Wheeler Languages

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Introduction

The Burrows-Wheeler Transform (BWT) of a given string is an invertible transformation with many important and deep properties (see [BW94]). It can be computed on a given string by marking the beginning of the string by the special character # and reading the first column of the matrix consisting of the co-lexicographically ordered circular permutations of the string (BW-matrix, see Figure 1-(a)).

The fact that the transform is invertible can be seen as one of its most basic and useful features, and it is a consequence of the fact that the BWT (actually the BW-matrix) enjoys the so-called First-Last property (FL-property, more on this below). Being invertible and, at the same time, rich of single-letter runs induced by the co-lexicographic order of prefixes, the BWT becomes the basis for a family of tools needing very little extra data-structures (see [NM07]).

The FL-property consists in the observation that in the first and last columns of the BW-matrix, the relative order of different occurrences of the same character is maintained. Consider, for example, the BWT of the string #banana, that is bnn#aaa, and notice that the First-Last property can be used to instruct us on how to reconstruct #banana: start from # on the first column, search the occurrence of # in the last column, move to the first column on the same row, and continue with the corresponding character (i.e. b, see Figure 1-(b)). The correctness of the reconstruction of the original string is a consequence of the FL-property: at each step the character read on the first column corresponds to the one determined on the last column.

Figure 1: Starting from # in the first (F) column of the BW-matrix in (a), reading the character, and moving to the corresponding position in the last (L) column, the original string can be reconstructed. The full procedure is encoded in the path (linear automaton) in (b).

The above graph can be seen as a very simple (linear) state-labelled finite automaton, with node labels organized in the order they appear in the F column (the BWT). With a slight twist, let us now use a different ordering: the one induced by the L column of the BW-matrix. The result, reflecting on the linear automaton the nice computational features of the BWT, is depicted in Figure 2.

Figure 2: The path automaton of Figure 1 reorganized according to the order of column L.

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1 In the “official” definition of the transform, the lexicographic ordering of circular permutations and a $-$-mark of the end of the string are used. Working with the co-lexicographic ordering is a bit more natural while studying formal languages and does not make any significant difference.

2 The co-lexicographic order of prefixes can be read on the right side of the BW-matrix.
In a sense, this layout seems more natural as it orders nodes according to the co-lexicographic ordering of the strings read from the source to each of the nodes. The graphs we obtain in this way are precisely paths encoding the procedure inverting the Burrows-Wheeler transform of a given string. Much more interestingly, one may ask the following question: can we generalise our considerations to the context of general ordered graphs (i.e. not being necessarily paths)? One may wonder which properties of graphs/automata/orderings enforce the above behaviour.

The objects resulting from this analysis are Wheeler graphs \cite{GMS17} and their characterising properties—working for general ordered graphs—are:

(i) the ordering of character-labelled states must be coherent with an (a priori fixed) order of characters, and

(ii) the ordering of states \( u, v \) bearing the same character-label must be coherent with the ordering of all the predecessor pairs \( u', v' \) with associated arcs \( (u', u), (v', v) \).

The main application of Wheeler graphs is that they admit an efficient index data structure for searching for subpaths with a given path label \cite{GMS17}. This is in contrast with recent results showing that in general, the subpath search problem can not be solved in subquadratic time, unless the strong exponential time hypothesis is false \cite{EGMT19}. The indexing version of the problem was also recently shown to be hard, unless the orthogonal vectors hypothesis is false \cite{EMT20}. The strong exponential time hypothesis implies the orthogonal vectors hypothesis.

In the big picture, Wheeler graphs lift the applicability of the Burrows-Wheeler transform from strings to languages.

In this paper we study the regular languages accepted by automata having a Wheeler graph as transition function. The study is carried out in both the deterministic and the non-deterministic case and shows that Wheeler Automata establish a deep link between intervals of states—in the Wheeler ordering imposed by the definition—and “intervals” of strings—in the co-lexicographic ordering of prefixes of elements in \( \mathcal{L} \). Our investigation starts from some results already appeared in \cite{ADPP20}, where we proved that the classic characterisation of regular languages based on Myhill-Nerode Theorem can be generalised and adapted to the Wheeler case. The generalisation is proved by introducing equivalence classes which are convex sets in the co-lexicographic ordering of prefixes of strings in \( \mathcal{L} \). This characterization allows also to prove that the (potential) exponential blow-up in the number of states observed in general when passing from a non-deterministic to a deterministic automaton, cannot take place in the Wheeler case.

In this paper we apply these results (which we add with complete proofs for the sake of completeness and readability) to find a solution to the problem of effectively testing for Wheelerness languages given by a deterministic or non-deterministic automaton. In addition, in the deterministic case we can show that the test takes polynomial time. The results on testing Wheelerness are based on a theorem that characterises minimal deterministic automata accepting Wheeler languages on a purely graph-theoretic property.

Next, we take the automata’s point of view on Wheelerness. More specifically, since the problem of deciding whether a given NFA can be endowed with a Wheeler order is obviously decidable, we tackle its complexity which, although polynomial in special cases (see \cite{ADPP20}), is known to be NP-complete in the general case (see \cite{GT19}). Here we prove that over a natural subclass of NFA, the reduced ones—that is, those in which no two states are reachable by the same set of strings—are, the problem can in fact be solved in polynomial time.

Finally, we take a closer look at classical operations among Wheeler Languages. Since Wheeler languages are a subclass of the class of Ordered Languages (see \cite{ST74}), they are star-free, namely they can be generated from finite languages by boolean operations and compositions only. As such, they are definable in the first order theory of linear orders \( FO(<) \) (see \cite{DG08} for a survey on FO-definable languages). However, as we shall see, there are very few classical operations preserving Wheelerness. While regular languages are closed for boolean and regular operations, we prove that, with a few exceptions, this is not the case for Wheeler languages.

The paper is organised as follows. Section \textit{Basics} contains basic notions and notations. Section \textit{Wheeler Automata and Convex Sets} introduces the notion of Wheeler Automata and links the natural linear orderings definable on states and strings, respectively. In this section we also introduce convex equivalences, allowing us to prove a precise “Wheeler version” of the classical Myhill-Nerode Theorem for
regular languages. Section Testing Wheelerness tackles the problem, discussed in two separate subsections, of whether a given language or a given automaton is Wheeler. The next section, Closure Properties for Wheeler Languages, considers regular operations and closure properties that are known to hold for regular languages and checks whether they also hold for Wheeler languages. In this section we further consider intervals on the co-lexicographic order, proving that they are Wheeler. We conclude the paper with the section Conclusions and Open Problems.

1 Basics

1.1 Automata

If \( \Sigma \) is a finite alphabet, we denote by \( \Sigma^*(\Sigma^*) \) the set of (non-empty) finite words over \( \Sigma \). If \( \mathcal{L} \subseteq \Sigma^* \) we denote by \( \text{Pref}(\mathcal{L}) \), \( \text{Suff}(\mathcal{L}) \), and \( \text{Fact}(\mathcal{L}) \) the set of prefixes, suffixes, and factors of strings in \( \mathcal{L} \), respectively. More formally:

\[
\text{Pref}(\mathcal{L}) = \{ \alpha : \exists \beta \in \Sigma^* \, \alpha \beta \in \mathcal{L} \}, \quad \text{Suff}(\mathcal{L}) = \{ \beta : \exists \alpha \in \Sigma^* \, \alpha \beta \in \mathcal{L} \}, \quad \text{Fact}(\mathcal{L}) = \{ \alpha : \exists \beta, \gamma \in \Sigma^* \, \gamma \alpha \beta \in \mathcal{L} \}.
\]

In the following we will denote by \( \mathcal{A} = (Q, s, \delta, F) \) a finite automaton (an NFA) accepting strings in \( \Sigma^* \), with \( Q \) as set of states, \( s \) unique initial state with no incoming transitions, \( \delta(\cdot, \cdot) : Q \times \Sigma \rightarrow \mathcal{P}(Q) \) transition function, and \( F \subseteq Q \) final states. Note that assuming that \( s \) has no incoming transitions is not restrictive, as any NFA can be made to satisfy this condition by just duplicating \( s \) into an initial \( s' \) with no incoming transitions and a non-initial \( s'' \) with all the incoming transitions of the original \( s \).

An automaton \( \mathcal{A} \) is deterministic (a DFA), if \( |\delta(q, a)| \leq 1 \), for any \( q \in Q \) and \( a \in \Sigma \). As customary, we extend \( \delta \) to operate on strings as follows: for all \( q \in Q, a \in \Sigma \), and \( \alpha \in \Sigma^* \):

\[
\delta(q, \epsilon) = \{ q \}, \quad \delta(q, a\alpha) = \bigcup_{v \in \delta(q, a)} \delta(v, \alpha).
\]

If the automaton is deterministic we write \( \delta(q, \alpha) = q' \) for the unique \( q' \) such that \( \delta(q, \alpha) = \{ q' \} \) (if defined). We denote by \( \mathcal{L}(\mathcal{A}) = \{ \alpha \in \Sigma^* : \delta(s, \alpha) \cap F \neq \emptyset \} \) the language accepted by the automaton \( \mathcal{A} \). \( \mathcal{A} \) is dubbed complete if for any \( q \in Q, a \in \Sigma \), \( \delta(q, a) \) is defined. In general, we do not assume \( \delta \) to be complete—to see why, wait for Example 2 below—, while we do assume that each state can reach a final state and also that every state is reachable from the (unique) initial state. Hence, \( \text{Pref}(\mathcal{L}(\mathcal{A})) \), the collection of prefixes of words accepted by \( \mathcal{A} \), consists of the set of words that can be read by \( \mathcal{A} \).

Using the terminology from [ADPP20], an input-consistent automaton is such that every state has incoming edges labeled by the same character. This class of automata is the one considered in the original definition of Wheeler graph in [CMS17]. It is fully general: any automaton can be converted into an input-consistent one recognizing the same language at the price of increasing \( |Q| \) by a multiplicative factor \( |\Sigma| \) [ADPP20]. Moving labels from an edge to its target state, input-consistent automata can be described as state-labeled automata (see Example 11). In this paper we will therefore use the term state-labeled in place of input-consistent. Given a state-labeled automaton, we denote by \( \lambda : Q \rightarrow \Sigma \cup \{ \# \} \) the function that returns the (unique) label of a state, so that \( \delta(v, c) \) is the set of \( c \)-labeled successors of \( v \). To make \( \lambda \) complete and to be consistent with the definition of Burrows-Wheeler Transform, we assign \( \lambda(s) = \# \notin \Sigma \), where \# is a character not labeling any other state. When for all \( u, v \in C \subseteq Q \) we have \( \lambda(u) = \lambda(v) \), let \( \lambda(C) \) be the unique character \( c = \lambda(u) \), for any \( u \in C \). To make notation consistent between edge-labeled and state-labeled automata, given a path \( v_0, \ldots, v_n \) we define its label as \( \lambda(v_1) \ldots \lambda(v_n) \), so that the first node \( v_0 \) does not contribute to the string labeling the path. All our results dealing with Wheeler automata will use state-labeled automata. In other results, however, we will need to work with standard edge-labeled automata. In this case, we will explicitly say that the automaton is edge-labeled and use the notation \( \lambda(u, v) \in \Sigma \) to denote the label of an automaton’s edge \((u, v)\) (note that, in the case of edge-labeled automata, no edge is labeled with \#).

1.2 Convex Sets

As we shall see, Wheeler automata and languages naturally lead to considering convex subsets of a linear order. We collect here a few definitions and results that will turn out handy while reasoning on convex sets.
**Definition 1.** Consider a linear order \((L, <)\).

1. A **convex set** in \((L, <)\) is a \(I \subseteq L\) such that
   \[
   (\forall x, x' \in I)(\forall y \in L)(x < y < x' \Rightarrow y \in I).
   \]
2. Given \(I, J\) convex in \((L, <)\) and \(I \subseteq J\), then:
   - \(I\) is a **prefix** of \(J\) if \((\forall x \in I)(\forall y \in J \setminus I)(x < y)\);
   - \(I\) is a **suffix** of \(J\) if \((\forall y \in J \setminus I)(\forall x \in I)(y < x)\).
3. A family \(C\) of non-empty convex sets in \((L, <)\) is said to have the **prefix/suffix property** if, for all \(I, J \in C\) such that \(I \subseteq J\), \(I\) is either a prefix or a suffix of \(J\).

In particular, if \(a, b \in L\) for a linear order \((L, <)\), then we denote by \([a, b]\) the convex set:
\[
[a, b] = \{ c \in L : a \leq c \leq b \}.
\]

\([a, b]\) is called the **closed interval based on** \(a, b\); other kinds of intervals, denoted by \((a, b), (a, b], (−\infty, b)\) are defined as usual. Notice that any convex set \(I\) having a maximum and a minimum is an interval:
\[
I = [\min_{<I}, \max_{<I}].
\]

In particular, all convex subsets of a finite linear order are intervals, and we shall use freely both names for them.

The most convenient feature of a family \(C\) enjoying the prefix/suffix property, is the fact that its elements can be easily ordered.

**Definition 2.** Let \(C\) be a family of non-empty convex sets of a linear order \((L, <)\) having the prefix/suffix property. Let \(<^1\) (or simply \(<\)) the binary relation over \(C\) defined by
\[
I <^1 J \iff (\exists x \in I)(\forall y \in J)(x < y) \lor (\exists y \in J)(\forall x \in I)(x < y).
\]

The following lemma is easily proved.

**Lemma 1.1.** \((C, <^1)\) is a **strict linear order**.

Note that whenever any non-empty convex set \(I\) has minimum \(m_I\) and maximum \(M_I\)—which is the case, for example, when the linear order \((L, <)\) is finite—, the above order \(<^1\) can be equivalently described on a family having the prefix/suffix property, by:
\[
I <^1 J \iff (m_I < m_J) \lor [(m_I = m_J) \land (M_I < M_J)] \iff m_I + M_I < m_J + M_J.
\]

The following lemma will allow us to bound (linearly) the blow-up of the number of states taking place when moving from a Wheeler NFA to a Wheeler DFA (see Definition 3 below).

**Lemma 1.2.** Let \((L, <)\) be a finite linear order of cardinality \(|L| = n\), and let \(C\) be a prefix/suffix family of non-empty convex sets in \((L, <)\). Then:

1. \(|C| \leq 2n - 1\).
2. The upper bound is tight: for every \(n\), there exists a prefix/suffix family of size \(2n - 1\).

**Proof.** (1) Since \(L\) is finite, for any \(I, J \in C\) we have
\[
I < J \iff m_I + M_I < m_J + M_J
\]
which implies
\[
I \neq J \iff m_I + M_I \neq m_J + M_J.
\]
Since the possible values of \(m_I + M_I\), for \(I \in C\), range between 2 and \(2n\), the bound \(|C| \leq 2n - 1\) follows.

(2) Consider the prefix/suffix family containing just one maximal interval and all its proper prefixes and suffixes: \(C = \{L[1, n], L[1, 1], \ldots, L[1, n-1], L[2, n], \ldots, L[n, n]\}\). This family satisfies \(|C| = 2n - 1\).
Definition 3. Consider a linear order \((L, <)\) and an equivalence relation \(\sim\) over its domain \(L\).

1. We say that \(\sim\) is convex if its equivalence classes are convex sets in \((L, <)\).

2. The convex refinement of \(\sim\) over \((L, <)\), is the relation \(\sim^c\) on \(L\) defined as follows. For all \(a, b \in L\):
   \[a \sim^c b \iff a \sim b \land (\forall d \in L)(\min\{a, b\} < d < \max\{a, b\} \rightarrow a \sim d).\]

Lemma 1.3. The convex refinement \(\sim^c\) of an equivalence relation \(\sim\) over \((L, <)\), is a convex equivalence relation.

