RADIAL SYMMETRY FOR A QUASILINEAR ELLIPTIC EQUATION WITH A CRITICAL SOBOLEV GROWTH AND HARDY POTENTIAL

FRANCESCANTONIO OLIVA, BERARDINO SCIUNZI AND GIUSI VAIRA

Abstract. We consider weak positive solutions to the critical $p$-Laplace equation with Hardy potential in $\mathbb{R}^N$

$$-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1}$$

where $1 < p < N$, $0 \leq \gamma < \left(\frac{N-p}{p}\right)^p$ and $p^* = \frac{Np}{N-p}$.

The main result is to show that all the solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ are radial and radially decreasing about the origin.

1. Introduction and statement of the main result

We study the doubly critical problem

$$\begin{aligned}
-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} &= u^{p^*-1} & \text{in } \mathbb{R}^N \\
u > 0 & & \text{in } \mathbb{R}^N \\
u \in \mathcal{D}^{1,p}(\mathbb{R}^N)
\end{aligned}$$

(1.1)

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator with $1 < p < N$, $0 \leq \gamma < \gamma_p := \left(\frac{N-p}{p}\right)^p$ and $p^* := \frac{Np}{N-p}$ is the critical exponent for the Sobolev embedding. Here $\mathcal{D}^{1,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$, the space of smooth functions with compact support, with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}.$$

By standard regularity theory, see [12, 23], it follows that solutions to (1.1) are of class $C^{1,\alpha}$ far from the origin.

We address the study of the classification of positive solutions to (1.1). As we shall discuss later on, this is a crucial issue since problem (1.1) naturally appears in the study of $p$-Hardy-Sobolev inequalities as well as it appears as a limiting problem in many applications. Our main effort is to show that all the positive solutions to (1.1) are radial (and radially decreasing) about the origin. Once the radial symmetry of the solution is proved it is easy to derive the associated ordinary differential equation fulfilled by the solution $u = u(r)$. The classification result reduces therefore to an ODE analysis that has been already carried out in [1] where the radial symmetry of the solutions was an assumption.
Let us start discussing the simpler case $\gamma = 0$. In this case the problem reduces to the following critical one

\[
\begin{align*}
- \Delta_p u &= u^{p^* - 1} \quad \text{in } \mathbb{R}^N, \\
u &= 0 \quad \text{in } \mathbb{R}^N, \\
u &\in D^{1,p}(\mathbb{R}^N).
\end{align*}
\]

For such a problem a huge literature is available and the classification of positive weak solutions of (1.2) is well understood. Indeed, for $\delta > 0$ and $x_0 \in \mathbb{R}^N$, an explicit family of solutions to (1.2) is given by

\[
V_{\delta,x_0}(x) := \left(\frac{\delta \frac{1}{p-1} \alpha_{N,p}}{\delta^{\frac{1}{p-1}} + |x - x_0|^{\frac{1}{p-1}}} \right)^{\frac{N-p}{p}},
\]

where $\alpha_{N,p} := \frac{N}{p-1} \left(\frac{N-p}{p-1}\right)^{\frac{1}{p-1}}$. The family of functions given by (1.3) are the minimizers to

\[
S_p := \inf_{\varphi \in D^{1,p}(\mathbb{R}^N), \varphi \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^p}{\left(\int_{\mathbb{R}^N} \varphi^{p^*} \right)^{\frac{p}{p^*}}},
\]

and the classification of the minimizers (see [21]) follows via symmetrization arguments. Note that such a technique can be applied in the same way both in the semilinear case $p = 2$ and in the quasilinear case $1 < p < \infty$.

Furthermore, if we restrict the attention to the class of radial solutions, then the analysis carried out in [14] shows that all the regular radial solutions to (1.2) are given by (1.3).

For $p = 2$ all the solutions to the equation are classified by (1.3) as a consequence of the results in [2] where the Kelvin transform is strongly exploited. A Kelvin type transformation is not applicable for the quasilinear case and this fact causes that a different proof is needed. When no a priori assumption are imposed, the classification of all the positive solutions to (1.2) (showing that all the solutions to (1.2) are given by (1.3)) has been in fact an open and challenging problem recently solved in [7, 19, 24] (see also [8, 9]). The techniques used are mainly based on a fine asymptotic analysis at infinity and refined versions of the moving plane procedure, see [13, 20].

Let us now turn to the case $0 < \gamma < \gamma_p$ but in the case $p = 2$ so that $\gamma_2$ is the best constant in the Hardy-Sobolev inequality for $p = 2$. For

\[
S_{2,\gamma} = \inf_{\varphi \in D^{1,2}(\mathbb{R}^N), \varphi \neq 0} \frac{\int_{\mathbb{R}^N} \left( |\nabla \varphi|^2 - \frac{\gamma}{|x|^2} \varphi^2 \right)}{\left(\int_{\mathbb{R}^N} |\varphi|^{2^*} \right)^{\frac{2}{2^*}}},
\]

it is known that $S_{2,\gamma}$ is attained and extremals for $S_{2,\gamma}$ have the form (up to a multiplicative constant)

\[
U_\delta(x) = \delta^{-\frac{2^*}{2}} U \left( \frac{x}{\delta} \right) = \frac{\alpha_N \delta^\Gamma}{|x|^{\beta - (\delta \frac{2^*}{2} - |x|^{-\frac{1}{2}} \frac{2^*}{2})}}, \quad \delta > 0,
\]

where

\[
U(x) = \frac{\alpha_N}{|x|^{\beta - (1 + |x|^{-\frac{1}{2}}) \frac{2^*}{2}}} = \frac{\alpha_N}{\left( |x|^{\frac{2^*}{2} \beta - |x|^{-\frac{1}{2}} \frac{2^*}{2}} + |x|^{\frac{2^*}{2} \beta - |x|^{-\frac{1}{2}} \frac{2^*}{2}} \right)^{\frac{2^*}{2}}}
\]
with
\[
\Gamma = \sqrt{\frac{(N - 2)^2}{4} - \gamma}, \quad \beta_\pm = \frac{N - 2}{2} \pm \Gamma, \quad \alpha_N = \left[ \frac{4\Gamma^2 N}{N - 2} \right]^{\frac{N-2}{4}},
\]
see [3, 12, 22]. Moreover (1.5) gives all the solutions of the problem (1.1) for \( p = 2 \) and \( \gamma \in (0, \gamma_2) \) and this has been proved in the celebrated paper [22]. In the case \( p = 2 \) it is also known that when \( \gamma < 0 \) then \( S_{2, \gamma} \) is not attained even if (1.5) are still solutions of the problem.

Here we are concerned with the quasilinear doubly critical case \( 1 < p < N \) and \( \gamma \in (0, \gamma_p) \). It is worth recalling that in [1] the authors considered minimization problem:

\[
S_{p, \gamma} = \inf_{\varphi \in D^{1, p}(\mathbb{R}^N), \varphi \neq 0} \left( \frac{\int_{\mathbb{R}^N} \left( |\nabla \varphi|^p - \frac{\gamma}{|x|^p} \varphi^p \right)}{\left( \int_{\mathbb{R}^N} |\varphi|^p \right)^{\frac{p}{p^*}}} \right).
\]

(1.6)

It follows that \( 0 < S_{p, \gamma} < S_p \) where \( S_p \) is defined in (1.4) and \( S_{p, \gamma} \) is attained by a function \( u_0(x) \) which is not explicit. It has been proved in [1] that all minimizers of (1.6) are radial. Also uniqueness up to scaling of the radial solutions as well as the asymptotic behavior are proved showing in particular that, given a radial solution \( u = u(r) \) to (1.1), then

\[
\lim_{r \to 0} r^{\gamma_1} u(r) = C_1, \quad \lim_{r \to +\infty} r^{\gamma_2} u(r) = C_2
\]

and

\[
\lim_{r \to 0} r^{\gamma_1+1} |u'(r)| = C_1 \gamma_1, \quad \lim_{r \to +\infty} r^{\gamma_2+1} |u'(r)| = C_2 \gamma_2,
\]

for some positive constants \( C_1, C_2 \). Here and hereafter \( \gamma_1, \gamma_2 \in [0, +\infty) \), \( \gamma_1 < \gamma_2 \) are defined as the two roots of the equation

\[
\mu^{p-2} \left[ (p-1) \mu^2 - (N-p) \mu \right] + \gamma = 0.
\]

(1.7)

We remark (for later use) that

\[
0 \leq \gamma_1 < \frac{N-p}{p} < \gamma_2 \leq \frac{N-p}{p-1}.
\]

Note that when \( p = 2 \) then \( \gamma_1 = \beta_- \) and \( \gamma_2 = \beta_+ \). Instead, when \( p \neq 2 \) but \( \gamma = 0 \) then \( \gamma_1 = 0 \) and \( \gamma_2 = \frac{N-p}{p-1} \). Moreover in [25, 26] the author extends the results on the asymptotic behavior proved for radial solutions in [1] to all weak positive solutions of (1.1).

