A NOTE ON THE PAPER “HOMOMORPHISMS WITH RESPECT TO A FUNCTION”

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ABSTRACT. We give a direct proof of the main result of the paper “Homomorphisms with respect to a function” by K. Boulabiar and F. Gdara without using the Axiom of Choice.

1. The Theorem

For a topological space $X$, the lattice-ordered ring (under the pointwise algebraic operations and pointwise ordering) of all real-valued continuous functions on $X$ is denoted by $C(X)$. Recall that a completely regular Hausdorff space $X$ is called realcompact if it is homeomorphic to a closed subspace of the product space of the reals. For details about the lattice ordered group $C(X)$ and the notion of realcompactness, we refer to [4].

For $r \in \mathbb{R}$, the map $r : \mathbb{R} \to \mathbb{R}$ is defined by $r(x) := r$. For a nonempty index set $I$ and $j \in I$, the mapping $P_j : \prod_{i \in I} \mathbb{R} \to \mathbb{R}$ is defined by

$$P_j((x_i)_{i \in I}) := x_j,$$

and $e_j := (x_i)_{i \in I}$ with $x_j = 1$ and $x_i = 0$ for $i \neq j$. We note that $P_j \in C(\prod_{i \in I} \mathbb{R})$.

For a topological space $X$, $H : C(X) \to \mathbb{R}$ denotes a positive group homomorphism with $H(1) = 1$. We note that, in this case, $H$ is linear. For a $\varphi \in C(\mathbb{R})$, the map $H$ is called a $\varphi$-homomorphism if

$$H \circ \varphi = \varphi \circ H,$$

that is, for every $f \in C(X)$, we have

$$H(\varphi \circ f) = \varphi(H(f)).$$

We say that $H$ is point evaluated if there exists $k \in X$ such that

$$H(f) = f(k)$$

for all $f \in C(X)$. 

2000 Mathematics Subject Classification. 54C30, 46E05, 46E25.

Key words and phrases. Continuous function, Evaluation, Homomorphism, Realcompact.
The following is the main result of [1].

**Theorem 1.1.** Let $X$ be a realcompact space. The following are equivalent:

(i) $H$ is point evaluated at some point of $X$.

(ii) $H$ is a $\varphi$-homomorphism for all $\varphi \in C(\mathbb{R})$ with $\varphi(r) > \varphi(0)$ for all $r \in \mathbb{R} \setminus \{0\}$.

(iii) There exists $\varphi \in C(\mathbb{R})$ with $\varphi(r) > \varphi(0)$ for all $r \in \mathbb{R} \setminus \{0\}$ such that $H$ is a $\varphi$-homomorphism.

## 2. Proof Of Theorem 1.1

In [1], using the notions of Stone-Čech compactification and Stone-extensions, the proof of the above theorem is given, whereby the Axiom of choice has implicitly been used. Following [3], we can give a direct proof of the above theorem without using the Axiom of Choice.

**Proof of Theorem 1.1.** In the proof we mainly follow the arguments of [3], so to be as self contained as possible, we repeat some parts of the proof therein. As indicated in [1], the implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are obvious. To see (iii) $\implies$ (i), suppose that for a $\varphi \in C(\mathbb{R})$, the map $H$ is a $\varphi$-homomorphism. Without loss of generality, we can suppose that $\varphi(0) = 0$. There are two cases:

**Case 1.** $X = \prod_{i \in I} \mathbb{R}$.

Let $c := (c_i) := (\varphi(P_i))$. We show that for each $0 \leq f \in C(X)$, one has $\varphi(f) = f(c)$. For each $f \in C(X)$, let $k_f : X \to \mathbb{R}$ be defined by $k_f := f - f(c) \mathbf{1}$. Then $\varphi(k_f) = \varphi(f) - f(c)$ and $H(k_f) = 0$ if and only if $H(f) = f(c)$. Hence it is enough to show that $f(c) = 0$ implies $H(f) = 0$. Let $0 \leq f \in C(X)$ be given.

