SIMPLICIAL COMPLEXES ASSOCIATED TO CERTAIN SUBSETS OF NATURAL NUMBERS AND ITS APPLICATIONS TO MULTIPLICATIVE FUNCTIONS

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1. INTRODUCTION

We call a set of positive integers closed under taking unitary divisors an unitary ideal. It can be regarded as a simplicial complex. Moreover, a multiplicative arithmetical function on such a set corresponds to a function on the simplicial complex with the property that the value on a face is the product of the values at the vertices of that face. We use this observation to solve the following problems:

A. Let \( r \) be a positive integer and \( c \) a real number. What is the maximum value that \( \sum_{s \in S} g(s) \) can obtain when \( S \) is a unitary ideal containing precisely \( r \) prime powers, and \( g \) is the multiplicative function determined by \( g(s) = c \) when \( s \in S \) is a prime power?

B. Suppose that \( g \) is a multiplicative function which is \( \geq 1 \), and that we want to find the maximum of \( g(i) \) when \( 1 \leq i \leq n \). At how many integers do we need to evaluate \( g \)?

C. If \( S \) is a finite unitary ideal, and \( g \) is multiplicative and \( \geq 1 \), then the maximum of \( g \) on \( S \) occurs at a facet, and any facet is optimal for some \( g \). If \( W_1, \ldots, W_\ell \) is an enumeration of the facets in some order, is there always a \( g \) as above so that \( g(W_1) \leq g(W_2) \leq \cdots \leq g(W_\ell) \)?

2. UNITARY IDEALS AND SIMPLICIAL COMPLEXES

Let \( \mathbb{N} \) denote the non-negative integers and \( \mathbb{N}^+ \) the positive integers, with subsets \( \mathbb{P} \) the prime numbers and \( \mathbb{P}^\mathbb{P} \) the set of prime powers. Recall that an unitary divisor (or a block factor) of \( n \in \mathbb{N}^+ \) is a divisor \( d \) such that \( \gcd(d, n/d) = 1 \). In this case, we write \( d \mid\mid n \) or \( n = d \oplus n/d \). If \( \gcd(a, b) > 1 \) we put \( a \oplus b = 0 \).

**Definition 2.1.** A subset \( S \subset \mathbb{N}^+ \) is a unitary ideal if

\[
s \in S, \quad d \in \mathbb{N}^+, \quad d \mid\mid s \quad \Rightarrow \quad d \in S
\]  

**Definition 2.2.** For any unitary ideal \( S \subset \mathbb{N}^+ \) with \( X = X(S) = \mathbb{P}^\mathbb{P} \cap S \), we define the simplicial complex \( \Delta(S) \) on \( X(S) \) by

\[
\Delta(S) \ni \sigma = \{a_1, \ldots, a_r\} \quad \iff \quad a_1a_2\cdots a_r \in S \quad \text{and} \quad \forall 1 \leq i < j \leq r : \gcd(a_i, a_j) = 1
\]

Clearly, \( \Delta(S) \) is finite iff \( X(S) \) is finite iff \( S \) is finite. Furthermore:

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Lemma 2.3. Any finite simplicial complex can be realized as $\Delta(S)$ for some $S$.

Proof. Take as many prime numbers as there are vertices in the simplicial complex, so that the vertex $v_i$ corresponds to the prime number $p_i$. For any $\sigma = \{v_{a_1}, \ldots, v_{a_r}\}$ in the simplicial complex we let $p_{a_1}p_{a_2}\cdots p_{a_r} \in S$. □

Note 2.4. In what follows, we will sometimes regard elements in $S$ as faces in $\Delta(S)$, without explicitly pointing this out. We trust that the reader will not be confused by this.

