Conical metrics on Riemann surfaces, I: the compactified configuration space and regularity

Rafe Mazzeo  
Stanford University  

Xuwen Zhu  
Stanford University

Abstract

We introduce a compactification of the space of simple positive divisors on a Riemann surface, as well as a compactification of the universal family of punctured surfaces above this space. These are real manifolds with corners. We then study the space of constant curvature metrics on this Riemann surface with prescribed conical singularities at these divisors. Our interest here is in the local deformation for these metrics, and in particular the behavior as conic points coalesce. We prove a sharp regularity theorem for this phenomenon in the regime where these metrics are known to exist. This setting will be used in a subsequent paper to study the space of spherical conic metrics with large cone angles, where the existence theory is still incomplete.

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1 Introduction

This paper is a sequel to [13] by the first author and Weiss concerning the
space of metrics with constant curvature and prescribed conic singularities
on a compact Riemann surface $M$. In that paper a careful analysis was made
of the deformation theory for such metrics provided all the cone angles are
less than $2\pi$. This assumption simplifies both the analytic and the geometric
considerations considerably. Consider the space of tuples $(c, p, \vec{\beta}, K, A)$,
where $c$ is a conformal structure on $M$, $p$ a collection of $k$ distinct points,
$\vec{\beta}$ a $k$-tuple of parameters prescribing the cone angles at the points $p_j$,
with each $\beta_j \in (0, 1)$ (this corresponds to all cone angles lying in $(0, 2\pi)$),
and constant $K$ specifying the Gauss curvature and $A > 0$ specifying the area,
all subject to the requirement forced by Gauss-Bonnet that

$$\chi(M, \vec{\beta}) := \chi(M) + \sum_{j=1}^{k} (\beta_j - 1) = \frac{1}{2\pi} KA. \quad (1)$$
It is known through the work of several authors that to each such tuple there exists a unique metric on $M$ which has constant curvature $K$, area $A$, and conical singularities at the points $p_j$ with cone angle $2\pi \beta_j$. There is a caveat when $K > 0$ and $k > 2$ which states that in this case an extra condition is needed on the cone angles, namely that they satisfy the so-called Troyanov condition

$$\min\{2, 2\beta_j\} + k - \chi(M) > \sum_{i=1}^{k} \beta_i, \quad j = 1, \ldots, k,$$

which is trivial when restricted to $\vec{\beta} \in (0, 1)^k$ except when $M = S^2$. The main result of [13] states that the Teichmüller space $T_{\gamma,k}^{\text{conic}}$ of all such solutions moduli the space of diffeomorphisms of $M$ isotopic to the identity is a smooth manifold.

It is known that the situation becomes much more complicated when some or all of the $\beta_j$ are greater than 1, at least in the case that $K > 0$. One classical inspiration to study this case is when each $\beta_j \in \mathbb{N}$, i.e., all cone angles are integer multiples of $2\pi$, in which case examples are easily obtained as ramified covers over other compact surfaces with metrics of constant curvature. Existence and uniqueness of spaces with arbitrary cone angles and curvature $K \leq 0$ subject to (1) is relatively easy, see [14]. Much more recent is the dramatic breakthrough by Mondello and Panov [16], which establishes through beautiful and purely geometric reasoning necessary and sufficient conditions on the possible set of values $\vec{\beta}$ for which there exists a metric with constant curvature 1 (a spherical metric) on $S^2$ with these prescribed cone angle parameters.

This last-cited paper leaves open some fundamental questions. The one which interests us here is to describe the space of points $p$ and cone angle parameters $\vec{\beta}$ for which there exist spherical metrics with this data prescribing the conic singularities. The answer is complicated and (at least to our understanding) not completely explicit. An initial hope might be to show that the space of all solutions (mod diffeomorphisms) is a smooth manifold. From this one might then further try to apply various techniques from geometric analysis to count solutions. Unfortunately, for spherical cone metrics with cone angles greater than $2\pi$, this space fails to be smooth on certain subvarieties. One of our goals, which will be addressed in a sequel to this paper, is to understand this failure more precisely. Briefly, however, the key observation is that if $g$ is a spherical metric for which the deformation theory
is obstructed, it is possible to consider this solution in a larger moduli space where the deformation theory is unobstructed. This broader setting consists of letting certain of the cone points $p_j$ split into clusters of cone points with smaller angles. This is an analytic manifestation of one of the important steps in the geometric arguments of Mondello and Panov [16].

The analysis needed to carry this out turns out to be somewhat complicated and requires the development of some machinery which will occupy a significant part of this paper. We regard this machinery of independent interest, and expect that it may be a useful tool in studying various other analytic problems involving geometric objects which are singular or otherwise distinguished at families of points which can cluster. We mention in particular the study of solutions of the two-dimensional vortex equation on a Riemann surface, as well as the study of analytic constructions related to holomorphic quadratic differentials in relationship to the Hitchin moduli space.

This paper has two main sections. In the first part we develop these general ideas, which involve the construction of a resolution via real blow-up of the configuration space of $k$ points on $M$ and of the universal family of marked surfaces over this blown-up configuration space. Similar constructions are classical in algebraic geometry if one uses complex blowups, but our use of real blow-ups and other $C^\infty$ methods here lead to spaces which are compact manifolds with corners which encode the different modes of clustering of these $k$ points. This construction is closely related to other recent work, notably the ongoing work of Kottke and Singer [10] on the compactification of the moduli space of monopoles in $\mathbb{R}^3$.

In the second main section we consider the space of metrics with constant curvature and prescribed conic singularities; in this paper we restrict attention to flat and hyperbolic metrics. Our main result here is a new regularity result: we show that the family of hyperbolic or flat metrics with conic singularities with arbitrary cone angle lifts to be polyhomogeneous, a natural generalization of smoothness, on the compactified universal family of curves over this extended configuration space. We also prove the analogous space of spherical conic surfaces with cone angles less than $2\pi$ and satisfying the Troyanov constraint has the same regularity properties.

These results are first steps in our program to understand the entire moduli space of constant curvature conical metrics on surfaces. The explanation of the extended configuration family, which is the setting for this regularity theory, is already of interest, and its definition is vindicated by our main
regularity theorem. In a second paper we will employ this machinery to understand features of the moduli space of spherical cone metrics where the cone angles are greater than $2\pi$. Our eventual goal is to understand the stratified nature of these moduli spaces in sufficient detail that we can produce a count of solutions. We also hope to reach a better correspondence between the classical results and tools used to study these problems and the ones developed here.

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## 2 Resolution of point configurations

The first part of this paper focuses on a rather intricate geometric construction, which is a resolution via real blow-up of the configuration space of $k$ points on a compact Riemann surface $M$, as well as the resolution of the universal family over this space.

To be more specific, let $D_k(M)$ denote the space of nonnegative divisors on $M$ of total degree $k$. Thus a point of $D_k(M)$ consists of an ordered $k$-tuple of not necessarily distinct points $p_1, \ldots, p_k \in M$. Although it is more common to study this using algebro-geometric ideas, we take a decidedly real and $C^\infty$ approach. Away from coincidences where two or more of the $p_j$ are the same, $D_k(M)$ is a copy of $M^k$ with all the partial diagonals removed. Of course, $\Sigma_k$ acts freely on this open set. Our first goal is to define a real compactification $E_k(M)$ of this open dense set in $D_k(M)$, which we call the extended configuration space. This compactification is a manifold with corners. We next consider the product $E_k(M) \times M$; this is a trivial bundle which has a tautological multi-valued section $\sigma$: if $p$ is a point in $E_k(M)$, then $\sigma(p)$ is the divisor $p$ considered as a subset of $M$. We shall also define a resolution of this object using a suitable blowup of the graph of $\sigma$, which as before is a manifold with corners, which we call the extended configuration family and denote $C_k(M)$. This is not quite a fibration over $E_k(M)$ since certain fibers are ‘broken’; instead it is a $b$-fibration, a natural extension of the notion of fibrations to the category of manifolds with corners.
2.1 The extended configuration space

Our first goal is to define a good compactification for \( D_k(M)^* \) the space of all ‘simple’ divisors. We begin with some notation. Suppose first that \( I \subset \{1, \ldots, k\} \) is an index set with \( |I| \geq 2 \). The \( I^{\text{th}} \) partial diagonal is the subset

\[
\Delta_I = \{ p \in D_k : p_i = p_j, \forall i, j \in I \}.
\]

There is a reverse partial order of diagonals corresponding to the inclusion of index sets,

\[
I \subset J \iff \Delta_I \supset \Delta_J.
\]

The union of the two index sets is defined in the usual sense. If \( I \) and \( J \) have at least one common element, one can identify the partial diagonal corresponding to their union as the intersection of their diagonals:

\[
\Delta_{I \cup J} = \Delta_I \cap \Delta_J, \text{ if } I \cap J \neq \emptyset.
\]

The assumption of nonempty intersection guarantees that the intersection of two diagonals is still a diagonal. Otherwise, if \( I \cap J = \emptyset \), there is a strict inclusion

\[
\Delta_{I \cup J} \subsetneq \Delta_I \cap \Delta_J.
\]

On the other hand, when \( |I \cap J| \geq 2 \), the set corresponding to the intersection of index sets \( \Delta_{I \cap J} \) is the smallest diagonal containing both \( \Delta_I \) and \( \Delta_J \).

We also let

\[
\Delta^0_I = \Delta_I \setminus (\cup_{J \supseteq I} \Delta_J).
\]

It is then clear that the ensemble \( \{\Delta^0_I\} \) is a stratification of \( M^k = D_k \); the dense open stratum equals

\[
D_k^* = M^k \setminus (\cup_I \Delta^0_I).
\]

To resolve the point collisions, we resolve all the partial diagonals; this is done by blowing up the diagonals iteratively in order of decreasing index set. In other words, if \( I \subsetneq J \), then \( \Delta_J \) is blown up before \( \Delta_I \). When carrying out an iterated blowup, there are two main issues about which one must be careful. Let \( X \) be a manifold with corners, containing two \( p \)-submanifolds, \( Y_1 \) and \( Y_2 \). (A \( p \)-submanifold \( Y \subset X \) is defined to be a submanifold for which some neighborhood \( U \supset Y \) is diffeomorphic as a manifold with corners to the
normal bundle $NY$.) The iterated blowup $[X; Y_1; Y_2]$ is the manifold with corners obtained as follows. First blow up $Y_1$ in $X$ to obtain a space $[X; Y_1]$. Now lift $Y_2 \setminus (Y_1 \cap Y_2)$ to this space and take its closure. Finally, take the blowup of this lift in $[X; Y_1]$. In general the resulting space depends on the order in which these blowups are taken; the reverse order may result in a nondiffeomorphic space. There are two special situations where the order does not matter: the first is if $Y_1 \subset Y_2$, and the second is if $Y_1$ and $Y_2$ meet transversely. In the prescription for blowing up the partially ordered sequence of partial diagonals, we are blowing up these partial diagonals by inclusion, i.e., we always blow up the ‘smaller’ submanifolds first. However, we must check that the second criterion is satisfied.

**Lemma 1.** Let $\mathcal{I}$ and $\mathcal{J}$ be any two index sets. Suppose that $X_{\mathcal{I}, \mathcal{J}}$ is the manifold with corners obtained by blowing up all the partial diagonals $\Delta_K$ in $M^k$ for which $K \supset I$ and $K \supset J$. Then the lifts of $\Delta_I$ and $\Delta_J$ are transverse in $X_{\mathcal{I}, \mathcal{J}}$.

**Proof.** When $\mathcal{I} \cap \mathcal{J} \neq \emptyset$, the intersection $\Delta_{\mathcal{I} \cup \mathcal{J}} = \Delta_I \cap \Delta_J$ is already blown up in $X_{\mathcal{I}, \mathcal{J}}$, so the closures of the lifts of $\Delta_I$ and $\Delta_J$ are disjoint, and the order obviously does not matter.

When $\mathcal{I} \cap \mathcal{J} = \emptyset$, then there exist (complex) coordinates $\{s_i\}_{i=1}^k$ such that, using obvious identifications of index sets, $\Delta_I = \{s_I = 0\}$, $\Delta_J = \{s_J = 0\}$, hence $\Delta_I$ and $\Delta_J$ intersect transversely. \qed

Using this Lemma, we may now proceed through this sequence of blowups to obtain the extended (ordered) configuration space

$$\mathcal{E}_k = [D_k; \cup_I \Delta_I].$$

(3)

One consequence of this operation is that the action of symmetric group $\Sigma_k$ on $D_k$ is resolved.

**Proposition 1.** The symmetric group $\Sigma_k$ acts freely on $\mathcal{E}_k$.

**Proof.** The fixed points of $\Sigma_k$ on $D_k$ are precisely the partial diagonals, and moreover, the isotropy group at $\Delta_I$ is a subgroup of the isotropy group at $\Delta_J$ when $\Delta_I \supset \Delta_J$, or equivalently when $\mathcal{I} \subset \mathcal{J}$. Thus our iterative blowup corresponds to the blowup which resolves this group action, define by blowing up the fixed point sets ordered by reverse isotropy type inclusion, and it is not hard to check in this case that the isotropy groups of the lifted
group action are all trivial. This is a special case of a more general iterated blowup considered by Albin and Melrose [1] which resolves a general Lie group action.

The space $E_k$ appears rather complicated at first glance, but the combinatorial structure of its faces mirrors the partially ordered set of subsets $\{I\}$ of $\{1, \ldots, k\}$. For each element $I$ of this set, there is a boundary hypersurface $F_I$ of $E_k$ generated by blowing up $\Delta_I$. We also denote by $\rho_I$ the boundary defining function for this face. Identifying the interior of $E_k$ with the nonsingular part of $M^k$ away from all the diagonals, we see that $\rho_I$ provides a measurement of the radius of a cluster of $|I|$ coalescing points.

2.2 The extended configuration family

We next consider the universal family over $E_k$. This is a space $C_k$ equipped with a $b$-fibration

$$
\beta : C_k \rightarrow E_k,
$$

such that for each $p \in E_k^{\text{reg}}$, the fiber $\beta^{-1}(p)$ is the surface $M$ blown up at the points of $p$. If $p$ lies in one of the boundary faces of $E_k$, this fiber is a union of surfaces with boundary which encode the various ways the corresponding cluster of points can come together.

To define this universal family, we begin with the trivial fibration $E_k \times M \rightarrow E_k$, and let $z$ be a generic point on the fiber, which we may as well assume is a local holomorphic coordinate there. We wish to resolve the graph of the canonical section $\sigma$ of this bundle:

$$
\{(p, z) \in E_k \times M : z \in \sigma(p)\}.
$$

Since $\sigma$ is multi-valued, we must first blow up the crossing loci, which are contained in the graphs of $\sigma$ over the lifted partial diagonals. More specifically, if $p$ lies in a partial diagonal $\Delta_I$, we write $\sigma^T(p)$ for the corresponding coincidence point $p_{i_1} = \ldots = p_{i_r}$, $I = (i_1, \ldots, i_r)$, and then define the coincidence set

$$
\Delta^C_I = \{\rho_I = 0, z = \sigma^T(p)\}
$$

The space $C_k$ may now be defined by iteratively blowing up this collection of submanifolds with respect to the partial order on index sets, culminating at the last step in the blowups of the nonsingular parts of the graph of $\sigma$. 

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i.e., the submanifolds $\Delta^C_i = \{z = p_i\}$, $i = 1, \ldots, k$, where $p$ does not lie in any partial diagonal. Altogether,

$$\mathcal{C}_k = [\mathcal{E}_k \times M; \{\Delta^C_I\}]$$

(5)

Completely analogously to what happens in $\mathcal{E}_k$, we have

**Lemma 2.** The lifts of $\Delta^C_I$ and $\Delta^C_J$ are transverse after $\Delta^C_{I \cup J}$ has been blown up. In particular, the lifts of $\Delta^C_I$ and $\Delta^C_J$ do not meet when $i \neq j$.

**Proof.** As before, this follows from the fact that

$$\Delta^C_I \cap \Delta^C_J = \Delta^C_{I \cup J}$$

when $I \cap J \neq \emptyset$, while $\Delta^C_I$ and $\Delta^C_J$ are transverse away from $\Delta^C_{I \cup J}$ when $I \cap J = \emptyset$. The last assertion is obvious. \qed

### 2.3 The simplest case, $k = 2$

The description of the boundary faces of $\mathcal{E}_k$ and $\mathcal{C}_k$ is somewhat complicated and at first glance confusing, so to warm up, we present the cases $k = 2$ and $3$ in some detail since it is possible to see what is going on without too much work then.

The space of ordered divisors $\mathcal{D}_2$ is simply $M^2$, and there is a single diagonal $\Delta_{12} = \{p_1 = p_2\}$, hence

$$\mathcal{E}_2 = [\mathcal{D}_2; \Delta_{12}].$$

Here and below we keep the subscript $12$ to foreshadow the general case. From local coordinates $(z_1, z_2)$ near $(p_0, p_0) \in \Delta$, we determine the center of mass $\zeta = \frac{1}{2}(z_1 + z_2)$ and displacement $w = \frac{1}{2}(z_1 - z_2)$, so that

$$z_1 = \zeta + w, \quad z_2 = \zeta - w.$$  

The blowup amounts to setting $w = \rho_{12} e^{i\theta}$, $\theta \in [0, 2\pi]$ and adding the face $\rho_{12} = 0$.

The front face $F_{12}$ is then a possibly nontrivial circle bundle over the diagonal. Indeed, it is the unit normal bundle of the diagonal in $M^2$, and hence has Euler characteristic equal to $\chi(M)$. In any case, we have coordinates $(\theta, \zeta)$ on $F_{12}$ and a full set of coordinates $(\rho_{12}, \theta, \zeta)$ near this face in the blowup.
The symmetric group $\Sigma_2$ interchanges the two coordinates $(z_1, z_2)$, and hence sends $\zeta \mapsto \zeta$, $w \mapsto -w$. In local coordinates, $(\rho_{12}, \theta, \zeta) \mapsto (\rho_{12}, \pi + \theta, \zeta)$, and it is easy to see that this is a free action.

