S.O.S. APPROXIMATION OF POLYNOMIALS, NONNEGATIVE ON A REAL ALGEBRAIC SET

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Abstract. With every real polynomial \( f \), we associate a family \( \{ f_{\epsilon, r} \} \) of real polynomials, in explicit form in terms of \( f \) and the parameters \( \epsilon > 0, r \in \mathbb{N} \), and such that \( \| f - f_{\epsilon, r} \|_1 \to 0 \) as \( \epsilon \to 0 \).

Let \( V \subset \mathbb{R}^n \) be a real algebraic set described by finitely many polynomials equations \( g_j(x) = 0, j \in J \), and let \( f \) be a real polynomial, nonnegative on \( V \). We show that for every \( \epsilon > 0 \), there exist nonnegative scalars \( \{ \lambda_j(\epsilon) \} \) such that, for all \( r \) sufficiently large,

\[
f_{\epsilon, r} + \sum_{j \in J} \lambda_j(\epsilon) g_j^2 \quad \text{is a sum of squares.}
\]

This representation is an obvious certificate of nonnegativity of \( f_{\epsilon, r} \) on \( V \), and very specific in terms of the \( g_j \) that define the set \( V \). In particular, it is valid with no assumption on \( V \). In addition, this representation is also useful from a computational point of view, as we can define semidefinite programming relaxations to approximate the global minimum of \( f \) on a real algebraic set \( V \), or a semi-algebraic set \( K \), and again, with no assumption on \( V \) or \( K \).

1. Introduction

Let \( V \subset \mathbb{R}^n \) be the real algebraic set

\[
V := \{ x \in \mathbb{R}^n \mid g_j(x) = 0, \quad j = 1, \ldots, m \},
\]

for some family of real polynomials \( \{ g_j \} \subset \mathbb{R}[x](= \mathbb{R}[x_1, \ldots, x_n]) \).

The main motivation of this paper is to provide a characterization of polynomials \( f \in \mathbb{R}[x] \), nonnegative on \( V \), in terms of a certificate of positivity. In addition, and in view of the many potential applications, one would like to obtain a representation that is also useful from a computational point of view.

In some particular cases, when \( V \) is compact, and viewing the equations \( g_j(x) = 0 \) as two opposite inequations \( g_j(x) \geq 0 \) and \( g_j(x) \leq 0 \), one may obtain Schmüdgen’s sum of squares (s.o.s.) representation \( [17] \) for \( f + \epsilon (\epsilon > 0) \), instead of \( f \). Under an additional assumption on the \( g_j \)'s that define \( V \), the latter representation may be even refined to become Putinar

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and Jacobi and Prestel s.o.s. representation, that is, \( f + \epsilon \) can be written

\[
(1.2) \quad f + \epsilon = f_0 + \sum_{j=1}^{m} f_j g_j,
\]

for some polynomials \( \{f_j\} \subset \mathbb{R}[x] \), with \( f_0 \) a s.o.s. Hence, if \( f \) is nonnegative on \( V \), every approximation \( f + \epsilon \) of \( f \) (with \( \epsilon > 0 \)) has the representation (1.2). The interested reader is referred to Marshall [10], Prestel and Delzell [12], and Scheiderer [15] for a nice account of such results.

**Contribution.** We propose the following result: Let \( \|f\|_1 = \sum_{\alpha} |f_{\alpha}| \) whenever \( x \mapsto f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha} \). Let \( f \in \mathbb{R}[x] \) be nonnegative on \( V \), as defined in (1.1), and let \( F := \{f_{\epsilon r}\}_{r \in \mathbb{N}} \) be the family of polynomials

\[
(1.3) \quad f_{\epsilon r} = f + \epsilon \sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_{i}^{2k}}{k!}, \quad \epsilon \geq 0, \quad r \in \mathbb{N}.
\]

(So, for every \( r \in \mathbb{N} \), \( \|f - f_{\epsilon r}\|_1 \rightarrow 0 \) as \( \epsilon \downarrow 0 \).)

Then, for every \( \epsilon > 0 \), there exist nonnegative scalars \( \{\lambda_j(\epsilon)\}_{j=1}^{m} \), such that for all \( r \) sufficiently large (say \( r \geq r(\epsilon) \)),

\[
(1.4) \quad f_{\epsilon r} = q_\epsilon - \sum_{j=1}^{m} \lambda_j(\epsilon) g_j^2,
\]

for some s.o.s. polynomial \( q_\epsilon \in \mathbb{R}[x] \), that is, \( f_{\epsilon r} + \sum_{j=1}^{m} \lambda_j(\epsilon) g_j^2 \) is s.o.s.

Thus, with no assumption on the set \( V \), one obtains a representation of \( f_{\epsilon r} \) (which is positive on \( V \) as \( f_{\epsilon r} > f \) for all \( \epsilon > 0 \)) in the simple and explicit form (1.4), an obvious certificate of positivity of \( f_{\epsilon r} \) on \( V \). In particular, when \( V = \mathbb{R}^n \), one retrieves the result of [13], which states that every nonnegative real polynomial \( f \) can be approximated as closely as desired, by a family of s.o.s. polynomials \( \{f_{\epsilon r(\epsilon)}\}_r \), with \( f_{\epsilon r} \) as in (1.2).

Notice that \( f + nr = f_{\epsilon 0} \). So, on the one hand, the approximation \( f_{\epsilon r} \) in (1.4) is more complicated than \( f + \epsilon \) in (1.2), valid for the compact case with an additional assumption, but on the other hand, the coefficients of the \( g_j \)'s in (1.4) are now scalars instead of s.o.s., and (1.4) is valid for an arbitrary algebraic set \( V \).

The case of a semi-algebraic set \( \mathbb{K} = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0, \quad j = 1, \ldots, m\} \) reduces to the case of an algebraic set \( V \subset \mathbb{R}^{n+m} \), by introducing \( m \) slack variables \( \{z_j\} \), and replacing \( g_j(x) \geq 0 \) with \( g_j(x) - z_j^2 = 0 \), for all \( j = 1, \ldots, m \).

Let \( f \in \mathbb{R}[x] \) be nonnegative on \( \mathbb{K} \). Then, for every \( \epsilon > 0 \), there exist nonnegative scalars \( \{\lambda_j(\epsilon)\}_{j=1}^{m} \) such that, for all sufficiently large \( r \),

\[
f + \epsilon \sum_{k=0}^{r} \left[ \sum_{i=1}^{n} \frac{x_{i}^{2k}}{k!} + \sum_{j=1}^{m} \frac{z_j^{2k}}{k!} \right] = q_\epsilon - \sum_{j=1}^{m} \lambda_j(\epsilon) (g_j - z_j^2)^2,
\]
for some s.o.s. $q_\epsilon \in \mathbb{R}[x,z]$. Equivalently, everywhere on $\mathbb{K}$, the polynomial
\[ x \mapsto f(x) + \epsilon \sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} + \epsilon \sum_{k=0}^{r} \sum_{j=1}^{m} \frac{g_j(x)^k}{k!} \]
coincides with the polynomial $x \mapsto q_\epsilon(x_1, \ldots, x_n, \sqrt{g_1(x)}, \ldots, \sqrt{g_m(x)})$, obviously nonnegative.

The representation (1.4) is also useful for computational purposes. Indeed, using (1.4), one can approximate the global minimum of $f$ on $V$, by solving a sequence of semidefinite programming (SDP) problems. The same applies to an arbitrary semi-algebraic set $\mathbb{K} \subset \mathbb{R}^n$, defined by $m$ polynomials inequalities, as explained above. Again, and in contrast to previous SDP-relaxation techniques as in e.g. [6, 7, 8, 11, 18], no compactness assumption on $V$ or $\mathbb{K}$ is required.

In a sense, the family $F = \{ f_\epsilon r \} \subset \mathbb{R}[x]$ (with $f_0r \equiv f$) is a set of regularizations of $f$, because one may approximate $f$ by members of $F$, and those members always have nice representations when $f$ is nonnegative on an algebraic set $V$ (including the case $V \equiv \mathbb{R}^n$), whereas $f$ itself might not have such a nice representation.

**Methodology.** To prove our main result, we proceed in three main steps.

1. We first define an infinite dimensional linear programming problem on an appropriate space of measures, whose optimal value is the global minimum of $f$ on the set $V$.

2. We then prove a crucial result, namely that there is no duality gap between this linear programming problem and its dual. The approach is similar but different from that taken in [9] when $V \equiv \mathbb{R}^n$. Indeed, the approach in [9] does not work when $V \not\equiv \mathbb{R}^n$. Here, we use the important fact that the polynomial $\theta_r$ is a moment function. And so, if a set of probability measures $\Pi$ satisfies $\sup_{\mu \in \Pi} \int \theta_r \, d\mu < \infty$, it is tight, and therefore, by Prohorov’s theorem, relatively compact. This latter intermediate result is crucial for our purpose.

