Acceleration in integro-differential combustion equations

Emeric Bouin ∗ Jérôme Coville † Guillaume Legendre ‡

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Abstract

We study acceleration phenomena in monostable integro-differential equations with ignition nonlinearity. Our results cover fractional Laplace operators and standard convolutions in a unified way, which is also a contribution of this paper. To achieve this, we construct a sub-solution that captures the expected dynamics of the accelerating solution, and this is here the main difficulty. This study involves the flattening effect occurring in accelerated propagation phenomena.

Keywords: integro-differential operators, fractional laplacian, acceleration, spreading.

1 Introduction.

In this paper, we are interested in describing quantitatively propagation phenomena in the following (non-local) integro-differential equation:

\[ u_t(t,x) = D[u](t,x) + f(u(t,x)) \quad \text{for} \quad t > 0, x \in \mathbb{R}, \tag{1.1} \]

where

\[ D[u](t,x) := \text{P.V.} \left( \int_{\mathbb{R}} [u(t,y) - u(t,x)] J(x-y) \, dy \right) \]

with \( J \) is a nonnegative function satisfying the following properties.

**Hypothesis 1.1.** Let \( s \in [0, \frac{1}{2}] \). The kernel \( J \) is symmetric and is such that there exists positive constants \( J_0, J_1 \) and \( R_0 \geq 1 \) such that

\[
\int_{|z| \leq 1} J(z)|z|^2 \, dz \leq 2J_1 \quad \text{and} \quad \frac{J_0}{|z|^{1+2s}} \mathbb{I}_{\{|z| \geq 1\}} \geq J(z) \geq \frac{J_0^{-1}}{|z|^{1+2s}} \mathbb{I}_{\{|z| \geq R_0\}}.
\]

The operator \( D[\cdot] \) describes the dispersion process of the individuals. Roughly, the kernel \( J \) gives the probability of a jump from a position \( x \) to a position \( y \), so that the tails of \( J \) are of crucial importance to quantify the dynamics of the population. As a matter of fact, the parameter \( s \) will thus appear in the rates we obtain later. One may readily notice that our hypothesis on \( J \) allows to cover the two broad types of integro-differential operators \( D[u] \) usually considered in the literature which are the fractional laplacian \((-\Delta)^su\) and the standard convolution operators with integrable kernels often written \( J \ast u - u \). This universality is one main contribution of this paper.

∗CEREMADE - Université Paris-Dauphine, UMR CNRS 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail: bouin@ceremade.dauphine.fr

†UR 546 Biostatistique et Processus Spatiaux, INRA, Domaine St Paul Site Agropolis, F-84000 Avignon, France. E-mail: jerome.coville@inra.fr

‡CEREMADE - Université Paris-Dauphine, UMR CNRS 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail: legendre@ceremade.dauphine.fr
Hypothesis 1.2. Take $\theta > 0$,
\[
    f(1) = 0, \quad f(u) = 0 \quad \text{if} \quad u \leq \theta \quad \text{and} \quad f(u) > 0 \quad \text{in} \quad [\theta, 1].
\]

The strong maximum principle implies that the solution to (1.1) takes values in $[0, 1]$ only. Moreover, since the initial data is decreasing, at all times $t \in \mathbb{R}^+$, the function $x \mapsto u(t, x)$ is decreasing over $\mathbb{R}$, from one to zero. To follow the propagation, we may thus follow level sets of height $\lambda \in (0, 1)$,
\[
x_\lambda(t) := \sup \{ x \in \mathbb{R}, u(t, x) \geq \lambda \}.
\]

Our main result is the following.

Theorem 1.3. Assume that $J$ satisfies Hypothesis 1.1 with $s < \frac{1}{2}$ and that $f$ is an ignition nonlinearity. For any $\lambda \in (0, 1)$, the level line $x_\lambda(t)$ accelerates with the following rate,
\[
t^{\frac{1}{s}} \lesssim x_\lambda(t) \lesssim t^{\frac{1}{s}+\epsilon}.
\]

The main purpose of this research report is to show the lower bound, the sharp upper bound will come in a later work.

Let us review some existing works around this issue. Existence of fronts have been obtained for the fractional laplacian for $s > 1/2$ by Mellet et al in [11] and for the convolution type operator $J \ast u - u$ provided $J$ has a first moment by Coville [7, 6]. See also some related works by Shen et al [12, 13]. Here we explore a situation where no first moment at infinity exists, that is $s \leq 1/2$. Thanks to the monostable results [8, 9], we already know that accelerated propagation can only occur in this region of parameter. This is drastic contrast with truly monostable nonlinearities [4, 3].

![Figure 1: Schematic view of the expected behaviour of solution at a given time $t$.](image)

The rest of the paper is organised as follows. The following Section 2 is about the behaviour of the linear problem. Then, Section 3 describes in broad lines the construction of the sub-solution.

## 2 Flattening in the linear problem

Finally, the flattening behaviour of linear anomalous diffusions is reminiscent from [1, 10, 5, 2], and we shall here recall and extend some principal computations from these papers.
The Fourier symbol of the operator $D[\cdot]$ is
\[ W(\xi) := \int_{\mathbb{R}} (\cos(\xi y) - 1) J(y) \, dy. \]

Note that one may recover two typical cases. If $J(y) \propto |y|^{-1-2s}$, that is $D[\cdot]$ is a fractional Laplacian, then $W(\xi) = |\xi|^{2s}$. If $J$ is an integrable function with unit mass, as in convolution models, then $W(\xi) = J(\xi) - 1$. The presence of a singularity at 0 for $J$ has an influence on large frequencies $\xi$, whereas the tail of $J$ influences small frequencies. As a consequence,

For small $\xi$, write,
\[ \int_{\mathbb{R}} (\cos(\xi y) - 1) J(y) \, dy = \int_{|y| \leq R_0} (\cos(\xi y) - 1) J(y) \, dy + \int_{|y| \leq R_0} (\cos(\xi y) - 1) J(y) \, dy \]

The first integral in the r.h.s is of order $|\xi|^{2s}$ with explicit constant by a direct Taylor expansion and using the hypothesis on $J$. The second one is estimated as follows. Since
\[ \int_{|y| \leq R_0} J_0^{-1} \frac{\cos(\xi y) - 1}{|y|^{1+2s}} \, dy \leq \int_{|y| \leq R_0} (\cos(\xi y) - 1) J(y) \, dy \leq \int_{|y| \leq R_0} J \frac{\cos(\xi y) - 1}{|y|^{1+2s}} \, dy, \]

we have that $\int_{|y| \leq R_0} (\cos(\xi y) - 1) J(y) \, dy$ is of order $-|\xi|^{2s}$ with explicit estimates. As a consequence, since $s \leq \frac{1}{2}$, $W$ is of order $-|\xi|^{2s}$ with explicit estimates.

