DELOCALIZED BETTI NUMBERS AND MORSE TYPE INEQUALITIES

M.E. ZADEH

Abstract. In this paper we state and prove Morse type inequalities for Morse functions as well as for closed differential 1-forms. These inequalities involve de localized Betti numbers. As an immediate consequence, we prove the vanishing of delocalized Betti numbers of manifolds fibering over the circle.

1. Introduction

Given a manifold $M$ and a real Morse function $f$ on $M$ the following Morse inequalities establish relations between the very basic topology of $M$ and the number of critical points of order $j$ denoted by $C_j$ (cf. [6])

$$C_k - C_{k-1} + \cdots + C_0 \geq \beta^k - \beta^{k-1} + \cdots + \beta^0$$

Here $\beta^j := \dim H^j(M, \mathbb{R})$ is the $j$-th Betti number of $M$. These relations have been the subject of many significant generalizations. S.Novikov and M.Shubin have proved in [8] that these inequalities hold if the Betti numbers being replaced by the $L^2$-Betti numbers. The $L^2$-Betti numbers(or von Neumann Betti numbers) were introduced by M.Atiyah in [1] in his investigation on equivariant index theorem. The Morse theory for closed 1-forms has been introduced by S.Novikov and he has proved in [7] that the Morse inequalities can be generalized to closed 1-forms if one replace the Betti numbers by the Novikov numbers. In [2, theorem 1] it is shown that the Novikov-Shubin inequalities hold as well for closed 1-forms. In this paper we are interested in the delocalized Betti numbers which are introduced by J.Lott in [4]. These delocalized Betti numbers are not yet well studied and enjoy properties which are not satisfied by the ordinary or $L^2$-Betti numbers, e.g the delocalized Betti numbers of any manifold with free abelian fundamental group and of hyperbolic manifolds vanish. However In this paper we show that some appropriate combinations of delocalized Betti numbers satisfy the Novikov-Shubin and the Farber inequalities (see theorems 2 and 5). We prove the delocalized Novikov-Shubin inequalities by following J.Roe’s account[10] of the E.Witten approach to Morse theory [11]. As a consequence of this method we reprove the vanishing of the delocalized Euler character of $M$. To prove the delocalized Morse inequalities for closed 1-forms we use these inequalities for Morse function and follow the method used by M.Farber. As a consequence of the Morse inequalities for closed 1-forms we prove that the delocalized Betti numbers of manifolds which fiber over the circle vanish. This vanishing theorem for $L^2$-Betti numbers was conjectured by M.Gromov and proved by W.Lück in [5].

In section 2 we state and prove the Morse inequalities in a very general analytic framework. And in section 3 we use these inequalities to prove the Morse inequalities for Morse functions and delocalized Betti numbers. In these two section we follow closely the methods used by J.Roe in [10] chapter 14. We use the main theorem of section 3 to prove the Morse inequalities for closed 1-forms and delocalized Betti numbers in section 4.

2. General analytic Morse inequalities

Let $(M, g)$ be a closed oriented riemannian manifold and denote by $\triangle^k$ the laplacian operator acting on differential $k$-forms on $M$. Let $(\tilde{M}, \tilde{g})$ be the riemannian universal covering of $M$ with $\tilde{g} = \pi^* g$, where $\pi$ is the covering map. We denote by $G$ the fundamental group of $M$ and by $\hat{\triangle}^k = d^* d + dd^*$

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the laplacian operator acting on $L^2$-elements of $\Omega^k(M)$. For $0 \leq k \leq n$ let $T_k$ be a real valued non-negative and continuous trace on the space of all smoothing $G$-invariant operators on $L^2(M, \Lambda^k T^* M)$. The continuity is understood with respect to the uniform convergence of Schwartz kernels on compact subsets of $M \times M$. We usually omit the subfix $k$ and denote these traces by the same symbol $T$. Let $\tilde{P}^k$ denotes the orthogonal projection on $\ker \tilde{\Delta}^k$ which is a smoothing operator on $L^2(M, \Lambda^k T^* M)$, c.f. [1]. We define the $k$-th T-Betti number by the following relation

$$\beta^k_T := T(\tilde{P}^k).$$

For $\phi$ a rapidly decreasing non-negative smooth function on $\mathbb{R}^2$ with $\phi(0) = 1$. The operator $\phi(\tilde{\Delta}^k)$ is a smoothing operator and so we can define $\mu^k_T = T(\phi(\tilde{\Delta}^k))$. Notice that $\beta^k_T$ and $\mu^k_T$ are both non-negative real numbers.

