Geometry of gauged LG-model with compatible boundary conditions

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Abstract

This paper introduces the geometry of the open string Floer theory of gauged Landau-Ginzburg model via gauged Witten equations. Given a $G$-invariant Morse-Bott holomorphic function $W$ on a Hamiltonian space $(M, \omega, G)$, Lefschetz thimbles are constructed from proper Lagrangian submanifolds of critical set of $W$. We study an energy functional on path space whose gradient flow equation corresponds to the gauged Witten equations with temporal gauge on a strip end, and whose critical points are Lagrangian intersections in the reduced critical space of $W$.

Keywords  LG model, Symplectic reduction, Lefschetz thimble, Fredholm theory

1 Introduction

In 1994, Kontsevich introduced the Homological Mirror Symmetry (HMS) conjecture [18] inspired by mirror symmetry. If $M$ and $M'$ are mirror Calabi-Yau manifolds, HMS conjecture predicts the categorical equivalence between Fukaya category of $M$ and the derived category of coherent sheaves on $M'$, and vice versa. HMS has been proved for many cases, for example, elliptic curve [25, 26], Abelian varieties [15, 1], Strominger-Yau-Zaslow fibrations [19], the quartic K3 surface [30] and the Calabi-Yau hypersurfaces [31].

On the other hand, physicists [32, 33] also consider more general $N = 2$ field theories and proposed another mirror symmetry which is connected LG singularity model [8] and Calabi-Yau (CY) geometry. It was not until 2010 the LG/CY correspondence could be made precise [4, 5]. Orlov constructed the open string LG B-model in [22, 24] and established LG/CY correspondence on the B-side in [23]. In fact, Witten [33] generalized the LG/CY correspondence to complete intersections in projective space, or more

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generally to toric varieties. He invented an important model called the Gauged Linear Sigma Model (GLSM). The model generalizes the Witten equation to the Gauged Witten equations

\[
\begin{align*}
\bar{\partial} A u_i + \frac{\partial W}{\partial u_i} &= 0, \\
* F_A - \mu + \tau &= 0,
\end{align*}
\]

where \( A \) is a connection of a certain principal bundle on a Riemann surface, and \( \mu \) is the moment map of the GIT-quotient. Under GLSM, the LG/CY correspondence can be interpreted as a variation of the moment map \( \mu \). Lately, Fan-Jarvis-Ruan constructed a rigorous mathematical theory for GLSM [9, 10].

The main purpose of this paper is to construct a gauged categorization theory by studying gauged Witten equations. The base space is a principal \( G \)-bundle \( P \) over a Riemann surface \( C \) with strip ends which is labeled by Lefschetz thimbles. Given a non-compact Kähler manifold \((M, \omega)\) with a Hamiltonian action of \( G \) and a holomorphic \( G \)-invariant function \( W \) on \( M \), and fix a total real section \( \sigma \) of the log-canonical bundle of \( C \), we define a operator \( X_\sigma \forall \phi \in C^\infty(P, M) \),

\[
X_\sigma(\phi) = \text{Re}(X_W(\phi) \otimes \sigma) \in \Omega^{0,1}_G(P, \varphi^* TM).
\]

We consider the following first order partial differential equations

\[
\begin{align*}
\bar{\partial} A(\varphi) + X_\sigma(\varphi) &= 0, \\
* F_A + \mu(\varphi) - \tau &= 0,
\end{align*}
\]

which is equivalent to (1), where \( A \) is a connection of \( P \) and \( \varphi \) is a \( G \)-equivariant map from \( P \) to \( M \).

To set up proper boundary conditions of (2), we use suitable Lagrangian submanifolds (satisfying (C2)-(C4) in §3.1) of stable critical space \( \text{Crit}^s(W) \) of \( W \) to construct Lefschetz thimbles as Lagrangian branes. The method used in [17] is as follows. Let \( \psi_t \) be the gradient flow on \( M \) relative to the Hamiltonian vector field of \( \text{Im}(W) \). Keeping a connected component \( B \) of \( \text{Crit}^s(W) \) fixed, we consider a fiber bundle \( W^s(B) \) over \( B \) which is given by

\[
W^s(B) = \bigcup_{p \in B} W^s(p),
\]

with the projection map \( \psi_\infty = \lim_{t \to +\infty} \psi_t : W^s(B) \to B \), where the fiber \( W^s(p) \) over \( p \) is the stable manifold of \( p \). Each Lagrangian submanifold \( \mathbb{L} \) of \( B \) offers a pull-back bundle \( \psi^{-1}(\mathbb{L}) \), that is called a Lefschetz thimble. Different from the boundary setting in [14], we require connections \( A \) in (2) to be compatible with fixed Lagrangian labels so that Proposition 15 and Proposition 16 hold. Precisely, if \( A = \Phi ds + \Psi dt \) under a local orthogonal coordinate \((s, t)\) around a boundary point in \( C_i \subset \partial C \), with \( (\frac{\partial}{\partial s}, \frac{\partial}{\partial t})|_{t=0} = 0 \) and \((s,0) \in C_i \), then we require \( \Phi(s,0) \in g_{L_i}, \Psi(s,0) \in g_{L_i}^\perp \) and \( \partial_s \Phi(s,0) \in g_{L_i}^\perp \). We will see that such a boundary setting depends on a choice of a section \( \kappa \in \Gamma(\partial \hat{C}, P_{|\partial \hat{C}}) \).
1.1 Outline

In §2 we mainly review the definition of gauged Witten equations, including some necessary data. In §3 we construct Lefschetz thimbles and set up boundary conditions for configuration space. Similar to symplectic vortex equations [6, 14], we introduce in §4.1 the Yang-Mills-Higgs energy of a pair \((A, \varphi)\) which is defined by

\[
E_W(A, \varphi) := \frac{1}{2} \int_C \left( |dA\varphi|^2 + |FA|^2 + |\mu(\varphi) - \tau|^2 + 2|X_\sigma(\varphi)|^2 \right) dV_C. \tag{3}
\]

We show that the equations (2) are the variational equation of \(E_W\). The energy of a solution to (2) depends only on its limiting paths and homotopy class. In §4.2 we study the Morse-Bott type action functional \(F_W\) on the path space \(\mathcal{P}(L_0, L_1)\) (not Novikov path space since we assume Lefschetz thimbles are simply connected) with respect to each marked point. The critical set of \(F_W\) modulo gauge transformations can be described as Lagrangian intersection in reduced critical space of \(W\). In §5 we study the Fredholm theory of (2).

2 Review of Gauged Witten Equations

2.1 Moment map in gauge theory

We start with a brief review of Hamiltonian group actions. Some references are [7] and [2]. Let \(G\) be a connected compact Lie group and let \(\mathfrak{g}\) be its Lie algebra with an invariant inner product \(\langle , \rangle_{\mathfrak{g}}\). Assume that \((M, \omega)\) is a symplectic manifold with a Hamiltonian \(G\)-action \(\rho: G \to \text{Sym}(M, \omega)\) which induces a homomorphism \(\rho_*: \mathfrak{g} \to \mathfrak{X}(M, \omega)\). Let \(\mu: M \to \mathfrak{g}\) be a moment map associated with \(\rho\) which satisfies

(i) \(\langle d\mu(x)Y, \eta \rangle_{\mathfrak{g}} = \omega(X_\eta, Y), \ \forall x \in M, \ Y \in T_xM, \ \eta \in \mathfrak{g}\),

(ii) \(\mu \circ \rho_g = \text{Ad}(g^{-1}) \circ \mu, \ \forall g \in G,\)

where \(X_\eta\) stands for \(\rho_*(\eta)\) and \(\text{Ad}\) is the adjoint action of \(G\). We use the notation \(gx\) for the action of \(g \in G\) on \(x \in M\), instead of the complete expression \(\rho(g)x\).

Let \(\tau\) be a regular value of \(\mu\). The isotropy subgroup \(G_x\) of \(\tau\) acts freely on \(\mu^{-1}(\tau)\). We have a Marsden-Weinstein quotient \(\bar{M}(\tau) := \mu^{-1}(\tau)/G_x\) carrying an inherited symplectic structure from \(M\) as follows. Let \(\mathfrak{g}_x\) denote be the Lie algebra of \(G_x\). For any \(x \in \mu^{-1}(\tau)\), there is a chain complex

\[
0 \longrightarrow \mathfrak{g}_x \xrightarrow{L_x} T_xM \xrightarrow{d\mu(x)} \mathfrak{g} \longrightarrow 0,
\]

where \(L_x\xi := X_\xi(x), \ \forall \xi \in \mathfrak{g}\). The tangent space of \(\bar{M}(\tau)\) at \([x]\) has a symplectic structure since it can be identified with \(\ker d\mu(x)/L_x(\mathfrak{g}_x)\).

Let \(\pi: P \to C\) be a trivial principal \(G\)-bundle over a Riemann surface \(C\) with boundary. We denote by \(C^\infty_G(P, M)\) the set of all \(G\)-equivariant maps from \(P\) to \(M\).
For each $\psi \in C^\infty_G(P,M)$, the infinitesimal transformation of group action provides two linear operators

\[ L_\psi : \Omega^0(C, g_P) \to \Omega^0_G(P, \psi^*TM), \quad \alpha \mapsto X_\alpha(\psi), \]

\[ L_\psi^1 : \Omega^1(C, g_P) \to \Omega^1_G(P, \psi^*TM), \quad \beta \mapsto X_{\beta(\psi)}, \]

where $g_P$ is the vector bundle associated with $P$ by the adjoint action of $G$, and $\Omega^i_G(P, \psi^*TM)$ is the space of $i$-forms with values in $P \times_G \psi^*TM$. Choose a metric $h_M$ on $M$ which is compatible with the almost complex structure $J$ and $G$, i.e., $h_M = \omega(\cdot, J\cdot)$ and $g^*J = J, \forall g \in G$. Then the following identities hold:

\[ L_\psi = -J(d\mu(\psi))^*, \quad L_\psi^* = d\mu(\psi)J. \tag{4} \]

Let $\mathcal{A}(P)$ be the space of all connections on $P$. Atiyah and Bott [2] noticed that the affine space $\mathcal{A}(P)$ is an infinite-dimensional symplectic manifold with symplectic form

\[ \alpha_1 \times \alpha_2 \mapsto \int_C (\alpha_1 \wedge \alpha_2)_g. \]

The action of gauge group $G(P) := \Gamma(C, P \times \text{Ad} G)$ on $\mathcal{A}(P)$ is Hamiltonian, with curvature as its moment map. The Marsden-Weinstein quotient space of the moment map is the moduli space of flat connections. As observed in [7], the space $\mathcal{B}(P) := \mathcal{A}(P) \times C^\infty_G(P,M)$ is also an infinite-dimensional symplectic manifold with symplectic form

\[ ((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \int_C (\alpha_1 \wedge \alpha_2)_g + \int_C \omega(\beta_1, \beta_2)dV_C. \]

The action of gauge group on $\mathcal{B}(P)$ giving by

\[ g(A, \psi) = (gAg^{-1} + gdg^{-1}, g\psi) \]

is Hamiltonian, whose moment map is given by

\[ \mathcal{A}(P) \times C^\infty_G(P,M) \to \Omega^0(C, g_P), \quad (A, \psi) \mapsto *F_A + \mu(\psi). \]

We note that the connections and gauge transformations of $P$ have special expressions. Since $P$ is trivial, one can choose a trivialization $\phi$ of $P$, i.e., $\phi : C \times G \to P$ is an isomorphism, and there is a global section $\kappa \in \Gamma(C, P)$ defined by $\kappa(x) = \phi(x, e)$, where $e$ is the identity element of $G$. Then the Maurer-Cartan form $\varpi$ on $G$ can be regarded as a connection to $P$ with respect to $\kappa$. Each connection $A$ on $P$ has the form $A = \kappa^*(A) + \varpi$ along $\kappa$, where $\kappa^*A \in \Omega^1(C, \mathfrak{g})$. Moreover, we have an isomorphism with respect to $\kappa$

\[ \mathcal{A}(P) \cong \varpi + \Omega^1(C, \mathfrak{g}). \tag{5} \]

Besides, each $\psi \in C^\infty_G(P,M)$ is determined by $\psi \circ \kappa$, we have another isomorphism

\[ C^\infty_G(P,M) \cong C^\infty(C, M) \tag{6} \]
with respect to $\kappa$. Similarly, a gauge transformation $g$ of $P$ can be considered as a function $\tilde{g} = g \circ \kappa : C \to G$ along $\kappa$ which acts on $P$ by

$$g\phi(x, h) = \phi(x, \tilde{g}(x)h), \ \forall (x, h) \in C \times G.$$ 

In the case of no ambiguity, we sometimes omit $\phi, \kappa$ and write $\kappa^* A, g \circ \kappa$ as $A, g$ respectively.