In this section, we discuss the fact that the solution to the following linearised problem
\[ G_t = D[G] \quad \text{for} \quad t > 0, x \in \mathbb{R}, \]
\[ G(0, \cdot) = \delta_{x=0}, \tag{2.2} \]
flattens, that is, there exists $C_0 \in \mathbb{R}^+$ such that
\[ \forall t \in \mathbb{R}^+, \exists x_0 \in \mathbb{R}^+, \quad G(t, x) \geq \frac{C_0 t}{|x|^{1+2s}}. \]

The proof takes its spirit in [10, Proposition 2.2] and we explain the arguments below. Note that then the solution to
\[ v_t = D[v] \quad \text{for} \quad t > 0, x \in \mathbb{R}, \]
\[ v(0, \cdot) = \mathbb{1}_{(-\infty, 0]}, \tag{2.3} \]
is then given by
\[ v(t, x) = G(t, \cdot) * \mathbb{1}_{(-\infty, 0]}(\cdot)(x) = \int_0^{+\infty} G(t, y) \, dy. \]

From this computation, observe that for any $t$, we have $\lim_{x \to -\infty} v(t, x) = 1$. Getting an estimate of $G$ for very large $y$ will yield an estimate for $v$.

Solving in the Fourier variable, this is
\[ \forall \xi \in \mathbb{R}, \quad \hat{G}(t, \xi) = \exp(W(\xi)t). \]

Observe that, formally,
\[ G(t, x) = \int_{\mathbb{R}} \exp(W(\xi)t - ix\xi) \, d\xi = 2\Re\left( \int_0^\infty \exp(W(\xi)t - ix\xi) \, d\xi \right). \]

Compute, for any $x \neq 0$,
\[ \int_0^R \exp(W(\xi)t - ix\xi) \, d\xi = \frac{1}{x} \int_0^{Rx} \exp\left( W\left( \frac{u}{x} \right) t - iu \right) \, du \]

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Following the same steps as in [], we shall use a contour integral in the complex plane. Define the holomorphic function
\[ \varphi(z) := \exp \left( W \left( \frac{t}{x} \right) t - z \right) \]
that we integrate on the contour ... . The Cauchy integral theorem gives
\[
\frac{1}{x} \int_0^{Rx} \exp \left( W \left( \frac{u}{ix} \right) t - u \right) du = \frac{1}{ix} \int_0^{Rx} \exp \left( W \left( \frac{u}{ix} \right) t - u \right) du + \frac{1}{ix} \int_0^{\pi} \exp \left( W \left( \frac{Rx e^{i\theta}}{ix} \right) t - Rx e^{i\theta} \right) iRx e^{i\theta} \, d\theta
\]
The second integral goes to zero by the Lebesgue dominated convergence theorem. Taking \( R \to \infty \), we get that the following integral exists and that
\[
2 \Re \left( \int_0^{\infty} \exp (W(\xi) t - ix\xi) \, d\xi \right) = -\frac{2}{x} \Re \left( i \int_0^{\infty} \exp \left( W \left( \frac{u}{ix} \right) t - u \right) du \right)
\]
From the latter, we deduce that
\[
\lim_{x \to \infty} \frac{2}{x} \int_0^{\infty} \Re \left( \exp (W(\xi) t - ix\xi) \, d\xi \right) = \frac{2}{x} \int_0^{\infty} \Re \left( \exp \left( W \left( \frac{u}{ix} \right) t \right) \right) e^{-u} \, du
\]
using the scaling of \( W \) near zero and the dominated convergence theorem.

3 The strategy for the construction of sub-solutions.

In this section, we present the way that we construct a sub-solution to prove the lower bounds in Theorem 1.3. We are looking for a sub-solution \( u \) to (1.1) that satisfies everywhere
\[
-u \leq D[u] + f(u) \quad \text{and} \quad u \leq \varepsilon,
\]
for some \( \varepsilon \in (\theta, 1) \) and \( t > t^* \). We construct an at least of class \( C^2 \) function \( u \) piecewise,
\[
\begin{align*}
u &:= \varepsilon, \quad \text{on} \{ x \leq X(t) \}, \\
u &:= \phi, \quad \text{else},
\end{align*}
\]
with \( \phi(t, X(t)) = \varepsilon \). The point \( X(t) \) is unknown at that stage. As previously for the monostable case, we expect \( \phi \) to look like a solution of the standard fractional Laplace equation with Heaviside initial data at the far edge. In this situation, a natural candidate would be given by
\[
w(t, x) := \left[ \frac{x^{2s}}{\kappa t} + \gamma \right]^{-1}.
\]
with \( \kappa, \gamma \) positive free parameter that will be determined later on. Note that this function is well defined for \( t \geq 1 \) and \( x > 0 \). The expected decay in space of a solution of the standard fractional Laplace equation
with Heaviside initial data being at least of order $tx^{-2s}$, such a $w$ would have the good asymptotics. For $\varepsilon \in (0, 1)$, let us define $X(t) > 0$ such that $w(t, X(t)) = \varepsilon$. For such $X(t)$ to be well defined, we need to impose that $\gamma < \frac{1}{\varepsilon}$, and thus for such $\varepsilon$ and $\gamma < \frac{1}{\varepsilon}$, $X(t)$ is then defined by the following formula

$$X(t) = \left[\varepsilon^{-1} - \gamma\right]^{\frac{1}{2s}} (\kappa t)^{\frac{1}{2s}}.$$  

(3.6)

One may observe that $X(t)$ moves with the speed that we expect in Theorem 1.3. However, taking $\phi$ as this $w$ would not lead to a $C^2$ function at $x = X(t)$. As in the monostable case, to remedy this issue, we complete our construction by taking $\phi$ such that

$$u(t, x) := \begin{cases} 
\varepsilon & \text{for all } x \leq X(t), \\
3 \left(1 - \frac{1}{\varepsilon} w(t, x) + \frac{1}{3\varepsilon^2} w^2(t, x) \right) w(t, x) & \text{for all } x > X(t),
\end{cases}$$  

(3.7)

for $t > 1$.

