Research Article

Asymptotic Solution for a Class of Epidemic Contagion Ecological Nonlinear Systems

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In this paper, a class of systems for epidemic contagion is considered. An epidemic virus ecological model is described. Using the generalized variation iteration method, the corresponding approximate solution to the nonlinear system is obtained and the method for this approximate solution is pointed out. The accuracy of approximate solution is discussed, and it can control the epidemic virus transmission by using the parameters of the system. Thus, it has the value for practical application.

1. Introduction

Many results on epidemic virus transmission have been studied in the medical science and ecological environment [1–4]. There is a model of the nonlinear differential system for its basic physical phenomena. Currently, the solving method of the nonlinear epidemic virus population system is an important field [5–8].

The nonlinear differential system is an attractive investigated subject. Adams et al. considered the possible “shock” patterns that can exist in the solution to a singular perturbed, third-order nonlinear ordinary differential equation arising as the travelling-wave reduction of the Kuramoto–Sivashinsky equation. In particular, the existence (or otherwise) of oscillatory shocks and multiple shocks made up of combinations of oscillatory and monotonic shocks is examined, using an optimal truncation strategy to track crucial exponential small terms lying beyond all orders of the (divergent) algebraic expansion. The results provide further understanding of numerical solutions previously obtained by others, as well as giving a methodology which is applicable to much broader classes of differential equations exhibiting multiscale phenomena; they also afford the same new insight into the multiscales technique [9]. Bell and Deng studied singular perturbation of N-front traveling waves in the Fitzhugh–Nagumo equations [10]. Many approximate methods have been developed, such as the boundary layer method, the matched asymptotic method, and the multiple scales method. Many scholars have performed a great deal of work [11–21]. The authors also studied a class of ecological environment systems, such as HIV transmission, the solitary wave, the reaction diffusion system, and the model of prey and predator [22–28].

He proposed a novel variational approach for limit cycles of a kind of nonlinear oscillators. Some examples are given to illustrate the effectiveness and convenience of the method. The obtained results are valid for the whole solution domain with high accuracy [29]. Tao et al. discussed the variational approach and the Hamiltonian approach to nonlinear oscillators systematically. They gave an extension of the methods to a singular nonlinear oscillator by the iteration perturbation method [30]. In this paper, using the generalized variational iteration method, we study a class of epidemic virus population systems.

The rest of this paper is organized as follows. In Section 2, we studied a class of epidemic virus transmission population systems and then introduced the names of the variables in the equation and the boundary conditions. In Section 3, we constructed the generalized variation iterations, obtained the first approximations, used the same method, from the generalized variation iterations and the variation principle, and obtained higher-order approximate solution sequences to the system, and these sequences converge uniformly. In Section 4, we gave an example, using
the generalized variation iteration method and we obtained
the approximate solution to the system. In Section 5, we
solved asymptotic expansions to the epidemic virus trans-
mittance population dynamic system by using the perturbation
method. Comparing the obtained approximate solutions by
using two methods, we see that the approximate solution to
the epidemic virus transmission population dynamic system
by using the method of the generalized functional variational
iteration possesses a good accuracy. In Section 6, we con-
sidered the virus transmission parameters, obtained the
simulated curves for the approximate solution, and took
measures to control the virus transmission. In Section 7, we
obtained a conclusion, simplified the basic model, and solved
it by using the approximate method. The method of the
generalized variational iteration is simple and valid.

2. Epidemic Virus Population System

The following is a class of epidemic virus transmission
population systems [5, 6]:
\[
\frac{dS}{dt} - \alpha + \beta K(S)I + aS = 0, \tag{1}
\]
\[
\frac{dI}{dt} - \beta K(S)I + (a + b + c)I - dR = 0, \tag{2}
\]
\[
\frac{dR}{dt} - \gamma K(S)I + (a + c + d)R = 0, \tag{3}
\]
where \(S, I, \text{ and } R\) are the numbers of the sufferer population,
susceptible population, and potential population in the
broken area, respectively, \(t\) is the time, \(\alpha\) is an immigrate rate
of the sufferer population, \(a\) is the natural death rate, \(c\) and
\(e\) are the death rate of susceptible and potential populations,
respectively, \(b\) is the change rate of the sufferer population
changed into the potential population, and \(d\) is the lost rate
of immunity in potential population. The abovementioned \(e\)
is a nonnegative constant, other parameters are positive
constants, \(a + b + c > d\), and \(K(S)\) and \(\gamma K(S)\) are nonlinear
infectiousness for the sufferer population and potential
population, respectively. Also, \(K(S)\) is a sufficiently smooth
function in \((-\infty, \infty)\). Specially, nonlinear function \(K(S)\)
possesses the following bounds:
\[
aS^b, \ b > 0; \tag{4}
\]
\[
\frac{bS^n}{a + S^n}, \ a, b > 0, \ n \geq 1;
\]
\[
a(1 - \exp(-bS)), \ a, b > 0; \tag{10}
\]
\[
\frac{aS}{1 + bS + \sqrt{1 + 2bS}}, \ a, b > 0;
\]
\[
\frac{1 - \exp(-bS)}{1 + a\exp(-bS)}, \ a, b > 0,
\]
and so on.

