A note on the hyperbolic singular value decomposition without hyperexchange matrices

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Abstract

We present a new formulation of the hyperbolic singular value decomposition (HSVD) for an arbitrary complex (or real) matrix without using the concept of hyperexchange matrices and using only the concept of pseudo-unitary (or pseudo-orthogonal) matrices. The new formulation allows us to present an algorithm for computing HSVD in the general case. The new formulation is more natural and useful for some applications.

Keywords: hyperbolic singular value decomposition, SVD, hyperexchange matrices, pseudo-unitary group, pseudo-orthogonal group

2010 MSC: 15A18, 15A23

1. Introduction

This paper contains two new results. The first result is the presentation of a new formulation of HSVD for an arbitrary complex (or real) matrix (see Theorems 2 and 3) without using the concept of hyperexchange matrices and using only the concept of pseudo-unitary (or pseudo-orthogonal) matrices. In the literature, you can find the formulation of HSVD with using pseudo-unitary matrices instead of hyperexchange matrices in the particular case of full column rank matrices. The same statement with using pseudo-unitary matrices will be not correct for the general case of an arbitrary complex (or real) matrix. We obtain a new formulation of HSVD without using hyperexchange matrices for the general case. The second result is the presentation

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of the algorithm for computing HSVD in the general case. The standard formulation of HSVD (see Theorem 1) does not allow us to obtain an algorithm for computing HSVD because the matrix $V$ is hyperexchange with five parameters $j, l, t, k,$ and $s$, some of which, as it turns out in this paper, are redundant. The new formulation of HSVD (with three parameters $j, l,$ and $t$) allows us to obtain such algorithm.

The paper is organized as follows. In Section 2 we present the well-known formulation of HSVD with some remarks. We discuss that only three parameters among $j, l, t, k,$ and $s$ are important. In Section 3 we discuss hyperexchange matrices and that if we replace hyperexchange matrices by corresponding pseudo-unitary matrices in the standard formulation of HSVD, then it will be not correct for the general case of an arbitrary complex (or real) matrix. In Section 4 we present a new formulation of HSVD without using the concept of hyperexchange matrices in the general case. In Section 5 we present the algorithm for computing HSVD in the general case. The conclusions follow in Section 6.

2. On the standard formulation of HSVD with some remarks

In the current paper, we give all statements for the complex case. All statements will be correct if we replace complex matrices by the corresponding real matrices, the operation of Hermitian conjugation $^H$ by the operation of transpose $^T$, the following unitary groups ($m = p + q$)

$$U(n) = \{A \in \mathbb{C}^{n \times n}, A^H A = I\}, \quad U(p, q) = \{A \in \mathbb{C}^{m \times m}, A^H JA = J\}$$

by the corresponding orthogonal groups

$$O(n) = \{A \in \mathbb{R}^{n \times n}, A^T A = I\}, \quad O(p, q) = \{A \in \mathbb{R}^{m \times m}, A^T JA = J\}.$$  

We denote the identity matrix of the size $n$ by $I = I_n = \text{diag}(1, \ldots, 1)$ and the diagonal matrix with $+1$ appearing $p$ times and $-1$ appearing $q$ times on the diagonal by $J = J_m = \text{diag}(I_p, -I_q)$, $p + q = m$. One calls the group $O(p, q)$ a pseudo-orthogonal group, an indefinite orthogonal group, or a group of $J$-orthogonal matrices [3], [5]. There are also various names of the group $U(p, q)$: a pseudo-unitary group, an indefinite unitary group, a group of $J$-unitary matrices, a group of hypernormal matrices [1].

The first formulation of HSVD was done by R. Onn, A. O. Steinhardt and A. W. Bojanczyk in [3] for the particular case $m \geq n$, $\text{rank}(AJA^H) =$
rank($A$) = $n$ (the notation as in Theorem 1). In this particular case, $j = 0$ and the matrix $\Sigma$ is diagonal with all positive diagonal elements. In [9], the same three authors formulate the statement for a slightly more general case of arbitrary $m$ and $n$, rank($AJA^H$) = rank($A$) = min($m, n$). In the third work of the same authors [1], there is a generalization of HSVD to the case rank($AJA^H$) < rank($A$). This generalization uses complex entries of the matrix $\Sigma$. H. Zha [15] indicated that this generalization seems rather unnatural and presented another generalization using only real entries of the matrix $\Sigma$. We discuss this generalization below (see Theorem 1). B. C. Levy [6] presented the statement of Zha’s result in another form using another proof. At the same time, his statement is weaker than the previous one: there are additional arbitrary diagonal matrices instead of the identity matrices $I_j$ in the matrix $\Sigma$; there is no explicit form of the matrix $\hat{J}$ (like (2) in Theorem 1); only the case $m \geq n$ is considered. Also note the results of S. Hassi [4] and V. Sego [12], [11] on other generalizations of SVD to the hyperbolic case.