**Theorem 1** (analytic Morse inequalities). For $0 \leq k \leq n = \dim M$ we have the following inequalities

$$\mu^k_T - \mu^{k-1}_T + \cdots \pm \mu^0_T \geq \beta^k_T - \beta^{k-1}_T + \cdots \pm \beta^0_T,$$

and the equality holds for $k = n$.

**proof** Let $\{\phi_m\}_m$ be a sequence of non-negative rapidly decreasing smooth functions on $\mathbb{R}^2$ which converges to zero outside of $0 \in \mathbb{R}$ and $\phi_m(0) = 1$. The operator $\phi_m(\Delta^k)$ consists of non-negative smoothing operators with smooth Schwartz kernel $K_m$. The sequence of kernels $K_m$ converges uniformly on compact subsets of $M \times M$ to the kernel $\tilde{K}$ of the projection $\tilde{P}^k$. In fact the spectral theorem for self-adjoint operator $\tilde{\Delta}^k$ (see, e.g. [9] theorem VIII. 5) implies that $\phi_m(\tilde{\Delta}^k) \rightarrow \tilde{P}^k$ for any $L^2$-differential $k$-form $\omega$, i.e. $\phi_m(\tilde{\Delta}^k) \rightharpoonup \tilde{P}^k$ weakly in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$. Since the strong and the weak topology of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ coincide on bounded subsets, e.g. converging sequence one conclude the convergence $\phi_m(\tilde{\Delta}^k) \rightharpoonup \tilde{P}^k$ in the topological space $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ with strong topology. The Schwartz kernel theorem asserts that this topological space is isomorphic to $\mathcal{D}'(M \times M, \Lambda^k T^* M \times \Lambda^k T^* M)$, and with respect to this isomorphism the The convergence $\phi_m(\tilde{\Delta}^k) \rightarrow \tilde{P}^k$ reads the convergence of the kernels $K_m \rightarrow \tilde{K}$ which implies the assertion. Therefore by continuity of $T$ and by finiteness of $|\langle g \rangle|$ we obtain

$$\beta^k_T = \lim_{m \rightarrow \infty} T(\phi_m(\tilde{\Delta}^k)).$$

The function $\phi - \phi_m$ is non-negative, rapidly decreasing and vanishing at $0$, so it takes the form $x \psi^k_m(x)$ where $\psi_m$ is non-negative and rapidly decreasing. We have

$$T(dd^* \psi_m(\tilde{\Delta}^k)2) = T(\psi_m(\tilde{\Delta}^k)d d^* \psi_m(\tilde{\Delta}^k))$$

$$= T(d^* \psi_m(\tilde{\Delta}^k)2 d)$$

$$= T(d^* \psi_m(\tilde{\Delta}^k)2).$$

In the last step we have used the commutation relation $\tilde{\Delta}^k d = d \tilde{\Delta}^k - 1$. Therefore

$$(\mu^k_T - T(\phi_m(\tilde{\Delta}^k)) - (\mu^{k-1}_T - T(\phi_m(\tilde{\Delta}^k))) + \cdots \pm (\mu^0_T - T(\phi_m(\tilde{\Delta}^k))) = T(d^* \psi_m(\tilde{\Delta}^k)2).$$

Since $d^* \psi_m(\tilde{\Delta}^k)2$ is a non-negative smoothing operator, the right side of the above equality is non-negative for $k < n$ and is zero for $k = n$. By tending $m$ toward infinity and using relation (2.2) we get the desired inequalities. 

### 3. Delocalized Novikov-Shubin inequalities

We recall the definition of *delocalized Betti numbers* as they are introduced by J. Lott in [3]. We keep the notation of the previous section. Let $\tilde{P}$ be a smoothing $G$-invariant operator acting on $\Lambda^k T^* M$ with Schwartz kernel $\tilde{K}$ which is rapidly decreasing far from the diagonal of $M \times M$. For each $\tilde{x} \in M$ and $h \in G$ one can identify both $\Lambda^k T^*_{\tilde{x}} M$ and $\Lambda^k T^*_{h \tilde{x}} M$ with $\Lambda^k T^* M$ where $x = \pi(\tilde{x})$. Hencefor $\tilde{K}(\tilde{x}, h \cdot \tilde{x})$ can be considered as an element of $\text{End}(\Lambda^k T^* M)$. Consequently, given a conjugacy class $\langle g \rangle$ of $G$ the sum $\sum_{h \in \langle g \rangle} \tilde{K}(h \cdot \tilde{x}, \tilde{x})$ is finite and, as a function of $\tilde{x} \in \tilde{M}$, is invariant with respect to
the action of $G$. Therefore this function pushes down and defines a smooth section $K_{(g)}$ of the bundle $\text{End}(\Lambda^k(T^*M))$ over $M$. The following relation defines the delocalized trace $\text{Tr}_{(g)}$

