Curvature Invariants and the Geometric Horizon Conjecture in a Binary Black Hole Merger

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January 26, 2021

Abstract

We study curvature invariants in a binary black hole merger. It has been conjectured that one could define a quasi-local and foliation independent black hole horizon by finding the level-0 set of a suitable curvature invariant of the Riemann tensor. The conjecture is the geometric horizon conjecture and the associated horizon is the geometric horizon. We study this conjecture by tracing the level-0 set of the complex scalar polynomial invariant, $D$, through a quasi-circular binary black hole merger. We approximate these level-0 sets of $D$ with level-$\varepsilon$ sets of $|D|$ for small $\varepsilon$. We locate the local minima of $|D|$ and find that the positions of these local minima correspond closely to the level-$\varepsilon$ sets of $|D|$ and we also compare with the level-0 sets of $\text{Re}(D)$. The analysis provides evidence that the level-$\varepsilon$ sets track a unique geometric horizon. By studying the behaviour of the zero sets of $\text{Re}(D)$ and $\text{Im}(D)$ and also by studying the MOTSs and apparent horizons of the initial black holes, we observe that the level-$\varepsilon$ set that best approximates the geometric horizon is given by $\varepsilon = 10^{-3}$.

1 Introduction

1.1 Black Hole Horizons

Black holes are solutions of general relativity and are most naturally characterized by their event horizon. The event horizon of a black hole (BH) is defined as the boundary of the causal past of future null infinity. Intuitively, this means that on the inner side of the event horizon, light cannot escape to null infinity.
Notice that event horizons require knowledge of the global structure of spacetime [8, 9, 10]. However, for numerical relativity it is more convenient to use an initial value formulation of GR (a 3+1 approach), where initial data is given on a Cauchy hypersurface and is then evolved forward in time. This approach requires an alternative description of BH horizons which is not dependent of the BH’s future. [56, 55, 8, 4, 31, 28).

Let \( \Sigma \) be a compact 2D surface without border and of spherical topology, and consider light rays leaving and entering \( \Sigma \), with directions \( l \) and \( n \), respectively. Let \( q_{ab} \) be the induced metric on \( \Sigma \) and denote the respective expansions as \( \Theta(l) = q^{ab} \nabla_a l_b \) and \( \Theta(n) = q^{ab} \nabla_a n_b \). Then, \( \Theta(l) \) and \( \Theta(n) \) are quantities which are positive if the light rays locally diverge, and negative if the light rays locally converge, and are zero if the light rays are locally parallel. We say that \( \Sigma \) is a trapping surface if \( \Theta(l) < 0 \) and \( \Theta(n) < 0 \) [8, 43, 48, 45]. \( \Sigma \) is a marginally outer trapped surface (MOTS) if it has zero expansion for the outgoing light rays, \( \Theta(l) = 0 \) [43, 15, 16, 52, 20, 29]. (\( \Sigma \) is a future MOTS if \( \Theta(l) = 0 \) and \( \Theta(n) < 0 \) and a past MOTS if \( \Theta(l) = 0 \) and \( \Theta(n) > 0 \) [46]). The outermost MOTS is called the apparent horizon (AH) [48, 45, 46, 52, 26, 29]. If one smoothly evolves the original MOTS forward in time, one obtains a world tube which is foliated by these MOTS, and this world tube is known as a dynamical horizon (DH) [14, 8, 9, 10].

The above definitions serve as a quasi-local description of BHs [22, 21]. For example, tracking an AH only requires knowledge of the intrinsic metric \( q_{ab} \) restricted to the spacetime hypersurface and the extrinsic curvature of that hypersurface at a given time [27, 10, 26]. Gravitational fields at the AH are correlated with gravitational wave signals [27, 34, 33, 30, 29, 50], so AHs are useful to study gravitational waves. AHs are also used to numerically simulate binary black hole (BBH) mergers and the collapse of a star to form a BH [13]. For example, AHs play a role in checking initial parameters and reading off final parameters of Kerr black holes in gravitational wave simulations at LIGO [13, 2, 1]. DHs are also useful, as they could contribute to our understanding of BH formation [8, 9, 10, 13]. In addition, MOTSs turn out to be well-behaved numerically, and also can be used to trace physical quantities of a BH as they evolve over time and through a BBH merger [52, 20, 29, 83, 19]. One possible disadvantage of AHs is that the definition of AHs as the "outermost MOTS" is only useful in practice when a foliation is given [7].

It has been conjectured that one can uniquely define a smooth, locally determined and foliation invariant horizon based on the algebraic (Petrov) classification of the Weyl tensor [22, 21]. The necessary conditions for the Weyl tensor to be of a certain Petrov type can be stated in terms of scalar polynomials in the Riemann tensor and its contractions which are called scalar polynomial (curvature) invariants (SPIs). The first aim of this work is to study certain SPIs numerically during a BBH merger.

The Petrov classification is an eigenvalue classification of the Weyl tensor, valid in 4 dimensions (D). Based on this classification, there are six different Petrov types for the Weyl tensor in 4D: types I, II, D, III, N and O (which is flat spacetime). One can also use the boost weight decomposition to classify the Weyl tensor, which is equivalent in 4D to the Petrov classification. One can also algebraically classify the symmetric trace free operator, \( S_{ab} \), that is the trace free Ricci tensor, which is equivalent to the Segre classification [54].

