Interpolating sequences and Carleson measures in the Hardy-Sobolev spaces of the ball in $\mathbb{C}^n$. 

E. Amar

En l’honneur de Aline Bonami, Orléans, Juin 2014.
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$$\|f\|_{s,p}^p := \sup_{r<1} \int_{\partial \mathbb{B}} |(I + R)^s f(rz)|^p \, d\sigma(z),$$

where $I$ is the identity, $d\sigma$ is the Lebesgue measure on $\partial \mathbb{B}$ and $R$ is the radial derivative $Rf(z) = \sum_{j=1}^n z_j \partial f/\partial z_j(z)$.

For $s \in \mathbb{N}$, this norm is equivalent to

$$\|f\|_{s,p}^p := \max_{0 \leq j \leq s} \int_{\partial \mathbb{B}} |R^j f(z)|^p \, d\sigma(z),$$

This means that $R^j f \in H^p_s(\mathbb{B})$, $j = 0, \ldots, s$.

For $s = 0$ these spaces are the classical Hardy spaces $H^p(\mathbb{B})$ of the unit ball $\mathbb{B}$.
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Let $p'$ the conjugate exponent for $p$; the Hilbert space $H_s^2$ is equipped with reproducing kernels:

$$\forall a \in \mathbb{B}, \quad k_a(z) = \frac{1}{(1 - \bar{a} \cdot z)^{n-2s}},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product of the Hilbert space $H_s^2$. In the case $s = n/2$ there is a log for $k_a$. 

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Interpolating sequences and Carleson m

En l’honneur de Aline Bonami, Orleans

3/25
Let $p'$ the conjugate exponent for $p$; the Hilbert space $H^2_s$ is equipped with reproducing kernels:

$$\forall a \in \mathbb{B}, \quad k_a(z) = \frac{1}{(1 - \overline{a} \cdot z)^{n-2s}}, \quad \|k_a\|_{s,p} := \|k_a\|_{H^p_s} \simeq (1 - |a|^2)^{s-n/p'}$$
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Definition

*The measure* \( \mu \) *in* \( \mathbb{B} \) *is Carleson for* \( H^p_s \), \( \mu \in C_{s,p} \), *if we have the embedding*

\[
\forall f \in H^p_s, \quad \int_{\mathbb{B}} |f|^p \, d\mu \leq C\|f\|_{s,p}^p.
\]

| \( H^p_s(\mathbb{B}) \) | \( H^p_0(\mathbb{B}) \) | \( H^p_s(\mathbb{B}) \) |
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| Characterized geometrically by L. Carleson | Characterized geometrically by L. Hörmander | Studied by C. Cascante & J. Ortega; characterized for \( n-1 \leq ps \leq n \). |
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| Amer. J. Math. (1958) | Math. Scand. (1967) | Amer. J. Math. (2012) |
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The measure $\mu$ in $\mathbb{B}$ is Carleson for $H^p_s$, $\mu \in C_{s,p}$, if we have the embedding

$$\forall f \in H^p_s, \int_B |f|^p d\mu \leq C\|f\|^{p}_{s,p}.$$ 

We have the table concerning the Carleson measures:

| $H^p_s(D)$ | $H^p_s(B)$ | $H^p_0(B)$ |
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Definition

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$$\forall \lambda \in \ell^p(S), \exists f \in H^p_s(\mathbb{B}) :: \forall a \in S, f(a) = \lambda_a \| k_a \|_{s,p'}, \| f \|_{H^p_s} \leq C \| \lambda \|_p.$$
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Definition

The sequence $S$ of points in $\mathbb{B}$ is dual bounded (or minimal, or weakly interpolating) in $H^p_s(\mathbb{B})$, DB, if there is a bounded sequence $\{\rho_a\}_{a \in S} \subset H^p_s$ such that
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Let $S$ be an interpolating sequence in $H^p_s$ we say that $S$ has a bounded linear extension operator, BLEO, if there is a a bounded linear operator $E : \ell^p(S) \to H^p_s$ and a $C > 0$ such that
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\forall \lambda \in \ell^p(S), E(\lambda) \in H^p_s, \|E(\lambda)\|_{H^p_s} \leq C\|\lambda\|_p : \forall a \in S, E(\lambda)(a) = \lambda a \|k_a\|_{s,p'}.
$$
We have the table

\[
\begin{array}{|c|c|c|}
\hline
H^p(\mathbb{D}) & H^p(\mathbb{B}) & H^p_s(\mathbb{B}), s > 0 \\
\hline
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We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
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| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|------------------|-----------------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields\(^4\) for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer\(^5\) for $p = 2$ with $n - 1 < 2s \leq n$ |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ | \(\text{P} \cdot \text{Beurling}^6 \text{ for } p = \infty\) |

\(^4\) Amer. J. Math. (1961)  
\(^5\) Mem. Amer. Math. Soc. (2006)  
\(^6\) Preprint Uppsala (1962)
We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|-------------------|-----------------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields\(^4\) for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer\(^5\) for $p = 2$ $n - 1 < 2s \leq n$ |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ P. Beurling\(^6\) for $p = \infty$ E. A. for $p < \infty$ | | |

