CARTIER AND WEIL DIVISORS ON VARIETIES WITH QUOTIENT SINGULARITIES

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Abstract. The main goal of this paper is to show that the notions of Weil and Cartier \(\mathbb{Q}\)-divisors coincide for \(V\)-manifolds and give a procedure to express a rational Weil divisor as a rational Cartier divisor. The theory is illustrated on weighted projective spaces and weighted blow-ups.

Contents

Introduction 1
1. \(V\)-Manifolds and Quotient Singularities 2
2. Weighted Projective Spaces 5
3. Weighted Blow-ups and Embedded \(\mathbb{Q}\)-Resolutions 6
4. Cartier and Weil Divisors on \(V\)-Manifolds: \(\mathbb{Q}\)-Divisors 10
5. Conclusions and Future Work 15
References 16

Introduction

Singularity Theory deals with the study of complex spaces with non smooth points. The most powerful tool to study singularities is resolution, i.e. a proper analytic morphism from a smooth variety which is an isomorphism outside the singular locus. The existence of such resolutions is guaranteed by the work of Hironaka. Usually the combinatorics of the exceptional divisor of the resolution is quite complicated.

There is a class of singularities which is well understood, namely normal singularities which are obtained as the quotient of a ball in \(\mathbb{C}^n\) by the linear action of a finite group. Spaces admitting only such singularities are called \(V\)-manifolds and were introduced by Bailly. These varieties share several properties with manifolds, e.g. projective \(V\)-manifolds also carry a Hodge structure and a natural notion of normal crossing divisors can be defined (called \(\mathbb{Q}\)-normal crossing divisors). As it will be developed in the second author’s Ph.D. thesis, the study of the so-called \(\mathbb{Q}\)-resolutions (allowing quotient singularities, especially abelian) provides a better understanding of some families of singularities.

Intersection theory is needed for this study; it is also required to deal with divisors on \(V\)-manifolds. Two kinds of divisors appear in the literature: Weil and Cartier divisors. Weil divisors are formal sums of hypersurfaces and Cartier divisors are global sections of the quotient sheaf of meromorphic functions modulo non-vanishing holomorphic functions. The relationship between Cartier divisors and line bundles provides a nice way to define the intersection multiplicity of two divisors.
In the smooth category, both notions coincide but this is not the case for singular varieties (not even for normal ones). One can also consider Weil and Cartier \( \mathbb{Q} \)-divisors (tensorizing the corresponding groups by \( \mathbb{Q} \)). The main result of this paper is that these two notions coincide for \( V \)-manifolds and hence an intersection theory can be developed for Weil \( \mathbb{Q} \)-divisors using the theory of line bundles.

This result is probably known for specialists but we have not found a proof in the literature. There are some partial results for toric varieties (defined with simplicial cones). Moreover, in this work we give an algorithm to represent a Weil \( \mathbb{Q} \)-divisor as a Cartier \( \mathbb{Q} \)-divisor. We illustrate the use of this algorithm in an example living in a space obtained after a weighted blow-up. Such blow-ups can be understood from toric geometry but in this work they are presented in a more geometric way, generalizing standard blow-ups. For this definition, we need to describe abelian quotient singularities. In a forthcoming work, we will develop an intersection theory for surfaces with abelian quotient singularities.

The paper is organized as follows. In §1 we give a general presentation of varieties with quotient singularities and list their basic properties. In §2 we deal with weighted projective spaces and in §3 we introduce weighted-blow-ups. The main results about rational Weil and Cartier divisors on \( V \)-manifolds are stated and proven in §4. We finish with some conclusions and the expected future work.

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1. \( V \)-Manifolds and Quotient Singularities

**Definition 1.1.** A \( V \)-manifold of dimension \( n \) is a complex analytic space which admits an open covering \( \{U_i\} \) such that \( U_i \) is analytically isomorphic to \( B_i/G_i \) where \( B_i \subset \mathbb{C}^n \) is an open ball and \( G_i \) is a finite subgroup of \( GL(n, \mathbb{C}) \).

The concept of \( V \)-manifolds was introduced in [9] and they have the same homological properties over \( \mathbb{Q} \) as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler, see [2]. They have been locally classified by Prill [8]. To state this local result we need the following.

**Definition 1.2.** A finite subgroup \( G \) of \( GL(n, \mathbb{C}) \) is called *small* if no element of \( G \) has \( 1 \) as an eigenvalue of multiplicity precisely \( n-1 \), that is, \( G \) does not contain rotations around hyperplanes other than the identity.

For every finite subgroup \( G \) of \( GL(n, \mathbb{C}) \) denote by \( G_{\text{big}} \) the normal subgroup of \( G \) generated by all rotations around hyperplanes. Then the \( G_{\text{big}} \)-invariant polynomials form a polynomial algebra and hence \( \mathbb{C}^n/G_{\text{big}} \) is isomorphic to \( \mathbb{C}^n \).

The group \( G/G_{\text{big}} \) maps isomorphically to a small subgroup of \( GL(n, \mathbb{C}) \), once a basis of invariant polynomials has been chosen. Hence the local classification of \( V \)-manifolds reduces to the classification of actions of small subgroups of \( GL(n, \mathbb{C}) \).

**Theorem 1.3.** ([8]). Let \( G_1, G_2 \) be small subgroups of \( GL(n, \mathbb{C}) \). Then \( \mathbb{C}^n/G_1 \) is isomorphic to \( \mathbb{C}^n/G_2 \) if and only if \( G_1 \) and \( G_2 \) are conjugate subgroups.

We are interested in \( V \)-manifolds where the quotient spaces \( B_i/G_i \) are given by (finite) abelian groups. In this case the following notation is used.

**1.4.** For \( d := (d_1 \ldots d_r) \) we denote \( \mu_d := \mu_{d_1} \times \cdots \times \mu_{d_r} \), a finite abelian group written as a product of finite cyclic groups, that is, \( \mu_{d_i} \) is the cyclic group of \( d_i \)-th
roots of unity in \( \mathbb{C} \). Consider a matrix of weight vectors

\[ A := (a_{ij})_{i,j} = [a_i | \cdots | a_n] \in \text{Mat}(r \times n, \mathbb{Z}), \quad a_j := \begin{pmatrix} a_{1j} & \cdots & a_{rj} \end{pmatrix} \in \text{Mat}(r \times 1, \mathbb{Z}), \]

and the action

\[ (\mu_{d_1} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (\xi_d, x) \mapsto (\xi_d^{a_{11}}x_1, \xi_d^{a_{12}}x_2, \ldots, \xi_d^{a_{1n}}x_n), \quad x := (x_1, \ldots, x_n). \]

Note that the \( i \)-th row of the matrix \( A \) can be considered modulo \( d_i \). The set of all orbits \( \mathbb{C}^n/G \) is called \((cyclic) \) quotient space of type \((d; A)\) and it is denoted by

\[ X(d; A) := \begin{pmatrix} d_1 & a_{11} & \cdots & a_{1n} \\
 & \vdots & \ddots & \vdots \\
 & a_{r1} & \cdots & a_{rn} \end{pmatrix}. \]

The orbit of an element \( x \in \mathbb{C}^n \) under this action is denoted by \([x]_{(d; A)}\) and the subindex is omitted if no ambiguity seems likely to arise. Using multi-index notation the action takes the simple form

\[ \mu_d \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (\xi_d, x) \mapsto \xi_d \cdot x := (\xi_d^{a_{11}}x_1, \ldots, \xi_d^{a_{1n}}x_n). \]

The following classic result shows that the family of varieties which can locally be written like \( X(d; A) \) is exactly the same as the family of \( V \)-manifolds with abelian quotient singularities.

**Lemma 1.5.** Let \( G \) be a finite abelian subgroup of \( GL(n, \mathbb{C}) \). Then \( \mathbb{C}^n/G \) is isomorphic to some quotient space of type \((d; A)\).

