Constraining the weights of Stokes Polytopes using BCFW recursions for $\phi^4$

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Abstract

The relationship between certain geometric objects called polytopes and scattering amplitudes has revealed deep structures in QFTs. It has been developed in great depth at tree and loop-level for $\mathcal{N} = 4$ SYM theory and has been extended to the scalar $\phi^3$ and $\phi^4$ theories at tree-level. In this paper, we use the generalized BCFW recursion relations for massless planar $\phi^4$ theory to constrain the weights of a class of geometric objects called Stokes polytopes, which manifest in the geometric formulation of $\phi^4$ amplitudes. We compute the weights of $n = 8$ and $n = 10$ Stokes polytopes corresponding to eight- and ten-point amplitudes respectively. We generalize our results to higher-point amplitudes and show that in general the weights of an $n$-dimensional Stokes polytopes is fixed precisely in terms of lower-point weights, by the BCFW recursion relations.

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1 Introduction

In recent years a lot of work has been done in understanding both the analytic and geometric structure of scattering amplitudes in various classes of quantum field theories [1–4]. On-shell methods such as the BCFW recursion relations [6–10] have been not only extremely successful in simplifying the calculations of an infinite class of amplitudes but also revealed deep connections between physics and broad areas of mathematics such as Algebraic Geometry and Combinatorics. In addition, generalizations of the BCFW recursion relations have been successfully devised and are used to incorporate the boundary contributions, which correspond to pole at infinity, in a broad class of QFTs [9, 10].

Furthermore, the seminal work of [1] dubbed as the ‘Amplituhedron’ program has been greatly successful in developing a geometric formulation of the $\mathcal{N} = 4$ SYM theory. An essential feature in this formulation was that the scattering amplitudes were considered not as functions of particle momentum but rather as differential forms on certain auxiliary spaces. It was shown that there is a relation between these differential forms and certain positive geometries which completely encapsulates the $\mathcal{N} = 4$ SYM amplitudes. A remarkable feature of this formulation is that it is gauge-invariant and makes no reference to underlying principles of standard formulation of QFTs such as locality and unitarity, which are emergent in the formulation.
In [2,3] the authors extended this program to scalar $\phi^3$ and $\phi^4$ theories. In particular, for the massless planar tree-level $\phi^3$ amplitudes, it was shown in [2] that there is a precise relation between so-called planar scattering forms on kinematic space and a polytope known as Associahedron. Further, in [3,4], the authors developed a similar formulation for the massless planar $\phi^4$ amplitudes at tree-level and in [5] it was extended to the generalized case of $\phi^p$ interactions. It was shown that the planar $\phi^4$ amplitudes can be obtained from the geometry of an object known as Stokes polytopes. However, it was found that the calculation of $\phi^4$ amplitudes from the geometry of Stokes polytopes presents a peculiarity. The peculiarity lies in the fact that for a given number of particles $n$ there does not exist a unique Stokes polytope which completely determines the $\phi^4$ amplitudes; in contrast, there exists a unique Associahedron for a given $n$ which completely encapsulate the $\phi^3$ amplitudes. Each Stokes polytope contains only partial information about the complete $\phi^4$ amplitudes.

In order to determine the complete $\phi^4$ amplitudes, a weighted sum over all the Stokes polytopes is taken, and in general, these weights are not equal. The problem is made simpler by the fact that a cyclic permutation of the labels of only a few Stokes polytopes, referred to as the primitive Stokes polytopes, determine all the Stokes polytopes for a given dimension $n$. As a result of this fact, the weights can be parametrized only by the primitive quadrangulation of Stokes polytopes. The weights can be assigned a unique numerical value, which makes the sum over Stokes polytopes equal to the $\phi^4$ amplitudes.

In this paper, we address the issue of the undetermined weights of Stokes polytopes using the generalized BCFW recursion relations. We make use of the fact that the higher-point amplitudes factorize into lower-point amplitudes at physical poles. The factorization constrains the weights of the higher-point amplitudes, calculated by summing up over Stokes polytopes, in terms of the weights of the lower-point amplitudes. The boundary terms in $\phi^4$, which corresponds to the $O(z^0)$-behaviour for large $z$, fix the numerical values of these weights uniquely.

The paper is organized as follows. In section 2.1, we give an overview of the calculation of amplitudes from the geometry of Stokes polytopes. We briefly review the important notions of quadrangulation, $Q$-compatibility and convex realization of Stokes polytopes. The overview is not exhaustive, and we refer the interested reader to [2–4] for complete details. In section 2.2, we review the generalized BCFW recursion relations for $\phi^4$ amplitudes as given in [10], and discuss the calculation of boundary terms in $\phi^4$ amplitudes. In section 3, we use the generalized BCFW recursion relations to constrain the weights of higher-point Stokes polytopes in terms of lower-point weights. Firstly, we use the boundary terms to show that the lowest-point weight $\alpha_6$ is fixed uniquely. Then, we calculate the weights of Stokes polytopes of eight- and ten-point amplitudes and show that these are determined exactly in terms of the six-point weights. By substituting the value of $\alpha_6$, we determine the numerical value of eight- and ten-point weights and show that these agree with the results in [3]. In section 4, we generalize our results and prove that the weights of Stokes polytopes for any number of particles $n$ can be determined.
exactly in terms of the six-point weights. We end the paper with a discussion of our results and a brief commentary on using other methods to determine the weights.

Before proceeding we want to add that while this manuscript was being prepared, we came across [15], which uses ‘BCFW’ like recursion relations as described in [12, 13] to determine the weights of the Stokes polytopes in $\phi^4$ and in general, $\phi^p$ theory.

## 2 Amplitudes for massless planar $\phi^4$ theory

In this section, we give an overview of the key results of [3], where the relationship between planar Feynman graphs in $\phi^4$ theory and positive geometries was established. We focus on the construction of planar massless $\phi^4$ amplitudes at tree-level by summing over the Stokes polytopes. Further, we also review the key results of [10], where the boundary behaviour of the $\phi^4$ amplitudes was analyzed, and the generalized BCFW recursion relations was presented. Throughout the paper, we have considered the tree-level massless planar $\phi^4$ amplitudes.

### 2.1 $\phi^4$ amplitudes from Stokes Polytopes

In [3,4], the authors extended the seminal work of [1] to show that the planar amplitudes for massless $\phi^4$ theory can be obtained from the positive geometries of a polytope of dimension \( \left( \frac{n-4}{2} \right) \) known as Stokes polytopes.

