Extending $\mathcal{PT}$ symmetry from Heisenberg algebra to E2 algebra

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The E2 algebra has three elements, $J$, $u$, and $v$, which satisfy the commutation relations $[u, J] = iv$, $[v, J] = -iu$, $[u, v] = 0$. We can construct the Hamiltonian $H = J^2 + gu$, where $g$ is a real parameter, from these elements. This Hamiltonian is Hermitian and consequently it has real eigenvalues. However, we can also construct the $\mathcal{PT}$ symmetric and non-Hermitian Hamiltonian $H = J^2 + igu$, where again $g$ is real. As in the case of $\mathcal{PT}$-symmetric Hamiltonians constructed from the elements $x$ and $p$ of the Heisenberg algebra, there are two regions in parameter space for this $\mathcal{PT}$-symmetric Hamiltonian, a region of unbroken $\mathcal{PT}$ symmetry in which all the eigenvalues are real and a region of broken $\mathcal{PT}$ symmetry in which some of the eigenvalues are complex. The two regions are separated by a critical value of $g$.

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I. INTRODUCTION

The Heisenberg algebra for a quantum mechanical system having one degree of freedom consists of three elements: the coordinate operator $x$, the momentum operator $p$, and the unit element 1. These three elements obey the single commutation relation

$$[x, p] = 1i \quad (1)$$

This algebra possesses two independent discrete symmetries under which the commutation relation remains invariant. The first symmetry, called parity (space reflection), is represented by the linear operator $\mathcal{P}$, where $\mathcal{P}^2 = 1$. Under the action of $\mathcal{P}$ both $x$ and $p$ change sign:

$$\mathcal{P} x \mathcal{P} = -x, \quad \mathcal{P} p \mathcal{P} = -p. \quad (2)$$

The second symmetry, called time reversal, is represented by the antilinear operator $\mathcal{T}$, where $\mathcal{T}^2 = 1$. Under the action of $\mathcal{T}$ both $p$ and $i$ change sign, but $x$ does not:

$$\mathcal{T} p \mathcal{T} = -p, \quad \mathcal{T} i \mathcal{T} = -i, \quad \mathcal{T} x \mathcal{T} = x. \quad (3)$$

In quantum mechanics the Hamiltonian operator $H$ is expressed in terms of the operators $x$ and $p$: $H = H(x, p)$. It is conventional to require that the Hamiltonian be Hermitian so that the eigenvalues of $H$ are real. However, in 1998 it was shown that the Hamiltonian

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need not be Hermitian for the eigenvalues to be real \[1,2\]. In that paper the family of non-Hermitian Hamiltonians
\[ H = p^2 + x^2(ix)^\epsilon, \] (4)
was introduced, and it was shown that the eigenvalues of these \(\mathcal{PT}\)-symmetric Hamiltonians are real when the parameter \(\epsilon \geq 0\). A rigorous proof of spectral reality is given in Refs. \[3,4\]. The parametric range \(\epsilon \geq 0\) is referred to as a region of unbroken \(\mathcal{PT}\) symmetry; in this region all the eigenfunctions of \(H\) are also eigenfunctions of the \(\mathcal{PT}\) operator. In the parametric region \(\epsilon < 0\) some of the eigenvalues are complex; this range of \(\epsilon\) is said to be a region of broken \(\mathcal{PT}\) symmetry.

A second conventional reason for requiring that the Hamiltonian \(H\) be Hermitian is that \(H\) determines the time evolution of the theory, and if \(H\) is Hermitian, then the time evolution is unitary (probability conserving). However, in 2002 it was shown that if the \(\mathcal{PT}\) symmetry of a non-Hermitian Hamiltonian is unbroken, then the time evolution is unitary \[5\].

In general, Hermitian Hamiltonians differ from \(\mathcal{PT}\)-symmetric Hamiltonians in that the spectrum of a Hermitian Hamiltonian is always real while the spectrum of a non-Hermitian \(\mathcal{PT}\)-symmetric Hamiltonian often has a parametric region of unbroken \(\mathcal{PT}\) symmetry where the eigenvalues are all real and a region of broken \(\mathcal{PT}\) symmetry where some of the eigenvalues are complex. The boundary between these two regions is a phase transition, and this phase transition has recently been observed in several different laboratory experiments \[6\|9\].

