Compactly Generated Shape Index for Infinite-Dimensional Local Dynamical Systems on Complete Metric Spaces

Jintao Wang†
Center for Mathematical Sciences, Huazhong University of Science and Technology, Wuhan 430074, China

Jinqiao Duan‡
Department of Applied Math., Illinois Institute of Technology, Chicago IL 60616, USA

Desheng Li§
Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, China

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†E-mail address: wangjt@hust.edu.cn
‡Corresponding author, E-mail address: duan@iit.edu
§E-mail address: lidsmath@tju.edu.cn
Abstract  We establish a theory of compactly generated shape index for local semiflows on complete metric spaces via more general shape index pairs. The main advantages are that the quotient space $N/E$ is not necessarily metrisable for the shape index pair $(N, E)$ and $N \setminus E$ need not to be a neighbourhood of the compact invariant set $K$. In this new index theory, we can calculate the shape index of $K$ in every closed subset that contains a local unstable manifold of $K$, and define the shape cohomology index of $K$ to develop the Morse equations. This provides a more effective way to calculate shape indices and Morse equations theoretically and specifically for infinite dimensional systems, without particular requirements on the index pairs or the unstable manifolds.

Keywords:  Local semiflows; Compactly generated shape; Shape index; Čech cohomology theory; Morse equations.

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1 Introduction

Index theory plays a significant role in the development of dynamical systems. The well-known indices includes topological degree, Morse index, Conley index, Maslov index and shape index. As a generalisation of Morse index, Conley index was first introduced by Conley and his group for flows on locally compact spaces in 1970s [3] to describe the topology of invariant sets. Later on this theory was extended to local semiflows on complete metric spaces by Rybakowski et al. [25]. And as a result, Conley index can be successfully applied to infinite-dimensional dynamical systems. We briefly present the basic idea of Conley index theory below.

Let $K$ be a compact isolated invariant set of a semiflow $\Phi$ on a complete metric space $X$. Some appropriate homotopies induced by the semiflow $\Phi$ can help to show that all the pointed quotient spaces $(N/E, [E])$ of Conley index pairs $(N, E)$ have the same homotopy type. Recall that a Conley index pair is a pair of suitable closed sets $(N, E)$, where $N$ is an isolating neighbourhood of $K$, and $E$ is an exit set of $N$. Then the homotopy Conley index $h(\Phi, K)$ of $K$ is defined to be the homotopy type of the pointed space $(N/E, [E])$.

For some more general invariant sets that may be very complicated in their topological structures, shape is another type of suitable concept to describe their topological properties. Shape is indeed a generalisation of homotopy type, and was first invented by Borsuk [2] for metric spaces. Since spaces with the same homotopy type have also the same shape, for the compact isolated invariant set $K$ given above, one can immediately define the shape index $s(K)$ of $K$ as

$$s(K) = \text{Sh}(N/E, [E]),$$

(1.1)

where $(N, E)$ is a suitable index pair and Sh denotes the shape functor. This setting is the basic idea of defining shape index of the compact isolated invariant set $K$ in [35] and this present paper.

Originally, shape (Borsuk’s shape) index was first introduced by Robbin and Salamon [23] for the flows on a compact smooth manifold. Their approach to the shape index theory was further developed in the works of Mrozek [22] and Sánchez-Gabites [26] for dynamical systems on locally compact spaces. We give a brief description of Robbin and Salamon’s approach in [23] as follows.

When they considered a compact isolated invariant set $K$ of a flow on a compact smooth manifold, Robbin and Salamon introduced a certain intrinsic topology for the unstable manifold $W^u(K)$ of $K$, by which they obtained a homeomorphism between
the quotient space \( N/E \) of an index pair \((N, E)\) and the one-point compactification of \( W^u(K) \). Hence the quotient space \( N/E \) and the one-point compactification have the same (Borsuk’s) shape, which was defined as the shape index of \( K \). This shape index can be obtained by either the unstable manifold \( W^u(K) \) or using index pairs; and so one can compute the shape index and Morse equation of an invariant set \( K \) by using the unstable manifold of \( K \) and its Morse sets.

The case where the phase space is non-locally compact is much more complicated. Kapitanski and Rodnianski in \([13]\) proved that the global attractor of a semiflow on complete metric spaces has Borsuk’s shape of the phase space and developed an elementary Morse theory for attractors. Their work was extended to isolated invariant sets of flows on locally compact metric spaces in a recent paper \([29]\) by Sanjurjo, in which the author also considered semiflows on non-locally compact spaces (see \([29]\), Section 6). It was shown that if a semiflow \( \Phi \) is two-sided when it is restricted on the unstable manifold \( W^u(K) \), then the shape index \( s(K) \) of \( K \) can be successfully calculated via its unstable manifold.

For more general cases, the authors \([35]\) in 2015 used the quotient flow to establish the shape Conley index theory via index pairs mentioned above. The semiflow \( \Phi \) is assumed to be local, asymptotically compact on complete metric spaces and more generally, is not supposed to be two-sided on the unstable manifolds. The index pair \((N, E)\) of isolated invariant sets used therein is a shape index pair, different from Conley index pairs. The shape index pair \((N, E)\) of a compact isolated invariant set \( K \) is a pair of closed sets \( N \) and \( E \) possessing the following properties:

(i) \( N \setminus E \) is strongly admissible with \( E \) being an exit set of \( N \);

(ii) \( K \) is the maximal compact invariant set in \( N \setminus E \); and

(iii) \( N \setminus E \) contains a local unstable manifold of \( K \).

To use the Borsuk’s shape to give the definition \((1.1)\), we need some additional assumptions to guarantee the metrisability of \( N/E \) under the quotient topology for a shape index pair \((N, E)\). It is clear that this shape index pairs can be constructed by using local unstable manifolds \( W^u_N(K) \) and their appropriate sections, since the compactness of \( W^u_N(K) \) \((25)\) can make the quotient space be metrisable.

In this paper, we employ the compactly generated shape defined for general Hausdorff spaces in Rubin and Sanders \([24]\), which allows us to remove the additional assumptions in \([35]\), such as the separability of the phase space and the compactness.
of the exit set (this was not mentioned clearly in [35]). Then we develop a new type of shape index theory for local semiflows on complete metric spaces via much more general index pairs \((N, E)\).

Since \(N/E\) is a normal Hausdorff space for a closed pair \((N, E)\) in a metric space, we adopt the compactly generated shape (H-shape for short, denoted by \(Sh_H\)). We still use the shape index pair \((N, E)\) stated above for a compact isolated invariant set without the condition that the quotient space \(N/E\) is metrisable. Thanks to the consequence in [6] that the compact global attractor and the whole phase space have the same H-shape for semiflows on Hausdorff spaces, we prove that the pointed spaces \((N/E, [E])\) have the same H-shape for all shape index pairs \((N, E)\), by the similar strategy in [35]. Thus we can define the compactly generated shape index (H-shape index) \(s(K)\) as the H-shape \(Sh_H(N/E, [E])\) of \((N/E, [E])\).

Thanks to the condition (iii) in the definition of shape index pair presented above, we do not require to pick index pairs of \(K\) in a neighbourhood of \(K\). This prompts us to relax the condition “S-continuity” of a family of semiflows and isolated invariant sets for the famous property “continuation” (Section 1.12 of [25]). For H-shape index, we replace S-continuity by a weaker condition — H-continuity (see Subsection 4.3 for details). This brings us a lot of convenience in application.

Based on this H-shape index and similar to the Conley index, we also hope to present the corresponding Morse theory of a compact isolated invariant set by means of certain cohomology theory, as the Čech cohomology theory for Conley index theory.

Note that, H-shape is defined via the direct systems of compact subspaces of the given Hausdorff space \(X\) and (ANR-)shape maps between them. When considering the Čech cohomology, which is ANR-shape (see Section 4 below) invariant (cf. [20]), we obtain an inverse system \(G^*\) of Čech cohomology groups of the compact subspaces of \(X\). We know from Appendix 3.F in Chapter 3 of [9] that, the inverse limit of the inverse system \(G^*\) may not be isomorphic to the Čech cohomology group of \(X\). But the inverse limits are equivalent for equivalent inverse systems. Then it arises a question that whether the Čech cohomology group is H-shape invariant. However, we can avoid answering this question by considering the H-shape cohomology theory.

Due to the fact that the pointed space \((N/E, [E])\) has an H-shape of compact pointed space for each shape index pair \((N, E)\) of a compact isolated invariant set \(K\), we are allowed to define the cohomology index of \(K\) as the Čech cohomology group \(\hat{H}^n(N/E, [E])\) for a compact shape index pair \((N, E)\). This is sufficient for us to develop the Morse theory.

In our situation, each bounded Conley index pair defined in [25] is a shape index pair.
and can be perfectly applied to giving the H-shape index. We see that the shape index $s(K)$ and Morse equations of an isolated invariant set $K$ can be calculated by using either the Conley index pairs or unstable manifolds of $K$ and its Morse sets. What is more, one can calculate the shape index $s(K)$ simply by using only a suitable closed set containing a local unstable manifold of $K$. This greatly increases the flexibility of the calculation of indices and Morse equations. This convenience will be revealed in the examples and applications in the latter sections.

This paper is organised as follows. In Section 2 we present some necessary notions and results in the theory of homotopy on quotient spaces and dynamical system on Hausdorff spaces. Some necessary properties of quotient flows defined on quotient spaces of Ważewski pairs are given in Section 3. Section 4 is the central part of this paper, in which we introduce the concept of shape index pairs, define compactly generated shape indices for isolated invariant sets and illustrate the continuation property by a simple example. Section 5 consists of the definition of H-shape cohomology index and the establishment of Morse equations. In Section 6, we apply the H-shape index theory to the $p$-Laplacian evolutionary system to illustrate that our approach here may increase the flexibility of the calculation of shape indices and offer a better way to calculate Morse equations of isolated invariant sets. In Section 7, we consider an application of H-shape index to an abstract retarded nonautonomous functional differential equation and obtain an interesting consequence for bounded full solutions.

2 Preliminaries

In this section we collect some necessary notions and results in the theory of topology and dynamical systems on Hausdorff spaces (see [17]). The reader is supposed to be familiar with basic knowledge of algebraic topology.

2.1 HEP and homotopy equivalence

Let $X$ be a topological space. Given a closed subset $A$ of $X$, the pair $(X, A)$ is said to have the homotopy extension property (HEP for short), if for every space $Y$ and continuous mapping $F : X \times \{0\} \cup A \times I \to Y$, there exists a continuous map $\tilde{F} : X \times I \to Y$ such that $\tilde{F}$ is an extension of $F$.

**Proposition 2.1** ([25]). The pair $(X, A)$ has the HEP if and only if $A$ is a strong deformation retract of one of its open neighbourhoods.
Let $A$ and $B$ be two closed subsets of $X$. The quotient space $B/A$ is defined as follows.

If $A \neq \emptyset$, then the space $B/A$ is obtained by collapsing $A$ to a single point $[A]$ in $B \cup A$. If $A = \emptyset$, we choose a single isolated point $\ast \notin B$ and define $B/A$ to be the space $B \cup \{\ast\}$ equipped with the sum topology. In the latter case we still use the notation $[A]$ to denote the base point $\ast$.

We have a homotopy equivalence of quotient spaces as follows, see [35].

**Proposition 2.2.** Let $A$ and $B$ be two closed subsets of $X$. Suppose $(X, A)$ has the HEP and that $B$ is a strong deformation retract of $A$. Then $(X/A, [A]) \simeq (X/B, [B])$.

### 2.2 Local semiflows

Let $X$ be a topological space. The space $X$ is always assumed to be a Hausdorff topological space and sometimes to be a metric space if necessary.

**Definition 2.3.** A local semiflow $\Phi$ on $X$ is a continuous map $\Phi : D(\Phi) \to X$, where $D(\Phi)$ is an open subset of $\mathbb{R}^+ \times X$, and $\Phi$ enjoys the following properties:

1. For each $x \in X$, there exists $0 < T_x \leq \infty$ such that
   $$(t, x) \in D(\Phi) \iff 0 \leq t < T_x;$$

2. $\Phi(0, x) = x$ for all $x \in X$;

3. If $(t + s, x) \in D(\Phi)$, where $t, s \in \mathbb{R}^+$, then $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$.

The number $T_x$ in (1) is called the maximal existence time of $\Phi(t, x)$, and $D(\Phi)$ is called the domain of $\Phi$.

In the case when $D(\Phi) = \mathbb{R}^+ \times X$, we simply call $\Phi$ a global semiflow.

Let $\Phi$ be a given local semiflow on $X$. For notational convenience, we will rewrite $\Phi(t, x)$ as $\Phi(t)x$.

A subset $N$ of $X$ is said to be admissible, if for arbitrary sequences $x_n \in N$ and $t_n \to +\infty$ with $\Phi([0, t_n])x_n \subset N$ for all $n$, the sequence of the end points $\Phi(t_n)x_n$ has a convergent subsequence. A subset $N \subset X$ is strongly admissible, if

1. $N$ is admissible; and

2. $\Phi$ does not explode in $N$, i.e., we have $T_x = \infty$ whenever $\Phi([0, T_x])x \subset N$ for all $x \in N$. 


Since $X$ may be an infinite-dimensional space, to overcome the difficulty due to the lack of compactness of $X$, we always assume that $\Phi$ is asymptotically compact, that is, each bounded subset of $X$ is admissible. It is well known that this condition is naturally satisfied by many important examples from applications, see [8,25,31,34].

