Numerical evaluation of a two loop diagram in the cut-off regularization

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The sunset diagram of $\lambda\phi^4$ theory is evaluated numerically in cutoff scheme and a nonzero finite term (in accordance with dimensional regularization (DR) result) is found in contrast to published calculations. This finding dramatically reduces the critical couplings for symmetry breaking in the two loop effective potential discussed in our previous work.

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I. INTRODUCTION

Dimensional regularization [1] has become an efficient regularization scheme for theoretical calculations in high energy physics. It makes analytical results more available, and its gauge invariance almost decisively defeated the other schemes (say cutoff scheme) in QCD and other gauge theories.

However, things are not so simple. It is very natural to anticipate that DR might fail sooner or later somewhere as it is just a special regularization but never a true theory of the short distance physics. Such event is now known: the $\overline{MS}$ scheme does not provide a physical prediction for low energy nucleons scattering in case of long scattering lengths [2] in nonperturbative context. To yield meaningful results, unconventional subtraction is devised and now known as PDS [2]1. Since this is obtained with hindsight, we should not discard the cutoff scheme without further exploiting its virtues. There have recently been some further investigations over the relations between the divergences in DR and cutoff schemes and the associated limitations [4,5]. The conventional virtues of the DR scheme were shown to be due to an implicit subtraction [4]. In nonperturbative contexts involving multiloop corrections the DR might even make renormalization infeasible [5].

Traditionally, it is very hard to obtain analytical results for multiloop amplitudes within the cutoff scheme. In this report we show that one could achieve this goal numerically at a pretty good precision. We will make use of the dynamical symmetry breaking of $\lambda\phi^4$ discussed in [6] to illustrate the numerical evaluation of the finite constant term in the two loop sunset diagram. Thus the report is organized as follows. First we quote some results of Ref. [6] in Sec. II as preparation. In Sec. III we present our strategy and procedures for doing the numerical analysis of the problem and its possible prospective significance. The result is also given there. In Sec. IV we revisit the problem introduced in Sec. II to show the significance of the new result obtained in Sec. II. The summary will also be given there.

II. RENORMALIZATION OF THE TWO LOOP EFFECTIVE POTENTIAL

In [6], the dynamical symmetry breaking of massless $\lambda\phi^4$ model with $Z_2$ symmetry was studied in two loop effective potential. The algorithm in use is well known according to Jackiw [7],

$$V(2l) \equiv \lambda\phi^4 + \frac{1}{2}I_0(\Omega) + 3\lambda I_1^2(\Omega) - 48\lambda^2\phi^2 I_2(\Omega),$$

$$\Omega \equiv \sqrt{12\lambda}\phi^2; \quad I_0(\Omega) = \int \frac{d^4k}{(2\pi)^4} \ln(1 + \frac{\Omega^2}{k^2}); \quad I_1(\Omega) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \Omega^2};$$

$$I_2(\Omega) = \int \frac{d^4kd4l}{(2\pi)^8} \frac{1}{(k^2 + \Omega^2)(l^2 + \Omega^2)((k + l)^2 + \Omega^2)}.$$  

Here we have Wick rotated all the loop integrals into Euclidean space. The loop integrals and the effective potential had been calculated in literature [7,8]. For comparison, we list out below only the results for the sunset diagram defined by $I_2(\Omega)$ in Eq. (3), the only two loop diagram with overlapping divergences.

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1There are other schemes that could yield similar predictions as PDS did, see, e.g., Ref. [3].
A. Loop amplitudes in DR and cutoff schemes

From [8] we find that
\[
\mu^{4}\mathcal{I}^{(D)}(\Omega) = - \frac{3\Omega^2}{2(4\pi)^{2}}\left(1 + (3 - 2L)\epsilon + (2L^2 - 6L + 7 + 6S - \frac{5}{3}\zeta(2))\epsilon^2\right)
\]  \hspace{1cm} (4)

with \( S = \sum_{n=0}^{\infty} \frac{1}{(2+3n)^{2}} \), \( L = L + \gamma - \ln 4\pi \) and \( L = \ln \frac{\Omega^2}{\mu^2} \). While from [7],
\[
\mathcal{I}^{(A)}(\Omega) = \frac{1}{(4\pi)^{4}}\left\{ 2\Lambda^2 - \frac{3\Omega^2}{2} \ln \frac{\Omega^2}{\Lambda^2} + 3\Omega^2 \ln \frac{\Omega^2}{\Lambda^2} + o(\Lambda^{-2}) \right\}. \hspace{1cm} (5)
\]

Note that the finite double logarithmic term (leading term) disagree between the two regularization schemes, the disagreement is removed only after all subdivergences are removed [5]. We note in particular that the term proportional to \( \Omega^2 \) in the two loop integral is not explicitly given in [7], and it is this finite term and its numerical estimation that is our main concern in this paper.

