Local copying of orthogonal entangled quantum states

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Abstract. In classical information theory one can, in principle, produce a perfect copy of any input state. In quantum information theory, the no cloning theorem prohibits exact copying of non-orthogonal states. Moreover, if we wish to copy multiparticle entangled states and can perform only local operations and classical communication (LOCC), then further restrictions apply. We investigate the problem of copying orthogonal, entangled quantum states with an entangled blank state under the restriction to LOCC. Throughout, the subsystems have finite dimension $D$. We show that if all of the states to be copied are non-maximally entangled, then novel LOCC copying procedures based on entanglement catalysis are possible. We then study in detail the LOCC copying problem where both the blank state and at least one of the states to be copied are maximally entangled. For this to be possible, we find that all the states to be copied must be maximally entangled. We obtain a necessary and sufficient condition for LOCC copying under these conditions. For two orthogonal, maximally entangled states, we provide the general solution to this condition. We use it to show that for $D = 2, 3$, any pair of orthogonal, maximally entangled states can be locally copied using a maximally entangled blank state. However, we also show that for any $D$ which is not prime, one can construct pairs of such states for which this is impossible.

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1. **Introduction**

The no cloning theorem of Wootters and Zurek [1] and Dieks [2] prohibits the creation of perfect copies of non-orthogonal quantum states. This famous result has profound implications for quantum communications, e.g. the security of quantum cryptography [3]. It is also well known that any set of orthogonal states can be perfectly copied in principle. However, it is not known how well this can be achieved if there are restrictions on the set of possible quantum operations.

A common scenario in quantum information processing and communications is where a multiparticle, possibly entangled state is distributed among a number of spatially separated parties. Each of these parties can perform arbitrary local operations on the subsystems they possess. However, they can only send classical information to each other. When this is the case, the parties are restricted to performing local (quantum) operations and classical communication (LOCC). There has been a considerable amount of activity devoted to understanding the properties of LOCC operations. Certain specific quantum information processing tasks, such as entanglement distillation and, more recently, state discrimination, have been the focus of a particularly large amount of attention with respect to the LOCC constraint. In this paper, we investigate the problem of copying orthogonal, entangled, quantum states under these conditions.

Quantum copying and quantum state discrimination are closely related operations [4]. In the study of state discrimination under the LOCC constraint, it has been found that any pair of orthogonal, entangled, pure, bipartite states can be perfectly discriminated by LOCC [5]. This is not generally possible for more than two states. Also, it has been found that any two non-orthogonal, entangled, pure, bipartite states can be optimally discriminated by LOCC [6]–[8]. Again, this is not generally possible for more than two states [9].

We see that the LOCC constraint imposes restrictions on the number of states for which certain discrimination tasks are possible. Given that copying is closely related to state discrimination, we might imagine that the LOCC constraint could also affect the number of states for which certain copying procedures are possible. We will show in this paper that this is indeed the case.
In fact, we shall see that some of the restrictions on LOCC copying are, if we wish to use entanglement efficiently, more severe than those on LOCC state discrimination. This will turn out to be a consequence of the fact that, when copying states by LOCC, there are certain factors we must take into account that do not apply to LOCC state discrimination. In LOCC state discrimination, the original state is typically destroyed. This is of no concern, since we only wish to know the state, not preserve it. However, in copying the state, not only do we wish to preserve the original state, we also wish to imprint it onto another system initialized in a ‘blank’ state. If we restrict ourselves to performing LOCC copying and the states we wish to copy are entangled, then the blank state must be entangled also. If this were not the case, then the copying procedure would create entanglement, which is well known to be impossible under LOCC [10, 11].

It was recently discovered by Ghosh et al [12] in an independent work, that some sets of orthogonal, maximally entangled states can be copied by LOCC and with a maximally entangled blank state. These authors considered LOCC copying of Bell states. The Bell states, which are maximally entangled states of two qubits, each contain one ebit of entanglement. These authors showed that LOCC copying of any two Bell states is possible with a blank state containing one ebit of entanglement. They found, however, that to copy all four Bell states requires one further ebit of entanglement. This is still less than the two further ebits that would be required to perform an arbitrary operation on the four qubits comprising the states $|\psi_j\rangle$ and $|b\rangle$ by LOCC [13]–[15].

In this paper, we obtain numerous results which relate to the problem of copying pure, bipartite, orthogonal, entangled states by LOCC. Throughout, we are interested in making perfect copies deterministically. In section 2, we set up the copying problem in general terms. In doing so, we acknowledge the fact that an LOCC operation may, in principle, involve an unlimited number of rounds of classical communication. As such, the operation may become unwieldy in formal terms. Rather than deal with this possibility directly, we take an alternative approach based on the fact that LOCC operations form a subset of the set of separable operations. The form of a general separable operation is well known and more convenient to work with.

For the reasons we gave above, the LOCC copying procedure must use an entangled blank state. Entanglement is a precious resource in quantum information processing. Consequently, it is highly desirable that entanglement is used efficiently and, if at all possible, conserved by the operation. To investigate this matter fully, we require a measure of entanglement. The problem of quantifying entanglement is central to quantum information theory. For pure, bipartite states in the asymptotic limit, where many copies of the states are available, a unique measure of entanglement can be provided [16, 17]. This is the entropy of entanglement. However, in the scenario considered in this paper, we only have one copy of each of the states to be processed: the state to be copied and the blank state.

In this ‘one-shot’ scenario, there exist pairs of incomparable states, for which one cannot unambiguously decide whether the entanglement of one state is greater than, less than or equal to that of the other. Ideally, we would like the entanglement of the blank state to equal that of the most entangled of the states to be copied, as this would represent the most efficient use of entanglement. However, our desire to use entanglement efficiently leads us to, in general, account for the possibility of incomparability of the blank state and some of the states to be copied.

We show that when all of the states to be copied are non-maximally entangled, accounting for this possible incomparability leads to scenarios where, although the LOCC copying procedure is possible, the blank state cannot be directly transformed into the state to be copied by LOCC. Instead, the original copy of the state serves as an entanglement catalyst [18] which facilitates...
the copying procedure. This point is illustrated in the simple case where we wish to copy just one state.

In section 3, we analyse in detail the problem of locally copying $N$ orthogonal, entangled states with $D$-dimensional subsystems. The blank state is also taken to be an entangled state whose subsystems are also $D$-dimensional. We focus in particular on the situation where one of the states to be copied is maximally entangled. This simplifies the problem in many respects. Firstly, the possibility of catalytic copying, with its attendant complications, does not arise, since a maximally entangled state cannot serve as an entanglement catalyst [18]. Consequently, the blank state must be maximally entangled also. Secondly, we show that the local Kraus operators for a separable copying operation must be proportional to unitary operators if they are to copy a maximally entangled state. This is very helpful, since any separable operation whose Kraus operators have this property can be performed by LOCC. Indeed, we find that we may, without loss of generality, take the entire copying operation to consist of just two local unitary operations, with one being carried out by each party, and no classical communication. This implies that if one of the states to be copied is maximally entangled, then they must all be maximally entangled. We then use the convenient form of these operators to obtain a general necessary and sufficient condition for LOCC copying of $N D$-dimensional maximally entangled states with a maximally entangled blank state.

This condition is difficult to solve for arbitrary $N$ and $D$. However, it can be solved exactly for $N = 2$ and all $D$. In section 4, we present this solution in detail and describe a number of its consequences. In particular, we find that for $D = 2, 3$, any pair of maximally entangled, bipartite pure states can be copied using the same LOCC operation and a maximally entangled blank state. However, we also show that for any $D$ which is not prime, there exist such pairs for which this copying operation is impossible. We conclude in section 5 with a discussion of our results.

2. The problem of LOCC copying

2.1. General considerations

Let us consider the following scenario, depicted in figure 1. We have two parties, Alice and Bob, occupying spatially separated laboratories $\alpha$ and $\beta$ respectively. Alice and Bob each have two $D$-dimensional quantum systems. Alice’s systems will be labelled 1 and 3 while Bob’s will be labelled 2 and 4. Associated with each of these systems is a copy of the $D$-dimensional Hilbert space $\mathcal{H}$. The tensor product Hilbert spaces of Alice’s and Bob’s pairs will be denoted by $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ respectively. Alice and Bob also possess ancillary quantum systems enabling them to carry out arbitrary local quantum operations. They also share a two-way classical channel allowing unlimited classical communication between them.