4 Proof of Theorem 1.3.

Start by observing that $\bar{u}$ defined in (3.7) satisfies (3.4) if and only if,

$$0 \leq D[u] + f(\varepsilon), \quad x \leq X(t),$$  

(4.8)

$$u_t \leq D[u] + f(u), \quad \text{else.}$$  

(4.9)

As a consequence, again the main work is to derive good estimates for $D[u]$ in both regions $x \leq X(t)$ and $x \geq X(t)$. The estimate in the first region will be rather direct to get and will rely mostly on the fact that $\bar{u}$ is constant there together with the tails of $J$. In the latter region, things are more intricate. We have to split it into two zones, as depicted on Figure 2 below, each one being the stage of one specific character of the model and thus demanding a specific way to estimate $D[u]$.

Let us now show that for the right choice of $\varepsilon$ and $\kappa$ the function $\bar{u}$ is indeed a subsolution to (3.4) for all $t \geq 1$.

4.1 Some preliminary estimates.

4.1.1 Facts and formulas on $X$ and $w$.

As in the previous construction, let us recall some useful facts. From direct computations we have:

$$u_t = u_x = u_{xx} = 0$$  

for all $t > 0, x < X(t)$  

(4.10)

$$u_t = 3w_t \left(1 - \frac{w}{\varepsilon}\right)^2, \quad u_x = 3w_x \left(1 - \frac{w}{\varepsilon}\right)^2,$$  

for all $t > 1, x > X(t)$  

(4.11)

$$u_{xx}(t, x) = 3 \left(1 - \frac{w}{\varepsilon}\right) \left[w_{xx} \left(1 - \frac{w}{\varepsilon}\right) - \frac{2w_x^2}{\varepsilon}\right]$$  

for all $t > 1, x > X(t)$  

(4.12)

Note crucially that $\bar{u}$ is then at a $C^2$ function in $x$ and $C^1$ in $t$. We will also need repeatedly the following information on derivatives of $w$ at any point $(t, x)$ where $w$ is defined.

$$w_t = \kappa w^2(t, x) \frac{x^{2s}}{(\kappa t)^2}$$  

(4.13)

$$w_x = -2sw^2(t, x) \frac{x^{2s-1}}{\kappa t}$$  

(4.14)

$$w_{xx} = 4s^2 w^3(t, x) \frac{x^{4s-2}}{(\kappa t)^2} + 2s(2s-2s)w^2(t, x) \frac{x^{2s-2}}{\kappa t}$$  

(4.15)
Figure 2: Schematic view of the sub-solution at a given time $t$. Several zones have to be considered. The exact expression of $X_\eta(t)$ will appear naturally later. The blue zone is where $\underline{u}$ is constant, making computations easier. In the orange zone, the fact that $u > \theta$ is crucial. In the green (far-field) zone, the decay imitating a fractional Laplace equation gives the right behaviour.

Observe that since $s \leq 1/2$, we deduce from the latter that $w$ is convex in $[X(t), +\infty)$. Moreover by using the definition of $X(t)$ we also deduce that for $t \geq 1$,

$$w_x(t, X(t)) = \frac{-2s\varepsilon [1 - \gamma\varepsilon]}{X(t)}.$$  \hfill (4.16)

4.1.2 An estimate for $w$ on $[X(t) + 1, +\infty)$.

**Proposition 4.1.** For all $\kappa > 0$ and all $\varepsilon \in (0, 1)$, $\gamma < \frac{1}{2}$, we have for all $x \geq X(t)$,

$$w(t, x) \leq \frac{(1 - \gamma\varepsilon)\kappa t}{x^{2s}}.$$  

*Proof.* By using (3.5), the definition of $w$, since we have

$$w(t, x) = \left(1 + \frac{\kappa t X(t)^{2s}}{x^{2s}} \right)^{-1}.$$  

Since $x \geq X(t)$,

$$w(t, x) \leq \frac{\kappa t (1 - \varepsilon\gamma)}{x^{2s}}.$$  

\hfill $\Box$

4.2 Estimating $D[\underline{u}]$ when $x \leq X(t)$.

On this region, by definition of $\underline{u}$, we have

$$D[\underline{u}](t, x) = \int_{y \geq X(t)} [\underline{u}(t, y) - \varepsilon] J(x - y) \, dy.$$
This section aims at showing (4.8). For the convenience of the reader, we shall state this is the following

**Proposition 4.2.** For all, \( 0 < s \leq \frac{1}{4}, 1 > \varepsilon > \theta, \gamma < \frac{1}{2} \) and \( \kappa \) there exists \( t_0(\varepsilon, \kappa, s, \gamma) \) such that for all \( t \geq t_0 \)

\[
D[u](t, x) + f(\varepsilon) \geq 0 \quad \text{for all} \quad x \leq X(t).
\]

**Proof.** Recall that by (??), we have

\[
D[u] \geq -\frac{f(\varepsilon)}{2} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z^{1-2s} dz \right) w_x(t, X(t))^2.
\]

We are now ready to choose \( B := \left( \frac{3(\varepsilon+1)}{s f(\varepsilon)} \right)^{\frac{1}{2}} \) above. We then get

\[
D[u] \geq -\frac{f(\varepsilon)}{2} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z^{1-2s} dz \right) w_x(t, X(t))^2.
\]

Set for legibility \( C_0 := 3 \left( J_1 + J_0 \int_1^B z^{1-2s} dz \right) \), and use the estimate (4.16) on \( w_x(t, X(t)) \), to get

\[
D[u] + f(\varepsilon) \geq \frac{f(\varepsilon)}{2} - \frac{C_0}{\varepsilon} w_x(t, X(t))^2,
\]

\[
\geq \frac{f(\varepsilon)}{2} - \frac{4C_0s^2\varepsilon^2 [1-\gamma\varepsilon]^2}{X^2(t)}.
\]

the proposition is proved by taking \( t \) large since \( X(t) \to +\infty. \)

\[ \square \]

### 4.3 Estimate of \( D[u] \) on \( x > X(t) \).

In this region, as exposed earlier and shown in Figure 2, we shall estimate \( D[u] \) in two separate intervals

\[
[X(t), \sup \{ X + R_0; X_\eta(t) \}], \quad [\sup \{ X_\eta(t), X(t) + R_0 \}, +\infty).
\]

with \( X_\eta \) to be chosen.