2.2 Totally real section of the log-canonical bundle

To define the gauged Witten equations on a Riemann surface, we need to choose a totally real section of the log-canonical bundle. Let $\mathcal{C} = (C, x_1, \cdots, x_k)$ be a closed Riemann surface with marked points $x_1, \cdots, x_k$. The log-canonical bundle of $\mathcal{C}$ is defined as

$$K_{\log, \mathcal{C}} := K_C \otimes \mathcal{O}(x_1) \otimes \cdots \otimes \mathcal{O}(x_k).$$

By Riemann-Roch theorem and Serre duality, we have the dimensional formula

$$\dim C H^0(C, K_{\log, \mathcal{C}}) = g(C) + k - 1.$$ 

When $k \geq 2$ or $k = 0, g(C) > 1$, the line bundle $K_{\log, \mathcal{C}}$ always admits non-trivial global sections.

However, when $\partial C \neq \emptyset$ and $x_1, \cdots, x_k$ are all boundary marked points, we equip $K_{\log, \mathcal{C}}$ with a totally real subbundle $K_{\log, \mathcal{C}}^R$ of $K_{\log, \mathcal{C}}$ over $\partial C$. A section $\sigma \in \Gamma(C, K_{\log, \mathcal{C}})$ is totally real if $\bar{\partial} \sigma = 0$ and $\sigma|_{\partial C} \in \Gamma(\partial C, K_{\log, \mathcal{C}}^R)$. It follows in [11] that

$$\dim \mathbb{R} H^0(C, \partial C, K_{\log, \mathcal{C}}) = k - 1.$$ 

Fix a nonzero totally real section $\sigma$ of $K_{\log, \mathcal{C}}$, we endow $C$ with a singular metric as $dV_C = \frac{i}{2} \sigma \wedge \bar{\sigma}$. For every $p \in C$ distinct from $x_1, \cdots, x_k$ with $\sigma(p) \neq 0$, one can choose an appropriate local coordinate $(U, z)$ at $p$ such that $\sigma|_U = -idz$.

2.3 Gauged Witten equations and perturbed equations

In this subsection, we establish gauged Witten equations on a punctured Riemann surface which is equipped with some extra data.

In order to study the holomorphic curves in $P \times_G M$, one need to define a twisted Cauchy-Riemann operator. Each connection $A \in \mathcal{A}(P)$ corresponds a covariant differentiation $d_A$ which can be thought of as a section of an infinite vector bundle. More specifically, consider the Fréchet bundle $\mathcal{V} \to C^\infty_G(P, M)$ whose fiber over $\varphi$ is given by $\Omega^1_G(P, \varphi^*TM)$. The covariant differentiation is defined by

$$d_A : C^\infty_G(P, M) \to \mathcal{V}, \ \varphi \mapsto d\varphi \circ h_A,$$

where $h_A$ is a horizontal projection on $TP$ determined by $A$. It is showed in [6, §2.1] that $d_A$ satisfies

$$d_A \varphi = d\varphi + L_\varphi A.$$  (7)
Since $dA\varphi$ is equivariant and horizontal, hence descends to a 1-form on $C$ with values in the quotient bundle $E_\varphi := P \times_G \varphi^*\text{TM} \to C$. We let $\tilde{\varphi} = \varphi \circ \kappa$, then $E_\varphi \cong \tilde{\varphi}^*\text{TM}$, where $\kappa$ is a fixed section of $P$. Define the twisted Cauchy-Riemann operator by

$$\bar{\partial}A\varphi := \frac{1}{2}(dA\varphi + J \circ dA\varphi \circ j_C) \in \Omega^{0,1}(C, E_\varphi),$$

where $j_C$ is the almost complex structure of $C$.

Let $(M, \omega)$ be a non-compact Kähler manifold with a Hamiltonian group action $(G, \mu)$. Suppose that $W$ is a $G$-invariant Morse-Bott holomorphic function on $M$. Denote by $X_{\text{Re}W}$, $X_{\text{Im}W}$ the Hamiltonian vector field corresponding to the real part and imaginary part of $W$. The Cauchy-Riemann equation about $W$ implies $X_{\text{Im}W} = J X_{\text{Re}W}$.

For any $G$-equivariant map $\varphi \in C^\infty_G(P, M)$, there are two $G$-equivariant pull-back vector fields $X_{\text{Re}W}(\varphi), X_{\text{Im}W}(\varphi) \in \Omega^0(C, E_\varphi)$. We define

$$X_\sigma(\varphi) := \text{Re}(X_W(\varphi) \otimes \sigma) \in \Omega^{0,1}(C, E_\varphi),$$

$$X_\sigma(\varphi) := \text{Im}(X_W(\varphi) \otimes \sigma) \in \Omega^{1,0}(C, E_\varphi),$$

where $\sigma$ is a fixed totally real section of $K_{\log, \varphi}$.

Just like symplectic vortex equations, combining the moment map on $B(P)$ and the perturbed twisted Cauchy-Riemann equation, the gauged Witten equations are built up as follows.

**Definition 1.** Given a $G$-invariant Morse-Bott holomorphic function $W$ on $M$, a non-zero totally real section $\sigma \in H^0(C, K_{\log, \varphi})$. The gauged Witten equation with Lagrangian boundary conditions (determined by a Lagrangian labels $\mathcal{L}$ for $\hat{C}$, see §3.2) is defined as

$$\begin{cases}
\bar{\partial}A(\varphi) + X_\sigma(\varphi) = 0, \\
*F_A + \mu(\varphi) = \tau, \\
(A, \varphi) \in B_L(P).
\end{cases} \quad (8)$$

**Proposition 2.** The gauged Witten equations (8) are invariant under the action of gauge group $G(P)$.

**Proof.** By the fact that $W$ is $G$-invariant and $\mu$ is $G$-equivariant, one can check that $G(P)$ preserves (8). \qed

We next calculate the local expression of (8). Choose a local coordinate $(U, z = s + it)$ of $C$ such that $\sigma = -idz$ and a trivialization of $P$. Every connection has the form

$$A = \Phi ds + \Psi dt + \varpi,$$

where $\Phi, \Psi : U \to \mathfrak{g}$ are smooth functions, and $\varpi$ is the Maurer-Cartan form on $G$. Sometimes we use locally $(\Phi, \Psi)$ for $A$. By (7), we have

$$dA\varphi = \left(\frac{\partial \varphi}{\partial s} + X_\Phi(\varphi)\right)ds + \left(\frac{\partial \varphi}{\partial t} + X_\Psi(\varphi)\right)dt,$$

where $j_C$ is the almost complex structure of $C$. 

Let $(M, \omega)$ be a non-compact Kähler manifold with a Hamiltonian group action $(G, \mu)$. Suppose that $W$ is a $G$-invariant Morse-Bott holomorphic function on $M$. Denote by $X_{\text{Re}W}$, $X_{\text{Im}W}$ the Hamiltonian vector field corresponding to the real part and imaginary part of $W$. The Cauchy-Riemann equation about $W$ implies $X_{\text{Im}W} = J X_{\text{Re}W}$.

For any $G$-equivariant map $\varphi \in C^\infty_G(P, M)$, there are two $G$-equivariant pull-back vector fields $X_{\text{Re}W}(\varphi), X_{\text{Im}W}(\varphi) \in \Omega^0(C, E_\varphi)$. We define

$$X_\sigma(\varphi) := \text{Re}(X_W(\varphi) \otimes \sigma) \in \Omega^{0,1}(C, E_\varphi),$$

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where $\Phi, \Psi : U \to \mathfrak{g}$ are smooth functions, and $\varpi$ is the Maurer-Cartan form on $G$. Sometimes we use locally $(\Phi, \Psi)$ for $A$. By (7), we have

$$dA\varphi = \left(\frac{\partial \varphi}{\partial s} + X_\Phi(\varphi)\right)ds + \left(\frac{\partial \varphi}{\partial t} + X_\Psi(\varphi)\right)dt,$$
In addition, $\varpi$ is flat, i.e., $F_{\varpi} = d\varpi + \frac{1}{2}[\varpi, \varpi] = 0$. We have

$$F_A = F_{\varpi} + d\varpi A + \frac{1}{2}[A, A] = \left(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi]\right) ds \wedge dt.$$  \hspace{1cm} (10)

Using $X_{\text{Im}W} = JX_{\text{Re}W}$, one obtains

$$2(\bar{\partial}_A (\varphi) + X_\sigma (\varphi)) = \Box \cdot ds + J(\Box) \cdot dt,$$

where $\Box = \frac{\partial \varphi}{\partial s} + X_\Phi (\varphi) + J(\frac{\partial \varphi}{\partial t} + X_\Psi (\varphi) + X_{2\text{Re}W} (\varphi))$. Locally the equations (8) have the following expression

$$\begin{cases}
\frac{\partial \varphi}{\partial s} + X_\Phi (\varphi) + J(\frac{\partial \varphi}{\partial t} + X_\Psi (\varphi) + X_{2\text{Re}W} (\varphi)) = 0, \\
\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] + \mu (\varphi) - \tau = 0.
\end{cases}$$  \hspace{1cm} (11)

We next study the perturbation of (8). The main reference is [6].

**Definition 3.** Let $Q \in C_\infty^G ([0, 1] \times M)$ be a family of smooth $G$-invariant functions on $M$. Denote $Q$ by $Q_t$ for $t \in [0, 1]$. We call $Q$ an integral of motion of the Hamiltonian system $(M, \omega, W)$ if

$$\forall t \in [0, 1], \{Q_t, \text{Re}W\} = \{Q_t, \text{Im}W\} = 0,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on $C_\infty (M)$ with respect to the symplectic structure $\omega$.

Let $H \in \Omega^1 (C, C_\infty^G (M))$. For any tangent vector $v \in TC$ we denote by $X_{H_v}$ the $G$-invariant Hamiltonian vector field of $H(v)$. Every map $\varphi \in C_\infty^G (P, M)$ determines a form $X_H (\varphi) \in \Omega^1_G (P, \varphi^* TM)$ by

$$X_H (\varphi)(v) = X_{H_{\varphi^*v}} (\varphi(p)), \quad \forall v \in T_p P.$$

Hence $X_H (\varphi)$ descends to a one-form in $\Omega^1 (C, \varphi^* TM/G)$.

**Definition 4.** Let $H \in \Omega^1 (C, C_\infty^G (M))$. We call $H$ a Hamiltonian perturbation with respect to $\sigma$ if for every $j = 1, \cdots, k$, there exists an integral of motion $Q^j$ such that

$$X_H^{0,1} |_{\Theta_j} = \frac{1}{2} JX_{Q^j} ds - \frac{1}{2} X_{Q^j} dt, \quad \sigma = dt - ids$$

on the $j$-th strip end $\Theta_j$.

Given a Hamiltonian perturbation $H$, we replace the first equation in (8) by the perturbed equation

$$\bar{\partial}_A (\varphi) + X_\sigma (\varphi) + X_H^{0,1} (\varphi) = 0.$$  \hspace{1cm} (12)

On each strip end $\Theta_j$, (12) has the expression

$$\frac{\partial \varphi}{\partial s} + X_\Phi (\varphi) + J(\frac{\partial \varphi}{\partial t} + X_\Psi (\varphi) + X_{2\text{Re}W} (\varphi) + X_{Q^j} (\varphi)) = 0.$$  \hspace{1cm} (13)
3 Lefschetz thimble and boundary conditions

3.1 Lefschetz thimble and reduced Lagrangian submanifold

In this section we construct Lefschetz thimbles from Lagrangian submanifolds of critical set of a Morse-Bott function. We say that $x \in M$ is a stable point if the stabilizer group $\{ g \in G | gx = x \}$ of $x$ is finite. Given a $G$-action on $M$, let $\text{Stab}_G(M)$ denote the set of all stable points in $M$.

Let $W$ be a $G$-invariant holomorphic function on the symplectic manifold $(M, \omega)$ and suppose that $W$ is of Morse-Bott type on $\text{Stab}_G(M)$. Clearly the imaginary part of $W$, denoted by $f$ throughout this subsection, is a $G$-invariant real function on $(M, \omega)$.

Let $\psi_t$ denote the flow on $M$ generated by the Hamiltonian vector field $X_f$, i.e.,

$$\frac{d\psi_t(x)}{dt} = X_f(\psi_t(x)), \forall x \in M.$$ 

We note some properties about $\psi_t$. It is obvious that the action of $\psi_t$ on $M$ commutes with $G$. The diffeomorphism $\psi_t$ preserves $f$ and $\omega$. Moreover, we contain the following lemmas.