We shall prove here that actually all positive solutions to (1.1) are radially symmetric thus allowing to deduce that the characterization of the solutions described here above do apply to all positive solutions. In particular, as a consequence of our result, we deduce uniqueness up to scaling of the positive solutions as well as the their asymptotic behavior at the origin and at infinity.

Our main result is the following:

**Theorem 1.1.** Assume \( \gamma \in (0, \gamma_p) \) and let \( u \) be a positive solution to (1.1). Then \( u \) is radial and radially decreasing with respect to the origin.

All the proofs of the classification results described above are based on the use of the *moving plane method*. When \( p \neq 2 \) this is completely not trivial because of the nonlinear degenerate nature of the operator. In our case, when trying to adapt the techniques developed in [9, 10, 19], an obstruction occurs due to the homogeneity of the Hardy potential. In particular this fact is related to the nonlinear nature of the operator that also obstructs the application of the...
techniques introduced in [22]. In fact, to face this fact, we exploit a different test function technique that, on the other hand, introduces several difficulties as the reader shall see. Let us also stress that, for the absence of the Kelvin transformation, an analysis on the behaviour at infinity is needed. We will in fact exploit the results in [25, 26] and in particular our Theorem 3.3.

1.1. Notations. Throughout the paper, we denote by Ω the complement of a domain Ω ⊂ \( \mathbb{R}^N \) in \( \mathbb{R}^N \), by 
\[ C^k_0(\mathbb{R}^N) = \{ u \in C^k(\mathbb{R}^N) : u(x) \to 0 \text{ as } |x| \to +\infty \} , \]
and by \( B_R(x_0) \) the ball of radius \( R \) centered at \( x_0 \in \mathbb{R}^N \).
Moreover \( \chi_\Omega \) is the characteristic function of the set \( \Omega \), \( (v-w)^+ := \max\{v-w,0\} \) and \( (v-w)^- := \min\{v-w,0\} \).
Finally we underline that we will denote by \( C, C_i, c_i \) several constants whose value may change from line to line and, sometimes, on the same line. However these values will be not relevant in the proofs.
We remark that the potential \( |x|^{-p} \) is related to the Hardy-Sobolev inequality. More precisely, for all \( u \in D^{1,p}(\mathbb{R}^N) \), one has
\[ \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \leq \frac{1}{\gamma_p} \int_{\mathbb{R}^N} |\nabla u|^p, \] (1.8)
where \( \gamma_p^{-1} \) is optimal and never achieved.
As a consequence of a Pohozaev type identity, one can see that problem (1.1) does not have non-trivial solutions in any bounded starshaped domain with respect to the origin (Lemma 3.7 in [15]).

2. Preliminaries and known technical results

In this section we first recall useful results such as the strong comparison principle, a weighted Hardy-Sobolev inequality and decay estimates.

Let us start the discussion on the strong comparison principles recalling the following

**Theorem 2.1** (Theorem 1.4 of [11]). Let \( u, v \in C^1(\overline{\Omega}) \) where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \) with \( \frac{2N+2}{N+2} < p < 2 \) or \( p > 2 \). Suppose that either \( u \) or \( v \) is a weak solution of
\[
\begin{align*}
-\Delta_p u &= f(x,u) \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]
(2.1)

with \( f : \overline{\Omega} \times [0, \infty) \to \mathbb{R} \) is a continuous function which is positive and of class \( C^1 \) in \( \overline{\Omega} \times (0, \infty) \).
Assume that
\[ -\Delta_p u + \Lambda u \leq -\Delta_p v + \Lambda v \quad \text{and} \quad u \leq v \quad \text{in } \Omega, \]
where \( \Lambda \in \mathbb{R} \). Then \( u \equiv v \) in \( \Omega \) unless \( u < v \) in \( \Omega \).

Actually the assumption that \( u \) or \( v \) fulfil the zero Dirichlet boundary datum can be removed and local versions of Theorem 2.1 are available, see [17, 18]. On the contrary there are no results removing the assumption \( p > \frac{2N+2}{N+2} \). Therefore in some cases we could prefer to exploit also the following result:
Theorem 2.2 (Theorem 1.4 of [5]). Suppose \( \Omega \) is a domain in \( \mathbb{R}^N \) and let \( u, v \in C^1(\Omega) \) weakly satisfy

\[-\Delta_p u + \Lambda u \leq -\Delta_p v + \Lambda v \quad \text{and} \quad u \leq v \quad \text{in} \ \Omega,\]

\[1 < p < \infty \] and denote by \( Z^u_v := \{ x \in \Omega : \nabla u(x) = \nabla v(x) = 0 \} \). Then if there exists \( x_0 \in \Omega \setminus Z^u_v \) with \( u(x_0) = v(x_0) \), then \( u \equiv v \) in the connected component of \( \Omega \setminus Z^u_v \) containing \( x_0 \). The same result holds if more generally

\[-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \quad \text{and} \quad u \leq v \quad \text{in} \ \Omega,\]

with \( f : \mathbb{R} \to \mathbb{R} \) locally Lipschitz continuous.

In the spirit of the moving plane procedure we shall exploit the strong comparison principle together with the weak comparison principle (that actually will be included in the proofs and we refer the readers to [10]) and improved Hardy inequalities proved in [16]. For convenience we summarize the following

Theorem 2.3 (Proposition 1.1 of [10]). Let \( r \geq 1 \), \( \tau > 0 \), \( \alpha, \gamma \in \mathbb{R} \) such that

\[\frac{1}{r} + \frac{\gamma}{\tau} = \frac{1}{r} + \frac{\alpha - 1}{N},\]

and with

\[0 \leq \alpha - \gamma \leq 1.\]

Let \( u \in C^1_0(\mathbb{R}^N \setminus \{0\}) \) and let \( \frac{1}{r} + \frac{\gamma}{N} < 0 \) then it holds

\[\left( \int_{\mathbb{R}^N} |x|^\tau |u|^\tau \right)^{\frac{1}{\tau}} \leq C \left( \int_{\mathbb{R}^N} |x|^{\alpha} |\nabla u|^\alpha \right)^{\frac{1}{\alpha}}\]

where \( C \) is a positive constant independent of \( u \).

Remark 2.4. In Theorem 2.3 it is assumed that \( u \in C^1_0(\mathbb{R}^N \setminus \{0\}) \). Actually it is clear from the proof, and via density arguments, that the same result applies if \( u \) is defined in exterior domains and has the right decay properties at infinity.

To exploit Theorem 2.3 for weak positive solutions to problem (1.1) we need to know the asymptotic behavior of the solution at infinity. Let us start recalling some results from [25] [26].

Theorem 2.5. Let \( u \in D^{1,p}(\mathbb{R}^N) \) be a weak positive solution to equation (1.1). Then there exist positive constants \( C, c \) depending on \( N, p, \gamma \) and the solution \( u \) such that

\[c|x|^{-\gamma_1} \leq u(x) \leq C|x|^{-\gamma_1} \quad \text{for} \ |x| < R_0,\]  \hspace{1cm} (2.2)

and

\[c|x|^{-\gamma_2} \leq u(x) \leq C|x|^{-\gamma_2} \quad \text{for} \ |x| > R_1.\]  \hspace{1cm} (2.3)

Moreover

\[|\nabla u(x)| \leq c|x|^{-(\gamma_1+1)} \quad \text{for} \ |x| < R_0,\]  \hspace{1cm} (2.4)

and

\[|\nabla u(x)| \leq c|x|^{-(\gamma_2+1)} \quad \text{for} \ |x| > R_1.\]  \hspace{1cm} (2.5)

Here \( \gamma_1, \gamma_2 \) are roots of (1.7) and such that

\[0 \leq \gamma_1 < \frac{N-p}{p} < \gamma_2 \leq \frac{N-p}{p-1},\]

while \( 0 < R_0 < 1 < R_1 \) are constants depending on \( N, p, \gamma \) and the solution \( u \).

Finally, we recall the following regularity result for solutions of (1.1).

Theorem 2.6 ([11] [12] [23]). Let \( u \) be any solution of (1.1), then \( u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \) with \( 0 < \alpha < 1 \).
3. Asymptotic estimates

Here we shall prove some new gradient estimates that we will use in the next section in order to apply the moving plane method. The moving plane procedure is strongly related to the use of suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to suitable comparison principles.

