EFFECT OF PREDATORS’ BEHAVIOR ON PREY-PREDATOR INTERACTION

MARIAM AL-MOQBALI, IBRAHIM M. ELMOJTABA*, NASSER AL-SALTI

Department of Mathematics, College of Science, Sultan Qaboos University,
PO Box 36, Al Khoudh, Muscat, Oman

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we study the effect of the behavior of predators on prey-predator interaction. We assume that there is one prey population that suffers from the fear of predators which could force it to move in to the refuge; and three are two predator populations, one of them is aggressive in its attack and the other one using sit-and-wait procedure, which is less aggressive. All the model’s equilibrium points have been found and their stability was established. The possibility of transcritical and Hopf bifurcation was also investigated and numerical simulations were given. The effect of prey refuge and fear also are detected. The cost of them is to allow the model to reach to double transcritical point. The effect of the competition between prey populations is to convert the model from the stable limit cycle to a spiral stable equilibrium point of afraid prey with predator. When it becomes large it converts the model to the stable trivial solution.

Keywords: predators’ behavior; prey refuge; Holling type-II functional response; bifurcation.
2010 AMS Subject Classification: 92B05, 34C23.

1. PRELIMINARIES

Theory predicts that organisms should modulate behavior to maximize their expected fitness [1]. Since death is the most definite negative effect on fitness, it is very important for individuals to avoid predation and therefore a strong selection for proper anti-predator response is to be

*Corresponding author
E-mail address: elmojtaba@squ.edu.om
Received May 30, 2021

8095
expected. Behavioral adjustments to predator presence are very widespread responses used by organisms in many taxa [10]. The optimal behaviorally mediated anti-predator response is of course highly dependent on what kind of predator an organism is encountering. For example, [11] showed that damselfly larvae that swim to escape fish predators have a very low chance of survival, while the same escape response in the presence of invertebrate predators resulted in much higher chances of survival. Their study illustrates that a good anti-predator response to one type of predator can be maladaptive in the presence of another predator type. It is therefore vital for the prey to be able to identify and correctly assess the risk linked to a specific predator. Since many organisms encounter several different types of predators, plasticity in the behavioral response to predators is expected to be beneficial for prey populations [4, 13]. In order to study the effects of preys’ behavior changes due to the fear of predators and the effect of predator populations, we developed a mathematical model with one prey population and two predator populations. It is assumed that the prey population fears the predator and each predator follows a different type of attack; one of them is very aggressive and the other one follows the sit-and-wait strategy which considered to be less aggressive in its nature [2, 3, 5].

2. Mathematical Mode Building and Analysis

To develop our model we assume that there is one prey population and two predator populations. It is assumed that the prey population suffers from fear of predator which affect its reproduction and cause it to move inside a refuge for sometimes which affects the availability of food and causes the rise of inter-species competition [7], therefore the reproduction term of the prey population takes the form

\[
\frac{bN}{1 + e_1 P_1 + e_2 P_2} - dN - sN^2
\]

Where the term

\[
f(e; P_1, P_2) = \frac{1}{1 + e_1 P_1 + e_2 P_2}
\]

represents the fear of the prey from the predator. It is also assumed that one predator is aggressive in nature and therefore its attack follows Holling type-II functional response, and the other one is less aggressive in its behavior and therefore it follows a modified Holling type-II
functional response in its attack [8]. Taking these assumption into consideration, our model is represented by the following set of differential equations

\[
\begin{align*}
\frac{dN}{dt} &= \frac{bN}{1 + e_1 P_1 + e_2 P_2} - dN - sN^2 - \frac{b_1 N(1 - m_1)P_1}{1 + \alpha_1 N} - \frac{b_2 N(1 - m_2)P_2}{1 + \alpha_2 N + \gamma P_2} \\
\frac{dP_1}{dt} &= -c_1 P_1 + \frac{b_1 \lambda_3 N(1 - m_1)P_1}{1 + \alpha_1 N} \\
\frac{dP_2}{dt} &= -c_2 P_2 + \frac{b_2 \lambda_4 N(1 - m_2)P_2}{1 + \alpha_2 N + \gamma P_2}
\end{align*}
\]

where \(N\) represents the prey which afraid and stay in refuges, \(P_1\) and \(P_2\) represent bold and aggressive predator respectively. The next table is demonstrated the meaning of each parameter:

### 2.1. Equilibrium Points.

The system (1) has the following equilibrium points:

1. \(E_0(0, 0, 0)\), the trivial equilibrium point.
2. \(E_1(\frac{b - d}{s}, 0, 0)\), representing the existence of prey which stay in refuge.
3. \(E_2(\frac{c_1}{b_1 \lambda_3(1 - m_1) - \alpha_1 c_1}, P_1^*, 0)\), representing the existence of afraid pray and first predator. Where \(P_1^*\) is the root of equations \(H_1 X^2 + H_1 X + H_0 = 0\), where \(H_2 = e_1(\alpha_1 c_1 - b_1 \lambda_3(1 - m_1))^2, H_1 = b_1 \lambda_3^2(1 - m_1)(b_1(1 - m_1) + e_1 d) - 2 \alpha_1 b_1(1 - m_1) + c_1 e_1 \lambda_3(\alpha_1 d - s) + \alpha_1^2 c_1 e_1, H_0 = \lambda_3(b_1(1 - m_1)(d - b) \lambda_3 + c_1(\alpha_1(b - d) + s))\)
4. \(E_3(N^*, 0, P_2^*)\), representing the existence of afraid pray and second predator which is faster than first predator. Where \(N^*\) represents the solution of equation \(N^* = AX^3 + BX^2 + CX + D = 0\), \(A = \lambda_4 \gamma e_2 s(\alpha_2 c_2 - b_2 \lambda_4(1 - m_2)), B = -(b_2(1 - m_2) + \gamma d)(1 - m_2)e_2 b_2 \lambda_4^2 + c_2 \lambda_4([2 \alpha_2(1 - m_2) + \gamma(\alpha_2 d + s)]e_2 - s \gamma^2) - \alpha_2^2 c_2 e_2, C = c_2 \left[\lambda_4((b - d) \gamma^2 - \gamma(b_2(1 - m_2) - e_2 d) + 2b_2 e_2(1 - m_2)) - 2 \alpha_2 c_2(e_2 - \frac{1}{2} \gamma)\right], D = c_2^2(\gamma - e_2)\). Consequently we can choose all factors to be positive or negative. We will consider the equation \(N^* = AX_2^3 + BX_2^2 + CX_2 + D = 0\). Consequently by using Descartes’s Sign Rule it can be chosen \(D > 0\) with \(B < 0\) and \(C > 0\) or \(D < 0\) with \(B > 0\) and \(C > 0\) to get positive solution and, it can be taken \(D > 0\), with any sign of \(b\) and \(c\) except \(B > 0\) and \(C < 0\) to get a unique positive solution.
The meaning

\( b \) The birth rate of prey.

\( d \) The death rate of prey.

\( s \) The competition rate between prey.

\( b_1, \alpha_1 \) (units; 1/time), (units; 1/prey) respectively are describe the effect of capture rate and handling time on the feeding rate of the bold predator.

\( b_2, \alpha_2 \) (units; 1/time), (units; 1/prey) respectively are describe the effect of capture rate and handling time on the feeding rate of the aggressive predator.

\( \gamma \) measures the effect of anti-bold predator behaviour of prey.

\( c_1, c_2 \) The death rate of bold predators and aggressive predators respectively.

\( e_1, e_2 \) fear parameter of prey from bold predator and aggressive predator respectively.

\( m_1, m_2 \) refuge parameter of prey according of fear from bold predator and aggressive predator respectively.

\( b_1 \lambda_3 \) The increment rate of the bold predator.

\( b_2 \lambda_4 \) The increment rate of the aggressive predator.