The extended configuration family is now obtained from the product $E_2 \times M$ by blowing up in succession the two submanifolds

$$\Delta^C_1 = \{(0, \theta, \zeta, z) : z = \zeta\} \subset F_{12} \times M, \quad \text{and} \quad \Delta^C_1 \cup \Delta^C_2 = \{(p, \sigma(p)) : p \in \mathcal{D}_2\}.$$

For the first of these blowups, introduce spherical coordinates $(R_{12}, \Omega)$ around the codimension three submanifold $\{\rho_{12} = 0, \zeta = z\}$, so $R_{12} \geq 0$ and $\Omega \in S^2_+$. We write

$$\Omega = (\rho_{12}, z - \zeta)/R_{12} = (\sin \omega, \cos \omega e^{i\phi})$$

where $\omega \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$, and so

$$\rho_{12} = R_{12} \sin \omega, \quad z = \zeta + R_{12} \cos \omega e^{i\phi}.$$

We also set $z - \zeta = re^{i\phi}$.

The face created by this blowup, which we call $\mathcal{C}_{12}$, is the total space of a fibration $\pi_{12} : \mathcal{C}_{12} \to F_{12}$, with each fiber a copy of $S^2_+$. The preimage of a point $(0, \theta, \zeta) \in F_{12}$ is the union of two manifolds with boundary: the first is the blowup of $M$ around the point $\zeta$, $[M, \{\zeta\}]$ and the second is $S^2_+$. These meet along their common boundary, which is a circle. From the remarks above,

$$\pi_{12}^{-1}(\rho_{12}) = A R_{12} \omega,$$

where $A$ is a strictly positive smooth function. The significance of this computation is that the lift of the defining function for $F_{12}$ equals the product of defining functions for the fiber $M$ blown up at $\zeta$ and the half-sphere, up to a nonvanishing smooth factor.

We now turn to the second blowup. Consider the graph of $\sigma$,

$$\Delta^C_1 = \{z = z_1\} = \{z = \zeta, \rho_{12}e^{i\theta}\} = \{\omega = \pi/4, \phi = \theta\}$$

$$\Delta^C_2 = \{z = z_2\} = \{z = \zeta - \rho_{12}e^{i\theta}\} = \{\omega = \pi/4, \phi = \theta + \pi\}.$$  

These two components intersect $\mathcal{C}_{12}$ at two disjoint copies of $S^1$. Their intersection with each $\pi_{12}^{-1}(0, \theta, \zeta)$ consists of two points on each $S^2_+$ fiber. In other words, the boundaries of these components are each circles, and there is one
Figure 1: The singular fibration of $C_2 \to \mathcal{E}_2$

point of each of these circles in each $\mathbb{S}_+^2$ fiber. These intersection points are given explicitly in (8). Observe that as $\theta$ goes from 0 to $\pi$, these two points interchange. The final configuration space is equal to

$$C_2 := [\mathcal{E}_2 \times M; \Delta_{12}^C; \Delta_1^C \cup \Delta_2^C]$$

This space is equipped with a $b$-fibration

$$\pi : C_2 \to \mathcal{E}_2.$$ 

Over a regular point $p \notin \Delta_{12}$, the preimage $\pi^{-1}(p)$ is a copy of $M$ blown up at the two points of $p$. On the other hand, the preimage $\pi^{-1}(0, \theta, \zeta)$ of a point on $F_{12}$ is the union of $M$ blown up at the single point $\zeta$ and the half-sphere $\mathbb{S}_+^2$ blown up at two points, $[\mathbb{S}_+^2; \{\omega = \pi/4, \phi = \theta\} \cup \{\omega = \pi/4, \phi = \theta + \pi\}]$.

Note finally that since each $\Delta_i^C$ projects surjectively to $\mathcal{E}_2$, the relation describing the pullback of boundary defining functions stays the same as for the space before this last blowup.

2.4 The case $k = 3$

The next case illustrates the iterative nature of the general construction.

As before, we work in local coordinates $(z_1, z_2, z_3) \in M^3$ near a point $(p_0, p_0, p_0) \in \Delta_{123}$. 

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The initial blowup of the central diagonal \( \Delta_{123} \) in \( \mathcal{D}_3 \) results in the space \([\mathcal{D}_3; \Delta_{123}]\). Just as for \( \mathcal{E}_2 \), this is a manifold with boundary; its boundary, or front face at this first step, is called \( F_{123} \) and is a sphere bundle over \( \Delta_{123} \), but now with three-dimensional spherical fibers. In choosing coordinates, it seems to be more convenient to break the symmetry by using classical Jacobi coordinates (introduced originally to study the \( N \)-body problem in celestial mechanics). Thus we define a center of mass \( \zeta \) in the first two variables, as well as two displacement variables

\[
\zeta = \frac{1}{2}(z_1 + z_2), \quad w_1 = \frac{1}{2}(z_1 - z_2), \quad w_2 = z_3 - \frac{1}{2}(z_1 + z_2).
\]

In these coordinates,

\[
\Delta_{123} = \{ \zeta + w_1 = \zeta - w_1 = \zeta + w_2 \} = \{ w_1 = w_2 = 0 \}.
\]

The resolution is the blowup of the origin in \( \mathbb{C}w_1w_2 \), which is captured by spherical coordinates in the fibers of the normal bundle:

\[
N_\zeta \Delta_{123} \cong \mathbb{R}^4_{w_1,w_2} \ni \rho_{123} \Theta, \quad (\rho_{123}, \Theta) \in \mathbb{R}^+ \times S^3.
\]

To proceed, write

\[
\Theta = (e^{i\phi_1} \cos \theta, e^{i\phi_2} \sin \theta), \quad (\theta, \phi_1, \phi_2) \in [0, \pi/2] \times [0, 2\pi]^2,
\]

corresponding to the fact that \( S^3 \) is a join of two circles, so that

\[
w_1 = \rho_{123} \cos \theta e^{i\phi_1}, \quad w_2 = \rho_{123} \sin \theta e^{i\phi_2}.
\]

These coordinates are singular at \( \theta = 0, \pi/2 \). To remedy this near \( \theta = \pi/2 \), for example, we use instead the projective coordinate \( \bar{w}_1 = w_1/\rho_{123} \) along with \( \phi_2 \).

In these new coordinates, the lifts of the partial diagonals in a neighborhood of \( \Delta_{123} \) have the form

\[
\Delta_{12} = \{ z_1 = z_2 \} = \{ w_1 = 0 \} = \{ \theta = \pi/2 \} = \{ \bar{w}_1 = 0 \}
\]

\[
\Delta_{13} = \{ z_1 = z_3 \} = \{ w_1 = w_2 \} = \{ \theta = \pi/4, \phi_1 = \phi_2 \}
\]

\[
\Delta_{23} = \{ z_2 = z_3 \} = \{ w_1 = -w_2 \} = \{ \theta = \pi/4, \phi_1 = -\phi_2 \}.
\]

The additional expression for the lift of \( \Delta_{12} \) is included because \( \theta = \pi/2 \) is a singular locus for the \( (\theta, \phi_1, \phi_2) \) coordinate system. Each of these lifts is
locally a product $S^1 \times \mathbb{R}^+$ near $S^3 \times \{0\}$, and their intersections with each fiber of $F_{123}$ are three disjoint circles. We then blow these up in any order to obtain

$$E_3 = [D_3; \Delta_{123}; \Delta_{12} \cup \Delta_{13} \cup \Delta_{23}]$$

There are three new front faces, $F_{ij}$, each with a boundary defining function $\rho_{ij}$, and each (locally) diffeomorphic to $S^1 \times S^1 \times \mathbb{R}^+$. The intersection $F_{ij} \cap F_{123}$ is a torus $S^1 \times S^1$.

Let us illustrate this geometry near $F_{12} \cap F_{123}$ in coordinates. From (9) and (10), $\varphi_2 \in S^1$ and $\rho_{123} \in \mathbb{R}^+$ parametrize $\Delta_{12}$, while we set $\tilde{w}_1 := w_1/\rho_{123}$ as a coordinate for the normal bundle. Blowing up at $\tilde{w}_1 = 0$ amounts to writing $\tilde{w}_1 = \rho_{12} e^{i\theta_{12}}$. This is of course really the same as setting $\rho_{12} = \cos \theta, \theta_{12} = \varphi_1$, but the new monikers have been introduced to conform with the general notation when $k > 3$. Altogether, we have a full set of local coordinates

$$((\zeta, \rho_{12}, \theta_{12}, \varphi_2, \rho_{123}) \in \mathbb{R}^2 \times \mathbb{R}^+ \times S^1 \times \mathbb{R}^+), \quad (11)$$

These are related to the original coordinates by

$$z_1 = \zeta + \rho_{123}\rho_{12} e^{i\theta_{12}}, \quad z_2 = \zeta - \rho_{123}\rho_{12} e^{i\theta_{12}}, \quad z_3 = \zeta + \rho_{123} \sqrt{1 - \rho_{12}^2} e^{i\varphi_2}. \quad (12)$$

Now consider the configuration space $\tilde{C}_3 = \tilde{E}_3 \times M$. Near $(F_{12} \cap F_{123}) \times M$ we extend the local coordinates (11) to

$$((\zeta, \rho_{12}, \theta_{12}, \varphi_2, \rho_{123}, z) \in \mathbb{R}^2 \times \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^2), \quad (13)$$

where $z$ is a local coordinate on $M$. The coordinate $\zeta$ is sometimes omitted below. The first step is to resolve the ‘central’ coincidence set, cf. (11),

$$\Delta_{123}^C = \{z - \zeta = \rho_{123} = 0\},$$

which we do by introducing spherical coordinates $(z - \zeta, \rho_{123}) = R_{123}\Omega_{123}$, where $R_{123} \geq 0$ and

$$S^2_+ \ni \Omega_{123} = (\cos \omega e^{i\phi}, \sin \omega), \quad \omega \in [0, \pi/2], \quad \phi \in S^1. \quad (14)$$

This introduces the front face $\mathcal{C}_{123} = \{R_{123} = 0\}$. Coordinates at this stage are

$$((\zeta, R_{123}, \Omega_{123}, \rho_{12}, \theta_{12}, \varphi_2) \in \mathbb{R}^2 \times \mathbb{R}^+ \times S^2_+ \times \mathbb{R}^+ \times S^1 \times S^1).$$
Note also that

\[ z_1 - \zeta = -(z_2 - \zeta) = R_{123} \sin \omega \rho_{12} e^{i\theta_{12}}, \]
\[ z_3 - \zeta = R_{123} \sin \omega \sqrt{1 - \rho_{12}^2} e^{i\phi_2}, z - \zeta = R_{123} \cos \omega e^{i\phi}. \]  

(15)

As before, the coordinates \((\omega, \phi)\) become degenerate at \(\omega = \pi/2\), so in a neighborhood of this locus we replace these by the projective coordinate \(\tilde{z} = (z - \zeta)/R_{123}\).

Now fix a point \(q = (\rho_{12}, \theta_{12}, \phi_2)\) on the front face \(F_{123}\) in the base \(E_3\). The fiber above \(q\) after the blowup above is a union of \(\mathbb{R}^2\) blown up at the origin (or more globally, the surface \(M\) blown up at the point \(\zeta\)) and a half-sphere, parametrized by the coordinates \((\omega, \phi)\) (or \(\tilde{z}\)) above. These meet along the circle \(\{\omega = 0\}\). If \(\pi_3\) is the blowdown map, then, analogous to (7),

\[ \pi_3^{\ast} \rho_{123} = R_{123} \sin \omega = AR_{123} \omega \]  

(16)

for some smooth nonvanishing function \(A\).

The partial coincidence sets \(\Delta_{ij}^C\) intersect the fiber over \(q\) only in the interior of the hemisphere. In local coordinates,

\[ \Delta_{12}^C = \{z = z_1 = z_2\} = \{\rho_{12} = 0, \omega = \pi/2\} = \{\tilde{w}_1 = 0, \tilde{z} = 0\} \]
\[ \Delta_{13}^C = \{z = z_1 = z_3\} = \{\rho_{12} = \sqrt{2}/2, \tan \omega = \sqrt{2}, \theta_{12} = \phi_2 = \phi\} \]
\[ \Delta_{23}^C = \{z = z_2 = z_3\} = \{\rho_{12} = \sqrt{2}/2, \tan \omega = \sqrt{2}, \theta_{12} + \pi = \phi_2 = \phi\} \]  

(17)

The final expression for the lift of \(\Delta_{12}^C\) is included because of the degeneracy of the other coordinate system at \(\rho_{12} = 0, \omega = \pi/2\).

Now blow up \(\Delta_{12}^C\). This introduces a new front face \(C_{12}\), the fibers of which (for each \((\zeta, q)\)) are hemispheres “on top” of the previous hemispheres. Carrying out the analogous blowup for the two other partial diagonals as well leads to the collection of front faces \(C_{ij}\), each diffeomorphic to \(\mathbb{R}^+ \times S^1 \times S^1 \times S^2_+\). Each \(C_{ij}\) fibers over the front face \(F_{ij}\) in the base \(E_3\) by projecting off the final \(S^2_+\). Again, this is very similar to the two-point case.

Let \(\Omega_{12} \in S^2_+\) be the variable in the fiber of \(C_{12}\) and \(R_{12}\) the corresponding boundary defining function. Thus

\[ (\cos \omega e^{i\phi}, \rho_{12}) = R_{12} \Omega_{12} \]

and

\[ \Omega_{12} = (\cos \omega_{12} e^{i\phi_{12}}, \sin \omega_{12}), \ \omega_{12} \in [0, \pi/2], \ \phi_{12} \in S^1. \]  

(18)
Figure 2: One of the singular fibers in $\mathcal{C}_3$, where two of the points collide faster than the third one

Coordinates near $\mathcal{C}_{12}$ are then

$$(\zeta, R_{123}, R_{12}, \Omega_{12}, \theta_{12}, \phi_2) \in \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \times S^2_+ \times S^1 \times S^1,$$

and these translate to the original coordinates by

$$z_1 - \zeta = -(z_2 - \zeta) = R_{123} \sqrt{1 - (R_{12} \cos \omega_{12})^2} R_{12} \sin \omega_{12} e^{i\theta_{12}},$$

$$z_3 = R_{123} \sqrt{1 - (R_{12} \cos \omega_{12})^2} \sqrt{1 - (R_{12} \sin \omega_{12})^2} e^{i\phi_2},$$

$$z - \zeta = R_{123} R_{12} \cos \omega_{12} e^{i\phi_{12}}. \quad (19)$$

The boundary defining function relation near the corner $\mathcal{C}_{123} \cap \mathcal{C}_{12}$ is given by

$$\pi^*_3 \rho_{123} = R_{123} \sqrt{1 - (R_{12} \cos \omega_{12})^2} \sim R_{123},$$

$$\pi^*_3 \rho_{12} = R_{12} \sin \omega_{12} \sim R_{12} \omega_{12}. \quad (20)$$

We conclude by blowing up the lifts of the three submanifolds $\Delta_i^\mathcal{C} = \{z = z_i\}$. In a generic fiber over $F_{123}$, (i.e., away from all the partial diagonals), $\Delta_i^\mathcal{C}$
meets the front face $E_{123}$ on the hemisphere at three distinct points. Using the coordinates (15),

\[
\begin{align*}
\Delta_1^C &= \{ \cot \omega = \rho_{12}, \theta_{12} = \phi \} \\
\Delta_2^C &= \{ \cot \omega = \rho_{12}, \theta_{12} = \phi + \pi \} \\
\Delta_3^C &= \{ \cot \omega = \sqrt{1 - \rho_{12}^2}, \phi_2 = \phi \}.
\end{align*}
\]  

On approach to the resolved partial diagonals, two of the points $z_i$ converge to one another on $M$. However, their lifts converge to distinct points on a hemispherical fiber of the innermost front face. In other words, $\Delta_1^C$ and $\Delta_2^C$ both intersect $C_{ij}$ while if $k$ is the third value distinct from $i, j$, then $\Delta_k^C$ does not. In terms of the coordinates (19), we have

\[
\begin{align*}
\Delta_1^C &= \{ \omega_{12} = \pi/4, \theta_{12} = \phi_{12} \} \\
\Delta_2^C &= \{ \omega_{12} = \pi/4, \theta_{12} = \phi_{12} + \pi \} \\
\Delta_3^C &= \{ \cot \omega = \sqrt{1 - \rho_{12}^2}, \phi_2 = \phi_{12} \}.
\end{align*}
\]  

The roles of $z_1$, $z_2$ and $z_3$ are interchangeable in this whole discussion, even though the Jacobi coordinates $w_1$ and $w_2$ break this symmetry. It is also not hard to check that the symmetric group $\Sigma_3$ acts freely on $C_3$.

As this case makes clear, the geometry illustrated in Figure 2 requires compound singular coordinate transformations, which quickly become quite involved. The optimistic interpretation is that the compound asymptotics of solutions to natural elliptic operators which appear later in this paper are captured entirely by the intricate but still quite comprehensible geometry of this iterated blowup.

### 2.5 Boundary faces of $E_k$ and $C_k$

The previous discussion makes clear that while it is possible to write out the iterated polar coordinate systems corresponding to the iterated blowups, it becomes prohibitively complicated after a few steps. We now discuss some general features of these blowups which should make their structure more apparent. In fact, we shall examine the geometry of the boundary faces of $E_k$ and $C_k$ more carefully, both invariantly but to a lesser extent using local coordinates. For simplicity, we work with the space of ordered divisors, but omit the tildes, and at the end pass to the quotient by $\Sigma_k$.

We first describe the interiors of each face of $E_k$. Thus fix any subset $I$ of $\{1, \ldots, k\}$, and set $\ell = |I|$. Recall that $\Delta_I^0$ is the open dense subset of $\Delta_I$
where only the points $p_i, i \in I$, coincide. Now consider the blowup of $\Delta_I$ over this subset:

$$[M_k^I; \Delta_0^I],$$

where $M_k^I$ is the subset of $M^k$ where we omit all partial diagonals $\Delta_0^J$ where $J \cap I \neq \emptyset$. The normal bundle of $\Delta_0^I$ in $M_k^I$ is naturally identified with $(TM)^{\oplus (\ell-1)}$ (over a suitable open dense subset of $M$), hence the front face $F_k^I$ of this blowup is a bundle over $M^k-\ell+1 \cong \Delta_I$ with fiber $S^{2\ell-3}$.

In local coordinates, assume that $I = \{1, \ldots, \ell\}$ and write $(z_1, \ldots, z_k)$ as the sum of a center of mass $\zeta^I(z')$ and $w' \in C^{[I]-1}$, where $\sum_{i \in I} w_i = 0$, so $w'$ lies in a space of real dimension $2|I| - 2$. The blowup is in $w'$ only, and we see once again that the new front face is fibered by copies of $S^{2\ell-3}$.

