Identification of Time-varying \textit{in situ} Signals in Quantum Circuits

Xi Cao,\textsuperscript{1,†} Yu-xi Liu\textsuperscript{2,3} and Rebing Wu\textsuperscript{1,3,∗}

\textsuperscript{1}Department of Automation, Tsinghua University, Beijing 100084, China
\textsuperscript{2}Institute of Micro-Nano Electronics, Tsinghua University, Beijing 100084, China
\textsuperscript{3}Center for Quantum Information Science and Technology, BNRist, Beijing 100084, China

The identification of time-varying \textit{in situ} signals is crucial for characterizing the dynamics of quantum information processes placed in highly isolated environments. Under certain circumstances, they can be identified from time-resolved measurements via Ramsey interferometry experiments, but only with very special probe systems can the signals be explicitly read out, and a theoretical analysis is lacking on whether the measurement data are sufficient for unambiguous identification. In this paper, we formulate this problem as the invertibility of the underlying quantum input-output system, and derive the algebraic identifiability criterion as well as the inversion algorithm for numerically identifying the signals. The criterion and algorithm can be applied to both closed and open quantum systems, and their effectiveness is demonstrated by numerical examples.

I. INTRODUCTION

The full characterization of quantum dynamics is crucial for high-precision modeling and manipulation of quantum information processing systems. In the literature, systematic studies have been casted to the identification of quantum states and operations (as known as quantum tomography) \cite{1-3} or quantum Hamiltonians \cite{4-10}. Most of these works focus on the estimation of constant but unknown quantities, e.g., a density or process matrix \cite{4,5} or some parameters in the Hamiltonian \cite{9,10}, based on maximum-likelihood, least-square or comprehensive sensing estimators. However, the identification of unknown time-varying signals has been rarely studied so far. Such problems broadly exist in low-temperature quantum information processing systems.

∗rbwu@tsinghua.edu.cn
Fig 1: Schematic diagram of a superconducting quantum computing platform. The in situ signals are delivered from ambient signal generators that may experience distortion along the transmission line. The signals are fed into a quantum chip of multiple qubits, whose measurement signal are conducted out for identifying the in situ signals.

where in situ signals are not reachable by ambient measurement devices. For the example of superconducting quantum chips [11] shown in Fig. 1, the DC or AC control signals always experience distortion along the attenuator and the control line [11][19], but the distorted signals can only be indirectly acquired by the qubit readout signals.

Since most in situ signals cannot be directly measured, they have to be directly extracted from a quantum probe (e.g., a qubit). Under special circumstances, in situ signals can be directly readout form the qubit phase measured by Ramsey experiments (see Section [11] [20][22], but such scheme is not generalizable to more complicated systems. More seriously, as will be shown in Sec. [11] there may exist non-unique estimations among
which it is uneasy to determine the true one. Therefore, whether the signals are theoretically identifiable, and how to uniquely identify them, are to be well understood.

From a system point of view, the identification of time-resolved signals from time-varying measurements can be thought of as reversing the system’s input-output mapping \([23]\). Whether the signal is identifiable is equivalent to the left invertibility of the system (Chapter 5, \([24]\)), i.e., the property that different inputs must produce different outputs. In parallel, the right invertibility (Chapter 5, \([24]\)) is referred to as the property that any desired time-varying output can be produced by some (non-unique) input function. For examples in the classical domain, the left invertibility was applied for estimating the source of heat conduction from temperature measurements \([25]\), while the right invertibility was often used for designing tracking control of a robot along a chosen trajectory \([26]\). All studies collectively showed that the invertibility of a general input-output system is determined by its relative degree that can be specified by an inversion algorithm.

In the quantum domain, the left invertibility was first studied by Ong, Clark and Tarn \([27]\) in a nonlinear filtering problem based on non-demolition continuous-time measurements. This work showed that, under adequate Lie algebraic conditions, the time-dependent input of a quantum system can be recovered online from a single measured output. Later on, the inversion (of right invertible systems) was also applied to the quantum control design as a reference tracking problem based on virtual feedback \([28–33]\) or to the estimation of quantum states and Hamiltonians \([7, 9]\).

In this paper, we will apply the inversion-based method to the identification of time-varying \textit{in situ} signals. This can be treated as a generalization of the work of Ref. \([27, 34]\), but the measurements do not have to be non-demolitional for offline identification because one can measure the time-resolved output via ensemble average. We will also extend the invertibility criterion and inversion algorithm from the single-input-single-output case to more complicated multi-input-multi-output cases, which are useful under circumstances where multiple signals are simultaneously coupled to a multi-qubit system.

