On pure spinor formalism for quantum superstring
and spinor moving frame

Igor A Bandos

Department of Theoretical Physics, University of the Basque Country UPV/EHU, PO Box 644, E-48080 Bilbao, Spain
IKERBASQUE, Basque Foundation for Science, E-48011, Bilbao, Spain
E-mail: igor.bandos@ehu.es

Received 22 May 2013, in final form 4 September 2013
Published 8 October 2013
Online at stacks.iop.org/CQG/30/235011

Abstract

The $D = 10$ pure spinor constraint can be solved in terms of spinor moving frame variables $v^{−\alpha}$ and eight-component complex null vector $\Lambda^{+}_{q}, \Lambda^{+}_{q} \Lambda^{+}_{q} = 0$, which can be related to the $\kappa$-symmetry ghost. Using this and similar solutions for the conjugate pure spinor and other elements of the non-minimal pure spinor formalism, we present a (hopefully useful) reformulation of the measure of the pure spinor path integral for superstring in terms of products of Cartan forms corresponding to the coset of 10D Lorentz group and to the coset of complex orthogonal group $SO(8, C)$. Our study suggests a possible complete reformulation of the pure spinor superstring in terms of new irreducible set of variable.

PACS numbers: 11.30.Pb, 11.25.-w, 04.65.+e, 11.10Kk

1. Introduction

The pure spinor approach [1] is very successful in the description of quantum superstring. It provides the way of covariant calculation of loop superstring amplitudes [2–4] (see [5, 6] for recent progress and more references). However, despite the certain progress reached in [7] and [8], its origin and relation with classical Green–Schwartz action is still to be clarified more. Furthermore, the origin and structure of the path integral measure [4, 9, 10], which is used to obtain the superstring loop amplitudes in the frame of pure spinor formalism, remains mysterious.

The recent study in [11] addressed this problem by trying to extract the pure spinor measure $d^{11}\lambda$ from the $C^{16} – \{0\}$ integration measure characteristic for the space of 10D Weyl spinors.

In this paper, we develop an approach which can be considered as ‘bottom–up’ with respect to [11]. To begin with, instead of trying to extract the pure spinor measure from the measure on a bigger space, we decompose $d^{11}\lambda$ on two factors which look less mysterious: one of these is the measure on the space $C^{7} – \{0\}$ of eight-dimensional complex null-vectors.
$\Lambda_q^+$ (described, in particular, in appendix of [11]) and second is constructed from four vielbein forms of $S^8$ space (which can be identified with a celestial sphere of 10D observer)\(^1\) with the use of the $SO(8)$ Klebsh–Gordan coefficients $\gamma_{qp}^{\lambda}$, the above complex null-vectors $\Lambda_q^+$ and its complementary $\Lambda_q^-$ (another complex null vector which obeys $\Lambda_q^+ \Lambda_q^- = 1$)\(^2\).

After finding the above described expression for the pure spinor $d^{11}\lambda$, we apply the similar method to the other elements of the path integral measure of the non-minimal pure spinor formalism, beginning from $d^{11}\bar{\lambda}$ for ‘conjugate spinor’ $\bar{\lambda}_\alpha$ which obeys, besides the pure spinor constraints, the condition $\lambda^\gamma \bar{\lambda}_\alpha \neq 0$. Already at this stage it is useful to complete the set of two complex eight component null-vectors $\Lambda_q^+$ to the $SO(8, \mathbb{C})$ valued matrix. We use this ‘$SO(8, \mathbb{C})$ frame’ matrix field to solve the constraints on the fermionic spinors $r_\alpha$ of the non-minimal pure spinor formalism. However, this solution includes a component defined up to an infinitely reducible symmetry. This hampers the straightforward way to writing the path integral measure, but suggests an alternative based on using the BRST charge and free conformal field theory (CFT) action of the pure spinor formalism for a step-by-step procedure to find, beginning from the solution of pure spinor constraints, the proper set of irreducible variables describing the degrees of freedom (dofs) of the non-minimal pure spinor formalism.

This paper is organized as follows. In section 2 we briefly (and schematically) review the pure spinor approach to quantum superstring. After describing the minimal pure spinor formalism in section 2.1, we briefly discuss (in section 2.2) the result of [11] on searching for the possibility to separate the pure spinor measure from the expression for the measure in the space $\mathbb{C}^{16}$ of unrestricted Weyl spinors. The non-minimal extension of the pure spinor formalism is reviewed in section 2.3. In section 3, we present the general solution ([14]) of the pure spinor constraints in terms of complex $SO(8)$ null-vectors $\Lambda_q^+$ and spinor moving frame variables.

The properties of these spinor moving frame variables are described in the appendix and briefly resumed in section 3.1. This is used in section 4 to find the expression for the pure spinor measure $d^{11}\lambda$. The first of such expressions, found in section 4.1, is further reformulated in section 4.3 with the use of Cartan forms corresponding to a coset of $SO(8, \mathbb{C})$ group. That, together with connection-type Cartan forms, provide a basis of cotangent space to the $SO(8, \mathbb{C})$ group manifold parametrized by complex null-vector $\Lambda_q^+$, entering the solution of the pure spinor constraint for $\lambda^\gamma$, its complementary complex null vector $\Lambda_q^-$, entering the solution of the constraint for the ‘conjugate pure spinor’ $\bar{\lambda}_\alpha$, and six complex vectors, $\Lambda_q^I$, orthogonal to those and among themselves. This set of variables helps us to solve, in section 4.2, the constraints for the fermionic spinor $r_\alpha$ of the non-minimal pure spinor formalism. In section 4.4, we present the measure $d^{11}\bar{\lambda}$, for integration over the space of conjugate pure spinors. However, as we discuss in section 5, to write in the similar term, the measure $d^{11}r$ for the fermionic pure spinor $r_\alpha$ is not so straightforward because our solution of the constraint for $r_\alpha$ is infinitely reducible; this is to say it is written in terms of larger number of parameters ($15 > 11$) restricted by an infinitely reducible symmetry. However, the previous stages of our procedure suggest a possible way out which we discuss in concluding section 6. There we present the wanted expression for $d^{11}r$ and discuss a possible reformulation of the pure spinor path integral description of quantum superstring in terms of irreducible variables.

---

1 The complex 11 dofs in 10D pure spinor $\lambda^\gamma$ are reproduced in our decomposition as $7+4$ where $7 = 8 - 1$ is the dimension of the space $\mathbb{C}^7 - \{0\}$ spanned by the complex null-vector $\Lambda_q^+$, obeying $\Lambda_q^+ \Lambda_q^- = 0$, while $4 = 8/2$ corresponds to eight real dimensions of $S^8$.\(^2\)

2 These two are complex and are used to obtain a kind of holomorphic 4-form spin tensor on $\mathbb{S}^8$ space (of eight real dimensions) which contributes to the holomorphic 11-form $d^{11}\lambda$. Note that this does not imply an introduction of complex structure on just $\mathbb{S}^8$ manifold; rather it suggests a possible complex structure on the space of pure spinors considered as 11-dimensional fiber bundle over $\mathbb{C}^7 - \{0\}$ with the fiber $\mathbb{S}^8$.\(^2\)
The appendix contains the detailed description of the spinor moving frame formalism and spinor moving frame formulation of the classical Green–Schwarz superstring\(^3\).

We conclude this section by describing our notation and conventions. Note that it is not obligatory (although probably convenient) to read it as far as all the necessary definitions are given and explained in the main text.

1.1. Notation and conventions

Latin symbols from the beginning of the alphabet denote the flat 10D vector indices, \(a, b, c, d = 0, 1, \ldots, 9\). Greek letters are used for 10D Weyl and Majorana–Weyl spinor indices \(\alpha, \beta, \gamma, \delta = 1, \ldots, 16\). Our metric conventions are ‘mostly minus’, \(\eta_{ab} = \text{diag}(1, -1, \ldots, -1)\). We denote the bosonic coordinate of 10D bosonic superspace by \(x^a\); as far as we are dealing with the flat superspace only, \(\mu = 0, 1, \ldots, 9\) can be considered as vector index of the Lorentz group \(SO(1, 9)\) and identified with the tangent space vector index \(a = 0, 1, \ldots, 9\) carried by bosonic supervielbein form \(E^a = dx^a + \frac{i}{2}\theta a^a d\theta\). Due to the same reason, the index of the fermionic coordinate of 10D superspace, \(\theta\) can be identified with 10D Weyl spinor index of the \(SO(1, 9)\); actually in the case of \(N = 1\), superstring \(\theta^a\) is real and hence is a Majorana–Weyl spinor, but we call the index ‘Weyl’ as far as it is also carried by some complex spinors, including the pure spinor \(\lambda^a\) and its complementary \(\bar{\lambda}_a\).

The real \(16 \times 16\) Klebsh–Gordan coefficients for \(SO(1, 9)\) group are denoted by \(\sigma_{\alpha\beta}\) and \(\bar{\sigma}_{\alpha\beta}\). These 10D Pauli matrices are symmetric and obey

\[
\sigma^a \sigma^b + \sigma^b \sigma^a = 2\eta_{ab} I_{16 \times 16}, \quad \sigma_{\alpha\beta} \sigma_{\gamma\delta} = 0 = \bar{\sigma}_{\alpha\beta} \bar{\sigma}_{\gamma\delta}.
\]

The vector indices of \(SO(8)\) are denoted by \(i, j, k = 1, \ldots, 8\), while \(q, p = 1, \ldots, 8\) and \(\dot{q}, \dot{p} = 1, \ldots, 8\) are s- and c-spinorial indices of this group. The \(SO(8)\) Pauli matrices are denoted by \(\gamma^i_{\dot{q}\dot{p}} = \tilde{\gamma}^i_{pq}\) and obey

\[
\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij} I_{8 \times 8}, \quad \tilde{\gamma}^i \gamma^j + \tilde{\gamma}^j \gamma^i = 2\delta^{ij} I_{8 \times 8}.
\]

The sign indices ‘\(^c\)’ and ‘\(^s\)’ describe the weight with respect to \(SO(1, 1)\) group; note that the upper plus index is equivalent to lower minus, \(^c = \_\) and vice versa.

Finally, \(I, J, K, L = 1, \ldots, 6\) are \(SO(6, \mathbb{C})\) vector indices.

The 10D pure spinor is a 16 component complex Weyl spinor \(\lambda^a\) which obeys the constraints

\[
\lambda_\alpha \lambda = 0.
\]

The spinor moving frame variables are two sets of eight constrained Majorana–Weyl spinors, \(v^a_q\) collected in s-spinor of \(SO(8)\), and \(v^{+a}_q\), of opposite \(SO(1, 1)\) weight, collected in c-spinor of \(SO(8)\). The constraints, explicit form of which can be found in the main text and appendix, guaranty that the \(16 \times 16\) matrix \(V_{(a)\beta} = (v^a_q \beta, v^{+a}_q \bar{\beta})\), spinor moving frame matrix, takes its values in \(\text{Spin}(1, 9)\) group, doubly covering of the \(SO(1, 9)\),

\[
V_{(a)\beta} = (v^a_q \beta, v^{+a}_q \bar{\beta}) \in \text{Spin}(1, 9).
\]

The \(SO(1, 9)\) matrix doubly covered by the spinor moving frame matrix is the moving frame matrix

\[
U_a^{(b)} = \left(\frac{i}{2} (u_a^+ + u_a^-), u_a^+ \frac{i}{2} (u_a^+ - u_a^-)\right) \in SO(1, 9)
\]

\(^3\) This is the place to note the existence of other attempts to reformulate the pure spinor approach and to derive it from classical description of the Green–Schwarz superstring [7, 12, 13]. They have their own advantages and issues, however, no one of these alternative approaches has reached the stage of development allowing to calculate loop superstring amplitudes, as it is the case with the non-minimal pure spinor formalism described in section 3.3.
composed of the complementary light-like vectors $u_8^a$ and $u_9^a$, normalized by $u^{a8}u_8^a = 2$ and eight mutually orthogonal normalized spacelike vectors $u_i^a$ which are also orthogonal to $u_8^a$ and $u_9^a$. These are called moving frame variables. In the theories with $SO(1, 1) \times SO(8)$ gauge symmetry, the (spinor) moving frame variables can be considered as homogenous coordinates for the coset

$$SO(1, 9) \overline{SO(1, 1) \times SO(8)} \Leftrightarrow \{ (v_q^{-\alpha}, v_q^{\alpha}) \}$$

(to be more precise, we should speak about coset of Spin(1, 9) but our level of accuracy here allows for the above type of simplifications).

The 16 covariant Cartan forms providing the basis of this coset are denoted by $\Omega^{\#_i} = u^{a8}du_i^a$ and $\Omega^{\#_i} = u^{a9}du_i^a$. The induced $SO(1, 1)$ and $SO(8)$ connection composed from the moving frame variables, which are denoted by $\Omega^{(0)} = \frac{1}{2}u^{a8}du_a$ and $\Omega^{(1)} = u^{a9}du_a$. These together with the above $\Omega^{\#_i}$ and $\Omega^{\#_i}$ provide the basis for the space tangential to the Lorentz group $SO(1, 9)$.

