Josephson current through a Luttinger liquid

Rosario Fazio(1), F. W. J. Hekking(2), and A. A. Odintsov(2,3)

(1) Istituto di Fisica, Università di Catania, viale A. Doria 6, 95129 Catania, Italy
(2) Institut für Theoretische Festkörperphysik, Universität Karlsruhe, Postfach 6980, 76128 Karlsruhe, FRG
(3) Present address: Electrotechnical Laboratory, 1-1-4 Umezono, Tsukuba-shi, Ibaraki 305, Japan
Permanent address: Nuclear Physics Institute, Moscow State University, Moscow 119899 GSP, Russia
(March 23, 2022)

Abstract

We study the Josephson effect through a one-dimensional system of interacting electrons, connected to two superconductors by tunnel junctions. The interactions are treated in the framework of the one-channel Luttinger model. At zero temperature, the Josephson critical current is found to decay algebraically with increasing distance between the junctions. The exponent is proportional to the strength of the Coulomb interaction. If the Luttinger liquid has a finite size, the Josephson current depends on the total number of electrons modulo 4. These parity effects are studied for the ring, coupled capacitively to a gate-voltage and threaded by a magnetic flux. The Josephson current changes continuously as a function of the gate voltage and stepwise as a function of the magnetic flux. The electron-electron interaction introduces qualitatively new features compared to the non-interacting case.

PACS numbers: 74.50 +r, 72.15 Nj

Typeset using REVTeX
Recent technological developments in the fabrication of semiconductor-superconductor (S-Sc) interfaces made the observation of supercurrent in S-Sc-S junctions possible. This progress was accompanied by a number of appealing theoretical predictions. The Josephson current through a narrow constriction in a high-mobility two-dimensional non-interacting electron gas (2DEG) should be quantized. Interference effects of two electrons entering the normal region from a superconductor are important, especially in mesoscopic samples. This interference influences the Josephson current through the normal region.

It is well-known that electron-electron interactions affect the transport properties of mesoscopic devices as well. The interaction manifests itself in the single charge effects. In particular, the Josephson current through SSS and SNS systems can be modulated by a gate-voltage. This has been described in terms of a phenomenological capacitance model. However, in low-dimensional semiconductor nanostructures, with a low electron density, electron-electron interactions should be treated microscopically. As a result, the transport properties of a (quasi-) one-dimensional quantum wire show deviations from Fermi-liquid behaviour.

In the present paper we analyze the Josephson current through a Luttinger liquid (LL). Specifically, we consider two geometries which can be realized experimentally: A long wire with tunnel contacts to two superconductors at a distance \( d \) (Fig. 1a), and a ring with circumference \( L \) and tunnel contacts at a distance \( L/2 \) (Fig. 1b). Various aspects of transport in mesoscopic systems (parity effects and interference combined with electron-electron interactions) and their interplay can be studied in this system. The aim of this paper is to study the influence of the Coulomb interaction on the Josephson critical current. Second, we would like to see how the parity effects which are present in the ring manifest themselves in the dependence of the Josephson current on magnetic flux and gate-voltage. The coupling of bulk superconductors to a one-dimensional (1D) system of interacting electrons was studied by Fisher. In contrast to our work, a chiral LL of spin-polarized electrons was considered and tunneling with spin flip was analyzed.

The systems under consideration (Fig. 1) can be described by the Hamiltonian \( \hat{H} = \hat{H}_{S1} + \hat{H}_{S2} + \hat{H}_{L} + \hat{H}_{T} \). Here, \( \hat{H}_{S1}, \hat{H}_{S2} \) are the BCS-Hamiltonians for the bulk superconductors with gap \( \Delta_1, \Delta_2 \) respectively, kept at a phase difference \( \chi = \chi_1 - \chi_2 \). The (1D) electron system is described by the Hamiltonian

\[
\hat{H}_{L} = \hbar \int \frac{dx}{\pi} \sum_{j} v_{j} \left[ \frac{g_{j}}{2} (\nabla \phi_{j})^{2} + \frac{2}{g_{j}} (\nabla \theta_{j})^{2} \right].
\]  

