Geometry of Quaternionic Kähler connections with torsion

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Abstract

The target space of a (4,0) supersymmetric two-dimensional sigma model with Wess-Zumino term has a connection with totally skew-symmetric torsion and holonomy contained in Sp(n).Sp(1), QKT-connection. We study the geometry of QKT-connections. We find conditions to the existence of a QKT-connection and prove that if it exists it is unique. Studying conformal transformations we obtain a lot of (compact) examples of QKT manifolds. We present a (local) description of 4-dimensional homogeneous QKT structures relying on the known result of naturally reductive homogeneous Riemannian manifolds. We consider Einstein-like QKT manifold and find closed relations with Einstein-Weyl geometry in dimension four.

Running title: Quaternionic Kähler with torsion

Keywords. Almost Quaternionic, Hyper Hermitian, Quaternionic Kähler, Torsion, Locally conformal Quaternionic Kähler, Naturally reductive homogeneous Riemannian spaces, Einstein-Weyl geometry.

AMS Subject Classification: Primary 53C25, Secondary 53C15, 53C56, 32L25, 57S25

1 Introduction and statement of the results

An almost hyper complex structure on a 4n-dimensional manifold $M$ is a triple $H = (J_\alpha)$, $\alpha = 1, 2, 3$, of almost complex structures $J_\alpha: TM \to TM$ satisfying the quaternionic identities $J_\alpha^2 = -id$ and $J_1J_2 = -J_2J_1 = J_3$. When each $J_\alpha$ is a complex structure, $H$ is said to be a hyper complex structure on $M$.

An almost quaternionic structure on $M$ is a rank-3 subbundle $Q \subset \text{End}(TM)$ which is locally spanned by almost hypercomplex structure $H = (J_\alpha)$; such a locally defined triple $H$ will be called an admissible basis of $Q$. A linear connection $\nabla$ on $TM$ is called quaternionic connection if $\nabla$ preserves $Q$, i.e. $\nabla_X \sigma \in \Gamma(Q)$ for all vector fields $X$ and smooth sections $\sigma \in \Gamma(Q)$. An almost

*The author is supported by Contract MM 809/1998 with the Ministry of Science and Education of Bulgaria and by Contract 238/1998 with the University of Sofia "St. Kl. Ohridski".
quaternionic structure is said to be a quaternionic if there is a torsion-free quaternionic connection. A \( Q \)-hermitian metric is a Riemannian metric which is Hermitian with respect to each almost complex structure in \( Q \). An almost quaternionic (resp. quaternionic) manifold with \( Q \)-hermitian metric is called an almost quaternionic Hermitian (resp. quaternionic hermitian) manifold.

For \( n = 1 \) an almost quaternionic structure is the same as an oriented conformal structure and it turns out to be always quaternionic. When \( n \geq 2 \), the existence of torsion-free quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated \( \text{GL}(n,H)\text{SP}(1) \) structure \([10, 32, 42]\). If the Levi-Civita connection of a quaternionic hermitian manifold \((M, g, Q)\) is a quaternionic connection then \((M, g, Q)\) is called Quaternionic Kähler (briefly QK). This condition is equivalent to the statement that the holonomy group of \( g \) is contained in \( \text{SP}(n)\text{SP}(1) \) \([1, 2, 39, 40, 25]\). If on a QK manifold there exist an admissible basis \((H)\) such that each almost complex structure \( (J_\alpha) \in (H), \alpha = 1, 2, 3 \) is parallel with respect to the Levi-Civita connection then the manifold is called hyper Kähler (briefly HK). In this case the holonomy group of \( g \) is contained in \( \text{SP}(n) \).

The notions of quaternionic manifolds arise in a natural way from the theory of supersymmetric sigma models. The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion \([14, 22]\). The geometry of \((4,0)\) supersymmetric two-dimensional sigma models without Wess-Zumino term (torsion) is a hyper Kähler manifold. In the presence of torsion the geometry of the target space becomes hyper Kähler with torsion (briefly HKT) \([23]\). This means that the complex structures \( J_\alpha, \alpha = 1, 2, 3 \), are parallel with respect to a metric quaternionic connection with totally skew-symmetric torsion \([23]\). Local \((4,0)\) supersymmetry requires that the target space of two dimensional sigma models with Wess-Zumino term be either HKT or quaternionic Kähler with torsion (briefly QKT) \([31]\) which means that the quaternionic subbundle is parallel with respect to a metric linear connection with totally skew-symmetric torsion and the torsion 3-form is of type \((1,2)+(2,1)\) with respect to all almost complex structures in \( Q \). The target space of two-dimensional \((4,0)\) supersymmetric sigma models with torsion coupled to \((4,0)\) supergravity is a QKT manifold \([24]\). If the torsion of a QKT manifold is a closed 3-form then it is called strong QKT manifold. The properties of HKT and QKT geometries strongly resemble those of HK and QK ones, respectively. In particular, HKT \([23]\) and QKT \([24]\) manifolds admit twistor constructions with twistor spaces which have similar properties to those of HK \([21]\) and QK \([34, 40, 41]\).

The main object of interest in this article is the differential geometric properties of QKT manifolds. We find necessary and sufficient conditions to the existence of a QKT connection in terms of the Kähler 2-forms and show that the QKT-connection is unique if dimension is at least 8 (see Theorem 2.2 below). We prove that the QKT manifolds are invariant under conformal transformations of the metric. This allows us to present a lot of (compact) examples of QKT manifolds. In particular, we show that the compact quaternionic Hopf manifolds studied in \([34]\), which do not admit a QK structure, are QKT manifolds. In the compact case we show the existence of Gauduchon metric i.e. the unique conformally equivalent QKT structure with co-closed torsion 1-form.

It is shown in \([24]\) that the twistor space of a QKT manifold is always complex manifold provided the dimension is at least 8. It admits complex contact (resp. Kähler) structure if the torsion 4-form is of type \((2,2)\) and some additional nondegeneracy (positivity) conditions are fulfilled \([24]\). Most of the known examples of QKT manifolds are homogeneous constructed in \([33]\). However, there are no homogeneous proper QKT manifolds (i.e. QKT which is not QK or HKT) with torsion 4-form of type \((2,2)\) in dimensions greater than four by the result of \([33]\). We generalise this result showing that there are no proper QKT manifolds with torsion 4-form of type \((2,2)\) provided that the torsion
is parallel and dimension is at least 8.

In dimension 4 a lot of examples of QKT manifolds are known [24, 33]. In particular, examples of homogeneous QKT manifolds are constructed in [33]. We notice that there are many (even strong) QKT structures in dimension 4, all depending on an arbitrary 1-form. We give a local description of 4-dimensional QKT manifolds with parallel torsion; namely such a QKT manifold is a Riemannian product of a real line and a 3-dimensional Riemannian manifold. We observe that homogeneous QKT manifolds are precisely naturally reductive homogeneous Riemannian manifolds, the objects which are well known. We present a complete local description (up to an isometry) of 4-dimensional homogeneous QKT manifold is of this type.

Acknowledgements. The research was done during the author’s visit at the Abdus Salam International Centre for Theoretical Physics, Trieste Italy. The author thanks the Abdus Salam ICTP for support and the excellent environment. The author also thanks to G. Papadopoulos for his interest, useful suggestions and remarks. He is grateful to S. Marchiafava for pointing out some incorrect statements and L. Ornea for finding the time to read and comment on a first draft of the manuscript.

2 Characterisations of QKT connection

Let \((M, g, (J_{\alpha}) \in Q, \alpha = 1, 2, 3)\) be a 4n-dimensional almost quaternionic manifold with \(Q\)-hermitian Riemannian metric \(g\) and an admissible basis \((J_{\alpha})\). The Kähler form \(F_{\alpha}\) of each \(J_{\alpha}\) is defined by \(F_{\alpha} = g(\cdot, J_{\alpha} \cdot)\). The corresponding Lee forms are given by \(\theta_{\alpha} = \delta F_{\alpha} \circ J_{\alpha}\).

For an \(r\)-form \(\psi\) we denote by \(J_{\alpha}\psi\) the \(r\)-form defined by

\[ J_{\alpha}\psi(X_1, \ldots, X_r) := (-1)^r \psi(J_{\alpha}X_1, \ldots, J_{\alpha}X_r), \alpha = 1, 2, 3. \]

Then \((d^r \psi)_\alpha = (-1)^r J_{\alpha} d\psi\). We shall use the notations \(d_{\alpha} F_{\beta} := (d^0 F_{\beta})_\alpha\), i.e.

\[ d_{\alpha} F_{\beta}(X, Y, Z) = -d F_{\beta}(J_{\alpha}X, J_{\alpha}Y, J_{\alpha}Z), \alpha, \beta = 1, 2, 3. \]

We recall the decomposition of a skew-symmetric tensor \(P \in \Lambda^2 T^* M \otimes TM\) with respect to a given almost complex structure \(J_{\alpha}\). The \((1, 1), (2, 0)\) and \((0, 2)\) part of \(P\) are defined by \(P^{1, 1}(J_{\alpha}X, J_{\alpha}Y) = P^{1, 1}(X, Y), P^{2, 0}(J_{\alpha}X, Y) = J_{\alpha} P^{2, 0}(X, Y), P^{0, 2}(J_{\alpha}X, Y) = -J_{\alpha} P^{0, 2}(X, Y)\), respectively.

