Investigation of problems of closing of geophysical cracks in thermoelastic media in the case of flow of fluids with impurities

A N Martirosyan*, A V Davtyan**, A S Dinunts***, and H A Martirosyan****
Goris State University, Goris, Armenia
E-mail: *ashot.martirosyan.14@gmail.com, **davtyananush@gmail.com, ***dinuntsas@gmail.com, ****hayk.martirosyan.75@mail.ru

Abstract. The purpose of this article is to investigate a problem of closing cracks by building up a layer of sediments on surfaces of a crack in an infinite thermoelastic medium in the presence of a flow of fluids with impurities. The statement of the problem of closing geophysical cracks in the presence of a fluid flow is presented with regard to the thermoelastic stress and the influence of the impurity deposition in the liquid on the crack surfaces due to thermal diffusion at the fracture closure. The Wiener–Hopf method yields an analytical solution in the special case without friction. Numerical calculations are performed in this case and the dependence of the crack closure time on the coordinate is plotted. A similar spatial problem is also solved. These results generalize the results of previous studies of geophysical cracks and debris in rocks, where the closure of a crack due to temperature effects is studied without taking the elastic stresses into account.

1. Introduction
The problems of increasing the impurity layer contained in a liquid entering a crack located in an infinite thermoelastic plane are considered. These problems were investigated in [1] as problems of closing cracks by injecting a hot hydrothermal mixture with crystallization of silica and their deposition on the crack surfaces under the assumption that the liquid flow and temperature at the point of entry into the crack are constant in time. In [2], the problem of closing a crack is considered when the liquid temperature and flow at the entry point into the crack are functions of time.

In [1, 2] and in numerous works of other foreign authors, the problems of closing a geophysical crack by deposition of crystals from a hydrothermal fluid due to thermal and thermal diffusion effects are studied without consideration of elastic stresses. In [3], the change of pressure and permeability of a fluid in a fracture in a rock array is studied with regard to elastic stresses and thermal diffusion, i.e., the effects of thermal stress and precipitation/dissolution of silicon dioxide are taken into account. The heat exchange between a crack and a rock array is modeled by taking only the thermal conductivity into account. The fluid pressure can vary with time, while the flow velocity is assumed to be constant. The experimental curves for closing bands filled with a fluid with inclusions in the Earth’s medium are given in [4].
Numerous numerical experiments have been performed to simulate the fluid injection/liquid production. A temporary change in the crack aperture is considered in response to the individual and combined effects of thermal stress and silica dissolution/deposition. The results show that for smaller initial fracture ruptures, a significant increase in the fracture permeability and the associated pressure drop at the injection point are mainly due to thermoelastic effects, while an increase in the rupture aperture near the injection point is mainly the result of silica dissolution/deposition. On the other hand, for large initial holes, the effect of precipitation/dissolution of silicon dioxide is minimal, and the thermoelastic effects predominate, since the intensity of the connection between fractures with high permeability and mass of rocks with low permeability becomes weaker. As follows from these results, it is necessary to take into account the thermoelastic stresses and the effects of thermal diffusion, which is the main subject of this work.

In the present paper based on the Wiener–Hopf [5] and integral transforms methods, the mathematical solutions are obtained for the important applied problems of closing geophysical cracks by increasing the sediment layer on the crack surfaces in an infinite thermoelastic medium in the presence of a flow of fluids with impurities. The problem of closing the geophysical crack in the presence of a fluid flow is formulated with regard to the thermoelastic stress and influence of the impurity deposition in the liquid on the crack surfaces due to thermal diffusion at the fracture closure. The Wiener–Hopf method yields an analytic solution in the special case without friction. The solution of the nonstationary mixed plane boundary-value problem for thermoelasticity based on the Laplace integral transforms with respect to time and the Fourier integral transforms with respect to the coordinate is reduced to the Wiener–Hopf equation with continuous coefficients. After the inverse integral transformation, the solution for displacements is given in Smirnov-Sobolev form [6]. In this case, for some values of the parameters, numerical calculations were made and the crack closing time versus the coordinate was plotted. A similar spatial problem is also solved. The solution results are compared with the results obtained by the Wiener–Hopf method in the case of stationary crack, and the coincidence of these approaches is shown. These results generalize the previously performed work for geophysical cracks and faults in rocks, where the closure of a crack due to temperature effects is studied without consideration of elastic stresses.