When only one of two spinor moving frame variables, say $v_q^{-\alpha}$, enters a model, this possesses the additional $K_8$ gauge symmetry, $\delta v_q^{\alpha} = k^{+i}\tilde{\gamma}_q^i \delta v_q^{-\alpha} = 0$. Using this together with $SO(1, 1)$ and $SO(8)$ as identification relations, one can identify $v_q^{-\alpha}$ with homogeneous coordinates of coset $SO(1, 1) \times SO(8) \otimes K_8$ isomorphic to the celestial sphere of the 10D observer, $S^8$.

$$\{ v_q^{-\alpha} \} \Leftrightarrow \frac{SO(1, 9)}{SO(1, 1) \times SO(8) \otimes K_8} = S^8.$$  

The vielbein form of this coset can be identified with $\Omega^{\#_i} = u^{a8}du_i^a$, while $\Omega^{\#_i}$ transforms inhomogeneously under $K_8$.

2. Pure spinor description of superstring

2.1. Minimal pure spinor formalism

In this section, we give a very brief and schematic description of minimal pure spinor approach to quantum superstring [1, 2], which is used to describe the tree superstring amplitudes. The main object of this formalism is the BRST charge of the following very simple form

$$Q = \int \lambda^\alpha D_\alpha.$$  

(2.1)

Here, $\lambda^\alpha$ is a bosonic spinor and $D_\alpha$ is a quantum fermionic operator representing the fermionic (Grassmann-odd) constraints characteristic of the Green–Schwartz superstring (and, hence, commuting with $\lambda^\alpha$, as this is not a dynamical variable of Green–Schwarz formulation of superstring) and $\alpha = 1, \ldots, 16$ is the 10D Weyl spinor index. As 10D Weyl spinor representation has no counterpart of the charge conjugation matrices, the upper spinor index cannot be lowered so that one can state that $D_\alpha$ carries the left chiral spinor representation of $SO(1, 9) 16_L$, while the pure spinor $\lambda^\alpha$ carries the right chiral spinor representation 16_R.

The fermionic constraints $D_\alpha$ anti-commute on a composed bosonic vector field $P_a$

$$[D_\alpha(\sigma), D_\beta(\sigma')] = -2i\sigma_a^{\alpha\beta}P_a\delta(\sigma - \sigma').$$  

(2.2)

The explicit form of $P_a$ is again not essential for our discussion here; $\sigma_a^{\alpha\beta} = \sigma_a^{\beta\alpha}$ is the 10D counterpart of the relativistic Pauli matrices or Klebsh–Gordan coefficients for $10$ in the decomposition of the product of two left chiral spinor representations $16_L \times 16_L = 10 + 120 + 126$. These obey

$$\left(\sigma^a^\alpha\sigma^b^\beta + \sigma^b^\alpha\sigma^a^\beta\right)u_\alpha^\beta = 2\eta^{ab}\delta_{\alpha\beta}.$$  

(2.3)
where $\eta^{\mu\nu} = \text{diag}(+1, -1, \ldots, -1)$ is the 10D Minkowski space metric and $\tilde{\sigma}_a^\beta = \tilde{\sigma}^\beta_a$ is the ‘dual’ 10D Pauli matrix, i.e. the Klebsh–Gordan coefficients for $10$ in the decomposition of the product of two right chiral spinor representations $16_R \times 16_R = 10 + 120 + 126$.

The custom of the pure spinor approach literature is to use (instead the Poisson brackets or anti-commutators) the OPE’s (operator product expansion) in terms of which (2.2) is represented by

$$D_\alpha(z)D_\beta(y) \mapsto \frac{1}{y - z} \sigma_a^{\alpha\beta} P_a.$$ (2.4)

Using this, or equation (2.2), one easily calculates $Q^2 = -i \int \lambda \sigma^a \lambda \overline{P_a}$ and the key observation is that, if we impose on the bosonic spinor $\lambda^\alpha$ the constraint

$$\lambda \sigma^a \lambda := \lambda^\alpha \sigma_a^\beta \lambda^\beta = 0,$$ (2.5)

then the BRST charge (2.1) is nilpotent

$$Q^2 = 0 \iff \lambda \sigma^a \lambda = 0.$$ (2.6)

Spinor obeying the constraint (2.5) is called ‘pure spinor’. The notion of pure spinor was introduced by Cartan [18]. In the context of supersymmetric theories, the $D = 10$ pure spinors were introduced and used in [19–21]. The treatment of the nilpotent operator (2.1) as BRST charge suggests that in the pure spinor approach to superstring the pure spinor is considered as a ghost variable. This also advocates to require the wavefunctions and vertex operators to depend on $\lambda$ polynomially.

Note that the pure spinors defined in such a way are complex (this is to say: Weyl but not Majorana–Weyl), as far as for real (Majorana–Weyl) 10D spinors, the constraint (2.5) has only trivial solution $\lambda_{MW} = 0$. For complex (Weyl) spinors, the space of solutions of the $D = 10$ pure spinor constraint (2.5) is 11-dimensional [1].

Another ingredient of the minimal pure spinor formalism is the free CFT action for all (the left moving$^4$) dofs

$$S_{\text{min}} = \int \left( \frac{1}{2} \partial \sigma_{\alpha} \overline{\partial} \theta^\alpha + p_\alpha \overline{\partial} \theta^\alpha - w^\alpha \overline{\partial} \lambda^\alpha \right).$$ (2.7)

Here, $p^\alpha$ is the momentum for the (left-moving) Grassmann coordinate function $\theta^\alpha$, a superpartner of the (left-moving part of the) bosonic coordinate function $x^\mu$ (this enters (2.7) together with its right-moving counterpart) and $w^\alpha$ is the momentum conjugate to the pure spinor $\lambda^\alpha$. The fact that this latter is constrained by (2.5) is reflected by the gauge symmetry acting on its momentum,

$$\delta w^\alpha = \Xi^\alpha (\sigma_{\alpha\beta}) w_{\beta}. \quad (2.8)$$

This symmetry leaves invariant set of currents $(\sigma_{\alpha\beta} := \frac{1}{2} (\sigma_{\alpha\nu} \sigma^\nu_{\beta} - \sigma_{\nu\beta} \sigma^\nu_{\alpha}))$

$$N_{\alpha\beta} := \frac{1}{2} \lambda \sigma_{\alpha\beta} w, \quad J_\lambda := \lambda w, \quad T_\lambda := \partial \lambda w, \quad (2.9)$$

the last of which, (2.10), is the 2D energy momentum tensor of the pure spinor field.

To avoid the doubling of the dofs, one imposes a requirement of analyticity, this is to say, to consider all the wavefunctions and amplitudes to have holomorphic dependence on $\lambda^\alpha$ (i.e., to be dependent on $\lambda^\alpha$ but not on $(\lambda^\alpha)^*$). Then, the superstring path integral [2–4] contains a chiral measure, which is based on the 11-form $d^{11} \lambda \wedge (d^{11} \lambda)^*$.

$^4$ After Wick rotation, usually assumed in the pure spinor approach, the left-moving fields become holomorphic and right-moving become anti-holomorphic.
A (relatively) simple expression for this 11-form [22, 23] (which is implicit in [3, 4])

$$d^{11}\lambda = \frac{1}{(\lambda\bar{\lambda})} \left( \hat{\lambda} \hat{\sigma}^{a_1 a_2} (\hat{\lambda} \hat{\sigma})^{a_3} (\hat{\sigma}_{abc})^{a_4 ... a_{11}} \epsilon_{\alpha_{1}...\alpha_{5}} \epsilon_{\beta_{1}...\beta_{6}} d\lambda_{\alpha_{1}} \wedge ... \wedge d\lambda_{\alpha_{5}} \right),$$  \hspace{1cm} (2.11)

is written with the use of a dual or ‘conjugate’ spinor \(\hat{\lambda}_a\) restricted by the condition that

$$\hat{\lambda}_a \hat{\lambda}_a \neq 0.$$  \hspace{1cm} (2.12)

This can be considered just as a reference spinor in the minimal pure spinor formalism, but appeared to be a necessary ingredient in the non-minimal pure spinor formalism introduced in [3].

2.2. Can one separate the pure spinor measure \(d^{11}\lambda\) from \(\mathbb{C}^{16}\) measure \(d^{16}\rho\)?

As it was mentioned in the introduction, the investigation of the possibility to extract the pure spinor measure (2.11) from the integration measure \(d^{16}\rho\) on the space of unconstrained bosonic 10D Weyl spinors \(\rho^a\) has been performed in [11]. To this end, besides the pure spinors \(\lambda^{\alpha}\), which carry 11 complex dofs and thus obey \(d\lambda^{\alpha_1} \wedge ... \wedge d\lambda^{\alpha_{11}} = 0\) the authors of [11] define the complementary constrained variables \(\hat{\zeta}^a = \frac{1}{2}(\rho\sigma^a\rho)\), obeying \(\hat{\zeta}^{\alpha_1} \wedge ... \wedge \hat{\zeta}^{\alpha_{11}} = 0\), which is designed to ‘measure’ a distance of spinor \(\rho\) from being pure. The bottom line of the study of [11] is the following expression for the measure on the space of unconstrained spinor calculated on the surface \(\hat{\zeta}^a = 0\):

$$[d^{16}\rho]_{\hat{\zeta}^a=0} = \frac{1}{5!11!3!12!} \frac{1}{(\lambda\bar{\lambda})} \hat{\sigma}_{bcd}^{a_1 a_2} (\hat{\lambda} \hat{\sigma}_{a_3})^{a_3} ... (\hat{\lambda} \hat{\sigma}_{a_{11}})^{a_{11}} \epsilon_{\alpha_{1}...\alpha_{5}} \epsilon_{\beta_{1}...\beta_{6}} d\lambda_{\alpha_{1}} \wedge ... \wedge d\lambda_{\alpha_{5}} \wedge \wedge \frac{1}{(\lambda\bar{\lambda})} (\hat{\lambda} \hat{\sigma}_{bcd} a_{14}) \hat{\xi}_{\alpha_{1}} \wedge ... \wedge \hat{\xi}_{\alpha_{5}}.$$ \hspace{1cm} (2.13)

The problem for the applications of this equation, noted already in [11], is the contraction on the indices \(bcd\) between two factors. However, the authors [11] expressed their hope to resolve the problem of factorizing \(d^{11}\lambda\) and \(d^5\zeta\) (2.13) using the constraints which restrict \(\lambda\) and \(\zeta\) variables.

In this paper our approach will be different. We will not try to factorize the pure spinor measure \(d^{11}\lambda\) from \(d^{16}\rho\) but instead will try to solve the pure spinor constraints in terms of spinor moving frame variables and some other variables, and to rewrite \(d^{11}\lambda\) in terms of these new variables. Furthermore, we will try to carry out this program for the integration measure of other constrained variable of the non-minimal pure spinor formalism, which we are going to describe in the next section 2.3.

2.3. Non-minimal pure spinor formalism

The non-minimal formalism [3] includes, in addition to the coordinate functions and the pure spinor \(\lambda^a\), also the above dual or conjugate spinor \(\hat{\lambda}_a\) obeying the pure spinor constraint

$$\hat{\lambda}_a \hat{\lambda}_a := \hat{\lambda}_a \hat{\sigma}^{a\beta} \hat{\sigma}_\beta = 0,$$ \hspace{1cm} (2.14)

as well as the fermionic spinor \(r_a\), obeying

$$\hat{\lambda}_a r_a := \hat{\lambda}_a \hat{\sigma}^{a\beta} r_\beta = 0.$$ \hspace{1cm} (2.15)

It also uses their canonically conjugate momenta: bosonic \(\bar{w}^a\) and fermionic \(s^a\). Due to the constraints (2.14) and (2.15), these are defined up to the gauge transformations

$$\delta \bar{w}^a = \hat{\xi}^a (\hat{\sigma}_a \hat{\lambda}^\beta - \xi^a (\hat{\sigma}_a \bar{\lambda})^\beta), \hspace{1cm} \delta s^a = \xi^a (\hat{\sigma}_a \hat{\lambda})^\beta.$$ \hspace{1cm} (2.16)
These leave invariant the (free CFT) action of the non-minimal pure spinor formalism,
\[ S_{\text{nommin}} = \int \left( \frac{1}{2} a_{\mu} \tilde{a} a^{\mu} + p_{\alpha} \tilde{a} \partial a^{\alpha} - \bar{w}_{\alpha} \tilde{a} \lambda^{\alpha} - \tilde{\bar{w}}_{\alpha} \tilde{\lambda}_{\alpha} \right) + s^\alpha \tilde{\partial} r_\alpha. \] (2.17)
and also the BRST charge of the non-minimal pure spinor formalism,
\[ Q = \int (\lambda^\alpha D_\alpha + \tilde{\bar{w}}^\alpha r_\alpha), \] (2.18)
which contains, besides (2.18), the additional contribution of the non-minimal sector, \( \int \tilde{\bar{w}}^\alpha r_\alpha \).

