Quantization of Emergent Gravity

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ABSTRACT: Emergent gravity is based on a novel form of the equivalence principle known as the Darboux theorem or the Moser lemma in symplectic geometry stating that the electromagnetic force can always be eliminated by a local coordinate transformation as far as spacetime admits a symplectic structure, in other words, a microscopic spacetime becomes noncommutative (NC). If gravity emerges from U(1) gauge theory on NC spacetime, this picture of emergent gravity suggests a completely new quantization scheme where quantum gravity is defined by quantizing spacetime itself, leading to a dynamical NC spacetime. Therefore the quantization of emergent gravity is radically different from the conventional approach trying to quantize a phase space of metric fields. This approach for quantum gravity allows a background independent formulation where spacetime as well as matter fields is equally emergent from a universal vacuum of quantum gravity.

KEYWORDS: Models of Quantum Gravity, Gauge-Gravity Correspondence, Non-Commutative Geometry.
1. Introduction

This paper grew out of the author’s endeavor, after fruitful interactions with his colleagues, to clarify certain aspects of the physics of emergent gravity [1, 2, 3] proposed by the author himself a few years ago. The most notable feedbacks (including some confusions and fallacies) may be classified into three categories:

(I) Delusion on noncommutative (NC) spacetime,
(II) Prejudice on quantization,
(III) Globalization of emergent geometry.
Let us first defend an authentic picture from the above actions (I) and (II). But we have to confess our opinion still does not win a good consensus. (III) will be one of main issues addressed in this paper.

We start with the discussion why we need to change our mundane picture about gravity and spacetime if emergent gravity picture makes sense. According to the general theory of relativity, gravity is the dynamics of spacetime geometry where spacetime is realized as a (pseudo-)Riemannian manifold and the gravitational field is represented by a Riemannian metric \[ g \]. Therefore the dynamical field in gravity is a Riemannian metric over spacetime and the fluctuations of metric necessarily turn a (flat) background spacetime into a dynamical structure. Since gravity is associated to spacetime curvature, the topology of spacetime enters general relativity through the fundamental assumption that spacetime is organized into a (pseudo-)Riemannian manifold. A main lesson of general relativity is that spacetime itself is a dynamical entity.

The existence of gravity introduces a new physical constant, \( G \). The existence of the gravitational constant \( G \), together with another physical constants \( c \) and \( \hbar \) originated from the special relativity and quantum mechanics, implies that spacetime at a certain scale known as the Planck length \( L_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6 \times 10^{-33} \text{cm} \), is no longer commuting, instead spacetime coordinates obey the commutation relation
\[
[y^\mu, y^\nu] = i\theta^{\mu\nu}.
\] (1.1)

Note that the NC spacetime (1.1) is a close pedigree of quantum mechanics which is the formulation of mechanics on NC phase space \([x^i, p_j] = i\hbar\delta^i_j\). Once Richard Feynman said (The Character of Physical Law, 1967) “I think it is safe to say that no one understands quantum mechanics”. So we should expect that the NC spacetime similarly brings about a radical change of physics. Indeed a NC spacetime is much more radical and mysterious than we thought. However we understood the NC spacetime too easily. The delusion (I) on NC spacetime is largely rooted to our naive interpretation that the NC spacetime (1.1) is an extra structure (induced by \( B \) fields) defined on a preexisting spacetime. This naive picture inevitably brings about the interpretation that the NC spacetime (1.1) necessarily breaks the Lorentz symmetry. See the introduction in [6] for the criticism of this viewpoint.

One of the reasons why the NC spacetime (1.1) is so difficult to understand is that the concept of space is doomed and the classical space should be replaced by a state in a complex vector space \( \mathcal{H} \). Since the NC spacetime (1.1), denoted by \( \mathbb{R}^{2n}_\theta \), is equivalent to the Heisenberg algebra of \( n \)-dimensional harmonic oscillator, the Hilbert space \( \mathcal{H} \) in this case is the Fock space (see eq. (6.2)). Since the Hilbert space \( \mathcal{H} \) is a complex linear vector space, the superposition of two states must be allowed which necessarily brings about the interference of states as in quantum mechanics. We may easily be puzzled by a gedanken experiment that mimics the two slit experiment or Einstein-Podolsky-Rosen experiment in quantum mechanics. Furthermore any object \( \mathcal{O} \) defined on the NC spacetime \( \mathbb{R}^{2n}_\theta \) becomes an operator acting on the Hilbert space \( \mathcal{H} \). Thus we can represent the object \( \mathcal{O} \) in the Fock space \( \mathcal{H} \), i.e., \( \mathcal{O} \in \text{End}(\mathcal{H}) \). Since the Fock space has a countable basis, the representation of \( \mathcal{O} \in \text{End}(\mathcal{H}) \) is given by an \( N \times N \) matrix where \( N = \dim(\mathcal{H}) \). In the case at
hand, $N \to \infty$. This is the point we completely lose the concept of space. And this is the pith of quantum gravity to define the final destination of spacetime.

To our best knowledge, quantum mechanics is the more fundamental description of nature. Hence quantization, understood as the passage from classical physics to quantum physics, is not a physical phenomenon. The world is already quantum and quantization is only our poor attempt to find the quantum theoretical description starting with the classical description which we understand better. The same philosophy should be applied to a NC spacetime. In other words, spacetime at a microscopic scale, e.g. $L_P$, is intrinsically NC and so spacetime at this scale should be replaced by a more fundamental quantum algebra such as the algebra of $N \times N$ matrices denoted by $\mathcal{A}_N$. Therefore the usual classical spacetime does not exist a priori. Rather it must be derived from the quantum algebra like classical mechanics is derived from quantum mechanics. But the reverse is not feasible. In our case the quantum algebra is given by the algebra $\mathcal{A}_N$ of $N \times N$ matrices. Since $\mathcal{A}_N \cong \text{End}(\mathcal{H})$, it is isomorphic to an operator algebra $\mathcal{A}_\theta$ acting on the Hilbert space $\mathcal{H}$. The algebra $\mathcal{A}_\theta$ is NC and generated by the Heisenberg algebra (1.1). The spacetime structure derived from the NC $\ast$-algebra $\mathcal{A}_\theta$ is dubbed emergent spacetime. But this emergent spacetime is dynamical and so gravity will also be emergent via the dynamical NC spacetime because gravity is the dynamics of spacetime geometry. It is called emergent gravity. But it turns out (1.1) that the dynamical NC spacetime is defined as a deformation of the NC spacetime (1.1) (see eq. (6.1)) and the deformation is related to $U(1)$ gauge fields on the NC spacetime (1.1). This picture will be clarified in section 4.

In this emergent gravity picture, any spacetime structure is not assumed a priori but defined by the theory. That is, the theory of emergent gravity must be background independent. Hence it is necessary to define a configuration in the algebra $\mathcal{A}_\theta$, for instance, like eq. (1.1), to generate any kind of spacetime structure, even for flat spacetime. A beautiful picture of emergent gravity is that the flat spacetime is emergent from the Moyal-Heisenberg algebra (1.1). See section 6.2 for the verification. Many surprising results immediately come from this dynamical origin of flat spacetime (2, 3), which is absent in general relativity. As a result, the global Lorentz symmetry, being an isometry of flat spacetime, is emergent too. If true, the NC spacetime (1.1) does not break the Lorentz symmetry. Rather it is emergent from the NC spacetime (1.1). This is the picture how we correct the delusion (I).

The dynamical system is described by a Poisson manifold $(M, \theta)$ where $M$ is a differentiable manifold whose local coordinates are denoted by $y^\mu$ ($\mu = 1, \cdots, d = \dim(M)$) and the Poisson structure

$$\theta = \frac{1}{2} \theta^{\mu \nu} \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu} \in \Gamma(\Lambda^2 TM) \quad (1.2)$$

is a (not necessarily nondegenerate) bivector field on $M$. Quantum mechanics is defined by quantizing the Poisson manifold $(M, \theta)$ where $M$ is the phase space of particles with local coordinates $y^\mu = (x^i, p_i)$. Let us call it $\hbar$-quantization. Similarly the NC spacetime (1.1) is defined by quantizing the Poisson manifold $(M, \theta)$ where $M$ is a spacetime manifold, e.g., $M = \mathbb{R}^{2n}$ and $y^\mu$’s are local spacetime coordinates. Let us call it $\theta$-quantization. The first order deviation of the quantum or NC multiplication from the classical one is given by the
Poisson bracket of classical observables. Thus the Poisson bracket of classical observables may be seen as a shadow of the noncommutativity in quantum world. Then the correct statement is that spacetime always supports the spacetime Poisson structure (1.2) if a microscopic spacetime is NC. And, if we introduce a line bundle $L \rightarrow M$ over a spacetime Poisson manifold $M$, the Poisson manifold $(M, \theta)$ also becomes a dynamical manifold like the particle Poisson manifold.

The reason is the following. For simplicity, let us assume that the Poisson bivector (1.2) is nondegenerate and define so-called a symplectic structure $B \equiv \theta^{-1} \in \Gamma(\Lambda^2 T^*M)$. In this case the pair $(M, B)$ is called a symplectic manifold where $B$ is a nondegenerate, closed two-form on $M$. If we consider a line bundle $L \rightarrow M$ over the symplectic manifold $(M, B)$, the curvature $F = dA$ of the line bundle deforms the symplectic structure $B$ of the base manifold. The resulting 2-form is given by $\mathcal{F} = B + F$ where $F = dA$ is the field strength of dynamical U(1) gauge fields. See appendix A for the origin of this structure. Note that the Bianchi identity $d\mathcal{F} = 0$ leads to $d\mathcal{F} = 0$ and $\mathcal{F}$ is invertible unless $\det(1 + B^{-1}F) = 0$. Then $\mathcal{F} = B + F$ is still a symplectic structure on $M$ and so the dynamical gauge fields defined on a symplectic manifold $(M, B)$ manifest themselves as a deformation of the symplectic structure. In section 2 we show this picture also holds for a general Poisson manifold.

In consequence the dynamical spacetime Poisson manifold is modeled by a U(1) gauge theory on $(M, \theta)$ with a fixed Poisson structure $\theta$ on a spacetime manifold $M$. Therefore we can quantize the dynamical Poisson manifold and its quantization leads to a dynamical NC spacetime described by a NC gauge theory. This $\theta$-quantization is neither quantum mechanics nor quantum field theory because the underlying Poisson structure refers to not a particle phase space but spacetime itself. Many people believe that NC gauge theory is a classical theory because the $\hbar$-quantization does not come into play yet. This attitude is based on a prejudice of long standing that quantization is nothing but the $\hbar$-quantization. So a routine desk work insists on the $\hbar$-quantization of the NC gauge theory to define a “quantum” NC field theory. But we want to raise a question: Is it necessary to have the “quantum” NC field theory?

First of all, NC gauge theory is not a theory of particles but a theory of gravity, called the emergent gravity [10]. We pointed out before that, after a matrix representation $A_N = \text{End}(\mathcal{H})$ of NC spacetime, we completely lose the concept of space and so the concept of “point” particles becomes ambiguous. But the $\hbar$-quantization is just the quantization of a particle phase space whatever it is finite-dimensional (quantum mechanics) or infinite-dimensional (quantum field theory) because the Planck constant $\hbar$ has the physical dimension of (length) $\times$ (momentum). Moreover the NC gauge theory describes a dynamical NC spacetime and so formulate a theory of quantum gravity as was shown in [2, 3] and also in this paper. As we argued before, the dynamical NC spacetime becomes an operator acting on a Hilbert space and so the spacetime structure in emergent gravity should be derived from the NC gauge theory. This picture leads to the concept of emergent spacetime. But if spacetime is emergent, everything supported on the spacetime should be emergent too for an internal consistency of the theory. For example, quantum mechanics
must be emergent together with spacetime \[1\]. We will illuminate in section 6 how matter fields can be realized as topological objects in NC \(*\)-algebra \(A_\theta\) which correspond to stable spacetime geometries. Recently a similar geometric model of matters was presented in \[2\]. To conclude, a NC gauge theory is already a quantum description because it is a quantized theory of a spacetime Poisson manifold and it is not necessary to further consider the \(\hbar\)-quantization. Rather quantum mechanics has to emerge from the NC gauge theory. This is our objection to the prejudice (II).

If general relativity emerges from a U(1) gauge theory on a symplectic or Poisson manifold, it is necessary to realize the equivalence principle and general covariance, the most important properties in the theory of gravity (general relativity), from the U(1) gauge theory. How is it possible? A remarkable aspect of emergent gravity is that there exists a novel form of the equivalence principle even for the electromagnetic force \[1, 2\]. This assertion is based on a basic property in symplectic geometry known as the Darboux theorem or the Moser lemma \[13, 14\] stating that the electromagnetic force can always be eliminated by a local coordinate transformation as far as spacetime admits a symplectic structure, in other words, a microscopic spacetime becomes NC. The Moser lemma in symplectic geometry further implies \[15, 16, 17\] that the local coordinate transformation to a Darboux frame is equivalent to the Seiberg-Witten (SW) map defining a spacetime field redefinition between ordinary and NC gauge fields \[18\]. Therefore the equivalence principle in general relativity is realized as a noble statement \[1\] that NC gauge fields can be interpreted as the field variables defined in a locally inertial frame and their commutative description via the SW map corresponds to the field variables in a laboratory frame represented by general curvilinear coordinates.\[2\] This beautiful statement is also true for a general Poisson manifold as will be shown in section 2.

It is possible to lift the novel form of the equivalence principle to a deformed algebra of observables using the “quantum” Moser lemma \[20, 21\]. The quantum Moser lemma demonstrates that a star product deformed by U(1) gauge fields and an original star product are in the same local gauge equivalence class. See eq. (4.8). In particular two star products in the local gauge equivalence are Morita equivalent and related by the action of a line bundle \[22, 23\]. In this sense we may identify the Morita equivalence of two star products with the “quantum” equivalence principle. This will be the subject of section 4.

The basic program of infinitesimal calculus, continuum mechanics and differential geometry is that all the world can be reconstructed from the infinitely small. For example, manifolds are obtained by gluing open subsets of Euclidean space where the notion of sheaf

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1Recently Nima Arkani-Hamed advocated this viewpoint in the Strings 2013 conference. The slides and video recording of his talk are available at http://strings2013.sogang.ac.kr.

2We think this picture may have an important implication to black hole physics. Since we are vague so far about it, we want to quote a remark due to Emil Martinec \[19\]: “The idea that the observables attached to different objects do not commute in the matrix model gave a realization of the notion of black hole complementarity. One can construct logical paradoxes if one attributes independent commuting observables to the descriptions of events by observers who fall through the black hole horizon to probe its interior, as well as observers who remain outside the black hole and detect a rather scrambled version of the same information in the Hawking radiation. If these two sets of observables do not commute, such paradoxes can be resolved.”
embodies the idea of gluing their local data. The concept of connection also plays an important role for the gluing. According to Cartan, connection is a mathematical alias for an observer traveling in spacetime and carrying measuring instruments. Global comparison devices are not available owing to the restriction of the finite propagation speed. So differential forms and vector fields on a manifold are defined locally and then glued together to yield a global object. The gluing is possible because these objects are independent of the choice of local coordinates. Infinitesimal spaces and the construction of global objects from their local data are familiar in all those areas where spaces are characterized by the algebras of functions on them. Naturally emergent gravity also faces a similar feature. The local data for emergent gravity consist of NC gauge fields on a local Darboux chart. They can be mapped to a Lie algebra of inner derivations on the local chart because the NC $\ast$-algebra $A_\theta$ always admits a nontrivial inner automorphism. Basically we need to glue these local data on Darboux charts to yield global vector fields which will eventually be identified with gravitational fields, i.e., vielbeins. This requires us to construct a global star product. The star product is obtained by a perturbative expansion of functions in a formal deformation parameter, e.g., the Planck's constant $\hbar$ which requires one to consider Taylor expansions at points of $M$.[24]. This suggests that a global version of the star product should be defined in terms of deformations of the bundle of infinite jets of functions.[25, 26, 27]. See appendix C for a brief review of jet bundles. In section 5 we will discuss how global objects for the Poisson structure and vector fields can be constructed from the Fedosov quantization of symplectic and Poisson manifolds[28, 29]. This will fill out the gap in our previous works which were missing the step (III).

Recent developments in string theory have revealed a remarkable and radical new picture about gravity. For example, the AdS/CFT correspondence[30, 31, 32] shows a surprising picture that a large $N$ gauge theory in lower dimensions defines a nonperturbative formulation of quantum gravity in higher dimensions. In particular, the AdS/CFT duality shows a typical example of emergent gravity and emergent space because gravity in higher dimensions is defined by a gravityless field theory in lower dimensions. For comprehensive reviews, see, for example, Refs.[33, 34, 35, 36]. In section 6, we show that the AdS/CFT correspondence is a particular example of emergent gravity from NC U(1) gauge fields. Since the emergent gravity, we believe, is a significant new paradigm for quantum gravity, it is desirable to put the emergent gravity picture on a rigorous mathematical foundation. See also some related works[37, 38, 39, 40] and references therein. We want to forward that direction in this paper although we touch only a tip of the iceberg.

The paper is organized as follows. Next three sections do not pretend to any originality. Essential results can be found mostly in[21, 21]. The leitmotif of these sections is to give a coherent exposition in a self-contained manner to clarify why the symplectic structure of spacetime is arguably the essence of emergent gravity realizing the duality between general relativity and NC U(1) gauge theory.

In section 2, we elucidate how U(1) gauge fields deform an underlying Poisson structure of spacetime. It is shown that these deformations in terms of U(1) gauge fields can be identified via the Moser lemma with local coordinates transformations. These coordinate transformations are represented by Poisson U(1) gauge fields and lead to the semi-classical
version of SW maps between ordinary and NC U(1) gauge fields [18].

In section 3, we review the Kontsevich’s deformation quantization [24] to understand how to lift the results in section 2 to the case of deformed algebras.

In section 4, it is shown that dynamical NC spacetime is modeled by a NC gauge theory via the quantum Moser lemma. It is straightforward to identify the SW map using the local covariance map in [20, 21]. An important point is that the star product defined by a Poisson structure deformed by U(1) gauge fields is Morita equivalent to the original undeformed one [22, 23]. This means that NC U(1) gauge theory describes their equivalent categories of modules. We suggest that the Morita equivalence between two star products can be interpreted as the “quantum” equivalence principle for quantum gravity.

In section 5, we discuss how (quantum) gravity emerges from NC U(1) gauge theory. First we identify local vector fields from NC U(1) gauge fields on a local Darboux chart. And we consider the extension of the local data to an infinitesimal neighborhood using normal coordinates and then present a prescription for global vector fields using the jet isomorphism theorem [11, 42] stating that the objects in the ∞-jet are represented by the covariant tensors only. We consider a global star product using the Fedosov quantization [25, 26] in order to verify the prescription for global vector fields. We also discuss a symplectic realization of Poisson manifolds [13, 44] and symplectic groupoids [15, 16].

In section 6, we show that the representation of a NC gauge theory in a Hilbert space is equivalent to a large N gauge theory which has appeared as a nonperturbative formulation of string/M theories. We also illuminate how time emerges together with spaces from a Hamiltonian dynamical system which is always granted by a background NC space, e.g. eq. (1.1), responsible for the emergent space. We emphasize that the background is just a condensate that must be allowed to develop and exist. Finally we argue that the AdS/CFT correspondence [30, 31, 32] can be founded on the emergent gravity from NC U(1) gauge fields.

In section 7, after a brief summary of the results obtained in this paper, we discuss possible implications to string theory, emergent quantum mechanics and quantum entanglements building up emergent spacetimes proposed by M. Van Raamdonk [17, 18].

In appendix A, we highlight the local nature of NC U(1) gauge fields using the relation between a Darboux transformation and SW map [13, 14, 17] to emphasize why we need a globalization of local vector fields obtained from them.

In appendix B, we discuss modular vector fields since emergent gravity requires unimodular Poisson manifolds [19]. Since Poisson manifolds can be thought of as semiclassical limits of operator algebras, it is natural to ask whether they have modular automorphism groups, like von Neumann algebras. It was shown [50, 51, 52] that it is the case. This confirms again that Poisson manifolds are intrinsically dynamical objects.

In appendix C, we give a brief exposition on jet bundles because they have been often used in this paper. A jet bundle can be regarded as the coordinate free version of Taylor expansions and so a useful tool for a geometrical covariant field theory though it is not widely used in physics so far. We refer to [53, 54, 55] for more detailed expositions.
2. U(1) gauge theory on Poisson manifold

In this section we recapitulate a fascinating picture that the Darboux theorem or the Moser lemma in symplectic geometry can be interpreted as a novel form of the equivalence principle for electromagnetic force. Fortunately all the essential details were greatly elaborated in [20, 21] where it was shown that the local deformations of a symplectic or Poisson structure can be transformed into a diffeomorphism symmetry using the Darboux theorem or the Moser lemma in symplectic geometry and lead to the SW map between commutative and NC gauge fields. Here we will review the results in [20, 21] to clarify why the symplectic structure of spacetime leads to the novel form of the equivalence principle stating that the electromagnetic force can be always eliminated by a local coordinate transformation. It has been emphasized in [1, 2] that the equivalence principle for the electromagnetic force should be the first principle for emergent gravity.

Consider an Abelian gauge theory on a smooth real manifold $M$ that also carries a Poisson structure (1.2). First we introduce the Schouten-Nijenhuis (SN) bracket for polyvector fields [44, 56, 57]. A polyvector field of degree $k$, or $k$-vector field, on a manifold $M$ is a section of the $k$-th exterior power $\Lambda^k TM$ of the tangent bundle and is dual to a $k$-form in $\Lambda^k T^* M$. If $\Pi = \frac{1}{k!} \sum_{\mu_1, \ldots, \mu_k} \Pi_{\mu_1 \cdots \mu_k} \partial \partial y_{\mu_1} \wedge \cdots \wedge \partial \partial y_{\mu_k}$ is a $k$-vector field, we will consider it as a homogeneous polynomial of degree $k$ in the odd variables $\zeta_\mu \equiv \partial \partial y_{\mu}$:

$$\Pi = \frac{1}{k!} \sum_{\mu_1, \ldots, \mu_k} \Pi_{\mu_1 \cdots \mu_k} \zeta_{\mu_1} \cdots \zeta_{\mu_k}. \tag{2.1}$$

If $P$ and $Q$ are $p$- and $q$-vector fields, the SN bracket of $P$ and $Q$ is defined by [24, 57]

$$[P, Q]_S = \sum_{\mu} \left( \frac{\partial P}{\partial \zeta_{\mu}} \frac{\partial Q}{\partial y_{\mu}} - (-)^{(p-1)(q-1)} \frac{\partial Q}{\partial \zeta_{\mu}} \frac{\partial P}{\partial y_{\mu}} \right). \tag{2.2}$$

Clearly the bracket $[P, Q]_S = -(-)^{(p-1)(q-1)}[Q, P]_S$ defined above is a homogeneous polynomial of degree $p + q - 1$, so it is a $(p + q - 1)$-vector field. The SN bracket (2.2) satisfies a general property [57] that, if $X$ is a vector field, then

$$[X, \Pi]_S = \mathcal{L}_X \Pi \tag{2.3}$$

for a $k$-vector field $\Pi$ where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field $X$. In particular, if $X$ and $Y$ are two vector fields, then the SN bracket of $X$ and $Y$ coincides with their Lie bracket. We adopt the following differentiation rule for the odd variables

$$\frac{\partial}{\partial \zeta_{\mu}} (P \wedge Q) = P \frac{\partial Q}{\partial \zeta_{\mu}} + (-)^q \frac{\partial P}{\partial \zeta_{\mu}} Q. \tag{2.4}$$

Then it is straightforward to verify the graded Jacobi identity for the SN bracket (2.2) [57]:

$$(-)^{(p-1)(r-1)}[[P, Q]_S, R]_S + (-)^{(q-1)(p-1)}[[Q, R]_S, P]_S + (-)^{(r-1)(q-1)}[[R, P]_S, Q]_S = 0 \tag{2.5}$$
where $P \in \Gamma(\Lambda^p TM)$, $Q \in \Gamma(\Lambda^q TM)$, $R \in \Gamma(\Lambda^r TM)$. The bivector $\theta = \frac{1}{2} \theta^{\mu \nu} \zeta_\mu \zeta_\nu \in \Gamma(\Lambda^2 TM)$ is called a Poisson structure if and only if it obeys
\[
[\theta, \theta]_S = \frac{1}{3} \left( \theta^{\mu \lambda} \partial_\lambda \theta^{\rho \nu} + \theta^{\nu \lambda} \partial_\lambda \theta^{\mu \rho} + \theta^{\rho \lambda} \partial_\lambda \theta^{\mu \nu} \right) \zeta_\mu \zeta_\nu \zeta_\rho = 0. \tag{2.6}
\]

Let us define the space $V^p(M) = \oplus_{p \geq 0} \Gamma(\Lambda^p TM)$ which forms a graded Lie algebra under the SN bracket \((2.2)\) if the grade of $V^p(M) \equiv \Gamma(\Lambda^p TM)$ is defined to be $p - 1$. Also introduce a differential operator $d_\theta : V^p(M) \to V^{p+1}(M)$ defined by
\[
d_\theta \Pi \equiv -[[\Pi, \theta]]_S \tag{2.7}
\]
for any $p$-vector field $\Pi$ in $V^p(M)$. Using eq. (2.6), one can show that the coboundary operator (2.7) is nilpotent, i.e.,
\[
d^2_\theta \Pi = [[[[\Pi, \theta]]_S, \theta]]_S = \frac{1}{2} [[[\theta, \theta]]_S, \Pi]_S = 0 \tag{2.8}
\]
where the Jacobi identity (2.5) was used. Then one can define a differential complex \((V^\bullet(M), d_\theta)\) given by
\[
\cdots \to V^{p-1}(M) \xrightarrow{d_\theta} V^p(M) \xrightarrow{d_\theta} V^{p+1}(M) \to \cdots \tag{2.9}
\]
which is called the Lichnerowicz complex. The cohomology of this complex is called the Poisson cohomology \([44]\) and is defined as the quotient groups
\[
H^\bullet_\theta(M) = \text{Ker } d_\theta / \text{Im } d_\theta. \tag{2.10}
\]

Given a Poisson structure $\theta \in \mathcal{V}^2(M)$ on a manifold $M$, the Poisson bivector $\theta$ induces a natural homomorphism $\rho : T^* M \to TM$ by
\[
A \mapsto \rho(A) = -\theta^{\mu \nu} A_\nu \frac{\partial}{\partial y^\mu} \tag{2.11}
\]
for $A = A_\mu(y) dy^\mu \in T^*_p M$. The bundle homomorphism (2.11) is called the anchor map of $\theta$. The anchor map (2.11) can be written as
\[
\rho(A) \equiv A_\theta = A_\mu d_\theta y^\mu = \theta^{\mu \nu} A_\mu \zeta_\nu. \tag{2.12}
\]

Note that, if the coboundary operator $d_\theta$ acts on a smooth function $f \in C^\infty(M)$, it generates a vector field in $\Gamma(TM)$ given by
\[
d_\theta f = -[f, \theta]_S = -\theta^{\mu \nu} \partial_\nu f \partial_\mu \equiv X_f \tag{2.13}
\]
which is called the Hamiltonian vector field. Thus the operator $d_\theta$ acting on the space $C^\infty(M)$ defines the correspondence $f \mapsto X_f$ and one can introduce a bilinear map $\{-, -\}_\theta : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, the so-called Poisson bracket, defined by
\[
\{f, g\}_\theta \equiv (d_\theta f)(g) = X_f(g) = -X_g(f) = \langle \theta, df \wedge dg \rangle \tag{2.14}
\]
for $f, g \in C^\infty(M)$. One can also act the coboundary operator $d_\theta$ on the vector field $A_\theta \in \Gamma(TM)$ and the corresponding bivector field is given by

$$F_\theta \equiv d_\theta A_\theta = \frac{1}{2} F_{\mu\nu} d_\theta y^\mu \wedge d_\theta y^\nu = \frac{1}{2} \theta^{\mu\rho} F_{\rho\sigma\nu} \zeta^\mu \zeta^\nu \in \Omega^2(M) \quad (2.15)$$

where $F \equiv dA = \frac{1}{2} F_{\mu\nu} dy^\mu \wedge dy^\nu \in \Gamma(\Lambda^2 T^*M)$ and the condition (2.6) was used to deduce $d^2_\theta y^\mu = 0$. Thus one can consider the bivector $F_\theta$ to be dual to the two-form $F = dA$.

We will identify the one-form $A$ in eq. (2.11) with a connection of line bundle $L \to M$ and $F = dA$ with its curvature. This identification is consistent with the dual description in terms of polyvectors: First note that, under an infinitesimal gauge transformation $A \mapsto A + d\lambda$, the vector field $A_\theta$ in (2.12) changes by a Hamiltonian vector field $d_\theta \lambda$, i.e., $A_\theta \mapsto A_\theta + d_\theta \lambda$. And, from the definition (2.15), we have $d_\theta F_\theta = 0$ due to $d^2_\theta = 0$ or $[\theta, \theta]_S = 0$, which is dual to the Bianchi identity $dF = 0$ in gauge theory.

Now we perturb the Poisson structure $\theta$ by introducing a one-parameter deformation $\theta_t$ with $t \in [0, 1]$ whose evolution obeys

$$\partial_t \theta_t = \mathcal{L}_{A_{\theta_t}} \theta_t \quad (2.16)$$

with initial condition $\theta_0 = \theta$. Here $A_{\theta_t} = A_\mu d_\theta y^\mu$ is a $t$-dependent vector field defined by the anchor map (2.12) of $\theta_t$. Note that the evolution (2.16) can be written according to eq. (2.3) as

$$\partial_t \theta_t = \mathcal{L}_{A_{\theta_t}} \theta_t = [A_{\theta_t}, \theta_t]_S = -d_{\theta_t} A_{\theta_t} = -F_{\theta_t}. \quad (2.17)$$

In terms of local coordinates, the evolution equation (2.16) thus takes the form

$$\partial_t \theta_{t}^{\mu\nu} = -(\theta_t F_{\theta_t})^{\mu\nu}, \quad \theta^{\mu\nu}_0 = \theta^{\mu\nu}. \quad (2.18)$$

The formal solution (defined in power series of $t$) is given by [20, 21]

$$\theta_t = \theta \frac{1}{1 + tF\theta}. \quad (2.19)$$

One can see that $\theta_t$ is a Poisson structure for all $t$, i.e. $[\theta_t, \theta_t]_S = 0$, because $[\theta_t, \theta_t]_S = 0$ at $t = 0$ and

$$\partial_t [\theta_t, \theta_t]_S = 2[[A_{\theta_t}, \theta_t]_S, \theta_t]_S = [A_{\theta_t}, [\theta_t, \theta_t]_S]_S = \mathcal{L}_{A_{\theta_t}} [\theta_t, \theta_t]_S$$

where we have used the Jacobi identity (2.3).

Suppose that $\theta$ is invertible and define $B \equiv \theta^{-1} \in \Gamma(\Lambda^2 T^*M)$. In this case the nondegenerate two-form $B$ defines a symplectic structure on $M$ because $[\theta, \theta]_S = 0$ implies $dB = 0$. Denoting $\Theta \equiv \theta_1$, the solution (2.19) can be written as the form

$$\Theta = \frac{1}{B + F} \quad (2.21)$$

and $dF = 0$ due to $[\Theta, \Theta]_S = 0$. Hence the initial symplectic structure $B$ evolves to a new symplectic structure $\mathcal{F} \equiv B + F$ according to the evolution equation (2.16) defining the one-parameter deformation. Since we have identified the one-form $A$ in the anchor map
with U(1) gauge fields, the fluctuations of U(1) gauge fields can be understood as the deformation of symplectic manifold \((M, B)\). Therefore the U(1) gauge theory, especially the Maxwell’s theory of electromagnetism, on a symplectic manifold \((M, B)\) can be understood completely in the context of symplectic geometry.

Since the \(t\)-evolution \(2.16\) is generated by the vector field \(A\theta\), it can be integrated to a flow of \(A\theta\) starting at identity where \(t\) plays a role of the affine parameter labeling points on the curve of the flow (see Appendix A in [21] for the derivation). The result is given by

\[
\rho_A^t = \exp(A\theta + \partial_t) \exp(-\partial_t)|_{t=0}
\]  
(2.22)

that relates the Poisson structures \(\Theta = \theta_1\) and \(\theta = \theta_0\), i.e.,

\[
\rho_A^t \Theta = \theta.
\]  
(2.23)

In order to calibrate the deformation in eq. (2.19) or (2.21) due to local gauge fields, let us represent the exponential map \(\rho_A^t\) by

\[
\rho_A^t = \text{id} + \mathfrak{A}_A
\]  
(2.24)

where the differential operator \(\rho_A^t\) basically pulls back functions on \(M\) to functions on \(TM\). Let \(\mathfrak{P} = (C^\infty(M), \{-, -\}_\theta)\) be a Poisson algebra on the vector space \(C^\infty(M)\) equipped with the Poisson bracket \((2.14)\). Then the diffeomorphism \(2.22\) defines a natural algebra homomorphism \(\rho_A^* : \mathfrak{P} \rightarrow \mathfrak{P}\) which acts on the Poisson algebra \(\mathfrak{P}\) and the equivalence relation \(2.23\) for Poisson structures can be translated into the equivalence relation of Poisson algebras defined by

\[
\rho_A^* \{f, g\}_\Theta = \{\rho_A^* f, \rho_A^* g\}_\theta
\]  
(2.25)

for \(f, g \in C^\infty(M)\). The Poisson algebra \(\mathfrak{P}' = (C^\infty(M), \{-, -\}_{\Theta})\) now depends on the gauge field \(F = dA\) and so physically the map \(2.24\) serves to incorporate a back-reaction of gauge fields on the Poisson algebra.

The separation \(2.24\) lucidly shows that \(\mathfrak{A}_A = \rho_A^t - \text{id}\) vanishes when turning off the U(1) gauge fields, i.e. \(A = 0\). To examine the response of \(\mathfrak{A}_A\) under an infinitesimal gauge transformation \(A \mapsto A + d\lambda\), first recall that the vector field \(A\theta\) changes by a Hamiltonian vector field \(d\lambda: A\theta \mapsto A\theta + d\lambda\). The effect of this gauge transformation on the flow \(2.22\) is given to first order in \(\lambda\) by \([21]\)

\[
\rho_A^{t+d\lambda} = (\text{id} + d\lambda) \circ \rho_A^t
\]  
(2.26)

or equivalently

\[
\mathfrak{A}_{A+d\lambda} = \mathfrak{A}_A + d\lambda + \{\mathfrak{A}_A, \lambda\}_\theta
\]  
(2.27)

The derivation employs a clever observation. Consider a \(t\)-dependent function \(f(t)\) whose \(t\)-evolution is governed by \((\partial_t + A(t)) f(t) = 0\) where \(A(t)\) is a differential operator of arbitrary degree. One can show that the function \(f(t)\) satisfies the recursion relation \(e^{\partial_t + A(t)} e^{-\partial_t} f(t+1) = f(t)\). As the Baker-Campbell-Hausdorff formula implies, there are no free \(t\)-derivatives acting on \(f(t+1)\) and so one can evaluate the recursion relation at \(t = 0\) and get \(e^{\partial_t + A(t)} e^{-\partial_t}|_{t=0} f(1) = f(0)\).
\[
\tilde{\lambda}(\lambda, A) = \sum_{n=0}^{\infty} \frac{(A\theta + \partial_t)^n(\lambda)}{(n + 1)!} \bigg|_{t=0}.
\]  
(2.28)

The gauge transformation \( \tilde{\lambda} \) shows that \( \rho_A^* \) is a coordinate-independent and manifestly covariant object under a (non-Abelian) gauge transformation generated by \( \tilde{\lambda} \) in (2.28). In this respect, \( A_A \) in eq. (2.24) plays the role of a generalized gauge connection and so we call it a Poisson gauge field. We also call it a symplectic gauge field for the case that the bivector field \( \theta \) is nondegenerate. The exponential map \( \rho_A^* \) is a semi-classical version of the covariantizing map \( D_A \) which is a formal differential operator acting on a function \( f \in C^\infty(M) \) such that \( D_A f = f + f_A \) becomes a covariant function under the NC version of the gauge transformation (2.27). The field strength of Poisson gauge fields evaluated on two functions \( f, g \) is defined by

\[
F_A(f, g) = \{\rho_A^* f, \rho_A^* g\}_\theta - \rho_A^* \{f, g\}_\theta = \rho_A^* \left( \{f, g\}_\Theta - \{f, g\}_\theta \right)
\]  
(2.29)

where eq. (2.25) was used. Abstractly, the 2-cochain field strength (2.29) can be regarded as a bidifferential operator acting on two functions and can be written in the form

\[
F_A = \rho_A^* \circ (\Theta - \theta) = \rho_A^* \circ F_\theta
\]  
(2.30)

where

\[
F_\theta = \frac{1}{2} F_{\mu\nu} dy^\mu \wedge dy^\nu
\]  
(2.31)

with

\[
F_{\mu\nu} = \left( \frac{1}{1 + F_\theta F} \right)_{\mu\nu}.
\]  
(2.32)

Now we come to an important picture about the diffeomorphism (2.23) between two different Poisson structures. The \( t \)-evolution (2.16) implies that all the Poisson structures \( \theta_t \) for \( t \in [0, 1] \) are related by coordinate transformations generated by the flow \( \rho^*_t(A) \) of \( A_t \) such that \( \rho^*_t(A)\theta_{t'} = \theta_t \). In particular, denoting \( \rho^*_{01}(A) = \rho_A^* \), we have the relation (2.23). This constitutes an appropriate generalization of Moser’s lemma from symplectic geometry to Poisson case [21]. According to the Weinstein’s splitting theorem for a \( d \)-dimensional Poisson manifold \((M, \theta)\),\(^5\) one can choose coordinates

---

\(^4\)It may be more enticing to define the field strength as \( F_A(f, g) = \{\rho_A^* f, \rho_A^* g\}_\theta = \rho_A^* \{f, g\}_\theta \). A motivation for this definition is simply to achieve a background independent object for the field strength [13, 3] because it may be ambiguous to discriminate dynamical fields from a background part in the case of a generic Poisson structure.

\(^5\)The splitting theorem [13, 4, 5] states that a \( d \)-dimensional Poisson manifold \((M, \theta)\) is locally equivalent to the product of \( \mathbb{R}^{2n} \) equipped with the canonical symplectic structure with \( \mathbb{R}^{d-2n} \) equipped with a Poisson structure of rank zero at a point in a local neighborhood of \( M \). That is, the Poisson manifold \((M, \theta)\) is locally isomorphic (in a neighborhood of \( P \in M \)) to the direct product \( S \times N \) of a symplectic manifold \((S, \sum_{i=1}^{n} dq^i \wedge dp_i)\) with a Poisson manifold \((N_P, \{-,\}_N)\) whose Poisson tensor vanishes at \( P \). Thus the Poisson structure \( \theta \) can be consistently restricted to a leaf \( S \) as nondegenerate and the leaves become \( 2n \)-dimensional symplectic manifolds.
\[ y^\mu := (q^1, \ldots, q^n, p_1, \ldots, p_n, r^1, \ldots, r^{d-2n}) \] on a neighborhood centered at a point \( P \in M \) such that

\[ \theta = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} + \frac{1}{2} \sum_{a,b=1}^{d-2n} \varphi_{ab}(r) \frac{\partial}{\partial r^a} \wedge \frac{\partial}{\partial r^b} \quad \text{and} \quad \varphi_{ab}(r)|_P = 0. \]  

(2.33)

We call such a coordinate system the Darboux-Weinstein frame. In the Darboux-Weinstein frame, the Poisson bracket \( \{ f, g \}_\theta = \theta(df, dg) \) is given by

\[ \{ f, g \}_\theta = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i} \right) + \sum_{a,b=1}^{d-2n} \{ r^a, r^b \}_\theta \frac{\partial f}{\partial r^a} \frac{\partial g}{\partial r^b} \]  

(2.34)

where \( \varphi_{ab}(r) = \{ r^a, r^b \}_\theta \). Therefore the existence of the Moser flow \( \rho^*_A \) implies that it is always possible to find a coordinate transformation to nullify the local deformation of Poisson structures by U(1) gauge fields. In other words, there always exists the Darboux-Weinstein frame where the underlying Poisson structure locally takes the form (2.33). Therefore we will locally use the exponential map (2.22) to define the Poisson algebra in the Darboux-Weinstein frame.

If the Poisson structure \( \theta \) is nondegenerate, i.e., it defines a symplectic structure \( B \equiv \theta^{-1} \) on \( M \), the equation (2.23) can be written as the form

\[ \rho^*_A(B + F) = B. \]  

(2.35)

This is the original statement of the Moser lemma in symplectic geometry [13, 14]. This rather well-known theorem in symplectic geometry suggests an important physics [1, 2]. The Moser lemma (2.33) implies that the electromagnetic force \( F = dA \) can always be eliminated by a local coordinate transformation like the gravitational force in general relativity as far as an underlying space admits a symplectic structure. Therefore there exists a novel form of the equivalence principle even for the electromagnetic force. Moreover we observed that the equivalence principle for the electromagnetic force \( F \) holds for a general Poisson manifold \( (M, \theta) \) where \( \theta \) is not necessarily nondegenerate. In the end a critical question is whether U(1) gauge theory on a symplectic manifold \( (M, B) \) or more generally a Poisson manifold \( (M, \theta) \) can be formulated as a theory of gravity. It was positively answered in [1, 2, 58] (see also reviews [3, 10, 59]) that Einstein gravity can emerge from electromagnetism if spacetime admits a symplectic or Poisson structure. We will elaborate this idea more concretely in the context of emergent gravity.

Using the bundle homomorphism \( \theta : T^*M \to TM \), we defined the anchor map (2.11) of \( \theta \in V^2(M) \) and introduced the correspondence \( A \mapsto A_\theta \) defined by eq. (2.12) between the connection \( A \) of line bundle (regarded as a one-form in \( T^*M \)) and a vector field \( A_\theta \). The perturbed vector field \( A_\theta \) generates a flow as an integral curve of \( A_\theta \) which defines a one-parameter group of diffeomorphisms obeying the relation (2.23). The dynamical diffeomorphism (2.24) then associates the U(1) gauge field \( A \) with a Poisson gauge field \( A_\theta \) which corresponds to a semi-classical version of the SW map [18]. But the transformation (field redefinition) from ordinary to Poisson gauge fields has to preserve the gauge
equivalence relation between ordinary and Poisson gauge symmetries. This condition is summarized by the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{SW}} & A_A \\
\delta_\lambda & \\nA + d\lambda & \xrightarrow{\text{SW}} & A_A + D_A \tilde{\lambda}
\end{array}
\]

where \( \delta_\lambda A \equiv d\lambda \) and \( \delta_\lambda A_A \equiv D_A \tilde{\lambda} = d\theta \tilde{\lambda} + \{A, \tilde{\lambda}\}_\theta \). The gauge transformation (2.27) shows that the Poisson gauge field \( A_A \) does satisfy the equivalence relation (2.36), i.e.,

\[
A_A + \delta_\tilde{\lambda} A_A = A_A + \delta_\lambda A.
\]

It is straightforward to derive the Lie algebra structure

\[
[\delta_{\tilde{\lambda}_1}, \delta_{\tilde{\lambda}_2}] = \delta_{\{\tilde{\lambda}_1, \tilde{\lambda}_2\}_\theta}
\]

by applying the gauge transformation \( \delta_{\tilde{\lambda}} A_A(f) = \{f, \tilde{\lambda}\}_\theta + \{A_A(f), \tilde{\lambda}\}_\theta \) and using the Jacobi identity for the Poisson bracket (2.14). It may be emphasized that the SW map \( \text{SW} : A \mapsto A_A \) can be basically obtained by finding the Moser flow (2.22) satisfying eq. (2.23) for the Poisson case or eq. (2.35) for the symplectic case.

Let us introduce local coordinates \( y^\mu \) on a patch \( U \subset M \) and consider the action of the diffeomorphism \( \rho_A^* \) on the coordinate function \( y^\mu \). We represent the action as

\[
\rho_A^*(y^\mu) \equiv x^\mu(y) = y^\mu + \mathcal{Y}^\mu(y)
\]

which plays the role of covariant (dynamical) coordinates. According to the Weinstein’s splitting theorem, one can always choose a local coordinate chart \( \{y^\mu\}, U \) such that the Poisson structure \( \theta \) on \( U \) is given by the Darboux-Weinstein frame (2.33). Such a coordinate chart \( \{y^\mu\}, U \) on a general Poisson manifold will be called Darboux-Weinstein coordinates or simply Darboux coordinates for the symplectic case. Therefore, for the symplectic case where the Poisson structure \( \theta \) is nondegenerate, one can assume, without loss of generality, that the initial Poisson structure \( \theta_0 = \theta \) is always constant. For the constant symplectic structure \( B = \theta^{-1} \), it is useful to represent the covariant coordinates in the form

\[
x^\mu(y) = y^\mu + \theta^\mu_\nu a_\nu(y) \in C^\infty(M)
\]

and define corresponding “covariant momenta” by

\[
C_\mu(y) \equiv B_{\mu\nu}x^\nu(y) = p_\mu + a_\mu(y) \in C^\infty(M)
\]

with \( p_\mu \equiv B_{\mu\nu}y^\nu \). In this case the field strength (2.29) evaluated on the coordinate basis \( \{y^\mu\} \) is given by

\[
F_A(y^\mu, y^\nu) = \{x^\mu, x^\nu\}_\theta - \theta^\mu_\nu = -\left(\theta f \theta\right)^\mu_\nu
\]
or
\[ \mathcal{F}_A(p_\mu, p_\nu) = \{C_\mu, C_\nu\}_\theta + B_{\mu\nu} = f_{\mu\nu} \] (2.43)

where
\[ f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + \{a_\mu, a_\nu\}_\theta. \] (2.44)

Evaluating the field strength \( \mathcal{F}_A(y^\mu, y^\nu) \) with the cochain map (2.30) and then identifying it with eq. (2.42), one can get the relation \[ f_{\mu\nu}(y) = \left(1 + F F_\theta \right)_{\mu\nu}(x) \] (2.45)

where the coordinate transformation (2.39) was assumed such that \( \rho^*_A(f(y)) = f(\rho^*_A(y)) = f(x) \) for a smooth function \( f \in C^\infty(M) \). For a general Poisson structure \( \theta \), the SW map (2.45) must be replaced by
\[ \theta^{\mu\nu}(y) + \Xi^{\mu\nu}(y) = \rho^*_A \left( \theta^{\mu\nu}(y) + (\theta F \theta)^{\mu\nu}(y) \right) \] (2.46)

where
\[ \Xi^{\mu\nu}(y) = \{y^\mu, \mathcal{Y}^{\nu}(y)\}_\theta - \{y^{\nu}, \mathcal{Y}^{\mu}(y)\}_\theta + \{\mathcal{Y}^{\mu}(y), \mathcal{Y}^{\nu}(y)\}_\theta \] (2.47)

and \( F_{\mu\nu}(y) \) is given by eq. (2.32). For the general case, the diffeomorphism \( \rho^*_A \) in eq. (2.46) now nontrivially acts on the Poisson tensor \( \theta^{\mu\nu}(y) \) too and so the SW map is rather complicated.

