On Sparse variational methods and the Kullback-Leibler divergence between stochastic processes

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Abstract

The variational framework for learning inducing variables \cite{Titsias2009} has had a large impact on the Gaussian process literature. The framework may be interpreted as minimizing a rigorously defined Kullback-Leibler divergence between the approximate and posterior processes. To our knowledge this connection has thus far gone unremarked in the literature. Many of the technical requirements for such a result were derived in the pioneering work of Seeger \cite{Seeger2003b, Seeger2003a}. In this work we give a relatively gentle and largely self-contained explanation of the result. The result is important in understanding the variational inducing framework and could lead to principled novel generalizations.

1 Introduction

The variational approach to inducing point selection of Titsias \cite{Titsias2009} has been highly influential in the active research area of scalable Gaussian process approximations. The chief advantage of this particular framework is that the inducing points positions are variational parameters rather than models parameters and as such are protected from overfitting. The original framework is applied to conjugate likelihoods and has been extended to non-conjugate likelihoods \cite{Chai2012, Hensman2013}. An important advance in the use of variational methods was their combination with stochastic gradient descent \cite{Hoffman2013} and the variational inducing point framework has been combined with such methods in the conjugate \cite{Hensman2013} and non-conjugate cases \cite{Hensman2015}.

The approach has also been successfully used to perform scalable inference in more complex models such as the Gaussian process latent variable model \cite{Titsias2010, Damianou2014} and the related Deep Gaussian process \cite{Damianou2012, Hensman2014}.

To be more concrete let us setup some notation, consider a function \( f \) mapping an index set \( X \) to the set of real numbers

\[
 f : X \mapsto \mathbb{R}.
\]  

(1)

Entirely equivalently we may write \( f \in \mathbb{R}^X \) or use sequence notation \((f(x))_{x \in X}\).

We also define set indexing of the function. If \( S \subseteq X \) is some subset of the index set then

\[
 f_S := (f(x))_{x \in S}
\]

(2)

and we may straightforwardly extend this definition to single elements of the index set \( f_x := f(x) \). We can put this notation to immediate use by defining a subset \( D \subseteq X \) of the index set, of size \( N \), that corresponds to those input points for which we have observed data. The corresponding function values will then be denoted \( f_D \). For simplicity we will assume that we have one, possibly noisy, possibly non-conjugate observation \( y \) per input data point which will together form a set \( Y \).
Gaussian processes allow us to define a prior over functions $f$. After we observe the data we will have some posterior which we wish to approximate with a sparse distribution. At the heart of the variational inducing point approximation is the idea of 'augmentation' that appears in the original paper and many subsequent ones. We choose to monitor a set $Z \subseteq X$ of size $M$. These points may have some overlap with the input data points $D$ but to give a computational speed up $M$ will need to be less than the number of data points $N$. The Kullback-Leibler divergence given as an optimization criterion in Titsias’ original paper is

$$ KL[q(f_{D|Z}, f_Z)||p(f_{D}\setminus Z, f_Z|Y)] $$

$$ = \int q(f_{D|Z}, f_Z) \log \frac{q(f_{D|Z}, f_Z)}{p(f_{D|Z}, f_Z|Y)} \, df_{D|Z} \, df_Z $$

The variational distribution at those data points which are not also inducing points is taken to have the form:

$$ q(f_{D\setminus Z}, f_Z) := p(f_{D\setminus Z}|f_Z)q(f_Z) $$

where $p(f_{D\setminus Z}|f_Z)$ is the prior conditional and $q(f_Z)$ is a variational distribution on the inducing points only. Under this factorization, for a conjugate likelihood, the optimal $q(f_Z)$ has an analytic Gaussian solution (Titsias, 2009). The non-conjugate case was then studied in subsequent work (Chai, 2012; Hensman et al., 2015). In both cases the sparse approximation requires only $O(NM^2)$ rather than the $O(N^3)$ required by exact methods in the conjugate case or many commonly used non-conjugate approximations that don’t assume sparsity.

The augmentation is justified by arguing that the model remains marginally the same when the inducing points are added. Thus the inducing point positions can be considered to be variational parameters and are consequently protected from overfitting.

