INDICATORS FOR PLURISUBHARMONIC FUNCTIONS
OF LOGARITHMIC GROWTH

Alexander RASHKOVSKII

Abstract. A notion of indicator for a plurisubharmonic function $u$ of logarithmic growth in $\mathbb{C}^n$ is introduced and studied. It is applied to evaluation of the total Monge-Ampère measure $(dd^c u)^n(\mathbb{C}^n)$. Upper bounds for the measure are obtained in terms of growth characteristics of $u$. When $u = \log |f|$ for a polynomial mapping $f$ with isolated zeros, the indicator generates the Newton polyhedron of $f$ whose volume bounds the number of the zeros.

1 Introduction

We consider plurisubharmonic functions $u$ of logarithmic growth in $\mathbb{C}^n$, i.e. satisfying the relation

$$u(z) \leq C_1 \log^+ |z| + C_2$$

with some constants $C_j = C_j(u) \geq 0$. The class of such functions will be denoted by $\mathcal{L}(\mathbb{C}^n)$ or simply by $\mathcal{L}$. (It is worth mentioning that in the literature the notation $\mathcal{L}$ is used sometimes for the class of functions satisfying (1) with $C_1 = 1$; for our purposes we need to consider the whole class of functions of logarithmic growth, and denoting it by $\mathcal{L}$ we follow, for example, [13], [14].) It is an important class containing, in particular, functions of the form $\log|P|$ with polynomial mappings $P : \mathbb{C}^n \to \mathbb{C}^N$. Various results concerning the functions of logarithmic growth can be found in [10]-[14], [20], [2], see also the references in [2] and [3]. For general properties of plurisubharmonic functions and the complex Monge-Ampère operators, we refer the reader to [11], [14], [3], and [4].

A remarkable property of functions $u \in \mathcal{L}$ is finiteness of their total Monge-Ampère measures

$$M(u; \mathbb{C}^n) = \int_{\mathbb{C}^n} (dd^c u)^n < \infty$$

as long as $(dd^c u)^n$ is well defined on the whole $\mathbb{C}^n$; we use the notation $d = \partial + \bar{\partial}$, $d^c = (\partial - \bar{\partial})/2\pi i$. Moreover, the total mass is tied strongly to the growth of the function. For example, if

$$\log^+ |z| + c \leq u(z) \leq \log^+ |z| + C,$$

then $M(u; \mathbb{C}^n) = M(\log |z|; \mathbb{C}^n) = 1$. The objectives for the present paper is to study $M(u; \mathbb{C}^n)$ when no regularity condition on $u$ like (2) is assumed. In case of
\( u = \log |P| \) with \( P : \mathbb{C}^n \to \mathbb{C}^N \) a polynomial mapping with isolated zeros, \( M(u ; \mathbb{C}^n) \) equals (if \( N = n \)) or dominates (if \( N > n \)) the number of the zeros counted with their multiplicities.

If \( u = v \) near the boundary of a bounded pseudoconvex domain \( \Omega \), then
\[
\int_{\Omega} (dd^c u)^n = \int_{\Omega} (dd^c v)^n,
\]
so the total measure \( M(u ; \mathbb{C}^n) \) is determined by the asymptotic behavior of \( u \) at infinity. For its evaluation we thus need precise characteristics of the behavior. The basic one is the logarithmic type
\[
\sigma(u) = \limsup_{z \to \infty} \frac{u(z)}{\log |z|}.
\]

Another known characteristic is the logarithmic multitype \( (\sigma_1(u), \ldots, \sigma_n(u)) \) [14]:
\[
\sigma_1(u) = \sup \{ \tilde{\sigma}_1(u; z') : z' \in \mathbb{C}^{n-1} \}
\]
where \( \tilde{\sigma}_1(u; z') \) is the logarithmic type of the function \( u_{1,z'}(z_1) = u(z_1, z') \in \mathcal{L}(\mathbb{C}) \) with \( z' \in \mathbb{C}^{n-1} \) fixed, and similarly for \( \sigma_2(u), \ldots, \sigma_n(u) \). For example, if \( P \) is a polynomial of degree \( d_k \) in \( z_k \), then \( \sigma_k(\log |P|) = d_k \).

Due to the certain symmetry between the behavior of \( u \in \mathcal{L} \) at infinity and the local behavior of a plurisubharmonic function at a fixed point of its logarithmic singularity, the type \( \sigma(u) \) can be regarded as the Lelong number of \( u \) at infinity:
\[
\sigma(u) = \nu(u, \infty).
\]

One can also consider the directional Lelong numbers at infinity with respect to directions \( a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \):
\[
\nu(u, a, \infty) = \limsup_{z \to \infty} \frac{u(z)}{S_a(z)},
\]
where
\[
S_a(z) = \sup_k a_k^{-1} \log |z_k|.
\]

In [17], the residual Monge-Ampère measure of a plurisubharmonic function \( u \) at a point \( x \in \mathbb{C}^n \), \( (dd^c u)^n |_{\{x\}} \), was studied by means of the local indicator of \( u \) at \( x \). Using the same approach, we introduce here a notion of the indicator of \( u \in \mathcal{L} \):
\[
\Psi_{u,x}(y) = \lim_{R \to +\infty} R^{-1} \sup \{ u(z) : |z_k - x_k| \leq |y_k|^R, 1 \leq k \leq n \}.
\]
It is a plurisubharmonic function of the class \( \mathcal{L} \) which is the (unique) logarithmic tangent to \( u \) at \( x \), i.e. the weak limit in \( L_{loc}^1(\mathbb{C}^n) \) of the functions \( m^{-1}u(x_1 + \)
If \((dd^c u)^n\) is defined on \(C^n\), the indicator also controls the total Monge-Ampère mass of \(u\) (Theorem 4):

\[
M(u; C^n) \leq M(\Psi_{u,x}; C^n).
\]

(8)

Since \(\Psi_{u,x}(y) = \Psi_{u,x}(|y_1|, \ldots, |y_n|)\), the evaluation of its mass is much more easy than that for the original function \(u\). It gives us, in particular, the bounds

\[
M(u; C^n) \leq \nu(u, a, \infty)_{a_1 \ldots a_n} \forall a \in R^n
\]

(Theorem 5) and

\[
M(u; C^n) \leq n! \sigma_1(u) \ldots \sigma_n(u)
\]

(Theorem 8). A particular case of the latter result (when \(u\) is the logarithm of modulus of an equidimensional polynomial mapping with isolated zeros of regular multiplicities) was obtained in [15].