In this paper we consider the algebra \(E_2\), which is more complicated than the Heisenberg algebra. This algebra has three elements, which are designated \(J\), \(u\), and \(v\), and these elements obey the commutation relations
\[ [u,J] = iv, \quad [v,J] = -iu, \quad [u,v] = 0. \] (5)
This operator algebra arises when one considers a two-dimensional quantum system restricted to a ring of radius \(r\). We can represent the operators \(J\), \(u\), and \(v\) in polar form as
\[ J = -i \frac{\partial}{\partial \theta}, \quad u = \sin \theta, \quad v = \cos \theta, \] (6)
where we have taken \(r = 1\). The \(E_2\) algebra is more complicated than the Heisenberg algebra \[1\], but it reduces to the Heisenberg algebra in the limit as \(r \to \infty \) \[10\].

Like the Heisenberg algebra, the \(E_2\) algebra is separately invariant under each of two different symmetries, parity \(\mathcal{P}\) and time reversal \(\mathcal{T}\). We define a parity transformation as a reflection through the center of the ring. Thus, a point on the ring is mapped to a point on the opposite side of the ring such that \(\theta \to \theta + \pi\). Then, under a parity transformation
\[ \mathcal{P}J\mathcal{P} = J, \quad \mathcal{P}u\mathcal{P} = -u, \quad \mathcal{P}v\mathcal{P} = -v, \] (7)
which clearly leaves \(5\) invariant \[11\]. Time reversal changes the sign of \(i\), and thus its effect is to reverse the sign of \(J\) but to leave \(u\) and \(v\) invariant. Thus, the time-reversal transformation
\[ \mathcal{T}J\mathcal{T} = -J, \quad \mathcal{T}u\mathcal{T} = u, \quad \mathcal{T}v\mathcal{T} = v \] (8)
also leaves the \(E_2\) algebra \(5\) invariant.

This paper is organized very simply: In Sec. \[11\] we construct a Hamiltonian in terms of the elements of the \(E_2\) algebra. This Hamiltonian contains a coupling-constant parameter
$g$; if $g$ is real, the Hamiltonian is Hermitian, and if $g$ is imaginary, the Hamiltonian is non-Hermitian but $\mathcal{PT}$ symmetric. We show that for real $g$ the eigenvalues are all real and that if $g$ is imaginary, there are regions of broken and unbroken $\mathcal{PT}$ symmetry. Finally, in Sec. III we give some brief concluding remarks and discuss possible future directions for research.

II. HERMITIAN AND $\mathcal{PT}$-SYMMETRIC HAMILTONIANS CONSTRUCTED FROM THE ELEMENTS OF E2

The operators $J$, $u$, and $v$ are Hermitian, and thus it is easy to construct a Hermitian Hamiltonian in terms of these operators. One such Hamiltonian is

$$H = J^2 + gv,$$

(9)

where $g$ is a real parameter. For this Hamiltonian, the Schrödinger eigenvalue differential equation takes the form

$$-\psi''(\theta) + g \cos(\theta) \psi(\theta) = E \psi(\theta),$$

(10)

which is the Mathieu equation [12]. This is the equation for a quantum pendulum.

To find the eigenvalues $E$ of (10) we must impose boundary conditions. The simplest such boundary conditions express the bosonic requirement that the eigenfunctions be single valued on the ring:

$$\psi(\theta + 2\pi) = \psi(\theta).$$

(11)

However, instead of imposing the $2\pi$-periodic boundary conditions in (11), we can also impose the fermionic boundary requirement that the eigenfunctions be $2\pi$ anti-periodic:

$$\psi(\theta + 2\pi) = -\psi(\theta).$$

(12)

Because these boundary conditions are homogeneous and the Schrödinger equation (10) is symmetric under $\theta \rightarrow -\theta$, the eigenfunctions will be either odd or even in $\theta$.