Consider now a set of entangled, bipartite, pure states $\{|\psi_j\rangle\}$, where $j \in \{1, \ldots, N\}$. Throughout this article, when $N > 1$, we shall take the $|\psi_j\rangle$ to be orthogonal. This implies that, without the LOCC restriction, the states could be perfectly copied. Particles 1 and 2 are initially prepared in one of these states although Alice and Bob do not know which one. Particles 3 and 4 are initially prepared in the known, bipartite, blank state $|b\rangle$. Alice and Bob aim to perform the transformation

$$|\psi^{12}_j\rangle \otimes |b^{34}\rangle \rightarrow |\psi^{12}_j\rangle \otimes |\psi^{34}_j\rangle$$

(2.1)

by LOCC. Here, the superscripts indicate the particles that have been prepared in each state.
General quantum state transformations are described using the quantum operations formalism [19, 20]. A quantum operation on a quantum system with Hilbert space $\mathcal{S}$ is represented mathematically by a completely positive, linear, trace non-increasing map from the set of linear operators on $\mathcal{S}$ to itself (when the input and output Hilbert spaces are identical, which is the case in the present context). Let us denote such a map by $\mathcal{E}$ and consider a quantum system whose initial state is described by a density operator $\rho$. This map transforms the density operator according to

$$\rho \rightarrow \frac{\mathcal{E}(\rho)}{\text{Tr}(\mathcal{E}(\rho))}.$$  \hfill (2.2)

A particularly useful representation of quantum operations is the operator-sum representation:

$$\mathcal{E}(\rho) = \sum_{k=1}^{K} F_k \rho F_k^\dagger,$$  \hfill (2.3)

where $K$ is some positive integer. For $\mathcal{E}$ to be a physically realizable quantum operation, the $F_k$, which are known as the Kraus operators, must be linear operators that satisfy

$$\sum_{k=1}^{K} F_k^\dagger F_k \leq \mathbb{I},$$  \hfill (2.4)

where $\mathbb{I}$ is the identity operator on $\mathcal{S}$. The equality holds when the map is trace preserving for all states, in which case the quantum operation is deterministic for all states. If the operation is not trace preserving for a particular initial state, then it can only be implemented with probability equal to the trace of the final state. Whether or not the operation has been implemented can be always be determined in principle, and this may be viewed as a generalized measurement. More generally, any experiment implements a trace preserving sum of trace non-increasing
quantum operations. The operation that has actually been carried out can always, in principle, be
determined, and it formally corresponds to a particular outcome of a generalized measurement.

There are many particular kinds of quantum operations of special interest. In the present
context, two kinds are particularly important. These are the separable operations [21] and
the LOCC procedures. In a separable operation acting on two systems in spatially separated
laboratories \( \alpha \) and \( \beta \), the \( F_k \) may be written as

\[
F_k = A_k \otimes B_k. \tag{2.5}
\]

Here, \( A_k \) and \( B_k \) are local Kraus operators acting on \( \mathcal{H}_\alpha \) and \( \mathcal{H}_\beta \) respectively. In this context, we
may refer to the \( F_k \) as the global Kraus operators.

LOCC procedures are sequences of trace preserving local quantum operations carried out in
the individual laboratories, interspersed with rounds of classical communication. The information
received at each laboratory is used to control the subsequent local operation at the same location.
The number of rounds of classical communication can be arbitrarily large and, consequently,
LOCC procedures can be difficult to work with. However, the set of such procedures is a subset
of the set of separable operations. It follows that separability is only a necessary condition for
a quantum operation to be implementable by LOCC [10]. It is not sufficient. Still, the fact that
the global Kraus operators for separable operations have the simple form shown in (2.5) often
makes such operations a useful starting point for investigating problems relating to LOCC. See,
for example, [9].

2.2. Catalytic copying

Due to the limitations on the LOCC manipulation of entanglement, it is a non-trivial matter to
determine the set of blank states which enable one to copy, by LOCC, even a single, known state \( |\psi\rangle \). In principle, the conditions under which this is possible can be obtained using Nielsen’s
theorem [22]. This result specifies the conditions under which one pure, bipartite, entangled state
can be transformed into another by deterministic LOCC.

Nielsen’s theorem involves the concept of majorization, which we will briefly review. Consider two real, \( R \)-component vectors \( v = (v_1, \ldots, v_R) \) and \( w = (w_1, \ldots, w_R) \). Furthermore,
let \( v^\downarrow \) and \( w^\downarrow \) be the vectors obtained from \( v \) and \( w \) by arranging their components in non-
increasing order. The vector \( w \) is said to majorize the vector \( v \) if

\[
\sum_{i=1}^{r} v_i^\downarrow \leq \sum_{i=1}^{r} w_i^\downarrow, \tag{2.6}
\]

for all \( r \in \{1, \ldots, R\} \) and with the equality holding for \( R = r \). This majorization relation is
usually written as \( w \succ v \) or \( v \prec w \).

Consider now two pure bipartite states \( |\phi_1\rangle \) and \( |\phi_2\rangle \). These may be written in Schmidt
decomposition form as

\[
|\phi_s\rangle = \sum_{i=1}^{R} \sqrt{\lambda_{si}} |x_{si}\rangle \otimes |y_{si}\rangle,
\]

where \( s \in \{1, 2\} \) and where the maximum subsystem Hilbert space dimension is \( R \). The Schmidt vectors \( \lambda_s = (\lambda_{s1}, \ldots, \lambda_{sR}) \) may, without loss of generality, be taken to have real, non-negative components.

Nielsen’s theorem states that \( |\phi_1\rangle \) can be transformed by deterministic LOCC into \( |\phi_2\rangle \) if and only if

\[
\lambda_1 \prec \lambda_2. \tag{2.7}
\]
Returning to the problem of LOCC copying, let the states $|\psi\rangle$ and $|b\rangle$ have the Schmidt vectors $\lambda_\psi$ and $\lambda_b$. We wish to implement the transformation

$$|\psi^{12}\rangle \otimes |b^{34}\rangle \rightarrow |\psi^{12}\rangle \otimes |\psi^{34}\rangle,$$

(2.8)

by LOCC. Nielsen’s theorem implies that this will be possible if and only if

$$\lambda_\psi \otimes \lambda_b < \lambda_\psi \otimes \lambda_\psi.$$

(2.9)

Clearly, this copying transformation will be possible if the transformation $|b\rangle \rightarrow |\psi\rangle$ is possible by LOCC, i.e., if $\lambda_b < \lambda_\psi$. However, what if $|b\rangle \rightarrow |\psi\rangle$ is impossible by LOCC? When $|b\rangle$ cannot be transformed into $|\psi\rangle$ by deterministic LOCC, there appear, at first sight, to be two cases to consider, corresponding to whether or not $|\psi\rangle \rightarrow |b\rangle$ is possible by LOCC. However, we shall now show that the possibility of the LOCC transformation $|\psi\rangle \rightarrow |b\rangle$, when combined with our assumptions that the copying transformation in (2.8) is possible by LOCC and that the transformation $|\psi\rangle \rightarrow |b\rangle$ is not, leads to a contradiction.

To do this, it is useful to introduce another relation between two vectors, the trumping relation. Consider two real vectors $v$ and $w$. If there exists a real vector $u$ such that

$$u \otimes v < u \otimes w,$$

(2.10)

then we say that ‘$w$ trumps $v$’ and write this relation as $v <_T w$ or $w >_T v$. From (2.9), we clearly see that

$$\lambda_b < _T \lambda_\psi.$$

(2.11)

The trumping relation is weaker than the majorization relation: that is, if $v < w$ then $v <_T w$, but not necessarily vice versa. We are assuming that $|\psi\rangle \rightarrow |b\rangle$ is possible by LOCC, which implies that $\lambda_\psi < \lambda_b$. Therefore,

$$\lambda_\psi < _T \lambda_b.$$

(2.12)

We shall now use the following theorem due to Jonathan and Plenio [18]: if $v <_T w$ and $w <_T v$, then $v^\dagger = w^\dagger$. Combining this result with (2.11) and (2.12), we see that $\lambda_\psi^\dagger = \lambda_b^\dagger$. When this is so, it follows that $\lambda_b < \lambda_\psi$ and, by Nielsen’s theorem, that the transformation $|b\rangle \rightarrow |\psi\rangle$ is actually possible by LOCC, which contradicts our premise.

The remaining possibility is that both $|b\rangle \rightarrow |\psi\rangle$ and $|\psi\rangle \rightarrow |b\rangle$ are impossible to perform by LOCC. When this is the case, the states $|\psi\rangle$ and $|b\rangle$ are said to be incomparable. Even though incomparable states cannot be transformed into each other by LOCC, there is the possibility that the transformation in (2.8) is possible. When this is so, $|\psi^{12}\rangle$, which is unchanged by the copying procedure, is said to act as a catalyst for the transformation $|b^{34}\rangle \rightarrow |\psi^{34}\rangle$.

