Quantum quincunx for walk on circles in phase space with indirect coin flip

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New Journal of Physics 10 (2008) 053025 (18pp)
Received 16 February 2008
Published 20 May 2008
Online at http://www.njp.org/
doi:10.1088/1367-2630/10/5/053025

Abstract. The quincunx, or Galton board, has a long history as a tool for demonstrating and investigating random walk processes, but a quantum quincunx (QQ) for demonstrating a coined quantum walk (QW) is yet to be realized experimentally. We propose a variant of the QQ in cavity quantum electrodynamics, designed to eliminate the onerous requirement of directly flipping the coin. Instead, we propose driving the cavity in such a way that cavity field displacements are minimized and the coin is effectively flipped via this indirect process. An effect of this indirect flipping is that the walker’s location is no longer confined to a single circle in the planar phase space, but we show that the phase distribution nonetheless shows quadratic enhancement of phase diffusion for the quantum versus classical walk despite this small complication. Thus our scheme leads to coined QW behaviour in cavity quantum electrodynamics without the need to flip the coin directly.
1. Introduction

The quantum walk (QW) has proven to be one of the most important developments in theoretical quantum information science, both as an intriguing generalization of the ubiquitous random walk (RW) in physics [1, 2] and for exponential algorithmic speed-ups [3–6]. In order to convey understanding of the RW and to study its properties, the quincunx, or ‘bean machine’, or ‘Galton board’ [7], was developed to exhibit the features of RWs in experiments. The quincunx has a long and distinguished history as a tool for studying and experimentally presenting RW behaviour.

More recently, the quincunx has been extended to the case that the walker is a wave rather than a particle and thereby manifests a ‘wave walk’, which has been demonstrated experimentally using laser beams [8]. Then, driven by the importance of the QW as a quantum information analogue of the RW, the quantum quincunx (QQ) has been proposed, for example, in the ion trap [9] and in cavity quantum electrodynamics [10]. However, realization of the QQ, with the desired features of a quantum coin, a single walker and controllable decoherence, has, so far, been elusive.

Elusiveness of the QQ is intrinsically tied with its value as a tool to study QWs. The QQ is important as a pedagogical tool to demonstrate QW behaviour, and both the QQ and QW are sensitive to open system effects, such as decoherence. Under typical conditions, the RW would be expected to emerge from the QW due to decoherence [11] so, similarly, the quincunx could emerge from the QQ under appropriate conditions. In theoretical studies of QWs, open system effects are typically ignored. Neglecting open systems is fine if the QW is to be implemented in a quantum computer with full fault tolerant error correction, but the QW is important beyond
this context alone. Consequently, open systems are important to QWs, and the QQ will help to explore and to understand open system effects in QWs.

To create a QQ, the following steps are followed: a particular QW is chosen, in our case a discrete bivalued coined QW with one walker who has one degree of freedom. Furthermore, a signature for QW behaviour is identified, such as enhanced diffusion or uniformity of the distribution for the walker’s degree of freedom. Then a physical system is chosen whose Hamiltonian dynamics match the evolution of the QW. Finally, open system dynamics are incorporated into the analysis in order to account for non-unitary evolution as well as to incorporate realistic measurement into the model.

In our case, we choose the QW over a circle in phase space \([0, \pi] \times [0, \pi]\), which arises naturally for a simple harmonic oscillator. Points in phase space correspond to the oscillator position–momentum pair \((x, p)\), which we henceforth refer to as the phase space ‘location’, and energy-conserving evolution of the oscillator guarantees that \(E = (x^2 + p^2)/2\) (for the oscillator of unit mass and unit frequency) is a conserved quantity, thereby constraining the phase space trajectory to a circle in phase space centered at the origin \((0, 0)\).

The discrete walk on the circle corresponds to phase jumps \(\Delta \theta = \theta_2 - \theta_1\) for

\[
\theta_i = \tan^{-1} \frac{p_i}{x_i},
\]

which is well defined provided that \(x \neq 0 \neq p\). The discrete RW on a circle, corresponding to phase jumps \(\pm \Delta \theta\), with \(\Delta \theta\) of fixed size and the sign \(\pm\) chosen randomly, has been used to provide a clear explanation of the phase diffusion of the laser field [12]. More recently, the RW on a circle in phase space has been generalized to the QW on a circle in phase space: in the quantum case the walker’s location as a point in phase space is replaced by a localized wavefunction centered at a location \((x, p)\), and the random flip of sign \(\pm\) is replaced by a quantum coin given by a qubit, which is flipped by a Hadamard operation and then entangled with the oscillator by free evolution. An example of a localized wavefunction is the coherent state

\[
|\alpha\rangle = D(\alpha)|0\rangle, \quad (1.2)
\]

with \(|0\rangle\) the ground state of the simple harmonic oscillator and

\[
D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \quad (1.3)
\]

the unitary displacement operator [13], and \(\alpha = (x + ip)/\sqrt{2}\) for localization at \((x, p)\). The full QW on a circle in phase space is described in detail in section 2.

In these QW-on-the-circle schemes, the coin qubit is directly controlled; however, in the context of cavity quantum electrodynamics, the coin qubit is an atom within a high-finesse resonator. The high-finesse nature of the cavity mitigates against direct control of the atom, but one viable option is to drive the coin qubit indirectly by pulsing the cavity, which in turn drives the atom. Here we show that this indirect coin flip indeed suffices to create a QQ, but the QW is no longer confined to a circle in phase space but rather undergoes a QW that hops between different circles in phase space. (Alternatively, we could consider directly driving the coin and reading the resonator state indirectly from the coin state [14], but this approach is quite distinct from the scheme proposed here.) In section 3, we explain this revised QW involving simultaneous driving of both the oscillator and the coin qubit.