Without this justification, however, the $KL$-divergence in equation could seem to be a strange optimization target. The $KL$-divergence has the inducing variables on both sides. Thus it might seem that in optimizing the inducing point positions we are trying to hit a ‘moving target’. It would thus seem desirable to rigorously formulate a ‘one sided’ $KL$-divergence that leads to Titsias’ formulation. Such a derivation could be viewed as a rigorous mathematical proof of the augmentation argument and could help correctly generalize this elegant framework. Such a derivation is the topic of this article. As we shall see it requires some fairly technical mathematical machinery which will also be elucidated from the various necessary sources.

In terms of existing work the major other references are the early work of Seeger (2003a, 2003b). In particular Seeger identifies the $KL$-divergence between processes (more commonly referred to as a relative entropy in those texts) as a measure of similarity and applies it to PAC-Bayes and to subset of data sparse methods. Crucially Seeger outlines the rigorous formulation of such a $KL$-divergence which is a large technical obstacle. The explanation of this point is somewhat distributed over the work and its references. One of the aims of this article is to pull a large proportion of the necessary knowledge together and to give more detail where helpful or expedient. Further, we extend the stochastic process formulation to inducing points which are not necessarily selected from the data and show that this is equivalent to Titsias’ formulation. In so far as we are aware this relationship has not previously been noted in the literature. The idea of using the $KL$-divergence between processes is also mentioned in the early work of Csato and Opper (2002, 2002) but the transition from finite dimensional multivariate Gaussians to infinite dimensional Gaussian processes is not covered at the level of detail discussed here. An optimization target that in intent seems to be similar to a $KL$-divergence between stochastic process is briefly mentioned in the work of Alvarez (2011). The notation used suggests that the integration is with respect to an ‘infinite dimensional dimensional Lebesgue measure’, which as we shall see is an argument that arrives at the right answer via a mathematically flawed route. Chai (2012) seems to have been at least partly aware of Seeger’s $KL$-divergence theorems (Seeger, 2003b) but instead uses them to bound the finite joint predictive probability of a non sparse process.

This article proceeds by first discussing the finite dimensional version of the full argument. This requires considerably less mathematical machinery and much of the intuition can be gained from this case. After reviewing some of the mathematical background we then proceed to give the full measure theoretic formulation. Finally we conclude.
2 Finite index set case

Consider the case where $X$ is finite. We introduce a new set $\ast := X \setminus (D \cup Z)$, in words: all points that are in the index set that aren’t inducing points or data points. These points might be of practical interest for instance when making predictions on hold out data.

We extend the variational distribution to include these points:

$$q(f_\ast, f_{D\setminus Z}, f_Z) := p(f_\ast, f_{D\setminus Z}|f_Z)q(f_Z).$$

We then consider the KL-divergence between this extended variational distribution and the full posterior distribution $p(f|Y)$

$$\mathcal{KL}[q(f_\ast, f_{D\setminus Z}, f_Z)||p(f|Y)]$$

$$= \mathcal{KL}[q(f_\ast, f_{D\setminus Z}, f_Z)||p(f_\ast, f_{D\setminus Z}, f_Z|Y)]$$

$$= \int q(f_\ast, f_{D\setminus Z}, f_Z) \log \frac{q(f_\ast, f_{D\setminus Z}, f_Z)}{p(f_\ast, f_{D\setminus Z}, f_Z|Y)} df_\ast df_{D\setminus Z} df_Z$$

Now we substitute the definitions of $q(f_\ast, f_{D\setminus Z}, f_Z)$ and $p(f_\ast, f_{D\setminus Z}, f_Z|Y)$ in terms of conditional distributions into the integral, observing some cancellation:

$$= \int p(f_\ast, f_{D\setminus Z}|f_Z)q(f_Z) \log \left\{ \frac{p(f_\ast|f_{D\setminus Z}, f_Z)p(f_{D\setminus Z}|f_Z)q(f_Z)p(Y)}{p(f_\ast|f_{D\setminus Z}, f_Z)p(f_{D\setminus Z}|f_Z)p(f_Z|Y)} \right\} df_\ast df_{D\setminus Z} df_Z$$

$$= \int p(f_\ast, f_{D\setminus Z}|f_Z)q(f_Z) \log \left\{ \frac{p(f_{D\setminus Z}|f_Z)q(f_Z)p(Y)}{p(f_{D\setminus Z}|f_Z)p(f_Z|Y)} \right\} df_\ast df_{D\setminus Z} df_Z$$