One can also find the Jacobian factor \( J = \left| \frac{\partial y}{\partial x} \right| \) for the coordinate transformation (2.39). For the symplectic case, it is easy to deduce from eq. (2.35) that
\[ J(x) = \sqrt{\det(1 + F \theta)}(x). \] (2.48)

But the general Poisson case (2.23) requires a careful treatment. One can first solve eq. (2.23) in a subspace where the Poisson tensor is nondegenerate and then extend the solution (2.48) to entire space such that it satisfies eq. (2.23). (Note that \( \theta \) is placed on both sides of eq. (2.23).) Then it ends with the result (2.48) again. The equations (2.45), (2.46) and (2.48) consist of a semiclassical version of the SW map \[ \text{describing a spacetime field redefinition between ordinary and symplectic or Poisson gauge fields in the approximation of slowly varying fields, } \sqrt{\theta} \frac{\partial F}{\partial x} \ll 1, \text{ in the sense keeping field strengths (without restriction on their size) but not their derivatives.} \]

We conclude this section with a brief summary. The electromagnetic force manifests itself as the deformation of an underlying Poisson structure and the deformation is described by a formal solution \( \frac{\partial \theta}{\partial t} \) of the evolution equation (2.16). But every Poisson structures \( \theta_t \) for \( t \in [0, 1] \) are related to the canonical Poisson structure (2.33) in the Darboux-Weinstein frame by a local coordinate transformation (2.22) generated by the vector field \( A_{\theta_t} \). This Darboux-Weinstein frame corresponds to a locally inertial frame in general relativity and so constitutes a novel form of the equivalence principle for the electromagnetic force as a viable analogue of the equivalence principle in general relativity.
3. Deformation quantization

Now we want to quantize the Poisson algebra \( \mathfrak{P} = (C^\infty(M), \{-, -\}_\theta) \) introduced in the previous section. The canonical quantization of the Poisson algebra \( \mathfrak{P} = (C^\infty(M), \{-, -\}_\theta) \) consists of a complex Hilbert space \( \mathcal{H} \) and of a quantization map \( Q : C^\infty(M) \to A_\theta \) by \( f \mapsto Q(f) \equiv \hat{f} \) should be \( \mathbb{C} \)-linear and an algebra homomorphism:

\[
    f \cdot g \mapsto \hat{f} \star \hat{g} = \hat{f} \cdot \hat{g}
\]

and

\[
    f \star g \equiv Q^{-1}(Q(f) \cdot Q(g))
\]

for \( f, g \in C^\infty(M) \) and \( \hat{f}, \hat{g} \in A_\theta \). The Poisson bracket \([2,14]\) controls the failure of commutativity

\[
    [\hat{f}, \hat{g}] \sim i\{f, g\}_\theta + \mathcal{O}(\theta^2).
\]

A natural question at hand is whether such quantization is always possible for general Poisson manifolds with a radical change in the nature of the observables. An essential step is to construct the Hilbert space for a general Poisson manifold, which is in general highly nontrivial. In order to postpone or rather circumvent difficult questions related to the representation theory, we will simply choose to work within the framework of deformation quantization \([62, 28]\) which allows us to focus on the algebra itself. Later (in section 6) we will consider a strict quantization with a Hilbert space.

M. Kontsevich proved \([24]\) that every finite-dimensional Poisson manifold \( M \) admits a canonical deformation quantization and the equivalence classes of Poisson manifolds modulo diffeomorphisms can be naturally identified with the set of gauge equivalence classes of star products on a smooth manifold. The existence of a star product on an arbitrary Poisson manifold follows from the general formality theorem: The differential graded Lie algebra of Hochschild cochains defined by polydifferential operators is quasi-isomorphic to the graded Lie algebra of polyvector fields.\(^6\) Let \( A \) be an arbitrary unital associative algebra with multiplication \( \ast \). The Hochschild \( p \)-cochains \( C^p(A, A) \equiv \text{Hom}(A^\otimes p, A) \) are the space of \( p \)-linear maps \( \mathcal{C}(f_1, \cdots, f_p) \) on \( A \) with values in \( A \) and the coboundary operator \( d_* : C^p \to C^{p+1} \) is defined by \([24]\)

\[
    (d_* \mathcal{C})(f_1, \cdots, f_{p+1}) = f_1 \ast \mathcal{C}(f_2, \cdots, f_{p+1})
    + \sum_{i=1}^{p} (-1)^i \mathcal{C}(f_1, \cdots, f_{i-1}, f_i \ast f_{i+1}, f_{i+2}, \cdots, f_{p+1})
    + (-)^{p+1} \mathcal{C}(f_1, \cdots, f_p) \ast f_{p+1}
\]

for \( \mathcal{C} \in C^p(A, A) \) and \( f_1, \cdots, f_{p+1} \in A \). The Hochschild complex admits a bracket \([-, -]_G : C^m(A, A) \otimes C^n(A, A) \to C^{m+n-1}(A, A) \), called the Gerstenhaber bracket or in abbreviation

\(^6\)A quasi-isomorphism is a morphism of (co)chain complexes such that the induced morphism of (co)homology groups is an isomorphism.
the $G$-bracket, and it is given by

$$[C_1, C_2]_G = C_1 \circ C_2 - (-)^{(m-1)(n-1)} C_2 \circ C_1 \quad (3.5)$$

where the composition $\circ$ for $C_1 \in C^m$ and $C_2 \in C^n$ is defined as

$$(C_1 \circ C_2)(f_1, \cdots, f_{m+n-1})$$

$$= \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} C_1(f_1, \cdots, f_{i-1}, C_2(f_i, \cdots, f_{i+n-1}), f_{i+n}, \cdots, f_{m+n-1}). \quad (3.6)$$

It is then straightforward to verify the graded Jacobi identity

$$(-)^{(p_1-1)(p_3-1)}[[C_1, C_2]_G, C_3]_G + (-)^{(p_1-1)(p_2-1)}[[C_2, C_3]_G, C_1]_G + (-)^{(p_2-1)(p_3-1)}[[C_3, C_1]_G, C_2]_G = 0 \quad (3.7)$$

for $C_i \in C^{p_i}$, $i = 1, 2, 3$. Using the definition (3.5) of the $G$-bracket, the action (3.4) of the coboundary operator $d_\ast$ on a $p$-cochain $C \in C^p$ can be compactly written as

$$d_\ast C = -[C, \ast]_G \quad (3.8)$$

where $\ast \in C^2 = \text{Hom}(A \otimes A, A)$ is the multiplication of functions

$$\ast(f_1, f_2) = f_1 \ast f_2. \quad (3.9)$$

Then it is easy to show that, for any $p$-cochain $C \in C^p$,

$$d^2_\ast C = [C, \ast \circ \ast]_G = \frac{1}{2} [C, [\ast, \ast]_G]_G \quad (3.10)$$

where we used the fact $\frac{1}{2}[[\ast, \ast]_G] = \ast \circ \ast \in C^3$. Note that $[\ast, \ast]_G$ measures the associativity of the product $\ast \in C^2$ and it should identically vanish because the algebra $A$ was assumed to be associative, i.e.,

$$\frac{1}{2}[[\ast, \ast]_G(f, g, h) = (f \ast g) \ast h - f \ast (g \ast h) = 0. \quad (3.11)$$

Therefore the associativity of $A$ implies that the differential $d_\ast : C^p \rightarrow C^{p+1}$ in (3.4) is nilpotent, i.e. $d^2_\ast = 0$ and the corresponding cohomology

$$H^\ast(A, A) = \text{Ker} d_\ast / \text{Im} d_\ast \quad (3.12)$$

is called the Hochschild cohomology of cochain complex $C^\ast(A, A)$.

We first take $A = C^\infty(M)$ to be the algebra of smooth functions on a real manifold $M$. Then the Hochschild cochains of the algebra $A = C^\infty(M)$ are given by polydifferential operators on $M$ denoted by $D_{\text{poly}}(M)$. There is another differential graded Lie algebra

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7An $n$-polydifferential operator in $D_{\text{poly}}(M)$ is a multilinear map that acts on a tensor product of $n$ functions and has degree $n - 1$. In a local chart $\{y^\mu\}$, all elements of $D_{\text{poly}}(M)$ look like

$$f_1 \otimes \cdots \otimes f_n \mapsto \sum_{I_1, \ldots, I_n} C^{I_1, \ldots, I_n} \partial_{I_1} f_1 \cdots \partial_{I_n} f_n$$

where the sum is finite and $I_k$ are multi-indices.
$T_{\text{poly}}(M)$ we introduced in section 2. That is the graded Lie algebra of polyvector fields on $M$

$$T_{\text{poly}}^k(M) = \Gamma(\Lambda^{k+1}TM), \quad k \geq -1$$

(3.13)

equipped with the SN bracket (2.2) and setting the differential $d_T \equiv 0$. To prove the formality conjecture, Kontsevich showed [24] that there exists an $L_{\infty}$-quasi-isomorphism $U : T_{\text{poly}}(M) \to D_{\text{poly}}(M)$. The $L_{\infty}$-morphism $U$ is a collection of skew-symmetric multilinear maps $U_n : \otimes^n T_{\text{poly}}(M) \to D^{m-1}_{\text{poly}}(M)$ from tensor products of $n$ $k_i$-vector fields to $m$-differential operators satisfying the formality equation [24, 33, 4]. The degree of the polydifferential operator matches with the overall degree of polyvector fields if

$$m = 2 - 2n + \sum_{i=1}^{n} k_i.$$  

(3.14)

Since we will use the explicit formula for the formality equation later, we present it here though it is rather complicated as well as not really inspirational:

$$\frac{1}{2} \sum_{I\cup J=(0,1,\cdots,n)} \epsilon(I,J) \hat{Q}_2(U_{|I|}(\alpha_I),U_{|J|}(\alpha_J))$$

$$= \frac{1}{2} \sum_{i\neq j} \epsilon_\alpha(i,j,1,\cdots,\hat{i},\cdots,\hat{j},\cdots,n) U_{n-1}(Q_2(\alpha_i,\alpha_j),\alpha_1,\cdots,\hat{\alpha}_i,\cdots,\hat{\alpha}_j,\cdots,\alpha_n).$$

(3.15)

Here $\hat{Q}_2(\Phi_1,\Phi_2) = (-1)^{d_1(d_2-1)}[\Phi_1,\Phi_2]_G$ where $d_i$ is the degree of the polydifferential operator $\Phi_i$, i.e., $d_i+1$ is the number of functions it is acting on and $Q_2(\alpha_1,\alpha_2) = (-1)^{d_i} [\alpha_1,\alpha_2]_S$ where $d_i$ is the degree of the polyvector field $\alpha_i$. Finally $|I|$ denotes the number of elements of multi-indices $I$ and $\epsilon(I,J)$ is an alternating sign depending on the number of transpositions of odd elements in the permutation of $(1,\cdots,n)$ associated with the partition $(I,J)$. As a special case, $U_0 \equiv \mu$ is defined to be the ordinary multiplication of functions:

$$\mu(f_1 \otimes f_2) = f_1 f_2.$$  

(3.16)

It is then useful to introduce the Hochschild differential $d_H : D^m_{\text{poly}} \to D^m_{\text{poly}}$ defined by

$$d_H \Phi = -[\Phi,\mu]_G$$

(3.17)

for $\Phi \in D^m_{\text{poly}}$. Note that the Hochschild differential $d_H$ is a particular case of the previous coboundary operator (3.4) when the multiplication $\star$ is given by $\mu$. It is easy to check that

$$d_H^2 \Phi = \frac{1}{2} [\Phi,[\mu,\mu]]_G = 0$$

(3.18)

because $[\mu,\mu]_G$ measures the associativity of $C^\infty(M)$, i.e.,

$$\frac{1}{2} [\mu,\mu]_G(f,g,h) = (fg)h - f(gh) = 0.$$  

(3.19)

8The underlying complex for the $L_{\infty}$-morphisms is shifted complexes. For a complex $C$, the shifted complex denoted by $C[1]$ means $C[1]^k = C^{k+1}$. 

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As another special case, in particular, $U_1$ is the natural map from a $k$-vector field to a $k$-differential operator:

$$U_1(\alpha)(f_1, \cdots, f_k) = \alpha^{\mu_1 \cdots \mu_k} \partial_{\mu_1} f_1 \cdots \partial_{\mu_k} f_k = \langle \alpha, df_1 \wedge \cdots \wedge df_k \rangle$$  \hspace{1cm} (3.20)

where $\alpha \in T^{k-1}_{\text{poly}}(M)$ is defined by (2.4). Note that the formality condition (3.15) implies that $d_H U_1(\alpha) = -[U_1(\alpha), \mu]_G = 0$. In other words, $U_1(\alpha)$ is a derivation, i.e.,

$$U_1(\alpha)(f_1, \cdots, f_i, f_{i-1}, f_{i+1}, f_{i+2}, \cdots, f_k + 1) = f_i U_1(\alpha)(f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_k + 1)$$

$$+ U_1(\alpha)(f_1, \cdots, f_i, f_{i+2}, \cdots, f_k + 1)f_{i+1}$$  \hspace{1cm} (3.21)

which is certainly satisfied by the representation (3.20).

Now we will try to digest the meaning of the formality theorem in the context of deformation quantization. Consider the following formal series [25, 21, 53]:

$$\ast \equiv \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_n(\theta, \cdots, \theta),$$  \hspace{1cm} (3.22)

$$\Phi(\alpha) \equiv \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+1}(\alpha, \theta, \cdots, \theta),$$  \hspace{1cm} (3.23)

$$\Psi(\alpha_1, \alpha_2) \equiv \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+2}(\alpha_1, \alpha_2, \theta, \cdots, \theta),$$  \hspace{1cm} (3.24)

where $\theta$ is a Poisson bivector and $\alpha, \alpha_1, \alpha_2$ are polyvector fields of some degrees and we introduced a formal deformation parameter $\hbar$. According to the matching condition (3.14), $\ast$ is a bidifferential operator whereas $\Phi(\alpha)$ is a $k$-differential operator if $\alpha$ is a $k$-vector field and $\Psi(\alpha_1, \alpha_2)$ is a $(k_1 + k_2 - 2)$-differential operator if $\alpha_1$ and $\alpha_2$ are $k_1$- and $k_2$-vector fields, respectively. Since $\ast$ is a bidifferential operator that maps two functions to a function, this can be used to define a product

$$f \ast g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_n(\theta, \cdots, \theta)(f, g) = fg + hB_1(f, g) + h^2 B_2(f, g) + \cdots,$$  \hspace{1cm} (3.25)

where $B_i$ are bidifferential operators. From now on we will refer the multiplication in the Hochschild complex to the above $\ast$-product. We denote by $A_\theta := C^\infty(M)[[\hbar]]$ a linear space of formal power series of the deformation parameter $\hbar$ with coefficients in $C^\infty(M)$, which is defined by the deformation quantization map $Q : C^\infty(M) \to A_\theta$.

Let us represent the $\ast$-product in eq. (3.22) as

$$\ast = \mu + \hbar \tilde{B}.$$  \hspace{1cm} (3.26)

Given an element $\tilde{B} \in D^1_{\text{poly}}$, we can interpret $\mu + \hbar \tilde{B}$ as a deformation of the original product (3.16). One can apply the formality equation (3.15) in each order of $\hbar^n$ to yield

$$d_\ast \ast = -[\mu + \hbar \tilde{B}, \mu + \hbar \tilde{B}]_G = h^2 \Phi(\theta, \theta)_G.$$  \hspace{1cm} (3.27)

Therefore the formality theorem says [24] that the formal multiplication $\ast$ in eq. (3.26) is associative, i.e. $[\ast, \ast]_G = 0$ if and only if the bivector $\theta$ is a Poisson structure, i.e.
$[\theta, \theta]_{S} = 0$. Since the original product $\mu$ is also associative, i.e. $[\mu, \mu]_{G} = 0$, the equation (3.27) can be written as

$$d_{H} \hat{B} - \frac{\hbar}{2} [\hat{B}, \hat{B}]_{G} = 0. \tag{3.28}$$

Thus one can see that the associativity of the $\star$-product (3.25) is equivalent to the Maurer-Cartan equation (3.28) for the element $\hat{B}$ in the differential graded Lie algebra $D_{\text{poly}}(M)$.

Two star products $\tilde{\star}$ and $\star$ are said to be equivalent if and only if there exists a linear operator $D : A_{\tilde{\theta}} \to A_{\theta}$ of the form

$$Df = f + \sum_{n=1}^{\infty} \hbar^{n} D_{n}(f) \tag{3.29}$$

such that

$$f \tilde{\star} g = D^{-1}(Df \star Dg). \tag{3.30}$$

The equivalence relation (3.30) can be depicted by the commutativity of the diagram

$$\begin{array}{ccc}
A_{\tilde{\theta}} \times A_{\tilde{\theta}} & \xrightarrow{\tilde{\star}} & A_{\tilde{\theta}} \\
\downarrow D \times D & & \downarrow D \\
A_{\theta} \times A_{\theta} & \xrightarrow{\star} & A_{\theta}
\end{array} \tag{3.31}$$

When substituting the expansion (3.25) into the Maurer-Cartan equation (3.28), one can get the conditions for the coefficients $B_{i}$ in the lowest orders:

$$d_{H} B_{1} = 0, \tag{3.32}$$

$$d_{H} B_{2} - \frac{1}{2} [B_{1}, B_{1}]_{G} = 0. \tag{3.33}$$

It may be instructive to explicitly check the above Maurer-Cartan equations by considering the following expansion for the associativity of the $\star$-product (3.25):

$$(f \star g) \star h - f \star (g \star h) \equiv m_{0} + \hbar m_{1} + \hbar^{2} m_{2} + \cdots \tag{3.34}$$

where

$$m_{0} = (fg)h - (gh)f = \frac{1}{2} [\mu, \mu]_{G}(f, g, h), \tag{3.35}$$

$$m_{1} = fB_{1}(g, h) - B_{1}(fg, h) + B_{1}(f, gh) - B_{1}(f, g)h = (d_{H}B_{1})(f, g, h), \tag{3.36}$$

$$m_{2} = fB_{2}(g, h) - B_{2}(fg, h) + B_{2}(f, gh) - B_{2}(f, g)h - B_{1}(B_{1}(f, g), h) + B_{1}(f, B_{1}(g, h)) = (d_{H}B_{2} - \frac{1}{2} [B_{1}, B_{1}]_{G})(f, g, h). \tag{3.37}$$

In passing from the commutative product to a NC product, there is always an ordering problem and the explicit form of the NC product depends on the ordering prescription. The notion of equivalence between the products $\tilde{\star}$ and $\star$ can be understood as an axiomatic and generalized notion of passing from one to another ordering prescription. For example, the standard ordered star product where one writes all momenta to the right of coordinates is equivalent to the totally symmetrized Weyl ordered star product. More generally, it was proven [24, 66, 67] that the set of all star products on a symplectic manifold $(M, \omega)$ up to the equivalence (3.30) is classified by the formal de Rham cohomology $H^{2}_{dR}(M)[[\hbar]]$ and there is a unique star product on $M$ for each element $\omega + h\alpha[[\hbar]]$ with $\alpha[[\hbar]] \in H^{2}_{dR}(M)[[\hbar]]$. In particular, there is only one equivalence class of symplectic star products on $\mathbb{R}^{2n}$. 

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Since the associativity of the $\star$-product requires $m_i = 0$ for $\forall i = 0, 1, 2, \ldots$, we recover eqs. (3.32) and (3.33) at $i = 1, 2$, respectively. Let us decompose the 2-cochain $B_1$ into the sum of the symmetric part and of the anti-symmetric part:

$$B_1 = B_1^+ + B_1^-,$$

$$B_1^+(f, g) = \frac{1}{2}(B_1(f, g) \pm B_1(g, f)).$$  \hspace{1cm} (3.38)

One can solve the cocycle condition (3.32) by

$$B_1 = B_1^H + d_H D_1$$  \hspace{1cm} (3.39)

where $B_1^H$ is a harmonic cochain in the second Hochschild cohomology group (3.12) (with the multiplication $\star = \mu$) and $D_1$ is an arbitrary differential operator. Note that

$$d_H D_1(f, g) = f D_1(g) - D_1(f g) + D_1(f) g$$  \hspace{1cm} (3.40)

and so $d_H D_1(f, g) = d_H D_1(g, f)$. Therefore it is possible to take the symmetric part $B_1^+$ such that $B_1^+ = d_H D_1$ by choosing an appropriate differential operator $D_1$. It is then easy to prove that $m_1 = 0$ in eq. (3.36) can be solved if and only if $B_1^H = B_1^-$ and $B_1^H$ is a derivation of $C^\infty(M)$, i.e.,

$$B_1^H(f, gh) = g B_1^H(f, h) + B_1^H(f, g) h.$$  \hspace{1cm} (3.41)

Thus, as we noticed in eq. (3.24), the harmonic cocycle $B_1^-$ comes from a Poisson bivector field $\theta$ on $M$:

$$B_1^-(f, g) = \{ \theta, df \wedge dg \} = \{ f, g \}_\theta, \quad \theta \in \Gamma(\Lambda^2 T M).$$  \hspace{1cm} (3.42)

And the second Hochschild cocycle $B_1^-$ is invariant under the gauge transformation (3.30) because eq. (3.30) implies that

$$B_1'(f, g) = B_1(f, g) + d_H D_1(f, g).$$  \hspace{1cm} (3.43)

In other words, the symmetric part $B_1^+$ in eq. (3.38) can be killed by the gauge transformation (3.30). The above argument in the first order approximation can be generalized to all orders by solving (3.27). Hence the formality theorem asserts that the deformation of the commutative multiplication $\mu$ on $C^\infty(M)$ is in bijection to deforming the Poisson bracket on $C^\infty(M)$ and so one can describe the deformations of the Poisson algebra $\mathfrak{P} = (C^\infty(M), \{ -, - \}_\theta)$ as the elements of the Hochschild complex obeying the formality equation (3.15). For that reason we call the $\star$-product (3.25) a formal quantization of the Poisson algebra $\mathfrak{P}$. The simplest example of deformation quantization is the Moyal star product defined by

$$(f \star g)(y) = e^{i \frac{1}{2} \theta^{\mu \nu} \partial_\mu \partial_\nu} f(x) g(y)|_{x=y}$$  \hspace{1cm} (3.44)

which is a particular case of the Kontsevich’s star product (3.25) with a constant Poisson structure $\theta^{\mu \nu} = \left( \frac{1}{T} \right)^{\mu \nu}$.

Given a Poisson structure $\theta$, i.e. $[\theta, \theta]_S = 0$, the second Hochschild cohomology (3.12) defines the equivalence class of star multiplications if

$$\widetilde{\star} = \star + d_\star \widetilde{C}$$  \hspace{1cm} (3.45)
where \( \widehat{C} \) is an arbitrary differential operator in \( D_{\text{poly}}^0(M) \). Note that eq. (3.45) is an infinitesimal version of eq. (3.30) where \( D = \text{id} + \widehat{C} \) is defined by eq. (3.29). Two star products in the same equivalence class obeying eq. (3.30) automatically satisfy the associativity condition

\[
[x, y]_G(f, g, h) = 2D^{-1}((Df * Dg) * Dh - Df * (Dg * Dh)) = 0 \tag{3.46}
\]

if the original \( * \)-product is associative.

Suppose that there are two associative star products \( *' \) and \( * \). We may consider the star product \( *' \) as a deformation of the star product \( * \) like as eq. (3.23), i.e.,

\[
*' = * + \hbar \hat{B} \tag{3.47}
\]

but keeping the associativity, i.e., \([*', *]_G = 0\). Since the initial \( * \)-product is associative, i.e., \([*, *]_G = 0\), the associativity condition of the new star product \( *' \) can be written as the following Maurer-Cartan equation

\[
d_* \hat{B} - \frac{\hbar}{2} [\hat{B}, \hat{B}]_G = 0. \tag{3.48}
\]

Therefore, we may connect every associative products by the deformation \( \hat{B} \in D_{\text{poly}}^1(M) \) which obeys the Maurer-Cartan equation (3.48). But they need not belong to the same equivalence class or the same Hochschild cohomology \( H^2(\mathcal{A}, \mathcal{A}) \) unless \( \hat{B} = d_* \widehat{C} \). In other words, the space of all associative algebras may be generated by the solutions of the Maurer-Cartan equation (3.48).

We know that \( \delta_X^\alpha \equiv \Phi(X) \) in eq. (3.22) is a linear differential operator if \( \alpha = X \) is a vector field in \( \Gamma(TM) \). And \( \hat{f} \equiv \Phi(f) \) and \( \Psi(X, Y) \) are functions in \( \mathcal{A}_\theta \) if \( f \in C^\infty(M) \) and \( X, Y \in \Gamma(TM) \). For a Poisson structure \( \theta \), the formality condition (3.15) leads to the relations [23, 26]

\[
d_* \Phi(X) = -[\Phi(X), *]_G = h\Phi(d_\theta X), \tag{3.49}
\]
\[
d_* \hat{f} = -[\Phi(f), *]_G = h\Phi(d_\theta f), \tag{3.50}
\]
\[
[\Phi(X), \Phi(Y)]_G + [\Psi(X, Y), *]_G = \Phi([X, X]_S) + h(\Psi(d_\theta X, Y) + \Psi(X, d_\theta Y)), \tag{3.51}
\]

where \( d_\theta X = -\mathcal{L}_X \theta \) and \( d_\theta f = X_f \) according to eqs. (2.3) and (2.13) and \( \Psi(d_\theta X, Y) \) and \( \Psi(X, d_\theta Y) \) are differential operators since \( d_\theta X \) and \( d_\theta Y \) are 2-vector fields. Note that \( d_* \hat{f} = h\Phi(d_\theta f) \) is an inner derivation of the \( * \)-product, i.e.,

\[
(d_* \hat{f}) (g) = (g * \hat{f} - \hat{f} * g) = -[\hat{f}, g]_*. \tag{3.52}
\]

A vector field \( X \) that preserves the Poisson bracket, i.e., \( d_\theta X = -\mathcal{L}_X \theta = 0 \), is called the Poisson vector field. Thus, it results from eq. (3.43) that, if \( X \) is a Poisson vector field, then \( \delta_X^* \) is a derivation of the \( * \)-product, viz.,

\[
[d_* \Phi(X)](f, g) = -\delta_X^* (f * g) + f * \delta_X^* g + (\delta_X^* f) * g = 0. \tag{3.53}
\]

\( ^{10} \)It may be remarked that \( D_{\text{poly}}^0(M) \) is the set of differential operators and so the generators in \( \text{Diff}(M) \) belong to \( D_{\text{poly}}^0(M) \). Therefore, if we change coordinates in star products, we obtain a gauge equivalent star product. Cf. the Theorem 2.3 in Ref. [24].
In this case, the relation (3.51) reads as

$$\left[ \delta^*_X, \delta^*_Y \right] G - \delta^*_\left[X,Y\right] = d_\mu \Psi(X,Y)$$  \hspace{1cm} (3.54)

where $[X,Y]$ is the ordinary Lie bracket of vector fields. The result (3.54) illustrates that the formality map $U : T_{poly}(M) \to D_{poly}(M)$ from the algebra of multivector fields to the algebra of multidifferential operators fails to preserve the Lie algebra structure but the difference between the two terms is an exact cochain in $D_{poly}(M)$. So the map $U$ induces an isomorphism between the cohomology groups of corresponding complexes. This is exactly the role played by the $L_\infty$-morphism $U$.

Given a Poisson vector field $X$, $d\theta X = 0$, one can solve eq. (3.49) by

$$\Phi(X) = \delta^*_X + d_\mu f$$  \hspace{1cm} (3.55)

where $\delta^*_X$ is an element of the first Hochschild cohomology in eq. (3.12) and a Hamiltonian vector field $d\theta f$ is now mapped to the inner derivation $d_\mu f \equiv \text{ad}_{\mu}^- f = -\left[ f, \cdot \right]$. Note that the first Poisson cohomology group $H_1^\theta(M)$ in eq. (2.10) is the quotient of the space of Poisson vector fields (i.e. vector fields $X$ such that $d\theta X = -[X, \theta]_S = 0$) by the space of Hamiltonian vector fields (i.e. the vector fields of the type $d\theta f = -[f, \theta]_S = X f$). Therefore the first Hochschild cohomology of the deformation quantization algebra $A_\theta$ is isomorphic to the first Poisson cohomology of the Poisson structure $\theta$.

It may be rewarding to check eqs. (3.49) and (3.50) up to next-to-leading order by considering the following expansions

$$f \star g = fg + \hbar \theta^{\mu\nu} \partial_\mu f \partial_\nu g + \cdots,$$
$$\delta^*_X = X + \hbar U_2(X, \theta) + \cdots,$$
$$\hat{f} = f + \hbar U_2(f, \theta) + \cdots.$$  \hspace{1cm} (3.56)

It is straightforward to derive the result

$$[d_\mu \Phi(X)](f,g) = -\hbar (L_X \theta + [U_2(X, \theta), \mu]_G)(f,g) + \mathcal{O}(\hbar^2).$$  \hspace{1cm} (3.57)

Hence one can see that $d_H U_2(X, \theta) = -[U_2(X, \theta), \mu]_G = 0$ for an arbitrary vector field $X$ (not necessarily a Poisson vector field obeying $L_X \theta = 0$) to satisfy the relation (3.49). In other words, the differential operator $U_2(X, \theta)$ in eq. (3.23) is a derivation with respect to the ordinary product $\mu$. Such a differential operator can be absorbed into the ordinary vector field $X$ in the leading term of eq. (3.23), which means that one can set $U_2(X, \theta) = 0$. Similarly eq. (3.50) requires that $\mu(U_2(f, \theta) \otimes g - g \otimes U_2(f, \theta)) = 0$. That is, $U_2(f, \theta)$ is still a usual commutative function at the first order in $\theta$ and so it can be gauged away by redefining the commutative function $f$ which results in $U_2(f, \theta) = 0$. (See also the footnote 10 in Ref. [21].) This fact leads to an interesting consequence in emergent gravity which will be discussed later.
4. Noncommutative gauge theory

Suppose that there exists a line bundle $L$ on a Poisson manifold $(M, \theta)$ whose Poisson bivector $\theta$ takes the Darboux-Weinstein frame \[ (2.33) \]. And recall that U(1) gauge fields arise as the one-form connection $A$ of the line bundle $L$ on the Poisson manifold and they manifest themselves as a deformation of Poisson structure as was shown in eq. (2.19). But we can use the exponential map \[ (2.22) \] to define the Poisson algebra on the Darboux-Weinstein chart \[ (2.33) \] again even in the presence of U(1) gauge fields as was verified in eq. (2.25). Then Poisson gauge fields are incarnated as a coordinate representation of the exponential map \[ (2.34) \] on the local chart. A NC gauge theory is basically defined by quantizing the Poisson algebra on such Darboux-Weinstein charts (the right-hand side of eq. \[ (2.25) \]). The construction is a rather straightforward application of the formality maps in eqs. \[ (3.22)-(3.24) \]. We will briefly review some essential points to frame important applications to emergent gravity, focusing only on the rank one (Abelian) case.

So far we have examined how the deformation quantization of any Poisson manifold $(M, \theta)$ can be derived from the formality theorem which stipulates the existence of an $L_\infty$-quasi-isomorphism $U : T_{\text{poly}}(M) \to D_{\text{poly}}(M)$ from the differential graded Lie algebra of polyvector fields on $M$ with vanishing differential on the SN bracket \[ (2.2) \] into the differential graded Lie algebra of polydifferential operators on $M$ with the Hochschild differential $d_*$ and the G-bracket \[ (3.3) \]. Since Poisson gauge fields are defined by the exponential map \[ (2.39) \] and they are subject to the Poisson bracket defined by the right-hand side of eq. \[ (2.25) \], we can try to quantize the Poisson algebra $(M, \{-,-\}_\theta)$ to define a NC gauge theory on the Poisson manifold $(M, \theta)$ \[ [20, 21] \]. The symplectic gauge fields and corresponding NC gauge fields will be a particular case in which the Poisson structure is nondegenerate. Using the anchor map \[ (2.11) \], one can always associate the U(1) gauge field $A$ with a vector field $A_\theta$ defined by eq. \[ (2.12) \]. Then the formality map \[ (3.23) \] maps the vector field $A_\theta$ to a differential operator defined by

$$A_* = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+1}(A_\theta, \theta, \cdots, \theta). \quad (4.1)$$

One can also apply the formula \[ (3.49) \] to yield the bidifferential operator $d_* A_* = \hbar F_*$ corresponding to the U(1) field strength $F = dA$:

$$F_* = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+1}(F_\theta, \theta, \cdots, \theta) \quad (4.2)$$

where $F_\theta = d_\theta A_\theta$ is given by eq. \[ (2.13) \]. In section 2, we have introduced the Poisson gauge fields $\mathfrak{A}_A$ in eq. \[ (2.24) \] through the Moser flow \[ (2.22) \]. Similarly the corresponding NC gauge fields are defined by introducing a $t$-dependent star product

$$\star_t = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_n(\theta_t, \cdots, \theta_t) \quad (4.3)$$
and a quantum evolution equation\footnote{Since the G-bracket $[A_{\ast t}, \ast_t]_G$ is a quantum generalization of the classical Lie bracket $[A_{\theta t}, \theta_t]$, we may define a quantum Lie derivative with respect to a differential operator $A_{\ast t}$ by $\hat{L}_{A_{\ast t}} \Theta \equiv [A_{\ast t}, \Theta]_G$ for $\Theta \in D_{poly}(M)$. Then the quantum evolution equation (4.4) takes a suggestive form $\partial_t \ast_t = -\hbar F_{\ast_t}$.}

$$\partial_t \ast_t = -\hbar F_{\ast_t}$$  \hspace{1cm} (4.4)

where $F_{\ast_t} = d_{\ast_t} A_{\ast_t} = -[A_{\ast_t}, \ast_t]_G$. First it may be instructive to consider an infinitesimal deformation of the star product (4.3) by taking the limit $t = \epsilon \to 0$. It is easy to show that the result is given by eq. (3.47) with $B = -F_*$. Therefore the bidifferential operator (4.2) has to satisfy the Maurer-Cartan equation

$$d_{\ast} F_{\ast} + \frac{\hbar}{2} [F_{\ast}, F_{\ast}]_G = 0$$  \hspace{1cm} (4.5)

in order for the new star product $\ast' := \ast_\epsilon$ to preserve the associativity. The evolution equation (4.4) is a quantum version of the classical evolution (2.16) and indeed it is derived from eq. (4.3) using eqs. (2.17) and (4.2). Since the quantum $t$-evolution is generated by the differential operator $A_{\ast_t}$, it can be integrated to a quantum Moser flow \cite{20, 21}

$$D_A = \exp(hA_{\ast_t} + \partial_t) \exp(-\partial_t)|_{t=0}$$  \hspace{1cm} (4.6)

which relates two star products $\ast' = \ast_1$ and $\ast = \ast_0$ such that $D_A (\ast') = \ast$.

The quantum flow $D_A$ plays an important role of covariantizing map $f \mapsto D_A f := f + f_A$ which maps a function $f \in A_\theta$ to a covariant function $D_A f \in A_\theta$ that transforms under NC gauge transformations by conjugation

$$D_A f \mapsto \Lambda \ast D_A f \ast \Lambda^{-1}.$$  \hspace{1cm} (4.7)

Note that the covariance map $D_A$ depends on gauge fields as eq. (4.6) clearly indicates. And, as we remarked before, it is also defined on local patches of a Poisson manifold where the local gauge field $A$ is defined. But it can be globalized by gluing local patches together using NC gauge (and coordinate) transformations between local patches \cite{22}. It was shown \cite{20, 21} that the quantum Moser flow (4.6) is defined as a quantization of the classical Moser flow $\rho_{\ast A}$, i.e., $D_A = D \circ \rho_{\ast A}$. With this property, the local covariance map $D_A$ can be used to define a new star product $\ast'$ via

$$f \ast' g = D_A^{-1}(D_A f \ast D_A g).$$  \hspace{1cm} (4.8)

Although the new star product $\ast'$ depends on gauge fields, they appear only via the gauge invariant field strength $F = dA$ and so it can be globally defined (after a globalization à la \cite{23}). Nevertheless the equivalence relation (4.8) between two star products $\ast'$ and $\ast$ holds locally because it is involved with the locally defined covariance map. Globally the star products $\ast'$ and $\ast$ are in general neither gauge equivalent nor in the same cohomology class. Instead it was shown in \cite{22, 24} that two star products are Morita equivalent if and only if they are, modulo diffeomorphisms, related by the action of an element $F \in \Pic(C^\infty(M)) \cong \cdots$
$H^2(M, \mathbb{Z})$, i.e., an element of equivalence classes of a line bundle $L$ over $M$. The closed two-form $F$ representing the first Chern class $c_1(L)$ of the line bundle $L \to M$ acts on the Poisson structure $\theta$ as eq. (2.19) to generate a new Poisson structure $\theta_1 = \Theta$. Therefore the Morita equivalent star products $\star'$ and $\star$ in the local gauge equivalence (4.8) are related by the action of a line bundle $L$, i.e.,

$$L : [\star] \to [\star']$$  \hspace{1cm} (4.9)

and the invertible covariantizing map (4.6) can be considered as a quantum lift of the exponential map (2.23).

We exactly mirror the classical case (2.24) to define NC gauge fields

$$D_A = \text{id} + \hat{A}_A.$$  \hspace{1cm} (4.10)

The NC gauge field $\hat{A}_A = D_A - \text{id}$ is a local 1-cochain in $D^0_{\text{poly}}(M)$, i.e., a formal differential operator depending on U(1) gauge fields and transforms under the NC gauge transformation (4.7) as

$$f_A \mapsto f_A' = \Lambda \star f_A \star \Lambda^{-1} + \Lambda \star [f, \Lambda^{-1}] \star,$$  \hspace{1cm} (4.11)

where $f_A \equiv \hat{A}_A(f)$. The star product (4.3) describes a formal associative deformation of the Poisson structure $\theta_t$ defined by eq. (2.19). The star product $\star'$ is a formal deformation of the original star product $\star$ in the sense of eq. (4.47) if an element $\hat{B} = \frac{1}{\hbar}(\star' - \star) \in D^1_{\text{poly}}(M)$ obeys the Maurer-Cartan equation (3.48). Note that in terms of NC gauge fields in eq. (4.10), the gauge equivalence (4.8) can be written as the form

$$D_A \circ (\star' - \star) = d_A \hat{A}_A + \hat{A}_A \star \hat{A}_A \equiv F_A.$$  \hspace{1cm} (4.12)

Therefore $\star'$ is gauge equivalent to $\star$ provided $\hat{B} = \frac{1}{\hbar}D_A^{-1} \circ F_A \in D^1_{\text{poly}}(M)$. Then the Morita equivalence between the formal deformations of Poisson structures $\theta_1 = \Theta$ and $\theta_0 = \theta$ means that the Maurer-Cartan element $\hat{B}$ describes an orbit of the action (4.4) of the formal diffeomorphism $D_A$.

All quantities in deformation quantization are defined by formal power series of a deformation parameter, e.g., typically $\hbar$. Hence it is necessary to keep track of $\hbar$ to control the expansion of the power series. We are applying this expansion to the formal series in eqs. (3.22)-(3.24) by simply taking the replacement $\theta \to \hbar \theta$ in the formality maps. Then the classical (or commutative) limit corresponds to the limit $\hbar \to 0$. In this classical limit, the covariance map $D_A = D \circ \rho_A^*$ in eq. (4.6) as well as the equivalence map $D$ in eq. (2.23) reduce to the identity operators and so the Moser flow $\rho_A^*$ in eq. (2.22) must also become an identity operator in the limit. But it is not obvious to see how the exponential map (2.22) reduces to the identity operator in the limit $\hbar \to 0$. In order to cure this situation, it may be necessary to introduce a formal vector field $X \in \Gamma(TM)[[\hbar]]$ and a formal Poisson tensor $\vartheta \in \Gamma(\Lambda^2 TM)[[\hbar]]$:

$$X = h \bar{X}_1 + h^2 \bar{X}_2 + \cdots \equiv hX,$$  \hspace{1cm} (4.13)

$$\vartheta = h \bar{\theta}_1 + h^2 \bar{\theta}_2 + \cdots \equiv h\theta.$$  \hspace{1cm} (4.14)
This can be simply implemented by replacing \( \theta \) by \( \hbar \theta \) and, accordingly, \( A_\theta \to \hbar A_\theta \), for all formulas in section 2. Important changes, for example, are given by

\[
(2.17) \quad \to \quad \partial_t \theta_t = -\hbar F_{\theta_t}, \quad (4.15)
\]

\[
(2.19) \quad \to \quad \theta_t = \theta \frac{1}{1 + \hbar F \theta}, \quad (4.16)
\]

\[
(2.22) \quad \to \quad \rho^*_A = \exp(\hbar A_{\theta_t} + \partial_t) \exp(-\partial_t)|_{t=0}. \quad (4.17)
\]

Now the Moser flow (4.17) shows the desired behavior, \( \rho^*_A \to \text{id} \), in the classical limit \( \hbar \to 0 \). In particular, the new formula (4.16) gives us the formal relation

\[
\Theta = \theta \frac{1}{1 + \hbar F \theta} \quad (4.18)
\]

between Poisson structures \( \theta_1 = \Theta \) and \( \theta_0 = \theta \), which suggests an attractive picture. In the classical limit, there is a primitive Poisson structure \( \theta \) and other Poisson structures can be obtained by deforming the primitive Poisson structure \( \theta \) in terms of line bundle \( L \to M \) such that the curvature \( F = dA \) acts on the Poisson structure. Then the formality map \( U : \Theta \mapsto \ast' \) in eq. (4.3) gives us the star product \( \ast' \) locally equivalent to the star product \( \ast \) of the original Poisson structure \( \theta = \rho^*_A(\Theta) \) which realizes the action (4.9) of the line bundle \( L \). Therefore it is not necessary to consider the set of all Poisson structures in the classical limit. Instead, without loss of generality, we can assume the primitive Poisson structure \( \theta \) to be in the Darboux-Weinstein frame (2.33) and then deform it by turning on U(1) gauge fields to generate the set of all possible Poisson structures with a fixed first order \( \partial \theta \). This approach provides a more unified description of formal deformations of a given Poisson structure \( \theta \).

The equivalence classes for the deformation quantization of a Poisson manifold \( M \) are characterized by geometric properties of the underlying manifold \( M \). For example, for a symplectic manifold \( M \), they are parameterized by the second de Rham cohomology space \( H^2(M, \mathbb{R}) \). See the footnote 9. Now we are considering a line bundle \( L \to M \) on a fixed Poisson manifold \( (M, \theta) \). Since the line bundle \( L \) acts on the Poisson structure as eq. (4.18), one can define NC U(1) gauge theory as the deformation quantization of the (dynamical) Poisson manifold \( (M, \Theta) \). The equivalence relation (4.18) then implies that the deformation quantization \( \ast' \) of the dynamical Poisson manifold \( (M, \Theta) \) is Morita-equivalent to the deformation quantization \( \ast \) of the base manifold \( (M, \theta) \) and the NC U(1) gauge theory describes their equivalent categories of modules. Thus, given two star products \( \ast' \) and \( \ast \) in the Morita equivalent class, we can always associate a NC U(1) gauge theory to realize

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12If we implemented the \( h \)-expansions (4.13) and (4.14) in the formality maps (4.1) and (4.2) (by the replacements \( A_\theta \to \hbar A_\theta \) and \( F_\theta \to \hbar^2 F_\theta \)) from the outset, the formulas (4.3) and (4.4) did not get the \( h \)-factor. But the differential operator \( A_\ast \) would instead start from an \( O(h) \)-term and so the classical limit of \( D_A \) in eq. (4.6) will be same.

13One may understand the formal power series (4.14) as the deformation in terms of \( U(1) \) gauge fields which is akin to eq. (4.18), i.e., \( \theta = \theta_1 \frac{1}{1 + \hbar F \theta_1} \) where \( f = d\bar{a} \). Substituting this expression into eq. (4.18) yields the result \( \Theta = \theta_1 \frac{1}{1 + \hbar (F + \bar{f}) \theta_1} \). Hence one may attribute the quantum \( h \)-corrections in eq. (4.18) to the deformation due to \( U(1) \) gauge fields by a simple shift \( F \to F + \bar{f} \).
the action \( \text{(1.9)} \). For example, the deformation quantization of any symplectic manifold is described by a NC U(1) gauge theory where the system can be described by the Moyal star product \( \text{(3.44)} \). But this description is local in nature because it is linked to a specific choice of coordinates known as the Darboux coordinates.

The topology of a line bundle \( L \) on commutative \( \mathbb{R}^{2n} \) is trivial and this fact leads to the conclusion in the footnote \( \text{[3]} \). But it becomes nontrivial on a symplectic or NC \( \mathbb{R}^{2n} \) because the NC space admits the existence of NC U(1) instantons \( \text{[68, 69]} \). Furthermore it was shown in \( \text{[3, 4]} \) that the singularities of U(1) instantons on commutative \( \mathbb{R}^{4} \) are blown up to noncontractible two cycles after turning on the noncommutativity of the space \( \mathbb{R}^{4} \) and so incorporating the backreaction of NC U(1) instantons brings about a topology change of \( \mathbb{R}^{4} \) in the context of emergent gravity. As a result, the first Chern class \( c_1(L) \) of the line bundle \( L \) supported on those two cycles is nonzero. Therefore it may be necessary to also include singular line bundles of U(1) instantons in the classification of star products on \( \mathbb{R}^{2n} \) because the singularity of U(1) instantons can be resolved by NC deformations.\(^{14} \)

That is, the Morita equivalence of star products may be more subtle due to the topology change of an underlying manifold triggered by NC U(1) instantons. This means that there exists a nontrivial class of star products if the line bundle \( L \rightarrow M = \mathbb{R}^{2n} \) describes (generalized) U(1) instantons (for instance, Hermitian U(1) instantons obeying the Donaldson-Uhlenbeck-Yau equations in six dimensions) whose singularities at a finite number of points are blown up to two cycles. In the end the classification of star products can be interpreted as the (Morita) equivalence classes of NC U(1) gauge theories which are parameterized by the first Chern class \( c_1(L) \) of the line bundle \( L \) replacing the second de Rham cohomology space \( H^2(M, \mathbb{R}) \) for Poisson manifolds.