In section 4, we consider the Jaynes–Cummings (JC) model [15] as the underpinning of the results in section 3. Whereas section 3 presents a generalized Hadamard transformation of the coin that involves driving both the oscillator and the coin, in this section we use the JC
model, which is one of the most important models in quantum optics to describe cavity quantum electrodynamic systems, and was originally used to explain the maser (and hence the laser) \[15\]. In the dispersive limit, and with judicious timing to achieve the right phase steps, we show in this section that a QW on circles in phase space can be well approximated by JC dynamics. The results are summarized in section 5.

2. Background

The RW on a circle in phase space, used to describe laser diffusion \[12\], comprises two coupled systems: the walker, which is physically a simple harmonic oscillator, and the unbiased two-sided coin, which is mathematically an unbiased random bit. The joint system of the coin + walker has a state space $L^1(\mathbb{R}) \times \{0, 1\}$. That is, the walker’s state corresponds to distributions in $L^1(\mathbb{R})$, and the coin can have either value $\zeta \in \{0, 1\}$. Evolution consists of alternating coin flips, which generates 0 or 1 randomly with equal probability, and then the walker’s distribution in phase space is rotated by an angle $\pm \Delta \theta$ with the sign $\pm$ given by $(-1)^\zeta$.

In quantizing the QW, the walker’s distribution is replaced by a state $\rho \in B(H_w)$ for $B(H_w)$ the Banach space of bounded operators on $H_w \cong L^2(\mathbb{R})$. The coin is replaced by a qubit with Hilbert space $H_c \cong P\mathbb{C}^2$, namely the projective space of two-component complex vectors. The joint coin + walker space $H_c \otimes H_w$ is spanned by a basis set comprising tensor products of Fock states $|n\rangle = \hat{a}^\dagger n |0\rangle / \sqrt{n!}$, $\hat{a}^\dagger = 2^{-1/2}(\hat{x} - i\hat{p})$ (2.1) for $|0\rangle$ the oscillator ground state in equation (2.1), and for $|0\rangle$ and $|1\rangle$ the two coin basis states. The Fock states are also known as number states, and Fock state $|n\rangle$ is an eigenstate of the number operator $\hat{n} \equiv \hat{a}^\dagger \hat{a}$ with eigenvalue $n \in \mathbb{N}$.

The QW is effected as an alternating sequence of two operations, namely the Hadamard transformation on the coin

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|, \quad |\pm\rangle = (|0\rangle \pm |1\rangle) / \sqrt{2}$$ (2.2)

and the free evolution

$$F(\Delta \theta) = \exp(i\hat{n}\hat{\sigma}_z \Delta \theta)$$ (2.3)

between coin flips. The free evolution effects a conditional rotation of the walker’s state by an angle $\pm \Delta \theta$ which is chosen given an initial walker state $|\alpha\rangle$ [10]:

$$\frac{1}{\sqrt{n}} < \Delta \theta < \frac{2\pi}{n + \sqrt{n}}.$$ (2.4)

In fact, the evolution of the walker can be entangled with the coin state by this evolution, and this entanglement between the coin and walker degrees of freedom underpins the dramatic differences between the classical RW versus the QW. The resultant evolution is achieved by repeated application of the QW unitary operator

$$U = F(H \otimes \mathbb{I});$$ (2.5)

after $N$ discrete time steps, the state of the coin+walker evolves according to the evolution operator $U^N$. 

New Journal of Physics 10 (2008) 053025 (http://www.njp.org/)
As we shall see, the QW signature will be evident in the phase distribution of the walker’s state \[ |\phi\rangle \]

The phase distribution for the walker’s reduced state \( \rho_w \), obtained by tracing out the joint coin+walker state over the coin’s degree of freedom, is

\[
P(\phi) = \lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} e^{in\phi} |n\rangle \langle n| \]

as constructed from phase states

\[
|\phi\rangle_M \equiv \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{in\phi} |n\rangle.
\]

Phase states are thus dual to the Fock states in the sense that

\[
\langle n| \phi \rangle_M = \frac{e^{in\phi}}{\sqrt{M}},
\]

if \( n < M \), and the overlap is zero otherwise, and, for \( \phi_m = 2m\pi/M \),

\[
\text{span}\{|\phi_m\rangle; m = 0, 1, \ldots, M - 1\} = \text{span}\{|n\rangle; n = 0, 1, \ldots, M - 1\},
\]

with \( \{|\phi_m\rangle\} \) an orthonormal basis of the subspace. For arbitrary phase states \( |\phi\rangle \) and \( |\phi + \delta\rangle \), their overlap is given by

\[
M|\phi \rangle |\phi + \delta\rangle_M = \frac{1}{M} \sum_{m=0}^{M-1} e^{im\delta} = \frac{1}{M} e^{i(M-1)\delta} U_{M-1}(\delta),
\]

for

\[
U_{M-1}(\delta) = \frac{\sin(M\delta/2)}{\sin(\delta/2)},
\]

the Chebyshev polynomial of the second kind.

In the coin + walker basis, with phase states as the walker basis states, the free evolution operator acts according to

\[
F|\xi, \phi\rangle = |\xi, \phi + (-1)^{e^i\theta}\rangle,
\]

so the phase states form a natural representation for studying this evolution. Furthermore, the signature of both the RW and QW, and their differences, is in the phase distribution (2.6) of the reduced walker state \( \rho_w \).

The dispersion of the phase distribution is especially important. As moments are not particularly useful for distributions over compact domains, other strategies are needed. For the phase distribution over the domain \([0, 2\pi)\), Holevo’s version of the standard deviation is particularly useful as it reduces to the ordinary standard deviation for small spreads and is sensible when the dispersion is large over the domain. Holevo’s standard deviation is

\[
\sigma_H = \sqrt{\langle e^{i\phi} \rangle^2 - 1},
\]

with respect to any phase distribution \( P(\phi) \).

The Holevo standard deviation has been shown to evolve according to \( \sigma_H \propto \frac{1}{t} \) for the QW, whereas \( \sigma_H \propto \sqrt{t} \) for the RW, at least for short times where the phase distribution has support over less than the circle. This quadratic speed-up of phase spreading in a unitary evolution is a hallmark of the QW on the circle. We will use this quadratic speed-up as the indication of QW in the system.