The boost weight algebraic classification generalizes the Petrov classification...
to $N$-dimensional spacetimes \cite{22,21,23,24,15,40}. In $N$ D, we start with the frame of $N$–vectors, \{l, n, \{m_i\}_{i=2}^{N-1}\}, where $l$ and $n$ are null, $l \cdot n = 1$, and the \{m_i\} are real, spacelike, mutually orthonormal, and span the orthogonal complement to the plane spanned by $l$ and $n$. The possible orthochronous Lorentz transformations are generated by null rotations about $l$, null rotations about $n$, spins (which involve rotations about $m_i$), and boosts \cite{40}. With respect to the given frame, boosts are given by the transformation:

$$l \rightarrow \lambda l$$
$$n \rightarrow \lambda^{-1} n$$
$$m_i \rightarrow m_i$$

for all $i \in \{2, \ldots, N-1\}$ and for some $\lambda \in \mathbb{R}\{0\}$. (The remaining transformations are given in \cite{23,24,15,40}.) It is possible to decompose the Weyl tensor into components organized by boost weight \cite{23,24,15,16,18}. Just as an SPI is a scalar obtained from a polynomial in the Riemann tensor and its contractions \cite{22,21}, an SPI of order $k$ is a scalar given as a polynomial in various contractions of the Riemann tensor and its covariant derivatives up to order $k$ \cite{22,21}. It turns out that BH spacetimes are completely characterized by their SPIs \cite{22,21,17}. The necessary discriminant conditions on the 4D Weyl tensor for the spacetime to be of type $\Pi/\mathcal{D}$ can be stated as two real conditions and are given in \cite{17}.

Contracting the 4D (complex) null tetrad, $(l, n, m, \bar{m})$ where $m$ and $\bar{m}$ are complex conjugates, with the Weyl tensor, $C_{abcd}$, one may form the complex scalars, $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and, in terms of these scalars, as in the Newman-Penrose (NP) formalism (discussed later), one may define the scalar invariants:

$$I = \Psi_0 \Psi_4 - 4 \Psi_1 \Psi_3 + 3 \Psi_2^2$$

(1)

$$J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}$$

(2)

It can be shown that the two aforementioned real scalar conditions are equivalent to the real and imaginary parts of the following complex syzygy \cite{54}:

$$\mathcal{D} \equiv I^3 - 27 J^2 = 0$$

(3)

Thus for Petrov types $\Pi$ and $\mathcal{D}$, equation (3) holds everywhere. It also turns out that for Petrov types $\mathcal{III}$, $\mathcal{N}$, and for $\mathcal{O}$, we have $I = J = 0$, so (3) is satisfied trivially.

1.2 The Geometric Horizon Conjecture

Having discussed the Petrov and boost weight classifications, we now turn to the Geometric Horizon Conjecture (GHC) in which we define the geometric horizon (GH) as the set on which the SPIs, defined in (3), vanish \cite{22,21}. The level-0 sets of these SPIs might not form a horizon with nice properties, however, since these SPIs could vanish additionally on axes of symmetry or fixed points
of isometries \[22, 21\]. We know from \[3\] that if the spacetime is algebraically special, then the given complex SPI vanishes. More precisely, the GHC is given as follows \[22, 21\]:

**GH Conjecture:** If a BH spacetime is zeroth-order algebraically general, then on the geometric horizon the spacetime is algebraically special. We can identify this geometric horizon using scalar curvature invariants.

**Comments:** Note that when studying the GHC, one would need to ensure that the GH exists and is unique. If the GHC were true in an algebraically general spacetime, then one could say on this horizon, the Weyl tensor is more algebraically special than its background spacetime and this horizon is at least of type \(\text{II}\). This horizon is then foliation independent and quasi-local \[22, 21\].

If the spacetime is algebraically special, one then considers the second part of the GHC, which is analogous to the algebraic GHC above, but involving differential SPIs. Differential SPIs (of order \(k \geq 1\)) are scalars obtained from polynomials in the Riemann tensor and its covariant derivatives and their contractions. This second part of the GHC thus states that if a BH spacetime is algebraically special (so that on any GH the BH spacetime is automatically algebraically special), and if the first covariant derivative of the Weyl tensor is algebraically general, then on the GH the covariant derivative of the Weyl tensor is algebraically special, and this GH can also be identified as the level–0 set of certain differential SPIs \[22, 21\]. In this case, the GH is identified as the set of points on which the covariant derivative of the Weyl tensor, \(C_{abcd,e}\) is of type \(\text{II}\) \[22\]. It follows that one may obtain a clearer picture of the GH by taking the level–0 sets of these differential SPIs.

In addition to SPIs, Cartan invariants can play a role within the frame approach and they are easier to compute. For example, Cartan invariants are useful in event horizon detection; indeed, it was demonstrated in \[20\] that in 4D and 5D, one can construct invariants in terms of the Cartan invariants which detect the event horizon of any stationary asymptotically flat BH solutions. One could rewrite the statement of the algebraic GHC in the language of the boost-weight classification \[10\] to say that "if there is some frame with respect to which the Weyl tensor in a BH spacetime has a vanishing boost-weight +2 term, then on the GH, there is some frame with respect to which the Weyl tensor has a vanishing boost-weight +1 term." This desired frame is called the *algebraically preferred null frame* (APNF). For example, in an algebraically general 4D spacetime, the APNF is the frame in which the Weyl tensor is of algebraic type \(\text{I}\) so that the boost weight +2 terms of the Weyl tensor are 0 with respect to this frame, which is always possible in 4D. Then, the GH is identified as the set of points on which the terms of boost weight +1 are zero. (If the Weyl tensor is type \(\text{II}\), then one can analyze the covariant derivative of the Weyl tensor and ask for it to be algebraically special). The task in this frame approach to study the GHC, therefore, is to first find this APNF, \((l, n, m, \overline{m})\) and thus the orthogonal AHs/DHs \[53\]. To this end, the Cartan algorithm can be used to completely fix this frame \[59\], and with respect to this frame, one obtains the associated Cartan scalars. From these scalars, one can identify the level–0 set of \(C_{abcd,e}\) with the GH and, via NP calculus, obtain the NP spin expansion coefficients with respect to this APNF. It is of particular interest to
study the spin coefficients, \(\rho\) and \(\mu\), as their level-0 set could be associated with the GH. This definition is related to that of AHs and DHs, but with the additional geometric/algebraic motivation afforded by the Cartan scalars. There are many examples, outlined in the next section, which motivate identifying the GH as the level-0 set of \(\rho\) and \(\mu\), but a more careful study of \(\rho, \mu\) and their evolution through a BBH merger is beyond the scope of this paper.

