The non-resonant, relativistic dynamics of circumbinary planets

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ABSTRACT

We investigate a non-resonant, three-dimensional (spatial) model of a hierarchical system composed of a point-mass stellar (or substellar) binary and a low-mass companion (a circumbinary planet or a brown dwarf). We take into account the leading relativistic corrections to the Newtonian gravity. The secular model of the system relies on the expansion of the perturbing Hamiltonian in terms of the ratio of semi-major axes $\alpha$, averaged over the mean anomalies. We found that a low-mass object in a distant orbit may excite a large eccentricity of the inner binary when the mutual inclination of the orbits is larger than about $60^\circ$. This is related to the strong instability caused by a phenomenon that acts similarly to the Lidov–Kozai resonance (LKR). The secular system may be strongly chaotic and its dynamics unpredictable over long-term time-scales. Our study shows that in the Jupiter- or brown-dwarf-mass regime of the low-mass companion, the restricted model does not properly describe the long-term dynamics in the vicinity of the LKR. The relativistic correction is significant for the parametric structure of a few families of stationary solutions in this problem, in particular for direct orbit configurations (with mutual inclination less than $90^\circ$). We found that the dynamics of hierarchical systems with small $\alpha \sim 0.01$ may be qualitatively different in the realms of Newtonian (classic) and relativistic models. This holds true even for relatively large masses of the secondaries.

Key words: methods: analytical – celestial mechanics – planetary systems.

1 INTRODUCTION

Extrasolar planets are discovered routinely.1 Currently, about 500 low-mass companions to stars of different spectral types are known. Most of them are bounded to single stars. Moreover, there is also a growing number of planetary candidates in binaries as well as in multistellar systems (see Eggenberger 2010 for the statistical properties of planets in binaries). Generally, following the nomenclature in Rabl & Dvorak (1988), we can consider two classes of such multiple systems. In the satellite case or S-type configuration, a planet revolves around one of the primaries in the binary and the second primary is much more distant. In the cometary or circumbinary configuration (C-type hereafter), the planet has a wide orbit around the inner, massive binary.

Current theories of planet formation in multiple stellar systems (e.g. Takeda, Kita & Rasio 2009, and references therein) show that the inclination of a planetary orbit to the orbital plane of the binary may be non-zero in both C-type and S-type systems. It is well known that in S-type configurations, when the mutual inclination of circular orbits is larger than the critical value $i_{\text{crit}} \sim 40^\circ$, the inner orbit undergoes large-amplitude oscillations of the eccentricity, which is in anti-phase with the mutual inclination. This dynamical phenomenon is well known as the Kozai (or Lidov–Kozai) resonance (Lidov 1962; Kozai 1962). We will call it the LKR hereafter. Keeping in mind the two types of possible orbital configurations, this instance of the LKR may also be understood as the inner LKR (Krasinskii 1972, 1974). Actually, many authors explain the large eccentricities of some planetary candidates as due to forcing by a distant star or massive companion (Fabrycky & Tremaine 2007; Tamuz et al. 2008). In fact, the amplification of eccentricity and inclination may also appear in C-type systems. The critical inclination is then $\sim 60^\circ$ and may be attributed to the outer LKR (Krasinskii 1972, 1974; Migaszewski & Goździewski 2010), and this will be addressed further in this work.

In the literature, the problem is most often considered as a restricted one, which means that the planet is a massless particle that does not perturb the motion of the binary. It has been studied in many papers, with different analytical and numerical techniques. The restricted model helps us to simplify the analysis; nevertheless, the assumption of negligible influence of the planet on the motion of primaries may be not valid if the planet is large. In fact, low-mass objects in the few Jupiter-mass range are quite common. If the mutual interactions are significant, as we will show further in this work, the binary orbit may be strongly perturbed by a distant,
It should be emphasized here that a large number of physical and orbital parameters fully characterizing the planetary configurations contradicts our desire to study the problem in a qualitative way, with the help of particular geometric tools. Hence, we restrict the work to specific ranges of these parameters, focusing on a ‘typical’ binary with a relatively large mass ratio of the primaries, as well as a circumbinary object in the Jupiter-/brown-dwarf-mass range. Moreover, analysing corrections to Newtonian point-mass gravity, we only consider relativistic effects that, in turn, limit the orbital parameters of the binary. Conservative and dissipative tidal distortion are neglected here, though they might dominate in compact binaries or in configurations with very hot Jupiter or super-Earth planets (Fabrycky & Tremaine 2007; Mardling 2007; Batygin, Bodenheimer & Laughlin 2009; Ragozzine & Wolf 2009; Mardling 2010). In the range of semi-major axes \( \sim 0.025 \) au, the planetary tidal bulge raises an apsidal rotation of the inner orbit, which may reach a few degrees per year, exceeding the effects of general relativity and the rotational stellar quadrupole by more than an order of magnitude (Ragozzine & Wolf 2009). However in general, as we explain below, rotational distortions introduce extra degrees of freedom to the model (assuming that the stellar and planetary spins may be arbitrary) that cannot be treated in terms of the geometric tools that are natural to investigate two-degrees-of-freedom Hamiltonian dynamics. Still, although the tidal effects could be treated basically in this formalism too, this would introduce new parameters (radians of bodies, Love numbers); hence we postpone investigations of this more general and complex model till future papers. Overall, as we show below, in the parameter ranges investigated here (semi-major axis of the binary \( \sim 0.1–0.2 \) au and larger), general relativity is dominant over the rotational and tidal corrections to the mutual Newtonian interactions. However, we shall also demonstrate that our results may be quite easily scaled down to the regime of masses and semi-major axes typical for multiplanet configurations, and investigated mostly in the coplanar case.

The layout of this paper is as follows. In Sections 2 and 3, we derive the 3D secular model of the planetary system, in which mean-motion resonances of low order are absent, following the coplanar case considered in Migaszewski & Goździewski (2008). We try to keep the presentation self-contained; therefore we recall basic facts regarding the dynamics of the secular model that might be found in other papers already published. Section 3 describes a test of the secular approximation, and recalls the notion of so-called representative planes of initial conditions, as well as a scheme of investigating families of stationary solutions in the secular model. Section 4 is for the analysis of the eccentricity evolution and chaotic dynamics. Section 5 is devoted to a parametric survey of equilibria in the classic (point-mass Newtonian) model. In Section 6, we study the influence of PN corrections on these solutions. In the concluding Section 7, we summarize our results and sketch some perspectives regarding further research.

2 A SECULAR 3D MODEL OF N BODIES

We consider a general, spatial model of an \( N \)-planet system around a central star. It may be described in terms of the Hamiltonian written with respect to canonical Poincaré variables (Michtchenko et al. 2006) in the form \( \mathcal{H} = \mathcal{H}_{\text{kepl}} + \mathcal{H}_{\text{pert}} \), where

\[
\mathcal{H}_{\text{kepl}} = \sum_{i=1}^{N} \left( \frac{p_{r,i}^2}{2r_i^2} - \frac{\mu_i}{r_i} \right)
\]

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stands for the integrable part comprising the direct sum of the relative, Keplerian motions of point-mass secondaries, \(m_i, i = 1, \ldots, N\), with respect to the primary mass \(m_0\). We also define the mass parameters \(\mu_i = k^2(m_0 + m_i)\), where \(k\) is the Gauss gravitational constant and \(\beta_i^* = (1/m_i + 1/m_0)^{-1}\) are the so-called reduced masses. The term \(H_{\text{pert}}\) stands for the perturbing function of the Keplerian motions. We assume that the perturbation is a sum of two terms:

\[
H_{\text{pert}} = H_{\text{NG}} + H_{\text{GR}},
\]

(2)

where \(H_{\text{NG}}\) is related to small Newtonian mutual interactions between \(m_i\) and \(m_j\) and we assume that \(H_{\text{pert}}/H_{\text{kepl}} \ll 1\). This may be accomplished either by keeping \(m_i\) small (we then have the planetary regime) or permitting that secondary masses are relatively large (even comparable with the central object) and simultaneously requiring large separations between particular orbits (the stellar regime). The \(H_{\text{GR}}\) term is for leading general relativity (PN) corrections to the potential of the central star and the innermost companion. Here we focus on non-resonant systems with well-separated orbits; hence we may neglect the relativistic post-Newtonian perturbations of the outer bodies. If the semi-major axis ratios \(\alpha_{ij} = a_i/a_j < 0.1\) are small, the relativistic corrections for distant objects are smaller by orders of magnitude than the PN perturbation of the inner binary.

Following the notion of Poincaré coordinates, the \(H_{\text{NG}}\) perturbation may be written as follows:

\[
H_{\text{NG}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left( \frac{k^2 m_i m_j}{\Delta_{ij}} + \frac{P_i \cdot P_j}{m_0} \right),
\]

(3)

where \(r_i, i = 1, \ldots, N\) are the position vectors of \(m_i\) relative to the central body, \(P_i\) are their conjugate momenta relative to the barycentre of the whole \((N + 1)\)-body system and \(\Delta_{ij} = \|r_i - r_j\|\) denote the relative distance between bodies \(i\) and \(j\).

After Richardson & Kelly (1988), or developing the PN Hamiltonian from the general Lagrangian in Brumberg (2007), the post-Newtonian potential in the PN formulation is \(H_{\text{GR}} \equiv \beta^2 H_{\text{GR}}\), where \(H_{\text{GR}}\) has the following form:

\[
H_{\text{GR}} = \gamma_1 P^2 + \gamma_2 \frac{P^2}{r} + \gamma_3 (r \cdot P)^2 + \gamma_4 \frac{1}{r^2},
\]

(4)

with coefficients \(\gamma_1, \gamma_2, \gamma_3, \gamma_4:\)

\[
\gamma_1 = -\frac{(1 - 3\nu)}{8c^2}, \quad \gamma_2 = -\frac{\mu^2 (3 + \nu)}{2c^2},
\]

\[
\gamma_3 = \frac{(\mu^*)^2}{2c^2}, \quad \gamma_4 = -\frac{\mu^* \nu}{2c^2},
\]

where \(c\) stands for the speed of light in a vacuum, \(\mu^* = k^2(m_0 + m_i)\), \(\nu \equiv m_0 m_i/(m_0 + m_i)^2\), \(P\) is the astero-centric momentum of the innermost secondary (normalized through the reduced mass),

\[
P = v + \frac{1}{c^2} \left[ 4\gamma_3 (v \cdot v) v + 2\gamma_2 v^2 + 2\gamma_4 (r \cdot v) r \right],
\]

(5)

and \(v \equiv p\) stands for the astero-centric velocity of the innermost object (still assuming that the relativistic corrections from the other bodies in the system are neglected). Hence, \(P = v\) with an accuracy of \(O(c^{-2})\) and the relativistic Hamiltonian is conserved up to the order of \(O(c^{-4})\).

