More On Critical Collapse of Axion-Dilaton System in Dimension Four

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Abstract

We complete our previous study of critical gravitational collapse in the axion-dilaton system by analysing the hyperbolic and parabolic ansätze. As could be expected, the corresponding Choptuik exponents in four-dimensions differ from the elliptic case.
1 Introduction

Choptuik scaling ([1], for reference on the relevant literature, see [2]) is a remarkable property in many cases of gravitational collapse that was discovered twenty years ago. A system that experiences gravitational collapse is the axion-dilaton system motivated in part by String Theory. The study of Choptuik scaling in such a system was started in [3, 4, 5]. In this paper we complete the extension of these papers started in [6], where we considered the elliptic case (see section 2) in four and higher dimensions. In that paper we also argued the existence of two more possibilities to satisfy the conditions of continuous self-similarity, but we did not perform the relevant computations to determine their corresponding critical exponents. This is what we do in this paper in the special case of four-dimensions, just to show that criticality also appears in those cases.

This paper is organized as follows. For convenience we briefly describe the axion/dilaton system and the different continuous self-similar ansätze in section two, where we also write down the equations of motion in four-dimensions and the initial conditions in each case. In section three we present the critical solutions in the hyperbolic and parabolic cases in some detail, and finally in section four we carry out the necessary perturbations to obtain the Choptuik exponent in each case. A number of appendices are dedicated to some technical details needed in the numerical analysis. More details and references can be found in [6].

2 The axion/dilaton system

The axion $a$ and dilaton $\phi$ field can be combined into a single complex field $\tau \equiv a + ie^{-\phi}$, its dynamics and coupling to gravity is described by the action:

$$ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \frac{\partial_a \tau \partial^a \bar{\tau}}{(\text{Im} \tau)^2} \right). $$

where $R$ is the scalar curvature. The equations of motion are:

$$ R_{ab} - \frac{1}{4(\text{Im} \tau)^2} (\partial_a \tau \partial_b \bar{\tau} + \partial_a \bar{\tau} \partial_b \tau) = 0 \quad (2) $$

$$ \nabla^a \nabla_a \tau + \frac{i \nabla^a \tau \nabla_a \bar{\tau}}{\text{Im} \tau} = 0. \quad (3) $$

As in our previous work [6] we look for critical solutions by assuming spherical symmetry and continuous self-similarity (CSS). Following [3, 4, 5] the metric is taken to be:

$$ ds^2 = (1 + u(t,r)) \left( -b(t,r)^2 dt^2 + dr^2 \right) + r^2 d\Omega^2. \quad (4) $$
We define a scale invariant variable as $z \equiv -r/t$ so CSS means that the dimensionless functions $u(t, r), b(t, r)$ in the metric are expressed only in terms of $z$: $b(t, r) = b(z), u(t, r) = u(z)$.

The CSS condition for $\tau$ was considered in detail in [6]. The axion-dilaton Lagrangian has a global $SL(2, R)$-symmetry (it is broken to a $SL(2, Z)$-subgroup by non-perturbative phenomena, see for instance [7]) hence we can compensate the action of the homothety vector field $(\xi = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r})$ by means of an $SL(2, R)$-transformation. There are in fact three possible ansätze, depending on whether the $SL(2, R)$-transformation chosen is elliptic, hyperbolic or parabolic. The elliptic ansatz is given by:

$$\tau(t, r) = i \frac{1 - (-t)^{i\omega} \mathcal{f}(z)}{1 + (-t)^{i\omega} \mathcal{f}(z)},$$

(5)

where $\omega$ is a real constant determined by regularity conditions of the critical solution. The critical solutions and their exponents were explored in [6] from four to ten dimensions.

The second ansatz is called hyperbolic and is given by:

$$\tau(t, r) = \frac{1 - (-t)^{\omega} \mathcal{f}(z)}{1 + (-t)^{\omega} \mathcal{f}(z)},$$

(6)

finally the parabolic ansatz is simply: $\tau(t, r) = \mathcal{f}(z) + \omega \log(-t)$.