For each \(\alpha = 1, 2, 3\), we denote by \(dF_{\alpha}^+\) (resp. \(dF_{\alpha}^-\)) the \((1, 2) + (2, 1)\)-part (resp. \((3, 0) + (0, 3)\)-part) of \(dF_{\alpha}\) with respect to the almost complex structure \(J_{\alpha}\). We consider the following 1-forms

\[ \theta_{\alpha, \beta} = -\frac{1}{2} \sum_{i=1}^{4n} dF_{\alpha}^+(X, e_i, J_{\beta} e_i), \quad \alpha, \beta = 1, 2, 3. \]

Here and further \(e_1, e_2, \ldots, e_{4n}\) is an orthonormal basis of the tangential space.

Note that \(\theta_{\alpha, \alpha} = \theta_{\alpha}\).

The Nijenhuis tensor \(N_{\alpha}\) of an almost complex structure \(J_{\alpha}\) is given by

\[ N_{\alpha}(X, Y) = [J_{\alpha}X, J_{\alpha}Y] - [X, Y] - J_{\alpha}[J_{\alpha}X, Y] - J_{\alpha}[X, J_{\alpha}Y]. \]

The celebrated Newlander-Nirenberg theorem [30] states that an almost complex structure is a complex structure if and only if its Nijenhuis tensor vanishes.

Let \(\nabla\) be a quaternionic connection i.e.

\[ \nabla J_{\alpha} = -\omega_{\beta} \otimes J_{\gamma} + \omega_{\gamma} \otimes J_{\beta}, \]

where the \(\omega_{\alpha}, \alpha = 1, 2, 3\) are 1-forms.
Here and henceforth $(\alpha, \beta, \gamma)$ is a cyclic permutation of $(1, 2, 3)$.

Let $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ be the torsion tensor of type $(1, 2)$ of $\nabla$. We denote by the same letter the torsion tensor of type $(0, 3)$ given by $T(X, Y, Z) = g(T(X, Y), Z)$. The Nijenhuis tensor is expressed in terms of $\nabla$ as follows

$$
(2.2) \quad N_\alpha(X, Y) = 4T^0_\alpha(X, Y)
+ (\nabla_{J_\alpha X} J_\alpha)(Y) - (\nabla_{J_\alpha Y} J_\alpha)(X) - (\nabla_Y J_\alpha)(J_\alpha X) + (\nabla_X J_\alpha)(J_\alpha Y),
$$

where the $(0,2)$-part $T^0_\alpha$ of the torsion with respect to $J_\alpha$ is given by

$$
(2.3) \quad T^0_\alpha(X, Y) = \frac{1}{4} (T(X, Y) - T(J_\alpha X, J_\alpha Y) + J_\alpha T(J_\alpha X, Y) + J_\alpha T(X, J_\alpha Y)).
$$

We recall that if a $3$-form $\psi$ is of type $(1,2)+(2,1)$ with respect to an almost complex structure $J$ then it satisfies the equality

$$
(2.4) \quad \psi(X, Y, Z) = \psi(JX, JY, Z) + \psi(JX, Y, JZ) + \psi(X, JY, JZ).
$$

**Definition.** An almost quaternionic hermitian manifold $(M, g, (H_\alpha) \in Q)$ is *QKT manifold* if it admits a metric quaternionic connection $\nabla$ with totally skew symmetric torsion which is $(1,2)+(2,1)$-form with respect to each $H_\alpha, \alpha = 1, 2, 3$. If the torsion $3$-form is closed then the manifold is said to be *strong QKT manifold*.

It follows that the holonomy group of $\nabla$ is a subgroup of $\text{SP}(n) \times \text{SP}(1)$.

By means of (2.1), (2.2) and (2.4), the Nijenhuis tensor $N_\alpha$ of $J_\alpha, \alpha = 1, 2, 3$, on a QKT manifold is given by

$$
(2.5) \quad N_\alpha(X, Y) = A_\alpha(Y)J_\beta X - A_\alpha(X)J_\beta Y - J_\alpha A_\alpha(Y)J_\gamma X + J_\alpha A_\alpha(X)J_\gamma Y,
$$

where

$$
(2.6) \quad A_\alpha = \omega_\beta + J_\alpha \omega_\gamma.
$$

**Remark 1.** The definition of QKT manifolds given above is equivalent to that given in [24] because the requirement the torsion to be $(1,2)+(2,1)$-form with respect to each $H_\alpha, \alpha = 1, 2, 3$, is equivalent, by means of (2.5), to the fourth condition of (4) in [24]. The torsion of $\nabla$ is $(1,2)+(2,1)$-form with respect to any (local) almost complex structure $J \in Q$ [24]. This follows also from (2.3) and the general formula (6) in [3] which expresses $N_\beta$ in terms of $N_{J_1}, N_{J_2}, N_{J_3}$. In fact, it is sufficient that the torsion is a $(1,2)+(2,1)$-form with respect to the only two almost complex structures of $(H)$ since the formula (3.4.4) in [3] gives the necessary expression of $N_{J_\alpha}$ by $N_{J_1}$ and $N_{J_2}$. Indeed, it is easy to see that the formula (3.4.4) in [3] holds for the $(0,2)$-part $T^0_\alpha, \alpha = 1, 2, 3$, of the torsion. Hence, the vanishing of the $(0,2)$-part of the torsion with respect to any two almost complex structures in $(H)$ implies the vanishing of the $(0,2)$-part of $T$ with respect to the third one.

On a QKT manifold there are three naturally associated $1$-forms to the torsion defined by

$$
(2.7) \quad t_\alpha(X) = -\frac{1}{2} \sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i), \quad \alpha = 1, 2, 3.
$$

We have

**Proposition 2.1** *On a QKT manifold $J_1t_1 = J_2t_2 = J_3t_3$.***
Proof. Applying \((2.4)\) with respect to \(J_\beta\) we obtain
\[
\begin{align*}
t_\alpha(X) &= -\frac{1}{2} \sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i) = -\frac{1}{2} \sum_{i=1}^{4n} T(X, J_\beta e_i, J_\gamma e_i) \\
&= \frac{1}{2} \sum_{i=1}^{4n} T(J_\beta X, e_i, J_\gamma e_i) - \frac{1}{2} \sum_{i=1}^{4n} T(J_\beta X, J_\beta e_i, J_\alpha e_i) + \frac{1}{2} \sum_{i=1}^{4n} T(X, e_i, J_\alpha e_i).
\end{align*}
\]
The last equality implies \(t_\alpha = J_\beta t_\gamma\) which proves the assertion. Q.E.D.

The 1-form \(t = J_\alpha t_\alpha\) is independent of the chosen almost complex structure \(J_\alpha\) by Proposition \([2.1]\). We shall call it the torsion 1-form of a given QKT manifold.

Remark 2. Every QKT manifold is a quaternionic manifold. This is an immediate consequence of \((2.3)\) and Proposition 2.3 in \([4]\).

However, the converse to the above property is not always true. In fact, we have

**Theorem 2.2** Let \((M, g, (J_\alpha \in \mathcal{Q})\) be a \(4n\)-dimensional \((n > 1)\) quaternionic manifold with Q-hermitian metric \(g\). Then \(M\) admits a QKT structure if and only if the following conditions hold

\[
(d_\alpha F_\alpha)^+ - (d_\beta F_\beta)^+ = \frac{1}{2} (K_\alpha \wedge F_\beta - J_\beta K_\alpha \wedge F_\alpha - (K_\beta - J_\alpha K_\alpha) \wedge F_\gamma),
\]
where \((d_\alpha F_\alpha)^+\) denotes the \((1,2)+(2,1)\) part of \((d_\alpha F_\alpha)\) with respect to the \(J_\alpha, \alpha = 1, 2, 3\). The 1-forms \(K_\alpha, \alpha = 1, 2, 3\), are given by

\[
K_\alpha = \frac{1}{1-n} (J_\beta \theta_\alpha + \theta_{\alpha, \gamma}).
\]

The metric quaternionic connection \(\nabla\) with torsion 3-form of type \((1,2)+(2,1)\) is unique and is determined by

\[
\nabla = \nabla^g + \frac{1}{2} \left( (d_\alpha F_\alpha)^+ - \frac{1}{2} (J_\alpha K_\alpha \wedge F_\gamma + K_\alpha \wedge F_\beta) \right),
\]
where \(\nabla^g\) is the Levi-Civita connection of \(g\).

