FORM FACTORS OF HADRONIC SYSTEMS IN VARIOUS FORMS OF RELATIVISTIC QUANTUM MECHANICS

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The form factor of hadronic systems in various forms of relativistic quantum mechanics is considered. Motivated by the agreement of the nucleon “point-form” results with experiment, results for a toy model corresponding to the simplest Feynman diagram are first presented. These ones include the results for this diagram, which plays the role of an experiment, for the front-form and instant-form in standard kinematics \((q^+ = 0\) and Breit frame), but also in unconventional kinematics and finally a Dirac’s point-form inspired approach. Results for an earlier “point-form” approach are reminded. Results are also presented for the pion charge form factor. Conclusions as for the efficiency of various approaches are given.

1. Introduction

The primary goal for studying baryon physics is to get insight on how QCD is realized in the non-perturbative regime. This includes for instance the structure of baryons in terms of constituent quarks and the properties of these ones. In this respect, form factors represent an important source of information since their momentum dependence allows one to probe baryons at different scales. To fully exploit the experimental data however, a safe implementation of relativity is required.

There are various ways to implement relativity. Ultimately, they should converge to unique predictions by incorporating two- or many-body currents beside the one-body current generally retained in calculations. In the frame of relativistic quantum mechanics, different forms have been proposed, following the work by Dirac \(^1\). They can be classified according to the symmetries of the hypersurface which physics is described on, determining at the same time the dynamical or kinematical character of the Poincaré-group generators \(^2\). For applications to baryons, some approaches do well by giving the constituent quarks some form factor \(^3\). Other ones do without \(^4,5\). This obviously calls for an independent check of the reliability
of the underlying formalisms.

A system that can provide a useful testing ground consists of two scalar particles of mass $m$ exchanging a scalar particle of mass $\mu$. In the two extremes, $\mu = 0$ (Wick-Cutkosky model) and $\mu = \infty$ (vertex function), the Bethe-Salpeter equation can be solved and solutions can be used to calculate form factors that are exact ones and can thus play the role of an “experiment”. It has also been shown that the mass spectrum provided by the first model is reasonably described by a simple mass operator whose solutions can be employed for the calculation of form factors in different forms of relativistic quantum mechanics. In the second case, the uncertainty due to the range of the interaction is reduced to a minimum and the solution is essentially known. The comparison of the results obtained in both ways (Bethe-Salpeter equation and mass operator) offers many advantages. The dynamics of the interaction is the simplest one that can be imagined and the uncertainty arising from that one on the effective interaction entering the mass operator is limited. Spin effects are absent, allowing one to check possibly large effects like those due to the Lorentz contraction or the “point-form” spectator approximation at high $Q^2$. Finally, the intrinsic form factors of the constituents, if any, cancel in the comparison while they have to be accounted for when an experiment is involved, which could actually contribute to their determination.

In this paper, we concentrate on the above schematic model in the case $\mu = \infty$. Beside the “experiment”, we consider form factors in various forms and, for some of them, with different kinematics. The forms of interest here include the instant and front ones as well as a Dirac’s inspired point form. An earlier “point-form” implementation, which differs from the Dirac’s one by the fact it involves a hyperplane perpendicular to the velocity of the system (and not a hyperboloid), is also considered. Some results similar to the above ones will be considered for the pion charge form factor. Apart from the fact that the pion represents a physical system, there are real data but, as already mentioned, one has then to worry about constituent form factors.

The plan of the paper is as follows. After reminding the relation of the constituent momenta to the total momentum of the system in different forms, we present and discuss results for the charge form factor in the schematic model. The emphasis is put on the differences and the similarities between various approaches. This is followed by a presentation of results for the pion charge form factor. Their discussion and a conclusion are finally given.
2. Form factors in a schematic model

Figure 1. Kinematics relative to the photon absorption on a two-body system.

The interaction of a two-body system with an external probe is represented in Fig. 1 in the single-particle current approximation. In the frame of relativistic quantum mechanics, which we are working in here, the particles in the intermediate state are on mass shell \( e = \sqrt{m^2 + p^2} \). In order to calculate the corresponding form factor, two ingredients are needed: 1) the relation of the constituent momenta to the total momentum of the system (respectively \( \vec{p}_1 \) and \( \vec{p}_2 \), and \( \vec{P} \)). This one characterizes the form under consideration and is closely related to the symmetries of the hypersurface which physics is described on. 2) a solution of the mass operator which can be taken as form-independent.

