Incomplete SMT Techniques for Solving Non-Linear Formulas over the Integers

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We present new methods for solving the Satisfiability Modulo Theories problem over the theory of Quantifier-Free Non-linear Integer Arithmetic, SMT(QF-NIA), which consists in deciding the satisfiability of ground formulas with integer polynomial constraints. Following previous work, we propose to solve SMT(QF-NIA) instances by reducing them to linear arithmetic: non-linear monomials are linearized by abstracting them with fresh variables and by performing case splitting on integer variables with finite domain. For variables that do not have a finite domain, we can artificially introduce one by imposing a lower and an upper bound, and iteratively enlarge it until a solution is found (or the procedure times out).

The key for the success of the approach is to determine, at each iteration, which domains have to be enlarged. Previously, unsatisfiable cores were used to identify the domains to be changed, but no clue was obtained as to how large the new domains should be. Here we explain two novel ways to guide this process by analyzing solutions to optimization problems: (i) to minimize the number of violated artificial domain bounds, solved via a Max-SMT solver, and (ii) to minimize the distance with respect to the artificial domains, solved via an Optimization Modulo Theories (OMT) solver. Using this SMT-based optimization technology allows smoothly extending the method to also solve Max-SMT problems over non-linear integer arithmetic. Finally we leverage the resulting Max-SMT(QF-NIA) techniques to solve $\exists\forall$ formulas in a fragment of quantified non-linear arithmetic that appears commonly in verification and synthesis applications.

CCS Concepts: • Mathematics of computing → Solvers; • Theory of computation → Logic and verification; Automated reasoning; • Computing methodologies → Equation and inequality solving algorithms; Theorem proving algorithms;

Additional Key Words and Phrases: Non-linear arithmetic, satisfiability modulo theories

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1. INTRODUCTION

Polynomial constraints are pervasive in computer science. They appear naturally in countless areas, ranging from the analysis, verification and synthesis of software and hybrid systems [Colón et al. 2003] [Sankaranarayanan et al. 2008] [Sankaranarayanan et al. 2004] [Platzer et al. 2009] to, e.g., game theory
In all these cases, it is crucial to have efficient automatic solvers that, given a formula involving polynomial constraints with integer or real variables, either return a solution to the formula or report that there is none.

Therefore, it is of no surprise that solving this sort of non-linear formulas has attracted wide attention over the years. A milestone result of Tarski’s [Tarski 1951] is a constructive proof that the problem is decidable for the first-order theory of real closed fields, in particular for the real numbers. Unfortunately the algorithm in the proof has non-elementary complexity, i.e., its cost cannot be bounded by any finite tower of exponentials, and is thus essentially useless from a practical point of view. For this reason, for solving polynomial constraints in \(\mathbb{R}\), computer algebra has traditionally relied on the more workable approach of cylindrical algebraic decomposition (CAD) [Collins 1975] [Basu et al. 2003]. Still, its applicability is hampered by its doubly exponential complexity.

Due to the limitations of the existing techniques, further research has been carried out in polynomial constraint solving, spurred in the last decade by the irruption of SAT and its extensions [Biere et al. 2009] [Nieuwenhuis et al. 2006]. Thus, several techniques have emerged in this period which leverage the efficiency and automaticity of this new technology. E.g., for solving polynomial constraints in \(\mathbb{R}\), interval constraint propagation has been integrated with SAT and SMT engines [Fränzle et al. 2007] [Gao et al. 2010] [Khanh and Ogawa 2012]. Other works preprocess non-linear formulas before passing them to an off-the-shelf SMT solver for quantifier-free linear real arithmetic [Ganai and Ivancic 2009], or focus on particular kinds of constraints like convex constraints [Nuzzo et al. 2010]. In the implementation of many of these approaches computations are performed with floating-point arithmetic. In order to address the ever-present concern that numerical errors can result in incorrect answers, the framework of \(\delta\)-complete decision procedures has been proposed [Gao et al. 2012] [Gao et al. 2013]. In another line of research, as opposed to numerically-driven approaches, symbolic techniques from algebraic geometry such as the aforementioned CAD [Jovanovic and de Moura 2012], Gröbner bases [Junges et al. 2013] [Passmore et al. 2010], Handelman’s representations [Marechal et al. 2016] or virtual substitution [Corzilius and Abraham 2011] have been successfully adapted to SAT and SMT. As a result, several libraries and toolboxes have been made publicly available for the development of symbolically-driven solvers [Corzilius et al. 2012] [de Moura and Passmore 2013] [Corzilius et al. 2015].

On the other hand, when variables have to take integer values, even the problem of solving a single polynomial equation is undecidable (Hilbert’s 10th problem, [Cooper 2004]). Despite this theoretical limitation, and following a similar direction to the real case, several incomplete methods that exploit the progress in SAT and SMT have been proposed for dealing with integer polynomial constraints. The common idea of these approaches is to reduce instances of this kind of formulas into problems of a simpler language that can be straightforwardly handled by existing SAT/SMT systems, e.g., propositional logic [Fuhs et al. 2007], linear bit-vector arithmetic [Zankl and Middeldorp 2010] or linear integer arithmetic [Borralleras et al. 2012]. All these techniques are oriented towards satisfiability, which makes them convenient in applications where finding solutions is more relevant than proving that none exists (e.g., in verification when generating ranking functions [Larraz et al. 2013], invariants [Larraz et al. 2013] or other inductive properties [Larraz et al. 2014] [Brockschmidt et al. 2015]).
In this article we build upon our previous method [Borralleras et al. 2012] for deciding SMT(QF-NIA) by reduction to SMT(QF-LIA). In that work, non-linear monomials are linearized by abstracting them with fresh variables and by performing case splitting on integer variables with finite domain. In the case in which variables do not have finite domains, artificial ones are introduced by imposing a lower and an upper bound. While the underlying SMT(QF-LIA) solver cannot find a solution (and the time limit has not been exceeded yet), domain relaxation is applied: some domains are made larger by weakening the bounds. To guide which bounds have to be relaxed from one iteration to the following one, unsatisfiable cores are employed: at least one of the artificial bounds that appear in the unsatisfiable core should be weaker. Unfortunately, although unsatisfiable cores indicate which bounds should be relaxed, they provide no hint on how large the new domains have to be made. This is of paramount importance, since the size of the new linearized formula (and therefore the time needed to determine its satisfiability) can increase significantly depending on the number of new cases that must be added.

A way to circumvent this difficulty could be to find alternative techniques to the unsatisfiable cores which, when a solution with the current domains cannot be found, provide more complete information for the domain relaxation. In this paper we propose such alternative techniques. The key idea is that an assignment of numerical values to variables that is “closest” to being a true solution (according to some metric) can be used as a reference as regards to how one should enlarge the domains. Thus, the models generated by the SMT(QF-LIA) engine are put in use in the search of solutions of the original non-linear problem, with a similar spirit to [de Moura and Bjørner 2008a] for combining theories or to the model-constructing satisfiability calculus of [de Moura and Jovanovic 2013].

However, as pointed out above, in our case we are particularly interested in minimal models, namely those that minimize a cost function that measures how far assignments are from being a true solution to the non-linear problem. Minimal models have long been studied in the case of propositional logic [Ben-Eliyahu and Dechter 1996; Ben-Eliyahu-Zohary 2005; Soh and Inoue 2010]. In SMT, significant advancements have been achieved towards solving the optimization problems of Maximum Satisfiability Modulo Theories (Max-SMT, [Nieuwenhuis and Oliveras 2006; Cimatti et al. 2010]) and Optimization Modulo Theories (OMT, [Oliver 2012; Sebastiani and Tomasi 2015]). Thanks to this research, several SMT systems are currently offering optimization functionalities (Sebastiani and Trentin 2015; Li et al. 2014; Bjørner et al. 2015). Taking advantage of these recent developments, in this paper we make the following contributions:

(1) In the context of solving SMT(QF-NIA), we present different heuristics for guiding the domain relaxation step by means of the analysis of minimal models. More specifically, we consider two different cost functions:
— the number of violated artificial domain bounds (leading to Max-SMT problems);
— the distance with respect to the artificial domains (leading to OMT problems).
We evaluate these model-guided heuristics experimentally with an exhaustive benchmark set and compare them with other techniques for solving SMT(QF-NIA). The results of this evaluation show the potential of the method.
(2) Based on the results of the aforementioned experiments, we extend our best approach for SMT(QF-NIA) to handle problems in Max-SMT(QF-NIA).
(3) Finally we apply our Max-SMT(QF-NIA) techniques to solve SMT and Max-SMT problems in the following fragment of quantified non-linear arithmetic: \( \exists \forall \) formulas

\footnote{This is the extended version of the conference paper presented at SAT ’14 [Larraz et al. 2014].}
where \( \exists \) variables are of integer type and \( \forall \) variables are of real type, and non-linear monomials cannot contain the product of two real variables. Formulas of this kind appear commonly in verification and synthesis applications [Dutertre 2015], for example in control and priority synthesis [Cheng et al. 2013], reverse engineering of hardware [Gascon et al. 2014] and program synthesis [Tiwari et al. 2015].

This paper is structured as follows. Section 2 reviews basic background on SMT, Max-SMT and OMT, and also on our previous approach in [Borralleras et al. 2012]. In Section 3 two different heuristics for guiding the domain relaxation step are presented, together with experiments and several possible variants. Then Section 4 proposes an extension of our techniques from SMT(QF-NIA) to Max-SMT(QF-NIA). In turn, in Section 5 our Max-SMT(QF-NIA) approach is applied to solving Max-SMT problems with \( \exists \forall \) formulas. Finally, Section 6 summarizes the conclusions of this work and sketches lines for future research.

2. PRELIMINARIES
2.1. SMT, Max-SMT and OMT

Let \( \mathcal{P} \) be a fixed finite set of propositional variables. If \( p \in \mathcal{P} \), then \( p \) and \( \neg p \) are literals. The negation of a literal \( l \), written \( \neg l \), denotes \( \neg p \) if \( l = p \), and \( p \) if \( l = \neg p \). A clause is a disjunction of literals \( l_1 \lor \cdots \lor l_n \). A (CNF) propositional formula is a conjunction of clauses \( C_1 \land \cdots \land C_n \). The problem of propositional satisfiability (abbreviated SAT) consists in, given a propositional formula, to determine whether it is satisfiable, i.e., if it has a model: an assignment of Boolean values to variables that satisfies the formula.

The satisfiability modulo theories (SMT) problem is a generalization of SAT. In SMT, one has to decide the satisfiability of a given (usually, quantifier-free) first-order formula with respect to a background theory. In this setting, a model (which we may also refer to as a solution) is an assignment of values from the theory to variables that satisfies the formula. Examples of theories are quantifier-free linear integer arithmetic (QF-LIA), where literals are linear inequalities over integer variables, and the more general quantifier-free non-linear integer arithmetic (QF-NIA), where literals are polynomial inequalities over integer variables. Unless otherwise stated, in this paper we will assume that variables are all of integer type.

Another generalization of SAT is Max-SAT [Li and Manyà 2009], which extends the problem by asking for more information when the formula turns out to be unsatisfiable: namely, the Max-SAT problem consists in, given a formula \( F \), to find an assignment such that the number of satisfied clauses in \( F \) is maximized, or equivalently, that the number of falsified clauses is minimized. This problem can in turn be generalized in a number of ways. For example, in weighted Max-SAT each clause of \( F \) has a weight (a positive natural or real number), and then the goal is to find the assignment such that the cost, i.e., the sum of the weights of the falsified clauses, is minimized. Yet a further extension of Max-SAT is the partial weighted Max-SAT problem, where clauses in \( F \) are either weighted clauses as explained above, called soft clauses in this setting, or clauses without weights, called hard clauses. In this case, the problem consists in finding the model of the hard clauses such that the sum of the weights of the falsified soft clauses is minimized. Equivalently, hard clauses can also be seen as soft clauses with infinite weight.

