Concentration of Measure for the Graphon Particle System

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May 11, 2022

Abstract

We study heterogeneously interacting diffusive particle systems with mean-field type interaction characterized by an underlying graphon and their finite particle approximations. Under suitable conditions, we obtain exponential concentration estimates over a finite time horizon for both 1 and 2 Wasserstein distances between the empirical measures of the finite particle systems and the averaged law of the graphon system, extending the work of Bayraktar-Wu [3].

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∗E. Bayraktar is supported in part by the National Science Foundation under grant DMS-2106556 and by the Susan M. Smith Professorship.

MSC 2020 subject classifications: 05C80, 60J60, 60K35.

Keywords and phrases: graphon, graphon particle system, heterogeneous interaction, mean-field interaction, networks, concentration bounds, transport inequalities, $L^1$-Fourier class
1 Introduction

In this article, we study concentration of measures related to the graphon particle system and its finite particle approximations. This work is a continuation of earlier papers\cite{2, 3, 4}. A graphon particle system consists of uncountably many heterogeneous particles $X_u$ for $u \in [0,1]$ with which their interactions are characterized by a graphon. More precisely, for a fixed $T > 0$ and $d \in \mathbb{N}$, we consider the following system

$$X_u(t) = X_u(0) + \int_0^t \left[ \int_0^1 \int_{\mathbb{R}^d} \phi(X_u(s), y) G(u, v) \mu_{u,s}(dy) dv + \psi(X_u(s)) \right] ds + \sigma B_u(t),$$

where $\{B_u\}_{u \in [0,1]}$ is a family of i.i.d. $d$-dimensional Brownian motions, $\{X_u(0)\}_{u \in [0,1]}$ is a collection of independent (but not necessarily identically distributed) $\mathbb{R}^d$-valued random variables with law $\mu_u(0)$, independent of $\{B_u\}_{u \in [0,1]}$ for each $u \in [0,1]$, defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Two functions $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ represent pairwise interactions between the particles and single particle drift, respectively. $\sigma \in \mathbb{R}^{d \times d}$ is a constant and $G : [0,1] \times [0,1] \to [0,1]$ is a graphon, that is, a symmetric measurable function.

Along with the graphon particle system, we introduce two finite particle systems with heterogeneous interactions, which approximate $\text{(1.1)}$. For a fixed, arbitrary $n \in \mathbb{N}$ and each $i \in [n] := \{1, \cdots , n\}$, we first consider the not-so-dense analogue of $\text{(1.1)}$ introduced in Section 4 of\cite{2}:

$$X^n_i(t) = X^n_i(0) + \int_0^t \left[ \frac{1}{np(n)} \sum_{j=1}^n \xi^n_{ij}\phi(X^n_i(s), X^n_j(s)) + \psi(X^n_i(s)) \right] ds + \sigma B^n_i(t),$$

where $\{p(n)\}_{n \in \mathbb{N}} \subset (0,1]$ is a sequence of numbers and $\{\xi^n_{ij}\}_{1 \leq i, j \leq n}$ are independent Bernoulli random variables satisfying

$$\xi^n_{ij} = \xi^n_{ji}, \quad \mathbb{P}(\xi^n_{ij} = 1) = p(n)G\left(\frac{i}{n}, \frac{j}{n}\right), \quad \text{for every } i, j \in [n],$$

independent of $\{B_{i/n}, X_{i/n}(0) : i \in [n]\}$. Here, $p(n)$ represents the global sparsity parameter and the strength of interaction between the particles in $\text{(1.2)}$ is scaled by $np(n)$, the order of the number of neighbors, as in mean-field systems on Erdős-Rényi random graphs\cite{7, 13, 27}; the convergence of $p(n) \to 0$ as $n \to \infty$ implies that the graph is sparse, but we shall consider the not-so-dense case $np(n) \to \infty$ of diverging average degree in the random graph.

The other finite particle approximation system is given by

$$\bar{X}^n_i(t) = \bar{X}^n_i(0) + \int_0^t \left[ \frac{1}{n} \sum_{j=1}^n \phi(\bar{X}^n_i(s), \bar{X}^n_j(s)) G\left(\frac{i}{n}, \frac{j}{n}\right) + \psi(\bar{X}^n_i(s)) \right] ds + \sigma B^n_i(t).$$

Since this system has a nonrandom coefficient for the interaction term (but still models heterogeneous interaction via the graphon), it is easier to analyze than the other finite particle system $\text{(1.2)}$. We note that three systems $\text{(1.1)} - \text{(1.3)}$ are coupled in a sense that they share initial particle locations $X_{i/n}(0)$ and Brownian motions $B_{i/n}$ for $i \in [n]$.

Law of large numbers (LLN)-type of convergence results for the systems $\text{(1.2)}$ and $\text{(1.3)}$ to the graphon particle system $\text{(1.1)}$ under suitable conditions are studied in\cite{2}. The exponential
ergodicity of the two systems (1.1) and (1.2), as well as the uniform-in-time convergence of (1.2) to (1.1) under a certain dissipativity condition, are presented in [4]. There are more studies of the graphon particle systems [3, 14], and works of associated heterogeneously interacting finite particle models [6, 13, 17, 25, 26, 27]. These studies are recently arisen since graphons have been widely applied in mean-field game theory for both static and dynamic cases, see e.g. [1, 2, 10, 11, 12, 20, 21, 28, 30, 31, 33] and references therein. Among these studies, our work is particularly linked to [3]. Denoting $W_1$ the 1-Wasserstein distance and defining the empirical measures of the three particle systems at time $t \in [0, T]$

$$L_{n,t} := \frac{1}{n} \sum_{i=1}^{n} \delta(X^i_n(t)), \quad \bar{L}_{n,t} := \frac{1}{n} \sum_{i=1}^{n} \delta(\bar{X}^i_n(t)), \quad \bar{L}_{n,t} := \frac{1}{n} \sum_{i=1}^{n} \delta(\bar{X}_{i/n}(t)),$$

along with the averaged law $\bar{\mu}_t := \int_0^1 \mu_{u,t} \, du$ of the graphon system (1.1), concentration bounds of the types $\mathbb{P}[\sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) > \epsilon], \sup_{t \geq 0} \mathbb{P}[W_1(\bar{L}_{n,t}, \bar{\mu}_t) > \epsilon]$ for $\epsilon > 0$ are computed in [2] under certain conditions. In particular, uniform-in-time concentration bound of the latter type is studied in an infinite time horizon setting under an extra dissipativity condition on $\psi$. These results are established by computing certain sub-Gaussian estimates rather directly with the moment generating function of the standard normal random vector (Lemmas 3.7 - 3.10 of [3]).

In contrast, our work focuses on the case of finite time horizon and deals with more general sparsity sequence $\{p(n)\}_{n \in \mathbb{N}} \subset (0, 1]$ for (1.2), whereas the results of [3] only cover the dense graphs, i.e., $p(n) \equiv 1$. Our argument adopts the method of [16] as follows. When $X^n = (X^n_1, \cdots, X^n_n)$ represents the state of the so-called Nash equilibrium of a symmetric $n$-player stochastic differential game and $\bar{\mu}$ is the measure flow of the unique equilibrium of the corresponding mean-field game, the authors of [16] compute concentration bounds of the probabilities

$$\mathbb{P}[\sup_{0 \leq t \leq T} W_p(\bar{L}_{n,t}, \bar{\mu}_t) > \epsilon], \quad p \in \{1, 2\},$$

with the notation of (1.4). Their argument uses transportation inequalities in [18] to show that Lipschitz functions of $\bar{X}^n$ concentrate around their means, and obtains the aforementioned bounds from this concentration property. We apply a similar approach to the finite particle system (1.3) to obtain the bound of the probability that Lipschitz function values of the particles on the space $(\mathcal{C}([0, T] : \mathbb{R}^d))_n$, deviate from their means in Theorems 3.1 and 3.2. Combining this bound with the facts in Section 3.1 that the expectations of the $W_2$-distances between the empirical measures in (1.4) converge to zero as the number of particles goes to infinity, we show the concentration result of (1.5). In particular, obtaining the same exponential bound in $n$ for the probability (1.5) in terms of $W_2$-metric in Theorem 3.5 as in the case of $p = 1$ in Theorem 3.3 is the new result.

Moreover, inspired by the proof in [27], we compare the particles $X^i_n$ and $\bar{X}^i_n$ to improve the exponential bounds in (1.5) for $\mathbb{P}[\sup_{0 \leq t \leq T} W_p(\bar{L}_{n,t}, \bar{\mu}_t) > \epsilon]$ when $p = 1$, at the expense of an assumption on the interaction function $\phi$, namely being a member of the $L^1$-Fourier class. When $p = 2$, we also present the similar exponential bound for the system (1.2) on the dense graphs ($p(n) \equiv 1$). Without such condition on $\phi$, we have the same bound, but in terms of the bounded Lipschitz metric ($d_{BL}$-metric), a weaker metric than $W_1$.

This paper is organized as follows. In Section 2 we introduce the notation, state the assumptions, and recall some of the relevant existing results concerning the particle systems (1.1)
We use $K$ and the other preliminary results. Section 3 provides our main results and Section 4 gives proofs of these results.

2 Preliminaries

In this section, we first introduce the notation which will be used throughout this paper. We then state several assumptions with some of the basic results on the particle systems (1.1) - (1.3) from 2, 3, and provide several well-known results regarding transportation cost inequalities without proof. Finally, Bernstein’s inequality with the concept of $L^1$-Fourier class will be introduced.

2.1 Notation

Given a metric space $(S,d)$ and a function $f : S \to \mathbb{R}$, we define

$$\|f\|_\infty := \sup\{|f(x)| : x \in S\},$$

$$\|f\|_{\text{Lip}} := \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x,y \in S, x \neq y \right\},$$

$$\|f\|_{BL} := 2(\|f\|_\infty + \|f\|_{\text{Lip}}),$$

and we say that $f$ is Lipschitz (bounded Lipschitz) if $\|f\|_{\text{Lip}} < \infty (\|f\|_{BL} < \infty)$, respectively. In particular, $f$ is called $a$-Lipschitz if $\|f\|_{\text{Lip}} = a$, and note that $\|f\|_{BL} \leq 1$ implies $|f(x) - f(y)| \leq |x - y| \wedge 1$.

Denote by $\mathcal{P}(S)$ the space of Borel probability measures on $S$ and we shall use the standard notation $\langle \mu, \varphi \rangle := \int_S \varphi \, d\mu$ for integrable functions $\varphi$ and measures $\mu$ on $S$. When $(S, \| \cdot \|)$ is a normed space, we write $\mathcal{P}^p(S, \| \cdot \|)$ for the set of $\mu \in \mathcal{P}(S)$ satisfying $\langle \mu, | \cdot |^p \rangle < \infty$ for a given $p \in [1, \infty)$. We denote by $\text{Lip}(S, \| \cdot \|)$ the set of 1-Lipschitz functions, i.e., $f : S \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq \|x - y\|$ for every $x, y \in S$.

