ON THE ESTIMATION OF LOCALLY STATIONARY FUNCTIONAL TIME SERIES

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Abstract. This study develops an asymptotic theory for estimating the time-varying characteristics of locally stationary functional time series (LSFTS). We investigate a kernel-based method to estimate the time-varying covariance operator and the time-varying mean function of an LSFTS. In particular, we derive the convergence rate of the kernel estimator of the covariance operator and associated eigenvalue and eigenfunctions and establish a central limit theorem for the kernel-based locally weighted sample mean. As applications of our results, we discuss methods for testing the equality of time-varying mean functions in two functional samples.

Keywords: functional time series, locally stationary process, principal component analysis, nonparametric estimation

1. Introduction

Functional time series analysis has been a growing interest in many scientific fields, such as environmetrics (Aue, Dubart Norinho, and Hörmann (2015)), biometrics (Chiou and Müller (2009)), demographics (Li, Robinson, and Shang (2020)), and finance (Kokoszka and Zhang (2012), Chen, Lei, and Tu (2016)). We refer to Ramsey and Silverman (2005), Ferraty and Vieu (2006), and Horváth and Kokoszka (2012) as the standard references on functional data and functional time series analysis.

In the literature of functional time series analysis, most of the studies are based on stationary models (e.g., Bosq (2000, 2002), Dehling and Sharipov (2005), Antoniadis, Paparoditis, and Sapatinas (2006), Horváth, Kokoszka, and Reeder (2013), Aue, Dubart Norinho, and Hörmann (2015), and Dette, Kokot, and Volgushev (2020)). However, many functional time series exhibit nonstationary behavior. Typical examples include the daily patterns of temperature observed in a region and daily curves of implied volatility of an option as a function of moneyness. We can also find other examples of nonstationary functional time series in van Delft and Eichler (2018). One way to model nonstationary behavior is provided by the theory of locally stationary processes.

A locally stationary process, as proposed by Dahlhaus (1997), is a nonstationary time series that allows its parameters to be time-varying and can be approximated by a stationary time series at each rescaled time point. This property enables us to develop asymptotic theories for the estimation of time-varying characteristics. The rigorous definition of local stationarity for functional data is given in Section 2. There is a vast literature on locally stationary (multivariate) time series. We mention Dahlhaus and Subba Rao (2006), Fryzlewicz, Sapatinas, and Subba Rao (2008), Kristensen (2009), Koo and Linton (2012), Vogt (2012), Zhou (2014), Zhang and Wu (2015), Truquet (2017, 2019), and Kurisu, Fukami, and Koike (2022), to name a few. Since the introduction of the notion of locally stationary processes,
stationary processes, it has been extended in several directions, such as spatial data (Pezo (2018), Kurisu (2022a)) and spatio-temporal data (Matsuda and Yajima (2018)). As recent important contributions in the literature of functional time series analysis, we refer to van Delft and Eichler (2018), Aue and van Delft (2020), Bücher, Dette, and Heinrichs (2020), and van Delft and Dette (2021). van Delft and Eichler (2018) extend the notion of locally stationary processes to functional time series that take values in a Hilbert space. The authors investigate frequency domain methods for locally stationary functional data. Aue and van Delft (2020) develop a method for testing the null hypothesis that the observed functional time series is stationary against the hypothesis that it is locally stationary. Bücher, Dette, and Heinrichs (2020) develop a similarity measure based on time-varying spectral density operators for cluster analysis of nonstationary functional time series. We also mention Kurisu (2022b), who investigates nonparametric regression for locally stationary functional time series and Dette and Wu (2021) for the construction of confidence surfaces of the mean function of locally stationary functional time series.

The objective of this paper is to develop an asymptotic theory for a kernel-based method to estimate the time-varying covariance operator and the time-varying mean function of the locally stationary functional time series at each rescaled time point. The first objective is related to the problem of estimating time-varying eigenvalues and eigenfunctions of the covariance operator, which is important to develop methods for functional principal component analysis (FPCA) that are based on those time-varying (or local) characteristics. It has been recognized that FPCA is a dimension reduction technique in many scientific research fields, such as chemical engineering, functional magnetic resonance imaging, and environmental science. We refer to Ramsey and Silverman (2005) for a detailed comparison between PCA in multivariate setting and FPCA. Specifically, we consider a kernel-based estimator of the time-varying covariance operator and derive its convergence rate (i.e. Theorem 3.1). The result also enables us to derive convergence rates of estimators of associated eigenvalues and eigenfunctions.

The second objective is to extend the statistical methods based on the sample mean of functional time series (e.g., testing the equality of mean functions in two functional samples (cf. Horváth, Kokoszka, and Reeder (2013) and Dette, Kokot, and Volgushev (2020))) developed under the assumption of stationarity to nonstationary cases. We consider a kernel-based locally weighted sample mean and show that it converges to a Gaussian process (i.e. Theorem 3.2). We also propose a kernel-based estimator of the long-run covariance kernel function of the Gaussian process and establish its consistency.

As possible applications of our results, we discuss problems on estimating local mean functions and testing the equality of local mean functions of locally stationary functional time series. For the implementation of the two sample test, we use FPCA-based projection methods to retain most of the information carried by the original process and provide a method for selecting the number of principal components. Consequently, we extend well-known results developed for stationary functional time series to our framework.

The organization of this paper is as follows. In Section 2, we introduce the notion of local stationarity for functional time series that take values in a Hilbert space and dependence structure of the functional time series. In Section 3, we present the main results. In Section 4, we discuss some applications of our results. All proofs are included in the Appendix.
1.1. Notations. For any positive sequences \( a_n \) and \( b_n \), we write \( a_n \preceq b_n \) if a constant \( C > 0 \) independent of \( n \) exists such that \( a_n \leq C b_n \) for all \( n \) and \( a_n \sim b_n \) if \( a_n \preceq b_n \) and \( b_n \preceq a_n \). For any \( a, b \in \mathbb{R} \), we write \( a \lor b = \max\{a, b\} \) and \( a \land b = \min\{a, b\} \). We use the notations \( \xrightarrow{d} \) and \( \xrightarrow{p} \) to denote convergence in distribution and convergence in probability, respectively. For a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \), let \( \|x\| = \sqrt{\langle x, x \rangle} \), \( x \in H \) and let \( L \) denote the space of bounded (continuous) linear operators on \( H \) with the norm

\[
\|\Psi\|_L = \sup\{\|\Psi(x)\| : \|x\| \leq 1, x \in H\}.
\]

An operator \( \Psi \in L \) is said to be compact if there exist two orthonormal basis \( \{v_j\} \) and \( \{f_j\} \), and a real sequence \( \{\lambda_j\} \) converging to zero such that

\[
\Psi(x) = \sum_{j=1}^{\infty} \lambda_j \langle v_j, x \rangle f_j, \quad x \in H.
\]

Further, a compact operator with the singular value decomposition (1.1) is said to be a Hilbert-Schmidt operator if \( \sum_{j=1}^{\infty} \lambda_j^2 < \infty \). Let \( S \) be the space of Hilbert-Schmidt operators on \( H \) with inner product

\[
\langle \Psi_1, \Psi_2 \rangle_S = \sum_{j=1}^{\infty} \langle \Psi_1(e_j), \Psi_2(e_j) \rangle,
\]

where \( \{e_j\} \) is an arbitrary orthonormal basis of \( H \). The Hilbert-Schmidt norm on \( S \) is defined as \( \|\Psi\|_S = \sqrt{\sum_{j=1}^{\infty} \lambda_j^2} \), which does not depend on the choice of the orthonormal basis. Note that \( (L, \| \cdot \|_L) \) is a Banach space and \( (S, \| \cdot \|_S) \) is a separable Hilbert space. We refer to [Bosq (2000)] for details on the properties of operators on a Hilbert space. For \( x \in H \), define the tensor product \( f \otimes g : H \otimes H \to H \) as the bounded linear operator \( (f \otimes g)(x) = \langle g, x \rangle f \).

2. Settings

In this section, we introduce the notion of a locally stationary functional time series that extends the notion of a locally stationary process introduced by [Dahlhaus (1997)]. Furthermore, we discuss dependence structures of the functional time series. For some probability space \((\Omega, \mathcal{A}, P)\) and for \( p \geq 1 \), let \( L^p(\Omega, \mathcal{A}, P) \) denote the space of real-valued random variables such that \( \|X\|_p = (E[|X|^p])^{1/p} < \infty \). Let \( L^2 = L^2([0,1]^d) \) denote a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) defined by

\[
\langle x, y \rangle = \int_{[0,1]^d} x(t)y(t)dt, \quad x, y \in H = L^2.
\]

Further, let \( L^p_H = L^p_H(\Omega, \mathcal{A}, P) \) denote the space of \( H \)-valued random functions \( X \) such that

\[
v_p(X) = (E[|X|^p])^{1/p} = \left( E \left[ \left( \int_{[0,1]^d} X^2(t)dt \right)^{p/2} \right] \right)^{1/p} < \infty.
\]

2.1. Local stationarity. Intuitively, a functional time series \( \{X_{t,T}\}_{t \in \mathbb{Z}} \) in \( L^p_H \) is locally stationary if it behaves approximately stationary in local time. We refer to [Dahlhaus and Subba Rao (2006)] and [Dahlhaus, Richter, and Wu (2019)] for the idea of locally stationary time series and its general theory, and [van Delft and Eichler (2018)] and [Au and van Delft (2020)] for the notion of local stationarity for a Hilbert space-valued time series. To ensure that it is locally stationary around
each rescaled time point \( u \in [0, 1] \), a process \( \{X_{t,T}\} \) in \( L^p_H \) can be approximated by a stationary functional time series \( \{X_{t}^{(u)}\} \) in \( L^p_H \) stochastically. This concept can be defined as follows.

**Definition 2.1.** A sequence of \( H \)-valued stochastic process \( \{X_{t,T}\}_{t=1}^T \) in \( L^p_H \) is locally stationary if, for each rescaled time point \( u \in [0, 1] \), there exists an associated \( H \)-valued process \( \{X_{t}^{(u)}\}_{t \in \mathbb{Z}} \) in \( L^p_H \) stochastically. This concept can be defined as follows.

(i) \( \{X_{t}^{(u)}\}_{t \in \mathbb{Z}} \) is strictly stationary.

(ii) It holds that

\[
\|X_{t,T} - X_{t}^{(u)}\| \leq \left( \frac{t}{T} - u + \frac{1}{T} \right) U_{t,T}^{(u)} \text{ a.s.,} \tag{2.1}
\]

for all \( 1 \leq t \leq T \) where \( \{U_{t,T}^{(u)}\} \) is a process of positive variables satisfying \( E[(U_{t,T}^{(u)})^\rho] < C \) for some \( \rho > 0 \), \( C < \infty \) independent of \( u, t, \) and \( T \).

Definition 2.1 is a natural extension of the notion of local stationarity for real-valued time series introduced in Dahlhaus (1997).

**2.2. Dependence structure.** For each \( u \in [0, 1] \), we assume that \( \{X_{t}^{(u)}\} \) is \( L^p-m \)-approximable, that is, \( X_{t}^{(u)} \in L^p_H \) is of the form

\[
X_{t}^{(u)} = f_u(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots),
\]

where \( \varepsilon_j \) are i.i.d. elements taking values in a measurable space \( S \), and \( f_u \) is a measurable function \( f_u : S^\infty \to H \). Note that \( X_{t}^{(u)} \) is strictly stationary. We also assume that if \( \{\varepsilon^{(m)}_{j,k}\} \) is an independent copy of \( \{\varepsilon_j\} \) defined on the same probability space, then letting

\[
X_{m,t}^{(u)} = f_u(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-m+1}, \varepsilon_{t-t-m}, \varepsilon^{(m)}_{t,t-m}, \varepsilon^{(m)}_{t,t-m-1}, \ldots), \tag{2.2}
\]

we have

\[
\sum_{m \geq 1} v_p(X_{t}^{(u)} - X_{m,t}^{(u)}) < \infty.
\]

We can show that a wide class of functional time series (e.g., functional AR(1) process and functional ARCH(1) process) is \( L^p-m \)-approximable under some regularity conditions. Hence the concept of \( L^p-m \) approximability of functional time series is popular in the literature of functional time series analysis. See Hörmann and Kokoszka (2010) and Horváth and Kokoszka (2012) for details on the properties of \( L^p-m \)-approximable random functions. We also refer to Zhou and Dette (2023) and Dette and Wu (2021) who consider physical representation of functional time series as a different dependence structure (see, for example, Section 2 in Zhou and Dette (2023) for the connection and difference between \( L^p-m \) approximability and physical dependence).

**3. Main results**

In this section, we introduce a kernel-based method to estimate the time-varying covariance operator and the time-varying mean function of a locally stationary time series at each rescaled time point \( u \in [0, 1] \).
3.1. Estimation of eigenvalues and eigenfunctions of local covariance operators. We make the following assumption on the process \( \{X_{t,T}\} \).

**Assumption 3.1.** The process \( \{X_{t,T}\} \) in \( L^4_H \) is locally stationary. In particular, there exists an \( L^4-m\)-approximable process \( \{X_{t}^{(u)}\} \) in \( L^4_H \) that satisfies Definition 2.1 with \( p = 4 \) and some \( \rho \geq 2 \). We also assume that \( E[X_{t}^{(u)}] = 0 \) in \( L^2 \).

