The Quantization of Geodesic Deviation.

Mark D. Roberts,

Department of Mathematics and Applied Mathematics, 
University of Cape Town, 
Rondebosch 7701, 
South Africa

roberts@gmunu.mth.uct.ac.za

March 24, 2022

Published: Gen.Rel.Grav. 28(1996)1385-1392.

Eprint: gr-qc/9903097

Comments: 9 pages, no diagrams, no tables, Latex2e.

3 KEYWORDS:

Geodesic Deviation: Spreading of the Wave Packet: one to many particle interpretation.

1999 PACS Classification Scheme:

http://publish.aps.org/eprint/gateway/pacslit

11.15q, 03.70+k, 04.60+n.

1991 Mathematics Subject Classification:

http://www.ams.org/msc

81T13, 83C10, 81T20, 83C45.
Abstract

There exists a two parameter action, the variation of which produces both the geodesic equation and the geodesic deviation equation. In this paper it is shown that this action can be quantized by the canonical method, resulting in equations which generalize the Klein-Gordon equation. The resulting equations might have applications, and also show that entirely unexpected systems can be quantized. The possible applications of quantized geodesic deviation are to: i) the spreading wave packet in quantum theory, ii) and also to the one particle to many particle problem in second quantized quantum field theory.

1 Introduction.

In relativity the path of a particle which only interacts with the gravitational field is given by a geodesic, and this can be quantized to give the Klein-Gordon equation. The paths of many particles which only interact with the gravitational field are described by the geodesic deviation equation. Both the geodesic and geodesic deviation equations have been derived from simultaneous actions, Bazanski (1977) [1]. How to quantize such systems is not immediately apparent as it is not clear which action to start with. The coordinate space action which produces the general geodesic and geodesic deviation equations has non-covariant associated momenta which depend on the Christoffel symbol and hence the coordinates. Quantization of a non-covariant system is not acceptable because the wave function would depend on the coordinates used. Instead we use a coordinate space action which has unit normalized momenta $p_\alpha p^\alpha = 1$ and is covariant; variations of the corresponding phase space action give the full non-normalized geodesic and geodesic deviation equations and thus describe the whole system. Quantization of the phase space action is carried out by the canonical method, Dirac (1963) [3].

There are at least three motives for studying the quantization of such actions, only the first is touched on here. The first is that the actions are gauge invariant systems with two gauge parameters, and as such provide a testing ground for the techniques for quantizing gauge systems.

The second is that in quantum theory the wave packet spreads, whereas in gravitational theory matter attracts. In Riemann geometry the Ricci identity relates the Riemann tensor to the commuted covariant derivatives of a vector field. The covariant derivative of a vector field can be decomposed into acceleration, expansion, shear and vorticity. If this decomposition is
inserted into the Ricci identity and then the resulting equations contracted and transvected one obtains Raychaudhuri’s equation, Hawking and Ellis (1973) p.84. Furthermore if an energy condition (the time-like convergence condition) is assumed, then all the terms (except the vorticity) show that the rate of expansion is negative: in other words the effect of gravitation is to make a fluid contract or a system of particles coalesce. On the other hand in non-relativistic quantum mechanics, as described by Schrödinger’s equation, see for example Schiff (1968) p.64-65; a wave packet can be constructed which is taken to correspond to a free non-interacting classical particle. As time increases the wave packet spreads. Thus for a system of particles described by classical gravitation there is a contraction of the system as opposed to a quantum non-relativistic single particle system where there is expansion. Now a quantization of geodesic deviation describes both many particles and is a quantum system - so would there be overall expansion or contraction of the wave function? There is a possibility that the two effects cancel out.