\( \lambda_3 \) is the root of equation

\( M_2X^2 + M_1X + M_0 = 0 \), such that,

\( M_2 = c_1c_2\lambda_4^2(\alpha_1c_1 - b_1\lambda_3(1 - m_1)) \),

\( M_1 = -\gamma \left[ -(c_2(\alpha_1(\gamma - e_2)) + \alpha_2e_2 - b_2\lambda_4c_2(1 - m_2))\alpha_1c_1^2 + \lambda_3(\alpha_5c_1(\alpha_1 - \alpha_2)c_2^2 + (2b_1c_1^2(1 - m_1)(\gamma - e_2)c_1 + \frac{1}{2}\alpha_2e_2)) \right] + \gamma [(\alpha_1(1 - m_1) - \gamma d) - x\gamma e_1\lambda_4c_2 - \lambda_2^2b_1b_2x_1(1 - m_1)(1 - m_2))c_1^2 + \lambda_2^2(2e_1c_2(\alpha_1 - \frac{1}{2}\alpha_2) + ((1 - m_1)(\gamma - e_2)b_1) - \lambda_2^2\gamma e_1(b_2(1 - m_2) + \gamma d)\lambda_3(1 - b_1c_1c_2 - \gamma e_1\lambda_3^2b_1^2c_2^2(1 - m_2)^2) \]
From equations (1), the Jacobian matrix of the system is given by:

\[
J(E_i) = \begin{bmatrix}
S_1^* & S_2^* & S_3^* \\
\frac{b_1 b_2 P_1(1-m_1)}{(1+\alpha_1 N)^*} & -c_1 + \frac{b_1 b_2 b_4 N(1-m_1)}{(1+\alpha_1 N)^2} & 0 \\
\frac{b_2 b_3 P_2(1-m_2)(1+yP_2)}{(1+\alpha_2 N+yP_2)^*} & 0 & -c_2 + \frac{b_2 b_4 N(1-m_2)(1+\alpha_2 N)}{(1+\alpha_2 N+yP_2)^2}
\end{bmatrix}.
\]

Where, \( S_1^* = \frac{b}{1+e_1 P_1 + e_2 P_2} - d - sN - \frac{b_1 P_1(1-m_1)}{1+\alpha_1 N} - \frac{b_2 P_2(1-m_2)}{1+\alpha_2 N+yP_2} + N(-s + \frac{b_1 \alpha_1 P_1(1-m_1)}{(1+\alpha_1 N)^2} + \]

Notice that it is need to take the negative solution of \( f(x) \) to get \( P_1^* \) is positive. So \( b_1 \lambda_3(1-m_1) - \alpha_1 c_1 \) is positive, this implies \( M_2 \) is negative. To get unique negative solution this need to take \( M_0 \) is positive. In addition it can be chosen \( M_0 \) is negative with \( M_1 \) is negative and \( M_1^2 - 4M_2M_0 > 0 \) to get two negative solution.

### 2.2. Local stability of the equilibrium points.

**Theorem 1.** The stability of the system (1) is given by:

(i) \( E_0(0,0,0) \) is locally asymptotically stable if \( b-d < 0 \).

(ii) \( E_1(\frac{b-d}{s},0,0) \) is locally asymptotically stable if \( -c_1 + \frac{b_1 \lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} < 0 \) and \( -c_2 + \frac{b_2 \lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)} < 0 \).

(iii) \( E_2(\frac{c_1}{b_1 \lambda_3(1-m_1) - \alpha_1 c_1}, P_1^*, 0) \) is locally asymptotically stable if \( -c_2 + \frac{b_2 \lambda_4 N^*(1-m_2)}{(1+\alpha_2 N^*)} < 0 \) and \( \alpha_1^2 sN^* + \alpha_1(2sN^* - b_1 P_1^*(1-m_1)) + s > 0 \).

(iv) \( E_3(N^*, 0, P_2^*) \) is locally asymptotically stable if \( -c_1 + \frac{b_1 \lambda_3 N^*(1-m_1)}{(1+\alpha_1 N^*)} < 0 \) and \( [\gamma^2 sP_2^* + (s\gamma(2\alpha_2 N^* + 2) + b_2(\gamma \lambda_4 - \alpha_2)(1-m_2)) P_2^* + s(1 + \alpha_1 N^*)^2] > 0 \).

(v) \( E_4(N^*, P_1^*, P_2^*) \), is locally asymptotically stable by using Qualitative Matrix Stability method.

**Proof**

From equations (1), the Jacobian matrix of the system is given by:
By evaluating the Jacobian matrix at each equilibrium points, we get:

(i) The Jacobian matrix at $E_0(0,0,0)$ is given by:

$$J(E_0) = \begin{bmatrix} b - d & 0 & 0 \\ 0 & -c_1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}.$$ 

clearly the eigenvalues of the matrix are $b - d$, $-c_1$ and $-c_2$ and it is obvious that $E_0(0,0,0)$ is asymptotically stable if $b - d < 0$.

(ii) The Jacobian matrix at $E_1\left(\frac{b - d}{s}, 0, 0\right)$ is given by:

$$J(E_1) = \begin{bmatrix} d - b & \frac{b - d}{s} & \frac{b - d}{s} \\ 0 & -c_1 + \frac{b_1 \lambda_3(b-d)(1-m_1)}{s + \alpha_1(b-d)} & 0 \\ 0 & 0 & -c_2 + \frac{b_2 \lambda_4(b-d)(1-m_2)}{s + \alpha_2(b-d)} \end{bmatrix},$$

clearly this is upper triangular matrix, so the eigenvalues of the matrix are $d - b$, $-c_1 + \frac{b_1 \lambda_3(b-d)(1-m_1)}{s + \alpha_1(b-d)}$ and $-c_2 + \frac{b_2 \lambda_4(b-d)(1-m_2)}{s + \alpha_2(b-d)}$ which implies that $E_1\left(\frac{b - d}{s}, 0, 0\right)$ is asymptotically stable if $b - d > 0$, $-c_1 + \frac{b_1 \lambda_3(b-d)(1-m_1)}{s + \alpha_1(b-d)} < 0$ and $-c_2 + \frac{b_2 \lambda_4(b-d)(1-m_2)}{s + \alpha_2(b-d)} < 0$.

(iii) The Jacobian matrix at $E_2(N^*, P^*_1, 0)$ is given by:

$$J(E_2) = \begin{bmatrix} N^*(-s + \frac{b_1 \alpha_1 P^*_1(1-m_1)}{(1+\alpha_1 N^*)^2}) & -N^*\frac{b_1(1-m_1)}{(1+\alpha_1 N^*)^3} & -N^*\frac{b_2(1-m_2)}{(1+\alpha_2 N^*)^3} \\ 0 & 0 & 0 \\ 0 & 0 & -c_2 + \frac{b_2 \lambda_4 N^*(1-m_2)}{(1+\alpha_2 N^*)} \end{bmatrix},$$

It is obvious that $-c_2 + \frac{b_2 \lambda_4 N^*(1-m_2)}{(1+\alpha_2 N^*)}$ is one of the eigenvalue of the Jacobian matrix.

