A potential-based formulation of the classical and relativistic Navier-Stokes equations

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Analogies drawn to Maxwell’s equations in tandem with complementary viscous flow theory, involving the introduction of a tensor potential, has been used to achieve exact integration of the Navier-Stokes equations. The same methodology facilitates the derivation of an elegant four-dimensional Lorentz-invariant first-integral formulation of the energy-momentum equations for viscous flow assuming a flat space-time, consisting of a single tensor equation. It represents a generalisation of corresponding Galilei-invariant theory associated with the classical incompressible Navier-Stokes equations, with the key features that it: (i) takes the same form as its two- and three-dimensional incompressible counterparts; (ii) applies to any physical model, in that it does not depend at outset on the constitutive fluid relationship forming the energy-momentum tensor, other than that the latter is taken as being symmetric. The new general theory is applied to the problem of propagating acoustic waves, with and without viscous damping, and shown to recover the well-known classical expressions for sound speed and damping rate consistent with those available in the open literature.

1 Introduction

In various branches of physics the judicious use of potential fields has become indispensable for the reformulation and elegant solution of a variety of problems, with Maxwell’s equations for the electric field, E, and magnetic flux density, B₁ in a vacuum being a prime example. Following the introduction of scalar and vector potentials \(\varphi\) and \(A₁\), respectively, such that \(B₁ = \varepsilon_{ijk} \partial_i A_k\) and \(E₁ = -\partial_t \varphi - \partial_i A_i\), two of the four equations are fulfilled identically, the remaining two being transformed into second-order partial differential equations (PDEs).

Recently a potential formulation of the Navier-Stokes (NS) equations has been developed [1], an essential underpinning being analogies drawn following the methodical reduction of Maxwell’s equations [2]: The continuity equation, \(\partial_t u_i = 0\), is shown to be fulfilled identically following the introduction a vector potential \(\Psi_k\) for the velocity, in accordance with \(u_i = \varepsilon_{ijk} \partial_j \Psi_k\) and \(\varepsilon_{ijk}\) denoting the 3D Levi-Civita symbol. Subsequent parallels drawn with Maxwell’s theory, involving the introduction of a tensor potential \(\alpha_{ij}\) and a vector potential \(\varphi\), facilitates the formulation of an exact first integral of the full, unsteady, incompressible NS equations in their most general, preliminary, form:

\[
\begin{align*}
\rho u_i u_j + (p + \eta) \delta_{ij} &= \frac{1}{2} \varepsilon_{ijk} j^{jqp} \partial_p [a_{kq} + a_{qk}] + \partial_i (\varepsilon_{ijk} \partial_j \varphi_k) + \partial_j (\varepsilon_{ijk} \partial_i \varphi_k), \\
\rho \partial_t \Psi_n &= \partial_n \partial_k \left[ \frac{1}{2} \varepsilon_{kqp} \alpha_{pq} + \varphi_k - \eta \psi_k \right] - \partial_k \partial_k [\varphi_n - \eta \psi_n].
\end{align*}
\]

Starting from the above exact integration of the NS equations, rather than the NS equations themselves, solutions (both analytical and numerical) have been obtained to a hierarchy of classical flow problems available in the open literature showing speed and damping rate consistent with those available in the open literature.

2 Relativistic generalisation

The methodology applied successfully to the 2D and 3D incompressible NS equations is now employed to address the case of 4D Lorentz-invariant viscous flow in the form of the energy-momentum equations. For this purpose a flat space-time is assumed and use made of the usual relativistic notation with metric signature \((+,−,−,−)\), that is, \(\eta_{\mu\nu} = (\eta^{\mu\nu}) = \text{diag}(1,−1,−1,−1)\), the coordinate vector \((x^\mu) = (ct, x_1, x_2, x_3)\), together with the respective 4-gradient in covariant form given by \(\partial_\mu (c^{-1} \partial_t, \partial_1, \partial_2, \partial_3)\) and in contra-variant form by \(\partial^\mu = (\eta^{\mu\nu} \partial_\nu) = (c^{-1} \partial_t, −\partial_1, −\partial_2, −\partial_3)\).

The basic equations of motion take the form of a combined energy-momentum balance:

\[
\partial_\alpha T^{\alpha\beta} = 0 ,
\]

supplemented by the continuity equation \(\partial_\beta (\rho u^\alpha) = 0\), with particle density \(\rho\), and a balance equation for the specific entropy \(s\) \((u^\alpha) = \gamma (1, u_1/c, u_2/c, u_3/c)\), with \(\gamma^{-1} = 1 - (u_1^2 + u_2^2 + u_3^2)/c^2\), is used to denote the 4-velocity, while the

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DOI: 10.1002/pamm.202000231

PAMM - Proc. Appl. Math. Mech. 2020;20:e202000231

www.gamm-proceedings.com

https://doi.org/10.1002/pamm.202000231

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energy-momentum tensor $T^{\alpha\beta}$ takes [3] the general form:

$$T^{\alpha\beta} = (nmc^2 + ne + p) u^{\alpha}u^{\beta} - n\gamma^2 + R^{\alpha\beta},$$  \hspace{1cm} (4)$$

where $\epsilon = \epsilon(n, s)$ is the internal energy density, $p = n^2 \partial \epsilon / \partial n$ is the pressure and $R^{\alpha\beta}$ a tensor taking irreversible effects such as viscosity and heat conduction into account. The latter two depend on the constitutive relationships chosen to underpin the fluid model.

