Fat Euclidean Gravity
with
Small Cosmological Constant

Raman Sundrum*

Department of Physics and Astronomy
Johns Hopkins University
3400 North Charles St.
Baltimore, MD 21218-2686

Abstract

The cosmological constant problem is usually considered an inevitable feature of any
effective theory capturing well-tested gravitational and matter physics, without regard
to the details of short-distance gravitational couplings. In this paper, a subtle effective
description avoiding the problem is presented in a first quantized language, consistent
with experiments and the Equivalence Principle. First quantization allows a minimal
domain of validity to be carved out by cutting on the proper length of particle worldlines.
This is facilitated by working in (locally) Euclidean spacetime, although considerations
of unitarity are still addressed by analytic continuation from Lorentzian spacetime. The
new effective description demonstrates that the cosmological constant problem is sensitive
to short-distance details of gravity, which can be probed experimentally. “Fat Gravity”
toy models are presented, illustrating how gravity might shut off at short but testable
distances, in a generally covariant manner that suppresses the cosmological constant.
This paper improves on previous work by allowing generalizations to massless matter,
non-trivial spins, non-perturbative phenomena, and multiple (metastable) vacua.

*email: sundrum@pha.jhu.edu
1 Introduction

Imagine integrating out all physics above a TeV to arrive at an effective field theory of gravity and Standard Model matter,

\[ Z = \int [\mathcal{D}g_{\mu\nu}]_{p_\mu < \text{TeV}} [\mathcal{D}\psi_{\text{SM}}]_{p_\mu < \text{TeV}} e^{i \int d^4x \mathcal{L}_{\text{eff}}} \quad (1.1) \]

This effective theory beautifully accounts for everything we have tested experimentally in fundamental physics, while obeying the sacred principles of Equivalence, relativity, unitarity, and locality (down to 1/TeV). However, to match the real world the theory requires extreme fine-tuning in the cosmological constant [1]. So it is probably wrong.

In looking for what to change, we see that there is a vast tract of completely unexplored gravitational physics between \(10^{-3} \text{ eV} \) (\(\sim (100 \text{ microns})^{-1}\)) [2] and a TeV which has, by default in Eq. (1.1), been assumed to be a desert. Could new gravitational physics in this regime eliminate cosmological constant fine-tuning? A reasonably model-independent way of approaching this is to imagine integrating out just the gravitational physics in this regime, yielding

\[ Z = \int [\mathcal{D}g_{\mu\nu}]_{10^{-3} \text{eV}} [\mathcal{D}\psi_{\text{SM}}]_{\text{TeV}} e^{i \int d^4x \mathcal{L}_{\text{eff}}} \quad (1.2) \]

This effective description has the following properties:

i) It economically accounts for everything we have experimentally tested.

ii) It is fully relativistic.

iii) It obeys the Equivalence Principle.

iv) The purely matter couplings are manifestly local down to distances of order 1/TeV.

v) The couplings of gravity to itself and to matter are manifestly local down to distances of \(1/10^{-3} \text{eV} \sim 100 \text{ microns}\).

vi) It is unitary in the regime of small momentum transfers less than \(10^{-3} \text{eV}\).

vii) It is unitary up to a TeV in the pure matter sector, when gravitational interactions are neglected.

However Eq. (1.2) does have its price. When matter coupled to gravity is considered above \(10^{-3} \text{ eV}\), unitarity is violated. To see this, recall that we are positing new gravitational physics far below a TeV. There is plenty of phase space for a process such as

\[ e^+ e^- \rightarrow \sqrt{s} = 10 \text{ GeV} \quad e^+ e^- + \text{new gravitational stuff}, \quad (1.3) \]

where states within the domain of Eq. (1.2) are taken to states outside the domain. In the fundamental theory there should be some probability for this to happen, and therefore there must be a violation of unitarity within Eq. (1.2). On this basis of a lack of what we might call “kinematic closure”, a purist would disqualify Eq. (1.2) as an effective field theory.
Of course the loss of probability in Eq. (1.2) need only be of gravitational strength, $G_N \cdot \text{TeV}^2$, which is far below our experimental precision. We only see gravitational effects experimentally because the smallness of $G_N$ is compensated by energies very much larger than a TeV, arranged by having sources containing enormous numbers of fundamental particles. Such sources are necessarily large and slowly varying and therefore the relevant gravitational degrees of freedom have predominantly large wavelengths, captured by Eq. (1.2). Thus there is a domain of validity for Eq. (1.2), encompassing all experiments and observations performed so far and with all the advantages of effective field theory, as long as we are willing to neglect $G_N \cdot \text{TeV}^2$ when uncompensated by macroscopic factors.

This “almost effective field theory” is a good starting point for studying the cosmological constant problem. It enjoys the properties of effective field theory to within experimental precision. Furthermore, after decades of looking maybe there is simply no full effective field theory in which the cosmological constant can be understood because the resolution lies outside of field theory, even at low energy. Of course, the highest standard for any resolution is embedding within some fundamental theory of gravity and matter, but perhaps this need not be the first step.

Alas, the rather minimal description of the world given by Eq. (1.2) still does not evade cosmological constant fine-tuning. In Feynman diagram language the dominant diagrams contributing to the cosmological constant are those with very soft gravitational fields on external lines and hard matter fields and couplings in the interior. All of these ingredients are still present in Eq. (1.2). Furthermore, it seems that there is now little room left for new physics to be hiding that might solve the problem. We apparently have already pared down the effective description to just encompass the experimentally tested regimes. It is this argument, in some form or another, that convinces many theorists that there can be no new local physics that resolves the cosmological constant problem.

Nevertheless, in earlier papers [3] [4] I have argued that Eq. (1.2) is not minimal enough, that it implicitly makes an untested assumption about the short distance coupling of gravity to matter, and that there is a different possibility, “fat gravity”, which eliminates the cosmological constant problem. Fat gravity has a distinct and exciting prediction for short-distance tests of gravity [2] [5] [6], namely a suppression of the gravitational force.

Some related ideas and issues in the literature are as follows. The general possibility of a connection between the cosmological constant problem and sub-millimeter gravity was pointed out many years ago in Ref. [7]. Failed attempts to explicitly realize cosmological constant relaxation mechanisms based on light scalars are reviewed in Ref. [1]. Ref. [8] argued that any resolution of the cosmological constant problem within local effective field theory would yield light scalars with sub-millimeter range, which should be sought experimentally. Ref. [9]
described an approach to the cosmological constant problem with sub-millimeter non-locality in
the gravitational couplings. While this shares some similarities with fat gravity, there are also
important qualitative differences which make satisfying the Equivalence Principle problematic.
Ref. [10] described an extra-dimensional approach to the cosmological constant problem which
invoked fat gravity as a natural corollary of a sub-millimeter higher-dimensional Planck length.

In this paper I want to rederive the fat-gravity loop-hole in two distinct steps:

I) Replace Eq. (1.2) with an effective description with the same virtues (i – vii), but
with naturally small cosmological constant. This step is obviously powerful, but nevertheless
conservative. I do not posit what the new gravitational physics above $10^{-3}$ eV is, I just make
manifest the subtle loop-hole in the cosmological constant problem that is already present
subject to only (i – vii).

II) Present a toy model of fat gravity, that is actually commit to what the gravitational
physics above $10^{-3}$ eV is like, such that the cosmological constant is indeed small. In fact this
step requires only a reinterpretation of the results of step I, the mathematics is the same. The
fat gravity model is useful in that it is simple and you can play with it, push it around and ask
some tough diagnostic questions. You can also see qualitatively what happens in short distance
tests of gravity. But it is a toy because although the graviton is an extended object, with a
size $\sim 100$ microns, the relativistic corollary, graviton excitations at $10^{-3}$ eV, have not been
included. Associated with their absence there are Planck-suppressed violations of unitarity,
for gravitational momentum transfers $\sim 10^{-3}$ eV. Well above this scale unitarity is trivially
satisfied simply because fat gravity does not contain large gravitational momentum transfers,
and well below this scale unitarity is non-trivially satisfied. It is a rather satisfying part of the
model that it gets this low-energy unitarity right.

Why is a fully-fledged fat gravity model with graviton excitations so hard to build? Because
it requires a relativistic theory of an extended object, one of whose modes is the massless
graviton. Therefore it is at least as tough a venture as string theory. But a fat gravity phase
of string theory (with relatively point-like matter) is still unknown. (Neither is there known to
be a general No-Go theorem.) Nevertheless I presume that if a fully unitary and relativistic fat
gravity exists it is probably within some as yet unknown phase of string theory.

There are two central reasons why the loop-hole I want to point out is easily missed. The first
is that we are used to thinking about the coarse-graining procedure (in order to get effective
field theories from more fundamental ones) in terms of momentum (or other de-localized)
modes. The subtlety in the present problem is simple to see in position space but appears like
a conspiracy in momentum space. Therefore I will do the coarse-graining in position space.
This is most easily done in Euclideanized field theory, so I will work in locally Euclidean
spacetime. Nevertheless, the statements about unitarity, (vi) and (vii), will be recovered by
analytic continuation from Lorentzian spacetime.

A second reason is that the usual formalism of second quantization, and associated Feynman diagrams of Eq. (1.2), automatically correlate certain effects in a way which would be justified if there were a single cutoff, as in Eq. (1.1), but not if there are unknown new degrees of freedom above the lower cutoff, as in Eq. (1.2). For this reason we will begin our discussion by treating matter in first quantized effective theory. Once we have learned the basic lesson, we will generalize back to the more familiar second quantized formalism. Strictly speaking, steps I and II of this paper are carried out subject to all conditions (i – vii) for perturbative matter comprised only of scalar fields with general couplings and masses, coupled to gravity. However, a more powerful “block-spin” [11] approach to coarse-graining is also presented which suggests a clear generalization to non-perturbative and general matter. The price for the latter approach is that the cutoff on gravitational physics, $10^{-3} \text{eV}$, is not manifestly Lorentz invariant (and hence not manifestly generally covariant), but yields Lorentz invariance order by order in a soft-graviton expansion. I believe that this feature is a purely technical rather than a conceptual obstacle, because there is no such issue for the restricted case of perturbative scalar matter. Purely matter interactions are manifestly Lorentz invariant.

