Big Data on the Rise?
(Testing monotonicity of distributions)

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Abstract

The field of property testing of probability distributions, or distribution testing, aims to provide fast and (most likely) correct answers to questions pertaining to specific aspects of very large datasets. In this work, we consider a property of particular interest, monotonicity of distributions. We focus on the complexity of monotonicity testing across different models of access to the distributions [CFGM13, CRS12, CR14, RS09]; and obtain results in these new settings that differ significantly from the known bounds in the standard sampling model [BKR04].

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1 Introduction

Before even the advent of data, information, records and insane amounts thereof to treat and analyze, probability distributions have been everywhere, and understanding their properties has been a fundamental problem in Statistics.\footnote{As well as – crucially – in crab population analysis [Pea04].} Whether it be about the chances of winning a (possibly rigged) game in a casino, or about predicting the outcome of the next election; or for social studies or experiments, or even for the detection of suspicious activity in networks, hypothesis testing and density estimation have had a role to play. And among these distributions, monotone ones have often been of paramount importance: is the probability of getting a cancer decreasing with the distance from, say, one’s microwave? Are aging voters more likely to vote for a specific party? Is the success rate in national exams correlated with the amount of money spent by the parents in tutoring?

All these examples, however disparate they may seem, share one unifying aspect: data may be viewed as the probability distributions it defines and originates from; and understanding the properties of this data calls for testing these distributions. In particular, our focus here will be on testing whether the data – its underlying distribution – happens to be monotone,\footnote{Recall that a distribution $D$ on $\{1, \ldots, n\}$ is said to be monotone (non-increasing) if $D(1) \geq \cdots \geq D(n)$, i.e. if its probability mass function is non-increasing.} or on the contrary far from being so. Since the seminal work of Batu et al. [BKR04], this fundamental property has been well-understood in the usual model of access to the data, which only assumes independent samples. However, a recent trend in distribution testing has been concerned with introducing and studying new models which provide additional flexibility in observing the data. In these new settings, our understanding of what is possible and what remains difficult is still in its infancy; and this is in particular true for monotonicity, for which very little is known. This work intends to mitigate this state of affairs.

We hereafter assume the reader’s familiarity with the broad field of property testing, and the more specific setting of distribution testing. For detailed surveys of the former, she or he is referred to, for instance, [Fis01, Ron08, Ron10, Gol10]; an overview of the latter can be found e.g. in [Rub12]. Details of the models we consider (besides the usual sampling oracle setting, denoted by SAMP) are described in [CFGM13, CRS12, CRS14] (for the conditional sampling oracle COND, and its variants INTCOND and PAIRCOND restricted respectively to interval and pairwise queries); [BKR04, GMV06, CR14] for the Dual and Cumulative Dual models; and [RS09] for the evaluation-only oracle, EVAL. The reader confused by the myriad of notations featured in the previous sentence may find the relevant definitions in Appendix A (as well as in the aforementioned papers).

Caveat. It is worth mentioning that we do not consider here monotonicity in its full general setting: we shall only focus on distributions defined on the line. In particular, the (many) works concerned with distributions over high-dimensional posets are out of the scope of this paper.

1.1 Techniques

Two main ideas are followed in obtaining our upper bounds: the first one, illustrated in Section 3 and Section 4.1, is the approach of Batu et al. [BKR04], which reduces monotonicity testing to uniformity testing on polylogarithmically many intervals. This relies on a structural result for
monotone distributions which asserts that they admit a succinct partition in intervals, such that on each interval the distribution is either close to uniform (in $\ell_2$ distance), or puts very little weight.

The second approach, on which Section 4.2, Section 5.1 and Section 6 are based, also leverages a structural result, due this time to Birgé [Bir87]. As before, this theorem states that each monotone distribution admits a succinct “flat approximation”, but in this case the partition does not depend on the distribution itself (see Section 2 for a more rigorous exposition). From there, the high-level idea is to perform two different checks: first, that the distribution $D$ is close to its “flattening” $\bar{D}$; and then that this flattening itself is close to monotone – where to be efficient the latter exploits the fact that the effective support of $\bar{D}$ is very small, as there are only polylogarithmically many intervals in the partition. If both tests succeed, then it must be the case that $D$ is close to monotone.

### 1.2 Results

A summary of results, including the best currently known bounds on monotonicity testing of distributions, can be found in Table 1 below. As noted in Section 3, many of the lower bounds are implied by the corresponding lower bound on testing uniformity.

| Model       | Upper bound                                                                 | Lower bound                                                                 |
|-------------|-----------------------------------------------------------------------------|-----------------------------------------------------------------------------|
| SAMP        | $\tilde{O}\left(\frac{\sqrt{n}}{\epsilon}\right)$                        | $\Omega\left(\frac{n}{\epsilon^2}\right)$                                |
| COND        | $\tilde{O}\left(\frac{1}{\epsilon^2}\right)$, $\tilde{O}\left(\frac{\log^2 n + \log n}{\epsilon^2}\right)$ | $\Omega\left(\frac{1}{\epsilon^2}\right)$                                |
| INTCOND     | $\tilde{O}\left(\frac{\log n}{\epsilon^4}\right)$                        | $\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$                   |
| EVAL        | $O\left(\max\left(\frac{\log n}{\epsilon}, \frac{1}{\epsilon^2}\right)\right)^*$ | $\Omega\left(\frac{n}{\epsilon^2}\right)^*$, $\Omega\left(\frac{\log n}{\log \log n}\right)^*$ |
| Cumulative Dual | $O\left(\frac{1}{\epsilon^2}\right)$                                    | $\Omega\left(\frac{1}{\epsilon}\right)$                                  |

Table 1: Summary of results for monotonicity testing. The highlighted ones are new; bounds with an asterisk* hold for non-adaptive testers.

The upper bounds for the conditional models, Theorem 4.1 and Theorem 4.2, are given in Section 4. Section 5 contains details of the results in the evaluation query model: the upper bound of Theorem 5.4, and the non-adaptive and adaptive lower bounds of respectively Theorem 5.5 and Theorem 5.7. Finally, in Section 6 we prove Theorem 6.1, our upper bound for the Cumulative Dual access model. We also note that, in the course of obtaining one of our upper bounds, we derive a result previously (to the best of our knowledge) absent from the literature: namely, that learning monotone distributions in the EVAL-only model can be accomplished using $O((\log n)/\epsilon)$ queries (Lemma 5.2).

Finally, we show in Appendix B that some of our techniques extend to tolerant testing, and describe in two of the models tolerant testers for monotonicity whose query complexity is only logarithmic in $n$ (Theorem B.1 and Corollary B.3).
2 Preliminaries

All throughout this paper, we denote by \([n]\) the set \(\{1, \ldots, n\}\), and by \(\text{log}\) the logarithm in base 2.

A probability distribution over a (finite) domain\(^3\) \(\Omega\) is a non-negative function \(D: \Omega \rightarrow [0, 1]\) such that \(\sum_{x \in \Omega} D(x) = 1\). We denote by \(\mathcal{U}(\Omega)\) the uniform distribution on \(\Omega\). Given a distribution \(D\) over \(\Omega\) and a set \(S \subseteq \Omega\), we write \(D(S)\) for the total probability weight \(\sum_{x \in S} D(x)\) assigned to \(S\) by \(D\). Finally, for \(S \subseteq \Omega\) such that \(D(S) > 0\), we denote by \(D_S\) the conditional distribution of \(D\) restricted to \(S\), that is \(D_S(x) = \frac{D(x)}{D(S)}\) for \(x \in S\) and \(D_S(x) = 0\) otherwise.

As is usual in property testing of distributions, in this work the distance between two distributions \(D_1, D_2\) on \(\Omega\) will be the total variation distance:

\[
d_{TV}(D_1, D_2) = \frac{1}{2} \|D_1 - D_2\|_1 = \frac{1}{2} \sum_{x \in \Omega} |D_1(i) - D_2(i)| = \max_{S \subseteq \Omega}(D_1(S) - D_2(S))
\]

which takes value in \([0, 1]\).

On the domain and parameters. Unless specified otherwise, \(\Omega\) will hereafter by default be the \(n\)-element set \([n]\). When stating the results, the accuracy parameter \(\varepsilon \in [0, 1]\) is to be understood as taking small values, either a tiny constant or a quantity arbitrarily close to 0; however, the actual parameter of interest will always be \(n\), viewed as “going to infinity”. Hence any dependence on \(n\), no matter how mild, shall be considered as more expensive than any function of \(\varepsilon\) only.

On monotone distributions. We now state here a few crucial facts about monotone distributions, namely that they admit a succinct approximation, itself monotone, close in total variation distance:

**Definition 2.1 (Oblivious decomposition).** Given a parameter \(\varepsilon > 0\), the corresponding oblivious decomposition of \([n]\) is the partition \(\mathcal{I}_\varepsilon = (I_1, \ldots, I_\ell)\), where \(\ell = \Theta\left(\frac{\ln(n+1)}{\varepsilon}\right) = \Theta\left(\frac{\ln n}{\varepsilon}\right)\) and\(^4\)

\(|I_k| = \left[(1 + \varepsilon)^k\right], 1 \leq k \leq \ell.\)

For a distribution \(D\) and parameter \(\varepsilon\), define \(\Phi_\varepsilon(D)\) to be the flattened distribution with relation to the oblivious decomposition \(\mathcal{I}_\varepsilon\):

\[
\forall k \in [\ell], \forall i \in I_k, \quad \Phi_\varepsilon(D)(i) = \frac{D(I_k)}{|I_k|}.
\]

Note that while \(\Phi_\varepsilon(D)\) (obviously) depends on \(D\), the partition \(\mathcal{I}_\varepsilon\) itself does not; in particular, it can be computed prior to getting any sample or information about \(D\).

**Theorem 2.2 ([Bir87]).** If \(D\) is monotone non-increasing, then \(d_{TV}(D, \Phi_\varepsilon(D)) \leq \varepsilon.\)

\(^3\)For the focus of this work, all distributions will be supported on a finite domain; thus, we do not consider the fully general definitions from measure theory.

\(^4\)We will often ignore the floors in the definition of the oblivious partition, to avoid more cumbersome analyses and the technicalities that would otherwise arise. However, note that this does not affect the correctness of the proofs: after the first \(O\left(\frac{1}{\varepsilon}\right)\) intervals (which will be, as per the above definition, of constant size), we do have indeed \(|I_{k+1}| \in [1 + \frac{\varepsilon}{2}, 1 + 2\varepsilon]|I_k|\). This multiplicative property, in turn, is the key aspect we shall rely on.
Remark 2.3. The first use of this result in this discrete learning setting is due to Daskalakis et al. [DDS12]. For a proof for discrete distributions (whereas the original paper by Birgé is intended for continuous ones), the reader is referred to [DDS+13] (Section 3.1, Theorem 5).