### 2.1 Equations of motion and initial conditions in four dimension

In [6] one can find the equations of motion in dimensions four up to ten for all three ansätze. Here for convenience we reproduce them for the hyperbolic and parabolic cases in four dimension. Having used spherical symmetry, there are no gravitational degrees of freedom left, thus $u(z), b(z)$ should be re-expressed just in terms of $f(z)$. If we use the Einstein equations for the angular variables we obtain:

$$u(z) = -\frac{z b'(z)}{b(z)}.$$  

(7)

The other equations of motion include $b(z), f(z)$ but the equation for $b(z)$ can be used to express this function in terms of $f(z), f'(z)$. The time coordinate is chosen so that the gravitational collapse takes place at $t = 0$. Using the scaling properties of the metric (1) we can set regular under time scaling so that by making use of this invariance $b(t, 0) = 1$ for $t < 0$. Regularity at the origin also implies $u(t, 0) = 0$.

$^2$Its value cannot vanish $\omega = 0$ otherwise we just get the trivial solution $f(z) = constant, b(z) = 1$. 
The equations of motion for the four dimension for hyperbolic case are:

\[
0 = \frac{b'}{b(f - f')^2} f' f'' + \frac{\omega (b^2 - z^2)}{b (f - f')^2} (f f' + \bar{f} f') + \frac{\omega^2 z |f|^2}{b(f - f')^2}
\]
\[
0 = -f'' - \frac{z(b^2 + z^2)}{b^2(f - f')^2} f'^2 f' + \frac{2}{(f - f') \left( 1 + \frac{\omega (b^2 + z^2)}{2 b^2 (f - f')} \right)} f'^2 + \frac{\omega (b^2 + 2 z^2)}{b^2 (f - f')^2} f f' \\
\quad + \frac{2}{z} \left( -1 + \frac{\omega z^2 (f + \bar{f})}{(b^2 - z^2) (f - f')} + \frac{\omega^2 z^4 |f|^2}{b^2 (f - f')^2} \right) f' - \frac{\omega^2 z}{b^2 (f - f')^2} f'^2 f \\
\quad + \frac{\omega}{(b^2 - z^2)} \left( -1 - \frac{\omega (f + \bar{f})}{(f - f')} + \frac{\omega^2 z^2 |f|^2}{b^2 (f - f')^2} \right) f .
\]

We find it convenient to represent split \( f(z) \) into its real and imaginary parts:

\[ f(z) = u(z) + iv(z). \]

These equations are invariant under a constant scaling \( f \to \lambda f \), therefore we have freedom to choose the real value of \( f(z) (u(z)) \) or its imaginary part \( (v(z)) \) as we wish at a particular value of \( z \). We use this freedom to set \( u(0) = 1 \). Requiring regularity at the origin we find that \( f'(z = 0) \) should vanish. Thus the initial conditions for the hyperbolic case are:

\[ b(0) = u(0) = 1, u'(0) = v'(0) = 0 \]

For the parabolic case the equations of motion in four dimensions are:

\[
0 = \frac{b'}{b(f - f')^2} f' f'' - \frac{\omega (z^2 - b^2)}{b (f - f')^2} (f' + f') + \frac{\omega^2 z}{b(f - f')^2}
\]
\[
0 = -f'' - \frac{z(b^2 + z^2)}{b^2(f - f')^2} f'^2 f' + \frac{2}{(f - f') \left( 1 + \frac{\omega (b^2 + z^2)}{2 b^2 (f - f')} \right)} f'^2 + \frac{\omega (b^2 + 2 z^2)}{b^2 (f - f')^2} f f' \\
\quad + \frac{2}{z} \left( -1 + \frac{\omega z^2}{(b^2 - z^2) (f - f')} + \frac{\omega^2 z^4}{b^2 (f - f')^2} \right) f' - \frac{\omega^2 z}{b^2 (f - f')^2} f'^2 f \\
\quad + \frac{\omega}{(b^2 - z^2)} \left( -1 - \frac{\omega}{(f - f')} + \frac{\omega^2 z^2}{b^2 (f - f')^2} \right) f .
\]