Let \( u, v \) be positive and \( C^1 \)-functions in a neighbourhood of some point \( x_0 \in \mathbb{R}^N \). Then it holds

\[
|\nabla u|^{p-2} \nabla u \cdot \nabla \left( u - \frac{v^p}{u^p} u \right) + |\nabla v|^{p-2} \nabla v \cdot \nabla \left( v - \frac{v^p}{v^p} v \right) \geq C_p \min\{v^p, u^p\} \left( |\nabla \log u| + |\nabla \log v| \right)^p |\nabla \log u - \nabla \log v|, \tag{3.1}
\]

near \( x_0 \) for some constant \( C_p \) depending only on \( p \).

**Proof.** The estimate (3.1) for \( 1 < p < 2 \) can be found in Lemma 3.1 of [25]. Then we just need to prove (3.1) for \( p \geq 2 \).

By making some simple computations we find that

\[
T := |\nabla u|^{p-2} \nabla u \cdot \nabla \left( u - \frac{v^p}{u^p} u \right) + |\nabla v|^{p-2} \nabla v \cdot \nabla \left( v - \frac{v^p}{v^p} v \right) = |\nabla u|^p + |\nabla v|^p - v^p \left( |\nabla \log u|^p + p |\nabla \log u|^{p-2} |\nabla \log u| \cdot (\nabla \log v - \nabla \log u) \right) - u^p \left( |\nabla \log v|^p + p |\nabla \log v|^{p-2} |\nabla \log v| \cdot (\nabla \log u - \nabla \log v) \right). \tag{3.2}
\]

Now let \( f(t) = |a + t(b-a)|^p \) for \( a, b \in \mathbb{R}^N \) then one has

\[
f(1) = f(0) + f'(0) + \int_0^1 (1-t)f''(t),
\]

which gives (recall that \( p \geq 2 \))

\[
|b|^p = |a|^p + p|a|^{p-2}a \cdot (b-a) + p(p-2) \int_0^1 (1-t)|a + t(b-a)|^{p-4}(|a + t(b-a)) \cdot (b-a)|^2 \, dt + p \int_0^1 (1-t)|a + t(b-a)|^{p-2}|b-a|^2 \, dt \geq |a|^p + p|a|^{p-2}a \cdot (b-a) + \int_0^1 (1-t)p|a + t(b-a)|^{p-2}|b-a|^2 \, dt. \tag{3.3}
\]

We apply (3.3) to (I) with \( a = \nabla \log u \) and \( b = \nabla \log v \) and to (II) with \( a = \nabla \log v \) and \( b = \nabla \log u \). Hence we get
\[ T \geq v^p \left[ \int_0^1 (1-t)p|\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2}|\nabla \log u - \nabla \log v|^2 \, dt \right] + u^p \left[ \int_0^1 (1-t)p|\nabla \log v + t(\nabla \log u - \nabla \log v)|^{p-2}|\nabla \log u - \nabla \log v|^2 \, dt \right] \geq \frac{3}{4}pv^p|\nabla \log u - \nabla \log v|^2 \left[ \int_0^1 |\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \, dt \right] + \frac{3}{4}pu^p|\nabla \log u - \nabla \log v|^2 \left[ \int_0^1 |\nabla \log v + t(\nabla \log u - \nabla \log v)|^{p-2} \, dt \right]. \quad (3.4) \]

Now suppose that \(|\nabla \log u| \geq |\nabla \log v|\). In order to estimate the first term on the right hand side of \((3.4)\) we distinguish two cases.

First of all let \(|\nabla \log v - \nabla \log u| \leq \frac{1}{2}|\nabla \log u|\) then (recall \(0 < t < 1\))
\[
|\nabla \log u + t(\nabla \log v - \nabla \log u)| \geq |\nabla \log u| - |\nabla \log v - \nabla \log u|,
\]
then
\[
\geq \frac{1}{2}|\nabla \log u| \geq \frac{1}{4}(|\nabla \log u| + |\nabla \log v|),
\]
namely
\[
|\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \geq \left( \frac{1}{4} \right)^{p-2}(|\nabla \log u| + |\nabla \log v|)^{p-2}.
\]

Otherwise if \(|\nabla \log v - \nabla \log u| > \frac{1}{2}|\nabla \log u|\) then we let
\[
t_0 := \frac{|\nabla \log u|}{|\nabla \log v - \nabla \log u|} \in (0, 2).
\]

Hence
\[
|\nabla \log u + t(\nabla \log v - \nabla \log u)| \geq |\nabla \log u| - t|\nabla \log u - \nabla \log v| = |t_0| |\nabla \log u - \nabla \log v| - t|\nabla \log u - \nabla \log v| = |t_0 - t||\nabla \log u - \nabla \log v| \geq \frac{1}{2}|t_0 - t||\nabla \log u| \geq \frac{1}{4}|t_0 - t|(|\nabla \log u| + |\nabla \log v|),
\]

since we are assuming that \(|\nabla \log u| \geq |\nabla \log v|\). Therefore
\[
|\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \geq \left( \frac{1}{4} \right)^{p-2}|t_0 - t|^p - (|\nabla \log u| + |\nabla \log v|)^{p-2}.
\]

Then, observing that \(\int_0^{\frac{1}{2}} |t_0 - t|^{p-2} \geq C_p\), one has
\[
\frac{3}{4}pv^p|\nabla \log u - \nabla \log v|^2 \left[ \int_0^1 |\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \, dt \right] \geq C_pv^p(|\nabla \log u| + |\nabla \log v|)^{p-2} |\nabla \log u - \nabla \log v|^2.
\]

In the case \(|\nabla \log u| \leq |\nabla \log v|\), arguing in the same way, we deduce that
\[
\frac{3}{4}pu^p|\nabla \log u - \nabla \log v|^2 \left[ \int_0^1 |\nabla \log v + t(\nabla \log u - \nabla \log v)|^{p-2} \, dt \right] \geq C_pu^p(|\nabla \log u| + |\nabla \log v|)^{p-2} |\nabla \log u - \nabla \log v|^2,
\]
which concludes the proof. □
As we have already observed, a key tool in our proofs is the moving plane technique. To exploit it we need the following notations. We will study the symmetry of the solutions in the $\nu-$direction for any $\nu \in S^{N-1}$ (i.e. $|\nu| = 1$). Since the problem is invariant up to rotations we fix $\nu = e_1$ and we let

$$T_\lambda = \{ x \in \mathbb{R}^N : x_1 = \lambda \},$$
$$\Sigma_\lambda = \{ x \in \mathbb{R}^N : x_1 < \lambda \},$$
$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1},$$
$$u_\lambda(x) = u(x_\lambda).$$

Now we state a result that will be used afterwards.

**Theorem 3.2.** Let $1 < p < N$ and let $v \in C^{1,\alpha}_\text{loc}(\mathbb{R}^N \setminus \{0\})$ with $0 < \alpha < 1$ be a positive solution to

$$-\Delta_p v - \frac{\gamma}{|x|^p} v^{p-1} = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

such that

$$\lim_{|x| \to 0} v(x) = \infty.$$

(3.5)

Then, if $v$ fulfills (3.3), it follows that $v$ is a radial (strict) decreasing function.

**Proof.** First of all we need to prove that $v$ is a radial non-increasing function by applying the moving plane technique. We fix a direction $\nu = e_1$ and, for $\lambda < 0$, we take as test function $\varphi_{1,\lambda} = v^{1-p} (v^p - v^p_\lambda)^+ \chi_\Sigma$ and $\varphi_{2,\lambda} = v_\lambda^{1-p} (v^p - v^p_\lambda)^+ \chi_\Sigma$ in the weak formulation solved, respectively, by $v$ and $v_\lambda$. We note that $v_\lambda$ solves

$$-\Delta_p v_\lambda - \frac{\gamma}{|x|^p} v_\lambda^{p-1} = 0.$$

(3.7)

We also remark that, by using (3.6),

$$\text{supp}(\varphi_{j,\lambda}) \subset \subset \Sigma_\lambda \setminus \{0\} \quad j = 1, 2.$$

It is easy to verify that $\varphi_{1,\lambda}$, $\varphi_{2,\lambda} \in D^{1,p}(\mathbb{R}^N)$ (here we also exploit (3.3)). Furthermore, since $\varphi_{1,\lambda}$, $\varphi_{2,\lambda}$ have compact support far from the singularities, we can use the weak formulations of (3.6) and (3.7) and, taking the difference, we deduce that

$$\int_{\Sigma_\lambda} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi_{1,\lambda} - |\nabla v_\lambda|^{p-2} \nabla v_\lambda \cdot \nabla \varphi_{2,\lambda} + \gamma \int_{\Sigma_\lambda} \left( \frac{1}{|x|^p} + \frac{1}{|x_\lambda|^p} \right) (v^p - v^p_\lambda)^+ = 0,$$