It is well known that the equations of motion of the \(N\)-body system with \(N \geq 3\) are not integrable. However, making use of the assumptions above, we may apply the averaging proposition (Arnold, Kozlov & Neishtadt 1993) to remove short-order perturbations and to derive the equations of secular dynamics governing the long-term evolution of the mean orbital elements.

To perform the averaging, the perturbation must be expressed in terms of the canonical action-angle variables \((I, \phi)\):

\[
\dot{H}(I, \phi) = H_{\text{kepl}}(I) + H_{\text{pert}}(I, \phi),
\]

(6)

and we assume that \(H_{\text{pert}}(I, \phi) \sim \epsilon H_{\text{kepl}}(I)\), where \(\epsilon \ll 1\) is a small parameter. Here, we use the mass-weighted Delaunay elements (e.g. Murray & Dermott 2000):

\[
l_i = M_i, \quad L_i = \beta_i^* \sqrt{\mu_i^*} \, a_i, \quad g_i = \omega_i, \quad G_i = L_i \sqrt{1 - e_i^2},
\]

\[
h_i = \Omega_i, \quad H_i = G_i \cos \upsilon_i,
\]

(7)

where \(M_i\) are the mean anomalies, \(a_i\) stand for the canonical semi-major axes, \(e_i\) are the eccentricities, \(\omega_i\) denote inclinations, \(\omega_0\) are the arguments of pericentre and \(\nu_i\) denote the longitudes of the ascending node. The Hamiltonian of an \(N\)-planet system written in terms of these Delaunay variables (equation 7) has the form

\[
H = \sum_{i=1}^{N} \left( \frac{\mu_i^*}{2L_i^2} + \frac{1}{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} H_{\text{pert}} \, dM_1 \cdots dM_N. \right)
\]

(8)

In this formulation, \(L_i\) are the fast angles. They may be eliminated through the averaging that is accomplished with

\[
H_{\text{sec}} = \frac{1}{(2\pi)^N} \sum_{i=1}^{N} \cdots \int_0^{2\pi} H_{\text{pert}} \, dM_1 \cdots dM_N.
\]

(9)

We should remember here that \(H_{\text{sec}}\) is valid only if

(i) \(H_{\text{pert}} \sim \epsilon H_{\text{kepl}}\) (where \(\epsilon \ll 1\) means a small parameter) and the averaging of the unperturbed Keplerian orbits is equivalent to performing the first step of the perturbation calculus (Ferraz-Mello 2007) and

(ii) there are no mixed resonances between the inner binary and the outer companion (e.g. between slow frequencies of the inner orbit and the mean motion of the outer orbit).

We checked that the planetary systems studied in this paper obey these assumptions within the respective parameter ranges. These calculations rely on the averaged model, and will be given below (see the end of Section 3). Because the secular Hamiltonian \(H_{\text{sec}}\) does not depend on the mean anomalies \(M_i\), the conjugate momenta \(L_i\) are integrals of the secular problem. Obviously, the mean semi-major axes are also constant; hence we get rid of \(N\) degrees of freedom (DOF).

### 3 AVERAGING THE 3D MODEL OF N BODIES

In Migaszewski & Goździewski (2008) we describe a simple scheme of averaging the perturbing function \(H_{\text{pert}}\) (equation 2) in the coplanar case, which makes use of the very basic properties of Keplerian motion: the mixed anomalies algorithm. This method may be easily applied to the 3D problem. At first, we consider the direct part of the mutual interaction between the planets, \(H_{\text{NG}}\) (equation 3). The indirect part averages out to a constant and does not contribute to the secular dynamics (Brouwer & Clemence 1961).

The secular Hamiltonian may be written as a sum of terms representing mutual interactions between all pairs of bodies \(i < j\), where \(a_i < a_j\):

\[
\langle H_{\text{NG}} \rangle = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left( \frac{H_{\text{NG}}^{i, j}}{2} \right).
\]

(10)
For a particular pair of planets, we calculate the following integral:

\[
\langle \mathcal{H}_{\text{SEC}}^{i,j} \rangle = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{k^2 m_i m_j}{\Delta_{i,j}} \, d\Omega_i \, d\Omega_j. \tag{11}
\]

Hence the problem may be reduced to averaging the inverse of the distance \(\Delta_{i,j}\) between two particular planets over their mean anomalies:

\[
\Delta_{i,j} = \sqrt{r_i^2 + r_j^2 - 2 r_i \cdot r_j}, \tag{12}
\]

where \(r_i\) and \(r_j\) must be expressed in a common reference frame \(\mathcal{F}\). The same vectors, written in the orbital reference frames \(\mathcal{F}_i\) of each planet, are \(r_i|_{\mathcal{F}_i} = [x_i, y_i, 0]^T\), and expressed in the common reference frame they have the form \(r_i = \hat{\theta}_i r_i|_{\mathcal{F}_i}\). Here, the rotation matrix \(\hat{\theta}_i \equiv \hat{\theta}_i(\alpha_i, \Omega_i, I_i)\) is the matrix product of elementary Eulerian rotations (Murray & Dermott 2000):

\[
\hat{\theta}_i(\alpha_i, \Omega_i, I_i) \equiv \mathcal{P}_3(-\Omega_i) \mathcal{P}_1(I_i) \mathcal{P}_3(-\alpha_i). \tag{13}
\]

Equation (12) may be rewritten as follows:

\[
\Delta_{i,j} = r_i \sqrt{1 - 2 \frac{1}{r_j} \cdot \frac{r_j}{r_i} + \left(\frac{r_j}{r_i}\right)^2}. \tag{13}
\]

Following Migaszewski & Go\’dziewski (2008), we express the radius vector \(r_i\) of the inner body in each planetary pair with respect to the eccentric anomaly. The radius vector of the outer body in the pair is parametrized by the true anomaly. This choice implies that \(\Delta_{i,j}\) expanded in Taylor series with respect to \(\alpha\) is a trigonometric polynomial. To compute the integral in equation (11), we also change the integration variables:

\[
dM_i \equiv \mathcal{I}_i \, dE_i, \quad \text{and} \quad dM_j \equiv \mathcal{J}_j \, df_j, \]

where auxiliary functions appear:

\[
\mathcal{I}_i = 1 - e_i \cos E_i, \quad \mathcal{J}_j = \frac{1 - e_j}{(1 + e_j \cos f_j)^2}. \tag{14}
\]

Finally, the averaged mutual perturbation has the same general form as in the coplanar case (Migaszewski & Go\’dziewski 2008):

\[
\langle \mathcal{H}_{\text{SEC}}^{i,j} \rangle = \frac{k^2 m_i m_j}{a_j} \left[ 1 + \sqrt{1 - e_j^2} \sum_{i=2}^{\infty} \lambda_{i,j}^{(2)} \mathcal{R}_j^{i,j}(e_i, e_j, \omega_i, \omega_j, \Omega_i, \Omega_j, I_i, I_j) \right], \tag{15}
\]

although explicit expressions for \(\mathcal{R}_j^{i,j}\) are obviously different in the 3D model. The zeroth-order term in equation (15) is reduced to a constant and does not influence the secular equations of motion. Also, the first-order term vanishes identically. The remaining terms \(\mathcal{R}_j^{i,j}\) have a rather complex form. In the simplest three-body system \((i = 1, j = 2)\), we may express them in the Laplace reference frame. In this case, \(\Delta \Omega = \pm \pi\) and \(G_3 \sin I_1 = G_3 \sin I_2\) (see e.g. Michtchenko et al. 2006). It is also natural to introduce the mutual inclination, \(i_{mut} \equiv I_1 + I_2\). Then the \(\mathcal{R}_j^{i,j}\) terms of the order of 2 and 3 may be expressed with the quadrupole and octupole terms, respectively (Ford et al. 2000; Lee & Peale 2003; Farago & Laskar 2010). The quadrupole-level term is the following:

\[
\mathcal{R}_2^{1,2} = \frac{1}{8} D_1 (2 + 3 e_1^2) - \frac{15}{16} e_1^3 D_2 \cos 2\omega_1, \tag{16}
\]

where \(C_1 \equiv \cos i_{mut}\) and \(D_1 = (3C_1^2 - 1)/2\), \(D_2 = C_1^2 - 1\). The third-order (octupole-level) term reads as follows:

\[
\mathcal{R}_3^{(1,2)} = \frac{15}{64} D_0 e_1 e_2 \cos \Delta \sigma \left(3e_1^4 + 4\right) + \frac{525}{256} D_1 D_2 e_1 e_2 \cos (3\omega_1 - \omega_2) + \frac{525}{512} D_4 e_1 e_2 \cos (3\omega_1 + \omega_2) + \frac{15}{128} D_6 e_1 e_2 \cos (\omega_1 + \omega_2) + \frac{45}{512} D_8 e_1 e_2 \cos (\omega_1 + \omega_2), \tag{17}
\]

where coefficients \(D_j\) are

\[
D_3 = (1 + C_1)/2, \quad D_4 = 1 - C_1, \quad D_5 = -15C_1^2 + 5C_1^2 + 11C_1 - 1, \quad D_6 = (15C_1^2 + 5C_1^2 - 11C_1 - 1)/8. \]

Equations (16) and (17) are written similarly to terms appearing in the coplanar model [see equations (22) and (23) in Migaszewski & Go\’dziewski (2008)]. Clearly, if \(i_{mut} = 0\) then \(D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = 0\) and formulae \(\mathcal{R}_2^{i,j}\), \(\mathcal{R}_3^{i,j}\) coincide with those in the coplanar problem. An explicit expansion of \(\mathcal{H}_{\text{SEC}}\) shows that the quadrupole-order term in \(\sigma\) introduces the evolution of eccentricity \(e_1\), and in this approximation the outer eccentric \(e_2\) is constant. The variation of the outer eccentricity may only be introduced through the third-order (octupole) and higher terms. Indeed, up to the quadrupole approximation, the secular Hamiltonian does not depend on \(\omega_2\) (the cyclic angle) and the eccentricity of the outer body becomes an additional integral of motion. In this case, the problem can be reduced to one DOF and is integrable (Lidov & Ziglin 1976). This feature has been accounted for in many recent papers; moreover, the apparently subtle third-order perturbation to the Keplerian model or the first-order perturbation to the integrable quadrupole-order approximation may introduce qualitative changes of the dynamics.