**Example.** We give an example of locally stationary functional time series that satisfies Assumption 3.1. Let \( \{H_{k}(u;F_{t}^{(k)})\}_{t \in \mathbb{Z}} \) be stationary time series for each \( k \in \mathbb{N} \) and \( u \in [0,1] \) that satisfy the following conditions:

- (E1) \( H_{k}(u;F_{t}^{(k)}) = G_{k}(u;\zeta_{t}^{(k)},\zeta_{t-1}^{(k)},\zeta_{t-2}^{(k)}, \ldots) \) where \( \zeta_{t}^{(k)} \) are i.i.d. elements taking values in a measurable space \( S \), and \( G_{k}(u;\cdot) \) is a measurable function \( G_{k}(u;\cdot) : S^{\infty} \rightarrow \mathbb{R} \).
- (E2) \( \zeta(t) = \{\zeta_{t}^{(k)}\}_{t \in \mathbb{Z}} \) are mutually independent.
- (E3) There exist random variables \( \bar{H}_{k}(F_{t}^{(k)}) \) such that
  \[
  |H_{k}(t/T;F_{t}^{(k)}) - H_{k}(u;F_{t}^{(k)})| \leq \left| \frac{t}{T} - u \right| \bar{H}_{k}(F_{t}^{(k)}),
  \]
  \[
  \sum_{k \geq 1} E \left[ \log \left( \frac{H_{k}(u;F_{t}^{(k)})}{H_{k}(u;F_{t}^{(k)})} \right)^{1/2} \right] < \infty, \sum_{k \geq 1} E \left[ \bar{H}_{k}(F_{t}^{(k)})^{2} \right] < \infty.
  \]
- (E4) Let \( \{\zeta_{t}^{(k,m)}\}_{m \geq 1} \) be an independent copy of \( \{\zeta_{t}^{(k)}\}. \) Assume that
  \[
  \sum_{m \geq 1} \left( \sum_{k \geq 1} E \left[ \left( H_{k}(u;F_{t}^{(k)}) - H_{k}(u;F_{t}^{(k,m)}) \right)^{1/2} \right] \right) < \infty,
  \]
  where \( H_{k}(u;F_{t}^{(k,m)}) = G_{k}(u;\zeta_{t}^{(k)},\zeta_{t-1}^{(k)};\cdots,\zeta_{t-m+1}^{(k)},\zeta_{t-m}^{(k,m)},\zeta_{t-m-1}^{(k,m)}, \ldots) \).

For a given orthogonal basis \( \{\phi_{k}(s)\}_{k = 1}^{\infty} \) of \( L^{2}([0,1]^{d}) \), consider the functional time series defined by

\[
X_{t}^{(u)}(s) = \sum_{k \geq 1} H_{k}(u;F_{t}^{(k)})\phi_{k}(s), \quad X_{m,t}^{(u)}(s) = \sum_{k \geq 1} H_{k}(u;F_{t}^{(k,m)})\phi_{k}(s),
\]

\[
X_{t,T}(s) = X_{t(t/T)}^{(u)}(s), \quad 1 \leq t \leq T.
\]

**Proposition 3.1.** Under Conditions (E1)-(E4), the processes \( X_{t,T} \) and \( X_{t}^{(u)} \) satisfy Assumption 3.1.

Now we move on to our main results. Let \( C_{u} : H \rightarrow H \) be the covariance operator of a zero-mean stationary process \( \{X_{t}^{(u)}\} \), that is, \( C_{u}(\cdot) = E[X_{t}^{(u)} \otimes X_{t}^{(u)}] \). As in the multivariate time series case, \( C_{u} \) admits the singular value decomposition

\[
C_{u}(x) = \sum_{j=1}^{\infty} \lambda_{u,j} \langle v_{u,j}, x \rangle v_{u,j},
\]

where \( \{\lambda_{u,j}\} \) are the non-negative eigenvalues (in descending order) and \( \{v_{u,j}\} \) are the corresponding normalized eigenfunctions, that is, \( C_{u}(v_{u,j}) = \lambda_{u,j} v_{u,j} \) and \( \|v_{u,j}\| = 1 \). The \( \{v_{u,j}\} \) form an orthonormal basis of \( H \). Hence \( X_{t}^{(u)} \) allows for the Karhunen-Loève representation \( X_{t}^{(u)} = \sum_{j=1}^{\infty} \lambda_{u,j} \langle v_{u,j}, X_{t}^{(u)} \rangle v_{u,j} \).
\[ \sum_{j=1}^{\infty} \langle X_t^{(u)}, v_{u,j} \rangle v_{u,j} \]. The coefficients \( \langle X_t^{(u)}, v_{u,j} \rangle \) in this expansion are called the FPC scores of \( X_t^{(u)} \).

Consider the empirical local covariance operator
\[
\hat{C}_u = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_{t,T} \otimes X_{t,T},
\]
where \( K_1 \) is a one-dimensional kernel function and we use the notation \( K_{1,h}(x) = K_1(x/h) \), and \( h \) is a sequence of positive constants (bandwidths) such that \( h = h_T \rightarrow 0 \) as \( T \rightarrow \infty \). When the mean function of \( X_t^{(u)} \) is unknown, that is, \( E[X_t^{(u)}] = m(u, \cdot) \), then we can estimate \( m(u, \cdot) \) by a kernel-based locally weighted sample mean \( \hat{X}_T^{(u)} = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h}(u - t/T)X_{t,T} \). In this case, \( \hat{C}_u \) is given by
\[
\hat{C}_u = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \left( X_{t,T} - \hat{X}_T^{(u)} \right) \otimes \left( X_{t,T} - \hat{X}_T^{(u)} \right).
\]

Under some regularity conditions, we can show \( E \left[ \| X_T^{(u)} - m(u, \cdot) \|^2 \right] = O \left( h^2 + \frac{1}{T} + \frac{1}{Th^2} \right) \). We discuss the asymptotic properties of \( \hat{X}_T^{(u)} \) in Section 3.2. See Theorem 3.2 below and its proof for details. We make the following assumption on the kernel function \( K_1 \).

**Assumption 3.2.** The kernel \( K_1 \) is symmetric around zero, bounded, and has a compact support, i.e., \( K_1(v) = 0 \) for all \( |v| > C_1 \) for some \( C_1 < \infty \). Moreover, \( \int K_1(z)dz = 1 \) and \( K_1 \) is Lipschitz continuous, i.e., \( |K_1(v_1) - K_1(v_2)| \leq C_2|v_1 - v_2| \) for some \( C_2 < \infty \) and all \( v_1, v_2 \in \mathbb{R} \).

The next result provides the convergence rate of the kernel estimator \( \hat{C}_u \) at each rescaled time point \( u \).

**Theorem 3.1.** Under Assumptions 3.1 and 3.2, we have
\[
\| \hat{C}_u - C_u \|_\mathcal{L} = O_P \left( h + \sqrt{\frac{1}{Th}} + \frac{1}{Th^2} \right) \tag{3.1}
\]
for \( u \in [C_1h, 1 - C_1h] \).

The first term \( h \) in the error bound (3.1) comes from the local stationarity of \( X_{t,T} \), i.e., the approximation error of \( X_{t,T} \) by \( X_t^{(u)} \), and the second and the third terms correspond to variance and bias terms, respectively.

Let \( \{\tilde{v}_{u,j}\}_{j=1}^{q} \) be the eigenfunctions of \( \hat{C}_u \) corresponding to the \( q \) largest eigenvalues and set \( \tilde{c}_{u,j} \) as
\[
\tilde{c}_{u,j} = \text{sign} \left( \langle \tilde{v}_{u,j}, v_{u,j} \rangle \right).
\]

Further, let \( \{\tilde{\lambda}_{u,j}\} \) be estimators of \( \{\lambda_{u,j}\} \) that satisfy
\[
\hat{C}_u(\tilde{v}_{u,j}) = \tilde{\lambda}_{u,j} \tilde{v}_{u,j}, \quad 1 \leq j \leq q.
\]

The next result immediately follows from Theorem 3.1. It shows that \( \{\tilde{\lambda}_{u,j}\}_{j=1}^{q} \) are consistent estimators of \( \{\lambda_{u,j}\}_{j=1}^{q} \), and \( \{\tilde{v}_{u,j}\}_{j=1}^{q} \) consistently estimates \( \{v_{u,k}\}_{j=1}^{q} \) up to sign change.
Corollary 3.1. Suppose $\lambda_{u,1} > \lambda_{u,2} > \cdots > \lambda_{u,q} > \lambda_{u,q+1}$. Under Assumptions 3.1 and 3.2, we have
\[ \max_{1 \leq j \leq q} |\hat{\lambda}_{u,j} - \lambda_{u,j}| = O_P \left( h + \frac{1}{Th} + \frac{1}{Th^2} \right), \quad (3.2) \]
\[ \max_{1 \leq j \leq q} |\hat{c}_{u,j} \hat{v}_{u,j} - v_{u,j}| = O_P \left( h + \frac{1}{Th} + \frac{1}{Th^2} \right) \quad (3.3) \]
for $u \in [C_1 h, 1 - C_1 h]$.

The convergence rates (3.2) and (3.3) are optimized by choosing $h \sim T^{-1/3}$ and the optimized rates are
\[ |\hat{\lambda}_{u,j} - \lambda_{u,j}| = O_P \left( T^{-1/3} \right), \quad |\hat{c}_{u,j} \hat{v}_{u,j} - v_{u,j}| = O_P \left( T^{-1/3} \right). \]

3.2. Asymptotic normality of locally weighted sample means. Now we make the following assumption instead of Assumption 3.1.

Assumption 3.3. The process $\{X_{t,T}\}$ in $L^2_H$ is locally stationary. In particular, there exists an $L^2$-m-approximable process $\{X_t^{(u)}\}$ in $L^2_H$ that satisfies $E[X_t^{(u)}] = 0$ in $L^2$ and satisfies Definition 2.1 with $p = 2$ and some $\rho \geq 2$.

Define
\[ \bar{X}_T^{(u)} = \frac{1}{Th} \sum_{t=1}^T K_{1,h} \left( u - \frac{t}{T} \right) X_{t,T}. \]

The next result is a central limit theorem of $\bar{X}_T^{(u)}$.

Theorem 3.2. Suppose $Th \wedge \frac{1}{Th^2} \rightarrow \infty$ as $T \rightarrow \infty$. Under Assumptions 3.2 and 3.3, for each $u \in (0, 1)$, we have
\[ \sqrt{Th} \bar{X}_T^{(u)} \xrightarrow{d} G^{(u)} \text{ in } L^2, \]
where $G^{(u)}$ is a Gaussian element in $L^2$ with $E[G^{(u)}(s)] = 0$ and with the covariance kernel function
\[ E[G^{(u)}(s_1)G^{(u)}(s_2)] = c^{(u)}(s_1, s_2) \int K_{1,h}^2(z)dz \text{ where } \]
\[ c^{(u)}(s_1, s_2) = E[X_0^{(u)}(s_1)X_0^{(u)}(s_2)] + \sum_{t \geq 1} E[X_0^{(u)}(s_1)X_t^{(u)}(s_2)] + \sum_{t \geq 1} E[X_0^{(u)}(s_2)X_t^{(u)}(s_1)]. \]

Note that the infinite sums in the definition of the kernel $c^{(u)}$ converge in $L^2([0, 1]^d \times [0, 1]^d)$, that is, $\int_{[0, 1]^d \times [0, 1]^d} (c^{(u)}(s_1, s_2))^2 ds_1 ds_2 < \infty$. See Appendix A for details. If $E[X_t^{(u)}] = m(u, \cdot) \neq 0$, the same result of Theorem 3.2 holds by replacing $\bar{X}_T^{(u)}$ and $X_t^{(u)}$ with $\bar{X}_T^{(u)} - m(u, \cdot)$ and $X_t^{(u)} - m(u, \cdot)$, respectively. In Section 4, we discuss applications of Theorem 3.2 and it plays an important role in deriving the limit distributions of statistics for testing the equality of time-varying mean functions of two functional time series.

For the inference on the local mean function $m(u, \cdot)$, we need to estimate the long-run local covariance kernel $c^{(u)}$. We propose the following estimator as an estimator of $c^{(u)}$.
\[ \hat{c}^{(u)}(s_1, s_2) = \hat{\gamma}^{(u)}_0(s_1, s_2) + \sum_{t=1}^{T-1} K_2 \left( \frac{t}{b} \right) \left\{ \hat{\gamma}^{(u)}_t(s_1, s_2) + \hat{\gamma}^{(u)}_t(s_2, s_1) \right\} \quad (3.4) \]
where
\[
\hat{\gamma}^{(u)}_t(s_1, s_2) = \frac{1}{T h} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \left( X_{j,T}(s_1) - \bar{X}^{(u)}_T(s_1) \right) \left( X_{j-t,T}(s_2) - \bar{X}^{(u)}_T(s_2) \right)
\]
for \(1 \leq t \leq T - 1\), \(b\) is a sequence of positive constants (bandwidths) such that \(b \to \infty\) as \(T \to \infty\), and \(K_2\) is a kernel function that satisfies the following condition.

**Assumption 3.4.** The kernel \(K_2\) is bounded, and has a compact support, i.e., \(K_2(v) = 0\) for all \(|v| > C_2\) for some \(0 < C_2 < \infty\). Moreover, we assume that \(K_2(0) = 1\).

For the bandwidth \(b\), we make the following assumption.

**Assumption 3.5.** As \(T \to \infty\), \(\min\{Th, \frac{1}{Th^p}, b, \frac{Th}{p}\} \to \infty\).

Under the above assumptions, we can establish the following result.

**Theorem 3.3.** Assume that Assumptions 3.1 (with \(p = 4\)), 3.2, 3.4, and 3.5 are satisfied. Additionally, assume that \(\lim_{m \to \infty} m v_2(X^{(u)}_0 - X^{(u)}_{m,0}) = 0\). Then for each \(u \in (0,1)\), we have
\[
\int_{[0,1]^d} \int_{[0,1]^d} \left( \hat{c}^{(u)}(s_1, s_2) - c^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 \to 0 \quad \text{as} \quad T \to \infty,
\]
where \(c^{(u)}\) is the covariance kernel that appears in Theorem 3.2.

Theorem 3.3 enables us to construct local FPCA-based statistics for testing the equality of the local mean functions. See Section 4 for details.

### 4. Applications

In this section, we discuss two possible applications of our results. The first is the estimation of the local mean function of the locally stationary functional time series. The second is a two sample test for the difference of local mean functions, which generalizes the results of Horváth, Kokoszka, and Reeder (2013) for the strictly stationary case to our settings.