Recall that an arithmetical function is a function $g : \mathbb{N}^+ \to \mathbb{C}$, and that an arithmetical function is multiplicative iff

$$g(ab) = g(a)g(b)$$

whenever $\gcd(a, b) = 1$. Hence, a multiplicative function is determined by its values on $\mathbb{P}^+$, and we have

Lemma 2.5. Let $S$ be a unitary ideal, and $g$ a multiplicative function. By abuse of notation, put $g(\sigma) = g(a_1a_2\cdots a_r)$ if $\sigma = \{a_1, \ldots, a_r\} \in \Delta(S)$. Then $g(\sigma) = g(a_1)g(a_2)\cdots g(a_r)$, so the value of $g$ at a simplex is the product of the values of $g$ at the vertices of said simplex.

We'll be interested in three problems:

1. Calculating the sum $\sum_{s \in S} g(s)$,
2. Maximizing $g$ on $S$,
3. Finding the total orders on $S$ induced by $g$.

3. Summing $g$ on $S$

We henceforth assume that $S$ is finite, with $S \cap \mathbb{P}^+$ containing $r$ elements, and that $g$ is a multiplicative function. Put $G(S) = \sum_{s \in S} g(s)$.

Let us start with the simplest cases. If $S$ consists entirely of prime powers then $\Delta(S)$ consists of $r$ isolated points, on which $g$ can take any values. The other extreme is that $s, t \in S$, $\gcd(s, t) = 1$ implies that $st \in S$, and that all prime powers in $S$ are in fact primes. Then $S$ consists of all square-free products of these $r$ primes, so $\Delta(S)$ is an $(r - 1)$-dimensional simplex. In this case, it is easy to see that $G(S) = \prod_{j=1}^r (1 + g(p_j))$, where $p_1, \ldots, p_r$ are the primes in $S$.

More generally, if $\Delta(S)$ have $\ell$ faces $W_1, \ldots, W_\ell$, with $2^\ell < |\Delta(S)|$, then the following formula might be useful. Put

$$\tilde{g}({a_1, \ldots, a_v}) = \prod_{j=1}^v (1 + g(a_i)) = \sum_{\sigma \in \{a_1, \ldots, a_v\}} g(\sigma).$$

The principle of Inclusion-Exclusion gives

$$G(S) = \sum_{i=1}^\ell \tilde{g}(W_i) - \sum_{1 \leq i < j \leq \ell} \tilde{g}(W_i \cap W_j) + \sum_{1 \leq i < j < k \leq \ell} \tilde{g}(W_i \cap W_j \cap W_k) - \ldots$$

If $S$ is arbitrary, but $g$ special in that it takes the same value on all prime powers, then $G(S)$ is also easily calculable.
Lemma 3.1. If there exists a $c$ such that $g(s) = c$ for all $s \in S \cap \mathbb{P}$, then
\[ G(S) = (1, c, c^2, \ldots, c^r) \cdot (1, f_0, f_1, \ldots, f_{r-1}), \] (5)
where $(f_0, f_1, \ldots, f_{r-1})$ is the $f$-vector of $\Delta(S)$, i.e. $f_i$ counts the number of $i$-dimensional (i.e. having $i + 1$ vertices) faces of $\Delta(S)$.

Proof. A $(v - 1)$-dimensional simplex of $\Delta(S)$ contributes $c^v$ to $G(S)$; there are $f_{v-1}$ such simplexes, so the total contribution is $c^v f_{v-1}$. Letting $v$ range from 0 to $r$ and summing yields the result. \qed