In this paper, if \(\Sigma\) consist of a finite number of letters ordered by \(\prec\), we denote, again by \(\prec\), the co-lexicographic order over \(\Sigma^*\), defined for \(\alpha = a_1 \ldots a_n, \beta = b_1 \ldots b_k\), as:
\[\alpha \prec \beta \iff (n < k \land (\forall j \leq n)(a_{n-j} = b_{k-j})) \lor (\exists i)(a_{n-i} < b_{k-i} \land (\forall j < i) a_{n-j} = b_{k-j}).\]

2 Wheeler Automata and Convex Sets

Wheeler languages will be defined below to be regular languages accepted by Wheeler automata, that is, automata equipped with an ordering among states. It will be proved in 2.2 that Wheeler languages are naturally given as finite families of non-empty convex sets on \(\prec\) enjoying the prefix/suffix property.

Let us begin giving the definition of Wheeler automaton.

Definition 4. A Wheeler NFA (WNFA) \(A = (Q, s, \delta, <, F)\) is an NFA endowed with a binary relation \(<\), such that: \((Q, <)\) is a linear order having the initial state \(s\) as minimum, \(s\) has no in-going edges, and the following two (Wheeler) properties are satisfied. Let \(v_1 \in \delta(u_1, a_1)\) and \(v_2 \in \delta(u_2, a_2)\):

(i) \(a_1 \prec a_2 \rightarrow v_1 < v_2;\)

(ii) \((a_1 = a_2 \land u_1 < u_2) \rightarrow v_1 \leq v_2.\)

A Wheeler DFA (WDFA) is a WNFA in which the cardinality of \(\delta(u, a)\) is always less than or equal to one.

Remark 2.1. A consequence of Wheeler property (i) is that \(A\) is input-consistent, that is all transitions entering a given state \(u \in Q\) bear the same label: if \(u \in \delta(v, a)\) and \(u \in \delta(w, b)\), then \(a = b.\)

On the grounds of the above remark, when drawing Wheeler automata we “move” labels from edges to nodes and therefore deal with state-labeled automata: all edges entering a node labelled \(e \in \Sigma\) would then be \(e\)-edges. As mentioned in the introduction, to make \(\lambda\) complete we set \(\lambda(s) = \# \notin \Sigma\), where \# labels just \(s\).

Unless explicitly stated, if we use an alphabet \(\Sigma\) containing alphabetical letters, we implicitly suppose \(\Sigma\) ordered alphabetically.

Example 1. The following automaton proves that the language \(ax^*b|zx^*d\) is Wheeler (states ordered from left to right):

![Wheeler Automaton Diagram](image-url)
A key consequence of (i) and (ii) above (already proved in [GMS17]), is the fact that the set of states reachable in a WNFA $A$ while reading a given string $\alpha$ is an interval in $(Q, \prec)$. This important fact will be re-proved below—in Lemma 2.4, together with what we may call a sort of its “dual”, that is, the set of strings read while reaching a given state is a convex set. More precisely, if $A = (Q, s, \delta, \prec, F)$ is a WNFA, $u \in Q$, and $\alpha \in \Sigma^*$, let $I_u = \delta(s, \alpha), I_u = \{\alpha : \delta(s, \alpha) = u\}$; then it easily follows that

$$\alpha \in I_u \text{ if and only if } u \in I_\alpha,$$

and in Lemma 2.4 we shall prove that $I_u$ is a convex set in $(Q, \prec)$, while $I_u$ is convex in $(\text{Pref}(_L(A)), \prec)$.

Preliminary to our result is the following lemma.

**Lemma 2.2.** [ADPP20] If $A = (Q, s, \delta, \prec, F)$ is a WNFA, $u, v \in Q$ are states, and $\alpha, \beta \in \text{Pref}(_L(A))$, then:

1. if $\alpha \in I_u, \beta \in I_v$, and $\{\alpha, \beta\} \not\subseteq I_v \cap I_u$, then $\alpha \prec \beta$ if and only if $u < v$;
2. if $u \in I_\alpha, v \in I_\beta$, and $\{u, v\} \not\subseteq I_\beta \cap I_\alpha$, then $\alpha \prec \beta$ if and only if $u < v$.

**Proof.**

(1) Suppose $\alpha \in I_u, \beta \in I_v$ and $\{\alpha, \beta\} \not\subseteq I_v \cap I_u$. From this we have that $\alpha \in I_u \setminus I_v$ or $\beta \in I_v \setminus I_u$, hence $u \neq v$ and $\alpha \neq \beta$ follows.

If $u = s$ or $v = s$, either $\alpha$ or $\beta$ is the empty string $\epsilon$ and the result follows easily. Hence, we suppose $u \neq s \neq v$ and (hence) $\alpha \neq \epsilon \neq \beta$.

To see the left-to-right implication, assume $\alpha \prec \beta$: we prove that $u < v$ by induction on the maximum between $|\alpha|$ and $|\beta|$. If $|\alpha| = |\beta| = 1$, then the property follows from the Wheeler-(i). If $\max(|\alpha|, |\beta|) > 1$ and $\alpha$ and $\beta$ end with different letters, then again the property follows from Wheeler-(i). Hence, we are just left with the case in which $\alpha = \alpha'e$ and $\beta = \beta'e$, with $e \in \Sigma$. Since $\alpha \prec \beta$, we have $\alpha' \prec \beta'$. Consider states $u', v'$ such that $\alpha' \in I_{u'}, \beta' \in I_{v'}$, and $u \in \delta(u', e), v \in \delta(v', e)$. Then $\alpha' \in I_{u'} \setminus I_{v'}$ or $\beta' \in I_{v'} \setminus I_{u'}$ because otherwise we would have $\alpha' \in I_{v'}$ and $\beta' \in I_{u'}$ which imply respectively $\alpha \in I_v$ and $\beta \in I_u$. By induction we have $u' < v'$ and therefore, by Wheeler-(ii), $u \leq v$. From $u \neq v$ it follows $u < v$.

Conversely, for the right-to-left implication, suppose $u < v$. Since $\alpha \neq \beta$, if it were $\beta < \alpha$ then, by the above, we would have $v < u$: a contradiction. Hence, $\alpha \prec \beta$ holds.

(2) Recall that, by definition, $\alpha \in I_u$ if and only if $u \in I_\alpha$ and $\beta \in I_v$ if and only if $v \in I_\beta$. Hence, the hypothesis that $u \in I_\alpha, v \in I_\beta$ and $\{u, v\} \not\subseteq I_\beta \cap I_\alpha$, is equivalent to say that $\alpha \in I_u, \beta \in I_v$ and $\{\alpha, \beta\} \not\subseteq I_v \cap I_u$. Therefore, (2) follows from (1).

\[\square\]

The following corollary, to be used in Section 3.1, observes that the sequence of states reached in a W DFA while reading a monotone sequence of strings, must “stabilise” to some specific state. As a matter of fact, it will be proved in Lemma 2.6 that a similar property holds also for a WNFA.

**Corollary 2.3.** [ADPP20] If $A = (Q, \delta, q, \prec, F)$ is a W DFA, then, for all $\alpha, \beta \in \text{Pref}(_L(A))$ it holds

$$\alpha \prec \beta \Rightarrow \delta(s, \alpha) \leq \delta(s, \beta), \text{ and } \delta(s, \alpha) < \delta(s, \beta) \Rightarrow \alpha \prec \beta$$

Moreover, any sequence of states $(\delta(s, \alpha_i))_{i \geq 1}$ for $(\alpha_i)_{i \geq 1}$ monotone sequences in $(\text{Pref}(_L(A)), \prec)$, is eventually constant. More precisely, if $(\alpha_i)_{i \geq 1}$ is a sequence in $(\text{Pref}(_L(A)), \prec)$ such that either

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_i \leq \ldots \text{ or } \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_i \geq \ldots$$

then there exists $u \in Q$ and $n \geq 1$ such that $\delta(s, \alpha_h) = \delta(s, \alpha_k) = u$, for all $h, k \geq n$. 
Proof. If \( A = (Q, \delta, q, <, F) \) is a W DFA and \( \alpha \in \text{Pref}(\mathcal{L}(A)) \) then, for all \( u \in Q \), it holds
\[
\alpha \in I_u \iff u = \delta(q, \alpha),
\]
and
\[
\alpha < \beta \Rightarrow \delta(s, \alpha) \leq \delta(s, \beta), \quad \text{and} \quad \delta(s, \alpha) < \delta(s, \beta) \Rightarrow \alpha < \beta
\]
easily follows from the previous lemma. If \( (\alpha_i)_{i \geq 1} \) is a monotone sequence in \( (\text{Pref}(\mathcal{L}(A)), \prec) \), then the first implication above implies that \( (\delta(s, \alpha_i))_{i \geq 1} \) is a monotone sequence in \( (Q, <) \). Since \( Q \) is a finite set, the corollary follows. \( \square \)

The following lemma refers to WNFA and proves that the collection of states reached reading a given string, turns out to be an interval in the Wheeler order of states. WDFA can be seen a particular case in which intervals degenerate in a single state. Let \( I_Q = \{ I_u : u \in Q \} \) and \( I_{\text{Pref}}(\mathcal{L}(A)) = \{ I_\alpha : \alpha \in \text{Pref}(\mathcal{L}(A)) \} \).

**Lemma 2.4.** [ADPP20] If \( A = (Q, s, \delta, <, F) \) is a WNFA and \( \mathcal{L} = \mathcal{L}(A) \), then:

1. for all \( u \in Q \), the set \( I_u \) is convex in \( (\text{Pref}(\mathcal{L}(A)), \prec) \);
2. \( I_Q \) is a prefix/suffix family of convex sets in \( (\text{Pref}(\mathcal{L}(A)), \prec) \);
3. for all \( \alpha \in \text{Pref}(\mathcal{L}(A)) \), the set \( I_\alpha \) is an interval in \( (Q, <) \) (already proved in [GMS17]);
4. \( I_{\text{Pref}}(\mathcal{L}(A)) \) is a prefix/suffix family of intervals in \( (Q, <) \).

**Proof.**

1. Suppose \( \alpha < \beta < \gamma \) with \( \alpha, \gamma \in I_u \) and \( \beta \in \text{Pref}(\mathcal{L}(A)) \); we want to prove that \( \beta \in I_u \). From \( \beta \in \text{Pref}(\mathcal{L}(A)) \) it follows that there exists a state \( v \) such that \( \beta \in I_v \). Suppose, for contradiction, that \( \beta \notin I_u \). Then \( \beta \in I_v \setminus I_u \) and from \( \alpha < \beta \) and Lemma 2.2 it follows \( u < v \). Similarly, applying again Lemma 2.2 from \( \beta < \gamma \) we have \( v < u \), which is a contradiction.

2. Suppose, for contradiction, that \( I_u, I_v \in I_Q \) are such that \( I_u \subseteq I_v \) and \( I_u \) is neither a prefix nor a suffix of \( I_v \). In these hypotheses there must exist \( \alpha, \alpha' \in I_v \setminus I_u \) and \( \beta \in I_u \) such that \( \alpha < \beta < \alpha' \). Lemma 2.2 implies \( v < u < v \), which is a contradiction.

Points (3), (4) follow similarly from Lemma 2.2. \( \square \)

**Remark 2.5.** Clearly, Lemma 2.4 given above continues to hold also in the case of complete Wheeler automata, with \( \text{Pref}(\mathcal{L}(A)) \) replaced by \( (\Sigma^*, \prec) \).

Since Definition 4 allows the transition function of Wheeler DFA’s to be incomplete, one could wonder why not forcing completeness in the definition of Wheeler automaton. We can now show, using the above remark, that incompleteness is somehow necessary: the class of languages would be different if completeness were required.

**Example 2.** A Wheeler language not recognised by any complete WDFA.

Let \( A = (Q, s, \delta, <, F) \) be the following (incomplete) WDFA such that \( \mathcal{L}(A) = \mathcal{L} = b^+a: \)

```
start ---- s ------- a ------- b
```

Suppose, for contradiction, that \( \mathcal{L} = \mathcal{L}(A') \), where \( A' = (Q', s', \delta', <', F') \) is a complete Wheeler DFA. Since the set \( Q' \) is finite, there exist \( i, k \in \mathbb{N} \) with \( i < k \) and \( \delta(s', b^i) = \delta(s', b^k) = u \), for some \( u \in Q' \). From \( b^ja \in \mathcal{L} \) it follows \( \delta(s', b^ja) = z \) for some \( z \in F' \). Consider now \( v \in Q' \) such that \( \delta(s', ab^j) = v \). By Remark 2.5, \( I_u \) is a convex set in the linear order consisting of all words read by Wheeler automaton \( A' \), ordered co-lexicographically, that is \( (\Sigma^*, \prec) \). Since \( b^j \prec ab^j \prec b^k \) and \( b^j, b^k \in I_u \) implies \( ab^j \in I_u \) and since \( A' \) is a DFA, \( v = u \) follows. But then \( \delta(s', ab^ja) = z \in F' \) and we would have \( ab^ja \in \mathcal{L} \), contradicting \( \mathcal{L} = b^+a \).
From Lemma 1.1 it follows that \((I_Q, \prec^i)\) and \((I_{\text{Pref}(L(A))}, \prec^i)\) are linear orders.

**Lemma 2.6.** [ADPP20] Let \(A = (Q, s, \delta, <, F)\) be a WNFA. Consider \(I_u, I_v \in I_Q\) and \(I_\alpha, I_\beta \in I_{\text{Pref}(L(A))}\).

1. \(I_u \prec^i I_v\) implies that \(u < v\) and \(u < v\) implies that \(I_u \prec^i I_v\).

2. \(I_\alpha \prec^i I_\beta\) implies that \(\alpha < \beta\) and \(\alpha < \beta\) implies that \(I_\alpha \prec^i I_\beta\).

3. Any sequence of intervals \((I_{\alpha_i})_{i \geq 1}\) where \((\alpha_i)_{i \geq 1}\) is a monotone sequence in \((\text{Pref}(L), \prec)\), is eventually constant.

**Proof.**

1. Suppose \(I_u \prec^i I_v\). Then, either there exists \(\alpha \in I_u\) such that for all \(\beta \in I_v\) it holds \(\alpha < \beta\), or there exists \(\beta \in I_v\) such that for all \(\alpha \in I_u\) it holds \(\alpha < \beta\). In the first case, we have \(\alpha \in I_u \setminus I_v\), while in the second case we have \(\beta \in I_v \setminus I_u\). In both cases \(u < v\) follows from Lemma 2.2.

For the second implication suppose, for contradiction, that \(u < v\) and \(I_u \prec^i I_v\) holds. Then, either there exists \(\alpha \in I_u\) such that for all \(\beta \in I_v\) it holds \(\alpha < \beta\), or there exists \(\beta \in I_v\) such that for all \(\alpha \in I_u\) it holds \(\alpha < \beta\). In the first case, \(\alpha \in I_v \setminus I_u\), while in the second case \(\beta \in I_u \setminus I_v\). In both cases we obtain \(v < u\) by Lemma 2.2, a contradiction.

2. This point is entirely similar to the above.

3. This is proved similarly to Corollary 2.3 using (2), so we provide just a sketch: since \(Q\) is finite, also the set of intervals on \(Q\) is finite, thus by property (2) \((I_{\alpha_i})_{i \geq 1}\) must stabilize, being \((\alpha_i)_{i \geq 1}\) a monotone sequence in \((\text{Pref}(L), \prec)\).

If \(A\) is a WNFA we can prove that the following construction, which is the “convex version” of the classic powerset construction for NFA, allows determinisation without exponential blow-up.

**Definition 5.** If \(A = (Q, s, \delta, <, F)\) is a WNFA we define its (Wheeler) **determinization** as the automaton \(A^d = (Q^d, s^d, \delta^d, <^d F^d)\), where:

- \(Q^d = I_{\text{Pref}(L(A))}\);
- \(s^d = I_s = \{s\}\)
- \(F^d = \{I_\alpha \mid \alpha \in L(A)\}\);
- \(\delta^d : I_{\text{Pref}(L(A))} \times \Sigma \rightarrow I_{\text{Pref}(L(A))}\) is the partial function defined as \(\delta^d(I_\alpha, a) = I_{\alpha a}\), for all \(a \in \Sigma\) and \(\alpha e \in \text{Pref}(L(A))\);
- \(<^d = \prec^i\).