#### 4.1. Estimation of local mean functions

Let \(D\) be a compact subset of \(\mathbb{R}^d\) and let \(H = L^2\), the space of real-valued square integrable functions on \(D\). Consider the following model:
\[
X_{t,T}(s) = m \left( \frac{t}{T}, s \right) + v_{t,T}(s), \quad s \in D
\]
for each \(u \in [0,1]\), \(\{X^{(u)}_t\}\) is a \(L^2\)-approximable random function and
\[
\|X_{t,T} - X^{(u)}_t\| \leq \|m(t/T, \cdot) - m(u, \cdot)\| + \|v_{t,T} - v^{(u)}_{t}\| \leq (C_m + 1) \left( \left| u - \frac{t}{T} \right| + \frac{1}{T} \right)(1 + U^{(u)}_{t,T}).
\]
Then \(\{X_{t,T}\}\) is a locally stationary \(H\)-valued functional time series in \(L^2_H\).
Remark 4.1. When $d = 1$, the model \((4.1)\), with constant mean function (i.e., \(m(u, s) = m(s)\)) and stationary error process (i.e., \(\{v_{t,T}\}\) itself is stationary), includes many examples, such as financial data (Li, Robinson, and Shang (2020)), biology (Chiou and Müller (2009)) and environmental data (Aue, Dubart Norinho, and Hörmann (2015)). When $d = 2$, the model \((4.1)\) can be seen as a spatio-temporal model with a time-varying mean function. The model is also called a surface time series. Surface data provide an alternative approach to analyzing spatial data, where the continuous realization of a random field is considered as a unit. This approach can have computational advantages, especially if the locations where data are observed are dense in space. For example, \(\{X_{t,T}\}\) would be an observation of functional surfaces of the daily records of temperature or precipitation with a time-varying mean function. Other examples include acoustic phonetic data (Aston, Pigoli, and Tavakoli (2017)), data from two surface electroencephalograms (EEG) (Crainiceanu et al. (2011)), and proteomics data (Morris et al. (2011)). We also refer to Martínez-Hernández and Genton (2020) as an overview of recent developments in functional data including surface time series.

Let \(\hat{m}(u, \cdot) = \hat{X}_T^{(u)}\). Applying Theorem 3.2, we can estimate the time-varying mean function \(m(u, \cdot)\):

\[
E \left[ \|\hat{m}(u, \cdot) - m(u, \cdot)\|^2 \right] = O \left( \frac{1}{Th} \right).
\]

If we do not assume $\frac{1}{Th} \to \infty$ as $T \to \infty$, then we can also show

\[
E \left[ \|\hat{m}(u, \cdot) - m(u, \cdot)\|^2 \right] = O \left( h^2 + \frac{1}{Th} + \frac{1}{T^2 h^4} \right).
\]

The term $h^2$ in the convergence rate arises from the local stationarity of $X_{t,T}$, i.e., the approximation error of $X_{t,T}$ by a stationary process $X_t^{(u)}$. The second and the third terms correspond to variance and bias terms, respectively. The convergence rate \((4.2)\) is optimized by choosing $h \sim T^{-1/3}$ and the optimized rate is $E \left[ \|\hat{m}(u, \cdot) - m(u, \cdot)\|^2 \right] = O(T^{-2/3})$.

4.2. Two sample problems. Let \(\{X_{t,T}\}\) be an $H$-valued locally stationary functional time series. In principle, we can test whether the underlying functional time series \(\{X_{t,T}\}\) is stationary or not. Aue and van Delft (2020) proposed a method for testing the null hypothesis that \(\{X_{t,T}\}\) is stationary against the alternative hypothesis that the process is locally stationary. Further, Horváth, Kokoszka, and Reeder (2013) proposed methods for testing the equality of (constant) mean functions of two strictly stationary functional time series. Once the null hypothesis, that \(\{X_{t,T}\}\) is stationary, is rejected, their methods for testing the equality of mean functions cannot be applied to the observations. We extend their methods to nonstationary functional time series by using our main results.

Assume that we have two samples of functional time series \(\{X_{t,T_1}\}_{t=1}^{T_1}\) and \(\{Y_{t,T_2}\}_{t=1}^{T_2}\) follow the location models

\[
X_{t,T_1}(s) = m_1 \left( \frac{t}{T_1}, s \right) + v_{t,T_1}(s),
\]
\[
Y_{t,T_2}(s) = m_2 \left( \frac{t}{T_2}, s \right) + w_{t,T_2}(s),
\]

where \(\{v_{t,T_1}\}\) and \(\{w_{t,T_2}\}\) are zero-mean $H$-valued error processes that satisfy Assumption 3.3 and the associated stationary processes \(\{v_t^{(u)}\}\) and \(\{w_t^{(u)}\}\) have local long-run covariance kernel functions...
where for random variables $A$ and $B$, $A \overset{d}{=} B$ denotes that $A$ and $B$ has the same distribution, \{η_{u,j}\} is a sequence of the eigenvalues of the covariance operator $\Phi^{(u)}$ defined by $\phi^{(u)}$, and \{N_{j}\} is a sequence of i.i.d. standard normal random variables. We can also estimate the covariance kernel function $\phi^{(u)}(s_1, s_2)$ by

\[ \tilde{\phi}^{(u)}(s_1, s_2) = (1 - \tilde{\theta})\widehat{c}_{1,u}(s_1, s_2) + \tilde{\theta}\widehat{c}_{2,u}(s_1, s_2), \]

where $\tilde{\theta} = \frac{T_1}{T_1 + T_2}$, and $\widehat{c}_{j,u}(s_1, s_2), j = 1, 2$ are defined analogously as $\widehat{c}(u)$ in (3.4). The next result establishes the consistency of $\tilde{\phi}^{(u)}$. 

Under $H_0$, the random variable $U^{(u)}_{\infty,\infty}$ has the representation

\[ U^{(u)}_{\infty,\infty} \overset{d}{=} \sum_{j=1}^{\infty} \eta_{u,j} N_{j}^2, \]

where for random variables $A$ and $B$, $A \overset{d}{=} B$ denotes that $A$ and $B$ has the same distribution, \{η_{u,j}\} is a sequence of the eigenvalues of the covariance operator $\Phi^{(u)}$ defined by $\phi^{(u)}$, and \{N_{j}\} is a sequence of i.i.d. standard normal random variables. We can also estimate the covariance kernel function $\phi^{(u)}(s_1, s_2)$ by

\[ \tilde{\phi}^{(u)}(s_1, s_2) = (1 - \tilde{\theta})\widehat{c}_{1,u}(s_1, s_2) + \tilde{\theta}\widehat{c}_{2,u}(s_1, s_2), \]

where $\tilde{\theta} = \frac{T_1}{T_1 + T_2}$, and $\widehat{c}_{j,u}(s_1, s_2), j = 1, 2$ are defined analogously as $\widehat{c}(u)$ in (3.4). The next result establishes the consistency of $\tilde{\phi}^{(u)}$. 

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Proposition 4.2. Suppose \( T_1 h \wedge \frac{1}{T_0} \to \infty, \frac{T_1}{T_1 + T_2} \to \theta \in (0,1) \) as \( T_1 \wedge T_2 \to \infty \). Assume that \( \{X_{i,T_1}\} \) and \( \{Y_{i,T_2}\} \) satisfy the same assumption of Theorem 3.3. Then we have

\[
\int_{[0,1]^d} \int_{[0,1]^d} \left( \hat{\phi}(s_1, s_2) - \phi(s_1, s_2) \right)^2 ds_1 ds_2 \to 0.
\]

Since the distribution of \( U_{\infty, \infty}^{(u)} \) depends on \( \{\eta_{u,j}\} \), for the implementation of the two sample test \((4.3)\), we approximate the distribution of \( U_{\infty, \infty}^{(u)} \) using functional principal components. Consider the following truncated version of \( U_{\infty, \infty}^{(u)} \):

\[
\tilde{U}_{\infty, \infty}^{(u)} = \sum_{j=1}^{q} \eta_{u,j} N_j^{(u)}.
\]  

Let \( \tilde{\eta}_{u,j} \) be the \( j \)-th largest eigenvalue of \( \hat{\Phi}^{(u)} \) defined by \( \hat{\phi}^{(u)} \) and let \( \tilde{\nu}_{u,j} \) denote the eigenfunction associated with \( \tilde{\eta}_{u,j} \). Since \( \hat{\phi}^{(u)} \) and \( \Phi^{(u)} \) are integral operators computed from covariance kernel functions \( \hat{\phi}^{(u)} \) and \( \phi^{(u)} \), Proposition 4.2 yields

\[
\| \hat{\Phi}^{(u)} - \Phi^{(u)} \|_S = \int_{[0,1]^d} \int_{[0,1]^d} \left( \hat{\phi}^{(u)}(s_1, s_2) - \phi^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 \to 0.
\]  

If \( \eta_{u,1} > \cdots > \eta_{u,q} > \eta_{u,q+1} \), then \((4.5)\) and the same argument in the proof of Corollary 3.1 yield

\[
\max_{1 \leq j \leq q} |\tilde{\eta}_{u,j} - \eta_{u,j}| \to 0, \quad \max_{1 \leq j \leq q} \| \tilde{\nu}_{u,j} - \nu_{u,j} \| \to 0,
\]  

where \( \tilde{\nu}_{u,j} = \tilde{\kappa}_{u,j} \tilde{\nu}_{u,j} \) and \( \tilde{\kappa}_{u,j} = \text{sign}(\tilde{\nu}_{u,j}, \nu_{u,j}) \).

Define two kinds of truncated version of \( U_{T_1,T_2}^{(u)} \) based on functional principal components as follows:

\[
U_{T_1,T_2}^{(u)} = \frac{T_1 T_2 h}{T_1 + T_2} \sum_{j=1}^{q} \left( \tilde{X}_{T_1}(u) - \tilde{Y}_{T_2}(u), \tilde{\nu}_{u,j} \right)^2, \quad U_{T_1,T_2}^{(u)} = \frac{T_1 T_2 h}{T_1 + T_2} \sum_{j=1}^{q} \tilde{\eta}_{u,j} \left( \tilde{X}_{T_1} - \tilde{Y}_{T_2}, \tilde{\nu}_{u,j} \right)^2.
\]

The next result characterizes asymptotic properties of \( \tilde{U}_{T_1,T_2}^{(u)} \) and \( \hat{U}_{T_1,T_2}^{(u)} \). It can be shown by using (4.6), Proposition 4.1, and the continuous mapping theorem.

Proposition 4.3. Suppose \( T_1 h \wedge \frac{1}{T_0} \to \infty, \frac{T_1}{T_1 + T_2} \to \theta \in (0,1) \) as \( T_1 \wedge T_2 \to \infty \) and \( \eta_{u,1} > \cdots > \eta_{u,q} > \eta_{u,q+1} \). Assume that \( \{X_{i,T_1}\} \) and \( \{Y_{i,T_2}\} \) satisfy the same assumption of Theorem 3.3. Then under \( H_0 \), we have

\[
\tilde{U}_{T_1,T_2}^{(u)} \to d \sum_{j=1}^{q} \eta_{u,j} N_j^{(u)} \text{,} \quad \hat{U}_{T_1,T_2}^{(u)} \to \chi^2(q),
\]

where \( \chi^2(q) \) is the \( \chi^2 \)-distribution with \( q \) degrees of freedom. Additionally, under \( H_1 \),

\[
\frac{T_1 + T_2}{T_1 T_2 h} \tilde{U}_{T_1,T_2}^{(u)} \to p \sum_{j=1}^{q} \left( m_1(u, \cdot) - m_2(u, \cdot), \nu_{u,j} \right)^2,
\]

\[
\frac{T_1 + T_2}{T_1 T_2 h} \hat{U}_{T_1,T_2}^{(u)} \to p \sum_{j=1}^{q} \eta_{u,j}^{-1} \left( m_1(u, \cdot) - m_2(u, \cdot), \nu_{u,j} \right)^2.
\]
Proposition 4.3 implies that \( \bar{U}(u^T_1, T_2) \rightarrow \infty \) under \( H_1 \) if at least one of the projections \( \langle m_1(u, \cdot) - m_2(u, \cdot), \nu_{u,j} \rangle \) is different from zero.

When \( \eta_{u,q_0} > \eta_{u,q_0+1} = 0 \) for some positive integer \( q_0 \), we can estimate \( q_0 \) based on a ratio method introduced in Lam and Yao (2012). The estimator \( \hat{q}_0 \) is given as follows:

\[
\hat{q}_0 = \arg \min_{1 \leq j \leq q} \frac{\eta_{u,j+1}}{\eta_{u,j}},
\]

where \( q \) is a prespecified positive number and set \( 0/0 = 1 \) for convenience. Let \( \epsilon_0 \) be a prespecified small positive number. We set \( |\hat{\eta}_{u,j}/\hat{\eta}_{u,1}| \) as 0 to reduce estimation error in the implementation, if its absolute value is smaller than \( \epsilon_0 \). The following result establishes the consistency of \( \hat{q} \).

**Corollary 4.1.** Let \( q \geq q_0 \). Under the same assumption of Proposition 4.2, we have \( \hat{q}_0 \rightarrow_q q_0 \) as \( T \rightarrow \infty \).

Applying the same rule as described above, it is straightforward to extend Corollary 4.1 to the case that \( \eta_{u,q_0} \) and \( \eta_{u,q_0+1} \) satisfy \( \eta_{u,q_0}/\eta_{u,1} \geq \epsilon_0 \) and \( \eta_{u,q_0+1}/\eta_{u,1} < \epsilon_0 \) for some prespecified small positive number \( \epsilon_0 \).

## 5. Concluding remarks

In this paper, we have studied a kernel method for estimating the time-varying covariance operator and the time-varying mean function of a locally stationary functional time series. We derived the convergence rate of the kernel estimator of the covariance operator and established a central limit theorem for the kernel-based locally weighted sample mean. As applications of these results, we extended methods for testing the equality of time-varying mean functions, which were developed for stationary functional time series to our framework.