Corollary 2.4 (Robustness). Suppose $D$ is $\varepsilon$-close to monotone non-increasing. Then $d_{TV}(D, \Phi_\alpha(D)) \leq 2\varepsilon + \alpha$; furthermore, $\Phi_\alpha(D)$ is also $\varepsilon$-close to monotone non-increasing.

Proof. Let $P$ be a monotone non-increasing distribution such that $d_{TV}(D, P) \leq \varepsilon$. By the triangle inequality,

$$d_{TV}(D, \Phi_\alpha(D)) \leq d_{TV}(D, P) + d_{TV}(P, \Phi_\alpha(P)) + d_{TV}(\Phi_\alpha(P), \Phi_\alpha(D)) \leq \varepsilon + \alpha + d_{TV}(\Phi_\alpha(P), \Phi_\alpha(D))$$

where the last inequality uses the assumption on $P$ and Theorem 2.2 applied to it. It only remains to bound the last term: by definition,

$$2d_{TV}(\Phi_\alpha(P), \Phi_\alpha(D)) = \sum_{i=1}^{n} |\Phi_\alpha(D)(i) - \Phi_\alpha(P)(i)| = \sum_{k=1}^{\ell} \sum_{i \in I_k} |\Phi_\alpha(D)(i) - \Phi_\alpha(P)(i)|$$

$$= \sum_{k=1}^{\ell} \sum_{i \in I_k} \left| \frac{D(I_k) - P(I_k)}{|I_k|} \right| = \sum_{k=1}^{\ell} |D(I_k) - P(I_k)|$$

$$= \sum_{k=1}^{\ell} \left| \sum_{i \in I_k} (D(i) - P(i)) \right| \leq \sum_{k=1}^{\ell} \sum_{i \in I_k} |D(i) - P(i)| = 2d_{TV}(P, D)$$

$$\leq 2\varepsilon$$

(showing in particular the second part of the claim, as $\Phi_\alpha(P)$ is monotone) and thus

$$d_{TV}(D, \Phi_\alpha(D)) \leq 2\varepsilon + \alpha$$

as claimed.

One can interpret this corollary as saying that the Birgé decomposition provides a tradeoff between becoming simpler (and at least as close to monotone) while not staying too far from the original distribution. Incidentally, the last step of the proof above implies the following easy fact:

Fact 2.5. For all $\alpha \in (0, 1]$,

$$d_{TV}(\Phi_\alpha(P), \Phi_\alpha(D)) \leq d_{TV}(P, D)$$

and in particular, for any property $\mathcal{P}$ preserved by the Birgé transformation (such as monotonicity)

$$d_{TV}(\Phi_\alpha(D), \mathcal{P}) \leq d_{TV}(D, \mathcal{P}).$$

Other tools. Finally, we will use as subroutines the following results of Canonne et al.. The first one, restated below, provides a way to “compare” the probability weight of disjoint subsets of elements in the COND model:
Lemma 2.6 ([CRS12, Lemma 2]). Given as input two disjoint subsets of points \( X, Y \subseteq \Omega \) together with parameters \( \eta \in (0, 1], K \geq 1, \) and \( \delta \in (0, 1/2], \) as well as \( \text{COND} \) query access to a distribution \( D \) on \( \Omega, \) there exists a procedure \( \text{COMPARE} \) that either outputs a value \( \rho > 0 \) or outputs \( \text{High} \) or \( \text{Low}, \) and satisfies the following:

(i) If \( D(X)/K \leq D(Y) \leq K \cdot D(X) \) then with probability at least \( 1 - \delta \) the procedure outputs a value \( \rho \in [1 - \eta, 1 + \eta] D(Y)/D(X); \)

(ii) If \( D(Y) > K \cdot D(X) \) then with probability at least \( 1 - \delta \) the procedure outputs either \( \text{High} \) or a value \( \rho \in [1 - \eta, 1 + \eta] D(Y)/D(X); \)

(iii) If \( D(Y) < D(X)/K \) then with probability at least \( 1 - \delta \) the procedure outputs either \( \text{Low} \) or a value \( \rho \in [1 - \eta, 1 + \eta] D(Y)/D(X). \)

The procedure performs \( O\left( \frac{K \log(1/\delta)}{\eta^2} \right) \) \( \text{COND} \) queries on the set \( X \cup Y. \)

The second allows one to estimate the distance between the uniform distribution and an unknown distribution \( D, \) given access to a conditional oracle to the latter:

Theorem 2.7 ([CRS12, Theorem 14]). Given as input \( \varepsilon \in (0, 1] \) and \( \delta \in (0, 1], \) as well as \( \text{PAIR-COND} \) query access to a distribution \( D \) on \( \Omega, \) there exists an algorithm that outputs a value \( \tilde{d} \) and has the following guarantee. The algorithm performs \( \tilde{O}(1/\varepsilon^{20}) \) queries and, with probability at least \( 1 - \delta, \) the value it outputs satisfies \( |\tilde{d} - d_{TV}(D, \mathcal{U})| \leq \varepsilon. \)

3 Previous work: Standard model

In this section, we describe the currently known results for monotonicity testing in the standard (sampling) oracle model. These bounds on the sample complexity, tight up to logarithmic factors, are due to Batu et al. [BKR04]; while not directly applicable to the other access models we will consider, we note that some of the techniques they use will be of interest to us in Section 4.1.

Theorem 3.1 ([BKR04, Theorem 10]). There exists an \( O\left( \frac{\sqrt{n}}{\varepsilon^6 \text{poly} n} \right) \)-query tester for monotonicity in the \( \text{SAMP} \) model.

Proof (sketch). Their algorithm works by taking this many samples from \( D, \) and then using them to recursively split the domain \( [n] \) in half, as long as the conditional distribution on the current interval is not close enough to uniform (or not enough samples fall into it). If the binary tree created during this recursive process exceeds \( O\left( \log^2 n/\varepsilon \right) \) nodes, the tester rejects. Batu et al. then show that this succeeds with high probability, the leaves of the recursion yielding a partition of \( [n] \) in \( \ell_{\text{max}} = O\left( \log^2 n/\varepsilon \right) \) intervals \( I_1, \ldots, I_{\ell_{\text{max}}}, \) such that either

(a) the conditional distribution \( D_{I_j} \) is \( O(\varepsilon) \)-close to uniform on this interval; or

(b) \( I_j \) is “light”, i.e. has weight at most \( O(\varepsilon/\ell_{\text{max}}) \) under \( D. \)

This implies this partition defines an \( \ell_{\text{max}} \)-flat distribution \( \tilde{D} \) which is \( \varepsilon/2 \)-close to \( D, \) and can be easily learnt from another batch of samples; once this is done, it only remains to test (e.g., via linear programming, which can be done efficiently) whether this \( \tilde{D} \) is itself \( \varepsilon/2 \)-close to monotone, and accept if and only this is the case. \( \square \)

\footnotesize
\begin{itemize}
  \item We observe that the dependence on \( \varepsilon \) could be brought down to \( \varepsilon^4, \) by using instead machinery from [DKN15, Theorem 11] to perform this step.
\end{itemize}
Theorem 3.2 ([BKR04, Theorem 11]). Any tester for monotonicity in the SAMP model must perform $\Omega\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$ queries.

Proof (sketch). To prove this lower bound, they reduce the problem of uniformity testing to monotonicity testing: from a distribution $D$ over $[n]$ (where $n$ is for the sake of simplicity assumed to be even), one can run a monotonicity tester (with parameter $\varepsilon' \overset{\text{def}}{=} \varepsilon/3$) on both $D$ and $\Omega$, where the latter is defined as $\Omega(i) \overset{\text{def}}{=} D(n - i)$, $i \in [n]$; and accept if and only if both tests pass. If $D$ is uniform, clearly $D = \Omega$ is monotone; conversely, one can show that if both $D$ and its “mirrored version” $\Omega$ pass the test (are $\varepsilon'$-close to monotone non-increasing), then it must be the case that $D$ is $\varepsilon$-close to uniform. The lower bound then follows from the $\Omega\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$ lower bound of [Pan08] for testing uniformity.

We note that the argument above extends to all models: that is, any lower bound for testing uniformity directly implies a corresponding lower bound for monotonicity in the same access model (giving the bounds in Table 1).

Open question. At a (very) high-level, the above results can be interpreted as “relating monotonicity to uniformity”. That is, the upper bound is essentially established by proving that monotonicity reduces to testing uniformity on polylogarithmically many intervals, while the lower bound follows from showing that it reduces from testing uniformity on a constant number of them. Thus, an interesting question is whether, qualitatively, the former or the latter is tight in terms of $n$. Are uniformity and monotonicity strictly as hard, or is there an intrinsic gap, even if only polylogarithmic, between the two?

Question 3.3. Can monotonicity be tested in the SAMP model with $O(\sqrt{n})$ samples, or are $\Omega(\sqrt{n} \log^c n)$ needed for some absolute constant $c > 0$?

4 With conditional samples

In this section, we focus on testing monotonicity with a stronger type of access to the underlying distribution, that is given the ability to ask conditional queries. More precisely, we prove the following theorem:

Theorem 4.1. There exists an $\tilde{O}\left(\frac{1}{\varepsilon^2}\right)$-query tester for monotonicity in the COND model.

Furthermore, assuming only a (restricted) type of conditional queries are allowed, one can still get an exponential improvement from the standard sampling model:

Theorem 4.2. There exists an $\tilde{O}\left(\frac{\log^2 n}{\varepsilon^4}\right)$-query tester for monotonicity in the INTCOND model.

We now prove these two theorems, starting with Theorem 4.2. In doing so, we will also derive a weaker, poly$(\log n, 1/\varepsilon)$-query tester for COND; before turning in Section 4.2 to the constant-query tester of Theorem 4.1.

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6[BKR04] actually only shows a $\Omega(\sqrt{n})$ lower bound, as they invoke in the last step the (previously best known) lower bound of [GR00] for uniformity testing; however, their argument straightforwardly extends to the result of Paninski.
4.1 A poly(log \(n, 1/\varepsilon\))-query tester for \textsc{IntCond}

Our algorithm (Algorithm 1) follows the same overall idea as the one from [BKR04], which a major difference. As in theirs, the first step will be to partition \([n]\) into a family of (polylogarithmically many) intervals, such that the conditional distribution \(D_I\) on each interval \(I\) is close to uniform; that is,

\[
d_{TV}(D_I, U_I) = \sum_{i \in I} \left| \frac{D(i)}{D(I)} - \frac{1}{|I|} \right| \leq \frac{\varepsilon}{4}.
\] (5)

The original approach (in the sampling model) of Batu and al. was based on estimating the \(\ell_2\) norm of the conditional distribution via the number of collisions from a sufficiently large sample; this yielded a \(\tilde{O}(\sqrt{n})\) sample complexity.