Our focus is on phase spreading as a signature for the QW, but phase is not directly measurable. However, phase can be inferred from homodyne or from optical homodyne tomography measurements.
3. QWs on circles

In the previous schemes, implementations of the QW on a circle have been proposed for ion traps [9] or cavity QED [10, 16], and each scheme relies on direct driving of the coin (i.e. directly flipping the coin without modifying the cavity field). In realistic systems this may not be possible, and instead the simple harmonic oscillator will be driven, which then drives the coin via the oscillator–coin coupling.

3.1. Generalized Hadamard transformation

In this section, we treat this strategy of indirectly driving the coin by generalizing the Hadamard transformation to

$$H \mapsto \exp \left\{ \frac{i}{4} \left[ \hat{\sigma}_x + \lambda \hat{\lambda} \right] \right\} = H \otimes D,$$  \hspace{1cm} (3.1)

for $D(\alpha)$ the unitary displacement operator (1.3) with $\alpha \mapsto i\lambda/\sqrt{2}$. Thus $\lambda$ is the kick the walker receives during the Hadamard pulse. In the next section, we will derive an approximation to the unitary operator (3.1) by beginning with the JC model Hamiltonian.

The generalized Hadamard transformation (3.1) nicely factorizes into a Hadamard transformation and a displacement operation. The Hadamard transformation effects the desired coin flip, but the displacement operator simultaneously moves the walker to another circle in phase space. As $\lambda$ in equation (3.1) is real, the kick is a displacement in $x$. The nature of the QW on circles in phase space is made clear in figure 1. The radius of a given circle in phase space is the modulus of the displacement parameter, namely $|\alpha|$. The phase spread of the initial walker state is independent of which circle the walker is on, hence the constant size of the black dots in the figure regardless of the specific circle.

In this geometric representation, the coin flip Hadamard operation is accompanied by a concomitant displacement that shifts the walker’s distribution (the large black dot in figure 1) from one circle of radius $n_j$ to another circle of radius $n_j'$. To understand the effect of hopping to different circles of phase space, let us consider a coin+walker state initially in the state $|0, x + ip\sqrt{2}\rangle$ with $\alpha = \frac{x+ip}{\sqrt{2}}$, which corresponds to the coin in the 0 state and the walker localized at $(x, p)$ in phase space.

3.2. The first step

The first step corresponds to the application of the unitary operator

$$U = F(H \otimes D).$$ \hspace{1cm} (3.2)

First, the generalized Hadamard transformation $H \otimes D(i\lambda/\sqrt{2})$ is applied:

$$H \otimes D \left| 0, \frac{x + ip}{\sqrt{2}} \right\rangle = \left| +, \frac{x + \lambda + ip}{\sqrt{2}} \right\rangle.$$ \hspace{1cm} (3.3)

This generalized Hadamard operator is then followed by the unitary conditional phase operator $F$ on the state (3.3), which yields the resultant state

$$U \left| 0, \frac{x + ip}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}} \left[ \left| 0, \frac{x + \lambda + ip}{\sqrt{2}} e^{i\Delta q} \right\rangle + \left| 1, \frac{x + \lambda + ip}{\sqrt{2}} e^{-i\Delta q} \right\rangle \right].$$ \hspace{1cm} (3.4)
Equation (3.4) has three important features. One is that the resultant state is an entanglement of a coherent state with a qubit of the type that is observed in microwave cavity quantum electrodynamics experiments [21]. The second important point is that each of the two walker states $|x + \frac{\lambda}{\sqrt{2}} + i\frac{p}{\sqrt{2}}\rangle$ are localized on the same circle in phase space, and third the rotation of the coherent state by angle $\Delta \theta$ is independent of which circle the walker is on.

Thus, although the walker is forced to hop between circles during the application of each Hadamard transformation (3.1), we will show that the QW survives this generalized action.

### 3.3. After $N$ steps

Consider an initial state of the coin + walker as

$$|\Phi(\alpha)\rangle = \frac{1}{\sqrt{2}}|0, \alpha\rangle + \frac{i}{\sqrt{2}}|1, \alpha\rangle. \quad (3.5)$$

The state after $N$ steps is $|\Phi(N)\rangle = U^N|\Phi\rangle$. The phase distribution for the walk after $N$ steps is $P(\phi)$ in equation (2.7) for $\rho_w = \text{Tr}_c(|\Phi(N)\rangle\langle\Phi(N)|)$. The first 3 steps for the walk and corresponding phase distributions are discussed in the appendix.

The state after $N$ steps is

$$|\Phi(N, \alpha, \lambda, \Delta \theta)\rangle = \sum_{i=1}^{2^N-1} [p_i(N)|0, \alpha_i(N, \alpha, \lambda, \Delta \theta)\rangle + q_i(N)|1, \beta_i(N, \alpha, \lambda, \Delta \theta)\rangle]. \quad (3.6)$$
The coin + walker state (3.6) adopts a simple form: it is an entanglement between orthogonal coin qubit states with superpositions of coherent states. The weights $p_i, p_j$ and coherent state amplitudes $\alpha_i, \beta_j$ are determined by recursion relations presented in the appendix. After tracing out the coin state,

$$
\rho_w(N, \alpha, \lambda, \Delta \theta) = \sum_{i,j} \left[ p_i(N) p_j^*(N) \langle \alpha_i(N, \alpha, \lambda, \Delta \theta) \beta_j(N, \alpha, \lambda, \Delta \theta) \rangle^M + q_i(N) q_j^*(N) \langle \beta_i(N, \alpha, \lambda, \Delta \theta) \alpha_j(N, \alpha, \lambda, \Delta \theta) \rangle^M \right]
$$

is obtained. The phase distribution for the state after $N$ steps is thus

$$
P(\phi; N, \alpha, \lambda, \Delta \theta) = \lim_{M \to \infty} \sum_{i,j} \left[ p_i(N) p_j^*(N) \langle \alpha_i(N, \alpha, \lambda, \Delta \theta) \beta_j(N, \alpha, \lambda, \Delta \theta) \rangle \times \langle \alpha_j(N, \alpha, \lambda, \Delta \theta) \beta_i(N, \alpha, \lambda, \Delta \theta) \rangle^M + q_i(N) q_j^*(N) \langle \beta_i(N, \alpha, \lambda, \Delta \theta) \alpha_j(N, \alpha, \lambda, \Delta \theta) \rangle \times \langle \beta_j(N, \alpha, \lambda, \Delta \theta) \alpha_i(N, \alpha, \lambda, \Delta \theta) \rangle^M \right],
$$

where the overlap of the phase state with the coherent state is given by

$$
\mathcal{M}(\phi|\alpha) = e^{-|\alpha|^2/2} \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \left( a e^{-i \phi} \right)^n, \quad (3.9)
$$

which is a function of both $\lambda$ and $\Delta \theta$.