In Theorems 6 and 7 we give a geometric description for the mass of an indicator. Denote \(\psi_{u,x}(t) = \Psi_{u,x}(e^{t_1}, \ldots, e^{t_n}), t = (t_1, \ldots, t_n) \in R^n\), and

\[
\Theta_{u,x} = \{a \in R^n : \langle a, t \rangle \leq \psi_{u,x}(t) \forall t \in R^n\}.
\]

Then

\[
M(\Psi_{u,x}; C^n) = n! Vol(\Theta_{u,x}).
\]

(9)

When \(u = \log |P|\) with \(P\) a polynomial mapping, the set \(\Theta_{u,0}\) is the Newton polyhedron for \(P\) at infinity (see, for example, [8]), i.e. the convex hull of the set \(\omega_0 \cup \{0\}\),

\[
\omega_0 = \{s \in Z_+^n : \sum_j \frac{\partial^s P_j}{\partial z^s}(0) \neq 0\},
\]

and so the right-hand side of (8) is the Newton number of \(P\) at infinity. Therefore, a hard result due to A.G. Kouchnirenko on the number of zeros of an equidimensional polynomial mapping [4] follows directly from (8) and (9).

2 Indicators as growth characteristics

Let \(u\) be a plurisubharmonic function in \(C^n\). Given \(x \in C^n\) and \(t \in R^n\), denote by \(g(u, x, t)\) the mean value of \(u\) over the set \(T_t(x) = \{z \in C^n : |z_k - x_k| = e^{t_k}, 1 \leq k \leq n\}\), and by \(g'(u, x, t)\) the maximum of \(u\) on \(T_t(x)\).
Proposition 1  Let \( u \in \mathcal{L} \), \( x \in \mathbb{C}^n \). Then for every \( t \in \mathbb{R}^n \) the following limits exist and coincide:

\[
\lim_{R \to +\infty} R^{-1}g(u, x, Rt) = \lim_{R \to +\infty} R^{-1}g'(u, x, Rt) =: \psi_{u,x}(t) < \infty.
\]

Moreover, if \( g'(u,x,0) \leq 0 \), the common limit \( \psi_{u,x}(t) \) is obtained by the increasing values.

Proof. For \( x \in \mathbb{C}^n \) and \( t \in \mathbb{R}^n \) fixed, the function \( f(R) := g(u, x, Rt) \) is convex on \( \mathbb{R} \) and has the bound \( f(R) \leq C_1R + C_2 \forall R > 0 \) with some \( C_1, C_2 > 0 \). Therefore, for all \( R_0 \in \mathbb{R} \), the ratio

\[
\frac{f(R) - f(R_0)}{R - R_0}
\]

is increasing in \( R > R_0 \) and bounded and thus has a limit as \( R \to +\infty \). It implies the existence of \( \hat{g}(u,x,t) = \lim_{R \to +\infty} R^{-1}g(u, x, Rt) \). In the same way we get the value \( \hat{g}'(u,x,t) = \lim_{R \to +\infty} R^{-1}g'(u, x, Rt) \). Evidently, \( \hat{g}(u,x,t) \leq \hat{g}'(u,x,t) \), and the standard arguments using Harnack’s inequality give us \( \hat{g}(u,x,t) = \hat{g}'(u,x,t) \). The last statement of the proposition follows from the increasing of \((10)\) with \( R_0 = 0 \).

Now we proceed, as in [1], to a plurisubharmonic characteristic of growth for \( u \in \mathcal{L} \). Denote \( \mathbb{C}^{\ast n} = \{ z \in \mathbb{C}^n : z_1 \ldots z_n \neq 0 \} \). The mappings \( \text{Log} : \mathbb{C}^{\ast n} \to \mathbb{R}^n \) and \( \text{Exp} : \mathbb{R}^n \to \mathbb{C}^{\ast n} \) are defined as \( \text{Log}(z) = (\log |z_1|, \ldots, \log |z_n|) \) and \( \text{Exp}(t) = (\exp t_1, \ldots, \exp t_n) \), respectively. Let \( \mathcal{L}^c \) be the subclass of \( \mathcal{L} \) formed by \( n \)-circled plurisubharmonic functions \( u \), i.e. \( u(z) = u(|z_1|, \ldots, |z_n|) \). By \( \mathcal{L}^c(\mathbb{R}^n) \) we denote the class of functions \( \varphi(t), t \in \mathbb{R}^n \), which are convex in \( t \), increasing in each \( t_k \) and such that there exists a limit \( \lim_{T \to +\infty} T^{-1}\varphi(T, \ldots, T) < \infty \).

The mappings \( \text{Exp} \) and \( \text{Log} \) generate an isomorphism between the cones \( \mathcal{L}^c \) and \( \mathcal{L}^c(\mathbb{R}^n) \) ([1], Th. 1): \( u \in \mathcal{L}^c \iff \text{Exp}^\ast u \in \mathcal{L}^c(\mathbb{R}^n) \), \( h \in \mathcal{L}^c(\mathbb{R}^n) \iff \text{Log}^\ast h \) extends to a (unique) function from the class \( \mathcal{L}^c \). Given \( u \in \mathcal{L}^c \), the function \( \text{Exp}^\ast u \) will be referred to as the convex image of \( u \).