**Lemma 2.7** (WNFA Determinization). [ADPP20] If \(A = (Q, s, \delta, <, F)\) is a WNFA with \(n\) states over an alphabet \(\Sigma\) (with at least one \(a\)-edge for each \(a \in \Sigma\)), then \(A^d\) is a WDFA with at most \(2n - 1 - |\Sigma|\) states, and \(L(A^d) = L(A)\).

**Proof.** The fact that \(L(A^d) = L(A)\) is seen as in the (classic) regular case: the reachable subsets of the powerset construction are exactly the ones in \(Q^d\).

We prove that \(<^d\) is a Wheeler order on the states of the automaton \(A^d\). By Lemma 2.4, the set \(Q^d = I_{\text{Pref}(L(A))}\) of states of \(A^d\) is a prefix/suffix family of intervals, so that, by Lemma 1.1 \(<^d\) is a linear order on \(Q^d\). Next, we check the Wheeler properties. The only vertex with in-degree 0 is \(I_s\), and it clearly precedes those with positive in-degree. For any two edges \((I_\alpha, I_{\alpha a_1}, a_1), (I_\beta, I_{\beta a_2}, a_2)\) we have:

(i) if \(a_1 < a_2\) then \(\alpha a_1 < \beta a_2\), and from Lemma 2.6 it follows \(I_{\alpha a_1} \leq^d I_{\beta a_2}\). Moreover, by the input consistency of \(A\), states in \(I_{\alpha a_1}\) are \(a_1\)-states, while states in \(I_{\beta a_2}\) are \(a_2\)-states; hence \(I_{\alpha a_1} \neq I_{\beta a_2}\), so that \(I_{\alpha a_1} <^d I_{\beta a_2}\) follows.
(ii) If $a = a_1 = a_2$ and $I_\alpha < I_\beta$, from Lemma 2.6 it follows $\alpha \prec \beta$, so that $\alpha a \sim \beta a$ and, using again Lemma 2.6, we obtain $I = I_{\alpha a} \leq^i I = I_{\beta a}$.

Finally, we prove that $|Q| \leq 2n - 1 - |\Sigma|$. By the Wheeler properties, we know that the only interval in $I_{\text{Pref}(\mathcal{L}(A))}$ containing the initial state $s$ of the automaton $A$ is $\{s\}$ and that the remaining intervals can be partitioned into $|\Sigma|$-classes, by looking at the letter labelling incoming edges. Let $\Sigma = \{a_1, \ldots, a_k\}$, and, for every $i = 1, \ldots, k$, let $m_i$ be the number of states of the automaton $A$ whose incoming edges are labelled $a_i$: then $\sum_{i=1}^k m_i = n - 1$. Using Lemma 2.8 we see that the intervals in $Q^d$ composed by $a_i$ states are at most $2m_i - 1$, so that the total number of intervals in $V^d$ is at most $1 + \sum_{i=1}^k (2m_i - 1) = 1 + 2(\sum_{i=1}^k m_i) - k = 1 + 2(n - 1) - k = 2n - 1 - k = 2n - 1 - |\Sigma|$. □

We will use the following Lemma in the next section.

**Lemma 2.8.** \[\text{ADPP20}\] Let $A = (Q, s, \delta, <, F)$ be a WNFA, $\alpha, \beta, \delta \in \text{Pref}(\mathcal{L}(A))$, $u, v, w \in Q$.

1. if $\alpha \prec \delta \prec \beta$ and $I_\alpha = I_\beta$, then $I_\alpha = I_\beta$;
2. if $u < w < v$ and $I_u = I_v$, then $I_u = I_w$.

**Proof.**

1. Suppose $\alpha \prec \delta \prec \beta$ and $I_\alpha = I_\beta$. If $u \in I_\alpha = I_\beta$ then $\alpha, \beta \in I_u$ and since by Lemma 2.4 $I_u$ is a convex set, $\delta \in I_u$ follows. Hence, $u \in I_\beta$, from which it follows that $I_\alpha \subseteq I_\beta$. Suppose, for contradiction, that $I_\alpha \not\subseteq I_\beta$ and let $v \in I_\beta \setminus I_\alpha$. It follows $\delta \in I_v$ and $\alpha \not\in I_v$. Consider $u$ such that $\alpha \in I_u$, then $\alpha \in I_u \setminus I_v \subseteq I_v$, from which it follows that $u < v$ by Lemma 2.2. On the other hand, $\beta \in I_u \setminus I_v$ as well, because $u \in I_\beta = I_\alpha$ and $v \not\in I_\beta = I_\alpha$, then $\delta \prec \beta$ and Lemma 2.2 implies $v < u$. A contradiction.

2. This point is entirely similar to the above.

\[\square\]

### 2.1 Convex Equivalences from Wheeler Automata

Given a WNFA $A$, we consider two convex equivalence relations, $\sim_A$ and $\approx_A$.

**Definition 6.** If $A = (Q, s, \delta, <, F)$ is an WNFA, $\alpha, \beta \in \text{Pref}(\mathcal{L}(A))$, and $u, v \in Q$, we define:

- $\alpha \sim_A \beta$ if and only if $I_\alpha = I_\beta$.
- $u \approx_A v$ if and only if $I_u = I_v$.

While we shall write $\approx$ instead of $\approx_A$ when the automaton $A$ is clear from the context. Note that, by Lemma 2.8, $\approx$-equivalence classes are in fact intervals of $(Q, <)$—that is, $\approx$ is a convex equivalence over $(Q, <)$. As we shall see in Lemma 2.9, the equivalence $\sim_A$ over $\text{Pref}(\mathcal{L}(A))$ is also convex, with respect to the co-lexicographic order on $\text{Pref}(\mathcal{L}(A))$.

**Definition 7.** Given a language $\mathcal{L} \subseteq \Sigma^*$, an equivalence relation $\sim$ over $\text{Pref}(\mathcal{L})$ is:

- right invariant, when for all $\alpha, \beta \in \text{Pref}(\mathcal{L})$ and $\gamma \in \Sigma^*$:
  
  $$\text{if } \alpha \sim \beta \text{ and } \alpha \gamma \in \text{Pref}(\mathcal{L}), \text{ then } \beta \gamma \in \text{Pref}(\mathcal{L}) \text{ and } \alpha \gamma \sim \beta \gamma;$$

- input consistent if all words belonging to the same $\sim$-class end with the same letter.

**Lemma 2.9.** \[\text{ADPP20}\] If $A = (Q, s, \delta, <, F)$ is an $n$-states WNFA such that $\mathcal{L} = \mathcal{L}(A)$, then:

1. $\sim_A$ is a right invariant, input consistent, convex equivalence relation over $\text{Pref}(\mathcal{L})$;
2. $\sim_A$’s index is less than or equal to $2n - 1 - |\Sigma|$;
3. $\mathcal{L}$ is a union of $\sim_A$-classes.
Proof.

1. We first check that $\sim_A$ equivalence classes are convex sets (convex sets) of $(\text{Pref}(L), \prec)$. If $\alpha \prec \beta \prec \gamma$ are such that $\alpha, \beta, \gamma \in \text{Pref}(L)$ and $\alpha \sim_A \gamma$, then $\beta \sim_A \alpha$ follows from Lemma 2.8.

As for right invariance, suppose $\alpha \sim_A \beta$. Then $I_\alpha = I_\beta$, from which it follows $I_{\alpha e} = I_{\beta e}$ because for any state $u \in I_{\alpha e}$ there exists a state $u' \in I_\alpha = I_\beta$ such that $u' \in \delta(u, e)$; hence $u \in I_{\beta e}$. This proves that $I_{\alpha e} \subseteq I_{\beta e}$. The reverse inclusion is proved similarly.

Input consistency of $\sim_A$ follows from Wheeler properties, since if two words end with different letters, then they cannot lead to the same state in a Wheeler automaton.

2. The index of $\sim_A$ is equal to the cardinality of $I_{\text{Pref}(L(A))}$ which is a prefix/suffix family of $(Q, \prec)$ by Lemma 2.3. By Lemma 2.4, this index is bounded by $2n - 1 - |\Sigma|$.

3. $\mathcal{L} = \bigcup_{\alpha \in \mathcal{L}} [\alpha]_{\sim_A}$.

If $\mathcal{A}$ is a W DFA, $\mathcal{L} = L(\mathcal{A})$, and $\alpha \in \text{Pref}(\mathcal{L})$, then $I_\alpha$ contains a single state: $\sim_A$’s index is equal to the number of states of the automaton $\mathcal{A}$.

Let us now consider the second equivalence, $\approx_A$ (or, simply, $\approx$).

**Definition 8.** Let $\mathcal{A} = (Q, s, \delta, \prec, F)$ be a WNFA. The quotient automaton $\mathcal{A}/\approx = (Q^\approx, s^\approx, \delta^\approx, \prec^\approx, F^\approx)$ is defined as follows:

- $Q^\approx = \{[u]^\approx | u \in Q\}$;
- $s^\approx = [s]^\approx = \{s\}$;
- $\delta^\approx([v]^\approx, e) = \{[u]^\approx | (\exists u' \in [u]^\approx)(\exists v' \in [v]^\approx)(u' \in \delta(v', e))\}$;
- $[u]^\approx \prec^\approx [v]^\approx$ if and only if $[u]^\approx \neq [v]^\approx \land u < v$;
- $F^\approx = \{[u]^\approx | [u]^\approx \cap F \neq \emptyset\}$.

Note that the relation $\prec^\approx$ on the equivalence classes is well defined because, by Lemma 2.3, the equivalence classes $[u]^\approx$ are (disjoint) intervals of $(Q, \prec)$.

**Lemma 2.10.** $\mathcal{A}/\approx$ is a Wheeler automaton and $L(\mathcal{A}) = L(\mathcal{A}/\approx)$.

**Proof.** The fact that the order on equivalence classes defined above is Wheeler follows easily from the definition and the fact that the equivalence classes are intervals.

To see that $L(\mathcal{A}) = L(\mathcal{A}/\approx)$, observe that, although in general the implication $[u]^\approx \in \delta^\approx([v]^\approx, e) \Rightarrow u \in \delta(v, e)$ does not hold, we do have that $[u]^\approx \in \delta^\approx(s^\approx, e) \Rightarrow u \in \delta(s, e)$ does hold. As a matter of fact, more generally, we can prove that for all $\alpha \in \Sigma^*$:

$$[u]^\approx \in \delta^\approx(s^\approx, \alpha) \text{ if and only if } u \in \delta(s, \alpha).$$  \(1\)

The direction from left to right of (1) is proved by induction on $|\alpha|$.

For the base case, suppose $[u]^\approx \in \delta^\approx(s^\approx, e) = s^\approx$; then, since $s^\approx = \{s\}$ we have $u = s \in \delta(s, e)$.

For the inductive step, suppose $[u]^\approx \in \delta^\approx(s^\approx, \alpha e)$; then let $v \in Q$ be such that $[v]^\approx \in \delta^\approx(s^\approx, \alpha)$, and $[u]^\approx \in \delta^\approx([v]^\approx, e)$. By inductive hypothesis, $v \in \delta(s, \alpha)$ and by definition of $\delta^\approx$ we know that there are $u', v' \in Q$, such that $u \approx u', v \approx v'$, and $u' \in \delta(v', e)$. From $[v]^\approx = [v']^\approx$ it follows that $v' \in \delta(s, \alpha)$, and so $u' \in \delta(s, \alpha e)$. Since $[u]^\approx = [u']^\approx$, it follows $u \in \delta(s, \alpha e)$.

The direction from right to left of (1) is easy to see.

From (1) $L(\mathcal{A}) = L(\mathcal{A}/\approx)$ follows. In fact: $\alpha \in L(\mathcal{A}) \Leftrightarrow (\exists u \in F)(u \in \delta(s, \alpha)) \Leftrightarrow (\exists u \in F)([u]^\approx \in \delta(s^\approx, \alpha)) \Leftrightarrow (\exists [u]^\approx \in F^\approx)([u]^\approx \in \delta(s^\approx, \alpha)) \Leftrightarrow \alpha \in L(\mathcal{A}/\approx)$.

When the $\approx$-classes are not singletons, two different states in a (W)NFA can be reached by exactly the same collection of $\alpha$’s in $\Sigma^*$. To avoid this trivial kind of redundancy, we introduce the following notion.
Definition 9. A Wheeler NFA $A = (Q, s, \delta, <, F)$ is reduced if for all $u, v \in Q$,
\[ u \neq v \text{ if and only if } I_u \neq I_v. \]

It is clear that the quotient automaton $A/\approx$ of a WNFA is reduced. As a consequence of Lemma 2.10 we have:

Corollary 2.11. Any WNFA is equivalent to a reduced one.

Our interest in reduced automata relies on the following result:

Lemma 2.12. [ADPP20] The Wheeler order of a reduced WNFA is unique.

Proof. Let $A = (Q, s, \delta, <, F)$ be a reduced WNFA. If $u \neq v \in Q$ then $I_u \neq I_v$ and either $I_u \setminus I_v \neq \emptyset$ or $I_v \setminus I_u \neq \emptyset$. If $I_u \setminus I_v \neq \emptyset$, consider $\alpha \in I_u \setminus I_v$, and $\beta \in I_v$. Then, by Lemma 2.2, if $\alpha < \beta$ then $u < v$. Similarly, if $\alpha \in I_u \setminus I_v$ and $\beta \in I_v$, we have that $\alpha < \beta$ implies $v < u$.

In both cases, the Wheeler order is (uniquely) determined. \hfill $\square$

In Corollary 2.10 we shall see that deciding whether a given Wheeler NFA is reduced is in $P$. Reduced NFAs are considered again in Section 3.2, where we prove that deciding Wheelerness for a reduced NFA can be done in polynomial time (contrary to the case of general NFA, see [GT19]).

2.2 A Myhill-Nerode Theorem for Wheeler Languages

Given $L \subseteq \Sigma^*$, we define the right context of $\alpha \in \Sigma$, as
\[ \alpha^{-1}L = \{ \gamma \in \Sigma^* : \alpha \gamma \in L \}, \]
and we denote by $\equiv_L$ the Myhill-Nerode equivalence (right syntactic congruence) on Pref($L$) defined as
\[ \alpha \equiv_L \beta \iff \alpha^{-1}L = \beta^{-1}L. \]

Definition 10. The input consistent, convex refinement $\equiv_L^c$ of $\equiv_L$ is defined as follows:
\[ \alpha \equiv_L^c \beta \iff \alpha \equiv_L \beta \land end(\alpha) = end(\beta) \land (\forall \gamma \in \text{Pref}(L))(\min\{\alpha, \beta\} < \gamma < \max\{\alpha, \beta\} \rightarrow \gamma \equiv_L \alpha), \]
where $\alpha, \beta \in \text{Pref}(L)$ and $end(\alpha)$ is the final character of $\alpha$ when $\alpha \neq \epsilon$, and $\epsilon$ otherwise.

Lemma 2.13. [ADPP20] If $L \subseteq \Sigma^*$, then $\equiv_L^c$ is a convex, right invariant, input consistent equivalence relation over (Pref($L$), $<$) and $L$ is a union of classes of $\equiv_L^c$.

Proof. The equivalence $\equiv_L^c$ is input consistent by definition. Moreover, it is convex, being a convex refinement of an equivalence over the ordered set $(\text{Pref}(L), <)$ (see Lemma 1.3).

To prove that $\equiv_L^c$ is right invariant, consider $\alpha, \alpha', \gamma \in \text{Pref}(L)$ and assume $\alpha \equiv_L^c \alpha'$. Note that:

- if $\alpha \gamma \in \text{Pref}(L)$ then there exists $\nu \in \Sigma^*$ such that $\alpha \gamma \nu \in L$, therefore $\alpha' \gamma \in \text{Pref}(L)$ follows from $\alpha \equiv_L \alpha'$;

- $\alpha \gamma \equiv_L \alpha' \gamma$ follows from $\alpha \equiv_L \alpha'$.