The problem of finding, for a general state $|\psi\rangle$, the set of blank states $|b\rangle$ for which $|\psi\rangle$ can be copied by entanglement catalysis is a challenging task. This is due to the fact that no analytical way of ordering the Schmidt coefficients of a general tensor product of two states has yet been discovered. Nevertheless, by numerical methods, one can easily check for particular states whether or not the majorization relation in (2.7) is satisfied. One can then search for pairs of pure, bipartite entangled states such that one cannot be transformed into another directly but for which the transformation is possible with a catalyst. A specific example of catalytic copying, which we obtained in this way, is as follows. Consider the case of $D = 5$ and a state $|\psi\rangle$ with Schmidt coefficients $\sqrt{0.39}$, $\sqrt{0.26}$, $\sqrt{0.18}$, $\sqrt{0.17}$ and 0. Consider also a blank state $|b\rangle$ with Schmidt coefficients $\sqrt{0.32}$, $\sqrt{0.28}$, $\sqrt{0.24}$, $\sqrt{0.085}$ and $\sqrt{0.075}$. For these two states, one can
readily verify using Nielsen’s theorem that the transformation $|b\rangle \rightarrow |\psi\rangle$ is impossible by LOCC while the transformation $|\psi^{12}\rangle \otimes |b^{34}\rangle \rightarrow |\psi^{12}\rangle \otimes |\psi^{34}\rangle$ can be carried out this way.

The main focus of this paper is on LOCC copying of multiple quantum states with efficient use of entanglement. Even for a single, known state, the problem is complicated by the possibility of catalytic copying as we have just demonstrated. To generalize this to multiple states, we would require an understanding of multi-state catalytic entanglement transformations, about which little, if anything, is currently known. Fortunately, there is a large class of states sets that we can consider for which the issue of catalysis does not arise. These are sets where at least one of the states to be copied is maximally entangled. Their preferential status is a consequence of the fact that maximally entangled states cannot serve as catalysts for pure, bipartite entanglement transformations [18]. Such sets will be the focus of our attention for the remainder of this paper.

3. LOCC copying of a pure orthogonal set including a maximally entangled state

3.1. Form of the local Kraus operators

Returning to the problem of locally copying the $N$ states $|\psi_j\rangle$, recall that we require the copying operation to be separable. This implies that the global Kraus operators will have the form shown in (2.5), where the $A_{k}^{13}$ and $B_{k}^{24}$ act on $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively. In terms of these operators, the copying transformation will have the form

$$A_{k}^{13} \otimes B_{k}^{24} |\psi_j^{12}\rangle \otimes |b^{34}\rangle = \sigma_{jk} |\psi_j^{12}\rangle \otimes |\psi_j^{34}\rangle. \quad (3.1)$$

Here, the superscripts on the operators indicate the particles on which they act. Also, the $\sigma_{jk}$ are some complex coefficients that satisfy $\sum_{k=1}^{K} |\sigma_{jk}|^2 = 1$.

Separability of the copying operation is, as we have noted above, only a necessary and not a sufficient condition for LOCC copying. However, the combination of the separability condition with specific features relating to particular sets of states can lead us to exact necessary and sufficient conditions for LOCC copying. The remainder of this paper is devoted to investigating the LOCC copying problem for a class of such sets. These are sets where at least one of the states to be copied is maximally entangled.

For the sake of definiteness, let the state $|\psi_1\rangle$ be maximally entangled. It is known [18] that a maximally entangled state cannot serve as a catalyst. Therefore, the transformation $|b\rangle \rightarrow |\psi_1\rangle$ must be possible by LOCC. Since we are restricting ourselves to blank states of a pair of $D$ dimensional particles, it follows from Nielsen’s theorem that the blank state is necessarily maximally entangled also.

This section is devoted to determining the conditions under which the $|\psi_j\rangle$ can be copied by LOCC when both $|\psi_1\rangle$ and the blank state $|b\rangle$ are maximally entangled. In the first part of this section, we will see how the requirements of our operation have interesting implications for the form of the local Kraus operators in equation (3.1). We will then obtain the general necessary and sufficient conditions under which our desired operation is physically possible.

To begin, let $\{|x_i\rangle\}$ be an orthonormal basis for the single particle Hilbert space $\mathcal{H}$. We will frequently work with the following reference maximally entangled state in $\mathcal{H}^{\otimes 2}$:

$$|\psi_{max}^{\otimes 2}\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |x_i^1\rangle \otimes |x_i^2\rangle. \quad (3.2)$$
We will also frequently encounter the product states $|x_r^i\rangle \otimes |x_s^j\rangle$, for particles $r, s$ where $r, s \in \{1, \ldots, 4\}$. As such, it is convenient to adopt a simpler notation for these states. Define

$$|X_{\mu}^{rs}\rangle = |x_r^i\rangle \otimes |x_s^j\rangle,$$

where $\mu = \mu(i, j) \in \{1, \ldots, D^2\}$. Each value of $\mu$ must correspond to unique values of $i$ and $j$. This can be achieved, for example, by letting $\mu = i + D(j - 1)$ with $i, j \in \{1, \ldots, D\}$. More generally, we will use Greek subscripts to index elements of this basis according to the same formula as for $\mu$.

The fact that the state $|\psi_1\rangle$ is maximally entangled implies that there exists a unitary operator $U_1$ on $\mathcal{H}$ such that

$$|\psi_1^{12}\rangle = (U_1^1 \otimes 1^2)|\psi_{\text{max}}^{12}\rangle.$$

When the particle pair $(3,4)$ is in this state, we replace the superscripts 1 and 2 with 3 and 4 respectively.

The blank state $|b\rangle$ is also maximally entangled, so there exists a unitary operator $U_b$ on $\mathcal{H}$ such that

$$|b^{34}\rangle = (U_b^3 \otimes 1^4)|\psi_{\text{max}}^{34}\rangle.$$

We now proceed to show that, without loss of generality, the $A_k^{13}$ and $B_k^{24}$ in (3.1) may be taken to be, up to multiplicative coefficients, unitary. To do this, we note that the most general LOCC procedure consists of an arbitrarily long sequence of local operations in Alice’s and Bob’s laboratories interspersed with rounds of classical communication. The entire LOCC operation is initiated by one party. For the sake of definiteness, and without loss of generality, let this party be Alice. Alice implements a deterministic local operation on her system. This operation, which is trace preserving, may be a sum of trace non-increasing operations in which Alice obtains (classical) information about which of these operations was carried out. The entire operation is then a generalized measurement. If it is, then the measurement result is communicated to Bob. Upon receiving this, Bob implements a local operation corresponding to this result. He then communicates a description of his operation to Alice (if she does not already know the operation he will perform given the classical information she sent him) together with any measurement results and the process can repeat an arbitrarily large number of times.

The crucial point is the fact that if Alice and Bob begin with the state $|\psi_1^{12}\rangle \otimes |b^{34}\rangle$, which is a maximally entangled state of the pairs $(1,3)$ and $(2,4)$, then the LOCC copying procedure will produce the state $|\psi_1^{12}\rangle \otimes |\psi_3^{34}\rangle$, which is also a maximally entangled state of these pairs of particles. No LOCC procedure can transform a maximally entangled state into a non-maximally entangled state, and then into another maximally entangled state. It follows that each step in their LOCC copying procedure can do no more than transform one maximally entangled state of these pairs of particles into another. So, let $|\chi_1\rangle$ and $|\chi_2\rangle$ be maximally entangled states of the pairs $(1,3)$ and $(2,4)$. We may write these states as

$$|\chi_r\rangle = (V_r^{13} \otimes 1^{24})|\psi_{\text{max}}^{12}\rangle \otimes |\psi_{\text{max}}^{34}\rangle,$$

where $r \in \{1, 2\}$ and the $V_r^{13}$ are unitary operators acting on $\mathcal{H}_\alpha$. We will now investigate the properties of a local operation in one laboratory that transforms $|\chi_1\rangle$ into $|\chi_2\rangle$. For the sake of definiteness, we let this operation be carried out by Alice in her laboratory $\alpha$. The following
argument applies equally well if the operation were to be carried out by Bob. Alice carries out a local operation, which we shall denote by $\mathcal{E}^{13}$. This takes the form of a completely positive, linear, trace non-increasing map on the space of linear operators on $\mathcal{H}_\alpha$. Interpreting this operation as corresponding to a generalized measurement outcome, whose probability is $p$ for the initial state $|\chi_1\rangle$, this operation must produce the state $|\chi_2\rangle$ according to

$$\mathcal{E}^{13} \otimes \mathbb{I}^{24}_2 (|\chi_1\rangle\langle \chi_1|) = p|\chi_2\rangle\langle \chi_2|.$$  

(3.7)