We calculated the secular expansion (equation 15) up to tenth order.\(^2\) One should be aware that by increasing the order of this expansion, we do not necessarily improve the approximation of the secular model of the real system, because this model is still limited by the first-order perturbation theory. In Section 3.2 we will examine the accuracy of the secular expansion in more detail.

Finally, the averaged relativistic correction possesses the same form as in the coplanar case (Migaszewski & Go\’dziewski 2008). Moreover, we include this perturbation only in the mutual interaction of masses \(m_0\) and \(m_j\):

\[
\langle \mathcal{H}_{\text{CR}} \rangle = -\frac{3}{8} \frac{\mu^3}{c^2} \frac{\beta \beta^*}{L_1} G_1 + \text{const}, \tag{18}
\]

as was explained above.

Having the averaged model in hand, we may calculate the secular frequencies of the inner companion and compare them with the mean motion of the outer object \(n_2\). For the relativistic advance of the inner periastron we have

\[
\frac{f_{1,rel}}{n_2} = \frac{3}{8} \frac{\mu_2^3}{c^2 \alpha} \sqrt{\frac{\mu_1}{\mu_2^2} \alpha^{-3/2} \frac{1}{1 - e_1^2}}, \tag{21}
\]

\(^2\)This expansion is available on request in the form of a raw MATHEMATICA input file; it will be also available online, after publishing this manuscript.
and for the apsidal motion forced by the mutual interaction of the inner and outer companion (in the quadrupole approximation):

\[
\frac{f_{1,\text{mat}}}{n_2} = \sqrt{\frac{\mu_2}{\mu_1} \frac{m_2}{m_0}} \frac{1}{\sqrt{1 - e_1^2}} \frac{1}{\Lambda_1} + \frac{m_1}{m_0} \frac{1}{(1 - e_1^2)^2} \Lambda_2,
\]

where \(\Lambda_{1,2}\) are the following functions of the geometric elements:

\[
\Lambda_1 = \frac{3}{8} \left(1 - e_1^2\right) \left[3C_1^2 - 1 - 5(C_1^2 - 1) \cos 2\omega_1\right] + \frac{3}{8} C_1^2 \left[2 + 3e_1^2\right] - 5e_1^2 \cos 2\omega_1,
\]

\[
\Lambda_2 = \frac{3}{8} C_1 \left[2 + 3e_1^2\right] - 5e_1^2 \cos 2\omega_1.
\]

Assuming now that \(\mu_2 \sim k m_0\) (the central mass dominates), we may obtain the following estimates of the secular frequencies in terms of the characteristic units in our model: the relativistic frequency relative to \(n_2\) is

\[
\frac{f_{1,\text{rel}}}{n_2} = 2 \times 10^{-5} \left(\frac{m_2}{100 m_1}\right) \left(\frac{1}{100 m_0} \right) \left(\frac{0.2 \text{ au}}{a_1} \right) \left(\frac{0.04 \text{ au}}{\alpha} \right)^{3/2} \frac{1}{1 - e_1^2},
\]

while the mutually forced frequency relative to \(n_2\) is

\[
\frac{f_{1,\text{mut}}}{n_2} = 8 \times 10^{-5} \left(\frac{m_2}{100 m_1}\right) \left(\frac{1}{100 m_0} \right) \left(\frac{\alpha}{0.04} \right)^{3/2} \frac{1}{(1 - e_1^2)^2} \Lambda_1 + 2 \times 10^{-4} \left(\frac{m_2}{100 m_1}\right) \left(\frac{1}{100 m_0} \right) \left(\frac{\alpha}{0.04} \right)^{3/2} \frac{1}{(1 - e_1^2)^2} \Lambda_2.
\]

These frequencies may be compared with the tidal apsidal frequency induced by the primary and the inner body bulge (Migaszewski & Goździewski, in preparation):

\[
\frac{f_{1,\text{tid}}}{n_2} = \left\{4 \times 10^{-8} \left(\frac{m_1}{100 m_1}\right) \left(\frac{1}{100 m_0} \right) \left(\frac{R_0}{1 R_0} \right)^5 \left(\frac{k_{1,0}}{0.03} \right) + 7 \times 10^{-6} \left(\frac{m_2}{100 m_1}\right) \left(\frac{100 m_1}{m_0} \right) \left(\frac{R_1}{2 R_1} \right)^5 \left(\frac{k_{1,1}}{0.15} \right) \right\} \times \left(\frac{0.2 \text{ au}}{a_1} \right)^5 \left(\frac{0.04 \text{ au}}{\alpha} \right)^{3/2} \frac{1}{(1 - e_1^2)^2} \Lambda_1 + 3e_1^2/2 + e_1^4/8 \right(\frac{1}{(1 - e_1^2)^2} \Lambda_2).
\]

where \(R_0\) is the stellar radius, \(R_1\) is the radius of the inner secondary and \(k_{1,0}, k_{1,1}\) are the tidal Love numbers of these bodies. Let us choose a reference model by setting characteristic parameters of \(a_1 \sim 0.2 \text{ au}, \alpha \sim 0.04, m_0 \sim 1 m_0, m_1 \sim 100 m_1, R_0 \sim 1 R_0, R_1 \sim 2 R_1\). Assuming that the bodies are modelled by polytopes with indices of 3 and 2, respectively, we compute their Love numbers, \(\sim 0.03\) for the primary and \(\sim 0.15\) for the inner secondary. Then setting \(e_1 \sim 0\), we obtain that the relativistic frequency is comparable with the mutual Newtonian frequency, while the mean motion of the outer secondary is orders of magnitude larger than both of them (hence no mixed resonances are possible). Simultaneously, the assumptions of the averaging principle are fulfilled well. This guarantees that the evolution of the mean (secular) system closely follows the real configuration over a time-scale of order \(\sim 1/\epsilon\), where \(\epsilon\) is the small parameter of the perturbation.

Under the same assumptions, the tidal frequency is orders of magnitude smaller than the leading frequencies of the mutual (Newtonian) and relativistic corrections. This means that the tidal effect is indeed negligible in as far as the model parameters do not deviate strongly from the characteristic values as defined above.

### 3.1 A global, 2D representation of the phase space

Because the general planetary N-body problem is very complex, we restrict further analysis to its simplest non-trivial case of three bodies (‘non-trivial’ in the sense of its non-integrability). We shall consider configurations of a host star and two planets or C-type systems comprised of a binary and a more distant body (a planet).

We recall that the secular Hamiltonian \(\mathcal{H}_{\text{sec}}\) of the three-body problem does not depend on \(M_1, M_2\); therefore the conjugate actions \((L_1, L_2)\) are constant. The Hamiltonian \(\mathcal{H}\) written in the Laplace reference frame depends on \(\Delta \Omega = \Omega_2 - \Omega_1 \equiv \pm \pi\) only, not on \(\Omega_1\) and \(\Omega_2\) separately. Hence, the canonical transformation (e.g. Migchelenko et al. 2006)

\[
(\omega_1, G_1) \quad (\omega_2, G_2) \quad \Rightarrow \quad (\theta_1 = \frac{1}{2}(\Omega_1 + \Omega_2), J_1 = H_1 + H_2) \quad (\Omega_2, H_2) \quad (\theta_2 = \frac{1}{2}(\Omega_1 - \Omega_2), J_2 = H_1 - H_2)
\]

removes \(\Omega_1, \Omega_2\) from the secular Hamiltonian. After this transformation it does not depend on \(\theta_1\), therefore \(J_1 = |C| = C = \text{const}\), where \(C\) is the total angular momentum of the system. Moreover, \(\theta_2 = \pm \pi/2 = \text{const}\) (after the Jacobi reduction of nodes) and \(J_2\) may be expressed as a function of \(G_1, G_2\) and \(J_1\) in the following form:

\[
J_2 = (G_1^2 - G_2^2)/J_1.
\]

Therefore, for constant values of angular momentum \(J_1 = C\) and secular energy \(\mathcal{H}_{\text{sec}}\), the secular dynamics are reduced to a two-DOF Hamiltonian system. Instead of the total angular momentum \(C\), we shall use the so-called angular momentum deficit (AMD; Laskar 2000):

\[
\text{AMD} = L_1 + L_2 - C,
\]

or its normalized value of \(A = \text{AMD}/(L_1 + L_2), A \in [0, 1]\) (Migaszewski & Goździewski 2009b). Because \(L_1, L_2\) and \(C\) are integrals of the secular system, the relativistic correction (18) does not change \(A\) and the DOF number does not change. Because (\(\mathcal{H}_{\text{sec}}\)) depends on \(G_1\) only, it affects only the temporal evolution of \(\omega_1\). We note here that the perturbation induced by the quadrupole moment of the star, which was discussed (Migaszewski & Goździewski 2009a) in the coplanar case, also depends on \(H_1\) in the 3D problem, i.e. on the orbital inclination to the equatorial plane of the star. This perturbation introduces an additional frequency to \(\Omega_1\) and then \(\Delta \Omega\) is not constant any more. This means that we cannot perform the reduction of nodes and the secular problem would have three DOF. It also means that the Laplace reference frame, defined in terms of the total orbital angular momentum, does not possess any constant orientation in space. Being aware of this problem, we do not consider the dynamical flattening of the star and/or of the innermost planet. The two-DOF model is then less general but the dynamics are more tractable, thanks to the geometrical tools that can be applied to study this basic, low-dimensional problem.

