Fuzzy Extension of Simplified variant of AHP

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Abstract. A class of methods for ranking a finite set of objects based on a multiplicative matrix of pairwise comparisons, when the comparison results are expressed by fuzzy values, is considered. A fuzzy extension of the simplified version of the AHP, which was introduced by the author earlier, is proposed. In this extension, the elements of the matrix of pairwise comparisons can be type 1 or type 2 fuzzy numbers.

1. Introduction

Pairwise comparison methods (see, for example, [1]) are based on the idea of comparing only two objects at a time, which is significantly less demanding on cognitive capabilities of Decision Makers (DMs) than comparing several objects at the same time. Considering multicriteria problem with a given problem hierarchy, a set of objects in one level of the hierarchy is always compared pair wisely with respect to the superior object from the higher level of the hierarchy. It is well known that pairwise comparison methods usually need \( n(n-1)/2 \) comparisons of objects, where \( n \) is the number of all objects.

The analytic hierarchy process (AHP) [2] is one of the most popular and widely employed multicriteria pairwise comparison methods. In this technique, the processes of rating alternatives and aggregating to find the most relevant alternatives are integrated. The approach by T.L. Saaty based on maximal eigenvalue and the corresponding eigenvector of the comparison matrix is employed for ranking a set of alternatives or for the selection of the best in a set of alternatives.

The AHP was repeatedly criticized by many researchers because it cannot conserve the ranking of the components of the priority vector when one of alternative is removed. In 2004 the author of this paper proposed a so-called simplified variant of AHP [3], where were revised important components of the AHP. Namely, was suggested a simplified procedure of forming the pairwise comparison matrix. According to this procedure we no need information about all the elements of comparison matrix lying above the principal diagonal, but only about certain \( n-1 \) “basis” elements; the priority vector (eigenvector) can then be easily and exactly calculated. The choice of a particular basis set corresponds to the scheme used to compare objects, which can be chosen to obtain the most reliable information from an expert. The method proposed by the author was considerably simpler than the original method, both at the stage of forming the comparison matrix and at the stage of computing the priority vector.

Section 2 of the paper contains appropriate information connected with the simplified variant of AHP. Basic notions in fuzzy sets are presented in Section 3. In particular, type 1 as well as type 2 fuzzy numbers and some arithmetic operations over them are considered. Fuzzy extension of
simplified variant of AHP is proposed and illustrated in Section 4.

2. Simplified variant of AHP

Let a set of \( n \) objects (alternatives) denoted by \( A_1, A_2, \ldots, A_n \) be given. It is required to find the weight of each of these objects; i.e., we must find the priority vector \( w \) with the components \( w_1, w_2, \ldots, w_n \).

For the pairwise comparison matrix \( A \) let us formulate the following properties:

1) All elements of \( A \) are positive; i.e., \( a_{ij} > 0 \) for all \( i, j = 1, 2, \ldots, n \).

2) The matrix \( A \) is reciprocal; i.e., \( a_{ij} = 1/a_{ji} \) for all \( i, j = 1, 2, \ldots, n \). In particular, \( a_{ii} = 1 \) for all \( i = 1, 2, \ldots, n \).

3) The matrix \( A \) is consistent; i.e., \( a_{ij} \cdot a_{jk} = a_{ik} \) for all \( i, j, k = 1, 2, \ldots, n \).

4) The number \( n \) is the principal eigenvalue of \( A \) and the equation \( A \cdot w = n \cdot w \) has a unique (normalized) solution (i.e., a priority vector) \( w = (w_1, w_2, \ldots, w_n)^T \) with positive components.

**Theorem 1** [3]. Let the matrix \( A = (a_{ij})_{nn} \) satisfying conditions 1) and 2) be constructed from its first row by formula

\[
a_{ij} = \frac{a_{1j}}{a_{1i}}, \quad i, j = 1, 2, \ldots, n. \tag{1}
\]

Then, \( A \) possesses properties 3) – 4); there is no other matrix with the same first row that has properties 3) – 4); and the right principal eigenvector \( w = (w_1, w_2, \ldots, w_n) \) corresponding to the principal eigenvalue of \( A \) has the components defined by formula

\[
w_i = a_{im} = \frac{a_{1m}}{a_{1i}}, \quad i = 1, 2, \ldots, n. \tag{2}
\]

**Definition 1** [3]. A set of elements of the positive matrix \( A = (a_{ij})_{nn} \) lying above the principal diagonal is called the determining set if all other elements of \( A \) can be uniquely found from it using properties 2) – 3), and the matrix thus constructed has properties 2) – 4).