**Claim ($\dagger$):** if $f \mid_U = 0$ for some open set $U$ which contains $c$, then $H(f) = 0$. There exists a family $(U_i)_{i \in I}$ of open subsets of $\mathbb{R}$ such that

$$c \in \prod_{i \in I} U_i = V \subset U \quad \text{and} \quad F = \{i \in I : U_i \neq \mathbb{R}\} \quad \text{is finite.}$$

Let $h : X \to \mathbb{R}$ be defined by

$$h := \sum_{i \in F} \varphi \circ (P_i - c_i).$$

Note that

$$H(h) = 0.$$ 

It is clear that $h(x) \neq 0$ whenever $x \notin V$. Define $g : X \to \mathbb{R}$ by

$$g(x) := \frac{f(x)}{h(x)} \chi_{X \setminus V}.$$
Then $g$ is continuous and $f = gh$. For each $n \in \mathbb{N}$, since 

$$0 \leq g - g \wedge n \leq \frac{1}{n} g^2,$$

we have

$$0 \leq gh - (g \wedge n)h = gh - (gh) \wedge nh \leq \frac{1}{n} g^2 h.$$

Notice also that for each $n \in \mathbb{N}$, since $H$ is positive, we have

$$H((g \wedge n)h) = H(gh \wedge nh) \leq H(nh) = nH(h) = 0.$$

This implies that

$$0 \leq H(f) = H(fg) = H(fg - gh \wedge nh) \leq \frac{1}{n}(g^2 h)$$

for each $n$, whence $H(f) = 0$.

Claim (‡). if $f(c) = 0$, then $H(f) = 0$. Suppose that it is not. Then we may suppose that $H(f) = 1$ and $f(c) = 0$. For each $n \in \mathbb{N}$, let

$$U_n = f^{-1}(-\frac{1}{n}, \frac{1}{n}).$$

Then $c \in U_n$ and

$$(f - f \wedge \frac{1}{n})|U_n = 0.$$

By Claim (‡), for all $n$, we have

$$1 = H(f) = H(f \wedge \frac{1}{n}) \leq H(\frac{1}{n}) = \frac{1}{n} \to 0,$$

which is a contradiction. Clearly this shows that $H(f) = 0$ whenever $f(c) = 0$.

Case 2. $X$ is homeomorphic to a closed subset $Y = \prod_{i \in I} \mathbb{R}$.

Let $\pi : C(Y) \to C(X)$ be defined by

$$\pi(f) := f|_X,$$

where $f|_X$ denotes the restriction of $f$ to $X$. Then there exists $c \in Y$ such that $H \circ \pi(f) = f(c)$ for each $f \in C(Y)$. Suppose that $c \notin X$. As $Y$ is a completely regular Hausdorff space, there exists $f \in C(Y)$ such that $f(c) = 1$ and $f|_X = 0$. This implies that

$$1 = f(c) = P \circ \pi(f) = P(f|X) = 0,$$

which is a contradiction. Hence $H(f) = f(c)$ for all $0 \leq f \in C(X)$. For arbitrary $f \in C(X)$, we then have

$$H(f) = H(f^+ - f^-) = H(f^+) - H(f^-) = f^+(c) - f^-(c) = f(c).$$

This completes the proof. □
In [2] it is noticed that if \( X \) is a completely regular Hausdorff space then a nonzero linear map, \( H : C(X) \to \mathbb{R} \) with \( H(1) = 1 \), is a ring homomorphism if and only if there exists a net \((x_\alpha)\) in \( Y \) such that
\[
H(f) = \lim f(x_\alpha)
\]
for all \( f \in C(X) \). By combining this with the above Theorem immediately we get the following theorem.

**Theorem 2.1.** Let \( X \) be a completely regular Hausdorff space and \( H : C(X) \to \mathbb{R} \) be a positive group homomorphism with \( H(1) = 1 \). Then the following are equivalent.

i.) \( H \) is a Ring homomorphism.

ii.) \( H \) is a \( \varphi \)-homomorphism for all \( \varphi \in C(\mathbb{R}) \) satisfying \( \varphi(x) > \varphi(0) \) for all \( x \neq 0 \).

iii.) \( H \) is a Riesz homomorphism.

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