The other eigenvalues is gotten from reduced matrix, which is:

$$\begin{bmatrix} N^*(-s + \frac{b_1 \alpha_1 P^*_1(1-m_1)}{(1+\alpha_1 N^*)^2}) & -N^*\frac{b_1(1-m_1)}{(1+\alpha_1 N^*)^3} & -N^*\frac{b_2(1-m_2)}{(1+\alpha_2 N^*)^3} \\ 0 & 0 & 0 \end{bmatrix},$$

The characteristic polynomial is:

$$P(\lambda) = \lambda^2 + \frac{b_1 \alpha_1 P^*_1(1-m_1)(1+\alpha_1 N^*)^2}{(1+\alpha_1 N^*)^3} N^* \lambda + \frac{b_1(1-m_1)(1+\alpha_1 N^*)^2 + b_2(1+\alpha_2 N^*)}{(1+\alpha_1 N^*)^3(1+\alpha_2 N^*)^3}.$$
Clearly, this point is locally asymptotically stable if
\[
(\alpha_1^2 sN^* + \alpha_1(2sN^* - b_1P_1^*(1 - m_1) + s) > 0.
\]

(iv) The Jacobian matrix at \( E_3(N^*, 0, P_2^*) \) is given by:
\[
J(E_3) = \begin{bmatrix}
N^* (-s + \frac{b_1\alpha_1P_1^*(1 - m_1)}{1 + \alpha_2N^* + \gamma P_2^*}) & -N^* (\frac{b_1(1 - m_1)}{1 + \alpha_2N^*}) & -N^* (\frac{b_1(1 - m_1)(1 + \alpha_2N^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} + \frac{b_2}{(1 + e_2P_2^*)^2}) \\
0 & -c_1 + \frac{b_1\lambda_3N^*(1 - m_1)}{(1 + \alpha_2N^* + \gamma P_2^*)} & 0 \\
\frac{b_2\lambda_2P_1^*(1 - m_2)}{(1 + \alpha_2N^* + \gamma P_2^*)} & 0 & -\frac{\gamma b_2\lambda_4P_1N^*(1 - m_2)}{(1 + \alpha_2N^* + \gamma P_2^*)^2}
\end{bmatrix}.
\]

It is clearly that \(-c_1 + \frac{b_1\lambda_3N^*(1 - m_1)}{(1 + \alpha_2N^*)}\) is one of the eigenvalue of the Jacobian matrix. The other eigenvalues is gotten from reduced matrix, which is:
\[
\begin{bmatrix}
N^* (-s + \frac{b_1\alpha_1P_1^*(1 - m_2)}{1 + \alpha_2N^* + \gamma P_2^*}) & -N^* (\frac{b_2(1 - m_2)(1 + \alpha_2N^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} + \frac{b_e_2}{(1 + e_2P_2^*)^2}) \\
\frac{b_2\lambda_4P_1^*(1 - m_2)(1 + \gamma P_2^*)}{(1 + \alpha_2N^* + \gamma P_2^*)^2} & -N^* (\frac{b_2(1 - m_2)(1 + \alpha_2N^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} + \frac{b_2}{(1 + e_2P_2^*)^2}) \\
\end{bmatrix}.
\]

The characteristic polynomial is:
\[
(2) \quad P(\lambda) = \lambda^2 + \gamma^2sP_2^* + (s\gamma(2\alpha_2N^* + 2) + b_2(\gamma\lambda_4 - \alpha_2)(1 - m_2))P_2^* + s(1 + \alpha_1N^*)^2N^*\lambda + \frac{b_2\lambda_4(1 - m_2)N^*P_2^*}{(1 + \alpha_2N^* + \gamma P_2^*)^2} (s\gamma\alpha_2(1 + e_2P_2^*)N^* + s(1 + e_2P_2^*)N^* [s\gamma(1 + e_2P_2^*)^2 + b_e_2\alpha_2]) + \frac{b_2\lambda_4(1 - m_2)N^*P_2^*}{(1 + e_2P_2^*)^2(1 + \alpha_2N^* + \gamma P_2^*)^3} (2be_2\gamma P_2^* + be_2^2(1 - m_2)P_2^* + 2b_2e_2(1 - m_2)P_2^* + be_2 + b_2(1 - m_2)).
\]

Clearly, this point is locally asymptotically stable if
\[
\left[\gamma^2sP_2^* + (s\gamma(2\alpha_2N^* + 2) + b_2(\gamma\lambda_4 - \alpha_2)(1 - m_2))P_2^* + s(1 + \alpha_1N^*)^2\right] > 0.
\]

(v) The Jacobian matrix at \( E_4(N^*, P_1^*, P_2^*) \) is given by:
\[
J(E_4) = \begin{bmatrix}
N^* (-s + \frac{b_1\alpha_1P_1^*(1 - m_1)}{1 + \alpha_2N^* + \gamma P_2^*}) & -N^* (\frac{b_1(1 - m_1)}{1 + \alpha_2N^*}) & -N^* (\frac{b_1(1 - m_1)(1 + \alpha_2N^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} + \frac{b_e_2}{(1 + e_2P_2^*)^2}) \\
\frac{b_1\lambda_3P_1^*(1 - m_1)}{(1 + \alpha_2N^* + \gamma P_2^*)} & 0 & 0 \\
\frac{b_1\lambda_3P_1^*(1 - m_1)(1 - \gamma P_2^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} & 0 & -\frac{\gamma b_2\lambda_4P_1N^*(1 - m_2)}{(1 + \alpha_2N^* + \gamma P_2^*)^2}
\end{bmatrix}.
\]

where \( B^* = -N^* \left( \frac{b_2(1 - m_2)(1 + \alpha_2N^*)}{(1 + \alpha_2N^* + \gamma P_2^*)} + \frac{b_e_2}{(1 + e_2P_2^*)^2} \right) \) To prove the stability of the coexistence equilibrium point we will use the Qualitative matrix stability method.

\[
Q_3 = \text{sign}(J_4) = \begin{bmatrix}
- & - & - \\
+ & 0 & 0 \\
+ & 0 & -
\end{bmatrix}.
\]

Then:
From \( Q_3 \) we have \( q_{11} = -, q_{22} = 0, q_{33} = - \). Thus condition (1) and (2) of theorem are satisfied.

(ii) Since, \( q_{12} = -q_{21} =, q_{13} = -q_{31}, q_{23} = q_{32} = 0 \) which means every pair of interacting nodes have opposite sign and this implies that condition (3) of theorem is satisfied.

(iii) From it is clear that condition (4) of theorem is satisfied.

(iv) \( \det(J_3) \neq 0 \) since the solution of \( Jx = 0 \) is only the trivial solution.

Thus from figure there are two predation links between \( N \) and \( P_1 \) and between \( N \) and \( P_2 \). Since these links are connected by node \( P_1 \) and \( P_2 \), then the entire signed digraph forms a predation community.

**Color test**

We color each node with a negative feedback loop with gray and the other nodes by white as in figure.

Applying the color test to our model we find that:

* The node \( N \) is gray and the other nodes \( P_1 \) and \( P_2 \) are white.

* However, condition (ii) is not satisfied because there is no predation link between white nodes.

* The gray node \( N \) is connected by a predation link to both white nodes \( P_1 \) and \( P_2 \).

In conclusion, the signed matrix \( Q_3 \) fails the color test. Hence the Jacobian matrix \( J_4 \) corresponding to the equilibrium point \( E_4(N^*, P_1^*, P_2^*) \), where \( Q_3 = sign(J_4) \) is qualitative stable. Then the equilibrium is locally asymptotically stable.
2.3. **Global stability of the equilibrium points.** The global stability of stability points will be analysed by transforming the system of equations (1) into a linear system and then choosing a suitable Lyapunov function to analyse each equilibrium point or by using Dulac’s criteria, following the same strategy as [?].

**Theorem 2.** The global stability of the system of equations (1) is given by: $E_1 = (N^*, 0, 0) = \left(\frac{b-d}{s}, 0, 0\right)$ is global asymptotically stable.

