On Quantum Obstruction Spaces and Higher Codimension Gauge Theories

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(Dated: August 20, 2020)

Using the quantum construction of the BV-BFV method for perturbative gauge theories, we show that the obstruction for quantizing a codimension 1 theory is given by the second cohomology group with respect to the boundary BRST charge. Moreover, we give an idea for the algebraic construction of codimension $k$ quantizations in terms of $E_k$-algebras and higher shifted Poisson structures by formulating a higher version of the quantum master equation.

Keywords: Quantum Field Theory, Gauge Theory, BV-BFV Formalism, Deformation Quantization, Extended Field Theory, Shifted Poisson Structures, Higher Categories

I. INTRODUCTION

The Batalin–Vilkovisky formalism [5–7] is a powerful method to deal with perturbative quantizations of local gauge theories. The extension of this formalism to manifolds with boundary combines the Lagrangian approach of the Batalin–Vilkovisky (BV) formalism in the bulk with the Hamiltonian approach of the Batalin–Fradkin–Vilkovisky (BFV) formalism [4, 19] on the boundary of the underlying source (spacetime) manifold. This construction is known as the BV-BFV formalism [12–14]. In particular, it describes a codimension 1 quantum gauge formalism. Within a classical gauge theory one is interested in describing the obstructions for it to be quantizable. The cohomological symplectic formulation suggests an operator quantization for the boundary action. To get a well-defined and consistent cohomology theory, one has to require that this induced operator squares to zero. This will lead to obstruction spaces for boundary theories by considering a deformation quantization of the boundary action in order to formulate a boundary version of the quantum master equation as the gauge-independence condition. We will show that the obstruction for the quantization of manifolds with boundary is controlled by the second cohomology group with respect to the cohomological vector field on the boundary fields. Moreover, we formulate a classical extension of higher codimension $k$ theories as in [12] which we call BF$^k$V theories. The coupling for each stratum, in fact, is easily extended in the classical setting (BV-BF$^k$V theories), whereas for the quantum setting it might be rather involved. In order to formulate a fully extended topological quantum field theory in the sense of Baez–Dolan [3] or Lurie [26], the coupling is indeed necessary. Since one layer of the quantum picture, namely the quantum master equation, is described in terms of deformation quantization, we can formulate an algebraic approach for the higher codimension extension in terms of $E_k$- and $P_k$-algebras [25, 31]. Here $E_k$ denotes the $\infty$-operad of little $k$-dimensional disks [20, 23, 25]. Moving to one codimension higher corresponds to the shift of the Poisson structure by $-1$ since the symplectic form is shifted by $+1$ (see [29] for the shifted symplectic setting). This is controlled by the operad $P_k$ on codimension $k$ which corresponds to $(1-k)$-shifted Poisson structures [9, 30].

Using this notion, we give some ideas for the quantization in higher codimension. Moreover, if one uses the notion of Beilinson–Drinfeld (BD) algebras [8, 16], in particular $\mathbb{BD}_0$- and $\mathbb{BD}_1$-algebras, one can try to consider the action of $P_0 \cong \mathbb{BD}_0(h)$ (for $h \to 0$) on $P_1 \cong \mathbb{BD}_1(h)$ (for $h \to 0$) in order to capture the algebraic structure of the classical bulk-boundary coupling (see also [30, Section 5]). Here $\cong$ denotes an isomorphism of operads. In general, one can define the $BD_k$ operads to provide a certain interpolation between the $P_k$ and $E_k$ operads in the sense that they are graded Hopf [24] differential graded (dg) operads over $K[h]$ where $h$ is of weight 1 and $K$ a field of characteristic zero, together with the equivalences

$$\mathbb{BD}_k/h \cong P_k, \quad \mathbb{BD}_k[h^{-1}] \cong E_k(h).$$

The formality of the $E_k$ operad [20, 23, 35] implies the equivalence $\mathbb{BD}_k \cong P_k[h]$. There is a formulation of a $\mathbb{BD}_{2r}$-algebra in terms of brace algebras [10, 31] and one can show that there is in fact a quasi-isomorphism $P_2 \cong \mathbb{BD}_2/h$ (for $h \to 0$). However, the notion of a $\mathbb{BD}_k$-algebra for $k \geq 3$ in terms of braces is currently not defined, but there should not be any obstruction to do this. Using these operads, one can define a deformation quantization of a $P_{k+1}$-algebra $A$ to be a $\mathbb{BD}_{k+1}$-algebra $A_h$ together with an equivalence of $P_{k+1}$-algebras $A_h/h \cong A$ (see [9, 28] for a detailed discussion).

Notation and conventions

We will denote functions on a space $X$ by $\mathcal{O}(X)$. Local functions on $X$ will be denoted by $\mathcal{O}_{loc}(X)$. We denote the space of vector fields on a manifold $M$ by $\mathfrak{X}(M)$ and the space of differential $k$-forms on $M$ by $\Omega^k(M)$. We denote by $A[t]$ the space of formal power series in a formal parameter $t$ with coefficients in some algebra $A$. The imaginary unit is denoted by $i := \sqrt{-1}$. If the manifolds are infinite-dimensional, they are usually Banach or Fréchet manifolds.