Finally, after going through the detailed derivations and understanding their point, there is a simple and manifestly generally covariant (and Lorentz invariant) prescription enforcing the results of the derivations in the fully realistic setting which avoids the cosmological constant fine-tuning. Without the derivations of this paper the prescription might appear ad hoc. This prescription was anticipated in Ref. [4].

The paper is organized as follows. In Section 2, the technical problem of the unboundedness of the gravitational action in Euclidean space [12] is discussed, including how the problem is avoided in perturbative gravity, which is all we consider here. We also discuss UV regulators to cut off gravity at $\sim 10^{-3} \text{eV}$. In Section 3, we introduce scalar matter in first quantized formalism. In Section 4, we show that for scalar matter there is an even more minimal effective description than Eq. (1.2) which continues to satisfy (i – vii) but has naturally small cosmological constant. The same mathematics is then turned into a toy fat gravity model. In Section 5, we discuss how the short distance static gravitational force is necessarily suppressed relative to Newton’s Law in the fat gravity model and how the transition distance scale is narrowly constrained by present experiments and cosmological constant naturalness. In Section 6, we discuss a different cutoff procedure that gives up manifest Lorentz invariance but makes the generalization to general non-perturbative second quantized matter much more obvious, and retains the small cosmological constant. We first present this in a simpler system where not just matter, but even gravity is replaced by a weakly coupled scalar field. This step is not strictly necessary but makes for easier reading. In Section 7, we repeat Section 6 but now
for real gravity, and deal with the fact that the cutoff procedure does not manifestly preserve general coordinate invariance. We are careful to distinguish the steps needed to recover this symmetry in the infrared from the cosmological constant problem. In Section 8, we generalize our results for real gravity coupled to second quantized scalar matter to the case of general second quantized matter such as the Standard Model. Thus issues such as the non-perturbative QCD effects on the cosmological constant can be easily dealt with. In Section 9, we study the case when there are multiple vacua in the matter sector and show that within our fat gravity model it is the true vacuum which has small cosmological constant while false vacua have large positive cosmological constants. In Section 10 we give a simple prescription which is manifestly generally coordinate invariant for practical computations in the Standard Model coupled to soft gravity, and which encapsulates the lessons of this paper and avoids cosmological constant fine-tuning. The prescription depends on the preceding sections for its justification. Section 11 provides conclusions. A technical proof appears in the appendix.

I imagine a reader, jaded by having witnessed many unsatisfactory attempts on the cosmological constant problem. Such a person needs to see what the essential point of the present paper is before committing to reading every secondary development. The essential point is obtained by reading Sections 2, 3, 4, 5 and perhaps 11. Readers who are content after reading these sections will certainly want to read the rest of the paper, which is however more technically involved.

2 Approach to Euclidean Quantum Gravity

2.1 Unboundedness of Euclidean Einstein action

In the Euclideanized version of quantum gravity [12], the partition functional is given by a sum over Riemannian metrics of positive weights,

\[ \mathcal{Z} = \int \mathcal{D}g_{\mu\nu} \, e^{-S_{\text{grav}}}. \]

(2.1)

The first important problem we must face is not a UV one but an IR one: the Euclidean Einstein action is unbounded from below [12], even when restricted to slowly varying geometries, and therefore the functional integral is ill-defined, even after UV regularization. Our approach to this deep problem is to avoid it and work completely perturbatively in the gravitational coupling and graviton field, \( h_{\mu\nu} \equiv g_{\mu\nu} - \delta_{\mu\nu} \). Perturbatively, inverting the quadratic terms of \( S_{\text{Einstein}} \) plus gauge fixing terms formally defines a Euclidean space propagator, while the remainder of the action plus ghost terms give Euclidean space interaction vertices. Thereby, one has a well-defined set of Feynman rules, despite the non-existence of the Euclidean path
integral. The associated Feynman diagrams have meaning, they are the analytic continuation of the Feynman diagrams from Lorentzian spacetime. From now on we will consider Eq. (2.1) to be just a formal mnemonic for the Euclidean quantum gravity perturbation expansion. At this level the only issue is the UV regularization.

2.2 Choice of gravity cutoff

We will find it convenient to switch language and denote the gravity cutoff by a distance scale \( \ell \) rather than an energy scale. Mostly \( \ell \) will be kept general, but later phenomenological considerations will constrain \( \ell \sim \mathcal{O}(100) \) microns. A procedure is required for coarse-graining the fundamental gravity sector and matching it to effective gravity, valid above distance scale \( \ell \),

\[
\mathcal{Z} = \int [\mathcal{D}g_{\mu\nu}]_\ell e^{-S_{\text{grav}}}. \tag{2.2}
\]

We will assume that there is a cutoff procedure (at least perturbatively about flat space), compatible with general covariance, which gives meaning to this expression. The detailed form is not required beyond knowing that it cuts off gravitational fluctuations below \( \ell \).

One concrete (but inelegant) example to keep in mind is just a momentum cutoff, corresponding to integrating out the high momentum modes of the gravitational field, \( h_{\mu\nu} \equiv g_{\mu\nu} - \delta_{\mu\nu} \), so that the remaining measure \( \int [\mathcal{D}g_{\mu\nu}]_\ell \) has a momentum cutoff \( |p| < 1/\ell \). While this preserves Euclidean “Poincare” invariance, it violates general coordinate invariance, or more precisely, after gauge fixing and adding ghost fields in some relativistic gauge such as de Donder, it violates BRST invariance. It is well known in quantum field theory that it is permissible to use a gauge symmetry violating regulator as long as one adds counter-terms to the regulated theory so that the BRST identities are recovered in the continuum limit \[13\] \[14\]. Here we are not trying to take the UV cutoff away as in a real continuum limit, but rather ensure that the BRST identities are recovered for soft gravitational momenta, \( |p| \ll 1/\ell \). This is done by adding and tuning BRST-violating counter-terms, order by order in gravitational momenta \( x \times \ell \), which restore the BRST symmetry of the soft amplitudes. This is analogous to restoring BRST symmetry in theories with a continuum limit, but with the technical difference that in the renormalizable case the number of counter-terms is fixed, while in the present case the number of counter-terms increases with the order to which one is working in the soft momentum expansion. In particular, the procedure breaks down for gravitational momentum transfers \( \sim 1/\ell \). This will not matter for us since BRST symmetry is important for (the Euclidean reflection of) unitarity, and we are already anticipating that it will be impossible to maintain exact unitarity for gravitational momentum transfers \( \sim 1/\ell \). The basic procedure of symmetry restoration systematically in the soft momentum expansion was discussed in Ref. \[15\].
The counter-terms to restore BRST invariance in the infrared may appear as a type of "fine-tuning", which adds to and complicates the cosmological constant fine-tuning problem we are trying to solve. However, there is a clear distinction. The BRST-restoring counter-terms are added according to a symmetry principle, which we believe is a principle of the fundamental theory, but mutilated by the coarse-graining procedure whereby some degrees of freedom are integrated out and some are left in the theory. These counter-terms are not a cheat when it comes to the cosmological constant, which is BRST-invariant and cannot be cancelled by the minimal set of BRST-restoring counter-terms.

There may of course be other more elegant cutoff procedures that we could use. Dimensional regularization is certainly an elegant and simple gravity regularization. For most of this paper we will not consider it, partly because I have not yet checked its applicability to the type of integrals arising from the novel way that matter is treated here. It will however be very useful in phrasing the simple prescription of Section 10, which gives a practical recipe for realistic computations while staying true to the results of the previous sections.

3 First Quantized Scalar Matter

3.1 Second-quantized “target”

Now let us add some matter to our theory. In this section we consider a "target" Euclidean field theory,

\[ Z = \int [\mathcal{D}g_{\mu\nu}] \mathcal{D}\psi \ e^{-S}, \]  

where the matter fields, \( \psi_n \), are all scalars. This will keep the first quantized form as simple as possible. They may have any masses, heavy or light, but of course light scalars will require fine-tuning. This is an entirely separate fine-tuning from the cosmological constant problem and I will assume that the scalar fine tuning is performed to obtain any desired spectrum of physical masses. Scalars are just a simplifying step in the analysis.

While gravity wavelengths are somehow cut off at \( \ell \), matter is of course cut off at much smaller distance. However by choosing a renormalizable form for the matter we can make the theory insensitive to the matter cutoff. In particular we will take an action of the form,

\[ S = \int d^4x \sqrt{g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \psi_n \partial_\nu \psi_n + V(\psi) + K(\psi)R + cR^2 + dR_{\mu\nu}R^{\mu\nu} \right\} + \text{sources}, \]  

where the potential \( V \) is a quartic polynomial in the \( \psi_n \), and \( K(\psi) \) is quadratic in the \( \psi_n \), with \( K(0) \approx M_{Pl}^2 \). We will allow \( V \) to be rather general except that we assume that super-renormalizable interactions such as \( \psi \) tadpoles and cubic potentials are weak enough relative...
to the relevant masses that their effects can be treated perturbatively. Also we assume that
the field basis of the $\psi_n$ has been chosen so as to diagonalize their mass-matrix, and that
we are expanding about an origin in $\psi$-space such that the mass-squared eigenvalues are all
positive. By power-counting, the couplings of Eq. (3.2) are sufficient to renormalize the matter
propagating in fixed geometries. The geometries themselves are being integrated over but with
cutoff $\ell$. Thus our scalar matter analog of Eq. (1.2) is well-defined (in perturbation theory).
The higher derivative gravity couplings $c$ and $d$ (needed to renormalize matter loop divergences)
are taken to be small enough to treated as perturbations as well.