Theorem 3.2. Let $\Psi(r, c)$ denote the maximum that $G(S)$ can obtain when $|S \cap \mathbb{P}| = r$ and $g(s) = c \in \mathbb{R}$ for all $s \in S \cap \mathbb{P}$. Then, if $r$ is odd,
\[ \Psi(r, c) = \begin{cases} 1 + \sum_{i=1}^{r} c^i \binom{r}{i} & \text{if } c > 0 \\ 1 + \sum_{i=1}^{2} c^i \binom{r}{i} & \text{if } \frac{1}{n-3} < c < 0 \\ 1 + \sum_{i=1}^{4} c^i \binom{r}{i} & \text{if } \frac{1}{n-5} < c < \frac{1}{n-3} \\ 1 + \sum_{i=1}^{6} c^i \binom{r}{i} & \text{if } \frac{2}{n-7} < c < \frac{1}{n-5} \\ \vdots & \vdots \\ 1 + \sum_{i=1}^{r-1} c^i \binom{r}{i} & \text{if } c < \frac{-(n-1)}{2} \end{cases} \] (6)
and if $r$ is even
\[ \Psi(r, c) = \begin{cases} 1 + \sum_{i=1}^{r} c^i \binom{r}{i} & \text{if } c > 0 \\ 1 + \sum_{i=1}^{2} c^i \binom{r}{i} & \text{if } \frac{1}{n-3} < c < 0 \\ 1 + \sum_{i=1}^{4} c^i \binom{r}{i} & \text{if } \frac{1}{n-5} < c < \frac{1}{n-3} \\ 1 + \sum_{i=1}^{6} c^i \binom{r}{i} & \text{if } \frac{2}{n-7} < c < \frac{1}{n-5} \\ \vdots & \vdots \\ 1 + \sum_{i=1}^{r-2} c^i \binom{r}{i} & \text{if } c < \frac{-(n-2)}{3} \\ 1 + \sum_{i=1}^{r} c^i \binom{r}{i} & \text{if } c < -n \end{cases} \] (7)

Proof. Put $c = (c, c^2, \ldots, c^r), f = (f_0, f_1, \ldots, f_{r-1})$. It follows from Lemma (2.3) that we must maximize $f \cdot c$ over all possible $f$-vectors $f$ of simplicial complexes on $r$ vertices. Since $f \cdot c$ is a linear function, it will suffice to evaluate $f \cdot c$ on a set of vertices that span the convex hull of $f$-vectors of simplicial complexes on $r$ vertices. Kozlov \[ \] showed that the set
\[ \left\{ \tilde{F}_1, \ldots, \tilde{F}_r \right\}, \quad \tilde{F}_i = \left( \binom{r}{1}, \binom{r}{2}, \ldots, \binom{r}{i}, 0, \ldots, 0 \right) \] (8)
is minimal with the property that its convex hull contains all $f$-vectors of simplicial complexes on $r$ vertices. Hence, it is enough to decide which of the $r$ numbers
\[ K_1 = \tilde{F}_1 \cdot c = cn \]
\[ K_2 = \tilde{F}_2 \cdot c = cn + c^2 \binom{n}{2} \]
\[ \vdots \]
\[ K_r = \tilde{F}_r \cdot c = \sum_{i=1}^{r} c^i \binom{n}{i} \] (9)
is the greatest.

Clearly, if \( c > 0 \), then \( K_r \) is the greatest. If \( c < 0 \) and \( r \) is odd then we always have the inequalities shown in Figure 1.

\[
K_{2i+2} - K_{2i} = c^{2i+1} \left( \frac{r}{2i+1} \right) + c^{2i+2} \left( \frac{r}{2i+2} \right) = c^{2i+1} \left( \frac{r}{2i+1} \right) + c \left( \frac{r}{2i+2} \right)
\]

which is > 0 iff
\[
c < -\left( \frac{r}{2i+1} \right) = -\frac{2i+2}{r-2i-1}
\]

Since
\[
-\left( \frac{r}{3} \right) > -\left( \frac{r}{4} \right) > \cdots > -\left( \frac{r}{r-1} \right)
\]

the result for the odd case follows. The even case is proved similarly; here the inequalities for \( c < 0 \) are as in Figure 2. \( \square \)

4. Maximizing \( g \) on \( S \)

As we noted at the start of the previous section, if \( S \) consists of all square-free products of a finite set \( \{p_1, \ldots, p_r\} \) of primes, then \( \Delta(S) \) is an \((r-1)\)-simplex. Hence, if \( g \) is real-valued and \( g(s) \geq 1 \) (we call such a \( g \) multiplicative and log-positive), then the maximum of \( g \) on \( S \) is \( g(p_1p_2\cdots p_r) \). More generally:

**Lemma 4.1.** Suppose that \( g \) is multiplicative and log-positive. Let \( W_1, \ldots, W_\ell \) be the facets (i.e. a simplexes maximal w.r.t inclusion) of \( \Delta(S) \). Then the maximum value \( g(\sigma) \) for \( \sigma \in \Delta(S) \) is obtained on some facet \( W_i \).