If \( h = \text{Exp}^\ast u \in \mathcal{L}^c(\mathbb{R}^n) \) satisfies the homogeneity condition

\[
h(ct) = c \cdot h(t) \quad \forall c > 0, \forall t \in \mathbb{R}^n,
\]

the function \( u \) will be called an indicator. We denote the collection of all indicators by \( \mathcal{I} \). It is easy to see that any indicator \( \Psi \) satisfies \( \Psi \leq 0 \) in the unit polydisk

\[
D = \{ z \in \mathbb{C}^n : |z_k| < 1, \ 1 \leq k \leq n \}
\]

and \( \Psi > 0 \) in

\[
D^{-1} = \{ z \in \mathbb{C}^n : |z_k| > 1, \ 1 \leq k \leq n \}
\]

if \( \Psi \neq 0 \).
Clearly, the function $\psi_{u,x}$ defined in Proposition 1 belongs to the class $\mathcal{L}(\mathbf{R}^n)$, so $\log^*\psi_{u,x}$ extends to a function $\Psi_{u,x} \in \mathcal{L}^c$:

$$\Psi_{u,x}(y) = \psi_{u,x}(|y_1|, \ldots, |y_n|), \quad y \in \mathbf{C}^n.$$ 

Moreover, $\Psi_{u,x} \in \mathcal{I}$. We will call it the indicator of $u \in \mathcal{L}$ at $x$.

The restriction of $\Psi_{u,x}$ to the polydisk $D$ coincides with the local indicator of $u$ at $x$ introduced in [17]. In particular, $\Psi_{u,x} \equiv 0$ in $D$ if and only if the Lelong number of $u$ at $x$ equals 0. Besides, the directional Lelong numbers of $\Psi_{u,x}$ at 0 are the same as those of $u$ at $x$.

**Proposition 2** Let $\Phi \in \mathcal{I}$, then

(a) $\Phi$ is continuous as a function $\mathbf{C}^n \to \mathbf{R} \cup \{-\infty\}$;

(b) $\Psi_{\Phi,x}(y) = \Phi(\tilde{y})$ where $\tilde{y}_k = \sup \{|y_k|, 1\}$ if $x_k \neq 0$, and $\tilde{y}_k = y_k$ otherwise.

**Proof.** (a) Since $\exp^*\Phi \in C(\mathbf{R}^n)$, $\Phi \in C(\mathbf{C}^n)$. Its continuity on $\mathbf{C}^n$ can be shown by induction in $n$. Let it be already proved for $n \leq l$ (the case $n = 1$ is obvious). Consider any point $z^0 \in \mathbf{C}^{l+1}$ with $z^0_j = 0$ for some $j$. If $\Phi(z^0) = -\infty$, then $\Phi(z^s) \to -\infty$ for every sequence $z^s \to z^0$. If $\Phi(z^0) > -\infty$, consider the projections $\tilde{z}^s$ of $z^s \to z^0$ to the subspace $L_j = \{z \in \mathbf{C}^{l+1} : z_j = 0\}$: $\tilde{z}^s_j = 0$ and $\tilde{z}^s_m = z^s_m \forall m \neq j$. Since $\Phi|_{L_j} \not\equiv -\infty$, the induction assumption implies $\Phi(\tilde{z}^s) \to \Phi(z^0)$. Therefore, $\liminf \Phi(\tilde{z}^s) \geq \liminf \Phi(z^s) = \Phi(z^0)$ that proves lower semicontinuity of $\Phi$ at $z^0$ and thus its continuity.

(b) For any $t \in \mathbf{R}^n$ and $R > 0$,

$$R^{-1}g'(\Phi, x, Rt) = R^{-1}\Phi(|x_1| + e^{Rt_1}, \ldots, |x_n| + e^{Rt_n})$$

$$= \Phi([|x_1| + e^{Rt_1}]^{1/R}, \ldots, [|x_n| + e^{Rt_n}]^{1/R}).$$

The argument $[|x_k| + e^{Rt_k}]^{1/R}$ tends to $\exp\{t_k^+\}$ if $x_k \neq 0$, and to $\exp\{t_k\}$ otherwise, so the statement follows from (a).

The growth characteristics (3), (4), (6) of functions $u \in \mathcal{L}$ can be expressed in terms of the indicators. We will use the following notation:

$$1 = (1, \ldots, 1), \quad 1_1 = (1, 0, \ldots, 0), \quad 1_2 = (0, 1, 0, \ldots, 0), \ldots, 1_n = (0, \ldots, 0, 1). \quad (11)$$

**Proposition 3** (a) $\nu(u, \infty) = \nu(\Psi_{u,x}, \infty) = \psi_{u,x}(1)$;

(b) $\nu(u, a, \infty) = \nu(\Psi_{u,x}, a, \infty) = \psi_{u,x}(a) \forall a \in \mathbf{R}^n_+$;

(c) $\sigma_k(u) = \sigma_k(\Psi_{u,x}) = \psi_{u,x}(1_k), \quad k = 1, \ldots, n.$
Proof. The relation $\nu(u,a,\infty) = \psi_{u,x}(a)$ follows directly from the definition of $\psi_{u,x}$. The equalities $\nu(u,\infty) = \psi_{u,x}(1)$ and $\sigma_k(u) = \psi_{u,x}(1_k)$ are proved in Theorems 1 and 2 of [14]. Being applied to the function $\Psi_{u,x}$ instead of $u$, they give us the first equalities in (a)-(c) in view of Proposition 2. The proof is complete.

Theorem 1 Let $u \in L$, $x \in \mathbb{C}^n$. Then

$$u(z) \leq \Psi_{u,x}(z - x) + C \quad \forall z \in \mathbb{C}^n$$

(12)

with $C = g'(u, x, 0)$. Moreover, $\Psi_{u,x}$ is the least indicator satisfying (12) with some constant $C$.