A positive geometry, for example polygons and polytopes, is a closed geometry with boundaries or facets of all co-dimensions. There is a unique meromorphic differential form $\Omega$ that is canonically associated with a positive geometry whose form is fixed by the requirement of having logarithmic singularities at the boundaries and the residue at the boundary is equal to the canonical form of the boundary [11]. These canonical forms link the positive geometries to the scattering amplitudes.

For the analysis of planar amplitudes, the planar kinematic variables are used. These variables are defined as

\[
X_{ij} = (P_{ij...j-1})^2 \equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2, \quad 1 \leq i < j \leq n, \quad (2.1)
\]

where $p_i$ represents the momentum of the $i$-th particle. The Mandelstam variables can be expressed in terms of planar variables as

\[
s_{ij} = 2p_i \cdot p_j = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}. \quad (2.2)
\]

For the $\phi^4$ amplitudes, the quadrangulations of an $n$-gon, where $n$ is always even, are considered (figure 2, 3). The total number of ways to completely quadrangulate an $n = (2I + 2)$-gon is equal to the Fuss-Catalan number $F_I = \frac{1}{2I+1} \left( \binom{2I}{I} \right)$. Each quadrangulation
of the \( n \)-gon is associated with a planar Feynman graph with propagators \( X_{a_1}, \ldots, X_{a_{n-4}} \).

Unlike the planar \( \phi^3 \) amplitudes\(^1\), the \( \phi^4 \) amplitudes cannot be constructed from a unique canonical form related to a positive geometry. In \( \phi^4 \), for a given number of particles \( n = 2I + 2 \), there is \( F_I \) number of Stokes polytopes whose weighted sum gives the full amplitude. These weights are constrained by the factorization of the amplitudes at the physical poles.

For the construction of Stokes polytopes, it is important to define the notion of compatibility of a quadrangulation with the reference quadrangulation. This follows from the notion of compatibility of a diagonal with the reference quadrangulation and is given in detail in [3, 4]. The vertices of a Stokes polytopes with reference quadrangulation \( Q \) are the quadrangulations which are compatible with \( Q \) (figure 2). A key result from [3] is that the construction of Stokes polytopes depends on the reference quadrangulation chosen and different reference quadrangulations give different Stokes polytopes denoted by \( S_n^Q \).

For a given quadrangulation \( Q \) of an \( n \)-gon, the \( Q \) dependent planar scattering form is given as

\[
\Omega^Q_n = \sum_{\text{Graphs}} (-1)^{\sigma(\text{flip})} d \ln X_{a_1} \wedge d \ln X_{a_2} \ldots \wedge d \ln X_{a_{n-4}} ,
\]

where \( \sigma(\text{flip}) = \pm 1 \). A single quadrangulation does not capture the contributions from all the \( \phi^4 \) propagators.

With the Stokes polytopes and its respective canonical differential form defined, the pullback of the canonical form on the polytopes gives a form proportional to the partial amplitude corresponding to the quadrangulation \( Q \). To define the pullback, a convex realization of Stokes polytopes is established by imposing a set of constraints\(^2\). This is done by embedding the Stokes polytope \( S_n^Q \) of dimension \( \left( \frac{n-4}{2} \right) \) inside an Associahedra \( A_n \) with dimension \( (n-3) \). This gives a set of constraints that locate the Stokes polytope in the kinematic space \( \mathcal{K}_n \).

For example, in \( n = 6 \) case the constraints corresponding to reference quadrangulation \( Q = (14) \) and \( Q \)-compatible quadrangulation \( Q_1 = (36) \) are given by

\[
\begin{align*}
s_{ij} &= -c_{ij} \quad \forall \ 1 \leq i < j \leq n - 1 = 5, \ |i - j| \geq 2 \quad (2.4a) \\
X_{1,3} &= d_{13}, \quad X_{1,5} = d_{15} \quad \text{with} \quad d_{13}, d_{15} \geq 2 . \quad (2.4b)
\end{align*}
\]

The constraints in (2.4a) locate the 3-D associahedron \( A_6 \) inside \( \mathcal{K}_6 \). The constraints in (2.4b) locates the 1-D Stokes polytopes \( S_6^{(14)} \) inside \( A_6 \).

The pullback of (2.3) on the space of \( S_n^Q \) gives

\[
\omega_n^Q = \left( m_n^Q \right) dX_{a_1} \wedge dX_{a_2} \ldots \wedge dX_{a_{n-4}} ,
\]

\(^1\)The massless planar \( \phi^3 \) amplitudes can be obtained from the canonical form associated to a polytope known as Associahedron [2].

\(^2\)A positive geometry known as the ABHY associahedra completely encapsulates the \( \phi^3 \) and \( \phi^4 \) amplitudes. The convex realization of Stokes polytopes never enters the ABHY formalism [4].
where \( m_n^Q \) is the rational canonical function associated with the Stokes polytope \( S_n^Q \) and determines partial amplitude. The weighted sum of these functions over all Stokes polytopes gives the full planar scattering amplitude.

The computation of \( m_n^Q \) is greatly aided by a curious fact about \( n \)-gon, i.e. all quadrangulations of an \( n \)-gon can be determined by cyclic permutation \( \sigma \) of a subset of quadrangulations. This subset is referred to as the primitive quadrangulation. What this implies is that, given \( n \) number of particles, once the rational canonical functions for a given set of primitives \( \{ Q_1, \ldots, Q_J \} \) have been calculated, all the other \( m_n^Q \)’s can be computed by a cyclical permutation of the labels of \( m_n^Q \), where \( Q_i \in \{ Q_1, \ldots, Q_J \} \).

The \( m_n^Q \)’s are referred to as primitives. Therefore the master formula for evaluating the amplitudes is given as

\[
\tilde{M}_n = \sum_{Q|\text{primitives}} \sum_{\sigma} \alpha_Q \ m_n^{(\sigma \cdot Q)},
\]

(2.6)

where \( \alpha_Q \) are the weights. These are parametrized only by the primitive quadrangulations, i.e.

\[
\alpha_Q = \alpha_{\overline{Q}} \quad \forall \quad \overline{Q} = \sigma \cdot Q
\]

(2.7)

There is a unique choice of weights \( \alpha_Q \), which is constrained by the factorization of amplitude at the physical poles, such that \( \tilde{M}_n = M_n \), where \( M_n \) is the \( \phi^4 \) amplitude.

