Analytical Expressions of Masses and Mixings
in a Democratic Seesaw Mass Matrix Model

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Abstract

On the basis of a seesaw-type mass matrix model $M_f \simeq m_L M_F^{-1} m_R$ for quarks and leptons $f$, analytical expressions of the masses and mixings of the fermions $f$ are investigated. Here, the matrices $m_L$ and $m_R$ are common to all $f$ (up- and down-; quarks and leptons), and the matrix $M_F$ characterizing the heavy fermion sector has the form $[(\text{unit matrix}) + (\text{democratic-type matrix})]$. An application to the quark sectors is discussed.

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§1. Introduction

Why is the top quark mass \( m_t \) so enhanced compared with the bottom quark mass \( m_b \)? Why is the \( u \)-quark mass \( m_u \) of the order of the \( d \)-quark mass \( m_d \)? In most models, in order to understand \( m_t \gg m_b \), it is inevitable to bring in a parameter which takes hierarchically different values between up- and down-quark sectors. However, from the point of view of the “democracy of families”, such a hierarchical difference seems to be unnatural. What is of great interest to us is whether we can find a model in which \( M_u \) and \( M_d \) are almost symmetric in their matrix structures and in their parameter values.

Recently, by applying the so-called “seesaw” mechanism\(^1\) to quark mass matrix,\(^2\) the authors\(^3\) have proposed a model which provides explanations of both \( m_t \gg m_b \) and \( m_u \sim m_d \), while keeping the model “almost” up-down symmetric. The essential idea is as follows: the mass matrices \( M_f \) of quarks and leptons \( f_i \) (\( i = 1, 2, 3 \): family index) are given by

\[
M_f \simeq -m_L M_F^{-1} m_R, \tag{1.1}
\]

where \( F_i \) denote heavy fermions \( U_i, D_i, N_i \) and \( E_i \), corresponding to \( f_i = u_i, d_i, \nu_i \) and \( e_i \), respectively. They have assumed that the mass matrix \( m_L \) (\( m_R \)) between \( f_L \) (\( f_R \)) and \( F_R \) (\( F_L \)) is common to all \( f = u, d, \nu, e \) (i.e., independently of up-/down- and quark-/lepton- sectors) and \( m_R \) is proportional to \( m_L \), i.e., \( m_R = \kappa m_L \). The variety of \( M_f \) (\( f = u, d, \nu, e \)) comes only from the variety of the heavy fermion matrix \( M_F \) (\( F = U, D, N, E \)). If we take a parametrization which gives \( \det M_U \simeq 0 \) in the up-quark sector, but which does not give \( \det M_D \simeq 0 \) in down-quark sector, the model can provide \( m_t \gg m_b \), keeping the model “almost” up-down symmetric because of the factor \( M_F^{-1} \) in the seesaw expression (1.1). On the other hand, they have taken \( M_F = m_0 \lambda O_f \) as the form of the heavy fermion mass matrix \( M_F \), where

\[
O_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b_f e^{i\beta_f} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \equiv 1 + 3 b_f e^{i\beta_f} X, \tag{1.2}
\]

and \( \lambda \) is an enhancement factor with \( \lambda \gg \kappa \gg 1 \). Note that the inverse of the matrix \( O_f \) is again given by the form\(^4\) \([(a \text{ unit matrix}) + (a \text{ democratic matrix})]\), i.e.,

\[
O_f^{-1} = 1 + 3 a_f e^{i\alpha_f} X, \quad \tag{1.3}
\]
with
\[ a_f e^{i\alpha_f} = -\frac{b_f e^{i\beta_f}}{1 + 3b_f e^{i\beta_f}}. \] (1.4)

Thus, we can provide top-quark mass enhancement \( m_t \gg m_b \) in the limit of \( b_u e^{i\beta_u} \to -1/3 \), because it leads to \( |a_u| \to \infty \). On the other hand, since a democratic mass matrix\(^5\) makes only one family heavy, we can keep \( m_u \sim m_d \).

They have taken
\[ m_L = \frac{1}{\kappa} m_R = m_0 Z = m_0 \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix}, \] (1.5)

where \( z_i \) are normalized as \( z_1^2 + z_2^2 + z_3^2 = 1 \) and given by
\[ \frac{z_1}{\sqrt{m_e}} = \frac{z_2}{\sqrt{m_\mu}} = \frac{z_3}{\sqrt{m_\tau}} = \frac{1}{\sqrt{m_e + m_\mu + m_\tau}}, \] (1.6)
in order to give the charged lepton mass matrix \( M_e \) for the case \( b_e = 0 \), i.e., \( M_e = m_0(\kappa/\lambda)Z^2 \). They have obtained\(^3\) reasonable quark mass ratios and Cabibbo-Kobayashi-Maskawa\(^6\) (CKM) matrix parameters by taking \( \kappa/\lambda = 0.02, b_u = -1/3, \beta_u = 0, b_d \simeq -1 \) and \( |\beta_d| \simeq 18^\circ \).

However although they numerically evaluated the behavior of the CKM matrix elements to the parameters \( \kappa/\lambda, b_f \) and \( \beta_f \), they did not give analytical expressions of the CKM matrix elements. Therefore, of their results, we cannot see which are results only for a special choice of the parameters and which are (almost) parameter-independent ones. For example, they predicted a value \( |V_{cb}| = 0.0598 \), which is somewhat large compared with the recent experimental value\(^7\) \( |V_{cb}| = 0.041 \pm 0.003 \). However, we cannot see whether the discrepancy is a fatal defect in this model or not.

What is of great interest to us is to clarify the general features of the democratic seesaw mass matrix, without confining ourselves to the phenomenology of the quark masses and CKM matrix elements. It is also interesting to apply the model to other fermion systems, for example, to neutrino sector, a hypothetical fermion system, and so on. For this purpose, it is inevitable to obtain analytic expressions
of the fermion masses $m^f_i$ and the family-mixing matrix $U^f_L$ for arbitrary values of the parameters $b_f$ and $\beta_f$, and not to give such the numerical study as in Ref. 3). In §3, we will give general expressions of the fermion masses $m^f_i$ for arbitrary $b_f$ and $\beta_f$, although the cases of $b_f = -1/3$ and $b_f \simeq -1$ have already been in Ref. 3). In §4, we will obtain a general expression of the $3 \times 3$ family-mixing matrix $U^f_L$ for arbitrary values of the parameters $b_f$ and $\beta_f$.

Since the previous paper\(^3\) put stress on the “economy of adjustable parameters” of the model, the predictions were done by adjusting only three parameters $\kappa/\lambda$, $b_d$ and $\beta_d$. As a result, some of the predictions were in poor agreement with experiment. In the present paper, we will loosen the parameter constraints in the previous model\(^3\) (we will bring two additional phase parameters into the model). As a result, the predictability of the model decreases. However, the purpose of the present paper is not to improve the previous quark mass matrix model, but to investigate more general features of a democratic seesaw mass matrix model without confining ourselves in the quark mass matrix phenomenology.

As an application of our general study to the quark sectors, in §5, we will discuss analytical expression of the CKM matrix. In §6, we will give re-fitting of the CKM matrix parameters. Also, a possible shape of the unitary triangle $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ in our model will be discussed.

The final section §7 will be devoted to the summary and discussion.

