BRANCHED COVERS AND RATIONAL HOMOLOGY BALLS

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Abstract. The concordance group of knots in $S^3$ contains a subgroup isomorphic to $(\mathbb{Z}_2)\infty$, each element of which is represented by a knot $K$ with the property that for every $n > 0$, the $n$-fold cyclic cover of $S^3$ branched over $K$ bounds a rational homology ball. This implies that the kernel of the canonical homomorphism from the knot concordance group to the infinite direct sum of rational homology cobordism groups (defined via prime-power branched covers) contains an infinitely generated two-torsion subgroup.

1. Introduction

There is a homomorphism

$$\varphi: \mathcal{C} \to \prod_{q \in \mathbb{Q}} \Theta_3^q,$$

where $\mathcal{C}$ is the smooth concordance group of knots in $S^3$, $\mathbb{Q}$ is the set of prime power integers, and $\Theta_3^q$ is the rational homology cobordism group. For a knot $K$ and $q \in \mathbb{Q}$, the $q$-component of $\varphi(K)$ is the class of $M_q(K)$, the $q$-fold cyclic cover of $S^3$ branched over $K$.

In the paper [1], Aceto, Meier, A. Miller, M. Miller, Park, and Stipsicz proved that $\ker \varphi$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^5$. Here we will prove that $\ker \varphi$ contains a subgroup isomorphic to $(\mathbb{Z}_2)^\infty$. Our examples are of the form $K_n # -K_n^r$, where $-K_n$ denotes the concordance inverse of $K_n$ (the mirror image of $K_n$ with string orientation reversed), and $K_n^r$ is formed from $K_n$ by reversing its string orientation. Such knots easily seen to be in the kernel of $\varphi$; the more difficult work is to find nontrivial examples of order two.

The first known example of a nontrivial element in $\ker \varphi$ was represented by the knot $K_1 = 8_{17} # -8_{17}$. That $K_1$ is of order at most two is elementary; that $K_1$ is nontrivial in $\mathcal{C}$ is one of the main results of [9], proved using twisted Alexander polynomials.

The results of [7] provide an infinitely generated free subgroup of $\ker \varphi$. Conjecturally, $\mathcal{C} \cong \mathbb{Z}\infty \oplus (\mathbb{Z}_2)^\infty$; if true, then $\ker \varphi \cong \mathbb{Z}\infty \oplus (\mathbb{Z}_2)^\infty$.

1.1. Main result. Figure 1 illustrates a knot $P_n$ in a solid torus, where $J_n$ represents the braid illustrated on the right in the case of $n = 5$; $n$ will always be odd. We let $K_n$ denote the satellite of $8_{17}$ built from $P_n$. In standard notation, $K_n = P_n(8_{17})$. For future reference, we illustrate the braid $J_n^*$ formed by rotating $J_n$ around the vertical axis.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The knot $P_n \subset S^1 \times B^2$, $J_n$, and $J_n^*$.}
\end{figure}

**Theorem 1.** Let $K_n = P_n(8_{17})$. For all odd $n$, the knot $L_n = K_n # -K_n^r$ satisfies $2L_n = 0 \in \mathcal{C}$ and $L_n \in \ker \varphi$. There is an infinite set of prime integers $\mathcal{P}$ for which $L_\alpha \neq L_\beta \in \mathcal{C}$ for all $\alpha \neq \beta$ in $\mathcal{P}$. In particular, the set of knots $\{L_n\}_{n \in \mathcal{P}}$ generates a subgroup of $\ker \varphi$ that is isomorphic to $(\mathbb{Z}_2)^\infty$.

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The rest of the paper presents a proof of this theorem. The first two claims are easily dealt with in Sections 2 and 3. The more difficult step of the proof calls on an analysis of twisted Alexander polynomials and their relevance to knot slicing; a review of twisted polynomials is included in Section 4. The proof of Theorem 1 is completed in Section 5, with the exception of a number theoretic result that is described in Appendix A.

