N=2 Supersymmetric Kinks and real algebraic curves

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Abstract

The kinks of the (1+1)-dimensional Wess-Zumino model with polynomic superpotential are investigated and shown to be related to real algebraic curves.

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The dimensional reduction of the (3+1)-dimensional Wess-Zumino model, produces an interesting (1+1)-dimensional Bose-Fermi system; this field theory enjoys N=2 extended supersymmetry provided that the interactions are introduced via a real harmonic superpotential, see [1]. In a recent paper [2] Gibbons and Townsend have shown the existence of domain-wall intersections in the (3+1)D WZ model, the authors relying on the supersymmetry algebra of the (2+1)D dimensional reduction of the system. Although the domain-wall junctions are two-dimensional structures, their properties are reminiscent of the one-dimensional kinks from which they are made. In this letter we shall thus describe the kinks of the underlying (1+1)-dimensional system.

The basic fields of the theory are:

• Two real bosonic fields, \( \phi^a(x^\mu), a = 1, 2 \) that can be assembled in the complex field: \( \phi(x^\mu) = \phi^1(x^\mu) + i\phi^2(x^\mu) \in \text{Maps}(\mathbb{R}^{1,1}, \mathbb{C}). \) \( x^\mu = (x^0, x^1) \) are local coordinates in the \( \mathbb{R}^{1,1} \) Minkowski space, where we choose the metric \( g^{\mu\nu}; g^{00} = -g^{11} = 1, g^{12} = g^{21} = 0. \)

• Two Majorana spinor fields \( \psi^a(x^\mu), a = 1, 2. \) We work in a Majorana representation of the Clifford algebra \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}, \gamma^0 = \sigma^2, \gamma^1 = i\sigma^1, \gamma^5 = \gamma^0\gamma^1 = \sigma^3 \) where \( \sigma^1, \sigma^2, \sigma^3 \) are the Pauli matrices, such that \( \psi^a\psi^a = \psi^a. \) We also define the adjoint spinor as \( \bar{\psi}(x^\mu) = \psi^\dagger(x^\mu)\gamma^0 \) and consider Majorana-Weyl spinors: \( \psi_\pm^a(x^\mu) = \frac{1 \pm \gamma^5}{2}\psi^a(x^\mu) \) with only one non-zero component.

Interactions are introduced through the holomorphic superpotential: \( W(\phi) = \frac{i}{2} (W^1(\phi^1, \phi^2) + iW^2(\phi^1, \phi^2)). \) One could in principle start from the supercharges:

\[
\hat{Q}_{\pm}^{BC} = \int dx^1 \sum_{a,b} [f^B]^{ab} \left[ (\partial_0 \phi^a \mp \partial_1 \phi^a)\psi^{b\pm} \right] \pm \sum_c [f^C]^{bc} \frac{\partial W^C}{\partial \phi^c} \psi^{ac}.
\]

where \( W^B, B = 1, 2, \) are respectively the real part if \( B = 1 \) and the imaginary part if \( B = 2 \) of \( W(\phi) \) and \( [f^B] \) is either the identity or the complex structure endomorphism in \( \mathbb{R}^2 \; [3]: \)

\[
[f^{B=1}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [f^{B=2}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Nevertheless, the Cauchy-Riemann equations:

\[ \frac{\partial W^1}{\partial \phi^1} = \frac{\partial W^2}{\partial \phi^2} \quad \frac{\partial W^1}{\partial \phi^2} = -\frac{\partial W^2}{\partial \phi^1}, \]  

tell us that the theory is fully described by choosing either \( W^1 \) or \( W^2 \). We thus set \( W^C = W^1 \) and find the basic SUSY charges to be \( \hat{Q}^B_\pm = Q^B_\pm \):

\[ Q^B_\pm = \int dx^1 \sum_{a,b} [f^B]^a_b \left[ (\partial_0 \phi^a \mp \partial_1 \phi^a) \psi^b_\pm \mp \frac{\partial W^1}{\partial \phi^b} \psi^a_\pm \right] \]  