\(^4\) Amer. J. Math. (1961)  
\(^5\) Mem. Amer. Math. Soc. (2006)  
\(^6\) Preprint Uppsala (1962)
We have the table

| $H^p(\mathbb{D})$                          | $H^p(\mathbb{B})$  | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------------------------------|-------------------|-----------------------------|
| IS characterized by                      | IS no characterized | IS characterized           |
| L. Carleson for $p = \infty$              |                   | by Arcozzi Rochberg &       |
| and by Shapiro & Shields$^4$ for any $p$  |                   | Sawyer$^5$ for $p = 2$      |
| Same for all $p$                          | Depending on $p$  | $n - 1 < 2s \leq n$        |
| IS $\Rightarrow$ BLEO for all $p$        |                   | Depending on $p$            |
| P . Beurling$^6$ for $p = \infty$         | IS $H^\infty \Rightarrow$ BLEO |
| E . A. for $p < \infty$                   |                   | A. Bernard$^7$              |

$^4$Amer. J. Math. (1961)
$^5$Mem. Amer. Math. Soc. (2006)
$^6$Preprint Uppsala (1962)
$^7$CRAS (1971)
We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|--------------------|-----------------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields$^4$ for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer$^5$ for $p = 2$ $n - 1 < 2s \leq n$ |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ P. Beurling$^6$ for $p = \infty$ E. A. for $p < \infty$ | IS $H^\infty \Rightarrow$ BLEO A. Bernard$^7$ | IS $H^p \Rightarrow ??$ |

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$^4$Amer. J. Math. (1961)
$^5$Mem. Amer. Math. Soc. (2006)
$^6$Preprint Uppsala (1962)
$^7$CRAS (1971)
We have the table

| $H^p(\mathbb{D})$       | $H^p(\mathbb{B})$       | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------------|--------------------------|----------------------------|
| IS characterized by    | IS no characterized      | IS characterized          |
| L. Carleson for $p = \infty$ and by Shapiro & Shields\(^4\) for any $p$ |                           | by Arcozzi Rochberg & Sawyer\(^5\) for $p = 2$ |
| Same for all $p$       | Depending on $p$         | $n - 1 < 2s \leq n$       |
| IS $\Rightarrow$ BLEO for all $p$ | IS $H^\infty \Rightarrow$ BLEO |                           |
| P. Beurling\(^6\) for $p = \infty$ | A. Bernard\(^7\)         | $?? \; p \neq 1, 2$ |
| E. A. for $p < \infty$ | IS $H^p \Rightarrow ??$ |                           |

\(^4\) Amer. J. Math. (1961)  
\(^5\) Mem. Amer. Math. Soc. (2006)  
\(^6\) Preprint Uppsala (1962)  
\(^7\) CRAS (1971)
We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B})$, $s > 0$ |
|-------------------|-------------------|-----------------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields\(^4\) for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer\(^5\) for $p = 2$ $n - 1 < 2s \leq n$ |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ P. Beurling\(^6\) for $p = \infty$ E. A. for $p < \infty$ | IS $H^\infty \Rightarrow$ BLEO A. Bernard\(^7\) IS $H^p \Rightarrow$ ?? | ?? $p \neq 1, 2$ |
| DB $H^p \Rightarrow$ IS $H^q$, $\forall q \leq \infty$ by Shapiro & Shieds | |

\(^4\)Amer. J. Math. (1961)  
\(^5\)Mem. Amer. Math. Soc. (2006)  
\(^6\)Preprint Uppsala (1962)  
\(^7\)CRAS (1971)
We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|-------------------|-------------------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields$^4$ for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer$^5$ for $p = 2$ $n - 1 < 2s \leq n$ |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ P. Beurling$^6$ for $p = \infty$ E. A. for $p < \infty$ | IS $H^\infty \Rightarrow$ BLEO A. Bernard$^7$ IS $H^p \Rightarrow ??$ | ?? $p \neq 1, 2$ |
| DB $H^p \Rightarrow IS H^q, \forall q \leq \infty$ by Shapiro & Shieds | DB $H^p \Rightarrow IS H^q, \forall q < p$ with BLEO ($q = p?$) by E. A |

$^4$Amer. J. Math. (1961)  
$^5$Mem. Amer. Math. Soc. (2006)  
$^6$Preprint Uppsala (1962)  
$^7$CRAS (1971)
We have the table

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B})$, $s > 0$ |
|-------------------|-----------------|------------------|
| IS characterized by L. Carleson for $p = \infty$ and by Shapiro & Shields\(^4\) for any $p$ | IS no characterized | IS characterized by Arcozzi Rochberg & Sawyer\(^5\) for $p = 2$ \[n - 1 < 2s \leq n\] |
| Same for all $p$ | Depending on $p$ | Depending on $p$ |
| IS $\Rightarrow$ BLEO for all $p$ P . Beurling\(^6\) for $p = \infty$ E . A. for $p < \infty$ | IS $H^\infty \Rightarrow$ BLEO A. Bernard\(^7\) IS $H^p \Rightarrow ??$ | $??$ $p \neq 1, 2$ |
| DB $H^p \Rightarrow IS$ $H^q$, $\forall q \leq \infty$ by Shapiro & Shieds | DB $H^p \Rightarrow IS$ $H^q$, $\forall q < p$ with BLEO ($q = p$?) by E. A | Next Theorem |