The new momentum variables, bosonic \( \tilde{\bar{w}}^\alpha \) and fermionic \( s^\alpha \), appear inside the five combinations which include new contributions to the bosonic currents (2.9) and (2.10),
\[ \tilde{N}_{ab} := \frac{1}{2} (\tilde{\bar{w}}_{ab} \tilde{\lambda} - \sigma_{ab} r), \quad \tilde{J}_\lambda := \tilde{\bar{w}} \tilde{\lambda} - s r, \] (2.19)
and the fermionic currents
\[ S_{ab} := \frac{1}{4} \bar{\sigma}_{ab} \bar{\lambda}, \quad S := \bar{\tilde{\lambda}}. \] (2.21)

The path integral of the non-minimal pure spinor formalism includes the integration over the fields \( \lambda^\alpha \) and \( r_\alpha \) which are based on the measures [22]
\[ d^{11} \tilde{\lambda} = \frac{1}{(\lambda, \bar{\lambda})} (\lambda^a)_{a_1} (\lambda^b)_{a_2} (\lambda^c)_{a_3} (\bar{\lambda}^\beta)_{\beta_1} \sigma_{abc} \epsilon^{a_1 \ldots a_5 \beta_1 \ldots \beta_5} \frac{\partial}{\partial \lambda_{\beta_1}} \land \frac{\partial}{\partial \bar{\lambda}_{\beta_1}} \] (2.22)
and
\[ d^{11} r = \frac{1}{(\lambda, \bar{\lambda})} (\lambda^a)_{a_1} (\lambda^b)_{a_2} (\lambda^c)_{a_3} (\bar{\lambda}^\beta)_{\beta_1} \sigma_{abc} \epsilon^{a_1 \ldots a_5 \beta_1 \ldots \beta_5} \frac{\partial}{\partial r_{\beta_1}} \ldots \frac{\partial}{\partial r_{\beta_1}}. \] (2.23)

To understand better the structure of the pure spinor path integral for quantum superstring, it may be useful to find another, although equivalent, form of the measure factors (2.11), (2.22), (2.23). Below we will try to express the measure, and also the fields of the pure spinor formalism, in terms of constrained but irreducible variables; this is to say we will try to solve the pure spinor condition and other constraints of the pure spinor formalism in such a way that no reducible symmetries appear.

3. Pure spinors and spinor moving frame

We begin by the general solution of the \( D = 10 \) pure spinor constraint (2.5) which was found in [14],
\[ \lambda^\alpha = \Lambda_q^+ v_q^-, \quad \Lambda_q^+ \Lambda_q^+ = 0. \] (3.1)
This involves a complex null vector \( \Lambda_q^+ \) (the measure in the space of which was considered in the appendix of [11]), which carries the ghost number characterizing the pure spinor \( \lambda^\alpha \), and also the set of eight Majorana–Weyl spinors constrained by
\[ v_q^a \sigma^a v_p = \delta_{qp} v_p^-, \quad u_q^a u_-^a = 0, \] (3.2)
\[ v_q^a \sigma^{a_1 \ldots a_5} v_p^- = 0. \] (3.3)
Due to the first equation in (3.2), the composite spinor (3.1) obeys \( \lambda \sigma^a \lambda = \Lambda_q^+ \Lambda_q^+ u_-^a \) and the rhs of this equation vanishes due to that the complex 8 component \( \Lambda_q^+ \) is null vector, \( \Lambda_q^+ \Lambda_q^+ = 0 \). Thus (3.1) indeed solves (2.5); moreover, it is the general solution of (2.5).
Indeed, as we discuss below in more detail (see section 4.1), the variables in the rhs of equation (3.1) carry 11 complex or 22 real not-pure-gauge dofs, the same as the number of \( \gamma \)-matrices, \( \sigma \)-matrices, and eight-dimensional space which doubly covers the 8-sphere \( S^8 \). Hence parametrizes is an eight-dimensional space which doubly covers the 8-sphere \( S^8 \) of the pure constraint interesting) one can show that the space of 8 spinors \( v_q^\alpha \) constrained by (3.2), (3.3) and considered modulo SO(8) and scaling transformations, 

\[
 v_q^\alpha \mapsto O_{pq} v_p^\alpha e^{-2\beta}, \quad O_{pq} O_{qr} = \delta_{pq},
\]

is an eight-dimensional space which doubly covers the 8-sphere \( S^8 \).

The easiest way to see this implies considering the light-like vector \( u_a^m \) as an element of a moving frame and the set of constrained spinors \( v_q^\alpha \) as an element of spinor moving frame. Although to this end we introduce a new set of constrained variables, this is convenient because in such a moving frame formulation it is possible to find a covariant set of irreducible constraints defining the basic variables [25] and thus to escape the necessity to use the reducible constraints and reducible symmetries.6

In the remaining part of this section we just present some formulas of spinor moving frame formalism which will be used below. In appendix, to make these natural and to clarify their origin and meaning, we review the spinor moving frame formulation of superstring and also present more details on spinor moving frame variables.

### 3.1. Useful equations of the spinor moving frame formalism

The set of 10D spinor moving frame variables includes, in addition to the above discussed \( v_q^\alpha \), also the set of eight constrained 10D spinors \( v_q^\alpha \) carrying c-spinor index of SO(8), \( q = 1, \ldots, 8 \). These can be considered as square roots of another light-like vector \( u_a^m \)

\[
 v_q^+ = \delta_{ij} v_p^+ = \delta_{ij} u_a^\alpha v_p^\alpha, \quad u_a^\alpha v^\alpha = 0,
\]

and also obey the constraints

\[
 v_q^- = -\gamma_{ij} v_q^+ = -\gamma_{ij} u_a^\alpha v_q^\alpha, \quad u_a^\alpha v^\alpha = 0,
\]

where \( \gamma_{ij} \) is the SO(8) gamma matrix, \( i = 1, \ldots, 8 \).

The constraints (3.2), (3.6) and (3.8) completely determine the moving frame vectors \( u_a^m, u_a^\alpha, u_a^\# \) and guarantee that these obey the following orthogonality and normalization conditions

\[
 u_a^m u_a^m = 0, \quad u_a^\alpha u_a^\alpha = 0, \quad u_a^\# u_a^\# = 2,
\]

\[
 u_a^m u_a^{\#} = 0, \quad u_a^\alpha u_a^{\#} = 0, \quad u_a^\alpha u_a^{\#} = -\delta^{ij},
\]

One can solve the light-likeness constraints by \( u_a^m = E(1, n) \), where \( I = 1, \ldots, 9 \) and \( n = (n^I) \) satisfy \( nI = 1 \) and hence parametrizes \( S^9 \). When \( u_a^m \) is related to the momentum of massless particle, this \( S^9 \) can be recognized as celestial sphere.

Due to the meaning of \( v_q^\alpha \), one can also refer to equation (3.1) as a decomposition of pure spinor. However, using this name one should keep in mind that the decomposition has such a simple form only for a particular moving frame, specially designed for a given pure spinor \( \lambda^a \). Actually our general solution of pure spinor constraint, equation (3.1), provides a definition of such special moving frame.
\[ \delta_u^b = \frac{1}{2} u^a_i D_u^a + \frac{1}{2} u^a_i u^b + u^a_i D_u^b. \]  

(3.10)

We will also need in the complementary set of constrained spinors \( v_{aq}^+, v_{aq}^- \) which form the inverse spinor moving frame matrix. These obey

\[ v_{p-a}^+ v_{aq}^+ = \delta_{pq}, \quad v_{p-a}^- v_{aq}^- = 0, \quad v_{p-a}^+ v_{aq}^- = 0, \quad v_{p-a}^+ v_{aq}^- = \delta_{pq} \]  

(3.11)

and also

\[ u^a\sigma_{a\beta} = 2 v_{aq}^+ v_{aq}^-, \quad v_{q}^+ \sigma_{a} v_{q}^+ = \delta_{aq} u^a, \]  

(3.12)

\[ u^a\sigma_{a\beta} = 2 v_{aq}^+ v_{aq}^-, \quad v_{q}^- \sigma_{a} v_{q}^- = \delta_{aq} u^a, \]  

(3.13)

\[ u^a\sigma_{a\beta} = 2 v_{aq}^+ v_{aq}^-, \quad v_{q}^+ \sigma_{a} v_{q}^+ = v_{q}^\prime u^a. \]  

(3.14)

The derivative of the spinor moving frame variables

\[ Dv_{q}^- := dv_{q}^- + \Omega^{0}(v_{q}^- v_{q}^-) + \frac{1}{2} \Omega^{ij} v_{j}^- v_{q}^- v_{q}^- = -\frac{1}{2} \Omega^{q\prime} v_{q}^- v_{q}^- v_{q}^-; \]

(3.15)

\[ Dv_{q}^+ := dv_{q}^+ - \Omega^{0}(v_{q}^+ v_{q}^+) + \frac{1}{2} \Omega^{ij} v_{j}^+ \gamma_{q}^+ \gamma_{q}^+ = -\frac{1}{2} \Omega^{q\prime} v_{q}^+ v_{q}^+ v_{q}^. \]  

(3.16)

are expressed in terms of \( SO(1, 9)/[SO(1, 1) \times SO(8)] \) Cartan forms

\[ \Omega^\alpha := u^a d\alpha_a, \quad \Omega^{-\alpha} := \eta^a d\alpha_a \]  

(3.17)

and the composite \( SO(1, 1) \) and \( SO(8) \) connection

\[ \Omega^{0} := \frac{1}{2} u^a d\alpha_a, \quad \Omega^{ij} := u^a d\alpha_a \]  

(3.18)

In (3.15), (3.16) we also defined the \( SO(1, 1) \times SO(8) \) covariant derivative \( D \) constructing with the use of these connections. The covariant derivatives of the moving frame vectors read

\[ Du_{q}^+ := du_{q}^+ + 2 u^a_{q} \Omega^{0} = u^a_{q} \Omega^{-q}, \]  

(3.19)

\[ Du_{q}^- := du_{q}^- - 2 u^a_{q} \Omega^{0} = u^a_{q} \Omega^{q}. \]  

(3.20)

\[ Du_{q}^{ij} := du_{q}^{ij} + 2 u^a_{q} \Omega^{ij} = \frac{1}{2} u^a_{q} \Omega^{q} + \frac{1}{2} u^a_{q} \Omega^{-q}. \]  

(3.21)

We will also need the derivatives in the space of spinor moving frame variables, \( D^{0}, D^{ij}, D^{\alpha}, \) \( D^{a} \), which can be defined by the following decomposition of the differential, acting in this space, on the above defined Cartan forms

\[ du_{0} := \Omega^{0} d\Omega^{0} + \frac{1}{2} \Omega^{ij} d\Omega^{ij} + \Omega^{-q} d\Omega^{q} + \Omega^{q} d\Omega^{-q}. \]  

(3.22)

Their action on moving frame and spinor moving frame variables are collected in the appendix (equations (A.31)–(A.35)). In Hamiltonian mechanics of dynamical system involving the (spinor) moving frame variables, the counterparts of \( D^{0}, D^{ij}, D^{\alpha}, \) and \( D^{a} \) are covariant monomials; we will denote them by the same symbols.

4. Pure spinor path integral measure and spinor moving frame

The straightforward way to relate the spinor moving frame formulation of superstring [16] with the pure spinor approach of [1–4] is to perform the covariant quantization of the former and to search for the place where the complexification, characteristic for the pure spinor approach [1–4], and the pure spinor constraint appear at the stage of regularization. For M0-brane, this is to say 11D massless superparticle, this program was completed in [14]. However, for the superstring this way seems to be very difficult technically. Thus here we will be rather trying to apply a more practical and intuitive approach, which was initiated by Siegel in 80th [24] and is used as basic in pure spinor formalism.

Namely, we will assume that the spinor moving frame variables and their covariant monomials (the classical mechanics incarnations of the above \( D^{0} \)) enter a free CFT action for superstring, similar to (2.7) or (2.17), and use them to construct the objects characteristic for pure spinor formalism and the measure of the path integral describing the loop amplitudes of the superstring.
4.1. Pure spinor measure \( d^{11}\lambda \) and holomorphic 4-form on \( S^8 \) space of spinor moving frame variables

Let us return to the general solution of the \( D = 10 \) pure spinor constraint (2.5) [14],

\[
\lambda^\alpha = \Lambda^+_q v^\alpha_q, \quad \Lambda^+_q \Lambda^+_q = 0.
\]

(4.1)

In addition to the spinor moving frame variable \( v^\alpha_q \), which can be considered as homogeneous coordinates of the coset \( \text{Spin}(1, 9)/[\text{SO}(1, 1) \otimes \text{SO}(8) \otimes K_4] = S^8 \), and, thus, carries just 8 dofs, equation (3.1) involves the eight-component complex light-like vector \( \Lambda^+_q \). This carries 7 complex or 14 real dofs, thus completing the real dimension 8 of \( S^8 \) to 22, which is equivalent to 11 complex dimension of the space of \( D = 10 \) pure spinor. Furthermore, the complex null-vector \( \Lambda^+_q \) carries the ghost number +1, characteristic for the pure spinor \( \lambda^\alpha \), and can be related to the ghost for the \( \kappa \)-symmetry of superstring (which is irreducible in the spinor frame formulation [16]).