§2. Assumptions for the model

In the present model, quarks and leptons $f_i$ belong to $f_L = (2, 1)$ and $f_R = (1, 2)$ of SU(2)$_L \times$ SU(2)$_R$ and heavy fermions $F_i$ are vector-like, i.e., $F_L = (1, 1)$ and $F_R = (1, 1)$. The vector-like fermions $F = (F_1, F_2, F_3)$ acquire masses $M_F$ at a large energy scale $\mu = m_0 \lambda$. The SU(2)$_L$ and SU(2)$_R$ symmetries are broken by Higgs bosons $\phi_L = (\phi^+_L, \phi^0_L)$ and $\phi_R = (\phi^+_R, \phi^0_R)$, which belong to (2,1) and (1,2) of SU(2)$_L \times$ SU(2)$_R$, at energy scales $\mu = m_0$ and $\mu = m_0 \kappa$, respectively.

Let us summarize the fundamental assumptions in the previous paper\(^3\) before starting our analytical study of the democratic seesaw mass matrix model.

[Assumption I] The $6 \times 6$ mass matrix $M$ for the fermions $(f, F)$ has a would-be “seesaw” form

$$
(\mathcal{F} F)_L M \begin{pmatrix} f \\ F \end{pmatrix}_R = (\mathcal{F} F)_L \begin{pmatrix} 0 & m_L \\ m_R & M_F \end{pmatrix} \begin{pmatrix} f \\ F \end{pmatrix}_R,
$$

(2.1)
there is no Higgs boson which belongs to $(2,2)$ of $SU(2)_L \times SU(2)_R$, in contrast with the conventional $SU(2)_L \times SU(2)_R$ model.\textsuperscript{8)}

[Assumption II] The structure of $m_R$ is the same as that of $m_L$ except for a constant coefficient $\kappa$, \textit{i.e.},

\[
m_R = \kappa m_L .
\]

[Assumption III] The heavy fermion mass matrix $M_F$ takes a form $[(a$ unit matrix$) + (a$ rank-one matrix$)]$, \textit{i.e.},

\[
M_F = m_0 \lambda_F (1 + 3b_f e^{i\beta_f} R_1) ,
\]

where $R_1$ is an arbitrary rank-one matrix.

The requirement that the matrix $R_1$ is a rank-one matrix is indispensable to realize that the choice $\text{det} M_F(b_f) = 0$ makes a mass of only one fermion heavy, \textit{i.e.}, $m_t \gg m_c > m_u$ with keeping $m_u \sim m_d$. Note that at this stage, it is not necessary to assume that the matrix $R_1$ has a democratic form as defined by (1.2). Without losing generality, we can take a favorite family-basis of the heavy fermions $F = (F_1, F_2, F_3)$. However, in order to obtain the successful fitting of the quark masses and CKM mixings in Ref. 3), the following assumption is essential.

[Assumption IV] When we choose the family-basis where $R_1$ takes the democratic form

\[
R_1 = X \equiv \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} ,
\]

the matrix $m_L$ takes a diagonal form $m_0 Z$, (1.5).

If we take another family-basis $(f', F') = (Af, AF)$, the mass matrix $M'$ for $(f', F')$ is given by

\[
M' = \begin{pmatrix} 0 & m'_L \\ m'_R & M'_F \end{pmatrix} = \begin{pmatrix} 0 & Am_L A^\dagger \\ Am_R A^\dagger & AM_F A^\dagger \end{pmatrix} .
\]

Without losing generality, we can choose a basis on which $M'_F$ takes a diagonal form

\[
M'_F = m_0 \lambda_F \text{diag}(1, 1, 1 + 3b_f e^{i\beta_f}) .
\]

Therefore, Assumption IV can be replaced with the following expression:
[Assumption IV'] On the family-basis on which $M'_F$ is diagonal, the mass matrices $m'_L$ and $m'_R$ are given on the family-basis $f' = (f'_1, f'_2, f'_3)$ which consists of representations of the permutation group $S_3$ of three elements, i.e.,

\[
\begin{align*}
  f'_1 &= \frac{1}{\sqrt{2}} (f_1 - f_2), \\
  f'_2 &= \frac{1}{\sqrt{6}} (f_1 + f_2 - 2f_3), \\
  f'_3 &= \frac{1}{\sqrt{3}} (f_1 + f_2 + f_3),
\end{align*}
\]

where $f_i$ are fermion states in which $m'_L$ and $m'_R$ are diagonalized.

In any expressions IV and IV', it is essential that $M_F$ is given by a form $[(a \text{ unit matrix}) + (a \text{ democratic matrix})]$ on the family-basis on which $m_L$ and $m_R$ take diagonal forms. For a mechanism which generates such a democratic mass matrix, some ideas have been proposed: a permutation symmetry of three elements $S_3$, a composite model based on an analogy of hadronic $\pi^0$-$\eta$-$\eta'$ mixing, a BCS-like mechanism, and so on. However, the purpose of the present paper is not to investigates the origin of the democratic mass matrix form. We do not touch the origin of the form (2.4).

In the numerical study for the quark sectors, the coefficient $\lambda_F$ will be assumed as $\lambda_U \simeq \lambda_D \equiv \lambda_Q \neq \lambda_E$, because the evolution effects of Yukawa coupling constants can be different according as the fermions have color or not, even if $\lambda_U = \lambda_D = \lambda_E$ at a unification energy scale.

As we stated in §1, in the present paper, we will loosen parameter constraints in the previous model and we will bring two additional phase parameters $\delta_2$ and $\delta_3$ into the CKM-matrix phenomenology. We assume that the Higgs bosons $\phi_L$ and $\phi_R$ couple to the fermions universally, but with the degree of freedom of their phases, as follows:

\[
H_{Yukawa} = \sum_{i=1}^{3} (\bar{u}_{L_i} \bar{d}_{L_i}) \left( y_{L_i} \exp(i\delta^d_{L_i}) \right) \begin{pmatrix} \phi^+_L \\ \phi^0_L \\ \phi^-_L \end{pmatrix} D_{R_i}
\]

\[
+ \sum_{i=1}^{3} (\bar{u}_{L_i} \bar{d}_{L_i}) (y_{L_i} \exp(i\delta^u_{L_i})) \begin{pmatrix} \phi^0_L \\ \phi^+_L \\ -\phi^-_L \end{pmatrix} U_{R_i}
\]

\[+ h.c. + (L \leftrightarrow R) + [(u, d, U, D) \to (\nu, e, N, E)] ,
\]
where \( y_{Li} \) and \( y_{Ri} \) are real parameters, and they are universal for the quark and lepton sectors. Therefore, the matrices \( m_L \) and \( m_R \) in (2.1) are replaced with

\[
m^f_L = m_0 P^f_L Z \equiv m_0 \text{diag} \left( z_1 \exp(i\delta^f_{L1}), z_2 \exp(i\delta^f_{L2}), z_3 \exp(i\delta^f_{L3}) \right),
\]

and \( m^f_R = m_0 \kappa P^f_R Z \), respectively, where \( P^f_L \) and \( P^f_R \) are phase matrices.

For these phase parameters, the CKM matrix is dependent only on \( P^u_L \dagger P^d_L \equiv P = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) \).

Of the three parameters \( \delta_i \) (\( i = 1, 2, 3 \)), only two are observable. Without losing generality, we can put \( \delta_1 = 0 \). In the present model, the nine observable quantities (five quark mass ratios and four CKM matrix parameters) are described by the seven parameters \( (\kappa / \lambda, b_u, b_d, \beta_u, \beta_d, \delta_2, \delta_3) \). Since we put the ansatz “maximal top-quark-mass enhancement” according to the Ref. 3), we fix \( b_u \) and \( \beta_u \) at \( b_u = -1/3 \) and \( \beta_u = 0 \). However, we still possess five free parameters. In order to economize in the number of the free parameters, we will give some speculation on these parameters in the final section. On the other hand, since the phases \( \delta_{Li}^e \) and \( \delta_{Ri}^e \) are not observable, we can put \( P^e_L = P^e_R = 1 \).