\(^4\) Amer. J. Math. (1961)  
\(^5\) Mem. Amer. Math. Soc. (2006)  
\(^6\) Preprint Uppsala (1962)  
\(^7\) CRAS (1971)
Definition

The sequence \( S \) is Carleson, \( CS \), in \( H^p_s(\mathbb{B}) \), if the associated measure
\[ \nu_S := \sum_{a \in S} \| k_{s,a} \|_{s,p}^{-p} \delta_a \]
is Carleson for \( H^p_s(\mathbb{B}) \).
Definition

The sequence $S$ is Carleson, CS, in $H^p_s(\mathbb{B})$, if the associated measure

$$\nu_S := \sum_{a \in S} \|k_{s,a}\|_{s,p}^{-p} \delta_a$$

is Carleson for $H^p_s(\mathbb{B})$.

Theorem

Let $S$ be a sequence of points in $\mathbb{B}$ such that
**Definition**

*The sequence* $S$ *is Carleson, CS, in* $H^p_s(B)$, *if the associated measure*

$$\nu_S := \sum_{a \in S} ||k_{s,a}||_{s,p'}^{-p} \delta_a$$

*is Carleson for* $H^p_s(B)$.

**Theorem**

*Let* $S$ *be a sequence of points in* $B$ *such that*

- **there is a sequence** $\{\rho_a\}_{a \in S}$ *in* $H^p_s$ *such that*

  $$\forall a, b \in S, \quad \rho_a(b) \simeq \delta_{ab} ||\rho_a||_{s,p} ||k_a||_{s,p'}.$$
Definition

The sequence $S$ is Carleson, CS, in $H^p_s(B)$, if the associated measure

$$\nu_S := \sum_{a \in S} \|k_{s,a}\|_{s,p'}^{-p}\delta_a$$

is Carleson for $H^p_s(B)$.

Theorem

Let $S$ be a sequence of points in $B$ such that

- there is a sequence $\{\rho_a\}_{a \in S}$ in $H^p_s$ such that
  \[\forall a, b \in S, \rho_a(b) \simeq \delta_{ab}\|\rho_a\|_{s,p}\|k_a\|_{s,p'}\cdot\]
- If $0 < s < \frac{n}{2} \min(\frac{1}{p'}, \frac{1}{q'})$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, i.e. $s < \frac{n}{2p'}$ and
  \[\frac{p}{2} < r < p,\] we have
Definition

The sequence $S$ is Carleson, CS, in $H^p_s(B)$, if the associated measure

$$\nu_S := \sum_{a \in S} \|k_{s,a}\|_{s,p'}^{-p} \delta_a$$

is Carleson for $H^p_s(B)$.

Theorem

Let $S$ be a sequence of points in $B$ such that

- there is a sequence $\{\rho_a\}_{a \in S}$ in $H^p_s$ such that
  $$\forall a, b \in S, \quad \rho_a(b) \simeq \delta_{ab} \|\rho_a\|_{s,p} \|k_a\|_{s,p'}.$$  

- If $0 < s < \frac{n}{2} \min\left(\frac{1}{p'}, \frac{1}{q'}\right)$ with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), i.e. $s < \frac{n}{2p'}$ and $\frac{p}{2} < r < p$, we have

  $$\forall j \leq s, \quad \|R^j(\rho_a)\|_p \lesssim \|R^j(k_a)\|_p \Rightarrow \|\rho_a\|_{s,p} \lesssim \|k_a\|_{s,p}.$$
**Definition**

The sequence $S$ is Carleson, CS, in $H^p_s(\mathbb{B})$, if the associated measure

$$\nu_S := \sum_{a \in S} \|k_{s,a}\|_{s,p'}^{-p} \delta_a$$

is Carleson for $H^p_s(\mathbb{B})$.

**Theorem**

Let $S$ be a sequence of points in $\mathbb{B}$ such that

- there is a sequence $\{\rho_a\}_{a \in S}$ in $H^p_s$ such that
  $$\forall a, b \in S, \quad \rho_a(b) \simeq \delta_{ab} \|\rho_a\|_{s,p} \|k_a\|_{s,p'}.$$  
- If $0 < s < \frac{n}{2} \min\left(\frac{1}{p'}, \frac{1}{q'}\right)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, i.e. $s < \frac{n}{2p'}$ and
  $$\frac{p}{2} < r < p,$$
  we have
  $$\forall j \leq s, \quad \|R^j(\rho_a)\|_p \lesssim \|R^j(k_a)\|_p \Rightarrow \|\rho_a\|_{s,p} \lesssim \|k_a\|_{s,p}.$$  
- $S$ is Carleson in $H^q_s(\mathbb{B})$. 

En l’honneur de Aline Bonami, Orleans 7/25
Definition

The sequence \( S \) is Carleson, \( \text{CS} \), in \( H^p_s(\mathbb{B}) \), if the associated measure

\[ \nu_S := \sum_{a \in S} \| k_{s,a} \|_{s,p'}^{-p} \delta_a \]

is Carleson for \( H^p_s(\mathbb{B}) \).

