Correspondence theory on $p$-Fock spaces with applications to Toeplitz algebras

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Abstract

We prove several results concerning the theory of Toeplitz algebras over $p$-Fock spaces using a correspondence theory of translation invariant symbol and operator spaces. The most notable results are: The full Toeplitz algebra is the norm closure of all Toeplitz operators with bounded uniformly continuous symbols. This generalizes a result obtained by J. Xia in the case $p = 2$, which was proven by different methods. Further, we prove that every Toeplitz algebra which has a translation invariant $C^*$ subalgebra of the bounded uniformly continuous functions as its set of symbols is linearly generated by Toeplitz operators with the same space of symbols.

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1 Introduction

In recent years, the study of Toeplitz algebras over Bergman or Segal-Bargmann-Fock spaces has experienced significant interest. Usually, one tries to understand the structure of an algebra, say a $C^*$ algebra, generated by Toeplitz operators with symbols from a subset of $L^\infty$ having some common property, which gives access to a deeper study. We mention [3, 11, 22] and also refer to references therein. A joint approach in all those works was that they only deal with subalgebras of the full Toeplitz algebra. By the full Toeplitz algebra we mean the Banach algebra generated by all Toeplitz operators with $L^\infty$ symbols. In contrast to that, J. Xia in his paper [24] proved a remarkable result about the full Toeplitz algebra. Let us fix some notation before going into details.

By $F^p_t$ we denote the $p$-Fock space, i.e. the space of holomorphic functions on $\mathbb{C}^n$ which are $p$-integrable with respect to a certain Gaussian measure with a parameter $t > 0$. We will denote the full Toeplitz algebra on $F^p_t$ by $\mathcal{T}^p_t$. By $\mathcal{T}^{p,t}(S) \subseteq \mathcal{L}(F^p_t)$ we mean the Banach algebra generated by Toeplitz operators with symbols in $S \subseteq L^\infty(\mathbb{C}^n)$, by $\mathcal{T}^{p,t}_{lin}(S) \subseteq \mathcal{L}(F^p_t)$ the closed linear space generated by such Toeplitz operators and by $\mathcal{T}^{2,t}(S) \subseteq \mathcal{L}(F^2_t)$ the $C^*$ algebra generated by these operators. The result by J. Xia is now the following:

Theorem (24). The following holds true:

$$\mathcal{T}^{2,1}_{lin} = \mathcal{T}^{2,1}_{lin}(L^\infty(\mathbb{C}^n)).$$
We want to stress that Xia also proved an analogous result for the Toeplitz algebra on the Bergman space over the unit ball in \( \mathbb{C}^n \) in his paper. Further, with well-known methods it is possible to improve Xia’s result to \( \mathcal{T}_{lin}^{2,1} = \mathcal{T}_{lin}^{0,1}(BUC(\mathbb{C}^n)) \). Here, \( BUC(\mathbb{C}^n) \) is the space of all bounded and uniformly continuous functions on \( \mathbb{C}^n \).

While the assumption \( t = 1 \) was not crucial in Xia’s proof, the restriction to \( p = 2 \) was important. Our first main theorem will be an improvement of that result:

**Theorem A.** Let \( 1 < p < \infty \) and \( t > 0 \). Then, we have

\[
\mathcal{T}^{p,t} = \mathcal{T}_{lin}^{p,t}(BUC(\mathbb{C}^n))
\]

We want to stress that our approach to this problem gives a more constructive result on how to approximate operators from \( \mathcal{T}^{p,t} \) by Toeplitz operators than Xia’s proof, which was based on an abstract \( C^* \) algebraic argument. Our method of proof of Theorem A will lead us naturally to the study of translation-invariant algebras, both on the side of symbols and on the side of operator algebras. Here, we say that \( \mathcal{D}_0 \subset BUC(\mathbb{C}^n) \) is translation-invariant if \( f(-z) \in \mathcal{D}_0 \) for all \( f \in \mathcal{D}_0, z \in \mathbb{C}^n \). These investigations will lead us to our second main result. In the following, \( U \) is the operator \( Uf(z) = f(-z) \).

**Theorem B.** Let \( \mathcal{D}_0 \subset BUC(\mathbb{C}^n) \) be closed, translation- and \( U \)-invariant. Then, the following are equivalent:

(i) \( \mathcal{D}_0 \) is a \( C^* \) algebra with respect to the standard operations and \( L^\infty \) norm;
(ii) \( \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) \) for all \( t > 0 \).

If the above equivalent conditions are fulfilled, then we have \( \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0) \) for all \( 1 < p < \infty, t > 0 \).

If further \( \mathcal{D}_0 \) is a closed, translation- and \( U \)-invariant \( C^* \) subalgebra of \( BUC(\mathbb{C}^n) \) and \( \mathcal{I} \subset \mathcal{D}_0 \) is a closed, translation- and \( U \)-invariant subset of \( \mathcal{D}_0 \), then the following are equivalent:

(i*) \( \mathcal{I} \) is an ideal in \( \mathcal{D}_0 \);
(ii*) \( \mathcal{T}_{lin}^{2,t}(\mathcal{I}) \) is a one-sided ideal in \( \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) \) for all \( t > 0 \);
(iii*) \( \mathcal{T}_{lin}^{2,t}(\mathcal{I}) \) is a two-sided ideal in \( \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) \) for all \( t > 0 \).

Under these assumptions, \( \mathcal{T}_{lin}^{p,t}(\mathcal{I}) = \mathcal{T}^{p,t}(\mathcal{I}) \) is a closed and two-sided ideal in \( \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) \) for all \( 1 < p < \infty, t > 0 \).

Let us add some words on the background of the method we will use. In the paper [23], R. Werner introduced his concept of Quantum Harmonic Analysis, as he called it. By this, he in essence means a combination of two things: First of all, he introduced a concept of convolution between objects from \( L^1(\mathbb{R}^{2n}) \times \mathcal{N}(L^2(\mathbb{R}^n)) \) and objects from \( L^\infty(\mathbb{R}^{2n}) \times \mathcal{L}(L^2(\mathbb{R}^n)) \). Here, \( \mathcal{N}(L^2(\mathbb{R}^n)) \) denotes the trace class operators on the underlying Hilbert space. His concept of convolutions naturally extends the convolution between functions from \( L^1(\mathbb{R}^{2n}) \) and \( L^\infty(\mathbb{R}^{2n}) \). Using this convolution formalism (and a theory of regular operators, which we will not need in this work), he then studies a natural correspondence between certain subspaces of \( L^\infty(\mathbb{R}^{2n}) \) and of \( \mathcal{L}(L^2(\mathbb{R}^n)) \).
Based on ideas from that work, and extending them, we obtain the above results for Toeplitz algebras. Further ideas, which go into the proof of Theorem B are results on quantization estimates [1, 2] and limit operators [4], which have been studied out of independent interest and now fit nicely into our theory. Let us also mention that the correspondence theory which we study in this work gives rise to several other results, both old and new, with simple proofs. We name a view examples:

1. The characterization of compact operators from $L(F^p_t)$, which was first proven in [4], is derived.

2. Generalized Schatten-von Neumann classes $S^p_0(F^p_t)$ over $F^p_t$ are characterized as the closure of the space of Toeplitz operators with $L^p_0(\mathbb{C}^n)$ symbols.

3. We show that the algebras of Lagrangian-invariant Toeplitz operators are linearly generated. This was first obtained in [11].

The work is organized as follows: In part 2 we study Werner’s Quantum Harmonic Analysis in the setting of $p$-Fock spaces. First, we introduce the appropriate Banach space version of Schatten-von Neumann ideals in Section 2.1. In Section 2.2 we discuss some basics on $p$-Fock spaces and certain natural operators on them. In Section 2.3, we introduce the notion of convolutions as motivated by Werner’s work. In Section 2.4, we see that the well-known Toeplitz- and Berezin-maps can be nicely studied in Werner’s convolution formalism. In Section 2.5 we fix several results, which are part of Werner’s Quantum Harmonic Analysis (as formulated in our scope of $p$-Fock spaces) but have not been noted directly in [23]. They will come in handy later in our work. Section 2.6 concludes the development of Werner’s theory in our setting by introducing his concept of correspondence theory. Part 3 is then dedicated to applying the theory to Toeplitz operators. Section 3.1 discusses several immediate consequences of the correspondence theory to the theory of Toeplitz algebras, e.g. the proof of Theorem A. Sections 3.2 and 3.3 are finally dedicated to the proof of Theorem B, which uses the already mentioned quantization estimates and limit operator techniques.

We want to mention the following: Section 2.3 discusses Werner’s convolution formalism in some detail. Most of the theory can be immediately carried over from the original work, yet one must spend some care at certain points. Our presentation of the material in that section was closely inspired by the work [16], which also discusses Werner’s convolution formalism in the Schrödinger representation, but in closer detail. Section 2.6 follows the corresponding part of Werner’s initial work closely and is included for the readers convenience.

2 Convolution formalism and correspondence theory over $p$-Fock spaces

2.1 Schatten-von Neumann ideals

Let $X$ be a complex Banach space. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators on $X$. Recall that an operator $A \in \mathcal{L}(X)$ is nuclear,
if there are sequences \((x_j) \subset X, (y_j) \subset X'\) with \(\sum_{j=1}^{\infty} \|y_j\|_{X'} \|x_j\|_X < \infty\) such that

\[
A = \sum_{j=1}^{\infty} y_j \otimes x_j.
\] (2.1)

For such an operator we define

\[
\|A\|_{\mathcal{N}} := \inf \sum_{j=1}^{\infty} \|y_j\|_{X'} \|x_j\|_X,
\]

where the infimum is taken over all possible representations (2.1). We denote by \(\mathcal{N}(X)\) the set of all nuclear operators on \(X\). Together with the norm \(\| \cdot \|_{\mathcal{N}}\), this is well-known to be a Banach ideal in \(L(X)\). If the underlying Banach space \(X\) has the approximation property, we can define the nuclear trace for \(A \in \mathcal{N}(X)\) through

\[
\text{Tr}(A) = \sum_{j=1}^{\infty} y_j(x_j),
\]

where the trace is independent of the choice of representation (2.1), cf. [13, Theorem V.1.2]. From now on, we assume that \(X\) has the approximation property and is also reflexive. Then, one can show that the duality relations

\[
(\mathcal{K}(X))' = \mathcal{N}(X), \quad (\mathcal{N}(X))' = L(X)
\]

hold true, where the duality is induced by the trace map:

\[
\langle A, B \rangle = \text{Tr}(AB).
\]

Here, \(\mathcal{K}(X)\) denotes the compact operators. For details on the general theory of operator ideals, we refer to the books [8, 13, 17].

We will now deal with the method of complex interpolation. For an introduction to that topic, we refer to [6], from which we also take our notation. We want to interpolate between the spaces \(\mathcal{N}(X)\) and \(L(X)\). Since \(\mathcal{N}(X) \subset L(X)\), we can use the method of complex interpolation to obtain new ideals between \(\mathcal{N}(X)\) and \(L(X)\). Using \(L(X)\) as the ambient Hausdorff topological vector space, in which we embed the compatible couple \(\mathcal{A} := (L(X), \mathcal{N}(X))\), we get

\[
\Delta(\mathcal{A}) := \mathcal{N}(X) \cap L(X) = \mathcal{N}(X) \quad \text{and} \quad \Sigma(\mathcal{A}) := \mathcal{N}(X) + L(X) = L(X),
\]

where equalities are understood as normed vector spaces. Using the complex interpolation method, we obtain a family of subspaces of \(L(X)\):

\[
(L(X), \mathcal{N}(X))_{[\theta]} = (L(X), \mathcal{N}(X))_{[\theta]}\] for \(0 \leq \theta \leq 1\).

Since \(\mathcal{N}(X) \subset L(X)\), the family of interpolation spaces is decreasing [6, Theorem 4.2.1]:

\[
\theta_0 \leq \theta_1 : (L(X), \mathcal{N}(X))_{[\theta_0]} \supset (L(X), \mathcal{N}(X))_{[\theta_1]}.\]

Further, since \(\Delta(\mathcal{A}) = \mathcal{N}(X)\) is dense in \((L(X), \mathcal{N}(X))_{[\theta]}\) [6, Theorem 4.22], we obtain using the approximation property:

\[
(L(X), \mathcal{N}(X))_{[\theta]} = \mathcal{K}(X).
\]
We also have
\((L(X), N(X))_{[1]} = N(X)\).

With [6, Theorem 4.2.2] we obtain
\((L(X), N(X))_{[\theta]} = (K(X), N(X))_{[\theta]},\)

i.e. each interpolation space consists of compact operators. Further, since \(L(X)\) and \(N(X)\) are ideals, for each \(A \in L(X)\) we obtain maps (which we denote by the same symbol):
\[
L_A : L(X) \rightarrow L(X), \quad B \mapsto AB,
\]
\[
L_A : N(X) \rightarrow N(X), \quad B \mapsto AB.
\]
Interpolating this map, we obtain
\[
L_A : (L(X), N(X))_{[\theta]} \rightarrow (L(X), N(X))_{[1/p_0]}, B \mapsto AB,
\]
i.e. the interpolated spaces are left ideals. Analogously, they are right ideals. For \(1 \leq p_0 < \infty\), we define the ideals of compact operators \(S^{p_0}(X)\) by
\[
S^{p_0}(X) := (L(X), N(X))_{[1/p_0]}.
\]
In particular,
\[
S^1(X) = N(X).
\]
One can show the norm inequalities
\[
\|A\|_{op} \leq \|A\|_{S^{p_0}} \leq \|A\|_{S^{q_0}}
\]
for \(p_0 \geq q_0\), where \(\| \cdot \|_{op}\) denotes the operator norm on \(L(X)\). If \(X\) is a Hilbert space, these interpolated ideals are just the usual Schatten-von Neumann ideals [19, 21]. Surprisingly, it seems that no concrete description of the ideals \(S^{p_0}(X)\) is available if \(X\) is not a Hilbert space [18, Section 6.6.6.1].