A solution (or trajectory) on an interval $J \subset \mathbb{R}^1$ is a map $\gamma : J \to X$ satisfying

$$
\gamma(t) = \Phi(t-s)\gamma(s), \quad \text{for all } s, t \in J, \ s \leq t.
$$

A full solution $\gamma$ is a solution defined on the whole line $\mathbb{R}^1$. If $x \in X$ is such that $\Phi(t)x = x$ for all $t \geq 0$, we say $x$ is an equilibrium.

The $\omega$-limit set and $\alpha$-limit set of a solution $\gamma$ are defined as follows. If $\gamma$ is defined on an interval containing $[0, \infty)$, it is defined that

$$
\omega(\gamma) = \{ y \in X : \text{there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y \}.
$$

If $\gamma$ is defined on an interval containing $(-\infty, 0]$, it is defined that

$$
\alpha(\gamma) = \{ y \in X : \text{there exists } t_n \to -\infty \text{ such that } \gamma(t_n) \to y \}.
$$

For an $x \in X$ with $T_x = \infty$, we define $\omega(x) = \omega(\gamma)$ with $\gamma(t) = \Phi(t)x$ for $t \geq 0$.

A set $A$ is said to be invariant if $\Phi(t)A = A$ for all $t \geq 0$. For $A \subset X$, we denote by $I(A)$ the maximal invariant set in $A$. When a closed set $A$ is strongly admissible, one can easily verify $I(A)$ is compact (see Theorem 4.5, Chap. 1 in [25]).

An invariant set $A \subset X$ is said to be isolated, if $A$ has a neighbourhood $N$ such that $A = I(N)$. Accordingly, a neighbourhood $N$ of $A$ such that $A = I(N)$ is called an isolating neighbourhood of $A$.

Given an invariant set $A$ with $A \subset N \subset X$, we define the local stable and unstable manifold, $W^s_N(A)$ and $W^u_N(A)$ of $A$ in $N$ as follows:

$$
W^s_N(A) := \bigcup_{\omega(\gamma) \subset K} \{ \gamma(t) : \gamma([0, \infty)) \subset N, \ t \in [0, \infty) \},
$$

and

$$
W^u_N(A) := \bigcup_{\alpha(\gamma) \subset K} \{ \gamma(t) : \gamma((-\infty, 0]) \subset N, \ t \in (-\infty, 0] \},
$$

where $\gamma$ is a solution and $\omega(\cdot)$ and $\alpha(\cdot)$ are limit sets. If $N = X$ is the whole phase space, we simply write $W^s(A) = W^s_X(A)$ and $W^u(A) = W^u_X(A)$.

### 2.3 Attractors

Here we use the attractor theory of topological spaces stated in [17], which is a generalisation of the attractor theory in metric spaces [8,21,31,34].
Let $X$ be a Hausdorff topological space and $A, B$ be two subsets of $X$. We say that $A$ attracts $B$, if $T_x = \infty$ for all $x \in B$ and moreover, for an arbitrary neighbourhood $U$ of $A$ there exists $T > 0$ such that

$$\Phi(t)B \subset U,$$

for all $t > T$.

**Definition 2.4.** A nonempty sequentially compact invariant set $A \subset X$ is said to be an **attractor** of $\Phi$, if it attracts a neighbourhood $U$ of $\overline{A}$ and $A$ is the maximal sequentially compact invariant set in $U$.

**Remark 2.5.** This definition of attractor differs from the setting in [17], where the authors consider $U$ to be a neighbourhood of $A$ if $\overline{A} \subset \text{int} U$. Here we will adopt the concept in common sense that $U$ is a neighbourhood of $A$ provided $A \subset \text{int} U$. In this sense, this definition of attractor is the same as that in [17] in essence.

Here we use sequential compactness over compactness, due to the fact that these two concepts are not equivalent in general topological spaces. Moreover, we can make good use of the convergence of sequences under sequential compactness (in comparison to [21]).

Particularly, for metric spaces, sequential compactness is equivalent to the compactness. Consequently, if $X$ is metrisable, this definition of attractors is equivalent to those given in [8, 17, 34]. Precisely, $A \subset X$ is an attractor of $\Phi$ in $X$, if and only if $A$ is nonempty, compact and invariant and attracts a neighbourhood of itself.

Let $A$ be an attractor. Set

$$\Omega(A) = \{x \in X : A \text{ attracts } x\}.$$

$\Omega(A)$ is called the **region of attraction** (or **attraction basin**) of $A$. One can easily verify that $\Omega(A)$ is open; moreover, $A$ attracts each compact subset of $\Omega(A)$, see [17]. In the case when $\Omega(A) = X$, we simply call $A$ the **global attractor** of $\Phi$.

Let $K \subset X$ be a closed subset and $U$ be a subset of $X$ with $K \subset U$. A continuous function $\zeta : U \to \mathbb{R}^+$ is called a $K_0$ function of $K$ on $U$, if

$$\zeta(x) = 0 \iff x \in K.$$

If moreover the level set

$$\zeta^a = \{x \in U : \zeta(x) \leq a\}$$

is closed in $X$ for every $a \geq 0$, we say $\zeta$ is a $K_\infty^\infty$ function of $K$ on $U$.

Let $A$ be an attractor and $\Omega := \Omega(A)$ be the region of attraction of $A$. 

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Definition 2.6. A nonnegative continuous function $\zeta : \Omega \to \mathbb{R}^+$ is said to be a Lyapunov function of $A$, if $\zeta$ is a $K_0$ function of $A$ on $\Omega$, and for $x \in \Omega \setminus A$ and $t > 0$, we have $\zeta(\Phi(t)x) < \zeta(x)$.

The existence of Lyapunov function for an attractor in Hausdorff spaces is given in the following proposition, a result similar to that for other spaces ([12, 13, 15, 17]).

Proposition 2.7. Let $X$ be a Hausdorff space. Assume that the attractor $A$ is closed and has a $K_0$ function $\psi$ on $\Omega$. Then $A$ has a Lyapunov function $\zeta$ on $\Omega$. What is more, if $\psi$ is $K_\infty^0$ of $A$ on $\Omega$, so is $\zeta$.

Proof. The first conclusion comes from Theorem 5.1 in our earlier paper [17]. In order to prove it, we defined for $x \in \Omega$,

$$
\psi_1(x) = \sup_{t \geq 0} \psi(\Phi(t)x) \quad \text{and} \quad \zeta(x) = \psi_1(x) + \int_{0}^{\infty} e^{-t} \psi_1(\Phi(t)x)dt. \quad (2.1)
$$

By a standard argument ([13, 17]), we showed that $\psi_1$ is continuous and $\zeta$ is the $K_0$ Lyapunov function we want.

Now we assume $\psi$ is $K_\infty^0$ of $A$ on $\Omega$ and show the second conclusion. It suffices to show that $\zeta$ defined in (2.1) is $K_\infty^0$ of $A$ on $\Omega$, i.e., for every $a > 0$, $\zeta^a$ is closed in $X$.

Indeed, by definition, we have

$$
\zeta(x) \geq \psi_1(x) \geq \psi(x) \quad \text{for every } x \in \Omega.
$$

This means $\zeta^a \subset \psi^a$. By the continuity of $\zeta$ on $\Omega$, $\zeta^a$ is a closed subset of $\psi^a$. The assumption that $\psi$ is $K_\infty^0$ of $A$ on $\Omega$ implies that $\psi^a$ is closed in $X$. Then we immediately obtain that $\zeta^a$ is closed in $X$. The proposition is proved. \qed

3 Wązewski Pairs and Quotient Flows

In this section the phase space $X$ is assumed to be a complete metric space. Given a subset $N \subset X$, define a function $t_N : X \to \mathbb{R}^+ \cup \{\infty\}$ as

$$
t_N(x) = \inf\{t \geq 0 : \text{either } t \geq T_x, \text{ or } \Phi(t)x \notin N\}, \quad \text{for } x \in X. \quad (3.1)
$$

Note that for each $x$, $t_N(x)$ is the maximal time such that $\Phi([0, t_N(x)]x \subset N$.

Let $N$ and $E$ be two closed subsets of $X$. The subset $E$ is said to be $N$-positively invariant, if for all $x \in E \cap N$ and $t \geq 0$, we have $\Phi([0, t])x \subset E$ whenever $\Phi([0, t])x \subset N$.

The subset $E$ is said to be an exit set of $N$, if
(1) $E$ is $N$-positively invariant; and

(2) for every $x \in N$ with $t_N(x) < T_x$, there exists $t \leq t_N(x)$ such that $\Phi(t)x \in E$. 

**Definition 3.1.** A pair of closed subsets $(N, E)$ of $X$ is called a Ważewski pair, if

(1) $E$ is an exit set of $N$; and

(2) $N \setminus E$ is strongly admissible.

Let there be given a Ważewski pair $(N, E)$. Now we consider the quotient space $N/E$. For notational simplicity, we denote $[A] = \pi(A)$ for $A \subset N \cup E$, where $\pi : N \cup E \to N/E$ is the usual quotient map. Define a quotient flow $\overline{\Phi}$ of $\Phi$ on $N/E$ as follows:

If $\overline{x} = [E]$, then
$$
\overline{\Phi}(t)\overline{x} \equiv \overline{x}
$$

for $t \in \mathbb{R}^+$; and if $\overline{x} = [x]$ for some $x \in N \setminus E$, then
$$
\overline{\Phi}(t)\overline{x} = \begin{cases} 
[\Phi(t)x], & \text{for } t < t_{N\setminus E}(x); \\
[E], & \text{for } t \geq t_{N\setminus E}(x).
\end{cases}
$$

Since $E$ is $N$-positively invariant, it can be easily seen that $\overline{\Phi}(t)$ is a well defined semigroup on $N/E$.

Observe that $N/E$ is a normal Hausdorff space. Lemma 3.2 in [35] applies here, and we obtain that $\overline{\Phi}$ is a global semiflow on $N/E$ as follows.

**Theorem 3.2.** The quotient flow $\overline{\Phi}$ is continuous on $\mathbb{R}^+ \times N/E$ and $N/E$ is strongly admissible.

Similar to [16][35], the following fact is immediate from the definition.

**Theorem 3.3.** If $I(N \setminus E) \cap E = \emptyset$, then the equilibrium $[E]$ is an attractor of $\overline{\Phi}$ in $N/E$.

Let $(N, E)$ be a Ważewski pair and $\overline{\Phi}$ be the quotient flow on $N/E$. Then we have the following conclusion.

**Theorem 3.4.** Every closed attractor $\mathcal{A}$ of $\overline{\Phi}$ containing $[E]$ has a $K_0^\infty$ Lyapunov function on $\Omega(\mathcal{A})$. 

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Proof. By Proposition 2.7, it is sufficient to find a $K_0^{\infty}$ function $\psi$ of $A$ on $\Omega(A)$. If $N \setminus E \cap E = \emptyset$ and $A = \{[E]\}$, we have that $\Omega(A) = \{[E]\}$. The function $\psi([E]) = 0$ is just what we desire.

Now we only consider the case when $N \setminus E \cap E \neq \emptyset$ or $A \neq \{[E]\}$.

In order to find the function $\psi$ in this case, we necessarily get back to the Ważewski pair $(N, E)$. Denote $Y = N \cup E$ and let $\pi : Y \to N/E$ be the quotient map. Denote

$$U = \pi^{-1}(\Omega(A)) \quad \text{and} \quad A = \pi^{-1}(A).$$

We see that $U$ is open and $A$ is closed in $Y$. Moreover, the attraction region $\Omega(A) = N/E$ if and only if $U = Y$.

We claim that there exists a $K_0^{\infty}$ function $\delta$ of $A$ on $U$. Assuming this claim, we define $\psi : \Omega(A) \to \mathbb{R}^+$ such that

$$\psi([E]) = 0 \quad \text{and} \quad \psi(x) = \delta(x) \quad \text{for} \quad x = \pi(x) \in \Omega(A) \ \text{with} \ x \in U \setminus E.$$

The map $\psi$ is well defined. The continuity of $\psi$ is guaranteed by the properties of $\delta$ and $\pi$. Furthermore, $\psi^a = \pi(\delta^a)$ is closed in $N/E$ by the closedness of $\pi$, which indicates that $\psi$ is $K_0^{\infty}$ of $A$ on $\Omega(A)$. This confirms Theorem 3.4.

Now it remains to show the claim. We define $\delta$ as follows,

$$\delta(x) = \begin{cases} d(x, A), & \text{if } U = Y; \\ \frac{d(x, A)}{\min\{d(x, Y \setminus U), 1\}}, & \text{otherwise.} \end{cases}$$

Clearly, the function $\delta$ is $K_0$ of $A$ on $U$ and $\delta(x) \geq d(x, A)$ for $x \in U$. To check that $\delta$ is $K_0^{\infty}$ of $A$ on $U$, we only need to show $\delta^a$ is closed in $Y$ for each $a > 0$.

