On the Equivalence of Three Complete Cyclic Systems of Integers

Wolfdieter Lang

Karlsruhe
Germany

wolfdieter.lang@partner.kit.edu

Abstract

The system of coaches by Hilton and Pedersen, the system of cyclic sequences of Schick, and Brändli-Beyne, related to diagonals in regular \(2n\)-gons, and the system of modified modular doubling sequences elaborated in this paper are proved to be equivalent. The latter system employs the modified modular equivalence used by Brändli-Beyne. A sequence of Euler tours related on Schick’s cycles of diagonals is also presented.

1 Introduction

A) Complete coach system \(\Sigma(b)\)

Hilton and Pedersen [6] found in the context of paper-folding the quasi-order theorem and introduced a complete system of coaches \(\Sigma(b)\), for \(b = 2n + 1, n \in \mathbb{N}\). Each coach \(\Sigma(b, i), \) for \(i \in \{1, 2, ..., c(b)\}\) consists of two rows of length \(r(b, i)\). There is the upper row \(A(b, i)\) with odd positive integers \(a(b, i)_j, \) for \(j = 1, 2, ..., r(b, i)\), that are relatively prime to \(b\), and the lower row \(K(b, i)\) with positive integers \(k(b, i)_j\), that are certain maximal exponents of 2. All odd numbers from the upper rows of \(\Sigma(b)\) constitute the smallest positive reduced residue system modulo \(b\) for odd numbers (here called \(RRS_{\text{odd}}(b)\), given in \(A216319\)). The recurrence for the \(a_j\)-numbers, defining also the \(k_j\) numbers, are (the arguments \((b, i)\) are suppressed)

\[
a_{j+1} = \frac{b - a_j}{2^{k_j}}, \text{ for } j \geq 1, \text{ and odd input } a_1 < \frac{b}{2}, \text{ with } \gcd(a_j, b) = 1, \tag{1}
\]

where \(2^{k_j}\) is the maximal power of 2 dividing \(b - a_j\), i.e., \(k_j\) is the 2-adic valuation of \(b - a_j\). The length of a coach is determined by the cyclic condition \(a(b, i)_{r(b, i)+1} = a(b, i)_1\) for the first time (the primitive period length is \(r(b, i)\)). The sum of the \(k\) values of the lower row is identical for all coaches of \(\Sigma(b)\): \(k(b) := \sum_{j=1}^{r(b, i)} k(b, i)_j\), for each \(i \in \{1, 2, ..., c(b)\}\). The number of coaches \(c(2n + 1)\) is given in OEIS[13] \(A135303(n)\), for \(n \geq 1\), and \(k(2n + 1)\), called the quasi-order of 2 \((\text{mod } (2n + 1))\), is given in \(A003558(n)\).

\(^{1}\)http://www.itp.kit.edu/~wl
Example 1. ([6], p. 262)

\[
\Sigma(65) = \begin{bmatrix}
1 & 3 & 31 & 17 & 7 & 29 & 9 & 11 & 27 & 19 & 23 & 21 \\
6 & 1 & 1 & 4 & 1 & 2 & 3 & 1 & 1 & 1 & 1 & 2
\end{bmatrix}.
\] (2)

In the following a coach is written like \(\Sigma(65, 2) = [A(65, 2), K(65, 2)] = [[3, 31, 17], [1, 1, 4]].\) Here \(c(65) = 4, k(65) = 6,\) and the \(r\)-tuple is \(\{r(65, 1), ..., r(65, 4)\} = \{1, 3, 3, 5\}.\)

\(k(b)\) satisfies the quasi-order theorem ([6], p. 102 and p. 261)

\[2^{k(b)} \equiv (-1)^{r(b,i)} \pmod{b}, \text{ for each } i \in \{1, 2, ..., c(b)\}.\] (3)

The \(r\)-tuples for \(b = \{3, 5, ..., 41\}\) are shown in A332434, with \((-1)^{r(2n+1)}\) given in A332433\((n), \text{ for } n \geq 1.\) The coach theorem is ([6], p. 262)

\[c(b)k(b) = \frac{\varphi(b)}{2}, \text{ for } b = 2n+1, n \in \mathbb{N},\] (4)

where Euler’s totient is \(\varphi = \text{A000010}.\) In the example \(4 \cdot 6 = 24 = \frac{\varphi(65)}{2} = \frac{20}{2}.\)

B) Complete cycle system SBB(b)

i) In Schick’s book [16] a geometric algorithm in a unit circle is proposed which leads to periodic integer sequences with the recurrence relation for \(\{q_j(b) = q_j(b; q_0)\}\) (in [16] \(p\) is used instead of the present odd \(b\)\)

\[q_j(b) = b - 2|q_{j-1}(b)|, \text{ for } j \in \mathbb{N},\] (5)

with a certain odd input \(q_0,\) with \(gcd(q_0, b) = 1,\) starting with \(q_0(1) = (-1)^{\frac{k+1}{2}}\). This sequence of distinct odd numbers is periodic with length of the (principal) period \(pes(b)\) (\(pes\) stands in [16] for \(periode \text{ spéctrale},\) because the construction is called \(Spektalalgorithmus\)).

The recurrence leads to identical period length for each input \(q_0.\) If not yet all odd members from \(RRS\odd(b),\) with \(#RRS\odd(b) = \delta(b) = \text{A055034}(b) = \varphi(b)/2,\) are present then a new periodic sequence with the smallest missing member of \(RRS\odd(b)\) as input \(q_0(2)\) is considered (where the sign is \((-1)^{\frac{k+1}{2}}(-1)^{2q_0(2)-1}\), etc., until all elements of \(RRS\odd(b)\) appear. The number of such disjoint periodic sequences with odd members coprime to \(b\) is in [16], Korollar, p. 14, called \(B.\) The recurrence shows that the primitive period length \(pes(b)\) satisfies the same formula like \(k(b)\) from the coach system, and it divides \(\varphi(b)/2.\) This leads to Schick’s theorem, and \(B(b) = c(b),\) the number of coaches, in accordance with the coach theorem 4.

\[B(b)pes(b) = \frac{\varphi(b)}{2}.\] (6)

See the tables 3.1 to 3.10 in [16], pp. 158 - 166 for these signed periodic sequences. These cycles will be called here \(SBB(b, i)\) for \(i \in 1, 2, ..., B(b)\) for the different inputs \(q_0(b, i).\)

The \(BB\) in the \(SBB\) notation comes from the following part ii) on the Brändli-Beyne paper using unsigned sequences. The complete cycle system is then \(SBB(b).\)
Example 2. ([16], p. 161)

\[ SBB(65) = \{(−1, 63, −61, −57, −49, −33), (3, 59 − 53, −41, −17, 31), \]
\[ (7, 51, −37, −9, 47, −29), (11, 43 − 21, 23, 19, 27)\}. \]

(7)

Here \( B(65) = 4, \) \( \text{pes}(65) = 6, \) and \( 4 \cdot 6 = \frac{\varphi(65)}{2} = 24. \) Note that \( \delta(b) = \frac{\varphi(2b)}{2} = \frac{\varphi(b)}{2}. \)

ii) In the paper by Brändli-Beyne [3] unsigned Schick sequences are considered (we use the same symbols \( q_j = q_j(b, i) \) but henceforth as positive integers)

\[ q_j(b, i) = |b − 2q_{j−1}(b, i)|, \quad \text{for} \quad j \in \mathbb{N}, \]

with certain positive odd inputs \( q_0(b, i) \), for \( i \in \{1, 2, ..., B(b)\} \), and \( \gcd(q_0(b, i), b) = 1 \), as explained above. The complete system of (sign-less) cycles is also named \( SBB(b) \) (without risk of confusion).

The numbers in these cycles correspond to certain length ratios diagonal/radius in a regular \((2b)\)-gon, viz \( d_j^{(2b)} = 2 \sin \left( \frac{\pi}{2b} j \right) \), for \( j \in \{1, 2, ..., 2b − 1\} \). The diagonals connect the vertices \( V_j^{(2b)} \), for \( j \in \{0, 1, ..., 2b\} \), with \( V_0^{(2b)} \) (Cartesian coordinates \((0, r)\), with radius \( r \) of the circumscribing circle). The vertices are taken in the positive (counter-clockwise) sense, starting with the length ratio for the side \( s(2b)/r = d_1^{(2b)} \).

The diagonals in the upper half plane (including the real axis) are labeled with \( j \in \{1, 2, ..., b\} \). The (odd) numbers \( k \) of each cycle stand for the labels of the diagonals \( d_k^{(2b)} \).

Example 3. For \( b = 17 \) the two cycles are \((B(17) = 2, \) \( \text{pes}(17) = 4)\)

\[ SBB(17, 1) = (1, 15, 13, 9), \quad \text{and} \quad SBB(17, 2) = (3, 11, 5, 7). \]

(9)

The elements of \( \text{RRS}_{\text{odd}}(17) \) are all odd numbers from 1 to 15.

The length ratios from the first cycle are (Maple [10] 10 digits): \( d_1^{(34)} \approx .1845367190, \) \( d_1^{(34)} \approx 1.965946199, \) \( d_3^{(34)} \approx 1.864944459, \) and \( d_9^{(34)} \approx 1.478017835. \) See Figure 1.

For \( b = 11, \) with \( B(11) = 1 \) and \( \text{pes}(11) = 5 \) see [3], Figure 1, with the diagonals called \( \sigma_{2n+1} \), for \( n \in \{1, 2, ..., 5\} \).

Each of the \( B(b) \) cycles, for each odd \( b \geq 3 \), interpreted as length ratios for diagonals, suggests to consider a trail in the \((2b)\)-gon by following the \( \text{pes}(b) \) diagonals in the order of the cycle repeatedly, starting from node (vertex) \( V_0^{(2b)} \) in the positive sense until a periodic sequence, an oriented Euler tour, is completed. That this is always possible will be proved in the next section. Periodicity will be reached after \( L(b, i) \) steps. For \( b = 7 \) see Figure 2.

Because of periodicity one can start at any vertex which is reached in the tour starting with vertex \( V_0^{(2b)} \). In this example all 14 vertices are visited.
Figure 1: The diagonals in the 34-gon for $b = 17$ for cycle $(1, 15, 13, 9)$ (solid blue), and $(3, 11, 5, 7)$ (dashed green).

Figure 2: The counter-clockwise Euler tour for $b = 7$, with $14 \cdot 3 = 42$ arrows in the 14-gon. Start at the vertex with label 0. The color sequence for the arrows is blue, green, magenta, black, yellow, steel blue, and a final red arrow pointing back to label 0. The colors are given in order to follow the trail without looking at the sequence of the 42 vertex labels given later in section 2. For an enlarged version see a link in A332439.

