The 2+1 Dirac equation with the $\delta$ potential

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Abstract

In this study the bound state of 2+1 Dirac equation with the cylindrically symmetric $\delta(r - r_0)$-potential is presented. It is shown that the energy can be obtained by the transcendental equation obtained from the matching condition in the configuration space. It is surprisingly found that the relation between the radial functions $F_{jE}^+, G_{jE}^+$ and $F_{jE}^-, G_{jE}^-$ can be established by $SO(2)$ group.

Key words: Dirac equation, $\delta$ potential, $SO(2)$ group

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1. Introduction

As we know, it is a curious and complex situation to solve the Dirac equation with the $\delta$ potential in comparison with the equivalent problem in non-relativistic quantum mechanics, i. e., the Schrödinger equation, which is discussed in any course on quantum mechanics. This is basically related to the fact that being the Dirac equation of first order, a singular potential, like the $\delta$ one, induces discontinuities at the level of the wave function themselves instead of the usual discontinuities that appear in the first derivative in the Schrödinger equation.

This puzzling situation has been discussed previously [1-2]. The partial wave operators have been constructed by making use of the self-adjoint extension theory. Recently, other authors have shown how to deal with the Dirac equation with the $\delta$-potential in the different spaces [3-6]. The position-space treatment of this problem has been carried out in [3, 4]. However, it has been discussed in the momentum space [5] as well as in the configuration-space treatment [6]. All of those papers mentioned above have mainly discussed this problem in 3+1 dimensional space. The study of this similar problem in one dimension has been completed in [7, 8]. On the other hand, the positron emission which is related with this problem has been studied in Ref. [9].

With the interest of the lower-dimensional field theory, it is necessary to study the 2+1 Dirac equation with the $\delta$-potential, which has never been addressed in the
literature, to our knowledge. The purpose of this paper is to study this problem in the configuration space.

This paper is organized as follows. Section 2 is devoted to the introduction of the 2+1 Dirac equation. The bound state will be discussed in Section 3. A conclusion will be given in Section 4.

2. Dirac equation in 2+1 dimensions

It is convenient to recall some general properties of the solution of the Dirac equation in a central potential. For more details the reader may consult the book [10].

For the present discussion we do not need the explicit form of the angular part. Consider the 2+1 Dirac equation

\[
\sum_{\mu=0}^{2} i\gamma^\mu (\partial_\mu + ieA_\mu) \psi = M\psi
\]

(1)

where \( M \) is the mass of the particle, and

\[
\gamma^0 = \sigma_3 \quad \gamma^1 = i\sigma_1 \quad \gamma^2 = i\sigma_2.
\]

(2)

Throughout this paper the natural units \( \hbar = c = 1 \) are employed if not explicitly stated otherwise. Discuss the special case where only the zero component of \( A_\mu \) is
non-vanishing and cylindrically symmetric:

\[ A_1 = A_2 = 0 \quad eA_0 = V(r) \]  

where \( V(r) \) is taken as the attractive cylindrically symmetric potential

\[ V(r) = -a\delta(r - r_0) \quad a > 0 \]  

Let

\[ \psi_{jE}(t, r) = e^{-iEt}r^{-1/2} \begin{pmatrix} F_{jE}(r)e^{i(j-1/2)\phi} \\ G_{jE}(r)e^{i(j+1/2)\phi} \end{pmatrix}, \]

where \( j \) denotes the total angular momentum, \( j = \pm 1/2, \pm 3/2, \ldots \). It is shown that the radial components \( F_{jE} \) and \( G_{jE} \) satisfy the following set of coupled differential equations [11]

\[ \frac{d}{dr}G_{jE}(r) + \frac{j}{r}G_{jE}(r) = (E - V(r) - M)F_{jE}(r), \]  

\[ -\frac{d}{dr}F_{jE}(r) + \frac{j}{r}F_{jE}(r) = (E - V(r) + M)G_{jE}(r). \]  

It is easy to see that the solutions with a negative \( j \) can be obtained from those with a positive \( j \) by interchanging \( F_{jE}(r) \leftrightarrow G_{-j-E}(r) \), so that in the following we only discuss the solutions with a positive \( j \).

The physically admissible solutions are finite, continuous, vanishing at the origin, and square integrable:

\[ F_{jE}(r) = G_{jE}(r) = 0, \quad \text{when} \quad r \rightarrow 0, \]
\[
\int_0^\infty dr \left\{ |F_{jE}(r)|^2 + |G_{jE}(r)|^2 \right\} < \infty.
\] (8)

The solutions for |E| > M describe the scattering states, and those for |E| ≤ M describe the bound states. In this note, what we are interested in is related with the bound state.