**Lemma 5.** For any $\xi \in g$, we have $\omega(X_f, X_\xi) = 0$.

**Proof.** For any point $x \in M$, by the fact that the function $f(\exp(t\xi) \cdot x)$ of $t$ is constant, it follows that

$$\frac{d}{dt}|_{t=0} f(\exp(t\xi) \cdot x) = df(X_\xi) = \omega(X_f, X_\xi) = 0.$$ 

$\square$

**Lemma 6.** The diffeomorphism $\psi_t$ preserves the moment map. Furthermore, $\psi_t$ induces a Hamiltonian action on $\mu^{-1}(\tau)/G$.

**Proof.** For any $x \in M$ and $\xi \in g$, by lemma 5, we have

$$\frac{d}{dt}|_{t=0} (\mu(\psi_t(x)), \xi)_g = (d\mu(X_f), \xi)_g = \omega(X_f, X_\xi) = 0.$$ 

This means $\mu \circ \psi_t = \mu$. $\square$

We refer to [3] especially §3 for some facts about Morse-Bott functions. Since $W$ is holomorphic, the imaginary part $f$ is also a Morse-Bott function and we have $\text{Crit}(f) = \text{Crit}(W)$. Assume that the stable critical set

$$\text{Crit}^s(W) := \text{Crit}(W) \cap \text{Stab}_G(M) = \bigcup_{\alpha=1}^{q} B_\alpha$$

has finite connected components. Consider the flow $\psi_t$ relative to the Hamiltonian vector field $X_f$. For $p \in \text{Crit}^s(W)$, the flow $\psi_t$ defines a stable manifold $W^s(p)$ and an unstable
manifold $W^u(p)$ by

$$W^s(p) := \{ x \in M \mid \lim_{t \to +\infty} \psi_t(x) = p \}, \quad W^u(p) := \{ x \in M \mid \lim_{t \to -\infty} \psi_t(x) = p \}.$$  

For each connected component $B \subset \text{Crit}(W)$, one can also define the stable and unstable manifolds to be

$$W^s(B) = \bigcup_{p \in B} W^s(p), \quad W^u(B) = \bigcup_{p \in B} W^u(p).$$

For each critical point $p \in B \subset \text{Crit}(f)$, the Hessian of $f$ determines a quadratic form on the tangent space

$$\text{Hess}_p(f) : T_p M \times T_p M \to \mathbb{R}.$$  

Moreover, the tangent space splits as

$$T_p M = T_p B \oplus \nu^+_p(B) \oplus \nu^-_p(B),$$

where $\text{Hess}_p(f)$ is positive on $\nu^+_p(B)$ and negative on $\nu^-_p(B)$. Let $\lambda^+_p = \dim \nu^+_p(B)$ and $\lambda^-_p = \dim \nu^-_p(B)$. Just like complex Morse theory, we have the following lemma.

**Lemma 7.** Let $W$ be a holomorphic Morse-Bott function with imaginary part $f$ on a complex manifold $M$ and let $B$ be a connected component of the critical set $\text{Crit}(W)$. Then for any $p \in B$, we have $\lambda^+_p = \lambda^-_p$.

Proof. We verify that the complex structure $J : TM \to TM$ of $M$ is an isomorphism between $\nu^+_p(B)$ and $\nu^-_p(B)$. In local holomorphic coordinates $z_j = x_j + \sqrt{-1} y_j, j = 1, \ldots, n$, the complex structure $J$ is given by

$$J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$  

The lemma will be proved by showing that

$$\text{Hess}_p(f)(u, u) + \text{Hess}_p(f)(Ju, Ju) = 0, \quad \forall u \in T_p M, \quad (14)$$

We let $I$ be the identity matrix, $Q = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{n \times n}, R = \left( \frac{\partial^2 f}{\partial x_i \partial y_j} \right)_{n \times n}$ and $S = \left( \frac{\partial^2 f}{\partial y_i \partial y_j} \right)_{n \times n}$. Then (14) can be written as

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} Q & -I \\ R^T & S \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = 0. \quad (15)$$

Now observe that $f$ and the real part of $W$ satisfy the Cauchy-Riemann equation, we have $R = R^T, Q + S = 0$. Hence (15) holds. The proof is completed.  

As a direct result, Lemma 7 tells us that $W^s(p)$ and $W^u(p)$ are of the same dimension $\frac{1}{2}(\dim M - \dim B)$. Since $p \in B$ is a fixed point of $\psi_t$, we have a family of linear maps $\psi_t : T_p M \to T_p M$. It is obvious that

$$\psi_t|_{T_p B} = id, \quad \lim_{t \to +\infty} \psi_t|_{\nu^+_p(B)} = 0, \quad \lim_{t \to -\infty} \psi_t|_{\nu^-_p(B)} = 0. \quad (16)$$

**Lemma 8.** For $p \in B \subset \text{Crit}^s(W)$, the linear space $T_p B$ and $\nu^+_p(B) \oplus \nu^-_p(B)$ are orthogonal with respect to the symplectic form $\omega$. Consequently, $(B, \omega)$ is a symplectic submanifold of $(M, \omega)$.

**Proof.** For any $X \in T_p B, Y \in \nu^+_p(B)$ and $Z \in \nu^-_p(B)$, it follows from (16) and $\psi_t \omega = \omega$ that

$$\omega(X, Y) = \lim_{t \to +\infty} \psi_t \omega(X, Y) = \omega(X, \lim_{t \to +\infty} \psi_t Y) = 0,$$

$$\omega(X, Z) = \lim_{t \to -\infty} \psi_t \omega(X, Z) = \omega(X, \lim_{t \to -\infty} \psi_t Y) = 0.$$

Hence, $\nu^+_p(B) \oplus \nu^-_p(B)$ is the symplectic orthogonal complement of $T_p B$. This signifies that the restriction of $\omega$ on $T_p B$ is nondegenerate. $\square$

Fix a connected component $B$ of $\text{Crit}^s(W)$. In order to get good symplectic quotients, we assume that the Lagrangian submanifold $L$ of $B$ and moment map $\mu$ satisfy the following conditions

(C1) $\mu$ is proper, $\tau \in Z(\mathfrak{g})$ is a regular value of $\mu$ and $G$ acts freely on $\mu^{-1}(\tau)$.

(C2) $L$ is the set of fixed points of an anti-symplectic involution which is compatible with $G$, i.e., there exists a diffeomorphism $\gamma \in \text{Diff}(B)$ such that

$$\gamma^* \omega = -\omega, \quad \gamma^2 = id, \quad L = \text{Fix}(\gamma),$$

and $S(\mathfrak{g}) = \rho^{-1}(\gamma \rho(\mathfrak{g}) \gamma)$ is an automorphism of $G$, where $\rho : G \to \text{Diff}(B)$ is the group action.

(C3) $L \cap \mu^{-1}(\tau) \neq \emptyset$.

(C4) $L$ is connected and simply connected.

Condition (C1) ensure that $B$ intersects transversally with $\mu^{-1}(\tau)$ and we get two symplectic quotient spaces $\bar{M} = \mu^{-1}(\tau)/G$ and $\bar{B} = (B \cap \mu^{-1}(\tau))/G$. Conditions (C2) and (C3) as in [14, §4] guarantee that the reduced space $\bar{L}$ from $L$ is still a Lagrangian submanifold in $\bar{B}$. Condition (C4) assure the functional $\mathcal{F}_W$ (defined in §4.2) on path space is well-defined.

With above assumptions (C1)-(C4), define $G_L := \{g \in G | S(g) = g\}$. One can check that $G_L$ is the isotropy subgroup of $L$. Since the derivative $S_*$ of $S$ is an automorphism of $\mathfrak{g}$, the Lie algebra of $G_L$ and its orthogonal complement are given by

$$\mathfrak{g}_L = \{\eta \in \mathfrak{g} | S_* \eta = \eta\}, \quad \mathfrak{g}_L^\perp = \{\eta \in \mathfrak{g} | S_* \eta = -\eta\}.$$
The Lie algebra $\mathfrak{g}$ decompose into $\mathfrak{g}_L \oplus \mathfrak{g}^\perp_L$ with relations

$$[\mathfrak{g}_L, \mathfrak{g}_L] \subset \mathfrak{g}_L, \quad [\mathfrak{g}_L, \mathfrak{g}_L^\perp] \subset \mathfrak{g}_L^\perp, \quad [\mathfrak{g}_L^\perp, \mathfrak{g}_L^\perp] \subset \mathfrak{g}_L.$$  \hspace{1cm} (17)

Frauenfelder [14, §4] shows that with the assumptions (C1), (C2) and (C3) $L$ intersects transversally with $\mu^{-1}(\tau)$ and $\bar{\mathcal{L}} := (\mathcal{L} \cap \mu^{-1}(\tau))/G_L$ is a Lagrangian submanifold of $\bar{B}$.

**Remark 9.** For $x \in \mathcal{L} \cap \mu^{-1}(\tau)$, the dimension of $T_x\mathcal{L} + T_x\mu^{-1}(\tau)$ equals $\dim B - \dim G_L$, not $\dim B$. It suffices to show that the map $d\mu(x) : T_x\bar{\mathcal{L}} \to \mathfrak{g}_L^\perp$ is surjective. See [14, §4] for details. Then

$$\dim \mathcal{L} \cap \mu^{-1}(\tau) = \dim \mathcal{L} + \dim G_L - \dim G.$$ \hspace{1cm} (18)

Hence, the dimension of $\bar{\mathcal{L}}$ equals half the dimension of $\bar{B}$.

In this paper, we use only stable manifolds to construct Lefschetz thimbles. The Stable and Unstable Manifold Theorem in [3, §3] shows that $\psi_\infty := \lim_{t \to +\infty} \psi_t : W^s(B) \to B$ is a fiber bundle. For a submanifold $X$ of $B$, the pull-back bundle $i^*_X W^s(B)$ is a submanifold of $M$, where $i_X$ is the embedding of $X$. Given a Lagrangian submanifold $\mathcal{L}$ of $B$, we consider a Lefschetz thimble $L$ associated with $\mathcal{L}$, which is a pull-back bundle

$$L := i^*_L W^s(B) = \psi_\infty^{-1}(\mathcal{L}) = \bigcup_{p \in \mathcal{L}} W^s(p).$$

When $\mathcal{L}$ is simply connected, $L$ is also simply connected. Note that the projection $\psi_\infty : L \to \mathcal{L}$ satisfies $\psi_\infty^* \omega = \omega$, we obtain the following lemma.

**Lemma 10.** Suppose $\mathcal{L}$ is a Lagrangian submanifold of $(B, \omega)$. Then the Lefschetz thimble $L$ associated with $\mathcal{L}$ is a Lagrangian submanifold of $(M, \omega)$.

**Proof.** Observe that $\dim L = \dim \mathcal{L} + \lambda^+_L = \frac{1}{2} \dim M$. It suffices to check $\omega|_L = 0$. For $x \in \mathcal{L}$, $y \in W^s(x)$ and $u, v \in T_yL$, we have

$$\omega(u, v) = \lim_{t \to +\infty} \psi_t^* \omega(u, v) = \omega(\psi_\infty^* u, \psi_\infty^* v) = 0.$$

$\square$

Notice that $\psi_t$ commutes with $G$, we have

**Lemma 11.** Let $g \in G$ and $p \in \text{Crit}^s(W)$. If $q \in W^s(p)$, then $gq \in W^s(gp)$.

If we consider the isotropy subgroup of $L$, defined to be $G_L := \{g \in G \mid gL = L\}$, Lemma 11 implies $G_L = G_L$. We next show that a Lefschetz thimble $L$ can reduce to a Lagrangian submanifold $\bar{L}$ in $\mu^{-1}(\tau)/G$. Now that $\psi_t$ preserves $\mu$ by Lemma 6, it follows that $W^s(p) \subset \mu^{-1}(\tau)$ for $p \in B$ if and only if $\mu(p) = \tau$. Due to the transversality of $\mathcal{L}$ and $\mu^{-1}(\tau)$, the Lefschetz thimble $L$ intersects $\mu^{-1}(\tau)$ transversally. Moreover, their intersection is a fiber bundle

$$L \cap \mu^{-1}(\tau) = \bigcup_{p \in \mathcal{L} \cap \mu^{-1}(\tau)} W^s(p).$$
over \( L \cap \mu^{-1}(\tau) \) and has dimension \( \frac{1}{2} \dim M - \dim G + \dim G_L \) by (18). Since \( G_L = G_L \), the reduced space \( \bar{L} := (L \cap \mu^{-1}(\tau))/G_L \) is a Lagrangian submanifold of the symplectic quotient space \((\bar{M}, \omega)\). The map \( \bar{\psi}_\infty : \bar{L} \to \bar{\mathbb{L}} \) reduced by \( \psi_\infty \) provides a bundle structure and \( \bar{\mathbb{L}} \) is a section of \( \bar{L} \). We describe the relations of these symplectic manifolds and Lagrangian submanifolds as the diagram

\[
\begin{align*}
(B, L, \omega) & \xrightarrow{\text{reduce}} (\bar{B}, \bar{\mathbb{L}}, \bar{\omega}) \\
\downarrow i & \downarrow \\
(M, L, \omega) & \xrightarrow{\text{reduce}} (\bar{M}, \bar{\mathbb{L}}, \bar{\omega})
\end{align*}
\]

where \( i \) is embedding.