- If $\alpha \gamma < \beta' < \alpha' \gamma$, for $\beta' \in \text{Pref}(L)$, then $\beta' = \beta \gamma$, and $\alpha < \beta < \alpha'$. Since $\alpha, \alpha'$ belong to the same $\equiv_L^c$ class, then $\beta \equiv_L \alpha$, and $\beta' = \beta \gamma \equiv_L \alpha \gamma$ follows.

Since $\alpha \gamma, \beta \gamma$ end with the same letter, the previous points imply that $\equiv_L^c$ is right invariant.

Finally, $L$ is a union of classes of $\equiv_L^c$ because $L$ is a union of $\equiv_L$ classes and $\equiv_L^c$ is a refinement of $\equiv_L$.

Lemma 2.14. [ADPP20] If $A = (Q, s, \delta, <, F)$ is a WNFA and $L = \mathcal{L}(A)$, then $\sim_A$ is a refinement of $\equiv_L^c$.

Proof. Suppose $\alpha \sim_A \beta$; then $\alpha \equiv_L \beta$ follows easily from the definition of $\sim_A$, and $\text{end}(\alpha) = \text{end}(\beta)$ follows from the input consistency of $A$. To prove that $\alpha \equiv_L^c \beta$ we only have to show that if $\gamma \in \text{Pref}(L)$ and $\min\{\alpha, \beta\} < \gamma < \max\{\alpha, \beta\}$ then $\gamma \equiv_L \alpha$. This holds because, by Lemma 2.8, from $\alpha < \gamma < \beta$ and $I_\alpha = I_\beta$, we have $I_\alpha = I_\gamma$, hence $\alpha \sim_A \gamma$ holds, and $\alpha \equiv_L \gamma$ follows.

\hfill $\square$
Corollary 2.15. [ADPP20] If $A = (Q, s, \delta, <, F)$ is a WNFA with $|Q| = n$ and $L = \mathcal{L}(A)$, then $\equiv^c_L$’s index is bounded by $2n - 1 - |\Sigma|$.

Proof. By Lemma 2.13 we know that $\sim_A$ is a refinement of $\equiv^c_L$, hence the number of classes of $\equiv^c_L$ is less than or equal to the number of classes of $\sim_A$, which is bounded by $2n - 1 - |\Sigma|$, as proved in the Lemma 2.9.

Note that, if $L$ is Wheeler, we cannot always extend $\equiv^c_L$ to the set $\Sigma^*$ maintaining the preceding corollary. For example, if $L$ is the Wheeler language of Example 2, then the equivalence relation $\equiv^c_L$ has an infinite number of classes over $\Sigma^*$.

Theorem 2.16 (Myhill-Nerode for Wheeler Languages). [ADPP20] Given a language $L \subseteq \Sigma^*$, the following are equivalent:

1. $L$ is a Wheeler language (i.e. $L$ is recognized by a WNFA).
2. $\equiv^c_L$ has finite index.
3. $L$ is a union of classes of a convex, input consistent, right invariant equivalence over $(\text{Pref}(L), \prec)$ of finite index.
4. $L$ is recognized by a WDFA.

Proof.

(1) $\Rightarrow$ (2) From Corollary 2.15.

(2) $\Rightarrow$ (3) $L$ is a union of $\equiv^c_L$ classes, which by Lemma 2.13 is a convex, input consistent, right invariant equivalence of finite index.

(3) $\Rightarrow$ (4) Suppose $L$ is a union of classes of a convex, input consistent, right invariant equivalence relation $\sim$ of finite index. We build a WDFA $A_\sim = (Q_\sim, s_\sim, \delta_\sim, <_\sim, F_\sim)$ such that $L = \mathcal{L}(A)$ as follows:

- $Q_\sim = \{[\alpha]\sim | \alpha \in \text{Pref}(L)\}$;
- $s_\sim = \{[\epsilon]\sim\}$ (note that, by input consistency, $[\epsilon]\sim = \{\epsilon\}$);
- if $I \epsilon \cap \text{Pref}(L) \neq \emptyset$ and $I \epsilon \subseteq J$, then $\delta_\sim(I, \epsilon) = J$ (note that $J$, if existing, is unique because equivalence classes are pairwise disjoint);
- $<_\sim = \prec^I$, that is: $I <_\sim J$ if and only if $(\forall \alpha \in I)(\forall \beta \in J) \alpha \prec \beta$.
- $F_\sim = \{I : I \subseteq L\}$.

For all $I \in Q_\sim$ and $\alpha \in \text{Pref}(L)$, observe that $\delta_\sim(I, \alpha)$ (if defined) is always a singleton set (i.e. $A_\sim$ is deterministic).

We prove that:

$$\alpha \in I \text{ if and only if } \delta_\sim(s_\sim, \alpha) = I.$$ 

by induction on the length of $\alpha \in \text{Pref}(L)$. If $\alpha = \epsilon$ then $\delta_\sim(s_\sim, \alpha) = [\epsilon]\sim$ and $[\epsilon]\sim = \{\epsilon\}$, by definition. If $\alpha = \alpha' \epsilon \in \text{Pref}(L)$ with $\epsilon \in \Sigma$, then $\alpha' \in \text{Pref}(L)$ and

$$\alpha' \epsilon \in I \Leftrightarrow \exists J(\alpha' \epsilon \in J \wedge \emptyset \neq Je \subseteq I) \Leftrightarrow \exists J(\delta(s_\sim, \alpha') = J \wedge \emptyset \neq Je \subseteq I) \Leftrightarrow \delta(s_\sim, \alpha) = I.$$ 

From the above claim and the definition of $F_\sim$, it easily follows that $L$ is the language recognised by $A_\sim$.

We conclude by checking that $A_\sim$ is Wheeler, proving the two Wheeler properties (i) and (ii). To see Wheeler-(i) assume $e \prec e'$ with $e, e' \in \Sigma$. Consider $I, J \in Q_\sim$ such that both $\delta_\sim(I, e)$ and $\delta_\sim(J, e')$ are defined and are equal to $H$ and $K$, respectively. By definition of $\delta_\sim$, there are $\alpha \in I,$
\( \alpha' \in J \) with \( \alpha e \in H \) and \( \alpha' e' \in K \). From \( e \prec e' \) it follows that \( H \prec^i K \) since all words in \( H \) end with \( e \), while all words in \( K \) end with \( e' \).

To see Wheeler-(ii) assume \( L \prec \), \( e \in \Sigma \), and both \( \delta_\prec(I,e) \) and \( \delta_\prec(J,e) \) are defined and equal to \( H \) and \( K \), respectively. In these hypotheses there are \( \alpha \in I \), \( \alpha' \in J \), with \( \alpha e \in H \) and \( \alpha' e \in K \). It follows \( \alpha \prec \alpha' \) and therefore, \( \alpha e \prec \alpha' e \) and \( H \preceq^i K \).

This ends the proof of the implication \( (3) \Rightarrow (4) \).

(4) \( \Rightarrow \) (1) Trivial.

Remark 2.17. If \( D \) is a WDFA with \( |Q| = n \) states, then the equivalence \( \sim_D \) over \( \text{Pref}(L) \) defined in Def. 4 has \( n \) classes, because each class \([\alpha]_\sim \) can be uniquely identified with the unique state \( u_\alpha = \delta(s, \alpha) \). Moreover, \( \sim_D \) is a convex, input consistent, right invariant equivalence (Lemma 2.9) and we may construct the WDFA \( A_\sim \) described in (3 \( \Rightarrow \) 4) of Theorem 2.16 note that \( A_\sim \) is isomorphic to \( D \), via the map \( \phi : Q_\sim \rightarrow Q \) defined as \( \phi([\alpha]_\sim) = u_\alpha \).

Corollary 2.18. \([\text{ADPP20}]\) If \( A = (Q, s, \delta, <, F) \) is a WNFA with \( |Q| = n \) and \( L = L(A) \), then there exists a unique, minimum-size (on the number of states) WDFA \( B \) such that \( L = L(B) \) and the number of \( B \)'s states is less than or equal to \( 2n - 1 - |\Sigma| \). Moreover, the construction of \( B \) is effective and can be done in polynomial time.

Proof. If \( L \) is recognized by an \( n \)-states WNFA, then from Lemma 2.13 we know that the equivalence \( \equiv \) is a convex, input consistent, right invariant equivalence relation of finite index, and \( L \) is a union of its classes. Hence, using the construction employed to prove (3 \( \Rightarrow \) 4) of Theorem 2.16 we can build a WDFA \( B = A_\equiv \), whose number of states is equal to the number of \( \equiv \)-classes. From Corollary 2.15 we know that the number of classes of \( \equiv \) is bounded by \( 2n - 1 - |\Sigma| \).

To see that \( B \) has the minimum number of classes observe that, by Lemma 2.13 any automaton \( D \) accepting \( L \) induces an equivalence relation \( \sim_D \) which is a refinement of \( \equiv \). If \( D \) is deterministic, the number of \( \sim_D \)-classes is equal to the number of \( D \)'s states which is, therefore, greater or equal than the number of \( \equiv \)-classes. It follows that, if \( D \) is a WDFA with the minimum number of states among WDFA’s recognising \( L \), then \( \sim_D \equiv \equiv \), this implies that \( A_\equiv = A_\sim \) so that

\[ B = A_\equiv = A_\sim = D. \]

where the last isomorphism follows from Remark 2.17. For the effectiveness of the construction of \( B \) we refer to \([\text{ADPP20}]\).

Corollary 2.19. We can decide in polynomial time whether a Wheeler NFA is reduced.

Proof. Let \( A \) be a Wheeler NFA. For each pair \( u, v \) of \( A \)-states, we consider the two Wheeler automata \( A^u, A^v \) which are obtained from \( A \) by considering, as set of final states, \( \{u\}, \{v\} \), respectively. Note that we can test in polynomial time whether \( L(A^u) = L(A^v) \), because, by Corollary 2.18 we can determine WNFA in polynomial time, and check if their languages are equal still in polynomial time (since they are deterministic). Then, \( A \) is reduced iff \( L(A^u) \neq L(A^v) \) for all pairs \( u \neq v \).

3 Testing Wheeleerness

3.1 Is a \( L \) Wheeler?

In this section we prove that, given a regular language \( L \) (say, by an NFA \( A \) recognizing it), it is decidable whether or not \( L \) is Wheeler. Moreover, if we start from a DFA recognizing \( L \), we describe a polynomial time algorithm to complete the task. Note that in this section we deal with standard edge-labeled automata.

We begin by giving an automata-free characterization of Wheeleerness.
Lemma 3.1. A regular language $L$ is Wheeler if and only if all monotone sequences in $(\text{Pref}(L), \prec)$ become eventually constant modulo $\equiv_L$. In other words, for all sequences $(\alpha_i)_{i \geq 1}$ in $\text{Pref}(L)$ with
\[
\alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_i \equiv \ldots \text{ or } \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_i \geq \ldots
\]
there exists an $n$ such that $\alpha_h \equiv_L \alpha_k$, for all $h, k \geq n$.

Proof. For the direction from left to right, suppose that $L$ is Wheeler and consider an infinite monotone sequence $(\alpha_i)_{i \geq 1}$ in $(\text{Pref}(L), \prec)$. By Theorem 2.16 there exist a W DFA $A = (Q, q, \delta, F, \prec)$ recognizing $L$ and from Corollary 2.3 it follows that there exists $n$ such that $\delta(s, \alpha_h) = \delta(s, \alpha_k)$, for all $h, k \geq n$. This, in turn, implies that $\alpha_h \equiv_L \alpha_k$, for all $h, k \geq n$.

For the direction from right to left, suppose the regular language $L$ is not Wheeler. By Theorem 2.16 we know that $\equiv_L$ has infinite index. However, $L$ is regular, the equivalence $\equiv_L$ has finite index; hence there exists a sequence $(\gamma_i)_{i \geq 1}$ of elements which are equivalent with respect to $\equiv_L$ but pairwise not $\equiv_L$-equivalent. From this sequence one can easily extract a subsequence $(\beta_i)_{i \geq 1}$ which is either monotone increasing or monotone decreasing and composed of $\equiv_L$-equivalent elements (either the set $\{i \geq 1 : \forall j > i (\gamma_j \prec \gamma_i)\}$ is finite, and we extract an infinite increasing subsequence, or is infinite and we extract an infinite decreasing sequence). Suppose the sequence $(\beta_i)_{i \geq 1}$ is decreasing (a similar argument can be used in case it is increasing). By possibly discarding a finite number of initial elements from such a sequence, we may assume that all $\beta_i$’s end with the same letter. Then, for all $i$, from $\beta_i \not\equiv_L \beta_{i+1}$ and $\beta_i \equiv_L \beta_{i+1}$ it follows that there exists $\eta_i \in \text{Pref}(L)$ such that:
\[
\beta_i \succ \eta_i \succ \beta_{i+1}
\]
and $\beta_i \not\equiv_L \eta_i$. If we define $(\alpha_i)_{i \geq 1} = (\beta_1, \eta_1, \beta_2, \eta_2, \ldots)$, then $(\alpha_i)_{i \geq 1}$ is monotone in $(\text{Pref}(L), \prec)$, but there exists no $n$ such that $\alpha_h \equiv_L \alpha_k$, for all $h, k \geq n$.

Example 3. If $\Sigma = \{a\}$, we see that the regular language $\{a^{2i+1} : i \geq 0\}$ is not Wheeler by considering the sequence $(\alpha_i)_{i \geq 1}$ with $\alpha_i = a^i$. Another example of application of Lemma 3.1 is the language $L = ax^*b \mid cx^*d$ which was proved to be non Wheeler in [GMS17]. Consider the sequence
\[
\alpha_i = \begin{cases} ax^i & \text{if } i \text{ is odd;} \\ cx^i & \text{if } i \text{ is even} \end{cases}
\]
Then $(\alpha_i)_{i \geq 1}$ is a monotone (increasing) sequence in $(\text{Pref}(L), \preceq)$ with $\alpha_i \not\equiv_L \alpha_{i+1}$, and from Lemma 3.1 it follows that $L$ is not Wheeler.

Remark 3.2. In the following theorem we shall use some simple properties of the co-lexicographic order:
\[
\xi \prec \zeta \Leftrightarrow \xi \rho \prec \zeta \rho, \quad (2)
\]
\[
\xi \prec \zeta \Rightarrow \xi \prec \rho \zeta, \quad (3)
\]
\[
|\xi| \geq |\zeta| \land \xi \succ \zeta \Rightarrow \xi \succ \rho \zeta, \quad (4)
\]

Theorem 3.3. Consider a regular language $L = \mathcal{L}(A)$, where $A$ is the minimum edge-labeled DFA recognizing $L$ with initial state $s$. Then $L$ is not Wheeler if and only if there exist strings $\mu, \nu$, and $\gamma$ such that:

1. $\mu$ and $\nu$ label paths from $s$ to states $u$ and $v$, respectively, with $u \neq v$;
2. $\gamma$ labels two cycles, one starting from $u$ and one starting from $v$;
3. $\mu, \nu \prec \gamma$ or $\gamma \prec \mu, \nu$;
4. $|\mu|, |\nu| \leq |\gamma| \leq 2 + |A| + 2|A|^2 + |A|^3$, where $|A|$ is the number of states of the automaton $A$.

Proof. We first prove that the four conditions above are sufficient to prove $L$ is not Wheeler. If $\mu, \nu$, and $\gamma$ are as above, then $\mu \neq \nu$ since they end in distinct states $u, v$ and $A$ is deterministic.

Suppose now, without loss of generality, that $\mu \prec \nu$ and:
a) if \( \mu \prec \nu \prec \gamma \), let \( \eta_i = \mu \gamma^i \), \( \beta_i = \nu \gamma^i \), while
b) if \( \gamma \prec \mu \prec \nu \), let \( \eta_i = \nu \gamma^i \), \( \beta_i = \mu \gamma^i \).

