Is there any polynomial upper bound for the universal labeling of graphs?

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Abstract A universal labeling of a graph $G$ is a labeling of the edge set in $G$ such that in every orientation $\ell$ of $G$ for every two adjacent vertices $v$ and $u$, the sum of incoming edges of $v$ and $u$ in the oriented graph are different from each other. The universal labeling number of a graph $G$ is the minimum number $k$ such that $G$ has universal labeling from $\{1, 2, \ldots, k\}$ denoted it by $\overrightarrow{\chi_u}(G)$. We have $2\Delta(G) - 2 \leq \overrightarrow{\chi_u}(G) \leq 2\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of $G$. In this work, we offer a provocative question that is: “Is there any polynomial function $f$ such that for every graph $G$, $\overrightarrow{\chi_u}(G) \leq f(\Delta(G))$?”. Towards this question, we introduce some lower and upper bounds on their parameter of interest. Also, we prove that for every tree $T$, $\overrightarrow{\chi_u}(T) = O(\Delta^3)$. Next, we show that for a given 3-regular graph $G$, the universal labeling number of $G$ is 4 if and only if $G$ belongs to Class 1. Therefore, for a given 3-regular graph $G$, it is an $\textbf{NP}$-complete to determine whether the universal labeling number of $G$ is 4. Finally, using probabilistic methods, we almost confirm a weaker version of the problem.

Keywords Universal labeling · Universal labeling number · 1-2-3-Conjecture · Regular graphs · Trees
1 Introduction

Throughout the paper we denote \( \{1, 2, \ldots, k\} \) by \( N_k \) and we use West (1996) for terminology and notations which are not defined here, also we consider only simple and finite graphs and digraphs. Karoński et al. (2004) initiated the study of \textit{neighbour-sum-distinguishing} labeling. They introduced an edge-labeling which is additive vertex-coloring that means for every edge \( uv \), the sum of labels of the edges incident to \( u \) is different from the sum of labels of the edges incident to \( v \). It was conjectured in Karoński et al. (2004) that every graph with no isolated edge has a \textit{neighbour-sum-distinguishing} labeling from \( N_3 \) (1-2-3-conjecture). This conjecture has been studied extensively by several authors, for instance see Ahadi and Dehghan (2016), Dehghan et al. (2013) and Karoński et al. (2004). Currently, we know that every connected graph with more than two vertices has a \textit{neighbour-sum-distinguishing} labeling, using the labels from \( N_5 \) (Kalkowski et al. 2010).

Various directed versions of the problem of vertex distinguishing colorings are of interest and have been recently investigated more intensively, see for instance Ahadi and Dehghan (2013) and Araujo et al. (2015, 2016). In this work we consider a new directed version of this problem. A \textit{universal labeling} of a graph \( G \) is a labeling of the edge set in \( G \) such that in every orientation \( \ell \) of \( G \) for every two adjacent vertices \( v \) and \( u \), the sum of incoming edges of \( v \) and \( u \) in the oriented graph are different from each other. The \textit{universal labeling number} of a graph \( G \) is the minimum number \( k \) such that \( G \) has \textit{universal labeling} from \( N_k \), denoted it by \( \overrightarrow{\chi}_u(G) \). For this notion, we conjecture that every graph should be weightable with a polynomial number of weights only, where the notion of polynomiality is with respect to the graph maximum degree parameter. As first steps towards the conjecture, we introduce some lower and upper bounds on their parameter of interest, then we prove the conjecture for trees, and partially for 3-regular and 4-regular graphs, and finally using probabilistic methods, we almost confirm a weaker version of the conjecture.

2 Universal labeling

A \textit{universal labeling} of a graph \( G \) is a labeling of the edge set in \( G \) such that in every orientation \( \ell \) of \( G \) for every two adjacent vertices \( v \) and \( u \), the sum of incoming edges of \( v \) and \( u \) in the oriented graph are different from each other. The \textit{universal labeling number} of a graph \( G \) is the minimum number \( k \) such that \( G \) has \textit{universal labeling} from \( N_k \), denoted it by \( \overrightarrow{\chi}_u(G) \). See Fig. 1

![Fig. 1](image-url)  
A universal labeling for the graph \( G = K_3 \cup K_{1,3} \)
Every graph has some universal labelings, for example one may put the different powers of two \( \{2^i : i \in \mathbb{N} \} \) on the edges of \( G \), where \( n \) is the number of vertices. Motivated by 1-2-3-conjecture, the following question posed by the second author on MathOverflow (Dehghan 2014).