The problem of Max-SMT merges Max-SAT and SMT, and is defined from SMT analogously to how Max-SAT is derived from SAT. Namely, the Max-SMT problem consists in, given a set of pairs \( \{ [C_1, \omega_1], \ldots, [C_m, \omega_m] \} \), where each \( C_i \) is a clause and \( \omega_i \) is its weight (a positive number or infinity), to find an assignment that minimizes the sum of the weights of the falsified clauses in the background theory. As in SMT, in this context we are interested in assignments of values from the theory to variables.
Finally, the problem of Optimization Modulo Theories (OMT) is similar to Max-SMT in that they are both optimization problems, rather than decision problems. It consists in, given a formula $F$ involving a particular numerical variable called $cost$, to find the model of $F$ such that the value assigned to $cost$ is minimized. Note that this framework allows one to express a wide variety of optimization problems (maximization, piecewise linear functions, etc.).

2.2. Solving SMT(QF-NIA) with Unsatisfiable Cores

In [Borralleras et al. 2012], we proposed a method for solving SMT(QF-NIA) problems based on encoding them into SMT(QF-LIA). The basic idea is to linearize each non-linear monomial in the formula by applying a case analysis on the possible values of some of its variables. For example, if the monomial $x^2yz$ appears in the input QF-NIA formula and $x$ must satisfy $0 \leq x \leq 2$, we can introduce a fresh variable $v_{x^2yz}$, replace the occurrences of $x^2yz$ by $v_{x^2yz}$ and add to the clause set the following three case splitting clauses:

$$
\begin{align*}
    x = 0 & \rightarrow v_{x^2yz} = 0, \\
    x = 1 & \rightarrow v_{x^2yz} = yz, \\
    x = 2 & \rightarrow v_{x^2yz} = 4yz.
\end{align*}
$$

In turn, new non-linear monomials may appear, e.g., $yz$ in this example. All non-linear monomials are handled in the same way until a formula in QF-LIA is obtained, for which efficient decision procedures exist [Dutertre and de Moura 2006; Dillig et al. 2011; Griggio 2012].

Note that, in order to linearize a non-linear monomial, there must be at least one variable in it which is both lower and upper bounded. When this property does not hold, new artificial domains can be introduced for the variables that require them (for example, for unbounded variables one may take $\{-1, 0, 1\}$). In principle, this implies that the procedure is no longer complete, since a linearized formula with artificial bounds may be unsatisfiable while the original QF-NIA formula is actually satisfiable. A way to overcome this problem is to proceed iteratively: variables start with bounds that make the size of their domains small, and then the domains are enlarged on demand if necessary, i.e., if the formula turns out to be unsatisfiable. The decision of which bounds are to be relaxed is heuristically taken based on the analysis of an unsatisfiable core (an unsatisfiable subset of the clause set) that is obtained when the solver reports unsatisfiability. There exist many techniques in the literature for computing unsatisfiable cores (see, e.g., [Asín Achá et al. 2010] for a sample of them). In [Borralleras et al. 2012] we employed the well-known simple and effective approach of [Zhang and Malik 2003], consisting in writing a trace on disk and extracting a resolution refutation, whose leaves form an unsatisfiable core. Note that the method tells which bounds should be weakened, but does not provide any guidance in regard to how large the change on the bounds should be. This is critical, as the size of the formula in the next iteration (and so the time required to determine its satisfiability) can grow significantly depending on the number of new case splitting clauses that have to be added. Therefore, in lack of a better strategy, a typical heuristic is to decrement or increment the bound (for lower bounds and for upper bounds, respectively) by a constant value.

Procedure $solve_{SMT,QF-NIA\_cores}$ in Algorithm 1 describes more formally the overall algorithm from [Borralleras et al. 2012] for solving SMT(QF-NIA). First, the required artificial bounds are computed (procedure $artificial\_bounds$, with pseudo-code in Algorithm 2). Then the linearized formula (procedure $linearize$, with pseudo-code in Algorithm 3) together with the artificial bounds are passed to an SMT(QF-LIA) solver.
ALGORITHM 1: Algorithm for solving SMT(QF-NIA) with unsatisfiable cores

status solve_SMT_QF_NIA_cores(formula \( F_0 \)) { // returns whether \( F_0 \) is satisfiable
  \( B = \text{artificial\_bounds}(F_0) \); // \( B \) are artificial bounds enough to linearize \( F_0 \)
  \( F = \text{linearize}(F_0, B) \);
  \( \text{while (not \ timed\_out())} \) {
    \((\text{ST}, \text{UC}) = \text{solve}_\text{SMT\_QF\_LIA}(F, B)\); // unsatisfiable core \text{UC} computed here for simplicity
    if \((\text{ST} == \text{SAT})\) return \text{SAT}; // is \( F \land B \) satisfiable?
    else if \((B \cap \text{UC} == \emptyset)\) return \text{UNSAT};
    else {
      \( B' = \text{new\_domains\_cores}(B, \text{UC}) \); // at least one in the intersection is relaxed
      \( F = \text{update}(F, B, B') \); // add case splitting clauses
      \( B = B' \);
    }
  }
  return \text{UNKNOWN};
}

ALGORITHM 2: Procedure \text{artificial\_bounds}

\text{set \artificial\_bounds(formula \( F_0 \))} { // returns the artificial bounds for linearization
  \( S = \text{choose\_linearization\_variables}(F_0) \); // choose enough variables to linearize \( F_0 \)
  \( B = \emptyset \); // set of artificial bounds
  \( \text{for (V in S)} \) {
    if \((\text{lower\_bound}(V, F_0) == \perp)\) // cannot find lower bound of \( V \) in \( F_0 \)
      \( B = B \cup \{V \geq L\}; \) // for a parameter \( L \), e.g. \( L = -1 \)
    if \((\text{upper\_bound}(V, F_0) == \perp)\) // cannot find upper bound of \( V \) in \( F_0 \)
      \( B = B \cup \{V \leq U\}; \) // for a parameter \( U \), e.g. \( U = 1 \)
  }
  return \( B \);
}

ALGORITHM 3: Procedure \text{linearize}

\text{formula linearize(formula \( F_0, \text{set} B \))} { // returns the linearization of \( F_0 \)
  \( N = \text{nonlinear\_monomials}(F_0) \);
  \( F = F_0 \);
  \( \text{while (N \neq \emptyset)} \) {
    let \( Q \in N \); // non-linear monomial to be linearized next
    \( V_Q = \text{fresh\_variable}() \);
    \( F = \text{replace}(Q, F, V_Q) \); // replace all occurrences of \( Q \) in \( F \) by \( V_Q \)
    \( C = \emptyset \); // clauses of the case splitting
    \( V = \text{linearization\_variable}(Q) \); // choose a finite domain variable in \( Q \) to linearize
    \( \text{for (K in [lower\_bound}(V, F_0 \cup B), \text{upper\_bound}(V, F_0 \cup B))} \)
      \( C = C \cup \{V = K \rightarrow V_Q = \text{evaluate}(Q, V, K)\}; \)
    \( F = F \cup C; \)
    \( N = N \setminus \{Q\} \cup \text{nonlinear\_monomials}(C) \); // new non-linear monomials may be introduced
  }
  return \( F \);
}

(procedure solve_SMT_QF_LIA), which tests if their conjunction is satisfiable\(^2\)\). If the solver returns SAT, we are done. If the solver returns UNSAT, then an unsatisfiable

\(^2\)Note that, in this formulation, the linearization consists of the clauses of the original formula after replacing non-linear monomials by fresh variables, together with the case splitting clauses. On the other hand, it does not include the artificial bounds, which for the sake of presentation are kept as independent objects.
domains

domains

(109x285)ing artificial bounds so as to linearize:

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exceeded). We refer the reader to [Borralleras et al. 2012] for further details.

again, and the process is repeated (typically, while a predetermined time limit is not

update

ALGORITHM 5: Procedure update

formula update(formula F, set B, set B’) { // adds cases when relaxing the bounds from B to B’

F’ = F;

for (V such that V ≥ L in B and V ≥ L’ in B' and L ≠ L’)

for (K in [L’, L – 1]w)

for (Q such that V == linearization_variable(Q)) // V was used to linearize monomial Q

F’ = F’ ∪ { V = K → VQ = evaluate(Q, V, K)}; // VQ is the variable standing for Q

for (V such that V ≤ U in B and V ≤ U’ in B’ and U ≠ U’)

for (K in [U + 1, U’]w)

for (Q such that V == linearization_variable(Q))

F’ = F’ ∪ { V = K → VQ = evaluate(Q, V, K)};

return F’;

core is also computed. If this core does not contain any of the artificial bounds, then

the original non-linear formula must be unsatisfiable, and we are done too. Otherwise,

at least one of the artificial bounds appearing in the core must be chosen for relaxation

(procedure new-domains_cores, with pseudo-code in Algorithm 4). Once the domains

are enlarged and the appropriate case splitting clauses are added (procedure update,

with pseudo-code in Algorithm 5), the new linearized formula is tested for satisfiability

again, and the process is repeated (typically, while a predetermined time limit is not

exceeded). We refer the reader to [Borralleras et al. 2012] for further details.

Example 2.1. Let F₀ be the formula

tx + y ≥ 4 ∧ t² + x² + w² + y² ≤ 12,

where variables t, x, w, y are integer. Let us also assume that we introduce the following

artificial bounds so as to linearize: B = –1 ≤ t, x, w, y ≤ 1. Now a linearization F of

F₀ could be for example:

vtx + y ≥ 4 ∧ vtx² + vₓ² + vw² + vy² ≤ 12 ∧

(t = -1 → vtx = -x) ∧

(t = 0 → vtx = 0) ∧

(t = 1 → vtx = x) ∧

(t = -1 → vₓ = 1) ∧ (w = -1 → vw = 1) ∧

(t = 0 → vₓ = 0) ∧ (w = 0 → vw = 0) ∧

(t = 1 → vₓ = 1) ∧ (w = 1 → vw = 1) ∧

(x = -1 → vx = 1) ∧ (y = -1 → vy = 1) ∧

(x = 0 → vx = 0) ∧ (y = 0 → vy = 0) ∧

(x = 1 → vx = 1) ∧ (y = 1 → vy = 1) ∧
where \( v_{tx}, v_{t^2}, v_{x^2}, v_{w^2}, v_{y^2} \) are fresh integer variables standing for the non-linear monomials in the respective subscripts.

In this case the formula \( F \land B \) turns out to be unsatisfiable. For instance, the SMT(QF-LIA) solver could produce the following unsatisfiable core:

\[
\{ v_{tx} + y \geq 4, \\
t = -1 \rightarrow v_{tx} = -x, \\
t = 0 \rightarrow v_{tx} = 0, \\
t = 1 \rightarrow v_{tx} = x, \\
y \leq 1, \\
t \leq -1, -1 \leq x, \\
t \leq 1, x \leq 1 \}
\]

Intuitively, if \(|t|, |x|, y \leq 1\), then it cannot be the case that \( tx + y \geq 4 \). At this stage, one has to relax at least one of the artificial bounds in the core, for example \( x \leq 1 \). Notice that, on the other hand, the core does not provide any help in regard to deciding the new upper bound for \( x \). If, e.g., we chose that it were \( x \leq 4 \), then \( x \leq 4 \) would replace \( x \leq 1 \) in the set of artificial bounds \( B \), and the following clauses would be added to the linearization \( F \):

\[
x = 2 \rightarrow v_{x^2} = 4 \\
x = 3 \rightarrow v_{x^2} = 9 \\
x = 4 \rightarrow v_{x^2} = 16
\]

In the next iteration one could already find solutions to the non-linear formula \( F_0 \), for instance, \( t = v_{t^2} = w = v_{w^2} = y = v_{y^2} = 1, x = v_{tx} = 3, \) and \( v_{x^2} = 9 \).

3. SOLVING SMT(QF-NIA) WITH MINIMAL MODELS

Taking into account the limitations of the method based on cores when domains have to be extended, in this section we present a model-guided approach to perform this step. Namely, we propose to replace the satisfiability check in linear arithmetic with an optimization call, so that the best model found by the linear solver can be used as a reference for relaxing the bounds.

