Recent Advances in Dyson-Schwinger Studies*

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(Dated: 11 February 2003)

There have been many demonstrations of the utility of the Dyson-Schwinger equations of QCD as a systematic, phenomenological framework for describing the perturbative and non-perturbative dynamics of hadrons in terms of Euclidean Green functions of quarks and gluons. Still, there remain some unanswered questions regarding the theoretical underpinnings of the approach. I review several studies that are shedding light on how such questions involving the connection between the DSEs and their use in phenomenological applications might be resolved, and then review predictions for some exotic meson states.

I. INTRODUCTION

The Dyson-Schwinger equations (DSEs) are an infinite set of coupled integral equations that relate all of the dressed n-point functions of a quantum field theory to each other. These n-point functions (which take their enumeration from the number of space-time points they depend upon) determine all of the dynamics of the theory. With no external particle sources present, the most elementary of the n-point functions of QCD are the dressed quark and gluon propagators \( S(x_1-x_2) \) and \( D_{\mu\nu}(x_1-x_2) \).

A central aspect of QCD is that the elementary degrees of freedom appearing in the Lagrangian (quarks and gluons) are not simply related to the observed asymptotic degrees of freedom, the hadrons (mesons and baryons). This is referred to as confinement and understanding its origin represents one of the outstanding theoretical problems in physics. It then follows that knowledge of elementary two-point functions, the dressed quark and gluon propagators might not, by itself, be very useful since one can never observe quarks and gluons propagating directly. Clearly, n-point functions such as the fully-interacting quark-antiquark \( M(x_1,\ldots,x_n) \) and three-quark scattering amplitudes \( H(x_1,\ldots,x_6) \), which are related to meson and baryon observables, may be more interesting to explore. However, the self-coupled nature of the Dyson-Schwinger equations entails that these four-point and six-point functions are related to other n-point functions including the two-point dressed quark and gluon propagators \( S(x_1-x_2) \) and \( D_{\mu\nu}(x_1-x_2) \). Understanding the nature of this set of coupled, non-linear integral equations is a daunting task, but advances are continually being made. In the following, a summary of some of the most recent advances is provided.

One traces the study of the DSEs of QCD back to 1950s and 1960’s. These studies were crucial to understanding connections between field theories and observables, but applications to hadron phenomena were only moderately successful due to the lack of computational power available at the time. In the 1980’s, models formulated using the DSEs to describe the quark dynamics within hadrons, were found to provide good and compact descriptions of the light pseudoscalar and vector mesons. After these early phenomenological success, there followed a period of theoretical studies of the DSEs which explored the role of gluons, gauge invariance and confinement, which continues to this day [1, 2].

However, since the mid 1990’s, much focus has turned to the study of physical processes which now permeates almost every subfield of hadron physics including: chiral physics and pion decays [3], \( \pi \pi \) scattering [4], electromagnetic form factors of the light pseudoscalar [5] and light-vector meson octets [6], inelastic scattering and quark-parton distributions [7], high-energy diffractive electro-production and hadron scattering [8], finite temperature studies of QCD [9], exotic mesons [10], baryon bound states, their properties and electromagnetic form factors [11], low-energy meson photo-production [12], and many others.

Yet, a complete understanding of some theoretical aspects of the framework has proven elusive. These unresolved questions include: How does one construct systematic and convergent expansions for the kernels of the DSEs? Are DSE and lattice QCD studies complementary approaches to the study of QCD phenomena? How does one calculate observables in Minkowski space from a quantum field theory based in Euclidean space? In the following, I give a brief account of recent studies that provide some answers to these questions. Finally, I report on an application of DSEs to exotic mesons.