**Example 12** (Hypersurfaces). Let \( f(x) \in \mathbb{C}[x_1, \cdots, x_n] \) be a nondegenerate quasi-homogeneous polynomial of weight \((b, b_1, \cdots, b_n) \in \mathbb{Z}^{n+1}_0 \) i.e.,

\[
f(\lambda^{b_1}x_1, \cdots, \lambda^{b_n}x_n) = \lambda^b f(x_1, \cdots, x_n), \ \forall \lambda \in \mathbb{C}.
\]

Let \( S^1 \) acts on \( \mathbb{C}^{n+1} \) by \( \lambda \cdot (x_0, x_1, \cdots, x_n) = (\lambda^{-b}x_0, \lambda^{b_1}x_1, \cdots, \lambda^{b_n}x_n) \). The moment map of this action is a quadratic function

\[
\mu = -\frac{1}{2}b|x_0|^2 + \frac{1}{2} \sum_{i=1}^n b_i|x_i|^2.
\]

Then the function \( W = x_0f(x) : \mathbb{C}^{n+1} \to \mathbb{C} \) is \( S^1 \)-invariant. The stable critical set \( \text{Crit}^s(W) \) has two parts

\[
B_1 = \{0\} \times \{x \in \mathbb{C}^n \mid f(x) = 0, x \neq 0\}, \quad B_2 = \mathbb{C}^* \times \{0\}.
\]

For every \( \tau \in \mathbb{R}\setminus\{0\} \), \( \mu^{-1}(\tau) \) cannot intersect \( B_1 \) and \( B_2 \) at the same time. Let \( S_{b,\tau} \) be the sphere \( \{x \in \mathbb{C}^n \mid \sum b_i|x_i|^2 = 2\tau\} \). When \( \tau > 0 \), the reduced critical space is

\[
B_1 = [f^{-1}(0) \cap S_{b,\tau}]/S^1 \cong (f^{-1}(0)\setminus\{0\})/\mathbb{C}^*,
\]

which is a hypersurface in the weighted projective space \( \mathbb{P}^{n-1} \). When \( \tau < 0 \), the reduced critical space \( B_2 \) has only one point.

One can generalize Example 12 to complete intersections \( W = \sum p_i f_i \).

**Definition 13.** Let \( F = (f_1, \cdots, f_k) : \mathbb{C}^n \to \mathbb{C}^k \) be a holomorphic map. \( F \) is called \textit{nondegenerate} if the Jacobian of \( F \) is of full rank outside the origin.

**Example 14** (Complete intersection). Let \( f_1, \cdots, f_k \in \mathbb{C}[x_1, \cdots, x_n] \) be a list of quasi-homogeneous polynomials with common weight \((b, b_1, \cdots, b_n) \in \mathbb{Z}^{n+k}_0 \) and suppose that \( F = (f_1, \cdots, f_k) \) is nondegenerate. Define an action of \( S^1 \) on \( \mathbb{C}^{n+k} \) by

\[
\lambda \cdot (p_1, \cdots, p_k, x_1, \cdots, x_n) = (\lambda^{-b}p_1, \cdots, \lambda^{-b}p_k, \lambda^{b_1}x_1, \cdots, \lambda^{b_n}x_n), \quad \lambda \in S^1.
\]
This action is hamiltonian with a moment map $\mu : \mathbb{C}^{k+n} \to \mathbb{R}$,

$$\mu(p, x) = -\frac{b}{2} \sum_{j=1}^{k} |p_j|^2 + \frac{1}{2} \sum_{j=1}^{n} b_j |x_j|^2.$$ 

Then the function $W(p, x) = \sum_{i=1}^{k} p_i f_i(x) : \mathbb{C}^{n+k} \to \mathbb{C}$ is $S^1$-invariant. The stable critical set of $W$ has two parts

$$B_1 = \{0\} \times \{x \in \mathbb{C}^n \setminus \{0\} | F(x) = 0\} \cong F^{-1}(0) \setminus \{0\},$$
$$B_2 = (\mathbb{C}^k \setminus \{0\}) \times \{0\} \cong \mathbb{C}^k \setminus \{0\}.$$ 

Since $dF$ is of full rank, it follows from implicit function theorem that $B_1$ is a smooth affine variety. We next discuss transversality of $F^{-1}(0) \cap \mu^{-1}(\tau)$. Let $v$ be a generator of $\text{Lie}(S^1)$. For every $\bar{x} \in \text{Crit}^s(W)$, observe that $F(g \cdot \bar{x}) = 0, \forall g \in S^1$, it is similar to Lemma 5 that for $i = 1, \cdots, k$,

$$\omega(X_v, X_{f_i}) = 0.$$ 

Then $d\mu(x), df_1(x), \cdots, df_k(x)$ are linearly independent over $\mathbb{C}$ for $x \neq 0$. In addition, the action of $S^1$ on $\text{Crit}^s(W)$ has no fixed point. Then the Jacobian $(d\mu, dF)$ of $(\mu, F)$ is also of full rank outside the origin. This implies that $B_1$ intersect $\mu^{-1}(\tau)$ transversally. When $\tau > 0$, the reduced critical space $B_1 \cong (F^{-1}(0) \setminus \{0\})/\mathbb{C}^*$ is a complete intersection in $\mathbb{P}^{n-1}_\mathbb{C}$. When $\tau < 0$, the reduced critical space is $B_2 = S^{2k-1}/S^1 \cong \mathbb{CP}^k$.

If $f_1, \cdots, f_k$ are quasi-homogeneous polynomials with real coefficients, then conjugation is an anti-symplectic involution on each critical set of $\sum p_i f_i$. One obtains two Lagrangian submanifolds from conjugation whose isotropy subgroup are both isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

### 3.2 Lagrangian boundary conditions

The configuration space $\mathcal{B}(P)$ should be endowed with appropriate Lagrangian boundary conditions to ensure that the dimension of the moduli space of (8) is finite. In this section we use the same notations as in [29].

Let $(C, x_1, \cdots, x_k)$ be a compact Riemann surface with boundary and finite boundary marked points. We divide these marked points into two parts $\{x_1, \cdots, x_k\} = \Sigma^+ \cup \Sigma^-$. We call the points of $\Sigma^+, \Sigma^-$ its incoming and outgoing points respectively at infinity. Let $\bar{C} = C \setminus \{x_1, \cdots, x_k\}$ be the pointed-boundary Riemann surface. We still denote by $\bar{P}$ the principal bundle restricting to $\bar{C}$. The boundary of $\bar{C}$ is decomposed into many connected components $\cup_{i=1}^{s} C_i$.

A Lagrangian labels for $\bar{C}$ is a family of Lagrangian submanifolds $\{L_i\}_{i=1}^{s}$ in $M$. Every $L_i$ is indexed by $C_i \subset \partial \bar{C}$. Each marked point $p$ corresponds to a pair of Lagrangian submanifolds $L_{p,0}, L_{p,1} \in \{L_i\}_{i=1}^{s}$ in the natural boundary orientation, such that $p \in \overline{C}_{p,0} \cap \overline{C}_{p,1}$.

Fix a connected component $B \subset \text{Crit}^s(W)$ and a list of Lagrangian submanifolds $L_1, \cdots, L_s$ of $B$, we take the associated Lefschetz thimbles $L_1, \cdots, L_s$ to form a Lagrangian labels $\mathcal{L} = \{L_i\}_{i=1}^{s}$ for $\bar{C}$. We say a connection $A \in \mathcal{A}(P)$ is compatible with $\mathcal{L}$
if \( A|_{C_i} \in A^1(C_i, g_{L_i}) \), \(*A|_{C_i} \in A^1(C_i, \mathfrak{g}_{L_i}^\perp)\) and \( F_A \) is a 2-form with values in \( \mathfrak{g}_{L_i}^\perp \) along \( C_i \) for all \( i = 1, \ldots, s \). Denote by \( \mathcal{A}_L(P) \) the set of all compatible connections with \( \mathcal{L} \). Such a Lagrangian labels \( \mathcal{L} \) determines a configuration space \( \mathcal{B}_L(P) \) by

\[
\mathcal{B}_L(P) = \mathcal{A}_L(P) \times \mathcal{C}^\infty_{\mathcal{L}}(P, M),
\]

where \( \mathcal{C}^\infty_{\mathcal{L}}(P, M) = \{ \varphi \in \mathcal{C}^\infty_{\mathcal{L}}(P, M) \mid \varphi \circ \kappa(C_i) \subset L_i, \forall i \} \).

It is simple to describe a connection \( A \in \mathcal{A}_L(P) \) on \( \partial \hat{C} \). Assume that \((s, t)\) is a local orthogonal coordinate, as in Figure 1, of a boundary point in \( C_i \) with \((\frac{\partial}{\partial s}, \frac{\partial}{\partial t})|_{t=0} = 0\) and \((s, 0) \in C_i \). Then \( A \) can be expressed by \( \Phi ds + \Psi dt \) with \( \Phi(s, 0) \in g_{L_i}, \Psi(s, 0) \in \mathfrak{g}_{L_i}^\perp \) and \( \partial_t \Phi(s, 0) \in \mathfrak{g}_{L_i}^\perp \). The gauge group with Lagrangian boundary condition is given by

\[
\mathcal{G}_L(P) = \{ g \in \mathcal{G}(P) \mid g(C_i) \subset G_{L_i}, g^{-1}N(g)(C_i) \subset \mathfrak{g}_{L_i}^\perp, \forall i \},
\]

where \( N \) is the unit normal vector field on \( \partial \hat{C} \). The infinitesimal action (or Lie algebra) of \( \mathcal{G}_L(P) \) is given by

\[
\Omega^0_L(\hat{C}, \mathfrak{g}_P) = \{ \alpha \in \Omega^0(\hat{C}, \mathfrak{g}_P) \mid \alpha(C_i) \subset \mathfrak{g}_{L_i}, N(\alpha(C_i)) \subset \mathfrak{g}_{L_i}^\perp, i = 1, \ldots, s \}.
\]

Similarly, fix a connection \( A \in \mathcal{A}_L(P) \), define \( \Omega^1_L(\hat{C}, \mathfrak{g}_P) := T_A \mathcal{A}_L(P) \). The following proposition shows that the group action of \( \mathcal{G}_L(P) \) on \( \mathcal{B}_L(P) \) is well-defined.

**Proposition 15.** Let \( A \in \mathcal{A}_L(P) \) and \( g \in \mathcal{G}_L(P) \). Then \( g^* A \in \mathcal{A}_L(P) \).

