FPT Inapproximability of Directed Cut and Connectivity Problems

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Abstract

Cut problems and connectivity problems on digraphs are two well-studied classes of problems from the viewpoint of parameterized complexity. After a series of papers over the last decade, we now have (almost) tight bounds for the running time of several standard variants of these problems parameterized by two parameters: the number \( k \) of terminals and the size \( p \) of the solution. When there is evidence of FPT intractability, then the next natural alternative is to consider FPT approximations. In this paper, we show two types of results for directed cut and connectivity problems, building on existing results from the literature: first is to circumvent the hardness results for these problems by designing FPT approximation algorithms, or alternatively strengthen the existing hardness results by creating "gap-instances" under stronger hypotheses such as the (Gap-)Exponential Time Hypothesis (ETH). Formally, we show the following results:

\textbf{Cutting paths between a set of terminal pairs, i.e., Directed Multicut}: Pilipczuk and Wahlstrom [TOCT ’18] showed that Directed Multicut is \( W[1] \)-hard when parameterized by \( p \) if \( k = 4 \). We complement this by showing the following two results:

- Directed Multicut has a \( k/2 \)-approximation in \( 2^{O(p^2)} \cdot n^{O(1)} \) time (i.e., a 2-approximation if \( k = 4 \)),
- Under Gap-ETH, Directed Multicut does not admit an \( (\frac{59}{58} - \varepsilon) \)-approximation in \( f(p) \cdot n^{O(1)} \) time, for any computable function \( f \), even if \( k = 4 \).

\textbf{Connecting a set of terminal pairs, i.e., Directed Steiner Network (DSN)}: The DSN problem on general graphs is known to be \( W[1] \)-hard parameterized by \( p + k \) due to Guo et al. [SIDMA ’11]. Dinur and Manurangsi [ITCS ’18] further showed that there is no FPT \( k^{1/4-o(1)} \)-approximation algorithm parameterized by \( k \), under Gap-ETH. Chitnis et al. [SODA ’14] considered the restriction to special graph classes, but unfortunately this does not lead to FPT algorithms either: DSN on planar graphs is \( W[1] \)-hard parameterized by \( k \). In this paper we consider the DSN\textsubscript{PLANAR} problem which is an intermediate version: the graph is general, but we want to find a solution whose cost is at most that of an optimal planar solution (if one exists). We show the following lower bounds for DSN\textsubscript{PLANAR}:

- DSN\textsubscript{PLANAR} has no \( (2 - \varepsilon) \)-approximation in FPT time parameterized by \( k \), under Gap-ETH. This answers in the negative a question of Chitnis et al. [ESA `18].
- DSN\textsubscript{PLANAR} is \( W[1] \)-hard parameterized by \( k + p \). Moreover, under ETH, there is no \( (1 + \varepsilon) \)-approximation for DSN\textsubscript{PLANAR} in \( f(k, p, \varepsilon) \cdot n^{o(k+\sqrt{p+1}/\varepsilon)} \) time for any computable function \( f \).

\textbf{Pairwise connecting a set of terminals, i.e., Strongly Connected Steiner Subgraph (SCSS)}: Guo et al. [SIDMA ’11] showed that SCSS is \( W[1] \)-hard parameterized by \( p + k \), while Chitnis et al. [SODA ’14] showed that SCSS remains \( W[1] \)-hard parameterized by \( p \), even if the input graph is planar. In this paper we consider the SCSS\textsubscript{PLANAR} problem which is an intermediate version: the graph is general, but we want to find a solution whose cost is at most that of an optimal planar solution (if one exists). We show the following lower bounds for SCSS\textsubscript{PLANAR}:

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• SCSS_{PLANAR} is W[1]-hard parameterized by \( k + p \). Moreover, under ETH, there is no \((1 + \varepsilon)\)-approximation for SCSS_{PLANAR} in \( f(k, p, \varepsilon) \cdot n^{o(\sqrt{k + p + \frac{1}{\varepsilon}})} \) time for any computable function \( f \).

Previously, the only known FPT approximation results for SCSS applied to general graphs parameterized by \( k \): a 2-approximation by Chitnis et al. [IPEC ’13], and a matching \((2 - \varepsilon)\)-hardness under Gap-ETH by Chitnis et al. [ESA ’18].

### 1 Introduction

Given a weighted directed graph \( G = (V, E) \) with two terminal vertices \( s, t \) the problems of finding a minimum weight \( s \leadsto t \) cut and a minimum weight \( s \leadsto t \) path can both be famously solved in polynomial time. There are two natural generalizations when we consider more than two terminals: either we look for connectivity/cuts between all terminals of a given set, or we look for connectivity/cuts between a given set of terminal pairs. This leads to the four problems of \textsc{Directed Multiway Cut}, \textsc{Directed Multicut}, \textsc{Strongly Connected Steiner Subgraph} and \textsc{Directed Steiner Network}:

- **Cutting all paths between a set of terminals:** In the \textsc{Directed Multiway Cut} problem, we are given a set of terminals \( T = \{t_1, t_2, \ldots, t_k\} \) and the goal is to find a minimum weight subset \( X \subseteq V \) such that \( G \setminus X \) has no \( t_i \leadsto t_j \) path for any \( 1 \leq i \neq j \leq k \).

- **Cutting paths between a set of terminal pairs:** In the \textsc{Directed Multicut} problem, we are given a set of terminal pairs \( T = \{(s_i, t_i)\}_{i=1}^k \) and the goal is to find a minimum weight subset \( X \subseteq V \) such that \( G \setminus X \) has no \( s_i \leadsto t_i \) path for any \( 1 \leq i \leq k \).

- **Connecting all terminals of a given set:** In the \textsc{Strongly Connected Steiner Subgraph} (SCSS) problem, we are given a set of terminals \( T = \{t_1, t_2, \ldots, t_k\} \) and the goal is to find a minimum weight subset \( X \subseteq V \) such that \( G[X] \) has a \( t_i \leadsto t_j \) path for every \( 1 \leq i \neq j \leq k \).

- **Connecting a set of terminal pairs:** In the \textsc{Directed Steiner Network} (DSN) problem, we are given a set of terminal pairs \( T = \{(s_i, t_i)\}_{i=1}^k \) and the goal is to find a minimum weight subset \( X \subseteq V \) such that \( G[X] \) has an \( s_i \leadsto t_i \) path for every \( 1 \leq i \leq k \).

All four of the aforementioned problems are known to be NP-hard, even for small values of \( k \). One way to cope with NP-hardness is to try to design polynomial time approximation algorithms with small approximation ratio. However, apart from \textsc{Directed Multiway Cut}, which admits a 2-approximation in polynomial time [36], all the other three problems are known to have strong lower bounds (functions of \( n \)) on the approximation ratio of polynomial time algorithms [16, 20, 26]. Another way to cope with NP-hardness is to design FPT algorithms. However, apart from \textsc{Directed Multiway Cut} which has an FPT algorithm parameterized by the size \( p \) of the cutset, all the other three problems are known to be W[1]-hard (and hence fixed-parameter intractable) parameterized by size \( p \) of the solution \( X \) plus the number \( k \) of terminals/terminal pairs. When neither of the paradigms of polynomial time approximation algorithms nor (exact) FPT algorithm seem to be successful, the next natural alternative is to try to design FPT approximation algorithms or show hardness of FPT approximation results.

In this paper, we consider the remaining three problems of \textsc{Directed Multicut}, \textsc{Strongly Connected Steiner Subgraph} and \textsc{Directed Steiner Network}, for which strong approximation and parameterized lower bounds exist, from the viewpoint of FPT approximation algorithms. We obtain two types of results for these three problems: the first is to circumvent the W[1]-hardness and polynomial-time inapproximability results for these problems by designing FPT approximation algorithms, and the second is to strengthen the existing W[1]-hardness by creating “gap-instances” under stronger hypotheses than \( \text{FPT} \neq \text{W}[1] \) such as (Gap-) Exponential Time Hypothesis (ETH). Throughout, we use \( k \) to denote number of terminals or terminal pairs and \( p \) to denote size of the solution. First, in Section 1.1, we give a brief overview of the current state-of-the-art results for each the three problems from the lens of polynomial time approximation algorithms, FPT algorithms, and FPT approximation algorithms followed by the formal statements of our results. Then, in Section 1.2 we describe the recent flux of results which have set up the framework of FPT hardness of approximation under (Gap-)ETH, and how we use it obtain our hardness results in this paper.
1.1 Previous work and our results

The Directed Multicut problem

Garg et al. [24] showed that Directed Multicut is NP-hard even for \( k = 2 \). The current best approximation ratio in terms of \( n \) is \( O(n^{1/23} \cdot \log^{O(1)} n) \) due to Agarwal et al. [1], and it is known that Directed Multicut is hard to approximate in polynomial time to within a factor of \( 2^{\Omega(\log^{1-\epsilon} n)} \) for any constant \( \epsilon > 0 \), unless \( \text{NP} \subseteq \text{ZPP} \) [16]. There is a simple \( k \)-approximation in polynomial time obtained by solving each terminal pair as a separate instance of min \( s \rightarrow t \) cut and then taking the union of all the \( k \) cuts. Chekuri and Madan [8] and later Lee [31] showed that this is tight: assuming the Unique Games Conjecture of Khot [29], it is not possible to approximate Directed Multicut better than factor \( k \) in polynomial time, for any fixed \( k \). On the FPT side, Marx and Razgon [35] showed that Directed Multicut is \( \text{W}[1] \)-hard parameterized by \( p \). For the case of bounded \( k \), Chitnis et al. [13] showed that Directed Multicut is FPT parameterized by \( p \) when \( k = 2 \), but Pilipczuk and Wahlstrom [38] showed that the problem remains \( \text{W}[1] \)-hard parameterized by \( p \) when \( k = 4 \). The status of Directed Multicut parameterized by \( p \) when \( k = 3 \) is an outstanding open question. We first obtain the following FPT approximation for Directed Multicut parameterized by \( p \), which beats any approximation obtainable when parameterizing by \( k \) (even in XP time) according to [8, 31]:

**Theorem 1.1.** The Directed Multicut problem admits an \([k/2]\)-approximation in \( 2^{O(p^2)} \cdot n^{O(1)} \) time.

The proof of the above theorem uses the FPT algorithm of Chitnis et al. [13, 15] for Directed Multiway Cut parameterized by \( p \) as a subroutine. Note that Theorem 1.1 gives an FPT \( 2 \)-approximation for Directed Multicut With 4 Pairs. We complement this upper bound with a constant factor lower bound for approximation ratio of any FPT algorithm for Directed Multicut With 4 Pairs.

**Theorem 1.2.** Under Gap-ETH, for any \( \epsilon > 0 \) and any computable function \( f \), there is no \( f(p) \cdot n^{O(1)} \) time algorithm that computes an \( \lfloor \frac{59}{38} - \epsilon \rfloor \)-approximation for Directed Multicut With 4 Pairs.

We did not optimize the constant \( 59/38 \) in order to keep the analysis simple: we believe it can be easily improved, but our techniques would not take it close to the upper bound of 2.

The Directed Steiner Network (DSN) problem

The DSN problem is known to be NP-hard, and furthermore even computing an \( O(2^{\log^{1-\epsilon} n}) \)-approximation is not possible [20] in polynomial time, unless \( \text{NP} \subseteq \text{DTIME}(n^{\text{polylog}(n)}) \). The best known approximation factors for polynomial time algorithms are \( O(n^{2/3+\epsilon}) \) and \( O(k^{1/2+\epsilon}) \) [5, 9, 22]. On the FPT side, Feldman and Ruhl [21] designed an \( n^{O(k^2)} \) algorithm for DSN (cf. [23]). Chitnis et al. [14] showed that the Feldman-Ruhl algorithm is tight: under ETH, there is no \( f(k) \cdot n^{o(k)} \) algorithm (for any computable function \( f \)) for DSN even if the input graph is a planar directed acyclic graph. Guo et al. [25] showed that DSN remains \( \text{W}[1] \)-hard even when parameterized by the larger parameter \( k + p \). Dinur and Manurangsi [19] further showed that DSN on general graphs has no FPT approximation algorithm with ratio \( k^{1/4-o(1)} \) when parameterized by \( k \), under Gap-ETH.

Chitnis et al. [11] considered two relaxations of the Directed Steiner Network problem: the \( B_1 \)-DSN problem where the input graph is bidirected\(^1\), and the DSN\(_{\text{PLANAR}} \) problem where the input graph is general but the goal is to find a solution whose cost is at most that of an optimal planar solution (if one exists). The main result of Chitnis et al. [11] is that although \( B_1 \)-DSN\(_{\text{PLANAR}} \) (i.e., the intersection of \( B_1 \)-DSN and DSN\(_{\text{PLANAR}} \)) is \( \text{W}[1] \)-hard parameterized by \( k + p \), it admits a parameterized approximation scheme: for any \( \epsilon > 0 \), there is a \( \max\{2^{2^{O(1/\epsilon)}}, 2^{2^{1/\epsilon}}\} \) time algorithm for \( B_1 \)-DSN\(_{\text{PLANAR}} \) which computes a \((1+\epsilon)\)-approximation. Such a parameterized approximation is not possible for \( B_1 \)-DSN as Chitnis et al. [11] showed that under Gap-ETH there is a constant \( \alpha > 0 \) such that there is no FPT \( \alpha \)-approximation.

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\(^1\)Bidirected graphs are directed graphs which have the property that for every edge \( u \rightarrow v \) in \( G \) the reverse edge \( v \rightarrow u \) exists in \( G \) as well and moreover has the same weight as \( u \rightarrow v \).
They asked whether a parameterized approximation scheme for the remaining variant of DSN, i.e., the DSN$_{\text{PLANAR}}$ problem, exists. We answer this question in the negative with the following lower bound

**Theorem 1.3.** Under Gap-ETH, for any $\varepsilon > 0$ and any computable function $f$, there is no $f(k) \cdot n^{O(1)}$ time algorithm that computes a $(2-\varepsilon)$-approximation for DSN$_{\text{PLANAR}}$, even if the input graph is a directed acyclic graph (DAG).

The W[1]-hardness proof of [14] for DSN on planar graphs parameterized by $k$ does not give hardness parameterized by $p$ since in that reduction the value of $p$ grows with $n$. Our next result shows that the slightly more general problem of DSN$_{\text{PLANAR}}$ (here the input graph is general, but we want to find a solution of cost $\leq p$ if there is a planar solution of size $\leq p$) is indeed W[1]-hard parameterized by $k+p$.

Also we obtain a lower bound for approximation schemes for this problem under ETH, i.e., under a weaker assumption than the one used for Theorem 1.3.\(^2\)

**Theorem 1.4.** The DSN$_{\text{PLANAR}}$ problem is W[1]-hard parameterized by $p+k$, even if the input graph is a directed acyclic graph (DAG). Moreover, under ETH, for any computable function $f$

- there is no $f(k,p) \cdot n^{o(k+\sqrt{p})}$ time algorithm for DSN$_{\text{PLANAR}}$, and
- there is no $f(k,\varepsilon,p) \cdot n^{o(k+\sqrt{p+1/\varepsilon})}$ time algorithm which computes a $(1+\varepsilon)$-approximation for DSN$_{\text{PLANAR}}$ for every $\varepsilon > 0$.

Note that just the W[1]-hardness of DSN$_{\text{PLANAR}}$ parameterized by $k+p$ already follows from [11] who showed that even the special case of B1-DSN$_{\text{PLANAR}}$ is W[1]-hard parameterized by $k+p$. However, this reduction from [11] was from $\ell$-Clique to an instance of B1-DSN$_{\text{PLANAR}}$ with $k = O(\ell^2)$ and $p = O(\ell^6)$, whereas Theorem 1.4 gives a reduction from $\ell$-Clique to DSN$_{\text{PLANAR}}$ with $k = O(\ell)$ and $p = O(\ell^2)$. This gives much improved lower bounds on the running times.

**The Strongly Connected Steiner Subgraph (SCSS) problem**

The SCSS problem is NP-hard, and the best known approximation ratio in polynomial time for SCSS is $k^\varepsilon$ for any $\varepsilon > 0$ [7]. A result of Halperin and Krauthgamer [26] implies SCSS has no $\Omega((\log^{2-\varepsilon} n)$-approximation for any $\varepsilon > 0$, unless NP has quasi-polynomial Las Vegas algorithms. On the FPT side, Feldman and Ruhl [21] designed an $n^{O(k)}$ algorithm for SCSS (cf. [23]). Chitnis et al. [14] showed that the Feldman-Ruhl algorithm is almost optimal: under ETH, there is no $f(k) \cdot n^{o(k/log k)}$ algorithm (for any computable function $f$) for SCSS. Guo et al. [25] showed that SCSS remains W[1]-hard even when parameterized by the larger parameter $k+p$. Chitnis et al. [11] showed that the SCSS problem restricted to bidirected graphs remains NP-hard, but is FPT parameterized by $k$. The SCSS problem admits a square-root phenomenon on planar graphs: Chitnis et al. [14] showed that SCSS on planar graphs has an $O(k \log k) \cdot n^{O(\sqrt{k})}$ algorithm, and under ETH there is a tight lower bound of $f(k) \cdot n^{o(\sqrt{k})}$ for any computable function $f$. The W[1]-hardness proof of [14] for SCSS on planar graphs parameterized by $k$ does not give hardness parameterized by $p$, since in that reduction the value of $p$ grows with $n$.

Our next result shows that the slightly more general problem of SCSS$\_{\text{PLANAR}}$ (here the input graph is general, but we want to find a solution of cost $\leq p$ if there is a planar solution of size $\leq p$) is indeed W[1]-hard parameterized by $k+p$. We also obtain a lower bound for approximation schemes for this problem under ETH:

**Theorem 1.5.** The SCSS$\_{\text{PLANAR}}$ problem is W[1]-hard parameterized by $p+k$. Moreover, under ETH, for any computable function $f$

- there is no $f(k,p) \cdot n^{o(\sqrt{k+p})}$ time algorithm for SCSS$\_{\text{PLANAR}}$, and
- there is no $f(k,\varepsilon,p) \cdot n^{o(\sqrt{k+p+1/\varepsilon})}$ time algorithm which computes a $(1+\varepsilon)$-approximation for SCSS$\_{\text{PLANAR}}$ for every $\varepsilon > 0$.

To the best of our knowledge, the only known FPT approximation results for SCSS applied to general graphs parameterized by $k$: a simple FPT 2-approximation due to Chitnis et al. [12], and a matching $(2-\varepsilon)$-hardness (for any constant $\varepsilon > 0$) under Gap-ETH due to Chitnis et al. [11].

\(^2\)In the following, $o(f(k,p,\varepsilon))$ means any function $g(f(k,p,\varepsilon))$ such that $g(x) \in o(x)$. 

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1.2 FPT inapproximability results under (Gap-)ETH

A standard hypothesis for showing lower bounds for running times of FPT and exact exponential time algorithms is the Exponential Time Hypothesis (ETH) of Impagliazzo and Paturi [27].

**Hypothesis 1.6. Exponential Time Hypothesis (ETH):** There exists a constant $\delta > 0$ such that no algorithm can decide whether any given 3-CNF formula is satisfiable in time $O(2^{\delta n})$ where $n$ denotes the number of clauses.

The original conjecture stated the lower bound as exponential in terms of the number of variables not clauses, but the above statement follows from the Sparsification Lemma of [28]. The Exponential Time Hypothesis has been used extensively to show a variety of lower bounds including those for FPT algorithms, exact exponential time algorithms, hardness of polynomial time approximation, and hardness of FPT approximation. We refer the interested reader to [32] for a survey on lower bounds based on ETH.

To show the W[1]-hardness of DSN$_{\text{PLANAR}}$ (Theorem 1.4) and SCSS$_{\text{PLANAR}}$ (Theorem 1.5) parameterized by $k + p$ we design parameterized reductions from $\ell$-Clique to these problems such that max$\{k, p\}$ is upper bounded by a function of $\ell$. Furthermore, by choosing $\varepsilon$ to be small enough such that computing an $(1 + \varepsilon)$-approximation is the same as computing the optimal solution, we also obtain runtime lower bounds for $(1 + \varepsilon)$-approximations for these two problems by translating the $f(\ell) \cdot n^{o(\ell)}$ lower bound for $\ell$-Clique [10] under ETH (for any computable function $f$).

Recently, a gap version of the ETH was proposed:

**Hypothesis 1.7. Gap-ETH [18, 33]:** There exists a constant $\delta > 0$ such that, given a 3CNF formula $\Phi$ on $n$ variables, no $2^{o(n)}$-time algorithm can distinguish between the following two cases correctly with probability at least $2/3$:

- $\Phi$ is satisfiable.
- Every assignment to the variables violates at least a $\delta$-fraction of the clauses of $\Phi$.