we write again \( f(z) = u(z) + iv(z) \) these equations are independent of \( u(z) \), and a similar analysis to the the previous case leads to the initial conditions:

\[ b(0) = 1, u'(0) = v'(0) = u(0) = 0. \]
In this case, the equations of motion are invariant under shifts of $f(z)$ by a real number. These conditions (and those of the hyperbolic case) can also be imposed in any number of dimensions to look for critical solutions.

In the elliptic case, writing $f(z) = f_m(z)e^{if_a(z)}$, the regularity conditions imply:

$$b(0) = 1, f'_m(0) = f'_a(0) = f_a(0) = 0$$

3 Properties of the critical solutions in the hyperbolic and parabolic cases in four dimension

In this section we discuss briefly the properties of the critical hyperbolic and parabolic solutions. We follow [4, 5].

In (8), (9), (10), (11) we do have five singular points, $z = \pm 0$ corresponds to origin of polar coordinates where we have studied the regularity conditions. The point $z = \infty$ describes the surface $t = 0$. The easiest way to study the neighbourhood of $z = \infty$ is to use a change of variables and a redefinition of the fields $f(z), b(z)$ [5]. This is discussed in detail in the Appendices.

The singularities $b(z_{\pm}) = \pm z_{\pm}$ correspond to the surfaces where the homothetic Killing vector becomes null. They are related to the backward (forward) light cones of the space-time origin. For $b(z_{+}) = z_{+}$ the solution should be smooth across this surface. However the forward cone $b(z_{-}) = -z_{-}$ represents the Cauchy horizon of the space-time and we should not require more than continuity of $f, b$ in this region. We need to impose smoothness of the space-time just a bit below the forward cone and then extend the solution by continuity.

Using regularity at the origin and at $z_{+}$ and using the initial conditions described, the critical solution is determined by the following four parameters (in both hyperbolic and parabolic cases):

$$|v(0)|, \omega, z_{+}, |v(z_{+})|$$

We explain our procedure to determine these parameters in the hyperbolic case in detail, and just give the results in the parabolic case.

Taylor expanding at the origin, and using the regularity conditions we obtain:

$$b(z) = 1 + \frac{\omega^2(1 + v(0)^2)}{12(v(0)^2)}z^2 + O(z^4)$$
\[ v(z) = v(0) + \frac{\omega(\omega - v(0)^2)}{3v(0)} z^2 + O(z^4) \]

\[ u(z) = 1 + \frac{-\omega(1 + \omega)}{3} z^2 + O(z^4) \] (13)

To obtain the remaining values of \( \omega, v(0) \), we integrate out from the origin to positive values of \( z \) and we also integrate in from \( z_+ \) towards the origin. We also Taylor expand at \( z_+ \) and impose regularity. Matching the two solutions at an intermediate point and requesting continuity in the functions and their first derivatives, completely determines all parameters in the critical solution. The solution for hyperbolic is given by the parameters:

\[
\begin{align*}
\omega &= 1.127, \\
 z_+ &= 1.561, \\
|v(0_+)| &= 0.397, \\
|v(z_+)| &= 1.016.
\end{align*}
\] (14)

In the parabolic case we obtain:

\[
\begin{align*}
\omega &= 1.200, \\
 z_+ &= 1.601, \\
|v(0_+)| &= 0.321, \\
|v(z_+)| &= 1.071
\end{align*}
\] (15)

We used rather moderate precision to determine the parameters in the solution. If one wants to determine \( z_-, f(z_-) \) as well, the precision has to be increased substantially as a consequence of the fact that the solution is relatively flat in the forward light-cone. Some details are given in Appendix A in the elliptic case.