**Proof.** For a boundary point \( p \in C_i \), we choose an orthogonal coordinate around \( p \) as in Figure 1. Let \( A = \Phi ds + \Psi dt \) with \( \Phi(s, 0) \in \mathfrak{g}_{L_i}, \Psi(s, 0) \in \mathfrak{g}_{L_i}^\perp, \partial_t \Phi(s, 0) \in \mathfrak{g}_{L_i}^\perp \) and \( g(s, 0) \in G_{L_i}, g^{-1} \partial_t g(s, 0) \in \mathfrak{g}_{L_i}^\perp \). Then

\[
g^* A = (g\Phi g^{-1} - \partial_s gg^{-1}) ds + (g\Psi g^{-1} - \partial_t gg^{-1}) dt.
\]

Firstly, it is easy to see that \((g\Phi g^{-1} - \partial_s gg^{-1})|_{t=0} = 0 \in \mathfrak{g}_{L_i} \). Secondly, note that \( \mathfrak{g}_{L_i}^\perp \) is an invariant subspace of the adjoint representation of \( G_{L_i} \), i.e., \( \text{Ad}(h)(\mathfrak{g}_{L_i}^\perp) \subset \mathfrak{g}_{L_i}^\perp, \forall h \in G_{L_i} \). Then \((g\Psi g^{-1} - \partial_t gg^{-1})|_{t=0} = 0 \in \mathfrak{g}_{L_i}^\perp \). Finally, we verify that \( \partial_t (g\Phi g^{-1} - \partial_s gg^{-1})|_{t=0} = 0 \in \mathfrak{g}_{L_i}^\perp \). By (17) we know that \([g^{-1} \partial_t g, \Phi]|_{t=0} = 0 \in \mathfrak{g}_{L_i}^\perp \). Using \( \partial_t gg^{-1} = \text{Ad}(g)(g^{-1} \partial_t g) \) we obtain that

\[
\partial_t (g\Phi g^{-1})|_{t=0} = \text{Ad}(g)([g^{-1} \partial_s g, \Phi] + \partial_t \Phi)|_{t=0} \in \mathfrak{g}_{L_i}^\perp.
\]

In addition, we have \( \partial_s (g\partial_t g^{-1})|_{t=0} = 0 \in \mathfrak{g}_{L_i}^\perp \). Observe that

\[
-\partial_t (\partial_s gg^{-1}) = \text{Ad}(g)[g^{-1} \partial_s g, g^{-1} \partial_t g] + \partial_s (g\partial_t g^{-1})
\]

and use (17) we obtain that \( \partial_t (\partial_s gg^{-1})|_{t=0} = 0 \in \mathfrak{g}_{L_i}^\perp \). \(\square\)
Proof. Choose an orthogonal coordinate of $\Theta$ as in Figure 1. Set $A|_{\Theta_p} = \Phi ds + \Psi dt$ with $\Phi(s,i) \in g_L$ and $\partial_t \Phi(s,i) \in g_L^1$ for $i = 0, 1$. Then $g^* \Phi = 0$ is equivalent to the equations about $g$

\[
\begin{align*}
\left\{\begin{array}{l}
\ g\Phi g^{-1} - \partial_s gg^{-1} = 0, \\
\ g(s,0) \in G_{L_0}, g(s,1) \in G_{L_1}, \\
\ g^{-1}\partial_t g|_{t=0} \in g_L^0, g^{-1}\partial_t g|_{t=1} \in g_{L_i}^1.
\end{array}\right.
\end{align*}
\]

Observe that (19) is a first order ordinary differential equation, there exists $h \in C^\infty(\Theta_p, G)$ such that $q(t)h(s,t)$ is the general solution of (19) where $q \in C^\infty([0, 1], G)$. For $i = 0, 1$, since $\Phi(s,i) \in g_L$, equation (19) can be solved in $G_{L_i}$. Without loss of generality, we assume that $h(s,i) \in G_{L_i}$ for $i = 0, 1$. Then (21) is equivalent to

\[
(\partial_t hh^{-1} + g^{-1}\partial_t q)|_{t=1} \in g_{L_i}^1, \quad i = 0, 1.
\]

Choose $s_0 > 0$ and $q : [0, 1] \rightarrow G$ such that $q(0) = q(1) = 1, \partial_t hh^{-1}(s_0, i) + \partial_t q(i) \in g_{L_i}^1$ for $i = 0, 1$. Then (22) holds due to the fact that

\[
\partial_s(\partial_t hh^{-1})|_{t=1} = Ad(h)(\partial_t \Phi)|_{t=1} \in g_{L_i}^1.
\]

\end{proof}

4 Energy functional

4.1 Yang-Mills-Higgs energy on $B_{L}(P)$

In this subsection we will show that the solutions of gauged Witten equations (8) minimize the Yang-Mills-Higgs energy functional with fixed limit paths. Similar to symplectic vortex equations [6, 14], we introduce the energy for gauged Witten equation. Let $p > 2$. The Yang-Mills-Higgs energy functional $E_W$ of a pair $(A, \varphi) \in B_{L}^{k,p}(P)$ is defined by

\[
E_W(A, \varphi) := \frac{1}{2} \int_{\dot{C}} (|d_A \varphi|^2 + |F_A|^2 + |\mu(\varphi) - \tau|^2 + 2|X_\sigma(\varphi)|^2) dV_{\dot{C}}.
\]

The functional $|X_\sigma(\varphi)|^2$ is then used as the potential at critical coupling in $E_W$. The remaining three terms in the functional are same as ones in symplectic vortex equations.
Be aware of that the function $\text{Im} W|_{L_1}$ is constant and equals $\text{Im} W(B)$, therefore $\text{Im}(W \circ \varphi)$ is independent of $\varphi \in C^\infty_P$. For a given pair $(A, \varphi) \in \mathcal{B}_L$, we define $\varphi_A^* \omega \in \Omega^2(\hat{C})$ as

$$\varphi_A^* \omega(u, v) = \omega(\varphi_*(\tilde{u}), \varphi_*(\tilde{v})), $$

where $\tilde{u}, \tilde{v}$ are the lifting horizontal vectors of $u, v$ with respect to the connection $A$. Since $\omega$ is $G$-invariant, $\varphi_A^* \omega$ is well-defined.

**Theorem 17.** Let $(A, \varphi) \in \mathcal{B}_L$, suppose that the limit of $\varphi$ exists at all marked points, and denote them by $\varphi_1, \ldots, \varphi_k$. Then the energy of $(A, \varphi)$ is given by

$$E_W(A, \varphi) = \int_{\hat{C}} |(\bar{\partial} A \varphi + X_\varphi(\varphi)|^2 + \frac{1}{2} |* F_A + \mu(\varphi) - \tau|^2) dV_C + F_W(A, \varphi), \quad (24)$$

where $F_W(A, \varphi) = \int_{\hat{C}} \varphi_A^* \omega - (F_A, \mu(\varphi) - \tau) - 2 \sum_{i=1}^k \int_0^1 \text{Re}(W \circ \varphi_i) dt + 2\text{Im} W(B) \cdot l(\partial \hat{C})$ and $l(\partial \hat{C})$ is the length of $\partial \hat{C}$.

**Proof.** We first show that

$$\frac{1}{2} |d_A \varphi|^2 dV_C = |\bar{\partial} A \varphi|^2 dV_C + \varphi_A^* \omega. \quad (25)$$

To see this, we choose an orthogonal basis $e_1, e_2$ of $T_\varphi C$, and let $\tilde{e}_1, \tilde{e}_2$ be the lifting horizontal basis of $T_\varphi P$. Then

$$|\bar{\partial} A \varphi|^2 = \sum_{i=1}^2 (\bar{\partial} A \varphi(\tilde{e}_i), \bar{\partial} A \varphi(\tilde{e}_i)) = \frac{1}{2} |d_A \varphi|^2 - \omega(d \varphi(\tilde{e}_1), d \varphi(\tilde{e}_2)).$$

Next, the following two equations are obvious

$$|F_A|^2 + |\mu(\varphi) - \tau|^2 = |* F_A + \mu(\varphi) - \tau|^2 - 2(* F_A, \mu(\varphi) - \tau),$$

$$|\bar{\partial} A \varphi|^2 + |X_\varphi(\varphi)|^2 = |\bar{\partial} A \varphi + X_\varphi(\varphi)|^2 - 2(\bar{\partial} A \varphi, X_\varphi(\varphi)). \quad (26)$$

Finally, we choose an isothermic coordinate $z = s + it$ with $\sigma = -id z$. Then

$$(\bar{\partial} A \varphi, X_\sigma(\varphi)) = \sum_{i=1}^2 (\bar{\partial} A \varphi(\tilde{e}_i), X_\sigma(\varphi)(e_i))$$

$$= \frac{1}{2} (d \varphi(\tilde{e}_1) + J d \varphi(\tilde{e}_2), X_\sigma(\varphi)(e_1)) + \frac{1}{2} (d \varphi(\tilde{e}_2) - J d \varphi(\tilde{e}_1), X_\sigma(\varphi)(e_2))$$

$$= \frac{1}{2} \frac{\partial \varphi}{\partial s} + J \frac{\partial \varphi}{\partial t}, X_{\text{Im} W}(\varphi) + \frac{1}{2} \frac{\partial \varphi}{\partial t} - J \frac{\partial \varphi}{\partial s}, X_{\text{Re} W}(\varphi)) \quad (27)$$

Combine (25),(26) and (27), the proof of (24) is complete. \qed
Remark 18. For a special case, when $C$ is a compact Riemann surface (see [6]), the energy of a symplectic vortex $(A, \varphi)$ on $C$ is given by

$$F_W(A, \varphi) = \int_C \varphi^* \omega - \langle F_A, \mu(\varphi) - \tau \rangle_g = ([\omega + \tau - \mu(\varphi)], [\varphi]).$$

Remark 19. Actually, for a solution $(A, \varphi)$ of (8), the energy $E_W(A, \varphi)$ has another form of expression. In an isothermic coordinate $z = s + it$ with $\sigma = -idz$, let $A = \Phi ds + \Psi dt$. A tedious calculation shows that

$$\varphi^* \omega - \langle F_A, \mu(\varphi) - \tau \rangle_g = \varphi^* \omega + \left[\frac{\partial}{\partial t}(\Phi, \mu(\varphi) - \tau)\right]_g - \frac{\partial}{\partial s}(\Psi, \mu(\varphi) - \tau)_g ds \wedge dt.$$

Observe a fact that the connection $A$ satisfies $\langle \Phi, \mu(\varphi) - \tau \rangle_{g}\mid_{\partial C} = 0$, then we have

$$E_W(A, \varphi) = \int_C \varphi^* \omega - \sum_{i=1}^k \int_0^1 \left[\langle \Phi_i, \mu(\varphi_i) - \tau \rangle_g + \text{2Re}(W \circ \varphi_i)\right] dt + c,$$

where $(\Phi_i, \varphi_i)$ is the limit path of $(A, \varphi)$ at $i$-th boundary marked point and $c$ is a constant.

When $C$ is a disk with strip ends, for a given Hamiltonian perturbation $H$, we define the energy of $(A = \Phi ds + \Psi dt, \varphi)$ as

$$E_H(A, \varphi) = \frac{1}{2} \int_C \left(\frac{\partial \varphi}{\partial s} + X_\Phi(\varphi)\right)^2 + \left|\frac{\partial \varphi}{\partial t} - X_\Psi(\varphi) + X_{2\text{Re}W(\varphi)} + X_{Q(\varphi)^2} + |F_A|^2 + |\mu(\varphi) - \tau|^2\right| ds dt.$$

Then the variational equation of $E_H$ is the perturbed equation (12). Assume that $(\Phi, \Psi, \varphi)$ is a solution of (12), then

$$E_H(\Phi, \Psi, \varphi) = \int_C \varphi^* \omega - \langle F_A, \mu(\varphi) - \tau \rangle_g - \sum_{i=1}^k \int_0^1 Q_i(\varphi_i) + 2\text{Re}(W(\varphi_i)) dt + c.$$

Remark 20. We mark that $E_W(A, \varphi) < \infty$ signifies $\lim_{s \to \pm \infty} |d_A \varphi| = \lim_{s \to \pm \infty} |X_{\sigma}(\varphi)| = 0$ and $\lim_{s \to \pm \infty} |F_A| = \lim_{s \to \pm \infty} |\mu(\varphi) - \tau| = 0$. If $(A, \varphi) \in B_C(P)$ is a solution of (8) with finite energy and $A\mid_{\Theta_p} = \Psi dt$ on a strip end $\Theta_p$. Then

$$\lim_{s \to \pm \infty} \partial_s \Psi = 0, \quad \lim_{s \to \pm \infty} \partial_s \varphi = 0.$$

4.2 Functional action on path space

In the rest of this paper, we take $M$ to be $\mathbb{C}^n$ with a fixed symplectic form $\omega$. Theorem 17 shows that the energy of a gauged Witten instanton is determined by its limit paths. We shall study the remainder energy functional on path space $P(L_{p,0}, L_{p,1})$ for each boundary marked point. Just like various versions of Floer theory [12, 14, 34, 21], the energy functional on path space contains abundant information about Lagrangian
intersection. For ease of notations, we use \( L_0, L_1 \) for \( L_{p,0}, L_{p,1} \) respectively. Recall that the space of path with respect to \( p \) is given by

\[
\Omega(g_{L_0}, g_{L_1}) = \{ \eta \in C^\infty([0, 1], G) | \eta(0) \in g_{L_0}^1, \eta(1) \in g_{L_1}^1 \},
\]

\[
\Omega(L_0, L_1) = \{ x \in C^\infty([0, 1], \mathbb{C}^n) | x(0) \in L_0, x(1) \in L_1 \},
\]

\[
\mathcal{P}(L_0, L_1) := \Omega(g_{L_0}, g_{L_1}) \times \Omega(L_0, L_1).
\]

The path space \( \mathcal{P}(L_0, L_1) \) is connected and simply connected due to the linear structure of \( \mathbb{C}^n \) and the assumption (C4) mentioned in §3.1 (for a proof see [13, Lemma 5.2]). Besides, \( \mathcal{P}(L_0, L_1) \) carries a group action of \( \mathcal{H}(L_0, L_1) \), simply denoted by \( \mathcal{H} \), which is defined to be

\[
\mathcal{H}(L_0, L_1) := \{ h \in C^\infty([0, 1], G) | h(i) \in G_{L_i}, h^{-1}\partial_th(i) \in g_{L_i}^1, i = 0, 1 \}
\]

and the action is given by

\[
h^\ast(\eta, x) = (h\eta h^{-1} + h\partial_th^{-1}, hx).
\]

Here we do not consider the Novikov space since the above action cannot lift to the universal covering space of \( \mathcal{P}(L_0, L_1) \).