3. In the final step, we use our recent result [9] which states that if a polynomial $h \in \mathbb{R}[x]$ is nonnegative on $\mathbb{R}^n$, then $h + \epsilon \theta_r$ ($\epsilon > 0$) is a sum of squares, provided that $r$ is sufficiently large.

The paper in organized as follows. After introducing the notation and definitions in §2, some preliminary results are stated in §3 whereas our main result is stated and discussed in §4. For clarity of exposition, most proofs are postponed in §5 and some auxiliary results are stated in an Appendix; in particular, duality results for linear programming in infinite-dimensional spaces are briefly reviewed.

2. Notation and definitions

Let $\mathbb{R}_+ \subset \mathbb{R}$ denote the cone of nonnegative real numbers. For a real symmetric matrix $A$, the notation $A \succeq 0$ (resp. $A > 0$) stands for $A$ positive semidefinite (resp. positive definite). The sup-norm $\sup_j |x_j|$ of a vector
Let $x \in \mathbb{R}^n$, is denoted by $\|x\|_\infty$. Let $\mathbb{R}[x]$ be the ring of real polynomials, and let

\begin{equation}
(2.1) \quad v_r(x) := (1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_1 x_n, x_2^2, x_2 x_3, \ldots, x_n^2)
\end{equation}

be the canonical basis for the $\mathbb{R}$-vector space $A_r$ of real polynomials of degree at most $r$, and let $s(r)$ be its dimension. Similarly, $v_\infty(x)$ denotes the canonical basis of $\mathbb{R}[x]$ as a $\mathbb{R}$-vector space, denoted $A$. So a vector in $A$ has always finitely many zeros.

Therefore, a polynomial $p \in A_r$ is written

\begin{equation}
x \mapsto p(x) = \sum_{\alpha} p_\alpha x^\alpha = \langle p, v_r(x) \rangle, \quad x \in \mathbb{R}^n,
\end{equation}

(\text{where } x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}) \text{ for some vector } p = \{p_\alpha\} \in \mathbb{R}^{s(r)}, \text{ the vector of coefficients of } p \text{ in the basis } (2.1).

Extending $p$ with zeros, we can also consider $p$ as a vector indexed in the basis $v_\infty(x)$ (i.e. $p \in A$). If we equip $A$ with the usual scalar product $\langle \cdot, \cdot \rangle$ of vectors, then for every $p \in A$,

\begin{equation}
p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha = \langle p, v_\infty(x) \rangle, \quad x \in \mathbb{R}^n.
\end{equation}

Given a sequence $y = \{y_\alpha\}$ indexed in the basis $v_\infty(x)$, let $L_y : A \to \mathbb{R}$ be the linear functional

\begin{equation}
p \mapsto L_y(p) := \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\alpha = \langle p, y \rangle.
\end{equation}

Given a sequence $y = \{y_\alpha\}$ indexed in the basis $v_\infty(x)$, the moment matrix $M_r(y) \in \mathbb{R}^{s(r) \times s(r)}$ with rows and columns indexed in the basis $v_r(x)$ in (2.1), satisfies

\begin{equation}
(2.3) \quad [M_r(y)](1,j) = y_\alpha \text{ and } M_r(y)(i,1) = y_\beta \Rightarrow M_r(y)(i,j) = y_{\alpha + \beta}.
\end{equation}

For instance, with $n = 2$,

\begin{equation}
M_2(y) = \begin{bmatrix}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{bmatrix}.
\end{equation}

A sequence $y = \{y_\alpha\}$ has a representing measure $\mu_y$ if

\begin{equation}
y_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu_y, \quad \forall \alpha \in \mathbb{N}^n.
\end{equation}

In this case one also says that $y$ is a moment sequence. In addition, if $\mu_y$ is unique then $y$ is said to be a determinate moment sequence.

The matrix $M_r(y)$ defines a bilinear form $\langle \cdot, \cdot \rangle_y$ on $A_r$, by

\begin{equation}
\langle q, p \rangle_y := \langle q, M_r(y)p \rangle = L_y(qp), \quad q, p \in A_r,
\end{equation}

where $\langle \cdot, \cdot \rangle_y$ is the usual inner product on $A_r$. This form is positive definite if and only if $\mu_y$ is unique.
and if $y$ has a representing measure $\mu_y$, then

\[
L_y(q^2) = \langle q, M_r(y)q \rangle = \int_{\mathbb{R}^n} q(x)^2 \mu_y(dx) \geq 0, \quad \forall q \in \mathcal{A},
\]

so that $M_r(y)$ is positive semidefinite, i.e., $M_r(y) \succeq 0$.

3. Preliminaries

Let $V \subset \mathbb{R}^n$ be the real algebraic set defined in (1.1), and let $B_M$ be the closed ball

\[
B_M = \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq M \}.
\]

**Proposition 3.1.** Let $f \in \mathbb{R}[x]$ be such that $-\infty < f^* := \inf_{x \in V} f(x)$. Then, for every $\epsilon > 0$, there is some $M_\epsilon \in \mathbb{N}$ such that

\[
f_M^* := \inf \{ f(x) \mid x \in B_M \cap V \} < f^* + \epsilon, \quad \forall M \geq M_\epsilon.
\]

Equivalently, $f_M^* \downarrow f^*$ as $M \to \infty$.

**Proof.** Suppose it is false. That is, there is some $\epsilon_0 > 0$ and an infinite sequence sequence $\{M_k\} \subset \mathbb{N}$, with $M_k \to \infty$, such that $f_{M_k}^* \geq f^* + \epsilon_0$ for all $k$. But let $x_0 \in V$ be such that $f(x_0) < f^* + \epsilon_0$. With any $M_k \geq \|x_0\|_\infty$, one obtains the contradiction $f^* + \epsilon_0 \leq f_{M_k}^* \leq f(x_0) < f^* + \epsilon_0$. \hfill $\square$

For every $r \in \mathbb{N}$, let $\theta_r \in \mathbb{R}[x]$ be the polynomial

\[
x \mapsto \theta_r(x) := \sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!}, \quad x \in \mathbb{R}^n,
\]

and notice that $n \leq \theta_r(x) \leq \sum_{i=1}^{n} e^{x_i^2} =: \theta_\infty(x)$, for all $x \in \mathbb{R}^n$. Moreover, $\theta_r$ is a moment function, as it satisfies

\[
\lim_{M \to \infty} \inf_{x \in B_M^c} \theta_r(x) = +\infty,
\]

where $B_M^c$ denotes the complement of $B_M$ in $\mathbb{R}^n$; see e.g. Hernandez-Lerma and Lasserre [3] p. 10.

Next, with $V$ as in (1.1), introduce the following optimization problems.

\[
\mathbb{P} : \quad f^* := \inf_{x \in V} f(x),
\]

and for $0 < M \in \mathbb{N}$, $r \in \mathbb{N} \cup \{\infty\}$,

\[
\mathbb{P}_M^r : \quad \begin{cases}
\inf_{\mu} \int f \, d\mu \\
\text{s.t.} \quad \int g_j^2 \, d\mu \leq 0, \quad j = 1, \ldots, m \\
\int \theta_r \, d\mu \leq n e M^2 \\
\mu \in \mathcal{P}(\mathbb{R}^n),
\end{cases}
\]

where $\mathcal{P}(\mathbb{R}^n)$ is the space of probability measures on $\mathbb{R}^n$ (with $\mathcal{B}$ its associated Borel $\sigma$-algebra). The respective optimal values of $\mathbb{P}$ and $\mathbb{P}_M^r$ are denoted $\inf \mathbb{P} = f^*$ and $\inf \mathbb{P}_M^r$ or $\min \mathbb{P}$ and $\min \mathbb{P}_M^r$ if the minimum is attained (in which case, the problem is said to be solvable).
Proposition 3.2. Let \( f \in \mathbb{R}[x] \), and let \( P \) and \( P_M^r \) be as in (3.4) and (3.5) respectively. Assume that \( f^* > -\infty \). Then, for every \( r \in \mathbb{N} \cup \{\infty\} \), \( \inf \ P_M^r \downarrow f^* \) as \( M \to \infty \). If \( f \) has a global minimizer \( x^* \in V \), then \( \min \ P_M^r = f^* \) whenever \( M \geq \|x^*\|_{\infty} \).