**Problem 1** Is there a polynomial function \( f \) such that for every graph \( G \), \( \chi_u^-(G) \leq f(\Delta(G)) \)?

Let \( G \) be a graph and \( e \) be an arbitrary edge in \( G \). Without lose of generality suppose that \( \ell \) is a universal labeling for \( G \), it is easy to see that \( \ell \) is also a universal labeling for \( G \backslash \{e\} \). Thus, in the set of graphs with \( n \) vertices, complete graph \( K_n \) has the maximum universal labeling number, so we have the following problem: Is there a polynomial function \( f \) in terms of \( n \) such that \( \chi_u^-(K_n) \leq f(n) \)? If the answer to this problem is no, then there is a graph \( G \) and an edge \( e \in E(G) \) such that the universal labeling number of \( G \backslash \{e\} \) is a polynomial in terms of \( n \), but the universal labeling number of \( G \) is not polynomial in terms of \( n \). Finding such graphs can be interesting.

For \( k \in \mathbb{N} \), a proper edge \( k \)-coloring of \( G \) is a function \( c : E(G) \rightarrow \mathbb{N}_k \), such that if \( e, e' \in E(G) \) share a common endpoint, then \( c(e) \) and \( c(e') \) are different. The smallest integer \( k \) such that the graph \( G \) has a proper edge \( k \)-coloring is called the chromatic index of the graph \( G \) and denoted by \( \chi'(G) \). Let \( f \) be a proper edge coloring for a given graph \( G \). Then the function \( \ell(e) = 2^{f(e)-1} \) is a universal labeling for the graph \( G \). By Vizing’s theorem (1964), the chromatic index of a graph \( G \) is equal to either \( \Delta(G) \) or \( \Delta(G) + 1 \). So, every graph \( G \) has a universal labeling from \( \{2^{i-1} : i \in \mathbb{N}_{\Delta(G)+1}\} \). On the other hand, note that every universal labeling for the edges of \( G \) is a proper edge coloring of \( G \). Therefore the universal labeling number is at least the chromatic index of a graph. Thus,

\[
\Delta(G) \leq \chi_u^-(G) \leq 2\Delta(G).
\]

Let \( G \) be a graph and \( f : E(G) \rightarrow \mathbb{N}_{\chi_u^-(G)} \) be a universal labeling for it. Suppose that \( v \) is a vertex with maximum degree \( \Delta \) and let \( \{v_1, \ldots, v_\Delta\} \) be the set of neighbors of the vertex \( v \). Since every universal labeling for the edges of \( G \) is a proper edge coloring, thus \( f(vv_1), f(vv_2), \ldots, f(vv_\Delta) \) are distinct numbers. Without loss of generality suppose that \( f(vv_1) < f(vv_2) < \cdots < f(vv_\Delta) = M \). Now, consider the following partition for \( \mathbb{N}_{M-1} : \{i, M-i : i \in \mathbb{N}_{[M/2]} \} \). The set \( \{f(vv_i) : i \in \mathbb{N}_{\Delta-1}\} \) contains at most one number from each of the above sets, so \( \Delta(G) \leq \lfloor M/2 \rfloor + 1 \), therefore,

\[
2\Delta(G) - 2 \leq \chi_u^-(G) \leq 2\Delta(G).
\]  

**Remark 1** Here we show that the lower bound in Eq. 1 is sharp. Consider the complete bipartite graph \( K_{1,m} \), by Eq. 1, \( 2m - 2 \leq \chi_u^-(G) \). Consider an arbitrary order for the edges of \( G \) and label them by \( m-1, m, \ldots, 2m-2 \), respectively. This labeling is a universal labeling. Thus \( \chi_u^-(K_{1,m}) = 2m - 2 \).

In the first theorem, we solve Problem 1, for trees.