Just for illustration, not for following the 68 arrows in both cases in detail, see the Figures 3 and 4 for the two Euler tours for $b = 17$. All 34 vertices are visited, in fact twice.
**Figure 3:** 68 arrows in the 34-gon for the directed *Euler* tour for \( b = 17 \) generated from the first cycle \((1, 15, 13, 9)\) of diagonals.

**Figure 4:** 68 arrows in the 34-gon for the *Euler* tour for \( b = 17 \) generated from the second cycle \((3, 11, 5, 7)\) of diagonals.

The underlying digraph with \( 2b \) nodes (vertices) and \( L(b, i) \) directed edges (arrows) which are trailed only once in the tour from, say, node \( V_0^{2b} \), is simple (neither parallel arrows nor loops), but may not be regular (nodes may be of different order). Also, not all of the \( 2b \) nodes may be involved in the tour. If one keeps the remaining nodes one will have an unconnected digraph of connectivity number \( \mathcal{N}(b) - 1 \). where the number of nodes of the *Euler* tour is \( \mathcal{N}(b) \), Alternatively, one can discard the remaining nodes in order to have a connected digraph. For example, for \( b = 21 \) with \( B(21) = 1 \) only 21 nodes, not 42, are involved in the tour, \( \mathcal{N}(21) = 21 \), and it is also an irregular graph. See **Figure 5**.

**Figure 5:** 42 arrows in the 42-gon for the directed *Euler* tour for \( b = 21 \) generated from the cycle \((1, 19, 17, 13, 5, 11)\). Only 21 nodes are visited. The order of these nodes is 2, 6, 4 for the labels 0, 1, 2 (mod 6), respectively.

In the next section a theorem on the length of the primitive periods of these *Euler* tours will be given. Then, in *Section 3*, the third complete system of cyclic sequences, the *MDS*(\( b \)) system (for Modified Diagonal Sequences), for odd \( b \geq 3 \), will be treated. This system is simpler than the mentioned ones, and will be proved to be equivalent to each of the other two systems in *Section 4*. A modified multiplicative modular relation will be employed, which has already been introduced in the Brändli and Beyne paper \([3]\). This modification of the doubling sequence with powers of 2 has already been used in the proof of the quasi-order theorem in \([6]\), p. 103, equation (7.12), and has also been proposed by Gary W. Adamson in a comment, Aug 20 2019, to *A003558*. 
2 Schick sequences and Euler tours

The cycles $SBB(b, i)$ for the $B(b) = \text{A135303}((b - 1)/2)$ different input values $q_0(i)$ ($B$ is identical with the coach number $c$), with period length $pes(b) = \text{A003558}((b - 1)/2)$ (identical with the quasi-order of 2 modulo $b$ of the coach system, called $k(b)$), define by their odd elements length ratios diagonal/radius in a regular $(2b)$-gon, as explained above. Following a trail from, say, vertex $V_{0}^{(2b)}$ in the positive (counter-clockwise) sense, defined by a repetition of the diagonals according to the cycle numbers, leads, as will be proved in the theorem, to a positively oriented Euler tour, named $ET(b, i)$, of length $L(b, i)$, with the given formula 10.

Theorem 4.

1. The trail on a regular $(2b)$-gon, with odd $b \geq 3$, starting at node (vertex) $V_{0}^{(2b)}$, with label 0 and arrows (directed diagonals) following repeatedly the cycle $SBB(b, i)$ of length $pes(b)$, for any $i \in \{1, 2, ..., B(b)\}$, in the positive sense, will lead to a periodic sequence of node labels with primitive length of the period $L(b, i)$. Therefore, this trail will be a counterclockwise oriented Euler tour $ET(b, i)$.

2. The formula for the length of $ET(b, i)$ is

$$L(b, i) = \frac{2bpes(b)}{\gcd(SUM(SBB(b, i)), 2b)},$$

where $SUM(SBB(b, i))$ is the sum of the elements of the (unsigned) cycle $SBB(b, i)$.

Proof.

1. Starting from the node (vertex) with label 0 the trail visits, in order, the nodes with labels given by the partial sums of the infinite periodic sequence $SBB(b, i)_{\infty} = \text{repeat}(SBB(b, i))$, evaluated modulo $2b$. After the first block of length $pes(b)$, starting with 0, the second block of this length starts with $SUM(SBB(b, i))$. Each new block starts then with a multiple of $SUM(SBB(b, i))$. Therefore the trail will become a periodic Euler tour if after some $m = m(b, i)$ blocks the next block would begin again with 0 modulo $2b$. This means that $SUM(SBB(b, i)) m \equiv 0 \pmod{2b}$. Because this linear congruence has always at least one solution periodicity is guaranteed.

2. The length $L(b, i)$ of the Euler tour is then $m(b, i)pes(b)$. The solution of a linear congruence is known (see e.g., [2], Ch. 5.3, pp. 110 - 113), and in the present case it is the trivial solution $m = 0$ if $g := \gcd(SUM(SBB(b, i)), 2b) = 1$, and if $g \geq 2$ there is besides the trivial solution at least one other solution in $\{1, 2, ..., 2b - 1\}$. If $g = 1$ the second appearance of the 0 at the beginning of a block will be after $2b$ blocks, and $L(b, i) = 2bpes(b, i)$. If $g \geq 2$ the smallest non-trivial solution is $(2b)/g$, and for this $m$ value periodicity appears for the first time.
The sum of numbers of the unsigned Schick cycles $SBB(2n + 1, q_0 = 1)$ in the case of \( B(2n+1) = \text{A135303}(n) = 1 \) are given in \( \text{A333848}(n) \), for \( n \geq 1 \). The corresponding numbers \( \gcd(SUM(SBB(b, 1)), 2b) \) are given in \( \text{A333849} \).

The \( b \) numbers with \( B(b) \geq 2 \) are listed increasingly in \( \text{A333855} \), and their \( B \) numbers are in \( \text{A333853} \). For these numbers \( b = 2n + 1 \) the sums of the cycles \( SBB(b, i) \), for \( i \in \{1, 2, ..., B(b)\} \), are given in the table \( \text{A333850} \) (where \( k \) is used instead of \( i \)). The corresponding \( \gcd(SUM(SBB(b, i)), 2(2n + 1)) \) values are given in table \( \text{A333851} \).

Note that the \( \gcd(SUM(SBB(b, i)), 2b) \) values are not independent of \( i \). See \( \text{A333851} \) for the first cases with \( b = 65, 133, ... \).

**Example 5. Euler tour \( ET(7, 1) \)**

\( b = 7, B = 1, \text{pes}(7) = 3, SBB(7, 1) = [1, 5, 3], \text{SUM}(SBB(7, 1)) = 9, \gcd(9, 2 \cdot 7) = 1, \text{length } L(7, 1) = \frac{2 \cdot 7 \cdot 3}{1} = 42 \)

\( ET(7, 1) = [0, 1, 6, 9, 10, 1, 4, 5, 10, 13, 0, 5, 8, 9, 0, 3, 4, 9, 12, 13, 4, 7, 8, 13, 2, 3, 8, 11, 12, 3, 6, 7, 12, 1, 2, 7, 10, 11, 2, 5, 6, 11] \).

*The corresponding digraph is shown in Figure 2, and enlarged as a link in \( \text{A332439} \). There are 42 arrows, and all 14 nodes are visited thrice (a regular graph of order 6).*

### 3 Interludium on Schick’s cycles for primes

Special attention is paid in Schick's book to prime \( b \) (remember that he uses \( p \) for odd numbers \( \geq 3 \)). The following theorem will show that \( \text{A268923} \) gives in fact the odd prime numbers with \( B \geq 2 \). This will be done by proving that the complement relative to the set of odd primes is \( \text{A216371} \) (the odd primes with \( B = 1 \)). In this proof the identity, equation 6 is used, and \( 2 \text{pes}(\text{prime}(n)) = \text{order}(2, 3 \text{prime}(n)) \), for \( n \geq 2 \), is derived.

**Theorem 6.**

1. \( 2 \cdot \text{order}(4, \text{prime}(n)) = \text{order}(2, 3 \text{prime}(n)), \text{ for } n \geq 2. \)

2. \( \text{pes}(\text{prime}(n)) = \text{order}(4, \text{prime}(n)), \text{ for } n \geq 2. \)

3. The set of all odd primes \( p \) such that all odd primes \( q \), with \( q < p \), satisfy

\[
\frac{\varphi(pq)}{2} / \text{order}(2, pq) > 1 \text{ is equal to the set of odd primes satisfying } B(\text{prime}) \geq 2.
\]

**Proof.**

1. It will be proved for odd primes \( p \) that \( 2^{2 \text{order}(4, p)} - 1 = 3kp \), with integer \( k \), and that this exponent is the least one \( \geq 1 \). Clearly \( 4^{\text{order}(4, p)} - 1 = k'p \) with some integer \( k' \) by definition of \( \text{order}(4, p) \), and this exponent is the least one \( \geq 1 \). Also, \( 2^n - 1 = (1 + 3)^n - 1 \equiv 0 \pmod{3} \), certainly for any positive \( n \), by the binomial theorem. Hence \( k' = 3k \). See \( \text{A082654}(n) = \text{order}(4, \text{prime}(n)), \text{ for } n \geq 2. \)
2. The Jonathan Skowera formula, Jun 29 2013, in \texttt{A003558} shows that \( \text{A003558}(\hat{n}) = \text{pes}(2\hat{n} + 1) \), for \( \hat{n} \geq 1 \). Hence for \( \text{prime}(n) = 2\hat{n} + 1 \), with \( \hat{n} = \hat{n}(n) \), one has \( \text{pes}(\text{prime}(n)) = \text{A003558}(\hat{n}(n)) \), for \( n \geq 2 \).

The direction \( \Rightarrow \) is then trivial from the definition of \texttt{A003558}(\hat{n}(n)) by just squaring. The direction \( \Leftarrow \) with \texttt{A082654}(n), for \( n \geq 2 \), means that \((2^{k(n)} + 1)(2^{k(n)} - 1) \equiv 0 \pmod{\text{prime}(n)}\). But this implies \( \text{prime}(n) \) divides one of the factors, hence \( 2^{k(n)} \equiv \pm 1 \pmod{\text{prime}(n)} \), for \( n \geq 2 \), with minimal \( k(n) > 1 \); this is the definition from \texttt{A003558}.

3. It is proved that the complement of \texttt{A268923} relative to the set of odd primes is \texttt{A216371}. For the complement one tries to find all odd primes \( p \) such that an odd prime \( q < p \) exists with \( \text{order}(2, pq) = 1 \).

As such a prime \( q = 3 \) is chosen. From the Schick’s theorem eq. 6 one knows that \( B(p) = \frac{p - 1}{2 \text{pes}(p)} \), for \( p \geq 3 \). With parts 1 and 2, just proved, this becomes \( B(p) = \frac{p - 1}{\text{order}(2, 3p)} \).