(3.8)

and, since $|x| > |x_\lambda|$ in $\Sigma_\lambda$, the second term on the left hand side of (3.8) is nonnegative. Then, exploiting (3.3), it follows that

$$C_p \int_{\Sigma_\lambda \cap \{v \geq v_\lambda\}} v_\lambda^p (|\nabla \log v| + |\nabla \log v_\lambda|)^{p-2} |\nabla (\log v - \log v_\lambda)|^2 = 0$$

which implies that $\log v - \log v_\lambda$ is constant $\Sigma_\lambda \cap \{v \geq v_\lambda\}$ and since $\log v - \log v_\lambda = 0$ on $T_\lambda$ we have $v \leq v_\lambda$ on $\Sigma_\lambda$ for any $\lambda < 0$. We repeat the same argument in the $-e_1$ direction deducing that $v$ is symmetric with respect to the $e_1$-direction. This procedure can be clearly performed in any direction $\nu \in S^{N-1}$ whence one gets the radial monotone nonincreasing behavior of $v$.

A simple application of the Hopf Lemma (that can be applied since the level sets are spheres) shows now that $v$ has no critical points and in particular the radial derivative is strictly negative. □
Next we provide the corresponding lower bound for the decay rate of $|\nabla u|$ of Theorem 2.5.

**Theorem 3.3.** Let $1 < p < N$ and let $u$ be a solution of (3.1). Then there exists $R_2 > 0$ and a constant $\bar{C} > 0$ such that

$$|\nabla u(x)| \geq \frac{\bar{C}}{|x|^{\gamma_2}} \quad \text{for } |x| > R_2. \quad (3.9)$$

**Proof.** Once that Theorem 3.2 is in force we can carry out the proof borrowing some ideas from Theorem 2.2 of [19]. We sketch it for the sake of completeness.

By contradiction let us assume that there exist sequences of radii $R_n$ and points $x_n$ with $R_n \to +\infty$ as $n \to +\infty$ and $|x_n| = R_n$, such that

$$|\nabla u(x_n)| \leq \frac{\theta_n}{R_n^{\gamma_2+1}}, \quad (3.10)$$

with $\theta_n \to 0$ as $n \to +\infty$. Without loss of generality we suppose $R_n > 1$ for any $n$ and we set $w_{R_n}(x) := R_n^{2\gamma} u(R_n x)$. One can observe that for fixed $0 < a < A$ then $|w_{R_n}|_{L^\infty(B_A \setminus B_a)}$ is bounded with respect to $n$. Otherwise if $|x| > \frac{A}{R_n}$ one deduces by Theorem 2.5 that

$$\frac{\bar{c}}{A^{\gamma_2}} \leq w_{R_n}(x) \leq \frac{\bar{C}}{a^{\gamma_2}},$$

and that

$$\begin{cases}
  w_{R_n}(x) \leq \frac{\bar{C}}{A^{\gamma_2}} & x \in \partial B_A, \\
  w_{R_n}(x) \geq \frac{\bar{c}}{a^{\gamma_2}} & x \in \partial B_a.
\end{cases} \quad (3.11)$$

Therefore, the above bound in $L^\infty(B_A \setminus B_a)$ implies that $w_{R_n}$ is also uniformly bounded in $C^{1,\alpha}(K)$ with $0 < \alpha < 1$ for any compact set $K \subset B_A \setminus B_a$. Finally, since $a > 0$, without loss of generality we suppose that the $C^{1,\alpha}$ estimates hold in the closure of $B_A \setminus B_a$. Hence, for $x \in B_A \setminus B_a$ and up to subsequences, one gets that $w_{R_n}(x) \to w_{a,A}(x)$ in $C^{1,\alpha'}$ for $0 < \alpha' < \alpha$. We also underline that $w_{a,A}(x)$ satisfies (3.11). Furthermore, since

$$-\Delta_p w_{R_n} - \frac{\gamma}{|x|^{p}} w_{R_n}^{p-1} \leq \frac{w_{R_n}^{p-1}}{R_n^{(p-1)\gamma_2-p}} \quad \text{in } \mathbb{R}^N,$$

then

$$-\Delta_p w_{a,A} - \frac{\gamma}{|x|^{p}} w_{a,A}^{p-1} = 0 \quad \text{in } B_A \setminus B_a. \quad (3.12)$$

Now, for $j \in \mathbb{N}$, one can take $a_j = \frac{j}{n}$ and $A_j = j$ and reasoning as above one constructs $w_{a_j,A_j}$. Then, for $j \to \infty$, a diagonal argument implies the existence of a limiting profile $w_\infty$ such that $w_\infty \equiv w_{a_j,A_j}$ in $B_{A_j} \setminus B_{a_j}$. In particular from (3.12) read for $w_{a_j,A_j}$ one has

$$-\Delta_p w_\infty - \frac{\gamma}{|x|^{p}} w_\infty^{p-1} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$ 

From (3.11) with $a = a_j$ and $A = A_j$, one gets that the limiting profile $w_\infty$ is such that

$$\lim_{|x| \to +\infty} w_\infty(x) = 0 \quad \text{and} \quad \lim_{|x| \to 0} w_\infty(x) = +\infty$$

and it satisfies (2.5). Therefore Theorem 3.2 can be applied providing that $w_\infty$ is radial with negative radial derivative.

To conclude let now $x_n$ be as in (3.10) and set $y_n = \frac{x_n}{R_n}$. Then, by (3.10), it follows that $|\nabla w_{R_n}(y_n)|$ tends to zero as $n \to +\infty$. Up to subsequences, since $|y_n| = 1$, we have that $y_n \to \bar{y} \in \partial B_1$. Consequently, by the uniform convergence of the gradients one has that $\nabla w_\infty(\bar{y}) = 0$. 
which is in contradiction with the definition of \( w_\infty \), since, by Theorem 6\textit{x}, this cannot happen.

\[ \square \]

4. Proof of the Symmetry Result

We are now able to prove Theorem 1\textit{x}. First of all we underline that it is easy to see that \( u_\lambda \) solves

\[
- \Delta_p u_\lambda - \frac{\gamma}{|x_\lambda|^p} u_\lambda^{p-1} = u_\lambda^{\alpha-1} \quad \text{in } \mathbb{R}^N. \tag{4.1}
\]

In what follows we set

\[
\Lambda^- = \{ \lambda < 0 : u \leq u_\mu \text{ in } \Sigma_\mu, \forall \mu \leq \lambda \}, \quad \Lambda^+ = \{ \lambda > 0 : u \geq u_\mu \text{ in } \Sigma_\mu, \forall \mu \leq \lambda \}.
\]

If \( \Lambda^- \neq \emptyset \) and \( \Lambda^+ \neq \emptyset \) we denote by \( \lambda^-_0 := \sup \Lambda^- \) and by \( \lambda^+_0 := \inf \Lambda^+ \).

Roughly speaking, the moving plane method consists of two main steps: first in reflecting the domain about a fixed hyperplane and proving that the value the solution at each reflected point is larger than the value at the point itself and secondly in moving the hyperplane to a critical position; finally the solution results to be symmetric with respect to this limit hyperplane.

\textbf{Proof of Theorem 1\textit{x}} We prove the result by analyzing, sometimes in different ways, the case \( 1 < p < 2 \) and the case \( p > 2 \). For \( p = 2 \) we refer to [22]. We divide the proof in two steps.

**Step 1:** \( \Lambda^- \neq \emptyset \) and \( \Lambda^+ \neq \emptyset \).

We only prove \( \Lambda^- \neq \emptyset \), which is the existence of \( \lambda < 0 \) with \( |\lambda| \) sufficiently large such that \( u \leq u_\mu \) in \( \Sigma_\mu \) for every \( \mu \leq \lambda \). The proof of the fact that \( \Lambda^+ \neq \emptyset \) is analogous and, at the end of the step, we outline the main changes in the proof in order to conclude it.