In this paper, we shall study the (algebraic) SPIs in relation to the first (algebraic) part of the GHC. More specifically, we will study the complex level-zero set of the invariant, \(D = I^3 - 27J^2\), as given in (3), in 4D during a BBH merger. This could possibly help provide insight as to whether one can define a proper unique horizon based on the algebraic classification of the Weyl tensor. This conjecture might have to be modified so that instead of analyzing level-0 sets of the real SPIs, we analyze instead level-\(\varepsilon\) sets for small \(\varepsilon\). Such an \(\varepsilon\) could be determined by locating the local minima of the SPIs. However, further evidence from the analysis of \(D\) below perhaps suggests that this is not the case.

### 1.3 Examples and Motivation

There are many examples to support the plausibility of the GHC [39, 20, 3, 9, 10]. For example, in the Kerr spacetime, by invoking the notion of a non-expanding weakly isolated null horizon and an isolated horizon, it can be proven, using the induced metric and induced covariant derivatives on the submanifold and assuming the dominant energy condition, that the Weyl and Ricci tensors are both of type \(\Pi/D\) on the event horizon. This means that one can extract a subset of the set of points where the Weyl and Ricci tensors are both of algebraic type \(\Pi/D\), to define a smooth BH horizon, namely the event horizon [5, 36, 8, 9, 10, 22, 21]. It can also be shown that the covariant derivatives of the Riemann tensor are also of type \(\Pi\) on this horizon [22, 21]. The Kerr geometry can be approximated by the spacetime of a BH formed by a collapsing star. Thus, there should be a horizon surrounding the event horizon for a collapsing BH that can be identified using the algebraic conditions on the Riemann tensor mentioned earlier. By continuity, the inside of the event horizon should also approximate the Kerr geometry, and the Kerr geometry admits an inner horizon. This inner horizon is shown to be a null surface, but is unstable, allowing for the possibility that the GH is unique at later times [8, 9, 10, 5, 36, 22, 21].

Another example to support the GHC comes from a family of exact closed universe solutions to the Einstein-Maxwell equations with a cosmological constant representing an arbitrary number of BHs, discovered by Kastor and Traschen (KT) [35]. Consider the merger of 2 BHs. At early times, there are two 3D disjoint GHs forming around each BH [22, 21]. However, at intermediate times, it turns out that the invariant, \(D = I^3 - 27J^2 = 0\) as in (3) only at the co-ordinate positions of each of the BHs, along certain segments of the symmetry axis, and along a 2D cylindrical surface, which expands to engulf the 2 BHs as they coalesce [22, 21]. During the intermediate process, there are 3D surfaces located at a finite distance from the axis of symmetry for which the traceless Ricci tensor (and hence the Ricci tensor, \(R_{ab}\)) is of algebraic type \(\Pi/D\). There is also evidence of a minimal 3D dynamically evolving surface where a scalar invariant, \(W_1\), assumes a constant minimum value. This suggests that there is a GH during the dynamical regime between the spacetimes [22, 21], but further
investigation is needed. At late times, the spacetime then settles down to a type D Reissner-Nordstrom-de-Sitter BH spacetime with mass \( M = m_1 + m_2 \), which turns out to have a GH \[39, 20\]. So a GH forms at the beginning and end of the coalescence. For further information on the two-BH solution, see \[35\]. The KT solution for multiple BHs was studied and GHs around each BH were found in \[38\]. The results were compared with the previously mentioned 2-BH solution. Additionally, three black-hole solutions were studied and GHs were found around these BHs also \[22, 21\]. For information on more than two BHs, see \[42\].

There are additional examples of spacetimes that support the GHC either by explicitly exhibiting GHs or by finding other established BH horizons on which the Weyl and Ricci tensor are algebraically special \[22, 21, 19, 54\]. One can also verify the GHC by identifying GHs with the level–0 sets of the NP spin coefficients, \( \rho \) and \( \mu \), as mentioned before. There are four examples to support this. The first example is in stationary spacetimes with stationary horizons (e.g. Kerr-Newman-NUT-AdS) \[39\]. In this spacetime, the Weyl and Ricci tensors are type D everywhere, and the APNF can be choosed to make this condition manifest. Here, the GH can be identified as the set on which the covariant derivatives of the Weyl and Ricci tensors are of type II, and this coincides with the level–0 set of \( \rho \) \[39\]. The second example is in spherically symmetric spacetimes such as vacuum solutions or known exact solutions (e.g. Vaidya or LTB dust models) \[22\]. In this case, the Weyl tensor is of type D and here the Ricci tensor provides no useful information. Adopting the APNF to the type-D condition of the Weyl tensor, we then find that the GH coincides with the set of points on which the covariant derivative of the Weyl tensor is of type II, which is also the level–0 set of \( \rho \). For the third example, consider quasi-spherical Szekeres spacetimes \[19\], where the Weyl tensor is of type D and the Ricci tensor is of type I. Studying the components of the covariant derivative of the Weyl tensor, shows that \( \rho = 0 \) precisely on the GH. A similar situation holds in the fourth example \[35\], which is the Kastor-Traschen solution for \( N > 1 \) BHs. Here, the Weyl tensor is of type I and the Ricci tensor is of type D. With respect to an adapted APNF, it follows that \( \rho = 0 \) and \( \mu = 0 \) precisely on the set \( \{ C_{abcd,e} \text{ is of type D} \} \), which identifies the GH. The authors are also currently studying vacuum solutions in the case of axisymmetry (so \( R_{ab} = 0 \)) and where the Weyl tensor is of algebraic type I. Based on the previous examples, it is natural to study the covariant derivative of the Weyl tensor in this setting and it is also of interest to study the level–0 sets of the NP spin coefficients \( \rho \) and \( \mu \).