However, using directly as a subroutine (in the \textsc{Cond} model) an algorithm for testing uniformity, one can perform this first step with \(\ell_{\text{max}} \log \frac{1}{\delta} = \ell_{\text{max}} \log \ell_{\text{max}}\) calls\(^7\) to this subroutine, each with approximation parameter \(\varepsilon^4\) (the proof of correctness from [BKR04] does not depend on how the test of uniformity is actually performed, in the partitioning step).

**Fact 4.3** ([CRS14]). One can test \(\varepsilon\)-uniformity of a distribution \(D_r\) over \([r]\) in the conditional sampling model:

- with sample complexity \(\tilde{O}\left(\frac{1}{\varepsilon^2}\right)\), given access to a \textsc{Cond}_{D_r} oracle;
- with sample complexity \(\tilde{O}\left(\frac{\log^3 r}{\varepsilon^3}\right)\), given access to a \textsc{IntCond}_{D_r} oracle.

**Remark 4.4.** The first item actually apply to a (more restricted) access to a \textsc{PairCond} oracle; however, this cannot be used for our purpose, as the subroutine will be called on subintervals \(I \subset [n]\), and will perform (standard) sampling on \(I\) – which a \textsc{PairCond} oracle for a distribution over \([n]\) cannot do.

As a corollary, we get:

**Corollary 4.5.** Given access to a conditional oracle \(O\) for a distribution \(D\) over \([n]\), the algorithm \textsc{TestMonCond}^\(O\) outputs ACCEPT when \(D\) is monotone and FAIL when it is \(\varepsilon\)-far from monotone, with probability at least 2/3. The algorithm uses

- \(\tilde{O}\left(\frac{\ell_{\text{max}}}{\varepsilon^2} + \frac{\ell_{\text{max}} + \log^4 n}{\varepsilon^4}\right)\) samples, when \(O = \textsc{Cond}_D\);
- \(\tilde{O}\left(\frac{\ell_{\text{max}}}{\varepsilon^4} + \frac{\ell_{\text{max}} + \log^4 n}{\varepsilon^3} + \frac{\log^4 n}{\varepsilon^4}\right)\) samples, when \(O = \textsc{IntCond}_D\).

which implies **Theorem 4.2**. Note that we make sure in Step 9 that each of the intervals we recurse on contains at least one of the “reference samples” \(h_i\): this is in order to guarantee all conditional queries made on a set with non-zero probability. Discarding the “light intervals” can be done without compromising the correctness, as with high probability each of them has probability weight at most \(\frac{\varepsilon}{\ell_{\text{max}}^2}\), and therefore in total the light intervals can amount to at most \(\varepsilon^4\) of the probability weight of \(D\) – as in the original argument of Batu et al., we can still conclude that with high probability \(\hat{D}\) is \(\varepsilon/2\)-close to \(D\).

\(^7\)Where the logarithmic dependence on \(\delta\) aims at boosting the (constant) success probability of the uniformity testing algorithm, in order to apply a union bound over the \(O(\ell_{\text{max}})\) calls.
Algorithm 1 General algorithm TestMonCond$^O$

**Require:** $O \in \{\text{COND}, \text{INTCOND}\}$ access to $D$

1. Define $\ell_{\max} \overset{\text{def}}{=} O\left(\frac{\log^2 n}{\varepsilon}\right)$, $\delta \overset{\text{def}}{=} O\left(\frac{1}{\ell_{\max}}\right)$.
2. Draw $m \overset{\text{def}}{=} O\left(\frac{\varepsilon}{\ell_{\max}} \log \frac{1}{\delta}\right)$ samples $h_1, \ldots, h_m$.
3. **PartitionStart**
4. Start with interval $I \leftarrow [n]$
5. **repeat**
6. Test (with probability $\geq 1 - \delta$) if $D_I$ is $\varepsilon/4$-close to the uniform distribution on $I$
7. if $d_{\text{TV}}(D_I, U_I) > \frac{\varepsilon}{4}$ then
8. bisect $I$ in half
9. recursively test each half that contains at least one of the $h_i$’s, mark them as “light”
10. else if $\ell_{\max}$ splits have been made then
11. return FAIL
12. end if
13. until all intervals are close to uniform or have been marked “light”
4. **PartitionEnd**
15. Let $\mathcal{I}_\ell = \{I_1, \ldots, I_{\ell}\}$ denote the partition of $[n]$ into intervals induced by the leaves of the recursion from the previous step.
16. Obtain an additional sample $T$ of size $O\left(\frac{\log 4 n}{\varepsilon^2}\right)$.
17. Let $\hat{D}$ denote the $\ell$-flat distribution described by $(w, \mathcal{I}_\ell)$ where $\omega_j$ is the fraction of samples from $T$ falling in $I_j$.
18. if $\hat{D}$ is $(\varepsilon/2)$-close to monotone then $\triangleright$ Can be tested in poly($\ell$)-time ([BKR04, Lemma 8])
19. return ACCEPT
20. else
21. return FAIL
22. end if
4.2 A poly(1/\varepsilon)-query tester for COND

The idea in proving Theorem 4.1 is to reduce the task of testing monotonicity to another property, but on a (related) distribution over a much smaller domain. We begin by introducing a few notations, and defining the said property:

4.2.1 Reduction from testing properties over [\ell]

For fixed \alpha and D, let \( D^{\text{red}}_\alpha \) be the reduced distribution on [\ell] with respect to the oblivious decomposition \( \mathcal{I}_\alpha \); i.e,

\[ \forall k \in [\ell], \quad D^{\text{red}}_\alpha (k) = D(I_k) = \Phi_\alpha (D)(I_k) \]

Note that given oracle access \( \text{SAMP}_D \), it is easy to simulate \( \text{SAMP}_{D^{\text{red}}_\alpha} \).

**Definition 4.6** (Exponential Property). Fix \( n, \alpha \), and the corresponding \( \ell = \ell(n, \alpha) \). For distributions over [\ell], let the property \( \mathcal{P}_\alpha \) be defined as “\( Q \in \mathcal{P}_\alpha \) if and only if there exists \( D \in \mathcal{M} \) over [n] such that \( Q = D^{\text{red}}_\alpha \).”

**Fact 4.7.** Given a distribution \( Q \) over [\ell], let \( \text{expand}_\alpha (Q) \) denote the distribution over [n] obtained by “spreading” uniformly \( Q(k) \) over \( I_k \) (again, considering the oblivious decomposition of [n] for \( \alpha \)). Then,

\[ Q \in \mathcal{P}_\alpha \iff \text{expand}_\alpha (Q) \in \mathcal{M} \tag{6} \]

**Fact 4.8.** Given a distribution \( Q \) over [\ell], the following also holds:\(^8\)

\[ \mathcal{P}_\alpha (Q) \iff \forall k < \ell, \quad Q(k + 1) \leq (1 + \alpha)Q(k) \tag{7} \]

**Remark 4.9.** It follows from Fact 4.7 that, for \( D \) over [n],

\[ \Phi_\alpha (D) \in \mathcal{M} \iff D^{\text{red}}_\alpha \in \mathcal{P}_\alpha \tag{8} \]

We shall also use the following result on flat distributions (adapted from [BKR04, Lemma 7]):

**Fact 4.10.** \( \Phi_\alpha (D) \) is \( \varepsilon \)-close to monotone if and only if it is \( \varepsilon \)-close to a \( \mathcal{I}_\alpha \)-flat monotone distribution (that is, a monotone distribution piecewise constant, according to the same partition \( \mathcal{I}_\alpha \)).

**Proof.** The sufficient condition is trivial; for the necessary one, assume \( \Phi_\gamma (D) \) is \( \varepsilon \)-close to monotone, and let \( Q \) be a monotone distribution proving it. We show that \( d_{TV}(\Phi_\gamma (D), \Phi_\gamma (Q)) \leq \varepsilon \):

\[
2d_{TV}(\Phi_\gamma (D), \Phi_\gamma (Q)) = \sum_{k=1}^{\ell} |\Phi_\gamma (D)(I_k) - \Phi_\gamma (Q)(I_k)| = \sum_{k=1}^{\ell} |\Phi_\gamma (D)(I_k) - Q(I_k)| \\
= \sum_{k=1}^{\ell} \sum_{i \in I_k} (\Phi_\gamma (D)(i) - Q(i)) \leq \sum_{k=1}^{\ell} \sum_{i \in I_k} |\Phi_\gamma (D)(i) - Q(i)| \\
= \sum_{i=1}^{n} |\Phi_\gamma (D)(i) - Q(i)| = 2d_{TV}(\Phi_\gamma (D), Q) \leq 2\varepsilon.
\]

*We point out that the equivalence stated here once again ignores, for the sake of conceptual clarity, technical details arising from the discrete setting. Taking these into account would yield a slightly weaker characterization, with a twofold implication instead of an equivalence; which would still be good enough for our purpose.*
Observe that Fact 4.10, Remark 4.9 and Fact 4.7 altogether imply that, for $\mathcal{I}_\alpha$-flat distributions, distance to monotonicity and distance to $\mathcal{P}_\alpha$ of the reduced distribution are equal.

4.2.2 Efficient approximation of distance to $\Phi(D)$

**Lemma 4.11.** Given $\text{COND}$ access to a distribution $D$ over $[n]$, there exists an algorithm that, on input $\alpha$ and $\varepsilon, \delta \in (0,1]$, makes $\tilde{O}\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right)$ queries and outputs a value $\hat{d}$ such that, with probability at least $1 - \delta$, $|\hat{d} - d_{\text{TV}}(D, \Phi_\alpha(D))| \leq \varepsilon$.