For each boundary marked point \( p \), we fix a path \( l_0 \in \Omega(L_0, L_1) \). Let \( Q_t \) be the integral of motion in a given Hamiltonian perturbation on the strip end \( \Theta_p \). For every \( l \in \Omega(L_0, L_1) \), one can choose a map \( x : [0, 1] \times [0, 1] \to \mathbb{C}^n \) such that

\[
x(s, 0) \in L_0, \quad x(s, 1) \in L_1, \quad x(0, t) = l_0(t), \quad x(1, t) = l(t).
\]

We define the action functional on \( \mathcal{P}(L_0, L_1) \) by

\[
\mathcal{F}_W(\eta, l) = -\int_{[0, 1]^2} x^\ast \omega + \int_0^1 \langle \mu(l) - \tau, \eta \rangle_g + 2\text{Re}(W(l)) + Q_t(l)dt,
\]

where \( x \) is a map connecting \( l \) and \( l_0 \). Since \( L_0, L_1 \) are simply connected (see §3.1) and \( \pi_2(\mathbb{C}^n) \) is trivial, the symplectic area \( \int_{[0, 1]^2} x^\ast \omega \) is well-defined. Then \( \mathcal{F}_W \) is invariant under the action of unit connected component of \( \mathcal{H}(L_0, L_1) \).

We introduce a metric on \( \mathcal{P}(L_0, L_1) \) in the following way. With the decomposition \( T_{(\eta, x)}\mathcal{P}(L_0, L_1) = \Omega(g_{L_0}, g_{L_1}) \oplus T_x\Omega(L_0, L_1), \) for any \( (\alpha_i, \beta_i) \in T_{(\eta, x)}\mathcal{P}(L_0, L_1), i = 1, 2, \) the metric of tangent space is defined as

\[
((\alpha_1, \beta_1), (\alpha_2, \beta_2)) := \int_0^1 \langle \alpha_1, \alpha_2 \rangle_g + (\beta_1, \beta_2)dt. \tag{29}
\]

Using the metric (29), we obtain a one-form \( \alpha \) on \( \mathcal{P}(L_0, L_1) \) defined as

\[
\alpha(\eta, l)(v_1, v_2) = \int_0^1 \langle \mu(l) - \tau, v_1 \rangle_g + \omega(l'(t) + X_\eta(l) + 2X_{\text{Re}W}(l) + X_Q(l), v_2 \rangle dt \tag{30}
\]
for each $(v_1, v_2) \in T_{(\eta, t)}\mathcal{P}(L_0, L_1)$. To compute the critical points of $\mathcal{F}_W$, we consider a two-parameters family of paths $(\eta_t, x_v) \in \mathcal{P}(L_0, L_1)$ parametrized by $(u, v) \in \mathbb{R}^2$, and compute $\frac{\partial \mathcal{F}_W}{\partial x}(\eta_t)$ and $\frac{\partial \mathcal{F}_W}{\partial r}(x_v)$. A straightforward calculation shows that $d\mathcal{F}_W = \alpha$. Hence, $(\eta, x)$ is a critical point of $\mathcal{F}_W$ if and only if it satisfies

$$\begin{cases}
  \frac{\partial r}{\partial t} + X_\eta(x) + X_{2\text{Re}W}(x) + X_Q(x) = 0, \\
  \mu(x) = \tau.
\end{cases} \quad (31)$$

Moreover, (31) is invariant under the action of $\mathcal{H}$.

To compute the gradient flow line of $\mathcal{F}_W$ we consider a one-parameter family of paths $r(s) = (\Psi(s, t), x(s, t))$ in $\mathcal{P}(L_0, L_1)$. For $Y = (U, V) \in T_{r(s)}\mathcal{P}$, we see

$$(r', Y) + d\mathcal{F}_W(Y) = \int_0^1 \left( \frac{\partial \Psi}{\partial s} + \mu(x) - \tau, U \right)_0 + \left( \frac{\partial x}{\partial s} + J(\frac{\partial x}{\partial t} + X_\Phi(x) + X_{2\text{Re}W}(x) + X_Q(x)), V \right) dt.$$ 

Hence $r(s)$ is a gradient flow line of $\mathcal{F}_W$ with respect to the metric (29) if and only if it satisfies the gauged Witten equations on $[0, 1] \times (0, +\infty)$ with temporal gauge

$$\begin{cases}
  \frac{\partial x}{\partial t} + J(\frac{\partial x}{\partial t} + X_\Phi(x) + X_{2\text{Re}W}(x) + X_Q(x)) = 0 \\
  \frac{\partial x}{\partial s} + \mu(x) - \tau = 0.
\end{cases} \quad (32)$$

We now turn to the quotient space $\text{Crit}(\mathcal{F}_W)/\mathcal{H}$ of the critical locus of $\mathcal{F}_W$. Denote by $\mathcal{O}_t$ the gradient flow of the Hamiltonian vector field $-2\text{Re}W$. Then $\mathcal{O}_t$ is commutative with $G$. Let us go back to the first part in (31). By lemma 5 and the fact that $\{Q_1, \text{Im}W\} = 0$, we have $(\text{Re}W(x), X_\eta(x) + X_Q(x)) = 0$. Then equations (31) leads to

$$\frac{\partial x}{\partial t} + X_\eta(x) + X_Q(x))^2 = -2(\frac{\partial x}{\partial t}, X_{\text{Re}W}(x)) = -2\frac{\partial}{\partial t}\text{Im}(W \circ x).$$

Since the equality $\text{Im}W|_{L_{p, 0}} = \text{Im}W|_{L_{p, 1}}$ holds for every boundary marked point $p$, we have

$$\int_0^1 \frac{\partial x}{\partial t} + X_\eta(x) + X_Q(x))^2 dt = 2\text{Im}(W \circ x)|_0^1 = 0.$$

Thus (31) are equivalent to the following equations

$$\begin{cases}
  \frac{\partial x}{\partial t} + X_\eta(x) + X_Q(x) = 0, \\
  dW|_{x(t)} = 0, \\
  \mu(x) = \tau.
\end{cases} \quad (33)$$

This means the critical point $(\eta, x)$ of $\mathcal{F}_W$ satisfies a Hamiltonian system with the image of $x$ lying in the manifold $B \cap \mu^{-1}(\tau)$. Let $\psi_t$ denote the flow on $\mathbb{C}^n$ generated by $X_Q$. Note that $\psi_t$ commutes with $G$ and preserves $W$, there is an induced Hamiltonian action $\psi_t$ on $\bar{B}$. In the next proposition, we give a description of the critical set modulo gauge equivalence. For this purpose, we consider a generalized exponential mapping

$$\text{Exp}: C^\infty([0, 1], \mathfrak{g}) \to \{ g \in C^\infty([0, 1], G) | g(0) = 1 \}$$
defined as follows: For a given $\eta(t) \in C^\infty([0, 1], g)$, the left invariant vector field of $\eta(t)$ is Lipschitz with respect to a bi-invariant connection on $G$. Therefore, there exists a unique $g(t) \in C^\infty([0, 1], G)$ such that $g(0) = 1$ and $g(t) \cdot \frac{\partial}{\partial t} g(t)^{-1} = \eta(t)$. Define $\text{Exp}(\eta) = g$. Moreover, we define a holonomy mapping as
\[
\text{Hol} : C^\infty([0, 1], g) \to G, \quad \eta \mapsto \text{Exp}(\eta)(1).
\]

**Proposition 21.** There is a natural bijection between $\text{Crit}(\mathcal{F}_W)/\mathcal{H}$ and $\tilde{\psi}_1(\mathbb{L}_0) \cap \mathbb{L}_1$. Moreover, if $(\eta, x) \in \text{Crit}(\mathcal{F}_W)/\mathcal{H}$, then $x(1) = \text{Hol}(\eta)x(0)$.

**Proof.** If $p \in \tilde{\psi}_1(\mathbb{L}_0) \cap \mathbb{L}_1$, then there exists $x_0 \in \mathbb{L}_0 \cap \mu^{-1}(\tau), x_1 \in \mathbb{L}_1 \cap \mu^{-1}(\tau)$ and $a \in G$ such that $x_1 = \psi_1(ax_0)$. Choose a path $g(t) \in C^\infty([0, 1], G)$ such that $g(0) = 1$ and $g(1) = a$. Let $x(t) = \psi_t \circ g(t)(x_0)$ and $\eta(t) = g \partial_t g^{-1}$. Due to $\psi_t$ commutes with $G$, it is obvious that $(\eta, x) \in \mathbb{P}$ satisfies (33) and $g(1) = \text{Hol}(\eta)$. If we choose another path $h(t) \in C^\infty([0, 1], G)$ with $h(0) = 1$ and $h(1) = a$, then $h(t)g(t)^{-1} \in \mathcal{H}$, and
\[
(hg^{-1})^*(g\psi_t(x_0), g\partial_t g^{-1}) = (h\psi_t(x_0), h\partial_t h^{-1}).
\]

Hence, the mapping $j : \tilde{\psi}_1(\mathbb{L}_0) \cap \mathbb{L}_1 \to \text{Crit}(\mathcal{F}_W)/\mathcal{H}$ is well-defined. One readily see that $j$ is injective. Conversely, the existence of the generalized exponential mapping $\text{Exp}$ implies that $j$ is surjective. This completes the proof.

Now we explain why $\mathcal{F}_W$ on $\mathbb{P}/\mathcal{H}$ is type of Bott-Morse. To see this, consider the linearized operator of (33) at a critical point $(\eta, x) \in \mathbb{P}$,
\[
D_{\eta,x} : T_{\eta,x}\mathbb{P} \to \mathcal{U}_{\eta,x}, \quad (b, \xi) \mapsto (d\mu(x)(\xi), \partial t \xi + L_x b - \nabla_\xi \partial t x),
\]
where $\mathcal{U}_{\eta,x} = C^\infty([0, 1], \{x^*TC^n\})$. Denote by $\mathfrak{h}$ the Lie algebra of $\mathcal{H}$. For any $h \in \mathfrak{h}$, the group action of $\mathcal{H}$ on $\mathbb{P}$ induces a vector field
\[
X_{\eta,x}(h) = (L_x h, -\partial_t h - [\eta, h]), \quad \forall (\eta, x) \in \mathbb{P}.
\]
For every critical point $(\eta, x)$ of $\mathcal{F}_W$, there is a chain complex
\[
0 \to \mathfrak{h} \overset{X_{\eta,x}}{\to} T_{\eta,x}\mathbb{P} \overset{D_{\eta,x}}{\to} \mathcal{U}_{\eta,x} \to 0
\]
which gives a description of the tangent space $T_{[\eta,x]}\text{Crit}(\mathcal{F}_W)/\mathcal{H} \cong \text{Ker}(X_{\eta,x}^* \oplus D_{\eta,x})$. This implies that $\mathcal{F}_W$ is Bott-Morse type. Furthermore, similar to the discussion in [13, Proposition 4.6], there is an isomorphism between $\text{Ker}(X_{\eta,x}^* \oplus D_{\eta,x})$ and $T_{\pi(x(0))}\mathbb{L}_0 \cap \mathbb{L}_1$, where $\pi$ is the projection $\mu^{-1}(\tau) \to \mu^{-1}(\tau)/G$.

5 Fredholm theory

5.1 The $G$-equivariant Maslov index

In this subsection we introduce a definition of the $G$-equivariant Maslov index that will appear in the index formula (38). We first review the definition of the Maslov index.
See [20] or [16]. Let $R(n) = \text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$ be the manifold of totally real subspaces of $\mathbb{C}^n$. Define a function $\rho : R(n) \rightarrow S^1$ by

$$A \cdot \text{GL}(n, \mathbb{R}) \mapsto \frac{\det(A^2)}{\det(A^*A)},$$

where $A^*$ is the conjugate transpose of $A$. The Maslov index of a loop $\gamma : S^1 \rightarrow R(n)$ is defined by

$$\mu(\gamma) = \deg(\rho \circ \gamma).$$

If $\gamma = \cup \gamma_i$ is a union of loops in $R(n)$, define $\mu(\gamma) = \sum \mu(\gamma_i)$.