Let us now define the following operation on particles 1 and 3 whose action on an arbitrary density operator $\rho_{13}$ of these particles is

$$\tilde{\mathcal{E}}^{13}_1(R) = V_{2}^{13} \mathcal{E}^{13}_1(V_1^{13}\rho_{13}V_1^{13\dagger})V_2^{13}.$$  

(3.8)

From (3.6)–(3.8) we see that

$$\tilde{\mathcal{E}}^{13}_1(|\psi_{12}\rangle\langle \psi_{12}| \otimes |\psi_{34}\rangle\langle \psi_{34}|) = p|\psi_{12}\rangle\langle \psi_{12}| \otimes |\psi_{34}\rangle\langle \psi_{34}|.$$  

(3.9)

We will now proceed to show that the above transformation implies that

$$\tilde{\mathcal{E}}^{13}_1(\cdot) = p\mathbb{I}^{13}_2 (\cdot)\mathbb{I}^{13}_2.$$  

(3.10)

To do so, let us expand (3.9) in terms of the $|X_{\mu}\rangle$ basis states, which gives

$$\sum_{\mu,\nu}^{D^2} \tilde{\mathcal{E}}^{13}_1(|X_{\mu}^{13}\rangle\langle X_{\nu}^{13}|) \otimes |X_{\mu}^{24}\rangle\langle X_{\nu}^{24}| = p \sum_{\mu,\nu}^{D^2} |X_{\mu}^{13}\rangle\langle X_{\nu}^{13}| \otimes |X_{\mu}^{24}\rangle\langle X_{\nu}^{24}|,$$  

(3.11)

where we have omitted the overall factor of $1/D^2$. Acting on the (2,4) states to the left with $|X_{\nu}^{24}\rangle$ and to the right with $|X_{\delta}^{24}\rangle$ and making use of their orthonormality, we obtain

$$\tilde{\mathcal{E}}^{13}_1(|X_{\nu}^{13}\rangle\langle X_{\delta}^{13}|) = p|X_{\nu}^{13}\rangle\langle X_{\delta}^{13}|.$$  

(3.12)

An arbitrary linear operator $\Omega$ acting on $\mathcal{H}^{\otimes 2}$ may be written as

$$\Omega = \sum_{\gamma, \delta = 1}^{D^2} \omega_{\gamma \delta} |X_{\gamma}\rangle\langle X_{\delta}|,$$  

(3.13)

having the matrix elements $\omega_{\gamma \delta}$ in the $|X_{\gamma}\rangle$ basis. From (3.12) and the linearity of $\tilde{\mathcal{E}}^{13}_1$, it readily follows that

$$\tilde{\mathcal{E}}^{13}_1(\Omega^{13}) = p\Omega^{13}.$$  

(3.14)

Since this is true for any linear operator $\Omega$ on $\mathcal{H}^{\otimes 2}$, we require that (3.10) is true.

Combining this with (3.8), we see that Alice’s operation has the form

$$\tilde{\mathcal{E}}^{13}_1(\rho^{13}) = p(V_2 V_1^{\dagger})^{13}(V_1 V_2^{\dagger})^{13}.$$  

(3.15)
In this local operation, the Kraus operators may be taken to be $\sqrt{p(V_1 V_2)}$, which are clearly proportional to unitary operators. Furthermore, Alice’s overall local Kraus operators $A_{13}^k$ are simply the products of the local Kraus operators corresponding to the elementary steps she carries out in the entire LOCC procedure. These must also be proportional to unitary operators, since the product of any number of unitary operators is also a unitary operator. Clearly, the above argument also applies if the elementary step is carried out by Bob. We are therefore led to the following conclusion: the local Kraus operators $A_{13}^k$ and $B_{24}^k$ for the entire LOCC procedure are, up to overall multiplicative coefficients, unitary. These coefficients are real and non-negative since they are, from our above definition of the elementary step local Kraus operators, products of the square roots of probabilities. We may then write

$$A_{13}^k = f_k \tilde{A}_{13}^k, \quad (3.16)$$

$$B_{24}^k = g_k \tilde{B}_{24}^k. \quad (3.17)$$

Here, $\tilde{A}_{13}^k$ and $\tilde{B}_{24}^k$ are unitary operators on $H_\alpha$ and $H_\beta$ respectively and the $f_k, g_k$ are the real, non-negative coefficients which satisfy

$$\sum_{k=1}^{K} (f_k g_k)^2 = 1, \quad (3.18)$$

as a consequence of (2.4) and the fact that our LOCC procedure is trace preserving.

This has several important consequences that we can take advantage of. The first is the fact that any separable quantum operation whose local Kraus operators have this property can be carried out by LOCC. This can be done in the following way. At one of the laboratories, say $\alpha$, a random variable $Y$ with $K$ possible values $y_k$ and probability distribution $p_k = (f_k g_k)^2$ is generated. On obtaining the result $y_k$, Alice carries out the local unitary operation $\tilde{A}_{13}^k$. She also communicates the value of $Y$ to Bob, who then proceeds to implement the transformation $\tilde{B}_{24}^k$.

The fact that the global Kraus operators $F_k$ are, up to multiplicative coefficients, unitary implies that each one can be implemented deterministically. Furthermore, they must each carry out the desired LOCC copying transformation, for each of the states to be copied. Otherwise, the final state would be mixed. This implies that a necessary and sufficient condition for implementing the copying transformation is that the copying procedure can be implemented by a single global Kraus operator $F = A_{13}^{13} \otimes B_{24}^{24}$, where $A_{13}^{13}$ and $B_{24}^{24}$ are unitary. When this is the case, the complex coefficients $\sigma_{jk}$ in (3.1), where we may drop the index $k$, have unit modulus. Implementing these observations, (3.1) becomes

$$A_{13}^{13} \otimes B_{24}^{24} |\psi_{12}^j\rangle \otimes |b_{34}^j\rangle = e^{i\theta_j} |\psi_{12}^j\rangle \otimes |\psi_{34}^j\rangle, \quad (3.19)$$

for some angles $\theta_j$. The fact that $A$ and $B$ are unitary implies that the states $|\psi_j\rangle$ must all be maximally entangled. The reason for this is that, if any non-maximally entangled state $|\psi_j\rangle$ could be perfectly copied, then particles 3 and 4, initially prepared in the maximally entangled state $|b\rangle$, would be left in the non-maximally entangled state $|\psi_j\rangle$. This is impossible to achieve with a pair of local unitary operators.

In the remainder of this section, we shall use the above findings to obtain a general necessary and sufficient condition for LOCC copying, with a maximally entangled blank state, of the states $|\psi_j\rangle$ when they are orthonormal and maximally entangled.
3.2. Condition for LOCC copying

We saw above that, if $|\psi_1\rangle$ is maximally entangled, then the $|\psi_j\rangle$ are all maximally entangled. Consequently, we may write all of these states in the same form as we did for $|\psi_1\rangle$ in (3.4), that is, as

$$|\psi_{12}^j\rangle = (U_j^1 \otimes \mathbb{1}_2)|\psi_{max}\rangle,$$  

(3.20)

for some unitary operators $U_j$ on $\mathcal{H}$. Again, when considering the particle pair (3,4) in one of these states, we will change the superscripts 1 and 2 to 3 and 4 respectively.

Let us now define the following unitary operators on $\mathcal{H}_\alpha$:

$$C_{13}^j = (U_j^1 \otimes U_j^3)A_{13}(U_j^1 \otimes U_j^3).$$  

(3.21)

With a small amount of algebra, it is easily seen that (3.19) is equivalent to

$$C_{13}^j \otimes B_{24}^j |\psi_{12}^j\rangle \otimes |\psi_{34}^j\rangle = e^{i\theta_j} |\psi_{12}^j\rangle \otimes |\psi_{34}^j\rangle.$$  

(3.22)

In terms of the two-particle basis set $\{|X_\mu\rangle\}$, this can be written as

$$C_{13}^j \otimes B_{24}^j \sum_{\mu=1}^{D^2} |X_{13}^\mu\rangle \otimes |X_{24}^\mu\rangle = e^{i\theta_j} \sum_{\mu=1}^{D^2} |X_{13}^\mu\rangle \otimes |X_{24}^\mu\rangle.$$  

(3.23)

Notice that the $|X_{13}^\mu\rangle \otimes |X_{24}^\mu\rangle$ form a basis for the total Hilbert space $\mathcal{H}_\alpha \otimes \mathcal{H}_\beta$. Acting to the left throughout with $\langle X_{13}^\mu | \otimes \langle X_{24}^\mu |$ we obtain,

$$\sum_{\mu=1}^{D^2} \langle X_{13}^\mu | C_{13}^j |X_{13}^\mu\rangle \langle X_{24}^\mu | B_{24}^j |X_{24}^\mu\rangle = e^{i\theta_j} \sum_{\mu=1}^{D^2} \delta_{\nu\mu}\delta_{\tau\mu} = e^{i\theta_j} \delta_{\nu\tau}.$$  