Proof. When \( M \) is sufficiently large, \( B_M \cap V \neq \emptyset \), and so, \( P_M^r \) is consistent, and \( \inf \ P_M^r < \infty \). Let \( \mu \in \mathcal{P}(\mathbb{R}^n) \) be admissible for \( P_M^r \). From \( \int g_j^2 \, d\mu \leq 0 \) for all \( j = 1, \ldots, m \), it follows that \( g_j(x)^2 = 0 \) for \( \mu \)-almost all \( x \in \mathbb{R}^n \), \( j = 1, \ldots, m \). That is, for every \( j = 1, \ldots, m \), there exists a set \( A_j \in \mathcal{B} \) such that \( \mu(A_j^c) = 0 \) and \( g_j(x) = 0 \) for all \( x \in A_j \). Take \( A = \cap_j A_j \in \mathcal{B} \) so that \( \mu(A^c) = 0 \), and for all \( x \in A \), \( g_j(x) = 0 \) for all \( j = 1, \ldots, m \). Therefore, \( A \subset V \), and as \( \mu(A^c) = 0 \),

\[
\int_{\mathbb{R}^n} f \, d\mu = \int_A f \, d\mu \geq f^* \quad \text{because } f \geq f^* \text{ on } A \subset V,
\]

which proves \( \inf \ P_M^r \geq f^* \).

As \( V \) is closed and \( B_M \) is closed and bounded, the set \( B_M \cap V \) is compact and so, with \( f_M^* \) as in Proposition 3.1, there is some \( \hat{x} \in B_M \cap V \) such that \( f(\hat{x}) = f_M^* \). In addition, let \( \mu \in \mathcal{P}(\mathbb{R}^n) \) be the Dirac probability measure at the point \( \hat{x} \). As \( \|\hat{x}\|_{\infty} \leq M \),

\[
\int \theta_r \, d\mu = \theta_r(\hat{x}) \leq n e^{M^2}.
\]

Moreover, as \( \hat{x} \in V \), \( g_j(\hat{x}) = 0 \), for all \( j = 1, \ldots, m \), and so

\[
\int g_j^2 \, d\mu = g_j(\hat{x})^2 = 0, \quad j = 1, \ldots, m,
\]

so that \( \mu \) is an admissible solution of \( P_M^r \) with value \( \int f \, d\mu = f(\hat{x}) = f_M^* \), which proves that \( \inf \ P_M^r \leq f_M^* \). This latter fact, combined with Proposition 3.1 and with \( f^* \leq \inf \ P_M^r \), implies \( \inf \ P_M^r \downarrow f^* \) as \( M \to \infty \), the desired result. The final statement is immediate by taking as feasible solution for \( P_M^r \), the Dirac probability measure at the point \( x^* \in B_M \cap V \) (with \( M \geq \|x^*\|_{\infty} \)). As its value is now \( f^* \), it is also optimal, and so, \( P_M^r \) is solvable with optimal value \( \min \ P_M^r = f^* \).

Consider now, the following optimization problem \( Q_M^r \), the dual problem of \( P_M^r \), i.e.,

\[
(3.6) \quad Q_M^r : \quad \max_{\lambda, \delta, \gamma} \quad \gamma - n \delta e^{M^2} \\
\text{s.t.} \quad f + \delta \theta_r + \sum_{j=1}^m \lambda_j g_j^2 \geq \gamma \\
\gamma \in \mathbb{R}, \delta \in \mathbb{R}_+, \lambda \in \mathbb{R}_+^m,
\]

with optimal value denoted by \( \sup Q_M^r \). Indeed, \( Q_M^r \) is a dual of \( P_M^r \) because weak duality holds. To see this, consider any two feasible solutions \( \mu \in \mathcal{P}(\mathbb{R}^n) \) and \( (\lambda, \delta, \gamma) \in \mathbb{R}_+^m \times \mathbb{R}_+ \times \mathbb{R} \), of \( P_M^r \) and \( Q_M^r \), respectively. Then,
integrating both sides of the inequality in $Q^r_M$ with respect to $\mu$, yields
\[
\int f d\mu + \delta \int \theta_r d\mu + \sum_{j=1}^m \lambda_j \int g_j^2 d\mu \geq \gamma,
\]
and so, using that $\mu$ is feasible for $P^r_M$,
\[
\int f d\mu \geq \gamma - \delta n e^{M^2}.
\]
Hence, the value of any feasible solution of $Q^r_M$ is always smaller than the value of any feasible solution of $P^r_M$, i.e., weak duality holds.

In fact we can get the more important and crucial following result.

**Theorem 3.3.** Let $M$ be large enough so that $B_M \cap V \neq \emptyset$. Let $f \in \mathbb{R}[x]$, and let $r_0 > \max\{\deg f, \deg g_j\}$. Then, for every $r \geq r_0$, $P^r_M$ is solvable, and there is no duality gap between $P^r_M$ and its dual $Q^r_M$. That is, $\sup_{Q^r_M} = \min_{P^r_M}$.

For a proof see §5.1. We finally end up this section by re-stating a result proved in [9], which, together with Theorem 3.3, will be crucial to prove our main result.

**Theorem 3.4** ([9]). Let $f \in \mathbb{R}[x]$ be nonnegative. Then for every $\epsilon > 0$, there exists $r(\epsilon) \in \mathbb{N}$ such that,
\[
(3.7) \quad f_{\epsilon r(\epsilon)} (= f + \epsilon \theta_r) \quad \text{is a sum of squares,}
\]
and so is $f_{\epsilon r}$, for all $r \geq r(\epsilon)$.

## 4. Main result

Recall that for given $(\epsilon, r) \in \mathbb{R} \times \mathbb{N}$, $f_{\epsilon r} = f + \epsilon \theta_r$, with $\theta_r \in \mathbb{R}[x]$ being the polynomial defined in [3.2]. We now state our main result:

**Theorem 4.1.** Let $V \subset \mathbb{R}^n$ be as in (1.1), and let $f \in \mathbb{R}[x]$ be nonnegative on $V$. Then, for every $\epsilon > 0$, there exists $r(\epsilon) \in \mathbb{N}$ and nonnegative scalars $\{\lambda_j\}_{j=1}^m$, such that, for all $r \geq r(\epsilon)$,
\[
(4.1) \quad f_{\epsilon r} = q - \sum_{j=1}^m \lambda_j g_j^2,
\]
for some s.o.s. polynomial $q \in \mathbb{R}[x]$. In addition, $\|f - f_{\epsilon r}\|_1 \rightarrow 0$, as $\epsilon \downarrow 0$.

For a proof see [5.2].

**Remark 4.2.** (i) Observe that (4.1) is an obvious certificate of positivity of $f_{\epsilon r}$ on the algebraic set $V$, because everywhere on $V$, $f_{\epsilon r}$ coincides with the s.o.s. polynomial $q$. Therefore, when $f$ is nonnegative on $V$, one obtains with no assumption on the algebraic set $V$, a certificate of positivity for any approximation $f_{\epsilon r}$ of $f$ (with $r \geq r(\epsilon)$), whereas $f$ itself might not have such a representation. In other words, the $(\epsilon, r)$–perturbation $f_{\epsilon r}$ of $f$, has a **regularization** effect on $f$ as it permits to derive nice representations.
(ii) From the proof of Theorem 4.1 instead of the representation (4.1), one may also provide the alternative representation

\[ f_{cr} = q - \lambda \sum_{j=1}^{m} g_j^2, \]

for some s.o.s. polynomial \( q \), and some (single) nonnegative scalar \( \lambda \) (instead of \( m \) nonnegative scalars in (4.1)).

4.1. **The case of a semi-algebraic set.** We now consider the representation of polynomials, nonnegative on a semi algebraic set \( K \subset \mathbb{R}^n \), defined as,

\[ K := \{ x \in \mathbb{R}^n \mid g_j(x) \geq 0, \ j = 1, \ldots, m \}, \]

for some family \( \{ g_j \}_{j=1}^{m} \subset \mathbb{R}[x] \).

One may apply the machinery developed previously for algebraic sets, because the semi-algebraic set \( K \) may be viewed as the projection on \( \mathbb{R}^n \), of an algebraic set in \( \mathbb{R}^{n+m} \). Indeed, let \( V \subset \mathbb{R}^{n+m} \) be the algebraic set defined as

\[ V := \{ (x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid g_j(x) - z_j^2 = 0, \ j = 1, \ldots, m \}. \]

Then every \( x \in K \) is associated with the point \( (x, \sqrt{g_1(x)}, \ldots, \sqrt{g_m(x)}) \in V \).