### 3.2 First quantized translation

Now let us convert the perturbative expansion for Eqs. (3.1) and (3.2) to a first quantized
form (for scalar matter). The simplest guess for how to do this is to employ path integrals over
particle worldlines of mass $m$ using the (Euclideanized) point-particle action,

$$\int \mathcal{D}X(\tau) \ e^{-ms[X]},$$

where $X^\mu(\tau)$ is the parametrized worldline and $s$ is its proper length as measured by $g_{\mu\nu}$. This
obviously has the right classical limit given by geodesics. However, the complicated square-root
arising from Pythagoras’ Theorem in infinitesimal contributions to $s$, and the inapplicability
to massless matter, $m = 0$, make this type of path integral problematic. These problems are
resolved when an auxiliary worldline “einbein” field is introduced, giving more tractable path
integrals of the form

$$\int \mathcal{D}X^\mu(\tau) \int \mathcal{D}e(\tau) \ e^{-\int_0^1 d\tau \{g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu/2e + m^2e/2\}}.$$  (3.4)

It is understood that the reparametrization invariance of these path integrals is to be treated
as a gauge symmetry in the measure, so the measure has the volume of the associated gauge
orbits divided out. The worldlines in such path integrals can either be taken as having fixed end-points in spacetime, thereby defining a scalar propagator in the background geometry, or
as closed loops, thereby yielding a one-loop effective action for the background gravitational
field. As long as the measure is generally coordinate invariant, the general covariance of the
path integrals is manifest.

Considerable effort has gone into taming path integrals of the form of Eq. (3.4), with special
attention to the measure, in order to demonstrate that they reproduce the standard second-quantized expectations up to renormalizations of matter and gravity couplings in Eq. (3.2). Refs. [16] and [17] discuss the first-quantized approach in flat background spacetime, Ref. [18]
reviews work on the first-quantized approach in non-trivial background geometry, and Ref. [19] describes some recent progress. I will take it for granted here that Eq. (3.4) is well-defined and agrees with standard second quantization, at least for perturbative gravitational fields. We will not need a detailed form of the measure. We will however need to maintain a compact notation, and for this reason (only) I will from now on write path integrals of the type in Eq. (3.3) as a mnemonic for path integrals of the type in Eq. (3.4). There is little danger of confusion since the central results we will need are some bounds on the Boltzmann weights for non-zero $m$, which apply equally to either action. These bounds all follow from the fact that for any $X(\tau)$, the Boltzmann weight of Eq. (3.4) is at most as big as the Boltzmann weight of Eq. (3.3), easily verified by first extremizing with respect to $e(\tau)$.

The other ingredients in the perturbative expansion for matter are the vertices connecting propagators. These can be read off from Eq. (3.2). Note that there are no derivative couplings (in matter) so the couplings associated with the allowed vertices are just numbers. I will call a “web”, $W$, a network of matter particle world-lines in Riemannian spacetime which end only at vertices following from the action Eq. (3.2), or at sources. We thereby arrive at the first quantized form of the perturbative expansion of Eq. (3.1),

$$Z = \int [\mathcal{D}g_{\mu\nu}] \ e^{-S_{\text{grav}}} \int \mathcal{D}We^{-\sum_{n} m_{n}s_{n}[W]} \times \lambda[W] \ ,$$  \hspace{1cm} (3.5)

where now

$$S_{\text{grav}} = \int d^{4}x \sqrt{g} \left\{ \Lambda_{\cos} + M_{Pl}^{2}R + cR^{2} + dR_{\mu\nu}R_{\mu\nu} \right\} + \text{gravitational source}, \hspace{1cm} (3.6)$$

$s_{n}[W]$ is the length of the part of the web made out of world-lines of particle species $n$, and $\lambda[W]$ is the product of all the coupling constants at the vertices of $W$ that follow from Eq. (3.2).

To summarize, Eq. (3.1) is an analog of the real world effective description, Eq. (1.2), but with perturbative scalar matter, and Eq. (3.5) is its first-quantized form, with gravity still being treated as a (coarse-grained) field. As claimed in the introduction, Eq. (3.1) is really not a sufficiently minimal description. It includes a strong prejudice about the short-distance coupling of gravity to matter. But to see this we must massage Eq. (3.5).

Eq. (3.5) can straightforwardly be rewritten,

$$Z = \int [\mathcal{D}g_{\mu\nu}] \ e^{-S_{\text{grav}}} \ e^{\int \mathcal{D}We^{-\sum_{n} m_{n}s_{n}[w]} \times \lambda[w]} \ ,$$ \hspace{1cm} (3.7)

where $w$ denotes connected webs (where “connected” means only using matter lines, not gravity lines), while $W$ denotes a general web (that is, some collection of connected webs). The exponentiation occurs for the same reason as in Feynman diagrams, because in the original
sum, when there are \( N \) connected components we have a factor \( 1/N! \) in order to identify the different possible permutations.

4 Minimal effective description and Fat Gravity model

We can decompose the path integral over connected webs into a sum over two classes, say \( A \) and \( B \),

\[
\int \mathcal{D}w = \int [\mathcal{D}w]_A + \int [\mathcal{D}w]_B. \tag{4.1}
\]

Here, class \( A \) consists of any “vacuum” web that has “diameter” less than \( \ell \). The precise definition of “vacuum” web is one where no worldlines connect to external sources, although they may end on perturbative tadpole couplings in the potential \( V \) in Eq. (3.2). This is very much the analog of second quantized vacuum diagrams. The quotation marks only remind us that we are treating the gravitational field as an aspect of the vacuum in this definition. We define the diameter of a connected vacuum web as the maximum geodesic distance (given the particular background metric \( g_{\mu\nu} \)) between any two points on the web (though the geodesic between these two points is generally off the web). Class \( B \) is simply the complement of class \( A \). Intuitively, class \( A \) are “small” vacuum webs and class \( B \) are the remaining webs.

Thus,

\[
\mathcal{Z} = \int [\mathcal{D}g_{\mu\nu}]_\ell e^{-S_{\text{grav}}} \int [\mathcal{D}w]_A e^{-\sum n m_n s_n w} \lambda[w] e^{\int [\mathcal{D}w']_B e^{-\sum n m_n s_n w'} \lambda[w']}
\]

\[
= \int [\mathcal{D}g_{\mu\nu}]_\ell e^{-S_{\text{grav}}} \int [\mathcal{D}W]_A e^{-\sum n m_n s_n W} \lambda[W]
\]

\[
\times \int [\mathcal{D}W']_B e^{-\sum n m_n s_n W'} \lambda[W'], \tag{4.2}
\]

where in the second equality we are going back to the sum over all webs without regard to connectedness. For the second equality to hold we must generalize the definition of classes \( A \) and \( B \) to webs with disconnected components. The generalized definition is (obviously) that class \( A \) consists of only vacuum webs, each connected component of which has diameter less than \( \ell \). Class \( B \) consists of webs where each connected component is either a non-vacuum web, or a vacuum web with diameter greater than \( \ell \).

The great virtue of this way, Eq. (4.2), of presenting the first quantized form of Eq. (3.1), is that it is now manifest that there was in fact some more coarse-graining that we should have done in order to get a truly minimal effective description. We have written the partition functional as a product of path integrals. Coarse-graining means doing some of the integrals over physics that is not experimentally accessible and leaving the remaining integrals. In second
quantized form, Eq. (1.2) or the scalar analog Eq. (3.1), it is not obvious that there is any more coarse-graining we could do without integrating out tested physics, but in Eq. (4.2) we clearly see that the integral over webs of class A can still be done. Since they are all vacuum webs they do not contribute to any amplitudes with matter external lines, but only to the pure gravitational effective action. In general such contributions to the gravitational effective action would be non-local, but since all the class A webs are “small”, their effect on the coarse-grained gravitational fields (without fluctuations smaller than £) can be matched by purely local terms in the effective action. Furthermore, the decomposition of webs into classes A and B is obviously generally covariant, so the dominant terms in the local effective action arising from integrating out class A is the same as the terms we have already included in the gravitational action, Eq. (3.6). That is, integrating out class A should be considered as part of the process of gravitational coarse-graining up to distance £.

After integrating out class A, our minimal effective description is

\[ Z = \int \mathcal{D}g_{\mu\nu} e^{-S_{\text{grav}}} \int \mathcal{D}W_B e^{-\sum_n m_n s_n(W)} \times \lambda[W]. \] (4.3)

Note that this description has the same power as the less minimal Eqs. (1.2) and (3.1). All non-gravitational matter amplitudes arise from class B webs and are unchanged from Eq. (3.1) or its first-quantized equivalent, Eq. (4.2). Matter couplings to a fixed background metric are also unchanged and obey the Equivalence Principle, again because we have not touched the non-vacuum webs. For gravitational momentum transfers far below $1/\ell$, we have the usual local and general coordinate invariant effective action, locality after integrating out class A being argued in the previous paragraph. The purely matter amplitudes match the analytic continuations of the usual unitary Lorentzian ones, as do the full amplitudes with soft gravitational momentum transfers $\ll 1/\ell$. In this way, our checklist from the introduction, (i – vii), is satisfied, if we make an appropriate allowance for scalar matter in interpreting (i).

Note that the soft gravitational amplitudes are not at all trivial. For example, they contain standard non-analyticities arising from “large” diameter class B vacuum loops of very light and soft scalar matter (the analog of soft photons say). That is they match the analytic continuation from Lorentzian spacetime of soft gravity amplitudes with imaginary parts corresponding to intermediate processes of the form,

\[ \text{soft gravitons} \rightarrow \text{soft matter}. \] (4.4)

The integrated out class A vacuum loops do not affect these non-analyticities. As argued above they only renormalize (analytic) gravity effective vertices for coarse-grained gravity.
4.1 Technically natural size of the cosmological constant

Of course, the important question is what is the size of the cosmological constant in Eq. (4.3) now that we have integrated out the class A webs. If we had started from (the scalar-matter analog of) Eq. (1.1), then the class A webs would contribute a large renormalization of the cosmological constant. But the central point of the present paper is that without committing to a particular model of short-distance (< ℓ) gravity-matter coupling we do not know the contribution of class A webs. They are simply not part of our minimal effective description. The most conservative thing we can do is to estimate the renormalization-stable size within this description, Eq. (4.3). For simplicity, let us do this for the case where any matter mass eigenvalue is either “heavy” or “light”, \( m_n \gg 1/\ell \) or \( m_n \ll 1/\ell \). There are two types of loops to consider when estimating the renormalization stable value of the cosmological constant, matter loops and loops involving graviton exchange. Let us first consider pure matter loops, arising from summing vacuum webs in class B with fixed topology (where particle-type and vertex-type are considered as features of the topology of the web). If a web consists of purely heavy matter, then it makes a negligible contribution to the infrared cosmological constant, suppressed by at least \( e^{-m_n \ell} \), since \( \ell \) is the minimal vacuum-web diameter in class B. Thus, these heavy matter contributions to the cosmological constant which are robustly large in the effective descriptions of Eqs. (1.2) and (3.1), are negligible in the effective description of Eq. (4.3)!