**Definition 2** [3]. Let a set of elements \( \{a_{i,p}\} \) of the matrix \( A \) lying above the principal diagonal be given. If this set does not include a triple of elements \( a_{i_i/a}, a_{i_j/b}, a_{i_k/c} \in \{a_{i,r}\} \) such that \( i_a = i_b, j_a = j_c, j_b = j_k \) the set \( \{a_{i,r}\} \) is called independent.

**Definition 3** [3]. The determining set of elements of the matrix \( A \) containing the minimum number of elements is called a basis set.

For the set of \( k \) elements \( \{a_{i,r}\} \) lying above the principal diagonal, let us introduce the undirected graph consisting of \( n \) vertices in which the pair of vertices \( i_1, i_2, i_3 \in \{1, 2, \ldots, n\}, i_1 < i_2, i_3 \) is adjacent if and only if the given set \( \{a_{i,r}\} \) contains the element \( a_{i,ij} \).

**Theorem 2** [3]. For any set of elements \( \{a_{i,r}\} \) of the positive matrix \( A = (a_{ij})_{nn} \) lying above the principal diagonal, the following two assertions are equivalent:

1) \( \{a_{i,r}\} \) is a basis set;

2) \( \{a_{i,r}\} \) consists of \( n-1 \) elements and is independent;

3) \( \{a_{i,r}\} \) consists of \( n-1 \) elements and the corresponding graph is connected.

The existence of a family of basis sets can be used to choose a set that enables an expert to give the most reliable comparisons. For example, the set consisting of the first-row elements corresponds
to the pattern comparison scheme. In this case, formula (1) is used and the last column of $A$ will be the (in general, no normalized) priority vector $w=(w_1,w_2,...,w_n)$ (see (2)).

Consider another basis set, namely, $a_{i_2},a_{i_3},a_{i_4},...,a_{i_{n-1},r}$. This set corresponds to the following sequential comparison scheme. First, an arbitrary object from the set is chosen and given the number 1. Then, the most convenient object (from the viewpoint of comparing it with the first object) from the remaining ones is chosen and given the number 2. The comparison of these two objects gives the element $a_{i_2}$. Proceeding in the same way, we choose the third object that is the most convenient object (from the viewpoint of comparing it with the second object) to produce the element $a_{i_3}$, and so on. Calculating the (no normalized) priority vector $w=(w_1,w_2,...,w_n)^T$ based on the sequential comparison scheme can be performed by the following formula [3]

$$w_k = a_{i_k+1} \cdot a_{i_k+2} \cdot a_{i_k+3} \cdot a_{i_{n-1},r}, \quad k=1,2,...,n-1; \quad w_n = 1.$$  

(3)

3. Basic notions in fuzzy sets

Let $A$ be a certain non-empty crisp set (the so-called universal set). A fuzzy set $X$ in $A$ is defined by a membership function $\lambda_X : A \rightarrow [0,1]$. For each element $x \in A$, the number $\lambda_X(x) \in [0,1]$ is interpreted as its grade of membership to the set $X$. If the membership function $\lambda_X(\cdot)$ takes values 0 and 1 only, it becomes the characteristic function of a crisp set $X$. Two fuzzy sets are equal to each other if they have the identical membership functions.

A fuzzy set is called normal if the sharp upper bound of its membership function is equal to one, otherwise it is called subnormal. A fuzzy quantity is usually understood as a fuzzy set defined on a set (subset) of real numbers $\mathbb{R}$, and a fuzzy number is a normal convex fuzzy quantity. In this case, a convex fuzzy quantity, by definition, is characterized by the fact that for it the set $\{x \in \mathbb{R} | \lambda_X(x) \geq \alpha\}$ of any level $\alpha$ is a certain interval on the real line, i.e. bounded convex subset of numbers. The set of all fuzzy numbers we will denote by $\text{FN}$.

Algebraic operations over $\text{FN}$ is introduced by means of the so-called Zadeh’s extension principle. Namely, for two fuzzy numbers with the membership functions $\lambda_X(x), \lambda_Y(x)$, we have

$$\lambda_{X \circ Y}(z) = \sup_{(x,y) \in \mathbb{R}^2, x+y = z} \{ \min(\lambda_X(x), \lambda_Y(y)) \} \quad \text{for any } z \in \mathbb{R},$$  

(4)

where $(\circ)$ is one of the algebraic operations, for instance, addition, subtraction, multiplication, or division.