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2. Proof: $2L_n = 0 \in \mathcal{C}$

Let $P_n^* \subset S^1 \times B^2$ denote the knot formed using the braid $J_n^*$ in Figure 2. For any knot $K$, let $P_n^*(K)$ denote the satellite of $K$ built using $P_n^*$. It should be clear that $P_n$ and $P_n^*$ are orientation preserving isotopic, and thus for all knots $K$, $P_n(K) = P_n^*(K)$.

Figure 2 illustrates for an arbitrary knot $K$, the connected sum $P_n(K) \# P_n^*(K) = P_n(K) \# P_n(K)$ in the case of $n = 5$. Performing $n - 1$ band moves in the evident way yields the $(0, n)$–cable of $K \# K$. Thus, if $K \neq K \neq 0 \in \mathcal{C}$, then the $n$ components of this link can be capped off with parallel copies of the slice disk for $K \# K$, implying that $P_n(K) \# P_n(K) = 0 \in \mathcal{C}$. In particular, $2K_n = 0 \in \mathcal{C}$ and $2K_n = 0 \in \mathcal{C}.$

3. Proof $L_n \in \ker \varphi$

We prove a stronger statement: For all odd $n$, and for all positive integers $q$, $M_q(L_n)$ is a rational homology sphere that represents $0 \in \Theta_3^*$.

The $q$–fold cyclic cover of $S^3$ branched over $K_n \neq -K_n^*$ is the same space as the $q$–fold cyclic cover of $S^3$ branched over $K_n \neq -K_n$. A slice disk for $K_n \neq -K_n$ is built from $(S^3 \times I, K_n \times I)$ by removing a copy of $B^3 \times I$. Taking the $q$–fold branched cover shows that the $q$–fold cyclic cover of $B^3$ branched over that slice disk is diffeomorphic to $M_q(K_n)^* \times I$, where $M_q(K_n)^*$ denotes a punctured copy of $M_q(K_n)$. It remains to show that $M_q(K_n)$ is a rational homology 3–sphere.

A formula of Fox [5] and Goeritz [6] states that the order of the first homology of $M_q(K_n)$ is given by the product of values $\Delta_{K_n}(\omega_q^i)$, where $\Delta_{K_n}(t)$ denotes the Alexander polynomial, $\omega_q$ is a primitive $q$–root of unity, and $i$ runs from 1 to $q - 1$.

A result of Seifert [11] shows that $\Delta_{K_n}(t) = \Delta_{8_{17}}(t^n)\Delta_{P_n(U)}$, where $U$ is the unknot. We have that $P_n(U) = U$. The Alexander polynomial for $8_{17}$ is

$$\Delta_{8_{17}}(t) = 1 - 4t + 8t^2 - 11t^3 + 8t^4 - 4t^5 + t^6.$$ 

A numeric computation confirms that this polynomial does not have roots on the unit complex circle, and hence $\Delta_{8_{17}}(t^n)$ has no roots on the unit complex circle. From this is follows that $\Delta_{K_n}(\omega_q^i) \neq 0$ for all $i$; thus the order of the homology of $M_q(K_n)$ is finite.

4. Review of twisted polynomials and $8_{17}$

In this section we review twisted Alexander polynomials and their application in [8] showing that $8_{17} \neq -8_{17} \neq 0 \in \mathcal{C}$.

Let $(X, B) \to (S^3, K)$ be the $q$–fold cyclic branched cover of a knot $K$ with $q$ a prime power. In particular, $X$ is rational homology sphere. There is a canonical surjection $\varepsilon: H_1(X - B) \to \mathbb{Z}$. Suppose that $\rho: H_1(X) \to \mathbb{Z}_p$ is homomorphism for some prime $p$. Then there is an associated twisted polynomial $\Delta_{K, \varepsilon, \rho}(t) \in \mathbb{Q}(\omega_p)[t]$. It is well-defined, up to factors of the form $a t^p$, where $a \neq 0 \in \mathbb{Q}(\omega_p)$. These polynomials are discriminants of Casson-Gordon invariants, first defined in [3].
In the case of $K = 8_{17}$ and $q = 3$, we have $H_1(X) \cong \mathbb{Z}_{13} \oplus \mathbb{Z}_{13}$, and as a $\mathbb{Z}_{13}$-vector space this splits as a direct sum of a 3–eigenspace and a 9–eigenspace under the order three action of the deck transformation. Both eigenspaces are 1–dimensional. We denote this splitting as $E_3 \oplus E_9$. There are corresponding characters $\rho_3$ and $\rho_9$ of $H_1(X)$ onto $\mathbb{Z}_{13}$; these are defined as the quotient maps onto $H_1(X)/E_3$ and onto $H_1(X)/E_9$. We let $\rho_0$ denote the trivial $\mathbb{Z}_{13}$-valued character.

