Fermions on quantized vortices in superfluids and superconductors

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Abstract

The bound states of fermions in cores of quantized vortices in superconductors and Fermi superfluids and their influence on the vortex dynamics are discussed. The role of the spectral flow of the fermions through the gap nodes is emphasized.
1 Introduction

Gapless fermions govern the low temperature behavior of condensed matter. In homogeneous condensed matter the distinctive feature of the spectrum of the gapless fermions is the dimension of zeroes in the spectrum: the dimension $D$ of the manifold in 3-dimensional momentum space, where the energy of the fermion vanishes: $E(p) = 0$. There are $D = 0$ point node, $D = 1$ line of zeroes, $D = 2$ Fermi surface and $D = 3$ Fermi band (the fermionic condensate – a flat plateau in the quasiparticle energy at the Fermi level [24, 25]). The topological objects lead to the zeroes of the gap in the real space (say, on the vortex axis).

At the moment several groups of experiments support the possibility of lines of of nodes in the electron spectrum of some of the high-$T_c$ materials (see Reviews [59, 3, 69]. The point nodes result in many interesting properties of the superfluid $^3$He-A [78]. The gapless fermions concentrated near the vortex axis are important for the low-temperature dynamics of vortices in all superconductors and Fermi superfluids. Here we discuss some consequences of the gap nodes in bulk superconductors and in the core of vortices.

2 Electron density of states due to the gap nodes.

2.1 DOS in bulk superconductors and superfluids.

The gap nodes in the superconducting gap lead to the power law dependence of the density of states (DOS) on the frequency $\omega$:

$$N(\omega) = 2 \int \frac{d^3p}{(2\pi)^3} \delta(E(p) - \omega) \sim N_F \left( \frac{\omega}{\Delta} \right)^\alpha,$$  

(2.1)

where $N_F$ is the DOS of the normal state, $E = \sqrt{\epsilon^2 + |\Delta(p)|^2}$ is the quasiparticle excitation spectrum. The exponent $\alpha = 2 - D$, ie $\alpha = 2$ for the point nodes, $\alpha = 1$ for lines of nodes and $\alpha = 0$ if the Fermi surface remains in the superconducting state.

Situation changes in the presence of impurities: eg intermediate values $1 > \alpha > 0$ are possible for the case of the lines of gap nodes in the presence of impurities [51].
2.2 DOS induced by supercurrent and texture.

The gap nodes in the presence of textures lead to the nonzero density of states at zero energy $N(0) \neq 0$. Texture is a slow space variation of the order parameter: in superconductors due to the crystal lattice the variation of the phase of the order parameter is only allowed, which gives the supercurrent with the superfluid velocity $v_s = (\hbar / 2m)(\vec{\nabla} \Phi - e\vec{A})$. In superfluid $^3$He the anisotropy axis $\hat{l}$ can vary slowly in space (in $^3$He-A the unit vector $\hat{l}$ shows also the direction to the point gap node in momentum space). The DOS in supercurrent can be obtained from Eq. (2.1) by substitution $\omega \rightarrow p_F v_s$. In $\hat{l}$-texture one should substitute $\omega^2 \rightarrow v_F \Delta |\vec{\nabla} l|$. As a result one has

$$N(0) \sim N_F \left( \frac{v_F |\vec{l}|}{\Delta} \right)^{\frac{\alpha}{2}}, \quad (2.2)$$

for the texture (see [84] for $\alpha = 2$) and

$$N(0) \sim N_F \left( \frac{p_F v_s}{\Delta} \right)^{\alpha}, \quad (2.3)$$

for supercurrent (see [50] for point nodes, where $\alpha = 2$, and [48] for lines of zeroes, where $\alpha = 1$). Thus the gap nodes in the presence of texture or supercurrent give the linear term in heat capacity $C \propto N(0)T$ at low $T$.

The physical origin of the nonzero density of states due to supercurrent is the same as that of the depairing current in a conventional superconductor. In the presence of a supercurrent the quasiparticle excitation spectrum is shifted by an amount $p \cdot v_s$. The density of states at the Fermi level is given by

$$N(0) = 2 \int \frac{d^3p}{(2\pi)^3} \delta(E(p) + p \cdot v_s). \quad (2.4)$$

For the superconductor with nodes, the shift transforms the nodes into the Fermi surfaces, at which the quasiparticle energy $E + p \cdot v_s$ is zero. The Fermi surface gives the finite local DOS $N(0) \propto |p_0 \cdot v_s|^\alpha$, where $p_0$ is the direction to the node.