It is known [3, 6] that Gap-ETH follows from ETH given other standard conjectures, such as the existence of linear sized PCPs or exponentially-hard locally-computable one-way functions. We refer the interested reader to [6, 18] for a discussion on why Gap-ETH is a plausible assumption. In a breakthrough result, Chalermsook et al. [6] used Gap-ETH to show that the two famous parameterized intractable problems of Clique and Set Cover are completely inapproximable in FPT time parameterized by the size of the solution. In this paper, we obtain two hardness of approximation results (Theorem 1.2 and Theorem 1.3) based on Gap-ETH. The starting point of our hardness of approximation results are based on the recent results on parameterized inapproximability of the DENSEST $k$-SUBGRAPH problem. Recall that, in the DENSEST $k$-SUBGRAPH (D$k$S) problem [30], we are given an undirected graph $G = (V, E)$ and an integer $k$ and the goal is to find a subset $S \subseteq V$ of size $\ell$ that induces as many edges in $G$ as possible. Chalermsook et al. [6] showed that, under randomized Gap-ETH, there is no FPT approximation (parameterized by $k$) with ratio $k^{o(1)}$. This was improved recently by Dinur and Manurangsi [19] who showed better hardness and under deterministic Gap-ETH. We state their result formally:

**Theorem 1.8 ([19, Theorem 2]).** Under Gap-ETH, for any function $h(\ell) = o(1)$, there is no $f(\ell) \cdot n^{O(1)}$ time algorithm that, given a graph $G$ on $n$ vertices and an integer $k$, can distinguish between the following two cases:

- (YES) $G$ contains at least one $\ell$-clique as a subgraph.
- (NO) Every $\ell$-subgraph of $G$ contains less than $\ell^{h(\ell)-1} \cdot \binom{\ell}{2}$ edges.

Note that this result is essentially tight: there is a simple $O(\ell)$ approximation since the number of edges induced by a $\ell$-vertex subgraph is at most $\frac{\ell^2}{2}$ and at least $\frac{\ell}{2}$ (without loss of generality, we can assume there are no isolated vertices). Instead of working with D$k$S, we will reduce from a “colored” version of the problem called MAXIMUM COLORED SUBGRAPH ISOMORPHISM, which can be defined as follows.

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Dinur and Manurangsi [19] actually state their result for 2-CSPs
Theorem 1.10. Then, it runs the given algorithm \(B\) that \(B\) \(\lambda\). Otherwise, \(B\) the algorithm from Theorem 1.10 on \(f\) in time \(B\) \(f\) distinguishing problem stated in Corollary 1.11 in following two cases:

- for every subset \(S\), we set \(\lambda = \max\{val(\phi) : \phi \in \Lambda_{n,q}\}\).

The following constructions of special families of splitters are due to [2] and [37].

**Definition 1.9. (splitters)** Let \(n \geq r \geq s\). An \((n,s,r)\)-splitter is a family \(\Lambda\) of functions \(n \mapsto [r]\) such that for every subset \(S \subseteq [n]\) of size \(s\) there is a function \(\lambda \in \Lambda\) such that \(\lambda\) is injective on \(S\).

This problem is referred to as LABEL COVER in the hardness of approximation literature [4]. However, Chitnis et al. [11] used the name MAXIMUM COLORED SUBGRAPH ISOMORPHISM to be consistent with the naming conventions in the FPT community: this problem is an optimization version of COLORED SUBGRAPH ISOMORPHISM [34]. The graph \(H\) is sometimes referred to as the supergraph of \(\Gamma\). Similarly, the vertices and edges of \(H\) are called supernodes and superedges of \(\Gamma\). Moreover, the size of \(\Gamma\) is defined as \(n = |V_G|\), the number of vertices of \(G\). Additionally, for each assignment \(\phi\), we define its value \(\text{val}(\phi)\) to be the fraction of superedges \(i - j \in E_H\) such that \(\phi(i) - \phi(j) \in E_G\); such superedges are said to be covered by \(\phi\). The objective of MCSI is now to find an assignment \(\phi\) with maximum value. We denote the value of the optimal assignment by \(\text{val}(\Gamma)\), i.e., \(\text{val}(\Gamma) = \text{max}_\phi \text{val}(\phi)\).

Using Theorem 1.8 we derive the following two corollaries regarding hardness of approximation for MAXIMUM COLORED SUBGRAPH ISOMORPHISM when the supergraph \(H\) has special structure. These corollaries follow quite straightforwardly from Theorem 1.8 using the idea of splitters, but we provide proofs here for completeness.

**Theorem 1.10.** There exists a 2\(^{O(q)} \cdot n^{O(1)}\)-time algorithm that takes in \(n,q \in \mathbb{N}\) such that \(n \geq q\) and outputs \(n\), \(q\)-splitter family of functions \(\Lambda_{n,q}\) such that \(\left|\Lambda_{n,q}\right| = 2^{O(q)} \cdot \log n\).

**Corollary 1.11.** Assuming Gap-ETH, for any function \(h(\ell) = o(1)\), there is no \(f(\ell) \cdot n^{O(1)}\)-time algorithm that, given a MCSI instance \(\Gamma\) of size \(n\) such that the supergraph \(H = K_\ell\), can distinguish between the following two cases:

- \((\text{YES})\) \(\text{val}(\Gamma) = 1\).
- \((\text{NO})\) \(\text{val}(\Gamma) < \ell^{h(\ell)-1}\)

**Proof.** Suppose for the sake of contradiction that there exists an algorithm \(B\) that can solve the distinguishing problem stated in Corollary 1.11 in \(f(\ell) \cdot n^{O(1)}\) time for some computable function \(f\). We will use this to construct another algorithm \(B'\) that can solve the distinguishing problem stated in Theorem 1.8 in time \(f'(\ell) \cdot n^{O(1)}\) for some computable function \(f'\), which will thereby violate Gap-ETH.

The algorithm \(B'\), on input \((G, \ell)\), proceeds as follows. We assume w.l.o.g. that \(V = [n]\). First, \(B'\) runs the algorithm from Theorem 1.10 on \((n, \ell)\) to produce an \((n, \ell, \ell)\)-splitter family of functions \(\Lambda_{n,\ell}\). For each \(\lambda \in \Lambda_{n,\ell}\), it creates a MCSI instance \(\Gamma^\lambda = (G^\lambda, H^\lambda, V_1^\lambda \cup \cdots \cup V_\ell^\lambda)\) where

- the graph \(G^\lambda\) is simply the input graph \(G\),
- for each \(i \in [\ell]\), we set \(V_i^\lambda = \lambda^{-1}(\{i\})\), and,
- the supergraph \(H^\lambda\) is simply the complete graph on \([\ell]\), i.e., \(H^\lambda = ([\ell], \binom{[\ell]}{2})\).

Then, it runs the given algorithm \(B\) on \(\Gamma^\lambda\). If \(B\) returns YES for some \(\lambda \in \Lambda\), then \(B'\) returns YES. Otherwise, \(B'\) outputs NO.

It is obvious that the running time of \(B'\) is at most \(O(2^{O(q)} f(\ell) \cdot n^{O(1)})\). Moreover, if \(G\) contains an \(\ell\)-clique, say \((v_1, \ldots, v_\ell)\), then by the properties of splitters we are guaranteed that there exists \(\lambda^* \in \Lambda_{n,\ell}\) such that \(\lambda^*(\{v_1, \ldots, v_\ell\}) = [\ell]\). Hence, the assignment \(i \mapsto v_i\) covers all superedges in \(E_{H^\lambda}\), implying that \(B\) indeed outputs YES on such \(\Gamma^\lambda\). On the other hand, if every \(\ell\)-subgraph of \(G\) contains less than \(\ell^{h(\ell)-1} \cdot \binom{\ell}{2}\) edges, then, for any \(\lambda \in \Lambda_{n,\ell}\) and any assignment \(\phi\) of \(\Gamma^\lambda\), \((\phi(1), \ldots, \phi(\ell))\) induces less than
\(\ell^{(\ell)-1} \cdot \left(\frac{\ell}{2}\right)\) edges in \(G\). This also upper bounds the number of superedges covered by \(\phi\), which implies that \(\Gamma^k\) is a NO instance of Corollary 1.11. Thus, in this case, \(B\) outputs NO on all \(\Gamma^k\)'s. In other words, \(B\)'s can correctly distinguish the two cases in Theorem 1.8 in \(f'(\ell) \cdot n^{O(1)}\) time where \(f'(\ell) = 2^{O(\ell)} \cdot f(\ell)\). This concludes our proof of Corollary 1.11.

\[\text{Corollary 1.12. Assuming Gap-ETH, for any function } h(\ell) = o(1), \text{ there is no } f(\ell) \cdot n^{O(1)}-time algorithm that, given a MCSI instance } \Gamma \text{ of size } n \text{ such that the supergraph } H \text{ is the complete bipartite subgraph } K_{\frac{\ell}{2}, \frac{\ell}{2}}\text{, can distinguish between the following two cases:}
\]

- (YES) \(\text{val}(\Gamma) = 1\).
- (NO) \(\text{val}(\Gamma) < \ell^{(\ell)-1}\).

\[\text{Proof. Suppose for the sake of contradiction that there exists an algorithm } B \text{ that can solve the distinguishing problem stated in Corollary 1.12 in } f(\ell) \cdot n^{O(1)} \text{ time for some function } f. \text{ We will use this to construct another algorithm } B' \text{ that can solve the distinguishing problem stated in Theorem 1.8 in time } f'(\ell) \cdot n^{O(1)} \text{ for some computable function } f, \text{ which will thereby violate Gap-ETH.}
\]

The algorithm \(B'\), on input \((G, \ell)\), proceeds as follows. We assume w.l.o.g. that \(V = [n]\). First, \(B'\) runs the algorithm from Theorem 1.10 on \((n, \ell)\) to produce an \((n, \ell, \ell)-splitter family of functions \(A_{n, \ell}\). For each \(\lambda \in A_{n, \ell}\), it creates a MCSI instance \(\Gamma^\lambda = (G^\lambda, H^\lambda, V_1^\lambda \cup \cdots \cup V_{\ell}^\lambda)\) where

- the graph \(G^\lambda\) is simply the input graph \(G\).
- for each \(i \in [\ell]\), we set \(V_i^\lambda = \lambda^{-1}(\{i\})\), and,
- the supergraph \(H^\lambda\) is simply \(K_{\frac{\ell}{2}, \frac{\ell}{2}}\) where one side of the bipartition is \(\{1, 2, \ldots, \frac{\ell}{2}\}\) and the other side is \(\{\frac{\ell}{2} + 1, \frac{\ell}{2} + 2, \ldots, \ell\}\).

Then, it runs the given algorithm \(B\) on \(\Gamma^\lambda\). If \(B\) returns YES for some \(\lambda \in A\), then \(B'\) returns YES. Otherwise, \(B'\) outputs NO.

It is obvious that the running time of \(B'\) is at most \(O(2^{O(\ell)} f(\ell) \cdot n^{O(1)})\). Moreover, if \(G\) contains a \(\ell\)-clique, say \((v_1, \ldots, v_{\ell})\), then by the properties of splitters we are guaranteed that there exists \(\lambda^* \in A_{n, \ell}\) such that \(\lambda^*([v_1, \ldots, v_{\ell}]) = [\ell]\). Hence, the assignment \(i \mapsto v_i\) covers all superedges in \(E_{H^\lambda}\), implying that \(B\) indeed outputs YES on such \(\Gamma^\lambda\). On the other hand, if every \(\ell\)-subgraph of \(G\) contains less than \(\ell^{(\ell)-1} \cdot \left(\frac{\ell}{2}\right)\) edges, then, for any \(\lambda \in A_{n, \ell}\) and any assignment \(\phi\) of \(\Gamma^\lambda\), the mapping \((\phi(1), \ldots, \phi(\ell))\) induces less than \(\ell^{(\ell)-1} \cdot \left(\frac{\ell}{2}\right)^2\) edges in \(G\). This also upper bounds the number of superedges covered by \(\phi\). Since \(\ell^{(\ell)-1} \cdot \left(\frac{\ell}{2}\right) \geq \ell^{(\ell)-1} \cdot \left(\frac{\ell}{2}\right)^2\), it follows implies that \(\Gamma^\lambda\) is a NO instance of Corollary 1.12. Thus, in this case, \(B\) outputs NO on all \(\Gamma^\lambda\)'s. In other words, \(B\)'s can correctly distinguish the two cases in Theorem 1.8 in \(f'(\ell) \cdot n^{O(1)}\) time where \(f'(\ell) = 2^{O(\ell)} \cdot f(\ell)\). This concludes our proof of Corollary 1.12.

We prove Theorem 1.2 and Theorem 1.3 via reductions from Corollary 1.11 and Corollary 1.12 respectively.

## 2 FPT (In)Approximability of DIRECTED MULTICUT

In this section we design an FPT 2-approximation for DIRECTED MULTICUT WITH 4 PAIRS parameterized by \(p\) (Section 2.1) and complement this with a lower bound (Section 2.2) showing that no FPT algorithm (parameterized by \(p\)) for DIRECTED MULTICUT WITH 4 PAIRS can achieve a ratio of \((\frac{59}{58} - \varepsilon)\) under Gap-ETH.

### 2.1 FPT approximation algorithm

It is well-known that a \(k\)-approximation can be computed in polynomial time by taking union of min cuts of each of the \(k\) terminal pairs. Chekuri and Madan [8] and later Lee [31] showed that this approximation ratio is best-possible for polynomial time algorithms under the Unique Games Conjecture of Khot [29]. The same lower bound also applies for any constant \(k\), i.e., even an XP algorithm parameterized by \(k\) cannot compute a better approximation than a polynomial time algorithm. We now design an FPT
[\[k/2\]-approximation for DIRECTED MULTICUT. The idea is borrowed from the proof of Chitnis et al. [13] that DIRECTED MULTICUT WITH 2 PAIRS is FPT parameterized by \( p \).

**Theorem 1.1.** The DIRECTED MULTICUT problem admits a \([k/2]\)-approximation in \(2^{O(p^2)} \cdot n^{O(1)}\) time.

Formally, the algorithm takes an instance \((G, \mathcal{T})\) of DIRECTED MULTICUT and in \(2^{O(p^2)} \cdot n^{O(1)}\) time either concludes that there is no solution of cost at most \( p \), or produces a solution of cost at most \( p[k/2] \).

**Proof.** Let the pairs be \( \mathcal{T} = \{(s_i, t_i) : 1 \leq i \leq k\} \), and let OPT be the optimum value for the instance \((G, \mathcal{T})\) of DIRECTED MULTICUT. For now, assume that \( k \) is even. Introduce \( k/2 \) new vertices \( r_j, q_j \), for \( 1 \leq j \leq k/2 \), of weight \( p + 1 \) each, and add the following edges:

- \( r_j \rightarrow s_{2j-1} \) and \( t_{2j-1} \rightarrow q_j \)
- \( q_j \rightarrow s_{2j} \) and \( t_{2j} \rightarrow r_j \)

Let the resulting graph be \( G' \), and note that \( G \) has an \( s_i \rightarrow t_i \) path for some \( 1 \leq i \leq k \) if and only if \( G' \) has a \( q_{i/2} \rightarrow r_{i/2} \) or \( r_{(i-1)/2} \rightarrow q_{(i-1)/2} \) path (depending on whether \( i \) is even or odd). Since the vertices \( r_j, q_j \) have weight \( p + 1 \) each, it follows that \( G \) has a solution of size at most \( p \) for the instance \((G, \{(s_{2j-1}, t_{2j-1}, (s_{2j}, t_{2j})\})\) of DIRECTED MULTICUT if and only if \( G' \) has a solution of size at most \( p \) for the DIRECTED MULTIWAY CUT instance with input graph \( G \) and terminals \( r_j, q_j \). We use the algorithm of Chitnis et al. [13, 15] for DIRECTED MULTIWAY CUT which checks in \(2^{O(p^2)} \cdot n^{O(1)}\) time\(^4\) if there is a solution of cost at most \( p \). If there is no solution of cost at most \( p \) between \( r_j \) and \( q_j \) in \( G' \) then this implies that \( G \) has no cut of size at most \( p \) separating \( (s_{2j-1}, t_{2j-1}) \) and \( (s_{2j}, t_{2j}) \) and hence \( OPT > p \). Otherwise, there is a cut \( C_j \) in \( G \) of cost at most \( p \) which separates \( (s_{2j-1}, t_{2j-1}) \) and \( (s_{2j}, t_{2j}) \).

The output of the algorithm is the cut \( C = \bigcup_{j=1}^{k/2} C_j \). Clearly, if \( k \) is even then \( C \) is a feasible solution for the instance \((G, \mathcal{T})\) of DIRECTED MULTICUT with cost at most \( \sum_{j=1}^{k/2} \text{cost}(C_j) \leq pk/2 \). In case \( k \) is odd we use the above procedure for the terminal pairs \( \{(s_i, t_i) : 1 \leq i \leq k - 1\} \), and finally add a min cut between the last terminal pair \( (s_k, t_k) \). This results in the desired \([k/2]\)-approximation. \( \Box \)

### 2.2 No FPT \((\frac{59}{38} - \epsilon)\)-approximation under Gap-ETH

With the parameterized hardness of approximating MCSI ready, we can now prove our hardness results for DIRECTED MULTICUT with 4 terminal pairs.

**Theorem 1.2.** Under Gap-ETH, for any \( \epsilon > 0 \) and any computable function \( f \), there is no \( f(p) \cdot n^{O(1)} \) time algorithm that computes an \((\frac{59}{38} - \epsilon)\)-approximation for DIRECTED MULTICUT WITH 4 PAIRS.

Our proof of the parameterized inapproximability of DIRECTED MULTICUT WITH 4 PAIRS is based on a reduction from MAXIMUM COLORED SUBGRAPH ISOMORPHISM whose properties are described below.

**Lemma 2.1.** There exists a polynomial time reduction that, given an instance \( \Gamma = (G, K_\ell, V_1 \cup \cdots \cup V_\ell) \) of MCSI, produces an instance \((G', \mathcal{T}')\) of DIRECTED MULTICUT WITH 4 PAIRS such that

- **(Completeness):** If \( \text{val}(\Gamma) = 1 \), then there exists a solution \( N \subseteq V(G') \) of cost \( 29\ell^2 \) for the instance \((G', \mathcal{T}')\) of DIRECTED MULTICUT WITH 4 PAIRS
- **(Soundness):** If \( \text{val}(\Gamma) < \frac{1}{10} \), then every solution \( N \subseteq V(G') \) for the instance \((G', \mathcal{T}')\) of DIRECTED MULTICUT WITH 4 PAIRS has cost more than \( 29.5\ell^2 \).
- **(Parameter Dependency):** The size of the solution is \( p = O(\ell^2) \).

In the proof of Lemma 2.1, we actually use the same reduction as from [38], but with different weights. We reduce to the vertex-weighted variant of DIRECTED MULTICUT WITH 4 PAIRS where we have four different types of weights for the vertices:

- **Light** vertices (shown using gray color) which have weight \( B = \frac{\ell^2}{\binom{\ell}{2}} \)
- **Medium** vertices (shown using green color) which have weight \( 2B \)
- **Heavy** vertices (shown using orange color) which have weight \( 20\ell \)
- **Super-heavy** vertices (shown using white color) which have weight \( 100\ell^2 \)

\(^4\)This is independent of number of the terminals
2.2.1 Construction of the Directed Multicut With 4 Pairs instance

Without loss of generality (by adding isolated vertices if necessary) we can assume that $|V_i| = n$ for each $i \in [\ell]$. For each $i \in [\ell]$ let $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,n}\}$. Then $|V(G)| = nf$. We now describe the construction of the (vertex-weighted) Directed Multicut With 4 Pairs instance $(G', \mathcal{T}')$.

- Introduce eight terminals, arranged in four terminal pairs as follows:
  \[
  \mathcal{T}' = \{(s_0^{<,0}, s_0^{>0}), (s_0^{>,0}, s_0^{<0}), (s_n^{<,0}, s_n^{>0}), (s_n^{>,0}, s_n^{<0})\}
  \]

  Each of the 8 terminals is super-heavy.

- For every $1 \leq i \leq \ell$, we introduce a bidirected path on $2n + 1$ vertices (see Figure 2)
  \[
  Z_i := s_0^{>,0} \leftrightarrow s_1^{>,0} \leftrightarrow s_2^{>,0} \leftrightarrow \ldots \leftrightarrow s_n^{>,0} \leftrightarrow s_0^{>,0},
  \]
  called henceforth the z-path for color class $i$. For each $0 \leq a \leq n$ the vertex $s_a^{>,0}$ is super-heavy and for each $1 \leq a \leq n$ the vertex $s_a^{>,0}$ is heavy.