The weighted sum over the rational canonical function (2.6)

### 2.2 \( \phi^4 \) amplitudes from generalized BCFW recursion relations

In [10] it was shown that a generalized BCFW recursion relation, which gives a prescription to compute the boundary contributions, can be written for \( \phi^4 \) theory.

Consider the following BCFW shifts for \( \phi^4 \) denoted as \( \langle i|j \rangle \)

\[
|i| \rightarrow \hat{i} = |i| + z|j|, \quad |j| \rightarrow \hat{j} = |j| - z|i|.
\]

(2.8)

Under the BCFW shifts the amplitude \( M_n \) becomes a meromorphic function of \( z \) and in general, develops poles at finite \( z \) and at \( z = \infty \). There are two categories of Feynman diagram for \( \langle i|j \rangle \) deformation. Category (a) is where the particles \( i, j \) are attached to the same vertex and category (b) where \( i, j \) are attached to different vertices (figure 1).

For category (b) of Feynman diagrams, there is at least one propagator on the line connecting \( i \) and \( j \) that depends linearly on \( z \). This gives a factor of \( \frac{1}{P_{\hat{j}j} - z[P_{\hat{i}i}]} \) in the expression. In the limit, \( z \to \infty \), such factors have a zero contribution, and therefore category (b) diagrams do not contribute to the boundary terms.

For category (a) diagrams, there is a cancellation of the \( z \) terms in the summation of momenta, and therefore it has no \( z \) dependence in the expression. Consequently, the boundary terms are equal to Feynman diagrams where particles \( i \) and \( j \) have a common vertex.
From the above analysis, the generalized on-shell BCFW recursion relations is given as
\[
\mathcal{M}_n = \mathcal{P}_n + \mathcal{B}_n ,
\] (2.9)
where \(\mathcal{P}_n\) denotes the pole part corresponding to category (b) diagrams and is given as
\[
\mathcal{P}_n = \sum_I \mathcal{M}_L(z_I) \frac{1}{p^2_I} \mathcal{M}_R(z_I) ,
\] (2.10)
and \(\mathcal{B}_n\) denotes the boundary contributions corresponding to category (a) diagrams and is given as
\[
\mathcal{B}_n = \sum_{I' \cup J' = \{n\}\setminus\{i,j\}} \mathcal{M}_{I'} \frac{1}{p^2_{I'}} \frac{1}{p^2_{J'}} \mathcal{M}_{J'} .
\] (2.11)

When the particles \(i, j\) are color ordered and separated by a distance of more than two, there are no diagrams where they will be attached to the same vertex, and the boundary terms will vanish. For example, in the \((1\mid4)\) shift the boundary terms in \(\mathcal{M}_n\) are absent.

3 Determining the weights of Stokes polytopes

In this section, we use the generalized BCFW recursion relations, as discussed in section 2.2, to constrain the weights of the Stokes polytopes. Firstly we give the results for six-point amplitudes, calculated in full detail in [3,4]. Then, we use the six-point amplitudes as input for the calculation of eight- and ten-point amplitudes. We read the coefficients of the individual terms appearing in the amplitudes calculated from generalized BCFW and match them with the coefficients of respective terms in the amplitudes calculated from the summation over Stokes polytopes. We show that this along with the six-point weight uniquely determines the weights. We use the notation \(\alpha_n^Q\) to denote the \(n\)-point
weights corresponding to the primitive quadrangulation $Q$. Also, we have referred to
the summation over rational canonical function $m_n^Q$ as in (2.6), as a summation over the
Stokes polytopes $S_n^Q$ throughout the text.

3.1 Six-point amplitudes

There are three possible quadrangulations of the 6-gon as shown in figure 2. The weighted
sum over the Stokes polytopes $S_n^Q$ where $Q \in \{(14),(25),(36)\}$ is given as

$$\tilde{M}_6 = \alpha_Q^{(14)} \left( \frac{1}{X_{1,4}} + \frac{1}{X_{3,6}} \right) + \alpha_Q^{(25)} \left( \frac{1}{X_{2,5}} + \frac{1}{X_{1,4}} \right) + \alpha_Q^{(36)} \left( \frac{1}{X_{3,6}} + \frac{1}{X_{2,5}} \right).$$

(3.1)

The $n = 6$ case has only one primitive$^3$, $\left( \frac{1}{X_{1,4}} + \frac{1}{X_{3,6}} \right)$, and its cyclic permutation gives
the canonical rational functions corresponding to other quadrangulations as can be seen
from (3.1). This implies that the three weights are equal and the six-point amplitude is
given as

$$\tilde{M}_6 = 2\alpha_6 \left( \frac{1}{X_{1,4}} + \frac{1}{X_{2,5}} + \frac{1}{X_{3,6}} \right) = 2\alpha_6 \left( \frac{1}{P_{123}^2} + \frac{1}{P_{234}^2} + \frac{1}{P_{345}^2} \right),$$

(3.2)

where we used the equation (2.2) to express the planar variables $X$ in terms of $P^2$. We
also dropped the label for quadrangulation in the weight $\alpha_6$. Note that (3.2) gives the
correct $\phi^4$ amplitude upon substituting the correct numerical value for $\alpha_6$. This value is
determined in the next section.

3.2 Eight-point amplitudes

To determine the eight-point amplitudes, we use the $\langle 1|2\rangle$-shift and then apply BCFW
recursion relations. The boundary term has three contributions from the diagrams

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$^3$The choice of a primitive is arbitrary. The only condition is that two members of the set of primitive
quadrangulations must be related unrelated by a cyclic permutation.
(3|45678), (345|678) and (34567|8) and is given as

\[
\tilde{\mathcal{B}}_8^{[12]}(1, 2, \ldots, 8) = \tilde{\mathcal{M}}_4(1, 2, 3, -P) \frac{1}{P^2_{123}} \tilde{\mathcal{M}}_6(P, 4, 5, 6, 7, 8)
+ \tilde{\mathcal{M}}_4(8, 1, 2, -P) \frac{1}{P^2_{128}} \tilde{\mathcal{M}}_6(P, 3, 4, 5, 6, 7)
+ \tilde{\mathcal{M}}_4(3, 4, 5, -P) \frac{1}{P^2_{345} P^2_{678}} \tilde{\mathcal{M}}_4(P, 6, 7, 8),
\]

which on substituting (3.2) simplifies as

\[
\tilde{\mathcal{B}}_8^{[12]} = 2\alpha_6 \left( \frac{1}{P^2_{178}} + \frac{1}{P^2_{234}} + \frac{1}{P^2_{567}} + \frac{1}{P^2_{678}} \right) + \frac{2\alpha_6}{P^2_{234}} \left( \frac{1}{P^2_{234}} + \frac{1}{P^2_{456}} + \frac{1}{P^2_{567}} + \frac{1}{P^2_{678}} \right) + \frac{1}{P^2_{345} P^2_{678}}. \tag{3.4}
\]