**Theorem 1** For every tree \( T \), \( \chi_u^-(T) = \mathcal{O}(\Delta^3) \), where \( \Delta \) is the maximum degree of the graph.

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Let $G$ be a connected graph and $\chi_u(G) \leq 3$. By Eq. 1, $\Delta(G) \leq 2$. Therefore $G$ is a cycle or path (note that there is no graph with $\chi_u = 3$). For $\chi_u(G) \leq 4$, we have the following hardness.

**Theorem 2** Let $G$ be a 3-regular graph, then $\chi_u(G) = 4$ if and only if $G$ belongs to Class 1. Also, for a given 4-regular graph $G$, $\chi_u(G) = 7$ if and only if $G$ belongs to Class 1.

It was shown in Leven and Galil (1983) that it is NP-hard to determine the chromatic index of an $r$-regular graph for any $r \geq 3$, therefore, for a given 3-regular graph $G$, it is an NP-complete to determine whether the universal labeling number of $G$ is 4. Also, for a given 4-regular graph $G$, it is an NP-complete to determine whether the universal labeling number of $G$ is 7.

An edge labeling $\ell$ for a graph $G$ is almost everywhere universal labeling if for a random orientation $\overrightarrow{O}$, $\Pr(\ell$ over $\overrightarrow{O}$ is proper) = 1. In other words, for the labeling $\ell$ of $G$ for every two adjacent vertices $v$ and $u$, the sum of incoming edges of $v$ and $u$ in the oriented graph which is obtained from the random orientation $\overrightarrow{O}$ are different from each other. Although we do not know any polynomial upper bound for the universal labeling number, but there is a Quasi-polynomial upper bound for almost every where universal labeling.

**Theorem 3** Every graph $G$ has an almost everywhere universal labeling from $\mathbb{N}_{\log n \log \log n}$, where $n$ is the number of vertices of the graph.

**Remark 2** For the complete graph $K_n$, let $f(n) = \omega(n^2)$. Label every edge of $K_n$ randomly and independently by one color from $\mathbb{N}_{f(n)}$ with the same probability. Let $\overrightarrow{O}$ be a random orientation for the graph. The probability that the two ends of an arbitrary edges $e$, have equal sums of incoming edges (mod $f(n)$), is $1/f(n)$. So the probability that this labeling is proper over $\overrightarrow{O}$ is $1 - \mathcal{O}(n^{-f(n)}) \approx 1$. Note that although the universal labeling number is an increasing property, but one cannot consider the above computing to obtain an upper bound for almost everywhere universal labeling number of all graphs.

**The universal labeling game**

A universal labeling game is a perfect-information game played between two players. The input of the game is an undirected graph $G$ and number $k$ (the input is denoted by $(G, k)$). The game consists of $|E(G)|$ rounds. In each round the first player chooses an unlabeled edge $e$ and label it with one of the numbers $\mathbb{N}_k$, afterwards, the next player chooses an orientation for $e$. Let $f$ be the labeling of $G$ after $|E(G)|$ rounds and $D$ be its oriented graph. If every two adjacent vertices of $D$ receive distinct sums of incoming labels in $f$, then first player wins. The universal labeling game number of $G$, denoted by $\chi_u^g(G)$, is the minimum number $k$ such that the first player has a winning strategy on $(G, k)$. It is clear that $\chi_u^g(G) \leq \chi_u(G)$, we prove the following upper bounds for $G$.

**Theorem 4** (i) For a connected graph $G$, $\chi_u^g(G) = 2$ if and if $G$ is a path or even cycle.
(ii) For every graph $G$, $\chi^\#_u(G) \leq 2\Delta(G)$.

(iii) For every tree $T$, $\Delta(T) \leq \chi^\#_u(T) \leq \Delta(T) + 1$.

In the above theorem we show that for every tree $T$, $\Delta(T) \leq \chi^\#_u(T) \leq \Delta(T) + 1$. For a given tree $T$, determining the complexity of computing $\chi^\#_u(T)$ is interesting. Finally, we ask the following about the computational complexity of the universal labeling game number of a given graph $G$.