**Proof**

By letting $N = N^* + n, P_1 = P_1^* + p_1$ and $P_2 = P_2^* + p_2$, where $n, p_1$ and $p_2$ are small perturbations about $N^*, P_1^*$ and $P_2^*$ respectively, the system of equations (1) is turned into a linear system which is of the form $n_i = J(E_i)n_i$, where $J(E_i)$ is the Jacobian matrix of equations (1). Thus, the linear system of equations (1) is,

$$\frac{dn}{dt} = \left[(-s + \frac{b_1\alpha_1 P_1^*(1-m_1)}{(1+\alpha_1 N^*)^2} + \frac{b_2\alpha_2 P_2^*(1-m_2)}{(1+\alpha_2 N^*+\gamma P_2^*)^2})\right]N^*n - \left[\frac{b_1(1-m_1)}{(1+\alpha_1 N^*)^2} + \frac{b_2(1-m_2)}{(1+\alpha_2 N^*+\gamma P_2^*)^2}\right]N^*p_1$$

$$- \left[\frac{b_2(1-m_2)(1+\alpha_2 N^*)}{(1+\alpha_2 N^*+\gamma P_2^*)^2} + \frac{b_3}{(1+\alpha_1 P_1^*+\gamma P_2^*)^2}\right]N^*p_2$$

(3) $$\frac{dp_1}{dt} = \frac{b_1\lambda_3 P_1^*(1-m_1)}{(1+\alpha_1 N^*)^2}n$$

$$\frac{dp_2}{dt} = \left[\frac{b_2\lambda_3 P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2 N^*+\gamma P_2^*)^2}\right]n - \left[\frac{b_2\lambda P_2^* N^*(1-m_2)}{(1+\alpha_2 N^*+\gamma P_2^*)^2}\right]p_2$$

Global stability of $E_1(N_1^*, 0, 0) = (\frac{b-d}{s}, 0, 0)$. We define a Lyapunov function as $V(n, p_1, p_2) = \frac{n^2}{2N^*} + \frac{p_1^2}{2} + \frac{p_2^2}{2}$. It is obvious that $V(n, p_1, p_2)$ is a positive definite function. Differentiating $V$ with respect to time $t$ we get, $\dot{V}(n, p_1, p_2) = \frac{nn}{N^*} + p_1p_1 + p_2p_2$. By substituting for $n, p_1$ and $p_2$ in equations of system (3) gives,

$\dot{V}(n, p_1, p_2) = -(\frac{r}{k}n_2 + a_2p)n_2$, which is negative semi-definite. Therefore, $E_1(N^*, 0, 0)$ is globally asymptotically stable.

2.3.1. **Hopf bifurcation.**

**Theorem 3.** The system (1) undergoes a Hopf bifurcation at the positive equilibrium:

(i) $E_2(N^*, P_1^*, 0)$ when $s = s_0 = \frac{b_1(1-m_1)\alpha_1 P_1^*}{(1+\alpha_1 N^*)^2}$
(ii) \( E_3(N^*, 0, P_2^*) \) when \( s = s_0 = \frac{b_2(1 - m_2)(\alpha_2 - \gamma\lambda_4)P_2^*}{(1 + \alpha_2 N^* + \gamma P_2^*)^2} \)

**Proof**

(i) The eigenvalues of the linearized system around the equilibrium point \( E_2 \) are:

\[ \mu_{1,2} = \alpha(s) \pm i\beta(s) \]

where:

\[ \alpha(s) = \frac{1}{2} \text{trac}(J) \]
\[ \beta(s) = \sqrt{\text{det}(J) - (\alpha(s))^2} \]

where \( J \) is the Jacobian of the linearized system at the equilibrium point \( E_2 \).

From 2.2, Trase \( J(E_2) = N^* \left( s - \frac{b_1(1 - m_1)\alpha_1 P_1^*}{(1 + \alpha_1 N^*)^2} \right) \), and determinant

\[ J(E_2) = \frac{b_1\lambda_3(1 - m_1)N^*P_1^*\left(b_1(1 - m_1)(1 + e_1 P_1^*)^2 + be_1(1 + \alpha_1 N^*)\right)}{(1 + \alpha_1 N^*)^3(1 + e_1 P_1^*)^2} \]

Now at \( s_0 \), \( \alpha(s_0) = 0 \),

\[ \beta(s_0) = \frac{b_1\lambda_3(1 - m_1)N^*P_1^*\left(b_1(1 - m_1)(1 + e_1 P_1^*)^2 + be_1(1 + \alpha_1 N^*)\right)}{(1 + \alpha_1 N^*)^3(1 + e_1 P_1^*)^2} \neq 0 \]

and \( \frac{d\alpha}{ds}\bigg|_{s=s_0} = \frac{-N^*}{2} \neq 0 \).

Therefore from Hopf Theorem the proof is concluded.

(ii) The eigenvalues of the linearized system around the equilibrium point \( E_3 \) are:

\[ \mu_{1,2} = \phi(s) \pm i\psi(s) \]

where:

\[ \phi(s) = \frac{1}{2} \text{trac}(J) \]
\[ \psi(s) = \sqrt{\text{det}(J) - (\phi(s))^2} \]

where \( J \) is the Jacobian of the linearized system at the equilibrium point \( E_3 \).

From 2.2, Trase \( J(E_3) = N^* \left( s - \frac{b_2(1 - m_2)(\alpha_2 - \gamma\lambda_4)P_2^*}{(1 + \alpha_2 N^* + \gamma P_2^*)^2} \right) \), and determinant

\[ J(E_3) = \frac{b_2(1 - m_2)N^*\left(e_2 P_2^*\left(\gamma^2 s^* P_2^* + \alpha_2 \gamma s N^* + \gamma N^* + b_2(1 - m_2)\right) + R^*\right)}{(1 + \alpha_2 N^* + \gamma P_2^*)^3(1 + e_2 P_2^*)^2} \]

where \( R^* \) is equal

\( e_2 \gamma P_2^* (2s^* N^* + b) + P_2^* (2\alpha_2 \gamma s N^* + \gamma N^*(\alpha_2 b + 2s) + 2b_2 + 2b_2(1 - m_2)) + b(1 + \alpha_2 N^*) \) \quad + \gamma^2 s^* N^* P_2^* + \alpha_2 \gamma s N^* + \gamma N^* + b_2(1 - m_2), \)

Now at \( s_0 \),

\[ \phi(s)(s_0) = 0, \quad \psi(s_0) > 0 \text{ if } (\alpha_2 - \gamma\lambda_4) > 0 \text{ and } \frac{d\phi}{ds}\bigg|_{s=s_0} = \frac{-N^*}{2} \neq 0. \]

Therefore from Hopf Theorem the proof is concluded.
2.3.2. Transcritical bifurcation.

**Theorem 4.** The system (1) undergoes a Transcritical bifurcation at the positive equilibrium

(i) \( E_1(N^*, 0, 0) = \left( \frac{b-d}{s}, 0, 0 \right) \) when \((a)\) \( m_1 = \frac{b_1\lambda_3(b-d) - c_1(\alpha_1(b-a) + s)}{b_1\lambda_3(b-d)} \) or

(b) \( m_2 = \frac{b_2\lambda_4(b-d) - c_2(\alpha_2(b-a) + s)}{b_2\lambda_4(b-d)} \)

(ii) \( E_2(N^*, P_1^*, 0) \) when \( m_2 = \frac{N^*(b_1\lambda_3 - \alpha_1c_1) - c_1}{b_1\lambda_3N^*} \)

(iii) \( E_3(N^*, 0, P_2^*) \) when \( m_1 = \frac{N^*(b_2\lambda_4 - \alpha_2c_2) - c_2}{b_2\lambda_4N^*} \)

**Proof**

(i)(a) The eigenvalues of the linearized system around the equilibrium point \( E_1 \) are:

\[
\mathcal{J}(\mathcal{E}_1) = \begin{bmatrix}
    d - b & \frac{b-d}{s} \left( \frac{b_1\alpha_1(d-b) - s(\alpha_1+\alpha_1)(1-m_1)}{s+\alpha_1(b-d)} \right) & \frac{b-d}{s} \left( \frac{b_2\alpha_2(d-b) - s(\alpha_2+\alpha_2)(1-m_2)}{s+\alpha_2(b-d)} \right) \\
    0 & -c_1 + \frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0 \\
    0 & 0 & -c_2 + \frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}
\end{bmatrix},
\]

(4) \( \mathcal{J}(\mathcal{E}_1, m_{10}) = \begin{bmatrix}
    d - b & \frac{b-d}{s} \left( \frac{b_1\alpha_1(d-b) - s(\alpha_1+\alpha_1)(1-m_1)}{s+\alpha_1(b-d)} \right) & \frac{b-d}{s} \left( \frac{b_2\alpha_2(d-b) - s(\alpha_2+\alpha_2)(1-m_2)}{s+\alpha_2(b-d)} \right) \\
    0 & -c_1 + \frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0 \\
    0 & 0 & -c_2 + \frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}
\end{bmatrix} \)