§3. General expressions of the fermion masses

The general case \( P^f_L \neq 1 \) does not change the previous results\(^3\) as far as the mass ratios are concerned. Quark masses in terms of charged lepton masses have already been given in Ref. 3). However, the previous expressions were only those for the cases of \( (b_f = -1/3, \beta_f = 0) \), and \( (b_f \simeq -1, \beta_f \simeq 0) \). In the present paper, we will give general expressions for arbitrary values of \( b_f \) and \( \beta_f \).

Note that for the case \( b_f = -1/3 \) the seesaw expression (1.1) is not valid any longer because of \( \det M_F = 0 \). In Fig. 1, we illustrate the numerical behavior of fermion masses \( m^f_i \) (\( i = 1, 2, \cdots, 6 \)) versus the parameter \( b_f \) which has been evaluated from the \( 6 \times 6 \) matrix (2.1) without approximation (the behavior of \( m^f_i \) with \( i = 1, 2, 3 \) has been illustrated in Ref. 3). As seen in Fig. 1, the third fermion is sharply enhanced at \( b_f = -1/3 \) for \( \beta_f = 0 \). The calculation for the case \( b_f \simeq -1/3 \) must be done carefully.

For the case in which the seesaw expression (1.1) is in a good approximation, i.e., except for \( b_f e^{i\beta_f} \simeq -1/3 \), we can obtain simpler expressions of \( m^f_i \) :

\[
m^f_1 = z_1^2 \frac{c^f_0}{c^f_1} \frac{\kappa}{\lambda} m_0 ,
\]

(3.1)
\[ m_2^f = z_2^2 \left| \frac{c_2^f}{c_1^f} \right| \frac{\kappa}{\lambda} m_0 , \]  
\[ m_3^f = z_3^2 \left| \frac{c_3^f}{c_2^f} \right| \frac{\kappa}{\lambda} m_0 , \]  
where the functions \( c_n^f \equiv c_n(b_f, \beta_f) \) \((n = 1, 2, 3)\) are defined by
\[ c_n^f \equiv c_n(b_f, \beta_f) = n + \frac{1}{b_f e^{i\beta_f}} . \]

Although the expressions (3.1) – (3.3) are not valid for the cases \( b_f e^{i\beta_f} = -1, -1/2 \) and \(-1/3\), these are still very useful for the case \( \beta_f \neq 0 \).

More precise expressions for arbitrary values of \( b_f \) and \( \beta_f \) are obtained as follows. For the case of \( \lambda \gg \kappa \gg 1 \), by expanding the eigenvalues \( m_i^f \) \((i = 1, 2, 3)\) of the mass matrix (2.1) in \( \kappa/\lambda \), we obtain the following expressions of \( m_i^f \):
\( g(b, \beta) = (1 + 2b)^2 - 2(1 + b)(1 + 3b)\rho - 8b(1 - 2\rho)\sin^2 \frac{\beta}{2} \), \hspace{1cm} (3.9)

\[ h(b, \beta) = (1 + 3b)^2 - 12b\sin^2 \frac{\beta}{2} \], \hspace{1cm} (3.10)

\[ \rho = z_1^2 + z_2^2 + z_3^2, \]
\[ \sigma = z_1^2 + z_2^2 + z_3^2, \]

and for simplicity we have denoted \( b_f \) and \( \beta_f \) as \( b \) and \( \beta \).

Now let us apply the results (3.5)–(3.7) to the quark masses. Our interest is in the cases \( b_f \approx -1/3 \) and \( b_f \approx -1 \) whose values are favorable to the fitting of the up- and down-quark masses, respectively. The explicit expressions are as follows:

\[ m_u \approx \frac{3\sigma}{2\rho} \left( 1 + \frac{\sigma}{4\rho^2} - \frac{3}{2}\varepsilon_u \right) \frac{\kappa}{\lambda_U} m_0 \approx \frac{3m_e}{2m_\tau} \frac{\kappa}{\lambda_U} m_0, \]

\[ m_c \approx 2\rho \left[ 1 - \frac{3\sigma}{4\rho^2} - \frac{9}{2} (1 - \frac{8}{3}\rho) \varepsilon_u \right] \frac{\kappa}{\lambda_U} m_0 \approx \frac{2m_\mu}{m_\tau} \frac{\kappa}{\lambda_U} m_0, \]

\[ m_t \approx \frac{1}{\sqrt{3}} \frac{1}{\sqrt{1 + 27\varepsilon_u^2 \lambda_U^2 / \kappa^2}} m_0 \approx \frac{1}{\sqrt{3}} m_0, \]

\[ m_d \approx \frac{\sigma}{2|\sin(\beta_d/2)|\rho} \left( 1 + \frac{1}{2}\varepsilon_d \right) \frac{\kappa}{\lambda_D} m_0 \approx \frac{1}{2|\sin(\beta_d/2)|} \frac{m_e}{m_\tau} \frac{\kappa}{\lambda_D} m_0, \]

\[ m_s \approx 2 \left( 1 + \frac{3}{2}\varepsilon_d - 2\sin^2 \beta_d \right) |\sin \beta_d/2| \frac{\rho}{\lambda_D} m_0 \approx 2 |\sin \beta_d/2| \frac{1}{2} \frac{m_\mu}{m_\tau} \frac{\kappa}{\lambda_D} m_0, \]

\[ m_b \approx \frac{1}{2} \left( 1 - \frac{1}{2}\varepsilon_d + \frac{5}{2}\sin^2 \beta_d \right) \frac{\kappa}{\lambda_D} m_0 \approx \frac{1}{2} \frac{\kappa}{\lambda_D} m_0, \]

where the small parameters \( \varepsilon_u \) and \( \varepsilon_d \) are defined by

\[ b_u = - \frac{1}{3} + \varepsilon_u, \]
\[ b_d = -1 + \varepsilon_d. \]

(3.19)
Here, we have taken $\beta_u = 0$, because top-quark enhancement is caused only for the case of $\beta_u = 0$ (see Fig. 1). For down-quark masses, we have shown only the expressions for $b_d \simeq -1$ and $1 \gg \sin \beta_d \neq 0$, because from the numerical study in Ref. 3), we know that the observed down-quark mass spectrum is in favor of $b_d \simeq -1$ and $|\beta_d| \simeq 20^\circ$.

The expressions (3.13) – (3.19) lead to the following relations which are almost independent of the parameters $\kappa/\lambda$ ($\lambda \equiv \lambda_U = \lambda_D$), $\varepsilon_u$, $\varepsilon_d$ and $\beta_d$:

$$
\frac{m_u}{m_c} \simeq \frac{3}{4} \frac{m_e}{m_\mu} , \quad (3.20)
$$

$$
\frac{m_c}{m_b} \simeq \frac{4}{9} \frac{m_\mu}{m_\tau} , \quad (3.21)
$$

$$
\frac{m_d m_s}{m_b^2} \simeq \frac{4}{3} \frac{m_e m_\mu}{m_\tau^2} , \quad (3.22)
$$

$$
\frac{m_u}{m_d} \simeq \frac{3}{4} \frac{m_s}{m_c} \simeq \frac{3}{4} \frac{m_d m_\tau}{m_b m_\mu} \simeq 3 \left| \sin \frac{\beta_d}{2} \right| . \quad (3.23)
$$

The expressions (3.20) and (3.23) have already been given in Refs. 12) and 3), respectively. However, note that these relations are valid only for small value of $\varepsilon_u$ and $\varepsilon_d$, and not for general value of $b_f$.