B. Prescription dependence

The renormalization were performed in Ref. [6] in several prescriptions, \( \overline{MS} \) for \( V^{(D)}_{(2l)}(\Omega) \) [8], Jackiw [7], Coleman-Weinberg (CW) [9] and \( \mu^2_{\Lambda} \) (defined in [6], Appendix B) for \( V^{(A)}_{(2l)}(\Omega) \) with the renormalized potential taking the following form
\[
V_{(2l)}(\Omega) = \Omega^{4}\left\{ \frac{1}{144\lambda} + \frac{L - 1/2}{(8\pi)^2} + \frac{3\lambda}{(4\pi)^4}[L^2 + 2(L - 1)^2 + \alpha] \right\}, \hspace{1cm} (6)
\]

where \( L = L, \ln \frac{\Omega^2}{\mu^2}, \ln \frac{\Omega^2}{\mu^2_{\Lambda}} \) and \( \ln \frac{\Omega^2}{\mu^2_{\Lambda}} - \frac{2\pi}{\lambda} \) in respectively \( \overline{MS}, \mu^2_{\Lambda}, \) Jackiw and Coleman-Weinberg schemes. In all the above formulas the scheme dependence of field strength and coupling constant are understood. The explicit intermediate renormalization prescriptions dependence of the effective potential expressed by \( \alpha \) is listed in Table I.

The dynamical symmetry breaking solution in the two loop effective potential defined in Eq. (6) could be obtained from the first order condition \( \frac{\partial V_{(2l)}(\Omega, \phi)}{\partial \phi} = 0 \) and the solutions read [6],
\[
\phi_{f}^2(\lambda; [\mu, \alpha]) = \frac{\mu^2}{12\lambda} \exp \left\{ \frac{1}{6} \left[ 1 - \frac{4\pi^2}{3\lambda} + \sqrt{\frac{1}{3}(4 - 36\alpha - (1 + \frac{4\pi^2}{\lambda})^2)} \right] \right\}, \hspace{1cm} (7)
\]

whose existence requires that
\[
\alpha < \frac{1}{12}, \hspace{0.5cm} \lambda \geq \lambda_{cr} = \frac{4\pi^2}{\sqrt{4 - 36\alpha - 1}}. \hspace{1cm} (8)
\]

That means not all schemes are consistent with symmetry breaking. In addition, for the symmetry breaking solutions to be stable, the coupling must be further constrained [6],
\[
\lambda \geq \tilde{\lambda}_{cr} = \frac{4\pi^2}{\sqrt{4 - 36\alpha - 27 - 1}}. \hspace{1cm} (> \lambda_{cr}). \hspace{1cm} (9)
\]

Both Eq. (8) and (9) are summarized in Table II.

If the finite term proportional to \( \Omega^2 \) in \( \mathcal{I}^{(2)}(\Omega) \) existed, the \( \alpha \) in \( \mu^2_{\Lambda} \) and Jackiw prescriptions would consequently be replaced by \(-2 - \frac{4\pi^2}{\lambda} \) and \(-\frac{4}{3} - \frac{4\pi^2}{\lambda} \) in the following parameterization of Eq. (5)^2
\[
\mathcal{I}^{(A)}(\Omega) = \frac{1}{(4\pi)^{4}}\left\{ 2\Lambda^2 - \frac{3\Omega^2}{2} \ln \frac{\Omega^2}{\Lambda^2} + 3\Omega^2 \ln \frac{\Omega^2}{\Lambda^2} + C\Omega^2 + o(\Lambda^{-2}) \right\}. \hspace{1cm} (10)
\]

^2In Coleman-Weinberg prescription the definition of \( \alpha = \frac{4\pi^2}{\lambda} \) is not altered due to its special renormalization condition [7,9].
It is not difficult to see that a different value of $\alpha$ means different symmetry breaking status, hence different physics, a nonperturbative scheme dependence problem as emphasized in ref. [6]. It is therefore important to determine the value of $C$ in cutoff scheme in order to be sure that the Jackiw and $\mu^2$ prescription are really consistent symmetry breaking in two loop effective potential. Thus the rest of this report will be devoted to estimate this constant term numerically.