- For every pair $(i, j)$ where $1 \leq i, j \leq \ell$, $i \neq j$, we introduce two bidirected paths (see Figure 2 and Figure 1) on $2n + 1$ vertices
  \[
  X_{i,j} := x_0^{i,j} \leftrightarrow x_1^{i,j} \leftrightarrow x_2^{i,j} \leftrightarrow \ldots \leftrightarrow x_n^{i,j} \leftrightarrow x_0^{i,j}
  \]
  and
  \[
  Y_{i,j} := y_0^{i,j} \leftrightarrow y_1^{i,j} \leftrightarrow y_2^{i,j} \leftrightarrow \ldots \leftrightarrow y_n^{i,j} \leftrightarrow y_0^{i,j}
  \]
  We call these paths the x-path and the y-path for the pair $(i, j)$. For each $0 \leq a \leq n$ the vertices $x_a^{i,j}$ and $y_a^{i,j}$ are super-heavy. For each $1 \leq a \leq n$ the vertices $x_a^{i,j}$ and $y_a^{i,j}$ are medium.

- For every pair $(i, j)$ with $1 \leq i, j \leq \ell$, $i \neq j$, and every $0 \leq a \leq n$, we add arcs $(x_a^{i,j}, z_a^{i,j})$ and $(z_a^{i,j}, x_a^{i,j})$. See Figure 2 for an illustration.

- Furthermore, we attach terminals to the paths as follows: (shown using magenta edges in Figure 1 and Figure 2)
  - for every pair $(i, j)$ with $1 \leq i, j \leq \ell$, $i \neq j$, we add arcs $(s_0^{<,0}, x_0^{i,j})$ and $(y_0^{i,j}, t_0^{<0})$;
  - for every $1 \leq i \leq \ell$ we add arcs $(s_0^{<,0}, x_0^{i,j})$ and $(z_0^{>,0}, t_0^{<0})$;
  - for every pair $(i, j)$ with $1 \leq i < j \leq \ell$ we add arcs $(s_n^{<,0}, x_0^{i,j})$ and $(y_0^{i,j}, t_n^{<0})$;
  - for every pair $(i, j)$ with $\ell \leq i > j \geq 1$ we add arcs $(s_n^{<,0}, x_0^{i,j})$ and $(y_0^{i,j}, t_n^{<0})$.

- For every pair $(i, j)$ with $1 \leq i < j \leq \ell$ we introduce an acyclic $n \times n$ grid $P_{i,j}$ with vertices $p^{i,j}_{a,b}$ for $1 \leq a, b \leq n$ and arcs $(p^{i,j}_{a,b}, p^{i,j}_{a+1,b})$ for every $1 \leq a < n$ and $1 \leq b \leq n$, as well as $(p^{i,j}_{a,b}, p^{i,j}_{a+1,b+1})$ for every $1 \leq a \leq n$ and $1 \leq b < n$. We call this grid $P_{i,j}$ as the p-grid for the pair $(i, j)$. We set the vertex $p^{i,j}_{a,b}$ to be a light vertex if $v_a^i v_b^j \in E(G)$, and super-heavy otherwise. Finally, for every $1 \leq a \leq n$ we introduce the following arcs (shown as dotted in Figure 1):
  \[
  (x_0^{i,j}, p^{i,j}_{a,n}), (p^{i,j}_{a,n}, y_0^{i,j}), (x_0^{i,j}, p^{i,j}_{a,1}), (p^{i,j}_{a,1}, y_0^{i,j}).
  \]

This concludes the construction of the instance $(G', \mathcal{T}')$ of Directed Multicut With 4 Pairs. Note that $|V(G')| = (n + \ell)^{O(1)}$, and also $G'$ can be constructed in $(n + \ell)^{O(1)}$ time.

2.2.2 Completeness of Lemma 2.1: $\text{val}(\Gamma) = 1 \Rightarrow \text{Multicut of cost} \leq 29\ell^2$

Suppose that $\text{val}(\Gamma) = 1$, i.e., $G$ has an $\ell$-clique which has exactly one vertex in each $V_i$ for $1 \leq i \leq \ell$. Let this clique be given by $\{v^{i}_{a(i)} : 1 \leq i \leq \ell\}$. Define

\[
X = \{\hat{x}^{i,j}_{a(i)} : 1 \leq i, j \leq \ell, i \neq j\} \cup \{\hat{z}^{i,j}_{a(i)} : 1 \leq i \leq \ell\} \cup \{p^{i,j}_{a(i), a(j)} : 1 \leq i < j \leq \ell\}.
\]

Note that $X$ consists of exactly $\ell$ heavy $\hat{x}^{i,j}_{a(i)}$ vertices, $4(\ell^2)$ medium $\hat{z}^{i,j}_{a(i)}$ vertices, and $\ell^2$ light $p^{i,j}_{a(i), a(j)}$ vertices (the fact that $p^{i,j}_{a(i), a(j)}$ is light for every $1 \leq i < j \leq \ell$ follows from the assumption that the vertices $v^i_{a(i)}$ induce a clique in $G$). Hence, the weight of $X$ is exactly $\ell \cdot 20\ell + \left(\frac{\ell}{2}\right) \cdot (4 \cdot 2\ell^2) + \left(\frac{\ell}{2}\right) \cdot B = 20\ell^2 + \left(\frac{\ell}{2}\right) \cdot 9B = 29\ell^2$. As shown in [38], this set $X$ is a cutset for the instance $(G', \mathcal{T}')$ of Directed Multicut With 4 Pairs. For the sake of completeness, we repeat the arguments in Section A.
Figure 1: Illustration of the reduction for DIRECTED MULTICUT WITH 4 PAIRS. For $1 \leq i < j \leq \ell$, the grid $P_{i,j}$ is surrounded by the bidirectional paths $X_{i,j}$ on the left, $X_{j,i}$ on the top, $Y_{i,j}$ on the right and $Y_{j,i}$ on the bottom. Edges incident on terminals are shown in magenta. Green vertices are medium, orange vertices are heavy and white vertices are super-heavy. A desired solution is marked by red circles.
Figure 2: Illustration of the reduction for DIRECTED MULTICUT WITH 4 PAIRS. For every $1 \leq i < j \leq \ell$, the $z$-path $Z_i$ corresponding to the color class $i$ is surrounded by the bidirectional paths $X_{i,j}$ on the left and $Y_{i,j}$ on the right. Edges incident on terminals are shown in magenta. Green vertices are medium, orange vertices are heavy and white vertices are super-heavy.
2.2.3 Soundness of Lemma 2.1: Multicut of cost \( \leq 29.5\ell^2 \Rightarrow \text{val}(\Gamma) \geq \frac{1}{10} \)

Let \( \mathcal{X} \) be a solution to the instance \((G', \mathcal{X}')\) of DIRECTED MULTICUT WITH 4 PAIRS such that weight of \( \mathcal{X} \) is \( 29.5\ell^2 \). We now show that \( \text{val}(\Gamma) \geq \frac{1}{10} \).

**Observation 2.2.** Note that every super-heavy vertex has weight \( 100\ell^2 \) and hence \( \mathcal{X} \) cannot contain any super-heavy vertex.

**Lemma 2.3.** For each \( i \in [\ell] \), the solution \( \mathcal{X} \) contains at least one heavy vertex from \( Z_i \).

**Proof.** Note that there is a \( s_0^i \to t_0^i \) path as follows:
- \( s_0^i \to t_0^i \to z_0^i \)
- Use the z-path for color class \( i \) in one direction from \( z_0^i \) to \( z_n^i \)
- From Observation 2.2, we know that \( \mathcal{X} \) cannot contain any super-heavy vertex. Each vertex from the set \( \{ s_0^i, t_0^i, t^i \} \) is super-heavy. Hence, \( X \) must contain at least one heavy vertex from \( Z_i \). □

**Lemma 2.4.** For each \( 1 \leq i \neq j \leq \ell \), the solution \( \mathcal{X} \) contains at least one medium vertex from \( X_{i,j} \) and at least one medium vertex from \( Y_{i,j} \).

**Proof.** There is a path from \( s_0^i \) to \( t_0^j \) that traverses the entire x-path for the pair \((i, j)\) up to the vertex \( x_{i,j} \), and then uses the arc \((x_{i,j}, z_{i,j})\) to reach \( t_{0}^{i} \). From Observation 2.2, we know that \( \mathcal{X} \) cannot contain any super-heavy vertex. Each vertex from the set \( \{ s_0^i, t_0^i, t^i \} \) is super-heavy. Hence, \( X \) must contain at least one medium vertex from \( X_{i,j} \).

There is a path from \( s_0^i \) to \( t_0^j \) that starts with using the arc \((z_{0}^{i}, x_{0}^{j})\), and then traverses the y-path for the pair \((i, j)\) up to the vertex \( y_{i,j} \). From Observation 2.2, we know that \( \mathcal{X} \) cannot contain any super-heavy vertex. Each vertex from the set \( \{ s_0^i, t_0^i, t^i \} \) is super-heavy. Hence, \( X \) must contain at least one medium vertex from \( Y_{i,j} \). □

**Definition 2.5.** An integer \( i \in [\ell] \) is *good* if \( \mathcal{X} \) contains exactly one heavy vertex from the z-path for the color class \( i \), i.e., \( |\mathcal{X} \cap Z_i| = 1 \). In this case, we say that \( v^h_i \) be the unique vertex from the z-path for class \( i \) in the solution \( \mathcal{X} \).

**Lemma 2.6.** Let \( \text{GOOD} = \{ i \in [\ell] : i \text{ is good} \} \). Then \( |\text{GOOD}| \geq \frac{37\ell}{40} \).

**Proof.** From Lemma 2.3 we have a contribution of at least \( \ell \cdot 20\ell = 20\ell^2 \) towards weight of \( \mathcal{X} \) by heavy vertices. From Lemma 2.4 we have a contribution of at least \( 2\ell(\ell - 1) \cdot 2B = 8\ell^2 \) towards weight of \( \mathcal{X} \) by medium vertices.

By Lemma 2.3, every \( i \not\in \text{GOOD} \) must contribute at least two heavy vertices to \( \mathcal{X} \). Hence, we have
\[
29.5\ell^2 \geq \text{weight of } \mathcal{X} \geq 20\ell^2 + 8\ell^2 + (\ell - |\text{GOOD}|) \cdot (20\ell)
\]
Hence, \( |\text{GOOD}| \geq \frac{37\ell}{40} \). □

**Definition 2.7.** Let \( 1 \leq i < j \leq \ell \). We say that the pair \((i, j)\) is *great* if \( \mathcal{X} \) contains
- exactly one medium vertex from the x-path for the pair \((i, j)\)
- exactly one medium vertex from the y-path for the pair \((i, j)\)
- exactly one medium vertex from the x-path for the pair \((j, i)\)
- exactly one medium vertex from the y-path for the pair \((j, i)\)
- exactly one light vertex from the p-grid for the pair \((i, j)\)

Let \( \text{GOOD}-P A I R S = \{ (i, j) : 1 \leq i < j \leq \ell, i, j \in \text{GOOD} \} \)

**Lemma 2.8.** Let \( 1 \leq i < j \leq \ell \). If both \( i \) and \( j \) are good, and the pair \((i, j)\) is great then \( v^h_i - v^h_j \in E(G) \).
Lemma 2.4 we have a contribution of at least $\ell\cdot 8 \ell^2$ towards weight of $OE$. From Lemma 2.4 we have a contribution of at least $\ell \cdot 20\ell = 20\ell^2$ towards weight of $EO$, i.e., for each $1 \leq i \neq j \leq \ell$ the set $OE_{i,j}$ has at least four medium vertices.

\begin{proof}
Fix a pair $(i, j)$ with $1 \leq i < j \leq \ell$. Since $i, j \in \text{GOOD}$ we have that $OE \cap B_i = \{z_{\beta_i}^i\}$ and $OE \cap B_j = \{z_{\beta_j}^j\}$. Since $(i, j)$ is great, let

\begin{itemize}
  \item $OE \cap X_{i,j} = \text{for some } z_{\alpha(i)}^{i,j}$
  \item $OE \cap Y_{i,j} = \text{for some } z_{\alpha(j)}^{i,j}$
  \item $OE \cap X_{j,i} = \text{for some } z_{\alpha(i)}^{j,i}$
  \item $OE \cap Y_{j,i} = \text{for some } z_{\alpha(j)}^{j,i}$
\end{itemize}

Observe that $a \leq \beta_i$, as otherwise the path from $s_0^{i,n}$ to $t_0^{i,n}$ that traverses the x-path for the pair $(i, j)$ up to the vertex $x_{a-1}^{i,j}$, uses the arc $(x_{a-1}^{i,j}, x_{a}^{i,j})$, and traverses the z-path for the color class $i$ up to the endpoint $z_{\alpha(i)}^{i,j}$ is not cut by $OE$, a contradiction. A similar argument for the terminal pair $(s_0^{j,n}, t_0^{j,n})$ implies that $\beta_j \leq c$. However, if $a < \beta_i$, then the path from $s_0^{j,n}$ to $t_0^{j,n}$ that traverses the x path for the pair $(i, j)$ up to the vertex $x_{a}^{i,j}$, uses the arc $(x_{a}^{i,j}, z_{\alpha(i)}^{i,j})$, traverses the z-path for the color class $i$ up to the endpoint $z_{\alpha(i)}^{i,j}$, and finally uses the arc $(z_{\alpha(i)}^{i,j}, y_{\alpha(j)}^{j,i})$, is not cut by $OE$, a contradiction. A similar argument gives a contradiction if $\beta_i > c$. Hence, we have that $a = \beta_i = c$. Similarly, we can show that $n' = \beta_j = c'$.

Recall that $OE \cap P_{i,j} = p_{i,j}^{i,j}$. Hence, this vertex $p_{i,j}^{i,j}$ must hit each of the following two paths which were not cut by the heavy or medium vertices in $OE$:

\begin{itemize}
  \item A path $P_1$ from $s_0^{i,n}$ to $t_0^{i,n}$ that traverses the x-path for the pair $(i, j)$ up to the vertex $x_{a(i)}^{i,j}$, uses the arc $(x_{a(i)}^{i,j}, p_{\alpha(i)}^{i,j})$, traverses the $\alpha(i)$-th row of the $p$-grid for the pair $(i, j)$ up to the vertex $p_{\alpha(i)}^{i,j}$, uses the arc $(p_{\alpha(i)}^{i,j}, y_{\alpha(i)}^{j,i})$, and traverses the y-path for the pair $(i, j)$ up to the endpoint $y_{\alpha(i)}^{j,i}$.
  \item A path $P_2$ from $s_0^{j,n}$ to $t_0^{j,n}$ that traverses the z-path for the pair $(j, i)$ up to the vertex $x_{a(j)}^{j,i}$, uses the arc $(x_{a(j)}^{j,i}, p_{\alpha(j)}^{j,i})$, traverses the $\alpha(j)$-th column of the p-grid for the pair $(i, j)$ up to the vertex $p_{\alpha(j)}^{j,i}$, uses the arc $(p_{\alpha(j)}^{j,i}, y_{\alpha(j)}^{i,j})$, and traverses the y-path for the pair $(j, i)$ up to the endpoint $y_{\alpha(j)}^{i,j}$.
\end{itemize}

However, the only vertex in common of the two aforementioned paths for a fixed choice of $(i, j)$, $1 \leq i < j \leq n$, is the vertex $p_{\beta(i),\beta(j)}^{i,j}$. Hence, $\mu = \beta_i$ and $\delta = \beta_j$. By Observation 2.2, it follows that $p_{\beta(i),\beta(j)}^{i,j}$ is light, i.e., $p_{\beta(i),\beta(j)}^{i,j} \in E(G)$. \hfill $\square$

\textbf{Definition 2.9.} Let $1 \leq i < j \leq \ell$. We define $OE_{i,j} = OE \cap (X_{i,j} \cup X_{j,i} \cup Y_{i,j} \cup Y_{j,i} \cup P_{i,j})$

\textbf{Lemma 2.10.} Let $1 \leq i < j \leq \ell$ be such that $i, j \in \text{GOOD}$. Then either

- the pair $(i, j)$ is great and weight of $OE_{i,j}$ is exactly $9B$, or
- weight of $OE_{i,j}$ is at least $10B$

\textbf{Proof.} By Lemma 2.4, we know that $OE_{i,j}$ contains at least one medium vertex from each of the four paths $X_{i,j}, X_{j,i}, Y_{i,j}, Y_{j,i}$. Hence, $OE_{i,j}$ contains at least four medium vertices. If $OE_{i,j}$ contains at least 5 medium vertices then its weight is at least $5 \cdot (2B) = 10B$. Hence, suppose that $OE_{i,j}$ contains exactly four medium vertices. We now consider how many light vertices from the p-grid for the pair $(i, j)$ are present in $OE_{i,j}$:

- $OE_{i,j}$ contains exactly one light vertex from the p-grid for the pair $(i, j)$, i.e., the pair $(i, j)$ is great.
  \begin{itemize}
    \item Note that in this case the weight of $OE_{i,j}$ is exactly $4 \cdot (2B) + B = 9B$
  \end{itemize}
- Otherwise $OE_{i,j}$ contains at least two light vertices from the p-grid for the pair $(i, j)$.
  \begin{itemize}
    \item In this case, the weight of $OE_{i,j}$ is at least $4 \cdot (2B) + B + B = 10B$.
  \end{itemize}
\hfill $\square$

\begin{lemma}
Let $\mathcal{E} = \{1 \leq i < j \leq \ell : i, j \in \text{GOOD and } (i, j) \text{ is great}\}$. Then $|\mathcal{E}| \geq \frac{1}{20} \cdot \binom{\ell}{2}$
\end{lemma}

\textbf{Proof.} From Lemma 2.3 we have a contribution of at least $\ell \cdot 20\ell = 20\ell^2$ towards weight of $OE$. From Lemma 2.4 we have a contribution of at least $\binom{\ell}{2} \cdot 8B = 8\ell^2$ towards weight of $OE$, i.e., for each $1 \leq i \neq j \leq \ell$ the set $OE_{i,j}$ has at least four medium vertices.
We again prove by contrapositive. Suppose that, for some constant \( \varepsilon > 0 \) and for some computable function \( f(p) \) independent of \( n \), there exists an \( f(p) \cdot n^{O(1)} \)-time \((\frac{9}{58} - \varepsilon)\)-approximation algorithm for \( G \). We create an algorithm \( B \) that can distinguish between the two cases of Corollary 1.11 with \( h(\ell) = 1 - \frac{\log(10)}{\log \ell} = o(1) \). Our new algorithm \( B \) works as follows. Given an instance \((G, H, V_1 \cup \cdots \cup V_\ell)\) of MCSI where \( H = K_\ell \), the algorithm \( B \) uses the reduction from Lemma 2.1 to create a Directed Multicut With 4 Pairs instance \((G', \mathcal{X})\) with 4 terminal pairs. \( B \) then runs \( A \) on this instance with \( p = 29\ell^2 \); if \( A \) returns a solution \( N \) of cost less than \( 29.5\ell^2 \), then \( B \) returns YES. Otherwise, \( B \) returns NO.

To see that algorithm \( B \) can indeed distinguish between the YES and NO cases, first observe that, in the YES case the completeness property of Lemma 2.1 guarantees that the optimal solution has cost at most \( 29\ell^2 \). Since \( A \) is a \((\frac{9}{58} - \varepsilon)\)-approximation algorithm, it returns a solution of cost at most \( (\frac{9}{58} - \varepsilon) \cdot 29\ell^2 < 29.5\ell^2 \): this means that \( B \) outputs YES. On the other hand, if \((G, H, V_1 \cup \cdots \cup V_\ell)\) is a NO instance, i.e., \( \text{val}(\Gamma) < \frac{1}{10} = \varepsilon h(\ell)^{-1} \), then the soundness property of Lemma 3.1 guarantees that the optimal solution in \( G' \) has cost more than \( 29.5\ell^2 \) (which is greater than \((\frac{9}{58} - \varepsilon) \cdot 29\ell^2 \)) and hence \( B \) correctly outputs NO.

Finally, observe that the running time of \( B \) is \( f(p) \cdot |V(G')|^{O(1)} + (|V(G)| + \ell)^{O(1)} \) time needed to construct \( G' \). Since \(|V(G')| = (|V(G)| + \ell)^{O(1)} \) and \( p = O(\ell^2) \) it follows that the total running time is \( g(\ell) \cdot |V(G)| \) for some computable function \( g \). Hence, from Corollary 1.11, Gap-ETH is violated.

### 3 FPT inapproximability for DSNP

#### 3.1 \((2 - \varepsilon)\)-hardness for FPT approximation under Gap-ETH

The goal of this section is to show the following theorem:
Theorem 1.3. Under Gap-ETH, for any $\varepsilon > 0$ and any computable function $f$, there is no $f(k) \cdot n^{O(1)}$ time algorithm that computes a $(2 - \varepsilon)$-approximation for DSN$_{PLANAR}$.

3.1.1 Reduction from Colored Biclique to DSN$_{PLANAR}$

Lemma 3.1. For every constant $\gamma > 0$, there exists a polynomial time reduction that, given an instance $\Gamma = (G, H, V_1 \cup \cdots \cup V_\ell, W_1, W_2, \ldots, W_\ell)$ of MCSI where the supergraph $H$ is $K_{\ell,\ell}$, produces an instance $(G', \mathcal{D}')$ of DSN$_{PLANAR}$, such that

- (Completeness) If $\text{val}(\Gamma) = 1$, then there exists a planar network $N \subseteq G'$ of cost $2(1 + \gamma^{1/5})$ that satisfies all demands.
- (Soundness) If $\text{val}(\Gamma) < \gamma$, then every network $N \subseteq G'$ that satisfies all demands has cost more than $2(2 - 4\gamma^{1/5})$.
- (Parameter Dependency) The number of demand pairs $k = |\mathcal{D}'|$ is $2\ell$.