Proof. By Proposition 1,

$$g'(u, x, Rt) \leq R\psi_{u,x}(t) + g'(u, x, 0) \quad \forall R > 0, \forall t \in \mathbb{R}^n,$$

that implies (12) since $u(z + x) \leq g'(u, x, \text{Log}(z))$.

If $\Phi \in I$ satisfies $u(z) \leq \Phi(z - x) + C$, then

$$\Psi_{u,x} \leq \Psi_{\Phi(-x),x} = \Phi_{x,0} = \Phi,$$

the latter equality being a consequence of Proposition 3. The theorem is proved.

The indicator $\Psi_{u,x}$ can be easily calculated in the algebraic case, i.e. when $u$ is the logarithm of modulus of a polynomial mapping. Recall that the index $I(P, x, a)$ of a polynomial $P$ at $x \in \mathbb{C}^n$ with respect to the weight $a \in \mathbb{R}^n$ is defined as

$$I(P, x, a) = \inf \{\langle a, J \rangle : J \in \omega_x\}$$

where

$$\omega_x = \{J \in \mathbb{Z}^n_+ : \frac{\partial^j P}{\partial z^j}(x) \neq 0\}$$

(see e.g. [9]). For any $t \in \mathbb{R}^n$ we define

$$I_{up}(P, x, t) = \sup \{\langle t, J \rangle : J \in \omega_x\},$$

(13)

the upper index of $P$ at $x \in \mathbb{C}^n$ with respect to $t \in \mathbb{R}^n$. Clearly, $I_{up}(P, x, t) = -I(P, x, -t)$ for all $t \in -\mathbb{R}^n$.

Proposition 4 Let $u = \log |P|$, $P : \mathbb{C}^n \rightarrow \mathbb{C}$ being a polynomial. Then

$$\psi_{u,x}(t) = I_{up}(P, x, t) \quad \forall t \in \mathbb{R}^n, \forall x \in \mathbb{C}^n.$$
Proof. Let
\[ P(z) = \sum_{J \in \omega_x} c_J (z - x)^J \]
and \( d = I_{up}(P, x, t) \), so \( b_J := (t, J) - d \leq 0 \ \forall J \in \omega_x \). Then
\[ R^{-1}g'(u, x, t) = d + R^{-1} \sup_{\theta} \{ \log \left| \sum_J c_J \exp[Rb_Ji(\theta, J)] \right| \}. \]
Since there exists \( J_0 \in \omega_x \) with \( b_{J_0} = 0 \), the second term here tends to 0 as \( R \to +\infty \), and the statement follows.

**Proposition 5** Let \( u_1, \ldots, u_m \in \mathcal{L} \), \( u = \sup_k u_k \), \( v = \log \sum_k \exp u_k \). Then
\[ \Psi_{u,x} = \Psi_{v,x} = \sup_k \Psi_{u_k,x}. \]

Proof. Since \( u \geq u_k \), we have \( \Psi_{u,x} \geq \sup_k \Psi_{u_k,x} \). On the other hand, by (12),
\[ u(z) \leq \sup_k \{ \Psi_{u_k,x} + C_k \} \leq \sup_k \Psi_{u_k,x} + \sup_k C_k, \]
and the equality \( \Psi_{u,x} = \sup_k \Psi_{u_k,x} \) results from Theorem 1.

Similarly, the relations \( \Psi_{v,x} \geq \Psi_{u,x} \) and
\[ v(z) \leq \log \sum_k \exp[\Psi_{u_k,x}(z - x) + C_k] \leq \Psi_{u,x}(z - x) + m + \sup_k C_k \]
imply \( \Psi_{u,x} = \Psi_{v,x} \), and the proof is complete.

As a corollary of Propositions 4 and 5 we get

**Proposition 6** Let
\[ u = \frac{1}{q} \log \sum_{k=1}^m |P_k|^q \]
with \( P_1, \ldots, P_m \) polynomials and \( q > 0 \). Then \( \psi_{u,x}(t) = \sup_k I_{up}(P_k, x, t) \).

The indicator \( \Psi_{u,x} \) can be described as a tangent (in logarithmic coordinates) to the original function \( u \in \mathcal{L} \). For \( z \in \mathbb{C}^n \) and \( m \in \mathbb{N} \), we set \( z^m = (z_1^m, \ldots, z_n^m) \) and define the function
\[ (T_m u)(z) = m^{-1}u(x + z^m) \in \mathcal{L}. \]

**Theorem 2** \( T_m u \to \Psi_{u,x} \) in \( L^1_{loc}(\mathbb{C}^n) \) as \( m \to +\infty \).
Proof. First, the family $\{T_{m,x}u\}_m$ is relatively compact in $L^1_{loc}(\mathbb{C}^n)$. Really, (12) implies
\[(T_{m,x}u)(z) \leq \Psi_{u,x}(z - x) + m^{-1}C \quad \forall m. \tag{14}\]
Therefore, the family is uniformly bounded above on each compact subset of $\mathbb{C}^n$. Besides, $g(T_{m,x}u, 0, 0) = m^{-1}g(u, x, 0) \to 0$ and hence $g(T_{m,x}u, 0, 0) \geq -1$ for all $m \geq m_0$, and the compactness follows.