We next consider how these faces meet one another. The initial step in the construction of $E_k$ is the blowup of $\Delta_1 \ldots k$, and in this case the spherical fiber of $F_1 \ldots k$ has maximal dimension. Now suppose we are at the step where all partial diagonals corresponding to $J$ with $|J| > \ell$ have been blown up, and suppose that $|I| = \ell$. Then for any $J \supset I$, the lift of $\Delta_I$ intersects the fibers of $F_J$ in spheres of dimension $2\ell - 3$, as above. Proceeding through all blowups in the construction of $E_k$, we see that the fibers of $F_1 \ldots k$ in $E_k$ are spheres blown up iteratively at the entire collection of great subspheres arising from these other partial diagonals, and these subspheres are all odd-dimensional.

The key to understanding the extended configuration space $E_k$ and the extended configuration family $C_k$ is that each point $p$, respectively $(p, z)$, has a neighborhood where the points admit a well-defined decomposition into clusters. Each cluster has a center of mass, and this collection of centers of mass can be treated as a divisor with less than $k$ elements, while each cluster itself is a configuration with less than $k$ points. This is the inductive structure of the overall construction.

We describe a small neighborhood of a point on $\Delta_1 \ldots k$ which does not lie on any of the partial diagonals. To reiterate part of what was said above, there is a fibration $[M^k; \Delta_1 \ldots k] \to M$ with fiber $[M^{k-1}; p' = \vec{p}']$ (here $p' = (p_1, \ldots, p_{k-1})$ and $\vec{p}' = (p, p, \ldots, p)$ - $(k-1)$ times), i.e., $M^{k-1}$ blown up at a single point of $\Delta_1 \ldots k-1$. The normal bundle to $\Delta_1 \ldots k$ in $M^k$ is naturally isomorphic to $(TM)^{\oplus (k-1)}$, and $\Delta_1 \ldots k \cong M$. The boundary of $[M^k; \Delta_1 \ldots k]$ is the front face $F_1 \ldots k$, which is also identified with the boundary of the blowup.
of \((TM)^{(k-1)}\) around the 0-section. Thus \(F_{1...k}\) is a bundle over \(M\) with each fiber a copy of \(S^{2k-3}\).

It is now convenient to restrict to a fiber \([M^{k-1}; \bar{p}']\) at \(\bar{p} \in \Delta_{1...k}\), which we write as \(H_{k,\bar{p}}\) for convenience, and then restrict the universal space \([M^k; \Delta_{1...k}] \times M\) to \(H_{k,\bar{p}} \times M\). This is tantamount to removing the center of mass. The restrictions of various quantities below to this slice are denoted by a superscript ‘0’.

The front face \(C_{01...k}\) arising from the blow up \(\{\rho_1...k = 0, z = p\}\) in \(H_{k,\bar{p}} \times M\) is a fibration over \(F_{01...k}\) with half sphere fiber “attached” to the regular part at \(z = p\). Restricted to a neighborhood of a boundary point in \(H_{k,\bar{p}}\) disjoint from the lift of any partial diagonal, the lifts of the sections \(\Delta_i^c = \{z = p_i\}\) are (locally) disjoint. Blowing these up, then each new front face \(C_i^0\) is a product of \(\Delta_i^0\) and a circle, and \(C_i^0\) intersects \(C_i^0\) in a circle. Therefore each slice \(C_{01...k}\) is a hemisphere blown up at \(k\) distinct points, the positions of which depend on the base point in the boundary of the extended configuration space. This is the generic picture, and the description here is a ‘depth 1 blowup’. A description of neighborhoods near points where the lifted partial diagonals intersect \(F_{01...k}\) requires a more careful inductive description. In other words, recalling the cluster description above, the geometry of the front faces is similar, while internal to each cluster, the partial diagonals are blown up iteratively.

Consider any point \(p\) in \(E_k\), and consider all the boundary faces \(\{F_{\mathcal{I}}\}\) that goes through \(p\). The indices of those front faces is a tree structure defined as below.

**Definition 1.** A tree \(T\) is a subset of the power set of \([1, \ldots, k]\) such that for any elements \(\mathcal{I}, \mathcal{J} \in T\), one and only one of the following happens: (1) \(\mathcal{I} \subset \mathcal{J}\), (2) \(\mathcal{J} \subset \mathcal{I}\), or (3) \(\mathcal{I} \cap \mathcal{J} = \emptyset\). Each element in \(T\) is called a node. Any node which does not have any of its subset contained in \(T\) is called a leaf. The set of all possible trees is denoted by \(\mathfrak{T}\).

**Lemma 3.** For any point \(p \in E_k\), consider all the boundary faces \(F_{\mathcal{I}}\) that contains \(p\), and let \(T\) be the set of all such indices \(\mathcal{I}\). Then \(T\) is a tree.

**Proof.** In the construction of \(E_k\), if \(\mathcal{I} \cap \mathcal{J} \neq \emptyset\) and neither is contained in the other, then the intersection of the two diagonals \(\Delta_{\mathcal{I} \cap \mathcal{J}} = \Delta_{\mathcal{I}} \cap \Delta_{\mathcal{J}}\) (which is strictly contained in \(\Delta_{\mathcal{I}}\) and \(\Delta_{\mathcal{J}}\)) is blown up before \(\Delta_{\mathcal{I}}\) and \(\Delta_{\mathcal{J}}\), and the front faces \(F_{\mathcal{I}}\) and \(F_{\mathcal{J}}\) lift to be disjoint. This proves that the only
Figure 3: A tree structure gives information of clustering bubbles

possible index sets that can have nonempty intersection at \( p \) need to satisfy the condition in Definition 1.

Note that the elements in a tree \( T \) can be placed into a nesting structure based on inclusion, where the top level nodes are mutually disjoint index sets, and on the next level under each node there is a sub-tree structure whose nodes are subsets of the node one level above. The definition of the tree structure will help us to describe the boundary faces and corners in \( \mathcal{E}_k \) inductively. The nodes under each node form a tree, which reflects the iterative structure of corners in \( \mathcal{E}_k \) and \( \mathcal{C}_k \). Given a point \( p \) with its associated tree \( T \), we can describe the fiber above \( p \). The half-sphere bubbles in this fiber are labeled exactly by the nodes in \( T \). For the top level nodes, each of them correspond to a bubble attached directly to \( M \) and they are disjoint from each other; whereas the bubbles corresponding to lower level nodes are attached to the bubble which corresponds to the node directly one level above. Hence, for a given fiber \( \pi_k^{-1}(p) \), there is a one-to-one correspondence between the nodes of \( T \) and boundary faces of this fiber. And the leaves of this tree correspond to the deepest (highest codimensional) boundary faces. And for each node, the subset obtained by removing all the elements contained in the nodes of the lower level corresponds to the free points.

Different operations of the tree structure correspond to moving in different directions on \( \mathcal{E}_k \). Removing one top node to get a new tree corresponds to the action of moving off from the boundary face with the highest codimension. On the other hand, removing all the subtrees and keeping only the top level nodes corresponds to the most crude cluster decomposition of a neighborhood.
of \( p \).

It is useful to introduce the following set of open covers defined by clustering. Let \( \Pi_k : \mathcal{E}_k \to M^k \) be the blow down map, then for any \( \epsilon > 0 \) and index sets \( \{ \mathcal{I}_j \}_{j=1}^\ell \) such that \( \{1, \ldots, k\} = \bigcup_{j=1}^\ell \mathcal{I}_j \), and \( \mathcal{I}_j \cap \mathcal{I}_i = \emptyset, \forall i \neq j \), we define the following open set

\[
\mathcal{E}_k \supset U_{\mathcal{I}_*, \epsilon} := \prod_{i}^{\ell} \{(p_1, \ldots, p_k) \in M^k : d(p_{i'}, p_{j'}) > \epsilon, \text{ if } i' \in \mathcal{I}_i, j' \in \mathcal{I}_j, \forall i \neq j\}. \tag{23}
\]

This set defines those configurations that the only merging happens within each cluster. And away from the central front face, for any point \( p \in \mathcal{E}_k \setminus F_1, \ldots, k \), there exists an open set such that \( p \in U_{\mathcal{I}_*, \epsilon} \). In fact, take its associated tree \( T \) and denote the top level elements of \( T \) by \( \mathcal{I}_1, \ldots, \mathcal{I}_\ell' \). If \( \bigcup_{j=1}^{\ell'} \mathcal{I}_j \neq \{1, \ldots, k\} \), let \( \ell = \ell' + 1 \) and \( \mathcal{I}_\ell = \{1, \ldots, k\} \setminus \bigcup_{j=1}^{\ell'} \mathcal{I}_j \) (those are the free cone points); otherwise let \( \ell = \ell' \). Then we have

**Lemma 4.** There exists \( \epsilon > 0 \) such that \( p \in U_{\mathcal{I}_*, \epsilon} \).

**Proof.** From the definition of \( T \), \( p \) is lifted from the intersection of diagonals \( \{\Delta_{\mathcal{I}_j}\}_{j=1}^{\ell-1} \). This implies that the clustering only happens inside each index set \( \mathcal{I}_j \). And since \( p \notin F_1, \ldots, k \), we have at least two clusters, or equivalently \( \ell \geq 2 \). Let \( 2\epsilon \) be the minimal distance between the \( \ell \) clusters, it is easy to see that \( p \) is contained in the open set \( U_{\mathcal{I}_*, \epsilon} \) where each cluster is at least \( \epsilon \) away from each other.

Note that the sublevel elements in the tree give finer clusterings, which is certainly compatible with the decomposition above. \( \square \)

With the cluster decomposition, we can describe the iterative behavior of the configuration family \( \mathcal{E}_k \) and \( C_k \). In particular, in the base of this fibration there is a product decomposition into boundary faces of lower dimensions. A similar structure can be seen in the fibration.

**Lemma 5.** (1) [Product decomposition of base manifold] Given \( p \) and its associated neighborhood \( U_{\mathcal{I}_*, \epsilon} \) given in Lemma 4, there exists a neighborhood \( p \in U \) such that

\[
U_{\mathcal{I}_*, \epsilon} \supset U = \prod_{i=j}^{\ell} U_j, \; U_j \subset \mathcal{E}_{|\mathcal{I}_j|}.
\]
(2) [Vertical decomposition of fibers] For the fibers above \( \pi_k^{-1}(U) \) with \( U \) given above, there exists \( \ell \) open sets \( V_i \subset C_k \) with \( \bigcup_{i=1}^\ell V_i = \pi_k^{-1}(U) \), and in each open set \( V_i \), the fibration \( \pi_k \) decomposes as a product

\[
(\pi_{|I_i|} : \tilde{V}_i \to U_i) \times \Pi_{j \neq i} U_j
\]

with \( \tilde{V}_i \subset C_{|I_i|} \).

(3) [Free point region] For any point \((z, p) \in \pi_k^{-1}(p)\) with its associated node \( J \in \mathcal{T} \) such that the point is in the interior of a boundary face, let \( \mathcal{J}' = \mathcal{J} \setminus \bigcup_{J \supset I \in \mathcal{T}} I \) be the set of free points in \( \mathcal{J} \). If \( \mathcal{J}' \neq \emptyset \), then there exists a neighborhood of \( z \) with product decomposition

\[
\pi_k^{-1}U_{\mathcal{I}_z,\epsilon} \supset V_C \times V_E, V_C \subset C_{|\mathcal{J}'|}, V_E \subset \mathcal{E}_{k-|\mathcal{J}'|}.
\]

(4) If \( \mathcal{J}' \) defined above is empty, then there is a neighborhood decomposition

\[
(z, p) \in V_C \times V_E, V_C \subset M, V_E \subset \mathcal{E}_k.
\]

Proof. For the proof of (1), the decomposition essentially separates points into independent clusters. If \( p \in U_{\mathcal{I}_z,\epsilon} \), then there is a neighborhood of \( p \) in \( \mathcal{E}_k \) which does not intersect with preimages of any diagonals except \( F_{\mathcal{I}_z}, \mathcal{I} \subset \mathcal{I}_z \) for some \( i \). And in this neighborhood all the possible blow up occurs inside each cluster, so one can write \( p = (p_1, \ldots, p_\ell) \), with \( p_j \in E_{\mathcal{I}_z} \).

For property (2), we just need to notice that the clusters are separated with mutual distance uniformly larger than \( \epsilon \) for each fiber above \( U_{\mathcal{I}_z,\epsilon} \). Therefore such a decomposition of neighborhood exists. When restricting to each neighborhood \( V_j \), only \( p_i, i \in \mathcal{I}_j \) are included therefore locally there is a splitting of fibration such that the lift of projection \((z, p) \to p\) restricts to the lift of \( \{(z, p_I) \to p_{I_j}\} \times p_{\{1, \ldots, k\} \setminus \mathcal{I}_j} \).

The decomposition in (2) can be even refined when moving deeper inside a tree. To prove (3) and (4), take a point \((z, p) \in C_k \) and let \( \mathcal{C}_{\mathcal{J}} \) be the boundary face of which \((z, p)\) is an interior point. By the definition of the tree, the only boundary faces intersecting \( \mathcal{C}_{\mathcal{J}} \) in this fiber correspond to the nodes \( \mathcal{I} \) under \( \mathcal{J} \). And since \( z \) is in the interior of a boundary face, there is a neighborhood which does not intersect any of these \( \mathcal{C}_{\mathcal{I}} \). If \( \mathcal{J}' \neq \emptyset \), then there are \(|\mathcal{J}'|\) free points contained in this region, hence only \( \mathcal{C}_i, i \in \mathcal{J}' \) has intersection with this neighborhood; otherwise there are no free points in this neighborhood and it does not contain any other boundary faces. In the first case, one
can write the base variable \( \mathbf{p} = (\mathbf{p}_{\mathcal{J}'}, \mathbf{p}_{\mathcal{J}''}) \) with \( \mathcal{J}'' = \{1, \ldots, k\} \ \setminus \mathcal{J}' \). Only \( z = p_i, i \in \mathcal{J}' \) has nonempty intersection with this neighborhood, hence given by a neighborhood of \( \mathcal{C}_{|\mathcal{J}'|} \) which does not intersect any partial diagonals. And other \( \mathbf{p}_{\mathcal{J}''} \in \mathcal{E}_{k-|\mathcal{J}'|} \) serves as the rest of base variables. In the case of (4), there are no free points in this neighborhood, hence the fiber is just a neighborhood of \( M \). Under the same reasoning as before we obtain the product decomposition.

The lemma above implies that the geometry of \( \mathcal{C}_k \) can be described as a product of lower dimensional spaces, except the boundary of boundary faces (corners) of which description is contained in the following result.

**Proposition 2.** The natural projection \( \pi_k : \mathcal{C}_k \to \mathcal{E}_k \) is a \( b \)-fibration.

**Proof.** We prove this by induction. Note that it suffices to restrict to the slices \( \mathcal{H}_{k,p} \).

When \( k = 1 \), \( \mathcal{C}_1 \) is the blowup of \( M \times M \) around the diagonal, \( \mathcal{E}_1 = M \), and \( \mathcal{H}_{1,p} \) is the blowup \( [M; \{p\}] \). The restriction of \( \pi_1 \) to \( \mathcal{H}_{1,p} \) is just the blowdown map \( [M; \{p\}] \to \{p\} \), so the result is obvious.

We have already written the explicit relationship between boundary defining functions in the case \( k = 2 \) in (7), and this proves the result in that case too.

Now suppose \( \mathcal{C}_\ell \to \mathcal{E}_\ell \) is a \( b \)-fibration for any \( \ell < k \). Our goal is to show that the pullback \( \pi_k^* \rho_I \) of any boundary defining function in \( \mathcal{E}_k \) is a product of boundary defining functions in \( \mathcal{C}_k \), up to a factor of a nonvanishing smooth function. If \((z, \mathbf{p})\) is an interior point of a boundary face, then by the statement in (3) and (4) of the previous lemma, the fibration locally is given by the product of

\[
(\pi_{|\mathcal{J}'|} : \mathcal{C}_{|\mathcal{J}'|} \to \mathcal{E}_{|\mathcal{J}'|}) \times \mathcal{E}_{k-|\mathcal{J}'|}.
\]  

(24)

If \( |\mathcal{J}'| = k \), then the tree \( T \) associated to \((z, \mathbf{p})\) is given by exactly one node, and only one boundary face \( F_{1...k} \) contains \( \mathbf{p} \). Also note that \((z, \mathbf{p})\) is in the interior of the boundary face \( \mathcal{E}_{1,...,k} \) away from all other sub-diagonals, therefore by a similar argument as in the case \( k = 2 \) and 3, we have

\[
\pi_k^* \rho_{1,...,k} = AR_{1,...,k},
\]

where \( R_{1,...,k} \) is the boundary defining function for \( \mathcal{E}_{1,...,k} \). If \( |\mathcal{J}'| < k \), then \( \pi_{|\mathcal{J}'|} \) in (24) is a \( b \)-fibration by induction assumption. Hence the boundary defining functions \( \rho_I \) with \( I \subset \mathcal{J}' \) pulls back to a product of boundary
defining functions in $C_{|J|}$ (up to a nonvanishing smooth function) and other boundary defining functions $\rho_I, I \not\subset J'$ pull back trivially.

On the other hand, if $(z, p)$ is on the boundary of a boundary face, let $J$ be the node associated to this point and $I \supset J$ be the node directly above (if there is nothing above then formally we denote $R_I = w$ where $w$ is the boundary defining function of the original punctured surface). Then using (1) of the previous lemma, locally there is a product decomposition such that the points associated to $I$ is separated from others and we have

$$(\pi_{|I|} : C_{|I|} \to E_{|I|}) \times E_{k-|I|}$$

And a similar relation as in (31) is given:

$$\pi^* \rho_J = AR_J w_I, \quad \pi^* \rho_{J'} = R_{J'}, J' \neq J \quad (25)$$

where $R_J$ is the boundary defining function of $C_J$ and $w_I$ is the boundary defining function of $C_I$ in that corner. And other boundary defining functions pull back trivially because of the product decomposition.

\[ \square \]

3 Geometry of merging cone points

3.1 Review of results

We study constant curvature metrics with conical singularities, which is defined by the following: a smooth metric on $M \setminus p$ with constant curvature, and near the punctures the metric is asymptotically conical, that is, there exist local coordinates such that the metric is given by

$$e^{\phi}|z|^{2(\beta-1)}|dz|^2$$

with $\phi$ being smooth. There is also a geodesic coordinate description. In particular, in each case with curvature of different signs, asymptotically it is given by the flat metric.