Next we exploit the marginalization property of the Gaussian process:

$$= \int p(f_{D\setminus Z}|f_Z)q(f_Z) \log \left\{ \frac{p(f_{D\setminus Z}|f_Z)q(f_Z)}{p(f_{D\setminus Z}|f_Z)p(f_Z|Y)} \right\} df_{D\setminus Z} df_Z$$

$$= \int q(f_{D\setminus Z}, f_Z) \log \left\{ \frac{q(f_{D\setminus Z}, f_Z)}{p(f_{D\setminus Z}, f_Z|Y)} \right\} df_{D\setminus Z} df_Z$$

The last line is exactly the KL-divergence used by Titsias [2009] that we already described in equation [3]. We thus see that for finite index sets considering the KL-divergence between the two distributions is equivalent to Titsias’ KL-divergence.

We might choose to optimize our choice of the $M$ by selecting them from the $|X|$ possible values in the index set and comparing the KL-divergence between distributions given in equation [5]. The equivalence with equation [3] that we have just derived shows us that in this case the appearance of the inducing values on both sides of the equation is just a question of ‘accounting’. That is to say whilst we are in fact optimizing the KL-divergence between the full distributions, we only need to keep track of the distribution over function values $f_Z$ and $f_{D\setminus Z}$. All the other function values $f_\ast$ marginalize. For different choices of inducing points we will need to keep track of different function values and be able to safely ignore different values $f_\ast$.

3 Infinite index set case

3.1 There is no useful infinite dimensional Lebesgue measure

One might hope to cope with not only finite index sets but also infinite index sets in the way discussed in section 2. Unfortunately when $X$ and hence $f_\ast$ are infinite sets we cannot integrate with respect to a ‘infinite dimensional vector’. That is to say the notation $\int f(\cdot) df_\ast$ can no longer be correctly used.
For a discussion of this see, for example, Hunt et al (1992). The crux of the issue is that to give sensible answers such a measure would need to be translation invariant and locally finite. Unfortunately the only measure that obeys these two properties is the zero measure which assigns zero to every input set.

Thus we see that it will be necessary to rethink our approach to a KL-divergence between stochastic processes. It will turn out that a reasonable definition will require the full apparatus of measure theory.

3.2 Measure spaces and Lebesgue integral.

In this section we discuss the fundamental definitions in measure theory. Readers familiar with this material may safely skip to the next section, although they may wish to review the notation used. Readers looking for a fuller exposition may wish to consult a textbook (Billingsley, 1995; Capinski and Kopp, 2004).

A \( \sigma \)-algebra \( \Sigma \) on a set \( \Omega \) is a set of subsets \( E \) of \( \Omega \) that obey the following axioms:

1. It contains the full set: \( \Omega \in \Sigma \)
2. It is closed under complementation: \( E \in \Sigma \implies \Omega \setminus E \in \Sigma \)
3. It is closed under countable unions: \( E_i \in \Sigma \), \( i \in I \subseteq \mathbb{N} \implies \bigcup_{i \in I} E_i \in \Sigma \)

Consider a set of subsets \( G \) of \( \Omega \) which is not necessarily a \( \sigma \)-algebra. The \( \sigma \)-algebra generated by \( G \) is the unique smallest \( \sigma \)-algebra containing every element of \( G \). It is denoted \( \sigma(G) \). As an example of such a generated \( \sigma \)-algebra we give the Borel \( \sigma \)-algebra \( \mathcal{B} \) on the real numbers \( \mathbb{R} \) which is generated by the set of all open intervals \((a, b)\) of the real line. The Borel \( \sigma \)-algebra may be extended to multiple dimensions.

A measure is a function mapping elements of a \( \sigma \)-algebra, \( \Sigma \), to the extended real number line \( \mathbb{R} \cup \{\infty, -\infty\} \).

It obeys the following axioms:

1. It is non-negative. \( \forall E \in \Sigma, \, \mu(E) \geq 0 \)
2. The measure of the empty set \( \emptyset \) is zero. \( \mu(\emptyset) = 0 \)
3. The measure is countably additive. For \( I \subseteq \mathbb{N} \), \( \mu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \mu(E_i) \).