### 2.2 \(p\)-Fock spaces

For \(t > 0\) consider the measure \(\mu_t\) on \(\mathbb{C}^n\) given by
\[
d\mu_t(z) = \frac{1}{(\pi t)^n} e^{-\frac{|z|^2}{t}} dV(z),
\]
where \(V\) denotes the Lebesgue measure on \(\mathbb{C}^n\) and \(| \cdot |\) is the Euclidean norm on \(\mathbb{C}^n \cong \mathbb{R}^{2n}\). \(\mu_t\) is well-known to be a probability measure. During the whole paper, let \(1 < p < \infty\). We consider the \(p\)-Fock space
\[
F^p_t := L^p(\mathbb{C}^n, \mu_{2t/\mu}) \cap \text{Hol}(\mathbb{C}^n).
\]
Each space \(F^p_t\) is a Banach space. The usual duality properties hold: For \(q > 1\) such that \(\frac{1}{p} + \frac{1}{q} = 1\) it holds \((F^p_t)^\prime \cong F^q_t\) under the usual duality pairing
\[
\langle f, g \rangle_{F^p_t} := \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\mu_t(z),
\]
and the norms of \((F^p_t)', F^p_t\) under this identification are equivalent. When there is no confusion about which duality pairing is meant, we will write \(\langle \cdot, \cdot \rangle\) instead of \(\langle \cdot, \cdot \rangle_{F^2_t}\). Of course, \((F^2_t, \langle \cdot, \cdot \rangle_{F^2_t})\) is a Hilbert space.

For \(\alpha \in \mathbb{N}^n_0\) we define \(e_\alpha \in F^p_t\) through

\[
e_\alpha(z) = \sqrt{\frac{1}{\alpha!}|z|^{\alpha}}.
\]

It is well-known that this is an orthonormal basis for \(F^2_t\). For general \(p\), it is easy to prove that this is still a Schauder basis using Taylor expansion around the origin for each \(f \in F^p_t\). Since every Banach space with Schauder basis has the approximation property, we obtain:

**Proposition 2.1.** \(F^p_t\) is a reflexive Banach space with approximation property.

In the following let \(t > 0\) and \(1 < p < \infty\) be fixed. For \(1 \leq p_0 < \infty\) we will denote by \(L^{p_0}(\mathbb{C}^n)\) the Lebesgue space of (equivalence classes of) \(p_0\)-integrable functions with respect to the Lebesgue measure. \(L^\infty(\mathbb{C}^n)\) refers to the space of measurable and essentially bounded functions on \(\mathbb{C}^n\). For \(1 \leq p_0 < \infty\) we define the Banach spaces

\[
A^{p_0} := L^{p_0}(\mathbb{C}^n) \oplus S^{p_0}(F^p_t)
\]

and

\[
A^\infty := L^\infty(\mathbb{C}^n) \oplus \mathcal{L}(F^p_t),
\]

where we do not mention \(t\) and \(p\) in the notation for readability. These spaces can be equipped with the norms

\[
\|f \oplus A\|_{A^{p_0}} := \max(\|f\|_{L^{p_0}}, \|A\|_{S^{p_0}})
\]

and

\[
\|f \oplus A\|_{A^\infty} := \max(\|f\|_{L^{\infty}}, \|A\|_{op}).
\]

Note that \(A^\infty\) is a Banach algebra. It has the subalgebra

\[
\mathcal{K} := C_0(\mathbb{C}^n) \oplus \mathcal{K}(F^p_t),
\]

where \(C_0(\mathbb{C}^n)\) denotes the continuous functions on \(\mathbb{C}^n\) vanishing at infinity. On \(A^1\) we have the trace map

\[
\text{Tr}(f \oplus A) := \text{Tr}(f) + \text{Tr}(A),
\]

where \(\text{Tr}(A)\) denotes the nuclear trace and \(\text{Tr}(f) := \int_{\mathbb{C}^n} f(z) dV(z)\). As is well-known, we can identify \((L^1(\mathbb{C}^n))' \cong L^\infty(\mathbb{C}^n)\) and, as we already mentioned earlier, \((\mathcal{N}(X))' \cong \mathcal{L}(X)\) under the dual pairing induced by the trace map.

Recall that \(F^2_t\) is a reproducing kernel Hilbert space with reproducing kernel

\[
K^2_t(z, w) := e^{\frac{z \cdot w}{2t}}
\]

for each \(z \in \mathbb{C}^n\). The normalized reproducing kernels are defined through

\[
k^2_t(z) := \frac{K^2_t(z, w)}{\|K^2_t\|_{F^2}} = e^{\frac{z \cdot w}{2t} - \frac{1}{4}|w|^2}.
\]
It is easily shown that \( k_z^p \in F_p^p \) for each \( p \). For \( z \in \mathbb{C}^n \) we define the Weyl operator \( W_z \in \mathcal{L}(F_p^p) \) as

\[
W_z f(w) = k_z^p(w) f(w - z).
\]

One can show that each \( W_z \) is an isometric isomorphism. It holds \( W_z^{-1} = W_{-z} \). Further, the Banach space adjoints \( W_z^* \) of \( W_z \in \mathcal{L}(F_p^p) \) is given by \( W_{-z} \in \mathcal{L}(F_q^q) \) under the standard dual pairing (where \( q \) is the exponent conjugate to \( p \)). For \( p = 2 \), these operators are unitary. One readily checks the identity

\[
W_z W_w = e^{-i \frac{\ln(w)}{p}} W_{z+w}.
\]  

(2.2)

With these operators, we can define the action \( \alpha \) of \( \mathbb{C}^n \) on the Banach algebras \( \mathcal{A}^{p_0} \) and \( \mathcal{A}^\infty \). For a measurable function \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) we set

\[
\alpha_z(f)(w) := f(w - z).
\]

Further, for an operator \( A \in \mathcal{L}(F_p^p) \) define

\[
\alpha_z(A) := W_z AW_{-z}.
\]

Finally, for \( f \oplus A \) from \( \mathcal{A}^{p_0} \) or \( \mathcal{A}^\infty \) we set

\[
\alpha_z(f \oplus A) := (\alpha_z f) \oplus (\alpha_z A).
\]

Using Equation (2.2), it is straightforward to show that

\[
\alpha_z(\alpha_z(f \oplus A)) = \alpha_z(f \oplus A),
\]  

(2.3)

which shows that \( \alpha \) is indeed a group action of \( \mathbb{C}^n \) on \( \mathcal{A}^{p_0} \) and \( \mathcal{A}^\infty \). Let us mention that \( \|\alpha_z(A)\|_{\mathcal{A}^\infty} = \|A\|_{\mathcal{A}^\infty} \) for all \( A \in \mathcal{N}(F_p^p) \), as one sees from the definition of the nuclear norm.

**Lemma 2.2.** (1) Let \( 1 \leq p_0 < \infty \) and \( f \oplus A \in \mathcal{A}^{p_0} \). Then, \( z \mapsto \alpha_z(f \oplus A) \) is norm-continuous on \( \mathcal{A}^{p_0} \).

(2) Let \( f \oplus A \in \mathcal{K} \). Then, \( z \mapsto \alpha_z(f \oplus A) \) is norm-continuous in \( \mathcal{K} \).

(3) Let \( f \in L^{\infty}(\mathbb{C}^n) \). Then, the map \( z \mapsto \alpha_z(f) \) is weak* continuous.

(4) Let \( A \in \mathcal{L}(F_p^p) \). Then, the map \( z \mapsto \alpha_z(A) \) is continuous with respect to the strong operator topology.

**Proof.** It suffices to verify continuity at \( z = 0 \). Norm-continuity of \( z \mapsto \alpha_z(f) \) on \( L^{p_0}(\mathbb{C}^n) \) \( (1 \leq p_0 < \infty) \), \( C_0(\mathbb{C}^n) \) and weak* continuity on \( L^{\infty}(\mathbb{C}^n) \) are well-known. Let

\[
A = y \otimes x
\]

be a rank one operator \( (y \in (F_p^p)^*, x \in F_p^p) \) and identify \( y \) with \( \overline{y} \in F_q^q \) under the usual duality pairing with equivalent norms. Hence, for \( f \in F_p^p \) it holds

\[
\alpha_z(y \otimes x)(f) = W_z(x) \int_{\mathbb{C}^n} W_{-z}(f)(w) \overline{y(w)} d\mu_4(w) = W_z(x) \int_{\mathbb{C}^n} f(w) W_{-z}^*(\overline{y})(w) d\mu_4(w) = ((W_{-z}^* y) \otimes (W_z x))(f).
\]  

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Now,
\[
\|\alpha_z(A) - z\|_{\mathcal{N}} = \| (W^*_z y) \otimes (W_z x) - y \otimes x \|_{\mathcal{N}} \\
\leq \| (W^*_z y - y) \otimes (W_z x) \|_{\mathcal{N}} + \| y \otimes (W_z x - x) \|_{\mathcal{N}} \\
\leq \| W^*_z y - y \|_{\mathcal{F}_t^p} \| W_z x \|_{\mathcal{F}_t^p} + \| y \|_{\mathcal{F}_t^p} \| W_z x - x \|_{\mathcal{F}_t^p} \\
\leq C \| W_z \tilde{y} - \tilde{y} \|_{\mathcal{F}_t^p} \| W_z x \|_{\mathcal{F}_t^p} + \| y \|_{\mathcal{F}_t^p} \| W_z x - x \|_{\mathcal{F}_t^p} \\
\rightarrow 0, \quad z \rightarrow 0,
\]
where we used that $W_z \rightarrow \text{Id}$ strongly as $z \rightarrow 0$ on $\mathcal{F}_t^p$ and $\mathcal{F}_t^q$, which is easily verified. This implies $\|\alpha_z(A) - A\|_{\mathcal{N}} \rightarrow 0$ for all finite rank operators. Now, (1) and (2) follow through approximation by finite rank operators and the norm inequalities between the operator ideals.

Let $A \in \mathcal{L}(\mathcal{F}_t^p)$. Then, the continuity of the map $z \mapsto \alpha_z(A)$ in strong operator topology follows from the continuity of $z \mapsto W_z$ with respect to the strong operator topology. \hfill \Box

We will also need to consider the subspace of $\mathcal{A}^\infty$, on which the action of $\alpha$ is "well-behaved" in a suitable sense. We define
\[
\mathcal{C}_0 := \{ f \in L^\infty(\mathbb{C}^n); \, z \mapsto \alpha_z(f) \text{ is } \| \cdot \|_{L^\infty}\text{-continuous} \}, \\
\mathcal{C}_1 := \{ A \in \mathcal{L}(\mathcal{F}_t^p); \, z \mapsto \alpha_z(A) \text{ is } \| \cdot \|_{\text{op}}\text{-continuous} \}.
\]

Set
\[
\mathcal{C} := \mathcal{C}_0 \oplus \mathcal{C}_1 \subset \mathcal{A}^\infty.
\]

One can show that $\mathcal{C}_0 = \text{BUC}(\mathbb{C}^n)$, the set of bounded and uniformly continuous functions. A precise description of $\mathcal{C}_1$ is not obvious, we will obtain one later. For the moment, it suffices to mention that $\mathcal{C}_0$ is a $C^*$ algebra, while $\mathcal{C}_1$ in general is a Banach algebra (being a $C^*$ algebra for $p = 2$).

### 2.3 Convolutions

In this section, we introduce the notion of convolution between objects from $\mathcal{A}^1$ and $\mathcal{A}^\infty$. First, we discuss how to define the convolution as a map $* : \mathcal{A}^1 \times \mathcal{A}^1 \rightarrow \mathcal{A}^1$ and derive certain properties. Afterwards, using the duality $(\mathcal{A}^1)' \cong \mathcal{A}^\infty$, we extend the convolution to a map $\mathcal{A}^1 \times \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty$.

We follow the same path as in the Hilbert space case \cite{16,23}. Some proofs follow analogously, yet several constructions and proofs need to be dealt with with some more care and several adaptations are necessary.

Convolution between two functions from $L^1(\mathbb{C}^n)$ is well known. For completeness, we define this to be
\[
f \ast g(z) := \int_{\mathbb{C}^n} f(w) g(z - w) dV(w),
\]
which is again in $L^1(\mathbb{C}^n)$. Our first goal is to extend this convolution to a convolution between two elements from $\mathcal{A}^1$. Let $f \in L^1(\mathbb{C}^n)$ and $A \in \mathcal{N}(\mathcal{F}_t^p)$. We are going to define the convolution $f \ast A$ as a Bochner integral. We refer to the literature (e.g. \cite{29}) for an introduction to Bochner integration.