**Proof.** The group \( G \) can be seen as a direct product of cyclic groups generated by matrices \( M_1, \ldots, M_r \) of orders \( d_1, \ldots, d_r \) such that

\[ G = \{ M_1^{i_1} \cdots M_r^{i_r} | i_k = 0, \ldots, d_k - 1 \}. \]

Each of these matrices \( M_i, i = 1, \ldots, r, \) is conjugated to \( \text{Diag}(\zeta_{d_1}^{a_{i1}}, \ldots, \zeta_{d_n}^{a_{in}}) \) where \( \zeta_{d_i} \) is a \( d_i \)-th primitive root of unity. Moreover, they are simultaneously diagonalizable because they commute. This proves \( \mathbb{C}^n/G \simeq X(d; (a_{ij})_{i,j}) \).

Different types \((d; A)\) can give rise to isomorphic quotient spaces, see Remark 1.6. We shall prove that they can always be represented by an upper triangular matrix of dimension \((n - 1) \times n\), see Lemmas 1.8 and 1.9. Finding a simpler type \((d; A)\) to represent a quotient space will lead us to the notions of normalized type and space, see Definition 1.11.

**Remark 1.6.** For \( n = 3 \) the simple group automorphism on \( \mu_d \times \mu_d \) given by \((\xi, \eta) \mapsto (\xi \eta^{-1}, \eta)\) shows that the following two spaces are isomorphic under the identity map:

\[ X \left( \begin{pmatrix} d & a_{11} & a_{12} & a_{13} \\
 & d & a_{21} & a_{22} & a_{23} \end{pmatrix} \right) = X \left( \begin{pmatrix} d & a_{11} & a_{12} & a_{13} \\
 & d & a_{21} - a_{11} & a_{22} - a_{12} & a_{23} - a_{13} \end{pmatrix} \right). \]

Note that the determinants of the minors of order 2 are the same in both side of the previous equation. There is an obvious generalization for higher dimensions, allowing row operations when the two rows correspond to the same cyclic group. This is not a strong condition since we can also multiply any row (of \( d \) and \( A \) simultaneously) by an arbitrary integer.

**Example 1.7.** When \( n = 1 \) all spaces \( X(d; A) \) are isomorphic to \( \mathbb{C} \). Dividing each row by \( \gcd(d_i, a_i) \) does not change the quotient and we can work with \( X(\mu(d_1, \ldots, d_r); (a_{ij})_{i,j}) \) where \( \gcd(d_i, a_i) = 1 \).
The map \([x] \mapsto x^{d_1}\) gives an isomorphism between \(X(d_1; a_1)\) and \(C\). For \(r = 2\) one has that (we write the symbol \(\sim\) when the isomorphism is induced by the identity map)

\[
\frac{C}{\mu_{d_1} / \mu_{d_2}} \cong \frac{C / \mu_{d_1}}{\mu_{d_2}} \cong \frac{X(d_2; a_2d_1)}{\mu_{d_2}} \cong C,
\]

\([x] \mapsto x^{d_1}, \quad [x] \mapsto x^{\frac{a_2d_1}{\mu_{d_2}}}.
\]

To see the equality (\(*\)) observe that \(\xi_{d_2} \cdot x^{d_1} \equiv \xi_{d_2} \cdot [x] = [e^{d_2}_d x] \equiv \xi_{d_2}^{a_2d_1} x^{d_1}\). It follows that the corresponding quotient space is isomorphic to \(C\) under the map \([x] \mapsto x^{\text{lcm}(d_1, d_2)}\).

From Remark 1.6 and Example 1.7 the following lemmas follow easily.

**Lemma 1.8.** The following operations do not change the isomorphism type of \(X(d; A)\):

1. Permutation \(\sigma\) of columns of \(A\), \([x] \mapsto [(x_{\sigma(1)}, \ldots, x_{\sigma(n)})]\).
2. Permutation of rows of \(\langle d; A \rangle\), \([x] \mapsto [x]\).
3. Multiplication of a row of \(\langle d; A \rangle\) by a positive integer, \([x] \mapsto [x]\).
4. Multiplication of a row of \(A\) by an integer coprime with the corresponding row in \(d\), \([x] \mapsto [x]\).
5. Replace \(a_{ij}\) by \(a_{ij} + kd_j\), \([x] \mapsto [x]\).
6. If \(e\) is coprime with \(a_{1,n}\), and divides \(d_1\) and \(a_{1,j}\), \(1 \leq j < n\), then replace, \(a_{i,n} \mapsto ea_{i,n}\), \([x] \mapsto [(x_1, \ldots, x_n)]\).
7. If \(d_1 = 1\) then eliminate the last row, \([x] \mapsto [x]\).

**Lemma 1.9.** The space \(X(d; A) = C^n / \mu_d\) can always be represented by an upper triangular matrix of dimension \((n - 1) \times n\). More precisely, there exist a vector \(e := (e_1, \ldots, e_{n-1})\), a matrix \(B = (b_{i,j})_{i,j}\), and an isomorphism \([(x_1, \ldots, x_n)] \mapsto [(x_1, \ldots, x_n)]\) for some \(k \in \mathbb{N}\) such that

\[
X(d; A) \cong \begin{pmatrix}
  e_1 & b_{1,1} & \cdots & b_{1,n-1} & b_{1,n} \\
  \vdots & \ddots & \ddots & \vdots & \vdots \\
  e_{n-1} & \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & b_{n-1,n-1} & b_{n-1,n}
\end{pmatrix} = X(e; B).
\]

1.10. The action shown in Equation (1) of 1.4 is free on \((C^*)^n\), i.e \(x \in (C^*)^n, \xi_d \cdot x = x \implies \xi_d = 1\), if and only if the group homomorphism \(\mu_d \to GL(n, C)\) given by

\[
\xi_d = (\xi_{d_1}, \ldots, \xi_{d_r}) \mapsto \text{Diag}(\xi_{d_1}^{a_1}, \ldots, \xi_{d_r}^{a_r})
\]

is injective. If this is not the case, let \(H\) be the kernel of this group homomorphism. Then \(C^n / H \cong C^n\) and the group \(\mu_d / H\) acts freely on \((C^*)^n\) under the previous identification. Thus one can always assume that the free and the small conditions are satisfied. This motivates the following definition.

**Definition 1.11.** The type \((d; A)\) is said to be normalized if the following two conditions hold:

1. The action is free on \((C^*)^n\).
2. The group \(\mu_d\) is identified with a small subgroup of \(GL(n, C)\) under the group homomorphism given in Equation (2) of 1.10.

By abuse of language we often say the space \(X(d; A)\) is written in a normalized form when we mean the type \((d; A)\) is normalized.

**Proposition 1.12.** The space \(X(d; A)\) is written in a normalized form if and only if the stabilizer subgroup of \(P\) is trivial for all \(P \in C^n\) with exactly \(n - 1\) coordinates different from zero.
In the cyclic case the stabilizer of a point as above (with exactly \( n - 1 \) coordinates different from zero) has order \( \gcd(d, a_1, \ldots, a_n) \).

Using Lemma \( \text{[13]} \) it is possible to convert general types \((d; A)\) into their normalized form. Theorem \( \text{[13]} \) allows one to decide whether two quotient spaces are isomorphic. In particular one can use this result to compute the singular points of the space \( X(d; A) \) and Proposition \( \text{[11]} \) to decide if a given cyclic type is normalized.

In Example \( \text{[17]} \) we explain the previous normalization process in dimension one. The two and three-dimensional cases are treated in the following examples.

**Example 1.13.** Following Lemma \( \text{[13]} \) all quotient spaces for \( n = 2 \) are cyclic. The space \( X(d; a, b) \) is written in a normalized form if and only if \( \gcd(d, a) = \gcd(d, b) = 1 \). If this is not the case, one uses the isomorphism \( \text{[1]} \) (assuming \( \gcd(d, a, b) = 1 \))

\[
X(d; a, b) \to X \left( \frac{d}{\gcd(d, a, b)} : \frac{a}{\gcd(d, a, b)} : \frac{b}{\gcd(d, a, b)} \right),
\]

\[
[(x, y)] \mapsto [(x^{(d, a)}, y^{(d, b)})]
\]
to convert it into a normalized one, see also Lemma \( \text{[18]} \).