The dependence of the density of states on the supercurrent leads to the nonlinear Meissner effect [88] and to the power-law dependence of heat capacity on the magnetic field in the mixed state of superconductors (see next Section).
2.3 Magnetic-field dependence of DOS.

The DOS averaged over the elementary cell area $A_{\text{cell}}$ of the Abrikosov vortex lattice is

$$N(0) = \frac{1}{A_{\text{cell}}} \int_{A_{\text{cell}}} d^2r \, N(0, r) \approx N_F \frac{1}{A_{\text{cell}}} \int_{A_{\text{cell}}} d^2r \left( \frac{p_F v_s(r)}{\Delta} \right)^\alpha.$$  

(2.5)

Since $v_s \propto 1/r$ outside the core of the vortices, the integral diverges. The main contribution to the integral comes from the region far from the vortex core, at the distance of order the intervortex spacing $\sqrt{A_{\text{cell}}} \propto 1/\sqrt{H}$, which serves as the upper cutoff in the integral. This gives the following DOS in the mixed state of the superconductors with gap nodes:

$$N(0) \propto N_F \left( \frac{H}{H_{c2}} \right)^{\alpha/2}. \quad (2.6)$$

For a superconducting state with lines of nodes ($\alpha = 2$) the dominant magnetic-field dependence is $N(0) \sim N_F \sqrt{H/H_{c2}}$, while in superconductors with point nodes $N(0) \sim N_F H/H_{c2}$ or $N(0) \sim N_F H/H_{c2} \ln(H_{c2}/H)$ depending on the orientation of the point nodes with respect to the magnetic field [75]. Recent measurements of the magnetic-field dependence of the heat capacity [46, 47, 10] are consistent with the square-root law (see also [57] for the heavy fermionic compound UPt$_3$). Note that the impurities decrease the exponent $\alpha$ [51].

2.4 Topological stability of nodes.

Point nodes and Fermi surfaces are topologically stable: they are described by integer topological invariants (charges) expressed in terms of Green’s function [77, 78]. Those point nodes and Fermi surfaces, which have nonzero topological charge, are robust: they survive under external perturbations and can disappear only by mutual annihilation with zeroes of the opposite charge. Of course, the conservation of the topological charges does not exclude the possibility of transformation of zeroes to that of higher dimension: the point node can transform into the closed fermi surface, in turn the fermi surface can transform into the fermi band [24, 25, 53]. The latter represents the fermi surface of finite thickness or a pair of half-quantum vortices in momentum...
space, which is described by the same topological charge as the Fermi-surface.

The lines of zeroes generally have no topological stability and an existence of the nodal lines can be prescribed only by the symmetry of the superconducting state \[13\]. The nontrivial classes of the superconductors, which symmetry leads to the nodal lines in symmetric positions, are enlisted in Ref. \[82\]. The symmetry violating external perturbation, such as impurities or deviation of the crystal symmetry from the tetragonal one, destroys the line zeroes \[13\]. One could expect different types of behavior, which should depend on the parameters of the system. Impurities can: (i) produce the gap in the fermionic spectrum \[56\]; (ii) renormalize the exponent \(\alpha\) in \(N(\omega) \propto \omega^\alpha\) \[51\]; (iii) lead to localization \[37\]; or (iv) lead to the finite density of states \[67\].

3 Fermions localized in the vortex core.

3.1 Fermion zero modes

The energy \(E(Q, p_z)\) of excitations localized in the axisymmetric vortex core depends on the linear momentum \(p_z\) along the symmetry axis and on the discrete quantum number \(Q\) \[8\]. \(Q\) is the generalization of the angular momentum (the eigenvalue of the generator of the ”axial” symmetry of the vortex \[58\]). In the most symmetric vortices the low-lying levels are

\[
E_0(p_z, Q) = Q\omega_0(p_z)
\]

where the interlevel distance \(\omega_0 = \partial E/\partial Q \sim \Delta^2/E_F\) is small compared to the gap amplitude \(\Delta\) in the bulk superconductor.

Depending on the type of the vortex and on the pairing state the quantum number \(Q\) can be either integer or half of odd integer. In conventional singly quantized vortices (ie with winding number \(n = 1\)) the quantum number \(Q\) is half of odd integer, which means that excitations have small but finite gap \((1/2)\omega_0(0)\). Since this gap is small, some perturbations can modify the spectrum in such a way, that \(E_0(p_z, \pm1/2)\) can cross zero energy as a function of \(p_z\). This is discussed for the vortices in superconductors \[41\] and \(^3\)He \[13, 76\].
In doubly quantized \( (n = 2) \) vortices and in some singly quantized \(^3\)He-A vortices the quantum number \( Q \) is integer \([15]\). This allows the fermionic states with \( Q = 0 \) which corresponds to the flat band: the energy \( E_0(p_z, 0) \) is zero for all \( p_z \). Such flat band in the vortex core has been first discussed in \([32]\).