### 3.4. The spread in phase

In order to observe a QW, the choice of parameters is critical. Therefore, we study how choices of $\Delta \theta$ and $\lambda$ can affect the quality of the phase distribution for revealing a signature of a QW. We expect that the choice of $\Delta \theta$ controls the rate of spreading of the phase distribution because $\Delta \theta$ corresponds to the size of the walker’s step. On the other hand, $\lambda$ is responsible for breaking the symmetry of $P(\phi)$ around $\phi = 0$, which is evident in the leftward shift of the peaks in the phase distribution. We can see these effects in figure 2.

Specifically, we observe that for increasing $\lambda$, the overall distribution becomes more skewed towards positive $\phi$. The skewing is due to the increasing contribution from $|\beta_i\rangle$, which can be higher in amplitude than the $|\alpha_i\rangle$ terms, hence the concomitant narrowing of some peaks.

The spread of the phase distribution provides an important signature of the QW, and we use the Holevo standard deviation $\sigma_H$ (2.13) to quantify this spread. The graphs of $\sigma_H$ versus log $N$ and its log-log version in figure 3 clearly reveal the square root spreading feature for the RW and the quadratic enhancement for the QW. Therefore, the QW behavior is clearly present despite having generalized the Hadamard transformation to equation (3.1) and used a Holevo standard deviation for phase as a quantifier. Figure 3 thus makes it clear that the QWs over different circles in phase space are actual QWs.

The last two points in figures 3(a) and (b) bend down, which will also be seen later in figures 4(a) and (b), and is due to the wrap-around effect when the walker has gone nearly full circle in the phase plane. Specifically the phase spreading quantity, namely the Holevo standard deviation, saturates when the phase spreading approaches $2\pi$; this saturation effect is an artefact of not having good moments of distributions for phase and not due to any physical change in dynamics.
Figure 2. The phase distribution $P(\phi; N, \lambda, \Delta\theta)$ for the walker’s location after $N = 4$ steps of the QW over the different circles in phase space with initial state $(|0, \alpha\rangle + i|1, \alpha\rangle)/\sqrt{2}$, $\Delta\theta = 0.35$ and (a) $\lambda = 0$, (b) $\lambda = 0.2$, (c) $\lambda = 0.3$ and (d) $\lambda = 0.4$.

Figure 3. The Holevo standard deviation $\sigma_H$ of the phase distribution for the RW and QW, for $\alpha = 3$, $\lambda = 0.4$ and $\Delta\theta = 0.35$, as a function of the number of steps $N$ presented as (a) $\sigma_H$ versus $N$ and as (b) $\log\sigma_H$ versus $\log N$ for the classical RW (solid line) and the QW (dots).

3.5. The photon number distribution

A complication of RW and QW over different circles is that the number distribution can vary as the walker is effectively moving nearer and farther from the origin in phase space with the application of each generalized Hadamard transformation (3.1). This hopping is responsible for the narrowing of individual peaks in figure 2 as discussed earlier. For the number distribution

$$P(n; N, \alpha, \lambda, \Delta\theta) \langle n | \rho_w(N, \alpha, \lambda, \Delta\theta) | n \rangle = \sum_{i,j} p_i(N) p_j^*(N) \langle n | \alpha_i(N, \alpha, \lambda, \Delta\theta) \rangle \langle \alpha_j(N, \alpha, \lambda, \Delta\theta) | n \rangle$$

$$+ q_i(N) q_j^*(N) \langle n | \beta_i(N, \alpha, \lambda, \Delta\theta) \rangle \langle \beta_j(N, \alpha, \lambda, \Delta\theta) | n \rangle,$$

(3.10)
the walker’s effective distance from the origin in phase space is given by

\[ \sqrt{n} = \sum_{n=0}^{\infty} n P(n), \]  

(3.11)

where \( N, \alpha, \lambda \) and \( \Delta \theta \) are suppressed from the expression for brevity, and the walker’s radial spread in phase space is given by

\[ \delta n = \sqrt{\langle \tilde{n}^2 \rangle - \langle \tilde{n} \rangle^2}. \]  

(3.12)

The expression for \( \tilde{n} \) is

\[ \tilde{n}(N, \alpha, \lambda, \Delta \theta) = \sum_{i,j} p_i(N)p_j^*(N)\alpha_i^*(N, \alpha, \lambda, \Delta \theta)\alpha_j(N, \alpha, \lambda, \Delta \theta) + q_i(N)q_j^*(N)\beta_i^*(N, \alpha, \lambda, \Delta \theta)\beta_j(N, \alpha, \lambda, \Delta \theta), \]  

(3.13)

which can be approximated by

\[ \tilde{n}(N, \alpha, \lambda, \Delta \theta) \approx -\frac{1}{2} \left\{ -\alpha^2 + \frac{\alpha^2 \cos \Delta \theta}{\sin^2(\Delta \theta/2)} \right\}. \]  

(3.14)

for large \( N \).