Lemma 3.1 is proven as follows: we construct the DSN$_{PLANAR}$ instance in Section 3.1.1.2. The proofs of completeness and soundness of the reduction are deferred Section 3.1.1.3 and Section 3.1.1.4 respectively. First, we construct a “path gadget” which we use repeatedly in our construction.

3.1.1.1 Constructing a directed “path” gadget

For every integer $n$ we define the following gadget $P_n$ which contains $2^n$ vertices (see Figure 3). Since we need many of these gadgets later on, we will denote vertices of $P_n$ by $P_n(v)$ etc., in order to be able to distinguish vertices of different gadgets. All edges will have the same weight $B$, which we will fix later during the reductions. The gadget $P_n$ is constructed as follows: $P_n$ has a directed path of one edge corresponding to each $i \in [n]$. This is given by $P_n(0_i) \rightarrow P_n(1_i)$

Figure 3: The construction of the path gadget for $P_n$. Note that the gadget has $2n$ vertices. Each edge of $P_n$ has the same weight $B$.

3.1.1.2 Construction of the DSN$_{PLANAR}$ instance

We give a reduction which transforms an instance $G = (V, E)$ of MCSI($K_{\ell,\ell}$) into an instance of DSN which has $2\ell$ demand pairs and an optimum which is planar. Let the partition of $V$ into color classes be given by $\{V_1, V_2, \ldots, V_\ell, W_1, W_2, \ldots, W_\ell\}$. Without loss of generality (by adding isolated vertices if necessary), we can assume that each color class has the same number of vertices. Let $|V_i| = |W_i| = n'$ for each $1 \leq i \leq \ell$. Then $n = |V(G)| = 2n' \ell$. For each $1 \leq i, j \leq \ell$ we denote by $E_{i,j}$ the set of edges with one end-point in $V_i$ and other in $W_j$. 

15
We design two types of gadgets: the main gadget and the secondary gadget. The reduction from MCSI($K_{\ell, \ell}$) represents each edge set $E_{i,j}$ with a main gadget $M_{i,j}$. This is done as follows: each main gadget is a copy of the path gadget $\mathcal{P}_{|E_{i,j}|}$ from Section 3.1.1.1 with $B = \frac{2}{\ell}$, i.e., there is a row in $M_{i,j}$ corresponding to each edge in $E_{i,j}$. Each main gadget is surrounded by four secondary gadgets: on the top, right, bottom and left. Each of these gadgets are copies of the path gadget from Section 3.1.1.1 with $B = 0$:

- For each $1 \leq i \leq \ell + 1, 1 \leq j \leq \ell$ the horizontal gadget $HS_{i,j}$ is a copy of $\mathcal{P}_{|w_i|}$.
- For each $1 \leq i \leq \ell, 1 \leq j \leq \ell + 1$ the vertical gadget $VS_{i,j}$ is a copy of $\mathcal{P}_{|v_j|}$.

We refer to Figure 4 (bird’s-eye view) and Figure 5 (zoomed-in view) for an illustration of the reduction. Fix some $1 \leq i, j \leq \ell$. The main gadget $M_{i,j}$ has four secondary gadgets surrounding it:

- Above $M_{i,j}$ is the vertical secondary gadget $VS_{i,j+1}$.
- On the right of $M_{i,j}$ is the horizontal secondary gadget $HS_{i+1,j}$.
- Below $M_{i,j}$ is the vertical secondary gadget $VS_{i,j}$.
- On the left of $M_{i,j}$ is the horizontal secondary gadget $HS_{i,j}$.

Hence, there are $\ell(\ell + 1)$ horizontal secondary gadgets and $\ell(\ell + 1)$ vertical secondary gadgets.

Red intra-gadget edges: Fix $(i, j)$ such that $1 \leq i, j \leq \ell$. Recall that $M_{i,j}$ is a copy of $\mathcal{P}_{|E_{i,j}|}$ with $B = \frac{2}{\ell}$ and each of the secondary gadgets are copies of $\mathcal{P}_5$ with $B = 0$. With slight abuse of notation, we assume that the rows of $M_{i,j}$ are indexed by the set $\{(x,y) : (x,y) \in E_{i,j}, x \in W_i, y \in V_j\}$. We add the following edges (in red color) of weight 0: for each $(x,y) \in E_{i,j}$

- Add the edge $VS_{i,j+1}(1_x) \rightarrow M_{i,j}(0_{(x,y)})$. These edges are called top-red edges incident on $M_{i,j}$.
- Add the edge $HS_{i,j}(1_x) \rightarrow M_{i,j}(0_{(x,y)})$. These edges are called left-red edges incident on $M_{i,j}$.
- Add the edge $M_{i,j}(1_{(x,y)}) \rightarrow HS_{i+1,j}(0_b)$. These edges are called right-red edges incident on $M_{i,j}$.
- Add the edge $M_{i,j}(1_{(x,y)}) \rightarrow VS_{i,j}(0_c)$. These edges are called bottom-red edges incident on $M_{i,j}$.

These are called the intra-gadget edges incident on $M_{i,j}$.

Introduce the following $4\ell$ vertices (which we call border vertices):

- $a_1, a_2, \ldots, a_\ell$
- $b_1, b_2, \ldots, b_\ell$
- $c_1, c_2, \ldots, c_\ell$
- $d_1, d_2, \ldots, d_\ell$

Orange edges: For each $i \in [\ell]$ add the following edges (shown as orange in Figure 4) with weight $2^{\ell/5} / 4\ell$:

- $a_i \rightarrow VS_{\ell,\ell+1}(0_v)$ for each $v \in V_i$. These are called top-orange edges.
- $VS_{i,1}(1_v) \rightarrow b_i$ for each $v \in V_i$. These are called bottom-orange edges.
- $c_j \rightarrow HS_{i,1}(0_w)$ for each $w \in W_j$. These are called left-orange edges.
- $HS_{\ell,\ell+1}(1_w) \rightarrow d_j$ for each $w \in W_j$. These are called right-orange edges.

Finally, the set of demand pairs $\mathcal{D}'$ is given by:

- Type I: the pairs $(a_i, b_i)$ for each $1 \leq i \leq \ell$.
- Type II: the pairs $(c_j, d_j)$ for each $1 \leq j \leq \ell$.

Clearly, the total number of demand pairs is $k = |\mathcal{D}'| = 2\ell$. Let the final graph constructed be $G'$. Note that $G'$ has size $N = (n + \ell)^{O(1)}$ and can be constructed in $(n + \ell)^{O(1)}$ time. It is also easy to see that $G'$ is actually a DAG.

3.1.1.3 Completeness of the reduction in Lemma 3.1

If val($\Gamma$) = 1, then there exist $(v_1, v_2, \ldots, v_\ell) \in V_1 \times V_2 \times \cdots \times V_\ell$ and $(w_1, w_2, \ldots, w_\ell) \in W_1 \times W_2 \times \cdots \times W_\ell$ that induce a $K_{\ell,\ell}$. We now build a planar solution $N$ for the DSNPLANAR instance $(G', \mathcal{D}')$ as follows:

- For each $i \in [\ell]$ pick the orange edges $a_i \rightarrow VS_{\ell,\ell+1}(0_v)$ and $VS_{i,1}(1_v) \rightarrow b_i$.
- For each $j \in [\ell]$ pick the orange edges $c_j \rightarrow HS_{i,1}(0_w)$ and $HS_{\ell,\ell+1}(1_w) \rightarrow d_j$.
- For each $1 \leq i, j \leq \ell$ pick the black edge $MS_{i,j}(0_{v,w}) \rightarrow MS_{i,j}(1_{v,w})$ in the main gadget. This edge is guaranteed to exist since $v_i - w_j \in E_{i,j}$. Also pick the four red edges with one endpoint in $M_{i,j}$ given by $VS_{i,j+1}(1_x) \rightarrow MS_{i,j}(1_{v,w}), HS_{i,j}(1_w) \rightarrow MS_{i,j}(0_{v,w}), MS_{i,j}(1_{v,w}) \rightarrow HS_{i+1,j}(0_w)$ and $MS_{i,j}(1_{v,w}) \rightarrow VS_{i,j}(0_v)$.
Figure 4: A bird’s-eye view of the instance of $G'$ with $\ell = 3$ and $n' = 4$ (see Figure 5 for a zoomed-in view). Additionally we have some red edges between each main gadget and the four secondary gadgets surrounding it which are omitted in this figure for clarity (they are shown in Figure 5 which gives a more zoomed-in view).
Figure 5: A zoomed-in view of the main gadget $M_{i,j}$ surrounded by four secondary gadgets: vertical gadget $VS_{i,j+1}$ on the top, horizontal gadget $HS_{i,j}$ on the left, vertical gadget $VS_{i,j}$ on the bottom and horizontal gadget $HS_{i+1,j}$ on the right. Each of the secondary gadgets is a copy of the uniqueness gadget $U_n$ (see Section 3.1.1.1) and the main gadget $M_{i,j}$ is a copy of the uniqueness gadget $U_{|S_{i,j}|}$. The only inter-gadget edges are the red edges: they have one end-point in a main gadget and the other end-point in a secondary gadget. We have shown four such red edges which are introduced for every $(x,y) \in E_{i,j}$. 
For each $1 \leq i \leq \ell, 1 \leq j \leq \ell + 1$ pick the edge $V S_{i,j}(v_i) \rightarrow V S_{i,j}(1_{v_i})$.

For each $1 \leq i \leq \ell, 1 \leq j \leq \ell$ pick the edge $HS_{i,j}(w_{0_j}) \rightarrow HS_{i,j}(1_{w_{0_j}})$.

Note that red edges and edges in secondary gadgets have weight 0. Hence, the weight of $N$ is $4\ell \cdot \left( \frac{2\gamma^{1/3}}{4\ell} \right) + \ell^2 \cdot \left( \frac{2\gamma}{\ell^2} \right) = 2(1 + \gamma^{1/5})$ since we pick $4\ell$ orange edges and one black edge from each of the $\ell^2$ main gadgets.

We next show that $N$ satisfies all the demand pairs. Consider the pair $(c_j, d_j)$ for some $j \in [\ell]$. There is a $c_j \rightsquigarrow d_j$ path as follows: start with the edge $c_j \rightarrow HS_{1,j}(0_{w_j})$. Then for each $1 \leq i \leq \ell$ use the following edges in this order:

- Traverse the gadget $HS_{i,j}$ using the edge $HS_{i,j}(0_{w_j}) \rightarrow HS_{i,j}(1_{w_j})$
- Reach the main gadget $MS_{i,j}$ using the edge $HS_{i,j}(1_{w_j}) \rightarrow MS_{i,j}(v_{w_j})$
- Traverse the main gadget $MS_{i,j}$ using the edge $MS_{i,j}(v_{w_j}) \rightarrow MS_{i,j}(1_{v_{w_j}})$
- Reach the gadget $MS_{i+1,j}$ using the edge $MS_{i,j}(1_{v_{w_j}}) \rightarrow MS_{i,j}(0_{w_j})$

This way we have reached the vertex $HS_{\ell+1,j}(0_{w_j})$. Finally use the two edges $HS_{\ell+1,j}(0_{w_j}) \rightarrow HS_{\ell+1,j}(1_{w_j})$ and $HS_{\ell+1,j}(1_{w_j}) \rightarrow d_j$. The proof for the satisfiability of $a_i \rightsquigarrow b_j$ paths is similar.

Finally, we show that $N$ is planar. It is easy to see that removing the red edges from $G'$ leads to a planar graph (see Figure 4 for a planar embedding of this graph). It remains to show that the red edges we add in $N$ do not destroy planarity. For any main gadget $M_{i,j}$: the only red edges from $G'$ which are added in $N$ are as follows: one left-red edge and one top-red edge incident on the same 0-vertex of $M_{i,j}$ and one bottom-red edge and one right-red edge incident on the same 1-vertex of $M_{i,j}$. This can clearly be done while preserving planarity: the only 4 red edges retained in $N$ are shown as in Figure 5 (note that Figure 5 is actually supposed to have many more red edges which are omitted for clarity).

3.1.1.4 Soundness of the reduction in Lemma 3.1

Our soundness proof will be by contrapositive. Suppose that there exists a planar solution $N$ of cost $\rho \leq 2(2 - 4\gamma^{1/5})$ that satisfies all the demand pairs. We first define the following sets:

- $L_j := \{ w \in W_j : c_j \rightarrow HS_{1,j}(0_{w_j}) \in E(N) \}$ for each $j \in [\ell]$.
- $L := \bigcup_{j=1}^{\ell} L_j$.
- $R_j := \{ w \in W_j : HS_{\ell+1,j}(1_{w_j}) \rightarrow d_j \in E(N) \}$ for each $j \in [\ell]$.
- $R := \bigcup_{j=1}^{\ell} R_j$.
- $T_i := \{ v \in V_i : a_i \rightarrow V S_{i+1}(0_{v_i}) \in E(N) \}$ for each $i \in [\ell]$.
- $T := \bigcup_{i=1}^{\ell} T_i$.
- $B_i := \{ v \in V_i : V S_{i+1}(1_{v_i}) \rightarrow b_i \in E(N) \}$ for each $i \in [\ell]$.
- $B := \bigcup_{i=1}^{\ell} B_i$.
- $\mathcal{M}_j := \{ MS_{i,j}(0_{x,y}) \rightarrow MS_{i,j}(1_{x,y}) \in E(N) : x \in V_i, y \in W_j, x - y \in E_{i,j} \}$ for each $1 \leq i, j \leq \ell$.

Let $\alpha_W = |L| + |R|$ and $\alpha_V = |T| + |B|$. Since each orange edge has weight $\frac{2\gamma^{1/5}}{4\ell}$ it follows that

$$\max\{\alpha_W, \alpha_V\} \leq \alpha_W + \alpha_V \leq \frac{\rho}{(2\gamma^{1/5}/4\ell)} \leq \frac{4\ell \gamma^{1/5}}{(2\gamma^{1/5}/4\ell)} \leq 8\ell \gamma^{-1/5}$$

Claim 3.2. For each $1 \leq i, j \leq \ell$ we have

- $L_j \cap R_j \neq \emptyset$
- $T_i \cap B_i \neq \emptyset$
- $\mathcal{M}_j \neq \emptyset$

Proof. Fix any $j \in [\ell]$, and let $P$ be any $c_j \rightsquigarrow d_j$ path in $N$. By orientation of the edges of $G'$, it is easy to see that $P$ cannot contain any vertex of the vertical gadget $V S_{i,j+1}$ for any $i \in [\ell]$. Also, $P$ cannot contain any vertex of a vertical gadget $V S_{i,j}$ for any $i \in [\ell]$; due to the orientation of the edges, there is no path from $V S_{i,j}$ to $d_j$. Hence, each internal (i.e., non-orange) edge of $P$ has both end-points in the vertex set given by $\bigcup_{i=1}^{\ell+1} HS_{i,j} \cup \bigcup_{i=1}^{\ell+1} MS_{i,j}$.

Since $P$ is a $c_j \rightsquigarrow d_j$ path, the first edge of $P$ is an orange edge incident on $c_j$. Let this edge be to the vertex $HS_{1,j}(0_{w_j})$ for some $w_j \in W_j$. Recall that edges which have one endpoint in a main gadget and other the end-point in a horizontal gadget (see Figure 5) are of one of the following two types:

19
Claim 3.3. Let $\phi$ be at least the probability that $y$ is a subset of the set $\{i, j\}$ such that $x - y \in E_{i,j}$

Moreover, every 0-vertex of a horizontal gadget or main gadget has exactly one out-neighbor, which is its corresponding 1-vertex. Hence, each edge in $P$ with both end-points in a horizontal gadget must correspond to $w^* \in W_j$, which implies that the last edge of $P$ must be $HS_{i+1,j}(1_{w^*}) \rightarrow d_j$. Therefore, $w^* \in L_j \cap R_j$, and so $L_j \cap R_j \neq \emptyset$. Similarly, for each $i \in [\ell]$ we have $T_i \cap B_i \neq \emptyset$.

The argument above shows that any $c_j \sim d_j$ path in $G'$ uses an edge from each $M_{i,j}$ for $1 \leq i \leq \ell$.

Hence, $\mathcal{H}_{i,j} \neq \emptyset$ for each $1 \leq i, j \leq \ell$.

Next, recall that each edge in a main gadget has weight $\frac{2}{\ell}$. Since $N$ has cost $\rho$, we have

$$\sum_{1 \leq i,j \leq \ell} |\mathcal{H}_{i,j}| \leq \frac{\ell^2}{2} \cdot \rho \Rightarrow \sum_{1 \leq i,j \leq \ell} (|\mathcal{H}_{i,j}| - 1) \leq \frac{\ell^2}{2} \cdot (\rho - 2)$$

Since $\mathcal{H}_{i,j} \neq \emptyset$ for every $1 \leq i, j \leq \ell$, the above inequality implies that, for at least $\ell^2 - \frac{\ell^2}{2} \cdot (\rho - 2) = \frac{\ell^2}{2} \cdot (4 - \rho) \geq 8\gamma^{1/5} \ell^2 = 4\gamma^{1/5} \ell^2$ pairs of $(i,j)$’s we have $|\mathcal{H}_{i,j}| = 1$. Let $\mathcal{P}_{\text{unique}}$ be the set of all such pairs of $(i,j)$’s.

We will argue that a random assignment defined from picking one vertex from each $L_j \cap R_j$ (for $j \in [\ell]$) uniformly independently at random and one vertex from each $T_i \cap B_i$ (for $i \in [\ell]$) uniformly independently at random covers many superedges in expectation. To do this, we need to first show that, for many $(i,j)$’s, there exist $x \in (T_i \cap B_i)$ and $y \in (L_j \cap R_j)$ such that $x - y \in E_G$. In fact, we can show this for every $(i,j) \in \mathcal{P}_{\text{unique}}$ as stated below.

Claim 3.3. Let $1 \leq i, j \leq \ell$. If $(i,j) \in \mathcal{P}_{\text{unique}}$, then there exists $x \in (T_i \cap B_i)$ and $y \in (L_j \cap R_j)$ such that $x - y \in E_{i,j} \subseteq E_G$.

Proof. Since $(i,j) \in \mathcal{P}_{\text{unique}}$, the solution $N$ contains exactly one edge from the main gadget $M_{i,j}$. Let this edge be $M_{i,j}(0_{x,y}) \rightarrow M_{i,j}(1_{x,y})$. We now claim that $y \in (L_j \cap R_j)$ and $x \in (T_i \cap B_i)$. This would complete the proof since the existence of the edge $M_{i,j}(0_{x,y}) \rightarrow M_{i,j}(1_{x,y})$ implies $x - y \in E_{i,j}$.

We now show that $y \in L_j \cap R_j$. The proof for $x \in T_i \cap B_i$ is similar. Recall that in the proof of Claim 3.2, we have (implicitly) shown that the only set of edges from $M_{i,j}$ that can be used as part of the solution $N$ is a subset of the set $\{M_{i,j}(0_{x,y}) \rightarrow M_{i,j}(1_{x,y}) : v \in V_i, w \in V_j, v - w \in E_{i,j}\}$. Since the only edge from $M_{i,j}$ in the solution $N$ is $M_{i,j}(0_{x,y}) \rightarrow M_{i,j}(1_{x,y})$, it follows that $y \in L_j$. Similarly, $y \in R_j$ and hence $y \in L_j \cap R_j$.

Now, let $\phi : [2\ell] = V_H \rightarrow V_G$ be a random assignment obtained as follows:

- For each $1 \leq i \leq \ell$, choose $\phi(i)$ independently uniformly at random from $T_i \cap B_i$
- For each $1 \leq j \leq \ell$, choose $\phi(j + \ell)$ independently uniformly at random from $L_j \cap R_j$

By Claim 3.3, for every $(i,j) \in \mathcal{P}_{\text{unique}}$, there exists $v \in (T_i \cap B_i)$ and $w \in (L_j \cap R_j)$ such that $v - w \in E_{i,j} \subseteq E_G$. This means that, for such $(i,j)$, the probability that the superedge $i - j \in E_H$ is covered is at least the probability that $\phi(i) = v$ and $\phi(j) = w$, which is equal to $\frac{1}{|T_i \cap B_i||L_j \cap R_j|}$. As a result,
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We can now easily prove Theorem 1.3 by combining Lemma 3.1 and Corollary 1.12.

Gap-ETH is violated.

solution as well, if it exists) in soundness property of Lemma 3.1 guarantees that the optimal solution (and hence the planar optimal γ returns a solution of cost at most 2

Corollary 1.12. In the YES case, i.e., val

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Proof of Theorem 1.3.