In all cases we consider, the relation of the constituent momenta to the total momentum can be cast into the form:

\[
\vec{p}_1 + \vec{p}_2 - \vec{P} = \left(\frac{\xi}{\xi^0}\right) (e_1 + e_2 - E_P),
\]

(1)

where \( \xi^\mu \) is specific of each approach. This relation is fulfilled by a Lorentz-type transformation that allows one to express the constituent momenta in terms of an internal variable, \( \vec{k} \), which enters the mass operator, and the total momentum, \( \vec{P} \). This transformation, which underlies the Bakamjian-Thomas construction of the Poincaré algebra in the instant form \(^{13}\), reads:

\[
\vec{p}_{1,2} = \pm \vec{k} \pm \vec{w} \frac{\vec{w} \cdot \vec{k}}{\sqrt{w^0 + 1}} + \vec{w} e_k, \quad e_{1,2} = w^0 e_k \pm \vec{w} \cdot \vec{k},
\]

(2)

with \( \vec{w} \) and \( w^0 = \sqrt{1 + \vec{w}^2} \) given by:

\[
w^\mu = \frac{P^\mu}{2e_k} + \frac{\xi^\mu}{2e_k} \frac{4e_k^2 - M^2}{\sqrt{(\xi \cdot P)^2 + (4e_k^2 - M^2) \xi^2 + \xi \cdot P}}.
\]

(3)
The 4-vector $\xi^\mu$ appearing in the above expression multiplies a term $(4e_k^2 - M^2)$ that can be transformed into an interaction one. This is in accordance with the expectation that changing the surface pertinent to each approach implies the dynamics. Apart from these features, it can be seen that the above expression is independent of the scale of the 4-vector $\xi^\mu$. Thus, up to an irrelevant scale, the 4-vector $\xi^\mu$ is given as follows:

- **instant form:** $\xi^0 = 1$, $\vec{\xi} = 0$,
- **front form:** $\xi^0 = 1$, $\vec{\xi} = \vec{n}$, ($|\vec{n}| = 1$, fixed direction),
- **Dirac’s point form:** $\xi^0 = 1$, $\vec{\xi} = \vec{u}$, ($|\vec{u}| = 1$, from $(p_1 + p_2 - P)^2 = 0$).

In the last case, $\vec{u}$ can point to any direction, consistently with the absence of any orientation on a hyperboloid. The boost transformation introduced in an earlier “point-form” approach is recovered from Eq. (1) by taking $\xi^\mu \propto P^\mu$. Thus, the calculation of a form factor in this last approach implies initial and final states that are described on different hyperplanes, contrary to all other cases where a unique hypersurface is involved.

The second ingredient needed for the calculation of form factors is the solution of a mass operator. For the interaction model considered here (exchange of an infinitely-massive boson), the solution can be taken as
\( \phi(\vec{k}) \propto (\sqrt{e_k^2 (4e_k^2 - M^2)})^{-1} \). The constituent and total masses entering this expression, \( m = 0.3 \) GeV and \( M = 0.14 \) GeV, are chosen in accordance with those used for the pion results presented in the following section.

Two form factors can be considered for the system under consideration, a charge one, \( F_1(Q^2) \), and a Lorentz-scalar one, \( F_0(Q^2) \). Their expressions for different forms, which can be cast into a unique one in terms of the 4-vector, \( \xi^\mu \), can be found in most cases in Ref. 7. Due to the lack of space, results are presented here for \( F_1(Q^2) \). Two aspects of charge form factors are of interest, the charge radius and the asymptotic behavior, which are determined by the low and high \( Q^2 \) parts respectively. The form factors are presented accordingly in the left and right parts of Fig. 2. They contain:
- the exact form factor (continuous line),
- the standard front-form one \( (q^+ = 0) \), identical to the exact result,
- the instant-form one (dotted line, I.F. (Breit frame)),
- a Dirac’s inspired point-form one (short-dashed line, D.P.F.), corresponding to a fully Lorentz-covariant result \(^9\),
- a front-form one in the configuration where the initial and final momenta, \( \vec{P}_i \) and \( \vec{P}_f \) are parallel to the front orientation (dashed line, F.F. (parallel)),
- an instant-form one in the parallel kinematics, \( \vec{P}_i \parallel \vec{P}_f \), with an average momentum going to \( \infty \) (I.F. (parallel)) (coincides with the previous curve),
- and an earlier “point-form” one (dash-dot line, “P.F.”).

As form factors scale in most cases like \( Q^{-2} \) (up to log terms), the quantities displayed in the right part of Fig. 2 are multiplied by the factor \( Q^2 \).