The central problem to study is: given $(p, \vec{\beta})$ on $M$, does there exist constant curvature conical metrics, and is it unique. The study of this singular uniformization problem has a long history and has been very active recently. When the curvature $K$ is nonpositive, the conclusion is relatively straightforward. By the results of McOwen [14] and Luo–Tian [11], for any fixed $(c, p, \vec{\beta}, K, A)$ that satisfy the Gauss–Bonnet formula and $K \leq 0$, there exists
a unique constant curvature conical metric prescribed by the tuple. When $K > 0$, the situation is complicated depending on the cone angles. When all the cone angles are smaller than $2\pi$, by the results of Troyanov [17] and Luo–Tian loc. cit., there is a unique constant curvature metric when the cone angles are in the “Troyanov region” [2]. And when all the cone angles are less than $2\pi$, there is a moduli space structure, obtained by the first author and Weiss.

When some of the cone angles are bigger than $2\pi$, there is no uniform result as in the previous cases. When the genus of $M$ is greater than 0, general existing results by Carlotto–Malchiodi [2, 3], Bartolucci–De Marchis–Malchiodi [4]. When $M$ is a sphere, there are some results depending on the number of cone points. When $k = 2$, Troyanov [18] gave the results. When $k = 3$, the characterization via complex analytical methods was given by Eremenko [6] and Umehara–Yamada [19]. When $k = 4$ with symmetry, complex analysis techniques can also be applied, see Eremenko–Gabrielov–Tarasov [8].

Recently, the breakthrough of Mondello–Panov shows the necessary condition of existence on a sphere by the following holonomy condition:

$$d_1(\vec{\beta}, Z_{\text{even}}) \geq 1. \tag{26}$$

They also showed that when the inequality holds (“non-coaxial” situation) there exists at least such a metric. The recent results by Dey [5], Kapovich [9], and Eremenko [7] determined the necessary condition of existence when the equality holds in (26).

When trying to extend the result of [13] to get a smooth manifold structure in this case, we found that there are obstructions, reflected in the fact that the linearized operator fails to be surjective anymore. A key component in the construction of Mondello and Panov is the splitting of cone points, which turns out to be the key to resolve the analytic obstruction. We are going to describe the geometry of this process (splitting, or equivalently, merging of cone points) in the part below.

### 3.2 Local geometry of merging cone points

In the next three sections, we consider the behavior of the constant curvature metrics with some of the cone angles merging together (or equivalently, when a cone splits into several cones). The cases we are going to study in this paper
include all hyperbolic and flat metrics, and spherical metrics with angles less than $2\pi$.

We first describe this process locally by looking at the following family of metric parametrized by $t \in [0, \epsilon)$

$$g(t) = |z - p_1(t)|^{2(\beta_1-1)}|z - p_2(t)|^{2(\beta_2-1)}|dz|^2. \quad (27)$$

where $p_1(t)$ and $p_2(t)$ are smooth coordinates that parametrize two moving points on $M$. We also require $p_1(0) = p_2(0)$. When $t \neq 0$, it gives a metric with two separated cone points with angles $2\pi\beta_1$ and $2\pi\beta_2$. And the distance between the two cone points decreases as $t$ goes to 0. Eventually in the limit $t = 0$, the metric is given by

$$|z|^{2(\beta_1+\beta_2-2)}|dz|^2$$

which is a conic metric of with a single cone angle $2\pi(\beta_1+\beta_2-1)$ if $\beta_1+\beta_2-1 > 0$.

This process of merging two cone points can be generalized to multiple points. After merging $j$ points with angle $2\pi\beta$, the angle we get is given by the following defect formula:

$$2\pi\beta_0 := 2\pi \left( \sum_{i=1}^{j} \beta_i - (k - 1) \right). \quad (28)$$

One thing to notice from the defect formula is that not all conic points can be merged; it only happens when the "admissible condition" below is satisfied, otherwise there is an obstruction to produce a new conic points. When $\beta < 0$, $|z|^{2|\beta|-1}|dz|^2$ is no longer conical, and we get some open ends, the form of which depend on the curvature constant. Therefore, we define the following condition for merging:

**Definition 2.** We say a set of cone angles $\{\beta_i\}_{i \in \mathcal{I}}$ is **admissible** if

$$\sum_{i \in \mathcal{I}} \beta_i > |\mathcal{I}| - 1. \quad (29)$$

In particular, this implies that when $k = 2$, the two cone points need to satisfy $\beta_1 + \beta_2 > 1$.
3.3 Global geometry: angle constraints

The metrics we are going to consider in this paper are the following:

- Flat or hyperbolic conical metrics with $\vec{\beta} \in \mathbb{R}^k_+$ such that the Gauss–Bonnet formula (1) is satisfied.

- Spherical conical metrics with $\vec{\beta} \in (0,1)^k$ satisfying (2) if $k \geq 3$ or $\text{genus}(M) > 0$; or $\beta_1 = \beta_2$ if $k = 2$ and $M = S^2$.

From [17, 11, 14], there is a unique constant curvature metric for each of the configuration $(p, \vec{\beta})$ with $\vec{\beta}$ satisfying the condition above. Now considering the family of metrics with varying cone points, We would like to understand the uniform behavior of the metrics when some of cone points merge together. Using the defect formula we can see that

$$\beta_0 - 1 = \sum_{i=1}^{j} (\beta_i - 1).$$

Together with the Gauss–Bonnet formula, this implies that the curvature remains the same in this merging process.

Because of the restriction of cone angles defined above, not all cones can be merged to produce new cones. Moreover, the Troyanov constraint for spherical metrics is not preserved in this merging process. Therefore, for a given set of cone angles $\vec{\beta}$, the fiber conical metrics might potentially only be defined in a subset of $C_k$. And we define the following admissible region.

**Definition 3.** For fixed $\vec{\beta} = \{\beta_i\}_{i=1}^k$ given in 3.3, the admissible extended configuration space $E_{k,\vec{\beta}}$ is defined to be the union of configurations in $E_k$ with which there exists a constant conical curvature metric on $M$, i.e.

$$E_{k,\vec{\beta}} := E_k^0 \bigcup_{\sum_{i \in I_j} \beta_i > |I_j| - 1, \forall j} \{ p \in \cap_{j=1}^l F_{I_j} : \text{there exists a constant curvature metric with configuration } (p, \{ \sum_{i \in I_j} (\beta_i - 1) + 1 \}_{j=1}^l) \}. \quad (30)$$

The admissible extended configuration family $C_{k,\vec{\beta}}$ is defined in a similar way by only considering the admissible combinations, or equivalently $C_k \supset C_{k,\vec{\beta}} = \pi_k^{-1}(E_{k,\vec{\beta}})$. 

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### 3.3.1 Flat and hyperbolic case

The metrics we consider will be any $\vec{\beta} \in \mathbb{R}_+^k$ such that the Gauss-Bonnet formula (1) is satisfied. In particular, while the Gauss-Bonnet constraint

$$\sum (\beta - 1) \leq \chi(M)$$

gives an upper bound of $\beta_i$’s, we do not require the cone angles to be uniformly small.

By the results of McOwen, the admissible extended configuration space is relatively easy in this case. As long as the merging cone angles are admissible in the sense of definition 2, there exists a flat or hyperbolic conical metric after merging. In particular, in this case we have

$${\mathcal{E}}_{k,\vec{\beta}} = \mathcal{E}_k^0 \bigcup \bigcup_{\sum_{i \in I} \beta_i > |I| - 1} F_I,$$

and

$${\mathcal{C}}_{k,\vec{\beta}} = \mathcal{C}_k^0 \bigcup \bigcup_{\sum_{i \in I} \beta_i > |I| - 1} C_I,$$

When $k = 2$ and $M = S^2$, there is neither flat nor hyperbolic conical metrics on $M$. When $k \geq 3$ or the genus of $M$ is greater than 0, $\mathcal{C}_{k,\vec{\beta}} \setminus \mathcal{C}_k^0$ is nonempty in general. In particular, when the genus of $M$ and the cone angles are all sufficiently large, all directions of merging will be allowed and in that situation $\mathcal{C}_{k,\vec{\beta}} = C_k$.

### 3.3.2 Spherical case

In this case all the cone angles are assumed to be less than $2\pi$, hence $\vec{\beta} \in (0, 1)^k$. Notice that by the relation (28), the cone angle obtained after merging, denoted by $2\pi \beta_0$, is also less than $2\pi$. Therefore during this merging process we stay inside the class defined in 3.3.

The extra rigidity in 3.3 and the fact that merging does not preserve this constraints make $\mathcal{E}_{k,\vec{\beta}}$ and $\mathcal{C}_{k,\vec{\beta}}$ much more complicated to describe, compared to the other two cases. We illustrate a few cases here, keeping in mind that the football case ($M = S^2, k = 2, \beta_1 = \beta_2$) is special for the reason that will be made clear in Section 4.

We start with $M = S^2$. From [18], there is no “tear-drop” metric on $S^2$ with only one conical point. Hence there is no admissible merging on $\mathcal{C}_2$,
which implies $C_{2,\vec{\beta}} = C_2^0$. On a sphere with 3 conical points and assuming $0 < \beta_1 \leq \beta_2 \leq \beta_3 < 1$, the Troyanov condition is given by

$$2\beta_1 + 1 > \sum_{i=1}^{3} \beta_i, \text{ or } \beta_1 > \beta_2 + \beta_3 - 1. \tag{31}$$

We show that it cannot merge to a football. Since all cone angles are less than $2\pi$, therefore the merging process decreases the angle strictly, i.e. $\beta_1 + \beta_2 - 1 < \min\{\beta_i, \beta_j\}$. Hence the only feasible way to achieve a football would be merging the two bigger angles, and this gives

$$\beta_2 + \beta_3 - 1 = \beta_1$$

which contradicts (31). Since the three points cannot be simultaneously merged into one point either, we have $C_{3,\vec{\beta}} = C_3^0$.

When $k \geq 4$, depending on different $\vec{\beta}$, the situation can be very different. Assuming $0 < \beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4 < 1$ satisfying the Troyanov condition

$$2\beta_1 + 2 > \sum_{i=1}^{4} \beta_i,$$

if $1 < \beta_1 + \beta_4 = \beta_2 + \beta_3$ (which can be achieved for example by taking $\vec{\beta} = (\frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6})$), then by merging the two groups $\{\beta_1, \beta_4\}$ and $\{\beta_2, \beta_3\}$ simultaneously, we get a football; however since we cannot split a football to get an admissible 3-point configuration, one cannot merge $\{\beta_1, \beta_4\}$ nor $\{\beta_2, \beta_3\}$ without the other group. That is, $C_{14} \cap C_{23} \subset C_{4,\vec{\beta}}$ but $C_{14}^0 \cup C_{23}^0 \not\subset C_{4,\vec{\beta}}$. However, it is possible to merge $\{\beta_2, \beta_4\}$ or $\{\beta_3, \beta_4\}$ in most of those cases (as in the example given above), hence $C_{24}^0 \cup C_{24}^0 \subset C_{4,\vec{\beta}}$. In contrast, if $\beta_1 + \beta_4 \neq \beta_2 + \beta_3$ then one cannot get a football but merging into a 3-point configuration is still possible.

When the genus of $M$ is greater than 1, the description of $C_{k,\vec{\beta}}$ still depends on $\vec{\beta}$ and the Troyanov condition, however since we will not get any football configuration in this case, it is analytically same to the flat or hyperbolic cases.

4 Some analysis

Our approach to the study of the geometric problems described in the last section involves the analysis of conic elliptic operators on spaces with isolated
conic singularities, as employed already in [13]. The new feature here is that we study families of such operators on spaces with coalescing conic singularities.

4.1 $b$-vector fields on $M$ and conic elliptic operators

Let $M$ be a manifold with isolated conic singularities at the collection of points $p = \{p_1, \ldots, p_k\}$, and denote by $\hat{M} = [M; p]$ the blowup of $M$ at these points. Thus $\hat{M}$ is a manifold with $k$ boundary components; when $\dim M = 2$, each boundary component is diffeomorphic to a circle. Choose local polar coordinates $(r, \theta)$ near any $p_j$, so $r$ is a boundary defining function for the boundary face created by blowing up $p_j$ and $\theta$ is a set of local coordinates on that face, e.g. $\theta$ is the angular coordinate if $\partial \hat{M}$ is a union of circles. We recall the space of $b$-vector fields, which consists of the space of all smooth vector fields on $\hat{M}$ which are tangent to the boundary. In these local coordinates,

$$\mathcal{V}_b = C^\infty\text{-span}\{r \partial_r, \partial_\theta\}$$

A differential operator is called a $b$-operator if it can be written locally as a finite sum of products of elements of $\mathcal{V}_b$,

$$L = \sum_{j+|\alpha| \leq m} a_{j\alpha}(r, \theta)(r \partial_r)^j \partial_\theta^\alpha$$

(here we continue to think of $\theta$ as possibly multi-dimensional and $\alpha$ a multi-index). This operator is called $b$-elliptic if its $b$-symbol is nonvanishing,

$$b\sigma_m(L) = \sum_{j+|\alpha|=m} a_{j\alpha}(r, \theta)\rho^j \eta^\alpha \neq 0 \text{ for } (\rho, \eta) \neq (0, 0).$$

Finally, we say that $L$ is an elliptic conic operator if $L = r^{-m}A$ where $A$ is an elliptic $b$-operator of order $m$. We are primarily concerned with elliptic conic operators of order 2, in particular the Laplacian on a surface with isolated conic singularities.

Now suppose that $X$ is a more general manifold with corners, i.e., any point $q \in X$ has a neighborhood $U$ diffeomorphic to a neighborhood of the origin in an orthant $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ (in which case $q$ lies on a codimension $k$ corner). As before we define the space of $b$-vector fields on $X$ to consist of the smooth vector fields which are tangent to all boundary faces. In local
coordinates \((x_1, \ldots, x_k, y_1, \ldots, y_{n-k})\) near a codimension \(k\) corner, with all \(x_j \geq 0\), we have that

\[
V_b(X) = C^\infty \text{-span} \{x_i \partial_{x_j}, \partial_{y_\ell}\}.
\]

Our main examples of manifolds with corners here, of course, are the extended configuration spaces \(E_k\) and the extended configuration families \(C_k\). We do not consider here \(b\)-differential operators on general manifolds with corners. Instead, as motivated by our problem, the fibers \(\pi^{-1}(q) \subset C_k\) are unions of two dimensional surfaces with boundary, possibly ‘tied’ along their boundaries, and we consider the families of elliptic conic operators on these fibers, parametrized by \(q \in E_k\). Nonetheless, it still turns out to be important to consider \(b\)-vector fields on the entire space \(C_k\). In doing so, it is convenient to organize the boundary hypersurfaces into three types, corresponding to the boundary faces \(C_I\) which resolve point coincidences, the faces \(C_i\) corresponding to individual conic points, and the faces corresponding to the blowups of each fiber \(M_p\). We denote the boundary defining functions for these by \(R_I, R_i\) and \(\rho\), respectively.

4.2 \(b\)-Hölder spaces on \(M\) and mapping properties of conic elliptic operators

Conic elliptic operators act naturally between weighted Sobolev and Hölder spaces defined relative to differentiations by the vector fields in \(V_b(M)\). Our ultimate problem is nonlinear, so we define only the \(b\)-Hölder spaces and state the key mapping properties on these. For simplicity, we restrict attention to the 2-dimensional case.

**Definition 4** (Weighted \(b\)-Hölder spaces). For any function defined on \(M_p\), define the seminorm \([u]_{b,\delta}\) in the usual way away from a neighborhood of the boundary faces, while in each such neighborhood, in local polar coordinates, we set

\[
[u]_{b,\delta} := \sup_{0 < r < r_0} \sup_{r \leq r' \leq 2r} \frac{|u(r, y) - u(r', y')|}{|r - r'|^{\delta}}.
\]

Then \(C^{0,\delta}_b(M)\) consists of the functions \(u\) which are bounded and for which \([u]_{b,\delta} < \infty\).

Next, for any \(\ell \in \mathbb{N}\), define \(C^{\ell,\delta}_b(M)\) to consist of all functions \(u\) such that \(V_1 \ldots V_j u \in C^{0,\delta}_b(M)\) for every \(j \leq \ell\) and \(V_i \in V_b(M)\).

Finally, \(r^\mu C^{k,\delta}_b(M) = \{u = r^\mu v : v \in C^{\ell,\delta}_b(M)\}\).
It is immediate from these definitions that if $L$ is a second order conic elliptic operator, then for every $\ell \geq 2$,

$$L : r^\mu C^\ell_\delta(M) \longrightarrow r^{\mu-2} C^{\ell-2,\delta}(M)$$

is bounded.

It can happen that this map does not have closed range for certain values of $\mu$. Indeed, $\mu$ is called an indicial root of $L$ if there exists some function $\phi(\theta)$ such that $L(r^\mu \phi(\theta)) = O(r^{\mu-1})$. The expected order of decay or blow-up is $r^{\mu-2}$, so $\mu$ is an indicial root only if there is some leading order cancellation. It is not hard to check that if $\mu$ is an indicial root, then an appropriate sequence of cutoffs of $r^\mu \phi(\theta)$ can be constructed to show that (32) does not have closed range. This is explained at length in [13]. The following is the basic Fredholm result, proved in [12] but see also [13], Proposition 3.

**Proposition 3.** If $\mu$ is not an indicial root, then (32) is Fredholm.

If $L = \Delta + V$ where $V \in C^0_\delta$ for example (or even just $V \in r^{-2+\epsilon} C^0_\delta$ for any $\epsilon > 0$), then at a conic point $p$ with cone angle $2\pi \beta$, the indicial roots consist of the set of values $j/\beta$, $j \in \mathbb{Z}$. The coefficient $\phi(\theta)$ corresponding to the indicial root $j/\beta$ can be any linear combination of $\sin j\theta$ and $\cos j\theta$.

We also consider $L = \Delta + V$, when $V \in C^\ell_\delta$, as an unbounded operator

$$L : C^{\ell,\delta}_b(M) \longrightarrow C^{\ell,\delta}_b(M).$$

We then seek to characterize its domain, i.e., the (nonclosed) subspace

$$D^{\ell,\delta}_b(L) = \{ u \in C^{\ell,\delta}_b(M) : Lu \in C^{\ell,\delta}_b(M) \}.$$

This is called the Hölder Friedrichs domain for $L$.

**Proposition 4 ([13]).** The space $D^{\ell,\delta}_b(L)$ consists of functions $u \in C^{\ell+2,\delta}_b(M)$ such that near each conic point $p$,

$$u = a_0 + \sum_{j=1}^{N(\beta)} (a_{j1} \cos j\theta + a_{j2} \sin j\theta) r^{j/\beta} + \tilde{u},$$

for some constants $a_{j1}, a_{j2}$, where $N(\beta)$ is the largest value $N$ such that $N/\beta < 2$, and $\tilde{u} \in r^2 C^{\ell+2,\delta}_b(M)$. 