**Example 1.14.** The quotient space \( X(d; a, b, c) \) is written in a normalized form if and only if \( \gcd(d, a, b) = \gcd(d, a, c) = \gcd(d, b, c) = 1 \). As above, isomorphisms of the form \([(x, y, z)] \mapsto [(x, y, z^k)]\) can be used to convert types \((d; a, b, c)\) into their normalized form.

In \( \text{[2]} \) Fujiki computes resolutions of these cyclic quotient singularities and also studies, among others, the properties shown above.

### 2. Weighted Projective Spaces

Weighted projective spaces are the main examples of \( V \)-manifolds. The main reference that has been used in this section is \( \text{[3]} \). Here we concentrate our attention on describing the analytic structure and singularities.

Let \( \omega := (q_0, \ldots, q_n) \) be a weight vector, that is, a finite set of coprime positive integers. There is a natural action of the multiplicative group \( \mathbb{C}^\times \) on \( \mathbb{C}^{n+1} \setminus \{0\} \) given by

\[
(x_0, \ldots, x_n) \mapsto (t^{q_0}x_0, \ldots, t^{q_n}x_n).
\]
The set of orbits \( \mathbb{P}^{n+1}_\omega \) under this action is denoted by \( \mathbb{P}^{n+1}_\omega \) (or \( \mathbb{P}^n(\omega) \) in case of complicated weight vectors) and is called the **weighted projective space** of type \( \omega \).
The class of a nonzero element \( (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \) is denoted by \( [x_0 : \ldots : x_n]_\omega \) and the weight vector is omitted if no ambiguity seems likely to arise. When \( (q_0, \ldots, q_n) = (1, \ldots, 1) \) one obtains the usual projective space and the weight vector is always omitted. For \( x \in \mathbb{C}^{n+1} \setminus \{0\} \), the closure of \([x]_\omega \) in \( \mathbb{C}^{n+1} \) is obtained by adding the origin and it is an algebraic curve.

#### 2.1. Analytic structure.

As in the classical case, weighted projective spaces can be endowed with an analytic structure. However, in general they contain cyclic quotient singularities. To understand this structure, consider the decomposition \( \mathbb{P}^n = U_0 \cup \cdots \cup U_n \), where \( U_i \) is the open set consisting of all elements \([x_0 : \ldots : x_n]_\omega \) with \( x_i \neq 0 \). The map

\[
\tilde{\psi}_0 : \mathbb{C}^n \to U_0, \quad \tilde{\psi}_0(x_1, \ldots, x_n) := [1 : x_1 : \ldots : x_n]_\omega
\]
is clearly a surjective analytic map but it is not a chart since injectivity fails. In fact, \([1 : x_1 : \ldots : x_n]_\omega = [1 : x'_1 : \ldots : x'_n]_\omega \) if and only if there exists \( \xi \in \mu_{q_0} \) such that

\[
(\xi^d, \xi^{a_i}, \xi^{b_i}) \in \mathbb{P}^{n+1}_\omega.
\]

\(^{1}\)The notation \((i_1, \ldots, i_k) = \gcd(i_1, \ldots, i_k)\) is used in case of complicated or long formulas.
that $x_i = \xi^i x_i$ for all $i = 1, \ldots, n$. Hence the map above induces the isomorphism
\[ \psi_0 : X(q_0, q_1, \ldots, q_n) \rightarrow U_0, \]
\[ [(x_1, \ldots, x_n)] \mapsto [1 : x_1 : \ldots : x_n]_\omega. \]
Analogously, $X(q_i, q_0, \ldots, \hat{q}_i, \ldots, q_n) \cong U_i$ under the obvious analytic map. Since the transition maps are analytic, $\mathbb{P}^n_\omega$ is an analytic space with cyclic quotient singularities as claimed.

### 2.2. Simplifying the weights.
For different weight vectors $\omega$ and $\omega'$ the corresponding spaces $\mathbb{P}^n_\omega$ and $\mathbb{P}^n_{\omega'}$ can be isomorphic. Consider
\[ d_i := \gcd(q_0, \ldots, \hat{q}_i, \ldots, q_n), \]
\[ e_i := \text{lcm}(d_0, \ldots, \hat{d}_i, \ldots, d_n). \]
Note that $e_i | q_i$, $\gcd(d_i, d_j) = 1$ for $i \neq j$, and $\gcd(e_i, d_i) = 1$.

**Proposition 2.3.** Using the notation above, the following map is an isomorphism:
\[ \mathbb{P}^n(q_0, q_1, \ldots, q_n) \rightarrow \mathbb{P}^n\left(\frac{q_0}{e_0}, \ldots, \frac{q_n}{e_n}\right), \]
\[ [x_0 : \ldots : x_n] \mapsto \left[\frac{x_0}{e_0} : \ldots : \frac{x_n}{e_n}\right]. \]

**Proof.** Since $\gcd(q_i, d_i) = 1$ we have $e_i = d_0 \ldots \hat{d}_i \ldots d_n$. Now from Lemma 1.8 one has the following sequence of isomorphisms of analytic spaces:
\[ X(q_0, q_1, \ldots, q_n) \cong X(q_0, \frac{q_1}{d_1} : \ldots : \frac{q_n}{d_n}) \]
\[ X(q_0, \frac{q_1}{d_1} : \ldots : \frac{q_n}{d_n}) \cong X(q_0, \frac{q_1}{d_2 d_3} : \ldots : \frac{q_n}{d_2 d_3 \ldots d_n}) \]
\[ \ldots \cong X(q_0, \frac{q_1}{e_n} : \frac{q_2}{e_1 e_2} : \ldots : \frac{q_n}{e_1 e_2 \ldots e_n}) \]
Observe that in the $i$-th step, we divide the corresponding weight vector by $d_i$ except the $i$-th coordinate and hence the associated map is
\[ [(x_1, \ldots, x_i, \ldots, x_n)] \mapsto [(x_1, \ldots, x_i^{d_i}, \ldots, x_n)]. \]
Therefore $[1 : x_1 : \ldots : x_n]_\omega \mapsto [1 : x_1^{d_i} : \ldots : x_n^{d_i}]_\omega$ is an isomorphism by composition. Analogously one proceeds with the other charts. \qed

**Remark 2.4.** Note that, due to the preceding proposition, one can always assume the weight vector satisfies $\gcd(q_0, \ldots, \hat{q}_i, \ldots, q_n) = 1$, for $i = 0, \ldots, n$. In particular, $\mathbb{P}^1(q_0, q_1) \cong \mathbb{P}^1$ and for $n = 2$ we can take $(q_0, q_1, q_2)$ relatively prime numbers. In higher dimension the situation is a bit more complicated.

### 3. Weighted Blow-ups and Embedded Q-Resolutions
Classically an embedded resolution of $\{f = 0\} \subset \mathbb{C}^n$ is a proper map $\pi : X \rightarrow (\mathbb{C}^n, 0)$ from a smooth variety $X$ satisfying, among other conditions, that $\pi^{-1}(\{f = 0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space $X$ to contain abelian quotient singularities and the divisor $\pi^{-1}(\{f = 0\})$ to have normal crossings over this kind of varieties. This notion of normal crossing divisor on $V$-manifolds was first introduced by Steenbrink in [10].

**Definition 3.1.** Let $X$ be a $V$-manifold with abelian quotient singularities. A hypersurface $D$ on $X$ is said to be with $Q$-normal crossings if it is locally isomorphic to the quotient of a union of coordinate hyperplanes under a group action of type $(d; A)$. That is, given $x \in X$, there is an isomorphism of germs $(X, x) \cong (X(d; A), [0])$ such that $\pi^{-1}(\{f = 0\})$ to have normal crossings over this kind of varieties. This notion of normal crossing divisor on $V$-manifolds was first introduced by Steenbrink in [10].
Let $M = \mathbb{C}^{n+1}/\mu_d$ be an abelian quotient space not necessarily cyclic or written in normalized form. Consider $H \subset M$ an analytic subvariety of codimension one.

**Definition 3.2.** An embedded $\mathbb{Q}$-resolution of $(H, 0) \subset (M, 0)$ is a proper analytic map $\pi : X \to (M, 0)$ such that:

1. $X$ is a $V$-manifold with abelian quotient singularities.
2. $\pi$ is an isomorphism over $X \setminus \pi^{-1}(\text{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on $X$.