II. ROBUSTNESS OF TRUNCATION SCHEMES

The simplest DSE is the Dyson equation,

\[
S^{-1}(p) = Z_2(\zeta, \Lambda) i\gamma \cdot p + Z_4(\zeta, \Lambda) m(\zeta) \\
+ Z_1(\zeta, \Lambda) \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \frac{\lambda^a}{2} \gamma_\mu \Sigma(q) \Gamma^a_\nu(q, p),
\]

which gives the dressed quark propagator \( S(p) \). In Eq. (1), \( \lambda^a \) for \( a = 1, \ldots, 8 \) are the Gell-Mann SU(3)-color matrices, \( \Gamma^a_\nu(q, p) \) is the dressed quark-gluon vertex and \( m(\zeta) \) is the current quark mass. A translation-invariant

*Plenary talk given at NStar-2002, Pittsburgh PA, October 2002.
regularization scheme with scale $\Lambda$ is employed to render the integral finite. The dressed quark propagator is of the form,

$$S(p) = \frac{Z(p^2)}{i\gamma \cdot p + M(p^2)},$$  

(2)

where $Z(p^2)$ is the quark wave-function renormalization and $M(p^2)$ is the running quark mass. The dependence of Eq. (1) on the ultraviolet scale $\Lambda$ is removed by renormalizing at $p^2 = \zeta^2$, subject to the boundary conditions $Z(\zeta^2) = 1$ and $M(\zeta^2) = m(\zeta)$, and absorbing the $\Lambda$-dependence into the multiplicative-renormalization constants $Z_i(\zeta, \Lambda)$.

The dressed quark propagator is determined from Eq. (1) only after the dressed gluon propagator $D_{\mu\nu}(p-q)$ and dressed quark-gluon vertex $\Gamma^a_\nu(q, p)$ are known. However, these two- and three-point functions are solutions of their own DSEs, which in turn depend on still higher $n$-point functions. This is the infinitely-coupled nature of the DSEs, making it impossible to obtain a complete solution of QCD directly.

In phenomenological applications, one may proceed by making assumptions regarding some subset of $n$-point functions such that the DSEs are reduced to a closed system of equations which may be solved directly. The simplest of such truncation schemes is the rainbow approximation. It is equivalent to making the substitution,

$$Z_i(\zeta, \Lambda) g^2 \Gamma^a_\nu(q, p) \rightarrow \frac{\lambda^2}{2} \gamma_\nu V((q - p)^2),$$  

(3)

in Eq. (1), where $V(k^2)$ is an effective quark-gluon vertex. The exact dressed quark-gluon vertex depends on eight distinct Dirac-$\gamma$ matrices and on three variables $q^2$, $p^2$ and $q \cdot p$, while the effective vertex in Eq. (3) maintains a single Dirac-$\gamma$ matrix and depends on $(q - p)^2$ only.

Assuming a given form for the dressed gluon propagator $D_{\mu\nu}(k)$ and effective vertex $V(k^2)$, the quark propagator can be calculated from Eq. (1). Then, one considers how a quark and antiquark combine to form a meson. This is described by the homogeneous Bethe-Salpeter equation (BSE),

$$\lambda(P^2)\Gamma(p; P) = \int \frac{d^4q}{(2\pi)^4} K(p, q; P) S(q_+)\Gamma(q; P)S(q_-),$$  

(4)

where $K(p, q; P)$ is the two-particle irreducible $q\bar{q}$ scattering kernel, $\Gamma(p; P)$ is the BS amplitude, and $q_\pm = q \pm P/2$ are the momenta of the quark and antiquark. This can be solved for any value of meson momentum $P$, but when the eigenvalue $\lambda(P^2) = 1$, Eq. (4) predicts the existence of a bound state meson with mass $m = \sqrt{-P^2}$.