Let us define \( v = (v_1, v_2, v_3)^T \) and \( w = (w_1, w_2, w_3)^T \) are the right and left eigenvectors of \( \lambda_2 = 0 \). From (4) and \( \mathcal{J}(\mathcal{E}_1, m_{10})v = 0 \) as well as \( \mathcal{J}^T(\mathcal{E}_1, m_{10})w = 0 \), then,

\[
\left( (d-b)v_1 + H_{3}^{**}, \frac{(\lambda_4b_2(1-m_2)(b-d) - (\alpha_2(b-d) + s)c_2)v_3}{s+\alpha_2(b-d)}v_3, 0 \right) = (0, 0, 0)^T
\]

where \( H_{3}^{**} = \frac{(b_1(1-m_2)^2)}{s\lambda_3} + \frac{(\alpha_2b_2^2e_2^2 - \alpha_2b_2^2d - s)(b_2(1-m_2)^2)}{s(\alpha_2b_2^2d - s)} \) and

\[
\left( (d-b)w_1 + \frac{b_1\lambda_3((d-b) - c_1s)w_1}{s\lambda_3}, H_{2}^{**} \right)^T = (0, 0, 0)^T
\]

where \( H_{2}^{**} = \frac{(\alpha_2b_2^2e_2 - \alpha_2b_2^2d - s)(b_2(1-m_2)^2)(d-b)}{s(\alpha_2b_2^2d - s)} + \frac{(\alpha_4b_2(1-m_2)(b-d) - (\alpha_2(b-d) + s)c_2)}{s+\alpha_2(b-d)}w_3 \). So the left eigenvector is \( (0, w_2, 0)^T \) and the right eigenvector is
By using the series expansion of the function \( f(X) \), we can rewrite system (1) as a vector form:

\[
\dot{X} = f(X),
\]

where \( X = (N, P_1, P_2)^T \) and

\[
f(X) = \begin{pmatrix}
\frac{bN}{1 + e_1P_1 + e_2P_2} - dN - sN^2 - \frac{b_1N(1 - m_1)P_1}{1 + \alpha_1N} & - \frac{b_2N(1 - m_2)P_2}{1 + \alpha_2N + \gamma P_2} \\
-c_1P_1 + \frac{b_1\lambda_3N(1 - m_1)P_1}{1 + \alpha_1N} & -c_2P_2 + \frac{b_2\lambda_4N(1 - m_2)P_2}{1 + \alpha_2N + \gamma P_2} \\
-b_1 \lambda_3 P_1 & 0
\end{pmatrix}.
\]

Taking derivative on \( f(X) \) with respect to \( m_1 \), we get

\[
f_{m_1}(X) = \begin{pmatrix}
\frac{b_1NP_1}{1 + \alpha_1N} \\
\frac{-b_1\lambda_3NP_1}{1 + \alpha_1N} \\
0
\end{pmatrix}, \text{ then } f_{E_1,m_10}(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Hence, \( w^T f_{E_1,m_10}(X) = 0 \).

Next, taking derivative on \( f_{m_1}(X) \) with respect to \( X = (N, P_1, P_2)^T \), then,

\[
Df_{E_1,m_10}(X) = \begin{pmatrix}
0 & \frac{b_1N^*}{1 + \alpha_1N^*} & 0 \\
0 & -\frac{b_1\lambda_3N^*}{1 + \alpha_1N^*} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We have, \( w^T \left( Df_{E_1,m_10}(X) \cdot v \right) = \frac{b_1\lambda_3N^*v_1 w_2}{1 + \alpha_1N^*} \neq 0 \) Furthermore, we find \( \mathcal{D}^2 f(X) = D(J(E_i)) = D(D(f(X))) \) and then substitute the value of \( m_{10} \) and \( E_1 \), so we end by

\[
w^T [D^2 f_{E_1,m_10}(X)(v,v)] = \frac{2c_1(\alpha_1(b-d)+s)v_1 v_2 w_2}{(b-d)(1+\alpha_1N^*)} \neq 0.
\]

where, \((v,v)\) is a Kronecker product of \((v_1,v_2,v_3)^T\). Therefore, according to the Sotomayor's theorem for local bifurcation [9], system (1) has a transcritical bifurcation at steady state \( E_1 \) when the parameter \( m_1 \) passes through the bifurcation value \( m_{10} \).
(i)(b) The eigenvalues of the linearized system around the equilibrium point \( E_1 \) are:

\[
\mathcal{J}(E_1) = \begin{bmatrix}
  d - b & \frac{b-d}{s} \left( \frac{b_1 \alpha_1 (d-b) - s (b_1 + b_1 (1-m_1))}{s + \alpha_1 (d-b)} \right) & \frac{b-d}{s} \left( \frac{b_2 \alpha_2 (d-b) - s (b_2 + b_2 (1-m_2))}{s + \alpha_2 (d-b)} \right) \\
 0 & -c_1 + \frac{b_1 \lambda_3 (d-b) (1-m_1)}{s + \alpha_1 (d-b)} & 0 \\
0 & 0 & -c_2 + \frac{b_2 \lambda_4 (d-b) (1-m_2)}{s + \alpha_2 (d-b)}
\end{bmatrix},
\]

\[\mathcal{J}(E_1, m_2) = \begin{bmatrix}
  d - b & \frac{b-d}{s} \left( \frac{b_1 \alpha_1 (d-b) - s (b_1 + b_1 (1-m_1))}{s + \alpha_1 (d-b)} \right) & \frac{b-d}{s} \left( \frac{b_2 \alpha_2 (d-b) - s (b_2 + b_2 (1-m_2))}{s + \alpha_2 (d-b)} \right) \\
 0 & -c_1 + \frac{b_1 \lambda_3 (d-b) (1-m_1)}{s + \alpha_1 (d-b)} & 0 \\
0 & 0 & -c_2 + \frac{b_2 \lambda_4 (d-b) (1-m_2)}{s + \alpha_2 (d-b)}
\end{bmatrix}.
\]

Let us define \( v = (v_1, v_2, v_3)^T \) and \( w = (w_1, w_2, w_3)^T \) are the right and left eigenvectors of \( \lambda_3 = 0 \). From (5) and \( \mathcal{J}(E_1, m_2) v = 0 \) as well as \( \mathcal{J}^T(E_1, m_1) w = 0 \), then,

\[
(d - b)v_1 + H_4^{**} \left( \frac{\lambda_3 b_1 (1-m_1) (d-b) - (\alpha_1 (b-d) + s) c_1 v_2}{s + \alpha_1 (d-b)} \right) v_2 = 0,
\]

where \( H_4^{**} = \frac{b_2 (d-b) \lambda_3 - c_2 s) v_1}{s \lambda_4} + \frac{(a_1 b_2 \alpha_1 (d-b) + b_1 (1-m_1) s) (d-b) v_2}{s (a_1 (b-d) + s)} \) and

\[
\left( (d - b)w_1, \frac{b_2 \lambda_3 ((d-b) - c_2 s) w_1}{s \lambda_4}, H_5^{**} \right)^T = (0, 0, 0)^T
\]

where \( H_5^{**} = \frac{(a_1 b_2 \alpha_1 (d-b) + b_1 (1-m_1) s) (d-b) w_1}{s (a_1 (b-d) + s)} + \frac{(\lambda_3 b_1 (1-m_1) (d-b) - (\alpha_1 (b-d) + s) c_1 w_2)}{s + \alpha_1 (d-b)} \). So the left eigenvector is \((0, w_2, 0)^T\) and the right eigenvector is

\[
\left( \frac{b_2 c_2 \lambda_4 - b_2 e_2 \lambda_4 + c_2 s) v_1}{s \lambda_4 (d-b)}, 0, v_3 \right)^T.
\]