## 4 Choptuik exponent in the hyperbolic case

We follow the standard methods to compute the critical exponent [2, 4, 6]. We perturb the critical solution:

\[ f(z, t) = f_{ss}(z) + \epsilon |t|^{-\kappa} f_{\text{pert}}(z), \]

to obtain a set of linear equations for the perturbations with an eigenvalue equation for \( \kappa \). By taking the biggest value for \( \text{Re}(\kappa) \), the critical exponent can be found as \( \gamma = \frac{1}{\text{Re}(\kappa)}. \)
The linear equations have singular points in the same places as the critical solution. Once regularity is imposed we can determine completely the most relevant mode and the value of \( \kappa \).

We give a few details in what follow on the method we use to derive the perturbation equations. First, we keep the general form of the metric in terms of the functions \( b(t, r) \) and \( u(t, r) \) in the equations of motion. In addition, we find it convenient to write for the axion-dilaton field:

\[
\tau(t, r) = \frac{1 - f(t, r)}{1 + f(t, r)}
\]

Then we introduce the variations of the fields \( u(t, r), b(t, r) \) and \( \tau(t, r) \), as follows:

\[
\delta u(t, r) = (-t)^{-\kappa} u_1(t, r), \quad \delta b(t, r) = (-t)^{-\kappa} b_1(t, r)
\]

\[
\delta \tau(t, r) = \frac{-2\delta f(t, r)}{(1 + f(t, r))^2}, \quad \delta f(t, r) = (-t)^{\omega - \kappa} f_1(t, r),
\]

where \( u_1(t, r), b_1(t, r) \) and \( f_1(t, r) \) are the perturbations. The CSS ansatz for the critical solution and the perturbations are:

\[
b(t, r) = b(-r/t), \quad u(t, r) = u(-r/t)
\]

\[
\delta u(t, r) = (-t)^{-\kappa} u_1(-r/t), \quad \delta b(t, r) = (-t)^{-\kappa} b_1(-r/t)
\]

\[
\tau(t, r) = \frac{1 - (-t)^{\omega} f(-r/t)}{1 + (-t)^{\omega} f(-r/t)}, \quad \delta \tau(t, r) = -2 \frac{(-t)^{\omega - \kappa} f_1(-r/t)}{(1 + (-t)^{\omega} f(-r/t))^2}.
\]

The perturbed equations are rather cumbersome, and there is not much point in spelling them out explicitly here. Some details appear in Appendix D. As with the critical solution, some algebraic manipulations allow us to determine \( u_1, u', u, u' \), in terms of \( b_1, f_1 \).

As in the critical case, \( b_1(z) \) is determined in terms of \( f_1(z) \) and the critical solution. The linear equation for \( f_1(z) \) has the same singularities as the the unperturbed equations at \( z = 0, z = \infty \) and at \( z_{\pm} \).

We do require that the perturbations be smooth at the singularities of the original equations and this provides all the perturbations up to some re scalings.

Thus we found a set of linear equations for the perturbations which have solutions for some values of \( \kappa \). The solution with the biggest value of \( Re(\kappa) \) corresponds to Choptuik exponent. As usual:

\[
\gamma = \frac{1}{Re(k)}.
\]
Therefore by analyzing the perturbations appeared in Appendix D, we obtain in the hyperbolic case

\[ \gamma = 0.363 \]  \hspace{1cm} (16)

5 Choptuik exponent in the parabolic case

In this section we simply mention the differences with the previous case. The ansatz for the critical solution and the perturbation are in this case:

\[
\begin{align*}
 b(t, r) &= b(-r/t), \quad u(t, r) = u(-r/t) \\
 \delta u(t, r) &= (-t)^{-\kappa} u_1(-r/t), \quad \delta b(t, r) = (-t)^{-\kappa} b_1(-r/t) \\
 \tau(t, r) &= f(-r/t) + \omega \log(-t), \quad \delta \tau(t, r) = (-t)^{-\kappa} f_1(-r/t).
\end{align*}
\]

Following the same steps as in the hyperbolic case, we find the following value for the critical exponent:

\[ \gamma = 0.324 \]  \hspace{1cm} (17)

Just for comparison, we recall that the value of \( \gamma \) for the elliptic case is \[ 4 \] \( \gamma = 0.264 \).