In addition a corollary of Theorem 64, p. 106, by Nagell [11] (or the Theorem 10.8, p. 211, of Apostol [2]) is needed, namely the improvement of the Euler-Fermat theorem for certain non-exceptional numbers (here \( pq \), with \( q < p \)): \( 2^{\varphi(pq)/2} \equiv 1 \pmod{pq} \), where \( \varphi = \texttt{A000010} \), Euler’s totient function. It is then clear that \( \text{order}(2, pq) \) has to divide \( \varphi(pq)/2 \). For \( q = 3 \) this becomes \( \frac{p - 1}{\text{order}(2, 3p)} \geq 1 \) for all \( p \geq 5 \).

From application of Schick’s theorem equation (6) it is clear for which odd primes \( p \geq 5 \) equality holds, namely for those with \( B(p) = 1 \). Hence the complement of \texttt{A216371} is 3, and the primes \( p \) with \( B(p) = 1 \), which is \texttt{A216371}. Therefore \texttt{A268923} gives the odd primes \( p \) with \( B(p) \geq 2 \).

\[ \square \]

4  Complete cycle system of modified modular doubling sequences MDS(\( b \))

A) Modified modular congruence (mod* \( n \))

This modified modular congruence considered by Brändli and Beyne can be formulated based on ordinary reduced residue systems modulo \( n \), whose smallest non-negative version is given by the set named RRS(\( n \)) shown in \texttt{A038566} (but with \( 0 \) for \( n = 1 \)). \( \text{RRS}(n) = \{r_1, r_2, ..., r_{\varphi(n)}\} \), with \( r_j \) representing the residue class \( _n r_j \), and \( \varphi(n) = \# \text{RRS}(n) = \texttt{A000010}(n) \), for \( n \in \mathbb{N} \). E.g., \( \text{RRS}(4) = \{1, 3\} \), with representative \( 4 \Gamma \) consisting of all integers \( a = 1 + 4k \), with \( k \in \mathbb{Z} \), i.e., \( a \equiv 1 \pmod{4} \), and similarly for \( 3 \). Thus \( \text{RRS}(4) \) represents all odd integers (the union of the sets congruent to \( 1 \) and \( 3 \) modulo \( 4 \)).
**Definition 7.** The smallest non-negative reduced residue system mod* \( n \) is denoted by \( \text{RRS}^*(n) \), and this is \( \text{RRS}^*(1) = \{0\} \), \( \text{RRS}^*(2) = \{1\} \), and \( \text{RRS}^*(n) \) consists of the elements of the first half of \( \text{RRS}(n) \), for \( n \geq 3 \).

Each element \( r_j^* \in \text{RRS}^*(n) \) represents the reduced residue class denoted by \( \overline{n_{r_j}^*} \), and these classes are given in terms of ordinary ones by

\[
1\overline{0}^* = 1\overline{0}, \quad 2\overline{1}^* = 2\overline{1}, \quad n\overline{r}_j^* := \{n\overline{r}_j\} \cup \{n(-r_j)\}, \quad \text{for } j = 1, 2, ..., \frac{\varphi(n)}{2}, \quad \text{and } n \geq 3. \tag{11}
\]

The number of elements of \( \text{RRS}^*(n) \) is \#\( \text{RRS}^*(n) =: \sigma(n) = \text{A023022}(n) \), but with extra \text{A023022}(1) = 1. Note that the mod* residue classes represent together the same set of integers as the ones of \( \text{RRS}(n) \), but \( \text{RRS}^*(n) = \text{RRS}(n) \) only for \( n = 1 \) and \( n = 2 \).

E.g., \( n = 4 \): \( \text{RRS}^*(4) = \{1\} \), and the reduced residue class \( \overline{4\overline{1}^*} \) consists of the union of the ordinary reduced residue classes \( \overline{4\overline{1}} \) and \( \overline{4\overline{3}} \), hence \( \text{RRS}^*(4) \neq \text{RRS}(4) \), but represents the same integers as \( \text{RRS}(4) \), namely the odd ones (see above).

It is elementary that the elements of \( \text{RRS}^*(2m) \) are only odd (and reduced) integers, and \( \text{RRS}^*(2^m) \) represents all odd numbers, for each \( m \in \mathbb{N} \).

If in \( \text{RRS}^*(2m - 1) \), with \( m \in \mathbb{N} \), all even integers are discarded, the remaining subsets of the residue classes could be called \( \text{RRS}^* \text{odd}(2m - 1) \). But it is clear that \( \text{RRS}^* \text{odd}(2m - 1) \) represents the same odd integers as \( \text{RRS}^*(2(2m - 1)) \).

The equivalence relation \( a \overset{\text{mod}^*}{\sim} b \) for integer numbers \( a \) and \( b \), coprime to \( n \in \mathbb{N} \), is defined by \( a \in \overline{n_{r_j}^*} \) and \( b \in \overline{n_{r_j}^*} \), with \( r_j^* \in \text{RRS}^*(n) \), for some \( j \in \{1, 2, ..., \sigma(n)\} \). This is based on the fact that the restricted residue classes mod* \( n \) are pairwise disjoint and cover the integers coprime to \( n \).

The notation \( \text{mod}^*(a, n) \) gives \( r_j^* \in \text{RRS}^*(n) \), if \( a \in \overline{n_{r_j}^*} \).

It is defined by

\[
\text{mod}^*(a, n) := \begin{cases} 
\text{mod}(a, n) & \text{if } \text{mod}(a, n) \leq \frac{n}{2}, \\
\text{mod}(-a, n) & \text{if } \text{mod}(a, n) > \frac{n}{2},
\end{cases} \tag{12}
\]

where \( \text{mod}(a, n) \in \{0, 1, ..., n - 1\} \) (0 can appear only for \( n = 1 \) because \( a \) is reduced).

**Example 8.**

1. \( \text{mod}^*(17, 9) = 1 \) because \( \text{mod}(17, 9) = 8 \) and \( \text{mod}(-8, 9) = 1 \).

2. \( \text{mod}^*(4, 9) = 4 \) because \( \text{mod}(4, 9) = 4 < 4.5 \).

3. \( \text{mod}^*(5, 9) = 4 \) because \( \text{mod}(5, 9) = 5 > 4.5 \).

Because \( \text{mod}^* n \) uses numbers coprime to \( n \) it is not additive for \( n \geq 2 \), contrary to \( \text{mod} n \).

If \( \gcd(a, n) = 1 = \gcd(b, n) \) then \( \gcd(a + b, n) \neq 1 \) in general if \( n \geq 2 \). Because \( \text{mod}^*(a, n) = \text{mod}^*(-a + n, n) \), even if \( \gcd(a + b, n) = 1 \) (e.g., \( n = 9, a = 1, b = 4 \)) \( \text{mod}^* \) cannot be well defined (e.g., \( 4 \overset{\text{mod}^*}{\sim} 5 \), and \( \gcd(1 + 5, 9) = 3 \)).
However, mod*, like mod, is multiplicative. This is based on the lemma that if gcd(a, n) = 1 and gcd(b, n) = 1 then gcd(a b, n) = 1 (see, e.g., the simple proofs in [4], p. 30, Exercise 16 (a), or [12], p. 8, Satz 1.8). Also, gcd(a (-a + n), n) = 1 provided gcd(a, n) = 1.

That each reduced residue system mod*n, with σ(n) ≥ 2, forms an Abelian multiplicative group of degree σ(n) follows from mod n properties. The identity element is the class congruent to n1. If the system RRS*(n) is used this is the element 1. That a unique inverse element exists is based on a theorem on the unique solution of the linear congruence ax ≡ b (mod n) if gcd(a, n) = 1 (see, e.g., [12], p. 41, Satz 2.13, [2], p. 114, Theorem 5.20), which generalizes to the mod* case. The cases with σ(n) = 1 are trivial one element groups.

In [3] this multiplicative group of representatives of RRS*(n) has been considered and named G∗ n. For σ(n) ≥ 2 this is a direct product of cyclic groups Ck of order k, because it is a finite Abelian group (see e.g., [15], Satz 43 and Satz 42 p. 49 and 47).

If n is a product of two odd primes the smallest noncyclic group appears for n = 65 = 5 · 13 with the cycle structure 122 62 (meaning 2 cycles of length 12 and also of length 6, not regarding subcycles), namely (3, 9, 27, 16, 17, 14, 23, 4, 12, 29, 22, 1), (11, 9, 31, 16, 19, 14, 24, 4, 21, 29, 6, 1), (2, 4, 8, 16, 32, 1) and (7, 16, 18, 4, 28, 1). See [3], Theorem 3, p. 10, for the criterion on cyclic mod* groups for odd n. G∗ 65 is the group C4 × C3 × C2. For the cycle graph see [8], Figure 4, the 7th plot, where cycle groups are called Zk (as a side remark note that the modified modular congruence Modd used there in the context of regular n-gons and diagonal/side ratios is related to a set of representatives, called there Mn (equation (65)) that is different from RSS*(n) considered here).

In a further paper Brändli and Beyne, with two collaborators [9], studied polynomials which have as Galois group G∗ n, now called Z∗ n/2 = Z∗ n/(n - 1). They call these polynomials Ψn re (x) (re stands for reduced), and give the zeros in terms of sk(n) := 2 cos(2 π k/n)

for k values from RRS*(n), and n ≥ 1. This can be rewritten in terms of the algebraic number ρ(n) := 2 cos(π/n) of degree δ(n) = A055034(n), which is fundamental for regular n-gons (see [8]), as sk(n) = R(2 k, ρ(n)), with the monic Chebyshev polynomials R shown in A127672. The minimal polynomials of ρ(n), called C(n, x) in [8] (see also Table 2 there) are used in order calculate the Galois polynomials Ψn re (x) modulo C(n) (in order to reduce powers of ρ(n) with exponents larger than δ(n) − 1).

These polynomials are then (the use of C(n, x = ρ(n)) is indicated by the vertical bar)

Ψn re (x) = \prod_{j=1}^{σ(n)} (x - R(2 RRS*(n), j, ρ(n)))_{C(n, ρ(n))=0}, for n ∈ \mathbb{N}.