In the limit of unbroken SU(2)$_L \times$SU(2)$_R$, i.e., $m_L = m_R = 0$, heavy fermion masses $m_{F_i'}$ are given by

$$
m_{F'_1} = m_{F'_2} = \lambda_F m_0 ,
$$

$$
m_{F'_3} = \sqrt{1 + 6 b_f \cos \beta_f + 9 b_f^2 \lambda_F m_0} , \quad (3.24)
$$

where $F'_i$ are mass-eigenstates for the mass matrix $M_F = m_0 \lambda_F O_f$. As seen from (3.24), the minimum condition of the sum of the up-heavy-quark masses leads to $\beta_u = 0$ and $b_u = -1/3$. Therefore, the ansatz “maximal top-quark-mass enhancement” can be replaced by another expression that the parameters $(b_u, \beta_u)$ are fixed such that the sum of the up-heavy-quark masses becomes a minimum.

For the case of $Z \neq 0$, the heavy fermion masses are given by

$$
m_4^e \simeq m_5^e \simeq m_6^e \simeq \lambda_E m_0 , \quad (3.25)
$$
\[ m_4^u \simeq \frac{1}{\sqrt{3}} \kappa m_0, \quad m_5^u \simeq m_6^u \simeq \lambda_U m_0, \]  
(3.26)

\[ m_4^d \simeq m_5^d \simeq \lambda_D m_0, \quad m_6^d \simeq 2\sqrt{1 + 3 \sin^2(\beta_d/2)} \lambda_D m_0, \]  
(3.27)

where the numbering of \( m_i^f \) has been defined as \( m_4^f \leq m_5^f \leq m_6^f \) in the mass eigenstates \( F_i' \) (\( i = 1, 2, 3 \)). Note that only the fourth up-quark \( u_4 \) (\( \equiv U'_3 \)) is remarkably light compared with other heavy fermions. The enhancement of the top-quark \( u_3 \) (\( \equiv t \)) is caused at the cost of the lightening of \( U'_3 \). Since the mass ratio \( m_4^u / m_3^u \) is given by

\[ m_4^u / m_t \simeq \kappa \]  
(3.28)

and \( \kappa \) is of the order of \( m(W_R) / m(W_L) \), we can expect the observation of the fourth up-quark \( u_4 \) at an energy scale at which we can observe the right-handed weak bosons \( W_R \).

\section*{§4. General expression of family-mixing matrix}

We diagonalize the \( 6 \times 6 \) mass matrix \( M, (2.1) \), by the following two steps. As the first step, we transform the mass matrix \( M \) into

\[ M' = \begin{pmatrix} M_{11}' & 0 \\ 0 & M_{22}' \end{pmatrix} \equiv \begin{pmatrix} M_f & 0 \\ 0 & M_F' \end{pmatrix}. \]  
(4.1)

At the second step, we diagonalize the \( 3 \times 3 \) matrix \( M_f \equiv M_{11}' \) with \( P_f^L = P_f^R = 1 \) (which we denote as \( \tilde{M}_f \)) by two unitary matrices \( U_f^L \) and \( U_f^R \) as follows:

\[ U_f^L \tilde{M}_f U_f^{R\dagger} = D_f, \]  
(4.2)

where \( D_f = \text{diag}(m_1^f, m_2^f, m_3^f) \). Then, the CKM matrix \( V \) is given by

\[ V \simeq U_f^L P U_f^{R\dagger}, \]  
(4.3)

where the phase matrix \( P \) is defined by (2.11) and terms with the order of \( \lambda^{-2} \) which come from the \( f-F \) mixing have been neglected.
We denote the unitary matrix $U_L^f$ as

$$U_L^f \simeq \begin{pmatrix}
1 - \varepsilon^f_1 & (1 - \varepsilon^f_{12})p_f \frac{z_1}{z_2} & (1 - \varepsilon^f_{13})p_f \frac{z_1}{z_3} \\
-(1 - \varepsilon^f_{21})p_f^* \frac{z_1}{z_2} & 1 - \varepsilon^f_2 & (1 - \varepsilon^f_{23})q_f \frac{z_2}{z_3} \\
-(1 - \varepsilon^f_{31})q_f^* \frac{z_1}{z_3} & -(1 - \varepsilon^f_{32})q_f^* \frac{z_2}{z_3} & 1 - \varepsilon^f_3
\end{pmatrix}, \quad (4.4)$$

where the functions $p_f \equiv p(b_f, \beta_f)$ and $q_f \equiv q(b_f, \beta_f)$ are given by

$$p_f \equiv p(b_f, \beta_f) = \frac{b_f e^{i\beta_f}}{1 + b_f e^{i\beta_f}} = \frac{1}{c^f_1}, \quad (4.5)$$

$$q_f \equiv q(b_f, \beta_f) = \frac{b_f e^{i\beta_f}}{1 + 2b_f e^{i\beta_f}} = \frac{1}{c^f_2}, \quad (4.6)$$

with the relation

$$c^f_2 - c^f_1 = 1 \quad (4.7)$$

The next leading terms $\varepsilon^f_i$ and $\varepsilon^f_{ij}$ are obtained by putting the expression (4.4) into the unitary condition $U_L^f U_L^{f\dagger} = 1$ and the diagonalization condition $U_L^f \tilde{M}_f \tilde{M}_f^\dagger U_L^{f\dagger} = D_f^2$:

$$\varepsilon^f_1 = \frac{1}{2} \left( \frac{z_1}{z_2} \right)^2 \frac{1}{|c^f_1|^2}, \quad (4.8)$$

$$\varepsilon^f_3 = \frac{1}{2} \left( \frac{z_2}{z_3} \right)^2 \frac{1}{|c^f_2|^2}, \quad (4.9)$$

$$\varepsilon^f_2 = \varepsilon^f_1 + \varepsilon^f_3, \quad (4.10)$$

$$\varepsilon^f_{12} = (3 - 2c^f_1 c^f_2)\varepsilon^f_1, \quad (4.11)$$

$$\varepsilon^f_{21} = (3 - 2c^f_1 c^f_2)\varepsilon^f_1 - (c^f_2 + c^f_2^*)\varepsilon^f_3, \quad (4.12)$$

$$\varepsilon^f_{23} = (c^f_1 + c^f_2)\varepsilon^f_1 + (3 - 2c^f_2 c^f_2)\varepsilon^f_3, \quad (4.13)$$

$$\varepsilon^f_{32} = (3 - 2c^f_2 c^f_2)\varepsilon^f_3, \quad (4.14)$$
\[
\varepsilon'_{13} = (3 - 2c_1^f)\varepsilon_1^f, \tag{4.15}
\]
\[
\varepsilon'_{31} = (3 + 2c_2^f)\varepsilon_3^f. \tag{4.16}
\]

The expression (4.4) is valid as far as we can regard \(\varepsilon_1^f\) and \(\varepsilon_3^f\) as \(\varepsilon_1^f \ll 1\) and \(\varepsilon_3^f \ll 1\), i.e.,
\[
|c_f^1|^2 = \left(\frac{1}{b_f} + 1\right)^2 - \frac{4}{b_f} \sin^2 \frac{\beta_f}{2} \gg \frac{1}{2} \left(\frac{z_1}{z_2}\right)^2 = 0.0024, \tag{4.17}
\]
\[
|c_f^2|^2 = \left(\frac{1}{b_f} + 2\right)^2 - \frac{8}{b_f} \sin^2 \frac{\beta_f}{2} \gg \frac{1}{2} \left(\frac{z_2}{z_3}\right)^2 = 0.030. \tag{4.18}
\]

Therefore, for the cases \(b_f = -1\) and \(b_f = -1/2\), the expression (4.4) is valid only for the cases
\[
\left|\sin \frac{\beta_f}{2}\right| \gg \frac{1}{2\sqrt{2}} \frac{z_1}{z_2} = 0.025, \quad (|\beta_f| \gg 2.8^\circ), \tag{4.19}
\]
and
\[
\left|\sin \frac{\beta_f}{2}\right| \gg \frac{1}{4\sqrt{2}} \frac{z_2}{z_3} = 0.043, \quad (|\beta_f| \gg 4.9^\circ), \tag{4.20}
\]
respectively. For down-quark sector, we know that \(b_d \simeq -1\) and \(|\beta_d| \simeq 20^\circ\) from the phenomenological study\(^3\) of the quark mass ratios. The value \(|\beta_d| \simeq 20^\circ\) satisfies the condition (4.19), so that we can use the expression (4.4) for the down-quark sector. The expression (4.4) is not valid for the cases \(b_f = -1\) and \(b_f = -1/2\) with \(\beta_f = 0\), which do not satisfy the conditions (4.19) and (4.20). The expressions for these cases are given in Appendix A.