From the canonical quantization rules

\[ [\phi^a(x_1), \phi^b(y_1)] = i\delta^{ab} \delta(x_1 - y_1) = \{\psi^a_\pm(x_1), i\psi^b_\pm(y_1)\} \]

one checks that the \( N = 2 \) extended supersymmetric algebra

\[ \{Q^B_\pm, Q^C_\pm\} = 2\delta^{BC} P_\pm \quad \{Q^B_+, Q^C_-\} = -(1)^B (\delta^{BC} 2T + \epsilon^{BC} 2\tilde{T}) \]

is closed by the four generators \( Q^B_{\pm}, \) defined in [3]. Here

\[ P_\pm = \frac{1}{2} \int dx^1 \sum_a \left[ (\partial_0 \phi^a \mp \partial_1 \phi^a) (\partial_0 \phi^a \pm \partial_1 \phi^a) \mp 2i\psi^a_\mp \partial_1 \psi^a_\mp \right] \]

\[ + \frac{1}{2} \int dx^1 \sum_a \left[ \frac{\partial W^1}{\partial \phi^a} \frac{\partial W^1}{\partial \phi^a} - 2i \sum_b \frac{\partial^2 W^1}{\partial \phi^a \partial \phi^b} \psi^a_\mp \psi^b_\mp \right] \]

are the light-cone momenta and

\[ T = \int dx^1 \left[ \frac{\partial W^1}{\partial \phi^1} \frac{\partial W^1}{\partial x^1} + \frac{\partial W^1}{\partial \phi^2} \frac{\partial W^1}{\partial x^1} \right] = \int dW^1 = W^1(\infty) - W^1(-\infty) \]

\[ \tilde{T} = \int dx^1 \left[ \frac{\partial W^1}{\partial \phi^2} \frac{\partial W^1}{\partial x^1} - \frac{\partial W^1}{\partial \phi^1} \frac{\partial W^1}{\partial x^1} \right] = \int *dW^1 = W^2(\infty) - W^2(-\infty) \]

the central extensions.

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From the SUSY algebra one deduces,

\[ 2P_0 = 2|T| + (Q^B_\pm \mp (-1)^B Q^B)^2 = 2|\tilde{T}| + (Q^B_{\pm} \mp (-1)^B \epsilon^{BC} Q^C_{\pm})^2, \]

see [4]. We thus define the charge operators on zero momentum states:

\[ \hat{Q}^1_\pm = Q^1_\pm + Q^1 = -\int dx^1 \sum_a \left( \partial_1 \phi^a \mp \frac{\partial W^1}{\partial \phi^a} \right) \left( \psi^a_\pm \pm \psi^a \right) \]

\[ \hat{Q}^2_\pm = Q^2_\pm + Q^2 = \mp \int dx^1 \sum_a \left( \partial_1 \phi^a \pm \sum_b [f^2]^a_b \frac{\partial W^1}{\partial \phi^b} \right) \left( \psi^a_\pm \mp \sum_c [f^2]^{ac}_b \psi^c \right) \]

Spatially extended coherent states built from the solutions of any of the two systems of first order equations, [4]:

\[ \frac{d\phi^1}{dx^1} = \pm \frac{\partial W^1}{\partial \phi^1} \quad \frac{d\phi^2}{dx^1} = \mp \frac{\partial W^1}{\partial \phi^2} \]  

\[ \frac{d\phi^1}{dx^1} = \pm \frac{\partial W^1}{\partial \phi^2} \quad \frac{d\phi^2}{dx^1} = \mp \frac{\partial W^1}{\partial \phi^1} \]
that support the solutions of (5) are orthogonal to the curves related to the solutions of (6).