$$\text{Tr}_{(g)}(\tilde{P}) := \int_M \text{tr} K_{(g)} \, d\mu_g.$$

As in the previous section let $\tilde{\Delta}^k$ denotes the laplacian operator acting on $L^2$-sections of $\Lambda^k T^*M$. The orthogonal projection $\tilde{P}^k$ on $\ker(\tilde{\Delta}^k)$ is a smoothing $G$-invariant operator on $L^2(M, \Lambda^k T^*M)$ with kernel $\tilde{K}^k$ (see, e.g. [1]). The $k$-th delocalized Betti number $\beta^k_{(g)}$ is defined as follows

$$\beta^k_{(g)} = \text{Tr}_{(g)}(\tilde{P}^k).$$

Equivalently if $\phi_m$ is a sequence of real functions as in the theorem [1] e.g. $\phi_m(x) = e^{-x/m}$ then by [2]

$$\beta^k_T = \lim_{m \to \infty} \text{Tr}_{(g)}(\phi_m(\tilde{\Delta}^k)).$$

**Remark 1.** Notice that the trace $\text{Tr}_{(g)}$ is not always finite since it is not known weather the kernel $\tilde{K}^k$ is rapidly decreasing far from the diagonal of $M \times M$. Nevertheless if $(g)$ is a finite set then the above delocalized Betti numbers are well defined. This is why we restrict ourselves, from now on, to the class $\mathcal{C}(G)$ of finite conjugacy classes.

Since the delocalized traces $\text{Tr}_{(g)}$ are not in general positive, we cannot apply the theorem [1] to these traces (except for $k = n$). Instead we consider the following linear functional which are introduced in [3] and are proved to be positive traces

$$\text{Tr}_{(g)} := \text{Tr}_{(e)} + \frac{1}{|\langle g \rangle|} \text{Tr}_{(g)}$$

**Lemma 2.** The linear functional $\text{Tr}_{(g)}$ is a positive trace on the space of $G$-invariant smoothing operator.

**proof** For each $g \in G$ the linear functional $\text{Tr}_{(g)}$ is a trace (cf. [3] lemma 2) so $\text{Tr}_{(g)}$ is also a trace. In below we show that it is positive, i.e. $\text{Tr}_{(g)}(\tilde{P}) \geq 0$ if $\tilde{P} = \tilde{Q}^* \tilde{Q}$ where $\tilde{Q}$ is a $G$ invariant smoothing operator on $H := L^2(M, \Lambda^k T^*M)$. Let $\{\theta_k\}_{k \in \mathbb{N}}$ be an orthonormal basis for $H$ and for $h \in G$ put $(h,\theta)(\tilde{x}) := \theta(h \cdot \tilde{x})$. We identify $M$ with a distinguished fundamental domain of $G$ in $\tilde{M}$. With this identification we have

$$\text{Tr}_{(e)}(\tilde{P}) = \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(\tilde{x}) \rangle \, d\mu_{\tilde{g}}$$

$$= \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(\tilde{x}) \rangle \, d\mu_{\tilde{g}}^{\frac{1}{2}} \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(\tilde{x}) \rangle \, d\mu_{\tilde{g}}^{\frac{1}{2}}$$

$$= \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(\tilde{x}) \rangle \, d\mu_{\tilde{g}}^{\frac{1}{2}} \sum_k \int_M \langle \tilde{Q} \theta_k(h \cdot \tilde{x}), \tilde{Q} \theta_k(h \cdot \tilde{x}) \rangle \, d\mu_{\tilde{g}}^{\frac{1}{2}}$$

$$\geq \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(\tilde{x}) \rangle \, d\mu_{\tilde{g}}^{\frac{1}{2}} \langle \int_M \langle \tilde{Q} \theta_k(h \cdot \tilde{x}), \tilde{Q} \theta_k(h \cdot \tilde{x}) \rangle \, d\mu_{\tilde{g}} \rangle^{\frac{1}{2}}$$