§5. CKM matrix elements

The CKM matrix elements \(V_{ij}\) are given by (4.3). Without losing generality, we can take
\[
P = \text{diag}(1, e^{i\delta_2}, e^{i\delta_3}). \tag{5.1}
\]

For the up-quark sector, we put an ansatz “maximal top-quark-mass enhancement”, i.e., we assume that \(b_u = -1/3\) and \(\beta_u = 0\). Then, from (4.5) –
(4.16), we obtain

\[ p_u = p \left(-\frac{1}{3}, 0 \right) = -\frac{1}{2} , \quad q_u = q \left(-\frac{1}{3}, 0 \right) = -1 \]  \hspace{1cm} (5.2)

and

\[ \varepsilon_u^3 = \frac{1}{2} \left( \frac{z_2}{z_3} \right)^2 = 0.030 \gg \varepsilon_1^u = \frac{1}{8} \left( \frac{z_1}{z_2} \right)^2 = 0.0006 \]  \hspace{1cm} (5.3)

\[ \varepsilon_{12}^u \simeq \varepsilon_{13}^u \simeq 0 , \quad \varepsilon_{21}^u \simeq 3 \varepsilon_3^u , \quad \varepsilon_{23}^u \simeq \varepsilon_{31}^u \simeq \varepsilon_{32}^u \simeq \varepsilon_3^u , \]  \hspace{1cm} (5.4)

with \( \varepsilon_3^u \simeq 0 \). Therefore, the unitary matrix \( U_L^u \) is given by

\[
U_L^u \simeq \begin{pmatrix}
1 & -\frac{1}{2} \frac{z_1}{z_2} & -\frac{1}{2} \frac{z_1}{z_3} \\
\frac{1}{2} \frac{z_1}{z_2} (1 - 3 \varepsilon_3^u) & 1 - \varepsilon_3^u & -\frac{z_2}{z_3} (1 - \varepsilon_3^u) \\
\frac{z_1}{z_3} (1 - \varepsilon_3^u) & \frac{z_2}{z_3} (1 - \varepsilon_3^u) & 1 - \varepsilon_3^u
\end{pmatrix}. \hspace{1cm} (5.5)
\]

For the down-quark sector, for a time, we use the general expression (4.4) without assuming \( b_d \simeq -1 \) and \( \beta_d^2 \ll 1 \).

First, by neglecting \( \varepsilon_1^d \) and \( \varepsilon_{ij}^d \) terms, let us give rough estimates of \( V_{ij} \):

\[ V_{us} \simeq -\frac{1}{2} \frac{z_1}{z_2} \frac{1}{c_1^d} e^{i \delta_2} \left( 2 e^{-i \delta_2} + c_1^d \right) \]  \hspace{1cm} (5.6)

\[ V_{cb} \simeq -\frac{z_2}{z_3} \frac{1}{c_2^d} e^{i \delta_2} \left( 1 + c_2^d e^{i (\delta_3 - \delta_2)} \right) \]  \hspace{1cm} (5.7)

\[ V_{ub} \simeq -\frac{1}{2} \frac{z_1}{z_3} \frac{1}{c_2^d} e^{i \delta_2} \left( 2 e^{-i \delta_2} - 1 + c_2^d e^{i (\delta_3 - \delta_2)} \right) \]  \hspace{1cm} (5.8)

\[ V_{td} \simeq \frac{z_1}{z_3} \frac{1}{c_1^d} e^{i \delta_2} \left( c_1^d e^{-i \delta_2} + 1 + e^{i (\delta_3 - \delta_2)} \right) \]  \hspace{1cm} (5.9)

In, (5.1), we have taken \( \delta_1 = 0 \) without losing generality. We suppose that \( \delta_2 \) is also \( \delta_2 \approx 0 \). For \( \delta_2^2 \ll \sin^2 (\beta_d/2) \) and \( \delta_2^2 \ll \varepsilon_3^2 \left[b_d \equiv -(1 - \varepsilon_d)\right] \), the relations
(5.6) – (5.9) lead to

\[ V_{us} \simeq -\frac{1}{2} \frac{z_1}{z_2} \frac{c_3}{c_1} , \quad (5.10) \]

\[ V_{cb} \simeq -\frac{z_2}{z_3} \frac{1}{c_2} \left( c_2 + e^{-i(\delta_3 - \delta_2)} \right) e^{i(\delta_3 - \delta_2)} , \quad (5.11) \]

\[ V_{ub} \simeq -\frac{1}{2} \frac{z_1}{z_3} \frac{1}{c_2} \left( c_2 + e^{-i(\delta_3 - \delta_2)} \right) e^{i(\delta_3 - \delta_2)} , \quad (5.12) \]

\[ V_{td}^* \simeq \frac{z_1}{z_2} \frac{1}{c_1} \left( c_2 + e^{-i(\delta_3 - \delta_2)} \right) . \quad (5.13) \]

We can readily obtain an approximate relation

\[ \left| \frac{V_{ub}}{V_{cb}} \right| \simeq \frac{1}{2} \frac{z_1}{z_2} = \frac{1}{2} \frac{\sqrt{m_e}}{m_\mu} = 0.035 , \quad (5.14) \]

which is valid for arbitrary values of the parameters \( b_d , \beta_d \) and \( (\delta_3 - \delta_2) \). However, the prediction (5.14) is somewhat small compared with the observed value\(^7\) \( |V_{ub}/V_{cb}| = 0.08 \pm 0.02 \). This discrepancy can be corrected by taking the small terms \( \varepsilon_i^f \) and \( \varepsilon_{ij}^f \) into consideration (see Appendix B). We also obtain the relation

\[ |V_{td}| \simeq 2 \left| \frac{c_2}{c_1} \right| |V_{ub}| \simeq 2 \frac{m_\mu / m_\tau}{m_s / m_b} \frac{2 |V_{us}|}{\sqrt{m_e / m_c}} |V_{ub}| , \quad (5.15) \]

for arbitrary values of \( b_d , \beta_d \) and \( (\delta_3 - \delta_2) \) by using the relations (3.2), (3.3), (5.10), (5.12) and (5.13).

From (5.10) and the relation

\[ \frac{m_d}{m_s} \simeq \left( \frac{z_1}{z_2} \right)^2 \frac{|c_3||c_2|}{|c_1|^2} , \quad (5.16) \]

we can see that if we want to derive the well-known relation\(^13\)

\[ |V_{us}| \simeq \sqrt{m_d / m_s} , \quad (5.17) \]
we must impose a constraint

$$4 \simeq |c^d_3|^2 / |c^d_2||c^d_2| ,$$

(5.18)

on the parameter $\beta_d e^{i\beta_d}$. The simplest one of the solutions of (5.18) is $b_d e^{i\beta_d} \simeq -1$, which yields reasonable down-quark mass ratios $m_d/m_s$ and $m_s/m_b$.