**Remark 3.3.** Let $f : (M, 0) \to (\mathbb{C}, 0)$ be a non-constant analytic function germ. Consider $(H, 0)$ the hypersurface defined by $f$ on $(M, 0)$. Let $\pi : X \to (M, 0)$ be an embedded $\mathbb{Q}$-resolution of $(H, 0) \subset (M, 0)$. Then $\pi^{-1}(H) = (f \circ \pi)^{-1}(0)$ is locally given by a function of the form $x_1^{m_1} \cdots x_k^{m_k} : X(d; A) \to \mathbb{C}$ for some type $(d; A)$.

**3.4. Classical blow-up of $\mathbb{C}^{n+1}$.** Using multi-index notation we consider

$$\hat{\mathbb{C}}^{n+1} := \{([x, [u]] \in \mathbb{C}^{n+1} \times \mathbb{P}^n | x \in [u])\}.$$

Then $\pi : \hat{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ is an isomorphism over $\hat{\mathbb{C}}^{n+1} \setminus \pi^{-1}(0)$. The exceptional divisor $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^n$. The space $\hat{\mathbb{C}}^{n+1} = U_0 \cup \cdots \cup U_n$ can be covered with $n + 1$ charts each of them isomorphic to $\mathbb{C}^{n+1}$. For instance, the following map defines an isomorphism:

$$\mathbb{C}^{n+1} \longrightarrow U_0 = \{u_0 \neq 0\} \subset \hat{\mathbb{C}}^{n+1},$$

$$x \quad \mapsto \quad ((x_0, x_0x_1, \ldots, x_0x_n), [1 : x_1 : \ldots : x_n]).$$

**3.5. Weighted $(p_0, \ldots, p_n)$-blow-up of $\mathbb{C}^{n+1}$.** Let $\omega = (p_0, \ldots, p_n)$ be a weight vector. As above, consider the space

$$\hat{\mathbb{C}}^{n+1}(\omega) := \{([x, [u_\omega]] \in \mathbb{C}^{n+1} \times \mathbb{P}^n | x \in [u_\omega])\}.$$

Here the condition about the closure means that $\exists t \in \mathbb{C}, x_i = tp_iu_i, i = 0, \ldots, n$. Then the natural projection $\pi : \hat{\mathbb{C}}^{n+1}(\omega) \to \mathbb{C}^{n+1}$ is an isomorphism over $\hat{\mathbb{C}}^{n+1}(\omega) \setminus \pi^{-1}(0)$ and the exceptional divisor $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^n_\omega$.

Again the space $\hat{\mathbb{C}}^{n+1}(\omega) = U_0 \cup \cdots \cup U_n$ can be covered with $n + 1$ charts. However, $\varphi_0 : \hat{\mathbb{C}}^{n+1} \to U_0$ given by

$$\mathbb{C}^{n+1} \xrightarrow{\varphi_0} U_0 = \{u_0 \neq 0\} \subset \hat{\mathbb{C}}^{n+1}(\omega),$$

$$x \quad \mapsto \quad ((x_0^{p_0}, x_0^{p_1}x_1, \ldots, x_0^{p_n}x_n), [1 : x_1 : \ldots : x_n]),$$

is surjective but not injective. In fact $\varphi_0(x) = \varphi_0(y)$ if and only if

$$\exists \xi \in \mu_{p_0} : \begin{cases} y_0 = \xi^{-1}x_0, \\ y_i = \xi^{p_i}x_i, & i = 1, \ldots, n. \end{cases}$$

Hence the previous map $\varphi_0$ induces an isomorphism

$$X(p_0; -1, p_1, \ldots, p_n) \longrightarrow U_0.$$ 

Note that these charts are compatible with the ones described in [24] for the weighted projective space. In $U_0$ the exceptional divisor is $\{x_0 = 0\}$ and the first chart of $\mathbb{P}^n_\omega$ is the quotient space $X(p_0; p_1, \ldots, p_n)$.

**3.6. Weighted $\omega$-blow-up of $X(d; A)$.** The action $\mu_d$ on $\mathbb{C}^{n+1}$ extends naturally to an action on $\hat{\mathbb{C}}^{n+1}(\omega)$ as follows,

$$\xi_d : ([x, [u_\omega]] \xrightarrow{\mu_d} ([\xi_d^{p_0}x_0, \ldots, \xi_d^{p_n}x_n], [\xi_d^{p_0}u_0 : \ldots : \xi_d^{p_n}u_n]_\omega).$$

Let $X(d; A)(\omega) := \hat{\mathbb{C}}^{n+1}(\omega)/\mu_d$ denote the quotient space under this action. Then the induced projection

$$\pi : X(d; A)(\omega) \longrightarrow X(d; A), \quad ([x, [u_\omega]]_{(d; A)} \mapsto [x]_{(d; A)}$$

is an isomorphism.
is an isomorphism over \( X(d; A)(\omega) \setminus \pi^{-1}(0) \) and the exceptional divisor \( E := \pi^{-1}(0) \) is identified with \( \mathbb{P}^n_{\omega}/\mu_d \).

The action \( \mu_d \) above respects the charts of \( \hat{\mathbb{C}}^{n+1}(\omega) \) so that the new ambient space can be covered as

\[
X(d; A)(\omega) = \hat{U}_0 \cup \cdots \cup \hat{U}_n,
\]

where \( \hat{U}_i := U_i/\mu_d = \{ u_i \neq 0 \} \).

Let us study for instance the first chart. By using \( \varphi_0 \) one identifies \( U_0 \) with

\[
X(p_0; -1, p_1, \ldots, p_n).
\]

Let us consider the space \( X(p_0 d; (a_0)p_0 a_1 - p_1 a_0 | \ldots | p_n a_n - p_n a_0)) \). Since the following diagram commutes:

\[
\begin{array}{c}
(x_0, x_1, \ldots, x_n) \quad \mapsto \quad (\xi_{p_0 d}^0 x_0, \xi_{p_0 d}^0 x_1, \ldots, \xi_{p_0 d}^0 x_n) \\
\downarrow \\
(x_0^p, x_1^p, \ldots, x_n^p) \quad \mapsto \quad (\xi_{p_0 d}^{p_0} x_0^p, \xi_{p_0 d}^{p_0} x_1^p, \ldots, \xi_{p_0 d}^{p_0} x_n^p)
\end{array}
\]

where the vertical arrows are the chart morphisms, the upper arrow corresponds to the action defining \( X(p_0 d; (a_0)p_0 a_1 - p_1 a_0 | \ldots | p_n a_n - p_n a_0) \) and the lower arrow to the one defining \( X(p_0 d; p_0 A) = X(d; A) \). Since the chart is an isomorphism for \( X(p_0; -1, p_1, \ldots, p_n) \) we conclude that

\[
X \left( \left( \begin{array}{c} p_0 \\ p_0 d \end{array} \right) : \begin{array}{c|c|c|c} -1 & 1 & \cdots & p_n \\ a_0 & p_0 a_1 - p_1 a_0 & \cdots & p_0 a_n - p_n a_0 \end{array} \right) \right)
\]

is isomorphic to \( \hat{U}_0 \) and the isomorphism is defined by

\[
[x] \mapsto (x^p_0, x^p_1, \ldots, x^p_n, |1 : x_1 : \ldots : x_n \omega|).
\]

For \( i = 1, \ldots, n \), one proceeds analogously. As for the the exceptional divisor \( E = \pi^{-1}(0) = \mathbb{P}^n_{\omega}/\mu_d \), it is as usual covered by \( \hat{V}_0 \cup \cdots \cup \hat{V}_n \) so that these charts are compatible with the ones of \( X(d; A)(\omega) \) in the sense that \( \hat{V} = \hat{U}_1 |_{\{x_i = 0\}}, \hat{V} = \hat{U}_1 |_{\{x_n = 0\}} \).

