Critical behavior of the two-dimensional icosahedron model

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(Dated: September 2, 2018)

In the context of a discrete analogue of the classical Heisenberg model, we investigate critical behavior of the icosahedron model, where the interaction energy is defined as the inner product of neighboring vector spins of unit length pointing to vertices of the icosahedron. Effective correlation length and magnetization of the model are calculated by means of the corner-transfer matrix renormalization group (CTMRG) method. Scaling analysis with respect to the cutoff dimension m in CTMRG reveals the second-order phase transition characterized by the exponents ν = 1.62 ± 0.02 and β = 0.12 ± 0.01. We also extract the central charge from the classical analogue of the entanglement entropy as c = 1.90 ± 0.02, which cannot be explained by the minimal series of conformal field theory.

I. INTRODUCTION

Statistical models with short range interactions on two-dimensional (2D) regular lattices exhibit no spontaneously symmetry breaking at finite temperature, if the symmetry in local degrees of freedom is continuous. The classical ferromagnetic XY model is a typical example, which has O(2) symmetry, where the thermal average of the magnetization is zero at finite temperature. An introduction of discrete nature to local degrees of freedom then induces an order-disorder transition in low temperature, where the universality class is dependent on the type of discretization. The q-state clock model, which has $Z_q$ symmetry, is a well-known discrete analogue of the XY model. For the case of $q ≤ 4$, the clock model exhibits a second-order phase transition described by unitary minimal series of conformal field theory (CFT). If $q > 4$, the clock model has an intermediate critical phase between the high-temperature disordered phase and low-temperature ordered phase [2, 3], where transitions to the critical phase are of Berezinskii-Kosterlitz-Thouless (BKT) type [4, 5]. As $q$ increases, the low-temperature ordered phase shrinks, and the O(2) symmetry is finally recovered in the limit $q → ∞$.

Discretization of the classical Heisenberg model, which has O(3) symmetry, is not straightforward, in the sense that there is no established route of taking continuous-symmetry limit. A possible manner of discretization is to introduce the polyhedral anisotropies, such as tetrahedral, cubic, octahedral, icosahedral, and dodecahedral ones, which correspond to the discrete subgroups of the O(3) symmetry group. Let us consider the discrete vector-spin models, where on each lattice site there is a unit vector spin that can point to vertices of a polyhedron. The tetrahedron model can be mapped to the four-state Potts model [6]. For the octahedron model, presence of weak first-order phase transition is suggested by Patrascioiu and Seiler [10], and afterward is numerically confirmed [11]. The cube model can be mapped to three decoupled Ising models. Patrascioiu et al reported a second-order transition for the icosahedron and dodecahedron models, respectively, which have 12 and 20 local degrees of freedom [11, 12, 13]. For the icosahedron model, the estimated transition temperatures is $1/T_c = 1.802 ± 0.001$ and its critical indices are $ν ∼ 1.7$ and $γ ∼ 3.0$, which are inconsistent with the minimal series of CFT. By contrast, Surungan et al gave another estimation $ν ∼ 1.31$ for the same transition temperature [14]. However, the system size of Monte Carlo simulations in previous works may be too small to conclude the universality of the icosahedron model. Finally, a possibility of an intermediate phase is suggested for the dodecahedron model in Refs. [12] and [13], whereas a solo second-order transition is suggested in Ref. [14].