(3.24)

This can be written as

$$C_j B^T = e^{i\theta_j} \mathbb{1}.$$  

(3.25)

Here, $\mathbb{1}$ is the identity operator on $\mathcal{H} \otimes 2$ and $T$ denotes the transpose in the $\{|X_\mu\rangle\}$ basis. Solving for $B$ and making use of unitarity, we find that

$$B = e^{i\theta_j} C_j^*,$$  

(3.26)

where * denotes complex conjugation in the $\{|X_\mu\rangle\}$ basis. From this, we see that the operators $e^{-i\theta_j} C_j$ are independent of $j$. Using the explicit expression for $C_{13}^j$ in (3.21), we see that

$$e^{i\theta_j} (U_j^1 \otimes U_j^3)A_{13}(U_j^1 \otimes U_j^3) = e^{i\theta_j} (U_{j'}^1 \otimes U_{j'}^3)A_{13}(U_{j'}^1 \otimes U_{j'}^3),$$  

(3.27)

for all $j, j' \in \{1, \ldots, N\}$. Acting throughout to the left with $U_j^1 \otimes U_j^3$ and to the right with $U_{j'}^1 \otimes U_{j'}^3$ we obtain

$$e^{-i\theta_j} A_{13}[(U_j U_j^\dagger)^1 \otimes \mathbb{1}_3] = e^{-i\theta_j} [(U_{j'} U_{j'}^\dagger)^1 \otimes (U_{j'} U_{j'}^\dagger)^3]A_{13}.$$  

(3.28)
Prior to proceeding, we shall make a brief digression. From this point onwards, we will be concerned with operator equations involving just two particles in a shared entangled state. Consequently, it will be convenient to drop the particle superscripts. We do this because the particles involved will follow the tensor product ordering convention we established for such particle pairs in (3.4) and the subsequent paragraph. Also, the analysis that follows in the next section will be quite intricate and will not benefit from unnecessary notation.

For the sake of notational convenience, define the unitary operators

$$T_{jj'} = U_j U_{j'}^\dagger.$$  \hspace{1cm} (3.29)

Using this and the unitarity of $A$, we find that (3.28) is equivalent to

$$A(T_{jj'} \otimes 1) A^\dagger = e^{i(\theta_j - \theta_{j'})} (T_{jj'} \otimes T_{jj'}).$$ \hspace{1cm} (3.30)

From the above argument, it follows that the existence of a unitary operator $A$ on $\mathcal{H} \otimes 2$ which satisfies this equation, for some angles $\theta_j$ and $\theta_{j'}$, is both necessary and sufficient for the existence of an LOCC copying procedure which, with a maximally entangled blank state $|b\rangle$, copies all of the $|\psi_j\rangle$.

The next section will be devoted to the case of $N = 2$. Prior to addressing this case, we shall make some further general observations. Having defined the operators $U_j$ in terms of the reference maximally entangled state $|\psi_{max}\rangle$ in (3.20), one might suspect that the $T_{jj'}$ also make implicit reference to this state. However, this is not so. We can, in fact, write these operators solely in terms of the states to be copied, $|\psi_j\rangle$, and $D$, the dimensionality of $\mathcal{H}$. To do so, consider

$$|\psi_j\rangle\langle\psi_j| = \frac{1}{D} \sum_{i,i'=1}^{D} U_j|x_i\rangle\langle x_{i'}|U_{j'}^\dagger \otimes |x_i\rangle\langle x_{i'}|,$$ \hspace{1cm} (3.31)

where we have used (3.2). Denoting by ‘PT’ the partial trace with respect to the second system, we find

$$D \times PT(|\psi_j\rangle\langle\psi_j|) = \sum_{i,i'=1}^{D} U_j|x_i\rangle\langle x_{i'}|U_{j'}^\dagger \otimes Tr(|x_i\rangle\langle x_{i'}|) = \sum_{i=1}^{D} U_j|x_i\rangle\langle x_{i}|U_{j'}^\dagger = T_{jj'}.$$ \hspace{1cm} (3.32)

Here we have used equation (3.29) and the completeness of the $|x_i\rangle$. We see that the copying condition in (3.30) can be expressed solely in terms of the states to be copied and the dimensionality of the single-particle Hilbert space.

Notice that, from (3.32), if we take the full trace of $|\psi_j\rangle\langle\psi_j|$ we obtain

$$\langle\psi_j|\psi_j\rangle = \frac{1}{D} Tr(T_{jj'}).$$ \hspace{1cm} (3.33)

It is known from the original no cloning theorem that, for perfect copying to be possible, we require the states $|\psi_j\rangle$ and $|\psi_{j'}\rangle$ to be either orthogonal or, up to a phase, identical. It is interesting to see how this fact also follows from (3.30). Taking the full trace throughout equation (3.30) and making use of the unitarity of $A$, we obtain

$$D Tr(T_{jj'}) = e^{i(\theta_j - \theta_{j'})}[Tr(T_{jj'})]^2.$$ \hspace{1cm} (3.34)
This is a simple quadratic equation in \( \text{Tr}(T_{jj}') \), whose roots are 0 and \( De^{-i(\theta_j-\theta_{j}')}. \) From (3.33), we easily see that these roots correspond to \( |\psi_j\rangle \) and \( |\psi_{j}'\rangle \) being orthogonal and, up to a phase, identical respectively.

The problem of determining when a unitary operator \( A \) on \( \mathcal{H} \otimes \mathcal{H} \) satisfying (3.30) exists appears to be quite challenging for arbitrary \( N \) and \( D \). However, for \( N = 2 \), the problem can be solved exactly for all \( D \). We will present the detailed solution to this problem and explore some of its consequences in the next section.

4. LOCC copying of two orthogonal maximally entangled states

4.1. A spectral copying condition

From the above discussion, it follows that a necessary and sufficient condition for LOCC copying of two maximally entangled states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) with a maximally entangled blank state is that there exists a two-particle unitary operator \( A \) which implements the transformation in (3.30) for \( j, j' \in \{1, 2\} \) and some angles \( \theta_1 \) and \( \theta_2 \). Notice from the definition of the \( T_{jj'} \) in (3.29) that \( T_{jj} = 1 \), the identity operator on \( \mathcal{H} \). Consequently, for \( j = j' \), (3.30) is trivially satisfied by any unitary operator \( A \) and any angles \( \theta_j \). Also, the equations for \( T_{12} \) and \( T_{21} \) are simply the Hermitian adjoints of each other, so if one is true then so is the other. It follows that for the case of \( N = 2 \), we need only consider one of these equations. For the sake of definiteness, we will focus on the operator \( T_{12} \), which we will write simply as \( T \). We also write \( \Delta \theta = \theta_1 - \theta_2 \). For suitable choices of \( \theta_1 \) and \( \theta_2 \), this can take any real value. Our condition then becomes

\[
A(T \otimes 1)A^\dagger = e^{i\Delta \theta} (T \otimes T),
\]

where 1 is again the identity operator on \( \mathcal{H} \). We can simplify this expression further by removing the phase factor in the following way: define

\[
\tilde{T} = e^{i\Delta \theta} T.
\]

Then by simple substitution we find that (4.1) is equivalent to

\[
A(\tilde{T} \otimes 1)A^\dagger = \tilde{T} \otimes \tilde{T}.
\]

A unitary operator \( A \) satisfying this equation exists if and only if \( \tilde{T} \otimes 1 \) and \( \tilde{T} \otimes \tilde{T} \) have the same eigenvalues, with the same multiplicities. So, we may write our condition for LOCC copying of the two states as

\[
\text{spec}(\tilde{T} \otimes \tilde{T}) = \text{spec}(\tilde{T} \otimes 1),
\]

where ‘spec’ denotes the spectrum.

Throughout this section, it will be convenient to group the eigenvalues according to multiplicity. So, let \( M \leq D \) be the number of distinct eigenvalues. We shall write these as \( \lambda_r \), where \( r \in \{1, \ldots, M\} \). It is easy to see from (4.4) that, for every integer \( R \geq 2 \), we have

\[
\text{spec}(\tilde{T} \otimes R) = \text{spec}(\tilde{T} \otimes 1 \otimes (R-1)),
\]

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This implies that
\[ \lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k} \in \text{spec}(\tilde{T}) \]  
for all \( r_j \in \{1, \ldots, M\} \) and \( j \in \{1, \ldots, R\} \). To determine which pairs of maximally entangled states can be simultaneously locally copied with a maximally entangled blank state, we must find out which unitary operators satisfy (4.4). The current section will focus on solving this problem and exploring some of the consequences of its solution.