**Example 3.7.** We study the weighted blow-up

\[
\pi = \pi(d, a, b, \omega) : X(d; a, b)_{\omega} \longrightarrow X(d; a, b)
\]

of the origin of a normalized space \( X(d; a, b) \) with respect to \( \omega = (p, q) \). The new space is covered by

\[\hat{U}_1 \cup \hat{U}_2 = X \left( \left( \begin{array}{c} p \\ pd \end{array} \right) : \begin{array}{c|c|c} -1 & q & 1 \\ a & pb - qa \end{array} \right) \cup X \left( \left( \begin{array}{c} q \\ pd \end{array} \right) : \begin{array}{c|c} p & a \\ qa - pb & b \end{array} \right) \]

and the charts are given by

First chart

\[
\begin{array}{c}
((x, y)) \mapsto ((x^0, x^1 y, |1 : y \omega|)) \end{array} \]

Second chart

\[
\begin{array}{c}
((x, y)) \mapsto ((x y^p, y^q, |x : 1 \omega|)) \end{array} \]
The exceptional divisor \( E = \pi^{-1}(d,a,b,0) \) is identified with the quotient space \( \mathbb{P}^1_d(d,a,b) \) which is isomorphic to \( \mathbb{P}^1 \) under the map
\[
\mathbb{P}^1_d(d,a,b) \rightarrow \mathbb{P}^1, \\
[x:y]_\omega \mapsto [x^{dq/e} : y^{dp/e}],
\]
where \( e := \gcd(d, pc - qa) \). Again the singular points are cyclic and correspond to the origins. To normalize these quotient spaces, note that \( e = \gcd(d, pb - qa) = \gcd(\omega, -q + \beta pb) = \gcd(\omega, p - qa) \), where \( \beta a \equiv \mu b \equiv 1 \mod d \).

Then another expression for the two charts are given below.

First chart
\[
X \left( \frac{pd}{e} ; 1, -q + \beta pb \right) \rightarrow \hat{U}_1, \\
\left[ (x^r, y) \right] \mapsto \left[ ((x^p, x^qy), [1 : y]_\omega) \right] \left( d,a,b \right).
\]
(3)

Second chart
\[
X \left( \frac{qd}{e} ; -p + \mu qa \right) \rightarrow \hat{U}_2, \\
\left[ (x, y^q) \right] \mapsto \left[ ((xy^p, y^q), [x : 1]_\omega) \right] \left( d,a,b \right).
\]

Both quotient spaces are now written in their normalized form.

**Example 3.8.** Let \( \pi := \pi_\omega : \hat{\mathbb{C}}_\omega^3 \rightarrow \mathbb{C}^3 \) be the weighted blow-up at the origin with respect to \( \omega = (p, q, r), \gcd(\omega) = 1 \). The new space is covered as
\[
\hat{\mathbb{C}}_\omega^3 = U_1 \cup U_2 \cup U_3 = X(p; -1, q, r) \cup X(q; p, -1, r) \cup X(r; p, q, -1),
\]
and the charts are given by
\[
\begin{align*}
X(p; -1, q, r) & \longrightarrow U_1 : \left( x, y, z \right) \mapsto \left( x^p, x^qy, x^r z, [1 : y : z]_\omega \right), \\
X(q; p, -1, r) & \longrightarrow U_2 : \left( x, y, z \right) \mapsto \left( x y^p, y^qz, y^r, [x : 1 : z]_\omega \right), \\
X(r; p, q, -1) & \longrightarrow U_3 : \left( x, y, z \right) \mapsto \left( x z^p, y z^q, z^r, [x : y : 1]_\omega \right).
\end{align*}
\]

In general \( \hat{\mathbb{C}}_\omega^3 \) has three lines of (cyclic quotient) singular points located at the three axes of the exceptional divisor \( \pi_\omega^{-1}(0) \cong \mathbb{P}^2_\omega \). Namely, a generic point in \( x = 0 \) is a cyclic point of type \( \mathbb{C} \times X(\gcd(q, r); p, -1) \).

Note that although the quotient spaces are written in their normalized form, the exceptional divisor can be simplified:
\[
\mathbb{P}^2(p, q, r) \rightarrow \mathbb{P}^2 \left( \frac{p}{(p, q, r)}, \frac{q}{(p, q, r)}, \frac{r}{(p, q, r)} \right), \\
[x : y : z] \mapsto \left[ x^{gcd(p, r)} : y^{gcd(p, r)} : z^{gcd(p, q)} \right].
\]

Using just a weighted blow-up of this kind, one can find an embedded \( \mathbb{Q} \)-resolution for Brieskorn-Pham surfaces singularities, i.e. \( x^a + y^b + z^c = 0 \). For simplicity, let us assume \( a, b, c \) are pairwise coprime and consider the weight \( \omega := (p, q, r) \) where \( p := bc, q := ac \) and \( r = ab \). The blow-up \( \pi_\omega \) gives a \( \mathbb{Q} \)-resolution of this singularity.

The exceptional divisor \( E \) is isomorphic to \( \mathbb{P}^2_\omega \cong \mathbb{P}^2 \). The intersection of the strict transform with \( E \) is a generic line in \( \mathbb{P}^2 \). Note that the space \( \hat{\mathbb{C}}_\omega^3 \) has three singular points of type \( (bc, 1, ac, ab), (ac, bc, -1, ab) \) and \( (ab, bc, ac, -1) \), and three strata (isomorphic to \( \mathbb{C}^* \)) with transversal singularities of type \( X(a; bc, -1), X(b; ac, -1) \) and \( X(c; ab, -1) \).
4. Cartier and Weil Divisors on \( V \)-Manifolds: \( \mathbb{Q} \)-Divisors

The aim of this section is to show that when \( X \) is a \( V \)-manifold there is an isomorphism of \( \mathbb{Q} \)-vector spaces between Cartier and Weil divisors, see Theorem 4.12 below. It is explained in 4.17 how to write explicitly a \( \mathbb{Q} \)-Weil divisor as a \( \mathbb{Q} \)-Cartier divisor. Also, the case of the exceptional divisor of a weighted blow-up is treated in Example 4.18.

4.1. Divisors on complex analytic varieties.

Let \( X \) be an irreducible complex analytic variety. As usual, consider \( \mathcal{O}_X \) the structure sheaf of \( X \) and \( \mathcal{K}_X \) the sheaf of total quotient rings of \( \mathcal{O}_X \). Denote by \( \mathcal{K}^*_X \) the (multiplicative) sheaf of invertible elements in \( \mathcal{K}_X \). Similarly \( \mathcal{O}^*_X \) is the sheaf of invertible elements in \( \mathcal{O}_X \). Note that \( \mathcal{O}_X \) (resp. \( \mathcal{K}_X \)) is the sheaf of holomorphic (resp. meromorphic) functions and \( \mathcal{O}^*_X \) (resp. \( \mathcal{K}^*_X \)) is the sheaf of nowhere-vanishing holomorphic (resp. non-zero meromorphic) functions.

Remark 4.1. By a complex analytic variety we mean a reduced complex space. A subvariety \( V \) of \( X \) is a reduced closed complex subspace of \( X \), or equivalently, an analytic set in \( X \), cf. [6]. An irreducible subvariety \( V \) corresponds to a prime ideal in the ring of sections of any local complex model space meeting \( V \).

Definition 4.2. A Cartier divisor on \( X \) is a global section of the sheaf \( \mathcal{K}^*_X / \mathcal{O}^*_X \), that is, an element in \( \Gamma(X, \mathcal{K}^*_X / \mathcal{O}^*_X) = H^0(X, \mathcal{K}^*_X / \mathcal{O}^*_X) \). Any Cartier divisor can be represented by giving an open covering \( \{U_i\}_{i \in I} \) of \( X \) and, for all \( i \in I \), an element \( f_i \in \Gamma(U_i, \mathcal{K}^*_X) \) such that

\[
\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}^*_X), \quad \forall i, j \in I.
\]

Two systems \( \{(U_i, f_i)\}_{i \in I}, \{(V_j, g_j)\}_{j \in J} \) represent the same Cartier divisor if and only if on \( U_i \cap V_j \), \( f_i \) and \( g_j \) differ by a multiplicative factor in \( \mathcal{O}_X(U_i \cap V_j)^* \).

The abelian group of Cartier divisors on \( X \) is denoted by \( \text{CaDiv}(X) \). If \( D := \{(U_i, f_i)\}_{i \in I} \) and \( E := \{(V_j, g_j)\}_{j \in J} \) then \( D + E = \{(U_i \cap V_j, f_i g_j)\}_{i \in I, j \in J} \).