The third is that the Klein-Gordon equation arises from the quantization of a single particle, and in quantum field theory second quantization of this produces a many particle theory; Itzykson and Zuber (1985) p.110; from a physical point of view it would be more consistent to go from a many particle theory to a many particle theory. When this is done the wave function depends upon both $x$ and $r$, and so takes account of both the positions of the particles and there relative motion. In the approach used here assessment of the contribution of $x$ and $r$ to the wave function is hampered because the wave function is not separable into $x$ and $r$ dependent parts. There are higher deviation equations Bazanski (1977) and their quantization can be done by the same methods. All the deviation equations can be produced from a Taylor series expansion Bazanski (1976), and it might be that there is a single wave equation which incorporates the wave equation corresponding to all the deviation equations. The conventions used are: signature $+ - - -$, Riemann tensor $-2X_{\alpha[\beta\gamma]} = X^\delta R_{\delta\alpha\beta\gamma}$, $D$ and $\nabla$ signify covariant differentiation.
2 The Coordinate Space Action.

The simultaneous dynamic coordinate space action is discussed in Bazanski (1977) [1] eq.2.28 and is

$$2W = \int_{\tau_0}^{\tau_1} u_\alpha \frac{Dr_\alpha}{d\tau} d\tau.$$  \hspace{1cm} (1)

The gauge invariances are

$$\tau \to \tilde{\tau} = \tau + \epsilon_1, \quad \tau^\mu \to \tilde{\tau}^\mu = \tau^\mu + \epsilon_2 u^m u.$$  \hspace{1cm} (2)

The first Noether theorem implies

$$u_\alpha u^\alpha = k_1, \quad u_\alpha \frac{Dr^\alpha}{d\tau} = k_2.$$  \hspace{1cm} (3)

Varying the action

$$2\delta W = \int_{\tau_0}^{\tau_1} \left[ (\Delta u_\alpha) \frac{Dr_\alpha}{d\tau} + u_\alpha \Delta \left( \frac{Dr_\alpha}{d\tau} \right) \right] d\tau,$$  \hspace{1cm} (4)

then using

$$\Delta u^\alpha = \frac{D}{d\tau} \delta x^\alpha, \quad \Delta \frac{Dr_\alpha}{d\tau} = \frac{D}{d\tau} r^\alpha + R^\alpha_{\lambda \mu \nu} r^\lambda \delta x^\mu u^\nu,$$  \hspace{1cm} (5)

gives

$$2\delta W = \int_{\tau_0}^{\tau_1} d\tau \left[ \frac{D\delta x_\alpha}{d\tau} \frac{Dr_\alpha}{d\tau} + u_\alpha \frac{D}{d\tau} \Delta r^\alpha + u_\alpha R^\alpha_{\lambda \mu \nu} r^\lambda \delta x^\mu u^\nu \right]$$

$$= \int_{\tau_0}^{\tau_1} d\tau \left[ \frac{D^2r_\alpha}{d\tau^2} + u_\gamma R^\gamma_{\lambda \mu \nu} r^\lambda u^\nu \right] \delta x^\alpha + \frac{Du_\alpha}{d\tau} \Delta r^\alpha$$

$$+ \left( u_\alpha \Delta r^\alpha + \frac{Dr_\alpha}{d\tau} \delta x_\alpha \right) \bigg|_{\tau_0}^{\tau_1},$$  \hspace{1cm} (6)

Thus

$$\frac{2\delta W}{\Delta r^\alpha} = \frac{Du_\alpha}{d\tau},$$  \hspace{1cm} (7)

and

$$\frac{2\delta W}{\delta x_\alpha} = \frac{D^2r^\alpha}{d\tau^2} - R^\alpha_{\beta \gamma \delta} u^\beta R^\gamma u^\delta.$$  \hspace{1cm} (8)
Taking these variations to vanish gives the geodesic equation and the geodesic deviation equation respectively. The conjugate Lagrangian momenta are

\[ p_\alpha = \frac{\partial L}{\partial r^\alpha} = u_\alpha, \quad \pi_\alpha = \frac{\partial L}{\partial u^\alpha} = \frac{Dr_\alpha}{d\tau}. \] (9)

Using these the first integrals derived from the first Noether theorem can be rewritten as

\[ \sigma_1 \equiv p_\alpha p^\alpha - m^2 = 0, \quad \sigma_2 \equiv p_\alpha \pi^\alpha - l^2 = 0, \] (10)

where \( k_1 = m^2 \) and \( k_2 = l^2 \). These expressions for the momenta and first integrals suggest the extended phase space action given in the next section.