In this article, we focus on the critical behavior of the icosahedron model. We calculate magnetization, effective correlation length and entanglement entropy in the bulk limit by means of the corner-transfer-matrix renormalization group (CTMRG) method [15, 16], which is based on Baxter’s corner-transfer matrix (CTM) scheme [17, 18]. An advantage of the CTMRG method is that we can treat sufficiently large system size to obtain the conventional bulk physical quantities. Actually, the system size of CTM in this work is up to $10^4 × 10^4$ sites, which can be viewed as a bulk limit in comparison with (effective) correlation length of the system. Instead, CTMRG results are strongly dependent on $m$, the number of states kept for the block-spin variables, near the transition point. Nevertheless, this $m$-dependence of CTMRG results provides a powerful tool of the scaling analysis with respect to $m$ [20, 23].
ments $\nu = 1.62 \pm 0.02$ and $\beta = 0.12 \pm 0.01$. Another interesting point on the CTMRG approach is that the classical analogue of the entanglement entropy can be straightforwardly calculated through a reduced density matrix constructed from CTMs. The $m$-dependence analysis of the entanglement entropy also yields the central charge $c = 1.90 \pm 0.02$, which cannot be explained by the minimal series of CFT.

This article is organized as follows. In the next section, we introduce the icosahedron model, and briefly explain its tensor-network representation and CTMRG method. We first show the temperature dependence of the magnetization to capture the nature of the phase transition. In Section III, we apply the finite-$m$ scaling to the effective correlation length, magnetization, and the entanglement entropy. Transition temperature, critical exponents, and the central charge are estimated in detail. The results are summarized in the last section.

II. ICOSAHEDRON MODEL

Let us consider the icosahedron model, which is a discrete analog of the classical Heisenberg model. On each site of the square lattice, there is a vector spin $\mathbf{v}^{(p)}$ of unit length, which points to one of the vertices of the icosahedron, shown in Fig. 1 (a), where $p$ is the index of vertices running from 1 to 12. Figure 1 (b) shows four vector spins $\mathbf{v}^{(p)}$, $\mathbf{v}^{(q)}$, $\mathbf{v}^{(r)}$, and $\mathbf{v}^{(s)}$, around a ‘black’ plaquette, where we have introduced the chess-board pattern on the lattice. We have omitted the lattice index of these spins, since they can be formally distinguished by the minimal series of CFT.

In Baxter’s CTM formulation, the whole lattice is divided into four quadrants, as shown in Fig. 2. The partition function of a square-shaped finite-size lattice is expressed by a trace of the fourth power of CTMs

$$Z = \text{Tr} C^4,$$

where $C$ denotes the CTM. Note that each matrix element of $C$ corresponds to the partition function of the quadrant where the spin configurations along the row and column edges are specified. We numerically obtain $Z$ by means of the CTMRG method, where area of CTM is increased iteratively by repeating the system-size extension and renormalization group (RG) transformation. Then, the matrix dimension of $C$ is truncated with cutoff dimension $m$, and under an appropriate normalization, $C$ converges to its bulk limit after a sufficient number of iterations, even if we assume a fixed boundary condition. All the numerical data shown in this article are obtained after such convergence. The numerical precision of CTMRG results are controlled by the cutoff $m$ for the singular value spectrum $\{\lambda_i\}$ of CTMs with a truncation error $\epsilon(m) = 1 - \sum_i^m \lambda_i^4$. The universal distribution of the spectrum suggests that the asymptotic behavior of $\epsilon(m)$ could be model independent.

In practical computations, we assume the fixed boundary condition, where all the spins are pointing to the direction $\mathbf{v}^{(1)}$ on the boundary of the system. We define an
order parameter as the magnetization $M$ at the center of the system

$$M = \frac{1}{2} \sum_{s=1}^{12} \left( \psi^{(1)} \cdot \psi^{(s)} \right)^T \left[ C^t \right],$$

where $\psi^{(s)}$ is the vector spin at the center, and $T^t$ represents partial trace except for $\psi^{(s)}$. Figure 3 shows the temperature dependence of the magnetization $M$ calculated with $m = 100, 200, 300, 400$, and $500$. The magnetization is well converged with respect to $m$ for $T < 0.55$ or $T > 0.57$, and the result supports emergence of the ordered phase in low-temperature region as reported by Patrascioiu et al. [10, 12, 13]. As shown in the inset, however, the curve of $M$ has the shoulder structure exhibiting the strong $m$ dependence in the region $0.55 < T < 0.57$.