To conclude, we shall mention two important future research topics. The first is to consider a data driven formula for selecting bandwidth parameters \( h \) and \( b \). This topic (at least on \( b \)) has not been addressed in previous works (e.g., Horváth, Kokoszka, and Reeder (2013) and Li, Robinson, and Shang (2020)). Second, the two sample problem considered in this paper is limited to the hypothesis testing on the equality of the time-varying mean functions. We are hopeful that our theory can be extended to cover relevant hypotheses considered in the recent works by Dette, Kokot, and Aue (2020) and Dette, Kokot, and Volgushev (2020).
Appendix A. Proofs

A.1. Proofs for Section 3

Proof of Proposition 3.1. First, we check $X_{t}^{(u)}$ is a $L^4$-$m$-approximable process in $L^4_H$. Observe that

$$||X_{t}^{(u)} - X_{m,t}||^2 = \sum_{k \geq 1} |H_k(u; F^{(k)}_t) - H_k(u; F^{(k,m)}_t)|^2.$$ 

Then from Condition (E4), we obtain

$$\sum_{m \geq 1} v_4(X_t^{(u)} - X_{m,t}^{(u)}) = \sum_{m \geq 1} E \left[ ||X_t^{(u)} - X_{m,t}^{(u)}||^4 \right]^{1/4}$$

$$= \sum_{m \geq 1} E \left[ \left( \sum_{k \geq 1} |H_k(u; F^{(k)}_t) - H_k(u; F^{(k,m)}_t)|^2 \right)^2 \right]^{1/4}$$

$$\leq \sum_{m \geq 1} \left( \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} E \left[ |H_{k_1}(u; F^{(k_1)}_t) - H_{k_1}(u; F^{(k_1,m)}_t)|^4 \right] \cdot E \left[ |H_{k_2}(u; F^{(k_2)}_t) - H_{k_2}(u; F^{(k_2,m)}_t)|^4 \right] \right)^{1/4}$$

$$= \sum_{m \geq 1} \left( \sum_{k \geq 1} E \left[ |H_k(u; F_{t,k}) - H_k(u; F_{t,k}^{(m)})|^4 \right] \right)^{1/2} < \infty. \quad (A.1)$$

From Condition (E3), we also have

$$E \left[ ||X_t^{(u)}||^4 \right] = E \left[ \left( \sum_{k \geq 1} |H_k(u; F^{(k)}_t)|^2 \right)^2 \right]$$

$$= \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} E \left[ |H_{k_1}(u; F^{(k_1)}_t)|^2 |H_{k_2}(u; F^{(k_2)}_t)|^2 \right]$$

$$\leq \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} E \left[ |H_{k_1}(u; F^{(k_1)}_t)|^4 \right]^{1/2} E \left[ |H_{k_2}(u; F^{(k_2)}_t)|^4 \right]^{1/2}$$

$$= \left( \sum_{k \geq 1} E \left[ |H_k(u; F^{(k)}_t)|^4 \right] \right)^{1/2} < \infty. \quad (A.2)$$

Combining (A.1) and (A.2), $X_t^{(u)}$ is a $L^4$-$m$-approximable process in $L^4_H$.

Next we check that $X_{t,T}$ is a locally stationary functional time series in $L^4_H$ that satisfies Definition 2.1 with $\rho = 2$. This can be verified from Condition (E3) and

$$||X_{t,T} - X_{t}^{(u)}||^2 = \sum_{k \geq 1} |H_k(t/T; F^{(k)}_t) - H_k(u; F^{(k)}_t)|^2 \leq \left| \frac{t}{T} - u \right|^2 \sum_{k \geq 1} |H_k(F^{(k)}_t)|^2.$$
Proof of Theorem 3.1. Define

\[ \tilde{C}_u = \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_t^{(u)} \otimes X_t^{(u)}. \]

Decompose

\[ \tilde{C}_u - \bar{C}_u = \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \left( X_t^{(u)} \otimes (X_{t,T} - X_t^{(u)}) \right) \]

\[ + (X_{t,T} - X_t^{(u)}) \otimes X_t^{(u)} + (X_{t,T} - X_t^{(u)}) \otimes (X_{t,T} - X_t^{(u)}) \]

\[ =: C_{u,1} + C_{u,2} + C_{u,3}. \]

For \( C_{1,u} \),

\[ \|C_{1,u}(x)\| \leq \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \|X_t^{(u)}\| \|x\| \|X_{t,T} - X_t^{(u)}\| \]

\[ \leq \left( h + \frac{1}{T} \right) \frac{\|x\|}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \|X_t^{(u)}\| U_{t,T}^{(u)}. \]

Then

\[ E[\|C_{1,u}\|_\mathcal{L}] \leq \left( h + \frac{1}{T} \right) \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) E[\|X_t^{(u)}\| U_{t,T}^{(u)}] \]

\[ \leq \left( h + \frac{1}{T} \right) \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (E[\|X_t^{(u)}\|^2])^{1/2} E[(U_{t,T}^{(u)})^2]^{1/2} \]

\[ = O \left( h + \frac{1}{T} \right). \]

This implies \( \|C_{1,u}\|_\mathcal{L} = O_P(h + T^{-1}) \). Likewise, we have

\[ \|C_{2,u}\|_\mathcal{L} = O_P \left( h + \frac{1}{T} \right), \quad \|C_{3,u}\|_\mathcal{L} = O_P \left( h^2 + \frac{1}{T^2} \right). \]

Therefore,

\[ \|\tilde{C}_u - \bar{C}_u\|_\mathcal{L} = O_P \left( h + \frac{1}{T} \right). \quad (A.3) \]

Note that

\[ \tilde{C}_u - E[\tilde{C}_u] = \frac{1}{T h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \left( X_t^{(u)} \otimes X_t^{(u)} - E[X_t^{(u)} \otimes X_t^{(u)}] \right). \]

Define \( B_{u,t}(y) = \langle X_t^{(u)}, y \rangle X_t^{(u)} - C_u(y) \) and \( \bar{B}_{u,t} = h^{-1} K_{1,h}(u - t/T) B_{u,t} \). Then

\[ Th \|\tilde{C}_u - E[\tilde{C}_u]\|_\mathcal{S}^2 = Th \left( \frac{1}{T h} \sum_{t=1}^{T} \bar{B}_{u,k} \right)_\mathcal{S}^2 = \frac{h}{T} \left( \sum_{t_1=1}^{T} (\bar{B}_{u,t_1}, \bar{B}_{u,t_2}) + \sum_{t_1 \neq t_2} (\bar{B}_{u,t_1}, \bar{B}_{u,t_2}) \right). \]
Then we have

\[
\frac{h}{T} E \left[ \sum_{t_1=1}^{T} (\bar{B}_{u,t_1}, \bar{B}_{u,t_2})_S + \sum_{t_1 \neq t_2} (\bar{B}_{u,t_1}, \bar{B}_{u,t_2})_S \right]
\]

\[
= \frac{1}{T h} \sum_{k=-(T-1)}^{T-1} \sum_{t_1=1+|k|}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right| K_{1,h} \left. \left( u - \frac{t_1 - |k|}{T} \right) \right| E \left[ (B_{u,t_1}, B_{u,t_2})_S \right]
\]

\[
= \frac{1}{T h} \sum_{k=-(T-1)}^{T-1} E \left[ (B_{u,0}, B_{u,|k|})_S \right] \sum_{t_1=1+|k|}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right| K_{1,h} \left. \left( u - \frac{t_1 - |k|}{T} \right) \right|
\]

\[
\leq \frac{1}{T h} \sum_{k=0}^{T-1} E \left[ (B_{u,0}, B_{u,k})_S \right] \sum_{t_1=1+k}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right| K_{1,h} \left. \left( u - \frac{t_1 - k}{T} \right) \right|
\]

Hence we have

\[
T h E \left[ \| \tilde{C}_u - E[\tilde{C}_u] \|_S^2 \right]
\]

\[
\leq \frac{2}{T h} \sum_{k=0}^{T-1} E \left[ (B_{u,0}, B_{u,k})_S \right] \sum_{t_1=1+k}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right| K_{1,h} \left. \left( u - \frac{t_1 - k}{T} \right) \right|
\]

\[
\leq \frac{2}{T h} \sum_{k=0}^{T-1} E \left[ (B_{u,0}, B_{u,k})_S \right] \sum_{t_1=1+k}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right|
\]

\[
\leq \left( \frac{2}{T h} \sum_{t_1=1}^{T} K_{1,h} \left. \left( u - \frac{t_1}{T} \right) \right| \right) \sum_{k=0}^{T-1} E \left[ (B_{u,0}, B_{u,k})_S \right].
\]

Observe that

\[
\langle B_{u,0}, B_{u,k} \rangle_S = \sum_{j=1}^{\infty} \langle B_{u,0}(v_{u,j}), B_{u,k}(v_{u,j}) \rangle
\]

\[
= \langle X_0^{(u)}, X_k^{(u)} \rangle \sum_{j=1}^{\infty} \langle X_0^{(u)}, v_{u,j} \rangle \langle X_k^{(u)}, v_{u,j} \rangle - \sum_{j=1}^{\infty} \lambda_{u,j} \langle X_0^{(u)}, v_{u,j} \rangle^2
\]

\[
- \sum_{j=1}^{\infty} \lambda_{u,j} \langle X_k^{(u)}, v_{u,j} \rangle^2 + \sum_{j=1}^{\infty} \lambda_{u,j}^2
\]

\[
= \langle X_0^{(u)}, X_k^{(u)} \rangle^2 - \sum_{j=1}^{\infty} \lambda_{u,j} \langle X_0^{(u)}, v_{u,j} \rangle^2 - \sum_{j=1}^{\infty} \lambda_{u,j} \langle X_k^{(u)}, v_{u,j} \rangle^2 + \sum_{j=1}^{\infty} \lambda_{u,j}^2.
\]

Then we have

\[
E \left[ (B_{u,0}, B_{u,k})_S \right] = E \left[ \langle X_0^{(u)}, X_k^{(u)} \rangle^2 \right] - \sum_{j=1}^{\infty} \lambda_{u,j}^2, \ k \geq 1. \quad (A.4)
\]
Recall that \( X_0^{(u)} \) and \( X_{k,k}^{(u)} \) are independent and identically distributed. Thus we have

\[
E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 \right] = \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \left( E \left[ \langle X_0^{(u)}, v_{u,j} \rangle \langle X_0^{(u)}, v_{u,\ell} \rangle \right]^2 \right) \\
= \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \left( E \left[ \langle X_0^{(u)}, v_{u,j} \rangle X_0^{(u)} \right] \right)^2 \\
= \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \left( (C_u(v_{u,j}), v_{u,\ell}) \right)^2 = \sum_{j=1}^{\infty} \lambda_{u,j}^2.
\]

Moreover, since

\[
\langle X_0^{(u)}, X_{k,k}^{(u)} \rangle - \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 = \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 - \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle - 2\langle X_0^{(u)}, X_{k,k}^{(u)} \rangle \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle,
\]

we have

\[
E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 \right] - \sum_{j=1}^{\infty} \lambda_{u,j}^2 = E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 \right] - \sum_{j=1}^{\infty} \lambda_{u,j}^2 + E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 \right] \\
+ 2E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle \right] \\
= E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle^2 \right] + 2E \left[ \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle \langle X_0^{(u)}, X_{k,k}^{(u)} \rangle \right].
\]

Together with (A.4) and (A.5), we have

\[
|E \left[ \langle B_{u,0}, B_{u,k} \rangle \right]| \leq v_4^2(X_0^{(u)})v_4^2(X_k^{(u)} - X_{k,k}^{(u)}) + 2v_4^3(X_0^{(u)})v_4(X_k^{(u)} - X_{k,k}^{(u)}) \\
= v_4^2(X_0^{(u)})v_4^2(X_1^{(u)} - X_{k,1}^{(u)}) + 2v_4^3(X_0^{(u)})v_4(X_1^{(u)} - X_{k,1}^{(u)}).
\]

Since \( X_t^{(u)} \) is \( L^4 \)-m-approximable, for \( u \in [C_1h, 1 - C_1h] \), we have

\[
\text{Th}E \left[ \| \tilde{C}_u - E[\tilde{C}_u] \| \right]_S^2 \\
\lesssim \left( \frac{2}{Th} \sum_{t_1=1}^{T} K_{1,h} \left( u - \frac{t_1}{T} \right) \right) \sum_{k=0}^{T-1} |E \left[ \langle B_{u,0}, B_{u,k} \rangle \right]| \\
\lesssim O(1) \times \left\{ \sum_{k=1}^{\infty} v_4^2(X_0^{(u)})v_4^2(X_1^{(u)} - X_{k,1}^{(u)}) + 2v_4^3(X_0^{(u)})v_4(X_1^{(u)} - X_{k,1}^{(u)}) \right\} < \infty.
\]

Therefore, we obtain

\[
\| \tilde{C}_u - E[\tilde{C}_u] \|_S = O_P \left( \frac{1}{\sqrt{Th}} \right).
\]
Further, for each $u \in [C_1 h, 1 - C_1 h]$, we have
\[
\| E[\tilde{C}_u] - C_u \|_S = \left\| \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) - 1 \right\| C_u \|_S \\
= \left\| \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) - 1 \right\| C_u \|_S \\
= O \left( \frac{1}{Th^2} \right) + o(h). \tag{A.7}
\]

For the third equality, we used Lemma B.3.