First, assume $f \in \mathcal{C}_c(\mathbb{C}^n)$, the space of continuous functions with compact support. By Lemma \cite{22} $z \mapsto f(z)\alpha_z(A)$ is continuous from $\mathcal{C}^0$ to $\mathcal{N}(\mathcal{F}_t^p)$.
and therefore weakly measurable. Further, the map is separable-valued (take e.g. \(f(z)\alpha_z(A); \ z \in \mathbb{Q}^n \times i\mathbb{Q}^n\) as a countable dense subset of its range). Hence, the Pettis measurability theorem states that \(z \mapsto f(z)\alpha_z(A)\) is strongly measurable. It is now standard to show that strong measurability carries over to \(z \mapsto f(z)\alpha_z(A)\) for each \(f \in L^1(\mathbb{C}^n)\) by approximation with \(C(\mathbb{C}^n)\)-functions. Since \(z \mapsto \|f(z)\alpha_z(A)\|_{\mathcal{N}}\) is Lebesgue integrable on \(\mathbb{C}^n\), the function is Bochner integrable in the Banach space \(\mathcal{N}(F^p_t)\). We define

\[
f \ast A := A \ast f := \int_{\mathbb{C}^n} f(z)\alpha_z(A)dV(z) \in \mathcal{N}(F^p_t), \ f \in L^1(\mathbb{C}^n), A \in \mathcal{N}(F^p_t).
\]

It is immediate that

\[
\|f \ast A\|_{\mathcal{N}} \leq \int_{\mathbb{C}^n} |f(z)|\|\alpha_z(A)\|_{\mathcal{N}}dV(z) \leq \|f\|_{L^1} \|A\|_{\mathcal{N}}.
\]

Let \(U \in \mathcal{L}(F^p_t)\) be the isometric operator

\[
UF(w) = f(-w).
\]

For two operators \(A, B \in \mathcal{N}(F^p_t)\), we define their convolution \(A \ast B\) to be the function

\[
A \ast B : z \mapsto \text{Tr}(A(\alpha_z(UBU)));
\]

\textbf{Lemma 2.3.} Let \(A, B \in \mathcal{N}(F^p_t)\). Then, the function \(A \ast B\) is continuous and we have

\[
\|A \ast B\|_{L^1} \leq C\|A\|_{\mathcal{N}}\|B\|_{\mathcal{N}}
\]

(2.4)

for some constant \(C > 0\) depending only on \(n, p\) and \(t\) and

\[
\text{Tr}(A \ast B) = (\pi t)^n \text{Tr}(A) \text{Tr}(B).
\]

(2.5)

In particular, \(A \ast B \in L^1(\mathbb{C}^n)\).

\textbf{Proof.} First, observe that continuity of \(A \ast B\) follows immediately from the continuity of \(z \mapsto \alpha_z(UBU)\) on \(\mathcal{N}(F^p_t)\).

Assume that \(A\) and \(B\) are both rank one operators. Hence,

\[A = y_1 \otimes x_1, \ B = y_2 \otimes x_2\]

with \(y_j \in (F^p_t)'\), \(x_j \in F^p_t\). We again identify \(y_j\) with \(\tilde{y}_j \in F^p_t\). Then,

\[A\alpha_z(UBU) = (y_1 \otimes x_1)\alpha_z((U^*y_2) \otimes (Ux_2)) = (y_1 \otimes x_1)((W^*_zU^*y_2) \otimes (W_zUx_2)).\]

This is again a rank one operator, and one readily checks

\[A\alpha_z(UBU) = ((W^*_zU^*y_2) \otimes x_1)(W_zUx_2, y_1).\]

Further,

\[
\text{Tr}(A\alpha_z(UBU)) = \langle x_1, W^*_zU^*y_2(W_zUx_2, y_1) \rangle
\]

\[= \int_{\mathbb{C}^n} W^*_z(x_1)(w)\overline{y_2(w)}d\mu_z(w) \int_{\mathbb{C}^n} W_zU(x_2)(v)\overline{y_1(v)}d\mu_t(v)
\]

\[= \int_{\mathbb{C}^n} x_1(w + z)k^z_{-z}(w)\overline{y_2(-w)}d\mu_z(w) \int_{\mathbb{C}^n} x_2(z - v)k^v_z(v)\overline{y_1(v)}d\mu_t(v).\]
Assume for the moment that \( x_j \) and \( \tilde{y}_j \) are polynomials (which are dense in \( F_t^p \) and \( F_t^q \), respectively). We can then apply Fubini’s theorem and obtain:

\[
\int_{\mathbb{C}^n} \text{Tr}(A y_j (U B U)) dV(z)
= \int_{\mathbb{C}^n} \frac{y_2(-w)}{y_1(v)} \int_{\mathbb{C}^n} x_1(w + z) x_2(z - v) k_{-z}^t(w) k_z^t(v) dV(z) d\mu_t(w) d\mu_t(v).
\]

Since \( x_1, x_2 \) are polynomials in \( z_1, \ldots, z_n \), they (and their product) are in \( F_1^2 \) as well and it holds

\[
\int_{\mathbb{C}^n} x_1(w + z) x_2(z - v) k_{-z}^t(w) k_z^t(v) dV(z) = (\pi t)^n \int_{\mathbb{C}^n} x_1(w + z) x_2(z - v) e^{(-w,v)} d\mu_t(z)
\]

\[
= (\pi t)^n \langle x_1(w + \cdot) x_2(\cdot - v), K_v^t \rangle
= (\pi t)^n x_1(v) x_2(-w).
\]

We therefore get

\[
\int_{\mathbb{C}^n} \text{Tr}(A y_j (U B U)) dV(z) = (\pi t)^n \int_{\mathbb{C}^n} x_1(v) y_j(v) d\mu_t(v) \int_{\mathbb{C}^n} x_2(-w) y_j(-w) d\mu_t(w)
= (\pi t)^n \text{Tr}(y_1 \otimes x_1) \text{Tr}(y_2 \otimes x_2).
\]

Setting \( x = x_1 = \tilde{y}_2 \), \( y = x_2 = \tilde{y}_1 \) (which is possible, since we still assume that they are polynomials) we have

\[
\int_{\mathbb{C}^n} \langle y, W_z U x \rangle^2 dV(z) = (\pi t)^n |\text{Tr}(y \otimes x)|^2
\]

and hence it holds \( \langle y, W_z U x \rangle \in L^2(\mathbb{C}^n) \) as a function of \( z \) with

\[
\| \langle y, W_z U x \rangle \|_{L^2} \leq (\pi t)^{n/2} |\text{Tr}(y \otimes x)| \leq (\pi t)^{n/2} \| y \|_{(F_t^p)^{'} \cdot x} \| x \|_{F_t^p}.
\]

Therefore,

\[
\text{Tr}(A y_j (U B U)) = \langle W_{-z} U^* y_2, x_1 \rangle \langle y_1, W_z U x_2 \rangle \in L^1(\mathbb{C}^n)
\]

(understood as a function of \( z \)) and Hölder’s inequality yields the estimate

\[
\| \text{Tr}(A y_j (U B U)) \|_{L^1} \leq C \| y_1 \|_{(F_t^p)^{'}} \| y_2 \|_{(F_t^p)^{'}} \| x_1 \|_{F_t^p} \| x_2 \|_{F_t^p}
\]

for some constant \( C \) depending only on \( n, t \) and \( p \). Now, let \( x_j \in F_t^p, y_j \in (F_t^p)^{'} \) be arbitrary. Let \( (x_m^{(n)})^n_m, (y_m^{(n)})^n_m \) be sequences of polynomials converging to \( x_j \) and \( \tilde{y}_j \) in \( F_t^p \) and \( F_t^q \), respectively. Then,

\[
\text{Tr}(A y_j (U B U)) = \langle W_{-z} U^* y_2, x_1 \rangle \langle y_1, W_z U x_2 \rangle
= \lim_{m \to \infty} \langle W_{-z} U^* y_2^m, x_1^m \rangle \langle y_1^m, W_z U x_2^m \rangle.
\]

By Fatou’s Lemma we get

\[
\| \text{Tr}(A y_j (U B U)) \|_{L^1} \leq C \| y_1 \|_{(F_t^p)^{'}} \| y_2 \|_{(F_t^p)^{'}} \| x_1 \|_{F_t^p} \| x_2 \|_{F_t^p}.
\]

Now, taking the infimum over all possible representations \((\ref{representation})\) gives

\[
\| \text{Tr}(A y_j (U B U)) \|_{L^1} \leq C \| A \|_N \| B \|_N
\]

for arbitrary rank one operators. Having this estimate, it is easy to derive Equation \((\ref{rankone})\) for arbitrary rank one operators. Finally, it is standard to generalize \((\ref{rankone})\) and \((\ref{rankonedual})\) from rank one operators to arbitrary nuclear operators. \(\square\)
Combining the last few results, we see that we obtain (by linear extension) a convolution
\[ * : \mathcal{A}^1 \times \mathcal{A}^1 \rightarrow \mathcal{A}^1. \]
We will denote by \( \langle f, g \rangle_{\text{tr}}, \langle A, B \rangle_{\text{tr}} \) the duality pairing induced by the trace maps, i.e. for \( f \in L^1(\mathbb{C}^n), \ g \in L^\infty(\mathbb{C}^n) \) and \( A \in \mathcal{N}(F^p_1), \ B \in \mathcal{L}(F^p_1) \) we have
\[
\langle f, g \rangle_{\text{tr}} = \int_{\mathbb{C}^n} f(z)g(z)dV(z), \quad \langle A, B \rangle_{\text{tr}} = \text{Tr}(AB).
\]

**Lemma 2.4.** The convolution is commutative and associative. Further, for \( f_1, f_2 \in L^1(\mathbb{C}^n) \), \( A_1, A_2 \in \mathcal{N}(F^p_1) \) we have
\[
\alpha_z(f_1 * f_2) = \alpha_z(f_1) * f_2,
\]
\[
\alpha_z(f_1 * A_2) = \alpha_z(f_1) * A_2 = f_1 * \alpha_z(A_2),
\]
\[
\alpha_z(A_1 * A_2) = \alpha_z(A_1) * A_2.
\]
and there is a constant \( C > 0 \) (depending on \( n, p \) and \( t \)) such that
\[
\| (f_1 \oplus A_1) * (f_2 \oplus A_2) \|_{\mathcal{A}^1} \leq C\| f_1 \oplus A_1 \|_{\mathcal{A}^1} \| f_2 \oplus A_2 \|_{\mathcal{A}^1}.
\]

**Proof.** The proof of commutativity, associativity and the three identities carries over from the Hilbert space proof and follows from properties of the trace functional and the Bochner integral. We refer to [16] for details.

Our next goal will be to extend the convolution such that we can convolve elements from \( \mathcal{A}^1 \) with elements from \( \mathcal{A}^\infty \). The following identities will be useful for this:

**Lemma 2.5.** Let \( f \in L^1(\mathbb{C}^n) \) and \( A_1, A_2 \in \mathcal{N}(F^p_1) \). Then, we have
\[
\langle f * A_1, B \rangle_{\text{tr}} = \langle f, B * (UA_1 U) \rangle_{\text{tr}}, \quad B \in \mathcal{N}(F^p_1),
\]
\[
\langle f * A_2, B \rangle_{\text{tr}} = \langle A_2, (Uf) * B \rangle_{\text{tr}}, \quad B \in \mathcal{N}(F^p_1),
\]
\[
\langle A_1 * A_2, g \rangle_{\text{tr}} = \langle A_1, g * (UA_2 U) \rangle_{\text{tr}}, \quad g \in L^1(\mathbb{C}^n).
\]

**Proof.** Follows again from simple properties of the trace map and the Bochner integral.

We now want to extend the convolution \( * : \mathcal{A}^1 \times \mathcal{A}^1 \rightarrow \mathcal{A}^1 \) to a larger class, at least in the second factor. In the end, we will obtain a convolution \( * : \mathcal{A}^1 \times \mathcal{A}^\infty \rightarrow \mathcal{A}^\infty \). For \( f_1 \in L^1(\mathbb{C}^n), \ A_1 \in \mathcal{N}(F^p_1) \) and \( A_2 \in \mathcal{L}(F^p_1) \), we could still define the convolutions as
\[
f_1 * A_2 = \int_{\mathbb{C}^n} f_1(z)W_{z}A_2W_{-z}dV(z),
\]
\[
A_1 * A_2(z) = \text{Tr}(A_1 \alpha_z(UA_2 U)).
\]
These are actually equivalent to the definitions that we will give below (the integral above has to be understood as a weak\(^*\) integral), but for \( f_2 * A_1 \) \( (f_2 \in L^\infty(\mathbb{C}^n), \ A_1 \in \mathcal{N}(F^p_1)) \) the analogous definition of \( f_2 * A_1 \) makes no sense right away. The problem can be solved by defining \( f_2 * A_1 \) via duality \( \mathcal{L}(F^p_1) \cong (\mathcal{N}(F^p_1))' \). To have a unified approach, we define the convolution in all three cases through duality:
Definition 2.6. Let \( f_1 \in L^1(\mathbb{C}^n), f_2 \in L^\infty(\mathbb{C}^n), A_1 \in \mathcal{N}(F^p_t) \) and \( A_2 \in \mathcal{L}(F^p_t) \).
We define the convolutions \( f_1 * A_2 \in \mathcal{L}(F^p_t), f_2 * A_1 \in \mathcal{L}(F^p_t) \) and \( A_1 * A_2 \in L^\infty(\mathbb{C}^n) \) through the following duality relations:
\[
\langle f_1 * A_2, B \rangle_{tr} = \langle A_2, U f_1 * B \rangle_{tr} \quad \text{for all } B \in \mathcal{N}(F^p_t)
\]
\[
\langle f_2 * A_1, B \rangle_{tr} = \langle f_2, B * (U A_1 U) \rangle_{tr} \quad \text{for all } B \in \mathcal{N}(F^p_t)
\]
\[
\langle A_1 * A_2, g \rangle_{tr} = \langle A_2, g * (U A_1 U) \rangle_{tr} \quad \text{for all } g \in L^1(\mathbb{C}^n)
\]
We extend this convolution linearly to a map \( * : \mathcal{A}^1 \times \mathcal{A}^\infty \to \mathcal{A}^\infty \).

