On Coupled Logical Bisimulation for the λ-Calculus

Ryan Kavanagh\textsuperscript{1,2} & Jean-Marie Madiot\textsuperscript{2}
ryan@cs.queensu.ca, jeanmarie.madiot@ens-lyon.fr

\textsuperscript{1}: School of Computing
Queen’s University at Kingston
Kingston, Ontario, Canada
\textsuperscript{2}: Laboratoire de l’Informatique du Parallélisme
École normale supérieure de Lyon
Lyon, France

Abstract

We study coupled logical bisimulation (CLB) to reason about contextual equivalence in the λ-calculus. CLB originates in a work by Dal Lago, Sangiorgi and Alberti, as a tool to reason about a λ-calculus with probabilistic constructs. We adapt the original definition to the pure λ-calculus. We develop the metatheory of CLB in call-by-name and in call-by-value, and draw comparisons with applicative bisimulation (due to Abramsky) and logical bisimulation (due to Sangiorgi, Kobayashi and Sumii). We also study enhancements of the bisimulation method for CLB by developing a theory of up-to techniques for cases where the functional corresponding to bisimulation is not necessarily monotone.

1. Introduction

Several coinductive methods to reason about equivalences between higher-order programs or processes have been proposed. The starting point in this direction is Abramsky’s Applicative Bisimulation (AB)\textsuperscript{[1]}. Several alternatives to AB have been proposed since, with two main objectives. A first objective is to be able to develop the metatheory of the bisimilarity in a simple way. The main question related to this objective is to prove that bisimilarity coincides with contextual equivalence. A second objective is to be able to equip the coinductive method with powerful proof techniques, that allow one to avoid including redundant pairs in the relations being studied. These so-called \textit{up-to techniques}\textsuperscript{[12]} can turn out to be very useful in developing proofs of equivalence between programs.

To address these objectives, Logical Bisimulation (LB)\textsuperscript{[10]} and, successively, Environmental Bisimulation (EB)\textsuperscript{[11]} have been proposed. LB and EB can both be seen as improvements of AB.

Recently, in a study of a λ-calculus enriched with probabilistic features\textsuperscript{[3]}, Dal Lago, Sangiorgi and Alberti have introduced Coupled Logical Bisimulation (CLB). In that work, CLB is motivated by technical considerations, related to the way probabilistic λ-terms evolve and can be observed. The main purpose of the present work is to understand how CLB compares to existing notions of bisimulation for higher-order calculi. To achieve this, we formulate CLB in the simpler setting of the pure λ-calculus, and study its basic metatheory, as well as some up-to techniques.

We consider CLB in both the call-by-name and the call-by-value λ-calculus, relating it with AB and LB. To define bisimulation enhancements for CLB, the existing theory of bisimulation
enhancements [12] cannot be reused directly. The reason is that, like in the case of LB, the functional associated to the definition of bisimulation is not monotone. We show how sound up-to techniques for bisimulation can be adapted in this setting, and how to compose them.

Another contribution of the paper is a written proof of the context lemma for the call-by-value $\lambda$-calculus, which says that contextual equivalence is preserved when restricting to evaluation contexts only. While this result seems to belong to folklore, we have not been able to find it in the literature. The proof is not a direct adaptation of the corresponding result in call-by-name [6].

Outline of the paper. In Section 2, we present a theory of up-to techniques in absence of monotonicity of the functional corresponding to bisimulation. Section 3 recalls some general notions about the $\lambda$-calculus. We study CLB for the call-by-name $\lambda$-calculus in Section 4. We then present the main properties of CLB for the call-by-value $\lambda$-calculus in Section 5. In passing, we present in Section 5.1 a proof of the “context lemma” for the call-by-value $\lambda$-calculus.

2. Up-to techniques in absence of monotonicity

As we shall see in Section 4, the functional defining CLB is not monotone. In this section, we present an axiomatic theory of up-to techniques which does not rely on this property.

Although our theory is motivated by up-to techniques for bisimulation relations, it applies equally well to any theory comprising a family $F$ of “accepted relations” contained in a universe $U$ of all possible “relations”. Although our theory is inspired from bisimulations, we deliberately avoid specifying the nature of the “relations” in $U$: these need not be relations in the traditional sense of sets of pairs of terms. For example, our theory applies equally well to coupled logical bisimulations, presented below, which are pairs of relations, and it could be used to reason about any desired subset $F$ of some universe $U$.

Definition 1. Given a family $F \subseteq U$ of accepted relations, a progression for $F$ is a relation $\rightarrow \subseteq U \times U$ such that $R \rightarrow R$ only if there exists an $R' \in F$ such that $R \subseteq R'$. If $R \rightarrow S$, then we say that $R$ progresses to $S$.

Although up-to techniques for coinductively-defined families of relations are often given as total functions over $U$, we relax this definition to the following:

Definition 2. Given a family of relations $F \subseteq U$ with a progression, we call a partial function $P : U \rightarrow U$ an up-to technique and say that it is sound if whenever $P(R)$ is defined and $R \rightarrow P(R)$, there exists an $R' \in F$ such that $R \subseteq R'$. We say that $P$ is monotone if whenever $R \subseteq S$ and both $P(R)$ and $P(S)$ are defined, then $P(R) \subseteq P(S)$.

Definition 3. We say that an up-to technique $P$ is finitely convergent if there exists an $N$ such that for all $n, m > N, P^n = P^m$; in such situation, we call $N$ the finite convergence constant. Finally, we say that two up-to techniques $P$ and $Q$ commute if $P \circ Q = Q \circ P$.

We also make use of the following definitions, which are based on those in [9]:

Definition 4. We say that an up-to technique $P$ is compatible if for all $R$ and $S$ such that $R \rightarrow S$, $P(R)$ and $P(S)$ are defined and $P(R) \rightarrow P(S)$. We say that it is respectfully compatible if for all $R$ and $S$ such that $R \subseteq S$ and $R \rightarrow S$, $P(R)$ and $P(S)$ are defined, and $P(R) \subseteq P(S)$ and $P(R) \rightarrow P(S)$ hold. We say that an up-to technique $P$ is extensive if for all $R$, if $P(R)$ is defined, then $R \subseteq P(R)$.

We remark that, although the definitions of “monotone, compatible up-to technique” and “respectfully compatible up-to technique” are similar, monotony is a much stronger condition than
respectfulness. We also observe that respectfully compatible up-to techniques need not be compatible and vice-versa.

From these straightforward definitions, we can deduce sufficient conditions for the soundness of up-to techniques. As we shall see below, imposing a condition known as “continuity” on our progression relation provides significantly simpler sufficiency conditions for soundness.

**Proposition 5.** A finitely convergent, extensive, (respectfully) compatible up-to technique $P$ is sound.

**Proof.** If $P(R)$ is defined and $R \rightarrow P(R)$, we prove by induction on $n$ that for all $n$, $P^n(R)$ is defined, $P^n(R) \rightarrow P^{n+1}(R)$ and $R \subseteq P^n(R)$. Hence, $P^{N+1}(R) \rightarrow P^{N+2}(R) = P^{N+1}(R)$ by finite convergence. Thus for some $R'$, $P^{N+1}(R) \subseteq R' \in I$, which implies $R \subseteq R' \in I$. 

As a corollary of the proof, we get that:

**Corollary 6.** If $P$ is finitely convergent, extensive, and (respectfully) compatible and $R \rightarrow P(R)$, then, where $N$ is the finite convergence constant, for all $n > N$ we have $P^n(R) \rightarrow P^m(R)$ and $P^n(R) \subseteq R'$ for some $R' \in I$.

**Proof.** Repeatedly applying compatibility to $P^{N+1}(R) \rightarrow P^{N+1}(R)$, we get that $P^{m}(R) \rightarrow P^{n}(R)$ for all $n > N$, and so $P^n(R) \subseteq R' \in I$ for all $n > N$.

The following propositions tell us that composition of up-to techniques is well-behaved. They follow straightforwardly from the corresponding definitions.

**Proposition 7.** If $P$ and $Q$ are two extensive up-to techniques, then so is $Q \circ P$.

**Proof.** Assume $(Q \circ P)(R)$ is defined, then the proposition is immediate by transitivity: $R \subseteq P(R) \subseteq Q(P(R)) = (Q \circ P)(R)$.

**Proposition 8.** If $P$ and $Q$ are two (respectfully) compatible up-to techniques, then so is $Q \circ P$.

**Proof.** We show that compatibility is composed by composition; respectful compatibility follows in an identical manner. By compatibility of $P$, for all $R \rightarrow S$, $P(R)$ and $P(S)$ are defined and $P(R) \rightarrow P(S)$. By compatibility of $Q$, for all $R \rightarrow S$, $Q(R)$ and $Q(S)$ are defined and $Q(R) \rightarrow Q(S)$. Combining these two facts, we get that for all $R \rightarrow S$, $(Q \circ P)(R)$ and $(Q \circ P)(S)$ are defined and $(Q \circ P)(R) \rightarrow (Q \circ P)(S)$, i.e., $Q \circ P$ is compatible.

**Proposition 9.** If $P$ and $Q$ are two finitely convergent up-to techniques that commute, then $Q \circ P$ is finitely convergent.

**Proof.** Let $M$ and $N$ be the convergence constants of $P$ and $Q$ respectively, and let $L = \max(M, N)$. By commutativity, $(Q \circ P)^k = Q^k \circ P^k$ for all $k$, and so for all $m, n > L$, $(Q \circ P)^m = Q^m \circ P^m = Q^n \circ P^n = (Q \circ P)^n$. Thus, $Q \circ P$ is finitely convergent.

**Corollary 10.** If $P$ and $Q$ are two extensive, (respectfully) compatible, and finitely convergent up-to techniques that commute, then $Q \circ P$ is sound.

**Proof.** The composition $Q \circ P$ satisfies the hypotheses of Proposition 5 by Propositions 7, 8, and 9.

When dealing with monotone up-to techniques, we can relax the commutativity requirement at the expense of additional hypotheses.

**Lemma 11.** If $P$ and $Q$ are two monotone functions such that $(Q \circ P)(R) \subseteq (P \circ Q)(R)$ for all $R$, then for all $R$ and all $k$, $(Q \circ P)^k(R) \subseteq (P \circ Q^k)(R)$.
Proof. We proceed by induction on \( k \). The case \( k = 1 \) is by hypothesis, so assume \( (\Omega \circ \mathcal{P})^k(\mathcal{R}) \subseteq (\mathcal{P}^k \circ \mathcal{Q}^k)(\mathcal{R}) \) for some \( k \). Clearly

\[
(\Omega \circ \mathcal{P})^{k+1}(\mathcal{R}) = \left( (\Omega \circ \mathcal{P}) \circ (\Omega \circ \mathcal{P})^k \right)(\mathcal{R}),
\]

and by hypothesis,

\[
\left( (\Omega \circ \mathcal{P}) \circ (\Omega \circ \mathcal{P})^k \right)(\mathcal{R}) \subseteq \left( (\mathcal{P} \circ \Omega) \circ (\mathcal{Q} \circ \mathcal{P})^k \right)(\mathcal{R}).
\]

Since the composition of monotone functions is monotone, we get by the induction hypothesis that

\[
\left( (\mathcal{P} \circ \Omega) \circ (\mathcal{Q} \circ \mathcal{P})^k \right)(\mathcal{R}) \subseteq \left( (\mathcal{P} \circ (\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{Q} \circ \mathcal{P})^{k-1} \circ \mathcal{Q}^k \right)(\mathcal{R}).
\]

However, using once again the hypothesis that \( (\Omega \circ \mathcal{P})(\mathcal{R}) \subseteq (\mathcal{P} \circ \Omega)(\mathcal{R}) \) for all \( \mathcal{R} \), we get that

\[
\left( (\mathcal{P} \circ (\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{Q} \circ \mathcal{P})^{k-1} \circ \mathcal{Q}^k \right)(\mathcal{R}) \subseteq \left( (\mathcal{P} \circ (\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{Q} \circ \mathcal{P})^{k-1} \circ \mathcal{Q}^k \right)(\mathcal{R}),
\]

and so by the monotony of \( \mathcal{P} \), we get

\[
(\mathcal{P} \circ (\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{Q} \circ \mathcal{P})^{k-1} \circ \mathcal{Q}^k)(\mathcal{R}) \subseteq (\mathcal{P} \circ (\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{Q} \circ \mathcal{P})^{k-1} \circ \mathcal{Q}^k)(\mathcal{R}).
\]

Repeating in this manner, we get that

\[
((\mathcal{P}^i \circ \Omega \circ \mathcal{Q}^{k+1-i}) \circ \mathcal{Q}^k)(\mathcal{R}) \subseteq ((\mathcal{P}^i+1 \circ \Omega \circ \mathcal{Q}^k)(\mathcal{R}) \quad (3)
\]

for all \( 0 \leq i \leq k \). Thus, by transitivity using the inclusions (1), (2), and (3) all the way up to \( i = k \), we get that \( (\mathcal{P} \circ \mathcal{Q})^{k+1}(\mathcal{R}) \subseteq (\mathcal{P}^k \circ \mathcal{Q}^k)(\mathcal{R}) \) as desired. We thus conclude the lemma by induction. \( \square \)

**Proposition 12.** If \( \mathcal{P} \) and \( \mathcal{Q} \) are two monotone, finitely converging, (respectfully) compatible, and extensive up-to techniques such that \( (\Omega \circ \mathcal{P})(\mathcal{R}) \subseteq (\mathcal{P} \circ \Omega)(\mathcal{R}) \) for all \( \mathcal{R} \), and \( \mathcal{R} \mapsto (\Omega \circ \mathcal{P})(\mathcal{R}) \) implies \( \mathcal{R} \mapsto \Omega(\mathcal{R}) \), then \( \mathcal{P} \circ \Omega \) is sound.