#### 4.3.1 The region \( X(t) \leq x \leq \sup \{ X(t) + R_0; X_\eta \} \)

For all \( \varepsilon > \theta \), let \( \eta(\varepsilon) > 0 \) be the smallest positive root of the polynomial function \( z \mapsto \frac{1}{\varepsilon^2} z^3 + z - (\varepsilon - \theta) \). Then for such \( \eta(\varepsilon) \),

\[
3(\varepsilon - \eta) \left( 1 - \frac{\varepsilon - \eta}{\varepsilon} + \frac{\varepsilon - \eta}{3\varepsilon^2} \right) = 3(\varepsilon - \eta) \left( \frac{1}{3} + \frac{\eta}{3\varepsilon} + \frac{\eta^2}{3\varepsilon^2} \right) = (\varepsilon - \eta) \left( 1 + \frac{\eta}{\varepsilon} + \frac{\eta^2}{\varepsilon^2} \right)
\]

\[ = \frac{\varepsilon - \eta^3}{\varepsilon^2} \]

\[ = \theta + \eta \]

Let \( X_\eta(t) \) be such that \( w(t, X_\eta(t)) = \varepsilon - \eta \), then by construction we have \( X_\eta > X(t) \) and from the above computation, \( w(t, x) \geq \theta + \eta \) for all \( x \leq X_\eta(t) \).

We start this with an estimate

**Lemma 4.3.** For all \( B > 1 \) and \( \delta \geq \sup \{ R_0, B + X(t) - x \} \),

\[
D[u](t, x) \geq -\frac{J_0 u(t, x)}{s B^{2s}} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z^{1-2s} dz \right) \sup_{-B \leq \xi \leq B, \ x + \xi > X(t)} (w_x(t, x + \xi))^2.
\]

(4.17)
Proof. By definition of \( u \), for any \( \delta \geq R_0 \) we have, using Hypothesis 1.1,
\[
D[u](t, x) = \int_{x+z \leq X(t)-\delta} \frac{\varepsilon - u(t, x)}{|z|^{1+2s}} J(z) |z|^{1+2s} dz + \int_{x+z \geq X(t)-\delta} [u(t, x+z) - u(t, x)] J(z) dz,
\]
\[
= \frac{J_0^{-1}}{2s} \frac{\varepsilon - u(t, x)}{(x - X(t) + \delta)^{1+2s}} + \int_{x+z \geq X(t)-\delta} [u(t, x+z) - u(t, x)] J(z) dz. \tag{4.18}
\]

We shall now estimate
\[
\int_{x+z \geq X(t)-\delta} [u(t, x+z) - u(t, x)] J(z) dz.
\]

For \( B \geq 0 \) to be chosen later on, we decompose,
\[
\int_{x+z \geq X(t)-\delta} [u(t, x+z) - u(t, x)] J(z) dz = \int_{x+z \geq X(t)-\delta, |z| \leq B} [u(t, x+z) - u(t, x)] J(z) dz
\]
\[
+ \int_{x+z \geq X(t)-\delta, |z| > B} [u(t, x+z) - u(t, x)] J(z) dz. \tag{4.19}
\]

The second integral in the right hand side of the above expression is the easiest. Since \( u \) is positive and \( J \) satisfies (1.1) we then have for \( B > 1 \),
\[
\int_{x+z \geq X(t)-\delta, |z| \geq B} [u(t, x+z) - u(t, x)] J(z) dz \geq -u(t, x) J_0 \int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}}.
\]

When \( X(t) - \delta \leq x - B \), a short computation shows that
\[
\int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}} = \int_{X(t)-x-\delta \leq z \leq -B} \frac{dz}{|z|^{1+2s}} + \int_{z \geq B} \frac{dz}{z^{1+2s}}
\]
\[
= \int_{X(t)-x-\delta \leq z \leq -B} \frac{dz}{|z|^{1+2s}} + \int_{z \geq B} \frac{dz}{z^{1+2s}}
\]
\[
= \frac{1}{2sB^{2s}} - \frac{1}{2s(x + \delta - X(t))^{2s}} + \frac{1}{2sB^{2s}}.
\]

On the other hand if \( X(t) - \delta \geq x - B \) then
\[
\int_{x+z \geq X(t)-\delta, |z| \geq B} \frac{dz}{|z|^{1+2s}} = \int_{z \geq B} \frac{dz}{|z|^{1+2s}} = \int_{z \geq B} \frac{dz}{z^{1+2s}} = \frac{1}{2sB^{2s}}.
\]

In each situation we then have
\[
\int_{x+z \geq X(t)-\delta, |z| \geq B} [u(t, x+z) - u(t, x)] J(z) dz \geq -\frac{u(t, x) J_0}{s B^{2s}}. \tag{4.20}
\]

Let us now estimate the first integral of the right hand side of (4.19), that is, let us estimate
\[
I := \int_{x+z \geq X(t)-\delta, |z| \leq B} [u(t, x+z) - u(t, x)] J(z) dz.
\]

Since \( \underline{u}(t, x) \) is \( C^1 \) in \( x \) we have, for all \( t \geq 1 \) and \( x \in \mathbb{R} \),
\[
\underline{u}(t, x+z) - \underline{u}(t, x) = z \int_0^1 \underline{u}_x(t, x + \tau z) d\tau
\]

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and therefore we can rewrite $I$ as follows:

$$I = \int_{x+z \geq X(t)-\delta, |z| \leq B} \int_0^1 u_x(t, x + \tau z)zJ(z)\,d\tau dz.$$  