The remainder of this paper will be arranged as follows. Section \([\text{II}]\) shows how the ambiguity issue arises in a direct Ramsey-based identification examples, following which we propose the inversion-based method for analyzing the identifiability (i.e., invertibility) and reconstructing the input signals. Section \([\text{IV}]\) provides two
numerical examples, a one-qubit system with single input and a two-qubit system with multiple inputs, to show the advantage of inversion-based method, and how the singularity problem can be solved by abundant measurements. Finally, conclusions are drawn in Section V.

II. A DIRECT IDENTIFICATION SCHEME VIA RAMSEY INTERFEROMETRY

Let us start from a simple case. Suppose that the signal $u(t)$ to be identified be coupled to a single qubit probe \[^{22}\], and we expect to read out $u(t)$ through the time-resolved measurement of the qubit. A simple model for the readout process can be described by the Schrödinger equation $\dot{\psi}(t) = -iH(t)\psi(t)$, where $\psi(t)$ is the quantum state of the qubit probe and

$$H(t) = u(t)\sigma_z. \quad (1)$$

Here, $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices. The qubit is prepared at the initial superposition state $\psi(0) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and evolves as follows:

$$\psi(t) = \frac{1}{\sqrt{2}}[e^{-i\theta(t)/2}|0\rangle + e^{i\theta(t)/2}|1\rangle], \quad (2)$$

where the information about $u(t)$ is transferred to the accumulated phase

$$\theta(t) = \int_0^t u(\tau)d\tau. \quad (3)$$

In the laboratory, the phase $\theta(t)$ can be conveniently measured via a Ramsey experiment that corresponds to the expectation value of $\sigma_x$:

$$y(t) = \langle \psi(t)|\sigma_x|\psi(t)\rangle = \frac{\cos\theta(t)}{2}. \quad (4)$$

Reversing the above processes, we obtain the identification formula:

$$u(t) = \frac{d}{dt} [\pm \arccos 2y(t) + k\pi] = \frac{\mp \dot{y}(t)}{\sqrt{1 - 4y^2(t)}}. \quad (5)$$

There are two issues in this identification scheme. First, the identification formula \[^{5}\] provides two solutions that one cannot determine which one is correct, because the involved cosine function is not 1-to-1. For example,
Fig 2: Ramsey experiment based identification. The measured output (a) corresponds to two different time traces of the qubit phase (b) in which one is false. The resulting identified *in situ* input (c) is thus undecidable.

as is shown in Fig. 2, the two different signals $u(t) = \pm \sin \omega_0 t$ ($\omega_0 = 1$) accumulate different traces of phases that lead to the same measured output $y(t)$, and we are not able to judge whether $u(t) = \sin \omega_0 t$ or $u(t) = -\sin \omega_0 t$ is the real *in situ* signal. Later we will see that the signal is actually identifiable but the above identification scheme is flawed.

Second, this direct identification scheme relies on the analytical solvability of the time-dependent Schrödinger
equation (1), which is usually impossible when $H(t)$ and $H(t')$ do not commute for all $t \neq t'$. For example, the *in situ* signal cannot be simply encoded into the qubit phase when there is a bias term in the qubit probe Hamiltonian:

$$H(t) = \omega_0 \sigma_x + u(t) \sigma_z. \quad (6)$$

This issue is even severer in multi-qubit systems that are coupled with multiple input signals.

III. IDENTIFICATION SCHEME BY QUANTUM SYSTEM INVERSION

In this section, we will take the identification problem as an inverse problem of solving the output of dynamical Schrödinger equation, and derive the invertibility criterion as well the inversion algorithm for extracting the signals.

A. Single-input single-output case

To facilitate the following derivation, we assume that the probe system is a closed or a Markovian open system, so that the evolution can be described by:

$$\dot{\rho}(t) = [\mathcal{L}_0 + u(t)\mathcal{L}_1] \rho(t), \quad (7)$$

where the density matrix is initially prepared at $\rho(0) = \rho_0$. The super-operators $\mathcal{L}_0$ and $\mathcal{L}_1$, which are assumed to be precisely known, are a Liouvillian ($\mathcal{L}_0 = -i[H, \rho]$ with $H$ being the interaction Hamiltonian) or a Lindbladian (i.e., $\mathcal{L}_0 = 2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L$ with $L$ being an coupling operator). We expect to identify the input signal $u(t)$ from the time-resolved ensemble measurement

$$y(t) = \langle O \rangle \equiv \text{Tr}[\rho(t)O],$$

where $O$ is the corresponding observable.