We focus on the case in which the potential is:

\[ W(\phi) = 1 \]

The flow in \( \mathbb{R}^2 \) of the solutions of (5) or (6) and integrates.

\[ \frac{d\phi^2}{d\phi^1} = \frac{\partial W^1}{\partial \phi^1} \left( \frac{\partial W^1}{\partial \phi^2} \right)^{-1} = \frac{\partial W^1}{\partial \phi^2} d\phi^1 - \frac{\partial W^1}{\partial \phi^1} d\phi^2 = dW^2 = 0 \]

If \( W(\phi) \) is polynomic in \( \phi \), the solutions of (5) live on the real algebraic curves determined by the equation:

\[ W^2(\phi^1, \phi^2) = \gamma_\perp \]

where \( \gamma_\perp \) is a real constant. The solution flow of (6) in \( \mathbb{C} \),

\[ \frac{d\phi^2}{d\phi^1} = -\frac{\partial W^1}{\partial \phi^1} \left( \frac{\partial W^1}{\partial \phi^2} \right)^{-1} = -\frac{\partial W^1}{\partial \phi^2} d\phi^1 + \frac{\partial W^1}{\partial \phi^1} d\phi^2 = dW^1 = 0 \]

runs on the real algebraic curves:

\[ W^1(\phi^1, \phi^2) = \gamma \]

where \( \gamma \) is another real constant. There are two observations: (I) Solutions of system (5) live on curves for which \( W^2 = \text{constant} \) and solutions of (5) have support on curves for which \( W^1 = \text{constant} \). (II) The curves that support the solutions of (5) are orthogonal to the curves related to the solutions of (6).

Assume that \( W(\phi) \) has a discrete set of extrema, forming the vacuum orbit of the system: \( \frac{\partial W}{\partial \phi} |_{\psi(i)} = 0 \), \( i = 1, 2, \ldots, n \). Kinks are solutions of (5) and/or (6) such that they tend to \( \psi(i) \) when \( x_1 \) reaches \( \pm \infty \). \( \psi^{(i+)} \) and \( \psi^{(i-)} \) thus belong either to curves (5) or (6), which fixes the values of \( \gamma \) or \( \gamma_\perp \) for which the real algebraic curves support kinks. In Reference (3), a general proof based on singularity theory of the existence of these soliton solutions, that counts its number, is achieved. The energies of the states grown from kinks are\( \gamma_\perp \) that supports the solutions (5).

Therefore, the fermionic charges \( \hat{Q}_1^\perp \) and \( \hat{Q}_2^\perp \) are annihilated on coherent states \( |K_1^\perp \rangle \) and \( |K_2^\perp \rangle \) that correspond to the tensor product of the quantum antikink/kink, living respectively on curves \( W^2 = \text{constant} \) and \( W^1 = \text{constant} \), with its supersymmetric partners (the translational mode times a constant spinor). We find:

\[ \hat{Q}_1^\perp |K_1^\perp \rangle = \int dx_1 \sum_a \left[ \partial_1 \phi_{K_1^\perp}^a + \partial W^1 \right]_{\phi_{K_1^\perp}^a} \left[ \frac{1}{2} + 1 \right] |K_1^\perp \rangle = 0 \]

\[ \hat{Q}_2^\perp |K_2^\perp \rangle = \int dx_1 \sum_a \left[ \partial_1 \phi_{K_2^\perp}^a + \sum_b e^{ab} \frac{\partial W^1}{\partial \phi^b} \phi_{K_2^\perp}^b \right] \left[ \frac{1}{2} - \sum_c e^{ac} \partial_1 \phi_{K_2^\perp}^c \right] |K_2^\perp \rangle = 0 \]

on solutions of (7) and/or (8); the SUSY kinks are thus \( \frac{1}{2} \)-BPS states. The energy of these states does not receive quantum corrections \( \hat{Q}_1^\perp \) because \( N = 2 \) supersymmetry forbids any anomaly in the central charges.