One can show that when the rainbow approximation is employed in Eq. (1), the ladder approximation for the Bethe-Salpeter kernel,

$$K(p, q; P) \rightarrow \frac{\lambda^2}{2} \gamma_\mu \otimes \frac{\lambda^2}{2} \gamma_\nu D_{\mu\nu}(q-p) V((q - p)^2),$$  

(5)

must be employed in order to satisfy the axial-vector Ward-Takahashi identity and Goldstone theorem\textsuperscript{[13]}. (A review may be found in Ref. \textsuperscript{[3]}. Furthermore, it can be argued that the rainbow-ladder approximation represents the zeroth order term in a systematic series of approximations that preserve the Goldstone theorem. However, what is truly amazing is that this series of truncations converges at a surprisingly rapid rate\textsuperscript{[13, 14]}!

To illustrate how such truncations might converge, one compares solutions for the dressed quark propagator, and $\pi$- and $\rho$-meson masses obtained using the ladder-rainbow truncations to those obtained when higher-order terms are maintained in Eqs. (4) and (5). In Ref. \textsuperscript{[14]}, this comparison was carried out using a simple model for the dressed gluon propagator $D_{\mu\nu}(q-p)$. The quark-gluon vertex was dressed by a one-gluon loop, two-gluon loops and then with infinitely-many-gluon loops (using a recursive technique). The BSE kernel was constructed using the approach outlined in Ref. \textsuperscript{[13]}, which ensured that the Goldstone theorem and axial-vector Ward-Takahashi identity were maintained at each level of the truncation.

The study demonstrates that $\pi$ and $\rho$ meson bound state masses are well-described by the rainbow-ladder approximation without significant change when dressing of the quark-gluon vertex is included\textsuperscript{[14]}. (In retrospect, this explains the success enjoyed by the many studies of light pseudoscalar and vector mesons which have employed the ladder-rainbow approximation.) The study also found that deeper in the time-like region, the dressing of the quark-gluon vertex may become more significant, but is still well-approximated by including only one-gluon or two-gluon loops; that is, results for the quark propagator and the BSE obtained by using the recursively dressed quark-gluon vertex were very similar to results obtained using just two-gluon loops to dress the quark-gluon vertex\textsuperscript{[14]}.

The conclusion to be drawn from this study is that there is a systematic method for improving truncation schemes employed in phenomenological studies that allows a quantitative measure of their robustness. This achievement is quite important and represents a leap forward in the use of Dyson-Schwinger equations as a phenomenological tool. However, there is no free ride. Not all channels are equally robust or insensitive to the truncation schemes. Rather, pseudoscalar and vector channels are particularly stable in this respect. There are other channels for which the inclusion of dressings arising from a higher number of gluons has significant impact. Examples of such channels are the scalar and vector colored-diquark channels\textsuperscript{[15, 14]}. Therefore, in practice one should check each channel of interest to verify that results are stable under changes to the level of truncation employed.
III. CONNECTION TO CALCULATION ON THE LATTICE

The similarities between studies of DSEs and numerical simulations of QCD on the lattice are clear \[1,13,16\]. Both explore QCD from its Lagrangian in terms of quarks and gluon degrees of freedom and both are formulated using the Euclidean metric where the square of the momentum \( p^2 = p_\mu p^\mu = p \cdot p = \sum_{i=1}^{4} p_i^2 > 0 \) is space-like. In principle, one might be able to compare results obtained from lattice QCD simulations to those obtained from the DSEs directly. In particular, lattice results may be used as input for DSE calculations or might provide further tests of the robustness of truncation schemes. Alternatively, Dyson-Schwinger studies may provide additional insights into the role of some symmetries damaged by lattice QCD simulations; for example, it is possible to use the Dyson equation \[11\] to extrapolate lattice results for the quark propagator to the chiral limit of vanishing quark mass.