**Problem 2** Is computing of the universal labeling game number of a given graph NP-h?

### 3 Proofs

**Proof of Theorem 1** Let $T$ be a tree with maximum degree $\Delta$ and $c : E(G) \to \mathbb{N}_{\Delta}$ be a proper edge coloring for $T$ (note that the chromatic index of every tree $T$ is $\Delta$ (West 1996)). Define the function $N(e) = c(e) + \Delta - 2$. We say that a set of numbers is sum-free if for every $i > 1$, the sum of each $i$ members of that set is not member of it. Note that for each vertex $v$ the set of numbers $\{N(e) : e \ni v\}$ is sum-free (Fact 1).

Choose an arbitrary vertex $z$ of $T$, and perform a breadth-first search algorithm from the vertex $z$. This defines a partition $V_0, V_1, \ldots, V_d$ of the vertices of $T$ where each part $V_i$ contains the vertices of $T$ which are at depth $i$ (at distance exactly $i$ from $z$). Each edge is between two parts $i - 1$ and $i$ for a natural number $i$. Say an edge is at even-level if $i - 1$ is even and odd-level otherwise. Each vertex at even depth is incident with just one odd-level edge (except the root which is not incident with any odd-level edge) and each vertex at odd depth is incident with just one even-level edge (Fact 2).

Let $K = (\Delta - 1) + \Delta + (\Delta + 1) + \cdots + (2\Delta - 2) = \frac{3}{2}\Delta(\Delta - 1)$. Now put the labels on the edges as follows: If an edge $e$ is at even-level, then put $N(e)$ on it and if an edge $e$ is at odd-level, then put $N(e) \times K$ on it. We claim that, this is a universal labeling from $\{\Delta - 1, \Delta, \ldots, 3\Delta(\Delta - 1)^2\}$.

For every orientation of $T$ and for each vertex $v$, if we divide the sum of incoming edges of $v$ to $K$, we have $K \times c(v) + r(v)$, where $c(v)$ is the sum of $N(e)$’s for odd-level incoming edges of $v$ and $r(v)$ is the sum of $N(e)$’s for even-level incoming edges of $v$ (Fact 3).

Consider an orientation on $T$. Suppose an edge $e$ with vertices $v_{i-1}$, $v_i$ at depth $i - 1$, $i$ respectively, such that the sum of labels of incoming edges of $v_{i-1}$ and $v_i$ are equal. We have two cases:

**Case 1** $i - 1$ is even. By Fact 3 and because $e$ is an incoming edge of $v_{i-1}$ or $v_i$, we have $r(v_i) = r(v_{i-1}) > 0$. But by Fact 2, $v_i$ has only $e$ as an even-level edge. Therefore $N(e)$ should be equal to sum of $N(e)$’s for incoming even-level edges of $v_{i-1}$. By Fact 1, this is a contradiction.

**Case 2** $i - 1$ is odd. By Fact 3 and since $e$ is an incoming edge of $v_{i-1}$ or $v_i$ we have $c(v_i) = c(v_{i-1}) > 0$. But by Fact 2, $v_i$ has only $e$ as an odd-level edge. Therefore $N(e)$ should be equal to sum of $N(e)$’s for incoming odd-level edges of $v_{i-1}$, by Fact 1, this is a contradiction.
Thus that labeling is a universal labeling with $O(\Delta^3)$ labels. This completes the proof.

\[ \square \]

Proof of Theorem 2 Let $G$ be a given 3-regular graph and $\ell : E(G) \rightarrow \mathbb{N}_4$ be a universal labeling for it. Also, suppose that $v$ is an arbitrary vertex and $u_1, u_2, u_3$ are its neighbors. First, note that every universal labeling for the edges of $G$ is a proper edge coloring. Therefore the universal labeling number is at least the chromatic index of a graph. So, $|\{\ell(vu_i) : i \in \mathbb{N}_3\}| = 3$. Also, note that the set of numbers $\{\ell(vu_i) : i \in \mathbb{N}_3\}$ is sum-free, thus this set is not 123 or 134. Therefore, the set of numbers $\{\ell(vu_i) : i \in \mathbb{N}_3\}$ is 124 or 234.