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We focus on the case in which the potential is:

\[ U(\phi) = \frac{1}{2} \sum_a \frac{\partial W^1}{\partial \phi^a} \frac{\partial W^1}{\partial \phi^a} = \frac{1}{2} \left( 1 - 2(\phi_1^2 + \phi_2^2) \right) \cos \left[ (n - 1) \arctan \left( \phi_1^2 + \phi_2^2 \right) \right] \]

see (3) and (2). In polar variables in the \( \mathbb{R}^2 \) internal space,
\[ \rho(x^\mu) = +\sqrt{[\phi^1(x^\mu)]^2 + [\phi^2(x^\mu)]^2}, \quad \chi(x^\mu) = \arctan \frac{\phi^2(x^\mu)}{\phi^1(x^\mu)} \]

the potential reads:
\[ U(\rho, \chi) = \frac{1}{2} \left( 1 - 2\rho^{n-1}\cos(n-1)\chi + \rho^{2(n-1)} \right) \]  \hspace{1cm} (9)

There is symmetry under the \( D_{2(n-1)} \equiv \mathbb{Z}_2 \times \mathbb{Z}_{n-1} \) dihedral group: \( \chi' = -\chi, \chi' = \chi + \frac{2\pi j}{n}, j = 0, 1, 2, ..., n-2 \).

In Cartesian coordinates, these transformations form the \( D_{2(n-1)} \) sub-group of \( O(2) \) given by:

1. \( \phi'_2 = -\phi_2, \quad \phi'_1 = \phi_1 \)
2. \( \phi'^1 = \cos \frac{2\pi j}{n-1} \phi^1 - \sin \frac{2\pi j}{n-1} \phi^2, \quad \phi'^2 = \sin \frac{2\pi j}{n-1} \phi^1 + \cos \frac{2\pi j}{n-1} \phi^2 \)

The vacuum orbit is the set of \((n-1)\)-roots of unity:
\[ \mathcal{M} = \left\{ v^{(k)} = e^{i\frac{2\pi j}{n}} \right\} = \frac{D_{2(n-1)}}{\mathbb{Z}_2} = \mathbb{Z}_{n-1}. \]  \hspace{1cm} (10)

When the \( v^{(k)} \) vacuum is chosen to quantize the theory, the symmetry under the \( D_{2(n-1)} \) group is spontaneously broken to the \( \mathbb{Z}_2 \) sub-group generated by \( \chi' = -\chi - \frac{2\pi j}{n} \); this transformation leaves a fixed point, \( v^{(k)} \), if \( n \) is even and two fixed points, \( v^{(k)} \) and \( v^{(k+\frac{n-1}{2})} \), if \( n \) is odd.

The \( \mathbb{Z}_{n-1} \)-symmetry allows for the existence of \((n-1)\) harmonic superpotentials that are equivalent:
\[ W^{(j)}(\phi) = \frac{1}{2} \left[ \phi(j) - \frac{(\phi(j))^n}{n} \right], \quad \phi(j) = e^{i\frac{2\pi j}{n}} \phi, \text{ all of them leading to the same potential } U. \]  \hspace{1cm} (11)

where \( \chi(j) = \chi + \frac{2\pi j}{n} \). There is room for closing the \( N = 2 \) supersymmetry algebra \( \{ \psi \} \) in \( n-1 \) equivalent forms: define the \( n-1 \) equivalent sets of SUSY charges:
\[ Q^{(j)}_{\pm} = \int dx \sum_{a,b} \left[ f^{(j)} \frac{\partial}{\partial \phi^{(j)a}} \psi^{(j)b}_{\pm} \right], \]  \hspace{1cm} (12)

also in terms of the "rotated" fermionic fields \( \psi^{(j)a}_{\mp} \), and the corresponding central charges \( T^{(j)} \) and \( \tilde{T}^{(j)} \). Observe that the \( N = 2 \) supersymmetry is unbroken, while the choice of vacuum that spontaneously breaks the \( \mathbb{Z}_{n-1} \) symmetry does not affect the physics, which is the same for different values of \( j \).