The pole contribution is from two diagrams and is given as

\[
\tilde{\mathcal{P}}_8^{[12]}(1, 2, \ldots, 8) = \tilde{\mathcal{M}}_4(7, 8, \tilde{1}, -\tilde{P}) \frac{1}{P^2_{178}} \tilde{\mathcal{M}}_6(\tilde{P}, \tilde{2}, 3, 4, 5, 6)
+ \tilde{\mathcal{M}}_6(5, 6, 7, 8, \tilde{1}, -\tilde{P}) \frac{1}{P^2_{234}} \tilde{\mathcal{M}}_4(\tilde{P}, \tilde{2}, 3, 4)
\]

\[
= 2\alpha_6 \left( \frac{1}{P^2_{178}} + \frac{1}{P^2_{234}} + \frac{1}{P^2_{567}} + \frac{1}{P^2_{678}} \right) + \frac{2\alpha_6}{P^2_{234}} \left( \frac{1}{P^2_{234}} + \frac{1}{P^2_{456}} + \frac{1}{P^2_{567}} + \frac{1}{P^2_{678}} \right). \tag{3.5}
\]

The above equation can be further simplified by using the following relations

\[
\frac{1}{P^2_{178} P^2_{234}} + \frac{1}{P^2_{178} P^2_{234}} = \frac{1}{P^2_{178} P^2_{234}} \left( \frac{1}{1 - \frac{z_i}{z_j}} + \frac{1}{1 - \frac{z_j}{z_i}} \right) = \frac{1}{P^2_{178} P^2_{234}}, \tag{3.6}
\]

where we used the identity

\[
\sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \frac{1}{1 - \frac{z_i}{z_j}} = 1, \tag{3.7}
\]

and \(z_i\) and \(z_j\) are the locations of poles. We get

\[
\tilde{\mathcal{P}}_8^{[12]} = 2\alpha_6 \left( \frac{1}{P^2_{178} P^2_{234}} + \frac{1}{P^2_{178} P^2_{234}} + \frac{1}{P^2_{178} P^2_{234}} + \frac{1}{P^2_{234} P^2_{567}} + \frac{1}{P^2_{234} P^2_{678}} \right). \tag{3.8}
\]

The complete eight-point amplitude \(\tilde{\mathcal{M}}_8 = \tilde{\mathcal{P}}_8 + \tilde{\mathcal{B}}_8\) is given as

\[
\tilde{\mathcal{M}}_8 = 2\alpha_6 \left( \frac{1}{P^2_{123} P^2_{567}} + \frac{1}{P^2_{123} P^2_{678}} + \frac{1}{P^2_{123} P^2_{567}} + \frac{1}{P^2_{123} P^2_{678}} + \frac{1}{P^2_{123} P^2_{567}} \right) + \frac{1}{P^2_{345} P^2_{678}}. \tag{3.9}
\]

Now, we calculate the eight-point amplitude by summing up over all the Stokes polytopes
Figure 3: The two primitives of $n = 8$ Stokes polytopes corresponding to quadrangulations $Q = (14, 58)$ and $\tilde{Q} = (14, 16)$ respectively.

$S^Q_8$. The $n = 8$ case has two primitives and in total twelve quadrangulations\(^4\) (figure 3), which are the cyclic permutation of labels of the either of the two primitives. The canonical function corresponding to these primitives are given as

\[
m^Q_8 = \left(\frac{1}{X_{1,4}X_{5,8}} + \frac{1}{X_{3,8}X_{4,7}} + \frac{1}{X_{1,4}X_{4,7}} + \frac{1}{X_{3,8}X_{5,8}}\right)
= \left(\frac{1}{P_{123}^2P_{567}^2} + \frac{1}{P_{128}^2P_{456}^2} + \frac{1}{P_{123}^2P_{456}^2} + \frac{1}{P_{128}^2P_{567}^2}\right)\]  \hspace{1cm} (3.10)

\[
m^{\tilde{Q}}_8 = \left(\frac{1}{X_{1,4}X_{1,6}} + \frac{1}{X_{3,6}X_{1,6}} + \frac{1}{X_{3,6}X_{3,8}} + \frac{1}{X_{5,8}X_{3,8}}\right)
= \left(\frac{1}{P_{123}^2P_{678}^2} + \frac{1}{P_{123}^2P_{567}^2} + \frac{1}{P_{345}^2P_{678}^2} + \frac{1}{P_{345}^2P_{128}^2} + \frac{1}{P_{567}^2P_{128}^2}\right) .
\]

Taking the weighted sum over all the $S^Q_8$ using (2.6) and comparing each term to respective term in equation (3.9), we get the following relations

\[2\alpha^Q_8 + 2\alpha^{\tilde{Q}}_8 = 2\alpha_6, \quad \alpha^Q_8 + 4\alpha^{\tilde{Q}}_8 = 2\alpha_6 . \] \hspace{1cm} (3.11)

Also, the last term in (3.9) gives the relation

\[\alpha^Q_8 + 4\alpha^{\tilde{Q}}_8 = 1 . \] \hspace{1cm} (3.12)

Using (3.11) and (3.12) we determined the weight of six-point Stokes polytopes as $\alpha_6 = \frac{1}{2}$ and the weights for eight-point Stokes polytopes as

\[
\alpha^Q_8 = \frac{2\alpha_6}{3} = \frac{1}{3}, \quad \alpha^{\tilde{Q}}_8 = \frac{\alpha_6}{3} = \frac{1}{6} , \] \hspace{1cm} (3.13)

where the $Q$ and $\tilde{Q}$ denote the primitive quadrangulations. Note that the boundary term in (3.9) uniquely fixes $\alpha_6$.