It is written as a sum of the contributions from the spin \((j = \sigma)\) and charge \((j = \rho)\) degrees of freedom. We use the standard notation for the parameters \( g_{j} \) of the interaction strengths and the velocities \( v_{j} \) of spin and charge excitations. We also introduce the bosonic fields \( \phi_{s} = \phi_{\sigma} + s \phi_{\rho} \) and \( \theta_{s} = \theta_{\rho} + s \theta_{\sigma} \) for spin up \((s = 1)\) and down \((s = -1)\) fermions. These fields obey the commutation relation \([\phi_{s}(x), \theta_{s'}(x')] = (i\pi/2) \text{sign}(x' - x) \delta_{s,s'}\). The tunneling is assumed to occur through two tunnel junctions at the points \( x = 0 \) and \( x = d \),

\[
\hat{H}_{T} = \sum_{s} T_{1} \Psi_{S1,s}^{\dagger}(x = 0) \Psi_{L,s}(x = 0) + T_{2} \Psi_{S2,s}^{\dagger}(x = d) \Psi_{L,s}(x = d) + \text{(h.c.)}.
\]  

The tunnel matrix elements \( T_{1,2} \) can be related to the tunnel conductances \( G_{1,2} \) of the junctions: \( G_{i} = (4e^{2}/h)N_{L}(0)N_{i}(0)T_{i}^{2} \), where \( N_{L}(0) = 1/\pi \hbar v_{F} \), \( N_{i} \) is the normal density of states of a superconductor \((i = 1, 2)\), and \( v_{F} \) is the Fermi velocity.
The fermionic field operators \( \hat{\Psi} \) can be expressed in terms of spin and charge degrees of freedom \[ \hat{\Psi}_{L,s}^\dagger(x, \tau) = \sqrt{\rho_{0,s}} \sum_{m, \text{odd}} \exp \{imk_F x\} \exp \{im\theta_s\} \exp \{i\phi_s\}, \] where \( k_F \) is the Fermi-wave vector and \( \rho_{0,s} \equiv N_s/L = k_F/(2\pi) \) is the average electron density for one spin direction. If the LL is confined to a ring, the latter is just the energy \( \hbar k_F \). Here, \( \Phi_0 \) is the normal flux quantum \( \hbar/e \). The fields \( \theta \) and \( \phi \) are decomposed in terms of bosonic fields and topological excitations in the following fashion:

\[
\begin{align*}
\theta_j(x) &= \bar{\theta}_j(x) + \theta_j^0 + \pi M_j(x/2L), \\
\phi_j(x) &= \bar{\phi}_j(x) + \phi_j^0 + \pi (J_j - 4\delta_{j,\rho}\Phi/\Phi_0)(x/2L). 
\end{align*}
\]

Here, \( \bar{\theta}_j \) and \( \bar{\phi}_j \) are the non-zero modes

\[
\begin{align*}
\bar{\theta}_j(x) &= \frac{i}{2} \sqrt{\frac{g_j}{2}} \sum_{q \neq 0} \frac{\pi}{qL} \left| q \right|^{1/2} \text{sign}(q) \exp \left[iq(x - \Phi/\Phi_0)\right], \\
\bar{\phi}_j(x) &= \frac{i}{2} \sqrt{\frac{2}{g_j}} \sum_{q \neq 0} \frac{\pi}{qL} \left| q \right|^{1/2} \exp \left[iq(x - \Phi/\Phi_0)\right], 
\end{align*}
\]

where \( \hat{b}_{j,q}, \hat{b}^\dagger_{j,q} \) are Bose operators. \( M_j \) and \( J_j \) denote the topological excitations. They are related to the usual topological excitations for fermions with spin \( s \): \( M_s = (1/2)[M_{\rho} + sM_{\sigma}] \) and \( J_s = (1/2)[J_{\rho} + sJ_{\sigma}] \). Using the topological constraints for \( M_s \) and \( J_s \) given in Ref.\[8\] we find constraints for \( M_j \) and \( J_j \): (i) The topological numbers \( M_j \) and \( J_j \) are either simultaneously even or simultaneously odd; (ii) when \( N_s \) is odd the sum \( M_\rho \pm M_\sigma + J_\rho \pm J_\sigma \) takes values \( \ldots, -4, 0, 4, \ldots \), when \( N_s \) is even the sum \( M_\rho \pm M_\sigma + J_\rho \pm J_\sigma \) takes values \( \ldots, -6, -2, 2, 6, \ldots \). Here, the number of electrons \( N_s (N_s = N_{s=1}) \) determines the linearization point \( k_F \equiv \pi N_s/L \) of the original electron spectrum.