The expression for the measure \( d^{11}\lambda \) in equation (2.11) involves the 'conjugate' pure spinor \( \bar{\lambda}_a \) which is restricted by \( \bar{\lambda} \bar{\lambda} \neq 0 \) and the pure spinor constraint (2.14). This can be solved in terms of complimentary spinor moving frame variable \( v_{aq}^+ \) and another complex vector \( \bar{\Lambda}_q^+ \),

\[
\bar{\lambda}_a = \bar{\Lambda}_q^- v_{aq}^+, \quad \bar{\Lambda}_q^- \bar{\Lambda}_q^- = 0.
\]

(4.2)

This latter is restricted by the condition \( \bar{\lambda}_a \bar{\lambda}_a \neq 0 \). This suggests to write it in the form \( \bar{\Lambda}_q^- = \bar{\Lambda} \Lambda_q^- \), where \( \bar{\Lambda} \Lambda_q^- = 1 \) and \( \bar{\Lambda} \neq 0 \) is equal to the product of the pure spinor and conjugate pure spinor, \( \bar{\lambda} \lambda \). To resume,

\[
\bar{\lambda}_a = \bar{\Lambda} \Lambda_q^- v_{aq}^+,
\]

(4.3)

\[
\Lambda_q^- \bar{\lambda}_a = 0, \quad \Lambda_q^- \Lambda_q^- = 1.
\]

(4.4)

An equivalent form of equation (4.3) is

\[
\frac{\bar{\lambda}_a}{\bar{\lambda} \lambda} = \Lambda_q^- v_{aq}^+.
\]

(4.5)

Then, using the properties of spinor moving frame variables (see equations (3.2), (3.6), (3.8), (3.11) and (3.10)), we find

\[
\frac{(\bar{\lambda} \bar{\sigma}^a) \lambda}{(\bar{\lambda} \lambda)^2} = u^{a\phi} \Lambda_q^- v^\alpha_q + u^{a\phi} \Lambda_q^- \gamma_\phi v^\alpha_q.
\]

(4.6)

Furthermore, after some algebra, we obtain the following expression for the algebraic factor in the pure spinor measure (2.11):

\[
\frac{(\bar{\lambda} \bar{\sigma}^a) \lambda}{(\bar{\lambda} \lambda)^2} = 2u_{a\phi} v_{\phi q}^+ v^\alpha_q + u_{a\phi} \gamma_\phi v^\alpha_q \gamma_\phi.
\]

(4.7)

Now, as the indices \( q \) and \( \dot{q} \) enumerating spinor moving frame variables take only eight values and the indices in \( d\bar{\lambda}^\beta_1 \wedge \cdots \wedge d\bar{\lambda}^\beta_{10} \) in (2.11) are antisymmetrized with the five ones in (4.7), one concludes that, after decomposing \( d\bar{\lambda}^\beta_1 \) on \( v^\beta_1 \) and \( v^\beta_1 \) (see equation (A.27) in the appendix), only the term \( \infty v_{\phi q}^+ \cdots v_{\phi q}^+ v^\alpha_1 v^\alpha_1 v^\alpha_1 \) in \( d\bar{\lambda}^\beta_1 \wedge \cdots \wedge d\bar{\lambda}^\beta_{10} \) contributes to \( d^{11}\lambda \).

\[7\] The relation between complex null-vector \( \Lambda^+_q \) and real bosonic ghost for \( \kappa \)-symmetry was studied for 11D massless superparticle in [14] (there \( q = 1, \ldots, 16 \)); the relation for the case of \( D = 10 \) superstring should be similar, although the work on establishing it promises to be quite involving technically.
Furthermore, as far as according to (3.1),
\[ \text{d}x^\alpha = \text{d}x_\lambda^\alpha v^-_{\alpha} + \Lambda^+_{\lambda} \text{d}v^+_{\alpha} = \text{D}A^+_{\lambda} v^-_{\alpha} = -\frac{1}{2} \text{v}^+_{\alpha} \Lambda^+_\lambda \gamma^{i}_{pq} v^+_{q}. \] (4.8)
where \(D\) are \(SO(1, 1) \times SO(8)\) covariant derivatives defined in (3.15)–(3.21), we find that the nonvanishing contribution from \(\text{d}x_\lambda^\alpha \) to the pure spinor measure should be proportional to \(\text{D}A^+_{\lambda} \times \ldots \times \text{D}A^+_{\lambda} \) and to \(\text{v}^+_{\alpha} \times \ldots \times \text{v}^+_{\alpha} \). Hence, we can rewrite the original pure spinor measure (2.11) in the form
\[ d^{11} \lambda \equiv \text{C} e^{\eta_{i} \ldots \eta_{p}} \text{D}A^+_{\lambda} \times \ldots \times \text{D}A^+_{\lambda} \times \text{v}^+_{\alpha} \times \ldots \times \text{v}^+_{\alpha} \] (4.9)
We intentionally have not fixed the inessential numerical coefficient in the above expression for the pure spinor measure and have written it with \(\alpha\) symbol, because this expression happens to be intermediate. Below we will change it by expressing \(\text{D}A^+_{\lambda} \) factors in terms of \(SO(8, C)\) Cartan forms which we are going to define in section 4.3 after introducing, in the next section 4.2, a set of variables parametrizing the \(SO(8, C)\) group and using them, together with spinor moving frame, to solve the constraints on the fermionic spinor \(r_{a}\) of the non-minimal pure spinor formalism.

4.2. \textit{SO}(8,C) frame and solution of the constraints for \(r_{a}\)

One can complete the set of complex null vectors \(\Lambda^+_{\lambda}\) to the complete basis in the space of \(8\)-vectors by introduced the set of six mutually orthogonal and normalized vectors \(\Lambda^i_{\lambda}\), which are orthogonal to \(\Lambda^+_{\lambda}\),
\[ \Lambda^-_{\lambda} \Lambda^+_{\lambda} = 0, \quad \Lambda^+_{\lambda} \Lambda^+_{\lambda} = 0, \quad \Lambda^-_{\lambda} \Lambda^+_{\lambda} = 1, \] \[ \Lambda^-_{\lambda} \Lambda^-_{\lambda} = 0, \quad i \Lambda^+_{\lambda} \Lambda^-_{\lambda} = \delta^{ij}. \] (4.10)
Such a set of vectors can be collected in an \([SO(8)]^* = SO(8, C)\) valued matrix
\[ \Lambda^+_{\lambda} = \left( \Lambda^+_{\lambda} \frac{1}{2}(\Lambda^+_{\lambda} + \Lambda^-_{\lambda}), \quad i \left( \Lambda^+_{\lambda} - \Lambda^-_{\lambda} \right) \right) \in SO(8, C). \] (4.11)
Equations (4.10) then appear as \(\Lambda^+_{\lambda} \Lambda^+_{\alpha} = \delta_{\lambda\alpha}\), which is an equivalent form of (4.11). Its another equivalent form, \(\Lambda^+_{\lambda} \Lambda^+_{\alpha} = \delta_{\lambda\alpha}\), gives rise to the completeness condition
\[ \delta_{\lambda\alpha} = \Lambda^-_{\lambda} \Lambda^+_{\alpha} + \Lambda^-_{\alpha} \Lambda^+_{\lambda} + \Lambda^+_{\lambda} \Lambda^+_{\alpha}. \] (4.12)
The additional vectors \(\Lambda^i_{\lambda}\) are useful, in particular, to solve the constraints (2.15) imposed on the fermionic spinor \(r_{a}\),
\[ r_{a} = \chi \Lambda^-_{\lambda} \gamma^+_{\alpha} + \chi^i \Lambda^i_{\lambda} \gamma^+_{\alpha} + \chi^i \Lambda^-_{\lambda} \gamma^i_{\alpha} \gamma^+_{\alpha}. \] (4.13)
Here, the complex fermionic scalar \(\chi\) and the complex fermionic SO(6) vector \(\chi^i := \chi^{-i}\) are unconstrained, while the complex fermionic SO(8) vector \(\chi^i = \chi^{+i}\), with eight component, is defined up to the local transformations
\[ \chi^i \sim \lambda^i + \Lambda^-_{\lambda} \gamma^i_{\alpha} \chi^1_{\alpha} \] (4.14)
with \(8\)-component parameter \(\chi^1_{\alpha} := \chi^{(1)+}+\); furthermore, this latter is defined up to the transformations
\[ \chi^1_{\alpha} \sim \chi^{(1)}_{\alpha} + \Lambda^-_{\lambda} \gamma^i_{\alpha} \chi^1_{\alpha} \] (4.15)
with \(8\)-component parameter \(\chi^{(1)} := \chi^{(1)+}++\) defined, in its turn, up to the transformations
\[ \chi^{(1)} \sim \chi^{(1)} + \Lambda^-_{\lambda} \gamma^i_{\alpha} \chi^{(1)} \] (4.16)
Multidots in (4.14) mark that this process of finding 8-parametric indefiniteness of the parameters of symmetry can be continued up to infinity so that, like in the case of $\kappa$-symmetry of the standard Green–Schwarz formulation of superstring [26], we are dealing with the infinitely reducible symmetry. This implies that the effective number of the parameters $\chi^I$ in (4.13), which is reduced by the above gauge symmetry, has to be calculated as an infinite sum $8 - 8 + 8 - \cdots$. As usually, regularizing this expression by identifying it with the limit of geometric progression, we find $8 - 8 + 8 - \cdots = 8 \lim_{q^{p-1} \to q^{+1}} (1 - q + q^2 - \cdots) = 8 \lim_{q^{p-1} \to q^{+1}} \frac{1}{1 - q} = 4$.

This completes the number of component of $\chi^I$ and $\chi$, $6+1=7$, to 11, which is the number of components of the spinor $r_\alpha$ constrained by (2.15) [3]. This simple calculation gives an evidence that our (4.13) is the general solution of (2.15).

4.3. SO(8, C) Cartan forms and the pure spinor measure $d^{11}\lambda$.

Furthermore, the above group theoretical interpretation of the basis involving the complex light-like vectors $\Lambda_+^\alpha$ and $\Lambda_-^\alpha$ allows to define in an easy way the derivatives with respect to this constrained variables and also the measure in their space.

Let us introduce the set of Cartan forms including $SO(6, \mathbb{C})$ and $SO(2, \mathbb{C})$ connections

$$\Omega^{(0)}_\Lambda := \Lambda^I_q d\Lambda^I_q, \quad \Omega^{(0)}_\Lambda = \Lambda^I_q d\Lambda^I_q,$$

as well as $SO(8, C)/[SO(8, C) \otimes SO(2, \mathbb{C})]$ vielbein forms

$$\Omega^{\pm}_{\Lambda} := \Lambda^{\pm}_q d\Lambda^{\pm}_q.$$

(The subindex $\Lambda$ is introduced to make a difference with (real) $SO(1, 9)$ Cartan forms in equations (A.24)–(A.28)).

The derivatives of the $\Lambda_+^\alpha$ and $\Lambda_-^\alpha$ vectors can be expressed as

$$d\Lambda_+^\alpha = \Lambda_+^\alpha \Omega^{(0)}_\Lambda - \Lambda_-^\alpha \Omega^{\pm}_{\Lambda},$$

$$d\Lambda_-^\alpha = -\Lambda_-^\alpha \Omega^{(0)}_\Lambda - \Lambda_+^\alpha \Omega^{\pm}_{\Lambda},$$

$$d\Lambda_0^\alpha = \Lambda_0^\alpha \Omega^{\pm}_{\Lambda} + \Lambda_-^\alpha \Omega^{\pm}_{\Lambda} - \Lambda_+^\alpha \Omega^{\pm}_{\Lambda}.$$

Decomposing the exterior derivative in the $SO(8, C)$ group manifold (i.e. in the space parametrized by $\Lambda_+^\alpha$ and $\Lambda_-^\alpha$) on the Cartan forms,

$$d\lambda = \Omega^{\pm}_{\Lambda} D^{\pm}_{\Lambda} + \Omega^{(0)}_\Lambda D^{(0)}_{\Lambda} + \Omega^{\pm}_{\Lambda} D^{\pm}_{\Lambda} + \frac{1}{2} \Omega^{\pm}_{\Lambda} d\lambda,$$

we obtain the covariant derivatives $D^{\pm}_{\Lambda}, D^{(0)}_{\Lambda}, D^{\pm}_{\Lambda}$ generating the $SO(8)$ algebra. This statement can be easily checked using their action on the basic variables $\Lambda_+^\alpha$ and $\Lambda_-^\alpha$,

$$D^{(0)}_{\Lambda} \Lambda_+^\alpha = \pm \Lambda_+^\alpha, \quad D^{(0)}_{\Lambda} \Lambda_-^\alpha = 0, \quad D^{\pm}_{\Lambda} \Lambda_+^\alpha = 0, \quad D^{\pm}_{\Lambda} \Lambda_-^\alpha = 2 \Lambda_-^\alpha \delta^{ij},$$

$$D^{\pm}_{\Lambda} \Lambda_+^\alpha = -\Lambda_-^\alpha, \quad D^{\pm}_{\Lambda} \Lambda_-^\alpha = 0, \quad D^{\pm}_{\Lambda} \Lambda_0^\alpha = \Lambda_0^\alpha \delta^{ij},$$

$$D^{\pm}_{\Lambda} \Lambda_0^\alpha = 0, \quad D^{\pm}_{\Lambda} \Lambda_0^\alpha = -\Lambda_0^\alpha, \quad D^{\pm}_{\Lambda} \Lambda_0^\alpha = \Lambda_0^\alpha \delta^{ij}.$$
spinor moving frame field $\psi^a_q$ by an unavoidable complex matrix\(^8\). Due to this reason, it is useful to define counterparts $\Xi^{(0)}, \Xi^H, \Xi^H, \Xi^H$ of the above Cartan forms $\Omega^{(0)}, \Omega^H, \Omega^H, \Omega^H$, that are related to the $SO(8, R) \otimes SO(1, 1) \otimes SO(6, C)$ covariant derivatives of the complex vectors $\Lambda^\pm_q$ and $\Lambda^\pm_q$ (note that, actually, $\Lambda^\pm_q$ has already appeared in equation (4.8)),

\[
\begin{align*}
DA^+_q &= \Lambda^+_q + \Lambda^+_q \Xi^{(0)} - \Lambda^+_q \Omega^H_
u \Xi^H_
u =: \Lambda^+_q \Xi^{(0)} - \Lambda^+_q \Xi^H \\
DA^-_q &= \Lambda^+_q - \Lambda^-_q \Xi^{(0)} - \Lambda^-_q \Omega^H_
u \Xi^H_
u =: -\Lambda^-_q \Xi^{(0)} - \Lambda^-_q \Xi^H \\
DA^\pm_q &= \Lambda^\pm_q \Omega^H_
u \Xi^H_
u =: \Lambda^\pm_q \Xi^H \\
\end{align*}
\]  
(4.26)