Theorem

Let \( S \) be a sequence of points in \( \mathbb{B} \) such that

- there is a sequence \( \{ \rho_a \}_{a \in S} \) in \( H^p_s \) such that
  \[ \forall a, b \in S, \rho_a(b) \simeq \delta_{ab} \| \rho_a \|_{s,p} \| k_a \|_{s,p'}^{r}. \]
- If \( 0 < s < \frac{n}{2} \min \left( \frac{1}{p'}, \frac{1}{q'} \right) \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), i.e. \( s < \frac{n}{2p'} \) and \( \frac{p}{2} < r < p \), we have
  \[ \forall j \leq s, \| R^j(\rho_a) \|_p \lesssim \| R^j(k_a) \|_p \Rightarrow \| \rho_a \|_{s,p} \lesssim \| k_a \|_{s,p}. \]
- \( S \) is Carleson in \( H^q_s(\mathbb{B}) \).

Then \( S \) is \( \text{H}^r_s \) interpolating with the bounded linear extension property, provided that \( p \leq 2 \).
The table relative to Carleson sequences is
The table relative to Carleson sequences is

| $H^p(D)$ | $H^p(B)$ | $H^p_s(B), s > 0$ |
|----------|----------|--------------------|
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|-------------------|-----------------------------|
| IS $H^p \Rightarrow$ CS by L. Carleson |                   |                             |
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B})$, $s > 0$ |
|------------------|------------------|-----------------------------|
| IS $H^p \Rightarrow$ CS by L. Carleson | IS $H^p \Rightarrow$ CS by P. Thomas\(^8\) | |

\(^8\)Indagationes Math. (1987)
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|------------------|------------------|------------------|
| IS $H^p \Rightarrow$ CS by L. Carleson | IS $H^p \Rightarrow$ CS by P. Thomas$^8$ | IS $H^2_s \Rightarrow$ CS $H^2_s$ for $n - 1 < 2s \leq n$ by A.R.S |

$^8$Indagationes Math. (1987)
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B})$, $s > 0$ |
|------------------|------------------|-------------------------------|
| IS $H^p \Rightarrow$ CS by L. Carleson | IS $H^p \Rightarrow$ CS by P. Thomas$^8$ | IS $H^2_s \Rightarrow$ CS $H^2_s$ for $n - 1 < 2s \leq n$ by A.R.S |
| DB $H^p \Rightarrow$ IS $H^q \Rightarrow$ CS by Shapiro & Shieds | | |

$^8$Indagationes Math. (1987)
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------|-------------------|---------------------------|
| IS $H^p \Rightarrow$ CS by L. Carleson | IS $H^p \Rightarrow$ CS by P. Thomas$^8$ | IS $H^2_s \Rightarrow$ CS $H^2_s$ for $n - 1 < 2s \leq n$ by A.R.S |
| DB $H^p \Rightarrow$ IS $H^q \Rightarrow$ CS by Shapiro & Shieds | DB $H^p \Rightarrow$ CS by E.A. | |

$^8$Indagationes Math. (1987)
The table relative to Carleson sequences is

| $H^p(\mathbb{D})$                   | $H^p(\mathbb{B})$                   | $H^p_s(\mathbb{B}), s > 0$ |
|-------------------------------------|-------------------------------------|-----------------------------|
| IS $H^p \Rightarrow$ CS            | IS $H^p \Rightarrow$ CS            | IS $H^s_2 \Rightarrow$ CS $H^s_2$ for $n - 1 < 2s \leq n$ by A.R.S |
| by L. Carleson                      | by P. Thomas$^8$                    |                             |
| DB $H^p \Rightarrow$ IS $H^q \Rightarrow$ CS | DB $H^p \Rightarrow$ CS           |                             |
| by Shapiro & Shieds                 | by E.A.                             |                             |

$^8$Indagationes Math. (1987)
Definition

The multipliers algebra $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \ m h \in H_s^p.$$
Definition

The **multipliers algebra** $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \ mh \in H_s^p.$$  

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$.  

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Theorem 9.25

E. Amar

Interpolating sequences and Carleson measures in the Hardy-Sobolev spaces of the ball in $\mathbb{C}^n$. En l’honneur de Aline Bonami, Orléans. 9/25
Definition

The multipliers algebra $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \ mh \in H_s^p.$$ 

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$. 

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H_s^p(\mathbb{B})$ |
**Definition**

The multipliers algebra $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \, mh \in H_s^p.$$ 

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$.

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| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H_s^p(\mathbb{B})$ |
|-------------------|-------------------|-------------------|
| $\mathcal{M}_0^p(\mathbb{D}) = H^\infty(\mathbb{D})$, $\forall p$ |                     |                   |
The multipliers algebra $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \ mh \in H_s^p.$$ 

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$.

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H_s^p(\mathbb{B})$ |
|-------------------|-------------------|-------------------|
| $\mathcal{M}_0^p(\mathbb{D}) = H^\infty(\mathbb{D}), \forall p$ | $\mathcal{M}_0^p(\mathbb{B}) = H^\infty(\mathbb{B}), \forall p$ |
**Definition**

The **multipliers algebra** $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p, \ mh \in H_s^p.$$  

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$.