This is the key idea of the procedure \texttt{solve\_SMT\_QF\_NIA\_min\_models} for solving SMT(QF-NIA) shown in Algorithm 8 (cf. Algorithm 1; note that all subprocedures except for \texttt{optimize\_QF\_LIA} and \texttt{new\_domains\_min\_models} are the same). Now the SMT(QF-LIA) black box (procedure \texttt{optimize\_QF\_LIA}) does not just decide satisfiability, but finds the minimal model of its input formula \( F \) according to a certain cost function. If this model does not satisfy the original non-linear formula, then it can be employed as a hint in the domain relaxation (procedure \texttt{new\_domains\_min\_models}, with pseudo-code in Algorithm 7) as follows. Since the non-linear formula is not satisfied, it must be the case that some of the artificial bounds are not respected by the minimal model. By gathering these bounds, a set of candidates to be relaxed is obtained, as in the approach of Section 2.2. However, and most importantly, unlike with unsatisfiable cores now for each of these bounds a new value can be guessed too: one just needs to take the corresponding variable and enlarge its domain up to the value assigned in the minimal model. That is, let us assume that \( V \leq U \) is an artificial upper bound on variable \( V \) that is falsified in the minimal model. If \( V \) is assigned value \( U' \) in that model (and, hence, \( U < U' \)), then \( V \leq U' \) becomes the new upper bound of \( V \). A similar construction applies for lower bounds.

The intuition behind this approach is that the cost function should measure how far assignments are from being a solution to the original non-linear formula. Formally, the function must be non-negative and have the property that the models of the linearized formula with cost 0 are true models of the original non-linear formula:
ALGORITHM 6: Algorithm for solving SMT(QF-NIA) with minimal models

```
solve_SMT_QF_NIA_min_models(formula F)
status solve_SMT_QF_NIA_min_models(formula F)

status = artifcial bounds(F);
F = linearize(F, B); // B are artificial bounds enough to linearize F

while (not timed.out()) {
    // If ST == UNSAT then F is UNSAT
    if (ST == UNSAT) return UNSAT;
    // If ST == SAT then M is a model of F minimizing function cost below among all models of F
    if (cost(M) == 0) return SAT;
    else {
        B' = new domains_min_models(B, M);
        F = update(F, B, B');
        B = B';
    }
}
return UNKNOWN;
```

THEOREM 3.1. Let F₀ be an arbitrary formula in QF-NIA, and F be any linearization of F₀ in QF-LIA obtained using the procedure linearize with artificial bounds B. A function cost that takes as input the models of F is admissible if:

1. cost(M) ≥ 0 for any M model of F;
2. If cost(M) = 0 then M is a model of F₀.

If the cost functions in procedure solve_SMT_QF_NIA_min_models are admissible then the procedure is correct. That is, given a formula F₀ in QF-NIA:

1. if solve_SMT_QF_NIA_min_models (F₀) returns SAT then formula F₀ is satisfiable; and
2. if solve_SMT_QF_NIA_min_models (F₀) returns UNSAT then formula F₀ is unsatisfiable.

Proof.

1. Let us assume that solve_SMT_QF_NIA_min_models (F₀) returns SAT. Then there is a set of artificial bounds B such that F, the linearization of F₀ using B, satisfies the following: optimize_QF_LIA (F, B) returns a model M of F such that cost(M) = 0. As cost is admissible we have that M is a model of F₀.

2. Let us assume that solve_SMT_QF_NIA_min_models (F₀) returns UNSAT. Then there is a set of artificial bounds B such that F, the linearization of F₀ using B, satisfies that optimize_QF_LIA (F, B) returns UNSAT. By the specification of optimize_QF_LIA, this means that F is unsatisfiable. But since only case splitting clauses are added in the linearization, any model of F₀ can be extended to a model of F. By reversing the implication we conclude that F₀ must be unsatisfiable.

Under the assumption that cost functions are admissible, note that, if at some iteration in procedure solve_SMT_QF_NIA_min_models there are models of the linearization with null cost (hence satisfying the original non-linear formula), then the search is over: optimize_QF_LIA will return such a model, as it minimizes a non-negative cost function.
**ALGORITHM 7:** Procedure `new_domains_min_models`

```plaintext
set new_domains_min_models(set B, map M) { // returns the new set of artificial bounds
  let S ⊆ {b | b ∈ B, M ≠ b} such that S ≠ ∅; // choose among bounds violated by the model
  B' = B;
  for (V ≥ L in S) B' = B' − {V ≥ L} ∪ {V ≥ M(V)}; // L ≥ M(V) as M ≠ V ≥ L
  return B';
}
```

**ALGORITHM 8:** Procedure `optimize_QF_LIA_Max_SMT` based on Max-SMT(QF-LIA)

```plaintext
⟨status, map⟩ optimize_QF_LIA_Max_SMT(formula F, set B) {
  F' = F;
  for (V ≥ L in B) F' = F' ∪ {V ≥ L}; // added as a soft clause, e.g. with weight 1
  for (V ≤ U in B) F' = F' ∪ {V ≤ U}; // added as a soft clause, e.g. with weight 1
  return solve_Max_SMT_QF_LIA(F'); // call to Max-SMT solver
}
```

In what follows we propose two different admissible (classes of) cost functions: the number of violated artificial bounds (Section 3.1), and the distance with respect to the artificial domains (Section 3.2). In both cases, to complete the implementation of `solve_SMT_QF_NIA_min_models` the only procedure that needs to be defined is `optimize_QF_LIA`, since procedure `new_domains_min_models` is independent of the cost function (see Algorithm 7).

### 3.1. A Max-SMT(QF-LIA) Approach to Domain Relaxation

As sketched out above, a possibility is to define the cost of an assignment as the number of violated artificial bounds. A natural way of implementing this is to transform the original non-linear formula into a linearized weighted formula and use a Max-SMT(QF-LIA) tool. In this setting, the clauses of the linearization are hard, while the artificial bounds are considered to be soft (e.g., with weight 1 if we literally count the number of violated bounds). Procedure `optimize_QF_LIA_Max_SMT` is described formally in Algorithm 8. It is worth highlighting that not only is the underlying Max-SMT(QF-LIA) solver required to report the optimum value of the cost function, but it must also produce an assignment in the theory for which this optimum value is attained (so that it can be used in the domain relaxation). A direct and effective way of accomplishing this task is by performing branch-and-bound on top of an SMT(QF-LIA) solver, as done in [Nieuwenhuis and Oliveras 2006](#).

The next lemma justifies, together with Theorem 3.1, that procedure `solve_SMT_QF_NIA_min_models`, when instantiated with `optimize_QF_LIA_Max_SMT`, is correct:

**Lemma 3.2.** Let $F_0$ be an arbitrary formula in QF-NIA, and $F$ be any linearization of $F_0$ in QF-LIA obtained using the procedure linearize with artificial bounds $B$.  

---

3 Other approaches could also be employed for solving Max-SMT(QF-LIA); for example, one could iteratively obtain unsatisfiable cores and add relaxation variables and cardinality or pseudo-Boolean constraints to the instance until a SAT answer is obtained [Fu and Malik 2006](#), [Ansótegui et al. 2013](#), [Morgado et al. 2013](#). Nevertheless, here we opted for branch-and-bound for its simplicity and because it can be easily adapted to meet the requirements for solving Max-SMT(QF-NIA); see Section 4.
The function \( \text{cost} \) that takes as an input a model \( M \) of \( F \) and returns the number of bounds from \( B \) that are not satisfied by \( M \) is admissible.

**Proof.** It is obvious that the function is non-negative. It remains to be proved that, if \( \text{cost}(M) = 0 \), then \( M \models F_0 \). Indeed, if \( \text{cost}(M) = 0 \) then all bounds in \( B \) are satisfied. But since \( F \) is a linearization of \( F_0 \) with artificial bounds \( B \), we have that all additional variables standing for non-linear monomials have values in \( M \) that are consistent with the theory. Hence, we conclude that \( M \models F_0 \). □

Regarding the weights of the soft clauses, as can be observed from the proof of Lemma 3.2 it is not necessary to have unit weights. One may use different values, provided they are positive, and then the cost function corresponds to a weighted sum. Moreover, note that weights can be different from one iteration of the loop of solve\_SMT\_QF\_NIA\_min\_models to the next one.

**Example 3.3.** Let us consider the same formula as in Example 2.1

\[
 tx + y \geq 4 \land t^2 + x^2 + w^2 + y^2 \leq 12.
\]

Recall that, in this case, the artificial bounds are \(-1 \leq t, x, w, y \leq 1\). We obtain the weighted formula consisting of the clauses of \( F \) (as defined in Example 2.1) as hard clauses, and

\[
\begin{align*}
 [-1 \leq t, 1] \land [-1 \leq x, 1] \land [-1 \leq w, 1] \land [-1 \leq y, 1] & \land \\
 [t \leq 1, 1] \land [x \leq 1, 1] \land [w \leq 1, 1] \land [y \leq 1, 1],
\end{align*}
\]

as soft clauses (written following the format [clause, weight]).

In this case minimal solutions have cost 1: at least one of the artificial bounds has to be violated so as to satisfy \( v_{tx} + y \geq 4 \). For instance, the Max-SMT(QF-LIA) solver could return the assignment: \( t = v_{tx} = 1 \), \( x = v_{x^2} = 4 \) and \( w = v_{w^2} = y = v_{y^2} = v_{x^2} = 0 \), where the only soft clause that is violated is \( [x \leq 1, 1] \). Note that, as \( x = 4 \) is not covered by the case splitting clauses for \( v_{x^2} \), the values of \( v_{y^2} \) and \( x \) are unrelated. Now the new upper bound for \( x \) would become \( x \leq 4 \) (so the soft clause \([x \leq 1, 1]\) would be replaced by \([x \leq 4, 1]\)), and similarly to Example 2.1, the following hard clauses would be added:

\[
\begin{align*}
 x = 2 & \rightarrow v_{x^2} = 4 \\
 x = 3 & \rightarrow v_{x^2} = 9 \\
 x = 4 & \rightarrow v_{x^2} = 16
\end{align*}
\]

As seen in Example 2.1 in the next iteration there are solutions with cost 0, e.g., \( t = v_{tx} = w = v_{w^2} = y = v_{y^2} = 1 \), \( x = v_{x^2} = 3 \) and \( v_{x^2} = 9 \). □

One of the disadvantages of this approach is that potentially the Max-SMT(QF-LIA) solver could return models with arbitrarily large numerical values: note that what the cost function takes into account is just whether a bound is violated or not, but not by how much. For instance, in Example 2.1 it could have been the case that the Max-SMT(QF-LIA) solver returned \( w = y = 0 \), \( t = 1 \), \( x = 10^2 \), \( v_{x^2} = 0 \), etc. Since the model is used for extending the domains, a large number would involve adding a prohibitive number of case splitting clauses, and at the next iteration the Max-SMT(QF-LIA) solver would not able to handle the formula with a reasonable amount of resources. However, having said that, as far as we have been able to experiment, this kind of behaviour is rarely observed in our implementation; see Section 3.3 for more details.

On the other hand, the cost function in Section 3.2 below does not suffer from this drawback.
ALGORITHM 9: Procedure optimize\_QF\_LIA\_OMT based on OMT(QF-LIA)

\begin{verbatim}
(status, map) optimize\_QF\_LIA\_OMT(formula F, set B) {

    F' = F;
    E = 0;  // expression for the cost function

    for (V ≥ L in B) {
        lV = fresh\_variable();
        F' = F' ∪ \{(lV ≥ 0, lV ≥ L - V); E = E + lV;
    }

    for (V ≤ U in B) {
        uV = fresh\_variable();
        F' = F' ∪ \{(uV ≥ 0, uV ≥ V - U}; E = E + uV;
    }

    F' = F' ∪ \{cost = E};  // cost is the variable to be minimized

    return solve\_OMT\_QF\_LIA((cost, F'));  // call to OMT solver

}
\end{verbatim}

3.2. An OMT(QF-LIA) Approach to Domain Relaxation

Another possibility of cost function for models of the linearization is to measure the distance with respect to the artificial domains. This can be cast as a problem in OMT(QF-LIA) as follows.