If we fix the secular Hamiltonian in the form \(\mathcal{H}_{\text{sec}} \equiv \mathcal{H}_{\text{sec}}(G_1, G_2, \omega_1, \omega_2)\) then \(A\) may be considered as a free parameter of the secular model. Moreover, the phase space is four-dimensional and to represent the phase space of the system globally in terms of two-dimensional sections, which are easy to visualize, we follow a concept of the representative plane of initial conditions (Michtchenko & Malhotra 2004), hereafter the \(\Sigma\) plane. The \(\Sigma\) plane may be chosen in different ways, although all representations

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may be fixed and defined through the following conditions:

\[ \frac{\partial H_{\text{sec}}}{\partial \omega_1} = 0, \quad \frac{\partial H_{\text{sec}}}{\partial \omega_2} = 0. \]  

(20)

These conditions imply that all phase trajectories of the secular system cross the \( \Sigma \) plane (Michchenko et al. 2006; Libert & Henrard 2007; see also our explanation in Migaszewski & Goździewski 2009b). In accord with the symmetries in the secular 3D model, the solutions to these equations are the following four pairs of angles:

\[ (\omega_1, \omega_2) = \{(0, 0); (0, \pm \pi); (\pm \pi/2, \pm \pi/2); (\pm \pi/2, \mp \pi/2)\}. \]  

(21)

which also define four distinct quarters of the \( \Sigma \) plane, numbered with Roman numerals II, I, IV and III, respectively (see Michchenko et al. 2006 for details). Let us note that no other combinations of the angles are permitted by equations (20). This feature of the secular system flows from the symmetry of the secular Hamiltonian with respect to the apsidal lines of the mean orbits, and may also be justified by the explicit form of the equations of motion derived from the expansion of the perturbing Hamiltonian (see Michchenko et al. 2006; Migaszewski & Goździewski 2009b for details).

In this sense, the \( \Sigma \) plane may be thought as an analogue of the Poincaré cross-section. The conditions fixing the characteristic plane may also be rewritten as follows:

\[
\cos \omega_1 = \cos \omega_2 = 0 \cup \sin \omega_1 = \sin \omega_2 = 0.
\]

Furthermore, we shall use three basically equivalent geometric representations of the \( \Sigma \) plane, which cover certain combinations of the quarters (the solution pairs of the pericentre arguments):

(i) the \( \mathcal{P}_S \) plane is defined by \( \cos \omega_1 = \cos \omega_2 = 0 \) and

\[
\mathcal{P}_S = \{x = e_1 \sin \omega_1, y = e_2 \sin \omega_2, e_1 \in [0, 1], e_2 \in [0, 1]\};
\]

(ii) the \( \mathcal{P}_C \) plane is defined by \( \sin \omega_1 = \sin \omega_2 = 0 \) and

\[
\mathcal{P}_C = \{x = e_1 \cos \omega_1, y = e_2 \cos \omega_2, e_1 \in [0, 1], e_2 \in [0, 1]\};
\]

(iii) finally, two incarnations of the \( S \) plane are defined:

\[
S = \{x = e_1 \cos \Delta \sigma, y = e_2 \cos 2\omega_1, e_1 \in [0, 1], e_2 \in [0, 1]\},
\]

(24)

\[
S' = \{x = e_1 \cos 2\omega_1, y = e_2 \cos \Delta \sigma, e_1 \in [0, 1], e_2 \in [0, 1]\}.
\]

(25)

These were defined originally in Michchenko et al. (2006). It can be shown that the \( \mathcal{P}_S \) and \( \mathcal{P}_C \) planes carry the same information as the \( S \) plane. However, the \( S \) plane has a discontinuity along the \( x \)-axis and the former representations are sometimes more convenient for the analysis of solutions of a secular system.

3.2 A test of the analytic model

We left a test of the accuracy of the secular expansion until this point, because the introduced \( \Sigma \) planes are useful to illustrate the results of this test in a global manner. We select initial conditions in the \( S \) plane, and the secular energy computed with the help of the analytic expansion is compared with the results of numerical averaging developed in Gronchi & Milani (1998) and Michchenko & Malhotra (2004), which are exact up to the numerical quadrature error. We consider the non-relativistic case only, because the secular relativistic correction is exact (with the first non-zero post-Newtonian term included) and does not influence the precision of analytic formulae. Fig. 1 shows the levels of \( H_{\text{sec}} \), marked with solid curves in the \( S \) plane. Each panel is for a different order of the analytic approximation. The relative difference between values of the mean Hamiltonians derived through the analytic (‘A’) and numerical (‘N’) algorithms may be defined as follows:

\[
\Delta_i = \left| \frac{H_{\text{sec}}^A (l) - H_{\text{sec}}^N}{H_{\text{sec}}^N} \right|,
\]

(26)

where \( l \) is the order of the analytic expansion in \( \alpha \). The results of this comparison are illustrated in Fig. 1, which shows the levels of \( \Delta_i \) for a hierarchical system with \( \alpha = 0.04 \) and \( \mu \equiv m_1/m_2 = 5 \). The quadrupole-level model reproduces the secular Hamiltonian as an even function with respect to both \( x \equiv e_1 \sin \), \( \cos \) \( \omega_1 \) and \( y \equiv e_2 \sin \), \( \cos \) \( \omega_2 \). The higher order approximations of \( H_{\text{sec}} \) broke this symmetry. We have shown in Migaszewski & Goździewski (2009b) that the shape of \( H_{\text{sec}} \) depends significantly on \( \alpha \). This is more important in the spatial problem, because even for relatively small \( \alpha \) the quadrupole model distorts the structure of the phase space (see Section 4.1). An inspection of Fig. 1 reveals that the octupole model reconstructs the secular Hamiltonian and its shape in the \( S \) plane very well, because the largest deviations \( \Delta_3 \sim 10^{-3} \) appear only for \( e_1 \sim 1 \). In other parts of the representative plane, \( \Delta_3 \sim 10^{-5} \). The high-order expansions are obviously even more precise. Following estimates of the secular frequencies in the relevant parameter ranges (see Section 3), the averaging principle assures us that the orbital evolution of the secular system follows the ‘real’ orbits closely. Hence we may quite safely skip a comparison of the results from both approaches by direct numerical integrations.

---

Figure 1. The precision of the analytic theory in terms of \( \Delta_i \), equation (26), which is colour-coded in the \( S \) plane. Parameters of the planetary system are as follows: \( \alpha = 0.04, \mu = 5, A = 0.32 \). Panels from (a)–(d) are for expansions in \( \alpha \) of the second, third, sixth and tenth order, respectively.
The octupole model introduces the first-order perturbation to the integrable quadrupole model; hence, because it is very precise in the range of small $\alpha$, we focus further on this most simple, non-trivial case.

### 3.3 Equilibria in the secular 3D problem

In this work, we focus on the simplest class of solutions, i.e. the equilibria (or stationary solutions). These solutions imply the basic structure of the phase space. By determining their stability, we may derive the local structure of neighbouring phase trajectories through relatively simple analysis. The equilibria of a non-resonant system in terms of quadrupole approximations have been studied in the past (Krasinskii 1974; Lidov & Ziglin 1976). In our earlier work (Migaszewski & Goździewski 2009b), we classified families of equilibria emerging in the two-planet, non-planar problem in terms of the quasi-analytic averaging and basically exact secular model. We studied a few families of equilibria known in the literature, e.g. the zero-eccentricity solutions and the Lidov–Kozai resonance.

We also found new families of these equilibria, in particular the so-called chained orbits solution, which could scarcely be derived with the perturbative approach, although the results in Gronchi & Milani (1998) might be applied here. This work focused on the planetary regime of parameters $\mu$ and $\alpha$; the mass ratio $\mu$ was restricted to the range $[0.1, 2]$ and $\alpha \in [0.1, 0.667]$. Moreover, we explored the whole permitted range of $\Delta \in [0, 1]$. Yet we learned that the semi-analytic approach has serious disadvantages. All calculations, including integrations of the equations of motion and the stability analysis, must be performed with the help of numerical algorithms.

This may introduce large errors and hinders the analytic, qualitative analysis of the problem.

In this section, we extend the study of stationary solutions in Migaszewski & Goździewski (2009b) to a wider range of mass ratio, $\mu > 2$. Simultaneously, we consider smaller $\alpha < 0.1$. This makes it possible to apply the analytic model described and tested in Section 3. Because planets most likely emerge from remnants of a thin protoplanetary disc, we also restrict our attention to mutual inclinations up to $i_{\text{mut}} \sim 90^\circ$ (direct orbits). In that range, we may find families of equilibria related to the zero-eccentricity orbits and the LK resonance classified in Migaszewski & Goździewski (2009b) as solutions of family IVa, accompanied by families IIIa, IIIb and IVb+. Our analysis is also restricted to the initial conditions in quarters III and IV of the $S$ plane, in which these solutions may only ‘reside’. This region of the parameter space has been studied (Krasinskii 1974) with the quadrupole-level model, which is in some sense the next-to-trivial, non-interacting Keplerian approximation of three-body orbits. However, as we show below, this approximation may introduce artefacts due to the generally non-realistic symmetry of the secular Hamiltonian. Obviously, to avoid this problem, higher order expansions are required. We also attempt to extend the results of Ford et al. (2000), who applied the octupole-level theory to the analysis of hierarchical planetary systems with very small $\alpha \sim 0.01$.