The Hamiltonian is decoupled in the non-zero modes and the topological excitations:

\[
\hat{H}_L = \hbar \sum_{j=\rho,\sigma} \left\{ \sum_{q \neq 0} v_j|q| \hat{b}^\dagger_{q,j} \hat{b}_{q,j} + \frac{\pi v_j}{4L} \left[ \frac{g_j}{2} (J_j - 4\delta_{j,\rho}\Phi)^2 + \frac{2}{g_j} (M_j - 4\delta_{j,\rho}\Phi)^2 \right] \right\}. \tag{6}
\]

We introduced the flux frustration \( f_\Phi \equiv \Phi/\Phi_0 \), as well as the parameter \( f_\mu = (g_\rho L/4\pi v_\rho)\Delta \mu \), which is related to the difference \( \Delta \mu \) between the electro-chemical potential \( \mu \) of the superconducting electrodes and the Fermi energy \( E_{F,0} \) of the ring. For non-interacting electrons the latter is just the energy \( \hbar k_F^2/2M \) in the linearization point. Generally, the reference point \( \Delta \mu = 0 \) is defined from the requirements that for \( \Phi = 0 \) there are \( 2N_s \) electrons in the ground state \( (N_1 = N_{-1}) \) and the energies to add and remove the electron(s) to/from the system are the same. The electro-chemical potential \( \mu \) can be controlled by a gate-voltage.

The stationary Josephson effect can be obtained by evaluating the phase-dependent part of the free energy \( F(\chi) \): The Josephson current is given by \( I_J = -(2e/\hbar) \partial F/\partial \chi \). We expand \( F = -(1/\beta) \ln Z \) where \( Z = \text{Tr} \exp \{-\beta \hat{H} \} \) in powers of \( \hat{H}_T \); the lowest order contribution arises in \( 4^{th} \) order. It is represented by the diagram in the inset of Fig.\[2\], which
Spin-independent electron-electron interaction we should fix
small if the tunnel junctions are large compared to the Fermi wave length. For
the leading contributions to the correlation function correspond to
the Josephson current at zero temperature we find

$$n_1 \text{pair, characterized by the anomalous function } \Pi(0; d, \tau_1, \ldots, \tau_4).$$

Finally, both electrons tunnel through the first junction (amplitude $T_2^*$) and enter the superconductor $S_1$ as a Cooper pair, characterized by the anomalous function $F_{S_1}^*(\tau_1, \tau_2)$. If the distance $d$ between the junctions is much larger than the coherence length in the superconductor, the characteristic energies $\hbar v_F/d$ of the electrons propagating through the 1D system are much less than the energies $\Delta$ of excitations in superconductors. At low temperatures ($k_B T \ll \Delta$), a generic process consists of fast tunneling of two electrons from the superconductor into the 1D system ($|\tau_1 - \tau_2| \sim \hbar/\Delta$) and their slow propagation through the LL ($|\tau_1 - \tau_3| \sim d/v_F$). The phase-dependent part of the free energy then simplifies to

$$F(\chi) = -2\pi^2 N_1(0) N_2(0) \Re[T_1^2(T_2^*)^2 e^{-i\chi} \int_0^\beta d\tau \Pi(d; \tau)],$$

whith the Cooperon

$$\Pi(d, \tau) = \langle T_1 \hat{\Psi}_{L,+}(0, 0) \hat{\Psi}_{L,-}(0, 0) \hat{\Psi}_{L,-}^\dagger(d, \tau) \hat{\Psi}_{L,+}^\dagger(d, \tau) \rangle,$$

where the average is taken over the eigenstates of $\hat{H}_L$. The evaluation of (8) with the help of bosonized field operators like (3) is straightforward.

