A NEW PROOF OF VÁZSONYI’S CONJECTURE

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ABSTRACT. We present a self-contained proof that the number of diameter pairs among \( n \) points in Euclidean 3-space is at most \( 2n - 2 \). The proof avoids the ball polytopes used in the original proofs by Grünbaum, Heppes and Straszewicz. As a corollary we obtain that any three-dimensional diameter graph can be embedded in the projective plane.

Let \( S \) be a set of \( n \) points of diameter \( D \) in \( \mathbb{R}^d \). Define the diameter graph on \( S \) by joining all diameters, i.e., point pairs at distance \( D \). The following theorem was conjectured by Vázsnyi, as reported in [2]. It was subsequently independently proved by Grünbaum [3], Heppes [4] and Straszewicz [7].

**Theorem 1.** The number of edges in a diameter graph on \( n \geq 4 \) points in \( \mathbb{R}^3 \) is at most \( 2n - 2 \).

All three proofs (see [6, Theorem 13.14]) use the ball polytope obtained by taking the intersection of the balls of radius \( D \) centred at the points. However, these ball polytopes do not behave the same as ordinary polytopes. In particular, their graphs need not be 3-connected, as shown by Kupitz, Martini and Perles in [5], where a detailed study of the ball polytopes associated to the above theorem is made. The proof presented here avoids the use of ball polytopes.

**Theorem 2.** Any diameter graph in \( \mathbb{R}^3 \) has a bipartite double covering that has a centrally symmetric drawing on the 2-sphere.

In fact, each point \( x \in S \) will correspond to an antipodal pair of points \( x_r \) and \( x_b \) on the sphere, with \( x_r \) coloured red and \( x_b \) blue. Each edge \( xy \) of the diameter graph will correspond to two antipodal edges \( x_ry_b \) and \( x_by_r \) on the sphere, giving a properly 2-coloured graph on \( 2n \) vertices. The drawing will be made such that no edges cross. By Euler’s formula there will be at most \( 4n - 4 \) edges, hence at most \( 2n - 2 \) edges in the diameter graph. By identifying opposite points of the sphere we further obtain:

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Corollary 3. Any diameter graph in \( \mathbb{R}^3 \) can be embedded in the projective plane such that all odd cycles are noncontractible.

Therefore, any two odd cycles intersect, and we regain the following theorem of Dol’nikov [1]:

Corollary 4. Any two odd cycles in a diameter graph on a finite set in \( \mathbb{R}^3 \) intersect.

Proof of Theorem 2. Without loss we assume from now on that \( D = 1 \). Let \( S^2 \) denote the sphere in \( \mathbb{R}^3 \) with centre the origin and radius 1. We may repeatedly remove all vertices of degree at most 1 in the diameter graph. Since such vertices can easily be added later, this is no loss of generality. For each \( x \in S \), let \( R(x) \) be the intersection of \( S^2 \) with the cone generated by \( \{ y - x : xy \text{ is a diameter} \} \). Each \( R(x) \) is a convex spherical polygon with great circular arcs as edges. (If \( x \) has degree 2 then \( R(x) \) is an arc). Colour \( R(x) \) red and \( B(x) := -R(x) \) blue. Assume for the moment the following two properties of these polygons:

Lemma 1. If \( x \neq y \), then \( R(x) \) and \( R(y) \) are disjoint.

Lemma 2. If \( R(x) \) and \( B(y) \) intersect, then \( xy \) is a diameter and \( R(x) \cap B(y) = \{y - x\} \).

For each \( x \in S \) we choose any \( x_r \) in the interior of \( R(x) \) and let \( x_b = -x_r \). (If \( R(x) \) is an arc we let \( x_r \) be in its relative interior.) Draw arcs inside \( R(x) \) from \( x_r \) to all the vertices of \( R(x) \), as well as antipodal arcs from \( x_b \) to the vertices of \( B(x) \). This gives a centrally symmetric drawing of a 2-coloured double covering of the diameter graph. By Lemmas 1 and 2 no edges cross, and the theorem follows. \( \square \)

The following proofs of Lemmas 1 and 2 are dimension independent, which gives a double covering on \( S^{d-1} \) of any diameter graph in \( \mathbb{R}^d \).