Equations (3.13) and (3.14) for the mean number and (3.12) for the spread of the walker quantify the degree of hopping between circles in phase space, and these expressions will be useful in the next section. Although there is hopping to different circles, the QW is clearly evident in the quadratic enhancement of phase spreading, with respect to the Holevo standard deviation, shown in figure 3. Thus, provided that the parameters \( \alpha, \lambda \) and \( \Delta \theta \) are chosen judiciously, the generalization of the Hadamard coin flip transformation from (2.2) to (3.1) does not destroy the QW, but it does modify the QW from being on a circle in phase space to being on circles in phase space. In the next section, we approach the generalized Hadamard transformation from the microscopic perspective, and the mean number \( \tilde{n} \) turns out to be important with respect to controlling the QW in order to ensure optimal enhancement of phase spreading.

4. From JC evolution to QWs

In the previous section, we treated the indirectly driven coin via the generalized Hadamard transformation (3.1), but this transformation was introduced by fiat. In this section, we consider the JC model Hamiltonian [15], which underpins so much of quantum optics and cavity quantum electrodynamics, as a foundation for obtaining the generalized Hadamard transformation, or at least a good approximation to this transformation under reasonable conditions.

In quantum optics, the simple harmonic oscillator is typically the single mode electromagnetic field within the cavity, and the coin is an atom transiting the cavity. Cavity quantum electrodynamic realizations of QWs on the circle in phase space have been suggested [10, 16].

New Journal of Physics 10 (2008) 053025 (http://www.njp.org/)
4.1. Driven JC model with large detuning

For a simple harmonic oscillator with angular resonant frequency \( \omega_r \), coupled with strength \( g \) to a qubit of angular resonant frequency \( \omega_a \), the JC dynamics for the joint system is given by

\[
\hat{H}_{JC} = \omega_r \left( \hat{n} + 1/2 \right) + \frac{\omega_a}{2} \hat{\sigma}_z + g (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+). \tag{4.1}
\]

The joint system is driven by a time-dependent driving force (or field) by directly driving the simple harmonic oscillator according to

\[
\hat{H}_{dr} = \epsilon(t) \left( \hat{a}^\dagger e^{-i\omega_dt} + \hat{a} e^{i\omega_dt} \right), \tag{4.2}
\]

with \( \epsilon(t) \) the amplitude and \( \omega_d \) the driving carrier frequency. For simplicity we let \( \epsilon(t) \) be a constant for some of the time and zero for other times.

For large detuning \( g \ll |\Delta| = |\omega_a - \omega_r| \), conjugating the JC Hamiltonian under the action of

\[ V = \exp \left[ \frac{g}{\Delta} (\hat{a}^\dagger \hat{\sigma}_- - \hat{a} \hat{\sigma}_+) \right] \tag{4.3} \]

yields the effective Hamiltonian

\[ \hat{H}_{JC} = V \hat{H}_{JC} V^\dagger \approx (\omega_r + \chi \hat{\sigma}_z) \hat{n} + \frac{1}{2} (\omega_a + \chi) \hat{\sigma}_z + O(\chi^2), \tag{4.4} \]

for \( \chi \equiv g^2 / \Delta \); the conjugated driving Hamiltonian is thus

\[ \hat{H}_{dr} = V \hat{H}_{dr} V^\dagger \approx \epsilon(t) \left[ \left( \hat{a} + \frac{\Delta}{\chi} \hat{\sigma}_- \right)^\dagger e^{-i\omega_dt} + \text{hc} \right], \tag{4.5} \]

for ‘hc’ designating the Hermitian conjugate. The time evolution of equation (4.5) leads to the generalized Hadamard transformation (3.1).

4.2. Implementation of the generalized Hadamard transformation

To implement a QW, first we turn on the driving force \( \epsilon(t) = \epsilon \) for the Hadamard transformation. In a frame rotating at the drive frequency \( \omega_d \), the effective Hamiltonian of the coin+walker system is thus

\[ \hat{H}_{eff} = \frac{1}{2} \left[ 2 \chi \left( \hat{n} + 1/2 \right) - \delta_{da} \right] \hat{\sigma}_z - \delta_{dr} \hat{n} + \frac{\Omega_R}{2} \hat{\sigma}_x + \epsilon (\hat{a}^\dagger + \hat{a}), \tag{4.6} \]

with the detuning of the qubit transition frequency from the driving force

\[ \delta_{da} = \omega_d - \omega_a, \tag{4.7} \]

the detuning of the resonator from the driving force

\[ \delta_{dr} = \omega_d - \omega_r, \tag{4.8} \]

and the Rabi frequency

\[ \Omega_R = 2g \epsilon / \delta_{dr}. \tag{4.9} \]

The first term in equation (4.6) expression effects the coin-induced walker phase shift. The unitary operator generated by the effective Hamiltonian \( \hat{H}_{eff} \) is

\[ \exp \left[ -i\hat{H}_{eff} t \right] = (H \otimes D) \Xi, \tag{4.10} \]

\[
\text{New Journal of Physics 10 (2008) 053025 (http://www.njp.org/)}
\]
which is a good approximation to the generalized Hadamard transformation in (3.1) for \( D(\alpha = -i \epsilon t_H) \) the displacement operator (1.3) and \( \Xi \) is a ‘small’ operator explicitly shown in equation (4.15).

Choosing [22] \( \omega_d = 2 \bar{n} \chi + \omega_a \), \( \hat{H}_{\text{eff}} \) then generates rotations of the qubit about the \( x \)-axis with Rabi frequency \( \Omega_R \). In particular, choosing

\[
\omega_d = 2 \bar{n} \chi - 2g \epsilon / \Delta + \omega_a
\]

and

\[
t_H = \pi / 2 \Omega_R = \frac{\pi}{4g \epsilon} \left[ \Delta + 2 \bar{n} \chi - 2g \epsilon / \Delta \right]
\]

generates the Hadamard transformation for the coin state

\[
H = e^{i \bar{n} \Omega_R / 2 \hat{n}}
\]

within the generalized Hadamard transformation (4.10). The choice of pulse duration \( t_H \) is critical in effecting a Hadamard transformation, but this duration itself is a function of \( \bar{n} \), which we know from the previous section is time-dependent because the walker is hopping between circles in phase space. Specifically, \( t_H \) depends inversely on \( \Omega_R \) (4.9), which is itself inversely proportional to the driving field detuning \( \delta_{dr} \) (4.8). The driving field detuning is a function of \( \omega_d \) (4.11), and \( \omega_d \) is dependent on \( \bar{n} \) (4.11). Therefore, the duration of each pulse \( t_H \) must be chosen in accordance with the value of \( \bar{n} \) for the system.