To show the generic properties of the relevant families of equilibria, we begin with an example that is illustrated in Fig. 2. Panels in this figure show levels of $H_{\text{sec}}$ at the $P_2$ plane for $\alpha = 0.01$ and $\mu = 20$ and a few different values of $\Delta$. Fig. 2a was derived for $\Delta = 0.08$ and it reveals the zero-eccentricity equilibrium. For

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Levels of $H_{\text{sec}}$ at the $P_2$ plane calculated for $\alpha = 0.01$ and $\mu = 20$. Panels from (a)-(f) are for different values of $\Delta = [0.08, 0.12, 0.16, 0.19, 0.2, 0.25]$, respectively. These values of $\Delta$ imply mutual inclination at the origin $i_0 = [49^\circ, 61^\circ, 70^\circ, 77^\circ, 79^\circ, 89^\circ]$, respectively. The intensity of the shaded areas encodes the mutual inclination in prescribed ranges: a darker shade is for larger $i_{\text{mut}}$. The inclination ranges are $i_{\text{mut}} = [35^\circ, 45^\circ], [40^\circ, 55^\circ], [50^\circ, 65^\circ], [60^\circ, 75^\circ], [70^\circ, 85^\circ]$ in subsequent panels.
Figure 3. Families of stationary solutions (IVa, IIIa, IIIb, IVb+). Semi-major axis ratio $a = 0.01$, mass ratio $\mu = 20$. Dark dots are for stable equilibria, grey dots are for unstable equilibria. Crossed and dotted circles mark bifurcations. Small arrows show the direction of a particular stationary solution with increasing $A$. Grey (green in the online version) dots are for the positions of equilibria for particular values of $A = 0.12, 0.16, 0.19, 0.2, 0.25$ (the energy levels are shown in Fig. 2). Solutions are for the classic model, and were obtained with the help of octuple theory.

larger $A = 0.12$ (Fig. 2b) the figure-of-eight structure of Lidov–Kozai equilibrium appears and is labelled with IVa. We recall that the mutual inclination of circular orbits corresponding to LK bifurcation will be called the critical inclination, $i_{\text{crit}}$, although in general any such value of the mutual inclination that leads to a bifurcation of equilibria has the sense of being ‘critical’ (Krasinskii 1972, 1974). For larger value of $A = 0.16$ (Fig. 2c) the LKR ‘moves’ towards larger $e_1$ (simultaneously, $e_2 \sim 0$) and a new saddle point appears. We call this solution a member of family IIIa. For larger $A = 0.19$ (Fig. 2d) these structures still expand, and for $A = 0.2$ (Fig. 2e) two new equilibria emerge: one is associated with a quasi-elliptic point (family IIIb hereafter) and a saddle of family IVb+. Apparently, it emerges from a point near the IVa solution in the $P_5$ plane; nevertheless, it does not correspond to a bifurcation of this equilibrium (Migaszewski & Gożdziewski 2009b). Solution IVa moves towards larger $e_2$ (see Fig. 2f).

The parametric paths of these equilibria in terms of $A$ may be depicted in the $P_5$ plane (Fig. 3). Black and grey curves are for stable and unstable equilibria, respectively. The relevant families of equilibria are labelled with Roman numerals and Latin letters. The direction of ‘motion’ of particular solutions along the $A$-axis is marked with arrows. For reference, the positions of the equilibria for a few discrete values of $A = 0.12, 0.16, 0.19, 0.2, 0.25$ (corresponding to subsequent panels in Fig. 2) are marked with grey (green in the online version) dots and labelled. Following a particular evolution path of equilibrium IVa, we see that it appears for $A \sim 0.1$ through a bifurcation of the origin. When $A$ increases, this solution moves along $e_2 \sim 0$ towards larger $e_1$. For $A \sim 0.17$, it reaches the maximal $e_1 \sim 0.4$ and turns back towards smaller $e_1$ with a simultaneous increase of $e_2$. When $A$ increases even more, this solution reaches the border of convergence of the analytic expansion. We call this border the the anti-collision line (Migaszewski & Gożdziewski 2008).

The parametric evolution of equilibria in quadrant III is more complex. The first non-trivial stationary solution appears for $A \sim 0.15$, which is unstable, saddle point IIIa. It evolves along $e_1 \sim 0$ and then $e_2$ increases to large values. For $A \sim 0.195$, two new solutions IIIb and IVb+ emerge from the elliptic structure related to the LKR (see Fig. 2c). One of them is stable (solution IVb+) and the other is unstable (solution IIIb). They appear around $(e_1 \sim 0.45, e_2 \sim 0.15)$. When $A$ increases, the solution IVb+ moves towards $e_1 \rightarrow 1$ and $e_2 \rightarrow 0$. Simultaneously, equilibrium IIIb is shifted towards increasing $e_2$ and decreasing $e_1$. Then IIIa also increases $e_2$ and leaves the $e_1 = 0$ axis. Finally, for $A \sim 0.26$, equilibria IIIa and IIIb merge at one point $(e_1 \sim 0.13, e_2 \sim 0.8)$.

4 SECULAR CHAOS

Following Michtchenko et al. (2006), we showed in Migaszewski & Gożdziewski (2009b) that the averaged 3D model may involve strongly chaotic motions on secular time-scales. To study in detail how the model parameters influence the structure of the $S$ plane, and how it relates to long-term chaotic phenomena, we apply the fast indicator approach. Among many numerical tools of this kind, we choose the so-called coefficient of the diffusion of fundamental frequencies $\sigma$ introduced by Laskar (1990). To check whether a phase-space trajectory of a quasi-integrable Hamiltonian system is regular (quasi-periodic) or irregular (chaotic), one integrates the equations of motion over two subsequent intervals of time, e.g. $[0, T]$ and $[T, 2T]$. Next, we resolve the frequencies in the discrete orbital signal with the help of refined Fast Fourier Transform (FFT) analysis (Laskar 1990), obtaining two estimates of a given frequency, say $f_r$ and $f_{2r}$, over these two intervals of time. The coefficient of diffusion of the fundamental frequency is then defined through

$$\sigma = \frac{|f_{2r} - 1|}{f_r}.$$  

Clearly, if the signal does not change over time, $\sigma \sim 0$ and this means that the phase trajectory is quasi-periodic (stable). If $\sigma$ is significantly different from 0, the trajectory is chaotic and regarded as unstable. In our calculations, we used a variant of the frequency analysis developed by Šidlichovský & Nesvorný (1996), which is called the Frequency Modified Fourier Transform (FMFT). We also used publicly available code of the FMFT algorithm, kindly provided by David Nesvorný on his personal web-page.3

Because the secular evolution is associated with $\omega_i$ angles, we compute the $\sigma$ coefficient on the basis of complex time-series $\{G_i(t) \exp i \omega_i(t)\}, \ldots, l = 1, 2$, where $i$ is the imaginary unit. In this signal, the osculating eccentricity and pericentre argument for each planet represent rescaled canonical action-angle variables. Hence, by resolving its Fourier components we may determine the leading amplitudes (proper eccentricities) and the fundamental frequencies of pericentre angles.

Next, we performed massive integrations of the secular equations of motion. The initial conditions were selected on a grid of 200 $\times$ 100 data points of the $P_5$ plane. At each point of the dynamical map, we integrated the secular trajectory over $\sim 10^4$ secular periods with respect to the smaller frequency (typically, one of the fundamental frequencies is much larger than the other one). Having obtained the computed phase trajectory, we then find an estimate $\sigma$, as well as the maximal eccentricities attained by both orbits during the integration time (the so-called max $e$ indicator) and the amplitude of variation of the mutual inclination $\Delta i = (\max i_{\text{max}} - \min i_{\text{min}})$ attained during the integration time. These geometrical characteristics are very useful to understand the source of instabilities indicated and detected by the diffusion coefficient $\sigma$.

Figs 4–6 illustrate the results derived for the same system whose energy levels are shown in Fig. 2. Subsequent figures are for

3 http://www.boulder.swri.edu/~davidn/.
Figure 4. Dynamical maps for the classic (point-mass Newtonian) model, shown in the $P_S$ plane. Panels from (a)–(d) are for the coefficient of diffusion of fundamental frequencies ($\sigma$), maximal eccentricity of the inner and outer planet ($\max e_1, \max e_2$), respectively, and the amplitude of variation of the mutual inclination ($\Delta i$). Dots mark areas of librations of (b) $\omega_1$, (c) $\omega_2$ and (d) $\Delta \sigma$. Stationary solutions are marked with circles: dotted circles are for stable equilibria (corresponding to the maximum of the secular Hamiltonian), crossed circles are for unstable equilibria and empty circles are for linearly stable equilibria. These solutions are labelled in accord with Migaszewski & Goździewski (2009b). Parameters of the system are $\alpha=0.01$, $\mu=20$, $A=0.12$ (see also Fig. 2b).

Figure 5. Dynamical maps for the Newtonian, point-mass model in the $P_S$ plane, $A=0.16$. See the caption to Fig. 4 for more details, and also Fig. 2(c).

$A = 0.12, 0.16$ and 0.2, respectively (see Figs 2b, c and e for the respective levels of $\mathcal{H}_{\sec}$). (We note that for $A = 0.08$ the view of the phase space is basically very simple and the motions are regular everywhere.) The right-hand panel of each figure is for $\sigma$. The dynamical character of the phase-space trajectories is colour-coded: black means quasi-regular evolution of the secular system ($\sigma \sim 10^{-6} - 10^{-8}$) and light grey (yellow in the online version) is for strongly chaotic motions ($\sigma \geq 10^{-4}$). The $\sigma$ maps reveal that almost the whole phase space is filled up with regular motions, and chaos appears only in some small regions in the $P_S$ plane. For reference, the coordinates of equilibria IVa, IIIa, IIIb, IVb+ and 0 (the equilibrium at the origin) are marked with circles. The dotted
Figure 6. Dynamical maps for the Newtonian, point-mass model in the $P_S$ plane, $A = 0.20$. See the caption to Fig. 4 for more details, also Fig. 2(e).

4.1 A model explaining the secular chaos

The origin of chaotic secular dynamics may be explained by the presence of separatrices, which encompass different types of motions, librations and rotations of angles $\omega_1, \omega_2$ and $\Delta \sigma$. A classification of these libration modes in the secular problem of two-planet modes has been introduced in Michtchenko et al. (2006). The separatrices appear due to equilibria and periodic solutions in the integrable, or close to integrable, secular models, which might be understood as the quadrupole 3D model or and/or the coplanar octupole model shows that small perturbations may cause extended, geometric changes of the mean orbits. The amplification of $\epsilon_2$, with simultaneously almost constant relative inclination, $\Delta i \sim 0$ (see Figs 6c and d), seems to appear due to the bifurcation of equilibria IVb+ and IIIb for $A \sim 0.195$. 

The dynamical maps in Figs 4–6 reveal some zones in which the outer eccentricity is strongly amplified. This eccentricity is obviously constant in terms of the quadrupole model. This feature of the octupole model shows that small perturbations may cause extended, geometric changes of the mean orbits. The amplification of $\epsilon_2$, with simultaneously almost constant relative inclination, $\Delta i \sim 0$ (see Figs 6c and d), seems to appear due to the bifurcation of equilibria IVb+ and IIIb for $A \sim 0.195$. 

The origin of chaotic secular dynamics may be explained by the presence of separatrices, which encompass different types of motions, librations and rotations of angles $\omega_1, \omega_2$ and $\Delta \sigma$. A classification of these libration modes in the secular problem of two-planet modes has been introduced in Michtchenko et al. (2006). The separatrices appear due to equilibria and periodic solutions in the integrable, or close to integrable, secular models, which might be understood as the quadrupole 3D model or and/or the coplanar configuration.

