Scale Free Subnetworks by Design and Dynamics

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This article addresses the degree distribution of subnetworks, namely the number of links between the nodes in each subnetwork and the remainder of the structure (cond-mat/0408076). The transformation from a subnetwork-partitioned model to a standard weighted network, as well as its inverse, are formalized. Such concepts are then considered in order to obtain scale free subnetworks through design or through a dynamics of node exchange. While the former approach allows the immediate derivation of scale free subnetworks, in the latter nodes are sequentially selected with uniform probability among the subnetworks and moved into another subnetwork with probability proportional to the degree of the latter. Comparison of the designed scale-free subnetworks with random and Barabási-Albert counterparts are performed in terms of a set of hierarchical measurements.

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I. INTRODUCTION

In a short period of time, complex network research progressed all the way from uniform random models [1], [2] to the scale free networks of Barabási [4]. A good deal of the motivation for such developments has been accounted for by the scale free distribution of node degrees observed in models such as that proposed by Barabási and Albert [4]. One of the principal consequences of such a type of distribution is that it promotes the appearance of hubs, namely nodes with particularly high degree. By concentrating connections, hubs play a critical role in defining the network connectivity as well as other topological properties such as minimal paths. Another concept which has been found to be particularly useful in understanding complex networks is that of community, which can be informally understood as a group of nodes which are intensely interconnected but loosely connected to the remainder of the network (e.g. [3], [4], [7], [8]).

The relationship between hubs and communities has motivated some recent works [4], [5] which considered hubs as references for obtaining communities. Another concept directly related, but not necessarily equivalent, to communities is that of a subnetwork [6]. Given a network $\Gamma$, a subnetwork of $\Gamma$ is defined as a graph including a subset of nodes of $\Gamma$ plus their respective interconnections. Therefore, each community in a network can be understood as a densely linked subnetwork which is loosely connected with the remainder of the network. Every community is a subnetwork, but not every subnetwork is a community, i.e. communities are special cases of subnetworks. Because of their generality, subnetworks represent an interesting resource for theoretical and practical investigations of complex networks which has only scanty been explored [7]. One particularly interesting situation is the partition of a network into several subnetworks, in the sense that every node belongs exactly to one and only subnetwork. The concept of subnetwork degree was recently formalized [8] as the number of edges linking nodes inside the subnetwork to nodes in the remainder network.

The present work addresses subnetwork-partitioned models characterized by scale free subnetwork degrees. More specifically, we introduce a transformation from scale free subnetworks to traditional weighted networks, as well as its inverse. Two approaches to obtain scale free subnetworks from the random network $\Gamma$ are proposed: (i) by design and (ii) by dynamics. The former approach starts from the desired log-log curve and applies a direct, non-interactive method in order to obtain a subnetwork partition having similar node degree distribution. In the second methodology, nodes are sequentially selected from a subnetwork and reinserted into (possibly) another subnetwork with probability proportional to the degree of the latter. The comparison between the design scale free subnetworks and traditional random and Barabási-Albert models is also considered in terms of a set of recently introduced hierarchical features [9].

II. BASIC CONCEPTS

An undirected, unweighted network can be represented in terms of its adjacency matrix $K$, such that $K(i,j) = K(j,i) = 1$ whenever there is a link between nodes $i$ and $j$, with $1 \leq i, j \leq N$, and $K(i,j) = K(j,i) = 0$ otherwise. Similarly, an undirected, weighted network can be represented in terms of its weight matrix, in the sense that $W(i,j) = W(j,i) \geq 0$ corresponds to the weight of the edge between nodes $i$ and $j$. The absence of edges between those nodes is represented by making $W(i,j) = W(j,i) = 0$. Random networks, in the sense of Erdős and Rényi [3], [4], can be obtained by selecting among the $N(N-1)/2$ possible edges with uniform probability $\gamma$, yielding average degree $\langle k \rangle = \gamma(N-1)$.

The network of interest $\Gamma$ can be partitioned into $n$ subnetworks, such that each subnetwork $\kappa_i$ includes $N_i$ nodes from $\Gamma$ as well as the respective interconnections.
In case $N \gg N_i$, we have $k(c_i) \approx N_i \langle k \rangle$.

Given the original random network $\Gamma$, it is possible to construct a subnetwork-partitioned version by assigning nodes of $\Gamma$ to each community $c_i$ according to some criterion. The opposite operation, namely the transformation of a partitioned network into a traditional weighted network, henceforth called the subsumption of $\Gamma$ is also possible through the following steps: (i) each community $c_i$ is subsumed into a single node $c_i$ and (ii) the weight of the edge linking two nodes $c_i$ and $c_j$ is defined as the number of edges between the respective subnetworks. Figure 1 illustrates the subsumption of the subnetwork partitioned structure in (a) into the weighted network in (b). The inverse transformation can be obtained by using the design approach described in the following.

### III. SCALE FREE BY DESIGN

In this section we present how scale free subnetwork partitions of a random network $\Gamma$ can be immediately obtained such that the subnetwork degree follows a pre-specified scale free distribution.