Let us recapitulate how we got the NC gauge field \( \text{(4.10)} \). We started with a U(1) gauge theory defined on a Poisson manifold \( (M, \theta) \). The U(1) gauge field \( A \) is mapped via the anchor map \( \text{(2.11)} \) to a general vector field \( A_\theta \) with the bivector field \( F_\theta = d_\theta A_\theta \) in \( T_{\text{poly}}(M) \). The vector field \( A_\theta \) is then mapped via the formality map \( \text{(4.1)} \) to an arbitrary differential operator \( A_* \) in \( D_{\text{poly}}(M) \) with the field strength \( F_* = \frac{i}{\hbar} d_* A_* \). The whole mapping is depicted by the diagram:

\[
\begin{array}{cccccc}
A & \xrightarrow{\rho} & A_\theta & \xrightarrow{U} & A_* \\
d & & d_\theta & & d_* \\
F & \xrightarrow{\rho} & F_\theta & \xrightarrow{U} & F_*
\end{array}
\]

(4.19)

The chain of maps in eq. \( \text{(4.19)} \) is in general defined only locally because closed two-forms are only locally exact, i.e., \( F = dA \). In this correspondence the role of U(1) gauge fields can be understood as a deformation of the Poisson structure \( \theta \) of a base manifold \( (M, \theta) \) which can be described by the evolution equation \( \text{(1.13)} \) in \( T_{\text{poly}}(M) \). According to the formality map \( \text{(1.3)} \), the classical evolution equation in \( T_{\text{poly}}(M) \) can be lifted to the quantum evolution equation \( \text{(1.4)} \) in \( D_{\text{poly}}(M) \). The NC U(1) gauge field \( \hat{A}_A \) in eq. \( \text{(1.11)} \)

\(^{14}\) We thank Stefan Waldmann for helpful discussions related to this aspect.
is defined through the integral curve (4.6) of the quantum evolution equation. Since \( A \) is a \( U(1) \) gauge field whose gauge transformation is given by \( A + d\lambda \), one can also consider the corresponding mapping (4.19) after the gauge transformation:

\[
A + d\lambda \xrightarrow{\rho} A_\theta + d_\theta \lambda \xrightarrow{U} A_\ast + \frac{1}{\hbar} d_\ast \tilde{\lambda}
\]  

(4.20)

where the relation (3.50) was used. Thus one can consider a quantum Moser flow generated by the gauge-transformed differential operator \( A_\ast + \frac{1}{\hbar} d_\ast \tilde{\lambda} \) \[21\]:

\[
\mathcal{D}_{A + d\lambda} = \exp(hA_\ast + d_\ast \tilde{\lambda} + \partial_\ast) \exp(-\partial_\ast)|_{t=0} = (\text{id} + d_\ast \hat{\Lambda}) \circ \mathcal{D}_{A + \mathcal{O}(\tilde{\lambda}^2)}
\]  

(4.21)

where \( \hat{\Lambda}(\lambda, A) \) is a quantum version of the gauge parameter (2.28) and is given by

\[
\hat{\Lambda}(\lambda, A) = \sum_{n=0}^{\infty} \frac{(hA_\ast + \partial_\ast)^n(\tilde{\lambda})}{(n + 1)!}|_{t=0}.
\]  

(4.22)

In terms of NC gauge fields, the NC gauge transformation (4.21) reads as

\[
\hat{A}_{A + d\lambda} = \hat{A}_{A} + (d_\ast \hat{\Lambda} + [\hat{A}_A, \hat{\Lambda}]_\ast)
\]  

(4.23)

or

\[
\hat{A}_{A + \delta_{\lambda} A} = \hat{A}_{A} + \hat{\delta}_{\Lambda} \hat{A}_{A},
\]  

(4.24)

where \( \delta_{\lambda} A = d\lambda \) and \( \hat{\delta}_{\Lambda} \hat{A}_A \equiv \hat{D}_\ast \hat{\Lambda} = d_\ast \hat{\Lambda} + [\hat{A}_A, \hat{\Lambda}]_\ast \). Note that the right-hand side of eq. (4.23) is an infinitesimal version of the finite NC gauge transformation (4.11) with \( \Xi \approx 1 - \hat{\Lambda} \). The gauge equivalence relation (4.24) constitutes the SW map from commutative \( U(1) \) gauge fields to NC \( U(1) \) gauge fields, which is a quantum version of (semi-)classical SW map (2.37). The SW map can be depicted by the commutativity of the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{SW}} & \hat{A}_A \\
& \downarrow_{\delta_{\lambda}} & \downarrow_{\hat{\delta}_{\Lambda}} \\
A + \delta_{\lambda} A & \xrightarrow{\text{SW}} & \hat{A}_A + \hat{\delta}_{\Lambda} \hat{A}_A
\end{array}
\]  

(4.25)

It is straightforward to show that the NC gauge transformations in eq. (4.24) form a Lie algebra under the \( \ast \)-product, i.e.,

\[
[\hat{\delta}_{\Lambda_1}, \hat{\delta}_{\Lambda_2}] = \hat{\delta}_{[\Lambda_1, \Lambda_2]}_\ast,
\]  

(4.26)

which of course recovers (2.38) in the (semi-)classical limit.

\[15\] In order to derive (4.21), it may be necessary to use the Leibniz rule of the Hochschild differential (3.8), \( d_\ast(C_1 \circ C_2) = (-1)^{|C_1|} (d_\ast C_1) \circ C_2 + C_1 \circ (d_\ast C_2) \), for \( C_1 \in C^n \) and \( C_2 \in C^n \) and \( [hA_\ast + \partial_\ast, d_\ast \tilde{\lambda}]_G = d_\ast [hA_\ast + \partial_\ast](\tilde{\lambda}) \) where \( [\partial_\ast, d_\ast \tilde{\lambda}]_G(f) = \partial_\ast([f, \tilde{\lambda}]_\ast) \).
In order to define the field strength of NC gauge fields, it is useful to introduce the Chevalley-Eilenberg (CE) complex \((C^\bullet(A,A), d_{CE} \equiv \widehat{d})\) and a cup product \(\circ\). The CE complex is defined as follows \([77]\). A \(p\)-cochain \(C^p = \text{Hom}(A^\wedge^p, A)\) is the space of \(p\)-linear skew-symmetric maps with values in an associative algebra \(A\) and the differential \(\widehat{d} : C^p \rightarrow C^{p+1}\) is defined on homogeneous elements by

\[
2h(\widehat{d}C)(f_1, \cdots, f_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} [f_i, C(f_1, \cdots, \hat{f}_i, \cdots, f_{p+1})]_\ast
+ \sum_{i<j} (-1)^{i+j} C([f_i, f_j]_\ast, f_1, \cdots, \hat{f}_i, \cdots, \hat{f}_j, \cdots, f_{p+1})
\]

(4.27)

where the hat stands for an omitted symbol. It is straightforward to prove that \(\widehat{d}^2 = 0\), using the Jacobi identity of the associative \(\ast\)-algebra \(A\). It may be instructive to explicitly check that \((\widehat{d}^2 C)(f_1, f_2, f_3) = 0\). The cup product \(\circ : C^m \times C^n \rightarrow C^{m+n}\) is defined by

\[
h(C_1 \circ C_2)(f_1, \cdots, f_{m+n}) = \frac{1}{(m+n)!} \sum_{\sigma \in \mathfrak{S}_{m+n}} \epsilon(\sigma) C_1(f_{\sigma_1}, \cdots, f_{\sigma_m}) \ast C_2(f_{\sigma_{m+1}}, \cdots, f_{\sigma_{m+n}})
\equiv h(C_1 \circ C_2)(f_1 \wedge \cdots \wedge f_{m+n})
\]

(4.28)

where we introduced the symmetrization map defined by

\[(f_1, \cdots, f_k) \mapsto (f_1 \wedge \cdots \wedge f_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)(f_{\sigma_1}, \cdots, f_{\sigma_k})\]

(4.29)

and \(\epsilon(\sigma)\) is an alternating sign depending on the number of transpositions in the permutation \(\sigma \in \mathfrak{S}_k\) of \(k\) elements.

We define the field strength of NC gauge fields evaluated on two functions \(f, g\) by lifting the case \([2.29]\) of Poisson gauge fields to the NC version:

\[
\widehat{F}_A(f, g) = \frac{1}{2h} \left( [D_A(f), D_A(g)]_\ast - D_A([f, g]_\ast) \right) = \left( D_A \circ D_A - \frac{1}{h} D_A \circ \ast \right)(f \wedge g)
= \widehat{\delta}_A(f \wedge g)
\]

(4.30)

where \(\widehat{\delta}_A \equiv \widehat{d}\widehat{A}_A + \widehat{A}_A \wedge \widehat{A}_A\) and \((D_A \circ \ast)(f \wedge g) = \frac{1}{2} D_A([f, g]_\ast)\).\(^{16}\) Note that the CE-differential \(\widehat{d}\) in \((4.27)\) is an antisymmetric version of the Hochschild differential \(d_\ast\) in \((3.4)\), e.g., \(h(\widehat{d}\widehat{A}_A)(f, g) = (d_\ast\widehat{A}_A)(f \wedge g)\) and \(2h(\widehat{d}\widehat{\delta}_A)(f, g, h) = 3(d_\ast\widehat{\delta}_A)(f \wedge g \wedge h)\). It leads to the relation that the field strength \((4.30)\) is the antisymmetrized version of the Hochschild field strength \((4.12)\), i.e.,

\[
\widehat{\delta}_A(f, g) = \frac{1}{h} F_A(f \wedge g).
\]

(4.31)

\(^{16}\)We combined the definition of the cup product \(\circ : C^1 \times C^1 \rightarrow C^2\) and the skew-symmetric property of \(f \wedge g\) to define the wedge product \((\widehat{A}_A \circ \widehat{A}_A)(f \wedge g) = (\widehat{A}_A \wedge \widehat{A}_A)(f \wedge g).\) The \(h\) factor in eqs. \((4.27)\) and \((4.28)\) and the factor 2 in eq. \((4.27)\) are originated from the fact that \([f, g]_\ast = 2h[f, g]_\ast + \mathcal{O}(h^3)\). Furthermore it may be necessary to replace the formal deformation parameter \(h\) to \(\frac{i}{\hbar}\) to match with the physicist’s convention such as eq. \((3.44)\).
Combining the field strength \( \hat{F}_A(f,g) = \frac{1}{2\pi} D_A((f,g) \star [f,g] \star) = \frac{1}{\hbar} (D_A \circ (\star' - \star))(f \wedge g) \), the NC field strength \( \hat{F}_A \) can compactly be written as
\[
\hat{F}_A = \frac{1}{\hbar} D_A \circ (\star' - \star). \tag{4.32}
\]

Using the formality map \( \hat{F}_A \) and \( \theta_1 = \Theta = \theta + F_\theta \) where \( F_\theta \) is given by eq. \( (2.31) \), one can calculate the right-hand side of eq. \( (4.32) \) and the result is given by
\[
\hat{F}_A = D_A \circ (\Phi(F_\theta) + \frac{\hbar}{2} \Psi(F_\theta, F_\theta) + \cdots). \tag{4.33}
\]

Note that both sides of eq. \( (4.33) \) are bidifferential operators. The map \( \hat{F}_A \) between ordinary and NC gauge fields constitutes the exact SW map for NC U(1) gauge theory \[18\] and represents a quantum version of the (semi-)classical SW map \( (2.45) \). By comparing eq. \( (4.32) \) with eq. \( (3.47) \) and identifying \( \hat{B} = D_A \circ \hat{F}_A \), one can also see that \( D_A \circ \hat{F}_A \) is a solution of the Maurer-Cartan equation \( (3.48) \) \[21\]. Finally the nilpotent property of the CE-differential, i.e. \( \hat{d}^2 = 0 \), leads to the integrability condition that the NC field strength \( \hat{F}_A \) in eq. \( (4.30) \) obeys the Bianchi identity, i.e.,
\[
\hat{d} \hat{F}_A + \hat{A}_A \wedge \hat{F}_A - \hat{F}_A \wedge \hat{A}_A = 0. \tag{4.34}
\]

It is worthwhile to notice that the CE-complex \( \{C^*(A, A), \hat{d}\} \) in the classical limit \( \hbar \to 0 \) reduces to the Lichnerowicz complex \( (2.9) \) for the Poisson cohomology. For example, one may check this property with the NC field strength \( (4.30) \). First notice that the field strength \( (2.29) \) of Poisson gauge fields can be written as
\[
F_A(f,g) = \langle d_\theta A_A + A_A \wedge A_A, df \wedge dg \rangle \tag{4.35}
\]
where
\[
\{A_A(f), A_A(g)\}_\theta \equiv \langle A_A \wedge A_A, df \wedge dg \rangle \tag{4.36}
\]
is a commutative analogue of the cup product \( (4.28) \). And look at the classical limit of the CE-differential \( (4.27) \) which reads as
\[
(\hat{d} \hat{A}_A)(f,g) \xrightarrow{\hbar \to 0} \{f, A_A(g)\}_\theta - \{g, A_A(f)\}_\theta - A_A(\{f, g\}_\theta) = \langle d_\theta A_A, df \wedge dg \rangle \tag{4.37}
\]
with the definition \( \hat{A}_A(f)|_{\hbar \to 0} \equiv A_A(f) \). Combining all together, one can get the relation
\[
\hat{F}_A(f,g)|_{\hbar \to 0} = (\hat{F}_A|_{\hbar \to 0}, df \wedge dg) = F_A(f,g) \tag{4.38}
\]
where
\[
\hat{F}_A = \hat{d} \hat{A}_A + \hat{A}_A \wedge \hat{A}_A \xrightarrow{\hbar \to 0} d_\theta A_A + A_A \wedge A_A. \tag{4.39}
\]

In this sense the CE-complex defined by eqs. \( (4.27) \) and \( (4.28) \) may be regarded as a (deformation) quantization of the Lichnerowicz complex \( (2.9) \) for the Poisson cohomology.
5. Emergent gravity from U(1) gauge fields

5.1 Interlude for local life

Let $M$ be a $d$-dimensional Riemannian manifold whose metric is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \quad \mu, \nu = 1, \cdots, d. \quad (5.1)$$

In general relativity, the metric (5.1) of the Riemannian manifold $M$ is also defined by introducing at each spacetime point in $M$ a local reference frame of tangent bundle $TM$ in the form of $d$ linearly independent vectors (so-called vielbeins) $E_a = E^\mu_a \partial_\mu \in \Gamma(TM)$. The frame basis $\{E_a : a = 1, \cdots, d\}$ defines dual vectors $E_a = E_a^\mu(x)dx^\mu \in \Gamma(T^*M)$ by a natural pairing $\langle E^a, E_b \rangle = \delta^a_b$. This pairing leads to the relation $E^a_\mu E^\mu_b = \delta^a_b$. In terms of the basis in $\Gamma(TM)$ or $\Gamma(T^*M)$, the metric (5.1) can be written as

$$ds^2 = \delta_{ab} E^a \otimes E^b = \delta_{ab} E^a_\mu(x)E^b_\nu(x)dx^\mu \otimes dx^\nu \quad (5.2)$$

or

$$\left( \frac{\partial}{\partial s} \right)^2 = \delta^{ab} E^a \otimes E^b = \delta^{ab} E^a_\mu(x)E^b_\nu(x)\partial_\mu \otimes \partial_\nu. \quad (5.3)$$

We remark that, mathematically, a vector field $X$ on a smooth manifold $M$ is a derivation of the algebra $A = C^\infty(M)$.

As was observed before, given a Poisson structure $\theta \in \Gamma(\Lambda^2 TM)$ on a manifold $M$, there is a natural homomorphism $\rho : T^*M \rightarrow TM$ induced by the Poisson bivector $\theta$. Since any smooth vector field generates a one-parameter family of deformations described by eq. (2.16), one can perturb the Poisson structure $(M, \theta)$ by a line bundle $L \rightarrow M$. Then the deformed Poisson bivector is given by eq. (2.19) where $F = dA$ is the curvature of the line bundle $L$. Conversely, one can find a one-parameter family of diffeomorphisms generated by vector fields $A_{\theta_t} = A_\mu(y)d\theta_t y^\mu$ such that all the Poisson structures $\theta_t$ for $t \in [0, 1]$ are related by coordinate transformations defined by the exponential map $\rho^*_t$, i.e., $\rho^*_t \theta_t = \theta_t$. In particular, denoting $\rho^*_0 = \rho^*_A$, we have the relation (2.23). Thus we conceive a novel form of the equivalence principle because the electromagnetic force is to the deformation of a Poisson manifold what the gravitational force is to the deformation of a Riemannian manifold. As the equivalence principle in general relativity beautifully explains why the gravitational force manifests itself as a spacetime geometry, one may ask whether a similar geometrization of the electromagnetic force can arise from the novel form of the equivalence principle based on the Darboux theorem or the Moser lemma in symplectic or Poisson geometry. For example, one may wonder whether the frame basis $E_a \in \Gamma(TM)$ in the metric (5.3) can arise from U(1) gauge fields. This poses a profound question about the possibility that a $d$-dimensional Riemannian manifold $(M, g)$ can be defined by U(1) gauge theory on a Poisson manifold $(M, \theta)$. We will address this issue in the context of emergent gravity but let us proceed with a general discussion.

So far we have mostly been stuck to a coordinate independent description. But the physics will be of benefit to an explicit description by introducing a local coordinate system. We will choose a local coordinate system $\{y^\mu\}$ on a local patch $U_i \subset M$ such that
\{y^\mu\}_{U_i} are in the Darboux-Weinstein frame (2.33). And consider the action of the formal diffeomorphism \(D_A\) in eq. (2.11) on the local coordinates \(\{y^\mu\}\) on \(U_i \subset M\) and define the so-called covariant (dynamical) coordinates by

\[
X^\mu \equiv D_A(y^\mu) = y^\mu + \hat{Y}^\mu(y)
\]

(5.4)

where \(\hat{Y}^\mu(y) \equiv \hat{A}_\mu(y^\mu)\). From now on we will adopt the physicist’s convention by replacing \(\hbar\) by \(\frac{\hbar}{2}\). (See the footnote 16.) And we will often set \(\hbar = 1\) whenever it is not necessary to keep in with it. The Darboux-Weinstein coordinates obey the following commutation relation

\[
[y^\mu, y^\nu]_* = i\theta^{\mu\nu}(y)
\]

(5.5)

where higher order terms except the \(O(\hbar)\)-term identically vanish. When the bivector \(\theta \equiv B^{-1}\) is an invertible symplectic structure, \(\theta^{\mu\nu}\) in eq. (5.5) become a nondegenerate constant \(d \times d\) matrix with \(d = \text{even}\). In this case, as in eq. (2.11), it is convenient to designate covariant “momentum” variables

\[
C_\mu(y) \equiv D_A(p_\mu) = p_\mu + \hat{A}_\mu(y)
\]

(5.6)

where \(p_\mu = B_{\mu\nu}y^\nu\) and \(\hat{A}_\mu(y) \equiv B_{\mu\nu}\hat{Y}^\nu(y)\) define NC U(1) gauge fields used to formulate a NC gauge theory. It is then straightforward to calculate the NC field strength (4.30) in the Darboux-Weinstein frame:

\[
\hat{F}_{\mu\nu}(y) \equiv \hat{\delta}_A(p_\mu, p_\nu)
\]

\[
= \partial_\mu \hat{A}_\nu(y) - \partial_\nu \hat{A}_\mu(y) - i[\hat{A}_\mu(y), \hat{A}_\nu(y)]_*
\]

(5.7)

Note that NC gauge fields can also be viewed as sections of a deformed vector bundle \(L[[\hbar]] \to M\) or naturally interpreted as connections over the NC algebra \(A_\theta\).

Here we intend to view the vielbeins \(E_a \in \Gamma(TM)\) on a Riemannian manifold \(M\) as a derivation of the algebra \(A = C^\infty(M)\). And we want to understand, if any, whether the vielbeins \(E_a \in \Gamma(TM)\) in the Riemannian metric (5.3) can arise from a derivation of NC \(*\)-algebra \(A_\theta = C^\infty(M)[[\hbar]]\) in commutative limit \(|\theta| \to 0\). Thereby let us again look at the chain of maps:

\[
A \xrightarrow{\rho} A_\theta \xrightarrow{U} A_*
\]

(5.8)

17 Although NC gauge theory construction in Refs. [20, 21] can be applied to a general Poisson structure \(\theta\), we will assume the unperturbed Poisson structure \(\theta\) to be of the form in the Darboux-Weinstein frame (2.33) keeping in mind the remark below (4.18). This minimal choice will be of great benefit to the construction of a Hilbert space of NC algebra \(A_\theta\) for some cases, e.g., the Heisenberg-Moyal algebra with constant \(\theta^{\mu\nu}\) and Lie algebras with a linear Poisson structure \(\theta^{\mu\nu} = f^{\mu\nu}y^\lambda\).

18 The most famous example of (quantized) Poisson algebra is the SO(3) algebra of angular momenta \(L_i = \varepsilon_{ij}^k x^j p_k\) in quantum mechanics. In this case or the Poisson algebra case in general, there is no distinction (polarization) between coordinates and momenta. Moreover this example illustrates that the generators \(L_i\) of Poisson algebra arise from the composite operators of \(x^i\) and \(p_i\) obeying the symplectic algebra \([x^i, p_j] = i\hbar \delta^{ij}\). It is known [43] that for a general Poisson manifold there exists globally an essentially unique symplectic realization of the Poisson manifold which possesses a local groupoid structure compatible with the symplectic structure. Later we will think of the possibility for the quantization of a Poisson manifold.
Given U(1) gauge fields viewed as a one-form \( A \) on a Poisson manifold \( (M, \theta) \), the first step is to associate a vector field \( A_\theta \in \Gamma(TM) \) using the anchor map \( (2.11) \) (though in general not injective except for the symplectic case). And the next step is then to apply the formality map \( U : T_{\text{poly}}(M) \to D_{\text{poly}}(M) \) to get a differential operator \( A_* \) acting on the NC algebra \( A_\theta \). Hence it is natural to consider whether it is possible to take the differential operator \( A_* \) as a candidate of the derivation for the metric fields. But it turns out that it is not a proper choice for the following two reasons. First note that combining eqs. (3.49) and (3.53) for \( X = A_\theta \) leads to the relation

\[
-\delta^*_X (f \star g) + f \star \delta^*_X g + (\delta^*_X f) \star g = \hbar [\Phi(d_\theta A_\theta)](f, g).
\] (5.9)

This result immediately implies that the differential operator \( \delta^*_X = \Phi(X) = A_* \) for \( X = A_\theta \) is not a derivation of \( \star \)-algebra \( A_\theta \) unless \( F_\theta = d_\theta A_\theta = 0 \). The bivector field \( F_\theta = d_\theta A_\theta \) is dual to the U(1) field strength \( F = dA \) which is usually non-vanishing in NC gauge theory. Another reason is that we want to define the derivation algebra relevant to NC gauge fields, but NC gauge fields are defined via the covariantizing map (4.6) instead of \( A_* \) itself. It should be also remarked for later arguments that the NC gauge fields in eq. (5.4) or (5.6) are locally defined because the covariantizing map \( D_A \) depends on gauge fields and their coordinate representation (3.4) or (5.6) is also defined in the Darboux-Weinstein frame (2.33) on a local patch \( U_i \subset M \). In appendix A we will illuminate this local nature of NC gauge fields by showing that they are basically defined by local coordinate transformations into a Darboux frame which corresponds to the SW map between commutative and NC gauge fields [18]. It is also worthwhile to remark that the Kontsevich’s star product is defined on an open subset of \( \mathbb{R}^d \) whose Poisson bivector is given by the Darboux-Weinstein frame (2.33) and so it should be used locally on a general Poisson manifold. But the local expressions can be glued together to obtain a global star product [25]. We will thus allow all compatible coordinate systems by an atlas on \( M \) as a family of local Darboux charts \( \{(U_i, \varphi_i) : i \in I\} \) where \( \varphi_i : U_i \to \mathbb{R}^d \) and assume for the moment that they are consistently glued together as in [2] to define a global star product à la [24].

Hence we may first define a derivation algebra \( \mathfrak{X}_i := \Gamma(TU_i)[[\hbar]] \) (or in general \( \mathfrak{X}_i := \mathcal{V}^*(U_i)[[\hbar]] \)) on a local Darboux chart \( (U_i, \varphi_i) \) as a subalgebra of \( D_{\text{poly}}(M) \) and then try to glue the algebras \( \{\mathfrak{X}_i : i \in I\} \) altogether to yield a globally defined algebra \( \mathfrak{X} = \bigcup_{i \in I} \mathfrak{X}_i \) of derivations. On a local Darboux chart \( (U_i, \varphi_i) \), we have NC gauge fields in eq. (5.4) or (5.6) and consider them as elements in the NC algebra \( A_\theta \). Then there is a natural map from the NC algebra \( A_\theta \) to the Lie algebra \( \mathfrak{X}_i \) defined by the inner derivation (3.50). To be specific, we define an adjoint action of the NC algebra \( A_\theta \) as

\[
\text{ad}_\hat{f} : \hat{g} \mapsto -i[\hat{f}, \hat{g}]_*
\] (5.10)

for \( \hat{f}, \hat{g} \in A_\theta \). Obviously the adjoint action (5.10) satisfies the Leibniz rule, i.e.,

\[
\text{ad}_\hat{f}(\hat{g} \star \hat{h}) = (\text{ad}_\hat{f}\hat{g}) \star \hat{h} + \hat{g} \star (\text{ad}_\hat{f}\hat{h})
\] (5.11)

for \( \hat{g}, \hat{h} \in A_\theta \) and so it defines a derivation of \( A_\theta \). Using the Kontsevich’s formula (3.28), one can expand the commutator in eq. (5.10) to get an explicit form of the polydifferential
operator \( X_f^* \equiv \text{ad}_f \in \mathfrak{x}_i \) given by
\[
X_f^* = X_f + \sum_{n=2}^{\infty} \xi_{f}^{\mu_1 \cdots \mu_n}(y)\partial_{\mu_1} \cdots \partial_{\mu_n} \tag{5.12}
\]
where \( X_f \) is an ordinary Hamiltonian vector field defined by \( X_f(g) = \{f, g\}_\theta \). An explicit formula for \( X_f^* \) up to second order can be found in \([65]\). The Jacobi identity for the \(*\)-commutator \([\hat{f}, \hat{g}]_*\) leads to the result that the polydifferential operator \(5.12\) on \( \mathcal{A}_\theta \) satisfies the deformed Lie algebra
\[
[X_f^*, X_g^*] = X_{[\hat{f}, \hat{g}]}^*. \tag{5.13}
\]
It should be noted that the polydifferential operator \(5.12\) recovers the usual vector field at leading order. Thus it is obvious that the left-hand side of eq. \(5.13\) is a deformation of the ordinary Lie bracket of vector fields.

For simplicity, let us first consider the symplectic case. On a local Darboux chart \( U_i \), we have the set of dynamical gauge fields given by eq. \(5.6\). Hence, according to the adjoint map \( A_\theta \to \mathfrak{x}_i \) in eq. \(5.10\), one can derive generalized vector fields for the set \( \{\hat{C}_a(y) \in \mathcal{A}_\theta : a = 1, \cdots, d = 2n\} \) on \( U_i \subset M \) and they are given by
\[
v_a^* \equiv X_{\hat{C}_a}^* \in \mathfrak{x}_i : v_a^*(\hat{f}) = -i[\hat{C}_a(y), \hat{f}]_* \tag{5.14}
\]
for any \( \hat{f} \in \mathcal{A}_\theta \). Using the \(*\)-commutator relations
\[
-i[\hat{C}_a(y), \hat{C}_b(y)]_* = -B_{ab} + \hat{F}_{ab}(y) \in \mathcal{A}_\theta, \tag{5.15}
\]
\[
-[\hat{C}_a(y), [\hat{C}_b(y), \hat{C}_c(y)]_\ast] = \hat{D}_a\hat{F}_{bc}(y) \in \mathcal{A}_\theta, \tag{5.16}
\]
one can apply the Lie algebra homomorphism \(5.13\) to the above gauge fields to yield the differential operators given by
\[
X_{\hat{F}_{ab}}^* = -i[v_a^*, v_b^*] \in \mathfrak{x}_i, \tag{5.17}
\]
\[
X_{\hat{D}_a\hat{F}_{bc}}^* = -[v_a^*, [v_b^*, v_c^*]] \in \mathfrak{x}_i. \tag{5.18}
\]
Then one can use the above relations to transform the equations of NC gauge fields in \( \mathcal{A}_\theta \) into the (geometric) equations of generalized vector fields in \( \mathfrak{x}_i \). For example, NC U(1) gauge fields in \( d = 4 \) dimensions admit (anti-)self-dual connections, the so-called NC U(1) instantons, obeying the self-duality equations
\[
\hat{F}_{ab}(y) = \pm \frac{1}{2}\varepsilon_{ab}^{cd} \hat{F}_{cd}(y). \tag{5.19}
\]
According to the map \(5.17\), the NC U(1) instantons can thus be understood as some (geometric) objects described by the self-duality equations
\[
[v_a^*, v_b^*] = \pm \frac{1}{2}\varepsilon_{ab}^{cd}[v_c^*, v_d^*]. \tag{5.20}
\]
\[19\] It may be convenient to distinguish local vector fields from global ones which will be considered later. For this purpose we use small letters to denote local vector fields and large letters to indicate global vector fields introduced later.
Similarly, according to the map (5.18), the equations of motion as well as the Bianchi identity for general NC U(1) gauge fields are transformed into the following differential equations:

\[ \hat{D}^a \hat{F}_{ab} = 0 \quad \Leftrightarrow \quad [v^a, [v^a, v^b]] = 0, \quad (5.21) \]
\[ \frac{1}{3!} \delta^{def}_{abc} \hat{D}_d \hat{F}_{ef} = 0 \quad \Leftrightarrow \quad \frac{1}{3!} \delta^{def}_{abc} [v^d, [v^e, v^f]] = 0. \quad (5.22) \]

In order to identify geometric objects described by the differential equations (5.21) and (5.22), it is necessary first to know the relation between the (inverse) vielbeins \( E_a \in \Gamma(TM) \) and the generalized vector fields \( v^a \in X_i \). Note that the vector fields (vielbeins) in the gravitational metric (5.3) are globally defined. Therefore in order to identify a gravitational metric from locally defined NC gauge fields we need to consider a global version of the Lie algebra of derivations. For this purpose, we can use NC U(1) gauge transformations as well as coordinate transformations to glue the locally defined derivations on overlapping regions of an open covering \( M = \bigcup_{i \in I} U_i \). First it will be instructive to understand a corresponding situation in general relativity. On relying on the equivalence principle in general relativity, which is mathematically tantamount to the simple statement that every manifold is locally flat, at every point \( x' \) of spacetime one can choose a set of coordinates \( \xi^a \) that are locally inertial at \( x' \). The metric components in any general non-inertial coordinate system are then given by

\[ \tilde{g}^{\mu\nu}(x) = \delta^{ab} e^\mu_a(x) e^\nu_b(x) \quad (5.23) \]

where

\[ e^\mu_a(x) = \frac{\partial x^\mu}{\partial \xi^a(x)}|_{x=x'}. \quad (5.24) \]

Therefore the equivalence principle always guarantees the existence of \( d \) linearly independent flat coordinates \( (\xi^1, \cdots, \xi^d) \) such that the metric locally becomes flat, i.e.,

\[ \tilde{g}^{\mu\nu}(x) \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} = \delta^{ab}. \quad (5.25) \]

Note that we have intentionally distinguished the locally defined inertial frame (5.24) from the globally defined basis \( E_a = E^\mu_a(x) \frac{\partial}{\partial x^\mu} \) in eq. (5.2) which in general cannot be written as the form (5.24) unless spacetime is flat. But the basis \( E_a \) can be restricted to an infinitesimal neighborhood \( U_{x'} \) centered at \( x' \) so that it can be represented in the locally inertial frame (5.24), i.e., \( E^\mu_a|_{U_{x'}} = e^\mu_a \). As the metric \( \tilde{g}^{\mu\nu}(x) \) varies smoothly with \( x \), there is no obstacle to find a more general basis by allowing the \( d \times d \) matrix \( e^\mu_a(x) \) to vary smoothly with \( x \). This should be the case because every manifold can be constructed by gluing together Euclidean domains. This suggests a simple recipe to get a globally defined frame \( E_a \) from a locally defined coordinate basis \( e_a \) on \( U_{x'} \):

\[ e_a|_{U_{x'}} \rightarrow E_a. \quad (5.26) \]

However the replacement (5.26) should be compatible with the orthonormality of the bases:

\[ e_a \cdot e_b = \delta_{ab} \quad \Leftrightarrow \quad E_a \cdot E_b = \delta_{ab}. \quad (5.27) \]
which means that \( g_{\mu \nu}(x)|_{U_1'} \rightarrow g_{\mu \nu}(x) = E_\mu^a(x)E_\nu^a(x) \). With this replacement the metric (5.23) in a locally inertial frame can be extended to an entire region with the metric (5.3) because the frames \( E_a \) are now coordinate independent and so globally defined.

We need a similar globalization for locally defined vector fields \( v_a^* \in \mathfrak{x}_i \). Deferring a detailed exegesis later, let us take a simple recipe analogous to the replacement (5.26). That is, we will implicitly assume that local Darboux charts \( (U_i, \varphi_i) \) and derivation algebras \( \mathfrak{x}_i \), defined over there are consistently glued together by coordinate transformations and NC U(1) gauge transformations on overlapping regions \([22, 25]\). We will denote by \( V_a^* \in \mathfrak{x} \) the global version of generalized vector fields obtained through the gluing of local data:

\[
v_a^*|_{U_i} = \tilde{v}_a + \sum_{n=2}^{\infty} \xi_{\mu_1^a \cdots \mu_n^a}(y) \partial_{\mu_1} \cdots \partial_{\mu_n} \in \mathfrak{x}_i \]

\[
\rightarrow V_a^* = V_a + \sum_{n=2}^{\infty} \Xi_{\mu_1^a \cdots \mu_n^a}(x) \partial_{\mu_1} \cdots \partial_{\mu_n} \in \mathfrak{x}. \tag{5.28}
\]

Note that the vector fields \( v_a^*|_{U_i} \in \mathfrak{x}_i \) are defined by the inner derivation (5.14) in a local Darboux frame and the local vector fields \( \tilde{v}_a = \tilde{v}_a^\mu(y) \partial_\mu \in \Gamma(TU_i) \) are divergence-free, i.e., \( \partial_\mu \tilde{v}_a^\mu(y) = 0 \). This means that the vector fields \( \tilde{v}_a \) are volume-preserving, i.e., \( \mathcal{L}_{\tilde{v}_a} \nu_D = 0 \) for the flat volume form \( \nu_D = \frac{\nu_D}{n!V^d} = d^2ny \) in a Darboux frame. As a parallel analogue of (5.27) in general relativity, the replacement (5.28) similarly needs to keep the volume-preserving condition for global vector fields \( V_a = V_a^\mu(x) \partial_\mu \in \Gamma(TM) \) such that

\[
\mathcal{L}_{\tilde{v}_a} \nu_D = 0 \quad \iff \quad \mathcal{L}_{V_a} \nu = 0 \tag{5.29}
\]

for some volume form \( \nu \). A Poisson manifold with the above property is called unimodular and any symplectic manifold is unimodular. We give a brief exposition in appendix B for modular vector fields, Poisson homology and their deformation quantization. Suppose that the volume form is given by \( \nu = \lambda^2 V^1 \wedge \cdots \wedge V^d \) where \( V^a = V_a^\mu(x)dx^\mu \in \Gamma(TM) \) are globally defined covectors, i.e., \( \langle V^a, V_b^\mu \rangle = \delta_b^a \). Then, by definition, we get

\[
\lambda^2 = \nu(V_1, \cdots, V_d). \tag{5.30}
\]

One can see that the right-hand side of eq. (5.29), when restricted to a local Darboux chart, reduces to the left-hand side, as it should be. It must be emphasized that the above globalization will be compatible with the derivation structure (5.11) as well as the Lie algebra structure (5.13) because the differential operators \( V_a^* \in \mathfrak{x} \) are realized as an inner derivation of global star product, as will be shown later. Nevertheless it turns out that the global vector fields \( V_a \in \Gamma(TM) \) will reproduce a general volume-preserving vector fields. Since we eventually want to achieve a background independent formulation of NC gauge theory in terms of the algebra of (large \( N \)) matrices, this property is actually desirable because any derivation of the matrix algebra is well-known to be inner.

One can choose the conformal factor \( \lambda \) such that the orthonormal vectors \( E_a \) preserve the volume form \( \tilde{\nu} = \lambda^{d-a} \nu_g \) where \( \nu_g = E^1 \wedge \cdots \wedge E^d = \sqrt{\det g_{\mu \nu}}d^d x \) is the Riemannian volume form \([49, 59]\). This means that the gauge theory vectors \( V_a = V_a^\mu(x) \partial_\mu \in \Gamma(TM) \)
are related to the basis of orthonormal tangent vectors $E_a = E^\mu_a(x)\partial_\mu$ by

$$V_a = \lambda E_a$$

and their covectors in $\Gamma(T^*M)$ are related by

$$E^a = \lambda V^a.$$ (5.32)

This can be checked as follows:

$$\mathcal{L}_{V_a} \nu = \mathcal{L}_{\lambda E_a}(\lambda^{2-d} \nu_g) = \mathcal{L}_{E_a}(\lambda^{3-d} \nu_g) = \mathcal{L}_{E_a} \tilde{\nu} = 0.$$ (5.33)

Using the relation (5.32), one can completely determine the Riemannian metric (5.2) in terms of the global vector fields defined by NC gauge fields via eqs. (5.14) and (5.28) and it takes the form

$$ds^2 = E^a \otimes E^a = \lambda^2 V^a \otimes V^a = \lambda^2 V^a_\mu dx^\mu \otimes dx^\nu.$$ (5.34)

Now we are ready to translate the equations of motion (5.21) for NC gauge fields together with the Bianchi identity (5.22) into some geometric equations related to Riemann curvature tensors determined by the metric (5.34) at leading order. To see what they are, recall that, in terms of covariant derivatives, the torsion $T$ and the curvature $R$ are given by well-known formulae

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$ (5.35) (5.36)

where $X,Y,Z$ are vector fields on $M$. Because $T$ and $R$ are multilinear differential operators, one can easily deduce the following relations

$$T(V_a, V_b) = \lambda^2 T(E_a, E_b),$$

$$R(V_a, V_b)V_c = \lambda^3 R(E_a, E_b)V_c.$$ (5.37) (5.38)

Since we want to recover the general relativity from the emergent gravity approach, we will impose the torsion free condition, $T(E_a, E_b) = 0$. Then it is straightforward, by using eqs. (5.35) and (5.36) and repeatedly converting $\nabla_a V_b - \nabla_b V_a$ into $[V_a, V_b]$, to derive the identity below

$$R(E_a, E_b)E_c + \text{cyclic}(a \rightarrow b \rightarrow c) = \lambda^{-3}\left( [V_a, [V_b, V_c]] + \text{cyclic}(a \rightarrow b \rightarrow c) \right).$$ (5.39)

Therefore we arrived at a pleasing result that the Bianchi identity (5.22) for NC gauge fields reduces at leading order to the first Bianchi identity for Riemann curvature tensors, i.e.,

$$\widehat{D}_a F_{bc} + \text{cyclic}(a \rightarrow b \rightarrow c) = 0 \quad \iff \quad \left( R_{abcd} + \text{cyclic}(a \rightarrow b \rightarrow c) \right) + \mathcal{O}(\theta^2) = 0.$$ (5.40)

---

20The standard formula for the covariant divergence $\nabla \cdot V$ of a vector field $V$ is given by $\mathcal{L}_V \nu_g = (\nabla \cdot V) \nu_g$. Therefore we get $\mathcal{L}_{V_a} \nu = (\nabla \cdot V_a + (2-d)V_a \log \lambda) \nu$. Unfortunately there was a stupid mistake for the divergence formula in the footnote 19 of [2]. But fortunately this remained a harmless slip and did not affect any results.
We will discuss later how the classical general relativity is corrected due to the NC structure of spacetime.

The transformation for the equations of motion (5.21) into gravitational equations requires more algebras. But it is natural to expect the Einstein equations from NC gauge fields at leading order\(^{21}\)

\[
\hat{D}^a \hat{F}_{ab} = 0 \quad \Leftrightarrow \quad R_{ab} - 8\pi G (T_{ab} - \frac{1}{2} \delta_{ab} T) + O(\theta^2) = 0. \tag{5.41}
\]

Thus the upshot of the analysis is to determine the form of energy-momentum tensor \(T_{ab}\).

It was determined in [2] only in lower dimensions \(d \leq 4\). Since we do not know the result in higher dimensions, let us focus on the four dimensions. First note that the Einstein gravity arises at the first order of NC gauge fields, i.e., \(R_{ab} \sim O(\theta)\) and the parameters, \(G_{YM}\) and \(\theta\), defining the NC gauge theory are related to the gravitational constant \(G\) by

\[
\frac{G\hbar^2}{c^2} \sim G_{YM}^2 |\theta| \tag{5.42}
\]

where \(|\theta| \equiv (\text{Pf} \theta)^{\frac{1}{n}}\). Therefore the Einstein equations (5.41) imply that \(T_{ab} \sim O(1)\). We know that NC U(1) gauge theory reduces to the ordinary Maxwell theory at \(O(1)\) and so \(T_{ab}\) in eq. (5.41) has to contain the Maxwell energy-momentum tensor. Indeed the detailed analysis reveals some surprise [2, 3]. In addition to the Maxwell energy-momentum tensor, it also contains an exotic energy-momentum tensor which is absent in Einstein gravity. The reason is as follows. Define the structure equation of vector fields \(V_a \in \Gamma(TM)\) as

\[
[V_a, V_b] = -g_{ab} V_c \tag{5.43}
\]

and take the canonical decomposition

\[
g_{abc} = g_c^{(+)} \eta_{ab}^{(+)} + g_c^{(-)} \eta_{ab}^{(-)} \tag{5.44}
\]

according to the Lie algebra splitting \(\text{so}(4) = \text{su}(2)_L \oplus \text{su}(2)_R\). It turns out [2] that the energy-momentum tensor \(T_{ab}\) consists of purely interaction terms between self-dual and anti-self-dual parts in eq. (5.44). However the energy-momentum tensor \(T_{ab}\) has a nonvanishing trace though it originates from the mixed sectors, i.e., \((\text{su}(2)_L, \text{su}(2)_R)\) and \((\text{su}(2)_R, \text{su}(2)_L)\) whose trace normally vanishes because the Ricci scalar in general relativity belongs to \((\text{su}(2)_L, \text{su}(2)_L)\) and \((\text{su}(2)_R, \text{su}(2)_R)\) sectors [73, 73, 74]. Consequently, the Ricci scalar deduced from eq. (5.41) becomes nonzero although the standard decomposition of curvature tensors implies that the Ricci tensor \(R_{ab}\) in eq. (5.41) should be traceless. (The traceless Ricci tensor and the Ricci scalar were denoted by \(B\) and \(s\), respectively, in eq. (4.29) in [74].) In addition, in a long-wavelength limit where the scalar modes are dominant, it reduces to

\[
T_{\mu\nu} \approx -\frac{R}{32\pi G} g_{\mu\nu} \tag{5.45}
\]

\(^{21}\)A frugal way to derive eq. (5.41) is to first calculate the Ricci tensor \(R_{ab}\) in terms of vector fields \(V_a\) [2] and then add appropriate terms by inspection on both sides of \([V^a, [V_a, V_b]] = 0\) so that the left-hand side together with the added terms yields \(R_{ab}\).
where $R$ is the Ricci scalar for the metric tensor (5.34). Hence, one can see that the mystic energy (5.45) cannot be realized in Einstein gravity. Moreover eq. (5.45) implies that the mystic energy behaves like the dark energy with $w = -1$ after the Wick rotation into the Lorentzian signature [3]. Actually it copies all the properties of dark energy and so it was suggested in Refs. [2, 3] as a possible candidate of dark energy.

5.2 Equivalence principle and Riemann normal coordinates

First let us understand how to realize the globalization (5.26) of vector fields from local data in an inertial frame. The underlying idea is that local invariants of a metric in Riemannian geometry are quantities expressible in local coordinates in terms of the metric and its derivatives and they have an invariance property under changes of coordinates. It is a fundamental result in Riemannian geometry that such invariants can be written in terms of the curvature tensor of the metric and its covariant derivatives. Hence the full Taylor expansion of the metric can be recovered from the iterated covariant derivatives of curvature tensors. As a consequence, any local invariant of Riemannian metrics has a universal expression in terms of the curvature tensor and its covariant derivatives. This is known as the jet isomorphism theorem [41, 42] stating that the space of infinite order jets of metrics modulo coordinate changes is isomorphic to a space of curvature tensors and their covariant derivatives modulo the orthogonal group. (One may view the jet bundle as a coordinate free version of Taylor expansions. See appendix C for a brief exposition of jet bundles.)

The Taylor expansion of a metric at a point $p \in M$ can be more explicit by considering a coordinate system which is locally flat at that point on a curved manifold. The coordinates in an open disk centered at the origin are normal coordinates [41, 75] arising from an orthonormal basis at the origin if and only if for each point $p$ in the disk,

$$g_{ab}(p) \xi^b = \delta_{ab} \xi^b.$$  \hspace{1cm} (5.46)

As one can see from eq. (5.46), geodesic normal coordinates are determined up to the orthogonal group $O(d)$, i.e., different normal coordinates are related by an element of $O(d)$. The basic idea behind the so-called Riemann normal coordinates (RNCs) is to use the geodesics through a given point to define the coordinates for nearby points. They have an appealing feature that the geodesic equations

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} = 0$$  \hspace{1cm} (5.47)

passing through the point have the same form as the equations in the Cartesian coordinate system in Euclidean geometry because the Levi-Civita connections $\Gamma^\mu_{\rho\sigma}$ vanish at that point. It may be useful to recall [41] that the projection onto any integral curve of a standard horizontal vector field of the bundle $F(M)$ of linear frames over $M$ is a geodesic and, conversely, every geodesic is obtained in this way.