For a separable Banach space $(S, \| \cdot \|)$, we endow $\mathcal{P}^p(S, \| \cdot \|)$ with the $p$-Wasserstein metric

$$W_{p,(S,\| \cdot \|)}(\mu, \nu) := \inf_\pi \left( \int_{S \times S} \|x - y\|^p \pi(dx, dy) \right)^{1/p}, \quad p \geq 1,$$

where the infimum is taken over all probability measures $\pi$ on $S \times S$ with first and second marginals $\mu$ and $\nu$. We also write the product space $S^n := S \times \cdots \times S$, equipped with the $\ell^p$ norm for any $p \geq 1$

$$\|x\|_{n,p} = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p},$$

for $x = (x_1, \cdots, x_n) \in S^n$. When the space $S$ or the norm $\| \cdot \|$ is understood, we sometimes omit it from the above notations.

Denote by $C([0, T] : S)$ the space of continuous functions from $[0, T]$ to $S$, and $\|x\|_{\ast, t} := \sup_{0 \leq s \leq t} |x_s|$, where $| \cdot |$ is the usual Euclidean norm on $\mathbb{R}^d$ for $x \in C([0, T] : \mathbb{R}^d)$ and $t \in [0, T]$. We write $\mathcal{L}(X)$ the probability law of a random variable $X$ and $[n] := \{1, \cdots, n\}$ for any $n \in \mathbb{N}$. We use $K$ to denote various positive constants throughout the paper and its value may change from line to line.
For a Polish space \((S,d)\) with Borel \(\sigma\)-field \(S\), we also consider the space of probability measures over \((S,\mathcal{S})\) endowed with the topology of weak convergence, which is metrized by the BL-metric, defined for \(\mu,\nu \in \mathcal{P}(S)\) as

\[
d_{BL}(\mu,\nu) := \sup \left\{ \left| \int_S f \, d(\mu - \nu) \right| : f : S \to \mathbb{R} \text{ with } \|f\|_{BL} \leq 1 \right\}.
\]

(2.1)

Note the dual representation of the 1-Wasserstein metric

\[
W_1(\mu,\nu) := \sup \left\{ \left| \int_S f \, d(\mu - \nu) \right| : f : S \to \mathbb{R} \text{ with } \|f\|_{Lip} \leq 1 \right\},
\]

along with the relationship \(d_{BL} \leq W_1\). We shall also use the notation for given \(\mu,\nu \in \mathcal{P}(C([0,T] : \mathbb{R}^d))\)

\[
W_{p,t}(\mu,\nu) := \inf_{\pi} \left( \int \|x - y\|^p_{p,t} \pi(dx, dy) \right)^{1/p}, \quad t \in [0,T], \quad p \geq 1,
\]

where the infimum is taken over all probability measures \(\pi\) with marginals \(\mu\) and \(\nu\).

Let us define three \(n \times n\) random matrices \(P^{(n)}, \bar{P}^{(n)}, \text{ and } D^{(n)}\), concerning the systems (1.2), (1.3) for every \(n \in \mathbb{N}\) with entries

\[
P_{i,j}^{(n)} := \frac{\xi_{ij}^n}{np(n)}, \quad i,j \in [n],
\]

\[
\bar{P}_{i,j}^{(n)} := \frac{1}{n} G\left(\frac{i}{n}, \frac{j}{n}\right), \quad i,j \in [n],
\]

\[
D^{(n)} := P^{(n)} - \bar{P}^{(n)}.
\]

(2.3)

For these matrices, we define the \(\ell_\infty \to \ell_1\) norm of an \(n \times n\) matrix \(A\)

\[
\|A\|_{\infty \to 1} := \sup \left\{ \langle x, Ay \rangle : x, y \in [-1,1]^n \right\}.
\]

(2.4)

This norm is known to be equivalent to the so-called cut norm (see (3.3) of [23]).

We denote the empirical measures of the approximation systems for each \(n \in \mathbb{N}\)

\[
 L_n := \frac{1}{n} \sum_{i=1}^n \delta_{(X^n_i)}, \quad \bar{L}_n := \frac{1}{n} \sum_{i=1}^n \delta_{(\bar{X}^n_i)},
\]

(2.5)

all of which are random elements of \(\mathcal{P}(C([0,T] : \mathbb{R}^d))\).

We conclude this subsection by recalling the relative entropy of two probability measures \(\mu,\nu\) over the same measurable space

\[
H(\mu|\nu) := \left\{ \begin{array}{ll}
\int \log \left( \frac{d\mu}{d\nu} \right) d\mu, & \text{if } \mu \ll \nu; \\
\infty, & \text{otherwise}.
\end{array} \right.
\]

(2.6)
2.2 Existence and uniqueness of the solutions

We state the existence and uniqueness of strong solutions to the systems (1.1) - (1.3).

Assumption 2.1.

(a) \( \phi \) is bounded; \( \phi \) and \( \psi \) are Lipschitz, i.e., there exists a constant \( K > 0 \) such that

\[
|\phi(x_1,y_1) - \phi(x_2,y_2)| + |\psi(x_1) - \psi(x_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|)
\]

holds. Moreover, the initial particles have finite second moments, i.e.,

\[
\sup_{u \in [0,1]} \mathbb{E}|X_u(0)|^2 < \infty.
\]  

(2.7)

(b) The map \([0,1] \ni u \mapsto \mu_u(0) = \mathcal{L}(X_u(0)) \in \mathcal{P}(\mathbb{R}^d)\) is measurable.

Lemma 2.1 (The existence and uniqueness of the particle systems).

(a) Under Assumption 2.1 (a), two systems (1.2), (1.3) have unique strong solutions.

(b) Under Assumption 2.1 (a) and (b), the graphon system (1.1) has a unique strong solution, and the map \([0,1] \ni u \mapsto \mu_u \in \mathcal{P}(C([0,T] : \mathbb{R}^d))\) is measurable.

Proof of Lemma 2.1 (a) is classical (see e.g. Theorem 5.2.9 of [24]). Part (b) follows from Proposition 2.1 of [2]. As pointed out in Remark 2.2 of [2], we note that the boundedness condition on \( \phi \) in Assumption 2.1 (a) can be removed throughout this paper at the expense of a stronger condition \( \sup_{u \in [0,1]} \mathbb{E}|X_u(0)|^{2+\epsilon} < \infty \) for some \( \epsilon > 0 \) than (2.7). We occasionally need an even stronger condition on the initial particles as in the following.

Assumption 2.2. The initial particles \( \{X_u(0)\}_{u \in [0,1]} \) are independent with law \( \mu_{u,0} \in \mathcal{P}(\mathbb{R}^d) \), which satisfies

\[
\sup_{u \in [0,1]} \int_{\mathbb{R}^d} e^{\kappa |x|^2} \mu_{u,0}(dx) < \infty, \quad \text{for some } \kappa > 0.
\]  

(2.8)

We observe later that the condition (2.8) is equivalent to (3.1) from Lemma 2.6. Under this stronger assumption, we have the finite fourth moment of the solution to (1.1). The proof is standard, hence is omitted (see, e.g. [20], or Proposition 2.1 of [3]).

Lemma 2.2. Under Assumptions 2.1, 2.2, the solution to (1.1) satisfies

\[
\sup_{u \in [0,1]} \sup_{t \in [0,T]} \mathbb{E}[|X_u(t)|^4] < \infty.
\]
2.3 Continuity of the graphon system

The following result, which states the continuity of the graphon system (1.1), is from Theorem 2.1 of [2].

**Assumption 2.3.** There exists a finite collection of subintervals \(\{I_i : i \in [N]\}\) for some \(N \in \mathbb{N}\), satisfying \(\bigcup_{i=1}^{N} I_i = [0, 1]\). For each \(i, j \in [N]\):

(a) The map \(I_i \ni u \mapsto \mu_u(0) \in \mathcal{P}(\mathbb{R}^d)\) is continuous with respect to the \(W_2\)-metric.

(b) For each \(u \in I_i\), there exists a Lebesgue-null set \(N_u \subset [0, 1]\) such that \(G(u, v)\) is continuous at \((u, v) \in [0, 1] \times [0, 1]\) for each \(v \in [0, 1] \setminus N_u\).

(c) There exists \(K > 0\) such that
\[
W_2(\mu_{u_1}(0), \mu_{u_2}(0)) \leq K|u_1 - u_2|, \quad u_1, u_2 \in [0, 1],
\]
\[
|G(u_1, v_1) - G(u_2, v_2)| \leq K\left(|u_1 - u_2| + |v_1 - v_2|\right), \quad (u_1, v_1), (u_2, v_2) \in I_i \times I_j.
\]

**Lemma 2.3.** Suppose that Assumption 2.3 holds.

(a) (Continuity) Under Assumption 2.3 (a) and (b), the map \(I_i \ni u \mapsto \mu_u \in \mathcal{P}(C([0, T] : \mathbb{R}^d))\) is continuous with respect to the \(W_{2,T}\) metric for every \(i \in [N]\).

(b) (Lipschitz continuity) Under Assumption 2.3 (c), there exists \(\kappa > 0\), which depends on \(T\), such that \(W_{2,T}(\mu_u, \mu_v) \leq \kappa|u - v|\) whenever \(u, v \in I_i\) for some \(i \in [N]\).

In Lemma 2.3(b), note that we have, in particular,
\[
\sup_{\|f\|_{L^p} \leq 1} \left| \int_{\mathbb{R}^d} f(x)\mu_{u,t}(dx) - \int_{\mathbb{R}^d} f(x)\mu_{v,t}(dx) \right| \leq W_{2,T}(\mu_u, \mu_v) \leq \kappa|u - v|, \quad \forall t \in [0, T].
\]

2.4 A law of large number of the mean-field particle system

Besides the assumptions introduced in this section, we will need the following assumption on the sparsity parameter for the system (1.2), as briefly mentioned in Section II.

**Assumption 2.4.** The sequence \(\{p(n)\}_{n \in \mathbb{N}}\) in (1.2) satisfies \(np(n) \to \infty\) as \(n \to \infty\).

We introduce the following law of large number result for the mean-field particle system (1.2), which is Theorem 4.1 of [2]. We write \(\mu_u\) the law of \(X_u\) in the graphon particle system (1.1) for each \(u \in [0, 1]\), and define
\[
\bar{\mu} := \int_0^1 \mu_u du.
\]

**Lemma 2.4.** Under Assumptions 2.1, 2.3 and 2.4,
\[
L_n \to \bar{\mu} \quad \text{in} \quad \mathcal{P}(C([0, T] : \mathbb{R}^d)) \quad \text{in probability, as} \quad n \to \infty.
\]
Moreover, we have
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\|X^n_i - X_{\frac{i}{n},T}\|_2^2 \to 0, \quad \text{as} \quad n \to \infty.
\]
2.5 Transportation inequalities

In this subsection, we present some preliminary results regarding transportation inequalities. The first result is from Theorem 9.1 of [32], illustrating the transportation inequality with the uniform norm for the laws of diffusion processes.