Lemma 2.7. (1) For \((f_1 \oplus A_1) \in \mathcal{A}^1 \) and \((f_2 \oplus A_2) \in \mathcal{A}^\infty \) it holds
\[
\|(f_1 \oplus A_1) * (f_2 \oplus A_2)\|_{\mathcal{A}^\infty} \leq C \|(f_1 \oplus A_1)\|_{\mathcal{A}^1} \|(f_2 \oplus A_2)\|_{\mathcal{A}^\infty}
\]
for some constant \( C > 0 \) which depends only on \( n, p \) and \( t \).

(2) The convolution \( * : \mathcal{A}^1 \times \mathcal{A}^\infty \to \mathcal{A}^\infty \) is associative in the sense that for \((f_1 \oplus A_1), (f_2 \oplus A_2) \in \mathcal{A}^1 \) and \((g \oplus B) \in \mathcal{A}^\infty \) we have
\[
[(f_1 \oplus A_1) * (f_2 \oplus A_2)] * (g \oplus B) = (f_1 \oplus A_1) * [(f_2 \oplus A_2) * (g \oplus B)].
\]

(3) Let \((f_1 \oplus A_1) \in \mathcal{A}^1 \) and \((f_2 \oplus A_2) \in \mathcal{A}^\infty \). Then, the following identities are valid:
\[
\alpha_z(f_1 * f_2) = \alpha_z(f_1) * f_2 = f_1 * \alpha_z(f_2),
\]
\[
\alpha_z(f_1 * A_2) = \alpha_z(f_1) * A_2 = f_1 * \alpha_z(A_2),
\]
\[
\alpha_z(A_1 * f_2) = \alpha_z(A_1) * f_2 = A_1 * \alpha_z(f_2),
\]
\[
\alpha_z(A_1 * A_2) = \alpha_z(A_1) * A_2 = A_1 * \alpha_z(A_2).
\]

Proof. The result follow immediately from the duality relations, which define the convolution, and corresponding relations for the convolution between \( \mathcal{A}^1 \) objects.

Since \( \mathcal{A}^{po} \) was set up such that it is an interpolation space between \( \mathcal{A}^1 \) and \( \mathcal{A}^\infty \), we can define the convolution as a map \( \mathcal{A}^1 \times \mathcal{A}^{po} \to \mathcal{A}^{po} \) through interpolation and all the properties of the previous lemma carry over with obvious modifications.

We have the following important observation:

Lemma 2.8. Let \((f_1 \oplus A_1) \in \mathcal{A}^1, (f_2 \oplus A_2) \in \mathcal{A}^\infty \). Then, we have
\[
(f_1 \oplus A_1) * (f_2 \oplus A_2) \in \mathcal{C},
\]
i.e. \( f_1 \ast f_2, A_1 * A_2 \in BUC(\mathbb{C}^n) \) and \( f_1 * A_2, f_2 * A_1 \in \mathcal{C}_1 \).

Proof. We have
\[
\|\alpha_z(f_1 * A_2) - \alpha_w(f_1 * A_2)\|_{op} = \|(\alpha_z(f_1) - \alpha_w(f_1)) * A_2\|_{op} \leq \|\alpha_z(f_1) - \alpha_w(f_1)\|_{L^1} \|A_2\|_{op} \to 0, \quad z \to w,
\]
where we used that \( \alpha \) acts continuously on \( L^1(\mathbb{C}^n) \). The other cases are proven analogously.
Although the convolutions are now defined through duality, we might still use their old definitions in two of the three cases, as we already mentioned above. For simplicity, we discuss the case $f_1 * A_2$ for $f_1 \in L^1(\mathbb{C}^n)$ only if $A_2 \in \mathcal{C}_1$, which gives some extra information.

**Lemma 2.9.** 1) Let $f_1 \in L^1(\mathbb{C}^n)$ and $A_2 \in \mathcal{C}_1$. Then, their convolution can be expressed as

$$f_1 * A_2 = \int_{\mathbb{C}^n} f_1(z) \alpha_z(A_2) dV(z) \in \mathcal{C}_1.$$  

2) For $A_1 \in \mathcal{N}(F^p_t)$ and $A_2 \in L(F^p_t)$, we have

$$A_1 * A_2(z) = \text{Tr}(A_1 \alpha_z(U A_2 U)).$$

**Proof.** Using properties of Bochner integrals and trace maps, one verifies that these objects satisfy the duality definition of the convolutions. In 1) $f_1 * A_2 \in \mathcal{C}_1$ follows, since the integral converges in $\mathcal{C}_1$ as a Bochner integral.

### 2.4 Toeplitz quantization, Berezin transform and convolution

Recall that for $p = 2$, the orthogonal projection

$$P_t : L^2(\mathbb{C}^n, \mu_t) \to F^2_t$$

is given by

$$P_t(f)(z) = \int_{\mathbb{C}^n} e^{\frac{i}{2} \langle w, z \rangle} f(w) d\mu_t(w).$$

As is well-known, the operator defined by the same integral expression defines a continuous projection

$$P_t : L^p(\mathbb{C}^n, \mu_{2t/p}) \to F^p_t,$$

$$P_t(f)(z) = \int_{\mathbb{C}^n} e^{\frac{i}{2} \langle w, z \rangle} f(w) d\mu_t(w)$$

$$= \left( \frac{2}{p} \right)^n \int_{\mathbb{C}^n} e^{\frac{i}{2} \langle w, z \rangle} (\frac{2}{p} |w|^2)^{p/2} d\mu_{2t/p}(w),$$

cf. [15, 25]. For $f \in L^\infty(\mathbb{C}^n)$ we will denote by $T^*_f$ (suppressing $p$ in the notation) the Toeplitz operator

$$T^*_f : F^p_t \to F^p_t, \quad T^*_f g = P_t(f g).$$

It is easy to see that $T^*_f$ is bounded if $f \in L^\infty(\mathbb{C}^n)$. Further, for $f \in L^1(\mathbb{C}^n)$ we define $T^*_f$ by the same formula. Here, boundedness is not entirely trivial. For a suitable measurable function $f : \mathbb{C}^n \to \mathbb{C}$ we will define its Berezin transform at $t > 0$ through

$$\tilde{f}^{(t)}(z) := \langle f k^t_z, k^t_z \rangle_{F^2_t},$$

if it exists. Further, for $A \in \mathcal{L}(F^p_t)$ we define the Berezin transform as

$$\tilde{A}(z) := \langle A k^t_z, k^t_z \rangle_{F^2_t}.$$
For $f \in L^\infty(\mathbb{C}^n)$, one readily checks that
\[ \tilde{T}_t^f = \tilde{f}^{(t)}. \]

It is our next goal to express the maps $f \mapsto T_t^f$ and $A \mapsto \tilde{A}$ using convolutions. For this, we consider the operator $P_C = 1 \otimes 1 \in \mathcal{N}(F_1^p)$, i.e.
\[ P_C f = f(0) \in F_1^p. \]

We will also need the following normalized version of $P_C$:
\[ \mathcal{R}_t := \frac{1}{(\pi t)^n} P_C. \]

**Lemma 2.10.** Let $A \in \mathcal{L}(F_1^p)$. Then, we have
\[ \tilde{A} = P_C \ast A. \]

**Proof.** One readily checks for $f \in F_1^p$ that
\[ AW_z U P_C U W_z^* f = Ak_z^t \cdot f(z) e^{-\frac{|z|^2}{2t}}. \]

Therefore, an eigenvector of that operator to a non-zero eigenvalue needs to be a multiple of $Ak_z^t$, and for these eigenvectors we obtain
\[ AW_z U P_C U W_z^*(Ak_z) = Ak_z^t \cdot Ak_z^t(z) e^{-\frac{|z|^2}{2t}}, \]

i.e. the only non-zero eigenvalue of $AW_z U P_C U W_z^*$ is
\[ Ak_z^t(z) e^{-\frac{|z|^2}{2t}} = \langle Ak_z^t, K_z^t \rangle e^{-\frac{|z|^2}{2t}} = \tilde{A}(z). \]

Since the trace of a finite rank operator coincides with the sum of its eigenvalues, the proof is finished. \qed

**Lemma 2.11.** For $f \in L^1(\mathbb{C}^n)$, it holds $T_t^f = \mathcal{R}_t \ast f \in \mathcal{N}(F_1^p)$. In particular, $T_t^f$ is a bounded linear operator.

**Proof.** For $f \in L^1(\mathbb{C}^n)$ we have
\[
\begin{align*}
P_C \ast f(g) &= \int_{\mathbb{C}^n} f(z) W_z P_C W_{z^*} g(z) \, dV(z) \\
&= \int_{\mathbb{C}^n} f(z) W_z(1) e^{-\frac{|z|^2}{2t}} g(z) \, dV(z) \\
&= \int_{\mathbb{C}^n} f(z) e^{-\frac{|z|^2}{2t}} g(z) \, dV(z) \\
&= (\pi t)^n T_t^f g.
\end{align*}
\]

The next goal is to extend the relation $\mathcal{R}_t \ast f = T_t^f$ to all $f \in L^\infty(\mathbb{C}^n)$.

**Proposition 2.12.** Let $f \in L^\infty(\mathbb{C}^n)$. Then, we have $\mathcal{R}_t \ast f = T_t^f$. \hfill 14
Proof. Recall that for \( f \in L^\infty(C^n) \) the convolution \( \mathcal{R}_t * f \) was defined through the duality relation

\[
\langle \mathcal{R}_t * f, B \rangle_{tr} = \langle f, B * (U \mathcal{R}_t U) \rangle_{tr}, \quad B \in \mathcal{N}(F_p^n).
\]

It is simple to check that \( U P_C U = P_C \), hence we obtain

\[
\langle \mathcal{R}_t * f, B \rangle_{tr} = \langle f, B * \mathcal{R}_t \rangle_{tr} = \frac{1}{(\pi t)^n} \langle f, \tilde{B} \rangle_{tr}.
\]

Letting \( B = k_z^2 \otimes k_z^2 \), we obtain

\[
\langle \mathcal{R}_t * f, k_z^2 \otimes k_z^2 \rangle_{tr} = \frac{1}{(\pi t)^n} \langle f, (k_z^2 \otimes k_z^2)^* \rangle_{tr}.
\]

For each \( A \in \mathcal{L}(F_p^n) \), the operator \( A(k_z^2 \otimes k_z^2) \) acts as

\[
A(k_z^2 \otimes k_z^2)(g) = Ak_z^2 \cdot (g, k_z^2)_{F_p^2},
\]

i.e. we obtain for the trace

\[
\langle A, k_z^2 \otimes k_z^2 \rangle_{tr} = \text{Tr}(A(k_z^2 \otimes k_z^2)) = \langle Ak_z^2, k_z^2 \rangle_{F_p^2} = \tilde{\Delta}(z).
\]

Further, the Berezin transform of \( k_z^2 \otimes k_z^2 \) is given by

\[
(k_z^2 \otimes k_z^2)^*(w) = \langle (k_z^2 \otimes k_z^2)^*(w), k_z^2 \rangle_{F_p^2} = \langle k_z^2, k_z^2 \rangle_{F_p^2}(k_z^2, k_z^2)_{F_p^2} = e^{-\frac{|w|^2}{t}},
\]

which yields

\[
\langle f, (k_z^2 \otimes k_z^2)^* \rangle_{tr} = \int_{C^n} f(w)e^{-\frac{|w|^2}{t}} dV(w) = (\pi t)^n \tilde{f}(u)(z).
\]

Therefore, the operator \( \mathcal{R}_t * f \) fulfills the relation

\[
(\mathcal{R}_t * f)^* = \tilde{f}(t),
\]

which then implies that \( \mathcal{R}_t * f = T_f^n \) as the Berezin transform is injective.

Let us consider the maps

\[
\Psi : L^1(C^n) \to \mathcal{N}(F_p^n), \quad \Psi(f) = T_f^n = \mathcal{R}_t * f
\]

and

\[
\Phi : \mathcal{N}(F_p^n) \to L^1(C^n), \quad \Phi(A) = \tilde{A} = P_C * A.
\]

Restating the duality relations for the convolutions, we obtain for \( f \in L^1(C^n) \) and \( B \in \mathcal{L}(F_p^n) \cong (\mathcal{N}(F_p^n))' \):

\[
\langle \Psi(f), B \rangle_{tr} = \langle \mathcal{R}_t * f, B \rangle_{tr} = \langle f, (U \mathcal{R}_t U) * B \rangle_{tr} = \langle f, \mathcal{R}_t * B \rangle_{tr},
\]

i.e. the Banach space adjoint of \( \Psi \) is given by

\[
(\Psi)' : \mathcal{L}(F_p^n) \to L^\infty(C^n), \quad (\Psi)'(A) = \frac{1}{(\pi t)^n} \tilde{A}.
\]

Analogously, one sees that for \( A \in \mathcal{N}(F_p^n) \) and \( g \in L^\infty(C^n) \) we have

\[
\langle \Phi(A), g \rangle_{tr} = \langle P_C * A, g \rangle_{tr} = \langle A, (U P_C U) * g \rangle_{tr} = (\pi t)^n \langle A, T_g^n \rangle_{tr},
\]

i.e.

\[
(\Phi)' : L^\infty(C^n) \to \mathcal{L}(F_p^n), \quad (\Phi)'(f) = (\pi t)^n T_f^n.
\]
Proposition 2.13. \( \{ T_f^t; f \in L^1(\mathbb{C}^n) \} \) is dense in \( \mathcal{N}(F_p^p) \) and \( \{ \tilde{A}; A \in \mathcal{N}(F_p^p) \} \) is dense in \( L^1(\mathbb{C}^n) \).