We may construct the normal coordinates around each point $p$ of $M$ using the exponential map $\exp_p : T_p M \rightarrow M$. Basically the normal coordinates are the coordinates of the tangent space at $p \in M$ pulled back to the base manifold. Recall that the exponential
map \( \exp_p : T_p M \to M \) is defined by \( \exp_p(v) := \gamma_v(1) \) where \( \gamma_v : [0,1] \to M \) is a geodesic curve for which \( \gamma_v(0) = p \) and \( \dot{\gamma}_v(0) = v \in T_p M \). Thus, for any \( p \in M \), there exists a neighborhood \( U \) of 0 in \( T_p M \) and a neighborhood \( U \) of \( p \) in \( M \) so that \( \exp_p : U \to U \) is a (local) diffeomorphism. By the construction, for every \( q \in U \), there exists a unique geodesic which joins \( p \) to \( q \) and lies entirely in \( U \). Given an orthonormal frame \( \{ e_a \}_{a=1}^d \) of \( T_p M \), the linear isomorphism \( \xi : \mathbb{R}^d \to T_p M \) by \( (\xi^1, \ldots, \xi^d) \mapsto \xi^a e_a \) defines a coordinate system in \( U \) in a natural manner. Therefore the map

\[
\exp_p \circ \xi : \xi^{-1}(U) \to U
\]

is a local chart for \( M \) around \( p \) and its inverse defines the normal coordinate system in \( U \). The normal coordinates on \( U \) are then given by \( \exp_p^{-1}(q) = y^\mu(q) e_\mu \) or equivalently

\[
y^\mu = l^\mu \circ \exp_p^{-1}
\]

where \( (l^1, \ldots, l^d) \) is the dual basis of \( e_a \). In terms of the normal coordinates, the vielbeins 

\[
\tilde{e}_\mu^a = \tilde{e}_\mu^a(y) dy^\mu \quad (\text{objects in the } \infty\text{-jet will be denoted with the tilde})
\]

are given by

\[
\tilde{e}_\mu^a = \delta_\mu^a - \frac{1}{6} R_{\rho\mu\sigma} y^\rho y^\sigma - \frac{1}{12} \nabla_\lambda R_{\rho\mu\sigma} y^\rho y^\sigma y^\lambda
\]

\[
- \left( \frac{1}{40} \nabla_\rho \nabla_\sigma R_{\alpha\mu\nu} - \frac{1}{120} R_{\alpha\nu\lambda} R_{\mu\rho\sigma} \right) y^\rho y^\lambda y^\mu y^\nu + O(y^5). \tag{5.50}
\]

Then the metric \( \tilde{g}_{\mu\nu} = \tilde{e}_\mu^a \tilde{e}_\nu^a \) in the \( \infty\text{-jet} \) is given by

\[
\tilde{g}_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\sigma\nu} y^\rho y^\sigma - \frac{1}{6} \nabla_\lambda R_{\mu\rho\sigma\nu} y^\rho y^\sigma y^\lambda
\]

\[
- \left( \frac{1}{20} \nabla_\rho \nabla_\sigma R_{\mu\alpha\nu\beta} - \frac{2}{45} R_{\mu\rho\lambda\sigma} R_{\nu\alpha\lambda\beta} \right) y^\rho y^\sigma y^\mu y^\nu + O(y^5), \tag{5.51}
\]

and so

\[
\det \tilde{g}_{\mu\nu} = 1 - \frac{1}{3} R_{\mu\nu\sigma\rho} y^\sigma y^\rho - \frac{1}{6} \nabla_\lambda R_{\mu\nu\sigma\rho} y^\rho y^\sigma y^\lambda
\]

\[
- \left( \frac{1}{20} \nabla_\rho \nabla_\sigma R_{\mu\rho} + \frac{1}{90} R_{\mu\rho\lambda\nu} R_{\nu\rho\lambda\sigma} - \frac{1}{18} R_{\mu\nu\rho\sigma} R_{\rho\sigma} \right) y^\rho y^\sigma y^\mu y^\nu + O(y^5). \tag{5.52}
\]

These formulas exhibit how the curvature and its derivatives locally affect the metric and volume form \( \nu_y = \sqrt{\det \tilde{g}_{\mu\nu}} dy \). A closed formula for the vielbein as well as the metric in the RNC expansion is now available due to a remarkable paper \[77\] which demonstrates the jet isomorphism theorem \[11, 12\].

Therefore we may compare local invariants at a point \( p \in M \) determined by the objects in the \( \infty\text{-jet} \) such as \( \tilde{e}_\mu^a \) and \( \tilde{g}_{\mu\nu} \) with those determined by the global quantities such as \( E_\mu^a \) and \( g_{\mu\nu} \) (if they are known). If they coincide each other up to any arbitrary order, we can identify these two quantities, i.e.,

\[
\tilde{e}_\mu^a \cong E_\mu^a \quad \text{and} \quad \tilde{g}_{\mu\nu} \cong g_{\mu\nu}. \tag{5.53}
\]

The identification \( (5.53) \) makes sense if the geodesic Taylor expansion in a patch around \( p \) converges and so the patch has to be taken sufficiently small in a strongly curved region.
The above prescription means that sections in the $\infty$-jet bundle $E$ (the infinite jet prolongations of the frame bundle and its symmetric tensor product—see appendix C) belong to the same equivalence class and thus it holds everywhere because the objects in the $\infty$-jet are represented by the covariant tensors only (or the natural tensors in the terminology of [11]) that respect an invariance property under changes of coordinates. Hence we can implement the identification (5.53) to define a prescription for the globalization (5.26).

5.3 Fedosov manifolds and global deformation quantization

Note that the NC gauge theory was defined by quantizing the Poisson algebra $\Phi = (C^\infty(M), \{-, -\}_\theta)$ of Poisson gauge fields on local Darboux-Weinstein charts. And we defined the inner derivation (5.14) from local NC gauge fields in $A_\theta$. Thus it is necessary to glue together local objects defined on Darboux charts to yield global objects. We will employ a similar prescription as eq. (5.53) for the globalization (5.28) using the Fedosov’s approach of deformation quantization. Let us first consider a symplectic manifold $(M, \omega)$ and $\{(U_i, \varphi_i) : i \in I\}$ an atlas on $M$. By $\omega$ we mean the symplectic 2-form. We introduce a connection $\partial^S$ on the symplectic manifold $(M, \omega)$ which preserves the symplectic form $\omega$. The Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are defined as usual by $\partial^S_{\mu} \varphi = \Gamma^\lambda_{\mu\nu} \partial^S_{\nu}$. The curvature tensor of a symplectic connection is also defined by the usual formula (5.36) and is given in the holonomic basis by

$$\mathfrak{R}^\mu_{\nu\rho\sigma} = \partial^S_{\rho} \Gamma^\mu_{\sigma\nu} - \partial^S_{\sigma} \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu}. \quad (5.54)$$

Here we use the Fraktur letter for the symplectic curvature tensor to avoid a confusion with the Riemann curvature tensor in the previous sections. The symplectic connection $\Gamma = (\Gamma^\lambda_{\mu\nu})$ on $M$ is a torsion free connection locally satisfying the condition

$$\partial^S \omega = 0. \quad (5.55)$$

In any Darboux coordinates on a local chart $(U_i, \varphi_i)$ where $\omega_{\mu\nu}$ are constants, (5.55) reduces to

$$\omega_{\mu\nu} \Gamma^\rho_{\lambda\nu} - \omega_{\nu\rho} \Gamma^\rho_{\lambda\mu} = \Gamma_{\mu\lambda\nu} - \Gamma_{\nu\lambda\mu} = 0 \quad (5.56)$$

where $\Gamma_{\mu\nu\lambda} \equiv \omega_{\mu\rho} \Gamma^\rho_{\nu\lambda}$. The connection (5.55) is thus a symplectic analogue of the Levi-Civita connection in Riemannian geometry. Combining the torsion free (i.e., symmetric) condition, i.e. $\Gamma_{\mu\nu\lambda} = \Gamma_{\lambda\nu\mu}$, the symmetric connections preserving $\omega$ are exactly the connections with the Christoffel symbols $\Gamma_{\lambda\mu\nu}$ which are completely symmetric with respect to all indices $\mu, \nu, \lambda$. Such a symplectic connection exists on any symplectic manifold. In particular, the triple $(M, \omega, \Gamma)$ is called a Fedosov manifold and the deformation quantization on a symplectic manifold $(M, \omega)$ is defined by the data $(M, \omega, \Gamma)$. Note that every Kähler manifold is a Fedosov manifold. Indeed Fedosov manifolds constitute a natural generalization of Kähler manifolds. However we will not refer to the existence of any Riemannian

---

22It is a well-known fact (e.g., see Remark 1.4 in [7]) that a symmetric connection preserving $\omega$ exists if and only if $\omega$ is closed. If $\omega$ is a symplectic 2-form, then locally we can take the trivial connection in Darboux coordinates. Globally we can glue symmetric connections preserving $\omega$ using a partition of unity. We will basically use this fact for constructing global vector fields on a symplectic manifold.
metric when considering a Fedosov manifold since we want to derive the former from the latter according to the spirit of emergent gravity.

For a symplectic manifold \((M, \omega)\), each tangent space \(T_pM\) at \(p \in M\) is a symplectic vector space and \((TM = \bigcup_{p \in M} T_pM, \omega)\) becomes a symplectic vector bundle over \(M\). Given a point \(p \in U \subset M\) we can construct the exponential map \(\exp_p : U \to U\) defined by the symplectic connection \(\partial_S\) where \(U\) is a small neighborhood of \(0\) in \(T_pM\). Let \(x(t)\) be a curve in \(U\) satisfying the geodesic equation \((5.47)\) defined in local coordinates \((x^1, \cdots, x^d)\) such that \(x(0) = p \in U\), \(x'(0) = v \in U\) and \(\exp_p(v) = x(1)\) where the Christoffel symbols \(\Gamma^\lambda_{\mu\nu}\) are now defined by eq. \((5.55)\). Using the exponential map, we can construct the normal coordinate system on \(U\) defined by \(\exp_p^{-1}(q) = y^\mu(q)e_\mu\) in the same way as \((5.49)\). In other words, if \(v = y^1 \frac{\partial}{\partial x^1} + \cdots + y^d \frac{\partial}{\partial x^d} \in T_pM\), then \((y^1, \cdots, y^d)\) are the normal coordinates of \(\exp_p v\). In this case the geodesic equation \((5.47)\) along the curve \((x^1, \cdots, x^d) = t(y^1, \cdots, y^d)\) enforces

\[
\Gamma^\lambda_{\mu\nu}(x)y^\nu y^\lambda = 0 \tag{5.57}
\]

and, taking the limit as \(t \to 0\), eq. \((5.57)\) in turn implies

\[
\Gamma^\mu_{\nu\lambda}(0) = 0. \tag{5.58}
\]

Let us take the Taylor expansion of \(\omega_{\mu\nu}\) in terms of these normal coordinates (the tilde denotes an \(\infty\)-jet object):

\[
\tilde{\omega}_{\mu\nu}(y) = B_{\mu\nu} + \sum_{n=2}^\infty \frac{1}{n!} \omega_{\mu\nu,\lambda_1\cdots\lambda_n}(0)y^{\lambda_1} \cdots y^{\lambda_n}
\equiv B_{\mu\nu} + F^x_{\mu\nu}(y) \tag{5.59}
\]

where \(B_{\mu\nu} = \tilde{\omega}_{\mu\nu}(0)\) are constant values of the symplectic two-form at \(p \in M\) which is assumed to be in the Darboux frame. An important point is that different normal coordinates with the same origin differ by a linear transformation and so the expansion coefficients in eq. \((5.59)\) define tensors, called affine normal tensors, on \(M\). Note that the \(\omega\)-preserving condition \((5.55)\) reduces to \(\partial_\lambda \omega_{\mu\nu} = \Gamma^\mu_{\lambda\nu} - \Gamma^\nu_{\lambda\mu}\) and so the condition \((5.55)\) was already imposed in the expansion \((5.59)\) (where \(y\)-dependent terms start from \(O(y^2)\)). It may be noted that the expansion \((5.59)\) can be formally written as the exponential map

\[
\tilde{\omega}_{\mu\nu}(y) = \left(\exp_p(v)\omega(0)\right)^{\mu\nu}. \tag{5.60}
\]

There exists an analogue of the jet isomorphism theorem for a Fedosov manifold (see, for example, Theorem 5.11 in \([78]\) and also \([79]\)) stating that any local invariant of a Fedosov manifold is a function of \(\omega_{\mu\nu}\) and a finite number of covariant derivatives of its curvature

\[\text{23Given a connection } \nabla, \text{ the covariant derivative of a tensor } T \text{ along a curve } \lambda(t) \text{ is defined by } \frac{dT}{dt} = \nabla_\lambda T \text{ and the tensor } T \text{ is said to be parallelly transported along } \lambda \text{ if } \frac{dT}{dt} = 0. \text{ And the curve } \lambda(t) \text{ is said to be a geodesic curve if } \frac{d}{dt}\left(\frac{\partial}{\partial t}\right)_\lambda = 0 \text{ for an affine parameter } t. \text{ If we choose local coordinates so that } \lambda(t) \text{ has the coordinates } x^\mu(t) \text{ and so } \frac{\partial}{\partial t} = \frac{dx^\mu(t)}{dt}\frac{\partial}{\partial x^\mu}, \text{ we get the geodesic equation } (5.47) \text{ for the connection } \nabla \text{ which might be either the Levi-Civita connection or a symplectic connection.} \]
tensor $\mathfrak{R}_{\mu\nu\rho\sigma} = \omega_{\mu\lambda} \mathfrak{R}^{\lambda}_{\nu\rho\sigma}$ which does not depend on the choice of local coordinates.\textsuperscript{24} Note that, for a Fedosov manifold, the curvature tensor (5.54) has the following symmetry property \textsuperscript{78}

$$\mathfrak{R}_{\mu\nu\rho\sigma} = -\mathfrak{R}_{\mu\nu\sigma\rho}, \quad \mathfrak{R}_{\mu\nu\rho\sigma} = \mathfrak{R}_{\nu\mu\rho\sigma}$$

(5.61)

that is slightly different from Riemannian manifolds. Using several identities for the curvature tensors (e.g., Proposition 5.2, Lemma 5.14 and Theorem 5.18 in \textsuperscript{78}), it can be shown that the Taylor expansion (5.59) starts as follows:

$$\tilde{\omega}_{\mu\nu}(y) = B_{\mu\nu} + \frac{1}{6} \mathfrak{R}_{\rho\sigma\mu\nu} y^\rho y^\sigma - \frac{1}{12} (\partial^S \mathfrak{R}_{\mu\nu\sigma\rho} - \partial^S \mathfrak{R}_{\mu\rho\sigma\nu}) y^\rho y^\sigma y^\lambda + O(y^4).$$

(5.62)

It is easy to invert the above result to yield the corresponding Poisson bivector $\tilde{\theta}^{\mu\nu}(y) = (\tilde{\omega}^{-1})^{\mu\nu}(y)$:

$$\tilde{\theta}^{\mu\nu}(y) = \theta^{\mu\nu} - \frac{1}{6} \mathfrak{R}_{\rho\sigma\mu\nu} \theta^{\rho\sigma} y^\rho y^\sigma$$

$$+ \frac{1}{12} (\partial^S \mathfrak{R}_{\rho\sigma\mu\nu} - \partial^S \mathfrak{R}_{\rho\mu\sigma\nu}) \theta^{\rho\sigma} y^\rho y^\sigma y^\lambda + O(y^4).$$

(5.63)

Using the first Bianchi identity $\mathfrak{R}_{\mu(\nu\rho\sigma)} = 0$,\textsuperscript{25} the second term in eq. (5.63) can be rewritten as $\mathfrak{R}_{\rho\sigma\mu\nu} \theta^{\rho\sigma} y^\rho y^\sigma = \mathfrak{R}^{\mu\rho\sigma\nu} \theta^{\mu\nu} y^\rho y^\sigma$ and so recovers eq. (3.9) in Ref. \textsuperscript{26}. Similarly we can consider the Taylor expansion of a vector field $v_a = v^a_{\mu}(x)\partial_{\mu} \in \Gamma(TM)$ in terms of normal coordinates. The leading order terms are given by\textsuperscript{26}

$$\tilde{v}^\mu_a(y) = v^a_{\mu}(x)\partial_{\mu} + \frac{1}{2} (\partial^S \mathfrak{R}^{\mu}_{\rho\sigma\nu} v^a_{\nu} + \frac{1}{3} \mathfrak{R}^{\mu\rho}_{\sigma\nu\lambda} v^a_{\lambda}) y^{\rho} y^{\sigma}$$

$$+ \frac{1}{6} (\partial^S \mathfrak{R}^{\mu}_{\rho\sigma\nu} \theta^{\rho\sigma} + \mathfrak{R}^{\mu\rho\nu}_{\sigma\lambda\nu} \theta^{\sigma\nu} + \frac{1}{6} \mathfrak{R}^{\mu\rho}_{\sigma\lambda\nu} \theta^{\sigma\nu} y^\rho y^\sigma) + O(y^4).$$

(5.64)

In order to derive the above result, we used the identities (4.9) and (4.10) in Ref. \textsuperscript{78} and the relation $\partial_{\lambda} \mathfrak{P}^{\mu}_{\rho\sigma} = \frac{1}{3} (\mathfrak{R}^{\mu}_{\rho\sigma\nu\lambda} + \mathfrak{R}^{\mu\nu}_{\sigma\rho\lambda})$ in the geodesic coordinates obeying (5.58) which is also true in Riemannian geometry.

Consider a Fedosov manifold $(M, \Omega, \partial^S)$ where $\Omega = \frac{1}{2} \Omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu \in \Gamma(\Lambda^2 \ast^k T^* M)$ is a globally defined symplectic two-form and $\partial^S$ is a symplectic connection, i.e., $\partial^S \Omega = 0$. Also introduce a complete set of global vector fields $V_a = V^\mu_a(x)\partial_{\mu} \in \Gamma(TM)$, $a = 1, \ldots, d$. Since a Fedosov manifold $(M, \Omega, \partial^S)$ has the connection $\partial^S$, it is possible to construct local invariants of the Fedosov manifold, e.g., curvature tensors and their covariant derivatives.

\textsuperscript{24}As is well-known, a symplectic manifold does not admit local invariants such as curvature due to the famous Darboux theorem. The dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. But, by introducing the concept of the symplectic connection, it is now possible to construct curvature tensors and their covariant derivatives. So the symplectic connection corresponds to the operation to make a Darboux chart be infinitesimally small. This “infinitesimal” approach will bring about another benefit to bypass the need of gluing together star-products defined on (large) Darboux charts.

\textsuperscript{25}We use the bracket notation for symmetrization and antisymmetrization over tensor indices: $X_{(\mu\nu\rho)} \equiv X_{\mu\nu\rho} + X_{\nu\rho\mu} + X_{\rho\mu\nu}$ and $X^{(\mu\nu\rho)} \equiv X^{\mu\nu\rho} - X^{\nu\mu\rho}$.

\textsuperscript{26}Note that $\tilde{v}_a \in \Gamma(TM)$ in eq. (5.28) are (locally) Hamiltonian vector fields, i.e., $\tilde{v}_a = \partial_{\mu} C_a = \partial_{a} + \cdot \cdot \cdot$. Since they will be identified with eq. (5.64) by definition, the vector fields $\tilde{v}_a$ describe the mutation from the flat basis, i.e., $v^a_{\mu} = \delta^a_{\mu}$. 

\textsuperscript{25}We use the bracket notation for symmetrization and antisymmetrization over tensor indices: $X_{(\mu\nu\rho)} \equiv X_{\mu\nu\rho} + X_{\nu\rho\mu} + X_{\rho\mu\nu}$ and $X^{(\mu\nu\rho)} \equiv X^{\mu\nu\rho} - X^{\nu\mu\rho}$.
It was shown in [78] (see, in particular, Theorem 5.11) and [79] that any local invariant of a Fedosov manifold is an appropriate function of the components of \( \Omega \) and of the covariant derivatives of the curvature tensor. The above Taylor expansions in terms of normal coordinates exhibit such local invariants at lowest orders. Therefore we can calculate local invariants at a point \( p \in M \) determined by the symplectic two-form \( \Omega \) and global vector fields \( V_a \) and compare them with those determined by \( \tilde{\omega}_{\mu\nu}(x; y) \) and \( \tilde{v}^a_{\mu}(x; y) \) on a geodesic extension of Darboux section. But there is an ambiguity coming from the symplectic connection. Unlike the Riemannian connection, the symplectic connection is not unique. Any two symplectic connections differ by a completely symmetric tensor \( S_{\mu\nu\lambda} \). See, for example, section 2.5 in [29]. So we may impose an additional condition requiring \( S_{\mu\nu\lambda} = \Gamma_{(\mu\nu\lambda)} = 0 \). Then the symplectic connection is uniquely determined by \( \Omega_{\mu\nu} \) as

\[
\Gamma_{\lambda\mu\nu} = \frac{1}{3} \left( \partial_{\mu} \Omega_{\lambda\nu} + \partial_{\nu} \Omega_{\lambda\mu} \right). \tag{5.65}
\]

Note that this choice is compatible with the geodesic condition (5.58) because \( \Gamma_{\lambda\mu\nu} \) are completely symmetric in Darboux coordinates. So we can consistently implement the following identification

\[
\tilde{\omega}_{\mu\nu} \cong \Omega_{\mu\nu} \quad \text{and} \quad \tilde{v}^a_{\mu} \cong V^a_{\mu} \tag{5.66}
\]

if their local invariants at \( p \in M \) coincide each other up to any arbitrary order. It should be globally well-defined because the Taylor expansion is independent of the choice of local coordinates. And the prescription (5.66) simply means the passage from local to global objects by gluing together the local data in the left-hand side. This prescription for the globalization constitutes a symplectic counterpart of the Riemannian case (5.53).

A standard mathematical device for patching the local information together to obtain a global theory is to use the notion of formal geometry [25, 26]. Formal geometry provides a convenient language to describe the global behavior of objects defined locally in terms of coordinates. Now we will explain how the above prescription (5.66) can be obtained by introducing formal local coordinates defined by a smooth map \( \phi : U \to M \) from a neighborhood \( U \) of the zero section of \( TM \) to \( M \). The smooth map, \( (x, y \in U_x) \mapsto \phi_x(y) \), is called a generalized exponential map if \( \phi_x(0) = x \) and \( d_y \phi_x(0) = \text{id}, \forall x \in M \). Here we shall look at the exponential map for a torsion free but not necessarily symplectic connection. If \( f \) is a smooth function on \( M \), we can define the pullback \( \phi^* f := f \circ \phi \in C^\infty(U) \) which satisfies \( d(\phi^* f) = df \circ d\phi \). Since we are interested in the Taylor expansion of \( \phi^* f(y) \) at \( y = 0 \in U \) which will be denoted by \( f_\phi(x; y) \), \(^{27}\) we define an equivalence relation for two generalized exponential maps, \( \phi \sim \psi \), if all partial derivatives of \( \phi_x \) and \( \psi_x \) at \( y = 0 \) coincide. A formal exponential map is an equivalence class of such maps. Choosing local coordinates \( \{ x^\mu \} \) on the base and \( \{ y^\mu \} \) on the fiber, we can write such a formal exponential

\(^{27}\) In [80], it was denoted by \( T\phi^* f \in \hat{ST}^*M \) where \( T \) means the Taylor expansion in the \( y \in U_x \)-variables around \( y = 0 \) and \( \hat{S} \) denotes the formal completion of the symmetric algebra. The bundle \( \hat{ST}^*M \) of formally completed symmetric algebra of the cotangent bundle \( T^*M \) is defined as a jet bundle whose sections are given by eq. (5.71). Instead we will denote this bundle by \( \mathcal{E} \) according to [26].
map generated by a tangent vector \( v = y^1 \frac{\partial}{\partial x^1} + \cdots + y^d \frac{\partial}{\partial x^d} \in T_p M \) as a formal power series

\[
(\exp_p(v) \phi)^\mu := \phi^\mu(y) = x^\mu + \sum_{n=1}^{\infty} \frac{1}{n!} \phi^\mu_{x,\lambda_1 \cdots \lambda_n}(x) y^{\lambda_1} y^{\lambda_2} \cdots y^{\lambda_n} \tag{5.67}
\]

that depends smoothly on \( x \in M \). The coefficients in the exponential map (5.67) can be determined using the geodesic flow of a torsion free connection defined by

\[
\hat{\Phi}_x + \Gamma^\mu_{\rho \sigma} (\Phi_x) \hat{\phi}_x^\rho \hat{\phi}_x^\sigma = 0 \tag{5.68}
\]

where \( \Phi_x(t, y), \ t \in [0, 1], \) is a formal curve with initial conditions \( \Phi_x(0, y) = x \) and \( \Phi_x(0, y) = y \). It is easy to show that the required formal exponential map \( \phi_x(y) = \Phi_x(1, y) \) is given in local coordinates by

\[
\phi_x^\mu(y) = x^\mu + y^\nu - \frac{1}{2} \Gamma^\mu_{\rho \sigma}(x) y^\rho y^\sigma - \frac{1}{3!} (\partial_\lambda \Gamma^\mu_{\rho \sigma}(x) - 2 \Gamma^\mu_{\nu \lambda}(x) \Gamma^\nu_{\rho \sigma}(x)) y^\rho y^\sigma y^\lambda + \cdots . \tag{5.69}
\]

By putting \( \phi_x^\mu(y) \) at the origin, i.e., \( x = 0 \), we note the similarity with the exponential map (2.33) determined by the Moser flow (2.16). We will see later that they are related to each other.

It is now straightforward to consider the Taylor expansion of the pullback \( f_\phi(x; y) = f(\phi_x(y)) \) of a smooth function \( f \in C^\infty(M) \) via the formal exponential map \( \phi : U \to M \). We write

\[
f_\phi(x; y) = f(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{\phi, \lambda_1 \cdots \lambda_n}(x) y^{\lambda_1} y^{\lambda_2} \cdots y^{\lambda_n} \tag{5.70}
\]

where the coefficient \( f^{(n)}_{\phi} \) is a covariant symmetric tensor of rank \( n \) and smoothly depends on \( x \in M \). It turns out \([26, 81]\) that \( f_\phi \) is a particular example of a section of the jet bundle \( \mathcal{E} \to M \) (where \( \mathcal{E} \) is the bundle \( F(M) \times_{GL(d, \mathbb{R})} \mathbb{R}[[y^1, \cdots, y^d]] \) associated to the frame bundle \( F(M) \) on \( M \)) with the fiber \( \mathbb{R}[[y^1, \cdots, y^d]] \) (i.e., formal power series in \( y \) with real coefficients) and transition functions induced from the transition functions of \( TM \). In general any section of \( \mathcal{E} \) is of the form

\[
\sigma(x; y) = \sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}_{\lambda_1 \cdots \lambda_n}(x) y^{\lambda_1} y^{\lambda_2} \cdots y^{\lambda_n} \tag{5.71}
\]

where \( a_{\lambda_1 \cdots \lambda_n} \) define covariant tensors on \( M \). In this way the variables \( y^\lambda \) may be thought of as formal coordinates on the fibers of the tangent bundle \( TM \). Also recall \([26]\) that a section \( \sigma \) of the jet bundle \( \mathcal{E} \) is the pullback of a function, i.e. \( \sigma = f_\phi \) if and only if

\[
D_{X}^{(0)} \sigma = 0, \quad \forall X \in \Gamma(TM) \tag{5.72}
\]

where \( D_{X}^{(0)} \) is the differential operator given by

\[
D_{X}^{(0)} = X - X^\mu(x) \frac{\partial \phi^\nu_x}{\partial x^\mu} \left( \frac{\partial \phi^\nu_x}{\partial y^\lambda} \right)^{-1}_\nu \frac{\partial}{\partial y^\lambda} =: X + \tilde{X}. \tag{5.73}
\]
It is easy, using the expansion (5.69), to yield the inverse of the Jacobian matrix \( \begin{pmatrix} \frac{\partial \phi^\mu}{\partial y^\nu} \end{pmatrix} \) which is given by
\[
\left[ \left( \frac{\partial \phi^\mu}{\partial y^\nu} \right)^{-1} \right]^{\lambda}_{\nu} = \delta^\lambda_\nu + \Gamma^\lambda_\nu_\rho y^\rho + \left( \frac{1}{2} \partial_\nu \Gamma^\lambda_\rho_\sigma - \frac{1}{3} \partial^\lambda_\mu_\rho_\sigma \right) y^\rho y^\sigma + \cdots .
\]
(5.74)

The property (5.72) is simply a result of the chain rule for the section \( \sigma = f \circ \phi \). By observing that
\[
D^{(0)}(0)X_\sigma(x; y) = \frac{d}{dt} \big|_{t=0} \sigma(x(t) ; \phi^{-1}_x(x(t)) \phi_x(y))
\]
(5.75)
for any curve \( t \mapsto x(t) \in M \) such that \( x(0) = x \) and \( \dot{x}(0) = X \in T_x M \), it can be proven [81] that \( [D^{(0)}(0)X, D^{(0)}(0)Y] = D^{(0)}(0)[X, Y] \). See also appendix C. Its immediate consequence is that the covariant derivative \( D^{(0)} = dx^\mu \partial_\mu \) defines a flat connection, i.e., \((D^{(0)})^2 = 0\). This is also called the Grothendieck connection.

Let us write the flat connection \( D^{(0)} = dx^\mu \partial_\mu \) as the form
\[
D^{(0)}_\mu = \frac{\partial}{\partial x^\mu} - R^\lambda_\mu_\nu(x; y) \frac{\partial}{\partial y^\lambda}
\]
(5.76)
where
\[
R^\lambda_\mu(x; y) \equiv \frac{\partial \phi^\nu_\mu}{\partial x^\rho} \left[ \left( \frac{\partial \phi^\lambda}{\partial y^\rho} \right)^{-1} \right]^{\lambda}_{\nu}
\]
(5.77)
is a formal power series in \( y \) which begins with \( \delta^\lambda_\nu \) and whose coefficients are smooth in \( x \). By these properties it immediately follows [26, 81] that a section of the jet bundle \( E \) is the Taylor expansion of a globally defined function if and only if it is \( D^{(0)} \)-closed. Obviously \( D^{(0)} \) is a derivation of the usual product of sections of \( E \), i.e., \( (D^{(0)}(\sigma \tau) = (D^{(0)}(\sigma) \tau + \sigma D^{(0)}(\tau) \) for \( \sigma, \tau \in \Gamma(E) \). Thus the algebra of global functions on \( M \) can be identified with the subalgebra of \( D^{(0)} \)-closed sections. A differential form with values in \( E \) is a section of the bundle \( E \otimes \Lambda^m T^* M \), which can be expressed locally as
\[
\Sigma(x; y) = \sum_{n=0}^{\infty} \frac{1}{m! n!} a^{(m, n)}_{\rho_1 \rho_2 \cdots \rho_m \lambda_1 \lambda_2 \cdots \lambda_n} y^{\lambda_1} y^{\lambda_2} \cdots y^{\lambda_n} dx^{\rho_1} \wedge dx^{\rho_2} \wedge \cdots \wedge dx^{\rho_m}.
\]
(5.78)

It is useful to define the total degree of a form on \( M \) taking values in sections of \( E \) as the sum of the form degree and the degree in \( y \) and then to decompose the Grothendieck connection (5.76) in the following way
\[
D^{(0)} = -\delta + d^S + A
\]
(5.79)
where
\[
\delta \equiv dx^\mu \frac{\partial}{\partial y^\mu}
\]
(5.80)
is the zero-degree part and
\[
d^S \equiv dx^\mu \left( \frac{\partial}{\partial x^\mu} - \Gamma^\lambda_\mu_\nu y^\nu \frac{\partial}{\partial y^\lambda} \right)
\]
(5.81)
is the degree-one part and finally
\[ A \equiv dx^\mu A_\mu(x; y) \frac{\partial}{\partial y^\mu} = dx^\mu \left( -\frac{1}{3} \mathcal{R}_\mu^\rho y^\rho y^\sigma + O(y^3) \right) \frac{\partial}{\partial y^\lambda} \] (5.82)
is at least of second degree in \( y \). The requirement of the vanishing of the curvature \( (D^{(0)})^2 \equiv \Upsilon \) yields the condition
\[ 0 = \Upsilon = -\mathcal{R}_\mu^\rho y^\rho \frac{\partial}{\partial y^\mu} + F - \delta A \] (5.83)
with \( \mathcal{R}_\mu^\rho = \frac{1}{2} \mathcal{R}_\mu^\rho dx^\rho \wedge dx^\sigma \) and \( F = dS A + A^2 \).

Define the “inverse” operator of \( \delta \) by
\[ \delta^{-1} \Sigma_{(n)} = \frac{1}{m+n} \nu \Sigma_{(n)} \] (5.84)
when \( m+n > 0 \) and \( \delta^{-1} \Sigma_{(n)} = 0 \) when \( m+n = 0 \), where \( \Sigma_{(n)} \) is a monomial with degree \( m+n \) in eq. (5.78) and \( \nu = y^n \frac{\partial}{\partial y^n} \in T_y M \). Then there is a Hodge-decomposition [28, 29] that any form \( \Sigma \in \Gamma(\mathcal{E}) \otimes \Lambda^* M \) has the representation
\[ \Sigma = \delta \delta^{-1} \Sigma + \delta^{-1} \delta \Sigma + a^{(0,0)} \] (5.85)
where \( a^{(0,0)} \) is a function on \( M \) (independent of \( y \)) in eq. (5.78). Note that, at least at leading order,
\[ \delta^{-1} A = 0 \] (5.86)
but it can be proven that it is generally true. Since \( \text{deg } (A) \geq 2 \) and so \( A^{(0,0)} = 0 \), the Hodge decomposition (5.85) together with eqs. (5.83) and (5.86) leads to the relation
\[ A = -\delta^{-1} \mathcal{R} + \delta^{-1} F \] (5.87)
where \( \mathcal{R} \equiv \mathcal{R}_\mu^\nu y^\nu \frac{\partial}{\partial y^\mu} \). By cohomological perturbation theory, it is not difficult to prove [28, 29] that the cohomology of \( D^{(0)} \) for the bundle \( \mathcal{E} \otimes \Lambda^* M \) is almost trivial and concentrated in degree 0, i.e., functions \( a^{(0,0)}(x) \) in eq. (5.78). This fact will be important later for the global version of deformation quantization for Poisson manifolds as well as our construction of global vector fields.

The above Taylor expansion can be generalized to a polyvector field \( \Xi \in \mathcal{V}^k(M) \) using the exponential map \( \phi_x \) again. Consider the push-forward \( (\phi_x)^{-1} \Xi \) of a \( k \)-vector field \( \Xi \) defined on \( M \). Its Taylor expansion denoted by \( \Xi_{\phi} \) becomes a formal multivector field in \( y \) for any \( x \in M \). For example, if \( X \) is a vector field on \( M \), then we get the coefficients
\[ X_\phi^\mu(x; y) = X^\lambda(\phi_x(y)) \left[ \left( \frac{\partial \phi_x}{\partial y} \right)^{-1} \right]^\mu_\lambda. \] (5.88)
The result is exactly the same as eq. (5.64) with the replacement \( \nu^\mu \rightarrow X^\mu \). Similarly, for a Poisson bivector \( \Pi \in \mathcal{V}^2(M) \), the corresponding Taylor expansion is given by
\[ \Pi_{\phi}^{\mu\nu}(x; y) = \Pi^{\mu\nu} (x) + \partial^\alpha_\mu \Pi^{\mu\nu}(x) y^\lambda + \frac{1}{2} \left( \partial^\rho_\mu \partial^\rho_\nu \Pi^{\mu\nu}(x) - \frac{1}{3} \mathcal{R}^{[\mu}_{\rho\lambda\sigma} \Pi^{\nu\lambda]}(x) \right) y^\rho y^\sigma + \cdots \] (5.89)
Note that the above result coincides with eq. (5.63) when \( \Pi^{\mu\nu} \) are constants in a Darboux frame obeying eq. (5.58). Recall that the tangent bundle of a manifold is an example of a vector bundle. Thus, given a Poisson manifold \((M, \Pi)\), the Poisson structure on \(M\) induces a Poisson structure on each fiber of the tangent bundle \(TM\) and so each tangent space \(T_xM\) for any \(x \in M\) can be considered as an affine space with the fiberwise Poisson structure. In this way, the tangent bundle \(TM\) becomes a Poisson manifold with the fiberwise Poisson bracket. In particular, for a symplectic manifold \((M, \Omega)\), the tangent space \(T_xM\) becomes a symplectic vector space equipped with a constant symplectic structure. Since \(\Pi^2\) and \(\Pi \Omega\) for any \(x \in M\), it is enough to evaluate the components of \(\Xi\) and \(\Omega\) at \(y = 0\) and to replace formally each \(dy^\mu\) by \(dx^\mu\) and each \(\partial y^\mu\) by \(\partial x^\mu\) in order to recover the global objects \(\Xi\) and \(\Omega\) [80].

Therefore we will regard the bivector \(\Pi^\phi\) as an induced Poisson structure on \(U^e \subset TM\). Choosing local coordinates \((x, y)\) for \(U^e\), its expression is locally given by

\[
\Pi^\phi = \frac{1}{2} \Pi^{\mu\nu}(x; y) \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu}. \tag{5.90}
\]

The local fiber coordinates \(\{y^\mu\}\) \(U^e\) by construction will be given by the Darboux-Weinstein coordinates on \(U^e\) such that the coefficients \(\Pi^{\mu\nu}(x; y)\) take the simplest form (2.34) and so they become constants (independent of \(y\) but \(x\)-dependent) for a symplectic vector space.

We introduce the Poisson bracket on sections of \(E\) by

\[
\{\sigma, \tau\}_\Pi^\phi(x; y) = \Pi^{\mu\nu}_\phi(x; y) \frac{\partial \sigma(x; y)}{\partial y^\mu} \frac{\partial \tau(x; y)}{\partial y^\nu}. \tag{5.91}
\]

for \(\tau, \sigma \in \Gamma(E)\). Since the set of flat sections obeying eq. (5.72), denoted by \(\text{ker } D^{(0)}\), forms a subalgebra, we can restrict the Poisson bracket (5.91) to \(\text{ker } D^{(0)}\). Using the one-to-one correspondence between \(C^\infty(M)\) and \(\text{ker } D^{(0)}\), we identify \(\sigma = f^\phi = f \circ \phi\) and \(\tau = g^\phi = g \circ \phi\). Then it is straightforward, using the chain rule [26]

\[
\frac{\partial f^\phi}{\partial x^\lambda} = \frac{\partial f}{\partial x^\lambda} \frac{\partial \phi^e}{\partial x^e}, \quad \frac{\partial f^\phi}{\partial y^\lambda} = \frac{\partial f}{\partial y^\lambda} \frac{\partial \phi^e}{\partial y^e}, \tag{5.92}
\]

to show that

\[
\{f^\phi, g^\phi\}_\Pi^\phi(x; y) = \{f, g\}_\Pi(\phi^e(y)) \tag{5.93}
\]

where the right-hand side is the Poisson bracket of global functions on \(M\). Similarly, using eq. (5.88), one can show that

\[
X^\phi(f^\phi) = (Xf)^\phi \tag{5.94}
\]

for a global vector field \(X \in \Gamma(TM)\) and its push-forward \(X^\phi := (\phi_x)^{-1}X\). Observe that by assumption \(d_y \phi_x(0) = \text{id}\) we can recover the global objects \(X \in \Gamma(TM)\) and \(\Pi \in \mathcal{V}^2(M)\) from \(X^\phi\) and \(\Pi^\phi\), respectively, by evaluating their components at \(y = 0\) and replacing formally each \(\partial y^\mu\) by \(\partial x^\mu\) as one can explicitly see from eqs. (5.88) and (5.89). In general, if \(\Xi^\phi\) and \(\Omega^\phi\) are in the image of \(T\phi^*\) (see the footnote [27]) of a polyvector field \(\Xi \in \mathcal{V}^k(M)\) and a \(k\)-form \(\Omega \in \Gamma(\Lambda^k T^* M)\), respectively, i.e.,

\[
\Xi^\phi = (\phi_x)^{-1}\Xi, \quad \Omega^\phi = (\phi_x)^*\Omega, \tag{5.95}
\]

it is enough to evaluate the components of \(\Xi^\phi\) and \(\Omega^\phi\) at \(y = 0\) and to replace formally each \(dy^\mu\) by \(dx^\mu\) and each \(\partial y^\mu\) by \(\partial x^\mu\) in order to recover the global objects \(\Xi\) and \(\Omega\) [80].
explicitly, if \( \Xi_\phi(x; y) = \Xi^{\mu_1 \cdots \mu_k}(x; y) \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \) is equal to \( T\phi^* \Xi = T\phi_*^{-1} \Xi \), then, in local coordinates,

\[
\Xi(x) = \Xi^{\mu_1 \cdots \mu_k}(x; 0) \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}},
\]

and, if \( \Omega_\phi(x; y) = \Omega_{\mu_1 \cdots \mu_k}(x; y) dy^{\mu_1} \wedge \cdots \wedge dy^{\mu_k} \) is equal to \( T\phi^* \Omega \), then

\[
\Omega(x) = \Omega_{\mu_1 \cdots \mu_k}(x; 0) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}.
\]

Let us summarize how we got global objects. Let us focus on the symplectic case for simplicity. At the outset we prepare the system of multivector fields \( \{ \Xi \in \mathcal{V}^k(M) : k = 0, 1, \cdots, d \} \) at a point \( p \in M \) whose local coordinates are \( x \). Then we develop the system along a geodesic curve described by eq. (5.66). We extend the system such that it goes into a Darboux frame (2.33) at the end point of the geodesic flow whose coordinates are \( \phi_x(y) \). We denote the system in the Darboux frame as \( \{ \Xi_\phi = (\phi_x)^{-1}_* \Xi \in \mathcal{V}^k_\phi(U) : k = 0, 1, \cdots, d \} \). In particular, we can construct the Grothendieck connection for an infinite jet bundle \( \mathcal{E} \) of functions using the formal coordinates \( \phi_x(y) \). This connection allows us to identify smooth functions on \( M \) with flat (or integrable) sections of the jet bundle \( \mathcal{E} \). But the Darboux frame is malleable because the torsion free connection will vanish there and so it can be further extended using normal coordinates as we discussed before. Indeed this extension corresponds to the situation that the system is initially prepared in the Darboux frame and so the exponential map is given by \( \phi_x(y) = x + y \). That is the reason why we get a parallel result with the normal coordinate system. In this case we attribute the infinitesimal development on a Darboux chart to \( U(1) \) gauge fields as applied in eq. (5.54). Hence we have the relation \( (\phi_x)^{-1}_* \theta = \phi_x^* \theta = \theta \) since \( \phi_x : \mathcal{U}_x \rightarrow M \) is a diffeomorphism and so we can identify the exponential map \( \phi_x^* \) with the Moser flow (2.22). Eventually we will quantize a symplectic manifold \( (M, \Omega) \) in the Darboux frame where the Poisson bivector \( \Pi_\phi \in \mathcal{V}^2_\phi \) takes the simplest form. See the footnote [17] for the advantage of this frame.

The previous identification (5.66) can now be well founded on this global approach. Consider two local Darboux charts \( (U_1, \phi_1) \) and \( (U_2, \phi_2) \) such that \( U_1 \cap U_2 \neq \emptyset \) and \( \phi_i : \mathcal{U}_i \rightarrow M \) is a formal exponential map given by (5.69) on \( U_i \subset M \) for \( i = 1, 2 \). On each chart the Poisson structure is defined from the global Poisson structure \( \Pi \equiv \Omega^{-1} \in \mathcal{V}^2(M) \) with its own exponential map: \( \Pi_{\phi_1} = (\phi_{1x})_*^{-1} \Pi \) and \( \Pi_{\phi_2} = (\phi_{2x})_*^{-1} \Pi \). On an overlap \( U_{21} = U_1 \cap U_2 \), they are definitely related to each other by

\[
\Pi_{\phi_2} = (\phi_{21x})_*^{-1} \Pi_{\phi_1}
\]

where \( \phi_{21x} \equiv \phi_{2x} \circ \phi_{1x}^{-1} : \mathcal{U}_x \rightarrow U_{21} \). Note that the exponential map \( \phi_{21} \) is a diffeomorphism between nearby Darboux charts and so it can be generated by a normal coordinate system. As a result two Poisson structures must be related to each other according to eq. (5.63) and so the exponential map \( \phi_{21} \) will be of the form (2.22). This gluing procedure was described in [22]. Similarly, for the exponential maps obeying eq. (5.94) on each Darboux

\[\text{Note that the exponential map in eq. (5.66) can be identified with } \phi_x(y), \text{ i.e., } \exp_p(v) = \phi_x(y) = x + y \text{ where the point } p \text{ was taken to be the origin, } x = 0.\]
chart, we have $X_{\phi_1}(f_{\phi_1}) = (Xf)_{\phi_1}$ and $X_{\phi_2}(f_{\phi_2}) = (Xf)_{\phi_2}$. Thus, on the intersection $U_{21} = U_1 \cap U_2$, the gluing condition for vector fields on local charts is given by

$$X_{\phi_2} = (\phi_{21})^{-1}X_{\phi_1}. \quad (5.99)$$

Note that, if $X_\phi$ is a Hamiltonian vector field on $U_\phi$, i.e. $X_\phi := X_{f_\phi}$ for any global function $f$ on $M$, the relation (5.94) reduces to

$$X_{f_\phi}(g_\phi) = (Xf)_{\phi}(g) \quad (5.100)$$

which precisely means eq. (5.93). Thus we see that a global Hamiltonian vector field $X_f \in \Gamma(TM)$ is mapped via the formal exponential map to a Hamiltonian vector field $X_\sigma$ of a flat section $\sigma = f_\phi$ on the jet bundle $E$. Finally, we can apply the rule (5.94) and (5.97) to identify the global objects in eq. (5.66):

$$\Omega_{\mu\nu}(x) = (\Pi^{-1})_{\mu\nu}(x; y = 0), \quad V^\mu_a(x) = (V_a)^\mu_\phi(x; y = 0) \quad (5.101)$$

where the set of Hamiltonian vector fields obeys the relation (5.100), i.e., $(V_a)_\phi(f_\phi) = (V_a(f))_{\phi}$. Now next step is to quantize the symplectic manifold $(M, \Omega)$. Since the emergent gravity claims that a symplectic manifold $(M, \Omega)$ gives rise to a Riemannian manifold $(M, g)$, according to the emergent gravity picture, we understand the quantization of the dynamical symplectic manifold $(M, \Omega)$ as the quantization of the corresponding (emergent) Riemannian manifold $(M, g)$. Thus the emergent gravity picture suggests a completely new quantization scheme of Riemannian manifolds where quantum gravity is defined by a dynamical NC spacetime. For example, U(1) gauge theory on a symplectic manifold can be identified with a Fedosov manifold which includes any Kähler manifolds, and so the NC gauge theory corresponds to the quantization of the Fedosov manifold which should contain “quantized Kähler manifolds”. In order to quantize the manifold $M$, it is important to note that we have an isomorphism of Poisson algebras

$$\iota : C^\infty(M) \to \ker D^{(0)} \quad (5.102)$$

from the algebra of smooth functions on $M$ onto the algebra of horizontal sections of $E$. Therefore one may try to quantize $(E, D^{(0)})$ and to identify a subalgebra of the quantized algebra $\hat{E} := E[[\hbar]]$ with the vector space $C^\infty(M)[[\hbar]]$ in such a way that the induced multiplication on $A_\hbar = C^\infty(M)[[\hbar]]$ gives a deformation quantization of $M$. For this program to work, we need the quantum Grothendieck connection $D := \sum_{n=0}^{\infty} \hbar^n D^{(n)}$ to be a derivation (so that the space of flat sections of $\hat{E}$ becomes a subalgebra) and to be flat (so that there is no obstruction to the integrability of the horizontal distribution defining the quantum connection). It is straightforward to quantize the space of sections of the jet bundle $E$ which is subject to the Poisson bracket (5.91) with the $y^\mu$ as quantized variables and $x^{\mu}$ as parameters. Using the Kontsevich’s formality map $[23, 26]$,

$$\hat{x} \equiv \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_n(\Pi_\phi, \cdots, \Pi_\phi), \quad (5.103)$$

where

$$\phi_{21}^{-1}X_{\phi_1}.$$
we define the star product for sections \( \sigma, \tau \in \Gamma(\mathcal{E}) \) by
\[
\hat{\star}(\sigma \otimes \tau) = \sigma \hat{\star} \tau.
\] (5.104)

The star product (5.103) is basically the Moyal star-product on \( T_xM \) and the pair \( (\Gamma(\mathcal{E})[[\hbar]], \hat{\star}) \) is known as the Weyl algebra and will be denoted by \( W_x \). The algebras \( W_x, x \in M \), can be smoothly patched and we get a fiber bundle \( W = \sqcup_{x \in M} W_x \) on \( M \), called the Weyl algebra bundle over \( M \). Hence the Weyl algebra bundle may be thought of as a “quantum tangent bundle”.

