Formulation of radiation temperature gradient equations with non-homogeneous density in stars

A Yasrina* and L L Firdausiyah

1Department of Physics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Malang, Jl. Semarang 5, Lowokwaru, Malang 65145, Indonesia.

*Corresponding author : atsnaita.yasrina.fmipa@um.ac.id

Abstract. The interior of a star consists of the convection region and radiation. The energy transfer process in the convection region and radiation is different which indicates the energy transfer process in the star. The energy transfer is affected by the density of the star. The radiation energy transfer equation with non-homogeneous density has been formulated. The density of a non-homogeneous star is the density that only depends on the distance. It has been obtained the formulation of an energy transfer equation in radiation with a non-homogeneous density in the form of a star temperature gradient. This equation is obtained by formulating the relationship between the specific intensity of the beam, the flux, and the luminosity of the star. The density of a non-homogeneous star affects the luminosity and flux of the star. Meanwhile, the star temperature gradient equation does not depend on the density of non-homogeneous stars.

1. Introduction

Research on the generation of energy in stars has been started since 1920. The energy source in stars comes from hydrogen fusion. The theory of the generation of energy in stars developed in 1938. Stars can generate energy through the carbon, nitrogen, and oxygen (CNO) cycle. The CNO cycle occurs when the atomic nucleus has a large kinetic energy to oppose the Coulomb force between atomic nuclei [1]. Stellar evolution is particularly affected by the generation of energy in the star's core. Energy delivery in the interior of a star has a profound influence on stellar evolution [2-3]. Energy generation in stars is influenced by the structure of the star [4]. The variable that affects is the mass of the star. A star with a small mass will produce a nonmetal atomic nucleus [5]. In the diagram Hertzprung Russel even depicts the evolution of stars which is influenced by star structure [6]. A star with a smaller mass will have mechanical and thermal differences with a star with a larger mass[7].

The generation of energy in a star can go through two events, namely convection and radiation. For example, energy delivery by convection occurs in the shroud of the Sun [8]. However, energy delivery to the Sun is dominated by radiation[9]. Between convection and radiation which is more dominant, one of them is influenced by the mass of the star. In some studies, such as in stellar evolution, the stellar structure and energy delivery in the star are assumed to be a constant variable star density. Meanwhile, star density can affect star temperature, star luminosity and star flux[10-11]. Star temperature, stellar luminosity and stellar flux are the quantities that can describe the delivery of energy in a star [12]. Therefore, it is necessary to formulate a mathematical energy delivery for a star with non-constant or non-homogeneous density and to compare it with the constant or homogeneous density that has been obtained in existing research.
2. Methods
In the formulation of the radiation energy delivery equation in a star with a non-homogeneous density, it is obtained from the general equation for the delivery of energy by radiation in a star. The initial step is to define the temperature gradient equation. Then proceed by defining the specific intensity of the emission \((I(r, \theta))\), the decrease in intensity equation \((I(r, \theta))\) with respect to distance \((r) (\partial I/\partial r)\), energy flux equation \((F(r))\), and the energy flux change equation \((F'(r))\) with respect to distance \((r) (\partial F(r)/\partial r)\).

These steps are used to describe the derivative equation of pressure \((P)\) against \((r) (\partial P/\partial r)\). The energy density equation for the radial function of the black body \((u(r))\), Luminosity equation \(L(r) = F(r)/4\pi R^2\), and the derivative equation of temperature \((T)\) against \((r) (\partial T/\partial r)\). Then from the equation that has been obtained, it is used to describe the temperature gradient equation \((\Delta T)\) related to luminosity \((L(r))\) and energy flux \((F)\). Furthermore, the final results of the equations for the delivery of radiation energy in stars in general and the results of equations for the delivery of radiation energy in stars with non-homogeneous densities are obtained. The comparison between the two results of the equation is calculated to find the comparison results for the energy flux, luminosity, and its temperature gradient. All of these steps are applied to star density \(\rho = 3M/(\pi R^3)\left[1 - \left(\frac{r}{R}\right)^2\right]\). The results obtained are compared to determine the energy flux, luminosity, and the temperature gradient, whether it depends on the density.