**Proof.** Assume \( \mathcal{R} \mapsto (\Omega \circ \mathcal{P})(\mathcal{R}) \). Let \( N \) and \( M \) be the convergence constants of \( \mathcal{P} \) and \( \mathcal{Q} \) respectively (cf. Definition 3), and let \( L = \max(N, M) + 1 \). Then by Corollary 6, we have that \( \Omega^L(\mathcal{R}) \mapsto \Omega^L(\mathcal{R}) \), and so by compatibility of \( \mathcal{P} \), we get that \( (\mathcal{P}^L \circ \Omega^L)(\mathcal{R}) \mapsto (\mathcal{P}^L \circ \Omega^L)(\mathcal{R}) \). Thus, we have that \( (\mathcal{P}^L \circ \Omega^L)(\mathcal{R}) \subseteq \mathcal{R}' \) for some \( \mathcal{R}' \in \mathcal{F} \). By Proposition 7, \( (\Omega \circ \mathcal{P}) \) is extensive, so by repeated application of extensiveness and transitivity, we get that \( \mathcal{R} \subseteq (\Omega \circ \mathcal{P})^L(\mathcal{R}) \). By Lemma 11, we get that \( (\Omega \circ \mathcal{P})^L(\mathcal{R}) \subseteq (\mathcal{P}^L \circ \Omega^L)(\mathcal{R}) \). Thus, since we have \( \mathcal{R} \subseteq (\mathcal{P}^L \circ \Omega^L)(\mathcal{R}) \subseteq \mathcal{R}' \in \mathcal{F} \) by transitivity, we conclude that \( \mathcal{P} \circ \Omega \) is sound. \( \square \)

By using a stronger notion of progression, we can drop the hypotheses of commutation and finite convergence when showing soundness of compositions and up-to techniques in general. This stronger notion may seem ad hoc, but we will see that it is satisfied by the canonical progressions of well known bisimulations.

**Definition 13.** A progression \( \mathcal{R} \) is said to be continuous if for all ascending chains of relations \( \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \cdots \) and \( S_0 \subseteq S_1 \subseteq \cdots \) such that \( \mathcal{R}_i \mapsto S_i \) for all \( i \), we have \( \forall \mathcal{R} \mapsto \forall S \), where \( \forall \mathcal{R} = \bigcup_{i \in \mathbb{N}} \mathcal{R}_i \) and \( \forall S = \bigcup_{i \in \mathbb{N}} S_i \).

**Proposition 14.** If \( \mathcal{P} \) is an extensive up-to technique, then for all \( \mathcal{R} \), and where \( \mathcal{P}^i \) is the \( i \)-th iterate of \( \mathcal{P} \), \( \forall \mathcal{P}(\mathcal{R}) = \bigcup_{i \in \mathbb{N}} \mathcal{P}^i(\mathcal{R}) \) is a fixpoint for \( \mathcal{P} \).
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Proof. We want to show that \( P(vP(R)) = vP(R) \). The inclusion \( vP(R) \subseteq P(vP(R)) \) is immediate by extensiveness, so assume \( S \in P(vP(R)) \). Then there exists an \( i \in \mathbb{N} \) such that \( S \in P(P^i(R)) \). Then \( S \in P^{i+1}(R) \), so \( S \in vP(R) \). We conclude the equality by double inclusion. \( \square \)

Proposition 15. If \( \mathcal{F} \) has a continuous progression \( \rightarrow \) and \( P \) is a (respectfully) compatible, extensive up-to technique, then \( P \) is sound and \( vP(R) \rightarrow vP(R) \) whenever \( R \rightarrow P(R) \).

Proof. Assume \( R \rightarrow P(R) \), then, by extensiveness, \( R \subseteq P(R) \). Then for all \( k \in \mathbb{N} \), \( P^k(R) \rightarrow P^{k+1}(R) \) by (respectful) compatibility, and \( P^k(R) \subseteq P^{k+1}(R) \) by extensiveness. Let \( R_i = P^i(R) \) and \( S_i = P^{i+1}(R) \), then by continuity, \( vR \rightarrow vS \). However, \( vR = vP = vS \), so there exists an \( \mathcal{R}' \in \mathcal{F} \) such that \( vR \subseteq \mathcal{R}' \). Thus, \( R \subseteq vR \subseteq \mathcal{R}' \in \mathcal{F} \) and we conclude that \( P \) is sound. \( \square \)

Corollary 16. If \( \mathcal{F} \) has a continuous progression and \( P \) and \( Q \) are two (respectfully) compatible, extensive up-to techniques, then \( Q \circ P \) is sound.

Proof. The up-to technique \( Q \circ P \) is extensive by Proposition 7 and compatible by Proposition 8, and so sound by Proposition 15. \( \square \)

Corollary 17. If \( \mathcal{F} \) has a continuous progression and \( P \) is a (respectfully) compatible, extensive up-to techniques, then \( vP \) is (respectfully) compatible, extensive, and sound.

Finally, it is often useful to consider up-to techniques which are in a certain manner “asymmetric”. For example, for many types of bisimulation, soundness of “weak bisimilarity up-to-bisimilarity” requires us to use strong bisimilarity on the side of the term making a small-step transition, and permits weak bisimilarity on the side of the term answering with a large-step transition; this discussion will be made more precise with examples in the following sections.

Definition 18. A one-sided progression is a relation \( \rightarrow \subseteq \mathbb{U} \times \mathbb{U} \) such that both \( R \rightarrow R \) and \( R^{\mathbb{O}P} \rightarrow R^{\mathbb{O}P} \) only if there exists \( \mathcal{R}' \in \mathcal{F} \) such that \( R \subseteq \mathcal{R}' \).

Definition 19. Given a family of relations \( \mathcal{F} \subseteq \mathbb{U} \) with a one-sided progression, we call a partial function \( P : \mathbb{U} \rightarrow \mathbb{U} \) an asymmetric up-to technique, and say that it is sound if whenever \( P(R) \) and \( P(R^{\mathbb{O}P}) \) are defined and \( R \rightarrow P(R) \) and \( R^{\mathbb{O}P} \rightarrow P(R^{\mathbb{O}P}) \), there exists an \( \mathcal{R}' \in \mathcal{F} \) such that \( R \subseteq \mathcal{R}' \).

We adapt the definitions of commutativity, extensiveness, finite convergence, and commutativity in the obvious manner, substituting one-sided progression for progression.

Proposition 20. Every finitely convergent, extensive, (respectfully) compatible asymmetric up-to-technique \( P \) is sound.

Proof. Adapt the proof of Proposition 5, starting from \( R \rightarrow P(R) \) and \( R^{\mathbb{O}P} \rightarrow P(R^{\mathbb{O}P}) \), reaching \( P^{N+1}(R) \rightarrow P^{N+1}(R) \) and \( P^{N+1}(R^{\mathbb{O}P}) \rightarrow P^{N+1}(R^{\mathbb{O}P}) \), and deducing that \( P^{N+1}(R) \) is contained in some \( \mathcal{R}' \in \mathcal{F} \). Conclude that \( P \) is sound since by extensiveness and transitivity, \( R \subseteq \mathcal{R}' \in \mathcal{F} \) and \( R \) was arbitrary. \( \square \)

Proposition 21. If \( \mathcal{F} \) is has a continuous one-sided progression, and \( P \) is an extensive, (respectfully) compatible and asymmetric up-to technique, then \( P \) is sound.

Proof. Adapt the proof of Proposition 15, starting from \( R \rightarrow P(R) \) and \( R^{\mathbb{O}P} \rightarrow P(R^{\mathbb{O}P}) \), reaching \( vR \rightarrow vR \) and \( (vR)^{\mathbb{O}P} \rightarrow (vR)^{\mathbb{O}P} \), and deducing that \( R \subseteq vR \subseteq \mathcal{R}' \in \mathcal{F} \). \( \square \)
3. Preliminaries on the $\lambda$-calculus

We establish notation for the $\lambda$-calculus and prove a few useful lemmas about its contexts. We denote by $\Lambda$ the set of all $\lambda$ terms, and by $\Lambda^\ast$ the set of all closed $\lambda$ terms, i.e., those with no free variables. If $R \subseteq \Lambda \times \Lambda$ is a relation, and $\tilde{M} = (M_1, \ldots, M_n)$ and $\tilde{N} = (N_1, \ldots, N_n)$ are vectors in $\Lambda^n$, then we write $\tilde{M} \ R \ \tilde{N}$ for $(M_1 \ R N_1) \wedge \cdots \wedge (M_n \ R N_n)$. By abuse of notation, if $x \in \Lambda$, $M \in \Lambda^m$, and $\tilde{N} \in \Lambda^n$, we write $\tilde{M}X\tilde{N}$ for the term $M_1 \cdot M_nXN_1 \cdot \cdot \cdot N_n$. Finally, if $R$ is a relation and $S \subseteq \Lambda \times \Lambda$, we denote by $R \upharpoonright_S := R \cap S$ the restriction of $R$ to $S$.

**Definition 22.** If $R, R' \subseteq \Lambda \times \Lambda$ are two relations, then write $R R'$ for their composition, i.e., $M \ R R' \ N$ if and only if there exists an $L$ such that $M \ R L \ R' \ N$.

We recall the familiar notion of context (see, e.g., [2]) and Gordon’s [4] canonical contexts:

**Definition 23.** A context $C$ is given by the following grammar,

$$C ::= x \mid \cdot \mid C_1 C_2 \mid \lambda x. C,$$

Let a canonical context be a context $\emptyset$ given by the grammar

$$\emptyset ::= \cdot \mid \lambda x. C,$$

where $C$ ranges over all contexts.

A context can contain multiple holes, which we label $[\cdot]_1, [\cdot]_2, \ldots, [\cdot]_n$; by convention, each hole is assigned a unique $i$, and these are assigned in sequential order from left to right. Then if $C$ is a context with $n$ holes and $\tilde{M} \in \Lambda^n$, $C[\tilde{M}] \in \Lambda$ is obtained by replacing $[\cdot]_i$ in $C$ with $M_i$, the $i$th projection of $\tilde{M}$. We further adopt the notation $C(M)$ to denote the context $C$ whose every hole is filled with the term $M$.

**Definition 24.** If $\tilde{C} \subseteq \Lambda \times \Lambda$, then its open contextual closure, $\tilde{C}^\circ$, is given by

$$\tilde{C}^\circ = \{ (C[\tilde{M}], C[\tilde{N}]) \mid C \text{ is a context and } \tilde{M} \ R \tilde{N} \},$$

and its closed contextual closure is $\tilde{C}^\ast = \tilde{C}^\circ \cap \Lambda^\ast \times \Lambda^\ast$.

Since we deal mostly with relations on closed terms, unless otherwise specified, we take contextual closure to be closed contextual closure.

The following two facts are immediate from the definition of contextual closure of binary relations on $\lambda$-terms.

**Lemma 25.** The contextual closures of the empty set are the respective identity relations, i.e., $\emptyset^\circ = \text{Id}_{\Lambda \times \Lambda}$ and $\emptyset^\ast = \text{Id}_{\Lambda^\ast \times \Lambda^\ast}$.

**Lemma 26.** For $\delta \in \{ \circ, \ast \}$, if $A \ R^\delta C$ and $B \ R^\delta D$, then $AB \ R^\delta CD$.

The following technical lemma will frequently be used in the soundness proofs for the up-to context technique and when proving that coupled logical relations are congruences.

**Lemma 27.** For all $R \subseteq \Lambda^\ast \times \Lambda^\ast$, $\tilde{M} \ R \tilde{N}$, $M, N \in \Lambda$, and contexts $C_1$:

1. if $C_1[\tilde{M}] = M \ R^\ast N = C_1[\tilde{N}]$, then for all $\delta \in \{ \circ, \ast \}$, $E$ and $F$ such that $E \ R^\delta F$, and variables $x$, we have $M[E/x] \ R^\ast N[F/x]$;
2. If $C_1[M] = M \, \R \cdot N = C_1[N]$ with $fv(M) = \{x\}$ for some $x$, then for all $\triangle \in \{\circ, *\}$ and $E$ and $F$ such that $E \, \R \cdot F$, we have $M[E/x] \, \R \cdot N[F/x]$.

3. If $C_1[M] = M \, \R \cdot N = C_1[N]$, then for all $\triangle \in \{\circ, *\}$, $E$ and $F$ such that $E \, \R \cdot F$, and variables $x$, we have $M[E/x] \, \R \cdot N[F/x]$.

Proof. Let $\tilde{E}$, $\tilde{F}$, and $C_2$ be such that $F = C_2[\tilde{F}]$, and $\tilde{E} \, \R \cdot \tilde{F}$. Since each entry of $\tilde{M}$ and $\tilde{N}$ is in $\Lambda^*$, $x \not\in fv(M) \cup fv(N)$. Assume first that $M \, \R^* \cdot N$, then $fv(M) \cup fv(N) = \emptyset$, so $M[E/x] = M$ and $N[F/x] = N$ for all $E,F$, and $x$. Thus, $M[E/x] = M \, \R^* \cdot N = N[F/x]$ as desired. Now assume that $M \, \R^* \cdot N$. If $C_1 = [\cdot]$, then $M = C_1[M] \in \Lambda^*$ and $N = C_1[N] \in \Lambda^*$, so $M[E/x] \, \R \cdot N[F/x]$. Since $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^0$, we conclude that $M[E/x] \, \R^* \cdot N[F/x]$. Now assume that $C_1 \not= [\cdot]$, then $(C_1[M])[E/x] = (C_1[E/x])[M]$ and similarly for $N$ and $F$. Now consider the context $C_3 = C_1[C_2/x]$, or more formally, the context $C_3 = \gamma(C_1, x, C_2)$ where $\gamma(C_1, x, C_2)$ is recursively given by

$$\gamma(C_1, x, C_2) = \begin{cases} C_2 & \text{if } C_1 = x \\ y & \text{if } C_1 = y \\ \cdot & \text{if } C_1 = [\cdot] \\ \lambda z.\gamma(C'_1, x, C_2) & \text{if } C_1 = \lambda z.\gamma(C'_1, x, C_2) \text{ and } z \not= x \\ \gamma(C'_1, x, C_2) & \text{if } C_1 = \gamma(C'_1, x, C_2) \\ \end{cases}$$

and let $\tilde{M}'$ be the vector obtained by interweaving $\tilde{M}$ and $\tilde{E}$ such that $C_3[\tilde{M}'] = C_1[C_2[\tilde{E}]/x][\tilde{M}]$. More formally, we can let $\tilde{M}' = \phi(C_1, x, \tilde{M}, \tilde{E})$ where

$$\phi(C, x, \tilde{M}, \tilde{E}) = \begin{cases} \tilde{E} & \text{if } C = x \\ y & \text{if } C = y \\ M_i & \text{if } C = [\cdot] \\ \phi(C', x, \tilde{M}, \tilde{E}) & \text{if } C = \lambda z.\gamma(C') \\ \phi(C', x, \tilde{M}, \tilde{E}) \cdot \phi(C'', x, \tilde{M}, \tilde{E}) & \text{if } C = C' \cdot C''. \\ \end{cases}$$

Similarly, let $\tilde{N}' = \phi(C_1, x, \tilde{N}, \tilde{F})$, then $N[F/x] = C_3[\tilde{N}]$. If $fv(M) = fv(N) = \{x\}$, then $C_3[\tilde{M}']$ and $C_3[\tilde{N}']$ are closed, so $C_3[\tilde{M}'] \, \R^* \cdot C_3[\tilde{N}]$. In all other cases, since $C_3[\tilde{M}'] \, \R^* \cdot C_3[\tilde{N}]$, this completes our case analysis and we conclude the lemma.