### III. STRATEGY FOR NUMERICAL ESTIMATION OF C

First let us put the sunset diagram into the following form after integrating out the angular variables,

$$I_2(M) = \frac{1}{2(4\pi)^4} \int_0^\beta \frac{dx}{x} \int_0^\beta \frac{dy}{y} \left\{ x + y + M - \sqrt{(x+y+M)^2 - 4xy} \right\}$$

(11)

with $M \equiv \Omega^2$, $x \equiv k^2$, $y \equiv l^2$ and $\beta \equiv \Lambda^2$. The asymptotic form of this integral as cutoff tends to infinity reads

$$I_2(M) = \frac{1}{(4\pi)^4} \left\{ 2\beta - \frac{3M}{2} \ln \frac{M}{\beta} + 3M \ln \frac{M}{\beta} + CM \right\} + o(M/\beta)$$

$$\equiv I_2^{(asy)}(M) + \frac{CM}{(4\pi)^4} + o(M/\beta).$$

(12)

In order to determine the constant $C$ we compare $I_2(M)$ and $I_2^{(asy)}(M)$ after we evaluated the former numerically.

The feasibility of numerical treatment of an apparently divergent integral is guaranteed by the observation that the cutoff need not be very large. Eq.( 12) tells us that the analytical source of error comes from computer. Thus we will first perform one integral analytically (this is feasible in most cases) while leave the remaining one to be spared both the computer time and numerical error we should perform the analytical calculation as far as we can.

The step too small will improve the precision analytically but cost a lot of time and might accumulate numerical errors, especially in personal computer. Usually a divergent integral has a smooth integrand except in a finite range where the integrand is steeply changing. The step must be small enough within such steep regions. The last concern is to spare both the computer time and numerical error we should perform the analytical calculation as far as we can.

In the following we will describe one approach of our numerical treatment in detail, the one with differentiation of the constant $C$. There is a main obstacle in the practice for numerical estimation of divergent integral with respect to $M$ in order to be sure that the Jackiw and $\mu^2$ prescription are really consistent symmetry breaking in the finite numbers. We must get rid of such pieces. The remaining logarithmic terms save most of the computer’s capacity in order to get a finite number at a precision of $10^{-4}$. The larger the cutoff is chosen, the less precision capacity is left over for finite numbers. We must get rid of such pieces. The remaining logarithmic terms save most of the computer capacity for the finite numbers, as even taking the cutoff as large as $10^{10}M$ will only yield a double log term at order $\sim 100 \ln 10 \sim 230$, which consumes almost no precision capacity. This goal could be readily achieved at least in two ways: for the sunset diagram, the power law term $\sim 2\beta$ could be removed either by subtracting it explicitly in the integrand or by differentiating the whole integral with respect to $M$. In fact we will report both approaches’ result.

There is another subtle point that need careful consideration, the step length for the numerical integration. Taking the step too small will improve the precision analytically but cost a lot of time and might accumulate numerical errors, especially in personal computer. Usually a divergent integral has a smooth integrand except in a finite range where the integrand is steeply changing. The step must be small enough within such steep regions. The last concern is to spare both the computer time and numerical error we should perform the analytical calculation as far as we can. Thus we will first perform one integral analytically (this is feasible in most cases) while leave the remaining one to be computed.

In the following we will describe one approach of our numerical treatment in detail, the one with differentiation with respect to $M$. That is, to extract $C$ we just consider the following double integral which at most possesses logarithmic divergence,

$$\frac{\partial I_2(M)}{\partial M} = - \frac{1}{(4\pi)^4} \left\{ I_1 - I_2 + \frac{1}{2}(I_3 - I_4) \right\} = \frac{\partial I_2^{(asy)}(M)}{\partial M} + \frac{C}{(4\pi)^4} + o(M/\beta).$$

(13)

$$I_1 \equiv \int_0^\beta \frac{dx}{(x+M)^2} \int_0^\beta \frac{dy}{y+M}(x+y+M),$$

(14)

$$I_2 \equiv \int_0^\beta \frac{dx}{(x+M)^2} \int_0^\beta \frac{dy}{y+M}(x+y+M)\sqrt{(x+y+M)^2 - 4xy},$$