This was carried out recently \[17\] using a parametrization of the Landau-gauge dressed gluon propagator \( D_{\mu\nu}(k) \) obtained from a lattice QCD simulation \[18\], and a simplified form of the dressed quark-gluon vertex of the form Eq. (3) and parametrized by

\[
V(k^2) = \left( 1 + \frac{a(m(\zeta))}{k^2} \right) \left( 1 + \frac{b}{k^2} \right)^{-1} + \cdots ,
\]

where logarithmic perturbative-QCD corrections have been suppressed, \( a \) and \( b \) are parameters determined by using this vertex and the lattice gluon propagator \( D_{\mu\nu}(k) \) in Eq. (1) to fit the dressed quark propagator functions \( Z(p^2) \) and \( M(p^2) \). The resulting running quark mass functions \( M(p^2) \) are compared to those obtained from the lattice simulation \[19\] for a collection of current quark masses \( m(\zeta) = 27, 50 \) and 102 MeV, as shown by the upper three curves in the left panel of Fig. 1. Self-consistency with lattice results required a linear extrapolation of the lattice results (circles).

This disagreement is caused by having employed too simple a form for the dressed quark-gluon vertex in Eq. (3), or from the inadequacy of a linear extrapolation of lattice data to the chiral limit. Using the Gell-Mann–Oakes–Renner relation as a check on self-consistency of the DSE approach suggests the latter may be true \[17\]. Reconciliation of these approaches requires additional investigation, but it is clear that such comparisons are important and will provide new insights for both Dyson-Schwinger and lattice communities in the future.

IV. ANALYTIC CONTINUATION INTO MINKOWSKI SPACE

Another issue, which has received less attention than those discussed above, concerns the mechanism by which a quantum field theory, formulated in Euclidean space, is related to hadron observables in Minkowski space. Analytic continuation is the method by which observables in Minkowski space are obtained from the Euclidean amplitudes or Green functions. For example, the \( \rho \)-meson bound state mass \( m_\rho \) corresponds to finding the value of \( P^2 \) for which the BSE (4) eigenvalue \( \lambda(P^2 = -m_\rho^2) = 1 \). Since the Dyson-Schwinger framework is defined for Euclidean momenta \( P^2 > 0 \), one must analytically continue the Bethe-Salpeter eigenvalue \( \lambda(P^2) \) to time-like momenta \( P^2 = -m_\rho^2 < 0 \).

One way in which this process of analytic continuation may be understood is by allowing the fourth component of the Euclidean vector \( P_\mu \) to become complex \( P_4 \rightarrow iE \) where \( E > 0 \). As a result of this continuation, during the integration over the (Euclidean) relative-quark momentum \( q_\mu \), the arguments of the quark propagators...
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continities and/or cuts in the complex-momentum plane [21].
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complex momentum plane. For light pseudoscalar (π and K) mesons, the excursion into the complex plane is minor
and so is easy to handle. For the light vector mesons (ρ, ω, K* and ϕ), this process can still be carried out
by means of straight-forward techniques, albeit numerically intensive [21]. However, for mesons with masses sig-
nificantly greater than 1 GeV, one may encounter poles and/or cuts in the quark propagator during this process
of analytic continuation.

It has long been thought that the dressed quark propagator may contain complex-conjugate pairs of singular-
ities and/or cuts in the complex-momentum plane [21].
To explore how one might carry out an analytic con-
tinuation of the BSE [4] in this event, we considered a simple model form for $S(q)$ by N pairs of complex-
conjugate poles,

$$S(q) = \sum_{n=1}^{N} \left( \frac{z_n^{*}}{i\gamma \cdot \bar{q} + m_n} + \frac{z_n}{i\gamma \cdot q + m_n^{*}} \right),$$  \hspace{1cm} (7)

where $m_n$ are complex-valued mass scales and $z_n$ are complex-valued magnitudes [22].