(13)

It is not surprising that these are the minimal polynomials of the algebraic number

2 cos(2 π/n) = R(2, ρ(n)) = ρ(n)^2 - 2 of degree σ(n), and they are given in A232624, where they are named MPR2(n, x).
Example 9.

\[ \Psi_{65}^{(5)}(x) = MPR2(65, x) = 1 - 12x - 180x^2 - 101x^3 + 2085x^4 + 1802x^5 - 9126x^6 - 7168x^7 + 20886x^8 + 13653x^9 - 28667x^{10} - 15001x^{11} + 25284x^{12} + 10282x^{13} - 14822x^{14} - 4540x^{15} + 5832x^{16} + 1292x^{17} - 1521x^{18} - 229x^{19} + 252x^{20} + 23x^{21} - 24x^{22} - x^{23} + x^{24}. \]  

(14)

B) Complete MDS(b) system

Instead of the two lines \( \{ \text{sequences are used, namely } MDSseq(i, j) \} \) and length \( r(b, i) \) in the complete coach system \( \Sigma(b) \) of Hilton and Pedersen [6] from section (A) of the Introduction, and the periodic positive integer sequence \( \{q(b, i)_j\} \) with \( i \in \{1, 2, \ldots, c(b)\} \) and length \( \{r(b, i) \} \) of the \( \Sigma(b) \) by Schick [16] and Brändli-Beyne [3] from section (B) \( \text{i) and ii) of the Introduction} \) periodic doubling sequences are used, namely \( \{\mod^*(a(b, i) 2^i, b)\} \), with certain odd numbers \( a(b, i) \) coprime to \( b \), where \( i \in \{1, 2, \ldots, c^*(b)\} \) and \( j \in \{1, 2, \ldots, pes(b)\} \).

It will turn out that \( c^*(b) = c(b) = B(b) \), and the length of the period \( P(b) \), independent of \( i \), is identical with \( k(b) = pes(b) \).

The short abbreviation \( MDS \) is used instead of \( MMDS \), for Modified Modular Doubling Sequence.

Definition 10.

1. A doubling sequence \( DSseq(b, i) \), for \( i \in \{1, 2, \ldots, c^*(b)\} \), is defined by

\[ \{\mod(a(b, i) 2^i, b)\}_{j \geq 1}, \text{ for } b = 2n + 1, \text{ with } n \in \mathbb{N}, \text{ and some odd element } a(b, i) \in RRS^*(b). \]

2. A modified modular doubling sequence \( MDSseq(b, i) \), for \( i \in \{1, 2, \ldots, c^*(b)\} \), is defined by \( \{\mod^*(a(b, i) 2^i, b)\}_{j \geq 1}, \text{ for } b = 2n + 1, \text{ with } n \in \mathbb{N}, \text{ and some odd element } a(b, i) \in RRS^*(b). \)

Note that the input \( a(b, i) \) appears not as first number (for \( j = 1 \)) of these sequences. In the \( MDSseq(b, i) \) case it will be shown to appear at the end of the first period (as it would also in the other sequence with a doubled length of the period).

The odd members of \( RRS^*(b) \) are denoted by \( RRS^{\text{odd}}(b) \). \#\( RRS^{\text{odd}}(b) \) = \( A332435((b - 1)/2) \), and \#\( RRS^{\text{even}}(b) \) = \( A332436((b - 1)/2) \).

Example 11. \( RRS^{\text{odd}}(33) = \{1, 5, 7, 13\}, MDSseq(33, 1) = \{\text{repeat}(2, 4, 8, 16, 1)\}, MDSseq(33, 2) = \{\text{repeat}(10, 13, 7, 14, 5)\}. \)

The two sequences of the example are periodic with the same length of period, i.e., \( P(33, 1) = P(33, 2) = 5 \).

In general one starts with input \( a(b, 1) = 1 \), then if not all odd numbers from \( RRS^{\text{odd}}(b) \) are present in the period of the sequence one takes as next input, \( a(b, 2) \), the smallest missing number in \( RRS^{\text{odd}}(b) \), in this example 5, etc., until all odd numbers from \( RRS^{\text{odd}}(b) \).
are present, and then the system $MDS(b)$ of $c^*(b)$ sequences is complete. In the example only two sequences are needed ($c^*(33) = 2$). This procedure is similar to the one of $SBB(b)$ systems with the different initial values for $q_0(b, i)$.

After it has been proved that $MDS_{seq}(b, i)$ is always periodic with the same period length for different $i$, hence called $P(b)$, the primitive period will be denoted by $MDS(b, i)$.

**Proposition 12. Complete $MDS(b)$ system**

1. Each sequence $MDS_{seq}(b, i)$ is periodic, and the length of the primitive period $MDS(b, i)$ is independent of any input $a(b, i) \in RRS^{\ast}\text{odd}(b)$, for $b \geq 3$. This length of the period will be called $P(b)$.

2. $P(b) = k(b) = pes(b) = A003558(b)$.

3. $MDS(b, i)_{P(b)} = a(b, i)$.

4. $\bigcup_{i=1}^{c^*(b)} MDS(b, i) = RRS^*(b)$.

5. $c^*(b) P(b) = \frac{\varphi(b)}{2}$. Hence $c^*(b) = c(b) = B(b) = A135302(b)$, for $b \geq 3$.

**Proof.**

1. The condition for the primitive length of the period $P(b, i)$ of $MDS_{seq}(b, i)$, if existing, is $\text{mod}^* (a(b, i) 2^{P(b, i) + 1}, b) = \text{mod}^* (a(b, i) 2^1, b)$ for the smallest $P(b, i) > 0$. Because $0 < a(b, i) < \left\lfloor \frac{b-1}{2} \right\rfloor$, hence $\text{mod}(a(b, i), b) = a(b, i)$, it follows that $\text{mod}^*(a(b, i), b) = a(b, i)$. Multiplicativity of $\text{mod}^*$ tells that the condition is $\text{mod}^*(2^{P(b, i)}, b) = 1$, with $1 \in RRS^*(b)$, representing the unique identity element of $\text{mod}^*$. Hence a solution exists, because there is always a solution for \( \text{mod}(2^{P(b, i)}, b) = +1 \), namely the order $(2, b)$. But this $P(b, i)$ may not give the primitive length of the period (see the next part). Therefore $MDS_{seq}(b, i)$ is certainly periodic for each $i$, independently of $a(b, i)$.

2. $\text{mod}^*(2^{P(b)}, b) = 1$ means that $2^{P(b)} \equiv +1 \pmod{b}$ or $2^{P(b)} \equiv -1 \pmod{b}$. For the primitive length of the period the least $P(b) > 0$ has to be chosen. This is the same congruence problem as the one for $k(b)$ of the coach system $\Sigma(b)$, and also for $pes(b)$ of the $SBB(b)$ system.

3. $MDS(b, i)_{P(b)} = \text{mod}^*(a(b, i) 2^{P(b)}, b)$. Because $0 < a(b, i) \leq \frac{b-1}{2}$, and if $\text{mod}(2^{P(b)}, b) = +1$ then $\text{mod}^*$ becomes mod, and the result is obviously $a(b, i)$. If $\text{mod}(2^{P(b)}, b) = -1$ then $\text{mod}(-a(b, i), b) \geq \frac{b-1}{2}$ and the result is $\text{mod}(+a(i, b), b) = a(i, b)$.

4. By construction of $MDS(b)$ all odd numbers of $RRS^*$ appear once in the union of $MDS(b, i)$, for $i = 1, 2, ..., c^*(b)$. Each even number $2k$ of the complete set $MDS(b)$ has a smaller odd ancestor $A000265(2k)$ (its odd part) in some cycle $MDS(b, i)$ with a certain odd initial value. Every even element of $RRS^*$, always $\leq \frac{b-1}{2}$, is in one of the $c^*(b)$ cycles of $MDS(b)$ because its odd ancestor, always $\leq \frac{b-1}{2}$, has to be in precisely one cycle $MDS(b, i)$, with a certain odd initial value.
5. A corollary of 1., 4. and 2. because \( \#RRS^*(b) = \sigma(b) = \frac{\varphi(b)}{2} \).

\[
\sum_{i=1}^{c^*(b)} P(b, i) = c^*(b) P(b) = \#RRS^*(b).
\]

Then 2. together with eqs. 4 and 6 are used.

For the cycle system \( MDS(b) \) for \( b = 2n + 1 \), with \( n = 1, 2, \ldots, 35 \), see Table 2, and also A334430.

\( MDS(b, i)_j \), for \( j = 1, 2, \ldots, P(b) \), are the elements of the cycle \( MDS(b, i) \). For simplicity the argument \( (b, i) \) will by suppressed, and instead of \( MDS_i \) just \( c_j \) (c for cycle) is used in the following Proposition.

**Proposition 13.** For each cycle \( MDS(b, i) \) the members \( c_j = c(b, i)_j \) satisfy the recurrence

\[
c_j = \begin{cases} 
2c_{j-1} & \text{if } c_{j-1} \leq \left\lfloor \frac{b-1}{4} \right\rfloor, \\
b - 2c_{j-1} & \text{if } c_{j-1} > \left\lfloor \frac{b-1}{4} \right\rfloor,
\end{cases}
\]

for \( j \geq 2 \), and the input is \( c_1 = \text{mod}^*(2a(b, i), b) \).

**Proof.** This uses the mod* definition of \( c_j \), and multiplicativity of mod to extract a factor 2.

The second line follows from the \( c_j \) definition in the case when \( (b-1)/2 < \text{mod}(2a2^{j-1}, b) \leq (b-1) \), i.e., \( (b-1)/4 < \text{mod}(a2^{j-1}, b) \leq \frac{b-1}{2} \). In this case the mod* definition of \( c_{j-1} \) tells that \( c_{j-1} = \text{mod}(a2^{j-1}, b) \). Because in this case \( c_{j} = \text{mod}(-2a2^{j-1}, b) = -2c_{j-1} \), that is \( b - 2c_{j-1} \) because the numbers \( c \) are always from \( RRS^*(b) \), hence \( \leq \frac{b-1}{2} \). The simpler first line is proved in a similar manner.

This can be written as a line formulae.

**Corollary 14.**

\[
c_j = \frac{1}{2}(b - \left| b - 4c_{j-1} \right|).
\]

\[
c_j = \text{mod}^*(2c_{j-1}, b).
\]

\[
c_j = \text{mod}((-1)^{p_j}2c_{j-1}, b), \text{ where } p_j = \text{parity}(c_j).
\]

for \( j = 2, 3, \ldots, P(b) \), and the initial value is \( c_1 = c(b, i)_1 \), that is \( 2a(b, i) \) if \( a(b, i) \leq \left\lfloor \frac{b-1}{4} \right\rfloor \), and \( b - 2a(b, i) \) if \( a(b, i) > \left\lfloor \frac{b-1}{4} \right\rfloor \). In the second equation \( c(b, i)_{j-1} \in RRS^*(b, i) \) from Proposition 12 4. has been used, hence \( \text{mod}(2c(b, i)_{j-1}, b) = 2c(b, i)_{j-1} \). In the last equation the recurrence is considered mod \( b \). In the upper alternative \( c_j \) is even, and the lower one odd.