2 Simulating a Binary Black Hole Merger

2.1 Previous Work

We wish to study the behaviour of the complex SPI, \( \mathcal{D} \), as defined in \[3\], through a BBH merger. Since the Kerr geometry is type D everywhere, it follows that \( \mathcal{D} = 0 \) everywhere for a Kerr BH. It is known that in a BBH merger the merged BHs at late times settle down to a solution well described by the Kerr metric \[22, 21\]. Thus, for a merger of 2 initially Kerr BHs, a plot of the real part and imaginary part of \( \mathcal{D} \) should be roughly zero everywhere at early and late times. However, in the intermediate “dynamical” region (during the actual
merger and coalescence at intermediate times), these same zero plots should yield important information. This is what we wish to study.

We highlight some known features of a binary black hole merger, as described in [48, 49, 27]. This also serves to set up our notation. In [48, 49], it was found that there is a connected sequence of MOTSs, which interpolate between the initial and final states of the merger (two separate BHs to one BH, respectively) [48, 49]. The dynamics are as follows: Initially, there are two BHs with disjoint MOTS (which are AHs at this point [27]), $S_1$, and $S_2$, one around each BH. Then, as the two BHs evolve, a common MOTS forms around the two separate BHs and bifurcates into an inner MOTS, $S_i$, which surrounds the MOTS and an outer MOTS, $S_c$. $S_c$ increases in area and encloses $S_1$, $S_2$ and $S_i$, and is the AH at the time of the merger [27, 48, 49]. The fate of $S_i$ and the bifurcation at the time of the merger is well understood [27, 48, 49, 32, 41, 47].
Figure 2: Comparing selected local minima of $|\mathcal{D}|$ along the $x$–coordinate direction with selected level sets of $|\mathcal{D}|$ at times $t = 12$ (upper plot), $t = 16$ (middle plot) and $t = 20$ (lower plot).
Figure 3: Comparing level $\pm 0.01$ contours of $\mathcal{D}_r = \text{Re}(\mathcal{D})$ and $\mathcal{D}_i = \text{Im}(\mathcal{D})$ with level $-0.001$ contours of $|\mathcal{D}|$. The upper, middle and lower left plots are plots of $\mathcal{D}_r = \text{Re}(\mathcal{D})$ at times $t = 12, 16, 20$, respectively and the upper, middle and lower right plots are plots of $\mathcal{D}_i = \text{Im}(\mathcal{D})$ at times $t = 12, 16, 20$, respectively.
Figure 4: Comparing the white level–0.001 sets of $|\mathcal{D}|$ with the MOTS as described in [48, 49] at times $t = 12$ (upper left), $t = 16$ (upper right) and $t = 20$ (lower left and right). The lower left panel shows the inner MOTS in purple whereas the lower right panel shows the outer MOTS in purple.
2.2 Present Work

Instead of studying a head-on collision, in this paper we shall study a quasi-circular orbit of two merging, equal mass and non-spinning BHs. This simulation has not been presented elsewhere and the results of this simulation are new. In these simulations, the Einstein toolkit infrastructure was used \[37\] and the simulations are run using \(4^{th}\) order finite differencing on an adaptive mesh grid, with adaptive refinement level of \(6\) \[51\] \[12\]. Brill-Lindquist initial data with BH positions and momenta set up to satisfy the QC-0 initial condition were used \[25\]. Instead of analyzing a sequence of MOTS throughout the merger, we seek to define and study a GH as it evolves through the merger, in accordance with the GHC. Since (3) sets necessary conditions for the Weyl tensor to be of algebraic type II, we seek to analyze the constant contours of the difference \(D = I_3 - 27 J^2\). In the simulations, the real and imaginary parts of \(I\) and \(J\) are calculated using the Cartan invariants, \(\{\Psi_i\}_{i=0}^5\), as given in equations (1) and (2), and the calculations are carried out using the orthonormal fiducial tetrad, as given by [11].

Note that in the rest of this paper, we will use the notation from [48, 49, 27] to describe the various MOTSs that appear in our simulations. To recapitulate, \(S_1\) and \(S_2\) are the 1st and 2nd initial MOTS and \(S_i\) and \(S_c\) describe the inner and common MOTS as they appear in our simulation, respectively. We will also be interested in the spherical approximations of \(S_1\) and \(S_2\). Let \(S_1\) be a spherical surface centred at the centroid of \(S_1\) and whose radius is the average radius of \(S_1\) at each time during the merger. Correspondingly, let \(S_2\) be a spherical surface centred at the centroid of \(S_2\) whose radius is the average radius of \(S_2\). We will also make use of the spherical surface, \(S_c\), which is centred at the centroid of \(S_c\) whose radius is the average radius of \(S_c\).

Figures 1-4 provide plots of various level sets of \(|D|\) as functions of \((x, y)\) \(\in \mathbb{R}^2\) at a fixed spatial coordinate value of \(z = 0.3125\) and at selected instances of the time parameter, \(t\), where \(t = 0\) indicates the start of the numerical computation. These level sets are also compared with \(S_1\) and \(S_2\). The full compliment of pictures describing this BBH merger are displayed in [44]. We present a subset of those figures to illustrate the essential features. In each of Figures 1-4, the data corresponding to \(x < 0\) was obtained by rotating the data corresponding to \(x > 0\) by 180 degrees about the \(x = y = 0\) axis. In Figures 1-4, we plot the centroids and outlines of \(S_1\) and \(S_2\) along with \(S_1\) and \(S_2\), and also \(S_i\) and \(S_c\), when they have formed. It is found that \(S_1\) and \(S_2\) provide valid spherical approximations to \(S_1\) and \(S_2\).