3 The Phase Space Action.

The extended phase space action which generalizes the action of Pavsic (1987) \( \mathfrak{F} \) for geodesics is

\[ I = \int_{\tau_0}^{\tau_1} d\tau \left[ p_\alpha \frac{Dr^\alpha}{d\tau} + u_\alpha \pi^\alpha + \lambda_1 (p_\alpha p^\alpha - m^2) + \lambda_2 (p_\alpha \pi^\alpha - l^2) \right]. \] (11)

Varying with respect to \( \lambda_1, \lambda_2, \pi^\alpha, p^\alpha \) is straightforward

\[ \frac{\delta I}{\delta \lambda_1} = p_\alpha p^\alpha - m^2, \quad \frac{\delta I}{\delta \lambda_2} = p_\alpha \pi^\alpha - l^2, \]

\[ \frac{\delta I}{\delta \pi^\alpha} = u_\alpha + \lambda_2 p_\alpha, \quad \frac{\delta I}{\delta p_\alpha} = \frac{Dr^\alpha}{d\tau} + 2\lambda_1 p^\alpha + \lambda_2 \pi^\alpha. \] (12)

Varying with respect to \( r^\alpha \) and \( x^\alpha \)

\[ \delta I_{(\text{rem.})} = \int_{\tau_0}^{\tau_1} d\tau \left[ p_\alpha \Delta \frac{Dr^\alpha}{d\tau} + \pi^\alpha \Delta u^\alpha \right], \] (13)

again using the expressions \( \mathfrak{F} \) gives

\[ \delta I_{(\text{rem.})} = \int_{\gamma_0}^{\gamma_1} d\tau \left[ p_\alpha \frac{D\Delta r^\alpha}{d\tau} + p_\gamma R^\gamma_{\alpha\nu\lambda} r^\lambda \delta x^\alpha u^\nu + \pi_\alpha \frac{D\Delta x^\alpha}{d\tau} \right] \]

\[ = \int_{\gamma_0}^{\gamma_1} d\tau \left( \left[ \frac{D\pi_\alpha}{dt} + p_\gamma R^\gamma_{\alpha\nu\lambda} r^\lambda u^\nu \right] \delta x^\alpha l + \frac{Dp_\alpha}{d\tau} \Delta r^\alpha \right) \]

\[ + (p_\alpha \Delta r^\alpha + \pi_\alpha \delta x^\alpha) \bigg|_{\tau_0}^{\tau_1}. \] (14)
Thus the remaining variations give

\[
\frac{\delta I}{\delta r^\alpha} = \frac{Dp_\alpha}{d\tau}, \quad \frac{\delta I}{\delta x^\alpha} = \frac{d\pi_\alpha}{d\tau} - R_{\alpha\beta\gamma\delta}u^\beta r^\gamma p^\delta.
\]

(15)

Setting these variations \[\text{12}\] and \[\text{15}\] equal to zero gives six equations, the first four can be used to give expressions for \(\lambda_1, \lambda_2, \pi_\alpha, p_\alpha\) thus:

\[
\lambda_1 = \pm \sqrt{\frac{u_\gamma u^\gamma}{2m}} \left( \frac{u_\gamma r^\gamma}{u_\gamma u^\gamma} - \frac{l^2}{m^2} \right), \quad \lambda_2 = \pm \sqrt{\frac{u_\gamma u^\gamma}{m}},
\]

\[
\pi^\alpha = \pm \frac{m}{\sqrt{u_\gamma u^\gamma}} \left( h^\alpha_{\beta\gamma} r^\gamma + \frac{l^2}{m^2} u^\alpha \right), \quad p^\alpha = \pm \frac{mu^\alpha}{\sqrt{u_\gamma u^\gamma}},
\]