2. Statement of the problem of crack in thermoelastic plane with a fluid flow

In this paper, the problem of closing cracks by building up sediments on crack surfaces is considered in infinite thermoelastic media in the case of flow of a fluid with impurities entering the crack at initial time \( t = 0 \).

In the plane \( x_1, x_2 \), the equation of crack profiles is \( x_2 = \pm h(x_1, t) \), where \( h(x_1, t) = h_0 + u_2(x_1, x_2, t), x_2 \approx 0 \), and the thickness \( 2h_0 \) is small. Due to symmetry of the problem, we later take only the upper sign and solve the problem for \( x_2 \geq 0 \); here \( u_1, u_2 \) are displacement components in elastic media.

Following [1, 2, 7], we write the following approximate relation for \( y = 0 \):

\[
\rho_s \frac{\partial h}{\partial t} = q\gamma \frac{\partial T}{\partial x} - K \sigma_{22} + \lambda_1 \frac{\partial C}{\partial y},
\]

where the left-hand side represents the mass of sediment deposited per unit time and proportional to the closure rate of the crack, the first term on the right-hand side is the mass flow density of inclusions in the fluid along the axis \( x_2 \) [2] transmitted by the fluid per unit time due to the temperature gradient along the \( x_1 \), the second term is the tribological term with stresses [8], the third term corresponds to the flux density of impurity concentration along the axis \( x_2 \), where \( K = \text{const} \) is a tribological experimental constant of the crack boundary or an increase in its surface due to symmetry, where one can approximately include the term from barodiffusion,
assuming that \( \sigma_{x2x2} = -P, \partial P/\partial x_2 \approx -\sigma_{x2x2}/h_0 \), \( \rho_s \) is the density of inclusions, \( C \) is the concentration of impurities in the fluid, \( \gamma = \partial C/\partial T \), \( \gamma \) = const [9], \( q = \rho_f v h_0 \) denotes the flow of entering fluid, \( \rho_f \) is the density of fluid, \( v \) is the fluid velocity along the axis \( x_1 \), \( \lambda_1 = -\rho_s D \), and \( D \) is the diffusion coefficient. The temperature difference \( T(x, y, t) - T_0 \) is assumed to be the same in the elastic medium and fluid, where it is approximately assumed that \( x_2 \approx 0 \), and the notation \( T(x, t) = T(x, 0, t) - T_0 \) is used.

The equations of motion in displacements for a thermoelastic medium [9] have the form

\[
\begin{align*}
\frac{a^2 \partial^2 u_1}{\partial x_1^2} + b^2 \frac{\partial^2 u_1}{\partial x_2^2} + (a^2 - b^2) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\delta}{2} \frac{\partial}{\partial x_1} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) &= \frac{\partial^2 u_1}{\partial t^2}, \\
\frac{a^2 \partial^2 u_2}{\partial x_2^2} + b^2 \frac{\partial^2 u_2}{\partial x_1^2} + (a^2 - b^2) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\delta}{2} \frac{\partial}{\partial x_2} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) &= \frac{\partial^2 u_2}{\partial t^2},
\end{align*}
\]

where \( u_i \) (i = 1, 2, 3) are displacement components, \( a \) and \( b \) are the velocities of longitudinal and transverse waves, \( \delta = K_1/\rho \cdot (C_\rho - C_v)/C_v \), \( K_1 = \lambda + \frac{2\mu}{3} \) is the bulk modulus, \( \lambda \) and \( \mu \) are the elastic Lame constants, \( t \) is the time, \( \rho \) is the density of thermoelastic medium, and \( C_\rho \) and \( C_v \) are the specific thermal capacities [9]. These formulas are obtained by neglecting the thermal conductivity in the temperature equation

\[
\frac{\partial T}{\partial t} + \frac{C_\rho - C_v}{\alpha C_v} \frac{\partial}{\partial t} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = k \left( \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} \right),
\]

where \( k \) is the coefficient of thermal conductivity and \( \alpha \) is the coefficient of thermal expansion.