A trapezoidal fuzzy number (TFN) $X$ is defined by means

$$\lambda_X(x) = \begin{cases} 
0, & x < a, \\
\frac{x-a}{b-a}, & a \leq x \leq b, \\
1, & b \leq x \leq c, \\
\frac{d-x}{d-c}, & c \leq x < d, \\
0, & x \geq d, 
\end{cases}$$

and denoted by $(a,b,c,d)$). A TFN is called positive if $a > 0$. When $b = c$ we deal with a triangle fuzzy number (TrFN). A TrFN can be represented as $(a,b,b,c)$ or shorty as a triple $(a,b,c)$.

For $X = (a,b,c,d), Y = (a',b',c',d')$, according to (4) we obtain

$$X(\circ)Y = (a+a', b+b', c+c', d+d')$$

and sum of two TFNs is TFN too. But the result of multiplication or division over TFNs, in general, is not TFN. For this reason, the so-called simplified standard arithmetic operations $(\circ),(\ast)$ are introduced (over positive TFNs)
Intervals are a particular case of trapezoidal fuzzy numbers. It is clear that interval \([\mu^-, \mu^+]\) can be written as TFN \((\mu^-, \mu^+, \mu^*, \mu^+)\). However, intervals are also a particular class of crisp sets on \(\mathbb{R}\) with a well-defined interval arithmetic for performing arithmetic operations on intervals. Unlike for the case of triangular and trapezoidal fuzzy numbers, the results of arithmetic operations with intervals are again intervals. Let \(\overline{x} = [\mu^-, \mu^+], \overline{y} = [\lambda^-, \lambda^+]\) be two positive intervals, i.e., \(\mu^-, \lambda^- > 0\). Then the arithmetic operations are defined by using standard interval arithmetic as

\[
\overline{x}(\overline{y}) = [\mu^- + \lambda^-, \mu^+ + \lambda^+], \quad \overline{x}(\overline{y}) = [\mu^- \cdot \lambda^-, \mu^+ \cdot \lambda^+], \quad \alpha(\overline{x}) = [\alpha \cdot \mu^-, \alpha \cdot \mu^+], \quad \alpha > 0,
\]

In practice, it can be complicated to assign exact membership degrees (i.e., crisp numbers) to all objects under consideration. When membership degrees in a fuzzy set \(X\) are themselves represented by fuzzy sets, \(X\) is called a type 2 fuzzy set \([4-5]\). If membership degrees of elements of fuzzy set \(X\) form a crisp interval, \(X\) is called an interval type 2 fuzzy set (IT2FS) \(([6-7])\).

An interval type-2 fuzzy set \(X\) can be represented by two (lower and upper) functions \(\mu^*_A(x)\) and \(\mu^*_A(x)\), which for any object \(x \in A\) provide an interval \([\mu^*_A(x), \mu^*_A(x)]\subseteq[0;1]\) of possible membership degrees of \(x\) in \(X\). An interval type-2 fuzzy number (IT2FN) is defined on a set (subset) of real numbers \(\mathbb{R}\). Arithmetic operations over IT2FNs are defined on the basis of interval arithmetic operations, which are applied to \(\mu^*_A(x)\) as well as \(\mu^*_A(x)\).

### 4. Fuzzy extension

In Section 2 we have dealt with matrix \(A\) whose elements are crisp. Analysis shows that all conclusions obtained for crisp matrix \(A\) will be true for fuzzy matrix \(A\). In particular, elements of matrix \(A\) can be TrFNs as well as IT2FNs.

Before forming a matrix of paired comparisons \(A\), it is necessary to choose the most convenient scheme of comparison. Theorem 2 gives necessary and sufficient conditions for a given set of elements \(\{a_{ir}\}\) to be a basis set. The two most evaluable schemes have been described above. These are a pattern comparison scheme and a sequential comparison scheme. The both schemes should contain \(n-1\) basis elements.

The evaluation of the components of the priority vector should be carried out on the basis of formulas (1) or (2), depending on the chosen comparison scheme. In any case these evaluations involve performing multiplication and division on fuzzy numbers. Then the resulting priority vector should be normalized. In order to normalize we will need to perform summation and division.