2.3 Discussion

Figure 1 provides the contour plots of the magnitude of \(D = I_3 - 27 J^2\), denoted \(|D|\), on a log scale (see [3]) for \(t = 8, 12, 16, 18, 20, 24\) in the upper left, upper right, middle left, middle right, lower left and lower right panels, respectively, for fixed \(z = 0.03125\). Since \(|D|\) is positive definite, the level-0 sets of \(|D|\) are impossible to find precisely due to discrete resolution and numerical error. Instead, we highlight the evolution of the level-\(\varepsilon\) sets, where \(\varepsilon = 3 \times 10^{-4}, 5 \times 10^{-4}, 1 \times 10^{-3}\). The overlaid green, red and white contours of each frame are the level-3 \(= 10^{-4}\), level-5 \(= 10^{-4}\) and level-1 \(= 10^{-3}\) sets of \(|D|\), respectively. The blue dots in Figure 1 give the centres of \(S_1\) and \(S_2\) (or, in other words, the
curves, each of which contains the centroids of corresponding initial BHs through the merging process. [44] has plots of $|\mathcal{D}|$ with the relevant level–$\varepsilon$ sets superimposed at all times $t = 0, \ldots, 30, 34, 38, 42$.

At early times (i.e., at $t = 8$ and $t = 12$ in the upper left and right panels, respectively), each of the level–$\varepsilon$ sets are partitioned into pairs of simple closed curves, each of which contains the centroids of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. At $t = 16$ (middle left panel), the red level–$5 \times 10^{-4}$ set and the white level–$1 \times 10^{-3}$ set of $|\mathcal{D}|$ each form a third simple closed curve between $\mathcal{S}_1$ and $\mathcal{S}_2$, and this third simple closed curve is centred at $(x, y) = (0, 0)$. (The green level–$3 \times 10^{-4}$ set forms its third simple closed curve at time $t = 14$, but this is not shown in the paper for brevity (see [44] for details)). At times $t = 18$ (middle right panel), $t = 20$ (bottom left panel) and $t = 24$ (bottom right panel), for each respective $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$, the multiple simple closed curves partitioning the level–$\varepsilon$ set of $|\mathcal{D}|$ have joined so that each level–$\varepsilon$ curve is now a single simple closed curve surrounding the 2 centroids of $\mathcal{S}_1$ and $\mathcal{S}_2$. It follows that the level–$\varepsilon$ curves for each $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$ at each $t$ form an invariantly defined, foliation invariant horizon that contains each separate BH at early times, and contains the merged BH at late times.

The evolution of the level–$\varepsilon$ curves through the quasi-circular BBH merger in Figure 1 is reminiscent of the sequence of MOTS that take place during the head-on collision simulation in [48, 49]. In particular, in [48, 49] after the two separate initial BHs start to merge together, a third MOTS forms and bifurcates into $\mathcal{S}_1$ and $\mathcal{S}_2$. This bifurcation, also summarized in [27], can be compared to our quasi-circular BBH merger simulations, particularly at $t = 16$ in the middle left panel of Figure 1 when each level–$\varepsilon$ set is partitioned into three simple closed curves for $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$. However our numerical computations are not precise enough to study the details of the bifurcation, as found in [48, 49, 27]. At time $t = 24$, in the lower right panel of Figure 1, it also seems that the centroids of $\mathcal{S}_1$ and $\mathcal{S}_2$ do not merge fully. Thus, it is possible that at late times, the level–$\varepsilon$ sets of $|\mathcal{D}|$ for $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$ may track $\mathcal{S}_1$ and $\mathcal{S}_2$, which have been found in [27] to overlap but not intersect at late time. However, our simulations did not run to late enough times to make this clear.

Consequently, Figure 1 provides strong evidence that for each $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$, the level–$\varepsilon$ sets of $|\mathcal{D}|$ track a unique GH, which can be identified by the level–0 set of $\mathcal{D}$. In [44], we also found that a subset of the level–0 sets of $\mathcal{D}_r$ and $\mathcal{D}_l$ both track the level–$\varepsilon$ sets of $|\mathcal{D}|$ for all $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$ at all times $t = 0$, ... $30, 34, 38, 42$, lending further support to the fact that these level–$\varepsilon$ sets of $|\mathcal{D}|$ approximate the level–0 sets of $\mathcal{D}$. From Figure 1, we have evidence of the fact that the level–$\varepsilon$ sets track the GHs through all stages of the BBH merger, including the time when the level–$\varepsilon$ sets of $|\mathcal{D}|$ start as disjoint simple closed curves with each surrounding the centroids of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, when the level–$\varepsilon$ sets of $|\mathcal{D}|$ are partitioned into three simple closed curves at intermediate times, and when the level–$\varepsilon$ sets of $|\mathcal{D}|$ form a single simple closed curve surrounding both centroids of $\mathcal{S}_1$ and $\mathcal{S}_2$ at late times.

While Figure 1 shows that the level–$\varepsilon$ sets of $|\mathcal{D}|$ are well behaved as GHs, the GHC implies the existence of a horizon on which the Riemann tensor is algebraically special which, in turn, implies the existence of a horizon characterized by the level–0 set of $\mathcal{D}$. One has by definition that the complex invariant, $\mathcal{D} = 0$
if and only if its magnitude, $|D| = 0$. However, numerically finding the level–0 sets of $|D|$ is extremely difficult because $|D|$ is a positive definite quantity, so that any numerical errors would provide a positive contribution. Furthermore, in the numerical simulation it is possible that the actual zeros of $|D|$ do not occur at any points which are sampled for the discrete mesh being used. Thus, the level–$\varepsilon$ sets of $|D|$ could indeed approximate the level–0 sets of the complex invariant, $D$. It remains to estimate the appropriate preferred value for $\varepsilon$.