Here the third equality follows from the $G$-invariance of the riemannian volume element, while the last inequality is the Cauchy-Schwarz inequality for $L^2$-sequences. The following pairing is a symmetric bilinear function on $H$

$$\langle \omega, \eta \rangle := \int_M \langle \omega(\tilde{x}), \eta(\tilde{x}) \rangle \, d\mu_{\tilde{g}}$$

By applying the Cauchy-Schwarz inequality arising from this bilinear function to the last expression in above, we get the following inequality

$$\text{Tr}_{(e)}(\tilde{P}) \geq \sum_k \int_M \langle \tilde{Q} \theta_k(\tilde{x}), \tilde{Q} \theta_k(h \cdot \tilde{x}) \rangle \, d\mu_{\tilde{g}}.$$
This inequality beside the following relation
\[ \text{Tr}_{(\xi)}(\hat{P}) = \sum_{h \in (\xi)} \sum_{k} \int_{M} \langle \hat{Q}\theta_k(\hat{x}), \hat{Q}\theta_k(\hat{h}, x) \rangle \, d\mu_{\hat{h}} \]
implies the following relation which prove the assertion of the lemma
\[ |\langle \xi \rangle \text{Tr}_{(\epsilon)}(\hat{P})| \geq |\text{Tr}_{(\xi)}(\hat{P})| \, . \]

Using the trace \( T_{(\xi)} \) we define the following combination of delocalized Betti numbers
\[ \gamma_{(\xi)}^k := T_{(\xi)}(\hat{P}^k) = \beta_{(\xi)}^k + \frac{1}{|\langle \xi \rangle|} \beta_{(\xi)}^k \, . \]

Now let \( f \) be a Morse function on \( M \) and denote by \( \hat{f} \) its lifting to \( \hat{M} \). For \( s > 0 \) put \( d_s := e^{-sf}de^{sf} \) and \( d_s^\ast := e^{sf}d^*e^{-sf} \) and define the Witten deformed laplacian, acting on \( L^2 \)-elements of \( \Omega^k(\hat{M}) \), by the following relation
\[ \tilde{\Delta}_s^k : d_s d_s^\ast + d_s^\ast d_s \]
This deformed laplacian is a perturbation of the laplacian \( \tilde{\Delta}^k \) by differential operators of order zero. So it is an elliptic second order differential operator and its \( L^2 \)-kernel consists of smooth differential \( k \)-forms and the projection on this kernel is a smoothing operator, c.f. \( [10] \). Just as in above we can define the deformed \( k \)-th delocalized Betti number \( \beta_{(\xi)}^k(s) \). In fact these deformed delocalized Betti numbers \( \beta_{(\xi)}^k(s) \) are independent of \( s \). To see that consider the deformed de-Rham complex of \( L^2 \)-differential forms
\[ \cdots \to \Omega^k(T^*\hat{M}) \xrightarrow{d^k} \Omega^{k+1}(T^*\hat{M}) \to \cdots \]
This complex is equivariant with respect to the action of the group \( G \), so the \( k \)-th homology vector space of this complex \( H^k_s(\hat{M}, \mathbb{R}) \) is a \( G \)-vector space. This real \( G \)-vector space is isomorphic to the kernel of \( \tilde{\Delta}_s^k \). The isomorphism associates to an element in ker \( \tilde{\Delta}_s^k \) its class in \( H^k_s(\hat{M}, \mathbb{R}) \) and is clearly \( G \)-equivariant. Moreover the above deformed complex is isomorphic to the ordinary de-Rham complex (corresponding to \( s = 0 \)) through conjugation with \( e^sf \) which is \( G \)-equivariant as well. We conclude that there is a \( G \)-equivariant isomorphism between ker \( \tilde{\Delta}_s^k \) and ker \( \tilde{\Delta}^k \). Since the action of \( G \) on \( L^2 \)-differential form is symmetric this implies that the orthogonal projections on ker \( \tilde{\Delta}_s^k \) and on ker \( \tilde{\Delta}^k \) are \( G \)-similar. Therefore by taking the \( \text{Tr}_{(\xi)} \) we conclude that the deformed Betti numbers are actually independent of \( s \). In particular
\[ (3.5) \quad \gamma_{(\xi)}^k(s) = \gamma_{(\xi)}^k : \quad 0 \leq k \leq n. \]
These relations and the theorem \( [10] \) applied to the deformed laplacian imply the following inequalities (equality holds for \( k = n \))
\[ (3.6) \quad \mu_{(\xi)}^k(s) - \mu_{(\xi)}^{k-1}(s) \geq \gamma_{(\xi)}^k - \gamma_{(\xi)}^{k-1} + \cdots + \gamma_{(\xi)}^0, \]
where \( \mu_{(\xi)}^k(s) := T_{(\xi)}(\phi(\tilde{\Delta}_s^k)) \). We shall to study the behavior of \( \mu_{(\xi)}^k(s) \) when \( s \) goes to infinity in order to prove the following theorem.