These details of the quasiparticle spectrum at low energy are important only at very low temperature \( T \sim \omega_0 \). At higher temperatures \( T \gg \omega_0 \) the discreteness of the spectrum is sometimes not very important and one can consider \( Q \) as a continuous variable. In what follows we assume that \( T_c \gg T \gg \omega_0 \).

The Eq.\((3.1)\) shows that there is a branch in the quasiparticle energy spectrum which as a function of the ”continuous” quantum number \( Q \) crosses zero of energy. Such branches of the fermions localized on vortices or other topological objects, which energy spectrum cross zero, are called fermion zero modes. The existence of the fermion zero modes in a background the topological object is usually deduced by the application of certain index theorems which relate the number of such modes to the topological charge of the object, say, to the vortex winding number \( n \). Originally such relation was found for the spectrum \( E(p_z) \) of fermions localized on strings in particle physics (see Refs. \([22, 86, 17]\)): the number \( N_{zm} \) of the branches which as a function of \( p_z \) cross the zero level equals \( n \).

For fermions in condensed matter vortices there is no such theorem, but a similar theorem exists if one considers the spectrum as a function of the quantum number \( Q \). The number \( N_{zm} \) of the branches which as a function of \( Q \) cross the zero level is \( N_{zm} = 2n \) \([80]\). The factor 2 corresponds to the spin degeneracy.

### 3.2 Fermion zero modes in semiclassical approach

The existence of fermionic zero modes does not depend on the details of the system (provided that the interlevel distance \( \omega_0 \) is small) and is completely defined by topology. So here we will consider the simplest and best known case of an axisymmetric vortex in superfluid or superconductor with \( s \)-wave pairing. The orbital quantum number \( Q \) is considered here as a continuous variable and so one can use the quasiclassical approximation for the fermions localized in the vortex core. The Bogoliubov Hamiltonian for the fermions...
with given spin projection is a $2 \times 2$ matrix

$$\mathcal{H} = -\frac{i}{m} \tilde{\tau}_3 \mathbf{q} \cdot \vec{\nabla} + \tilde{\tau}_1 \text{Re} \Psi(\mathbf{r}) - \tilde{\tau}_2 \text{Im} \Psi(\mathbf{r}) \, .$$  (3.2)

Here $\tilde{\tau}$ are the Bogoliubov-Nambu matrices in the particle-hole space; $\mathbf{q} = \mathbf{p} - \mathbf{\hat{z}}(\mathbf{p} \cdot \mathbf{\hat{z}})$ is the quasiparticle momentum in the transverse plane and

$$\Psi(\mathbf{r}) = e^{i m \Phi} \Delta(\mathbf{r})$$  (3.3)

is the gap function (order parameter) in the axisymmetric vortex with winding number $n$.

The quantum numbers, which characterize the fermionic levels in this approximation, are (i) the magnitude of transverse momentum of quasiparticle $q$, which is related to the longitudinal projection of momentum $q^2 = p_x^2 - p_z^2$, (ii) the radial quantum number $n_r$, and (iii) the continuous impact parameter $y = \mathbf{\hat{z}} \cdot (\mathbf{r} \times \mathbf{q})/q$. It is related to the angular momentum $\hbar Q$ by $Q = qy$.

Introducing the coordinate $x = \mathbf{r} \cdot \mathbf{q}$ along $\mathbf{q}$, such that $r^2 = x^2 + y^2$, and assuming that in the important regions one has $|y| \ll |x|$, one obtains the dependence of the gap function on $x$ and $y$:

$$\Psi(\mathbf{r}) \approx \Delta(|x|) (\text{sign}(x) - in \frac{y}{|x|}) \, ,$$  (3.4)

and the Hamiltonian:

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \, ,$$

$$\mathcal{H}^{(0)}(x) = -i \frac{q}{m} \nabla_x + \tilde{\tau}_1 \Delta(|x|) \text{sign}(x) \, , \quad \mathcal{H}^{(1)}(x, y) = n \tilde{\tau}_2 y \frac{\Delta(|x|)}{|x|} \, .$$  (3.5)