We observe that for $\varepsilon = 3 \times 10^{-4}$, $5 \times 10^{-4}$, $1 \times 10^{-3}$, the level–$\varepsilon$ contours are very close to each other, showing that the level–$\varepsilon$ sets vary continuously with $\varepsilon$. We also observe that if $\varepsilon_1 \leq \varepsilon_2$, then the 2D area enclosed by the level–$\varepsilon_1$ curve encloses the 2D area enclosed by the level–$\varepsilon_2$ curve. Thus, each panel of Figure 1 indicates that $|D|$ decreases on average with average distance from the centroids of $S_1$ and $S_2$, which means that the plots of $|D|$ show no global minima. However, Figures 2 and 3 indicate that the plots of $|D|$ do have local minima which approximately coincide with the level–0 sets of $D_v$ and with the level–$\varepsilon$ sets for $\varepsilon = 3 \times 10^{-4}$, $1 \times 10^{-3}$.

In order to investigate further the level–0 sets of $|D|$ (or equivalently the level–$\varepsilon$ sets of $D$), we find out where $|D|$ takes a local minimum value. If the value of $|D|$ itself is small, then these locations of the local minima could possibly indicate positions of the actual zeros of $|D|$, which would be caused by numerical errors. For example, the zero of $|D|$ could be “missed by the discretization”—i.e. the contour plots of $|D|$ could display local minimum values on the mesh when $|D|$ is theoretically zero. It could also be the case that the GHC should be modified so that the GH is defined as the set of points where $|D|$ reaches the local minimum instead of being identically zero. If this were the case, then locating the local minima of $|D|$ would locate the GH precisely instead of approximating it. However, further evidence from the analysis of $D_v$ below perhaps suggests that this is not the case.

To this end, we examine 1D plots of $|D|$ as functions of $y$ for fixed $x$, henceforth referred to as “slice plots.” Along each slice plot, we find the values of $y = y_{\text{min}}$, where $|D|$ assumes a local minimum value, and record the corresponding point $(x_{\text{min}}, y_{\text{min}})$. In [44], we show explicitly the slice plots of the contour plots of $|D|$ at times $t = 12, 16, 20$ and demonstrate explicitly the process of finding the local minima of $|D|$. In Figure 2, we restrict our attention to finding the positions of the local minima of $|D|$ whose corresponding $|D|$ values lie within the range $[1 \times 10^{-4}, 1.2 \times 10^{-3}]$ which contains $\{3 \times 10^{-4}, 5 \times 10^{-4}, 1 \times 10^{-3}\}$, the set of values of $\varepsilon$ being considered for level–$\varepsilon$ sets of $|D|$. The positions of the local minima of $|D|$ corresponding to this desired range are plotted with green dots in Figure 2. The remaining features of Figure 2 are the same as in Figure 1.

In summary, Figure 2 gives the contour plots of $|D|$ at times $t = 12, 16, 20$ in the upper, middle and lower panels, respectively, with green level–$3 \times 10^{-4}$ sets of $|D|$, white level–$1 \times 10^{-3}$ sets of $|D|$ and the blue points marking the centroids of $S_1$ and $S_2$, as in Figure 1. The points where $|D|$ assumes a local minimum value and lies in the range $[1 \times 10^{-4}, 1.2 \times 10^{-3}]$ are plotted using green dots. [44] also presents these same plots of $|D|$, along with the green local minima, at magnified resolution.

Technically speaking, the green points in Figure 2 correspond to local minima of $|D|$ only along slice plots of $|D|$ vs $y$ for a fixed $x$. In [44], we have also
examined 1D plots of $|D|$ as functions of $x$ for fixed $y$ and found that the set of points, $(x_{\text{min}}, y_{\text{min}}) \in \mathbb{R}^2$ where $|D|$ assumes a local minimum value along each fixed $y = y_{\text{min}}$, “slice plot” corresponds closely with the set of points, $(x_{\text{min}}, y_{\text{min}}) \in \mathbb{R}^2$ where $|D|$ assumes a local minimum value along each fixed $x = x_{\text{min}}$ slice plot. Thus, the points where $|D|$ assumes a local minimum value along its corresponding slice plot accurately approximate the points corresponding to an overall local minima of $|D|$.

We observe that at times $t = 12$ and $t = 16$, these local minima appear to track the green level $3 \times 10^{-4}$ sets, while at time $t = 20$, these local minima appear to track more closely the white level $1 \times 10^{-3}$ sets. In any case, the positions of the local minima of $|D|$ align closely with the level-$\varepsilon$ sets of $|D|$ for $\varepsilon = 3 \times 10^{-4}$, $1 \times 10^{-3}$. This is also described in [44] in more detail. Therefore, Figure 2 demonstrates that positions of the local minima of $|D|$ accurately track the GH and provides supporting evidence that the level-$\varepsilon$ sets track the GH for $\varepsilon = 3 \times 10^{-4}$, $1 \times 10^{-3}$. In [44], we also track the level-0 sets of $D_{\varepsilon}$ and compare with the locations of the local minima of $|D|$ at original resolution, and also find that all local minima of $|D|$ track closely the level-0 sets of $D_{\varepsilon}$.

The problem of estimating the level-0 sets of the complex $D$ cannot be definitively resolved by analyzing $|D|$ because, as previously noted, $|D|$ is a positive definite quantity and the discrete resolution imposed by the numerical simulation does not necessarily allow one to accurately find level-0 sets of $|D|$. Thus, to gain further insight into the GH through the BBH merger, it is therefore helpful to analyze quantities which change sign through a zero. By continuity, the level-0 sets of these quantities can be deduced from adjacent positive and negative level-\(\varepsilon\) sets for small $\varepsilon_0$. This avoids some of the problems of numerical resolution that occur when using the local minima of $|D|$ to estimate the zeros of $|D|$ (and hence $D$). The quantities we choose to analyze are $D_{\varepsilon} = \text{Re}(D)$ and $D_i = \text{Im}(D)$.