Here, \( w_2 \) and \( v_3 \) are any non-zero real numbers. Now, system (1) can be rewritten in the following vector form:

\[
\dot{X} = f(X),
\]

where \( X = (N, P_1, P_2)^T \) and

\[
f(X) = \begin{pmatrix}
  -dN - sN^2 - \frac{b_1 N (1-m_1) P_1}{1 + \alpha_1 N} & - \frac{b_2 N (1-m_2) P_2}{1 + \alpha_2 N} \\
  -c_1 P_1 + \frac{b_1 \lambda_3 N (1-m_1) P_1}{1 + \alpha_1 N} & -c_2 P_2 + \frac{b_2 \lambda_4 N (1-m_2) P_2}{1 + \alpha_2 N}
\end{pmatrix}.
\]

Taking derivative on \( f(X) \) with respect to \( m_2 \), then,

\[
D f_{E_1, m_2}(X) = \begin{pmatrix}
  0 & 0 & \frac{b_2 N^*}{1 + \alpha_2 N^*} \\
  0 & 0 & 0 \\
  0 & 0 & - \frac{b_2 \lambda_4 N^*}{1 + \alpha_2 N}
\end{pmatrix}.
\]
We have, \( w^T \left( Df_{\xi_1, m_{20}}(X) \cdot v \right) = -\frac{b_2 \lambda_4 N^3 v_3 w_3}{(1 + \alpha_1 N^3)} \neq 0 \).

Furthermore, we find \( D^2 f(X) = D(J(E_1)) = D(D(f(X))) \) and then substitute the value of \( m_{20} \) and \( E_1 \), so we end by

\[
\begin{align*}
\left[ x^T D^2 f_{\xi_1, m_{20}}(X) (v, v) \right] &= \frac{2c_2 (v_1 - \gamma N^3 v_3) (\alpha_2 (b - d) + s) v_3 w_3}{(b - d)(1 + \alpha_2 N^3)^2} \neq 0.
\end{align*}
\]

where, \((v, v)\) is a Kronecker product of \((v_1, v_2, v_3)^T\). Therefore, according to the Sotomayor’s theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state \( E_1 \) when the parameter \( m_2 \) passes through the bifurcation value \( m_{20} \).

(ii) The eigenvalues of the linearized system around the equilibrium point \( E_2 \) are:

\[
J(\xi_2) = \begin{bmatrix}
N^* \left(-s + \frac{b_1 N^3 p_1 (1-m)}{1+\alpha_1 N^3} \right) & -N^* \left( \frac{b_1 (1-m)}{1+\alpha_1 N^3} + \frac{b e_1}{1+e_1 p_1^2} \right) & -N^* \left( \frac{b_2 (1-m)}{1+\alpha_1 N^3} + \frac{b e_2}{1+e_1 p_1^2} \right) \\
\frac{-b_2 \lambda_4 N^3}{(1+\alpha_1 N^3)^2} & 0 & 0 \\
0 & 0 & -c_2 + \frac{b_2 \lambda_4 N^3 (1-m_2)}{1+\alpha_1 N^3}
\end{bmatrix}
\]

\[
(6) \ J(\xi_2, m_{20}) = \begin{bmatrix}
N^* \left(-s + \frac{b_1 N^3 p_1 (1-m_1)}{1+\alpha_4 N^3} \right) & -N^* \left( \frac{b_1 (1-m_1)}{1+\alpha_4 N^3} + \frac{b e_1}{1+e_1 p_1^2} \right) & -N^* \left( \frac{c_2}{\lambda_4 N^3} + \frac{b e_2}{1+e_1 p_1^2} \right) \\
\frac{-b_2 \lambda_4 N^3 (1-m_1)}{1+\alpha_4 N^3} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Let us define \( v = (v_1, v_2, v_3)^T \) and \( w = (w_1, w_2, w_3)^T \) are the right and left eigenvectors of \( \lambda_3 = 0 \). From (6) and \( J(\xi_2, m_{20}) v = 0 \) as well as \( J^T(\xi_2, m_{20}) w = 0 \), then,

\[
\left( N^* \left(-s + \frac{b_1 N^3 p_1 (1-m_1)}{1+\alpha_4 N^3} \right) w_1 - N^* \left( \frac{b_1 (1-m_1)}{1+\alpha_4 N^3} + \frac{b e_1}{1+e_1 p_1^2} \right) w_2 - N^* \left( \frac{c_2}{\lambda_4 N^3} + \frac{b e_2}{1+e_1 p_1^2} \right) v_3, \frac{b_2 \lambda_4 N^3 (1-m_1)}{1+\alpha_4 N^3} v_1, 0 \right) = (0,0,0)^T
\]

and

\[
\left( N^* \left(-s + \frac{b_1 N^3 p_1 (1-m_1)}{1+\alpha_4 N^3} \right) w_1 + \frac{b \lambda_4 N^3 (1-m_1)}{1+\alpha_4 N^3} w_2, -N^* \left( \frac{b_1 (1-m_1)}{1+\alpha_4 N^3} + \frac{b e_1}{1+e_1 p_1^2} \right) w_1 - N^* \left( \frac{c_2}{\lambda_4 N^3} + \frac{b e_2}{1+e_1 p_1^2} \right) v_2 \right) = 0
\]

So the left eigenvector is \((0,0,w_3)^T\) and the right eigenvector is

\[
\left(0, v_2, -\frac{\lambda_4 N^3 \left( \frac{b_1 e_1 b (1+\alpha_1 N^3)}{1+\alpha_1 N^3} \right) \left( 1+e_1 p_1^2 \right) w_1 + b_1 b_1 \left( 1+e_1 p_1^2 \right) w_2}{\left( 1+e_1 p_1^2 \right) ^2} \right)^T \]

Here, \( w_3 \) and \( v_2 \) are any non-zero real numbers. Now, system (1) can be rewritten as in the following vector form:

\[
\dot{X} = f(X),
\]

where \( X = (N, p_1, p_2)^T \) and
Theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state \( w \). Hence, \( f_m(X) = \begin{pmatrix} \frac{b_2NP_2}{1 + \alpha_2N + \gamma P_2} \\ 0 \\ -\frac{b_2\lambda_4NP_2}{1 + \alpha_2N + \gamma P_2} \end{pmatrix} \), then \( f_{\mathcal{E}^2,m_0}(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \).

Hence, \( w^T f_{\mathcal{E}^2,m_0}(X) = 0 \).

Next, taking derivative on \( f_m(X) \) with respect to \( X = (N, P_1, P_2)^T \), we get,

\[
Df_{\mathcal{E}^2,m_0}(X) = \begin{pmatrix} 0 & \frac{b_2N^*}{1 + \alpha_2N^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{b_2\lambda_4N^*}{1 + \alpha_2N} \end{pmatrix}.
\]

We have, \( w^T(Df_{\mathcal{E}^2,m_0}(X) \cdot v) = \frac{[b_1(1-m_1)(1+\alpha_1P_2^2)+bc(1+\alpha_1N^*)]b_2\lambda_2^2N^2v_2v_3}{(1+\alpha_1N^*)(1+\alpha_2N^*)c_2(1+\alpha_1P_2^2)+bc\lambda_4N^*} \neq 0 \). Furthermore, we find \( \mathcal{D}^2 f(X) = D(D(f(X))) \) and then substitute the value of \( m_2 \) and \( E_2 \), so we end by

\[
w^T[D^2f_{\mathcal{E}^2,m_0}(X)(v,v)] = \frac{2c_2(v_1 - \gamma v_3N^*)v_3w_3}{N^*(1 + \alpha_1N^*)} \neq 0.
\]

where, \((v,v)\) is a Kronecker of \((v_1, v_2, v_3)^T\). Therefore, according to the Sotomayor’s theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state \( \mathcal{E}^2 \) when the parameter \( m_2 \) passes through the bifurcation value \( m_2^0 \).