Proof. To prove the 'if' part, let \(\nabla\) be a metric quaternionic connection satisfying \((2.1)\) which torsion \(T\) has the required properties. We follow the scheme in \([17]\). Since \(T\) is skew-symmetric we have

\[
\nabla = \nabla^g + \frac{1}{2} T.
\]
We obtain using \((2.4)\) and \((2.1)\) that

\[
\begin{align*}
\frac{1}{2} (T(X, J_\alpha Y, Z) + (T(X, Y, J_\alpha Z)) &= -g \left((\nabla^g_X J_\alpha)Y, Z\right) \\
&+ \omega_\beta(X)F_\gamma(Y, Z) - \omega_\gamma(X)F_\beta(Y, Z).
\end{align*}
\]

The tensor \(\nabla^g J_\alpha\) is decomposed by parts according to \(\nabla J_\alpha = (\nabla J_\alpha)^{2,0} + (\nabla J_\alpha)^{0,2}\), where \([17]\)

\[
\begin{align*}
g \left((\nabla^g_X J_\alpha)^{2,0}Y, Z\right) &= \frac{1}{2} \left((d_\alpha F_\alpha)^+ (X, J_\alpha Y, J_\alpha Z) - (d_\alpha F_\alpha)^+ (X, Y, Z) \right) \\
g \left((\nabla^g_X J_\alpha)^{0,2}Y, Z\right) &= \frac{1}{2} \left(g(N_\alpha(X, Y), J_\alpha Z) - g(N_\alpha(X, Z), J_\alpha Y) - g(N_\alpha(Y, Z), J_\alpha X) \right)
\end{align*}
\]
Taking the $(2,0)$ part in (2.12) we obtain using (2.13) that

\begin{equation}
T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Y) = (dF^+_\alpha(X, J_\alpha Y, J_\alpha Z) - (dF^+_\alpha(X, Y, Z) + C_\alpha(X)F_\gamma(Y, Z) + C_\alpha(J_\alpha X)F_\beta(Y, Z),
\end{equation}

where

\begin{equation}
C_\alpha = \omega_\beta - J_\alpha \omega_\gamma.
\end{equation}

The cyclic sum of (2.15) and the fact that $T$ and $(dF_\alpha)^+$ are $(1,2)+(2,1)$-forms with respect to each $J_\alpha$, gives

\begin{equation}
T = (d_\alpha F_\alpha)^+ - \frac{1}{2} (J_\alpha C_\alpha \wedge F_\gamma + C_\alpha \wedge F_\beta).
\end{equation}

Further, we take the contractions in (2.17) to get

\begin{equation}
J_\alpha t_\alpha = -\theta_\alpha - J_\beta C_\alpha, \quad J_\alpha t_\alpha = -J_\gamma \theta_\alpha - nJ_\gamma C_\beta, \quad J_\alpha t_\alpha = J_\beta \theta_\gamma - nJ_\alpha C_\gamma,
\end{equation}

Using Proposition 2.1, (2.6) and (2.16), we obtain consequently from (2.18) that

\begin{equation}
A_\alpha = J_\alpha C_\beta + J_\gamma C_\gamma = J_\beta (\theta_\gamma - \theta_\beta),
\end{equation}

\begin{equation}
(n - 1)J_\beta C_\alpha = \theta_\alpha - J_\beta \theta_\alpha, \quad \gamma.
\end{equation}

Then (2.8) and (2.9) follow from (2.17) and (2.20).

For the converse, we define $\nabla$ by (2.10). To complete the proof we have to show that $\nabla$ is a quaternionic connection. We calculate

\begin{equation}
g(\nabla X J_\alpha)Y, Z) = g((\nabla^2 X J_\alpha)Y, Z) + \frac{1}{2} (T(X, J_\alpha Y, Z) + T(X, Y, J_\alpha Z))
\end{equation}

\begin{equation}
= \omega_\beta(X)F_\gamma(Y, Z) - \omega_\gamma(X)F_\beta(Y, Z),
\end{equation}

where we used (2.13), (2.14), (2.19), (2.9), (2.7), (2.16) and the compatibility condition (2.8) to get the last equality. The uniqueness of $\nabla$ follows from (2.10) as well as from Theorem 10.3 in [32] which states that any quaternionic connection is entirely determined by its torsion (see also [18]).

Q.E.D.

In the case of HKT manifold, $K_\alpha = dF_\alpha = 0$ and Theorem 2.2 is a consequence of the general results in [17] (see also [20]) which imply that on a hermitian manifold there exists a unique linear connection with totally skew-symmetric torsion preserving the metric and the complex structure, the Bismut connection. This connection was used by Bismut [1] to prove a local index theorem for the Dolbeault operator on non-Kähler manifold. The geometry of this connection is referred to KT-geometry by physicists. Obstructions to the existence of (non-trivial) Dolbeault cohomology groups on a compact KT-manifold are presented in [5].

We note that (2.19) and (2.20) are also valid in the case $n = 1$.

We get, as a consequence of the proof of Theorem 2.2, the following integrability criterion which is discovered in dimension 4 in [19].

**Proposition 2.3** The Nijenhuis tensors of a QKT manifold depend only on the difference between the Lie forms. In particular, the almost complex structures $J_\alpha$ on a QKT manifold $(M, (J_\alpha) \in Q, g, \nabla)$ are integrable if and only if

\begin{equation}
\theta_\alpha = \theta_\beta = \theta_\gamma.
\end{equation}
Proof. The Nijenhuis tensors are given by (2.5) and (2.19). Q.E.D.

**Corollary 2.4** On a 4n-dimensional QKT manifold the following formulas hold

\[ J_\beta \theta_{\alpha,\gamma} = -J_\gamma \theta_{\alpha,\beta}, \]

\[ (n^2 + n)\theta_\alpha - n\theta_\beta - n^2\theta_\gamma + J_\gamma \theta_{\beta,\alpha} + nJ_\alpha \theta_{\gamma,\beta} = n(n + 1)J_\beta \theta_{\alpha,\gamma} = 0. \]

If \( n = 1 \) then
\[ \theta_\alpha = J_\beta \theta_{\alpha,\gamma} = -J_\gamma \theta_{\alpha,\beta}. \]

**Proof.** The first formula follows directly from the system (2.18). Solving the system (2.18) with respect to \( C_\alpha \) we obtain
\[ (n^3 - 1)J_\beta C_\alpha = (\theta_\alpha - J_\gamma \theta_{\beta,\alpha}) + n(\theta_\beta - J_\alpha \theta_{\gamma,\beta}) + n^2(\theta_\gamma - J_\beta \theta_{\alpha,\gamma}). \]

Then (2.21) is a consequence of (2.22) and (2.20). The last assertion follows from (2.20). Q.E.D.

**Corollary 2.5** On a 4n-dimensional \((n > 1)\) QKT manifold the sp\((1)\)-connection 1-forms are given by
\[ \omega_\beta = \frac{1}{2} J_\beta \left( \theta_\gamma - \theta_\beta + \frac{1}{1-n} \theta_\alpha \right) + \frac{1}{2(1-n)} \theta_{\alpha,\gamma}. \]

**Proof.** The proof follows in a straightforward way from (2.19), (2.20), (2.6) and (2.16). Q.E.D.

Theorem 2.2 and the above formulas lead to the following criterion

**Proposition 2.6** Let \((M, g, (H))\) be a 4n-dimensional \((n > 1)\) QKT manifold. The following conditions are equivalent:

i) \((M, g, (H))\) is a HKT manifold;

ii) \(d_\alpha F_\beta^+ = d_\beta F_\gamma^+ = d_\gamma F_\alpha^+\);

iii) \(\theta_\alpha = J_\beta \theta_{\alpha,\gamma}\).

**Proof.** If \((M, g, (H))\) is a HKT manifold, the connection 1-forms \(\omega_\alpha = 0, \alpha = 1, 2, 3\). Then ii) and iii) follow from (2.16), (2.20), (2.6) and (2.3).

If iii) holds, then (2.20) and (2.19) yield \(C_\alpha = A_\alpha = 0, \alpha = 1, 2, 3\), since \(n > 1\). Consequently, 
\[ 2\omega_\alpha = J_\beta C_\beta - J_\beta A_\beta = 0 \] by (2.16) and (2.7). Thus the equivalence of i) and iii) is proved.

Let ii) holds. Then we compute that \(\theta_\alpha = J_\beta \theta_{\beta,\alpha}\). Since \(n > 1\), the equality (2.22) leads to \(C_\alpha = 0, \alpha = 1, 2, 3\), which forces \(\omega_\alpha = 0, \alpha = 1, 2, 3\) as above. This completes the proof. Q.E.D.

The next theorem shows that QKT manifolds are stable under a conformal transformations.