It is possible to extend this definition to signed measures but we will not require this in what follows. A probability measure is a measure for which \( \mu(\Omega) = 1 \). The triple \((\Omega, \Sigma, \mu)\) is called a measure space.

Given two \( \sigma \)-algebras on two different sets, namely \( \Sigma_1 \) on \( \Omega_1 \) and \( \Sigma_2 \) on \( \Omega_2 \) a function \( g : \Omega_1 \mapsto \Omega_2 \) is called a measurable function if for any set \( E \in \Sigma_2 \) the pre-image of that set \( g^{-1}(E) \) is a member of \( \Sigma_1 \).

The Lebesgue integral is a measure theoretic formulation of the notion of integral. A non-negative measurable function \( \psi : \Omega_1 \mapsto \mathbb{R} \) is a simple function if its range is a finite set \( \{y_1, y_2, \ldots, y_C\} \). We may write:

\[
\psi(\omega) = \sum_{i=1}^{C} y_i \mathbb{1}_{A_i}(\omega)
\]  

(12)

where the sets \( A_i = g^{-1}(y_i) \), \( i = 1, \ldots, C \) are by assumption a sequence of measurable sets and \( \mathbb{1} \) is the indicator function.

The integral of a simple function with respect to a measure \( \mu \) on \( \Sigma_1 \) and some element \( E \in \Sigma_1 \) is:

\[
\int_{E} \psi \, d\mu = \sum_{i=1}^{C} y_i \mu(A_i \cap E)
\]  

(13)

Using the definition for simple functions we extend the definition of Lebesgue integral to general non-negative measurable functions \( g \) thus:
\[ \int_E g \, d\mu = \sup \left\{ \int_E \psi \, d\mu : 0 \leq \psi \leq g, \psi \text{ is simple} \right\} \] (14)

For those readers more familiar with the Riemann integral it may be helpful to give an intuitive motivation of the relationship between the two. In the case of the Riemann integral we partition the domain of the function into increasingly small sets. We then evaluate the function to provide an increasingly accurate approximation of the area under the curve. In the case of the Lebesgue integral we divide the range of the function into increasingly small sets, take the pre-image of each element of the partition and then use the measure \( \mu \) to measure it. The Lebesgue integral will make it possible for us to take the integral with respect to relatively complex probability measures such as those corresponding to Gaussian processes, as we shall see in the sections that follow.

Consider two measures \( \mu, \eta \) on \((\Omega, \Sigma)\). \( \mu \) is absolutely continuous with respect to \( \eta \) if the null sets of \( \eta \) are null sets of \( \mu \) that is to say \( \eta(A) = 0 \iff \mu(A) = 0 \). If both measures allocate finite measure to the whole space, as will be the case with probability measures, then it is a consequence of the Radon-Nikodym theorem that there exists some measurable function \( \frac{d\mu}{d\eta} : \Omega \mapsto [0, \infty) \) known as the Radon-Nikodym derivative with the property that for all \( E \in \Sigma \):

\[ \mu(E) = \int_E \frac{d\mu}{d\eta} \, d\eta \] (15)

Suppose we have a third measure \( \lambda \) with the property:

\[ \eta(E) = \int_E \frac{d\eta}{d\lambda} \, d\lambda, \] (16)

then the following relation holds:

\[ \mu(E) = \int_E \frac{d\mu}{d\eta} \frac{d\eta}{d\lambda} \, d\lambda \] (17)

### 3.3 The product \( \sigma \)-algebra and the Kolmogorov extension theorem

The presentation in this section follows the notes of Sengupta (2014) fairly closely. We now return to considering functions from an index set to the reals \( f : X \mapsto \mathbb{R} \) which we can also denote in sequence notation \((f(x))_{x \in X}\). Consider a projection map \( \pi_{U \to V} : \mathbb{R}^U \mapsto \mathbb{R}^V \) that for \( V \subset U \subseteq X \) has the following property:

\[ \pi_{U \to V} : (f(x))_{x \in U} \mapsto (f(x))_{x \in V} \] (18)

A cylinder set is the pre-image of the projection \( \pi_{X \to V} \) of a Borel set \( E \in \mathbb{R}^V \) for finite \( V \), denoted:

\[ \pi_{X \to V}^{-1}(E). \] (19)

Clearly it is a subset of the set of all functions from \( X \) to \( \mathbb{R} \). Let \( \mathcal{C} \) be the set of all cylinder sets. We may generate a \( \sigma \)-algebra using this set which we denote \( \sigma(\mathcal{C}) \). This \( \sigma \)-algebra is known as the product \( \sigma \)-algebra.