Proof. As is well-known, the symbol map \( f \mapsto T_f^t \) (\( f \in L^\infty(\mathbb{C}^n) \)) and the Berezin transform \( A \mapsto \tilde{A} \) (\( A \in \mathcal{L}(F_p^p) \)) are injective. Therefore, the Banach space adjoints \( (\Psi)' \) and \( (\Phi)' \) of the maps \( \Psi \) and \( \Phi \) are injective. This in turn implies that \( \Psi \) and \( \Phi \) have dense range, which is just the statement of the proposition. \( \square \)

We have seen that convolution by \( R_t \) yields the maps
\[
L^1(\mathbb{C}^n) \to \mathcal{N}(F_p^p), \quad f \mapsto T_f^t
\]
\[
L^\infty(\mathbb{C}^n) \to \mathcal{L}(F_p^p), \quad f \mapsto T_f^t.
\]
Further, convolution by \( P_C \) yields maps
\[
\mathcal{N}(F_p^p) \to L^1(\mathbb{C}^n), \quad A \mapsto \tilde{A}
\]
\[
\mathcal{L}(F_p^p) \to L^\infty(\mathbb{C}^n), \quad A \mapsto \tilde{A}.
\]
Applying now complex interpolation to both maps, we obtain the following result. At least part (i) is already well-known in the case \( p = 2 \) with different proof, compare e.g. [14, 25].

Lemma 2.14. Let \( 1 \leq p_0 < \infty \).

(i) For \( f \in L^{p_0}(\mathbb{C}^n) \) we have \( T_f^t \in \mathcal{S}^{p_0}(F_p^p) \).

(ii) For \( A \in \mathcal{S}^{p_0}(F_p^p) \) we have \( \tilde{A} \in L^{p_0}(\mathbb{C}^n) \).

Simple approximation arguments yield now the following:

Lemma 2.15. Let \( 1 \leq p_0 < \infty \).

(i) \( \{ T_f^t; f \in L^{p_0}(\mathbb{C}^n) \} \) is dense in \( \mathcal{S}^{p_0}(F_p^p) \).

(ii) \( \{ \tilde{A}; A \in \mathcal{S}^{p_0}(F_p^p) \} \) is dense in \( L^{p_0}(\mathbb{C}^n) \).

2.5 Characterizations of \( \mathcal{C}_1 \)

In the following, we will denote by \( f_s \) the function
\[
f_s(z) = \frac{1}{(\pi s)^n} e^{-\frac{|z|^2}{s}},
\]
where \( s > 0 \). The result of this section is the following:

Proposition 2.16. The following equalities hold true:
\[
\mathcal{C}_1 = \overline{\text{span}\{ g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{L}(F_p^p) \}}
\]
\[
= \{ B \in \mathcal{L}(F_p^p); \ f_s \ast B \to B \text{ in operator norm as } s \to 0 \}
\]
\[
= R_t \ast \text{BUC}(\mathbb{C}^n)
\]
\[
= \{ g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{C}_1 \},
\]
where closures are taken in \( \mathcal{L}(F_p^p) \).
Remark 2.17. Once we have proven the first three equalities, the last equality follows directly from the Cohen-Hewitt factorization theorem. Since we will not need this equality, we do not discuss the factorization theorem and refer to the literature (e.g. [10]). The equality

\[ C_1 = \{ B \in \mathcal{L}(F^p_t); \ g_s * B \rightarrow B \text{ in operator norm as } s \to 0 \} \]

is well-known in the theory of Banach modules over locally compact groups. We give a short proof below for completeness.

We prove the following lemma as a preparation.

Lemma 2.18. Convolution by \( f_s \) is an approximate identity in \( \mathcal{A}^{p_0} \) for \( 1 \leq p_0 < \infty \) and in \( \mathcal{C} \), i.e.

\[
\| f_s * (g_1 \oplus A_1) - (g_1 \oplus A_1) \|_{\mathcal{A}^{p_0}} \to 0, \quad s \to 0
\]
\[
\| f_s * (g_2 \oplus A_2) - (g_2 \oplus A_2) \|_{\mathcal{A}^{\infty}} \to 0, \quad s \to 0
\]

for each \((g_1 \oplus A_1) \in \mathcal{A}^{p_0}\) and \((g_2 \oplus A_2) \in \mathcal{C}\).

Proof. For \( g_1 \in L^{p_0}(\mathbb{C}^n) \) and \( g_2 \in BUC(\mathbb{C}^n) \) it is well-known and not hard to prove that

\[
\| f_s * g_1 - g_1 \|_{L^{p_0}} \to 0, \quad s \to 0,
\]
\[
\| f_s * g_2 - g_2 \|_{L^{\infty}} \to 0, \quad s \to 0.
\]

Consider operators of the form

\[ A_1 = \mathcal{R}_t * g_1 \]

with, as above, \( g_1 \in L^{p_0}(\mathbb{C}^n) \). For \( g \in L^1(\mathbb{C}^n) \), one easily establishes the identity

\[ f_s * (\mathcal{R}_t * g) = \mathcal{R}_t * (f_s * g) \]

using Lemma 2.4, which then carries over to the case \( g \in L^{p_0}(\mathbb{C}^n) \). We therefore obtain

\[
\| f_s * (\mathcal{R}_t * g_1) - \mathcal{R}_t * g_1 \|_{\mathcal{S}^{p_0}} = \| \mathcal{R}_t * (f_s * g_1 - g_1) \|_{\mathcal{S}^{p_0}}
\]
\[
\leq \| \mathcal{R}_t \|_{\mathcal{N}} \| f_s * g_1 - g_1 \|_{L^{p_0}}
\]
\[
\to 0, \quad s \to 0
\]

by the \( \mathcal{A}^{p_0} \)-version of Lemma 2.7. By Lemma 2.15(i), \( \mathcal{R}_t * L^{p_0}(\mathbb{C}^n) \) is dense in \( \mathcal{S}^{p_0}(F^p_t) \), hence the result for \( \mathcal{S}^{p_0}(F^p_t) \) follows from some standard density argument.

Let \( B \in \mathcal{C}_1 \), i.e. \( z \mapsto W_z BW_{-z} \) is continuous with respect to the operator norm. We claim that \( f_s * B \rightarrow B \) in operator norm as \( s \to 0 \). Using basic properties of the Bochner integral, we obtain

\[
\| B - f_s * B \|_{\text{op}} = \left\| \int_{\mathbb{C}^n} f_s(w)B - f_s(w)W_z BW_{-z}dV(w) \right\|_{\text{op}}
\]
\[
\leq \int_{\mathbb{C}^n} f_s(w)\| B - W_z BW_{-z} \|_{\text{op}} dV(w).
\]
An easy consequence of \( B \in \mathcal{C}_1 \) and the inverse triangle inequality is the fact that \( w \mapsto \|B - W_0 BW_{-w}\| \) is in \( \text{BUC}(\mathbb{C}^n) \). Therefore, we have

\[
\int_{\mathbb{C}^n} f_s(w)\|B - W_0 BW_{-w}\|_{op} \, dV(w) \to \|B - W_0 BW_{-0}\|_{op} = 0, \quad s \to 0,
\]

and thus \( f_s \ast B \to B \).

**Proof of Proposition 2.16.** The following inclusions hold true due to Lemma 2.8:

\[
\mathcal{C}_1 \supseteq \text{span}\{g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{L}(F^p_t)\}
\]

\[
\supseteq \{B \in \mathcal{L}(F^p_t); \ f_s \ast B \to B \text{ in operator norm as } s \to 0\}.
\]

The previous lemma proves the inclusion

\[
\mathcal{C}_1 \subseteq \{B \in \mathcal{L}(F^p_t); \ f_s \ast B \to B \text{ in operator norm as } s \to 0\},
\]

i.e. we obtain

\[
\mathcal{C}_1 = \text{span}\{g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{L}(F^p_t)\}
\]

\[
= \{B \in \mathcal{L}(F^p_t); \ f_s \ast B \to B \text{ in operator norm as } s \to 0\}.
\]

Lemma 2.8 also yields that \( \mathcal{R}_t \ast \text{BUC}(\mathbb{C}^n) \subseteq \mathcal{C}_1 \). It remains to show that \( \mathcal{R}_t \ast \text{BUC}(\mathbb{C}^n) \) is dense in \( \mathcal{C}_1 \). Let \( B \in \mathcal{C}_1 \). Then, we can choose \( g \in L^1(\mathbb{C}^n) \) such that \( \|B - g \ast B\|_{op} < \varepsilon \) according to Lemma 2.18. Since \( \mathcal{R}_t \ast \mathcal{N}(F^p_t) \) is dense in \( L^1(\mathbb{C}^n) \) by Proposition 2.13, we can choose \( C \in \mathcal{N}(F^p_t) \) such that \( \|g - \mathcal{R}_t \ast C\|_{L^1} < \varepsilon \). Combining this, we obtain

\[
\|B - \mathcal{R}_t \ast (C \ast B)\|_{op} \leq \|B - g \ast B\|_{op} + \|(g - \mathcal{R}_t \ast C) \ast B\|_{op} \leq (1 + \|B\|_{op})\varepsilon,
\]

and \( C \ast B \in \text{BUC}(\mathbb{C}^n) \) due to Lemma 2.8. Therefore, \( \mathcal{R}_t \ast \text{BUC}(\mathbb{C}^n) \) is dense in \( \mathcal{C}_1 \). Hence, we have proven

\[
\mathcal{C}_1 = \text{span}\{g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{L}(F^p_t)\}
\]

\[
= \{B \in \mathcal{L}(F^p_t); \ g_s \ast B \to B \text{ in operator norm as } s \to 0\}
\]

\[
= \mathcal{R}_t \ast \text{BUC}(\mathbb{C}^n).
\]

As already mentioned, the equality

\[
\mathcal{C}_1 = \{g \ast B; \ g \in L^1(\mathbb{C}^n), \ B \in \mathcal{C}_1\}
\]

is now a consequence of the Cohen-Hewitt factorization theorem.

### 2.6 Correspondence theory

This section closely follows the setup introduced by R. Werner in [23]. Definitions and proofs are as presented in that paper, but we added a certain amount of details for convenience.

**Definition 2.19.** A subspace \( \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1 \subset \mathcal{A}^\infty \) is said to be a pair if \( \mathcal{N}(F^p_t) \ast \mathcal{D} \subseteq \mathcal{D} \). In this case, \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) are called corresponding spaces.

Before we turn to the main result of correspondence theory below, we will need the following lemma:
Lemma 2.20. Let $\mathcal{D} \subset \mathcal{C}$ be closed and $\alpha$-invariant. Then, we have

$$L^1(\mathbb{C}^n) \ast \mathcal{D} \subset \mathcal{D}.$$ 

Proof. As above, for $(f \oplus A) \in \mathcal{D}$ and $g \in L^1(\mathbb{C}^n)$ the convolution $g \ast (f \oplus A)$ is defined as a Bochner integral with integrand $z \mapsto g(z)\alpha_z(f \oplus A)$ taking values in the Banach space $\mathcal{D}$, hence the integral is naturally contained in $\mathcal{D}$. \hfill \square

The following result is [23, Theorem 4.1].

Theorem 2.21. (1) If $\mathcal{D}$ is a pair, then $\overline{\mathcal{D}}$ is also a pair.

(2) Let $\mathcal{D}$ be a pair. Then, $\mathcal{R}_t \ast \mathcal{D}_0$ is $\| \cdot \|_{op}$-dense in $\mathcal{D}_1 \cap \mathcal{C}_1$ and $P_{\mathcal{C}} \ast \mathcal{D}_1$ is $\| \cdot \|_{L_{\infty}}$-dense in $\mathcal{D}_0 \cap \mathcal{C}_0$.

(3) Let $\mathcal{D}$ be a pair. Then,

$$A \in \mathcal{C}_1 \text{ and } P_{\mathcal{C}} \ast A \in \mathcal{D}_0 \implies A \in \overline{\mathcal{D}_1},$$

$$f \in \mathcal{C}_0 \text{ and } \mathcal{R}_t \ast f \in \mathcal{D}_1 \implies f \in \overline{\mathcal{D}_0},$$

where closures are taken with respect to $\| \cdot \|_{op}$ and $\| \cdot \|_{L_{\infty}}$, respectively.

(4) For each closed $\alpha$-invariant subspace $\mathcal{D}_0 \subset \mathcal{C}_0$ there is a unique closed and $\alpha$-invariant corresponding subspace $\mathcal{D}_1 \subset \mathcal{C}_1$ and vice versa. For the unique correspondences of closed, $\alpha$-invariant subspaces in part (4) of the theorem we write $\mathcal{D}_0 \leftrightarrow \mathcal{D}_1$.

Although the theorem was initially formulated only for the Hilbert space case, its proof carries over to our setting of $p$-Fock spaces unchanged. We present the initial proof from [23]:

Proof. (1) Let $f \in \overline{\mathcal{D}_0}$ and $f_n \in \mathcal{D}_0$ such that $f_n \to f$ and $A \in \mathcal{N}(F_p^0)$. Since $\mathcal{D}$ is a pair, $A \ast f_n \in \mathcal{D}_1$ for all $n$. By Lemma 2.7

$$\|A \ast (f - f_n)\|_{op} \leq \|A\|_{\mathcal{N}}\|f - f_n\|_{L_{\infty}} \to 0, \quad n \to \infty,$$

hence $A \ast f \in \overline{\mathcal{D}_1}$. $\mathcal{N}(F_p^0) \ast \overline{\mathcal{D}_1} \subset \overline{\mathcal{D}_0}$ follows analogously.