Given a non-linear formula \(F_0\), let us consider a linearization \(F\) obtained after applying procedure linearize with artificial bounds \(B\). Now, let \(vars(B)\) be the set of variables \(V\) for which an artificial domain \([l_V, U_V]\) ∈ \(B\) is added for the linearization. Formally, the cost function is \(\sum_{V \in vars(B)} \delta(V, [l_V, U_V])\), where \(\delta(z, [l, U])\) is the distance of \(z\) with respect to \([l, U]\):

\[
\delta(z, [l, U]) = \begin{cases} 
  l - z & \text{if } z < l \\
  0 & \text{if } l \leq z \leq U \\
  z - U & \text{if } z > U 
\end{cases}
\]

Note that, in the definition of the cost function, one can safely also include bounds which are not artificial but derived from the non-linear formula: the contribution to the cost of these is null, since they are part of the original formula and therefore must always be respected.

The approach is implemented in the procedure optimize\_QF\_LIA\_OMT shown in Algorithm 9. In this procedure, an OMT(QF-LIA) solver is called (procedure solve\_OMT\_QF\_LIA). Such a system can be built upon an existing SMT(QF-LIA) solver by adding an optimization simplex phase II [Schrijver 1998] when the SAT engine reaches a leaf of the search space. For the OMT(QF-LIA) solver to handle the cost function, the problem requires the following reformulation. Let \(cost\) be the variable that the solver minimizes. For each variable \(V \in vars(B)\) with domain \([l_V, U_V]\), let us introduce once and for all two extra integer variables \(l_V\) and \(u_V\) (meaning the distance with respect to the lower and to the upper bound of the domain of \(V\), respectively) and the auxiliary constraints \(l_V ≥ 0, u_V ≥ l_V - V, u_V ≥ 0, u_V ≥ V - U_V\). Then the cost function is determined by the equation \(cost = \sum_{V \in vars(B)} (l_V + u_V)\), which is added to the formula together with the aforementioned auxiliary constraints.
The following result claims that the proposed cost function is admissible. Hence, by virtue of Theorem 3.1 if procedure optimize_QF-LIA_OMT is implemented as in Algorithm 9 then procedure solve_SMT_QF-LIA_min_models is sound:

**Lemma 3.4.** Let \( F_0 \) be an arbitrary formula in QF-NIA, and \( F \) be any linearization of \( F_0 \) in QF-LIA obtained using the procedure linearize with artificial bounds \( B \).

The function \( cost \) that takes as an input a model of \( F \) and returns its distance to the artificial domains:

\[
\sum_{V \in \text{vars}(B)} \delta(V, [l_V, u_V])
\]

is admissible.

**Proof.** The proof is analogous to that of Lemma 3.2. \( \square \)

Intuitively the proposed cost function corresponds to the number of new cases that will have to be added in the next iteration of the loop in solve_SMT_QF-NIA_min_models. However, it is also possible to consider slightly different cost functions: for instance, one could count the number of new monomials that were linearized by case splitting on \( V \). In general, similarly to Section 3.1 one may have a template of cost function of the form \( cost = \sum_{V \in \text{vars}(B)} (\alpha_V l_V + \beta_V u_V) \), where \( \alpha_V, \beta_V > 0 \) for all \( V \in \text{vars}(B) \). Further, again these coefficients may be changed from one iteration to the next one.

**Example 3.5.** Yet again let us take the same non-linear formula from Example 2.1

\[
t x + y \geq 4 \land t^2 + x^2 + w^2 + y^2 \leq 12.
\]

Let us also recall the artificial bounds: \(-1 \leq t, x, w, y \leq 1\). By using the linearization \( F \) as defined in Example 2.1 one can express the resulting OMT(QF-LIA) problem as follows:

\[
\min \delta(t, [-1, 1]) + \delta(x, [-1, 1]) + \delta(w, [-1, 1]) + \delta(y, [-1, 1]) \quad \text{subject to } F,
\]

or equivalently,

\[
\min \text{cost } \text{subject to } \begin{align*}
F & \land \\
\quad l_t & \geq 0 \land l_t \geq -1 - t \land u_t \geq 0 \land u_t \geq t - 1 \land \\
\quad l_x & \geq 0 \land l_x \geq -1 - x \land u_x \geq 0 \land u_x \geq x - 1 \land \\
\quad l_w & \geq 0 \land l_w \geq -1 - w \land u_w \geq 0 \land u_w \geq w - 1 \land \\
\quad l_y & \geq 0 \land l_y \geq -1 - y \land u_y \geq 0 \land u_y \geq y - 1 \land \\
\text{cost} & = l_t + u_t + l_x + u_x + l_w + u_w + l_y + u_y
\end{align*}
\]

In this case, it can be seen that minimal solutions have cost 1. For example, the OMT(QF-LIA) solver could return the assignment: \( x = v_{x2} = 1, t = 2, v_{t2} = 4 \) and \( w = v_{w2} = y = v_{y2} = v_{t2} = 0 \). Note that, as \( t = 2 \) is not covered by the case splitting clauses, the values of \( v_{t2} \) and \( v_{t2} \) are unrelated to \( t \). Now the new upper bound for \( t \) is \( t \leq 2 \), and clauses

\[
t = 2 \rightarrow v_{t2} = 2x \\
t = 2 \rightarrow v_{t2} = 4
\]
are added to the linearization.

At the next iteration there is still no solution with cost 0, so at least another further iteration is necessary before a true model of the non-linear formula can be found. ■

One of the drawbacks of this approach is that, as the previous example suggests, domains may be enlarged very slowly. This implies that, in cases where solutions have large numbers, many iterations are needed before one of them is discovered. See Section 3.3 below for more details on the performance of this method in practice.

3.3. Experimental Evaluation of Model-guided Approaches

Here we evaluate experimentally our approaches for SMT(QF-NIA) and compare them with other non-linear solvers, namely those participating in the QF-NIA division of the 2016 edition of SMT-COMP (http://smtcomp.sourceforge.net). More in detail, we consider the following tools:

— AProVE-NIA [Fuhs et al. 2007] with its default configuration;
— CVC4 [Barrett et al. 2011] version of 05-27-2016;
— ProB [Krings et al. 2015];
— SMT-RAT [Corzilius et al. 2015];
— yices-2 [Dutertre 2014] version 2.4.2;
— raSAT [Tung et al. 2016], with two different versions: raSAT-0.3 and raSAT-0.4 exp;
— z3 [de Moura and Bjørner 2008b] version 4.4.1;
— bcl-cores, our core-based algorithm [Borralleras et al. 2012];
— bcl-maxsmt, our Max-SMT-based algorithm from Section 3.1;
— bcl-omt, our OMT-based algorithm from Section 3.2.

All bcl-* solvers share essentially the same underlying SAT engine and QF-LIA theory solver. Moreover, some strategies are also common:

— procedure artificial bounds uses a greedy algorithm for approximating the minimum set of variables that have to be introduced in the linearization (as shown in [Borralleras et al. 2012], computing a set with minimum size is NP-complete). For each of these variables we force the domain \([-1, 1]\), even if variables have true bounds (for ease of presentation, we will assume here that true bounds always contain \([-1, 1]\)). This turns out to be useful in practice, as quite often satisfiable formulas have solutions with small coefficients. By forcing the domain \([-1, 1]\), unnecessary case splitting clauses are avoided and the size of the linearized formula is reduced.
— the first time a bound is chosen to be relaxed is handled specially. Let us assume it is the first time that a lower bound (respectively, an upper bound) of \(V\) has to be relaxed. By virtue of the remark above, the bound must be of the form \(V \geq -1\) (respectively, \(V \leq 1\)). Now, if \(V\) has a true bound of the form \(V \geq L\) (respectively, \(V \leq U\)), then the new bound is the true bound. Otherwise, if \(V\) does not have a true lower bound (respectively, upper bound), then the lower bound is decreased by one (respectively, the upper bound is increased by one). Again, this is useful to capture the cases in which there are solutions with small coefficients.
— from the second time on, domain relaxation of bcl-maxsmt and bcl-omt follows basically what is described in Section 3, except for a correction factor aimed at instances in which solutions have some large values. Namely, if \(V \leq u\) has to be relaxed and in the minimal model \(V\) is assigned value \(U\), then the new upper bound is \(U + \alpha \cdot \min(\beta, m)\), where \(\alpha\) and \(\beta\) are parameters, \(n\) is the number of times the upper bound of \(V\) has been relaxed, and \(m\) is the number of occurrences of \(V\) in the original formula. As regards bcl-cores, a similar expression is used in which the current

\(^4\)Available at http://www.lsi.upc.edu/~albert/tocl2017.tgz.
bound $u$ is used instead of $U$, since there is no notion of “best model”. The analogous strategy is applied for lower bounds.

The experiments were carried out on the StarExec cluster [Stump et al. 2014], whose nodes are equipped with Intel Xeon 2.4GHz processors. The memory limit was set to 60 GB, the same as in the 2016 edition of SMT-COMP. As regards wall clock time, although in SMT-COMP jobs were limited to 2400 seconds, in our experiments the timeout was set to 1800 seconds, which is the maximum that StarExec allowed us.

Two different sources of benchmarks were considered in this evaluation. The first benchmark suite (henceforth referred to as Term) was already used in the conference version of this paper [Larraz et al. 2014] and consists of 1934 instances generated by the constraint-based termination prover described in [Larraz et al. 2013]. In these problems non-linear monomials are quadratic.

The other benchmarks are the examples of QF-NIA in the SMT-LIB [Barrett et al. 2016], which are grouped into the following families:

— AProVE: 8829 instances
— calypto: 177 instances
— LassoRanker: 120 instances
— leipzig: 167 instances
— mcm: 186 instances
— UltimateAutomizer: 7 instances
— UltimateLassoRanker: 32 instances
— LCTES: 2 instances

Results are displayed in two tables (Tables I and II) for the sake of presentation. Rows represent systems and distinguish between SAT and UNSAT outcomes. Columns correspond to benchmark families. For each family, the number of instances is indicated in parentheses. The cells either show the number of problems of a given family that were solved by a particular system with a certain outcome, or the total time (in seconds) to process all of them. The best solver for each family (for SAT and for UNSAT examples) is highlighted in bold face.

Due to lack of space, the results for family LCTES do not appear in the tables. This family consists of just two benchmarks, one of which was not solved by any system. The other instance was only solved by CVC4, which reported UNSAT in 0.5 seconds.

As the tables indicate, overall our techniques perform well on SAT instances, being the results particularly favourable for the Term family. This is natural: linearizing by case splitting is aimed at finding solutions quickly without having to pay the toll of heavy-weight non-linear reasoning. If satisfiable instances have solutions with small domains (which is often the case, for instance, when they come from our program analysis applications), our techniques usually work well. On the other hand, for families Aprove, leipzig and mcm the results are only comparable or slightly worse than those obtained with other tools\[5\]. One of the reasons could be that, at least for Aprove and leipzig, formulas have a very simple Boolean structure: they are essentially conjunctions of literals and few clauses (if any). For this particular kind of problems CAD-based techniques such as those implemented in yices-2 and z3, which are precisely targeted at conjunctions of non-linear literals, may be more adequate.

\[5\] However, it must be remarked that we detected several inconsistencies between raSAT-0.3 and the rest of the solvers in the family mcm, which makes the results of this tool unreliable.
Table I. Experimental evaluation of SMT(QF-NIA) solvers on benchmark families Term, AProVE, calypto and Lasso-Ranker.

| Family         | AProVE-NIA | SMT-RAT | yices-2 | raSAT-0.3 | raSAT-0.4 exp | z3 | bcl-cores | bcl-maxsmt | bcl-omt |
|----------------|------------|---------|---------|-----------|---------------|----|----------|-----------|---------|
|                | term (1934) | AProVE (8829) | calypto (177) | LassoRanker (120) |
|                | # p. | time | # p. | time | # p. | time | # p. | time | # p. | time |
| SAT            | 161 | 1,459.27 | 11 | 5,816.44 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 763.89 | 0 | 11.07 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 162 | 231.63 | 48 | 22,889.02 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 286 | 0 | 0.00 | 1 | 1.631 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 160 | 2,827.37 | 21 | 2,516.21 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 286 | 0 | 0.00 | 1 | 1.425 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 32 | 15,758.07 | 99 | 187,204.34 | 1 | 5.28 | 1 | 1,351.68 |
| UNSAT          | 0 | 1,800.07 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 134 | 17,857.21 | 8 | 3,309.13 | 0 | 0.00 | 3 | 1.60 |
| UNSAT          | 162 | 1,472.00 | 23 | 3,906.84 | 0 | 0.00 | 6 | 0.34 |
| SAT            | 153 | 4,978.91 | 17 | 1,004.25 | 0 | 0.00 | 6 | 0.31 |
| UNSAT          | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 148 | 7,351.45 | 19 | 2,957.99 | 0 | 0.00 | 6 | 0.34 |
| UNSAT          | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |

Table II. Experimental evaluation of SMT(QF-NIA) solvers on benchmark families leipzig, mcm, UltimateAutomizer (UA) and UltimateLassoRanker (ULR).

| Family         | leipzig (167) | mcm (186) | UA (7) | ULR (32) |
|----------------|--------------|-----------|--------|----------|
|                | # p. | time | # p. | time | # p. | time | # p. | time |
| SAT            | 161 | 1,459.27 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 763.89 | 0 | 11.07 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 162 | 231.63 | 48 | 22,889.02 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 286 | 0 | 0.00 | 1 | 1.631 | 0 | 0.00 |
| SAT            | 160 | 2,827.37 | 21 | 2,516.21 | 0 | 0.00 | 0 | 0.00 |
| UNSAT          | 0 | 286 | 0 | 0.00 | 1 | 1.425 | 0 | 0.00 |
| SAT            | 32 | 15,758.07 | 99 | 187,204.34 | 1 | 5.28 | 1 | 1,351.68 |
| UNSAT          | 0 | 1,800.07 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 134 | 17,857.21 | 8 | 3,309.13 | 0 | 0.00 | 3 | 1.60 |
| UNSAT          | 162 | 1,472.00 | 23 | 3,906.84 | 0 | 0.00 | 6 | 0.34 |
| SAT            | 153 | 4,978.91 | 17 | 1,004.25 | 0 | 0.00 | 6 | 0.31 |
| UNSAT          | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
| SAT            | 148 | 7,351.45 | 19 | 2,957.99 | 0 | 0.00 | 6 | 0.34 |
| UNSAT          | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 | 0 | 0.00 |
Regarding UNSAT instances, it can be seen that our approaches, while often competitive, can be outperformed by other tools in some families. Again, this is not surprising: linearizing may not be sufficient to detect unsatisfiability when deep non-linear reasoning is required. On the other hand, sometimes there may be a purely linear argument that proves that an instance is unsatisfiable. Our techniques can be effective in these situations, which may be relatively frequent depending on the application. This would be the case of families Term, calypo, LassoRanker and ULR.

Comparing our techniques among themselves, overall bcl-maxsmt tends to give the best results in terms of number of solved SAT and UNSAT instances and timings. For example, we can see that bcl-cores proves many fewer unsatisfiable instances than model-guided approaches. The reason is the following. Let $F_0$ be a formula in QF-NIA, and $F$ be a linearization of $F_0$ computed with artificial bounds $B$. Let us assume that $F$ is unsatisfiable. In this case, when the algorithm in bcl-cores tests the satisfiability of $F \land B$, it finds that it is unsatisfiable. Then, if we are lucky and an unsatisfiable core that only uses clauses from $F$ is obtained, then it can be concluded that $F_0$ is unsatisfiable immediately. However, there may be other unsatisfiable cores of $F \land B$, which use artificial bounds. Using such a core leads to performing yet another (useless) iteration of domain relaxation. Unfortunately the choice of the unsatisfiable core depends on the way the search space is explored, which does not take into account whether bounds are original or artificial so as not to interfere with the Boolean engine heuristics. On the other hand, model-guided approaches always detect when the linearization is unsatisfiable. As for SAT instances, the number of solved problems of bcl-cores is similar to that of bcl-maxsmt, but the latter tends to be faster.

Regarding bcl-omt, it turns out that, in general, the additional iterations required in the simplex algorithm to perform the optimization are too expensive. Moreover, after inspecting the traces we have confirmed that as Example 3.5 suggested, bcl-omt enlarges the domains too slowly, which is hindering the search.

### 3.4. Variants

According to the experiments in Section 3.3 altogether the approach based on MaxSMT(QF-LIA) gives the best results among our methods. In this section we propose several ideas for improving it further.

#### 3.4.1. Non-incremental Strategy

A common feature of the procedures for solving SMT(QF-NIA) described in Sections 2.2, 3.1 and 3.2 is that, when no model of the linearization is found that satisfies all artificial bounds, the domains are enlarged. Thus, iteration after iteration, the number of case splitting clauses increases. In this sense, the aforementioned methods are incremental. A disadvantage of this incrementality is that, after some iterations, the formula size may easily become too big to be manageable.

On the other hand, instead of enlarging a domain, one can follow a non-incremental strategy and replace the domain by another one that might not include it. For example, in the model-guided approaches, when computing the new domain for a variable one may discard the current domain and for the next iteration take an interval centered at the value in the minimal model (procedure new_domains_min_models_non_inc, described in Algorithm 10). The procedure for updating the formula has to be adapted accordingly too, so that the case splitting clauses correspond to the values in the artificial domains (procedure update_non_inc, shown in Algorithm 11). In this fashion one

---

6For the sake of efficiency, bcl-cores does not guarantee that cores are minimal with respect to subset inclusion: computing minimal unsatisfiable sets [Belov et al. 2012] to eliminate irrelevant clauses implies an overhead that in our experience does not pay off. But even if minimality were always achieved, there could still be unsatisfiable cores in $F \land B$ using artificial bounds.
ALGORITHM 10: Procedure new
 domains
 models
 non
 jnc

 set new_domains
 min
 models
 non
 jnc(set B, map M) { // returns the new set of artificial bounds
 let S \subseteq \{b | b \in B, M \neq b\} such that S \neq \emptyset; // choose among bounds violated by the model
 W = \{\text{var}(b) | b \in S\}; // variables with domain to be relaxed
 B' = \{b | b \in B \land \text{var}(b) \notin W\};
 for (V in W)
   B' = B' \cup \{M(V) - R \leq V \leq M(V) + R\}; // for a parameter R > 0
 return B';
 }

 ALGORITHM 11: Procedure update
 non
 jnc

 formula update
 non
 jnc(formula F, set B, set B') { // variables whose domain has changed
 W = \{\text{var}(b) | b \in B' - B\}; // variables whose domain has changed
 F' = remove_case_splits(F, W); // remove outdated case splitting clauses
 for (V in W)
   let L, U such that L \leq V \leq U in B';
   for (K in [L, U])
     for (Q such that V = \text{linearization_variable}(Q)) // V was used to linearize monomial Q
       F' = F' \cup \{V = K \rightarrow V_Q = \text{evaluate}(Q, V, K)\}; // V_Q is the variable standing for Q
 return F' \cup \{\forall b \in B \neg b\}; // forbid B: no solution there
 }

can control precisely the number of case splitting clauses, and therefore the size of the formula.

Since monotonicity of domains from one iteration to the next one is now not maintained, this approach requires bookkeeping so as to avoid repeating the same choice of artificial domains. One way to implement this is to add clauses that forbid each of the combinations of domains that have already been tried and with which no model of the original formula was found. Namely, let $B$ be such a combination of artificial bounds. We add the (hard) clause $\bigvee_{b \in B} \neg b$, which forces that at least one of the bounds in $B$ cannot hold. Now, in any following iteration, if the minimal model $M$ of the linearization does not satisfy the non-linear formula, the new set of artificial bounds $B'$ is such that $M \models B'$. This together with $M \models B$ implies $B' \models B$, and therefore artificial bounds cannot be repeated. See procedure update
 non
 jnc in Algorithm 11.

Finally note that, although this alternative strategy for producing new artificial bounds can in principle be adapted to either of the model-guided methods, it makes the most sense for the Max-SMT(QF-LIA)-based procedure. The reason is that, being model-guided, in this approach the next domains to be considered are determined by the minimal model and, as already observed in Section 3.1, this minimal model may assign large values to variables and thus lead to intractable formula growth.

Example 3.6. Let us take the formula and artificial bounds of the running example. We resume Example 3.3, where the following minimal solution of cost 1 was shown: $t = 6$, $x = 1$, $y = 2$, $z = 4$, and $w = 0$, $y = 0$, $w = 0$, $z = 0$, being $x \leq 1$ the only violated artificial bound. Now, taking a radius $R = 2$ for the interval around $x = 4$, in the next iteration the following artificial bounds would be considered: $-1 \leq t, w, y \leq 1$ and $2 \leq x \leq 6$. Moreover, the following clause would be added to the linearization:

$$ -1 > t \vee 1 < t \vee -1 > x \vee 1 < x \vee -1 > w \vee 1 < w \vee -1 > y \vee 1 < y $$
together with

\[(x = 2 \rightarrow v_{x \cdot 2} = 4) \quad \land \quad (x = 3 \rightarrow v_{x \cdot 2} = 9) \quad \land \quad (x = 4 \rightarrow v_{x \cdot 2} = 16) \quad \land \quad (x = 5 \rightarrow v_{x \cdot 2} = 25) \quad \land \quad (x = 6 \rightarrow v_{x \cdot 2} = 36),\]

while clauses

\[(x = -1 \rightarrow v_{x \cdot 2} = 1) \quad \land \quad (x = 0 \rightarrow v_{x \cdot 2} = 0) \quad \land \quad (x = 1 \rightarrow v_{x \cdot 2} = 1)\]

would be removed.

\[\blacksquare\]

### 3.4.2. Optimality Cores.

When following the approach presented in Section 3.4.1, one needs to keep track of the combinations of domains that have already been attempted, in order to avoid repeating work and possibly entering into cycles. As pointed out above, this can be achieved for instance by adding clauses that exclude these combinations of domains. From the SMT perspective, these clauses can be viewed as conflict explanations, if one understands a conflict as a choice of artificial domains that does not lead to a solution to the original non-linear problem. Following the SMT analogy, it is important that explanations are as short as possible. In this section we present a technique aimed at reducing the size of these explanations.

Let us focus on the model-guided procedures, in which at each iteration one minimizes a cost function.

**Definition 3.7.** Let \(F\) be a clause set with variables \(X\), and let \(\text{cost} : X \rightarrow \mathbb{R}\) be a function. Let \(C \subseteq F\) be a subset of the clauses and \(\text{cost}' : X \rightarrow \mathbb{R}\) be another function. We say that the pair \((C, \text{cost}')\) is an optimality core of the minimization problem \(\min\{\text{cost}(M) \mid M \models F\}\) if the following conditions hold:

1. \(\text{cost}' \leq \text{cost}\); and
2. \(\min\{\text{cost}'(M) \mid M \models C\} = \min\{\text{cost}(M) \mid M \models F\}\).