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4.3 Families of conic elliptic operators

The previous subsection reviews a few standard results about conic elliptic operators on surfaces. Our interest is in families of such operators, particularly as the conic points coalesce. In particular, suppose $L_p$ is such a family where the cone points are located at some simple divisor $p \in D_k$. A key difficulty in extending the theory for individual operators to families is that the function space on which $L_p$ acts vary with $p$. We use the geometric machinery developed above to handle this.

More specifically, we first consider weighted Hölder spaces on $C_k$; the restrictions of these spaces to the fibers $\pi^{-1}(q)$, $q \in E_k$, then define the appropriate families of weighted Hölder spaces on which we may describe extensions of the mapping properties.

In the following, let $\vec{\nu}$ be a weight vector, with components indexed by the hypersurfaces of $C_k$. We then define in the obvious way the weighted Hölder spaces $r^\vec{\nu}C^{\ell,\delta}_b(C_k)$. To make sense of the restrictions of these spaces to each fiber, we need an easy result.

**Lemma 6.** The restriction of $r^\vec{\nu}C^{\ell,\delta}_b(C_k)$ to each fiber $M_p = \pi^{-1}(p)$ is precisely the weighted space $r^\vec{\nu}C^{\ell,\delta}_b(M_p)$, where (abusing notation slightly), the weight vector $\vec{\nu}$ here has components indexed by the boundary components of $M_p$.

**Proof.** Observe first that the boundary faces of $M_p$ are the components of the intersection $(\cup_{\Sigma} C_{\Sigma} \cup \cup_{i} C_{i}) \cap M_p$.

The fact that the restriction of $C^{0,\delta}_b(C_k)$ equals $C^{0,\delta}_b(M_p)$ is straightforward from the definitions since the boundary defining functions for the faces of $C_k$ restrict to the boundary defining functions for the faces of each fiber, and there are coordinates tangent to the faces of $C_k$ which also restrict to the the $\theta$ coordinates on each fiber.

Next, the weight functions restrict naturally as well. Thus we must finally show that $C^{\ell,\delta}_b(C_k)$ restricts to $C^{\ell,\delta}_b(M_p)$. For this, note that if $V \in V_b(M_p)$, then there is an extension of $V$ to $V \in V_b(C_k)$. Thus if $u \in C^{\ell,\delta}_b(C_k)$ and $V_j \in V_b(M_p)$, $i \leq \ell$, and if $\hat{V}_i$ are the lifts, then $\hat{V}_1 \ldots \hat{V}_\ell u \in C^{0,\delta}_b$, and the restriction of this expression is just $V_1 \ldots V_\ell u$, which by the first step lies in $C^{0,\delta}_b(M_p)$.

Next, for any fiber $M_p$ in $C_k$, if $\vec{\beta}$ is the set of cone angles, then we construct the Friedrichs-Hölder domain by including at each $p_j$ the terms with local expressions $r^{j/\beta_j} \phi_j(\theta)$, $0 \leq j < 2\beta_i$. 

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Definition 5. If $\bar{\beta}$ is fixed, then the fiberwise Hölder-Friedrichs domain associated to a family of conic metrics $g_p$ is given by

$$D_{Fr}^{\ell,\delta}(C_k) = \{ u \in C_b^{\ell,\delta}(C_k) : \Delta_{g_p}(u|_{M_p}) \in C_b^{\ell,\delta}(M_p), \; p \in E_k \}.$$ 

It is clear that $D_{Fr}^{\ell,\delta}(C_k)$ varies smoothly with $p \in E_k$ over the regular fibers (where all the $p_i$ are distinct). We may proceed just as in [13] to obtain that

$$u = a_0 + \sum_{j=1}^{[2\beta]} r^{\bar{\beta}} f_j(\theta) + \tilde{u}, \; \tilde{u} \in r^{2}C_b^{2,\delta}(C_k), \tag{34}$$

where as before $f_j = a_{j1} \cos j\theta + a_{j2} \sin j\theta$. In this free region, smoothness follows from the smoothness of the boundary defining functions with respect to $p$.

When $p$ approaches a face $F_I$ of $E_k$, then the aggregate cone angle is $\beta = \sum_{i \in I} (\beta_i - 1) + 1$, and functions in $D_{Fr}^{\ell,\delta}(C_k)$ have the form

$$u = a_0 f_0(w) + \sum_{j=1}^{[2\beta]} \rho^{\bar{\beta}} f_j(w) + \tilde{u}, \; \tilde{u} \in \rho^2C_b^{2,\delta}(C_k); \tag{35}$$

here $\rho = \rho_I$ is the boundary defining function for the half sphere $C_I$ and $f_j(w)$, $j = 0, \ldots, [2\beta]$, are functions on $C_I$ such that each $\rho^{\bar{\beta}} f_j(w)$ is (formally) annihilated by the rescaled operator $\rho^{2\beta} \Delta_{g_p}$ at $C_I$. On this front face, the conic points are all separated. Therefore, functions in the Friedrichs-Hölder domain annihilated by $\rho^{2\beta} \Delta_{g_p}$ are as in (34), where each cone points on this face has the obvious cone angle extended from the interior of $C_k$. This means that functions on fibers $M_p$ near this face extend smoothly to this face.

Near each boundary component of $C_I$, where $\rho^\theta = r$, choose the coordinates $w = (s, \theta)$ on the bubble. Then $r^{2} \Delta_{g_0,p}$ is a $b$-operator of the form below

$$(r \partial_r - s \partial_s)^2 + \beta^{-2} \partial_\theta^2.$$ 

Finally, $r^{2} \Delta_{g_0,p}(r^{\bar{\beta}} f_j) = 0$ implies that

$$\left( r^{\bar{\beta}} - s \partial_s \right) f_j = 0. \tag{36}$$
5 Flat conical metrics

We now begin our analysis of the space of constant curvature conic metrics by studying the simplest case, when the problem is linear. Thus we fix closed surface $M$ and a set of cone angles $\vec{\beta}$ such that

$$\chi(M, \vec{\beta}) := \chi(M) + \sum_{j=1}^{k} (\beta_j - 1) = 0. \quad (37)$$

It is standard that if (37) is satisfied, then for each marked conformal structure $(c, p)$ there exists a unique flat conic metric with cone angle $2\pi \beta_i$ at $p_i$. Our goal in this section is to show that this family of flat conic metrics is polyhomogeneous on $C_k$.

This is a local regularity theorem, so we fix a smooth family $g_0(c)$ of smooth constant curvature metrics on $M$ representing a neighborhood in the space of (unmarked) conformal structures. For each $p \in D_k(M)^s$, consider the linear problem

$$\Delta g_0(c) G = 2\pi \sum_{i=0}^{k} (\beta_i - 1) \delta_{p_i}.$$ 

Then

$$-\int_M K = -2\pi \chi(M) = \int_M 2\pi (\sum_{i} (\beta_i - 1)) \delta_{p_i},$$

we see that the Liouville equation

$$\Delta g_0(c) u + K_{g_0(c)} = 0$$

has a solution $u = G$ which is unique if we require that $\int_M G = 0$. This solution $G$ is essentially the Green function for $\Delta_{g_0}$. It clearly depends smoothly on $c, p \in D_k^s$ and $z \in M \setminus p$, and near each $p_i$ has the form

$$G \sim (\beta_i - 1) \log |z| + \tilde{G}_i$$

where each $\tilde{G}_i$ is $C^\infty$ in a neighborhood of $p_i$. We then define

$$g_0(c, p, \vec{\beta}) = e^{2G} g_0(c). \quad (38)$$

Each of these metrics is flat and, because of the asymptotic structure of $G$, has a conic singularity with cone angle $2\pi \beta_i$ at $p_i$. This family of metrics is smooth when all the points $p_i$ are distinct.
Proposition 5. Fix $\vec{\beta}$ satisfying $[37]$. Then the family of metrics $g_0(c, p, \vec{\beta})$ extends to a polyhomogeneous family of fiber metrics on $C_k$.

Because $g_0(c)$ is smooth, it suffices to show that the scalar function $G$ extends to be polyhomogeneous on $C_k$. Note that by the remarks above, $G$ is $C^\infty$ on the interior of the extended configuration family, so our task is to examine its behavior near each of the boundary faces and corners of $C_k$. In other words, we must prove that there exists an index family $\{E_I\}$ such that

$$G \sim \sum_{(j,\ell) \in E_I} (\rho_I)^j (\log \rho_I)^\ell a_{j,\ell}(w_I),$$

where $w_I$ are variables in the interior of each $C_I$ and each $a_{j,\ell}$ is polyhomogeneous with index family $\{E_J\}_{J \neq I}$. Note that polyhomogeneity of $G$ near the simplest faces $C_i$ is obvious. We also suppress the smooth dependence of $G$ on $c$.

The proof of Proposition 5 is by induction on $k$. We begin with the proof when $k = 2$.

Proposition 6. When $k = 2$, $G(z, p)$ is polyhomogeneous on $C_2$.

Proof. Suppose that the two points $p_1$ and $p_2$ converge at the point $p_{12}$, which we may as well assume is fixed and is the center of mass of these two points. Referring to the local coordinates in §2.3, we may as well restrict to a slice where $\zeta = \zeta_0 = 0$, and also work on the space of ordered pairs $\tilde{C}_2$ rather than $C_2$. Then

$$G(z, p) = (\beta_1 - 1) \log |z - w| + (\beta_2 - 1) \log |z + w| = (\beta_1 - 1) \log |re^{i \phi} - re^{i \theta}| + (\beta_2 - 1) \log |re^{i \phi} + re^{i \theta}|.$$

By $[6]$, $r = R_{12} \cos \omega, \rho = R_{12} \sin \omega$, so

$$G(z, p) = (\beta_{12} - 1) \log R_{12} + (\beta_1 - 1) \log |\cos \omega e^{i(\phi - \theta)} - \sin \omega| + (\beta_2 - 1) \log |\cos \omega e^{i(\phi - \theta)} + \sin \omega|.$$  \hspace{1cm} (40)

Here, and later in this paper, we set

$$\beta_{12} = \beta_1 + \beta_2 - 1,$$ \hspace{1cm} (41)

i.e., $2\pi \beta_{12}$ is the cone angle which results when two cone points with cone angles $2\pi \beta_1$ and $2\pi \beta_2$ merge. The expression in (40) is certainly polyhomogeneous as $R_{12} \to 0$ away from $\omega = 0$ (the corner, where $C_{12}$ meets $[M; \{p_{12}\}]$).
and the points where $e^{i(\phi-\theta)} = \pm 1$. To understand behaviour near the corner, write $s = \rho/r = \tan \omega$, so that when $\omega < \pi/4$, say, and recalling that $r = R_{12} \cos \omega$, we have

$$G(z, p) = (\beta_{12} - 1) \log r + \frac{1}{2} (\beta_1 - 1) \log (1 - 2s \cos (\theta - \phi) + s^2) + \frac{1}{2} (\beta_2 - 1) \log (1 + 2s \cos (\theta - \phi) + s^2), \quad (42)$$

The second and third terms on the right are smooth and vanish $s = 0$. Note that there is an apparent asymmetry in the indices 1 and 2 here; however, when the points $p_1$ and $p_2$ are switched, the angle $\theta$ changes to $\theta + \pi$, so this expression is actually symmetric after all. Finally, near $\omega = \pi/4$ and $\theta = \phi$, for example, write $\tan \omega = 1 + \sigma$, so that $\cos \omega = 1/\sqrt{2 + 2\sigma + \sigma^2}$. Then

$$G(z, p) = (\beta_{12} - 1) \log (R_{12}/\sqrt{2 + 2\sigma + \sigma^2}) + (\beta_1 - 1) \log |e^{i(\theta-\phi)} - 1 - \sigma| + (\beta_2 - 1) \log |e^{i(\theta-\phi)} + 1 + \sigma|,$$

and this is obviously polyhomogeneous around the face $\mathcal{C}_1$ created by blowing up $\sigma = 0$. The argument is the same near $\mathcal{C}_2$.

The assertion about polyhomogeneity of $G$ on $\mathcal{C}_2$ is now proved. \hfill \Box

**Proof of Proposition 5.** Suppose that the result has been proven for $\mathcal{C}_j$ with any $j < k$. Without loss of generality, we can restrict to the slice with the fixed center of mass $\zeta = \zeta_0 = 0$.

We first consider the case that is away from the central diagonal $\mathcal{C}_{1 \ldots k}$, that is, at most $k - 1$ points can merge together. This is the case for example when the configuration $\vec{\beta} \in \mathbb{R}^k$ is such that $\sum_{i=1}^k (\beta_i - 1) \leq 1$. Then we can cover $\mathcal{E}_{k, \vec{\beta}}$ by open sets $\{U_{\mathcal{I}, \epsilon}\}$ defined in (23). That is, the only possible merging happens within the sub-clusters, and the distance between any clusters is bounded away from 0. From Lemma 5, $U_{\mathcal{I}, \epsilon}$ locally has a product structure, identified with an open subset $\Pi_{j=1}^\ell U_{\mathcal{I}_j} \subset \Pi_{j=1}^\ell \mathcal{E}_{|\mathcal{I}_j|}$. The total space fibers over $U_{\mathcal{I}, \epsilon}$, and is given locally by a product of fibrations. In this case, the conformal factor can be written as a sum

$$G = \sum_{j=1}^\ell (\sum_{i \in \mathcal{I}_j} (\beta_i - 1) \log |z - z_i|).$$

Since $\{z_i\}_{i \in \mathcal{I}_j}$ is bounded away from any other clusters $\mathcal{P}_{\mathcal{I}_j', j' \neq j}$, the term $\sum_{i \in \mathcal{I}_j} (\beta_i - 1) \log |z - z_i|$ is only singular near $V_i$ as defined in Lemma 5. By
induction, this term is polyhomogeneous on $C_{I^j}$, hence is polyhomogeneous on $U_{\mathcal{I}^j\cdot e}$. Same argument can be applied to other terms. By considering all the open covers, we get polyhomogeneity of $G$ on $\mathcal{C}_k$ in this case.

Now we consider the behavior near the central face, and all the $k$ points can merge together. We now prove that away from all sub-diagonals, $G$ is polyhomogeneous near $\mathcal{C}^0_{1,...,k}$. In this region we write $p_i = z_i + \zeta$ and assume that the center of mass $\zeta = 0$. Then writing $(z, z_1, \ldots, z_k) = R_{1...k} \Omega$, $\Omega = (\Omega_0, \ldots, \Omega_k)$, we have

$$G = \sum_{i=1}^{k} (\beta_i - 1) \log |z - z_i| = (\sum_{i=1}^{k} \beta_i - k) \log R_{1...k} + \sum (\beta_i - 1) \log |\Omega_0 - \Omega_i|$$

and since $z$ remains away from the sub-diagonals, only the term $\log R_{1...k}$ is singular here and this is obviously polyhomogeneous on the interior of $\mathcal{C}_{1...k}$. And each term $(\beta_i - 1) \log |\Omega_0 - \Omega_i|$ is singular only near $\mathcal{C}_i$ and is polyhomogeneous.

Near the outer boundary of $\mathcal{C}_{1...k}$, set $z = r e^{i\phi}$ and $w_i = z_i / r$. Then

$$G = (\sum_{i=1}^{k} \beta_i - k) \log r + \sum (\beta_i - 1) \log |e^{i\phi} - w_i|.$$  

Notice that all the faces $\mathcal{C}_i$ occur along the submanifolds \{z = z_i\} $\subset$ \{|w_i| = 1\}, so provided we stay away from these submanifolds, then only the first term on the right is singular, and it is polyhomogeneous. At the principal diagonal $\mathcal{C}_i$, however, $w_i = e^{i\theta_i}$ and the additional singular term is $\log |e^{i\phi} - e^{i\theta_i}|$, which is polyhomogeneous there.

Finally, if $p$ is near any one of the partial diagonals, including near their intersection with $\mathcal{C}_{1...k}$, then it is in a neighborhood of some intersection of front faces $\{\mathcal{C}_{I_j}\}_{j=1}^{T}$, where each $\mathcal{I}_j$ is a proper subset of \{1, $\ldots$, $k$\}, and the $\mathcal{I}_j$ have no elements in common. The resolution ensures that the faces $\mathcal{C}_{I_i}$ and $\mathcal{C}_{I_j}$ are disjoint, so we can once again factor out the defining function $R_{1...k}$ and separate out the indices $i$ which do not lie in any of the $\mathcal{I}_j$, and write

$$G = (\sum_{i=1}^{k} \beta_i - k) \log R_{1...k} + \sum_{i \notin \cup_{j} \mathcal{I}_j} (\beta_i - 1) \log |w - w_i| + \sum_{j=1}^{\ell} f_j,$$  \hspace{1cm} (43)

where here $w = z / R_{1...k}$, $w_i = z_i / R_{1...k}$. Here $f_j$ is the rescaled factor

$$f_j = \sum_{i \in \mathcal{I}_j} (\beta_i - 1) \log |w - w_i| = \sum (\beta_i - 1) \log R_{\mathcal{I}_j} + \sum (\beta_i - 1) \log |\Omega_0^{\mathcal{I}_j} - \Omega_i^{\mathcal{I}_j}|$$

\[37\]
where \( R_{I_j} \) is the boundary defining function for \( \mathcal{C}_{I_j} \) and the coordinates over this face is given by \((w, w_i)_{i \in I_j} = R_{I_j}(\Omega^{T_j}_0, \Omega^{T_j}_i)\). By induction, each rescaled factor \( f_j \) is up to a smooth summand the Green function near \( I_j \) and hence is polyhomogeneous near the collection of faces which constitute the resolution near this cluster. This behavior is uniform as \( R_{I...k} \to 0 \).

It is perhaps wise to illustrate this induction for \( \mathcal{C}_3 \). In this case, near \( \mathcal{C}_{12} \cap \mathcal{C}_{123} \) we can write

\[
G = (\beta_1 - 1) \log |z - \epsilon_1(1 + \epsilon_2)| + (\beta_2 - 1) \log |z - \epsilon_1(1 - \epsilon_2)| + (\beta_3 - 1) \log |z + \epsilon_1| 
\]

We wish to examine the region \( R_{123} \to 0 \) and \( R_{12} \to 0 \), and in this region, \( \epsilon_1 \sim R_{123}, \epsilon_2 \sim R_{12} \). Therefore, using \( w_{12} = \frac{w_3 - 1}{\epsilon_2} \) as a coordinate on \( \mathcal{C}_{12} \),

\[
G = (3 \sum_{i=1}^{3} \beta_i - 3) \log \epsilon_1 + (2 \sum_{i=1}^{2} \beta_i - 2) \log \epsilon_2 \\
+ (\beta_1 - 1) \log |w_{12} - 1| + (\beta_2 - 1) \log |w_{12} + 1| \\
+ (\beta_3 - 1) \log |w_3 + 1|. \tag{44}
\]

Since \( w_3 \sim 1 \) here, this is polyhomogeneous.