First, suppose that $G$ has the chromatic index 3 and let $f$ be a such proper edge coloring with colors 0, 1, 2. The function $2^f$ is a universal labeling for its edges. Next, assume that $G$ has a universal labeling $\ell$. For every vertex $v$ and its neighbors $u_1, u_2, u_3$, the set of numbers $\{\ell(vu_1), \ell(vu_2), \ell(vu_3)\}$ is 124 or 234. So the following function is a proper edge coloring for $G$.

$$f(v) = \begin{cases} \ell(v), & \text{if } \ell(v) \neq 3, \\ 1, & \text{if } \ell(v) = 3. \end{cases}$$

This completes the proof for the first part of theorem.

Now, assume that $G$ is a 4-regular graph. by Eq. 1, $\overline{\chi}_u(G) \geq 6$. First, we show that $\overline{\chi}_u(G) \geq 7$. To the contrary, suppose that $\overline{\chi}_u(G) = 6$ and let $f : E(G) \rightarrow \mathbb{N}_6$ be a universal labeling for $G$. Suppose that there is vertex $v \in V(G)$ such that $6 \notin \{f(uv)uv \in E(G)\}$. So, $\{f(uv)uv \in E(G)\} \subset \mathbb{N}_5$. It is clear that $1 \notin \{f(uv)uv \in E(G)\}$, therefore, the set of numbers $\{f(uv)uv \in E(G)\}$ is 2345. But that set of numbers is not sum-free, so this is a contradiction. Consequently, for every vertex $v \in V(G)$, $6 \in \{f(uv)uv \in E(G)\}$. Since $6 \in \{f(uv)uv \in E(G)\}$, thus, at most, one of the numbers 2, 4 is in $\{f(uv)uv \in E(G)\}$. Also, one of the numbers 1, 5 is in $\{f(uv)uv \in E(G)\}$. Therefore, $3 \in \{f(uv)uv \in E(G)\}$. There are four cases for the set $\{f(uv)uv \in E(G)\}$: 6321, 6341, 6325 and 6345. The three former cases are not sum-free, so the set of numbers for every vertex $v$ is $\{f(uv)uv \in E(G)\}$ is 6345. Let $vu \in E(G)$, Since $3 + 6 = 4 + 5$ therefore, the two adjacent vertices $v$ and $u$ can have equal in degree, that is a contradiction. So, $\overline{\chi}_u(G) \geq 7$. Next, we show that every vertex $v, 7 \in \{f(uw)uw \in E(G)\}$. Suppose that there is vertex $v$ such that $7 \notin \{f(uw)uw \in E(G)\}$. Without loss of generality suppose that $vu \in E(G)$ and $f(uv) = 4$. We have $7 \in \{f(uw)uw \in E(G)\}$. In this situation $7 = 4 + 3$ and this a contradiction. Therefore, every vertex $v, 7 \in \{f(uw)uw \in E(G)\}$. One of the numbers 1, 6 is in $\{f(uw)uw \in E(G)\}$, also, one of the numbers 2, 5 is in $\{f(uw)uw \in E(G)\}$ and one of the numbers 3, 4 is in $\{f(uw)uw \in E(G)\}$. Hence, there are eight cases for $\{f(uw)uw \in E(G)\}$. Among those sets only 1357, 4567, 3567, 2367 are sum-free.

Since for every vertex $v, 7 \in \{f(uv)uv \in E(G)\}$, thus if $a + b = c + 7$, then the set of numbers $\{f(uv)uv \in E(G)\}$ is not $abc7$. By this fact the set of numbers $\{f(uv)uv \in E(G)\}$ is not 1357, 4567, 2367. Thus $\{f(uv)uv \in E(G)\}$ is 3567. So $f$ is a proper 4-edge coloring for $G$. On the other hand if $G$ belongs to class 1, then let $\ell$ be one of its proper 4-edge coloring with colors 3567. This coloring is also a universal labeling for $G$. This completes the proof. \[ \square \]
Next, we prove that every graph $G$ has an almost every where universal labeling from $\mathbb{N}_{\frac{\lg n}{\lg \lg n}}$, where $n$ is the number of vertices of the graph. The key idea is: partitioning the edges of the graph into two parts based on the degrees of the vertices and labeling each part independently.