The system is said to be left invertible (or functional observable) if for any admissible $u(t) \neq u'(t)$, their resulting outputs $y(t) \neq y'(t)$. To decide whether the system is invertible, we can differentiate the measurement
\[ \dot{y}(t) = \langle \mathcal{L}^*_0 O \rangle + \langle \mathcal{L}^*_1 O \rangle u(t), \]  

(8)

where the \( \mathcal{L}^*_k O \) \((k = 0, 1)\) represents the adjoint operation of \( \mathcal{L}_k \) on the observable \( O \), i.e., \( \text{Tr}[\mathcal{L}_k \rho O] = \text{Tr}[\rho \mathcal{L}^*_k O] = \langle \mathcal{L}^*_k O \rangle \).

If it happens that \( \langle \mathcal{L}^*_1 O \rangle \equiv 0 \), the differentiation can be repeated for \( \alpha \) times until \( \langle \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \rangle \neq 0 \), which gives

\[ y^{(\alpha)}(t) = \langle (\mathcal{L}^*_0)^\alpha O \rangle + \langle \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \rangle u(t). \]  

(9)

Then, we can formally write

\[ u(t) = \phi[\rho, y^{(\alpha)}(t)] = \frac{y^{(\alpha)}(t) - \langle (\mathcal{L}^*_0)^\alpha O \rangle}{\langle \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \rangle}, \]  

(10)

and replace it back to Eq. (7), which leads to the differential equation

\[ \dot{\rho}(t) = \left\{ \mathcal{L}_0 + \phi[\rho(t), y^{(\alpha)}(t)] \mathcal{L}_1 \right\} \rho(t). \]  

(11)

This nonlinear equation forms the inverse system of (7) because \( y(t) \) becomes the input function while the original input \( u(t) \) becomes the output through Eq. (10).

The above inversion process can be taken as a Taylor expansion (as the function of the time \( t \)) of the output function \( y(t) \). The index \( \alpha \), which is called the relative degree of the quantum system, indicates that \( u(t) \) affects \( y(t) \) via its \( \alpha \)th-order time-derivative. It can be proven that the system is invertible if and only if \( \alpha \) is a finite integer [27].

The above inversion process also provides an inversion algorithm that extracts \( u(t) \) by numerically solving the inverse system (11) from the known measurement data \( y(t) \) and the prepared initial state \( \rho(0) = \rho_0 \). Note that the condition \( \langle \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \rangle \neq 0 \) is hardly verifiable because \( \rho(t) \) is not analytically solvable. In practice, we can relax this condition to the operator \( \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \) instead of its expectation value, i.e., the system’s relative degree is \( \alpha \) if \( \mathcal{L}^*_0 (\mathcal{L}^*_1)^k O \) vanishes for \( k = 0, \ldots, \alpha - 1 \) but not for \( k = \alpha \). Under this condition, the input signals are identifiable at least on a non-empty time interval as long as \( \langle \mathcal{L}^*_1 (\mathcal{L}^*_0)^{\alpha-1} O \rangle \) is nonzero at \( t = 0 \). It
is possible that \((\mathcal{L}_1^*\mathcal{L}_0^{\alpha-1})O\) crosses zero at some nonzero time, which makes Eq. (10) singular, the multiple solutions may exist after this time instant \([30]\). Therefore, the relaxed algebraic condition is only necessary for invertibility.

Let us revisit the example discussed in Section II. The derivation of \(y(t) = \langle \sigma_x \rangle\) yields
\[
u(t) = -\frac{\dot{y}(t)}{\langle \psi(t) | \sigma_y | \psi(t) \rangle},
\]
which indicates that the system’s relative degree is 1 and hence the system is invertible at least on a non-empty time interval as long as \(\langle \psi(t) | \sigma_y | \psi(t) \rangle \neq 0\) at \(t = 0\). Therefore, the failure of Ramsey-based scheme is not due to the algorithm design but the system’s invertibility.

In comparison, we can think of the case that the measurement is chosen to be \(y(t) = \langle \psi(t) | \sigma_z | \psi(t) \rangle\). It can be verified that \(y^{(\alpha)}(t) \equiv 0\) for any integer \(\alpha\), i.e., the system is not invertible because the relative degree is infinite. In such case, there exists no algorithms by which \(u(t)\) can be uniquely identified from \(y(t)\).