**Example 15.** \( b = 63 \), \( c^*(63) = 3 \), \( P(63) = 6 \), \( \left\lfloor \frac{63 - 1}{4} \right\rfloor = 15 \). \( MDS(63, 3) = [22, 19, 25, 13, 26, 11]. \) \( c_2(63, 3) = 19 > 15, c_3(63, 3) = 63 - 2 \cdot 19 = \frac{1}{2}(63 - (4 \cdot 19 - 63)) = 25. \)
Like in the complete system $SBB(b)$ the elements (numbers) of the cycles of $MDS(b)$ are interpreted as length ratios diagonal/radius in regular $(2b)$-gons as follows. A number $k$ (even or odd) from a cycle $MDS(b,i)$, hence $k \in [1, (b - 1)/2]$, for $i \in \{1, 2, ..., c^*(b)\}$, is mapped to $2 \cos \left( \frac{\pi k}{b} \right) = R(k, \rho(b)) > 0$, where $R$ are monic Chebyshev polynomials shown in A127672, and $\rho(b) = 2 \cos \left( \frac{\pi}{b} \right)$, the length ratio diagonal/side of the diagonal $d_2^{(b)} = V_0^{(b)} V_2^{(b)}$ in the regular $b$-gon. [The $R$ polynomials are in [1], Table 22.7, p. 797, called $C$. In [7], Table 2, p. 72, they are called Lucas polynomials.]

The connection to length ratios diagonal/radius in the regular $(2b)$-gon comes from the trigonometric identity

$$2 \cos \left( \frac{\pi k}{b} \right) = 2 \sin \left( \frac{\pi (b \pm 2k)}{2b} \right).$$

(17)

For the positive sign see also the table and a comment in A082375. But here the negative sign will be chosen. As mentioned above all values are positive. This is an advantage over considering cosine arguments $a(b,i) 2^k$ like in the later discussed complete iteration system $1coS(b)$.

E.g., $b = 5$, $MDS(5, 1) = (2, 1)$: $k = 1$ corresponds to the two diagonal/radius ratios $d_7^{(10)}$ and $d_3^{(10)}$ both of length $\varphi$, with the golden ratio $\varphi = 1.61803398... = A001622$ (in the lower and upper half-plane, respectively). For $k = 2$ these are $d_9^{(10)}$ and $d_1^{(10)}$, the side/radius ratios of length $\varphi - 1$.

The numbers $2 \cos \left( \frac{\pi k}{b} \right) = R(k, \rho(b))$, with $k \in RRS^*(b)$, can be taken as zeros of a monic polynomial $P^*(b,x)$ of degree $\sigma(b) = A023022(b) = A055034(b) = \delta(b)$. It follows that this is a polynomial with coefficients from the ring of integers of the simple algebraic field extension $\mathbb{Q}(\rho(b))$, denoted by $\mathcal{O}_{\mathbb{Q}(\rho(b))} =: \mathbb{Z}[\rho(b)]$. The following proposition gives the formula. See Table 1 for $b = 2n + 1$ for $n = 1, 2, ..., 10$.

**Proposition 16. $P^*(b,x)$ polynomials**

The monic polynomials

$$P^*(b,x) := \prod_{j=1}^{\sigma(b)} \left( x - 2 \cos \left( \frac{\pi RRS^*(b)j}{b} \right) \right) = \prod_{j=1}^{\sigma(b)} \left( x - R(RRS^*(b)j, \rho(b)) \right)_{C(b, \rho(b)) = 0}.$$  

(18)

with $b = 2n + 1$, $n \in \mathbb{N}$, have coefficients which are integers in the simple field extension $\mathbb{Q}(\rho(b))$, where $\rho(b) = 2 \cos \left( \frac{\pi}{b} \right)$.

In the evaluation the minimal polynomial $C(b,x)$ of $\rho(b)$, given in A187360, is used to eliminate powers of $\rho(b)$ with exponents larger than $\delta(b) - 1 = \sigma(b) - 1$.

**Proof.** The first equation gives the definition. The last equation follows from the definition of the $R$ polynomials in terms of Chebyshev $T$ polynomials, and their trigonometric definition. Then the coefficients will be expressed in the power basis of $\mathbb{Q}(\rho(b))$ of degree $[\mathbb{Q}(\rho(b)) : \mathbb{Q}] = \delta(b)$ with rational integer coefficients, because $R \in \mathbb{Z}[x]$. \[QED\]
Example 17.

\[
\begin{align*}
RRS^*(9) &= [1, 2, 4], \sigma(9) = 3, \\
P^*(9, x) &= (x - 2 \cos(\pi/9)) (x - 2 \cos(\pi 2/9)) (x - 2 \cos(\pi 4/9)), \\
&= x^3 - 2 \rho(9) x^2 + (-3 + 2 \rho(9)^2) x - 1, \\
&\text{with } \rho(9) = 2 \cos(\pi/9) = \text{A332437} = 1.87938524. \cdots 
\end{align*}
\]

C) Complete IcoS-system

An alternative approach to the modified modular doubling sequence has been considered by Gary W. Adamson and J. Kappraff [7] by studying iteration of the quadratic polynomial \( R(2, x) = x^2 - 2 \) with certain seeds (see above for Chebyshev \( R \) polynomials, and the name Lucas polynomial in [7]). It has also been used as \( r-t \) table \( (r-t \) for root trajectory) by Gary W. Adamson in his Aug 25 2019 comment in A065941. As will be shown this is very similar to the MDS-system. The iteration of \( x^2 - 2 \) is proposed in a procedure 1. by Devaney [5], on p. 25, for investigation, and in chapter 7.1, pp. 69 - 71, this function is called \( Q_{-2}(x) \). It is shown in chapter 10.2, pp. 121 - 126 to be semiconjugate to the function \( V(x) = 2|\cdot x| - 2 \), and also to the angle doubling function \( D(\theta) = 2\theta \) on the unit circle.

If one starts the iteration of \( R(2, x) \) with a seed \( 2 \cos(\pi 1/b) \), for \( b = 2n + 1 \), with \( n \in \mathbb{N} \), one does not obtain a purely periodic sequence if signs are respected. In order to obtain purely periodic sequences one starts the iteration with \( R^{[1]}(2, x) \), not with the identity map \( R^{[0]}(2, x) = x \). The later given theorem shows periodicity with primitive period length \( P(b) \) like for MDS(b), and also the number of seeds \( c^*(b) \) necessary to obtain finally all \( 2 \cos(\pi j/b) \), values with \( j \in RRS^*(b) \). Note that not all iterations will be positive. Hence for the interpretation as length ratios diagonal/radius in \( (2b) \)-gons the signs will have to be ignored.

This complete system of periodic signed cosine sequences obtained by iteration will be called IcoS(b), and its elements are denoted by sets IcoS(b, i), for \( i = 1, 2, ..., c^*(b) \).

Lemma 18. The iteration with the Chebyshev polynomial \( R(2, x) = x^2 - 2 \) is periodic for any seed \( x = 2 \cos(\pi j/b) = R(j, \rho(b)) \), where \( \rho(b) = 2 \cos(\pi/b) \), and odd \( j \in RRS^*(b) \), and the primitive length of the period is \( P(b) = A003558(b - 1)/2 \).

Proof. The standard formula for the Chebyshev \( T \)-polynomials (see A053120, and [14], p. 5, Exercise 1.1.6) \( T(n, T(m, x)) = T(n m, x) \) translates for \( R(n, x) = 2T(n, x/2) \) to \( R(n, R(m, x)) = R(n m, x) \). The \( T \) and \( R \) polynomials with negative index are identical with the ones for positive index: \( R(-n, x) = R(n, x) \), for \( n \in \mathbb{N}_0 \).

From the periodicity of the cosine function, and \( R(j, \rho(n)) = 2 \cos(\pi j/n) \), follows that \( R(k, R(j, \rho(n)) = R(k j - q 2 n, \rho(n)) \), for \( n \in \mathbb{N} \) and \( q \in \mathbb{Z} \). Together with the symmetry under sign flip of the index, periodicity requires that

\[
R(k j, \rho(n)) = R(\pm(k j - q 2 n), \rho(n)), \text{ for } n \in \mathbb{N}, \text{ and } q \in \mathbb{Z}. \tag{20}
\]

For the iteration of \( R^{[k]}(2, x) = R(2^k, x) \), for \( k \in \mathbb{N} \), periodicity therefore means in the present context, that, with any seed \( x = 2 \cos(\pi j/b) \), and positive odd \( j \) with \( \gcd(j, b) = 1 \)
and \( j \leq \frac{b-1}{2} \), i.e., \( j \in RRS^\ast\) odd(b),
\[
R(2^k, \rho(b)) = R(\pm(2^k j - q 2 b), \rho(b)) = R(j 2^1, \rho(b)),
\]
(21)

For the smallest \( k > 1 \), and then the primitive length of the period is \( P(b) = k - 1 \). This leads to \( q b = j (2^{k-1} + 1) > 0 \), i.e., \( q = q' j \), with positive integer \( q' \) because \( j \not| b \).
Therefore, \( j \) drops out and \( 2^{k-1} \equiv \pm 1 \mod(b) \). Thus, \( P(b) = pes(b) = A003558(b - 1)/2) \). \( \square \)

In order to see the close relationship to the MDS-system the following lemma will establish the connection to the mod\(^r\) congruence.

Lemma 19.
\[
R(a(b, i) 2^k, \rho(b)) = 2 \cos \left( \frac{\pi a(b, i) 2^k}{b} \right) = \pm 2 \cos \left( \frac{\pi \hat{j}}{b} \right) \text{ with } \hat{j} = \text{mod}^r(a(b, i) 2^k, b),
\]
(22)

and the sign is + for \( x := \text{mod}(a(b, i) 2^k, 2 b) \in I := (0, \frac{b}{2}) \) or \( x \in IV := \left[ \frac{b}{2}, b \right) \), and the sign is − if \( x \in II := \left[ b, \frac{3b}{2} \right) \) or \( x \in III := \left[ \frac{3b}{2}, 2b - 1 \right) \).

The identification of the unsigned \( 2 \cos(\frac{\pi \hat{j}}{b}) \) values with length ratios diagonal/radius in a regular \((2 b)\)-gon are then obtained with the help of equation (15).

Proof. First note that the first term \((k = 1)\) is only positive if \( 0 < a(b, i) \leq \frac{b-1}{4} \). The only negative case for \( a(b, 1) = 1 \) is IcoS(3,1) = \{-1\}. Larger seeds are only needed for \( b = 17, 31, 33, \ldots \) (A333855), and seem to lead always to positive values of the first term. E.g., even for \( b = 127 \) with \( P(127) = 7 \) and \( c^\ast(127) = 9 \) the largest seed is \( 21 < 31 = \left\lfloor \frac{b-1}{4} \right\rfloor \).