This shows clearly that the exponent in the three cases are different. It is important to point out that it would be useful to use more powerful numerical methods in order to get more accurate expressions for the exponents. Our interests was to use enough precision so as to distinguish the exponents in the parabolic, hyperbolic and elliptic cases \[ 9 \]. As remarked in that paper, if instead of assuming CSS one only requires discrete self-similarity, the discrete scale transformation can then be compensated by an element of \( SL(2, \mathbb{Z}) \in SL(2, \mathbb{R}) \). It would be interesting to know how the critical exponents depend on the modular transformations.

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Appendix A : The change of variables at infinity and the solutions for the elliptic case

Here we show how using field redefinitions we can study the equations at \( z = \infty \). Using the representation \( f(z) = f_m(z)e^{if_a(z)} \) we can divide e.o.m’s to the modulus and phase parts, and find their leading behaviour at infinity. By replacing \( f_m(z) = uz^k \) and \( f_a(z) = vz^s \) in the equations of motion we obtain near infinity: \( f(z) = z^{-i\omega}e^{if_0} \). Thus the equations of motion for the elliptic case become regular there by the following change of variables:

\[
\begin{align*}
dw &= b(z) \frac{dz}{z^2}, \quad w = 0 \quad \text{at} \quad z = \infty, \\
F(w) &= z^{-i\omega}f(z), \\
v(w) &= \frac{b(z)}{z}, \\
u(w) &= u(z).
\end{align*}
\] (18)

In four dimensions it is shown in [5] how to analyze the equations of motion in terms of new independent variable \( w \). It is straightforward to extend their arguments to five dimensions, and we find:

\[
0 = v' + \frac{2(v^2 - 1)}{3(1 - |F|^2)^2}F'F'' + 1 - \frac{2\omega^2|F|^2}{3(1 - |F|^2)^2},
\] (19)

\[
0 = F'' - \frac{2(2vF' + i\omega F)}{3(1 - |F|^2)^2}F'F'' + \frac{2F^2}{1 - |F|^2} + \frac{v}{(v^2 - 1)} \left( 1 + \frac{2i\omega(1 + |F|^2)}{(1 - |F|^2)^2} + \frac{4\omega^2|F|^2}{3(1 - |F|^2)^2} \right) F' - \frac{i\omega}{v^2 - 1} \left( -2 - \frac{i\omega(1 + |F|^2)}{1 - |F|^2} - \frac{2\omega^2|F|^2}{3(1 - |F|^2)^2} \right) F.
\] (20)

Near \( z_- \) the space-time becomes nearly flat so one has to increase the accuracy of the parameters. To be able to distinguish \( z_- \) in all dimensions of space-time we have to be more precise as we will see in the table 1. Therefore near \( z_- \) we increase the precision of the numerical computation. Then we can use of the new coordinate \( w, v, F \) to integrate from \( w = w_+ \) until the next singularity at \( w = w_- \). It is important to clarify that \( F'' \) and all higher derivatives of \( F \) are discontinuous at \( w_- \). This in principle physically acceptable.

To find the second singularity we need to take into account the region \( 0 > z > z_- \). Since \( f \) and \( f' \) are continuous at \( z = z_- \), we start integration at \( z = -\epsilon_0 \) and integrate out to \( z_- \). Time rescaling allows us to impose once more \( b(0_-) = 1 \) and requiring regularity at \( z = 0_- \), we find the solutions in all ten dimensions of space time for elliptic case in table 1.
\[
\begin{array}{ccccccc}
\text{dimension} & \omega & z_- & |f(0_-)| & |f(z_-)| \\
4 & 1.1769527 & -1.0000372 & .0117429 & .0076030 \\
5 & 1.2976152 & -1.0000302 & .0100758 & .0071049 \\
6 & 1.4695887 & -1.0000403 & .0111984 & .0081036 \\
7 & 1.6107643 & -1.0000450 & .0127263 & .0095849 \\
8 & 1.7218682 & -1.0000534 & .0132912 & .0102488 \\
9 & 1.7910669 & -1.0000563 & .0155778 & .0122424 \\
10 & 1.8524703 & -1.0000747 & .0155778 & .0122424 \\
\end{array}
\]

Table 1: Finding solutions for the elliptic case at \( z_- \) in dimensions 4, 5, 6, 7, 8, 9, 10, respectively.