The Kolmogorov extension theorem concerns circumstances under which a collection of consistent distributions on functions for finite index sets \( U \subset X \) implies the existence of a corresponding probability measure on functions with countably or uncountably infinite index set \( X \). To be more concrete consider a family of Borel probability measures, labelled by their corresponding finite index set, \( \mu_V : \mathcal{B}(V) \mapsto [0, 1] \) which obeys the following consistency relation for all \( V \subset X \) and all second finite index sets with \( U \supset V \):

\[ \mu_V(\pi_{U \to V}^{-1}(E)) = \mu_V(E) \] (20)

over all Borel sets \( E \subset \mathbb{R}^V \). If such a set of consistent finite dimensional distributions exists then Kolmogorov’s theorem states that there is a unique probability measure on the product \( \sigma \)-algebra, \( \mu_X : \sigma(\mathcal{C}) \mapsto [0, 1] \) with the property:

\[ \mu_X(\pi_{X \to V}^{-1}(E)) = \mu_V(E) \] (21)
for all finite index sets $V$ and Borel sets $E \subset \mathbb{R}^V$.

A common example of the application of this theorem is to argue for the existence of a Gaussian process with a given mean and covariance function on the basis of the marginalization property of multivariate Gaussian distributions.

### 3.4 The $KL$-divergence between processes

We have now reviewed all the mathematical background necessary and we can return to the original questions: “How does one rigorously define the $KL$-divergence between stochastic processes” and “How does this relate to Titsias’ $KL$-divergence?” which are answered in the next two sections.

Suppose we have two measures $\mu$ and $\eta$ for $(\Omega, \Sigma)$ and that $\mu$ is absolutely continuous with respect to $\eta$. Then there exists a Radon-Nikodym derivative $\frac{d\mu}{d\eta}$ and the correct definition for $KL$-divergence between these measures is:

$$KL[\mu || \eta] = \int_\Omega \log \left\{ \frac{d\mu}{d\eta} \right\} d\mu \quad (22)$$

In the case where $\mu$ is not absolutely continuous with respect to $\eta$ we let $KL[\mu || \eta] = \infty$. In the case where the measure is Borel on $\mathbb{R}^B$ for some finite $B$ and both measures are dominated by Lebesgue measure $m$ this reduces to the more familiar definition:

$$KL[\mu || \eta] = \int_\Omega u \log \left\{ \frac{u}{v} \right\} dm \quad (23)$$

where $u$ and $v$ are the respective densities with respect to Lebesgue measure. The first definition is more general and allows us to deal with the problem of there being no sensible infinite dimensional Lebesgue measure by instead integrating with respect to the measure $\mu$.

### 3.5 Recovering the sparse variational inducing framework

In this section we are now interested in three types of probability measure on sets of functions $f : X \mapsto \mathbb{R}$. The first is the prior measure $P$ which will be assumed to be a Gaussian process. The second is the approximating measure $Q$ which will be assumed to be a sparse Gaussian process and the third is the posterior process $\hat{P}$ which may be Gaussian or non-Gaussian depending on whether we have a conjugate likelihood. We specify the densities $q$ with respect to Lebesgue measure on any finite index set $V \subset X$ to avoid ever trying to express a density with respect to infinite dimensional Lebesgue measure. The Kolmogorov extension theorem then straightforwardly confirms the existence of the measures in question on the product $\sigma$-algebra.

To compute the $KL$-divergence between the approximating process and the posterior we need to find the Radon-Nikodym derivative. We start with the answer and then show that it is correct. The Radon-Nikodym derivative is:

$$\frac{dQ}{d\hat{P}}(f) = \frac{q(f_{\mathbb{R}^V \cup D}, f_{\mathbb{R}^V})}{p(f_{\mathbb{R}^V \cup D}, f_{\mathbb{R}^V})} \quad (24)$$

Here $q$ and $p$ are the corresponding densities with respect to Lebesgue measure of the corresponding finite dimensional marginals. In this case the Radon-Nikodym derivative only depends on finitely many of the function points, namely the values at the data points and the inducing points.