**Theorem 2.7** Let \((M, g, (J_\alpha), \nabla)\) be a 4n-dimensional QKT manifold. Then every Riemannian metric \(\tilde{g}\) in the conformal class \([g]\) admits a QKT connection. If \(\tilde{g} = fg\) for a positive function \(f\) then the QKT connection \(\widetilde{\nabla}\) corresponding to \(\tilde{g}\) is given by
\[ \tilde{g}(\nabla_X Y, Z) = fg(\nabla_X Y, Z) + \frac{1}{2} (df(X)g(Y, Z) + df(Y)g(X, Z) - df(Z)g(X, Y)) + \frac{1}{2} (J_\alpha df \wedge F_\alpha + J_\beta df \wedge F_\beta + J_\gamma df \wedge F_\gamma)(X, Y, Z). \]

If \(M\) is compact then there exists a unique (up to homotety) metric \(g_G \in [g]\) with co-closed torsion 1-form.
Proof. First we assume $n > 1$. We shall apply Theorem 2.2 to the quaternionic Hermitian manifold $(M, \bar{g} = fg, (J_\alpha) \in Q)$. We denote the objects corresponding to the metric $\bar{g}$ by a line above the symbol e.g. $\bar{F}_\alpha$ denotes the Kähler form of $J_\alpha$ with respect to $\bar{g}$. An easy calculation gives the following sequence of formulas

\[
d\alpha \bar{F}_\alpha^+ = J_\alpha df \wedge F_\alpha + fd\alpha F_\alpha^+; \quad \bar{\theta}_\alpha = \theta_\alpha + (2n - 1)d\ln f; \quad \bar{\theta}_{\alpha,\gamma} = \theta_{\alpha,\gamma} - J_\beta d\ln f.
\]

We substitute (2.25) into (2.9), (2.19) and (2.23) to get

\[
\bar{K}_\alpha = K_\alpha - 2J_\beta d\ln f, \quad \bar{A} = A, \quad \bar{\omega}_\alpha = \omega_\alpha - J_\beta d\ln f.
\]

Using (2.25) and (2.26) we verify that the conditions (2.8) with respect to the metric $\bar{g}$ are fulfilled.

Theorem 2.2 implies that there exists a QKT connection $\bar{\nabla}$ with respect to $(\bar{g}, Q)$. Using the well known relation between the Levi-Civita connections of conformally equivalent metrics, (2.25) and (2.26), we obtain (2.24) from (2.10).

If $n = 1$ we define the new QKT connection with respect to $(\bar{g}, Q)$ by (2.24).

Using (2.24), we find that the torsion tensors $T$ and $\bar{T}$ of $\nabla$ and $\bar{\nabla}$ are related by

\[
\bar{T} = fT + J_\alpha df \wedge F_\alpha + J_\beta df \wedge F_\beta + J_\gamma df \wedge F_\gamma.
\]

Consequently, we obtain from (2.27) for the torsion 1-forms $t$ and $\bar{t}$ that

\[
\bar{t} = t - (2n + 1)d\ln f.
\]

If $M$ is compact, we may apply to (2.28) the theorem of Gauduchon for the existence of a Gauduchon metric on a compact Weyl manifold [15, 16] to obtain the desired metric $g_G$. Q.E.D.

We shall call the unique metric with co-closed torsion 1-form on a compact QKT manifold the Gauduchon metric.

Corollary 2.8 On a compact QKT manifold with closed (non exact) torsion 1-form the Gauduchon metric $g_G$ cannot have positive definite Riemannian Ricci tensor. In particular, if it is an Einstein manifold then it is of non-positive scalar curvature.

Further, if the Gauduchon metric is Ricci flat then the corresponding torsion 1-form $t_G$ is parallel with respect to the Levi-Civita connection of $g_G$.

Proof. The two form $dt$ is invariant under conformal transformations by (2.28). Then the Gauduchon metric has harmonic torsion 1-form i.e $dt = \delta t = 0$. The claim follows from the Weitzenböck formula (see e.g. [8]) $\int_M (|dt|^2 + |\delta t|^2) dV = \int_M (|\nabla^g t|^2 + Ric^g(t^#, t^#)) dV = 0$, where $t^#$ is the dual vector field of $t$, $|.|$ is the usual tensor norm and $dV$ is the volume form. Q.E.D.

Theorem 2.7 allows us to supply a large class of (compact) QKT manifold. Namely, any conformal metric of a QK, HK or HKT manifold will give a QKT manifold. This leads to the notion of locally conformally QK (resp. locally conformally HK, resp. locally conformally HKT) manifolds (briefly l.c.QK (resp. l.c.HK, resp. l.c.HKT) manifolds) in the context of QKT geometry.

The l.c.QK and l.c.HK manifolds have already appeared in the context of Hermitian-Einstein-Weyl structures [36] and of 3-Sasakian structures [12]. These two classes of quaternionic manifolds are studied in detail (mostly in the compact case) in [34, 35].

We recall that a quaternionic Hermitian manifold $(M, g, Q)$ is said to be l.c.QK (resp. l.c.HK, resp. l.c.HKT) manifold if each point $p \in M$ has a neighbourhood $U_p$ such that $g|_{U_p}$ is conformally equivalent to a QK (resp.HK, resp.HKT) metric. There are compact l.c.QK manifold which do not
admit any QK structure \( [34] \). Typical examples of compact l.c. QK manifolds without any QK structure are the quaternionic Hopf spaces \( H = (H^n - \{0\})/\Gamma \), where \( \Gamma \) is an appropriate discrete group acting diagonally on the quaternionic coordinates in \( H^n \) (see \([34]\)).

We recall that on a l.c.QK manifold the 4-form \( \Omega = \sum_3 \alpha = 1 F_\alpha \wedge F_\alpha \) satisfies \( d\Omega = \omega \wedge \Omega \), \( d\omega = 0 \), where \( \omega \) is locally defined by \( \omega = 2d\ln f \). On a l.c.QK manifold viewed as a QKT manifold by Theorem 2.7 the torsion 1-form is equal to \( t = (2n + 1)^{-1} \omega \) by (2.28). The QK manifolds are Einstein provided the dimension is at least 8 [1, 7]. Then, the Gauduchon Theorem \([16]\) applied to l.c.QK manifold in \([34]\) can be stated in our context as follows.

**Corollary 2.9** Let \((M, g)\) be a compact 4n-dimensional \((n > 1)\) QKT manifold which is l.c.QK and assume that no metric in the conformal class \([g]\) of \( g \) is QK. Then the torsion 1-form of the Gauduchon metric \( g_G \) is parallel with respect to the Levi-Civita connection of \( g_G \).

**Theorem 2.7, Theorem 2.2 together with Proposition 2.3 and Proposition 2.6 imply the following**

**Corollary 2.10** Every l.c.QK manifold admits a QKT structure.

Further, if \((M, g, (J_\alpha), \nabla)\) is a 4n-dimensional \( n > 1 \) QKT manifold then:

i) \((M, g, (J_\alpha), \nabla)\) is a l.c.QK manifold if and only if

\[
T = \frac{1}{2n + 1} (t_\alpha \wedge F_\alpha + t_\beta \wedge F_\beta + t_\gamma \wedge F_\gamma), \quad dt = 0;
\]

ii) \((M, g, (J_\alpha), \nabla)\) is a l.c.HKT manifold if and only if the 1-form \( \theta_\alpha - J_\beta \theta_\alpha, \gamma \) is closed i.e.

\[
d(\theta_\alpha - J_\beta \theta_\alpha, \gamma) = 0;
\]

iii) \((M, g, (J_\alpha), \nabla)\) is a l.c.HK manifold if an only if (2.29) holds and

\[
\theta_\alpha - J_\beta \theta_\alpha, \gamma = \frac{2(1-n)}{2n + 1} t.
\]

### 3 Curvature of a QKT space

Let \( R = [\nabla, \nabla] - \nabla [\nabla, \nabla] \) be the curvature tensor of type \((1,3)\) of \( \nabla \). We denote the curvature tensor of type \((0,4)\) \( R(X, Y, Z, V) = g(R(X, Y)Z, V) \) by the same letter. There are three Ricci forms given by

\[
\rho_\alpha(X, Y) = \frac{1}{2} \sum_{i=1}^{4n} R(X, Y, e_i, J_\alpha e_i), \quad \alpha = 1, 2, 3.
\]

**Proposition 3.1** The curvature of a QKT manifold \((M, g, (J_\alpha), \nabla)\) satisfies the following relations

\[
R(X, Y)J_\alpha = \frac{1}{n} (\rho_\gamma(X, Y) J_\beta - \rho_\beta(X, Y) J_\gamma),
\]

\[
\rho_\alpha = d\omega_\alpha + \omega_\beta \wedge \omega_\gamma.
\]

**Proof.** We follow the classical scheme (see e.g. \([3, 25, 8]\)). Using (2.1) we obtain

\[
R(X, Y)J_\alpha = -(d\omega_\beta + \omega_\gamma \wedge \omega_\alpha)(X, Y) J_\gamma + (d\omega_\gamma + \omega_\alpha \wedge \omega_\beta)(X, Y) J_\beta.
\]
Taking the trace in the last equality, we get
\[
\rho_\alpha(X, Y) = \frac{1}{2} \sum_{i=1}^{4n} R(X, Y, J_\alpha e_i, J_\alpha e_i) = \frac{1}{2} \sum_{i=1}^{4n} R(X, Y, J_\beta e_i, J_\gamma e_i) = -\frac{1}{2} \sum_{i=1}^{4n} R(X, Y, e_i, J_\alpha e_i) + 2n (d\omega_\alpha + \omega_\beta \wedge \omega_\gamma)(X, Y) J_\beta.
\]

Q.E.D.