Combining (A.3), (A.6), (A.7) and $\| \cdot \|_S \leq \| \cdot \|_L$, we have
\[
\| \hat{C}_u - C_u \|_L \leq \| \hat{C}_u - \tilde{C}_u \|_L + \| \tilde{C}_u - E[\tilde{C}_u] \|_S + \| E[\tilde{C}_u] - C_u \|_S \\
= O_P \left( h + \sqrt{\frac{1}{Th}} \right) + O_P \left( \frac{1}{Th^2} \right) + o(h) \\
= O_P \left( h + \sqrt{\frac{1}{Th}} + \frac{1}{Th^2} \right)
\]
for each $u \in [C_1 h, 1 - C_1 h]$. \hfill \Box

**Proof of Corollary 3.1.** Combining Theorem 3.1 and Lemmas B.1 and B.2, we have
\[
| \hat{\lambda}_{u,j} - \lambda_{u,j} | \leq \| \hat{C}_u - C_u \|_L = O_P \left( h + \sqrt{\frac{1}{Th}} + \frac{1}{Th^2} \right),
\]
\[
\| \hat{v}_{u,j} \hat{v}_{u,j} - v_{u,j} \| \leq \frac{2\sqrt{2}}{\alpha_{u,j}} \| \hat{C}_u - C_u \|_L = O_P \left( h + \sqrt{\frac{1}{Th}} + \frac{1}{Th^2} \right),
\]
where $\alpha_{u,1} = \lambda_{u,1} - \lambda_{u,2}$, $\alpha_{u,j} = \min \{ \lambda_{u,j-1} - \lambda_{u,j}, \lambda_{u,j} - \lambda_{u,j+1} \}$, $2 \leq j \leq q$. \hfill \Box

**Proof of Theorem 3.2.** Define
\[
\tilde{X}_T^{(u)} = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_t^{(u)}.
\]
(Step1) In this step, we will show
\[
ThE[\| \tilde{X}_T^{(u)} - \bar{X}_T^{(u)} \|_2^2] = o(1). \tag{A.8}
\]
Observe that
\[
\| \tilde{X}_T^{(u)} - \bar{X}_T^{(u)} \| \leq \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \| X_t - X_t^{(u)} \| \\
\leq \left( h + \frac{1}{T} \right) \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) U_t^{(u)}.
\]
Then we have
\[
\begin{align*}
\sqrt{T} h E[\|X_T^{(u)} - \tilde{X}_T^{(u)}\|^2] & \leq \sqrt{T} h \left( h + \frac{1}{T} \right)^2 \frac{1}{T^2 h^2} E \left[ \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) U_{t_1,T}^{(u)} U_{t_2,T}^{(u)} \right] \\
& \leq \sqrt{T} h^3 \frac{1}{T^2 h^2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) E[(U_{t_1,T}^{(u)})^2] \\
& \lesssim \sqrt{T} h^3 \left( \frac{1}{T} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \right)^2 = o(1).
\end{align*}
\]
This implies that
\[
\sqrt{T} h X_T^{(u)} = \sqrt{T} h \tilde{X}_T^{(u)} + o_P(1).
\]

(Step 2) In this step, we give a sketch of the rest of the proof.
In Step 3, we will show
\[
\lim_{m \to \infty} \lim_{T \to \infty} E \left[ \left( \frac{1}{\sqrt{T} h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_t^{(u)}(s) - X_{m,t}^{(u)}(s)) \right)^2 ds \right] = 0 \quad (A.9)
\]
where the variables \(X_{m,t}^{(u)}\) are defined in (2.2).
In Step 4, we will show that for any \(m \geq 1\),
\[
\frac{1}{\sqrt{T} h} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_{m,t}^{(u)} \overset{d}{\to} G_m^{(u)} \text{ in } L^2 \quad (A.10)
\]
where \(G_m^{(u)}\) is a Gaussian process with \(E[G_m^{(u)}(s)] = 0\) and with the covariance kernel function
\[
E[G_m^{(u)}(s_1) G_m^{(u)}(s_2)] = c_m^{(u)}(s_1, s_2) \int K_2^2(z) dz
\]
where
\[
c_m(s_1, s_2) = E[X_0^{(u)}(s_1) X_0^{(u)}(s_2)] + \sum_{t=1}^{m} E[X_0^{(u)}(s_1) X_t^{(u)}(s_2)] + \sum_{t=1}^{m} E[X_0^{(u)}(s_2) X_t^{(u)}(s_1)].
\]
In Step 5, we will show
\[
G_m^{(u)} \overset{d}{\to} G^{(u)} \text{ in } L^2. \quad (A.11)
\]
Combining Theorem 3.2 in [Billingsley (1999)] with (A.9)-(A.11), we have
\[
\sqrt{T} h \tilde{X}_T^{(u)} \overset{d}{\to} G^{(u)} \text{ in } L^2. \quad (A.12)
\]
Therefore, (A.8) and (A.12) yield \(\sqrt{T} h \tilde{X}_T^{(u)} \overset{d}{\to} G^{(u)} \text{ in } L^2\).

(Step 3) In this step, we will show (A.9). By stationarity,
\[
\begin{align*}
E \left[ \left( \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_t^{(u)}(s) - X_{m,t}^{(u)}(s)) \right)^2 \right] \\
& = \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) E \left[ (X_t^{(u)}(s) - X_{m,t}^{(u)}(s))^2 \right] \\
& + 2 \sum_{1 \leq t_1 < t_2 \leq T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) E \left[ (X_{t_1}^{(u)}(s) - X_{m,t_1}^{(u)}(s))(X_{t_2}^{(u)}(s) - X_{m,t_2}^{(u)}(s)) \right].
\end{align*}
\]
Note that
\[
\frac{1}{Th} \int \sum_{t=1}^{T} K_{1,h}^{2} \left( u - \frac{t}{T} \right) E \left[ (X_{1}^{(u)}(s) - X_{m,1}^{(u)}(s))^2 \right] ds
\]
\[
= v_{2}^{2} (X_{1}^{(u)} - X_{m,1}^{(u)}) \left( \frac{1}{Th} \sum_{t=1}^{T} K_{1,h}^{2} \left( u - \frac{t}{T} \right) \right) \sim v_{2}^{2} (X_{1}^{(u)} - X_{m,1}^{(u)}) \to 0 \text{ as } m \to \infty. \quad (A.13)
\]

Observe that if \( t_{2} > t_{1} \), then \( (X_{t_{1}}^{(u)}, X_{m,t_{1}}^{(u)}) \) is independent of \( X_{t_{2}-t_{1},t_{1}}^{(u)} \) because
\[
X_{t_{2}-t_{1},t_{2}}^{(u)} = f_{u}(\varepsilon_{t_{2}}, \varepsilon_{t_{2}-1}, \ldots, \varepsilon_{t_{1}+1}, \varepsilon_{t_{2},t_{1}}, \varepsilon_{t_{2},t_{1}-1}, \ldots).
\]

Then \( E[(X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))X_{t_{2}-t_{1},t_{2}}^{(u)}] = 0 \) and so
\[
\sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))X_{t_{2}}^{(u)}(s) \right]
\]
\[
= \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))(X_{t_{2}}^{(u)}(s) - X_{t_{2}-t_{1},t_{2}}^{(u)}(s)) \right].
\]

Applying the Cauchy-Schwarz inequality, we have
\[
\left| \int \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))(X_{t_{2}}^{(u)}(s) - X_{t_{2}-t_{1},t_{2}}^{(u)}(s)) \right] ds \right|
\]
\[
\leq \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right)
\]
\[
\times \int E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))^2 \right]^{1/2} E \left[ (X_{t_{2}}^{(u)}(s) - X_{t_{2}-t_{1},t_{2}}^{(u)}(s))^2 \right]^{1/2} ds
\]
\[
\leq \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) \left( E[\|X_{t_{1}}^{(u)} - X_{m,t_{1}}^{(u)}\|^2] \right)^{1/2} \left( E[\|X_{t_{2}}^{(u)} - X_{t_{2}-t_{1},t_{2}}^{(u)}\|^2] \right)^{1/2}
\]
\[
= \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) \left( E[\|X_{1}^{(u)} - X_{m,1}^{(u)}\|^2] \right)^{1/2} \left( E[\|X_{1}^{(u)} - X_{t_{2}-t_{1},1}\|^2] \right)^{1/2}
\]
\[
\lesssim v_{2} (X_{1}^{(u)} - X_{m,1}^{(u)}) \left( \sum_{t_{1}=1}^{T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) \right) \sum_{k \geq 1} v_{2} (X_{1}^{(u)} - X_{k,1}^{(u)}) \lesssim v_{2} (X_{1}^{(u)} - X_{m,1}^{(u)}) \times O(Th).
\]

For the last inequality, we used the \( L^{2}-m \)-approximability of \( X_{t}^{(u)} \). This yields
\[
\limsup_{m \to \infty} \limsup_{T \to \infty} \frac{1}{Th} \left| \int \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))X_{t_{2}}^{(u)}(s) \right] ds \right| = 0.
\]

Likewise, we can show that
\[
\limsup_{m \to \infty} \limsup_{T \to \infty} \frac{1}{Th} \left| \int \sum_{1 \leq t_{1} < t_{2} \leq T} K_{1,h} \left( u - \frac{t_{1}}{T} \right) K_{1,h} \left( u - \frac{t_{2}}{T} \right) E \left[ (X_{t_{1}}^{(u)}(s) - X_{m,t_{1}}^{(u)}(s))X_{m,t_{2}}^{(u)}(s) \right] ds \right| = 0.
\]

Combining these results and (A.13) yield (A.9).
(Step 4) In this step, we will show (A.10). Recall that for every integer \( m \geq 1 \), \( \{X_{m,t}^{(u)}\} \) is an \( m \)-dependent sequence of functions. Let \( N > 1 \) be an integer and let \( \{v_{m,j}\} \) and \( \{\lambda_{m,j}\} \) are the orthonormal eigenfunctions and the corresponding eigenvalues of the integral operator with the kernel \( c_m \). Then by the Karhunen-Loève expansion, we have

\[
X_{m,t}^{(u)}(s) = \sum_{j \geq 1} \langle X_{m,t}^{(u)}, v_{m,j} \rangle v_{m,j}(s).
\]

Define \( X_{m,t}^{(u,N)}(s) = \sum_{j=1}^{N} \langle X_{m,t}^{(u)}, v_{m,j} \rangle v_{m,j}(s) \). By the triangle inequality, we have

\[
\left\{ E \left[ \int \left( \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \right)^2 \right] \right\}^{1/2}
\leq \left\{ E \left[ \int \left( \sum_{t \in I(0)} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \right)^2 \right] \right\}^{1/2}
\]

\[+ \cdots + \left\{ E \left[ \int \left( \sum_{t \in I(m-1)} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \right)^2 \right] \right\}^{1/2}\]

where \( I(k) = \{ t : 1 \leq t \leq T, t = k \mod m \}, 0 \leq k \leq m - 1 \). Since \( X_{m,t}^{(u)} \) is \( m \)-dependent, \( \sum_{t \in I(\ell)} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \) is a sum of independent random variables. Thus we have

\[
\frac{1}{Th} E \left[ \int \left( \sum_{t \in I(\ell)} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \right)^2 \right] \leq \sum_{j > N} E[(X_{m,1}^{(u)}, v_{m,j})^2] \leq \sum_{j > N} E[(X_{m,1}^{(u)}, v_{m,j})^2].
\]

Since \( \lim_{N \to \infty} \sum_{j > N} E[(X_{m,1}^{(u)}, v_{m,j})^2] \to 0 \), we conclude that for any \( r > 0 \),

\[
\limsup_{N \to \infty} \limsup_{T \to \infty} \left( \int \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_{m,t}^{(u)}(s) - X_{m,t}^{(u,N)}(s)) \right)^2 ds > r \right) = 0. \quad (A.14)
\]

Observe that

\[
\frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_{m,t}^{(u,N)} = \sum_{j=1}^{N} v_{m,j} \frac{1}{\sqrt{Th}} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \langle X_{m,t}^{(u)}, v_{m,j} \rangle.
\]
Utilizing the orthogonality of \( \{v_{m,j}\} \) and applying the central limit theorem for stationary \( m \)-dependent sequences (Theorem 6.4.2 in Brockwell and Davis (1991)) and the Cramér-Wald theorem, we have

\[
\left( \frac{1}{\sqrt{T h}} \sum_{t=1}^{T} K_{1,t} \left( u - \frac{t}{T} \right) \langle X_{m,t}^{(u)}, v_{m,j} \rangle, 1 \leq j \leq N \right) \xrightarrow{d} Z_N(0, \Lambda_N)
\]

where \( Z_N(0, \Lambda_N) \) is a \( N \)-dimensional Gaussian random variable with zero mean and covariance matrix \( \Lambda_N = \text{diag}(\lambda_{1,1}, \ldots, \lambda_{m,N}) \int K_{1}^2(z)dz \). Then we have

\[
\sum_{j=1}^{N} v_{m,j} \frac{1}{\sqrt{T h}} \sum_{t=1}^{T} K_{1,t} \left( u - \frac{t}{T} \right) \langle X_{m,t}^{(u)}, v_{m,j} \rangle \xrightarrow{d} \left( \int K_{1}^2(z)dz \right)^{1/2} \sum_{j=1}^{N} \lambda_{m,j}^{1/2} Z_j v_{m,j} \text{ in } L^2,
\] (A.15)

where \( Z_j \) are independent standard Gaussian random variables. It is easy to see that

\[
\int \left( \sum_{j>N}^{N} \lambda_{m,j}^{1/2} Z_j v_{m,j} \right)^2 ds = \sum_{j>N}^{N} \lambda_{m,j} Z_j^2 \xrightarrow{p} 0 \text{ as } N \to \infty. \tag{A.16}
\]

Combining Theorem 3.2 in Billingsley (1999) with (A.14)-(A.16), we have

\[
\frac{1}{\sqrt{T h}} \sum_{t=1}^{T} K_{1,t} \left( u - \frac{t}{T} \right) X_{m,t}^{(u)} \xrightarrow{d} \sum_{j=1}^{N} \lambda_{m,j}^{1/2} Z_j v_{m,j} \text{ in } L^2
\]

for any \( m \geq 1 \). Since \( \sum_{j=1}^{\infty} \lambda_{m,j}^{1/2} Z_j v_{m,j} \) has the same distribution as \( G_m^{(u)} \), we obtain (A.10).