However, when there is a bias term in the qubit probe Hamiltonian, as follows:
\[
\dot{\psi}(t) = -i [\omega_a \sigma_x + u(t) \sigma_z] \psi(t),
\]
we can differentiate \(y(t)\) twice to obtain:
\[
u(t) = \omega_a \langle \psi(t) | \sigma_x | \psi(t) \rangle - \frac{\omega_a^{-1} \dot{y}(t)}{\langle \psi(t) | \sigma_x | \psi(t) \rangle}.
\]
Thus, this system becomes invertible with relative degree \(\alpha = 2\) under the same measurement.

**B. Multi-input Multi-output case**

Suppose that the quantum system has multiple input signals that are to be identified from multiple time-resolved measurements, as follows:
\[
\dot{\rho}(t) = [\mathcal{L}_0 + \bar{u}(t) \cdot \vec{L}_c] \rho(t),
\]
in which \textit{in situ} signals \(\bar{u}(t) = \begin{bmatrix} u_1(t), \cdots, u_m(t) \end{bmatrix}^T\) are coupled to the system via Liouvillians (or Lindbladians) \(\vec{L}_c = (\mathcal{L}_1, \cdots, \mathcal{L}_m)\). The dot product is referred to as the inner product \(\bar{u}(t) \cdot \vec{L} = \sum_{k=1}^{m} u_k(t) \mathcal{L}_k\). We expect to
identify these signals from n measurements $\vec{y}(t) = [y_1(t), \cdots, y_n(t)]^T$ with $y_i(t) = \langle O_\ell \rangle$ being the expectation value of the corresponding observable $O_\ell$.

Similarly, the quantum system is said to be invertible if for any two different input vectors $\vec{u}(t) \neq \vec{u}'(t)$, the resulting output vectors $\vec{y}(t) \neq \vec{y}'(t)$. According to Eq. (15), we differentiate $\vec{y}(t)$ and obtain:

$$\dot{\vec{y}} = \langle L_0^* \vec{O} \rangle + \langle \vec{L}_c^* \vec{O} \rangle \vec{u},$$

where

$$\langle \vec{L}_c^* \vec{O} \rangle = \left( \langle \vec{L}_c^* O_1 \rangle, \cdots, \langle \vec{L}_c^* O_n \rangle \right)^T,$$

with

$$\langle \vec{L}_c^* O_k \rangle = (\langle L_1^* O_k \rangle, \cdots, \langle L_m^* O_k \rangle)$$

for $k = 1, 2, \cdots, n$. Similarly, to avoid the evaluation of expectations, we analyze the corresponding operators. First, operator arrays $\vec{L}_c^* O_i$ and $\vec{L}_c^* O_p$ are said to be linearly independent if

$$\lambda_1 \vec{L}_c^* O_i + \cdots + \lambda_p \vec{L}_c^* O_p \neq 0$$

for any nonzero real numbers $\lambda_1, \cdots, \lambda_p$, and the rank of a group of operator arrays is referred to as the maximal number of mutually independent arrays in them. If there exist $m$ mutually linearly independent arrays among $\vec{L}_c^* O_1, \cdots, \vec{L}_c^* O_n$, i.e., $\text{rank}(\vec{L}_c^* \vec{O}) = m$ then the signals can be formally calculated as a least-square solution of

$$\vec{u}(t) = \phi[\rho, \dot{\vec{y}}] = \left( \langle \vec{L}_c^* \vec{O} \rangle^T \langle \vec{L}_c^* \vec{O} \rangle \right)^{-1} \langle \vec{L}_c^* \vec{O} \rangle \left[ \dot{\vec{y}} - \langle L_0^* \vec{O} \rangle \right],$$

This formula is then replaced back to Eq. (15) to obtain the following inverse system

$$\dot{\rho}(t) = \left\{ L_0 + \dot{\phi}[\rho(t), \dot{\vec{y}}(t)] \cdot \vec{L}_c \right\} \rho(t).$$

Similar to the single-input-single-output systems, the expectation $\langle \vec{L}_c^* \vec{O} \rangle$ may become rank deficient at some time instance even when the operator rank condition $\text{rank} \left[ \vec{L}_c^* \vec{O} \right] = m$ is satisfied. Moreover, the operator rank
of $\hat{L}_c^* \hat{O}$ may also be lower than $m$, under which circumstance Eq. [19] has no unique solutions for all time $t$. In such case, we need to extract $\hat{u}(t)$ via higher-order derivatives of $\hat{y}(t)$. To do this, we first divide $\hat{O} = (\hat{O}_1, \hat{O}_1)$ such that

$$\text{rank} \left[ \hat{L}_c^* \hat{O}_1 \right] = \text{rank} \left[ \hat{L}_c^* \hat{O} \right],$$

and $\hat{L}_c^* \hat{O}_1$ is linearly dependent with the arrays of $\hat{L}_c^* \hat{O}_1$, i.e., there exists a matrix $V_{11}$ such that $\hat{L}_c^* \hat{O}_1 = V_{11} \hat{L}_c^* \hat{O}_1$.