Here, we found a simple explanation for the mechanism generating chaos, which is in fact the same as in a perturbed pendulum. To show this, let us recall that the expansion of $H_{sec}$ to the second order in $\sigma$ is the celebrated integrable quadrupole-level model. The energy levels of this model are illustrated in Fig. 7(a). Because $\epsilon_2$ is the integral of motion, the representative plane may be constructed.
similarly to the integrable coplanar problem, $S^e \equiv \{e_1 \cos 2\omega_1 \times e_2\}$. In the region marked in light grey (green in the online version), any constant level of $e_2$ crosses the energy curve at two turning points, limiting the range of variation of $e_1$. If the $e_2$ level is tangent to the given level of the energy, the dynamics must be then confined to fixed $e_1$; hence we obtain an equilibrium point (stable or unstable). A set of these equilibria for increasing, fixed values of $e_2$ is marked by two thick curves that meet around $\{e_1 = 0, e_2 = 0\}$. This is the bifurcation point at which two families of equilibria emerge – the stable branch of the LKR and the unstable origin $\{e_1 = 0, e_2 = 0\}$.

For a given value of constant $e_2$, we may then find the maximum range of $e_1$, which corresponds to librations of the canonical angle $\omega_1$, and this value determines the position of the separatrix; see Fig. 7(b) for the separatrix shown in the phase diagram in the $\{e_1 \cos 2\omega_1 \times e_1 \sin 2\omega_1\}$ plane, corresponding to the energy level marked by the thicker curve on the level diagram (Fig. 7a). Collecting such points along increasing $e_2$, we construct the red dashed curve in Fig. 7(a), showing the separatrix region globally in the whole $e_2$ range (see the shaded area in Fig. 7a). The dashed line that marks the unstable origin might also be understood as the other branch of the global separatrix, which, for any value of $e_2$, corresponds to the equilibrium point.

Now, considering the perturbed quadrupole model in terms of the octupole-level secular expansion, we may expect that chaotic motions will appear close to the separatrix, due to the perturbation. Indeed, this is illustrated in two panels of Fig. 8. The left panel is for the frequency diffusion $\sigma$ computed for initial conditions in the $S^e$ plane. This plane shows the energy levels of the octupole model, with stable (thick solid curve) and unstable (thick dashed line) equilibria in the quadrupole model overplotted. The grey dashed curve is for the global separatrix of the LK resonance in the quadrupole model, and the light grey region is for the librations of $\omega_1$ in the integrable case. Clearly, chaotic orbits are found along the branch of unstable equilibria as well as close to the red separatrix curve. This may be better seen in the Poincaré cross-section computed for the fixed energy level marked with a thicker curve that is shown.

Figure 7. Left panel: the levels of the secular energy (thin curves) depicted in the representative plane of the quadrupole problem, $S'' = \{e_1 \cos 2\omega_1 \times e_2\}$, $\alpha = 0.01$, $\mu = 20$, $A = 0.12$. Recall that this problem is integrable and $e_2 \equiv \text{constant}$ is a parameter. The black, thick curve marks LKR equilibrium for different energies and $e_2$ integral. The red, dotted curve marks the separatrix between librations and rotations of $\omega_1$ for fixed values of integrals. The shaded (green) zone indicates librations of $\omega_1$ around $\pi/2$. Right panel: the phase diagram in the $\{e_1 \cos 2\omega_1 \times e_1 \sin 2\omega_1\}$ plane and at a fixed energy level marked by the thicker curve in the left panel. The separatrix and the region of libration of $\omega_1$ are also marked with green.

Figure 8. Left panel: the secular energy levels (thin curves) shown in the $S'$ plane of the octupole model, $\alpha = 0.01$, $\mu = 20$, $A = 0.12$. Shaded areas are for initial conditions leading to chaotic evolution of the secular model. The red curve is for the separatrix of the LKR resonance in the quadrupole model; the black thick curve is for the equilibria in this model. Right panel: the Poincaré cross-section $\Delta \sigma = \pi$ in the $\{e_1 \cos 2\omega_1 \times e_1 \sin 2\omega_1\}$ plane. It corresponds to a fixed energy level marked with the blue curve in the left panel. The separatrix of the quadrupole problem and a region of librating $\omega_1$ are also marked.
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in the right-hand panel of Fig. 8(b). This panel in fact comprises two sections \(\Delta \sigma = \pi\) in the \(\{e_1 \cos 2\omega_1 \times e_1 \sin 2\omega_1\}\) plane, for \(\Delta \sigma > 0\) (the left half-plane) and \(\Delta \sigma < 0\) (the right half-plane). In place of the separatrix curves, wide chaotic regions appear.

A careful inspection of Fig. 8(a) reveals additional narrow banana-shaped chaotic structures. They are related to the separatrix of the librations of angle \(\Delta \sigma\) around \(\pi\) and the simultaneous circulation of angle \(\omega_1\), which is classified as mode 2 in Michtchenko et al. (2006, see their fig. 3). Further, the plots of fundamental frequencies computed along the \(x\)-axis of the \(S^2\) plane in Fig. 8(a), which are shown in Fig. 9, reveal that indeed in this region the fundamental frequencies decrease to 0, indicating the separatrix region.

This example shows clearly that an apparently very small perturbation (recall that \(\alpha = 0.01\)) to the integrable model introduces extended chaotic behaviour and a qualitative change of the secular dynamics of the system.

This interpretation is helpful to understand the source of chaotic motions shown in all dynamical maps (Figs 4–6): usually they appear close to the separatrices associated with unstable equilibria or resonances of the secular angles (unstable periodic orbits).

Fig. 10 shows the temporal evolution of the mean orbits with initial conditions close to the origin in Fig. 4 (see its caption for details). The eccentricity of the inner orbit is significantly perturbed, while the outer orbit is almost unchanged. The mutual inclination evolves in antiphase with \(e_1\), and \(\omega_1\) librates around \(\pi/2\). These are typical features of the Lidov–Kozai resonance; still, we recall that in this instance the smaller outer body forces the LK cycles on the orbit of much more (20 times) massive inner secondary.

**5 EQUILIBRIA IN THE CLASSIC OCTUPOLE MODEL**

The dynamical maps and their analysis show that the equilibria constrain the secular dynamics of the perturbed model. In terms of the Newtonian, point-mass formulation, the stationary solutions depend on parameters \(\alpha\) and \(\mu\). Limiting our survey to \(i_{\text{max}} < \pi/2\) (direct orbits), we perform a parametric survey of the equilibria in terms of the octupole expansion, in such a manner that a comparison with the results derived for the more general, relativistic model will be possible. Here, we consider a more extended range of mass ratio \(\mu\) than in Migaszewski & Goździewski (2009b), covering a transition from the planetary regime (small \(\mu\)) to the circumbinary case (large \(\mu\)). However, the assumption of small \(\alpha\) makes it possible to use the analytic formulation of the secular Hamiltonian, which is very precise in terms of the octupole-level approximation, instead of the numerical approach in Migaszewski & Goździewski (2009b).

The parameter dependence of the equilibria is illustrated in Fig. 11, which shows the \(P_2\) plane (note that due to the symmetry only the upper half-plane is illustrated, see also Fig. 3). We set \(\alpha = 0.04\), \(\mu \in [1.5, 3, 5, 10, 20, 25, 50]\), and then positions of the equilibria may be traced along increasing \(A\), which might be understood as the natural curve parameter.

In fact, instead of \(\mu\), we choose a new parameter \(\beta\) (see Krasinskii 1974; Migaszewski & Goździewski 2010):

\[
\beta(\mu, \alpha) \equiv L_1/L_2 \sim \mu \sqrt{\alpha}.
\]
which better reflects the dependence of the dynamics on both $\mu$ and $\alpha$ than on one of these parameters alone (we did a few numerical tests for different $\alpha$ that confirm this scaling very well). Actions $L_i$ reflect the angular momentum partition between both components. If $\beta \sim 1 (\mu \sim 5)$ both secondaries are dynamically equivalent, if $\beta > 1 (\mu > 5)$ then the inner body ‘dominates’ dynamically in the system and if $\beta < 1$ then the hierarchy is reversed. To illustrate the stability of the equilibria, the curves are marked with black dots for Lyapunov stable solutions and grey dots mean unstable solutions. The curves are labelled with both $\mu$ and $\beta$ parameters.

In the right-hand half-plane (quarter IV, $\omega_1 = \omega_2 = \pi/2$), for $i_{\text{mut}} < \pi/2$ only one solution appears, which is the LK resonance. For small $\mu$, it evolves with increasing $\mathcal{A}$ along the axis of $e_2 \sim 0$ between $e_1 = 0$ (when the first LK bifurcation appears, see Fig. 2b) and $e_1 \sim 1$. After the second bifurcation of the LKR (Fig. 2e), two solutions emerge. One of them is the LKR for $i_{\text{mut}} > \pi/2$ (see Migaszewski & Goździewski 2009b, this solution is not discussed here). The second new solution moves along the axis $e_1 \sim 1$ up to large $e_2$.

For $\mu \geq 2$, the LKR does not reach $e_1 = 1$ but ‘turns back’ with decreasing $e_1$ and increasing $e_2$. For larger mass ratio, the maximal $e_1$ is smaller. The families of LKR for $\mu = 1.5, 3$ correspond to systems with $\beta < 1$, hence the dynamical hierarchy is reversed and the eccentricity of the inner, more massive body is forced by the outer companion. The position of the LKR family moves to the region of smaller $e_1$, and for large enough $\mu$ this solution is confined to the $e_1 \sim 0$ axis and tends to large $e_2$. The direction of parametric evolution of this equilibrium, corresponding to increasing $\mathcal{A}$, is marked with an arrow. We see that for all $\mu$ this direction is the same.