3. Generalized Variation Iteration System

In order to obtain the approximate solution to systems
\((1)\)–\((3)\), we introduce a set of functional \(F_i (i = 1, 2, 3)\)
[27, 28]:
\[
F_1 = S - \int_0^t \lambda_1(\tau) \left[ \frac{dS}{d\tau} - \alpha + \beta K(S)I + aS \right] d\tau, \tag{5}
\]
\[
F_2 = I - \int_0^t \lambda_2(\tau) \left[ \frac{dI}{d\tau} - \beta K(S)I + (a + b + c)I - dR \right] d\tau, \tag{6}
\]
\[
F_3 = R - \int_0^t \lambda_3(\tau) \left[ \frac{dR}{d\tau} - \gamma K(S)I + (a + d + e)R \right] d\tau, \tag{7}
\]
where \(\tilde{S}, \tilde{I}, \text{ and } \tilde{R}\) are the restricted variables of \(S, I, \text{ and } R\),
respectively, while \(\lambda_i \ (i = 1, 2, 3)\) are the Lagrange
multiplicators.

Compute the variations \(\delta F_i \ (i = 1, 2, 3)\) of functional and
equate to zero as follows:
\[
\delta F_1 = \delta S - (\lambda_1 \delta S) \bigg|_{t=\tau} + \int_0^t \left( \frac{\partial \lambda_1}{\partial \tau} + a\lambda_1 \right) \delta S \, d\tau = 0, \tag{8}
\]
\[
\delta F_2 = \delta I - (\lambda_2 \delta I) \bigg|_{t=\tau} + \int_0^t \left( \frac{\partial \lambda_2}{\partial \tau} + (a + b + c)\lambda_2 \right) \delta I \, d\tau = 0, \tag{9}
\]
\[
\delta F_3 = \delta R - (\lambda_3 \delta R) \bigg|_{t=\tau} + \int_0^t \left( \frac{\partial \lambda_3}{\partial \tau} + (a + d + e)\lambda_3 \right) \delta R \, d\tau = 0. \tag{10}
\]

From equations (8)–(10), we have
\[
\frac{\partial \lambda_1}{\partial \tau} + a\lambda_1 = 0, \tag{11}
\]
\[
\frac{\partial \lambda_2}{\partial \tau} + (a + b + c)\lambda_2 = 0,
\]
\[
\frac{\partial \lambda_3}{\partial \tau} + (a + d + e)\lambda_3 = 0,
\]
\[
\lambda_i (\tau) \bigg|_{t=\tau} = 1, \ i = 1, 2, 3. \tag{12}
\]

From problems (11) and (12), we obtain
\[
\lambda_1 (t) = \exp \{a (t - \tau)\}, \tag{13}
\]
\[
\lambda_2 (t) = \exp \{(a + b + c)(t - \tau)\}, \tag{14}
\]
\[
\lambda_3 (t) = \exp \{(a + d + e)(t - \tau)\}. \tag{15}
\]

From equations (5)–(7) and (13)–(15), we can construct
the following generalized variation iterations:
In the variation principle, we can obtain sequences solution to systems (1)–(3). From the generalized variation iterations (16)–(18), we can obtain higher-order approximate solutions to systems (19)–(21), so we have

\[
S_{n+1} = S_n - \int_0^t \exp \left[ a(t - \tau) \left( \frac{dS_n}{d\tau} - \alpha + \beta K(S_n)I_n + aS_n \right) \right] d\tau, \quad n = 0, 1, \ldots, \tag{16}
\]

\[
I_{n+1} = I_n - \int_0^t \exp \left[ (a + b + c)(t - \tau) \left( \frac{dI_n}{d\tau} - \beta K(S_n)I_n + (a + b + c)I_n - dR_n \right) \right] d\tau, \quad n = 0, 1, \ldots, \tag{17}
\]

\[
R_{n+1} = R_n - \int_0^t \exp \left[ (a + d + e)(t - \tau) \left( \frac{dR_n}{d\tau} - \gamma K(S_n)I_n + (a + d + e)R_n \right) \right] d\tau, \quad n = 0, 1, \ldots \tag{18}
\]