The functions \( f_i \) above are called local equations of the divisor on \( U_i \). A Cartier divisor on \( X \) is effective if it can be represented by \( \{(U_i, f_i)\}_i \) with all local equations \( f_i \in \Gamma(U_i, \mathcal{O}_X) \).

Any global section \( f \in \Gamma(X, \mathcal{K}^*_X) \) determines a principal Cartier divisor \( (f)_X := \{(X, f)\} \) by taking all local equations equal to \( f \). That is, a Cartier divisor is principal if it is in the image of the natural map \( \Gamma(X, \mathcal{K}^*_X) \to \Gamma(X, \mathcal{K}^*_X / \mathcal{O}^*_X) \). Two Cartier divisors \( D \) and \( E \) are linearly equivalent, denoted by \( D \sim E \), if they differ by a principal divisor. The Picard group \( \text{Pic}(X) \) denotes the group of linear equivalence classes of Cartier divisors.

The support of a Cartier divisor \( D \), denoted by \( \text{Supp}(D) \) or \( |D| \), is the subset of \( X \) consisting of all points \( x \) such that a local equation for \( D \) is not in \( \mathcal{O}^*_X, x \). The support of \( D \) is a closed subset of \( X \).

Definition 4.3. A Weil divisor on \( X \) is a locally finite linear combination with integral coefficients of irreducible subvarieties of codimension one. The abelian group of Weil divisors on \( X \) is denoted by \( \text{WeDiv}(X) \). If all coefficients appearing in the sum are non-negative, the Weil divisor is called effective.

Remark 4.4. In the algebraic category the locally finite sum of Definition 4.3 is automatically finite. Therefore \( \text{WeDiv}(X) \) is the free abelian group on the codimension one irreducible algebraic subvarieties of \( X \). Similar considerations hold if \( X \) is a compact analytic variety.
4.5. Order function. Let $V \subset X$ be an irreducible subvariety of codimension one. It corresponds to a prime ideal in the ring of sections of any local complex model space meeting $V$. The local ring of $X$ along $V$, denoted by $O_{X,V}$, is the localization of such ring of sections at the corresponding prime ideal; it is a one-dimensional local domain.

For a given $f \in O_{X,V}$ define $\text{ord}_V(f)$ to be

$$\text{ord}_V(f) := \text{length}_{O_{X,V}} \left( \frac{O_{X,V}}{(f)} \right).$$

This determines a well-defined group homomorphism $\text{ord}_V : \Gamma(X, K_X^*) \to \mathbb{Z}$ satisfying, for a given $f \in \Gamma(X, K_X^*)$, the following local finiteness property:

$$\forall x \in X, \exists U_x \subset X \text{ open neighborhood of } x \mid \# \{\text{ord}_V(f) \neq 0 \mid V \cap U_x \neq \emptyset\} < +\infty.$$

The previous length, $X$ being a complex analytic variety of dimension $n \geq 2$, can be computed as follows. Choose $x \in V$ such that $x$ is smooth in $X$ and $(V,x)$ defines an irreducible germ. This germ is the zero set of an irreducible $g \in O_{X,x}$. Then $\text{ord}_V(f) = \text{ord}_{V,x}(f)$, where $\text{ord}_{V,x}(f)$ is the classical order of a meromorphic function at a smooth point with respect to an irreducible subvariety of codimension one; it is known to be given by the equality $f = g^{\text{ord}} \cdot h \in O_{X,x}$ where $h \nmid g$. The same applies if $X$ is 1-dimensional and smooth.

Remark 4.6. The order $\text{ord}_{V,x}(f)$ does not depend on the defining equation $g$, as long as we choose $g$ irreducible. In fact, two irreducible $g,g' \in O_{X,x}$ with $V(g) = V(g')$ only differ by a unit in $O_{X,x}$. Moreover, $\text{ord}_{V,x}(f)$ does not depend on $x$, since the set $V_{\text{red}}$ of regular points is connected if $V$ is irreducible.

Now if $D$ is a Cartier divisor on $X$, one writes $\text{ord}_V(D) = \text{ord}_V(f_i)$ where $f_i$ is a local equation of $D$ on any open set $U_i$ with $U_i \cap V \neq \emptyset$. This is well defined since $f_i$ is uniquely determined up to multiplication by units and the order function is a homomorphism. Define the associated Weil divisor of a Cartier divisor $D$ by

$$T_X : \text{CaDiv}(X) \to \text{WeDiv}(X),$$

$$D \mapsto \sum_{V \subset X} \text{ord}_V(D) \cdot [V],$$

where the sum is taken over all codimension one irreducible subvarieties $V$ of $X$. The previous sum is locally finite, i.e. for any $x \in X$ there exists an open neighborhood $U$ such that the set $\{\text{ord}_V(D) \neq 0 \mid V \cap U \neq \emptyset\}$ is finite. By the additivity of the order function, the mapping $T_X$ is a homomorphism of abelian groups.

A Weil divisor is principal if it is the image of a principal Cartier divisor under $T_X$; they form a subgroup of $\text{WeDiv}(X)$. If $\text{Cl}(X)$ denotes the quotient group of their equivalence classes, then $T_X$ induces a morphism $\text{Pic}(X) \to \text{Cl}(X)$.

These two homomorphisms ($T_X$ and the induced one) are in general neither injective nor surjective. In this sense one has the following result.

Theorem 4.7. (cf. [7, 21.6].) If $X$ is normal (resp. locally factorial) then the previous maps $\text{CaDiv}(X) \to \text{WeDiv}(X)$ and $\text{Pic}(X) \to \text{Cl}(X)$ are injective (resp. bijective). The image of the first map is the subgroup of locally principal Weil divisors.

---

2A Weil divisor $D$ on $X$ is said to be locally principal if $X$ can be covered by open sets $U$ such that $D|_U$ is principal for each $U$. 
Remark 4.8. Locally factorial essentially means that every local ring \( \mathcal{O}_{X,x} \) is a unique factorization domain. In particular, every smooth analytic variety is locally factorial. In such a case, Cartier and Weil divisors are identified and denoted by \( \text{Div}(X) := \text{CaDiv}(X) = \text{WeDiv}(X) \). Their equivalence classes coincide under this identification and we often write \( \text{Pic}(X) = \text{Cl}(X) \).

Example 4.9. Let \( X \) be the surface in \( \mathbb{C}^3 \) defined by the equation \( z^2 = xy \). The line \( V = \{ x = z = 0 \} \) defines a Weil divisor which is not a Cartier divisor. In this case \( \text{Pic}(X) = 0 \) and \( \text{Cl}(X) = \mathbb{Z}/(2) \). Note that \( X \) is normal but not locally factorial. However, the associated Weil divisor of \( \{(X, x)\} \) is

\[
T_X \left( \{(X, x)\} \right) = \sum_{Z \subset X, \text{irred}} \text{ord}_Z(x) \cdot [Z] = 2[V].
\]

Thus \( [V] \) is principal as an element in \( \text{WeDiv}(X) \otimes \mathbb{Q} \) and corresponds to the \( \mathbb{Q} \)-Cartier divisor \( \frac{1}{2}\{(X, x)\} \).

This fact can be interpreted as follows. First note that identifying our surface \( X \) with \( X(2; 1, 1) \) under \( [(x, y)] \mapsto (x^2, y^2, xy) \), the previous Weil divisor corresponds to \( D = \{ x = 0 \} \). Although \( f = x \) defines a zero set on \( X(2; 1, 1) \), it does not induce a function on the quotient space. However, \( x^2 : X(2; 1, 1) \to \mathbb{C} \) is a well-defined function and gives rise to the same zero set as \( f \). Hence as \( \mathbb{Q} \)-Cartier divisors one writes \( D = \frac{1}{2}\{(X(2; 1, 1), x^2)\} \).

4.2. Divisors on V-manifolds.

Example 4.10 above illustrates the general behavior of Cartier and Weil divisors on \( V \)-manifolds, namely Weil divisors are all locally principal over \( \mathbb{Q} \). To prove it we need some preliminaries.