Prior to giving this solution, we make the following intriguing observation. The physical problem of LOCC copying leads to the mathematical problem expressed in (4.4), where physical considerations require that \( \tilde{T} \) is unitary. However, if we are interested in this equation from a purely mathematical perspective, then there is the question of what properties a general linear operator \( \tilde{T} \) must have in order to solve (4.4). We will now show that the eigenvalues of any linear operator, if they are all non-zero, must have unit modulus in order to satisfy (4.4).

To prove this, we make use of the fact that we may, without loss of generality, take the \( \lambda_{r_j} \) to be arranged in non-increasing order in terms of their moduli:
\[ 0 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_M|. \]  
(4.7)

Let us notice that equation (4.4) implies that \( \lambda_1^2 \in \text{spec}(\tilde{T}) \). We now assume that \( |\lambda_1| = \min_r \{|\lambda_r|\} < 1 \). It immediately follows that \( |\lambda_1^2| = |\lambda_1|^2 < \min_r \{|\lambda_r|\} \) for \( \lambda_1 \neq 0 \), contradicting this assumption. Our assumption must therefore be false. Similarly, we see that (4.4) implies that \( \lambda_M^2 \in \text{spec}(\tilde{T}) \). Let us assume that \( |\lambda_M| = \max_r \{|\lambda_r|\} > 1 \). We then obtain \( |\lambda_M^2| = |\lambda_M|^2 > \max_r \{|\lambda_r|\} \), which also leads to a contradiction. This argument implies that \( |\lambda_r| = 1 \) and leads to the conclusion that the non-zero eigenvalues must be of the form
\[ \lambda_r = e^{i \phi_r} \]  
(4.8)
for some angles \( \phi_r \in [0, 2\pi) \). Without loss of generality, we may take these angles to be ordered according to
\[ 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_M < 2\pi. \]  
(4.9)

We will now prove that a unitary operator \( \tilde{T} \), whose eigenvalues are of course all non-zero, satisfies (4.4) if and only if the following two conditions are satisfied:

(i) The distinct eigenvalues of \( \tilde{T} \) are the \( M \)th roots of unity, for some positive integer \( M \) which is a factor of \( D \) and which may be equal to \( D \) itself.

(ii) The distinct eigenvalues of \( \tilde{T} \) have equal degeneracy.

We will first prove the necessity of condition (i), following which we will see that when this condition is satisfied, condition (ii) is necessary and sufficient for (4.4) to hold.

Our proof of the necessity of (i) begins by establishing that, for each \( r \), there is a positive integer \( k_r \in \{1, \ldots, M\} \) such that
\[ \lambda_{k_r}^r = 1. \]  
(4.10)
To prove this, notice that, from (4.8), we obtain
\[ \lambda_r^n = e^{in\phi_r}. \]  
(4.11)
for every integer \( n \). When \( n \) is non-negative, we see from (4.6) that we must have \( \lambda^n_r \in \text{spec}(\tilde{T}) \).

However, the spectrum of \( \tilde{T} \) is finite. In view of this, consider a particular eigenvalue \( \lambda_r \) and two arbitrary positive integers \( n_r \) and \( n'_r \). From (4.6), we see that \( \lambda^n_r, \lambda^n_{r'} \in \text{spec}(\tilde{T}) \). The spectrum of \( \tilde{T} \) has precisely \( M \) distinct eigenvalues. So, for fixed \( n_r \), let us define \( n'_r = n_r + k_r \), where \( k_r \in \{1, \ldots, M\} \). There clearly must be at least one value of \( k_r \) for which \( \lambda_{n'_r} = \lambda^n_r \). When these are equal, we have

\[
e^{i(n'_r-n_r)\phi_r} = e^{ik_r\phi_r} = \lambda^n_r = 1,
\]

as required.

One important consequence of (4.10) is the fact that

\[
1 \in \text{spec}(\tilde{T}).
\]

(4.13)

This follows from (4.6), which tells us that any product of eigenvalues of \( \tilde{T} \) is also an eigenvalue of \( \tilde{T} \). We simply apply this to (4.10), taking \( R = k_r \) and \( r_1, \ldots, r_M = r \).

From this, we see that the ordering of the angles in (4.9) implies that \( \phi_1 = 0 \). We can then update (4.9) in the light of (4.13) to obtain

\[
0 = \phi_1 \leq \phi_2 \leq \cdots \leq \phi_M < 2\pi.
\]

(4.14)

Another consequence of (4.6) is the fact that, for each \( r \in \{1, \ldots, M\} \),

\[
\lambda^{-1}_r = \lambda^*_r \in \text{spec}(\tilde{T}).
\]

(4.15)

We obtain this in the following way. We know from (4.6) and, in the case of \( k_r = 1 \), equation (4.13), that \( \lambda^{-1}_r \in \text{spec}(\tilde{T}) \). However, it follows from (4.10) that \( \lambda^{-1}_r = \lambda^{-1}_r \), so we get (4.15).

Let us now use the above observations to prove that the \( \lambda_r \) must be the \( M \)th roots of unity. From (4.6) and (4.15), we easily obtain

\[
\lambda_r, \lambda^*_r \in \text{spec}(\tilde{T}),
\]

(4.16)

for all \( r, r' \in \{1, \ldots, M\} \). We now set \( r' = (r \mod M) + 1 \). We also write the angular spacings between neighbouring eigenvalues as

\[
\delta_r = \begin{cases} 
\phi_{r+1} - \phi_r & : r \in \{1, \ldots, M - 1\}, \\
2\pi + \phi_1 - \phi_M & : \ r = M.
\end{cases}
\]

(4.17)

Combining these definitions and making use of (4.16), we obtain

\[
e^{ik} \in \text{spec}(\tilde{T}).
\]

(4.18)

The mean value of the \( \delta_r \) is \( 2\pi/M \). Consider now the smallest of these angular spacings, which we shall denote by \( \delta_{\text{min}} \), which must be non-zero because we are working with distinct eigenvalues. To fit the \( M \) distinct eigenvalues around the unit circle, we require that \( \delta_{\text{min}} \leq 2\pi/M \). However, we know from (4.10) that \( e^{ik_{\text{min}}} = 1 \) for some \( k \in \{1, \ldots, M\} \). It is impossible to satisfy this requirement for non-zero \( \delta_{\text{min}} \) unless \( \delta_{\text{min}} \geq 2\pi/M \). Combining these two
inequalities gives
\[ \delta_{\text{min}} = \frac{2\pi}{M}. \]  
(4.19)

It is now easy to see that the \( \lambda_r \) must be the \( M \)th roots of unity. Given that \( e^{i\delta_{\text{min}}} \) is an eigenvalue of \( \tilde{T} \), which we know to be the case from (4.18), we can apply (4.6) to conclude that the \( e^{i\delta_{\text{min}}} \), for all \( r \in \{1, \ldots, M\} \), are also eigenvalues of \( \tilde{T} \). These \( M \) complex numbers, which are distinct, are the \( M \)th roots of unity. Since \( \tilde{T} \) has exactly \( M \) distinct eigenvalues, we conclude that the spectrum of \( \tilde{T} \) consists precisely of these \( M \)th roots of unity. This completes the proof of the necessity of condition (i).

Let us now show that when condition (i) is satisfied, condition (ii) is necessary and sufficient for \( \tilde{T} \) to satisfy (4.4). We will begin by proving its necessity. The eigenvalues \( \lambda_r \) of \( \tilde{T} \) have been grouped according to their multiplicity. So, let us denote the degeneracy of \( \lambda_r \), as an eigenvalue of \( \tilde{T} \), by \( d_{\tilde{T}}^r \). Combining the fact that the \( \lambda_r \) are the \( M \)th roots of unity for some integer factor \( M \) of \( D \) with the phase ordering in (4.14), we see that the distinct eigenvalues of \( \tilde{T} \) are given by
\[ \lambda_r = \exp \left[ \frac{2\pi i (r - 1)}{M} \right]. \]  
(4.20)

Furthermore, must have
\[ \sum_{r=1}^{M} d_{\tilde{T}}^r = D. \]  
(4.21)

Of course, the \( \lambda_r \) are also the eigenvalues of \( \tilde{T} \otimes \tilde{T} \). However, they will have different degeneracies. So, let us denote by \( d_{\tilde{T} \otimes \tilde{T}}^r \) the degeneracy of \( \lambda_r \) as an eigenvalue of \( \tilde{T} \otimes \tilde{T} \). For these degeneracies, we have
\[ \sum_{r=1}^{M} d_{\tilde{T} \otimes \tilde{T}}^r = D^2. \]  
(4.22)