In order to choose the appropriate pulse duration \( t_H \) for each step, we employ the following protocol, which depends on the time-dependent mean number \( \bar{n} \). In this protocol, \( \bar{n} \) is obtained from a theoretical analysis rather than continuous measurements or sampling, which could disturb the system. In the first step, we let

\[
\bar{n} = |\alpha|^2
\]

and use this value to determine \( \omega_d \) according to equation (4.11). Then this value of \( \omega_d \) is used to compute \( \Omega_R \) and, from this, \( t_H \). The duration of the generalized Hadamard pulse is precisely this value of \( t_H \). In subsequent steps \( \bar{n} \) will have changed due to the walker hopping to other circles in phase space, so \( \bar{n} \) has to be computed and used in a protocol described in section 4.4.

We now have expressions for \( H \) and \( D \) in equation (4.10) and require

\[
\Xi = \prod_{n=0}^{\infty} \exp \left[ \frac{-t_H}{2} ( -i t_H \hat{\chi} \hat{n} )^{2n+1} \Omega_R \hat{\sigma}_x + \frac{i t_H}{2} ( -i t_H \chi )^{2n+1} ( \hat{\sigma}^\dagger - \hat{\sigma} ) \hat{\sigma}_z \right]
\]

\[
\times \exp \left[ \frac{i t_H}{2} ( -i t_H \hat{\chi} \hat{n} )^{2n+2} \Omega_R \hat{\sigma}_x + \frac{i t_H}{2} ( -i t_H \chi )^{2n+2} ( \hat{\sigma}^\dagger + \hat{\sigma} ) \right].
\]

We can see that \( \Xi \) is close to unity for our choice of parameters; thus equation (4.10) tends to the generalized Hadamard of equation (3.1). The spectrum of the operators \( \hat{\sigma}_x \) and \( \hat{\sigma}_z \) is 0, 1, and the relative size of \( \hat{\sigma}^\dagger + \hat{\sigma} \) and \( i ( \hat{\sigma}^\dagger - \hat{\sigma} ) \) is never much more than \( |\alpha| \) because \( |\langle \hat{\sigma} \rangle| = |\alpha| \). In the case \( t_H \hat{\chi} = \pi g / 4 \epsilon \ll 1 \), we neglect the higher orders of \( t_H \hat{\chi} \). In the case of large detuning, that is \( g / \delta_{dr} \ll 1 \), the term

\[
- \frac{t_H}{2} \Omega_R \hat{\sigma}_x = \frac{i t_H \pi g^2}{4 \delta_{dr} \hat{n}} \hat{\sigma}_y
\]

can also be neglected. Thus \( \Xi \) can be approximated by

\[
\Xi \approx \exp \left[ \frac{\pi}{8} t_H g ( \hat{\sigma}^\dagger - \hat{\sigma} ) \hat{\sigma}_z \right].
\]
The evolution of initial states under $\Xi$ are shown as

$$\Xi|j, \alpha \rangle \approx |j, \alpha + (-1)^j \pi t_H g/8), \quad j = 0, 1.$$  \tag{4.18}

As $\Xi$ in equation (4.17) is close to an identity operation for the restricted choices of parameters, the resultant generalized Hadamard transformation (4.10) is quite close to the ideal (3.1) in the previous section. It is thus important to choose parameters for which $\Xi$ can be neglected. In this case, the displacement operator $D$ in equation (4.10) is responsible for displacing the walker’s distance from the origin in phase space by $|\alpha| \mapsto |\alpha|(1 + \epsilon t_H/2)$. Fortunately, even the effects of this induced jump in $|\alpha|$ can be minimized by varying the duration of successive generalized Hadamard pulses.

### 4.3. Implementation of the first step

In the previous subsection, we have seen how the generalized Hadamard transformation generated by the JC model is very close to the ideal Hadamard transformation of section 3. The importance of choosing the appropriate duration of the generalized Hadamard pulse was noted in section 4.2. Each step of the QW corresponds to first performing the generalized Hadamard transformation and then the conditional phase shift operation given by $F$ (2.3). In this subsection, we concentrate solely on the walker’s first step, which is the generalized Hadamard transformation followed by $F$.

The conditional phase shift $\Delta \theta$ has a size that is constrained by (2.4). In terms of parameters in the JC model, the step size is

$$\Delta \theta = \pm \chi (\tau + t_H),$$  \tag{4.19}

for $\tau$ the time between generalized Hadamard pulses. Because the JC Hamiltonian applies to the dynamics both during the generalized Hadamard pulse, which has duration $t_H$, and during the period between these pulses, which has duration $\tau$, the step size (4.19) is proportional to the total time for each step, namely $\tau + t_H$.

At time $\tau + t_H$ the first step is completed, but $\bar{n}$ has changed. The new $\bar{n}$ after the completion of the first step is required to calculate the appropriate $t_H$ for the second step. The value of $\bar{n}$ after the first step is readily obtained from equation (3.13) by inserting the relevant parameters as well as $N = 1$. From this value of $\bar{n}$, the pulse duration for the next generalized Hadamard transformation is given by equation (3.13). This knowledge of $t_H$ for the next generalized Hadamard transformation prepares us for the second step.

### 4.4. N steps

The previous subsection describes how to perform the first step and obtain the information required to set the duration for the subsequent generalized Hadamard transformation. In this subsection, we describe the transformations required for the walker to go an arbitrary number $N$ steps. Unlike the case of the QW on a single circle or the case of QWs on circles described in section 3, here the choice of $t_H$ for each circle is more complicated but quite important.