First, we select a set of initial approximations \((S_0(t), I_0(t), R_0(t))\). Also, it is a set of solutions to the linear system as follows:

\[
\frac{dS_0}{dt} = \alpha - aS_0, \tag{19}
\]

\[
\frac{dI_0}{dt} = -(a + b + c - d)I_0 + dR_0, \tag{20}
\]

\[
\frac{dR_0}{dt} = -(a + d + e)R_0. \tag{21}
\]

From equations (16)–(18), we can obtain the first approximations:

\[
S_0(t) = C_0^a e^{(-\alpha + a)t}, \tag{22}
\]

\[
I_0(t) = C_0^b e^{(-(a + b + c - d)t) - \frac{dc_0^b}{a + d + e} e^{-(a + d + e)t}}, \tag{23}
\]

\[
R_0(t) = C_0^c e^{(-(a + d + e)t)}, \tag{24}
\]

where \(C_0^i (i = 1, 2, 3)\) are the arbitrary constants, which can be decided by the conditions of systems (1)–(3).

Using the same method, from generalized variation iterations (16)–(18), we can obtain higher-order approximate solution to systems (1)–(3).

From the generalized variation iterations (16)–(18) and the variation principle, we can obtain sequences \(\{S_n(t)\}, \{I_n(t)\}\), and \(\{R_n(t)\}\). Also, these sequences converge uniformly [31, 32].

Using the same method, from generalized variation iterations (16)–(18), we can obtain higher-order approximate solution to systems (1)–(3).

Let

\[
S(t) = \lim_{n \to \infty} S_n(t), \tag{28}
\]

\[
I(t) = \lim_{n \to \infty} I_n(t), \tag{28}
\]

\[
R(t) = \lim_{n \to \infty} R_n(t). \tag{28}
\]

Thus, we have the following theorem.
Theorem 1. If $K(S)$ is a sufficiently smooth bounded function in $(-M, M)$, where $M$ is a sufficiently large constant, and

$$ aS^b, \, b > 0, $$

$$ \frac{bS^n}{a + S^n}, \, a, b > 0, \, n \geq 1, $$

$$ a(1 - \exp(-bS)), \, a, b > 0, $$

then $(S(t), I(t), R(t))$ is the exact solution to systems (1)–(3) in $(-M, M)$.

4. Example

For convenience, from (1)–(3), we set that $a = b = c = d = a = \delta = 1, \, \epsilon = 0, \, 0 < \beta = \gamma = \epsilon \ll 1$, and $K(S) = S$. Thus, we consider the following example:

$$ S_1(t) = \exp(-t) + t - \epsilon \int_0^t \exp(-t - \tau) \left[ \left( \exp(-\tau) + \frac{1}{2} \exp(-2\tau) + 1 \right) \right] \, d\tau, $$

$$ I_1(t) = \frac{1}{2} \exp(-2t) + t \int_0^t \exp(3(t - \tau)) \left[ \left( \exp(-\tau) + \frac{1}{2} \exp(-2\tau) + 1 \right) \right] \, d\tau, $$

$$ R_1(t) = \exp(-2t) + \epsilon \int_0^t \exp(2(t - \tau)) \left[ \left( \exp(-\tau) + \frac{1}{2} \exp(-2\tau) + 1 \right) \right] \, d\tau. $$

Using the same method, from generalized variation iterations (33)–(35), we can obtain the higher-order approximate solution $(S_n(t), I_n(t), R_n(t))$ ($n = 2, 3, \cdots$) to systems (5)–(7), and their constructions are omitted.

5. Accuracy of Approximate Solution

Now, we assume the absolute nondimensional parameters to systems (30)–(32). From Figures 1–3, we show the simulated curves: the real curve is the exact solution $(S(t), I(t), R(t))$ and the approximation curve is the first approximation solution $(S_1(t), I_1(t), R_1(t))$.

On the other hand, we can also solve asymptotic expansions to epidemic virus transmission population dynamic systems (30)–(32) by using the perturbation method.

Using the generalized variation iteration method, we can obtain the approximate solution to systems (30)–(32).

From equations (22)–(24), we have the initial approximate solution $(S_0(t), I_0(t), R_0(t))$ to systems (30)–(32). Also, we have

$$ S_0(t) = \exp(-t) + t, $$

$$ I_0(t) = \frac{1}{2} \exp(-2t), $$

$$ R_0(t) = \exp(-2t). $$

where we can take $c_i^0 = 1, i = 1, 2, 3$.