3. Results and discussion
It is assumed that a star with mass \(M\) and radius \(R\) has a star density that continues to increase from the star’s core to the surface of the star as a radial function as in equation (1)\([13]\)

\[\rho = \frac{3M}{\pi R^3} \left[1 - \left(\frac{r}{R}\right)^2\right].\]  

In terms of points \(r\) from the star's core. The amount of energy flowing in a cylindrical element \(dl\) which forms an angle \(\theta\) to the radial vector as in Figure (1). The difference in radiation intensity is equal to \([I(r + dr, \theta + d\theta) - I(r, \theta)] d\Omega dS\). The magnitude \(I(r + dr, \theta + d\theta) - I(r, \theta)\) is

\[I(r + dr, \theta + d\theta) - I(r, \theta) = e \rho \frac{dI}{4\pi} - \kappa \rho I(r, \theta) dl,\]  

Equation (3.2) can be written as

\[\frac{\partial I}{\partial r} \cos \theta - \frac{\partial I}{\partial \theta} \sin \theta - e \rho \frac{dI}{4\pi} + \kappa \rho I = 0.\]  

The energy flux of a star is

\[F = L/(4\pi r^2),\]  

The energy flux of a star in terms of intensity is

\[F(r) = \int_{\Omega} I(r, \theta) \cos \theta d\Omega.\]  

Equation (5) is integrated in terms of the space angle obtained

\[\frac{\partial}{\partial r} \int I \cos \theta d\Omega - \frac{1}{r} \int \frac{\partial I}{\partial \theta} \sin \theta d\Omega - e \rho \frac{1}{4\pi} \int d\Omega + \kappa \rho \int I d\Omega = 0.\]  

Meanwhile for \(\int \frac{\partial I}{\partial \theta} \sin \theta d\Omega\) is

\[\int \frac{\partial I}{\partial \theta} \sin \theta d\Omega = \int \frac{\partial I}{\partial \theta} \sin \theta \sin \theta d\theta d\varphi = \int \sin^2 \theta d\theta d\varphi = -2F,\]  

with

\[\int I d\Omega = c \ u(r).\]  

Equation (7) substituted for equation (6) is obtained

\[\frac{\partial F}{\partial r} + \frac{2F}{r} - e \rho + \kappa \rho c u(r) = 0.\]  

The flux of radiant energy for a given frequency is

\[F_\nu = -\frac{4\pi}{3\kappa_\nu \rho} \frac{\partial v_\nu(T)}{\partial T} \frac{dT}{dr},\]  

and the energy flux for all frequencies is
Figure 1. The basic geometric quantity used to derive a gradient in radiation temperature [10]

\[ F = -\frac{c}{\kappa R \rho} \frac{dP_r}{dr} \]  

(11)

Equation (11) can be written as

\[ \frac{dP_r}{dr} + \frac{\kappa R \rho}{c} F = 0. \]  

(12)

Equation (12) is the diffusion equation of radiation. Equation (12) is also obtained

\[ F = -\frac{4a c}{3 \kappa R \rho} T^3 \frac{\partial T}{\partial r}, \]  

(13)

with

\[ \frac{\partial T}{\partial r} = \frac{T}{P} \frac{\partial P}{\partial r} \nabla_{\text{rad}}. \]  

(14)

The magnitude \( \nabla_{\text{rad}} \) is an expression of the temperature gradient in the star’s interior when energy is transmitted by radiation. Meanwhile the luminosity of a star is

\[ L = 4\pi r^2 F = -\frac{16\pi a c}{3 \kappa R \rho} r^2 T^3 \frac{\partial T}{\partial r}. \]  

(15)

or

\[ F = -\frac{4a c}{3 \kappa R \rho} T^3 \left( \frac{T}{P} - \frac{G \rho}{r^2} \right) \nabla_{\text{rad}}. \]  

(16)

with the temperature gradient of the star is [10]

\[ \nabla_{\text{rad}} = \frac{3 \kappa R}{16 \pi a c G m} L. \]  

(17)

Equation (15) substituted into equation (12) is obtained

\[ \frac{dP_r}{dr} = -\frac{\kappa R \rho}{c} F = -\frac{\kappa R \rho}{c} \frac{L}{4\pi r^2}. \]  

(18)

Equation (12) has a relationship with time because according to equation (15) that luminosity is the energy emitted by a star per unit time.

The difference in radiation intensity for the non-homogeneous density is

\[ I(r + dr, \theta + d\theta) - I(r, \theta) = \frac{3M}{\pi r^2} \left( 1 - \left( \frac{r}{R} \right)^2 \right) \left( e/4\pi - \kappa I(r, \theta) \right) d\theta. \]  

(19)

The differential equation for radiation intensity in a nonhomogeneous star.

\[ \frac{\partial I}{\partial r} \cos \theta - \frac{\partial I}{\partial \theta} \sin \theta - \frac{3M}{\pi r^2} + \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \left( e/4\pi + \kappa I(r, \theta) \right) = 0. \]  

(20)

There is a relationship between intensity and radiation energy flux. The equation showing the relationship between this flux of radiant energy and the intensity for a star with a non-homogeneous density is

\[ \frac{\partial I}{\partial r} \cos \theta d\Omega - \frac{\partial I}{\partial \theta} \frac{\sin \theta}{r} d\Omega - \frac{3M}{\pi r^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \left( e/4\pi - \kappa I(r, \theta) \right) d\Omega = 0, \]  