Now let $v$ be a partial weak limit of $T_{m,x}u$, i.e. $T_{m,x}u \to v$ for some subsequence $m_s$. By (14),
\[v \leq \Psi_{u,x}. \tag{15}\]
On the other hand, the convergence of $T_{m_s,x}u$ to $v$ implies
\[g(T_{m_s,x}u, 0, t) \to g(v, 0, t) \quad \forall t \in \mathbb{R}^n.\]
At the same time, by the definition of $\psi_{u,x}$,
\[g(T_{m_s,x}u, 0, t) = m^{-1}g(u, x, mt) \to \psi_{u,x}(t) \quad \forall t \in \mathbb{R}^n,\]
so $g(v, 0, t) = \psi_{u,x}(t)$ and thus $g(v, 0, t) = g(\Psi_{u,x}, 0, t)$. Being compared to (15) it gives us $v = \Psi_{u,x}$, that completes the proof.

We conclude this section by studying dependence of $\Psi_{u,x}$ on $x$.

**Proposition 7** Let $u \in \mathcal{L}$. Then

(a) $\Psi_u(z) := \sup \{\Psi_{u,x}(z) : x \in \mathbb{C}^n\} \in \mathcal{L}$;

(b) for any $z \in \mathbb{C}^n$, $\Psi_{u,x}(z) = \Psi_u(z)$ for all $x \in \mathbb{C}^n \setminus E_z$, $E_z$ being a pluripolar subset of $\mathbb{C}^n$;

(c) for any $z \in D^{-1}$, $\Psi_{u,x}(z) = \Psi_u(z)$ for all $x \in \mathbb{C}^n$;

(d) $\Psi_u(z) \geq 0 \quad \forall z \in \mathbb{C}^n$, $\Psi_u \equiv 0$ in $D$.

Proof. Since $u \in \mathcal{L}$, there is a constant $A > 0$ such that $u(z) \leq AS^+(z) \forall z \in \mathbb{C}^n$, where $S^+(z) = S^+_1(z) = \sup_k \log^+ |z_k|$.

We fix a point $z \in \mathbb{C}^n$ and consider the function
\[u_R(x) = R^{-1}g'(u, x, R \log(z)), \quad R > 0.\]
It is plurisubharmonic in $\mathbb{C}^n$, and
\[u_R(x) \leq R^{-1}AS^+ (|x_1| + |z_1|^R, \ldots, |x_n| + |z_n|^R).\]
Therefore, the family \( \{u_R\}_{R>1} \) is uniformly bounded above on compact subsets of \( \mathbb{C}^n \), and

\[
u(x) := \limsup_{R \to +\infty} u_R(x) \leq A S^+(z) \quad \forall x \in \mathbb{C}^n.
\]

(16)

Its regularization \( u^*_\infty(x) = \limsup_{y \to x} u_\infty(y) \) is then plurisubharmonic in \( \mathbb{C}^n \) and bounded and so \( u^*_\infty = \text{const} \). We have \( u_\infty(x) \leq u^*_\infty(x) \) for all \( x \in \mathbb{C}^n \) with the equality outside a pluripolar set \( E_z \subset \mathbb{C}^n \). We observe now that \( u_\infty(x) = \Psi_{u,x}(z) \) and \( u^*_\infty(x) = \Psi_u(z) \), so (b) is proved.

Let \( z^{(j)} \to z \), then the set

\[
E = \bigcup_{j=1}^{\infty} E_{z^{(j)}} \cup E_z
\]

is pluripolar. For \( x \in \mathbb{C}^n \setminus E \),

\[
\Psi_u(z) = \Psi_{u,x}(z) = \lim_{j \to \infty} \Psi_{u,x}(z^{(j)}) = \lim_{j \to \infty} \Psi_u(z^{(j)}),
\]

that proves continuity of \( \Psi_u \). Therefore, \( \Psi_u = \Psi_u^* \) is plurisubharmonic and belongs to \( L \) in view of (16), that gives us (a).

If \( x, y \in \mathbb{C}^n \) and \( a \in \mathbb{R}^n \), we have for any \( \varepsilon > 0 \), \( g'(u, x, Ra) \leq g'(u, y, (1+\varepsilon)Ra) \) for all \( R > R_0(\varepsilon, x, y) \), so \( \psi_{u,x}(a) \leq \psi_{u,y}(a) \) that implies (c).

Finally, (d) follows from the relation \( \Psi_{u,x}|_D = 0 \) provided \( \nu(u, x) = 0 \).

3 Monge-Ampère measures

Now we pass to study the Monge-Ampère measures of functions \( u \in L \). We can benefit by the plurisubharmonicity of the growth characteristic \( \Psi_u \) as well as by its specific properties established in the previous section.

Any indicator \( \Phi \) belongs to \( L^\infty_{loc}(\mathbb{C}^n) \), so \( (dd^c \Phi)^n \) is well defined on \( \mathbb{C}^{*n} \). If \( \Phi \in L^\infty_{loc}(\mathbb{C}^n \setminus \{0\}) \), then \( (dd^c \Phi)^n \) is defined on the whole space \( \mathbb{C}^n \); the class of such indicators will be denoted by \( \mathcal{I}_0 \).

Let \( T \) denote the distinguished boundary \( \{ z \in \mathbb{C}^n : |z_1| = \ldots = |z_n| = 1 \} \) of the unit polydisk \( D \).

**Proposition 8** Let \( \Phi \in \mathcal{I} \). Then

(a) \( (dd^c \Phi)^n = 0 \) on \( \mathbb{C}^{*n} \setminus T \);