The \( j \) pairs of first-order systems of equations:
\[ \frac{d\rho}{dx_1} = \sin \chi(j) - \rho^{n-1} \sin n\chi(j), \quad \rho^2 \frac{d\chi}{dx_1} = \rho \cos \chi(j) - \rho^n \cos n\chi(j) \]  \hspace{1cm} (11)
\[ \frac{d\rho}{dx_1} = \cos \chi(j) - \rho^{n-1} \cos n\chi(j), \quad \rho^2 \frac{d\chi}{dx_1} = -\rho \sin \chi(j) + \rho^n \sin n\chi(j) \]  \hspace{1cm} (12)

correspond to \[ \{ \psi \} \] and \[ \{ \tilde{\psi} \} \] for this particular case. The solutions lie respectively on the algebraic curves
\[ \rho \sin \chi(j) - \frac{1}{n} \rho^n \sin n\chi(j) = \gamma_\perp \]  \hspace{1cm} (13)
\[ \rho \cos \chi(j) - \frac{1}{n} \rho^n \cos n\chi(j) = \gamma \]  \hspace{1cm} (14)

which form two families of orthogonal lines in \( \mathbb{R}^2 \). In the family of curves \( \{ \gamma \} \) there are kinks joining the vacua \( v^{(k)} \) and \( v^{(k')} \) if and only if:
\[ \sin \frac{2\pi (k+j)}{n-1} - \frac{1}{n} \sin \frac{2\pi (k+j)n}{n-1} = \sin \frac{2\pi (k'+j)}{n-1} - \frac{1}{n} \sin \frac{2\pi (k'+j)n}{n-1} = \gamma_\perp \]  \hspace{1cm} (15)
This fixes the value of $\gamma_\perp = \gamma^K_\perp$ for which the algebraic curve supports a topological kink. Simili modo,

$$\cos \frac{2\pi(k + j)}{n - 1} - \frac{1}{n} \cos \frac{2\pi(k + j)n}{n - 1} = \cos \frac{2\pi(k' + j)}{n - 1} - \frac{1}{n} \cos \frac{2\pi(k' + j)n}{n - 1} = \gamma^K$$  

is the value of the constant if the kink belong to the orthogonal family \([14]\). Solutions of \([13]\) and/or \([16]\) exist, respectively, if and only if

$$2(k + k' + 2j) = n - 1 \mod 2(n - 1)$$  

and/or

$$k + k' + 2j = 0 \mod n - 1$$  

Given the kink curves, the kink form factors are obtained in the following way:

One solves for $\chi$ in \([13]\) or \([14]\),

$$\chi + \frac{2\pi j}{n - 1} = h(\gamma^K, \rho), \quad \chi + \frac{2\pi j}{n - 1} = h(\gamma^K_\perp, \rho)$$

and plugs these expressions into the first equation of \([12]\) or \([11]\),

$$\frac{d\rho}{dx_1} = \sin h(\gamma^K, \rho) - \rho^{n-1} \sin[nh(\gamma^K, \rho)], \quad \frac{d\rho}{dx_1} = \cos h(\gamma^K_\perp, \rho) - \rho^{n-1} \cos[nh(\gamma^K_\perp, \rho)]$$

which are immediately integrated by quadratures: if $a$ is an integration constant

$$\int \frac{d\rho}{\sin h(\gamma^K, \rho) - \rho^{n-1} \sin[nh(\gamma^K, \rho)])} = (x_1 + a)$$

$$\int \frac{d\rho}{\cos h(\gamma^K_\perp, \rho) - \rho^{n-1} \cos[nh(\gamma^K_\perp, \rho)])} = (x_1 + a)$$

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We first consider the lower odd cases, only for $W(j=0)$. The other kinks are obtained by application of a $\mathbb{Z}_{n-1}$ rotation.