**Corollary 28.** Suppose $\mathcal{R} \subseteq \Lambda^* \times \Lambda^* \times \Lambda^* \times \Lambda^* \times \Lambda^* \times \Lambda^*$. Then, we have $(C[\tilde{P}])[M/x] \, \R^* \cdot (C[\tilde{Q}])[N/x]$.

We use the definition of congruence given by Selinger [13]:

**Definition 29.** A relation $\mathcal{R} \subseteq \Lambda \times \Lambda$ is said to be a congruence if it is an equivalence relation and additionally respects the rules for constructing $\lambda$-terms, i.e., if it satisfies:

$$\begin{array}{c c c} A \, \mathcal{R} \, C & B \, \mathcal{R} \, D & M \, \mathcal{R} \, N \\ AB \, \mathcal{R} \, CD & \lambda x.\mathcal{R} \, \lambda x.\mathcal{R} \end{array}$$

Finally, the following definitions will serve to define coupled logical bisimulations in the next section:

**Definition 30.** A paired relation $\mathcal{R}$ is a pair of relations $(\mathcal{R}_1, \mathcal{R}_2)$ with $\mathcal{R}_1, \mathcal{R}_2 \subseteq \Lambda^* \times \Lambda^*$. A coupled relation is a paired relation $\mathcal{R}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

We define the usual set theoretic operations on coupled relations in a pointwise manner, e.g., for two coupled relations $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{R}$ if $\mathcal{R}_1' \subseteq \mathcal{R}_1$ and $\mathcal{R}_2' \subseteq \mathcal{R}_2$, we let $\mathcal{R} \cup \mathcal{R}' = (\mathcal{R}_1 \cup \mathcal{R}_1', \mathcal{R}_2 \cup \mathcal{R}_2')$, etc.
4. CLB in the Call-by-name $\lambda$-calculus

We begin by considering the theory of coupled logical bisimulations for the call-by-name (cbn) $\lambda$-calculus. Many proofs are omitted or only sketched since they can be seen as simplifications of the proofs given in the call-by-value case (Section 5).

**Definition 31.** The call-by-name $\lambda$-calculus is defined by the following reduction rules:

- $M \rightarrow M'$
- $MN \rightarrow M'N$
- $(\lambda x. P)N \rightarrow P[N/x']$

We take the set $V$ of values to be the set of all abstractions $\lambda x. P \in \Lambda^*$. We write $\Rightarrow$ for the reflexive and transitive closure of $\rightarrow$, and say that a term $M$ converges, written $M \Downarrow$, if there exists a value $\lambda x. P$ such that $M \Rightarrow \lambda x. P$; we may also write $M \Downarrow M'$ and $M' \Downarrow$. Similarly, we say $M$ diverges, written $M \Uparrow$, if it does not converge; in this case, it will sometimes be useful to write $M \Uparrow M'$ if $M \Rightarrow M'$.

**Definition 32.** A cbn evaluation context $C$ is given by the following grammar ($M$ ranges over $\Lambda^*$):

$C ::= [ \cdot ] | C M$

**Definition 33.** Two terms $M, N \in \Lambda^*$ are contextually equivalent, written $M \simeq^R N$, if for all contexts $C$, $C[M] \Downarrow$ if and only if $C[N] \Downarrow$. Similarly, we say that two terms $M$ and $N$ are evaluation-contextually equivalent, written $M \cong^R N$, if for all evaluation contexts $C$, $C[M] \Downarrow$ if and only if $C[N] \Downarrow$.

Although contextual equivalence appears to be a stronger notion of equivalence than evaluation-contextual equivalence, Milner’s [6] “context lemma” (see also Gordon [4, Proposition 4.18]) tells us otherwise:

**Theorem 34 ([6]).** Evaluation-contextual equivalence and contextual equivalence coincide, i.e., $M \cong^R N$ if and only if $M \simeq^R N$.

The following definition is based on [3, Lemma 5.8]:

**Definition 35.** If $R, R' \subseteq \Lambda^* \times \Lambda^*$ are relations, then the evaluation-contextual closure of $R$ under $R'$, written $(R \otimes R')$, is given by

$$R \otimes R' = \{ (E\bar{M}, F\bar{N}) \mid E \in R F \text{ and } \bar{M} R' \bar{N} \}$$

where $\bar{M}$ and $\bar{N}$ are potentially empty.

We remark that for all $R$ and $R'$, $R \subseteq (R \otimes R')$.

**Definition 36.** If $R$ is a coupled relation, then let its contextual closure $R^C$ be given by

$$R^C = (R_1^+, (R_2 \otimes R_3^+) \cup R_4^+)$$

4.1. Coupled Logical Bisimulation

The following definition is extracted from the corresponding notion in [3]:

**Definition 37.** A coupled relation $R$ is a coupled logical bisimulation (CLB) if whenever $M \not\in R N$, we have:

1. if $M \rightarrow M'$, then there exists an $N'$ such that $N \Rightarrow N'$ and $M \not\in R N'$;
2. if \( M = \lambda x. M' \), then \( N \implies \lambda x. N' \) and for all \( P, Q \in \textbf{A}^* \) such that \( P \ R_1^* Q \), we have \( M'[P/x] \ R_2 \ N'[Q/x] \); 

3. and the converses of the two previous conditions for \( N \).

Coupled logical bisimilarity, written \( \approx^m = (\approx_1^m, \approx_2^m) \), is the pairwise union of all CLBs.

As one would hope, CLBs have a continuous progression:

**Definition 39** (CLB progressions, call-by-name case). Given coupled relations \( R \) and \( S \), we say \( R \) **progresses to** \( S \), written \( R \rightarrow S \), if whenever \( M \ R_2 \ N \), then:

1. whenever \( M \longrightarrow M' \) then \( N \longrightarrow N' \) and \( M' \ S_2 \ N' \);

2. whenever \( M = \lambda x. P \) then \( N \longrightarrow \lambda x. Q \) such that for all \( X \ R_1^* Y \), \( P[X/x] \ S_2 \ Q[Y/x] \);

3. the converses of the previous two conditions for \( N \).

**Proposition 39.** For all coupled relations \( R, R \rightarrow R \) if and only if \( R \) is a CLB. Thus, the relation \( \rightarrow \) is a progression for CLBs in the universe of coupled relations.

**Proposition 40.** The relation \( \rightarrow \) is continuous.

**Proof.** Assume \( \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \cdots \) and \( S_0 \subseteq S_1 \subseteq \cdots \) are two ascending chains of relations such that \( \mathcal{R}_i \rightarrow S_i \) for all \( i \), and assume \( M \ (\forall i) \ R_2 \ N \) and \( X \ (\forall i) \ R_1^* Y \) for arbitrary \( M, N, X, Y \). Then \( X = C[M] \) and \( Y = C[N] \) for some \( M \ (\forall i) \ N \). Moreover, there exists a \( K \) such that for all \( n > K \), \( M \ (\forall n) \ R_2 \ N \) and \( M \ (\forall n) \ 

If \( M \rightarrow M' \), then, since \( \mathcal{R}_n \rightarrow S_n \), \( N \rightarrow N' \) such that \( M' \ (S_n) \ R_2 \ N' \). Since \( S_n \subseteq \forall \mathcal{S} \), we then get that \( M' \ (\forall \mathcal{S}) \ R_2 \ N' \) and we are done.

If \( M = \lambda x. M' \), then, since \( \mathcal{R}_n \rightarrow S_n \), \( N \rightarrow \lambda x. N' \) such that \( M'[X/x] \ (S_n) \ R_2 \ N'[Y/x] \) for all \( X \ (\forall n) \ Y \). Since \( S_n \subseteq \forall \mathcal{S} \), we then get that \( M'[X/x] \ (\forall \mathcal{S}) \ R_2 \ N'[Y/x] \) and we are done.

The symmetric cases for \( N \) follow symmetrically.

In the notation of Section 2, the family of desired relations \( \mathcal{F} \) is the set of all CLBs, and the universe \( \mathcal{U} \) of relations we are working in is the set of all coupled relations. Unfortunately, we cannot extend our notion of progression to all paired relations since it is not the case that for all paired relations \( \mathcal{R}, \mathcal{R} \rightarrow \mathcal{R} \) only if there exists a CLB \( \mathcal{R}' \) such that \( \mathcal{R} \subseteq \mathcal{R}' \). To see why, consider the paired relation \( \mathcal{R} = ([1, \Omega]), ([\Omega, \Omega]) \), where \( \Omega = (\lambda x.xx)(\lambda x.xx) \) is the standard divergent term and \( I = \lambda x.x \) the identity. It is easy to see that \( \mathcal{R} \rightarrow \mathcal{R} \), so assume there existed a CLB \( \mathcal{R}' \) such that \( \mathcal{R} \subseteq \mathcal{R}' \). Since every CLB is a coupled relation, we would have \( (I, \Omega) \in \mathcal{R}_1 \subseteq \mathcal{R}_1' \subseteq \mathcal{R}_2 \). Since \( I = \lambda x.x \), by the second clause of the definition of CLB, \( \Omega \) would converge.

We further develop properties of progression that will be useful in showing the soundness of various up-to techniques:

**Proposition 41.** If \( (\mathcal{R}_i)_{i \in I} \) and \( (S_i)_{i \in I} \) are two families of paired relations such that \( \mathcal{R}_i \rightarrow S_i \) for all \( i \in I \), then \( \bigcap_{i \in I} (\mathcal{R}_i) \cup \bigcup_{i \in I} (\mathcal{R}_i) \rightarrow \bigcup_{i \in I} S_i \).

**Proof.** Let \( \mathcal{U} = \bigcap_{i \in I} (\mathcal{R}_i) \cup \bigcup_{i \in I} (\mathcal{R}_i) \), and assume \( M \mathcal{U}_2 N \). Then there exists some \( \mathcal{R}_i \) such that \( M \ (\mathcal{R}_i) \mathcal{U}_2 N \).

If \( M \rightarrow M' \), then \( N \rightarrow N' \) such that \( M' \ (S_i) \mathcal{U}_2 N' \), and by inclusion, we get that \( M' \ (\cup_{i \in I} S_i) \mathcal{U}_2 N' \).
If \( M = \lambda x.M' \) and \( X \cup_1 Y \), then since \( \cup_1 \subseteq (R_1)_1 \), we have \( X (R_1)_1^* Y \) by the monotonicity of contextual closure. Then since \( R_1 \vdash S_1, N \Rightarrow \lambda x.N' \) and \( M'[X/x] \cup_1 (S_1)_2 N'[Y/x] \), and by inclusion, we get that \( M'[X/x] \cup_1 (\bigcup_{i \in 1} (S_1)_2) N'[Y/x] \).

The symmetric cases for \( N \) follow symmetrically, and we’re done.

**Proposition 42.** Progression \( \Rightarrow \) is closed under left intersection and right union.

**Proposition 43.** For all \( R, S \) such that \( R_1^* \subseteq R_2 \) and \( S_1^* \subseteq S_2 \), if \( R \Rightarrow S \) then \( (R_1^*, R_2) \Rightarrow (S_1^*, S_2) \).

**Proof.** Immediate by the idempotence of contextual closure.

**Corollary 44.** If \( R \) is a CLB such that \( R_1^* \subseteq R_2 \), then \((R_1^*, R_2)\) is also a CLB.

**Proof.** By Proposition 39, \( R \Rightarrow R \), so by Proposition 43, \((R_1^*, R_2) \Rightarrow (R_1^*, R_2)\) and \((R_1^*, R_2)\) is a coupled relation. Then again, by Proposition 39, \((R_1^*, R_2)\) is a CLB.

**Lemma 45.** If \((R_1, R_2)\) is a CLB, then so is \((R_1^*, R_2)\) for all \( R_1^* \subseteq R_1 \).

**Proof.** Check the definition and use the monotonicity of contextual closure.

**Corollary 46.** If \((R_1, R_2)\) is a CLB, then so is \((\emptyset, R_2)\).

**Lemma 47.** Let \( A, B \subseteq R_2 \) such that \((A, R_2)\) and \((B, R_2)\) are both CLBs. Then \((A \cup B, R_2)\) is also a CLB.

**Proof.** Follows from a straightforward check of the definition.

**Lemma 48.** If \( R \) is a CLB, \( M \vdash_2 N \) and \( M \Rightarrow M' \), then there exists \( N' \in \Lambda^* \) such that \( N \Rightarrow N' \) and \( M' \vdash_2 N' \).