(b) if \( \Phi \in \mathcal{I}_0 \), then \( (dd^c \Phi)^n = \tau'_\Phi \delta(0) + \tau''_\Phi dm_T \) where \( \tau'_\Phi, \tau''_\Phi \geq 0, \delta(0) \) is the Dirac measure at 0, and \( dm_T = (2\pi)^{-n}d\theta_1 \ldots d\theta_n \) is the normalized Lebesgue measure on \( T \).
Proof. (a) It suffices to show that for every \( y \in \mathbb{C}^n \) there exists an analytic disk \( \gamma_y \) containing \( y \) such that the restriction of \( \Phi \) to \( \gamma_y \) is harmonic near \( y \) ([3], Lemma 6.9). Let \( y = (|y_1|e^{i\theta_1}, \ldots, |y_n|e^{i\theta_n}) \in \mathbb{C}^n \). Consider the mapping \( \lambda_y : \mathbb{C} \to \mathbb{C}^n \) given by 
\[
\lambda_y(\zeta) = (|y_1|e^{i\theta_1}, \ldots, |y_n|e^{i\theta_n});
\]
note that \( \lambda_y(1) = y \). Since \( y \in \mathbb{C}^n \setminus T \), \( \lambda_y \) is not constant. Set \( \Delta = \{ \zeta \in \mathbb{C} : |\zeta - 1| < 1/2 \} \) and \( \gamma_y = \lambda_y(\Delta) \subset \mathbb{C}^n \). Then \( \Phi(\lambda_y(\zeta)) = \Re \zeta \cdot \Phi(\lambda_y(1)) \), so the restriction of \( \Phi \) to \( \gamma_y \) is harmonic.

(b) follows from (a) since locally plurisubharmonic functions cannot charge pluripolar sets and \( \Phi(y) \) is independent of \( \arg y_k \), \( 1 \leq k \leq n \).

We will say that the unbounded locus of \( u \in PSH(\mathbb{C}^n) \) is separated at infinity if there exists an exhaustion of \( \mathbb{C}^n \) by bounded pseudoconvex domains \( \Omega_k \) such that \( \inf \{ u(z) : z \in \partial \Omega_k \} > -\infty \) for each \( k \). The collection of all functions \( u \in \mathcal{L} \) whose unbounded loci are separated at infinity will be denoted by \( \mathcal{L}_s \). By [3], Corollary 2.3, the Monge-Ampère current \( (dd^c u)^n \) is well defined on \( \mathbb{C}^n \) for any function \( u \in \mathcal{L}_s \).

We are going to compare the total Monge-Ampère mass 
\[
M(u; \mathbb{C}^n) = \int_{\mathbb{C}^n} (dd^c u)^n
\]
of \( u \in \mathcal{L}_s \) with that of its indicator. The key result is the following comparison theorem (which is actually a variant of B.A. Taylor’s theorem [21]).

**Theorem 3** Let \( u, v \in \mathcal{L}_s \), \( v \geq 0 \) outside a bounded set, and 
\[
\lim \sup_{z \to \infty} \frac{u(z)}{v(z) + \eta \log |z|} \leq 1 \quad \forall \eta > 0.
\]
Then \( M(u; \mathbb{C}^n) \leq M(v; \mathbb{C}^n) \).

**Proof.** By the definition of the class \( \mathcal{L}_s \), there exist numbers \( 0 \leq m_1 \leq m_2 \leq \ldots \) such that \( u(z) > -m_k \) near \( \partial \Omega_k \), \( \{ \Omega_k \} \) being the pseudoconvex exhaustion of \( \mathbb{C}^n \). Let \( w(z) = \sup \{ u(z), -m_k \} \) for \( \Omega_k \setminus \Omega_{k-1} \) (assuming \( \Omega_0 = \emptyset \)), then \( w \in \mathcal{L} \cap L_{\text{loc}}^\infty(\mathbb{C}^n) \) and satisfies the same asymptotic relation at infinity as \( u \) does. Besides, 
\[
\int_{\Omega_k} (dd^c u)^n = \int_{\Omega_k} (dd^c w)^n \quad \forall k,
\]
so
\[
M(u; \mathbb{C}^n) = M(w; \mathbb{C}^n). \tag{17}
\]
Denote \( v_\eta(z) = v^+(z) + \eta \log^+ |z|, \eta > 0 \). Let \( \epsilon > 0 \) and \( C > 0 \), then \( w(z) \leq (1 + \epsilon) v_\eta - 2C \) for all \( z \in \mathbb{C}^n \setminus B_\alpha \) with \( B_\alpha \) a ball of the radius \( \alpha = \alpha(\eta, C, \epsilon) \). Therefore, 
\[
E(\eta, C, \epsilon) := \{ z \in \mathbb{C}^n : (1 + \epsilon) v_\eta - C < w(z) \} \subset B_\alpha.
\]
By the Comparison Theorem for bounded plurisubharmonic functions,

\[
\int_{E(\eta,C,\epsilon)} (dd^c w)^n \leq \int_{E(\eta,C,\epsilon)} (dd^c [(1 + \epsilon)v_\eta - C])^n \leq (1 + \epsilon) \int_{C^n} (dd^c v_\eta)^n.
\] (18)

For any compact \( K \subset C^n \) one can find \( C > 0 \) such that \( K \subset E(\eta,C,\epsilon) \), so (18) gives us

\[
M(w; C^n) \leq (1 + \epsilon) M(v_\eta; C^n).
\]

Since \( v_\eta \) decreases to \( v^+ \) as \( \eta \to 0 \) and in view of the arbitrary choice of \( \epsilon \), we then get

\[
M(w; C^n) \leq M(v^+; C^n) = M(v; C^n),
\]

which by (17) completes the proof.

As an immediate consequence we have

**Theorem 4** For any \( u \in L_* \),

\[
M(u; C^n) \leq M(\Psi_u, x; C^n) \leq M(\Psi_u; C^n).
\]

To get effective bounds for \( M(u; C^n) \), we estimate the Monge-Ampère masses of the indicators.

**Proposition 9** Let \( \Phi \in \mathcal{I} \), \( z^0 \in D^{-1} \). Then

\[
\Phi(z) \leq \Phi(z^0) \sup_k \frac{\log^+ |z_k|}{\log |z^0_k|} \forall z \in C^n.
\]

**Proof.** Denote \( \psi = [\Phi(z^0)]^{-1} Exp^* \Phi \), \( \alpha = Log(z^0) \in R^n_+ \). It suffices to prove the relation \( \psi(t) \leq s^a_+ (t) \) for all \( t \) with \( |t_k| < a_k \), \( 1 \leq k \leq n \); here \( s_a = Exp^* S_a \) with \( S_a \) defined in (7), so \( s^a_+ (t) = \sup_k t_k^+ / a_k \).