It is always possible to simplify to \( \pm 2 \cos(\frac{\pi \hat{j}}{b}) \) based on the periodicity of cosine, which implies that only \( \text{mod}(a(b, i) 2^k, 2 b)\) enters. Consider the four intervals \( I \) to \( IV \), given in the lemma, with signs +, −, −, + of \( 2 \cos(\pi x/b) \), respectively. Here \( x = \text{mod}(a(b, i) 2^k, 2 b) \) is a positive integer \( \in [1, 2(b - 1)] \). If \( x_1 \in I \) then \( x_1 = \text{mod}(x_1, b) = \text{mod}^r(x_1, b) \) because \( x_1 \leq \left\lfloor \frac{b}{2} \right\rfloor < \frac{b}{2} \), and its cosine value is positive. If \( x_4 \in IV \) one has \( \text{mod}(x_4, b) = x_4 - b \in II \). Hence \( \text{mod}^r(x_4, b) = \text{mod}(-x_4, b) = -x_4 + 2 b \in I \). Similarly for \( x_2 \in II \) and for \( x_3 \in III \), both with negative cosine values, leading to \( \text{mod}^r(x_2, b) = -x_2 + b \in I \), and \( \text{mod}^r(x_3, b) = -x_3 + b \in I \), respectively. \( \square \)

Definition 20. Complete system IcoS
The iteration procedure \( \{R^k(2, x)\}_{k \geq 1} \) uses first the seed \( x = 2 \cos(\frac{\pi}{b}) = \rho(b) \). If in the period IcoS(b, 1) = \{R(1 \cdot 2^k, \rho(b))\}_{k \geq 1}, consisting after trigonometric simplifications of terms \( \pm 2 \cos(\frac{\pi \hat{j}}{b}) \) (shown in the preceding lemma), all odd and even numbers \( \hat{j} \in RRS^\ast(b) \) appear then only this seed \( a(b, 1) \) is needed, and IcoS(b) = IcoS(b, 1) is complete. Otherwise
another iteration is started using a seed \(2 \cos(\pi a(b, 2)/b)\) with the smallest odd member \(a(b, 2) \in RRS^*(b)\) that did not appear in \(IcoS(b, 1)\). This gives the period \(IcoS(b, 2) = \{R(a(b, 2) \cdot 2^k, \rho(b))\}_{k \geq 1}\), etc., and the \(IcoS(b)\) system is complete after all numbers of \(RRS^*(b)\) have been reached.

Example 21.

\[
b = 17, P(b) = 4, c^*(b) = 2:
\]

\[
IcoS(17, 1) = \{2 \cos(\pi (2/17)), 2 \cos(\pi (4/17)), 2 \cos(\pi (8/17)), -2 \cos(\pi (1/17))\}
= \{1.8649..., 1.4780..., .1845..., -1.9659...\}.
\]

\[
IcoS(17, 2) = \{2 \cos(\pi (6/17)), -2 \cos(\pi (5/17)), -2 \cos(\pi (7/17)), -2 \cos(\pi (3/17))\}
= \{.8914..., -1.2052..., -5.473... - 1.7004...\}. \tag{23}
\]

Theorem 22. The complete system \(IcoS(b)\), with \(b = 2n + 1\), consist of \(c(b) = \frac{\varphi(b)}{2P(b)} = \frac{A135303(n)}{2P(b)}\) disjoint periods \(IcoS(b, i)\) of length \(P(b) = A003558(n)\), and each number \(\hat{j} \in RRS^*(b)\) appears once in the elements \(\pm 2 \cos\left(\frac{\pi j^2}{b}\right)\) of this set of complete periods.

Proof. By construction of \(IcoS(b)\) (definition 20) every odd number \(\hat{j}\) of \(RRS^*(b)\) is in one of the periods \(IcoS(B, i)\). The proof for the even numbers \(\hat{j}\) can be taken over, mutatis mutandis, from Theorem 12, 4 and 5.

This shows that \(IcoS(b)\) is very similar to the \(MDS(b)\) (see the recurrence 14), but works with signed cosine values. The signs have to be ignored for the interpretation as diagonal/radius lengths ratios in the regular \((2b)\)-gon. Therefore, the \(MDS(b)\) system is preferred because it works with positive integers and maps to positive cosine and sine values.

5 Equivalence of the three complete systems

A1) \(MDS(b) \implies \Sigma(b)\)

The proof will be given element-wise: \(MDS(b, i) \implies \Sigma(b, i)\), for odd \(b \geq 3\) and \(i = 1, 2, ..., c^*(b)\). The elements of \(MDS(b, i)\) are \(c(b, i)\), abbreviated as \(c_j\), for \(j = 1, 2, ..., P(b)\). It has been shown in Theorem 12 5. that \(c^*(b) = c(b)\), and in 12 2. that \(P(b) = k(b)\).

For the proof the recurrence relation of the odd members of \(MDS(b, i)\) will be needed. For this one considers the cycle \(MDS'(b, i)\) with the input \(a(b, i)\) as first element \(c'_0\), and \(c'_j = c_j\), for \(j = 1, 2, ..., P(b) - 1\). This \(MDS'\) appears (in a different notation) in [6], at the top of p. 103, in the proof of the quasi-order theorem.

Lemma 23. Recurrence for odd elements of \(MDS'(b, i)\)

The odd elements of \(MDS'(b, i)\), collected in the list \(C\theta'(b, i)\) with elements \(c\theta'(b, i)\), for \(j = 0, 1, ..., r^*(b, i) - 1\), satisfy the recurrence relation (with \(l'_j = l'(b, i)\))

\[
c\theta'_j = b - 2^l c\theta'_{j-1}, \quad \text{for } j = 1, 2, ..., r^*(b, i) - 1, \tag{24}
\]
with input $c_{0} = a(b, i)$, and $i \in \{1, 2, ..., c^{*}(b)\}$. The $l'$-tuple of length $r^{*}(b, i)$ gives the differences of the indices $\text{ind}$ of consecutive odd numbers in $MDS'(b, i)$. Its elements are

$$l_{j}' = \text{ind}(co_{j}') - \text{ind}(co_{j'-1}')$$

for $j = 1, 2, ..., r^{*}(b, i)$,

(25)

with $\text{ind}(co_{r^{*}(b, i)'})) = P(b)$.

Proof. Between the two consecutive odd numbers $co_{j-1}' = c'_{\sum_{i=1}^{j-1} l_{i}'}$ and $co_{j}' = c'_{\sum_{i=1}^{j-1} l_{i}'+l_{j}'}$ (empty sums vanish) there are $l_{j}' - 1$ doublings in $MDS'$, and for the next odd number the second version of the recurrence of the $c' = c$ elements given in equation (15) of Proposition 13, applies. This means that $co_{j}' = b - 2(2^{l_{j}'-1})co_{j-1}' = b - 2^{l_{j}'}co_{j-1}'$.

For the odd elements of of $MDS(b, i)$, given by $Co(b, i) = [co(b, i), ..., co(b, i)_{r^{*}(b, i)}]$, this translates to

**Corollary 24. Recurrence for odd elements of $MDS(b, i)$**

$$co(b, i)_{j} = b - 2^{l'(b, i)}co(b, i)_{j-1}, \text{ for } j = 2, 3, ..., r^{*}(b, i),$$

and $co(b, i)_{1} = b - 2^{l'(b, i)}a(b, i).$

(26)

**Example 25.** $b = 63, i = 2$: $P(63) = 6, r^{*} = 4, MDS = [10, 20, 23, 17, 29, 5]$, $MDS' = [5, 10, 20, 23, 17, 29], Co' = [5, 23, 17, 29], l' = [3, 1, 1, 1]$, because e.g., $l_{1}' = \text{ind}(23) - \text{ind}(5) = 3 - 0 = 3$, and $l_{4}' = P(63) - \text{ind}(29) = 6 - 5 = 1$.

Recurrences: $17 = co_{2}' = 63 - 2^{1}co_{1}' = 63 - 2 \cdot 23 = 17$.

$29 = co_{3}' = 63 - 2^{1}co_{2}' = 63 - 2 \cdot 17 = 29$.

**Definition 26. two lists $U(b, i)$ and $L(b, i)$**

Let $MBS(b, i)$, for odd integer $b \geq 3$, and $i \in \{1, 2, ..., c^{*}(b) = c(b)\}$, be the period of length $P(b) = k(b)$. The elements of the list $MBS(b, i)$ are denoted by $c(b, i)_{j}$ (abbreviated as $c_{j}$). Let the sub-list of odd numbers of $MBS(b, i)$ be denoted by $Co(b, i)$ of length $r^{*}(b, i)$ with elements $co_{j}$ (no interchange of the odd elements is allowed). The position (index) of $co_{j}$ in the list $MBS(b, i)$ is denoted by $\text{ind}(co_{j})$.

1. The list $U(b, i)$ has elements $u_{j}$, for $j = \{1, 2, ..., r^{*}(b, i)\}$, given by

$$u_{j} = co_{r^{*}(b, i)-j+1},$$

(27)

i.e., the odd numbers of $MBS(b, i)$ are read backwards.

2. The list $L(b, i)$ has elements $l_{j}$, for $j = \{1, 2, ..., r^{*}(b, i)\}$, given by

$$l_{j} = \text{ind}(u_{j}) - \text{ind}(u_{j+1}), \text{ with } \text{ind}(u_{r^{*}(b, i)+1}) := 0,$$

(28)

i.e., the number of steps to go in $MDS(b, i)$ backwards from one odd number to the next is counted (in a cyclic manner).
Example 27. \( b = 63, c^*(63) = 3, P(63) = 6 \).

\( MDS(63, 3) = [22, 19, 25, 13, 26, 11], Co(63, 3) = [19, 25, 13, 11], r^*(63, 3) = 4, \)

\( \text{ind}(19) = 2, \text{ind}(25) = 3, \text{ind}(13) = 4, \text{ind}(11) = P(63) = 6. \)

\( U(63, 3) = [11, 13, 25, 19], \text{length}(U(63, 3)) = r(63, 3) = 4. \)

\( L(63, 3) = [2, 1, 1, 2], \text{length}(L(63, 3)) = \text{length}(U(63, 3)). \)

Proposition 28. Let \( MBS(b, i), U(b, i) \) and \( L(b, i) \) be as in the definition above, then

\[
U(b, i) = A(b, i), \quad \text{and} \quad U(b, i) = K(b, i),
\]

with the upper and lower lines \( A(b, i) \) and \( K(b, i) \) of the coach \( \Sigma(b, i) \), respectively.

Proof. By definition of \( U(b, i) \) the length \( r^*(b, i) \) has been identified with \( r(b, i) \), the length of coach \( \Sigma(b, i) \). This is possible because the recurrence of the \( MDS(b, i) \) elements \( c_j \), when considered modulo \( b \), satisfy the third equation of the Corollary 14. This implies, with \( \text{Proposition 12, 3}, \) that \( a(b, i) = c_P(b) = \text{mod}(a(b, i)(-1)^{r^*(b, i)} 2^p(b), b) \) because the number of sign flips starting with \( c_1 \), ending with \( c_P(b) \) is the number \( r^*(b, i) \) of odd numbers of \( MDS(b, i) \). The input \( a(b, i) < (b-1)/2 \) and coprime to \( b \) drops out, and from \( P(b) = k(b) \) and the quasi-order theorem 3 one can identify \( r^*(b, i) \) with \( r(b, i) \).