Let $\hat{y}_1 = \langle \hat{O}_1 \rangle$ and $\hat{y}_1 = \langle \hat{O}_1 \rangle$. We differentiate them

$$\langle L_0^* \hat{O}_1 \rangle + \langle \hat{L}_c^* \hat{O}_1 \rangle \hat{u} = \dot{\hat{y}}_1 \quad (21)$$

$$\langle L_0^* \hat{O}_1 \rangle + \langle \hat{L}_c^* \hat{O}_1 \rangle \hat{u} = \dot{\hat{y}}_1 \quad (22)$$

and then eliminate $\hat{u}$ in the second equation using the relation $\hat{L}_c^* \hat{O}_1 = V_{11} \hat{L}_c^* \hat{O}_1$, which gives $\hat{y}_2 = \langle \hat{O}_2 \rangle$, where

$$\hat{O}_2 = L_0^* \hat{O}_1 - V_{11} L_0^* \hat{O}_1, \quad \hat{y}_2 = \dot{\hat{y}}_1 - V_{11} \dot{\hat{y}}_1.$$ 

This equation can further differentiated to produce a new group of linear equations of $\hat{u}$:

$$\langle L_0^* \hat{O}_2 \rangle + \langle \hat{L}_c^* \hat{O}_2 \rangle \hat{u} = \dot{\hat{y}}_2 \quad (23)$$

If $\hat{L}_c^* \hat{O}_1$ and $\hat{L}_c^* \hat{O}_2$ includes $m$ linearly independent rows of operators, we can let $\hat{O}_2 = \hat{O}_2$ and halt the process. Otherwise, we can do the same operation on $\hat{O}_2$ by separating its linearly independent part. Inductively, if the system is invertible, we can obtain a transformation $\hat{O}' = V \hat{O} = (\hat{O}_1, \cdots, \hat{O}_n)^T$ of the observables after repeatedly doing the above differentiation process, which leads to the following group of equations:

$$\langle L_0^* \hat{O}_1 \rangle + \langle \hat{L}_c^* \hat{O}_1 \rangle \hat{u} = f_1 [\hat{y}^{(1)}, \cdots, \hat{y}^{(\alpha_r)}]$$

$$\vdots$$

$$\langle L_0^* \hat{O}_n \rangle + \langle \hat{L}_c^* \hat{O}_n \rangle \hat{u} = f_n [\hat{y}^{(1)}, \cdots, \hat{y}^{(\alpha_r)}],$$

in which

$$\text{rank} \left( \hat{L}_c^* \hat{O}' \right) = m \quad (24)$$

10
and \( \tilde{g}_k = f[\tilde{g}^{(1)}, \ldots, \tilde{g}^{(\alpha_k)}] \), \( k = 1, \ldots, r \), are linear functions of the derivatives of \( \tilde{y} \) with \( \alpha_1 < \cdots < \alpha_r \). The required highest order of differentiation, \( \alpha_r \), is defined as the relative degree of the multi-input-multi-output system. The system is invertible if and only if the relative degree is finite.

From the above inversion process, an obvious fact is that, to guarantee the transformation of observables exists, the number of measurement outputs must not be less than \( m \). In practice, one can introduce more (redundant) measurement outputs (i.e., \( n > m \)), which will help release the singularity issue occurring when solving the linear equation because the coefficient matrix is less singular when more rows are involved. This will be shown in the following numerical simulations.

IV. NUMERICAL SIMULATIONS

In this section, we carry out numerical simulations to show how the inversion-based algorithm can be applied for identifying \textit{in situ} signals in quantum circuits.

We simulated the example discussed in Section II using the inversion algorithm, where the input signal is chosen as a standard sinusoidal signal \( u(t) = \sin \omega_0 t \) (\( \omega_0 = 1 \)). As shown in Fig. 3, the identification result is uniquely determined and is perfectly identical to the actual input signal. However, the test of other signals, e.g., \( u(t) = u_0 + \sin \omega_0 t \) with a DC component, encounters singularity due to the vanishing of the denominator in (19) at some critical time \( t \neq 0 \). The numerical simulation became unstable near the critical time and converges to a false solution. In other simulations, the system may also come back to the true solution.

