POSITIVE MASS THEOREM AND FREE BOUNDARY MINIMAL SURFACES

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Abstract. Built on a recent work of Almaraz, Barbosa, de Lima on positive mass theorems on asymptotically flat manifolds with a noncompact boundary, we apply free boundary minimal surface techniques to prove their positive mass theorem and study the existence of positive scalar curvature metrics with mean convex boundary on a connected sum of the form $(T^{n-1} \times [0,1])#M_0$.

1. Introduction

An asymptotically flat manifold is used to model an isolated gravitational system in physics. The positive mass conjecture states that if the system has nonnegative local mass density, then the system must have nonnegative total mass measuring at spatial infinity. Schoen and Yau [SY79] in 1970s established the positive mass theorem for time-symmetric case of the conjecture using minimal surfaces. They proved the three dimensional case. It is also called the Riemannian positive mass theorem. When the dimension is less than eight, the positive mass theorem can be reduced down to dimension three, see [Sch89]. Witten [Wit81] found an elegant proof for the non-time-symmetric case (spacetime version) using spinor techniques and a mathematically rigorous account of his proof can be found in [PT82]. Witten’s proof applies for spin manifolds of all dimensions. As for spacetime version without spin assumption, there is a recent work of Eichmair, Huang, Lee and Schoen [EHLS15] which uses marginally outer trapped surface (abbr. MOTS) to replace minimal surfaces in the argument.

There is also a lot of work to extend positive mass theorem to hyperbolic settings. [Wan01] and [CH03] use Witten-type arguments to prove a positive mass theorem for asymptotically hyperbolic manifolds. Later Andersson, Cai, and Galloway [ACG08] uses the BPS brane action to give a proof of the non-spin case.

We first recall the definition of a standard asymptotically flat manifold.

Definition 1. (Asymptotically flat) We say that $(M^n, g)$ is asymptotically flat with decay rate $\tau > 0$ if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi : M \setminus K \rightarrow \mathbb{R}^n \setminus \bar{B}_1(0)$ such that the following asymptotics
holds as $r \to +\infty$:

$$|g_{ij}(x) - \delta_{ij}| + r|g_{ij,k}| + r^2|g_{ij,kl}| = o(r^{-\tau})$$

where $\tau > \frac{n-2}{2}$.

**Definition 2.** (ADM mass, after Arnowitt, Deser and Misner [ADM60])

Let $(M^n, g)$ be the manifold specified above, assume that the scalar curvature $R_g$ of $M$ is integrable, then the quantity defined for an asymptotically flat manifold $M$ below

$$E_{ADM} = \lim_{r \to +\infty} \left\{ \int_{S_r} (g_{ij,j} - g_{jj,i}) \mu^i dS_r \right\}$$

is called the ADM mass. Here, $x = (x_1, \cdots, x_n)$ is the coordinate system induced by $\Psi$, $r = |x|$, $g_{ij}$ are the components of $g$ with respect to $x$ and the comma denotes partial differentiation. $S_r$ denotes the standard sphere of radius $r$, $\mu^i$ normal to $S_r$ under Euclidean metric and the comma denotes partial differentiation.

The definition of the ADM mass relies on the choice of coordinates and its geometric invariance of ADM mass is proved by Bartnik [Bar86]. Then the positive mass theorem says that

**Theorem 1.** (Positive mass theorem [SY79b, SY17]) If $(M, g)$ is asymptotically flat with scalar curvature $R_g \geq 0$ and $R_g$ is integrable, then $E_{ADM} \geq 0$. $E_{ADM} = 0$ if and only if the manifold $(M, g)$ is isometric to the standard Euclidean space $(\mathbb{R}^n, \delta)$.

The seminal work [SY79b] reveals deep connections between the geometry of non-compact minimal surface in asymptotically flat 3-manifolds and non-negative scalar curvature. We briefly outline their ideas below.

Assume the ADM mass $E_{ADM} < 0$, then the metric can be perturbed into a metric which is conformally flat at infinity (i.e. outside a compact set) such that the scalar curvature $R_g$ has a strict positive sign and negativity of the mass is preserved. Then choose a large number $\sigma > 0$, let

$$\Gamma_{\sigma,a} = \{(\hat{x}, x_n) \in \mathbb{R}^n : |\hat{x}| = \sigma, x_n = a\},$$

we find a minimal hypersurface $\Sigma_{\sigma,a}$ solving the Plateau problem in $M$ with boundary $\Gamma_{\sigma,a}$. The deformed metric allows fixing two coordinate slabs $\{x_n = \pm \Lambda\}$ such that any $\Sigma_{\sigma,a}$ realizing the minimum among all $\{|\Sigma_{\sigma,a}| \}_{a \in [-a_0, a_0]}$ lies strictly between the slabs. Moreover, it is possible to choose a number $a = a(\sigma) \in (-a_0, a_0)$ such that $\Sigma_\sigma = \Sigma_{\sigma,a(\sigma)}$ has the least area among all $\{|\Sigma_{\sigma,a}| \}_{a \in [-a_0, a_0]}$. We can take a subsequence $\sigma_i \to \infty$ such that $\Sigma_\sigma$ converge to a strongly stable minimal hypersurface. The contradiction will follow from the strong stability and Gauss-Bonnet theorem.

The technical part of their proof is to handle asymptotics. Lohkamp [Loh99] observed that if the mass is negative, the metric can be transformed further into a metric which is Euclidean at infinity. This allows compactification by identifying edges of a large cube. The compactified manifold
$M'$ has nonnegative scalar curvature and is not flat. By argue that the $(n - 1)$-th homology group of $M'$ is non-trivial and that there exists an area-minimizing hence stable minimal hypersurface in $M'$. The stability of this hypersurface allows a dimension reduction argument from dimension seven down to dimension three, and the proof of dimension three finishes with the Gauss-Bonnet theorem.

Recently Almaraz, Barbosa and de Lima in [ABdL16] introduce a notion of an asymptotically flat manifold with a non-compact boundary as well as an ADM mass:

**Definition 3.** (Asymptotically flat with a noncompact boundary) We say that $(M,g)$ is asymptotically flat with decay rate $\tau > 0$ if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi : M \setminus K \to \mathbb{R}^n_+ \setminus \bar{B}^+_1(0)$ such that the following asymptotics holds as $r \to +\infty$:

$$|g_{ij}(x) - \delta_{ij}| + r|g_{ij,k}| + r^2|g_{ij,kl}| = o(r^{-\tau})$$

where $\tau > \frac{n-2}{2}$.

**Definition 4.** (ADM mass with a noncompact boundary) ADM mass for $M$ is given by

$$m_{ADM} = \lim_{r \to +\infty} \left\{ \int_{S^{n-1}_{r,+}} (g_{ij,j} - g_{jj,i})\mu^i dS_{r,+}^{n-1} + \int_{S^{n-2}_{r}} g_{1n} g^{1} dS_{r}^{n-2} \right\}$$

where $\mathbb{R}^n_+ = \{ x \in \mathbb{R} : x_1 \geq 0 \}$ and $\bar{B}^+_1(0) = \{ x \in \mathbb{R}^n_+ : |x| \leq 1 \}$. We also use the Einstein summation convention with index ranges $i,j,k = 1, \cdots , n$ and $a,b,c = 2, \cdots , n$. Observe that along $\partial M$, $\{ \partial_a \}$ spans $T\partial M$ while $\partial_1$ points inwards. $S^{n-1}_{r,+} \subset M$ is a large coordinate hemisphere of radius $r$ with outward unit normal $\mu$, and $\vartheta$ is the outward pointing unit co-normal to $S^{n-2}_{r} = \partial S^{n-1}_{r,+}$, viewed as the boundary of the bounded region $\Sigma_r \subset \Sigma$.

We write $m(M,g)$ if we want to emphasize the dependence on the manifold and the metric, and we write $m_g$ for short if the manifold $M$ is clear from the context. See Fig. [Fig. 1] for a hemisphere in such an asymptotically flat manifold.

Motivated by the proof by Schoen and Yau [SY79b] using minimal hypersurface techniques, S. Almaraz, E. Barbosa and L. de Lima proved a positive mass theorem for asymptotically flat manifolds with a non-compact boundary, more specifically,

**Theorem 2.** [ABdL16, Theorem 1.3] When $3 \leq n \leq 7$ and if $(M,g)$ is asymptotically flat $R_g \geq 0$ and $H_g \geq 0$ then $m_{ADM} \geq 0$, with the equality occurring if and only if $(M,g)$ is isometric to $(\mathbb{R}^n_+, \delta)$.

Their method is to perturb the metric, making the manifold $M$ conformally flat at infinity and the mean curvature of $\partial M$ strictly positive, therefore $\partial M$ serves a barrier for the area-minimizing hypersurface to exist.
We provide a different approach which uses free boundary minimal hypersurfaces instead.

We also study the geometry of \((\mathbb{T}^{n-1} \times [0,1])\#M_0\) where \(M_0\) is non-flat. We prove the following

**Theorem 3.** There does NOT exist a metric on \((\mathbb{T}^{n-1} \times [0,1])\#M_0\) with nonnegative scalar curvature and nonnegative mean curvature along the boundary where \(3 \leq n \leq 7\).