**Proof of Theorem 3** Let $G$ be a graph with $n$ vertices. Without loss of generality assume that $\frac{\lg n}{\lg \lg n}$ is an integer number (if $\frac{\lg n}{\lg \lg n}$ is not integer, we can consider $\lfloor \frac{\lg n}{\lg \lg n} \rfloor$ in the proof). Consider the edge-induced subgraph of $G$ over all edges like $e$ such that both ends of $e$ have degrees less than $\lg n$ in $G$ and call it $H$. Let $f$ be a proper edge coloring of $H$ by labels $\{2^i : i \in \mathbb{N}_{\lg n}\}$. Now, assume that $2n < p_1 < p_2 < \cdots < p_{\frac{\lg n}{\lg \lg n}}$ are prime numbers such that for every $k$, $p_{2k-1}, p_{2k} = \Theta(n^k)$ and also $\frac{p_{2k+1}}{p_{2k}} > n$. For example, let $(1 + i \epsilon)n^{\lfloor \frac{i+1}{2} \rfloor} \leq p_i \leq (1 + (i + 1) \epsilon)n^{\lfloor \frac{i+1}{2} \rfloor}$; where $\epsilon$ is a positive number and $\epsilon = o\left(\frac{\lg \lg n}{\lg n}\right)$. Label every edge of $G \setminus H$ randomly and independently by one prime numbers form $p_1, \ldots, p_{\frac{\lg n}{\lg \lg n}}$. Next multiply each label of $G \setminus H$ by $2^{1 + \lfloor \lg n \rfloor}$.

Assume that $\vec{O}$ is a random orientation for $G$ such that there is an edge $e = uv$ in $G$ with equal sums of incoming edges at both ends of the edge.

Now, we compute the probability of the event $B_e$ that is “$v$ and $u$ have equal sums of incoming edges”. Let $d_i(v)$ be the number of incoming edges incident with $v$ with label $i$. Therefore $\sum_i i d_i(v) = \sum_i i d_i(u)$. According to the definition of prime numbers, for every $k$ we have $d_{2k-1}(v) + d_{2k}(v) = d_{2k-1}(u) + d_{2k}(u)$ and consequently $d_i(v) = d_i(u)$ for every $i \leq \frac{\lg n}{\lg \lg n}$. If $e$ is an edge of $H$, since the edges of $H$ have proper edge coloring with distinct powers of two, and since the labels used in $G \setminus H$ are multiplied by $2^{1 + \lfloor \lg n \rfloor}$, the event $B_e$ cannot be occur. If $e$ is not in $H$, then at least one of the vertices $u$ and $v$ has degree more than or equal to $\lg n$. Assume that the degree of $v$ is more than or equal to $\lg n$ (note that all of edges incident with $v$, appear in $G \setminus H$). Let $K = \frac{\lg n}{\lg \lg n}$.

$$
\Pr(B_e) \leq \Pr\left( \bigwedge_i (d_i(v) = d_i(u)) \right) = \sum_i \Pr\left( \bigwedge_i (d_i(v) = d_i(u) = a_i) \right).
$$

The above summation is over all vectors $(a_1, \ldots, a_K)$ with $\sum_i a_i \leq d(v)$.

$$
\Pr(B_e) \leq \sum_i \left( \sum_i \Pr\left( \left( \vec{u}v \text{ has label } p_i \right) \land (d_i(u) = d_i(v) + 1) \land \bigwedge_{j,j \neq i} (d_j(u) = d_j(v)) \right) \right)
$$

$$
+ \sum_i \Pr\left( \left( \vec{v}u \text{ has label } p_i \right) \land (d_i(u) = d_i(v) - 1) \land \bigwedge_{j,j \neq i} (d_j(u) = d_j(v)) \right) \right).
$$
\[ = \sum \left( \frac{1}{2K} \sum_{a_j} \sum_{j \leq K} \Pr(d_j(u) = b_j) \times \Pr(d_j(v) = a_j) + \frac{1}{2K} \sum_{a_j} \sum_{j \leq K} \Pr(d_j(u) = c_j) \times \Pr(d_j(v) = a_j) \right). \]