The most general version of the hyperbolic singular value decomposition (HSVD) is given in [15] by H. Zha. We start this paper with the formulation of Zha’s result (Theorem 1). Later we will discuss a new formulation of HSVD, which is more useful for some applications from our point of view.

**Theorem 1.** Let we have $J = \text{diag}(I_p, -I_q)$, $p + q = m$. For an arbitrary matrix $A \in \mathbb{C}^{n \times m}$, there exist matrices $U \in U(n)$ and $V \in \mathbb{C}^{m \times m}$,

$$V^HJV = \hat{J} := \text{diag}(-I_j, I_j, -I_t, I_{l-t}, I_s, -I_{k-s}),$$

(2)

such that

$$A = U\Sigma V^H,$$

$$\Sigma = \begin{pmatrix}
I_j & I_j & 0 & 0 \\
0 & 0 & D_l & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \in \mathbb{R}^{n \times m},$$

(3)

where $D_l \in \mathbb{R}^{l \times l}$ is a diagonal matrix with all positive diagonal elements, which are uniquely determined. Here we have

$$j = \text{rank}(A) - \text{rank}(AJA^H), \quad l = \text{rank}(AJA^H),$$

$t$ is the number of negative eigenvalues of the matrix $AJA^H$. 

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Remark 1. Note that the statement of Theorem 1 contains parameters $j$, $l$, $t$, $k$, and $s$. H. Zha in his work [15] (see Remark 6) says that there are four important parameters (invariants) $j$, $l$, $k$, $s$. In our opinion, it is more correct to say about not four but three invariants $j$, $l$, and $t$, which we mention in Theorem 1 (or, alternatively, $j$, $l$, and $s$). The numbers $j$, $l$ and $t$ with the diagonal elements of the matrix $D$ uniquely determine HSVD for the fixed $p$ and $m = p + q$. At the same time, the matrices $U$ and $V$ are not uniquely determined in HSVD. The numbers $k$ and $s$ are uniquely determined by $j$, $l$, and $t$:

$$k = m - 2j - l = m - 2\text{rank}(A) + \text{rank}(AJA^H), \quad (4)$$

$$s = p - j - l + t = p - \text{rank}(A) + t. \quad (5)$$

Later we will see this fact again: a new formulation of HSVD (Theorems 2 and 3) does not contain parameters $k$ and $s$. Thus there are three important for HSVD parameters: $j$, $l$, and $t$, which depend on $A$ and $J$.

Really, because of the law of inertia the number $p$ of $+1$ and the number $q$ of $-1$ in the matrices $J$ and $\hat{J}$ are the same. Using $j + l - t + s = p$, we get (5). For determining $k$, we have $2j + l + k = m$ and obtain (4).

Remark 2. Positive numbers on the diagonal matrix $D_l$ (the number of them equals $l$) and zeros on the continuation of this diagonal in the matrix $\Sigma$ (the number of such zeros equals min($m - 2j, n - j$) - $l$) are called hyperbolic singular values. Thus the number of hyperbolic singular values equals min($m - 2j, n - j$) in the general case.

In this paper, we give a generalization of Theorem 1, which is more natural and useful for applications from our point of view.

3. On hyperexchange matrices and HSVD

In [8], a complex matrix $A$ with the condition

$$A^HJA = \hat{J}, \quad (6)$$

where $J = J_m = \text{diag}(I_p, -I_q)$, $p + q = m$, and $\hat{J} = \hat{J}_m$ is a diagonal matrix with entries $\pm1$ in some order, is called a hyperexchange matrix.
Remark 3. Note that sometimes you can find another definition of a hyper-
exchange matrix: $AJA^H = \hat{J}$ (see [6]). The second definition is not equivalent
to the first one (6). Let us multiply the both sides of (6) on the left by $A\hat{J}$
and on the right by $A^{-1}J$. We obtain $A\hat{J}A^H = J$, which differs from the
second definition. Note that the matrix $B = A^{-1}$ satisfies $BJB^H = \hat{J}$.