(iii) The eigenvalues of the linearized system around the equilibrium point \( E_3 \) are:

\[
\mathcal{J}(\mathcal{E}_3^2) = \begin{pmatrix}
N^*(-s + \frac{b_2\lambda_1P_1^2(1-m_1)}{1+\alpha_1N^*+\gamma P_1^2}) & -N^*(-\frac{b_1(1-m_1)}{1+\alpha_1N^*+\gamma P_1^2}) & -N^*(-\frac{b_2(1-m_2)(1+\alpha_1N^*)}{1+\alpha_2N^*+\gamma P_2^2}) \\
0 & -c_1 + \frac{b_1\lambda_2N^*(1-m_1)}{1+\alpha_2N^*} & 0 \\
\frac{b_2\lambda_4P_2^2(1-m_2)(1+\gamma P_2^2)}{(1+\alpha_2N^*+\gamma P_2^2)} & 0 & -\frac{b_2\lambda_4P_2^2N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^2)}
\end{pmatrix}.
\]

\( \mathcal{J}(\mathcal{E}_3^2,m_1) = \begin{pmatrix}
N^*(-s + \frac{b_2\lambda_1P_1^2(1-m_1)}{1+\alpha_1N^*+\gamma P_1^2}) & -N^*(-\frac{b_1(1-m_1)}{1+\alpha_1N^*+\gamma P_1^2}) & -N^*(-\frac{b_2(1-m_2)(1+\alpha_1N^*)}{1+\alpha_2N^*+\gamma P_2^2}) \\
0 & 0 & 0 \\
\frac{b_2\lambda_4P_2^2(1-m_2)(1+\gamma P_2^2)}{(1+\alpha_2N^*+\gamma P_2^2)} & 0 & 0
\end{pmatrix} \)
Let us define $v = (v_1, v_2, v_3)^T$ and $w = (w_1, w_2, w_3)^T$ are the right and left eigenvectors of $\lambda_2 = 0$. From (7) and $\mathcal{J}(\delta, m_{10})v = 0$ as well as $\mathcal{J}^T(\delta, m_{10})w = 0$, then,

$\left(N^T \left[ (-s + \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2}) v_1 - \left( a_1 N^* + \gamma P_2 \right) v_2 - \left( \frac{b_2(1-m_2)}{(1+a_1 N^*)^2} + \frac{be_2}{(1+e_2 P_1)^2} \right) v_3 \right] - \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2} v_1 - \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2} v_2 + \frac{b_1 a_1 N^*}{(1+e_2 P_1)^2} v_3 \right] \cdot G_1 = (0,0,0)^T,$

where $G_1 = \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2} v_1 + \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2} v_2 + \frac{b_2 a_1 P_1(1-m_1)}{(1+a_1 N^*)^2} v_3$ and

$\left(N^T \left[ (-s + \frac{b_1 a_1 P_1(1-m_1)}{1+a_1 N^*}) w_1 + \frac{b_1 a_1 P_1(1-m_1)}{1+a_1 N^*}) w_2, -N^T \left( b_1(1-m_1) + \frac{be_2}{(1+e_2 P_1)^2} \right) w_1 + \frac{b_1 a_1 N^*}{(1+e_2 P_1)^2} \right] \cdot w_1, -N^T \left( \frac{c_2}{\lambda_2 N^*} + \frac{be_2}{(1+e_2 P_1)^2} \right) w_1 \right)^T = 0$

So the left eigenvector is $(0, w_2, 0)^T$ and the right eigenvector is

$$\left( v_1, \frac{\lambda_1 v_1 \left[ e_2 P_2 (\gamma^2 sN^* + \gamma N^* (1 + \alpha_2 N^* + b_2 (1-m_2)) + \gamma N^* (1 + \alpha_2 N^* + b_2 (1-m_2)) \right]}{\gamma (1 + \alpha_2 N^* + \gamma P_2)} \right)^T$$

, where $G_2^* = P_2^* (\gamma^2 P_2^* (2sN^* + b) + \gamma (2\alpha_2 N^* + N^* (2s + \alpha_2 b)) + 2b_2 (1 - m_2)) + e_2 b (1 + \alpha_2 N^*) + e_2 N^* + \gamma s N^* (1 + \alpha_2 N^*) + b_2 (1 - m_2)$. Here, $w_2$ and $v_1$ are any non-zero real numbers. Now, similarly system (1) can be rewritten as in the following vector form:

$$\dot{X} = f(X),$$

where $X = (N, P_1, P_2)^T$ and

$$f(X) = \begin{pmatrix} \frac{bN}{1 + e_1 P_1 + e_2 P_2} - dN - sN^2 - \frac{b_1 N (1 - m_1) P_1}{1 + \alpha_1 N} - \frac{b_2 N (1 - m_2) P_2}{1 + \alpha_2 N + \gamma P_2} \\ \frac{b_1 a_1 N^*}{1 + \alpha_1 N} \\ \frac{b_1 \lambda_3 N (1 - m_1) P_1}{1 + \alpha_1 N} - \frac{be_2}{(1+e_2 P_1)^2} + \frac{b_2 a_1 N (1 - m_2) P_2}{1 + \alpha_2 N + \gamma P_2} - \frac{c_2}{\lambda_2 N^*} \end{pmatrix}.$$  

Taking derivative on $f(X)$ with respect to $m_1$, we get

$$f_{m_1}(X) = \begin{pmatrix} \frac{b_1 N P_1}{1 + \alpha_1 N} \\ - \frac{b_1 \lambda_3 N P_1}{1 + \alpha_1 N} \\ 0 \end{pmatrix}, \text{ then } f_{\delta, m_{10}}(X) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

Hence, $w^T f_{\delta, m_{10}}(X) = 0$.

Next, taking derivative on $f_{m_1}(X)$ with respect to $X = (N, P_1, P_2)^T$. Then,

$$D f_{\delta, m_{10}}(X) = \begin{pmatrix} 0 & - \frac{b_1 N^*}{1 + \alpha_1 N^*} & 0 \\ - \frac{b_1 \lambda_3 N^*}{1 + \alpha_1 N^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$
We have, \( w^T \left( D f_{E_3, m_{10}}(X) \right) \cdot v = \frac{\left( e_2 P_2^2 \left[ \gamma s N^* P_2 + \gamma s N^* \right] \right) + G_1 + \lambda_3 N v_{1 w_2}}{1 + \alpha_1 N^*} \neq 0 \), where

\[
G_3 = e_2 \left[ P_2^2 (2s N^* + b) + P_2^2 (2s \alpha_2 N^* + b + 2) + 2b_2 (1 - m_2) + 2b_2 (1 - m_2) + b_2 (1 + \alpha_2 N^*) \right] + \gamma s N^* P_2^2 + \gamma s N^* (1 + \alpha_2 N^*) + b_2 (1 - m_2)
\]

Furthermore, we find \( D^2 f(X) = D(J(E_i)) = D(D(f(X))) \) and then substitute the value of \( m_{10} \) and \( E_3 \), so we end by

\[
w^T \left[ D^2 f_{E_3, m_{10}}(X)(v, v) \right] = \frac{c_1 (v_1 + v_2) v_2 w_2}{N^* (1 + \alpha_1 N^*)} \neq 0.
\]

where, \( (v, v) \) is a Kronecker product of \( (v_1, v_2, v_3) \). Therefore, according to the Sotomayor’s theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state \( E_3 \) when the parameter \( m_1 \) passes through the bifurcation value \( m_{10} \).