(2) $\mathcal{R}_t \ast \mathcal{D}_0 \subset \mathcal{D}_1 \cap \mathcal{C}_1$ follows since $\mathcal{D}$ is a pair and by Lemma 2.8. Let $A \in \mathcal{D}_1 \cap \mathcal{C}_1$ be arbitrary and $\varepsilon > 0$. By Lemma 2.18 there exists $h \in L^1(\mathbb{C}^n)$ such that $\|A - h \ast A\|_{op} < \varepsilon$. Further, by Proposition 2.13 there exists $B \in \mathcal{N}(F_p^0)$ such that $\|h - \mathcal{R}_t \ast B\|_{L_{1}} < \varepsilon$. Together, we obtain

$$\|A - (\mathcal{R}_t \ast B) \ast A\|_{op} < \varepsilon(1 + \|A\|_{op}).$$

Finally, since $\mathcal{N}(F_p^0) \ast \mathcal{D}_1 \subset \mathcal{D}_0$, it holds $B \ast A \in \mathcal{D}_0$. Density of $P_{\mathcal{C}} \ast \mathcal{D}_1$ in $\mathcal{D}_0$ follows analogously.

(3) The reasoning of (2) with the assumption $P_{\mathcal{C}} \ast A \in \mathcal{D}_0$ instead of $A \in \mathcal{D}_1$ proves the first implication, the second follows analogously.
(4) Assume $D_0 \subset BUC(\mathbb{C}^n)$ is closed and $\alpha$-invariant. Define the spaces
\[
D_1^- = D_0 \ast N(F_p^0) \subset C_1 \\
D_1^+ = \{ A \in C_1; \ A \ast B \in D_0 \text{ for each } B \in N(F_p^0) \}.
\]

By Lemma 2.20 we have $L^1(\mathbb{C}^n) \ast D_0 \subset D_0$. $D_0 \oplus D_1^-$ is a pair since $N(F_p) \ast D_1^- = N(F_p) \ast N(F_p^0) \ast D_0 \subset L^1(\mathbb{C}^n) \ast D_0 \subset D_0$.

Further, by definition we have $N(F_p) \ast D_1^+ \subset D_0$. One easily checks $D_1^- \subset D_1^+$, hence $D_0 \oplus D_1^+$ is also a pair. If $E_1 \subset L(F_p^0)$ is any other subspace such that $D_0 \oplus E_1$ is a pair, then
\[
D_1^- \subset E_1 \subset D_1^+.
\]

Let $A \in D_1^+$. Then, part (3) of the theorem applied to the pair $D_0 \oplus D_1^-$ yields $A \in D_1^-$, which proves the result.

The other direction of the correspondence is proven analogously.

\[\square\]

We want to note that there is an analogous correspondence theory for pairs $D = D_0 \oplus D_1 \subset A^p$, $1 \leq p < \infty$, with identical statements and proofs (up to the obvious changes). Further, Werner formulated his version of the above theorem using the notion of regular operators, which we did not introduce. Our operators $P_z$ and $R_s$ are such regular operators. We will not dwell on this and refer to Werner’s original work [23].

### 3 Applications to Toeplitz algebras

Let $S \subset L^\infty(\mathbb{C}^n)$. Then, we denote by $T^{p,t}(S) \subset L(F_p^0)$ the Banach algebra generated by all Toeplitz operators with symbols in $S$. Let us denote by $T^{p,t} := T^{p,t}(L^\infty(\mathbb{C}^n))$ the full Toeplitz algebra. By $T^{p,t}_0 \subset L(F_p^0)$ we denote the closed linear span of Toeplitz operators with symbols in $S \subset L^\infty(\mathbb{C}^n)$. Finally, in the case of $p = 2$ we will use $T^{2,t}_0 \subset L(F_2^0)$ for the $C^\ast$ algebra generated by Toeplitz operators with symbols in $S$. The following result is an immediate consequence of Proposition 2.16.

**Theorem 3.1.** We have
\[
T^{p,t} = T^{p,t}_0(BUC(\mathbb{C}^n))
\]
\[
= \{ A \in L(F_p^0); \ z \mapsto W_z A W_{-z} \text{ is continuous w.r.t. the operator norm} \}
\]
\[
= \{ A \in L(F_p^0); \ f_s \ast A \to A \text{ in operator norm as } s \to 0 \}
\]
\[
= \{ g \ast A; \ g \in L^1(\mathbb{C}^n), \ A \in C_1 \}.
\]

**Proof.** Using $R_s \ast f = T_f^s$, Proposition 2.16 gives $C_1 = \overline{R_t \ast BUC(\mathbb{C}^n)} = T^{p,t}_0(BUC(\mathbb{C}^n))$. Since we have $T_f^s \subset C_1$ by Lemma 2.8 and $C_1$ is a Banach algebra, we also obtain $T^{p,t}_0(BUC(\mathbb{C}^n)) \subseteq T^{p,t} \subseteq C_1$.  

\[\square\]
Remark 3.2. The result $\mathcal{T}^{2.1} = \mathcal{T}^{2.1}_{lin}(L^\infty(\mathbb{C}^n))$ was obtained by J. Xia in [24]. While it is no serious problem to extend Xia’s proof to general $t > 0$, the assumption $p = 2$ was crucial in an important step of his proof. Hence, the above theorem improves that result.

Theorem 3.1 tells us: For each $A \in \mathcal{T}^{p,t}$ there is a sequence of Toeplitz operators $T_{f_n}$ such that $T_{f_n} \to A$ in operator norm. So far, we have no information how the symbols $f_n$ are related to the operator $A$. A careful investigation of the underlying theory can give us the answer. Recall that convolution by 
\[ f_s(z) = \frac{1}{(\pi s)^n} e^{-|z|^2} \]
is an approximate identity in $\mathcal{C}$. Let $N \in \mathbb{N}$. Since the span of functions of the form 
\[ \alpha_z(f_t) = \alpha_z(\mathcal{R}_t \ast P_C) = \mathcal{R}_t \ast (\alpha_z(P_C)) \]
is dense in $L^1(\mathbb{C}^n)$ by Wiener’s Theorem [20 Theorem 9.5], it follows that there are constants $c_j^N$ and points $z_j^N$ such that
\[
\left\| f_{t/N} - \mathcal{R}_t \ast \left( \sum_{j=1}^{MN} c_j^N \alpha_{z_j^N}(P_C) \right) \right\|_{L^1} \leq \frac{1}{N}. \tag{3.1}
\]

Then, by the usual norm estimates for convolutions,
\[
\| A - \sum_{j=1}^{MN} c_j^N \alpha_{z_j^N}(\tilde{A}) \|_{op} = \| A - \mathcal{R}_t \ast \left( \sum_{j=1}^{MN} c_j^N \alpha_{z_j^N}(\tilde{A}) \right) \|_{op}
\]
\[
= \| A - \mathcal{R}_t \ast \left( \sum_{j=1}^{MN} c_j^N \alpha_{z_j^N}(P_C) \right) \ast A \|_{op}
\]
\[
\leq \| A - f_{t/N} \ast A \|_{op}
\]
\[
+ \| f_{t/N} \ast A - \mathcal{R}_t \ast \left( \sum_{j=1}^{MN} c_j^N \alpha_{z_j^N}(P_C) \right) \ast A \|_{op}
\]
\[
\leq \| A - f_{t/N} \ast A \|_{op} + C \| A \|_{op} \frac{1}{N} \to 0, \quad N \to \infty.
\]

Hence, we can approximate the operator $A$ by a sequence of Toeplitz operators, where each Toeplitz operator has a weighted version of the Berezin transform of $A$ as its symbol.

The following version of Werner’s Correspondence Theorem $\mathcal{T}^{2.1}$ in the Toeplitz operator setting gives a different perspective on Theorem 3.1.

**Proposition 3.3.** Let $\mathcal{D}_0 \subset BUC(\mathbb{C}^n)$ be a closed $\alpha$-invariant subspace. Then we have
\[
\mathcal{D}_0 \leftrightarrow \mathcal{T}^{p,t}_{\lin}(\mathcal{D}_0).
\]

**Proof.** By Proposition $\mathcal{T}^{2.12}$, $\mathcal{R}_t \ast f = T^t_f$ for $f \in \mathcal{D}_0$. By Theorem $\mathcal{T}^{2.21}$ the set of these operators is dense in the closed and $\alpha$-invariant subspace of $\mathcal{C}_1$ corresponding to $\mathcal{D}_0$. \(\square\)
Remark 3.4. It is important to note and easy to prove that $D_0$ is invariant under the operator $U$ if and only if $T_{lin}^{p,t}(D_0)$ is invariant under adjoining $U$ from left and right, i.e. for $A \in T_{lin}^{p,t}(D_0)$ we have $UAU \in T_{lin}^{p,t}(D_0)$.

As a simple application of this correspondence, we obtain the following result. Part (2) is well known, see e.g. [5] for the Hilbert space case. Part (3) was the essential results in [4], yet our proof is now significantly shorter now. Observe that part (1) was also mentioned in [23] in the Schrödinger representation.

Proposition 3.5. (1) We have

$$C_0(\mathbb{C}^n) \leftrightarrow K(F_t^p).$$

(2) The full ideal of compact operators is generated by Toeplitz operators, i.e.

$$K(F_t^p) = \{T_f^1; f \in C_0(\mathbb{C}^n)\} \|\cdot\|_{op}.$$  

(3) An operator $A \in \mathcal{L}(F_t^p)$ is compact if and only if $A \in \mathcal{T}^{p,t}$ and $\tilde{A} \in C_0(\mathbb{C}^n)$.

Proof. Since $C_0(\mathbb{C}^n)$ is closed and translation invariant, we have

$$C_0(\mathbb{C}^n) \leftrightarrow \mathcal{T}_{lin}(C_0(\mathbb{C}^n)) \subset K(F_t^p).$$

Further, $K(F_t^p)$ is also closed and $\alpha$-invariant. For $A \in K(F_t^p)$ it is well-known that

$$\tilde{A} \in C_0(\mathbb{C}^n),$$

hence

$$K(F_t^p) \leftrightarrow D_0 \subset C_0(\mathbb{C}^n)$$

which proves the correspondence. The second result is an immediate consequence of Proposition 3.3. The third result now follows easily from this correspondence and Theorem 2.21.

3.1 Linearly generated Toeplitz algebras

If one is thinking about possible generalizations of Theorem 3.1 one could ask if a given Banach algebra of Toeplitz operators is linearly generated by some set of functions. If the algebra is $\alpha$-invariant, the answer to this is positive. The following statement is in a sense dual to Proposition 3.3.

Theorem 3.6. Let $S \subset \mathcal{T}^{p,t}$ be an $\alpha$-invariant closed subspace (not necessarily an algebra). Then, there is some closed and $\alpha$-invariant subspace $D_0 \subset BUC(\mathbb{C}^n)$ such that

$$S = T_{lin}^{p,t}(D_0).$$

Further, $D_0$ is given by $\text{span}\{\tilde{A}; A \in S\}$.

Proof. Since $S \subset \mathcal{T}^{p,t}$, $\alpha$ acts norm-continuously on $S$. Theorem 2.21 asserts that there is some closed and $\alpha$-invariant $D_0 \subset BUC(\mathbb{C}^n)$ such that

$$S \leftrightarrow D_0 = \text{span}\{\tilde{A}; A \in S\}.$$

As we have seen before,

$$D_0 \leftrightarrow T_{lin}^{p,t}(D_0).$$
Before we continue with the general theory, want to present two examples of linearly generated Toeplitz algebras:

**Examples.**

1. **The CCR algebra.** In [7], L. Coburn studied the CCR algebra in the Bargmann representation. For this, let $p = 2, t = 1/2$ and define

   $$ CCR(C^n) := C^*(\{W_z; \ z \in C^n\}). $$

   Here, $C^*(A)$ denotes the $C^*$ algebra generated by the set $A \subset L(F_{2}^{1/2})$. It is well-known that each Weyl operator $W_z$ coincides with a certain Toeplitz operator, hence $CCR(C^n) \subset T^{2,1/2}$. Coburn showed

   $$ CCR(C^n) = T^{2,1/2}(AP) = T^{2,1/2}_{lin}(AP), $$

   where $AP$ denotes the Banach algebra of almost periodic functions. Note that $AP$ can be seen to be $\alpha$-invariant. Hence, certain $\alpha$-invariant closed symbol spaces may generate the same space of operators linearly and as an algebra.

2. **Radial Toeplitz operators.** In the study of commutative $C^*$ algebras generated by Toeplitz operators on the Fock spaces $F^2_t$, there are two relevant model cases: The radial case and the horizontal case. The $C^*$ algebra generated by Toeplitz operators with radial symbols on $F^2_t$ is linearly generated by radial Toeplitz operators. Yet, the space of radial functions is not $\alpha$-invariant. Hence, there are interesting aspects of the problem of linearly generated Toeplitz algebras outside the scope of our approach.

$CCR(C^n)$ is an example of a Toeplitz algebra which is generated as an algebra and as a closed linear space by the same set $D_0 \subset BUC(C^n)$. We will further investigate such subspaces of $T^{p,t}$.

**Lemma 3.7.** If $D_0 \subset BUC(C^n)$ is closed and $\alpha$-invariant then $D_0$ is closed under the operation $f \mapsto \tilde{f}(t)$ for all $t > 0$.