The Hamiltonian $\mathcal{H}^{(0)}(x)$ is "supersymmetric" - there is an operator $\tilde{\tau}_2$ which anticommutes with $\mathcal{H}^{(0)}(x)$, i.e. $\{ \mathcal{H}^{(0)}, \tau_2 \} = 0$, and it has an integrable eigenfunction corresponding to zero eigenvalue:

$$\mathcal{H}^{(0)} \chi^{(0)}(x) = 0 \, , \quad \chi^{(0)}(x) \propto (\tilde{\tau}_0 - \tilde{\tau}_2) \exp \left[ -\frac{m}{q} \int_0^{|x|} dr \Delta(r) \right] \, .$$  (3.6)

Here $\tilde{\tau}_0$ is the diagonal $2 \times 2$ matrix.

Using first order perturbation theory in $\mathcal{H}^{(1)}$ one obtains the lowest energy levels:

$$E_0(Q, p_z) \equiv E_{n_r = 0}(Q, p_z) \approx <0|\mathcal{H}^{(1)}|0> = -ny <\frac{\Delta(r)}{r}> = -Q n \omega_0(p_z) \, ,$$  (3.7)
\[ \omega_0(p_z) = \frac{1}{q(p_z)} \frac{\int_0^\infty dr |\chi^{(0)}(r)|^2 \Delta(r)/r}{\int_0^\infty dr |\chi^{(0)}(r)|^2} . \]  

(3.8)

For \( n = 1 \) this is the anomalous branch of the low-energy localized fermions obtained in Ref. [7]. If the energy spectrum is considered as a continuous function of \( Q \), this anomalous branch crosses zero at \( Q = 0 \).

4 Vortices with broken symmetry.

4.1 Broken symmetry in \( ^3\)He-B vortices. Point gap nodes in the broken symmetry core.

Four types of linear defects have been experimentally verified to exist in \( ^3\)He-B in a rotating container [35]. Among them there are two types of \( n = 1 \) vortices: in both of them the parity is broken within the core and in one of them in addition the axial symmetry is broken [58].

The discrete symmetry breaking leads to a new phenomenon (see Review [58]). In conventional vortices without broken parity (Eq.(3.3)) the order parameter and thus the superfluid gap vanishes on the vortex axis, i.e. the vortex axis consists of the "normal" Fermi liquid with the Fermi surface. In the broken symmetry core such Fermi surface on the axis splits into the gap nodes which occupy some finite region of the vortex core, as a result the singularity on the axis is smoothened: everywhere in the core the system is in the superfluid state. Now at each point \( r \) of such a smooth core there is one or several pairs of the gap nodes, i.e. the energy \( E(p, r) \) becomes zero at \( p = \pm p_a(r) \hat{l}_a(r) \). Here the unit vector \( \hat{l}_a(r) \) shows the direction of the \( a \)-th pair of the gap nodes. There is an important relation between the topology of the \( \hat{l}_a(r) \)-field and the winding number \( n \) of the vortex:

\[ n = \sum_a \int_{\text{over core}} \frac{dx \, dy}{4\pi} \left( \hat{l}_a \cdot \frac{\partial \hat{l}_a}{\partial x} \times \frac{\partial \hat{l}_a}{\partial y} \right) . \]  

(4.1)

This means that, for dissolving the singularity on the axis of \( n \)-quantum vortex, the point gap nodes within the core should cover the unit sphere \( n \) times. In Sec.5 we shall use this equation for the calculation of the spectral flow force on the moving vortex.
4.2 Vortices and vortex sheet in $^3$He-A.

The $^3$He-A is the superfluid with point gap nodes directed along the axis $\hat{l}$ of the spontaneous orbital momentum of the Cooper pair. According to Eq.(4.1) this means that the vortices in $^3$He-A can be continuous with an arbitrary size of the vortex core. Depending on the type of the vortex the core region is limited by the size of the container, by spin-orbital coupling or by external magnetic field. The Eq.(4.1) is the generalization of the Mermin-Ho relation first found for the $\hat{l}$ vector in $^3$He-A [44]. Until now four different types of vorticity have been observed and investigated in $^3$He-A in a rotating container [53]. Among them there are two types of continuous $n = 2$ vortex lines, the singular $n = 1$ vortex, and the vortex sheet [54]. All of them have broken axial and discrete symmetries.