For the stress components, we have the relations

\[
\begin{align*}
\frac{\sigma_{x2x2}}{\rho} &= (a^2 - 2b^2) \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2b^2 \frac{\partial u_2}{\partial x_1} + \frac{\delta}{2} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right), \\
\frac{\sigma_{x1x2}}{\rho} &= b^2 \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right).
\end{align*}
\]

After linearization, the boundary conditions of contact between the medium and the liquid, i.e., for \( x_1 > 0 \) and \( x_2 = 0 \), with the mechanical upbuilding taken into account, can also be written as

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= -\frac{K_2}{\rho_s} \sigma_{x1x2}, \\
\frac{\partial u_2}{\partial t} &= -q \frac{\gamma \rho(C_\rho - C_v)}{\rho_s \alpha C_v l} \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) - \frac{K}{\rho_s} \sigma_{x2x2} - \frac{i_0}{\rho_s} H(t) H(v t - x_1),
\end{align*}
\]

where the diffusion flux \( \lambda_1 \partial C/\partial y \) is assumed to be known and approximately equal to \(-i_0 H(v t - x) H(t)\), \( i_0 \) is the constant diffusion flow of inclusions along the axis \( x_2 \) at the entrance to the crack, and approximately is taken \( \partial T/\partial x_1 \approx (T - T_0)/l \), where \( l \) is a constant, \( H(t) \) is the Heaviside unit step function, \( K_2 \) is the coefficient of the horizontal wear of the medium. In the linear approximation, the value \( q \) and \( i_0 \) are the same as in the initial section \( x_1 = 0 \), the term in the boundary condition containing \( i_0 \) corresponds to the flux density of the impurity concentration along the axis \( x_2 \), which occurs when the fluid enters the fracture.

In the general case, we have to add the diffusion equation for the impurity concentration, as well as the thermal diffusion equations for the fluid inside the crack \( |x_2| \leq h(x_1, t) \). The above assumptions here we made to simplify the problem.
3. Solution of problem of building up the crack in the absence of friction

Consider the following simplified mixed problem for a thermoelastic isotropic medium with the boundary conditions of the form (figure 1):

\[ \sigma_{x_1 x_2} = 0, \quad -\infty < x_1 < \infty \]

\[ u_2 = 0, \quad x_1 < 0 \]

\[ \frac{\partial u_2}{\partial t} = -K_3 \frac{\partial u_1}{\partial x_1} - K_4 \frac{\partial u_2}{\partial x_2} - \frac{i_0}{\rho_s} H(v t - x_1) H(t), \quad x_1 > 0 \]

where \( a^2 = a^2 + \delta, \) \( \xi = (C_p - C_v)/(C_v l) \cdot \nu h_0 \rho/\rho_s, \) \( K_3 = \rho/\rho_s \cdot K(a^2 - b^2) + \xi, \) \( K_4 = \rho/\rho_s \cdot K a^2 + \xi. \)

In the plane case, the equations of motion for a thermoelastic isotropic medium in displacements have the form (2). The initial conditions are zero:

\[ u_i \big|_{t=0} = 0, \quad \frac{\partial u_i}{\partial t} \big|_{t=0} = 0, \quad i = 1, 2. \]

We use the Laplace integral transformations with respect to \( t \) and the Fourier integral transformations with respect to \( x_1, \) denote the Laplace integral transformations of \( u_i \) by \( u_i^L \) and the Fourier integral transformations of \( u_i^L \) by \( u_i^{LF}, \) and seek the solution of the problem in the form

\[ u_i^L = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} e^{i \alpha x_1 + i \beta_n x_2} u_i^{LF} d\tilde{\alpha}, \quad \tilde{\beta}_n = \sqrt{\frac{\omega^2}{c_n^2} - \tilde{\alpha}^2}, \quad c_1 = \tilde{a}, \quad c_2 = b, \quad i = 1, 2, \]

where \( s = -i \omega \) is a parameter of plane Laplace transformation. Then problem (2), (6) is reduced to solving the Wiener–Hopf equation

\[ \Omega^+ - \frac{i_0}{2 \pi \rho_s (s/v + i \tilde{\alpha})} = \frac{R(\tilde{\alpha})}{i \omega^2 b^2 \tilde{\beta}_1} V^-, \]

where

\[ R(\tilde{\alpha}) = \tilde{\beta}_1 \frac{\omega^3}{b^2} - 2 \tilde{\alpha}^2 \tilde{\beta}_1 \tilde{\beta}_2 (K_4 - K_3) - \left[ K_4 \frac{\omega^2}{\tilde{a}^2} - (K_4 - K_3) \tilde{a}^2 \right] \left( \frac{\omega^2}{b^2} - 2 \tilde{a}^2 \right), \]