In fact, the only metric with nonnegative scalar curvature and nonnegative mean curvature is flat which in turn will force \(M_0\) to be flat. The existence of positive scalar curvature metrics on \(\mathbb{T}^n\#M_0\) were studied most notably by works of Schoen, Yau [SY79a] and Gromov, Lawson [GL83]. This non-existence result Theorem 3 was essentially due to Gromov and Lawson [GL83]. For the convenience of the reader, we include our sketch of their proof. Note that their proof used minimal surfaces techniques and we use instead free boundary minimal surfaces.

Also, other results of the latter require a spin assumption on the manifold \(\mathbb{T}^n\#M_0\). For Theorem 3 the corresponding spin versions can be established via the analysis of Dirac spinors with integrated Bochner formula (see Hijazi, Motiel and Zhang [HMZ01, Eq. (2.3)]). We focus here on the non-spin case, and the dimension assumption \(3 \leq n \leq 7\) is a technical one.

Recently, Schoen and Yau [SY17] develop a minimal slicing theory and use it to settle the non-spin higher dimensional positive mass theorem and also established a non-existence result of positive scalar curvature metrics on \(\mathbb{T}^n\#M_0\) with \(n\) greater than seven. We intend to generalize the theory to the boundary setting in future works.
The article is organized as follows:

In Section 2, we record some basics of free boundary minimal hypersurfaces and that of conformal changes. In Section 3, we present another approach which resembles more formally with Schoen and Yau’s original approach by replacing minimal hypersurface with free boundary minimal hypersurface instead. We follow closely to Schoen’s article [Sch89]. In Section 4, we study the relationship of the geometry of $(\mathbb{T}^{n-1} \times [0, 1])\# M_0$ and the positive mass theorem. In the appendix, we give details of some of the computations.

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2. Free boundary minimal hypersurfaces and conformal geometry

In this section, $(M^n, g)$ is a smooth manifold of dimension $n$ with nonempty boundary $\partial M$ and $\Sigma^{n-1}$ is an immersed hypersurface whose boundary lies in $\partial M$. Let $\Sigma$ be a free boundary minimal hypersurface which is a critical point of the volume functional $V(\Sigma)$ among all surfaces whose boundary lies in $\partial M$. We compute the first and second variation of the functional $V(\Sigma)$ with respect to variational vector field $X$ whose restriction on $\partial \Sigma$ is tangent to $\partial M$. This computation also fills some calculational details missing in [Sch89].

We adopt the following notations.

Notations. We use $(\Sigma, \partial \Sigma) \leftrightarrow (M, \partial M)$ to denote that $\Sigma$ is a hypersurface in $M$ with boundary lying on $\partial M$. $B(X, Y) := \langle \nabla_X \nu, Y \rangle$ is the second fundamental form of $\Sigma$ in $M$ where $\nu$ is a fixed unit normal to $\Sigma$ in $M$ and $X, Y$ are tangent to $\Sigma$ when restricted to $\Sigma$. Let $A(Y, Z) := \langle \nabla_Y \eta, Z \rangle$ where $\eta$ is the outward normal of $\partial M$ in $M$ and $Y, Z$ are tangent to $\partial M$ in $M$. When $\eta$ is also normal to $\partial \Sigma$ in $\Sigma$, the second fundamental form $A$ evaluated on $T\partial \Sigma \times T\partial \Sigma$ can be expressed as the second fundamental form of $\partial \Sigma$ in $\Sigma$ as well.

2.1. Basics of free boundary minimal hypersurfaces. We have the following definition

Definition 5. (Free boundary minimal hypersurface) $(\Sigma, \partial \Sigma) \leftrightarrow (M, \partial M)$ is said to be a free boundary minimal hypersurface if the first variation of the volume functional $V(\Sigma)$ vanishes along any vector field $X$ which only has components tangential to $\partial M$ along $\partial \Sigma$.

It is well known that the first variation of any hypersurface $\Sigma$ is given by

$$
\delta \Sigma(X) := \delta V(X)|_{\Sigma} = \int_{\Sigma} \text{div}_\Sigma X = \int_{\Sigma} H(X, \nu) + \int_{\partial \Sigma} \langle X, \eta \rangle,
$$
here \( \nu \) is a fixed normal of \( \Sigma \), \( H = \text{div}_\Sigma \nu \) is the mean curvature of \( \Sigma \) and \( \eta \) is the outward normal of \( \partial \Sigma \) in \( \Sigma \). We see immediately from the definition of a free boundary minimal hypersurface, \( \Sigma \) is free boundary minimal if and only if

\[
H \equiv 0 \text{ on } \Sigma \text{ and } \eta \perp \partial M \text{ along } \partial \Sigma.
\]

We record the second variation of a free boundary minimal hypersurface in the following theorem and postpone the calculation to the appendix.

**Theorem 4.** Given a free boundary minimal hypersurface \( \Sigma \) in \( M \), let \( \nu \) be a normal to \( \Sigma \) in \( M \) and \( X \) be a variational vector field. \( X \) admits the decomposition that

\[
X = \phi \nu + T
\]

where \( T \) is tangent to \( \Sigma \) and \( \nu \) is normal to \( \Sigma \). Let the normal component of \( \nabla_X X \) be \( \phi \nu \) and tangent component be \( \hat{Z} \). Assume that \( X \) is tangent to \( \partial M \) along \( \partial \Sigma \) then the second variation of volume is

\[
\delta^2 \Sigma(X) := \delta^2 V(X)|_{\Sigma} = \int_{\Sigma} F_X \, d\text{vol}_\Sigma
\]

where the density is given by

\[
F_X = -\phi^2 \text{Ric}(\nu) - \phi^2 |B|^2 + |\nabla \phi|^2 + \text{div}(T \text{div} T - \nabla_T T) + \text{div} \hat{Z} - 2(\phi B_{ij} T_i) T_j.
\]

If the variational vector field \( X \) is normal to \( \Sigma \), then we can write the second variation in a simpler form

\[
\delta^2 \Sigma(f \nu) = \int_{\Sigma} |\nabla f|^2 - (\text{Ric}(\nu) + |B|^2) f^2 - \int_{\partial \Sigma} f^2 A(\nu, \nu)
\]

where \( f \in C^\infty_c(\Sigma) \).

**Definition 6.** We say that \( \Sigma \) is a stable free boundary minimal surface if for any \( f \in C^\infty_c(\Sigma) \), the second variation \( \delta^2 \Sigma(f \nu) \geq 0 \). The inequality \( \delta^2 \Sigma(f \nu) \geq 0 \) is called the stability inequality.

We write down the stability inequality for free boundary minimal hypersurfaces in full,

\[
(1) \quad \delta^2 \Sigma(f \nu) = \int_{\Sigma} (-f^2 \text{Ric}(\nu) - f^2 |B|^2 + |\nabla f|^2) - \int_{\partial \Sigma} f^2 A(\nu, \nu) \geq 0.
\]

With the Gauss-Codazzi equation,

\[
\text{Ric}(\nu) + |B|^2 = \frac{1}{2} R_M - \frac{1}{2} R_\Sigma + \frac{1}{2} |B|^2.
\]

This is the fundamental observation made by Schoen and Yau [SY79b]. Decomposition along \( \partial \Sigma \) of the mean curvature \( H_{\partial M} \) gives that

\[
H_{\partial M} = \sum_{j=1}^{n-2} (\nabla_{e_j} \eta, e_j) + A(\nu, \nu) = H_\Sigma + A(\nu, \nu)
\]
where the orthonormal frame $e_j$ is tangent to $\partial \Sigma$, $e_{n-1} = \eta$ and $e_n = \nu$. We insert these identities back to (1) and get
\[
\int_\Sigma \left[ |\nabla f|^2 - f^2 \left( \frac{1}{2} R_M - \frac{1}{2} R_\Sigma + \frac{1}{2} |B|^2 \right) \right] - \int_{\partial \Sigma} f^2 (H_{\partial M} - H_{\partial \Sigma}) \geq 0.
\]
A rewrite of this inequality assuming that $R_M > 0$ and $H_{\partial M} \geq 0$,
\[
(2) \quad \int_\Sigma |\nabla f|^2 + \frac{1}{2} f^2 R_\Sigma + \int_{\partial \Sigma} f^2 H_{\partial \Sigma} > 0 \quad \text{for all } 0 \neq f \in C^\infty(\Sigma)
\]
using $R_M$ is strictly positive everywhere is sufficient for our purpose.

2.2. Conformal changes. Given any manifold $(M^n, g)$, take any $u > 0$ on $M$, the conformal changed metric $\hat{g} = u^{\frac{4}{n-2}} g$ gives a law for the change of the scalar curvature and the mean curvature of the boundary.

Denote $c_n = \frac{n-2}{4(n-1)}$. We define the conformal Laplacian by
\[
L = c_n R_g - \Delta_g u \quad \text{in } M,
\]
and the scalar curvature under $\hat{g}$ is given by
\[
R_{\hat{g}} = c_n^{-1} u^{-\frac{n+2}{2}} (c_n R_g - \Delta_g u) = c_n^{-1} u^{-\frac{n+2}{2}} Lu.
\]
We define also an operator acting along the boundary
\[
B = \partial_\nu + 2 c_n H_g \quad \text{on } \partial M,
\]
and the mean curvature $H_{\hat{g}}$ of the boundary $\partial M$ under $\hat{g}$ is given by
\[
H_{\hat{g}} = \frac{1}{2} c_n^{-1} u^{-\frac{n-2}{2}} (2 c_n H_g + \partial_\nu u) = \frac{1}{2} c_n^{-1} u^{-\frac{n-2}{2}} Bu \quad \text{along } \partial M.
\]
where $\partial_\nu$ denotes the derivative along the outward unit normal $\nu$ to $\partial M$ in $M$. We write often $L(M, g) = L$ and $B(M, g) = B$ to avoid confusion.