Conversely, there exists a multiplicative and log-positive \( h \) so that \( h(W_1) \) is maximal.
Proof. If $\sigma \subset \tau$ then $g(\sigma) \leq g(\tau)$, so the maximum is attained on a facet.

For the converse, define $h$ on $X(S)$ by

$$h(p) = \begin{cases} 1 & \text{if } p \notin W_1 \\
2 & \text{if } p \in W_1 \end{cases}$$  \hspace{1cm} (13)$$

We extend $h$ to a multiplicative function on $\Delta(S)$. It is then clear that $h(W_1) = 2^{|W_1|}$ whereas $h(W_i) = 2^{|W_1 \cap W_i|} < 2^{|W_1|}$ for $i > 1$; the last inequality follows since $W_1, W_i$ are facets and hence maximal w.r.t. inclusion. If we want a multiplicative $h$ which is strictly $> 1$ on non-empty simplexes, we can define $h(p) = 1 + \varepsilon$ for $p \notin W_1$, where $\varepsilon$ is some small positive number. \hfill \Box

We let $[n] = \{1, 2, \ldots, n\}$. Then $[n]$ is a unitary ideal, so we have

**Corollary 4.2.** If $g$ is multiplicative and log-positive function then the maximum $g(s)$ with $1 \leq s \leq n$ is obtained on a facet of $\Delta([n])$.

As an example, if $n = 30$ then the $\Delta([30])$ looks like Figure 3, so the facets are

$$12, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30.$$  \hspace{1cm} (14)

Thus about 57% of the simplicies in $\Delta([30])$ are facets. In general, we have the following:

**Theorem 4.3** (Snellman). Let $p_i$ denote the $i$'th prime number. For large $n$, the number of facets in $\Delta([n])$ is approximatively $\gamma n$, where

$$\gamma = 1 - \frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{p_i} - \frac{1}{p_{i+1}} \prod_{j=1}^{i} p_j \approx 0.607714359516618$$  \hspace{1cm} (14)$$

**Proof.** See [5]. \hfill \Box

So, if we are to maximize (on $[n]$) a large number of different $g_i$'s which are multiplicative and log-positive, it makes sense to precompute the facets of $\Delta([n])$, and their factorizations. If $q_1, \ldots, q_r$ are the prime powers $\leq n$, ...
and \( w_1, \ldots, w_\ell \) are the facets of \( \Delta([n]) \), let \( A = (a_{ij}) \) be the \( \ell \times r \) integer matrix defined by
\[
\forall 1 \leq i \leq \ell : \quad w_i = \prod_{j=1}^{r} q_j^{a_{ij}} \quad (15)
\]
Then, if \( g \) is a log-positive multiplicative function,
\[
\forall 1 \leq i \leq \ell : \quad g(w_i) = \prod_{j=1}^{r} g(q_j)^{a_{ij}} \quad (16)
\]
This means that in order to find the maximum for \( g \) we need to perform \( r \) evaluations to find the \( g(q_i) \)'s, then calculate \( \ell \approx \gamma n \) numbers, each of which is the product of at most \( r \) terms, and then find the maximum of those numbers.