If $U = Y$, we are already done by the continuity of $\delta$. If $U \neq Y$, we prove it by contradiction. Actually, because the level set $\delta^a$ is closed in $U$, if $\delta^a$ is not closed in $Y$ for some $a > 0$, we have

$$\overline{\delta^a} \subset U \quad \text{but} \quad \overline{\delta^a} \nsubseteq U,$$

i.e., there is $x_0 \in (Y \setminus U) \cap \overline{\delta^a}$. Then there exists $x_n \in \delta^a \setminus A$ such that $x_n \to x_0$ as $n \to \infty$. Denote

$$B := \{x_n : n \in \mathbb{N}\}.$$

We have that $B \subset Y \setminus A$ is compact and $\varepsilon := d(B, A) > 0$. We may as well assume $d(x_n, Y \setminus U) < 1$ for all $n \in \mathbb{N}^+$ and hence

$$\delta(x_n) = \frac{d(x_n, A)}{d(x_n, Y \setminus U)} \geq \frac{\varepsilon}{d(x_n, Y \setminus U)}. \quad (3.2)$$

It is deduced that $d(x_n, Y \setminus U) \to 0$ from that $x_n \to x_0 \in Y \setminus U$. By (3.2), we obtain that $\delta(x_n) \to \infty$ as $n \to \infty$. But it follows from $x_n \in \delta^a$ that $\delta(x_n) \leq a$, which is a contradiction. The proof is complete. \qed
4 Compactly Generated Shape Index

In this section we introduce the notion of shape index pairs and define compactly generated shape indices for isolated invariant sets in metric spaces.

For the presentation hereafter, we first introduce the definition of H-shape of pairs of Hausdorff spaces here, see [6] for the case of Hausdorff spaces and originally [24]. The reader is supposed to be familiar with the basic homotopy theory.

4.1 H-Shape for Pairs of Hausdorff Spaces

Shape theory was first introduced by Borsuk [2] for metric spaces in 1968, and later Mardešić and Segal [20] gave an extension of Borsuk’s shape theory via ANR-systems to include compact Hausdorff topological spaces. We refer to this definition of shape given by Mardešić and Segal as ANR-shape, denoted by Sh\textsubscript{ANR}. In 1974 Rubin and Sanders gave a different extension to the realm of Hausdorff spaces, called “compactly generated shape”, shortly H-shape, see [24]. The establishment of H-shape theory is based on the ANR-shape theory of compact Hausdorff spaces.

In the following, we introduce the definition of H-shape for pairs of Hausdorff spaces in detail as an extension of H-shape for Hausdorff spaces (\[6, 24\]).

Let \((X, X_0)\) and \((Y, Y_0)\) be pairs of Hausdorff spaces. If the pairs satisfy that the relation \(Y \subset X\) and \(Y_0 \subset X_0\), we denote this relation by \((Y, Y_0) \subset (X, X_0)\). A compact pair \((X, X_0)\) is a pair with both \(X\) and \(X_0\) being compact Hausdorff spaces.

Given two compact pairs \((K, K_0)\) and \((L, L_0)\), there is a shape map \(f : (K, K_0) \to (L, L_0)\), which is defined as follows, see [6, 20, 24]. The shape map \(f\) assigns to every pair \((Q, Q_0)\) having the homotopy type of a CW-complex pair and to every homotopy class \(\eta : (L, L_0) \to (Q, Q_0)\), a homotopy class \(f(\eta) : (K, K_0) \to (Q, Q_0)\), such that, if \((Q', Q'_0)\) is another pair having the homotopy type of a CW-complex pair and \(\eta' : (L, L_0) \to (Q', Q'_0)\) is a homotopy class, then if \(\xi : (Q, Q_0) \to (Q', Q'_0)\) is a homotopy class, the commutativity (up to homotopy) of

\[
\begin{array}{ccc}
(L, L_0) & \xrightarrow{\eta} & (Q, Q_0) \\
\downarrow{\xi} & & \downarrow{\xi} \\
(Q, Q_0) & \xrightarrow{f(\eta)} & (Q', Q'_0)
\end{array}
\]

Let \(\Lambda\) be a directed set. A \(CS^2\)-system is a direct system \(X^* = \{(X_\lambda, X_{0\lambda}), p_{\lambda\lambda'}, \Lambda\}\) in the compact shape category of pairs of Hausdorff spaces (see [20]), that is to say,
each \((X_\lambda, X_{0\lambda})\) is a compact pair and if \(\lambda \leq \lambda'\) in \(\Lambda\), then \(p_{\lambda\lambda'} : (X_\lambda, X_{0\lambda}) \rightarrow (X_{\lambda'}, X_{0\lambda'})\) is a shape map such that

(i) \(p_{\lambda\lambda} = 1_{(X_\lambda, X_{0\lambda})}\) is the identity shape map,

(ii) if \(\lambda \leq \lambda' \leq \lambda''\), then \(p_{\lambda'\lambda''}p_{\lambda\lambda'} = p_{\lambda\lambda''}\).

A \(CS^2\)-morphism \(F : X^* \rightarrow Y^* = \{(Y_\mu, Y_0), q_{\mu\mu'}, M\}\) is a pair \(F = (f, \lambda)\) consisting of an increasing function \(f : \Lambda \rightarrow M\) and a collection of shape maps \(f_\lambda : (X_\lambda, X_{0\lambda}) \rightarrow (Y_{f(\lambda)}, Y_{0f(\lambda)})\) such that if \(\lambda \leq \lambda'\) then \(q_{f(\lambda)f(\lambda')}f_\lambda = f_{\lambda'}p_{\lambda\lambda'},\) that is to say, the following diagram commutes.

\[
\begin{array}{ccc}
(X_\lambda, X_{0\lambda}) & \xrightarrow{f_\lambda} & (Y_{f(\lambda)}, Y_{0f(\lambda)}) \\
p_{\lambda\lambda'} & & q_{f(\lambda)f(\lambda')} \\
\downarrow & & \downarrow \\
(X_{\lambda'}, X_{0\lambda'}) & \xrightarrow{f_{\lambda'}} & (Y_{f(\lambda')}, Y_{0f(\lambda')})
\end{array}
\]

Defining the identity \(1_{X^*}\) and compositions in the usual way, we finally have a category of \(CS^2\)-systems and \(CS^2\)-morphisms between them, denoted by \(CS^2\).

Two \(CS^2\)-morphisms \(F, G : X^* \rightarrow Y^*\) are homotopic, \(F \simeq G\), if for each \(\lambda \in \Lambda\), there is \(\mu \in M\) with \(f(\lambda), g(\lambda) \leq \mu\) such that \(q_{f(\lambda)\mu}f_\lambda = q_{g(\lambda)\mu}g_\lambda\), i.e., the following commutative diagram.

\[
\begin{array}{ccc}
(Y_{f(\lambda)}, Y_{0f(\lambda)}) & \xrightarrow{f_\lambda} & (Y_{g(\lambda)}, Y_{0g(\lambda)}) \\
\downarrow q_{f(\lambda)\mu} & & \downarrow q_{g(\lambda)\mu} \\
(X_\lambda, X_{0\lambda}) & \xrightarrow{g_\lambda} & (Y_\mu, Y_{0\mu})
\end{array}
\]

Surely the homotopy relation \(\simeq\) is a morphism equivalence, see [24]. We say \(X^*\) and \(Y^*\) have the same homotopy type, provided there are \(CS^2\)-morphisms \(F : X^* \rightarrow Y^*\) and \(G : Y^* \rightarrow X^*\) such that \(GF \simeq 1_{X^*}\) and \(FG \simeq 1_{Y^*}\); and we say \(F\) is a homotopy equivalence from \(X^*\) to \(Y^*\).

Given a pair \((X, X_0)\) of Hausdorff spaces, let \(c(X, X_0)\) be the set of all compact pairs of \((K, K_0) \subset (X, X_0)\) ordered by inclusions, which makes \(c(X, X_0)\) a directed set. Then one has a \(CS^2\)-system

\[
C(X, X_0) = \{(K, K_0), i_{(K, K_0)(K', K_0')}, c(X, X_0)\}
\]

such that \((K, K_0) \in c(X, X_0)\) and if \((K, K_0) \subset (K', K_0')\) then \(i_{(K, K_0)(K', K_0')}\) is the inclusion shape map.
Definition 4.1. Let $(X, X_0)$ and $(Y, Y_0)$ be pairs of Hausdorff spaces. If $C(X, X_0)$ and $C(Y, Y_0)$ have the same homotopy type, we say $(X, X_0)$ and $(Y, Y_0)$ have the same shape, denoted by $\text{Sh}_H(X, X_0) = \text{Sh}_H(Y, Y_0)$.

Remark 4.2. Since ANR-shape is defined by the inverse systems of neighbourhoods of a given metric space (compact Hausdorff space) in an ANR and the homotopy classes between them, it mainly describes the space from outside, see [13, 20]. Here ANR means the absolute neighbourhood retract of metric spaces (or compact Hausdorff spaces) (see [20]). By comparison, H-shape defined above is via the direct systems of compact subsets of a given Hausdorff space and the (ANR-)shape maps between them (see also [6, 24]). Correspondingly, H-shape provides an inner description of the Hausdorff space. In spite of the definitions in different means and distinct descriptions of the space, ANR-shape and H-shape coincide for compact Hausdorff spaces ([24]).

If $\varphi : (X, X_0) \to (Y, Y_0)$ is a continuous map, let $f : c(X, X_0) \to c(Y, Y_0)$ and $f_{(K, K_0)} : (K, K_0) \to (\varphi(K), \varphi(K_0))$ such that $f((K, K_0)) = (\varphi(K), \varphi(K_0))$, which is increasing, and $f_{(K, K_0)}$ is the shape map induced by $\varphi_{(K, K_0)} : (K, K_0) \to (\varphi(K), \varphi(K_0))$. Thus we obtain a CS$^2$-morphism $F = (f_{(K, K_0)}, f)$ induced by $\varphi$. If $F$ is a homotopy equivalence from $C(X, X_0)$ to $C(Y, Y_0)$, we say $\varphi$ induces an H-shape equivalence.

We can see in a straightforward way that H-shape is a homotopy invariant for pairs of Hausdorff spaces, as stated in Rubin and Sanders [24], i.e.,

$$(X, X_0) \simeq (Y, Y_0) \Rightarrow \text{Sh}_H(X, X_0) = \text{Sh}_H(Y, Y_0).$$

The following result is a pointed version of Theorem 4.2 in [6], and the similar results in metric spaces can be found in [7, 13, 30].

Theorem 4.3. Let $X$ be a Hausdorff space, and $\Phi$ be a global semiflow on $X$. Suppose that $\Phi$ has a compact global attractor $\mathcal{A}$, and that the system has an equilibrium $e \in \mathcal{A}$. Then the inclusion $(\mathcal{A}, e) \to (X, e)$ induces an H-shape equivalence.

Let $(X, x_0)$ and $(Y, y_0)$ be two pointed spaces. The wedge sum $(X, x_0) \vee (Y, y_0)$ and smash product $(X, x_0) \wedge (Y, y_0)$ are defined, respectively, as follows:

$$(X, x_0) \vee (Y, y_0) = (\mathcal{W}, (x_0, y_0)),$$

$$(X, x_0) \wedge (Y, y_0) = ((X \times Y)/\mathcal{W}, [\mathcal{W}]).$$
where $W = X \times \{y_0\} \cup \{x_0\} \times Y$. Similar to the definition for homotopy type (see Section 1.10 in [25]), the operations “∨” and “∧” can be also defined to H-shape of pointed spaces, by the definition of H-shape. If both $(X, x_0)$ and $(Y, y_0)$ have the HEP, their H-shapes satisfy the following equations,

$$\text{Sh}_H(X, x_0) \lor \text{Sh}_H(Y, y_0) = \text{Sh}_H((X, x_0) \lor (Y, y_0)),$$

$$\text{Sh}_H(X, x_0) \land \text{Sh}_H(Y, y_0) = \text{Sh}_H((X, x_0) \land (Y, y_0)).$$

This is also a natural generalisation of the definitions of wedge sum and smash product for H-shape of Hausdorff spaces (see [24]).

### 4.2 H-Shape Index

From now on we always consider $X$ as a complete metric space. Let $\Phi$ be a local semiflow on $X$.

**Definition 4.4.** Let $K \subset X$ be a compact isolated invariant set of $\Phi$. A Ważewski pair $(N, E)$ is said to be a shape index pair of $K$, if

1. there is a closed admissible neighbourhood $U$ of $K$ such that $W^n_u(K) \subset \overline{N \setminus E}$;
2. $K \cap E = \emptyset$; and
3. $K = I(\overline{N \setminus E})$.

**Remark 4.5.** (1) A Conley index pair $(N, E)$ of a compact isolated invariant set $K$ is a Ważewski pair such that $N \setminus E$ is an isolating neighbourhood of $K$ (see [3, 25]). The set $N \setminus E$ surely contains a local unstable manifold of $K$ and $K \cap E = \emptyset$. Therefore, a Conley index pair is naturally a shape index pair in Definition 4.4.

(2) The shape index pairs $(N, E)$ given in [35] are also specific examples of the shape index pairs defined above, since besides the conditions (1) (2) (3) in Definition 4.4 it is also implicated therein that the quotient space $N/E$ is metrisable in quotient topology.

We are now in position to define the H-shape index of $K$ via shape index pairs introduced here.

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Definition 4.6. Let \((N, E)\) be a shape index pair of \(K\). The H-shape index \(s(\Phi, K)\) of \(K\) is defined as
\[
s(\Phi, K) = \text{Sh}_H(N/E, [E]).
\]

When the semiflow \(\Phi\) is clear, we will simply write \(s(\Phi, K)\) as \(s(K)\).