The values of $\Delta_{8_{17},\epsilon,\rho_1}(t)$ are given in [9], duplicated here in Appendix 3. For $i = 0$ it is polynomial in $\mathbb{Q}[t]$. For $i = 3$ and $i = 9$ it is in $\mathbb{Q}(\omega_{13})[t]$ and is not in $\mathbb{Q}[t]$. An essential observation is that for $8_{17}$, the roles of $\rho_3$ and $\rho_9$ are reversed. All three of the polynomials are irreducible in their respective polynomial rings, once any factors of $(1 - t)$ and $t$ are removed.

In [9] the proof that $8_{17} \# -8_{17}$ is not slice comes down to the observation that no product of the form

$$\sigma_3(\Delta_{8_{17},\epsilon,\rho_1}(t))\sigma_9(\Delta_{8_{17},\epsilon,\rho_1}(t)) \text{ or } \sigma_3(\Delta_{8_{17},\epsilon,\rho_9}(t))\sigma_9(\Delta_{8_{17},\epsilon,\rho_9}(t))$$

is of the form $af(t)f(t^{-1})(1 - t)^j$ for some $f(t) \in \mathbb{Q}(\omega_{13})[t]$. (That is, these products are not norms in the polynomial ring $\mathbb{Q}(\omega_{13})[t,t^{-1}]$ modulo powers of $(1 - t)$ and $t$.) Here $i = 0$ or $i = 9$ and $j = 0$ or $j = 3$. The number $a$ is in $\mathbb{Q}(\omega)$ and the $\sigma_3$ are Galois automorphisms of $\mathbb{Q}(\omega_p)$ (which acts by sending $\omega_p$ to $\omega_p'$).

Showing that the product of the polynomials does not factor in this way is elementary once it is established that $\Delta_{8_{17},\epsilon,\rho_3}(t)$ and $\Delta_{8_{17},\epsilon,\rho_9}(t)$ are irreducible and not Galois conjugate.

5. Main Proof

Using the fact that $-\mathbb{P}_n(8_{17}^r) = \mathbb{P}_n(8_{17})^r$, the knot $L_{\alpha} \# L_{\beta}$ can be expanded as

$$\mathbb{P}_n(8_{17}) \# \mathbb{P}_n(8_{17})^r \# \mathbb{P}_n(8_{17}) \# \mathbb{P}_n(8_{17})^r.$$

We begin by analyzing the 3–fold cover of $S^3$ branched over $\mathbb{P}_n(8_{17})$ and assume that 3 does not divide $n$. This cover is $M_3(\mathbb{P}_n(8_{17}))$ and we denote the branch set in the cover by $\hat{B}$.

There is the obvious separating torus $T$ in $S^3 \setminus \mathbb{P}_n(8_{17})$. Since 3 does not divide $n$, $T$ has a connected separating lift $\tilde{T} \subset M_3(\mathbb{P}_n(8_{17}))$. One sees that $\tilde{T}$ splits $M_3(\mathbb{P}_n(8_{17}))$ into two components: $X$, the 3–fold cyclic cover of $S^3 \setminus \mathbb{P}_n(8_{17})$ and $Y$, the 3–fold cyclic branched cover of $S^1 \times B^2$, branched over $\mathbb{P}_n$. A simple exercise shows that since $P_n(U)$ is unknotted, $Y$ is the complement of some knot $\tilde{T}_n \subset S^3$.