From equations (25)–(27) and (33) and (34), we can obtain the first approximations:

$$ \frac{dS}{dt} - 1 + S + \epsilon IS = 0, $$

$$ \frac{dI}{dt} + 3I - R - \epsilon IS = 0, $$

$$ \frac{dR}{dt} + 2R - \epsilon IS = 0. $$

Let the solution $(\bar{S}, \bar{T}, \bar{R})$ to systems (30)–(32) have the following form:

$$ \bar{S} = \sum_{n=0}^{\infty} \bar{S}_n(t) \epsilon^n, $$

$$ \bar{T} = \sum_{n=0}^{\infty} \bar{T}_n(t) \epsilon^n, $$

$$ \bar{R} = \sum_{n=0}^{\infty} \bar{R}_n(t) \epsilon^n, $$

$$ 0 < \epsilon \ll 1. $$

Substituting equation (39) into systems (30)–(32), developing nonlinear terms in $\epsilon$, equating coefficients of the same powers of $\epsilon$, and letting initial values be $(S(0), I(0), R(0)) =$
(1, 1, 1) for systems (30)–(32), we can obtain $\bar{S}_n, T_n, R_n$ ($n = 0, 1, \ldots$) successively. For $n = 0$, we have
\begin{align*}
\bar{S}_0(t) &= \exp(-t) + t, \\
T_0(t) &= \frac{1}{2} \exp(-2t). \\
R_0(t) &= \exp(-2t).
\end{align*}
(40) (41) (42)

Substituting equations (39)–(42) into expansions (30)–(32), we obtain the first-order perturbed solution $(\bar{S}_{app}, T_{app}, R_{app})$ to systems (30)–(32):
Comparing obtained approximate solutions (36)–(38) with (43)–(45) by using the above two methods, respectively, we observe that they are the same for the first-order precision \(O(\varepsilon), \ 0 < \varepsilon \ll 1\). Therefore, we can see that the
approximate solution \((S(t), I(t), R(t))\) to epidemic virus transmission population dynamic systems (30)–(32) by using the method of the generalized functional variational iteration possesses a good accuracy.

6. Change Virus Transmission Parameters

We can obtain the simulated curves for the approximate solution and take measures to control the virus transmission.

Now, we assume the absolute nondimensional parameters of systems (25)–(27) as follows:

1. Take the virus transmission parameters as \(a = b = c = 1\), \(\alpha = d = e = 0\), \(\beta = \gamma = 0.5\) or \(\beta = \gamma = 5\), and \(K(S) = S\), respectively. We can obtain the simulated curves \(S(t), I(t),\) and \(R(t)\) (see Figures 4–6). Also, we show the simulated curves of \(S(t), I(t),\) and \(R(t)\), respectively, where the real curve is the case as \(\beta = \gamma = 0.5\) and the ‘+’ curve is the case as \(\beta = \gamma = 5\).

2. Take the virus transmission parameters as \(a = b = c = 1\), \(\alpha = d = e = 0\), \(\beta = \gamma = 5\), and \(K(S) = S\) or \(K(S) = S^5\), respectively. We can obtain the simulated curves \(S(t), I(t),\) and \(R(t)\) (see

![Figure 5: Illustration for the curve of \(I(t)\) (real curve as \(\beta = \gamma = 0.5\) and ‘+’ curve as \(\beta = \gamma = 5\)).](image)

![Figure 6: Illustration for the curve of \(R(t)\) (real curve as \(\beta = \gamma = 0.5\) and ‘+’ curve as \(\beta = \gamma = 5\)).](image)

![Figure 7: Illustration for the curve of \(S(t)\) (real curve as \(K(S) = S\), and ‘o’ curve as \(K(S) = S^5\)).](image)

![Figure 8: Illustration for the curve of \(I(t)\) (real curve as \(K(S) = S\), and ‘o’ curve as \(K(S) = S^5\)).](image)
Figures 7–9). Also, we show the simulated curves of \( S(t) \), \( I(t) \), and \( R(t) \), respectively, where the real curve is the case as \( K(S) = S \) and the ‘+’ curve is the case as \( K(S) = S^5 \).

Thus, from Figures 4–9, we can use the parameters to control the virus transmission.

7. Conclusions

The epidemic virus transmission is a complicated phenomenon. Hence, we need to simplify the basic model and solve it by using the approximate method. The method of the generalized variational iteration is simple and valid.

The generalized variational iteration is an approximate method, which differs from the general numerical method. The expansion of the solution can also be used analytically. Thus, we can further study the qualitative and quantitative behaviors of the epidemic virus transmission population for the infected and the susceptible populations in the broken area.

Data Availability

The authors declare that the data in the article are available, shareable, and referenced. Readers can check each article at http://apps.webofknowledge.com/.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Feng Yi-Hu and Hou Lei wrote the original draft.

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