The vortex sheet is represented by the soliton with accumulated vorticity. Each vortex in this structure is continuous $n = 1$ vortex, which can live only within the soliton; it represents some kind of the Bloch line which separates the parts of the soliton having degenerate mirror-reflected structures [16]. If the soliton is present in the vessel, then under acceleration of the container the "Bloch" vortices enter the soliton from the surface of container earlier than the bulk vortices can form. As a result, instead of formation of the conventional lattice of the isolated vortices, the vortex sheet grows and finally occupies the whole container. Such "Bloch" vortices within the domain walls or grain boundaries in superconductors were discussed in relation to the fractional magnetic flux [61, 62].

4.3 Vortices in heavy fermionic superconductors.

Different possible types of the broken symmetry in the vortices were discussed for the heavy fermionic superconductors (see Review [39]). An interesting possibility is the dissociation of the $n = 1$ vortex into two $n = 1/2$ vortices [89]. More on $1/2$ vortices see in Conclusion.

4.4 Broken symmetry within the cosmic strings.

The spontaneous breaking of the continuous symmetry in the cosmic strings has been discussed by Witten [87]. In this case the spontaneously broken symmetry is the electromagnetic symmetry $U(1)$. This implies that the core
of the string is superconducting: there are nondissipative current states in which the superconducting electric current along the vortex axis is concentrated within the core. In the $^{3}\text{He-B}$ vortices with broken axial symmetry such current states correspond to the twisting of the nonaxisymmetric vortex core; this spiral structure of the vortex core has been observed in NMR experiments \cite{27}.

The strings in the electroweak model are similar to the vortices in $^{3}\text{He-A}$ (see \cite{85}). Electroweak strings with different winding number $n$ have been considered in Ref. \cite{1}, they correspond to the $^{3}\text{He-A}$ vortices with broken parity.

4.5 Effect of the fermion zero modes on the core structure.

The fermion zero modes on vortices can also trigger the instability towards the symmetry breaking within the core. The vortex-core fermions which spectrum $E(p_z)$ crosses zero as function of $p_z$, form the 1D Fermi-liquid. The instabilities experienced by 1D fermionic system at low $T$ lead to the symmetry breaking in the vortex core. In the case of the electroweak strings this effect has been investigated in \cite{49,38}. The similar effect for the Abrikosov vortices in superconductors has been discussed in \cite{1}. Such instability is enhanced if instead of the 1D Fermi liquid there is a flat band of fermions in the vortex core. Such a flat band can exist in different types of symmetric vortices \cite{143}. It has been also discussed for the 0 − π Josephson contacts in high-temperature superconductors \cite{18}. The instability of the flat band can lead to the breaking of the time inversion with formation of the state with the big ferromagnetic moment in the Josephson junction \cite{18}.

The fermion zero modes, which spectrum $E(Q)$ crosses zero as a function of $Q$, also influence the core structure at low $T \ll T_c$ (though without symmetry breaking). An anomalously large slope of the gap amplitude $\Delta(r)$ at $r = 0$ was analytically found in \cite{34} and numerically confirmed in \cite{14}.

5 Spectral flow force on moving vortex

The fermion zero modes also influence the dynamics of the vortex leading to the special nondissipative force on the moving vortex \cite{80} which is differ-
ent from the conventional Magnus force. Let us first consider this force for the simple case when the singularity on the vortex core is smoothed by the formation of the region of the point gap nodes within the vortex core.

### 5.1 Point gap nodes and axial anomaly.

Near the $a$-th point gap node at $\p = \pm p_a \hat{l}_a$ the energy spectrum of the Bogoliubov excitation has the following general form:

$$E^2(p, r) = g^{jk}(r)(p_i - qA_i(r))(p_k - qA_k(r)) . \quad (5.1)$$

Here $A = p_a \hat{l}_a$, and $q = \pm 1$. This energy corresponds to that of the massless particle, with “electric” charge $q$, which moves in the “electromagnetic” field $A$ and in the gravity field described by the metric tensor $g^{ik}(r)$.

It also appears that these massless (gapless) quasiparticles in the vicinity of the point gap nodes are chiral: like neutrino they are either left-handed or right-handed [78]. For such gapless chiral fermions the phenomenon of the axial anomaly takes place [2, 4] which gives rise to the production of, say, left particles from the vacuum in the presence of the “electric” and “magnetic” fields. The total number of left particles produced by these fields per unit time is given by the Adler-Bell-Jackiw anomaly equation

$$\frac{1}{2\pi^2} \int d^3r \partial_t A \cdot (\nabla \times A) . \quad (5.2)$$