The interest of optimality cores in our context is that they allow identifying a subset of the artificial bounds that is enough for soundly discarding the current combination of domains (and possibly many more). Let us now describe how optimality cores and these subsets of artificial bounds may be obtained in the Max-SMT approach. In this case, one searches the minimum number of violated artificial bounds. In a similar way to resolution refutations obtained from unsatisfiable instances, after each call to the Max-SMT(QF-LIA) solver on the linearization with soft artificial bounds one may retrieve a lower bound certificate [Larrosa et al. 2011]. This certificate consists essentially of a tree of cost resolution steps, and proves that any model of the linearization will violate at least as many artificial bounds as the reported optimal model. Now consider the leaves of this tree. Let \(H\) be those leaves which are hard clauses (corresponding to clauses of the linearization), and \(S\) be those which are soft (corresponding to soft bounds). If we call \(\text{cost}_S\) the function that counts the number of bounds from \(S\) that are not satisfied, from the definition it follows that \((H, \text{cost}_S)\) is an optimality core. In particular, we are interested in the set of artificial bounds \(S\): if the optimal model of the Max-SMT problem has positive cost, then there is no model of the linearization that can also satisfy \(S\). So we can soundly add the clause \(\lor_{b \in S} \neg b\), which replaces the (potentially much) longer clause \(\lor_{b \in B} \neg b\) in Algorithm 11.
Example 3.8. Once more let us consider the running example. We proceed as in Example 3.6 but instead of adding the clause
\[-1 > t \lor 1 < t \lor -1 > x \lor 1 < x \lor -1 > w \lor 1 < w \lor -1 > y \lor 1 < y\]
we add
\[-1 > t \lor 1 < t \lor -1 > x \lor 1 < x \lor -1 > w,\]
i.e., we discard the literals \(-1 > w, 1 < w\) and \(-1 > y\). Following the above notation, we can do that since \((H, \text{cost}_S)\) is an optimality core, where \(H\) is
\[\{v_{tx} + y \geq 4, \ t = -1 \rightarrow v_{tx} = -x, \ t = 0 \rightarrow v_{tx} = 0, \ t = 1 \rightarrow v_{tx} = x\}\]
and \(S\) is
\[\{-1 \leq t, 1,\ [t \leq 1, 1], \ [-1 \leq x, 1], \ [x \leq 1, 1], \ [y \leq 1, 1]\}.
\]

3.5. Experimental Evaluation of Max-SMT(QF-LIA)-based Approaches

In this section we evaluate experimentally the variations of the Max-SMT(QF-LIA) approach proposed in Sections 3.4.1 and 3.4.2. In addition to the benchmarks used in Section 3.4, we have also considered instances produced by our constraint-based termination prover VeryMax (http://www.cs.upc.edu/~albert/VeryMax.html) on the divisions of the termination competition termCOMP 2016 [Termination Competition] in which it participated, namely Integer Transition Systems and C Integer. Since internally VeryMax generates Max-SMT(QF-NIA) rather than SMT(QF-NIA) problems, soft clauses were removed. Given the huge number of obtained examples, of the order of tens of thousands, we could not afford carrying out the experiments with all the tools considered in Section 3.3 and had to restrict the evaluation to the competing solvers that overall performed the best, namely z3 and yices-2. Hence, in addition to these two, the following solvers are considered here:

- bcl-maxsmt, the Max-SMT(QF-LIA)-based approach as in Section 3.3
- bcl-ninc, the non-incremental algorithm from Section 3.4.1
- bcl-ninc-cores, the non-incremental algorithm that uses optimality cores from Section 3.4.2

Moreover, to further reduce the time required by the experiments, we decided to discard those benchmarks which could be solved both by yices-2 and bcl-maxsmt in negligible time (less than 0.5 seconds). After this filtering, finally 20354 and 2019 benchmarks were included in families Integer Transition Systems and C Integer, respectively. Results are displayed in Tables III, IV and V, following the same format as in Section 3.3.

These results confirm that, in general, our techniques work well on SAT instances: except for families leipzig, mcm and UA, the best tool is one of the bcl-* solvers. The gap with respect to yices-2 and z3 is particularly remarkable on benchmarks coming from our termination proving applications (families Term, Integer Transition Systems and C Integer).

On the other hand, as was already justified in Section 3.3 regarding UNSAT problems, in some families the bcl-* solvers are clearly outperformed by the CAD-based
Incomplete SMT Techniques for Solving Non-Linear Formulas over the Integers

Table III. Experimental evaluation of SMT(QF-NIA) solvers on benchmark families Term, AProVE, calypto and Lasso-Ranker.

| Term (1934) | AProVE (8829) | calypto (177) | Lasso-Ranker (120) |
|-------------|--------------|--------------|-----------------|
| Term        | # p. | time | # p. | time | # p. | time | # p. | time |
| SAT         | 1,830 | 79,764.09 | 7,959 | 5,293.65 | 79 | 6.53 | 4 | 0.16 |
| UNSAT       | 69 | 940.15 | 764 | 4,964.66 | 97 | 488.88 | 97 | 87.54 |
| z3          | SAT | 194 | 77,397.16 | 8,023 | 14,790.21 | 79 | 540.03 | 4 | 13.16 |
| UNSAT       | 70 | 3,459.77 | 286 | 7,989.62 | 97 | 388.38 | 97 | 10.04 |
| bcl-maxsmt  | SAT | 1,857 | 811.54 | 8,027 | 1,763.70 | 80 | 4.74 | 4 | 0.08 |
| UNSAT       | 67 | 31.33 | 202 | 51.72 | 97 | 994.17 | 103 | 2.96 |
| bcl-ninc    | SAT | 1,857 | 276.20 | 8,028 | 1,825.93 | 80 | 5.50 | 4 | 0.10 |
| UNSAT       | 67 | 184.41 | 202 | 57.52 | 97 | 155.32 | 103 | 9.31 |
| bcl-ninc-cores | SAT | 1,857 | 349.48 | 8,028 | 1,825.93 | 80 | 5.50 | 4 | 0.10 |
| UNSAT       | 67 | 184.41 | 202 | 57.52 | 97 | 155.32 | 103 | 9.31 |

Table IV. Experimental evaluation of SMT(QF-NIA) solvers on benchmark families leipzig, mcm, UltimateAutomizer (UA) and UltimateLassoRanker (ULR).

| leipzig (167) | mcm (186) | UA (7) | ULR (32) |
|--------------|-----------|--------|---------|
| # p. | time | # p. | time | # p. | time | # p. | time |
| yices-2 SAT | 92 | 715.04 | 11 | 5,816.44 | 0 | 0.00 | 6 | 0.05 |
| UNSAT | 1 | 0.01 | 0 | 0.00 | 2 | 0.02 | 26 | 11.07 |
| z3 SAT | 162 | 1,472.00 | 23 | 3,064.84 | 0 | 0.00 | 6 | 0.34 |
| UNSAT | 0 | 0.00 | 0 | 0.00 | 2 | 0.02 | 26 | 45.20 |
| bcl-maxsmt SAT | 153 | 4,978.91 | 17 | 1,004.25 | 0 | 0.00 | 6 | 0.31 |
| UNSAT | 0 | 0.00 | 0 | 0.00 | 1 | 0.02 | 26 | 25.96 |
| bcl-ninc SAT | 155 | 5,193.20 | 23 | 3,983.74 | 0 | 0.00 | 6 | 0.33 |
| UNSAT | 0 | 0.00 | 0 | 0.00 | 1 | 0.02 | 26 | 28.47 |
| bcl-ninc-cores SAT | 156 | 3,602.32 | 19 | 2,031.77 | 0 | 0.00 | 6 | 0.40 |
| UNSAT | 0 | 0.00 | 0 | 0.00 | 1 | 0.02 | 26 | 31.28 |

Table V. Experimental evaluation of SMT(QF-NIA) solvers on benchmark families LCTES, Integer Transition Systems (ITS) and C Integer (CI).

| LCTES (2) | ITS (20354) | CI (2019) |
|-----------|-------------|----------|
| # p. | time | # p. | time | # p. | time |
| yices-2 SAT | 0 | 0.00 | 8,408 | 471,160.33 | 714 | 84,986.50 |
| UNSAT | 0 | 0.00 | 4,085 | 142,965.19 | 246 | 24,498.79 |
| z3 SAT | 0 | 0.00 | 5,993 | 784,681.66 | 566 | 16,827.79 |
| UNSAT | 0 | 0.00 | 2,004 | 504,022.31 | 148 | 17,919.88 |
| bcl-maxsmt SAT | 0 | 0.00 | 11,321 | 262,793.96 | 895 | 6,530.07 |
| UNSAT | 0 | 0.00 | 2,318 | 51,699.62 | 129 | 1,722.72 |
| bcl-ninc SAT | 0 | 0.00 | 11,522 | 246,918.87 | 943 | 12,174.59 |
| UNSAT | 0 | 0.00 | 2,502 | 49,572.32 | 142 | 7,399.77 |

techniques of yices-2 and z3. This suggests that a mixed approach that used our methods as a filter and that fell back to CAD after some time threshold could possibly take the best of both worlds.

Comparing our techniques among themselves, there is not an overall clear winner. For SAT examples, it can be seen that the non-incremental approach is indeed a useful heuristic: bcl-ninc tends to perform better, being the biggest difference in the Integer Transition Systems and C Integer families. As regards optimality cores, as could be expected on SAT instances they do not prove profitable and result into a slight over-
ALGORITHM 12: Algorithm for solving Max-SMT(QF-NIA)

\{(status, map) solve\_Max\_SMT\_QF\_NIA(formula $F_0$) \} // returns if $H_0$ is satisfiable and best model wrt $S_0$

- $B = \text{artificial\_bounds}(F_0)$;
- $(H, S) = \text{linearize}(F_0, B)$;
- best\_so\_far = ⊥;
- max\_soft\_cost = ∞;

while (not timed\_out()) {
  \((ST, M) = \text{optimize\_QF\_LIA\_Max\_SMT\_threshold}(H, S, B, \text{max\_soft\_cost});
  \)
  if (ST == UNSAT) \{ return (UNSAT, ⊥); \}
  else if (\text{cost}_B(M) == 0) \{
    \text{best\_so\_far} = M;
    \text{max\_soft\_cost} = \text{cost}_S(M) - 1;
    \}
  else \{
    B' = \text{new\_domains\_min\_models}(B, M);
    H = \text{update}(H, B, B');
    B = B';
  \}
  return (UNKNOWN, ⊥);
\}

head of bcl-ninc-cores with respect to bcl-ninc. On the other hand, on UNSAT examples quite often (namely, families Term, LassoRanker, Integer Transition Systems and C Integer) the shorter conflict clauses discarding previous combinations of artificial domains help in detecting unsatisfiability more efficiently. Still, for this kind of instances bcl-maxsmt is usually the best of the three, since fewer iterations of the loop in procedure solve\_SMT\_QF\_NIA\_min\_models are required to prove that the formula is unsatisfiable.

4. SOLVING MAX-SMT(QF-NIA)

This section is devoted to the extension of our techniques for SMT(QF-NIA) to Max-SMT(QF-NIA), which has a wide range of applications, e.g. in termination and non-termination proving [Larraz et al. 2013] Larraz et al. 2014 as well as safety analysis [Brockschmidt et al. 2015]. Taking into account the results of the experiments in Sections 3.3 and 3.5 we will choose the Max-SMT(QF-LIA) approaches as SMT(QF-NIA) solving engines for the rest of this article. In particular, in the description of the following algorithms we will take as a reference the first version explained in Section 3.1 since adapting the algorithms to the variations from Sections 3.4.1 and 3.4.2 is easy.

4.1. Algorithm

We will represent the input $F_0$ of a Max-SMT(QF-NIA) instance as a conjunction of a set of hard clauses $H_0 = \{C_1, \ldots, C_n\}$ and a set of soft clauses $S_0 = \{[D_1, \Omega_1], \ldots, [D_m, \Omega_m]\}$. The aim is to decide whether there exist assignments $\alpha$ such that $\alpha \models H_0$, and if so, to find one such that $\sum \{\Omega \mid D, \Omega \in S_0, \alpha \not\models D\}$ is minimized.

Procedure solve\_Max\_SMT\_QF\_NIA for solving Max-SMT(QF-NIA) is shown in Algorithm 12. In its first step, as usual the initial artificial bounds $H$ are chosen (procedure artificial\_bounds), with which the input formula $F_0 = H_0 \land S_0$ is linearized (procedure

\footnote{We will abuse notation and represent with $B$ both the set of artificial bounds and also the corresponding set of weighted clauses. The exact meaning will be clear from the context.}
ALGORITHM 13: Procedure optimize\textsubscript{QF-LIA Max-SMT threshold}(formula H, formula S, set B, number msc) {

\begin{align*}
\text{\textbf{status, map}} &\quad \text{optimize\textsubscript{QF-LIA Max-SMT threshold}}(H, \text{formula} \ S, \text{set} \ B, \text{number} \ msc) \{ \\
F' &= H \cup S; \quad \text{\footnotesize // } H \text{ are hard clauses, } S \text{ are soft} \\
\text{for} \ (V \geq L \text{ in } B) &\quad \text{\footnotesize // } \text{added as a soft clause, e.g. with weight 1} \\
F' &= F' \cup \{ [V \geq L, 1] \}; \\
\text{for} \ (V \leq U \text{ in } B) &\quad \text{\footnotesize // } \text{added as a soft clause, e.g. with weight 1} \\
F' &= F' \cup \{ [V \leq U, 1] \}; \\
\text{return} &\quad \text{solve\textsubscript{Max-SMT QF-LIA}}(F', \text{msc}); \quad \text{\footnotesize // call to Max-SMT solver}
\}
\end{align*}

— $H$ results from replacing the non-linear monomials in $H_0$ by their corresponding fresh variables, and adding the case splitting clauses;
— $S$ results from replacing the non-linear monomials in $S_0$ by their corresponding fresh variables.