The view of the left-hand half-plane is more complex. As shown in Migaszewski & Goździewski (2009b), the first solution appearing in quarter III of the $\mathcal{P}_\alpha$ plane is unstable equilibrium IIIa emerging due to the second bifurcation of the origin ($e_1 = e_2 = 0$) (marked with $I_1^*$ in our paper). Then, with increasing $\mathcal{A}$, this solution moves along $e_1 \sim 0$ towards large $e_2$. For some value of $\mathcal{A}$ (e.g. $\sim 0.195$ for $\alpha = 0.01, \mu = 20$; see Fig. 3) solutions IIIb and IVb+ appear at the same point of phase space (this is illustrated with a crossed circle in the left-hand half-plane of Fig. 3). Solution IVb+ then moves along $e_1 \to 1$ and $e_2 \to 0$. Simultaneously, solution IIIb evolves towards a point marked with a dotted circle. Solution IIIa also moves to that point and for some critical $\mathcal{A}$ both solutions merge and vanish. Fig. 11 reveals that the parametric paths of equilibria depend on parameter $\mu$ ($\alpha = 0.04$ was fixed in this test). The path of the LKR becomes closer to the $e_1 \sim 0$ axis for larger mass ratio. Also, families of stationary solutions that are present in quarter III become closer to the origin. The range of eccentricities corresponding to this equilibrium is very small for $\mu \geq 25$. We may also notice that the bifurcation points (crossed and dotted circles, respectively) tend to each other with increasing mass ratio, which is consistent with the transition between the planetary and the circumbinary regime. For $\mu \geq 25$, these two points are merged. In this particular case, the structure of equilibria in quarter III of the $\mathcal{P}_\alpha$ plane is more simple than in the general case because only one solution persists, a stable equilibrium corresponding to a nearly circular outer orbit.

For different $\alpha < 0.1$, the general, global view of the families of equilibria is very similar to the results presented in Fig. 11. In fact, as we noticed previously, the parametric evolution of the equilibrium is reflected by parameter $\beta(\mu, \alpha)$, and basically does not depend on the individual values of $\alpha$ and $\mu$.
The results presented in the previous section illustrate the already well-known features of the classic model (Michodzchenko & Malhotra 2004; Michodzchenko et al. 2006). In the approximation of small masses, the secular dynamics of the Newtonian two-planet hierarchical model depend on the semi-major axis ratio and planetary mass ratio and not on individual system parameters (\(a_1, a_2, m_1, m_2\)). In our works (Migaszewski & Goździewski 2009a, 2010), we have shown that this feature is not preserved in the more general model, including relativistic, rotational and tidal corrections to Newtonian point-mass interactions. In these settings, the secular dynamics depend on the individual semi-major axes as well as the individual planetary masses. Because the overall structure of phase space is characterized by the equilibria, we now attempt to show that deviations between these equilibria in the classic and relativistic models become more important when the system dimension scales down \((a_1, a_2 \text{ decrease when } \alpha \text{ is constant})\) and masses \(m_1, m_2\) are smaller when their ratio \(\mu = \frac{m_1}{m_2}\) is kept constant.

The differences between the two coplanar models manifest themselves through the shapes and localization of stationary solutions and depend on the individual masses and semi-major axes (Migaszewski & Goździewski 2009a). Now we can observe the same features in a spatial planetary system, corrected for the relativistic perturbation. Fig. 12 presents families of equilibria in the same manner as Fig. 11 (due to the symmetry, only the \(\gamma\)-positive half-plane of the \(P_3\) plane is presented). Families of stationary solutions in the classic model are drawn with dark grey and grey curves for stable and unstable solutions, respectively. Solutions in the relativistic model are plotted with black (stable equilibria) and grey (unstable equilibria) curves. In this experiment, we varied the individual masses of the secondary bodies, still keeping their ratio \(\mu = 10\) as a constant. The masses are varied between \((m_1 = 100 m_1, m_2 = 10 m_1)\) and \((m_1 = 3 m_1, m_2 = 0.3 m_1)\) and the primary mass is \(m_0 = 1 m_\odot\).

The results illustrated in Fig. 12 confirm that even for large secondary masses the parametric curves of stationary solutions in the realm of the classic model depend weakly on the individual masses. Yet similarly to the coplanar case, curves of equilibria calculated with the relativistic corrections differ significantly from the results obtained for the classic model. The deviations between both models are more significant for smaller masses of the secondary bodies. For instance, the family IVa moves to the range of much smaller \(e_1\). Solutions of this type exist even for very small \(m_1\) and \(m_2\), which is just not possible in the Newtonian model (Migaszewski & Goździewski 2009b). When the mutual perturbations between secondaries are small enough, the critical inclination in the relativistic model that leads to the LK bifurcation becomes larger than \(\pi/2\). Thus \(\xi_{\text{crit}}\) increases with decreasing masses.

In quadrant III of the \(P_3\) plane \((\omega_2 = -\omega_1 = \pm \pi/2)\), the structure of the equilibria is even more complex. For some critical values of masses \((m_2 \sim 2.07 m_1)\) the parametric paths divide into two parts. The top part, characterized by larger \(e_2\), comprises two unstable equilibria emerging from one bifurcation point, which then meet at another bifurcation point. The bottom part of the equilibria curve represents a saddle point that changes its stability from an unstable IIIa-type solution to the stable solution IVb+1. This branch is very similar to the equilibria in the classic system with large \(\mu\) (or large \(\beta > 5\)) observed already in Fig. 11; however this takes place for quite a different value of \(\beta = 2\).

The stationary solutions are determined by the shape of the secular Hamiltonian. Its levels are plotted in the \(P_3\) plane in Fig. 13. Each

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**Figure 12.** Families of stationary solutions presented in the \(P_3\) plane, calculated for \(\alpha = 0.04, \mu = 10, a_1 = 0.2 \text{ au}, a_2 = 5.0 \text{ au}, m_0 = 1 m_\odot\) and \(m_2\) varied in range \(10, 4, 2.7, 2.07, 1.85, 1.3, 1.0, 0.3 m_1\). Equilibria in the classic model are compared with equilibria in the relativistic model. The mass of the outer body \(m_2\) labels each particular curve. Black dots are for stable equilibria, grey dots are for unstable equilibria of the relativistic model. Equilibria of the classic model are plotted with blue and violet dots for stable and unstable solutions, respectively. Positions of equilibria were calculated with the help of octupole theory.
panel in this figure is calculated for the same parameters \( \mu, \alpha, A \) (but in this case particular values of masses and semi-major axes are varied). Let us recall that Fig. 13(a) shows the phase space calculated in the classic model, while subsequent Figs 13(b), (c) and (d) are for the relativistic model, and for different semi-major axes and masses, labelled in subsequent panels.

If the masses are large (Fig. 13b), we can see four elliptic points separated by four saddles (let us recall that the \( PS \) plane is redundant and the energy levels are reflected with respect to the origin, thus in fact we have only two unique elliptic points and only two saddles). The elliptic points may be identified with solutions IVa \((\omega_1 = \omega_2 = \pm \pi/2)\) and IIIb \((\omega_1 = -\omega_2 = \pm \pi/2)\). The saddles correspond to solutions IIIa \((e_1 \sim 0)\) and IVb+ \((e_2 \sim 0)\). When the masses decrease \((\mu\) is still kept constant), the structure surrounding solution IIIb becomes smaller and moves towards solution IIIa. Simultaneously, the saddle point IVb+ tends to the origin. For masses small enough (Fig. 12), solutions IIIa and IIIb merge and vanish while IVb+ ‘falls’ into the origin.

Figs 14–16 shows the dynamical maps for the relativistic model, and are constructed in the same manner as the maps in Figs 4–6. The figures correspond to the masses and \( A \) used to plot the energy levels in Figs 13(a), (b) and (c) respectively: Fig. 14 is for the classic model, Figs 15 and 16 are for the relativistic model with \( m_2 = 10m_1 \) and \( m_2 = 1.85m_1 \), respectively. The mass ratio is \( \mu = 10 \) in both instances. The order of panels and symbol-coding of equilibria, as well as the coding for libration zones of the angles \( \omega_1, \omega_2, \Delta \varpi \), are the same as in Figs 4–6; in particular, dotted, crossed and empty circles mark the Lyapunov stable, unstable and linearly stable equilibria, respectively. Clearly, the overall view of the phase space is different in all cases. The regions of chaotic motion (light grey areas in the \( \sigma \) maps) obtained for the classic and relativistic model are significantly different. Also the dependence on the masses of the secondaries in the relativistic models is evident. The structure of chaotic/regular secular evolution is reflected in the max \( e_{1,2} \) maps and through the librational regions of \( \omega_1 \) and \( \omega_2 \).

We note that in the case of regular solutions, \( \omega_1 \) librates around \( \pm \pi/2 \) when \( i_{\text{mut}} > i_{\text{crit}} \). In some part of this region \( \omega_2 \) also librates around \( \pm \pi/2 \).

Fig. 17 illustrates the temporal evolution for an initial condition written in the caption. The parameters of this model are the same as in Fig. 14. We choose the same initial eccentricities \( e_1 = 0.45 \) and \( e_2 = 0.001 \) and integrate the secular equations of motion of classic (grey curves) and relativistic (black curves) models. We note the qualitative differences between both configurations. The classic
model leads to much larger variations of the elements. In particular, the outer eccentricity $e_2$ is strongly amplified compared with the variations in the relativistic model. Still, this is not a rule. Inspecting the bottom row of Fig. 16, we can find regions in the $P_S$ plane in which, for the same initial condition, max $e_2$ becomes larger in the relativistic model than in the Newtonian model. Also the secularly chaotic configurations appear in quite different zones of the phase space in both models. Moreover, the relativistic corrections may transform regular evolution in the classic model into chaotic evolution in the relativistic systems, and vice versa. Remarkably, the configuration illustrated in Fig. 14 has very large masses $m_1 = 100 m_1$ and $m_2 = 10 m_1$. 

**Figure 14.** Dynamical maps for the classic, Newtonian model in the $P_S$ plane. The model parameters are $\alpha = 0.04$, $\mu = 10$, $A = 0.215$, $i_0 = 84^\circ$, $a_1 = 0.2$ au, $a_2 = 5.0$ au, $m_0 = 1 m_\odot$, $m_1 = 100 m_1$, $m_2 = 10 m_1$. See the text and caption to Fig. 4 for details.