We again prove by contrapositive. Suppose that, for some constant ε > 0 and for some function f(k) independent of n, there exists an f(k) · N0(1)-time (2 − ε)-approximation algorithm for DSNPLANAR where k is the number of terminal pairs and N is the size of the instance. Let us call this algorithm $\mathcal{A}$.

Given ε > 0, it is easy to see that there exists a sufficiently small $\gamma' = \gamma'(\epsilon)$ such that $\frac{2(2-4\gamma^{1/5})}{2(1+\gamma^{1/5})} \geq (2-\epsilon)$. We create an algorithm $\mathcal{B}$ that can distinguish between the two cases of Corollary 1.12 with $h(\ell) = 1 - \log log(1/\gamma') = o(1)$. Our new algorithm $\mathcal{B}$ works as follows. Given an instance $(G, H, V_1 \cup \cdots \cup V_{\ell}, W_1 \cup \cdots \cup W_{\ell})$ of MCSI of size n where $H = K_{\ell, \ell}$, the algorithm $\mathcal{B}$ uses the reduction from Lemma 3.1 to create in $(n + \ell)^{O(1)}$ time a DSNPLANAR instance on the graph $G'$ with $k = 2\ell$ terminal pairs and size $N = (\ell + n)^{O(1)}$. The algorithm $\mathcal{B}$ then runs $\mathcal{A}$ on this instance; if $\mathcal{A}$ returns a solution $N$ of cost at most $2(2-4\gamma^{1/5})$, then $\mathcal{B}$ returns YES. Otherwise, $\mathcal{B}$ returns NO.

We now show that the algorithm $\mathcal{B}$ can indeed distinguish between the YES and NO cases of Corollary 1.12. In the YES case, i.e., val($\Gamma$) = 1, the completeness property of Lemma 3.1 guarantees that the optimal planar solution has cost at most $2(1 + \gamma^{1/5})$. Since $\mathcal{A}$ is a $(2-\epsilon)$-approximation algorithm, it returns a solution of cost at most $2(1 + \gamma^{1/5}) \cdot (2-\epsilon) \leq 2(2-4\gamma^{1/5})$ where the inequality comes from our choice of $\gamma'$; this means that $\mathcal{B}$ outputs YES. On the other hand, in the NO case, i.e., val($\Gamma$) < $\gamma$, the soundness property of Lemma 3.1 guarantees that the optimal solution (and hence the planar optimal solution as well, if it exists) in $G'$ has cost more than $2(2-4\gamma^{1/5})$, which implies that $\mathcal{B}$ outputs NO.

Finally, observe that the running time of $\mathcal{B}$ is $f'(k) \cdot N^{O(1)} + \text{poly}(\ell + n)^{O(1)}$ which is bounded by $f''(\ell) \cdot n^{O(1)}$ for some computable function $f''$ since $k = 2\ell$ and $N = (n + \ell)^{O(1)}$. Hence, from Corollary 1.12, Gap-ETH is violated.

\[ \sum_{(i,j) \in \mathcal{P}_{\text{unique}}} \frac{1}{|T_i \cap B_j| \cdot |L_j \cap R_j|} \geq \frac{1}{|\mathcal{P}_{\text{unique}}|} \left( \frac{|\mathcal{P}_{\text{unique}}|^3}{(\ell \cdot |T|) \cdot (\ell \cdot |L|)} \right) \]

Hence, there exists an assignment of $\Gamma$ with value at least $\gamma$, which implies that val($\Gamma$) ≥ $\gamma$. This completes the proof of Lemma 3.1.

3.1.2 Finishing the proof of Theorem 1.3

We can now easily prove Theorem 1.3 by combining Lemma 3.1 and Corollary 1.12.

Proof of Theorem 1.3. We again prove by contrapositive. Suppose that, for some constant $\epsilon > 0$ and for some function $f(k)$ independent of $n$, there exists an $f(k) \cdot N^{O(1)}$-time $(2-\epsilon)$-approximation algorithm for DSNPLANAR where $k$ is the number of terminal pairs and $N$ is the size of the instance. Let us call this algorithm $\mathcal{A}$.

Given $\epsilon > 0$, it is easy to see that there exists a sufficiently small $\gamma' = \gamma'(\epsilon)$ such that $\frac{2(2-4\gamma^{1/5})}{2(1+\gamma^{1/5})} \geq (2-\epsilon)$. We create an algorithm $\mathcal{B}$ that can distinguish between the two cases of Corollary 1.12 with $h(\ell) = 1 - \log log(1/\gamma') = o(1)$. Our new algorithm $\mathcal{B}$ works as follows. Given an instance $(G, H, V_1 \cup \cdots \cup V_{\ell}, W_1 \cup \cdots \cup W_{\ell})$ of MCSI of size $n$ where $H = K_{\ell, \ell}$, the algorithm $\mathcal{B}$ uses the reduction from Lemma 3.1 to create in $(n + \ell)^{O(1)}$ time a DSNPLANAR instance on the graph $G'$ with $k = 2\ell$ terminal pairs and size $N = (\ell + n)^{O(1)}$. The algorithm $\mathcal{B}$ then runs $\mathcal{A}$ on this instance; if $\mathcal{A}$ returns a solution $N$ of cost at most $2(2-4\gamma^{1/5})$, then $\mathcal{B}$ returns YES. Otherwise, $\mathcal{B}$ returns NO.

We now show that the algorithm $\mathcal{B}$ can indeed distinguish between the YES and NO cases of Corollary 1.12. In the YES case, i.e., val($\Gamma$) = 1, the completeness property of Lemma 3.1 guarantees that the optimal planar solution has cost at most $2(1 + \gamma^{1/5})$. Since $\mathcal{A}$ is a $(2-\epsilon)$-approximation algorithm, it returns a solution of cost at most $2(1 + \gamma^{1/5}) \cdot (2-\epsilon) \leq 2(2-4\gamma^{1/5})$ where the inequality comes from our choice of $\gamma'$; this means that $\mathcal{B}$ outputs YES. On the other hand, in the NO case, i.e., val($\Gamma$) < $\gamma$, the soundness property of Lemma 3.1 guarantees that the optimal solution (and hence the planar optimal solution as well, if it exists) in $G'$ has cost more than $2(2-4\gamma^{1/5})$, which implies that $\mathcal{B}$ outputs NO.

Finally, observe that the running time of $\mathcal{B}$ is $f'(k) \cdot N^{O(1)} + \text{poly}(\ell + n)^{O(1)}$ which is bounded by $f''(\ell) \cdot n^{O(1)}$ for some computable function $f''$ since $k = 2\ell$ and $N = (n + \ell)^{O(1)}$. Hence, from Corollary 1.12, Gap-ETH is violated. □
3.2 Lower Bounds for FPT Approximation Schemes for DSN_{PLANAR}

We obtain the following result regarding the parameterized complexity of DSN_{PLANAR} parameterized by $k + p$.

**Theorem 1.4.** The DSN_{PLANAR} problem is W[1]-hard parameterized by $p + k$. Moreover, under ETH, for any computable function $f$ and any $\varepsilon > 0$

- There is no $f(k, p) \cdot n^{O(k + \sqrt[p]{p})}$ time algorithm for DSN_{PLANAR}, and
- There is no $f(k, \varepsilon, p) \cdot n^{O(k + \sqrt[p+1]{p} + 1/\varepsilon)}$ time algorithm which computes a $(1 + \varepsilon)$-approximation for DSN_{PLANAR}

We reduce from the Grid Tiling problem:

\[
(\ell, n) - \text{GRID TILING}
\]

**Input:** Integers $\ell, n$, and $\ell^2$ non-empty sets $S_{i,j} \subseteq [n] \times [n]$ where $1 \leq i, j \leq \ell$

**Question:** For each $1 \leq i, j \leq \ell$ does there exist a value $y_{i,j} \in S_{i,j}$ such that

- If $y_{i,j} = (x, y)$ and $y_{i,j+1} = (x', y')$ then $x = x'$.
- If $y_{i,j} = (x, y)$ and $y_{i+1,j} = (x', y')$ then $y = y'$.

![Figure 6](image-url) An instance of Grid Tiling with $\ell = 3, n = 5$ with a solution highlighted in red. Note that in a solution, all entries from a row agree in the second coordinate and all entries from a column agree in the first coordinate.

See Figure 6 for an example of an instance of Grid Tiling. We use the same construction as for Lemma 3.1, but with different weights. We design two types of gadgets: the main gadget and the secondary gadget. We represent each set $S_{i,j}$ with a main gadget $M_{i,j}$ as follows: each main gadget is a copy of the path gadget $\mathcal{P}_{|S_{i,j}|}$ from Section 3.1.1.1 with $B = 1$, i.e., there is a row in $M_{i,j}$ corresponding to each element from $S_{i,j}$. Each main gadget is surrounded by four secondary gadgets: on the top, right, bottom and left. Each of these gadgets are copies of the path gadget from Section 3.1.1.1:

- For each $1 \leq i \leq \ell + 1, 1 \leq j \leq \ell$ the horizontal gadget $HS_{i,j}$ is a copy of $\mathcal{P}_n$ with $B = 1$
- For each $1 \leq i \leq \ell, 1 \leq j \leq \ell + 1$ the vertical gadget $VS_{i,j}$ is a copy of $\mathcal{P}_n$ with $B = 1$

We refer to Figure 4 (bird’s-eye view) and Figure 5 (zoomed-in view) for an illustration of the reduction. Fix some $1 \leq i, j \leq \ell$. The main gadget $M_{i,j}$ has four secondary gadgets surrounding it:

- Above $M_{i,j}$ is the vertical secondary gadget $VS_{i,j+1}$
- On the right of $M_{i,j}$ is the horizontal secondary gadget $HS_{i+1,j}$
- Below $M_{i,j}$ is the vertical secondary gadget $VS_{i,j}$
- On the left of $M_{i,j}$ is the horizontal secondary gadget $HS_{i+1,j}$

Hence, there are $\ell(\ell + 1)$ horizontal secondary gadgets and $\ell(\ell + 1)$ vertical secondary gadgets.

**Red intra-gadget edges:** Fix $(i, j)$ such that $1 \leq i, j \leq \ell$. Recall that $M_{i,j}$ is a copy of $\mathcal{P}_{|S_{i,j}|}$ with $B = 1$ and each of the secondary gadgets are copies of $\mathcal{P}_n$ with $B = 1$. With slight abuse of notation, we assume that the rows of $M_{i,j}$ are indexed by the set $\{(x, y) \in S_{i,j} : x, y \in [n]\}$. We add the following edges (in red color) of weight 1: for each $(x, y) \in S_{i,j}$

- Add the edge $VS_{i,j+1}(1_x) \rightarrow M_{i,j}(0_{x,y})$. These edges are called top-red edges incident on $M_{i,j}$. 

22
• Add the edge $HS_{i,j}(1_{x_j}) \rightarrow M_{i,j}(0_{(x,y)})$. These edges are called left-red edges incident on $M_{i,j}$.
• Add the edge $M_{i,j}(1_{(x,y)}) \rightarrow HS_{i+1,j}(0_y)$. These edges are called right-red edges incident on $M_{i,j}$.
• Add the edge $M_{i,j}(1_{(x,y)}) \rightarrow VS_{i,j}(0_1)$. These edges are called bottom-red edges incident on $M_{i,j}$.

These are called the intra-gadget edges incident on $M_{i,j}$.

Introduce the following 4 edges (which we call border vertices):

- $a_1, a_2, \ldots, a_{\ell}$
- $b_1, b_2, \ldots, b_{\ell}$
- $c_1, c_2, \ldots, c_{\ell}$
- $d_1, d_2, \ldots, d_{\ell}$

**Orange edges:** For each $i \in [\ell]$ add the following edges (shown as orange in Figure 4) with weight 1:

- $a_i \rightarrow VS_{i,\ell+1}(0_v)$ for each $v \in V_i$. These are called top-orange edges.
- $VS_{i,1}(1_v) \rightarrow b_i$ for each $v \in V_i$. These are called bottom-orange edges.
- $c_j \rightarrow HS_{1,j}(0_w)$ for each $w \in W_j$. These are called left-orange edges.
- $HS_{\ell+1,j}(1_w) \rightarrow d_j$ for each $w \in W_j$. These are called right-orange edges.

Finally, the set of demand pairs $\mathcal{D}'$ is given by:

- **Type I:** the pairs $(a_i, b_i)$ for each $1 \leq i \leq \ell$.
- **Type II:** the pairs $(c_j, d_j)$ for each $1 \leq j \leq \ell$.

Let the final graph constructed be $G'$. Note that $G'$ has size $N = (n + \ell)O(1)$ and can be constructed in $(n + \ell)O(1)$ time. It is also easy to see that $G'$ is actually a DAG.

Fix the budget $B' = 6\ell + 7\ell^2 = O(\ell^2)$. We now show that the instance $(\ell, n, \{S_{i,j} : i, j \in [\ell]\})$ of $(\ell, n)$-GRID TILING answers YES if and only the instance $(G', \mathcal{D}')$ of DSNPLANAR has a solution of cost at most $B'$.

### 3.2.1 Grid Tiling answers YES $\Rightarrow$ Instance $(G', \mathcal{D}')$ of DSNPLANAR has a planar solution of cost $\leq B'$

Suppose that GRID TILING has a solution, i.e., for each $1 \leq i, j \leq \ell$ there is a value $(x_{i,j}, y_{i,j}) = \gamma_{i,j} \in S_{i,j}$ such that

- for every $i \in [\ell]$, we have $x_{i,1} = x_{i,2} = x_{i,3} = \ldots = x_{i,\ell} = \alpha_i$, and
- for every $j \in [\ell]$, we have $y_{1,j} = y_{2,j} = y_{3,j} = \ldots = y_{\ell,j} = \beta_j$.

We now build a planar solution $N$ for the $B_1$-DSN instance $(G', \mathcal{D}')$ and show that it has weight at most $B'$. In the edge set $N$, we take the following edges:

1. For each $i \in [\ell]$ pick the edges
   - Top-orange edge $a_i \rightarrow VS_{i,\ell+1}(0_{\alpha_i})$
   - Bottom-orange edge $VS_{i,1}(1_{\alpha_i}) \rightarrow b_i$
   - Left-orange edge $c_j \rightarrow HS_{1,j}(0_{\beta_j})$
   - Right-orange edge $HS_{\ell+1,j}(1_{\beta_j}) \rightarrow d_j$
   This incurs a cost of $4\ell$ since each orange edge has cost 1.

2. For each $1 \leq i, j \leq \ell$ for the main gadget $M_{i,j}$, pick the edge $MS_{i,j}(0_{\alpha_i, \beta_j}) \rightarrow MS_{i,j}(1_{\alpha_i, \beta_j})$ of weight 1. Note that this edge exists because $(\alpha_i, \beta_j) \in S_{i,j}$ for each $1 \leq i, j \leq \ell$ because GRID TILING answers YES. Additionally we also pick the following four red edges (each of which has weight 1):
   - $VS_{i,j}(1_{\alpha_i}) \rightarrow M_{i,j}(0_{\alpha_i, \beta_j})$
   - $HS_{i,j}(1_{\beta_j}) \rightarrow M_{i,j}(0_{\alpha_i, \beta_j})$
   - $VS_{i,j}(0_{\alpha_i}) \leftarrow M_{i,j}(1_{\alpha_i, \beta_j})$
   - $HS_{\ell+1,j}(0_{\beta_j}) \leftarrow M_{i,j}(1_{\alpha_i, \beta_j})$
   This incurs a cost of $\ell^2 + 4\ell^2 = 5\ell^2$.

3. For each $1 \leq j \leq \ell + 1$ and $1 \leq i \leq \ell$ for the vertical secondary gadget $VS_{i,j}$, pick the edge $VS_{i,j}(0_{\alpha_i}) \rightarrow VS_{i,j}(1_{\alpha_i})$ which has weight 1. This incurs a cost of $\ell(\ell + 1)$.

4. For each $1 \leq j \leq \ell$ and $1 \leq i \leq \ell + 1$ for the vertical secondary gadget $VS_{i,j}$, pick the edge $HS_{i,j}(0_{\beta_j}) \rightarrow HS_{i,j}(1_{\beta_j})$ which has weight 1. This incurs a cost of $\ell(\ell + 1)$.
Hence, the weight of $N$ is $4\ell + \ell^2 + 4\ell^2 + \ell(\ell + 1) + \ell(\ell + 1) = B^*$. We now argue that $N$ is planar. It is easy to see that removing the red edges from $G'$ leads to a planar graph (see Figure 4 for a planar embedding of this graph). It remains to show that the red edges we add in $N$ do not destroy planarity. For any main gadget $M_{i,j}$: the only red edges from $G'$ which are added in $N$ are as follows: one left-red edge and one top-red edge incident on the same 0-vertex of $M_{i,j}$ and one bottom-red edge and one right-red edge incident on the same 1-vertex of $M_{i,j}$. This can clearly be done while preserving planarity: the only 4 red edges retained in $N$ are shown as in Figure 5 (note that Figure 5 is actually supposed to have many more red edges which are omitted for clarity).

It remains to show that $N$ is indeed a solution for the $\text{DSN}_{\text{PLANAR}}$ instance $(G', \mathcal{D}')$. We show that each demand pair of Type I is satisfied. Fix $i \in [\ell]$. Then there is an $a_i \rightarrow b_i$ path in $N$ given by the following edges:

- $a_i \rightarrow VS_{i,\ell+1}(0_{\alpha})$
- For each $\ell + 1 \geq r \geq 2$ use the path $VS_{i,r}(0_{\alpha}) \rightarrow VS_{i,r}(1_{\alpha}) \rightarrow M_{i,r-1}(0_{\alpha},\beta_{r-1}) \rightarrow M_{i,r-1}(1_{\alpha},\beta_{r-1}) \rightarrow VS_{i,r-1}(0_{\alpha})$.
- Finally use the path $VS_{i,1}(0_{\alpha}) \rightarrow VS_{i,1}(1_{\alpha}) \rightarrow b_i$

The argument showing that each demand pair of Type II is satisfied in $N$ is very similar, and we omit the details here.

3.2.2 Instance $(G', \mathcal{D}')$ of $\text{DSN}_{\text{PLANAR}}$ has a solution of cost $\leq B^* \Rightarrow \text{GRID TILING}$ answers YES

Suppose that the instance $(G', \mathcal{D}')$ of $\text{DSN}_{\text{PLANAR}}$ has a solution $N$ of cost at most $B^* = 6\ell + 7\ell^2$. We will now show that this implies that $\text{GRID TILING}$ answers YES. This implies that if $\text{GRID TILING}$ answers NO then the cost of an optimal solution (and hence the cost of an optimal planar solution, if one exists) is greater than $B^*$.

**Lemma 3.4.** $N$ contains at least $4\ell$ orange edges. In fact, for each $1 \leq i \leq \ell$ we have that $N$ contains at least one

- outgoing orange edge from $a_i$
- incoming orange edge into $b_i$
- outgoing orange edge from $c_j$
- incoming orange edge into $d_j$

**Proof.** The terminal pair $(a_i, b_i)$ is in $\mathcal{D}'$ for each $i \in [\ell]$. Since the only outgoing edges from $a_i$ are top-orange edges, it follows that $N$ contains at least one orange edge outgoing from $a_i$. The other three claims follow by similar arguments. $\square$

For each $j \in [\ell]$, we define

$$\text{HORIZONTAL}(j) = \{c_j, d_j\} \cup \left( \bigcup_{i \in [\ell]} M_{i,j} \right) \cup \left( \bigcup_{i \in [\ell+1]} HS_{i,j} \right)$$

**Lemma 3.5.** For every $j \in [\ell]$ any $c_j \rightarrow d_j$ path must have all edges in $G'[\text{HORIZONTAL}(j)]$.

**Proof.** Observe that the graph $G'$ is a DAG. Any $c_j \rightarrow d_j$ path starts with $c_j$ which is a vertex from $\text{HORIZONTAL}(j)$. The only incoming edges into $\text{HORIZONTAL}(j)$ are the in-vertical red edges from the set of vertices $\bigcup_{i=1}^{\ell} VS_{i,j+1}$ and the only outgoing edges from $\text{HORIZONTAL}(j)$ are the out-vertical red edges to the set of vertices $\bigcup_{i=1}^{\ell} VS_{i,j}$. Hence, no $c_j \rightarrow d_j$ path can leave the vertex set $\text{HORIZONTAL}(j)$. $\square$

For each $i \in [\ell]$, we define

$$\text{VERTICAL}(i) = \{a_i, b_j\} \cup \left( \bigcup_{j \in [\ell]} M_{i,j} \right) \cup \left( \bigcup_{j \in [\ell+1]} VS_{i,j} \right)$$

The proof of the next lemma is analogous to that of Lemma 3.5:
Lemma 3.6. For every \( i \in [\ell] \) any \( a_i \rightsquigarrow b_i \) path must have all edges in \( G'[\text{VERTICAL}(i)] \).