Lemma 2.5. For a fixed \( T > 0 \) and \( k \in \mathbb{N} \), suppose that \( X^x = \{ X^x_t \}_{t \in [0, T]} \) is the unique strong solution of the SDE

\[
dX^x_t = b(t, X^x_t)dt + \Sigma \, dW_t, \quad \forall t \in [0, T], \quad X_0 = x \in \mathbb{R}^k,
\]

on a probability space \( C([0, T] : \mathbb{R}^k) \) supporting a \( k \)-dimensional Brownian motion \( W \).

Here, \( b: [0, T] \times C([0, T] : \mathbb{R}^k) \rightarrow \mathbb{R}^k \) satisfies for any \( \xi, \eta \in C([0, T] : \mathbb{R}^k) \)

\[
|b(t, \xi) - b(t, \eta)| \leq L \sup_{0 \leq s \leq t} |\xi(s) - \eta(s)| = L\|\xi - \eta\|_{s, t}, \quad \forall t \in [0, T],
\]

for some constants \( L > 0 \) and \( \Sigma \in \mathbb{R}^{k \times k} \). Let \( P^x \in \mathcal{P}(C([0, T] : \mathbb{R}^k)) \) be the law of \( X^x \) for any \( x \in \mathbb{R}^k \). Then for any \( Q \in \mathcal{P}(C([0, T] : \mathbb{R}^k)) \), we have

\[
W_{1, C([0, T] : \mathbb{R}^k), \|\cdot\|_{k, 2}}(P^x, Q) \leq W_{2, C([0, T] : \mathbb{R}^k), \|\cdot\|_{k, 2}}(P^x, Q) \leq 6e^{15L^2} H(Q|P^x),
\]

where \( H(Q|P) \) is the relative entropy of \( Q \) with respect to \( P \), defined in (2.6).

The following result (Theorem 5.1 of [16]) characterizes concentration of a probability measure with a transportation cost inequality and Gaussian integrability property. The equivalence between (2.12) and (2.13) is originally from Theorem 3.1 of [8], and the equivalence between (2.12) and (2.15) is due to Theorem 2.3 of [18].

Lemma 2.6. For a probability measure \( \mu \in \mathcal{P}^1(S) \) on a separable Banach space \( (S, \| \cdot \|) \), the following statements are equivalent up to a universal change in the positive constant \( c \).

\( (i) \) The transportation cost inequality

\[
W_{1, S}(\mu, \nu) \leq \sqrt{2eH(\nu|\mu)}
\]

holds for every \( \nu \in \mathcal{P}(S) \).
\( (ii) \) For every 1-Lipschitz function \( f \) on \( S \) and \( \lambda \in \mathbb{R} \)

\[
\int_S e^{\lambda(f - \langle \mu, f \rangle)} \, d\mu \leq \exp \left( \frac{e\lambda^2}{2} \right)
\]

holds.
\( (iii) \) For every 1-Lipschitz function \( f \) on \( S \) and \( a > 0 \)

\[
\mu(f - \langle \mu, f \rangle > a) \leq \exp \left( -\frac{a^2}{2c} \right).
\]

\( (iv) \) \( \mu \) is sub-Gaussian, i.e.,

\[
\int_S e^{c\|x\|^2} \mu(dx) < \infty.
\]
The next result is well-known tensorization of transportation cost inequalities from Corollary 5 of [22]. The major difference between (i) and (ii) is that the inequality (2.16) is dimension-free, i.e., the right-hand side does not depend on \( n \).

**Lemma 2.7.** For each \( n \in \mathbb{N} \), consider a set of probability measures \( \{\mu_i\}_{i \in [n]} \subset \mathcal{P}(S) \) on a separable Banach space \( (S, \| \cdot \|) \).

(i) If the inequality \( W_{1,S}(\mu_i, \nu) \leq \sqrt{2cH(\nu|\mu_i)} \) holds for every \( i \in [n] \) and \( \nu \in \mathcal{P}^1(S) \), then

\[
W_{1,(S^n, \| \cdot \|_{n,1})}(\mu_1 \otimes \cdots \otimes \mu_n, \rho) \leq \sqrt{2ncH(\rho|\mu_1 \otimes \cdots \otimes \mu_n)},
\]

holds for every \( \rho \in \mathcal{P}^1(S^n) \).

(ii) If the inequality \( W_{2,S}(\mu_i, \nu) \leq \sqrt{2cH(\nu|\mu_i)} \) holds for every \( i \in [n] \) and \( \nu \in \mathcal{P}^2(S) \), then

\[
W_{2,(S^n, \| \cdot \|_{n,2})}(\mu_1 \otimes \cdots \otimes \mu_n, \rho) \leq \sqrt{2cH(\rho|\mu_1 \otimes \cdots \otimes \mu_n)}, \tag{2.16}
\]

holds for every \( \rho \in \mathcal{P}^2(S^n) \).

We finally mention the following result on the Wasserstein distance of the empirical measures of independent but not necessarily identically distributed random variables. This is Lemma A.1 of [3], a generalization of Theorem 1 of [19] where i.i.d. random variables are considered. This result will be used in proving Proposition 3.2.

**Lemma 2.8.** Let \( \{Y_i\}_{i \in \mathbb{N}} \) be independent \( \mathbb{R}^d \)-valued random variables and define

\[
\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \quad \bar{\nu}_n := \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(Y_i).
\]

For a fixed \( p > 0 \), assume that \( \sup_{i \in \mathbb{N}} \mathbb{E}|Y_i|^q < \infty \) holds for some \( q > p \). Then there exists a constant \( K > 0 \) depending only on \( p, q, \) and \( d \) such that for every \( n \geq 1 \)

\[
\mathbb{E}[W_p^n(\nu_n, \bar{\nu}_n)] \leq K \alpha_{p,q}(n) \left( \int_{\mathbb{R}^d} |x|^q \bar{\nu}_n(dx) \right)^{p/q},
\]

where

\[
\alpha_{p,q}(n) := \begin{cases} 
  n^{-1/2} + n^{-(q-p)/q}, & \text{if } p > d/2 \text{ and } q \neq 2p, \\
  n^{-1/2} \log(1 + n) + n^{-(q-p)/q}, & \text{if } p = d/2 \text{ and } q \neq 2p, \\
  n^{-p/d} + n^{-(q-p)/q}, & \text{if } p < d/2 \text{ and } q \neq d/(d-p).
\end{cases}
\]

### 2.6 Bernstein’s inequality and \( L^1 \)-Fourier class

When comparing two approximation systems \[1.2\] and \[1.3\], controlling the matrix \( D^{(n)} \) of (2.3) is essential. Thus, we introduce the following concentration of \( D^{(n)} \) in terms of \( \| \cdot \|_{\infty \to 1} \) norm, which is from Lemma 2 of [27]. Its proof is straightforward application of Bernstein’s inequality (Lemma 2.10 or Bennett’s inequality) with the distribution of independent \( n^2 \) entries of the matrix \( D^{(n)} \). We will use Bernstein’s inequality again in Section 3.3 to prove Lemma 3.1, an elaboration of Lemma 2.9.
Lemma 2.9. For any $0 < \eta \leq n$, we have

$$
P \left[ \frac{||D^{(n)}||_{\infty \to 1}}{n} > \eta \right] \leq \exp \left( -\frac{\eta^2 n^2 p(n)}{2 + \frac{2}{3}} \right).
$$

In particular, under Assumption 2.4, we have for every $\eta > 1/n \log n > \eta$,

$$
\frac{1}{n} \log P \left[ \frac{||D^{(n)}||_{\infty \to 1}}{n} > \eta \right] \rightarrow -\infty, \quad \text{as } n \to \infty.
$$

Lemma 2.10 (Bernstein’s inequality, Theorem 2.9 of [9]). Let $X_1, \cdots, X_k$ be independent random variables with finite variance such that $X_i \leq b$ for some $b > 0$ almost surely for each $i \in [k]$. Let $v = \sum_{i=1}^k \mathbb{E}[X_i^2]$, then we have

$$
P \left[ \sum_{i=1}^k (X_i - \mathbb{E}(X_i)) \geq u \right] \leq \exp \left( -\frac{u^2}{2(v + \frac{bu^3}{3})} \right).
$$

When the interaction function $\phi$ belongs to a special class of functions, we shall see in the proof of Theorem 3.4 that the distance $W_1(L_n, \bar{L}_n)$ can be easily expressed in terms of the quantity $||D^{(n)}||_{\infty \to 1}$. This observation is inspired by the work of [27]. To state more precisely, we introduce the notion of $L^1$-Fourier class of functions.

Definition 2.1. Identifying $\mathbb{R}^{2d}$ with $\mathbb{R}^d \times \mathbb{R}^d$, we say that a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ belongs to the $L^1$-Fourier class, if there exists a finite complex measure $m_f$ over $\mathbb{R}^{2d}$ such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$
f(x, y) = \int_{\mathbb{R}^{2d}} \exp \left( 2\pi \sqrt{-1} \langle (x, y), z \rangle \right) m_f(dz).
$$

We recall that a finite complex measure $m$ over $\mathbb{R}^{2d}$ is a set function $m : \mathcal{B}(\mathbb{R}^{2d}) \to \mathbb{C}$ of the form $m = m_+^x - m_-^x + \sqrt{-1}(m_+^y - m_-^y)$, where each $m_+^x, m_-^x, m_+^y, m_-^y$ is a finite, $\sigma$-additive (nonnegative) measure over $\mathbb{R}^{2d}$. We define the total mass of $m$

$$
||m||_{TM} := m_+^x(\mathbb{R}^{2d}) + m_-^x(\mathbb{R}^{2d}) + m_+^y(\mathbb{R}^{2d}) + m_-^y(\mathbb{R}^{2d}).
$$

If a function $f$ is an inverse $L^1$-transform of a function in $L^1(\mathbb{R}^{2d})$, then $f$ belongs to the $L^1$-Fourier class. In particular, any Schwartz function belongs to the $L^1$-Fourier class. An example of such function is the Kuramoto interaction; if $d = 1$ and $\phi(x - y) = K \sin(y - x)$ for some constant $K$, then the corresponding complex measure is equal to

$$
m_{\phi} = \frac{K}{2\sqrt{-1}} (\delta_{(-1,1)} + \delta_{(1,-1)}).
$$

The finite system (1.2) of “oscillators” with the Kuramoto interaction function is studied in [13].
3 Main results

This section consists of three parts. The first part shows that expectations of the $W_2$-distances between two empirical measures on $\mathbb{R}^d$ related to the systems (1.1) - (1.3) converge to zero as the number of particles goes to infinity. The second part gives exponential bounds of the probabilities that Lipschitz function values of the particles $\bar{X}^n$ of the system (1.3) on $(C([0,T] : \mathbb{R}^d))^n$ deviate from their means; the stronger norm we use for the space $(C([0,T] : \mathbb{R}^d))^n$, the stronger assumption on the initial distribution of the particles is needed. These results for the first two parts will be used in proving the results in the last subsection. In the last part, we derive several concentration results of the finite particle systems (1.2), (1.3) toward the graphon particle system (1.1) under different metrics.