**Proof.** We describe such algorithm for a constant probability of success; boosting the success probability to $1 - \delta$ at the price of a multiplicative $\log \frac{1}{\delta}$ factor can then be achieved by standard techniques (repetition, and taking the median value). Let $D, \varepsilon$ and $\mathcal{I}_\alpha$ be defined as before; define $Z$ to be a random variable taking values in $[0,1]$, such that, for $k \in [\ell]$, $Z$ is equal to $d_{\text{TV}}(D_{I_k}, \mathcal{U}_{I_k})$ with probability $\omega_k = D(I_k)$. It follows that

\[
\mathbb{E}[Z] = \sum_{k=1}^{\ell} \omega_k d_{\text{TV}}(D_{I_k}, \mathcal{U}_{I_k}) = \frac{1}{2} \sum_{k=1}^{\ell} \sum_{i \in I_k} \left| D_{I_k}(i) - \frac{1}{|I_k|} \right| = \frac{1}{2} \sum_{k=1}^{\ell} \sum_{i \in I_k} \left| D(i) - \frac{D(I_k)}{|I_k|} \right| = \frac{1}{2} \sum_{i=1}^{n} |D(i) - \Phi_\alpha(D)| = d_{\text{TV}}(D, \Phi_\alpha(D)).
\]

Putting aside for now the fact that we only have (using as a subroutine the $\text{COND}$ algorithm from Theorem 2.7 to estimate the distance to uniformity) access to additive approximations of the $d_{\text{TV}}(D_{I_k}, \mathcal{U}_{I_k})$’s, one can simulate independent draws from $Z$ by taking each time a fresh sample $i \sim D$, looking up the $k$ for which $i \in I_k$, and calling the $\text{COND}$ subroutine to get the corresponding value. Applying a Chernoff bound, only $O(1/\varepsilon^2)$ such draws are needed, each of them costing $\tilde{O}(1/\varepsilon^{20})$ $\text{COND}$ queries.

Dealing with approximation. It suffices to estimate $\mathbb{E}[Z]$ within an additive $\varepsilon/2$, which can be done with probability $9/10$ by simulating $m = O(1/\varepsilon^2)$ samples from $Z$. To get each sample, for the index $k$ drawn we can call the $\text{COND}$ subroutine with parameters $\varepsilon/2$ and $\delta = 1/(10m)$ to obtain an estimate of $d_{\text{TV}}(D_{I_k}, \mathcal{U}_{I_k})$. By a union bound we get that, with probability at least $9/10$, all estimates are within an additive $\varepsilon/2$ of the true value, incurring only a $O(\log 1/\varepsilon)$ additional factor in the overall sample complexity $\tilde{O}(1/\varepsilon^{20})$. Conditioned on this, we get that the approximate value we compute instead of $\mathbb{E}[Z]$ is off by at most $\varepsilon/2 + \varepsilon/2 = \varepsilon$. \hfill $\square$

4.2.3 The algorithm

**Algorithm 2** Algorithm TestMonCond

**Require:** $\text{COND}$ access to $D$

1: Simulating $\text{COND}_{\Phi_\alpha}$, check if $\Phi_\alpha(D)$ is $(\varepsilon/4)$-close to monotone by testing $(\varepsilon/4)$-farness (of $D_{\alpha}^{\text{red}}$) to $\mathcal{P}_\alpha$; return FAIL if not.

2: Test whether $\Phi_\alpha(D)$ is $(\varepsilon/4)$-close to $D$ using the sampling approach discussed above; return FAIL if not.

3: return ACCEPT
The tester is described in Algorithm 2. The second step, as argued in Lemma 4.11, uses \( \tilde{O}(1/\varepsilon^{22}) \) samples; we will show in Section 4.2.4 that efficiently testing \( \varepsilon \)-farness to \( P_\gamma \) is also achievable with \( \tilde{O}(1/\varepsilon^6) \) COND queries – concluding the proof of Theorem 4.1.

Correctness of Algorithm 2. Assume we can efficiently perform the two steps, and condition on their execution being correct (as each of them is run with for instance parameter \( \delta = 1/10 \), this happens with probability at least \( 3/4 \)).

- If \( D \) is monotone non-increasing, so is \( \Phi_\alpha(D) \); by Remark 4.9, this means that \( P_\alpha(D_\alpha^{\text{red}}) \) holds, and the first step passes. Theorem 2.2 then ensures that \( D \) and \( \Phi_\alpha(D) \) are \( \alpha \)-close, and the algorithm outputs ACCEPT;
- If \( D \) is \( \varepsilon \)-far from monotone, then either (a) \( \Phi_\alpha(D) \) is \( \varepsilon_2 \)-far from monotone or (b) \( d_{\text{TV}}(D, \Phi_\alpha(D)) > \varepsilon_2 \); if (b) holds, no matter how the algorithm behaves in first step, the algorithm not go further that the second step, and output FAIL. Assume now that (b) does not hold, i.e. only (a) is satisfied. By putting together Fact 4.10, Remark 4.9 and Fact 4.7, we conclude that (a) implies that \( D^{\text{red}}_\alpha \) is \( \varepsilon_2 \)-far from \( P_\alpha \), and the algorithm outputs FAIL in the first step.

4.2.4 Testing \( \varepsilon \)-farness to \( P_\gamma \)

To achieve this objective, we begin with the following lemma, which relates the distance of between a distribution \( Q \) and \( P_\alpha \) to the total weight of points that violate the property.

Lemma 4.12. Let \( Q \) be a probability distribution over \( [\ell] \), and \( W = \{i : Q(i) > (1 + \alpha)Q(i - 1)\} \) be the set of witnesses (points which violate the property). Then, the distance from \( Q \) to the property \( P_\alpha \) is \( O(1/\alpha)Q(W) \).

Proof. One can define the procedure \( \text{Fixup}_\alpha \) which, given a distribution \( Q \) and the corresponding \( W \), acts as follows:

Ensure: \( \text{Fixup}_\alpha(Q) \) is a distribution satisfying \( P_\alpha \)

\[
Q' \leftarrow Q, W' \leftarrow W
\]

while \( W' \neq \emptyset \) do

- Let \( i > 1 \) be the smallest (leftmost) point in \( W' \), set \( \Delta \leftarrow 0 \) and \( d \leftarrow 1 \).
- while \( Q(i - d) \geq Q'(i)(1 + \alpha)^{-d} \) do

  \[
  \Delta \leftarrow \Delta + \left( Q'(i)(1 + \alpha)^{-d} - Q'(i - d) \right)
  
  \]

- end while

- while \( \Delta > 0 \) do

  \[
  Q'(k) \leftarrow Q'(k) - \min(\Delta, Q'(k))
  \]

  \[
  \Delta \leftarrow \Delta - \min(\Delta, Q'(k))
  \]

- end while

end while

return \( Q' \)

If \( \Delta \) denotes the total probability weight reassigned (i.e, the sum of the \( \Delta_i \)'s, where \( \Delta_i \) is the total weight reassigned for witness \( i \)), then we have that \( 2d_{\text{TV}}(Q, \text{Fixup}_\alpha(Q)) = \sum_{i=1}^\ell |\text{Fixup}_\alpha(Q)(i) - Q(i)| \leq \)
Lemma 4.12 implies that when $Q$ is $\varepsilon$-far from having the property, it suffices to sample $O\left(\frac{1}{\alpha\varepsilon}\right)$ points according to $Q$ and compare them to their neighbors to detect a violation with high probability. Note that this last test would be easy, granted access to an exact $\text{EVAL}$ oracle; for the purpose of this section, however, we can only use an approximate one. The lemma below addresses this issue, by ensuring that there will be many points “patently” violating the property.

**Lemma 4.14.** Let $Q$ be as above, and, for $\tau > 0$, let $W_\tau = \{ i : Q(i) > (1 + \alpha + \tau)Q(i-1) \}$ be the set of $\tau$-witnesses (so that $W = \bigcup_{\tau > 0} W_\tau$). Then, the distance from $Q$ to the property $\mathcal{P}_\alpha$ is at most $O(1/(\alpha + \tau))Q(W_\tau) + O(\tau/\alpha^2)$.

**Corollary 4.15.** Taking $\alpha = \Theta(\varepsilon)$ and $\tau = \varepsilon^2$, we get that if $Q(W_\tau) \leq \varepsilon^2$, then $Q$ is $O(\varepsilon)$-close to $\mathcal{P}_\alpha$.

**Proof of Lemma 4.14.** We first apply the “fix-up” as defined in the proof of Lemma 4.12 to get $Q'$ such that $Q'(i) \leq (1 + \alpha + \tau)Q'(i-1)$ for all $i$, at a cost of $O\left(\frac{1}{\alpha + \tau}\right)Q(W_\tau)$. Next, we obtain a distribution $Q''$ satisfying $\mathcal{P}_\alpha$ by apply the fix-up to all $i$ such that $Q'(i) > (1 + \alpha)Q'(i-1)$. If we start from some violating $i$ (until we reach some $k_i = i - d$ such that $Q'(k)$ does not need to be fixed since $Q''(k+1) \leq (1 + \alpha)Q''(k)$), we know that before the fix-up, for each $1 \leq d \leq k_i$, $Q'(i - d) \geq \frac{Q'(i)}{(1 + \alpha + \tau)^d}$, and now, after the fix-up $Q''(i) = Q'(i)$ and $Q''(i - d) = \frac{Q'(i)}{(1 + \alpha + \tau)^d}$. The cost of this increase is:

$$Q''(i - d) - Q'(i - d) \leq Q'(i) \cdot \left(\frac{1}{(1 + \alpha)^d} - \frac{1}{(1 + \alpha + \tau)^d}\right)$$

Using the fact that $(1 + \alpha + \tau) = (1 + \alpha)(1 + \tau / (1 + \alpha)) < (1 + \alpha)(1 + \tau)$ so that $\frac{1}{1 + \alpha + \tau} > \frac{1}{(1 + \alpha)(1 + \tau)}$, we get

$$Q''(i - d) - Q'(i - d) \leq Q'(i) \cdot \frac{1}{(1 + \alpha)^d} \cdot \left(1 - \frac{1}{(1 + \tau)^d}\right)$$

$$= Q'(i) \cdot \frac{1}{(1 + \alpha)^d} \cdot \frac{(1 + \tau)^d - 1}{(1 + \tau)^d}$$

$$\leq Q'(i) \cdot \frac{\tau^d}{(1 + \alpha)^d}$$

(where the last inequality uses the fact that $(1 + \tau)^d - 1 = d\tau + \binom{d}{2}\tau^2 + \cdots + d\tau^{d-1} + \tau^d$ which is less than $d\tau \cdot (1 + \tau)^d = d\tau + d^2\tau^2 + d\binom{d}{2}\tau^3 + \cdots + d^d\tau^d + d^{d+1} \tau^{d+1}$). Since, for $x \in [0, 1]$

$$\sum_{i=1}^{n} i \cdot x^i \leq \sum_{i=1}^{\infty} i \cdot x^i = \frac{x}{(1 - x)^2},$$

we get that $d_{TV}(Q, \text{FIXUP}_{\alpha}(Q)) = O\left(\frac{1}{\alpha}\right)Q(W)$ (and $\text{FIXUP}_{\alpha}(Q)$ clearly satisfies $\mathcal{P}_\alpha$).
we get that
\[
\sum_{1 \leq d \leq k_i} (Q''(i - d) - Q'(i - d)) \leq Q'(i) \cdot \tau \sum_{d=1}^{\infty} d \cdot \frac{1}{(1 + \alpha)^d} < Q'(i) \cdot \frac{\tau(1 + \alpha)}{\alpha^2}.
\] (12)