(15)

$$I_3 \equiv \int_0^\beta \frac{dx}{x+M} \int_0^\beta \frac{dy}{y+M},$$

(16)

$$I_4 \equiv \int_0^\beta \frac{dx}{x+M} \int_0^\beta \frac{dy}{y+M}\frac{x+y+M}{\sqrt{(x+y+M)^2 - 4xy}}.$$
It is not difficult to see that the $2\beta$ piece is removed in all the four integrals $I_1, I_2, I_3$ and $I_4$ there is at most double log terms. Then it is also easy to perform two folds integrals once. Moreover, $I_1$ and $I_3$ can be done analytically at all. Then the results read
\begin{align}
I_1 &= \ln_{\frac{\beta + M}{M}} \left\{ \ln_{\frac{\beta + M}{M}} + \frac{M}{\beta + M} - 1 \right\} + \frac{\beta^2}{M(M + \beta)}, \\
I_2 &= \int_0^\beta \frac{dx}{x + M} \left\{ -\frac{\sqrt{R}}{\sqrt{a}} \sinh^{-1} \frac{a - xY}{2\sqrt{xM}} + \sinh^{-1} \frac{Y - x}{2\sqrt{xM}} \right\} \|_{Y = \beta + M} \|_{Y = M}, \\
I_3 &= \ln_{\frac{2}{M}} \beta + M, \\
I_4 &= \int_0^\beta \frac{dx}{x + M} \left\{ -\frac{x}{\sqrt{a}} \sinh^{-1} \frac{a - xY}{2\sqrt{xM}} + \sinh^{-1} \frac{Y - x}{2\sqrt{xM}} \right\} \|_{Y = \beta + M} \|_{Y = M}
\end{align}
with $R \equiv a - 2xY + Y^2$, $a \equiv x^2 + 4xM$. There is a seemingly singularity in the integrands ($\sim \frac{1}{\sqrt{xM}}$ as $x \sim 0$) in both $I_2$ and $I_4$ which does not materialize after one completes the operations on the variable $Y$,
\begin{align}
I_2 &= \int_0^\beta \frac{dx}{x + M} \left\{ -\frac{\sqrt{R}}{\sqrt{a}} \sinh^{-1} \frac{a - xY}{2\sqrt{xM}} + \sinh^{-1} \frac{Y - x}{2\sqrt{xM}} \right\} \|_{Y = \beta + M} \|_{Y = M}, \\
I_4 &= \int_0^\beta \frac{dx}{x + M} \left\{ -\frac{x}{\sqrt{a}} \sinh^{-1} \frac{a - xY}{2\sqrt{xM}} + \sinh^{-1} \frac{Y - x}{2\sqrt{xM}} \right\} \|_{Y = \beta + M} \|_{Y = M}, \\
u &= \frac{x(x + 3M - \beta)}{2(\beta + M)\sqrt{xM}}, \quad v = \frac{x^2 + 3xM}{2M\sqrt{xM}}, \quad s = \frac{\beta + M - x}{2\sqrt{xM}}, \quad t = \frac{M - x}{2\sqrt{xM}}.
\end{align}

The only necessary numerical works to be done are the two one fold integrals defined in Eq. (22, 23). As the integrand become steeper as $x$ is closer to the origin, thus to reduce the error from numerical integration we separate the integral into several intervals in order to match between the step size and the steepness of the integrand. Taking $M$ to be 1 will make the numerical integration easy and the $C$ has been estimated with two cutoff scale, $\beta/M \equiv \Lambda^2/\Omega^2 = 10^4$, $10^5$. The match between the interval and step size is shown in Table III and IV respectively for $\beta/M = 10^4$ and $\beta/M = 10^5$.

Subtracting $\frac{\partial I_{\text{asy}}(\nu)}{\partial M}$ from the numerically obtained integral defined in Eq. (13) we find the constant $C$: $C = 4.13048$ for $\beta/M = 10^4$ and $C = 4.15412$ for $\beta/M = 10^5$. The interval $(0, 10^{-6})$ is not included as its contribution could at most be of order $10^{-4}$. Noting that one cutoff is order greater than the other, the agreement of the two cases is striking. Moreover, if one try another approach, i.e., by directly subtracting $I_3$'s integrand a term that upon integration will yield the $2\beta$ piece, one could also obtain an estimate of $C$ with the results read: for $\beta/M = 10^4$, $C = 4.14128$ with a homogeneous step $10^{-4}$, $C = 4.14274$ with a homogeneous step $10^{-3}$, $C = 4.156178$ with a homogeneous step $10^{-2}$; for $\beta/M = 10^5$, $C = 4.160739$ with a homogeneous step $10^{-2}$, $C = 4.1633$ with a homogeneous step 0.005; for $\beta/M = 10^6$, $C = 4.162993$ with a homogeneous step $10^{-2}$. Combining the two approaches' results we could safely conclude that $C$ is not zero. To be more conservative we anticipate that the true value should lie in the interval [4, 4.3].