**Theorem 3** (delocalized Novikov-Shubin inequalities). Let \( f \) be a Morse function on \( M \) and denote by \( C_k \) the number of critical point of Morse index \( k \). For \( 0 \leq k \leq n = \dim M \) we have the following inequalities
\[ (3.7) \quad C_k - C_{k-1} + \cdots + C_0 \geq \gamma_{(\xi)}^k - \gamma_{(\xi)}^{k-1} + \cdots + \gamma_{(\xi)}^0, \]
and the equality holds for \( k = n \).

**proof** At first we recall from \( [10] \) that the deformed laplacian has the following form
\[ (3.8) \quad \tilde{\Delta}_s^k = \tilde{\Delta}^k + sL_0 + s^2|df|^2 \]
Lemma 4. Let $\phi$ be a rapidly decreasing function as above such that the Fourier transform of the function $\psi$ defined by $\psi(t) := |t^2|$ is supported in $(-r, r)$. Let $\hat{K}$ denote the kernel of the smoothing operator $\phi(\Delta_k^\beta)$. Then $\hat{K}(., .)$ tends uniformly to zero on $\bar{M}\backslash \bar{U}_{2r} \times \bar{M}\backslash \bar{U}_{2r}$ when $s$ goes to infinity.

On the other hand the Schwartz kernel of $\phi(\Delta_k^\beta)$ is supported in the distance $r$ of the diagonal of $M \times M$. So if $\omega$ is a differential $k$-form which is supported within the distance $2r$ of a critical point of $\beta$ then $\phi(\Delta_k^\beta)\omega$ is supported within the distance $3r$ of the same critical point. To see this let $\hat{D}_s := d_s + d_s^*$, then $\nabla_s = (\hat{D}_s)^2$ and by the condition on the support of $\psi$ we have

$$\phi(\Delta_k^\beta)\omega(\tilde{x}) = \psi(D_x)\omega(\tilde{x}) = \int_{-r}^r \psi(t)e^{-it\hat{D}_s}\omega(\tilde{x})\, dt$$

This relation and the unite propagation speed property for the deformed Dirac operator $\hat{D}_s$ imply that $\phi(\Delta_k^\beta)\omega$ is supported within distance $3r$ of the support of $\omega$. Since the action of $G$ on $M$ is uniformly properly discontinuous, for $r$ sufficiently small and for a non-trivial element $h \in G$, the element $(\tilde{x}, h.\tilde{x})$ is not in the distance $2r$ of the diagonal of $M \times M$. Consequently the previous lemma and the above discussion show that for a non-trivial conjugacy class $(g)$ the delocalized traces $\text{Tr}_{(g)}$ has no contribution in the value of $\lim_{s \to \infty} T_{(g)}(\phi(\Delta_k^\beta))$ when $s$ goes to infinity, so

$$\lim_{s \to \infty} \mu_{(g)}^k(s) = \lim_{s \to \infty} \text{Tr}_{(g)}(\phi(\Delta_k^\beta)).$$