For any $\delta \geq B + X(t) - x$, let us observe that by symmetry we have

$$\int_{x+z \geq X(t)-\delta, |z| \leq B} \int_0^1 J(z)z\,d\tau dz = 0.$$  

As a consequence we can rewrite $I$ as follows:

$$I = \int_{x+z \geq X(t)-\delta, |z| \leq B} \int_0^1 [u_x(t, x + \tau z) - u_x(t, x)]zJ(z)\,d\tau dz.$$  

Since $u_x$ is a $C^1$ function, by using the Taylor expansion

$$[u_x(t, x + \tau z) - u_x(t, x)] = \tau z \int_0^1 u_{xx}(t, x + \tau \sigma z)\,d\sigma$$

we have

$$I = \int_{x+z \geq X(t)-\delta, |z| \leq B} \int_0^1 u_{xx}(t, x + \sigma z)\tau^2 z^2 J(z)\,d\tau d\sigma dz,$$

$$\geq \min_{-B \leq x \leq B} u_{xx}(t, x + \xi) \left( \int_{|z| \leq B} \int_0^1 \tau^2 z^2 J(z)\,d\tau d\sigma dz \right)$$

$$\geq \min_{-B \leq x \leq B} u_{xx}(t, x + \xi) \left( \int_{|z| \leq 1} \int_0^1 \tau^2 z^2 J(z)\,d\tau d\sigma dz + \int_{1 \leq |z| \leq B} \int_0^1 \tau^2 z^2 J(z)\,d\tau d\sigma dz \right).$$

By using (4.19), (4.20) and (4.21) and the convexity of $w$, we deduce that

$$I \geq -\frac{6}{\varepsilon} \sup_{x+\xi > X(t)} w_x(t, x + \xi)^2 \left( \int_{|z| \leq 1} \int_0^1 \tau^2 z^2 J(z)\,d\tau d\sigma dz + \int_{1 \leq |z| \leq B} \int_0^1 \tau^2 z^2 J(z)\,d\tau d\sigma dz \right).$$

Hence we have

$$I \geq -\frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z^{-2s} \,dz \right) \sup_{x+\xi > X(t)} w_x(t, x + \xi)^2. \quad (4.21)$$

Collecting (4.19), (4.20) and (4.21), we get the following estimate for all $B > 1$ and $\delta \geq \sup\{R_0, B + X(t) - x\},$

$$\mathcal{D}[u](t, x) \geq J_0^{-1} \frac{\varepsilon - u(t, x)}{2s (x - X(t) + \delta)^{2s}} - \frac{J_0(u(t, x))}{s B^{2s}} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z^{-1-2s} \,dz \right) \sup_{x+\xi > X(t)} (w_x(t, x + \xi))^2 \quad (4.22)$$

with ends the proof of the lemma since $u \leq \varepsilon$. \hspace{1cm} \Box

**Proposition 4.4.** For all $s \leq \frac{1}{4}$, $1 > \varepsilon > \theta, \kappa$ and $\gamma < \frac{1}{4}$ there exists $t_1$ such that

$$\mathcal{D}[u] + f(u) \geq \frac{1}{2} f(u) \quad \text{for all} \quad t \geq t_1, \quad X(t) < x < \sup\{X(t) + R_0; X_0\}.$$
Proof. Recall that (4.17) is still all the way valid in this new context. Thus, for any \( \delta > \sup\{ R_0, X(t) - x + B\} \), using that \( u \leq \varepsilon \), when \( X(t) < x \leq \sup\{ X(t) + R_0; X_\eta(t) \} \),

\[
\mathcal{D}[w](t, x) \geq - \frac{J_0 \varepsilon}{s} - \frac{3}{\varepsilon} \sup_{-B < \xi < B, x + \xi > X(t)} (w_x(t, x + \xi))^2 \left( J_1 + J_0 \int_1^B z^{1-2s} \, dz \right)
\]

Recall that since \( x \leq X_\eta(t) \), we have \( u \geq \theta + \eta \). On the other hand, observe that since \( x < X(t) + R_0 \) and since \( u \) is smooth we have

\[
\begin{align*}
\hat{u}(t, x) & \geq \hat{u}(t, X(t) + R_0) = \hat{u}(t, X(t)) + \hat{u}(t, X(t) + R_0) - \hat{u}(t, X(t)) \\
& = \hat{u}(t, X(t)) + R_0 \int_0^1 \hat{w}_x(t, X(t) + \tau) \, d\tau,
\end{align*}
\]

and by using the definition of \( \hat{w}_x \) in (4.11), and the convexity of \( x \rightarrow w(t, x) \) at any time, we deduce that

\[
\hat{u}(t, x) \geq \hat{u}(t, X(t)) + 3R_0 w_x(t, X(t)) = \varepsilon + 3R_0 w_x(t, X(t)).
\]

From the estimate of \( w_x(t, X(t)) \), (4.16), it follows that

\[
\hat{u}(t, x) \geq \varepsilon - \frac{6R_0 s \varepsilon}{X(t)},
\]

which, thanks to \( \lim_{t \to +\infty} X(t) = 0 \), enforces for \( t \geq t' \),

\[
\hat{u}(t, x) \geq \theta + \eta.
\]

In both cases, we then have for \( t \geq t' \),

\[
\hat{u}(t, x) \geq \theta + \eta.
\]

As a consequence \( f(u) > 0 \) for all \( x \leq \sup\{ X(t) + R_0, X_\eta \} \) and \( t > t' \). Specify \( B = \nu J_0 \tau \) with \( \nu > 1 \) to be chosen later on. Then from the above inequality we deduce that

\[
\mathcal{D}[\hat{u}](t, x) \geq - \frac{f(u)}{\nu^{2s}} - \frac{3}{\varepsilon} \sup_{-B < \xi < B, x + \xi > X(t)} (w_x(t, x + \xi))^2 \left( J_1 + J_0 \int_1^B z^{1-2s} \, dz \right) \text{ when } x > X(t),
\]

from which yields for \( X(t) < x < X(t) + R_0 \) and \( t \geq t^* \),

\[
\mathcal{D}[\hat{u}](t, x) + f(\hat{u}) \geq f(\hat{u}) \left( 1 - \frac{1}{\nu^{2s}} \right) - \frac{3}{\varepsilon} \sup_{-B < \xi < B, x + \xi > X(t)} (w_x(t, x + \xi))^2 \left( J_1 + J_0 \int_1^{\nu^{-1-s} J_0} z^{1-2s} \, dz \right).
\]

Recall that \( w \) is convex in \( x \), so that

\[
\sup_{-B < \xi < B, x + \xi > X(t)} (w_x(t, x + \xi))^2 = w_x(t, X(t))^2,
\]

and choose now \( \nu > \nu_0 := \sup \left\{ \frac{4}{\nu^{1-s}}, \frac{s(f(u))}{J_0 \tau} + 1 \right\} \), we then get using (4.16),

\[
\mathcal{D}[\hat{u}] + \frac{1}{2} f(\hat{u}) \geq \frac{1}{4} f(\hat{u}) - 4s^2 \varepsilon(1 - \gamma \varepsilon)^2 C_1 \left( \frac{1}{X(t)} \right)^2,
\]

with \( C_1 := 3 \left[ J_1 + J_0 \int_1^B z^{1-2s} \, dz \right] \).