A Mayer-Vietoris argument shows that $H_1(M_3(\mathbb{P}_n(8_{17}))) \cong \mathbb{Z}_{13} \oplus \mathbb{Z}_{13}$ and the two canonical representations $\rho_3$ and $\rho_9$ that are defined on $X$ extend trivially on $Y$ and so to $M_3(\mathbb{P}_n(8_{17}))$. We denote these extension $\rho_3'$ and $\rho_9'$. Let $e'$ be the canonical surjective homomorphism $e': H_1(M_3(\mathbb{P}_n(8_{17}))) \setminus B) \rightarrow \mathbb{Z}$. Restricted to $X$ we have $e'(x) = e(nx)$, where $e$ was the canonical representation to $\mathbb{Z}$ defined for the cover of $S^3 \setminus \mathbb{P}_n(8_{17})$.

In [8] Theorem 3.7 there is a discussion of twisted Alexander polynomials of satellite knots in $S^3$, working in the greater generality of homomorphisms to the unitary group $U(m)$. (A map to $\mathbb{Z}_{n}$ can be viewed as a representation to $U(1)$). The proof of that theorem, which relies on the multiplicativity of Reidemeister torsion, applies in the current setting, yielding the following lemma.

Lemma 2.

$$\Delta_{\mathbb{P}_n(8_{17}),\epsilon,\rho_1}(t) = \Delta_{8_{17},\epsilon,\rho_3}(t^n)\Delta_{\hat{T}_n}(t).$$

Similar results hold for the knot $\mathbb{P}_n(8_{17})^r$ and for the character $\rho_0$.

As described in [8] Casson-Gordon theory implies that if $L_{\alpha} \# L_{\beta}$ is slice, then for some 3–eigenvector or for some 9–eigenvector the corresponding twisted Alexander polynomial is a norm; that is, it factors as $at^kf(t)f(t^{-1})$, modulo multiples of $(1 - t)$. If it is a 3–eigenvector, the relevant polynomial is of the form

$$\Delta(t) = \sigma_3(\Delta_{8_{17},\epsilon,\rho_3}(t^n))\sigma_9(\Delta_{8_{17},\epsilon,\rho_9}(t^n))^\epsilon\sigma_9(\Delta_{8_{17},\epsilon,\rho_9}(t^3))^\epsilon\sigma_3(\Delta_{8_{17},\epsilon,\rho_3}(t^3))^\epsilon(\Delta_{\hat{T}_n}(t)\Delta_{\hat{T}_n}(t))2,$$

where one of $x, y, z$, or $w$ is equal to 1, and each of the others are either 1 or 0.

The four $\mathbb{Q}(\omega_{13})[t]$–polynomials that appear here,

$$\Delta_{8_{17},\epsilon,\rho_3}(t^n), \Delta_{8_{17},\epsilon,\rho_9}(t^n), \Delta_{8_{17},\epsilon,\rho_3}(t^3), \text{ and } \Delta_{8_{17},\epsilon,\rho_9}(t^3),$$

and all their Galois conjugates are easily seen to be distinct for any pair $\alpha \neq \beta$. The following number theoretic result implies that there is an infinite set of primes such that if $\alpha \in P$ and $\beta \in P$, then no product as given in Equation 1 can be a norm in $\mathbb{Q}(\omega_{13})[t]$, proving that the connected sum $L_{\alpha} \# L_{\beta}$ is not slice. We will present a proof in Appendix A.
Lemma 3. Let \( f(t) \in \mathbb{Z}(\omega_p)[t] \) be an irreducible monic polynomial. If there exists \( \zeta \in \mathbb{C} \) such that \( f(\zeta) = 0 \) and \( \zeta^n \neq 1 \) for all \( n > 0 \), then the set of primes \( p \) for which \( f(t^p) \) is reducible is finite.

Proof of Theorem 1. The last factor in Equation 1 involving the \( \tilde{J}_n \) is a norm, so it can be ignored in determining if the product is a norm.

A numeric computation shows that the twisted polynomials \( \Delta_{8n, e, p}(t) \) for \( i = 3 \) and \( i = 9 \) do not have roots on the unit circle, so Lemma 3 can be applied with \( F = \mathbb{Q}(\omega_{13}) \). Let \( P \) be the infinite set of primes with the property that if \( p \in P \) then \( \Delta_{8n, e, p}(t) \) and \( \Delta_{8n, e, p}(t^p) \) are irreducible. Consider the case of \( x = 1 \) in Equation 1. Then, assuming that \( \alpha \in P \) and \( \beta \in P \), the term \( \sigma_\alpha(\Delta_{8n, e, p}(t^\alpha)) \) that appears in Equation 1 is relatively prime to the remaining factors and all the factors are irreducible, modulo powers of \( t \) and \( (1 - t) \). Hence, the product cannot be of the form \( t^k(1 - t)^l f(t)f(t^{-1}) \) for any \( f(t) \in \mathbb{Q}(\omega_{13})[t] \). The cases of \( y, z \), or \( w = 1 \) are the same.