Similarly, from (5.10), (5.11) and (5.13), we obtain the relation

$$\frac{|V_{td}|}{|V_{cb}| |V_{us}|} \simeq 2 \left| \frac{c^d_2}{c^d_3} \right| .$$

(5.19)

Since $c^d_2 \simeq 1$ and $c^d_3 \simeq 2$ for $b_d e^{i\beta_d} \simeq -1$, the ratio $(2 |c^d_2/c^d_3|)$ approximately takes one, so that we obtain the relation for the case of $b_d e^{i\beta_d} \simeq -1$,  

$$\frac{|V_{td}|}{|V_{cb}|} \simeq |V_{us}| \simeq \sqrt{\frac{m_d}{m_s}} ,$$

(5.20)

which is valid for arbitrary value of $(\delta_3 - \delta_2)$. On the other hand, values of $|V_{td}|$, $|V_{cb}|$ and $|V_{us}|$ must be carefully estimated because those contain the small factor $c^d_1$

$$c^d_1 = \left[ \varepsilon_d - 2ie^{-i\beta_d/2} \sin^2 (\beta_d/2) \right] / (1 - \varepsilon_d) ,$$

(5.21)

which is sensitive to the values of $\varepsilon_d$ and $\beta_d$.

The rephasing invariant\textsuperscript{14} $J$ is expressed in terms of $|V_{ij}|$ as follows:\textsuperscript{15}

$$J^2 = |V_{us}|^2 |V_{cb}|^2 |V_{ub}|^2 \left( 1 + |V_{us}|^2 - |V_{cb}|^2 - \omega \right)$$

$$-\frac{1}{4} \left[ |V_{us}|^2 |V_{cb}|^2 - \left( |V_{us}|^2 + |V_{cb}|^2 \right) |V_{ub}|^2 + \left( 1 - |V_{ub}|^2 \right) \omega \right] ,$$

(5.22)

where

$$\omega = |V_{cd}|^2 - |V_{us}|^2 = |V_{ts}|^2 - |V_{cb}|^2 = |V_{ub}|^2 - |V_{td}|^2 .$$

(5.23)

By using (5.20) and the observed fact $|V_{us}|^2 \gg |V_{cb}|^2 \gg |V_{ub}|^2$, we obtain

$$|J| \simeq \sqrt{1 - \frac{1}{4} \frac{|V_{ub}/V_{cb}|^2}{|V_{us}|^2} |V_{us}| |V_{cb}| |V_{ub}|}$$
\[ \simeq \frac{1}{2} \sqrt{\frac{m_e m_d}{m_\mu m_s}} \left(1 - \frac{m_e/m_\mu}{4 m_d/m_s}\right) |V_{cb}|^2. \]  

(5.24)

For \( \delta_2^2 \ll \sin^2 \beta_d + \varepsilon_d^2 \ll 1 \), we obtain

\[ |V_{cb}| \simeq \frac{Z_2}{Z_3} \left| c_2^d + e^{-i(\delta_3 - \delta_2)} \right|. \]  

(5.25)

In order to explain the observed value\(^7\) of \( |V_{cb}| \)

\[ |V_{cb}| = 0.041 \pm 0.003, \]  

(5.26)

the case \( \delta_3 - \delta_2 \simeq 0 \) is obviously ruled out because of \( z_2/z_3 = \sqrt{m_\mu/m_\tau} = 0.244 \) and \( c_2^d \simeq 1 \), and, rather, the case \( \delta_3 - \delta_2 \simeq \pi \) is favorable to (5.26). By putting

\[ \delta_3 = \delta + \delta_2 + \pi, \]  

(5.27)

we obtain

\[ |V_{cb}| \simeq \frac{Z_2}{Z_3} \sqrt{\varepsilon_d^2 + (\sin \beta_d + \sin \delta)^2}. \]  

(5.28)

Similarly, for the case \( |\delta_2|^2 < |\delta|^2 \ll 1 \), we obtain

\[ |V_{td}| \simeq \frac{Z_1}{Z_3} \sqrt{\varepsilon_d^2 + (\sin \beta_d + \sin \delta)^2 \varepsilon_d^2 + \sin^2 \beta_d}. \]  

(5.29)

So far, we have not assumed \( b_d = -1 \). However, considering our parametrization \( b_e = 0 \) and \( b_u = -1/3 \), it is likely that the value of \( b_d \) is given not by \( b_d \simeq -1 \), but by a simple rational number \( b_d = -1 \). In Appendix B, we will show the more precise expressions of \( |V_{ij}| \), in which we take the small terms \( \varepsilon_i^f \) and \( \varepsilon_{ij}^f \) given in (4.8) – (4.16) into consideration, but we assume \( b_d = -1 \).

\section*{§6. Numerical results of the CKM matrix parameters}

Numerical results of \( |V_{ij}| \) for \( \delta_2 = \delta = 0 \) have already been given in Ref. 3). Although the purpose of the present paper is not to give the numerical estimates, in order to complement the study of the previous section, in the present section, we shall give a numerical study of \( |V_{ij}| \) without the restriction \( \delta_2 = \delta = 0 \).

As the numerical inputs, according to Ref. 3), we use \( \kappa/\lambda = 0.02 \), \( b_u = -1/3 \), \( \beta_u = 0 \), \( b_d = -1 \) and \( \beta_d = 18^\circ \), which are required for a reasonable fit with the
observed quark masses. Our interest is in the behavior of $|V_{ij}|$ versus the phase parameters $\delta_2$ and $\delta_3$ defined by (2.10) [5.1], because in the previous study, the degree of freedom of the phases ($\delta_2$, $\delta_3$) was not taken into consideration. In Fig. 2, we illustrate the allowed regions of ($\delta_2$, $\delta_3$) which give the observed values of $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$

\[
|V_{us}| = 0.2205 \pm 0.0018 , \\
|V_{cb}| = 0.041 \pm 0.003 , \\
|V_{ub}/V_{cb}| = 0.08 \pm 0.02 .
\]

(6.1)

We have two allowed regions of ($\delta_2$, $\delta_3$): we obtain the predictions

\[
|V_{us}| = 0.2195 , \\
|V_{cb}| = 0.0388 , \\
|V_{ub}| = 0.0028 , \\
|V_{ub}/V_{cb}| = 0.072 , \\
|V_{id}| = 0.0105 , \\
J = 1.8 \times 10^{-5} ,
\]

(6.2)

for ($\delta_2$, $\delta_3$) = (0°, 174°) and

\[
|V_{us}| = 0.2211 , \\
|V_{cb}| = 0.0411 , \\
|V_{ub}| = 0.0027 , \\
|V_{ub}/V_{cb}| = 0.065 , \\
|V_{id}| = 0.0092 , \\
J = 2.4 \times 10^{-5} ,
\]

(6.3)

for ($\delta_2$, $\delta_3$) = (−4°, 152°).

In Fig. 3, we show the possible unitary-triangle shape of the present model on the ($\rho$, $\eta$) plane, where ($\rho$, $\eta$) are the Wolfenstein parameters defined by $V_{ub} \equiv |V_{us}| |V_{cb}| (\rho - i\eta)$, $V_{us} = |V_{us}|$ and $V_{cb} = |V_{cb}|$.

The vertex ($\rho$, $\eta$) moves on the circle which is denoted by the solid line in Fig. 3 according as the parameter $\delta_3$ varies from 0° to 360°. For reference, we have shown the constraints from the observed values $|V_{ub}/V_{cb}|$, $\Delta m_{B_d}$ and $\varepsilon_K$. Both triangles which correspond to the cases ($\delta_2$, $\delta_3$) = (0°, 174°) and (−4°, 152°) satisfy these constraints safely.