Although the covariant derivative (5.73) is a derivation of the usual product of sections of \( \mathcal{E} \), it is not a derivation of \( \hat{\star} \). So we introduce a quantum covariant derivative in the direction of \( X \in T_xM \) defined by
\[
\tilde{D}_X = X + A(\tilde{X}) = X^\mu(x) \bar{D}_\mu
\] (5.105)
where \( \tilde{X} = -X^\mu(x) R^\mu_{\lambda}(x; y) \frac{\partial}{\partial y} \) is given by eq. (5.73) and the formality map (3.23) for the quantum connection is defined by
\[
A(\tilde{X}) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+1}(\tilde{X}, \Pi_\phi, \cdots, \Pi_\phi).
\] (5.106)

Since \( A(\tilde{X}) = \tilde{X} + \cdots \) and \( U_2(\tilde{X}, \Pi_\phi) = 0 \) (see eq. (3.57) and comments below),
\[
\tilde{D}_X = D^{(0)}_X + \sum_{n=2}^{\infty} \frac{\hbar^n}{n!} U_{n+1}(\tilde{X}, \Pi_\phi, \cdots, \Pi_\phi).
\] (5.107)

The formality identity (3.49) applied to the quantum covariant derivative (5.105) implies the crucial statement that \( \tilde{D} \) is a derivation of \( \hat{\star} \) (see Proposition 4.2 in [26] for the proof), i.e.,
\[
\tilde{D}(\sigma \hat{\star} \tau) = (\tilde{D}\sigma) \hat{\star} \tau + \sigma \hat{\star}(\tilde{D}\tau)
\] (5.108)
for \( \sigma, \tau \in \Gamma(W) \).

But the quantum connection \( \tilde{D} = dx^\mu \bar{D}_\mu \) is not flat in general but it has a curvature given by [26]
\[
\tilde{D}^2 \sigma = [F^M, \sigma]_{\hat{\star}}
\] (5.109)
where \( F^M \) is a 2-form on \( M \) with values in the section of \( W \) defined using the formality identity (3.51) as
\[
F^M(X, Y) = \Psi(\tilde{X}, \tilde{Y}) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} U_{n+2}(\tilde{X}, \tilde{Y}, \Pi_\phi, \cdots, \Pi_\phi).
\] (5.110)

Hence we need to modify \( \tilde{D} \) by adding more “quantum corrections” to have a flat quantum connection so that the new covariant derivative
\[
\mathcal{D} = \tilde{D} + [\gamma, \cdot]_{\hat{\star}}
\] (5.111)

\[29\text{Furthermore, } U_n(\xi, \Pi_\phi, \cdots, \Pi_\phi) = 0 \text{ for } n \geq 2 \text{ if } \xi \text{ is a linear vector field } [21]. \text{ Thus } \tilde{X} \text{ in eq. (5.107) can be replaced by } \tilde{X} = \iota_X(-\delta + A). \]
is again a derivation where \( \gamma \) is a one-form on \( M \) with values in the section of \( W \). Note that \( \mathfrak{D} \) is certainly a derivation of \( \hat{\star} \) because the adjoint action \( \text{ad}(\gamma) \) in eq. (5.111) is automatically a derivation of \( \hat{\star} \) and so we only need to find the one-form \( \gamma \) such that the quantum corrected connection \( \mathfrak{D} \) is flat, i.e., \( \mathfrak{D}^2 = 0 \). The flatness condition \( \mathfrak{D}^2 = 0 \) can be stated as the form
\[
G^M \equiv F^M + \mathfrak{D} \gamma + \hat{\gamma} \hat{\star} \gamma = \omega
\] (5.112)
where \( \omega \) is a central element, i.e., \( [\omega, \sigma]_\mathfrak{D} = 0, \forall \sigma \in \Gamma(W) \) and the wedge product between forms was implicitly assumed. From the definition (5.109), it is obvious that the Bianchi identity, \( \mathfrak{D} F^M = 0 \), is guaranteed [26, 29]. This in turn leads to the Bianchi identity \( \mathfrak{D} G^M = 0 \).

When we write \( \mathfrak{D} \) in eq. (5.107) as the form \( \mathfrak{D} = D^{(0)} + A'(\bar{x}) \) (see the footnote 29 for the definition of \( \bar{x} \)) where \( A'(\bar{x}) \) at most starts from \( \mathcal{O}(h^2) \), the curvature \( F^M \) is given by
\[
F^M = D^{(0)} A'(\bar{x}) + A'(\bar{x}) \hat{\star} A'(\bar{x})
\] (5.113)
which starts from \( \mathcal{O}(h^2) \) too. The connection \( \mathfrak{D} \) with the above properties is called the Fedosov connection [28, 29].

If we are able to find the one-form \( \gamma \) so that \( G^M = \omega \), then \( \mathfrak{D} \)-closed sections, denoted by \( \ker \mathfrak{D} \), will form a nontrivial subalgebra of \( W \). A basic observation to determine \( \gamma \) from the equation (5.112) is that the \( D^{(0)} \)-cohomology is trivial [28, 29]. The procedure is similar to that to yield eq. (5.87). For this purpose it is convenient to split the Fedosov connection as the form \( \mathfrak{D} = -\delta + \mathfrak{D}' \) and then write the equation (5.112) as the following form
\[
\delta \gamma = F^M - \omega + \mathfrak{D}' \gamma + \gamma \hat{\star} \gamma.
\] (5.114)
In particular it is enough to choose \( \gamma \) so that it starts from \( \mathcal{O}(h^2) \) because \( F^M \) at most starts from \( \mathcal{O}(h^3) \). Then the one-form \( \gamma \) is uniquely determined by the lowest term \( F^M - \omega \) using the iteration method with the filtration defined by the grading \( \text{deg}(y) = 1 \) and \( \text{deg}(\hbar) = 2 \).

Consequently we have a quantum version of the classical isomorphism (5.102) stating that there is a module isomorphism between \( \mathcal{A}_\theta = C^\infty(M)[[\hbar]] \) and \( \ker \mathfrak{D} \) such that the star product in \( \mathcal{A}_\theta \) inherits from the star product \( \hat{\star} \) in ker \( \mathfrak{D} \). More precisely there exists a quantization map \( \rho : \Gamma(W) \to \Gamma(W) \) so that the formal series \( \rho = \text{id} + \sum_{n=1}^\infty \hbar^n \rho_n \) obeys the relation
\[
\mathfrak{D} \rho(\sigma) = \rho( D^{(0)} \sigma )
\] (5.115)
for every \( \sigma \in \Gamma(W) \). The quantization map \( \rho \) can be uniquely determined by solving eq. (5.113) using the same iteration method as eq. (5.114). In particular it is easy to show that \( \rho_1 = 0 \). Therefore the image under \( \rho \) of the space of \( D^{(0)} \)-flat sections of \( W \) is the subalgebra of \( \mathfrak{D} \)-flat sections of \( W \). Since \( \ker D^{(0)} \) is isomorphic to the space of formal series of functions on \( M \), we can finally define a global star product on \( M \) by [25, 26]
\[
f \hat{\star} g = \left[ \rho_1^{-1} \left( \rho(f) \hat{\star} \rho(g) \right) \right]_{y=0}.
\] (5.116)
This constitutes the quantum version of the first part for the globalization (5.101) where we regard the left-hand side of eq. (5.116) as the star product of the global Poisson structure \( \Pi = \Omega^{-1} \).

It recovers the classical result when \( h \to 0 \).
Now it is straightforward to prescribe the quantum version of the second part for the globalization (5.101). Let \( \{ \hat{\mathcal{C}}_a \in \ker \mathcal{D} : a = 1, \cdots, d \} \) be the set of global \( \mathcal{D} \)-flat sections of \( W \) which may be obtained from the space \( \ker \mathcal{D}^{(0)} \) with the quantization map \( \rho \) obeying the relation (5.115) or gluing the local covariant momentum variables \( \hat{\mathcal{C}}_a \) \( \text{`a la} \) [22]. Then the adjoint action

\[
\text{ad}(\hat{\mathcal{C}}_a) \equiv -i[\hat{\mathcal{C}}_a, \cdot]_\hat{\star} \quad (5.117)
\]

defines a derivation of \( \ker \mathcal{D} \) since \( \text{ad}(\hat{\mathcal{C}}_a) \) definitely satisfies the derivation property and preserves the space \( \ker \mathcal{D} \), i.e.,

\[
\mathcal{D}(\text{ad}(\hat{\mathcal{C}}_a)(\sigma)) = -i[\mathcal{D}\hat{\mathcal{C}}_a, \sigma]_\hat{\star} - i[\hat{\mathcal{C}}_a, \mathcal{D}\sigma]_\hat{\star} = 0 \quad \text{for any } \sigma \in \ker \mathcal{D}.
\]

Actually more is true; it is enough for \( \mathcal{D}\hat{\mathcal{C}}_a := \psi_a \) to be central closed one-forms on \( M \), i.e.,

\[
[\psi_a, \sigma]_\hat{\star} = 0, \quad \forall \sigma \in \ker \mathcal{D} \quad \text{and} \quad d\psi_a = \mathcal{D}\psi_a = \mathcal{D}^2 \hat{\mathcal{C}}_a = 0.
\]

(See Appendix A in [82] for a succinct summary of derivation algebras from the Fedosov quantization approach.) The global vector fields \( V_a^\star \) can then be obtained by applying the rule (5.96) to \( \text{ad}(\hat{\mathcal{C}}_a) \) [26]. Explicitly they are given by

\[
V_a^\star \equiv \text{ad}(\hat{\mathcal{C}}_a)|_{y=0; \partial/\partial y^\mu \rightarrow \partial/\partial x^\mu}. \quad (5.118)
\]

Together with the globally defined star product (5.114), we realize the prescription (5.28) for the globalization of the vector fields \( V_a^\star \in \mathfrak{X} \).

In order to complete the globalization, it is also necessary to understand how to lift the volume preserving condition (5.29) to quantum vector fields \( V_a^\star \in \mathfrak{X} \). To understand this issue, we need to look at the modular class of Poisson manifolds in the first Poisson cohomology space of the manifold [52], i.e., the equivalence class of Poisson vector fields modulo Hamiltonian vector fields which is an infinitesimal Poisson automorphism. We devote appendix B to a brief review of modular vector fields and Poisson homology. Recall that, on an orientable manifold, there exists a volume form invariant under all Hamiltonian vector fields if and only if there exists a modular vector field which vanishes. Such a Poisson manifold is called unimodular [52]. It turns out (see appendix B) that it is possible to define a trace as a NC version of integration and to lift the modular vector fields up to a quantized Poisson manifold if a Poisson manifold was originally unimodular. Thus it is necessary to restrict to unimodular Poisson manifolds in order to realize the volume preserving condition (5.29) even in the quantum level. It is in no way a restriction to symplectic manifolds since any symplectic manifold is unimodular [50, 51]. In fact, the Liouville volume form is invariant under all Hamiltonian vector fields. However, up to our best knowledge, it is not well understood yet how to define a trace for a general (non-unimodular) Poisson manifold. Henceforth we will consider only unimodular Poisson manifolds.

Let us write the NC vector field (5.28) as \( V_a^\star = V_a + \Xi_a \) where \( V_a \) represents a global Hamiltonian vector field defined by eq. (5.101) while \( \Xi_a \) is a polydifferential operator comprising derivative corrections due to the NC structure of spacetime. Definitely the NC vector field \( V_a^\star \) represents a NC deformation of the usual vector field \( V_a \). As we observed before, Einstein gravity arises from the vector fields \( V_a \) at leading order. Therefore it should be obvious that the polydifferential operator \( \Xi_a \) will generate the derivative corrections of Einstein gravity. It is thus remained to determine the precise form of the derivative corrections (which may be a challenging problem). Nevertheless we expect the NC emergent
gravity may be very similar to the NC gravity in Refs. [83, 84] as was conjectured in [1] since the NC gravity is also based on a NC deformation of the diffeomorphism symmetry. But it should be remarked that the emergent gravity does not allow a coupling of cosmological constant like \( \int d^d x \sqrt{g} \Lambda \) which is of prime importance to resolve the cosmological constant problem [3]. We will not in this paper calculate the corrections except to note that the leading NC corrections will identically vanish. We showed in eq. (3.57) (see also the footnote 10 in [21]) that any arbitrary vector field \( X \) is stable at least up to the first order of NC deformations, i.e., \( U_2(X, \theta) = 0 \). This statement is similarly true for a smooth function \( f \in C^\infty(M) \), i.e., \( U_2(f, \theta) = 0 \). Indeed it must be a generic property for the deformation quantization because it comes only from the associativity of an underlying algebra. This fact implies that the NC corrections of the vector field \( V_\alpha \) start at most from the second order, i.e., \( \Xi^\mu\nu_\alpha(x) \partial_\mu \partial_\nu = 0 \). As a result, the general relativity, which is emergent from the leading order of NC gauge fields, receives no next-to-leading order corrections. In other words, the emergent gravity from NC gauge fields predicts an intriguing result that Einstein gravity corresponds to a (local) minimum of moduli space of Poisson (or Riemannian) structure deformations and is stable up to first order against quantum deformations due to the NC structure of spacetime.

5.4 Generalization to Poisson manifolds

So far we have mostly kept symplectic manifolds in mind for the construction of global vector fields and their quantization. Now we will think of the generalization to Poisson manifolds. In the context of Poisson geometry \((M, \Pi)\), generally speaking, one cannot find a “Poisson connection” \( \nabla^P \) such that \( \nabla^P \Pi = 0 \) since parallel transport preserves the rank of the Poisson tensor and so the Poisson manifold must be regular in order for such a connection to exist. But it turns out [25, 26, 27, 81] that it is enough to consider only a torsion free linear connection to formulate a star product on any Poisson manifold. Thus one can work only with an affine torsion free connection and construct the formal exponential map \((5.67)\) for the torsion free connection. Actually the previous formalism except the earlier discussion referring to the Fedosov manifold can also be applied to any Poisson manifold with impunity. So let us recapitulate the essential steps for the deformation quantization of Poisson manifolds we have discussed starting from the paragraph containing \((5.67)\). Given a torsion free connection on a Poisson manifold \((M, \Pi)\), one builds an identification of the commutative algebra \( C^\infty(M) \) of smooth functions on \( M \) with the algebra of flat sections of the jet bundle \( E \rightarrow M \), for the Grothendieck connection \( D^{(0)} \). And then one quantizes this situation to yield a quantum jet bundle \( \widehat{E} \rightarrow M \). A deformed algebra structure on \( \Gamma(\widehat{E}) \) is obtained through fiberwise quantization of the jet bundle using Kontsevich star product on \( \mathbb{R}^d \), and a deformed flat connection \( \mathcal{D} \) which is a derivation of this deformed algebra structure is constructed “à la Fedosov”. Again one constructs an identification between

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30 It must be remarked that, although we pretend the vector field \( V_\alpha \) is independent of \( \theta \), \( V_\alpha \) is actually \( O(\theta) \) since it is a Hamiltonian vector field defined by the Poisson bracket with the Poisson gauge fields \((2.41)\). (It may be noted that the Poisson gauge fields in eq. \((2.41)\) are given by \( C_\mu(y) = B_{\mu\nu} x'_{\nu}(y) \) which cancels out \( \theta^{\mu\nu} \) in the Poisson bracket.) Therefore, precisely speaking, the \( O(\theta^2) \) correction identically vanishes and the nontrivial NC corrections start from \( O(\theta^3) \).
the formal series of functions on $M$ and the algebra of flat sections of this quantized bundle of algebras. Finally this identification defines the star product on $M$. This quantization procedure can also be implemented to a general Poisson manifold $[23, 24]$. Hence the globalization in eq. (5.101) can be simply generalized to the Poisson case if the first one is replaced by

$$
\Pi_{\mu\nu}(x) = (\Pi_\phi)_{\mu\nu}(x; y = 0).
$$

(5.119)

However we also need to lift the volume preserving condition (5.29) to a quantum Poisson algebra. Thus it is necessary to restrict Poisson manifolds to unimodular ones as we discussed before.

But, as we discussed in the footnote 18, there is an interesting realization of Poisson manifolds in terms of symplectic realizations. A symplectic realization of a Poisson manifold $(M, \Pi)$ is a Poisson map

$$
\varphi : (S, \Omega) \rightarrow (M, \Pi)
$$

(5.120)

from a symplectic manifold $(S, \Omega)$ to $(M, \Pi)$. More precisely there is a collection of functions of the canonical variables $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ which is a subalgebra under the canonical Poisson bracket and generated by a finite number of independent functions $\varphi_1, \cdots, \varphi_r$. This means that $\mathbb{R}^r$ has a Poisson structure induced from the canonical symplectic structure $\mathbb{R}^{2n}$ in the sense that $\Phi = (\varphi_1, \cdots, \varphi_r) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^r$ is a Poisson map. For the $\text{SO}(3)$ algebra of angular momenta in the footnote 18, for example, we have the Poisson map $\varphi_i := L_i = \varepsilon_{ijk} x^j p_k$ from the symplectic manifold $(S, \Omega) = (\mathbb{R}^6, \sum_{i=1}^3 dx_i \wedge dp_i)$ to the Poisson manifold $(M, \Pi) = (\mathbb{R}^3, \frac{1}{2} \varepsilon_{ijk} \varphi_k \frac{\partial}{\partial \varphi_i} \wedge \frac{\partial}{\partial \varphi_j})$. The symplectic realization is a very natural object in Poisson geometry, in particular, from the point of view of the quantization theory of Poisson manifolds. For Poisson manifolds which are the classical analogue of associative algebras, symplectic realizations play a similar role as representations do for associative algebras. The symplectic realization of a Poisson manifold was first introduced by Lie who proved that such a realization always exists locally for any Poisson manifold of constant rank. After almost a century, Weinstein proved [13] the local existence theorem of symplectic realizations for general Poisson manifolds and later found [15, 16] that there exists globally a unique symplectic realization which possesses a local groupoid structure (though there is in general an obstruction for the existence of a global groupoid structure) compatible with the symplectic structure. There is also a direct global proof for the existence of symplectic realizations of arbitrary Poisson manifolds [53].

In mathematics literatures, the Poisson map $\varphi$ is assumed to be a surjective submersion. But we will not assume it because there are some examples in quantum mechanics with $\dim M \geq \dim S$. We will illustrate such kind of symplectic realizations known as the boson realization of Lie algebras or the Schwinger representation [31]. Suppose that the Poisson structure of $2n$-dimensional NC space (1.1) is given by $\theta^{2i-1, 2i} = \zeta^i > 0$, $i = 1, \cdots, n$, otherwise $\theta^{\mu\nu} = 0$. Using this canonical pairing (polarization) of symplectic structure, define $n$-dimensional annihilation and creation operators as

$$
a_i = \frac{y^{2i-1} + iy^{2i}}{\sqrt{2\zeta^i}}, \quad a_i^\dagger = \frac{y^{2i-1} - iy^{2i}}{\sqrt{2\zeta^i}}.
$$

(5.121)
Then the Moyal-Heisenberg algebra (1.1) reduces to the Heisenberg algebra $\mathcal{A}$ of $n$-dimensional harmonic oscillator, i.e.,

$$[a_i, a_j^\dagger] = \delta_{ij}. \quad (5.122)$$

The Schwinger representation of Lie algebra generators for $G = SU(n)$ is defined by

$$Q^I = a_i^\dagger T^I_{ij} a_j, \quad I = 1, \cdots, r, \quad (5.123)$$

where $r = \dim G = n^2 - 1$ and $T^I_{ij}$s are constant $n \times n$ Hermitian matrices satisfying the $su(n)$ Lie algebra $[T^I, T^J] = i f^{IJK} T^K$. It is easy to verify that the operators $Q^I$ obey the commutation relation of $su(n)$ Lie algebra

$$[Q^I, Q^J] = i f^{IJK} Q^K. \quad (5.124)$$

In this example, we have the Poisson map $\varphi^I := Q^I$ from the symplectic manifold $(S, \Omega) = (\mathbb{R}^{2n}, \frac{1}{2} \theta^{-1} dy^\mu \wedge dy^\nu)$ to an $r$-dimensional Poisson manifold $(M, \Pi) = (\mathbb{R}^r, \frac{1}{2} f^{IJK} \varphi^K \frac{\partial}{\partial \varphi^I} \wedge \frac{\partial}{\partial \varphi^J})$. In the case $n \geq 3$, $\dim M > \dim S$ and so it is possible to have a generalized Poisson map which is not necessarily a surjective submersion.

The above Schwinger representation can be generalized to any semi-simple Lie algebras. Suppose that $A$ is the Heisenberg algebra defined by eq. (5.122) and $g$ is an arbitrary Lie algebra of $G$ and $(\rho, V)$ is an $n$-dimensional representation of $g$ where $V$ is the representation space and $\rho$ is the homomorphic mapping from $g$ to $\text{End}(V)$. Then the Schwinger representation of $g$ is given by a Poisson map $\varphi$ defined by

$$X \mapsto \varphi(X) = a_i^\dagger (\rho X)_{ij} a_j, \quad \forall X \in g. \quad (5.125)$$

It is easy to check that the Poisson map is a Lie algebra homomorphism from $g$ to $A$, i.e.,

$$[[\varphi(X), \varphi(Y)] = \varphi([X, Y]), \quad \forall X, Y \in g. \quad (5.126)$$

The Poisson map (5.123) provides a symplectic realization (which is not necessarily a surjective submersion) to all semi-simple Lie group families, including the five exceptional groups. Note that we already quantized Poisson manifolds via the symplectic realization (5.123) or (5.125) although the complete classification of irreducible representations for the quantized Poisson algebras (5.124) and (5.126) still remains. Fortunately the 20th century had been completed the latter problem at least for semi-simple Lie algebras [87, 88]. Later we will discuss why the symplectic realization of a Poisson manifold supplies a great benefit for the quantization of Poisson manifolds.

There is another important realization of Poisson manifolds by the so-called symplectic groupoids which was initiated with the works [15, 46]. This can be seen as a generalization of the famous Lie’s third theorem in the theory of Lie groups in the sense that the correspondence between symplectic groupoids and Poisson manifolds is a natural extension of the one between Lie groups and Lie algebras. We refer to Chapters 8 and 9 in [14] for a nice exposition of symplectic realizations and symplectic groupoids. The symplectic groupoid is a natural object in Poisson geometry for the following reason. We know that a Poisson algebra $\mathfrak{P} = (C^\infty(M), \{-, -\}_g)$ is a Lie algebra on the vector space $C^\infty(M)$ with respect
to the Poisson bracket \{−, −\}θ on a manifold M. As every finite-dimensional Lie algebra \(\mathfrak{g}\) over \(\mathbb{R}\) is associated to a Lie group \(G\), a natural question is then whether there is a Lie group integrating this Poisson-Lie algebra. We may pose the issue with the following basic construction [43]. Given a finite dimensional real Lie algebra \(\mathfrak{g}\), its dual space \(\mathfrak{g}^*\) carries a Poisson structure, the Kirillov-Kostant structure. Let \(G\) be any Lie group whose Lie algebra is \(\mathfrak{g}\), and let \(T^*G\) be its cotangent bundle with its canonical symplectic structure. Then \(\mathfrak{g}^*\) may be embedded as the cotangent space at the identity, a Lagrangian submanifold of \(T^*G\). Thus the Lie group \(G\) leads to a symplectic realization of \(\mathfrak{g}^*\) by the cotangent bundle \(T^*G\) with the symplectic form \(\Omega = d\Lambda\), where \(\Lambda\) is the Liouville one-form of \(T^*G\).

For a general Poisson manifold \(M\), the program is to embed \(M\) as a Lagrangian submanifold of a symplectic groupoid \(\mathcal{G}\) in such a way that \(\mathcal{G}\) integrates the cotangent bundle \(T^*M\) of the Poisson manifold \(M\). For a given Poisson manifold \(M\), the Poisson bracket on its functions extends to a Lie bracket among all differential one-forms, which is the contravariant analogue of the Lie bracket on vector fields. For exact one-forms, it can be defined by \([df, dg] = d\{f, g\}_\theta\). For arbitrary one-forms, the so-called Koszul bracket is generalized to the formula [44]

\[
[\xi, \eta] = L_{\xi\theta}\eta - L_{\eta\theta}\xi - d(\theta(\xi, \eta)),
\]

(5.127)

where \(\xi\theta = \rho(\xi)\) and \(\eta\theta = \rho(\eta)\) are the anchor map (2.11) for the one-forms \(\xi\) and \(\eta\), respectively. This bracket of differential forms satisfies the following two important properties

\[
[\xi, f\eta] = f[\xi, \eta] + (\xi\theta f)\eta,
\]

\[
[\rho(\xi), \rho(\eta)] = \rho([\xi, \eta]),
\]

(5.128)

where \(f \in C^\infty(M)\). All these properties described by the triple \((T^*M, [−, −], \rho)\) make the cotangent bundle of a Poisson manifold \(M\) a special case of a more general object in differential geometry, called a Lie algebroid. A Lie algebroid is a straightforward generalization of a Lie algebra and the Lie algebroid of a symplectic groupoid \(\mathcal{G}\) is canonically isomorphic to \(T^*M\). Therefore the Lie groupoid integrating a Poisson manifold has a natural symplectic structure.

To define a general Lie algebroid, one can simply replace \(T^*M\) by a general vector bundle \(E\). A Lie algebroid \(\mathcal{L}\) is then a triple \((E, [−, −], \rho)\) consisting of a vector bundle \(E\) over a manifold \(M\), together with a Lie algebra structure \([−, −]\) on the vector space \(\Gamma(E)\) of the smooth global sections of \(E\), and the anchor map of vector bundles \(\rho : E \to TM\). The above properties (5.128) are generalized in an obvious way [44] to

\[
[X, fY] = f[X, Y] + (\rho(X)f)Y,
\]

\[
[\rho(X), \rho(Y)] = \rho([X, Y]),
\]

(5.129)

for all \(X, Y \in \Gamma(E)\) and \(f \in C^\infty(M)\). Here \(\rho(X)f\) is the derivative of \(f\) along the vector field \(\rho(X)\). The anchor map \(\rho\) defines a Lie algebra homomorphism from the Lie algebra of global sections of \(E\), with Lie bracket \([−, −]\), into the Lie algebra of vector fields on \(M\). Hence Lie algebroids can be thought as “infinite dimensional Lie algebras of geometric
type”, or “generalized tangent bundles”. To every Lie groupoid there is an associated Lie algebroid. But the converse is not true because there are obstructions to the integrations of Lie algebroids to Lie groupoids. For the case where \( E = T^*M \) and \( M \) is a Poisson manifold, for example, there are topological obstructions encoded in what are called the monodromy groups \([89]\).

The notion of symplectic groupoid provides a framework for studying the collection of all symplectic realizations of a given Poisson manifold. A Poisson manifold \( M \) is called integrable if a symplectic groupoid \( \mathcal{G} \) exists such that its infinitesimal version corresponds to a given Lie algebroid \((T^*M, [-,-], \rho)\). And it turns out that symplectic realizations contain a lot of information about integrability. For instance, it was shown in \([90]\) that a Poisson manifold is globally integrable if and only if it admits a complete symplectic realization. It is also interesting to note that Morita equivalent Poisson manifolds have equivalent categories of complete symplectic realizations \([91]\). It was proven in \([92]\) that the reduced phase space of the Poisson sigma model under certain boundary conditions has a natural groupoid structure, assuming that it is a smooth manifold and the symplectic groupoid integrating a given Poisson manifold is explicitly constructed for the integrable case. Furthermore it was shown in \([92]\) that the perturbative quantization of this model yields the Kontsevich star product formula. A formal version of the integration of Poisson manifolds by symplectic groupoids was also given in \([93]\). Therefore the symplectic realization of Poisson manifolds in terms of symplectic groupoids provides an efficient route of quantization of Poisson manifolds \([94]\) so that the deformation or geometric quantization of a symplectic groupoid \( \mathcal{G} \) descends to the quantization of a Poisson manifold \( M \) though it is nontrivial to quantize \( \mathcal{G} \) in such a way that the quantization descends to a quantization of \( M \).

5.5 Towards a global geometry

Now we will apply our globalization in eq. \((5.101)\) to the formulation of a global geometry in emergent gravity. First let us consider a symplectic structure \( \Omega = \Pi^{-1} \) on open subsets of \( M = \mathbb{R}^{2n} \) (see section 7 in \([95]\)). In this case we can take the formal exponential map \((5.69)\) given by

\[
\phi_\mu^\mu(y) = x^\mu + y^\mu. \tag{5.130}
\]

Then the flat connection \((5.76)\) takes the form \([95]\)

\[
D^{(0)} = dx^\mu \left( \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu} \right) \tag{5.131}
\]

and thus the algebra of flat sections of the jet bundle \( \mathcal{E} \) is generated by the set of smooth functions on \( M \) of the form

\[
\ker D^{(0)} = \{(C_a)_\phi(x; y) = C_a(x + y) : C_a \in C^\infty(M), \; d = 1, \cdots, 2n \}. \tag{5.132}
\]

Using the relations \((5.93)\) and \((5.94)\), it is easy to find the global vector fields defined by eq. \((5.101)\) and one yields the result

\[
V_a = \Pi^{\alpha\nu}(x) \frac{\partial C_a(x)}{\partial x^\mu} \frac{\partial}{\partial x^\nu}. \tag{5.133}
\]
On an open subset of \( M = \mathbb{R}^{2n} \), we can identify the exponential map \( \phi^\mu_x \) with the Moser flow (2.22) as we discussed before (see the argument below the footnote 28) and represent the symplectic form \( \Omega = \Pi^{-1} \) as the form (5.33), i.e., \( \Omega = B + F \) where \( B \) is a constant asymptotic value in the Darboux frame such that \( F \to 0 \) at \( |x| \to \infty \). In this case, the exponential map \( \phi^\mu_x(y) \) can be identified with covariant coordinates on the symplectic vector space \( T_xM \) defined by eq. (2.39), i.e., \( \phi^\mu_x(y) = \rho_A^\mu(y^\mu) \) and so \( (C_a)_\phi(x; y) = B_{a\mu}(x + y)^\mu \).\(^{31}\)

In the end the vector fields (5.133) are given by

\[
V_a = B_{a\mu}\Pi^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} \in \Gamma(TM).
\]

The dual one-forms \( V^a = V^a_\mu(x)dx^\mu \) are defined by the natural pairing \( \langle V^a, V_b \rangle = \delta^a_b \) and so they are given by

\[
V^a = dx^\mu \Omega_{\mu\nu} \theta^{\nu a} = dx^\mu \left( \delta_{\mu a} + (F\theta)_{\mu a}(x) \right) \in \Gamma(T^*M).
\]

Given the vector fields (5.134), we can solve the volume preserving condition (5.29) which is equivalent to finding a volume form \( \nu = \lambda^2 V^1 \wedge \cdots \wedge V^{2n} \) such that the modular vector field in (3.3) identically vanishes, i.e.,

\[
\phi_\nu = -X_{\log \nu} - \partial_\mu \Pi^{\mu\nu} \frac{\partial}{\partial x^\nu} = 0,
\]

where \( \nu(x) = \lambda^2 \det V^a_\mu \). The condition (5.136) can be written as

\[
0 = \partial_\mu \Pi^{\mu\nu} + \Pi^{\mu\nu} \partial_\mu \log \nu
= \Pi^{\mu\nu} \left( \partial_\mu \log \nu - \Pi^{\sigma\rho} \partial_\sigma \Omega_{\mu\rho} \right)
= \Pi^{\mu\nu} \left( \partial_\mu \log \nu - \frac{1}{2} \Pi^{\sigma\rho} \partial_\sigma \Omega_{\mu\rho} \right)
= \Pi^{\mu\nu} \left( \partial_\mu \log \nu - \frac{1}{2} \partial_\mu \log \det V^a_\mu \right)
= \Pi^{\mu\nu} \partial_\mu \log \left( \lambda^2 \sqrt{\det V^a_\mu} \right)
\]

where we used the Bianchi identity \( \partial_{[\sigma} \Omega_{\mu\rho]} = 0 \) in the third step and the formula \( \partial_\mu \log \det A = \text{Tr} A^{-1} \partial_\mu A \) for a matrix \( A \) in the fourth step. Therefore we get

\[
\lambda^2(x) = \frac{1}{\sqrt{\det V^a_\mu}} \quad \text{or} \quad \nu(x) = \sqrt{\det V^a_\mu}.
\]

The above result can be understood as follows. On an open subset of \( M = \mathbb{R}^{2n} \), the invariant volume form is certainly given by \( \nu_\phi = d^{2n}y = \frac{\text{Pf} B}{\text{Pf} \Omega}. \) Using the formula (5.95) for the volume form, we have the relation \( \nu_\phi = (\phi_x)^*\nu \). Since \( (\phi_x)^*\Omega = B \) as we argued above (see also eq. (2.33)), we get the volume form \( \nu = \frac{\text{Pf} \Omega}{\text{Pf} B} d^{2n}x = \frac{\text{Pf} B}{\text{Pf} \Omega} V^1 \wedge \cdots \wedge V^{2n} = \frac{1}{\sqrt{\det V^a_\mu}} V^1 \wedge \cdots \wedge V^{2n} \)

\[\text{Note that } x \text{ simply refers to local coordinates of a base point } p \in M \text{ of the tangent bundle } T_pM \text{ and so we may put the tangent space } T_pM \text{ at the origin}, x = 0. \text{ But it is convenient to consider the tangent space } T_pM \text{ at an arbitrary base point } p \in M \text{ for the construction of global vector fields (5.133).}\]
where we used the result (5.135). This is consistent with the result (5.138).

Finally the emergent metric (5.34) determined by the one-form basis (5.135) is given by

\[ g_{\mu \nu}(x) = \frac{V_\mu^a(x) V_\nu^a(x)}{\sqrt{\det V_{\mu}^a}} \]

\[ = \frac{\left( \delta_{\mu a} + (F \theta)_{\mu a}(x) \right) \left( \delta_{\nu a} + (\theta F)_{\nu a}(x) \right)}{\sqrt{\det(1 + (F \theta)(x))}}. \]  

(5.140)

This form of the metric was also appeared in different contexts in [61] (see eq. (4.15) which can be identified with \( g_{\mu \nu} = e^{-\phi} g_{\mu \nu} \)) and in [39] (see eq. (50) which coincides with eq. (5.140) in four dimensions). Note that \( \sqrt{\det g_{\mu \nu}} = \det(1 + (F \theta))^{1/4} \) and so \( \sqrt{\det g_{\mu \nu}} = 1 \) in four dimensions (\( n = 2 \)). One can expand the metric (5.140) in powers of \( F \) or \( \theta \) which leads to the expansion

\[ g_{\mu \nu}(x) = \delta_{\mu \nu} + (F \theta)_{\mu \nu} + (\theta F)_{\mu \nu} + (F \theta \theta F)_{\mu \nu} - \frac{1}{2} \delta_{\mu \nu} \left( \text{Tr}(F \theta) - \frac{1}{2} \text{Tr}(F \theta)^2 - \frac{1}{4} (\text{Tr}(F \theta))^2 \right) \]

\[ - \frac{1}{2} (F \theta + \theta F)_{\mu \nu} \text{Tr}(F \theta) + \cdots. \]  

(5.141)

The linear order metric of the above expanded form looks like the gravitational metric derived from the SW map (see eq. (50) in [6]) except the trace term \( -\frac{1}{2} \delta_{\mu \nu} \text{Tr}(F \theta) \). Interestingly, for symplectic U(1) instantons, all the trace terms in \( O(F^m) \) coming from the determinant of the denominator in eq. (5.140) are canceled by the diagonal components of next higher order terms in \( O(F^{m+1}) \). For this cancelation, it is crucial to use the identity \( \sqrt{\det(1 + F \theta)} = 1 + \frac{1}{4} \text{Tr} F \theta \) (that is eq. (20) in [96] derived from the instanton equation (18)). To be more specific, the metric (5.140) can be simplified as

\[ g_{\mu \nu}(x) = (\delta_{\mu \nu} + (F \theta + \theta F)_{\mu \nu} + (F \theta \theta F)_{\mu \nu}) \left( 1 + \frac{1}{4} \text{Tr}(F \theta)^{m-1} \right) \]

\[ = \left( \frac{1}{4} (F \theta + \theta F)_{\mu \nu} \text{Tr}(F \theta) \right) + O(F^{m+1}) \]  

After this cancelation, the metric (5.140) can be written as the form

\[ g_{\mu \nu}(x) = \delta_{\mu \nu} + (F \theta + \theta F)^\parallel_{\mu \nu} + \frac{(F \theta + \theta F)^\perp_{\mu \nu} + (F \theta \theta F)^\perp_{\mu \nu}}{1 + \frac{1}{4} \text{Tr}(F \theta)}. \]  

(5.142)

where \( \parallel \) and \( \perp \) denote the diagonal and off-diagonal parts, respectively. Thus the four-dimensional gravitational metric for symplectic U(1) instantons is approximated up to linear order by

\[ g_{\mu \nu}(x) \approx \delta_{\mu \nu} + (F \theta)_{\mu \nu} + (\theta F)_{\mu \nu} \]  

(5.143)

which is symmetric for arbitrary U(1) field strengths. This metric is precisely the form identified by the SW map [6, 97]. This analysis suggests that higher order terms, \( O(F^2) \), in the metric (5.142) and its inverse metric must be regarded as derivative corrections of general relativity that certainly exist as we discussed at the end of section 5.3. Moreover only tensor (spin 2) modes generate the higher order corrections. This is a desirable property since there is no scalar graviton in general relativity thanks to general covariance.
For a general symplectic manifold other than $\mathbb{R}^{2n}$, the Grothendieck connection $D^{(0)}$ becomes more complicated whose general form is given by eq. (5.76) and so it is nontrivial to find the set, ker $D^{(0)}$, of flat sections. (It is nontrivial even in two dimensions. See [98] for the deformation quantization of two-dimensional constant curvature spaces.) In this case the global vector fields $V_a$ will also take a complicated form than eq. (5.133). And we also need to solve the volume preserving condition (5.29) given by

$$0 = \mathcal{L}_{V_a} \nu = (\partial_\mu V_\mu^a + V_\mu^a \partial_\mu \log \nu) \nu = (\nabla \cdot V_a + 2(1 - n)V_a \log \lambda) \nu$$

(5.144)

where $\nu(x) = \lambda^2 \det V^a_\mu$. There is some interesting class of metrics, e.g., the Gibbons-Hawking metric [99] and the real heaven [100], which obeys $\partial_\mu V_\mu^a = 0$ [101]. In this case the condition (5.144) reduces to

$$\nu(x) = 1 \quad \text{or} \quad \lambda^2(x) = \frac{1}{\det V^a_\mu}$$

(5.145)

and so the emergent metric (5.34) is given by

$$g_{\mu\nu}(x) = \frac{V^a_\mu(x) V^a_\nu(x)}{\det V^a_\mu}.$$  

(5.146)

(There is a confusing flip of indices, $(\mu, \nu) \leftrightarrow (a, b)$, between eq. (5.146) and eq. (35) in [101]. We hope readers are not bothered by it.) For a general class of symplectic manifolds, it may be necessary to fully solve the volume preserving condition (5.144). In general it is not easy to find a solution such that the first term, $\partial_\mu V_\mu^a$, cancels out the second term, $V_\mu^a \partial_\mu \log \nu$, because they occasionally have a quite different functional form. One way to circumvent this difficulty is to attach a nonlocal dipole-like object to vector fields $V_a$ similar to an open Wilson line such that the inverse vielbeins $E_a = \lambda^{-1} V_a$ still become local by compensating the nonlocal object in $V_a$ by $\lambda$. It was argued in [19] that such kind of a nonlocal object is necessary for the LeBrun metric [102], that is the most general scalar-flat Kähler metric with a U(1) isometry and contains the Gibbons-Hawking metric, the real heaven as well as the multi-blown up Burns metric which is a scalar-flat Kähler metric on $\mathbb{C}^2$ with $n$ points blown up.

It may be worthwhile to point out some (surmisable) aspect of emergent Riemannian metrics. As in general relativity, the explicit form of a global metric depends on the choice of coordinates. For example, the usual spherical coordinate representation of Eguchi-Hanson metric [103, 104] is equivalent to the two-center Gibbons-Hawking metric [99] by a coordinate transformation [105] though their bare appearance looks very different. A similar feature also arises in emergent gravity. It was recently shown [4, 37] that the Cartesian coordinate representation of Eguchi-Hanson metric takes the form (5.143) and so it belongs to the class (5.140). However the Gibbons-Hawking representation of the Eguchi-Hanson metric takes the form (5.146) as was shown in [106, 107, 108, 101]. This implies that there exists a coordinate transformation relating two types of metrics if they describe the same manifold. And this coordinate transformation needs to be globally
Figure 1: Deformation quantization and covariance [21]

defined because it has to relate one globally defined metric to the other global metric. Actually the general covariance for the choice of coordinates is reflected in deformation quantization [24] and clearly discussed in section 3.3 in [21] in the context of NC gauge theory. The covariance in deformation quantization can be summarized with Fig. 1 which illustrates how the semi-classical and quantum constructions are affected by a change of coordinates $\sigma^*$. The vertical arrow in Fig. 1 indicates a quantization map defined by eq. (3.2). The deformation quantization of $\theta$ and $\theta'$ in new coordinates leads to star products $\star$ and $\star'$ that are related to the star products $\star$ and $\star'$ in old coordinates by the equivalence maps $\Sigma$ and $\Sigma'$, respectively. It might be emphasized that the coordinate transformation $\sigma^*$ is globally defined as we remarked above whereas the coordinate transformation in the Moser flow $\rho^*_A$ is locally defined as was verified by eq. (2.35). Hence it is naturally expected that their quantization maps $\Sigma (\Sigma')$ and $\mathcal{D}_A (\mathcal{D}_A)$ will also keep the property. The relation between local covariance maps $\mathcal{D}_A$ and $\mathcal{D}_A$ in the old and new coordinates can be deduced from the commutativity of the diagram of the horizontal plane in Fig. 1 [21]:

\[
\mathcal{D}_A = \Sigma^{-1} \circ \mathcal{D}_A \circ \Sigma',
\]
\[
\mathcal{A}_A = \Sigma^{-1} \circ (\Sigma' - \Sigma) + \Sigma^{-1} \circ \mathcal{A}_A \circ \Sigma'.
\]

It is well-known that there are three kinds of geometries (see, for example, Chapter 12 in [103]):

A. Symplectic geometry $(M, \Omega)$: geometry of a closed, nondegenerate, skew-symmetric bilinear form $\Omega$.

B. Riemannian geometry $(M, g)$: geometry of a positive-definite symmetric bilinear map $g$.

C. Complex geometry $(M, J)$: geometry of an integrable linear map $J$ with $J^2 = -1$. 

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Each category has a generalization to a more general geometry. For example, in the category A, we can relax the closedness condition, then we get an almost symplectic manifold, i.e., manifold $M$ with a nondegenerate, not necessarily closed, exterior 2-form $\Omega$. Or we can relax the nondegenerate condition. In this case we get a Poisson geometry $(M,\Pi)$ where the closedness condition is replaced by a generalized version \( [\Pi,\Pi]_S = 0 \).

The generalization of the category B is historical and still going on. The most famous (and rational) attempt is the Cartan geometry based on the Cartan connection generalizing the Levi-Civita connection.

The category C also admits a generalization to an almost complex manifold $(M,J)$ which is a smooth manifold $M$ having an endomorphism $J: TM \to TM$ of the real tangent bundle which satisfies $J^2 = -1$ but is not necessarily integrable. The integrability of an almost complex structure $J$ is measured by the Nijenhuis tensor defined by

$$
N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]
$$

for any two vector fields $X,Y$. The Newlander-Nirenberg theorem \[110\] states that an almost complex structure $J$ is integrable if and only if $N_J = 0$.

The emergent gravity is to aim for the derivation of a Riemannian geometry from the symplectic geometry or more generally the Poisson geometry. The question is what kind of Riemannian geometry arises from a given symplectic geometry. An onset is to think of a role of complex geometry for the emergent gravity. Though there is a close similarity between the symplectic geometry and the complex geometry, there do not exist any definite inclusion relations between them. For example, there exists a symplectic manifold with no complex structure: A circle bundle over circle bundle over a 2-torus \[111\]. And there is a complex manifold that admits no symplectic structure: the Hopf surface $S^3 \times S^1 \[109\]$. Thus it may be necessary to start with either a Poisson geometry or an almost symplectic geometry in order to derive a general Riemannian geometry. For instance, it can be shown \[3\] that the Kähler condition is equivalent to the Bianchi identity, $d\Omega = 0$, for a $U(1)$ field strength $\Omega$ and so it is required to consider the almost symplectic or Poisson structure to construct the four-dimensional Euclidean Schwarzschild black-hole geometry \[112\] since it is not a Kähler manifold (though it is a Ricci-flat manifold). Unfortunately it is very nontrivial to derive the emergent Riemannian geometry from a general almost symplectic or Poisson geometry. Therefore let us focus on the symplectic geometry. See \[58\] for emergent Riemannian geometries with a constant curvature derived from the Poisson geometry.