Using Proposition 3.1 we find a simple necessary and sufficient condition a QKT manifold to be a HKT one, i.e. the holonomy group of \(\nabla\) to be a subgroup of \(\text{Sp}(n)\).

**Proposition 3.2** A 4n-dimensional \((n > 1)\) QKT manifold is a HKT manifold if and only if all the three Ricci forms vanish, i.e \(\rho_1 = \rho_2 = \rho_3 = 0\).

**Proof.** If a QKT manifold is a HKT manifold then the holonomy group of \(\nabla\) is contained in \(\text{Sp}(n)\). This implies \(\rho_\alpha = 0\), \(\alpha = 1, 2, 3\).

For the converse, let the three Ricci forms vanish. The equations (3.31) mean that the curvature of the \(\text{Sp}(1)\) connection on \(Q\) vanish. Then there exists a basis \((I_\alpha, \alpha = 1, 2, 3)\) of almost complex structures on \(Q\) and each \(I_\alpha\) is \(\nabla\)-parallel i.e. the corresponding connection 1-forms \(\omega_{I_\alpha} = 0\), \(\alpha = 1, 2, 3\). Then each \(I_\alpha\) is a complex structure, by (2.5) and (2.6). This implies that the QKT manifold is a HKT manifold. Q.E.D.

We denote by \(\text{Ric}, \text{Ric}^g\) the Ricci tensors of the QKT connection and of the Levi-Civita connection, respectively. In fact
\[
\text{Ric}(X, Y) = \sum_{i=1}^{4n} R(e_i, X, Y, e_i).
\]

Our main technical result is the following

**Proposition 3.3** Let \((M, g, (J_\alpha), \nabla)\) be a 4n-dimensional QKT manifold. The following formulas hold

\[
\begin{align*}
(3.32) & \quad n \rho_\alpha(X, J_\alpha Y) + \rho_\beta(X, J_\beta Y) + \rho_\gamma(X, J_\gamma Y) = \\
& \quad -n \text{Ric}(X, Y) + \frac{n}{4} (dT)_\alpha(X, J_\alpha Y) + \frac{n}{2} (\nabla T)_\alpha(X, J_\alpha Y); \\
(3.33) & \quad (n - 1) \rho_\alpha(X, J_\alpha Y) = -\frac{n(n - 1)}{n + 2} \text{Ric}(X, Y) \\
& \quad + \frac{n}{4(n + 2)} \{(n + 1)(dT)_\alpha(X, J_\alpha Y) - (dT)_\beta(X, J_\beta Y) - (dT)_\gamma(X, J_\gamma Y)\} \\
& \quad + \frac{n}{2(n + 2)} \{(n + 1)(\nabla T)_\alpha(X, J_\alpha Y) - (\nabla T)_\beta(X, J_\beta Y) - (\nabla T)_\gamma(X, J_\gamma Y)\}, \\
(3.34) & \quad \text{where} (dT)_\alpha(X, Y) = \sum_{i=1}^{4n} dT(X, Y, e_i, J_\alpha e_i), \quad (\nabla T)_\alpha(X, Y) = \sum_{i=1}^{4n} (\nabla X T)(Y, e_i, J_\alpha e_i).
\end{align*}
\]

**Proof.** Since the torsion is a 3-form, we have
\[
(3.35) \quad (\nabla^g_X T)(Y, Z, U) = (\nabla_X T)(Y, Z, U) + \frac{1}{2} \sigma_{XYZ} \{g(T(X, Y), T(Z, U))\},
\]

where \(\sigma_{XYZ}\) denote the cyclic sum of \(X, Y, Z\).
The exterior derivative $dT$ is given by

$$
(3.36) \quad dT(X,Y,Z,U) = \sigma_{XYZ} \{ (\nabla_X T)(Y,Z,U) + g(T(X),T(Z),T(U)) \}
$$

$$
- (\nabla_U T)(X,Y,Z) + \sigma_{XYZ} \{ g(T(X),T(Y),T(Z)) \}.
$$

The first Bianchi identity for $\nabla$ states

$$
(3.37) \quad \sigma_{XYZ} R(X,Y,Z,U) = \sigma_{XYZ} \{ (\nabla_X T)(Y,Z,U) + g(T(X),T(Z),T(U)) \}.
$$

We denote by $B$ the Bianchi projector i.e. $B(X,Y,Z,U) = \sigma_{XYZ} R(X,Y,Z,U)$.

The curvature $R^g$ of the Levi-Civita connection is connected by $R$ in the following way

$$
(3.38) \quad R^g(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)(Y,Z,U) + \frac{1}{2}(\nabla_Y T)(X,Z,U)
$$

$$
- \frac{1}{2} g(T(X,Y),T(Z,U)) - \frac{1}{4} g(T(Y,Z),T(X,U)) - \frac{1}{4} g(T(Z,X),T(Y,U)).
$$

Define $D$ by $D(X,Y,Z,U) = R(X,Y,Z,U) - R(Z,U,X,Y)$, we obtain from (3.38)

$$
(3.39) \quad D(X,Y,Z,U) = \frac{1}{2}(\nabla_X T)(Y,Z,U) - \frac{1}{2}(\nabla_Y T)(X,Z,U) - \frac{1}{2}(\nabla_Z T)(U,X,Y) + \frac{1}{2}(\nabla_U T)(Z,X,Y),
$$

since $D^g$ of $R^g$ is zero.

Using (3.30) and (3.37) we find the following relation between the Ricci tensor and the Ricci forms

$$
(3.40) \quad \rho_\alpha(X,Y) = -\frac{1}{2} \sum_{i=1}^{4n} \{ R(Y,e_i,X,J_\alpha e_i) + R(e_i,X,Y,J_\alpha e_i) \} + \frac{1}{2} \sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i)
$$

$$
= -\frac{1}{2} Ric(Y,J_\alpha X) + \frac{1}{2} Ric(X,J_\alpha Y) + \frac{1}{2} \sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i)
$$

$$
+ \frac{1}{2n} \{ \rho_\beta(J_\gamma Y) - \rho_\beta(J_\gamma X,Y) + \rho_\gamma(J_\beta X,Y) - \rho_\gamma(X,J_\beta Y) \}.
$$

On the other hand, using (3.30), we calculate

$$
(3.41) \quad \sum_{i=1}^{4n} D(X,e_i,J_\alpha e_i,Y) = \sum_{i=1}^{4n} \{ R(X,e_i,J_\alpha e_i,Y) + R(Y,e_i,J_\alpha e_i X) \}
$$

$$
= -Ric(Y,J_\alpha X) - Ric(X,J_\alpha Y)
$$

$$
+ \frac{1}{n} \{ \rho_\beta(X,J_\gamma Y) + \rho_\beta(Y,J_\gamma X) - \rho_\gamma(Y,J_\beta X) - \rho_\gamma(X,J_\beta Y) \}.
$$

Combining (3.40) and (3.41), we derive

$$
(3.42) \quad n \rho_\alpha(X,J_\alpha Y) + \rho_\beta(X,J_\beta Y) + \rho_\gamma(X,J_\gamma Y) =
$$

$$
- nRic(X,Y) + \frac{n}{2} B_\alpha(X,J_\alpha Y) + \frac{n}{2} D_\alpha(X,J_\alpha Y),
$$

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where the tensors $B_\alpha$ and $D_\alpha$ are defined by $B_\alpha(X,Y) = \sum_{i=1}^{4n} B(X,Y,e_i,J_\alpha e_i)$ and $D_\alpha(X,Y) = \sum_{i=1}^{4n} D(X,e_i,J_\alpha e_i,Y)$. Taking into account (3.33), we get the expression

$$D_\alpha(X,Y) = \frac{1}{2} \sum_{i=1}^{4n} (\nabla_X T)(Y,e_i,J_\alpha e_i) + \frac{1}{2} \sum_{i=1}^{4n} (\nabla_Y T)(X,e_i,J_\alpha e_i) \quad \alpha = 1, 2, 3. \quad (3.43)$$

To calculate $B_\alpha + D_\alpha$ we use (3.36) twice and (3.43). After some calculations, we derive

$$B_\alpha(X,Y) + D_\alpha(X,Y) = \frac{1}{2} \sum_{i=1}^{4n} dT(X,Y,e_i,J_\alpha e_i) + \sum_{i=1}^{4n} (\nabla_X T)(Y,e_i,J_\alpha e_i), \quad \alpha = 1, 2, 3. \quad (3.44)$$

We substitute (3.44) into (3.42). Solving the obtained system, we obtain

$$\frac{n}{2} \{(dT)_\alpha(X,J_\beta Y) - (dT)_\beta(X,J_\alpha Y)\} + \frac{n}{2} \{(\nabla T)_\alpha(X,J_\alpha Y) - (\nabla T)_\beta(X,J_\beta Y)\}. \quad (3.45)$$

Finally, (3.42) and (3.43) imply (3.33). Q.E.D.