Finally recalling that \( \lim_{t \to \infty} \frac{1}{X(t)} = 0 \), we may find \( t_1 \geq t^* \) such that for all \( t \geq t_1 \), the right hand side of the above expression is positive ending thus the proof of this proposition. \( \square \)
4.3.2 The region $x > \sup\{X(t) + R_0; X_0\}$

**Lemma 4.5.** For any time $t > 1$ and $x \geq \sup\{X(t) + R_0, X_2(t)\}$,

$$D[u](t, x) \geq \frac{\varepsilon - u(t, x)}{2sJ_0 x^{2s}} + J_1 \min_{-1 < \xi < 1} \frac{\partial u_x(t, x + \xi)}{2sB^{2s}} + \frac{3J_0}{4} \left( \int_1^B z^{-2s} dz \right) w_x(t, x). \quad (4.23)$$

**Proof.** Let us go back to the definition of $D[u](t, x)$ that we split into three parts:

$$D[u](t, x) = \int_{-\infty}^{-1} [u(t, x + z) - u(t, x)] J(z) \, dz + \int_{-1}^{1} [u(t, x + z) - u(t, x)] J(z) \, dz + \int_1^{\infty} [u(t, x + z) - u(t, x)] J(z) \, dz.$$

Since $x \geq X(t) + R_0$ and $u$ is decreasing, the first integral can be estimated as follows:

$$\int_{-\infty}^{-1} [u(t, x + z) - u(t, x)] J(z) \, dz \geq J_0^{-1} \int_{-\infty}^{X(t)-x} \frac{u(t, x + z) - u(t, x)}{z^{1+2s}} \, dz + \int_{X(t)-x}^{-1} [u(t, x + z) - u(t, x)] J(z) \, dz \geq \frac{J_0^{-1}}{2s} \frac{\varepsilon - u(t, x)}{(x - X(t))^{2s}}. \quad (4.24)$$

To obtain an estimate of the second integral, we actually follow the same steps as several times previously to obtain via Taylor expansion,

$$\int_{-1}^{1} [u(t, x + z) - u(t, x)] J(z) \, dz = \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} \frac{\partial u_x(t, x + \tau \sigma z)}{\tau \sigma z^2} J(z) \, d\tau d\sigma dz \geq J_1 \min_{-1 < \xi < 1} \frac{\partial u_x(t, x + \xi)}{2sB^{2s}}. \quad (4.25)$$

Finally, let us estimate the last integral

$$I := \int_{1}^{+\infty} [u(t, x + z) - u(t, x)] J(z) \, dz = \int_{1}^{B} [u(t, x + z) - u(t, x)] J(z) \, dz + \int_{B}^{+\infty} [u(t, x + z) - u(t, x)] J(z) \, dz,$$

for $B > 1$ to be chosen later on. Since $u$ is positive we have

$$\int_{B}^{+\infty} [u(t, x + z) - u(t, x)] J(z) \, dz \geq - \frac{J_0 u(t, x)}{2sB^{2s}}. \quad (4.26)$$

By using again a Taylor formula, the last integral rewrites

$$\int_{1}^{B} [u(t, x + z) - u(t, x)] J(z) \, dz = \int_{1}^{B} \int_{0}^{1} \frac{\partial u_x(t, x + \tau z)}{\tau z} J(z) \, d\tau dz.$$

Observe that since $x \geq X_2$ and $w$ is convex, (4.12) implies

$$\frac{\partial u_x(t, x + \tau z)}{\tau z} \geq \frac{3}{4} w_x(t, x).$$

It follows that

$$\int_{1}^{B} [u(t, x + z) - u(t, x)] J(z) \, dz \geq \frac{3}{4} \left( \int_{1}^{B} z J(z) \, dz \right) w_x(t, x) \geq \frac{3J_0}{4} \left( \int_1^B z^{-2s} dz \right) w_x(t, x). \quad (4.27)$$
using Hypothesis 1.1. Collecting (4.24), (4.25), (4.26), (4.27), we find for $x \geq X(t) + R_0$,

$$
\mathcal{D}[u](t, x) \geq \frac{\varepsilon - u(t, x)}{2sJ_0x^{2s}} + J_1 \min_{-1<\xi<1} u_{xx}(t, x + \xi) - \frac{J_0u(t, x)}{2sB^{2s}} + \frac{3J_0}{4} \left( \int_1^B z^{-2s} \, dz \right) w_x(t, x),
$$

which ends the proof of the lemma.

Recall that by Lemma 4.5, we have in the range $x \geq X(t) + R_0$,

$$
\mathcal{D}[u](t, x) \geq \frac{\varepsilon - u(t, x)}{2sJ_0x^{2s}} + J_1 \min_{-1<\xi<1} u_{xx}(t, x + \xi) - \frac{J_0u(t, x)}{2sB^{2s}} + \frac{3J_0}{4} \left( \int_1^B z^{-2s} \, dz \right) w_x(t, x).
$$

(4.28)

Let us now estimate $\mathcal{D}[u]$ when $x \geq \sup \{X(t) + R_0, X_\eta(t)\}$.