**Wire geometry.** For an infinitely long wire (Fig. 1a), the topological excitations play no role since their energies $\sim \pi v_F/L$ are vanishingly small. The Cooperon propagator $\Pi(d, \tau)$ is given by

$$\Pi_w(d, \tau) = \rho_0^2 \sum_{n_1, n_2, \text{odd}} e^{i(n_1+n_2)(k_F d + \eta)} \times \left[ \frac{\alpha^2}{d^2 + v_F^2 \tau^2} \right]^{g_\sigma(n_1-n_2)^2/16} \left[ \frac{\alpha^2}{d^2 + v_F^2 \tau^2} \right]^{1/g_\rho + g_\rho(n_1+n_2)^2/16},$$

where $e^{i\eta} = [(d + i v_F \tau)/(d - i v_F \tau)]^{1/2}$, and $\alpha$ is a cut-off parameter of the order $1/k_F$. If $n_1 + n_2 \neq 0$, the rapid oscillations related to $e^{i(n_1+n_2)(k_F d)}$ make the Josephson term vanishingly small if the tunnel junctions are large compared to the Fermi wavelength. For $n_1 + n_2 = 0$ the leading contributions to the correlation function correspond to $n_1 - n_2 = \pm 2$. For the spin-independent electron-electron interaction we should fix $g_\sigma = 2$ and $v_\rho = (2/g_\rho)v_F$. For the Josephson current at zero temperature we find

$$I_J = \frac{2\pi e v_F}{d} G_1 G_2 \frac{F_w(g_\rho, d)}{4e^2/h} F_w(g_\rho, d) \sin(\chi),$$

where

$$F_w = \left[ \frac{1}{k_F d} \right]^{2/g_\rho - 1} \int \frac{dx}{\pi} \frac{1}{\sqrt{1 + x^2}} \left[ \frac{1}{1 + (2x/g_\rho)^2} \right]^{1/g_\rho}$$

($F_w = 1$ for noninteracting electrons).

**Ring geometry.** For the ring (Fig. 1b) we get
\[ I_J = \frac{2\pi e v_F}{L} \frac{G_1 G_2}{(4e^2/h)^2} \sum_{\epsilon = \pm 1} \langle F_\epsilon(g_\rho, L, \epsilon, M, J_\rho) \sin (\chi + \epsilon \pi M/2 + \pi J_\rho/2) \rangle_{J,M}, \]  

with

\[ F_\epsilon = \left[ \frac{\pi}{k_F L} \right]^{2/g_\rho - 1} \int dx \frac{1}{\cosh(x)} \left[ \frac{1}{\cosh(2x/g_\rho)} \right]^{2/g_\rho} \cosh \left[ \left( \frac{2}{g_\rho} \right)^2 (M - 4f_\mu) x + \epsilon J_\sigma x \right]. \]  

Here, \( \langle \ldots \rangle_{J,M} \) means evaluation with respect to the ground-state configuration of the topological excitations \( J_J, M_J \) of the LL subject to the topological constraints.

In Fig. 2 the dependence of the critical current on \( g_\rho \) for the two geometries is shown. The Josephson current is suppressed by the repulsive interaction, \( I_J \propto d^{-2/g_\rho} \). We estimate the magnitude of the current, using typical numbers from the experiment of Mailly et al., to be of the order of several nA in the non-interacting case.

We discuss now the flux and gate-voltage dependence in the ring geometry. The Josephson current Eq. (13) depends on the gate-voltage via the parameter \( f_\mu \). The flux does not enter explicitly into Eq. (13). However, the Josephson current depends on \( f_\Phi \) and \( f_\mu \) implicitly via the topological numbers \( (J_\rho, J_\sigma, M_\rho, M_\sigma) \). Therefore, one can expect that the Josephson current changes stepwise as a function of the flux (the jumps correspond to the change in the topological numbers) and shows a continuous dependence (with jumps) on the gate-voltage (see Figs. 3, 4).