In the above formula if \( j \neq i \), then \( b_j = c_j = a_j \) otherwise \( b_i = a_i + 1, c_j = a_j - 1. \)

\[ \Pr(B_e) \leq \sum (1 + o(1)) \Pr\left( \bigwedge_i (d_i(u) = a_i) \right) \times \Pr\left( \bigwedge_i (d_i(v) = a_i) \right) \leq \sum (1 + o(1)) \Pr\left( \bigwedge_i (d_i(u) = a_i) \right) \times \max_{a_i, i \leq K} \Pr\left( \bigwedge_i (d_i(v) = a_i) \right) = (1 + o(1)) \max_{a_i, i \leq K} \Pr\left( \bigwedge_i (d_i(v) = a_i) \right). \]

Let \( M := \max_{a_i, i \leq K} \Pr\left( \bigwedge_i (d_i(u) = a_i) \right) \). We have:

\[ M = \Pr\left( d(v) = \lg n, d^+(v) = \frac{\lg n}{2}, d_1(v) = \cdots = d_K(v) \right) = \left( \frac{\lg n}{2}, \frac{\lg \lg n}{2}, \ldots, \frac{\lg \lg n}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2K} \right)^K = (\lg n)! \left( (\lg n/2)! \right)\left( \left( \frac{\lg \lg n}{2} \right)! \right)^K (\frac{1}{4K}) \frac{\lg n}{2}. \]

By Stirling’s approximation \( \sqrt{2\pi n} (\frac{n}{e})^n \leq n! \leq \sqrt{e^2 n} (\frac{n}{e})^n \), we have:
the second player orients \(vv\) to \(u\). Consider the following strategy for the second player. For each edge \(e = v_i \rightarrow u\), \(1 \leq i \leq 2k + 1, u \notin V(C)\) the second player orients \(e\) from \(v_i\) to \(u\). Also for each edge \(e = v_i \rightarrow v_{i+1}\) the second player orients \(e\) from \(v_i\) to \(v_{i+1}\). By this approach, without any attention to the labels of the vertices, the second player wins the game. So if \(G\) contains an odd cycle, then \(\overrightarrow{\chi_0}(G) > 2\).

Now, let \(G\) be a graph without any odd cycle. The graph \(G\) is a bipartite graph. If \(\Delta(G) > 2\), then the second player has a winning strategy. Let \(v \in V(G), d(v) > 2\) and \(v_1, v_2, v_3 \in N(v)\). Consider the following strategy for the second player. For each edge \(e = vu|u \notin \{v_1, v_2, v_3\}\), the second player orients \(e\) from \(v\) to \(u\). Also for each edge \(e = v_i \rightarrow u|u \neq v, 1 \leq i \leq 3\), the second player orients \(e\) from \(v_i\) to \(u\). The orientations of three edges \(vv_1, vv_2, vv_3\) remain unknown (the orientations of other edges are not important). Note that there is no edge between \(v_1, v_2, v_3\) (\(G\) is a bipartite graph). With no loss of generality suppose that the first player chooses \(vv_1\) before \(vv_2\) and \(vv_3\), also chooses \(vv_2\) before \(vv_3\). If the first player labels \(vv_1\) by 1. The second player orients \(vv_1\) from \(v\) to \(v_1\). Next if the first player labels \(vv_2\) by 1. The second player orients \(vv_2\) from \(v\) to \(v_2\), and orients \(vv_3\) from \(v\) to \(v_3\). otherwise, the second player orients \(vv_2\) from \(v\) to \(v_2\), and orients \(vv_3\) from \(v_3\) to \(v\). So the second player wins the game. If the first player labels \(vv_1\) by 2, by a similar strategy, the second player wins the game. Therefore, \(G\) does not contain any odd cycle and \(\Delta(G) \leq 2\). So every connected component of \(G\) is an even cycle or a path. It is easy to see that in this case the first player has a winning strategy.

\[\Pr(B_e) \leq (1 + o(1))M\]

\[= (1 + o(1)) \left(\frac{\lg n}{4 \lg n}\right)^{\frac{\lg n}{4}} (1 + o(1)) \left(\frac{4 \lg n}{(\lg \lg n)^{1+0.00001}}\right)^{\frac{\lg n}{4}}\]

\[= o(10^{-\frac{\lg n}{2}})\]

Consequently,

\[\Pr(B_e) \leq (1 + o(1))M\]