§7. Summary and discussion

In conclusion, we have obtained the analytical expressions of the masses and mixings of the light fermions $f$ in the democratic seesaw mass matrix model (2.1). The fermion mass ratios are controlled by the parameters $b_f$, $\beta_f$ and $\kappa/\lambda_F$, as shown in Fig. 1. We have fixed the parameters ($z_1$, $z_2$, $z_3$) by taking $b_e = 0$ as given in (1.6). The model can yield a large enhancement of top-quark mass, $m_t \gg m_b$ (keeping $m_u \sim m_d$), without taking hierarchically different values of
mass matrix parameters in the up-quark sector. In the region of \((b_u \simeq -1/3, \beta_u \simeq 0)\) in which large top-quark-mass enhancement occurs, the mass relation (3.20), \(m_u/m_c \simeq 3m_c/4m_\mu\), is valid almost independently of the parameter \(\kappa/\lambda_F\) \((F = U)\). The value of \(\kappa/\lambda_U\) is fixed by the observed values of \(m_c/m_t\). The observed down-quark mass values are in favor of \(b_d \simeq -1\) with a small \(\beta_d\) (but \(\beta_d \neq 0\)). The mass relations (3.21)–(3.23) have been obtained for \(b_d \simeq -1\) with a small \(\beta_d^2\) and with \(\lambda_D = \lambda_U\) \((\equiv \lambda)\). Those relations are insensitive to value of \(|\beta_d|\). The value of \(|\beta_d|\) can be fixed by the observed mass ratio \(m_u/m_d\) (or \(m_s/m_c\)) as shown in (3.23).

As an application of the results, we have discussed the CKM matrix \(V\). For the up-quark sector, we have assumed “maximal top-quark-mass enhancement”, i.e., \(b_u = -1/3\) and \(\beta_u = 0\). We also suppose \(\delta_2 \simeq 0\) in (5.1). Then, the relations (5.14) and (5.15) are valid almost independently of the parameters \(b_d\), \(\beta_d\) and \((\delta_3 - \delta_2)\). The observed down-quark mass ratios is favorable to \(b_de^{i\beta_d} \simeq -1\). When we take \(b_de^{i\beta_d} \simeq -1\), the relations (5.17), (5.20) and (5.24) are valid almost independently of the value of \((\delta_3 - \delta_2)\). In order to fit \(|V_{cb}|\) and \(|V_{ub}|\) to the observed values, it is required that \(\delta_3 - \delta_2 \simeq \pi\).

Thus, in order to obtain a good fitting of the quark mass ratios and CKM mixings, we must take \(b_ue^{i\beta_u} = -1/3\) and \(b_de^{i\beta_d} \simeq -1\) together with \(b_e^{i\beta_e} = 0\). The choice \((b_u = -1/3, \beta_u = 0)\) is described by the ansatz of “maximal top-quark-mass enhancement” or “minimal up-heavy-quark mass”. However, the same ansatz cannot apply to the down-quark sector (it leads to a wrong solution \(b_de^{i\beta_d} = -1/3\)).

As an application of the fermion mass expressions (3.1)–(3.3), let us note the fermion mass ratio

\[
r_f \equiv \frac{m_1^f m_3^f}{m_2^f} = \frac{m_1^f/m_2^f}{m_2^f/m_1^f} = \frac{m_1^f/m_2^f}{m_2^f/m_3^f},
\]

which is expressed as

\[
r_f = \left( \frac{z_1 z_3}{z_2^2} \right)^2 \left| \frac{c_1^f}{c_3^f} \right|^2 \left( \frac{\left| c_2^f \right|}{\left| c_1^f \right|} \right)^3.
\]

Since

\[
0 = \frac{\partial r_f}{\partial b_f} = -\frac{2|c_2^f|}{|c_3^f|^2 |c_1^f|^5} b_f^2 (2b_f + \cos \beta_f)
\]
\[(3b_f + 2 \cos \beta_f + \sqrt{9 - 8 \cos^2 \beta_f})(3b_f + 2 \cos \beta_f - \sqrt{9 - 8 \cos^2 \beta_f}) \times (3b_f + 2 \cos \beta_f + \sqrt{9 - 8 \cos^2 \beta_f})(3b_f + 2 \cos \beta_f - \sqrt{9 - 8 \cos^2 \beta_f}) \] (7.3)

the maximal points of \(r_f\) are given by

\[b_f = -\frac{1}{3}(2 \cos \beta_f - \sqrt{9 - 8 \cos^2 \beta_f}) \] (7.4)

and

\[b_f = -\frac{1}{3}(2 \cos \beta_f + \sqrt{9 - 8 \cos^2 \beta_f}) \] (7.5)

For \(\beta_f^2 \ll 1\), the former and the latter give \(b_f \simeq -1/3\) and \(b_f \simeq -1\), respectively. Although the expressions (3.1)–(3.3) are not valid for \(b_f = -1/3\), \(b_f = -1/2\) and \(b_f = -1\) with \(\beta_f = 0\), we can see, from the numerical study of the \(6 \times 6\) mass matrix, that the results (7.4) and (5.5) are valid even for \(|\beta_f| \to 0\). Therefore, it is interesting to put an ansatz that the parameter \(b_f\) takes its values such that the ratio \(r_f\) becomes maximal. Since \(\partial r_f/\partial |\beta_f| < 0\), the ansatz leads to \(|\beta_f| \to 0\). The solution \((b_f = -1/3, \beta_f = 0)\) is favorable to the up-quark sector, but, for down-quark sector, the choice \(\beta_f = 0\) is not favorable. We will have to consider an additional reason for \(\beta_f \neq 0\). If we accept such the additional condition \(\beta_d \neq 0\) (but \(\beta_d^2 \ll 1\)), we can obtain the desirable choice \(b_d \simeq -1\).

In addition to the solutions (7.4) and (7.5), the remaining solutions of (7.3), \(b_f = 0\) and \(b_f = -\cos \beta_f/2 \simeq -1/2\), are also interesting. The former \(b_f = 0\) corresponds to the case of the charged lepton sector. The latter \(b_f \simeq -1/2\) is favorable to understanding a large neutrino mixing which has been suggested from the atmospheric neutrino data,\(^{19}\) as pointed out in Ref. 20 (also see (A.1) in Appendix A).

Thus, although the ansatz for the mass ratio \(r_f\) brings very interesting results, at present, we cannot find any plausible mechanism which justifies such the ansatz. Considering naively, it is strange that the parameter value of \(b_f e^{i\beta_f}\) is controlled by the ratio \(m_1^f m_3^f/m_2^f\), because \(b_f e^{i\beta_f}\) is a parameter in the heavy-fermion mass matrix \(M_F\) which is generated at \(\mu = m_0 \lambda_F \gg m_W\), while the masses \(m_i^f\) \((i = 1, 2, 3)\) are generated at \(\mu = m_0 \sim m_W\). We consider that there is a fundamental law (mechanism) which controls the value of \(b_f e^{i\beta_f}\) in \(M_F\), and the law indirectly affects the mass ratio \(r_f\), too. As a result, the points which yield \(\partial r_f/\partial b_f = 0\) can correspond to the physical values of \(b_f\) for quarks and leptons. However, it is a future task to clarify whether this scenario is true or not.
The purpose of the present paper is to obtain analytical expressions of fermion masses and mixings for convenience of further investigating of the democratic seesaw mass matrix model. The model brings many new aspects beyond the conventional mass matrix models, and it seems to be worth while investigating the model furthermore.