3.1 Concentration in mean of the $W_2$-distance

Let us recall the law $\mu_{u,t}$ of (1.1), the empirical measures (1.4) of the three systems and the averaged law $\tilde{\mu}_t := \frac{1}{T} \int_0^T \mu_{u,t} du$ for every $t \in [0,T]$. We give two expectations converging to zero as $n \to \infty$ in the following. The proofs are provided in Section 4.1.

**Proposition 3.1.** Under Assumptions 2.1 and 2.4,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \tilde{L}_{n,t}) \right] \to 0,$$

as $n \to \infty$.

**Proposition 3.2.** Under Assumptions 2.1, 2.2, and 2.3(c),

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2(\tilde{L}_{n,t}, \tilde{\mu}_t) \right] \to 0,$$

as $n \to \infty$.

By virtue of Lemma 2.4, we have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2^2(L_{n,t}, \tilde{L}_{n,t}) \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{n} \sum_{i=1}^{n} |X^n_i(t) - X^n_i(t)|^2 \right]$$

$$\leq \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| X^n_i - \bar{X}^n \right\|^2_{\ast,T} \right] \to 0, \quad \text{as } n \to \infty.$$

Combining the last convergence with Propositions 3.1 and 3.2, we immediately have other convergences of the expectations.

**Corollary 3.1.** Under assumptions of Propositions 3.1, 3.2

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \tilde{\mu}_t) \right] \to 0, \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2(\tilde{L}_{n,t}, \tilde{\mu}_t) \right] \to 0,$$

as $n \to \infty$. 

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3.2 Concentration around mean

We present in this subsection the concentration of 1-Lipschitz function of the particles $\bar{X}^n$ around its mean, under two different norms $\ell^1$ and $\ell^2$. Proofs of the results rely on the transportation inequalities presented in Section 2.5 and they will be given in Section 4.2.

From Lemma 2.6, we note that the condition (2.8) of Assumption 2.2 in Theorem 3.1 below is equivalent to the condition

$$W_1(\mu_{u,0}, \nu) \leq \sqrt{2\kappa H(\nu|\mu_{u,0})}, \quad \text{for every } u \in [0,1], \quad \nu \in P^1(S). \tag{3.1}$$

**Theorem 3.1.** Under Assumptions 2.1 and 2.2 there exists a constant $\delta > 0$, independent of $n$, such that for every $F \in Lip\left((C([0,T]:\mathbb{R}^d))^n, \|\cdot\|_{n,1}\right)$ and every $a > 0$

$$P\left[F(\bar{X}^n) - \mathbb{E}(F(\bar{X}^n)) > a\right] \leq 2 \exp\left(-\frac{\delta a^2}{n}\right) \tag{3.2}$$

holds.

We have the following analogous result to Theorem 3.1 when the condition (3.1) is replaced by (3.3). For any $u \in [0,1]$, if the initial law takes the form $\mu_{u,0}(dx) = e^{-U(x)}dx$ for some $U \in C^2(\mathbb{R}^d)$ with Hessian bounded below in semidefinite order by $cI$ for some $c > 0$, then $\mu_{u,0}$ satisfies the condition (3.3) with $\kappa = 1/c$. In particular, if $\mu_{u,0}$ has the standard normal distribution on $\mathbb{R}^d$, then (3.3) holds with $\kappa = 1$. We emphasize that the concentration inequality (3.4) is dimension-free; the right-hand side does not depend on $n$.

**Theorem 3.2.** Suppose that the initial particles $\{X_u(0)\}_{u \in [0,1]}$ are independent with law $\mu_{u,0} \in \mathcal{P}(\mathbb{R}^d)$, satisfying for some $\kappa > 0$

$$W_2(\mu_{u,0}, \nu) \leq \sqrt{2\kappa H(\nu|\mu_{u,0})}, \quad \text{for every } u \in [0,1], \quad \nu \in P^2(S). \tag{3.3}$$

Under Assumption 2.1 there exists a constant $\delta > 0$, independent of $n$, such that for every $F \in Lip\left((C([0,T]:\mathbb{R}^d))^n, \|\cdot\|_{n,2}\right)$ and every $a > 0$

$$P\left[F(\bar{X}^n) - \mathbb{E}(F(\bar{X}^n)) > a\right] \leq 2 \exp(-\delta a^2) \tag{3.4}$$

holds.

3.3 Concentration toward the graphon system

Recalling the notations in (1.4), we now provide concentration in terms of (1 and 2)-Wasserstein distance of the empirical measures of the finite particle systems toward the averaged measure $\tilde{\mu}_t$ of the graphon system. Proofs will be given in Section 4.3.

First, we have the following concentration result of $\bar{X}^n_t$ toward $\tilde{\mu}_t$ in terms of the $W_1$-distance, due to Theorem 3.1.
Theorem 3.3. Under Assumptions 2.1, 2.2, and 2.3(c), there exist constants \( \delta > 0 \), which is independent of \( n \), and \( N \in \mathbb{N} \) such that

\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \tilde{\mu}_t) > a \right] \leq 2 \exp \left( - \frac{\delta a^2 n}{4} \right)
\]

(3.5)

holds for every \( a > 0 \) and every \( n \geq N \).

Remark 3.1. Theorem 3.3 gives the same exponential bound as in Theorem 2.1 of [3]. Their proof mainly focuses on computing certain sub-Gaussian estimates, whereas our argument relies on the concentration property (3.2) of the system (1.3). Applying the same latter argument, we can even deduce the exponential bound in terms of the \( W_2 \)-metric in Theorem 3.5 below.

In Section 2.6, the concept of \( L^1 \)-Fourier class, along with Bernstein’s inequality was introduced to express \( W_1(L_n, \bar{L}_n) \) in terms of \( \| D^{(n)} \|_{\infty \rightarrow 1} \). This gives rise to the following concentration result of the particle system (1.2) toward the graphon system.

Theorem 3.4. Suppose that the components of the interaction function \( \phi \) belong to the \( L^1 \)-Fourier class (Definition 2.1). Under Assumptions 2.1, 2.2, 2.3(c), and 2.4, there exist constants \( \delta > 0 \), which is independent of \( n \), and \( N \in \mathbb{N} \) such that

\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(L_{n,t}, \bar{\mu}_t) > a \right] \leq 3 \exp \left( - \frac{\delta a^2 n}{16} \right)
\]

(3.6)

holds for every \( a > 0 \) and every \( n \geq N \). For general interaction functions \( \phi \), we have instead

\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} d_{BL}(L_{n,t}, \bar{\mu}_t) > a \right] \leq 3 \exp \left( - \frac{\delta a^2 n}{16} \right).
\]

(3.7)

The following result gives the concentration of \( \bar{L}_{n,t} \) toward \( \bar{\mu}_t \) as Theorem 3.3, but in terms of the \( W_2 \)-metric. Its proof is similar to that of Theorem 3.3 but Theorem 3.2 is used in place of Theorem 3.1.

Theorem 3.5. Under Assumptions 2.1, 2.2, and 2.3, together with the condition (3.3), there exist constants \( \delta > 0 \), independent of \( n \), and \( N \in \mathbb{N} \) such that

\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \bar{\mu}_t) > a \right] \leq 2 \exp \left( - \frac{\delta a^2 n}{4} \right)
\]

(3.8)

holds for every \( a > 0 \) and every \( n \geq N \).

Since we have the exponential bound in (3.8) in the \( W_2 \)-metric, one naturally expects to obtain a similar bound to (3.6) in the \( W_2 \)-metric as well. In order to achieve this, we need to find the exponential bound for the probability \( \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, L_{n,t}) > a \right] \), which requires us to handle the quantity \( \|(D^{(n)})^\top D^{(n)}\|_{\infty \rightarrow 1} \), instead of \( \|D^{(n)}\|_{\infty \rightarrow 1} \) as in the proof of Theorem 3.4. Controlling this quantity is done in Lemma 3.1 under an extra condition on the sparsity parameter \( p(n) \), more restrictive condition than the one in Assumption 2.4.

Assumption 3.1. The sparsity parameter sequence \( \{p(n)\}_{n \in \mathbb{N}} \subset (0, 1] \) of the system (1.2) satisfies either one of the following:
(a) \( p(n) \to 0 \) and \( np(n)^2 \to \infty \) as \( n \to \infty \), or
(b) \( p(n) \equiv 1 \) for every \( n \in \mathbb{N} \).

Recalling the notations of (2.3), the following lemma is needed when proving Theorem 3.6. Its proof in Section 4.3 is similar to that of Lemma 2.9 but requires more involved applications of Bernstein’s inequality.

**Lemma 3.1.** Under Assumption 3.1, there exists \( N \in \mathbb{N} \) such that
\[
P \left[ \frac{\| (D^{(n)})^T D^{(n)} \|_{\infty}^2}{n} > \eta \right] \leq 3n^2 \exp \left( - \frac{2n^2}{9 + 4\eta} \right)
\]
holds for every \( n \geq N \) and \( \eta > 0 \).

**Theorem 3.6.** Suppose that the components of the interaction function \( \phi \) belong to the \( L^1 \)-Fourier class. Under Assumptions 2.1, 2.2, 2.3, and 3.1(a), together with the condition (3.3), there exist constants \( K > 0 \), which is independent of \( n \), and \( N \in \mathbb{N} \) such that
\[
P \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \bar{\mu}_t) > a \right] \leq 4n^2 \exp \left( - \frac{a^4np(n)^4}{72K^2 + 8a^2K} \right)
\]
holds for every \( a > 0 \) and every \( n \geq N \).

Furthermore, if Assumption 3.1(a) is replaced by Assumption 3.1(b), we have the exponential bound in \( n \); there exist constants \( \delta > 0 \), which is independent of \( n \), and \( N \in \mathbb{N} \) such that
\[
P \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \bar{\mu}_t) > a \right] \leq \exp \left( - \frac{\delta a^4n}{a^2 + \delta} \right)
\]
holds for every \( a > 0 \) and every \( n \geq N \).

4 Proofs

In this section, we provide proofs of the results stated in Section 3.