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H_s^p(\mathbb{B})$ |
|-------------------|-------------------|-------------------|
| $\mathcal{M}_0^p(\mathbb{D}) = H^\infty(\mathbb{D})$, $\forall p$ | $\mathcal{M}_0^p(\mathbb{B}) = H^\infty(\mathbb{B})$, $\forall p$ | $\mathcal{M}_s^p = H^\infty(\mathbb{B}) \cap C.C.$ characterized for $n - 1 \leq ps \leq n$ |
Definition

The multipliers algebra $\mathcal{M}_s^p$ of $H_s^p$ is the algebra of functions $m$ on $\mathbb{B}$ such that

$$\forall h \in H_s^p,\; mh \in H_s^p.$$ 

The norm of a multiplier is its norm as an operator from $H_s^p$ into $H_s^p$.

| $H^p(\mathbb{D})$ | $H^p(\mathbb{B})$ | $H_s^p(\mathbb{B})$ |
|-------------------|-------------------|-------------------|
| $\mathcal{M}_0^p(\mathbb{D}) = H^\infty(\mathbb{D}), \; \forall p$ | $\mathcal{M}_0^p(\mathbb{B}) = H^\infty(\mathbb{B}), \; \forall p$ | $\mathcal{M}_s^p = H^\infty(\mathbb{B}) \cap C.C.$ characterized for $n - 1 \leq ps \leq n$ and for $p = 2$ by V. W. Depending on $p$ |
Definition

The sequence $S$ of points in $\mathbb{B}$ is **interpolating**, IS, in the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ if there is a $C > 0$ such that

\[
\forall \lambda \in \ell^\infty(S), \; \exists m \in \mathcal{M}_s^p :: \forall a \in S, \; m(a) = \lambda a \text{ and } \|m\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_\infty.
\]
Definition

The sequence $S$ of points in $\mathbb{B}$ is interpolating, IS, in the multipliers algebra $\mathcal{M}^p_s$ of $H^p_s(\mathbb{B})$ if there is a $C > 0$ such that

$$\forall \lambda \in \ell^\infty(S), \exists m \in \mathcal{M}^p_s:: \forall a \in S, \ m(a) = \lambda a \text{ and } \|m\|_{\mathcal{M}^p_s} \leq C\|\lambda\|_\infty.$$ 

Definition

Let $S$ be an interpolating sequence in $\mathcal{M}^p_s$;
Definition

The sequence $S$ of points in $\mathbb{B}$ is interpolating, IS, in the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ if there is a $C > 0$ such that
$$\forall \lambda \in \ell^\infty(S), \exists m \in \mathcal{M}_s^p : \forall a \in S, m(a) = \lambda a \text{ and } \|m\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_\infty.$$ 

Definition

Let $S$ be an interpolating sequence in $\mathcal{M}_s^p$; we say that $S$ has a bounded linear extension operator, BLEO, if there is a a bounded linear operator $E : \ell^\infty(S) \to \mathcal{M}_s^p$ and a $C > 0$ such that
$$\forall \lambda \in \ell^\infty(S), \ E(\lambda) \in \mathcal{M}_s^p, \ \|E(\lambda)\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_\infty : \forall a \in S, \ E(\lambda)(a) = \lambda a.$$
\[ H^\infty(\mathbb{D}) \quad H^\infty(\mathbb{B}) \quad M^p_s(\mathbb{B}) \]

Characterized by L. Carleson for \( p = 2 \) and \( n - 1 < 2s \leq n \) by A.R.S. and the Pick property. Characterized for \( p \geq 2 \) by E.A.

Theorem: If \( S \) is interpolating for \( M^p_s(\mathbb{B}) \) and \( p \geq 2 \), then \( S \) has a bounded linear extension operator.
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_p^s(\mathbb{B})$ |
|--------------------------|--------------------------|-----------------|
| IS characterized by L. Carleson | | |

Theorem

If $S$ is interpolating for $\mathcal{M}_p^s(\mathbb{B})$ and $p \geq 2$, then $S$ has a bounded linear extension operator.
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|----------------------|----------------------|----------------------|
| IS characterized by L. Carleson | No characterisation | |

Theorem

If $S$ is interpolating for $M^p_s(\mathbb{B})$ and $p \geq 2$, then $S$ has a bounded linear extension operator.
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $M^p_s(\mathbb{B})$ |
|------------------------|------------------------|------------------------|
| IS characterized by L. Carleson | No characterisation | Characterized for $p = 2$ and $n - 1 < 2s \leq n$ by A.R.S. and the Pick property |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}^p_s(\mathbb{B})$ |
|----------------------|----------------------|----------------------|
| IS characterized by L. Carleson | No characterisation | Characterized for $p = 2$ and $n - 1 < 2s \leq n$ by A.R.S. and the Pick property |
| IS $\Rightarrow$ BLEO by P. Beurling |                      |                      |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|-----------------|-----------------|-----------------|
| IS characterized by L. Carleson | No characterisation | Characterized for $p = 2$ and $n - 1 < 2s \leq n$ by A.R.S. and the Pick property |
| IS $\Rightarrow$ BLEO by P. Beurling | IS $\Rightarrow$ BLEO by A. Bernard | |
| $H^\infty(\mathbb{D})$                 | $H^\infty(\mathbb{B})$                      | $\mathcal{M}_s^p(\mathbb{B})$ |
|---------------------------------------|---------------------------------------------|---------------------------------|
| IS characterized by L. Carleson       | No characterisation                         | Characterized for $p = 2$ and $n - 1 < 2s \leq n$ by A.R.S. and the Pick property |
| IS $\Rightarrow$ BLEO by P. Beurling  | IS $\Rightarrow$ BLEO by A. Bernard         | IS $\Rightarrow$ BLEO for $p \geq 2$ by E. A. |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|---------------------|---------------------|---------------------|
| IS characterized by L. Carleson | No characterisation | Characterized for $p = 2$ and $n - 1 < 2s \leq n$ by A.R.S. and the Pick property |
| IS $\Rightarrow$ BLEO by P. Beurling | IS $\Rightarrow$ BLEO by A. Bernard | IS $\Rightarrow$ BLEO for $p \geq 2$ by E. A. |