Example 4.7. (1) If the compact isolated invariant set \(K = \emptyset\), we can take \((\emptyset, \emptyset)\) to be the shape index pair of \(\emptyset\), and thus we have \(s(\emptyset) = \overline{0}\), where \(\overline{0}\) is the H-shape of a pointed singleton. Applying this fact, we can determine that \(K \neq \emptyset\), if \(s(K) \neq 0\). This property is analogous to Conley index.

(2) By Theorem 4.3 if \(X\) is a complete metric space and \(A\) is the global attractor, it is clear that
\[
s(A) = \text{Sh}_H(X \cup \{\ast\}, \ast),
\]
where \(\ast \notin X\) is a single isolated point and \(X \cup \{\ast\}\) is endowed the sum topology. Particularly, if \(X\) is a normed linear space, \(s(A) = \Sigma^0\), since \(X\) is contractible. Here and in the sequel, we always use \(\Sigma^n\) to denote the H-shape of pointed \(n\)-dimensional sphere.

Taking the smash product into account, we have that
\[
s(\Phi_1 \times \Phi_2, K_1 \times K_2) = s(\Phi_1, K_1) \land s(\Phi_2, K_2),
\]
for two disjoint semiflows \(\Phi_1\) and \(\Phi_2\) with compact isolated invariant sets \(K_1\) and \(K_2\), respectively, where
\[
(\Phi_1 \times \Phi_2)(t)(x_1, x_2) := (\Phi_1(t)x_1, \Phi_2(t)x_2). \tag{4.1}
\]

The following result indicates that the shape index of an isolated invariant set is well defined.

Theorem 4.8 (Main Theorem). The H-shape index \(s(K)\) of \(K\) is independent of the choice of shape index pairs.

To show Theorem 4.8 we need the following lemma (we omit the proof), which is a new version of Lemma 4.6 and 4.7 in [35], with the framework of topological dynamical systems (given in Section 2) on the quotient space.

Lemma 4.9. Let \(K\) be a compact isolated invariant set with its shape index pair \((N, E)\) and let \(\Phi\) be the quotient flow on \(N/E\). Then \(\Phi\) has a compact global attractor \(A\) such that
\[
A = W^u([K]) \cup \{|E|\} \quad \text{and} \quad (A, |E|) \simeq (W^u_K(K)/E, [E]).
\]
The proof of Theorem 4.8 is indeed a modification of that of Theorem 4.5 in [35]. For the reader’s convenience, we present the main sketch of the proof as follows. Nevertheless, we need to keep in mind that the referred conclusions in [35] involved in the following proof is under the framework of the shape index pair and the quotient flow defined in this paper. And the proofs of the referred conclusions in [35] also work well here, with the Lyapunov functions used therein following from Theorem 3.4.

**Proof of Theorem 4.8** In the case when $W^u(K) = K$, for every shape index pair $(N, E), W^u_N(K) = K$. By Lemma 4.9 the quotient flow on $N/E$ has a global attractor $\mathcal{A}$. Moreover, we have $\mathcal{A} = [K] \cup \{[E]\}$, and so

$$Sh_H(N/E, [E]) = Sh_H([K] \cup \{[E]\}, [E]) = Sh_H(K \cup \{*\}, *),$$

which implies the conclusion of Theorem 4.8.

Now we only consider the case when $W^u(K) \neq K$. In this case, for all shape index pairs $(N, E)$ of $K$, we infer that $W^u_N(K) \cap E \neq \emptyset$ (see Lemma 4.8 in [35]).

Let $(N_1, E_1)$ and $(N_2, E_2)$ be two shape index pairs of $K$. Let $N = N_1 \cap N_2$ and $E = E_1 \cup E_2$. Then we have that $(N, E)$ is also a shape index pair of $K$. We aim to show that $(N_1/E_1, [E_1])$ and $(N_2/E_2, [E_2])$ have the same H-shape, for which we only need to prove that

$$Sh_H(N_k/E_k, [E_k]) = Sh_H(N/E, [E]), \quad k = 1, 2. \tag{4.2}$$

In the following we show that (4.2) holds true for $k = 1$.

Set $E^u = W^u_N(K) \cap E$, and define

$$\Gamma_1 = \{x \in N_1 : \text{ there exist } t \geq 0 \text{ and } y \in E^u \text{ such that } \Phi([0, t])y \subset N_1, \text{ and } \Phi(t)y = x\}.$$

It is easy to check that $E^u$ and $\Gamma_1$ are $N$-positively invariant and $N_1$-positively invariant, respectively. Moreover, we have

$$W^u_N(K) \cup \Gamma_1 = W^u_{N_1}(K). \tag{4.3}$$

Applying Lemma 4.10 in [35], we have an open neighbourhood $U$ of $K$ such that $\Gamma_1 \cap U = \emptyset$.

By using Lemma 4.9 in [35], there is a closed neighbourhood $F$ of $E$ in $N \cup E$ with $K \cap F = \emptyset$ such that $(N, F)$ is a shape index pair and has HEP; moreover,

$$W^u_N(K) \setminus F \subset U, \quad (N/E, [E]) \simeq (N/F, [F]). \tag{4.4}$$
Hence $\text{Sh}_H(N/E, [E]) = \text{Sh}_H(N/F, [F])$. On the other hand, by Theorem 4.3 we deduce

$$\text{Sh}_H(N/F, [F]) = \text{Sh}_H(\mathcal{A}', [F]),$$

where $\mathcal{A}'$ is the global attractor of the quotient flow on $N/F$. Therefore by Lemma 4.9 we find that

$$\text{Sh}_H(N/E, [E]) = \text{Sh}_H(\mathcal{A}', [F])$$

$$\text{Sh}_H(W_N(K)/F, [F]) = \text{Sh}_H(W_N(K)/F^u, [F^u]),$$

where $F^u = W_N(K) \cap F$.

Let $\Gamma = F^u \cup \Gamma_1$. Based on the fact that $\Gamma_1 \cap U = \emptyset$ and (4.4), we have

$$W_N(K) \cap \Gamma = F^u \cup (W_N(K) \cap \Gamma_1)$$

$$= F^u \cup (W_N(K) \cap (F \cup F^c) \cap \Gamma_1) = F^u \cup (W_N(K) \cap F \cap \Gamma_1) = F^u,$$

where $F^c = X \setminus F$. Then we have

$$((W_N(K) \cup \Gamma)/\Gamma, [\Gamma]) \cong (W_N(K)/\Gamma, [\Gamma]).$$

Therefore by (4.5) we obtain that

$$\text{Sh}_H(N/E, [E]) = \text{Sh}_H(W_{N_1}(K)/\Gamma, [\Gamma]).$$

(4.6)

Let $E_1^u = W_{N_1}(K) \cap E_1$. Consider the quotient space $W_{N_1}(K)/E_1^u$ along with the quotient flow $\tilde{\phi}_1$. Let $\pi$ be the quotient map from $W_{N_1}(K)$ to $W_{N_1}(K)/E_1^u$. It is clear that

$$\pi(W_{N_1}(K)/[\Gamma, [\Gamma]]) \cong (\pi(W_{N_1}(K))/[\pi(\Gamma), [\pi(\Gamma)]) .$$

(4.7)

Theorem 3.3 asserts that $[E_1^u]$ is an attractor of $\tilde{\phi}_1$. Since $\pi(\Gamma)$ is positively invariant and contained in the attraction basin of $[E_1^u]$ with $[E_1^u] \in [\pi(\Gamma)]$, then $[E_1^u]$ is a strong deformation retract of $[\pi(\Gamma)]$ (see Proposition 2.5 in [35]). Because $(N, F)$ has HEP, by Proposition 2.2 it is easy to see that $(W_N(K), F^u)$ has HEP. Consequently, $(W_{N_1}(K), \Gamma)$ and $(\pi(W_{N_1}(K)), \pi(\Gamma))$ have HEP as well. Therefore by Proposition 2.2 we have

$$(\pi(W_{N_1}(K))/[\pi(\Gamma), [\pi(\Gamma)]) \cong (\pi(W_{N_1}(K))/[\pi(E_1^u), [\pi(E_1^u)])$$

$$\cong (W_{N_1}(K)/E_1^u, [E_1^u]) \cong (W_{N_1}(K)/E_1, [E_1]).$$

By (4.6) and (4.7), we have that

$$\text{Sh}_H(N/E, [E]) = \text{Sh}_H(W_{N_1}(K)/E_1, [E_1])$$

$$= \text{Sh}_H(\mathcal{A}_1, [E_1]),$$

(by Lemma 4.9 = Sh$_H(\mathcal{A}_1, [E_1])$,
where $A_1$ is the global attractor of the quotient flow on $N_1/E_1$. Furthermore, by Theorem 4.3 we conclude that

$$\text{Sh}_H(N/E, [E]) = \text{Sh}_H(A_1, [E_1]) = \text{Sh}_H(N_1/E_1, [E_1]).$$

With a similar argument, the equality (4.2) also holds true for $k = 2$. We finish the proof now.

\begin{flushright}
\Box
\end{flushright}

### 4.3 Continuation Property

Similar to the Conley index (see [25]) and shape Conley index, H-shape index also has the continuation property, which involves a continuous family of local semiflows. We follow the basic concepts given for Conley index theory in [25] below.

Let $X$ be a complete metric space. For a sequence of local semiflows $\Phi_n$ on $X$, we write $\Phi_n \rightarrow \Phi_0$, if for all sequences $x_n \in X$ and $t_n \in \mathbb{R}^+$ with $x_n \rightarrow x_0$ and $t_n \rightarrow t_0$, $\Phi_n(t_n)x_n \rightarrow \Phi_0(t_0)x_0$.

Let $\Phi_n$ be a sequence of local semiflows on $X$. A set $N \subset X$ is said to be strongly $\{\Phi_n\}$-admissible if for two arbitrary sequences $x_n \in X$ and $0 \leq t_n \rightarrow \infty$ satisfying $\Phi_n([0, t_n])x_n \subset N$ for all $n \in \mathbb{N}^+$, the sequence of endpoints $\Phi_n(t_n)x_n$ has a convergent subsequence and furthermore, $\Phi_n$ does not explode in $N$ for every $n \in \mathbb{N}^+$.

Let $\Lambda$ be a metric space. We write $\Phi_{\lambda_n} \rightarrow \Phi_{\lambda_0}$, if $\Phi_{\lambda_n} \rightarrow \Phi_{\lambda_0}$ for every sequence $\lambda_n \in \Lambda$ with $\lambda_n \rightarrow \lambda_0$. A continuous family of local semiflows $\Phi_\lambda$ on $X$ is a family of local semiflows such that $\Phi_\lambda \rightarrow \Phi_{\lambda_0}$ for each $\lambda_0 \in \Lambda$ with $\lambda \rightarrow \lambda_0$.

Now we give the definition of H-continuity for the pair $(\Phi_\lambda, K_\lambda)$, where the ‘H’ is to mark the H-shape. This is a generalisation of S-continuity given in [25] for Conley index.

**Definition 4.10.** The pair $(\Phi_\lambda, K_\lambda)$ is said to be **H-continuous** at $\lambda_0 \in \Lambda$, if there is $\delta > 0$ and a closed subset $N$ of $X$ such that the following two conditions are fulfilled:

1. for every $\Phi_\lambda$ and $\lambda$ with $\rho(\lambda, \lambda_0) < \delta$, $N$ is strongly admissible and contains a local unstable manifold $W^u_{U_\lambda}(K_\lambda)$ for some closed neighbourhood $U_\lambda$ of $K_\lambda$;  
2. Whenever $\lambda_n \rightarrow \lambda_0$, then $\Phi_{\lambda_n} \rightarrow \Phi_{\lambda_0}$ and $N$ is $\{\Phi_{\lambda_n}\}$-admissible.

If $(\Phi_\lambda, K_\lambda)$ is H-continuous at each point $\lambda \in \Lambda$, $(\Phi_\lambda, K_\lambda)$ is said to be **H-continuous** in $\Lambda$.

With a similar process of discussion for Conley index ([25]), one immediately concludes that H-shape index possesses the following property, which is just what we call the **continuation property**.
Theorem 4.11 (\cite{25}). Let $K_\lambda$ be a compact isolated invariant set of $\Phi_\lambda$ for each $\lambda$ lying in a connected component $\Lambda_0$ of $\Lambda$. Suppose $(\Phi_\lambda, K_\lambda)$ is $H$-continuous in $\Lambda_0$. Then $s(\Phi_\lambda, K_\lambda)$ is constant for $\lambda \in \Lambda_0$.

Remark 4.12. Although both Conley index and $H$-shape index possess the continuation property, the assumption of $H$-continuity for $H$-shape index is much more relaxed. This is due to the relaxation of shape index pairs in comparison of Conley index pairs. Whereas, this simple relaxation makes great convenience for our calculation and discussion, which can be seen in the example and applications in the rest of the paper.

For a better understanding of continuation property and the shape index pair, we consider the following example, in which we also avoid the separability of the phase space.

Example 4.13. Consider the initial-boundary problem of the equation

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = \beta u(1 - u^p - 2), & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t \geq 0, \\
u(x, 0) = u_0(x), & x \in \Omega, 
\end{cases}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p > 2$ and $\beta > 0$ are constants. Such a problem has an invariant domain

$$X = \{u \in L^\infty(\Omega) | 0 \leq u \leq 1\}.$$ 

Hence we have a semiflow on $X$. Note that $X$ is not separable. Now we compute the $H$-shape index of 0 in $X$.