Lemma 3. Let \( x_1, \ldots, x_k \) and \( \sum_{i=1}^{k} \lambda_i x_i \) be unit vectors in \( \mathbb{R}^d \), with all \( \lambda_i \geq 0 \). Suppose that for some \( y \in \mathbb{R}^d \), \( \|y - x_i\| \leq 1 \) for all \( i = 1, \ldots, k \). Then \( \|y - \sum_{i=1}^{k} \lambda_i x_i\| \leq 1 \).

Proof. By the triangle inequality,

\[
1 \leq \| \sum_{i=1}^{k} \lambda_i x_i \| \leq \sum_{i=1}^{k} \lambda_i. \tag{1}
\]

Expanding \( \|y - x_i\|^2 \leq 1 \) by inner products,

\[
-2 \langle x_i, y \rangle \leq -\|y\|^2. \tag{2}
\]
Therefore,
\[ \|y - \sum_{i=1}^{k} \lambda_i x_i\|^2 = \|y\|^2 - 2 \sum_{i=1}^{k} \langle x_i, y \rangle + 1 \]
\[ \leq \left(1 - \sum_{i=1}^{k} \lambda_i\right) \|y\|^2 + 1 \quad \text{by (2)} \]
\[ \leq 1 \quad \text{by (1)}. \]

\[ \square \]

Proof of Lemma 1. Let the neighbours of \( x \) be \( x + x_i \), and the neighbours of \( y \) be \( y + y_j \), with the \( x_i \) and \( y_j \) unit vectors. Suppose that
\[ \sum_{i} \lambda_i x_i = \sum_{j} \mu_j y_j \in R(x) \cap R(y) \text{ with } \lambda_i, \mu_j \geq 0. \]

Since \( \|x + x_i - y\| \leq 1 \) for all \( i \), Lemma 3 gives
\[ \|x + \sum_{i} \lambda_i x_i - y\| \leq 1. \]

Similarly, Lemma 3 applied to \( \|x - y - y_j\| \leq 1 \) gives
\[ \|x - y - \sum_{j} \mu_j y_j\| \leq 1. \]

By the triangle inequality,
\[ 2 = \|2 \sum_{i} \lambda_i x_i\| \]
\[ = \|(x + \sum_{i} \lambda_i x_i - y) - (x - y - \sum_{j} \mu_j y_j)\| \]
\[ \leq \|x + \sum_{i} \lambda_i x_i - y\| + \|x - y - \sum_{j} \mu_j y_j\| \]
\[ \leq 2. \]

Since we have equality throughout, \( x + \sum_{i} \lambda_i x_i - y \) and \( -x + y + \sum_{j} \mu_j y_j \) are unit vectors in the same direction, hence are equal, which gives \( x = y. \)

\[ \square \]

Proof of Lemma 2. Since \( \|x_i - x_j\| \leq 1 \) for all \( i, j \), \( R(x) \) is properly contained in an open hemisphere of \( S^2 \), hence \( R(x) \cap B(x) = \emptyset \). Thus without loss of generality, \( x \neq y \). As before, let the neighbours of \( x \) be \( x + x_i \), and the neighbours of \( y \) be \( y + y_j \), with the \( x_i \) and \( y_j \) unit vectors. Suppose that \( \sum_{i} \lambda_i x_i = -\sum_{j} \mu_j y_j \in R(x) \cap B(y) \) with
\[\lambda_i, \mu_j \geq 0.\] For a fixed \(j\) we have that \(\|x + x_i - y - y_j\| \leq 1\) for all \(i\). Lemma 3 then gives
\[\|x + \sum_i \lambda_i x_i - y - y_j\| \leq 1\] for all \(j\).

Again by Lemma 3,
\[\|x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j\| \leq 1.\]

By the triangle inequality,
\[
2 = \|2 \sum_i \lambda_i x_i\|
= \|(x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j) + (y - x)\|
\leq \|x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j\| + \|y - x\|
\leq 2.
\]

Since we have equality throughout, \(x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j\) and \(y - x\) are unit vectors in the same direction, hence are equal, which gives \(x + \sum_i \lambda_i x_i = y\) and \(R(x) \cap B(y) = \{y - x\}\).

\[\square\]

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