APPENDIX A. FACTORING \( f(t^p) \)

In this appendix we prove Lemma 3 stated in somewhat more generality as Lemma 4 below. We first summarize some background material. Further details can be found in any graduate textbook on algebraic number theory.

- \( A \subset \mathbb{C} \) denotes the ring of algebraic integers. This is the ring consisting of all roots of monic polynomials in \( \mathbb{Z}[t] \).
- For an extension field \( F/\mathbb{Q} \), the ring of algebraic integers in \( F \) is defined by \( \mathcal{O}_F = F \cap A \).
- The property of transitivity states that if \( f(t) \in \mathcal{O}_F[t] \) is monic and \( f(\zeta) = 0 \), then \( \zeta \in \mathcal{O}_F \).
- \( \mathcal{O}_F^\times \) is defined to be the set of units in \( \mathcal{O}_F \).
- The norm of an element \( x \in \mathcal{O}_F \) is defined as \( N(x) = \prod x_i \in \mathbb{Z} \) where the \( x_i \) are the complex Galois conjugates of \( x \). This map satisfies \( N(xy) = N(x)N(y) \) for all \( x, y \in \mathcal{O}_F \). An element \( x \in \mathcal{O}_F \) is in \( \mathcal{O}_F^\times \) if and only if \( N(x) = \pm 1 \).
- The Dirichlet Unit Theorem states that for a finite extension \( F/\mathbb{Q} \), the abelian group \( \mathcal{O}_F^\times \) is finitely generated, isomorphic to \( G \oplus \mathbb{Z}^r \), where \( G \) is the number of embeddings of \( F \) in \( \mathbb{R} \), and \( 2s \) is the number of non-real embeddings of \( F \) in \( \mathbb{C} \).

Lemma 4. Let \( F \) be a finite extension of \( \mathbb{Q} \) and let \( f(t) \in \mathcal{O}_F[t] \) be an irreducible monic polynomial. If there exists \( \zeta \in \mathbb{C} \) such that \( f(\zeta) = 0 \) and \( \zeta^n \neq 1 \) for all \( n > 0 \), then the set of primes \( p \) for which \( f(t^p) \) is reducible is finite.

Proof. Step 1. If \( f(\zeta) = 0 \), then \( \zeta \in \mathcal{O}_{F(\zeta)} \).

This follows immediately from the assumption that \( f(t) \) is monic.

Step 2. Suppose that \( f(t) \in F[t] \) is irreducible and \( f(\zeta) = 0 \). If for some prime \( p \), \( f(t^p) \) is reducible over \( F \), then \( \zeta = \eta^p \) for some \( \eta \in \mathcal{O}_{F(\zeta)} \).

Let \( \xi \in \mathbb{C} \) satisfy \( \xi^p = \zeta \). Since \( f(t) \) is irreducible of degree \( n \) and \( f(t^p) \) is reducible, we have the degrees of extensions satisfying \( [F(\xi) : F] = n \) and \( [F(\xi) : F] < np \). It follows from the multiplicity of degrees of extensions that \( [F(\xi) : F] \leq p \). The polynomial \( t^p - \zeta \in F(\zeta)[t] \) has \( \xi \) as a root. For all \( i \), \( \omega_p^i\xi \) is also a root, so \( t^p - \zeta \) factors completely in \( \mathbb{C}[t] \) as

\[
t^p - \zeta = (t - \xi)(t - \omega_p\xi) \cdots (t - \omega_p^{p-1}\xi).
\]