(16)

where \(h^\alpha_{\beta\gamma}\) is the projection tensor

\[
h^\alpha_{\beta\gamma} = g^\alpha_{\beta\gamma} - \frac{u_\alpha u_\beta u_\gamma u^\gamma}{u_\gamma u^\gamma},
\]

(17)

and the \(\pm\) arises from taking the square root when solving for \(\lambda_2\), using the \(\delta \pi_\alpha\) and \(\delta \lambda_1\) equations, the lower sign should be used if the momentum and velocity are to be co-directional. These four equations \[\text{16}\] can be substituted into the two remaining equations \[\text{15}\] to give the general non-normalized geodesic and geodesic deviation equations in the form of Bazanski (1977) [1] eq.1.6 and 2.4

\[
\frac{D}{d\tau} \frac{mu^\alpha}{\sqrt{u_\gamma u^\gamma}} = 0,
\]

\[
\frac{D}{d\tau} \left( \frac{h^\alpha_{\beta\gamma}}{\sqrt{u_\gamma u^\gamma}} \frac{Dr^\gamma}{d\tau} \right) - \frac{1}{\sqrt{u_\gamma u^\gamma}} R_{\alpha\beta\gamma\delta} u^\beta r^\gamma p^\delta = 0,
\]

(18)

The Poisson brackets for variables \(A\) and \(B\) at equal \(\tau\) are defined by

\[
\{A, B\} = \frac{\Delta A \Delta B}{\partial r^\gamma \partial p^\gamma} + \frac{\Delta A \Delta B}{\partial x^\gamma \partial \pi^\gamma} - \frac{\Delta A \Delta B}{\partial \pi^\gamma \partial x^\gamma},
\]

(19)

where the \(\nabla / \partial x^\gamma\) signifies that covariant derivatives must be taken. The total Hamiltonian is

\[
H = \pm \frac{\sqrt{u_\gamma u^\gamma}}{2m} \left[ \left( \frac{u_\gamma r^\alpha}{u_\gamma u^\gamma} - \frac{l^2}{m^2} \right) (p_\beta r^\beta - m^2) + 2(p_\alpha \pi^\alpha - l^2) \right],
\]

(20)
Taking the two $\lambda$ variations of the extended phase space action to vanish implies that the two terms in the total Hamilton vanish separately. The Hamiltonian equations of motion are
\[
\dot{x}_\alpha = \{x_\alpha, H\} = \frac{\Delta H}{\partial \pi^\alpha}, \quad \dot{p}_\alpha = \{p_\alpha, H\} = -\frac{\Delta H}{\partial r^\alpha},
\]
where the dot as in $\dot{p}_\alpha$, again signifies that covariant derivatives are taken, for example $\dot{\pi}_\alpha = \frac{D\pi_a}{d\tau}$. The last equation is proved here, the others are similar and simpler. $\delta/\partial x^\alpha$ acting on $u$ and $p_\alpha$ vanishes, thus the equation reduces to $\dot{\pi}_\alpha = \mp u_\gamma u^\gamma m^{-1} \Delta \pi_\alpha/\partial x^\beta p^\beta$, using the expression for $p_\alpha$ and that $u_\beta \pi^{\alpha}_{\beta} = D\pi_\alpha/d\tau$ gives $\dot{\pi}_\alpha = D\pi_\alpha/d\tau$. Conjugate $(p, x)$ and $(\pi, r)$ in the Poisson bracket the Hamiltonian equations of motion are not recovered, as can be seen immediately from the equation for $\dot{x}_\alpha$.