5. Total orderings on \( S \) induced by \( g \)
As previously noted, if \( g \) is log-positive and \( \sigma \subset \tau \), then \( g(\sigma) \leq g(\tau) \). Moreover, if \( g \) is strictly log-positive, so that \( g(\sigma) > 1 \) for \( \sigma \neq \emptyset \), then
\[
\sigma \subsetneq \tau \quad \implies \quad g(\sigma) < g(\tau) \quad (17)
\]
Assume that \( g \) has this property, that \( S \) is a unitary ideal, and that furthermore \( g \) is injective when restricted to \( S \). Then \( S \), and hence \( \Delta(S) \), is totally ordered by
\[
x > y \iff g(x) > g(y) \quad (18)
\]
It is clear, by (17), that such a total order on \( \Delta(S) \) is a linear extension of the partial order given by inclusion of subsets. However, not all such linear extensions may occur.

**Definition 5.1.** Let \( r \) be a positive integer, and let \( V = \{v_1, \ldots, v_r\} \) be a linearly ordered set with \( r \) elements. Following MacLagan [2] we call a total order \( \succ \) on \( 2^V \) a boolean termorder if
\[
\emptyset \prec \sigma \quad \text{if} \quad \emptyset \neq \sigma \subset V \quad (19)
\]
\[
\sigma \cup \gamma \prec \tau \cup \gamma \quad \text{if} \quad \sigma \prec \tau \quad \text{and} \quad \gamma \cap (\sigma \cup \tau) = \emptyset \quad (20)
\]
We say that \( \prec \) is sorted if
\[
v_1 \prec v_2 \prec \cdots \prec v_r \quad (21)
\]
Furthermore, \( \prec \) is coherent if there exist \( r \) positive integers \( w_1, \ldots, w_r \) such that
\[
\alpha \prec \beta \iff \sum_{v_i \in \alpha} w_i < \sum_{v_j \in \beta} w_j. \quad (22)
\]

**Lemma 5.2.** Suppose that \( S \) is a finite unitary ideal, and let \( r \) be the number of prime powers in \( S \). Label these prime powers \( v_1, \ldots, v_r \). Consider the set \( M \) of all multiplicative \( g \) that are strictly log-positive, injective when restricted to \( S \), and let \( M^* \) denote the subset of those \( g \) that in addition fulfills
\[
g(v_1) < g(v_2) < \cdots < g(v_r). \quad (23)
\]
Let \( Y \) be the partial order on \( 2^{\{ v_1, \ldots, v_r \}} \) which is generated by the following relations:
\[
\{ v_i, \ldots, v_k \} \prec \{ v_i, \ldots, v_j \} \text{ if } k \notin \{ i, \ldots, j \},
\]
\[
\{ v_i, \ldots, v_j, v_k \} \prec \{ v_i, \ldots, v_j, v_{j+1}, \ldots, v_k \}
\text{ if } i_j + 1 \in [r] \setminus \{ v_i, \ldots, v_j, \ldots, v_k \} \tag{25}
\]

Let \( T \subseteq S \), and let \( Y_T \) be the induced subposet on \( T \subseteq \Delta(S) \subseteq 2^{\{ v_1, \ldots, v_r \}} \).
Then
(i) Any total order on \( T \) induced by a \( g \in \mathcal{M} \) (by a \( g \in \mathcal{M}^s \)) is the restriction of a (sorted) coherent boolean termorder.
(ii) Conversely, the restriction to \( T \) of a (sorted) coherent boolean termorder on \( 2^{\{ v_1, \ldots, v_r \}} \) is induced by some \( g \in \mathcal{M} \) (\( g \in \mathcal{M}^s \)).
(iii) Any total order on \( T \) induced by a \( g \in \mathcal{M}^s \) is a linear extensions of \( Y_T \).

Proof. We can W.L.O.G. assume that \( T = 2^{\{ v_1, \ldots, v_r \}} \). If \( \prec \) is induced by \( g \in \mathcal{M} \) then
\[
\alpha \prec \beta \iff \prod_{v_i \in \alpha} g(v_i) < \prod_{v_j \in \beta} g(v_j) \iff \sum_{v_i \in \alpha} \log g(v_i) < \sum_{v_j \in \beta} \log g(v_j).
\]

We can replace the \( \log g(v_i)'s \) by positive rational numbers that closely approximate them, and then, by multiplying out by a common denominator, by positive integers. Thus \( \prec \) is a coherent boolean termorder. It is clear that if \( g \in \mathcal{M}^s \) then \( \prec \) is sorted.