**Theorem**

*If $S$ is interpolating for $\mathcal{M}_s^p$ and $p \geq 2$, then $S$ has a bounded linear extension operator.*
**Definition**

The sequence $S$ of points in $\mathbb{B}$ is **dual bounded** (or minimal, or weakly interpolating) in the multipliers algebra $\mathcal{M}^p_s$ of $H^p_s(\mathbb{B})$ if there is a bounded sequence $\{\rho_a\}_{a \in S} \subset \mathcal{M}^p_s$ such that

$\forall a, b \in S, \rho_a(b) = \delta_{ab}$ and $\exists C > 0: \forall a \in S, \|\rho_a\| \leq C$.
Definition

The sequence $S$ of points in $\mathbb{B}$ is dual bounded (or minimal, or weakly interpolating) in the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ if there is a bounded sequence $\{\rho_a\}_{a \in S} \subset \mathcal{M}_s^p$ such that
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The sequence $S$ of points in $\mathbb{B}$ is dual bounded (or minimal, or weakly interpolating) in the multipliers algebra $\mathcal{M}^p_s$ of $H^p_s(\mathbb{B})$ if there is a bounded sequence $\{\rho_a\}_{a \in S} \subset \mathcal{M}^p_s$ such that

$$\forall a, b \in S, \rho_a(b) = \delta_{ab} \text{ and } \exists C > 0 : \forall a \in S, \|\rho_a\| \leq C.$$ 

If $S$ is interpolating in $\mathcal{M}^p_s$ then it is clearly dual bounded.
**Definition**

The sequence $S$ of points in $\mathbb{B}$ is **dual bounded** (or minimal, or weakly interpolating) in the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ if there is a bounded sequence $\{\rho_a\}_{a \in S} \subset \mathcal{M}_s^p$ such that

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If $S$ is interpolating in $\mathcal{M}_s^p$ then it is clearly dual bounded.

**Definition**

The sequence $S$ of points in $\mathbb{B}$ is **$\delta$ separated** in $H_s^p$ if

$$\forall a, b \in S, a \neq b, \exists f \in H_s^p : f(a) = 0, f(b) = \|k_a\|_{s,p'}, \|f\|_{s,p} \leq \delta^{-1}.$$
\[ H^\infty(\mathbb{D}) \quad H^\infty(\mathbb{B}) \quad \mathcal{M}_s^p(\mathbb{B}) \]
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|---------------------|---------------------|---------------------|
| DB $H^\infty \Rightarrow IS H^p$ | | |
| $\forall p \leq \infty$ with BLEO by Carleson, Shapiro & Shields | | |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}^p_s(\mathbb{B})$ |
|-----------------------|-----------------------|-----------------------|
| DB $H^\infty \Rightarrow IS H^p$  
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|------------------------|------------------------|------------------------|
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$\forall p < \infty$ with BLEO  
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for $p \geq 2$ with BLEO  
by E. A. |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|----------------|----------------|----------------|
| DB $H^\infty \Rightarrow IS H^p$ | DB $H^\infty \Rightarrow IS H^p$ | IS $\mathcal{M}_s^p \Rightarrow IS H_s^p$ for $p \geq 2$ with BLEO by E. A. |
| $\forall p \leq \infty$ with BLEO by Carleson, Shapiro & Shields | $\forall p < \infty$ with BLEO by E. A. | |
| IS $H^\infty \Rightarrow CS$ by Carleson | | |
| \(H^\infty(\mathbb{D})\) | \(H^\infty(\mathbb{B})\) | \(\mathcal{M}^p(\mathbb{B})\) |
|-----------------|-----------------|-----------------|
| DB \(H^\infty \Rightarrow IS H^p\) \(\forall p \leq \infty\) with BLEO by Carleson, Shapiro & Shields | DB \(H^\infty \Rightarrow IS H^p\) \(\forall p < \infty\) with BLEO by E. A. | IS \(\mathcal{M}^p \Rightarrow IS H^p_s\) for \(p \geq 2\) with BLEO by E. A. |
| IS \(H^\infty \Rightarrow CS\) by Carleson | IS \(H^\infty \Rightarrow CS\) by Varopoulos\(^9\) | |