Let $C$ be an oriented compact surface with boundary and let $(E, F)$ be a complex bundle pair over $(C, \partial C)$. There is a boundary Maslov index $\mu(E, F)$ defined as follows. Choose a trivialization $E \xrightarrow{\cong} C \times \mathbb{C}^n$ of $E$, the totally real bundle $F$ gives a union of loops $\gamma_F : \partial C \rightarrow R(n)$. Define $\mu(E, F) := \mu(\gamma_F)$ which is independent of the choice of trivialization of $E$.

**Definition 22.** Let $S$ be a compact Riemann surface with boundary. We say that $(E, F)$ is a $G$-equivariant bundle pair over $(S, \partial S)$ if

(i) $\pi : P \rightarrow S$ is a trivial principal $G$-bundle;

(ii) $E \rightarrow P$ is a $G$-equivariant complex vector bundle;

(iii) $F \rightarrow \pi^{-1}(\partial S)$ is a $G$-equivariant totally real subbundle of $E|_{\pi^{-1}(\partial S)}$.

We next define the $G$-equivariant boundary Maslov index of a $G$-equivariant bundle pair which depends only on the isomorphism class of pair.

**Definition 23.** Let $(E, F)$ be a $G$-equivariant bundle pair over $(S, \partial S)$. The $G$-equivariant boundary Maslov index $\mu^G(E, F)$ of $(E, F)$ is defined to be

$$\mu^G(E, F) := \mu(\alpha^*E, \alpha^*F),$$

where $\alpha$ is a global section of $P$. This integer is independent of $\alpha$.

**Example 24.** Suppose that $C$ is a compact Riemann surface without marked point and $\varphi \in C^\infty_C(P, \mathbb{C}^n)$. We treat $T\mathbb{C}^n$ as a holomorphic tangent bundle with respect to the holomorphic structure of $\mathbb{C}^n$. Let $\varphi^*T\mathbb{C}^n|_{xg} = g_*(T_{\varphi(x)}L_i)$ for $x \in \kappa(C_i), g \in G, i = 1, 2, \cdots, k$. Then $(\varphi^*T\mathbb{C}^n, \varphi^*T\mathbb{C}^n)$ is a $G$-equivariant bundle pair. Let $D$ be a union of $k$ disks. Within the category of smooth structure, we extend $C$ to a smooth closed Riemann surface $C'$ by gluing $C$ and $D$ along $\partial C$. Notice that $P$ is trivial over $C$, one can get a principal bundle $P'$ over $C'$ by gluing $P$ and a trivial bundle over disks. The extended bundle $P'$ is not unique up to isomorphism. Since the homotopy group of $\mathbb{C}^n$ is trivial, every $\varphi \in C^\infty_C(P, \mathbb{C}^n)$ can extend to a unique $G$-equivariant map $\tilde{\varphi} \in \text{Map}_G(P', \mathbb{C}^n)$ up to homotopy such that $\tilde{\varphi} \circ i = \varphi$, where $i : P \rightarrow P'$ is the embedding. Then $\tilde{\varphi}$ determines an equivariant homology class

$$[\tilde{\varphi}] = \tilde{\varphi}_*\pi_1^{-1}[C'] \in H_2^G(\mathbb{C}^n, \mathbb{Z}).$$
Here \( \tilde{\varphi} : H^G_2(P', \mathbb{Z}) \to H^G_2(C^n, \mathbb{Z}) \) and \( \pi_* : H^G_2(P', \mathbb{Z}) \to H^G_2(C', \mathbb{Z}) \) denote the induced homomorphisms (see [7, §2]). Due to the associativity of \( \mu^G \), we have

\[
\mu^G(\varphi^*T\mathbb{C}^n, \varphi^*T\mathcal{L}) = 2\langle c_1^G(T\mathbb{C}^n), [\tilde{\varphi}] \rangle - \mu^G(\tilde{\varphi}^*T\mathbb{C}^n|_D, \tilde{\varphi}^*T\mathcal{L}|_{\partial D}).
\]  

(34)

### 5.2 The elliptic complex

The configuration space \( \mathcal{B}_\mathcal{L} \) is a Fréchet space. Using the sobolev norm \( ||\cdot||^k,p \) we can complete the Fréchet spaces to obtain various Sobolev spaces \( \mathcal{B}^k_{\mathcal{L}} \). In order to study the linearization of (8), we consider the Fréchet bundle \( \mathcal{E} \to \mathcal{B}_\mathcal{L} \) whose fiber over \( (A, \varphi) \) is given by \( \mathcal{E}_{A,\varphi} = \Omega^0(\hat{\mathcal{C}}, g_P) \oplus \Omega^{0,1}(\hat{\mathcal{C}}, E_\varphi) \). Then equations (8) provides a smooth section \( S : \mathcal{B}_\mathcal{L} \to \mathcal{E} \) given by

\[
S(A, \varphi) = (*F_A + \mu(\varphi) - \tau, \tilde{\partial}_A(\varphi) + X_\varphi(\varphi)).
\]

Let \( \Lambda = (A, \varphi) \in \mathcal{B}_\mathcal{L} \) be a solution of (8). The action of the gauge group \( \mathcal{G}_\mathcal{L} \) on \( \mathcal{B}_\mathcal{L} \) induces a linear operator \( \mathcal{L} : \Omega^0_\mathcal{L}(\hat{\mathcal{C}}, g_P) \to T\Lambda \mathcal{B}_\mathcal{L} \) at \( \Lambda \) given by \( \mathcal{L}_\Lambda(\alpha) = (-d_A \alpha, L_\varphi \alpha) \).

The formal conjugate operator of \( \mathcal{L}_\Lambda \) is given by

\[
\mathcal{L}^*_\Lambda : \Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P) \oplus \Omega^0(\hat{\mathcal{C}}, E_\varphi) \to \Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P), \ (\alpha, \xi) \mapsto -d_A^* \alpha + L^*_\varphi \xi.
\]

Due to the gauge invariance of gauged Witten equation, there is a naturally associated deformation complex at \( \Lambda \) which incorporates \( \mathcal{L} \) and the linearization of \( S \). Denote by \( D_\Lambda : C^\infty(\hat{\mathcal{C}}, E_\varphi) \to \Omega^{0,1}(\hat{\mathcal{C}}, E_\varphi) \) the linearized operator of \( \tilde{\partial}_A \). Then the deformation complex of (8) is given by

\[
\Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P) \xrightarrow{\mathcal{L}^*_\Lambda} \Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P) \oplus \Omega^0(\hat{\mathcal{C}}, E_\varphi) \xrightarrow{D_\Lambda} \Omega^0(\hat{\mathcal{C}}, g_P) \oplus \Omega^{0,1}(\hat{\mathcal{C}}, E_\varphi),
\]

(35)

where \( D_\Lambda \) is the linearization of \( S \) given by the matrix

\[
\begin{pmatrix}
* d_A \\
\mu^G
\end{pmatrix}
\]

(36)

and \( \pi^* \pi \) is a section of \( \pi^*(K_{\text{LOG}_\mathcal{L}}) \). In order to compute its Euler characteristic, we consider the linear operator

\[
\Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P) \oplus \Omega^0(\hat{\mathcal{C}}, E_\varphi) \xrightarrow{\mathcal{L}^*_\Lambda \oplus D_\Lambda} \Omega^1_\mathcal{L}(\hat{\mathcal{C}}, g_P) \oplus \Omega^0(\hat{\mathcal{C}}, g_P) \oplus \Omega^{0,1}(\hat{\mathcal{C}}, E_\varphi),
\]

(37)

and we denote by \( \mathcal{L}_\Lambda = \mathcal{L}^*_\Lambda \oplus D_\Lambda \).

When \( \mathcal{C} \) is a compact Riemann surface without marked point, i.e., \( \hat{\mathcal{C}} = \mathcal{C} \), we have the following index theorem. Before this, we state a simple lemma, which is useful for the proof of Theorem 26.

**Lemma 25.** Let \( (X^\bullet, d_X), (Y^\bullet, d_Y) \) be two complexes consisting of vector spaces, and let \( f^* : X^\bullet \to Y^\bullet \) be a surjective morphism. Then the cohomology groups of the complex \( (\text{Ker}(f^*), d_X) \) are given by

\[
H^k(\text{Ker}(f^*)) \cong \ker(f^k) \oplus \text{coker}(f^{k-1}),
\]

where \( f^* \) is the induced homomorphisms between cohomology groups.
Proof. Since $f^*$ is surjective, the sequence of complexes

$$S: 0 \to \text{Ker}f^* \to X^* \to Y^* \to 0$$

is exact. Lemma 25 is a consequence of the standard long exact sequence of $S$. □

We next study the twisted de Rham complex $(\Omega^*(C, g), d)$ on $C$ with boundary values. For $i = 0, 1, 2$, we define two spaces

$$\Omega^i(\partial C, g_L) := \bigoplus_{j=1}^k \Omega^i(C_j, g_L), \quad \Omega^i(\partial C, g_L^2) := \bigoplus_{j=1}^k \Omega^i(C_j, g_L^2),$$

where $\bigcup_{j=1}^k C_j$ is a connected decomposition of $\partial C$. There are two projections

$$p : \Omega^i(\partial C, g) \to \Omega^i(\partial C, g_L), \quad q : \Omega^i(\partial C, g) \to \Omega^i(\partial C, g_L^2).$$

Consider the twisted de Rham complex $(\Omega^*(\partial C), d)$ on $\partial C$ and a natural exact sequence

$$ES : \Omega^1(\partial C, g_L) \to \Omega^0(\partial C, g_L) \oplus \Omega^1(\partial C, g_L) \to \Omega^0(\partial C, g_L).$$

Let $\ell : \partial C \hookrightarrow C$ be the embedding. Then we have a surjective morphism $\ell^*$ from $\Omega^*(C, g)$ to $\Omega^*(\partial C, g_L^2) \oplus ES$

$$\begin{array}{c}
\Omega^0(C, g) \xrightarrow{d} \Omega^1(C, g) \xrightarrow{d} \Omega^2(C, g) \\
\downarrow q\ell^* \oplus p\ell^* d \downarrow q\ell^* \oplus p\ell^* d \downarrow p\ell^*
\end{array}$$

$$\begin{array}{c}
\Omega^0(\partial C, g_L^2) \oplus \Omega^1(\partial C, g_L) \xrightarrow{d} \Omega^1(\partial C, g_L^2) \oplus \Omega^0(\partial C, g_L) \oplus \Omega^1(\partial C, g_L) \\
\downarrow q\ell^* \oplus p\ell^* d \oplus p\ell^* 
\end{array}$$

We note that $H^*(\Omega^*(C, g)) \cong H^*(C) \otimes g$ and $H^*(\Omega^*(\partial C, g_L^2)) \cong \bigoplus H^*(C_i, g_L^2)$.

Theorem 26. Assume $C$ has no marked point and $\partial C \neq \emptyset$. Let $p > 2$ and $k \geq 1$. Then $\hat{\mathcal{D}}_\Lambda : T\mathcal{N}^k_{\mathbb{C}} \to \mathcal{E}^{k-1,p} \oplus W^{k-1,p}(C, g_P)$ is a Fredholm operator for every $\Lambda = (A, \varphi) \in \mathcal{B}^{k,p}$. It has real index

$$\text{index} \hat{\mathcal{D}}_\Lambda = (n - \dim G)\chi(C) + \mu_G(\varphi^*TC^n, \varphi^*TL), \quad (38)$$

where $\chi(C)$ denotes the Euler characteristic of $C$. Here $\varphi^*TL$ is the totally real bundle on $\pi^{-1}(\partial C)$ defined by $\varphi^*TL|_{xg} = g_* (T_{\varphi(x)}L_i)$ for $x \in \kappa(C_i), g \in G$.

Proof. It is shown in [6] that the operator $-(d_A^*, d_A, D_A)$ is a compact perturbation of $\hat{\mathcal{D}}_\Lambda$. Combine $-d_A^*$ and $d_A$, we consider the operator

$$(-d_A^*, d_A) : \Omega^1_L(C, g_P) \to \Omega^0_L(C, g_P) \oplus \Omega^2_L(C, g_P)$$

whose index equals $-\chi(\text{Ker}(\ell^*))$, where $\text{Ker}(\ell^*)$ is the twisted de Rham complex with boundary values

$$\begin{array}{c}
\Omega^0_L(C, g) \to \Omega^1_L(C, g) \to \Omega^2_L(C, g) \\
\end{array}$$
From Lemma 25, we obtain that
\[ H^0(\text{Ker}(\ell^*)) = \cap g_{L_i}, H^1(\text{Ker}(\ell^*)) = \ker(\ell^1) \oplus \text{coker}(\ell^0), H^2(\text{Ker}(\ell^*)) = \text{coker}(\ell^1). \]

A simple calculation shows that
\[ \dim \text{coker} \ell^1 - \dim \text{coker} \ell^0 = \sum \dim g_{L_i}^+ - \dim G \cdot b_1(C), \]
\[ \dim \text{coker} \ell^0 = \sum \dim g_{L_i}^+ + \dim \cap g_{L_i} - \dim G. \]

Hence we have \( \chi(\text{Ker}(\ell^*)) = \dim G \cdot \chi(C). \) On the other hand, by Riemann-Roch theorem the operator \( D_\Lambda \) has index \( n\chi(C) + \mu(G) \cdot P, \varphi^*T\mathbb{C}^n, \varphi^*T\mathcal{L} \). The proof of the theorem is now complete.