By summing over all \(i\) from which we start the (second) “fix-up”, we get an increase of at most \(\frac{\tau(1 + \alpha)}{\alpha^2}\). By the triangle inequality, the total distance from \(Q\) to \(Q''\) is therefore at most
\[
\frac{1 + \alpha + \tau}{\alpha + \tau} Q(W_\tau) + \frac{\tau(1 + \alpha)}{\alpha^2}.
\] (13)

By leveraging Corollary 4.15, we are able to obtain efficient approximation of the distance of a distribution to the “exponential property”:

**Theorem 4.16.** There exists a constant \(0 < c < 1\) such that, for any \(\varepsilon > 0\): if \(Q\) satisfies \(P_\alpha\) (where \(\alpha = c\varepsilon\)), then with probability at least \(2/3\) Algorithm TestingExponentialProperty returns ACCEPT, and if \(Q\) is \(O(\varepsilon)\)-far from \(P_\alpha\), then with probability at least \(2/3\) Algorithm TestingExponentialProperty returns FAIL. The number of PAIRCOND queries performed by the algorithm is \(O\left(\frac{1}{\varepsilon^6}\right)\).

**Proof (sketch).** The algorithm can be found in Algorithm 3. We here prove its correctness, before turning to its sample complexity.

**Correctness.** Conditioning on the events of all calls to COMPARE returning a correct value (by a union bound, this happens with probability at least 9/10), we have that:

- if \(Q\) satisfies \(P_\alpha\), then for any sample \(s_i > 1\), COMPARE can only return Low or a value \(\rho\). In the latter case, since \(s_i \notin W = \emptyset\), it holds that \(Q(s_i - 1) \leq (1 + \alpha)Q(s_i)\), and therefore \(\rho \geq (1 - \eta)\frac{Q(s_i)}{Q(s_i - 1)} \geq \frac{1 - \eta}{1 + \alpha} > \frac{1 + \eta}{1 + \alpha + \tau}\) (where the last inequality holds because of the choice of \(\eta\)), and the algorithm does not reject;
- if however \(Q\) is \(O(\varepsilon)\)-far from \(P_\alpha\), Corollary 4.15 ensures that with probability at least 9/10 one of the samples will belong to \(W_\tau\). For such a \(s_i\), COMPARE will either return High (and the algorithm will reject) or a value \(\rho\). In the latter case, it will be the case that \(Q(s_i - 1) > (1 + \alpha + \tau)Q(s_i)\), and thus \(\rho < (1 + \eta)\frac{Q(s_i)}{Q(s_i - 1)} \geq \frac{1 + \eta}{1 + \alpha + \tau}\), and the algorithm will reject.

The outcome of the algorithm will hence be correct with probability at least 3/4.

**Sample complexity.** By choice of \(\alpha\), \(m = \Theta(1/\varepsilon^2)\) and \(\tau = \Theta(1/\varepsilon^3)\); each of the \(m\) calls to COMPARE costs \(O\left(\frac{K \log(1/\delta)}{\eta^2}\right) = O\left(\frac{\log m}{\tau^2}\right) = \tilde{O}\left(\frac{1}{\varepsilon^6}\right)\). \(\square\)
Algorithm 3 TestingExponentialProperty

Require: PAIRCOND access to $Q$, $\alpha \in [0, 1)$ \hspace{1em} \Comment{Useful for $\alpha = \Theta(\varepsilon) < 1$}
Ensure: with probability at least $3/4$ returns FAIL if $Q$ is $O(\varepsilon)$-close to $P_\alpha$, and ACCEPT if it satisfies $P_\alpha$.

Set $\tau \overset{\text{def}}{=} \varepsilon \alpha^2$

Draw $m \overset{\text{def}}{=} \Theta\left(\frac{1}{\varepsilon \alpha}\right)$ samples $s_1, \ldots, s_m$ from $Q$ \hspace{1em} \Comment{Contains an element from $W_\tau$ w.h.p.}

for $i = 1$ to $m$

if $s_i \geq 2$ then

Call COMPARE (from Lemma 2.6) on $\{s_i - 1\}, \{s_i\}$ with $\eta = \frac{\tau}{2}$, $K = 2$ and $\delta = \frac{1}{10m}$.

if the procedure outputs High then return FAIL

else if it outputs a value $\rho$ then

if $\rho < \frac{1+n}{1+\alpha+\tau}$ then return FAIL

end if

end if

end for

return ACCEPT

5 With EVAL access

In this section, we describe a poly(log $n, 1/\varepsilon$)-query tester for monotonicity in the Evaluation Query model (EVAL), in which the testing algorithm is granted query access to the probability mass function unknown distribution – but not the ability to sample from it.

Remark 5.1 (On the relation to $\ell_p$-testing for functions on the line). We observe that the results of Berman et al. [BRY14] in testing monotonicity of functions with relation to $\ell_p$ distances do not directly apply here. Indeed, while their work is indeed concerned with functions $f: [n] \rightarrow [0, 1]$ to which query access is granted, two main differences prevent us from using their techniques for EVAL access to distributions: first, the distance they consider is normalized, by a factor $n$ in the case of $\ell_1$ distance. A straightforward application of their result would therefore imply replacing $\varepsilon$ by $\varepsilon' = \varepsilon/n$ in their statements, incurring a prohibitive sample complexity. Furthermore, even adapting their techniques and structural lemmata is not straightforward, as distance to monotone $[0, 1]$-valued functions is not directly related to distance to monotone distributions: specifically, the main tool leveraged in their reduction to Boolean Hamming testing ([BRY14, Lemma 2.1]) does no longer hold for distributions.

5.1 A poly(log $n, 1/\varepsilon$)-query tester for EVAL

We start by stating two results we shall use as subroutines, before stating and proving our theorem.

Lemma 5.2. Given EVAL access to a monotone distribution $D$ over $[n]$, there exists a (non-adaptive) algorithm\footnote{Recall that a non-adaptive tester is an algorithm whose queries do not depend on the answers obtained from previous ones, but only on its internal randomness. Equivalently, it is a tester that can commit “upfront” to all the} that, on input $\varepsilon$, makes $O\left(\frac{\log n}{\varepsilon}\right)$ queries and outputs a monotone distribution $\hat{D}$ such that $d_{\text{TV}}(\hat{D}, D) \leq \varepsilon$. Furthermore, $\hat{D}$ is an $O\left(\frac{\log n}{\varepsilon}\right)$-histogram.
Proof. This follows from adapting the proof of Theorem 2.2 as follows: we consider the same oblivious partition of \([n]\) in \(\ell = O(\log n/\epsilon)\) intervals, but instead of taking as in (2) the weight of a point \(i \in I_k\) to be the average \(D(I_k)/|I_k|\), we consider the average of the endpoints of \(I_k = (a_k, a_{k+1})\):

\[
\forall k \in [\ell], \forall i \in I_k, \quad \tilde{D}_\epsilon(i) = \frac{D(a_k) + D(a_{k+1})}{2}.
\]

Clearly, this hypothesis can be (exactly) computed by making \(\ell\) \textsc{eval} queries. The result directly follows from observing that, in the proof of his theorem, Birgé first upperbounds \(\|\Phi_\epsilon(D) - D\|_1\) by \(\|\tilde{D}_\epsilon - D\|_1\), before showing the latter – which is the quantity we are interested in – is at most \(2\epsilon\) (see [Bir87, Eq. (2.4)–(2.5)]). The last step to be taken care of is the fact that \(\tilde{D}_\epsilon\), as defined, might not be a distribution – i.e., it may not sum to one. But as \(\tilde{D}_\epsilon\) is fully known, it is possible to efficiently (and without taking any additional sample) compute the \(\ell\)-histogram monotone distribution \(\hat{D}_\epsilon\) which is closest to it. We are guaranteed that \(\hat{D}_\epsilon\) will be at most \(4\epsilon\)-far from \(\tilde{D}_\epsilon\) in \(\ell_1\) distance, as there exists one particular distribution, namely \(\Phi_\epsilon(D)\), that is (being at a distance at most \(2\epsilon\) of \(D\) as well). Therefore, overall \(\hat{D}_\epsilon\) is a monotone distribution that is at most \(6\epsilon\)-far from \(D\) in \(\ell_1\) distance, i.e. \(d_{TV}(D, \hat{D}_\epsilon) \leq 3\epsilon\). \(\square\)

**Theorem 5.3** (Tolerant identity testing ([CR14, Remark 3 and Corollary 1])). Given \textsc{eval} access to a distribution \(D\) over \([n]\), there exists a (non-adaptive) algorithm that, on input \(\epsilon, \delta \in (0,1]\) and the full specification of a distribution \(D^*\), makes \(O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)\) queries and outputs a value \(\hat{d}\) such that, with probability at least \(1 - \delta\), \(\big|\hat{d} - d_{TV}(D, D^*)\big| \leq \epsilon\).

**Theorem 5.4.** There exists an \(O\left(\max\left(\log \frac{n}{\epsilon}, \frac{1}{\epsilon^2}\right)\right)\)-query tester for monotonicity in the \textsc{eval} model.

Proof. Calling Lemma 5.2 with accuracy parameter \(\epsilon/4\) enables us to learn a histogram \(\hat{D}\) guaranteed, if \(D\) is monotone, to be \((\epsilon/4)\)-close to \(D\); this by making \(\ell \overset{\text{def}}{=} O\left(\frac{\log n}{\epsilon^2}\right)\) queries. Using Theorem 5.3, we can also get with \(O(1/\epsilon^2)\) queries an estimate \(\hat{d}\) of \(d_{TV}(D, \hat{D})\), accurate up to an additive \(\epsilon/4\) with probability at least \(2/3\). Combining the two, we get, with probability at least \(2/3\),

(i) \(\hat{D}, (\epsilon/4)\)-close to \(D\) if \(D\) is monotone;

(ii) \(\hat{d} \in \left[d_{TV}(D, \hat{D}) - \epsilon/4, d_{TV}(D, \hat{D}) + \epsilon/4\right]\).