For the entire proof we denote by \( R_0, R_1 \) and \( R_2 \) the radii given by (2.2), (2.3) and (3.9) and we firstly observe that for \( |x_\lambda| > \max(R_1, R_2) \) one has, by (2.2) and (2.3), that there exists \( \hat{R}_0 := \hat{R}_0(\lambda) \) such that \( \hat{R}_0 < R_0, B_{\hat{R}_0}(\lambda) \subset \Sigma_\lambda \) and

\[
\sup_{x \in B_{\hat{R}_0}(\lambda)} u(x) < \inf_{x \in B_{\hat{R}_0}(\lambda)} u_{\lambda}(x). \tag{4.2}
\]

Therefore, exploiting also (2.3), we deduce that

\[
\sup_{x \in B_{\hat{R}_0}(\lambda)} u(x) \leq \inf_{x \in B_{\hat{R}_0}(\lambda)} u_{\lambda}(x),
\]

which gives that \( u < u_\lambda \) in \( B_{\hat{R}_0}(\lambda) \subset \Sigma_{\lambda} \) for every \( \lambda \leq \lambda \) and with \( \hat{R}_0 \) independent of \( \lambda \). Moreover we also denote by \( \eta \in C^\infty_0(B_{2R}(0)) \) a cut-off function such that \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( B_R(0) \) and \( |\nabla \eta| \leq \frac{2}{\pi} \).

In what follows we employ the following notation: \( \Sigma_\lambda' = \Sigma_\lambda \setminus B_{\hat{R}_0}(\lambda) \) and \( \hat{B}_\rho := B_\rho(0) \cap \Sigma_{\lambda} \) for \( \rho > 0 \).

If \( \alpha > \max\{2, p\} \) and \( \lambda \leq \lambda \), we consider

\[
\varphi_{1, \lambda} = \eta^\alpha u_\lambda^{1-p}(u_\lambda^{p})^+ \chi_{\Sigma_\lambda}, \quad \varphi_{2, \lambda} = \eta^\alpha u_\lambda^{1-p}(u_\lambda^{p})^+ \chi_{\Sigma_\lambda}. \tag{4.3}
\]

We remark that \( \text{supp}(\varphi_{j, \lambda}) \subset \hat{B}_{2R} \) for \( j = 1, 2 \). Then we take \( \varphi_{1, \lambda} \) as a test function in (1.1), \( \varphi_{2, \lambda} \) in (4.1) and we subtract. Hence, denoting by \( \psi_\lambda := (u_\lambda^{p})^+ \) and by \( \varphi_\lambda := (u - u_\lambda)^+ \) one
We start by estimating $I_1$. By using $(3.1)$ it yields that for $p > 2$ one has

$$I_1 \geq C_p \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^\alpha u_\lambda^p (|\nabla \log u| + |\nabla \log u_\lambda|)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2$$

$$\geq C_p \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^\alpha \left( \frac{u_\lambda}{u} \right)^p u^2 (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2$$

(4.6)

while for $1 < p < 2$ we obtain

$$I_1 \geq C_p \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^\alpha u_\lambda^p \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla \log u| + |\nabla \log u_\lambda|)^{2-p}}$$

$$\geq C_p \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^\alpha u_\lambda^p \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla u| + |\nabla u_\lambda|)^2-2p}.$$  (4.7)

We remark that in (4.6) we used that

$$\frac{u_\lambda}{u} \geq \tilde{c} \quad \text{in } \Sigma_\lambda,$$  (4.8)

and $c_1 := C_p \tilde{c}^p$. Indeed if $x \in \Sigma_\lambda \setminus B_{R_1}(0,\lambda)$ then from (4.6) and (4.8) one has (recall that $|x| \geq |x_\lambda|$)

$$\frac{u_\lambda}{u} \geq \tilde{c}_1 \frac{|x|^\gamma}{|x_\lambda|^\gamma} \geq \tilde{c}_1.$$  

Otherwise if $x \in \Sigma_\lambda \cap B_{R_1}(0,\lambda)$ then

$$\frac{u_\lambda}{u} \geq \tilde{c}_2 \inf_{x \in B_{R_1}(0)} u(x) \geq \tilde{c}_2,$$
and we set \( \tilde{c} = \min(\tilde{c}_1, \tilde{c}_2) \). Now it follows from (2.3) and (2.4) that
\[
I_2 \leq \alpha \int_{B_2 \setminus \{u \geq u_\lambda\}} \eta^{p-1} u \left( 1 - \left( \frac{u_\lambda}{u} \right)^p \right) |\nabla u|^{p-1}|\nabla \eta|
\leq \frac{2\alpha}{R} \int_{B_2 \setminus B_R} u |\nabla u|^{p-1} \leq \frac{C}{R} \int_{B_2 \setminus B_R} \frac{1}{|x|^{(\gamma_2+1)(p-1)+\gamma_2}} \leq \frac{C}{R^\beta}
\] (4.9)
where, from here on, \( \beta := p\gamma_2 + p - N \) which is strictly positive since \( \gamma_2 > \frac{N-p}{p} \). For \( I_3 \), using (2.3) and (4.8), we deduce that
\[
I_3 \leq C \int_{B_2 \setminus B_R} \alpha \eta^{p-1} u_\lambda^{1-p} (u^p - u_\lambda^p)^+ |\nabla u|^{p-1}|\nabla \eta|
\leq \frac{2}{R} \int_{B_2 \setminus B_R \cap \{u \geq u_\lambda\}} u_\lambda \left( \frac{u}{u_\lambda} \right)^p - 1 |\nabla u|^{p-1}
\leq \frac{2}{R} \int_{B_2 \setminus B_R \cap \{u \geq u_\lambda\}} u_\lambda^p |\nabla u|^{p-1} \leq \frac{C}{R} \int_{B_2 \setminus B_R \cap \{u \geq u_\lambda\}} u |\nabla u|^{p-1}
\leq \frac{C}{R} \left( \int_{R^N} |\nabla u_\lambda|^p \right)^{\frac{p-1}{p}} \left( \int_{B_2 \setminus B_R} u^p \right)^{\frac{1}{p}} \leq \frac{C}{R^\beta}
\] (4.10)
For the term \( I_4 \) we first note that (since \( u \geq u_\lambda \))
\[
I_4 = \int_{B_2} \left( \frac{u^p - u_\lambda^p}{u^p - u_\lambda^p} \right) \eta^p \psi \lambda \leq \int_{B_2} \frac{1}{u^{p-1}} \left( u^p - u_\lambda^p \right)^+ \eta^p \psi \lambda,
\]
then applying twice the Lagrange Theorem and using (2.3) one has that in case \( p^* > 2 \)
\[
I_4 \leq c_p \int_{B_2} u^{p^*-2} \eta^p \phi^2_\lambda \leq c_p \int_{B_2} \frac{1}{|x|^p(p^*-2)} \eta^p \phi^2_\lambda,
\]
while for \( 1 < p^* < 2 \) (recall (4.8))
\[
I_4 \leq c_p \int_{B_2} \eta^p \phi^2_\lambda u_\lambda^{2-p} \int_{B_2} \left( \frac{u}{u_\lambda} \right)^{2-p^*} \eta^p \phi^2_\lambda \leq c_p \int_{B_2} \eta^p \phi^2_\lambda \frac{1}{u^{2-p^*}} \leq c_p \int_{B_2} \frac{1}{|x|^{(\gamma_2+p^*-2)}} \eta^p \phi^2_\lambda,
\]
which gives for any \( p > 1 \)
\[
I_4 \leq c_p \frac{1}{|x|^{(\gamma_2+p^*-2)}} \eta^p \phi^2_\lambda.
\] (4.11)
Let us now consider \( f(t) = \log(a + t(b - a)) \) where \( a, b > 0 \) (\( b \geq a \)) then
\[
\log b = \log a + (b - a) \int_0^1 \frac{1}{a + t(b - a)},
\]
and since \( t \in [0, 1] \) we get
\[
b - a = \log b - \log a \int_0^1 \frac{1}{a + t(b - a)} \leq b(\log b - \log a).
\] (4.12)
We use (4.12) with \( b = u \) and \( a = u_\lambda \) and estimate the right hand side of (4.11) (by using also (2.3)) as
\[
I_4 \leq C \int_{B_2 \setminus \{u \geq u_\lambda\}} \frac{1}{|x|^{(\gamma_2+p^*-2)}} \eta^p u^2 (\log u - \log u_\lambda)^2
\leq C \int_{B_2} \frac{1}{|x|^{\gamma_2+p}} \eta^p ((\log u - \log u_\lambda)^+)^2.
\]
Moreover

\[ I_4 \leq C \int_{B_{2R}} \frac{1}{|x|^{3-2\alpha+2}} (\eta^{\frac{p}{2}} (\log u - \log u_\lambda)^+)^2 \]

\[ \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R}} |x|^{2\alpha-2} (\eta^{\frac{p}{2}} (\log u - \log u_\lambda)^+)^2, \] (4.13)

where

\[ \beta^* := \gamma_2 (p^* - p) - p; \quad 2\alpha := - ([\gamma_2 + 1](p - 2) + 2\gamma_2]. \]