Corollary 3.7. For every \( 1 \leq i, j \leq \ell \) the edge set \( N \) contains at least one intra-gadget edge from the main gadget \( M_{i,j} \).

Proof. Fix \( j \in [\ell] \). By Lemma 3.5, there is an \( c_j \rightsquigarrow d_j \) path contained in \( G'[\text{HORIZONTAL}(j)] \). Hence, this path must contain at least one intra-gadget edge from each main gadget \( M_{i,j} \) for each \( 1 \leq i \leq \ell \).

Analogous lemmas hold also for the horizontal secondary gadgets and the vertical secondary gadgets:

Corollary 3.8. For every \( 1 \leq i \leq \ell + 1, 1 \leq j \leq \ell \) the edge set \( N \) contains at least one intra-gadget edge from the horizontal secondary gadget \( HS_{i,j} \).

Proof. Fix \( j \in [\ell] \). By Lemma 3.5, there is an \( c_j \rightsquigarrow d_j \) path contained in \( G'[\text{HORIZONTAL}(j)] \). Hence, this path must contain at least one intra-gadget edge from each horizontal secondary gadget \( HS_{i,j} \) for each \( 1 \leq i \leq \ell + 1 \).

Corollary 3.9. For every \( 1 \leq i \leq \ell, 1 \leq j \leq \ell + 1 \) the edge set \( N \) contains at least one intra-gadget edge from the vertical secondary gadget \( VS_{i,j} \).

Proof. Fix \( i \in [\ell] \). By Lemma 3.6, there is an \( a_i \rightsquigarrow b_i \) path contained in \( G'[\text{VERTICAL}(j)] \). Hence, this path must contain at least one intra-gadget edge from each vertical secondary gadget \( VS_{i,j} \) for each \( 1 \leq j \leq \ell + 1 \).

Corollary 3.10. For each \( 1 \leq i, j \leq \ell \), the solution \( N \) contains at least one
- top-red edge incident on \( M_{i,j} \)
- right-red edge incident on \( M_{i,j} \)
- bottom-red edge incident on \( M_{i,j} \)
- left-red edge incident on \( M_{i,j} \)

Proof. Fix some \( 1 \leq i, j \leq \ell \). By Lemma 3.5, there is an \( c_j \rightsquigarrow d_j \) path contained in \( G'[\text{HORIZONTAL}(j)] \). The only way to enter \( M_{i,j} \) by edges within \( \text{HORIZONTAL}(j) \) is via left-red edges incident on \( M_{i,j} \), and the only way to exit \( M_{i,j} \) by edges within \( \text{HORIZONTAL}(j) \) is via right-red edges incident on \( M_{i,j} \). Hence, \( N \) contains at least one left-red edge and at least one right-red edge incident on \( M_{i,j} \).

By Lemma 3.6, there is an \( a_i \rightsquigarrow b_i \) path contained in \( G'[\text{VERTICAL}(j)] \). The only way to enter \( M_{i,j} \) by edges within \( \text{VERTICAL}(j) \) is via top-red edges incident on \( M_{i,j} \), and the only way to exit \( M_{i,j} \) by edges within \( \text{VERTICAL}(j) \) is via bottom-red edges incident on \( M_{i,j} \). Hence, \( N \) contains at least one top-red edge and at least one bottom-red edge incident on \( M_{i,j} \).

We show now that there is no slack, i.e., weight of \( N \) must be exactly \( B^* \).

Lemma 3.11. The weight of \( N \) is exactly \( B^* \), and hence it is minimal (under edge deletions) since no edges have zero weights.

Proof. We have the following collection of pairwise disjoint sets of edges which are guaranteed to be contained in \( N \):
- \( 4\ell \) orange edges (from Lemma 3.4). This incurs a cost of at least \( 4\ell \).
- A cost of at least \( 1 \) from intra-gadget edges of vertical secondary gadgets (from Corollary 3.9). This incurs a cost of at least \( \ell(\ell + 1) \).
- A cost of at least \( 1 \) from intra-gadget edges of horizontal secondary gadgets (from Corollary 3.8). This incurs a cost of at least \( \ell(\ell + 1) \).
- A cost of at least \( 1 \) from intra-gadget edges of main gadgets (from Corollary 3.7). This incurs a cost of \( \ell^2 \).
- A cost of \( \geq 4 \) from inter-gadget edges of main gadgets (from Corollary 3.10). This incurs a cost of \( 4\ell^2 \).
Hence, the cost of \( N \) is at least \( 4\ell + \ell(\ell + 1) + \ell(\ell + 1) + \ell^2 = B^* \). But, we are given that cost of \( N \) is at most \( B^* \). Hence, the cost of \( N \) is exactly \( B^* \).

The following corollary follows from Lemma 3.11:

**Corollary 3.12.** The solution \( N \) contains exactly one intra-gadget edge from each gadget (main, vertical secondary or horizontal secondary). Hence,

- for each \( 1 \leq i \leq \ell + 1, 1 \leq j \leq \ell \), the unique intra-gadget edge from the vertical secondary gadget \( V_{S_{ij}} \) in \( N \) is \( V_{S_{ij}}(0_{x_{ij}}) \rightarrow V_{S_{ij}}(1_{x_{ij}}) \) for some \( x_{ij} \in [n] \)
- for each \( 1 \leq i \leq \ell, 1 \leq j \leq \ell + 1 \), the unique intra-gadget edge from the horizontal secondary gadget \( H_{S_{ij}} \) in \( N \) is \( H_{S_{ij}}(0_{y_{ij}}) \rightarrow H_{S_{ij}}(1_{y_{ij}}) \) for some \( y_{ij} \in [n] \)
- for each \( 1 \leq i, j \leq \ell \), the unique intra-gadget edge from the main gadget \( M_{i,j} \) in \( N \) is \( M_{i,j}(0_{\lambda_{i,j}, \delta_{i,j}}) \rightarrow M_{i,j}(1_{\lambda_{i,j}, \delta_{i,j}}) \) for some \( \lambda_{i,j}, \delta_{i,j} \in S_{i,j} \)

The following corollary follows from Lemma 3.11:

**Corollary 3.13.** For each \( 1 \leq i, j \leq \ell \), the solution \( N \) contains exactly one
- top-red edge incident on \( M_{i,j} \)
- right-red edge incident on \( M_{i,j} \)
- bottom-red edge incident on \( M_{i,j} \)
- left-red edge incident on \( M_{i,j} \)

Consider a main gadget \( M_{i,j} \). The main gadget has four secondary gadgets surrounding it: \( V_{S_{ij}} \) below it, \( V_{S_{i,j+1}} \) above it, \( H_{S_{i,j}} \) to the left and \( H_{S_{i+1,j}} \) to the right.

**Lemma 3.14.** \((propagation)\) For every main gadget \( M_{i,j} \), we have \( x_{ij} = \lambda_{i,j} = x_{i,j+1} \) and \( y_{ij} = \delta_{i,j} = y_{i+1,j} \).

**Proof.** Due to symmetry, it suffices to only argue that \( x_{ij} = \lambda_{i,j} \). By Corollary 4.16, the only intra-gadget edge from the vertical secondary gadget is \( V_{S_{ij}} \) in \( N \) is \( V_{S_{ij}}(0_{x_{ij}}) \rightarrow V_{S_{ij}}(1_{x_{ij}}) \) and the only intra-gadget edge from the main gadget \( M_{i,j} \) in \( N \) is \( M_{i,j}(0_{\lambda_{i,j}, \delta_{i,j}}) \rightarrow M_{i,j}(1_{\lambda_{i,j}, \delta_{i,j}}) \). By Corollary 3.13, there is exactly one bottom-red incident edge on \( M_{i,j} \). Moreover, this is the only incoming edge into \( V_{S_{ij}} \). Hence, it follows that \( x_{ij} = \lambda_{i,j} \).

**Lemma 3.15.** The **GRID TILING** instance \((\ell, n, \{S_{ij} : i, j \in [\ell]\})\) has a solution.

**Proof.** By Lemma 4.18, it follows that for each \( 1 \leq i, j \leq \ell \) we have \( x_{ij} = \lambda_{i,j} = x_{i,j+1} \) and \( y_{ij} = \delta_{i,j} = y_{i+1,j} \) in addition to \((\lambda_{i,j}, \delta_{i,j}) \in S_{i,j}\) (by the definition of the main gadget). This implies that **GRID TILING** has a solution.

### 3.2.3 Finishing the proof of Theorem 1.4

There is a simple reduction [17, Theorem 14.28] from \( \ell\)-**CLIQUE** on \( n \) vertex graphs to \((\ell, n)\)-**GRID TILING**. Combining the two directions from Section 3.2.1 and Section 3.2.2 gives a parameterized reduction from \((\ell, n)\)-**GRID TILING** to \textsc{DSN} on \((n + \ell)^{O(1)}\) vertex graphs with \( k = O(\ell) \) and \( p = O(\ell^2) \). Composing the two reductions, we get a parameterized reduction from an \( \ell\)-**CLIQUE** instance on \( n \) vertices to an instance \textsc{DSN} on \((n + \ell)^{O(1)}\) vertices, \( k = O(\ell) \) \( p = O(\ell^2) \). Hence, the \( W[1]\)-hardness of \textsc{DSN} parameterized by \((k + p)\) follows from the \( W[1]\)-hardness of \( \ell\)-**CLIQUE** parameterized by \( \ell \). Moreover, Chen et al. [10] showed that, for any function \( f \), the existence of an \( f(\ell) \cdot n^{o(\ell)} \) time algorithm for **CLIQUE** violates ETH. Hence, we obtain that, under ETH, there is no \( f(k, p) \cdot n^{o(k + \sqrt{p + 1}/\ell)} \) time algorithm for \textsc{DSN}.

Suppose now that there is an algorithm \( k \) which runs in time \( f(k, p, \epsilon) \cdot n^{o(k + \sqrt{p + 1}/\ell)} \) (for some computable function \( f \)) and computes an \((1+\epsilon)\)-approximate solution for **DSN**. Recall that our reduction works as follows: **GRID TILING** answers YES if and only if **DSN** has a solution of cost \( B^* \leq 6\ell + 7\ell^2 \leq 13\ell^2 < 14\ell^2 \). Consequently, consider running \( k \) with \( \epsilon = \frac{1}{14\ell^2} \) implies that
We obtain the following result regarding the parameterized complexity of SCSS

\[ \text{Theorem 1.5.} \]

The SCSS problem is W[1]-hard parameterized by \( p + k \). Moreover, under ETH, for any computable function \( f \) and any \( \varepsilon > 0 \)

- there is no \( f(k, p) \cdot n^{o(\sqrt{k + p})} \) time algorithm for SCSS, and
- there is no \( f(k, \varepsilon, p) \cdot n^{o(\sqrt{k + p + 1/\varepsilon})} \) time algorithm which computes an \((1 + \varepsilon)\)-approximation for SCSS.

To prove Theorem 1.5, we give a reduction which transforms an instance \((\ell, n, \{S_{i,j} : i, j \in [\ell]\})\) of \( \ell \times \ell \) GRID TILING into an instance of \((G, \mathcal{T})\) of SCSS which has \(|\mathcal{T}| = O(\ell^2)\) terminals and an optimum which is planar and has size \( O(\ell^2)\). First we construct a “uniqueness” gadget which is used repeatedly as a building block in our construction.

### 4.1 Constructing a “uniqueness” gadget

For every integer \( n \) we define the following gadget \( \mathcal{G}_n \) which contains \( 4n + 4 \) vertices (see Figure 7). Since we need many of these gadgets later on, we will denote vertices of \( \mathcal{G}_n \) by \( \mathcal{G}_n(v) \) etc., in order to be able to distinguish vertices of different gadgets. All edges will have the same weight \( B \), which we will fix later during the reductions. The gadget \( \mathcal{G}_n \) is constructed as follows (we first construct an undirected graph, and then bidirect each edge):

- For each \( i \in [n] \) introduce four vertices \( \mathcal{G}_n(0_i), \mathcal{G}_n(1_i), \mathcal{G}_n(2_i), \mathcal{G}_n(3_i) \).
- Introduce two terminal vertices \( \mathcal{G}_n(s_1) \) and \( \mathcal{G}_n(s_2) \).
- \( \mathcal{G}_n \) has a path of three edges corresponding to each \( i \in [n] \).
  - Let \( i \in [n] \). Then we denote the path in \( \mathcal{G}_n \) corresponding to \( i \) by \( P_{\mathcal{G}_n}(i) := \mathcal{G}_n(0_i) \rightarrow \mathcal{G}_n(1_i) \rightarrow \mathcal{G}_n(2_i) \rightarrow \mathcal{G}_n(3_i) \).
  - Each of these edges is called as a “base” edge and has weight \( B \).
- We add the following edges:
  - \( \mathcal{G}_n(s_1) \rightarrow \mathcal{G}_n(1_i) \) for each \( i \in [n] \)
  - \( \mathcal{G}_n(s_2) \leftarrow \mathcal{G}_n(2_i) \) for each \( i \in [n] \)
  - Each of these edges is called a “connector” edge and has weight \( B \).
- We also add the edge \( \mathcal{G}_n(s_2) \rightarrow \mathcal{G}_n(s_1) \) with weight \( B \) and call it as a “bridge” edge.

**Definition 4.1.** We define the set of left boundary vertices of \( \mathcal{G}_n \) to be \( \bigcup_{i=1}^n \mathcal{G}_n(0_i) \) and the set of right boundary vertices of \( \mathcal{G}_n \) to be \( \bigcup_{i=1}^n \mathcal{G}_n(3_i) \)

**Definition 4.2.** A set of edges \( E' \) of \( \mathcal{G}_n \) satisfies the “in-out” property if each of the following four conditions is satisfied

- \( \mathcal{G}_n(s_1) \) can reach some right boundary vertex via a path contained in the gadget \( \mathcal{G}_n \)
- \( \mathcal{G}_n(s_1) \) can be reached from some left boundary vertex via a path contained in the gadget \( \mathcal{G}_n \)
- \( \mathcal{G}_n(s_2) \) can reach some right boundary vertex via a path contained in the gadget \( \mathcal{G}_n \)
- \( \mathcal{G}_n(s_2) \) can be reached from some left boundary vertex via a path contained in the gadget \( \mathcal{G}_n \)

**Definition 4.3.** We say that a set of edges of \( \mathcal{G}_n \) is represented by \( i \in [n] \) if it contains exactly the following six edges

- the connector edges \( \mathcal{G}_n(s_1) \rightarrow \mathcal{G}_n(1_i) \) and \( \mathcal{G}_n(s_2) \leftarrow \mathcal{G}_n(2_i) \)
Figure 7: The construction of the uniqueness gadget for $U_n$. Note that the gadget has $4n + 4$ vertices. Each base edge is denoted by a filled edge and each connector edge is denoted by a dotted edge in the figure.

- the base edges given by the directed path $P_{U_n}(i) := U_n(0_i) \rightarrow U_n(1_i) \rightarrow U_n(2_i) \rightarrow U_n(3_i)$.
- the bridge edge $U_n(s_2) \rightarrow U_n(s_1)$

We denote this set of edges by $E_{U_n}(i)$.

**Observation 4.4.** For any $i \in [n]$, the set of edges $E_{U_n}(i)$ forms a planar graph.

Note that for any $i \in [n]$ the set of edges $E_{U_n}(i)$ of $U_n$ represented by $i \in [n]$ satisfies the “in-out” property. We now show a lower bound on the cost/weight of edges we need to pick from $U_n$ to satisfy the “in-out” property.

**Lemma 4.5.** Let $E'$ be a set of edges of $U_n$ which satisfies the “in-out” property. Then we have that either

1. the weight of $E'$ is at least $7B$

**OR**

2. the weight of $E'$ is exactly $6B$ and there is an integer $i \in [n]$ such that $E'$ is represented by $i$

**Proof.** First we observe that $E'$ must contain the bridge edge $U_n(s_2) \rightarrow U_n(s_1)$: this is because $U_n(s_1)$ must be reached from some left boundary vertex and the only incoming edge into $U_n(s_1)$ is the bridge edge. This incurs a cost of $B$. We also clearly need at least two connector edges in $N$:
- One outgoing edge from $U_n(s_1)$ so that it can reach some right boundary vertex.
- One incoming edge into $U_n(s_2)$ so that it can be reached from some left boundary vertex.

This incurs a cost of $2B$ in $E'$. We now see how many base edges must be present in $E'$. We define the following:
- “0-1” edges: This is the set of edges \{ $U_n(0_i) \rightarrow U_n(1_i)$ : $1 \leq i \leq n$ \}
- “1-2” edges: This is the set of edges \{ $U_n(1_i) \rightarrow U_n(2_i)$ : $1 \leq i \leq n$ \}
- “2-3” edges: This is the set of edges \{ $U_n(2_i) \rightarrow U_n(3_i)$ : $1 \leq i \leq n$ \}

We have the following cases:
We design two types of gadgets: the main gadget and the secondary gadget. The reduction from GRID TILING represents each cell of the grid with a copy of the main gadget, and each main gadget is surrounded by four secondary gadgets: on the top, right, bottom and left. Each of these gadgets are actually copies of each gadget is a copy of \( \mathcal{U} \), and each of the secondary gadgets are copies of \( \mathcal{U} \) (with \( B = 1 \)). With slight abuse of notation, we assume that the rows of \( M_{i,j} \) are indexed by the set \( \{(x,y): (x,y) \in S_{i,j}\} \). We add the following edges (in red color) of weight 1: for each \( 1 \leq i, j \leq \ell \):

- Add the edge \( V_{S_{i,j+1}}(3_x) \rightarrow M_{i,j}(0_{x,y}) \). These edges are called top-red edges incident on \( M_{i,j} \).
- Add the edge \( H_{S_{i,j}}(3_y) \rightarrow M_{i,j}(0_{x,y}) \). These edges are called left-red edges incident on \( M_{i,j} \).

This incurs a cost of \( 3B \). Therefore, the solution \( E' \) has cost \( \geq B + 2B + 3B = 6B \). If \( E' \) contains one more edge than the ones listed above, then the cost of \( E' \) is \( \geq 7B \) since each edge of \( \mathcal{U} \) has weight \( B \).

Suppose that the solution \( E' \) has cost exactly \( 6M \). Hence, it follows from the previous arguments that \( E' \) contains exactly the following six edges:

- A connector edge outgoing from \( \mathcal{U}(s_1) \). Let this edge be \( \mathcal{U}(s_1) \rightarrow \mathcal{U}(1_\lambda) \) for some \( \lambda_1 \in \{n\} \)
- A connector edge incoming into \( \mathcal{U}(s_2) \). Let this edge be \( \mathcal{U}(s_2) \leftarrow \mathcal{U}(1_\lambda) \) for some \( \lambda_2 \in \{n\} \)
- The bridge edge \( \mathcal{U}(s_2) \rightarrow \mathcal{U}(s_1) \)
- An 0 – 1 edge given by \( \mathcal{U}(0_\beta) \rightarrow \mathcal{U}(0_\beta) \) for some \( \beta_1 \in \{n\} \)
- An 1 – 2 edge given by \( \mathcal{U}(0_\beta) \rightarrow \mathcal{U}(0_\beta) \) for some \( \beta_2 \in \{n\} \)
- An 2 – 3 edge given by \( \mathcal{U}(0_\beta) \rightarrow \mathcal{U}(0_\beta) \) for some \( \beta_3 \in \{n\} \)

We now show that \( \lambda_1 = \lambda_2 = \beta_1 = \beta_2 = \beta_3 \) for some \( i \in \{n\} \).

- How does \( \mathcal{U}(s_1) \) reach a right boundary vertex: This path must use a connector edge outgoing from \( \mathcal{U}(s_1) \) followed by an 1 – 2 edge and an 2 – 3 edge. This implies that \( \lambda_1 = \beta_2 = \beta_3 \).
- How is \( \mathcal{U}(s_2) \) reached from a left boundary vertex: This path must use an 0 – 1 edge followed by a 1 – 2 edge and then a connector edge incoming into \( \mathcal{U}(s_2) \). This implies that \( \beta_1 = \beta_2 = \lambda_2 \).
- Hence, we have that \( \lambda_1 = \lambda_2 = \beta_1 = \beta_2 = \beta_3 \) for some \( i \in \{n\} \), i.e., \( E' \) is represented by some \( i \in \{n\} \).

The following corollary follows immediately from the second part of proof of the previous lemma.

**Corollary 4.6.** For every \( i \in \{n\} \) there is a set of edges \( E_{\mathcal{U}}(i) \) of cost exactly \( 6B \) which represents \( i \) (and hence also satisfies the “in-out” property).

### 4.2 Construction of the instance \((G, \mathcal{P})\) of SCSS\_PLANAR

We design two types of gadgets: the main gadget and the secondary gadget. The reduction from GRID TILING represents each cell of the grid with a copy of the main gadget, and each main gadget is surrounded by four secondary gadgets: on the top, right, bottom and left. Each of these gadgets are actually copies of the “uniqueness gadget” \( \mathcal{U} \) from Section 4.1 with weight of each edge set to \( B = 1 \): each secondary gadget is a copy of \( \mathcal{U} \), and for each \( 1 \leq i, j \leq \ell \) the main gadget \( M_{i,j} \) (corresponding to the set \( S_{i,j} \)) is a copy of \( \mathcal{U}(S_{i,j}) \). We refer to Figure 8 (bird’s-eye view) and Figure 9 (zoomed-in view) for an illustration of the reduction.