A key feature associated with the first integral of equations (3), as derived below, is that it applies to any physical model; the only necessary assumption is one of symmetry $T^{\alpha\beta} = T^{\beta\alpha}$. In the accompanying analysis parentheses around an index pair denotes symmetrisation, e.g. $\partial (\mu u_{\nu}) = (\partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu}) / 2$, with square brackets indicating skew-symmetrisation.

### 2.1 First integral: preliminary form

Proceeding as in [1], a corresponding fourth rank tensor potential, $a_{\nu\lambda\kappa\rho}$, satisfying:

$$T^{\alpha\beta} = \varepsilon^{\alpha\nu\lambda\kappa}\epsilon^{\beta\gamma\kappa\rho} \partial_{\gamma} a_{\nu\lambda\kappa\rho},$$  \hspace{1cm} (5)$$
is employed which fulfils equation (3) identically. The tensor potential has to be symmetric with respect to the exchange $\nu \leftrightarrow \lambda$ or $\kappa \leftrightarrow \rho$. Any contributions being symmetric with respect to the exchange $\nu \leftrightarrow \lambda$ or $\kappa \leftrightarrow \rho$ are filtered out by the Levi-Civita symbols, thus one can write: $a_{(\nu\lambda)(K\rho)} = a_{(\nu\lambda)(K\rho)} = 0$. Considering both symmetry expressions, it follows that the tensor $a_{\nu\lambda\kappa\rho}$ contains $6 \cdot (6 + 1)/2 = 21$ independent entries. The preliminary form (5) corresponds to equation (1) apropos the first integral of the 3D-NS equations.

### 2.2 Re-ordered, final utilitarian form

By the identity $\varepsilon^{\alpha\mu\nu}\epsilon^{\beta\gamma\kappa\rho} = -4\eta^{\beta\gamma} \eta^{\kappa\rho} \eta^{\nu\mu} \delta^{\alpha}_{\beta} \delta^{\gamma}_{\nu} \delta^{\kappa}_{\rho}$, Eq. (5) can be written in the concise re-ordered form:

$$T^{\alpha\beta} = \Box \tilde{a}^{\alpha\beta} - \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} + \eta^{\beta\alpha} \partial_{\mu} A^{\mu} + 4\eta^{\beta\alpha} \partial_{\mu} \partial^{\mu} a_{\nu\lambda}^{\ \alpha\mu},$$  \hspace{1cm} (6)$$
in terms of a symmetric second rank tensor field $\tilde{a}^{\alpha\beta}$ and a vector field $A^{\mu}$ defined as:

$$\tilde{a}^{\alpha\beta} := 4\eta^{\beta\mu} a_{\nu\lambda}^{\ \lambda\nu} - \eta^{\beta\mu} a_{\nu\lambda}^{\ \nu\lambda},$$  \hspace{1cm} (7)$$
$$A^{\mu} := \partial_{\nu} \tilde{a}^{\nu\mu} = 4\partial^{\lambda} a_{\nu\lambda}^{\ \nu\mu} - \partial^{\mu} a_{\nu\lambda}^{\ \nu\lambda},$$  \hspace{1cm} (8)$$

with the d’Alembertian $\Box := \partial_{\mu} \partial^{\mu} = \eta^{\mu\nu} \partial_{\nu} \partial_{\mu} = \frac{1}{c^2} \partial^2 - \nabla^2$. Next, a rigorous analysis [4] reveals that by a proper gauging of the fourth rank tensor potential $a_{\nu\lambda\kappa\rho}$ elimination of the last term in (6), $4\eta^{\beta\lambda} \partial_{\mu} \partial^{\mu} a_{\nu\lambda}^{\ \alpha\mu}$ is possible, enabling equation (6) to be written as the final, utilitarian, form of the first integral:

$$T^{\alpha\beta} = \Box \tilde{a}^{\alpha\beta} - \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} + \eta^{\beta\alpha} \partial_{\mu} A^{\mu},$$  \hspace{1cm} (9)$$

containing only the second rank tensor $\tilde{a}^{\alpha\beta}$ and its divergence $A^{\mu} = \partial_{\nu} \tilde{a}^{\nu\mu}$. The original fourth rank tensor $a_{\nu\lambda\kappa\rho}$ need be considered no longer, reducing the relevant number of potential fields from 21 to 10.

In [4], the new approach is exemplified by solving the classical problem, involving compressibility, of propagating acoustic waves starting from the above 4D generalisation. The associated analysis reveals that: (i) in the case of inviscid flow a d’Alembert type equation results directly, via a simplified procedure compared to the classical approach, as a linear combination of the linearised field equations with the expression obtained for the speed of sound being in accordance with the same reported in the open literature; (ii) when viscous effects are present, spatial exponential damping of the sound waves arises, once more in full agreement with earlier results.

**Acknowledgements**  We wish to thank the German Research Foundation (DFG), for their support via research grant SCHO 767/6-3. Open access funding enabled and organized by Projekt DEAL.

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