4 Quantization.

The gauges could be fixed by introducing
\[
\sigma_3 = x^I - \tau = 0, \quad \sigma_4 = p_\alpha r^\alpha = 0,
\]
and then calculating $C_{\alpha\beta} = \{\sigma_\alpha l, \sigma_\beta l\}$ in order to produce Dirac brackets, but here just Poisson brackets are used. The coordinates and momenta in phase space obey
\[
\{p_\alpha, r^\beta\} = \{\pi_\alpha, x^\beta\} = -\delta^\beta_\alpha.
\]
To quantize the system the Poisson brackets are replaced by Heisenberg brackets, applied to this implies the operator substitutions
\[
p_\alpha \rightarrow -\frac{i\hbar \nabla}{\partial r^\alpha}, \quad \pi_\alpha \rightarrow -\frac{i\hbar \nabla}{\partial x^\alpha}.
\]
Assuming that both terms in the total Hamiltonian vanish separately these operator substitutions give
\[
\left(\frac{\nabla}{\partial r^\alpha} \frac{\nabla}{\partial r_\alpha} + \frac{m^2}{\hbar^2}\right) \psi = 0, \quad \left(\frac{\nabla}{\partial x^\alpha} \frac{\nabla}{\partial x_\alpha} + \frac{l^2}{\hbar^2}\right) \psi = 0,
\]
where the wave function $\psi$ is dependent on both $x^\alpha$ and $r^\alpha$. The operator substitutions are not applied to equation for example in the quantization.
of just a free particle the geodesic equation can be written as \( u^\beta p_{\alpha;\beta} = 0 \) or \( p^\beta p_{\alpha;\beta} = 0 \), and the operator substitutions would give \( u^\beta \psi_{\alpha;\beta} = 0 \) or \( \psi_\beta R^\beta_\alpha + m^2 \psi_\alpha = 0 \), the first of these gives an unusual restriction on the wave function, the flat space limit shows that the second of these is incorrect; because the operator substitutions are not applied to the Riemann tensor does not explicitly occur in the differential equation for the wave function. Taking

\[
\psi = A \exp\left(\frac{i}{\hbar}S\right),
\]

(26)

and multiplying by \( \hbar^2/\psi \), (25) becomes

\[
\begin{align*}
&i\hbar S_{r\alpha} r^\alpha - S_{r\alpha} S^{r\alpha} + m^2 = 0, \\
&i\hbar S_{x\alpha} x^\alpha - S_{x\alpha} S^{x\alpha} + l^2 = 0,
\end{align*}
\]

(27)

noting from ref eq:3.4 that the principle Hamiltonian \( S \) obeys \( p_\alpha = \nabla S/\partial r^\alpha \) and \( \pi_\alpha = \nabla S/\partial x^\alpha \) shows that \( \sigma_1 \) and \( \sigma_2 \) in equation (11) are recovered in the limit \( \hbar \to 0 \). Defining

\[
S \equiv r^\gamma U_\gamma + V,
\]

(28)

where \( U \) and \( V \) are functions of only \( x^\alpha \), shows that (27) can be written in the form

\[
\begin{align*}
u_\alpha u^\alpha &= m^2, \\
-i\hbar U_\alpha + U_\alpha V^\alpha &= l^2.
\end{align*}
\]

(29)

No general separation of the wave function \( \psi \) into \( x^\alpha \) and \( r^\alpha \) dependent parts is known; however a particular case is obtained by defining

\[
\psi = A \exp\left(\frac{r^\beta r^\gamma \phi_{\beta\gamma}}{\phi}\right),
\]

(30)

where \( \phi \) is a function of \( x^\alpha \) only, then using the quantum analog of \( p_\alpha r^\alpha = 0 \) given by

\[
r^\alpha \phi_\alpha = a,
\]

(31)

where \( a \) is a constant, and \( l = 0 \), the first equation in (25) reduces to the Klein-Gordon equation

\[
\phi_\alpha + \frac{m^2}{\hbar^2} \phi = 0,
\]

(32)

and the second vanishes identically.
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