Let $V = H^1_0(\Omega)$. The inner product $\langle \cdot, \cdot \rangle$ on $V$ is defined by

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v \, dx, \quad u, v \in V,$$

and the corresponding norm $\| \cdot \|$. We know that the equation (4.8) has a weak solution (\cite{34})

$$u \in L^\infty(0, T; X) \cap L^2(0, T; V),$$

such that

$$\int_{\Omega} \frac{\partial u}{\partial t} \, dx + \langle u, v \rangle = \int_{\Omega} \beta u(1 - u^p - 2) \, dx,$$

for every $v \in V$. 

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We first consider the case when the phase space is $V \cap X$. To compute the shape index of $0$ for (4.8), we consider the following equation related to (4.8),

$$\begin{cases}
\frac{\partial u}{\partial t} - (\Delta + \beta)u = -\lambda \beta u^{p-1}, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t \geq 0,
\end{cases}$$

(4.9)

where $\lambda \in [0, 1]$. When $\lambda = 0$, (4.9) is the linearisation of (4.8) at $u = 0$; when $\lambda = 1$, (4.9) is just (4.8). It is well known that the linear operator $-(\Delta + \beta)$ has only finitely many negative eigenvalues. Denote the eigenvalues of $-\Delta$ in $V$ by $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots \to \infty$. Let $\Phi_\lambda$ be the semiflow generated by (4.9). We first consider the case when $\beta \neq \mu_k$ for all $k \in \mathbb{N}^+$. By the standard argument (see [14, 25]), the family of semiflows $\Phi_\lambda$ is continuous in $\lambda \in [0, 1]$, and $(\Phi_\lambda, \{0\})$ is H-continuous in $[0, 1]$. By the continuation property (Theorem 4.11), we have

$$s(\Phi_1, \{0\}) = s(\Phi_0, \{0\}) = \begin{cases}
\emptyset, & \beta > \mu_1, \ \beta \neq \mu_k, \\
\Sigma^0, & 0 < \beta < \mu_1.
\end{cases}$$

(4.10)

The second equality in (4.10) results from the following discussion.

Since $\beta \neq \mu_k$, we let $V_0$ be the subspace of $V$ spanned by the eigenfunctions of $\mu_k$ with $0 < \mu_k < \beta$. Pick $R \in (0, 1)$ and let

$$N := \{u \in V_0 \cap L^\infty(\Omega) : 0 \leq u \leq R\} \quad \text{and} \quad E := \{u \in N : u = R\}.$$ 

We know that $N$ is a local unstable manifold of 0 in $V \cap X$ and $E$ is the exit set of $N$. Hence the pair $(N, E)$ is a shape index pair of $\{0\}$ for $\Phi_0$. When $0 < \beta < \mu_1$, we see that $(N, E) = (\{0\}, \emptyset)$ and so $s(\Phi_0, \{0\}) = \Sigma^0$. When $\beta > \mu_1$, the space $N/E$ is contractible and then $s(\Phi_0, \{0\}) = \emptyset$.

If $\beta = \mu_k$ for some $k \in \mathbb{N}^+$, we need to consider the centre manifold of 0. It is well known (see Chapter 6, [10]) that 0 is asymptotically stable in its centre manifold. As a consequence, we obtain that

$$s(\Phi_0, \{0\}) = \begin{cases}
\emptyset, & \beta = \mu_k, \ k > 1, \\
\Sigma^0, & \beta = \mu_1.
\end{cases}$$

(4.11)

Now we consider the case when the phase space is $X$. Let $\Phi$ be the semiflow generated by (4.8). Since there is a natural embedding from $L^\infty(\Omega)$ into $L^p(\Omega)$, then $\Phi(t)u_0 \in V$ for all $u_0 \in X$. The unstable manifold of 0 in $V \cap X$ is hence also that
of 0 in $X$. Note that the local unstable manifold $W^u_U(0)$ of 0 is finite-dimensional, for some closed neighbourhood $U$ of 0 in $X$. The topologies of $W^u_U(0)$ induced by $X$ and $V \cap X$ are equivalent. As a result, by summarising (4.10) and (4.11), we conclude the following result

$$s(\Phi, \{0\}) = \begin{cases} 
\Pi, & \beta > \mu_1, \\
\Sigma^0, & 0 < \beta \leq \mu_1.
\end{cases}$$

**Remark 4.14.** If we replace the nonlinear term $\beta u(1-u^{p-2})$ in (4.8) by $\beta u(1-u^{|p-2|})$, we are allowed to set the phase space to be

$$X = \{u \in L^\infty(\Omega) | -1 < u < 1\}.$$

With almost the same argument, we obtain that $s(\{0\}) = \Sigma^r$, where $r$ is defined as

$$r = r(\beta) := \sum_{0<\mu_k<\beta} r_k,$$

where $r_k$ denotes the multiplicity of the eigenvalue $\mu_k$.

## 5 Establishment of Morse Equations

In this section we study the Morse equations of a Morse decomposition for a compact isolated invariant set associated to H-shape. To do this, we need to consider some H-shape invariant for these invariant sets. In our theory, we need the H-shape invariant to possess the following property:

**(P)** The pair $(N, E)$ and its quotient space $(N/E, [E])$ have the same H-shape invariant.

By Theorem 2.3 of [27] and Theorem 2.9 of [28], we know the shape groups (obtained from direct limits of homotopy groups) are H-shape invariants for the pointed Hausdorff spaces. However, homotopy groups do not meet (P), let alone the shape groups. Note that the Čech cohomology groups satisfy the property (P) and the exactness property ([5,32]). But we do not know whether the Čech cohomology group is H-shape invariant. Hence it is necessary to develop some new type of cohomology theory based on Čech cohomology groups.

### 5.1 H-shape Cohomology Groups and Indices

Given a pair $(X, X_0)$ of Hausdorff spaces, an abelian group $G$ and each $q \in \mathbb{N}$, there exists a Čech cohomology group $\check{H}^q(X, X_0; G)$, which is also abelian. It is found
from the third section of Chapter II in [20] that Čech cohomology groups are shape (by the inverse systems) invariant for pairs of topological spaces, including compact pairs. Thus when two compact pairs $(X, X_0)$ and $(Y, Y_0)$ have the same ANR-shape, for an arbitrary abelian group $G$, we have that

$$\tilde{H}^*(X, X_0; G) \approx \tilde{H}^*(Y, Y_0; G).$$

We omit the coefficient group $G$ in the following and the reader can take the coefficient group as the integer group. By considering the Čech cohomology groups of each compact pair $(K, K_0) \subset (X, X_0)$, we can obtain an inverse system of Čech cohomology groups

$$\tilde{H}^*(C(X, X_0)) = \{\tilde{H}^*(K, K_0), i^*_\lambda(K, K_0)(K', K'_0), c(X, X_0)\}.$$

Recall the inverse limit (see [5]) of an inverse system of groups $G = (G_\lambda, p_\lambda, \lambda)$ consists of a group $G_\infty$ and homomorphisms $p_\lambda : G_\infty \to G_\lambda$ such that

$$p_{\lambda'}p_\lambda = p_\lambda, \quad \lambda \leq \lambda'.$$

Moreover, if $p_\lambda' : G' \to G_\lambda$ is another collection of homomorphisms with property (5.1), then there is a unique homomorphism $g : G' \to G_\infty$ such that

$$p_{\lambda'}g = p_\lambda', \quad \lambda \in \Lambda.$$

We denote $G_\infty = \lim_{\leftarrow} G_\lambda$. Clearly, the inverse limit $G_\infty$ of $G_\lambda$ is unique up to a natural isomorphism. According to Theorem 3.14 in Chapter VIII of [5], the inverse limit $\lim_{\leftarrow} G_\lambda$ exists for every inverse system $G_\lambda$ of groups.

**Definition 5.1.** Let $(X, X_0)$ be a pair of Hausdorff spaces. The H-shape cohomology group $C\tilde{H}^q(X, X_0)$ for each $q \in \mathbb{N}$ is defined as

$$C\tilde{H}^q(X, X_0) = \lim_{\leftarrow} \tilde{H}^q(C(X, X_0)).$$

Since Čech cohomology groups satisfy (P), so do the H-shape cohomology groups via the inverse limit. Furthermore, by the properties of inverse limit, H-shape cohomology group is H-shape invariant, i.e., for each $q \in \mathbb{N},$

$$\text{Sh}_H(X, X_0) = \text{Sh}_H(Y, Y_0) \Rightarrow C\tilde{H}^q(X, X_0) \approx C\tilde{H}^q(Y, Y_0).$$

It is trivial that H-shape cohomology groups satisfy Eilenberg-Steenrod Axioms except the exactness property. But in our situation, we only need the exactness property for pairs of Hausdorff spaces with H-shape of compact pairs. This is confirmed by the following theorem.
**Theorem 5.2.** Suppose that \((X, X_0)\) and a compact pair have the same H-shape. Then \(C\check{\mathcal{H}}^*(X, X_0)\) satisfies the exactness property.

**Proof.** Let \((K, K_0)\) is a compact pair. We assume \((X, X_0)\) and \((K, K_0)\) have the same H-shape.

First we prove \(C\check{\mathcal{H}}^*(K, K_0)\) satisfy the exactness property. Since \((K, K_0)\) is compact, we have another direct system \(K^* = \{(K, K_0), 1_K\}\) besides \(C(K, K_0)\), where the index set is a singleton and \(1_K\) is the identity shape map. It is a simple result in [24] that \(C(K, K_0)\) and \(K^*\) have the same homotopy type and so do \(\check{\mathcal{H}}^q(C(K, K_0))\) and \(\check{\mathcal{H}}^q(K^*)\) as inverse systems of groups for \(q \in \mathbb{N}\). Thus by considering the inverse limit, we have

\[
C\check{\mathcal{H}}^q(K, K_0) \approx \lim_{\leftarrow} \check{\mathcal{H}}^q(K^*) = \check{\mathcal{H}}^q(K, K_0). \tag{5.3}
\]

As we know from [5,32], Čech cohomology groups \(\check{\mathcal{H}}^*(K, K_0)\) satisfy the exactness property. And hence by the isomorphism (5.3), \(C\check{\mathcal{H}}^*(K, K_0)\) also has the exactness property.

By the supposition and (5.2), we obtain the exactness of \(C\check{\mathcal{H}}^*(X, X_0)\). The proof is complete. \(\square\)

Now we consider a local semiflow \(\Phi\) on a complete metric space \(X\). Let \(K\) be a compact isolated invariant set of \(\Phi\).

**Definition 5.3.** Let \(K\) be a compact isolated invariant set with \((N, E)\) being a shape index pair of \(K\). The **H-shape cohomology index** of \(K\) is defined for each \(q \in \mathbb{N}\), as

\[
C\check{\mathcal{H}}^q(s(\Phi, K)) = C\check{\mathcal{H}}^q(N/E, [E]).
\]

If the semiflow \(\Phi\) is clear, we will simply write the H-shape cohomology index of \(K\) as \(C\check{\mathcal{H}}^q(s(K))\).

As H-shape cohomology groups satisfy the property (P) and (5.2), by Theorem 4.8 it is easy to see that \(C\check{\mathcal{H}}^q(s(K))\) is independent of the choice of the shape index pairs. Therefore, H-shape cohomology index is well defined as above.

### 5.2 Morse Decompositions of Invariant Sets

For the reader’s convenience, we recall briefly the definition of Morse decompositions of invariant sets for the dynamical systems on topological spaces, see, [17] or more classically [3,13,15,25].
Let $X$ be a topological space and $K$ a compact invariant set. Then the restriction $\Phi|_K$ of $\Phi$ on $K$ is a semiflow on $K$. A set $A \subset K$ is called an attractor of $\Phi$ in $K$, if it is an attractor of $\Phi|_K$. Note that this attractor in $K$ is a restricted one that is defined in an invariant set.

Let $A$ be an attractor of $\Phi$ in $K$. Set

$$A^* = \{ x \in K : \omega(x) \cap A = \emptyset \}.$$ 

$A^*$ is called the repeller dual to $A$ relative to $K$. Accordingly, $(A, A^*)$ is called an attractor-repeller pair in $K$.

**Definition 5.4.** Let $K$ be a compact invariant set. An ordered collection

$$\mathcal{M} = \{ M_1, \cdots, M_n \}$$

of subsets $M_k \subset K$ is called a Morse decomposition of $K$, if there exists an increasing sequence $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = K$ of attractors in $K$ such that

$$M_k = A_k \cap A_{k-1}^*, \quad 1 \leq k \leq n.$$ 

The attractor sequence of $A_k$ ($k = 0, 1, \cdots, n$) is often called the Morse filtration of $K$, and each $M_k$ is called a Morse set of $K$.

**Remark 5.5.** It is well known that each Morse set is compact and invariant, and moreover, if $K$ is isolated, then so are the Morse sets $M_k$ (see [25]).

### 5.3 Morse Equations

Morse equation is one of the most interesting and important topics of compact invariant sets for dynamical systems ([3, 13, 23, 25, 29, 35]). It can reflect a lot of information of the inner structure of compact invariant set, such as the dimensions, the topological structure of the Morse sets and the connected trajectories between Morse sets. We establish the Morse equations in the framework of H-shape cohomology index in a standard way as follows.