It is important that the left quasiparticle carries the linear momentum $p_a \hat{l}_a$ and the equal momentum is carried by the left quasihole. As a result the counterpart of the axial anomaly in condensed matter leads to the net product of the quasiparticle linear momentum in the time dependent texture [78]:

$$\partial_t \mathbf{P}_{qp} = \frac{1}{2\pi^2} \sum_a \int d^3r p_a \hat{l}_a (\partial_t A \cdot (\nabla \times A)) . \quad (5.3)$$

Since the total linear momentum is nevertheless conserved, this equation means that momentum is transferred from the collective variables describing the inhomogeneous superconductng state to the system of quasiparticles. This is the linear-momentum anomaly related to the fermions in superfluids, superconductors and ferromagnets, which was first identified in $^3$He-A [73, 50]. It is caused by the spectral flow of fermions through the point gap nodes.
5.2 Momentum exchange between the moving continuous vortex and the heat bath.

The spectral flow of fermions results in a curious exchange of the linear momentum between the moving vortex and the heat bath. Let us consider first the vortex with smoothened core. When the vortex moves with velocity $v_L$, the position of the gap nodes $\hat{\ell}_a$ texture becomes time dependent: $\hat{\ell}_a = \hat{\ell}_a(r - v_L t)$. As a result the “electric” field arises

$$E = \partial_t A = p_0 \partial_t \hat{l}_a = -p_0 (v_L \cdot \nabla) \hat{\ell}_a. \quad (5.4)$$

Here we for simplicity consider the isotropic case with coordinate independent magnitude of the momentum $p_a = p_0$. According to the Eq.(5.3) this leads to the production of quasiparticle momentum. This momentum is absorbed by the heat bath which moves with the velocity $v_n$ (for superconductors this is the velocity of the crystal lattice, while for superfluids the $v_n$ is the velocity of the normal component of the liquid). The force per unit length of the vortex acting on the vortex from the heat bath is

$$\mathbf{F}_{\text{spectral flow}} = \frac{p_0^3}{2\pi^2} \sum_a \int dx \, dy \, \hat{l}_a \cdot ((\nabla \times \hat{l}_a) \cdot ((v_L - v_n) \cdot \nabla) \hat{l}_a), \quad (5.5)$$

Simple transformation of this equation using the integration by parts gives $^7_9$

$$\mathbf{F}_{\text{spectral flow}} = \frac{p_0^3}{6\pi^2} (v_L - v_n) \times \hat{z} \sum_a \int dx \, dy \, \hat{l}_a \cdot (\partial_x \hat{l}_a \times \partial_y \hat{l}_a). \quad (5.6)$$

Finally using the Eq.(4.1) one obtains the anomaly contribution to the force acting on the vortex with winding number $n$:

$$\mathbf{F}_{\text{spectral flow}} = n\pi \frac{p_0^3}{3\pi^2} \hat{z} \times (v_n - v_L). \quad (5.7)$$

This force is reactive, i.e. nondissipative, since it is even under time inversion ($v \rightarrow -v$, $n \rightarrow -n$). It does not depend on the details of the vortex structure and is defined by the winding number $n$ of the vortex and by the magnitude of the momentum $p_0$ at which the gap node takes place. This stresses the topological origin of this anomalous force.
Similar nondissipative force on the magnetic vortices and skyrmions in magnets, first found in \[52\], was also interpreted in terms of the spectral flow of fermions in \[74, 65\].

The Eq.(5.7) can be extended to case of singular core, i.e. to the limit of zero radius of the region of gap nodes. In this limit the point gap nodes are collected on the vortex axis to make there the gapless Fermi surface and thus the normal Fermi liquid. The parameter \(p_0\) in Eq.(5.7) becomes the Fermi momentum \(p_F\) of this Fermi surface, and the factor \(p_F^3/3\pi^2\) transforms to the particle density on the vortex axis in accordance with suggestion made in \[1\]. However, as was shown by Kopnin \[29\], this result is model dependent.

### 5.3 Spectral flow in singular vortices

The above analysis is valid only in the model when the core size exceeds the coherence length in superconductors, which is always true for the \(^3\)He-A vortices. Here we calculate this spectral-flow force without using of the smooth model for vortex core. In the quantum-mechanical approach the effect occurs due to the spectral flow of the \(E(Q)\) zero modes and is suppressed if the spectral flow is not allowed due to the discrete nature of the quantum number \(Q\). If the spectral flow is not suppressed, the result for this force remains the same (if one neglects the small possible deviation of \(p_0\) from \(p_F\)) and does not depend on the core structure.