Note that, in both cases, all \( \eta_i \)'s and \( \beta_i \)'s belong to \( \text{Pref}(L) \) and \( \eta_i \not\equiv_L \beta_i \), because \( \eta_i, \beta_i \) end in different nodes \( u, v \) of the minimum automaton. Moreover, for any \( i, \eta_i \prec \beta_i \), in case a) while \( \eta_i \succ \beta_i \) in case b). Finally, it can easily be checked that \( \beta_i \prec \eta_{i+1} \) holds in case a), while \( \beta_i \succ \eta_{i+1} \) holds in case b) since \( \mu \succ \gamma \) and \( \gamma \) is not a suffix of \( \mu \), being \( |\gamma| > |\mu| \).

Hence we have:

a) \( \eta_1 \prec \beta_1 \prec \ldots \prec \eta_i \prec \beta_i \prec \ldots \)
b) \( \eta_1 \succ \beta_1 \succ \ldots \succ \eta_i \succ \beta_i \succ \ldots \)

In both cases we have a monotone sequence in \( \text{Pref}(L) \) which is not eventually constant modulo \( \equiv_L \), so that \( L \) is not Wheeler by Lemma 3.1.

We now prove the converse of our main statement: if \( L \) is not Wheeler we can find \( \mu, \nu \) and \( \gamma \) satisfying conditions (1)-(4) above.

Claim 1. If \( L \) is not Wheeler, there exist words \( \alpha, \beta, \alpha', \gamma' \in \text{Pref}(L) \) such that:

- \( \alpha \prec \beta \prec \alpha' \);
- \( \alpha, \alpha' \) end in a state \( u \) and \( \beta \) ends in \( v \) with \( u \neq v \);
- \( |\gamma'| \geq |A|^2 \) and \( \gamma' \) labels two cycles starting from \( u \) and \( v \), respectively;
- \( |\alpha|, |\beta|, |\alpha'| \leq 2 + |A| + |A|^2 + |A|^3 \).

To prove the above claim we apply Lemma 3.1. Consider a monotone sequence \( (\alpha_i)_{i \geq 1} \) which is not eventually constant modulo \( \equiv_L \). Assume that \( \alpha_i \prec \alpha_{i+1} \), for all \( i \) (the case \( \alpha_i \succ \alpha_{i+1} \), for all \( i \), is analogous). By possibly erasing a finite number of initial elements in the sequence, we can assume that all \( \alpha_i \)'s end with the same \( |A|^2 + 1 \) letters (this is possible by the finiteness of \( \Sigma \) and by the fact that the monotone sequence \( (\alpha_i)_{i \geq 1} \) is not eventually constant). Let \( \theta \in \Sigma^* \) be such that \( |\theta| = |A|^2 + 1 \) and \( \alpha_i = \alpha'_i \theta \), with \( \alpha_i \prec \alpha'_{i+1} \prec \ldots \). Since \( \alpha'_i \equiv_L \alpha'_{i+1} \) implies \( \alpha_i \equiv_L \alpha_{i+1} \), the monotone sequence \( (\alpha'_i)_{i \geq 1} \) is also not eventually constant modulo \( \equiv_L \). Since the set of \( A \)'s states is finite, and \( (\alpha'_i)_{i \geq 1} \) is not eventually constant modulo \( \equiv_L \), by possibly considering a subsequence of \( (\alpha'_i)_{i \geq 1} \) we can further suppose that all elements of odd index end in the same state \( x' \), all elements of even index end the same state \( y' \), and \( x' \neq y' \).

Let \( m = |A|^2 \), and consider the last \( |A|^2 + 1 \) states \( x' = x_0, x_1, \ldots, x_m \) of the \( \alpha_1 \)-labelled path from the initial state \( s \). Note that all \( \alpha_i \)'s with odd \( i \) share this path. Similarly, consider the last \( |A|^2 + 1 \) states \( y' = y_0, y_1, \ldots, y_m \) of the \( \alpha_2 \)-labelled path from the initial state \( s \). Again, all \( \alpha_i \)'s with even \( i \) share this path. Moreover, both paths are labelled by the same word \( \theta \) and \( x_k \neq y_k \), for all \( k = 0, \ldots, m \) (otherwise the sequence \( (\alpha_i)_{i \geq 1} \) would be eventually constant, which is not).

Since \( |\theta| = m + 1 = |A|^2 + 1 \), we can find \( i_0, n_0 \) with \( 0 \leq i_0 < n_0 \leq m \) such that \( (x_{i_0}, y_{i_0}) = (x_{n_0}, y_{n_0}) \), that is, the two subpaths
\[
\begin{align*}
x_{i_0}, x_{i_0+1}, \ldots, x_{n_0} &= x_{i_0}, \\
y_{i_0}, y_{i_0+1}, \ldots, y_{n_0} &= y_{i_0},
\end{align*}
\]
are cycles of the same length labelled by the same word, say \( \gamma' \). Note that \( |\gamma'| \leq |A|^2 \).

Since \( \gamma' \) is a factor of \( \theta \), there exist \( \eta, \delta \in \Sigma^* \) such that \( \theta = \eta \gamma' \delta \). All \( \alpha'_i \eta \)'s with \( i \) odd end in \( x_{i_0} \) and all \( \alpha'_i \eta \)'s with \( i \) even end in \( y_{i_0} \), with \( x_{i_0} \neq y_{i_0} \). Moreover, \( \gamma' \) labels two cycles starting in \( x_{i_0} \) and \( y_{i_0} \), respectively.

Let \( \alpha = \alpha'_1 \eta, \beta = \alpha'_2 \eta, \alpha' = \alpha'_3 \eta, \) and note that \( \alpha, \beta, \alpha' \) satisfies the first three properties of our Claim, with \( u = x_{i_0} \) and \( v = y_{i_0} \).

We now prove the last point of Claim 1 that is, we can also limit, effectively, the lengths of \( \alpha, \beta, \) and \( \alpha' \). Given a word \( \varphi \in \Sigma^* \) and \( k \geq 1 \) we denote by \( \varphi(k) \) the \( k \)-th letter from the right, whenever \( |\varphi| \leq k \), or the empty word \( \varepsilon \), otherwise (e.g. \( \varphi(1) \) is the last letter of \( \varphi \)).

Given \( \alpha, \beta, \alpha', \gamma, u, v \) as defined above, let \( d_{\alpha', \beta} \) be the first position from the right in which \( \alpha' \) and \( \beta \) differ. Since \( \beta' \succ \beta \), we have \( |\alpha'| \geq d_{\alpha', \beta} \). Similarly, let \( d_{\beta, \alpha} \) be the first position from the right in which \( \beta \) and \( \alpha \) differ. Again, since \( \beta \succ \alpha \), we have \( |\beta| \geq d_{\beta, \alpha} \). Proceeding by cases, consider:
i) \( d_{\alpha', \beta} \leq d_{\beta, \alpha} \) (so that the position in \( \beta \) from the right are: \( \ldots, d_{\beta, \alpha} \ldots, d_{\alpha', \beta} \ldots, 2, 1 \));

ii) \( d_{\beta, \alpha} < d_{\alpha', \beta} \) (so that the position in \( \beta \) from the right are: \( \ldots, d_{\alpha', \beta} \ldots, d_{\beta, \alpha} \ldots, 2, 1 \)).

In case i), since \(|\alpha'| \geq d_{\alpha', \beta}, |\beta| \geq d_{\beta, \alpha} \geq d_{\alpha', \beta}, \) the words \( \alpha', \beta, \) and \( \alpha \) end with the same word \( \xi \) with \(|\xi| = d_{\alpha', \beta} - 1 \). Since \( \beta \prec \alpha' \) it must be that \( \beta(d_{\alpha', \beta}) \prec \alpha'(d_{\alpha', \beta}) \). That is, for some \( \phi, \phi', \psi \in \Sigma^* \):

\[
\alpha = \phi\alpha(d_{\alpha', \beta}) \xi \prec \beta = \psi\beta(d_{\alpha', \beta}) \xi \prec \alpha' = \phi'\alpha'(d_{\alpha', \beta})\xi,
\]

with \( \phi\alpha(d_{\alpha', \beta}) = \epsilon \), whenever \(|\alpha| < d_{\alpha', \beta} \).

See Figure 3.

\[ \begin{array}{c}
\alpha' \\
\gamma \\
\beta \\
\gamma \\
\alpha \\
\end{array} \]

\[
\alpha' \equiv \ldots \phi' \ldots \alpha'(d_{\alpha', \beta}) \ldots \xi \ldots
\]

\[
\beta \equiv \ldots \psi \ldots \beta(d_{\beta, \alpha}) \ldots \xi \ldots
\]

\[
\alpha \equiv \ldots \phi \ldots \alpha(d_{\alpha', \beta}) \ldots \xi \ldots
\]

Figure 3: Case i) with \( d_{\alpha', \beta} < d_{\beta, \alpha} \).

We can assume, without loss of generality, that \(|\xi| \leq |A|^3 \). In fact, if \(|\xi| > |A|^3 \) then, considering the triples of states visited simultaneously while reading the last \( d_{\alpha', \beta} \)'s letters of \( \alpha, \beta, \alpha' \), respectively, we should meet a repetition. If this were the case, we could erase a common factor from \( \phi, \psi \), obtaining a shorter word \( \xi_1 \) such that \( \phi\alpha(d_{\alpha', \beta})\xi_1 \prec \psi\beta(d_{\alpha', \beta})\xi_1 \prec \phi'\alpha'(d_{\alpha', \beta})\xi_1 \), with the three paths still ending in \( u, v \) and \( \xi \), respectively. Hence, we may suppose \(|\xi| \leq |A|^3 \) in (5) above.

Consider now the case \( d_{\alpha', \beta} = d_{\beta, \alpha} \). Let \( s_1, s_2, \) and \( s_3 \) be the states reached from \( s \) by reading \( \phi, \psi \), and \( \phi' \), respectively. Since \( d_{\beta, \alpha} = d_{\alpha', \beta} \) is a position on the right of \( \phi, \psi \) and \( \phi' \) in \( \alpha, \beta, \alpha' \), respectively, we may suppose w.l.o.g. that \( \phi, \psi_1, \phi_1 \) label simple paths leading from \( s \) to \( s_1, s_2, s_3 \), so that \( |\phi|, |\psi_1|, |\phi_1| \leq |A| \). Hence in this case we have \(|\alpha|, |\beta|, |\alpha'| \leq 1 + |A| + |A|^3 \).

Next, consider the case \( d_{\alpha', \beta} < d_{\beta, \alpha} \). In this case, \( \phi, \psi \) end with the same word \( \xi' \) with \(|\xi'| = d_{\beta, \alpha} - d_{\alpha', \beta} - 1 \), and

\[
\phi = \phi_1\alpha(d_{\beta, \alpha})\xi' \prec \psi = \psi_1\beta(d_{\beta, \alpha})\xi'
\]

(see the picture above). We may assume, without loss of generality, that \(|\xi'| \leq |A|^2 \). In fact, if \(|\xi| > |A|^2 \) then, reasoning as above but considering pairs of states instead of triples, we could erase a common factor from \( \xi' \), obtaining a shorter word \( \xi'_1 \) such that

\[
\phi_1\alpha(d_{\beta, \alpha})\xi'_1 \alpha(d_{\alpha', \beta})\xi \prec \psi_1\beta(d_{\beta, \alpha})\xi'_1 \beta(d_{\alpha', \beta})\xi \prec \alpha' = \phi'(d_{\alpha', \beta})\xi,
\]

where the words above still end in \( u, v, \) and \( \xi \), respectively. By repeating the same argument, we see that we may suppose \(|\xi_1| \leq |A|^2 \) and \(|\xi| \leq |A|^3 \) in (6).

Consider now the states \( s_1, s_2, \) and \( s_3 \) reached by reading \( \phi_1, \psi_1, \) and \( \phi' \) from \( s \), respectively. Since \( d_{\beta, \alpha} \) is a position on the right of \( \phi_1, \psi_1, \) and \( d_{\alpha', \beta} \) is a position on the right of \( \psi_1 \) and \( \phi' \), we may assume, without loss of generality, that \( \phi_1, \psi_1, \) and \( \phi' \) label simple paths leading from \( s \) to \( s_1, s_2, \) and \( s_3 \), respectively. Hence \(|\phi_1|, |\psi_1|, |\phi'| \leq |A| \).

Summarising, we may suppose \(|\alpha|, |\beta|, |\alpha'| \leq 2 + |A| + |A|^2 + |A|^3 \), ending the proof of case i).

Case ii), in which \( d_{\alpha', \beta} > d_{\beta, \alpha} \), can be treated analogously. The skeptical reader can consult the following graphic proof (see Figure 4) and this ends the proof of Claim 1. See Figure 4.
and hence π node Q no path of length ℓ such that π satisfies the required properties.

We consider two cases:

1. γ < β; in this case we define µ = β and ν = α′, so that γ < µ < ν;
2. β < γ; in this case we define µ = α and ν = β, so that µ < ν < γ.

From |α|, |β|, |α′| < 2 + |A| + |A|^2 + |A|^3 it follows that

\[ |γ| \leq \max\{|α|, |β|, |α′|\} + |γ′| \leq (2 + |A| + |A|^2 + |A|^3) + |A|^2 = 2 + |A| + 2|A|^2 + |A|^3 \]

and hence µ, ν, γ satisfy the required properties.

We now use the preceding theorem to prove the decidability of being a Wheeler language.

**Theorem 3.4.** We can decide whether the regular language L accepted by a given edge-labeled DFA A is Wheeler in polynomial time.

**Proof.** Since the construction of the minimum automaton recognizing a language can be done in polynomial time starting from a DFA recognizing it, we may suppose that A is minimum. We exhibit a dynamic programming algorithm that finds µ, ν, and γ satisfying Theorem 3.3 if and only if such strings exist. Let N = 2 + |A| + 2|A|^2 + |A|^3 be the (polynomial) upper bound to the length of those strings.

We consider only the case µ, ν < γ, as the other can be solved symmetrically. Let π_{u,ℓ}, with u ∈ Q \(\{s\}\) and 2 ≤ ℓ ≤ N, denote the predecessor of u such that the co-lexicographically smallest path of length (number of nodes) ℓ connecting the source s to u passes through π_{u,ℓ} as follows: s ↼ π_{u,ℓ} ↼ u. The node π_{u,ℓ} coincides with s if ℓ = 2 and u is a successor of s; in this case, the path is simply s ↼ u. If there is no path of length ℓ connecting s with u, then π_{u,ℓ} = ⊥. Note that the set \{π_{u,ℓ} : 2 ≤ ℓ ≤ N, u ∈ Q \(\{s\}\)\} stores in just polynomial space all co-lexicographically smallest paths of any fixed length ℓ ≤ N from the source to any node u. We denote such path with α_{ℓ}(u), and the corresponding sequence of labels with λ(α_{ℓ}(u)) (that is, the sequence of ℓ − 1 symbols labeling the path’s edges). Note that α_{ℓ}(u) can be obtained recursively (in O(ℓ) steps) as α_{ℓ}(u) = α_{ℓ−1}(π_{u,ℓ}) ↼ u, where α_{1}(s) = s by convention.

Clearly, each π_{u,ℓ} can be computed in polynomial time using dynamic programming. First, we set π_{u,2} = s for all successors u of s. Then, for ℓ = 3, ..., N:

\[ π_{u,ℓ} = \arg\min_{v ∈ Pred(u)} λ(α_{ℓ−1}(v)) \cdot λ(v, u) \]

where Pred(u) is the set of all predecessors of u and the argmin operator compares strings in co-lexicographic order. In the equation above, if none of the α_{ℓ−1}(v) are well-defined (because there is no path of length ℓ − 1 from s to v), then π_{u,ℓ} = ⊥.

The second (similar) ingredient is to compute pairs ψ_{u,u',v,v',ℓ} = (u'', v''), with u, u', v, v' ∈ Q \(\{s\}\) and 2 ≤ ℓ ≤ N, such that:

![Figure 4: Case ii).](image-url)
1. $u''$ is a predecessor of $u'$ and $v''$ is a predecessor of $v'$.
2. $\lambda(u'', u') = \lambda(v'', v') = c$, for some $c \in \Sigma$,
3. there exist two paths of length (number of nodes) $\ell - 1$ from $u$ to $u''$ and from $v$ to $v''$ labelled with the same string $\beta$ (if $\ell = 2$, then $\beta = e$), and
4. $(u'', v'')$ is chosen so that $\beta \cdot c$ is co-lexicographically maximum.