Let $X$ be a complete metric space and $K$ a compact isolated invariant set. Suppose $K$ has a Morse decomposition $\mathcal{M} = \{ M_1, \cdots, M_n \}$ with the corresponding Morse filtration

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = K.$$
Let \((N, E)\) be a shape index pair of \(K\). Let \(\tilde{\Phi}\) be the quotient flow on \(N/E\) defined as in Section 3. Then it can be shown that \(\tilde{\mathcal{M}} = \{\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_n\}\) forms a Morse decomposition of the global attractor \(A\) of \(\Phi\), where

\[
\tilde{M}_0 = \{[E]\}, \quad \tilde{M}_k = \pi(M_k) \quad (1 \leq k \leq n),
\]

and \(\pi : N \cup E \to N/E\) is the quotient map. Then we have a corresponding Morse filtration \(\{0, A_0, \ldots, A_n\}\) of \(\tilde{\mathcal{M}}\).

By Theorem 3.4 for \(k = 0, 1, \ldots, n - 1, A_k\) has a \(K^\infty_0\) Lyapunov function \(\zeta_k\) on the region of attraction \(\Omega_k := \Omega(A_k)\). Pick \(a_k > 0\) and set

\[
\tilde{N}_k := \bigcap_{i=k}^{n-1} \zeta_i^{a_i}, \quad k = 0, 1, \ldots, n - 1.
\]

Let \(N_n = N\) and \(N_k = \pi^{-1}(\tilde{N}_k), k = 0, 1, \ldots, n - 1\). Then we have a sequence of closed subsets satisfying

\[
N_0 \subset N_1 \subset \cdots \subset N_n.
\]

It is easy to verify that \((N_k, N_{k-1})\) and \((N_k, N_0)\) are shape index pairs of \(M_k\) and \(A_k\), respectively, \(k = 1, \ldots, n\). By very standard argument (see e.g. [25, 29]) one can obtain the following Morse equation associated with H-shape cohomology theory:

\[
\sum_{k=1}^{n} \sum_{q=0}^\infty t^q \text{rank} \tilde{C}H^q(N_k, N_{k-1}) = \sum_{q=0}^\infty t^q \text{rank} \tilde{C}H^q(N_n, N_0) + (1 + t)Q(t), \quad (5.4)
\]

where

\[
Q(t) = \sum_{k=1}^{n} \sum_{q=1}^\infty t^{q-1} \text{rank} \delta_k^q,
\]

and \(\delta_k^q\) is the coboundary operator from \(\tilde{C}H^{q-1}(N_{k-1}, N_0)\) to \(\tilde{C}H^q(N_k, N_{k-1})\).

Since \(N_k\) is a closed subset of the compact set \(N\), each \(N_k\) is compact for \(k = 0, 1, \ldots, n\). Then referring to the property (P), we know for \(0 \leq l \leq k \leq n, \tilde{C}H^q(N_k, N_l)\) is isomorphic to \(\tilde{C}H^q(N_k/N_l, [N_l])\). As \((N_k, N_{k-1})\) and \((N_k, N_0)\) are compact shape index pairs of \(M_k\) and \(A_k\), respectively, we have

\[
\tilde{C}H^q(N_k, N_{k-1}) \approx \tilde{C}H^q(s(M_k)), \quad \tilde{C}H^q(N_k, N_0) \approx \tilde{C}H^q(s(A_k)).
\]

Hence (5.4) can be rewritten as follows:

\[
\sum_{k=1}^{n} \sum_{q=0}^\infty t^q \text{rank} \tilde{C}H^q(s(M_k)) = \sum_{q=0}^\infty t^q \text{rank} \tilde{C}H^q(s(K)) + (1 + t)Q(t). \quad (5.5)
\]
For each compact isolated invariant set \( M \), set

\[
p(t, s(M)) = \sum_{q=0}^{\infty} t^q \text{rank} C\tilde{H}^q(s(M)).
\]

\( p(t, s(M)) \) is called the formal Poincaré polynomial of \( s(M) \). Now the Morse equation (5.5) can be restated in terms of formal Poincaré polynomials:

\[
\sum_{k=1}^{n} p(t, s(M_k)) = p(t, s(K)) + (1 + t)Q(t).
\]

6Applications to a Nonlinear Evolutionary System

In this section we apply the compactly generated shape index theory to the Cauchy problem of evolutionary equations. Consider a general nonlinear evolutionary equation

\[
\frac{du}{dt} + Nu + f(u) = 0, \quad u \in X,
\]

where \( X \) is a Banach space, \( N : X \to X^* \) is a nonlinear operator and \( f : X \to X^* \) is a nonlinear functional. In a large number of applications (see, \([10, 25, 34]\)), the maximal compact invariant set \( K \) (such as the global attractor) of (6.1) is contained in some subspace \( X_0 \) of \( X \).

Let us consider the calculation of the Morse equation of \( K \). Essentially, a Morse decomposition of \( K \) is independent of the total space \( X \) and the subspace \( X_0 \). Hence Morse equation does not depend on these spaces either, but only relies on the unstable manifold of \( K \). Because of this, shape index pairs’ independence of neighbourhoods of the isolated invariant set brings great flexibility to the relevant calculations.

For the special case when \( K \) is the global attractor, the topological structure of \( K \) may be very complicated, let alone the dynamical behaviors of the restricted systems on \( K \). Besides, the natural Lyapunov function of (6.1) (whenever existing) is often defined on some proper subspace or even some lower-dimensional invariant manifold in \( X \). As a result, one can hardly calculate the Conley index pairs of the invariant set \( K \) for such systems generated by (6.1). However, our shape index pairs can well avoid such difficulties.

In the following, we give an example of the application to a \( p \)-Laplacian evolutionary equation to illustrate the flexible calculations of our shape index pairs and Morse theory.
6.1 Basic Results on $p$-Laplacian Evolutionary Equations

We consider the following Cauchy problem of a $p$-Laplacian evolutionary equation with a nonlinear term of polynomials.

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) + f(u) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (6.2) \\
u(x, t) &= 0, \quad \text{on } \partial \Omega \times \mathbb{R}^+, \quad (6.3) \\
u(x, 0) &= u_0(x), \quad \text{in } \Omega, \quad (6.4)
\end{aligned}
\]

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^m$, and

\[f(s) = |s|^{q-2}s + b_1 s + b_0,\]

with $m, p, q > 2$ and $b_1, b_0 \in \mathbb{R}$. It is known that, by the nonlinear eigenvalue property of $p$-Laplacian operator (see [1]) and the Galerkin’s method (see e.g. [34]), for each $u_0 \in L^2(\Omega)$, the problem (6.2)-(6.4) has a unique weak solution $u \in C(\mathbb{R}^+; L^2(\Omega))$ with

\[u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)) \cap L^q(0, T; L^q(\Omega)),\]

for every $T > 0$; see Yang, Sun and Zhong [37]. Moreover, the solution mapping $u_0 \to u(t)$ generates a global semiflow $\Phi$ on $L^2(\Omega)$ and

\[\Phi(t)u_0 \in W^{1,p}_0(\Omega) \cap L^q(\Omega) \quad \text{for all } t > 0.\]

The semiflow $\Phi$ is asymptotically compact and has a global attractor $\mathcal{A}$ in $L^2(\Omega)$.

6.2 Calculating the Global Attractor’s Morse Equation

Suppose the system possesses only finitely many equilibria: $e_1, e_2, \cdots, e_n$. Then $\mathcal{M} = \{e_1, e_2, \cdots, e_n\}$ constructs a Morse decomposition of $\mathcal{A}$. Now we want to present explicitly the Morse equation of $\mathcal{A}$. Then one has to figure out all the Čech cohomology groups $C\check{H}^r(s(e_k))$ and $C\check{H}^r(s(\mathcal{A}))$.

We first consider the global attractor $\mathcal{A}$. It follows from Example 4.7 that

\[s(\mathcal{A}) = \Sigma^0,\]

and hence we find that $C\check{H}^0(s(\mathcal{A})) = \mathbb{Z}$ and $C\check{H}^r(s(\mathcal{A})) = 0$ for all $r > 0$.

Now we try to calculate $C\check{H}^r(s(e_k))$. If $e_k$ is hyperbolic, then $C\check{H}^r(s(e_k))$ is completely determined by the local unstable manifold of $e_k$, and the calculation of $C\check{H}^r(s(e_k))$ is somewhat trivial. However, when $e_k$ is not hyperbolic, the situation
seems to be complicated, since in this case the local unstable manifolds of \( e_k \) usually remain unknown. Luckily the system is a gradient system, and hence it has a natural Lyapunov function \( J \) defined as

\[
J(u) := \frac{1}{p}\|u\|_{1,p}^p + \int_{\Omega} g(u)dx,
\]

where

\[
\|u\|_{1,p} = \left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p}, \quad g(s) = \frac{1}{q}|s|^q + \frac{b_1}{2}s^2 + b_0s.
\]

Traditionally, it is supposed to find appropriate Conley index pairs via the level sets \( J^c \) of the function \( J \) to calculate the shape index \( s(e_k) \). But as we see, \( J \) is only defined on a subspace \( W^{1,p}_0(\Omega) \cap L^q(\Omega) \) of \( L^2(\Omega) \), which can not help to construct an isolated neighbourhood of \( e_k \). In this situation, one usually restricts the system on the attractor \( \mathcal{A} \) and considers \( \mathcal{A} \) that is compact in \( L^2(\Omega) \), as the phase space. But if we do this, firstly, it is necessary to calculate the attractor \( \mathcal{A} \). What is more, the topological structure of the global attractor is rather complicated and difficult to figure out. Thus the continuity of the restricted semigroup on \( \mathcal{A} \) is hardly checked. Consequently, it is almost an impossible task to make the required calculations via Conley index pairs.

However, the shape index pair presented in Definition 4.4 can be chosen in the subspace \( W^{1,p}_0(\Omega) \cap L^q(\Omega) \) of \( L^2(\Omega) \). For this reason, we now consider to give shape index pairs by using the Lyapunov function (6.5).

We may assume for simplicity that

\[
J(e_1) < J(e_2) < \cdots < J(e_n).
\]

Pick two numbers \( \alpha \) and \( \beta \) with

\[
J(e_{k-1}) < \alpha < J(e_k) < \beta < J(e_{k+1}).
\]

We claim that \((J^\beta, J^\alpha)\) is a shape index pair of \( e_k \).

To see this, we first show that for each \( c \in \mathbb{R}^1 \), the level set \( J^c \) is closed in \( L^2(\Omega) \). Indeed, let \( u_n \in J^c \), and \( u_n \to u_0 \) in \( L^2(\Omega) \). Since \( J(u_n) \leq c \) for all \( n \), the sequence \( u_n \) is bounded in both the spaces \( W^{1,p}_0(\Omega) \) and \( L^q(\Omega) \) by the fact

\[
g(s) \geq \frac{1}{q}|s|^q + l, \quad \text{for some } l \in \mathbb{R}, \quad \text{and}
\]

\[
\frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q} \int_{\Omega} |u|^q dx \leq J(u) - l|\Omega| \leq c - l|\Omega|,
\]

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where $|\Omega|$ is the Lebesgue measure of $\Omega$. Hence we can obtain a subsequence of $u_n$ (still denoted by $u_n$), such that

$$u_n \overset{w}{\rightharpoonup} v \ (\text{in } W_0^{1,p}(\Omega)), \quad \text{and} \quad u_n \overset{w}{\rightarrow} w \ (\text{in } L^q(\Omega)).$$

One can verify that $u_0 = v = w$ in the sense of distribution. Thus we have $u_0 \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$. By the boundedness of $u_n$ in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$, one can pick a subsequence $u_{n_k}$ such that both the sequences $\|u_{n_k}\|_{1,p}$ and $|u_{n_k}|_q$ ($| \cdot |_q$ denotes the norm of $L^q(\Omega)$) converge due to the boundedness. Then

$$c \geq \limsup_{n \to \infty} J(u_n) \geq \limsup_{n_k \to \infty} J(u_{n_k})$$

$$= \frac{1}{p} \lim_{n_k \to \infty} \|u_{n_k}\|_{1,p}^p + \frac{1}{q} \lim_{n_k \to \infty} |u_{n_k}|_q^q + \lim_{n_k \to \infty} \int_{\Omega} \left( \frac{b_1}{2} u_{n_k}^2 + b_0 u_{n_k} \right) \, dx$$

$$\geq \frac{1}{p} \|u_0\|_{1,p}^p + \frac{1}{q} |u_0|_q^q + \lim_{n_k \to \infty} \int_{\Omega} \left( \frac{b_1}{2} u_{n_k}^2 + b_0 u_{n_k} \right) \, dx$$

$$= \frac{1}{p} \|u_0\|_{1,p}^p + \frac{1}{q} |u_0|_q^q + \int_{\Omega} \left( \frac{b_1}{2} u_0^2 + b_0 u_0 \right) \, dx = J(u_0).$$

Therefore $u_0 \in J^c$, which assures the closedness of $J^c$.

As $J^\beta_\alpha = \overline{J^\beta \setminus J^\alpha}$ is a bounded subset of $L^2(\Omega)$, it follows by the asymptotic compactness of $\Phi$ that $J^\beta_\alpha$ is strongly admissible (Recall that $\Phi$ is a global semiflow, hence no solutions explode). Since both $J^\alpha$ and $J^\beta$ are positively invariant for $\Phi$, clearly $J^\alpha$ is $J^\beta$-positively invariant. The verification of that $J^\beta_\alpha$ contains a local unstable manifold of $e_k$ and that $J^\alpha$ is an exit set of $J^\beta$ is also trivial. We omit the details. Hence we see that $(J^\beta, J^\alpha)$ is indeed a shape index pair.