The only webs that contribute to the gravitational effective action must come from light matter straddling distances of \( \ell \) or larger, with any heavy matter propagating over short distances. Such short lines can be treated as approximate local effective vertices from the perspective of the light matter propagating over \( \ell \) distances. That is, for this class of webs we could get the same answer by having first integrated out all heavy masses at the very beginning of our story, Eq. (3.1), so that their sole effect is to renormalize the vertices for light matter. Then we can just focus on light matter loops as if there were no heavy particles. One can imagine light matter diagrams regulated by heavy Pauli-Villars fields, in which case we immediately see that these regulators decouple from the gravitational effective action for the same reason as heavy physical particles do, other than a renormalization of light matter vertices. Thus after light matter renormalization, class B contributions to the gravitational effective action must be UV cut off by \( 1/\ell \). There is no other scale since we can neglect the light masses. The light matter contribution to the cosmological constant must therefore be of order \( 1/\ell^4 \). Of course, graviton momenta, and therefore their corrections to the cosmological constant, are also cut off by \( 1/\ell \).

In summary, the technically natural size of the cosmological constant in the effective description of Eq. (4.3) is \( \sim 1/\ell^4 \). This completes step I described in the introduction for the simple case of perturbative scalar matter.
4.2 Euclidean Fat Gravity Model

Although we know the technically natural size of the cosmological constant, arising from the physics described by Eq. (4.3), we do not know the contributions from the physics integrated out, namely the gravitational sector at short distances $< \ell$ and its couplings to “small” class A matter webs. It is now easy to give a toy model of what this physics might look like which has the defining features of what I have previously called “fat gravity”. We will take the gravitational sector to simply not allow fluctuations below distances of $\ell$, much as perturbative strings cannot have meaningful fluctuations below $\ell_{\text{string}}$. Our coarse-graining cutoff provides a crude model of this. Secondly, we assume that fat gravity is blind to the small class A matter webs. That is, the renormalization of the gravitational effective action from integrating out class A webs is negligible. Finally, we will take the tree-level cosmological constant, $\Lambda_{\text{cos}}$, to vanish or be small $< 1/\ell^4$.

With these assumptions, our toy model of fat Euclidean gravity is simply to take Eq. (4.3), not as an effective description, but as the full model! Clearly, in the toy model then, our estimate of the technically natural size of the cosmological constant is its real size. Another way to write the fat gravity model is

$$Z_{\text{fat gravity}} = \int [Dg_{\mu\nu}] e^{-S_{\text{grav}}} \int DW e^{-\sum_n m_n s_n[W]} \times \lambda[W] \times \theta_B[W],$$

(4.5)

where now we are formally integrating over all webs, but the usual weight for a web is multiplied by $\theta_B[W]$, which is one if the web is in class B, and zero if the web is in class A. This rather trivial re-writing makes more manifest that the short-distance modification of gravity coupling to matter, represented by $\theta_B[W]$, has absolutely no dependence on the matter couplings and masses, which appear in the other factors in the weight of a web. There is no secret fine-tuning with respect to the matter parameters. Yet it suppresses the cosmological constant corrections. That is remarkable from the usual viewpoint. Furthermore, $\theta_B[W]$ is completely generally coordinate invariant. It is not absolutely local as a functional of the web, but it is local down to the distance $\ell$. This is possible in a fat graviton model, the graviton should really be viewed as spread over a distance $\ell$.

Approaches to the cosmological constant problem which take non-locality as an underlying physical principle are Ref. [9] and Refs. [20]. Ref. [9] entertains non-locality only up to distances of order 100 microns, and in this sense crudely resembles the approach of this paper, without however having safe-guarded the Equivalence Principle for quantum matter. The non-locality in Refs. [20] on the other hand is present at the largest distances. This approach seems quite orthogonal to that pursued here.

Our toy model has sharp transitions at distance $\ell$, which in any fully realistic relativistic model would be smoothed out and correlated with gravitational excitations at $1/\ell$. These
excitations are obviously missing in our toy model and mean that for precisely momentum transfers of order $1/\ell$ it has no unitary analytic continuation to Lorentzian spacetime. For much higher momentum transfers, in our model only pure matter interactions are possible, and these are obviously the continuation of unitary matter interactions from Lorentzian spacetime since we have not tampered with the non-vacuum webs. Also for very small momentum transfers $\ll 1/\ell$ the model amplitudes (now both gravitational and light matter) are continuations of unitary Lorentzian amplitudes, just as when Eq. (4.3) is read as a coarse-grained effective description.

This completes step II of the program discussed in the introduction, for the simplest type of perturbative scalar matter.

5 Phenomenology of Fat Gravity

A rather straightforward consequence of fat gravity, Eq. (4.5), is that the static gravitational force, which is Newtonian at distances $\gg \ell$, must be suppressed at short distances $< \ell$. The usual long distance behavior follows because we are not modifying gravity in that regime. In momentum space the static (zero energy) behavior is just given by the graviton propagator $\sim 1/q^2$. At short distances $< \ell$ however, the gravitational field is cut off and the static force is suppressed. This is what makes the fat gravity model phenomenologically interesting, it has this rather unique testable prediction. The suppression of the gravitational force at short distance in effective field theory can only be accomplished by adding short distance repulsive forces of equal strength. This equality of force strengths is technically unnatural in non-supersymmetric effective field theory. Thus seeing a significant suppression would indicate non-field-theoretic gravitational physics, such as fat gravity.

One might wonder whether we really had to impose the gravity cutoff at $\ell$, thereby modifying short-distance static gravity. If we had not done this and continued the Newtonian force to short distances and high momenta, we could rotate such graviton exchanges using $SO(4)$-symmetry into the Euclidean continuation of hard timelike graviton exchanges. Unitarity would then require non-vanishing gravitational effective action sensitive to hard/heavy matter loops, with Lorentzian continuation having imaginary parts corresponding to

$$\text{timelike graviton} \rightarrow \text{hard/heavy matter.} \quad (5.1)$$

But we have shown that such sensitivity is absent in fat gravity. Consistency of the fat graviton model therefore requires that the gravitational coupling to matter be suppressed at short distance.
There is a very narrow window for fat gravity to solve the cosmological constant problem. Gravity is tested down to \( \sim 100 \) microns \(^2\) without deviation from Newtonian predictions, so \( \ell < 100 \) microns. On the other hand, the natural size of the radiatively stable cosmological constant in fat gravity is \( \sim \frac{1}{16\pi^2\ell^4} \), to be compared with the observed dark energy \( \sim (10^{-3} \text{ eV})^4 \). Therefore, naturalness imposes \( \ell > 20 \) microns \(^4\).

6 “Block-Spin” coarse-graining: all-scalar warm-up

6.1 Gravity \( \rightarrow \) scalar model

We will now pursue a technically different coarse-graining procedure which will allow us to generalize our first quantized observations to standard second quantization and general matter content, thereby allowing us to tackle realistic and non-perturbative matter phenomena, including issues such as multiple vacua. We proceed in two stages. As a warm-up, but not strictly necessary, we look at a theory without any type of gauge symmetry, involving two sectors both containing only scalars. That is, we not only consider matter to be scalars, \( \psi \), but replace the gravity sector by a single light real scalar \( \phi \). The metric in the model is now fixed to flat space. Our starting point in second-quantized language is then

\[
\mathcal{Z} = \int [\mathcal{D} \phi] e^{-S},
\]

where the action has scalar interactions, and the \( \phi \) sector has been cut off at \( \ell \) in the spirit of coarse-graining. For example, the \( \phi \) cutoff can be taken as a momentum cutoff \( p^2 < \frac{1}{\ell^4} \). Unlike the case of real gravity discussed in Section 2.2, this cutoff violates no essential gauge or global symmetries. We imagine that \( \phi \) couplings to \( \psi \) matter and to itself are extremely weak and observable only at small momentum transfers compared to \( \ell \) for similar reasons to the real gravity case. To parallel the usual gravity notation we will take \( \phi \) to be dimensionless, with a large “Planck scale” normalization, suppressing its interactions (with other dimensionful scales set by TeV or less). For a technical reason, not present for realistic matter, we assume that there is an exact symmetry \( \psi \rightarrow -\psi \) for light matter fields.

We can now write the effective description Eq. \((6.1)\) with \( \psi \) matter in first quantized form,

\[
\mathcal{Z} = \int [\mathcal{D} \phi] e^{-S_\phi} e^{\int [\mathcal{D} w] C} e^{-\sum_n m_n s_n [w] \times \lambda [w]} e^{\int [\mathcal{D} w'] D} e^{-\sum_n m_n s_n [w'] \times \lambda [w']},
\]

\[
= \int [\mathcal{D} \phi] e^{-S_\phi} \int [\mathcal{D} W] C e^{-\sum_n m_n s_n [W] \times \lambda [W]} \int [\mathcal{D} W'] D e^{-\sum_n m_n s_n [W'] \times \lambda [W']},
\]

where we make a decomposition of webs into two classes \( C \) and \( D \), which is technically different from our earlier decomposition into classes \( A \) and \( B \), but morally the same. Connected
vacuum webs (where now “vacuum” includes the $\phi$ background just as it did the gravitational background earlier) with diameters $\ll \ell$ are all in C, while connected non-vacuum webs and connected vacuum webs with diameters $\gg \ell$ are all in D, just as was the case before for A and B respectively. The detailed differences concern vacuum webs with diameters $\sim \mathcal{O}(\ell)$.