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Appendix A

The expressions of $U^f_L$ for the cases $b_f = -1/2$ and $b_f = -1$ with $\beta_f = 0$ are given in Ref. 20). The results are as follows:

$$U^f_L \simeq \begin{pmatrix}
1 & -\frac{z_1}{z_2} & -\frac{z_1}{z_3} \\
\frac{1}{\sqrt{2}} \left( \frac{z_1}{z_2} - \frac{z_1}{z_3} \right) & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \left( \frac{z_1}{z_2} + \frac{z_1}{z_3} \right) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} 
\end{pmatrix}, \quad (A.1)$$

for $b_f = -1/2$ and $\beta_f = 0$, and

$$U^f_L \simeq \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \left( \frac{z_2}{z_3} - \frac{z_1}{z_3} \right) \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \left( \frac{z_2}{z_3} + \frac{z_1}{z_3} \right) \\
-\frac{z_1}{z_3} & -\frac{z_2}{z_3} & 1 
\end{pmatrix}, \quad (A.2)$$
for $b_f = -1$ and $\beta_f = 0$.

### Appendix B

For $b_d = -1$, $\beta_d^2 \ll 1$ (but $\beta_d^2 \neq 0$), $\delta^2 \ll 1$ and $\delta_d^2 \ll 1$, from (4.4) with $b_f = -1$ and (5.5), we obtain the following analytical expressions of $|V_{ij}|$:

\[
|V_{us}| \simeq \frac{z_1}{2z_2} |\sin(\beta_d/2)| \left[ 1 - 3\varepsilon_1^d + \varepsilon_3^d + \sin \frac{\beta_d}{2} \sin \left( \frac{\beta_d}{2} - \frac{\delta_2}{2} \right) + \frac{1}{2} \sin^2 \frac{\beta_d}{2} \right], \quad (B.1)
\]

\[
|V_{cb}| \simeq \frac{z_2}{z_3} |\sin \beta_d + \sin \delta| \left( 1 + \eta_{cb} \right), \quad (B.2)
\]

\[
|V_{ub}| \simeq \frac{z_1}{z_3} \frac{1 - \varepsilon_3^d}{\sqrt{1 + 8 \sin^2(\beta_d/2)}} |\sin \beta_d + \sin \delta + 2 \sin \delta_2| \left( 1 + \eta_{ub} \right), \quad (B.3)
\]

\[
|V_{td}| \simeq 2 \frac{z_1}{z_3} |\sin \beta_d + \sin \delta| \left( 1 + \eta_{td} \right), \quad (B.4)
\]

where

\[
\varepsilon_1^d = \frac{1}{8} \left( \frac{z_1}{z_2} \right)^2 \frac{1}{\sin^2(\beta_d/2)} \approx \frac{1}{2} \frac{m_d}{m_s} \approx \frac{1}{2} |V_{us}|^2, \quad (B.5)
\]

\[
\varepsilon_3^d = \frac{1}{2} \left( \frac{z_1}{z_3} \right)^2 \frac{1}{\sqrt{1 + 8 \sin^2(\beta_d/2)}} \approx \frac{1}{2} \frac{m_\mu}{m_\tau}, \quad (B.6)
\]

\[
\eta_{cb} = 2 \left[ \left( \sin^2 \frac{\beta_d}{2} + \sin^2 \frac{\delta}{2} \right)^2 - 2 \left( \varepsilon_1^d + \varepsilon_3^d \right) \sin^2 \frac{\beta_d}{2} + 4 \varepsilon_3^d \sin^2 \frac{\delta}{2} + \frac{1}{4} \left( \varepsilon_1^d - \varepsilon_3^d \right)^2 \right]
\]

\[
\times \left[ \left( 1 - \varepsilon_3^d - \frac{5}{2} \varepsilon_3^d \right) \left( \sin \frac{\beta_d}{2} + \sin \frac{\delta}{2} \right) \right]^{-2}, \quad (B.7)
\]

\[
\eta_{ub} = 2 \left( \sin^2 \frac{\beta_d}{2} + \sin^2 \frac{\delta}{2} \right)^2 + 2 \sin^2 \frac{\beta_d}{2} \left( \sin \frac{\delta}{2} + \sin \frac{\delta_2}{2} \right)^2
\]
\[-8 \sin \frac{\beta_d}{2} \sin \frac{\delta_2}{2} \sin \frac{\delta_2}{2} \left( \sin \frac{\delta}{2} + \sin \frac{\delta_2}{2} \right) - 4 \left( \sin^2 \frac{\delta}{2} - \sin^2 \frac{\delta_2}{2} \right) \sin^2 \frac{\delta_2}{2} \]

\[-8 \varepsilon_b \left\{ -2 \sin \frac{\beta_d}{2} \left( \sin \frac{\beta_d}{2} - \sin \frac{\delta}{2} \right) + \left( \sin \frac{\delta}{2} + \sin \frac{\delta_2}{2} \right)^2 \right\} \]

\[\times \left[ \left( 1 - \varepsilon_d^3 \right) (\sin \beta_d + \sin \delta + 2 \sin \delta_2) \right]^{-2}, \quad \text{(B.8)}\]

\[\eta_{pd} = 2 \left[ \left( \sin^2 \frac{\beta_d}{2} + \sin^2 \frac{\delta}{2} \right)^2 + 4 \sin \frac{\beta_d}{2} \sin \frac{\delta}{2} \sin \frac{\delta_2}{2} \left( \sin \frac{\beta_d}{2} - \sin \frac{\delta}{2} - \sin \frac{\delta_2}{2} \right) \right. \]

\[-\varepsilon_1^d \left( \sin \frac{\beta_d}{2} + \sin \frac{\delta}{2} \right)^2 \left[ \left( 1 - \varepsilon_1^d \right) (\sin \beta_d + \sin \delta) \right]^{-2}. \quad \text{(B.9)}\]
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Figure Captions

Fig. 1. Masses \( m_i^f \) \( (i = 1, \cdots, 6) \) versus \( b_f \) for the case of \( \kappa = 10 \) and \( \kappa/\lambda = 0.02 \). The solid and broken lines denote for the cases of \( \beta_f = 0 \) and \( \beta_f = 18^\circ \), respectively. At \( b_f = 0 \), the charged lepton masses \( m_e, m_\mu \) and \( m_\tau \) have been used as input values for the parameters \( z_i \). For up- and down-quark sectors, the values \( b_u = -1/3 \) and \( b_d = -1 \) are chosen from the phenomenological study\(^3\) of the observed quark masses.

Fig. 2. Constraints on the phase parameters \( (\delta_2, \delta_3) \) from the experimental values \( |V_{us}| = 0.2205 \pm 0.0018 \) (dotted lines), \( |V_{cb}| = 0.041 \pm 0.003 \) (solid lines) and \( |V_{ub}/V_{cb}| = 0.08 \pm 0.02 \) (dashed lines). The hatched areas denote the allowed regions.

Fig. 3. Trajectories of the vertex \( (\rho, \eta) \) of the unitary triangle for the cases \( \delta_2 = 0^\circ \) and \( \delta_2 = -4^\circ \). The points ◦, □, ◇ and △ denote the vertex \( (\rho, \eta) \) for \( \delta_3 = 150^\circ, 160^\circ, 170^\circ \) and \( 180^\circ \), respectively. The other parameters are fixed to \( \kappa = 10, \kappa/\lambda = 0.02, b_u = -1/3, \beta_u = 0, b_d = -1, \) and \( \beta_d = 18^\circ \) from the observed quark mass ratios. The solid, broken and dot-dashed lines denote constraints from \( |V_{ub}/V_{cb}|, |\Delta m_{B_d}| \) and \( \varepsilon_K \). The two triangles correspond to the cases \( (\delta_2, \delta_3) = (0^\circ, 174^\circ) \) and \( (-4^\circ, 152^\circ) \), respectively.
Fig. 2

\( \delta_3 \) (deg)

\( \delta_2 \) (deg)
Fig. 3