Given a symplectic manifold $(M,\Omega)$, we can introduce a symplectic connection (see the footnote \[22\] obeying eq. (5.55) which reads as

$$
Z(\Omega(X,Y)) = \Omega(\partial_S X, Y) + \Omega(X, \partial_S Y)
$$

(5.150)

for any vector fields $X,Y,Z$. In this case, if we can find a compatible complex structure $J$ such that $g_J(X,Y) = \Omega(X,JY)$ for any $X,Y \in \Gamma(TM)$ is a positive-definite symmetric bilinear map, we can identify $g_J(X,Y)$ with a Riemannian metric \[5.1\]. Indeed, if we take $X = \partial_\mu, Y = J\partial_\nu, Z = \partial_\lambda$, the above condition (5.150) can be written as

$$
\partial_\lambda g_{\mu\nu} = \Gamma^\rho_\lambda g_{\rho\nu} + \Gamma^\rho_\lambda g_{\rho\mu} + \Omega(\partial_\mu, (\partial_S J)(\partial_\nu)).
$$

(5.151)
Therefore, if we assume that \( \partial_S^\lambda J = 0 \), the Christoffel symbol \( \Gamma^\mu_{\rho\nu} \) in eq. (5.151) should be regarded as the Levi-Civita connection, \( \nabla^{LC} \), in general relativity because \( \nabla^{LC} g = 0 \). Thus, the condition, \( \partial_S^\lambda J = 0 \), can be interpreted as the one that parallel translation should preserve the almost complex structure, i.e., \( \nabla^{LC} J = 0 \) or equivalently the metric is Kähler (113). Consequently, in order to define an emergent geometry from a symplectic manifold \((M, \Omega)\), the problem at hand is whether it is possible to find the compatible complex structure \( J \) within the context of symplectic geometry such that \( g_I(X, Y) \equiv \Omega(X, JY) \) becomes a positive-definite Riemannian metric. Note that to any symplectic manifold \((M, \Omega)\) one can always find almost complex structures \( J \) that are compatible with \( \Omega \) (see section 12.3 in [109]).

Actually this compatible complex structure may be inherited from the symplectic vector space \((TM, \Omega_\phi)\) where \( \Omega_\phi(x; y) = \frac{1}{2} \Omega_{\mu\nu}(x; y) dy^\mu \wedge dy^\nu \) is the symplectic structure dual to the Poisson bivector \((5.90)\). Since the global vector fields \( (V^a)_\phi \in \Gamma(TM) \) and their dual covectors \( (V^a)_\phi \in \Gamma(T^*M) \) form a complete basis for \( TM \) and \( T^*M \), respectively, we use them to represent

\[
\Omega_\phi(x; y) = -\frac{\lambda^2}{2} I_{ab} V^a(x; y) \wedge V^b(x; y), \tag{5.152}
\]

\[
\Pi_\phi(x; y) = \frac{1}{2\lambda^2} I^{ab} V_a(x; y) \wedge V_b(x; y), \tag{5.153}
\]

where the constant symplectic matrix \( I_{ab} \) is given by

\[
I_{2i-1,2j} = \delta_{ij} = -I_{2j,2i-1}, \quad i, j = 1, \ldots, n. \tag{5.154}
\]

Here we supposed to choose the smooth function \( \lambda(x; y) \) so that \( \Omega_\phi(x; y) \) becomes a closed two-form. It turns out that \( \lambda = \lambda(x; 0) \) can be identified with eq. (5.30) for Kähler manifolds. The complex structure \( J_\phi = (\phi_x)^{-1} J = (\phi_x)^* J \) on \( T_p M, \ p \in M \), is a smooth tensor field of type \((1, 1)\) and is locally in a coordinate chart given by

\[
J_\phi(x; y) = J^{\mu}_\nu(x; y) \frac{\partial}{\partial y^\mu} \otimes dy^\nu \tag{5.155}
\]

where smooth functions \( J^{\mu}_\nu(x; y) \) on \( T_p M \) satisfy a matrix relation

\[
J^{\lambda}_\mu(x; y) J^{\nu}_\lambda(x; y) = -\delta^{\nu\mu}. \tag{5.156}
\]

We can borrow the complex structure \((5.155)\) from the symplectic structure \((5.152)\) as follows:

\[
J_\phi(x; y) = I_{ab} V_a(x; y) \otimes V^b(x; y). \tag{5.157}
\]

It is then easy to check that \( g_\phi(V_a, V_b) = \Omega_\phi(V_a, JV_b) = \lambda^2 \delta_{ab} \). If we define the metric \( g_{\mu\nu}(x) = (g_\phi)_{\mu\nu}(x; 0) \) according to the rule used for eqs. (5.90) and (5.97), we finally get the emergent metric \((5.34)\) given by

\[
g_{\mu\nu}(x) = \lambda^2 V^a(x) V^b(x) \delta_{ab}. \tag{5.158}
\]
Depending on the choice of $\lambda^2$, the above metric reproduces either eq. (5.140) or (5.146). But it is interesting to note that the complex structure (5.157) is immune from the different choice of $\lambda$ because it is the tensor field of type $(1,1)$.

In order for the metric (5.158) to be Kähler, there are several constraints. First we have the volume preserving condition (5.144) which can be represented as the form $g_{ab} = V_a \log \lambda^2$ [49] using the structure equation (5.43). Of course it is in principle solved if the complete set of vector fields $V_a$ and the invariant volume form $\nu = \lambda^2 V^1 \wedge \cdots \wedge V^{2n}$ are determined. Essential constraints are given by the integrability of the almost complex structure (5.157), $N_f(V_a, V_b) = 0$, and the closedness condition of the symplectic two-form (5.152), $d\Omega_\phi(x; 0) = 0$. In this case, $\lambda^2$ in eq. (5.30) should be identified with that in $\Omega_\phi$ because the compatibility condition, $g(X, Y) = \Omega(X, JY)$, must be obeyed for Kähler metrics. But they do not have to be identified for non-Kähler metrics since the compatibility condition is not necessarily satisfied. Let us summarize all these constraints:

$$\mathcal{L}_V \nu = 0 \quad \iff \quad g_{ab} = V_a \log \lambda^2,$$

$$d\Omega_\phi(x; 0) = 0 \quad \iff \quad I_{(ab} V_{c)} \log \lambda^2 = -I_{d(a g_{bc})},$$

$$N_f(V_a, V_b) = 0 \quad \iff \quad (I \lrcorner g)_{(ab} = (\Gamma^3 \lrcorner g)_{ab},$$

where the contraction symbols mean that $(I \lrcorner g)_{ab} = I_{ad} g_{db} + I_{bd} g_{da} + I_{cd} g_{ab}$ and $(\Gamma^3 \lrcorner g)_{ab} = I_{ad} I_{bc} I_{ef} g_{de}$. To derive (5.160), we used the Maurer-Cartan equation, $dV^a = \frac{1}{2} g_{bc} V^b \wedge V^c$, dual to the structure equation (5.43). If we introduce the structure equation for orthonormal frames defined by

$$[E_a, E_b] = -f_{ac} E_c,$$

the structure functions are related each other by [3]

$$g_{ab} = \lambda (f_{ac} - E_a \log \lambda \delta_{bc} + E_b \log \lambda \delta_{ac}).$$

In terms of the structure functions $f_{ac}$ in eq. (5.162), the above three constraints take more simpler forms given by

$$\mathcal{L}_{E_a} \tilde{\nu} = 0 \quad \iff \quad f_{bab} = (3 - 2n) E_a \log \lambda,$$

$$d\Omega_\phi(x; 0) = 0 \quad \iff \quad I_{d(a f_{bc})} = 0,$$

$$N_f(E_a, E_b) = 0 \quad \iff \quad (I \lrcorner f)_{(ab} = (\Gamma^3 \lrcorner f)_{ab}.$$

The first condition is consistent with the fact (5.33) that the vector fields $E_a$ preserve the volume form $\tilde{\nu} = \lambda^{3-2n} \nu$. The third condition should be expected since the complex structure (5.157) is invariant under the Weyl scaling $V_a \rightarrow E_a = \lambda^{-1} V_a$.

By definition a Kähler manifold is a symplectic manifold $(M, \Omega)$ equipped with an integrable compatible almost complex structure. The symplectic form $\Omega$ is then called a Kähler form. That is, any Kähler manifold must be both symplectic and complex. We also know any symplectic or complex manifold admits almost complex structures. Then one might ask: If $M$ is both symplectic and complex, is it necessarily Kähler? It is known [114] that any noncompact almost complex manifold has a compatible symplectic structure. But the situation is very different for closed symplectic manifolds. The answer for them is
strikingly no. The Kodaira-Thurston example \[109\] demonstrates such a case which is given by a manifold \(M = \mathbb{R}^4/\Gamma\) where \(\Gamma\) is a discrete group generated by symplectomorphisms acting on \(\mathbb{R}^4\). The manifold \(M\) is a flat torus bundle over a torus and is both symplectic and complex. But it admits no Kähler structure. Thus a question is to what extent the emergent geometry obeying eqs. (5.159)-(5.161) describes an almost complex manifold. Since the almost complex structure (5.157) was already chosen to be compatible with \(\Omega\), the above argument implies that non-Kähler manifolds may not solve both eqs. (5.159) and (5.160) simultaneously.\(^{32}\) There is a useful object to detect an obstruction for a symplectic manifold \(M\) to be Kähler which is the Massey products in \(H^\bullet(M, \mathbb{R})\) (for the definition of the Massey products, see, for example, \[113\]). Symplectic manifolds can have non-trivial rational Massey triple products, but all the Massey triple products on closed Kähler manifolds are zero \[113\]. Therefore a symplectic manifold with nontrivial Massey products must be non-Kähler. In this regard, it is also worthwhile to recall an important theorem \[113\] stating that all compact Kähler manifolds are zero \[113\]. Therefore a symplectic manifold (5.154) is formal, all its Massey products vanish. The formality of a space \(M\) means that its real homotopy type of \(M\) is completely defined by the real cohomology ring \(H^\bullet(M, \mathbb{R})\). Hence the existence of nontrivial Massey products indicates that there is more to the rational homotopy type of the manifold than can be seen from the cohomology algebra. Based on these observations, we conjecture that it is necessary to generalize global objects in eq. (5.101) to “locally” Hamiltonian vector fields to describe non-Kähler manifolds emergent from a generic symplectic geometry. But we are not adept in adding any remark for a global Riemannian geometry emergent from a general Poisson algebra.

We conclude this section with the calculation of the Kähler constraints eqs. (5.159)-(5.161) in four dimensions. In four dimensions, the constant symplectic matrix (5.154) is equal to the third self-dual ’t Hooft matrix, i.e., \(I_{ab} = \eta_{ab}^3\) and we utilize the canonical splitting (5.44). The result is given by

\[
\begin{align*}
(5.159) & \quad \eta_{ab}g_b^{(+)(i)} + \eta_{ab}g_b^{(-)(i)} = -V_a \log \lambda^2, \\
(5.160) & \quad \eta_{i(ab} \eta_{c)d}g_d^{(+)(i)} + \eta_{i(ab} \eta_{c)d}g_d^{(-)(i)} = \eta_{i(ab}^3 V_c) \log \lambda^2, \\
(5.161) & \quad \eta_{i(ab} \eta_{c)d}g_d^{(+)(i)} - \eta_{i(ab} \eta_{c)d}g_d^{(+)(i)} = -\epsilon^{ij3} \eta_{ijab} g_c^{(+)(i)},
\end{align*}
\]

where \(i, j = 1, 2, 3\). Contracting \(\eta_{ab}^3\) with eq. (5.168) and using eq. (5.167) leads to the result, \(g_a^{(+)(3)} = 0, \forall a\). After solving eq. (5.169) in a similar way, we yield the Kähler constraints in four dimensions:

\[
g_a^{(+)(1)} = \epsilon_{ab}^3 g_b^{(+)(2)}, \quad g_a^{(+)(3)} = 0.
\]

An interesting point is that the anti-self-dual parts, \(g_a^{(-)(i)}\), are not constrained when \(I_{ab} = \eta_{ab}^3\). But, if we chose a different complex structure, e.g., \(I_{ab} = \eta_{ab}^3\), the situation would be flipped so that instead the self-dual parts, \(g_a^{(+)(i)}\), are not constrained.

\(^{32}\)In this case, it may be necessary to take a more general form for the global symplectic structure (5.152) in such a way that \(\tilde{\Omega}_a(x; 0) = -\frac{i}{2} \epsilon_{ab}^3 V^a(x) \wedge V^b(x)\) becomes a closed two-form. In addition it has to be required that the symplectic form \(\tilde{\Omega}_a\) is compatible with the other two conditions which will determine \(\lambda = \lambda(\lambda)\). But the compatibility condition will be failed in the sense that \(g_\alpha(V_\alpha, V_\beta) = \frac{i}{2} \epsilon_{ab}^3 \tilde{\Omega}_a(V_\alpha, J V_\beta)\).
6. Noncommutative geometry and quantum gravity

The quantization of a classical system has proved to be a delicate as well as difficult problem. (See, for example, [116] for some obstructions and difficulties in quantization theory.) There are two main approaches to the quantization of a general symplectic or Poisson manifold. The formal deformation quantization [12, 28, 29] gives rise to a NC deformation of the algebra of smooth functions, whereas the emphasis of geometric quantization [117, 118, 44] is on the construction of a Hilbert space, the "space of states". The deformation quantization embodies quantum dynamics as much as possible in terms of deformed algebra structures without using the customary representations in Hilbert spaces. However a physically reasonable concept for the states is necessary in order to understand the spectral structure of the observable algebra, which is missing in deformation quantization. Moreover many classical concepts such as the concept of spaces and points still remain even after (formal) quantization. Therefore the deformation quantization should not be regarded as a complete quantization [119] but rather an intermediate stage of an ultimate quantization. See also [120] for various aspects of representation theory in deformation quantization and the concepts of states as positive functionals and the GNS construction.

6.1 Quantum geometry and matrix models

We will use deformation quantization to find a canonical method to quantize a general symplectic or Poisson manifold. First recall the remark below eq. (4.18) that a general Poisson manifold \((M, \Theta)\) can be constructed by the deformation in terms of a line bundle \(L \rightarrow M\) on a primitive Poisson manifold \((M, \theta)\). Without loss of generality, we can assume the primitive Poisson structure \(\theta\) to be in the Darboux-Weinstein frame (2.33). It was shown [20, 21] that the deformation of a Poisson manifold \((M, \theta)\) in terms of a line bundle \(L \rightarrow M\) can be formulated by a NC gauge theory using the local covariance map \(D_A\) obeying the property (1.8). In the end the quantization of a general Poisson manifold is modeled by a NC gauge theory whose star product is defined by the (formal) deformation quantization of the primitive Poisson manifold \((M, \theta)\). In order to construct a natural Hilbert space \(\mathcal{H}\), the space of states, on which the deformed algebra acts, we will consider the representation theory of the quantized Poisson algebra \((A_\theta, \star)\) which is Morita-equivalent to the dynamical quantum Poisson algebra \((A_\Theta, \star')\) as was shown in Refs. [22, 23]. We discussed this aspect before in section 4.

Of course the representation theory of the quantized Poisson algebra \((A_\theta, \star)\) is in general nontrivial except as the case of symplectic manifolds for which the primitive Poisson structure \(\theta\) is reduced to the canonical one, familiar in quantum mechanics, in the Darboux-Weinstein frame (2.33). In this case the quantization of the primitive Poisson manifold \((M, \theta)\) gives rise to the Moyal-Heisenberg algebra (1.1) and we will regard it as a vacuum algebra. In the presence of a line bundle \(L \rightarrow M\), the vacuum coordinates \(y^\mu\) should be promoted to the covariant dynamical coordinates defined by eq. (5.4) and a general NC space

\[
[X^\mu, X^\nu]_\star = i\theta^{\mu\nu} - i(\theta \hat{F}_\theta)^{\mu\nu} := i\Theta^{\mu\nu}
\]  (6.1)
generated by the dynamical coordinates is regarded as a (large) deformation of the vacuum NC space (1.1) due to NC gauge fields. As we discussed before, the vacuum algebra (1.1) is equivalent to the Heisenberg algebra (5.122) of n-dimensional harmonic oscillator. Hence the underlying Hilbert space on which the deformed algebra (6.1) acts is given by the representation space of the Heisenberg algebra (5.122). It is the Fock space defined by

\[ \mathcal{H} = \{ |\vec{n}\rangle \equiv |n_1, \cdots, n_n\rangle | n_i \in \mathbb{Z}_{\geq 0}, i = 1, \cdots, n \}, \]

(6.2)

which is orthonormal, i.e., \( \langle \vec{n}|\vec{m}\rangle = \delta_{\vec{n},\vec{m}} \) and complete, i.e., \( \sum_{\vec{n}=0}^{\infty} |\vec{n}\rangle\langle \vec{n}| = 1_{\mathcal{H}} \), as is well-known from quantum mechanics. Since the Fock basis (6.2) is a countable basis, it is convenient to introduce a one-dimensional basis using the “Cantor diagonal method” to put the n-dimensional non-negative integer lattice in \( \mathcal{H} \) into one-to-one correspondence with the infinite set of natural numbers (i.e., 1-dimensional positive integer lattice) \[ ]:

\[ \mathbb{Z}^n_{\geq 0} \leftrightarrow \mathbb{Z}_{>0} : |\vec{n}\rangle \leftrightarrow |n\rangle, n = 1, \cdots, N \to \infty. \]

(6.3)

In this one-dimensional basis, the completeness relation of the Fock space (6.2) is now given by \( \sum_{n=1}^{\infty} |n\rangle\langle n| = 1_{\mathcal{H}} \).

Consider two arbitrary dynamical fields \( \hat{C}_1(y) \) and \( \hat{C}_2(y) \) on the NC space (1.1) which are elements of the deformed algebra \( \mathcal{A}_\theta \). In quantum mechanics physical observables are considered as operators acting on a Hilbert space. Similarly the dynamical variables on NC space \( \mathbb{R}_\theta^{2n} \) can be regarded as operators acting on the Hilbert space (6.2). Thus we can represent the operators acting on the Fock space (6.2) as \( N \times N \) matrices in \( \text{End}(\mathcal{H}) \equiv \mathcal{A}_N \) where \( N = \dim(\mathcal{H}) \to \infty \):

\[
\hat{C}_1(y) = \sum_{n,m=1}^{\infty} |n\rangle\langle |\hat{C}_1(y)|m\rangle\langle m| := \sum_{n,m=1}^{\infty} (\Phi_1)_{nm}|n\rangle\langle m|,
\]

\[
\hat{C}_2(y) = \sum_{n,m=1}^{\infty} |n\rangle\langle |\hat{C}_2(y)|m\rangle\langle m| := \sum_{n,m=1}^{\infty} (\Phi_2)_{nm}|n\rangle\langle m|,
\]

(6.4)

where \( \Phi_1 \) and \( \Phi_2 \) are \( N \times N \) matrices in \( \mathcal{A}_N = \text{End}(\mathcal{H}) \). Then we get a natural composition rule

\[
(\hat{C}_1 \star \hat{C}_2)(y) = \sum_{n,l,m=1}^{\infty} |n\rangle\langle |\hat{C}_1(y)|l\rangle\langle l|\hat{C}_2(y)|m\rangle\langle m| = \sum_{n,l,m=1}^{\infty} (\Phi_1)_{nl}(\Phi_2)_{lm}|n\rangle\langle m|. \]

(6.5)

The above composition rule implies that the ordering in the NC algebra \( \mathcal{A}_\theta \) is perfectly compatible with the ordering in the matrix algebra \( \mathcal{A}_N \). Thus we can straightforwardly translate multiplications of NC fields in \( \mathcal{A}_\theta \) into those of matrices in \( \mathcal{A}_N \) using the matrix representation (6.4) without any ordering ambiguity. Furthermore, since symplectic manifolds are always unimodular as we discussed in appendix B, we can define a trace on the deformed algebra \( \mathcal{A}_\theta \) as the integral (5.23) over \( M = \mathbb{R}^{2n} \). Using the map (5.4) between the NC \( \star \)-algebra \( \mathcal{A}_\theta \) and the matrix algebra \( \mathcal{A}_N \), the trace over \( \mathcal{A}_\theta \) can be transformed into the trace over \( \mathcal{A}_N \), i.e.,

\[
\int_M \frac{d^{2n}y}{(2\pi)^n|\text{Pf}\theta|} = \text{Tr}_\mathcal{H} = \text{Tr}_N. \]

(6.6)
Let us apply the matrix representation (6.4) to a NC gauge theory. For this purpose, consider a $d = (m + 2n)$-dimensional NC U(1) gauge theory on $\mathbb{R}^m \times \mathbb{R}^{2n}_\theta$ whose coordinates are $X^M = (x^\mu, y^a)$, $M = 0, 1, \ldots, d - 1$, $\mu = 0, 1, \ldots, m - 1$, $a = 1, \ldots, 2n$ where $\mathbb{R}^m \ni x^\mu$ is an $m$-dimensional either Minkowski or Euclidean spacetime and $\mathbb{R}^{2n}_\theta \ni y^a$ is a $2n$-dimensional NC space obeying the commutation relation

$$[y^a, y^b]_\theta = ig^{ab}. \quad (6.7)$$

The $d$-dimensional U(1) connections are similarly split as

$$D_M(X) = \partial_M - iA_M(x, y) = (D_\mu, D_a)(x, y) \quad (6.8)$$

where

$$D_\mu(x, y) = \partial_\mu - iA_\mu(x, y), \quad (6.9)$$

$$D_a(x, y) = -i(B_{ab}y^b + A_a(x, y)) \equiv -iC_a(x, y). \quad (6.10)$$

Here we used the fact (5.10) that $\partial_a = adp_a = -i[B_{ab}y^b, -]_\theta$ and we omitted the hat symbol to indicate NC gauge fields for notational simplicity. Using the matrix representation (6.4) defined by $A_\theta \rightarrow A_N : D_M(x, y) \mapsto (D_\mu, -i\Phi_a)(x)$, the $d$-dimensional NC U(1) gauge theory is exactly mapped to the $m$-dimensional $U(N \rightarrow \infty)$ Yang-Mills theory:

$$S = -\frac{1}{4g_{YM}^2} \int d^dX (F_{MN} - B_{MN})^2 \quad (6.11)$$

$$= -\frac{1}{g_{YM}^2} \int d^m\!x\! Tr_N \left( \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} D_\mu\Phi_a D^\mu\Phi^a - \frac{1}{4} [\Phi_a, \Phi_a]^2 \right) \quad (6.12)$$

where $F_{MN} - B_{MN} = i[D_M, D_N]_\ast$, $g_{YM}^2 = (2\pi)^n |\text{Pf}g_{YM}^2|$ and $B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}$. Note that, according to the matrix representation (6.4), the NC U(1) gauge symmetry in $d$ dimensions is also exactly mapped to the ordinary $U(N \rightarrow \infty)$ gauge symmetry in $m$ dimensions, i.e.,

$$D'_M(X) = U(X) \ast D_M(X) \ast U(X)\!^\dagger, \quad U(X) \in U(1)_\ast$$

$$\leftrightarrow \quad (D_\mu, \Phi_a)^\dagger(x) = U(x)(D_\mu, \Phi_a)(x)U(x)\!^\dagger, \quad U(x) \in U(N \rightarrow \infty). \quad (6.13)$$

According to our construction, the above large $N$ gauge theory can be regarded as a (strict) quantization of a $d$-dimensional manifold $\mathcal{M}$ along the directions of Poisson structure $\theta = B^{-1}$ which is extended only along the $2n$-dimensional subspace. A remarkable point is that the resulting matrix models or large $N$ gauge theories described by the action (6.12) arise as a nonperturbative formulation of string/M theories. For instance, we get the IKKT matrix model for $m = 0$ [21], the BFSS matrix quantum mechanics for $m = 1$ [22] and the matrix string theory for $m = 2$ [23]. The most interesting case arises for $m = 4$ and $n = 3$ which suggests an engrossing connection that the 10-dimensional NC U(1) gauge theory on $\mathbb{R}^{3.1} \times \mathbb{R}_\theta^6$ is equivalent to the bosonic action of 4-dimensional $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory, which is the large $N$ gauge theory of
AdS/CFT duality [30, 31, 32]. According to the large $N$ duality or gauge/gravity duality, the large $N$ matrix model (6.12) is dual to a higher dimensional gravity or string theory. Hence it should not be surprising that the $d$-dimensional NC U(1) gauge theory should describe a theory of gravity (or a string theory) in $d$ dimensions.\footnote{We may emphasize that the equivalence between the $d$-dimensional NC U(1) gauge theory (6.11) and $m$-dimensional U($N \rightarrow \infty$) Yang-Mills theory (6.12) is a mathematical identity and has been known long ago, for example, in [7, 8]. Nevertheless the possibility that gravity can emerge from NC U(1) gauge fields has been largely ignored until recently. But the emergent gravity picture based on NC U(1) gauge theory debunks that this coincidence did not arise by some fortuity and so we want to quote an epigram due to John H. Schwarz [124]: “Take coincidences seriously”.} In other words, the emergent gravity from NC gauge fields is actually the manifestation of the gauge/gravity duality or large $N$ duality in string/M theories. Therefore the emergent gravity from NC gauge fields opens a lucid avenue to understand the gauge/gravity duality such as the AdS/CFT correspondence. While the large $N$ duality is still a conjectural duality and its understanding is far from being complete to identify an underlying first principle for the duality, we are reasonably understanding the first principle for the emergent gravity from NC U(1) gauge fields and we know how to derive gravitational variables from gauge theory quantities. Later we will show that the 4-dimensional $\mathcal{N} = 4$ supersymmetric U(N) Yang-Mills theory is equivalent to the 10-dimensional $\mathcal{N} = 1$ supersymmetric NC U(1) gauge theory on $\mathbb{R}^{3,1} \times \mathbb{R}^6$ if we consider the Moyal-Heisenberg vacuum (6.7) which is a consistent solution of the former – the $\mathcal{N} = 4$ super Yang-Mills theory. We showed above that the $m$-dimensional U($N \rightarrow \infty$) Yang-Mills theory is equivalent to the $d = (m+2n)$-dimensional NC U(1) gauge theory on $\mathbb{R}^m \times \mathbb{R}^{2n}$. Thus we can apply the emergent gravity picture in section 5 to the $d = (m+2n)$-dimensional NC U(1) gauge theory to derive a $d$-dimensional Einstein gravity which is certainly expected to be dual to the $m$-dimensional U($N \rightarrow \infty$) Yang-Mills theory. We think this trinity relation between large $N$ gauge theories, NC U(1) gauge theories and gravitational theories in various dimensions will shed light on the gauge/gravity duality or large $N$ duality. For this reason, let us focus on the commutative limit of the NC gauge theory in (6.11) which corresponds to a planar limit ($N \rightarrow \infty$) of large $N$ gauge theory. Suppose that the global Poisson structure (see eqs. (5.89) and (5.90) where $y^\mu$’s are fiber coordinates and are not related to $y^a$ in eq. (6.7)) is given by $\Pi = \frac{1}{2} \Pi^{ab}(x,y) \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b} \in \Gamma(\wedge^2 TM_{2n})$ obtained by gluing together local Poisson structures ($U_i \subset M_{2n}, \Theta$) on open subsets in $2n$-dimensional symplectic manifold $M_{2n} = \bigcup_i U_i$. Let $\mathcal{M}$ be an emergent $d$-dimensional manifold which locally looks like $\mathcal{M} \approx \mathbb{R}^m \times M_{2n}$ and so may be regarded as a regular Poisson manifold. We can follow the procedure in section 5 to derive $d$-dimensional global vector fields $V_A = (V_\mu, V_a) \in \Gamma(TM)$ using the map (5.101) from the $d$-dimensional NC U(1) connections (6.8). For example, on open subsets of $\mathbb{R}^m \times \mathbb{R}^{2n}$ (where life becomes simple as was illustrated in eq. (5.133)), they are given by (6.14)

$$V_A(f) = \begin{cases} 
\frac{\partial f(x,y)}{\partial y^\mu} + \{A_\mu(x,y), f(x,y)\}_\Pi, & A = \mu; \\
\{C_a(x,y), f(x,y)\}_\Pi, & A = a \end{cases}$$

We may emphasize that the equivalence between the $d$-dimensional NC U(1) gauge theory (6.11) and $m$-dimensional U($N \rightarrow \infty$) Yang-Mills theory (6.12) is a mathematical identity and has been known long ago, for example, in [7, 8]. Nevertheless the possibility that gravity can emerge from NC U(1) gauge fields has been largely ignored until recently. But the emergent gravity picture based on NC U(1) gauge theory debunks that this coincidence did not arise by some fortuity and so we want to quote an epigram due to John H. Schwarz [124]: “Take coincidences seriously”.}
for any $f \in C^\infty(M)$ where

$$V_\mu(X) = \partial_\mu + A^a_\mu(X) \frac{\partial}{\partial y^a}, \quad V_a(X) = C^b_a(X) \frac{\partial}{\partial y^b}. \quad (6.15)$$

Then the dual covectors $V^A = V^A_M(X) dX^M \in \Gamma(T^*M)$ are given by

$$V^A(X) = \left(dx^\mu, V^a_b(X)(dy^b - A^b_\mu(X) dx^\mu)\right), \quad (6.16)$$

where $C^c_a V^b_c = \delta^b_a$. The vector fields $V_A$ are volume preserving as before, i.e. $\mathcal{L}_{V_A} \nu = 0$, with respect to the volume form

$$\nu = \lambda^2 V^1 \wedge \cdots \wedge V^d = d^{2n}x \wedge \nu_{2n} \quad (6.17)$$

where $\nu_{2n} = \lambda^2 V^1 \wedge \cdots \wedge V^{2n}$. Therefore the $d$-dimensional Lorentzian metric on $M$ emergent from the NC U(1) gauge fields or large $N$ matrices is given by

$$ds^2 = \eta_{AB} E^A \otimes E^B = \lambda^2 \eta_{AB} V^A \otimes V^B$$

$$= \lambda^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + \delta_{ab} V^a_c V^b_d (dy^c - A^c) (dy^d - A^d)\right) \quad (6.18)$$

where $A^a = A^a_\mu dx^\mu$ and

$$\lambda^2 = \nu(V_1, \cdots, V_d). \quad (6.19)$$

The $d$-dimensional emergent gravity described by the metric (6.18) is completely determined by the configuration of $d$-dimensional symplectic U(1) gauge fields $A_M(x,y)$ or the $m$-dimensional gauge-Higgs system $(A_\mu, \Phi_a(x))$ in U(N) gauge theory. In other words, the equations of motion and the Bianchi identity for dynamical gauge fields in the NC U(1) gauge theory or U(N) gauge theory can be mapped to the corresponding equations for the $d$-dimensional Lorentzian metric (6.18) in a similar way as eqs. (5.40) and (5.41). As expected, it will be difficult to complete the mission for general gauge fields and indeed we do not yet know the precise form of Einstein equations determined by symplectic or large $N$ gauge fields except lower dimensions $d \leq 4$. Hence it may be instructive to consider a more simpler system. For this purpose, we may introduce linear algebraic conditions of $d$-dimensional field strengths $F_{AB}$ as a higher-dimensional analogue of four-dimensional self-duality equations such that the Yang-Mills equations of motion for the action (6.11) follow automatically. These are of the following type [125, 126, 127]:

$$\frac{1}{2} T_{ABCD} F_{CD} = \zeta F_{AB} \quad (6.20)$$

with a constant 4-form tensor $T_{ABCD}$. The relation (6.20) clearly implies via the Bianchi identity, $D[A F_{BC}] = 0$, that the equations of motion, $D^A F_{AB} = 0$, are satisfied provided $\zeta$ is nonzero. Keeping in with the action (6.12), a particularly interesting choice for the tensor $T_{ABCD}$ will be the case; $T_{abcd} \neq 0$, otherwise $T_{ABCD} = 0$. In this case, nontrivial gauge fields are mapped to adjoint Higgs fields $\Phi_a(x)$ in U(N) Yang-Mills theory that obey the commutation relation

$$-i [\Phi_a(x), \Phi_b(x)] = -i [C_a(x, y), C_b(x, y)]_* = -B_{ab} + F_{ab}(x, y). \quad (6.21)$$
For instance, the important examples in four \(n = 2\) and six \(n = 3\) dimensions are given by

\[
\begin{align*}
n &= 2 : & T_{abcd} &= \varepsilon_{abcd}, & \zeta &= \pm 1, \\
n &= 3 : & T_{abcd} &= \frac{1}{2} \varepsilon_{abcdef} I_{ef}, & \zeta &= -1,
\end{align*}
\]

(6.22)

(6.23)

where \(I_{ab}\) is the constant symplectic matrix (5.154) in six dimensions. In the case (6.22), we recover the self-duality equation (5.19) for NC U(1) instantons. In the 6-dimensional case (6.23), we have the so-called Hermitian Yang-Mills equations given by

\[
\begin{align*}
F_{ab} &= -\frac{1}{4} \varepsilon_{abcdef} F_{cd} I_{ef}, \\
I^{ab} F_{ab} &= 0.
\end{align*}
\]

(6.24)

Actually the second equation needs not be imposed separately because it can be derived from the first one by using the identity \(\frac{1}{8} \varepsilon_{abcdef} I_{cd} I_{ef} = I_{ab}\). The above Hermitian Yang-Mills equations can be understood as follows. For \(d > 4\), the 4-form tensor \(T_{ABCD}\) cannot be invariant under \(G = SO(d)\) rotations and the equation (6.20) breaks the rotational symmetry to a subgroup \(H \subset SO(d)\). In the 6-dimensional case, the 4-form tensor (6.23) breaks the rotational symmetry \(G = SO(6) = SU(4)/\mathbb{Z}_2\) to a subgroup \(H = U(3) \subset SO(6)\). Then we can decompose the 15-dimensional vector space of 2-forms \(\Lambda^2 T^* M\) under the unbroken symmetry group \(H\) into three subspaces [128]:

\[
\Lambda^2 T^* M = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8
\]

(6.25)

where \(\Lambda^2_1, \Lambda^2_6, \Lambda^2_8\) are a one-dimensional (singlet), six-dimensional and eight-dimensional vector spaces taking values in \(U(1) \subset U(3)\), \(G/H = \mathbb{C}P^3\), and \(SU(3) \subset U(3)\), respectively. The Hermitian Yang-Mills equations (6.20) project the vector space \(\Lambda^2 T^* M\) into the eight-dimensional subspace \(\Lambda^2_8\) which preserves the \(SU(3)\) rotational symmetry [128].

Using the map [5.28] or [5.101] (whose simplest version will be defined by eq. (6.14)), we can identify the emergent metric (6.18) for gauge fields obeying the self-duality equations, (5.19) and (6.24), in four and six dimensions, respectively. It was shown [1, 2, 3, 101] that the classical limit of 4-dimensional NC U(1) instantons, called symplectic U(1) instantons, is equivalent to gravitational instantons which are Ricci-flat, Kähler manifolds and so Calabi-Yau 2-folds \(M_4 = CY_2\). If we consider 6-dimensional NC Hermitian U(1) instantons defined by (6.24), the first condition is translated into the Kähler condition of a six-dimensional manifold \(M_6\) and the second condition demands a Ricci-flat condition on \(M_6\). In the end, the classical limit of 6-dimensional NC Hermitian U(1) instantons will be mapped to 6-dimensional Ricci-flat and Kähler manifolds, namely, Calabi-Yau 3-folds \(M_6 = CY_3\) [129]. Remember that NC U(1) gauge fields in extra dimensions (i.e., along the space \(M_{2n}\)) are mapped to the adjoint scalar fields in U(N) gauge theory and obey the commutation relation (5.21). According to our scheme, we thus expect that NC (Hermitian) U(1) instantons correspond to the quantization of symplectic U(1) instantons in four and six dimensions and so equivalently “quantized” Calabi-Yau manifolds. They will be represented by some topological objects made out of large \(N\) matrices \(\Phi_a \in A_N\) in
the Hilbert space (6.2). It was claimed in [2, 3] that the topological objects take values in the K-theory $K(A_\theta)$ for NC $\star$-algebra $A_\theta$. Via the Atiyah-Bott-Shapiro isomorphism [130] that relates complex and real Clifford algebras to K-theory, combined with the trinity relation [74] between NC U(1) instantons, $SU(n)$ Yang-Mills instantons and Calabi-Yau $n$-folds, it was conjectured there that the topological objects made out of large $N$ matrices $\Phi_a \in A_N$ should be realized as leptons and quarks in the fundamental representation of the holonomy group $SU(n)$ of Calabi-Yau $n$-folds. Recently a similar geometric model of matters was advocated in [12, 131]. Later we will further discuss this geometric model of matters (or emergent matters).

We remark closely related approaches for the quantization of symplectic (or Poisson) manifolds. Bressler and Soibelman [132] studied some relationship between mirror symmetry and deformation quantization and suggested that the A-model is related to deformation quantization in the sense that there is a category of holonomic modules (that are the modules with smallest possible characteristic varieties) over the quantized algebra of smooth functions on a symplectic manifold and it becomes equivalent (at least locally) to the Fukaya category of the same symplectic manifold. [34] Kapustin [134] argued that for a certain class of symplectic manifolds the category of A-branes is equivalent to a NC deformation of the category of B-branes on the same manifold and so A-branes can also be described in terms of modules over a NC algebra. He also observed that generalized complex manifolds are in some sense a semi-classical approximation to NC complex manifolds with $B$-fields. In particular he showed that the equivalence arises from the SW transformation that relates gauge theories on commutative and NC spaces. Later this suggestion has been extended and made more precise in [133, 134], partly using the framework of generalized complex geometry. Gukov and Witten [119] formulated the problem of quantizing a symplectic manifold $(M, \omega)$ in terms of the A-model of a complexification of $M$ where the Hilbert space obtained by the quantization of $(M, \omega)$ is the space of strings connecting an ordinary Lagrangian A-brane and a space-filling coisotropic A-brane. Recently Kay [137] showed how affine and projective special Kähler manifolds (arising as moduli spaces of vector multiplets in 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories) emerge from the structure of Fedosov quantization of symplectic manifolds.

6.2 Emergent time

So far we have kept silent to a notorious issue in quantum gravity known as the “emergent time” [133]. We have considered only the emergence of spaces from NC $\star$-algebra $A_\theta$. But the special relativity unifies space and time into a single entity – spacetime. Furthermore the general relativity dictates that space and time should be subject to the general covariance and they must be coalesced into the form of Minkowski spacetime in a locally inertial frame. Hence, if we want to realize the (quantum) general relativity from a NC $\star$-algebra $A_\theta$, it is desirable to put space and time on an equal footing. If a space is emer-

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gent, so should time. But the concept of time is more stringent since it is difficult to give up the causality and unitarity. So we want to define the emergent time together with the emergent space. The craving picture is that time is entangled with spaces to unfold into spacetime and to take the shape of Lorentz covariance. Essentially our leitmotif is to understand what is time. We will see soon that quantum mechanics, a close pedigree of NC space, gives us a decisive lesson for this question.

Before addressing the issue of emergent time, it will be important to identify where the flat space, i.e. the space with the metric $g_{\mu\nu} = \delta_{\mu\nu}$, comes from. In order to trace out the origin of flat space, let us look at the emergent metric (5.34). Definitely the emergent metric becomes flat when $V^a = \delta^a_\mu dx^\mu \in \Gamma(T^*M)$ or equivalently $V^a = \delta^a_\mu \frac{\partial}{\partial x^\mu} \in \Gamma(TM)$ in which $\lambda^2 = 1$. Then the definition (5.14) of vector fields immediately implies that the (flat) vector field $V^{(0)}_\mu = \partial_\mu$ is coming from the (vacuum) gauge field given by $\hat{C}^{(0)}_a = p_a = B_{a\mu} y^\mu$, turning off all fluctuations. Note that the vielbein $E^{(0)}_\mu = V^{(0)}_\mu = \partial_\mu$ in this case can be extended to entire space and so we do not need the globalization (5.28). We now get into the most beautiful and remarkable point of emergent gravity that is the underlying key point to resolve the cosmological constant problem [2, 3, 59]: The vacuum algebra (6.7) is responsible for the generation of flat spacetime that is not an empty space unlike general relativity. Instead the flat spacetime is emergent from a uniform condensation of gauge fields in vacuum. Here we have embraced time too because we will eventually describe the evolution of spacetime geometry in terms of derivations of an underlying NC algebra generated by the vacuum algebra (6.7).

In quantum mechanics, the time evolution of a dynamical system is defined as an inner automorphism of NC algebra $A_\hbar$ generated by the NC phase space

$$[x^i, p_j] = i\hbar \delta^i_j.$$  

(6.26)

It is worthwhile to realize that the mathematical structure of emergent gravity is basically the same as quantum mechanics. The former is based on the NC space (6.7) while the latter is based on the NC phase space (1.21). Another fundamental fact for the concept of emergent time is that any Poisson manifold $(M, \Pi)$ always admits a dynamical Hamiltonian system on $M$ where the Poisson structure $\Pi$ is a bivector in $\Gamma(\Lambda^2 TM)$ and the dynamics of the system is described by the Hamiltonian vector field $X_f = \Pi(df)$ for any energy function $f \in C^\infty(M)$ of an underlying Poisson algebra [13, 14]. Since the concept of emergent time has been explored in [2, 3, 59] along this viewpoint, let us here consider this issue from different perspective.

For this reason, let us look at the $d = (m + 2n)$-dimensional emergent metric (6.18). According to the gauge/gravity duality, we regard the $d$-dimensional emergent spacetime described by the metric (6.18) as a bulk geometry $\cal{M}$ dual to the $m$-dimensional large
$N$ gauge theory \((6.12)\). However we have to note that the \(m\)-dimensional commutative spacetime \(\mathbb{R}^{m-1,1}\) was not emergent but preexisted from the beginning. Of course this spacetime also becomes dynamical when the gauge fields \(A_\mu(x)\) are nontrivial fluctuations. But the original background spacetime \(\mathbb{R}^{m-1,1}\) was preexisting\(^{36}\) unlike the entirely emergent space \(M_{2n}\). Initially the emergent space \(M_{2n}\) was not existent in the large \(N\) gauge theory \((6.12)\). This space is only emergent as a result of the vacuum condensate described by \(\Phi_0^a = B_{ab}y^b\) where \(g^a\)'s satisfy the commutation relation \((6.7)\). Note that the configuration of vacuum gauge fields \(\Phi_0^a\) is a consistent solution of \(U(N)\) Yang-Mills theory \((6.12)\) and is achieved by turning off all fluctuations, i.e., \(A_\mu = A_a = 0\). It might be emphasized that the vacuum expectation value \(\langle \Phi_a \rangle_{\text{vac}} = B_{ab}y^b\) of adjoint scalar fields does not break the Lorentz symmetry \(SO(m-1,1)\) as in the Higgs mechanism \(\langle \phi \rangle_{\text{vac}} = v\) because \(\langle \Phi_a \rangle_{\text{vac}}\) are \(SO(m-1,1)\) scalars. Even it should not be interpreted as the breaking of Lorentz symmetry \(SO(2n)\) in extra dimensions since the extra space \(\mathbb{R}^{2n}\) is newly emergent from the vacuum condensate \((7)\). For this vacuum solution, the \(d\)-dimensional metric \((6.18)\) precisely reduces to \(\mathbb{R}^{d-1,1} = \mathbb{R}^{m-1,1} \times \mathbb{R}^{2n}\) where \(\mathbb{R}^{2n}\) is the emergent space triggered by the Moyal-Heisenberg algebra \((6.7)\). The enticing point for us is that the NC space \((6.7)\) plays a similar role in doing the NC phase space \((6.26)\) in quantum mechanics. To be precise, we can introduce a Hamiltonian system, i.e., Heisenberg equations, describing the evolution of spacetime geometry using the NC algebra \((6.7)\) in the exactly same way as quantum mechanics.

To illuminate this aspect, let us reconsider the action \((6.12)\) for the case \(m = 1\): \(S = -\frac{1}{g^2_{YM}} \int dt \text{Tr}_N\left(\frac{1}{2}(D_0 \Phi_a)^2 - \frac{1}{4}[\Phi_a, \Phi_a]^2\right)\)

\[= -\frac{1}{4G^2_{YM}} \int d^dX (F_{MN} - B_{NM})^2 \quad (6.27)\]

where we derived the \(d = (2n + 1)\)-dimensional NC gauge theory using the fact that the Moyal-Heisenberg algebra \((6.7)\) is a solution of the matrix quantum mechanics. The \((2n + 1)\)-dimensional Lorentzian metric emergent from the matrix quantum mechanics \((6.27)\) is simply given by the metric \((6.18)\) for the case of \(m = 1\):

\[ds^2 = \lambda^2\left(-dt^2 + \delta_{ab}V^a_cV^b_d(dy^c - \mathbf{A}^c)(dy^d - \mathbf{A}^d)\right) \quad (6.28)\]

where \(\mathbf{A}^a = A^a_0 dt\) and \(\lambda^2\) is determined by an invariant volume form \(\nu = dt \wedge \nu_{2n}\). The above metric is generated by vector fields \(V_A = (V_0, V_a)(t, y) \in \Gamma(T(\mathbb{R} \times M_{2n}))\) which are, for example, on open subsets of \(\mathbb{R} \times \mathbb{R}^{2n}\), given by eq. \((6.14)\):

\[V_0(f) = \frac{\partial}{\partial t} f(t, y) + \{A_0, f\}_\Pi(t, y), \quad (6.29)\]

\[V_a(f) = \{C_a, f\}_\Pi(t, y), \quad (6.30)\]

for any smooth function \(f \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n})\). If all fluctuations are turned off, we can see that the emergent geometry \((6.28)\) reduces to flat Minkowski spacetime \(\mathbb{R}^{d-1,1}\) and

\(^{36}\)It is interesting to notice that this part of geometry is described by \(\partial_\mu\) in the covariant derivative \((6.9)\).
the global Lorentz symmetry \( SO(d - 1, 1) \) is emergent too as an isometry of the vacuum geometry \( \mathbb{R}^{d-1,1} \). Note that, if we identity \( A_0(t, y) := -H(t, y) \) with a Hamiltonian \( H(t, y) \) of a dynamical system whose phase space is characterized by the global Poisson structure \( \Pi = \Pi^{ab}(t, y) \partial_a \wedge \partial_b \), the first equation (6.29) for a temporal vector field \( V_0 \) is precisely the Hamilton’s equation of the dynamical system. It is obvious that the dynamical system in our case is a spacetime geometry described by the Lorentzian metric (6.28) and the quantization of the dynamical system should be described by the action (6.27). In this sense, the matrix quantum mechanics, known as the BFSS matrix model \([122]\), should describe a quantum geometry of space and time.