The C/D decomposition is obtained by first imposing a hypercubic lattice structure on the Euclidean spacetime, with lattice spacing of $\ell$, and elementary blocks of size $\ell^4$. These blocks will provide a fixed basis for distinguishing large and small webs, rather than using web diameters. Although the diameter approach is certainly the more elegant and spacetime-symmetric one, the approach we now take, related to standard “block-spin” coarse-graining [11], makes the exercise in accounting we are about to do more tractable. Since the precise definitions of C and D, and the statement and proofs of their properties, is rather technical, I will first give a sloppy description to get across the basic idea. “Small” class C webs are typically contained in some particular hypercubic block, denoted by integer coordinates, $N^\mu$. Thus,

$$\int [\mathcal{D}w]_C e^{-\sum_n m_n s_n[w]} \times \lambda[w] \approx \sum_N \int [\mathcal{D}w]_N e^{-\sum_n m_n s_n[w]} \times \lambda[w],$$

where the integral on the right-hand side is over all connected vacuum webs which are entirely within block $N$. Now such an integral over webs with restricted domain has a simple and obvious second quantized form,

$$\int [\mathcal{D}w]_N e^{-\sum_n m_n s_n[w]} \times \lambda[w] = \ln \{ \int_{\psi(\partial N) = 0} \mathcal{D}\psi e^{-S_{\text{matter}}} \},$$

where the integral on the right hand side is over fields inside block $N$ with Dirichlet boundary conditions on the boundary of the block, and $S_{\text{matter}}$ is the part of the action depending on matter fields (and possibly $\phi$ too). The logarithm arises because we have restricted to connected webs. Thus we can write a fully second quantized version of our ultimate coarse graining procedure (I) and the associated “fat $\phi$” model (II). The only problem is that a small but non-negligible fraction of small webs will not sit entirely within some hypercubic block, but will straddle a boundary between two blocks, hence the “$\approx$” in Eq. (6.3). We will be much more careful below.

6.2 Pure mathematics of blocks and webs

The results we need are given here and proven in the appendix. There exists a subset of connected vacuum webs, C, such that any integral over C (with arbitrary integrand) can be re-written,

$$\int [\mathcal{D}w]_C \ldots = \sum_N \sum_{b \in \mathcal{B}} \epsilon(b) \int [\mathcal{D}w]_b N \ldots,$$
where $C$ contains every connected vacuum web with diameter $< \ell$, some with $\ell \leq$ diameter $\leq 4\ell$, and none with diameter $> 4\ell$ (that is, $C$ defines a precise notion of “small” vacuum webs). The set $\mathcal{S}$ is a finite set of “composite blocks”, $b$, all contained in the region of spacetime $[0, 2\ell]^4$. Composite blocks are simply unions of some of the elementary $\ell^4$ blocks of the hyper-cubic lattice. $\epsilon(b) = \pm 1$ depending in some way on the block $b$. The composite block $b_N$ is simply the block $b$ translated by $N^\mu \ell$ from the vicinity of the origin to the vicinity of the elementary block $N$. The integral on the right hand side is over all connected vacuum webs entirely contained inside block $b_N$.

A corollary of the generality of Eq. (6.5) for general integrands is that even though the right hand side sums over webs with varying signs and multiplicities (since the blocks $b \in \mathcal{S}$ can overlap), in net any web gets summed once or not at all. Those that are summed are in $C$ and those that are not (or are non-vacuum connected webs) are defined to be in class D.

In the case where the web integrand is the standard one, we can clearly write a simple second-quantized translation analogous to Eq. (6.4),

\begin{equation}
\int [\mathcal{D}w]_{b_N} e^{-\sum_n m_n s_n[w]} \times \lambda[w] = \ln \left\{ \int_{\psi(\partial b_N) = 0} \mathcal{D}\psi e^{-S_{\text{matter}}} \right\},
\end{equation}

where the integral on the right-hand side is over fields within $b_n$ which obey Dirichlet boundary conditions on the boundary of $b_N$.

Two more results we will need involve integrals over the insides and boundaries of our blocks, $b_N$,

\begin{equation}
\sum_{N^\mu} \sum_{b \in \mathcal{S}} \epsilon(b) \int_{b_N} d^4x... = \int d^4x..., \end{equation}

where the right hand side is the integral over the entire Euclidean spacetime, and

\begin{equation}
\sum_{N^\mu} \sum_{b \in \mathcal{S}} \epsilon(b) \int_{\partial b_N} d^3x... = 0.
\end{equation}

The explicit construction of the composite blocks $b \in \mathcal{S}$ that ultimately define classes C and D and their properties is detailed in the appendix. However, other than their existence we will not need their explicit form here.

### 6.3 The more minimal effective description

By the above results we have the second quantized translation,

\begin{equation}
\int [\mathcal{D}w]_{C} e^{-\sum_n m_n s_n[w]} \times \lambda[w] = \sum_{N} \sum_{b \in \mathcal{S}} \epsilon(b) \ln \left\{ \int_{\psi(\partial b_N) = 0} \mathcal{D}\psi e^{-S_{\text{matter}}} \right\}.
\end{equation}
The minimal effective description upon integrating out class C is therefore given by

$$
\mathcal{Z} = \int [\mathcal{D}\phi] e^{-S_{\phi}} \int [\mathcal{D}W] e^{-\sum_n m_n s_n[W]} \times \lambda[W]
$$

$$
= \int [\mathcal{D}\phi] e^{-S_{\phi} - \sum N \sum_{b \in S} \epsilon(b) \ln \{\int \psi'[\partial N = 0] \mathcal{D}\psi e^{-S_{\text{matter}}}\} \int \mathcal{D}\psi e^{-S_{\text{matter}}}}. \quad (6.10)
$$

In writing the second-quantized form in the second line we have used the fact that the connected webs of class D are all connected webs minus those in class C. That is relative to Eq. (6.1), there are a set of “counterterms” in the $\phi$ action coming from the absence of class C.

These “counterterms” are manifestly local down to $\sim \ell$ in $\phi$, that is a sum over the lattice of terms which depends on $\phi$ in regions of size of order $\ell^4$. These counterterms are not independent of matter couplings and masses, so from the second quantized viewpoint it looks as if we are including some finely tuned counter-terms, but in fact we have derived them from a first-quantized theory where we are simply integrating out short-distance gravity-matter physics about which we are ignorant (step I of our program). As before, we can consider Eq. (6.10) as a toy model of a “fat $\phi$” (step II of our program), but this does not affect the mathematics.

Since the $\phi$ fields are cut off to only fluctuate on distances $> \ell$ these counter-terms can be matched to exactly local ones, that is, just renormalizing the couplings of $S_{\phi}$. Of course the class C counter-terms only respect hypercubic lattice symmetries, but full Euclidean spacetime “Poincare” symmetry is an automatic accidental symmetry of the leading operators in $\phi$. This is a standard infrared feature of many lattice theories. We will not be so fortunate in the case of real gravity.

Note that since the only difference between Eq. (6.10) and Eq. (6.1) are $\phi$ counterterms which are local (down to $\ell$), Eq. (6.10) has precisely the same non-analyticities in the low-energy, $\ll 1/\ell$, amplitudes, including those involving $\phi$. These are the Euclidean reflection of the imaginary parts of the Lorentzian diagrams corresponding to very low energy unitarity. If we go through the checklist (i – vii) given in the introduction we see that we have now lost exact (Euclidean) Poincare invariance because of our cutoff. But the symmetry is respected in the absence of the $\phi$ sector and is also recovered accidentally in the soft $\phi$ limit. There is no Equivalence Principle or real world data to test for. Other than these necessary exceptions other properties clearly hold.

### 6.4 Suppression of $\psi$ contributions to $\Gamma_{\text{eff}}[\phi]$

To quickly see the power of Eq. (6.10), let us first specialize to the case where all matter is heavy, $m_n \gg 1/\ell$. Therefore we can integrate it out completely in the integrals on the right hand side of Eq. (6.9) to yield an effective lagrangian for $\phi$ which is local. This local lagrangian
can be divided up into bulk terms in each $b_N$ as well as terms localized on the boundary of the block, $\partial b_N$,

$$
\int [Dw] e^{-\sum_n m_n s_n[w]} \times \lambda[w] = -\sum_N \sum_{b \in S} \epsilon(b) \left\{ \int_{b_N} d^4 x L_{\text{bulk}}(\phi) + \int_{\partial b_N} d^3 x L_{\text{boundary}}(\phi) \right\}. \tag{6.11}
$$

Note that since the lagrangians are local, the bulk lagrangian has no dependence on the nature of the block $b_N$. That is $L_{\text{bulk}}(\phi)$ is just the standard result in infinite Euclidean spacetime from integrating out the heavy matter.

By Eqs. (6.8) and (6.7), the sum of all boundary terms cancel and the sum over bulk terms gives an integral over all Euclidean spacetime,

$$
\int [Dw] e^{-\sum_n m_n s_n[w]} \times \lambda[w] = -\int d^4 x L_{\text{bulk}}(\phi). \tag{6.12}
$$

Thus class C contains precisely the corrections to the $\phi$ sector arising from integrating out matter in standard effective field theory. It follows that to all orders in the effective field theory expansion in $1/(m_\psi \ell)$, the class D contribution to the $\phi$ effective action vanishes! This agrees with our old first-quantized argument: since all connected vacuum webs in D have diameter $> \ell$ and heavy particle lines, their contributions are necessarily suppressed by at least $e^{-m_\psi \ell}$. Thus the minimal effective description, Eq. (6.10), does not suffer from a fine-tuning problem. Only coarse-grained $\phi$ loops can contribute to the $\phi$ potential, but this is all cut off at $1/\ell$.

Now let us consider a more general situation where some of the $\psi_n$ are very light, $m_n \ll 1/\ell$ and the rest are heavy, $m_n \gg 1/\ell$. To compute the right-hand side of Eq. (6.9) we first integrate out all the heavy $\psi$’s, leaving a local effective lagrangian in each block $b_N$, for the $\phi$ and the light $\psi$’s. The story of the local terms which are independent of the light $\psi$’s is identical to the case above where these fields were absent. The boundary terms dependent on the light $\psi$’s must have derivatives into the bulk acting on them because the $\psi$’s themselves vanish at the boundary. Using this and the $\psi \rightarrow -\psi$ symmetry, by power-counting the only local lagrangian terms which can depend on positive powers of $m_{\text{heavy}}$ are the bulk light $\psi$ mass terms. However, these are precisely the terms which we are fine-tuning to ensure that our light scalar matter is in fact very light. We have accepted this purely matter fine tuning as the price for playing with scalar matter at all. Once done there are no terms in the bulk or boundary local lagrangians with couplings which depend on positive powers of $m_{\text{heavy}}$, only $\ln(m_{\text{heavy}})$ or negative powers of $m_{\text{heavy}}$.