### 3. Numerical solution

In this section we will present some numerical simulations in order to show the theoretically established results. The values of the parameters are taken from the following table, and some of them will be varied in order to see their effect.

| Parameters | value |
|------------|-------|
| \( b \)   | 0.4   |
| \( d \)   | 0.2   |
| \( s \)   | 0.0009|
| \( b_1, \alpha_1 \) | 0.05, 0.002 |
| \( b_2, \alpha_2 \) | 0.05, 0.002 |
| \( \gamma \) | 0.002 |
| \( c_1, c_2 \) | 0.1, 0.1 |
| \( e_1, e_2 \) | 0.002, 0.002 |
| \( m_1, m_2 \) | 0.6, 0.6 |
| \( \lambda_3 \) | 0.1 |
| \( \lambda_4 \) | 0.1 |

**Table 2.** Table of value of parameters
As we have seen in figure 2, which demonstrates the effect stability of the trivial solution when the interspecies competition is very big and \( b - d < 0 \), i.e. when the death rate is greater than the birth rate. In Figure 2 (i) \( s = 0.0009 \) and in (ii) \( s = 0.8 \) with \( b > d \). While in Figure 2(iii) \( s = 0.0009, b < d \). Figure 3 illustrates the stability of the existence of prey only \((N^*, 0, 0)\) under some conditions. The stability of the solution which reach the point \((N^*, P_1^*, 0)\), which represents the existence of prey and the first predator, is shown in figure 4. Figure 5 shows the stability of the solutions when it reach the point \((N^*, 0, P_2^*)\) which represents the existence of prey and second predator. The stability of the solutions when it reaches the coexistence point is illustrated in figure 6. The limit cycles are illustrated in figures 7, 8 and 9 for the solutions of existence of prey with first predator, with the second predator and when it reaches the coexistence point, respectively. Figure 10 shows the effect of transcritical parameter which is refuge parameters \( m_1 \) and \( m_2 \) as illustrated in next table:

| value of \( m_1 \) | The existence point |
|------------------|-------------------|
| [0,0.6)           | \((N^*, P_1^*, 0)\) |
| 0.6              | \((N^*, P_1^*, P_2^*)\) |
| (0.6,1]          | \((N^*, 0, P_2^*)\) |

| value of \( m_2 \) | The existence point |
|------------------|-------------------|
| [0,0.6)           | \((N^*, 0, P_2^*)\) |
| 0.6              | \((N^*, P_1^*, P_2^*)\) |
| (0.6,1]          | \((N^*, P_1^*, 0)\) |
The effect of hopf bifurcation parameter which is the inter-species competition parameter is demonstrated in figures 11 and 12. The next table explains this result. We can see that when the value of this parameter changes the solutions changes from a stable limit cycle to an asymptotically stable point of the prey with one predator where we have the phenomenon of hopf bifurcation. When it further changes the solutions moves towards the point where the prey exists alone, where we have a transcritical bifurcation.

Figures 15-18 show the effect of fear parameters $e_1$ and $e_2$. As we can see when $e_1$ is small the solution reach the point of existence of prey with the first predator. If the value of fear parameter $e_1$ increases, the solution reaches same point but with decreasing in amount of predator population. If the value of the fear continues to increase, then the solution will reach to the point of existence prey only. Same situation happens if the value $e_2$ increased but here the solution reach the point of existence of prey with the second predator.

| value of $s$            | The existence point     | The situation            |
|-------------------------|-------------------------|--------------------------|
| $[0,0.000314)$          | $(N^*, P_1^*, 0)$       | limit cycle.             |
| $(0.000314,0.0036]$     | $(N^*, P_1^*, 0)$       | Local asymptotically stable point. |
| $(0.0036,1)$            | $(N^*, 0, 0)$           | Local asymptotically stable point. |

| value of $s$            | The existence point     | The situation            |
|-------------------------|-------------------------|--------------------------|
| $(0,0.000279995]$       | $(N^*, 0, P_2^*)$       | limit cycle.             |
| $(0.000279995,0.0036]$  | $(N^*, 0, P_2^*)$       | Local asymptotically stable point. |
| $(0.0036,1]$            | $(N^*, 0, 0)$           | Local asymptotically stable point. |
Figure 2. The stability of trivial point $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.6, b = 0.4, d = 0.41$

Figure 3. The stability of prey $N = P_1 = P_2 = 70, s = 0.004, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.6, b = 0.4, d = 0.2, b_1 = b_2 = 0.05$
Figure 4. The stability of prey with first predator $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.8$

Figure 5. The stability of prey with second predator $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m_2 = 0.6$
FIGURE 6. The stability of coexistence point $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.6$

FIGURE 7. The limit cycle of first predator with prey $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.8, b = 0.4, d = 0.2$
Figure 8. The limit cycle of second predator with prey $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m_2 = 0.6, b = 0.4, d = 0.2$

Figure 9. The limit cycle of coexistence point $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.6, b = 0.4, d = 0.2$
Figure 10. Effect of transcriptional refuge parameter on model, \( N = P_1 = P_2 = 70, s = 0.0004, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m_2 = 0.6 \)

Figure 11. Effect of hopf bifurcation parameter on model on the point of first predator with prey.
Figure 12. Hopf bifurcation diagram of model on the point of first predator with prey.

Figure 13. Effect of hopf bifurcation parameter on model on the point of second predator with prey.
**Figure 14.** Hopf bifurcation diagram of the point of second predator with prey.

**Figure 15.** Effect of fear parameter on model $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.8$
Figure 16. Phase plane of effect of fear parameter on model $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m_2 = 0.8$.

Figure 17. Effect of fear parameter on model, $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m_2 = 0.6$.

Figure 18. Phase plane of effect of fear parameter on model, $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m_2 = 0.6$. 
4. CONCLUSION

In this paper we formulated model of one prey with two predators population; aggressive predator, bold predator and prey which is afraid and stay in refuge. We use holing type II functional response and Beddington-DeAngelis functional response[12]. We proved the locally and globally stabilities of equilibria. In addition we hold a hopf and double transcritical bifurcations of some parameters. The cost of fear and prey refuge is allow model to reach to double transcritical. we notice that if fear is good when it is small. The effect of competition of prey population is to convert the model from the stable limit cycle to a spiral stable equilibrium point of prey with predator. When it becomes large it converts model to stable trivial solution.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] P.A. Abrams, Foraging time optimisation and interactions in food webs, Amer. Naturalist, 120(1984), 382-390.
[2] B.A. Belgrad, B.D. Griffen, Predator and prey interactions mediated by prey personality and predator hunting mode, Proc. R. Soc. B: Biol. Sci. 283 (2016), 20160408.
[3] T. Brodin, Predator effects on behaviour and life-history of prey, Doctoral thesis, Umea University, Sweden, 2005.
[4] L. Chen, F. Chen, L. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge, Nonlinear Anal.: Real World Appl. 11 (2010), 246-252.
[5] D. Tomšič, J. Sztarker, M. Berón de Astrada, D. Oliva, E. Lanza, The predator and prey behaviors of crabs: from ecology to neural adaptations, J. Exper. Biol. 220 (2017), 2318–2327.
[6] J.H.P. Dawes, M. O. Souza, A derivation of Holling’s type I, II and III Functional responses in predator prey systems, J. Theor. Biol. 327 (2013), 11-22.
[7] T.K. Kar, Stability analysis of a prey-predator model incorporating a prey refuge, Commun. Nonlinear Sci. Numer. Simul. 10 (2005), 681-691.
[8] W. Ko, K. Ryu, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a prey refuge, J. Differ. Equ. 231 (2006), 534-550.
[9] Y.A. Kuznesov, Elements of applied bifurcation theorem, Springer, 1998.
[10] S.L. Lima, Nonlethal effects in the ecology of predator-prey interactions, BioScience, 48 (1998), 25-34.
[11] R. Stoks, M. De Block, Fitness effects from egg to reproduction: bridging the life history transition, Ecology, 86 (2005), 185-197.

[12] X. Wang, Studying both direct and indirect effects in predator-prey interaction, The University of Western Ontario, Thesis, 2016.

[13] H. Zhang, Y. Cai, S. Fu, W. Wang, Impact of the fear effect in a prey-predator model incorporating a prey refuge, Appl. Math. Comput. 356 (2019), 328–337.