**Figure 15.** The dynamical maps for the octupole model with relativistic corrections. Masses of the secondary bodies are $m_1 = 100 m_1$, $m_2 = 10 m_1$. See the text and caption to Fig. 4 for more details.
7 CONCLUSIONS

In this work, we attempt to show that the global features of the secular dynamics of a 3D, non-resonant planetary system depend qualitatively on the apparently subtle relativistic corrections to Newtonian gravity. A lesson that we learned when studying the coplanar case (Migaszewski & Goździewski 2009a) is that non-Newtonian point-mass interactions might be very important for the global dynamics because, contrary to the intuition one may have, the corrections to the Newtonian interactions might be not small, in comparison with the mutual point-to-point gravity. The numerical analysis of this multi-parameter problem is complex, hence the underlying idea of this paper lies in the construction of possibly a precise analytical model. Essentially simple averaging of the perturbing Hamiltonian in Migaszewski & Goździewski (2008) makes it possible to derive analytic secular theory up to the order of 10 in the ratio of semi-major axes $\alpha$. The accuracy of this model may be compared with the results of numerical averaging of the perturbation (Michchenko & Malhotra 2004). Moreover, for the class of hierarchical systems considered in our paper, the third-order model is already precise enough to find and investigate the qualitative features of the system. In this work we focus on three-body configurations, e.g. a star and two massive planets or a binary star and one planet. 'Fortunately', the second-order model is integrable, hence the octupole-level approximation might be considered as the first-order perturbation to this analytically soluble case. This makes it possible to understand the sources of instabilities appearing in the full (non-averaged) model.

The averaging over mean anomalies reduces the dynamics to a system having two DOF, which may then be investigated with the help of rich geometrical tools like the Poincaré cross-section and the representative planes of initial conditions introduced in Michchenko et al. (2006). Unfortunately, all additional corrections that increase the DOF number, such as for instance the rotational quadrupole moment of the star and/or of the planets, must be neglected here. This is the price that must be paid for a possibly global model of the dynamics. In the assumed range of $a_1$ and masses, the relativistic ‘corrections’ in fact compare with the Newtonian point-mass mutual interactions and are much larger than other perturbations, such as tidal and rotational distortions of the bodies. The analysis of the secular frequencies introduced by various corrections justifies the fact that the model only takes into account the PN perturbation to the potential of the star and the inner secondary.

Having the two-DOF model, we investigate the simplest class of solutions, which are equilibria. We focus on the Lidov–Kozai resonance (LKR) in systems characterized by a large range of the mass ratio $\mu$. This part extends the results derived for small $\mu$ in Migaszewski & Goździewski (2009b). We found that even a much smaller outer body (planetary mass $m_2$ with respect to substellar values of $m_1$) moving in a wide, highly inclined ($i_{\text{mut}} \sim 60^\circ$) orbit may significantly perturb the inner orbit. In turn, a restricted model of the circumbinary planet, in which we assume that the planet does not influence the binary, is not generally valid, even if the inner mass $m_1$ is 100 times larger than the outer body $m_2$.

We also studied the parametric structure of families of particular equilibria classified as IVa, IIIa, IIb and IVb+ in Migaszewski & Goździewski (2009b) for small $\alpha$. Thanks to this assumption, the analytic model makes it possible to investigate the transition between the planetary regime (small $\mu$) and the circumbinary regime ($\mu \sim 50, 100$). This study shows that solutions in the planetary problem may disappear for large mass ratio.

A particularly interesting feature of the octupole model is the appearance of secular chaos. We found that if $i_{\text{mut}}$ exceeds the critical inclination $i_{\text{crit}}(\mu, \alpha)$ the long-term evolution of the system may be strongly chaotic, leading to large amplification of the eccentricities. In the regular regions of the phase space, the mean angle $\omega_2$ librates around $\pm \pi/2$. In some parts of these regions the second secular angle $\omega_2$ also librates around $\pm \pi/2$. The initial conditions satisfying $i_{\text{mut}} > i_{\text{crit}}$ lead to a strong amplification of the inner eccentricity $e_1$. 

Figure 16. The dynamical maps for the octupole model with relativistic corrections. Masses of the secondary bodies are $m_1 = 18.5 \, m_1, m_2 = 1.85 \, m_1$. See the text and caption to Fig. 4 for more details.
Simultaneously, for the same values of angular momentum of the system, we may observe strong amplification of $e_2$ in some region of phase space, with almost constant relative inclination of the orbits. This behaviour may be attributed to unstable equilibrium IIIb emerging in the secular system. The amplification of $e_1$ happens not only for librations of $\omega_1$ around $\pm \pi/2$ (in the LK regime) but also and particularly when this angle behaves chaotically, varying in the whole range $[0, \pi]$. The dynamical maps reveal that the primary source of the chaotic motion is the unstable equilibria and unstable periodic orbits in the full system, following the appearance of separatrices in the integrable, quadrupole-level model.

Thanks to the simple analytic model, the influence of relativistic correction $\mathcal{H}_{\text{GR}}$ on the global secular dynamics of the problem can be clearly demonstrated. A simple proof of this influence is provided by analysis of the equilibria in the perturbed model. The differences between the Newtonian and relativistic models are larger when the mutual interactions between the secondaries are weaker, e.g. when companion masses are smaller. However, the dynamics are basically very simple up to the limit of the critical inclination, when the first bifurcation of the origin ($e_1 = e_2 = 0$) occurs, and this feature of the dynamics is preserved in both models.

We stress that although the analysis is done for specific, discrete mass ratios, the results are valid in as far the assumptions of the averaging theorem are fulfilled and the corrections besides general relativity are negligible. We demonstrated that, similarly to the coplanar problem, the global 3D dynamics of the classic, Newtonian model essentially depend only on the ratios of semi-major axes and masses of the secondaries. However, although we consider mostly circumbinary configurations, the results may be are easily extrapolated to the ‘typical’ planetary regime already investigated (Michtchenko et al. 2006). Moreover, when the PN corrections are added to the model the dynamics are much more complex. Still, the global picture of phase space is determined by the ratio of these corrections and the Newtonian mutual interactions. Hence, if the system scales down while this ratio is roughly preserved, the structure of phase space determined by stationary solutions of the secular system should not essentially change.

Our approach may be generalized for other perturbations, such as rotational and conservative tidal distortion of the bodies in the system. Unfortunately, in the most general case the dimension of the hierarchical system cannot generally be reduced to two DOF. Moreover, these perturbations lead to even more interesting and intriguing dynamics, which have been investigated in the coplanar and spatial case (Fabrycky & Tremaine 2007; Mardling 2007; Migaszewski & Goździewski 2009a; Ragozzine & Wolf 2009). We work on a global approach suitable for 3D systems, aiming to publish our results in future papers.

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REFERENCES

Adams F. C., Laughlin G., 2006, Int. J. Mod. Phys. D, 15, 2133
Arnold V. I., Kozlov V. V., Neishtadt A. I., 1993, Dynamical Systems III. Mathematical Aspects of Classical and Celestial Mechanics, Encyclopaedia of Mathematical Sciences. Springer Verlag, Berlin
Batygin K., Bodenheimer P., Laughlin G., 2009, ApJ, 704, L49
Brouwer D., Clemence G. M., 1961, Methods of Celestial Mechanics. Academic Press, New York
Brumberg V., 2007, Cel. Mech. Dyn. Astron., 99, 245
Eggenberger A., 2010, in Goździewski K., Niedzielski A., Schneider J., eds, EAS Publ. Ser. Vol. 42, Extrasolar Planets in Multi-Body Systems: Theory and Observations. EDP Sciences, Les Ulis, p. 19
Fabrycky D., Tremaine S., 2007, ApJ, 669, 1298
Farago F., Laskar J., 2010, MNRAS, 401, 1189
Ferraz-Mello S., ed., 2007, Astrophys. Space Sci. Libr. Vol. 345, Canonical Perturbation Theories – Degenerate Systems and Resonance. Springer, New York
Ferrer S., Osácar C., 1994, Cel. Mech. Dyn. Astron., 60, 187
Ford E. B., Kozinsky B., Rasio F. A., 2000, ApJ, 535, 385
Gronchi G. F., Milani A., 1998, Cel. Mech. Dyn. Astron., 71, 109
Harrington R. S., 1968, AJ, 73, 190
Kozai Y., 1962, AJ, 67, 579
Krasinskii G. A., 1972, Cel. Mech., 6, 60
Krasinskii G. A., 1974, in Wyller A. A., ed., Proc. IAU Symp. Vol. 62, The Stability of the Solar System and of Small Stellar Systems. Reidel, Dordrecht, p. 95
Laskar J., 1990, Icarus, 88, 266
Laskar J., 2000, Phys. Rev. Lett., 84, 3240
Lee M. H., Peale S. J., 2003, ApJ, 592, 1201
Libert A.-S., Henrard J., 2007, Icarus, 191, 469
Lidov M. L., 1962, Planet. Space Sci., 9, 719
Lidov M. L., Ziglin S. L., 1976, Cel. Mech., 13, 471
Mardling R. A., 2007, MNRAS, 382, 1768
Mardling R. A., 2010, MNRAS, 407, 1048

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Michtchenko T. A., Malhotra R., 2004, Icarus, 168, 237
Michtchenko T. A., Ferraz-Mello S., Beaugé C., 2006, Icarus, 181, 555
Migaszewski C., Goździewski K., 2008, MNRAS, 388, 789
Migaszewski C., Goździewski K., 2009a, MNRAS, 392, 2
Migaszewski C., Goździewski K., 2009b, MNRAS, 395, 1777
Migaszewski C., Goździewski K., 2010, in Goździewski K., Niedzielski A., Schneider J., eds, EAS Publ. Ser. Vol. 42, Extrasolar Planets in Multi-Body Systems: Theory and Observations. EDP Sciences, Les Ulis, p. 385
Murray C. D., Dermott S. F., 2000, Solar System Dynamics. Cambridge Univ. Press, Cambridge

Rabl G., Dvorak R., 1988, A&A, 191, 385
Ragozzine D., Wolf A. S., 2009, ApJ, 698, 1778
Richardson D. L., Kelly T. J., 1988, Cel. Mech., 43, 193
Takeda G., Kita R., Rasio F. A., 2009, in Pont F., Sasselov D., Holman M., eds, Proc. IAU Symp. Vol. 253, Transiting Planets. Cambridge Univ. Press, Cambridge, p. 181
Tamuz O. et al., 2008, A&A, 480, L33
Šidlichovský M., Nesvorný D., 1996, Cel. Mech. Dyn. Astron., 65, 137

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