**Proof.** Follows by a straightforward induction on the length of the reduction \( M \Rightarrow M' \).

Having introduced the necessary material for CLB, we can work with the theory of Section 2 to analyse up-to techniques for CLB in the cbn case. We study two versions of the up-to contexts technique. This will allow us to deduce congruence properties for CLB.

**Definition 49.** We call **up-to evaluation context** the up-to technique \( P_{ev} \) given by:

\[
P_{ev}(R) = (R_1, R_2) \cup \{(EM, FN) \mid E R_2 F, M R_1^* N\}.
\]

**Proposition 50.** We have that \( \forall P_{ev}(R) = (R_1, (R_2 \otimes_n R_1^*)) \).

**Proof.** Simple double inclusion.

**Proposition 51.** Up-to evaluation context is extensive and respectfully compatible.

**Proof.** Extensiveness is immediate by definition, so assume \( R \Rightarrow S \) and \( R \subseteq S \). Clearly \( P_{ev}(R) \subseteq P_{ev}(S) \). We want to show that \( P_{ev}(R) \Rightarrow P_{ev}(S) \), so assume \( M P_{ev}(R) N \). If \( M \not\vdash_2 N \), then we’re done since \( R \Rightarrow S \), so assume instead that \( M = EM' \) and \( N = FN' \) for \( E R_2 F \) and \( M' R_1^* N' \). Since \( R \subseteq S \) and contextual closure is monotone, we have \( M' \delta_1^* N' \).

Assume first that \( M = M'' \), then we fall into two cases. The first is that \( EM' \Rightarrow E'M' \). Since \( E R_2 F \), then \( F \Rightarrow F' \) such that \( E' R_2 F' \). Then \( E'M' P_{ev}(S) \) \( F' N' \) and we’re done. The second is that \( E = \lambda x.E' \) and that \( EM' \Rightarrow E'[M'/x] \). Then since \( E R_2 F \), then \( F \Rightarrow \lambda x.F' \) such that \( \lambda x.E' S_2 \lambda x.F' \) and for all \( X R_1^* Y \), \( E'[X/x] S_2 F'[Y/x] \). Thus, \( FN' \Rightarrow F'[N'/x] \) and \( E'[M'/x] P_{ev}(S) \) \( F'[N'/x] \) and we’re done.

The case that \( EM' = \lambda x.M'' \) is impossible, so we’re done and conclude respectful compatibility.
Corollary 52. Up-to evaluation context is sound.

Corollary 53. If 𝑅 is a CLB, then so is \( (𝑅_1, (𝑅_2 \otimes_𝑛 𝑅^*_1)) \).

Proof. By Proposition 39, \( 𝑅 \Rightarrow 𝑅 \). Then by compatibility, \( 𝑃^l_{𝑒𝑣}(𝑅) \Rightarrow 𝑃^l_{𝑒𝑣}(𝑅) \) for all \( i \in \mathbb{N} \), and by extensiveness, we have an ascending chain of relations \( 𝑅 \subseteq 𝑃^l_{𝑒𝑣}(𝑅) \subseteq 𝑃^l_{𝑒𝑣}(𝑅) \subseteq \cdots \). Then by continuity (Proposition 40), \( 𝑉𝑃^l_{𝑒𝑣}(𝑅) \Rightarrow 𝑉𝑃^l_{𝑒𝑣}(𝑅) \), and so by Proposition 39, \( 𝑉𝑃^l_{𝑒𝑣}(𝑅) \) is a CLB. Since \( 𝑃^l_{𝑒𝑣}(𝑅) = (𝑅_1, (𝑅_2 \otimes_𝑛 𝑅^*_1)) \), we conclude that \( (𝑅_1, (𝑅_2 \otimes_𝑛 𝑅^*_1)) \) is a CLB.

Definition 54. We call up-to context the up-to technique given by \( 𝑅 \Rightarrow 𝑅^C \).

Lemma 55. If \( E 𝑅^C_2 F \) and \( \overline{𝑀} 𝑅^*_1 \overline{𝑁} \), then \( \overline{𝐸𝑀} 𝑅^C_2 𝐹\overline{𝑁} \).

Proof. By case analysis on \( E 𝑅^C_2 F \). If \( E 𝑅^C_2 F \) because \( E 𝑅^*_1 F \), then the conclusion follows by Lemma 26. Finally, if \( E 𝑅^C_2 F \) because there exist \( E’, F’, K, L \) such that \( E’ 𝑅_2 F’, 𝑅^*_1 𝑅^*_1, E = E’K’, F = F’L \), then by Corollary 28, \( \overline{𝐾𝑀} 𝑅^*_1 \overline{𝐿𝑁} \), and so by construction, \( \overline{𝐸𝑀} = E’\overline{𝐾𝑀 Rif} F’\overline{𝐿𝑁} = 𝐹\overline{𝑁} \).

Proposition 56. Up-to context is extensive and respectfully compatible.

Proof. Extensiveness is immediate by definition of up-to context.

Assume \( 𝑅 \Rightarrow 𝑆 \) and \( 𝑅 \subseteq 𝑆 \); we want to show that \( 𝑅^C \subseteq 𝑆^C \) and \( 𝑅^C \Rightarrow 𝑆^C \). The inclusion \( 𝑅^C \subseteq 𝑆^C \) is obvious. Since \( (𝑅_1, (𝑅_2 \otimes_𝑛 𝑅^*_1)) \Rightarrow (𝑆_1, (𝑆_2 \otimes_𝑛 𝑆^*_1)) \) by Propositions 51 and 50 and Corollary 17, it is sufficient to show that \( (𝑅_1, 𝑅^*_1 \setminus (𝑅_2 \otimes_𝑛 𝑅^*_1)) \Rightarrow 𝑆^C \), since then, by Proposition 41,

\[
(𝑅_1 \cap 𝑅_1, (𝑅_2 \otimes_𝑛 𝑅^*_1) \cup (𝑅^*_1 \setminus (𝑅_2 \otimes_𝑛 𝑅^*_1))) \Rightarrow (𝑆 \cup 𝑆^C),
\]

i.e., \( (𝑅_1, 𝑅^*_C) \Rightarrow 𝑆^C \) by extensiveness and Proposition 50. Then, since contextual closure is idempotent and \( 𝑅^*_1 \subseteq 𝑅^*_2 \) and \( (𝑆_1)^* \subseteq 𝑆_2 \), we get \( (𝑅^*_1, 𝑅^*_C) \Rightarrow 𝑆^C \) by Proposition 43, i.e., \( 𝑅^C \Rightarrow 𝑆^C \).

We show that \( (𝑅_1, 𝑅^*_1 \setminus (𝑅_2 \otimes_𝑛 𝑅^*_1)) \Rightarrow 𝑆^C \). Assume \( M 𝑅^*_1 N \). Then \( M = C[\overline{𝑀}] \) and \( N = C[\overline{𝑁}] \) for \( \overline{𝑀} 𝑅^*_1 \overline{𝑁} \). We proceed by induction on \( C \).

If \( C = x \), then \( M = M_1, 𝑅_1 N_1 \), and \( N_1 = N \), and since \( 𝑅_1 \subseteq 𝑅_2 \), and \( 𝑅 \Rightarrow 𝑆 \subseteq 𝑆^C \) and \( \Rightarrow \) is closed under right union, we're done.

The case \( C = x \) is impossible: it is never the case that \( x 𝑅^*_1 x \) since \( x \notin Λ^* \).

Consider the case where \( C = λx.C’ \), i.e., \( λx.C’[\overline{𝑀}] \) \( 𝑅^*_1 \) \( λx.C’[\overline{𝑁}] \). Since there exists no \( M’ \) such that \( λx.C’[\overline{𝑀}] \Rightarrow M’ \), we need only check the second clause of the definition of CLB, namely that for all \( X R_1^* Y, (C’[\overline{𝑀}])[X/x] S_1^C (C’[\overline{𝑁}])[Y/x] \). By Corollary 28, \( (C’[\overline{𝑀}])[X/x] R_1^* (C’[\overline{𝑁}])[Y/x] \), so \( X \subseteq X \且 S_j^C (C’[\overline{𝑁}])[Y/x] \) and we’re done.

Now assume that \( C = C_1C_2 \), and let \( \overline{𝑀} = M_1M_2 \) be such that \( C[\overline{𝑀}] = C_1[\overline{M}][C_2[M_2] \) and similarly for \( \overline{𝑁} \). We show the first part of the definition, namely, that if \( M \Rightarrow M’ \) then \( N \Rightarrow N’ \) such that \( M’ S_2^C N’ \). We fall into one of four mutually exclusive cases:

(i) \( C_1[M_1] \Rightarrow C_1’[\overline{M}_1] \)

(ii) \( C_1[M_1] \Rightarrow C_1[M_1’] \)

(iii) \( C_1[M_1] \Rightarrow β M_1’ \)

(iv) \( C_1[M_1] = λx.P \) and \( M \Rightarrow P[C_2[M_2]/x] \).
If we fall into case (i), then there is a transition internal to the context $C_1$ not involving any of the $M_k$, that is to say, "$C_1[n] \rightarrow C'_1[n']". Then $M \rightarrow C_1[M_1]C_2[M_2]$ and $N \Rightarrow C'_1[N_1]C_2[N_2]$. Since $C_1C_2$ is again a context, we have

$$C'_1[M_1]C_2[M_2] \subseteq C'_1[N_1]C_2[N_2],$$

and since $R^*_1 \subseteq S^*_1 \subseteq S^*_2$,

$$C'_1[M_1]C_2[M_2] \subseteq S^*_2 C'_1[N_1]C_2[N_2],$$

as desired.

Case (ii) is impossible: take $M_1 = (M_1, \ldots, M_n)$ and $N_1 = (N_1, \ldots, N_n)$. If $C_1[M_1] \rightarrow C_1[M'_1]$, then by call-by-name reduction implies that $C_1$ is of the form $C_1 = \tilde{n}_1C'_1$. Thus, $M = [M_2]C'_1[M_2, \ldots, M_n]C_2[M_2] \subseteq [R_2 \odot n R^*_1] [N_1]C'_1[N_2, \ldots, N_n]C_2[N_2] = N$, a contradiction.

If we fall into case (iii), that’s to say, if there’s a $\beta$-reduction involving the context $C_1$ and $M_1$, then $C_1[M] \rightarrow_\beta M'_1$, so $M \rightarrow M'_1C_2[M_2]$. By the induction hypothesis, $C_1[N] \Rightarrow N'_1$ such that $M'_1 \subseteq S^*_2 N'_1$. Thus, $N \Rightarrow N'_1C_2[N_2]$. Since $C_2[M_2] \subseteq C'_1[N_2]$ and $R^*_1 \subseteq S^*_1$, by Lemma 55, we conclude that $M'_1C_2[M_2] \subseteq S^*_2 N'_1C_2[N_2]$ as desired.

Finally, we consider case (iv). By the induction hypothesis, $C_1[N_1] \Rightarrow \lambda x . Q$ such that $\lambda x . P \subseteq S^*_2 \lambda x . Q$ and for all $X \ni R^*_1 Y$, $P[X/x] \subseteq S^*_2 Q[Y/x]$. Thus, since $C_2[M_2] \subseteq C'_1[N_2]$, we get $P[C_2[M_2]/x] \subseteq S^*_2 Q[C_2[N_2]/x]$ as desired.

This completes the induction on $C$. Thus, we conclude that $(R_1, R^*_1 \setminus (R_2 \odot n R^*_1)) \Rightarrow S^*_2$, and thus that $R^C \Rightarrow S^C$.

**Corollary 57.** Up-to context is sound.

**Corollary 58.** If $R$ is a CLB, then so is $R^C$.

**Proof.** Immediate by Proposition 39 and respectful compatibility.

**Corollary 59.**

1. If $M \equiv^!_1 N$, then $C[M] \equiv^!_1 C[N]$ for all contexts $C$.

2. If $E \equiv^!_2 F$, then $\exists[E] \equiv^!_2 \exists[F]$ for all evaluation contexts $\exists$.

**Proof.** We show the first statement. If $M \equiv^!_1 N$, then there exists a CLB $R \subseteq \equiv^!$ such that $M \equiv R_1 N$. Then $R^C \subseteq \equiv^!$ also and $R^*_1 \subseteq R^C \subseteq \equiv^!$.

We proceed in a similar fashion to show the second statement: if $E \equiv^!_2 F$ then there exists a CLB $R \subseteq \equiv^!$ such that $E \equiv R_2 F$. Then again, $R^C \subseteq \equiv^!$. Since $(R_2 \odot n R^*_1) \subseteq R^C$ and $Id \subseteq R^*_1$, we have $(R_2 \odot n Id) \subseteq R^C$. Thus, $E \equiv R_2 F$ implies $\exists[E] \equiv R^C \exists[F]$ for all evaluation contexts $\exists$, and since $R^C \subseteq \equiv^!$, we conclude the second statement.

**Corollary 60.** We have the inclusion $\equiv^\sim \subseteq (\equiv^!, \equiv^\approx)$, i.e., if $M \equiv^!_1 N$, then $M \equiv^\sim N$, and if $E \equiv^!_2 F$, then $E \equiv^\sim F$.

**Proof.** Assume first that $M \equiv^!_1 N$, then $C[M] \equiv^!_1 C[N]$, and so for some CLB $R$, $C[M] \subseteq C[N]$. By Lemma 48, this implies that $C[M] \Rightarrow \lambda x . P$ for some $P$ if and only if $C[N] \Rightarrow \lambda x . Q$ for some $Q$, and conversely. But this is exactly the definition of $M \equiv^\sim N$. The case of $E \equiv^!_2 F$ follows in an identical manner.

**Lemma 61.** If $E \Rightarrow E'$, then $E \equiv^\approx E'$.
Proof. It is easy to verify that \( (\emptyset, \rightarrow) \) is a CLB. By Corollary 60, we get \( \implies \subseteq \cong \subseteq \equiv_n \), so if \( E \rightarrow E' \), \( E \equiv_n E' \).