\[ \Omega^+ = \frac{1}{2 \pi} \int_{-\infty}^{0} \left( s u_2^L + K_3 \frac{\partial u_1^L}{\partial x_1} + K_4 \frac{\partial u_2^L}{\partial x_2} \right) \bigg|_{x_2=0} e^{-i \alpha x_1} d x_1, \]

\[ V^- = \frac{1}{2 \pi} \int_{0}^{\infty} u_2^L \big|_{y=0} e^{-i \alpha x_1} d x_1. \]
Here the index (+) stands for analytic functions of $\alpha$ in the upper half-plane, and the index $(-)$ stands for analytic functions in the lower half-plane $\alpha$. Applying the Noble method [5], factorizing the function (9), solving equation (8), and making the inverse transformations, we obtained the solution of problem (2), (6) and the displacement component along the axis $x_2$ in the following form:

$$u_2 = \frac{C_0i\alpha}{\pi \rho_s(\alpha_0 + \alpha/v)D_+ (\alpha/v)} \left\{ \frac{\alpha - \alpha_0 x_1 / \alpha}{(\alpha_0 - \alpha/v)D_-(\alpha_0)} \ln \left| \frac{\sqrt{\alpha t/x_1 - 1} - \sqrt{\alpha/v - 1}}{\sqrt{\alpha t/x_1 - 1} + \sqrt{\alpha/v - 1}} \right| \right\} ,$$

where

$$F_2(u) = -\mu(u) \exp[-\chi(u)],$$

$$\chi(u) = \frac{1}{2\pi i} \int_{1/a}^{1/b} \frac{R(\zeta)}{R(\zeta) - u} d\zeta - \frac{1}{\pi} \int_{1/a}^{1/b} \frac{1}{\arctan \frac{\zeta^2 - \frac{1}{\alpha^2}}{2(K_3 - K_4)\zeta^2 - \frac{1}{\alpha^2} \sqrt{\frac{1}{\alpha^2} - \frac{1}{\zeta^2}}}} \frac{d\zeta}{\zeta - u},$$

$$\mu(u) = \frac{1}{\sqrt{\left[ \frac{1}{\alpha} \left( u_2^2 - \frac{1}{\alpha^2} \right) + 2(K_3 - K_4)u^2 \left( u_2^2 - \frac{1}{\alpha^2} \sqrt{\frac{1}{\alpha^2} - u_2^2} \right) \right] - 2(K_3 - K_4)u^2 \left( \frac{1}{\alpha^2} - u_2^2 \right)}},$$

$$D_{\pm}(\alpha) = \exp \left\{ \frac{1}{\pi} \int_{1/a}^{1/b} \frac{\sqrt{\frac{1}{\alpha^2} - \frac{1}{\zeta^2}} - 2(K_4 - K_3)\zeta^2 \sqrt{\frac{1}{\alpha^2} - \frac{1}{\alpha^2} - \frac{1}{\zeta^2}}}{\left( \frac{\alpha^2}{\alpha^2} - \frac{2}{\alpha} \right) \left( \frac{K_4}{\alpha} \zeta^2 + \frac{K_3}{\alpha} \left( 1 - \frac{1}{\alpha^2} \right) \right)} d\zeta \right\} ,$$

here $\pm \alpha_0 = \pm$ stand for zeros of the function $R(\alpha)$.

Calculating the function (12) for $\frac{a}{b} = \sqrt{3}$, $\frac{K_4}{\alpha} = 0.8$, $\frac{K_3}{\alpha} = 0.3$, $\frac{\rho_s a}{\alpha} = 1/(2 \times 10^5)$, $\alpha = 1000 \text{ cm/s}$, $\beta = 100$, we obtain $\pm 0.981431$ for $\pm \alpha_0$ and the graph (figure 2) of the dependence of $u_2/\alpha t$ on $x_1/\alpha t$. As a result of calculations, it is shown that for $x_1/\alpha t \leq 2.1 \times 10^{-7}$, $u_2/\alpha t > 0$, and for $x_1/\alpha t \geq 2.1 \times 10^{-7}$, $u_2/\alpha t < 0$, i.e., the crack width $h = h_0 + u_2$ decreases. Taking a very thin microcrack ($h_0 = 10^{-5} \text{ cm}$), it is possible to construct the crack closure time $t$ dependence on the coordinate $x_1$ (figure 3). At the times $t = 4s$ and $t = 6s$, the profile of the upper face of the fracture in the section $10^{-7} \text{ cm} < x_1 < 0.008 \text{ cm}$ has the shape shown in figure 4.