In order to see the nature of the observed shoulder structure in $M$, we calculate the probability $P^{(s)}$ of finding $\psi^{(s)}$ at the center of the system. Figure 4 shows the temperature dependence of $P^{(s)}$ with $m = 500$. In the region $T < 0.55$ the probability $P^{(1)}$ is dominant. Around $T \sim 0.56$, the value of $P^{(s)}$ for $s = 2, 3, 4, 5$, and $6$ are comparable to $P^{(1)}$, and the sum $P^{(2)} + P^{(3)} + P^{(4)} + P^{(5)} + P^{(6)}$ is larger than $P^{(1)}$. Such a marginal behavior might suggest a possibility of an intermediate (or floating) critical phase, like the two succeeding phase transitions in clock-like models $29, 35$. We perform the scaling analysis with respect to $m$ to clarify the nature of the phase transition in the next section.

**III. SCALING ANALYSIS**

As described above, the calculated results of the magnetization $M$ exhibit the finite-$m$ dependence near the transition point. In general, the cutoff dimension $m$ for the CTM introduces an effective correlation length in the critical region $39, 40$, which corresponds to a regularization for the infrared divergence. By controlling the cutoff $m$, we can systematically analyze the critical behavior in the vicinity of the critical point, which we call the finite-$m$ scaling $21, 23$, which shares many aspects in common with the finite-size scaling analysis $24, 23$.

In the scaling analysis, one generally assumes that an observable $A$ with the scaling dimension $x_A$ obeys the following scaling function,

$$A(b, t) = b^x A_{b, t} \left( b^{1/\nu} t \right),$$

where $t = T/T_c - 1$ is the scaled temperature and $b$ is a characteristic length scale intrinsic to the system, which basically corresponds to the correlation length. In finite-size scaling analysis, $b$ is replaced by the linear dimension of system $\ell$ and then, the scaling function $f_A$ is extracted by a systematic control of $\ell$. Note that the asymptotic forms $f_A(y) \sim y^{-x_A}$ for $y \gg 1$ and $f_A(y) \sim \text{const.}$ for $y \to 0$ are also assumed in Eq. (5), in order to reproduce the proper scaling law in the bulk limit $\ell \to \infty$.

For the finite-$m$ scaling in CTMRG, meanwhile, we introduce a well-controllable length scale through the cutoff dimension $m$, instead of $\ell$. After sufficient number of iterations in CTMRG, we have a renormalized row-to-row transfer matrix. We can then define an effective correlation length

$$\xi(m, t) = \left[ \ln(\zeta_1/\zeta_2) \right]^{-1},$$

where $\zeta_1$ and $\zeta_2$ are the largest and second-largest eigenvalues of the renormalized row-to-row transfer matrix. Note that the unit of length is set as the lattice constant. An essential point is that the following scaling relation can be assumed,

$$\xi(m, t) \sim m^\kappa g \left( m^{\kappa/\nu} t \right),$$

with the asymptotic forms $g(y) \sim |y|^{-\nu}$ for $y \gg 1$ and $g(y) \sim \text{const.}$ for $y \to 0$. Each limit yields the behavior $\xi(m, t) \sim t^{-\nu}$ for a finite $t$ under the condition $m^\kappa \gg t^{-\nu}$, and $\xi(m, t) \sim m^\kappa$ for a finite $m$ under $m^\kappa \ll t^{-\nu} [21, 22]$. Note that $\kappa$ is an independent dimension scaling, which
is characteristic to the matrix-product-state (MPS) description of the eigenvector of the row-to-row transfer matrix. Combining $b \sim \xi(m, t)$ and Eq. (5), we obtain the finite-$m$ scaling formula as

$$A(m, t) = m^{x_{A} \nu / \nu} \chi_{A}\left(m^{\nu / \nu} t\right),$$  
(8)

where $\chi_{A}$ is a new scaling function satisfying $\chi_{A}(y) \sim |y|^{-x_{A}}$ for $y \gg 1$. For a finite $t$ under the condition $m^{\nu / \nu} t \gg 1$, Eq. (8) reproduces $A(m, t) \sim |t|^{-x_{A}}$, while for a finite $m$ under $m^{\nu / \nu} t \ll 1$, Eq. (8) gives $A(m, t) \sim m^{-x_{A} \nu / \nu}$.