In the particular case $J = \hat{J}$, $A$ becomes a $J$-unitary matrix $A^HJA = J$.
$J$-unitary matrices form a pseudo-unitary group $U(p, q)$ (1). From
$A^HJA = J$, it follows that $A$ is invertible and $A^{-1} = JA^HJ$. We conclude that $A^HJA = J$ is equivalent to $A^HJA = J$ for a $J$-unitary matrix $A$.

Remark 4. It is not difficult to show that hyperexchange matrices and $J$-
unitary matrices are closely connected: for an arbitrary hyperexchange matrix $A$,
there exists a permutation matrix $S$ such that $AS$ is $J$-unitary.

Really, from the law of inertia, it follows that matrices $J$ and $\hat{J}$ have the
same numbers of 1 and $-1$ on the diagonal. It means that these two matrices
are connected with the aid of some permutation matrix $S$: $\hat{J} = SJS^T$. Let
us remind that a permutation matrix has exactly one nonzero element, equal
to 1, in each column and in each row. A permutation matrix is orthogonal
$S^TS = I$. We get $SJS^T = A^HJA$, i.e. $(AS)^HJ(AS) = J$ and $AS$ is $J$-unitary.

From our point of view, $J$-unitary matrices are more natural than hyper-
exchange matrices, they are widely used in different applications. In the next
section, we give a generalization of Theorem 1 using only $J$-unitary matrices,
without using hyperexchange matrices.

Let us note the following important fact.

Remark 5. If we replace $V^HJV = \hat{J}$ by $V^HJV = J$ in Theorem 1, then the
statement of the theorem will not be correct in the general case.

In other words, we can not change the condition for matrix $V$ from hyper-
exchange to $J$-unitary in the formulation of Theorem 1 in the general case.
Let us give a counterexample for the real case $A \in \mathbb{R}^{n \times m}$.

Let us consider

$$A_{1 \times 2} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad J = \text{diag}(1, -1), \quad n = 1, \quad m = 2.$$
We have \( \text{rank}(A) = 1 \) and \( \text{rank}(AJA^T) = 1 \). Let us prove that there are no matrices \( D, U, \) and \( V \) of the following form

\[
D = \begin{pmatrix} d & 0 \end{pmatrix} \in \mathbb{R}^{1 \times 2}, \quad U = \begin{pmatrix} u \end{pmatrix} \in \mathbb{R}^{1 \times 1}, \quad U^T U = 1,
\]

\[
V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}, \quad V^T J V = J
\]

such that \( A = U D V^T \).

The condition \( V^T J V = J \) is equivalent to

\[
v_{11}^2 = 1 + v_{21}^2, \quad v_{12}^2 = 1 + v_{22}^2, \quad v_{11} v_{12} = v_{21} v_{22}.
\]

We obtain

\[
\begin{pmatrix} 0 & 1 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} u \\ \vdots \end{pmatrix} \begin{pmatrix} d & 0 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix},
\]

i.e. \( udv_{11} = 0 \) and \( udv_{21} = 1 \). Using \( d \neq 0 \) and \( u \neq 0 \), we get \( v_{11} = 0 \), which is in a contradiction with \( v_{11}^2 = 1 + v_{21}^2 \).

**Remark 6.** If we add condition that \( A_{n \times m} \) is a full column rank matrix (we have also \( n \geq m \) and \( j = 0 \) in this case) to the formulation of Theorem [7] then we can replace condition \( V^H J V = \hat{J} \) by \( V^H J V = J \) and the statement of the theorem will be correct.

This particular case is usually considered in the literature (see, for example, [14, 7, 10]). That is why sometimes the formulation of HSVD is given without using the concept of hyperexchange matrix. It seems to us that if the formulation of HSVD is not given for the general case, then it is better to indicate clearly which particular case is considered, because this confuses the unprepared reader. In this section, we try to distinguish the general case and the particular cases for the convenience of the reader. The difference in the use of hyperexchange and \( J \)-unitary matrices in some another interesting generalization of HSVD (so-called two-sided HSVD) is discussed in [11, 12].