\(^4\)The complete set of quadrangulations for $n = 8$ Stokes polytopes are given as

$\sigma \cdot Q(14, 58) \Rightarrow C_1 = \{(14, 58), (25, 16), (36, 27), (47, 38)\}$ and

$\sigma \cdot \tilde{Q}(14, 16) \Rightarrow C_2 = \{(14, 16), (25, 27), (36, 38), (47, 14), (58, 25), (16, 36), (27, 47), (38, 58)\}$
3.3 Ten-point amplitudes

To determine the relation between the weights of \( n = 10 \) Stokes polytopes and \( n = 6 \) weights, firstly we express the \( n = 8 \) amplitude in terms of \( \alpha_8^Q \) and \( \alpha_8^\tilde{Q} \) i.e. as weighted sum over \( \mathcal{S}_8^Q \) as \(^5\)

\[
\widetilde{\mathcal{M}}_8(1, 2, ..., 8) = (2\alpha_8^Q + 2\alpha_8^\tilde{Q}) \sum_{\sigma \in \mathcal{Z}_8} \left( \frac{1}{P_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)\sigma(6)\sigma(7)}} + (\alpha_8^Q + 4\alpha_8^\tilde{Q}) \sum_{\sigma \in \mathcal{Z}_8} \left( \frac{1}{P_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)\sigma(6)\sigma(7)}} \right) \right). \tag{3.14}
\]

Again, we use the \( \langle 1|2 \rangle \)-shift and apply generalized BCFW recursion relations. The boundary terms has four contributions form the diagrams \((23|456789(10)), (345|6789(10)), (3456789|8(10)) \) and \((3456789|10) \) and is given as

\[
\widetilde{B}_{10}^{[1|2]}(1, 2, ..., 10) = \widetilde{\mathcal{M}}_4(1, 2, 3, -P) \frac{1}{P_{123}^2} \widetilde{\mathcal{M}}_8(P, 4, ..., 10)
+ \widetilde{\mathcal{M}}_4(1, 2, -P_1, -P_2) \left( \frac{1}{P_{234}^2} \widetilde{\mathcal{M}}_4(P, 1, 3, 4, 5) \right) \left( \frac{1}{P_{1234}^2} \widetilde{\mathcal{M}}_6(P, 2, 6, ..., 10) \right)
+ \widetilde{\mathcal{M}}_4(1, 2, -P_1, -P_2) \left( \frac{1}{P_{234}^2} \widetilde{\mathcal{M}}_6(P, 1, 3, ..., 7) \right) \left( \frac{1}{P_{1234}^2} \widetilde{\mathcal{M}}_4(P, 2, 8, 9, 10) \right)
+ \widetilde{\mathcal{M}}_4(10, 1, 2, -P) \frac{1}{P_{12(10)}^2} \widetilde{\mathcal{M}}_8(P, 3, ..., 9). \tag{3.15}
\]

Using \((3.2) \) and \((3.14) \) the boundary term is simplified to

\[
\widetilde{B}_{10}^{[1|2]} = \frac{1}{P_{123}^2} \left[ (2\alpha_8^Q + 2\alpha_8^\tilde{Q}) \left( \frac{1}{P_{567}^2 P_{45678}^2} + \frac{1}{P_{678}^2 P_{45678}^2} + \frac{1}{P_{789}^2 P_{45678}^2} + \frac{1}{P_{789}^2 P_{45678}^2(10)} + \frac{1}{P_{89}^2 P_{789}^2(10)} \right) \right.
+ (\alpha_8^Q + 4\alpha_8^\tilde{Q}) \left( \frac{1}{P_{567}^2 P_{45678}^2} + \frac{1}{P_{567}^2 P_{45678}^2} + \frac{1}{P_{567}^2 P_{45678}^2} + \frac{1}{P_{567}^2 P_{45678}^2(10)} \right) + \frac{1}{P_{89}^2 P_{789}^2(10)} \right]
+ \frac{2\alpha_6}{P_{567}^2 P_{56789}^2} \left[ \left( \frac{1}{P_{456}^2 P_{56789}^2} + \frac{1}{P_{456}^2 P_{56789}^2} \right) + \frac{1}{P_{567}^2 P_{1234}^2} \left( \frac{2\alpha_8^Q + 2\alpha_8^\tilde{Q}}{P_{456}^2 P_{56789}^2} + \frac{1}{P_{567}^2 P_{56789}^2} \right) \right]
+ \frac{2\alpha_6}{P_{567}^2 P_{56789}^2} \left( \frac{1}{P_{456}^2 P_{56789}^2} + \frac{1}{P_{456}^2 P_{56789}^2} \right) \left( \alpha_8^Q + 4\alpha_8^\tilde{Q} \right) \left( \frac{1}{P_{456}^2 P_{56789}^2} + \frac{1}{P_{567}^2 P_{56789}^2} \right) \left( \frac{1}{P_{567}^2 P_{12(10)}^2} \right). \tag{3.16}
\]

\(^5\)Note that we are using the weighted sum of Stokes polytopes as our input in BCFW to determine higher-point weights in terms of lower-points weights.
The pole part is given as
\[ \hat{P}^{(12)}_{10}(1, 2, \ldots, 10) = \hat{M}_4(9, 10, \hat{1}, -\hat{P}) \frac{1}{P^2_{19(10)}} \hat{M}_8(\hat{P}, 2, 3, \ldots, 8) \]
\[ + \hat{M}_6(7, 8, 9, 10, \hat{1}, -\hat{P}) \frac{1}{P^2_{23456}} \hat{M}_8(\hat{P}, 2, 3, 4, 5, 6) \]
\[ + \hat{M}_8(5, 6, 7, 8, 9, 10, \hat{1}, -\hat{P}) \frac{1}{P^2_{234}} \hat{M}_4(\hat{P}, 2, 3, 4) . \]
which is expanded as
\[ \hat{P}^{(12)}_{10} = \frac{1}{P^2_{19(10)}} \left[ \left( 2\alpha_8^Q + 2\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{456}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{45678}} + \frac{1}{P^2_{345}P^2_{23456}} + \frac{1}{P^2_{678}P^2_{234}} \right) \right. \]
\[ + \left( \alpha_8^Q + 4\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{567}P^2_{234}} + \frac{1}{P^2_{345}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{45678}} + \frac{1}{P^2_{456}P^2_{23456}} + \frac{1}{P^2_{567}P^2_{234}} \right) \]
\[ + \frac{1}{P^2_{234}} \left( \frac{1}{P^2_{567}P^2_{234}} + \frac{1}{P^2_{345}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{234}} \right) \right] \left( 2\alpha_8^Q + 2\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{678}P^2_{56789}} + \frac{1}{P^2_{789}P^2_{6789(10)}} + \frac{1}{P^2_{89(10)}P^2_{234}} \right) \right] \times \]
\[ \left( \frac{1}{P^2_{456}} + \frac{1}{P^2_{9(10)\hat{1}}} \right) \right] + \left( \alpha_8^Q + 4\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{678}P^2_{234}} + \frac{1}{P^2_{567}P^2_{56789}} + \frac{1}{P^2_{789}P^2_{6789(10)}} + \frac{1}{P^2_{89(10)}P^2_{234}} \right) \right] \times \]
\[ \left( \frac{1}{P^2_{456}} + \frac{1}{P^2_{9(10)\hat{1}}} \right) \right] . \]
\[ (3.17) \]