We derive a simple consequence of (2) encoded in the following lemma,

**Lemma 1.** Given any compact manifold $(M^n, g)$ with boundary $\partial M$, suppose that
\[
\int_M |\nabla f|^2 + \frac{1}{2} R_M f^2 + \int_{\partial M} f^2 H_{\partial M} > 0
\]
for all $0 \neq f \in C^\infty(M)$. Then $M$ admits a positive scalar curvature metric $\hat{g}$, and under this metric the boundary $\partial M$ is minimal.

**Proof.** The eigenvalue problem
\[
\begin{cases}
Lu = \lambda u & \text{in } M, \\
Bu = 0 & \text{on } \partial M
\end{cases}
\]
admits a positive solution $u > 0$. 

Using Rayleigh quotient and that $2c_n < 1$, 
\[ \lambda = \frac{\int_M (|\nabla u|^2 + c_n R_M u^2) + 2c_n \int_{\partial M} H_{\partial M} u^2}{\int_M u^2} \geq 2c_n \left[ \frac{\int_M (|\nabla u|^2 + \frac{1}{2} R_M u^2) + \int_{\partial M} H_{\partial M} u^2}{\int_M u^2} \right] > 0. \]

Let $\tilde{g} = u^{4-n} g$, then 
\[ R_{\tilde{g}} = c_n^{-1} u^{-\frac{n+2}{n-2}} (c_n R_g - \Delta_g u) \]
\[ = c_n^{-1} u^{-\frac{n+2}{n-2}} \lambda u = \lambda c_n^{-1} u^{-\frac{4}{n-2}} > 0 \]
and 
\[ H_{\tilde{g}} = u^{-\frac{2}{n-2}} (H_g + \frac{1}{2} c_n^{-1} \partial_n u) = 0 \text{ along } \partial M \]
give that the metric $\tilde{g}$ is the desired metric. \hfill \Box

Similarly, we have

**Lemma 2.** Given any compact manifold $(M^n, g)$ with boundary $\partial M$, suppose that 
\[ \int_M |\nabla f|^2 + \frac{1}{2} R_M f^2 + \int_{\partial M} f^2 H_{\partial M} > 0 \]
for all $0 \neq f \in C^\infty(M)$. Then $M$ admits a scalar-flat metric $\hat{g}$, under this metric the boundary $\partial M$ is strictly mean convex.

**Proof.** The proof is similar to the previous lemma, except that we consider the following Steklov-type eigenvalue problem instead:
\[ L \phi = 0 \text{ in } M, \quad B \phi = \lambda \phi \text{ on } \partial M \]
We omit the details. \hfill \Box

3. An alternate proof of Theorem 2

In this section, we provide another proof of the positive mass theorem (Theorem 2) using free boundary minimal hypersurfaces.

3.1. Step 1: Existence of area-minimizing hypersurface with free boundary. We assume on the contrary that $m_g < 0$. By the density theorem [ABdL16 Proposition 4.1], we can assume that $g = h^{\frac{4}{n-2}} \delta, R_g > 0$ on $M$ and $H_g > 0$ on $\partial M$ where $h(x) = 1 + C(n) m_g |x|^{2-n} + O(|x|^{1-n})$, where $C(n)$ is a constant depending only on the dimension. Consider the vector field $\eta = h^{-\frac{2}{n-2}} \partial_n$. We compute the divergence of $\eta$ with respect to $g$
\[ \text{div}_g \eta = -2(n-1) C(n) m_g \frac{x^n}{|x|^n} + O(|x|^{-n}) \]
In particular we see that $\text{div}_x \eta > 0$ for $x^n \geq a_0$ and $\text{div}_x (-\eta) > 0$ for $x^n \leq -a_0$ for some constant $a_0$. Let $\sigma$ be a large real number, let

$$\Gamma_{\sigma,a} = \{ x = (\bar{x}, x^n) : x^n = a, |\bar{x}| = \sigma, x^1 \geq 0 \}$$

and

$$C_{\sigma} = \{ x = (\bar{x}, x^n) : |\bar{x}| = \sigma, x^1 \geq 0 \text{ or } x^1 = 0 |\bar{x}| \leq \sigma \}.$$ 

The half cylinder $C_{\sigma}$ with $\partial M$ bounds to the interior a region $\Omega_\sigma$ in $M$. We solve a Plateau problem within the class of hypersurfaces with partially free boundary on $\partial M$ and fixed boundary $\Gamma_{\sigma,a}$ and obtain an $(n-1)$-dimensional hypersurfaces $\Sigma_{\sigma,a}$ with least area among all such hypersurfaces. Where the free boundary and fixed boundary meet is called the corner of the hypersurface. In our situation, the corner is the following set

$$\Lambda_{\sigma,a} = \{ x = (\bar{x}, x^n) : |\bar{x}| = \sigma, x^1 = 0, x^n = a \}.$$ 

When solving a Plateau problem, regularity issues will often arise. The requirement that the dimension $3 \leq n < 8$ is one of them (see [Fed96]), and in particular regularity is a problem at the corner. We now list standard known facts about the regularity of $\Sigma_{\sigma,a}$.

The interior regularity of $\Sigma_{\sigma,a}$ is just classical geometric measure theory (see [Fed96]). The regularity at the free boundary of the boundary away from the corner $\Lambda_{\sigma,a}$ is shown by Gruter [Gru87a, Gru87b] and the regularity near $\Gamma_{\sigma,a} \sim \Lambda_{\sigma,a}$ follows from the work of Hardt and Simon [HS79]. Although Gruter [Gru90] claimed some regularity results at the corner, but we have not seen those get published.

In conclusion, regularity can only be an issue at the corners. However, we are able to bypass this when taking limits.
For any large $\sigma > 0$, let
$$V(\sigma) = \min\{\text{Vol}(\Sigma_{\sigma,a}) : a \in [-a_0, a_0]\}.$$ 
First, we note that

**Claim 1:** The function $a \mapsto \mathcal{H}^{n-1}(\Sigma_{\sigma,a})$ is continuous.

To prove this, take any two numbers $a_1, a_2 \in [-a_0, a_0]$, we can further assume that $a_1 < a_2$. $\Sigma_{\sigma,a}$ is area-minimizing with free boundary on $\partial M$ and fixed boundary $\Gamma_{\sigma,a'}$. The union of $\Sigma_{\sigma,a_1}$ and $\cup_{a \in [a_1, a_2]} \Gamma_{\sigma,a}$ serves as a comparison surface (which is only piecewise smooth) for $\Sigma_{\sigma,a_1}$. By volume comparison,
$$\mathcal{H}^{n-1}(\Sigma_{\sigma,a_1}) \leq \mathcal{H}^{n-1}(\Sigma_{\sigma,a_2}) + \mathcal{H}^{n-1}(\bigcup_{a \in [a_1, a_2]} \Gamma_{\sigma,a}),$$
where the last term will be bounded by $C(a_2 - a_1)\sigma^{n-2}$ and the constant $C$ depend only on the dimension $n$ and the decay properties of the metric $g$. Switching roles of $\Sigma_{\sigma,a_1}$ and $\Sigma_{\sigma,a_2}$, we get
$$|\mathcal{H}^{n-1}(\Sigma_{\sigma,a_1}) - \mathcal{H}^{n-1}(\Sigma_{\sigma,a_2})| \leq C|a_2 - a_1|\sigma^{n-2},$$
and hence finish the proof of the **Claim 1**.

Hence, the $V(\sigma)$ is attained by some hypersurface $\Sigma_{\sigma,a}$ where $a = a(\sigma) \in [-a_0, a_0]$. Also, we claim further that

**Claim 2.** There exists $a = a(\sigma) \in (-a_0, a_0)$ such that $\text{Vol}(\Sigma_{\sigma,a}) = V(\sigma)$.

To show that $a(\sigma) < a_0$, let $\Omega_{\sigma,a}$ be the region lying below $\Sigma_{\sigma,a}$ and
$$U_{\sigma,a} = \{(\bar{x}, x^n) \in \Omega_{\sigma,a} : x^n > a_0 - \delta\}$$
where $\delta$ is chosen small such that $\text{div}_g \eta > 0$ for $x^n > a_0 - \delta$. We show by contradiction that $U_{\sigma,a} = \emptyset$. We have by the divergence theorem,
$$0 < \int_{U_{\sigma,a}} \text{div}_g \eta = \int_{\Sigma_{\sigma,a} \cap \{x^n > a_0 - \delta\}} \langle \eta, \nu \rangle + \int_{\Omega_{\sigma,a} \cap \{x^n = a_0 - \delta\}} \langle \eta, -\eta \rangle = \int_{\Sigma_{\sigma,a} \cap \{x^n > a_0 - \delta\}} \langle \eta, \nu \rangle - \text{Vol}(\Omega_{\sigma,a} \cap \{x^n = a_0 - \delta\})$$
where on the right hand side integration on other pieces vanish because $\eta$ is tangent to $C_\sigma$. By Cauchy-Schwarz inequality we arrive
$$\text{Vol}(\Omega_{\sigma,a} \cap \{x^n = a_0 - \delta\}) < \text{Vol}(\Sigma_{\sigma,a} \cap \{x^n \geq a_0 - \delta\}).$$
In the case when $a < a_0 - \delta$, we can project pieces of $\Sigma_{\sigma,a}$ above $\{x^n = a_0 - \delta\}$ down to $\{x^n = a_0 - \delta\}$ and get a smaller area. This contradicts the minimality of $\Sigma_{\sigma,a}$ among all hypersurfaces with $\Gamma_{\sigma,a}$ as boundary in $\Omega_\sigma$. When $a > a_0 - \delta$ we can do the same projection and we get a surface with fixed boundary $\Gamma_{\sigma,a_0-\delta}$ and its area is strictly less that $V(\sigma)$, and this contradicts the choice of $a$. Therefore $U_{\sigma,a} = \emptyset$. We obtain similarly the lower bound for $a(\sigma)$. In conclusion, $a(\sigma) \in (-a_0, a_0)$. 