In Figure 3 we plot, with magnified resolution, the contour plots of $D_{\varepsilon}$ (left panels) and $D_i$ (right panels) along with their level--0.01 sets in yellow and their level--0.01 sets in lime green in contours of $D_{\varepsilon}$ (left panels) and $D_i$ (right panels) at times $t = 12$, $16$, $20$ in the upper, middle and lower panels, respectively. In [44], we also analyzed the quantities $\text{Re}(D^2) = D^2_{\varepsilon} - D^2_i$ and $\text{Im}(D^2) = 2 D_{\varepsilon} D_i$ with their associated level--0.01 sets, in addition to the level--0.01 sets of $D_{\varepsilon}$ and $D_i$. The grey regions in each of the frames in Figure 3 correspond to regions where $-0.01 < D_{\varepsilon} < 0.01$ (resp. $-0.01 < D_i < 0.01$), as indicated in the colorbar, but the black regions correspond to regions where $D_{\varepsilon} \geq 1$ (resp. $D_i \geq 1$) and the white regions correspond to regions where $D_{\varepsilon} \leq -1$ (resp. $D_i \leq -1$). As before, the blue dots mark the centroids of $S_{\varepsilon}$ and $S_i$.

Furthermore, in Figure 3 (and in the relevant figures in [44]), we compared the level--0.01 sets of $D_{\varepsilon}$ and $D_i$ with the white level--$1 \times 10^{-3}$ sets of $|D|$ of Figures 1--2. We know that a positive level set and a nearby negative level set indicates a sign change of the quantity being plotted. Hence, there must be a surface among the level $\pm 0.01$ sets of $D_{\varepsilon}$ (resp. $D_i$) across which $D_{\varepsilon}$ (resp. $D_i$) change sign. This surface gives the level-0 set of $D_{\varepsilon}$ (resp. $D_i$). By estimating the zeros of $D_{\varepsilon}$ or $D_i$ in this manner, we reduce the possibility of numerical noise that comes with plotting the level-0 sets of $D_{\varepsilon}$ or $D_i$ directly. Upon inspection of each frame in Figure 3, we see that the level--0.01 sets of $D_{\varepsilon}$ (resp. $D_i$) occur in close proximity with but are contained in the interior of the level--$1 \times 10^{-3}$ sets.
of $|D|$. Thus, each of $D_r$ and $D_i$ have level–0 sets which occur in close proximity to the level $\pm 0.01$ sets of $D_r$ and $D_i$, respectively, and closely approximate the level $1 \times 10^{-3}$ sets of $|D|$. It follows that an examination of the contours of $D_r$ and $D_i$ in Figure 3 provides strong evidence that the level $1 \times 10^{-3}$ set of $|D|$ well approximates the elusive level–0 sets of the complex invariant, $D$.

We next explicitly compare the level $1 \times 10^{-3}$ sets of $|D|$ with the corresponding MOTSs $S_{1,2,i,c}$ and the spherical approximations, $S_{1,2,c}$, as described below, in Figure 4. We display the 2D contour plots of $|D|$ with magnified resolution at times $t = 12$ and $t = 16$ in the upper left and upper right panels, respectively, and we display the 2D contour plots of $|D|$ at $t = 20$ and in the lower left and lower right panels. As in Figures 1–3, the white curves denote the white level $1 \times 10^{-3}$ sets of $|D|$. The dark blue ellipses in all frames track the outline of the $(x,y)$ coordinates of a $z = 0.03125$ section of $S_1 \cup S_2$ and the blue points in all frames of Figure 4 mark the centroids of $S_1$ and $S_2$, as before. The light sky blue curves mark the $(x,y)$ coordinates of points on $S_1 \cup S_2$ whose corresponding $z$ coordinate values lie in the range $[0.02,0.04]$. For completeness, the centroids of $S_1$ and $S_2$ are plotted with light sky blue dots. By definition, the centroids of $S_1$ and $S_2$ align exactly with the centroids of $S_1$ and $S_2$, respectively.

In the present quasi-circular simulation, the bifurcation of the third MOTS into $S_i$ and $S_c$ occurs between times $t = 18.5$ and $t = 18.75$. Once this happens, $S_1$ and $S_2$ are no longer AHs, as the MOTSs, $S_c$, now surrounds $S_1$, $S_2$ and $S_c$. Thus, in order to compare our level–$\varepsilon$ sets of $|D|$ with AHs (the outermost MOTSs), we have included plots of $S_1$ and $S_c$ here. The purple dots on the bottom left (resp. bottom right) panel of Figure 4 label the $(x,y)$ coordinates of the points on $S_1$ (resp. $S_c$) whose corresponding $z$ value lies in the range, $[-0.1,+0.1]$. In the bottom right-hand corner of Figure 4, we also plot the $z = 0.03125$ section of the outermost MOTS (now the AH) which we label as $S_c$. Note that in the bottom right-hand corner, the outer MOTS at late times is so big that the entire two dots and the scale for scale of the white level–0.001 sets of $|D|$ and $S_{1,2}$ are squashed at the origin.

From each panel of Figure 4, we notice that the MOTSs, $S_1$ and $S_2$, are closely mimicked by $S_1$ and $S_2$, respectively. Hence, $S_1$ and $S_2$ are nearly spherically symmetric surfaces. We also find that the white level $1 \times 10^{-3}$ set of $|D|$ coincides closely with $S_1$ and $S_2$, especially at early times. Since the MOTSs $S_c$ and $S_i$ have not yet formed at such early times, $S_1$ and $S_2$ are AHs at early times. Therefore, Figure 4 shows us that the white level $1 \times 10^{-3}$ sets of $|D|$ well approximate the AH at early times. At later times, however, it appears that the AH, $S_c$, diverges from the white level $1 \times 10^{-3}$ sets, so that the white level $1 \times 10^{-3}$ sets of $|D|$ no longer approximate the AH in this regime. (Note that this AH, $S_c$, is well approximated by its spherical approximation, $S_{c}$. ) Instead, these white level $1 \times 10^{-3}$ sets of $|D|$ are approximated reasonably closely by $S_1$ and especially $S_1$ and $S_2$. This lends support to the choice of the white level $1 \times 10^{-3}$ sets of $|D|$ as a representative approximation to the level–0 set of $|D|$ (and hence of $D$).