Now, making repeated use of the identity (3.7), the above equation can be further simplified to
\[ \hat{P}^{(12)}_{10} = \frac{1}{P^2_{19(10)}} \left[ \left( 2\alpha_8^Q + 2\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{456}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{45678}} + \frac{1}{P^2_{345}P^2_{23456}} + \frac{1}{P^2_{678}P^2_{234}} \right) \right. \]
\[ + \left( \alpha_8^Q + 4\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{567}P^2_{234}} + \frac{1}{P^2_{345}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{45678}} + \frac{1}{P^2_{456}P^2_{23456}} + \frac{1}{P^2_{567}P^2_{234}} \right) \]
\[ + \frac{1}{P^2_{234}} \left( \frac{1}{P^2_{567}P^2_{234}} + \frac{1}{P^2_{345}P^2_{34567}} + \frac{1}{P^2_{567}P^2_{234}} \right) \right] \left( 2\alpha_8^Q + 2\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{678}P^2_{56789}} + \frac{1}{P^2_{789}P^2_{6789(10)}} + \frac{1}{P^2_{89(10)}P^2_{234}} \right) \right] \times \]
\[ \left( \frac{1}{P^2_{456}} + \frac{1}{P^2_{9(10)\hat{1}}} \right) \right] + \left( \alpha_8^Q + 4\alpha_8^\tilde{Q} \right) \left( \frac{1}{P^2_{678}P^2_{234}} + \frac{1}{P^2_{567}P^2_{56789}} + \frac{1}{P^2_{789}P^2_{6789(10)}} + \frac{1}{P^2_{89(10)}P^2_{234}} \right) \right] \times \]
\[ \left( \frac{1}{P^2_{456}} + \frac{1}{P^2_{9(10)\hat{1}}} \right) \right] , \]
\[ (3.18) \]

where we have used the relations in (3.11), (3.12), and the fact that \(2\alpha_6 = 4\alpha_6^2\) since \(\alpha_6 = \frac{1}{2}\), in making the simplifications.
Now, for the \( n = 10 \) case there are seven primitive Stokes polytopes and in total fifty-five quadrangulations. These correspond to Cube type, Snake type, Lucas type and Mixed type Stokes polytopes [3]. Taking the weighted sum over all the Stokes polytope \( S_{10}^Q \) and comparing to the amplitude \( M_{10} = \tilde{P}_{10}^{[12]} + \tilde{B}_{10}^{[12]} \) computed by BCFW recursions, we get the following set of equations that constrain the ten-point weights as

\[
\begin{align*}
4\alpha_{10}^Q + 2\alpha_{10}^Q + 2\alpha_{10}^Q + 2\alpha_{10}^Q + 2\alpha_{10}^Q + 2\alpha_{10}^Q &= 2\alpha_6, \\
2\alpha_{10}^a + \alpha_{10}^a + \alpha_{10}^a + 2\alpha_{10}^a + 2\alpha_{10}^a &= 2\alpha^Q + 2\tilde{\alpha}^Q, \\
\alpha_{10}^a + 2\alpha_{10}^a + \alpha_{10}^a + 2\alpha_{10}^a + 2\alpha_{10}^a &= 2\alpha_6, \\
2\alpha_{10}^b + \alpha_{10}^b + \alpha_{10}^b + 4\alpha_{10}^b + 2\alpha_{10}^b + 2\alpha_{10}^b &= 2\alpha^Q + 2\tilde{\alpha}^Q, \\
\alpha_{10}^a + 4\alpha_{10}^a + 3\alpha_{10}^a + 2\alpha_{10}^a + 2\alpha_{10}^a &= 2\alpha_6, \\
\alpha_{10}^a + 2\alpha_{10}^a + \alpha_{10}^a + 2\alpha_{10}^a + 2\alpha_{10}^a &= 4\alpha^Q, \\
\alpha_{10}^a + 4\alpha_{10}^a + 2\alpha_{10}^a + 3\alpha_{10}^a + 2\alpha_{10}^a + 2\alpha_{10}^a &= \alpha^Q + 4\tilde{\alpha}^Q
\end{align*}
\]

where \( \{Q_a, Q_b, Q_c, Q_d, Q_e, Q_f, Q_g\} \) correspond to set of quadrangulations that form the primitives of \( n = 10 \) Stokes polytopes.

Substituting (3.11) in the above equation and solving for the seven undetermined \( \alpha_{10} \)'s in terms of \( \alpha_6 \) we get

\[
\begin{align*}
\alpha_{10}^a &= \frac{1}{12} (12\alpha_6^2 - \alpha_6) = \frac{5}{24}, \\
\alpha_{10}^b &= \frac{1}{12} (12\alpha_6^2 - 5\alpha_6) = \frac{1}{24}, \\
\alpha_{10}^c &= \frac{1}{12} (19\alpha_6 - 36\alpha_6^2) = \frac{1}{24}, \\
\alpha_{10}^d &= \frac{1}{12} (12\alpha_6^2 - 5\alpha_6) = \frac{1}{24}, \\
\alpha_{10}^e &= \frac{1}{6} = \frac{2}{24}, \\
\alpha_{10}^f &= \frac{1}{4} (4\alpha_6^2 - \alpha_6) = \frac{3}{24}, \\
\alpha_{10}^g &= \frac{1}{4} (3\alpha_6 - 4\alpha_6^2) = \frac{3}{24},
\end{align*}
\]

where we substituted \( \alpha_6 = \frac{1}{2} \) in the end. The weights determined in (3.13) and (3.21) are in perfect agreement with the results in [3].