Let us use these observations to estimate matter contributions to the $\phi$ effective potential in the minimal effective description, Eq. (6.10). Clearly,

$$
\Gamma_{\text{eff}}[\phi] = \ln(\int D\psi e^{-S_{\text{matter}}}) - \sum_{N^\nu} \sum_{b \in S} \epsilon(b) \ln \left\{ \int_{\psi(\partial b_N)=0} D\psi e^{-S_{\text{matter}}} \right\}, \tag{6.13}
$$

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again since class D connected webs are all webs minus class C. By the procedure of integrating out heavy matter and matching to a matter theory with only light \( \psi \)'s, we can take the functional integral to be over only these light fields, with \( S \) replaced by the matched effective vertices discussed above. By our discussion above, the diagrams involving only heavy particle lines cancel out. Furthermore, the diagrams involving light \( \psi \) lines are UV finite after matter renormalization. The reason is that any divergence is local, just as pure heavy loops are local. Bulk divergences from class D (=all connected webs minus class C) then cancel by Eq. (6.7), and boundary localized divergences cancel by Eq. (6.8). Finally, the finite effective potential for \( \phi \) is set by the scale \( 1/\ell \), when we neglect light masses, and non-negative powers of \( 1/m_{\text{heavy}} \) as power-counted above. That is, the corrections are all at most of order \( 1/\ell^4 \). (Recall that we normalized \( \phi \) to be dimensionless like the metric.)

7 Block-spin coarse-graining with real gravity and scalar matter

If matter is kept restricted to scalar fields (with \( \psi \rightarrow -\psi \) symmetry for light matter as before), but \( \phi \) is now replaced by real gravity, we can re-do the steps of the previous section, and must only deal with the extra steps arising from the explicit breaking of the gravitational BRST symmetry by the coarse-graining.

The minimal effective description parallel to Eq. (6.10) is then,

\[
Z = \int [\mathcal{D}h_{\mu \nu} \mathcal{D}\text{ghosts}] \, e^{-S_{\text{grav}}-S_{\text{c.t.}}} \int [\mathcal{D}W] D e^{-\sum m_n s_n [W]} \times \lambda [W] \\
= \int [\mathcal{D}h_{\mu \nu} \mathcal{D}\text{ghosts}] \, e^{-S_{\text{grav}}-S_{\text{c.t.}}-\sum N \sum b \in S \times b} \ln \{ \int \psi' (\partial b_N) = 0 \mathcal{D}\psi e^{-S_{\text{matter}}} \} \\
\times \int \mathcal{D}\phi e^{-S_{\text{matter}}},
\]

where \( S_{\text{c.t.}} \) represents the BRST-violating counter-terms to be tuned to recover BRST symmetry in the limit of soft gravity processes, already discussed in Section 2.2 in the case where a momentum cutoff is used for gravity. The only difference here is that the set of required counter-terms is larger, respecting only hypercubic lattice symmetries, but otherwise the procedure and philosophy is the same as described in Section 2.2. A similar and well-known procedure for restoring BRST symmetry associated to chiral gauge invariance in lattice-regulated theory was described in Ref. [23]. (This procedure was only established perturbatively, and indeed there is some controversy on the non-perturbative applicability. Of course here we are only proceeding perturbatively in \( G_N \).) Analogously to the \( \phi \) case it is manifest that the non-analyticity in
amplitudes for momenta below $1/\ell$ are the same as in Eq. (3.1), since the difference consists of gravitational vertices which are local down to $\ell$.

Let us consider the matter contributions to the gravitational effective action, the analog of Eq. (6.13),

$$
\Gamma_{\text{eff}}[h_{\mu\nu}] = \ln(\int D\psi e^{-S_{\text{matter}}}) - \sum_N \sum_{b \in S} \epsilon(b) \ln\left\{ \int_{\psi(\partial b) = 0} D\psi e^{-S_{\text{matter}}} \right\}.
$$

(7.2)

Once again we can integrate out heavy matter, the only sensitive terms to this in the light effective theory being BRST-violating or the light mass terms, all of which we are content to fine-tune away. There are no truly divergent BRST-violations since they would be local, and by Eqs. (6.8) and (6.7) these would cancel out of the class D contributions. Therefore integrating out the light matter now gives contributions to the gravitational effective action which are set by the scale $1/\ell$, whether BRST conserving or not, this being the only scale since divergences cancel as usual using Eqs. (6.7) and (6.8), and light masses are neglected. For the cosmological constant the dominant contributions must be of order $1/\ell^4$ with corrections by powers of $1/(m_{\text{heavy}})$, BRST and continuous spacetime symmetry violations are suppressed in the infrared by tuning of $S_{\text{c.t.}}$.

When matter consists only of heavy fields, the separation of BRST violation and the cosmological constant problem is particularly clear. In this case the right hand sides of the matter contributions to the gravitational effective action, Eq. (7.2), after integrating out the heavy matter is given by integrals of local effective lagrangians (for $h_{\mu\nu}$), of either bulk or boundary type. By Eqs. (6.7) and (6.8) the entire matter contribution to the gravity effective action vanishes! In particular there is no correction to the cosmological constant and there is no violation of BRST symmetry from the matter. The only violation of BRST invariance is at most from gravity loops due to a non-symmetric gravity regulator itself, as discussed in Section 2.2, and the counterterms needed to cancel these violations in the infrared are totally independent of matter couplings and masses.

8 Generalization to realistic non-perturbative matter

The beauty of the second quantized version of Eq. (7.1) is that it has a simple generalization to more general and realistic matter, replacing the $\psi$ scalar functional integration by one over the Standard Model fields with Standard Model action,

$$
Z = \int [Dh_{\mu\nu} D\text{ghosts}] e^{-S_{\text{grav}} - S_{\text{c.t.}} - \sum_N \sum_{b \in S} \epsilon(b) \ln\{ \int_{\psi(\partial b) = 0} D\psi e^{-S_{\text{matter}}} \}] \\
\times \int D\psi_{\text{SM}} e^{-S_{\text{SM}}}. 
$$

(8.1)
This effective description makes sense even when non-perturbative matter effects are important, such as in QCD. As long as there are no light scalars $\ll 1/\ell$ with large non-derivative couplings in the matter sector we do not need the $\psi \to -\psi$ symmetry to repeat our analysis that the stable size of the cosmological constant is order $1/\ell^4$. That symmetry only helped to eliminate local terms sensitive to heavy masses from arising on the boundaries of blocks, $b_N$, upon integrating out heavy matter. (Of course here, heavy means in the sense of the physical spectrum, so for example the proton or pion is heavy even though one of their constituents is the “massless” gluon.) However, by power-counting there are no such operators with spinor or vector light matter that preserve enough spacetime and gauge symmetry to be induced on the boundaries and have couplings of positive dimension (which could then be set by heavy masses).

Once $S_{c.t.}$ is tuned to recover relativistic and BRST invariance in the gravitational couplings in the infrared, Eq. (8.1), interpreted either as a minimal effective description (I) or as a fat gravity toy model (II), satisfies the checklist from the introduction (i – vii). In the non-gravitational limit the matter sector has exact relativistic invariance.

Eq. (8.1) is essentially our final formulation, to be interpreted as an effective coarse-grained description of Nature, or as a fat gravity toy model. It must be confessed that we have not derived Eq. (8.1) in its full generality, but rather we have explicitly done the special case of perturbative scalar matter and then simply jumped to the natural generalization. Hopefully we are still on the right track.

9 Multiple Matter Vacua

In this section, we will be putting our faith in Eq. (8.1) interpreted as a fat gravity toy model. In particular, note that there is an important feature, namely that there appear to be two inequivalent ways we could try to add a purely classical cosmological constant term to Eq. (8.1). One is to add an arbitrary constant vacuum energy density to the Standard Model action. It is straightforward to see that this way fails to modify the physical cosmological constant, since $S_{SM}$ appears in two places in Eq. (8.1) in just such a way that a constant vacuum energy cancels out. The second way is to simply add a cosmological constant to $S_{grav}$. This of course, works. However, the decision we made in setting up the fat gravity toy model is that we imagine pure matterless fat gravity as unable to produce cosmological constant larger than $1/\ell^4$, that is, $S_{grav}$ contains a cosmological constant of at most $1/\ell^4$. These observations will be important in what follows. Of course, it is possible that fat gravity operates in a different way in Nature, but we will at least be able to examine one consistent possibility below.

A useful probe of any resolution of the cosmological constant problem is to imagine how
things work when the matter sector contains long-lived metastable vacua in addition to the true vacuum. One way of phrasing the cosmological constant problem is that in the absence of gravity one can identify a Poincare-invariant matter vacuum and the problem is how to maintain this invariance once gravity is “added”, at least approximately over large spacetime regions. Metastable matter vacua in the absence of gravity can be approximately Poincare-invariant over large regions of spacetime, so the question is whether whatever mechanism one is considering for dealing with the cosmological constant suppresses it in the metastable phase as well as in the true vacuum. If this is the case it would appear to make inflation impossible. This question is useful because it appears in standard effective field theory that if a metastable phase has very small cosmological constant, then the true vacuum must have lower energy density, which translates into a negative cosmological constant. If it is the true vacuum that has zero cosmological constant then the metastable vacuum must have higher energy density, translating into positive cosmological constant. It appears inconsistent for any mechanism to eliminate the cosmological constant in all matter vacua, so it is interesting to check in any proposal how the decision is made as to which vacuum’s cosmological constant to suppress, or whether in some mysterious way our standard expectations are violated.

Here we will argue that in the toy model of fat Euclidean gravity given above, the answer is that it is the true vacuum’s cosmological constant that is suppressed, while metastable vacua have positive cosmological constant. That is metastable vacua are in an inflationary phase.

For simplicity, let us consider the case of (gravity coupled to) a single scalar matter field, $\psi$, with a double-well potential $V(\psi)$ with two inequivalent local minima, $V(\psi_t) < V(\psi_f)$. For this simple situation we can use Eq. (7.1) as a stripped down version of Eq. (8.1). We can expand about either the true vacuum at $\psi_t$ or the false one at $\psi_f$. We assume that about either vacuum the physical mass is large, $m_\psi \gg 1/\ell$. Since the decay of the false vacuum is a non-perturbative process, it will not appear at any order in the perturbative expansion about $\psi_f$. This will simplify things since we are not interested here in the decay but what happens in the long period (in real Lorentzian time) before the decay. Perturbation theory applied to the matter interactions allows us to focus on this.