(Step5) In this step, we will show (A.11). Since \( G_m^{(u)} \) is a zero-mean Gaussian process, it is sufficient to show

\[
\int \int (c_m^{(u)}(s_1, s_2) - c_m^{(u)}(s_1, s_2))^2 ds_1 ds_2 \to \infty \text{ as } m \to \infty. \tag{A.17}
\]

First, we show that the kernel \( c^{(u)} \) converge in \( L^2([0,1]^d \times [0,1]^d) \). Observe that

\[
(c_m^{(u)}(s_1, s_2))^2 \leq 4 \left( \text{E}[X_0^{(u)}(s_1)X_0^{(u)}(s_2)] \right)^2 + \left( \sum_{i=1}^{m} \text{E}[X_0^{(u)}(s_1)X_i^{(u)}(s_2)] \right)^2
\]

\[
= \left( \sum_{i=1}^{m} \text{E}[X_0^{(u)}(s_2)X_i^{(u)}(s_1)] \right)^2
\]

\[
=: 4 (C_{m,1}(s_1, s_2) + C_{m,2}(s_1, s_2) + C_{m,3}(s_1, s_2)) =: \tilde{c}_m(s_1, s_2).
\]

For \( C_{m,1}(s_1, s_2) \),

\[
\int \int C_{m,1}(s_1, s_2) ds_1 ds_2 \leq \int \int \text{E}[X_0^{(u)}(s_1)]^2 \text{E}[X_0^{(u)}(s_2)]^2 ds_1 ds_2 = v_2^4(X_0^{(u)}). \tag{A.18}
\]
For $C_{m,2}(s_1, s_2)$, applying the Cauchy-Schwarz inequality yields

$$\int \int C_{m,2}(s_1, s_2) ds_1 ds_2 \leq \sum_{t_1=1}^{m} \sum_{t_2=1}^{m} \int \int \left| E[X_0^{(u)}(s_1)X_{t_1}^{(u)}(s_2)] - E[X_0^{(u)}(s_1)X_{t_2}^{(u)}(s_2)] \right| ds_1 ds_2 \leq \sum_{t_1=1}^{m} \sum_{t_2=1}^{m} \int \int \left| E[X_0^{(u)}(s_1)(X_{t_1}^{(u)}(s_2) - X_{t_2}^{(u)}(s_2))] - E[X_0^{(u)}(s_1)(X_{t_2}^{(u)}(s_2) - X_{t_1}^{(u)}(s_2))] \right| ds_1 ds_2 \leq \sum_{t_1=1}^{m} \sum_{t_2=1}^{m} \int \int \left| E[(X_0^{(u)}(s_1))^2] - E[(X_{t_1}^{(u)}(s_2) - X_{t_2}^{(u)}(s_2))^2] \right|^{1/2} ds_1 ds_2 \leq v_2(X_0^{(u)}) \left( \sum_{t=1}^{\infty} v_2(X_0^{(u)} - X_{t,0}^{(u)}) \right)^2 < \infty. \quad (A.19)$$

Likewise,

$$\int \int C_{m,3}(s_1, s_2) ds_1 ds_2 \leq v_2(X_0^{(u)}) \left( \sum_{t=1}^{\infty} v_2(X_0^{(u)} - X_{t,0}^{(u)}) \right)^2. \quad (A.20)$$

Then we have $\lim_{m \to \infty} \int \int c_m^{(u)}(s_1, s_2) ds_1 ds_2 < \infty$.

Combining (A.19) and (A.20) and applying the monotone convergence theorem, we have

$$\int \int (\tilde{c}_m^{(u)}(s_1, s_2))^2 ds_1 ds_2 = \int \int \lim_{m \to \infty} (c_m^{(u)}(s_1, s_2))^2 ds_1 ds_2 \leq \int \int \lim_{m \to \infty} \tilde{c}_m^{(u)}(s_1, s_2) ds_1 ds_2 = \lim_{m \to \infty} \int \int \tilde{c}_m^{(u)}(s_1, s_2) ds_1 ds_2 < \infty. \quad (A.21)$$

Applying almost the same argument to show (A.21), we can show (A.17). Therefore, we complete the proof. \qed

**Proof of Theorem 2.3** (Step1) In this step, we give a sketch of the proof.

In Step2, we will show

$$\int \int \left( \tilde{\gamma}_{0}^{(u)}(s_1, s_2) - E[X_0^{(u)}(s_1)X_0^{(u)}(s_2)] \right)^2 ds_1 ds_2 = o_P(1). \quad (A.22)$$

Define

$$\gamma_{t,11}(s_1, s_2) = \frac{1}{T_h} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{j}^{(u)}(s_1)X_{j}^{(u)}(s_2).$$

In Step3, we will show

$$\int \int \left( \sum_{t=1}^{T-1} K_{2}(t/b)\tilde{\gamma}_{t}^{(u)}(s_1, s_2) - \sum_{t=1}^{T-1} K_{2}(t/b)\gamma_{t,11}(s_1, s_2) \right)^2 ds_1 ds_2 = o_P(1). \quad (A.23)$$
In Step 4, we will show
\[
\int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,11}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 = o_P(1), \tag{A.24}
\]
where \(c_1^{(u)}(s_1, s_2) = \sum_{t=1}^{\infty} E[X_0^{(u)}(s_2)X_t^{(u)}(s_1)]\). Hence, \(A.23\) and \(A.24\) yields
\[
\int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \tilde{\gamma}_t^{(u)}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 = o_P(1). \tag{A.25}
\]
Likewise, we can show
\[
\int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \tilde{\gamma}_t^{(u)}(s_2, s_1) - c_1^{(u)}(s_2, s_1) \right)^2 ds_1 ds_2 = o_P(1). \tag{A.26}
\]
Combining \(A.22\), \(A.25\) and \(A.26\), we complete the proof.

(Step 2) In this step, we will show
\[
\int \int \left( \tilde{\gamma}_0^{(u)}(s_1, s_2) - E[X_0^{(u)}(s_1)X_0^{(u)}(s_2)] \right)^2 ds_1 ds_2 = o_P(1). \tag{A.27}
\]
Decompose
\[
\tilde{\gamma}_0^{(u)}(s_1, s_2) = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \left( X_{t,T}(s_1) - \bar{X}_T^{(u)}(s_1) \right) \left( X_{t,T}(s_2) - \bar{X}_T^{(u)}(s_2) \right)
\]
\[
= \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_{t,T}(s_1)X_{t,T}(s_2) - \bar{X}_T^{(u)}(s_1)\bar{X}_T^{(u)}(s_2)
\]
\[
= \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_t^{(u)}(s_1)X_t^{(u)}(s_2)
\]
\[
+ \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) X_t^{(u)}(s_1)(X_{t,T}(s_2) - X_t^{(u)}(s_2))
\]
\[
+ \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_{t,T}(s_1) - X_t^{(u)}(s_1))X_t^{(u)}(s_2)
\]
\[
+ \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) (X_{t,T}(s_1) - X_t^{(u)}(s_1))(X_{t,T}(s_2) - X_t^{(u)}(s_2))
\]
\[
- \bar{X}_T^{(u)}(s_1)\bar{X}_T^{(u)}(s_2)
\]
\[
= : \gamma_{0,1}(s_1, s_2) + \gamma_{0,2}(s_1, s_2) + \gamma_{0,3}(s_1, s_2) + \gamma_{0,4}(s_1, s_2) - \bar{X}_T^{(u)}(s_1)\bar{X}_T^{(u)}(s_2).
\]
Define
\[
w_T^{(u)} = \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right), \quad w_{t,T}^{(u)} = \frac{K_{1,h}(u - t/T)}{\sum_{t=1}^{T} K_{1,h}(u - t/T)}.
\]
Note that $\sum_{t=1}^{T} w_{t,T}^{(u)} = 1$. Observe that

$$
\int \int \left\{ \zeta_{0}^{(u)}(s_1, s_2) - E[X_{0}^{(u)}(s_1)X_{0}^{(u)}(s_2)] \right\}^2 ds_1 ds_2 \\
\leq 32 \int \int \left\{ \gamma_{0,1}^{(u)}(s_1, s_2) - w_{T}^{(u)} E[X_{0}^{(u)}(s_1)X_{0}^{(u)}(s_2)] \right\}^2 ds_1 ds_2 + 32 \int \int \left( \gamma_{0,2}^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 \\
+ 32 \int \int \left( \gamma_{0,3}^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 + 32 \int \int \left( \gamma_{0,4}^{(u)}(s_1, s_2) \right)^2 ds_1 ds_2 \\
+ 32 \int \int \left( (1 - w_{T}^{(u)}) E[X_{0}^{(u)}(s_1)X_{0}^{(u)}(s_2)] \right)^2 ds_1 ds_2 + 32 \int \int \left( X_{T}^{(u)}(s_1)X_{T}^{(u)}(s_2) \right)^2 ds_1 ds_2 \\
= \sum_{j=1}^{6} \Gamma_{j,T}^{(u)}.
$$

For $\Gamma_{6,T}^{(u)}$, applying Theorem 8.2 we have

$$
\Gamma_{6,T}^{(u)} = 32 \left( \int \left( X_{T}^{(u)}(s) \right)^2 ds \right)^2 = \| X_{T}^{(u)} \|^4 = O_P \left( \frac{1}{T^2 h^2} \right) . \quad (A.28)
$$

For $\Gamma_{5,T}^{(u)}$, applying Lemma 1.3 we have

$$
\Gamma_{5,T}^{(u)} = 32(1 - w_{T}^{(u)})^2 \int \int \left( E[X_{0}^{(u)}(s_1)X_{0}^{(u)}(s_2)] \right)^2 ds_1 ds_2 \\
\leq 32(1 - w_{T}^{(u)})^2 \left( E \left[ \int (X_{0}^{(u)}(s))^2 ds \right] \right)^2 \lesssim o \left( h^2 \right) . \quad (A.29)
$$

For $\Gamma_{2,T}^{(u)}$, observe that

$$
E \left[ \Gamma_{2,T}^{(u)} \right] \\
\lesssim \frac{1}{(Th)^2} \sum_{t=1}^{T} K_{t,h}^2 \left( u - \frac{t}{T} \right) E \left[ \left( \int (X_{t}^{(u)}(s_1))^2 ds_1 \right) \left( \int (X_{t,T}(s_2) - X_{t}^{(u)}(s_2))^2 ds_2 \right) \right] \\
+ \frac{1}{(Th)^2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} K_{1,h} \left( u - \frac{t_1}{T} \right) K_{1,h} \left( u - \frac{t_2}{T} \right) \\
\times E \left[ \left( \int X_{t_1}^{(u)}(s_1)X_{t_2}^{(u)}(s_1) ds_1 \right) \left( \int (X_{t_1,T}(s_2) - X_{t_1}^{(u)}(s_2))(X_{t_2,T}(s_1) - X_{t_2}^{(u)}(s_1)) ds_2 \right) \right].
$$

Note that

$$
E \left[ \left( \int (X_{t}^{(u)}(s_1))^2 ds_1 \right) \left( \int (X_{t,T}(s_2) - X_{t}^{(u)}(s_2))^2 ds_2 \right) \right] = E \left[ \| X_{t}^{(u)} \|^2 \| X_{t,T} - X_{t}^{(u)} \|^2 \right] \\
\leq \left( h + \frac{1}{T} \right)^2 E \left[ \| X_{t}^{(u)} \|^2 (U_{t,T}^{(u)})^2 \right] \\
\lesssim h^2 v_{3}^{2}(X_{0}^{(u)})
$$

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Then we have

\[ |E \left( \left( \int X_{t_1}^{(u)}(s_1)X_{t_2}^{(u)}(s_1)ds_1 \right) \left( \int (X_{t_1,T}(s_2) - X_{t_1}^{(u)}(s_2))(X_{t_2,T}(s_1) - X_{t_2}^{(u)}(s_1))ds_2 \right) \right) | \]

\[ \leq E \left[ \|X_{t_1}^{(u)}\|\|X_{t_2}^{(u)}\|\|X_{t_1,T} - X_{t_1}^{(u)}\|\|X_{t_2,T} - X_{t_2}^{(u)}\| \right] \]

\[ \leq \left( h + \frac{1}{T} \right)^2 E \left[ \|X_{t_1}^{(u)}\|\|X_{t_2}^{(u)}\|\|U_{t_1,T}^{(u)}U_{t_2,T}^{(u)} \right] \lesssim h^2 e^2(X_0^{(u)}). \]

Then we have

\[ E [\Gamma_{2,T}^{(u)}] \lesssim \frac{h^2}{Th} \left( \frac{1}{Th} \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) \right) + h^2 \left( \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \right)^2 \lesssim h^2. \] (A.30)

Likewise, we have

\[ E [\Gamma_{3,T}^{(u)}] \lesssim h^2, \quad E [\Gamma_{4,T}^{(u)}] \lesssim h^4. \] (A.31)

For \( \Gamma_{1,T}^{(u)} \), applying the ergodic theorem for weighted random variables in a Hilbert space (see Hanson and Pledger (1969) for example),

\[ \Gamma_{1,T}^{(u)} = 32 \left( w_T^{(u)} \right)^2 \int \int \left\{ (w_T^{(u)})^{-1} \gamma_{0,1}^{(u)}(s_1,s_2) - E[X_0^{(u)}(s_1)X_0^{(u)}(s_2)] \right\}^2 ds_1 ds_2 = o_P(1). \] (A.32)

Combining (A.28) - (A.32), we obtain (A.27).