Remark 3. The Ricci tensor of a QKT connection is not symmetric in general. From (3.37), (3.35) and the fact that $T$ is a 3-form we get the formula $Ric(X,Y) - Ric(Y,X) = \sum_{i=1}^{4n} (\nabla_{e_i} T)(e_i,X,Y) = -\delta T(X,Y)$. Hence, the Ricci tensor of a linear connection with totally skew-symmetric torsion is symmetric if and only if the torsion 3-form is co-closed.

4 QKT manifolds with parallel torsion and homogeneous QKT structures

Let $(G/K,g)$ be a reductive (locally) homogeneous Riemannian manifold. The canonical connection $\nabla$ is characterised by the properties $\nabla g = \nabla T = \nabla R = 0$ [20, p.193]. A homogeneous quaternionic Hermitian manifold (resp. homogeneous hyper Hermitian) manifold $(G/K,g,Q)$ is a homogeneous Riemannian manifold with an invariant quaternionic Hermitian subbundle $Q$ (resp. three invariant anti commuting complex structures ). This means that the bundle $Q$ (resp. each of the three complex structures) is parallel with respect to the canonical connection $\nabla$. The torsion of $\nabla$ is totally skew-symmetric if and only if the homogeneous Riemannian manifold is naturally reductive [20] (see also [14, 33]. Homogeneous QKT (resp. HKT) manifolds are homogeneous quaternionic Hermitian (resp. homogeneous hyper Hermitian) manifold which are naturally reductive. Examples of homogeneous HKT and QKT manifolds are presented in [33]. The homogeneous QKT manifolds in [33] are constructed from homogeneous HKT manifolds.

In this section we generalise the result of [33] which states that there are no homogeneous QKT manifold with torsion 4-form $dT$ of type $(2,2)$ in dimensions greater than four. First, we prove the following technical result

**Proposition 4.1** Let $(M,g,(J_\alpha),\nabla)$ be a $4n$-dimensional $(n > 1)$ QKT manifold with 4-form $dT$ of type $(2,2)$ with respect to each $J_\alpha, \alpha = 1, 2, 3$. Suppose that the torsion is parallel with respect to the QKT-connection. Then the Ricci forms $\rho_\alpha$ are given by

$$\rho_\alpha(X,J_\alpha Z) = \lambda g(X,Y), \quad \alpha = 1, 2, 3, \quad (4.46)$$

where $\lambda$ is a smooth function on $M$. 

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Proof. Let the torsion be parallel i.e. $\nabla T = 0$. Remark 3 shows that the Ricci tensor is symmetric. The equalities (3.36) and (3.37) imply

$$B(X,Y,Z,U) = \sigma_{XYZ} \{g(T(X,Y),T(Z,U)) = \frac{1}{2}dT(X,Y,Z,U).$$

We get $D = 0$ from (3.39).

Suppose now that the 4-form $dT$ is of type $(2,2)$ with respect to each $J_\alpha$, $\alpha = 1,2,3$. Then it satisfies the equalities

$$dT(X,Y,Z,U) = dT(J_\alpha X,J_\alpha Y,Z,U) + dT(J_\alpha X,Y,J_\alpha Z,U) + dT(X,J_\alpha Y,J_\alpha Z,U).$$

The similar arguments as we used in the proof of Proposition 2.1 but applying (4.48) instead of (2.4), yield

**Lemma 4.2** On a QKT manifold with 4-form $dT$ of type $(2,2)$ with respect to each $J_\alpha$, $\alpha = 1,2,3$, the following equalities hold:

$$d(T)_{1}(X,J_1Y) = (dT)_2(X,J_2Y) = (dT)_3(X,J_3Y),$$

$$d(T)_\alpha(X,J_\alpha Y) = -(dT)_\alpha(J_\alpha X,Y), \quad \alpha = 1,2,3.$$  

We substitute (4.49), (4.47) and $D = 0$ into (3.45) and (3.33) to get

$$\rho_1(X,J_1Y) = \rho_2(X,J_2Y) = \rho_3(X,J_3Y),$$

$$\rho_\alpha(X,J_\alpha Y) = -\frac{n}{n+2}\text{Ric}(X,Y) + \frac{n}{4(n+2)}(dT)_\alpha(X,J_\alpha Y), \quad \alpha = 1,2,3.$$  

The equality (4.50) shows that the 2-form $dT_\alpha$ is a $(1,1)$-form with respect to $J_\alpha$. Hence, the $dT_\alpha$ is $(1,1)$-form with respect to each $J_\alpha$, $\alpha = 1,2,3$, because of (4.49). Since the Ricci tensor $\text{Ric}$ is symmetric, (4.52) shows that the Ricci tensor $\text{Ric}$ is of hybrid type with respect to each $J_\alpha$ i.e. $\text{Ric}(J_\alpha X,J_\alpha Y) = \text{Ric}(X,Y), \alpha = 1,2,3$ and the Ricci forms $\rho_\alpha, \alpha = 1,2,3$ are $(1,1)$-forms with respect to all $J_\alpha, \alpha = 1,2,3$. Taking into account (3.39), we obtain

$$R(X,J_\alpha X,Z,J_\alpha Z) + R(X,J_\alpha X,J_\beta Z,J_\gamma Z) + R(J_\beta X,J_\gamma X,Z,J_\alpha Z) + R(J_\beta X,J_\gamma X,J_\beta Z,J_\gamma Z) = \frac{1}{n}(\rho_\alpha(X,J_\alpha X) + \rho_\alpha(J_\beta X,J_\gamma X))g(Z,Z) = \frac{2}{n}\rho_\alpha(X,J_\alpha X)g(Z,Z),$$

where the last equality of (4.53) is a consequence of the following identity

$$\rho_\alpha(J_\beta X,J_\gamma X) = -\rho_\beta(J_\beta X,X) = \rho_\alpha(X,J_\alpha X).$$

The left side of (4.53) is symmetric with respect to the vectors $X,Z$ because $D = 0$. Hence, $\rho_\alpha(X,J_\alpha X)g(Z,Z) = \rho_\alpha(Z,J_\alpha Z)g(X,X), \alpha = 1,2,3$. The last equality together with (4.51) implies (4.46).

**Theorem 4.3** Let $(M,g,(J_\alpha))$ be a $4n$-dimensional ($n > 1$) QKT manifold with 4-form $dT$ of type $(2,2)$ with respect to each $J_\alpha, \alpha = 1,2,3$. Suppose that the torsion is parallel with respect to the QKT-connection. Then $(M,g,(J_\alpha))$ is either a HKT manifold with parallel torsion or a QK manifold.

Q.E.D.
Proof. We apply Proposition 4.1. If the function \( \lambda = 0 \) then \( \rho_\alpha = 0, \alpha = 1, 2, 3 \), by (4.46) and Proposition 3.2 implies that the QKT manifold is actually a HKT manifold.

Let \( \lambda \neq 0 \). The condition (4.46) determines the torsion completely. We proceed involving (3.31) into the computations as in [24]. We calculate using (2.1) and (4.46) that

\[
(\nabla_Z \rho_\alpha)(X,Y) = \lambda \left\{ \omega_\beta(Z)F_\gamma(X,Y) - \omega_\gamma(Z)F_\beta(X,Y) \right\} - d\lambda(Z)F_\alpha(X,Y).
\]

(4.54)

Applying the operator \( d \) to (3.30), we get taking into account (4.46) that

\[
d\rho_\alpha = \lambda(F_\beta \wedge \omega_\gamma - \omega_\beta \wedge F_\gamma)
\]

(4.55)

On the other hand, we have

\[
d\rho_\alpha = \sigma_{XYZ} \{(\nabla_Z \rho_\alpha)(X,Y) + \lambda(T(X,Y,J_\alpha Z))\}, \quad \alpha = 1, 2, 3.
\]

(4.56)

Comparing the left-hand sides of (4.55) and (4.56) and using (4.54), we derive

\[
\lambda \sigma_{XYZ} \{(T(X,Y),J_\alpha Z)\} = d\lambda \wedge F_\alpha(X,Y,Z), \quad \alpha = 1, 2, 3.
\]

The last equality implies \( \lambda T = J_\alpha d\lambda \wedge F_\alpha, \quad \alpha = 1, 2, 3 \). If \( \lambda \) is a non zero constant then \( T = 0 \) and we recover the result of [24]. If \( \lambda \) is not a constant then there exists a point \( p \in M \) and a neighbourhood \( V_p \) of \( p \) such that \( \lambda_{|V_p} \neq 0 \). Then

\[
T = J_\alpha d\ln \lambda \wedge F_\alpha, \quad \alpha = 1, 2, 3.
\]

(4.57)

We take the trace in (4.57) to obtain

\[
4(n - 1)J_\alpha d\ln \lambda = 0, \quad \alpha = 1, 2, 3.
\]

(4.58)

The equation (4.58) forces \( d\lambda = 0 \) since \( n > 1 \) and consequently \( T = 0 \) by (4.57). Hence, the QKT space is a QK manifold which completes the proof.

Q.E.D.

Theorem 4.4 shows that there are no homogeneous (proper) QKT manifolds with torsion 4-form of type (2,2) in dimensions greater than four which is proved in [33] by different methods using the Lie algebra arguments.