Without loss of generality we restrict the further consideration by odd values of \( N_\rho \). We start from the non-interacting case \( g_\rho = 2 \). The ground-state configurations \( (J_\rho, J_\sigma, M_\rho, M_\sigma) \) are displayed in Fig. 3a. Each state in the ring below the chemical potential \( \mu \) of the superconductors is occupied by two electrons with opposite spin. Hence, the topological numbers \( M_\sigma \) and \( J_\sigma \) are always zero. The increase of the flux shifts down (up) the energy levels of the electrons moving in the positive (negative) direction along the ring. Increasing \( f_\mu \) corresponds to a uniform shift of all energy levels down with respect to the chemical potential \( \mu \). A pair of electrons tunnels into (out of) the ring each time when an empty (filled) energy level crosses the chemical potential. This leads to a change in the topological number \( M_\rho, \Delta M_\rho = 2(-2) \). Simultaneously, the topological number \( J_\rho \) changes, \( \Delta J_\rho = \pm 2 \), since the system acquires orbital momentum. With the increase of the flux \( \Phi \) by one flux quantum, a pair of electrons enters the positive branch of the spectrum and (for a different value of \( \Phi \)) another pair leaves the negative branch. For each process \( \Delta J_\rho = 2 \). Accordingly, the Josephson critical current \( I_{J,c} = \text{max}_\chi |I_J(\chi)| \) (Fig. 3) changes stepwise with the flux making two jumps per period \( \Phi_0 \). One of the jumps occurs at \( f_\Phi = -f_\mu + 1/2 \) in Fig. 3.

For \( f_\rho = 0 \), the chemical potential of the superconductors \( \mu \) is in the middle of the gap between the last occupied and first unoccupied energy levels \( (E_{F,0} = \mu) \). For this reason, the two processes mentioned above occur at the same point \( f_\sigma = 1/2 \). One can say that a pair of electrons jumps from the negative branch to positive branch (e.g., the transition \( (0,0,0,0) \rightarrow (4,0,0,0) \) in Fig. 3a) and the number of electrons in the system is conserved.

The situation changes drastically for interacting electrons. The interaction lifts the degeneracy of the single particle states with different spin and the system acquires additional stiffness with respect to the change in the electron number (for the repulsive interactions \( g_\rho < 2 \) the coefficient \( 2/g_\rho \) of the term with the particle number \( M_\rho \) is larger than the coefficients of the other topological terms in Eq. (8)). For this reason, the electrons tunnel one by one...
into the ring and new ground-state configurations (e.g. \((J_\rho, J_\sigma, M_\rho, M_\sigma) = (1,1,1,1)\) and \((3,-1,11)\)) with an odd number of electrons arise (see Fig. 4a). In particular, the parity of the electron number on the ring can be changed not only by changing the gate-voltage, but also by changing the flux. Depending on the value of the gate-voltage the increase of the flux by one flux quantum is accompanied by the tunneling of two, one or zero electrons into the ring and the same number of tunneling events out of it. Accordingly, the Josephson current (Fig. 4b) shows four, two or zero jumps per period \(\Phi_0\).

The size of the intervals of the gate-voltage, where the electron number (characterized by \(M_\rho\)) does not depend on the flux, increases with the increase of the repulsive interaction. In the limit of strong interaction the number of particles is fixed for any value of the gate-voltage except for small intervals near the points \(f_\mu = 1/8 + n/4\) with integer \(n\).

We have also found that for some ground-state configurations, the Josephson current changes sign and the ring acts as a \(\pi\)-junction. This happens, for example, in the regions with \(J_\rho = 2 \mod(4)\) both in the interacting and in the non-interacting cases (e.g. in the region \((2,0,2,0)\) in Figs. 3a, 4a). Such behaviour can be tested through the interference pattern of a SQUID, consisting of the ring and a standard Josephson junction. A more detailed analysis of all these features will be presented elsewhere.\[17\]

In this letter we studied the Josephson effect through a 1D system of interacting electrons: a quantum wire and a ring. The Josephson current was found to be suppressed by the Coulomb interaction. The interaction qualitatively modifies the parity effects in the ring and effects the dependences of the critical current on the flux and gate-voltage. An anomalous sign of the Josephson current was found in some range of parameters.