We fix such a point \( t \) and denote \( \alpha = s^a_+ (t) < 1 \), \( \beta = (1 - \alpha)^{-1} > 1 \). Consider the segment \( l_t = \{a + \lambda (t - a) : 0 \leq \lambda \leq \beta\} \subset R^n \); observe that

\[
a_k + \beta (t_k - a_k) = \beta a_k (t_k a_k^{-1} - \alpha) \leq 0, \ 1 \leq k \leq n.
\] (19)

Let \( u(\lambda) \) be the restriction of \( \psi \) to \( l_t \), \( v(\lambda) = (\alpha - 1) \lambda + 1 \). The function \( u \) is convex on \( l_t \), and \( v \) is linear. Besides, \( u(0) = v(0) = 1 \), \( v(\beta) = 0 \), and \( u(\beta) \leq 0 \) in view of (19). Therefore, \( u \leq v \) on \( l_t \). In particular, \( \psi(t) = u(1) \leq v(1) = s^a_+ (t) \), and the proposition is proved.
Theorem 5  Let $\Phi \in I_0$, $z^0 \in D^{-1}$. Then

$$M(\Phi; C^n) \leq \frac{[\Phi(z^0)]^n}{\log |z_1^0| \ldots \log |z_n^0|}.$$ 

In particular, for any $u \in \mathcal{L}$,

$$M(\Psi_u; C^n) \leq \left[ \nu(u, a, \infty) \right]^n_{a_1 \ldots a_n} \forall a \in \mathbb{R}^n_+.$$ 

Proof. The first relation follows from Proposition 9 and Theorem 3, since (taking $a = \log(z^0) \in \mathbb{R}^n_+$)

$$M(S^+_a; C^n) = M(S_a; C^n) = [\log |z_1^0| \ldots \log |z_n^0|]^{-1}.$$ 

The second inequality results now from Proposition 3 (b).

We can give a geometric interpretation for the total Monge-Ampère masses of indicators, which in many cases leads to their exact calculation.

Let $\Phi \in \mathcal{I}$, $\varphi = \text{Exp}^* \Phi$. Denote

$$\Theta_\Phi^+ = \{a \in \mathbb{R}^n : \langle a, t \rangle \leq \varphi^+(t) \; \forall t \in \mathbb{R}^n \}.$$ 

Proposition 10  $\Theta_\Phi^+$ is a convex compact subset of $\mathbb{R}^n_+$, $\Theta_\Phi^+ \subset \{a \in \mathbb{R}^n : 0 \leq a_k \leq \varphi^+(1_k), \; 1 \leq k \leq n \}$. 

Proof. Convexity of $\Theta_\Phi^+$ is evident. Further, if $a \in \Theta_\Phi^+$, then $\langle a, \pm 1_k \rangle \leq \varphi^+(\pm 1_k)$, $1_k$ being defined by (11). and the statement follows because $\varphi^+(-1_k) = 0$.

By $Vol(P)$ we denote the Euclidean volume of $P \subset \mathbb{R}^n$.

Theorem 6  For any $\Phi \in \mathcal{I}_0$,

$$M(\Phi; C^n) = n! \text{Vol}(\Theta_\Phi^+).$$

Proof. By Proposition 8,

$$M(\Phi; C^n) = M(\Phi^+; C^n) = \int_T (dd^c \Phi^+)^n.$$ 

It can be easily checked that the complex Monge-Ampère operator $(dd^c U)^n$ of an $n$-circled locally bounded plurisubharmonic function $U$ is related to the real Monge-Ampère operator $\mathcal{M}A[u]$ of its convex image $u$ by the equation

$$\int_G (dd^c U)^n = n! \int_{\text{Log}(G)} \mathcal{M}A[u].$$
for every \( n \)-circled Borelean set \( G \subset \mathbb{C}^n \) (see e.g. [19]). Since \( \text{Log}(T) = \{0\} \),

\[
M(\Phi^+; \mathbb{C}^n) = n! \mathcal{M}_A[\varphi^+](\{0\}).
\] (20)

As was established in [18], for any convex function \( v \) on a domain \( \Omega \subset \mathbb{R}^n \),

\[
\mathcal{M}_A[v](F) = \text{Vol}(\omega(F,v)) \quad \forall F \subset \Omega,
\] (21)

where

\[
\omega(F,v) = \bigcup \{a \in \mathbb{R}^n : v(t) \geq v(t_0) + \langle a, t - t_0 \rangle \forall t \in \Omega\}
\]
is the gradient image of the set \( F \) for the surface \( \{y = v(x), x \in \Omega\} \). In our situation, it means that

\[
\omega(\{0\}, \varphi^+) = \{a \in \mathbb{R}^n : \varphi^+(t) \geq \langle a, t \rangle \forall t \in \mathbb{R}^n\} = \Theta^+_\Phi,
\] (22)

so the statement follows from (20)–(22).