**Proposition 4.6.** For any $s < \frac{1}{2}$, $\kappa > 0$, $\varepsilon > \theta$ there exists $\gamma_0$ and $t_3 > 0$ such that for all $t \geq t_3$ and $\gamma_0 \leq \gamma < \frac{1}{\varepsilon}$

$$
\mathcal{D}[u](t, x) \geq \frac{\eta}{2sJ_0x^{2s}} \left( 1 - \frac{\mathcal{J}_0^2\kappa t(1 - \varepsilon\gamma)}{\eta B^{2s}} \right) + J_1 \min_{-1<\xi<1} u_{xx}(t, x + \xi) + 3J_0 \int_1^B z^{-2s} \, dz w_x(t, x).
$$

Let us rewrite $\gamma := \frac{-2s}{\varepsilon}$ with $\sigma \in (0, 1)$, then we have

$$
\mathcal{D}[u](t, x) \geq \frac{\eta}{2sJ_0x^{2s}} \left( 1 - \frac{\mathcal{J}_0^2\kappa t\sigma}{\eta B^{2s}} \right) + J_1 \min_{-1<\xi<1} u_{xx}(t, x + \xi) + 3J_0 \int_1^B z^{-2s} \, dz w_x(t, x).
$$

Let us now estimate from below $\min_{-1<\xi<1} u_{xx}(t, x + \xi)$. Using (4.12) and the convexity of $w$, we see that

$$
\min_{-1<\xi<1} u_{xx}(t, x + \xi) \geq -\frac{6}{\varepsilon} (w_x(t, x - 1))^2,
$$

which thanks to (4.14) and that $w(t, x - 1) \leq w(t, X(t)) = \varepsilon$ leads to

$$
\min_{-1<\xi<1} u_{xx}(t, x + \xi) \geq -24s^2\varepsilon^2 w^2(t, x - 1) \frac{(x - 1)^{4s - 2}}{(\kappa t)^2}.
$$

By proposition 4.1 since $X(t) + 1 \leq X(t) + R_0 \leq x$, we have

$$
w(t, x - 1) \leq \frac{\kappa t\sigma}{(x - 1)^{2s}}.
$$

and thus since $x \geq X_\eta(t)$

$$
\min_{-1<\xi<1} u_{xx}(t, x + \xi) \geq -24s^2\varepsilon^2 \frac{1}{(x - 1)^2} \geq -48s^2\varepsilon^2 \frac{1}{(X_\eta(t) - 1)^{2-2s}} \frac{1}{x^{2s}}.
$$

Now by using that $s \leq \frac{1}{2}$, the definition of $X_\eta(t)$, since $X_\eta(t) > (\kappa t)^{\frac{1}{2}} (\frac{\varepsilon}{\eta})^{\frac{1}{2}}$ we may find $t'(\kappa, s, \varepsilon)$ independent of $\sigma$ such that for all $t \geq t'$

$$
J_1 \min_{-1<\xi<1} u_{xx}(t, x + \xi) \geq -\frac{\eta}{8J_0x^{2s}}.
$$
Therefore for all \( t \geq t' \) we then get
\[
\mathcal{D}(\underline{u})(t,x) \geq \frac{\eta}{2s \mathcal{J}_0 x^{2s}} \left( \frac{3}{4} - \frac{\mathcal{J}_0^2 \kappa t \sigma}{\eta B^{2s}} \right) + 3 \mathcal{J}_0 \int_1^B z^{-2s} dz w_x(t,x).
\]
Again, by using Proposition 4.1, we deduce that
\[
w_x(t,x) = -2s \frac{x^{2s-1}}{\kappa t} w(t,x) \geq -2s \frac{(\kappa t)^2 \sigma^2 x^{2s-1}}{\kappa t} x^{1-2s} = -\frac{\kappa t}{x^{2s+1}} (2s \sigma^2),
\]
and we then have
\[
\mathcal{D}(\underline{u})(t,x) \geq \frac{\eta_0}{2s \mathcal{J}_0 x^{2s}} \left( \frac{3}{4} - \frac{\mathcal{J}_0^2 \kappa t \sigma}{\eta B^{2s}} - \frac{12s^2 \mathcal{J}_0^2 \sigma^2 \kappa t}{\eta x} \int_1^B z^{-2s} dz \right).
\]
Let \( C_1 := \frac{\mathcal{J}_0^2}{\eta} \) and \( C_2 := 12s^2 \), we then have
\[
\mathcal{D}(\underline{u})(t,x) \geq \frac{\eta}{2s \mathcal{J}_0 x^{2s}} \left( \frac{3}{4} - C_1 \kappa t \sigma \left[ \frac{1}{B^{2s}} + \frac{C_2 \sigma}{x} \int_1^B z^{-2s} \right] \right).
\]
We now treat the case \( s = \frac{k}{2} \) and \( s < \frac{k}{2} \)

Recall now that \( s < \frac{k}{2} \), so for \( B > 1 \) we have
\[
\int_1^B z^{-2s} dz \leq \frac{B^{1-2s}}{1-2s},
\]
and the above expression reduce to
\[
\mathcal{D}(\underline{u})(t,x) \geq \frac{\eta}{2s \mathcal{J}_0 x^{2s}} \left( \frac{3}{4} - C_1 \kappa t \sigma \left[ \frac{1}{B^{2s}} + \frac{C_2 \sigma}{x} B^{1-2s} \right] \right).
\]
with \( \mathcal{C}_2 := \frac{C_2}{1-2s} \).

Let us now take \( B = \frac{2s}{(1-2s)\kappa \sigma} \) and check whether \( B > 1 \). Since \( x > X_0(t) \), this means that
\[
B \geq \frac{2s}{\mathcal{C}_2 (1-2s) \sigma} \kappa \left( \frac{\eta}{\varepsilon^2} \right)^{\frac{1}{2}},
\]
so for \( t \geq t^* := \left( \frac{\mathcal{C}_2 (1-2s) \sigma}{s} \right)^{2s} \kappa \left( \frac{\eta}{\varepsilon^2} \right)^{\frac{1}{2}} \) we then have \( B > 2 \) and a short computations shows
\[
\left[ \frac{1}{B^{2s}} + \frac{\mathcal{C}_2 \sigma}{x} B^{1-2s} \right] = (1-2s) \frac{\mathcal{C}_2^2 \sigma^2 x^{2s}}{(2s \kappa x)^2} + \frac{\mathcal{C}_2 \sigma}{x} \frac{(2s \kappa x)^{1-2s}}{\sigma^{1-2s} \mathcal{C}_2^{1-2s} (1-2s)^{1-2s}} = \left( 1-2s \right) \frac{\mathcal{C}_2^2 \sigma^2 x^{2s}}{(2s \kappa x)^2} + \frac{\mathcal{C}_2^2 \sigma^2 x^{2s} (2s \kappa x)^{1-2s}}{\sigma^{1-2s} (1-2s)^{2-2s}} = \frac{\sigma^{2s}}{x^{2s}} C_3
\]
with \( C_3 := \frac{\mathcal{C}_2}{2} \left( \frac{(1-2s) \sigma}{2s \kappa x} + \frac{(2s \kappa x)^{1-2s}}{(1-2s)^{1-2s}} \right) \).