3.2. Step 2: Strong stability and limiting behavior as $\sigma \to \infty$. Let $\Sigma_{\sigma} := \Sigma_{\sigma, a(\sigma)}$ be one of the hypersurfaces which realizes the minimum volume $V(\sigma)$. Let $X_1$ be a fixed vector field on $M$ which is equal to $\partial_n$ outside a compact set. Let $X_0$ be a vector field of compact support, and let $X = X_0 + \alpha X_1$ where $\alpha$ is a real number. The vector field $X$ generates a one-parameter group of diffeomorphism $F_t$. $F_t$ gives a variation of $\Sigma_{\sigma}$ and

$$\frac{d}{dt} \text{Vol}(F_t(\Sigma_{\sigma}))|_{t=0} = 0, \frac{d^2}{dt^2} \text{Vol}(F_t(\Sigma_{\sigma}))|_{t=0} \geq 0$$

We choose a sequence $\sigma_i \to \infty$ such that $\Sigma_{\sigma_i}$ converge to a limiting volume minimizing hypersurface $\Sigma \subset M$. We see that the possible singularity at the corner goes away by taking limits.

We claim that $\Sigma$ is a graph of a function $f$ near infinity, that is to say $\Sigma$ outside a compact set is given by $x^n = f(x^1, \ldots, x^{n-1})$ with $x^1 \geq 0$. We prove this claim by scaling techniques. Denote $\bar{x} = (x^1, \ldots, x^{n-1})$. Take $p = (\bar{x}, x^n) \in \Sigma$ with $|\bar{x}| = 2\sigma$ for $\sigma$ sufficiently large. $\Lambda$ is a number greater than 1, let $S_\Lambda = \{ x \in M : x^n = \Lambda \}$. Consider $\Omega_{\sigma}$ which is bounded by $\partial M$, $C_{\sigma}$ and the slabs $S_{-a_0}$ and $S_{a_0}$. Using volume comparison, we see that

$$|\Sigma \cap \Omega_{\sigma}| \leq \frac{\omega_{n-1}}{2} \sigma^{n-1} + O(\sigma^{-1})\sigma^{n-1}.$$ 

Outside a large enough compact set, $f$ satisfies the minimal surface equation

$$\sum_{ij} \left( \delta_{ij} - \frac{f_{ij} f_{ij}}{1 + |\nabla f|^2} \right) f_{,ij} + 2\frac{(n-1)}{n-2} \sqrt{1 + |\nabla f|^2} \frac{\partial}{\partial \nu_0} \log h = 0,$$

and on the boundary $\partial \Sigma$ (also outside a compact set)

$$\partial_1 f = 0.$$

Recall that under the metric $g = h^{n-2} \delta$, $h$ is the conformal factor

$$h(x) = 1 + C(n) m_{ADM} |x|^{2-n} + O(|x|^{1-n})$$

and

$$\nu_0 = (1 + |\nabla f|^2)^{-1/2}(-\partial f, 1).$$

We record the calculation of decay rate of $f$ in Lemma $\ref{lem:decay}$. Let $D_{\sigma}$ denote the portion of $\Sigma$ bounded by $C_{\sigma}$ to the interior. According to Lemma $\ref{lem:second_variation}$, the density of the second variation can be written as

$$F_X = -\varphi^2 \text{Ric}(\nu) - \varphi^2 |B|^2 + |\nabla \varphi|^2 + G$$

where

$$G = \text{div}(T \text{div} T - \nabla T T) + \text{div} \hat{Z} - 2(\varphi B_{ij} T_i)_{ij}.$$ 

Here, $T$ is the tangential component of $X$, $\nu$ is the normal of $\Sigma$ in $M$ and $\eta$ is the normal of $\partial D_{\sigma}$ in $\Sigma$.

The integral of $F_X$ over a large bounded region $D_{\sigma}$ is then by divergence theorem.
\[
\int_{D_\sigma} F_X = \int_{D_\sigma} \left( -\varphi^2 \text{Ric}(\nu) - \varphi^2 |B|^2 + |\nabla \varphi|^2 \right) + \int_{\partial D_\sigma} \left[ (T \div T - \nabla_T T, \eta) - 2\varphi B(T, \eta) + (\hat{Z}, \eta) \right].
\]

Along the free boundary \( \partial \Sigma \) we have

\[
(\hat{Z}, \eta) = (\nabla_X X, \eta) = (\nabla_T + \varphi \nu)(T + \varphi \nu, \eta)
= (\nabla_T \nu + \varphi \nabla_T T, \eta) = (\nabla_T T, \eta) + \varphi B(T, \eta) - \varphi^2 A(\nu, \nu) + \varphi (\nabla_T T, \eta)
= (\nabla_T T, \eta) + 2\varphi B(T, \eta) - \varphi^2 A(\nu, \nu)
\]

where \( A(Y, Z) := (\nabla_Y \eta, Z) \) for \( Y, Z \in T \partial M \) i.e. the second fundamental form of \( \partial M \) in \( M \). Since \( (T, \eta) = 0 \), hence the term left for \( \partial \Sigma \) is \(-\varphi^2 A(\nu, \nu)\).

From the decay conditions on \( h \) and \( f \), we see that the boundary terms along \( \partial D_\sigma \cap \Sigma \) decay faster than \( \sigma^{2-n} \), then integral over \( \partial D_\sigma \cap \Sigma \) tends to zero as \( \sigma \to \infty \). See Lemma 7 in the appendix. By letting \( \sigma \to \infty \), we arrive the strong stability inequality

\[
\int_{\Sigma} (-\varphi^2 \text{Ric}(\nu) - \varphi^2 |B|^2 + |\nabla \varphi|^2) - \int_{\partial \Sigma} \varphi^2 A(\nu, \nu) \geq 0
\]

Applying the same trick as in deriving (2), we see the inequality (3) becomes

\[
\int_{\Sigma} |\nabla \varphi|^2 + \frac{1}{2} \varphi^2 R_\Sigma + \int_{\partial \Sigma} \varphi^2 H_{\partial \Sigma} \geq \int_{\Sigma} \frac{1}{2} \varphi^2 (R_M + |B|^2) + \int_{\partial \Sigma} \varphi^2 H_{\partial M} > 0.
\]

The condition on \( \varphi \) can be derived from the condition on \( X \), since

\[
\varphi = \alpha(X, \nu) = \alpha(\partial_n, \nu) = \alpha h^{\frac{2}{n-2}} (1 + |\partial f|^2)^{-1/2}
\]

outside a compact set for a constant \( \alpha \).

Since \( \varphi - \alpha = O(|x'|^{3-n}) \) we see that \( \varphi - \alpha \) has finite mass and therefore we can take \( \varphi \) to be any function for which \( \varphi - \alpha \) has compact support or finite mass for some constant \( \alpha \).

### 3.3. Step 3: Strong stability and Gauss-Bonnet theorem.

We use the above to obtain a contradiction when \( n = 3 \). Specifically, taking \( \varphi \equiv 1 \) in the stability inequality (1), we have

\[
\int_{\Sigma} K + \int_{\partial \Sigma} k_g = \int_{\Sigma} \frac{1}{2} R_\Sigma + \int_{\partial \Sigma} H_{\partial \Sigma} > 0
\]
We use the large region $D_{\sigma}$ to approximate the integral, by Gauss-Bonnet theorem, we have
\[
\int_{\Sigma} K + \int_{\partial \Sigma \cap D_{\sigma}} k_{g} = 2\pi \chi(D_{\sigma}) - 2\pi - \int_{\partial D_{\sigma} - \partial \Sigma \cap D_{\sigma}} k_{g} + \alpha_1 + \alpha_2
\]
where $\alpha_i$ are the inner angles. Note that $D_{\sigma}$ has at least one boundary component and possibly has positive genus, hence $\chi(D_{\sigma}) = 2 - 2g - b \leq 2 - 2 \cdot 0 - 1 = 1$. So the right hand side converge a number less than to zero, yet the left hand side converges to a positive number by stability (4).

We see a contradiction. Therefore, when $n = 3$, $m_{ADM} \geq 0$.