It should be noted that the time evolution (6.29) for a general time-dependent system is not completely generated by an inner automorphism since the first term is not an inner but an outer derivation. But it is well-known \([14]\) that the time evolution of a time-dependent system can be defined by the inner automorphism of an extended phase space whose extended Poisson bivector is given by

\[
\tilde{\Pi} = \Pi + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial H}.
\]  

(6.31)

Then one can see that the temporal vector field (6.29) is equal to the generalized Hamiltonian vector field defined by

\[
V_0 = \tilde{X}_H = -\tilde{\Pi}(dH) = \Pi(dA_0) + \frac{\partial}{\partial t}.
\]  

(6.32)

However the spatial vector fields (6.30) remain intact because of the relation \( \tilde{V}_a = \tilde{\Pi}(dC_a) = \Pi(dC_a) = V_a \). Here we remark that the extended Poisson structure (6.31) raises a serious issue if the time variable might also be quantized, i.e., time also becomes an operator obeying the commutation relation \([t, H] = i\), for a general time-dependent system. We will not dwell into this issue since it is a challenging open question even in quantum mechanics. See \([139]\) for a comprehensive, up-to-date review of this and related topics. This retreat admits that we do not have any clear understanding on the issue of emergent time for a general “time-dependent” geometry. For the moment, we want to evade this perverse quantization issue of time by simply tolerating that the evolution of a spacetime geometry in nonequilibrium (we intentionally elude the tautology with the “evolution” by saying “in nonequilibrium” instead of “time-dependent”) is generated by both inner and outer automorphisms.

Our argument so far implies that the BFSS matrix model (6.27) can be interpreted as a Hamiltonian system of IKKT matrix model whose action is given by \([21]\)

\[
S = -\frac{1}{4g_{YM}^2} \text{Tr}_N[\Phi_a, \Phi_a]^2.
\]  

(6.33)

The above matrix model is a 0-dimensional theory and so it does not assume any kind of spacetime structures from the beginning. The theory is defined only with a bunch of \( N \times N \) matrices (as objects) which are subject to the following algebraic relations (as morphisms):

\[
[\Phi_a, [\Phi_b, \Phi_c]] + \text{cyclic}(a \to b \to c) = 0,
\]  

(6.34)

\[
[\Phi^a, [\Phi_a, \Phi_b]] = 0.
\]  

(6.35)
Physical solutions consist of all possible matrix configurations obeying the above matrix morphisms up to U(N) gauge transformations. We adopt a traditional picture so that general matrix configurations are constructed by considering all possible deformations over a vacuum solution, especially, the most primitive vacuum. Hence the prime step is to find the primitive vacuum on which all fluctuations are supported. In particular, we are interested in large $N$ limit, typically, $N \to \infty$. In this limit, a most natural primitive vacuum is given by the Moyal-Heisenberg algebra (6.7), i.e.,

$$\Phi^{(0)} \equiv \langle \Phi_a \rangle_{\text{vac}} = B_{ab} y^b \in \mathcal{A}_N$$  \hspace{1cm} (6.36)

where $B_{ab} = (\theta^{-1})_{ab}$. We now consider all possible deformations of the vacuum (6.36) and parameterize them as

$$\Phi_a(y) = B_{ab} y^b + A_a(y) \in \mathcal{A}_N.$$  \hspace{1cm} (6.37)

We notice that $-i\Phi_a(y)$ becomes a covariant derivative, $D_a(y) = \partial_a - iA_a(y)$, because the matrix model (6.33) contains only the adjoint operation between matrices under which we can identify $p_a \equiv B_{ab} y^b$ with $\text{ad} p_a = -i[B_{ab} y^b, -] = \partial_a$. Moreover, using the map (6.4) between the matrix algebra $\mathcal{A}_N$ and NC gauge fields in $\mathcal{A}_\theta$, we can realize a NC field theory representation of the matrix model (6.33). In particular, the adjoint scalar fields (6.37) are mapped to NC U(1) gauge fields (6.10):

$$\mathcal{A}_N \to \mathcal{A}_\theta : \Phi_a \mapsto C_a(y) = iD_a(y).$$  \hspace{1cm} (6.38)

Thus we can represent the matrix action (6.33) using NC U(1) gauge fields and the resulting action is given by [7, 8]

$$S = \frac{1}{4 G_{YM}^2} \int d^2y (F_{ab} - B_{ab})^2.$$  \hspace{1cm} (6.39)

Recall that the NC U(1) gauge fields $A_a(y)$ in eq. (6.37) were introduced as fluctuations around the vacuum (6.36) which supports an intrinsic symplectic or Poisson structure represented by the Heisenberg algebra (6.7). Therefore the deformations of the vacuum (6.36) in terms of NC U(1) gauge fields must be regarded as a dynamical system. The corresponding Heisenberg equation for an observable $f \in \mathcal{A}_\theta$ is defined by

$$\frac{df(y)}{dt} = -i[A_0(y), f(y)]^\star,$$  \hspace{1cm} (6.40)

that is precisely an analogue of quantum mechanics defined by the symplectic structure (6.26). Here we implicitly assumed that the dynamical mechanism we have considered is in the conservative process. For general time-dependent fluctuations, the above Heisenberg equation has to be replaced by

$$\frac{df(t, y)}{dt} = \frac{\partial f(t, y)}{\partial t} - i[A_0(t, y), f(t, y)]^\star.$$  \hspace{1cm} (6.41)

Its commutative limit will recover the Hamilton’s equation (6.29) that is organized into the temporal vector field, i.e., $\frac{df}{dt} := V_0$. It should be remarked that, if gravity is emergent from a more fundamental theory, for an internal consistency of the theory, spacetime as well as...
gravity should be simultaneously emergent from some fundamental degrees of freedom in
the theory. We observed that the emergent gravity from NC gauge fields is indeed the case.
Consequently, the emergent (quantum) gravity derived from the NC algebra (6.7) provides
a natural concept of emergent time via the Hamiltonian system of spacetime geometry
though the time-dependent case is still elusive.

Finally we want to point out that the above picture of emergent time is consistent
with that in general relativity. In the Hamiltonian formulation of general relativity, in
particular, in the ADM formalism [4, 14], the Hamiltonian $H$ is a constraint rather than
a dynamical variable. We claim that this should be the case too in NC gauge theory
because the $\Lambda$-symmetry in NC gauge theory is equivalent to diffeomorphism symmetry as
we showed in appendix A. Unfortunately this diffeomorphism symmetry is not manifest in
the action (6.27) because it has been represented in a particular vacuum state, e.g., eq.
(6.7). However we observed in section 4 that the (dynamical) diffeomorphism symmetry in
NC gauge theory is realized as the (local) gauge equivalence (4.8) between star products
or Morita equivalence (4.9) in representation theory (i.e., the ring-theoretic equivalence of
bimodules). In general relativity this choice of a particular vacuum corresponds to a par-
ticular background manifold whose metric is $\mathcal{g}$. In this case, the diffeomorphism symmetry
is reduced to a Killing symmetry, $\mathcal{L}_X\mathcal{g} = 0$, of the background metric $\mathcal{g}$. Precisely the corre-
spending situation also arises in NC gauge theory. In a particular vacuum characterized
by a specific symplectic 2-form $\omega$, e.g. $\omega = B$, the $\Lambda$-symmetry is (spontaneously) broken
to the symplectomorphism, $\mathcal{L}_X B = 0$, which is equivalent to NC U(1) gauge symmetry [1].
To be specific, the NC or large $N$ gauge theory (6.27) respects the NC U(1) or U($N \to \infty$)
gauge symmetry (1.13). Thus the temporal gauge field $A_0$ becomes a Lagrange multiplier
rather than a dynamical variable. The local gauge transformations will be generated by
the first class constraints which leave the physical states invariant like as general relativity
[40]. Of course, one should not expect that the temporal gauge field $A_0$ is directly
related to the Hamiltonian in general relativity since we do not take into account the full
diffeomorphism symmetry in NC gauge theory. Nonetheless we want to put forward that
the structure of gauge symmetries and constraints is compatible each other in two theories
and so the concept of emergent time congruous with general relativity will ensue too.

6.3 Matrix representation of Poisson manifolds

In this subsection we will briefly discuss the matrix representation of quantized Poisson
manifolds in a Hilbert space on which deformed Poisson algebra acts. We will focus on its
physical correspondence rather than a mathematical scrutiny.

As Kontsevich proved, every finite dimensional Poisson manifold admits a deformation
quantization. But its representation in a Hilbert space on which quantized Poisson algebra
acts is in general a challenging open problem. Fortunately the most important examples
of Poisson manifolds in physics occur in semi-simple Lie groups and their representation
theory had been mathematically completed in the 20th century. The Hilbert space for
an irreducible representation of a compact Lie algebra is a “finite” dimensional (complex)
vector space unlike the Moyal-Heisenberg algebra (1.1) whose irreducible representation
is infinite dimensional. We may also apply the Schwinger representation (5.123) of Lie
algebras. Indeed we will see later that, when we try to realize matter fields such as leptons and quarks and their non-Abelian interactions, i.e., weak and strong forces in the context of emergent geometry, this symplectic realization of quantum Poisson algebras becomes more relevant [2, 3]. For a general Poisson manifold, first we can employ the symplectic realization (5.120) of it and then quantize an ambient symplectic manifold or a symplectic groupoid as before, i.e., represent a corresponding NC algebra \( \mathcal{A}_\theta \) in the Hilbert space (6.2) which can also be modeled by either a NC gauge theory or a large \( N \) gauge theory. Finally, guided by the Poisson map from the ambient symplectic manifold to the original Poisson manifold, we can try to find an irreducible representation \((\rho, V)\) of a quantized Poisson algebra \( \mathcal{A}_P \subset \mathcal{A}_\theta \) where \( V \subset \mathcal{H} \) is an \( n \)-dimensional representation space and \( \rho \) is a Lie algebra homomorphism from \( \mathcal{A}_P \) to End(\( V \)).

We will consider two situations which incorporate a Poisson manifold. The first situation is that a Poisson manifold directly arises as a vacuum solution of a NC gauge theory or a large \( N \) gauge theory. In some cases the relevant action needs to contain a mass deformation. One can show that the IKKT matrix model (6.33) cannot admit a compact vacuum such as the Lie algebra (5.124). This must be true too for the action (6.12) because the latter can be obtained by applying the “matrix T-duality” [141] to the former. It was shown [58] that the mass deformation is actually required to realize constant curvature spacetimes such as \( d \)-dimensional sphere, de Sitter and anti-de Sitter spaces. However there seems to be a novel realization of constant curvature spacetimes as was recently verified in [70] with explicit examples. This realization is involved with a large topology change concomitant with the change of the compactness of spacetime geometry. This novel mechanism for compactifications is realized as follows. Consider the Moyal-Heisenberg vacuum (6.7) as a vacuum solution and then incorporate generic U(1) gauge fields whose field strength does not necessarily vanish at asymptotic infinity. For instance, the gauge fields \( A_\mu(y) \) in eq. (6.37) can be arranged to breed further vacuum condensates \( \langle F_{ab}(y) \rangle_{\text{vac}} \neq 0 \), which are superposed on the original background field \( B_{ab} \) for the Moyal-Heisenberg vacuum (6.7). The analysis in [70] shows that the additional condensate triggered by the U(1) field strength \( F_{ab}(y) \) leads to the topology change of spacetime geometry from a noncompact space to a compact space. We may envisage a generalization so that the extra vacuum condensates in \( \langle F_{ab}(y) \rangle_{\text{vac}} \neq 0 \) occur only in a subspace of \( \mathbb{R}^{2n} \), i.e., \( \text{rank} F_{|y|\to \infty} \leq \text{rank} B \). In this case the compactification of spacetime geometry will arise only in the subspace. If so, it may be possible to realize Poisson manifolds by turning on general U(1) gauge fields with a nontrivial asymptotic behavior. This mechanism may be called “dynamical symplectic realizations”. We think that the dynamical symplectic realization is physically more enticing than the mass deformation because the mass deformed matrix model does not reproduce the usual massless U(1) gauge theory in a commutative limit and so it is phenomenologically unviable.

The second situation is largely motivated by the speculation in [2, 3] realizing matter fields such as leptons and quarks in terms of NC U(1) instantons along extra dimensions. A similar geometric model of matters was recently appeared in [12, 13] where matters such as electron, proton, neutron and neutrino are realized in terms of four-manifolds such as Taub-NUT, Atiyah-Hitchin, \( \mathbb{C}P^2 \) and \( S^4 \). Note that all these four-manifolds in
our case arise from NC U(1) gauge fields \[101, 70\]. We start with the relation (6.21). It demonstrates that the adjoint scalar fields in U(N \to \infty) gauge theory over the Moyal-Heisenberg vacuum \[6.7\] are mapped to higher dimensional NC U(1) gauge fields. We are interested in a time-independent “stable” solution in U(N \to \infty) gauge theory. To construct such a stable solution, consider a NC gauge field configuration described by the generalized self-duality equation (6.20) where \(T^{abcd}\) are only nonvanishing structure constants and others identically vanish. For example, in four \((n = 2)\) and six \((n = 3)\) dimensions, they are given by eq. (6.22) and eq. (6.23), respectively. In these cases, the NC gauge field configurations describe NC U(1) instantons in four and six dimensions obeying (5.19) and (6.24), respectively. But these solutions partially break the Lorentz symmetry SO(2n) which is the isometry of \(\mathbb{R}^{2n}\) emergent from the vacuum gauge fields \(\Phi^{(0)}_a = B_{ab} y^b\).

In four dimensions, on one hand, \(T^{abcd}\) in eq. (6.22) does not break the Lorentz symmetry \(SO(4) = SU(2)_L \times SU(2)_R/\mathbb{Z}_2\). But the self-duality equation (5.19) breaks it into \(SU(2)_L\) or \(SU(2)_R\) depending on the self-duality. On the other hand, in six dimensions, \(T^{abcd}\) in (6.23) breaks the Lorentz symmetry \(SO(6) = SU(4)/\mathbb{Z}_2\) into \(U(3) \subset SO(6)\) because \(I_{ab}\) was inherited from the background Kähler form \(B_{ab} = |\theta|^{-1} I_{ab}\). The Hermitian U(1) instantons obeying (6.24) further break \(U(3)\) into \(SU(3)\).

The NC U(1) instantons in four or six dimensions (though it can be generalized to higher dimensions, we want to mainly focus on these two cases for simplicity and they seem to be mostly relevant to physics) will be realized as four or six dimensional submanifolds in \(d\)-dimensional spacetime described by the metric (5.18). As we argued in section 6.1, they are Calabi-Yau \(n\)-folds. And the unbroken Lorentz symmetry of NC U(1) instantons, e.g., \(SU(2)\) or \(SU(3)\), precisely coincides with the holonomy group of Calabi-Yau \(n\)-folds.

From a theoretical perspective, when there is a symmetry breaking, order parameters arise as is well-known in condensed matter systems such as superconducting or ferromagnetic materials. An example of the order parameter is the net magnetization in a ferromagnetic system, whose direction is spontaneously chosen when the system cooled below the Curie temperature. A similar phenomenon should happen in NC U(1) instantons or Calabi-Yau \(n\)-folds. In our case they are either \(SU(2)\) or \(SU(3)\) variables depending on solutions in extra dimensions. To be specific, let us consider NC U(1) gauge fields on \(\mathbb{R}^{p,1}\) with \(p = m - 1\) that appear in the covariant derivative (6.4). As we argued above, the NC coordinates \(y^a \in A_\theta\) should arrange themselves in the form of \(SU(2)\) or \(SU(3)\) variables due to the internal structure of \(\mathbb{R}^{p,1}\) originated from NC U(1) instantons or Calabi-Yau \(n\)-folds (see the footnote 37). Certainly they are given by eq. (6.124), which can be regarded as low-energy order parameters (or collective modes) in the vicinity of the solution of eq. (6.20). For this reason, let us expand NC U(1) gauge fields on \(\mathbb{R}^{p,1}\) in terms of the order parameters [2, 3]:

\[
A_\mu(x, y) = A_\mu(x) + A_\mu^I(x)Q^I + A_\mu^{IJ}(x)Q^I Q^J + \cdots,
\]

\[6.42\]

We may simplify the situation by assuming that NC U(1) gauge fields \(A_a(x, y)\) on emergent space \(\mathbb{R}^{2n}\) depend only on NC coordinates \(y^a\), i.e., \(A_a(y)\). Thus NC U(1) instantons in this case are extended along \(\mathbb{R}^{p,1}\) with \(p = m - 1\) whose thickness is set by \(\zeta = \sqrt{|\theta|}\). See Fig. 2. Therefore we may identify the NC U(1) instantons with \(D_p\)-branes extended along \(\mathbb{R}^{p,1}\) but their internal structures depend on their substances, i.e., NC U(1) instantons or equivalently Calabi-Yau \(n\)-folds with different dimensionality.
where we assumed that each term in (6.42) belongs to an irreducible representation of \( \rho = \text{End}(V) \) and \( V = L^2(\mathbb{C}^n) \). Remarkably \( SU(n) \) gauge fields \( A^I_\mu(x) \) as well as ordinary \( U(1) \) gauge fields \( A_\mu(x) \) arise as low lying excitations on \( \mathbb{R}^{p,1} \) of NC \( U(1) \) gauge fields when there exists a nontrivial solution obeying eq. (6.20) in extra dimensions.

As usual, the Poisson algebra appears as symmetry generators which are composite operators, namely a symplectic realization (5.125), rather than fundamental variables in emergent spacetime. Furthermore it was conjectured \( [2, 3] \) for the case of \( m = 4 \) (and \( n = 3 \)) considered in next subsection that the representation of four or six dimensional NC \( U(1) \) instantons in a (subspace of) Hilbert space \( \mathcal{H} \), e.g. eq. (6.2), has an incarnation in terms of chiral fermions in four-dimensional spacetime \( \mathbb{R}^{3,1} \). The curious conjecture was motivated by a very mysterious (at least for us) connection between homotopy groups, K-theory and Clifford modules \( [130, 142, 143] \). An underlying reasoning is the following \( [3] \): NC \( U(1) \) instantons made out of time-independent adjoint scalar fields (6.21) in \( U(N \to \infty) \) Yang-Mills theory can be regarded as a homotopy map

\[
\Phi_a : S^3 \to GL(N, \mathbb{C})
\]

from \( S^3 \) to the group of nondegenerate complex \( N \times N \) matrices. Thus the topological class of (perturbatively) stable solutions can be characterized by the homotopy group \( \pi_3(GL(N, \mathbb{C})) \).\(^{39}\) As is well-known, in the stable regime where \( N > 3/2 \), the homotopy group of \( GL(N, \mathbb{C}) \) or \( U(N) \) defines a generalized cohomology theory, known as the K-theory \( K(X) \). In our case where \( X = \mathbb{R}^{3,1} \), this group with compact support is given by

\[
K(\mathbb{R}^{3,1}) = \pi_3(U(N)) = \mathbb{Z}.
\]

We now come to the connection with K-theory, via the celebrated Atiyah-Bott-Shapiro isomorphism \( [130] \) that relates complex and real Clifford algebras to K-theory. It turns out \( [3] \) that the chiral fermions representing (or emergent from) the K-theory state (6.44) are in the fundamental representation of gauge group \( SU(2) \) or \( SU(3) \) that is coming from the unbroken Lorentz symmetry in extra dimensions or the holonomy group of Calabi-Yau \( n \)-folds. Through the minimal coupling of the (coarse-grained) fermion with \( SU(2) \) or \( SU(3) \) gauge fields in eq. (6.42), it was claimed in \( [3] \) that four (six)-dimensional NC \( U(1) \) instantons or Calabi-Yau 2 (3)-folds give rise to leptons (quarks). This phenomenon is very reminiscent of low-energy phenomenology via Calabi-Yau compactifications in string theory.

\(^{38}\) It is amusing to note that the Clifford algebra from a modern viewpoint can be thought of as a quantization of the exterior algebra, in the same way that the Weyl algebra is a quantization of the symmetric algebra. In this correspondence the “volume operator” \( \gamma_{d+1} = \gamma_1 \cdots \gamma_d \) in the Clifford algebra corresponds to the Hodge-dual operator \( * \) in the exterior algebra. Note also that any physical force is represented by a 2-form in the exterior algebra taking values in a classical Lie algebra and 2-forms are in one-to-one correspondence with Lorentz symmetry generators \( J^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \) in the Clifford algebra whose irreducible representations are spinors, in particular, chiral fermions in even dimensions. Hence the most engrossing connection is that the chiral fermions in the Clifford algebra correspond to self-dual instantons in the exterior algebra. Useful references for this point of view, for example, are Wikipedia (http://en.wikipedia.org/wiki/Clifford_algebra) and Ref. (14).

\(^{39}\) Any Lie group deformation retracts onto a maximal compact subgroup by the Iwasawa decomposition. In particular, we have homotopy equivalences \( GL(N, \mathbb{C}) \cong U(N) \), \( GL(N, \mathbb{R}) \cong O(N) \).
since a Calabi-Yau manifold serves as an internal geometry whose shapes and topology determine a detailed structure of the multiplets for elementary particles and gauge fields through the compactification. But it might be remarked that the Calabi-Yau manifolds in our case are non-compact and we do not yet know how to construct compact Calabi-Yau manifolds although we discussed a possible dynamical compactification mechanism earlier in this subsection.

6.4 Noncommutative field theory representation of AdS/CFT correspondence

Consolidating all the results obtained so far, here we want to argue that the AdS/CFT correspondence \cite{30, 31, 32} is a particular case of emergent gravity from NC U(1) gauge fields. But we will address only some essential features and any extensive progress along this approach will be reported elsewhere. The AdS/CFT correspondence implies that a wide variety of quantum field theories provide a nonperturbative realization of quantum gravity. In the AdS/CFT duality, the dynamical variables are large $N$ matrices and so gravitational physics at a fundamental level is described by NC operators. A field theory of gravity like Einstein’s general relativity defined in higher dimensions is a purely low-energy or large-distance approximation to some large $N$ gauge theory in lower dimensions where the relevant observables are approximately commutative. Conventional geometry and general relativity arise as collective phenomena, akin to fluid dynamics arising out of molecular dynamics. A key point to the AdS/CFT correspondence is that the dynamical variables belong to the $\mathcal{N} = 4$ vector multiplet in the adjoint representation of U(N) and so they are all $N \times N$ matrices. In particular, classical geometries or a supergravity limit appears in the planar limit $N \to \infty$. This is a motive why we have to stare again the equivalence between higher-dimensional NC U(1) gauge theory (6.11) and lower-dimensional U($N \to \infty$) Yang-Mills theory (6.12).

Keeping this picture in mind, let us consider four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group U(N). The $\mathcal{N} = 4$ super Yang-Mills theory is consisted only of a vector multiplet $(A_\mu, \lambda_i^a, \Phi_a)$, $i = 1, \cdots, 4$, $a = 1, \cdots, 6$ which contains 4-dimensional gauge fields $A_\mu$, four Majorana-Weyl gauginos $\lambda_i^a$ and six adjoint scalar fields $\Phi_a$ in the adjoint representation of gauge group U(N) \cite{45}. The action is given by

$$S = \int d^4x \text{Tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi_a D^\mu \Phi^a + \frac{g^2}{4} [\Phi_a, \Phi_a]^2 - \frac{i}{2} \bar{\lambda}_i \sigma^{\mu\nu} D_{\mu} \lambda_i ight. \\
+ \frac{g^2}{2} C_{i}^{\alpha} \lambda^i \Phi_a \Phi_a + \frac{g}{2} C_{\alpha}^{i,j} \bar{\lambda}_i \Phi_\alpha \Phi_j \right\}, \quad (6.45)$$

where $g$ is a gauge coupling constant and $C_{i}^{\alpha}$, $C_{\alpha}^{i,j}$ are Clebsch-Gordon coefficients related to the Dirac matrices $\rho^a$ for $SO(6)_R \cong SU(4)_R$. A crucial point for a sound progress is that the $\mathcal{N} = 4$ super Yang-Mills action (6.45) and supersymmetry transformations are a dimensional reduction of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory to four dimensions.
where $C_{10}$ and $C_{11}$ are the charge conjugation operators for 10 and 4 dimensions, respectively.

Consider a vacuum configuration of the action \((6.45)\):

\[
\langle \Phi_a \rangle_{\text{vac}} = b_{ab} y^b, \quad \langle A_\mu \rangle_{\text{vac}} = 0, \quad \langle \lambda^i \rangle_{\text{vac}} = 0.
\]

(6.47)

Assume that the vacuum expectation value $y^a \in A_N \ (N \to \infty)$ satisfies the Moyal-Heisenberg algebra

\[
[y^a, y^b] = i \theta^{ab} I_N \times N
\]

(6.48)

where the NC parameters $\theta^{ab} = (B^{-1})^{ab}$ are associated with the Poisson structure $\theta = \frac{1}{2} \theta^{ab} \frac{\partial}{\partial y^b} \wedge \frac{\partial}{\partial y^a} \in \Gamma(\Lambda^2 TM)$ of $M = \mathbb{R}^6$. Of course the commutation relation \((6.48)\) is meaningful only when we take the limit $N \to \infty$. It is obvious that the vacuum configuration \((6.47)\) in this limit is definitely a solution of the theory and preserves four-dimensional Lorentz symmetry. Now consider fluctuations of large $N$ matrices around the vacuum \((6.47)\):

\[
D_\mu(x, y) = \partial_\mu - i A_\mu(x, y), \quad D_a(x, y) \equiv -i \Phi_a(x, y) = -i(b_{ab} y^b + A_a(x, y)),
\]

(6.49)

\[
\Psi(x, y) = \left( \begin{array}{c} P^+_i \lambda_i^a \\ P^-_i \lambda_i \end{array} \right)(x, y),
\]

(6.50)

where we assumed that fluctuations also depend on vacuum moduli $y^a$. This procedure is exactly reverse to the previous matrix representation. Indeed, if we apply the matrix representation \((6.4)\) to fluctuations again, we can recover the original large $N$ gauge fields. Therefore let us introduce 10-dimensional coordinates $X^M = (x^\mu, y^a)$ and 10-dimensional connections defined by

\[
D_M(X) = \partial_M - i A_M(x, y) = (D_\mu, D_a)(x, y)
\]

(6.51)

whose field strength is given by

\[
F_{MN}(X) = i[D_M, D_N]_*= \partial_M A_N - \partial_N A_M - i[A_M, A_N]_*.
\]

(6.52)

As a consequence of the Moyal-Heisenberg vacuum \((6.48)\), according to the map between the NC $*$-algebra $A_\theta$ and the matrix algebra $A_N = \text{End}(\mathcal{H})$, large $N$ matrices in $\mathcal{N} = 4$ vector multiplet on $\mathbb{R}^{3,1}$ are mapped to NC gauge fields and their superpartners in $\mathcal{N} = 1$ vector multiplet on $\mathbb{R}^{3,1} \times \mathbb{R}_\theta^6$ where $\mathbb{R}_\theta^6$ is a NC space whose coordinate generators $y^a \in A_\theta$ obey the commutation relation \((6.48)\).
As we remarked before, the $\mathcal{N} = 4$ super Yang-Mills action (6.45) and the supersymmetry transformations (6.46) are obtained by a dimensional reduction of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory to four dimensions. Moreover the orderings in U(N) and NC U(1) gauge theories are compatible with each other as we verified in eq. (6.5). Hence it is straightforward to show that the 4-dimensional $\mathcal{N} = 4$ U(N) super Yang-Mills theory (6.45) can be organized into the 10-dimensional $\mathcal{N} = 1$ NC U(1) super Yang-Mills theory with the action

$$S = \int d^{10}X \left\{ -\frac{1}{4G_{YM}^2} (F_{MN} - B_{NM})^2 + i \frac{1}{2} \bar{\Psi} \Gamma^M D_M \Psi \right\} \quad (6.53)$$

where $B$-fields take the same form as eq. (6.11). The action (6.53) is invariant under $\mathcal{N} = 1$ supersymmetry transformations given by

$$\delta A_M = i \alpha \Gamma_M \Psi, \quad \delta \Psi = (F_{MN} - B_{MN}) \Gamma^{MN} \alpha. \quad (6.54)$$

We want to emphasize that the relationship between the 4-dimensional U(N) super Yang-Mills theory (6.45) and 10-dimensional NC U(1) super Yang-Mills theory (6.53) is not a dimensional reduction but they are exactly equivalent to each other. Therefore any quantity in lower-dimensional U(N) gauge theory can be transformed into an object in higher-dimensional NC U(1) gauge theory using the compatible ordering (6.5).

For example, a Wilson loop in U(N) gauge theory

$$W_N = \frac{1}{N} \text{Tr} P \exp \left( i \oint (A_\mu \dot{x}^\mu + \Phi_a \dot{y}^a) ds \right) \quad (6.55)$$

can be translated into a corresponding NC U(1) Wilson “line” defined by

$$\hat{W} = \frac{1}{V_6} \int d^6y P_\star \exp \left( i \oint_{\Gamma} (B_{a\dot{b}} \dot{y}^a \dot{y}^\dot{b} + A_M \dot{x}^M) ds \right) \quad (6.56)$$

where $V_6$ is a volume of extra 6-dimensional space. The gauge invariance requires the Wilson loop (6.55) to be closed. But the Wilson line (6.56) is defined in higher dimensions than (6.55) and so it need not be closed. It is enough to choose a path $\Gamma$ such that its projection onto $x$-space becomes a closed loop. (This possibility in $y$-space was already pointed out in [146].) Actually it perfectly makes sense because, in NC gauge theories, open Wilson lines constitute a set of gauge invariant operators [147, 148, 149]. $P$ denotes a path ordering which is taken only for loop variables $x^M(s)$ satisfying $\dot{x}^2 - \dot{y}^2 = 0$ to preserve supersymmetry (a minimal surface on the boundary of $AdS_5$) [146]. This path ordering with respect to large $N$ matrices recasts in the path ordering $P_\star$ with respect to star product. Then the phase factor $B_{a\dot{b}} \dot{y}^a \dot{y}^\dot{b}$ vanishes at leading order because of $\dot{y}^a = \frac{|\dot{x}^a|}{p} \dot{x}^a$ with $p^2 = \sum_{a=1}^6 \dot{y}^a \dot{y}^a$ but there will be a NC correction $\sim \frac{|\dot{x}|}{p}$ at next-to-leading order. So it will be interesting to see how NC $y$-space affects a singular behavior related to a correction coming from cusps or intersections of loops.

To recapitulate, one can regard the large $N$ matrices in $\mathcal{N} = 4$ supersymmetric gauge theory as operators in $\text{End}(\mathcal{H})$ acting on a separable Hilbert space $\mathcal{H}$ that is the Fock space of the Moyal-Heisenberg vacuum (6.48). An important point is that any field $\Phi(x) \in \mathcal{A}_N$
in the limit $N \to \infty$ on four-dimensional spacetime $\mathbb{R}^{3,1}$ can then be mapped to a NC field $\hat{\Phi}(x,y)$ defined on $(4+6)$-dimensional space $\mathbb{R}^{3,1} \times \mathbb{R}^{6}_\theta$ where $\mathbb{R}^{6}_\theta$ is a NC space defined by the Heisenberg algebra (6.43). In the end one sees that a large $N$ matrix $\Phi(x)$ can be represented by its master field $\hat{\Phi}(x,y) \in \mathcal{S}(\mathcal{C}^\infty(\mathbb{R}^{3,1}) \otimes \mathcal{A}_\theta)$ which is a higher-dimensional NC U(1) gauge field or its superpartner. Therefore the correspondence between the NC $\star$-algebra $\mathcal{A}_\theta$ and the matrix algebra $\mathcal{A}_N = \text{End}(\mathcal{H})$ leads to the equivalence between 4-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with U($N \to \infty$) gauge group and 10-dimensional $\mathcal{N} = 1$ supersymmetric NC U(1) gauge theory.

Since the large $N$ gauge theory (6.45) defined on a coherent vacuum (6.47) is mathematically equivalent to the NC U(1) gauge theory described by the action (6.53), we can try to derive a 10-dimensional gravity dual to the $\mathcal{N} = 4$ super Yang-Mills theory directly from the 10-dimensional NC U(1) gauge theory. First, if we turn off fermions, i.e. $\Psi = 0$, the equivalence is precisely the case with $m = 4$ and $n = 3$ in eqs. (6.11) and (6.12). Thereby the Lorentzian metric on an emergent 10-dimensional spacetime $\mathcal{M}$ is given by eq. (6.18). First consider a vacuum geometry with $A_M = 0$ and $\Psi = 0$ whose metric is given by

$$ds^2 = \lambda^2 (\eta_{\mu\nu} dx^\mu dx^\nu + dy^a dy^a).$$

(6.57)

So we see that the vacuum geometry for the $\mathcal{N} = 4$ super Yang-Mills theory should be conformally flat and its conformal class depends on the choice of volume form $\nu = d^4x \wedge \nu_6$.

There are two interesting cases which are conformally flat:

$$\nu_6 = d^6y \Rightarrow \lambda^2 = 1, \quad \mathcal{M} = \mathbb{R}^{9,1},$$

(6.58)

$$\nu_6 = \frac{d^6y}{\rho^2} \Rightarrow \lambda^2 = \frac{1}{\rho^2}, \quad \mathcal{M} = \text{AdS}_5 \times S^5,$$

(6.59)

where $\rho^2 = \sum_{a=1}^{6} y^a y^a$. What makes this difference? In order to pose this question, we need to address the uniqueness of the supersymmetric or BPS vacuum which is consistent with the isometry of the vacuum geometry (6.57), in particular, preserving $SO(6)_R$ Lorentz symmetry.

Naturally the supersymmetric or BPS vacuum preserving $SO(6)_R$ Lorentz symmetry is not unique. The vacuum (6.47) is only one of them. This vacuum recovers the first case (6.58)–10-dimensional flat Minkowski spacetime. Then what is the candidate for the second case (6.59)? An educated guess is to consider the stack of instantons at origin in the internal space $\mathbb{R}^{6}_\theta$ so that the directions picked up by instantons are randomly distributed in $SO(6) \cong SU(4)/\mathbb{Z}_2$. Note that $\mathbb{C}^3_\theta \cong \mathbb{Z}_{\geq 0}^3 \times \mathbb{T}^3$ has a natural lattice structure defined by the Fock space (3.2) whose lattice spacing is set by $\zeta = \sqrt{|\theta|}$. We noticed before that the Hermitian NC U(1) instanton obeying eq. (1.23) lives in the 8-dimensional vector space $\Lambda^2_8$ in eq. (8.23) picking up a particular value in the complex Grassmannian $\mathbb{C}P^3 = SU(4)/U(3)$ which is the set of complex 1-dimensional linear subspaces of $\mathbb{C}^4$. Thus we need to consider a small integer lattice around the origin on which instantons are compactly distributed taking all possible values in $\mathbb{C}P^3$ like a nucleus containing a lot of nucleons. Since NC U(1) instantons are BPS states, this superposition of instantons is probably allowed as a vacuum solution of the $\mathcal{N} = 4$ super Yang-Mills theory. (We noticed a closely related
work [150] which constructs a crystal configuration made of multi-Taub-NUT solutions which are equivalent to NC U(1) instantons in our case.) In the classical limit of infinitely many instantons, the stack of instantons at origin will look spherically symmetric without any preferred direction and preserve $SO(6)_R$ Lorentz symmetry. As we pointed out in footnote 37, these NC U(1) instantons are extended along $\mathbb{R}^{3,1}$ and are supposed to be D3-branes. See Fig. 2. Therefore, the configuration of the instanton lattice corresponds to the stack of $N$ D3-branes in the limit $N \to \infty$. Considering the fact that the near horizon geometry of the stack of $N$ D3-branes with $N \to \infty$ gives rise to the $AdS_5 \times S^5$ geometry [34], we speculate that the instanton lattice of infinitely many NC U(1) instantons obeying eq. (6.24) corresponds to the second case (6.58) for the $AdS_5 \times S^5$ geometry. It will be interesting to understand how our realization of the $AdS_5 \times S^5$ geometry is related to the picture in [151, 152] where $AdS_5 \times S^5$ space emerges as the geometry of the subspace of multi-instanton collective coordinates which dominates the path integral in the large $N$ limit.

If we include fluctuations generated by 10-dimensional NC U(1) gauge fields $A_M \neq 0$ and gauginos $\Psi \neq 0$ around a background geometry, either eq. (6.58) or (6.59), these excitations will deform the background geometry. For the bosonic excitations, the deformed geometry will be described by the metric (6.18). For example, the resulting geometry deformed by BPS states is expected to be similar to the bubbling geometry in Refs. [153, 154]. If we consider the fermionic excitations $\Psi(x,y)$, we can define fermionic vector fields in the exactly same way as the bosonic case (5.14):

$$\Psi^*_\alpha(f) \equiv -i[\Psi_\alpha(X), f]_* = \Psi^M_\alpha(X)\partial_M f + \cdots,$$

(6.60)

where $\alpha = 1, \cdots, 16$ is the spinor (Majorana-Weyl) index of $SO(9,1)$. Note that $\Psi^M_\alpha(X)$ carries both a vector index $M$ and a spinor index $\alpha$ of $SO(9,1)$ and so we can inter-
pret $\Psi_\alpha^M(X)$ as a 10-dimensional gravitino field. Combining bosonic and fermionic vector fields together, $(V_A^M, \Psi_\alpha^M(X))$ forms a 10-dimensional $\mathcal{N} = 1$ supergravity fields. But this result is not satisfactory to explain the dual supergravity theory on the $AdS_5 \times S^5$ background because we need the $\mathcal{N} = 2$ supergravity multiplet for this case. This failure may not be surprising for the following reason. We know that the global symmetry group of $\mathcal{N} = 4$ super Yang-Mills theory is $SU(2, 2|4)$ which contains the conformal symmetry and conformal supersymmetries $[34]$. From the field theory side, the presence of conformal supersymmetries results from the fact that the Poincaré supersymmetries and the special conformal transformations do not commute. Since both are symmetries, their commutator must also be a symmetry and these new generators are responsible for the conformal supersymmetries. From the gravity side, the conformal symmetry $SO(4, 2) = SU(2, 2)/\mathbb{Z}_2$ is the isometry of $AdS_5$. Thus we expect that such a symmetry enhancement for the background geometry (6.58) does not happen because the isometry of $\mathbb{R}^{9,1}$ is the usual Poincaré symmetry. So there is no supersymmetry enhancement either. Presumably the 10-dimensional $\mathcal{N} = 1$ supergravity multiplet is eligible in this case. Then the question is how to get the $\mathcal{N} = 2$ supergravity multiplet from the 10-dimensional $\mathcal{N} = 1$ vector multiplet. We conjectured before that the $AdS_5 \times S^5$ geometry arises from a nontrivial instanton condensate in vacuum which corresponds to a stack of infinitely many D3-branes. In this case we need to use a corresponding Poisson structure induced by the instanton condensate to define the supergravity fields, e.g. eq. (6.60), which has to explain why the Poincaré symmetry $ISO(3, 1)$ is enhanced to the conformal symmetry $SO(4, 2)$ in the instanton background. Unfortunately we do not know the Poisson structure describing the instanton condensate. It may be highly nontrivial to find it. Thus the question remains open.

7. Discussion

Emergence is the essential paradigm for our age (Emergence: The connected lives of ants, brains, cities, and software, Steven Johnson, 2001). The contemporary physics has revealed growing evidences that this paradigm can be applied to not only biology and condensed matter systems but also gravity and spacetime. It might be emphasized again that if gravity is emergent, then spacetime should be emergent too according to the general theory of relativity. However the concept of emergent spacetime is very stringent because spacetime plays a very fundamental role in physics; spacetime plays the background for everything to exist and develop. Without spacetime, nothing can exist and develop. A remarkable point in emergent gravity is that the spacetime background is just a condensate in vacuum that must be allowed for any existence to develop. If true, everything supported on the spacetime must simultaneously emerge together with emergent spaces for an internal consistency of the theory. We argued that matter fields such as leptons and quarks have to emerge too from topological objects in NC $\mathcal{A}_\theta$.

The basic picture of emergent gravity is that gravity and spacetime are collective manifestations of $U(1)$ gauge fields on NC spacetime. Thus, in this approach, any spacetime geometry is defined by NC $U(1)$ gauge fields. It is well-known $[155, 156]$ that the topology of NC $U(1)$ gauge fields is nontrivial and rich and NC $U(1)$ instantons $[68, 69]$ represent
the pith of their nontrivial topology. With this perspective we look into these topological solutions applying a deep connection (see the footnote 38) between K-theory, the homotopy groups of classical Lie groups, and Clifford algebras [142, 143]. In particular, the Atiyah-Bott-Shapiro isomorphism [130] suggests that these NC U(1) instantons formed along the extra four and six dimensions can be realized as leptons and quarks. Moreover NC U(1) instantons are stable excitations over the Moyal-Heisenberg vacuum (6.7) and so originally a part of spacetime geometry, namely, Calabi-Yau n-folds, according to the map (6.14). Consequently, we get a remarkable picture, if any, that leptons and quarks in the Standard model simply arise as stable localized geometries, which are topological objects in the defining algebra (NC \( \star \)-algebra) of quantum gravity. If it is not just a dream, emergent gravity may beautifully embody the emergent quantum mechanics through a novel unification between spacetime and matter fields [3].

We remark that the overall picture of emergent gravity is very parallel to string theory as observed in (footnote 3 of) [2] and (section 6 of) [59]. We may grasp an intuitive (though naive) picture for this parallelism. A Riemannian geometry defined by a pair \((M, g)\) defines an invariant distance in \(M\) which may be identified with a geodesic worldline of a particle propagating in the Riemannian manifold. Instead a symplectic geometry defined by a pair \((M, \Omega)\) measures an area embedded in \(M\) which may be identified with a worldsheet minimal surface of a string propagating in the symplectic manifold. If \(M\) is a Kähler manifold, i.e., the triple \((g, J, \Omega)\) is compatible, then there is a different, but closely related point of view. A (real) geodesic curve in a Riemannian manifold \(M\) is a path in \(M\). Then the minimal surface in \(M\) from the symplectic perspective is the complex analogue of the real geodesic, which is called a \(J\)-holomorphic curve introduced by Gromov [157].

To be precise, let \((M, J)\) be an almost complex manifold and \((\Sigma, j)\) be a Riemann surface. A smooth map \(u : \Sigma \rightarrow M\) is called \(J\)-holomorphic if the differential \(du : T\Sigma \rightarrow TM\) is a complex linear map with respect to \(j\) and \(J\):

\[
    du \circ j = J \circ du.
\]

This statement is equivalent to \(\bar{\partial} J u = 0\) where \(\bar{\partial} J u := \frac{1}{2}(du + J \circ du \circ j)\). Then it can be shown [158, 159] that, when a smooth map \(u : \Sigma \rightarrow M\) is \(J\)-holomorphic, we have the identity

\[
    \text{Area}_{g,J}(u) \equiv \int_{\Sigma} u^* \Omega = \frac{1}{2} \int_{\Sigma} |du|^2_J.
\]

This means that any \(J\)-holomorphic curve minimizes the “harmonic energy” \(S_P(u) = \frac{1}{2} \int_{\Sigma} |du|^2_J\) in a fixed homology class which is nothing but the Polyakov action in string theory. In other words, any \(J\)-holomorphic curve is a solution of the worldsheet Polyakov action. Using the \(J\)-holomorphic curves, Gromov proved a surprising non-squeezing theorem [157, 158, 159] stating that, if \(\phi : B_{2n}(r) \subset \mathbb{C}^n \rightarrow (M, \omega)\) is a symplectic embedding of the ball \(B_{2n}(r)\) of radius \(r\) in \((M, \omega)\) with the standard symplectic form \(\omega\) into the cylinder \(Z_{2n}(R) = B_2(R) \times \mathbb{R}^{2n-2}\), then \(r \leq R\).

We think that the non-squeezing theorem may be responsible for unexpected (nonlocal) effects in emergent gravity, e.g., the dark energy. First of all, it may be regarded as a classical manifestation of the UV/IR mixing in NC field theories [160]. Actually it was
conjectured in [2, 3] that the UV/IR mixing in NC spacetime may be the origin of the dark energy in our universe incarnated as the mystic energy (5.15).

We have constructed a global emergent geometry by gluing local data prescribed by NC gauge fields on local Darboux charts. From the Fedosov quantization approach, it corresponds to solving the cohomology of quantum Grothendieck connection $\mathcal{D}$. As we argued in section 6, the deformation quantization is not a complete quantization but an intermediate stage of a strict quantization with a Hilbert space. For the former case, the gluing basically means that two star products defined on nearby overlapping charts are Morita equivalent [22]. In this case, we can construct Hilbert spaces on local Darboux charts, e.g. eq. (6.2), on which dynamical NC gauge fields act. But, for the latter case, it is not obvious how to construct a “global” Hilbert space. Even it is not clear whether such a global Hilbert space exists or not. (This difficulty was also pointed out in [119].) Therefore, the strict quantization practically asks for the former approach. Then the question is: What is the operation in the Hilbert space corresponding to the gluing of local data on Darboux charts? A reasonable answer was put forward by M. Van Raamsdonk [17, 18]. Based on some well-understood examples of gauge/gravity duality, he argued for a picture of quantum gravity that the emergence of classically connected spacetimes is intimately related to the quantum entanglement of degrees of freedom in a nonperturbative description of the corresponding quantum system. This means that if a gravitational description with a large enough geometry emerges, the quantum system must develop highly entangled low-energy states. In this picture, quantum spacetimes are roughly analogous to manifolds obtained by gluing local patches together, with the physics in particular patches of spacetime described by particular quantum systems. Quantum entanglements between the non-perturbative degrees of freedom corresponding to different parts of spacetime play a critical role in connecting up the emergent spacetime. In our case each Darboux chart has its own Hilbert space and on overlaps they need to be connected each other by unitary transformations obeying the cocycle condition [22]. Therefore the Hilbert space representation of Morita equivalent star products may be closely related to the quantum entanglements building up emergent spacetimes.

Acknowledgments

We are very grateful to Joakim Arnlind, Kuerak Chung, Harald Dorn, Hoiil Kim, Sunggeun Lee, Albert Much, Raju Roychowdhury, Alexander Schenkel and Peter Schupp for helpful discussions. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No. 2011-0010597). This work was also supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number 2005-0049409.
A. Darboux coordinates and NC gauge fields

In this appendix we will formulate the deformation complex of symplectic structures discussed in section 2 in terms of open string theory. Through this analysis we want to emphasize the local nature of NC gauge fields by cultivating the Moser flow \( (2.35) \) to a local description in terms of Darboux coordinates.