We underline that \( \beta^* - 2\alpha + 2 = 2\gamma_2 p^* \) and that \( \beta^* > 0 \) since \( \gamma_2 > \frac{N-p}{p} \). For the right hand side of (4.13) we can apply Theorem 2.3 where \( r = 2, \tau = 2 \) which implies that

\[ \gamma := \alpha - 1 = - (\gamma_2 + 1)p \]

and that

\[ \frac{1}{2} + \frac{\gamma}{N} = \frac{N - \gamma_2 p - p}{2N} < 0 \]

since \( \gamma_2 > \frac{N-p}{p} \). Hence we obtain

\[ I_4 \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R}} |x|^{2\alpha} |\nabla (\eta^{\frac{p}{2}} (\log u - \log u_\lambda)^+)|^2, \] (4.14)

and now, in order to estimate the right hand side of (4.14), we distinguish between the case \( p > 2 \) and the case \( 1 < p < 2 \). From (4.13) and for \( p > 2 \) we get

\[ I_4 \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} \frac{1}{|x|^{(\gamma_2 + 1)(p-2)}} \eta^2 u_\lambda^2 |\nabla \log u - \nabla \log u_\lambda|^2 \]

\[ + \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} |x|^{2\alpha} (\log u - \log u_\lambda)^2 |\nabla \eta|^2 \]

\[ \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^2 u_\lambda^2 |\nabla u|^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 + \frac{C}{|\lambda|^{p\alpha} R^2} \int_{B_{2R} \setminus B_R} |x|^{2\alpha} \]

\[ \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^2 u_\lambda^2 (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 + \frac{C}{|\lambda|^{p\alpha} R^2}. \] (4.15)

Then, by using the estimates (4.9), (4.10), (4.11) and (4.15) in (4.15), we have

\[ \left( c_1 - \frac{C}{|\lambda|^{p\alpha}} \right) \int_{B_{2R} \cap \{u \geq u_\lambda\}} \eta^2 u_\lambda^2 (|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \leq \frac{C}{R^p} \]

\[ + \frac{C}{|\lambda|^{p\alpha} R^2} + \frac{C}{R^3}. \]

For \( |\lambda| \) sufficiently large, as \( R \) goes to \( +\infty \), we deduce that

\[ \int_{B_{2R} \cap \{u \geq u_\lambda\}} u^2 ((|\nabla u| + |\nabla u_\lambda|)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \leq 0. \]

Now we have to estimate the right hand side of (4.14) in the case \( 1 < p < 2 \).

We first remark that \( 2\alpha < 0 \) (for \( N > 2 \)) and, since \( |x| \geq |x_\lambda| \), one has that \( |x|^{2\alpha} \leq |x_\lambda|^{2\alpha} \). Then

\[ I_4 \leq \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} |x_\lambda|^{2\alpha} \eta^2 |\nabla \log u - \nabla \log u_\lambda|^2 \]

\[ + \frac{C}{|\lambda|^{p\alpha}} \int_{B_{2R} \cap \{u \geq u_\lambda\}} |x|^{2\alpha} (\log u - \log u_\lambda)^2 |\nabla \eta|^2. \] (4.16)
Let $\hat{R} = \max\{R_1, R_2\}$ and let $A_{\hat{R}, \hat{R}_0} = \overline{B_{\hat{R}}(0)} \setminus \overline{B_{\hat{R}_0}(0)}$. Then we get

$$B_{2\hat{R}} = A_{\hat{R}, \hat{R}_0} \cup \left( B_{2\hat{R}} \setminus A_{\hat{R}, \hat{R}_0} \right).$$

Exploiting (2.3) we deduce that

$$\int_{B_{2\hat{R}} \setminus A_{\hat{R}, \hat{R}_0}} |x_\lambda|^{2\alpha} \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \leq C \int_{B_{2\hat{R}} \setminus A_{\hat{R}, \hat{R}_0}} |x_\lambda|^\gamma \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \leq C \int_{B_{2\hat{R}} \setminus A_{\hat{R}, \hat{R}_0}} u_\lambda \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p}. \tag{4.17}$$

In $A_{\hat{R}, \hat{R}_0}$ it holds that $|x_\lambda| \geq \hat{R}_0$ and, since we are far from $0$, we also get that $|\nabla u_\lambda|$ is bounded. Let $L := \inf\limits_{B_{\hat{R}(0)} \setminus B_{\hat{R}_0}(0)} u$. Hence we get (by using (4.3) and the fact that $(|\nabla u| + |\nabla u_\lambda|)^{2-p} \leq C$ away from $0, 0$)

$$\int_{A_{\hat{R}, \hat{R}_0}} |x_\lambda|^{2\alpha} \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \leq \hat{C} \hat{R}_0^{2\alpha} \int_{A_{\hat{R}, \hat{R}_0}} \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \leq \frac{\hat{C} \hat{R}_0^{2\alpha}}{L^2} \int_{A_{\hat{R}, \hat{R}_0}} u_\lambda \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p}. \tag{4.18}$$

Gathering (4.17) and (4.18) in the first term of (4.16) and reasoning as in (4.15) for the second term of (4.16) one yields to

$$I_4 \leq \frac{C}{|\lambda|^{3\beta}} \int_{B_{2\hat{R}} \setminus \{ u \geq u_\lambda \}} u_\lambda ^2 \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p} + \frac{C}{|\lambda|^{3\beta} \hat{R}^3}. \tag{4.19}$$

Hence, by collecting (4.7), (4.9), (4.10) and (4.19) in (4.9), we get

$$\left( c_1 - \frac{C}{|\lambda|^{3\beta}} \right) \int_{B_{2\hat{R}} \setminus \{ u \geq u_\lambda \}} \eta^\alpha \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p} \leq \frac{C}{\hat{R}^p} + \frac{C}{R^p} + \frac{C}{|\lambda|^{3\beta} \hat{R}^3}. \tag{4.20}$$

Once again we can choose $|\lambda|$ large enough so that, as $R$ goes to $+\infty$, it yields

$$\int_{\Sigma_\lambda \cap \{ u \geq u_\lambda \}} u_\lambda \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p} = \limsup\limits_{R \to +\infty} \int_{B_{\hat{R}}} \left| \nabla \log u - \nabla \log u_\lambda \right|^2 \left( |\nabla u| + |\nabla u_\lambda| \right)^{2-p} \leq 0.$$

Hence, in both cases, $\log u - \log u_\lambda$ is constant and since $\log u - \log u_\lambda = 0$ on $T_\lambda$ then $\log u - \log u_\lambda = 0$ on the set $\Sigma_\lambda \cap \{ u \geq u_\lambda \}$. Therefore we get $u \leq u_\lambda$ on $\Sigma_\lambda$. Hence $\Lambda^+ \neq \emptyset$ and $\lambda_0^-$ exists and it is also finite.

In order to show that $\Lambda^+ \neq \emptyset$ then we take as test functions

$$\phi_{1, \lambda} = u^{1-p}(u^p - u_\lambda^p)^- \lambda \Sigma_\lambda, \quad \phi_{2, \lambda} = u^{1-p}(u^p - u_\lambda^p)^- \lambda \Sigma_\lambda$$

and, analogously to what already done, we are able to prove the claim so that there exists $\lambda_0^+$ which is also finite.

**Step 2:** $\lambda_0^- = \lambda_0^+ = 0$.

We argue by contradiction assuming that $\lambda_0^+ \neq 0$. Arguing as in the proof of Step 1 we will get the contradiction proving that $u \leq u_{\lambda_0^- + \varepsilon}$ in $\Sigma_{\lambda_0^- + \varepsilon}$ for all $0 \leq \varepsilon \leq \varepsilon$ for some $\varepsilon > 0$. 
In what follows we shall exploit the *strong comparison principle*. To do this we start noticing that from Step 1 and by continuity it holds that

\[ u \leq u_{\lambda_0^+} \quad \text{in} \quad \Sigma_{\lambda_0^+}. \]

By Theorem 2.2 we deduce that \( u \equiv u_{\lambda_0^-} \) or \( u < u_{\lambda_0^-} \) in any connected component \( C \) of \( \Sigma_{\lambda_0^-} \setminus Z_u \) \((Z_u = \{ \nabla u = 0 \})\). We will frequently use the fact that \( Z_u \) has zero Lebesgue measure [10].