Now we shall to prove the following equality which prove the desired inequalities of the theorem

$$\lim_{s \to \infty} \text{Tr}_{(g)}(\phi(\Delta_k^\beta)) = C_k.$$

Here $C_k$ is the number of the critical points of $f$ with index $k$. For this purpose Let $\tilde{\beta} = p^*\beta$ be a $G$-invariant smooth function on $\bar{M}$ which is supported in $\bar{U}_{3r}$ and is equal to 1 on $\bar{U}_{2r}$. The above lemma shows that

$$\lim_{s \to \infty} \text{Tr}_{(g)}(\phi(\Delta_k^\beta)) = \lim_{s \to \infty} \text{Tr}_{(g)}(\tilde{\beta}\phi(\Delta_k^\beta)),$$

where $\tilde{\beta}$ is the pointwise multiplication by $\tilde{\beta}$. So, the next step is to study the asymptotic behavior of $\text{Tr}_{(g)}(\tilde{\beta}\phi(\Delta_k^\beta))$ when $s$ goes to infinity. Since the kernel of $\phi(\Delta_k^\beta)$ is supported in the distance $r$ of the diagonal, the differential forms which are supported outside $\bar{U}_{4r}$ has no contribution in the value of the expression in the right hand side of (3.12), so to evaluate the value of this expression we can consider only those differential forms which are supported in $\bar{U}_{4r}$. In fact the previous lemma and the condition on the support and values of $\beta$ show that we may consider those differential forms which are supported in $\bar{U}_{3r}$. As for $\Delta_k^\beta$, the kernel of $\phi(\Delta_k^\beta)$ is supported in the distance $r$ of the diagonal of $M \times M$. So for $r$ sufficiently small one can lift the smoothing operator $\phi(\Delta_k^\beta)$ to $\bar{M}$. We have the following equality

$$\phi(\Delta_k^\beta)\omega = p^*\phi(\Delta_k^\beta)\omega; \quad \text{supp}(\omega) \subset \bar{U}_{3r},$$

To prove this equality, by relations (3.9), it suffices to show that

$$e^{-it\hat{D}_s}\omega = p^*(e^{-it\hat{D}_s})\omega; \quad -r \leq t \leq r.$$
wave equation $\partial_t + i\dot{D}_s = 0$. Moreover they have the same initial condition $\omega$ for $t = 0$. Therefore by uniqueness of wave operator we get the desired equality. Consequently

$$\text{Tr}_{(\omega)}(\beta\phi(\Delta^k_s)) = \text{Tr}_{(\omega)} p^*(\beta\phi(\Delta^k_s))$$

$$= \text{Tr}(\beta\phi(\Delta^k_s)).$$

(3.14)

Now the argument leading to relation (3.12) can be applied to $\Delta^k_s$ to deduce

$$\lim_{s \to \infty} \text{Tr}(\beta\phi(\Delta^k_s)) = \lim_{s \to \infty} \text{Tr}\phi(\Delta^k_s) = C_k.$$

The last equality is the main step of the analytical proof of the Morse inequalities for the Morse function $f$ on $M$, c.f. \cite[page 192]{10}. This equality and the above discussion prove the relation (3.11). This complete the proof of the theorem.

□

The inequality (3.7) can be written in the following form

$$(3.15)$$

$$C_k - C_{k-1} + \cdots + C_0 \geq \gamma_k - \gamma_{k-1} + \cdots + \gamma_0 + B^k_{(g)}.$$

where

$$B^k_{(g)} := \frac{1}{\langle g \rangle} (\beta^k - \beta^{k-1} + \cdots + \beta_0).$$

Inequalities (3.15) without the term $B^k_{(g)}$ at the right hand side are the Morse inequalities for $L^2$-Betti numbers established by S.Novikov and M.Shubin \cite{8}. The proof of the theorem \cite{3} can be applied to the localized trace $\text{Tr}_{(\omega)}$ and provides an analytical proof for the Novikov-Shubin inequalities.

**Remark 2.** It is clear that the $k = n$-case equality of the theorem \cite{1} holds even if the trace $T$ is not real. If we apply this equality to $\text{Tr}_{(g)}$ and follow the proof of the theorem \cite{3} we get the vanishing of the delocalized Euler character

$$\chi_{(g)}(M) := \beta^n - \beta^{n-1} + \cdots + \beta_0 = 0.$$

Of course this result can be proved by the heat equation approach to compute the Euler characteristic.

4. **Delocalized Morse inequalities for closed 1-forms**

In this section we use the main theorem of the previous section to prove a delocalized version of the Morse inequalities for closed 1-forms. Following \cite{2} we use then these inequalities to prove the vanishing of the delocalized Betti numbers for spacies which fiberat over the circle. Vanishing of the $L^2$-Betti numbers of such spaces was conjectured by M.Gromov and proved by W.Lück \cite{5}.