Fix some \( 1 \leq i, j \leq \ell \). The main gadget \( M_{i,j} \) has four secondary gadgets\(^5\) surrounding it:

- Above \( M_{i,j} \) is the vertical secondary gadget \( V_{S_{i,j+1}} \)
- On the right of \( M_{i,j} \) is the horizontal secondary gadget \( H_{S_{i+1,j}} \)
- Below \( M_{i,j} \) is the vertical secondary gadget \( V_{S_{i,j}} \)
- On the left of \( M_{i,j} \) is the horizontal secondary gadget \( H_{S_{i,j}} \)

Hence, there are \( \ell (\ell + 1) \) horizontal secondary gadgets and \( \ell (\ell + 1) \) vertical secondary gadgets. Recall that \( M_{i,j} \) is a copy of \( \mathcal{U}(S_{i,j}) \) and each of the secondary gadgets are copies of \( \mathcal{U} \) (with \( B = 1 \)). With slight abuse of notation, we assume that the rows of \( M_{i,j} \) are indexed by the set \( \{(x,y): (x,y) \in S_{i,j}\} \). We add the following edges (in red color) of weight 1: for each \( 1 \leq i, j \leq \ell \):

- Add the edge \( V_{S_{i,j+1}}(3_x) \rightarrow M_{i,j}(0_{x,y}) \). These edges are called top-red edges incident on \( M_{i,j} \).
- Add the edge \( H_{S_{i,j}}(3_y) \rightarrow M_{i,j}(0_{x,y}) \). These edges are called left-red edges incident on \( M_{i,j} \).

\(^5\)Half of the secondary gadgets are called “horizontal” since their base edges are horizontal (as seen by the reader), and the other half of the secondary gadgets are called “vertical”.

29
Figure 8: A bird’s-eye view of the instance of $G^*$ with $\ell = 3$ and $n = 4$ (see Figure 9 for a zoomed-in view). From the secondary gadgets, we have shown only the base edges: the connector edges and bridge edges are omitted for clarity. Similarly, the vertices and edges within each main gadget are not shown here either. Additionally we have some red edges between each main gadget and the four secondary gadgets surrounding it which are omitted in this figure for clarity (they are shown in Figure 9) which gives a more zoomed-in view. Also missing in this picture are the two special terminals $s^*, t^*$ in addition to the source edges, sink edges and the strong edge.
• Add the edge \( M_{t,j}(3(x,y)) \to HS_{t+1,j}(0,y) \). These edges are called right-red edges incident on \( M_{t,j} \).
• Add the edge \( M_{t,j}(3(x,y)) \to VS_{t,j}(0,y) \). These edges are called bottom-red edges incident on \( M_{t,j} \).

Introduce the following 4\( \ell \) vertices (which we call border vertices):

- \( a_1, a_2, \ldots, a_{\ell} \)
- \( b_1, b_2, \ldots, b_{\ell} \)
- \( c_1, c_2, \ldots, c_{\ell} \)
- \( d_1, d_2, \ldots, d_{\ell} \)

For each \( i \in [\ell] \) add the following edges (shown using orange color in Figure 8) with weight 1:

- \( a_i \to VS_{t+1,j}(0) \) for each \( j \in [n] \). We call these edges top-orange edges.
- \( b_i \to VS_{t+1,j}(3) \) for each \( j \in [n] \). We call these edges bottom-orange edges.
- \( c_i \to HS_{t+1,j}(0) \) for each \( j \in [n] \). We call these edges left-orange edges.
- \( d_i \to HS_{t+1,j}(3) \) for each \( j \in [n] \). We call these edges right-orange edges.

Introduce two new vertices \( s^*, t^* \) and add the following edges (not shown in Figure 8) with weight 1 each:

- \( s^* \to a_i \) and \( s^* \to c_i \) for each \( i \in [\ell] \). We call these edges source edges
- \( d_i \to t^* \) and \( b_i \to t^* \) for each \( i \in [\ell] \). We call these edges sink edges
- The edge \( t^* \to s^* \). We call this edge the strong edge.

This completes the construction of the graph \( G \). Note that \( G \) has size \( N = (n + \ell)O(1) \) and can be constructed in \( (n + \ell)O(1) \) time. We now define the set of terminals \( \mathcal{T} \) as follows:

- **Vertical terminals**: The set of vertical terminals is given by \( \bigcup_{1 \leq i \leq \ell, 1 \leq j \leq \ell+1} \{VS_{i,j}(s_1), VS_{i,j}(s_2)\} \)
- **Horizontal terminals**: The set of horizontal terminals is given by \( \bigcup_{1 \leq i \leq \ell+1, 1 \leq j \leq \ell} \{HS_{i,j}(s_1), HS_{i,j}(s_2)\} \)
- **Main terminals**: The set of main terminals is given by \( \bigcup_{1 \leq i \leq \ell} \{M_{i,j}(s_1), M_{i,j}(s_2)\} \)
- **Special terminals**: There are only two special terminals, namely \( s^* \) and \( t^* \).

We have \( \ell^2 \) main gadgets, \( \ell(\ell + 1) \) vertical secondary gadgets and \( \ell(\ell + 1) \) horizontal secondary gadgets. In addition to the two special terminals \( s^* \) and \( t^* \), we add two terminals corresponding to each of these gadgets. Hence, the total number of terminals is \( k = |\mathcal{T}| = O(\ell^2) \).

Fix the budget \( B^* = 1 + 20\ell + 24\ell^2 = O(\ell^2) \). We will show that any solution has weight at least \( B^* \) using the following intuition:

- The strong edge \( t^* \to s^* \) must be present
- 2\( \ell \) source edges and 2\( \ell \) sink edges must be present
- 4\( \ell \) orange edges must be present (one for each boundary vertex)
- Each of the \( \ell(\ell + 1) \) vertical secondary gadgets must satisfy “in-out” property and hence have weight at least 6
- Each of the \( \ell(\ell + 1) \) horizontal secondary gadgets must satisfy “in-out” property and hence have weight at least 6
- Each of the \( \ell^2 \) main gadgets must satisfy “in-out” property and hence have weight at least 6
- Each of the \( \ell^2 \) main gadgets must contribute at least four red edges (each of which have exactly one endpoint in the main gadget)

In the other direction, we will show that a solution having cost exactly \( B^* \) forces enough structure to allow us to conclude that GRID TILING answers YES.

### 4.3 GRID TILING answers YES ⇒ instance \((G, \mathcal{T})\) of SCSS PLANAR has a planar solution of cost \( \leq B^* \)

Suppose that GRID TILING has a solution, i.e., for each \( 1 \leq i, j \leq \ell \) there is a value \( (x_{i,j}, y_{i,j}) = (x_{i,j}, y_{i,j}) \in S_{i,j} \) such that

- for every \( i \in [\ell] \), we have \( x_{i,1} = x_{i,2} = x_{i,3} = \ldots = x_{i,\ell} = \alpha_i \) and
- for every \( j \in [\ell] \), we have \( y_{1,j} = y_{2,j} = y_{3,j} = \ldots = y_{\ell,j} = \beta_j \).
Figure 9: A zoomed-in view of the main gadget $M_{i,j}$ surrounded by four secondary gadgets: horizontal gadget $HS_{i,j+1}$ on the top, vertical gadget $VS_{i,j}$ on the left, horizontal gadget $HS_{i,j}$ on the bottom and vertical gadget $VS_{i+1,j}$ on the right. Each of the secondary gadgets is a copy of the uniqueness gadget $\mathbb{U}_n$ (see Section 4.1) and the main gadget $M_{i,j}$ is a copy of the uniqueness gadget $\mathbb{U}_{S_{i,j}}$. The only inter-gadget edges are the red edges: they have one end-point in a main gadget and the other end-point in a secondary gadget. We have shown four such red edges which are introduced for every $(x,y) \in S_{i,j}$. 
We now build a planar solution \( N \) for the instance \((G^*, \mathcal{T})\) of SCSS. In the edge set \( N \), we take the following edges:

1. The strong edge \( t^* \rightarrow s^* \). This incurs a cost of 1.
2. For each \( i \in [\ell] \) add the source edges \( s^* \rightarrow a_i \) and \( s^* \rightarrow c_i \). This incurs a cost of \( 2\ell \) since each source edge has weight 1.
3. For each \( i \in [\ell] \) add the sink edges \( b_i \rightarrow t^* \) and \( d_i \rightarrow t^* \). This incurs a cost of \( 2\ell \) since each sink edge has weight 1.
4. For each \( i \in [\ell] \) add the top orange edge \( a_i \rightarrow V S_{i, \ell+1}(0_{\alpha_i}) \) and the bottom orange edge \( V S_{i,1}(0_{\alpha_i}) \rightarrow b_i \). This incurs a cost of \( 2\ell \) since each orange edge has weight 1.
5. For each \( j \in [\ell] \) add the left orange edge \( c_j \rightarrow H S_{1,j}(0_{\alpha_i}) \) and the right orange edge \( H S_{\ell+1,j}(0_{\alpha_i}) \rightarrow d_j \). This incurs a cost of \( 2\ell \) since each orange edge has weight 1.
6. For each \( 1 \leq i \leq \ell + 1, 1 \leq j \leq \ell \), use Corollary 4.6 to add the set of edges \( E_{H S_{i,j}}(\beta_j) \) from \( H S_{i,j} \) of weight 6 which represents \( \beta_j \). This incurs a cost of \( 6\ell(\ell + 1) \) since there are \( \ell(\ell + 1) \) horizontal secondary gadgets.
7. For each \( 1 \leq i \leq \ell, 1 \leq j \leq \ell + 1 \), use Corollary 4.6 to add the set of edges \( E_{V S_{i,j}}(\alpha_i) \) from \( V S_{i,j} \) of weight 6 which represents \( \alpha_i \). This incurs a cost of \( 6\ell(\ell + 1) \) since there are \( \ell(\ell + 1) \) vertical secondary gadgets.
8. For each \( 1 \leq i, j \leq \ell \), use Corollary 4.6 to add the set of edges \( E_{M_{i,j}}((\alpha_i, \beta_j)) \) from \( M_{i,j} \) of weight 6 which represents \( (\alpha_i, \beta_j) \). Note that this is possible since the solution of the GRID TILING instance guarantees that \( (\alpha_i, \beta_j) \in S_{i,j} \) for each \( 1 \leq i, j \leq \ell \). This incurs a cost of \( 6\ell^2 \) since there are \( \ell^2 \) main gadgets.
9. For each \( 1 \leq i, j \leq \ell \), add the four edges (each of which has weight 1)
   
   \[
   \begin{align*}
   &V S_{i,j+1}(3_{\alpha_i}) \rightarrow M_{i,j}(0_{\alpha_i, \beta_j}) \\
   &H S_{i,j}(3_{\beta_j}) \rightarrow M_{i,j}(0_{\alpha_i, \beta_j}) \\
   &M_{i,j}(3_{\alpha_i, \beta_j}) \rightarrow H S_{i+1,j}(0_{\alpha_i}) \\
   &M_{i,j}(3_{\alpha_i, \beta_j}) \rightarrow V S_{i,j}(0_{\beta_j})
   \end{align*}
   \]
   
   Note that this is possible since the solution of the GRID TILING instance guarantees that \( (\alpha_i, \beta_j) \in S_{i,j} \) for each \( 1 \leq i, j \leq \ell \). This incurs a cost of \( 4\ell^2 \) since there are \( \ell^2 \) main gadgets.

It follows that the weight of \( N \) is exactly \( 1 + 4\ell + 4\ell + 6\ell(\ell + 1) + 6\ell(\ell + 1) + 6\ell^2 + 4\ell^2 = B^* \). We next show that \( N \) is in fact a planar solution of the instance \((G, \mathcal{T})\) of SCSS.

### 4.3.1 \( N \) is planar

We use the following two definitions to help us argue about planarity of \( N \);

**Definition 4.7.** We call the set of edges \( E_{\text{intra}} \) which have both end-points in the same gadget (either main, vertical secondary or vertical secondary) as intra-gadget edges. We call the set of edges \( E_{\text{inter}} \) which one end-point in a main gadget and other end-point in a secondary gadget as inter-gadget edges.

The source edges, sinks edges and the strong edge can be drawn on the “outside” of Figure 8. Hence, Figure 8 gives a planar embedding of \( G \backslash (E_{\text{intra}} \cup E_{\text{inter}}) \). We now consider the set of edges \( E_{\text{inter}} \cap N \). For any gadget (either main, horizontal secondary or vertical secondary), Corollary 4.6 implies that the edges which have both end-points in this gadget form a planar graph (see Observation 4.4). Finally, we now consider the set of edges \( E_{\text{inter}} \cap N \). For each main gadget \( M \), there are exactly four inter-gadget edges incident on \( M \); one each to the four secondary gadgets surrounding it. These four (red) inter-gadget edges do not destroy planarity either since two of them are incident on one 0-vertex of the main gadget and other two are incident on another 3-vertex of the main gadget (see Figure 9). Hence, \( N \) is planar.
4.3.2 $N$ is a solution for the instance $(G, \mathcal{T})$ of SCSS

Note that there are only two special terminals: $s^*$ and $t^*$. For each non-special terminal $x$, i.e., $x \in \mathcal{T} \setminus \{s^*, t^*\}$, we will show below that $N$ contains an $s^* \leadsto x$ path and an $x \leadsto t^*$ path. Since $(t^*, s^*) \in N$, this is sufficient to show that $N$ is indeed a solution for the instance $(G, \mathcal{T})$ of SCSS because:

- The strong edge $(t^*, s^*)$ gives a $t^* \rightarrow s^*$ path. There are many $s^* \leadsto t^*$ paths: choose any non-special terminal and concatenate the $s^\ast \leadsto x$ path with the $x \leadsto t^*$ path.
- Let $x$ be any non-special terminal. Then we are guaranteed existence of an $s^* \leadsto x$ path and an $x \leadsto t^*$ path. Since the strong edge $(t^*, s^*)$ is in $N$ it follows that there also exists an $x \leadsto s^*$ path and an $t^* \leadsto x$ path.
- For any two terminals $x, y$ there is an $x \leadsto y$ path as follows: take the $x \leadsto t^*$ path followed by the strong edge $(t^*, s^*)$ followed by the $s^* \leadsto y$ path.

Hence, it remains to show that for any non-special terminal $x$ the set $N$ contains an $s^\ast \leadsto x$ path and an $x \leadsto t^*$ path. We have three cases:

1. $x$ is a vertical terminal: Suppose $x$ is a terminal in $VS_{i,j}$ for some $1 \leq i \leq \ell, 1 \leq j \leq \ell + 1$.
   - We first show the existence of an $VS_{i,j}(s_i) \leadsto t^*$ path which also contains the vertex $VS_{i,j}(s_1)$:
     - $VS_{i,j}(s_2) \rightarrow VS_{i,j}(s_1)$
     - $VS_{i,j}(s_1) \rightarrow VS_{i,j}(1_\alpha) \rightarrow VS_{i,j}(2_\alpha) \rightarrow VS_{i,j}(3_\alpha)$
     - If $j \neq 1$, then use the $VS_{i,j}(3_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ path given by concatenating the following edges/paths:
       * $VS_{i,j}(3_\alpha) \rightarrow M_{i,j-1}(0_\alpha, 1_{\beta_i})$
       * The $M_{i,j-1}(0_\alpha, 1_{\beta_i}) \leadsto M_{i,j-1}(3_\alpha, 1_{\beta_i})$ path given by $P_{M_{i,j-1}}(\alpha_i, \beta_j)$
       * $M_{i,j-1}(3_\alpha, 1_{\beta_i}) \rightarrow VS_{i,j-1}(0_\alpha)$
       * The $VS_{i,j-1}(0_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ given by $P_{VS_{i,j-1}}(\alpha_i)$
     We do this for each $j$ (decreasing $j$ by 1 each time) each time until we reach the vertex $VS_{i,1}(0_\alpha)$
     - Then use the path $VS_{i,1}(0_\alpha) \rightarrow VS_{i,1}(1_\alpha) \rightarrow VS_{i,1}(2_\alpha) \rightarrow VS_{i,1}(3_\alpha) \rightarrow b_i \rightarrow t^*$
   - We now show existence of an $s^\ast \leadsto VS_{i,j}(s_1)$ path which also contains the vertex $VS_{i,j}(s_2)$:
     - $s^\ast \rightarrow a_i \rightarrow VS_{i,\ell+1}(0_\alpha)$
     - If $j \neq \ell + 1$, then use the $VS_{i,j}(3_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ path given by concatenating the following edges/paths:
       * $VS_{i,j}(3_\alpha) \rightarrow M_{i,j-1}(0_\alpha, 1_{\beta_i})$
       * The $M_{i,j-1}(0_\alpha, 1_{\beta_i}) \leadsto M_{i,j-1}(3_\alpha, 1_{\beta_i})$ path given by $P_{M_{i,j-1}}(\alpha_i, \beta_j)$
       * $M_{i,j-1}(3_\alpha, 1_{\beta_i}) \rightarrow VS_{i,j-1}(0_\alpha)$
       * The $VS_{i,j-1}(0_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ given by $P_{VS_{i,j-1}}(\alpha_i)$
     - $VS_{i,j}(s_2) \rightarrow VS_{i,j}(s_1)$
     - $VS_{i,j}(s_1) \rightarrow VS_{i,j}(1_\alpha) \rightarrow VS_{i,j}(2_\alpha) \rightarrow VS_{i,j}(3_\alpha)$
     - If $j \neq 1$, then use the $VS_{i,j}(3_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ path given by concatenating the following edges/paths:
       * $VS_{i,j}(3_\alpha) \rightarrow M_{i,j-1}(0_\alpha, 1_{\beta_i})$
       * The $M_{i,j-1}(0_\alpha, 1_{\beta_i}) \leadsto M_{i,j-1}(3_\alpha, 1_{\beta_i})$ path given by $P_{M_{i,j-1}}(\alpha_i, \beta_j)$
       * $M_{i,j-1}(3_\alpha, 1_{\beta_i}) \rightarrow VS_{i,j-1}(0_\alpha)$
       * The $VS_{i,j-1}(0_\alpha) \leadsto VS_{i,j-1}(3_\alpha)$ given by $P_{VS_{i,j-1}}(\alpha_i)$
     We do this for each $j$ (decreasing $j$ by 1 each time) each time until we reach the vertex $VS_{i,1}(0_\alpha)$
     - Then use the path $VS_{i,1}(0_\alpha) \rightarrow VS_{i,1}(1_\alpha) \rightarrow VS_{i,1}(2_\alpha) \rightarrow VS_{i,1}(3_\alpha) \rightarrow b_i \rightarrow t^*$

2. $x$ is a horizontal terminal: Suppose $x$ is a terminal in $HS_{i,j}$ for some $1 \leq i \leq \ell + 1, 1 \leq j \leq \ell$.
   - We first show the existence of an $HS_{i,j}(s_i) \leadsto t^*$ path which also contains the vertex $HS_{i,j}(s_1)$. If $i = \ell + 1$ then we can use the path $HS_{\ell+1,i}(s_2) \rightarrow HS_{\ell+1,i}(1_\beta_i) \rightarrow HS_{\ell+1,i}(2_\beta_i) \rightarrow HS_{\ell+1,i}(3_\beta_i) \rightarrow d_j \rightarrow t^*$. Otherwise if $i < \ell + 1$ then we use the path obtained

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Footnote 6: Here, by paths we technically mean walks since vertices and edges may repeat (all we care about is directed connectivity)
by concatenating the following paths (in order):
- \( HS_{i,j}(s_2) \rightarrow HS_{i,j}(s_1) \)
- \( HS_{i,j}(s_1) \rightarrow HS_{i,j}(1\beta_j) \rightarrow HS_{i,j}(2\beta_j) \rightarrow HS_{i,j}(3\beta_j) \)
- \( HS_{i,j}(3\beta_j) \rightarrow M_{i,j}(0\alpha_j,\beta_j) \rightarrow M_{i,j}(1\alpha_j,\beta_j) \rightarrow M_{i,j}(2\alpha_j,\beta_j) \rightarrow M_{i,j}(s_2) \)
- Now use the \( M_{i,j}(s_2) \rightarrow t^* \) path guaranteed by Case 1