6 Hyperbolic conic metrics

We next turn to the analytic description of the space of hyperbolic cone metrics which, as explained earlier, exist whenever

\[
\chi(M) + \sum_{i=1}^{k} (\beta_i - 1) < 0.
\]

The problem is now genuinely nonlinear and the proof of polyhomogeneity correspondingly more difficult. Indeed, the proof is directly inductive on the number of cone points. We now explain the strategy, which requires several steps.

For the case \( k = 2 \), we construct a family of background metrics which is hyperbolic away from the merging points and flat near these points, with a transitional region in between. Let \( \rho \) be the degeneration parameter which measures the distance to the fiber where the points coincide. We show that this family of background metrics can be chosen so that it solves the curvature...
equation up to an error of order $\rho^2$. We then solve the curvature equation on each fiber using the implicit function theorem and the invertibility of the linearized curvature equation on the Friedrichs-Hölder domain. Once the theorem has been established for $k = 2$, we follow an inductive procedure to construct families of background metrics in the general case with the same properties, and once again solve away the error terms using the implicit function theorem.

The case $k = 2$ already contains essentially all of the substantial difficulties, so this case is presented in careful detail.

6.1 The case of two merging cone points

Consider a family of simple divisors $p$ which converge to a point $q \in F_{12} \subset \mathcal{E}_k$. We may as well assume that $p_3, \ldots, p_k$ remain fixed, but $p_1$ and $p_2$ merge at a point $p_{12}$ which, for simplicity, we assume is the center of mass of $p_1$ and $p_2$ and also remains fixed. We write $p'$ for the (simple) $(k - 1)$-tuple $(p_{12}, p_3, \ldots, p_k)$. We are working locally near $q$, and this point is far from any of the other partial diagonals, so we use the local coordinates on $\mathcal{E}_2$ and $\mathcal{C}_2$: $\rho = \rho_{12}$ and $R = R_{12}$, and for simplicity we set $\theta_{12} = 0$, which amounts to fixing the direction through which $p_1$ and $p_2$ approach one another. If $\beta_i$ are the cone parameters at $p_i$, then as noted earlier, $\beta_1$ and $\beta_2$ determine the limiting cone parameter $\beta_{12} = \beta_1 + \beta_2 - 1$ at $p_{12}$. In order for the two points to merge, it is necessary that

\[ \beta_1 + \beta_2 > 1 \Leftrightarrow \beta_{12} > 0. \]

The fiber $\pi^{-1}(q) \subset \mathcal{C}_k$ consists of two surfaces with boundary, $M_{p'} = [M; \{p'\}]$ (the surface $M$ blown up at the points in $p'$) and the face $\mathcal{C}_{12}$, and these meet along a common circle.

The initial metric

We now construct a family of metrics on $M$ with $k$ conic singularities at the family of divisors $p$ above, which extends as a smooth family of fiberwise metrics on $\mathcal{C}_k$. This family is obtained locally near the fiber $\pi^{-1}(q)$ by gluing the fixed hyperbolic metric $h_{0,p'}$ with conic singularities at $p'$, with cone parameters $\beta_{12}, \beta_3, \ldots, \beta_k$, to the degenerating family of flat metrics $g_{0,p'}^f$ in (38). To do this, define

\[ g_{0,p} = \chi g_{0,p}^f + (1 - \chi)h_{0,p'}, \]  

(45)
where
\[
\chi(z, p) = \begin{cases} 
1 & \text{if } \rho < \bar{\rho} \text{ and } |z| < 2\bar{\rho} \\
0 & \text{if } \rho > 2\bar{\rho} \text{ or } |z| > 4\bar{\rho}
\end{cases}
\]
is a smooth nonnegative cutoff function for some small \( \bar{\rho} > 0 \). We usually drop \( p \) and \( p' \) from the subscripts for simplicity, and also write
\[
K_{0,\rho} = \begin{cases} 
0 & \text{if } \rho < \bar{\rho} \text{ and } |z| < 2\bar{\rho} \\
-1 & \text{if } \rho > 2\bar{\rho} \text{ or } |z| > 4\bar{\rho}.
\end{cases}
\]
for the curvature of the metrics in this family.

Our goal is to obtain precise analytic control of the solution to the conformal curvature equation
\[
\Delta g_{0,\rho} u + e^{2u} + K_{0,\rho} = 0 \tag{46}
\]
as \( \rho \to 0 \). In the neighborhood where \( \chi = 1 \), (46) becomes
\[
\Delta g_{0,\rho} u + e^{2u} = 0. \tag{47}
\]
(\( \Delta g_0 \) is the Laplace-Beltrami operator with nonnegative spectrum). In this region we can write
\[
g_{0,\rho} = e^{2v_0} |dz|^2
\]
where
\[
v_0 = (\beta_1 - 1) \log |z - \rho| + (\beta_2 - 1) \log |z + \rho| + \bar{v}_0
\]
for some harmonic function \( \bar{v}_0 \) which is uniformly bounded as \( \rho \to 0 \).

Our goal is to prove that the solution \( u \) to (46) is polyhomogeneous on \( \mathcal{E}_k \) near the point \( p' \in C_k, z = 0 \). To do this we construct the solution anew (even though its existence is guaranteed by standard barrier arguments) by first constructing an approximate solution which satisfies (46) to any fixed arbitrarily high order as \( \rho \to 0 \), and then correcting this to the exact solution using an analytic construction which guarantees that this additional correction term also vanishes to that same high order.

The specifics of the first part of this are that we construct the entire Taylor series for \( u \) along each of the two faces \( M_{p'} \) and \( C_{12} \). These series expansions are related to one another and must satisfy a set of matching conditions along the corner where these faces intersect. The fact that we can correct any finite part of this Taylor series to an exact solution with a
term which vanishes to that order means that these series represent the true expansions for this exact solution.

**Expansion at** $M_p' \cap C_{12}$ **within** $M_p'$

Recall the coordinates $z = re^{i\phi}$ and $\rho$ near $C_{12}$ in $E_k$ (as before, an angular coordinate is suppressed), and set

$$r = \vert z \vert, \text{ and } s = \rho/r.$$ 

Thus $s = 0$ defines the surface $M_p'$ while $r = 0$ defines $C_{12}$. There is a freedom in the conformal coordinate $z$ on $M_p'$ by holomorphic reparametrization, and we fix this below. We also define the coordinate

$$r = \frac{1}{\beta_{12}} \vert z \vert^{\beta_{12}}.$$ 

on $M_p'$, which is the radial distance function for the background flat conic metric. It follows from (42) that near the corner $r = s = 0$,

$$g_{0,p} = \alpha(s, r, \phi, \theta) (dt^2 + \beta_{12}^2 r^2 d\phi^2),$$

where $\alpha(s, r)$ is polyhomogeneous with $\alpha(0, r) = 1$ near $r = 0$. Our first result is that by choosing the coordinate $z$ carefully, the expansion for $\alpha$ has a particularly simple form; it simultaneously also gives the first term in the expansion of $u$ at $M_p'$.
Lemma 7. There is a unique bounded solution $u'_0 \in \bigcap_{m \geq 0} C^{m,\delta}_b(M_{p'})$ to the restriction of (46) to $M_{p'}$. This solution is polyhomogeneous as $r \to 0$, and if this defining function is chosen appropriately, then

$$u'_0 \sim \sum_{j \in \mathbb{N}_0} a_j r^{2j}.$$ 

Proof. All of this except the last assertion, i.e., the existence and polyhomogeneous regularity, is contained in [14] and [13]. For simplicity set $\beta = \beta_{12}$. Existence and uniqueness of the solution is proved in [14] by constructing bounded sub- and supersolutions, and this method also leads to the uniqueness of $u'_0$ amongst bounded solutions. The regularity theorem is proved in [13]. First, (scale-invariant) local elliptic regularity shows that $u'_0 \in C^{m,\delta}_b$ for every $m \geq 0$. The refined regularity theorem in §3 of that paper states that $u'_0$ is polyhomogeneous, with $u'_0 \sim \sum_{\ell,j \geq 0} a_{\ell j}(\phi) r^{\ell/\beta + j}$ as $r \to 0$, where $a_{\ell j}$ is linear in $\cos \ell \phi, \sin \ell \phi$.

It remains to prove the last assertion, that all $a_{\ell j} = 0$ when $\ell \neq 0$ in some choice of coordinates. To this end, recall that near a cone point, there are geodesic coordinates $(\tilde{r}, \theta)$ in terms of which the hyperbolic metric takes the canonical polar form

$$g = d\tilde{r}^2 + \beta^2 \sinh^2 \tilde{r} d\phi^2. \quad (48)$$

On the other hand, there is a local holomorphic coordinate $z$ centered at the conical point for which, in the region near $p_{12}$ in which it is flat, $g_{0,\rho} = |z|^{2(\beta-1)}|dz|^2$, so that

$$g = e^{2u'_0}|z|^{2(\beta-1)}|dz|^2$$

there. Since $\frac{1}{\beta}|z|^\beta = r$, we have

$$d\tilde{r}^2 + \beta^2 \sinh^2 \tilde{r} d\phi^2 = e^{2u'_0(z)}|z|^{2(\beta-1)}|dz|^2 = e^{2u'_0(z)}(d\tau^2 + \beta^2 \tau^2 d\phi^2).$$

This gives

$$\frac{d\tilde{r}}{d\tau} = e^{u'_0}, \sinh \tilde{r} = e^{u'_0} \tau,$$

or equivalently

$$\frac{d\tilde{r}}{\sinh \tilde{r}} = \frac{d\tau}{\tau},$$

and hence

$$\frac{\tilde{r}}{2} = c \tau.$$
We finally scale $z$ so that $c = 1/2$.
We have now shown that
\[
\tilde{r} \sim r \left( 1 + \sum_{j=1}^{\infty} \tilde{a}_j r^{2j} \right),
\]
i.e., $\tilde{r}$ is an odd function of $r$, so that $e^{u'_0} = r^{-1} \sinh \tilde{r}$ is even in $r$ and equals 1 when $r = 0$. We conclude that $u'_0$ is even and vanishes at $r = 0$, hence
\[
u_0' \sim \sum_{j=1}^{\infty} \tilde{a}_{0j} r^{2j} = \sum_{j \geq 1} a_{0j} r^{2j \beta} \quad (49)
\]
where each $a_{0j}$ is constant.

\begin{proof}

Expansion at $M_{p'}$

We next turn to the complete expansion of $u$ at the face $M_{p'}$. Away from the cone points the variable $\rho$ is a defining function for this face; we write the expansion in terms of $\rho$ and observe that the coefficients are valid on $M_{p'} \setminus \{p_{12}\}$. For simplicity, we assume here $2k\beta \notin \mathbb{N}$ for any $k \in \mathbb{N}$. The special case when $\beta$ is a rational number is discussed before Lemma 10.

\textbf{Proposition 7.} As $\rho \to 0$, there is an expansion

\[
u \sim u'_0 + \sum_{j=1}^{\infty} \rho^j u'_j(z)
\quad (50)
\]

where $u'_1$ has an expansion

\[
u'_1 \sim \sum_{\ell,k \in \mathbb{N}} r^{\ell+2k\beta} a_{1\ell k}(\phi)
\quad (51)
\]

where each coefficient $a_{1\ell k}$ is a linear combination of $\cos \ell \phi$ and $\sin \ell \phi$. For any $j > 1$, each $u'_j$ in turn has an expansion

\[
u'_j \sim \sum_{\ell,k \in \mathbb{N}} r^{\ell+2k\beta}(\log r)^{j-1} a_{j\ell k}(\phi) + \sum_{\ell,k \in \mathbb{N}, s \leq j-2} r^{\ell+2k\beta}(\log r)^s b_{j\ell ks}(\phi),
\quad (52)
\]

where each $a_{j\ell k}$ and $b_{j\ell ks}$ is a linear combination of $\cos(\ell \phi)$ and $\sin(\ell \phi)$, and any other $b_{j\ell ks}$ is a trigonometric polynomial of degree at most $\ell$. 

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Proof. Clearly $u$ is $C^\infty$ up to $M_{p'}$ away from $z = 0$, and thus has an expansion in nonnegative integer powers of $\rho$. Expand $K_{g_0,p} = \sum_{j=0}^\infty \rho^j K_j$ in (46) and insert a formal series expansion for $u$, as in the statement of this theorem. Notice here $K_j \equiv 0$ near $C_{12}$. Since $\Delta_{g_0,p}$ commutes with $\rho$ away from $C_{12}$, we obtain a recursive set of equations which successively determine all of the $u'_j$. The first of these is the curvature equation

$$\Delta_{g_0,p} u'_0 + e^{2u'_0} + K_0 = 0,$$

on $M_{p'}$. By the previous lemma, $u'_0$ has an expansion involving only the powers $r^{2k\beta}$, $k \in \mathbb{N}$.

The equation for $u'_j$, $j \geq 1$, is

$$e^{-2u'_0} \Delta_{g_0,p} u'_j + 2u'_j = e^{-2u'_0} \left(-K_j - \left(e^{2u^{(j-1)}} - 1 - 2u^{(j-1)}\right)\right),$$

(53)

where $u^{(j-1)} = \sum_{i=0}^{j-1} \rho^i u'_i$ and the notation $(w)_j$ means that we take the coefficient of $\rho^j$ in the expansion of $w$. Note that $e^{-2u'_0} \Delta_{g_0,p}$ is the Laplacian of the hyperbolic cone metric on $M_{p'}$, so if we use the special coordinates $(r, \phi)$ from Lemma $\overline{7}$ for $e^{2u'_0} g_{0,p}$, then the bounded formal solutions of $e^{2u'_0} g_{0,p} + 2$ are $r^\ell e^{\pm i\ell \phi}$, $\ell \in \mathbb{N}$. In other words, the nonnegative indicial roots are $\ell = 0, 1, 2, \ldots$.

When $j = 1$, (53) can be written out:

$$e^{-2u'_0} \Delta_{g_0,p} u'_1 + 2u'_1 = -e^{2u'_0} K_1$$

(54)

where the right hand side is 0 when solving near $C_{12}$. So solving this eigen equation we get Bessel functions for each Fourier mode, and $u'_1$ has an expansion as (51), where each $a_{1\ell k}$ is a linear combination of $\cos(\ell\phi)$ and $\sin(\ell\phi)$.

We now prove by induction that the expansion for any $u'_j$ has precisely the same form as in (52). The equation for $u'_2$ can be expanded out as

$$e^{-2u'_0} \Delta_{g_0,p} u'_2 + 2u'_2 = -e^{-2u'_0} \left(K_2 + 2(u'_1)^2\right).$$

So the right hand side has an expansion involving only

$$\prod_{\ell,k \in \mathbb{N}, \ell' \leq \ell} r^{\ell+2k\beta} e^{\pm i\ell' \phi}.$$  (55)

The appearance of a different $\ell'$ from $\ell$ is because of the product of linear combinations of $\cos(\ell\phi)$ and $\sin(\ell\phi)$ which gives a trigonometric polynomial
of order at most $\sum \ell$, or more precisely, from $\prod \ell e^{i\ell \phi}$. Solving this equation we get

$$u'_2 \sim \sum_{\ell, k \in \mathbb{N}} r^{\ell+2k\beta} (\log r) a_{2\ell k}(\phi) + \sum_{\ell, k \in \mathbb{N}, s \leq 0} r^{\ell+2k\beta} (\log r)^s b_{2\ell ks}(\phi).$$

The first term appears from the coincidence of terms in the right hand side expansion (55), $r^\ell e^{i\ell \phi}$, with the indicial roots of the operator $e^{2u'0} g_{0p} + 2$. Since it is a linear equation, at most one log $r$ appears here, and the lower order term $r^{\ell+2k\beta} e^{i\ell \phi}$ in the solution is put into the second term. The rest of the second term appears when solving for $r^{\ell+2k\beta} e^{i\ell' \phi}$ in (55) when $\ell' \neq \ell$ or $k \neq 0$. Since no indicial root is hit in this part, no log is produced. Note that the terms $b_{j00s}$ in the solution are always produced from indicial roots, therefore $b_{j00}$ has a simpler form and is a linear combination of $\cos(\ell \phi)$ and $\sin(\ell \phi)$.

Now suppose (52) is true for $u'_i$, $i < j$, then the terms in the right hand side of (53) are given by the form $\prod \sum_{i=j} u'_i$. By tracking the terms in (52) especially the term with the highest order of log’s, the right hand side has an expansion with terms given by the form of

$$\sum_{\ell, k \in \mathbb{N}, \ell' \leq \ell} r^{\ell+2k\beta} (\log r)^{j-2} e^{i\ell' \phi} + \sum_{\ell, k \in \mathbb{N}, s \leq j-3, \ell' \leq \ell} r^{\ell+2k\beta} (\log r)^s e^{i\ell' \phi}.$$  \hfill (56)

Solving the indicial terms in the first term of (56), that is, $r^\ell (\log r)^{j-2} e^{i\ell \phi}$, we get the first part of the expansion in (52). And solving the rest of the first term as well as the second term in (56), we get the second term in (52). Again, the terms $r^\ell (\log r)^s b_{j0s}(\phi)$ are only produced for solving terms in the same Fourier mode $r^\ell (\log r)^s e^{i\ell \phi}$, therefore $b_{j0s}$ is a linear combination of $\cos(\ell \phi)$ and $\sin(\ell \phi)$.

By induction, this concludes the proof of the proposition. \hfill \Box

**Expansion at $C_{12}$**

We next consider the expansion at $C_{12}$. Unlike the construction above at $M_p'$, some terms in this expansion can only be determined once we take into account their compatibility with the previous expansion. Write

$$u \sim \sum_{\alpha \in \mathcal{E}} R^\alpha (\log R)^\gamma u_{\alpha, \gamma}(s, \phi),$$  \hfill (57)

near this face, where $\mathcal{E}$ is an index set which is determined in the course of the argument below, see (61). And $\gamma \in \mathbb{N}$ will also be determined at the same
time. As usual, $R = \sqrt{\rho^2 + r^2}$, and we now use $s = \sin \omega$ where $R \sin \omega = \rho$ as the defining function for $M_p$; this is a good coordinate away from the pole of this hemisphere. However, it is for many purposes simpler to use the projective coordinates $\hat{\omega} = \omega / \rho$ and $\rho$, which are valid on the interior of $\mathcal{C}_{12}$; $|\hat{\omega}| \to \infty$ at the outer boundary of this face and $\rho$ is only a defining function for this face away from its outer boundary.