Now we are interested in the most general case and the counterexample above shows us that we must use the concept of hyperexchange matrices in the formulation of Theorem [1] In the next section, we give a generalization of Theorem [1] without using the concept of hyperexchange matrices for the general case. Such formulation is more natural and useful for applications.
4. A new formulation of HSVD

**Theorem 2.** Let we have $J = \text{diag}(I_p, -I_q)$, $p + q = m$. For an arbitrary matrix $A \in \mathbb{C}^{n \times m}$, there exist $U \in \mathbb{U}(n)$ and $V \in \mathbb{U}(p, q)$ such that

$$A = U\Sigma V^H, \quad (7)$$

where

$$\Sigma = \begin{pmatrix}
P_{l-t} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_t & 0 \\
0 & I_j & 0 & 0 & I_j \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{n \times m}, \quad (8)$$

where the first block has $p$ columns and the second block has $q$ columns, $P_{l-t}$ and $Q_t$ are diagonal matrices of the corresponding sizes with all positive uniquely determined diagonal elements (up to a permutation).

Moreover, choosing $U$ and $V$, we can change the order of all rows of the matrix $\Sigma$. Also we can change the order of columns in each of two blocks of the matrix $\Sigma$, but we can not change the order of two columns in different blocks. Thus we can always arrange diagonal elements of the matrices $P_{l-t}$ and $Q_t$ in decreasing (or ascending) order\(^1\).

Here we have

$$j = \text{rank}(A) - \text{rank}(AJA^H), \quad l = \text{rank}(AJA^H),$$

and $t$ is the number of negative eigenvalues of the matrix $AJA^H$ (note that $l - t$ is the number of positive eigenvalues of the matrix $AJA^H$).

**Proof.** To prove this theorem, we use Theorem [1] and remarks from the previous section of the paper. If

$$V^HJV = \tilde{J} = \text{diag}(-I_j, I_j, -I_t, I_{l-t}, I_s, -I_{k-s}), \quad (9)$$

then $VS = F$ is a $J$-unitary matrix for some permutation matrix $S^T = S^{-1}$ (see Remark[4]). From $A = U\Sigma V^H$ (3), we obtain $A = U\Sigma SF^H$. Multiplying the matrix $\Sigma$ on the right by $S$, we change the order of its columns. We omit

\(^1\)Alternatively, we can change the order of the first $l$ rows of the matrix $\Sigma$ and obtain all nonzero elements of the first $l$ rows of the matrix $\Sigma$ in decreasing (or ascending) order.
the detailed proof because of its cumbersomeness. The plan is the following. Using \( S^T \hat{J} S = J \) and the explicit form of the matrix \( \hat{J} \) (9), we get the explicit form of the matrix \( S \). Then we calculate the explicit form of the matrix \( \Sigma S \), where \( \Sigma \) is from (3). Finally, we obtain the explicit form of the new matrix \( \Sigma \) (8).

Note that we can also multiply the matrix \( \Sigma \) by an arbitrary permutation matrix \( S' \) on the left \( S' \Sigma \) because \( S' \in O(n) \). Thus we can change the order of rows of the matrix \( \Sigma \) as we want. We can multiply the matrix \( \Sigma \) on the right by an arbitrary permutation matrix of the form

\[
\begin{pmatrix}
S_1 & 0 \\
0 & S_2
\end{pmatrix} \in O(p, q),
\]

where \( S_1 \) and \( S_2 \) are arbitrary permutation matrices of order \( p \) and \( q \) respectively. Thus we can change the order of columns in each of two blocks of the matrix \( \Sigma \).

**Remark 7.** Note that there are no indices \( k \) and \( s \) in the formulation of Theorem (2) (but they were in the formulation of Theorem (1)). These indices do not have any important information on HSVD. HSVD depends only on hyperbolic singular values and three parameters \( j \), \( l \), \( t \) as we have already discussed in Remark (1). At the same time, we note that the matrices \( U \) and \( V \) are not uniquely determined in HSVD.

**Remark 8.** We can change \( V^H \) to \( V \) in (7), because if \( V \in U(p, q) \), then \( V^H \in U(p, q) \). Since analogous reasoning is not correct for hyperexchange matrices, we can not do the same in (3) (see Theorem (2)).