Assume first that \( \Sigma_{\lambda_0^-} \setminus Z_u \) has only one connected component. We observe that \( u \equiv u_{\lambda_0^-} \) is not possible in this case since, by (2.2), there exists \( B_{\bar{R}_0}(0, \lambda^-_0) \) where \( u < u_{\lambda_0^-} \); this means that \( u < u_{\lambda_0^-} \) in \( \Sigma_{\lambda_0^-} \setminus Z_u \).

Assume now that there are at least two connected components of \( \Sigma_{\lambda_0^-} \setminus Z_u \). Our Theorem 3.3 implies that \( Z_u \) is bounded so that only one component can be unbounded. We refer to such a unbounded connected component as \( C_1 \) and set

\[ C_\lambda := (C_1 \cap \Sigma_{\lambda_0^-}) \cup R_\lambda (C_1 \cap \Sigma_{\lambda_0^-}) \]

If \( u \equiv u_{\lambda_0^-} \) in \( C_1 \) it is easy to see that, by symmetry, \( C_\lambda \) contains at least one connected component of \( \mathbb{R}^N \setminus Z_u \). But this is not possible as it has been shown in [10] Theorem 1.4 and [6] Lemma 5. If else \( u \equiv u_{\lambda_0^-} \) in \( C_2 \) for some bounded component \( C_2 \), then in this case we set

\[ C_\lambda := C_2 \cup R_\lambda (C_2), \]

and also in this case, by symmetry, \( C_\lambda \) would contain at least one connected component of \( \mathbb{R}^N \setminus Z_u \) thus providing a contradiction. Resuming we just proved that

\[ u < u_{\lambda_0^-} \quad \text{in} \quad \Sigma_{\lambda_0^-} \setminus Z_u. \]

Now, recalling that \( Z_u \) is bounded by Theorem 3.3 we fix \( \overline{R} > 0 \) in such a way that

\[ Z_u \subseteq B_{\overline{R}}(0), \]

and, for \( \tau > 0 \), we let \( Z_u^\tau \) be an open set containing \( Z_u \) such that \( \mathcal{L}(Z_u^\tau) < \tau \) (that exists since \( \mathcal{L}(Z_u) = 0 \)). Then, for \( \delta, \varepsilon, \overline{R}, \overline{r} > 0 \), we denote by

\[ B_{R, \varepsilon} := B_{\overline{R}}(0) \cap \Sigma_{\lambda_0^+ + \varepsilon}, \quad S_\delta^\varepsilon := \left( (\Sigma_{\lambda_0^- + \varepsilon} \setminus \Sigma_{\lambda_0^- - \delta}) \cap B_{\overline{R}}(0) \right) \cup \left( Z_u^\tau \cap \Sigma_{\lambda_0^- - \delta} \right), \]

\[ K_\delta := B_{\overline{R}}(0) \cap \Sigma_{\lambda_0^- - \delta} \cap (Z_u^\tau)^c, \]

where \( \delta \leq \varepsilon \) so that \( K_\delta \) is nonempty. We underline that this construction gives

\[ \Sigma_{\lambda_0^- + \varepsilon} = B_{R, \varepsilon} \cup S_\delta^\varepsilon \cup K_\delta. \]

We also remark that, since \( K_\delta \) is compact, then by the uniform continuity of \( u \) and \( u_{\lambda_0} \), for \( \varepsilon > 0 \) small enough one has that \( u < u_{\lambda_0^+ + \varepsilon} \) in \( K_\delta \) for every \( \varepsilon \leq \tau \). Moreover we underline the existence of \( \bar{R}_0 \) such that \( u < u_{\lambda_0^- + \varepsilon} \) in \( B_{\bar{R}_0}(0, \lambda_0^- + \varepsilon) \subseteq \Sigma_{\lambda_0^- + \varepsilon} \) for every \( \varepsilon \leq \tau \) and with \( \bar{R}_0 \) independent of \( \varepsilon \) as done in Step 1.

From now on, for \( R > \overline{R} \), we consider \( \eta \in C_0^\infty(B_{2\overline{R}}(0)) \) a cut-off function with \( 0 \leq \eta \leq 1, \eta \equiv 1 \) on \( B_R(0) \) and \( |\nabla \eta| \leq \frac{2}{\overline{R}} \). Then, leting \( a > \max\{2, p\} \), we consider the following test functions

\[ \varphi_{1, \lambda_0^- + \varepsilon} = \eta^a (u^p - u_{\lambda_0^- + \varepsilon}^p)^+ \chi_{\Sigma_{\lambda_0^- + \varepsilon}}, \quad \varphi_{2, \lambda_0^- + \varepsilon} = \eta^a (u^p - u_{\lambda_0^+ + \varepsilon}^p)^+ \chi_{\Sigma_{\lambda_0^- + \varepsilon}}. \]
and, analogously to Step 1, \( \psi_{\lambda_0^+ + \varepsilon} := (u^p - u_{\lambda_0^+ + \varepsilon}^p)^+ \) and by \( \varphi_{\lambda_0^+ + \varepsilon} := (u - u_{\lambda_0^+ + \varepsilon})^+ \).

Let us take \( \varphi_{1, \lambda_0^+ + \varepsilon} \) as a test function in (1.1), \( \varphi_{2, \lambda_0^- + \varepsilon} \) in (4.1) and, reasoning as in Step 1, one yields to

\[
c_1 \int_{B_{2R} \cap (u \geq u_{\lambda_0^+ + \varepsilon})} \eta^\alpha u^2 \left( |\nabla u| + |\nabla u_{\lambda_0^- + \varepsilon}| \right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^- + \varepsilon}|^2 \\
\leq \int_{B_{2R} \cap B_{2R}^\varepsilon} (u^p - u_{\lambda_0^+ + \varepsilon}) \eta^\alpha \psi_{\lambda_0^- + \varepsilon} + \int_{B_{2R} \cap S_{\lambda_0^- + \varepsilon}^R} (u^p - u_{\lambda_0^- + \varepsilon}) \eta^\alpha \psi_{\lambda_0^+ + \varepsilon} + \frac{C}{R^2} + \frac{C}{R^3}.
\]  

(4.20)

Here we have used once again the fact that \( \frac{u_{\lambda_0^+ + \varepsilon}}{R} \geq \bar{c} \) for every \( 0 \leq \varepsilon \leq \bar{\varepsilon} \) as to deduce (1.8).

In order to estimate the first term on the right hand side of (4.20) we argue exactly as to estimate \( I_4 \) in (4.2) (taking into account Remark 2.4) where here \( R \) plays the role of \( \lambda \) in Step 1. Hence we get

\[
\int_{B_{2R} \cap B_{2R}^\varepsilon} (u^p - u_{\lambda_0^+ + \varepsilon}) \eta^\alpha \psi_{\lambda_0^- + \varepsilon} \leq \frac{C}{R^2} \\
+ \frac{C}{R^3} \int_{B_{2R} \cap B_{2R}^\varepsilon \cap (u \geq u_{\lambda_0^+ + \varepsilon})} \eta^\alpha u^2 \left( |\nabla u| + |\nabla u_{\lambda_0^- + \varepsilon}| \right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^- + \varepsilon}|^2.
\]

For the second term on the right hand side of (4.20) we reason as in Step 1, getting

\[
\int_{B_{2R} \cap S_{\lambda_0^+ + \varepsilon}^R} (u^p - u_{\lambda_0^- + \varepsilon}) \eta^\alpha \psi_{\lambda_0^- + \varepsilon} \leq C_u \int_{B_{2R} \cap S_{\lambda_0^- + \varepsilon}^R \cap (u \geq u_{\lambda_0^- + \varepsilon})} (\log u - \log u_{\lambda_0^- + \varepsilon})^2,
\]  

(4.21)

where

\[
C_u := \begin{cases} \sup \frac{u^{p^*} - 2}{u} & \text{if } p^* \geq 2, \\ \inf \frac{u^{p^*} - 2}{u} & \text{if } p^* < 2. \end{cases}
\]