3.4. Step 4: Dimension reduction argument. When $4 \leq n \leq 7$, we have the induced metric $\hat{g}$ on $\Sigma$ in terms of coordinates $x^1, \ldots, x^{n-1}$
\[
\hat{g}_{ij} = h(x, f(x)) \frac{4}{n-2} (\delta_{ij} + f_i f_j) = \delta_{ij} + O(|x|^{2-n})
\]
where $i, j$ ranges from 1 to $n - 1$. Therefore, $(\Sigma, \hat{g})$ is asymptotically flat and has mass zero. We consider the pair of operators $(\hat{L}_{\hat{g}}, B_{\hat{g}})$ defined by $\hat{L}_{\hat{g}} = -\Delta_{\hat{g}} + c_{n-1} R^\Sigma_{\hat{g}}$ in $\Sigma$ and $B_{\hat{g}} = \partial_{\eta} + 2 c_{n-1} H^{\Sigma}_{\hat{g}}$ on $\partial \Sigma$ where $\eta$ is the outward normal of $\partial \Sigma$ in $\Sigma$ under the metric $\hat{g}$.

Now we want to find a positive solution $u$ to the boundary value problem
\[
\begin{cases}
\hat{L}_{\hat{g}} u = 0 & \text{in } \Sigma, \\
B_{\hat{g}} u = 0 & \text{on } \partial \Sigma
\end{cases}
\]
satisfying also $u > 0$ on $\Sigma$ and $u \to 1$ at infinity.

Following from (4), we have that for any domain $D \subset \Sigma$, we have that $\lambda_1(D) > 0$. Using Fredholm alternative and [ABdL16, Proposition 3.3], we have a unique solution $v \in C^{2,\alpha}_{\gamma}(\Sigma)$ (see definitions of such weighted Holder spaces in [ABdL16]) to
\[
\begin{cases}
\hat{L}_{\hat{g}} v = -c_{n-1} R^\Sigma_{\hat{g}} & \text{in } \Sigma, \\
B_{\hat{g}} v = -c_{n-1} H^{\Sigma}_{\hat{g}} & \text{on } \partial \Sigma.
\end{cases}
\]
Then $u = v + 1$ is the desired solution.

We turn to positivity of $u$. Suppose now that the set $\Omega = \{ x \in \Sigma : u < 0 \}$ is not empty. Since $u \to 1$ at infinity then $\Omega$ must be a bounded domain of $\Sigma$. On $\partial \Omega$, $u = 0$. Such $u$ restricted to $\Omega$ will give an eigenfunction $u$ with zero eigenvalue. However $\lambda_1(\Omega) > 0$. This is a contradiction. So $u \geq 0$. The strict positivity in the interior $\Sigma$ follows by applying the usual maximum principle on a domain whose boundary is away from $\partial \Sigma$. The strict positivity on the boundary $\partial \Sigma$ follows from Hopf’s maximum principle on a ball tangent to the boundary at our chosen point. In conclusion, $u > 0$.

We see that $u$ has the asymptotics $u = 1 + m_0 |\hat{x}|^{3-n} + O(|\hat{x}|^{2-n})$, in particular has finite mass. Note that the dimension of $\Sigma$ is $n - 1$. Take $\varphi = u$ in (4), we see that
\[
-2 \int_{\partial \Sigma} H^\Sigma_{\hat{g}} u^2 - \int_{\Sigma} R^\Sigma_{\hat{g}} u^2 < 2 \int_{\Sigma} |\nabla u|^2 \leq \frac{1}{c_{n-1}} \int_{\Sigma} |\nabla u|^2.
\]
Let $\bar{g} = u^{4/(n-3)} \hat{g}$, then $(\Sigma^{n-1}, \bar{g})$ is scalar flat with minimal boundary according to [5]. We turn to the mass of $(\Sigma^{n-1}, \bar{g})$. The mass of $(\Sigma^{n-1}, \hat{g})$ and $(\Sigma^{n-1}, \bar{g})$ are denoted respectively $\hat{m}$ and $\bar{m}$. We have

$$\int_{\Sigma} |\nabla u|^2 = \lim_{\sigma \to \infty} \int_{D_\sigma} |\nabla u|^2$$

$$= \lim_{\sigma \to \infty} \int_{D_\sigma} -u\Delta_\hat{g}u + \int_{\partial D_\sigma \cap \partial \Sigma} u \frac{\partial u}{\partial \eta} + \int_{\partial D_\sigma \cap \partial \Sigma} u \frac{\partial u}{\partial \eta}$$

$$= \lim_{\sigma \to \infty} \int_{D_\sigma} -R^\Sigma u^2 - 2 \int_{\partial D_\sigma \cap \partial \Sigma} u^2 H^\Sigma_\hat{g} + \int_{\partial D_\sigma \cap \partial \Sigma} u \frac{\partial u}{\partial \eta}.$$ 

Considering the decay of $u - 1$ and [6],

$$\lim_{\sigma \to \infty} \int_{\partial D_\sigma \cap \Sigma} \frac{\partial u}{\partial \eta} = \lim_{\sigma \to \infty} \int_{\partial D_\sigma \cap \Sigma} \frac{\partial u}{\partial \eta} > 0.$$ 

Note that

$$\bar{m} = \bar{m} - \hat{m} = \lim_{\sigma \to \infty} \int_{\partial D_\sigma \cap \Sigma} -4 \frac{\partial u}{\partial \eta} < 0.$$ 

We infer as well that $m_0 < 0$.

In conclusion, $(\Sigma, u^{4/(n-3)} \hat{g})$ is asymptotically flat, and scalar flat with minimal boundary and has negative mass $\bar{m} < 0$. The contradiction follows inductively from the case $n = 3$. Here we finish the proof with rigidity statement given by [ABdL16, Lemma 3.3, 3.4].

**Remark 1.** In the case of dimension 3, we can avoid choosing a height $a$ to finish the proof. Since in dimension 3, we could utilize a logarithm cutoff trick on the stability inequality as in [SY79b]. In order to be consistent with higher dimensions, we use the strong stability in every dimension $3 \leq n < 8$.

4. **Geometry of $(\mathbb{T}^{n-1} \times [0, 1])#M_0$**

In this section, we study the geometry of $(\mathbb{T}^{n-1} \times [0, 1])#M_0$. More specifically, we settle the non-existence of metrics with positive scalar curvature and minimal boundary, and non-existence of scalar-flat metrics with mean convex boundary on this manifold $(\mathbb{T}^{n-1} \times [0, 1])#M_0$. This non-existence result was essentially due to Gromov and Lawson [GL83]. For the convenience of the reader, we include our sketch of their proof. Note that their proof used minimal surfaces techniques and we use instead free boundary minimal surfaces. We then adopt an idea of [Loh99] to modify an asymptotic flat manifold with a noncompact boundary into a manifold of such form. By keeping track of the scalar curvature and the mean curvature, we can provide a proof of Theorem [3]. This proof is simpler in the sense that we are doing analysis on a compact manifold, and avoiding the analysis of asymptotic behaviors.

**Lemma 3.** Given any compact manifold $(M^n, g)$ with $R_g \geq 0$ in $M$ and $H_g \geq 0$ on $\partial M$. Then $g$ can be conformally changed to a metric $\tilde{g}$ satisfying...
$R_{\bar{g}} > 0$ everywhere in $M$ and $H_{\bar{g}} \equiv 0$ on $\partial M$ unless $M$ is Ricci flat with totally geodesic boundary.

Proof. We have two cases to consider.

Case i: $R_{\bar{g}} > 0$ somewhere in $M$ or $H_{\bar{g}} > 0$ somewhere on $\partial M$. The existence of $\bar{g}$ follows from Lemma 1 by considering the eigenvalue problem

$$\begin{cases}
L(M, g)u = \lambda u & \text{in } M,
B(M, g)u = 0 & \text{on } \partial M.
\end{cases}$$

Case ii: $R_{\bar{g}} \equiv 0$ in $M$ and $H_{\bar{g}} \equiv 0$ on $\partial M$. We consider a family of metrics $g_t = g + t\gamma$ with $\gamma$ to be chosen later. Note that $t = 0$, $R_0 \equiv 0$ and $H_0 \equiv 0$.

Suppose that $\lambda_t$ is the first eigenvalue of $L_t := L_{g_t}$ and $B_t := B_{g_t}$, by variational characterization of eigenvalues

$$\lambda_t = \frac{\int_M (|\nabla_t u_t|^2 + c_n R_t u_t^2) d\mu_t + 2c_n \int_{\partial M} H_t u_t^2 d\sigma_t}{\int_M u_t^2 d\mu_t}.$$ 

Note that $u_0 = 1$ and $\lambda_0 = 0$. We use the dot notation to denote differentiation with respect to $t$ and evaluation at $t = 0$. For instance, $\dot{\lambda} = \lambda'(0)$.