However, when \( C \) has boundary marked points, the index of linearized operator depends on the asymptotic behavior of equation. We now only care about the situation on the strip ends. In this case we have \( \Omega^0(\Theta, E_\varphi) \equiv \Omega^0(\Theta, E_\varphi). \) We denote by \( P_\beta \) the principal bundle \( P \) restricting to the strip ends of \( \hat{C}. \) Let \( \Lambda = (\Psi dt, \varphi) \in \mathcal{B}_C(P_\beta) \) be a solution of (32) defined on strip ends with temporal gauge. In a local coordinate \( z = s + it \) with \( \sigma = -idz, \) the derivative of equation \( \partial \Lambda \phi + X_\sigma(\phi) = 0 \) at \( \Lambda \) with respect to \( \phi \) has the explicit form
\[ (D_\Lambda + \text{Re}(\nabla X_W(\varphi) \otimes \pi^* \sigma))(\beta) = \nabla_{\partial_n} \beta + J(\nabla_{\partial t} \beta + \nabla_\beta Q) + \nabla_\beta J(\partial_\beta \varphi + \varphi_2 + Q), \quad (39) \]
where \( Q = X_\Psi(\varphi) + X_2 \text{Re} W(\varphi). \) We rewrite the operator \( \hat{\partial}_\Lambda \) of deformation complex (35) restricting to \( P_\beta. \) Applying (36) and (39) we have \( \hat{\partial}_\Lambda|P_\beta = \partial_s + S_\Lambda, \) where \( S_\Lambda \) can be expressed in a square matrix form as
\[ S_\Lambda = \begin{pmatrix} 0 & \frac{\partial}{\partial t} + \text{ad}(\Psi) & L^s_\varphi \\ -\frac{\partial}{\partial \varphi} - \text{ad}(\Psi) & 0 & \frac{\partial \mu(\varphi)}{\partial \varphi} \\ L^s_\varphi & J \circ L_\varphi & R_\Lambda \end{pmatrix} \quad (40) \]
where \( R_\Lambda \xi = \nabla_\xi (JQ) + J(\frac{\partial \xi}{\partial t}) + (\nabla_\xi J) \frac{\partial \xi}{\partial \varphi}. \) We next show that \( S_{\Lambda(s)} \) is a family of self-adjoint operators with \( s \) varying. In general, for \( u = (\eta, x) \in \mathcal{P}(L_0, L_1) \) associated with some marked point, the domain of \( S_u \) is a path space
\[ \mathcal{P}_u = \{ a \in 
abla C^\infty([0,1], \mathbb{g})g(a(i)) \in \mathbb{g}_{L_i}, \partial_t a(i) \in \mathbb{g}_{L_i}^+, i = 0, 1 \} \times T_u \mathcal{P}(L_0, L_1) \]
\[ = \{ (a, b, \xi) \in C^\infty([0,1], \mathbb{g}^2 \times x^* T\mathbb{C}^n)|a(i) \in \mathbb{g}_{L_i}, \partial_t a(i) \in \mathbb{g}_{L_i}^+, b(i) \in \mathbb{g}_{L_i}^+, \xi(i) \in T_{x(i)}L_i, i = 0, 1 \} \]

**Proposition 27.** Let \( \Lambda = (\Psi dt, \varphi) \in \mathcal{B}_{C}^{1,2}(P_\beta) \) be a solution of (8) on a strip end and \( \Lambda(s) = (\Psi_s(t), \varphi_s(t)) \) is a path in \( \mathcal{P}^{1,2}(L_0, L_1). \) Then \( S_{\Lambda(s)} \) is an \( L^2 \)-self-adjoint operator with domain \( \mathcal{P}^{1,2}_{\Lambda(s)} \) for all \( s > 0. \)
Proof. We refer to the proof of Theorem 4.1 in [13]. We first show that 
\((\frac{\partial}{\partial t} + \text{ad}(\Psi))^* = -\frac{\partial}{\partial t} - \text{ad}(\Psi)\). By calculation,
\[
(\frac{\partial}{\partial t} + [\Psi, b], a) + (b, \frac{\partial}{\partial t} + [\Psi, a])
= \int_0^1 (\frac{\partial}{\partial t} + [\Psi, b], a) + (b, \frac{\partial}{\partial t} + [\Psi, a]) dt
= \int_0^1 \frac{\partial}{\partial t} (a, b) dt = 0.
\]
Next, it follows from (4) that 
\(d\mu(\varphi)^* = J \circ L_\varphi\). Finally we have to show that \(R_\Lambda\) is self-adjoint. Since \(\nabla\) is the Levi-Civita connection with respect to \(\omega(J, J)\), then for \(\forall u, v \in T_x \mathbb{C}^n\),
\[
(u, \nabla_u (JX_\varphi)) = (u, (\nabla_u J)X_\varphi + J\nabla_u X_\varphi)
= -((\nabla_u J)u, X_\varphi) - v(Ju, X_\varphi) + (\nabla_u (J\nabla_u u), X_\varphi)
= -\omega(X_\varphi, \nabla_u u) + v\omega(X_\varphi, u)
= -d(\mu - \tau, \Psi)(\nabla_u u) + v\omega(X_\varphi, u)
= -d(\mu - \tau, \Psi)(\nabla_u u) + ud(\mu - \tau, \Psi)(v)
= (v, \nabla_u (JX_\varphi)).
\]
Similarly, for \(\forall f \in C^\infty(\mathbb{C}^n)\), we have
\[
(u, \nabla_u (JX_f)) = -\omega(X_f, \nabla_u u) + v\omega(X_f, u)
= -df(\nabla_u u) + vdf(u)
= -df(\nabla_u u) + ud\omega(v)
= (v, \nabla_u (JX_f)).
\]
According to (41) and (42), it follows that \(\nabla(JQ)\) is self-adjoint. The proposition will be proved if we can show
\[
\int_0^1 (J \frac{\partial \xi_1}{\partial t} + (\nabla_{\xi_1} J)\varphi', \xi_2) dt = \int_0^1 (J \frac{\partial \xi_2}{\partial t} + (\nabla_{\xi_2} J)\varphi', \xi_1) dt. \tag{43}
\]
On the one hand, by calculation,
\[
(J \frac{\partial \xi_1}{\partial t}, \xi_2) = -\frac{\partial \xi_1}{\partial t} (\xi_1, \xi_2) - (\xi_1, \nabla_{\partial t}(J\xi_2))
= -\frac{\partial}{\partial t} (\xi_1, \xi_2) + (\xi_1, (\nabla_{\partial t}(J\xi_2)). \tag{44}
\]
On the other hand, notice that
\[
((\nabla_{\xi_1} J)\varphi', \xi_2) - ((\nabla_{\xi_2} J)\varphi', \xi_1)
= ((\nabla_{\xi_1} J)\varphi', \xi_2) + (\varphi', (\nabla_{\xi_2} J)\xi_1)
= -(\xi_1, (\nabla_{\varphi'} J)\xi_2), \tag{45}
\]
and
\[ \int_0^1 \frac{\partial}{\partial t} (\xi_1, J\xi_2) dt = (\xi_1, J\xi_2)|^1_0 = -\omega(\xi_1, \xi_2)|^1_0 = 0. \] (46)

Combine (44), (45) and (46) together, then (43) holds. The proof is complete. \( \square \)

Due to regularity, one only need to consider \( W^{1,p} \)-solutions of (8). The associated moduli space \( \mathcal{M}_L \) is given by
\[ \mathcal{M}_L := S^{-1}(0) / \mathcal{G}^2_{L, p}, \]
where \( S^{-1}(0) \) is the set of all irreducible \( W^{1,p} \)-solutions of (8). Now take \( C \) to be a disk with \( k \) boundary marked points. To highlight marked points, let \( \mathcal{F}_{W,i} \) denote the action functional on the \( i \)-th path space, \( \Theta_i \) the \( i \)-th strip end and let \( \mathcal{H}_i \) be the \( i \)-th group with respect to the \( i \)-th marked point. Every gauge transformation in \( \mathcal{G}_L \) tends to a limit in \( \mathcal{H}_i \) on the strip end \( \Theta_i \) and we get \( k \) evaluation homomorphisms
\[ ev_i : \mathcal{G}_L \to \mathcal{H}_i, \quad g \mapsto \lim_{s \to +\infty} g|_{\Theta_i}, \]
Meanwhile there are \( k \) evaluation maps on \( S^{-1}(0) \)
\[ ev_i : S^{-1}(0) \to \text{crit}(\mathcal{F}_{W,i}), \quad (A, \varphi) \mapsto \lim_{s \to +\infty} (\Psi, \varphi)|_{\Theta_i} \]
where \( A|_{\Theta_i} = \Psi dt \). Similarly, each \( ev_i \) induces an evaluation map on the moduli space
\[ \tilde{ev}_i : \mathcal{M}_L \to \text{crit}(\mathcal{F}_{W,i}) / \mathcal{H}_i. \]

For \( \Lambda \in \mathcal{M}_L \), let \( \Lambda_i(s) = \Lambda|_{\Theta_i} \). Due to the fact that \( \lim_{s \to +\infty} S_{\Lambda_i(s)} = \text{Hess} \tilde{ev}_i(\Lambda) \), the operator \( \lim_{s \to +\infty} S_{\Lambda_i(s)} \) is invertible if and only if \( \bar{I}_{x,0} \) and \( \bar{I}_{x,1} \) intersect transversally at \( \tilde{ev}_i(\Lambda) \). Here \( \text{Hess}_u = X_u \oplus D_u \) is the Hessian operator of \( \mathcal{F}_{W,i} \) at \( u \) modulo gauge transformations defined in §4.2.

A \( k \)-tuple \( (y_1, \cdots, y_k) \in \prod_{i=1}^k \bar{I}_{x,0} \cap \bar{I}_{x,1} \) is called transversal if \( \bar{I}_{x,0} \) and \( \bar{I}_{x,1} \) are transversal at \( y_i \) for all \( i \). We define the subspace \( \mathcal{M}_L(y_1, \cdots, y_k) \subset \mathcal{M}_L \) of all irreducible solutions \( (A, \varphi) \) with \( ev_{x_i}(A, \varphi) = y_i \). For the case \( C = [0, 1] \times \mathbb{R} \), the Fredholm property of \( \mathcal{G}_\Lambda : W^{r,p} \to W^{r-1,p} \) was proved in [27] for \( p = 2 \) and in [28] for \( p > 1 \). For general case, the operator \( \mathcal{G}_\Lambda \) is Fredholm since the Cauchy-Riemann operator is nondegenerate (see [29]) when \( (ev_1(\Lambda), \cdots, ev_k(\Lambda)) \) is transversal.

When \( C \cong \mathbb{R} \times [0, 1] \), there is a weight map (see [13, §5.1])
\[ I : \text{crit}(\mathcal{F}_W) \times \text{crit}(\mathcal{F}_W) \to \mathbb{Z}, \quad I(a, b) := \text{index} \mathcal{G}_\Lambda, \]
where \( \Lambda \) is a smooth gradient flow line of \( \mathcal{F}_W \) connecting the two critical points \( a \) and \( b \). This weight map has the following properties from gluing theorem: \( \forall a, b, c \in \text{crit}(\mathcal{F}_W), h \in \mathcal{H} \) and \( h' \in \mathcal{H}_0 \),
\[ I(a, b) + I(b, c) = I(a, c), \quad I(ha, hb) = I(a, b), \quad I(h'a, b) = I(a, b). \] (47)
Then there is also a weight map

\[ I^k : \prod_{i=1}^k \text{crit}(\mathcal{F}_{W,x_i}) \to \mathbb{Z}, \quad I^k(y_1, \cdots, y_k) := \text{index} \mathcal{D}_\Lambda, \]

where \( \Lambda \in \mathcal{M}_\mathcal{L}(y_1, \cdots, y_k) \). In the end we give a proposition of \( I^k \) generalizing (47).

**Proposition 28.** Suppose \( g \in \mathcal{G}_\mathcal{L} \) and let \( h_i = \text{ev}_i(g) \in \mathcal{H}_i \). Then

\[ I^k(h_1y_1, \cdots, h_ky_k) = I^k(y_1, \cdots, y_k). \]

**Proof.** It follows from gauge invariance.

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