We can now describe the testing algorithm:

queries it will make to the oracle.
Algorithm 4 Algorithm TestMonEval

Require: EVAL access to \( D \)

1: Set \( \alpha \) \( \equiv \) \( \varepsilon/4 \), and compute \( I_\alpha \).
2: Get a candidate approximation of \( D \) and test it for monotonicity, by:
   (a) Applying Lemma 5.2 with parameter \( \alpha \) to obtain \( \hat{D} \), histogram on \( I_\alpha \);
   (b) Checking (offline) whether \( \hat{D} \) is \( (\varepsilon/4) \)-close to monotone; \textbf{return} \text{FAIL} if not.
3: Get an estimate \( \hat{d} \) of \( d_{\text{TV}}(D, \hat{D}) \) up to additive \( \varepsilon/4 \), as per (ii); \textbf{return} \text{FAIL} if \( \hat{d} > \varepsilon/2 \).
4: \textbf{return} \text{ACCEPT}

To argue correctness, it suffices to observe that, conditioning on the estimate being as accurate as required (which happens with probability at least 2/3):

- if \( D \in \mathcal{M} \), then \( \hat{D} \in \mathcal{M} \) as well and we pass the first step. We also know by Lemma 5.2 that in this case \( d_{\text{TV}}(D, \hat{D}) \leq \alpha \), so that our estimate satisfies \( \hat{d} \leq \alpha + \varepsilon/4 = \varepsilon/2 \). Therefore, the algorithm does not reject here either, and eventually outputs \text{ACCEPT}.
- conversely, if the algorithm outputs \text{ACCEPT}, then we have both that (a) the distance of \( \hat{D} \) to \( \mathcal{M} \) is at most \( \varepsilon/4 \), and (b) \( d_{\text{TV}}(D, \hat{D}) \leq \hat{d} + \varepsilon/4 \leq 3\varepsilon/4 \); so overall \( d_{\text{TV}}(D, \mathcal{M}) \leq \varepsilon \).

As for the query complexity, it is straightforward from the setting of \( \alpha = \Theta(\varepsilon) \) and the foregoing discussion (recall that Step (b) can be performed efficiently, e.g. via linear programming ([BKR04, Lemma 8])).

\[ \square \]

5.2 An \( \Omega(\log n) \) (non-adaptive) lower bound for EVAL

In this section, we show that, when focusing on \textit{non-adaptive} testers, Theorem 5.4 is tight (note that the tester described in the previous section is, indeed, non-adaptive).

**Theorem 5.5.** For any \( \varepsilon \in (0, 1/2) \), any non-adaptive \( \varepsilon \)-tester for monotonicity in the EVAL model must perform \( \frac{1}{4} \log n \) queries.

**Proof.** We hereafter assume without loss of generality that \( n/3 \) is a power of two; and shall define a distribution over pairs of distributions, \( \mathcal{D} \), such that the following holds. A random pair of distributions \( (D_1, D_2) \) drawn from \( \mathcal{D} \) will have \( D_1 \) monotone, but \( D_2 \) \( \varepsilon \)-far from monotone. Yet, no non-adaptive deterministic testing algorithm can distinguish with probability 2/3 (over the draw of the distributions) between \( D_1 \) and \( D_2 \), unless it performs \( c \log n \) EVAL queries. By Yao’s minimax principle, this will guarantee that any non-adaptive randomized tester must perform at least \( c \log n \) queries in the worst case.

More specifically, a pair of distributions \( (D_1, D_2) \) is generated as follows. A parameter \( m \) is chosen uniformly at random in the set

\[ M \equiv \left\{ \frac{2}{\kappa \varepsilon}, \frac{2}{\kappa \varepsilon} (1 + \kappa), \ldots, \frac{2}{\kappa \varepsilon} (1 + \kappa \varepsilon)^k, \ldots, \frac{n}{3} \right\}, \]

where \( \kappa \) \( \equiv \) \( \frac{4}{1-2\varepsilon} \). \( D_1 \) is then set to be the uniform distribution on \( \{1, \ldots, (2 + \kappa \varepsilon) m\} \); as for \( D_2 \), it is defined as the histogram putting weight:

- \( \frac{1}{2} - \varepsilon \) on \( \{1, \ldots, m\} \);
• 0 on $I_m \overset{\text{def}}{=} \{m + 1, \ldots, \lfloor (1 + \frac{\kappa \epsilon}{2})m \rfloor \}$;
• $2\epsilon$ on $J_m \overset{\text{def}}{=} \{\lfloor (1 + \frac{\kappa \epsilon}{2})m \rfloor + 1, \ldots, \lfloor (1 + \kappa \epsilon)m \rfloor \}$;
• and $\frac{1}{2} - \epsilon$ on $\{\lfloor (1 + \kappa \epsilon)m \rfloor + 1, \ldots, \lfloor (2 + \kappa \epsilon)m \rfloor \}$.

It is not hard to see that $D_1$ is indeed monotone, and that the distance of $D_2$ from monotone is exactly $\epsilon$.

$D_j(i)$

Figure 1: Construction of $D_1$ (dotted) and $D_2$.

The key of the argument is to observe that if too few queries are made, then with high probability over the choice of $m$ no queries will hit the interval $I_m \cup J_m$; and that conditioning on this, what the tester sees in the yes- and no-cases is indistinguishable.

**Claim 5.6.** Let $T$ be a deterministic, non-adaptive algorithm making $q \leq \frac{\log n}{\epsilon}$ queries to the EV AL oracle. Then, the probability (over the choice of $m$) that a query hits $I_m \cup J_m$ is less than $1/3$.

**Proof.** This follows from observing that the probability that any fixed point $x \in [n]$ belongs to $I_m \cup J_m$ is at most $\frac{1}{|M|} (1 + o(1)) \frac{\epsilon}{\log n}$, as this can only happen for at most one value of $m$ (among $|M|$ equiprobable choices). By Markov’s inequality, this implies that the probability of any query falling into $I_m$ is at most $\frac{q}{|M|} < \frac{1}{3}$ (for $n$ big enough). □

To see why the claims directly yields the theorem, observe that the above implies that for any such algorithm, $\left| \Pr_{D_1} [T_{D_1} = \text{yes}] - \Pr_{D_2} [T_{D_2} = \text{yes}] \right| < \frac{1}{3}$. But then $T$ cannot be a successful monotonicity tester, as otherwise it would accept $D_1$ with probability at least $2/3$, and $D_2$ with probability at most $1/3$. □

### 5.3 An $\tilde{\Omega}(\log n)$ (adaptive) lower bound for EV AL

While the above lower bound is tight, it only applies to non-adaptive testers; and it is natural to ask whether allowing adaptivity enables one to bypass this impossibility result – and, maybe, to get constant-query testers. The following result shows that it is not the case: even for constant $\epsilon$, adaptive testers must also make (almost) logarithmically many queries in the worst case.
**Theorem 5.7.** There exist absolute constants $\varepsilon_0 > 0$ and $c > 0$ such that the following holds. Any $\varepsilon_0$-tester for monotonicity in the EVAL model must perform $c \frac{\log n}{\log \log n}$ queries. (Furthermore, one can take $\varepsilon_0 = 1/2$.)

Intuitively, if one attempts to design hard instances for this problem against adaptive algorithms, one has to modify a yes-instance to get a no-instance by “removing” some probability weight and “hiding” it somewhere else (where it will then violates monotonicity). The difficult part in doing so does not lie in hiding that extra weight: one can always choose a random element $k \in \{n/2, \ldots, n\}$ and add some probability to it. Arguing that any EVAL algorithm cannot find $k$ unless it makes $\Omega(n)$ queries is then not difficult, as it is essentially tantamount to finding a needle in a haystack.

Thus, the key is to take some probability weight from a subset of points of the support, in order to redistribute it. Note that this cannot this time be a local modification, as in a monotone distribution one cannot obtain $\Omega(1)$ weight from a constant number of points unless these are amongst the very first elements of the domain; and such case is easy to detect with $O(1)$ queries. Equivalently, we want to describe how to obtain two non-negative monotone sequences that are hard to distinguish, one summing to one (i.e., already being a probability distribution) and the other having sum bounded away from one (the slack giving us some “weight to redistribute”). To achieve this, we will rely on the following result due to Sariel Har-Peled [Har15], whose proof is reproduced below.\(^{10}\)

**Proposition 5.8.** Given query access to a sequence of non-negative numbers $a_n \geq \cdots \geq a_1$ and $\varepsilon \in (0,1)$, along with the promise that either $\sum_{k=1}^n a_k = 1$ or $\sum_{k=1}^n a_k \leq 1 - \varepsilon$, any (possibly adaptive) randomized algorithm that distinguishes between the two cases with probability at least 2/3 must make $\Omega\left(\frac{\log n}{\log \log n}\right)$ queries in the worst case. (Moreover, the result even holds for $\varepsilon = 1/2$.)

**Proof.** Let $(a_k)_{k \in [n]}$ be a sequence defined as follows: we partition the sequence into $L$ blocks. In the $i$-th block there are going to be $n_i$ elements (i.e., $\sum_i n_i = n$). Set the $i$-th block size to be $n_i = L^i$, where $L = \Theta(\log n / \log \log n)$ is the number of blocks. Let $\beta = \frac{(2L - 1)}{(2L)}$ be a normalizing factor; an element in the $i$-th block has value $\alpha_i = \frac{\beta}{2Ln_i}$, so that the total sum of the values in the sequence is $\beta/2 < 1/2$.

From $(a_k)_{k \in [n]}$, we obtain another sequence $(b_k)_{k \in [n]}$ by picking uniformly at random an arbitrary block, say the $j$-th one, and set all values in its block to be $\alpha_{j-1} = L\alpha_j$ (instead of $\alpha_j$). This increases the contribution of the $j$-th block from $\beta/2L$ to $\beta/2$, and increase the total sum of the sequence to $\beta(1 - \frac{1}{2L}) = 1$. Furthermore, it is straightforward to see that both sequences are indeed non-decreasing.