Let \( \mathbb{R}[z] := \mathbb{R}[z_1, \ldots, z_m] \), and \( \mathbb{R}[x, z] := \mathbb{R}[x_1, \ldots, x_n, z_1, \ldots, z_m] \), and for every \( r \in \mathbb{N} \), let \( \varphi_r \in \mathbb{R}[z] \) be the polynomial

\[ z \mapsto \varphi_r(z) = \sum_{k=0}^{r} \sum_{j=1}^{m} \frac{z_j^{2k}}{k!}. \]

We then get:

**Corollary 4.3.** Let \( K \) be as in (4.2), and \( \theta_r, \varphi_r \) be as in (3.2) and (4.4). Let \( f \in \mathbb{R}[x] \) be nonnegative on \( K \). Then, for every \( \epsilon > 0 \), there exist nonnegative scalars \( \{ \lambda_j \}_{j=1}^{m} \) such that, for all \( r \) sufficiently large,

\[ f + \epsilon \theta_r + \epsilon \varphi_r = q_\epsilon - \sum_{j=1}^{m} \lambda_j (g_j - z_j^2)^2, \]

for some s.o.s. polynomial \( q_\epsilon \in \mathbb{R}[x, z] \).

Equivalently, everywhere on \( K \), the polynomial

\[ x \mapsto f(x) + \epsilon \sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} + \epsilon \sum_{k=0}^{r} \sum_{j=1}^{m} \frac{g_j(x)^k}{k!} \]

coincides with the nonnegative polynomial \( x \mapsto q_\epsilon(x, \sqrt{g_1(x)}, \ldots, \sqrt{g_m(x)}) \).

So, as for the case of an algebraic set \( V \subset \mathbb{R}^n \), (4.5) is an obvious certificate of positivity on the semi-algebraic set \( K \), for the polynomial \( f_{cr} \in \mathbb{R}[x, z] \)

\[ f_{cr} := f + \epsilon \theta_r + \epsilon \varphi_r, \]
and in addition, viewing $f$ as an element of $\mathbb{R}[x,z]$, one has $\|f - f_\epsilon\|_1 \to 0$ as $\epsilon \downarrow 0$. Notice that no assumption on $K$ or on the $g_j$'s that define $K$, is needed.

Now, assume that $K$ is compact and the $g_j$'s that define $K$, satisfy Putinar’s condition, i.e., (i) there exits some $u \in \mathbb{R}[x]$ such that $u$ can be written $u_0 + \sum_j u_j g_j$ for some s.o.s. polynomials $\{u_j\}_{j=0}^m$, and (ii), the level set $\{x | u(x) \geq 0\}$ is compact.

If $f$ is nonnegative on $K$, then $f + \epsilon \theta_r$ is strictly positive on $K$, and therefore, by Putinar’s theorem (4.7)

$$f + \epsilon \theta_r = q_0 + \sum_{j=1}^m q_j g_j,$$

for some s.o.s. family $\{q_j\}_{j=0}^m$. One may thus either have Putinar’s representation (4.7) in $\mathbb{R}^n$, or (4.5) via a lifting in $\mathbb{R}^{n+m}$.

One may relate (4.5) and (4.7) by

$$q_\epsilon(x,z) = q_\epsilon^1(x) + q_\epsilon^2(x,z^2),$$

with

$$x \mapsto q_\epsilon^1(x) := q_0(x) + \sum_{j=1}^m (q_j(x) g_j(x) + \lambda_j g_j(x)^2),$$

and

$$(x, z) \mapsto q_\epsilon^2(x,z^2) := \epsilon \varphi_r(z) + \sum_{j=1}^m \lambda_j z_j^4 - 2g_j(x)z_j^2.$$

4.2. Computational implications. The results of the previous section can be applied to compute (or at least approximate) the global minimum of $f$ on $V$. Indeed, with $\epsilon > 0$ fixed, and $2r \geq \max[\deg f, \deg g_j^2]$, consider the convex optimization problem

$$Q_{\epsilon r} \left\{ \begin{array}{ll}
\min_y L_y(f_\epsilon), \\
\text{s.t.} \\
M_r(y) \succeq 0 \\
L_y(g_j^2) \leq 0, & j = 1, \ldots, m \\
y_0 = 1,
\end{array} \right.$$ (4.8)

where $\theta_r$ is as in (3.2), $L_y$ and $M_r(y)$ are the linear functional and the moment matrix associated with a sequence $y$ indexed in the basis (2.1); see (2.2) and (2.3) in [2].
\( Q_{\varepsilon r} \) is called a semidefinite programming (SDP) problem, and its associated dual SDP problem reads

\[
Q_{\varepsilon r}^* = \begin{cases} 
\max_{\lambda, \gamma, q} & \gamma \\
\text{s.t.} & f_{\varepsilon r} - \gamma = q - \sum_{j=1}^{m} \lambda_j g_j^2 \\
& \lambda \in \mathbb{R}^m, \quad \lambda \geq 0, \\
& q \in \mathbb{R}[x], \quad q \text{ s.o.s. of degree } \leq 2r.
\end{cases}
\] (4.9)

Their optimal values are denoted \( \inf Q_{\varepsilon r} \) and \( \sup Q_{\varepsilon r}^* \), respectively (or \( \min Q_{\varepsilon r} \) and \( \max Q_{\varepsilon r}^* \) if the optimum is attained, in which case the problems are said to be solvable). Both problems \( Q_{\varepsilon r} \) and its dual \( Q_{\varepsilon r}^* \) are nice convex optimization problems that, in principle, can be solved efficiently by standard software packages. For more details on SDP theory, the interested reader is referred to the survey paper [19].

That weak duality holds between \( Q_{\varepsilon r} \) and \( Q_{\varepsilon r}^* \) is straightforward. Let \( y = \{y_0\} \) and \( (\lambda, \gamma, q) \in \mathbb{R}_+^m \times \mathbb{R} \times \mathbb{R}[x] \) be feasible solutions of \( Q_{\varepsilon r} \) and \( Q_{\varepsilon r}^* \), respectively. Then, by linearity of \( L_y \),

\[
L_y(f_{\varepsilon r}) - \gamma = L_y(f_{\varepsilon r} - \gamma) = L_y(q - \sum_{j=1}^{m} \lambda_j g_j^2) = L_y(q) - \sum_{j=1}^{m} \lambda_j L_y(g_j^2) \\
\geq L_y(q) \quad \text{[because } L_y(g_j^2) \leq 0 \text{ for all } j = 1, \ldots, m] \\
\geq 0 \quad \text{[because } q \text{ is s.o.s. and } M_r(y) \geq 0; \text{ see (2.5).]}
\]

Therefore, \( L_y(f_{\varepsilon r}) \geq \gamma \), the desired conclusion. Moreover, \( Q_{\varepsilon r} \) is an obvious relaxation of the perturbed problem

\[
P_{\varepsilon r} : f_{\varepsilon r}^* := \min_x \{f_{\varepsilon r} \mid x \in V\}.
\]

Indeed, let \( x \in V \) and let \( y := v_2(x) \) (see (2.1)), i.e., \( y \) is the vector of moments (up to order 2r) of the Dirac measure at \( x \in V \). Then, \( y \) is feasible for \( Q_{\varepsilon r} \) because \( y_0 = 1, M_r(y) \geq 0 \), and \( L_y(g_j^2) = g_j(x)^2 = 0 \) for all \( j = 1, \ldots, m \). Similarly, \( L_y(f_{\varepsilon r}) = f_{\varepsilon r}(x) \). Therefore, \( \inf Q_{\varepsilon r} \leq f_{\varepsilon r}^* \).

**Theorem 4.4.** Let \( V \subset \mathbb{R}^n \) be as in (1.8), and \( \theta_r \) as in (1.9). Assume that \( f \) has a global minimizer \( x^* \in V \) with \( f(x^*) = f^* \). Let \( \varepsilon > 0 \) be fixed. Then

\[
f^* \leq \sup Q_{\varepsilon r}^* \leq \inf Q_{\varepsilon r} \leq f^* + \varepsilon \theta_r(x^*) \leq f^* + \varepsilon \sum_{i=1}^{n} e(x_i^2),
\]

(4.10)

provided that \( r \) is sufficiently large.