Lemma 4.10. Let \( B \subset \mathbb{C}^n \) be an open ball and let \( G \) be a finite group acting on \( B \). Then one has \( \text{Cl}(B/G) \otimes \mathbb{Q} = 0 \).

Proof. Let \( V \subset B/G := U \) be an irreducible subvariety of codimension one. We shall prove that there exists \( k \geq 1 \) such that \( k[V] \subset \text{WeDiv}(U) \) is principal.

Consider the natural projection \( \pi : B \to U \). Then \( W := \pi^{-1}(V) \) gives rise to an effective Weil divisor on the open ball \( B \). There exists \( f : B \to \mathbb{C} \) a holomorphic function such that \( W = \{ f = 0 \} \subset B \). Thus, \( V = \pi(W) = \{ [x] \mid x \in B, f(x) = 0 \} = \{ f = 0 \} \subset U \).

Moreover, by construction the holomorphic function \( f \) satisfies the following property

\[
\forall P \in U, \ [f(P) = 0 \implies f(\sigma \cdot P) = 0, \forall \sigma \in G].
\]

Note that \( f \) does not necessarily defines an analytic function on \( U \). This reflects the fact that, although \( V \) is given by just one equation, \( [V] \in \text{WeDiv}(U) \) is not principal, see Example 4.9. Now the main idea is to change \( f \) by another holomorphic function \( F \) such that \( V = \{ F = 0 \} \) but now with \( F \in \Gamma(U, \mathcal{O}_U) \).

Let us consider \( F = \prod_{\sigma \in G} f^\sigma \) where \( f^\sigma(x) = f(\sigma \cdot x) \); clearly it verifies the previous conditions. Then \( \{ (U, F) \} \) is a principal Cartier divisor and its associated Weil divisor is

\[
T_U \left( \{(U, F)\} \right) = \sum_{Z \subset U, \text{irred}} \text{ord}_Z(F) \cdot [Z] = \text{ord}_V(F) \cdot [V].
\]

Note that \( \text{ord}_Z(F) \neq 0 \) implies \( Z = V \), since \( V \) is irreducible. \( \square \)

Remark 4.11. Note that the proof of this result is based on an idea extracted from [2] Ex. 1.7.6.
Theorem 4.12. Let $X$ be a $V$-manifold. Then the notion of Cartier and Weil divisor coincide over $\mathbb{Q}$. More precisely, the linear map

$$T_X \otimes 1 : \text{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor $D$ on $X$ there always exists $k \in \mathbb{Z}$ such that $kD \in \text{CaDiv}(X)$.

Proof. The variety $X$ is normal and then Theorem 4.7 applies. Therefore the linear map $T_X \otimes 1$ is injective and its image is the $\mathbb{Q}$-vector space generated by the locally principal Weil divisors on $X$.

Let $V \subset X$ be an irreducible subvariety of codimension one. Consider $\{U_i\}_i$ an open covering of $X$ such that $U_i$ is analytically isomorphic to $B_i/G_i$ where $B_i \subset \mathbb{C}^n$ is an open ball and $G_i$ is a finite subgroup of $GL(n, \mathbb{C})$. By Lemma 4.10, $\text{Cl}(U_i) \otimes \mathbb{Q} = 0$ for all $i$.

Thus $[V|_{U_i}]$ is principal as an element in $\text{WeDiv}(U_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ which implies that $V$ is locally principal over $\mathbb{Q}$ and hence belongs to the image of $T_X \otimes 1$. □

Definition 4.13. Let $X$ be a $V$-manifold. The vector space of $\mathbb{Q}$-Cartier divisors is identified under $T_X$ with the vector space of $\mathbb{Q}$-Weil divisors. A $\mathbb{Q}$-divisor on $X$ is an element in $\text{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The set of all $\mathbb{Q}$-divisors on $X$ is denoted by $\text{Q-Div}(X)$.

4.3. Writing a Weil divisor as a $\mathbb{Q}$-Cartier divisor.

Following the proofs of Lemma 4.10 and Theorem 4.12 every Weil divisor on $X$ can locally be written as a $\mathbb{Q}$-Cartier divisor like

$$[V|_{U}] = \frac{1}{\text{ord}_V(F)} \{(U, F)\},$$

where $F = \prod_{\sigma \in G} f^\sigma$ and $V \cap U = \{f = 0\}$ with $f : B \to \mathbb{C}$ being holomorphic on an open ball and satisfying (3).

The rest of this section is devoted to explicitly calculating $\text{ord}_V(F)$. First, in Proposition 4.14 it is shown that $F$ is essentially a power of $f$, if the latter is chosen properly. Then, $\text{ord}_V(F)$ is computed in Proposition 4.16.

Proposition 4.14. Let $f : B \to \mathbb{C}$ be a non-zero holomorphic function on an open ball $B \subset \mathbb{C}^n$ such that the germ $f_x \in \mathcal{O}_{B, x}$ is reduced for all $x \in B$. Let $G$ be a finite subgroup of $GL(n, \mathbb{C})$ acting on $B$. As above consider $F = \prod_{\sigma \in G} f^\sigma$ where $f^\sigma(x) = f(\sigma \cdot x)$. The following conditions are equivalent:

1. $\forall P \in B, \quad [f(P) = 0 \iff f(\sigma \cdot P) = 0, \forall \sigma \in G]$.
2. $\forall \sigma \in G, \exists h_\sigma \in \Gamma(B, \mathcal{O}_B) \text{ such that } f^\sigma = h_\sigma f$.
3. $\exists h \in \Gamma(B, \mathcal{O}_B) \text{ such that } F = hf^{|G|}$.
4. $\exists k \geq 1, \exists h \in \Gamma(B, \mathcal{O}_B) \text{ such that } hf^k \in \Gamma(B/G, \mathcal{O}_{B/G})$.

Proof. For (1) $\Rightarrow$ (2), first note that $f^\sigma \in IV(f)$ (the ideal of the zero locus of $f$).

Now fix $x \in B$. Since $f_x$ is reduced, there exists a holomorphic function $h$ on a small enough open neighborhood of $x$ such that as germs $(f^\sigma)_x = h_x f_x$. The order of the converging power series $(f^\sigma)_x$ and $f_x$ are equal because the action is linear. Thus $h_x$ is a unit in $\mathcal{O}_{B, x}$. In particular $f^\sigma_x$ is holomorphic and does not vanish at $x \in B$.

For (2) $\Rightarrow$ (3), consider $h = \prod_{\sigma \in G} h_\sigma$. Then one has

$$F = \prod_{\sigma \in G} f^\sigma = \prod_{\sigma \in G} (h_\sigma f) = \left(\prod_{\sigma \in G} h_\sigma\right) \cdot f^{|G|} = hf^{|G|}.$$

For (3) $\Rightarrow$ (1), since $F : B/G \to \mathbb{C}$ is analytic, take $k = |G|$. Finally, note that $\forall P \in B, \ f(P) = 0 \iff (hf^k)(P) = (hf^k)(\sigma \cdot P) = 0 \iff f(\sigma \cdot P) = 0$. Hence (1) $\Rightarrow$ (1) follows and the proof is complete. □
This example shows that the reduceness condition in the statement of the previous result is necessary.

**Example 4.15.** Let \( f = (x^2 + y)(x^2 - y)^3 \in \mathbb{C}[x, y] \) and consider the cyclic quotient space \( M = X(\mathbb{C}[x, y]) \). Then \( \{f = 0\} \subset M \) defines a zero set, i.e. condition (1) holds, but there are no \( k \geq 1 \) and \( h \in \Gamma(B, \mathcal{O}_B^* ) \) such that \( hf^k \) is a well-defined function over \( M \).

**Proposition 4.16.** Let \( B \subset \mathbb{C}^n \) be an open ball and \( G \) a finite subgroup of \( GL(n, \mathbb{C}) \) acting on \( B \). Let \( V \subset B/G =: U \) be an irreducible subvariety of codimension one and consider

\[
F = \prod_{\sigma \in G} f^\sigma
\]

where \( f : B \to \mathbb{C} \) is a holomorphic function defining \( V \).