Lemma 62. If \( \lambda x. P \equiv_n (\lambda x.Q) \), then \( P[M/x] \equiv_n Q[M/x] \) for all \( M \).

Proof. If \( \lambda x. P \equiv_n (\lambda x.Q) \), then by Corollary 59, \( (\lambda x.P)M \equiv_n (\lambda x.Q)M \) for all \( M \). Moreover, \( (\lambda x.P)M \implies P[M/x] \) and \( (\lambda x.Q)M \implies Q[M/x] \). Thus, by Lemma 61, \( P[M/x] \equiv_n (\lambda x.P)M \equiv_n (\lambda x.Q)M \equiv_n Q[M/x] \) and the conclusion follows by transitivity of \( \equiv_n \).

Lemma 63. If \( M \equiv_n N \), then for all \( P \in \Lambda \), \( P[M/x] \equiv_n P[N/x] \) for all \( x \notin \text{bv}(P) \).

Proof. Let \( C = P[\cdot]/x \) be the context obtained by replacing every occurrence of \( x \) with a hole, and so \( C[M] = P[M/x] \) and similarly for \( N \). Then, by Corollary 59, \( C[M] \equiv_n C[N] \), so \( P[M/x] \equiv_n P[N/x] \). Since \( \equiv_n \subseteq \equiv \), we conclude \( P[M/x] \equiv_n P[N/x] \) as desired.

Proposition 64. The coupled relation \( (\sim_n, \sim_n) \) is a CLB, that is to say, \( (\sim_n, \sim_n) \subseteq \approx_n \).

Proof. We show the first clause of the definition of CLB. Assume \( M \equiv_n N \) and that \( M \rightarrow M' \) (so \( M \rightarrow M' \), and thus \( M \equiv_n M' \)). Then \( N \rightarrow N \) and \( M' \equiv_n N \) as desired follows by the transitivity and symmetry of \( \equiv_n \).

We now show that \( \lambda x. P \equiv_n N \) satisfies the second clause of the definition of CLB. By definition of \( \equiv_n \), \( N \rightarrow \lambda x. Q \). Let \( V \equiv_n W _1 \), then by Lemma 62, we have \( P[V/x] \equiv_n Q[W/x] \), and by Lemma 63, we have \( Q[V/x] \equiv_n Q[W/x] \). By transitivity of \( \equiv_n \), we conclude that \( P[V/x] \equiv_n Q[W/x] \) as desired.

Corollary 65. The contextual equivalences and coupled logical bisimilarity coincide; that is, we have \( (\equiv_n, \equiv_n) = (\equiv_n, \equiv_n) \).

Corollary 66. The two components of coupled logical bisimilarity coincide with each other and with contextual equivalences, i.e., \( \equiv_n \equiv_n = \equiv_n = \equiv_n = \equiv_n \).

Proof. Immediate by Corollary 65 and Theorem 34.

4.1.1. Further up-to techniques

We present further up-to techniques, in addition to the up-to evaluation and up-to context techniques presented above.

Definition 67. We call up-to reduction the up-to technique \( R \mapsto \Rightarrow \), where \( (\Rightarrow) ) = R_1 \) and \( (\Rightarrow) ) = R_2 \), i.e., \( M \Rightarrow N \), if there exist \( M' \) and \( N' \) such that \( M \rightarrow M' \), \( N \rightarrow N' \), and \( M' \preceq N' \).

Proposition 68. Up-to reduction is extensive and respectfully compatible.

Proof. Extensiveness is immediate by the reflexivity of \( \Rightarrow \): by definition, \( R_1 = (\Rightarrow) \). Moreover, \( R_2 \subseteq (\Rightarrow) \) since whenever \( M \Rightarrow N \), \( M \rightarrow M' \), \( N \Rightarrow N' \), i.e., \( M \rightarrow M' \), \( N \rightarrow N' \). Hence, \( R \subseteq (\Rightarrow) \).

We now show compatibility. Assume \( R \Rightarrow S \) and \( R \subseteq S \), then we want to show that \( \Rightarrow S \).

Assume \( M \Rightarrow N \), and \( M' \) and \( N' \) be such that \( M \rightarrow M' \), \( N \rightarrow N' \). First assume that \( M \rightarrow M' . \) If \( M \neq M' \), then we’re done, since \( N \rightarrow N \) and \( M' \rightarrow M'' \), \( N' \rightarrow N' \), and by inclusion of \( R \) in \( S \), \( M' \rightarrow M'' \). \( S \rightarrow N' \), \( N \). Otherwise, if \( M = M' \), then the fact that \( R \Rightarrow S \)
and $M'' \Rightarrow S_2 N''$ implies that $N'' \Rightarrow N'$ such that $M' \Rightarrow S_2 N'$, so $N \Rightarrow N'$ by transitivity of $\Rightarrow$, and $M' (\Rightarrow \forall \subseteq \gamma_2) N'$ by reflexivity of $\Rightarrow$, as desired.

Now assume that $M = \lambda x.P$, then $M'' = \lambda x.P$ as well. Then since $\lambda x.P \Rightarrow S_2 N''$ and $R \Rightarrow S$, we have that $N'' \Rightarrow \lambda x.Q$ such that for all $X : \forall \Rightarrow Y$, $P[X/x] \Rightarrow S_2 Q[Y/x]$. By transitivity of $\Rightarrow$, we thus have that $N \Rightarrow \lambda x.Q$ and for all $X (\Rightarrow \forall \subseteq \gamma_1) Y$, that $P[X/x] (\Rightarrow \forall \subseteq \gamma_2) Q[Y/x]$ as desired.

The symmetric cases follow symmetrically and we conclude respectful compatibility. 

**Corollary 69.** Up-to reduction is sound.

### 4.2. Applicative Bisimulation

We recall the big-step version of applicative bisimulation, as originally presented by Abramsky [1]:

**Definition 70.** A relation $\mathcal{R} \subseteq \Lambda^* \times \Lambda^*$ is called an **applicative bisimulation** if $M \mathcal{R} N$ implies that whenever $M \Rightarrow \lambda x.P, N \Rightarrow \lambda x.Q$ for some $Q$, and $P[W/x] \mathcal{R} Q[W/x]$ for all $W \in \Lambda^*$, and conversely for $N$. We call the union of all applicative bisimulations, written $\approx^n_A$, **applicative bisimilarity**.

It is not hard to show that applicative bisimilarity is itself an applicative bisimulation.

**Proposition 71.** If $(\mathcal{R}', \mathcal{R})$ is a CLB for some $\mathcal{R}' \subseteq \mathcal{R} \cap \text{Id}$, then $\mathcal{R}$ is an applicative bisimulation.

**Proof.** Immediate by Lemma 48 and the definitions of applicative bisimulation and coupled logical bisimulation.

**Corollary 72.** The relation $\approx^n_A$ is an applicative bisimulation.

**Lemma 73.** Applicative bisimilarity is an equivalence relation.

**Proof.** To show reflexivity, observe that $\text{Id}$ is an applicative bisimulation, so $M \approx^n_A M$ for all $M$.

To show symmetry, observe that if $M \approx^n_A N$, then $M \mathcal{R} N$ for some applicative bisimulation. Then $\mathcal{R}^\circ \mathcal{P} \subseteq \approx^n_A$ is also an applicative bisimulation, so $N \approx^n_A M$.

To show transitivity, observe that if $L \approx^n_A M$ and $M \approx^n_A N$, then $L \mathcal{R} M$ and $M \mathcal{R} N$ for some applicative bisimulations, $\mathcal{R}, \mathcal{R}' \subseteq \approx^n_A$. Then $L \Rightarrow \lambda x.L'. \text{iff} \; M \Rightarrow \lambda x.M'. \text{such that} \; L'[V/x] \mathcal{R} M'[V/x]$ for all $V$, and $M \Rightarrow \lambda x.M'. \text{iff} \; N \Rightarrow \lambda x.N'. \text{such that} \; M'[V/x] \mathcal{R}' N'[V/x]$ for all $V$. Thus, $L \Rightarrow \lambda x.L'. \text{iff} \; N \Rightarrow \lambda x.N'. \text{such that} \; L'[V/x] \mathcal{R} \mathcal{R}' N'[V/x]$ for all $V$. The converse follows in an identical fashion. It is thus clear that $\mathcal{R} \mathcal{R}'$ is an applicative bisimulation, so $L \approx^n_A N$ as desired.

**Lemma 74.** We have the following containment of relations: $\Rightarrow \subseteq \approx^n_A$.

**Proof.** Assume $M \Rightarrow N$ and that $M \Rightarrow \lambda x.P$. Then by the determinacy of the call-by-name semantics, $N \Rightarrow \lambda x.P$ and for all $V, P[V/x] \Rightarrow P[V/x]$ since $\Rightarrow$ is a reflexive relation. The converse follows identically. Thus, $\Rightarrow$ is an applicative bisimulation, so $\Rightarrow \subseteq \approx^n_A$.

Although the big-step formulation prevents every applicative bisimulation from being seen as a CLB via the mapping $\mathcal{R} \mapsto (\text{Id}, \mathcal{R})$, we at the very least have that every applicative bisimulation is, in a certain sense, contained in a CLB:

**Proposition 75.** The coupled relation $(\text{Id}, \approx^n_A)$ is a CLB.

**Proof.** By Lemma 74 and the fact that $\Rightarrow \subseteq \Rightarrow, \Rightarrow \subseteq \approx^n_A$. So if $M \Rightarrow M'$ and $M \approx^n_A N$, then since $N \Rightarrow N$, by transitivity, $M' \approx^n_A N$ as desired. If $M = \lambda x.P$, then since $M \approx^n_A N$, $N \Rightarrow \lambda x.Q$ such that $P[W/x] \approx^n_A Q[W/x]$ for all $W$ (recall that $\text{Id}^* = \text{Id}$). Thus, $(\text{Id}, \approx^n_A)$ is a CLB.
Corollary 76. Applicative bisimilarity, coupled logical bisimilarity, and contextual equivalences coincide, i.e.,
\( \approx^1_A = \approx^2_A = \equiv^n = \equiv^1 = \equiv^1 = \approx^1_n \). Thus, applicative bisimilarity is a congruence.

Proof. We have that \( \approx^1_A = \approx^2_A \) by double inclusion via Proposition 75 and Corollary 72. The other equalities follow from Corollary 66.

4.3. Logical Bisimulation

[10] introduced the notion of logical bisimulation, a notion we show to be subsumed by coupled logical bisimulation.

Definition 77. A relation \( R \subseteq \Lambda^* \times \Lambda^* \) is called a logical bisimulation if whenever \( M \mathrel{R} N \):

1. if \( M \rightarrow M' \), then \( N \Rightarrow N' \) and \( M' \mathrel{R} N' \);
2. if \( M = \lambda x.M' \), then \( N \Rightarrow \lambda x.N' \) and for all \( X \mathrel{R} Y, M'[X/x] \mathrel{R} N'[Y/x] \);
3. the converses for \( N \).

The union of all logical bisimulations is called logical bisimilarity and is denoted \( \approx^\text{ln} \).

As one would expect, we have the following proposition, the proof of which may be found in [10, Corollary 1 and Lemma 4]:

Proposition 78 ([10]). Logical bisimilarity is the largest logical bisimulation and is a congruence relation.

The following proposition follows from a straightforward check of the definitions and tells us that the notion of logical bisimulation is subsumed by that of CLB:

Proposition 79. A relation \( R \) is a logical bisimulation if and only if \( (R, R) \) is a CLB.

Corollary 80. Logical bisimilarity, coupled logical bisimilarities, applicative bisimilarity, and contextual equivalences coincide, i.e., \( \approx^\text{ln} = \approx^1_A = \approx^2_A = \equiv^n = \approx^1_n \).

4.4. Applicative and Logical Bisimulations

5. CLB in the Call-by-value \( \lambda \)-calculus

We now move to the study of call-by-value.

Definition 81. The call-by-value \( \lambda \)-calculus is defined by the following reduction rules:

\[
\begin{align*}
N & \rightarrow N' & MN & \rightarrow MN' \\
M & \rightarrow M' & V \in V & \rightarrow V' \\
\lambda x.P & \rightarrow \lambda x.P' & (\lambda x.P)V & \rightarrow P[V/x]
\end{align*}
\]

where we take the set \( V \) of values to be the set of all abstractions \( \lambda x.P \in \Lambda \) and all variables \( x \).

Since our theory is centered around closed terms, we restrict the set \( V \) to the set of all \( \lambda x.P \in \Lambda^* \) throughout our development. We let the relations \( \Rightarrow \), \( \downarrow \), and \( \uparrow \), and the predicates \( \downarrow \) and \( \uparrow \) be as before, except using call-by-value reduction instead of call-by-name reduction.

Definition 82. We say that two terms \( M, N \in \Lambda^* \) are contextually equivalent, written \( M \simeq^V N \), if for all contexts \( C, C[M] \downarrow \) if and only if \( C[N] \downarrow \).
Definition 83. A call-by-value evaluation context \(\mathcal{C}\) is given by the following grammar,

\[
\mathcal{C} ::= [] | M\mathcal{C} | \mathcal{C}V,
\]

where \(V\) ranges over \(V\) and \(M\) ranges over \(\Lambda^*\). Two terms \(M\) and \(N\) are evaluation-contextually equivalent, written \(M \equiv^\mathcal{C} N\), if for all evaluation contexts \(\mathcal{C}\), \(\mathcal{C}[M]\Downarrow\) if and only if \(\mathcal{C}[N]\Downarrow\).