As before, if such two paths and such a $\beta$ do not exist, then $\psi_{u,u',v,v',\ell} = \bot$. Moreover, if $(u, u')$ and $(v, v')$ are edges with $\lambda(u, u') = \lambda(v, v')$, then $\psi_{u,u',v,v',2} = \langle u, v \rangle$ and the two associated paths are $u \rightarrow u'$ and $v \rightarrow v'$.

Analogously to the (simpler) case seen before, these pairs store in polynomial space, for each $u, u', v, v'$ and length $\ell$, the co-lexicographically largest string of length $\ell - 1$ labeling two paths $u \rightsquigarrow u'$ and $v \rightsquigarrow v'$, as well as the two paths themselves. We denote these two paths as $\beta_\ell(u, u', v, v')$ and $\beta_\ell(u, u', v, v')$, respectively. Note that, by our definition, $\lambda(\beta_\ell(u, u', v, v')) = \lambda(\beta_\ell(u, u', v, v'))$. Again, these paths can be obtained in a recursive fashion using the pairs.

Pairs $\psi_{u,u',v,v',\ell} = \langle u'', v'' \rangle$ can be computed in polynomial time using dynamic programming as follows. We set all $\psi_{u,u',v,v',2} = \langle u, v \rangle$ whenever $(u, u')$ and $(v, v')$ are edges with $\lambda(u, u') = \lambda(v, v')$ ($\bot$ otherwise) and, for $\ell = 3, \ldots, N$:

$$
\psi_{u,u',v,v',\ell} = \arg\max_{\langle u'', v'' \rangle \in \text{Pred}(u') \times \text{Pred}(v') : \lambda(u'', u') = \lambda(v'', v')} \lambda(\beta_{\ell-1}(u, u'', v, v'')) \cdot \lambda(u'', u')
$$

where the argmax operator compares strings in co-lexicographic order.

To conclude, in order to check the conditions of Theorem [3.4], we proceed as follows. First, we guess the nodes $u$ and $v$ and the lengths $|\mu|, |\nu| < |\gamma| \leq 2(2 + |\mathcal{A}| + |\mathcal{A}|^2 + |\mathcal{A}|^3)$ (there are only polynomially-many choices to try). Then:

1. We compute the co-lexicographically smallest $\mu' = \lambda(\alpha_{|\mu|}(u))$ labeling a path of length $|\mu|$ from $s$ to $u$,
2. we compute the co-lexicographically smallest $\nu' = \lambda(\alpha_{|\nu|}(v))$ labeling a path of length $|\nu|$ from $s$ to $v$,
3. we compute the co-lexicographically largest $\gamma' = \lambda(\beta_{|\gamma|}(u, u, v, v))$ labeling two paths of length $|\gamma|$ from $u$ to $u$ and from $v$ to $v$ (that is, two cycles), and
4. we check if $\mu', \nu' \prec \gamma'$. We declare $\mathcal{L}(\mathcal{A})$ non Wheeler if and only if this test succeeds for at least one choice of $u, v, |\mu|, |\nu|, |\gamma|$.

Clearly, the existence of $\mu$, $\nu$, and $\gamma$ implies that $\mu', \nu', \gamma'$ exist and that they satisfy the conditions of Theorem [3.3] we have $\mu' \preceq \mu$, $\nu' \preceq \nu$, $\gamma' \succeq \gamma$, and $\mu,\nu \prec \gamma$, therefore $\mu', \nu' \prec \gamma'$ holds. Conversely, the theorem states that if we find such $\mu', \nu'$, and $\gamma'$ then the original language is not Wheeler.

In [ADPP20] it is presented a procedure for obtaining the minimum WDFA equivalent to a given acyclic DFA. We now show that, while a more general procedure for converting any DFA recognizing a Wheeler language into the minimum equivalent WDFA would solve the problem of Theorem [3.4] it would take exponential time in the worst case (as opposed to Theorem [3.3]) just to produce the output WDFA (or to decide that such a WDFA does not exist): there exists a family of regular languages where the size of the smallest WDFA is exponential in the size of the smallest equivalent DFA. Consider the family of languages $L_1, L_2, \ldots, \ M$, where $L_m = \{ \text{coa} \in \{a, b\}^m \} \cup \{ \text{da}f \in \{a, b\}^m \}$. Figure 5 shows a DFA and the smallest WDFA for the language $L_3$. In general, we can build a DFA for $L_m$ by generalizing the construction in the figure: the source node has outgoing edges labeled with $c$ and $d$, followed by simple linear size "universal gadgets" capable of generating all binary strings of length $m$, with one gadget followed by an $e$ and the other by an $f$. The two sink states are the only accepting states.

The smallest WDFA for $L_n$ is an unraveling of the described DFA, such that all paths up to (but not including) the sinks end up in distinct nodes, i.e. the universal gadgets are replaced by full binary
trees (see Figure 5). It is easy to see that the automaton is Wheeler as the only nodes that have multiple incoming paths are the sinks, and the sinks have unique labels.

By [ADPP20, Thm. 4.2], to prove that this is the minimum WDFA we need to check that all colexicographically consecutive pairs of nodes with the same incoming label are Myhill-Nerode inequivalent. As labels $c, d, e$ and $f$ occur only once, it is enough to focus on nodes that have label $a$ or $b$. Let $B_1, B_2, B_{2m+1-1}$ be the colexicographically sorted sequence of all possible binary strings with lengths $1 \leq |B_i| \leq m$ from the alphabet $\{a, b\}$. Observe that the nodes with incoming label $a$ and $b$ correspond to path labels of the form $cB_i$ and $dB_i$ for all $1 \leq i \leq 2^{m+1} - 1$. The co-lexicographically sorted order of these path labels is:

$$cB_1 < dB_1 < cB_2 < dB_2 < \ldots < cB_{2m+1-1} < dB_{2m+1-1}$$

Here we can see that all consecutive pairs have a different first character: they therefore lead to a different sink in the construction and hence are not Myhill-Nerode equivalent. We therefore conclude that the automaton is the minimum WDFA. The DFA has $n = 4m + 5$ states and the WDFA has $1 + 2^m + 2 = 1 + 2^{(n-5)/4+2}$ states, so we obtain the following result:

**Theorem 3.5.** The minimum WDFA equivalent to a DFA with $n$ states has $\Omega(2^{n/4})$ states in the worst case.

### 3.2 Is a $\mathcal{A}$ Wheeler?

In this section we consider the problem of deciding whether a given NFA can be endowed with a Wheeler order. In this case, since the problem is obviously decidable, we are interested in its complexity. Since input-consistency is a necessary condition for Wheelerness, without loss of generality in this section we will assume that the input NFA is state-labeled.

The problem has already been considered in [ADPP20, GT19], where the following results can be found: let d-NFA denote the class of NFA’s with at most $d$ equally-labelled transitions leaving any state.

1. ([ADPP20]) The problem of recognizing and sorting Wheeler d-NFA’s is in $P$ for $d \leq 2$ (in particular, it is in $P$ for deterministic automata, which correspond to the class of 1-NFA).

2. ([GT19]) shows that the problem is NP-complete for $d \geq 5$.

Here we see that NP completeness depends on redundancies of NFA: in fact, we shall prove that the problem of deciding whether a given reduced NFA (see Def. 9) can be endowed with a Wheeler order is in $P$. 

Figure 5: Left: a DFA recognizing $L_3$. Right: the minimum WDFA recognizing $L_3$. For clarity the labels are drawn on the nodes: the label of an edge is the label of the destination node.
Let \( A = (Q, s, \delta, F) \), with \(|Q| = n\), be an input-consistent NFA automaton (with no edges entering in the initial state \( s \)) over a finite ordered alphabet \( \Sigma = \{a_1, \ldots, a_k\} \), with \( a_1 \prec \ldots \prec a_k \). Let \( \lambda(u) \) be the label of (all) the edges entering \( u \), \( Q_u = \{ u \in Q : \lambda(u) = a \} \), \( Q_s = \{ s \} \); if \( C \subseteq Q\) then let \( \delta_u[C] = \{ q' \in Q : \exists q \in C \; q' \in \delta(q, a) \} \).

**Definition 11.** We say that a partition \( C = \{ C_1, \ldots, C_n \} \) of the set of the automaton states is \( a\text{-forward-stable} \), for \( a \in \Sigma \), if and only if for all \( C_1, C_j \in C \), either \( \delta_u[C_1] \supseteq C_j \) or \( \delta_u[C_j] \cap C_i = \emptyset \).

\( C \) is \textit{forward-stable} with respect to \( \delta \) if and only if is \( a\text{-forward-stable} \) for all \( a \in \Sigma \).

Consider the algorithm \[ \text{below, the \"Forward Algorithm\".} \]

**Algorithm 1: Forward Algorithm**

\[
\begin{align*}
\text{input} & : \text{A state-labeled NFA } A \\
\text{output} & : \text{The coarsest forward-stable partition of } A\text{'s states and (possibly) a Wheeler order of its states.}
\end{align*}
\]

1. \( C \leftarrow (Q, Q_{a_1}, \ldots, Q_{a_k}); \)
2. repeat
3. \( \quad \text{Set} \; \neg R(C), \text{for all } C \in C; \) \( \triangleright \; R(\cdot) \text{ stands for \"reached\"} \)
4. \( \quad C_{\text{old}} \leftarrow C; \)
5. \( \quad C \leftarrow \text{first}(C); \)
6. \( \quad \text{while } C = C_{\text{old}} \text{ and } C \neq \text{null do} \)
7. \( \quad \quad \text{for } C' \in C \text{ do} \)
8. \( \quad \quad \quad e = \lambda(C'); \)
9. \( \quad \quad \quad \text{if } R(C') \text{ then} \)
10. \( \quad \quad \quad \quad C'_1 \leftarrow C' \setminus \delta_e(C); \)
11. \( \quad \quad \quad \quad C'_2 \leftarrow \delta_e(C) \cap C'; \)
12. \( \quad \quad \quad \quad R(C'_1); R(C'_2); \)
13. \( \quad \quad \quad \text{else} \)
14. \( \quad \quad \quad \quad C'_1 \leftarrow \delta_e(C) \cap C'; \)
15. \( \quad \quad \quad \quad C'_2 \leftarrow C' \setminus \delta_e(C); \)
16. \( \quad \quad \quad \quad R(C'_1); \neg R(C'_2); \)
17. \( \quad \quad \quad \text{Insert}(C'_1, C'_2); \) \( \triangleright \text{ replace } C' \text{ with } C'_1, C'_2 \text{ (in order), ignoring empty sets} \)
18. \( \quad \quad \quad C = \text{next}(C, C); \)
19. \( \quad \text{until } C = C_{\text{old}}; \)

**Lemma 3.6.** The Forward Algorithm terminates in \( O(|Q|^2 \cdot |\delta|) \) steps.

**Proof.** After every iteration of the \texttt{repeat} command, the resulting partition is a refinement of the previous one, and the algorithm stops when we obtain the same partition of the previous iteration. Since the original partition can be refined at most \( |Q| \) times, we have at most \( |Q| \) iteration of the \texttt{repeat} command.

The \texttt{while} loop runs for at most \( |Q| \) times as well: by Line 18 and by the \texttt{while} condition, in the worst case we perform one iteration per element of \( C \). Being \( C \) a partition of \( Q \), its cardinality is bounded by \( |Q| \).

For each iteration of the \texttt{while} loop, in line 14 we compute the outgoing arcs labeled \( e \) of \( C' \), for each \( C \in C \). Overall, this amortizes to \( O(|\delta|) \) time per \texttt{while} iteration. Similarly, in the \texttt{for} loop we visit all the nodes in \( C' \), for each \( C' \in C \). This amortizes to \( O(|Q|) \) time per \texttt{while} iteration.

Overall, we obtain complexity \( O(|Q|^2 \cdot |\delta|) \).

**Lemma 3.7.** If \( C_{\text{out}} \) is the output of the Forward Algorithm and \( u, v \in C \in C_{\text{out}} \), then

\[ \{ \alpha : u \in \delta(s, \alpha) \} = \{ \alpha : v \in \delta(s, \alpha) \} \]
Proof. Suppose, by way of a contradiction, that there exists a word \( \alpha \in \Sigma^* \), an element \( C \in \mathcal{C}_{\text{out}} \), and two states \( u, v \in C \) such that \( u \notin \delta(s, \alpha) \), \( v \notin \delta(s, \alpha) \). Consider a word \( \alpha \) of minimal length having this property.

Let \( \alpha = \alpha' e \) and consider \( u' \in \delta(s, \alpha') \) such that \( u \notin \delta(u', e) \). Let \( C' \subset \mathcal{C}_{\text{out}} \) be such that \( u' \in C' \). Since \( \mathcal{C}_{\text{out}} \) is the output of the algorithm, \( C' \) cannot be a modifier for \( \mathcal{C}_{\text{out}} \); in particular, since \( u \notin \delta_e(C') \), we must have \( C' \subset \delta(C') \). Being \( v \in C' \), there must exist \( v' \in C' \) with \( v \in \delta(v', e) \). Since \( u', v \in C' \) with \( u' \in \delta(s, \alpha') \), by the minimality of \( \alpha \) we have \( v' \in \delta(s, \alpha') \). This implies \( v \in \delta(s, \alpha) \), which contradicts our hypothesis.

\[ R(C') \]

Let \( C' \subset \mathcal{C}_{\text{out}} \), \( y \in C_2' \), \( C_1' \subset \mathcal{C}_{\text{out}} \). We prove that \( x < y \). We begin observing that, for all \( k < h \), we must have \( \delta_k(C_k) \subset C' \). In fact, if this were not the case, we would have had \( C' \subset \delta_k(C_k) \) (or \( C' \) would have been “split” in a previous step). But then, when \( k = 1 \), \( C_1 \) was considered in the while loop, at line 13 or 17 the algorithm would have set \( R(C') \): a contradiction. Hence, \( \delta_k(C_k) \subset C' \) for all \( k < h \) and any edge entering in \( y \) must start from an element of \( y' \in C_j \) such that \( j > h \), that is: \( y \in \delta(y', e) \). Since \( x \in C_1 \), \( C_1 \subset \delta_k(C_k) \), there exists \( x' \in C_k \) with \( x \in \delta(x', e) \). Then \( x' < y' \), since by hypothesis the partition \( C \) agrees with the Wheeler order \( < \). Finally, by the Wheeler properties, \( x < y \) follows from \( x' < y' \), \( x \in \delta(x', e) \), and \( y \in \delta(y', e) \).

\[ R(C') \]

In this case, let \( x \in C_1' \), \( y \in C_2' \), \( y \in \delta(C), y \). We prove that \( x < y \). From \( R(C') \) it follows that there exists \( k < h \) with \( \delta_k(C_k) \subset C' \), hence, there exists \( x' \in C_k \) with \( x \in \delta(x', e) \). From \( y \in \delta_k(C_k) \subset C' \) it follows that there exists \( y' \in C_k \) with \( y \in \delta(y', e) \). From \( x' \in C_k \) and \( y' \in C_k \) it follows \( x' < y' \), since by hypothesis the partition \( C \) agrees with the Wheeler order \( < \). Finally, \( x < y \) follows from \( x' < y' \), \( x \in \delta(x', e) \), \( y \in \delta(y', e) \), and Wheeler properties.

The above analysis shows the following.

Remark 3.9. The equivalence relation \( \approx_{\text{out}} \) corresponding to the output partition \( \mathcal{C}_{\text{out}} \) of the Forward Algorithm can be a proper refinement of the equivalence \( \approx_{\mathcal{A}} \) described in Definition 6 as the automaton in Fig. 4 shows: the two last states are \( \approx_{\mathcal{A}} \)-equivalent and not \( \approx_{\text{out}} \)-equivalent.

Figure 6: An NFA for which the output relation \( \approx_{\text{out}} \) given by the Forward Algorithm is a proper refinement of \( \approx_{\mathcal{A}} \).