This bifurcation shows that multiple solutions exist after the critical time [30], and the inversion algorithm itself cannot determine which one is the actual signal to be identified. To further verify this point, we independently apply the least-square identification method for the same problem [35], which is done by minimizing the following least-square error:

\[
D[u(t)] = \| y(t) - F[u(t)] \|^2
\]  

(25)

where \( y(t) \) is the experimental measurement data and \( F(u(t)) \) is the calculated output signal through Eq. [13]. The least-square error is minimized by a gradient algorithm starting from some initial guess on \( u(t) \). We
simulated this method with three initial guesses $u^{(1)}(t) = 0$, $u^{(2)}(t) = 7 - 0.5t$ and $u^{(3)}(t) = \begin{cases} 7 & t < 15 \\ -7 & t \geq 15 \end{cases}$, which converge to distinct trajectories like in the inversion-based scheme. The identified signals match the actual signal before the critical time $t = 15$, but the signal iterated from initial guess $u^{(3)}(t)$ diverges at the critical time. Its shape is different from that via the inversion algorithm due to the numerical error caused by discretization of the time interval. The signal iterated from initial guess $u^{(2)}(t)$ diverges after the critical time, which means more than two solutions exist after the singularity point.

Therefore, as the system’s intrinsic property, the inversion-based algorithm may also fail to deterministically identify the signal. But different from the Ramsey-experiment based scheme, the inversion algorithm can at least identify the signal on a non-empty time interval. The singularity cannot be removed by any particular identification algorithm. One must collect more information to uniquely determine the input signal, e.g., introducing redundant measurements. In the simulation, we introduce the second measurement $y'(t) = \langle \psi(t)|\sigma_y|\psi(t)\rangle$ the two measurements are never simultaneously zero. As is seen in Fig. 3, we use the inversion-based solution to reproduce the input signal from two outputs, and the identification results perfectly matches the actual signal.

A. Simultaneous readout of multiple inputs

The inversion-based method is also advantageous in that multiple inputs can be simultaneously readout through multiple measurements. For illustration, we consider the following two-qubit system:

$$H(t) = u_1(t)\sigma_{1z} + u_2(t)\sigma_{2z} + u_3(t)\sigma_{1z}\sigma_{2z} + g\sigma_{x1}\sigma_{x2},$$

which involves three in situ signals $u_1(t)$, $u_2(t)$ and $u_3(t)$ corresponding to the qubits’ bias signal and coupling strength. Since $[H(t),H(t')] = 0$ for all $t,t'$, one may also encode the in situ signals into the qubit phases that can be readout by Ramsey interferometry experiments, but the ambiguity issue is still present.

Let us see how the inversion-based algorithm works. We need at least three time-resolved measurements, say

$$\vec{O} = (\sigma_{1x}, \sigma_{1y}, \sigma_{2x})^T,$$
Fig 3: Identification of the *in situ* signal $u(t) = 5 + \sin \omega_0 t$ ($\omega_0 = 1$) with (a) a single measurement, (b) two measurement by using inverse-system based method and with initial guess, (c) $u^1(t)$, (d) $u^2(t)$, (e) $u^3(t)$ by using least-square method. The single measurement case identification becomes singular when $\omega_0 t = 15$.

and the corresponding observable arrays

$$\vec{L}_c^* O_1 = (\sigma_{1y}, 0, \sigma_{1y} \sigma_{2z}),$$

$$\vec{L}_c^* O_2 = (-\sigma_{1x}, 0, \sigma_{1x} \sigma_{2z}),$$

$$\vec{L}_c^* O_3 = (0, \sigma_{2y}, \sigma_{1z} \sigma_{2y})$$

can be examined to be linearly independent with each other. Hence, the system is invertible at least on a non-
Fig 4: The identified \textit{in situ} signals $u_1(t) = u_2(t) = u_3(t) = 5 + \sin(\omega_0 t)$ ($\omega_0 = 1$) of the two-qubit system based on three (red dash), four (green dash) and five (blue dash) measurement outputs by using inverse-system based method; The identified signals from least-square method (black dash).

empty time interval with relative degree being $\alpha = 1$. We simulate the identification process with \textit{in situ} signals $u_1(t) = u_2(t) = u_3(t) = 5 + \sin(\omega_0 t)$ ($\omega_0 = 1$). As is shown in Fig. 4, the identified signals based on inverse-system method are all identical to the actual ones until $\omega_0 t \approx 15$rad. The third input $u_3(t)$ firstly diverges, followed by $u_1(t)$ and $u_2(t)$ at $\omega_0 t \approx 18$rad and $\omega_0 \approx 25$rad, respectively due to the singularity of $\langle \vec{L}^* \vec{O} \rangle$. To alleviate the singularity issue, we introduce two additional time-resolved measurements $O_4 = \sigma_{2y}$ and $O_5 = \sigma_{1z}\sigma_{2y}$. The
simulation results show that using four measurements does not make the identification better (converging to the other solution faster), but using five measurement can completely remove the singularity. Compared to the inverse-model based method, the identified signals by using least-square method diverge a lot from the input signals when there are multi-inputs. It means the least-square method is not stable for this input identification issue, which can easily converge to other solutions.