Definition 84. If \(R, R' \subseteq \Lambda^* \times \Lambda^*\) are relations, then \(R\)'s evaluation-contextual closure under \(R', (R \otimes_v R')\), is the least relation closed forward under the following rules:

\[
\begin{align*}
X R Y & \Rightarrow X(R \otimes_v R') Y, \\
M R' N & \Rightarrow M(R \otimes_v R') NY, \\
V R'\downarrow Y W & \Rightarrow XV(R \otimes_v R') YW.
\end{align*}
\]

Definition 85. If \(R\) is a coupled relation, then let its contextual closure \(R^\mathcal{C}\) be given by

\[
R^\mathcal{C} = (R_1^\mathcal{C}, (R_2 \otimes v R_3^\mathcal{C}) \cup R_4^\mathcal{C})
\]

5.1. Proving the Context Lemma for call-by-value

As for the call-by-name \(\lambda\)-calculus, we have a Milner-style context lemma. Although allusions to a proof exist in the literature, e.g., in a footnote in [7], the authors have been unable to find a published proof. We present ours below.

Theorem 86. We have the following equality of relations: \(\equiv^V = \equiv^\mathcal{C}\).

Lemma 87. The relation \(\equiv^\mathcal{C}\) is closed forward under the following rules:

\[
\begin{align*}
A \equiv^\mathcal{C} B & \quad M \in \Lambda^* & \quad A \equiv^\mathcal{C} B & \quad V \in V \\
MA \equiv^\mathcal{C} MB & & AV \equiv^\mathcal{C} BV.
\end{align*}
\]

Proof. We consider the first rule. Let \(\mathcal{C}\) is an evaluation context, we prove that \(\mathcal{C}[MA]\Downarrow\) if and only if \(\mathcal{C}[MB]\Downarrow\), which is equivalent to \(\mathcal{C}'[A]\Downarrow\) if and only if \(\mathcal{C}'[B]\Downarrow\) with \(\mathcal{C}' = \mathcal{C}[\{\}V]\). The latter holds since \(A \equiv^\mathcal{C} B\). The second rule follows in an identical manner with \(\mathcal{C}' = \mathcal{C}[\{\}V]\).

Corollary 88. The relation \(\equiv^\mathcal{C}\) is closed forward under the following rule:

\[
\begin{align*}
M \equiv^\mathcal{C} N & \quad V \equiv^\mathcal{C}\downarrow V X Y \\
MV \equiv^\mathcal{C} NW.
\end{align*}
\]

Proof. Assume \(M \equiv^\mathcal{C} N\) and \(V \equiv^\mathcal{C}\downarrow V X Y\). Then by Lemma 87, we have \(MV \equiv^\mathcal{C} NV\) and \(NV \equiv^\mathcal{C} NW\). Then by transitivity of \(\equiv^\mathcal{C}\), we get \(MV \equiv^\mathcal{C} NW\).

Lemma 89. If \(M \implies N\), then \(M \equiv^\mathcal{C} N\).

Proof. If \(\mathcal{C}\) is an evaluation-context, then \(\mathcal{C}[M] \implies \mathcal{C}[N]\), and so by determinism of \(\implies\), we know that \(\mathcal{C}[M]\) converges if and only if \(\mathcal{C}[N]\) does.

Proposition 90. If \(M \Downarrow V\) then \(M \equiv^\mathcal{C} V\).

Proof. Consider the relation:

\[
R = \{(C[M], C(V)) | C\ is \ a\ n-holed\ context, n \in N\}.
\]

We show that whenever \(P R Q\), if \(P \Downarrow V\), then \(Q\Downarrow\), and if \(P \implies P'\), then \(P' \implies R \implies Q\). We show the same property for \(R^{op}\) which is enough to conclude by determinism of \(\implies\).
We now handle the cases where both $C$ prove by induction on $M$. From this, and the clause about $M$, we proceed with an induction on $C$. If $C = C_1 C_2$ and we prove by induction on $C$ that there exists $C'$ such that $C(V) \rightarrow C'(V)$ and $C(M) \rightarrow C'(M)$ where $\rightarrow$ is the transitive closure of $\rightarrow$. Suppose first that $C_1$ or $C_2$ is not canonical.

- If $C_2$ is not canonical, then by induction $C_2(V) \rightarrow C'_2(V)$ and $C_2(M) \rightarrow C'_2(M)$ and thus $C_1 C_2(V) \rightarrow C_1 C'_2(V)$ and $C_1 C_2(M) \rightarrow C_1 C'_2(M)$. (Context $C'$ is then $C_1 C'_2$.)

- If $C_1$ is not canonical and $C_2$ is but $C_2 \neq []$ then by induction $C_1(V) \rightarrow C'_1(V)$ and $C_1(M) \rightarrow C'_1(M)$. Thus, $C_1 C_2(V) \rightarrow C'_1 C'_2(V)$ and $C_1 C_2(M) \rightarrow C'_1 C'_2(M)$ because $C_2(M)$ is not canonical. (Context $C'$ is then $C'_1 C'_2$.)

- If $C_1$ is not canonical and $C_2 = []$ then by induction $C_1(V) \rightarrow C'_1(V)$ and $C_1(M) \rightarrow C'_1(M)$. Thus, $C_1(V) V \rightarrow C'_1(V) V$ and $C_1(M) M \rightarrow C'_1(M) V$. (Context $C'$ is then $C'_1 V$.)

We now handle the cases where both $C_1$ and $C_2$ are canonical. Let $R$ be such that $V = \lambda x. R$.

- $C_1 = [] = C_2$: then $VV \rightarrow R[V/x]$ and $MM \rightarrow MV \rightarrow VV \rightarrow R[V/x]$. (Context $C'$ is the 0-holed context $R[V/x]$.)

- $C_1 = [] \neq C_2$: then $VC_2(V) \rightarrow R[C_2(V)/x]$ and $MC_2(M) \rightarrow V C_2(M) \rightarrow R[C_2(M)/x]$ since $C_2(M) \in \mathcal{V}$. (Then $C'$ is $R[C_2/x]$.)

- $C_1 = \lambda y . D_1$ and $C_2 = []$: then $(\lambda y . D_1(V)) V \rightarrow D_1[V/y](V)$ and $(\lambda y . D_1(M)) M \rightarrow (\lambda y . D_1(M)) V \rightarrow D_1[V/y](M)$. (Then $C'$ is $D_1[V/y]$.)

- $C_1 = \lambda y . D_1$ and $C_2 \neq []$: then $(\lambda y . D_1(V)) C_2(V) \rightarrow D_1[C_2(V)/y](V)$ and $(\lambda y . D_1(M)) C_2(M) \rightarrow D_1[C_2(M)/y](M)$. (Then $C'$ is $D_1[C_2/y]$.)

This case analysis shows us that if $P S Q$ and $P \rightarrow P'$ then $P' \Rightarrow S \Leftarrow Q$ for both $S \in \{R, R^\text{op}\}$. From this, and the clause about $P \in \mathcal{V}$, we can easily prove that $P \Rightarrow Q$ implies $(P \downarrow$ iff $Q \downarrow)$.

**Proposition 91.** If $V, W \in \mathcal{V}$ and $V \equiv^v W$ then $V \equiv^v W$.

**Proof.** We prove $C[V] \downarrow$ if $C[W] \downarrow$. Consider the evaluation context $C = (\lambda x . C[x]) []$ yielding $C[V] \rightarrow C[V]$, and $C[W] \rightarrow C[W]$. Since $V \equiv^v W$, $C[V] \downarrow$ iff $C[W] \downarrow$, hence $C[V] \downarrow$ iff $C[W] \downarrow$.

The above proposition can be strengthened as follows:

**Proposition 92.** If $M \downarrow$ and $N \downarrow$ then $M \equiv^v N$ implies $M \equiv^v N$.

**Proof.** Let $V$ and $W$ such that $M[V], N[W]$. By Proposition 90, $M \equiv^v V$ and $N \equiv^v W$. Then, if $M \equiv^v N$, then $V \equiv^v W$, then $V \equiv^v W$ by Proposition 91 and finally $M \equiv^v N$.

**Proposition 93.** If $M \downarrow$ and $N \downarrow$, then $M \equiv^v N$.
Proof. Consider the symmetric relation:
\[ \mathcal{R} = \{(C(M), C(N)) \mid C \text{ is a } n\text{-holed context}) \cup \{(\mathcal{O}_1[M'], \mathcal{O}_2[N']) \mid M' \uparrow \text{ and } N' \uparrow \}. \]

Suppose \( P \mathcal{R} Q \). Trivially, if \( P \in \mathcal{V} \) then \( Q \in \mathcal{V} \). Remains to prove that \( P \rightarrow P' \), then \( Q \rightarrow Q' \) and \( P' \mathcal{R} Q' \) for some \( Q' \). The second part of the relation is trivial. Regarding the first part, if \( M \) appears in evaluation position \( (P = \mathcal{O}_1[M]) \) then so does \( N = \mathcal{O}_2[N] \), the pair progressing to the second part of the relation. If \( M \) does not, then \( C(M) \rightarrow C'(M) \) and \( C(N) \rightarrow C'(N) \).

\[ \square \]

Corollary 94. \( M \simeq N \) if and only if \( M \not\simeq N \).

Proof. Clearly \( M \simeq N \) implies \( M \not\simeq N \). Considering \( \mathcal{O} = [\cdot] \), either \( M\downarrow \) and \( N\uparrow \) or \( M\uparrow \) and \( N\downarrow \) and using Propositions 92 and 93 we derive the other implication.

\[ \square \]

Lemma 95. If \( M \mathcal{R} \mathcal{R}' \mid N \) and \( M \rightarrow M' \), then one of the following cases holds

1. \( M \not\mathcal{R} N \);
2. \( M = E_M(\alpha), N = E_N(\beta), \alpha \mathcal{R} \beta, \|E_M\| = \|E_N\| = n, E_{Mn} = \{\cdot\}V, V \mathcal{R}' \mid \mathcal{V}_V \mathcal{V} \mathcal{W}, \) and \( \alpha \rightarrow \alpha' \);
3. \( M = E_M(\alpha), N = E_N(\beta), \alpha \mathcal{R} \beta, \|E_M\| = \|E_N\| = n, E_{Mn} = \{\cdot\}V, V \mathcal{R}' \mid \mathcal{V}_V \mathcal{W}, \) \( \alpha = \lambda x.\alpha', \) and \( M \rightarrow E_{M1} \cdots [E_{M(n-1)}(\alpha'[\mathcal{V}/x])] \cdots \);
4. \( M = E_M(\alpha), N = E_N(\beta), \alpha \mathcal{R} \beta, \|E_M\| = \|E_N\| = n, E_{Mn} = X[\cdot], E_{Nn} = Y[\cdot], X \mathcal{R}' \mathcal{Y}, \) and \( \alpha \rightarrow \alpha' \);
5. \( M = E_M(\alpha), N = E_N(\beta), \alpha \mathcal{R} \beta, \|E_M\| = \|E_N\| = n, E_{Mn} = X[\cdot], E_{Nn} = Y[\cdot], X \mathcal{R}' \mathcal{Y}, X = \lambda x.X', \alpha = \lambda y.\alpha', \) and \( M \rightarrow E_{M1} \cdots [E_{M(n-1)}(X'[\mathcal{V}/x])] \cdots \).

Moreover, where \( E_{M1} \) and \( E_{N1} \) are such that \( E_M = E_{M1}, \ldots, E_{Mn} \) and \( E_N = E_{N1}, \ldots, E_{Nn} \), whenever \( E_{M1} = \{\cdot\}V_i \) then \( E_{N1} = \{\cdot\}W_i \) for some \( W_i \) with \( V_i \mathcal{R}_1 \mid \mathcal{V}_V \mathcal{W}_i \) and conversely, and whenever \( E_{M1} = X_i[\cdot] \) then \( E_{N1} = Y_i[\cdot] \) for some \( Y_i \) with \( X_i \mathcal{R}_1 \mathcal{Y}_i \) and conversely.

Lemma 96. If \( \mathcal{R}, \mathcal{R}' \subseteq \mathcal{A}^* \times \mathcal{A}^* \) are relations, \( \vec{E} = E_1, \ldots, E_n \) and \( \vec{F} = F_1, \ldots, F_n \) are lists of experiments such that whenever \( E_i = \{\cdot\}V_i \) then \( F_i = \{\cdot\}W_i \) for some \( V_i \) with \( V_i \mathcal{R}_1 \mid \mathcal{V}_V \mathcal{W}_i \) and conversely, and whenever \( E_i = M_i[\cdot] \) then \( F_i = N_i[\cdot] \) for some \( N_i \) with \( M_i \mathcal{R}' \mathcal{N}_i \) and conversely, then for all \( \alpha \mathcal{R} \beta, \) \( \vec{E}[\alpha] \mathcal{R}_1 \mathcal{R}_2 \vec{F}[\beta] \).

Proof. By induction on \( n \). The case of \( n = 0 \) is trivial, so we assume true for some \( n - 1 \), and let \( E = E_1, \vec{E}' \) and \( F = F_1, \vec{F}' \) be two lists of experiments of length \( n \) satisfying the hypotheses. Then by the induction hypothesis, since \( \vec{E}' \) and \( \vec{F}' \) are appropriately related lists of length \( n - 1 \), for all \( \alpha \mathcal{R} \beta, \vec{E}'[\alpha] \mathcal{R}_1 \mathcal{R}_2 \vec{F}'[\beta] \). If \( E_1 = \{\cdot\}V_1 \) then \( F_1 = \{\cdot\}W_1 \) for some \( V_1 \mathcal{R}_1 \mid \mathcal{V}_V \mathcal{W}_1 \), and so by definition of \( \mathcal{R}_1 \mathcal{R}_2 \), we get \( \vec{E}[\alpha] = \vec{E}'[\alpha]V_1 \mathcal{R}_1 \mathcal{R}_2 \vec{F}'[\beta]W_1 \) if \( \vec{F}'[\beta] \) for all \( \alpha \mathcal{R} \beta \) as desired. The case of \( E_1 = M_1[\cdot] \) follows in a similar manner. We thus conclude the lemma by induction.

\[ \square \]

Lemma 97. If \( \vec{E} = E_1(E_2(\cdots(E_n(\cdots))) \) and \( \vec{F} = F_1(F_2(\cdots(F_n(\cdots)))) \) and \( E_i \mathcal{R}_1 F_i \) for \( 1 \leq i \leq n \) and \( \alpha \mathcal{R}_2 \beta \), then \( \vec{E}[\alpha] \mathcal{R}_2 \vec{F}[\beta] \).
we have:

The definition of coupled logical bisimulation for the call-by-value calculus differs from that for the

Proposition 100.