- $n = 3$:  
  - **Superpotential:** $W(\phi) = \frac{1}{2} \left( \phi - \phi^3 \right)$
    $$W^1 = \phi_1 - \frac{\phi_1^3}{3} + \phi_1 \phi_2^2 \quad W^2 = \phi_2 - \phi_1^2 \phi_2 + \frac{\phi_2^3}{3}$$
  - **Potential:** $U(\phi_1, \phi_2) = \frac{1}{2}[(\phi_1^4 - 1)^2 + 4\phi_2^4]$
  - **Vacuum orbit:** $M = \frac{\partial U}{\partial \phi} = \{v^0 = 1, v^1 = -1\}$
  - **Real algebraic curves:**
    $$\phi_1 - \frac{\phi_1^3}{3} + \phi_1 \phi_2^2 = \gamma; \quad \phi_2 - \phi_1^2 \phi_2 + \frac{\phi_2^3}{3} = \gamma_\perp$$
  - **Kink curve:** $\gamma_\perp = 0(\equiv W^2 = 0)$, tantamount to $\phi_2 = 0$.
  - **Kink form factor:**
    a) Solutions of $\frac{d\phi}{dx_1} = \pm(1 - \phi_1^2)$ on $\phi^2 = 0$: $\phi^K_1(x_1) = \pm \tanh(x_1 + a)$
  - **Kink energy:** $P_0[\phi^K_1] = |T| = |W^1(v^0) - W^1(v^1)| = \frac{4}{3}$
  - **Conserved SUSY charge:** $\hat{Q}_1^K = \frac{1}{4}$
• $n = 5$:
  
  – Superpotential: $W(\phi) = \frac{1}{2} \left( \phi - \frac{\phi^5}{\phi^4} \right)$

  \[ W^1 = \phi_1 \left( 1 - \frac{\phi_1^4}{5} + 2\phi_1^2\phi_2 - \phi_2^4 \right) \quad W^2 = \phi_2 \left( 1 - \phi_1^4 + 2\phi_1^2\phi_2^2 - \frac{\phi_2^4}{5} \right) \]

  – Potential: $U(\phi_1, \phi_2) = \frac{1}{2} \left( (\phi_1^* + 1)^2 - 4\phi_1^2 \right) \left( (\phi_2^* + 1)^2 - 4\phi_2^2 \right)$

  – Vacuum orbit: $M = \frac{\phi_1}{\phi_2} = \{ v^0 = 1, v^1 = i, v^2 = -1, v^3 = -i \}$

  – Real algebraic curves:

  \[ \phi_1 \left( 1 - \frac{\phi_1^4}{5} + 2\phi_1^2\phi_2 - \phi_2^4 \right) = \gamma; \quad \phi_2 \left( 1 - \phi_1^4 + 2\phi_1^2\phi_2^2 - \frac{\phi_2^4}{5} \right) = \gamma_\perp \]

  – Kink curves: a) $\gamma_\perp = 0 \equiv \phi_2 = 0$, b) $\gamma = 0 \equiv \phi_1 = 0$.

  – Kink form factor:

  a) Solutions of $\pm \frac{\phi_1}{\phi_1} = 1 - \phi_1^4$ on $\phi_2 = 0$: $\arctan \phi_1^K = \arctan \phi_1^K = \pm 2(x_1 + a)$

  b) Solutions of $\pm \frac{\phi_2}{\phi_2} = 1 - \phi_2^4$ on $\phi_1 = 0$: $\arctan \phi_2^K = \arctan \phi_2^K = \pm 2(x_1 + a)$

  – Kink energies:

  a) $P_0[\phi^K] = |T| = |W^1(v^0) - W^1(v^2)| = \frac{8}{7} \phi$

  b) $P_0[\phi^K] = |T| = |W^2(v^1) - W^2(v^3)| = \frac{8}{7}$

  – Conserved SUSY charges:

  a) $\tilde{Q}^1_\perp |K^1_\perp\rangle = 0$; b) $\tilde{Q}^2_\perp |K^2_\perp\rangle = 0$

• $n = 7$:

  – Superpotential: $W(\phi) = \frac{1}{2} \left( \phi - \frac{\phi^7}{\phi^6} \right)$

  \[ W^1 = \phi_1 - \frac{\phi_1^7}{7} + 3\phi_1^5\phi_2 - 5\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3 \quad W^2 = \phi_2 - \phi_1^6\phi_2 + 5\phi_1^4\phi_2^3 - 3\phi_1^2\phi_2^5 + \frac{\phi_2^7}{7} \]

  – Potential: $U(\phi_1, \phi_2) = \frac{1}{2} \left( (\phi_1^* + 1)^2 - 2(\phi_1^2 - \phi_2^2) \right) \left( (\phi_2^* + 1)^2 - 16\phi_1^2\phi_2^2 \right) + 1 \}

  – Vacuum orbit:

  $M = \frac{\phi_1}{\phi_2} = \{ v^0 = 1; v^1 = \frac{1}{2} + i\sqrt{3}; v^2 = -\frac{1}{2} + i\sqrt{3}; v^3 = -1; v^4 = -\frac{1}{2} - i\sqrt{3}; v^5 = \frac{1}{2} - i\sqrt{3} \}$

  – Real algebraic curves:

  \[ \phi_1 - \frac{\phi_1^7}{7} + 3\phi_1^5\phi_2 - 5\phi_1^3\phi_2^2 + 3\phi_1\phi_2^3 = \gamma; \quad \phi_2 - \phi_1^6\phi_2 + 5\phi_1^4\phi_2^3 - 3\phi_1^2\phi_2^5 + \frac{\phi_2^7}{7} = \gamma_\perp \]

  – Kink curves: there are two choices of $\gamma_\perp$ and three choices of $\gamma$ for which one finds kink curves. The other kinks associated with the other superpotentials can be obtained by $\mathbb{Z}_6$ rotations.

  a) $\gamma_\perp = \frac{3\sqrt{3}}{7}$: kink curve joining $v^1$ with $v^2$

  b) $\gamma = \frac{\sqrt{2}}{2}$: kink curve joining $v^4$ with $v^5$

  c) $\gamma = 0$: kink curve joining $v^0$ with $v^3$

  – Kink energies:

  a) $P_0[\phi^K] = |T| = |W^1(v^k) - W^1(v^{k+1})| = \frac{6}{7}$

  b) $P_0[\phi^K] = |T| = |W^2(v^k) - W^2(v^{k+2})| = \frac{6\sqrt{7}}{7}$

  c) $P_0[\phi^K] = |T| = |W^2(v^k) - W^1(v^{k+3})| = \frac{12}{7}$

  – Conserved SUSY charges:

  a) $\tilde{Q}^1_\perp |K^1_\perp\rangle = 0$; (b) and (c) $\tilde{Q}^2_\perp |K^2_\perp\rangle = 0$
We now study two even cases.

- The first and most interesting model occurs for \( n = 4 \). Here, we find that the kink curves are straight lines in \( W \)-space (true for any \( n \)) and curved in \( \phi \)-space, in agreement with Reference [8]:

  - Superpotential: \( W[\phi] = \frac{1}{2} \left( \phi - \frac{\phi^4}{6} \right) \)

    \[ W^1 = \phi_1 - \frac{\phi_1^4}{4} + \frac{3}{2} \phi_1^2 \phi_2^2 - \frac{\phi_2^4}{4}, \quad W^2 = \phi_2 \left( 1 - \phi_1^3 + \phi_1 \phi_2^3 \right) \]

  - Potential: \( U(\phi_1, \phi_2) = \frac{1}{2} \left[ (\phi \phi^*)^3 - 2 \phi_1 (\phi_1^2 - 3 \phi_2^2) + 1 \right] \)

  - Vacuum orbit: \( M = \frac{D\phi}{D\phi} = \{ v^0 = 1, v^1 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}, v^2 = -\frac{1}{2} - i \frac{\sqrt{3}}{2} \} \)