As a consequence, we have
\[
\mathcal{D}(\underline{u})(t,x) \geq \frac{\eta}{2s \mathcal{J}_0 x^{2s}} \left( \frac{3}{4} - C_1 C_3 \kappa t \sigma^{1+2s} x^{2s} \right).
\]
Since \( x \geq X(t) \), by exploiting (3.6), the definition of \( X(t) \), we then achieve
\[ D[\underline{u}](t, x) \geq \frac{\eta}{2s^{0}x^{2s}} \left( \frac{3}{4} - \frac{C_{1}C_{3}\kappa t^{1+2s}}{X^{2s}(t)} \right) \]

\[ \geq \frac{\eta}{2s^{0}x^{2s}} \left( \frac{3}{4} - C_{1}C_{3}\sigma^{2s} \right) \]

and the proposition is then proved by taking \( \sigma \) small and \( t \geq t_{3} := \sup\{t', t^{*}\} \).

\[ \square \]

4.4 Tuning the parameter \( \kappa \)

In this last part of the proof, we choose our parameter \( \kappa \) in order that for some \( t^{*} > 0 \), \( \underline{u} \) is indeed a sub-solution to (1.1) for \( t \geq t^{*} \). Recall that \( \underline{u} \) is a sub-solution if and only if (4.8) and (4.9) hold simultaneously. Since (4.8) holds unconditionally for \( t \) sufficiently large, the only thing left to check is that (4.9) holds for a suitable choice of \( \kappa \).

By using (4.11) and (4.13), (4.9) holds if particular

\[ 3 \frac{x^{2s}}{\kappa t^{2}} w^{2}(t, x) \leq D[\underline{u}](t, x) + f(\underline{u}), \quad x > X(t), \]

Set \( t^{*} := \sup\{t_{0}, t_{1}, t_{2}, t_{3}\} \), where \( t_{0}, t_{1}, t_{2} \) and \( t_{3} \) are respectively determined by Proposition 4.2, Proposition 4.4 and Proposition 4.6. To make our choice, let us decompose the set \( [X(t), +\infty) = I_{1} + I_{2} \) into two subsets defined as follows

\[ I_{1} := [X(t), X_{\eta}(t)], \quad I_{2} := [X_{\eta}(t), +\infty). \]

On the first interval, we have

**Lemma 4.7.** For all \( 0 < s \leq \frac{1}{4}, \varepsilon > \theta \), there exists \( t_{4}(\varepsilon) \geq t^{*} \) such that for all \( \kappa \), one has

\[ 3 \frac{x^{2s}}{\kappa t^{2}} w^{2}(t, x) \leq D[\underline{u}](t, x) + f(\underline{u}), \quad \text{for all} \quad x \in I_{1}. \]

**Proof.** First observe that since \( x \geq X_{\eta} \), by the previous proofs, we know that \( \underline{u} \geq \theta + \eta \) and thus \( f(\underline{u}) > \min_{s \in [\theta + \eta, \varepsilon]} f(s) =: m_{0} > 0 \) for all \( t \geq t^{*} \) and \( x \in I_{1} \).

Now by exploiting the definition of \( X_{\eta}(t) \) and \( X(t) \) it follows that \( x \leq \left( \kappa t \left[ 1 + \frac{1}{\varepsilon - \eta} \right] \right)^{-1} \), and that

\[ 3 \frac{x^{2s}}{\kappa t^{2}} w^{2}(t, x) \leq \frac{3}{t} \varepsilon^{2} \left[ 1 + \frac{1}{\varepsilon - \eta} \right] , \]

which by taking \( t \) large then yields, says \( t \geq t' \),

\[ 3 \frac{x^{2s}}{\kappa t^{2}} w^{2}(t, x) \leq \frac{f(\underline{u})}{2} . \]

Recall that by Proposition 4.4, we have for all \( x \in I_{1} \) and \( t \geq t^{*} \)

\[ D[\underline{u}] + f(\underline{u}) \geq \frac{f(\underline{u})}{2} . \]

We then end our proof by taking \( \gamma^{*} := \inf\{\gamma_{0}, \frac{1}{4\kappa} \} \) and \( t \geq t_{4} := \sup\{t', t^{*}\} \).

Finally, let us check what happens on \( I_{2} \),

**Claim 4.8.** There exists \( \kappa^{*} \) and \( t_{5} \geq t^{*} \) such that for all \( \kappa \leq \kappa^{*} \),

\[ 3 \frac{x^{2s}}{\kappa t^{2}} w^{2}(t, x) \leq D[\underline{u}] + f(\underline{u}), \quad \text{for all} \quad x \in I_{2}. \]
Proof. By Proposition 4.1, we have for $x \in I_2$ and $t \geq t^*$

$$w(t, x) \leq \frac{kt}{x^{2s}}[1 + \varepsilon],$$

Therefore, we have

$$3\frac{x^{2s}}{kt^2}w^2(t, x) \leq 3\frac{x^{2s}}{kt^2} \frac{(kt)^2}{x^{4s}}[1 + \varepsilon]^2 = \frac{\kappa[1 + \varepsilon]^2}{x^{2s}}.$$  

Now recall that by Proposition 4.2, we have for all $x \in I_2$ and $t \geq t^*$

$$\mathcal{D}[u] + f(u) \geq \frac{\eta}{16\mathcal{J}_0 x^{2s}}.$$  

The claim is then proved by taking $\kappa \leq \kappa^* := \frac{1}{48\mathcal{J}_0 x^{1+\varepsilon}}$.  

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