From \( U(b, i) \) and \( L(b, i) \) the recurrence for coaches, equation (1), can be proved by identifying \( l(b, i)_j \) with \( k(b, i)_j \). The reversed \( l \)-tuple is \( l' \), with \( l'(b, i)_j = l(b, i)^{r^*(b, i)-j+1} \).

The input of \( MDS(b, i) \) is \( A(b, i)_1 = a(b, i) = u(b, i)_1 = co(b, i)^{r^*(b, i)}. \) With equation (25) one has to prove

\[
b - co(b, i)^{r^*(b, i)-j+1} = 2^{l(b, i)_j} co(b, i)^{r^*(b, i)-j}, \quad \text{for} \quad j = 1, 2, \ldots, r^*(b, i) - 1.
\]

The input is now \( co(b, i)^{r^*(b, i)} = a(b, i) \). An index change \( \bar{j} = r^*(b, i) - j + 1 \), and the relation between \( l \) and \( l' \) gives

\[
co(b, i)_j = b - 2^{l'(b, i)_{\bar{j}}} co(b, i)_{\bar{j}-1}, \quad \text{for} \quad \bar{j} = 2, 3, \ldots, r^*(b, i).
\]

and the input is now \( co(b, i)_1 = b - 2^{l'(b, i)_1} a(b, i) \). But this is the proved recurrence from Corollary 24 equation (26). \( \square \)

A2) \( \Sigma(b) \implies MDS(b) \)

The proof for this direction has essentially been given by Hilton and Pedersen [6], pp. 102 - 103, in their proof of the quasi-order theorem. It will be recapitulated here in a different notation.

Definition 29. [6] Modified Coach Symbol \( MCSy(b, i) \)

For fixed odd \( b \geq 3 \) and \( i \in \{1, 2, \ldots, c(b)\} \) the modified coach symbol \( MCSy(b, i) \) has upper line \( Cy(b, i) \) with odd numbers (in the following the argument \( b, i \) will be mostly suppressed)

\[
cy_j = a_{r+2-j}, \quad \text{for} \quad j = 1, 2, \ldots, r + 1,
\]

with the upper line coach numbers \( a_j \) from \( A(b, i) \), of length \( r = r(b, i) \), and \( a_{r+1} = a_1 \), hence \( cy_1 = a_1 = cy_{r+1} \).
The lower line \( L_y(b, i) \) of length \( r \) has elements

\[
   l_y_j = k_{r+1-j}, \text{ for } j = 1, 2, \ldots, r,
\]

with the lower line coach numbers \( k_j = k(b, i)_j \) from \( K(b, i) \).

\[
   \sum_{j=1}^{r} l_y_j = \sum_{j=1}^{r} k_j = k(b), \text{ for each } i.
\]

**Definition 30.** [6] Extended Modified Coach Symbol EMCSy(b, i)

Between each \( c_{y_j} \) and \( c_{y_{j+1}} \) of the upper line \( C_y(b, i) \) of \( MCS_y(b, i) \) \( l_y_j - 1 \) consecutive doublings of \( c_y_j \) are inserted. This results in a one line list \( EMCS_y(b, i) \) of length \( k(b) + 1 \) but now with offset index 0.

With \( s_j := \sum_{q=1}^{j} l_y_q \) (where \( s_0 = 0 \)), the elements \( n_q \) of \( EMCS_y(b, i) \), for \( q = 0, 1, \ldots, k(b) - 1 \), are formally

\[
   n_{s_j + p(j)} = 2^{p(j)} c_{y_{j+1}}, \text{ for } p(j) = 0, 1, \ldots, l_y_{j+1} - 1, \text{ and } n_{k(b) + 1} = c_y_1,
\]

for \( j \in \{0, 1, \ldots, r(b, i) - 1\} \).

**Example 31.** \( b = 63, i = 2, k(63) = 6, r(63, 2) = 4 \).

\( A = [5, 29, 17, 23], K = [1, 1, 1, 3]; C_y = [5, 23, 17, 29, 5], L_y = [3, 1, 1, 1]. \)

\( EMCS_y(63, 2) = [5, 2^1 \cdot 5, 2^2 \cdot 5, 23, 17, 29, 5] = [5, 10, 20, 23, 17, 29, 5]. \)

This shows, with the Example 25, that \( EMCS_y(63, 2) \) without its last entry coincides with \( MDS'(63, 2) \), and without its first entry it becomes \( MDS(63, 2) \).

**Proposition 32.** MDS(b, i) from EMCSy(b, i)

For odd \( b \geq 3 \), and \( i \in \{1, 2, \ldots, c(b)\} \), the extended modified list \( EMCS_y(b, i) \), derived from the coach symbol \( \Sigma(b, i) \), gives without its first term \( n_0 = a(b, i) \) the modified modular doubling sequence \( MDS(b, i) \).

**Proof.** The recurrence relation with input \( a(b, i) \) for \( EMCS_y(b, i) \) given in [6]. p. 103, equation (7.13), for the entries \( n_q \) (here with offset \( q = 0 \)), coincides with the one given in Corollary 14, equation (16) for the \( MDS(b, i) \) entries \( c_j \).

\( \Box \)

**B1) MDS(b) \implies SBB(b)\)**

The proof will be given element-wise: \( MDS(b, i) \implies SBB(b, i) \), for odd \( b \geq 3 \) and \( i \in \{1, 2, \ldots, c^*(b)\} \). It has been shown in Proposition 12.5 that \( c^*(b) = B(b) \), and in 12.2 that \( P(b) = pes(b) \). The fixed arguments \( (b, i) \) will be mostly suppressed.

The \( SBB(b, i) \) recurrence for elements \( q_j \) from equation (8) with input \( q_0 = a(b, i) \), an odd integer from \( RRS^*(b) \), will be proved to follow from the recurrence Proposition 13, equation (15), for the \( MDS(b, i) \) cycle elements \( c_j \) with input \( a(b, i) = c_{P(b)} \) from Proposition 12.3.
Proposition 33. SBB(b, i) from MDS(b, i)
The elements of the (positive) SBB(b, i) cycle are given by

\[ q_j = b - 2c_{P(b)-1+j}, \quad \text{for} \quad j = 0, 1, ..., \text{pes}(b), \quad (36) \]

where \( c_j \) are the elements of the cycle MDS(b, i) of length \( P(b) = \text{pes}(b) \). \( c_{P(b)+l} = c_l \), for \( l \in \mathbb{N} \) (cyclicity). Hence the input is \( q_0(b, i) = c(b, i)_{P(b)} = a(b, i) \).

Example 34. \( b = 63, i = 2, P = 6 = \text{pes}, \) MDS(63, 2) = [10, 20, 23, 17, 29, 5]:
\( q_0 = 63 - 2c_5 = c_6 = 5, q_1 = 63 - 2c_6 = 53, q_2 = 63 - 2c_7 = 63 - 2c_1 = 43, q_3 = 63 - 2c_2 = 23, q_4 = 63 - 2c_3 = 17, q_5 = 63 - 2c_4 = 29, \) hence SBB(63, 2) = (5, 53, 43, 23, 17, 29).

Proof. The recurrences fit because \( 2c_j = b - |b - 4c_{j+1}| \) from Corollary 14. Thus

\[
\begin{align*}
q_j &= b - (b - |b - 4c_{P(b)-1+j}|) = |b - 4c_{P(b)-2+1+j}| \\
&= |b - 2(b - 2c_{P(b)-1+j})| = |b - 2q_{j-1}|. 
\end{align*}
\]  \( (37) \)

The input \( a(b, i) \) relation holds because of the recurrence equation (15), the lower line, because \( c_{P(b)} = a(b, i) \) is odd, hence \( q_0 = b - 2c_{P(b)-1} = c_{P(b)} = a(b, i) \).

\( \square \)

B2) SBB(b) \( \implies \) MDS(b)
This direction is clear from reading B1 backwards. One can use cyclicity of \( SBB(b, i) \) and replace \( q_{j+1-\text{pes}(b)} \) by \( q_{j+1} \), and use \( q_{\text{pes}(b)} = q_0 \) and \( q_{\text{pes}(b)+1} = q_1 \).

Proposition 35. MDS(b, i) from SBB(b, i)
The elements of the cycle MDS(b, i) are given by

\[ c_j = \frac{1}{2}(b - q_{j-1}), \quad \text{for} \quad j = 1, 2, ..., P(b), \quad (38) \]

where \( c_j \) are the elements of the cycle SBB(b, i) of length \( \text{pes}(b) = P(b) \). \( c_{\text{pes}(b)+l} = q_l \), for \( l \in \mathbb{N}_0 \) (cyclicity). Hence the input is \( a(b, i) = c(b, i)_{P(b)} = q_0(b, i) \).

Proof. Use in B1) the index change \( j' = P(b) - 1 + j \), cyclicity of SBB(b, i), and the recurrence for \( q_j \). For example, the equation for the input \( a(b, i) \) is now \( c_{P(b)} = \frac{1}{2}(b - q_{P(b)+1}) = \frac{1}{2}(b - q_1) = q_0 = a(b, i) \) from the recurrence \( q_1 = |b - 2q_0| = b - 2a(b, i) \), because \( RRS^*(b) \ni a(b, i) \leq \frac{b-1}{2} \).

\( \square \)

Example 36. \( b = 63, i = 3, P = 6 = \text{pes}, \) SBB(63, 3) = (11, 41, 19, 25, 13, 37):
\( c_1 = \frac{1}{2}(63 - q_2) = \frac{1}{2}(63 - 19) = 22, ..., c_5 = \frac{1}{2}(63 - q_6) = \frac{1}{2}(63 - 9) = 26, c_6 = \frac{1}{2}(63 - q_1) = 11, \) hence MDS(63, 3) = [22, 19, 25, 13, 26, 11].
6 Acknowledgments

This paper would not have been written without Gary W. Adamson’s suggestion to study the coach system of Hilton and Pedersen, the book by Schick, and the paper by Brändli and Beyne. He also introduced the author to his modified doubling sequence and his paper with Kappraff on iterations. We had an extensive e-mail exchange. The author is very grateful to him. He also likes to thank Carl Schick and Gerold Brändli for an inspiring e-mail exchange.

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AMS MSC numbers: 11B37, 11A07, 05C45.

Keywords: Coach system, modified modular congruence, diagonals in regular \((2n)\)-gons, cycles of integers, polynomial iteration, Euler tours.