These are

\[
\Xi^{(0)} = \Lambda^+_q \Xi^H - \Lambda^-_q \Xi^H, \quad \Xi^H = \Lambda^+_q \Xi^H + \Lambda^-_q \Xi^H
\]  
(4.29)

Using this, we obtain

\[
d\sigma^a = -\frac{1}{2} \Omega^e_{\nu\sigma} \Lambda^+_q \gamma^q \psi^\nu_{\psi} + (\Lambda^+_q \Xi^{(0)} - \Lambda^+_q \Xi^H) \psi^\nu_{\psi}.
\]  
(4.33)

Note also that, as far as we defined $\Lambda = \lambda^a \lambda_q$, using (4.2) we find

\[
d\sigma^a \lambda_q = d\Lambda - \lambda^a \lambda_q = \Lambda \Xi^{(0)}.
\]  
(4.34)

Now, using (4.26) we find $\epsilon^{\psi q_1 \cdots q_k} DA^+_p \wedge \cdots \wedge DA^+_p = 6 \Xi^{(0)} \wedge \Xi^H \wedge \cdots \wedge \Xi^H + \epsilon^{\psi q_1 \cdots q_k} \Lambda^+_q \Lambda^+_q \Lambda^+_q \Lambda^+_q$ which, together with (4.12), results in $\epsilon^{\psi q_1 \cdots q_k} \Lambda^+_q \Lambda^+_q \Lambda^+_q \Lambda^+_q = -2 \lambda^a_{\psi q} \psi^\nu_{\psi}$. Using this, we obtain $\epsilon^{\psi q_1 \cdots q_k} DA^+_p \wedge \cdots \wedge DA^+_p = \alpha \Lambda^+_q \epsilon^{\psi q_1 \cdots q_k} \Xi^{(0)} \wedge \Xi^H \wedge \cdots \wedge \Xi^H$. Substituting this for the first 7-form in the 11-form (4.9), we arrive at the following form of the pure spinor measure:

\[
d^{11}\lambda = \epsilon^{\psi q_1 \cdots q_k} \Xi^{(0)} \wedge \Xi^H \wedge \cdots \wedge \Xi^H \wedge \Omega^{-\xi_1} \wedge \cdots \wedge \Omega^{-\xi_1} M^{++i_1 \cdots i_4}(\Xi)\]
\]  
(4.35)

where

\[
M^{++i_1 \cdots i_4}(\Lambda) := (\Lambda^+ g_{i_1}) (\Lambda^+ g_{i_2}) (\Lambda^+ g_{i_3}) (\Lambda^+ g_{i_4}) \psi^{i_1 \cdots i_4} p_{i_1 \cdots i_4} M^{--}(\Lambda),
\]  
(4.36)

and

\[
M^{--}(\Lambda) := (\Lambda^+ g_{i_1} \psi(\Lambda^+ g_{i_2} \psi(\Lambda^+ g_{i_3} \psi(\Lambda^+ g_{i_4} \psi(\Lambda^+ g_{i_5} \psi(\Lambda^+ g_{i_6} \psi(\Lambda^+ g_{i_7} \psi(\Lambda^+ g_{i_8} \psi(\Lambda^+ g_{i_9} \psi(\Lambda^+ g_{i_10} \psi(\Lambda^+ g_{i_11} \psi(\Lambda^+ g_{i_12} \psi(\Lambda^+ g_{i_13} \psi(\Lambda^+ g_{i_14} \psi(\Lambda^+ g_{i_15} \psi(\Lambda^+ g_{i_16} \psi(\Lambda^+ g_{i_17} \psi(\Lambda^+ g_{i_18} \psi(\Lambda^+ g_{i_19} \psi(\Lambda^+ g_{i_20} \psi(\Lambda^+ g_{i_21} \psi(\Lambda^+ g_{i_22} \psi(\Lambda^+ g_{i_23} \psi(\Lambda^+ g_{i_24} \psi(\Lambda^+ g_{i_25} \psi(\Lambda^+ g_{i_26})\]
\]  
(4.37)

is obtained by contracting the second line of (4.9) with $\Lambda^+_q$.

In (4.35), all the 11 directions of the integration are represented by the generalized Cartan forms, seven of which correspond to a coset of $SO(8, C)$ and 4 to the coset of the Lorentz group Spin(1, 9). Let us stress that, although the latter 4-form in (4.35) contains wedge product $\Omega^{-\xi_1} \wedge \cdots \wedge \Omega^{-\xi_1}$ of real vielbein of $\mathbb{R}^8$, these are contracted with the complex $SO(8)$ tensor $M^{++i_1 \cdots i_4}(\Lambda)$ so that the corresponding contribution $\Omega^{-\xi_1} \wedge \cdots \wedge \Omega^{-\xi_1} M^{++i_1 \cdots i_4}(\Lambda)$ can be considered as a complex, holomorphic measure. Furthermore, as far as the complex $SO(8)$ tensor $M^{++i_1 \cdots i_4}(\Lambda)$ depends on complex null vector $\Lambda^+_q$ and its dual $\Lambda^-_q$, that is the complex holomorphic measure on the fiber $\mathbb{R}^8$ of the space of 10D pure spinors considered as fiber bundle with the base $\mathbb{C}^7 - \{0\}$.

\(^8\) In this respect it is interesting that, in a recent [35], Movshev has begun to develop in 11D the line similar to our basic construction in [14], but with the complexification of the 11D Lorentz group $SO(1, 10)$. The aim in [35] is to construct an 11D twistor transform related to the proposition of Cederwall [36] for the off-shell action of 11D supergravity in 11D pure spinor superspace. We should also note that, to our best knowledge, for the first time the twistor transform in $D = 11$ was discussed in [37], although without relation to pure spinors.
4.4. The ‘conjugate’ pure spinor measure $d^{11}\tilde{\lambda}$

Let us turn to the measure for the ‘conjugate’ pure spinor of the non-minimal formalism, equation (2.22). Its derivative reads

$$d^{11}\tilde{\lambda} = \frac{1}{2} \Omega^{ji} \tilde{\lambda} \Lambda^q_{\bar{r}} \gamma^q \bar{v}_{aq} + [(d\tilde{\Lambda} - \tilde{\Lambda} \Xi^{(0)}) \Lambda^q_{\bar{r}} - \Lambda^q_{\bar{r}} \Xi^{-1}] v_{aq}^+.$$  \hspace{1cm} (4.38)

This is related to (4.33) by reversing the $SO(1, 1)$ weight, substituting the upper Spin(1, 9) index by lower one (thus passing from spinor moving frame variables to the variables forming the inverse spinor moving frame matrix) and replacing $\Xi^{(0)}$ by $(d\tilde{\Lambda} - \tilde{\Lambda} \Xi^{(0)})$ (see equation (4.34)). The algebraic factors in (2.11) and (2.22) are related in a similar manner so that, as far as we are not interested in an overall numerical multiplier, we can write immediately the final answer for the ‘conjugate’ pure spinor measure (2.22). It reads

$$d^{11}\tilde{\lambda} = \frac{\tilde{\Lambda}^6}{6!} d\tilde{\Lambda} \wedge e^{h_{\bar{r}...k} \Xi^{-h} \wedge \ldots \wedge \Xi^{-h} \wedge \Omega^{\delta i}_4 \wedge \ldots \wedge \Omega^{\delta i}_4 M^{i_1\ldots i_4} (\Lambda^{\pm}),$$ \hspace{1cm} (4.39)

with

$$M^{i_1\ldots i_4} (\Lambda^{\pm}) := (\Lambda^\gamma \gamma^1 \gamma^2 \ldots \wedge (\Lambda^\gamma \gamma^1 \gamma^2 \ldots \wedge \Omega^{\delta i}_4 \wedge \ldots \wedge \Omega^{\delta i}_4 M^{i_1\ldots i_4} (\Lambda^{\pm}),$$ \hspace{1cm} (4.40)

$$M^{i_1\ldots i_4} (\Lambda^{\pm}) := (\Lambda^\gamma \gamma^1 \gamma^2 \ldots \wedge (\Lambda^\gamma \gamma^1 \gamma^2 \ldots \wedge \Omega^{\delta i}_4 \wedge \ldots \wedge \Omega^{\delta i}_4 M^{i_1\ldots i_4} (\Lambda^{\pm},$$ \hspace{1cm} (4.41)

Actually, to be more precise, one has to substitute $(d\tilde{\Lambda} - \tilde{\Lambda} \Xi^{(0)})$ for $d\tilde{\Lambda}$ in (4.39). However, as far as we expect to use $d^{11}\tilde{\lambda}$ only together with the original pure spinor measure (4.35), and the term $\propto \Xi^{(0)}$ does not contribute into $d^{11}\tilde{\lambda} \wedge d^{11}\tilde{\lambda}$, we prefer to write a simpler expression (4.35) from the very beginning.

Note that the complete measure of the non-minimal pure spinor formalism includes $d^{11}\tilde{\lambda} \wedge d^{11}\tilde{\lambda}$ and hence contains both $\Omega^{\omega i}$ and $\Omega^{\omega i}$. These enters inside the 8-form

$$\Omega^{\omega_1} \wedge \ldots \wedge \Omega^{\omega_i} \wedge \Omega^{\delta i}_4 \wedge \ldots \wedge \Omega^{\delta i}_4 M^{\omega_1\ldots i_4} (\Lambda^{\pm}) \wedge M^{++i_1\ldots i_4} (\Lambda^{\pm})$$ \hspace{1cm} (4.42)

which provides a kind of holomorphic measure on the noncompact coset $SO(1, 9)/[SO(1, 1) \otimes SO(8)]$ considered as a fiber of a bundle over $SO(8, \mathbb{C})/[U(1) \otimes SO(6, \mathbb{C})]$ coset. Note the difference with the case of minimal pure spinor formalism where, as discussed above, the measure includes only $\Omega^{\omega i}$ forms entering inside $\Omega^{\omega_1} \wedge \ldots \wedge \Omega^{\omega_i} M^{++i_1\ldots i_4} (\Lambda^{\pm})$ which provides the holomorphic measure on the compact fiber $\mathbb{S}^8$ of the fiber bundle over $\mathbb{C}^+ - \{0\}.

5. A problem with the fermionic measure $d^{11}r$ and a possible wayout

It is natural to try to reproduce as well the measure (2.23) for the constrained fermionic variable $r_{\bar{a}}$. However, the available solution for the constraints imposed on this variable, equation (4.13), includes $\chi$ defined up to an infinitely reducible symmetry transformations (4.14)–(4.16) (resembling the $\kappa$-symmetry of the original Green–Schwarz formulation of the superstring). This suggests to search for an alternative way to write the counterpart of the fermionic measure $d^{11}r$ in the non-minimal pure spinor path integral for the quantum superstrings.

To this end let us first concentrate on the contribution of the constrained fermionic spinor $r_{\bar{a}}$ into the BRST charge of the non-minimal spinor formalism, equation (2.18). It enters in the product with the momentum of the conjugate pure spinor, $\bar{u}^{\bar{a}} r_{\bar{a}}$, and the constraints for $r_{\bar{a}}$ comes from the requirement of the preservation of the $\Xi^\alpha$ gauge symmetry (2.16) acting on $\bar{u}^{\bar{a}}$. 