A pair of relations

Proposition 101.

reads as:

\[ \xi \]

Definition 99.

A coupled relation

Definition 98.

\( \approx \)

Coupled logical bisimilarity, written \( \xi \), is the pairwise union of all CLBs.

As in the call-by-name case, CLBs for the call-by-value \( \lambda \)-calculus have a continuous progression:

Definition 99. Given pairs of relations \( R \) and \( S \), we say \( R \) progresses to \( S \), written \( R \rightarrow S \), if whenever \( M \not\rightarrow N \), then:

1. whenever \( M \rightarrow M' \) then there exists an \( N' \) such that \( N \rightarrow N' \) and \( M' \not\rightarrow N' \);

2. whenever \( M = \lambda x.M' \) then \( N \rightarrow \lambda x.N' \) and \( \lambda x.M' \not\rightarrow \lambda x.N' \), and for all \( P, Q \in \Lambda^* \) such that \( P \not\rightarrow Q \), we have \( M' \not\rightarrow[P/x] S_2 \) \( N' \not\rightarrow[Q/x] \);\n
3. the converses of the previous two conditions for \( N \).

Coupled logical bisimilarity, written \( \approx \), is a CLB if and only if \( R \rightarrow R \). Thus, \( \rightarrow \) is a progression for coupled logical bisimulations.

Proposition 100. A pair of relations \( R \) is a CLB if and only if \( R \rightarrow R \). Thus, \( \rightarrow \) is a progression for coupled logical bisimulations.

Proposition 101. The relation \( \rightarrow \) is continuous.

Proof. The proof is identical to that of Proposition 40 apart from the case \( M = \lambda x.M' \), which now reads as:

If \( M = \lambda x.M' \) then, since \( R_n \rightarrow S_n \), \( N \rightarrow \lambda x.N' \) such that \( \lambda x.M' \not\rightarrow \lambda x.N' \), and so since \( S_n \subseteq \forall S \), we get that \( \lambda x.M' \not\rightarrow S_1 \) \( \lambda x.N' \). Moreover, \( M' \not\rightarrow[\lambda x.S]_2 N'[\lambda x] \) for all \( X \not\rightarrow Y \), so \( M' \not\rightarrow[X/x] Y \) and we're done.

Proposition 102. If \([R_i]\) and \([S_i]\) are two families of paired relations such that \( R_i \rightarrow S_i \) for all \( i \), then \( \bigcap_{i \in I}(R_i)_1 \bigcup_{i \in I}(R_i)_2 \rightarrow \bigcup_{i \in I} S_i \).
\textbf{Lemma 105.} If $\mathcal{R}$ is a CLB and $M \not\in \mathcal{R} \subseteq N$, then there exists a $V$ such that $M \downarrow V$ if and only if there exists a $W$ such that $N \downarrow W$.

\textbf{Proof.} Immediate by the definition of CLBV.

\textbf{Lemma 106.} The coupled relation $(\text{Id}|_{V \times V}, \rightarrow)$ is a CLB.

\textbf{Proof.} Assume $M \rightarrow M'$ and that $M \rightarrow M''$. If $M = M'$, then $M' \rightarrow M''$ also, and $M'' \rightarrow M''$. Conversely, if $M'' \rightarrow M'$, then $M \rightarrow M$ and by transitivity of $\rightarrow$, $M \rightarrow M'$.

Now assume $M = \lambda x. P$, then $M'' = \lambda x. P$, so $M'' \rightarrow \lambda x. P$ by reflexivity, and $\lambda x. P \ |_{V \times V} \lambda x. P$. Clearly for all $V \ |_{V \times V} W, P[V/x] \rightarrow P[W/x]$ by reflexivity, since $P[V/x] = P[W/x]$. Conversely, if $M' = \lambda x. P$, then $M \rightarrow \lambda x. P$ and the same argument applies.

As in Section 4, we study the up-to context technique, which allows us to deduce congruence.

\textbf{Definition 105.} We call \textbf{up-to context} the up-to technique given by $\mathcal{R} \rightarrow \mathcal{R}^V$. We say a coupled relation $\mathcal{R}$ is a \textbf{CLB up-to context} if $\mathcal{R} \rightarrow \mathcal{R}^V$.

\textbf{Lemma 106.} If $\mathcal{R} \rightarrow S$ and $\mathcal{R} \subseteq S$, then $(\mathcal{R}_1, \mathcal{R}_2^V) \rightarrow S^V$.

\textbf{Proof.} Suppose $M \mathcal{R}_1 \tilde{N}$. It is sufficient to show that: (i) if $M \rightarrow M'$, then there exists an $N'$ such that $N \rightarrow N'$ and $M' \triangleleft N'$, and symmetrically if $N \rightarrow N'$; (ii) if $M = \lambda x. M'$, then there exists an $N'$ such that $N \rightarrow \lambda x. N'$, $\lambda x. M' \triangleleft \lambda x. N'$, and for all $X \mathcal{R}_1^V \subseteq Y$, $M'[X/x] \triangleleft \lambda x. N'[x]$.

Let $M \mathcal{R}_1 \tilde{N}$ such that $M = C[M]$ and $N = C[N]$ and assume first that $M \rightarrow M'$. Then we proceed by induction on $C$, and observe that it is sufficient to show either of $M' \triangleleft N'$ or $M' \triangleleft \lambda x. N'$.

The case $C = [\ ]$ is trivial since it implies $M \mathcal{R}_1 N$, and we have as hypothesis that $\mathcal{R} \rightarrow S$.

The cases $C = x$ and $C = \lambda x. C'$ does not arise since $C[M] \rightarrow$.

Finally, we consider the case $C = C_1C_2$ and let $\tilde{M}_1, \tilde{M}_2$ be such that $\tilde{C}[\tilde{M}] = C_1[\tilde{M}_1]C_2[\tilde{M}_2]$ and similarly for $\tilde{N}$. By monotonicity of contextual closure, we observe that $A \mathcal{R}_1^V B$ implies $A \mathcal{R}_1B$ for all $A, B$ since $\mathcal{R} \subseteq S$. The reduction $M \rightarrow M'$ is due to one of the following mutually exclusive subcases:

1. $C_2[\tilde{M}_2] \rightarrow M'_2$, so $M \rightarrow C_1[\tilde{M}_1]M'_2$;
2. $C_2[\tilde{M}_2] \in Y$, and $C_1[\tilde{M}_1] \rightarrow M'_2$, so $M \rightarrow M'_1C_2[\tilde{M}_2]$;
3. $C_1[\tilde{M}_1] = \lambda x. P, C_2[\tilde{M}_2] \in Y$, so $M \rightarrow P[C_2[\tilde{M}_2]/x]$.
In the first case, since \( C_2[M_2] \not\rightarrow \alpha \) \( C_2[N_2] \), by induction hypothesis, \( C_2[N_2] \rightarrow N'_2 \) for some \( N'_2 \) such that \( M'_2 S_2^Y N'_2 \) and \( N \rightarrow N' \). If \( M'_2 S_2^Y N'_2 \) holds because of \( S'_1 \), then Lemma 26 gives us \( C_1[M_1]M'_2 S_1^Y C_1[N_1]N'_2 \) and we're done. If it holds because of \( S_2 \), then we're done because \( C_1[M_1]M'_2 (S_2 \circ \alpha) S_1^Y C_1[N_1]N'_2 \). Finally, if it holds because of \( (S_2 \circ \alpha) S_1^Y \), then \( C_1[M_1]S_1^Y C_1[N_1] \) implies \( C_1[M_1]M'_2 (S_2 \circ \alpha) S_1^Y C_1[N_1]N'_2 \) as desired.

In the second case, \( C_2[M_2] \in V \) implies \( C_2 \) is a canonical context. If \( C_2 = [], \) then \( M_2 = M_2 \) is a single element list, and since \( M_2 \not\rightarrow \alpha \) \( N_2 \not\rightarrow \alpha \) \( S_2 \), we have \( M_2 \not\rightarrow \alpha \) \( N_2 \not\rightarrow \alpha \) \( S_2 \). In either case, \( C_2[M_2] S_1^Y V \). By the induction hypothesis, since \( C_1[M_1] \not\rightarrow \alpha \) \( C_1[N_1] \) and \( C_1[M_1] \not\rightarrow \alpha \) \( C_1[N_1] \), \( C_1[N_1] \rightarrow N'_1 \) such that \( M'_2 S_2^Y N'_1 \); thus, \( N \rightarrow N' \). If \( M'_2 S_2^Y N'_1 \) because of \( S'_1 \), then Lemma 26 gives us \( M'_2 S_1^Y C_1[N_1]N'_2 \) and we're done. If the relation holds because of \( S_2 \), then \( M'_2 C_2[M_2] S_1^Y N'_2 \) and again we're done. Finally, if the relation holds because of \( (S_2 \circ \alpha) S_1^Y \), then \( C_2[M_2] S_1^Y N'_2 \) implies \( C_2[M_2] S_1^Y N'_2 \) \( N'_2 \) by Lemma 96 as desired.

Finally, in the third case, the same argument as in the second case gives us that \( C_2[N_2] \rightarrow N'_2 \), and similarly that \( C_1[N_1] \rightarrow \alpha \). Since \( C_2[M_2] S_1^Y \) \( N'_2 \), and \( \text{fv}(P) = \text{fv}(Q) = \{x\} \), by Lemma 27, we then conclude \( P[C_2[M_2]/x] S_1^Y Q[N'_2/x] \) and we're done.

This exhausts all possible reductions in the case of \( C = C_1C_2 \) and completes the induction on \( C \). The symmetric case follows symmetrically. We thus conclude the first half of the lemma.

Now assume \( M = \alpha \cdot M' \). Then we proceed by case analysis on \( M \), and observe that it is sufficient to show that \( N \rightarrow N' \), and that for all \( X \not\rightarrow \alpha \) \( Y \), either \( M'[X/x] S_1^Y N'[Y/x] \) or \( M'[X/x] S_2^Y N'[Y/x] \).

Clearly, \( M = \alpha \cdot M' \) implies \( C \) is a canonical context. If \( C = [], \) then \( \alpha \cdot M' \not\rightarrow \alpha \) \( N \) and the claim follows immediately from the fact that \( \not\rightarrow \alpha \) \( S \). Otherwise, we have \( C = \alpha \cdot C' \), and so \( N = \alpha \cdot C[N] \) implies \( N \rightarrow \alpha \cdot C'[N] \) by reflexivity, and since \( S_1^Y \leq S_2^Y \), \( \lambda x.C' \circ \not\rightarrow \alpha \) \( C' \cdot \not\rightarrow \alpha \) \( C' \cdot \not\rightarrow \alpha \) \( N \). Clearly, \( M = C'[M] \) and \( N' = C'[N] \) and \( \text{fv}(M') = \text{fv}(N') = \{x\} \). But by Lemma 27, we get that for all \( X \not\rightarrow \lambda x. \) \( Y \), \( M'[X/x] S_1^Y N'[Y/x] \) and thus conclude \( M'[X/x] S_1^Y N'[Y/x] \) as desired. This exhausts all possible cases for \( C \), and since the symmetric case follows symmetrically, we conclude the lemma.

\[ \square \]

**Lemma 107.** If \( \not\rightarrow \) \( S \), then \( \not\rightarrow \) \( S \). Suppose \( M \not\rightarrow \alpha \) \( N \). It is sufficient to show that: (i) if \( M \rightarrow M' \), then \( N \rightarrow N' \) such that \( M' S_1^Y N' \) and symmetrically for \( N \); (ii) if \( M = \alpha \cdot M' \), then there exists an \( N' \) such that \( N \rightarrow \alpha \cdot N' \), \( \lambda x.M' S_1^Y \lambda x.N' \), and for all \( X \not\rightarrow \alpha \) \( Y \), \( M'[X/x] S_1^Y N'[Y/x] \) and symmetrically if \( N = \lambda x.N' \).

The hypothesis \( M \rightarrow M' \) implies one of the six cases given by Lemma 95. Let \( \alpha, \beta, E_1, \text{etc.} \), be as in Lemma 95, and observe that since \( R_i \subseteq S_i, E_{M_1} S_1 E_{N_1} \) for all \( i \).

In the first case, we're done, for \( \not\rightarrow \). In the second and fourth case, if \( \alpha \rightarrow \alpha' \), then since \( \alpha \not\rightarrow \alpha' \) \( \not\rightarrow \alpha' \) \( \not\rightarrow \beta' \) such that \( \alpha' S_1^Y \beta' \). Then this implies \( M \\ E_1 M_1 \not\rightarrow \alpha' \), \( E_1 N_1 \not\rightarrow \beta' \), and by Lemma 97, \( E_1 \alpha' S_1^Y E_1 \beta' \). In the third case, if \( \alpha = \lambda x.\alpha' \), then since \( \alpha \not\rightarrow \alpha' \) \( \not\rightarrow \beta' \) such that for all \( X \not\rightarrow \lambda x. \) \( Y \), \( \alpha'[X/x] S_1^Y \beta'[Y/x] \). Thus, \( M \\ E_1 M_1 \not\rightarrow \alpha' [V/x] \), \( E_1 N_1 \not\rightarrow \beta' [V/x] \), and by Lemma 97 we get

\[ E_1 M_1 \not\rightarrow \alpha' [V/x] \not\rightarrow \beta' [V/x] \not\rightarrow \beta' [V/x] \not\rightarrow \beta' [V/x] \not\rightarrow \beta' [V/x] \not\rightarrow \beta' [V/x] \],

as desired.
In the fifth case, since $\mathcal{R}_2 \beta$ and $\alpha = \lambda x.\alpha'$, $\beta \Rightarrow \lambda x.\beta'' = \beta'$ such that $\alpha S^*_N[X/V] \beta'$. Let $X, \mathcal{R}_1 Y$ be such that $E_{Mn} = X[i]$, then by Lemma 106, since $X \Rightarrow X', Y \Rightarrow Y'$ such that $X' S^*_V Y'$. Let $E'_{Mn} = [\cdot][x]$, and then $M \Rightarrow M_1 \cdots [E_{M_1} [X/V]] \cdots$ and $N \Rightarrow E_{N1} \cdots [E_{N(n-1)} [E_N [Y/V]]] \cdots$. Then by Lemma 97 we get

$$E_{M1} \cdots [E_{M_{(n-1)}} [E_{M} [X/V]]] \cdots \approx_{\alpha} E_{N1} \cdots [E_{N(n-1)} [E_N[Y/V]]] \cdots$$

and we're done.