### 4 Generalization to higher-point amplitudes

#### 4.1 Overview of the proof

It is useful to introduce the following notations. Let \( N = \binom{n-4}{2} \) denote the dimensions of the Stokes polytopes. For example, six and eight particles correspond to \( N = 1 \) and \( N = 2 \) dimensional Stokes polytope respectively. Let \( \mathcal{F}_n \) denote the complete set of primitive quadrangulations of Stokes polytopes of dimension \( N \). Let \( \alpha_n^{Q^n} \) be the set of
weights corresponding to the primitive quadrangulations $Q_n \in F_n$ of the $N$-dimensional Stokes polytopes $S^Q_{n}$ and let $\tilde{\alpha}_6 = 2\alpha_6$. Then we would like to prove that if the $\langle i|j \rangle$-shift uniquely fixes the weights of $\tilde{M}_{n-2}$ in terms of $\tilde{\alpha}_6$, then the weights of the higher-point amplitudes $\tilde{M}_n$ is uniquely fixed in terms of $\tilde{\alpha}_6$ as well. Without any loss of generality, we chose the $\langle 1|4 \rangle$-shift to prove our statement. This is because for the $\langle 1|4 \rangle$-shift the boundary terms are absent in color-ordered $\phi^4$, and the amplitude $\tilde{M}_n$ follows a simple factorization scheme.

For the purpose of the proof, it is important to determine the dependence of amplitudes on the six-point weight $\tilde{\alpha}_6$. Consider the six-point amplitude $\tilde{M}_6$, which is $\propto \tilde{\alpha}_6$ as can be seen in (3.2). Since we take the six-point amplitude as the input to recursively construct the higher-point amplitudes, each product of lower-point amplitudes appearing in the factorization of the $n$-point amplitude must be proportional to some power of $\tilde{\alpha}_6$.

For example, the eight-point amplitude obey a factorization which is schematically given as $\tilde{M}_8 \sim \tilde{M}_4 \times \tilde{M}_6$ and is therefore $\propto \tilde{\alpha}_6$. The ten-point amplitude has the following factorization, schematically given as

$$\tilde{M}_{10} \sim \tilde{M}_4 \times \tilde{M}_8 + \tilde{M}_6 \times \tilde{M}_6,$$  \hspace{1cm} (4.1)

where the first term is $\propto \tilde{\alpha}_6$ and the second terms is $\propto (\tilde{\alpha}_6)^2$. Consequently, we have that the linear combinations of weights $\alpha^Q_8$ and $\alpha^Q_{10}$ that appear in the sum over all $S^Q_8$ and $S^Q_{10}$ respectively, are equal to some power of $\tilde{\alpha}_6$ as in (3.11) and (3.20). Let us assume that the above statement is true for $(n-2)$-point amplitude, i.e. each product of lower-point amplitudes appearing in the factorization of $\tilde{M}_{n-2}$ is proportional to some power of $\tilde{\alpha}_6$. And, the linear combinations of weights that appear in the summation over all $S^Q_{n-2}$ are equal to some power of $\tilde{\alpha}_6$.

We prove the above statement for a general $n$-point amplitude in the next section. The proof follows from the following steps

- We use the $\langle 1|4 \rangle$-shift to obtain the correct factorization of the $n$-point amplitude $\tilde{M}_n$.

- We show by induction that each term appearing in the factorization of $\tilde{M}_n$ can only be proportional to some power of $\tilde{\alpha}_6$.

- Using the fact that the weights of the primitives are parametrized only by $F_n$, we show that there are $J$ unique linear combinations of $J$ number of weights $\alpha^Q_n$ that appear in the weighted sum over all Stokes polytopes $S^Q_n$, where $J$ is the number of primitives for an $n$-point amplitude.

- Using the factorization property of $\tilde{M}_n$, proved by induction as in the second bullet point above, we show that the $J$ linear combinations of $\alpha^Q_n$ are equal to powers of $\tilde{\alpha}_6$.

\footnote{It does not matter which power of $\tilde{\alpha}_6$ because $\tilde{\alpha}_6$ raised to any power is equal to one, since $\tilde{\alpha}_6 = 1.$}
Finally, we use the above to prove our claim that the $\alpha_Q^n$ can be uniquely determined in terms of the six-point weight $\tilde{\alpha}_6$ using BCFW factorization.

4.2 $n$-point amplitudes

Consider the above statement for dimension $N = 2$ and $N = 3$. The statement holds true for these as can be seen from equations (3.11), (3.13), (3.20), and (3.21). Let us assume that the statement holds true for dimension $N = (k - 1)$, i.e. the weights of Stokes polytopes $S_{Q-2}^n$ are fixed uniquely in terms of $\tilde{\alpha}_6$ by the $\langle 1|4\rangle$-shift.

We prove that this statement is true for $N = k$. The amplitude $\tilde{\mathcal{M}}_n$ obeys the following factorization schematically given as

$$\tilde{\mathcal{M}}_{n}^{[4]} \sim \tilde{\mathcal{M}}_4 \times \tilde{\mathcal{M}}_{n-2} + \tilde{\mathcal{M}}_6 \times \tilde{\mathcal{M}}_{n-4} + \ldots + \tilde{\mathcal{M}}_q \times \tilde{\mathcal{M}}_{n-q+2} ,$$

(4.2)

where

$$q = \begin{cases} \frac{n}{2} & \text{if } \frac{n}{2} \text{ is even} \\ \left(\frac{n}{2} + 1\right) & \text{if } \frac{n}{2} \text{ is odd} . \end{cases}$$

(4.3)

Every term in (4.2) has an additional $\left(\frac{1}{P^2}\right)$ factor which we have omitted safely for the purpose of the proof.

Substituting (2.6) for each $\tilde{\mathcal{M}}$ in the factorization in (4.2) we get

$$\tilde{\mathcal{M}}_{n}^{[1|4]} \sim \sum_{l=4,6,\ldots}^{q} \left( \sum_{Q_l,\sigma} \alpha_{Q_l} m_{l}^{(\sigma \cdot Q_l)} \right) \times \left( \sum_{\tilde{Q}_l,\sigma'} \tilde{\alpha}_{n-l+2} m_{n-l+2}^{(\sigma' \cdot \tilde{Q}_l)} \right) ,$$

(4.4)

where $Q_l \in \mathcal{F}_l$ and $\tilde{Q}_l \in \mathcal{F}_{n-l+2}$. The $l = 4$ case in (4.4) corresponds to $\tilde{\mathcal{M}}_4$ and is equal to one. The $\alpha_{Q_l}$'s are polynomial functions in $\tilde{\alpha}_6$ by our assumption, and therefore the product $\alpha_{Q_l} \tilde{\alpha}_{n-l+2}$ will be polynomial functions in $\tilde{\alpha}_6$ as well.