As a consequence of (4.4), we see that
\[ d_{\tilde{T} \otimes \tilde{T}}^r = D d_{\tilde{T}}^r. \]  
(4.23)

Making use of (4.20), we find that the \( d_{\tilde{T} \otimes \tilde{T}}^r \) can be explicitly expressed in terms of the \( d_{\tilde{T}}^r \) in the following way: define
\[ G_{rsr'} = \begin{cases} 1 : & (s + s' - r) \mod M = 1, \\ 0 : & (s + s' - r) \mod M \neq 1, \end{cases} \]  
(4.24)

where \( s, s' \in \{1, \ldots, M\} \). After some algebra, we find that we may write
\[ d_{\tilde{T} \otimes \tilde{T}}^r = \sum_{s,s'=1}^{M} G_{rsr'} d_{\tilde{T}}^s d_{\tilde{T}}^{s'}. \]  
(4.25)
Combining (4.23) and (4.25), we see that the degeneracies $d^\hat{T}_r$ must satisfy

$$
\sum_{s,s'=1}^M G_{rs,s'}^T d^T_s d^T_{s'} = D d^T_r.
$$

This is a necessary and sufficient condition for the $\lambda_r$ to satisfy (4.4). It is evident from this expression that, for each $r$, the left-hand side is a quadratic form. For example, for $r = 1$, we have

$$
(d^T_1 \cdots d^T_M) \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0
\end{pmatrix}
(d^T_1 \cdots d^T_M) = D d^T_1.
$$

The corresponding quadratic forms for $r = 2, \ldots, M$ are obtained from (4.27) by cyclically shifting the elements of each column in this matrix down by $r - 1$ places. Let us define

$$
\sigma(r, s) = (r - s) \text{mod} M + 1.
$$

Using this and equation (4.24), one can readily verify that

$$
\sum_{s=1}^M G_{rs,s'}^T d^T_s = d^T_{\sigma(r,s')},
$$

from which we obtain

$$
\sum_{s,s'=1}^M G_{rs,s'}^T d^T_s \delta_{s'1} = d^T_r.
$$

Here, $\delta_{s'1}$ is the usual Kronecker delta. Combining this equation with (4.26), we get

$$
\sum_{s,s'=1}^M G_{rs,s'}^T (d^T_s - D \delta_{s'1}) = 0.
$$

Making use of (4.29), we find that this equation leads to

$$
\sum_{s'=1}^M d^T_{\sigma(r,s')} d^T_{s'} = d^T_r \sum_{s'=1}^M d^T_{s'}.
$$

We will now use this expression to show that the degeneracies $d^T_r$ must all be equal to $D/M$. Notice, from (4.21), that $D/M$ is the average of the $d^T_r$. They must all be equal if the maximum degeneracy is equal to this average degeneracy. Let $r_{\text{max}}$ be a value of $r$ such that $d^T_{r_{\text{max}}}$ is the maximum degeneracy. As a consequence of the positivity of the $d^T_r$, the following inequality...
must be satisfied:

$$\sum_{s'=1}^{M} d_{\sigma(r_{\text{max}},s')} \tilde{T}_{s'} \leq d_{\text{max}} \sum_{s'=1}^{M} d_{s'}$$

(4.33)

with the equality holding only if \(d_{\sigma(r_{\text{max}},s')} = d_{\text{max}}\) for all \(s'\). Now, for any fixed \(r, \sigma(r, s')\) merely permutes the integers \(s' \in \{1, \ldots, M\}\), so that all degeneracies must, from (4.32), be equal to the maximum degeneracy. This completes the proof of necessity.

Let us finally prove that when the distinct eigenvalues of \(\tilde{T}\) are the \(M\)th roots of unity, it is also sufficient that they have equal degeneracies \(d_{\tilde{T}} = D/M\) to satisfy (4.4). This is simple to show. For \(\lambda_r\) given by (4.20), (4.26) is equivalent to the spectral copying condition in (4.4). When \(d_{\tilde{T}} = D/M\), (4.26) is equivalent to

$$\sum_{s,s'=1}^{M} G_{rs's'} = M.$$  

(4.34)

To show that this equation is satisfied, we note that, when the \(d_{\tilde{T}}\) are all equal, then (4.29) gives

$$\sum_{s=1}^{M} G_{rs's'} = 1.$$  

(4.35)

Summing this expression over the index \(s'\) and making use of (4.24) leads to (4.34), completing the proof of sufficiency.

Let us take the opportunity here to discuss the above results, in their physical context, prior to exploring some of their consequences. For two orthogonal, maximally entangled bipartite states \(|\psi_1\rangle\) and \(|\psi_2\rangle\), having \(D\)-dimensional subsystems, to be locally copyable with a \(D\)-dimensional maximally entangled blank state, it is necessary and sufficient that the eigenvalues of the associated unitary operator \(\tilde{T}\), defined through (3.29) and (4.2), are, for some integer factor \(M\) of \(D\), the \(M\)th roots of unity and that these eigenvalues are equally degenerate.

We defined the operator \(\tilde{T}\) in (4.2) in terms of the operator \(T\) which contains all of the information about the relationship between \(|\psi_1\rangle\) and \(|\psi_2\rangle\). This definition amounted to the removal of the phase factor \(e^{i\Delta \theta}\) in (4.2). This factor was removed in order to simplify the above proofs of the LOCC copying conditions. However, for a particular pair of states, it is \(T\), rather that \(\tilde{T}\), that arises naturally. As such, it is important to formulate these LOCC copying conditions in terms of the spectrum of the \(T\) operator also. This is easily done. The incorporation of this arbitrary phase factor is equivalent to an arbitrary rotation of the spectrum in the complex plane. So, LOCC copying of \(|\psi_1\rangle\) and \(|\psi_2\rangle\) is possible if and only if the eigenvalues of \(T\) are, up to an overall rotation, equally degenerate \(M\)th roots of unity for some integer factor \(M\) of \(D\). In other words, they must have equal angular spacing and be equally degenerate.

Clearly, for any particular pair of orthogonal maximally entangled states \(|\psi_1\rangle\), \(|\psi_2\rangle\) and a particular maximally entangled blank state \(|b\rangle\) for which the LOCC copying operation is possible, it is important to have an explicit prescription for carrying out this procedure. This amounts to knowing two suitable local unitary operators \(A\) and \(B\) for which (3.19) is satisfied. From the results we have obtained here and in the preceding section, it is possible to obtain specific operators which carry out the required task.
Our starting point is the three states involved in the copying procedure, and also the arbitrary reference maximally entangled state $|\psi_{\text{max}}\rangle$. These are presumably known. From these, we deduce the operator $T$ using (3.32) and the fact that $T = T_{12}$. The operator $\tilde{T}$ is obtained using (4.2) and by setting $-\Delta\theta$ equal to the smallest among the arguments of the eigenvalues of $T$. From (4.3) and the unitarity of $A$, we see that we may write

$$\tilde{T} \otimes I = \sum_{r=1}^{M} \lambda_r P_r,$$

(4.36)

$$\tilde{T} \otimes \tilde{T} = \sum_{r=1}^{M} \lambda_r Q_r.$$

(4.37)

Here, $P_r$ and $Q_r$ are the projectors onto the eigenspaces of $\lambda_r$, which is an $M$th root of unity given by (4.20), as an eigenvalue of $\tilde{T} \otimes I$ and $\tilde{T} \otimes \tilde{T}$ respectively. Let us denote these eigenspaces by $\mathcal{H}_{\tilde{T}} \otimes I$ and $\mathcal{H}_{\tilde{T}} \otimes \tilde{T}$. These spaces have dimension $Dd_{\tilde{T}}$. Using these notions, we can obtain a unitary operator $A$ that satisfies (4.3) in the following way. Let $\{ |\xi_{rl}\rangle \}$ and $\{ |\eta_{rl}\rangle \}$ be orthonormal bases for $\mathcal{H}_{\tilde{T}} \otimes I$ and $\mathcal{H}_{\tilde{T}} \otimes \tilde{T}$ respectively. We clearly have $l \in \{ 1, \ldots, Dd_{\tilde{T}} \}$. Now consider the unitary operator

$$A = \sum_{r=1}^{M} \sum_{l=1}^{Dd_{\tilde{T}}} |\xi_{rl}\rangle \langle \eta_{rl}|.$$

(4.38)

One can easily show that $AP_r A^\dagger = Q_r$, which implies that $A$ satisfies (4.3) as required.