5 Four dimensional QKT manifolds

In dimension 4 the situation is completely different from that described in Theorem 2.2 and Theorem 4.3 in higher dimensions. For a given quaternionic structure on a 4-dimensional manifold \((M, g(H))\) (or equivalently, given an orientation and a conformal class of Riemannian metrics [19]) there are many QKT structures [23]. More precisely, all QKT structures associated with \((g, (H))\) depend on a 1-form \( \psi \) due to the general identity

\[
* \psi = -J\psi \wedge F,
\]

(5.59)
where \( \ast \) is the Hodge \( \ast \)-operator, \( J \) is an \( g \)-orthogonal almost complex structure with Kähler form \( F \) (see [19]). Indeed, for any given 1-form \( \psi \) we may define a QKT-connection \( \nabla \) as follows:
\[
\nabla = \nabla^g + \frac{1}{2} \ast \psi.
\]
Conversely, any 3-form \( T \) can be represented by \( T = -\ast \ast T \) and the connection given above is a quaternionic connection with torsion \( T = \ast \psi \). Hence, a QKT structure on a 4-dimensional oriented manifold is a pair \((g, t)\) of a Riemannian metric \( g \) and an 1-form \( t \). The choice of \( g \) generates three almost complex structures \((J_\alpha), \alpha = 1, 2, 3\), satisfying the quaternionic identities [19]. The torsion 3-form \( T \) is given by
\[
T = \ast t = t_\alpha \wedge F_\alpha = t_\beta \wedge F_\beta = t_\gamma \wedge F_\gamma.
\]
As consequence of (5.54), we obtain \( \ast dT = \ast d \ast t = -\delta t \). The last identity means that the torsion 3-form \( T \) is closed if and only if the 1-form \( t \) is co-closed. Thus, in dimension 4 there are many strong QKT structures.

In higher dimensions the conformal change of the metric induces a unique QKT structure by Theorem 2.7. We may define a QKT connection corresponding to a conformally equivalent metric \( \bar{g} = fg \) in dimension 4 by (2.24) and call this conformal QKT transformation. In the compact case, taking the Gauduchon metric of Theorem 2.7, we obtain

**Proposition 5.1** Let \((M, g, (H), \nabla)\) be a compact 4-dimensional QKT manifold. In the conformal class \([g]\) there exists a unique (up to homotety) strong QKT structure conformally equivalent to the given one.

Further, we consider QKT structures with parallel torsion. We have

**Theorem 5.2** A 4-dimensional QKT manifold \( M \) with parallel torsion 3-form is a strong QKT manifold, the torsion 1-form is parallel with respect to the Levi-Civita connection and \( M \) is locally isometric to the product \( N^3 \times \mathbb{R} \), where \( N^3 \) is a three dimensional Riemannian manifold admitting a Riemannian connection \( \nabla \) with totally skew-symmetric torsion, parallel with respect to \( \nabla \).

**Proof.** The proof is based on the following

**Lemma 5.3** A 4-dimensional QKT manifold has parallel torsion 3-form if and only if it has parallel torsion 1-form with respect to the Levi-Civita connection.

**Proof of Lemma 5.3.** We calculate using (5.60) and (2.1) that
\[
(\nabla_Z T)(X, Y, U) = \begin{cases} t_\alpha(U) & (\omega_\beta(Z)F_\gamma(Y, X) - \omega_\gamma(Z)F_\beta(Y, X)) \\
- t_\alpha(X) & (\omega_\beta(Z)F_\gamma(Y, U) - \omega_\gamma(Z)F_\beta(Y, U)) \\
+ t_\alpha(Y) & (\omega_\beta(Z)F_\gamma(X, U) - \omega_\gamma(Z)F_\beta(X, U)) \\
+ F_\alpha(Y, U)(\nabla_Z t_\alpha)X + F_\alpha(X, Y)(\nabla_Z t_\alpha)U - F_\alpha(X, U)(\nabla_Z t_\alpha)Y. 
\end{cases}
\]

Taking the trace in (5.61), we obtain
\[
(5.62) \quad \sum_{i=1}^4 (\nabla_Z T)(X, e_i, J_\alpha e_i) = -2(\nabla_Z t_\alpha)X - 2(\omega_\beta(Z)t_\gamma(X) - \omega_\gamma(Z)t_\beta(X)).
\]

Using (2.1), we get
\[
(5.63) \quad (\nabla_Z t_\alpha)X = (\nabla_Z t)J_\alpha X - (\omega_\beta(Z)t_\gamma(X) - \omega_\gamma(Z)t_\beta(X)).
\]
The equation (5.63) and (5.62) yield

\[ \sum_{i=1}^{4} (\nabla_Z T)(J_\alpha X, e_i, J_\alpha e_i) = 2(\nabla_{Zt} t)(X, \alpha = 1, 2, 3). \]  

Then \( \nabla t = 0 \) since the torsion is parallel. But \( \nabla g t = \nabla t \) by (2.11) and (5.60). Hence, \( \nabla g t = 0 \).

For the converse, we insert (5.63) into (5.61) to get

\[ (\nabla_Z T)(X,Y,U) = F_\alpha(Y,U)(\nabla g t)(J_\alpha X) + F_\alpha(X,Y)(\nabla g t)(J_\alpha Y) + F_\alpha(U,X)(\nabla g t)(J_\alpha Y). \]

since the dimension is equal to four. If \( \nabla g t = 0 \) then \( \nabla t = 0 \) and (5.63) leads to \( \nabla T = 0 \) which proves the lemma.

Lemma 5.3 shows that \((M,g)\) is locally isometric to the Riemannian product \( \mathbb{R} \times \mathbb{N}^3 \) of a real line and a 3-dimensional manifold \( \mathbb{N}^3 \) (see e.g. [26]). Using (5.60) we see that \( T(t^\# , X^ \perp, Y^ \perp) = 0 \) for every vector fields \( X^ \perp, Y^ \perp \) orthonormal to the vector field \( t^\# \) dual to the torsion 1-form \( t \). Hence, the torsion \( T \) and therefore the connection \( \nabla \) descend to \( \mathbb{N}^3 \).

In particular, \( \delta t = 0 \) and therefore the QKT structure is strong.

As a consequence of Theorem 5.2, we recover the following two results proved in [27] in the setting of naturally reductive homogeneous 4-manifolds

**Theorem 5.4** A (locally) homogeneous 4-dimensional QKT manifold is locally isometric to the Riemannian product \( \mathbb{R} \times \mathbb{N}^3 \) of a real line and a naturally reductive homogeneous 3-manifold \( \mathbb{N}^3 \).

**Theorem 5.5** Let \((M,g)\) be a 4-dimensional compact homogeneous QKT manifold. Then the universal covering space \( \tilde{M} \) of \( M \) is isometric to the Riemannian product \( \mathbb{R} \times \mathbb{N}^3 \) of a real line and the three dimensional space \( \mathbb{N}^3 \) is one of the following

i) \( \mathbb{R}^3, S^3, H^3 \);

ii) isometric to one of the following Lie groups with a suitable left invariant metric:

1. \( SU(2) \);

2. \( SL(2, \mathbb{R}) \), the universal covering of \( SL(2, \mathbb{R}) \);

3. the Heisenberg group.

Theorem 5.5 is based on the classification of 3-dimensional simply connected naturally reductive homogeneous spaces given in [44].

### 5.1 Einstein-like QKT 4-manifolds

It is well known [7, 1] that a 4n-dimensional \((n > 1)\) QK manifold is Einstein and the Ricci forms satisfy \( \rho_\alpha(X, J_\beta Y) = \rho_\beta(X, J_\gamma Y) = \rho_\gamma(X, J_\alpha Y) = \lambda g(X,Y) \), where \( \lambda \) is a constant. However, the assumptions that these properties hold on a QKT manifold \((n > 1)\) force the torsion to be zero [24] and the QKT manifold is a QK manifold. Actually, we have already generalised this result proving that if \( \lambda \) is not a constant the torsion has to be zero (see the proof of Theorem 4.3).

If the dimension is equal to 4 the situation is different. In this section we show that there exists a 4-dimensional (proper) QKT manifold satisfying similar curvature properties as those mentioned above.

We denote by \( K \) the following \((0,2)\) tensor

\[ K(X,Y) := \rho_\alpha(X, J_\alpha Y) + \rho_\beta(X, J_\beta Y) + \rho_\gamma(X, J_\gamma Y). \]

The tensor \( K \) is independent of the chosen local almost complex structures \((J_\alpha)\) because of the following
Proposition 5.6 Let \((M, g, (J_\alpha), \nabla)\) be a 4-dimensional QKT manifold. Then:

\[
K = -\text{Ric} + \nabla^g t - \frac{\delta t}{2} g;
\]

\[
\text{Skew}(\text{Ric}) = -\frac{1}{4} \langle dt, F_\alpha \rangle F_\alpha + \frac{1}{2}(dT)_{\alpha}, \quad \alpha = 1, 2, 3;
\]

\[
\text{Ric}^g = \text{Sym}(\text{Ric}) + \frac{1}{2}(\langle t^2 g - t \otimes t \rangle),
\]

where \(\langle, \rangle\) is the scalar product of tensors induced by \(g\), \(\text{Skew}\) (resp. \(\text{Sym}\)) denotes the skew-symmetric (resp. symmetric) part of a tensor.