**Remark 9.** The new formulation of HSVD does not use hyperexchange matrices in the general case. However, we have two diagonal matrices \( P \) and \( Q \) in (3) instead of one diagonal matrix \( D \) in (3). This fact has geometrical (or, we can also say, physical) meaning. This becomes more clear after application of HSVD and SVD in physics (see (13)). The matrix \( A \) may describe some tensor field, the matrices \( U \) and \( V \) may describe some (coordinate, gauge) transformations. The matrix \( \Sigma \) describes the same tensor field, but in some new coordinate system and with a new gauge fixing. The blocks \( P \) and \( Q \) of the matrix \( \Sigma \) describe the contributions of the tensor field to (using physical terminology for the case \( p = 1 \) and \( q = 3 \)) “time” (the first \( p \)) and “space” (the last \( q \)) coordinates. Such contributions depend on the number of positive
l − t and negative t eigenvalues of the matrix AJA^H respectively. From the statement of Theorem 1, it was unclear why there are exactly two blocks I_j in (3) in the degenerate case j ≠ 0. Now we know that each of two blocks I_j carries information about degeneration in each of two blocks of the matrix Σ.

For the convenience of the reader, let us give a reformulation of Theorem 2 to the case when a J-unitary matrix is on the left side and a unitary matrix is on the right side (as in [6] but now without using hyperexchange matrices and for the general case).

Theorem 3. Let we have J = diag(I_p, -I_q), p + q = m. For an arbitrary matrix B ∈ C^{m×n}, there exist R ∈ U(n) and L ∈ U(p, q) such that

\[ L^H BR = Σ, \]

where

\[ Σ = \begin{pmatrix} P_{l-t} & 0 & 0 & 0 \\ 0 & 0 & I_j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & Q_t & 0 & 0 \\ 0 & 0 & I_j & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{m×n}, \]

where the first block has p rows and the second block has q rows, P_{l-t} and Q_t are diagonal matrices of the corresponding sizes with all positive uniquely determined diagonal elements (up to a permutation).

Moreover, choosing L and R, we can change the order of all columns of the matrix Σ. Also we can change the order of rows in each of two blocks of the matrix Σ, but we can not change the order of two rows in different blocks. Thus we can always arrange diagonal elements of the matrices P_{l-t} and Q_t in decreasing (or ascending) order.

Here we have

\[ j = \text{rank}(B) - \text{rank}(B^H JB), \quad l = \text{rank}(B^H JB), \]

and t is the number of negative eigenvalues of the matrix B^H JB (note that l − t is the number of positive eigenvalues of the matrix B^H JB).

2Alternatively, we can change the order of the first l columns of the matrix Σ and obtain all nonzero elements of the first l columns of the matrix Σ in decreasing (or ascending) order.
Proof. Using $A = UΣV^H$ (3), we get $A^H = VΣ^TU^H$. Multiplying both sides on the left by $V^{-1}$ and on the right by $U$, we get $V^{-1}A^HU = Σ^T$. Using notation $B = A^H$, $L^H = V^{-1} ∈ U(p,q)$, $R = U ∈ U(n)$, we obtain the statement of the theorem.

5. Computing of HSVD

The new formulation of HSVD (Theorem 2 or 3) allows us to obtain an algorithm for computing HSVD (see Remark 10 and Lemma 1). The standard formulation of HSVD (see Theorem 1) does not allow us to obtain an algorithm for computing HSVD because the matrix $V$ is hyperexchange with five parameters $j$, $l$, $t$, $k$, and $s$, some of which, are redundant (see Section 4).

In this section, we use the formulation of HSVD from Theorem 3. For arbitrary matrix $B ∈ C^{m×n}$, we can easily find matrices $L ∈ U(p,q)$, $R ∈ U(n)$, $Σ ∈ R^{m×n}$ of the form (11) such that $L^HBR = Σ (10)$.

Lemma 1. For the matrices $B$, $L$, $R$, and $Σ$ from Theorem 3, we have the following equations:

$$(B^HJB)R = R(Σ^TJΣ), \quad (JBB^H)L = L(JΣΣ^T).$$

(12)

Proof. From (10), we obtain

$$R^HB^HL = Σ^T.$$  

(13)

Multiplying on the left by $R$ and on the right by $J$, we get

$$B^HLJ = RΣ^TJ.$$  

(14)

Using (14) and (10), we obtain the first equation from (12).