We shall consider here the conventional Abrikosov vortex in s-wave superconductor, but the result remains the same for arbitrary vortices. It can be applied to vortex in the p-wave \(^3\)He-B \[28\] as well as to the vortex in the d-wave superconductors: the complicated structure of the latter vortex \[84, 5, 11, 63, 19\] should not influence the topological result. Note that for the s-wave superconductor the result was first obtained long ago in Ref. \[30\] in a rigorous microscopic theory.

If the vortex moves with the velocity \(v_L\) with respect to the heat bath, the coordinate \(r\) is replaced by \(r - v_L t\) and the impact parameter \(y\) which enters the quasiparticle energy in Eq.(3.4) shifts with time. So the energy of zero mode becomes

\[
E(Q, p_z, t) = -n(y - \frac{\epsilon(q)}{q}t)q\omega_0(p_z) = -n(Q - \epsilon(q)t)\omega_0(p_z) \quad . \tag{5.8}
\]

Here \(\epsilon(q) = \hat{z} \cdot ((v_L - v_n) \times \mathbf{q})\) acts on fermions localized in the core in the
same way that an electric field acts on the fermions localized on a string in relativistic quantum theory. The only difference is that under this “electric” field the spectral flow occurs along $Q$ rather than along $p_z$. Along this path the fermionic levels cross the zero energy level at the rate

$$\partial_t Q = \epsilon(q) = \hat{z} \cdot ((v_L - v_n) \times q).$$

This leads to the quasiparticle momentum transfer from the vacuum (from the levels below zero) along the anomalous branch into the heat bath. This occurs at the rate

$$\partial_t P = \sum_p p \partial_t Q = \frac{1}{2} N_{zm} \int_{-p_F}^{p_F} \frac{dp_z}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} q\epsilon(q) = \pi n \frac{p_F^2}{3\pi^2} \hat{z} \times (v_n - v_L),$$

where the factor $\frac{1}{2}$ compensates the double counting of particles and holes, and we used the index theorem that the number of anomalous branches (fermion zero modes) is related to the winding number: $N_{zm} = 2n$. The Eq.(5.10) is the result, which was obtained in Sec.5.2 within the model of smooth core.

Here it is implied that all the quasiparticles, created from the negative levels of the vacuum state, finally become the part of the normal component outside the core. This means that there is nearly reversible transfer of linear momentum from the core fermions to the heat bath, which is valid only in the limit of large scattering rate: $\omega_0 \tau \ll 1$, where $\tau$ is the lifetime of the fermion on the level $Q$. The irreversibility of the momentum transfer leads to the small friction force $F_{\text{friction}} \propto \omega_0 \tau (v_n - v_L)$. The condition $\omega_0 \tau \ll 1$, which states that the interlevel distance on the zero mode branch is small compared to the life time of the level, is the crucial requirement for spectral flow to exist. In the opposite limit $\omega_0 \tau \gg 1$ the spectral flow is suppressed and the corresponding spectral flow force is exponentially small.

5.4 Magnus and spectral-flow forces

The reactive force $F_{\text{spectral flow}}$ from the heat bath on the moving vortex is the consequence of the reversible flux of momentum from the vortex into the region near the axis, i.e. into the core region. Within the core the linear momentum of the vortex transforms to the linear momentum of the fermions in the heat bath when the fermionic levels on anomalous branches cross zero.
The Magnus force comes from the flux of linear momentum from the vortex to infinity. The topological origin of this force is similar to the Aharonov-Bohm effect. This force can be obtained as a sum of the forces acting on the individual particles according to the equation

$$\partial_t p = (\vec{\nabla} \times \vec{v}_s) \times p, \quad (5.11)$$

where $p$ is the particle momentum and the vorticity $\vec{\nabla} \times \vec{v}_s = n \frac{\pi}{m} \delta_2(\mathbf{r})$ is concentrated in a thin tube (vortex core). The force on the vortex is

$$-\sum_p n(p) \partial_t p = -\left( \int d^2r (\vec{\nabla} \times \vec{v}_s) \right) \times \int \frac{d^3p}{4\pi^2} p n(p) = n\pi \hat{z} \times \mathbf{j}, \quad (5.12)$$

Here $n(p)$ is the particle distribution function and $\mathbf{j}$ is the total particle current in the frame of the moving vortex. This force can be expressed through the superfluid $\vec{v}_s$ and normal $\vec{v}_n$ velocities as a sum of the Magnus and Iordanskii forces \[20,21\]

$$\mathbf{F}_{\text{Magnus}} + \mathbf{F}_{\text{Iordanskii}} = n\pi \frac{\rho_s}{m} \hat{z} \times (\vec{v}_L - \vec{v}_s) + n\pi \frac{\rho_n}{m} \hat{z} \times (\vec{v}_L - \vec{v}_n). \quad (5.13)$$

On the connection between the Iordanskii force and the Aharonov-Bohm effect has been pointed out in Ref. \[64\]. Similar effect for the spinning cosmic string was discussed in \[15\].