**Appendix B : The change of variable at infinity for the hyperbolic case**

The analysis of \( z = \infty \) can be extended for the hyperbolic and parabolic cases. In the hyperbolic case, the equations of motion become regular at \( z = \infty \) through the change of variables:

\[
\begin{align*}
\frac{d}{dw} &= b(z) \frac{dz}{z^2}, \quad w = 0 \quad \text{at} \quad z = \infty, \\
F(w) &= z^{-\omega} f(z), \\
v(w) &= \frac{b(z)}{z}, \\
u(w) &= u(z).
\end{align*}
\]

leading to the equations:

\[
\begin{align*}
0 &= v' - \frac{v^2 - 1}{(F - F')^2} F' \dot{F}' + 1 + \frac{\omega^2 |F|^2}{(F - F')^2}, \\
0 &= -F'' - \frac{2vF'}{(F - F')^2} F' \dot{F}' + \frac{2F'^2}{(F - F')^2} + \frac{2\omega v}{(v^2 - 1)(F - F')} \left( (F + \dot{F}) + \frac{\omega |F|^2}{(F - F')} \right) F' \\
&\quad + \frac{\omega}{v^2 - 1} \left( -1 + \frac{\omega (F + \dot{F})}{(F - F')} + \frac{\omega^2 |F|^2}{(F - F')^2} \right) F.
\end{align*}
\]
Appendix C: The change of variable at infinity for the parabolic case

In this case the change of variables near $z = \infty$ is:

\[
\begin{align*}
    dw &= b(z) \frac{dz}{z^2}, \quad w = 0 \text{ at } z = \infty, \\
    F(w) &= f(z), \\
    v(w) &= \frac{b(z)}{z}, \\
    u(w) &= u(z).
\end{align*}
\]

leading to following equations of motion in $d = 4$:

\[
0 = v' + \frac{v^2 - 1}{(F - F')^2} \left( -F'F' + \frac{\omega(F' + \bar{F}')}{v} \right) + 1 + \frac{\omega^2}{v^2(F - F')},
\]

\[
0 = -F'' - \frac{2vF'}{(F - F')^2}F'\bar{F}' + \frac{2F'}{(F - F')^2} \left( F' + \frac{\omega(F' + \bar{F}')}{(F - F')} \right) + \frac{2\omega}{(v^2 - 1)(F - F')} \left( \frac{2}{v} \right.
\]

\[
+ \frac{\omega}{v(F - F')F'} + \frac{\omega}{v^2 - 1} \left( -\frac{1}{v^2} - \frac{2\omega}{v^2(F - F')} - \frac{\omega^2}{v^4(F - F')^2} \right)
\]

\[
- \frac{\omega}{v^2(F - F')^2} \left( \frac{\omega(F' + \bar{F}')}{v} - F'\bar{F}' \right).
\]

Appendix D: Perturbed equations for hyperbolic and parabolic case in four dimension