We apply the scaling analysis to several quantities calculated by CTMRG, with help of the Bayesian analysis for fitting \[41\]. We consider the temperature region $0.520 \leq T \leq 0.619$ for $m = 100, 200, 300, 400$, and $500$ in the following scaling analysis. We first apply the analysis to $\xi(m, t)$ in Eq. (6) and estimate critical temperature $T_{c}$, exponents $\kappa$ and $\nu$. Figure 5 (a) shows the scaling plot of $\xi(m, t)$ with the best fit values, $T_{c} = 0.555048(43)$, $\nu = 1.617(13)$, and $\kappa = 0.8983(17)$, where all data collapse on the scaling function $g$ in Eq. (7). The fitting errors in the Bayesian analysis are shown in the round brackets. If we use the data for $200 \leq m \leq 500$, we obtain $T_{c} = 0.554940(42)$, $\nu = 1.623(13)$, and $\kappa = 0.8830(19)$. Comparing these two fitting results, we adopt $T_{c} = 0.5550 \pm 0.0001$, $\nu = 1.62 \pm 0.02$, and $\kappa = 0.89 \pm 0.02$. This result of $T_{c}$ is consistent with the values $T_{c} \approx 0.555$ reported by both Patrascioiu et al.\[10, 12, 13\] and Surungan et al.\[14\]. While, the critical exponent $\nu$ is consistent with the value $\nu = 1.7^{+0.3}_{-0.1}$ in Refs. \[10, 12, 13\], but has a discrepancy from $\nu = 1.31 \pm 0.01$ in Ref. \[14\].

On the basis of the above $T_{c}$, $\nu$, and $\kappa$, moreover, we perform the finite-$m$ scaling analysis for the magnetization $M$ shown in Fig. 3. A particular point is that the shoulder structure in the inset of Fig. 3 directly reflects on the scaling function of Fig. 5 (b). Moreover, such shoulder structures of the scaling functions are consistently observed in Figs. 5 (a) and (c). These behaviors imply that the transition of the icosahedron model is described by a solo second-order transition, unlike to the clock models of $q > 4$ where the intermediate critical region emerges. Using the Basian fitting, then, we obtain $\beta = 0.1293(27)$ for $m = 100 \sim 500$ and $\beta = 0.1234(33)$ for $m = 200 \sim 500$. Taking into account the discrepancy, we adopt $\beta = 0.12 \pm 0.01$. We however think that this value should be improved in further extensive calculations.

In order to obtain additional information for the scaling universality, we calculate the classical analogue of the entanglement entropy. The concept of entanglement can be introduced to two-dimensional statistical models through the quantum-classical correspondence \[12, 43\]. Then, an essential point is that the fourth power of CTM, which appears in Eqs. (3) and (4), can be interpreted as a density matrix of the corresponding one-dimensional quantum system \[40\]. From the normalized density matrix

$$\rho = \frac{C^{4}}{Z},$$
(9)

we obtain the classical analogue of the entanglement entropy, in the form of Von Neumann entropy \[47, 48\]

$$S_{E} = -\text{Tr} \rho \ln \rho.$$  
(10)

In the context of CTMRG, the following relation

$$S_{E}(m, t) \sim \frac{c}{6} \ln \xi(m, t) + \text{const.},$$
(11)

is satisfied around the criticality \[40, 50\], where $c$ is the central charge. Taking the exponential of both sides of this equation, and substituting Eq. (7), we obtain

$$e^{S_{E}} \sim a \left[\xi(m, t)^{c/6}\right] = a \left[m^{\kappa} g(m^{\nu / \nu} t)^{c/6}\right] = m^{\kappa c/6} \tilde{g}(m^{\nu / \nu} t),$$
(12)

where $a$ is a non-universal constant, and $\tilde{g} \equiv a g^{c/6}$. Thus the critical exponent for $e^{S_{E}}$ is identified as $c/6$.