Informally, the idea is that to distinguish $(a_k)$ from $(b_k)$, any randomized algorithm must check the value in each one of the blocks. As such, it must read at least $\Omega(L)$ values of the sequence. To make the above argument more formal, with probability $p = 1/2$, give the original sequence of sum 1 as the input (we refer to this as original input). Otherwise, randomly select the block that has the increased values (modified input). Clearly, if the randomized algorithm reads less than, say, $L/8$ entries, it has probability (roughly) $1/8$ to detect a modified input. As such, the probability this algorithm fails, if it reads less than $L/8$ entries, is at least $(1 - p)(7/8) > 7/16 > 1/3$. \(\Box\)

**Proof of Theorem 5.7.** To get Theorem 5.7 from Proposition 5.8, we define a reduction in the obvious way: any EVAL monotonicity tester $T$ can be used to solve the promise problem above by

\(^{10}\)For more results on approximating the discrete integral of sequences, including an upper bound for monotone sequences reminiscent of Birgé’s oblivious decomposition, one may consult [Har06].
first choosing uniformly at random an element \( k \) in \( \{2, \ldots, n\} \), and then answering any query \( j \in [n] \setminus \{k\} \) from \( \mathcal{A} \) by returning the value \( a_j \). (This indeed defines a probability distribution that is either monotone (if \( \sum_k a_k = 1 \)) or far from it (if \( \sum_k a_k = 1/2 \)): \( k \) is the index where the – possibly – extra weight \( 1/2 \) would have been “hidden”, in a no-instance; and is therefore the only query point we cannot answer.) Conditioning on \( k \) not being queried (which occurs with probability \( 1 - O(1/n) \) given the random choice of \( k \), it is straightforward to see that outputting the value returned by \( T \) yields the correct answer with probability \( 2/3 \). From the above, any such \( T \) must therefore have query complexity \( \Omega \left( \frac{\log n}{\log \log n} \right) \).

Open question. It is worth noting that a different construction, also due to [Har15], yields a different lower bound of \( \Omega(1/\varepsilon) \) for the promise problem of Proposition 5.8. Combining the two (and applying the same reduction as above), we obtain a lower bound of \( \Omega(\max(\log n/\log \log n, 1/\varepsilon)) \) for testing monotonicity in the EVAL model. However, we do conjecture the right dependence on \( n \) to be logarithmic; more specifically, the author believe the above upper bound to be tight:

Conjecture 5.9. Monotonicity testing in the EVAL model has query complexity \( \Omega \left( \frac{\log n}{\varepsilon} \right) \).

6 With Cumulative Dual access

Theorem 6.1. There exists an \( O\left(\frac{1}{\varepsilon^4}\right) \)-query (independent of \( n \)) tester for monotonicity in the Cumulative Dual model.

Proof. We first give the overall structure of the tester – without surprise, very similar to the ones in Section 4.2 and Section 5.1:

| Algorithm 5 Algorithm TestMonCumulative |
|----------------------------------------|
| **Require:** CEVAL and SAMP access to \( D \) |
| 1: Set \( \alpha \overset{\text{def}}{=} \varepsilon/4 \), and compute \( I_\alpha \). |
| 2: Test if \( \Phi_\alpha(D) \) is \((\varepsilon/4)\)-close to monotone by testing \((\varepsilon/4)\)-closeness (of \( D^{\text{red}}_\alpha \)) to \( P_\alpha \); return \( \text{FAIL} \) if the tester rejects. |
| 3: Get an estimate \( \hat{d} \) of \( d_{TV}(D, \Phi_\alpha(D)) \) up to additive \( \varepsilon/4 \); return \( \text{FAIL} \) if \( \hat{d} > \varepsilon/2 \). |
| 4: return \( \text{ACCEPT} \) |

Before diving into the actual implementation of Steps 2 and 3, we first argue that, conditioned on their outcome being correct, TestMonCumulative outputs the correct answer. The argument is almost identical as in the proof of Theorem 5.4:

- if \( D \in \mathcal{M} \), then \( \Phi_\alpha(D) \in \mathcal{M} \) as well; by Remark 4.9 it follows that \( D^{\text{red}}_\alpha \in P_\alpha \) and we pass the first step. We also know (Theorem 2.2) that \( d_{TV}(D, \Phi_\alpha(D)) \leq \alpha \), so that our estimate satisfies \( \hat{d} \leq \alpha + \varepsilon/4 = \varepsilon/2 \). Therefore, the algorithm does not reject here either, and eventually outputs \( \text{ACCEPT} \).

- conversely, if the algorithm outputs \( \text{ACCEPT} \), then we have both that (a) the distance of \( \Phi_\alpha(D) \) to \( \mathcal{M} \) is at most \( \varepsilon/4 \), and (b) \( d_{TV}(D, \Phi_\alpha(D)) \leq \hat{d} + \varepsilon/4 \leq 3\varepsilon/4 \); so overall \( d_{TV}(D, \mathcal{M}) \leq \varepsilon \).

It remains to show how to perform steps 2 and 3 – namely, testing \( D^{\text{red}}_\alpha \) for \( P_\alpha \) given CEVAL and SAMP access to \( D \), and approximating \( d_{TV}(D, \Phi_\alpha(D)) \).
Testing $\gamma$-closeness to $P_\alpha$  This part is performed similarly as in Section 4.2.4, observing that one can easily simulate access to PAIRCOND$_Q$ from a CEVAL$_Q$ oracle. Indeed, Lemma 4.12 implies that when $Q$ is $\varepsilon$-far from having the property, it suffices to sample $O\left(\frac{1}{\alpha \varepsilon^2}\right)$ points according to $Q = D^\text{red}_\alpha$ and compare them to their neighbors to detect a violation with probability at least $9/10$. Note that this last test is easy, as we have query access to $Q$ (recall that we have a CEVAL$_D$ oracle, and that $D^\text{red}_\alpha(k) = D(I_k)$).

Efficient approximation of distance to $\Phi(D)$ Let $D$, $\varepsilon$ and $I_\alpha$ be as before; define $Z$ to be a random variable taking values in $[0,1]$, such that, for $k \in [\ell]$, $Z$ is equal to $d_{TV}(D_{I_k},U_{I_k})$ with probability $w_k = D(I_k)$. It follows that

$$
\mathbb{E}Z = \sum_{k=1}^{\ell} w_k d_{TV}(D_{I_k},U_{I_k}) = \frac{1}{2} \sum_{k=1}^{\ell} w_k \sum_{i \in I_k} |D_{I_k}(i) - \frac{1}{|I_k|}|
= \frac{1}{2} \sum_{k=1}^{\ell} \sum_{i \in I_k} |D(i) - \frac{D(I_k)}{|I_k|}| = \frac{1}{2} \sum_{i=1}^{n} |D(i) - \Phi_\alpha(D)|
= d_{TV}(D,\Phi_\alpha(D)).
$$

Furthermore, one can simulate $m = O(1/\varepsilon^2)$ i.i.d. draws from $Z$ by repeating independently the following for each of them:

- draw $i \sim D$ by calling SAMP$_D$, and look up the $k$ for which $i \in I_k$;
- get the value $D(I_k)$ with 2 CEVAL queries (note that $D(I_k) > 0$, as we just got a sample from $I_k$);
- estimate $d_{TV}(D_{I_k},U_{I_k})$ up to $\pm \varepsilon$ (with failure probability at most $\frac{1}{10m}$) by drawing $\tilde{O}(1/\varepsilon^2)$ uniform samples from $I_k$ and querying the values of $D$ on them, to estimate

$$
d_{TV}(D_{I_k},U_{I_k}) = \sum_{j \in I_k} \left| D_{I_k}(j) - \frac{1}{|I_k|} \right| \cdot \mathbb{I}_{\{D_{I_k}(j) < 1\}} = \sum_{j \in I_k} \left| \frac{D(j)}{D(I_k)} \right| \cdot \frac{|I_k|}{|I_k|} \cdot \mathbb{I}_{\{D_{I_k}(j) < 1\}}
= \mathbb{E}_{j \sim U_{I_k}} \left[ \left| \frac{D(j)}{D(I_k)} \right| \cdot \frac{|I_k|}{|I_k|} \cdot \mathbb{I}_{\{D_{I_k}(j) < 1\}} \right].
$$

Applying a Chernoff bound (and a union bound over all such simulated draws), performing this only $m$ times is sufficient to get an $\varepsilon$-additive estimate of $\mathbb{E}Z$ with probability at least $9/10$, for an overall $\tilde{O}(1/\varepsilon^4)$ number of queries (sample and evaluation).

Proof of Theorem 6.1  Correctness has already been argued, provided both subroutines do not err; by a union bound, the “good event” that the two steps produce such an outcome happens with probability at least $4/5$. The query complexity is the sum of $O\left(\frac{1}{\alpha \varepsilon^2}\right) = O\left(\frac{1}{\varepsilon^5}\right)$ (Step 2) and $\tilde{O}(1/\varepsilon^4)$ (Step 3), yielding overall the claimed $\tilde{O}(1/\varepsilon^4)$ sample complexity.  

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A Models of distribution testing: definitions

In this appendix, we define precisely the notion of testing of distributions over a domain $\Omega$ (e.g., $\Omega = [n]$), and the different models of access covered in this work.

Recall that a property $\mathcal{P}$ of distributions over $\Omega$ is a set consisting of all distributions that have the property. The distance from $D$ to a property $\mathcal{P}$, denoted $d_{TV}(D, \mathcal{P})$, is then defined as $\inf_{D' \in \mathcal{P}} d_{TV}(D, D')$. We use the standard definition of testing algorithms for properties of distributions over $\Omega$, where $n$ is the relevant parameter for $\Omega$ (i.e., in our case, its size $|\Omega|$). We chose here to phrase it in the most general setting possible with regard to how the unknown distribution is “queried”: and will specify this aspect further in the next paragraphs (sampling access, conditional access, etc.).

**Definition A.1.** Let $\mathcal{P}$ be a property of distributions over $\Omega$. Let $\text{ORACLE}_D$ be an oracle providing some type of access to $D$. A $q$-query $\text{ORACLE}$ testing algorithm for $\mathcal{P}$ is a randomized algorithm $\mathcal{T}$ which takes as input $n$, $\varepsilon \in (0, 1]$, as well as access to $\text{ORACLE}_D$. After making at most $q(\varepsilon, n)$ calls to the oracle, $\mathcal{T}$ either outputs $\text{ACCEPT}$ or $\text{REJECT}$, such that the following holds:

- if $D \in \mathcal{P}$, $\mathcal{T}$ outputs $\text{ACCEPT}$ with probability at least $2/3$;
- if $d_{TV}(D; \mathcal{P}) \geq \varepsilon$, $\mathcal{T}$ outputs $\text{REJECT}$ with probability at least $2/3$.

**A.1 The SAMP model**

In this first and most common setting, the testers access the unknown distribution by getting independent and identically distributed (i.i.d.) samples from it.

**Definition A.2 (Standard access model (sampling)).** Let $D$ be a fixed distribution over $\Omega$. A *sampling oracle* for $D$ is an oracle $\text{SAMP}_D$ defined as follows: when queried, $\text{SAMP}_D$ returns an element $x \in \Omega$, where the probability that $x$ is returned is $D(x)$ independently of all previous calls to the oracle.