In the absence of gravity, or with fixed background metric, the partition functional has the form,

$$Z = \int \mathcal{D}\psi e^{-S}.$$  \hfill (9.1)

We approximate this integral by expanding about the minima of $S$,

$$Z \sim \int_{\psi \sim \psi_t} \mathcal{D}\psi e^{-S} + \int_{\psi \sim \psi_f} \mathcal{D}\psi e^{-S},$$  \hfill (9.2)

where the action in the first term is taken to be expanded perturbatively about the Gaussian
approximation centered on $\psi_t$, and the action in the second term is taken to be expanded perturbatively about the Gaussian approximation centered on $\psi_f$. That is, the partition function is the sum of the perturbative expansions about the two possible vacua. If we are interested in the theory around the false vacuum, $\psi_f$, then we only keep sources in the second term and the first integral is just some constant. If we think of $e^{-S}$ as a relative probability in the statistical interpretation of Euclidean field theory, then the first integral drops out of conditional probabilities, conditional on being in the vicinity of $\psi_f$. Similarly if we are interested in the true vacuum, $\psi_t$, we only keep sources in the first term and the second integral drops out of conditional probabilities, conditional on being in the vicinity of $\psi_t$.

Now let us turn gravity back on. Eq. (7.2) no longer automatically applies, we must re-think it starting from the fat gravity partition functional, Eq. (7.1). Suppressing all details of gauge fixing, BRST violation and related counterterms,

$$Z_{fat\ gravity} = \int [Dg_{\mu\nu}]_\ell e^{-S_{grav}-\sum N \sum_{b \in S} \epsilon(b) \ln\{\int_{\psi'(|\partial_N|=0)} D\psi' e^{-S_{matter}}\}} \int D\psi e^{-S_{matter}}$$

$$= \int [Dg_{\mu\nu}]_\ell e^{-S_{grav}-\sum N \sum_{b \in S} \epsilon(b) \ln\{\int_{\psi'(|\partial_N|=0)} D\psi' e^{-S_{matter}}\}}$$

$$\times \{ \int_{\psi \sim \psi_t} D\psi e^{-S_{matter}} + \int_{\psi \sim \psi_f} D\psi e^{-S_{matter}} \} \text{,} \tag{9.3}$$

where in the second line we have expanded the matter integral containing sources about the two possible vacua. The sources are in the $\psi_t$ ($\psi_f$) portion if we are interested in the true (false) vacuum, the non-source term dropping out of conditional probabilities as before. However, in either case all of the sourceless block-wise functional integral appears in the gravitational Boltzmann weight, $\exp(-\sum N \sum_{b \in S} \epsilon_b \ln\{\int_{\psi'(|\partial_N|=0)} D\psi' e^{-S_{matter}}\})$. In this term the expansion about the true vacuum (minimal action) dominates exponentially over any false contribution. Integrating out the heavy matter about the true vacuum again gives local terms in the form of local bulk and boundary lagrangians. By Eqs. (6.7) and (6.8) the boundary terms cancel and the bulk terms add up to a local gravitational counter-term subtracting the vacuum energy at the true minimum. Thus whether one is expanding the correlators of the fat gravity theory about the true vacuum or the false vacuum, the fat gravity effect is to subtract the true vacuum energy up to $\sim 1/\ell^4$ gravitational corrections. Thus the true vacuum has small cosmological constant while the false vacuum has large positive cosmological constant.

10 “Quenched Gravity”

The results derived in this paper can be presented in a simple, manifestly general coordinate invariant prescription, valid either in Euclidean or Lorentzian spacetime. This prescription,
which I will call “quenched gravity” satisfies (i – vii), and was discussed in Ref. [4]. However, quenched gravity is stated directly in terms of the amplitudes and does not, at first sight, follow from a path integral in a simple way. To that extent, it may have seemed rather ad hoc, were it not for the preceding sections, which show how it arises naturally from a suppression of small vacuum loops in the path integral.

Quenched gravity applies when expanding about some matter (metastable) vacuum, with gravitational momentum transfers restricted to be below $1/\ell$, for some $\ell$. The general amplitudes are constructed in the usual way from tree diagrams, whose vertices are taken from the quantum 1PI effective action. This can be decomposed as

$$\Gamma_{\text{eff}} = \Gamma_{\text{grav}}[g_{\mu\nu}] + \Gamma_{\text{matter}}[g_{\mu\nu}, \psi],$$

(10.1)

where all the vertices in the second term contains some matter lines. The rule is to compute $\Gamma_{\text{matter}}$ in exactly the standard way, but $\Gamma_{\text{grav}}$ as follows. First work in the absence of gravity about the matter vacuum of choice, and match to an effective Lagrangian valid below $1/\ell$, by integrating out heavier physics. For example, in the realistic case, if $1/\ell \sim 10^{-3}$ eV, all hadronic physics should be integrated out, because “heavy” refers to the physical spectrum not to “massless” gluons. This effective Lagrangian is then coupled to gravity below $1/\ell$ and used to compute $\Gamma_{\text{grav}}[g_{\mu\nu}]$, using dimensional regularization and minimal subtraction.

The size of the cosmological constant in $\Gamma_{\text{grav}}[g_{\mu\nu}]$ is then obviously of order $1/\ell^4$ or less. We now add to $\Gamma_{\text{grav}}[g_{\mu\nu}]$ two local cosmological constant contributions. One is at the least of order $1/\ell^4$ with unknown coefficient, representing the vacuum energy contributions of new gravitational degrees of freedom which are at least of mass $1/\ell$, or if one prefers, it represents the quartic sensitivity to the $1/\ell$ cutoff of the low-energy effective field theory which is missed by dimensional regularization. The second contribution is from the vacuum energy density difference between the vacuum about which one is expanding and the true vacuum.

Of course if one is expanding about the true vacuum, quenched gravity yields a natural size of the effective cosmological constant of order $1/\ell^4$, while describing gravitational momentum transfers below $1/\ell$ (and general matter momenta). If new gravity experiments probe shorter distances $\ell' < \ell$ and reveal only Newtonian behavior, then the relativistic completion of General Relativity must continue to hold as a quantum effective field theory up to $1/\ell'$. That is, we must replace $\ell \to \ell'$ in our prescription. Now the natural size of the cosmological constant is $1/\ell'^4$. If this is much larger than the dark energy density of the universe then the cosmological constant problem has returned, but not until then. In the realistic case we are not there yet.
11 Conclusions

This paper has taken a first quantized Euclidean path to showing that a minimal description of the present experimental domain does not imply a cosmological constant fine-tuning problem. But if gravity described by effective quantum general relativity (that is, a point-like nature for the graviton) is extrapolated to distances much shorter than 100 microns, the cosmological constant problem does emerge. Naturalness then suggests that new gravitational physics should be revealed in future short-distance gravitational tests. The fat gravity toy models in this paper give a concrete illustration of the qualitative features of such new physics: suppression of the cosmological constant (of the true vacuum, if there are metastable vacua) in conjunction with suppression of the short-distance gravitational force.

Given that new gravitational physics is close at hand, it would be very useful to identify, estimate and parametrize accessible precision effects at distances above the transition scale $\ell$, within effective descriptions such as discussed here and in Ref. [4]. It also remains to search for a consistent stringy realization of fat gravity.

I suspect that there is a more elegant “block-spin” coarse-graining than presented here, based on the Regge calculus [24] which manifestly retains a discrete subgroup of general coordinate invariance. However, several technical issues remain to be worked through before this is clear.

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A Appendix

In this appendix, we will work for convenience in units in which $\ell = 1$. Most of the results on webs and blocks used in the paper will be proven by induction on the number of Euclidean spacetime dimensions, $d$. Of course in the end we will be interested in $d = 4$. The results for which we use induction are restated for general $d$ as follows:

In $d$ dimensions, there exists a subset, $C$, of connected vacuum webs, a set, $S$, of composite blocks (composed of unit lattice cells), $b$, which are contained in the region $[0, 2]^d$, and a binary-valued function on $S$, $\epsilon(b) = \pm 1$, $b \in S$ such that
(a)  
\[ \int [Dw]_C \ldots = \sum_N \sum_{b \in S} \epsilon(b) \int [Dw]_{bN} \ldots . \]  
(A.1)

(b) If the diameter of a vacuum web is < 1 then it must be an element of \( C \).

(c)  
\[ \sum_N \sum_{b \in S} \epsilon(b) \int d^d x \ldots = \int d^d x \ldots . \]  
(A.2)

(d)  
\[ \sum_N \sum_{b \in S} \epsilon(b) \int d^{d-1} x \ldots = 0. \]  
(A.3)

We will first prove this for the case where the metric is exactly flat, as is the case in the \( \phi \) model without real gravity, and then briefly discuss real gravity afterwards.

We begin by giving a proof for a single spacetime dimension, \( d = 1 \). Here we define \( C \) to be the connected vacuum webs which are entirely contained within some composite interval (one-dimensional block) of length 2, that is \([N_1, N_1 + 2]\) for some integer \( N_1 \). \( S \) consists of two blocks, \([0, 2]\) with \( \epsilon = +1 \), and \([0, 1]\) with \( \epsilon = -1 \). Now let us prove (a – d).

(a) Naively, one might think that
\[ \int [Dw]_C \ldots = \sum_{N_1} \int [Dw]_{[N_1, N_1 + 2]} \ldots . \]  
(A.4)

But this is wrong because some webs in \( C \) might sit inside two overlapping length-2 intervals, \([N_1 - 1, N_1 + 1]\) and \([N_1, N_1 + 2]\), so that they get counted twice on the right-hand side but just once on the left. But in precisely these cases, the web must be within the intersection, \([N_1, N_1 + 1]\). Thus the double-counting is removed by subtracting the integral over webs which fit into such a unit interval. That is,
\[ \int [Dw]_C \ldots = \sum_{N_1} (\int [Dw]_{[N_1, N_1 + 2]} - \int [Dw]_{[N_1, N_1 + 1]}) \ldots , \]  
(A.5)

which is just what we had to prove.

(b) If the diameter of a web is less than 1 then clearly the web must be contained in some length-2 interval of the form \([N_1, N_1 + 2]\), and hence is in \( C \).