In these projective coordinates, still writing $\beta = \beta_{12},$

$$g_{0,p} = \rho^{2\beta} e^{2\beta |d\hat{\omega}|^2}, \quad \hat{u} = (\beta_1 - 1) \log |\hat{\omega} - 1| + (\beta_2 - 1) \log |\hat{\omega} + 1| + \tilde{u}, \quad (58)$$

where $\tilde{u}$ is harmonic as a function of $\hat{\omega}$, so in particular is smooth across the singular points $\hat{\omega} = \pm 1$ in this face. It is also bounded in a neighborhood of $\mathcal{C}_{12}$, so in fact its restriction to $\mathcal{C}_{12}$ must be constant. In other words, $\rho^{-2\beta} g_{0,p}$ restricts to a flat metric $g_{\hat{\omega}}$ on the interior of $\mathcal{C}_{12}$ which has two conic singularities at $w = \pm 1$ and is asymptotic to the large end of a cone with cone angle $2\pi \beta$ as $\hat{\omega} \to \infty$.

We first focus on the part of expansion with no logs. The first result is the expansion with $\alpha \leq 2\beta$:

**Lemma 8.** The expansion $[57]$ with $\alpha \in (0, 2\beta]$ and $\gamma = 0$ is given by

$$\sum_{\alpha \in \mathcal{E} \cap (0, 2\beta]} R^\alpha u_\alpha(s, \phi), \quad \mathcal{E} \cap [0, 2\beta] = \{0, 1, 2, \ldots, [2\beta], 2\beta\}. \quad (58)$$

When $\alpha$ is an integer, $u_\alpha$ is determined by coefficients $b_{(\alpha-j)j0}(\phi)$ in $[52]$ and $a_{1j0}$ in $[51]$. When $\alpha = 2\beta$, $u_\alpha$ is determined by $a_{01}$ in $[49]$.

**Proof.** Each summand in $[57]$ can be rewritten as $(u_\alpha s^{-\alpha})(R s)^{\alpha} = u_\alpha' \rho^\alpha$ (the double prime superscript indicates that this is a term in the expansion near $\mathcal{C}_{12}$). Write the fiberwise Laplacian of $g_{0,p}$ in this neighborhood as $\rho^{-2\beta} e^{-2\beta \hat{\Delta}}$, where $\hat{\Delta}$ is the Laplacian for the flat metric $|d\hat{\omega}|^2$. Inserting $[57]$ into the equation, and recalling that $\hat{\Delta}$ commutes with $\rho$, we obtain that $\Delta u_\alpha'' = 0$ for $\alpha < 2\beta$. We require that $u_\alpha$ is bounded as $\hat{\omega} \to \infty$, or equivalently $u_\alpha'' = u_\alpha s^{-\alpha}$ grows at most like $s^{-\alpha} \sim |\hat{\omega}|^\alpha$. However, $\hat{\Delta} = e^{-2\beta \hat{\Delta}}$, so $u_\alpha''$ is harmonic with respect to the Euclidean Laplacian $\Delta_{\hat{\omega}}$, which means that it is a harmonic polynomial $p_\alpha(\hat{\omega})$ of degree $k_\alpha \leq \alpha$. If $\alpha$ is not an integer, then $s^\alpha p_\alpha(\hat{\omega})$ is a sum of terms each of which vanish at nonintegral rates as $s \to 0$. Therefore, since $u$ is smooth as $\rho \to 0$ away from $R = 0$, the only allowable exponents $\alpha < 2\beta$ are nonnegative integers. Let $\alpha$ be any one of
these values. We next narrow down the possible coefficient $p_\alpha(\hat{z})$. Note that as $s \to 0$, each term in $R^\alpha p_\alpha(\hat{z}) s^\alpha$ behaves like $\rho^{\alpha-j} R^j e^{ij\phi}$ for some $j \leq \alpha$. When $j = \alpha$, this becomes $R^\alpha e^{ij\phi}$, which does not match any term in the expansion of $u_0^\alpha$ in (49). Thus only terms with $j < \alpha$ are possible. Compatibility at the corner means that this must match the coefficient $a_{1j0}$ (if $\alpha - j = 1$) or $b_{(\alpha-j)j00}(\phi)$ in (52), which is a linear combination of cos $j\phi$ and sin $j\phi$. There is a unique homogeneous harmonic polynomial which satisfies this boundary condition. We have now shown that for any integer $\alpha < 2\beta$,

$$R^\alpha u_\alpha = \sum_{j=0}^{\alpha-1} R^\alpha p_{\alpha j}(\hat{z}) s^\alpha = \sum_{j=0}^{\alpha-2} \rho^{\alpha-j} R^j b_{(\alpha-j)j00}(\phi) + \rho^1 R^j a_{1j0}.$$  

Next consider the exponent $\alpha = 2\beta$. Essentially the same calculation as above yields

$$\Delta_\beta u_{2\beta}'' + 1 = 0, \quad u_{2\beta}'' = u_{2\beta} s^{-2\beta}.$$

Multiplying by

$$e^{2U} = e^{2((\beta_1-1)\log |\hat{z}| + (\beta_2-1)\log |\hat{z}| + 1))},$$

this becomes $\Delta_{2\beta} u_{2\beta}'' + e^{2U} = 0$. Since $e^{2U} \sim |\hat{z}|^{2\beta-2} \text{ as } |\hat{z}| \to \infty$, there exists a solution $u_{2\beta}''$ to this equation asymptotic to $A|\hat{z}|^{2\beta}$ where $A$ is constant. This solution is unique up to harmonic polynomials of degree strictly less than $2\beta$. However, as before, the presence of any such harmonic polynomial would lead to a term in the expansion of $u_{2\beta} = u_{2\beta}'' s^{2\beta}$ which is not smooth at $s = 0$, which is impossible. In addition, $R^{2\beta} u_{2\beta}'' s^{2\beta} \to A R^{2\beta}$ as $s \to 0$, so $A$ must equal the constant $a_{01}$ in (49). This determines $u_{2\beta}''$ uniquely. Furthermore, we have also realized the first part of the index set:

$$\mathcal{E} \cap [0, 2\beta] = \{0, 1, 2, \ldots, [2\beta], 2\beta\}.$$  

We now consider the range of indices $\alpha \in (2\beta, 4\beta)$ and obtain a similar result:

**Lemma 9.** When $\alpha \in (2\beta, 4\beta)$ and $\gamma = 0$, the index set $\mathcal{E}$ is given by

$${1 + 2\beta, 2 + 2\beta, \ldots, [2\beta] + 2\beta} \bigcup \{[2\beta] + 1, [2\beta] + 2, \ldots, [4\beta]\} \bigcup \{4\beta\}.$$  

When $\alpha = \ell + 2\beta$, $u_\alpha$ is determined by $b_{(\ell-1)10}(\phi)$ in (52) and $a_{1(\ell-1)1}$ in (51). When $\alpha \in \mathbb{N}$, $u_\alpha$ is determined by $a_{1j0}(\phi)$ and $b_{(\alpha-j)j00}(\phi)$. And when $\alpha = 4\beta$, $u_\alpha$ is determined by $a_{02}$ in (49).
Proof. When $\alpha < 4\beta$, based on whether the right hand side is trivial, there are two cases. The first is when $\alpha - 2\beta = \ell \in \{1, 2, \ldots, [2\beta]\}$. As before, write $u_\alpha R^\alpha = (u_\alpha s^{-\alpha})\rho^\alpha = u''_\alpha \rho^\alpha$. Putting this into (47), commuting the Laplacian past $\rho$, and then matching the coefficients as above, we get the inhomogeneous equation

$$\hat{\Delta}_\alpha u''_\alpha = \left(\exp \sum_{j=0}^{[2\beta]} 2\rho^j u''_j\right)\ell$$

(59)

where as before, $(\cdot)_\ell$ is the coefficient of $\rho^\ell$ in the expansion of the expression in parentheses. The right side here equals

$$-\sum 2^k u''_{j_1} \ldots u''_{j_k}$$

(60)

where the sum is over all partitions $(j_1, \ldots, j_k)$ with $\sum_{i=1}^k j_i = \ell$. Each $u''_j$ is a sum of homogeneous harmonic polynomials in $\hat{z}$ of degrees strictly less than $j$. When there is only one term in the sum, i.e. $k = 1$, then $u''_{j_1}$ is a harmonic polynomial of degree $\ell - 1$ and near the “infinity” $s = 0$ the angular coefficient of $s^{-i(\ell - 1)}$ is a linear combination of $\cos((\ell - 1)\phi)$ and $\sin((\ell - 1)\phi)$. On the other hand, when there are at least factors in the summand, i.e. $k \geq 2$, then each of these products is a homogeneous polynomial of degree less than or equal to $\ell - 2$, and near infinity the angular coefficients is a trigonometric polynomial with all terms $e^\pm i\ell' \phi$ with $\ell'$ no more than the degree of $s$. To combine these two situations, the right hand side is given by

$$s^{-(\ell - 1)} e^{\pm i(\ell - 1)\phi} + \sum_{0 \leq i \leq \ell - 2, \ell' \leq i} s^{-i} e^{\pm i\ell' \phi}.$$

The solution $u''_\alpha$ is the sum of an inhomogeneous term $\sum_{i=0}^{\ell - 1} q_i$ and potential homogeneous terms. Here each inhomogeneous term $q_i(\hat{z})$ solves away the $s^{-i}$ term in the above expansion, hence $q_i \sim A_i s^{-i + 2\beta}$. For the top degree $i = \ell - 1$, $A_{\ell - 1}$ is a linear combination of $\cos((\ell - 1)\phi)$ and $\sin((\ell - 1)\phi)$. And for the lower degrees $i < \ell - 1$, $A_i$ is a trigonometric polynomial of degree at most $i$. Altogether, these terms gives $\rho^\alpha u''_\alpha = R^{2\beta + \ell} s^{2\beta + \ell} \sum_{0 \leq i \leq \ell - 1} q_i(\hat{z})$. The term $R^{2\beta + \ell} s^{2\beta + \ell} q_{\ell - 1}$ would lead to a term $\rho^1 R^{\ell - 1 + 2\beta} A_{\ell - 1}$ which matches the coefficient $a_1(\ell - 1)$ in (51). On the other hand, each of the lower order terms in the expansion $R^{2\beta + \ell} s^{2\beta + \ell} q_i, i \leq \ell - 2$, leads to a term $\rho^{\ell - i} R^{i + 2\beta} A_i(\phi)$ which matches the coefficient $b_{(\ell - i)10}(\phi)$ in (52).
Regarding the potential homogeneous terms, $u''_{\alpha}$ is unique up to addition by harmonic polynomials of degree strictly less than $\alpha - 2\beta$. However this would give a term in $u_{\alpha} = u''_{\alpha}s^\alpha$ which is not smooth. By the same reasoning as in the case $\alpha = 2\beta$, we have shown that $u''_{\alpha}$ is uniquely determined by coefficients listed above.

The other case for $\alpha \in (2\beta, 4\beta)$ is given by the homogeneous equation
\[ \Delta u''_{\alpha} = 0. \]
By the same reasoning as before, $\alpha$ must be an integer, and $u''_{\alpha} = \sum p_{\alpha}(\hat{z})$ where each $p_{\alpha}$ is a harmonic polynomial of degree $j < \alpha$. And for each $j$, the boundary asymptotic of the term $\hat{R}\hat{s}^\alpha p_{\alpha}(\hat{z})$ is given by $\rho^{\alpha-j}R^jA_{\alpha-j}(\phi)$ which is a linear combination of $\sin \ell\phi$ and $\cos \ell\phi$, hence is matched by the coefficient $a_{(\alpha-j)j0}(\phi)$ (if $\alpha-j = 1$) or $b_{(\alpha-j)j00}(\phi)$.

When $\alpha = 4\beta$, the term $R^{4\beta}u_{4\beta}$ solves $\Delta_{\ell}\hat{u}_{4\beta} = 2u''_{2\beta} \sim A|\hat{z}|^{2\beta}$. Using the same argument as for $u''_{2\beta}$, $u''_{4\beta}$ is unique and asymptotic to $B|\hat{z}|^{4\beta}$ where $B$ is given by the constant $a_{02}$ in (49). Hence we have shown
\[ \mathcal{E} \cap (2\beta, 4\beta) = \{1 + 2\beta, 2 + 2\beta, \ldots, [2\beta] + 2\beta\} \cup \{[2\beta] + 1, [2\beta] + 2, \ldots, [4\beta]\} \cup \{4\beta\}. \]

Iteratively we can repeat the argument for $\alpha \in (2(n-1)\beta, 2n\beta]$. As before there are two cases: $\alpha - 2\beta = \sum \alpha''$ for some $\alpha' \in \mathcal{E}$, which by induction means $\alpha = j + 2k\beta$ with $n \geq k \geq 1$, and $k = n$ if and only if $j = 0$; or $\alpha \in \mathbb{N}$. Note that $\alpha = 2n\beta$ is included in the first case.

When in the first case, if $k > 0$, then $u''_{\alpha}$ solves an inhomogeneous equation (59) where the right hand side is a sum of terms of order $s^{-\ell-2\beta}e^{\ell\phi}$ where $\ell < j$, and for the same reason $u''_{\alpha}$ has a unique solution asymptotic to $\sum_{\ell < j} A_{\ell} |\hat{z}|^{\ell+2k\beta}$ hence lead to a term $A_{\ell} \rho^{\ell-j} R^{\ell+2k\beta}$, and each $A_{\ell}(\phi)$ is determined by coefficients $a_{(j-1)k}$ or $b_{(j-\ell)k0}$.

On the other hand, if in the first case $j = 0$ and $k = n$, then $\alpha = 2n\beta$. Then $u''_{\alpha}$ satisfies the same equation (59) with the special requirement that all the $\gamma_i$ in (60) are of the form $2j\beta$. Then by induction the right hand side is given by $A_1 |\hat{z}|^{2(n-1)\beta}$, hence $u''_{2n\beta}$ is unique and asymptotic to $B|\hat{z}|^{2n\beta}$ which is matched by coefficient $a_{00}$.

In the second case ($\alpha \in \mathbb{N}$), $u''_{\alpha}$ solves the homogeneous equation $\Delta u''_{\alpha} = 0$ and is a combination of harmonic polynomials of degree $j < \alpha$, and by the same argument as before, each term is determined by $a_{(1j0)}(\phi)$ (if $\alpha-j = 1$) or $b_{(\alpha-j)00}(\phi)$ (if $\alpha-j \geq 2$).
With the discussion above, we have

**Proposition 8.** When $\gamma = 0$, the index set is given by

\[ \mathcal{E} = \{ j + 2k\beta : j, k \in \mathbb{N} \} \subset \mathbb{R}. \]  

(61)

A further careful matching of the log terms higher up in the series is possible, and we shall return to this issue soon. The details presented here determine the approximate solution up to some finite order and indicate precisely what is needed to carry this out to all orders.

For each term $R^\alpha u_\alpha$ with $\alpha = j + 2k\beta$, $u_\alpha(s, \phi)$ is smooth up to $s = 0$ and asymptotically given by a sum of terms with growth \( \{ s^\ell : \ell \in \mathbb{N}, \ell \leq j \} \). That is, for the solution near the corner there is a double series expansion

\[ u \sim \sum_{\alpha=j+2k\beta} R^\alpha s^\ell u_{\alpha\ell}(\phi). \]

The growth of the indices matches the definition of the polyhomogeneity.

**The approximate solution**

**Lemma 10.** For any $N \in \mathbb{N}$, there is an approximate solution $\tilde{u}$ with a polyhomogeneous expansion

\[ \tilde{u} \sim \sum R^\alpha s^\ell u_{\alpha\ell}(\phi) + \mathcal{O}(\rho^{N+\epsilon}), \]

where the sum is over $\alpha \in \mathcal{E}$, $\alpha \leq N$ or $\ell \leq N$, where $\tilde{u}$ satisfies

\[ \Delta_{g_{0,\rho}} \tilde{u} + e^{2\tilde{u}} + K_{g_{0,\rho}} = \mathcal{O}(\rho^N). \]

as $\rho = Rs \to 0$.

**Proof.** We just need to construct approximate solution to the correct order. As explained above, for any $N \in \mathbb{N}$, near $\rho = 0$ there exists

\[ \sum_{\alpha \in \mathcal{E}, \alpha < N} R^\alpha u_\alpha(s, \phi) \]

such that the difference with the actual solution is bounded by $\rho^N$. Apply to the nonlinear operator to the difference, we get the error term bounded by $\rho^N$. \hfill \Box
Hence we have constructed a solution \( \tilde{u} \) such that
\[
\Delta_{g_0,p} \tilde{u} + e^{2\tilde{u}} + K_{g_0,p} = O(\rho^N)
\]
for arbitrarily high power \( N \). Let \( \bar{g}_0 = e^{2\tilde{u}}g_0,p \) and use the relation
\[
\Delta_{g_0,p} \tilde{u} - K_{\bar{g}_0}e^{2\tilde{u}} + K_{g_0,p} = 0,
\]
we get
\[
(K_{\bar{g}_0} + 1)e^{2\tilde{u}} = O(\rho^N).
\]
Since \( \tilde{u} \) is bounded, we have
\[
K_{\bar{g}_0} + 1 = O(\rho^N).
\]

**Correction to an exact solution**

Now we want to solve the actual curvature equation to get a solution \( v \) for each \( \rho \), such that
\[
\Delta_{\bar{g}_0} v + e^{2v} + K_{\bar{g}_0} = 0 \quad (62)
\]
which is equivalent to
\[
\Delta_{\bar{g}_0} v + 2v = -(K_{\bar{g}_0} + 1) - (e^{2v} - 1 - 2v).
\]

**Proposition 9.** For a given \( N > 0 \), for each fixed \( \rho > 0 \), there is a unique bounded solution \( v_{\rho,N} \) that solves the equation (62).