This definition immediately implies that all algorithms in this model are by essence non-adaptive: indeed, any tester or tolerant tester can be converted into a non-adaptive one, without affecting the sample complexity. (This is a direct consequence of the fact that all an adaptive algorithm can do when interacting with a $\text{SAMP}$ oracle is deciding to stop asking for samples, based on the ones it already got, or continue.)

**A.2 The COND model**

**Definition A.3 (Conditional access model [CFGM13, CRS12]).** Fix a distribution $D$ over $\Omega$. A *COND oracle* for $D$, denoted $\text{COND}_D$, is defined as follows: the oracle takes as input a query set $S \subseteq \Omega$, chosen by the algorithm, that has $D(S) > 0$. The oracle returns an element $i \in S$, where the probability that element $i$ is returned is $D_S(i) = D(i)/D(S)$, independently of all previous calls to the oracle.

Note that as described above the behavior of $\text{COND}_D(S)$ is undefined if $D(S) = 0$, i.e., the set $S$ has zero probability under $D$. Various definitional choices could be made to deal with this: e.g., Canonne et al. [CRS12] assume that in such a case the oracle (and hence the algorithm) outputs “failure” and terminates, while Chakraborty et al. [CFGM13] define the oracle to return
in this case a sample uniformly distributed in \( S \). In most situations, this distinction does not make any difference, as most algorithms can always include in their next queries a sample previously obtained; however, the former choice does rule out the possibility of non-adaptive testers taking advantage of the additional power \( \text{COND} \) provides over \( \text{SAMP} \); while such testers are part of the focus of [CFGM13].

Testing algorithms can often only be assumed to have the ability to query sets \( S \) that have some sort of “structure”, or are in some way “simple”. To capture this, one can define specific restrictions of the general \( \text{COND} \) model, which do not allow arbitrary sets to be queried but instead enforce some constraints on the queries: [CRS12] introduces and studies two such restrictions, “PAIRCOND” and “INTCOND”.

**Definition A.4** (Restricted conditional oracles). A PAIRCOND (“pair-cond”) oracle for \( D \) is a restricted version of \( \text{COND}_D \) that only accepts input sets \( S \) which are either \( S = \Omega \) (thus providing the power of a \( \text{SAMP}_D \) oracle) or \( S = \{x, y\} \) for some \( x, y \in \Omega \), i.e. sets of size two.

In the specific case of \( \Omega = [n] \), an INTCOND (“interval-cond”) oracle for \( D \) is a restricted version of \( \text{COND}_D \) that only accepts input sets \( S \) which are intervals \( S = [a, b] = \{a, a+1, \ldots, b\} \) for some \( a \leq b \in [n] \) (note that taking \( a = 1, b = n \) this provides the power of a \( \text{SAMP}_D \) oracle).

### A.3 The EVAL, Dual and Cumulative Dual models

**Definition A.5** (Evaluation model [RS09]). Let \( D \) be a fixed distribution over \( \Omega \). An evaluation oracle for \( D \) is an \( \text{EVAL}_D \) defined as follows: the oracle takes as input a query element \( x \in \Omega \), and returns the probability weight \( D(x) \) that the distribution puts on \( x \).

**Definition A.6** (Dual access model [BDKR05, GMV06, CR14]). Let \( D \) be a fixed distribution over \( \Omega \). A dual oracle for \( D \) is a pair of oracles \((\text{SAMP}_D, \text{EVAL}_D)\) defined as follows: when queried, the sampling oracle \( \text{SAMP}_D \) returns an element \( x \in \Omega \), where the probability that \( x \) is returned is \( D(x) \) independently of all previous calls to any oracle; while the evaluation oracle \( \text{EVAL}_D \) takes as input a query element \( y \in \Omega \), and returns the probability weight \( D(y) \) that the distribution puts on \( y \).

**Definition A.7** (Cumulative Dual access model [BKR04, CR14]). Let \( D \) be a fixed distribution over \( \Omega = [n] \). A cumulative dual oracle for \( D \) is a pair of oracles \((\text{SAMP}_D, \text{CEVAL}_D)\) defined as follows: the sampling oracle \( \text{SAMP}_D \) behaves as before, while the evaluation oracle \( \text{CEVAL}_D \) takes as input a query element \( j \in [n] \), and returns the probability weight that the distribution puts on \( \{j\} \), that is \( D([j]) = \sum_{i=1}^j D(i) \).

(Note that, as the latter requires some total ordering on the domain, it is only defined for distributions over \( [n] \); as was the \( \text{INTCOND} \) oracle from Definition A.4.)

### B Tolerant testing with Dual and Cumulative Dual access

In this appendix, we show how similar ideas can yield tolerant testers for monotonicity, as long as the access model admits both an agnostic learner for monotone distributions and an efficient distance approximator. As this is the case for both the Dual and Cumulative Dual oracles, this allows us to derive such tolerant testers with logarithmic query complexity. (Note that these results
imply that, in both models, tolerant testing monotonicity of an arbitrary distribution is no harder than learning an actually monotone distribution.)\textsuperscript{11}

**Theorem B.1.** Fix any constant $\gamma > 0$. In the dual access model, there exists an $(\varepsilon_1, \varepsilon_2)$-tolerant tester for monotonicity with query complexity $O\left(\frac{\log n}{\varepsilon_2^2}\right)$, provided $\varepsilon_2 > (3 + \gamma)\varepsilon_1$.

**Proof.** We will use here the result of Theorem 5.3 (with its probability of success slightly increased to 5/6 by standard techniques), as well as the Birgé decomposition of $[n]$ (with parameter $\Omega(\varepsilon_2)$) from Definition 2.1. The tester is described in Algorithm 6, and follows a very simple idea: leveraging the robustness of the Birgé flattening, it first agnostically learns an approximation $\bar{D}$ of the distribution and computes (offline) its distance to monotonicity. Then, using the (efficient) tolerant identity tester available in both dual and cumulative dual models, it estimates the distance between $\bar{D}$ and $D$: if both distances are small, the triangle inequality allows us to conclude $D$ must be close to monotone.

**Algorithm 6 Algorithm TOLERANTTESTMONOTONICITY**

| Require: | SAMP$_D$ and EVAL$_D$ oracle access, parameters $0 \leq \varepsilon_1 < \varepsilon_2$ such that $\varepsilon_2 > (3 + \gamma)\varepsilon_1$ |
|---|---|
| 1: | Set $\alpha \overset{\text{def}}{=} \frac{\gamma}{6(3 + \gamma)}$, $\gamma_1 \overset{\text{def}}{=} 2\varepsilon_1 + 2\alpha\varepsilon_2$ and $\gamma_2 \overset{\text{def}}{=} (1 - \alpha)\varepsilon_2 - \varepsilon_1$. $\triangleright$ So that $\gamma_2 - \gamma_1 = \Omega(\varepsilon_2)$ |
| 2: | Learn $\Phi_{\alpha\varepsilon_2}(D)$ to distance $\alpha\varepsilon_2$, getting a (piecewise constant) $\bar{D}$ $\triangleright$ $O\left(\frac{\log n}{\varepsilon_2^2}\right)$ samples |
| 3: | Test if $\bar{D}$ is $(\varepsilon_1 + \alpha\varepsilon_2)$-close to monotone; return FAIL if not. $\triangleright$ no sample needed (LP) |
| 4: | Test if $\bar{D}$ is $\gamma_1$-close to $D$ vs. $\gamma_2$-far; return FAIL if far. $\triangleright$ Tester from Theorem 5.3. |
| 5: | return ACCEPT |

**Correctness.** Suppose the execution of each step is correct; by standard Chernoff bounds, this event for Step 2 (that is, our estimate $\bar{D}$ of $\Phi_{\varepsilon_1}(D)$ being indeed $\varepsilon_1$-accurate) happens with probability at least 5/6; furthermore, Step 3 is deterministic, and Step 4 only fails with probability 1/6 – so the overall “good event” happens with probability at least 2/3. We hereafter condition on this.

- if $d_{TV}(D, M) \leq \varepsilon_1$, then according to Corollary 2.4:
  
  - (a) $d_{TV}($\Phi_{\alpha\varepsilon_2}(D), M$) \leq \varepsilon_1$, so that $d_{TV}($\bar{D}, M$) \leq \varepsilon_1 + \alpha\varepsilon_2$ and we pass Step 3; and
  
  - (b) $d_{TV}(D, \Phi_{\alpha\varepsilon_2}(D)) \leq 2\varepsilon_1 + \alpha\varepsilon_2$, which in turn implies that $d_{TV}($\bar{D}, D$) \leq 2\varepsilon_1 + 2\alpha\varepsilon_2 = \gamma_1$; and we pass Step 4.

Therefore, the tester eventually outputs ACCEPT.

- Conversely, if the tester accepts, it means that
  
  - (a) $d_{TV}($\bar{D}, M$) \leq \varepsilon_1 + \alpha\varepsilon_2$ and
  
  - (b) $d_{TV}($D, $\bar{D}$\right) \leq \gamma_2 = (1 - \alpha)\varepsilon_2 - \varepsilon_1$

hence, by the triangle inequality $d_{TV}(D, M) \leq \varepsilon_2$.

\textsuperscript{11}It is however worth noting that, because of the restriction $\varepsilon_2 > 3\varepsilon_1$ it requires, our tolerant testing result does not imply a distance estimation algorithm.
Query complexity. The algorithm makes $O\left(\frac{\log n}{\varepsilon^2}\right)$ SAMP queries in Step 2, and $m = O\left(\frac{1}{(\gamma_2 - \gamma_1)^2}\right) = O\left(\frac{1}{\varepsilon^2}\right)$ EVAL queries in Step 4.

Remark B.2. With CEVAL access (instead of EVAL), the cost of Step 2 would be reduced from $O(\log n/\varepsilon^3)$ sample queries to get an $(\alpha\varepsilon_2)$-approximation to $O(\log n/\varepsilon_2)$ cdf queries to get the exact flattened distribution (as we have only that many quantities of the form $D(I_k)$ to learn, and each of them requires only 2 cdf queries). This leads to the following corollary:

**Corollary B.3.** Fix any constant $\gamma > 0$. In the Cumulative Dual access model, there exists an $(\varepsilon_1, \varepsilon_2)$-tolerant tester for monotonicity with query complexity $O\left(\frac{1}{\varepsilon_2^2} + \frac{\log n}{\varepsilon_2}\right)$, provided $\varepsilon_2 > (3 + \gamma)\varepsilon_1$. 

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