The perturbation equations used in the hyperbolic case in four-dimensions to get the Chop-tuik exponent are:

\[
\begin{align*}
    b'_1(r) &= L_1 \left( b_1(r)(f(r) - \bar{f}(r))^3b'(r)(1 + k + b'(r)) - 2b(r)^2b_1(r)(f(r) - \bar{f}(r))(1 + b'(r)) \right.
\]

\[
\times f'(r)\bar{f}'(r) + wb(r)(1 + b'(r)) \left[ \bar{f}(r)^2((k - w)f_1(r) - 2b_1(r)f'(r)) + f(r)\bar{f}(r) \right.
\]

\[
\times(-(k + w)(f_1(r) - \bar{f}_1(r)) + 2b_1(r)(f'(r) - \bar{f}'(r)) + f(r)^2[(-k + w)\bar{f}_1(r) + \\
+ 2b_1(r)\bar{f}'(r)] \right)
\]
Similarly, in the parabolic case:

\[
f_1'(r) = L_2 \left( b(r)((k-w)f_1(r)\bar{f}(r)(1+k-w-b'(r)) - w f(r)^2 b_1'(r) + 2b(r)f_1(r) \\
- (2k-w+b'(r)) - \bar{f}_1(r)(w+b'(r)) - \bar{f}(r)b_1'(r)f'(r) + f(r)(-k+k^2 \\
+ 2kw - 2w(1+w))f_1(r) + (k-2w)f_1(r)b'(r) + w\bar{f}_1(r)(-1+w+b'(r)) \\
+ b_1'(r)(w\bar{f}(r) + 2b(r)f'(r))) + b_1(r)(w(1-k+w)f(r)^2 + b(r)(-4b(r)f'(r)^2 \\
+ \bar{f}(r)(-4k + 2w + 2b'(r))f'(r) - 2b(r)f''(r)) + f(r) \\
\times (w(-1+k+w)\bar{f}(r) + b(r)((4+k-2w+2b'(r))f'(r) + 2b(r)f''(r)))) \right)
\]

Similarly, in the parabolic case:

\[
b_1'(r) = L_3 \left( b_1(r)(f(r) - \bar{f}(r))^3 b'(r)(1+k+b'(r)) - 2b(r)^2 b_1(r)(f(r) - \bar{f}(r))(1+b'(r)) \\
\times f'(r)\bar{f}'(r) - wb(r)(1+b'(r))\left[ \bar{f}_1(r)(-2w + kf(r) - k\bar{f}(r)) + f_1(r)(2w + kf(r) \\
- k\bar{f}(r)) - 2b_1(r)(f(r) - \bar{f}(r))(f'(r) + \bar{f}'(r)) \right] \right)
\]

\[
f_1'(r) = L_4 \left( (k(1+k)b(r)f_1(r) - krb'(r)f_1(r) + b(r)b'(r)b_1(r)f'(r) - (b(r))^2 b_1'(r)f'(r) \\
+ rb_1'(r)(w - rf'(r)) + 2wb(r)(f_1(r) - \bar{f}_1(r))(w - 2b(r)f'(r)) \\
+ \frac{2kb(r)f_1(r)(w - b(r)f'(r))}{f(r) - \bar{f}(r)} + \frac{4b_1(r)(w - b(r)f'(r))^2}{f(r) - \bar{f}(r)} \right) \\
+ b_1(r)(-w + kw \\
+ (2 - k)rf'(r) + r^2 f''(r)) - 3b_1(r)(-w + wb'(r) - b(r)(-2 + b'(r))f'(r) \\
+ (b(r))^2 f''(r)) \right)
\]

where \( L_1, L_2, L_3, L_4 \) are defined as:

\[
L_1 = \frac{1}{b(r)(f(r) - \bar{f}(r))^3(-1 + k - b'(r))},
\]

\[
L_2 = \frac{1}{2b(r)^2 \left( f(r)(k + w - b'(r)) + \bar{f}(r)(-k + w + b'(r)) \right)}
\]

\[
L_3 = \frac{1}{b(r)(f(r) - \bar{f}(r))^3(-1 + k - b'(r))}
\]

11
\[ L_4 = \frac{1}{\left( -2(1 + k)rb(r) + 2\frac{b(r)^3}{r} + b(r)^2\left( -f(r) + f'(r) \right) + b'(r) + r^2b'(r) \right)} \]

where \( r \) in the definitions of \( L_i, i = 1, \ldots \) is \( z_+ \).

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