Finally, in the sixth case, since $\alpha \mathcal{R}_2 \beta$ and $\alpha = \lambda x.\alpha'$, $\beta \Rightarrow \lambda x.\beta'' = \beta'$ such that $\alpha S^*_N[X/V] \beta'$. Let $X, \mathcal{R}_1 Y$ be such that $E_{Mn} = X[i]$ and $E_{Nn} = Y[i]$, then by Lemma 106, since $X = \lambda x.\lambda y. X', Y = \lambda x.\lambda y. Y'$ such that $X'[v/x] \approx S^*_Y Y'[\mu/x]$ for all $\mathcal{R}_1^* \subseteq S^*_N$. Since $\mathcal{R}_1^* \subseteq S^*_N$, we get $X'[\alpha/x] \approx S^*_Y Y'[\beta'/x]$. Thus, $M \Rightarrow E_{M1} \cdots [E_{M_{(n-1)}} [X'[\alpha/x]]] \cdots$ and $N \Rightarrow E_{N1} \cdots [E_{N(n-1)} [Y'[\beta'/x]]] \cdots$. By Lemma 97, we conclude

$$E_{M1} \cdots [E_{M_{(n-1)}} [X'[\alpha/x]]] \cdots \approx_{\alpha} E_{N1} \cdots [E_{N(n-1)} [Y'[\beta'/x]]] \cdots$$

The symmetric case follows symmetrically.

Now assume $M = \lambda x. \mathcal{M}'$. Clearly, $M = \lambda x. \mathcal{M}'$ implies $E_N = [\cdot]$. Then $\lambda x. \mathcal{M}' \mathcal{R}_2 N$ and the claim follows immediately from the fact that $\mathcal{R} \Rightarrow S$. The symmetric case follows symmetrically, and so we conclude the lemma.

**Theorem 108.** The up-to-context technique is extensive and respectfully compatible, and hence sound.

**Proof.** Extensiveness is obvious. Assume $\mathcal{R} \Rightarrow S$ and $\mathcal{R} \subseteq S$, then by Lemma 106 we have $(\mathcal{R}_1, \mathcal{R}_1^*) \Rightarrow S^V$, and by Lemma 107, $(\mathcal{R}_1, (\mathcal{R}_2 \triangleleft \circ \mathcal{R}_1^*)) \Rightarrow S^V$. Then, by Lemma 102, $(\mathcal{R}_1, \mathcal{R}_1^* \cup (\mathcal{R}_2 \triangleleft \circ \mathcal{R}_1^*)) \Rightarrow S^V$. By the call-by-value analog of Proposition 43, we deduce that $\mathcal{R}^V \Rightarrow S^V$ and we are done.

**Corollary 109.** If $\mathcal{R}$ is a CLB, then so is $\mathcal{R}^V$.

**Corollary 110.**

1. If $M \approx^\mathcal{N}_N$, then for all contexts $C$, $C[M] \approx^\mathcal{N}_N C[N]$.
2. If $E \approx^\mathcal{N}_F$, then for all evaluation contexts $\mathcal{C}$, $\mathcal{C}[E] \approx^\mathcal{N}_F \mathcal{C}[F]$.

**Proof.** In the first case, if $M \approx^\mathcal{N}_N$, then there exists a CLB $\mathcal{R} \subseteq S$ such that $M \mathcal{R}_1 N$. Then by Corollary 109, $\mathcal{R}^V \subseteq S^V$. Since $C[M] \approx^\mathcal{N}_N C[N]$ and $\mathcal{R}^V \mathcal{R}_1 \mathcal{R}_1^*$, we get $C[M] \approx^\mathcal{N}_N C[N]$ as desired.

In the second case, if $E \approx^\mathcal{N}_F$, then there exists a CLB $\mathcal{R} \subseteq S^V$ such that $E \mathcal{R}_2 F$. Then again, $\mathcal{R}^V \subseteq S^V$. Since $(\mathcal{R}_2 \triangleleft \circ \mathcal{R}_1^*) \subseteq S^V$ and $\mathcal{I}d \subseteq \mathcal{R}_1^*$, we have $(\mathcal{R}_2 \triangleleft \circ \mathcal{R}_1^*) \subseteq S^V$. Thus, $E \mathcal{R}_2 F$ implies $\mathcal{C}[E] \mathcal{R}_2^V \mathcal{C}[F]$ for all evaluation contexts $\mathcal{C}$, and since $\mathcal{R}_2^V \subseteq S^V$, we deduce the second statement.

**Corollary 111.** We have the following inclusion of coupled relations: $\approx^\mathcal{N}_V \subseteq (\approx^\mathcal{N}_V, \approx^\mathcal{N}_V)$.

**Lemma 112.** If $\lambda x. P \equiv^\mathcal{N} \lambda x. Q$, then for all $V \in \mathcal{V}$, $P[V/x] \equiv^\mathcal{N} Q[V/x]$.

**Proof.** Since $\equiv^\mathcal{N} \subseteq \approx^\mathcal{N}_V$, $(\lambda x. P)[V/x] \Rightarrow P[V/x]$ implies $(\lambda x. P)[V/x] \equiv^\mathcal{N} P[V/x]$ and similarly $(\lambda x. Q)[V/x] \Rightarrow Q[V/x]$ since $\approx^\mathcal{N}_V$ is a congruence relation, $(\lambda x. P)[V/x] \equiv^\mathcal{N} (\lambda x. Q)[V/x]$, and so the lemma follows.

**Theorem 113.** The coupled relation $(\approx^\mathcal{N}_V, \approx^\mathcal{N}_V)$ is a CLB.

**Proof.** Assume $M \approx^\mathcal{N}_V N$. If $M \Rightarrow M'$, then $M \Rightarrow M'$, and so since $\Rightarrow \subseteq \approx^\mathcal{N}_V$, $M \approx^\mathcal{N}_V M'$. Then $N \Rightarrow N$ and by transitivity and symmetry, $M' \approx^\mathcal{N}_V N$.

If $M = \lambda x. P$, then by definition of $\approx^\mathcal{N}_V$, $N \Rightarrow \lambda x. Q$ for some $Q$, and $\lambda x. P \equiv^\mathcal{N} \lambda x. Q$ by the fact that $\approx^\mathcal{N}_V$ is an equivalence relation and $\Rightarrow \subseteq \approx^\mathcal{N}_V$. Then by Theorem 86, $\lambda x. P \approx^\mathcal{N} \lambda x. Q$ as desired. Since
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\( \simeq^v \subseteq \simeq^v \) and \( (\simeq^v)^* = \simeq^v \), by Lemma 27, we get that \( P[V/x] \simeq^v Q[W/x] \) for all \( V \) \( (\simeq^v)^* \) \( W \), as desired.

Since the symmetric cases follow symmetrically, we derive the theorem.

**Corollary 114.**
Coupled logical bisimilarity coincides with the contextual equivalences, i.e., \( (\simeq^v, \simeq^v) = (\simeq^v, \simeq^v) \).

**Proof.** By double inclusion via Corollary 111 and Theorem 113.

**Corollary 115.** Coupled logical bisimilarity, \( \approx^v \), is a CLB.

**Proof.** Immediate by Theorems 113 and 86.

### 5.3. Applicative Bisimulation

The call-by-value version of applicative bisimulation is nearly identical to the call-by-name version, apart from the obvious restriction to values in the substitution clause:

**Definition 116.** A relation \( R \subseteq \Lambda^* \times \Lambda^* \) is called an **applicative bisimulation** if \( M \ R N \) implies whenever \( M \Rightarrow \lambda x. P, N \Rightarrow \lambda x. Q \) for some \( Q \) and \( P[W/x] \ R Q[W/x] \) for all \( W \in \mathcal{V} \), and conversely for \( N \). We call the union of all applicative bisimulations, \( \approx^v_{\Lambda} \) Ass \textit{applicative bisimilarity}.

As in the call-by-name case, applicative bisimilarity is an applicative bisimulation and we seek to show that applicative bisimilarity coincides with contextual equivalence, i.e., is a congruence. In contrast to the call-by-name case, it is not easy to give a direct proof that applicative bisimilarity can be seen as a CLB for the call-by-value \( \lambda \)-calculus. Assume we tried the naive approach we used in the call-by-name case, and claimed that the coupled relation \( (\text{Id}_{\mathcal{V} \times \mathcal{V}}, \approx^v_{\Lambda}) \) is a CLB. We show that this claim is false: one can easily show that \( \lambda x. (\lambda y. y) x \approx^v_{\Lambda} \lambda x. x \). However, clause 2 of the definition of CLB then requires that \( \lambda x. (\lambda y. y) x \approx^v_{\Lambda} \lambda x. x \), which is clearly false. Thus, the proposed embedding of applicative bisimilarity into a CLB is incorrect. We could correct this deficiency by proposing instead the embedding \( (\approx^v_{\Lambda} \mid \mathcal{V} \times \mathcal{V}, \approx^v_{\Lambda}) \). However, this embedding is no longer faithful to the spirit of applicative bisimulation, since we now permit the substitution of non-identical pairs into values related by \( \approx^v_{\Lambda} \). This problem motivates the need for an analogue of the up-to environment proposed in [10] for logical bisimulation:

**Definition 117.** A coupled relation \( R \) is said to be a CLB **up-to environment** if it satisfies all clauses of the definition of CLB except for the requirement that \( \lambda x. M' \ R_1 \lambda x. N' \) in clause 2.

### 5.4. Logical Bisimulation

Although we cannot faithfully embed applicative bisimulation into call-by-value CLBs, we can still embed logical bisimulations (LBs). We present the call-by-value version of the logical bisimulation as introduced by [10].

**Definition 118.** A relation \( R \subseteq \Lambda^* \times \Lambda^* \) is called a **logical bisimulation** if whenever \( M \ R N \):

1. if \( M \rightarrow M' \), then \( N \rightarrow N' \) and \( M' \ R N' \);
2. if \( M = \lambda x. M' \), then \( N \rightarrow \lambda x. N' \), and for all \( \mathcal{V} \ R^* \mathcal{V} \ W, M'[\mathcal{V}/x] \ R N'[\mathcal{W}/x] \);
3. the converses for \( N \).

The union of all logical bisimulations is called **logical bisimilarity** and is denoted \( \approx_{lv} \).
Although it is claimed in [10] that to have soundness, we must additionally require that \( \lambda x. M \upharpoonright \lambda x. N' \) in clause 2 of the definition, that is to say, that clause 2 should read as “if \( M = \lambda x. M' \), then \( N \Rightarrow \lambda x. N' \), and for all \( V \upharpoonright [V \times V] W. M'[V/x] R N'[W/x]' \), the following proposition shows that this additional requirement is redundant:

**Proposition 119.** If \( R \) is a relation such that whenever \( M \upharpoonright \lambda x. M \), we have that \( N \Rightarrow \lambda x. N' \) and \( \lambda x. M \upharpoonright \lambda x. N' \).

**Proof.** Suppose \( P \upharpoonright Q. N \). By induction on \( N \Rightarrow Q \), using 1, we get \( P \Rightarrow P' \) such that \( P' \Rightarrow Q \). With \( P = \lambda x. M \) and \( Q' = \lambda x. N' \) for some \( N' \) (thanks to 2) we must have \( P' = \lambda x. M \) as well. \( \square \)

Unfortunately, we have not been able to similarly drop the requirement that \( \lambda x. M \upharpoonright \lambda x. M \) in the second clause of the definition of call-by-value CLBs.

As one would expect, we still have the following proposition, due to [10, Corollary 1, Lemma 4, and p. 12]:

**Proposition 120 ([10]).** Logical bisimilarity is the largest logical bisimulation and is a congruence relation.

Moreover, as in the call-by-name case, and thanks to Proposition 119, we have that:

**Proposition 121.** A relation \( R \) is a logical bisimulation if and only if \((R, R)\) is a CLB.

**Corollary 122.** Logical bisimilarity, coupled logical bisimilarities, and contextual equivalences coincide, i.e.,

\( \approx^{bL} = \approx^{v}_{\lambda} = \approx^{v} = \approx^{v}_{\lambda} = \approx^{v}_{\lambda} \).

### 6. Concluding Remarks

Logical bisimulations build upon applicative bisimulations and make proofs of congruence simpler, without relying on Howe’s method [5, 8]. Up-to techniques for logical bisimulations can be defined [10], in order to bisimulation proofs easier. Their definition, however, is rather ad hoc.

Coupled logical bisimulations bridge the gap between applicative and logical bisimulations: indeed, the latter are special cases of CLBs. One can reach applicative bisimulation by making the first component of a CLB as small as possible. For it to correspond to logical bisimulation, the first component needs to be larger, to the point of being equal to the second component.

We need to study further coupled logical bisimulations, in order in particular to draw a comparison with environmental bisimulation. While in the latter, intuitively, we need to make environments grow along the development of an equivalence proof, an interesting feature of CLB is the possibility to keep the first component of a coupled relation small.

Possible extensions of this work include treating richer \( \lambda \)-calculi, like a \( \lambda \)-calculus with imperative features, for which a notion of state have to be introduced.

**Acknowledgements** Discussions with Davide Sangiorgi, Damien Pous and Daniel Hirschkoff have been helpful in the development of this work. The authors acknowledge the support of the ANR 12IS02001 PACE project.
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