We are now ready to prove that deciding Wheelerness for reduced NFA is in \( P \).

Corollary 3.10. We can decide in polynomial time whether a reduced state-labeled NFA \( \mathcal{A} \) admits a Wheeler order.

Proof. This follows by the previous lemmas and the uniqueness of the Wheeler order on a reduced NFA (see Lemma 1.12). If we start the Forward Algorithm from a reduced NFA, by Lemma 3.7 we know that
the output partition \( C_{\text{out}} \) consists of singleton classes. By Lemma 3.8 we also know that if \( A \) is Wheeler then the unique possible Wheeler order is given by the (ordered) partition \( C_{\text{out}} \). Hence, to decide whether a reduced NFA \( A \) is Wheeler we can apply the algorithm, produce \( C_{\text{out}} \) in polynomial time, and test whether the induced order is Wheeler (this can be done in polynomial time, see [ADPP20]).

Moreover, the Forward Algorithm achieves the following: if \( A/ \approx_{\text{out}} \) is defined as in Definition 8 (but using relation \( \approx_{\text{out}} \) instead of \( \approx_{A} \)), it holds:

**Corollary 3.11.** Let \( A \) be a state-labeled NFA. If \( A \) is Wheeler, then then the Forward Algorithm builds and sorts, in polynomial time, the equivalent Wheeler NFA \( A/ \approx_{\text{out}} \).

**Proof.** By Lemma 3.7 \( \approx_{\text{out}} \) is a refinement of \( \approx_{A} \) (Definition 9). Using the same construction of Definition 8 and Lemma 2.10 we can moreover see that \( A/ \approx_{\text{out}} \) (having elements of \( C_{\text{out}} \) as states) is equivalent to \( A \). By Lemma 3.8 if \( A \) is Wheeler then \( C_{\text{out}} \) agrees with any Wheeler order \(<\) of \( A \).

It easily follows that the order \(<_{\text{out}} \) defined by \( C_{i} <_{\text{out}} C_{j} \) if and only if \( i < j \) is a Wheeler order on \( A/ \approx_{\text{out}} \). To see this, first note that if \( \lambda(C_{i}) < \lambda(C_{j}) \) then \( C_{i} <_{\text{out}} C_{j} \) since the Forward Algorithm preserves the order of the labels (Wheeler (i)). To prove Wheeler (ii), let \( C_{i} <_{\text{out}} C_{j} \) and \( C_{i}', C_{j}' \) be successors of \( C_{i} \) and \( C_{j} \), respectively, such that \( \lambda(C_{i}') = \lambda(C_{j}') \). Then, by definition of \( A/ \approx_{\text{out}} \) there exist \( u \in C_{i}, v \in C_{j}, u' \in C_{i}', \) and \( v' \in C_{j}' \) such that \( u', v' \) are successors of \( u, v \), respectively, with \( \lambda(u') = \lambda(v') = \lambda(C_{i}') = \lambda(C_{j}') \). Since \( C_{i} <_{\text{out}} C_{j} \), by Lemma 3.8 we have that \( u < v \). By Wheeler (ii) on \( A \), it follows that \( u < v \). Then, it must be the case that \( C_{i}' <_{\text{out}} C_{j}' \): if this were not the case, i.e. \( C_{j}' <_{\text{out}} C_{i}' \), then by Lemma 3.8 we would have \( v' < u' \), a contradiction. It follows that also Wheeler (ii) holds, therefore \( A/ \approx_{\text{out}} \) is Wheeler with order \(<\).

**Figure 7:** Left: non-Wheeler NFA \( A \) (the two states labeled \( b \) cannot be ordered). Right: Wheeler NFA \( A/ \approx_{\text{out}} \) and corresponding order output by the Forward Algorithm. The two states labeled \( a \) have been merged into a single state.

## 4 Closure Properties for Wheeler Languages

In this section we classify operations on languages depending on whether they preserve Wheelerness or not. The first observation is that Wheeler languages, being a subclass of the class of Ordered Languages (see [SITD04]), are star-free (that is, they can be generated from finite languages by Boolean operations and compositions only). As such, they can be definable in the so-called the first order theory of linear orders \( FO(<) \). However, as we shall see, there are very few “classical” operations which preserve Wheeler Languages.

### 4.1 Booleans

**Lemma 4.1.**

1. Finite and co-finite languages are Wheeler.
2. The union of a Wheeler language with a finite set is Wheeler.

3. The intersection of two Wheeler languages is Wheeler.

4. If \( \mathcal{L} \) is Wheeler, then Pref(\( \mathcal{L} \)) is Wheeler.

5. If \( \mathcal{L} \) is Wheeler, then Pref(\( \mathcal{L} \)) \( \setminus \mathcal{L} \) is Wheeler.

Proof.

1. If \( \mathcal{L} \) is finite (co-finite) and \((\alpha_i)_{i \geq 1}\) is a monotone sequence in Pref(\( \mathcal{L} \)), then there is a \( k \geq 1 \) such that, for all \( i > k \), the length of \( \alpha_i \) is longer than the length of any word in \( \mathcal{L} \) (\( \mathcal{L}^c \), respectively). This shows that any word having \( \alpha_i \) as prefix does not belong to \( \mathcal{L} \) (\( \mathcal{L}^c \), respectively), so that \( \alpha_i \equiv \alpha_j \) for all \( i, j > k \), and \( \mathcal{L} \) is Wheeler by Lemma 3.1.

2. If \( \mathcal{L} \) is Wheeler, \( F \) is a finite set, and \((\alpha_i)_{i \geq 1}\) is a monotone sequence in Pref(\( \mathcal{L} \cup F \)), then there is a \( k \geq 1 \) such that, for all \( i > k \), the length of \( \alpha_i \) is longer than the length of any word in the finite set \( F \). This implies that \( \alpha_i \in \text{Pref}(\mathcal{L}) \), for \( i > k \), and, since \( \mathcal{L} \) is Wheeler, there exists \( h \geq k \) such that \( \alpha_j \equiv \text{Pref}(\mathcal{L} \cup F) \alpha_{j+1} \), for all \( j \geq h \). Then \( \alpha_j \equiv \text{Pref}(\mathcal{L} \cup F) \alpha_{j+1} \), because, for length reasons, no words in \( F \) can have an \( \alpha_j \) as prefix.

3. Suppose \( \mathcal{L}_1, \mathcal{L}_2 \) are Wheeler, and consider a monotone sequence \((\alpha_i)_{i \geq 1}\) in Pref(\( \mathcal{L}_1 \cap \mathcal{L}_2 \)). Since

\[
\text{Pref}(\mathcal{L}_1 \cap \mathcal{L}_2) \subseteq \text{Pref}(\mathcal{L}_1) \cap \text{Pref}(\mathcal{L}_2)
\]

and \( \mathcal{L}_1, \mathcal{L}_2 \) are Wheeler, by Lemma 3.1 there exists \( h \) such that \( \alpha_i \equiv \alpha_j \), and \( \alpha_i \equiv \alpha_j \) both hold for \( i, j > h \). It follows that \( \alpha_i \equiv \alpha_j \), for all \( i, j > h \), and so \( \mathcal{L}_1 \cap \mathcal{L}_2 \) is Wheeler by Lemma 3.1.

4. Obvious, by considering a WDFA recognizing \( \mathcal{L} \), and considering all states as final.

5. Obvious, by considering a WDFA recognizing \( \mathcal{L} \), and changing non final with final states.

\[ \square \]

Corollary 4.2. The only Wheeler Languages on the one letter alphabet \( \Sigma = \{a\} \) are the finite or co-finite ones.

Proof. Suppose \( \mathcal{L} \subseteq \{a\}^* \) is neither finite nor co-finite. Since \( \mathcal{L} \) is not finite and the alphabet contains only one letter, we have Pref(\( \mathcal{L} \)) = \( \Sigma^* \), and, since \( \mathcal{L} \) is not co-finite, we have that Pref(\( \mathcal{L} \)) \( \setminus \mathcal{L} \) = \( \Sigma^* \setminus \mathcal{L} \) is infinite. Let \( \alpha = \alpha_1 \) be a word in \( \mathcal{L} \). Since there are only a finite number of words which are co-lexicographically smaller than \( \alpha \), there exists \( \alpha_2 \in \text{Pref}(\mathcal{L}) \setminus \mathcal{L} \) such that \( \alpha_1 \prec \alpha_2 \). Suppose we already have

\[
\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_m,
\]

\( m \) even, with \( \alpha_i \in \mathcal{L} \), for odd \( i \)'s, and \( \alpha_i \notin \mathcal{L} \) for even \( i \)'s. Then, since \( \mathcal{L} \) is infinite and there are only a finite number of words which are co-lexicographically smaller than \( \alpha_m \), there exists \( \alpha_{m+1} \in \mathcal{L} \) such that \( \alpha_m \prec \alpha_{m+1} \). Hence, we can define a monotone sequence which is not eventually constant modulo \( \equiv \mathcal{L} \), and \( \mathcal{L} \) is not Wheeler by Lemma 3.1. \( \square \)

We now turn to boolean operation not preserving Whelerness:

Lemma 4.3. Wheeler Languages are not closed for:

- Unions.
- Complements.

Proof. Unions. The languages \( \mathcal{L}_1 = ax^*b \), \( \mathcal{L}_2 = cx^*d \) are easily seen to be Wheeler, but their union is not (see Example 5).

Complements. Let \( \Sigma = \{a, b\} \) and \( \mathcal{L} = b^* \). Then \( \mathcal{L} \) is easily seen to be Wheeler, but its complement

\[
\overline{\mathcal{L}} = \{ \alpha \in \Sigma^* : \alpha \text{ contains at least an occurrence of the letter } a \}
\]

is not Wheeler.
is not Wheeler: consider the monotone sequence \( \text{Pref}( \mathcal{L} ) \) given by

\[
\alpha_i = \begin{cases} 
  b^i & \text{if } i \text{ is odd;} \\
  ab^i & \text{if } i \text{ is even.}
\end{cases}
\]

If \( i \) is odd, \( \alpha_i = b^i \notin \mathcal{L} \), while \( \alpha_{i+1} = ab^{i+1} \in \mathcal{L} \), so that \( \alpha_i \not\equiv _{\mathcal{L}} \alpha_{i+1} \), and \( \mathcal{L} \) is not Wheeler by Lemma 3.1.

\[\square\]

4.2 Concatenation

In general, the concatenation of two Wheeler languages is not necessarily Wheeler, as the following example shows:

Example 4. The languages \( \mathcal{L}_1 = b^*a \), \( \mathcal{L}_2 = b^+a \) are easily seen to be Wheeler, but their concatenation \( \mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \) is not: consider the monotone sequence in \((\text{Pref}(\mathcal{L}), \leq)\) given by

\[
\alpha_i = \begin{cases} 
  ab^i & \text{if } i \text{ is odd;} \\
  b^i & \text{if } i \text{ is even.}
\end{cases}
\]

If \( i \) is odd, we have \( \alpha_i \notin \mathcal{L} \), while \( \alpha_{i+1} \notin \mathcal{L} \). Hence, \( \alpha_i \not\equiv _{\mathcal{L}} \alpha_{i+1} \) for infinite \( i \)'s, and \( \mathcal{L} \) is not Wheeler.

On the positive side, we prove that the right concatenation of a Wheeler language with a finite set, is Wheeler. This is not true if consider left concatenation, even if the finite set is a single-letter word, as the following example shows.

Example 5. The language \( \mathcal{L} = \{ a^i : i \geq 1 \} \cup \{ ba^i : i \geq 1 \} \) is easily seen to be Wheeler but its concatenation on the left with the letter \( c \) is not. Indeed \( c \cdot \mathcal{L} = \{ ca^i : i \geq 1 \} \cup \{ cba^i \} \) and there exists a monotone sequence in \( \text{Pref}(c \cdot \mathcal{L}) \) which is not eventually constant modulo \( c \cdot \mathcal{L} \):

\[
ca > cba > cca > \ldots > cba^1 > ca^2 > \ldots
\]

From Lemma 3.1 it follows that \( c \cdot \mathcal{L} \) is not Wheeler.

\[\text{Lemma 4.4.} \quad \text{If } \mathcal{L} \text{ is Wheeler and } F \text{ is a finite set, then } \mathcal{L} \cdot F \text{ is Wheeler.}\]

**Proof.** Suppose \( \mathcal{L} \) is Wheeler, \( F \) is a finite set, and \( n = \max \{|w| : w \in F\} \) is the maximum of all lengths of words in \( F \). If \( (\alpha_i)_{i \geq 1} \) is a monotone sequence in \( \text{Pref}(\mathcal{L} \cdot F) \), then by possibly erasing an initial finite sequence we may suppose w.l.o.g. that \( |\alpha_i| \geq 2n \), and all \( \alpha_i \) end with the same \( 2n \)-suffix \( \gamma_1 \gamma_2 \), with \( |\gamma_1| = |\gamma_2| = n \). Let \( \alpha_i', \alpha_i'' \in \text{Pref}(\mathcal{L}) \) be such that

\[
\alpha_i = \alpha_i'' \gamma_1 \gamma_2 = \alpha_i' \gamma_2.
\]

Then both \( (\alpha_i')_{i \geq 1} \) and \( (\alpha_i'')_{i \geq 1} \) are monotone sequences in \( \text{Pref}(\mathcal{L}) \) and, since \( \mathcal{L} \) is Wheeler, there exists \( k \) such that \( \alpha_i' \equiv _{\mathcal{L}} \alpha_j' \) and \( \alpha_i'' \equiv _{\mathcal{L}} \alpha_j'' \), for all \( i, j \geq k \). We next prove that, for all \( i, j \geq k \), we also have \( \alpha_i' \equiv _{\mathcal{L} \cdot F} \alpha_j' \), from which \( \alpha_i \equiv _{\mathcal{L} \cdot F} \alpha_j \) follows. We must prove that for all \( \beta, \alpha_i' \beta \in \mathcal{L} \cdot F \Leftrightarrow \alpha_j' \beta \in \mathcal{L} \cdot F \). Suppose \( \alpha_i' \beta \in \mathcal{L} \cdot F \). Then

\[
\alpha_i' \beta = \alpha_i'' \cdot \beta' \cdot f,
\]

with \( \alpha_i'' \cdot \beta' \in \mathcal{L} \) and \( f \in F \). From \( \alpha_i'' \equiv _{\mathcal{L}} \alpha_j'' \) it follows \( \alpha_j'' \cdot \beta' \in \mathcal{L} \), so that

\[
\alpha_j' \beta = \alpha_j'' \cdot \beta' \cdot f \in \mathcal{L} \cdot F
\]

Summarizing, we proved that all elements of the monotone sequence \( (\alpha_i)_{i \geq 1} \) end eventually in the same \( \equiv _{\mathcal{L} \cdot F} \)-class, hence \( \mathcal{L} \cdot F \) is Wheeler.

\[\square\]
4.3 Kleene Star

In general, Wheeler languages are not closed for Kleene star, as the following example shows.

Example 6. The language $\mathcal{L} = \{aa\}$ is Wheeler (as any finite language), but $\mathcal{L}^* = \{a^{i+2} : i \geq 0\}$ is not Wheeler (see Example 5).

On the other hand, we can characterise which words $\alpha$ have a Kleene star $\alpha^*$ which is Wheeler, and, more generally, when a regular language of the form $\alpha_1 \alpha^* \alpha_2$ is Wheeler.

Definition 12. We say that $\alpha \in \Sigma^*$ is primitive if there exists no $\beta \neq \varepsilon$ and $i > 1$, such that $\alpha = \beta^i$.

Primitive words are important for Wheeler automata and languages as seen in the following results.

Lemma 4.5. If $A = (Q, s, \delta, F, <)$ is a WDFA and $\alpha$ is the label of a simple cycle in $A$, then $\alpha$ is primitive.