In differentiation, all terms drop out except for terms involving $\dot{R}$ and $\dot{H}$, hence

$$\dot{\lambda} = \frac{c_n}{\text{vol}(M, g)} \left[ \int_M \dot{R} d\mu + 2 \int_{\partial M} \dot{H} d\sigma \right]$$

$$= \frac{c_n}{\text{vol}(M, g)} \left[ \int_M \langle \text{Ric}_g \frac{1}{2} R_{\bar{g}g}, \gamma \rangle d\mu + \int_{\partial M} (A_{ij} - H_{\bar{g}g_{ij}}) \delta g^{ij} d\sigma \right]$$

$$= \frac{c_n}{\text{vol}(M, g)} \left[ \int_M \langle \text{Ric}_g, \gamma \rangle d\mu + \int_{\partial M} A_{ij} \delta g^{ij} d\sigma \right].$$

If $M$ is not Ricci flat, we choose $\gamma_{ij} = \varphi R_{ij}$ where $\varphi$ is a supported away from the boundary and positive in the sufficiently neighborhood around a point where $R_{ij}$ is not zero. This makes $\dot{\lambda} > 0$. If $M$ is Ricci flat, but the boundary is not totally geodesic i.e. $A_{ij} \neq 0$, choosing $\gamma$ such that $\delta g^{ij} = A^{ij}$ on the boundary will lead to $\dot{\lambda} > 0$. Therefore, we can do the same thing now as Case i. \hfill $\square$

A similar argument gives the following lemma,

**Lemma 4.** Assume that $(M^n, g)$ satisfies the conditions in Lemma 1, then the metric $g$ can be conformally changed to a metric with $R_{\bar{g}} \equiv 0$ in $M$ and $H_{\bar{g}} > 0$ everywhere on $\partial M$ unless $M$ is Ricci flat with totally geodesic boundary.

Proof. The proof is similar to the previous lemma, except that we consider the Steklov-type eigenvalue problem instead:

$$L(M, g)\phi = 0 \quad \text{in } M, \quad B(M, g)\phi = \lambda \phi \quad \text{on } \partial M$$

We omit the details. \hfill $\square$
In the following, we generalize Bochner’s theorem [Pet98, Chapter 7, Section 3] to manifolds with boundary. Denote
\[ H_T = \{ \omega \in \Lambda^1 M : d\omega = 0, \delta \omega = 0, \nu \cdot \omega = 0 \text{ on } \partial M \}, \]
\[ H_N = \{ \omega \in \Lambda^1 M : d\omega = 0, \delta \omega = 0, \nu^\flat \wedge \omega = 0 \text{ on } \partial M \}. \]

For more details, refer to a good exposition of Hodge-Morrey theory in [GMS98, Chapter 5].

**Lemma 5.** Given any compact manifold \((M^n, g)\) with nonnegative Ricci curvature \( \text{Ric}_g \geq 0 \) with boundary whose second fundamental form is non-negative, then every harmonic 1-form \( \omega \in H_T \cup H_N \) is parallel.

**Proof.** Let \( e_i \) be orthonormal frame where \( e_1 = \nu \) on \( \partial M \) and \( \theta^i \) be its dual frame. Recall the Bochner-Wietzenbock formula for \( \omega \),
\[ 0 = (d \delta + \delta d) \omega = \nabla^2_{e_i, e_i} \omega + \theta^i \wedge (e_j \lrcorner R(e_i, e_j) \omega) \]
where \( R(X, Y) = \nabla X \nabla Y - \nabla Y \nabla X - \nabla [X, Y] \).

Using integration by parts and direct calculation,
\[ 0 = \langle \Delta \omega, \omega \rangle = \int_M \langle \nabla^2_{e_i, e_i} \omega, \omega \rangle + \langle \omega, \theta^i \wedge (e_j \lrcorner R(e_i, e_j) \omega) \rangle = \int_{\partial M} \langle \nabla_\nu \omega, \omega \rangle - \int_M (\text{Ric}(\omega, \omega) + |\nabla \omega|^2). \]

If \( \omega \in H_N \), since \( \omega \) is a 1-form, we can assume that \( \omega = \phi \theta^1 \) on \( \partial M \). Since \( 0 = \delta \omega = \sum_{i=1}^n \theta^i \lrcorner \nabla_{e_i} \omega = \text{div}_M \omega \), we have
\[ \langle \nabla_\nu \omega, \omega \rangle = \phi \langle \nabla_\nu \omega, \nu \rangle = \phi \text{div}_M \omega - \sum_{i=2}^n \phi \langle \nabla_{e_i} \omega, e_i \rangle = -\phi^2 H_{\partial M} = -|\omega|^2 H_{\partial M} \]
and then
\[ -\int_{\partial M} H_{\partial M} |\omega|^2 - \int_M (\text{Ric}(\omega, \omega) + |\nabla \omega|^2) = 0. \]

If \( \omega \in H_T \), we have \( \omega^\flat \) is tangent to \( \partial M \), extend \( \nu \) to all of \( M \) such that \( \omega^\flat \) is orthogonal to \( \nu \) in an open neighborhood of \( \partial M \). Then along \( \partial M \),
\[ 0 = d\omega(\omega^\flat, \nu) = \omega^\flat(\omega(\nu)) - \nu(\omega(\omega^\flat)) - \omega(\nabla_\omega \nu - \nabla_\nu \omega^\flat) = -\langle \nabla_\omega \nu, \omega^\flat \rangle - \langle \nabla_\nu \omega, \omega^\flat \rangle = -A^\partial_M(\omega^\flat, \omega^\flat) - \langle \nabla_\nu \omega, \omega \rangle \]
will give
\[ - \int_{\partial M} A^{\partial M}(\omega^\sharp, \omega^\sharp) - \int_M (\text{Ric}(\omega, \omega) + |\nabla \omega|^2) = 0. \]
In either case, by nonnegativity of Ric and second fundamental form of \( \partial M \), it is necessary that \( |\nabla \omega| = 0 \) i.e. \( \omega \) is parallel. \( \square \)

In order to apply these lemmas, we still need that the connected sum \( (\mathbb{T}^{n-1} \times [0,1]) \# M_0 \) is not Ricci flat with totally geodesic boundary. We now denote the manifold \( (\mathbb{T}^{n-1} \times [0,1]) \# M_0 \) by \( (M^n, g) \). We assume the contrary i.e. \( \text{Ric}_g \equiv 0 \) and \( A_g \equiv 0 \). Since the boundary is totally geodesic, we can glue two copies of \( M \) along the boundary to get a Ricci flat manifold and reduce to the closed case. However, in line with previous lemmas, we apply again Hodge-Morrey theory with boundary.

**Lemma 6.** The manifold \( M \) is NOT Ricci flat with totally geodesic boundary.

**Proof.** Assume the contrary, we look at the degree 1 map from \( M \) to \( \mathbb{T}^{n-1} \times [0,1] \):
\[
\pi : M \to \mathbb{T}^{n-1} \times [0,1].
\]
Given the standard coordinates on \( \mathbb{T}^{n-1} \times [0,1] \), choose the form \( dx^i \), we pull it back using \( \pi \) to \( M \) i.e. \( \theta_i = \pi^* dx^i \), we have that \( \{ \theta_i \} \in H^1(M, \mathbb{Z}) \) for \( 2 \leq i \leq n \) and \( \theta_1 \) is a representative element of a relative necessary class in \( H^1(M, \partial M, \mathbb{Z}) \). Since the degree of \( \pi \) is 1,
\[
\int_M \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n = 1.
\]
Using Hodge-Morrey theory [GMS98, Chapter 5, Section 2], for \( 2 \leq i \leq n \), we can modify \( \theta_i \) to its harmonic representative \( \theta_i^H \in H_T \) and \( \{ \theta_i^H \} \in H^1(M, \mathbb{Z}) \). Similarly, we can modify \( \theta_1 \) to its harmonic representative \( \theta_1^H \in H_N \) and \( \{ \theta_1^H \} \in H^1(M, \partial M, \mathbb{Z}) \). By the previous lemma, these \( \theta_i^H \) are parallel. Let \( \theta_i = \theta_i^H + da_i \) where \( a_i \) are functions. Let \( i : \partial M \to M \) denotes the canonical injection, then along \( \partial M \), \( \nu \wedge \theta_i^H = 0, \ i^* \theta_i^H = 0 \) and \( i^* a_1 = a_1|_{\partial M} = 0 \). Since these forms \( \{ \theta_i \}, \{ \theta_i^H \} \) are closed, the difference
\[
\int_M \theta_1 \wedge \cdots \wedge \theta_n - \int_M \theta_1^H \wedge \cdots \wedge \theta_n^H
\]
by Stokes theorem can be transformed into integrals on the boundary. Every one of those integrals contains either \( i^* \theta_i^H \) or \( i^* a_i \), therefore vanishes. So
\[
\int_M \theta_1^H \wedge \cdots \wedge \theta_n^H = 1.
\]
This says that \( \{ \theta_i^H \} \) are non-trivial and form a parallel basis for the cotangent bundle \( T^*M \). So \( M \) is flat and this contradiction with our initial assumption finishes our proof. \( \square \)
4.1. Non-existence results. We now assume on the contrary that $M = (\mathbb{T}^{n-1} \times [0, 1]) \# M_0$ admits a metric $g$ with $R_g \geq 0$ and $H_g \geq 0$. By previous lemmas, we can further assume that $R_g$ is strictly positive.

Before we get to the proof, recall that a well known duality result in algebraic topology

**Theorem 5.** (Poincare-Lefschetz duality) Let $M$ be a manifold with boundary with fundamental class $z \in H_n(M, \partial M)$. Then the duality maps

\begin{align}
(7) \quad & D : H^k(M) \to H_{n-k}(M, \partial M) \\
\text{and} \quad & D : H^k(M, \partial M) \to H_{n-k}(M)
\end{align}

given by taking cap product with $z$ are both isomorphisms.