(c) We have,
\[ \sum_{N_1} (\int_{N_1}^{N_1 + 2} dx_1 - \int_{N_1}^{N_1 + 1} dx_1) \ldots = \sum_{N_1} \int_{N_1 + 1}^{N_1 + 2} dx_1 \ldots \]
\[ = \int_{-\infty}^{\infty} dx_1 \ldots . \]  
(A.6)
(d) If we have some function of 1-dimensional spacetime, \( f(x_1) \), then clearly
\[
\sum_{N_1}(f(N_1 + 2) + f(N_1) - f(N_1 + 1) - f(N_1)) = 0. \tag{A.7}
\]

Now let us make the inductive assumption that our results are true for \( d \) Euclidean spacetime dimensions. We will prove that they must then also be true in \((d + 1)\) dimensions. In order to not get confused between \( d \)-dimensional and \((d + 1)\)-dimensional objects we will denote the latter using barred symbols, \( \overline{C}, \overline{S}, \overline{b}, \overline{r}, \ldots \).

We define \( \overline{C} \) to contain any vacuum web \( \overline{w} \) which is contained in some thickness-2 slab of the form, \( N_{d+1} \leq x_{d+1} \leq N_{d+1} + 2 \), and where the projection of \( \overline{w} \) to \( d \) dimensions, \( \overline{w}_P \in C \).

Here “projection” has the straightforward interpretation that if one throws away the \( d + 1 \)-th coordinate information for a \((d + 1)\)-dimensional web, one is left with a \( d \)-dimensional web.

We define \( \overline{S} \) as consisting of composite blocks \( \overline{b} \equiv b \times [0, 2], b \in S \) with \( \overline{r}(\overline{b}) \equiv \epsilon(b) \), and \( \overline{b} \equiv b \times [0, 1], b \in S \) with \( \overline{r}(\overline{b}) \equiv -\epsilon(b) \). Obviously the \( \overline{b} \) are contained within \([0, 2]^{d+1}\) since the \( b \) are contained within \([0, 2]^d\).

(\( \overline{a} \)) We have,
\[
\sum_{\overline{b} \in \overline{S}} \sum_{b \in S} \epsilon(b) \int \overline{D} \overline{w}[\overline{b}]_{\overline{C}} \ldots = \sum_{N_{d+1}} \sum_{b \in S} \epsilon(b) \left( \int \overline{D} \overline{w}_b[N_{d+1}, N_{d+1} + 2] - \int \overline{D} \overline{w}_b[N_{d+1}, N_{d+1} + 1] \right) \ldots
\]
\[
= \sum_{N_{d+1}} \sum_{b \in S} \epsilon(b) \int \overline{D} w_b[N_{d+1}, N_{d+1} + 2] - \int \overline{D} x_{d+1}(w)[N_{d+1}, N_{d+1} + 1] \ldots
\]
\[
= \text{by (a)} \sum_{N_{d+1}} \int \overline{D} w_b[N_{d+1}, N_{d+1} + 2] - \int \overline{D} x_{d+1}(w)[N_{d+1}, N_{d+1} + 1] \ldots
\]
\[
= \int \overline{D} w \overline{C} \ldots. \tag{A.8}
\]

In the second line we have used the fact that a \((d + 1)\)-dimensional web, \( \overline{w} \), is specified by lifting a \( d \)-dimensional web, \( w \), by supplementing it with a function, \( x_{d+1}(w) \), giving the “elevation” of the web in the \((d + 1)\)-th dimension. In the third line we have used the inductive assumption (a) in \( d \) dimensions. In the fourth line we are using the obvious fact that the projection of the lift takes you back to the starting web, \( \overline{w}_P = w \), and the definition of \( \overline{C} \) requiring that webs fit into thickness-2 slabs. The possible non-uniqueness of such a slab for a given web in \( \overline{C} \) and the related issue of double-counting of webs is in exact analogy to the \( d = 1 \) case, that is the second term on the fourth line precisely removes the doubly-counted webs from the first term.

(\( \overline{b} \)) If the diameter of \( \overline{w} \) is less than \( \ell \), then it must certainly lie in some thickness-2 slab, \( N_{d+1} \leq x_{d+1} \leq N_{d+1} + 2 \). Furthermore, the diameter of \( \overline{w}_P \) must be at most that of \( \overline{w} \) because of the Euclidean metric. Therefore by the inductive assumption (b) and the definition of \( \overline{C} \), \( \overline{w} \in \overline{C} \).
We have,
\[
\sum_N \sum_{b \in S} \int_{\mathbb{R}} d^{d+1}x \ldots = \sum_{N_{d+1}} \sum_N \sum_{b \in S} \epsilon(b) \int_{b_N} d^d x (\int_{N_{d+1}}^{N_{d+1}+1} dx_{d+1} - \int_{N_{d+1}}^{N_{d+1}+1} dx_{d+1}) \ldots
\]
\[
= \sum_{N_{d+1}} \int d^d x \int_{N_{d+1}}^{N_{d+1}+1} dx_{d+1} \ldots
\]
\[
= \int d^{d+1}x \ldots \tag{A.9}
\]

We have,
\[
\sum_N \sum_{b \in S} \int_{\partial b_N} d^{d}x \ldots = \sum_{N_{d+1}} \sum_N \sum_{b \in S} \epsilon(b) \left( \int_{b_N,x_{d+1}=N_{d+1}+2} d^d x + \int_{b_N,x_{d+1}=N_{d+1}} d^d x \right.
\]
\[
+ \int_{\partial b_N} d^{d-1} x \int_{N_{d+1}}^{N_{d+1}+2} dx_{d+1} - \int_{b_N,x_{d+1}=N_{d+1}+1} d^d x
\]
\[
- \int_{b_N,x_{d+1}=N_{d+1}} d^d x - \int_{\partial b_N} d^{d-1} x \int_{N_{d+1}}^{N_{d+1}+1} dx_{d+1} \ldots
\]
\[
= \sum_{N_{d+1}} \sum_N \sum_{b \in S} \epsilon(b) \left( \int_{b_N,x_{d+1}=N_{d+1}+2} d^d x - \int_{b_N,x_{d+1}=N_{d+1}+1} d^d x \right) \ldots
\]
\[
= 0. \tag{A.10}
\]

This completes the proof by induction.

There is one more result which we need and can now prove directly, without induction on the number of dimensions, namely that any web with diameter greater than 4 is not contained in \( C \). We have (directly in 4 dimensions), using the result (a), that any web in \( C \) is certainly contained in the region \([0,2]^4\) or one of its translations. But the greatest distance between any two points within \([0,2]^4\) (or its translations) is obviously \( 4 = \sqrt{2^2 + 2^2 + 2^2 + 2^2} \). Therefore any web in \( C \) cannot have diameter greater than 4.

Now, all the above proofs have assumed a flat Euclidean metric. In the case of real gravity (as opposed to \( \phi \)) this does not hold. The issue of the metric is only relevant to those results giving lower and upper bounds on the diameter of class C webs. These in turn are used in the paper to show that all vacuum webs in class D have diameter > \( \ell \) and therefore heavy vacuum loops in class D are exponentially suppressed, and that class C webs have small enough diameter to be matched to local operators. These results continue to hold in the case of real gravity since we are only considering perturbative gravity in this paper, the metric is always being expanded about, and is close to, flat space.

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References

[1] For a review, see S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.

[2] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. 86 (2001) 1418, hep-ph/0011014; E. G. Adelberger [EOT-WASH Group Collaboration], hep-ex/0202008.

[3] R. Sundrum, JHEP 9907 (1999) 001, hep-ph/9708329.

[4] R. Sundrum, hep-th/0306106.

[5] J. C. Long, H. W. Chan and J. C. Price, Nucl. Phys. B539 (1999) 23, hep-ph/9805217 for a review see J. C. Long, J. C. Price, hep-ph/0303057.

[6] E. G. Adelberger, B. R. Heckel and A. E. Nelson, hep-ph/0307284.

[7] T. Banks, Nucl.Phys. B309 (1988) 493.

[8] S. R. Beane, Gen. Rel. Grav. 29 (1997) 945, hep-ph/9702419.

[9] J.W. Moffat, hep-ph/0102088.

[10] G. Dvali, G. Gabadadze and M. Shifman, Phys. Rev. D67 (2003) 044020, hep-th/0202174.

[11] L. P. Kadanoff, Physics 2 (1966) 263; for a review see C. Itzykson and J.-M. Drouffe, “Statistical Field Theory”, Cambridge University Press, Cambridge 1989.

[12] S. W. Hawking in “General Relativity: An Einstein Centenary Survey”, edited by S. W. Hawking and W. Israel, Cambridge University Press, Cambridge 1979.

[13] G. ’t Hooft, Nucl. Phys. B 33 (1971) 173.

[14] N. N. Bogoliubov and D. V. Shirkov, “Introduction to the theory of quantized fields”, John Wiley and Sons, New York 1980.

[15] K. Symanzik, Nucl. Phys. B226 (1983) 187.

[16] A. M. Polyakov, “Gauge Fields and Strings”, Harwood, Chur, Switzerland 1987.

[17] A. G. Cohen, G. W. Moore, P. Nelson and J. Polchinski, Nucl. Phys. B267 (1986) 143.

[18] C. Schubert, Phys. Rept. 355 (2001) 73, hep-th/0101036.
[19] F. Bastianelli and A. Zirotti, Nucl. Phys. B642 (2002) 372, hep-th/0205182.

[20] A. Adams, J. McGreevy, E. Silverstein, hep-th/0209226; N. Arkani-Hamed, S. Dimopoulos and G. Dvali, hep-th/0209227.

[21] A. G. Riess et al., Astron. J. 116 (1998) 1009, astro-ph/9805201; P. M. Garnavich et al., Astrophys. J. 509 (1998) 74, astro-ph/9806396; S. Perlmutter et al., Astrophys. J. 517 (1999) 565, astro-ph/9812133.

[22] K. Hagiwara et al., Phys. Rev. D66 (2002) 010001.

[23] A. Borrelli, L. Maiani, R. Sisto, G. C. Rossi and M. Testa, Nucl. Phys. B333 (1990) 335.

[24] T. Regge, Nuovo Cim. 19 (1961) 558.