Proof. Suppose, by way of a contradiction, that there exists a simple cycle labelled by $\alpha$ and there exists $i > 1$, such that $\alpha = \beta^i$. Then there exists $n < m < r$ such that $\beta^n, \beta^r$ are both labels of cycles starting from the same vertex $u$, while $\delta(u, \beta^m) \neq u$. Let $\gamma$ be a word such that $\delta(s, \gamma) = u$. Consider the sequence $(\gamma \beta^n)_{n \in \mathbb{N}}$, and note that it is monotone: if $\gamma \prec \gamma \beta^k$, then $\gamma \beta^k \prec \gamma \beta^{k+1}$ holds for any $k$, and similarly by transitivity of $\prec$ we obtain that $\gamma \beta^k \prec \gamma \beta^h$ holds for any $h > k$. Thus, the sequence is monotonically increasing. Conversely, if $\gamma \succ \gamma \beta$ then the sequence is monotonically decreasing. It follows that either $\gamma \beta^n \prec \gamma \beta^m \prec \gamma \beta^r$, or $\gamma \beta^n \succ \gamma \beta^m \succ \gamma \beta^r$. Since $\gamma \beta^n, \gamma \beta^r \in I_u$, by Lemma 3.4 we should also have $\gamma \beta^m \in I_u$. We shall use the following:

Notation. $\alpha' \vdash \alpha$ stands for $\alpha'$ is a prefix of $\alpha$ and $\alpha' \vdash \alpha$ stands for $\alpha'$ is a suffix of $\alpha$.

Lemma 4.6. Let $\alpha_1, \alpha, \alpha_2 \in \Sigma^*$. Then

$$\alpha_1 \alpha^* \alpha_2 \text{ is Wheeler } \Leftrightarrow \alpha \text{ is primitive}.$$  

Proof. ($\Rightarrow$) Suppose $\alpha$ is not primitive, say $\alpha = \beta^k$ with $k > 1$, $\beta \neq \varepsilon$, and consider the sequence

$$\alpha_1 \beta^k, \alpha_1 \beta^k+1, \alpha_1 \beta^{2k}, \alpha_1 \beta^{2k+1}, \alpha_1 \beta^{3k}, \alpha_1 \beta^{3k+1}, \ldots$$

in Pref$(\alpha_1 \alpha^* \alpha_2)$. As in the previous lemma, we can prove that the sequence is monotone: if $\alpha_1 \prec \alpha_1 \beta$, the sequence is monotonically increasing, while, if $\alpha_1 \succ \alpha_1 \beta$ then the sequence is monotonically decreasing. However, this sequence does not become eventually constant modulo $\equiv_{\alpha_1 \alpha^* \alpha_2}$, because, for all $n$,

$$\alpha_1 \beta^k \alpha_2 \in \alpha_1 \alpha^* \alpha_2 \text{ while } \alpha_1 \beta^{n+1} \alpha_2 \notin \alpha_1 \alpha^* \alpha_2$$

From the above and Lemma 3.4 it follows that $\alpha_1 \alpha^* \alpha_2$ is not Wheeler.

($\Leftarrow$) If $\alpha$ is primitive we first show that $\alpha_1 \alpha^*$ is Wheeler. Suppose not. Then there is a monotone sequence in Pref$(\alpha_1 \alpha^*)$ which does not become eventually constant modulo $\equiv_{\alpha_1 \alpha^*}$. By erasing an opportune prefix of the sequence we may suppose that it has the form

$$\alpha_1 \alpha \beta_1, \alpha_1 \alpha \beta_2, \alpha_1 \alpha \beta_3, \ldots$$

with $\beta_i \vdash \alpha$, for all $i$, and that all elements of the sequence end with the same $3|\alpha|$ characters. Notice that, since the sequence is not eventually constant modulo $\equiv_{\alpha_1 \alpha^*}$, there must be infinite $i$’s such that $\beta_i \neq \beta_{i+1}$. Hence, there are two different $\alpha$-prefixes, $\beta, \beta'$ such that $\alpha \beta^i \beta$ and $\alpha \beta^i \beta'$ end with the same $3|\alpha|$-characters, which implies that there exists an $\alpha$-prefix $\gamma$ such that $\alpha$ and $\alpha \gamma$ end with the same $|\alpha|$-characters; but then there exists $\delta$ such that $\alpha = \delta \gamma$, where $\delta, \gamma$ are both proper prefixes and proper suffixes of $\alpha$. This implies $\alpha = \delta \gamma = \gamma \delta$ which in turn implies (see (LS62)) that $\alpha$ is not primitive, a contradiction.

Hence, If $\alpha$ is primitive then $\alpha_1 \alpha^*$ is Wheeler, and $\alpha_1 \alpha^* \alpha_2$ is also Wheeler, being a concatenation of a Wheeler language with a finite set on the right.
4.4 Factors, Suffixes, and Inverses

Wheeler Languages are not closed for factors, suffixes, or inverses:

Example 7. Factors and Suffixes. The language \( L_1 = ax^*b \mid zx^*d \) is Wheeler (see Example 11), but \( \mathcal{L} = \text{Fact}(L_1) \) is not: consider the monotone sequence in \( (\text{Pref}(\mathcal{L}), \preceq) \) given by

\[
\alpha_i = \begin{cases} 
  x^i & \text{if } i \text{ is odd;} \\
  ax^i & \text{if } i \text{ is even.}
\end{cases}
\]

if \( i \) is odd, \( \alpha_i \notin \mathcal{L} \alpha_{i+1} \), because \( \alpha_i d = x^i d \in \mathcal{L} \) whereas \( \alpha_{i+1} d = ax^{i+1} d \notin \mathcal{L} \); hence \( \mathcal{L} = \text{Fact}(L_1) \) is not Wheeler by Lemma 3.1. Similarly, \( \text{Suff}(\mathcal{L}) \) is not Wheeler: considering the same monotone sequence above we have \( \alpha_i \in \text{Pref}(\text{Suff}(\mathcal{L})) \) and \( \alpha_i \notin \text{Suff}(\mathcal{L}) \alpha_{i+1} \), for odd \( i \)'s, because \( \alpha_i d = x^i d \in \text{Suff}(\mathcal{L}) \) whereas \( \alpha_{i+1} d = ax^{i+1} d \notin \text{Suff}(\mathcal{L}) \).

Inverses. Suppose, by way of a contradiction, that Wheeler languages were closed under inverses, and consider again the Wheeler language \( L = ax^*b \mid zx^*d \); then, by Lemma 3.1, \( \text{Pref}(L^{-1})^{-1} = \text{Suff}(L) \) would be Wheeler, while we proved the opposite in the previous point.

4.5 Morphisms

We now consider preservation under inverse image of monoid morphisms. Wheeler Languages are not closed in general under inverse images of morphisms. E.g. consider

\[
\varphi : \{a, b, c, d, x, z\} \to \{a, b, d, x, z\}, \quad \mathcal{L} = \text{ax}^*b \mid \text{zx}^*d \subseteq \Sigma^*,
\]

and the morphism \( \phi \) defined by \( \phi(c) = z \) and the identity on the other letters. Then \( \mathcal{L} \) is Wheeler (see Example 11), while \( \phi^{-1}(\mathcal{L}) = \text{ax}^*b \mid \text{cx}^*d \) is not Wheeler (see Example 8). We next prove that Wheeler languages are closed under inverse images of co-lex monoid morphisms:

Definition 13. Let \( \Sigma, \Sigma' \) be two finite alphabet. A co-lex morphism between \( (\Sigma^*, \preceq), (\Sigma'^*, \preceq) \) is a monoid morphism \( \phi : \Sigma^* \to \Sigma'^* \) such that

\[
\alpha \preceq \alpha' \Rightarrow \phi(\alpha) \preceq \phi(\alpha')
\]

Lemma 4.7. Suppose \( \Sigma, \Sigma' \) are finite alphabets and \( \phi : (\Sigma^*, \preceq) \to (\Sigma'^*, \preceq) \) is a co-lex morphism. If \( \mathcal{L} \subseteq \Sigma^* \) is a Wheeler language, then \( \phi^{-1}(\mathcal{L}) \subseteq \Sigma'^* \) is a Wheeler language.

Proof. If \( \phi : (\Sigma^*, \preceq) \to (\Sigma'^*, \preceq) \) is a morphism of ordered monoids and \( \phi^{-1}(\mathcal{L}) \) is not Wheeler, we prove that \( \mathcal{L} \) is not Wheeler. Since regular languages are closed by inverse images of morphisms, \( \phi^{-1}(\mathcal{L}) \) is a regular, non Wheeler language; by Lemma 3.1 there exists a strictly monotone sequence \( (\gamma_i)_{i \in \mathbb{N}} \) in \( \text{Pref}(\phi^{-1}(\mathcal{L})) \) with \( \gamma_i \notin \phi^{-1}(\mathcal{L}) \gamma_{i+1} \). Since \( \phi \) is a morphism, we obtain \( \phi(\gamma_i) \in \text{Pref}(\mathcal{L}) \). Moreover, since \( \phi \) is a co-lex morphism, we obtain that \( (\phi(\gamma_i))_{i \in \mathbb{N}} \) is monotone and \( \phi(\gamma_i) \notin \phi(\gamma_{i+1}) \) for all \( i \). Hence, \( (\phi(\gamma_i))_{i \in \mathbb{N}} \) is strictly monotone and Lemma 3.1 implies that \( \mathcal{L} \) is not Wheeler.

The closure of Wheeler languages under the inverse image of co-lex morphisms may suggest a natural generalization of the algebraic characterization of regular languages. Remember that a language \( \mathcal{L} \subseteq \Sigma^* \) is said to be recognized by a monoid morphism \( \phi : (\Sigma^*, \cdot) \to (M, \cdot) \) if \( \mathcal{L} = \phi^{-1}(\phi(\mathcal{L})) \) (or, equivalently, if \( \alpha \in \mathcal{L} \) and \( \phi(\alpha) = \phi(\beta) \) implies \( \beta \in \mathcal{L} \)). The algebraic characterization of regular languages states that these languages are exactly the ones which are recognized by morphisms over finite monoids.

Suppose now we add a total order \( \preceq \) over the elements of the monoid \( M \); we say that a monoid morphism \( \phi : (\Sigma^*, \cdot) \to (M, \cdot) \) respect the corresponding orders \( \prec, \preceq \) if, for all \( \alpha, \beta \in \Sigma^* \) it holds:

\[
\alpha \preceq \beta \Rightarrow \phi(\alpha) \preceq \phi(\beta).
\]

Lemma 4.8. If a language \( \mathcal{L} \subseteq \Sigma^* \) is recognized by a morphism over a finite monoid \((M, \cdot)\) and \( \preceq \) is an order over \( M \) such that \( \phi \) respect the orders \( \prec, \preceq \), then \( \mathcal{L} \) is Wheeler.
Proof. $\mathcal{L}$ is regular, since it is recognized by a morphism over a finite monoid $(M, \cdot)$. Suppose it is not Wheeler. Then by Lemma 4.8 there exists a monotone (say increasing) sequence $(\alpha_i)_{i \in \mathbb{N}}$ in $\text{Pref}(\mathcal{L})$ which is not eventually constant. Since the morphism respect the order, we have $\phi(\alpha_i) \leq \phi(\alpha_{i+1})$, for all $i$. Moreover, $\phi(\alpha_i) \neq \phi(\alpha_{i+1})$, for every $i$ such that $\alpha_i \not\in L \alpha_{i+1}$: from the previous inequality it follows that there exists $\delta \in \Sigma^*$ with $\alpha_i \delta \in \mathcal{L}$ and $\alpha_{i+1} \delta \not\in \mathcal{L}$ (or viceversa); if $\phi(\alpha_i) = \phi(\alpha_{i+1})$ then
\[
\phi(\alpha_i \delta) = \phi(\alpha_i) \phi(\delta) = \phi(\alpha_{i+1}) \phi(\delta) = \phi(\alpha_{i+1} \delta),
\]
and from $\alpha_i \delta \in \mathcal{L}$ it then follows $\alpha_{i+1} \delta \in \mathcal{L}$, a contradiction. Hence, $(\phi(\alpha_i))_{i \in \mathbb{N}}$ should be a monotone sequence which is strictly increasing for infinitely many index $i$, which is impossible, since $M$ is finite. \hfill $\square$

Unfortunately, Lemma 4.8 is too strong and cannot be reversed: the class of languages which are recognized by morphism as in Lemma 4.8 is closed under complements and factors, while Wheeler languages are not.

4.6 Intervals

Definition 14. If $\alpha_0 \preceq \alpha_1 \in \Sigma^+$, we define the intervals $(\alpha_0, \alpha_1), [\alpha_0, \alpha_1), (-\infty, \alpha_1) \ldots$ based on $\alpha_0, \alpha_1$ as usual, e.g.:
\[
(\alpha_0, \alpha_1) = \{ \beta \in \Sigma^* : \alpha_0 \prec \beta \prec \alpha_1 \}, \quad [\alpha_0, \alpha_1) = \{ \beta \in \Sigma^* : \alpha_0 \preceq \beta \prec \alpha_1 \}, \quad (-\infty, \alpha_1) = \{ \beta \in \Sigma^* : \beta \prec \alpha_1 \} \ldots
\]

Lemma 4.9. Suppose $\alpha_0 \preceq \alpha_1 \in \Sigma^+$. Then all intervals based on $\alpha_0, \alpha_1$ are Wheeler.

Proof. Let $\alpha_1 \in \Sigma^+$, and consider the interval $(-\infty, \alpha_1)$. If $F = \{ \beta : \beta \prec \alpha_1, |\beta| = |\alpha_1| \}$ we have
\[
(-\infty, \alpha_1) = \Sigma^+ \setminus F \cup \{ \gamma : \gamma \prec \alpha_1, |\gamma| \leq |\alpha_1| \}
\]
which is Wheeler by Lemma 4.1.

Similarly,
\[
(\alpha_0, +\infty) = \Sigma^+ \setminus \{ \beta : \alpha_0 \prec \beta, |\beta| \leq |\alpha_0| \},
\]
is Wheeler. If $\alpha_0 \prec \alpha_1$, then the interval $(\alpha_0, \alpha_1) = (\alpha_0, +\infty) \cap (-\infty, \alpha_1)$ is Wheeler, as intersection of Wheeler languages. Finally, the (half)-closed intervals $(-\infty, \alpha_0], [\alpha_0, \alpha_1), (\alpha_0, \alpha_1], \ldots$ are obtained from the open versions by adding one or two words, hence they are Wheeler by Lemma 4.1. \hfill $\square$

Note that Wheelerness does not generalize from interval to convex sets, as the following example shows.

Example 8. The regular language
\[
\mathcal{L} = ax^*a \mid bx^*b \mid b
\]
is convex in $\text{Pref}(\mathcal{L})$ but it is not Wheeler.

5 Conclusions and Open Problems

Wheeler Languages represent a formal tool to elegantly and fruitfully cast the notion of ordering of strings of a regular language $\mathcal{L}$ on an ordering of the states of an automaton $\mathcal{A}$ recognising $\mathcal{L}$. The key property, made explicit by the definition of Wheeler graphs, allows to doubly-link the co-lexicographic order of strings read while reaching a state $q$ with the position of $q$ in the Wheeler order of $\mathcal{A}$’s states. This is obtained by the initial fixing of an ordering of the alphabet $\Sigma$, which is the marking difference between the approach on ordering of states developed here and the work on ordered automata carried out in [ST74].

Many questions remain open, especially on the operational characterisation of Wheeler languages. Among the problems left open we mention:

1. Theorem 5.3 allow us to prove that the problem of deciding a regular language accepted by a given finite deterministic automaton is Wheeler in polynomial time. Can we generalise this theorem to NFA’s, in order to show that we can decide in polynomial time if a regular language accepted by a $NFA$ is Wheeler?
2. Is there a natural fragment of $FO(<)$ describing Wheeler Languages, or, is there a natural logic describing Wheeler Languages?

3. Can we find a finite number of “Wheeler operations” and a finite number of “basic Wheeler Languages” such that all Wheeler languages are obtained from the basic ones using the Wheeler operations?

4. Can we characterise Wheeler languages using monoids or other algebraic structures?

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