(21)
\[ \int \frac{\partial l}{\partial r} \cos \theta \, d\Omega - \int \frac{\partial l}{\partial \theta} \sin \theta \, d\Omega - \int \frac{3M}{\pi R^2} \left[ 1 - \frac{\left( \frac{r}{R} \right)^2}{4\pi} \right] d\Omega = 0. \] (22)

Equation (22) is integrated with \( \rho(r) \), then the \( \rho(r) \) component can come out of the integral, and we get

\[ \frac{\partial}{\partial r} \int l \cos \theta \, d\Omega - \frac{1}{r} \int \frac{\partial l}{\partial \theta} \sin \theta \, d\Omega - e \left( \frac{3M}{\pi R^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \right) \int d\Omega = 0, \] (23)

Equation (23) can be written as

\[ \frac{\partial F}{\partial r} - \frac{1}{r^2} \left( -2F \right) - e \left( \frac{3M}{\pi R^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \right) \int d\Omega = 0, \] (24)

or

\[ \frac{\partial F}{\partial r} + 2F - e\rho + \kappa \rho \int d\Omega = 0. \] (25)

The radiation flux is not affected by the calculation results, because \( I_\nu(r, \theta) \) depends on \( r \) and \( \theta \) components, while \( F \) is integrated for all frequencies neither \( r \) nor \( \theta \). The magnitude \( r \) is the distance from the center of the star. Therefore equation (25) has the same form as equation (9). Equations (9) and (25) depend on the intensity \( I_\nu(r, \theta) \). The radiation energy flux \( F \) is integrated for all frequencies. Therefore, the density value is considered a constant quantity in the integral process. In other words, a constant density (homogeneous) or dependent on \( r \) (non-homogeneous) does not affect the result of the integral flux of radiation energy concerning frequency.

The energy flux is the energy delivered by the star every one-unit second and each one-unit area. Radiant energy is carried in the interior of the star from the center of the star. The total radiant energy flux for all frequencies in the non-homogeneous density star is

\[ F = -\frac{4\pi^2 R^2}{9M} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} \frac{1}{\kappa} \frac{1}{\kappa} \frac{\partial B_\nu(T)}{\partial T} \int d\Omega, \] (26)

or

\[ F = -\frac{\pi c R^2}{3\kappa R} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} \int d\Omega. \] (27)

Equation (27) can be written as

\[ \frac{dF}{dr} + \frac{3\kappa R}{\pi c R^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] F = 0. \] (28)

Equation (28) is called the radiation diffusion equation for a non-homogeneous star. The total energy flux for a non-homogeneous density can also be written as

\[ F = -\frac{4ac\pi^2 r^2}{9M\kappa R} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} T^3 \frac{\partial T}{\partial r}. \] (29)

The luminosity of a star with a radiant energy flux as in equation (29) is obtained

\[ L = -\frac{16ac\pi^2 r^2}{9M\kappa R} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} T^3 \frac{\partial T}{\partial r}. \] (30)

or

\[ L = -\frac{16ac\pi^2 r^2}{9M\kappa R} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} T^3 \left( \frac{T}{P} \right) \left( \frac{Gm\rho}{r^2} \right) \nabla \text{rad}. \] (31)

Furthermore, the energy flux equation in equation (3.27) becomes

\[ F = -\frac{4ac}{3\kappa R \rho} T^3 \left( \frac{Gm\rho}{r^2} \right) \nabla \text{rad}. \] (32)

or

\[ \nabla \text{rad} = \frac{3\kappa R}{4acGm T^4} r^2 F. \] (33)

Since the flux of radiant energy is \( \frac{L}{4\pi r^2} \), equation (3.29) becomes

\[ \frac{L}{4\pi r^2} \text{rad} = \frac{3\kappa R}{16aGcGm T^4} L. \] (34)

The temperature gradient inside a star with a non-homogeneous density does not depend on the density inside the star. The temperature gradient inside an inhomogeneous star is the same as the temperature
gradient inside a star with a homogeneous type of mass. The temperature gradient inside the star has the same form as the state equation (EOS) for each star density $\rho = \rho (P, T, \mu)$ as energy is transported out through radiation [10]. However, the radiation energy flux and luminosity of a star with a non-homogeneous density depend on the density of the star.

4. Conclusion
The temperature gradient ($\nabla_{rad}$) inside a star with a non-homogeneous density equal to a homogeneous density is independent of the density inside the star. However, radiation energy flux and luminosity depending on the star density. The equations obtained are suitable for the radiant temperature, radiant energy flux, and luminosity for the equation of state (EOS), which provides one of the thermodynamic quantities in terms of the others for instance, $\rho = \rho (P, T, \mu)$.

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