Before ending this section, we should point out and stress that in order to evaluate divergent multiloop integrals numerically in cutoff regularization: (i) one does not need a very large cutoff scale, (ii) the power law term can be easily removed in order to save capacity for more important task (like determination of the finite parts). The numerical workload could further be reduced if one varies the step size with steepness of the integrand.

IV. NEW PREDICTIONS AND SUMMARY

With the new constant we could reevaluate the critical couplings for symmetry breaking with the result summarized in Table V and VI. We take $C$ to be 4 when calculating the critical couplings. It is not difficult to see that the critical couplings in $\mu_2^2$ and Jackiw prescriptions are dramatically lowered, or the symmetry breaking might take place at much lower values of the coupling. The interesting thing is, the $\Delta S$ critical coupling, which is smaller than the ones in the prescriptions based on old calculation of the sunset diagram [7], now becomes a larger one than the obtained with the new numerically determined constant $C$, and the larger the $C$ is, the smaller the critical couplings are. In this sense, the cutoff regularization is preferable to DR, just like the case in the EFT applications in nucleon interactions [10].
Generally one would expect the critical coupling to be close to 1 instead of being very large (like 10.7) for symmetry breaking to take place. Therefore it is a pleasing finding that constant $C$ is larger than 0 and is no less than 4. If $C = 20$, a reasonable value, the critical coupling could be 1.5238 and 1.5542. Of course, as the potential is only calculated at two loop level, the precise value of the critical coupling could not be taken very seriously. However, one should still be pleased to see that the critical couplings are dramatically lowered after more careful evaluation in the regularization scheme that is widely held as inferior to DR, even though the computation is not exact one.

In summary, we reevaluated the sunset diagram in $\lambda \phi^4$ in the cutoff regularization half analytically and half numerically and found that the finite local constant, which is usually taken to be zero, is not zero and this constant could dramatically lower the critical couplings for dynamical symmetry breaking in the two loop effective potential in a number of renormalization prescriptions. The important byproduct is that the numerical calculation could be efficiently done in cutoff regularization for divergent multiloop diagrams and we have illustrated that the cutoff scale need not be too large.

### TABLE I. $\alpha$ in various schemes

| Scheme       | $\alpha$   |
|--------------|------------|
| $\overline{MS}$ | $-2.6878$ |
| $\mu^2_\Lambda$ | $-2$      |
| Jackiw       | $-\frac{5}{4}$ |
| Coleman-Weinberg | $\frac{5}{4}$ |

### TABLE II. Critical values of $\lambda$

| Scheme       | $\lambda_{cr}$ | $\hat{\lambda}_{cr}$ |
|--------------|-----------------|-----------------------|
| $\overline{MS}$ | 4.368           | 5.2024                |
| $\mu^2_\Lambda$ | 5.1152          | 6.5797                |
| Jackiw       | 6.5797          | 10.698                |

### TABLE III. Match I for $\beta/M = 10^4$

| Interval     | $10^{-8}$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |
|--------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Step Size    | $10^{-8}$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |

### TABLE IV. Match II for $\beta/M = 10^5$

| Interval     | $10^{-8}$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |
|--------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Step Size    | $10^{-8}$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-3}$ | $10^{-2}$ |
TABLE V. New $\alpha$’s

| Scheme          | $\alpha$  |
|-----------------|-----------|
| $\overline{MS}$ | -2.6878   |
| $\mu_A^2$       | $-7^1_3$  |
| Jackiw          | $-6^1_2$  |
| Coleman-Weinberg| 16        |

TABLE VI. New critical values of $\lambda$

| Scheme | $\lambda_{cr}$ $\hat{\lambda}_{cr}$ |
|--------|----------------------------------------|
| $\overline{MS}$ | 4.368 5.2024 |
| $\mu_A^2$     | 2.5684 2.7181 |
| Jackiw        | 2.7181 2.8967 |

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