Due to the linear discontinuity of the spinor function given by the \(\delta\)-potential, we only need to fix the boundary conditions in the neighborhood of the shell \(r = r_0\). Multiplying Eq. (6a) by \(F_{jE}\) and Eq. (6b) by \(G_{jE}\) and calculating their summation, one can obtain

\[
F'_{jE}F_{jE} + G'_{jE}G_{jE} = -2MF_{jE}G_{jE} + j \frac{(F_{jE}^2 - G_{jE}^2)}{r}
\] (9)

where and hereafter we denote \(F_{jE}(r)\) and \(G_{jE}(r)\) by \(F_{jE}\) and \(G_{jE}\) respectively and the primes denote the radial derivatives with respect to the variable \(r\). Integrating between \(r_0 - \varepsilon\) and \(r_0 + \varepsilon\) and taking the limit \(\varepsilon \to 0\) we get

\[
\lim_{\varepsilon \to 0} \int_0^{r_0+\varepsilon} (F'_{jE}F_{jE} + G'_{jE}G_{jE}) dr = \lim_{\varepsilon \to 0} \int_{r_0-\varepsilon}^{r_0+\varepsilon} \left( -2MF_{jE}G_{jE} + j \frac{(F_{jE}^2 - G_{jE}^2)}{r} \right) dr,
\] (10)

which implies that

\[
\lim_{\varepsilon \to 0} (F_{jE}^2 + G_{jE}^2) \bigg|_{r_0-\varepsilon}^{r_0+\varepsilon} = 0.
\] (11)

It is found that the radial functions \(F_{jE}\) and \(G_{jE}\) can be regarded as the real and imaginary parts of a function in \(\mathbb{C}\). From the geometrical point of view, it is found that the norm of the two components spinor \(F\) and \(G\) are constant when crossing
the support of the \( \delta \)-potential, which coincides with the results established in [1, 2].

In this way, one can write the following boundary condition for all finite \( r \)

\[
F^2_{jE} + G^2_{jE} = F^2_{jE} + G^2_{jE}. \tag{12}
\]

where \( F^\pm_{jE} \equiv F(r_0 \pm \epsilon) \) and \( G^\pm_{jE} \equiv G(r_0 \pm \epsilon) \). Let us now multiply Eq. (6a) by \( G_{jE} \) and Eq. (6b) by \( F_{jE} \) respectively and calculate their difference,

\[
F'_{jE}G_{jE} - F_{jE}G'_{jE} = -(E - M)G^2_{jE} + (E + M)F^2_{jE} + \frac{2jG_{jE}F_{jE}}{r} - a\delta(r - r_0)(F^2_{jE} + G^2_{jE}) \tag{13}
\]

which is divided by \( F^2_{jE} + G^2_{jE} \), one can integrate in the neighborhood of the shell radius

\[
\lim_{\epsilon \to 0} \int_{r_0 - \epsilon}^{r_0 + \epsilon} \frac{F'_{jE}G_{jE} - F_{jE}G'_{jE}}{(F^2_{jE} + G^2_{jE})} dr = -a \lim_{\epsilon \to 0} \int_{r_0 - \epsilon}^{r_0 + \epsilon} \delta(r - r_0) dr. \tag{14}
\]

By using

\[
\frac{F'_{jE}G_{jE} - F_{jE}G'_{jE}}{(F^2_{jE} + G^2_{jE})} = \left( \frac{F_{jE}}{G_{jE}} \right)' \frac{1}{(G_{jE}/F_{jE})^2 + 1} \tag{15}
\]

and since

\[
\int \frac{1}{1 + h^2(x)} d|h(x)| = \arctan(h(x))
\]

we have

\[
\lim_{\epsilon \to 0} \left( \arctan \frac{F_{jE}}{G_{jE}} \right)_{r_0 - \epsilon}^{r_0 + \epsilon} = -a. \tag{16}
\]

In this way, this boundary condition can be written as

\[
\arctan \frac{F^+_{jE}}{G^+_{jE}} - \arctan \frac{F^-_{jE}}{G^-_{jE}} = -a. \tag{17}
\]
Define the dimensionless parameter \( \alpha \equiv \tan(a) \), equation (17) can then be expressed as

\[
\frac{F^+_{jE}}{G^+_{jE}} = \frac{(F^-_{jE}/G^-_{jE}) - \alpha}{1 + \alpha(F^-_{jE}/G^-_{jE})}.
\] (18)