B. The affection of noises

The prevalently existing noises in realistic quantum systems can affect the quality of identification or even destroy it. Taking the one-qubit probe as an example, we consider two typical classes of noises present in the following system:

\[
\dot{\psi}(t) = -i \left[ n_e(t) \sigma_z + u(t) \sigma_x \right] \psi(t), \\
y(t) = \langle \psi(t)|\sigma_x|\psi(t) \rangle + n_m(t).
\]

The noise \(n_e(t)\) comes from unwanted coupling to unspecified signals (e.g., crosstalk via some other qubit’s input), and \(n_m(t)\) comes from the imprecise measurement due to the randomness of quantum measurements or imperfect devices. According to Eq. 10, the measurement noise, especially its high-frequency components, can have fatal effect on the quality of readout results because it can be greatly amplified by the differentiation of \(y(t)\). The system is less affected by the high-frequency components of noise \(n_e(t)\) because they tend to be filtered.

In the simulation, we simulated low frequency (comparable with the frequency of the signals) and high frequency (25-30 times of the frequency of the signals) random noise in system \([n_e(t)]\). The variance of the noise are taken to be \(10^{-2}\) for \(n_e(t)\) and \(10^{-6}\) for \(n_m(t)\), respectively. As shown in Fig. 5, the measurement noise distorts the calculated in situ signal dramatically even with a smaller variance, especially when the frequency is high. By contrast, the high frequency noises in the system has minor affection on the identification. Therefore, in practice, the measured output should be carefully filtered to reduce the noise affect while keeping the signal undistorted as much as possible.
Fig 5: (a). The calculated in situ signal when there is low frequency system noise ($n_s$). (b). The calculated in situ signal when there is high frequency system noise. (c). The calculated in situ signal when there is low frequency measurement noise ($n_m$). (d). The calculated in situ signal when there is high frequency measurement noise.

V. CONCLUSION

To conclude, we propose the inverse-system based method to unambiguously identify time-varying in situ signals from time-resolved measurements. Although the signals are usually locally identifiable (i.e., likely diverge at some critical time due to the singularity), the proposed method still greatly generalizes the existing Ramsey-experiment-based schemes to arbitrary multi-input-multi-output systems, as long as the algebraic invertibility condition is satisfied. The simulation results show that it can perfectly extract the in situ signal in both single-
input and multiple-input systems by integrating the nonlinear Schrödinger equation. Although, the algorithm is applicable only on a finite time interval due to potential singularity, one can properly introduce redundant measurements to prolong the applicable time interval. The affection and limitation brought by system’s noises are also analyzed through numerical simulations.

The method we developed can be naturally generalized to any other quantum systems, no matter closed or open, as long as the probe system can be precisely modeled and the modeled system is invertible. In principle, one can freely choose the time-resolved measurements for extracting the in situ signals according to the invertibility condition. However, in practice one should pick those with lowest relative degree, so as to minimize the influence of measurement noise.

VI. ACKNOWLEDGMENTS

This work is supported by the National Key R&D Program of China (Grants No. 2017YFA0304304), NSFC (Grants No. 61833010 and No. 61773232.

[1] K. Banaszek, M. Cramer, and D. Gross, New Journal of Physics 15, 5020 (2013).
[2] A. Anis and A. I. Lvovsky, New Journal of Physics 14, 105021 (2012).
[3] A. M. Brączyk, D. H. Mahler, L. A. Rozema, A. Darabi, A. M. Steinberg, and D. F. V. James, 14, 085003 (2012).
[4] K. Maruyama, D. Burgarth, A. Ishizaki, T. Takui, and K. B. Whaley, Quantum Information & Computation 12, 763 (2012).
[5] Y. Wang, B. Qi, D. Dong, and I. R. Petersen, in 2016 IEEE 55th Conference on Decision and Control (CDC) (2016), pp. 2523–2528.
[6] S.-Y. Hou, H. Li, and G.-L. Long, Science Bulletin 62, 863 (2017).
[7] C. Le Bris, M. Mirrahimi, H. Rabitz, and G. Turinici, Esaim-control Optimisation and Calculus of Variations 13, 378 (2007).
[8] J. Zhang and M. Sarovar, Physical Review Letters 113, 080401 (2014).
[9] O. F. Alis, H. Rabitz, M. Q. Phan, C. Rosenthal, and M. Pence, Journal of Mathematical Chemistry 35, 65 (2004).
[10] R. B. Wu, T. F. Li, A. G. Kofman, J. Zhang, Y.-X. Liu, Y. A. Pashkin, J.-S. Tsai, and F. Nori, Phys. Rev. A 87, 022324 (2013).