\(^9\)CRAS (1972)
\[
\begin{array}{|c|c|c|}
\hline
H^\infty(\mathbb{D}) & H^\infty(\mathbb{B}) & \mathcal{M}_s^p(\mathbb{B}) \\
\hline
\text{DB } H^\infty \Rightarrow IS \ H^p \\
\forall p \leq \infty \text{ with BLEO} \\
\text{by Carleson, Shapiro & Shields} & \text{DB } H^\infty \Rightarrow IS \ H^p \\
\forall p < \infty \text{ with BLEO} & \forall p < \infty \text{ with BLEO} \\
\text{by E. A.} & IS \ M_s^p \Rightarrow IS \ H_s^p \\
& \text{for } p \geq 2 \text{ with BLEO} \\
& \text{by E. A.} \\
\hline
\text{IS } H^\infty \Rightarrow CS \\
\text{by Carleson} & IS \ H^\infty \Rightarrow CS \\
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\hline
\end{array}
\]

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\[
\begin{array}{|c|c|c|}
\hline
H^\infty(\mathbb{D}) & H^\infty(\mathbb{B}) & \mathcal{M}^p_s(\mathbb{B}) \\
\hline
\text{DB } H^\infty \Rightarrow \text{IS } H^p & \text{DB } H^\infty \Rightarrow \text{IS } H^p & \text{IS } \mathcal{M}^p_s \Rightarrow \text{IS } H^p_s \\
\forall p \leq \infty \text{ with BLEO} & \forall p < \infty \text{ with BLEO} & \text{for } p \geq 2 \text{ with BLEO} \\
\text{by Carleson, Shapiro & Shields} & \text{by E. A.} & \text{by E. A.} \\
\hline
\text{IS } H^\infty \Rightarrow \text{CS} & \text{IS } H^\infty \Rightarrow \text{CS} & \text{IS } \mathcal{M}^p_s \Rightarrow \text{CS } H^p_s \\
\text{by Carleson} & \text{by Varopoulos}^9 & \text{by E. A.} \\
\hline
\end{array}
\]

\(^9\text{CRAS (1972)}\)
Theorem

Let $S$ be an interpolating sequence for the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ then $S$ is also an interpolating sequence for $H_s^p$ provided that $p \geq 2$. 
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Let $S$ be an interpolating sequence for $\mathcal{M}_s^p$ then $S$ is Carleson $H^p_s(\mathbb{B})$. 
Theorem

Let $S_1$ and $S_2$ be two interpolating sequences in $M_{ps}$ such that $S := S_1 \cup S_2$ is separated, then $S$ is still an interpolating sequence in $M_{ps}$, provided that $s = 1$.\[10\]

CRAS (1971)
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|----------------------|----------------------|----------------------|

Theorem

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(CRAS (1971) 15/25)

E. Amar

Interpolating sequences and Carleson measures in the Hardy-Sobolev spaces of the ball in $\mathbb{C}^n$. En l'honneur de Aline Bonami, Orleans
| $H^\infty(D)$ | $H^\infty(B)$ | $\mathcal{M}_s^p(B)$ |
|-------------|-------------|-------------|
| Separated union of IS is IS, by L. Carleson | | |
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}_s^p(\mathbb{B})$ |
|---------------------|---------------------|----------------------------|
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\textsuperscript{10} CRAS (1971)
| $H^\infty(\mathbb{D})$ | $H^\infty(\mathbb{B})$ | $\mathcal{M}^p_s(\mathbb{B})$ |
|------------------------|------------------------|-------------------------------|
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**Theorem**

Let $S_1$ and $S_2$ be two interpolating sequences in $\mathcal{M}_s^p$ such that $S := S_1 \cup S_2$ is separated, then $S$ is still an interpolating sequence in $\mathcal{M}_s^p$.

---

$^{10}$CRAS (1971)
| \(H^\infty(\mathbb{D})\) | \(H^\infty(\mathbb{B})\) | \(\mathcal{M}_s^p(\mathbb{B})\) |
|------------------|------------------|------------------|
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**Theorem**

*Let \(S_1\) and \(S_2\) be two interpolating sequences in \(\mathcal{M}_s^p\) such that \(S := S_1 \cup S_2\) is separated, then \(S\) is still an interpolating sequence in \(\mathcal{M}_s^p\), provided that \(s = 1\).*

\(^{10}\)CRAS (1971)
Theorem

Let $\sigma_1$ and $\sigma_2$ be two interpolating sequences in the spectrum of the commutative algebra of operators $A$, such that $\sigma := \sigma_1 \cup \sigma_2$ is separated, then $\sigma$ is an interpolating sequence for $A$. 
Theorem

Let $\sigma_1$ and $\sigma_2$ be two interpolating sequences in the spectrum of the commutative algebra of operators $A$, such that $\sigma := \sigma_1 \cup \sigma_2$ is separated, then $\sigma$ is an interpolating sequence for $A$.

Corollary

Let $S_1$ and $S_2$ be two interpolating sequences in $M^2_s$ such that $S := S_1 \cup S_2$ is separated, then $S$ is still an interpolating sequence in $M^2_s$. 

HarmonicAnalysis
Thank you!
An ounce of probability.
An ounce of probability.

We shall prove:

**Theorem**

Let $S$ be a dual bounded sequence of points in $\mathbb{B}$ for $H^p(\mathbb{B})$. 

Proof. We already know that $S$ is Carleson, which means 

$$\|\sum_{a \in S} \nu a k_{a,q}\|_{H^q} \lesssim \|\nu\|_{\ell^q},$$

with the reproducing kernel:

$$k_a := (1 - \bar{a} \cdot z^n), \quad k_{a,q} := \frac{k_a}{\|k_a\|_{H^q}}.$$
An ounce of probability.