4.1 Proofs of results in Section 3.1

4.1.1 Proof of Proposition 3.1

Let us recall the identity (4.7), along with the notations (2.3). Using Hölder’s inequality, there exists \( K > 0 \), depending on \( \phi \) and \( \psi \), such that for every \( t \in [0, T] \)
\[
\sup_{0 \leq t \leq T} \left| X^n_i(t) - \bar{X}_i^n(t) \right|^2 \leq KT \int_0^T \left| \sum_{j=1}^n D^{(n)}_{i,j} \phi(X^n_i(s), X^n_j(s)) \right|^2 ds
\]
\[
+ KT \int_0^T \left| \sum_{j=1}^n \bar{P}^{(n)}_{i,j} \left( \phi(X^n_i(s), X^n_j(s)) - \phi(\bar{X}_i^n(s), \bar{X}_j^n(s)) \right) \right|^2 ds
\]
\[
+ KT \int_0^T \left| X^n_i(s) - \bar{X}_i^n(s) \right|^2 ds.
\]
Taking the expectation to the first term, and using the independence of \( \{D_{i,j}^{(n)}\}_{j \in [n]} \) and the boundedness of \( \phi \), we have
\[
KT \mathbb{E} \int_0^T \left| \sum_{j=1}^n D_{i,j}^{(n)} \phi(X_i^n(s), X_j^n(s)) \right|^2 ds \leq KT \int_0^T \sum_{j=1}^n \mathbb{E}[(D_{i,j}^{(n)})^2] ds \leq \frac{KT^2}{np(n)}.
\]

For the second term, Hölder’s inequality and the Lipschitz continuity of \( \phi \) give
\[
KT \mathbb{E} \int_0^T \sum_{j=1}^n \overline{P}_{i,j}^{(n)} \left( \phi(X_i^n(s), X_j^n(s)) - \phi(\bar{X}_i^n(s), \bar{X}_j^n(s)) \right)^2 ds
\leq KT \mathbb{E} \int_0^T \frac{1}{n} \sum_{j=1}^n (|X_i^n(s) - \bar{X}_i^n(s)|^2 + |X_j^n(s) - \bar{X}_j^n(s)|^2) ds.
\]

Combining above inequalities and averaging over \( i \in [n] \), we obtain
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_i^n(t) - \bar{X}_i^n(t)|^2 \right] \leq \frac{KT^2}{np(n)} + KT \int_0^T \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_i^n(u) - \bar{X}_i^n(u)|^2 \right] ds.
\]

Grönwall’s inequality yields
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_i^n(t) - \bar{X}_i^n(t)|^2 \right] \leq \frac{KT^2 \exp(KT^2)}{np(n)},
\]
and thus
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2^2(\bar{L}_{n,t}, \bar{\mu}_{n,t}) \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_i^n(t) - \bar{X}_i^n(t)|^2 \right] \leq \frac{KT^2 \exp(KT^2)}{np(n)} \to 0, \text{ as } n \to \infty.
\]

4.1.2 Proof of Proposition 3.2

We partition the interval \([0, T]\) into \( M := \left\lceil \frac{T}{\Delta} \right\rceil \) subintervals of length \( \Delta > 0 \):
\[
[0, T] = [0, \Delta] \cup [\Delta, 2\Delta] \cup \cdots \cup [(M - 1)\Delta, T] =: \bigcup_{h=1}^M [\Delta_h],
\]
where \( \Delta_h := [(h - 1)\Delta, h\Delta) \) for \( h = 1, \cdots, M - 1 \) and \( \Delta_M = [(M - 1)\Delta, T] \), and we choose the value of \( \Delta \) later. With the notation
\[
\bar{\mu}_{n,t} := \frac{1}{n} \sum_{i=1}^n \mu_{n,t} = \frac{1}{n} \sum_{i=1}^n \mathcal{L}(X_i^n(t)),
\]

triangle inequality gives
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2(\bar{L}_{n,t}, \bar{\mu}_t) \right] = \mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} W_2(\bar{L}_{n,t}, \bar{\mu}_t) \right]
\leq \mathbb{E} \left[ \max_{h \in [M]} W_2(\bar{L}_{n,t}, \bar{L}_{n,(h-1)\Delta}) \right] + \mathbb{E} \left[ \sup_{h \in [M]} W_2(\bar{L}_{n,(h-1)\Delta}, \bar{\mu}_{n,(h-1)\Delta}) \right]
+ \mathbb{E} \left[ \sup_{h \in [M]} W_2(\bar{\mu}_{n,(h-1)\Delta}, \bar{\mu}_{(h-1)\Delta}) \right] + \mathbb{E} \left[ \sup_{h \in [M]} \sup_{t \in \Delta_h} W_2(\bar{\mu}_{(h-1)\Delta}, \bar{\mu}_t) \right].
\]
\[
=: E_1 + E_2 + E_3 + E_4.
\]
For the first term $E_1$, we note that there exists $K > 0$, depending on the bounds of $\phi$, $\psi$, and $\sigma$, such that

$$|X_n(u) - X_n(s)|^2 \leq K|s - u|^2 + K|B_n(u) - B_n(s)|^2$$

holds for every $0 \leq u \leq s \leq T$, and thus we have

$$(E_1)^2 \leq \mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} W^2_2(\tilde{L}_{n,t}, \tilde{L}_{n,(h-1)\Delta}) \right] \leq \mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} \frac{1}{n} \sum_{i=1}^{n} \left| X_n(t) - X_n((h-1)\Delta) \right|^2 \right]$$

$$\leq \mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} \frac{1}{n} \sum_{i=1}^{n} \left( K \Delta^2 + K \left| B_n(t) - B_n((h-1)\Delta) \right|^2 \right) \right]$$

$$\leq K \Delta^2 + K \mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} \frac{1}{n} \sum_{i=1}^{n} \left| B_n(t) - B_n((h-1)\Delta) \right|^2 \right].$$

Applying Hölder’s inequality twice, the last expectation is bounded above by

$$\mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} \left( \frac{1}{n} \sum_{i=1}^{n} \left| B_n(t) - B_n((h-1)\Delta) \right|^4 \right)^{\frac{1}{2}} \right]$$

$$\leq \sqrt{\mathbb{E} \left[ \max_{h \in [M]} \sup_{t \in \Delta_h} \frac{1}{n} \sum_{i=1}^{n} \left| B_n(t) - B_n((h-1)\Delta) \right|^4 \right]}$$

$$\leq \sqrt{\sum_{h \in [M]} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \sup_{t \in \Delta_h} \left| B_n(t) - B_n((h-1)\Delta) \right|^4 \right]} \leq \sqrt{MC_1 \mathbb{E}[\Delta^2]} \leq \sqrt{(T + 1)C_4 \Delta}.$$

The second-last inequality uses the properties of the increments of Brownian motion and the Burkholder-Davis-Gundy inequality with the positive constant $C_4$. Therefore, we have the bound

$$(E_1)^2 \leq K \Delta^2 + K \sqrt{(T + 1)C_4 \Delta}.$$

For the second expectation $E_2$, Hölder’s inequality and Lemma 2.8 give

$$E_2 \leq \sum_{h \in [M]} \mathbb{E} \left[ W_2^2(\tilde{L}_{n,(h-1)\Delta}, \tilde{\mu}_{n,(h-1)\Delta}) \right] \leq \sum_{h \in [M]} \left( \mathbb{E} \left[ W_2^3(\tilde{L}_{n,(h-1)\Delta}, \tilde{\mu}_{n,(h-1)\Delta}) \right] \right)^{1/3}$$

$$\leq K \sum_{h \in [M]} \left( \int_{\mathbb{R}^d} |x|^4 \tilde{\mu}_{n,(h-1)\Delta}(dx) \right)^{1/4} \alpha_{3,4}^{1/3}(n) \leq KM \alpha_{3,4}^{1/3}(n) \longrightarrow 0,$$ 

as $n \to \infty$, where the last inequality follows from Lemma 2.2.

On the other hand, using the convexity of $W_2^2(\cdot, \cdot)$ and Lemma 2.3(b), there exists $K > 0$ satisfying

$$E_3^2 \leq \mathbb{E} \left[ \sup_{h \in [M]} W_2^2(\tilde{\mu}_{n,(h-1)\Delta}, \tilde{\mu}_{n,(h-1)\Delta}) \right] \leq \mathbb{E} \left[ \sup_{h \in [M]} \int_0^1 W_2^2(\mu_{[n\Delta]}(u)_{n,(h-1)\Delta}, \mu_{u,(h-1)\Delta}) \, du \right] \leq \frac{K}{n^2}.$$

Finally, for the last term $E_4$, we note from a straightforward computation that there exists $K > 0$ satisfying

$$\mathbb{E} \left| X_u(t) - X_u(s) \right|^2 \leq K|t - s|^2 + K \mathbb{E} \left| B_u(t) - B_u(s) \right|^2 \leq K|t - s|$$
for every \( u \in [0, 1] \) and \( s, t \in [0, T] \) satisfying \(|t - s| \leq 1\). Thus, we have

\[
E_4^2 \leq \sup_{h \in [M]} \sup_{t \in [\Delta, 0]} \int_0^1 W_2^2(\mu_{u, (h-1)\Delta}, \mu_{u,t}) \, du
\]

\[
\leq \sup_{h \in [M]} \sup_{t \in [\Delta, 0]} \int_0^1 \mathbb{E}|X_u((h-1)\Delta) - X_u(t)|^2 \, du \leq K\Delta.
\]

Let us combine all the bounds from \( E_1 \) to \( E_4 \). For any given \( \epsilon > 0 \), we can choose small enough \( \Delta \) such that \( E_1 + E_4 < \epsilon/2 \). Then, we can choose large enough \( N \in \mathbb{N} \) satisfying \( E_2 + E_3 < \epsilon/2 \) for every \( n \geq N \), which implies \( \mathbb{E}[\sup_{0 \leq t \leq T} W_2(\mathcal{L}_{n,t}, \bar{\mu}_t)] \leq \epsilon \) for every \( n \geq N \).

\[\square\]

### 4.2 Proofs of results in Section 3.2

#### 4.2.1 Proof of Theorem 3.1

Let us fix an arbitrary \( n \in \mathbb{N} \). We shall naturally identify elements of \((\mathbb{R}^d)^n\) with those of \(\mathbb{R}^{dn}\), identify elements of \((C([0, T] : \mathbb{R}^d))^n\) with those of \(C([0, T] : (\mathbb{R}^d)^n)\), and we shall specify which norm we use for each space. We can express the SDE (1.3) in the form of (2.10) with \( k = dn \), by setting

1. \((\mathbb{R}^d)^n \ni x = (x_i)_{i \in [n]}, \) where \( x_i = X_{i/n}(0) \);
2. \( C([0, T] : (\mathbb{R}^d)^n) \ni X^x = (\bar{X}_i^n)_{i \in [n]} \) where \( \bar{X}_i^n = (\bar{X}_{i,k})_{k \in [d]} \);
3. \( b : [0, T] \times C([0, T] : (\mathbb{R}^d)^n) \to (\mathbb{R}^d)^n \) such that \( b = (b_i)_{i \in [n]} \), \( b_i = (b_{i,k})_{k \in [d]} \) where \( b_{i,k}(t, X^x) = \frac{1}{n} \sum_{j=1}^n \phi_k(X_i^n(t), X_j^n(t)) G(\frac{i}{n}, \frac{j}{n}) + \psi_k(X_i^n(t)) \);
4. \( W = (W_i)_{i \in [n]} \) is a \((dn)\)-dimensional Brownian motion, where \( W_i \equiv B_{i/n} \);
5. \( \Sigma \) is a block-diagonal \((dn) \times (dn)\) matrix with block diagonal entries \( \sigma \).