4. Solution of a spatial problem of closing a geophysical crack in a thermoelastic medium in the presence of a fluid flow

In this section, we consider a spatial problem similar to the plane problem solved in Sec. 3, where the main attention was paid to the effects of thermal diffusion and thermoelasticity in the medium. A crack at the initial time is an infinite half-strip confined by half-planes $x_3 = \pm h_0$, $x_1 > 0$, i.e., it is a gap of solid elastic medium of uniform width $2h_0$. The equations of crack profiles are $x_3 = \pm h(x_1, x_2, t) > 0$. The fluid with impurities is injected into the crack and
moves at a constant speed in the direction of the axis $x_1$. Due to cooling and thermal diffusion, as well as mechanical buildup, the precipitation of dissolved components occurs, which leads to a change in the crack profile.

In the three-dimensional problem, the equations of thermoelasticity in displacements for an isotropic medium, similarly to the equations obtained in the plane case, have the form

$$(a^2 - b^2) \frac{\partial}{\partial x_i} \nabla + b^2 \Delta u_i = \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2, 3, \quad \nabla = \text{div} \ u, \quad \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}, \quad (12)$$

where $u_i$ ($i = 1, 2, 3$) are the displacement components and the other notation is the same. The initial conditions are zero:

$$u_i \big|_{t=0} = 0, \quad \frac{\partial u_i}{\partial t} \big|_{t=0} = 0, \quad i = 1, 2, 3.$$

With the crack build up taken into account, the boundary conditions on the crack ($x_1 < 0$) and beyond it ($x_1 > 0$) have the form ($|x_2| < \infty$)

$$\sigma_{x_1x_3} \big|_{x_3=0} = \rho b^2 \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \big|_{x_3=0} = 0 \quad \text{for } -\infty < x_1 < \infty, \quad (13)$$

$$\sigma_{x_2x_3} \big|_{x_3=0} = \rho b^2 \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \big|_{x_3=0} = 0 \quad \text{for } -\infty < x_1 < \infty, \quad (14)$$
\[ u_3\big|_{x_3=0} = 0 \quad \text{for} \quad x_1 < 0, \]
\[ \left[ \frac{\partial u_3}{\partial t} + K_3 \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \right]_{x_3=0} = -\frac{i_0}{\rho_s} H(vt-x_1)H(c-x_2)H(x_2)H(t) \quad \text{for} \quad x_1 > 0, \]  
(15)
(16)

where \( u_{1,2,3} = o(R_1^{1/2}) \), \( R_1 = x_1^2 + x_2^2 \to 0 \) (the condition on the edge), \( c = \text{const} \), \( i_0 \) is the constant diffusion flow of inclusions along the axis \( x_3 \) at the entrance to the crack, \( v \) is fluid velocity along the axis \( x_2 \), and the other notation is the same. By virtue of the boundary conditions, the problem can be considered as symmetric one with respect to the plane \( Ox_1x_2 \); moreover \( u_{1,2} \) are even functions and \( u_3 \) is an odd function of \( x_3 \), and therefore we can formulate the problem in the half-space \( x_3 \geq 0 \).

The solution of the problem is constructed by combining [6] the Wiener–Hopf methods, the Laplace integral transformations with respect to \( t \), the Fourier integral transformations with respect to \( x \), and analytic methods of the theory of functions of a complex variable. After cumbersome calculations, the exact solution of the problem is obtained in polar coordinates in the Smirnov–Sobolev form [6], and it is shown that there is no singularity at \( x_1 \to +0, \ x_3 = 0 \).

Conclusions
The study of closing of cracks is very actual and practical geophysical, technological and biological problem. In numerous works the problems of closing a geophysical crack due to thermal diffusion effects are studied without consideration of elastic stresses. In the present paper the statement of the problem of closing geophysical cracks in the presence of a fluid flow is presented with regard to the thermoelastic stress and the thermal diffusion effects. The nonstationary mixed plane boundary-value problem in the special case without friction based on the integral transforms and the Wiener–Hopf methods is solved. A similar spatial problem is also solved. The developed in the present paper analytical methods are rather general and obtained formulas are easy for calculations, which can be carried out in wide variety of engineering and medical, practical problems of closing of cracks.

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