As described in the previous section, provided $N \gg N_i$, the average degree of a subnetwork $c_j$ can be approximated as $k_j \approx N_j \langle k \rangle$, i.e. this degree becomes independent of the overall size of the random network $\Gamma$. This fact allows the immediate design of subnetwork partitions following virtually any subnetwork degree distribution, including the particularly important case of scale free models. The generic scale free log-log distribution of the degrees of a network is illustrated in Figure 2. In order to have the subnetwork degree histogram $h(k)$ such that $h(k) \propto k^\gamma$, we start by imposing that $\ln(h(k_j)) = (m - j)\Delta a$ for some pre-specified $\Delta a$, with $j = 1, 2, \ldots, m$, so that the values of $\ln(h(k))$ are uniformly distributed from $a$ down to $0$ with step $\Delta a = a/(m-1)$ along the $y$-axis, as $k_j$ varies from $k_1$ to $k_m$. It follows that $h(k_j) = \exp((m - j)\Delta a)$ and $\Delta k = -\Delta a/\xi$. Without loss of generality, we impose that $\ln(k_1) = 0$, which implies $\ln(k_j) = (j - 1)\Delta k$ and $\ln(k_m) = (m-1)\Delta k = -a/\xi$. So, $\ln(h(k_j)) = \xi\ln(k_j) + a$.

From the above developments, we have that $k_j = \exp((j - 1)\Delta k)$. In other words, it is desired that community $c_j$ has degree $k(c_j) = k_j$. We have from Section 11 that $k(c_j) \approx N_j \langle k \rangle$. Therefore, in order to have $k(c_j) = k_j$, we must have $N_j \approx k_j \langle k \rangle$. The total required communities is $n = \text{round}((\sum_{j=1}^m h(k_j))/\xi)$ and, because $h(k_j)$ communities with $N_j$ nodes each are needed, with $j = 1, 2, \ldots, m$, the total number of nodes in the random network is given as $N = \sum_{j=1}^m h(k_j)N_j$.

Observe that, for a specified $h(k_j)$, the total number $N$ of nodes can be increased by reducing $\Delta a$.

Figure 3 illustrates the average $\pm$ standard deviation of log-log node degree distributions obtained for 50 realizations of a designed subnetwork assuming $\xi = -1.0$, $a = 4$, $\Delta a = 0.5$ and $\langle k \rangle = 2$, implying $m = 9$, $n = 137$ and $N = 275$. The obtained average curve falls reason-
corresponding to the clustering coefficient of $R_d(i)$. Figure 4 presents the average ± standard deviations of such measurements obtained for the above 50 simulations as well as for random and Barabási-Albert scale free models with the same number of nodes and average degree. It is clear from such results that the designed models have topological properties strikingly similar to those of the respective Barabási-Albert models, except for the hierarchical common degree, which resulted remarkably distinct, exhibiting a peak near at the higher hierarchical levels. Slightly higher values of clustering coefficient are also observed for the design models.

IV. SCALE FREE BY DYNAMICS

The concepts and methods described in the previous sections can also be used to implement a dynamics of node exchange between the subnetworks in a partitioned system. Among the several possibilities, we investigate the scheme starting with a uniform subnetwork partition of a random network $\Gamma$ (i.e. each community $i$ initially has $N_i = N/n$ nodes) and involving sequential random selection of a subnetwork $c_i$, from which a node is randomly selected (uniform probability) and moved to (possibly) another subnetwork $c_j$ chosen with probability proportional to its respective degree $k(c_j)$. It is suggested that such a dynamical node exchange can be used to model several real-world phenomena such as the continuous exchange of individuals between institutions, e.g., music performers moving from an ensemble to another, animal species changing their environment, and so on. Figure 5 shows the log-log plot of the subnetwork degree distributions for three successive steps — i.e. $t = 1$, $t = 50$ and $t = 185$ — along the node exchange interactions. It is clearly perceived that the left-hand side of the log-log distribution tends to increase as the nodes are redistributed among the subnetworks.

V. CONCLUDING REMARKS

The concepts of subnetwork degree as well as the presently introduced notion of subnetwork partitions, have allowed interesting developments such as the design and evolution of scale free subnetworks. The hierarchical characterization of experimental results of a designed subnetwork partitioned model indicates that such networks present similar features to equivalent Barabási-Albert models, except for the hierarchical common degree, which tended to present a peak at higher hierarchical levels. Although we have concentrated attention on scale free degree distribution, the proposed concepts and methods can be immediately applied to many other situations including the design of community organized networks with generic degree distribution. Because for large values of $N$ the subnetwork degree can be well-approximated by the product between the number of
FIG. 4: The average ± standard deviation of the 5 hierarchical measurements in terms of $d$ considering the 50 design simulations (a-e) and random (f-j) and Barabási-Albert (k-o) models with the same number of nodes and average degree.

FIG. 5: Three stages of the subnetwork degree evolution by using the suggested node exchange dynamics.

nodes inside the subnetwork and the average degree of the underlying random network, the subnetwork degree distribution ultimately follows the distribution of the number of nodes in the subnetworks. As a consequence, geographical networks where nodes are uniformly distributed along the space and the subnetworks cover areas which follow a power law will result naturally scale free. Another issue deserving further attention is the dynamical redistribution of nodes among the subnetworks.

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