**Proof.** We use maximum principle to show the existence and uniqueness of bounded solutions. Note that, to show the (fiberwise) existence and uniqueness, we only need to consider each \( \rho \) independently. First start by solving the linear equation
\[
(\Delta_{\bar{g}_0} + 2)v_0 = -(K_{\bar{g}_0} + 1)
\]
where the right hand side is \( O(\rho^N) \) as discussed above. By using sup- and sub- solutions we show the existence and uniqueness of \( v_0 \in O(\rho^N) \). Once we solve out \( v_0 \), we can iteratively solve
\[
(\Delta_{\bar{g}_0} + 2)v_i = Q(v_{i-1})
\]
where the quadratic term \( Q(v) = -(e^{2v} - 1 - 2v) = O(v^2) \). Hence use the same maximum principle we get \( v_i \in O(\rho^N) \). And therefore \( \sum_{i=0}^{\infty} v_i \) converges to the solution \( v \). \( \square \)
Then we discuss the conormality of the solution obtained above. Let $U$ be any neighborhood of $F_{12} \subset \mathcal{H}_k$ such that $U$ does not intersect any other $F_I, I \neq \{1, 2\}$. That is, the only singular fiber in this neighborhood is the one that consists of $C_{12}$. We first show that we can approximate the solution to arbitrarily high power, and the remaining term is still infinitely differentiable (conormal).

**Proposition 10.** For any $N \in \mathbb{N}$ and the approximate solution $u_N$ constructed above, the solution $v$ to the equation

$$\Delta_{g_{0,p}} (u_N + v) + e^{2u_N + v} + K_{g_{0,p}} = 0$$

is conormal on $\pi_k^{-1}(U) \subset \mathcal{C}_k$.

**Proof.** The function $v$ satisfies the equation

$$\Delta_{g_0} v + e^{2v} + K_{g_0} = O(\rho^N).$$

And the nonlinear terms are of higher power. The Laplacian is given by the following form near the corner

$$\Delta_{g_0} = -(1 + f(s, \theta)) r^{-2} ((r \partial_r - s \partial_s)^2 + \beta^{-2} \partial_\theta^2)$$

where $f$ is a smooth function. Now apply any b-vector field $V \in \mathcal{V}_b(C_k)$ to $u$, and notice the commutator relation of the operator with the basis of b-vector fields:

$$[r \partial_r, \Delta_{g_0}] = -2 \Delta_{g_0}, \quad [s \partial_s, \Delta_{g_0}] = (\partial_s f) \Delta_{g_0}$$

we then have

$$\Delta_{g_0} (V v) + 2e^{2u_N} (V v) = V K_{g_0} - [V, \Delta_{g_0}] v \in \rho^N C_b^{l,\delta}(C_k).$$

So on face I the nonlinear operator

$$\tilde{N}(f) := e^{-2u} \Delta_{g_0} f + 2f$$

has the same linearization as $[62]$. Therefore repeating the proof of Proposition 9 we have $V v \in \rho^N C_b^{l,\delta}(C_k)$, hence $v \in \rho^N C_b^{l+1,\delta}(C_k)$. Iteratively we get

$$v \in \bigcap_{l \in \mathbb{N}_0} \rho^N C_b^{l,\delta}(C_k),$$

so $u$ is conormal. \hfill \Box

**Proposition 11.** The solution $u$ to the equation $[46]$ is polyhomogeneous on $\pi_k^{-1}(U) \subset \mathcal{C}_k$ where only two cone points collide.

**Proof.** This follows from the definition of polyhomogeneity by combining Lemma 10 and Proposition 10. \hfill \Box
6.2 The general case of hyperbolic metrics with merging cone points

With the complete information of metric degeneration when two cone points coalesce, we will now do an induction argument to show that, when more points merge, the behavior of the fiber metric is analogous, and the metric is polyhomogeneous. The metric is characterized by a similar expansion, possibly with more depth in a corner, depending on the clustering of the cone points.

**Theorem 1.** For any $k \geq 2$ and fixed $\vec{\beta}$, the fiberwise conical hyperbolic metrics $G_{p,\vec{\beta}}$ are polyhomogeneous on $C_k$.

**Proof.** The actual metric will be given by

$$e^{2u}g_{0,p}$$

We first show that the proof for 2 points also works for $k$ generic points, i.e. when the collision does not hit any sub-diagonals. In that case, one can construct approximate solutions to arbitrarily high order as before. The only difference is the operator on face $C_{1\ldots k}$ is a flat conical operator with $k$ points. The rest of the proof is the same. In particular, we have the same index set $E$ as in (61) and an asymptotic expansion of the conformal factor $u$ in the hyperbolic metric $e^{2u}g_{0,p}$:

$$u \sim \sum_{\alpha=j+2k\beta} R^\alpha u_\alpha(s,\phi), \quad u_\alpha \sim \sum_{\ell<j} A_{j\ell k} s^\ell e^{i\ell\phi}.$$  

The proof for the higher codimensional corner is done by induction. Let $q$ be any point in $C_k$ and consider its associated tree $T$. Let $I$ be the node associated with $q$, and boundary face denoted as $C_I$. We will do an induction argument based on the depth of the node, denoted by $N$, which is the height of the tree $T$ from the root up to this node. Trace the node from the root, we take the following sequence of subsets $I_N \subset \cdots \subset I_1$, where $I_N$ is the top node which is associated to the highest codimensional boundary face i.e. the one $q$ is on, and $I_1$ is the bottom node, which is “biggest bubble”. We have already discussed the case $N = 1$.

For convenience, we denote $C_{I_0}$ to be the original punctured surface $M_p'$. Similar to the depth-one case, We introduce the following notation: $R_{I_j}$ is the boundary defining function for face $C_{I_j}$, $s_{I_j}$ is the local coordinates on $C_{I_j}$.
near the intersection with $\mathcal{C}_{I_{j-1}}$ (which serves as a boundary defining function for $\mathcal{C}_{I_{j-1}}$), and $\rho_{I_j} = R_{I_j} s_{I_j}$ is the global boundary defining function for the blown-down image of $\mathcal{C}_{I_{j-1}}$. Note here $\{\rho_{I_j}\}_{j=1}^N$ are independent variables.

Figure 5: coordinates near the corner of codimension two. The cone points on each fiber are marked red.

We also denote the cone angle associated to each boundary face

$$\beta_{I_j} := \sum_{i \in I_j} \beta_i - (|I_j| - 1).$$

The asymptotic expansion in the deepest corner will be given by

$$u \sim \sum_{\alpha_i \in \mathcal{E}_i} R_{I_1}^{\alpha_1} \ldots R_{I_N}^{\alpha_N} u_{\alpha}(z_N) \tag{64}$$

where $\mathcal{E}_i = \{j + 2k\beta_I : j, k \in \mathbb{N}\}$.

Now assume the result holds true for any depth $N' < N$. This would imply that, away from the deepest corner, i.e. the preimage of $\{\rho_{I_1} = \ldots \rho_{I_N} = 0\}$
under the blow down map, the polyhomogeneity of $u$ is assumed to hold with the expansion \[ [64]. \]

We construct the approximate solution in the following four steps.

Step 1: Solve out the exact solution $u'_0$ away from face $\mathcal{E}_I_1$ on the original punctured singular surface $\mathcal{E}_I_0 = \mathcal{M}'$. By the same process as before, the fiber conformal factor has an expansion of

\[
\sum_j \rho^{j_1}_I \cdots \rho^{j_N}_I u'_j(r_I, \phi_I)
\]

where we can choose a conformal coordinate on $\mathcal{E}_I_0$ such that

\[
u'_0 \sim \sum_{k \geq 1} a_{0,0k}(\phi_I) r^{2k\beta_1}_I,
\]

\[
u'_j \sim \sum_{\ell, k \in \mathbb{N}} a_{\ell, k}(\phi_I) r^{\ell + 2k\beta_1}_I (\log r_I)^{j_1 - 1} + \sum_{\ell, k \in \mathbb{N}, s \leq |j|-2} b_{\ell, k,s}(\phi_I) r^{\ell + 2k\beta_1}_I (\log r_I)^s.
\]

Step 2: Construct the solution over face $\mathcal{E}_I_1$ and use the previous coefficients $\{a_*, b_*\}$ for boundary conditions. We get the following putative expansion

\[
u \sim \sum_{\alpha_1 \in \mathcal{E}_I_1, j \in \mathbb{N}^{N-1}} R^{\alpha_1}_{I_1, j} \rho^{j_1}_{I_2} \cdots \rho^{j_N}_{I_N} u'_{1,\alpha_1,j}(s_I, \phi_I)
\]

where

\[
\mathcal{E}_I_1 = \{ j + 2k\beta_1 : j, k \in \mathbb{N} \},
\]

and

\[
u'_{1,\alpha_1,j} \sim \sum_{\ell < j} A_{(j-\ell)\ell k}(\phi_I) s^\ell_I,
\]

and $A_{(j-\ell)\ell k}$ is determined by $\{a_*, b_*\}$ above in the same way as the two point case.

Using the Borel series, we construct an approximate solution to arbitrarily high order of $\rho_{I_1}$.

Step 3: Extend the solution above and we get the expansion of $u$ near face $\mathcal{E}_I_1$ away from $\mathcal{E}_I_2$:

\[
u \sim \sum_{\alpha_1 \in \mathcal{E}_I_1, j \in \mathbb{N}^{N-1}} R^{\alpha_1}_{I_1, j} \rho^{j_1}_{I_2} \cdots \rho^{j_N}_{I_N} u'_{1,\alpha_1,j}(s_I, \phi_I)
\]

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As before, this expansion does not provide a good approximation near $C_1$. As shown in Lemma 7, the choice of $r_1$ makes the first constant term to be 0, hence when $\alpha = \beta = 0$, $u_1'(0,0) = 0$. And by a similar argument, we can choose the conformal coordinate $r_2$ on face $C_1$ (which is a punctured compactified $\mathbb{R}^2$) centered at $C_2$, so that for integer $j$, $u_1',\alpha,j \sim \sum b_{j,k} r_2^{j+2k\beta_1}$.

Then we solve for the expansion

$$\sum_{\alpha_1 \in E_1, \alpha_2 \in E_2, j \in \mathbb{N}^{-2}} R_{I_1}^{\alpha_1} R_{I_2}^{\alpha_2} \rho_{I_3}^j \cdots \rho_{I_N}^j u_2',\alpha_1,\alpha_2, j(s_{I_2}, \phi_{I_2}),$$

on face $C_{I_1}$. The equation we need to solve is again

$$\Delta_{0,p} u + e^{2u} + K_g = 0$$

and notice $\Delta(r_j) = 0, \forall j$. We have the relation

$$R_{I_1}^{\alpha_1} R_{I_2}^{\alpha_2} u_2',\alpha,j = \rho_{I_1}^{\alpha_1} \rho_{I_2}^{\alpha_2} u_2''',\alpha,j$$

where $u_2''' = s_{I_1}' s_{I_2}' u_2',\alpha,j = C s_{I_2}' u_2',\alpha,j, 0 \neq C \in \mathbb{C}$. Here we used the fact that we are solving on face $C_{I_2}$ so $s_{I_1}$ is a nonzero complex number. And notice the Laplace operator scales near face $C_{I_2}$ as

$$\Delta_{0,p} = \rho_{I_1}^{-2\beta_1} \rho_{I_2}^{-2\beta_2} \Delta_{fl}$$

where $\Delta_{fl} = e^{-2u} \hat{\Delta}$ is the Laplace operator with respect to a flat conical metric on a punctured $\mathbb{R}^2$. Then the equation is the same as before, and we have

$$\Delta_{fl} u_2''',\alpha,\alpha, j = 0, \text{ if } \alpha_2 \in \mathbb{N}$$

so $u_2'''$ is a sum of harmonic polynomials in $z$, or

$$\rho_{I_2}^{\alpha-2\beta_2} \Delta_{fl} u_2''',\alpha = \sum \prod u_2,\alpha'$$

and there is a unique solution given by sum of terms with growth $s_2^j$. So we have

$$E_{I_2} = \{ j + 2k_2^j \beta_2 \}.$$ 

And near the corner we have an expansion of

$$u \sim \sum_{\alpha_1 \in E_1, \alpha_2 \in E_2, j \in \mathbb{N}^{-2}} R_{I_1}^{\alpha_1} R_{I_2}^{\alpha_2} \rho_{I_3}^j \cdots \rho_{I_N}^j u_2',\alpha_1,\alpha_2, j(s_{I_2}, \phi_{I_2}).$$
Step 4: Iterate the step 3 to get the expansion in deeper corners, each time replacing $\rho_{I_i}^j$ to $R_{I_i}^{\alpha_i}$. And by the same reasoning, we have

$$\mathcal{E}_{I_i} = \{j + 2k\beta_{I_i} : j, k \in \mathbb{N}\}. \quad (65)$$

We note here that this expansion matches with nearby fibers: when some of the $\rho_{I_i} \neq 0$, the terms in the expansion matches $[64]$.

Step 5: Conormality and polyhomogeneity. From the discussion above, for any positive integers $\hat{M} \in \mathbb{N}^N$, we can find approximate solution up to order $\rho^M$ by taking the Borel sum

$$u_M = \sum_{\alpha_i \in \mathcal{E}_{I_i}, \alpha_1 < M_1, \ldots, \alpha_N < M_N} R_{I_i}^{\alpha_1} \ldots R_{I_N}^{\alpha_N} u(\hat{z}_N)$$

then one can use maximal principle to show that there exists a unique correction term $v_M$ such that

$$\Delta g_{0,p} (u_M + v) + e^{2(u_M + v)} + K_{g_{0,p}} = 0.$$

By the same argument as before, using commutator relation of $R\partial_R, s\partial_s$ with $\Delta g_{0,p}$, $v$ is conormal and is contained in $\cap_{j=0}^{\infty} \rho^M C_b^j(C_k)$. By repeating the argument for arbitrarily large $M$, we get the polyhomogeneity of $u$.

7 The spherical case with all cone angles less than $2\pi$

We consider the same merging process for spherical metrics with small cone angles here. The spherical case with all cone angles less than $2\pi$ follows a similar construction as in the hyperbolic case. If $\beta \in (0, 1)^k$, then the cone angle obtained by merging any of those would still be less than $2\pi$:

$$2\pi \left( \sum_{i \in \mathcal{I}} (\beta_i - 1) + 1 \right) \in (0, 2\pi), \forall \mathcal{I} \subset \{1, \ldots, k\}.$$

The only difference in the proof from the hyperbolic case, is that the invertibility of the linearized operator does not always hold. We recall the following lemma from [13] regarding the spectrum of the Friedrichs extension of Laplacian:
Lemma 11 ([13], Proposition 13). For a spherical conical metric \( g \) on a sphere with all cone angles less than \( 2\pi \), unless it is a football, the first eigenvalue of the Laplacian associated to \( g \) is always strictly bigger than 2.

Here a spherical football is a spherical-suspension of a circle of length \( 2\pi \beta \) and is a genus zero surface with two antipodal conical points, each with angle \( 2\pi \beta \).

Remark. When the merging limit is a spherical football, it is the only case that the linearized operator \( \Delta_g - 2 \) is non-invertible on the singular fiber, and it exhibits a different analytic behavior. This case will be discussed separately in an upcoming paper.

We also recall the following existence and uniqueness result.

Lemma 12 ([17, 11]). When \( k \geq 3 \), there exists a unique spherical conical metric on \( M \) when \( \chi(M, \vec{\beta}) > 0 \). When \( k = 2 \) and \( M = S^2 \), there is a unique metric up to conformal dilation, if and only if \( \beta_1 = \beta_2 \).

We will follow the same strategy as in the hyperbolic case in Theorem 1 to show that the fiberwise metrics lift to be polyhomogeneous.

Theorem 2. For any family of spherical metrics with fixed cone angles \( \{2\pi \beta_i\} \) where cone points merge with the appropriate angle limits, except the case when the limiting shape is a spherical football, the fiberwise spherical metrics are polyhomogeneous on \( C_k \).

Proof. As illustrated in Theorem 1, we only need to show that for any point \( q \) in a corner \( \cap_{i=1}^\ell F_{I_i} \) where \( I_1 \supset \cdots \supset I_\ell \) are the nodes associated to \( q \) in the tree, the fiber spherical metrics \( G_p \) are polyhomogeneous near \( q \).

We start with a model metric \( g_{0,p} \), defined similarly to (45), by glueing the lifted flat conical metrics with spherical metrics away from the boundary face:

\[
g_{0,p} = \chi g_{fl}^{I_1} + (1 - \chi) g^{sph}_{p'}.
\]

Then we construct approximate solutions \( u_N \) using the same argument as before. The only difference is in the initial step solving the expansion on the original surface, we use uniqueness and existence result from Lemma 12. And by replacing \( \sinh \tilde{r} \) to \( \sin \tilde{r} \) in (48), the same argument as in Lemma 7 holds for this case, showing that the exact solution restricting to the singular fiber \( M_p' \) has an expansion near the conical point given by

\[
u_0' \sim \sum_{j \in \mathbb{N}} a_j \tilde{r}^{2j}.
\]
Since the rescaled flat conical metrics on the boundary faces and the equations to solve on those faces are the same as in the hyperbolic case, the rest of the construction of approximate solutions follows. And we obtain an asymptotic expansion

\[
G_p = e^{2u}g_{0,p}, \quad u \sim \sum_{\alpha_i \in \mathcal{E}_{I_i}} R_{I_i}^{\alpha_1} \ldots R_{I_N}^{\alpha_N} u_G(z_N). \tag{66}
\]

where \(\mathcal{E}_{I_i}\) is defined in (65). This way we get approximate solutions to arbitrarily high order.

Then for \(N \gg 0\), we apply implicit function theorem to get the remainder \(V_N\)

\[
\Delta_{g_{0,p}}(u_N + v_N) - e^{2(u_N + v_N)} + K_{g_{0,p}} = 0.
\]

By the “no football” assumption, the linearized operator for the above equation, \(\Delta e^{2u_N}g_{0,p} - 2\), is invertible on the singular fiber, since \(e^{2u_N}g_{0,p}\) restricted to \(M_{p'}\) is the exact spherical conical metric and hence Lemma 11 applies. And for any nearby fiber, by continuity of eigenvalue, the operator \(\Delta e^{2u_N}g_{0,p} - 2\) is also invertible. Hence by implicit function theorem, there exists a correction term \(v_N\) for any large \(N\).

Then the same commutator argument in Lemma 10 can be applied to show that \(v_N \in \bigcap_{\ell \in \mathbb{N}} \rho^N \mathcal{C}_b^{\ell,\alpha}\) is conormal for any \(N\). And together with the expansion (66), this shows that \(u\) is polyhomogeneous and hence the fiber metrics given by

\[
G_p = e^{2u}g_{0,p}
\]

is polyhomogeneous. \(\square\)

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