**Proof.** For $f \in D_0$ we have $T^t_f = R_{f} * f \in D_1$ and $P_\mathbb{C} * (R_{f} * f) = P_\mathbb{C} * T^t_f = \tilde{f}(t) \in D_0$, where $D_0 \leftrightarrow D_1$. \qed

We obtain the following general criterion for a Toeplitz algebra being linearly generated by the same class of symbols:

**Theorem 3.8.** Let $D_0 \subset BUC(C^n)$ be closed and $\alpha$-invariant. Then, we have

   $$ T^{p,t}_{lin}(D_0) = T^{p,t}(D_0) $$

   if and only if for each $k \in \mathbb{N}$ the range of the map

   $$ D_0^k \mapsto BUC(C^n), \quad (f_1, \ldots, f_k) \mapsto (T^t_{f_1} \cdots T^t_{f_k})^\sim $$

   (3.2)

   is contained in $D_0$. Further, we have

   $$ T^{2,t}_{lin}(D_0) = T^{2,t}_{*}(D_0) $$

   if and only if $D_0$ is closed under the product maps (3.2) and under taking complex conjugates.
Proof. Assume \( \mathcal{T}_{\text{lin}}^{p,t}(D_0) = \mathcal{T}^{p,t}(D_0) \). Then, the operator product \( T_{f_1}^t \ldots T_{f_k}^t \) is contained in \( \mathcal{T}_{\text{lin}}^{p,t}(D_0) \). The Berezin transform of that operator product, which is just convolution by \( P_C \), is therefore contained in \( D_0 \).

Now, assume that the range of the map \((3.2)\) is contained in \( D_0 \). We need to prove that \( T_{f_1}^t \ldots T_{f_k}^t \in \mathcal{T}_{\text{lin}}^{p,t}(D_0) \) for \( f_1, \ldots, f_k \in D_0 \). Since \( T_{f_1}^t \ldots T_{f_k}^t \in \mathcal{C}_1 \) and \((T_{f_1}^t \ldots T_{f_k}^t)^* = P_C \ast (T_{f_1}^t \ldots T_{f_k}^t) \in D_0 \) by assumption, this follows from Theorem \(2.24(3)\).

Finally, let \( p = 2 \) and let \( D_0 \) be closed under the product map \((3.2)\). If \( D_0 \) is further closed under taking complex conjugates, then the adjoint of each generator of \( \mathcal{T}^{2,t}(D_0) \) is also contained in \( \mathcal{T}^{2,t}(D_0) \), hence it is a \( C^* \) algebra. If on the other hand \( \mathcal{T}_{\text{lin}}^{2,t}(D_0) = \mathcal{T}^{2,t}(D_0) \), then for each \( f \in D_0 \) it holds

\[
(T_f^t)^* = T_f^{t*} = R_f \ast \overline{f} \in \mathcal{T}_{\text{lin}}^{2,t}(D_0).
\]

By Theorem \(2.24(3)\) we therefore obtain \( \overline{f} \in D_0 \).

While the previous result gives a characterization of all \( D_0 \) which have the desired property, it seems that the property of being closed under the above product map is in general difficult to verify. Yet, there is an important consequence: Since the Toeplitz operators are integral operators and the integral expression defining them does not depend on \( p \), and also the formula for the Berezin transform does not depend on \( p \), the property that the range of \((3.2)\) is contained in \( D_0 \) does not depend on \( p \). Hence, we obtain:

**Corollary 3.9.** Let \( D_0 \subset BUC(\mathbb{C}^n) \) be closed and \( \alpha \)-invariant. Then, we have

\[
\mathcal{T}_{\text{lin}}^{p,t}(D_0) = \mathcal{T}^{p,t}(D_0)
\]

for one \( p \) if and only if it holds true for all \( p \).

Analogous to the above reasoning one proves:

**Corollary 3.10.** Let \( D_0 \subset BUC(\mathbb{C}^n) \) be closed and \( \alpha \)-invariant such that

\[
\mathcal{T}_{\text{lin}}^{p,t}(D_0) = \mathcal{T}^{p,t}(D_0).
\]

If further \( I \subset D_0 \) is also closed and \( \alpha \)-invariant, then \( \mathcal{T}_{\text{lin}}^{p,t}(I) \) is a (left/right/two sided) ideal in \( \mathcal{T}_{\text{lin}}^{p,t}(D_0) \) for one \( p \) if and only if it is a (left/right/two sided) ideal for all \( p \).

While there is no \( C^* \) algebraic structure on the operator side (at least for \( p \neq 2 \)), the fact that \( \mathcal{T}^{p,t} \) depends “almost not on \( p \)” (without making this a precise statement) still allows us to obtain results typical for \( C^* \) algebras. Here is one example:

**Proposition 3.11.** Let \( D_0 \subset BUC(\mathbb{C}^n) \) be an \( \alpha \)-invariant \( C^* \) subalgebra. If \( \mathcal{T}_{\text{lin}}^{p,t}(D_0) \) contains at least one nontrivial compact operator, then it contains all compact operators.

The proof of this proposition is based on the following well-known fact, which is in turn a simple exercise using the Stone-Weierstrass Theorem.

**Lemma 3.12.** Let \( D_0 \) be an \( \alpha \)-invariant \( C^* \) subalgebra of \( BUC(\mathbb{C}^n) \). If \( D_0 \) contains a nontrivial function from \( C_0(\mathbb{C}^n) \), it holds \( C_0(\mathbb{C}^n) \subset D_0 \).
Proof of Proposition 3.11. Let $K \in \mathcal{T}_{\text{lin}}^{p,t}(D_0)$ be compact and $\neq 0$. Then, $0 \neq P_C^* K \in C_0(\mathbb{C}^n) \cap D_0$. The previous lemma implies $C_0(\mathbb{C}^n) \subset \mathcal{D}_0$, therefore $R_t C_0(\mathbb{C}^n) \subset \mathcal{T}_{\text{lin}}^{p,t}(D_0)$. Since $R_t C_0(\mathbb{C}^n) = K(F_p)$ (Proposition 3.5), this yields $K(F_p) \subset \mathcal{T}_{\text{lin}}^{p,t}(D_0)$.

The remaining part of this paper will deal with a proof of our second main theorem, which is the following:

**Theorem 3.13.** Let $D_0 \subset \text{BUC}(\mathbb{C}^n)$ be closed, $\alpha$- and $U$-invariant. Then, the following are equivalent:

(i) $D_0$ is a $C^*$ algebra with respect to the standard operations and $L^\infty(\mathbb{C}^n)$ norm;

(ii) $\mathcal{T}_{\text{lin}}^{2,t}(D_0) = \mathcal{T}_{\text{lin}}^{2,t}(D_0)$ for all $t > 0$.

If the above equivalent conditions are fulfilled, then we have $\mathcal{T}_{\text{lin}}^{p,t}(D_0) = \mathcal{T}_{\text{lin}}^{p,t}(D_0)$ for all $1 < p < \infty$, $t > 0$.

If $D_0$ is a closed, $\alpha$- and $U$-invariant $C^*$ subalgebra of $\text{BUC}(\mathbb{C}^n)$ and $I \subset D_0$ is closed, $\alpha$- and $U$-invariant, then the following are equivalent:

(i*) $I$ is an ideal in $D_0$;

(ii*) $\mathcal{T}_{\text{lin}}^{2,t}(I)$ is a left or right ideal in $\mathcal{T}_{\text{lin}}^{2,t}(D_0)$ for all $t > 0$;

(iii*) $\mathcal{T}_{\text{lin}}^{2,t}(I)$ is a two-sided ideal in $\mathcal{T}_{\text{lin}}^{2,t}(D_0)$ for all $t > 0$.

Under these assumptions, $\mathcal{T}_{\text{lin}}^{p,t}(I) = \mathcal{T}_{\text{lin}}^{p,t}(\mathcal{C})$ is a closed and two-sided ideal in $\mathcal{T}_{\text{lin}}^{p,t}(D_0)$ for all $1 < p < \infty$ and $t > 0$.

In this theorem, recall that $U \in \mathcal{L}(F_p)$ is the operator $Uf(z) = f(-z)$.

### 3.2 Applying quantization estimates

Certain quantization estimates for the Toeplitz quantization on Fock spaces have been studied intensively in recent years. We apply these estimates now for proving the first part of our theorem.

**Proposition 3.14.** 1) Let $D_0 \subset \text{BUC}(\mathbb{C}^n)$ be a closed and $\alpha$-invariant subspace such that for all $t > 0$ it holds

$$\mathcal{T}_{\text{lin}}^{p,t}(D_0) = \mathcal{T}_{\text{lin}}^{p,t}(D_0).$$

Then, $D_0$ is a Banach algebra under pointwise multiplication. If it even holds

$$\mathcal{T}_{\text{lin}}^{2,t}(D_0) = \mathcal{T}_{\text{lin}}^{2,t}(D_0)$$

for all $t > 0$, then $D_0$ is even a $C^*$ algebra.

2) Let $D_0$ be such that it fulfills all assumptions from part 1). Further, let $I \subset D_0$ be closed and $\alpha$-invariant such that $\mathcal{T}_{\text{lin}}^{p,t}(I)$ is either a left- or a right-ideal for all $t > 0$. Then, $I$ is an ideal in $D_0$. 25
Proof. 1) By Corollary 3.9, we may assume that $p = 2$. We need to prove that the pointwise product of two elements is contained in $D_0$. Therefore, let $f, g \in D_0$. For each $t > 0$, $D_0$ is closed under the product 

$$(f, g) \mapsto \tilde{T}_f T_g.$$ 

Since $f, g \in BUC(C^n)$, it holds \[2\] \[\|T_f T_g - T_{fg}\|_{op} \to 0, \quad t \to 0\] which implies \[\|\tilde{T}_f T_g - \tilde{T}_{fg}\|_{L^\infty} \to 0, \quad t \to 0.\] Since $fg \in BUC(C^n)$, we obtain \[1\] \[\|fg - \tilde{fg}(t)\|_{L^\infty} \to 0, \quad t \to 0.\] Finally, \[\|\tilde{T}_f T_g - fg\|_{L^\infty} \leq \|\tilde{T}_f T_g - \tilde{T}_{fg}\|_{L^\infty} + \|\tilde{fg}(t) - fg\|_{L^\infty} \to 0, \quad t \to 0.\] Since $\tilde{T}_f T_g \in D_0$ for each $t > 0$, and $D_0$ is assumed to be norm-closed, the result follows.

2) Follows from analogous reasoning as part 1), assuming $f \in D_0$ and $g \in I$ or vice versa.

This proposition of course implies $(ii) \Rightarrow (i)$ and $(ii^\ast) \Rightarrow (i^\ast)$ of Theorem 3.13.

3.3 Drawing information from limit operators

In what follows, we will denote by a non-separating compactification of $C^n$ a pair $(\psi, X)$, where $X$ is a compact topological space $X$ and a continuous map $\psi : C^n \to X$ such that $\psi(C^n)$ is dense in $X$. Observe that, in contrast to the standard definition of a compactification, we do not assume $\psi$ to be injective - this is why we use the notion of non-separating compactifications. If $(\psi, X)$ and $(\varphi, Y)$ are two non-separating compactifications of $C^n$, we say that $(\psi, X)$ and $(\varphi, Y)$ are equivalent (and write $(\psi, X) \sim (\varphi, Y)$) if there is a homeomorphism $\theta : X \to Y$ such that $\theta \circ \psi = \varphi$.

In this sense, there is a 1-1 correspondence between unital $C^*$ subalgebras of $C_b(C^n)$ and equivalence classes of non-separating compactifications: Given a unital $C^*$ subalgebra $A$ of $C_b(C^n)$, a representative of the equivalence class of non-separating compactifications is given by $(ev, M(A))$, where $M(A)$ is the maximal ideal space of $A$ with $w^*$ topology, and $ev$ is the map \[ev : C^n \to M(A), \quad ev(x)(f) = f(x).\] On the other hand, given any non-separating compactification $(\psi, X)$, we define $A_{(\psi, X)} \subset C_b(C^n)$ through \[A_{(\psi, X)} := \{f \circ \psi; \quad f \in C(X)\},\]
which is a unital $C^*$ subalgebra of $C_b(\mathbb{C}^n)$. One readily checks that for each unital $C^*$ subalgebra $A$ of $C_b(\mathbb{C}^n)$ and each non-separating compactification $(\psi, X)$ of $\mathbb{C}^n$ we have

\[
A_{(\psi,\mathcal{M}(A))} = A, \\
(\psi, X) \sim (\psi, \mathcal{M}(A_{(\psi,X)})).
\]  

(3.3)

Observe that this correspondence restricts to a 1-1 correspondence between compactifications in the usual sense (i.e. where the map $\psi$ is further assumed to be injective) and unital $C^*$ subalgebras of $C_b(\mathbb{C}^n)$ which separate points.