In particular, the Ricci tensor is symmetric if and only if the torsion 1-form is closed.

Proof. We use (3.42). From (5.64) and (3.43), we obtain

\[
D_\alpha(X, J_\alpha Y) = (\nabla X t) Y - (\nabla J_\alpha Y)' t J_\alpha X, \quad \alpha = 1, 2, 3.
\]

To compute \(B_\alpha\) we need the following general identity

**Lemma 5.7** On a 4-dimensional QKT manifold we have \(\nabla_{XJ}^\sigma g(T(X, Y), T(Z, U)) = 0\).

Proof of Lemma 5.7. Since \(\nabla_{XJ}^\sigma g(T(X, Y), T(Z, U))\) is a 4-form it is sufficient to check the equality for a basis of type \(\{X, J_\alpha X, J_\beta X, J_\alpha J_\beta X\}\). The last claim is obvious because of (5.60).

For each \(\alpha \in \{1, 2, 3\}\), Lemma 5.6 and (5.64) yield

\[
(5.70) B_\alpha(X, J_\alpha Y) = \sum_{i=1}^4 XJ_\alpha Y e_i (\nabla X t) (J_\alpha Y, e_i, J_\alpha e_i) = (\nabla X t) Y + (\nabla J_\alpha Y)' t J_\alpha X - \delta t g(X, Y).
\]

Substituting (3.64), (5.71) into (3.42) and putting \(n = 1\), we derive (5.66) since \(\nabla^g t = \nabla t\). Taking the trace in (5.65), we get \(\sum_{i=1}^4 (\nabla e_i) (e_i, X, Y) = \frac{1}{2} \sum_{i=1}^4 dt(e_i, J_\alpha e_i) F_\alpha (X, Y) + dt(J_\alpha X, J_\alpha Y), \alpha = 1, 2, 3\). Then (5.67) follows from the last equality and Remark 3. The equation (5.68) is a direct consequence of (3.38) and (5.66).

Q.E.D.

A 4n-dimensional QKT manifold \((M, g, (J_\alpha), \nabla)\) is said to be an Einstein QKT manifold if the symmetric part \(\text{Sym}(\text{Ric})\) of the Ricci tensor of \(\nabla\) is a scalar multiple of the metric \(g\) i.e. \(\text{Sym}(\text{Ric}) = \frac{\text{Scal}}{4n} g\), where \(\text{Scal} = \text{tr}_g \text{Ric}\) is the scalar curvature of \(\nabla\).

We note that the scalar curvature \(\text{Scal}\) of an Einstein QKT manifold may not be a constant.

We shall say that a 4-dimensional QKT manifold is \(\text{sp}(1)\)-Einstein if the symmetric part \(\text{Sym}(K)\) of the tensor \(K\) is a scalar multiple of the metric \(g\) since the tensor \(K\) is determined by the \(\text{sp}(1)\)-part of the curvature. On a \(\text{sp}(1)\)-Einstein QKT manifold \(\text{Sym}(K) = \frac{\text{Scal}^K}{4} g\), where \(\text{Scal}^K = \text{tr}_g K\).

For a given QKT manifold with torsion 1-form \(t\) we consider the corresponding Weyl structure \(\nabla^W\), i.e. the unique torsion-free linear connection determined by the condition

\[
(5.71) \nabla^W g = -t \otimes g.
\]

Conversely, in dimension 4, to a given Weyl structure \(\nabla^W g = \psi \otimes g\) we associate the QKT connection with torsion \(T = *(-\psi)\). Note that a given Weyl structure on a conformal manifold \((M, [g])\) does not depend on the particularly chosen metric \(g \in [g]\) but depends on the conformal class \([g]\). A Weyl structure is said to be Einstein-Weyl if the symmetric part \(\text{Sym}(\text{Ric}^W)\) of its Ricci tensor is a scalar multiple of the metric \(g\). Weyl structures and especially Einstein-Weyl structures have been much studied. For a nice overview of Einstein-Weyl geometry see [13]. The next theorem shows the link between Einstein-Weyl geometry and \(\text{sp}(1)\)-Einstein QKT manifolds in dimension 4.
Theorem 5.8 Let \((M, g, (J_\alpha), \nabla)\) be a 4-dimensional QKT manifold with torsion 1-form \(t\). The following conditions are equivalent:

i) \((M, g, (J_\alpha), \nabla)\) is a sp(1)-Einstein QKT manifold.

ii) The corresponding Weyl structure is an Einstein-Weyl structure.

Proof. The Weyl connection \(\nabla^W\) determined by (5.71) is given explicitly by

\[
\nabla^W_X Y = \nabla^g_X Y + \frac{1}{2} t(X)Y + \frac{1}{2} t(Y)X - \frac{1}{2} g(X,Y)t^\#.
\]

The symmetric part of its Ricci tensor is equal to

\[
\text{Sym}(\text{Ric}^W) = \text{Ric}^g - \text{Sym}(\nabla^g t) - \frac{1}{2} \left( |t|^2 g - t \otimes t \right) + \frac{\delta t}{2} g.
\]

(5.72)

Keeping in mind that \(\nabla^g t = \nabla t\), we get from (5.66), (5.68) and (5.72) that \(\text{Sym}(\text{Ric}^W) = -\text{Sym}(K)\). The theorem follows from the last equality. Q.E.D.

It is well known [6, 42] that on a 4-dimensional conformal manifold there exists a hypercomplex structure iff the conformal structure has anti-self-dual Weyl tensor (see also [19]). Every 4-dimensional hypercomplex manifold \((M, g, (H_\beta))\), i.e. (an oriented anti-self-dual 4-manifold) carries a unique HKT structure in view of the results in [19, 17]. Indeed, let \(\theta = \theta_\alpha = \theta_\beta = \theta_\gamma\) be the common Lee form. The unique HKT structure is defined by \(\nabla = \nabla^g - \frac{1}{2} \theta\) [19] (the uniqueness is a consequence of a general result in [17], see also [20]). The HKT structure on a 4-dimensional hypercomplex manifold is sp(1)-Einstein since the tensor \(K\) vanishes. The corresponding Weyl structure to the given HKT structure on a 4-dimensional hyperhermitian manifold is the Obata connection [19], i.e. the unique torsion-free linear connection which preserves each of the three hypercomplex structures. As a consequence of Theorem 5.8, we recover the result in [38] which states that the Obata connection of a hypercomplex 4-manifold is Einstein-Weyl and the symmetric part of its Ricci tensor is zero.

Theorem 5.8 and (5.66) show that every Einstein-Weyl structure determined by (5.71) on a 4-dimensional conformal manifold whose vector field dual to the 1-form \(t\) is Killing, induces an Einstein and sp(1)-Einstein QKT structure.

Corollary 5.9 Let \((M, [g], \nabla^W)\) be a compact 4-dimensional Einstein-Weyl manifold. Then the corresponding QKT structure to the Gauduchon metric of \(\nabla^W\) is Einstein and sp(1)-Einstein.

Proof. On a compact Einstein-Weyl manifold the vector field dual to the Lee form of the Gauduchon metric is Killing by the result of Tod [43]. Then the claim follows from Theorem 5.8 and (5.66). Q.E.D.

The Ricci tensor of a 4-dimensional QKT manifold is symmetric iff the torsion 1-form is closed by Proposition 5.6. Applying Theorem 3 in [16] and using Theorem 5.8, we obtain

Corollary 5.10 Let \((M, g, (J_\alpha), \nabla)\) be a 4-dimensional compact sp(1)-Einstein QKT manifold with symmetric Ricci tensor. Suppose that the torsion 1-form is not exact. Then the torsion 1-form corresponding to the Gauduchon metric \(g_G\) of \((M, g, (J_\alpha), \nabla)\) is parallel with respect to the Levi-Civita connection of \(g_G\) and the universal cover of \((M, g_G)\) is isometric to \(\mathbb{R} \times S^3\). In particular, the quaternionic bundle \((J_\alpha)\) admits hypercomplex structure.

A lot is known about Einstein-Weyl manifolds (see a nice survey [13]). There are many (compact) Einstein-Weyl 4-manifolds (e.g. \(S^2 \otimes S^2\)). Among them there are (anti)-self-dual as well as non (anti)-self-dual. We mention here the Einstein-Weyl examples of Bianchi IX type metric.
All these Einstein-Weyl 4-manifolds admit sp(1)-Einstein QKT structures by Theorem 5.8.

It is also known that there are obstructions to the existence of Einstein-Weyl structures on compact 4-manifold \([37]\). If the manifold \(M\) is finitely covered by \(T^2 \otimes S^2\) which cannot be Einstein-Weyl then \(M\) does not admit Einstein-Weyl structure and therefore there are no sp(1)-Einstein structures on \(M\).

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