Consider a general open string action defined by

\[
S = \frac{1}{4\pi\alpha'} \int_{\Sigma} |dX|^2 + \int_{\Sigma} B + \int_{\partial\Sigma} A
\]

(A.1)

where \( X : \Sigma \to M \) is a map from an open string worldsheet \( \Sigma \) to an ambient spacetime \( M \) and \( B(\Sigma) = X^*B(M) \) and \( A(\partial\Sigma) = X^*A(M) \) are pull-backs of spacetime fields to the worldsheet \( \Sigma \) and the worldsheet boundary \( \partial\Sigma \), respectively. The string action (A.1) respects two local gauge symmetries:

(I) \( \text{Diff}(M) \)-symmetry: \( X \mapsto X' = \phi(X) \in \text{Diff}(M) \),

(II) \( \Lambda \)-symmetry: \( (B, A) \mapsto (B - d\Lambda, A + \Lambda) \),

where the gauge parameter \( \Lambda \) is a one-form in \( M \). Note that the \( \Lambda \)-symmetry is present only when \( B \neq 0 \). The ordinary U(1) gauge symmetry, \( A \to A + d\lambda \), is a particular case with \( \Lambda = d\lambda \). When \( B = 0 \), the symmetry is reduced to the U(1) gauge symmetry. Therefore, in the presence of \( B \)-fields, the underlying local gauge symmetry is rather enhanced. Unfortunately the enhanced gauge symmetry due to \( B \)-fields had not been taken seriously. Only recently it starts to get some high interests under the names of generalized geometry, double field theory and higher spin gauge theory.

Let \( B \) be a symplectic structure on \( M \) and \( \theta \equiv B^{-1} \) be a Poisson structure on \( M \). Then the symplectic structure defines a bundle isomorphism \( B : T^*M \to TM \) by \( X \mapsto \Lambda = -\iota_X B \). This is the inverse of the anchor map we introduced in eq. (2.11). As a result, the \( B \)-field transformation, \( B \to B + d\iota_X B = (1 + \mathcal{L}_X)B \), can be understood as a coordinate transformation generated by the vector field \( X \in \Gamma(TM) \). In other words, the \( \Lambda \)-symmetry can be considered on par with diffeomorphisms which basically comes from the Darboux theorem or the Moser lemma in symplectic geometry. Thus the open string theory on a symplectic manifold \((M, B)\) admits two independent diffeomorphism symmetries. This enhancement of gauge symmetry already advocates a reason why there must be a radical change of physics as far as spacetime admits a symplectic structure, namely, a microscopic spacetime becomes NC.

A low energy effective field theory deduced from the open string action (A.1) is obtained by integrating out all the massive modes, keeping only massless fields which are slowly varying at the string scale \( l_s^2 = 2\pi\alpha' \). From the string action (A.1), one can infer that the data of low energy effective field theory will be specified by the triple \((g, B, A)\) where \( g \) denotes a Riemannian metric of the ambient spacetime \( M \). (We are ignoring a possible dilaton coupling in string theory since it does not play any crucial role in our argument.) Naturally the low energy effective theory should also respect the above local gauge symmetries in open string theory. Since the \( \Lambda \)-symmetry essentially acts as diffeomorphisms and so two local gauge symmetries should be treated on an equal footing, one
can expect that the low energy effective theory appears with the combination \((g + l_s^2 B, A)\) which signifies an advent of generalized geometry. The Λ-symmetry further constrains that the theory has to depend only on the gauge invariant quantity \(F = B + F\) where \(F = dA\). In the end the low energy effective theory of open strings will be determined by the quantity \(\mathcal{G} \equiv g + l_s^2 (B + F)\) and the action will be given by the obvious diffeomorphism invariant measure

\[
S_{DBI} = \mu_p \int d^{p+1}x \sqrt{\det \mathcal{G}}. \tag{A.2}
\]

The above action is called the Dirac-Born-Infeld (DBI) action.

Now let us apply the chain of symmetry transformations to the quantity \(G\):

\[
g + l_s^2 (B + F) \xrightarrow{\mathcal{F} = \mathcal{L}_B} g + l_s^2 (1 + \mathcal{L}_B)B \xrightarrow{\phi \in \text{Diff}(M)} \phi^*[g + l_s^2 (1 + \mathcal{L}_B)B] = G + l_s^2 B, \tag{A.3}
\]

where \(G = \phi^*(g)\). In the last operation of the above chain, the coordinate transformation \(\phi \in \text{Diff}(M)\) was chosen such that \(\phi^* = (1 + \mathcal{L}_B)^{-1} \approx e^{-\mathcal{L}_B}\). The Moser lemma (2.35) implies that the exponential map \(\phi^* \approx e^{-\mathcal{L}_B}\) can be identified with (the leading order of) the Moser flow (2.22). In terms of local coordinates \(\phi : y \mapsto x = x(y)\), the diffeomorphism between \(\mathcal{G}\) and \(\mathcal{G}' \equiv G + l_s^2 B\) in the map (A.3) reads as

\[
\mathcal{G}'_{\mu\nu}(y) = \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} \mathcal{G}_{ab}(x) \tag{A.4}
\]

where

\[
G_{\mu\nu}(y) = \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} g_{ab}(x). \tag{A.5}
\]

Consequently we get the equivalence between two different DBI actions

\[
\int d^{p+1}x \sqrt{\det[g + l_s^2 (B + F)]} = \int d^{p+1}y \sqrt{\det[G + l_s^2 B]}. \tag{A.6}
\]

Note that, though the coordinate transformation to a Darboux frame is defined only locally, the identity (A.6) holds globally because both sides are coordinate independent and so local Darboux charts in the right-hand side can be consistently glued together. As a result it is possible to obtain a global action for the right-hand side of eq. (A.6) by patching the local Darboux charts where the metric (A.4) will now be globally defined, i.e.,

\[
G_{\mu\nu}(x) = E^a_\mu(x) E^b_\nu(x) \delta_{ab}. \tag{A.7}
\]

Let us represent the coordinate transformation \(\phi : y \mapsto x = x(y) \in \text{Diff}(M)\) by eq. (2.40). Note that the dynamical variables on the right-hand side of eq. (A.6) are metric fields \(\{G_{\mu\nu}(y) : y \in M\}\) while they on the left-hand side are U(1) gauge fields \(\{F_{\mu\nu}(x) : x \in M\}\) in a specific background \((g, B)\). After substituting the expression (2.40) of dynamical coordinates into eq. (A.6), one can expand the right-hand side of eq. (A.6) around the background B-field. The result is given by [15]

\[
\int d^{p+1}y \sqrt{\det[G + l_s^2 B]} = \int d^{p+1}y \sqrt{\det(l_s^2 B)} \left(1 + \frac{l_s^4}{4} g^{ac.bd} \{C_a, C_b\}_\theta \{C_c, C_d\}_\theta + \cdots\right) \tag{A.8}
\]
where $\tilde{g}^{ab} = \frac{1}{l_s^2} (\theta g \theta)^{ab}$ is a constant open string metric and $C_a(y) = B_{ab}x^b(y)$ are covariant connections introduced in eq. (2.41). As was shown in eq. (2.43), $f_{ab} = \{C_a, C_b\} + B_{ab}$ are field strengths of symplectic gauge fields. Therefore we will get NC U(1) gauge theory from the right-hand side of eq. (A.8) after (deformation) quantization. In this respect, the equivalence (A.6) of DBI actions represents the SW map between commutative and NC gauge fields.

Some comments are in order to grasp some aspects of emergent gravity. Note that symplectic or NC gauge fields have been introduced to compensate local deformations of an underlying symplectic structure by U(1) gauge fields, i.e., the Darboux coordinates in $\phi : y \mapsto x = x(y) \in \text{Diff}(M)$ obey the relation $\phi^*(B + F) = B$. This local nature of NC gauge fields is also obvious from the identity (A.8) that they manifest themselves only in a locally inertial frame (in free fall) with the local metric (A.5). If the global metric (A.7) were used in the left-hand side of eq. (A.8), the identification of symplectic or NC gauge fields certainly became ambiguous. Nevertheless, it may be entertaining to see how the action looks like in terms of the global vector fields in (A.7). The same calculation as eq. (A.8) leads to the result

$$\int d^{p+1}y \sqrt{\det(G + l_s^2 B)} = \int d^{p+1}y \sqrt{\det(l_s^2 B)} \left( 1 + \frac{1}{4} J_{ab} J^{ab} + \cdots \right)$$

(A.9)

where

$$J^{ab} = \frac{\theta^{\mu\nu}}{l_s^2} E^a_\mu E^b_\nu = E^a_\mu J^{\mu\nu} E^b_\nu.$$  

(A.10)

Note that the frame fields in the expression (A.10) are the incarnation of symplectic or NC gauge fields in Darboux frames. But it may be illusory to find an imprint of symplectic or NC gauge fields in the expression (A.10). Rather they manifest themselves as a generic deformation of vacuum complex structure if we intent to interpret $J^{\mu\nu} = \theta^{\mu\nu}/l_s^2$ as a complex structure of $\mathbb{R}^{2n}$, i.e., $J^2 = -1$ and consider $E^a_\mu \approx \delta^a_\mu + h^a_\mu$. Therefore, with a layman’s conviction that the only consistent theory of dynamical metrics is general relativity (i.e. Einstein gravity) [161, 162], it should be a sensible idea to derive a spacetime geometry from NC gauge fields. The problem is how to patch together the local deformations to produce a global spacetime metric such as eq. (A.7). Of course the precise procedure is in general intricate. A useful mathematical device for patching the local information together to obtain a global theory is to use the theory of jet bundles, which is the subject reviewed in appendix C.

### B. Modular vector fields and Poisson homology

The aim of this appendix is to explain the property of the modular class of a Poisson manifold and of its quantization and to introduce a homology theory on Poisson manifolds, using differential forms, which is to a certain extent dual to the Poisson cohomology (2.10). Some of results will be stated without proofs because a rigorous proof may take us too far away from our purposes. Instead we will refer to useful references which must fill out the gap for the rigorous proof.
Let \( M \) be a Poisson manifold with Poisson tensor \( \theta \) with a trivial canonical class. Since the bundle \( \Lambda^d T^\ast M \) on \( M \) is trivial, a nowhere vanishing regular section of the canonical bundle always exists and so choose a smooth volume form \( \nu \). But the volume form \( \nu \) is defined up to a multiplication by any positive nonvanishing function \( a \in C^\infty(M) \):

\[
\nu \rightarrow \tilde{\nu} = a\nu. \tag{B.1}
\]

Take any \( f \in C^\infty(M) \). Due to dimensional reasons, the Lie derivative of the volume form \( \nu \) in the direction of the Hamiltonian vector \( X_f \) must be proportional to itself and thus there exists a smooth function \( \phi_\nu(f) \in C^\infty(M) \) such that

\[
\mathcal{L}_X f \nu = \phi_\nu(f) \nu. \tag{B.2}
\]

In local coordinates where the volume form is given by \( \nu = \upsilon(x) dx^1 \wedge \cdots \wedge dx^d \), it is easy to calculate the vector field \( \phi_\nu \) which is given by

\[
\phi_\nu = -X_{\log \upsilon} - \partial_{\mu} \theta^{\mu \nu} \frac{\partial}{\partial x^\nu}. \tag{B.3}
\]

From the definition (B.2), it is straightforward to check the following properties:

A: The map \( \phi_\nu : f \mapsto \phi_\nu(f) \) is a derivation of \( C^\infty(M) \), i.e., \( \phi_\nu(f \cdot g) = f \phi_\nu(g) + g \phi_\nu(f) \) for \( f, g \in C^\infty(M) \), and thus it is a vector field.

B: The map \( \phi_\nu \) is a derivation of \( \{-,-\}_\theta \), thus a Poisson vector field, i.e., \( d \theta \phi_\nu = 0 \).

C: Under the scale transformation (B.1), the vector field \( \phi_\nu \) changes as follows

\[
\phi_{\tilde{\nu}} = \phi_\nu + X_{-\log a}. \tag{B.4}
\]

The above properties can most easily be checked using the local expression (B.3). In particular, it is straightforward to check \( d g \phi_\nu = -[\phi_\nu, \theta]_S = 0 \) for which it may be necessary to use eq. (2.6). If \( \theta \) is a Poisson tensor, the vector field \( \phi_\nu \) in eq. (B.2) is called the modular vector field of \( \theta \) with respect to the volume form \( \nu \). These three facts together imply that the modular vector defines the first Poisson cohomology class \([\phi_\nu] \in H^1(\theta) \) of \( M \). This class is called the Poisson modular class. A Poisson manifold \((M, \theta)\) with \([\phi_\nu] = 0\) will be called unimodular. It is well known that any symplectic manifold \((M, \omega)\) is unimodular. (One may use the symplectic volume form \( \nu = \omega^n \) to prove that it gives the zero modular class.)

The Poisson modular class has an interesting interpretation, so-called the infinitesimal KMS condition. For a compactly supported function \( g \in C^\infty(M) \), the following chain of equalities holds:

\[
\int_M \{f, g\} \phi_\nu = \int_M (\mathcal{L}_X f g) \nu = \int_M (\mathcal{L}_X g \nu) - g \mathcal{L}_X f \nu
\]

\[
= \int_M (d(g \iota_{X_f} \nu) - g \mathcal{L}_X f \nu)
\]

\[
= -\int_M g \phi_\nu(f) \nu, \tag{B.5}
\]

where, when going from the second line to the third one, we used the Stokes’ theorem and the fact that the function \( g \) is compactly supported. Considering the integral with respect
to ν as a trace Trν on the associative algebra C∞(M), the above condition can be written as the form
\[ \text{Tr}_\nu \{ f, g \}_\theta = -\text{Tr}_\nu g \phi_\nu(f). \] (B.6)

Therefore the Poisson modular class measures the failure for the trace Trν to be also a trace at the Poisson algebra level. For a unimodular Poisson manifold, it is possible to find a volume form such that
\[ \int_M \{ f, g \}_\theta \nu = 0, \quad \forall f, g \in C^\infty(M) \] (B.7)
with at least one entry compactly supported. The existence of a Poisson trace is a nontrivial condition (there are many Poisson manifolds having no Poisson trace), although any symplectic manifold admits a trace. Therefore the modular class of a Poisson manifold is the obstruction to the existence of a density invariant under the flows of all Hamiltonian vector fields.

The modular vector fields are also related to the canonical homology of a Poisson manifold, given by the complex in which the chains are differential forms \( \Omega^\bullet(M) = \bigoplus_{k=0}^d \Omega^k(M) \). We define a homology operator \( \partial_\theta : \Omega^k(M) \to \Omega^{k-1}(M) \) by
\[ \partial_\theta = \iota_\theta \circ d - d \circ \iota_\theta \] (B.8)
where \( \iota_\gamma \) is the contraction with a k-vector field \( \gamma \in \mathcal{V}^k(M) \) with the rule \( \iota_{\gamma_1} \iota_{\gamma_2} = \iota_{\gamma_1 \wedge \gamma_2} \) for \( \gamma_i \in \mathcal{V}^k_i(M) \). It can be shown that \( \partial^2_\theta = 0 \). The corresponding homology of the complex \( (\Omega^\bullet(M), \partial_\theta) \) will be called the Poisson homology of \( M \) and denoted by \( H_\theta^\bullet(M) = \bigoplus_{k=0}^d H_k^\theta(M) = \text{Ker} \partial_\theta / \text{Im} \partial_\theta \). For example, it is easy to show that \( \{ f, g \}_\theta \) is an image of \( \partial_\theta \) and so the zeroth Poisson homology is represented by
\[ H_\theta^0(M) = C^\infty(M) / \{ C^\infty(M), C^\infty(M) \}_\theta. \] (B.9)
Hence the zeroth Poisson homology can be seen as dual to the space of Poisson traces. Suppose that \( M \) is oriented, so that we can identify densities with differential forms of top degree. A density \( \nu \) is thus a top-dimensional chain for Poisson homology. Its boundary is given by
\[ \partial_\theta \nu = -d(\iota_\theta \nu) = -\iota_{\phi_\nu} \nu \] (B.10)
which one can check by an explicit calculation using the result (B.3). Thus the modular field corresponds to the \( (d \text{ and } \partial_\theta \text{ exact}) \) \( (d - 1) \)-form \( \partial_\theta \nu = -\iota_{\phi_\nu} \nu \). As a result, a Poisson manifold \( (M, \theta) \) is unimodular if and only if there exists a volume form \( \nu \) such that \( \partial_\theta \nu = 0 \). This means that such a volume form defines a nontrivial cycle for the higher Poisson homology and so implies \( H_\theta^d(M) \neq 0 \).

It may be helpful to consider an instructive example [52]. Consider a regular Poisson structure on \( \mathbb{R}^2 \times \mathbb{S}^1 \), with coordinates \( (x, y, t) \), of the form
\[ \theta = \frac{\partial}{\partial y} \wedge \left( \frac{\partial}{\partial t} + g(x) \frac{\partial}{\partial x} \right) \] (B.11)
where \( g(x) = 0 \) just at \( x = 0 \). The symplectic leaves for this structure consist of the cylinder \( C \) defined by \( x = 0 \) and a family of planes which spiral around this cylinder. For
\[ \nu = dt \wedge dx \wedge dy, \] we have \( \iota_\nu = dx - g(x) dt, \) \( d(\iota_\nu) = -g'(x) dx \wedge dt \) and hence \( \phi_\nu = g'(x) \frac{\partial}{\partial y}. \)

Note that the modular vector field \( \phi_\nu \) in this case is coming from the second part of eq. (B.3) with \( \theta^{y\underline{t}} = g(x) \) and \( \theta^{y\underline{t}} = 1, \) which is not Hamiltonian unless \( g(x) \) is constant, so the modular class of this Poisson structure is nonzero \([52]\).

There is another nice formulation for modular vector fields. Let \( \nu \) be a volume form on a \( d \)-dimensional manifold \( M. \) Then there is a natural pairing \( \nu^\flat : \Omega^k(M) \to \Omega^{d-k}(M) \) between a \( k \)-vector field in \( \Omega^k(M) \) and a \((d - k)\)-form in \( \Omega^{d-k}(M) \) via the volume form defined by

\[ \nu^\flat(\Xi) = \iota_\Xi \nu, \quad (B.12) \]

that is, for a given \( k \)-vector field \( \Xi, \) there exists any \((d - k)\)-vector field \( \Pi \) such that \( \langle \nu^\flat(\Xi), \Pi \rangle = \langle \nu, \Xi \wedge \Pi \rangle. \) Thus \( \nu^\flat \) corresponds to the Hodge-* operator acting on polyvector fields in \( \Omega^\bullet(M). \) Define the operator \( \delta : \Omega^k(M) \to \Omega^{k-1}(M) \) by

\[ \delta = (\nu^\flat)^{-1} \circ d \circ \nu^\flat \quad (B.13) \]

which is dual to the codifferential operator \( d^\flat : \Omega^k(M) \to \Omega^{k-1}(M) \) in the Hodge-de Rham cohomology. Hence \( \nu^\flat \) intertwines \( \delta \) with \( d, \) namely, \( \nu^\flat \circ \delta = d \circ \nu^\flat \) which leads to the relation \( \nu^\flat \circ \delta^2 = d^2 \circ \nu^\flat. \) Since \( d^2 = 0, \) we also have \( \delta^2 = 0. \) In a local system of coordinates \((x^1, \cdots, x^d)\) with \( \nu = v(x) dx^1 \wedge \cdots \wedge dx^d \) and denoting \( \frac{\partial}{\partial x^\mu} = \zeta_\mu, \) we have the following simple formula for the divergence operator:

\[ \delta \Xi = v(x)^{-1} \frac{\partial^2}{\partial x^\mu \partial \zeta_\mu} (v(x) \Xi). \quad (B.14) \]

For example, \( \delta X = (\nu^\flat)^{-1} \circ \mathcal{L}_X \nu \equiv \text{div } X \) of a vector field \( X \) is nothing but the divergence of \( X \) with respect to the volume form \( \nu. \) Hence we call \( \delta \) in (B.13) the divergence operator.

Using the result (B.14), it is straightforward to show that the SN bracket (2.2) can be expressed by

\[ [P, Q]_S = (-)^q [\delta(P \wedge Q) - (\delta P) \wedge Q - (-)^{q-1} P \wedge \delta Q]. \quad (B.15) \]

This result immediately leads to the fact that the divergence operator is a graded derivation of the SN bracket:

\[ \delta [P, Q]_S = [P, \delta Q]_S + (-)^{q-1} [\delta P, Q]_S. \quad (B.16) \]

For a Poisson tensor \( \theta, \) it is then easy to derive from eq. (B.16) and the definition (B.13) the following properties

\[ \delta[\theta, \theta]_S = -2[\delta \theta, \theta]_S = 0, \quad (B.17) \]

\[ \mathcal{L}_{\delta \theta} \nu = d(\iota_{\delta \theta}) = d^2 \circ \iota_{\theta} \nu = 0. \quad (B.18) \]

The first property (B.16) implies that the vector field \( \delta \theta \) is a Poisson vector field and, according to eq. (B.18), it preserves the volume form \( \nu. \) Note that the divergence operator transforms under the scale transformation (B.1) as follows

\[ \delta \Xi = \delta \Xi + \langle \Xi, \log a \rangle |_S \quad (B.19) \]
where \( \tilde{\delta} \) is the divergence operator with respect to the volume form \( \tilde{\nu} = a\nu \). In particular, if \( \theta \) is a Poisson structure, then the transformation (B.19) is equal to

\[
\tilde{\delta}\theta = \delta\theta + X_{-\log a}.
\]  

Therefore we see that the vector field \( \delta\theta \) obeys all the properties (A-C) for the modular vector field. Indeed the straightforward calculation shows that \( \delta\theta = \phi\nu \) in eq. (B.3). This proves that the vector field \( \delta\theta \) is a modular vector field. Indeed the straightforward calculation shows that

\[
\delta\theta = \phi\nu
\]  

in eq. (B.3). This proves that the vector field \( \delta\theta \) is a modular vector field with respect to \( \nu \) and a Poisson manifold \((M,\theta)\) is unimodular if \( H^1_\theta(M) \ni [\delta\theta] = 0 \).

For a compact symplectic manifold \((M,\omega)\) of dimension \( d = 2n \) whose Poisson structure is given by \( \theta = \omega^{-1} \), there exist natural isomorphisms between the de Rham cohomology \( H^k_{dR}(M) \), Poisson cohomology \( H^k_\theta(M) \) and Poisson homology \( H^k_\theta(M) \) for \( k = 0, \cdots, d \) :

\[
H^k_\theta(M) \cong H^d_{dR} - k(M), \quad (B.21)
\]

\[
H^k_\theta(M) \cong H^d_{dR} - k(M), \quad (B.22)
\]

\[
H^k_\theta(M) \cong H^d_{dR} - k(M). \quad (B.23)
\]

Combining (B.21) with (B.22), we get the following Poincaré duality between Poisson homology and cohomology

\[
H^0_\theta(M) \cong H^{d-k}_\theta(M). \quad (B.24)
\]

However the above isomorphisms do not hold for non-symplectic Poisson manifolds and it is very hard to compute the Poisson (co)homology for them. But, if a Poisson manifold \((M,\theta)\) is unimodular, it turns out \([164, 165]\) that the Poincaré duality (B.24) still holds true.

A trace on a deformed algebra \( A_\theta = C^\infty(M)[[\hbar]] \) is by definition a linear functional \( \mu : C^\infty_c(M) \to \hbar^{-n}C[[\hbar]] \) on the compactly supported functions whose formal extension to \( C^\infty_c(M)[[\hbar]] \) satisfies the usual condition \( \mu(f \ast g) = \mu(g \ast f) \). For example, when \((M = \mathbb{R}^{2n},\omega)\) is a symplectic manifold, the natural trace coming via the Weyl correspondence from the trace of operators is

\[
f \mapsto \mu(f) = (2\pi\hbar)^{-n} \int_M f \omega^n. \quad (B.25)
\]

Therefore the trace \( \mu \) on a Poisson manifold can exist only when the Poisson manifold is unimodular obeying (B.7). A star product is called in \([50]\) strongly closed if the functional (B.25) still defines a trace. The existence of a strongly closed star product on an arbitrary symplectic manifold was shown in \([51]\). Moreover the classification in \([166]\) implies that every star product on a symplectic manifold is equivalent to a strongly closed star product and the set of traces for a star product on a symplectic manifold forms a 1-dimensional module over \( C[[\hbar]] \), so the trace is essentially unique. The existence of a strongly closed

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40Basically the trace by definition has to preserve the physical dimension of operators. That is the reason why there is \( \hbar^{-n} \) in the trace (B.25). If the symplectic structure \( \omega = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu \) refers to a plain Euclidean space \( \mathbb{R}^{2n} \) rather than a particle phase space, it is not necessary to include \( \hbar^n \) in the denominator of the trace (B.25) because \( [\omega] \) in this case is dimensionless in itself, i.e., \( [B_{\mu\nu}] = (\text{length})^{-2} \). Nevertheless it may be convenient to keep the deformation parameter \( \hbar \) to control the order of deformations.
star product was generalized in [167, 168] to the case of any unimodular Poisson manifolds with an arbitrary volume form. In particular, it was shown in [167] that the divergence of a Poisson bivector field (like the second term in eq. (B.3)) is involved with tadpoles (edges with both ends at the same vertex) in Feynmann diagrams and the anomalous terms vanish for divergence free Poisson bivector fields. Hence, if we insist on using traces as a NC version of integration, then we are forced to restrict ourselves to the quantization of unimodular Poisson manifolds. In other words, it is unreasonable to expect a trace on $A_\theta$ if we start with a non-unimodular Poisson manifold.

Now our concern is how the modular vector fields can be lifted to a (deformation) quantization. Note that by definition (the properties A-C below (B.3)) the modular vector field is a derivation of $C^\infty(M)$ (Property A) as well as a Poisson vector field (Property B). Therefore there is no essential obstruction for the lift of the modular vector fields to derivations of a $\star$-algebra if and only if a Poisson manifold is unimodular which belongs to a particular class of generic Poisson manifolds. For example, for a $\star$-product on a symplectic manifold which is always unimodular, any symplectic vector field $X$ extends to a derivation of the quantum algebra $A_\theta = (C^\infty(M)[[h]], \star)$. If $X$ is a Hamiltonian vector field, it can be chosen as an inner derivation which is precisely the case in eq. (5.117) or (5.118). In general any symplectic vector field can be quantized as a derivation of the quantum algebra $A_\theta$. See, e.g., Lemma 8.4 in [82] for the proof. For a general Poisson manifold, the quantization problem of the modular class in the formal case was recently obtained by Dolgushev in [168]. In particular, it was shown [168] that, if $D_X$ is a derivation of $A_\theta$ constructed from a modular vector field $X$ of Poisson structure $\theta$ via the Kontsevich’s formality theorem, then the modular (outer) automorphism of $A_\theta$ is generated by the exponential map $\exp(D_X)$ up to an inner automorphism.

C. Jet bundles

In this appendix we briefly review jet bundles which have been often used in this paper. We refer to [53, 54, 55] for more detailed exposition.

Suppose that $\pi : E \to M$ is a fiber bundle with fiber $F$. Introduce local coordinates $x^\mu$ for $M$ and $(x^\mu, z^i)$ for $E$ with coordinates $z^i$ of its standard fiber $F$ and let $\Gamma_p(E)$ denote the set of all local sections whose domain contains $p \in M$. We say that two sections $\sigma$ and $\sigma'$ of $\pi$ have a first-order contact at a point $x \in M$ if $\sigma^i(x) = \sigma'^i(x)$ and $\partial_\mu \sigma^i(x) = \partial_\mu \sigma'^i(x)$. This defines an equivalence relation on the space of local sections. They are called the first order jets $j^1_x \sigma$ of sections at $x$. One can justify that the definition of jets is coordinate independent. The set of all the 1-jets of local sections of $E \to M$, denoted by $J^1(E)$, has a natural structure of a differentiable manifold with respect to the adopted coordinates $(x^\mu, z^i, z^i_\mu)$ such that
\begin{equation}
  z^i(j^1_x \sigma) = \sigma^i(x), \quad z^i_\mu(j^1_x \sigma) = \partial_\mu \sigma^i(x).
\end{equation}
We call $z^i_\mu$ the jet coordinates. They posses the transition functions
\begin{equation}
  z^i_\mu = \frac{\partial x^\lambda}{\partial x'^\mu}(\partial_\lambda + z^j_\lambda \partial_j)z^i
\end{equation}
with respect to the bundle morphism $z^\mu = \phi^\mu(x, z)$, $x^\mu = \phi^\mu(x)$. The jet manifold $J^1E$ admits the natural fibrations

$$\pi^1 : J^1E \to M \text{ by } j^1_x \sigma \mapsto x,$$

$$\pi^1_0 : J^1E \to E \text{ by } j^1_x \sigma \mapsto \sigma(x).$$

Any section $\sigma$ of $E \to M$ has the jet prolongation to the section of the jet bundle $J^1E \to M$ defined by

$$(J^1\sigma)(x) = j^1_x \sigma, \quad z^i_\mu \circ J^1\sigma = \partial_\mu \sigma^i(x).$$

An important fact is that there is a one-to-one correspondence between the connections on a fiber bundle $E \to M$ and the global sections of the affine jet bundle $J^1E \to E$, as will be discussed later.

The notion of first jets $j^1_x \sigma$ of sections of a fiber bundle can naturally be extended to higher order jets. Let $I = (I_1, \cdots, I_n)$ be a multi-index (an ordered $n$-tuple of integers) and $\partial_I \equiv \tfrac{\partial^{|I|}}{\partial x^I}$, where $|I| = \sum_{i=1}^n I_i$. Define the local sections $\sigma, \sigma' \in \Gamma_p(E)$ to have the same $k$-jet at $p \in M$ if

$$\tfrac{\partial^{|I|}\sigma}{\partial x^I}|_p = \tfrac{\partial^{|I|}\sigma'}{\partial x^I}|_p, \quad 0 \leq |I| \leq k.$$  

A $k$-jet is an equivalence class under this relation and the $k$-jet with representative $\sigma$ is denoted by $j^k_p \sigma$. The holonomic sections $j^k_p \sigma$ are called $k$th-order jet prolongations of sections $\sigma \in \Gamma_p(E)$. In brief, one can say that sections of $E \to M$ are identified by the $k + 1$ terms of their Taylor series at points of $M$. The particular choice of coordinates does not matter for this definition. In this respect, jets may also be seen as a coordinate free version of Taylor expansions. The $k$-order jet manifold $J^kE$ is then defined by the set of all $k$-jets $j^k_x \sigma$ of all sections $\sigma$ of $\pi$. Therefore the points of $J^kE$ may be thought of as coordinate free representations of $k$th-order Taylor expansions of sections of $E$. The $k$-jet manifold $J^kE$ is endowed with an atlas of the adapted coordinates

$$x^\mu, z^i_1, \quad z^i_j \circ J^k \sigma = \partial_I \sigma^i(x), \quad 0 \leq |I| \leq k,$$

$$z^i_{\mu + I} = \frac{\partial x^\lambda}{\partial x^\mu} d^i_\lambda z^i_I,$$

where the symbol $d_\mu$ stands for the higher order total derivative defined by

$$d_\mu = \partial_\mu + \sum_{0 \leq |I| \leq k-1} z^i_{\mu + I} \partial_i^I, \quad d_\mu' = \frac{\partial x^\lambda}{\partial x'^\mu} d^\lambda$$

and $\partial_i^I \equiv \tfrac{\partial}{\partial z^i_I}$. We call the coordinates in eq. (C.7) the natural coordinates on the jet space.

There is a natural projection from $J^2E$ to $J^1E$, the truncation $\pi_1$, characterized by dropping the second-order terms in the Taylor expansion. In general, one has the natural truncations $\pi^m_n : J^nE \to J^mE$ for all $0 < m < n$ and $\pi^n : J^nE \to M$ by

$$\pi^m_n : j^m_x \sigma \mapsto j^m_x \sigma, \quad \pi^n : j^n_x \sigma \mapsto x.$$  

(C.10)
The coordinates \( (C.7) \) are compatible with the natural surjections \( \pi^n_m \) \((n > m)\) which form the composite bundle

\[
\pi^n : J^n E \xrightarrow{\pi^n_{n-1}} J^{n-1} E \xrightarrow{\pi^{n-2}_{n-2}} \ldots \xrightarrow{\pi^1_{1}} E \xrightarrow{\pi} M
\]

with the properties

\[
\pi^n_m \circ \pi^n_k = \pi^n_m, \quad \pi^k \circ \pi^n_m = \pi^n.
\]

(C.12)

The composite bundle \((C.11)\) is constructed by defining \( J^{k+1} E \) as the first jet bundle of \( J^k E \) over \( M \) and iterating this construction. Then each jet bundle \( J^{k+1} E \) becomes a vector bundle over \( J^k E \) and a fiber bundle over \( E \). The inductive limit \( E \equiv J^\infty E \) of the inverse sequence of eq. \((C.11)\) is defined as a minimal set such that there exist surjections

\[
\pi^\infty : E \to M, \quad \pi^\infty_0 : E \to E, \quad \pi^\infty_k : E \to J^k E
\]

obeying the relations \( \pi^\infty_n = \pi^k_n \circ \pi^\infty_k \) for all admissible \( k \) and \( n < k \). One can think of elements of \( E \) as being infinite order jets of sections of \( \pi : E \to M \) identified by their Taylor series at points of \( M \). Therefore a fiber bundle \( E \) is a strong deformation retract of the infinite order jet manifold \( E \). A bundle coordinate atlas \( \{U_E, (x^\mu, z^i)\} \) of \( E \to M \) provides \( E \) with the manifold coordinate atlas

\[
\{(\pi^\infty_0)^{-1}(U_E), (x^\mu, z^i)\}_{0 \leq |I|}, \quad z^i_{\mu+I} = \frac{\partial x^\lambda}{\partial x'^\mu} d_\lambda z^i_I.
\]

(C.14)

The tangent vectors to the fibers \( F \) form a vector subbundle of \( TE \) (because they have good transformational character) and it is called the vertical vector space denoted by \( T^\perp E \). Note that \( \Upsilon \) is tangent to the fiber if and only if \( \pi_* \Upsilon = 0 \), hence \( T^\perp E = \ker \pi_* \). But, although vectors tangent to \( M \) locally complement the vertical vector space, they do not transform properly on \( M \). Thus \( TM \) is not a subbundle of \( TE \). A nonlinear connection needs to be introduced to care a selection of complementary vector bundle to \( T^\perp E \) in \( TE \). This bundle is usually called the horizontal vector space and denoted by \( T^\parallel E \). The nonlinear connection may be defined via the short exact sequence of vector bundles over \( E \):

\[
0 \to T^\perp E \to TE \xrightarrow{\pi_*} \pi^* TM \to 0
\]

(C.15)

where \( \pi^* TM \) is the pull-back bundle \( E \times_M TM \) of \( TM \) onto \( E \). A nonlinear connection is a splitting of this short exact sequence: \( TE \cong T^\perp E \oplus T^\parallel E \) where \( T^\parallel E \cong \pi^* TM \).

A vector field \( X \) on a fiber bundle \( \pi : E \to M \) is called projectable if it projects onto a vector field on \( M \), i.e., there exists a vector field \( \tau \) on \( M \) such that

\[
\tau \circ \pi = T \pi \circ X.
\]

(C.16)

A projectable vector field takes the coordinate form

\[
X = X^\mu(x) \partial_\mu + X^i(x, z) \partial_i, \quad \tau = X^\mu(x) \partial_\mu.
\]

(C.17)

Its flow generates a local one-parameter group of automorphisms of \( E \to M \) over a local one-parameter group of diffeomorphisms of \( M \) whose generator is \( \tau \). A projectable vector
field is called vertical if its projection onto \( M \) vanishes, i.e., if it lives in \( T^\perp E \). Any projectable vector field \( X \) has the following \( k \)-order jet prolongation to a vector field on \( J^k E \):

\[
J^k X = X^\mu \partial_\mu + X^i \partial_i + \sum_{1 \leq |I| \leq k} \left( d_I (X^i - z^i_{\mu} X^\mu) + z^i_{\mu+1} X^\mu \right) \partial^I_i
\]  

(C.18)

where we used the compact notation \( d_I = d_{\mu_k} \circ \cdots \circ d_{\mu_1} \). If \( X \) is a vertical vector field on \( E \rightarrow M \), i.e., \( X^\mu = 0 \), one can see that \( J^k X \) is also a vertical vector field on \( J^k E \rightarrow M \). Indeed any vector field \( \rho_k(X) \equiv J^k X \) on \( J^k E \) admits the canonical decomposition

\[
\rho_k(X) = X_H + X_V
\]

(C.19)

over \( J^{k+1} E \) into the horizontal and vertical parts. There are also canonical bundle monomorphisms (embeddings) over \( J^k E \):

\[
\eta_k : J^{k+1} E \rightarrow T^* M \bigotimes_{J^k E} T J^k E,
\]

(C.20)

\[
\psi_k : J^{k+1} E \rightarrow T^* J^k E \bigotimes_{J^k E} T^\perp J^k E,
\]

(C.21)

The one-forms

\[
\psi^I_i \equiv dz^i_I - z^i_{\mu+1} dx^\mu
\]

(C.22)

in eq. (C.21) are called the local contact forms. A differential one-form \( \psi \) on the space \( J^k E \) is called a contact form if it is pulled back to the zero form on \( M \) by all prolongations. In other words, the one form \( \psi \in T^* J^k E \) is a contact form if and only if, for every open submanifold \( U \subset M \) and every \( \sigma \in \Gamma_p(E) \),

\[
(j^k_{x+1} \sigma)^* \psi = 0.
\]

(C.23)

Thus contact forms provide a characterization of the local sections of \( \pi^{k+1} \) which are prolongations of sections of \( \pi \). The distribution on \( J^k E \) generated by the contact forms is called the Cartan distribution.

It is also possible to consider the limit \( k \rightarrow \infty \) in eq. (C.13) for vector fields and differential forms on a jet bundle \( J^k E \rightarrow M \). Let \( (x^\mu, z^i_I; U) \) be a standard local chart of \( U \subset E \) and denote by \( \mathfrak{F}(U) := C^\infty(E) \) the algebra of functions on \( U \). Smooth functions on \( U \) may be defined through some finite order \( J^n E \):

\[
F : U \rightarrow \mathbb{R}
\]

(C.24)

by \( F = f \circ \pi_k^\infty \) for some smooth \( f : J^k E \rightarrow \mathbb{R} \). Let us write \( F(x, z) \in \mathfrak{F}(U) \) for a function on \( U \). The tangent bundle \( TE \) of \( E \) is the projective limit of \( \{(\pi^k)^* T J^k E\} \). And the space
\( \Gamma(T\mathcal{E}) \) of the sections of \( T\mathcal{E} \) is by definition the projective limit of \( \{ \Gamma((\pi^n_k)^*T^kJ^kE); n \geq k \} \). \( \Gamma(T\mathcal{E}) \) acts on the algebra \( \mathfrak{G}(\mathcal{U}) \) as derivations in the obvious way and hence carries a natural Lie algebra structure.

The exterior derivative on \( \mathfrak{G}(\mathcal{U}) \) is defined as usual and the result is given by

\[
dF = d_\mu F dx^\mu + \partial_i^J F \psi_i^J \\
\equiv d_H F + d_V F
\]

(C.25)

for \( F(x, z) \in \mathfrak{G}(\mathcal{U}) \). In general, the bundle of \( p \)-forms \( \wedge^p T^* \mathcal{E} \) is the injective limit of \( \{ (\pi^k)^* \wedge^p T^*J^kE \} \) and the space \( \Omega^p \mathcal{E} \) of \( p \)-forms consists of its sections. Hence a basic differential \( p \)-form on \( \mathcal{E} \) has a local coordinate expression

\[
F(x, z) dx^\mu_1 \wedge \cdots \wedge dx^\mu_r \wedge \psi^i_1 \wedge \cdots \wedge \psi^i_s
\]

(C.26)

with \( r + s = p \). A general \( p \)-form is a finite sum of such terms. Define the space \( \Omega^{r,s} \mathcal{E} \) of forms of type \( (r, s) \) to be all linear combinations of the form (C.26). Similarly to eq. (C.25), for \( \omega \in \Omega^{r,s} \mathcal{E} \), there exists a splitting

\[
d\omega = d_H \omega + d_V \omega
\]

(C.27)

where \( d_H \omega \in \Omega^{r+1,s} \mathcal{E} \) and \( d_V \omega \in \Omega^{r,s+1} \mathcal{E} \). In particular, consider a local contact form \( \psi^i_1 \in \Omega^{0,1} \mathcal{E} \). Then \( d\psi^i_1 = -\psi^i_{\mu+1} dx^\mu \) or in succinct form \( \delta \psi^i_1 = 0 \) and so \( d_V \psi^i_1 = 0 \). By virtue of \( d^2 = 0 \) we have \( d_H^2 = d_V^2 = d_H d_V + d_V d_H = 0 \). Thus, like the Dolbeault differential complex on a complex manifold, there exists a bicomplex \( (\Omega^* \mathcal{E}, d_H, d_V) \) of differential forms with the bigrading on \( \Omega^* \mathcal{E} = \bigoplus_{r,s} \Omega^{r,s} \mathcal{E} \).

A connection on a fiber bundle \( \pi: \mathcal{E} \rightarrow M \) is defined as a linear bundle monomorphism \( \Gamma: \mathcal{E} \times_M TM \rightarrow T\mathcal{E} \) over \( E \) by

\[
\Gamma: \dot{x}^\mu \partial_\mu \mapsto \dot{x}^\nu (\partial_\nu + \Gamma^i_\mu \partial_i)
\]

(C.28)

which splits the exact sequence (C.13), i.e., \( \pi_\sigma \circ \Gamma = \text{Id}_{E \times_M TM} \). Any connection in a fiber bundle defines a covariant derivative of sections. If \( \sigma: M \rightarrow E \) is a section, its covariant derivative is defined by

\[
\nabla^\Gamma \sigma = D_{\Gamma} \circ J^1 E: M \rightarrow T^* M \times T^\perp \mathcal{E}, \\
\nabla^\Gamma \sigma: (\partial_\mu \sigma^i - \Gamma^i_\mu \circ \sigma) dx^\mu \otimes \partial_i,
\]

(C.29)

where \( D_{\Gamma} \) is the covariant differential relative to the connection \( \Gamma \) defined by

\[
D_{\Gamma} : J^1 E \rightarrow T^* M \times T^\perp \mathcal{E}, \\
D_{\Gamma} = (\dot{\sigma}^i - \Gamma^i_\mu) dx^\mu \otimes \partial_i.
\]

(C.30)

A section \( \sigma \) is called an integrable section of a connection \( \Gamma \) if it belongs to the kernel of the covariant differential \( D_{\Gamma} \), i.e.,

\[
\nabla^\Gamma \sigma = 0 \quad \text{or} \quad J^1 \sigma = \Gamma \circ \sigma.
\]

(C.31)
The connection $\Gamma$ can also be seen as a global section $\Gamma : E \rightarrow J^1E$ of the jet bundle $\pi_0^1 : J^1E \rightarrow E$ satisfying
\[ \pi_0^1 \circ \Gamma = \text{Id}_E, \] (C.32)
whose coordinate representation is given by
\[ (x^\mu, z^i, z^i_\mu) \circ \Gamma = (x^\mu, z^i, \Gamma^i_\mu). \] (C.33)
Then the generalization to all higher order jets is obvious. A $k$-th order connection in a fiber bundle $E \rightarrow M$ is a section $\Gamma : E \rightarrow J^kE$ which satisfies
\[ \pi_0^k \circ \Gamma = \text{Id}_E. \] (C.34)
For an integrable section $\sigma \in \Gamma(E)$, according to eq. (C.33), the connection $\Gamma$ is given by
\[ \Gamma_\mu^i \circ \sigma = \partial_\mu \sigma^i(x) \] (C.35)
or for $\sigma \in \Gamma(J^kE)$, in general, according to eq. (C.4),
\[ \Gamma_\mu^i \circ \sigma = \partial_I \sigma^i(x). \] (C.36)
Now we introduce a flat connection $\nabla^H$ on $\pi^\infty : E \rightarrow M$. For each $y \in E$ define a linear subspace $H_y$ of $T_yE$ by
\[ H_y = \{d_H(\partial_\mu) = \partial_\mu + \Gamma^i_{\mu+I} \partial_I^i \equiv \nabla_\mu \} \] (C.37)
where $\nabla_\mu := d_\mu$ is the alias of the total derivative (C.9) and
\[ \Gamma^i_{\mu+I} = z^i_{\mu+I}. \] (C.38)
The property (C.38) implies that $H_y = \text{Im } d_\mu \sigma_\infty$ for some $\sigma_\infty \in \Gamma(E)$ where $d_\mu \sigma_\infty : T_yM \rightarrow T_yE$ with $x = \pi^\infty(y)$ is the differential. Hence $H = \bigcup_y H_y$ is a subbundle of $T^0E$. Because $d_\mu^2 = 0$, one can see that $\nabla_\mu$ is a flat connection, i.e., $[\nabla_\mu, \nabla_\nu] = 0$. Let $H^\perp = \bigcup_y H^\perp_y$ be the conormal bundle, where
\[ H^\perp_y = \{ \omega \in T^*_yE : \omega|_{H_y} = 0 \} \] (C.39)
for $y \in U \subset E$. One can easily check that $\psi^\perp_I(d_\mu) = 0$ and $d_\mu^\perp(d_\lambda) = \delta_\lambda^\mu$ for a frame $\{dx^\mu, \psi^\perp_I\}$ of $T^U$. Therefore, $H^\perp_y$ is spanned by $\{\psi^\perp_I(y)\}$, i.e., $H^\perp \subset \Omega^0E$. For an integrable (or a flat) section $\sigma_\infty \in \Gamma(E)$, the connection (C.38) can be written as
\[ \Gamma^I_\sigma \circ \sigma_\infty \equiv \sigma^I_\infty \circ \sigma_\infty = \partial_I \sigma_i(x) \] (C.40)
which means that $\sigma^\infty \psi^\perp_I = d_\sigma^\perp_I - \sigma^\infty_{\mu+I} dx^\mu = 0$ or $\sigma^\infty(I(H^\perp)) = 0$.

The flat connection $\nabla^H$ lifts $X \in \Gamma(TM)$ up to $\tilde{X} \in \Gamma(H) \subset \Gamma(TE)$. Denote this map $\tau : \Gamma(TM) \rightarrow \Gamma(TE)$. Note that $\tilde{X}$ is uniquely characterized by
\[ \tilde{X} = d_H(X) = X^\mu(x) \nabla_\mu \] (C.41)
where $X = X^\mu(x) \partial_\mu \in \Gamma(TM)$. Then one can easily check that, for $X, Y \in \Gamma(TM)$,
\[ [\tilde{X}, \tilde{Y}] = [X, Y]^\mu(x) \nabla_\mu = [X, Y]^\perp. \] (C.42)
This means that $\tau : \Gamma(TM) \rightarrow \Gamma(TE)$ is a Lie algebra homomorphism. For example it is an immediate consequence of (C.42) that $[\nabla_\mu, \nabla_\nu] = 0$. It should be the case because $d_\mu^2 = \frac{1}{2} dx^\mu \wedge dx^\nu [d_\mu, d_\nu] = 0$ where $d_\mu := \nabla_\mu$. 

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