By the degree calculation just given, \( t^p - \zeta \) has an irreducible factor \( g(t) \in F(\zeta)[t] \) of degree \( l < p \). We can write \( g(t) = \prod (t - \omega_p^j\xi) \) where the product is over some proper subset of \( \{0, \ldots, p - 1\} \). Multiplying this out, one finds that the constant term is of the form \( \omega_p^j\xi^l \in F(\zeta) \) for some \( j \) and \( l < p \). Since \( l \) and \( p \) are relatively prime, there are integers \( u \) and \( v \) such that \( ul + vp = 1 \). Thus, \( (\omega_p^j\xi^l)^u(\xi^{vl})^v = \omega_p^{ul}\xi \) for some \( s \). In particular, for some \( s \), we have \( \omega_p^{ul}\xi \in F(\zeta) \). We let \( \eta = \omega_p^{ul}\xi \) and find that \( \eta^p = (\omega_p)^p\xi^p = \zeta \). Finally, \( \eta \) satisfies the monic polynomial \( f(t^p) \) and thus is in \( \mathcal{O}_{F(\zeta)} \).

Step 3. The set of primes \( p \) such that \( \zeta = \eta^p \) for some \( \eta \in \mathcal{O}_{F(\zeta)} \) is finite.

If \( \zeta = \eta^p \) then \( N(\zeta) = N(\eta)^p \). If \( N(\zeta) \neq \pm 1 \), then the set of \( p \) for which \( N(\zeta) = a^p \) for some integer \( a \) is finite.
If $N(\zeta) = \pm 1$, then $\zeta \in O_{F(\zeta)}^\times$. Hence $\zeta$ represents a non-torsion element in a finitely generated abelian group, and thus it has a finite number of roots.

**Comments.** The argument just given is based on a summary of the proof for the case $F = \mathbb{Q}$ presented in MathOverflow by Dimitrov [4]. Step (2) is a special case of the *Vahlen-Capelli Theorem*, proved in the case of $F = \mathbb{Q}$ by Vahlen and for fields of characteristic 0 by Capelli [2]. A proof for fields of finite characteristic is given in the book by Rédei [10].

**Appendix B. Twisted polynomials of $8_{17}$**

Here are the three needed polynomials. We write $\omega$ for $\omega_{13}$.

$$
\Delta_{8_{17}, e, \rho_3}(t) = 1 - t - 34t^2 - 101t^3 - 34t^4 - t^5 + t^6.
$$

$$
\Delta_{8_{17}, e, \rho_3}(t)/(1 - t) =
1 +
t(2\omega + 2\omega^2 + 2\omega^3 + 4\omega^4 + 2\omega^5 + 2\omega^6 + \omega^7 + \omega^8 + 2\omega^9 + 4\omega^{10} + \omega^{11} + 4\omega^{12}) +
t^2(-15\omega - 10\omega^2 - 15\omega^3 - 15\omega^4 - 10\omega^5 - 10\omega^6 - 10\omega^7 - 10\omega^8 - 15\omega^9 - 15\omega^{10} - 10\omega^{11} - 15\omega^{12}) +
t^3(4\omega + 2\omega^2 + 2\omega^3 + 2\omega^4 + \omega^5 + \omega^6 + 2\omega^7 + 2\omega^8 + 4\omega^9 + 2\omega^{10} + 2\omega^{11} + 2\omega^{12}) +
t^4
$$

$$
\Delta_{8_{17}, e, \rho_3}(t)/(1 - t) =
1 +
t(6\omega + 5\omega^2 + 6\omega^3 + 6\omega^4 + 5\omega^5 + 5\omega^6 + 5\omega^7 + 5\omega^8 + 6\omega^9 + 6\omega^{10} + 5\omega^{11} + 6\omega^{12}) +
t^2(-13\omega - 12\omega^2 - 13\omega^3 - 13\omega^4 - 12\omega^5 - 12\omega^6 - 12\omega^7 - 12\omega^8 - 13\omega^9 - 13\omega^{10} - 12\omega^{11} - 13\omega^{12}) +
t^3(6\omega + 5\omega^2 + 6\omega^3 + 6\omega^4 + 5\omega^5 + 5\omega^6 + 5\omega^7 + 5\omega^8 + 6\omega^9 + 6\omega^{10} + 5\omega^{11} + 6\omega^{12}) +
t^4
$$

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