Now notice that there are two kinds of weights: those from the original soft clauses, and those introduced in the linearization. As they have different meanings, it is convenient to consider them separately. Thus, given an assignment $\alpha$, we define its (total) cost as $\text{cost}(\alpha) = (\text{cost}_B(\alpha), \text{cost}_S(\alpha))$, where $\text{cost}_B(\alpha) = \sum\{ \omega \mid [b, \omega] \in B, \alpha \neq b \}$ is the bound cost, i.e., the contribution to the total cost due to artificial bounds, and $\text{cost}_S(\alpha) = \sum\{ \Omega \mid [D, \Omega] \in S, \alpha \neq D \}$ is the soft cost, corresponding to the original soft clauses. Equivalently, if weights are written as pairs, so that artificial bound clauses become of the form $[C, (\omega, 0)]$ and soft clauses become of the form $[C, (0, \Omega)]$, we can write $\text{cost}(\alpha) = \sum\{ (\omega, \Omega) \mid [C, (\omega, \Omega)] \in S \cup B, \alpha \neq C \}$, where the sum of the pairs is component-wise. In what follows, pairs $(\text{cost}_B(\alpha), \text{cost}_S(\alpha))$ will be lexicographically compared, so that the bound cost (which measures the consistency with respect to the theory of QF-NIA) is more relevant than the soft cost. Hence, by taking this cost function and this ordering, we have a Max-SMT(QF-LIA) instance in which weights are not natural or non-negative real numbers, but pairs of them.

In the following step of the algorithm solve\textsubscript{Max-SMT QF-NIA}, the procedure optimize\textsubscript{QF-LIA Max-SMT threshold} (described in Algorithm [13]) dispatches this Max-SMT instance. This procedure is like that presented in Algorithm [8] with the only difference that now a parameter max\textsubscript{soft cost} is passed to the Max-SMT(QF-LIA) solver. This parameter restrains the models of the hard clauses the solver will consider: only assignments $\alpha$ such that $\text{cost}_S(\alpha) \leq \text{max soft cost}$ will be taken into account. That is, this adapted Max-SMT solver computes, among the models $\alpha$ of the hard clauses such that $\text{cost}_S(\alpha) \leq \text{max soft cost}$ (if any), one that minimizes $\text{cost}(\alpha)$. Thus, the search can be pruned when it is detected that it is not possible to improve the best soft cost found so far. This adjustment is not difficult to implement if the Max-SMT solver follows a branch-and-bound scheme (see Section [6.1]), as it is our case.

Then the algorithm examines the result of the call to the Max-SMT solver. If it is UNSAT, then there are no models of the hard clauses with soft cost at most max\textsubscript{soft cost}. Therefore, the algorithm can stop and report the best solution found so far, if any.

Otherwise, $M$ satisfies the hard clauses and has soft cost at most max\textsubscript{soft cost}. If it has null bound cost, and hence is a true model of the hard clauses of the original formula, then the best solution found so far and max\textsubscript{soft cost} are updated, in order to search for a solution with better soft cost. Finally, if the bound cost is not null, then
domains are relaxed as described in Section 3.1 in order to widen the search space. In any case, the algorithm jumps back and a new iteration is performed.

For the sake of simplicity, Algorithm 12 returns \((\text{UNKNOWN}, \bot)\) when time is exhausted. However, the best model found so far best_so_far can also be reported, as it can still be useful in practice.

The following theorem states the correctness of procedure solve\_Max\_SMT\_QF\_NIA:

**Theorem 4.1.** Procedure solve\_Max\_SMT\_QF\_NIA is correct. That is, given a weighted formula \(F_0\) in QF-NIA with hard clauses \(H_0\) and soft clauses \(S_0\):

1. If \(\text{solve\_Max\_SMT\_QF\_NIA}(F_0)\) returns \((\text{SAT}, M)\) then \(H_0\) is satisfiable, and \(M\) is a model of \(H_0\) that minimizes the sum of the weights of the falsified clauses in \(S_0\); and
2. If \(\text{solve\_Max\_SMT\_QF\_NIA}(F_0)\) returns \((\text{UNSAT}, \bot)\) then \(H_0\) is unsatisfiable.

**Proof.**

Let us assume that \(\text{solve\_Max\_SMT\_QF\_NIA}(F_0)\) returns \((\text{SAT}, M)\). The assignment \(M\) is different from \(\bot\), and hence has been previously computed in a call to \(\text{optimize\_QF\_LIA\_Max\_SMT\_Threshold}(H, S, B, \text{max\_soft\_cost})\) such that \(\text{cost}_B(M) = 0\). So \(M\) respects all artificial bounds in \(B\). Thanks to the case splitting clauses in \(H\), this ensures that auxiliary variables representing non-linear monomials have the right values. Therefore \(M\) satisfies \(H_0\), which is what we wanted to prove. Now we just need to check that indeed \(M\) minimizes the sum of the weights of the falsified clauses in \(S_0\).

Notice that, from the specification of \(\text{optimize\_QF\_LIA\_Max\_SMT\_Threshold}\), we know that there is no model of \(H\) such that its soft cost is strictly less than \(\text{cost}_S(M)\). Now let \(M'\) be a model of \(H_0\). By extending \(M'\) so that auxiliary variables representing non-linear monomials are assigned to their corresponding values, we have \(M' \models H\). By the previous observation, \(\text{cost}_{S_0}(M') = \text{cost}_S(M') \geq \text{cost}_S(M) = \text{cost}_{S_0}(M)\).

Now let us assume that \(\text{solve\_Max\_SMT\_QF\_NIA}(F_0)\) returns \((\text{UNSAT}, \bot)\). Let us also assume that there exists \(M'\) a model of \(H_0\), and we will get a contradiction. Indeed, again extending \(M'\) as necessary, we have that \(M' \models H\). If \(\text{solve\_Max\_SMT\_QF\_NIA}(F_0)\) returns \((\text{UNSAT}, \bot)\), then the previous call to \(\text{optimize\_QF\_LIA\_Max\_SMT\_Threshold}(H, S, B, \text{max\_soft\_cost})\) has returned \((\text{UNSAT}, \bot)\), and moreover no previous call to \(\text{optimize\_QF\_LIA\_Max\_SMT\_Threshold}\) has produced a model with null bound cost. This means that \(\text{max\_soft\_cost}\) has not changed its initial value, namely \(\infty\). Therefore \(H\) must be unsatisfiable, a contradiction. \(\square\)

**Example 4.2.** Let \(F_0\) be the weighted formula with hard clauses

\[
H_0 \equiv tx + y \geq 4 \land t^2 + x^2 + w^2 + y^2 \leq 12
\]

(the same of previous examples) and a single soft clause

\[
S_0 \equiv [t^2 + x^2 + y^2 \leq 1, 1].
\]

Let us take \(-1 \leq t, x, w, y \leq 1\) as artificial bounds. After linearization, we get a weighted linear formula with hard clauses:
unsatisfiability will be detected by optimize\_QF\_LIA\_Max\_SMT\_threshold. Then procedure solve\_Max\_SMT\_QF\_NIA will terminate reporting that the minimum cost (with respect to the original soft clauses \(S_0\)) is 1, and that a model with that cost is given by \(t = w = y = 1\) and \(x = 3\).
4.2. Experimental Evaluation

In this section we evaluate experimentally the approach proposed in Section 4.1 for solving Max-SMT(QF-NIA). We adapt the method to each of the three Max-SMT(QF-LIA)-based variants for solving SMT(QF-NIA). Thus, following the same names as in Section 3.5 here we consider the solvers bcl-maxsmt, bcl-ninc and bcl-ninc-cores. We also include in the experiments z3, which is the only competing tool that, up to our knowledge, can handle Max-SMT(QF-NIA) too. As regards benchmarks, we use the original Max-SMT(QF-NIA) versions (that is, keeping soft clauses) of the examples Integer Transition Systems and C Integer employed in Section 3.5.

Tables VI and VII show the results of the experiments on the families Integer Transition Systems and C Integer, respectively. In each table, row UNSAT indicates the number of instances that were proved to be unsatisfiable, and row OPT counts the instances for which optimality of the reported model could be established. A third row OPT + SAT adds to row OPT the number of problems in which a model was found, but could not be proved to be optimal. For the sake of succinctness, as in previous tables other outcomes (timeouts, UNKNOWN answer, etc.) are not made explicit. Columns represent systems and show either the number of problems that were solved with a certain outcome, or the total time (in seconds) to process all of them. The best solver in each case is highlighted in bold face.

From the tables it can be observed that bcl-ninc-cores is more effective than bcl-ninc for Max-SMT. This is natural: proving the optimality of the best model found so far implicitly involves proving unsatisfiability, more precisely that there cannot be a model with a better cost. And as was already remarked in Section 3.5, optimality cores help the non-incremental approach to detect unsatisfiability more quickly. Regarding the incremental approach, the results are inconclusive: depending on the benchmarks, bcl-maxsmt may perform better than bcl-ninc-cores, or the other way around. Finally, z3 is competitive or even superior when dealing with unsatisfiable problems, while it significantly lags behind for the rest of the instances.

5. SOLVING SMT AND MAX-SMT(∃Z ∀R-NIRA)

In this section we will extend our techniques for SMT and Max-SMT(QF-NIA) to the theory of ∃Z ∀R-NIRA. In this fragment of first-order logic, formulas are of the form ∃x ∀y F(x, y), where F is a quantifier-free formula whose literals are polynomial in-
equalities. Moreover, the existentially quantified variables have integer type, whereas the universally quantified ones are real. In particular we will focus on a subset of this logic, namely, those formulas in which monomials never contain the product of two universally quantified variables.

This fragment of quantified non-linear arithmetic is relevant to many applications. For example, it appears in verification and synthesis problems when the so-called template-based method \cite{Colon and Sipma 2002} is employed. In this framework, one attempts to discover an object of interest (e.g., an invariant, or a ranking function) by introducing a template, usually a linear inequality or expression, and solving a formula that represents the conditions the object should meet. For instance, let us find an invariant for the next loop:

\begin{verbatim}
double y = 0; while (y ≤ 2)y = y+1;
\end{verbatim}

A loop invariant $I(y)$ must satisfy the following initiation and inductiveness conditions:

\begin{itemize}
  \item \textbf{Initiation:} $\forall y_0 \ ((y_0 = 0 \rightarrow I(y_0))$
  \item \textbf{Inductiveness:} $\forall y_1, y_2 \ ((I(y_1) \land y_1 \leq 2 \land y_2 = y_1 + 1 \rightarrow I(y_2))$
\end{itemize}

Now a linear template $x_0 \ y \leq x_1$ is introduced as a candidate for $I(y)$, where $x_0$, $x_1$ are unknowns and $y$ is the program variable. Then the conditions needed for $I(y)$ to be an invariant can be expressed in terms of template unknowns and program variables as an $\exists\forall$ formula:

\[
\exists x_0, x_1 \ \forall y_0, y_1, y_2 \ ((y_0 = 0 \rightarrow x_0 \ y_0 \leq x_1) \land (x_0 \ y_1 \leq x_1 \land y_1 \leq 2 \land y_2 = y_1 + 1 \rightarrow x_0 \ y_2 \leq x_1))
\]

This falls into the logical fragment considered here. Indeed note that, since the template is linear, the non-linear monomials in the formula always consist of the product of a template unknown and a program variable. Moreover, we can regard that we are interested in integer coefficients, so the existential variables are integers, while the universal variables are reals, since the program variable is a double. On the other hand, if one is interested in finding models with other type patterns, the following can be taken into account: in general, if a formula

\[
\exists x \in \mathbb{Z} \ \forall y \in \mathbb{R} \ F(x, y)
\]

is satisfiable, then so are

\begin{itemize}
  \item $\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \ F(x, y)$,
  \item $\exists x \in \mathbb{Z} \ \forall y \in \mathbb{Z} \ F(x, y)$,
  \item $\exists x \in \mathbb{R} \ \forall y \in \mathbb{Z} \ F(x, y)$,
\end{itemize}

since the same witness $x$ can be taken.