4.2. Proof of Theorem 3. Now we are ready to finish the proof of Theorem 3.

**Proof of Theorem 3.** There exists a map $\pi : M \to \mathbb{T}^{n-1} \times [0, 1]$ we choose standard normalized forms $dx^i$ and its pull back $\theta_i := \pi^* dx^i$ to $M$. The fundamental class $z \in H_n(M, \partial M, \mathbb{Z})$, then the cap product $[\theta_n] \lrcorner z$ is in $H_{n-1}(M, \partial M, \mathbb{Z})$ by Lefschetz duality and nonzero. We have the following minimization procedure,

$$|\Sigma| = \min \{|\Sigma_0| : \Sigma_0 \in H_{n-1}(M, \partial M, \mathbb{Z})\}$$

and

$$\int_\Sigma \theta_1 \wedge \theta_2 \cdots \wedge \theta_{n-1} = 1.$$ 

Note that $\Sigma$ is area minimizing, hence stable. When $4 \leq n < 7$, the corresponding Rayleigh quotient, the stability inequality (2) and Lemma 1 give a metric $\mathring{g}$ on $\Sigma$ such that $R_{\mathring{g}} > 0$ and $H_{\mathring{g}} = 0$.

Hence we have a $(n-1)$ dimensional manifold $(\Sigma^{n-1}, \mathring{g})$ whose scalar curvature is positive and boundary is minimal together with forms $\theta_1, \cdots, \theta_{n-1}$.

Inductively doing this, we get down to a compact oriented surface $\Sigma^2$ with at least two boundary components. This will lead to $\chi(\Sigma^2) \leq 0$. However, by taking $\varphi \equiv 1$ in the stability inequality (2),

$$2\pi \chi(\Sigma^2) = \int_{\Sigma^2} \frac{1}{2} R_{\Sigma^2} + \int_{\partial \Sigma^2} H_{\partial \Sigma^2} > 0$$

gives positive Euler characteristic. This is a contradiction. When $n = 3$, we get directly a surface $\Sigma^2$ with boundary and we apply stability (2) by inserting $\varphi \equiv 1$ directly. □
4.3. Another proof of Theorem 3.

Gromov and Lawson’s proof. In our last proof, we make use of (7) of Poincare-Lefschetz duality (Theorem 5). Also note that by Lemma 4, it is possible to deform the manifold $M$ into a scalar-flat manifold whose mean curvature of the boundary is strictly positive. In fact, we can give another proof using the other part of the duality (8).

As in Lemma 6, we have that $[\theta^i] \in H^1(M, \mathbb{Z})$ for $2 \leq i \leq n$ and $\theta_1$ is a representative element of a relative cohomology class in $H^1(M, \partial M, \mathbb{Z})$. Then $[\theta_1] \sim z \in H_{n-1}(M, \mathbb{Z})$, by the same reasoning, $[\theta_1] \sim z$ is not trivial.

We then have the following minimization procedure,

$$|\Sigma| = \min \{|\Sigma_0| : \Sigma_0 \in H_{n-1}(M, \mathbb{Z})\}$$

and

$$\int_{\Sigma} \theta_2 \wedge \ldots \wedge \theta_n = 1.$$ 

The existence of a minimizer is due to nontriviality of $H_{n-1}(M, \mathbb{Z})$ and that two boundaries of $M$ has strictly positive mean curvature thus acting as barriers. Then $\Sigma$ has no boundary and does not intersect the boundary $\partial M$.

Case $n = 3$:

By construction, on $\Sigma$ we have two closed 1-forms $\theta_2, \theta_3$ such that $\Sigma$ is dual to $\theta_2 \wedge \theta_3$ and that

$$\int_{\Sigma} \theta_2 \wedge \theta_3 = 1.$$ 

This fact leads to that $\Sigma$ has positive genus. Because otherwise $\Sigma$ was a 2-sphere, the closed 1-forms $\theta_2, \theta_3$ would have to be exact by cohomology of the 2-sphere. But then by the Stokes theorem, $\int_{\Sigma} \theta_2 \wedge \theta_3 = 0$.

If $n = 3$ by the stability inequality (2), we also have that the Euler characteristic

$$\chi(\Sigma^2) = \int_{\Sigma} \frac{1}{2} R_{\Sigma} + \int_{\partial \Sigma} H_{\partial \Sigma} = \frac{1}{2} \int_{\Sigma} R_{\Sigma} > 0$$

gives that $\Sigma^2$ has zero genus. This is a contradiction.

Case $4 \leq n < 8$:

Since $\Sigma$ has no boundary, we consider the eigenvalue problem

$$L_{\Sigma} \phi = \sigma \phi \quad \text{in } \Sigma.$$ 

Let the first eigenfunction be $u$. The function $u$ is positive on $\Sigma$.

Using the Rayleigh quotient

$$\sigma_1 = \frac{\int_{\Sigma} (|\nabla u|^2 + c_{n-1} R_{\Sigma} u^2)}{\int_{\Sigma} u^2},$$

the stability (2) (without integrals over the boundary) and that $c_{n-1} < \frac{1}{2}$, we see that $\sigma_1 > 0$. 

Let $\hat{g} = u^{\frac{4}{n-2}}g_{\Sigma}$ (note here that the dimension of $\Sigma$ is $n-1$) with this conformal change we have a closed 2-surface $(\Sigma, \hat{g})$, whose scalar curvature is positive with forms $\theta_2, \cdots, \theta_n$. This is the standard case. See for example the earlier work of Schoen and Yau on positive scalar curvature [SY79a].

Inductively doing this, we get down to a compact oriented surface $\Sigma^2$ just like when $n = 3$.

4.4. Relation with positive mass theorem. We recall the density theorem [ABdL16, Proposition 4.1]:

**Proposition 1.** Given any asymptotically flat manifold $(M^n, g)$ with a non-compact boundary, given $\epsilon > 0$, there exists metric $\tilde{g}$ such that

1. $(M^n, \tilde{g})$ is asymptotically flat.
2. $(M^n, \tilde{g})$ satisfies $R_{\tilde{g}} = 0$ and $H_{\tilde{g}} = 0$.
3. $\tilde{g}$ is conformally flat near infinity i.e. $\tilde{g}$ is of the form $u^{\frac{4}{n-2}}g_{\delta}$ near infinity with $\Delta u = 0$ in $\mathbb{R}^n_+$ and $\frac{\partial u}{\partial x_1} = 0$ on $\partial \mathbb{R}^n_+$ for $|x|$ large.
4. $|m_g - m_{\tilde{g}}| < \epsilon$.

We further modify $\tilde{g}$ as follows

**Proposition 2.** Suppose that $M$ is given as above, if $m_g < 0$, we can deform $\tilde{g}$ into $\bar{g}$ such that

1. $(M, \bar{g})$ is scalar flat with zero mean curvature boundary: $R_{\bar{g}} = 0$ and $H_{\bar{g}} = 0$.
2. $\bar{g}$ is exactly Euclidean outside a compact set.

**Proof.** The proof of the case of the standard asymptotically flat manifold as in [Loh99, Proposition 6.1] carries over. It is sufficient to take into consideration the fact that the functions $h$ and $f$ constructed in [Loh99, Lemma 6.2] satisfy $\frac{\partial h}{\partial x_1} = \frac{\partial f}{\partial x_1} = 0$ along the boundary. □

**Lohkamp style proof of the positive mass theorem** Theorem 2. We assume on the contrary that $m_g < 0$. By the last two propositions, we modify the metric $g$ into $\tilde{g}$. We take a large $\Lambda > 0$ such that the region $\{x \in M : |x_i| \geq \Lambda\}$ is Euclidean, we identify $\{x_i = \Lambda\}$ and $\{x_i = -\Lambda\}$ for all $2 \leq i < n$ and then cut off the region outside $\{x_1 > \Lambda\}$, we obtain a compact manifold $M$ with boundary $\partial M$ (with two components at least) with $R_M \geq 0$ and $H_{\partial M} \geq 0$, and at some point $R_M > 0$. Then we see that this contradicts the non-existences results of Theorem 3. Hence, we have yet another proof of the positive mass theorem. For the rigidity statement, see the article [ABdL16, Lemma 4.3, 4.4]. □

**Appendix A. Details of Computations**

A.1. Second variation of minimal hypersurfaces with free boundary.
Proof of Theorem 4. Let $F = F(x, t) : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a 1-parameter family of diffeomorphisms of $\Sigma$ induced by $X$. We consider coordinates $x^i$ near a point $p \in \Sigma$, let

$$e_i = dF \left( \frac{\partial}{\partial x^i} \right), \quad X = dF \left( \frac{\partial}{\partial t} \right).$$

We can further assume that $x^i$ form a normal coordinate system at $p \in \Sigma$, hence

$$g_{ij}(p, 0) = \delta_{ij} \quad \text{and} \quad \nabla e_i e_j(p, 0) = 0.$$ 

Now we calculate the variation of $d\text{vol}_\Sigma$. First, under local coordinates, we have

$$\partial_t \sqrt{g} = g^{ij} \langle \nabla e_i X, e_j \rangle \sqrt{g}.$$ 

We calculate the variation $\partial_t g^{ij}$ and $\partial_t \langle \nabla e_i X, e_j \rangle$.