Let us examine first the simple case $g = 0$. The general solution to (10) for this case is

$$\psi(x) = A \sin \left(\sqrt{E} \theta\right) + B \cos \left(\sqrt{E} \theta\right).$$

(13)

Thus, there are four sets of eigenvalues: For odd bosonic eigenfunctions $B = 0$, and we get

$$E_n = n^2 \quad (n = 1, 2, 3, \ldots);$$

(14)

for even bosonic eigenfunctions $A = 0$, and we get

$$E_n = n^2 \quad (n = 0, 1, 2, 3, \ldots);$$

(15)

for odd fermionic eigenfunctions $B = 0$, and we get

$$E_n = \frac{1}{4} n^2 \quad (n = 1, 3, 5, 7, \ldots);$$

(16)

for even fermionic eigenfunctions $A = 0$, and we get

$$E_n = \frac{1}{4} n^2 \quad (n = 1, 3, 5, 7, \ldots).$$

(17)
FIG. 1: Odd bosonic eigenvalues for the Schrödinger equation (7) plotted as a function of real $g$. The spectrum for $g = 0$ is given in (14).

FIG. 2: Even bosonic eigenvalues for the Schrödinger equation (7) plotted as a function of real $g$. The spectrum for $g = 0$ is given in (15).

For $g \neq 0$ we use Mathematica to plot the eigenvalues as functions of $g$. The odd bosonic eigenvalues are shown in Fig. 1 and the even bosonic eigenvalues are shown in Fig. 2. Note that because the Hamiltonian is Hermitian, the eigenvalues are all real.

Now let us see what happens if we take the parameter $g$ in the Hamiltonian (9) to be pure imaginary. This choice of $g$ makes the Hamiltonian non-Hermitian but $\mathcal{PT}$-symmetric. For $\text{Im } g \neq 0$ we again use Mathematica to plot the eigenvalues as functions of $\text{Im } g$. Because
FIG. 3: Odd bosonic eigenvalues for the $\mathcal{PT}$-symmetric Hamiltonian (9) in which the parameter $g$ is pure imaginary. The eigenvalues are plotted as functions of $\text{Im} \, g$. The real (imaginary) parts of the eigenvalues are shown in the left (right) panel. Observe that the eigenvalues are all real when $-3.4645 < \text{Im} \, g < 3.4645$; this is the region of unbroken $\mathcal{PT}$ symmetry. There is an infinite sequence of critical points; the next critical points are at $\text{Im} \, g = \pm 15.0485$ and at $\pm 34.7994$.

the Hamiltonian is no longer Hermitian, some of the eigenvalues are complex. The real (left panel) and imaginary (right panel) parts of the eigenvalues are shown in Figs. 3 and 4 with the odd bosonic eigenvalues in Fig. 3 and the even bosonic eigenvalues in Fig. 4. The key feature of the spectrum is that all of the eigenvalues are real if $\text{Im} \, g$ lies between the critical values $-3.4645$ and $3.4645$ for the odd bosonic eigenvalues and between $-0.7344$ and $0.7344$ for the even bosonic eigenvalues. This is the region of unbroken $\mathcal{PT}$ symmetry. As $|\text{Im} \, g|$ increases past these critical points, the lowest two eigenvalues become degenerate and move into the complex plane as a complex-conjugate pair. Thus, we have entered the regions of broken $\mathcal{PT}$ symmetry. In fact, there is an infinite sequence of critical points: The next two lowest pairs of eigenvalues become degenerate and move into the complex plane at the critical points $\pm 15.0485$ and $\pm 34.7994$ for the odd bosonic eigenvalues and at $\pm 8.2356$ and $\pm 23.9030$ for the even bosonic eigenvalues.

Figures 3 and 4 give detailed plot of the transition from unbroken to broken $\mathcal{PT}$ symmetry. Observe that the eigenvalues become degenerate in pairs and that the real and imaginary parts of the eigenvalues make $90^\circ$ turns at the critical points. This is a clear indication that the critical points are square-root branch points.

III. CONCLUDING REMARKS

We have shown in this paper that the well studied properties of $\mathcal{PT}$-symmetric quantum mechanical Hamiltonians that are constructed from the elements of the Heisenberg algebra extend to Hamiltonians that are constructed from the elements of the E2 algebra. Both algebras are individually invariant under parity reflection $\mathcal{P}$ and under time reversal $\mathcal{T}$. If a Hamiltonian that is constructed from the elements of either of these algebras is Hermitian, then its eigenvalues are all real. However, if the Hamiltonian is non-Hermitian and $\mathcal{PT}$ symmetric, then there may be regions of unbroken and unbroken $\mathcal{PT}$ symmetry.