For an arbitrary $i$th step, we can calculate the average photon number $\bar{n}(i, \alpha, \lambda, \Delta \theta)$ based on the analytical result (3.13), and then decide the pulse duration of the $i$th step (4.12)

$$t_{H}^{i} = \frac{\pi}{4g_\epsilon} \left[ \Delta + 2\bar{n}(i, \alpha, \lambda, \Delta \theta) \chi - 2g_\epsilon / \Delta \right].$$  \tag{4.20}
Figure 4. (a) The Holevo standard deviation $\sigma_H$ of phase for both the quantum and classical RWs up to $N = 15$ for $\alpha = 3$ and $d = 21$. (b) The numerically simulated Holevo standard deviation for phase distribution in log–log scale is shown to be approximately linear in log $t$.

We apply the generalized Hadamard transformation $\exp[-i\hat{H}_{\text{eff}}t]$ followed by the unitary operator of the free evolution $\exp[i\chi(\tau + t_{\text{eff}})]\hat{n}\hat{\sigma}_z$. These two applications together affect the unitary operation

$$U_{\text{eff}} \approx F(H \otimes D).$$

(4.21)

Using our protocol for choosing durations of generalized Hadamard pulses, we obtain numerically the Holevo standard deviation for the phase distribution of the reduced walker state as a function of time $t$. In contrast to the related plots in figure 3 of section 3, which depend on the number of steps $N$, these plots explicitly depend on $t$. In section 3, the choice of $N$ versus $t$ is not significant because $t \propto N$; here, however, $t$ is not proportional to $N$ because of the varying duration of each step due to the variability of $\bar{n}$. In physical systems, the RW is characterized by its time dependence so, in that spirit, we also use time $t$, rather than the number of pulses $N$, to show the quadratic enhancement of the phase spreading for the QW versus the RW.

This quadratic enhancement is evident in figure 4. Figure 4(a) presents the Holevo standard deviation $\sigma_H$ of phase for both the quantum and classical RWs with achievable system parameters [24]

$$\frac{\omega_a, \omega_r, g, \epsilon}{2\pi} = (7000, 5000, 100, 1000) \text{ MHz.}$$

(4.22)

These numerical simulations, based on the given parameters, reveal that $\sigma_H$ is almost independent of the initial state of the charge qubit and approximately linear in $N$:

$$\sigma_H = (1.3964 \pm 0.0180)t + (0.1208 \pm 0.0018).$$

(4.23)

To show this more explicitly, we apply linear regression techniques to the log–log plot, which theoretically should be linear with a slope of 1/2 for the RW (depicted as a solid line in figure 4(b)) and slope 1 for the QW for small phase spreading. The numerically simulated Holevo standard deviation for phase distribution in log–log scale is shown to be approximately linear in log $t$:

$$\log \sigma_H = (0.924 \pm 0.009) \log t + (0.442 \pm 0.004),$$

(4.24)

and the $r = 0.99$. As $r$ is close to unity, this confirms the linear relationship between $\sigma_H$ and $t$. The slope is 0.924, which is quite close to unity. Together the slope being close to unity and the high value of $r$ demonstrate that this protocol does indeed lead to an enhancement of phase spreading that is very close to quadratic and is thus a signature of QW behaviour.
A superconducting circuit quantum electrodynamics realization of such a protocol has been proposed [25].

5. Conclusions

Motivated by the challenge of directly driving a coin qubit in a high-$Q$, strongly coupled cavity quantum electrodynamical realization of the QQ, we generalized the Hadamard coin flip to be indirect by instead kicking the resonator. The reason for our modification of the coin flipping protocol is that the resonator is already leaky so it allows for indirect flipping without further opening the system.

Our modification to the cavity quantum electrodynamical QQ has the effect that the walker is no longer confined to one circle in phase space; i.e. the walker’s amplitude is not a constant of motion. However, we show that the walker’s location in phase space is indeed localized in amplitude. Moreover, the effect of the deviation in amplitude can be ameliorated by adjusting the timing of the Hadamard pulses according to the expected mean photon number in the system.

By our technique, the indirectly flipped coin protocol yields a QQ that is almost as good as the cavity quantum electrodynamics model with a directly flipped coin, and our modified QQ is certainly good enough to manifest QW behaviour. We show the QW behaviour in the QQ by demonstrating a quadratically enhanced phase spreading. Somewhat counter-intuitively, phase diffusion is reduced by decoherence [10], and the amount of phase reduction can be controlled by adjusting the resonator loss rate. We also show the efficacy of our approach by using realistic superconducting circuit quantum electrodynamic parameters [25].

The signature of the QW is in the phase distribution. Although phase is not directly measured, its cosine and sine can be inferred from homodyne measurements [26], or from full optical homodyne tomography [27]. We also note that our QQ demonstrates the discrete coined QW and quadratic enhancement. Exciting new developments with exponentially faster algorithms [3]–[6] would need their own quincunxes to demonstrate these distinct effects.

Acknowledgments

We are grateful to A Blais for numerous helpful comments and suggestions. This work has been supported by the NSERC, MITACS, CIFAR Associateship, QuantumWorks and iCORE.

Appendix

We calculate the state of the coin+walker system after $N$ steps for small $N$. For the initial state $|\Phi\rangle$ in equation (3.5), after the $N$th step of walking on the circles, the state $|\Phi(N)\rangle = U^N |\Phi\rangle$ is shown in equation (3.6), where the coefficients $p_i(N)$ and $q_i(N)$ are obtained from the following recursion relations (for $N \geq 2$):

\[
p_i(N) = \begin{cases} p_i(N-1)/\sqrt{2}, & \text{if } 1 \leq i \leq 2^{N-2}, \\ q_{i-2^{N-2}}(N-1)/\sqrt{2}, & \text{if } 2^{N-2} < i \leq 2^{N-1} \end{cases}
\]

(A.1)

and

\[
q_i(N) = \begin{cases} p_i(N-1)/\sqrt{2}, & \text{if } 1 \leq i \leq 2^{N-2} \\ -q_{i-2^{N-2}}(N-1)/\sqrt{2}, & \text{if } 2^{N-2} < i \leq 2^{N-1} \end{cases}
\]

(A.2)
For the case $N = 0$, we have $p_0(0) = 1/\sqrt{2}$ and $q_0(0) = -i/\sqrt{2}$. We will show the case $N = 1$ below.