Let us illustrate the evaluation of the components of the fuzzy priority vector by the following two examples.

**Example 1** (a pattern comparison scheme). Let \(n = 4\). Assume that all elements of the matrix \(A\) are TrFNs and \(a_{12} = (2.5, 3, 3.5), a_{13} = l(+)(1.5, 2, 2.5) = (0.4, 0.5, 0.67)\) and \(a_{14} = (1.5, 2, 2.5)\). Due to formula (2), we have

\[
w_1 = a_{14}(+)a_{11} = (1.5, 2, 2.5)(+) = (1.5, 2, 2.5),
\]

\[
w_2 = a_{14}(+)a_{12} = (1.5, 2, 2.5)(+) = (0.43, 0.67, 2.33),
\]

\[
w_3 = a_{14}(+)a_{13} = (1.5, 2, 2.5)(+) = (0.24, 0.43, 0.73), \quad w_4 = 1.
\]

Calculate the sum

\[
w = w_1 + w_2 + w_3 + w_4 = (5.17, 0.67, 8.56)
\]

and the components of normalized priority vector in the form of TrFNs:

\[
\hat{w}_1 = w_1(+)w = (0.17, 0.26, 0.48), \quad \hat{w}_2 = w_2(+)w = (0.05, 0.09, 0.45),
\]

\[
\hat{w}_3 = w_3(+)w = (0.51, 0.13, 0.08), \quad \hat{w}_4 = 1.
\]
\[ \hat{w}_3 = w_3(+)w = (0.26, 0.52, 0.72), \hat{w}_4 = l(+)w = (0.12, 0.13, 0.19). \]

If it is necessary, the obtained TrFNs can be converted into crisp numbers using appropriate defuzzification method [8].

**Example 2** (a sequential comparison scheme). Let \( n = 4 \). Suppose that all elements of the matrix \( A \) are triangle IT2FNs and
\[ a_{12} = ((2.5, 2.8, 3.5), (2.5, 3, 3.5)), a_{34} = ((3.5, 3.8, 4.5), (3.5, 4, 4.5)), \]
\[ a_{23} = l((5.5, 5.8, 6.5), (5.5, 6, 6.5)) = ((0.15, 0.16, 0.18), (0.15, 0.17, 0.18)). \]

According to (3) we have \( w_1 = 1, w_3 = a_{34}, w_2 = a_{23} \cdot a_{34}, w_1 = a_{12} \cdot a_{23} \cdot a_{34}, \) or
\[ w_4 = ((1, 1), (1, 1)), w_3 = ((3.5, 3.8, 4.5), (3.5, 4, 4.5)), \]
\[ w_2 = ((0.15, 0.16, 0.18), (0.15, 0.17, 0.18)), (3.5, 3.8, 4.5), (3.5, 4, 4.5)) = ((0.52, 0.6, 0.81)), \]
\[ (0.52, 0.6, 0.81), w_1 = a_{12} \cdot a_{23} \cdot a_{34} = ((0.52, 0.6, 0.81)), \]
\[ ((0.52, 0.6, 0.81), (2.5, 2.8, 3.5), (2.5, 3.3, 3.5)) = ((1.25, 1.68, 2.83), (1.25, 2.04, 2.83)). \]

Hence,
\[ w = w_1 + w_2 + w_3 + w_4 = ((6.27, 7, 08, 9.14), (6.27, 7, 72, 9.14)) \]
and the components of normalized priority vector are
\[ \hat{w}_1 = ((0.14, 0.24, 0.45), (0.14, 0.26, 0.45)), \hat{w}_2 = ((0.06, 0.08, 0.13), (0.06, 0.09, 0.13)), \]
\[ \hat{w}_3 = ((0.38, 0.54, 0.72), (0.38, 0.56, 0.72)), \hat{w}_4 = ((0.11, 0.13, 0.16), (0.11, 0.14, 0.16)). \]

**5. Conclusion**

A simplified version of the AHP was extended to the case of a fuzzy multiplicative matrix of pairwise comparisons. According to this version, only \( n-1 \) basis elements of the square matrix \( A \) of order \( n \) are required to calculate the priority vector. The elements of this matrix can be arbitrary fuzzy numbers, for example, trapezoidal type 1 fuzzy numbers or interval type 2 fuzzy numbers. Formulas for calculating the priority vector have given, necessary arithmetic operations have described, and two illustrative examples have presented.

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