**Proof.** Observe that the polynomial \( f - f^* \) is nonnegative on \( V \). Therefore, by Theorem 1.1 for every \( \varepsilon \) there exists \( r(\varepsilon) \in \mathbb{N} \) and \( \lambda(\varepsilon) \in \mathbb{R}_+^m \), such that

\[
f - f^* + \varepsilon \theta_r + \sum_{j=1}^{m} \lambda_j(\varepsilon) g_j^2 = q_\varepsilon,
\]
for some s.o.s. polynomial $q_\epsilon \in \mathbb{R}[x]$. But this shows that $(\lambda(\epsilon), f^*, q_\epsilon) \in \mathbb{R}^n_+ \times \mathbb{R} \times \mathbb{R}[x]$ is a feasible solution of $Q^*_r$ as soon as $r \geq r(\epsilon)$, in which case, $\sup Q^*_r \geq f^*$. Moreover, we have seen that $\inf Q^*_r \leq f^*(x)$ for any feasible solution $x \in V$. In particular, $\inf Q^*_r \leq f^* + \epsilon\theta_r(x^*)$, from which (4.10) follows. \hfill \Box

Theorem 4.4 has a nice feature. Suppose that one knows some bound $\rho$ on the norm $\|x^*\|_\infty$ of a global minimizer of $f$ on $V$. Then, one may fix a priori the error bound $\eta$ on $|\inf Q^*_r - f^*|$. Indeed, let $\eta$ be fixed, and fix $\epsilon > 0$ such that $\epsilon \leq \eta(n\rho^2)^{-1}$. By Theorem 4.4, one has $f^* \leq \inf Q^*_r \leq f^* + \eta$, provided that $r$ is large enough.

The same approach works to approximate the global minimum of a polynomial $f$ on a semi-algebraic set $K$, as defined in [12]. In view of Corollary 4.3 and via a lifting in $\mathbb{R}^n+m$, one is reduced to the case of a real algebraic set $V \subset \mathbb{R}^n+m$, so that Theorem 4.4 still applies. It is important to emphasize that one requires no assumption on $K$, or on the $g_j$’s that define $K$. This is to be compared with previous SDP-relaxation techniques developed in e.g. [6, 7, 8, 11, 18], where the set $K$ is supposed to be compact, and with an additional assumption on the $g_j$’s to ensure that Putinar’s representation [13] holds.

5. Proofs

5.1. Proof of Theorem 4.3 To prove the absence of a duality gap, we first rewrite $P^*_M$ (resp. $Q^*_M$) as a linear program in (standard) form

$$\min_x \{\langle x, c \rangle \mid Gx = b, x \in C\}, \quad \text{resp.} \quad \max_w \{\langle w, b \rangle \mid c - G^*w \in C^*\},$$

don appropriate dual pairs of vector spaces, with associated convex cone $C$ (and its dual $C^*$), and associated linear map $G$ (and its adjoint $G^*$). Then, we will prove that $G$ is continuous, and the set $D := \{(Gx, (x, c)) \mid x \in C\}$ is closed, in some appropriate weak topology. This permits us to conclude by invoking standard results in infinite-dimensional linear programming, that one may find in e.g. Anderson and Nash [1]. For a brief account see [6, 1] and for more details, see e.g. Robertson and Robertson [14], and Anderson and Nash [1].

Let $\theta_r$ be as in [6, 2], and let $M(\mathbb{R}^n)$ be the $\mathbb{R}$-vector space of finite signed Borel measures $\mu$ on $\mathbb{R}^n$, such that $\int \theta_r d|\mu| < \infty$ (where $|\mu|$ denotes the total variation of $\mu$). Similarly, let $H^r$ be the $\mathbb{R}$-vector space of continuous functions $h : \mathbb{R}^n \to \mathbb{R}$, such that $\sup_{x \in \mathbb{R}^n} |h(x)|/\theta_r(x) < \infty$. With the bilinear form $(\cdot, \cdot) : M(\mathbb{R}^n) \times H^r$, defined as

$$(\mu, h) \mapsto \langle \mu, h \rangle = \int h d\mu, \quad (\mu, h) \in M(\mathbb{R}^n) \times H^r,$$

$(M(\mathbb{R}^n), H^r)$ forms a dual pair of vector spaces (See [6, 1]). Introduce the dual pair of vector spaces $(X, Y)$,

$$X := M(\mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}, \quad Y := H^r \times \mathbb{R}^m \times \mathbb{R},$$
and \((Z, W)\)
\[ Z := \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}, \quad W := \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}. \]

Recall that \(2r > \deg g_j^2\), for all \(j = 1, \ldots, m\), and let \(G : \mathcal{X} \to Z\) be the linear map
\[ (\mu, u, v) \mapsto G(\mu, u, v) := \begin{bmatrix} \langle \mu, g_1^2 \rangle + u_1 \\ \vdots \\ \langle \mu, g_m^2 \rangle + u_m \\ \langle \mu, 1 \rangle \end{bmatrix}, \]
with associated adjoint linear map \(G^* : W \to \mathcal{Y}\)
\[ (\lambda, \delta, \gamma) \mapsto G^*(\lambda, \delta, \gamma) := \begin{bmatrix} \sum_{j=1}^m \lambda_j g_j^2 + \delta \theta_r + \gamma \\ \lambda \\ \delta \end{bmatrix}, \]

Next, let \(M(\mathbb{R}^n)_+ \subset M(\mathbb{R}^n)\) be the convex cone of nonnegative finite Borel measures on \(\mathbb{R}^n\), so that the set \(C := M(\mathbb{R}^n)_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+ \subset \mathcal{X}\) is a convex cone in \(\mathcal{X}\). If \(H^r_+\) denotes the nonnegative functions of \(H^r\), then
\[ C^* = H^r_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+ \subset \mathcal{Y}. \]

is the dual cone of \(C\) in \(\mathcal{Y}\).

As \(2r > \max[\deg f, \deg g_j^2]\) it follows that \(f \in H^r\) and \(g_j^2 \in H^r\), for all \(j = 1, \ldots, m\). Then, by introducing slack variables \(u \in \mathbb{R}^m_+, v \in \mathbb{R}_+\), rewrite the infinite-dimensional linear program \(P^r_M\) defined in (3.6), in equality form, that is,
\[ P^r_M : \begin{align*}
\inf_{\mu, u, v} \langle (\mu, u, v), (f, 0, 0) \rangle \\
\text{s.t.} \quad G(\mu, u, v) &= \begin{bmatrix} 0 \\
 & ne^{M^2} \\
 & 1 \end{bmatrix} \\
(\mu, u, v) &\in C.
\end{align*} \]

The LP dual \((P^r_M)^*\) of \(P^r_M\) now reads
\[ (P^r_M)^* : \begin{align*}
\sup_{\lambda, \delta, \gamma} \langle (\lambda, \delta, \gamma), (0, ne^{M^2}, 1) \rangle \\
\text{s.t.} \quad (f, 0, 0) - G^*(\lambda, \delta, \gamma) &\in C^*.
\end{align*} \]

Hence, every feasible solution \((\lambda, \delta, \gamma)\) of \((P^r_M)^*\) satisfies
\[ f - \sum_{j=1}^m \lambda_j g_j^2 - \delta \theta_r - \gamma \geq 0; \quad \lambda, \delta \leq 0. \]

As \(\lambda, \delta \leq 0\) in (5.2), one may see that the two formulations (5.2) and (3.6) are identical, i.e., \(Q^r_M = (P^r_M)^*\).

As \(2r > \max[\deg f, \deg g_j^2]\), it follows that \(f - \sum_{j=1}^m \lambda_j g_j^2 - \delta \theta_r - \gamma \in H^r\), for all \((\lambda, \delta, \gamma) \in W\). Therefore, \(G^*(W) \subset \mathcal{Y}\), and so, by Proposition 6.2, the
linear map $G$ is weakly continuous (i.e. is continuous with respect to the weak topologies $\sigma(\mathcal{X},\mathcal{Y})$ and $\sigma(\mathcal{Z},\mathcal{W})$).

We next prove that the set $D \subset \mathcal{Z} \times \mathbb{R}$, defined as

\[(5.4) \quad D := \{ (G(\mu, u, v), \langle (\mu, u, v), (f, 0, 0) \rangle) | (\mu, u, v) \in C \}, \]

is weakly closed.

For some directed set $(A, \supseteq)$, let $\{ (\mu_\beta, u_\beta, v_\beta) \}_{\beta \in A}$ be a net in $C$, such that

\[\langle G(\mu_\beta, u_\beta, v_\beta), (\mu_\beta, u_\beta, v_\beta), (f, 0, 0) \rangle \rightarrow (a, b, c), d)\]

weakly, for some element $(a, b, c, d) \in \mathcal{Z} \times \mathbb{R}$. In particular

\[\mu_\beta(\mathbb{R}^n) \rightarrow c; \quad \langle \mu_\beta, \theta_r \rangle + v_\beta \rightarrow b; \quad \langle \mu_\beta, g^2_j \rangle + (u_\beta)_j \rightarrow a_j, j = 1, \ldots, m, \]

and $\langle \mu_\beta, f \rangle \rightarrow d$. As $(\mu_\beta, u_\beta, v_\beta) \in C$, and $\theta_r, g^2_j \geq 0$, it follows immediately that $a, b, c \geq 0$. We need to consider the two cases $c = 0$ and $c > 0$.