Multiplying (10) on the left by $LJ$, we get

$$JBR = LJΣ.$$  

(15)

Using (15) and (13), we obtain the second equation from (12).

Remark 10. If we denote

$$P_{l-t} = \text{diag}(p_1, \ldots, p_{l-t}), \quad Q_t = \text{diag}(q_1, \ldots, q_t),$$
then it can be easily verified that
\[ \Sigma^T J \Sigma = \text{diag}(p_1^2, \ldots, p_{t-1}^2, -q_1^2, \ldots, -q_t^2, 0, \ldots, 0). \]

From this equation and the first equation (12), it follows that singular values of the matrix \( B \) are square roots of the modules of the eigenvalues of the matrix \( B^H J B \). The columns of the matrix \( R \) are eigenvectors of the matrix \( B^H J B \). The matrix \( L \) can be found using the second equation (12).

Let us give one example for the following matrices
\[
B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad J = \text{diag}(1,-1), \quad m = 2, \quad n = 1.
\]

In this case, we have
\[
B^T J B = -3, \quad \text{rank}(B) = \text{rank}(B^T J B) = 1, \quad j = 0, \quad l = 1.
\]

Since eigenvalue of the matrix \( B^T J B \) equals \(-3\), it follows that \( t = 1 \) and a singular value of the matrix \( B \) equals \( \sqrt{3} \). We can choose the following matrix \( R \in O(1) \), the matrix \( \Sigma \) is determined uniquely:
\[
\Sigma = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}, \quad R = \begin{pmatrix} 1 \end{pmatrix}.
\]

Using \((JBB^T)L = L(J\Sigma \Sigma^T)\), we get
\[
\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} L = L \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}.
\]

Note that 0 and \(-3\) are eigenvalues of the matrix \( JBB^T \). Calculating eigenvectors of the matrix \( JBB^T \) and choosing correct multipliers, we get
\[
L = \begin{pmatrix} -2 & -1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix} \in O(1,1).
\]

Finally, we have
\[
L^T BR = \Sigma, \quad \begin{pmatrix} -2 & -1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 0 \sqrt{3} \end{pmatrix}.
\]

Note that the matrices \( L \) and \( R \) in (16) are not determined uniquely. For example, we can change the signs of these matrices at the same time.
Remark 11. In the case $J = I$ ($p = m$, $q = 0$), we obtain $j = 0$, $t = 0$, and the ordinary singular value decomposition \cite{2} as the particular case of Theorem 3 with $L \in U(m)$, $R \in U(n)$. In this case, the matrix $\Sigma$ is diagonal with all nonnegative diagonal elements. In this case, we obtain from (12) the well-known formulas

$$(B^H B) R = R(\Sigma^T \Sigma), \quad (BB^H) L = L(\Sigma \Sigma^T)$$

for finding $\Sigma$, $R$, and $L$. In this case, singular values of the matrix $B$ are square roots of the eigenvalues of the positive-definite Hermitian matrices $B^H B$ and $BB^H$, the columns of the matrix $L$ are eigenvectors of the matrix $BB^H$, and the columns of the matrix $R$ are eigenvectors of the matrix $B^H B$.

6. Conclusions

In this paper, we present a new formulation of HSVD for an arbitrary complex (or real) matrix without using the concept of hyperexchange matrices and using only the concept of pseudo-unitary (or pseudo-orthogonal) matrices. We present an algorithm for computing HSVD in the general case. The expressions (8) and (11) can be regarded as new useful canonical forms of an arbitrary complex (or real) matrix.

In our opinion, the statement of Theorem 2 (or Theorem 3) is more natural and useful for some applications (see Remarks 7, 9, 11). We expect a wide use of these theorems in computer science, image and signal processing, and physics. We use results of this paper to generalize results on Yang-Mills equations in Euclidean space $\mathbb{R}^n$ \cite{13} to the case of pseudo-Euclidean space $\mathbb{R}^{p,q}$ of an arbitrary dimension $p + q$.

Acknowledgments

The author is grateful to N. G. Marchuk for fruitful discussions.

This work is supported by the Russian Science Foundation (project 18-71-00010).

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