14
The momentum $\tilde{u}^\alpha$ enters also the free CFT action (2.17), in this case in the product with $\delta \lambda_\alpha$. At this stage let us note that, with our solution of the pure spinor constraints (4.3), $d\delta \lambda_\alpha = dz \delta \lambda_\alpha + d\bar{z} \delta \lambda_\alpha$ has the form (4.38) and, hence,

$$d\delta \lambda_\alpha = \frac{1}{2} \Omega^\Lambda_z \Lambda_\Lambda^{-1}_q \gamma_q v_{aq}^{-} + [i(\tilde{\delta} \Lambda - \Lambda \Xi_{(0)}^v \Lambda_\Lambda^{-1}_q - \Lambda_\Lambda^{-1}_q \Xi_{(0)}^v)]v_{aq}^{-}. \quad (5.1)$$

Here, $\Omega^\Lambda_z$ appear in the decomposition $\Omega^\Lambda_z = dz \Omega^\Lambda_z + d\bar{z} \Omega^\Lambda_z$ and, similarly, $\Xi^{-1} = dz \Xi^{-1} + d\bar{z} \Xi^{-1}$, $\Xi = dz \Xi + d\bar{z} \Xi$ etc.

Equation (5.1) implies that the free CFT action (2.17) contains $\tilde{u}^\alpha$ only in the following combinations (as far as only these contribute to $d\delta \lambda_\alpha \tilde{u}^\alpha$)

$$\Lambda^{-1}_q \gamma_q v_{aq}^{-}, \quad \tilde{u}^\alpha v_{aq}^{-}, \quad \tilde{u}^\alpha v_{aq}^{+}, \quad \Lambda^{-1}_q. \quad (5.2)$$

Let us observe that these combinations are invariant under the $\tilde{\Xi}^\alpha$ gauge symmetries of equation (2.16). Indeed, using the general solution (4.3) one sees that under this symmetry $\delta \tilde{u}^\alpha v_{aq}^{+} = \tilde{\Xi}^\alpha v_{aq}^{+}$ and $\delta \tilde{u}^\alpha v_{aq}^{-} = \tilde{\Xi}^\alpha v_{aq}^{-}$, so that the variations of the expressions in (2.17) vanish due to the algebraic properties of $\Lambda_\Lambda^{-1}_q$ and $\Lambda^{-1}_q$.

Now observe that the gauge invariant combinations in equation (2.17) are in one-to-one correspondence to the ‘covariant momenta’, $w_\lambda$, dual to $d\Lambda$ (which is to say, conjugate to $\Lambda$), and $D^{\alpha \beta}$ and $\bar{D}^{\alpha \beta}$ dual to the Cartan forms of the cosets of $SO(1,9)$ and $SO(8, C)$ groups, which enter the rhs of equation (4.38). (We denote these covariant momenta by the same symbols as the covariant derivatives in (4.22)–(4.25)).

The above observation encourages us to propose the following prescription of changing the dynamical variables and the free CFT action (presently, in its part related to conjugate pure spinor)

$$d\delta \lambda_\alpha \tilde{u}^\alpha \mapsto (\delta \Lambda - \Xi_{(0)}^v \Lambda_\Lambda^{-1}_q) \tilde{u}^\alpha + \Xi^{-1}_z D^{\alpha \beta}_\Lambda + \Omega^\Lambda_z \bar{D}^{\alpha \beta}. \quad (5.3)$$

Furthermore, we have to reformulate the non-minimal pure spinor model in such a way that it involves $w_\lambda, D^{\alpha \beta}_\Lambda$ and $\bar{D}^{\alpha \beta}$ instead of $\tilde{u}^\alpha$ defined modulo $\tilde{\Xi}^\alpha$ gauge symmetry (2.16),

$$\tilde{u}^\alpha \mapsto (w_\lambda, D^{\alpha \beta}_\Lambda, \bar{D}^{\alpha \beta}). \quad (5.4)$$

To check once more that equations (5.3) and (5.4) are reasonable, one can calculate the number of dofs. On the left, being a momentum conjugate to a pure spinor, $\tilde{u}^\alpha$ has to carry 11 complex dofs. On the right we have 1 complex dof in $w_\lambda, 6$ complex dofs in $D^{\alpha \beta}_\Lambda$ and 4 complex (8 real) dofs in $\bar{D}^{\alpha \beta}$.

In its turn, equations (5.3) and (5.4) suggest to make the following substitution in the BRST charge of the non-minimal pure spinor approach (2.18):

$$r_\alpha \tilde{u}^\alpha \mapsto r^{(0)}_\alpha \tilde{u}^\alpha + r^{-1}_\alpha D^{\alpha \beta}_\Lambda + r^{\beta}_\alpha \bar{D}^{\alpha \beta}. \quad (5.5)$$

and to consider the modification of the original formalism which involves the 11 complex (22 real) fermionic fields: 1 complex $r^{(0)}_\alpha, 6$ complex in $r^{-1}_\alpha$ and 8 real in $r^{\beta}_\alpha$, instead of the constrained $r_\alpha$,

$$r_\alpha \mapsto (r^{(0)}_\alpha, r^{-1}_\alpha, r^{\beta}_\alpha). \quad (5.6)$$

Then the similarity between the bosonic $d^{11} \lambda$ and fermionic $d^{11} r$ measures of the non-minimal pure spinor formalism, which are related by the map $d\lambda^\alpha \mapsto \frac{d}{d \lambda^\alpha}$, suggests to obtain the fermionic measure $d^{11} r$ of our approach by mapping

$$d \Xi^{(0)} \mapsto \frac{\partial}{\partial r^{(0)}}, \quad \Xi^{-1} \mapsto \frac{\partial}{\partial r^{-1}}, \quad \Omega^{\beta \epsilon} \mapsto \frac{\partial}{\partial r^{\beta \epsilon}}. \quad (5.7)$$

The result is

$$d^{11} r = M^{+++-11}((\Lambda^\pm) e^{11} \cdot \frac{\partial}{\partial r^{(0)}} \cdots \frac{\partial}{\partial r^{-1}} \frac{\partial}{\partial r^{\beta \epsilon}} \cdots \frac{\partial}{\partial r^{\beta \epsilon}}) \quad (5.8)$$
where \( M^{+-...} \) is defined in equations (4.36) and (4.37).

As far as non-minimal fermion contribution to the free CFT action is concerned, the natural prescription would be

\[
\bar{s}^{\mu} \partial r_{\mu} \mapsto s^{(0)} \bar{\partial} r^{(0)} + s^{(+)} \bar{\partial} r^{(+)} + s^{(-)} \bar{\partial} r^{(-)},
\]

so that

\[
s^{\mu} \mapsto (s^{(0)}, s^{(+)}, s^{(-)})
\]

with complex \( s^{(0)} \) and \( s^{(+)}, s^{(-)} \), and real \( s^{(-)} \) (all unconstrained).

6. Discussion and outlook. Toward quantum superstring formulation without reducible symmetries?

The discussion above suggests a possible reformulation of the quantum superstring theory, in particular the prescription to calculate loop superstring amplitudes reached in the frame of pure spinor approach [1–10], in terms of new set of variables. In such a formulation the elements of the ghost sector of the non-minimal pure spinor formalism

\[
(\lambda^\alpha, w_\alpha), \quad (\bar{\omega}^{\dot{\alpha}}, \bar{\lambda}_{\dot{\alpha}}), \quad (r_\alpha s^\alpha)
\]

are replaced by the constrained complex variables \( \Lambda q \) parameterizing the coset of \( \text{SO}(8, \mathbb{C}) \) group (equations (4.11) and (4.10)) and its Lie algebra, complex scalar and its momentum \((\Lambda, \bar{\omega}, \lambda)\), and the spinor moving frame variables parameterizing the coset \( \text{SO}(1, 9)/[\text{SO}(1, 1) \times \text{SO}(8)] \) (equations (A.10)–(A.15)) and their covariant momenta

\[
\frac{\text{SO}(8, \mathbb{C})}{\text{SO}(6, \mathbb{C})} = \left\{ (\Lambda^+, \Lambda^-) \right\}, \quad \left\{ (D_{\Lambda}^{(0)}, D_{\Lambda}^{\pm}) \right\}, \quad \left\{ (\bar{D}^{\dot{\Lambda}}, \bar{D}^{\mp}) \right\}.
\]

(Note by pass that, following the line of appendix A.3, we can replace the spinor moving frame variables by two sets of spinor moving frame variables parametrizing a product of two cosets isomorphic to 8-spheres, \( S^8 \times S^8 \). We however, do not elaborate on this possibility here).

Then the non-minimal pure spinor CFT action (2.17) is replaced by

\[
S_{new} = \int (1/2) \tilde{x}_\mu \tilde{x}^{\mu} + p_\alpha \tilde{\theta}^{\alpha} - \int (\Omega^{\beta}_{\alpha} \tilde{D}^{\beta} + \Omega^{\beta}_{\alpha} \tilde{D}^{\beta})
\]

\[
- \int \left( \tilde{\Omega}^{\beta}_{\alpha} D_{\Lambda}^{\beta} + \tilde{\Omega}^{\beta}_{\alpha} D_{\Lambda}^{(0)} + (\tilde{\partial} \Lambda - \Lambda \tilde{\partial} \Lambda) \tilde{w}_{\dot{\alpha}} \right)
\]

\[
+ \int (s^{(0)} \tilde{\partial} r^{(0)} + s^{(+)} \tilde{\partial} r^{(+)}) + s^{(-)} \tilde{\partial} r^{(-)}.
\]

The spinor moving frame momentum operators \( \tilde{D}^{(0)} \) and \( \tilde{D}^{(j)} \) are currents of the \( \text{SO}(1, 1) \) and of the \( \text{SO}(8) \) gauge symmetry of the model when acting on the spinor moving frame variable. The complete currents of \( \text{SO}(1, 1) \) and of the \( \text{SO}(8) \) act also on the fermionic ghosts and (in the case of \( \text{SO}(1, 1) \)) on \( \text{SO}(8, \mathbb{C}) \) parameters. Together with the \( \text{SO}(6, \mathbb{C}) \) and \( \text{SO}(2, \mathbb{C}) \) currents they replace \( J \) and \( N_{ab} \) (equations (2.9)–(2.19)) of pure spinor approach, schematically,

\[
N_{ab} + \tilde{N}_{ab} \mapsto \begin{cases} 
\text{D}^{ij} + \alpha (r^{ij} s^{(-)} - r^{ij} s^{(+)}) \\
\text{D}^{(0)} + \alpha (r^{(0)} s^{(-)} - r^{(0)} s^{(+)}) \\
\text{D}^{(0)} + \alpha (r^{(0)} s^{(-)} - r^{(0)} s^{(+)}) \\
\text{D}^{(0)} + \alpha (r^{(0)} s^{(-)} - r^{(0)} s^{(+)}) \\
\end{cases},
\]

\[
J \mapsto \tilde{\Lambda} \tilde{w}_{\dot{\alpha}} + \alpha (r^{\dot{\alpha}} s^{(-)} + r^{\dot{\alpha}} s^{(+)})
\]

(6.3)

(6.4)
The energy–momentum tensor of the reformulated CFT (6.3) should read
\[
T(z, \bar{z}) = -\frac{1}{2} \partial_x \mu \partial_x \mu - p_\alpha \partial \theta^\alpha + \Omega^i \bar{\Omega}^{\ast i} + \Omega^i \bar{\Omega}^{\ast i} + \bar{\Xi}_i D^{\ast i} + \Xi^i D_i
\]
(6.5)

The BRST operator of the non-minimal pure spinor formalism is replaced by
\[
Q = \int \left( \Lambda^i \bar{X}^i + \bar{r}^{(0)} \bar{w}_\Lambda \right) + \bar{r}^i \right) \cdot \Lambda^i \bar{X}^i + \bar{r}^i \cdot \left( \bar{X}^{(0)} - \bar{r}^{(0)} - \bar{r}^i \right).
\] (6.6)

The above presented reformulations of the elements of the non-minimal pure spinor path integral measure, equations (4.35), (4.39) and (5.8), actually correspond to the hypothetical quantum superstring formulation with the CFT action (6.3) and the BRST charge (6.6). Let us stress that these are 10D Lorentz invariant; despite forms, variables and momenta, like \(\Omega^i\), \(\bar{r}^{\ast i}\), \(\bar{X}^{\ast i}\) and others, carry the indices of \(SO(8)\) and \(SO(1, 1)\) groups, these are independent gauge symmetry groups of the spinor moving frame approach, which also possesses \(SO(1, 9)\) symmetry acting on \(x^\mu, \theta^\alpha\) and \(p_\alpha\) (and leaving invariant \(\Omega^i\), \(\bar{r}^{\ast i}\), \(\bar{X}^{\ast i}\) etc).

Certainly, the consistency of the reformulated theory needs to be checked. This implies the study of the possible conformal anomalies of the CFT (6.3), as well as of the cohomologies of the BRST operator (6.6). We plan to address these problems in the nearest future.

Then, if consistency and nontriviality is proved, the next stages to develop our approach would be the construction of the b-ghost (see [10]) and of the vertex operators in terms of new variables, obtaining the complete expression for simplest amplitudes thus making examples of the tree and loop amplitude calculations (beginning from the ones already calculated in the pure spinor formalism, e.g. in [38] and [6]).

We hope to turn to these stages in our future publications.

Acknowledgments

The author is thankful to Dima Sorokin and Mario Tonin for reading the paper and useful comments. A partial support from the research grants FIS2008-1980 and FPA2012-35043-C02-01 from Spanish MINECO, as well as from the Basque Government Research Group grant ITT559-10 is greatly acknowledged.