Case $c = 0$. From $\mu_\beta(\mathbb{R}^n) \rightarrow c$, it follows that $\mu_\beta \rightarrow \mu := 0$ in the total variation norm. But in this case, observe that $G(\mu, a, b) = (a, b, c)$.

It remains to prove that we also have $\langle \mu_\beta, f \rangle \rightarrow d = 0$, in which case, $(G(\mu, a, b), \langle \mu, f \rangle) = ((a, b, c, d))$, as desired.

Recall that $r \geq \text{deg} f$. Denote by $\{ y_\alpha(\beta) \}_{|\alpha| \leq 2r}$ the sequence of moments of the measure $\mu_\beta$, i.e.,

\[y_\alpha(\beta) = \int x^\alpha d\mu_\beta, \quad \alpha \in \mathbb{N}^n, \quad |\alpha| \leq 2r.\]

In particular, $y_0(\beta) = \mu_\beta(\mathbb{R}^n)$. From $\langle \mu_\beta, \theta_r \rangle + v_\beta \rightarrow b$, there is some $\beta_0 \in A$, such that $\langle \mu_\beta, \theta_r \rangle \leq 2b$ for all $\beta \geq \beta_0$. But this implies that

\[y_{2k}(i, \beta) := \int x_i^{2k} d\mu_\beta \leq 2r! b, \quad k \leq r, \quad i = 1, \ldots, n.\]

By Lemma 6.3 it follows that $y_{2\alpha}(\beta) \leq 2br!$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$, and $|y_\alpha(\beta)| \leq \sqrt{2} y_0(\beta) b r!$ for all $|\alpha| \leq r$. But then, as $y_0(\beta) = \mu_\beta(\mathbb{R}^n) \rightarrow c = 0$, we thus obtain $y_{\alpha}(\beta) \rightarrow 0$ for all $|\alpha| \leq r$. Therefore,

\[\langle \mu_\beta, f \rangle = \int f d\mu_\beta = \sum_{|\alpha| \leq r} f_\alpha \int x^\alpha d\mu_\beta = \sum_{|\alpha| \leq r} f_\alpha y_\alpha(\beta) \rightarrow 0,\]

the desired result.

Case $c > 0$. From $\mu_\beta(\mathbb{R}^n) \rightarrow c$ and $\langle \mu_\beta, \theta_r \rangle + v_\beta \rightarrow b$, there is some $\beta_0 \in A$, such that $\mu_\beta(\mathbb{R}^n) \leq 2c$ and $\langle \mu_\beta, \theta_r \rangle \leq 2b$ for all $\beta \geq \beta_0$. But as $\theta_r$ is a moment function, this implies that the family $\Delta := \{ \nu_\beta := \mu_\beta/\mu_\beta(\mathbb{R}^n) \}_{\beta \geq \beta_0}$ is a tight family of probability measures, and as $\Delta$ is a set of probability measures on a metric space, by Prohorov’s theorem, $\Delta$ is relatively compact (see [5] Chap. 1 and section [6.2].) Therefore, there is some probability measure $\nu^* \in M(\mathbb{R}^n)$, and a sequence $\{ n_k \} \subset \Delta$, such that $\nu_{n_k}$ converges to $\nu^*$, for the weak convergence of probability measures, i.e.,

\[\langle \nu_{n_k}, h \rangle \rightarrow \langle \nu^*, h \rangle, \quad \forall h \in C_c(\mathbb{R}^n)\]
(where $C_b(\mathbb{R}^n)$ denotes the space of bounded continuous functions on $\mathbb{R}^n$); see e.g. Billingsley [3]. Hence, with $\mu^* := c\nu^*$, we also conclude

\[ \mu_{nk}, h \to \mu^*, h, \quad \forall h \in C_b(\mathbb{R}^n). \]

Next, as $2r > \max\{\deg f, \deg g^2_j\}$, the functions $f/\theta_{r-1}$ and $g^2_j/\theta_{r-1}$, $j = 1, \ldots, m$, are all in $C_b(\mathbb{R}^n)$. Therefore, using Lemma 6.5 we obtain

\[ \langle \mu_{nk}, f \rangle \to \langle \mu^*, f \rangle, \text{ and } \langle \mu_{nk}, g^2_j \rangle \to \langle \mu^*, g^2_j \rangle, \quad j = 1, \ldots, m. \]

And, therefore,

\[ \langle \mu_{nk}, f \rangle \to \langle \mu^*, f \rangle = d, \quad \text{and } \langle \mu_{nk}, g^2_j \rangle \to \langle \mu^*, g^2_j \rangle, \quad j = 1, \ldots, m. \]

Finally, from the weak convergence (5.5), and as $\theta_r$ is continuous and non-negative,

\[ \langle \mu^*, \theta_r \rangle \leq \liminf_{k \to \infty} \langle \mu_{nk}, \theta_r \rangle \leq b, \]

see e.g. [3] Prop. 1.4.18.

So, let $v := b - \langle \mu^*, \theta_r \rangle \geq 0$, and $u_j := a_j - \langle \mu^*, g^2_j \rangle \geq 0$, $j = 1, \ldots, m$, and recalling that $c = \mu^*(\mathbb{R}^n)$, we conclude that $G(\mu^*, u, v) = (a, b, c)$, and $\langle (\mu^*, u, v), (f, 0, 0) \rangle = d$, which proves that the set $D$ in (5.4) is weakly closed.

Finally, by Proposition 3.2 $P^r_M$ is consistent with finite value as soon as $M$ is large enough to ensure that $B_M \cap V \neq \emptyset$. Therefore, one may invoke Theorem 6.4 and conclude that there is no duality gap between $P^r_M$ and its dual $Q^r_M$, the desired result. \(\square\)

### 5.2. Proof of Theorem 4.1

It suffices to prove the result for the case where $\inf_{x \in V} f(x) = f^* > 0$. Indeed, suppose that $f^* = 0$. Then with $\epsilon > 0$ fixed, arbitrary, $f^* + \epsilon > 0$ and so, suppose that (4.1) holds for $\hat{f} := f + \epsilon \nu$. There is some $r(\epsilon) \in \mathbb{N}$ such that, for all $r \geq r(\epsilon)$,

\[ \hat{f} = f + \epsilon \nu + \epsilon \theta_r = \epsilon q_{r} + \sum_{j=1}^{m} \lambda_j g_j^2, \]

for some s.o.s. polynomial $q_{r}$, and some nonnegative scalars $\{\lambda_j\}$. Equivalently,

\[ f + 2\epsilon \theta_r = \epsilon q_{r} + \epsilon \sum_{k=1}^{r} \sum_{j=1}^{n} \frac{x_j^{2k}}{k!} - \sum_{j=1}^{m} \lambda_j g_j^2 = \hat{q}_{r} - \sum_{j=1}^{m} \lambda_j g_j^2, \]

where $\hat{q}_{r}$ is a s.o.s. polynomial. Equivalently, $f_{2r} = \hat{q}_{r} - \sum_{j=1}^{m} \lambda_j g_j^2$, so that (4.1) also holds for $f$. Therefore, from now on, we will assume that $f^* > 0$.

So let $\epsilon > 0$ (fixed) be such that $f^* - \epsilon > 0$, and let $r \geq r_0$ with $r_0$ as in Theorem 3.3. Next, by Proposition 3.2 let $M$ be such that $f^* \leq \inf P^r_M \leq f^* + \epsilon$. By Theorem 3.3 we then have $\sup Q^r_M \geq f^*$. So, by considering a
maximizing sequence of $Q^r_M$, there is some $(\lambda, \delta, \gamma) \in \mathbb{R}_+^m \times \mathbb{R}_+ \times \mathbb{R}$, such that
\begin{equation}
0 < f^* - \epsilon < \gamma - n\delta e^{M^2} \leq f^* + \epsilon;
\end{equation}
and so,
\begin{equation}
f - (\gamma - n\delta e^{M^2}) + \sum_{j=1}^m \lambda_j g_j^2 \geq \delta (ne^{M^2} - \theta_t).
\end{equation}
By Proposition 3.1, we may choose $M$ such that there is some $x_M \in B_M/2 \cap V$ such that
\begin{equation}
f(x_M) \leq f^* + \epsilon.
\end{equation}
Evaluating (5.7) at $x = x_M$ yields
\begin{equation}
2\epsilon \geq f(x_M) - (\gamma - n\delta e^{M^2}) \geq \delta (ne^{M^2} - \theta_t(x_M)),
\end{equation}
and so, using $\|x_M\|_\infty \leq M/2$,
\begin{equation}
2\epsilon \geq \delta n (e^{M^2} - e^{M^2/4}),
\end{equation}
which yields $\delta \leq 2\epsilon/n (e^{M^2} - e^{M^2/4})$. Therefore, given $\epsilon > 0$, one may pick $(\lambda, \delta, \gamma)$ in a maximizing sequence of $Q^r_M$, in such a way that $\delta \leq \epsilon$. For such a choice of $(\lambda, \delta, \gamma)$, and in view of (5.6), we have
\begin{equation}
f + \delta \theta_t + \sum_{j=1}^m \lambda_j g_j^2 \geq (\gamma - n\delta e^{M^2}) + n\delta e^{M^2} \geq f^* - \epsilon + n\delta e^{M^2} \geq 0,
\end{equation}
so that the polynomial $h := f + \delta \theta_t + \sum_{j=1}^m \lambda_j g_j^2$ is nonnegative.