**Remark 6.1.** Note that the pair $(J^\beta, J^\alpha)$ in the above argument may fail to be a Conley index pair of $e_k$, because $J^\beta_\alpha$ is not a neighbourhood of $e_k$ in $L^2(\Omega)$ in general.

### 7 Applications to a Retarded Nonautonomous System

#### 7.1 Basic Notations and Results

Let $\mathcal{H}$ be a compact metric space with metric $d(\cdot, \cdot)$. A given dynamical system $\theta$ is defined on $\mathcal{H}$, i.e., a continuous mapping $\theta : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ satisfying the following group property:

$$\theta_0 h = h, \quad \theta_{s+t} h = \theta_s \theta_t h$$

for all $h \in \mathcal{H}$ and $s, t \in \mathbb{R}$. (Here we have written $\theta(t, h)$ as $\theta_t h$.)
We assume $H$ is minimal for $\theta$, that is, the dynamical system $\theta$ has no nonempty compact invariant proper subsets of $H$.

Let $X$ be a real Banach space with norm $\| \cdot \|$ and $L$ be a sectorial operator on $X$ with compact resolvent. Pick a number $a > 0$ such that

$$\Re z \geq a_0 > 0, \quad \text{for all } z \in \sigma(L + aI) \text{ and a constant } a_0. \quad (7.1)$$

Set $L = L + aI$. Then we can define the fractional powers $L^\alpha$ for $\alpha \in (0, 1)$; see [10] for details. For each $\alpha \geq 0$, define the fractional power $X^\alpha$ of $X$ to be the space $D(L^\alpha)$, which is equipped with the norm $\| \cdot \|_\alpha$ defined by

$$\|x\|_\alpha = \|L^\alpha x\|, \quad x \in X^\alpha.$$

Note that the definition of $X^\alpha$ is independent of the choice of the number $a$. We denote by $C_\alpha$ the embedding constant from $X^\alpha$ into $X$, i.e., $\|x\| \leq C_\alpha \|x\|_\alpha$, for $x \in X^\alpha$.

Suppose that $L$ has a spectral decomposition $\sigma(L) = \sigma^− \cup \sigma^+$, where

$$\Re z \leq -\beta < 0 (z \in \sigma^−), \quad \Re z \geq \beta > 0 (z \in \sigma^+) \quad (7.2)$$

for some $\beta > 0$. Let $X = X_1 \oplus X_2$ be the corresponding direct sum decomposition of $X$ with $X_1$ and $X_2$ being invariant subspaces of $A$. Let $\Pi_k : X \rightarrow X_k (k = 1, 2)$ be the projection from $X$ to $X_k$. Denote $L_k = L|_{X_k}$ and by $\{e^{-L_k t}\}_{t \geq 0}$ the semigroup generated by $-L_k$. By the basic knowledge on sectorial operators (see Henry [10]), we know that there exists $M \geq 1$ such that

$$\|L^\alpha e^{-L_1 t}\| \leq Me^{\beta t}, \quad \|e^{-L_1 t}\| \leq Me^{\beta t}, \quad t \leq 0,$n\|L^\alpha e^{-L_2 t}\Pi_2 L^{-\alpha}\| \leq Me^{-\beta t}, \quad \|L^\alpha e^{-L_2 t}\| \leq Mt^{-\alpha}e^{-\beta t}, \quad t > 0. \quad (7.3)$$

### 7.2 Main Problem and Conclusion

Consider the following retarded cocycle system in $X$:

$$\begin{cases}
\frac{du}{dt} + Lu = f(\theta_t h, u, u(t - \tau)), & t > 0, h \in H, \\
u(t) = \varsigma(t), & t \in [-\tau, 0],
\end{cases} \quad (7.4)$$

where $f(h, x, y)$ is a continuous mapping from $H \times X^\alpha \times X^\alpha$ to $X$ for some $\alpha \in [0, 1)$, $\varsigma : [-\tau, 0] \rightarrow X$ is continuous and the nonnegative number $\tau$ is the time delay. The space $H$ and the system $\theta$ are usually called the base space and the driving system of (7.4), respectively. We make the following assumption.
(F1) The nonlinear term $f(h, x, y)$ is globally Lipschitzian in $(x, y)$ in a uniform manner with respect to $h \in \mathcal{H}$, namely, there exists $l > 0$ such that
\[
\|f(h, x, y) - f(h, x', y')\| \leq l(\|x - x'\|_\alpha + \|y - y'\|_\alpha)
\]
for all $h \in \mathcal{H}$ and $x, y, x', y' \in X^\alpha$.

Under this assumption we infer from the basic theory on retarded evolution equations in Banach spaces (see [18, 36]) that the system (7.4) has a unique global mild solution $u : (-\tau, \infty) \to X$ for each initial data $\varsigma \in C := C([-\tau, 0], X)$. Here a global mild solution $u = u(t)$ for (7.4) is a mapping $u$ from $[-\tau, \infty)$ to $X$ such that $u(t) \in X^\alpha = D(L^\alpha)$ for $t > 0$ and $u$ satisfies the integral equation
\[
\begin{aligned}
\begin{cases}
  u(t) = e^{-Lt}\varsigma(0) + \int_0^t e^{-L(t-\nu)} f(\theta_{\nu}h, u(\nu), u(\nu - \tau))d\nu, & t > 0, \\
  u(t) = \varsigma(t), & -\tau \leq t \leq 0.
\end{cases}
\end{aligned}
\] (7.5)

Furthermore, we assume that $f$ satisfies the following condition.

(F2) The nonlinear term $f$ is sublinear at the infinity in $X^\alpha \times X^\alpha$ uniformly on $\mathcal{H}$, i.e.,
\[
\frac{\|f(h, x, y)\|}{\|x\|_\alpha + \|y\|_\alpha} \to 0 \quad \text{as} \quad \|x\|_\alpha + \|y\|_\alpha \to \infty, \quad \text{for all} \quad h \in \mathcal{H}.
\]

Then we have the existence of bounded full solutions for (7.4) as follows.

Theorem 7.1. Suppose $\mathcal{H}$ is minimal for $\theta$. Under the assumptions (F1) and (F2) on $f$, the system (7.4) has a bounded full solution pertaining to each $h$.

As a generalisation of a similar result for non-autonomous systems in [11], this theorem can be verified by using Banach contraction mapping principle in complete metric spaces. In our situation, we will employ H-shape index to prove this theorem in the following subsection.

7.3 The Proof of Theorem 7.1

First we consider the following equation dependent on $\lambda \in [0, 1]$,
\[
\begin{aligned}
\begin{cases}
  \frac{du}{dt} + Lu = \lambda f(\theta_{h\tau}h, u, u(t - \tau)), & t > 0, \quad h \in \mathcal{H}, \\
  u(t) = \varsigma(t), & -\tau \leq t \leq 0,
\end{cases}
\end{aligned}
\] (7.6)

where $\lambda \in [0, 1]$. Note that when $\lambda = 0$, (7.6) is a linear equation; when $\lambda = 1$, (7.6) is our original equation (7.4). Let $\phi_\lambda : [-\tau, \infty) \times \mathcal{H} \times C \to X$ be the solution mapping
of (7.6) for each \( \lambda \in [0, 1] \). Then by the classical theory of retarded, nonautonomous functional differential equations (see [10, 23, 36]), \( \varphi_\lambda(t; h, \varsigma) \) is continuous in \( \lambda, t, h, \varsigma \) respectively.

Concerning the equation (7.6), we have the following result.

**Lemma 7.2.** There exists \( R > 0 \) such that, for every bounded full solution \( u : \mathbb{R} \to X \) of (7.6) with each \( \lambda \in [0, 1] \), we have that

\[
u(t) \in X^\alpha \quad \text{and} \quad \|u(t)\|_\alpha \leq R \quad \text{for all} \ t \in \mathbb{R}.
\]  

(7.7)

**Proof.** This conclusion is a generalisation of a relevant one for autonomous dynamical systems (Theorem 5.1 in Chapter II of [25]).

By the theory of functional differential equations (see [10, 36]), if \( u \) is a full solution of (7.6) in \( X \), \( u(t) \) is in \( X^\alpha \) for all \( t \in \mathbb{R} \).

We prove (7.7) by contradiction. Suppose that for every \( R > 0 \), there exists a full solution \( u(t) \) of (7.6) such that \( \|u(t)\|_\alpha > R \) for some \( t \in \mathbb{R} \) and \( \lambda \in [0, 1] \). Then there is a sequence \( \lambda_n \in [0, 1] \) and a sequence of full bounded solutions \( t \to u_n(t) \) of (7.6) with \( \lambda = \lambda_n \) such that

\[
c_n := \sup_{t \in \mathbb{R}} \|u_n(t)\|_\alpha \to \infty, \quad \text{as} \ n \to \infty, \quad (7.8)
\]

and \( \|u_n(0)\|_\alpha > c_n - 1 \).

Let \( v_n = c_n^{-1}u_n \) and \( f_n : \mathcal{H} \times X^\alpha \times X^\alpha \to X \) be defined as

\[
f_n(h, x, y) = c_n^{-1}\lambda_n f(h, c_n x, c_n y). \quad (7.9)
\]

We claim that for each \( R_0 \geq 0 \),

\[
S_n := \sup \{ \|f_n(h, x, y)\| : \|x\|_\alpha + \|y\|_\alpha \leq R_0, h \in \mathcal{H} \} \to 0 \quad \text{as} \ n \to \infty. \quad (7.10)
\]

Assume that this claim holds. Noting that \( \|v_n(t)\|_\alpha \) is bounded by 1 for \( t \in \mathbb{R} \), we take \( R_0 \geq 2 \) in (7.10). By [11], the bounded full solution \( u(t) \) of (7.4) satisfies the following integral equation,

\[
u(t) = \int_{-\infty}^{t} e^{-L_2(t-\nu)} \Pi_2 f(\theta_\nu h, u(\nu), u(\nu - \tau)) d\nu
\]

\[
- \int_{t}^{\infty} e^{-L_1(t-\nu)} \Pi_1 f(\theta_\nu h, u(\nu), u(\nu - \tau)) d\nu. \quad (7.11)
\]

By substitution of \( u, f \) by \( v_n, f_n \), respectively, (7.11) also satisfies (7.6). As a result, by the inequalities (7.3),

\[
\|v_n(0)\|_\alpha \leq S_n M \left( \int_{-\infty}^{0} (\nu)^{-\alpha} e^{3\nu} d\nu + \int_{0}^{\infty} e^{-\beta\nu} d\nu \right) \to 0, \quad \text{as} \ n \to \infty.
\]
However, by (7.8),

\[ 1 \geq \|v_n(0)\|_\alpha = c_n^{-1}\|u_n(0)\|_\alpha > \frac{c_n - 1}{c_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty, \]

which leads to a contradiction! This asserts the lemma.

Now it remains to show the claim (7.10). Indeed, by the assumption (F2), for every \( \varepsilon > 0 \), there is \( R_1 > 0 \) such that for all \( h \in \mathcal{H} \),

\[ \|f(h, x, y)\| \leq \varepsilon(\|x\|_\alpha + \|y\|_\alpha) \quad \text{for } \|x\|_\alpha + \|y\|_\alpha > R_1. \]

By the assumption (F1), for \( \|x\|_\alpha + \|y\|_\alpha \leq R_1 \), we have \( \|f(h, x, y)\| \leq lR_1 + m \), where \( m = \max_{h \in \mathcal{H}} \|f(h, 0, 0)\| \). When \( \|x\|_\alpha + \|y\|_\alpha \leq R_0 \), we obtain from (7.9) that

\[ \|f_n(h, x, y)\| \leq \left\{ \begin{array}{ll}
    c_n^{-1}(lR_0 + m), & \text{if } c_n(\|x\|_\alpha + \|y\|_\alpha) \leq R_1, \\
    \varepsilon R_0, & \text{if } c_n(\|x\|_\alpha + \|y\|_\alpha) > R_1.
\end{array} \right. \]

This implies the claim. The proof is thus complete. \( \square \)

For sake of applying H-shape index, we take the space \( \mathcal{C} \) into consideration as the phase space for the system generated by (7.6) as follows (see [36]).