The left-hand side of (4.2) is equal to the sum over $S_{Q}^n$ and is given as

$$\tilde{\mathcal{M}}_n = \sum_{Q_n} \sum_{\sigma} \alpha_{Q_n} m_{n}^{(\sigma \cdot Q_n)} ,$$

(4.5)

where $Q_n \in \mathcal{F}_n$.

Under the $\langle 1|4\rangle$-shift, the amplitude $\tilde{\mathcal{M}}_{n}^{[1|4]}$ has two types of terms. Terms of type (A) do not depend on the shifted variables $|\hat{1}\rangle$ and $|\hat{4}\rangle$. Terms of type (B) are functions of these shifted variables. For terms of type (B), repeated use of the identity (3.7) removes the dependence on $|\hat{1}\rangle$ and $|\hat{4}\rangle$. After explicitly performing the sums in (4.4) and (4.5), we can choose a term with a particular momentum dependence in the denominator and match its coefficients in both the equations\(^8\).

---

\(^7\)The case $N = 1$ is the trivial case of six-point amplitude itself.

\(^8\)This is done after putting the correct factors of $\frac{1}{P^2}$ in (4.4).
This gives us the relations

\[ \sum_{Q_n} C_{Q_n} \alpha^{Q_n}_{n} = \left( \sum_{Q_l} C_{Q_l} \alpha^{Q_l}_{l} \right) \left( \sum_{\bar{Q}_l} C_{\bar{Q}_l} \alpha^{\bar{Q}_l}_{n-l+2} \right), \tag{4.6} \]

for some \( l \in \{4, 6, ..., q\} \), where \( q \) is defined in (4.3) and, \( C_{Q_n} \) are positive constants. The summation is over a subset of primitive quadrangulations. This is because a given term with a particular momentum dependence in the denominator appears only in a subset of the primitives. The coefficients \( C_{Q_n} \) count the number of times a particular term with weight \( \alpha^{Q_n}_{n} \) appears in the summation over Stokes polytopes of a given dimension \( N \).

Further, for \( n \) particles there are \( F_I \) quadrangulations of the Stokes polytopes \( S^Q_n \), where \( n = (2I + 2) \). A subset of these quadrangulations forms the primitive quadrangulations\(^9\). Since the weights of the Stokes polytopes are parametrized only by the primitive quadrangulations, there are precisely \( J \) number of weights, where \( J \) is the number of primitives and is strictly less than the total number of quadrangulations \( F_I \). Thus, equation (4.6) encodes \( J \) linear equations. Each linear equation is a unique linear combination of weights \( \alpha^Q_{n} \), where \( Q \in \mathcal{F}_n \), and is equal to some power of \( \alpha_6 \). This follows from our assumption about \( \mathcal{M}_{n-2} \) that the products of the lower-point amplitudes appearing in its factorization are equal to some power of \( \alpha_6 \), implying that the linear combinations of weights that appear in the sum over \( S^Q_{n-2} \) are equal to some power of \( \alpha_6 \). Consequently, each of the sums in the right side of equation (4.6) is equal to some power of \( \alpha_6 \) implying that the sum on the left is equal to a power of \( \alpha_6 \).

The \( J \) linear equations encoded in (4.6) can be solved for \( J \) weights \( \alpha^Q_{n} \) in terms of \( \alpha_6 \). This determines the \( \alpha^Q_{n} \)'s exactly, and upon substituting \( \alpha_6 = 1 \) fixes their numerical values for which \( \widetilde{\mathcal{M}}_n = \mathcal{M}_n \), completing our proof.

Q.E.D.

5 Discussion

The geometric formulation of scattering amplitudes is opening up new ways of thinking about QFTs. The results have been striking in the supersymmetric theories such as the \( \mathcal{N} = 4 \) SYM [1], where the geometry of the polytope referred to as ‘Amplituhedron’, completely encapsulates the amplitudes at all orders. A great understanding of amplitude at tree-level in non-supersymmetric theories such as the scalar massless planar \( \phi^3 \), \( \phi^4 \) and in general, \( \phi^p \) theories has been propelled by the work in [2–5]. However, it was shown in [3] that there is no single polytope structure for a given dimension that completely encapsulates the \( \phi^4 \) amplitudes at tree-level. Rather there is a family of Stokes polytopes, whose weighted sum gives the complete \( \phi^4 \) amplitude.

In this paper, we addressed the issue of computing the weights. We showed that the factorization of the \( \phi^4 \) amplitudes at the physical poles put strong constraints on the

\(^9\)A method to count the number of primitives in a given dimension \( n \ (n = 1, 2, 3) \) is given in [5].
weights. We showed that the boundary terms of $n = 8$ amplitudes uniquely fixed the value of the lowest-point weight as $\alpha_6 = \frac{1}{2}$. Further, we explicitly calculated the weights for $n = 8$ and $n = 10$ cases in section 3.2 and 3.3 and showed that the weights can be solved in terms of the six-point weight $\alpha_6$. In section 4, we generalized our result to higher-point amplitudes. Using mathematical induction, we proved that the correct factorization of an $n$-point amplitude fixes the weights $\alpha^Q_n$ exactly in terms of $\alpha_6$.

A key feature of our analysis of the weights relied on the boundary terms of $\phi^4$ amplitudes, which correspond to the $O(z^0)$-behaviour of the amplitudes at large $z$. As we saw that the higher-point weights are fixed uniquely in terms of the lowest-point weight $\alpha_6$, it is crucial to determine the value of $\alpha_6$. Further, the value of the weight $\alpha_6 = \frac{1}{2}$ is highly non-trivial, as it is only for this particular value that the dependence of the shifted variables in the factorization of a general $n$-point amplitude as in (4.4), can be removed.

A shortcoming of our analysis was that we could not derive an explicit formula for the higher-point weights. One of the limiting factors was that there does not exist a general method to count the number of primitive Stokes polytopes for dimensions greater than three. However, despite this limitation, we could make a general statement about the weights in our proof in section 4, which is a powerful result. Also, the computation of $\alpha^Q_n$ relied on the correct factorization of the amplitudes at the poles $\vec{P} = 0$. This is a step back from the Amplituhedron program, where the geometry of the polytopes is sufficient to determine the amplitudes fully. Further, we believe that the extension of the ‘BCFW’-type recursion relations for $\phi^3$ amplitudes, as presented in [12–14], to $\phi^4$ amplitudes can help to fix the weights by purely geometrical inputs. A key feature of these recursion relations is the fact that the $\phi^3$ amplitudes factorize at the poles in correspondence with the geometric factorization of the associahedron $A_n$. However, the extension to $\phi^4$ is not immediately evident.

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