Let $\omega$ be a closed differential 1-form on $M$. In a small open subset $U$ one has $\omega = df_U$, where $f_U$ is a smooth function on $U$ uniquely determined up to an additive constant. A point $p \in U$ is a non-degenerate critical point of $\omega$ with index $j$ if it is a non-degenerate critical point of $f_U$ with index $j$. As in the previous section we denote by $C_j$ the number of these points. Since $\omega$ is closed the map

$$(4.1)$$

$$\gamma \mapsto \int_\gamma \omega$$

defines a homeomorphism between the fundamental groups $G$ and the additive group $(\mathbb{R}, +)$.

As in the previous section let $C_j$ denotes the number of critical points of index $j$. The following theorem reduces to the theorem \cite{3} if $\omega$ is an exact form.

**Theorem 5** (delocalized Morse inequalities for closed 1-forms). For $0 \leq k \leq n = \dim M$ the following inequalities hold

$$(4.2)$$

$$C_k - C_{k-1} + \cdots + C_0 \geq \gamma_k - \gamma_{k-1} + \cdots + \gamma_0,$$

and the equality takes place for $k = n$. 
**Proof** For $m \in \mathbb{Z}^\geq 0$ define the normal subgroup $G_m$ of $G$ by $G_m := \xi^{-1}(m\mathbb{Z})$. Let $M_m$ denote the corresponding cyclic $m$-sheeted normal covering space of $M$ with $\pi_1(M_m) = G_m$. Denote by $\omega_m$ the lifting of $\omega$ to $M_m$. We have clearly

$$C_j(\omega_m) = mC_j(\omega).$$

The relevance of these covering spaces is that the number of critical points of $\omega_m$ can be approximated by the number of critical points of an exact form on $M_m$. More precisely there is a constant $C$, independent of $m$, and an exact Morse 1-form $\tilde{\omega}_m$ on $M_m$ such that

$$(4.3) \quad C_j(\omega_m) \leq C_j(\tilde{\omega}_m) \leq C_j(\omega_m) + C; \quad \text{for } 0 \leq j \leq n.$$

For the proof of this claim we refer to [2, section 2]. The Morse theory for the exact 1-form $\tilde{\omega}_m = df_m$ is exactly the Morse theory for the Morse function $f_m$. If in relation $(3.1)$ we were to integrate over the $m$-sheeted covering $M_m$ of $M$ instead of $M$ then we would get a positive trace say $T_m$ which is actually $m$-times of the trace $T$. The arguments leading to the theorem can be applied to this situation as well. Since the delocalized Betti numbers defined by $T_m$ are $m$-times the delocalized Betti numbers defined by $T$ the resulting Morse inequalities will be the following ones with the equality for $k = n$

$$C_k(\tilde{\omega}_m) - C_{k-1}(\tilde{\omega}_m) + \cdots \pm C_0(\tilde{\omega}_m) \geq m\gamma_k^m - m\gamma_{k-1}^m + \cdots \pm m\gamma_0^m.$$

Using these inequalities and relations $(4.3)$ we get

$$\sum_{j=0}^{k} (-1)^{k-j} C_j(\omega) = \frac{1}{m} \sum_{j=0}^{k} (-1)^{k-j} C_j(\omega_m)$$

$$\geq \frac{1}{m} \sum_{j=0}^{k} (-1)^{k-j} C_j(\tilde{\omega}_m) - \frac{k+1}{m} C$$

$$\geq \sum_{j=0}^{k} (-1)^{k-j} \gamma_j^m - \frac{k+1}{m} C.$$

Now taking the limit when $m$ goes to infinity, we obtain the claim of the theorem. □

As a corollary of the above theorem we show that the delocalized Betti numbers of a manifold, fibering over the circle, vanish.

**Corollary 6** (Vanishing theorem). All delocalized Betti numbers of a manifold fibering over $S^1$ vanish.

**Proof** Let $M \to S^1$ be a fibration. The pull-back 1-form $\omega = p^*(d\theta)$ on $M$ has no critical point. Applying the inequalities of the theorem we obtain the following vanishing result for $0 \leq j \leq n$

$$\gamma_j^m(M) = \beta_j^m(\omega) + \frac{1}{|\langle g \rangle|} \beta_j^m(\omega) = 0$$

Since $\beta_j^m(\omega) = 0$ by the previously mentioned result of W. Lück, we conclude that $\beta_j^m(\omega) = 0$. □

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Mathematisches Institut Georg-August-Universität Göttingen, Germany and Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

E-mail address: zadeh@uni-math.gwdg.de