Since the graph \(G \setminus H\) contains \(O(n^2)\) edges, so by linearity of expectation, with the probability \(1 - o(1)\) the labeling is proper. This completes the proof. \(\square\)

Proof of Theorem 4 (i) First, we show that if \(G\) is a graph with the universal labeling game number two then \(G\) does not contain an odd cycle. To the contrary suppose that \(G\) is a graph with some odd cycles. Let \(C = v_1v_2...v_{2k+1}v_1\) be a smallest odd cycle that \(G\) contains. The induced graph on the set of vertices \(\{v_1, v_2, ..., v_{2k+1}\}\) is an odd cycle, otherwise the graph \(G\) contains an odd cycle of a size smaller than \(2k + 1\). In this graph, if the first player only uses numbers one and two, then the second player has a winning strategy. For each edge \(e = v_i \rightarrow u, 1 \leq i \leq 2k + 1, u \notin V(C)\) the second player orients \(e\) from \(v_i\) to \(u\). Also for each edge \(e = v_i \rightarrow v_{i+1}\) the second player orients \(e\) from \(v_i\) to \(v_{i+1}\). By this approach, without any attention to the labels of the vertices, the second player wins the game. So if \(G\) contains an odd cycle, then \(\overrightarrow{\chi_0}(G) > 2\).
(ii) Let $G$ be a graph with the set of edges $\{e_1, \ldots, e_{|E(G)|}\}$. We present a strategy for the first player to win the game in $(G, 2\Delta(G))$. For each $i$, the first player chooses the edge $e_i$ in the round $i$ and assume that $f_{i-1} : \{e_j : j \in \mathbb{N}_{i-1}\} \rightarrow \mathbb{N}_{2\Delta(G)}$ is the partial labeling which is produced by the first player until the round $i-1$. Also let $D_{i-1}$ be the orientation for the set of edges $\{e_j : j \in \mathbb{N}_{i-1}\}$ which is produced by the second player until the round $i-1$. Define:

$$S_{i-1}(v) = \sum_{e_j = uv \in D_{i-1}, j \leq i-1} f(e_j).$$

The first player plays such that in each round like $k$, for every two adjacent vertices $w$ and $z$, $S_k(w) \neq S_k(z)$. Now, we present the strategy of the first player in the round $i$. The first player chooses the edge $e_i$ in the round $i$, let $e_i = uv$. The first player labels the edge $e_i$ from $\mathbb{N}_{2\Delta(G)}$ such that for every two adjacent vertices $w$ and $z$ of $G$, $S_i(w) \neq S_i(z)$. Note that each edge can produce at most one restriction for the value of $f_i(e_i)$. Thus, in order to make sure that no two adjacent vertices $w$ and $z$ of $G$ have $S_i(w) = S_i(z)$, there are at most $2\Delta(G) - 1$ restrictions for the value of $f_i(e_i)$. So the first player can find a proper value for $f_i(e_i)$ from $\mathbb{N}_{2\Delta(G)}$, thus has a winning strategy. This completes the proof.

(iii) Let $T$ be a tree. First, we show that $\chi_d^{\mathbf{8}}(T) \leq \Delta(T) + 1$. Choose an arbitrary vertex $v$ of $T$, and perform a breadth-first search algorithm from the vertex $v$. This defines a partition $V_0, V_1, \ldots, V_d$ of the vertices of $T$ where each part $V_i$ contains the vertices of $T$ which are at depth $i$ (at distance exactly $i$ from $v$). Let $e_1, \ldots, e_{|E(T)|}$ be an ordering of the edges according to their distance from the vertex $v$. Define the functions $f_i$, $D_i$ and $S_i$ similar to the part (ii). The first player plays such that in each round like $k$, for every edge $e_i = wz$, $i \leq k$, we have $S_k(w) \neq S_k(z)$. According to the order of the edges and the first player’s strategy there are at most $\Delta(T)$ restrictions for $f_k(e_k)$. Therefore, the first player can find a proper value for $f_k(e_k)$ from $\mathbb{N}_{\Delta(T)+1}$. It is easy to see that the second player has a winning strategy in $(T, \Delta(T) - 1)$ (the winning strategy is similar to part (i)). This completes the proof.

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