We must now find a suitable operator $B$. To do so, we are required to know the operator $U_b$. This can be deduced from (3.5) to be

$$U_b = D \times PT(|b\rangle \langle \psi_{\text{max}}|).$$

(4.39)

If we now combine (3.21) and (3.26), we find that $B$ is given by

$$B = e^{i\omega j}(U_j \otimes U_b)^T A^T (U_j^\dagger \otimes U_b^\dagger)^T,$$

(4.40)

for either $j = 1, 2$ and where $T$ again denotes the transpose in the $|X_\mu\rangle$. We may neglect the phase factor here entirely as it has no effect on the physical nature of the transformation.

We shall now explore some of the consequences of the local copying condition in (4.4), paying particular regard to the relationship between orthogonality and local copyability of two maximally entangled states with a maximally entangled blank state.

4.2. Consequences

Having established the LOCC copying condition for a pair of orthogonal, maximally entangled, bipartite, pure states with a maximally entangled blank state, it is natural to enquire as to when this condition is satisfied. We shall find that the dimensionality $D$ of the single particle Hilbert space $\mathcal{H}$ plays a prominent role here.

We will show that for $D = 2, 3$, every pair of orthogonal, maximally entangled, bipartite, pure states can be locally copied with a maximally entangled blank state. However, we will then
show that for every \( D \) which is not prime, one can construct pairs of such states for which this is impossible.

The proof for \( D = 2 \) is a simple matter. From (3.33) and (4.2), we know that the condition of orthogonality is \( \text{Tr}(\bar{T}) = \text{Tr}(\tilde{T}) = 0 \). For \( D = 2 \), \( \tilde{T} \) has just two, non-degenerate eigenvalues, implying that \( \tilde{T} \) having zero trace is equivalent to these summing to zero. Writing these two eigenvalues as \( e^{i\phi_1} \) and \( e^{i\phi_2} \), where we take \( 0 = \phi_1, \phi_2 < 2\pi \) as in (4.14), it is easily shown that this orthogonality condition can only be satisfied if \( \phi_2 = \pi \). When this is so, they are the 2nd roots of unity. So, for \( D = 2 \), any pair of orthogonal, maximally entangled states can be locally copied. This finding is in accord with the results of Ghosh et al [12] who showed that with 1 ebit of entanglement in the blank state, it is possible to copy, by LOCC, any pair of Bell states.

Let us now consider the case of \( D = 3 \). Here, the \( \tilde{T} \) operator has three eigenvalues, \( e^{i\phi_1}, e^{i\phi_2} \) and \( e^{i\phi_3} \). Again we take the phase ordering \( 0 = \phi_1 \leq \phi_2 \leq \phi_3 < 2\pi \). If the states are orthogonal, then

\[
1 + e^{i\phi_2} + e^{i\phi_3} = 0. \tag{4.41}
\]

Clearly, this is equivalent to \( e^{i\phi_2} + e^{i\phi_3} = -1 \). Separating the real and imaginary parts of this equation gives

\[
\cos(\phi_2) + \cos(\phi_3) = -1, \tag{4.42}
\]

\[
\sin(\phi_2) + \sin(\phi_3) = 0. \tag{4.43}
\]

From (4.43) we see that \( \sin^2(\phi_2) = \sin^2(\phi_3) \), which in turn gives \( \cos^2(\phi_2) = \cos^2(\phi_3) \) and so \( \cos(\phi_2) = \pm \cos(\phi_3) \). It is easily seen that we cannot have the minus sign here, since this would contradict (4.42). We therefore obtain

\[
\cos(\phi_2) = \cos(\phi_3). \tag{4.44}
\]

Substituting this into (4.42) gives

\[
\cos(\phi_2) = \cos(\phi_3) = -\frac{1}{2}. \tag{4.45}
\]

which implies that \( \phi_2 \) and \( \phi_3 \) must individually be equal to either \( 2\pi/3 \) or \( 4\pi/3 \). It follows from (4.43) that these two angles must be different, because (4.45) implies that the sines of these two possible angles are non-zero. Combining this with the fact that \( \phi_3 \geq \phi_2 \), we conclude that \( \phi_2 = 2\pi/3 \) and \( \phi_3 = 4\pi/3 \). The three eigenvalues are then the non-degenerate third roots of unity and (4.4) is satisfied as desired. It follows that the two states are locally copyable with a maximally entangled blank state.

The above analysis shows that for \( D = 2, 3 \), any pair of orthogonal, maximally entangled states can be copied using the same LOCC operation and a maximally entangled blank state. However, as we shall now see, this does not hold for arbitrary \( D \). In fact, we will now demonstrate that for any \( D \) which is not prime, one can construct pairs of orthogonal, maximally entangled states for this is impossible.

If \( D \) is not prime, then, by definition, there exist positive integers \( D_1, D_2 \geq 2 \) such that

\[
D = D_1D_2. \tag{4.46}
\]
Consider now the $D_1$th roots of unity $\exp(2\pi i(j - 1)/D_1)$, where $j \in \{1, \ldots, D_1\}$. The angular spacing between these complex numbers is $2\pi/D_1$. Consider now some small angular interval $\delta$ and an operator $T$ with the following set of distinct eigenvalues:

$$
\lambda_{jj'} = e^{2\pi i(j - 1)/D_1} e^{i(j' - 1)\delta},
$$

(4.47)

where $j' \in \{1, \ldots, D_2\}$. It should be noted at this point that every unitary operator $T$ on $\mathcal{H}$ corresponds to a set of pairs of maximally entangled bipartite states $|\psi_1\rangle$ and $|\psi_2\rangle$. Indeed, for arbitrary, fixed $T$ and $|\psi_2\rangle$, we can see from (3.20) and (3.29) that $|\psi_1\rangle$ is obtained using

$$
|\psi_1\rangle = (T \otimes \mathbb{I})|\psi_2\rangle.
$$

(4.48)

A set of eigenvalues of the form given in (4.47) is depicted in figure 2, with $D_1 = 4$ and $D_2 = 5$. We can easily choose $\delta$ in such a way that these will not be equally spaced. We may simply take any $\delta < 2\pi/D_1$ to achieve this. However, any unitary operator $T$ whose eigenvalues are the $\lambda_{jj'}$, with these being non-degenerate, can be seen to be traceless. We have

$$
\text{Tr}(T) = \exp\left[-i\left(\frac{2\pi i}{D_1} + \delta\right)\right] \sum_{j=1}^{D_1} \exp\left[\frac{2\pi i j}{D_1}\right] \sum_{j'=1}^{D_2} \exp(i j' \delta) = 0,
$$

(4.49)

because the first sum vanishes. So, the corresponding states are orthogonal. However, the fact that the eigenvalues are not equally spaced implies that the LOCC copying procedure is impossible.

So, we have seen that for $D = 2, 3$, any pair of orthogonal, maximally entangled, bipartite, pure states can be locally copied with a maximally entangled blank state. However, this is not generally the case when $D$ is not prime. As a consequence of this finding, a natural question to
ask is: for a fixed value of $D$, is a necessary and sufficient condition for LOCC copying of every pair of orthogonal, maximally entangled, bipartite, pure states, with a maximally entangled blank state, the primality of $D$? We were unable to determine whether or not this is so.

5. Discussion

In this paper, we have addressed the problem of LOCC copying of entangled states with an entangled blank state. We were concerned mainly with the situation where one of the states to be copied is maximally entangled. When this is the case, we must have at least one additional maximally entangled state, and this may be taken to be the blank state. When none of the states to be copied are maximally entangled, it is possible that the most efficient use of entanglement occurs when the blank state is incomparable with the states to be copied. We illustrated this in section 2. This is an application of the well-known phenomenon of entanglement catalysis. There is much work still to be done on entanglement catalysis before we can have a full understanding of the process of catalytic copying.

Fortunately, when one of the states to be copied is maximally entangled, this issue does not arise. In section 3, we derived a necessary and sufficient condition for LOCC copying a set of $N$ states including a maximally entangled state and with a maximally entangled blank state. This condition is, in general, difficult to solve for arbitrary $N$ and subsystem dimension $D$. However, we were able to make some interesting general observations about the set of states that can be copied and the associated copying transformations. Firstly, if one of the states to be copied is maximally entangled, then they must all be maximally entangled. Secondly, without loss of generality, the copying transformation may be taken to consist of just two unitary operations, with one being implemented in each laboratory.

For $N = 2$, this condition could be solved exactly for all $D$. We found that it relates to the eigenvalues of a certain unitary operator associated with the pair of states to be copied. These eigenvalues must, up to a phase, be the $M$th roots of unity, for some factor $M$ of $D$, and they must be equally degenerate. Having this information enabled us to show that for $D = 2, 3$, any pair of maximally entangled, orthogonal states can be copied by LOCC with a maximally entangled blank state. However, we were also able to show that for every $D$ which is not prime, there exist pairs of such states for which this is not possible.

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