- We now show existence of an \( s^* \sim HS_{i,j}(s_1) \) path which also contains the vertex \( HS_{i,j}(s_2) \). If \( i = 1 \) then we can use the path \( s^* \rightarrow c_j \rightarrow HS_{i,j}(0\beta_j) \rightarrow HS_{i,j}(1\beta_j) \rightarrow HS_{i,j}(2\beta_j) \rightarrow HS_{i,j}(s_2) \) into the gadget. Otherwise if \( i > 1 \) then we can use the path obtained by concatenating the following paths (in order):
  - The \( s^* \sim VS_{i-1,j+1}(s_1) \) path guaranteed by Case 1
  - \( VS_{i-1,j+1}(s_1) \sim VS_{i-1,j+1}(1\alpha_{i-1}) \rightarrow VS_{i-1,j+1}(2\alpha_{i-1}) \rightarrow VS_{i-1,j+1}(3\alpha_{i-1}) \)
  - \( VS_{i-1,j+1}(3\alpha_{i-1}) \rightarrow MS_{i-1,j}(0\alpha_{i-1},\beta_j) \rightarrow MS_{i-1,j}(1\alpha_{i-1},\beta_j) \rightarrow MS_{i-1,j}(2\alpha_{i-1},\beta_j) \rightarrow MS_{i-1,j}(3\alpha_{i-1},\beta_j) \)
  - \( MS_{i-1,j}(3\alpha_{i-1},\beta_j) \rightarrow HS_{i,j}(0\beta_j) \rightarrow HS_{i,j}(1\beta_j) \rightarrow HS_{i,j}(2\beta_j) \rightarrow HS_{i,j}(s_2) \rightarrow HS_{i,j}(s_1) \)

3. \( x \) is a main terminal: Suppose \( x \) is a terminal in \( MS_{i,j} \) for some \( 1 \leq i, j \leq \ell \).
- We first show the existence of an \( M_{i,j}(s_2) \sim t^* \) path which also contains the vertex \( M_{i,j}(s_1) \).
  - This path is obtained by concatenating the following paths (in order):
    - \( M_{i,j}(s_2) \rightarrow M_{i,j}(s_1) \rightarrow M_{i,j}(1\alpha_j,\beta_j) \rightarrow M_{i,j}(2\alpha_j,\beta_j) \rightarrow M_{i,j}(3\alpha_j,\beta_j) \)
    - \( M_{i,j}(3\alpha_j,\beta_j) \rightarrow HS_{i+1,j}(0\beta_j) \rightarrow HS_{i+1,j}(1\beta_j) \rightarrow HS_{i+1,j}(2\beta_j) \rightarrow HS_{i+1,j}(s_2) \)
    - The \( HS_{i+1,j}(s_2) \sim t^* \) path guaranteed by Case 2
- We now show the existence of an \( s^* \sim M_{i,j}(s_1) \) path which also contains the vertex \( M_{i,j}(s_2) \).
  - This path is obtained by concatenating the following paths (in order):
    - The \( s^* \sim HS_{i,j}(s_1) \) path guaranteed by Case 2
    - \( HS_{i,j}(s_1) \rightarrow HS_{i,j}(1\beta_j) \rightarrow HS_{i,j}(2\beta_j) \rightarrow HS_{i,j}(3\beta_j) \rightarrow M_{i,j}(0\alpha_j,\beta_j) \)
    - \( M_{i,j}(0\alpha_j,\beta_j) \rightarrow M_{i,j}(1\alpha_j,\beta_j) \rightarrow M_{i,j}(2\alpha_j,\beta_j) \rightarrow M_{i,j}(s_2) \rightarrow M_{i,j}(s_1) \)

4.4 Instance \((G, \mathcal{T})\) of \text{SCSS}_{\text{PLANAR}} has a solution of cost \( \leq B^* \Rightarrow \) \text{GRID TILING} answers \( \text{YES} \)

Suppose that the instance \((G, \mathcal{T})\) of \text{SCSS}_{\text{PLANAR}} has a solution \( N \) of cost at most \( B^* \). We will now show that this implies that \text{GRID TILING} answers \( \text{YES} \). This implies that if \text{GRID TILING} answers \( \text{NO} \) then the cost of an optimal solution (and hence the cost of an optimal planar solution, if one exists) is greater than \( B^* \).

**Lemma 4.8.** \( N \) contains the strong edge \((t^*, s^*)\)

**Proof.** The vertex \( t^* \) is a terminal and the only outgoing edge incident on \( t^* \) is the strong edge \((t^*, s^*)\).

**Lemma 4.9.** \( N \) contains at least \( 4\ell \) orange edges. In fact, for each \( 1 \leq i \leq \ell \) we have that \( N \) contains at least one
- outgoing orange edge from \( a_i \)
- incoming orange edge into \( b_i \)
- outgoing orange edge from \( c_j \)
- incoming orange edge into \( d_j \)

**Proof.** Fix \( i \in [\ell] \). The gadget \( VS_{i,\ell+1} \) has two terminals \( VS_{i,\ell+1}(s_1) \) and \( VS_{i,\ell+1}(s_2) \), and the only incoming edges into the gadget \( VS_{i,\ell+1} \) are the top orange edges outgoing from \( a_i \). Hence, for strong connectivity it follows that \( N \) contains at least one top-orange edge outgoing from \( a_i \) for each \( i \in [\ell] \), i.e., \( N \) contains at least \( \ell \) top-orange edges.

The other three claims follow by similar arguments.

**Lemma 4.10.** \( N \) contains each of the \( 2\ell \) source edges and each of the \( 2\ell \) sink edges

**Proof.** Fix \( i \in [\ell] \). The gadget \( VS_{i,\ell+1} \) has two terminals \( VS_{i,\ell+1}(s_1) \) and \( VS_{i,\ell+1}(s_2) \), and the only incoming edges into the gadget \( VS_{i,\ell+1} \) are the top orange edges outgoing from \( a_i \). Moreover, the only
incoming edge into \( a_i \) is the source edge \((s^*, a_i)\). Hence, for strong connectivity it follows that \( N \) contains the source edge \((s^*, a_i)\) for each \( i \in [\ell] \). Similarly, one can show that \( N \) contains all the other source edges and sink edges as well. 

**Lemma 4.11.** For every \( 1 \leq i, j \leq \ell \) the edge set \( N \) restricted to the main gadget \( M_{i,j} \) satisfies the “in-out” property. Hence, \( N \) has weight at least 6 in \( M_{i,j} \).

**Proof.** The main gadget \( M_{i,j} \) has two terminals \( M_{i,j}(s_1) \) and \( M_{i,j}(s_2) \). The only incoming edges into \( M_{i,j} \) are the top-red and left-red edges which are incident on the 0-vertices of \( M_{i,j} \). Hence, each terminal of \( M_{i,j} \) has to be reachable from some 0-vertex of \( M_{i,j} \). Similarly, the only outgoing edges from \( M_{i,j} \) are the bottom-red and right-red edges which are incident on the 3-vertices of \( M_{i,j} \). Hence, each terminal of \( M_{i,j} \) has to be able to reach some 3-vertex of \( M_{i,j} \). Hence, \( N \) restricted to the main gadget \( M_{i,j} \) satisfies the “in-out” property (recall Definition 4.2). By Lemma 4.5, the claim follows.

Analogous lemmas hold also for the horizontal secondary gadgets and the vertical secondary gadgets:

**Lemma 4.12.** For every \( 1 \leq i \leq \ell, 1 \leq j \leq \ell + 1 \) the edge set \( N \) restricted to the horizontal secondary gadget \( HS_{i,j} \) satisfies the “in-out” property. Hence, \( N \) has weight at least 6 in \( HS_{i,j} \).

**Lemma 4.13.** For every \( 1 \leq i \leq \ell + 1, 1 \leq j \leq \ell \) the edge set \( N \) restricted to the vertical secondary gadget \( VS_{i,j} \) satisfies the “in-out” property. Hence, \( N \) has weight at least 6 in \( VS_{i,j} \).

**Lemma 4.14.** For each \( 1 \leq i, j \leq \ell \), the solution \( N \) contains at least one

- top-red edge incident on \( M_{i,j} \)
- right-red edge incident on \( M_{i,j} \)
- bottom-red edge incident on \( M_{i,j} \)
- left-red edge incident on \( M_{i,j} \)

**Proof.** Fix some \( 1 \leq i, j \leq \ell \). We now show that \( N \) contains a top-red edge incident on \( M_{i,j} \) (the other 3 claims can be shown analogously). The vertical gadget \( VS_{i,j+1} \) has two terminals \( VS_{i,j+1}(s_1) \) and \( VS_{i,j+1}(s_2) \). The only outgoing edges incident on \( VS_{i,j+1} \) are the top-red edges incident on \( M_{i,j} \). Hence, strong connectivity of \( N \) implies that it contains at least one top-red edge incident on \( M_{i,j} \).

We show now that we have no slack, i.e., the weight of \( N \) must be exactly \( B^* \).

**Lemma 4.15.** The weight of \( N \) is exactly \( B^* \), and hence it is minimal (under edge deletions) since no edges have zero weight.

**Proof.** We have the following collection of pairwise disjoint sets of edges which are guaranteed to be contained in \( N \)

- The strong edge \((r^*, s^*)\) (from Lemma 4.8). This incurs a cost of 1.
- \( 4\ell \) orange edges (from Lemma 4.9). This incurs a cost of \( 4\ell \).
- \( 2\ell \) sources edges and \( 2\ell \) sink edges (from Lemma 4.10). This incurs a cost of \( 4\ell \).
- A cost of at least 6 from edges which have both endpoints in each vertical secondary gadget (from Lemma 4.13). This incurs a cost of \( 6\ell(\ell + 1) \)
- A cost of at least 6 from edges which have both endpoints in each horizontal secondary gadget (from Lemma 4.12). This incurs a cost of \( 6\ell(\ell + 1) \)
- A cost of at least 6 from edges which have both endpoints in each main gadget (from Lemma 4.11). This incurs a cost of \( 6\ell^2 \)
- A cost of at least 4 from edges which have exactly one endpoint in each main gadget (from Lemma 4.14). This incurs a cost of \( 4\ell^2 \)

Hence, the cost of \( N \) is at least \( 1 + 4\ell + 4\ell + 6\ell(\ell + 1) + 6\ell(\ell + 1) + 6\ell^2 + 4\ell^2 = B^* \). But, we are given that cost of \( N \) is at most \( B^* \). Hence, cost of \( N \) is exactly \( B^* \). 

The following corollary follows from Lemma 4.15 and Lemma 4.5:
Corollary 4.16. The weight of $N$ restricted to each gadget (main, vertical secondary or horizontal secondary) is exactly 6. Moreover,

- for each $1 \leq i \leq \ell + 1, 1 \leq j \leq \ell$, the vertical secondary gadget $VS_{i,j}$ is represented by some $x_{i,j} \in [n]$,
- for each $1 \leq i \leq \ell, 1 \leq j \leq \ell + 1$, the horizontal secondary gadget $HS_{i,j}$ is represented by some $y_{i,j} \in [n]$,
- for each $1 \leq i, j \leq \ell$, the main gadget $M_{i,j}$ is represented by some $(\lambda_{i,j}, \delta_{i,j}) \in S_{i,j}$.

The following corollary follows from Lemma 4.15 and Lemma 4.14:

Corollary 4.17. For each $1 \leq i, j \leq \ell$, the solution $N$ contains exactly one

- top-red edge incident on $M_{i,j}$
- right-red edge incident on $M_{i,j}$
- bottom-red edge incident on $M_{i,j}$
- left-red edge incident on $M_{i,j}$

Consider a main gadget $M_{i,j}$. The main gadget has four secondary gadgets surrounding it: $VS_{i,j}$ below it, $VS_{i,j+1}$ above it, $HS_{i,j}$ to the left and $HS_{i+1,j}$ to the right. By Corollary 4.16, these gadgets are represented by $x_{i,j}, x_{i,j+1}, y_{i,j}$ and $y_{i+1,j}$ respectively. The main gadget $M_{i,j}$ is represented by $(\lambda_{i,j}, \delta_{i,j})$.

Lemma 4.18. (propagation) For every main gadget $M_{i,j}$, we have $x_{i,j} = \lambda_{i,j} = x_{i,j+1}$ and $y_{i,j} = \delta_{i,j} = y_{i+1,j}$.

**Proof.** Due to symmetry, it suffices to only argue that $x_{i,j} = \lambda_{i,j}$. By Corollary 4.16, the main gadget $M_{i,j}$ is represented by $(\lambda_{i,j}, \delta_{i,j})$ and the vertical secondary gadget is represented by $x_{i,j}$. Hence, it follows that (recall Definition 4.3) the only $2 - 3$ edge from $M_i,j$ in $N$ is $M_{i,j}/2(\lambda_{i,j}, \delta_{i,j}) \rightarrow M_{i,j}(3(\lambda_{i,j}, \delta_{i,j}))$ and the only $0 - 1$ edge from $VS_{i,j}$ in $N$ is $VS_{i,j}(0_{x_{i,j}}) \rightarrow VS_{i,j}(1_{x_{i,j}})$. By Corollary 4.17, $N$ contains exactly one bottom-red edge incident on $M_{i,j}$. Since this is the only incoming edge into $VS_{i,j}$ it follows that $x_{i,j} = \lambda_{i,j}$. \hfill $\square$

Lemma 4.19. The Grid Tiling instance $(\ell, n, \{S_{i,j} : i, j \in [\ell]\})$ has a solution.

**Proof.** By Lemma 4.18, it follows that for each $1 \leq i, j \leq \ell$ we have $x_{i,j} = \lambda_{i,j} = x_{i,j+1}$ and $y_{i,j} = \delta_{i,j} = y_{i+1,j}$ in addition to $(\lambda_{i,j}, \delta_{i,j}) \in S_{i,j}$ (by the definition of the main gadget). This implies that Grid Tiling has a solution. \hfill $\square$

4.5 Finishing the proof of Theorem 1.5

There is a simple reduction [17, Theorem 14.28] from $\ell$-CLIQUE on $n$ vertex graphs to $(\ell, n)$-GRID TILING. Combining the two directions from Section 4.3 and Section 4.4 gives a parameterized reduction from $(\ell, n)$-GRID TILING to SCSSPLANAR on graphs with $(n + \ell)^{O(1)}$ vertices and $k = O(\ell^2) = p$. Composing the two reductions, we get a parameterized reduction from $\ell$-CLIQUE on $n$-vertex graphs to SCSSPLANAR on $(n + \ell)^{O(1)}$ vertex graphs with $k = O(\ell^2) = p$. Hence, the W[1]-hardness of SCSSPLANAR parameterized by $(k + p)$ follows from the W[1]-hardness of $\ell$-CLIQUE parameterized by $\ell$. Moreover, Chen et al. [10] showed that, for any function $f$, the existence of an $f(\ell) \cdot n^{o(\ell)}$ algorithm for CLIQUE violates ETH.

Hence, we obtain that, under ETH, there is no $f(k, p) \cdot n^{o(\sqrt{k + p})}$ time algorithm for SCSSPLANAR.

Suppose now that there is an algorithm $A$ which runs in time $f(k, p, \varepsilon) \cdot n^{o(\sqrt{k + p} + 1/f)}$ (for some computable function $f$) and computes an $(1 + \varepsilon)$-approximate solution for SCSSPLANAR. Recall that our reduction works as follows: Grid Tiling answers YES if and only if SCSSPLANAR has a solution of cost $B^* < 43\ell^2 < 44\ell^2$. Consequently, running $A$ with $\varepsilon = \frac{1}{44\ell^2}$ implies that $(1 + \varepsilon) \cdot B^* < B^* + 1$. Every edge of our constructed graph $G$ has weight at least 1, and hence a $(1 + \varepsilon)$-approximation is in fact forced to find a solution of cost at most $B^*$, i.e., $A$ finds an optimum solution. Since $k = O(\ell^2)$, $p = O(\ell^2)$ and $1/\varepsilon = O(\ell^2)$ it follows $f(k, p, \varepsilon) \cdot n^{o(\sqrt{k + p} + 1/f)} = g(\ell) \cdot n^{o(\ell)}$ for some computable function $g$. By the previous paragraph, this is not possible under ETH.

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We start with a simple observation about the structure of the graph $G'$. While the $x$-, $y$-, and $z$-paths are bidirected, they — together with the $p$-grids — are arranged in a DAG-like fashion. That is, there are directed arcs from $x$-paths to $z$-paths, from $z$-paths to $y$-paths, from $x$-paths to $p$-grids, and from $p$-grids to $y$-paths, but all cycles in $G'$ are contained in one $x$-, $y$-, or $z$-path.

Consider first the terminal pair $(s_{0-n}^y, t_{0-n}^y)$. The out-neighbors of $s_{0-n}^y$ are the endpoints $s_{0}^{x_i}$ for every pair $(i, j)$ with $1 \leq i, j \leq n$, $i \neq j$; the only in-neighbors of $t_{0-n}^y$ are the endpoints $z_{0}^{x_j}$ for every $1 \leq i \leq n$. Thus, by the previous observation, the only paths from $s_{0-n}^y$ to $t_{0-n}^y$ in the graph $G'$ start by going to some vertex $x_{0}^{x_i}$, traverse the $x$-path for the pair $(i, j)$ up to some vertex $x_{a}^{x_i}$, use the arc $(x_{a}^{x_i}, z_{a}^{x_j})$ to fall to the $z$-path for the color class $i$, and then traverse this $z$-path to the endpoint $z_{0}^{x_j}$. However, all such paths for $a \geq \alpha(i)$ are cut by the vertex $x_{a}^{x_i}(i) \in X$, while all such paths for $a < \alpha(i)$ are cut by the vertex $z_{a}^{x_j}(i) \in X$. Consequently, the terminal pair $(s_{0-n}^y, t_{0-n}^y)$ is separated in $G' \setminus X$.

A similar argument holds for the pair $(s_{0-n}^y, t_{0-n}^y)$. By the same reasoning, the only paths between $s_{0-n}^y$ and $t_{0-n}^y$ in the graph $G'$ are paths that start by going to some vertex $z_{0}^{x_j}$, traverse the $z$-path for the color class $i$, use the arc $(z_{a}^{x_j}, y_{a}^{x_j})$ for some $j \neq i$ to fall to the $y$-path for the pair $(i, j)$, and then continue along this $y$-path to the vertex $y_{0}^{x_j}$. However, all such paths for $a \geq \alpha(i)$ are cut by the vertex $y_{a}^{x_j}(i) \in X$, while all such paths for $a < \alpha(i)$ are cut by the vertex $y_{a}^{x_j}(i) \in X$. Let us now focus on the terminal pair $(s_{0-n}^y, t_{0-n}^y)$. Observe that there are two types of paths from $s_{0-n}^y$ to $t_{0-n}^y$ in the graph $G'$. The first type consists of paths that starts by going to some vertex $x_{0}^{x_j}$ where $i < j$, traverse the $x$-path for the pair $(i, j)$ up to some vertex $x_{a}^{x_j}$, use the arc $(x_{a}^{x_j}, p_{a,1})$ to fail to the $p$-grid for the pair $(i, j)$, traverse this $p$-grid up to a vertex $p_{b,n}$ where $b \geq a$, use the arc $(p_{b,n}, y_{b-1})$ to fall to the $y$-path for the pair $(i, j)$, and then traverse this path to the endpoint $y_{0}^{x_j}$. These paths are cut by $X$ as follows: the paths where $a < \alpha(i)$ are cut by $x_{a}^{x_j}(i) \in X$, the paths where $b > \alpha(i)$ are cut by $y_{a}^{x_j}(i) \in X$, while the paths where $a + b = \alpha(i)$ are cut by the vertex $p_{a,1}^{x_j}(i, \alpha(j)) \in X$; note that the $\alpha(i)$-th row of the grid is the only path from $p_{a,1}^{x_j}(i, \alpha(j))$ to $p_{a,1}^{x_j}(i, \alpha(j))$. Please observe that the terminal $t_{0-n}^y$ cannot be reached from the $p$-grid for the pair $(i, j)$ by going to the other $y$-path reachable from this $p$-grid, namely the $y$-path for the pair $(j, i)$; the $y$-path for the pair $(j, i)$ has only outgoing arcs to the terminal $t_{0-n}^y$ since $j > i$.

A similar argument holds for the pair $(s_{0-n}^y, t_{0-n}^y)$. The paths going through an $x$-path for a pair $(i, j)$, $i > j$, the $z$-path for the color class $i$, and the $y$-path for a pair $(i, j)$, $i > j$, are cut by vertices $x_{a}^{x_j}(i), y_{a}^{x_j}(i), p_{a,1}^{x_j}(i, \alpha(j)) \in X$. The paths going through an $x$-path for a pair $(i, j)$, $i > j$, the $p$-grid for the pair $(i, j)$, and the $y$-path for a pair $(i, j)$, are cut by the vertices $x_{a}^{x_j}(i), y_{a}^{x_j}(i), p_{a,1}^{x_j}(i, \alpha(j))$. Again, it is essential that the other $y$-path reachable from the $p$-grid for the pair $(j, i)$, namely the $y$-path for the pair $(j, i)$, does not have outgoing arcs to the terminal $t_{0-n}^y$, but only to the terminal $t_{0-n}^y$.

We infer that $X$ is a solution for the instance $(G', \mathcal{F}')$ of DIRECTED MULTICUT WITH 4 PAIRS.