In order to apply Lemma 2.5, it suffices to check the condition (2.11): for any \( X, Y \in C([0, T] : \mathbb{R}^{dn}) \), Hölder inequality and the Lipschitz continuity of \( \phi, \psi \) indeed yield for every \( t \in [0, T] \)

\[
|b(t, X) - b(t, Y)|^2
\]

\[
= \sum_{i=1}^n \sum_{k=1}^d \left| \frac{1}{n} \sum_{j=1}^n \left( \phi_k(X_i(t), X_j(t)) - \phi_k(Y_i(t), Y_j(t)) \right) G(\frac{i}{n}, \frac{j}{n}) + \psi_k(X_i(t)) - \psi_k(Y_i(t)) \right|^2
\]

\[
\leq 2 \sum_{i=1}^n \sum_{k=1}^d \left[ \left| \frac{1}{n} \sum_{j=1}^n \left( \phi_k(X_i(t), X_j(t)) - \phi_k(Y_i(t), Y_j(t)) \right) G(\frac{i}{n}, \frac{j}{n}) \right|^2 + \left| \psi_k(X_i(t)) - \psi_k(Y_i(t)) \right|^2 \right]
\]

\[
\leq 2 \sum_{i=1}^n \left[ K^2 |X_i(t) - Y_i(t)|^2 + \sum_{k=1}^d \frac{1}{n} \sum_{j=1}^n \left| \phi_k(X_i(t), X_j(t)) - \phi_k(Y_i(t), Y_j(t)) \right|^2 \right]
\]

\[
\leq (4K^2 + 1) \sum_{i=1}^n |X_i(t) - Y_i(t)|^2 = (4K^2 + 1) \sum_{i=1}^n \sum_{k=1}^d |X_{i,k}(t) - Y_{i,k}(t)|^2 \leq (4K^2 + 1)||X - Y||_{s,t}.
\]
Let \( P^x \in \mathcal{P}(C([0, T] : \mathbb{R}^{dn})) \) be the law of the solution of (1.3) in the notations of (1)-(5) above, then we have from Lemma 2.5 for any \( Q \in \mathcal{P}(C([0, T] : \mathbb{R}^{dn})) \),

\[
W_{1,1}(C([0, T] : \mathbb{R}^{dn}), || \cdot ||_{dn, 2}) (P^x, Q) \leq \sqrt{2c_1 H(Q) P^x},
\]

for some \( c_1 > 0 \).

For an arbitrary \( F \in \text{Lip} \left( (C([0, T] : \mathbb{R}^d))^n, || \cdot ||_{n, 1} \right) \), Hölder’s inequality shows that \( F \) is \( \sqrt{n} \)-Lipschitz function of the space \( (C([0, T] : \mathbb{R}^{dn}), || \cdot ||_{dn, 2}) \); we indeed obtain for \( X, Y \in (C([0, T] : \mathbb{R}^d))^n, || \cdot ||_{n, 1} \)

\[
|F(X) - F(Y)| \leq ||X - Y||_{n, 1} = \sum_{i=1}^d \|X_i - Y_i\|_{*,T}
\]

\[
\leq \sqrt{n \sum_{i=1}^d \|X_i - Y_i\|_{*,T}^2} \leq \sqrt{n \sum_{i=1}^d \sum_{k=1}^d \|X_{i,k} - Y_{i,k}\|_{*,T}^2} = \sqrt{n} ||X - Y||_{dn, 2}.
\]

Thus, Lemma 2.6 implies

\[
P^x (F - \langle P^x, F \rangle > a) \leq \exp \left( -\frac{a^2}{2c_1 n} \right), \tag{4.1}
\]

for any \( a > 0 \).

We now claim that there exists a positive constant \( c_2 \), which does not depend on \( n \), such that the map \( x \mapsto \langle P^x, F \rangle \) is \( c_2 \)-Lipschitz on \( (\mathbb{R}^d)^n \) with respect to the Euclidean \( \ell^p \)-norm for any \( F \in \text{Lip} \left( (C([0, T] : \mathbb{R}^d))^n, || \cdot ||_{n, p} \right) \) and for any \( p = 1, 2 \). Given any \( x, y \in (\mathbb{R}^d)^n \), we couple \( P^x \) and \( P^y \) by solving the system (1.3) from the two initial states \( x, y \) with the same Brownian motion, and denote the coupling by \( \pi_{x,y} \). We deduce that for \( \mathcal{L}(X) = P^x \) and \( \mathcal{L}(Y) = P^y \)

\[
|\langle P^x, F \rangle - \langle P^y, F \rangle|^p \leq \int |F(X) - F(Y)|^p \pi_{x,y}(dX, dY) \leq \int ||X - Y||_{n,p}^p \pi_{x,y}(dX, dY). \tag{4.2}
\]

When \( p = 2 \), we use a standard argument (the trivial inequality \((a + b)^2 \leq 2(a^2 + b^2)\), the Lipschitz continuity from Assumption 2.1(a), and a series of Hölder inequalities) to derive

\[
\sum_{i=1}^n \|X_i(t) - Y_i(t)\|^2 \leq 2||x - y||_{n,2}^2 + 2Kt \sum_{i=1}^n \int_0^t \left( \|X_i(s) - Y_i(s)\| + \frac{1}{n} \sum_{j=1}^n \|X_j(s) - Y_j(s)\| \right)^2 ds
\]

\[
\leq 2||x - y||_{n,2}^2 + 8Kt \sum_{i=1}^n \int_0^t \|X_i(s) - Y_i(s)\|^2 ds, \quad \forall t \in [0, T].
\]

Grönwall’s inequality yields that the last integrand of (4.2) for \( p = 2 \) is bounded by

\[
||X - Y||_{n,2}^2 \leq c_2^2 ||x - y||_{n,2}^2
\]

for some constant \( c_2 > 0 \), which depends on \( \phi, \psi \), and \( T \), but not on \( n \). When \( p = 1 \), proving \( ||X - Y||_{n,1} \leq c_2 ||x - y||_{n,1} \) is easier, and the claim follows.
On the other hand, we apply Lemmas 2.7, 2.6 to the assumption (3.1) to obtain for every $f \in Lip((\mathbb{R}^d)^n, \| \cdot \|_{n,1})$ and for any $a > 0$

$$
\mu_0^n(f - \langle \mu_0^n, f \rangle) > a \leq \exp\left(-\frac{a^2}{2\kappa n}\right).
$$

We conclude from (4.1), the above claim, and (4.3)

$$
\begin{align*}
\mathbb{P}\left[F(\bar{X}^n) - \mathbb{E}(F(\bar{X}^n)) > a\right] &\leq \mathbb{E}\left[\mathbb{P}\left(F(\bar{X}^n) - \langle P^x, F \rangle > \frac{a}{2} \mid \bar{X}^n(0) = x\right) \right] \\
&\quad + \mathbb{P}\left(\langle P^{\bar{X}^n(0)}, F \rangle - \mathbb{E}[\langle P^{\bar{X}^n(0)}, F \rangle] > \frac{a}{2}\right) \\
&\leq \exp\left(-\frac{a^2}{8c_1 n}\right) + \exp\left(-\frac{a^2}{8\kappa c_2^2 n}\right). 
\end{align*}
$$

The assertion (3.4) follows by choosing $1/\delta = 8 \max(c_1, \kappa c_2^2)$. \hfill \square

### 4.2.2 Proof of Theorem 3.2

We follow the proof of Theorem 3.1. Identifying the elements of $(C([0,T] : \mathbb{R}^d))^n$ with those of $C([0,T] : \mathbb{R}^{dn})$, expressing the SDE (1.3) in the form of (2.10), and applying Lemma 2.5, there exists a positive constant $c_1 > 0$ such that

$$
W_{1, C([0,T] : \mathbb{R}^{dn}), \| \cdot \|_{dn,2}}(P^x, Q) \leq 2c_1 \sqrt{H(Q||P^x)}
$$

holds for any $Q \in \mathcal{P}(C([0,T] : \mathbb{R}^{dn}))$. Here, $P^x$ is the law of the solution of (1.3). Moreover, Lemma 2.6 implies

$$
P^x(F - \langle P^x, F \rangle) > a \leq \exp\left(-\frac{a^2}{2c_1}\right),
$$

for any $a > 0$ and every $F \in Lip(C([0,T] : \mathbb{R}^{dn}), \| \cdot \|_{dn,2})$. It is easy to check that every function in $Lip(C([0,T] : \mathbb{R}^{dn}), \| \cdot \|_{dn,2})$ also belongs to $Lip\left((C([0,T] : \mathbb{R}^d))^n, \| \cdot \|_{n,2}\right)$, thus the inequality (4.4) also holds for every $F \in Lip\left((C([0,T] : \mathbb{R}^d))^n, \| \cdot \|_{n,2}\right)$.

We now apply Lemmas 2.7(ii), 2.6 to the assumption (3.3) to deduce

$$
\mu_0^n(f - \langle \mu_0^n, f \rangle) > a \leq \exp\left(-\frac{a^2}{2\kappa}\right),
$$

for every $f \in Lip((\mathbb{R}^d)^n, \| \cdot \|_{n,2})$ and for any $a > 0$.

From (4.4), (4.5), and the claim in the proof of Theorem 3.1 we conclude that

$$
\begin{align*}
\mathbb{P}\left[F(\bar{X}^n) - \mathbb{E}(F(\bar{X}^n)) > a\right] &\leq \mathbb{E}\left[\mathbb{P}(F(\bar{X}^n) - \langle P^x, F \rangle > \frac{a}{2} \mid \bar{X}^n(0) = x\right) \\
&\quad + \mathbb{P}\left(\langle P^{\bar{X}^n(0)}, F \rangle - \mathbb{E}[\langle P^{\bar{X}^n(0)}, F \rangle] > \frac{a}{2}\right) \\
&\leq \exp\left(-\frac{a^2}{8c_1}\right) + \exp\left(-\frac{a^2}{8\kappa c_2^2}\right). 
\end{align*}
$$

The result (3.4) follows by choosing $1/\delta = 8 \max(c_1, \kappa c_2^2)$. \hfill \square
4.3 Proofs of results in Section 3.3

4.3.1 Proof of Theorem 3.3

Note that
\[ Y \mapsto \sup_{0 \leq t \leq T} W_1 \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i(t)}, \bar{\mu}_t \right) \]
is \((1/n)\)-Lipschitz from \((C([0, T] : \mathbb{R}^d))^n, \| \cdot \|_{n,1} \) to \(\mathbb{R}\). Then, for any \(a > 0\),
\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) > a \right] \leq \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) - \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) \right] > \frac{a}{2} \right] + \mathbb{P} \left[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) \right] > \frac{a}{2} \right].
\]
The first term is bounded by the right-hand side of (3.5) from Theorem 3.1.