Therefore, invoking Theorem 3.4 proved in Lasserre [9], there is some $r(\epsilon) \in \mathbb{N}$ such that, for all $s \geq r(\epsilon)$, the polynomial $q_\epsilon := h + \epsilon \theta_s$ is a s.o.s. polynomial. But then, take $s > \max[r, r(\epsilon)]$ and observe that
\begin{equation}
\delta \theta_t + \epsilon \theta_s = (\delta + \epsilon) \theta_s - \delta \sum_{k=r+1}^s \sum_{j=1}^m \frac{x_j^2}{k!},
\end{equation}
and so
\begin{equation}
q_\epsilon = h + \epsilon \theta_s = f + \sum_{j=1}^m \lambda_j g_j^2 + (\delta + \epsilon) \theta_s - \delta \sum_{k=r+1}^s \sum_{j=1}^m \frac{x_j^2}{k!},
\end{equation}
or, equivalently,
\begin{equation}
f + \sum_{j=1}^m \lambda_j g_j^2 + (\delta + \epsilon) \theta_s = q_\epsilon + \delta \sum_{k=r+1}^s \sum_{j=1}^m \frac{x_j^2}{k!} = \hat{q}_\epsilon,
\end{equation}
where $\hat{q}_\epsilon$ is a s.o.s. polynomial.

As $\delta$ was chosen to satisfy $\delta \leq \epsilon$, we obtain
\begin{equation}
f + \sum_{j=1}^m \lambda_j g_j^2 + 2\epsilon \theta_s = \hat{q}_\epsilon + (\epsilon - \delta) \theta_s = \hat{q}_\epsilon,
\end{equation}
where again, $\hat{q}_\epsilon$ is a s.o.s. polynomial. □
In this section, we first briefly recall some basic results of linear programming in infinite-dimensional spaces, and then present auxiliary results that are used in some of the proofs in §5.

### 6.1. Linear programming in infinite dimensional spaces.

#### 6.1.1. Dual pairs.

Let \(X, Y\) be two arbitrary (real) vector spaces, and let \(\langle \cdot, \cdot \rangle\) be a bilinear form on \(X \times Y\), that is, a real-valued function on \(X \times Y\) such that

- The map \(x \mapsto \langle x, y \rangle\) is linear on \(X\) for every \(y \in Y\).
- The map \(y \mapsto \langle x, y \rangle\) is linear on \(Y\) for every \(x \in X\).

Then the pair \((X, Y)\) is called a **dual pair** if the bilinear form separates points in \(X\) and \(Y\), that is,

- For each \(0 \neq x \in X\), there is some \(y \in Y\) such that \(\langle x, y \rangle \neq 0\), and
- For each \(0 \neq y \in Y\), there is some \(x \in X\) such that \(\langle x, y \rangle \neq 0\).

Given a dual pair \((X, Y)\), we denote by \(\sigma(X, Y)\) the **weak topology** on \(X\) (also referred to as the \(\sigma\)-topology on \(X\)), namely the coarsest or weakest topology on \(X\), under which all the elements of \(Y\) are continuous when regarded as linear forms \(\langle \cdot, y \rangle\) on \(X\).

Equivalently, the base of neighborhoods of the origin of the \(\sigma\)-topology is the family of all sets of the form

\[ N(I, \epsilon) := \{ x \in X \mid \langle x, y \rangle \leq \epsilon, \forall y \in I \}, \]

where \(\epsilon > 0\) and \(I\) is a **finite** subset of \(Y\). (See for instance Robertson and Robertson [14, p. 32].) In this case, if \(\{x_n\}\) is a net or a sequence in \(X\), then \(x_n\) **converges** to \(x\) in the weak topology \(\sigma(X, Y)\) if

\[ \langle x_n, y \rangle \to \langle x, y \rangle, \quad \forall y \in Y. \]

**Definition 6.1.** Let \((X, Y)\) and \((Z, W)\) be two dual pairs of vector spaces, and \(G : X \to Z\), a linear map.

(a) \(G\) is said to be **weakly continuous** if it is continuous with respect to the weak topologies \(\sigma(X, Y)\) and \(\sigma(Z, W)\); that is, if \(\{x_n\}\) is a net in \(X\) such that \(x_n \to x\) in the weak topology \(\sigma(X, Y)\), then \(Gx_n \to Gx\) in the weak topology \(\sigma(X, Y)\), i.e.,

\[ \langle Gx_n, v \rangle \to \langle Gx, v \rangle, \quad \forall v \in W. \]

(b) The **adjoint** \(G^* : W \to Y\) of \(G\) is defined by the relation

\[ \langle Gx, v \rangle = \langle x, G^*v \rangle, \quad \forall x \in X, \, v \in W. \]

The following proposition gives a well-known (easy to use) criterion for the map \(G\) in Definition 6.1 to be weakly continuous.

**Proposition 6.2.** The linear map \(G\) is weakly continuous if and only if its adjoint \(G^*\) maps \(W\) into \(Y\), that is, \(G^*(W) \subset Y\).
6.1.2. Positive and dual cones. Let \((X, \mathcal{Y})\) be a dual pair of vector spaces, and \(C\) a convex cone in \(X\), that is, \(x + x'\) and \(\lambda x\) belong to \(C\) whenever \(x\) and \(x'\) are in \(C\) and \(\lambda > 0\). Unless explicitly stated otherwise, we shall assume that \(C\) is not the whole space, that is, \(C \neq X\), and that the origin (the zero vector in \(X\)) is in \(C\). In this case, \(C\) defines a partial order \(\geq\) in \(X\), such that
\[ x \geq x' \iff x - x' \in C, \]
and \(C\) is referred to as a positive cone in \(X\). The dual cone of \(C\) is the convex cone \(C^*\) in \(\mathcal{Y}\) defined by
\[ C^* := \{ y \in \mathcal{Y} \mid \langle x, y \rangle \geq 0, \forall x \in C \}. \]

6.1.3. Infinite linear programming (LP). An infinite linear program requires the following components:
- two dual pairs of vector spaces \((X, \mathcal{Y})\).
- a weakly continuous linear map \(G : X \to Z\), with adjoint \(G^* : W \to \mathcal{Y}\).
- a positive cone \(C\) in \(X\), with dual cone \(C^*\) in \(\mathcal{Y}\); and
- vectors \(b \in Z\) and \(c \in \mathcal{Y}\).

Then the primal linear program is
\[
\begin{align*}
\mathbb{P} : \quad & \text{minimize } \langle x, c \rangle \\
& \text{subject to: } Gx = b, \ x \in C.
\end{align*}
\]
The corresponding dual linear program is
\[
\begin{align*}
\mathbb{P}^* : \quad & \text{maximize } \langle b, w \rangle \\
& \text{subject to: } c - G^* w \in C^*, \ w \in W.
\end{align*}
\]
An element of \(x \in X\) is called feasible for \(\mathbb{P}\) if it satisfies (6.1), and \(\mathbb{P}\) is said to be consistent if it has a feasible solution. If \(\mathbb{P}\) is consistent then its value is defined as
\[ \inf \mathbb{P} := \inf \{ \langle x, c \rangle \mid x \text{ is feasible for } \mathbb{P} \}; \]
otherwise, \(\inf \mathbb{P} = +\infty\). The linear program \(\mathbb{P}\) is solvable if there is some feasible solution \(x^* \in X\), that achieves the value \(\inf \mathbb{P}\); then \(x^*\) is an optimal solution of \(\mathbb{P}\), and if one then writes \(\inf \mathbb{P} = \min \mathbb{P}\). The same definitions apply for the dual linear program \(\mathbb{P}^*\).

The next result can be proved as in elementary (finite-dimensional) LP.