[11] B. R. Johnson, *Controlling photons in superconducting electrical circuits* (Yale University, 2011).

[12] I. N. Hincks, C. E. Granade, T. W. Borneman, and D. G. Cory, Phys. Rev. Applied 4, 024012 (2015).

[13] F. Motzoi, J. M. Gambetta, S. T. Merkel, and F. K. Wilhelm, Phys. Rev. A 84, 022307 (2011).

[14] P. E. Spindler, Y. Zhang, B. Endeward, N. Gershernzon, T. E. Skinner, S. J. Glaser, and T. F. Prisner, Journal of Magnetic Resonance 218, 49 (2012).

[15] S. J. Glaser, U. Boscain, T. Calarco, C. P. Koch, W. Kckenberger, R. Kosloff, I. Kuprov, B. Luy, S. Schirmer, T. Schulte-Herbrueggen, et al., The European Physical Journal D 69 (2015).

[16] W. Rose, H. Haas, A. Q. Chen, N. Jeon, L. J. Lauhon, D. G. Cory, and R. Budakian, Phys. Rev. X 8, 011030 (2018).

[17] R. Patterson, A. Hammoud, and M. Elbuluk, Cryogenics 46, 231 (2006).

[18] R. Patterson, A. Hammoud, J. Dickman, S. Gerber, M. Elbuluk, and E. Overton (2004).

[19] X. Cao, B. Chu, H. Ding, L. Sun, Y.-x. Liu, and R. Wu, arXiv preprint (2018).

[20] M. Hofheinz, E. Weig, M. Ansmann, R. C. Bialczak, E. Lucero, A. Neeley, M. and, H. Wang, J. M. Martinis, and A. Cleland, Nature 454, 310 (2008).

[21] J. M. Gambetta, A. D. Córcoles, S. T. Merkel, B. R. Johnson, J. A. Smolin, J. M. Chow, C. A. Ryan, C. Rigetti, S. Poletto, T. A. Ohki, et al., Phys. Rev. Lett. 109, 240504 (2012).

[22] D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier, C. Urbina, D. Esteve, and M. Devoret, Fortschritte der Physik 51, 462 (2003).

[23] R. Hirschorn, IEEE Transactions on Automatic Control 24, 855 (1979).

[24] H. Sussmann, ed., *Nonlinear Controllability and Optimal Control* (Taylor & Francis Group, 1990).

[25] L. F. CAUDILL, H. RABITZ, and A. ASKAR, Inverse Problems 10, 1099 (1994).

[26] R. M. Robinson, D. R. R. Scobee, S. A. Burden, and S. Shankar Sastry, p. 98361X (2016).

[27] C. K. ONG, G. M. HUANG, T. J. TARN, and J. W. CLARK, Mathematical Systems Theory 17, 335 (1984).

[28] P. Gross, H. Singh, H. Rabitz, K. Mease, and G. M. Huang, Phys. Rev. A 47, 4593 (1993).

[29] Z. M. LU and H. RABITZ, Journal of Physical Chemistry 99, 13731 (1995).

[30] A. Jha, V. Beltran, C. Rosenthal, and H. Rabitz, Journal of Physical Chemistry A 113, 7667 (2009).

[31] E. Brown and H. Rabitz, Journal of Mathematical Chemistry 31, 17 (2002).
[32] W. Zhu, M. Smit, and H. Rabitz, The Journal of Chemical Physics 110, 1905 (1999).

[33] W. Zhu and H. Rabitz, The Journal of Chemical Physics 119, 3619 (2003).

[34] J. W. CLARK, C. K. ONG, T. J. TARN, and G. M. HUANG, Mathematical Systems Theory 18, 33 (1985).

[35] J. Huang, H. Chen, and L. I. Luoqing, International Journal of Wavelets Multiresolution & Information Processing 10, 337 (2012).