$$\partial_t g^{ij} = -g^{il} g^{jk} \partial g_{kl} = -\partial_t g_{ij},$$ 

and similarly

$$\partial_t \langle \nabla e_i X, e_j \rangle = \langle \nabla_X e_i, e_j \rangle - \langle \nabla_X e_j, e_i \rangle - \langle \nabla_X X, e_i \rangle e_j.$$

Hence we have, evaluating at $(p, 0)$

$$\partial_t \sqrt{g} = \langle R(X, e_i)X, e_i \rangle + \langle \nabla_X \nabla e_i X, e_j \rangle.$$

Let $X = T + \varphi \nu$ and $Z = \nabla_X X = \hat{Z} + \phi \nu$, since $\Sigma$ is minimal, we have that $\text{div}(\chi \nu) = 0$ for any function $\chi$, so $\text{div} X = \text{div} T$ and $\text{div} Z = \text{div} \hat{Z}$. Calculating term by term

$$\langle R(X, e_i)X, e_i \rangle = \langle R(T, e_i)T, e_i \rangle + 2\varphi \langle R(T, e_i)\nu, e_i \rangle - \varphi^2 \text{Ric}(\nu)$$

and

$$\langle \nabla_X \nabla e_i X, e_j \rangle = \langle \nabla_X (T + \varphi \nu) \nu, e_i \rangle + \langle \nabla e_i (T + \varphi \nu), \nu \rangle - \varphi^2 \text{Ric}(\nu)$$

$$= \left[ [\nabla e_i T, \nu] + a e_i \varphi ] [ [\nabla e_i T, \nu] + \nabla e_i \varphi \right]$$

$$= (B(T, e_i))^2 - 2B(T, \nabla \varphi) + |\nabla \varphi|^2.$$
and
\[- \langle \nabla_{e_j} X, e_i \rangle \langle \nabla_{e_i} X, e_j \rangle \]
\[= - \langle \nabla_{e_j} (T + \varphi \nu), e_i \rangle \langle \nabla_{e_i} (T + \varphi \nu), e_j \rangle \]
\[= - \langle \nabla_{e_j} T, e_i \rangle + \varphi \langle \nabla_{e_j} \nu, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle + \varphi \langle \nabla_{e_i} \nu, e_i \rangle \rangle \]
\[= - \langle \nabla_{e_j} T, e_i \rangle \langle \nabla_{e_i} T, e_j \rangle - \varphi^2 B(e_i, e_j)^2 - 2\varphi \langle \nabla_{e_i} T, e_j \rangle B(e_i, e_j). \]

The density of the second variation can therefore be written
\[F_X = -\varphi^2 \text{Ric}(\nu) - \varphi^2 |B|^2 + |\nabla \varphi|^2 + G \]
where
\[G = [\langle R(T, e_i) T, e_i \rangle + (\text{div } X)^2 + B(T, e_i)^2 - \langle \nabla_{e_i} T, e_j \rangle \langle \nabla_{e_j} T, e_i \rangle] + \text{div } \hat{Z} + [2\varphi \langle R(T, e_i) \nu, e_i \rangle - 2B(T, \nabla \varphi) - 2\varphi \langle \nabla_{e_i} T, e_j \rangle B(e_i, e_j)] \]
\[= \text{div}(T \text{div } T - \nabla_T T) + \text{div } \hat{Z} - 2(\varphi B_{ij} T_i)_{ij} \]

The last line is the consequence of the following two identities.
\[\text{div}(T \text{div } T - \nabla_T T) \]
\[= \langle R(T, e_i) T, e_i \rangle + (\text{div } X)^2 + B(T, e_i)^2 - \langle \nabla_{e_i} T, e_j \rangle \langle \nabla_{e_j} T, e_i \rangle \]
and
\[- 2(\varphi B_{ij} T_i)_{ij} \]
\[= 2\varphi \langle R(T, e_i) \nu, e_i \rangle - 2B(T, \nabla \varphi) - 2\varphi \langle \nabla_{e_i} T, e_j \rangle B(e_i, e_j). \]

Indeed, we calculate as follows
\[\text{div}(T \text{div } T - \nabla_T T) \]
\[= \langle \text{div } T \rangle^2 + \nabla_T \langle \nabla_{e_i} T, e_i \rangle - \text{div}(\nabla_T T) \]
\[= \langle \text{div } T \rangle^2 + \langle \nabla_T \nabla_{e_i} T, e_i \rangle - \langle \nabla_{e_i} \nabla_T T, e_i \rangle + \langle \nabla_{e_i} T, \nabla_T e_i \rangle \]
\[= \langle \text{div } T \rangle^2 + [\langle R(T, e_i) T, e_i \rangle + \langle \nabla_{[T, e_i]} T, e_i \rangle + \langle \nabla_{e_i} T, e_j \rangle \langle \nabla_T e_i, e_j \rangle + \langle \nabla_{e_i} T, \nu \rangle \langle \nabla_T e_i, \nu \rangle] \]
\[= \langle R(T, e_i) T, e_i \rangle + (\text{div } X)^2 + B(T, e_i)^2 - \langle \nabla_{e_i} T, e_j \rangle \langle \nabla_{e_j} T, e_i \rangle. \]

And
\[- 2(\varphi B_{ij} T_i)_{ij} \]
\[= -2\varphi \langle B_{ij} T_i \rangle - 2\varphi B_{ij} T_{i;j} - 2\varphi B_{ij;j} T_i \]
\[= -2\varphi \langle B_{ij} T_i \rangle - 2\varphi B_{ij} T_{i;j} - 2\varphi R(T, e_i, e_j, \nu) \]

where we use the Gauss-Codazzi equation
\[B_{ij;j} = B_{ij} - \nabla_i H = B_{i;jj} - B_{jj;i} = R(e_i, e_j, e_j, \nu) \]
and the minimality of \( \Sigma. \) \( \square \)
A.2. Decay estimates of Minimal surface equations.

Lemma 7. Suppose outside a large enough compact set $K$, $f$ satisfies the minimal surface equation
\[
\sum_{ij} \left( \delta_{ij} - \frac{f_if_j}{1 + |\partial f|^2} \right) f_{,ij} + \frac{2(n-1)}{n-2} \sqrt{1 + |\partial f|^2} \frac{\partial}{\partial v_0} \log h = 0
\]
and on the boundary $\partial \Sigma \sim K$,
\[
\partial_1 f = 0
\]
with the decay
\[
|f| + |x'||\partial f| + |x'|^2 |\partial^2 f| = O(|x'|^{-\alpha})
\]
where $0 < \alpha < 1$. The we can improve this decay rate to
\[
f(x') = a_0 + a_1 \log |x'| + O(|x'|^{-1+\epsilon})
\]
if $n = 3$; and
\[
f(x') = f = a_0 + a_1 |x'|^{3-n} + O(|x'|^{2-n+\epsilon})
\]
if $n \geq 4$.

Proof. $f$ satisfies a simpler Poisson equation outside a compact set
\[
\Delta f = - \frac{f_if_j}{1 + |\partial f|^2} f_{,ij} - \frac{2(n-1)}{n-2} \sqrt{1 + |\partial f|^2} \frac{\partial}{\partial v_0} \log h =: g.
\]
Using the asymptotics of $h$, we see that $g = O(|x'|^{\max(-1-3\alpha,-n+1-\alpha)})$. We extend $g$ to all of $\mathbb{R}^{n-1}$. Using [ABdL16, Lemma A.1], we can solve the following PDE
\[
\left\{ \begin{array}{l}
\Delta w = g & \text{in } \mathbb{R}^{n-1}_+ \\
\partial_1 w = 0 & \text{on } \partial \mathbb{R}^{n-1}_+
\end{array} \right.
\]
where $w$ satisfies the bound
\[
w = O(|x'|^{\max(1-3\alpha,3-n-\alpha)+\epsilon})
\]
with any $\epsilon > 0$. The we have that $v := f - w$ satisfies for large $|x'|$ the following PDE
\[
\left\{ \begin{array}{l}
\Delta v = 0 & \text{in } \mathbb{R}^{n-1}_+ \\
\partial_1 v = 0 & \text{on } \partial \mathbb{R}^{n-1}_+
\end{array} \right.
\]
When $4 \leq n < 8$, as already proved in [ABdL16, Section 5], we see that
\[
v = a_0 + a_1 |x'|^{3-n} + O(|x'|^{2-n}).
\]
Combining (9) and (10), we obtain an improved decay rate for $f$ and this decay rate can be further improved to decay rates similar to that of $v$ by [ABdL16, Lemma A.1], i.e. for any given $\epsilon > 0$,
\[
f = a_0 + a_1 |x'|^{3-n} + O(|x'|^{2-n+\epsilon}).
\]
For the dimension three case, we replace the kernel $\Gamma(x, y) := \Gamma(x - y)$ in [MEY63] by
\[
\Gamma(x, y) = \log |x - y| + \log |x - \tilde{y}|,
\]
we then can proceed in the same way as in dimensions $4 \leq n < 8$, and we arrive the decay
\[ f = a_0 + a_1 \log |x'| + O(|x'|^{-1+\varepsilon}). \]
Hence we finish proving the decay estimates for $f$. 

Remark 2. Since by construction $f$ is actually bounded, in dimension 3, $a_1$ has to be 0, i.e.
\[ f = a_0 + O(|x'|^{-1+\varepsilon}). \]
Moreover, there is a slightly different case handle by Schoen [Sch83].

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