Now we need to divide the estimate by the value of \( p \); indeed if \( p > 2 \) we apply a suitable weighted Poincaré inequality to the right hand side of (4.21) which can be found in Theorem 3.2 of [10]. Hence in this case one has

\[
\int_{B_{2R} \cap S_{\lambda_0^+ + \varepsilon}^R} (u^p - u_{\lambda_0^+ + \varepsilon}) \eta^\alpha \psi_{\lambda_0^- + \varepsilon} \\
\leq C_u \frac{2}{S_{\lambda_0^- + \varepsilon}} \int_{B_{2R} \cap S_{\lambda_0^- + \varepsilon}^R \cap (u \geq u_{\lambda_0^- + \varepsilon})} |\nabla u|^2 |\nabla \log u - \nabla \log u_{\lambda_0^- + \varepsilon}|^2 \\
\leq \frac{C_u}{S_{\lambda_0^- + \varepsilon}} \int_{B_{2R} \cap S_{\lambda_0^- + \varepsilon}^R \cap (u \geq u_{\lambda_0^- + \varepsilon})} u^2 \left( |\nabla u| + |\nabla u_{\lambda_0^- + \varepsilon}| \right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^- + \varepsilon}|^2.
\]
where $C_p(E)$ is the Poincaré constant which goes to zero as $|E| \to 0$. Otherwise if $1 < p < 2$ one can apply the classical Poincaré inequality in order to deduce

\[
\int_{B_{2R} \cap \Sigma_{\lambda_0}^\pm} (u^{p^*} - u^{p^*} - u^{p^*} - u^{p^*}) \partial u \partial \psi \lambda \\
\leq C_p^2(S_\lambda^\pm) C_u \int_{B_{2R} \cap \Sigma_{\lambda_0}^\pm} \left| \nabla \log u - \nabla \log u_{\lambda_0} \right|^2 \\
\leq \frac{CC_p^2(S_\lambda^\pm) C_u}{\inf u^2} \int_{B_{2R} \cap \Sigma_{\lambda_0}^\pm} u^2 \left( \left| \nabla u \right| + \left| \nabla u_{\lambda_0} \right| \right)^{p-2} \left| \nabla \log u - \nabla \log u_{\lambda_0} \right|^2,
\]

which can be deduced since in $\Sigma_{\lambda_0}^\pm \setminus B_{\bar{R}_0}(0)\lambda_0^\pm$ one has that

\[
\left( \left| \nabla u \right| + \left| \nabla u_{\lambda_0} \right| \right)^{2-p} \leq C,
\]

for some constant $C$ which does not depend on $\varepsilon \leq \varepsilon$. Hence in both cases one has that

\[
\frac{C_1}{R^2} + \frac{C}{R^3} \leq \frac{C_1}{R^2} + \frac{C}{R^3} \int_{(B_{2R} \cap \Sigma_{\lambda_0}^\pm) \setminus B_{\bar{R}_0}(0)\lambda_0^\pm} \partial u \partial \psi \lambda \\
+ \frac{CC_p^2(S_\lambda^\pm) C_u}{\inf u^2} \int_{\Sigma_{\lambda_0}^\pm \setminus B_{\bar{R}_0}(0)\lambda_0^\pm} \left( \left| \nabla u \right| + \left| \nabla u_{\lambda_0} \right| \right)^{p-2} \left| \nabla \log u - \nabla \log u_{\lambda_0} \right|^2.
\]

Now we take care of the variable parameters $\bar{R}, \delta, \varepsilon$. First we fix $\bar{R}$ large such that

\[
\frac{C}{\frac{C_1}{R^2}} < 1.
\]

Then, since $C_p^2(\Omega)$ goes to zero if the Lebesgue measure of $\Omega$ goes to zero, we choose $\delta, \varepsilon, \tau$ small so that

\[
\frac{CC_p^2(S_\lambda^\pm) C_u}{C_1 \inf u^2} < 1
\]

for every $0 \leq \varepsilon \leq \varepsilon$. Hence it follows that

\[
\frac{C_1}{R^2} + \frac{C}{R^3} \leq \frac{C_1}{R^2} + \frac{C}{R^3} \int_{(B_{2R} \cap \Sigma_{\lambda_0}^\pm) \setminus B_{\bar{R}_0}(0)\lambda_0^\pm} \partial u \partial \psi \lambda \\
+ \frac{CC_p^2(S_\lambda^\pm) C_u}{\inf u^2} \int_{\Sigma_{\lambda_0}^\pm \setminus B_{\bar{R}_0}(0)\lambda_0^\pm} \left( \left| \nabla u \right| + \left| \nabla u_{\lambda_0} \right| \right)^{p-2} \left| \nabla \log u - \nabla \log u_{\lambda_0} \right|^2 = 0,
\]

which gives that $u \leq u_{\lambda_0}^\pm$ in $\Sigma_{\lambda_0}^\pm$ which contradicts the definition of $\lambda_0^\pm$. This proves that $\lambda_0^- = 0$. In an analogous way we deduce that $\lambda_0^+ = 0$, which gives the symmetry of $u$ along the $e_1$-direction. Repeating the same arguments in the remaining $N - 1$ linearly independent directions of $\mathbb{R}^N$ then one deduces that $u$ is symmetric about the origin and that is a radially decreasing function. \qed
References

[1] B. Abdellaoui, V. Felli and I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 9 (2006), no. 2, 445–484.
[2] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth, Comm. Pure Appl. Math. 42 (1989), n. 3, 271–297.
[3] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001), no. 2, 229–258.
[4] K.S. Chou, C.W. Chu, On the best constant for a weighted Sobolev-Hardy inequality, J. London Math. Soc. (2) 48 (1993), no. 1, 137–151.
[5] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, Ann. Inst. H. Poincaré Anal. Non Linéaire. 15 (1998), no. 4, 493–516.
[6] S. Merchán, L. Montoro, I. Peral and B. Sciuini, Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy-Leray potential, Ann. Inst. H. Poincaré Anal. Non Linéaire. 31 (2014), no. 1, 1–22.
[7] L. Damascelli, S. Merchán, L. Montoro and B. Sciuini, Radial symmetry and applications for a problem involving the \(-\Delta_p(\cdot)\) operator and critical nonlinearity in \(\mathbb{R}^N\), Adv. Math. 256 (2014), 313–335.
[8] L. Damascelli, F. Pacella and M. Ramaswamy, Symmetry of ground states of p-Laplace equations via the moving plane method, Arch. Ration. Mech. Anal. 148 (1999), no. 4, 291–308.
[9] L. Damascelli and M. Ramaswamy, Symmetry of \(C^{1,\alpha}\) solutions of p-Laplace equations in \(\mathbb{R}^N\), Adv. Nonlinear Stud. 1 (2001), no. 1, 49–64.
[10] L. Damascelli and B. Sciuini, Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations, J. Differential Equations. 206 (2004), no. 2, 483–515.
[11] L. Damascelli and B. Sciuini, Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of m-Laplace equations, Calc. Var. Partial Differential Equations. 25 (2006), no. 2, 139–159.
[12] E. Di Benedetto, \(C^{1,\alpha}\) local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal., 7 (1983), no. 8, 827–850.
[13] B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle. Comm. Math. Phys., 68 (1979), no. 3, 209–243.
[14] M. Guedda and L. Veron, Local and global properties of solutions of quasilinear elliptic equations, J Differential Equations. 76 (1988), no.1, 159–189.
[15] J. García Azorero and I. Peral, Hardy Inequalities and some critical elliptic and parabolic problems, J. Differential Equations. 144 (1998), 441–476.
[16] H.M. Nguyen, M. Squassina. On Hardy and Caffarelli-Kohn-Nirenberg inequalities, Journal d’Analyse Mathematique, to appear.
[17] B. Sciuini, Some results on the qualitative properties of quasilinear elliptic equations. NoDEA. Nonlinear Differential Equations and Applications, 14 (2007), no. 3-4, 315–334.
[18] B. Sciuini, Regularity and comparison principles for p-Laplace equations with vanishing source term. Comm. Cont. Math., 16 (2014), no. 6, 1450013, 20.
[19] B. Sciuini, Classification of positive \(D^{1,p}(\mathbb{R}^N)\)-solutions to the critical p-Laplace equation in \(\mathbb{R}^N\), Advances in Mathematics, 291 (2016), 12–23.
[20] J. Serrin, A symmetry problem in potential theory. Arch. Rational Mech. Anal., 43 (1971), no. 4, 304–318.
[21] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976), no. 4, 353–372.
[22] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations. 1 (1996), no. 2, 241–264.
[23] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations. 51 (1984), no. 1, 126–150.
[24] J. Vétois, A priori estimates and application to the symmetry of solutions for critical p-Laplace equations, J. Differential Equations. 259 (2015), no. 8, 3929–3954.

Francescantonio Oliva, Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Via Scarpa 16, 00161 Roma, Italy
E-mail address: francesco.oliva@sbai.uniroma1.it
Berardino Sciunzi, Dipartimento di Matematica e Informatica, UNICAL, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy
E-mail address: sciunzi@mat.unical.it

Giusi Vaira, Dipartimento di Matematica e Fisica, Università degli studi della Campania “Luigi Vanvitelli”, Viale Lincoln 5, 81100 Caserta, Italy
E-mail address: giusi.vaira@unicampania.it