Form factors clearly fall into two sets: close or even identical to the “experiment” and far apart. The difference in the behavior can be ascribed to the dependence on the total mass of the system, \( M \). Rather weak in the former case, it becomes important in the latter one. Actually, results in this last case depend on the momentum transfer \( Q \) through the combination \( Q/2M \). This produces a charge radius that scales like the inverse of the mass of the system, hence the steep slope of the corresponding form factors at small \( Q^2 \). This feature is also responsible for the suppression of the form factors at high \( Q^2 \) (a factor \( \sim (M/2m)^2 \)). In the “P.F.” case, further suppression occurs, the dependence on \( Q^2 \) involving an extra factor \( (1 + Q^2/4M^2) \) at high \( Q^2 \) \(^8\), hence the approximate asymptotics \( Q^{-4} \) of the corresponding form factor in the present case.

Results very similar qualitatively to the above ones are obtained for the Lorentz-scalar form factors. The same conclusion holds to a lesser degree for interaction models involving the exchange of a zero-mass boson (Wick-Cutkosky model) \(^7\). In this case, some uncertainty affects the determination
of the effective interaction entering the mass operator. With the simplest possible interaction, the discrepancy does not however exceed a factor 2 in the cases where an identity was previously obtained. The discrepancy between the two sets of results mentioned above is considerably increased, in relation with a different asymptotic behavior, $Q^{-4}$ instead of $Q^{-2}$.

3. Pion charge form factor

In this section, we consider the pion charge form factor for which experimental data are known. Calculations similar to those of the previous section are presented. There are however two main differences that make worthwhile to consider this system. The pion consists of two quarks that carry a 1/2-spin. The interaction has a finite range and is dominated by a one-gluon exchange at small distances. Moreover, there are predictions concerning both the charge radius and the asymptotic behavior that should be ultimately recovered \(^{15,16}\). Expressions of form factors in the single-particle current approximation, which have to account for the quark spin, can be obtained from Ref. 17. As for the interaction entering the mass operator, it is taken as the sum of a confining potential with a standard string tension (1 GeV/fm) and a gluon exchange with strength $\alpha_s = 0.35$. No attempt is made in the present work to optimize the results which are presented in
Fig. 3. Their examination shows they are very similar to the scalar-particle ones, confirming those obtained in works with a different scope. We however notice that, in the best case, the $Q^{-2}$ QCD asymptotic behavior is not recovered and that relatively standard two-body currents are needed in this order. As the discrepancy with experiment evidenced by the other approaches we considered reaches orders of magnitude, one can safely discard them as efficient ways to implement relativity.

4. Discussion and conclusion
In this paper, we compared different forms of relativistic quantum mechanics to calculate form factors. We first considered a schematic interaction model. The very good or complete agreement with an exact calculation in some cases, the disagreement in other ones leaves no doubt about which approach is appropriate or inappropriate to get the bulk contribution of form factors. A similar conclusion can be inferred from considering the pion form factor. Thus, front- and instant-form approaches with standard kinematics ($q^+ = 0$ and Breit frame respectively) appear as quite convenient to calculate the dominant contribution to form factors. The same approaches with unconventional kinematics or the point-form approach require a large contribution from non-standard two-body currents (of the type 0/0).

One may wonder why approaches based on a single-particle current work well while other ones, fully covariant in some cases, don’t. Taking into account that front- and instant-form approaches with unconventional kinematics or a Dirac’s point-form inspired approach give relatively similar results (same asymptotic behavior and dependence on $Q$ through the factor $Q/2M$), we are tempted to consider that the approaches that work are an exception. A simple argument explaining the above observation would be helpful. One often has advantage to break some symmetry to get closer to the properties of a physical system (think to a deformed mean field for calculating the binding energy of a nucleus with $J = 0$). Thus, among the different approaches considered here, the Lorentz-covariant one (point form) may not be necessarily the best one. This approach has been advocated because the corresponding kinematical character of boosts makes easy to get wave functions of states with momenta different from the rest-frame one. However, one also has to relate the transferred momentum to the momenta carried by the struck particle. In the point-form approach, this relation involves the dynamics, while it has a kinematical character in the field-theory models underlying physics of interest here. With this respect, using
the point-form approach represents a bad strategy as interaction effects introduced in the above relation will have to be removed later on, under the form of two-body contributions. Instead, the standard front- and instant-form approaches are those which better fulfill the kinematical character of the above momentum relation in field theory. This is perhaps the reason why they are more appropriate for the calculation of form factors.

We began this presentation by reminding that the goal for studying form factors is to learn about hadronic physics. Present results for the pion charge form factor are still premature. However, taking into account that only the standard front- and instant-form approaches provide a reliable implementation of relativity when a single-particle current is considered, it is found that the corresponding form factors can \textit{a priori} accommodate a reasonable constituent form factor. Its precise nature has to be determined.

Acknowledgments

We are very grateful to A. Amghar, S. Noguera and L. Theußl for their collaboration at the early stage of the development of this work.

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