In what follows, let $A$ denote an $\alpha$- and $U$-invariant unital $C^*$ subalgebra of $BUC(\mathbb{C}^n)$, i.e. for each $f \in A$ and $z \in \mathbb{C}^n$ we have $\alpha_z(f) \in A$ and $Uf \in A$. We will always identify $\mathbb{C}^n$ with its image in the compactification corresponding to $A$. Let $f \in A$ and $z \in \mathbb{C}^n$. For $z, w \in \mathbb{C}^n$ we have

\[
\alpha_z(f)(w) = ev(z)(U\alpha_w(f)).
\]

(3.4)

For $x \in \mathcal{M}(A)$ and $w \in \mathbb{C}^n$ we define

\[
f_x(w) = x(U\alpha_w(f)),
\]

where we will abuse the notation $f_z = f_{ev(z)}$ for $z \in \mathbb{C}^n$, which is justified by Equation (3.4). As in [4], one proves that $f_x \in BUC(\mathbb{C}^n)$ for all $x \in \mathcal{M}(A)$. Let $(z_n)$, $\gamma \in \mathbb{C}^n$ be any net converging to $x \in \mathcal{M}(A)$. Then, $\alpha_{z_n}(f) = f_{z_n} \to f_x$ pointwise. An easy application of the Arzelà-Ascoli theorem shows that this convergence is even uniformly on compact subsets. On the level of Toeplitz operators, one can then show that $\alpha_{z_n}(T_{f}) = T_{\alpha_{z_n}(f)} \overset{SOT^*}{\to} T_{f_x}$, where the convergence is in strong operator topology [4]. This has the following important consequence, which follows as in [4] with only minor changes (since we are using a possibly smaller compactification of $\mathbb{C}^n$ than $\mathcal{M}(BUC(\mathbb{C}^n))$):

**Proposition 3.15.** For each $A \in \mathcal{T}_{lin}^{p,t}(A)$, the map

\[
\mathbb{C}^n \to \mathcal{T}_{lin}^{p,t}(A), \quad z \mapsto \alpha_z(A)
\]

extends to a continuous map

\[
\mathcal{M}(A) \to (\mathcal{T}_{lin}^{p,t}(BUC(\mathbb{C}^n)), \text{SOT}^*), \quad x \mapsto A_x,
\]

which is norm-bounded by $\|A\|_{op}$. Here, SOT$^*$ denotes the strong$^*$ operator topology, i.e. both functions $x \mapsto A_x$ and its Banach space adjoint (understood in the dual pairing coming from $F^2_\mathbb{C}$) $x \mapsto A_x^*$ are continuous with respect to the strong operator topology.

**Remark 3.16.** For each $A \in \mathcal{T}_{lin}^{p,t}(A)$, the operators $A_x$ ($x \in \mathcal{M}(A) \setminus \mathbb{C}^n$) are called the limit operators of $A$. They are of independent interest, as they can be used to investigate the Fredholm property and essential spectrum of $A$ [12].

Let $A, B \in \mathcal{T}_{lin}^{p,t}(A)$. Then, both $z \mapsto \alpha_z(A)$ and $z \mapsto \alpha_z(B)$ are uniformly bounded in norm by $\|A\|_{op}$ and $\|B\|_{op}$ respectively. Therefore,

\[
z \mapsto \alpha_z(AB) = \alpha_z(A)\alpha_z(B)
\]
extends to a strongly continuous map
\[ M(A) \ni x \mapsto (AB)_x = A_x B_x \in T^{p,t}, \]
i.e. passing to the limit operators is multiplicative. Further, since \((L(F^p_t))^* \ni W^*_2 = W_{-2} \in L(F^q_t)\) \((1/p + 1/q = 1)\), one sees that \((A_x)^* = (A^*)_x\).

**Proposition 3.17.** Let \( A \) be a unital, \( \alpha \)- and \( U \)-invariant \( C^* \) subalgebra of \( BUC(C^n) \). Then we have
\[ T^{p,t}_{lin}(A) = T^{p,t}(A) \]
and
\[ T^{2,t}_{lin}(A) = T^{2,t}_*(A) \]
for all \( t > 0 \).

**Proof.** Observe that the second statement and the first statement are equivalent, since the first is \( p \)-independent by Corollary 3.9 and \( T^{2,t}_{lin}(A) = T^{2,t}_*(A) \) follows since \( A \) is closed under taking complex conjugates. Therefore, it suffices to prove that \( T^{2,t}_{lin}(A) \) is closed under taking products.

Let \( A, B \in T^{2,t}_{lin}(A) \). As noted above,
\[ z \mapsto \alpha_z(AB) \]
extends to a strongly continuous map
\[ M(A) \ni x \mapsto (AB)_x \in T^{2,t}. \]
For the Berezin transform of the product, we have
\[ (AB)^*(z) = (\alpha_z(AB))^*(0) = ((AB)_z 1,1)_{F^2_t}, \]
which extends by strong continuity of \( x \mapsto (AB)_x \) to \((AB)^* \in C(M(A))\). Hence, \((AB)^* \in A\) by (3.5). Therefore, Theorem 2.11(3) yields that \( AB \in T^{2,t}_{lin}(A) \leftrightarrow A \), which finishes the proof.

We say that \( I \subset A \) is an \( \alpha \)- and \( U \)-invariant ideal of \( A \) if it is a closed ideal of \( A \) such that \( f \in I \) and \( z \in C^n \) imply \( \alpha_z(f) \in I \) and \( Uf \in I \). We will classify such ideals now.

Recall that the closed ideals \( I \) in \( A \cong C(M(A)) \) are in 1-1 correspondence with closed subsets \( I \) of \( M(A) \) via
\[ I_f = \{ f \in C(M(A)); f(x) = 0, x \in I \}. \]
One easily sees that for \( I \subset M(A) \) such that \( I \cap C^n \neq \emptyset \) and \( I_f \) translation-invariant, one necessarily has \( I = M(A) \) or equivalently \( I_f = \{ 0 \} \). Therefore, proper translation-invariant ideals “live at the boundary”. A classification of them can be given in terms of the following property of boundary points:

**Definition 3.18.** Let \( A \) be a unital \( \alpha \)- and \( U \)-invariant \( C^* \) subalgebra of \( BUC(C^n) \). We say that \( x \in M(A) \setminus C^n \) has Property A if the following holds true:

For each \( f \in A \) and \( z \in C^n \) we have \( x(f) = x(\alpha_z(f)) \) and \( x(Uf) = x(f) \).
Example. If we let $A = C_0(\mathbb{C}^n) \oplus \mathbb{C}1 \subset BUC(\mathbb{C}^n)$, then $\mathcal{M}(A)$ is the one point compactification of $\mathbb{C}^n$. One easily sees that the point at infinity has this property.

**Proposition 3.19.** (i) Maximal $\alpha$- and $U$-invariant closed ideals of $A$ are in 1-1 correspondence with points $x \in \mathcal{M}(A) \setminus \mathbb{C}^n$ which have Property A.

(ii) For a closed subset $I \subset \mathcal{M}(A) \setminus \mathbb{C}^n$ the ideal $I/I$ is $\alpha$- and $U$-invariant if and only if each $x \in I$ has Property A.

**Proof.** (i) This follows easily from the definitions.

(ii) Each closed ideal $I/I$ can be written as the intersection of maximal closed ideals, i.e.

$$I/I = \cap_{x \in I} I/I_x.$$  

If $I/I$ is further $\alpha$- and $U$-invariant, one easily sees that each $I/I_x$, $x \in I$, needs to be $\alpha$- and $U$-invariant as well. Therefore, the result follows from (i).

\[ \square \]

**Proposition 3.20.** Let $A \subset BUC(\mathbb{C}^n)$ be a unital, $\alpha$- and $U$-invariant $C^*$ subalgebra. If $I \subset A$ is a closed $\alpha$- and $U$-invariant ideal of $A$, then $\mathcal{T}_{lin}^{p,t}(I)$ is a closed and two-sided $\alpha$- and $U$-invariant ideal in $\mathcal{T}_{lin}^{p,t}(A)$ for all $t > 0$.

**Proof.** By an argument analogous to Corollary 3.10 it suffices to prove that $\mathcal{T}_{lin}^{p,t}(I)$ is an ideal in $\mathcal{T}_{lin}^{p,t}(A)$.

Let $I = I/I$ with $I \subset \mathcal{M}(A) \setminus \mathbb{C}^n$ where each point in $I$ has Property A. Let $f \in I$. Using Property A, one easily sees that $f_x = 0$ for each $x \in I$. Hence, for each operator $A \in \mathcal{T}_{lin}^{2,t}(I)$ one obtains $A_x = 0$ for $x \in I$. Let $B \in \mathcal{T}_{lin}^{2,t}(A)$. Then, we have $(BA)_x = B_x A_x = 0$ and $(AB)_x = A_x B_x = 0$ for each $x \in I$.

Now finally, by the above discussion, both $(AB)^\sim$ and $(BA)^\sim$ extend to functions in $C(\mathcal{M}(A))$ which vanish on $I$, i.e. $(AB)^\sim, (BA)^\sim \in I$. Thus, Theorem 2.21 (3) yields $AB, BA \in \mathcal{T}_{lin}^{2,t}(I)$, which is therefore an ideal in $\mathcal{T}_{lin}^{2,t}(A)$.

\[ \square \]

We are now in the position to drop the assumption that the $C^*$ algebra $A$ contains the unit element:

**Corollary 3.21.** Let $A \subset BUC(\mathbb{C}^n)$ be an $\alpha$- and $U$-invariant $C^*$ subalgebra. Then, we have

$$\mathcal{T}_{lin}^{p,t}(A) = \mathcal{T}_{lin}^{p,t}(A)$$

and

$$\mathcal{T}_{lin}^{2,t}(A) = \mathcal{T}_{lin}^{2,t}(A).$$

If further $I \subset A$ is an $\alpha$- and $U$-invariant ideal, then $\mathcal{T}_{lin}^{p,t}(I)$ is a two-sided ideal in $\mathcal{T}_{lin}^{p,t}(A)$ for all $t > 0$.

**Proof.** Assume $A$ is not unital. $A \oplus \mathbb{C}1 \subset BUC(\mathbb{C}^n)$ is an $\alpha$- and $U$-invariant unital $C^*$ subalgebra of $BUC(\mathbb{C}^n)$ in which $A$ is an $\alpha$- and $U$-invariant ideal. Therefore, $\mathcal{T}_{lin}^{p,t}(A)$ is an ideal in $\mathcal{T}_{lin}^{p,t}(A \oplus \mathbb{C}1)$ and in particular a $C^*$ algebra and needs to agree with $\mathcal{T}_{lin}^{p,t}(A)$. Further, $I$ is also an ideal in $A \oplus \mathbb{C}1$, therefore $\mathcal{T}_{lin}^{p,t}(I)$ is an ideal in $\mathcal{T}_{lin}^{p,t}(A \oplus \mathbb{C}1)$ and also in $\mathcal{T}_{lin}^{p,t}(A)$.

Combing all results, we have obtain a proof of Theorem 3.13
4 Discussion

First, let us provide a class of examples which give proper linearly generated subalgebras of $T_{p,t}^{lin}$ according to Theorem 3.13.

Let $O \subset \mathbb{C}^n$ be a closed, nonempty subset. We set

$$A_O := \{ f \in BUC(\mathbb{C}^n); \alpha_w(f) = f \text{ for all } w \in O \}.$$

We might assume without loss of generality that $O$ is an additive subgroup of $\mathbb{C}^n$. One easily sees that $A_O$ is a unital $\alpha$- and $U$-invariant $C^*$ algebra. Hence, the results from Theorem 3.13 apply:

$$T_{p,t}^{lin}(A_O) = T_{p,t}(A_O).$$

It is now not hard to prove that

$$T_{p,t}^{lin}(A_O) = T_{p,t}^{lin}(\{ f \in \mathcal{L}\infty(\mathbb{C}^n); \alpha_w(f) = f, \ w \in \mathcal{L} \}).$$

Remark 4.1. If we let $O = \mathcal{L}$ be a Lagrangian subspace of $\mathbb{C}^n$ (with the special case $\mathcal{L} = \{0\} \times i\mathbb{R}^n \subset \mathbb{C}^n$) we obtain the class of Lagrangian-invariant (respectively horizontal) Toeplitz operators. These have been studied in [11]. Among other results, it has been proven there using different methods that

$$T_{2,1}^{lin}(\{ f \in \mathcal{L}\infty(\mathbb{C}^n); \alpha_w(f) = f, \ w \in \mathcal{L} \}) = T_{2,1}^*(\{ f \in \mathcal{L}\infty(\mathbb{C}^n); \alpha_w(f) = f, \ w \in \mathcal{L} \}),$$

which turns out to be a special case of the above proposition.

Let $O \subset \mathbb{C}^n$ be as above and fix $z \in \overline{\{0\} \times \mathbb{R}^n} \setminus \{0\}$. Let us denote by

$$I_{O,z} := \{ f \in A_O; \alpha_w(f)(\lambda z) \to 0, \ \lambda \in \mathbb{R}, \ |\lambda| \to \infty \text{ for all } w \in \text{span} \ O \}$$

the set of all functions in $A_O$ which vanish in the direction $z$ orthogonal to $O$ at infinity. $I_{O,z}$ is an $\alpha$- and $U$-invariant ideal of $A_O$ to which the results from Theorem 3.13 apply, i.e. $T_{lin}^{p,t}(I_{O,z})$ is a two-sided ideal in $T_{lin}^{p,t}(A_O)$ for all $t > 0$.

We end by mentioning several open problems:

1) Is Theorem 3.13 still valid if we remove the condition of $D_0$ or $I$ being $U$-invariant? One part of the proof, Proposition 3.14, does not depend on that property.

2) Can we get a general correspondence between Banach algebras on the symbol side and on the operator side? I.e. does $D_0$ an $\alpha$- (and possibly $U$-) invariant Banach algebra already imply that $T_{lin}^{p,t}(D_0) = T_{p,t}(D_0)$ for all $t > 0$? Again, the inverse to this is true due to Proposition 3.14.

3) In [23], J. Xia proved equality of the Toeplitz algebra $T_{2,1}$ with a certain $C^*$ algebra of sufficiently localized operators. Does such an equality also hold in the case $p \neq 2$?
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