If \( G \) is small and \( f \) is chosen so that \( f_x \in \mathcal{O}_{B,x} \) is reduced \( \forall x \in B \), then \( \text{ord}_U(F : U \to \mathbb{C}) = |G| \).

**Proof.** Choose \([P] \in V \) such that \([P] \) is smooth in \( U \) and \((V, [P]) \) defines an irreducible germ; then \( \text{ord}_U(F) = \text{ord}_U([P], F) \), see 4.3.

By Theorem 4.13, since \( G \) is small and \([P] \in U \) is smooth, using the covering \( \pi : B \to U \), one finds an isomorphism of germs \((U, [P]) \cong (B/G, [P]) = (B, P)\) induced by the identity map. The germ \((V, [P])\) is converted under this isomorphism into \((W, P)\) where \( W \) is the zero set of \( f_P \in \mathcal{O}_{B,P} \).

By Proposition 4.14 there exists \( h \in \Gamma(B, \mathcal{O}_B^* ) \) such that \( F = hf^{|G|} \). Then the required order is \( \text{ord}_V([P], F : U \to \mathbb{C}) = \text{ord}_V([P], P(hf^{|G|} : B \to \mathbb{C}) = |G| \) as claimed. \( \square \)

**4.17.** Here we summarize how to write a Weil divisor as a \( \mathbb{Q} \)-Cartier divisor where \( X \) is an algebraic \( V \)-manifold.

1. Write \( D = \sum_{i \in J} a_i[V_i] \in \text{WeDiv}(X) \), where \( a_i \in \mathbb{Z} \) and \( V_i \subset X \) irreducible.
   Also choose \( \{U_j\}_{j \in J} \) an open covering of \( X \) such that \( U_j = B_j/G_j \) where \( B_j \subset \mathbb{C}^n \) is an open ball and \( G_j \) is a small finite subgroup of \( GL(n, \mathbb{C}) \).

2. For each \((i, j) \in I \times J\) choose a reduce polynomial \( f_{i,j} : U_j \to \mathbb{C}\) such that \( V_i \cap U_j = \{f_{i,j} = 0\} \), then
   \[
   [V_i|_{U_j}] = \frac{1}{|G_j|}\{(U_j, f_{i,j}^{\lfloor |G_j|\rfloor})\}.
   \]

3. Identifying \( \{(U_j, f_{i,j}^{\lfloor |G_j|\rfloor})\} \) with its image \( \text{CaDiv}(U_j) \hookrightarrow \text{CaDiv}(X) \), one finally writes \( D \) as a sum of locally principal Cartier divisors over \( \mathbb{Q} \).
   \[
   D = \sum_{(i, j) \in I \times J} \frac{a_i}{|G_j|}\{(U_j, f_{i,j}^{\lfloor |G_j|\rfloor})\}.
   \]

We finish this section with an example where the exceptional divisor of a weighted blow-up (which is in general just a Weil divisor) is explicitly written as a \( \mathbb{Q} \)-Cartier divisor.

**Example 4.18.** Let \( X \) be a surface with abelian quotient singularities. Let \( \pi : \tilde{X} \to X \) be the weighted blow-up at a point of type \((d; a, b)\) with respect to \( \omega = (p, q) \). In general, the exceptional divisor \( E := \pi^{-1}(0) \cong \mathbb{P}^1(d; a, b) \) is a Weil divisor on \( \tilde{X} \) which does not correspond to a Cartier divisor. Let us write \( E \) as an element in \( \text{CaDiv}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \).

As in 4.7 assume \( \pi := \pi_{(d, a, b), \omega} : X(d; a, b)_\omega \to X(d; a, b) \). Assume also that \( \text{gcd}(p, q) = 1 \) and \((d; a, b)\) is normalized. Using the notation introduced in 4.7, the
isomorphism. Hence, using the third chart, 

\[ Q_1 := X \left( \frac{ord_x}{e}, 1, \frac{-q + pb}{e} \right) \rightarrow \hat{U}_1, \]

\[ \left[ (x^e, y) \right] \rightarrow \left[ ((x^p, x^q y), [1 : y^\omega]) \right]_{(d,a,b)}, \]

where \( e := \gcd(d, pb - qa) \), see 3.7 for details.

In the first chart, \( E \) is the Weil divisor \( \{ x = 0 \} \subset Q_1 \). Note that the type representing the space \( Q_1 \) is in a normalized form and hence the corresponding subgroup of \( GL(2, \mathbb{C}) \) is small.

Following the discussion 4.17, the divisor \( \{ x = 0 \} \subset Q_1 \) is written as an element in \( \text{CaDiv}(Q_1) \otimes \mathbb{Q} \) like \( \frac{e}{dp} \{ (Q_1, x^d) \} \), which is mapped to \( \frac{e}{dp} \{ (\hat{U}_1, x^d) \} \in \text{CaDiv}(\hat{U}_1) \otimes \mathbb{Q} \) under the isomorphism \( \theta \).

Analogously \( E \) in the second chart is \( \frac{e}{dq} \{ (\hat{U}_2, y^d) \} \). Finally one writes the exceptional divisor of \( \pi \) as claimed,

\[ E = \frac{e}{dp} \{ (\hat{U}_1, x^d), (\hat{U}_2, 1) \} + \frac{e}{dq} \{ (\hat{U}_1, 1), (\hat{U}_2, y^d) \} = \frac{e}{dpq} \{ (\hat{U}_1, x^{dp}), (\hat{U}_2, y^{dp}) \}. \]

5. Conclusions and Future Work

In order to define the intersection number between two divisors the following fact is essential. Let \( X \) be a compact connected Riemann surface and let \( \pi : E \rightarrow X \) be a complex line bundle. Such a line bundle admits non-trivial meromorphic sections and the degree of such a section (the sum of the orders of the zeroes minus the sum of the orders of the poles) only depends on \( E \). In order to define the intersection number between two divisors on a \( V \)-surface (one of them with compact support), we can assume (up to multiplication by an integer) that they are Cartier divisors. Then, it is possible to define the intersection number as the degree of the pull-back in the normalization of the compact divisor of the line bundle associated with the other divisor. These ideas will be developed in a [1]. The following is an illustrative example.

**Example 5.1.** Let \( X = \mathbb{P}^2_{(2,1,1)} \) and consider the Weil divisors \( D_1 = \{ x = 0 \} \) and \( D_2 = \{ y = 0 \} \). Let us compute the Weil divisor associated with \( j_{D_1}^* D_2 \), where \( j_{D_1} : D_1 \hookrightarrow X \) is the inclusion. Following 4.17 the divisor \( D_2 \) can be written as

\[ D_2 = \frac{1}{2} (2D_2) = \frac{1}{2} \{ (U_1, y^2/x^2), (U_2, 1), (U_3, y^2/z) \} \in \text{CaDiv}(X) \otimes \mathbb{Q}, \]

where the covering is the one given in Section 2. Recall that \( U_3 \cong X(2;1,1) \). By definition, since \( D_1 \not\subset D_2 \), the pull-back is

\[ j_{D_1}^* D_2 = \frac{1}{2} \{ (U_2 \cap D_1, 1), (U_3 \cap D_1, y^2/z) \} = \frac{1}{2} E \]

and its associated Weil divisor on \( D_1 \) is

\[ T_{D_1}(j_{D_1}^* D_2) = \frac{1}{2} \sum_{P \in E} \text{ord}_P(E) \cdot |P| = \frac{1}{2} \text{ord}_{[0:0:1]}(E) \cdot [0 : 0 : 1] = \frac{1}{2} [0 : 0 : 1]. \]

Note that there is an isomorphism \( U_3 \cap D_1 = X(2;1) \simeq \mathbb{C} \), \([y] \mapsto y^2\), and the function \( y^2 : X(2;1) \rightarrow \mathbb{C} \) is converted into the identity map \( \mathbb{C} \rightarrow \mathbb{C} \) under this isomorphism. Hence, using the third chart, \( \text{ord}_{[0:0:1]}(E) = 1 \). It is natural to define the (global and local) intersection multiplicity as \( D_1 \cdot D_2 = (D_1 \cdot D_2)_{[0:0:1]} = \frac{1}{2} \).
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