For the simple form given in Eq. (7) and a phenomenologically successful BSE kernel [20], it was shown that one can deform the contour integrations thereby avoiding quark singularities, a necessary step to obtain the analytically continued eigenvalue $\lambda(P^2)$ for all values of momenta $-\infty < P^2 < +\infty$ [22]. The $P^2$ dependence of the resulting eigenvalue for the ground-state vector meson is shown as a solid curve in the right panel of Fig. 2. For comparison, the dashed curve in Fig. 2 was obtained by completely ignoring the need to deform the contour around poles when carrying out the integration in the BSE [4]; as this is not the correct implementation of analytic continuation, the resulting dashed curve is not correct. The two curves are in perfect agreement for momenta $P^2 > 4Re(m_1^2)$, where $m_1$ is the lightest quark mass parameter from Eq. (4). For such momenta, no quark singularities are encountered during the integration and so, the curves coincide here. For momenta more time-like than this, the two curves diverge and the solid curve is the correct analytically-continued eigenvalue $\lambda(P^2)$. The ground state vector meson mass $m_\rho$ is the solution of $\lambda(P^2 = -m_\rho^2) = 1$.

The Dyson-Schwinger framework is a renormalizable, Euclidean approach to the study of QCD. As such, one makes contact with observables by analytically continuing obtained amplitudes into Minkowski space. Using recently developed numerical methods, one can perform analytic continuations of the BSE, even in the presence of singularities in the quark propagators [22]. We now have the means to explore meson and baryon bound states of much heavier masses than previously possible.

V. EXOTIC MESONS FROM THE BETHE-SALPETER EQUATION

Before summarizing, I provide one example of a simple model calculation of the meson spectrum. Studies of hadrons (both baryons and mesons) within the Dyson-Schwinger framework are too numerous to describe adequately in the space provided here. An extensive review, appearing recently, provides an excellent summary of the current status of such studies [3]. Rather than reiterate aspects of these studies, I give a brief accounting of a recent study of exotic mesons not covered therein.

The most famous of exotic mesons is probably the $\sigma_1(1400)$, the iso-vector, vector meson with positive charge-conjugation parity and quantum numbers $J^{PC} = 1^{-+}$. The classification of “exotic” belongs to those mesons with quantum numbers that can not be constructed as a quark-antiquark state. A quark-antiquark state $|q\bar{q}\rangle$ with relative spin $S$ and angular momentum $L$ will have a spatial parity $P = (-1)^{L+1}$ and charge-
conjugation parity of $C = (-1)^{L+S}$. To generate such a state, one must include additional degrees of freedom. In the quark model, this may be accomplished by introducing either constituent gluons or flux-tube excitations or perhaps, by considering composite systems like $|qqqq\rangle$.

In a quantum field theoretic framework, such particles are generated by constructing operators with the desired $J^{PC} = 1^{-+}$ quantum numbers. One such operator is

$$\bar{\psi}(x)\gamma^\mu D^\mu_\nu \psi(x)$$

where $\psi(x)$ is the quark field and $D^\mu_\nu$ is the fourth component of the covariant derivative $D^\mu_\nu = \partial^\mu + igA^\mu_\nu$. The first part of this operator may be used to project out the exotic meson bound state solution from the Bethe-Salpeter equation. Predictions for the masses of two exotic vector mesons were obtained [10] using the simple separable model [23], which provides an excellent description of the light pseudoscalar, vector and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The vector-meson ground state BSE eigenvalue $\lambda(P^2)$ for a range of meson momenta $P^2$ (solid curve). The meson mass $m_\rho$ is calculated using $\lambda(-m_\rho^2) = 1$, as indicated by the vertical line [22].}
\end{figure}
axial-vector meson spectra as shown in Fig. 3. One exotic pseudoscalar and two exotic vector mesons are predicted.

VI. SUMMARY

Recent theoretical advances are putting the Dyson-Schwinger framework on firmer ground and in closer contact with complementary frameworks, such as lattice QCD simulations. Future studies will continue to increase our ability to use this framework as a phenomenological tool for exploring the dynamics of quarks and gluons in hadron processes.

Acknowledgements: This work is supported by the National Science Foundation under contract number PHY-0071361. Some original work reported herein resulted from collaborations with M.S. Bhagwat and P.C. Tandy.

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