Except for an arbitrary phase, the last expression can be written as a matrix relation between the radial functions at both sides of the potential,

\[
\begin{bmatrix}
F^+_{jE} \\
G^+_{jE}
\end{bmatrix} = \begin{bmatrix}
\cos(a) & -\sin(a) \\
\sin(a) & \cos(a)
\end{bmatrix} \begin{bmatrix}
F^-_{jE} \\
G^-_{jE}
\end{bmatrix} \equiv A \begin{bmatrix}
F^-_{jE} \\
G^-_{jE}
\end{bmatrix}
\] (19)

where

\[
A = \begin{bmatrix}
\cos(a) & -\sin(a) \\
\sin(a) & \cos(a)
\end{bmatrix}.
\]

We note that \( A \) is unitary and orthogonal, i.e., \( \det A = 1 \) and contains the information for finding the eigenvalue equation for the bound states. On the other hand, we see that matrix \( A \) can construct the \( SO(2) \) group, that is to say, the relation between the radial functions \( F^+_{jE}, G^+_{jE} \) and \( F^-_{jE}, G^-_{jE} \) at the both sides of the potential can be established by \( SO(2) \) group, which is a new point, to our knowledge. However, for the complex valued function with real and imaginary part given by \( F_{jE} \) and \( G_{jE} \) respectively, it is found that the \( \delta \) function manifests itself by a phase change of this function expressed by \( \tan(a) \).

3. Bound states
We now solve Eq. (6) for the energy $|E| \leq M$. In the region $[0, r_0]$, we have

$$F^{-}_{jE} = e^{-i(j-1/2)\pi/2} [(M + E)\pi kr/2]^{1/2} J_{j-1/2}(ikr)$$
$$G^{-}_{jE} = e^{-i(j-3/2)\pi/2} [(M - E)\pi kr/2]^{1/2} J_{j+1/2}(ikr)$$

(20)

where $J_m(x)$ is the Bessel function, and

$$\kappa = \left(M^2 - E^2\right)^{1/2}.\quad (21)$$

The ratio at $r = r_0-$ when $\lambda = 0$ is

$$\left|\frac{F^{-}_{jE}}{G^{-}_{jE}}\right|_{r=r_0-} = -i \left(\frac{M + E}{M - E}\right)^{1/2} \frac{J_{j-1/2}(ikr_0)}{J_{j+1/2}(ikr_0)} \quad (22)$$

In the region $[r_0, \infty)$, we have $V(r) = 0$ and

$$F^{+}_{jE} = e^{i(j+1/2)\pi/2} [(M + E)\pi kr/2]^{1/2} H^{(1)}_{j-1/2}(ikr)$$
$$G^{+}_{jE} = e^{i(j+3/2)\pi/2} [(M - E)\pi kr/2]^{1/2} H^{(1)}_{j+1/2}(ikr)$$

(23)

where $H^{(1)}_m(x)$ is the Hankel function of the first kind. The ratio at $r = r_0+$ can be expressed as

$$\left|\frac{F^{+}_{jE}}{G^{+}_{jE}}\right|_{r=r_0+} = -i \left(\frac{M + E}{M - E}\right)^{1/2} \frac{H^{(1)}_{j-1/2}(ikr_0)}{H^{(1)}_{j+1/2}(ikr_0)} \quad (24)$$

In this case, we can rearrange Eq. (11) as

$$i \left(\frac{M + E}{M - E}\right)^{1/2} \left[ \frac{H_{j-1/2}(ikr)}{H_{j+1/2}(ikr)} \right] = \alpha \left[ 1 - \frac{M + E}{M - E} \cdot \frac{J_{j-1/2}(ikr)}{J_{j+1/2}(ikr)} \cdot \frac{H_{j-1/2}(ikr)}{H_{j+1/2}(ikr)} \right] \quad (25)$$

from which, we can obtain the energy eigenvalue $E$ with respect to the different angular momentum $j$. 

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4. Concluding remarks

In this paper we have carried out the 2+1 Dirac equation with the $\delta$-potential. It is found that the ratio $F_{jE}/G_{jE}$ of the radial wave functions $F_{jE}$ and $G_{jE}$ at $r_0$ plays an important role in establishing the second boundary condition (18) from which we can obtain the energy eigenvalue $E$. This is a very simple method to study the 2+1 Dirac equation in configuration space. On the other hand, it is surprisingly found that the relation between the radial functions $F_{jE}^+, G_{jE}^+$ and $F_{jE}^-, G_{jE}^-$ at the both sides of the potential can be established by $SO(2)$ group, which is a very new point, to our knowledge.

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