Concerned with OEIS sequences: A000010, A000265, A001622, A003558, A023022, A038566, A053120, A055034, A065942, A082375, A082654, A127672, A135303, A187360, A216319, A216371, A232624, A268923, A332433, A332434, A332435, A332436, A332437, A332439, A333848, A333849, A333850, A333851, A333853, A333855, A334430.
Table 1: $P^*(b, x)$ for $b = 2n + 1$, $n = 1, 2, ..., 10$, with $\rho = \rho(b)$

| n | b   | $P^*(b, x)$                                                                                     |
|---|-----|--------------------------------------------------------------------------------------------------|
| 1 | 3   | $x - 1$                                                                                          |
| 2 | 5   | $x^2 + (1 - 2 \rho) x + 1$                                                                         |
| 3 | 7   | $x^4 + (3 - 2 \rho^2) x^2 + 2 \rho x - 1$                                                          |
| 4 | 9   | $x^4 - 2 \rho x^2 + (-3 + 2 \rho^2) x - 1$                                                         |
| 5 | 11  | $x^5 + (-1 + 6 \rho^2 - 2 \rho^3) x^4 + (-4 - 2 \rho + 2 \rho^2 + 2 \rho^3) x^3$                  |
|    |     | $+(1 + 8 \rho - 8 \rho^2 - 4 \rho^3 + 2 \rho^4) x^2 + (3 + 4 \rho - 12 \rho^2 - 2 \rho^3 + 4 \rho^4) x - 1$ |
| 6 | 13  | $x^6 + (1 - 6 \rho + 8 \rho^2 - 4 \rho^3 - 2 \rho^4) x^5 + (-3 - 2 \rho - 4 \rho^2 + 2 \rho^3 + 2 \rho^4) x^4$ |
|    |     | $+(-6 + 14 \rho + 10 \rho^2 - 12 \rho^3 - 4 \rho^4 + 2 \rho^5) x^3 + (2 + 16 \rho - 24 \rho^2 + 6 \rho^3 + 14 \rho^4) x^2$ |
|    |     | $+(1 - 14 \rho + 10 \rho^2 - 2 \rho^3 + 2 \rho^4) x + 1$                                           |
| 7 | 15  | $x^7 + (3 - 8 \rho - 2 \rho^2 + 2 \rho^3) x^6 + (-18 \rho + 6 \rho^2) x^5 + (-2 - 10 \rho + 2 \rho^2) x + 1$ |
| 8 | 17  | $x^8 + (1 + 8 \rho - 20 \rho^2 + 12 \rho^3 - 12 \rho^4) x^7 + (-7 + 4 \rho + 8 \rho^2 - 8 \rho^3 - 8 \rho^4 + 2 \rho^5 + 2 \rho^6) x^6$ |
|    |     | $+(-26 \rho - 38 \rho^2 + 60 \rho^3 + 36 \rho^4 - 30 \rho^5 - 8 \rho^6 + 4 \rho^7) x^5 + (9 - 46 \rho + 124 \rho^2 - 6 \rho^3 - 6 \rho^4 - 80 \rho^5 + 2 \rho^6 + 14 \rho^7) x^4$ |
|    |     | $+(52 \rho + 32 \rho^2 - 14 \rho^3 - 24 \rho^4 + 84 \rho^5 + 4 \rho^6 - 14 \rho^7) x^3 + (-12 + 8 \rho + 68 \rho^2 - 52 \rho^3 - 52 \rho^4 + 36 \rho^5 + 10 \rho^6 - 6 \rho^7) x^2$ |
|    |     | $+(-20 \rho + 6 \rho^2 + 20 \rho^3 - 2 \rho^4 - 4 \rho^5) x + 1$                                  |
| 9 | 19  | $x^9 + (-1 + 20 \rho^2 - 30 \rho^3 + 14 \rho^4 - 2 \rho^5) x^8 + (-8 - 4 \rho + 8 \rho^2 + 14 \rho^3 - 8 \rho^4 - 10 \rho^5 + 2 \rho^6 + 2 \rho^7) x^7$ |
|    |     | $+(3 + 22 \rho - 78 \rho^2 - 60 \rho^3 + 96 \rho^4 + 42 \rho^5 - 36 \rho^6 - 8 \rho^7 + 4 \rho^8) x^6$ |
|    |     | $+(29 - 14 \rho - 156 \rho^2 + 38 \rho^3 + 244 \rho^4 - 30 \rho^5 - 120 \rho^6 + 6 \rho^7 + 18 \rho^8) x^5$ |
|    |     | $+(-17 - 8 \rho + 254 \rho^2 - 8 \rho^3 - 350 \rho^4 + 8 \rho^5 + 156 \rho^6 - 2 \rho^7 - 22 \rho^8) x^4$ |
|    |     | $+(-20 - 30 \rho + 112 \rho^2 + 90 \rho^3 - 136 \rho^4 - 56 \rho^5 + 52 \rho^6 + 10 \rho^7 - 6 \rho^8) x^3$ |
|    |     | $+(8 + 22 \rho - 90 \rho^2 - 28 \rho^3 + 84 \rho^4 + 20 \rho^5 - 24 \rho^6 - 4 \rho^7 + 2 \rho^8) x^2$ |
|    |     | $+(5 + 10 \rho - 54 \rho^2 - 10 \rho^3 + 54 \rho^4 + 2 \rho^5 - 18 \rho^6 + 2 \rho^7) x + 1$         |
| 10| 21  | $x^{10} + (-1 - 12 \rho + 10 \rho^2 - 2 \rho^3 - 2 \rho^5) x^9 + (-10 - 26 \rho - 2 \rho^2 + 16 \rho^3 + 2 \rho^4 - 2 \rho^5) x^8$ |
|    |     | $+(36 \rho + 16 \rho^2 - 26 \rho^3 - 6 \rho^4 + 4 \rho^5) x^7 + (18 + 60 \rho - 10 \rho^2 - 44 \rho^3 + 2 \rho^4 + 8 \rho^5) x^6 + (10 + 10 \rho - 18 \rho^2 - 4 \rho^3 + 4 \rho^4) x + 1$ |
Table 2: MDS(b) for $b = 2n + 1$, $n = 1, 2, \ldots, 35$.

| $n$ | $b$ | $\text{MDS}(b, i)$, $i = 1, 2, \ldots, c^*(b)$ |
|-----|-----|-----------------------------------|
| 1   | 3   | [1]                                |
| 2   | 5   | [2, 1]                             |
| 3   | 7   | [2, 3, 1]                          |
| 4   | 9   | [2, 4, 1]                          |
| 5   | 11  | [2, 4, 3, 5, 1]                    |
| 6   | 13  | [2, 4, 5, 3, 6, 1]                 |
| 7   | 15  | [2, 4, 7, 1]                       |
| 8   | 17  | [2, 4, 8, 1], [6, 5, 7, 3]         |
| 9   | 19  | [2, 4, 8, 3, 6, 7, 5, 9, 1]        |
| 10  | 21  | [2, 4, 8, 5, 10, 1]                |
| 11  | 23  | [2, 4, 8, 7, 9, 5, 10, 3, 6, 11, 1]|
| 12  | 25  | [2, 4, 8, 9, 7, 11, 3, 6, 12, 1]   |
| 13  | 27  | [2, 4, 8, 11, 5, 10, 7, 13, 1]     |
| 14  | 29  | [2, 4, 8, 13, 3, 6, 12, 5, 10, 9, 11, 7, 14, 1] |
| 15  | 31  | [2, 4, 8, 15, 1], [6, 12, 7, 14, 3], [10, 11, 9, 13, 5] |
| 16  | 33  | [2, 4, 8, 16, 1], [10, 13, 7, 14, 5] |
| 17  | 35  | [2, 4, 8, 16, 3, 6, 12, 11, 13, 9, 17, 1] |
| 18  | 37  | [2, 4, 8, 16, 5, 10, 17, 3, 6, 12, 13, 11, 15, 7, 14, 9, 18, 1] |
| 19  | 39  | [2, 4, 8, 16, 7, 14, 11, 17, 5, 10, 19, 1] |
| 20  | 41  | [2, 4, 8, 16, 9, 18, 5, 10, 20, 1], [6, 12, 17, 7, 14, 13, 15, 11, 19, 3] |
| 21  | 43  | [2, 4, 8, 16, 11, 21, 1], [6, 12, 19, 5, 10, 20, 3], [14, 15, 13, 17, 9, 18, 7] |
| 22  | 45  | [2, 4, 8, 16, 13, 19, 7, 14, 17, 11, 22, 1] |
| 23  | 47  | [2, 4, 8, 16, 15, 17, 13, 21, 5, 10, 20, 7, 14, 19, 9, 18, 11, 22, 3, 6, 12, 23, 1] |
| 24  | 49  | [2, 4, 8, 16, 17, 15, 19, 11, 22, 5, 10, 20, 9, 18, 13, 23, 3, 6, 12, 24, 1] |
| 25  | 51  | [2, 4, 8, 16, 19, 13, 25, 1], [10, 20, 11, 22, 7, 14, 23, 5] |
| 26  | 53  | [2, 4, 8, 16, 21, 11, 22, 9, 18, 17, 19, 15, 23, 7, 14, 25, 3, 6, 12, 24, 5, 10, 20, 13, 26, 1] |
| 27  | 55  | [2, 4, 8, 16, 23, 9, 18, 19, 17, 21, 13, 26, 3, 6, 12, 24, 7, 14, 27, 1] |
| 28  | 57  | [2, 4, 8, 16, 25, 7, 14, 28, 1], [10, 20, 17, 23, 11, 22, 13, 26, 5] |
| 29  | 59  | [2, 4, 8, 16, 27, 5, 10, 20, 19, 21, 17, 25, 9, 18, 23, 13, 26, 7, 14, 28, 3, 6, 12, 24, 11, 22, 15, 29, 1] |
| 30  | 61  | [2, 4, 8, 16, 29, 3, 6, 12, 24, 13, 26, 9, 18, 25, 11, 22, 17, 27, 7, 14, 28, 5, 10, 20, 21, 19, 23, 15, 30, 1] |
| 31  | 63  | [2, 4, 8, 16, 31, 1], [10, 20, 23, 17, 29, 5], [22, 19, 25, 13, 26, 11] |
| 32  | 65  | [2, 4, 8, 16, 32, 1], [6, 12, 24, 17, 31, 3], [14, 28, 9, 18, 29, 7], [22, 21, 23, 19, 27, 11] |
| 33  | 67  | [2, 4, 8, 16, 32, 3, 6, 12, 24, 19, 29, 9, 18, 31, 5, 10, 20, 27, 13, 26, 15, 30, 7, 14, 28, 11, 22, 23, 21, 25, 17, 33, 1] |
| 34  | 69  | [2, 4, 8, 16, 32, 5, 10, 20, 29, 11, 22, 25, 19, 31, 7, 14, 28, 13, 26, 17, 34, 1] |
| 35  | 71  | [2, 4, 8, 16, 32, 7, 14, 28, 15, 30, 11, 22, 27, 17, 34, 3, 6, 12, 24, 23, 25, 21, 29, 13, 26, 19, 33, 5, 10, 20, 31, 9, 18, 35, 1] |