Appendix. Moving frame, spinor moving frame and twistor-like formulation of classical superstring

This appendix contains a brief review on moving frame and spinor moving frame variables and their applications to classical description of superstring. We begin in section A1. by describing moving frame formulation of classical Green–Schwarz superstring and then, in section A2, generalize it to twistor-like or spinor moving frame formulation [16]. In section A3, we review the spinor moving frame formulation of the superparticle [15], where the spinor moving frame variables can be considered as coordinates of the coset of the Lorentz group isomorphic to \(S^8\), which can be identified with the celestial sphere of a 10D observer (see [17]). We also discuss there a reformulation of the twistor-like action for superstring in which the generalized spinor moving frame sector parameterize the direct product \(S^8 \times S^8\) (instead of the noncompact coset \(Spin(1, 9)/SO(1, 1)\times SO(3)\) in [16]).
The moving frame associated with a massless (super)particle and (super)string is described by the $SO(1, 9)$ valued matrix

$$U_a^{(b)} = \left( \frac{1}{2} (u_a^\mu + u_a^-), \ u_a^\mu, \frac{1}{2} (u_a^- - u_a^\mu) \right) \in SO(1, 9). \quad (A.1)$$

Condition (A.1) implies

$$u_a^- u_a^\mu = 0, \quad u_a^\mu u_a^\nu = 0, \quad u_a^- u_a^- = 2, \quad u_a^\mu u_a^\mu = 0, \quad u_a^- u_a^\nu = -\delta^\nu_\mu,$$

as well as $U \eta U^T = \eta$ which reads as a unity decomposition

$$\delta_{ab} = \frac{1}{2} u_a^\mu u_b^\nu + \frac{1}{2} u_a^- u_b^- - u_a^\mu u_b^\nu. \quad (A.3)$$

The decomposition in (A.1) is invariant under the left action of the 10D Lorentz group $SO(1, 9)$ and under the right action of $SO(1, 1) \times SO(8)$ subgroup of $SO(1, 9)$. If dynamical model under study possesses gauge symmetry under these right $SO(1, 1) \times SO(8)$ transformations, one can use them as an identification relation in the $SO(1, 9)$ group manifold parametrized by the vectors (A.1) constrained by (A.2). Then one can consider these constrained vectors as homogeneous coordinates of the $SO(1, 9)/[SO(1, 1) \times SO(8)]$ coset (see [16]). This makes moving frame variables similar to the so-called harmonic variables of the R-symmetry groups [27] useful to formulate the $N = 2, 3$ supersymmetric theory in terms of unconstrained superfields. Such a similarity was the reason to call the moving frame and spinor moving frame variable ‘Lorentz harmonics’ (‘spinorial harmonics’) [16, 28]; also the name ‘light-cone harmonics’ was used in [29].

The early applications of the other versions of Lorentz harmonic approaches, close but not identical to [16] can be found in [30, 31] as well as in [17].

### A.1. Moving frame formulation of superstring

The Goldstone fields in the coset $SO(1, 9)/[SO(1, 1) \times SO(8)]$ describe the spontaneous breaking of $SO(1, 9)$ symmetry down to its $SO(1, 1) \times SO(8)$ subgroup. Such a symmetry breaking is characteristic for a 10D string (and superstring) model so that it is not surprising that one can write a formulation of superstring action with the use of the moving frame variables. Such an action was proposed and studied in [16]. In an arbitrary 10D supergravity background, the action of the moving frame formulation can be written as

$$S_{\text{moving frame}} = \frac{1}{2} \int_{W^3} \tilde{E}^\# \wedge \tilde{E}^\# - \int_{W^2} \tilde{B} \wedge \tilde{B}, \quad (A.4)$$

where $\wedge$ is the exterior product of differential forms ($\tilde{E}^\# \wedge \tilde{E}^\# = -\tilde{E}^\# \wedge \tilde{E}^\#$),

$$\tilde{E}^\# := \tilde{E}^\mu u_\mu^\#, \quad \tilde{E}^- := \tilde{E}^\mu u_\mu^-,$$  

(A.5)

where $u_\mu^\# (\tau, \sigma)$ and $u_\mu^- (\tau, \sigma)$ are light–like moving frame vector fields obeying (A.2) and

$$\tilde{E}^\# := \tilde{d} \tilde{Z}^M (\xi) E_M^a (\tilde{Z} (\xi)) = d^a \tilde{E}_m^a, \quad \tilde{E}^- := \partial_\alpha \tilde{Z}^M (\xi) E_M^a (\tilde{Z} (\xi))$$

(A.6)

is the pull–back of the bosonic supervielbein of the 10D superspace, $E^a (Z) := dZ^M E_M^a (Z)$, to the worldsheet $W^2$ with local coordinates $\xi^m = (\tau, \sigma)$. In the case of flat $N = 1$ superspace

$$E^a = \Pi^\# \delta^a_\mu = d\tau^\alpha \delta^a_\mu + \frac{1}{4} \theta^\alpha \sigma^a_\mu d\theta^\beta$$

(A.7)

(here we try to follow the notation of pure spinor literature [1–4, 9, 23]). Finally, $-\int_{M^4} \tilde{B} \wedge \tilde{B}$ is the Wess–Zumino term, the same as in the original Green–Schwarz formulation of the superstring action; that reads

$$S_{\text{GS}} = \frac{1}{2} \int_{W^2} d^2 \xi \sqrt{|\det (g_{mn})|} - \int_{W^2} \tilde{B} \wedge \tilde{B} \quad (A.8)$$
with \( g_{\mu \nu} := \dot{E}^m_{\nu} g_{\mu \nu} \dot{E}^b_m \) and \( d^2 \xi = d\tau \wedge d\sigma \). We do not need in explicit form of \( B_2 = \frac{1}{2} dZ^M \wedge dZ^N B_{NM}(Z) \) in this paper.

One can also introduce auxiliary worldsheet vielbein \( e^\# = d \xi^m e^\#_m(\xi) \), \( e^- = d \xi^m e^-_m(\xi) \) and write the moving frame action in the first order form \(^{16}\text{9}\),

\[
S_{\text{moving frame}}' = \frac{1}{2} \int_{W^2} (e^\# \wedge \dot{E}^- - e^- \wedge \dot{E}^\# - e^\# \wedge e^-) - \int_{W^2} \dot{B}_2, \tag{A.9}
\]

### A.2. Spinor moving frame

The above moving frame formulation of superstring, characterized by the action \((A.9)\) or \((A.4)\), was also called ‘twistor-like’ and ‘spinor moving frame’ formulation. This is related to the fact that one can introduce a set of constrained spinors \( v_q^a = v^-_q^a \) and \( v_q^\# = v^+_q^\# \) which can be considered as square roots of the light–like vectors \( u_q^a \) and \( u_q^\# \) in the sense of (the trace part of) equations \((3.2)\) and \((3.6)\). These two sets of constrained spinors can be considered as \(16 \times 8\) blocks of the spinor moving frame matrix

\[
V^\alpha_\beta = \left( v_q^{-\beta}, v_q^+\beta \right) \in \text{Spin}(1, 9). \tag{A.10}
\]

The \( \text{Spin}(1, 9) \) valuedness can be expressed as a statement of Lorentz invariance of the \(10\)-D Pauli matrices \( \sigma_{\alpha \beta} = \sigma_{\beta \alpha} \) and \( \tilde{\sigma}^{\alpha \beta} = \tilde{\sigma}^{\beta \alpha} \) which obey

\[
\sigma_{\alpha \beta} \tilde{\sigma}^{\alpha \beta} + \sigma_{\beta \alpha} \tilde{\sigma}^{\beta \alpha} = \eta_{ab} I_{16 \times 16}. \tag{A.11}
\]

Namely, equation \((A.10)\) \((V \in \text{Spin}(1, 9))\) states that the similarity transformation of the \(10\)-D Pauli matrices with the matrix \( V \) results in a linear combination of these Pauli matrices,

\[
V \sigma_a V^T = U^a_b \sigma_b, \quad V^T \tilde{\sigma}^{(a)} V = U^a_b \tilde{\sigma}^b. \tag{A.12}
\]

The \(10 \times 10\) matrix of the coefficients in the rhs of equations \((A.12)\) takes its values in the Lorentz group \( SO(1, 9) \). Identifying it with the moving frame matrix \((A.2)\) we find that \((A.12)\) implies \((3.2), (3.6)\) and a set of similar relations,

\[
v^\#_q^a \sigma^{a \alpha} v^\#_p^\beta = \delta_{\alpha \beta} u^\#_q^a, \quad u^\#_q^a \tilde{\sigma}^{a \alpha} = 2 v^\#_q^a v^\#_q^\beta, \tag{A.13}
\]

\[
v^-_q^\alpha \sigma^{\alpha \beta} v^-_p^\gamma = \delta_{\alpha \beta} u^-_q^\gamma, \quad u^-_q^\gamma \tilde{\sigma}^{\gamma \alpha} = 2 v^-_q^\gamma v^-_q^\beta, \tag{A.14}
\]

\[
v^-_q^\alpha \sigma^{a \alpha} v^+_q^\beta = -\gamma^a_{\alpha \beta} u^-_q^a, \quad u^+_q^a \tilde{\sigma}^{a \alpha} = -2 v^-_q^\gamma v^+_q^\gamma - v^-_q^\gamma v^+_q^\gamma. \tag{A.15}
\]

Note that \((3.3)\) and \((3.7)\) are obeyed by the second equation in \((3.13)\) and of the second equation in \((A.14)\), respectively.

To write the similar square root type relation in terms of matrices \( u_q^- \sigma_{a \beta} \) and \( u_q^\# \sigma_{a \beta} \), one has to introduce the inverse spinor moving frame matrix

\[
V^\alpha_\gamma = \left( v^-_\gamma, v^\#_\gamma \right) \in \text{Spin}(1, 9), \quad V^\alpha_\gamma V^{\beta}_\gamma = \delta^{(\beta)}_\gamma = \begin{pmatrix} \delta_{\alpha}^\beta & 0 \\ 0 & \delta_{\gamma}^\beta \end{pmatrix}. \tag{A.16}
\]

Its elements obey the constraints

\[
v^-_p^\alpha v^-_q^\gamma = \delta_{pq}, \quad v^-_p^\gamma v^-_q^\gamma = 0, \quad v^+_p^\alpha v^+_q^\gamma = 0, \quad v^+_p^\gamma v^+_q^\gamma = \delta_{pq} \tag{A.17}
\]

and also \((3.12)–(3.14)\).

\(^9\) The formulations of superstring model proposed and developed in \([30]\) and \([31]\) introduced Lorentz harmonics (moving frame variables and (counterparts of) spinor moving frame variables) as additional variables in the Hamiltonian approach to superstring developed on the basis of the standard Lagrangian action \((A.8)\).
A.3. Spinor moving frame and celestial sphere

The (spinor) moving frame formulation of the massless superparticle is based on the action [15, 28]

\[ S' = \int W (\tau) \hat{E}^a u_a^m (\tau) \] (A.18)
involving only one light–like spinor moving frame vector \( u_a^m \). Then the natural \( SO(1,1) \otimes SO(8) \) gauge symmetry of the splitting (A.1) in the superparticle action is enlarged to the semi-direct product \( [SO(1, 1) \otimes SO(8)] \otimes K_S \) [17], where \( K_S \) transformations act on the moving frame as

\[ \delta u_a^m = 0, \quad \delta u_a^m = 2k^m u_a^m, \quad \delta u_a^i = k^m u_a^m, \] (A.19)

and on the spinor moving frame variables by

\[ \delta v^{-} q = 0, \quad \delta v^{+} q = -k^m v^{-} p q, \] (A.20)

As it was stressed in [17], \( [SO(1, 1) \otimes SO(8)] \otimes K_S \) is the maximal parabolic subgroup of \( SO(1, 9) \) so that the coset \( Spin(1, 9)/[SO(1, 1) \otimes SO(8)] \otimes K_S \) is a compact space which is actually isomorphic to the 8-sphere \( S^8 \),

\[ Spin(1, 9)/[SO(1, 1) \otimes SO(8)] \otimes K_S = S^8. \] (A.21)

In the superparticle model (A.18) this \( S^8 \) can be identified with celestial sphere of the 10D observer [17].

Furthermore, taking into account equation (3.13) one can write the functional (A.18) in the form

\[ S' = \frac{1}{8} \int \rho (\tau) \hat{E}^a u_a^m \] (A.22)

which makes transparent that we are dealing with 10D generalization of the \( D = 4 \) Ferber–Schirafuji action functional [32] which provides a Lagrangian basis for the Penrose twistor approach [33]. This is why the spinor moving frame formulation of superparticle [15, 28] is also called twistor-like.

Using equations (3.13) and (3.12) one can also write the action (A.9) as the sum of two terms similar to (A.22) and a cosmological term; this observation suggested to call ‘twistor-like’ also the superstring spinor moving frame action (A.9) [16].