We break the proof that this is the correct Radon-Nikodym derivative into two parts. First we need to show that our candidate function is measurable, then we show that when integrated with respect to $\hat{P}$ over all members of the product $\sigma$-algebra it gives the approximating measure $Q$.

To show that $\frac{q(f_{\mathbb{R}^V \cup D}, f_{\mathbb{R}^V})}{p(f_{\mathbb{R}^V \cup D}, f_{\mathbb{R}^V})}$ is measurable we need to show that the pre-images of all intervals $(x, \infty)$ lie in the product $\sigma$-algebra denoted $\sigma(C)$. To do this we note that the function only depends on the finite set of function values $f_{\mathbb{R}^V \cup D}$. The pre-images are thus precisely cylinder sets of the form $\pi_{X \mapsto \mathbb{R}^V}(E)$ where $E \subset \mathbb{R}^{\mathbb{R}^V \cup D}$ will certainly be Borel for all $\hat{P}$ and $Q$ we consider.

Next we need to show that the measure $\hat{\mu}$ defined by:
\[
\hat{\mu}(E) = \int_{E} \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} d\hat{P}
\]

where \( E \in \sigma(C) \), is equal to the measure \( Q \). We can show this if we can show that both measures have the same finite dimensional marginals, since we can appeal to the uniqueness of the Kolmogorov extension. Thus consider any finite index subset \( V \subset X \), a Borel set \( C \in \mathbb{R}^V \) and the corresponding cylinder set \( \pi_{X \to V}^{-1}(C) \). For clarity of exposition we assume that \( V \supseteq Z \cup D \) but by careful, though not difficult, book keeping this may be extended to the general case. We have

\[
\hat{\mu}(\pi_{X \to V}^{-1}(C)) = \int_{\pi_{X \to V}^{-1}(C)} \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} d\hat{P}
\]

\[
\Rightarrow \hat{\mu}_{V}(C) = \int_{C} \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} d\hat{P}_{V}
\]

where the subscripting of the measures indicates the relevant finite dimensional measure on \( \mathbb{R}^V \). Now that we have a measure for a finite index set it is dominated by Lebesgue measure \( m_{V} \) on \( \mathbb{R}^V \) and thus we may write

\[
\hat{\mu}_{V}(C) = \int_{C} \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} p(f_{V\setminus(Z\cup D)}|f_{D\setminus Z}, f_{Z}) dm_{V}
\]

\[
= \int_{C} q(f_{D\setminus Z}, f_{Z}) p(f_{V\setminus(Z\cup D)}|f_{D\setminus Z}, f_{Z}) dm_{V}
\]

\[
= Q_{V}(C)
\]

as required. Now that we have confirmed we have the correct Radon-Nikodym derivative we show that the \( KL \)-divergence between stochastic processes reduces to the \( KL \)-divergence of Titsias.

\[
KL[Q || \hat{P}] = \int_{\mathbb{R}^X} \log \left\{ \frac{dQ}{d\hat{P}} \right\} dQ
\]

\[
= \int_{\mathbb{R}^X} \log \left\{ \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} \right\} dQ
\]

\[
= \int_{\mathbb{R}^{Z\cup D}} \log \left\{ \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} \right\} dQ_{Z\cup D}
\]

\[
= \int_{\mathbb{R}^{Z\cup D}} q(f_{D\setminus Z}, f_{Z}) \log \left\{ \frac{q(f_{D\setminus Z}, f_{Z})}{p(f_{D\setminus Z}, f_{Z}|Y)} \right\} dm_{Z\cup D}
\]

QED.

4 Conclusion and acknowledgements

In this work we have elucidated the connection between the variational inducing point framework (Titsias, 2009) and a rigorously defined \( KL \)-divergence between stochastic processes. As will be evident, many of the required technical results were derived early on in the use of Gaussian processes for machine learning by Seeger (2003a; 2003b).

It seems reasonable to hope that elucidating the measure theoretic roots of the formulation will help the community to generalise the framework and lead to even better practical results.

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