Using $T_{c}$, $\kappa$ and $\nu$ previously obtained by the finite-$m$ scaling for $\xi(m, t)$, we can estimate the central charge $c$. 

FIG. 5. (Color online) Finite-$m$ scalings for (a) correlation length in Eq. (6), (b) magnetization $M$ in Eq. (4), and (c) entanglement entropy in Eq. (10).
Figure 5 (c) shows the scaling plot of Eq. (12) for the data of \( m = 100, 200, 300, 400, \) and \( 500 \). The central charge is estimated as \( c = 1.894(12) \). If we exclude the case \( m = 100 \) for the scaling analysis, we obtain \( c = 1.900(15) \). Considering the discrepancy between the above values of \( c \), we adopt \( c = 1.90 \pm 0.02 \).

Here, it should be noted that this value is consistent with the relation
\[
\kappa = \frac{6}{c(\sqrt{12}/c + 1)},
\]
which is derived from the MPS description of one-dimensional critical quantum system.\cite{winter12} Substituting \( c = 1.90 \) and \( \kappa = 0.89 \) to Eq. (13), we actually have \( 6/(c(\sqrt{12}/c + 1)) - \kappa = 0.009 \), which provides a complementary check of the finite-\( m \) scaling in CTMRG.

**IV. SUMMARY AND DISCUSSION**

We have investigated the phase transition and its critical properties of the icosahedron model on a square lattice, where the local vector spin has twelve degrees of freedom. We have calculated the magnetization, the effective correlation length, and the classical analogue of the entanglement entropy by means of the CTMRG method. The CTMRG results are strongly dependent on \( m \), which is the cutoff dimension of CTMs, near the critical point. We have then performed the finite-\( m \) scaling analysis and found that the all numerical data can be well fitted with the scaling functions including the shoulder structures. We have thus confirmed that the icosahedron model exhibits the second-order phase transition at \( T_c = 0.5550 \pm 0.0001 \), below which the icosahedral symmetry is broken to a five-fold axial symmetry. Also, the scaling exponents are estimated as \( \nu = 1.62 \pm 0.02 \), \( \kappa = 0.89 \pm 0.02 \), and \( \beta = 0.12 \pm 0.01 \). From the relation between entanglement entropy and the effective correlation length, moreover, we have extracted the central charge as \( c = 1.90 \pm 0.02 \), which cannot be described by the minimal series of CFT. To clarify the mechanism of such a non-trivial critical behavior in the icosahedron model is an important future issue.

Our original motivation wasfrom the systematical analysis of the continuous-symmetry limit toward the \( O(3) \) Heisenberg spin. In this sense, the next target is the dodecahedron model having twenty local degrees of freedom, which requires massive parallelized computations of CTMRG. In addition, it is an interesting problem to introduce the XY-like uniaxial anisotropy to the icosahedron and dodecahedron models; A crossover of universality between the icosahedron/dodecahedron model and the clock models can be expected, where the shoulder structures of the scaling functions may play an essential role.

**V. ACKNOWLEDGMENT**

This research was partially supported by Grants-in-Aid for Scientific Research under Grant No. 25800221, 26400387, 17H02931, and 17K14359 from JSPS and by VEGA 2/0130/15 and APVV-16-0186. It was also supported by MEXT as “Challenging Research on Post-K computer” (Challenge of Basic Science: Exploring the Extremes through Multi-Physics Multi-Scale Simulations). The numerical computations were performed on the K computer provided by the RIKEN Advanced Institute for Computational Science through the HPCI System Research project (Project ID:hp160262).

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