It is interesting that for the fermionic eigenvalues of the Hamiltonian (9) there is no region of unbroken $\mathcal{PT}$ symmetry; that is, the eigenvalues are all complex when $g$ is nonzero.
FIG. 4: Even bosonic eigenvalues for the $\mathcal{PT}$-symmetric Hamiltonian (9) plotted as functions of $\text{Im} g$. The real (imaginary) parts of the eigenvalues are shown in the left (right) panel. The eigenvalues are all real when $-0.7344 < \text{Im} g < 0.7344$; this is the region of unbroken $\mathcal{PT}$ symmetry. In the regions of broken $\mathcal{PT}$ symmetry there is an infinite sequence of critical points; the next critical points are at $\text{Im} g = \pm 8.2356$ and at $\pm 23.9030$.

FIG. 5: Blow-up of the region near the critical points at $\text{Im} g = \pm 3.4645$ on Fig. 3. The imaginary parts of the two lowest energy levels vanish until $\text{Im} g$ passes a critical point. At this point the two energy levels become degenerate and $\text{Im} E(g)$ for each energy level suddenly makes a $90^\circ$ turn. This is the typical behavior of a function near a square-root singularity.

and purely imaginary, as shown in Fig. 7. Note from (16) and (17) that the odd and even fermionic eigenvalues are degenerate when $g = 0$. When $\text{Im} g \neq 0$, this degeneracy is split, and the eigenvalues become complex-conjugate pairs. Thus, the condition of $\mathcal{PT}$ symmetry seems to exclude real fermionic eigenvalues. The nonexistence of fermionic eigenvalues was already observed in earlier studies of $\mathcal{PT}$-symmetric crystal lattices [13, 14]. In these studies the discriminant was calculated as a function of the energy $E$. For Hermitian periodic potentials the discriminant $D(E)$ is a smooth, real oscillatory function of $E$. The inequality $|D(E)| < 2$ identifies a band of allowed energies and the inequality $|D(E)| > 2$ defines a gap of forbidden energies. At the band edge $D(E) = 2$ the eigenfunction is bosonic ($2\pi$ periodic) and at the band edge $D(E) = -2$ the eigenfunction is fermionic ($2\pi$ antiperiodic). In Refs. [13, 14] it was found that for $\mathcal{PT}$-symmetric periodic potentials half of the gaps
Fig. 6: Blow-up of the region near the critical points at $\text{Im } g = 0.7344$ on Fig. 4. As in Fig. 5, the imaginary part of the energies of the two lowest states is 0 until $\text{Im } g$ reaches a critical point. At this point the energy levels merge and become a complex-conjugate pair.

Fig. 7: Fermionic eigenvalues for the $\mathcal{PT}$-symmetric Hamiltonian (9) plotted as functions of $\text{Im } g$. The real (imaginary) parts of the eigenvalues are shown in the left (right) panel. The eigenvalues are all complex when $\text{Im } g \neq 0$; thus, there is no region of unbroken $\mathcal{PT}$ symmetry. The eigenvalues for both even and odd eigenfunctions are shown; the even and odd eigenvalues form complex-conjugate pairs. Five pairs of eigenvalues are shown in the figure.

Disappeared and at the band edges the eigenfunction is only bosonic.

The current work complements the work in Refs. [13, 14]. Rather than calculating the discriminant as a function of the energy $E$, we have calculated the energies of periodic bosonic and antiperiodic fermionic eigenfunctions as functions of the coupling constant $g$. We find while bosonic eigenfunctions have a region of unbroken $\mathcal{PT}$ symmetry, fermionic eigenfunctions do not have such a region.

There are some natural continuations of this research. It is important to verify that the higher algebras E3, E4, and so on, are also invariant under $\mathcal{P}$ and $\mathcal{T}$ transformations and that non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians constructed from the elements of these algebras have regions of unbroken and broken $\mathcal{PT}$ symmetry. Furthermore, it would be most interesting to calculate the $\mathcal{C}$ operator [5] in the unbroken $\mathcal{PT}$-symmetric regions.
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