In [44], we compare $S_1$ and $S_2$ to the level $5 \times 10^{-4}$ sets of $|D|$ at all times $t = 0, \ldots, 26$ and find that these red level $5 \times 10^{-4}$ sets track the spherically averaged AH extremely closely, especially at early times. Since the white level $1 \times 10^{-3}$ sets and red level $5 \times 10^{-4}$ sets track each other extremely closely, this lends support to the fact that the white level $1 \times 10^{-3}$ sets of $|D|$ are well
approximated by the AHs of the initial BHs at all times. We presently choose the white level–1 $\times 10^{-3}$ set of $|D|$ as our representative level– set of $|D|$ to approximate the level–0 set of $|D|$ and thus the level–0 set of $D$. In [44], we also have studied the contour plots of $D_r$ and $D_i$ and with their level–0 sets at all times $t = 0, \ldots, 30, 34, 38, 42$. Here we have illustrated the plot features present in Figures 2–4 in [44] for times $t = 12, 16, 20$. The simulation whose figures we present in [44] has not been presented elsewhere and the results of this simulation are new.

Therefore, Figures 1–4 provide strong evidence that one can define a unique smooth GH, theoretically given by the level–0 set of the complex invariant $D = I^3 - 27J^2$, which we have found is best approximated in the numerics by the level–1 $\times 10^{-3}$ sets of $|D|$.

3 Conclusions

We have studied the algebraic properties of the Weyl tensor by analyzing the time evolution of various level– sets of $|D|$ through a quasi-circular merger of two non-spinning, equal mass BHs where, in particular, $\varepsilon = 3 \times 10^{-4}, 5 \times 10^{-4}, 1 \times 10^{-3}$. These level– contours are superimposed on the contour plots of $|D|$ in Figure 1. In these plots, the locations of the two initial AHs were tracked by using the centroids of the initial AHs. We found that at early times, each such level set is partitioned into two disjoint simple closed curves, each of which contains one of the two centroids of the AHs of the 2 separate initial BHs. Then each level set, at some intermediate time, forms a third simple closed curve which is centred at the origin and positioned between the centroids of the AHs of the two initial BHs. These three simple closed curves then join and form one simple closed curve for each level set, which contains the centroids of both initial BHs.

The plots for $|D|$ in Figure 1 provide strong evidence that the level sets of $|D|$ identify the GH. However, it is impossible to identify the level–0 sets of $|D|$ precisely, since $|D|$ is a sum of positive definite terms, so numerical errors and discrete resolution cause $|D|$ to be strictly positive. Thus, to further study the zeros of $|D|$, which would indicate the zeros of the complex quantity $D$, we studied the positions of the local minima of $|D|$ along “slice plots” of $|D|$ vs $y$ for a fixed $x$ in Figure 2. Figure 2 demonstrates that the level– sets of $|D|$ correspond closely to the local minima of $|D|$, where $\varepsilon = 3 \times 10^{-4}, 1 \times 10^{-3}$. Since the local minima of $|D|$ approximate the zeros of $|D|$, Figure 2 provides supporting evidence that the level– sets of $|D|$ for $\varepsilon = 3 \times 10^{-4}, 1 \times 10^{-3}$ track the GH of the BBH merger.

Since $|D|$ is positive definite, its zeros cannot be traced by positive and negative level sets. Therefore, we have also analyzed quantities which change sign through a zero. In Figure 3, we examined contour plots of $D_r = \text{Re}(D)$ and $D_i = \text{Im}(D)$ with their associated level–0.01 sets and compared these plots with the white level–1 $\times 10^{-3}$ sets of $|D|$. We found surfaces surrounding the union of the level–0.01 contours of $D_r$ (resp. $D_i$) across which $D_r$ (resp. $D_i$) change sign, and are hence a subset of the level–0 sets of $D_r$ (resp. $D_i$). We also found these particular zeros of $D_r$ (and those of $D_i$) to be well approximated by the level–1 $\times 10^{-3}$ contours of $|D|$ in our plots and suggest that this approximation is valid in this setting.
In Figure 4, we compare the level $1 \times 10^{-3}$ contours of $|D|$ with the AHs. These AHs are given by $S_1$ and $S_2$ at early times and by $S_c$ after $S_c$ has formed. Figure 4 shows that $S_1$ and $S_2$ are reasonably approximated as spherically symmetric surfaces at all times. We also find that the level $1 \times 10^{-3}$ sets of $|D|$ are very well approximated by the AHs, $S_1$ and $S_2$, at early times, but at later times the AH diverges from this level set of $|D|$. However, even at late times, $S_1$ and $S_2$ continue to track a subset of the level $1 \times 10^{-3}$ set of $|D|$, as does $S_c$.

Therefore, in the binary black hole merger, as displayed in Figures 1–43 in [44] and summarized in Figures 1–4 above, the algebraic structure of the Weyl tensor is clearly identified by the level $\varepsilon$ sets of $|D|$, and it is plausible that the level set with $\varepsilon = 1 \times 10^{-3}$ accurately identifies the geometric horizon.

4 Acknowledgements

This work was supported financially by NSERC (AAC and ES). JMP would like to thank AAC for supervising his masters thesis and ES for numerical assistance and useful discussions, and the Perimeter Institute for Theoretical Physics for hospitality during this work.

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