\section{5.1. Algorithm}

Let us first describe how to deal with the satisfiability problem given a formula $\exists x \forall y \ F(x, y)$, and then the technique will extend to the more general Max-SMT($\exists\forall\mathbb{R}$-NIRA) problem naturally. Note that the requirement that monomials cannot contain the product of two universal variables allows writing the literals in $F$ as linear polynomials in variables $y$, i.e., in the form $a_1(x) \ y_1 + \cdots + a_n(x) \ y_n \leq b(x)$. Hence, if for instance $F$ is a clause, we can write it as

\[
\neg \left( \bigwedge_{i=1}^{m} a_{i1}(x) \ y_1 + \cdots + a_{in}(x) \ y_n \leq b_i(x) \land \bigwedge_{j=1}^{l} c_{j1}(x) \ y_1 + \cdots + c_{jn}(x) \ y_n < d_j(x) \right),
\]
or more compactly using matrix notation as $\neg\left( A(x) y \leq b(x) \land C(x) y < d(x) \right)$.

The key idea (borrowed from [Colón and Sipma 2002]) is to apply the following result from polyhedral geometry to eliminate the quantifier alternation and transform the problem into a purely existential one:

**Theorem 5.1 (Motzkin’s Transposition Theorem [Schrijver 1998]).** Let $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{l \times n}$ and $d \in \mathbb{R}^l$. The system $A(x) y \leq b(x) \land C(x) y < d(x)$ is unsatisfiable if and only if there are $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^l$ such that $\lambda \geq 0$, $\mu \geq 0$, $\lambda^T A(x) + \mu^T C(x) = 0$, $\lambda^T b(x) + \mu^T d(x) \leq 0$, and either $\lambda^T b(x) < 0$ or $\mu \neq 0$.

Thus Motzkin’s Transposition Theorem, we have that formulas

$$\exists x \forall y \neg\left( A(x) y \leq b(x) \land C(x) y < d(x) \right)$$

and

$$\exists x \exists \lambda \exists \mu \left( \lambda, \mu \geq 0 \land \lambda^T A(x) + \mu^T C(x) = 0 \land \lambda^T b(x) + \mu^T d(x) \leq 0 \land (\lambda^T b(x) < 0 \lor \mu \neq 0) \right)$$

are equisatisfiable. In general, if the formula $F$ in $\exists x \forall y F(x, y)$ is a CNF, this transformation is applied locally to each of the clauses with fresh multipliers.

Note that the formula resulting from applying Motzkin’s Transposition Theorem is non-linear, but the existentially quantified variables $\lambda$ and $\mu$ have real type. Fortunately, our techniques from Section 3 do not actually require that all variables are integer; it suffices that there are enough finite domain variables to perform the linearization. And this is indeed the case, since every non-linear monomial of the transformed formula has at most one occurrence of a $\lambda$ or a $\mu$ variable, and all other variables are integer. All in all, we have reduced the problem of satisfiability of the fragment of $\exists \forall \mathbb{R}$-NIRA under consideration to satisfiability of non-linear formulas that our approach can deal with.

Finally, regarding Max-SMT the technique extends clause-wise in a natural way. Given a weighted CNF, hard clauses are transformed using Motzkin’s Transposition Theorem as in the SMT case. As for soft clauses, let $[S, \Omega]$ be such a clause, where $S$ is of the form $\neg\left( A(x) y \leq b(x) \land C(x) y < d(x) \right)$. Then a fresh propositional symbol $p_S$ is introduced, and $[S, \Omega]$ is replaced by a soft clause $[p_S, \Omega]$ and hard clauses corresponding to the double implication

$$\left( \lambda, \mu \geq 0 \land \lambda^T A(x) + \mu^T C(x) = 0 \land \lambda^T b(x) + \mu^T d(x) \leq 0 \land (\lambda^T b(x) < 0 \lor \mu \neq 0) \right) \leftrightarrow p_S.$$

Therefore, similarly to satisfiability, we can solve the Max-SMT problem for the fragment of $\exists \forall \mathbb{R}$-NIRA of interest by reducing it to instances that can be handled with the techniques presented in Section 4.

**Example 5.2.** Let us consider again the problem of finding an invariant for the loop:

```
double y = 0; while (y <= 2) y = y+1;
```

However, now we will make the initiation condition soft, say with weight 1, while the inductiveness condition will remain hard (as done in [Brockschmidt et al. 2015]). The rationale is that, if the initiation condition can be satisfied, then we have a true invariant; and if it is not, then at least we have a conditional invariant: a property that, if at some iteration holds, then from that iteration on it always holds.

---

*In [Colón and Sipma 2002], Farkas’ Lemma is used instead of the generalization presented here.*
Using the same template as above, the formula to be solved is (quantifiers are omitted for the sake of presentation):

\[
\begin{align*}
[y_0 = 0 \rightarrow x_0 y_0 & \leq x_1, 1] \land \\
(x_0 y_1 & \leq x_1 \land y_1 \leq 2 \land y_2 = y_1 + 1 \rightarrow x_0 y_2 \leq x_1)
\end{align*}
\]

After moving the right-hand side of the implication to the left, and applying some simplifications, it results into:

\[
\begin{align*}
[0 & \leq x_1, 1] \land \\
\neg(x_0 y_1 & \leq x_1 \land y_1 \leq 2 \land x_0 (y_1 + 1) > x_1)
\end{align*}
\]

Now the transformation is performed clause by clause. Since the first clause \([0 \leq x_1, 1]\) does no longer contain universally quantified variables, it can be left as it is. As regards the second one, we introduce three fresh multipliers \(\lambda_1, \lambda_2, \) and \(\mu\) and replace \(\neg(x_0 y_1 \leq x_1 \land y_1 \leq 2 \land x_0 (y_1 + 1) > x_1)\) by

\[
\begin{align*}
(\lambda_1 & \geq 0 \land \lambda_2 \geq 0 \land \mu \geq 0 \land \lambda_1 x_0 + \lambda_2 - \mu x_0 = 0 \land \\
\lambda_1 x_1 + 2\lambda_2 + \mu(x_0 - x_1) & \leq 0 \land (\lambda_1 x_1 + 2\lambda_2 < 0 \lor \mu \neq 0)
\end{align*}
\]

All in all, the following Max-SMT instance must be solved:

\[
\begin{align*}
[0 & \leq x_1, 1] \land \\
(\lambda_1 & \geq 0 \land \lambda_2 \geq 0 \land \mu \geq 0 \land \lambda_1 x_0 + \lambda_2 - \mu x_0 = 0 \land \\
\lambda_1 x_1 + 2\lambda_2 + \mu(x_0 - x_1) & \leq 0 \land (\lambda_1 x_1 + 2\lambda_2 < 0 \lor \mu \neq 0)
\end{align*}
\]

There exist many solutions with cost 0, each of them corresponding to a loop invariant; for instance, \(x_0 = 1, x_1 = 3, \lambda_1 = 0, \lambda_2 = 1, \mu = 1\) (which represents the invariant \(y \leq 3\)).

\[\blacksquare\]

5.2. Experimental Evaluation

In this section we evaluate experimentally the approach proposed in Section 5.1 for solving Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{R}\)-NIRA). Similarly to Section 4.2, again we instantiate the method for the three Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{LIA}\))-based variants for solving SMT(\(\exists \mathbb{Z} \forall \mathbb{NIA}\)). So, using the same names as in Sections 3.5 and 4.2 in this evaluation we consider the solvers bcl-maxsmt, bcl-ninc and bcl-ninc-cores. Unfortunately, as far as we know no competing tool can handle the problems of Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{R}\)-NIRA) effectively. Hence, we have to limit our experiments to our own tools.

Regarding benchmarks, again we use the weighted formulas of the families Integer Transition Systems and C Integer, employed in Section 4.2. However, here problems are expressed in Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{R}\)-NIRA) rather than in Max-SMT(NIA); that is, Motzkin’s Transposition Theorem is applied silently inside the solver, and not in the process of generating the instances. Moreover, as illustrated in Example 5.2 Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{R}\)-NIRA) problems coming from the application of the template-based method can usually be simplified, e.g., by using equations to eliminate variables. In order to introduce some variation with respect to the evaluation in Section 4.2, we decided to experiment with the Max-SMT(\(\exists \mathbb{Z} \forall \mathbb{R}\)-NIRA) problems in raw form, without simplifications. Another difference is that, while in Section 4.2 multipliers were considered integer variables (so that purely integer problems were obtained), in this evaluation they have real type.
Table VIII. Experimental evaluation of Max-SMT(∃∀¬NIRA) solvers on benchmark family Integer Transition Systems (20354 benchmarks).

|        | bcl-maxsmt | bcl-ninc | bcl-ninc-cores |
|--------|------------|----------|---------------|
| UNSAT  | 2,196      | 89,259.58| 2,031         |
| OPT    | 6,707      | 1,002,816.92| 5,856     |
| OPT + SAT | 7,537    | 1,071,480.43| 6,485     |

Table IX. Experimental evaluation of Max-SMT(∃∀¬NIRA) solvers on benchmark family C Integer (2019 benchmarks).

|        | bcl-maxsmt | bcl-ninc | bcl-ninc-cores |
|--------|------------|----------|---------------|
| UNSAT  | 88         | 10,095.79| 76            |
| OPT    | 360        | 11,928.57| 379           |
| OPT + SAT | 429     | 13,983.45| 447           |

Results are shown in Tables VIII and IX following the same format as in Section 4.2. It is worth noticing that the number of solved instances is significantly smaller than in Tables VI and VII, respectively. This shows the usefulness of the simplifications performed when generating the Max-SMT(NIA) instances. Regarding which tool for Max-SMT(∃∀¬NIRA) among the three is the most powerful, on SAT instances there is not a global winner, while on unsatisfiable ones bcl-maxsmt has the best results for both families.

6. CONCLUSIONS AND FUTURE WORK

In this article we have proposed two strategies to guide domain relaxation in the instantiation-based approach for solving SMT(QF-NIA) [Borralleras et al. 2012]. Both are based on computing minimal models with respect to a cost function, namely: the number of violated artificial domain bounds, and the distance with respect to the artificial domains. We have experimentally argued that the former gives better results than the latter and previous techniques, and have devised further improvements, based on weakening the invariant that artificial domains should grow monotonically, and exploiting optimality cores. Finally, we have developed and implemented algorithms for Max-SMT(QF-NIA) and for Max-SMT(∃∀¬NIRA), logical fragments with important applications to program analysis and termination but which are missing effective tools.

As for future work, several directions for further research can be considered. Regarding the algorithmics, it would be interesting to look into different cost functions following the model-guided framework proposed here, as well as alternative ways for computing those minimal models (e.g., by means of minimal correction subsets [Marques-Silva et al. 2013; Bjørner and Narodytska 2015]). On the other hand, one of the shortcomings of our instantiation-based approach for solving Max-SMT/SMT(QF-NIA) is that unsatisfiable instances that require non-trivial non-linear reasoning cannot be captured. In this context, the integration of real-gaoleed CAD techniques adapted to SMT [Jovanovic and de Moura 2012] as a fallback or run in parallel appears to be a promising line of work.

Another direction for future research concerns applications. So far we have applied Max-SMT/SMT(QF-NIA/∃∀¬NIRA) to array invariant generation [Larraz et al. 2013], safety [Brockschmidt et al. 2015], termination [Larraz et al. 2013] and non-termination [Larraz et al. 2014] proving. Other problems in program analysis where we envision these techniques could help in improving the state-of-the-art are, e.g., the analysis of worst-case execution time and resource analysis. Also, so far we have only considered sequential programs. The extension of
Incomplete SMT Techniques for Solving Non-Linear Formulas over the Integers

Max-SMT-based techniques to concurrent programs is a promising line of work with a potentially high impact in the industry.

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