(Step3) Define

\[ \gamma_{t,11}(s_1,s_2) = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1)X_{j-1}^{(u)}(s_2). \]

In this step, we will show

\[ \int \int \left( \sum_{t=1}^{T-1} K_2(t/b)\tilde{\gamma}_{t}^{(u)}(s_1,s_2) - \sum_{t=1}^{T-1} K_2(t/b)\gamma_{t,11}(s_1,s_2) \right)^2 ds_1 ds_2 = o_P(1). \] (A.33)

Observe that

\[ \tilde{\gamma}_{t}^{(u)}(s_1,s_2) = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{j,T}(s_1)X_{j-T,T}(s_2) \]

\[ - \left\{ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{j,T}(s_1) \right\} \bar{X}_T^{(u)}(s_2) \]

\[ - \left\{ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{j-T,T}(s_2) \right\} \bar{X}_T^{(u)}(s_1) \]

\[ + \left\{ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \right\} \bar{X}_T^{(u)}(s_1)\bar{X}_T^{(u)}(s_2) \]

\[ =: \gamma_{t,1}(s_1,s_2) - \gamma_{t,2}(s_1,s_2) - \gamma_{t,3}(s_1,s_2) + \gamma_{t,4}(s_1,s_2). \]
For \( \gamma_{t,1}(s_1, s_2) \), decompose

\[
\gamma_{t,1}(s_1, s_2) = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1)X_{j-t}^{(u)}(s_2) \\
+ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j,T}(s_1) - X_j^{(u)}(s_1))X_{j-t}^{(u)}(s_2) \\
+ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j-t,T}(s_2) - X_j^{(u)}(s_2))X_j^{(u)}(s_1) \\
+ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j,T}(s_1) - X_j^{(u)}(s_1))(X_{j-t,T}(s_2) - X_j^{(u)}(s_2)) \\
=: \gamma_{t,11}(s_1, s_2) + \gamma_{t,12}(s_1, s_2) + \gamma_{t,13}(s_1, s_2) + \gamma_{t,14}(s_1, s_2).
\]

For \( \gamma_{t,11}(s_1, s_2) \), using the triangle inequality, we have

\[
E \left[ \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,12}(s_1, s_2) \right)^2 \, ds_1 ds_2 \right)^{1/2} \right] \\
\leq \sum_{t=1}^{T-1} |K_2(t/b)| E \left[ \left( \int \int (\gamma_{t,12}(s_1, s_2))^2 \, ds_1 ds_2 \right)^{1/2} \right].
\]

Since we have

\[
E \left[ \int \int (\gamma_{t,12}(s_1, s_2))^2 \, ds_1 ds_2 \right] \\
\leq \frac{1}{(Th)^2} \sum_{j=t+1}^{T} K_{1,h}^2 \left( u - \frac{j}{T} \right) E \left[ \int \int (X_{j,T}(s_1) - X_j^{(u)}(s_1))^2(X_j^{(u)}(s_2))^2 \, ds_1 ds_2 \right] \\
+ \frac{1}{(Th)^2} \sum_{j_1=t+1}^{T} \sum_{j_2=t+1}^{T} K_{1,h} \left( u - \frac{j_1}{T} \right) K_{1,h} \left( u - \frac{j_2}{T} \right) \\
\times \left| E \left[ \int \int (X_{j_1,T}(s_1) - X_{j_1}^{(u)}(s_1))(X_{j_2,T}(s_1) - X_{j_2}^{(u)}(s_1))(X_{j_1-t}^{(u)}(s_2)X_{j_2-t}^{(u)}(s_2) \, ds_1 ds_2 \right] \right| \\
\lesssim \frac{h^2}{Th} \left( \sum_{t=1}^{T} K_{1,h}^2 \left( u - \frac{t}{T} \right) \right) + h^2 \left( \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \right)^2 \lesssim h^2,
\]

we then have

\[
E \left[ \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,12}(s_1, s_2) \right)^2 \, ds_1 ds_2 \right] \lesssim h^2 \sum_{t=1}^{T-1} |K_2(t/b)| \lesssim h^2 b \to 0. \quad (A.34)
\]

Likewise, we can show

\[
E \left[ \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,13}(s_1, s_2) \right)^2 \, ds_1 ds_2 \right] \lesssim h^2 \sum_{t=1}^{T-1} |K_2(t/b)| \lesssim h^2 b \to 0 \quad (A.35)
\]

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and
\[
E \left[ \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,14}(s_1, s_2) \right)^2 ds_1 ds_2 \right] \lesssim h^2 \sum_{t=1}^{T-1} |K_2(t/b)| \lesssim h^4 b \to 0. \tag{A.36}
\]

(Step 3-2) For \( \gamma_{t,2}(s_1, s_2) \), decompose
\[
\gamma_{t,2}(s_1, s_2) = \left\{ \begin{array}{l}
\frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1) \\
+ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j,T}(s_1) - X_j^{(u)}(s_1)) \end{array} \right\} \bar{X}_T^{(u)}(s_2)
\]
\[
=: \gamma_{t,21}(s_1, s_2) + \gamma_{t,22}(s_1, s_2).
\]

For \( \gamma_{t,21}(s_1, s_2) \),
\[
\int \int (\gamma_{t,21}(s_1, s_2))^2 ds_1 ds_2 = \left( \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1) \right)^2 ds_1 \right) \left( \int (\bar{X}_T^{(u)}(s_2))^2 ds_2 \right).
\tag{A.37}
\]

Since
\[
E \left[ \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s) \right)^2 \right] = \frac{1}{(Th)^2} \sum_{j=t+1}^{T} K_{1,h}^2 \left( u - \frac{j}{T} \right) E \left[ \int (X_j^{(u)}(s))^2 ds \right]
\]
\[
+ \frac{2}{(Th)^2} \sum_{t+1 \leq j_1 < j_2 \leq T} K_{1,h} \left( u - \frac{j_1}{T} \right) K_{1,h} \left( u - \frac{j_2}{T} \right) E \left[ \int X_{j_1}^{(u)}(s) X_{j_2}^{(u)}(s) ds \right]
\]
\[
= \frac{v_2^2(X_0^{(u)})(Th)^2}{(Th)^2} \sum_{j=t+1}^{T} K_{1,h}^2 \left( u - \frac{j}{T} \right)
\]
\[
+ \frac{2}{(Th)^2} \sum_{t+1 \leq j_1 < j_2 \leq T} K_{1,h} \left( u - \frac{j_1}{T} \right) K_{1,h} \left( u - \frac{j_2}{T} \right) E \left[ \int X_{j_1}^{(u)}(s) X_{j_2}^{(u)}(s) ds \right]
\]
\[
\lesssim \frac{1}{(Th)^2} \sum_{j=t+1}^{T} K_{1,h}^2 \left( u - \frac{j}{T} \right)
\]
\[
+ \frac{1}{(Th)^2} \sum_{j=1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \sum_{k \geq 1} v_2(X_0^{(u)})v_2(X_0^{(u)} - X_{k,0}^{(u)}) = O \left( \frac{1}{Th} \right),
\]
we then have
\[
\max_{1 \leq t \leq T} E \left[ \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s) \right)^2 ds \right] = O \left( \frac{1}{Th} \right). \tag{A.38}
\]
Combining (A.37) and (A.38), we have
\[ E \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,21}(s_1, s_2) \right)^2 ds_1 ds_2 \right)^{1/2} \]
\[ \leq \sum_{t=1}^{T-1} |K_2(t/b)| \max_{1 \leq t \leq T} E \left[ \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s) \right)^2 ds \right]^{1/2} \leq \frac{b}{Th} \to 0. \]  
(A.39)

Likewise, we can show
\[ E \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,22}(s_1, s_2) \right)^2 ds_1 ds_2 \right)^{1/2} \lesssim \sqrt{\frac{b^2 h}{T}} \to 0. \]  
(A.40)

(Step-3) For \( \gamma_{t,3}(s_1, s_2) \), decompose
\[ \gamma_{t,3}(s_1, s_2) = \left\{ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_2) \right\} \tilde{X}_T^{(u)}(s_1) \]
\[ + \left\{ \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j-t,T}(s_2) - X_{j-t}^{(u)}(s_2)) \right\} \tilde{X}_T^{(u)}(s_1) \]
\[ =: \gamma_{t,31}(s_1, s_2) + \gamma_{t,32}(s_1, s_2). \]

For \( \gamma_{t,32}(s_1, s_2) \), applying almost the same argument to show (A.34), we have
\[ E \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,32}(s_1, s_2) \right)^2 ds_1 ds_2 \right)^{1/2} \]
\[ \leq \sum_{t=1}^{T-1} |K_2(t/b)| E \left[ \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_{j-t,T}(s) - X_{j-t}^{(u)}(s)) \right)^2 ds \right]^{1/2} \leq \frac{b}{Th} \to 0. \]  
(A.41)

Likewise, applying almost the same argument to show (A.39), we have
\[ E \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,31}(s_1, s_2) \right)^2 ds_1 ds_2 \right)^{1/2} \]
\[ \leq \sum_{t=1}^{T-1} |K_2(t/b)| \max_{1 \leq t \leq T} E \left[ \int \left( \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s) \right)^2 ds \right]^{1/2} \leq \frac{b}{Th} \to 0. \]  
(A.42)
Then we have

\[
\left( \int \int (\gamma_{t,4}(s_1, s_2))^2 \, ds_1 \, ds_2 \right)^{1/2} \leq \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \| X_T^{(u)} \|^4
\]

\[
\leq \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \| X_T^{(u)} \|^2 \lesssim \| X_T^{(u)} \|^2.
\]

Then we have

\[
E \left[ \left( \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,4}(s_1, s_2) \right)^2 \, ds_1 \, ds_2 \right)^{1/2} \right] \leq \sum_{t=1}^{T-1} |K_2(t/b)| E[\| X_T^{(u)} \|^2] \lesssim \frac{b}{Th} \to 0.
\]

(Step3-4) For \( \gamma_{t,4}(s_1, s_2) \), we have

\[
\left( \int \int (\gamma_{t,4}(s_1, s_2))^2 \, ds_1 \, ds_2 \right)^{1/2} = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \| X_T^{(u)} \|^4
\]

\[
\leq \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \| X_T^{(u)} \|^2 \lesssim \| X_T^{(u)} \|^2.
\]

(Step4) In this step, we will show

\[
\int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,11}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2 = o_P(1).
\]

Define

\[
\gamma_{t,11}^{(m)}(s_1, s_2) = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{m,j}^{(u)}(s_1) X_{m,j-t}^{(u)}(s_2),
\]

\[
c_1^{(u)}(s_1, s_2) = \sum_{t=1}^{\infty} E[X_0^{(u)}(s_2) X_t^{(u)}(s_1)],
\]

\[
c_1^{(u)}(s_1, s_2) = \sum_{t=1}^{m} E[X_{m,0}^{(u)}(s_2) X_{m,t}^{(u)}(s_1)],
\]

where \( \omega_{t,T}^{(u)} = \frac{1}{Th} \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) \). Note that \( \max_{1 \leq t \leq T-1} \omega_{t,T}^{(u)} \approx 1 \) uniformly in \( u \). Let \( m \geq 1 \) be a fixed constant. Observe that

\[
\int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,11}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2
\]

\[
\leq 64 \int \int \left( \sum_{t=1}^{T-1} K_2(t/b) \gamma_{t,11}^{(m)}(s_1, s_2) - \gamma_{t,11}^{(m)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2
\]

\[
+ 64 \int \int \left( \sum_{t=m+1}^{T-1} K_2(t/b) \gamma_{t,11}^{(m)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2 + 64 \int \int \left( \sum_{t=1}^{m} (K_2(t/b) - 1) \gamma_{t,11}^{(m)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2
\]

\[
+ 64 \int \int \left( \sum_{t=1}^{m} \gamma_{t,11}^{(m)}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2 + 64 \int \int \left( c_1^{(u)}(s_1, s_2) - c_1^{(u)}(s_1, s_2) \right)^2 \, ds_1 \, ds_2
\]

\[
= 64(q_{1,T} + q_{2,T} + q_{3,T} + q_{4,T} + q_{5,T} + q_{6,T}).
\]
(Step 4-1) For $q_{6,T}$, since $\int \int (c_m^{(u)}(s_1,s_2) - c_1^{(u)}(s_1,s_2))^2 ds_1 ds_2 \to 0$ as $m \to \infty$, we have

$$q_{5,T} = \int \int (c_1^{(u)}(s_1,s_2) - c_1^{(u)}(s_1,s_2))^2 ds_1 ds_2 \to 0 \text{ as } m \to \infty. \quad (A.45)$$

(Step 4-2) For $q_{5,T}$,

$$\frac{1}{2} q_{5,T} \leq \sum_{t=1}^{m} |\omega_{t,T}^{(u)} - 1| \left( \int \int (c_1^{(u)}(s_1,s_2))^2 ds_1 ds_2 \right)^{1/2} \to 0 \text{ as } T \to \infty \quad (A.46)$$

since $\lim_{T \to \infty} |\omega_{t,T}^{(u)} - 1|$ for each $1 \leq t \leq m$.