We shall prove:

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Let $S$ be a dual bounded sequence of points in $\mathbb{B}$ for $H^p(\mathbb{B})$. Then $S$ is $H^r$ interpolating with the bounded linear extension property, provided that $r < p \leq 2$. 
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An ounce of probability.

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**Proof.**

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The hypothesis means that there is a sequence \( \{\rho_a\}_{a \in S} \subset H^p \) such that
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\exists C > 0, \forall a \in S, \|\rho_a\|_p \leq C, \forall b \in S, \rho_a(b) = \delta_{a,b} \|k_a\|_{p'}.
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Let \( \lambda \in \ell^r(S) \) to have that \( S \) is IS \( H^r \) means that there is an 
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h \in H^r : \quad \forall a \in S, \quad h(a) = \lambda_a \| k_a \|_{r'}.
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Choose \( q \) such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) which is possible because \( r < p \), and set
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h(z) := \sum_{a \in S} \lambda_a \rho_a(z) k_{a,q}(z).
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We have \( h(b) = \lambda_b\rho_b(b)k_{b,q}(b) \simeq \lambda_b\|k_b\|_{r'} \) by a simple computation.
The hypothesis means that there is a sequence \( \{\rho_a\}_{a \in S} \subset H^p \) such that
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\]

We have \( h(b) = \lambda_b \rho_b(b) k_{b,q}(b) \approx \lambda_b \|k_b\|_{r'} \) by a simple computation. So it remains to evaluate the norm of \( h \) in \( H^r \).
Write $\lambda_a = \mu_a \nu_a$, with

$\mu_a := |\lambda_a|^{r/p}$, $\nu_a := |\lambda_a|^{r/q} \frac{\lambda_a}{|\lambda_a|} \Rightarrow \|\mu\|_{\ell^p} = \|\nu\|_{\ell^q} = \|\lambda\|_{\ell^r}$; then

$$h(z) := \sum_{a \in S} \mu_a \rho_a(z) \nu_a k_{a,q}(z)$$
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\[ h(z) := \sum_{a \in S} \mu_a \rho_a(z) \nu_a k_{a,q}(z) \]

and the idea is to write this sum of products as a product of sums
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$$h(z) = \mathbb{E}((\sum_{a \in S} \mu_a \epsilon_a \rho_a(z)) \times (\sum_{a \in S} \nu_a \epsilon_a k_{a,q}(z)))$$
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because $\mathbb{E}(\epsilon_a \epsilon_b) = \delta_{a,b}$. 

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because $\mathbb{E}(\epsilon_a \epsilon_b) = \delta_{a,b}$.

So, by Fubini and Hölder,

$$\|h\|_{H^r} = \mathbb{E}(\int_{\partial \mathbb{B}} |f|^r |g|^r \ d\sigma) = \int_{\Omega \times \partial \mathbb{B}} |f|^r |g|^r \ dP \otimes d\sigma.$$
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because $\mathbb{E}(\epsilon_a \epsilon_b) = \delta_{a,b}$.

So, by Fubini and Hölder,

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$$\leq \left( \int_{\Omega \times \partial \mathbb{B}} |f|^p \ dP \otimes d\sigma \right)^{r/p} \left( \int_{\Omega \times \partial \mathbb{B}} |g|^q \ dP \otimes d\sigma \right)^{r/q}.$$
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$$I = \int_{\Omega \times \partial B} |f|^p \, dP \otimes d\sigma = \int_{\partial B} \mathbb{E}(|f|^p),$$

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$$\mathbb{E}(|f|^p) = \mathbb{E}\left(\left| \sum_{a \in S} \mu_a \epsilon_a \rho_a(z) \right|^p\right) \sim \left(\sum_{a \in S} |\mu_a|^2 |\rho_a(z)|^2\right)^{p/2}$$

and, for $p \leq 2$,

$$\left(\sum_{a \in S} |\mu_a|^2 |\rho_a(z)|^2\right)^{p/2} \lesssim \sum_{a \in S} |\mu_a|^p |\rho_a(z)|^p$$
For $I$ we have

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and
\[ \|h\|_{H^{r}} \leq I^{1/p} J^{1/q} \lesssim (\|\mu\|_{\ell^{p}}^{1/p}(\|\nu\|_{\ell^{q}}^{1/q})^{1/q} \leq \|\lambda\|_{\ell^{r}}. \]
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\gamma(l, \cdot) := \frac{1}{N} \sum_{j=1}^{N} \theta^{-jl} \beta(j, \cdot) \in \mathcal{M}_s^p \Rightarrow \| \gamma(l, \cdot) \|_{\mathcal{M}_s^p} \leq C(S).
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This is the Fourier transform, on the group of $n$th roots of unity, of the function $\beta(j, \cdot)$, i.e. $\gamma(l, z) = \hat{\beta}(l, z)$, the parameter $z \in \mathbb{B}$ being fixed. We also have convolution becoming product by Fourier:

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**Lemma**

*We have, for \( j \leq s, \ k \in \mathbb{N}, \)*

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And this is the ”miracle lemma” we use to get our results.
Thank you!