The two forces in Eq.(5.13) exist both in Bose and Fermi superfluids, as distinct from the spectral-flow force which originates from the fermion zero modes in the core. The latter is temperature independent \[40\], if the condition $\omega_0 \tau \ll 1$ is satisfied. When $\omega_0 \tau$ is not small, then far from $T_c$ the spectral-flow force can be approximated by the following equation \[33,31\]:

$$\mathbf{F}_{\text{spectral flow}} = n\pi \frac{\rho_0^3}{3\pi^2} \frac{1}{1 + \omega_0^2 \tau^2} \hat{z} \times (\vec{v}_n - \vec{v}_L). \quad (5.14)$$

The balance of three nondissipative forces in Eqs.(5.14-15) acting on the vortex and of the friction force, which is $\propto \vec{v}_n - \vec{v}_L$, governs the low-frequency dynamics of the vortex. The spectral-flow force disappears in the limit $\omega_0 \tau \gg 1$. On the other hand, if the spectral-flow force has its largest value, it nearly compensates the Magnus force, since the total mass density $\rho = \rho_s + \rho_n$, which enters Magnus+Iordanskii force, is very close to the parameter $mp_L^2/(3\pi^2)$,
which enters the spectral-flow force. That is why the spectral flow within the core plays a very important part in the vortex dynamics and can lead to the change of the sign of the Hall effect [70, 9].

Note that the spectral-flow parameter $p_0/(3\pi^2)$ is close to $\rho/m$ only in the weak coupling limit, where the difference between these parameters is caused by a tiny asymmetry between particles and holes near the Fermi surface. If the interaction which leads to the Cooper pairing is strong enough, $p_0/(3\pi^2)$ can essentially deviate from $\rho/m$ and even become zero [78]. In the latter case the spectral-flow force disappears completely. The situation is exactly the same as for the problem of the intrinsic angular momentum in $^3$He-A, which magnitude $L_0$ is essentially modified by the spectral flow: instead of the nominal value $L_0 = (\hbar/2)\rho/m$ in the Bose liquid with the symmetry of $^3$He-A one has $L_0 = (\hbar/2)[\rho/m - p_0/(3\pi^2)]$ in the Fermi liquid.

In $^3$He-B the spectral-flow contribution to the nondissipative force has been measured [6]: the experimental temperature dependence follows the theoretical one found in [33, 28].

The spectral-flow force has been also calculated for Josephson vortices in SNS junction [42]: it almost cancels the Magnus force, again due to approximate particle-hole symmetry. This possibly provides an explanation for the experimental evidence of the negligible Magnus force in 2D Josephson junction arrays [7, 36, 72].

6 Conclusions

The fermion zero modes in the bulk condensed matter and within the topological objects are of great importance at low $T$, where they lead to different anomalies. In particle physics the anomaly related to the spectral flow could give rise to the baryogenesis at the early stage of the expanding universe [88]. In condensed matter the similar phenomenon is responsible for the “momentogenesis - production of the linear momentum during the vortex motion, which results in additional force acting on the vortex line.

An open question is the behavior of the fermions in the presence of new topological object in high-temperature superconductor – the half-quantum vortex, which has been discussed in [12] and recently observed in [26]. One may expect different interesting phenomena related to such vortices. For the case of superfluid $^3$He-A, where such vortices can also exist [83] but
still have not been identified, they are the counterpart of the Alice string in some models of particle physics. The person travelling around Alice string finds himself in the mirror reflected world \[60\]. In superfluid \(^3\)He-A, the analogous effect is the reversal of the spin of the quasiparticle upon circling the 1/2 vortex. The peculiar Aharonov-Bohm effect for 1/2 vortices has been discussed for \(^3\)He-A \[23, 58\] and modified for the Alice string \[43, 8\].

The 1/2 vortex observed in high-temperature superconductor carries 1/2 of the magnetic flux quantum \(\Phi_0 = h/2e\) \[26\]. In principle the objects with the fractional flux below \(\Phi_0/2\) are also possible \[82, 61, 62\]. They can arise if the time inversion symmetry is broken. The behavior of the fermions in the background of these and other extended topological objects in superconductors, superfluids, magnets and Quantum Hall systems is of primary interest.

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