Note that the superstring model can be formulated not only with moving frame variables parametrizing the noncompact coset, as in (A.9), but also with the action functional

\[ S_{moving \ frame}' = \frac{1}{2} \int_y (e^a \wedge \hat{E}^a u_a^m - e^a \wedge \hat{E}^a \tilde{u}^a - e^a \wedge \tilde{E}^a \tilde{u}^a - e^a \wedge \tilde{E}^a \tilde{u}^a) - \int \hat{B}_2 \] (A.23)

involving two light-like vectors from different moving frames; this is to say we do not require the contraction \( u_a^m \tilde{u}^a \) to be equal to a fixed constant. The auxiliary field sector of this model is the direct product of compact spaces, \( S^8 \otimes S^8 \).

A.4. Derivatives of the moving frame variables

Although the moving frame and spinor moving frame variables are highly constrained, their transparent group theoretical meaning allows to calculate easily their derivatives and variations [15, 16]10.

10 The authors of the relatively recent [34] prefer to work instead with explicit solutions of the constraints on the moving frame variables.
Referring to the original references for details, here we present here just the results. The $SO(1, 9) \times SO(1, 1) \otimes SO(8)$ covariant derivatives of the moving frame and spinor moving frame variables read

\begin{align}
Du_a^\nu := & \, du^\nu_a + 2u^\rho_a \Omega^{\rho(0)} = u^\nu_a \Omega^{-\nu}, \\
Du_a^\rho := & \, du^\rho_a - 2u^\sigma_a \Omega^{\sigma(0)} = u^\rho_a \Omega^\rho, \\
Du_a^\sigma := & \, du^\sigma_a + u^a_\rho \Omega^{\rho} = \frac{1}{2} u^\sigma_a \Omega^\rho + \frac{1}{2} u^\rho_a \Omega^{-\sigma}.
\end{align}

Here $\Omega^\rho := u^\mu_\rho du^\mu_a$ and $\Omega^{-\sigma} := u^\mu_\sigma du^\mu_a$ are covariant Cartan forms providing the vielbein for the coset $SO(1, 9)/[SO(1, 1) \times SO(8)]$. The Cartan forms $\Omega^\rho := u^\mu_\rho du^\mu_a$ and $\Omega^{0(0)} := \frac{1}{2} u^\rho_a du^\rho_a$ transform as $SO(8)$ and $SO(1, 1)$ connections.

The $SO(1, 1) \otimes SO(8)$ covariant derivatives of the spinor moving frame variables are expressed through the same Cartan forms by

\begin{align}
Dv^{-\alpha} := & \, dv^{-\alpha} + \Omega^{0(0)} v^{-\alpha} + \frac{1}{2} \Omega^i v^{-\alpha} v^i, \\
Dv^{\alpha} := & \, dv^{\alpha} - \Omega^{0(0)} v^{\alpha} + \frac{1}{2} \Omega^i v^{\alpha} v^i, \\
Dv^{+\alpha} := & \, \frac{1}{2} \Omega^0 v^{+\alpha} v^{+\alpha}, \quad Dv^{-\alpha} := \frac{1}{2} \Omega^{-\alpha} v^{+\alpha} v^{+\alpha}.
\end{align}

Now, using (A.17), is not hard to find that for the elements of the inverse spinor moving frame the covariant derivatives read

\begin{align}
Dv^{-\alpha} := & \, dv^{-\alpha} + \Omega^{0(0)} v^{-\alpha} + \frac{1}{2} \Omega^i v^{-\alpha} v^i, \\
Dv^{\alpha} := & \, dv^{\alpha} - \Omega^{0(0)} v^{\alpha} + \frac{1}{2} \Omega^i v^{\alpha} v^i,
\end{align}

The differential (derivative) in the space of moving frame variables is decomposed on the above Cartan forms

\begin{align}
d_{u,v} := & \, \Omega^{0(0)} d_{u,v} + \frac{1}{2} \Omega^j d_{u,v} + \Omega^{-\alpha} d_{u,v} + \Omega^{\alpha} d_{u,v}.
\end{align}

This decomposition defines the 'covariant harmonic derivatives' (cf [27]) which obey the Lorentz group algebra\(^{11}\). Using the above explicit expressions one can easily find their action on the moving frame variables,

\begin{align}
\begin{aligned}
\Omega^{0(0)} u^\nu_a &= 2u^\nu_a, \quad \Omega^{0(0)} v^\alpha &= -2v^\alpha, \quad \Omega^{0(0)} \mu_a &= 0, \\
\Omega^j u^\nu_a &= 0, \quad \Omega^j v^\alpha &= 0, \quad \Omega^j \mu_a &= 2\delta^j_i \mu_a, \\
\Omega^{\alpha} u^\nu_a &= 0, \quad \Omega^{\alpha} v^\alpha &= 0, \quad \Omega^{\alpha} \mu_a &= \frac{1}{2} \Omega^{\alpha} \mu_a, \\
\Omega^{\nu} u^\nu_a &= u^\nu_a, \quad \Omega^{\nu} v^\alpha &= 0, \quad \Omega^{\nu} \mu_a &= 0, \\
\Omega^{\sigma} u^\nu_a &= u^\nu_a, \quad \Omega^{\sigma} v^\alpha &= 0, \quad \Omega^{\sigma} \mu_a &= 0,
\end{aligned}
\end{align}

and on the spinor moving frame variables,

\begin{align}
\begin{aligned}
\Omega^{0(0)} v^+ &= v^+, \quad \Omega^{0(0)} v^- = -v^-, \\
\Omega^j v^+ &= \frac{1}{2} v^+ v^j, \quad \Omega^j v^- = \frac{1}{2} v^- v^j, \\
\Omega^{\alpha} v^+ &= 0, \quad \Omega^{\alpha} v^- = -\frac{1}{2} v^- v^\alpha, \\
\Omega^{\nu} v^+ &= 0, \quad \Omega^{\nu} v^- = -\frac{1}{2} v^- v^\nu, \quad \Omega^{\sigma} v^+ = 0.
\end{aligned}
\end{align}

When we discuss the spinor moving frame fields depending on worldsheet coordinates, we can introduce their covariant momentum, which can be denoted by the same symbols $\Omega^{\nu}$, $\Omega^{-\sigma}$, $\Omega^j$ and $\Omega^{0(0)}$ and obey the straightforward OPE generalization of equations (A.31)–(A.35).

\(^{11}\) One can write the general expression $d_{u,v} = \frac{1}{2} \Omega^{(ab)} d_{u,v}^{(ab)}$ with $d_{u,v}^{(ab)} = u_i \frac{1}{2} \delta_{ij} d_{u,v}^{(ij)}$. But then the last part of equation (A.30) would include some strange looking coefficients, $\Omega^{(ab)} \rightarrow -\frac{1}{2} \Omega^{(ab)}$, $\Omega^{-\sigma} \rightarrow -\frac{1}{2} \Omega^{-\sigma}$, and this is what we would like to escape. With our choice $\Omega^{-\sigma} = \frac{1}{2} \Omega^{(ab)} + \frac{1}{2} \Omega^{\alpha} \mu_a$ $\Omega^{(ab)} = \frac{1}{2} \Omega^{(ab)} + \frac{1}{2} \Omega^{\alpha} \mu_a$. But it is much more practical to calculate the action of $\Omega^{0(0)}$ and $\Omega^j$ on functions of moving frame and spinor moving frame variables by using equations (A.24)–(A.30).
References

[1] Berkovits N 2000 Super-Poincare covariant quantization of the superstring J. High Energy Phys. JHEP04(2000)018 (arXiv:hep-th/0001035)

[2] Berkovits N 2004 Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring J. High Energy Phys. JHEP09(2004)047 (arXiv:hep-th/0406055)

[3] Berkovits N 2005 Pure spinor formalism as an \( N = 2 \) topological string J. High Energy Phys. JHEP10(2005)089 (arXiv:hep-th/0509120)

[4] Berkovits N and Nekrasov N 2006 Multiloop superstring amplitudes from non-minimal pure spinor formalism J. High Energy Phys. JHEP12(2006)029 (arXiv:hep-th/0609012)

[5] Gomez H and Mafra C R 2010 The overall coefficient of the two-loop superstring amplitude using pure spinors J. High Energy Phys. JHEP09(2010)047 (arXiv:1003.0678 [hep-th])

[6] Mafra C R and Schlotterer O 2012 The structure of \( n \)-point one-loop open superstring amplitudes arXiv:1203.6215 [hep-th] and references therein

[7] Matone M, Mazzucato L, Oda I, Sorokin D and Tonin M 2002 The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings Nucl. Phys. B 639 182 (arXiv:hep-th/0206104)

[8] Bandos I A 2008 Spinor moving frame, M0-brane covariant BRST quantization and intrinsic complexity of the pure spinor approach Phys. Lett. B 659 388–98 (arXiv:0707.2336 [hep-th])

Bandos I A 2008 \( D = 11 \) massless superparticle covariant quantization, pure spinor BRST charge and hidden symmetries Nucl. Phys. B 779 63 (arXiv:0704.1219 [hep-th])

Bandos I A and Nurmagambetov A Y 1997 Generalized action principle and extrinsic geometry for \( N = 1 \) superparticle Class. Quantum Grav. 14 L579 (arXiv:hep-th/9610098)

Bandos I A and Zheltukhin A A 1993 Lorentz harmonics and new formulations of superstrings, and kappa symmetry JETP Lett. 54 421–4

Bandos I A and Zheltukhin A A 1994 \( D = 10 \) superstring: Lagrangian and Hamiltonian mechanics in twistor-like Lorentz harmonic formulation Phys. Part. Nucl. 25 453–77

[10] Howe P S 1986 Classical superstring mechanics Nucl. Phys. B 263 93

Bandos I A and Zheltukhin A A 1993 Lorentz harmonics and new formulations of superstrings in \( D = 10 \) and supermembranes in \( D = 11 \) Phys. At. Nucl. 56 113
Bandos I A and Zheltukhin A A 1993 Yad. Fiz. 56N1 198 (in Russian)
[26] Green M B and Schwarz J H 1984 Covariant description of superstrings Phys. Lett. B 136 367
[27] Galperin A, Ivanov E, Kalitizin S, Ogievetsky V and Sokatchev E 1984 Unconstrained $N = 2$ matter, Yang–Mills and supergravity theories in Harmonic superspace Class. Quantum Grav. 1 469–98
Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E S 2001 Harmonic Superspace (Cambridge: Cambridge University Press) pp 306
[28] Bandos I A 1990 A superparticle in Lorentz-Harmonic superspace Sov. J. Nucl. Phys. 51 906–14
Bandos I A 1990 Yad. Fiz. 51 1429–44 (in Russian)
Bandos I A 1990 JETP Lett. 52 205–7
[29] Sokatchev E 1986 Light cone harmonic superspace and its applications Phys. Lett. B 169 209–14
Sokatchev E 1987 Harmonic superparticle Class. Quantum Grav. 4 237–46
[30] Nissimov E R and Pacheva S J 1988 Manifestly superpoincare covariant quantization of the Green–Schwarz superstring Phys. Lett. B 202 325
Nissimov E, Pacheva S and Solomon S 1988 Covariant canonical quantization of the Green–Schwarz superstring Nucl. Phys. B 297 349
Nissimov E, Pacheva S and Solomon S 1989 Off-shell superspace $D = 10$ super Yang–Mills from covariantly quantized Green–Schwarz superstring Nucl. Phys. B 317 344
[31] Kallosh R and Rakhmanov M 1988 Covariant quantization of the Green–Schwarz superstring Phys. Lett. B 209 233
Kallosh R and Rakhmanov M 1988 Consistency of covariant quantization of Gs string Phys. Lett. B 214 549
[32] Ferber A 1978 Supertwistors and conformal supersymmetry Nucl. Phys. B 132 55–64
Shirafuji T 1983 Lagrangian mechanics of massless particles with spin Prog. Theor. Phys. 70 18–35
[33] Penrose R 1977 The twistor program Rep. Math. Phys. 12 65
Penrose R and MacCallum M A H 1972 Twistor theory: an approach to the quantization of fields and space-time Phys. Rep. 6 241 and reference therein
[34] Gomis J, Kamimura K and West P 2006 The construction of brane and superbrane actions using non-linear realisations Class. Quantum Grav. 23 7369–82 (arXiv:hep-th/0607057)
[35] Mourshev M V 2012 The odd twistor transform in eleven-dimensional supergravity arXiv:1206.0057 [hep-th]
[36] Cederwall M 2010 Towards a manifestly supersymmetric action for 11-dimensional supergravity J. High Energy Phys. JHEP01(2010)117 (arXiv:0912.1814 [hep-th])
Cederwall M 2010 $D = 11$ supergravity with manifest supersymmetry Mod. Phys. Lett. A 25 3201 (arXiv:1001.0112 [hep-th])
[37] Galperin A S, Howe P S and Townsend P K 1993 Twistor transform for superfields Nucl. Phys. B 402 531
[38] Mafra C R, Schlotterer O and Stieberger S 2011 Complete N-point superstring disk amplitude: I. Pure spinor computation arXiv:1106.2645 [hep-th]
Mafra C R, Schlotterer O and Stieberger S 2011 Complete N-point superstring disk amplitude: II. Amplitude and hypergeometric function structure arXiv:1106.2646 [hep-th]