If on the other hand \( \prec \) is a coherent boolean term order on \( 2^{\{ v_1, \ldots, v_r \}} \), then there are positive integers \( w_1, \ldots, w_r \) such that
\[
\alpha \prec \beta \iff \sum_{v_i \in \alpha} w_i < \sum_{v_j \in \beta} w_j \iff \prod_{v_i \in \alpha} \exp(w_i) < \prod_{v_j \in \beta} \exp(w_j).
\]

If we define \( g(v_i) = w_j \) and extend this multiplicatively, then \( g \) induces \( \prec \).
If \( \prec \) is sorted, clearly \( w_1 > w_2 > \cdots > w_r \), so \( g \in \mathcal{M}^s \).

If \( g \) is strictly log-positive and fulfills (23) then clearly
\[
g(\{ v_i, \ldots, v_k \}) < g(\{ v_i, \ldots, v_k \}) \tag{26}
\]
for \( k \notin \{ i, \ldots, j \} \), and likewise
\[
g(\{ v_i, \ldots, v_j, v_k \}) < g(\{ v_i, \ldots, v_{j+1}, \ldots, v_k \}) \tag{27}
\]
for \( i_j + 1 \in [r] \setminus \{ v_i, \ldots, v_j, \ldots, v_k \} \). Thus any total order induced by a \( g \in \mathcal{M} \) is a linear extension of \( Y \). \( \square \)

The symmetric group \( S_r \) acts transitively on \( \mathcal{M} \), and \( \{ \pi(\mathcal{M}^s) | \pi \in S_r \} \) is a partition of \( \mathcal{M} \) into \( \frac{r!}{|S_r|} = r! \) blocks. Hence

**Corollary 5.3.** Let \( t(T) \) denote the number of total orders on \( T \subseteq \Delta(S) \) that are induced by multiplicative functions \( g \in \mathcal{M}^s \), and let \( \ell(Y_T) \) denote the number of linear extensions of \( Y_T \). Then
\[
t(T) \leq r! \ell(Y_T) \tag{28}
\]
In the following example, we show that although any facet of $\Delta(S)$ is maximal w.r.t. some total order induced by a log-positive multiplicative $g$ (Lemma 4.1), there are only certain orderings among those facets that are possible.

**Example 5.4.** Let us consider the poset $Y$ with $r = 4$, and in particular the induced poset on the 2-subsets, which looks like Figure 5(b).

**Figure 4.** The poset $Y$

We see that this poset has exactly 2 linear extensions, corresponding to the two different ways of ordering the antichain \{23, 14\}. Thus, if

$$g(v_1) < g(v_2) < g(v_3) < g(v_4)$$

then there are two possible orderings for

$$\{g(v_1v_2), g(v_1v_3), g(v_1v_4), g(v_2v_3), g(v_2v_4), g(v_3v_4)\}.$$

If we remove the restriction (29), then there are $2 \times 4! = 48$ different orderings, out of the $\binom{4}{2}! = 720$ a priori possibilities. For instance,

$$g(v_1v_2) > g(v_2v_4) > g(v_1v_3) > g(v_2v_3) > g(v_1v_4) > g(v_3v_4)$$

is impossible, since $g(v_2v_4) > g(v_1v_4) \implies g(v_2) > g(v_1)$ but $g(v_1v_3) > g(v_2v_3) \implies g(v_1) > g(v_2)$. Hence, there is no multiplicative arithmetic function $g$ such that

$$g(6) > g(21) > g(10) > g(15) > g(14) > g(35).$$
The whole poset $Y$ on 4 letters looks like Figure 5(a). It consists of 16 elements and has, as the reader may easily verify, 78 linear extensions.

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