**Acknowledgements** We would like to thank L. Glazman, D. Khmel’nitskii, A.I. Larkin, D. Loss, and G.T. Zimanyi for useful discussions. The financial support of the European Community (HCM-network CHRX-CT93-0136) and of the Deutsche Forschungsgemeinschaft through SFB 195 is gratefully acknowledged.
REFERENCES

1 J. Nitta et al., Phys. Rev. B 46, 14286 (1992); A. Dimoulas, J.P. Heida, B.J. van Wees, T.M. Klapwijk, W. v.d. Graaf, and G. Borghs (unpublished)

2 C.W.J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991); A. Furusaki et al., ibid. 67, 132 (1991)

3 B.J. van Wees et al., Phys. Rev. Lett. 69, 510 (1992); F.W.J. Hekking and Yu.V. Nazarov, Phys. Rev. Lett. 71, 1625 (1993)

4 D.V. Averin and K.K. Likharev, in “Mesoscopic Phenomena in Solids”, edited by B.L. Altshuler, P.A. Lee, and R. Webb (North- Holland, Amsterdam, 1991); G. Schön and A.D. Zaikin, Phys. Rep. 198, 237 (1990)

5 K.A. Matveev et al., Phys. Rev. Lett. 70, 2940 (1993); P. Joyez et al., ibid. 72, 2458 (1994); R. Bauerschmitt et al., Phys. Rev. B 49, 4076 (1994)

6 C.L. Kane and M.P.A. Fisher, Phys. Rev. Lett. 68, 1220 (1992); K.A. Matveev and L.I. Glazman, ibid. 70, 990 (1993)

7 M.P.A. Fisher (unpublished)

8 C.L. Kane and M.P.A. Fisher, Phys. Rev. B 46, 15233 (1992)

9 F.D.M. Haldane, Phys. Rev. Lett. 47, 1840 (1981); J. Phys. C 14, 2585 (1981)

10 D. Loss, Phys. Rev. Lett. 69, 343 (1992)

11 N. Byers and C.N. Yang, Phys. Rev. Lett. 7, 46 (1961). Note that the corresponding gauge transformation leads to an additional phase factor for the tunnel matrix element $T_2$: $T_2 \rightarrow T_2 \exp \{-i\pi \Phi/\Phi_0\}$ for the set-up depicted in Fig. IIb.

12 S. Fujimoto and N. Kawakami (unpublished)

13 A. Luther and I. Peschel, Phys. Rev. B 9, 2911 (1974)

14 D. Mailly et al. Phys. Rev. Lett. 70, 2020 (1993)

15 The case of even $N_s$ leads to a similar picture apart from a relative shift along the $f_\phi$ and $f_\mu$ axes.

16 This regime resembles the situation in small metallic tunnel junctions where the charging energy is much larger than the electron level spacing.

17 R. Fazio, F.W.J. Hekking, and A.A. Odintsov (unpublished)
FIGURES

FIG. 1. The geometries discussed in the text: (a) one-dimensional wire connected to two superconductors by tunnel junctions, separated by a distance \(d\). (b) Ring with circumference \(L\), threaded by a magnetic flux \(\Phi\). It is connected to two superconductors by tunnel junctions, separated by a distance \(L/2\).

FIG. 2. Critical current as a function of \(g_\rho\) for the ring (solid line, normalized to \(I_{J,c}^{(0)} = (4\pi e v_F/ L)(G_1 G_2/(4e^2/\hbar)^2))\) and the wire (dashed line, normalized to \(I_{J,c}^{(0)} = (\pi e v_F/ d)(G_1 G_2/(4e^2/\hbar)^2))\). Inset: Lowest-order contribution to the Josephson effect.

FIG. 3. (a) Ground-state configurations for topological quantum numbers \((J_\rho, J_\sigma, M_\rho, M_\sigma)\) as a function of gate-voltage \((\mu)\) and magnetic flux \((\Phi)\), for \(g_\rho = 2\). (b) Critical current at \(T = 0\) for \(g_\rho = 2\) as a function of \(\Phi\) and \(\mu\).

FIG. 4. The same as Fig. 3, but for \(g_\rho = 1.75\).