We endow a norm \( \| \cdot \|_\mathcal{C} \) on \( \mathcal{C} \), such that \( \|s\|_\mathcal{C} = \max_{-\tau \leq s \leq 0} \|s(s)\| \). For two real numbers \( t' \leq t'' \), \( t \in [t', t''] \) and a continuous function \( u : [t' - \tau, t''] \rightarrow X \), we denote by \( u_t \) the element of \( \mathcal{C} \) given by \( u_t(s) = u(t + s) \) for \( t \in [t', t''] \) and \( s \in [-\tau, 0] \). Similar to (7.5), the mild solution of (7.6) can be written as

\[ \begin{align*}
    u_t(s) &= e^{-L(t+s)}\xi(0) + \lambda \int_0^{t+s} e^{-L(t+s-\nu)} f(\theta_\nu h, u_\nu(0), u_\nu(-\tau)) d\nu, \\
    t > 0, -\tau &\leq s \leq 0,
\end{align*} \tag{7.12} \]

In this framework, we denote the solution mapping of (7.6) by \( \tilde{\varphi}_\lambda : \mathbb{R}^+ \times \mathcal{H} \times \mathcal{C} \rightarrow \mathcal{C} \) such that for \( s \in [-\tau, 0] \),

\[ \tilde{\varphi}_\lambda(t; h, \xi)(s) = \varphi_\lambda(t + s; h, \xi) = u_t(s) \]

with \( u_t(s) \) defined as (7.12). The continuity of \( \tilde{\varphi}_\lambda \) over \( \mathbb{R}^+ \times \mathcal{H} \times \mathcal{C} \) follows immediately from that of \( \varphi_\lambda \) and the compactness of \([-\tau, 0]\) for each \( \lambda \in [0, 1] \). Also \( \varphi_\lambda \) depends on \( \lambda \in [0, 1] \) continuously. This allows us to define a skew product flow \( \Phi_\lambda \) on \( \mathcal{H} \times \mathcal{C} \) for each \( \lambda \in [0, 1] \) as

\[ \Phi_\lambda(t)(h, \xi) = (\theta_t h, \tilde{\varphi}_\lambda(t; h, \xi)), \tag{7.13} \]
where we endow the space $\mathcal{H} \times \mathcal{C}$ a metric $	ilde{d}(\cdot, \cdot)$ such that
\[
\tilde{d}((h_1, \varsigma_1), (h_2, \varsigma_2)) = d(h_1, h_2) + \|\varsigma_1 - \varsigma_2\|_C.
\]
Furthermore, we define a subspace $C^\alpha$ of $\mathcal{C}$ such that
\[
C^\alpha = \{\varsigma \in \mathcal{C} : \varsigma(t) \in X^\alpha, \text{ for all } t \in [-\tau, 0]\},
\]
with the norm $\|\varsigma\|_{C^\alpha} = \max_{-\tau \leq s \leq 0} \|\varsigma(s)\|_\alpha$ for every $\varsigma \in C^\alpha$.

Let $\mathcal{K}_\lambda$ be the union of all full bounded orbits of $\Phi_\lambda$ in $\mathcal{H} \times \mathcal{C}$ for each $\lambda \in [0, 1]$. By Lemma 7.2, there is $R > 0$ such that
\[
\mathcal{K}_\lambda \subset \mathcal{H} \times B^\alpha(R), \quad \text{for all } \lambda \in [0, 1],
\]
and applying the representation (7.12), we only need to verify that the product set $\mathcal{H} \times C$ is positively invariant for $\Phi_\lambda$. By the property of mild solutions, the unstable manifold of $\mathcal{K}_\lambda$ is surely contained in $\mathcal{H} \times C^\alpha$. Thus, we have the following conclusion.

**Lemma 7.3.** The pair $(\Phi_\lambda, \mathcal{K}_\lambda)$ is H-continuous in $[0, 1]$.

**Proof.** By the definition of H-continuity, we are allowed to work in the space $\mathcal{H} \times C^\alpha$, which is positively invariant for $\Phi_\lambda$. Here we denote the restricted semiflow on $\mathcal{H} \times C^\alpha$ still by $\Phi_\lambda$.

Since $f$ is globally Lipschitzian on $X^\alpha \times X^\alpha$ uniformly in $h$, by the classical results ([10], [25], [36]), we know that $\Phi_\lambda$ does not explode in every bounded set of $\mathcal{H} \times C^\alpha$ for $\lambda \in [0, 1]$. Then by the definition of H-continuity and the continuity of $\Phi_\lambda$ in $\lambda$, we only need to verify that the product set $\mathcal{H} \times \overline{B}^\alpha(R)$ is $\{\Phi_{\lambda_n}\}$-admissible for every sequence $\lambda_n \in [0, 1]$ with $\lambda_n \to \lambda$.

Let $h_n \in \mathcal{H}$, $s_n \in \overline{B}^\alpha(R)$, $0 < t_n \to \infty$ such that $\Phi_{\lambda_n}([0, t_n]) s_n \subset \overline{B}^\alpha(R)$. We will show that $\Phi_{\lambda_n}(t_n)(h_n, s_n)$ has a convergent subsequence in $\mathcal{H} \times \mathcal{C}$. Note that $\mathcal{H}$ is compact and $\tilde{\varphi}_{\lambda_n}(t_n, h_n, s_n) \in \overline{B}^\alpha(R)$. By Arzela-Ascoli theorem, it is sufficient to show that the sequence $\varphi_{\lambda_n}(t_n + \tau; h_n, s_n)$ is equi-continuous on $[-\tau, 0]$ in $X$ for all $n$, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $s_1, s_2 \in [-\tau, 0]$ with $|s_1 - s_2| < \delta$,
\[
\|\varphi_{\lambda_n}(t_n + s_1; h_n, s_n) - \varphi_{\lambda_n}(t_n + s_2; h_n, s_n)\| < \varepsilon \quad \text{for all } n \in \mathbb{N}^+.
\]

Indeed, denoting $u_{t_n}^n(s) = \varphi_{\lambda_n}(t + s; h_n, s_n)$ and applying the representation (7.12), we have for $-\tau \leq s_1 \leq s_2 \leq 0$,
\[
u_{t_n}^n(s_2) - u_{t_n}^n(s_1) = [e^{-L(s_2-s_1)} - I] u_{t_n}^n(s_1) + \lambda \int_{t_n+s_1}^{t_n+s_2} e^{-L(t_n+s_2-\nu)} f(\theta, h_n, u_{t_n}^n(0), u_{t_n}^n(-\tau)) d\nu
\]
\[=: J_1 + J_2,
\]

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where \( I : X \to X \) is the identity.

It is deduced from Theorem 1.4.3 in \([10]\) that, if \( s \geq 0 \) and \( x \in X^\alpha \) with arbitrary \( \alpha \in (0, 1) \), then there is \( \kappa_\alpha > 0 \) such that
\[
\|(e^{-Ls} - I)x\| \leq \alpha^{-1} \kappa_\alpha s^\alpha \|x\|_\alpha.
\]
Hence we have
\[
\|(e^{-Ls} - I)x\| \leq (e^{-\alpha s} - 1) \|x\|_\alpha + C_\alpha (e^{-\alpha s} - 1) \|x\|_
\]
\[
\leq (m + 2lR) \int_{s_1}^{s_2} e^{\beta_0(s_2 - \nu)} \|f(\theta_n h_n, u^n_\tau, (-\nu, -\tau))\| \, \nu
\]
\[
\leq (m + 2lR) C'' |s_1 - s_2| < \varepsilon / 2,
\]
when \( |s_1 - s_2| < \varepsilon_{2C''(m+2R)} \) is sufficiently small.

Combining (7.15) and (7.16), we obtain the desired equi-continuity for the sequence \( u^n_\tau \). This makes \( \Phi_{\lambda_n}(t_n)(h_n, s_n) \) possess a convergent subsequence. Hence \( \mathcal{H} \times \mathcal{C} \) is strongly \( \{\Phi_{\lambda_n}\} \)-admissible, which indicates that \( (\Phi_{\lambda}, \mathcal{K}) \) is \( \mathcal{H} \)-continuous in \([0, 1]\).

The proof is complete. \( \square \)

**Remark 7.4.** For Theorem 7.1 if we want to adopt Conley index of \( (\Phi_{\lambda}, \mathcal{K}) \) and its continuation property, it should be noticed that the checking of S-continuity in the space \( \mathcal{H} \times \mathcal{C} \) is a rather challenging task.

Since \( A \) has a compact resolvent, according to (7.1) and (7.2), the set \( \sigma^- \) contains only finitely many eigenvalues. Let \( r \) be the sum of all multiplicities of the eigenvalues in \( \sigma^- \). It is known that \( r > 0 \).

Now we calculate the \( \mathcal{H} \)-shape index \( s(\Phi_1, \mathcal{K}) \) as follows.
Lemma 7.5. The H-shape index of $\mathcal{K}_1$ for $\Phi_1$ is

$$s(\Phi_1, \mathcal{K}_1) = \mathcal{H}^* \wedge \Sigma^r \neq \emptyset,$$

where $\mathcal{H}^*$ denotes the H-shape of the pointed space $(\mathcal{H} \cup \{\ast\}, \ast)$ with $\ast \notin \mathcal{H}$.

Proof. We split the proof into two steps.

Step 1. We first consider the linear equation, that is, (7.6) when $\lambda = 0$,

$$\frac{du}{dt} + Lu = 0, \quad t > 0, \quad u(s) = \varsigma(s), \quad -\tau \leq s \leq 0.$$

In consideration of the phase space $C$ and recalling the consequences in [36], we obtain that the function $w(t) := u_t$ satisfies the linear system,

$$\frac{dw}{dt} + L_w w = 0, \quad t > 0, \quad w(0) = \varsigma,$$ (7.17)

where $L_w$ is a sectorial operator on $C$ corresponding to $L$ on $X$. Furthermore, the operator $L_w$ has the same eigenvalues as $L$ with the same multiplicities, respectively.

We denote by $\phi$ the semiflow generated by the linear equation (7.17). Observe that the origin 0 of $C$ is the maximal compact invariant set of $\phi$ in $C$, and 0 is a hyperbolic point for (7.17). Therefore, the unstable subspace of 0 in $C$ is $r$-dimensional. By this fact, it is easy to obtain that

$$s(\phi, \{0\}) = \Sigma^r.$$

Step 2. We consider the continuous family of semiflows $\Phi_\lambda$, $\lambda \in [0, 1]$, defined in (7.13), and compute the H-shape index of $\mathcal{K}_1$ for $\Phi_1$ via the continuation property.

When $\lambda = 0$, we see that $\Phi_0 = \theta \times \phi$, which is defined as (4.1). By the results of Step 1, we have that for every $R > 0$,

$$s(\Phi_0, \mathcal{K}_0) = s(\Phi_0, \mathcal{H} \times \{0\}) = s(\theta, \mathcal{H}) \wedge s(\phi, \{0\}) = \mathcal{H}^* \wedge \Sigma^r.$$

By the H-continuity of $(\Phi_\lambda, \mathcal{K}_\lambda)$ in $\lambda \in [0, 1]$ stated in Lemma 7.3 we infer from Theorem 4.11 that

$$s(\Phi_1, \mathcal{K}_1) = s(\Phi_0, \mathcal{K}_0) = \mathcal{H}^* \wedge \Sigma^r.$$

Then it is clear that $\mathcal{H}^* \wedge \Sigma^r \neq \emptyset$ (see [25, 33]). The calculation is finished. □

In the following, we use the framework of $\Phi = \Phi_1 : \mathcal{H} \times C \to \mathcal{H} \times C$ and denote simply

$$\Phi(t)(h, \varsigma) = (\theta_t h, \varphi(t; h, \varsigma))$$ and $\mathcal{H}' = \mathcal{K}_1$.
Based on Lemmas 7.2, 7.3, and 7.5, we now prove the main result Theorem 7.1.

**Proof of Theorem 7.1.** By Lemma 7.5, Example 4.7, and Lemma 7.2, we know that \( \mathcal{K} \neq \emptyset \) and \( \mathcal{K} \subset \mathcal{H} \times \overline{B}(R) \) for some \( R > 0 \).

We claim that for each \( h \in \mathcal{H} \), there is \( \varsigma \in \mathcal{C} \) such that \( (h, \varsigma) \in \mathcal{K} \).

Suppose that this claim is true. Noting that \( \mathcal{K} \) is an invariant set of \( \Phi \). For each \( h \in \mathcal{H} \), there is a full solution \( \tilde{\gamma}_h \) of \( \Phi \) contained in \( \mathcal{K} \) such that

\[
\tilde{\gamma}_h(t) = (\theta_t h, u^h_t), \quad \text{for all } t \in \mathbb{R},
\]

with \( u^h_t \) satisfying \( u^h_t = \tilde{\varphi}(t - t'; \theta_{t'} h, u^h_{t'}) \) for all \( t, t' \in \mathbb{R} \) and \( t \geq t' \). Converting the phase space from \( \mathcal{C} \) back to \( \mathcal{X} \), we have a full solution \( u^h \) of (7.4) such that

\[
u^h(t) = u^h(0).
\]

Then \( u^h \) is a full solution of (7.4) pertaining to \( h \) with \( \|u^h(t)\| \leq \|u^h_t\|_C \leq C_\alpha R \) for all \( t \in \mathbb{R} \). This leads to the final conclusion of Theorem 7.1.

Now it remains to prove the claim. Since the driving system \( \theta \) is independent of the phase space, the projection \( P : \mathcal{H} \times \mathcal{C} \to \mathcal{H} \) and the systems \( \Phi, \theta \) satisfy the following commutativity,

\[
\theta_t \circ P = P \Phi(t).
\]

Because \( \mathcal{K} \) is invariant for \( \Phi \), we have

\[
\theta_t(P \mathcal{K}) = P \Phi(t) \mathcal{K} = P \mathcal{K}, \quad \text{for all } t \geq 0,
\]

which implies \( P \mathcal{K} \) is invariant for \( \theta \). Moreover, by the compactness of \( \mathcal{K} \), we know that \( P \mathcal{K} \) is compact in \( \mathcal{H} \). Therefore, \( P \mathcal{K} \) is a compact invariant set for \( \theta \) in \( \mathcal{H} \).

By the minimality of \( \mathcal{H} \) for \( \theta \), the compact invariant sets in \( \mathcal{H} \) are only \( \emptyset \) and \( \mathcal{H} \) itself. Whereas, \( \mathcal{K} \) is nonempty, and so is \( P \mathcal{K} \). As a result, we must have that \( P \mathcal{K} = \mathcal{H} \), which implies the claim. \( \square \)

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