  - Real algebraic curves:

    \[ \phi_1 - \frac{\phi_1^4}{4} + \frac{3}{2} \phi_1^2 \phi_2^2 - \frac{\phi_2^4}{4} = \gamma; \quad \phi_2 \left( 1 - \phi_1^3 + \phi_1 \phi_2^3 \right) = \gamma \]

    - Kink curve: \( \gamma = -\frac{3}{8} \)

    - Kink form factor: on the kink curve we find \( \phi_1^{K^2} = f^{-1}[\pm (x + a)] \) where

      \[
      f(\phi_1) = \frac{d\phi_1}{\sqrt{\frac{3}{2} + 4 \phi_1 + 8 \phi_1^4 \left( 3 \phi_1^2 - \sqrt{\frac{3}{2} + 4 \phi_1 + 8 \phi_1^4} \right)}}
      \]

    - Kink energy: \( P_0[\phi^{K^2}] = |\tilde{T}| = |W^2(\nu^k) - W^2(\nu^{k+1})| = \frac{3\sqrt{3}}{4} \)

    - Conserved SUSY charge: \( \tilde{Q}_\pm^2 |K^2_\pm| = 0 \)

- \( n = 6 \):

  - Superpotential: \( W[\phi] = \frac{1}{2} \left( \phi - \frac{\phi^6}{6} \right) \)

    \[ W^1 = \phi_1 - \frac{\phi_1^6}{6} + \frac{5}{2} \phi_1^2 \phi_2^2 - \frac{5}{2} \phi_1^2 \phi_2^4 + \frac{\phi_2^6}{6}, \quad W^2 = \phi_2 \left( 1 - \phi_1^5 + \frac{10}{3} \phi_1^3 \phi_2^2 - \phi_1 \phi_2^4 \right) \]

  - Potential: \( U(\phi_1, \phi_2) = \frac{1}{7} \left[ (\phi \phi^*)^5 - 2 \phi_1 (\phi_1^4 + 5 \phi_2^4 - 10 \phi_1^2 \phi_2^2) + 1 \right] \)

  - Vacuum orbit: \( M = \frac{D\phi}{D\phi} = \{ v^0 = 1, v^1 = e^{i\frac{2\pi}{6}}, v^2 = e^{i\frac{4\pi}{6}}, v^3 = e^{i\frac{6\pi}{6}}, v^4 = e^{i\frac{8\pi}{6}} \} \)

  - Real algebraic curves:

    \[ \phi_1 - \frac{\phi_1^6}{6} + \frac{5}{2} \phi_1^2 \phi_2^2 - \frac{5}{2} \phi_1^2 \phi_2^4 + \frac{\phi_2^6}{6} = \gamma; \quad \phi_2 \left( 1 - \phi_1^5 + \frac{10}{3} \phi_1^3 \phi_2^2 - \phi_1 \phi_2^4 \right) = \gamma \]

    - Kink curves: there are two values of \( \gamma \) giving kink curves: a) \( \gamma = -\frac{1}{4} (1 + \sqrt{5}) \): kink curve joining \( v^2 \) with \( v^3 \), b) \( \gamma = \frac{5}{24} (-1 + \sqrt{5}) \): kink curve joining \( v^1 \) with \( v^4 \). The other kink curves are obtained through \( Z_5 \) rotations.

    - Kink energies: a) \( P_0[\phi^{K^2}] = |\tilde{T}| = |W^2(\nu^k) - W^2(\nu^{k+1})| = \frac{5}{6} \sqrt{\frac{5 - \sqrt{5}}{2}} \)

        \[ b) P_0[\phi^{K^2}] = |\tilde{T}| = |W^2(\nu^k) - W^2(\nu^{k+2})| = \frac{5}{6} \sqrt{\frac{5 + \sqrt{5}}{2}} \]

    - Conserved SUSY charges: (a) and (b) \( \tilde{Q}_\pm^2 |K^2_\pm| = 0 \)
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Figure 1: Kink curves in the $n = 4$, $n = 5$, $n = 6$ and $n = 7$ models