**Proposition 6.3** (Weak duality). If \(\mathbb{P}\) and \(\mathbb{P}^*\) are both consistent, then their values are finite and satisfy \(\sup \mathbb{P}^* \leq \inf \mathbb{P}\).

There is no duality gap if \(\sup \mathbb{P}^* = \inf \mathbb{P}\), and strong duality holds if \(\max \mathbb{P}^* = \min \mathbb{P}\), i.e., if there is no duality gap, and both \(\mathbb{P}^*\) and \(\mathbb{P}\) are solvable.

**Theorem 6.4.** Let \(D\) be the set in \(Z \times \mathbb{R}\), defined as
\[
D := \{ (Gx, \langle x, c \rangle) \mid x \in C \}.
\]
If \(\mathbb{P}\) is consistent with finite value, and \(D\) is weakly closed (i.e., closed in the weak topology \(\sigma(Z \times \mathbb{R}, W \times \mathbb{R})\)), then \(\mathbb{P}\) is solvable and there is no duality gap, i.e., \(\sup \mathbb{P}^* = \min \mathbb{P}\).
6.2. Auxiliary results. Let $\mathcal{B}$ be the Borel sigma-algebra of $\mathbb{R}^n$, $C_0(\mathbb{R}^n)$ be the space of bounded continuous functions that vanish at infinity, and let $\theta_r$ be as in (3.2). Let $M(\mathbb{R}^n)$ be the space of finite signed Borel measures on $\mathbb{R}^n$.

Lemma 6.5. Let $r \geq 1$, and let $\{\mu_j\}_{j \in J} \subset M(\mathbb{R}^n)$ be a sequence of probability measures, such that

\[ \sup_{j \in J} \int \theta_r \, d\mu_j < \infty. \]  

Then there is a subsequence $\{j_k\} \subset J$ and a probability measure $\mu$ on $\mathbb{R}^n$ (not necessarily in $M$), such that

\[ \lim_{k \to \infty} \int f \, d\mu_{j_k} = \int f \, d\mu, \]

for all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$, such that $f/\theta_{r-1} \in C_b(\mathbb{R}^n)$.

Proof. $\theta_r$ is a moment function (see (3.3)), and so, (6.4) implies that the sequence $\{\mu_j\}$ is tight. Hence, as $\mathbb{R}^n$ is a metric space, by Prohorov’s Theorem [5, Theor. 1.4.12], there is a subsequence $\{j_k\}$ and a measure $\mu \in M(\mathbb{R}^n)$ such that $\mu_{j_k} \Rightarrow \mu$, i.e.,

\[ \int h \, d\mu_{j_k} \to \int h \, d\mu, \]

for all $h \in C_b(\mathbb{R}^n)$, with $C_b(\mathbb{R}^n)$ being the space of bounded continuous functions $h : \mathbb{R}^n \to \mathbb{R}$. Next, let $\nu_{j_k}$ be the measure obtained from $\mu_{j_k}$ by:

\[ \nu_{j_k}(B) := \int_B \theta_{r-1} \, d\mu_{j_k}, \quad B \in \mathcal{B}. \]

Observe that from the definition of $\theta_r$, the function $\theta_r/\theta_{r-1}$ is a moment function, for every $r \geq 1$. And one has,

\[ \sup_k \int \theta_r/\theta_{r-1} \, d\nu_{j_k} = \sup_k \int \theta_r \, d\mu_{j_k} < \infty, \]

because of (6.4). Observe that $\nu_{j_k}(\mathbb{R}^n) \leq \rho$ for all $k$, and so, we may consider a subsequence of $\{j_k\}$ (still denoted $\{j_k\}$ for simplicity of notation) such that $\nu_{j_k}(\mathbb{R}^n) \to \rho (>0)$ as $k \to \infty$. With $\nu_{j_k} := \nu_{j_k}/\nu_{j_k}(\mathbb{R}^n)$, for all $k$, it follows that the sequence of probability measures $\{\nu_{j_k}\}$ is tight, which implies that there is a subsequence $\{j_n\}$ of $\{j_k\}$, and a measure $\nu \in M(\mathbb{R}^n)$, such that

\[ \int h \, d\nu_{j_n} \to \int h \, d\nu, \quad \forall h \in C_b(\mathbb{R}^n). \]

Since $\nu_{j_k}(\mathbb{R}^n) \to \rho$ as $k \to \infty$, we immediately get

\[ \int h \, d\nu_{j_n} = \int h (\rho + \nu_{j_n}(\mathbb{R}^n) - \rho) \, d\nu_{j_n} \to \int h \rho \, d\nu, \quad \text{as } n \to \infty, \]
for all $h \in C_b(\mathbb{R}^n)$. Equivalently, with $\nu := \rho \hat{\nu}$,

\begin{equation}
\text{as } n \to \infty, \quad \int h \, d\nu_{j_n} \to \int h \, d\nu, \quad \forall h \in C_b(\mathbb{R}^n).
\end{equation}

But as $h/\theta_{r-1} \in C_b(\mathbb{R}^n)$ whenever $h \in C_b(\mathbb{R}^n)$, (6.6) yields

\[ \int h/\theta_{r-1} \, d\nu = \lim_{n \to \infty} \int h/\theta_{r-1} \, d\nu_{j_n} = \lim_{n \to \infty} \int h \, d\mu_{j_n} = \int h \, d\mu, \]

for all $h \in C_b(\mathbb{R}^n)$.

As both $\mu$ and $\theta_{r-1} \, d\nu$ are finite measures, this implies that

\begin{equation}
\mu(B) := \int_B (1/\theta_{r-1}) \, d\nu, \quad B \in \mathcal{B}.
\end{equation}

Next, let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and such that $f/\theta_{r-1} \in C_b(\mathbb{R}^n)$. As $k \to \infty$, from (6.6),

\[ \int (f/\theta_{r-1}) \, d\nu_{j_k} \to \int (f/\theta_{r-1}) \, d\nu, \]

and so,

\[ \int f \, d\mu_{j_k} = \int (f/\theta_{r-1}) \, d\mu_{j_k} = \int (f/\theta_{r-1}) \, d\nu_{j_k} \to \int (f/\theta_{r-1}) \, d\nu = \int f \, d\mu, \quad \text{[by (6.7)]}, \]

the desired result. \hfill \Box

**Lemma 6.6.** Let $\mu$ be a measure on $\mathbb{R}^n$ (with $\mu(\mathbb{R}^n) = y_0$) be such that

\begin{equation}
\sup_{i=1, \ldots, n} \sup_{0 \leq k \leq r} \int x_i^{2k} \, d\mu \leq S.
\end{equation}

Then,

\begin{equation}
\sup_{\alpha \in \mathbb{N}^n : |\alpha| \leq r} |\int x^\alpha \, d\mu| \leq \sqrt{y_0 S}.
\end{equation}

**Proof.** Let $y = \{y_\alpha\}_{|\alpha| \leq 2r}$ be the sequence of moments, up to order $2r$, of the measure $\mu$, and let $M_r(y)$ be the moment matrix defined in (2.3), associated with $\mu$. Then, (6.8) means that those diagonal elements of $M_r(y)$, denoted $y^{(i)}_{2k}$ in Lasserre [9], are all bounded by $S$. Therefore, by Lemma 6.2 in [9], all diagonal elements of $M_r(y)$ are also bounded by $S$, i.e.,

\begin{equation}
y_{2\alpha} \leq S, \quad \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq r,
\end{equation}

and so are all elements of $M_r(y)$ (because $M_r(y) \succeq 0$). Next, consider the two columns (and rows) $1$ and $j$, associated with the monomials $1$ and $x^\alpha$, and all of the columns (and rows) indexed by $\alpha \in \mathbb{N}^n : |\alpha| \leq r$, and so are all elements of $M_r(y)$ (because $M_r(y) \succeq 0$). Next, consider the two columns (and rows) $1$ and $j$, associated with the monomials $1$ and $x^\alpha$, and so are all elements of $M_r(y)$ (because $M_r(y) \succeq 0$).
respectively, and with $|\alpha| \leq r$, that is, $M_r(y)(1, 1) = y_0$ and $M_r(y)(1, j) = y_\alpha$. As $M_r(y) \succeq 0$, we immediately have

$$M_r(y)(1, 1) \times M_r(y)(j, j) \succeq M_r(y)(1, j)M_r(y)(j, 1) = M_r(y)(1, j)^2,$$

that is, $y_0y_{2\alpha} \geq y_\alpha^2$. Using that $|\alpha| \leq r$ and (6.10), we obtain $y_0S \geq y_\alpha^2$, for all $\alpha$, $|\alpha| \leq r$, the desired result (6.9). □

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