The coherent state with $\alpha_c(N, \lambda, \Delta \theta)$ and $\beta_c(N, \lambda, \Delta \theta)$ can also be obtained from the following recursion relations for $N \geq 1$:

\[
\alpha_c(N, \lambda, \Delta \theta) \begin{cases} 
\alpha_c(N - 1, \lambda, \Delta \theta) + \lambda e^{i\Delta \theta}, & \text{if } 1 \leq i \leq 2^{N-2} \\
\beta_{i-2^{N-2}}(N - 1, \lambda, \Delta \theta) + \lambda e^{i\Delta \theta}, & \text{if } 2^{N-2} < i \leq 2^{N-1}
\end{cases}
\]  
\tag{A.3}

and

\[
\beta_c(N, \lambda, \Delta \theta) \begin{cases} 
\alpha_c(N - 1, \lambda, \Delta \theta) + \lambda e^{-i\Delta \theta}, & \text{if } 1 \leq i \leq 2^{N-2} \\
\beta_{i-2^{N-2}}(N - 1, \lambda, \Delta \theta) + \lambda e^{-i\Delta \theta}, & \text{if } 2^{N-2} < i \leq 2^{N-1}.
\end{cases}
\]  
\tag{A.4}

For the case $N = 0$, $\alpha_0(0) = \beta_0(0) = \alpha$.

After the first step, the state of the system is

\[ |\Phi(1)\rangle = p_1(1)|0, \alpha_c(1)\rangle + q_1(1)|1, \beta_c(1)\rangle, \]  
\tag{A.5}

with

\[ p_1(1) = \frac{1 + i}{2}, \quad q_1(1) = \frac{1 - i}{2}, \quad \alpha_c(1) = (\alpha + \lambda)e^{i\Delta \theta}, \quad \beta_c(1) = (\alpha + \lambda)e^{-i\Delta \theta}. \]  
\tag{A.6}

After the second step, the state is

\[ |\Phi(2)\rangle = \sum_{i=1}^{2} p_i(2)|0, \alpha_c(2)\rangle + q_i(2)|1, \beta_c(2)\rangle, \]  
\tag{A.7}

with

\[ p_1(2) = \frac{1 + i}{2\sqrt{2}}, \quad p_2(2) = \frac{1 - i}{2\sqrt{2}}, \quad q_1(2) = \frac{1 + i}{2\sqrt{2}}, \quad q_2(2) = \frac{1 - i}{2\sqrt{2}}, \]

\[ \alpha_c(2) = \alpha e^{2i\Delta \theta} + \lambda(e^{2i\Delta \theta} + e^{i\Delta \theta}), \quad \alpha_c(2) = \alpha + \lambda(e^{i\Delta \theta} + 1), \]  
\tag{A.8}

\[ \beta_c(2) = \alpha + \lambda(1 + e^{-i\Delta \theta}), \quad \beta_c(2) = \alpha e^{-2i\Delta \theta} + \lambda(e^{-i\Delta \theta} + e^{-2i\Delta \theta}). \]

The third step leads the state to

\[ |\Phi(3)\rangle = \sum_{i=1}^{4} p_i(3)|0, \alpha_c(3)\rangle + q_i(3)|1, \beta_c(3)\rangle, \]  
\tag{A.9}

with

\[ p_1(3) = \frac{1 + i}{4}, \quad p_2(3) = \frac{1 - i}{4}, \quad p_3(3) = \frac{1 + i}{4}, \quad p_4(3) = \frac{1 - i}{4}, \]

\[ q_1(3) = \frac{1 + i}{4}, \quad q_2(3) = \frac{1 - i}{4}, \quad q_3(3) = \frac{1 + i}{4}, \quad q_4(3) = \frac{1 - i}{4}, \]

\[ \alpha_c(3) = \alpha e^{3i\Delta \theta} + \lambda(e^{6i\Delta \theta} + e^{2i\Delta \theta} + e^{i\Delta \theta}), \quad \alpha_c(3) = \alpha e^{i\Delta \theta} + \lambda(e^{2i\Delta \theta} + 2e^{i\Delta \theta}), \]  
\tag{A.10}

\[ \alpha_c(3) = \alpha e^{4i\Delta \theta} + \lambda(2e^{2i\Delta \theta} + 1), \quad \alpha_c(3) = \alpha e^{-i\Delta \theta} + \lambda(e^{2i\Delta \theta} + 1 + e^{-i\Delta \theta}), \]

\[ \beta_c(3) = \alpha e^{3i\Delta \theta} + \lambda(2e^{i\Delta \theta} + e^{-i\Delta \theta}), \quad \beta_c(3) = \alpha e^{-3i\Delta \theta} + \lambda(e^{-i\Delta \theta} + e^{-2i\Delta \theta} + e^{-3i\Delta \theta}). \]
Figure A1. The phase distribution for the walker’s location after the first three steps of the QW on the different circles with initial state $((|0⟩ + i|1⟩) |α = 3⟩)/\sqrt{2}$, $Δθ = 0.35$ and different $λ$. (a) $λ = 0$ and (b) $λ = 0.4$. The yellow line is for the case $N = 0$, the blue one for $N = 1$, the green one for $N = 2$ and the red one for $N = 3$.

The entanglement between the coin qubit and the superposition of coherent states leads to the signature of QW compared to RW, that is the quadratic in phase spreading. From figure A1, for the given $α$ and fixed $Δθ$, the hopping between circles, i.e. $λ$ leads the phase distribution to be skewed towards positive $φ$ and individual peaks can become narrower or broader. However, for the case $λ \ll α$, we still obtain the characteristic quadratic enhancement in phase spreading for QW.

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