(Step 4-3) In this step, we will show that for every $m \geq 1$, $q_{4,T} = o_P(1)$ as $T \to \infty$. Applying the ergodic theorem for weighted random variables in a Hilbert space, for $1 \leq t \leq m$, we have

$$\omega_{t,T}^{(u)} = \int \int \left( (\omega_{t,T}^{(u)})_{(t_{11})}^{(m)}(s_1,s_2) - E[X_{0_m}^{(u)}(s_1)X_{m,t}^{(u)}(s_2)] \right)^2 ds_1 ds_2 = o_P(1).$$

Then using the triangular inequality, we have

$$\frac{1}{2} q_{4,T} \leq \sum_{t=1}^{m} \left| K_2(t/b) - 1 \right| \left( \int \int \left( \gamma_{t_{11}}^{(m)}(s_1,s_2) \right)^2 ds_1 ds_2 \right)^{1/2}$$

$$\leq \sum_{t=1}^{m} \left| K_2(t/b) - 1 \right| \left| \omega_{t,T}^{(u)} \right| \left( \int \int \left( E[X_{0_m}^{(u)}(s_2)X_{m,t}^{(u)}(s_1)] \right)^2 ds_1 ds_2 \right)^{1/2}$$

$$+ \sum_{t=1}^{m} \left| K_2(t/b) - 1 \right| \left| \omega_{t,T}^{(u)} \right| \left( \int \int \left( (\omega_{t,T}^{(u)})_{(t_{11})}^{(m)}(s_1,s_2) - E[X_{0_m}^{(u)}(s_2)X_{m,t}^{(u)}(s_1)] \right)^2 ds_1 ds_2 \right)^{1/2}.$$ 

Assumption 5.4 yields that $\max_{1 \leq t \leq m} |K_2(t/b) - 1| \to 0$ as $T \to \infty$. Then combining the results in Steps 4-2 and 4-3, we have

$$q_{3,T} = o_P(1) \text{ as } T \to \infty. \quad (A.48)$$

(Step 4-5) Let $[x]$ denote the integer part of $x \in \mathbb{R}$. For $q_{2,T}$, observe that

$$E[q_{2,T}]$$

$$= \sum_{t_1 = m+1}^{C} \sum_{t_2 = m+1}^{C} K_2(t_1/b) K_2(t_2/b) \frac{1}{(Th)^2} \sum_{j_1 = t_1+1}^{T} \sum_{j_2 = t_2+1}^{T} K_{1,h} \left( u - \frac{j_1}{T} \right) K_{1,h} \left( u - \frac{j_2}{T} \right)$$

$$\times \int \int E \left[ X_{m,j_1}^{(u)}(s_1) X_{m,j_1-t_1}^{(u)}(s_1) X_{m,j_2}^{(u)}(s_2) X_{m,j_2-t_2}^{(u)}(s_2) \right] ds_1 ds_2.$$

Note that $\{X_{m,t}^{(u)}\}_{t \in \mathbb{Z}}$ is an $m$-dependent sequence and $t_1, t_2 \geq m+1$. Then $X_{m,j_1}^{(u)}$ and $X_{m,j_1-t_1}^{(u)} (X_{m,j_2}^{(u)}$ and $X_{m,j_2-t_2}^{(u)}$) are independent. This implies that the number of terms when the term $E \left[ X_{m,j_1}^{(u)}(s_1) X_{m,j_1-t_1}^{(u)}(s_1) X_{m,j_2}^{(u)}(s_2) X_{m,j_2-t_2}^{(u)}(s_2) \right]$ is not zero is $O(bTh)$. Consequently,

$$E[q_{2,T}] \leq \frac{b}{Th} \to 0 \text{ as } T \to \infty. \quad (A.49)$$
(Step 4-6) In this step, we will show that for any \( r > 0 \),
\[
\lim_{{m \to \infty}} \lim_{{T \to \infty}} P(q_{1,T} > r) = 0. \tag{A.50}
\]

Observe that
\[
\sum_{{t=1}}^{T-1} K_2(t/b) (\gamma_{{t,11}}(s_1, s_2) - \gamma_{{t,11}}^{(m)}(s_1, s_2))
\]
\[
= \frac{1}{Th} \sum_{{t=1}}^{T-1} K_2(t/b) \sum_{{j=t+1}}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_j^{(u)}(s_1) X_{j-t}(s_2) - X_{{m,j}}^{(u)}(s_1) X_{{m,j-t}}^{(u)}(s_2))
\]
\[
= \frac{1}{Th} \left( \sum_{{t=1}}^{m} + \sum_{{t=m+1}}^{[C_2b]+1} \right) K_2(t/b) \sum_{{j=t+1}}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_j^{(u)}(s_1) X_{j-t}^{(u)}(s_2) - X_{{m,j}}^{(u)}(s_1) X_{{m,j-t}}^{(u)}(s_2))
\]
\[
= q_{{11,T}}^{(m)}(s_1, s_2) + q_{{12,T}}^{(m)}(s_1, s_2).
\]

Decompose
\[
X_j^{(u)}(s_1) X_{j-t}^{(u)}(s_2) - X_{{m,j}}^{(u)}(s_1) X_{{m,j-t}}^{(u)}(s_2)
\]
\[
= (X_j^{(u)}(s_1) - X_{{m,j}}^{(u)}(s_1)) X_{j-t}^{(u)}(s_2) + (X_{j-t}^{(u)}(s_2) - X_{{m,j-t}}^{(u)}(s_2)) X_{{m,j}}^{(u)}(s_1).
\]

Then applying the triangular inequality, we have
\[
E \left\{ \int \int \left( \frac{1}{Th} \sum_{{t=1}}^{m} K_2(t/b) \sum_{{j=t+1}}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_j^{(u)}(s_1) - X_{{m,j}}^{(u)}(s_1)) X_{j-t}^{(u)}(s_2) \right)^2 ds_1 ds_2 \right\}^{1/2}
\]
\[
\leq \frac{1}{Th} \sum_{{t=1}}^{m} |K_2(t/b)| \sum_{{j=t+1}}^{T} K_{1,h} \left( u - \frac{j}{T} \right)
\times E \left[ \left( \int (X_j^{(u)}(s_1) - X_{{m,j}}^{(u)}(s_1))^2 ds_1 \right)^{1/2} \left( \int (X_{j-t}^{(u)}(s_2))^2 ds_2 \right)^{1/2} \right]
\]
\[
\lesssim \left( \frac{1}{Th} \sum_{{t=1}}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \right) m E \left[ \| X_0^{(u)} - X_{{m,0}}^{(u)} \| \right]^{1/2} E \left[ \| X_0^{(u)} \| \right]^{1/2}
\]
\[
\lesssim m v_2 (X_0^{(u)} - X_{{m,0}}^{(u)}) \to 0 \text{ as } m \to \infty. \tag{A.51}
\]

Likewise, we can show
\[
E \left\{ \int \int \left( \frac{1}{Th} \sum_{{t=1}}^{m} K_2(t/b) \sum_{{j=t+1}}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_j^{(u)}(s_1) - X_{{m,j}}^{(u)}(s_1)) X_{{m,j-t}}^{(u)}(s_1) \right)^2 ds_1 ds_2 \right\}^{1/2}
\]
\[
\lesssim m v_2 (X_0^{(u)} - X_{{m,0}}^{(u)}) \to 0 \text{ as } m \to \infty. \tag{A.52}
\]

Combining \( \text{(A.51)} \) and \( \text{(A.52)} \), we have
\[
\lim_{{m \to \infty}} \lim_{{T \to \infty}} P \left( \int \int (q_{{11,T}}^{(m)}(s_1, s_2))^2 ds_1 ds_2 > r_1 \right) = 0 \tag{A.53}
\]
for any \( r_1 > 0 \).
Further, decompose
\[ X_j^{(u)}(s_1)X_j^{(u)}(s_2) = (X_j^{(u)}(s_1) - X_{t,j}^{(u)}(s_1))X_j^{(u)}(s_2) + X_{t,j}^{(u)}(s_1)X_j^{(u)}(s_2). \]
Then we have
\[
E\left\{ \int \int \left( \frac{1}{Th} \sum_{t=m+1}^{[C_2/b]+1} K_2(t/b) \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) (X_j^{(u)}(s_1) - X_{t,j}^{(u)}(s_1))X_j^{(u)}(s_2) \right)^2 ds_1 ds_2 \right\}^{1/2}
\]
\[
\leq \frac{1}{Th} \sum_{t=m+1}^{[C_2/b]+1} |K_2(t/b)| \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right)
\times E\left[ \left( \int (X_j^{(u)}(s_1) - X_{t,j}^{(u)}(s_1))^2 ds_1 \right)^{1/2} \left( \int (X_j^{(u)}(s_2))^2 ds_2 \right)^{1/2} \right]
\]
\[
\leq \left( \frac{1}{Th} \sum_{t=1}^{T} K_{1,h} \left( u - \frac{t}{T} \right) \right) \sum_{t=m+1}^{[C_2/b]+1} E\left[ \|X_0^{(u)} - X_{t,0}^{(u)}\|^2 \right]^{1/2} E\left[ \|X_0^{(u)}\|^2 \right]^{1/2}
\]
\[
\leq \sum_{t=m+1}^{\infty} v_2(X_0^{(u)} - X_{m,0}^{(u)}) \to 0 \text{ as } m \to \infty. \tag{A.54}
\]
Applying almost the same argument to show (A.49), we have
\[
E\left\{ \int \int \left( \frac{1}{Th} \sum_{t=m+1}^{[C_2/b]+1} K_2(t/b) \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_{t,j}^{(u)}(s_1)X_j^{(u)}(s_2) \right)^2 ds_1 ds_2 \right\}^{1/2}
\]
\[
\leq \frac{b}{Th} \to 0 \text{ as } T \to \infty. \tag{A.55}
\]
Combining (A.54) and (A.55), we have
\[
\lim_{m \to \infty} \lim_{T \to \infty} P\left( \int \int \left( \frac{1}{Th} \sum_{t=m+1}^{T-1} K_2(t/b) \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1)X_j^{(u)}(s_2) \right)^2 ds_1 ds_2 > r_2 \right) = 0 \tag{A.56}
\]
for any \( r_2 > 0 \). Likewise, we can show
\[
\lim_{m \to \infty} \lim_{T \to \infty} P\left( \int \int \left( \frac{1}{Th} \sum_{t=m+1}^{T-1} K_2(t/b) \sum_{j=t+1}^{T} K_{1,h} \left( u - \frac{j}{T} \right) X_j^{(u)}(s_1)X_j^{(u)}(s_2) \right)^2 ds_1 ds_2 > r_2 \right) = 0 \tag{A.57}
\]
for any \( r_2 > 0 \). Combining (A.56) and (A.57), we have
\[
\lim_{m \to \infty} \lim_{T \to \infty} P\left( \int \int (q_{12,T}^{(m)}(s_1,s_2))^2 ds_1 ds_2 > r_1 \right) = 0 \tag{A.58}
\]
for any \( r_1 > 0 \). Consequently, (A.53) and (A.58) yield (A.50).

(Step 4-7) Combining (A.45), (A.46), (A.47), (A.48), (A.49), and (A.50) yield (A.44). \( \square \)
A.2. Proofs for Section 4. We omit the proofs of Propositions 4.1, 4.2 and 4.3 since they immediately follow from the results in Section 3.

Proof of Corollary 4.1. For $1 \leq j \leq q_0 - 1$, from (4.6) with $q = q_0$, we have
\[
\frac{\hat{\eta}_{u,j+1}}{\hat{\eta}_{u,j}} \xrightarrow{p} \frac{\eta_{u,j+1}}{\eta_{u,j}} > 0. \tag{A.59}
\]
For $j \geq q_0$, applying Lemma B.1, we have
\[
|\frac{\hat{\eta}_{u,j+1}}{\hat{\eta}_{u,q_0}}| \leq \|\hat{\Phi}^{(u)} - \Phi^{(u)}\|_S \xrightarrow{p} 0. \tag{A.60}
\]
Thus we have
\[
\frac{\hat{\eta}_{u,q_0}+1}{\hat{\eta}_{u,q_0}} \rightarrow 0. \tag{A.61}
\]
For $j > q_0$, (A.60) yields
\[
P\left(\frac{\hat{\eta}_{u,j}}{\hat{\eta}_{u,1}} < \varepsilon_0, \frac{\hat{\eta}_{u,j+1}}{\hat{\eta}_{u,1}} < \varepsilon_0\right) \rightarrow 1.
\]
Hence
\[
P\left(\frac{\hat{\eta}_{u,j+1}}{\hat{\eta}_{u,j}} = 0/0 = 1\right) \rightarrow 1. \tag{A.62}
\]
With (A.59), (A.61) and (A.62), we complete the proof. □

Appendix B. Auxiliary Lemmas

In this section, we provide some auxiliary lemmas used in the proofs of main results. Recall that $\mathcal{L}$ denotes the space of bounded (continuous) linear operators on $H$ with the norm
\[
\|\Psi\|_\mathcal{L} = \sup\{\|\Psi(x)\| : \|x\| \leq 1\}.
\]
Let $C_1, C_2 \in \mathcal{L}$ be two compact operators with the following singular value decomposition:
\[
C_1(x) = \sum_{j \geq 1} \lambda_{1,j} \langle x, v_{1,j} \rangle f_{1,j}, \quad C_2(x) = \sum_{j \geq 1} \lambda_{2,j} \langle x, v_{2,j} \rangle f_{2,j} \quad x \in H, \tag{B.1}
\]
where $\{\lambda_{1,j}\}$ and $\{\lambda_{2,j}\}$ are sequences of nonnegative constants, and $\{v_{1,j}\}$, $\{v_{2,j}\}$, $\{f_{1,j}\}$ and $\{f_{2,j}\}$ are orthonormal bases of $H$.

Lemma B.1 (Lemma 1.6 in Golberg, Golberg and Kaashoek (1990)). Suppose $C_1, C_2 \in \mathcal{L}$ are two compact operators with singular value decompositions (B.1). Then, for each $j \geq 1$, $|\lambda_{1,j} - \lambda_{2,j}| \leq \|C_1 - C_2\|_\mathcal{L}$.

Now we assume that a compact operator $C_2$ with the singular value decomposition (B.1) is symmetric and $C_2(v_{2,j}) = \lambda_{2,j} v_{2,j}$, that is, $f_{2,j} = v_{2,j}$. Note that any covariance operator $C$ satisfies these conditions. Define
\[
v_{2,j}' = c_{2,j} v_{2,j}, \quad c_{2,j} = \text{sign}(\langle v_{1,j}, v_{2,j} \rangle).
\]

Lemma B.2 (Lemma 2.3 in Horváth and Kokoszka (2012)). Suppose $C_1, C_2 \in \mathcal{L}$ are two compact operators with singular value decompositions (B.1). If $C_2$ is symmetric and $f_{2,j} = v_{2,j}$ in (B.1), and its eigenvalues satisfy $\lambda_{2,1} > \lambda_{2,2} > \cdots > \lambda_{2,q} > \lambda_{2,q+1}$, then
\[
\|v_{1,j} - v_{2,j}'\| \leq \frac{2\sqrt{2}}{\alpha_{2,j}} \|C_1 - C_2\|_\mathcal{L}, \quad 1 \leq j \leq q,
\]
where $\alpha_{2,1} = \lambda_{2,1} - \lambda_{2,2}$ and $\alpha_{2,j} = \min(\lambda_{2,j-1} - \lambda_{2,j}, \lambda_{2,j} - \lambda_{2,j+1}), \ 2 \leq j \leq q$.

The proof of following Lemma B.3 is straightforward and thus omitted.

**Lemma B.3.** Suppose that kernel $K_1$ satisfies Assumption [3.2. Then, $$\sup_{u \in [C_1 h, 1 - C_1 h]} \left| \frac{1}{Th} \sum_{t=1}^{T} K_{1,h}(u - \frac{t}{T}) - 1 \right| = O\left(\frac{1}{Th^2}\right) + o(h).$$

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