The set \( \Theta^+_\Phi \) for \( \Phi = \Psi_{u,x}, u \in \mathcal{L}, x \in \mathbb{C}^n \), will be denoted by \( \Theta^+_u \), and by \( \Theta^+_\Phi \) for \( \Phi = \Psi_u \). Then Theorems 4 and 6 give us

**Theorem 7** For any \( u \in \mathcal{L}_* \) and \( x \in \mathbb{C}^n \),

\[
M(u; \mathbb{C}^n) \leq n! \text{Vol}(\Theta^+_u) \leq n! \text{Vol}(\Theta^+_u).
\] (23)

**Remark.** Let \( u = \log |P|, P = (P_1, \ldots, P_N) \) being a polynomial mapping. By Proposition 8,

\[
\psi_{u,x}(t) = I_{u,p}(P, x, t) = \sup_{1 \leq j \leq n} I_{u,p}(P_j, x, t),
\]

the upper indices \( I_{u,p}(P_j, x, t) \) defined by (13). In this case,

\[
\Theta^+_u = \{a \in \mathbb{R}^n : \langle a, t \rangle \leq I_{u,p}(P, x, t) \forall t \in \mathbb{R}^n\},
\]

so \( \Theta^+_{u,0} \) coincides with the Newton polyhedron for \( P \) at infinity (see Introduction). If \( N = n \) and \( P^{-1}(0) \) is discrete, then \( M(u; \mathbb{C}^n) \) is the number of zeros of \( P \) counted with the multiplicities. For this case, (23) gives the bound due to Kouchnirenko [7].

Theorem 7 produces also an upper bound for \( M(u; \mathbb{C}^n) \) via the multitype \((\sigma_1(u), \ldots, \sigma_n(u))\) of the function \( u \):

**Theorem 8** Let \( u \in \mathcal{L}_* \), then

\[
M(u; \mathbb{C}^n) \leq n! \sigma_1(u) \ldots \sigma_n(u);
\] (24)
in particular,

\[
\sum_{x}[\nu(u, x)]^n \leq n! \sigma_1(u) \ldots \sigma_n(u).
\] (25)

**Proof.** By Propositions 3 and 10, \( \Theta^+_u \subset \{a \in \mathbb{R}^n : 0 \leq a_k \leq \sigma_k(u), 1 \leq k \leq n\} \), and (24) follows from Theorem 4. It implies (25) in view of the known inequality \( (dd^c u)^n|_x \geq [\nu(u, x)]^n \).

**Remark.** It can be shown that inequality (24) implies Dyson’s lemma for algebraic hypersurfaces with isolated singular points (see [22]).
References

[1] E. Bedford and B.A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier 38 (1988), 133-171.

[2] E. Bedford and B.A. Taylor, *Uniqueness for the complex Monge-Ampère equation for functions of logarithmic growth*, Indiana Univ. Math. J. 38 (1989), 455-469.

[3] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex Analysis and Geometry (Univ. Series in Math.), ed. by V. Ancona and A. Silva, Plenum Press, New York 1993, 115-193.

[4] J.E. Fornaess and N. Sibony, *Complex dynamics in higher dimensions. II*, Modern Methods in Complex Analysis, Princeton Univ. Press, 1995, pp. 135-182.

[5] C.O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variables*, U.U.D.M. Report 1998:1, Uppsala University.

[6] M. Klimek, Pluripotential theory. Oxford University Press, London, 1991.

[7] A.G. Kouchnirenko, *Newton polyhedron and the number of solutions of a system of k equations with k indeterminates*, Uspekhi Mat. Nauk 30 (1975), no. 2, 266-267.

[8] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. 32 (1976), 1-31.

[9] S. Lang, Fundamentals of Diophantine Geometry, Springer, New York, 1983.

[10] P. Lelong, *Fonctions entières de type exponentiel dans $\mathbb{C}^n$*, Ann. Inst. Fourier, 16 (1966), 271-318.

[11] P. Lelong, Plurisubharmonic functions and positive differential forms, Gordon and Breach, New York, and Dunod, Paris, 1969.

[12] P. Lelong, *Théorème de Banach-Steinhaus pour les polynômes; applications entières d’espaces vectoriels*, Séminaire Pierre Lelong (Analyse), Année 1970. Lecture Notes Math. 205, pp. 87-112. Springer, Berlin, 1971.

[13] P. Lelong, *Fonctions plurisousharmoniques de croissance logarithmique dans $\mathbb{C}^n$, parti principale, extension du résultat des polynômes*. Semin. Conf. 8, pp. 211-229. Cetraro, Editel Italie, 1991. (1991)
[14] P. Lelong, *Mesure de Mahler et calcul de constantes universelles pour les polynômes de N variables*, Math. Ann. 299 (1994), 673-695.

[15] P. Lelong, *Remarks on pointwise multiplicities*, Linear Topologic Spaces and Complex Analysis 3 (1997), 112-119.

[16] P. Lelong and L. Gruman, *Entire Functions of Several Complex Variables*, Springer-Verlag, Berlin - Heidelberg - New York, 1986.

[17] P. Lelong and A. Rashkovskii, *Local indicators for plurisubharmonic functions*, J. Math. Pures Appl. 78 (1999), 233-247.

[18] J. Rauch and B. A. Taylor, *The Dirichlet problem for the multidimensional Monge-Ampère equation*, Rocky Mountain Math. J. 7 (1977), 345-364.

[19] A. Rashkovskii, *Newton numbers and residual measures of plurisubharmonic functions*, El. preprint [http://xxx.lanl.gov/abs/math/990562](http://xxx.lanl.gov/abs/math/990562).

[20] J. Siciak, *Extremal plurisubharmonic functions in Cⁿ*, Ann. Polon. Math. 39 (1981), 175-211.

[21] B.A. Taylor, *An estimate for an extremal plurisubharmonic function on Cⁿ*, Séminaire P. Lelong, P. Dolbeault, H. Skoda, Année 1982/1983. Lecture Notes in Math. 1028 (1983), 318-328.

[22] C. Viola, *On Dyson’s lemma in several variables*, In: Diophantine Approximations and Transcendental Numbers, Lumini 1990. Walter de Gruyter & Co., Berlin - New York, 1992, 281-284.

Mathematical Division, Institute for Low Temperature Physics
47 Lenin Ave., Kharkov 310164, Ukraine
E-mail: rashkovskii@ilt.kharkov.ua, rashkovs@rashko.ilt.kharkov.ua