Let us consider the auxiliary particle system (1.2) satisfying Assumption 2.4. Corollary 3.1 shows that the last probability vanishes for all but finitely many \(n\) and the result follows.

\[ \square \]

4.3.2 Proof of Theorem 3.4

We first prove (3.6). From the triangle inequality, we obtain
\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(L_{n,t}, \bar{\mu}_t) > a \right] \leq \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(\bar{L}_{n,t}, \bar{\mu}_t) > \frac{a}{2} \right] + \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_1(L_{n,t}, L_{n,t}) > \frac{a}{2} \right]. \tag{4.6}
\]
In what follows, we compute the bound for the last probability on the right-hand side. For fixed \(t \in [0, T]\) and \(i \in [n]\), we use the notation (2.3) to obtain
\[
X^n_i(t) - \bar{X}^n_i(t) = \int_0^t \sum_{j=1}^{n} D_{ij}^{(n)}(s) \phi(X^n_i(s), X^n_j(s)) \, ds \tag{4.7}
+ \int_0^t \sum_{j=1}^{n} P_{ij}^{(n)} \left( \phi(X^n_i(s), X^n_j^*(s)) - \phi(X^n_i(s), X^n_j(s)) \right) + \psi(X^n_i(s)) - \psi(\bar{X}^n_i(s)) \, ds.
\]
We define \(\Delta(t) := \frac{1}{n} \sum_{i=1}^{n} \|X^n_i(t) - \bar{X}^n_i(t)\|_{*,t}\), then deduce from the continuity of \(X^n_i(\cdot) - \bar{X}^n_i(\cdot)\) that there exists \(t_i \in [0, t]\) for each \(i \in [n]\) satisfying
\[
\Delta(t) = \frac{1}{n} \sum_{i=1}^{n} \left| X^n_i(t_i) - \bar{X}^n_i(t_i) \right| \leq \int_0^t \frac{1}{n} \sum_{i,j=1}^{n} \left| D_{ij}^{(n)} 1_{[0,t_i]}(s) \phi(X^n_i(s), X^n_j(s)) \right| \, ds \tag{4.8}
+ \frac{1}{n} \int_0^t \sum_{i,j=1}^{n} P_{ij}^{(n)} 1_{[0,t_i]}(s) \phi(X^n_i(s), X^n_j(s)) - \phi(\bar{X}^n_i(s), \bar{X}^n_j(s)) \, ds \tag{4.9}
+ \frac{1}{n} \int_0^t \sum_{i=1}^{n} 1_{[0,t_i]}(s) \left| \psi(X^n_i(s)) - \psi(\bar{X}^n_i(s)) \right| \, ds. \tag{4.10}
\]
Since each component \(\phi_k\) of \(\phi\) belongs to the \(L^1\)-Fourier class, there exists a finite complex measure \(m_{\phi_k}\) so that we can write for every \(k \in [d]\)
\[
\phi_k(X^n_i(s), X^n_j(s)) = \int_{\mathbb{R}^2d} a^k_i(z, s) b^j_i(z, s) m_{\phi_k}(dz), \quad z = (z_1, z_2), \tag{4.11}
\]

for some complex functions \( a_k^i, b_j^k \) of the form
\[
a_k^i(z, s) := \exp \left( 2\pi \sqrt{-1} \langle X_i^n(s), z \rangle \right), \quad b_j^k(z, s) := \exp \left( 2\pi \sqrt{-1} \langle X_j^n(s), z \rangle \right).
\]

Using the representation (4.11) with an application of Hölder inequality, the integral of (4.8) is bounded above by
\[
\frac{\max_{1 \leq k \leq d} \| m_{\phi_k} \|_{T M} }{n} \sum_{i, j=1}^{n} \sum_{k=1}^{d} \int_0^t \int_{\mathbb{R}^d} \left| D_{i,j}^{(n)} \mathbf{1}_{[0,t]}(s) a_i^k(z, s) b_j^k(z, s) m_{\phi_k}(dz) \right| ds
\]
\[
= \frac{K t}{n} \sum_{k=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle a^k(z, s), D^{(n)} b^k(z, s) \rangle m_{\phi_k}(dz) ds,
\]
where we defined the complex vectors
\[
a^k(z, s) := \left( \mathbf{1}_{[0,t]}(s) a_i^k(z, s) \right)_{i \in [n]}, \quad b^k(z, s) := \left( b_j^k(z, s) \right)_{j \in [n]}, \quad \text{for each } k \in [d].
\]

Since \( \ell^\infty \)-norms of these vectors are bounded by 1, decomposing them into real and complex parts gives for each \( k \in [d] \)
\[
\left| \langle a^k(z, s), D^{(n)} b^k(z, s) \rangle \right| \leq 4 \sup \left\{ \langle x, D^{(n)} y \rangle : x, y \in [-1, 1]^n \right\} = 4\|D^{(n)}\|_{\infty \rightarrow 1}.
\]

Thus, the right-hand side of (4.12) is bounded above by
\[
\frac{K t t^2}{n} \left( \max_{1 \leq k \leq d} \| m_{\phi_k} \|_{T M} \right) \|D^{(n)}\|_{\infty \rightarrow 1}.
\]

For the integrals of (4.9) and (4.10), we use the Lipschitz continuity of \( \phi \) and \( \psi \), thus there exists a constant \( K > 0 \) such that
\[
\Delta(t) \leq K \int_0^t \Delta(s) ds + \frac{\|D^{(n)}\|_{\infty \rightarrow 1}}{n} K t^2, \quad \forall t \in [0, T].
\]

Grönwall’s inequality yields
\[
\Delta(T) \leq \frac{K \|D^{(n)}\|_{\infty \rightarrow 1}}{n}, \quad (4.15)
\]

where \( K \) is now a positive constant depending on the time horizon \( T \). Recalling the notation \( \Delta(t) \), we obtain
\[
\sup_{0 \leq t \leq T} W_1(L_{n,t}, \bar L_{n,t}) \leq \Delta(T) \leq \frac{K \|D^{(n)}\|_{\infty \rightarrow 1}}{n},
\]
and finally Lemma 2.9 gives the bound for the last probability of (4.6)
\[
P\left[ \sup_{0 \leq t \leq T} W_1(L_{n,t}, \bar L_{n,t}) > \frac{a}{2} \right] \leq \exp \left( - \frac{a^2 n^2 p(n)}{8 K^2 + 2aK/3} \right).
\]

For the first probability on the right-hand side of (4.6), Theorem 3.3 yields for every \( n \geq N \)
\[
P\left[ \sup_{0 \leq t \leq T} W_1(\bar L_{n,t}, \bar m_t) > \frac{a}{2} \right] \leq 2 \exp \left( - \frac{\delta a^2 n}{16} \right).
\]
Thanks to Assumption 2.4 by choosing a larger value for $N \in \mathbb{N}$ than the one in Theorem 3.3 we can make $\exp \left( -\frac{a^2n^2p(n)}{8K^2+2nK/\beta} \right) \leq \exp \left( -\frac{40n^2}{16} \right)$ for every $n \geq N$, and the assertion (3.6) follows.

For the result (3.7), we can approximate general $\phi$ with those in $L^1$-Fourier class by the approximation method in Section 5.1.3 of [27], to find the exponential bound for the probability $\mathbb{P}[\inf_{0 \leq t \leq T} d_{BL} (L_{n,t}, \hat{L}_{n,t}) > a/2]$ similar to (4.16). By recalling the fact $d_{BL} \leq W_1$ and replacing all the $W_1$-metrics with the $d_{BL}$-metrics in (4.6), we arrive at (3.7). \hfill \Box

4.3.3 Proof of Theorem 3.5

It is easy to verify the $(\frac{1}{\sqrt{n}})$-Lipschitz continuity of the map

$$Y \mapsto \sup_{0 \leq t \leq T} W_2 \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i(t)}, \tilde{\mu}_t \right),$$

from $\left( (C([0,T]:\mathbb{R}^d))^n, \| \cdot \|_{n,2} \right)$ to $\mathbb{R}$. Then, for any $a > 0$,

$$\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2 (L_{n,t}, \tilde{\mu}_t) > a \right] \leq \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2 (L_{n,t}, \tilde{\mu}_t) - \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2 (L_{n,t}, \tilde{\mu}_t) \right] > \frac{a}{2} \right] + \mathbb{P} \left[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} W_2 (L_{n,t}, \tilde{\mu}_t) > \frac{a}{2} \right] \right].$$

The first term is bounded by the right-hand side of (3.8) from Theorem 3.2. As in the proof of Theorem 3.3, the last probability vanishes for all but finitely many $n$ from Corollary 3.1 \hfill \Box

4.3.4 Proof of Lemma 3.1

We note from (2.3) that $\{D^{(n)}_{i,j}\}_{1 \leq i,j \leq n}$ are independent zero-mean random variables and for every $i,j \in [n]$

$$\mathbb{E} \left[ (D^{(n)}_{i,j})^2 \right] = \frac{p(n)G(\frac{i}{n}, \frac{j}{n})(1 - p(n)G(\frac{i}{n}, \frac{j}{n}))}{(np(n))^2}. \quad (4.17)$$

In particular, since $p(n) \leq 1$, we have $0 \leq p(n)G(\frac{i}{n}, \frac{j}{n}) \leq 1$, and thus $\mathbb{E} \left[ (D^{(n)}_{i,j})^2 \right] \leq 1/(4n^2p(n)^2)$. Let us fix any $n \in \mathbb{N}$. For arbitrary $n$-dimensional vectors $\mathbf{x}, \mathbf{y} \in [-1,1]^n$, we have

$$\langle \mathbf{x}, (D^{(n)})^T D^{(n)} \mathbf{y} \rangle = \sum_{i=1}^{n} (D^{(n)} \mathbf{x})_i (D^{(n)} \mathbf{y})_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D^{(n)}_{i,j} D^{(n)}_{i,k} x_j y_k + \sum_{i=1}^{n} \sum_{j=1}^{n} (D^{(n)}_{i,j})^2 x_j y_j \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D^{(n)}_{i,j} D^{(n)}_{i,k} x_j y_k + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( (D^{(n)}_{i,j})^2 x_j y_j - \mathbb{E} \left[ (D^{(n)}_{i,j})^2 x_j y_j \right] \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{x_i y_j}{4n^2p(n)^2}. $$

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Thus, we have for fixed arbitrary $\eta > 0$
\[
\Pr \left[ \left\| \frac{(D^{(n)})^\top D^{(n)}}{n} \right\|_{\infty} > \eta \right] \leq \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1 \atop k \neq j}^{n} D_{i,j}^{(n)} D_{i,k}^{(n)} x_j y_k > \frac{\eta}{3} \right] + \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( (D_{i,j}^{(n)})^2 x_j y_j - \mathbb{E} \left[ (D_{i,j}^{(n)})^2 x_j y_j \right] \right) > \frac{\eta}{3} \right] + \Pr \left[ \frac{1}{4np(n)^2} > \frac{\eta}{3} \right] =: P_1 + P_2 + P_3. \tag{4.18}
\]

From Assumption 3.1 there exists $N \in \mathbb{N}$ such that $P_3$ vanishes for every $n \geq N$. In the following, we find the bounds for $P_1$ and $P_2$. Using the distribution
\[
D_{i,j}^{(n)} = \begin{cases} \frac{1-p(n)G(i/n,j/n)}{np(n)} & \text{with probability } p(n)G(i/n,j/n), \\ -\frac{1}{n}G(i/n,j/n) & \text{with probability } 1 - p(n)G(i/n,j/n), \end{cases}
\]
for each $i, j \in [n]$, we derive for $P_1$
\[
P_1 \leq \sum_{i=1}^{n} \Pr \left[ \sum_{j=1}^{n} \sum_{k=1 \atop k \neq j}^{n} D_{i,j}^{(n)} D_{i,k}^{(n)} x_j y_k > \frac{\eta}{3} \right] \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr \left[ \sum_{k=1 \atop k \neq j}^{n} D_{i,j}^{(n)} D_{i,k}^{(n)} x_j y_k > \frac{\eta}{3n} \right] \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \Pr \left[ \sum_{k=1 \atop k \neq j}^{n} D_{i,j}^{(n)} x_j y_k > \frac{np(n)}{3} \right] + \Pr \left[ \sum_{k=1 \atop k \neq j}^{n} D_{i,j}^{(n)} x_j y_k < -\frac{\eta}{3} \right] \right).
\]

The summands $D_{i,k}^{(n)} x_j y_k$ in the last two probabilities are independent zero-mean random variables, bounded above by $1/(np(n))$, bounded below by $-1/n$, and satisfies
\[
\sum_{k=1 \atop k \neq j}^{n} \mathbb{E} \left[ (D_{i,k}^{(n)} x_j y_k)^2 \right] \leq \frac{n - 1}{4n^2 p(n)^2} \leq \frac{1}{4np(n)^2},
\]
from (4.17). From Bernstein’s inequality (Lemma 2.10), we have
\[
\Pr \left[ \sum_{k=1 \atop k \neq j}^{n} D_{i,k}^{(n)} x_j y_k > \frac{np(n)}{3} \right] \leq \exp \left( -\frac{2np(n)^4 \eta^2}{9 + 4np(n)^2} \right),
\]
\[
\Pr \left[ -\sum_{k=1 \atop k \neq j}^{n} D_{i,k}^{(n)} x_j y_k > -\frac{\eta}{3} \right] \leq \exp \left( -\frac{2np(n)^2 \eta^2}{9 + 4np(n)^2} \right),
\]
thus
\[
P_1 \leq 2n^2 \exp \left( -\frac{2np(n)^4 \eta^2}{9 + 4np(n)^2} \right). \tag{4.19}
\]
We now compute the bound for $P_2$. Since we have

$$
P_2 \leq \sum_{i=1}^{n} \mathbb{P} \left[ \sum_{j=1}^{n} \left( (D_{i,j}^{(n)})^2 x_j y_j - \mathbb{E}[ (D_{i,j}^{(n)})^2 x_j y_j ] \right) > \frac{\eta}{3} \right]$$

(4.20)

and the summands $(D_{i,j}^{(n)})^2 x_j y_j - \mathbb{E}[ (D_{i,j}^{(n)})^2 x_j y_j ]$ in the probability are independent zero-mean random variables bounded above by $5/(2np(n))^2$. Moreover, we easily obtain the bound $\mathbb{E}[ (D_{i,j}^{(n)})^4 ] \leq 1/(np(n))^4$, and thus the sum of variances of summands are

$$\sum_{j=1}^{n} \mathbb{E} \left[ (D_{i,j}^{(n)})^2 - \mathbb{E}(D_{i,j}^{(n)})^2 \right]^2 \leq \sum_{j=1}^{n} \left( \mathbb{E}[ (D_{i,j}^{(n)})^4 ] + \mathbb{E}( (D_{i,j}^{(n)})^2 ) \right)^2 \leq \frac{17}{16n^3p(n)^4}.$$ 

Applying Bernstein’s inequality (Lemma 2.10) to each probability of (4.20) yields

$$P_2 \leq n \exp \left( - \frac{2n^3p(n)^4 \eta^2}{153 + 20np(n)^2 \eta} \right).$$

(4.21)

Comparing the bounds of (4.19), (4.21), modifying the value of $N$ if necessary, and plugging into (4.18), the result follows.

4.3.5 Proof of Theorem 3.6

The argument is similar to the proof of Theorem 3.4. The triangle inequality gives

$$\mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \tilde{\mu}_t) > a \right] \leq \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2(\tilde{L}_{n,t}, \tilde{\mu}_t) > \frac{a}{2} \right] + \mathbb{P} \left[ \sup_{0 \leq t \leq T} W_2(L_{n,t}, \tilde{L}_{n,t}) > \frac{a}{2} \right].$$

(4.22)

Recalling the identity (4.7), applying a series of Hölder’s inequality, and using the Lipschitz property yield

$$|X_i^n(t) - \bar{X}_i^n(t)|^2 - 2t \int_0^t \left| \sum_{j=1}^{n} D_{i,j}^{(n)} \phi(X_i^n(s), X_j^n(s)) \right|^2 ds$$

$$\leq 2K^2 t \int_0^t \left( \sum_{j=1}^{n} \tilde{P}_{i,j}^{(n)} \left( \left| X_i^n(s) - \bar{X}_i^n(s) \right| + \left| X_j^n(s) - \bar{X}_j^n(s) \right| \right)^2 + \left| X_i^n(s) - \bar{X}_i^n(s) \right|^2 \right) ds$$

$$\leq 2K^2 t \int_0^t 6 \left| X_i^n(s) - \bar{X}_i^n(s) \right|^2 + \frac{2}{n} \sum_{j=1}^{n} \left| X_j^n(s) - \bar{X}_j^n(s) \right|^2 ds, \quad \forall t \in [0, T].$$

For a fixed $t \in [0, T]$, by using the continuity of $X_i^n(\cdot) - \bar{X}_i^n(\cdot)$, there exists $t_i \in [0, t]$ for each $i \in [n]$ satisfying $\Box(t) := \frac{1}{n} \sum_{i=1}^{n} \| X_i^n - \bar{X}_i^n \|^2_{t, i} = \frac{1}{n} \sum_{i=1}^{n} |X_i^n(t_i) - \bar{X}_i^n(t_i)|^2$. Combining with the last inequality, we have

$$\Box(t) \leq 2t \int_0^t \frac{1}{n} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} D_{i,j}^{(n)} \mathbb{1}_{[0,t_i]}(s) \phi(X_i^n(s), X_j^n(s)) \right|^2 ds + 16K^2 t \int_0^t \Box(s) ds.$$ 

(4.23)
We recall the representations (4.11) and (4.13) and use Hölder’s inequality to derive for the first integral on the right-hand side

\[
\int_0^t \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^d D_{i,j}^{(n)} 1_{[0,t]}(s) \varphi(X_i^n(s), X_j^n(s)) \, ds
\]

\[
\leq d \int_0^t \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^d \left| \sum_{j=1}^n D_{i,j}^{(n)} 1_{[0,t]}(s) a_i^k(z(s), b_j^k(z(s)) \right| m_{\phi_k}(dz) \, ds
\]

\[
\leq d \int_0^t \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^d \|m_{\phi_k}\|_{TM} \int_{\mathbb{R}^d} \left| (a^k(z(s)), (D^{(n)} b^k(z(s))_i) \right| m_{\phi_k}(dz) \, ds
\]

\[
\leq d \int_0^t \sum_{k=1}^d \|m_{\phi_k}\|_{TM} \int_{\mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \| (D^{(n)} b^k(z(s))_i \right| m_{\phi_k}(dz) \, ds
\]

\[
\leq \max_{1 \leq k \leq d} \|m_{\phi_k}\|^2_{TM} n \| D^{(n)} D^{(n)}\|_{\infty \to 1} \frac{\| \sum_{k=1}^d \| (D^{(n)} b^k(z(s))_i \right| m_{\phi_k}(dz) \, ds}{n}
\]

In the last two inequalities, we used the fact that $\ell^\infty$-norms of two vectors $a^k(z(s))$ and $b^k(z(s))$ are bounded by 1. Thus, from (4.23), there exists a constant $K > 0$ such that

\[
\square(t) \leq K t^2 \left( (D^{(n)})_\top D^{(n)} \right)_{\infty \to 1} \frac{a^2}{n} + K t \int_0^t \square(s) \, ds
\]

holds for every $t \in [0, T]$ and applying Grönwall’s inequality gives

\[
\square(T) \leq K \left( (D^{(n)})_\top D^{(n)} \right)_{\infty \to 1} \frac{a^2}{n},
\]

where the constant $K$ now depends on $T$.

Since we have

\[
\sup_{0 \leq t \leq T} W_2(L_n, \bar{L}_n) \leq \sup_{0 \leq t \leq T} \sqrt{\frac{1}{n} \sum_{i=1}^n \left| X_i^n(t) - \bar{X}_i^n(t) \right|^2} \leq \sqrt{(\square(T)}
\]

Lemma 3.1 shows that there exists $N \in \mathbb{N}$ such that the last probability in (4.22) has the bound

\[
P \left( \sup_{0 \leq t \leq T} W_2(L_n, \bar{L}_n) > \frac{a}{2} \right) \leq P(\square(T) > \frac{a^2}{4}) \leq P \left( \left( (D^{(n)})_\top D^{(n)} \right)_{\infty \to 1} > \frac{a^2}{4K} \right)
\]

\[
\leq 3n^2 \exp \left( -\frac{a^4 n \eta(t)^4}{72K^2 + 8a^2 K} \right),
\]

for every $n \geq N$.

On the other hand, Theorem 3.5 gives the bound for the other probability in (4.22). By comparing both of the bounds under Assumption 3.1(a), the assertion 3.9 follows. The result 3.10 is now clear under Assumption 3.1(b), by setting $p(n) \equiv 1$ and redefining the constants $\delta > 0$ and $N \in \mathbb{N}$ appropriately.
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