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II. OBSTRUCTION SPACES FOR
QUANTIZATION ON MANIFOLDS WITH
BOUNDARY

A. Obstruction space in the bulk

We start with the BV approach for the bulk theory. A BV manifold is a triple

\((\mathcal{F}, S, \omega)\)

such that \(\mathcal{F}\) is a \(\mathbb{Z}\)-graded supermanifold, \(S \in \mathcal{O}_{loc}(\mathcal{F})\) is an even function of degree 0, and \(\omega \in \Omega^2(\mathcal{F})\) an odd symplectic form of degree \(-1\). The \(\mathbb{Z}\)-grading corresponds to the \textit{ghost number} which we will denote by “gh”. The BV space of fields \(\mathcal{F}\) is usually given as the \((-1)\)-shifted cotangent bundle of the BRST space of fields, i.e. \(\mathcal{F}_{BV} := T^*[\mathbb{Z}]\mathcal{F}_{BRST}\). In many cases, \(\mathcal{F}\) is an infinite-dimensional Fréchet manifold. Denote by \(Q\) the Hamiltonian vector field of \(S\) of degree +1, i.e. \(L_Q \omega = \delta S\), where \(\delta\) denotes the de Rham differential on \(\mathcal{F}\). If we denote by \((\cdot, \cdot)\) the odd Poisson bracket induced by the odd symplectic form \(\omega\) (also called the \textit{anti bracket}, or \textit{BV bracket}), we get

\[ Q = (S, \cdot). \]

Note that, by definition, \(Q\) is \textit{cohomological}, i.e. \(Q^2 = 0\). Moreover, \(Q\) is a symplectic vector field, i.e. \(L_Q \omega = 0\), where \(L\) denotes the Lie derivative. For a BV theory we require the \textit{classical master equation (CME)}

\[ Q(S) = (S, S) = 0 \quad (\text{II.1}) \]

to hold. It is well known that the obstruction space for quantization in the BV formalism is given by the first cohomology group with respect to \(Q\). We call the assignment of a source manifold to a BV manifold a BV theory.

**Theorem II.1.** The obstruction space for a BV theory to be quantizable is given by

\[ H^1_Q(\mathcal{O}_{loc}(\mathcal{F})). \quad (\text{II.2}) \]

**Proof.** Consider a deformation of the BV action \(S\), denoted by \(S_h\), depending on \(h\) and consider its expansion as a formal power series

\[ S_h := S_0 + hS_1 + h^2S_2 + \mathcal{O}(h^3) \]

\[ = \sum_{k \geq 0} h^k S_k \in \mathcal{O}_{loc}(\mathcal{F})[h], \quad (\text{II.3}) \]

where each \(S_k \in \mathcal{O}_{loc}(\mathcal{F})\) for all \(k \geq 0\) and \(\lim_{h \to 0} S_k = S\), i.e. \(S_0 := S\). Note that \(ghS_k = 0\) for all \(k \geq 0\) since \(ghS = 0\). For the quantum BV picture one should note that there is a canonical second order differential operator \(\Delta\) on \(\mathcal{O}_{loc}(\mathcal{F})\) such that \(\Delta^2 = 0\). It is called BV Laplacian (see [21, 38] for a mathematical exposure).

In particular, if \(\Phi^i\) and \(\Phi^+_i\) denote field and anti-field respectively, one can define \(\Delta\) as

\[ \Delta f = \sum_i (-1)^{gh \Phi^i + 1} f \left( \begin{array}{c} \delta \\ \delta \Phi^i \end{array} \right), \quad f \in \mathcal{O}_{loc}(\mathcal{F}). \]

We have denoted by \(\frac{\delta}{\delta \Phi^i}\) and \(\frac{\delta}{\delta \Phi^+_i}\) the left and right derivatives with respect to \(\Phi^i\). An analogue version also holds for the anti-fields \(\Phi^+_i\). In fact, we have

\[ \frac{\delta}{\delta \Phi^i} f = (-1)^{gh \Phi^i} (gh f + 1) \frac{\delta}{\delta \Phi^i}, \quad (\text{II.4}) \]

\[ \frac{\delta}{\delta \Phi^+_i} f = (-1)^{(gh \Phi^i + 1)(gh f + 1)} \frac{\delta}{\delta \Phi^+_i}, \quad (\text{II.5}) \]

To observe gauge-independence in the BV formalism, one requires the \textit{quantum master equation (QME)}

\[ \Delta \exp(S_h/h) = 0 \iff (S_h, S_h) + 2h\Delta S_h = 0 \quad (\text{II.6}) \]

to hold. Here we denote by \(\Delta\) the BV Laplacian. Solving (II.6) for each order in \(h\), we get the system of equations

\[ (S_0, S_0) = 0, \quad (\text{II.7}) \]

\[ \Delta S_0 = (S_0, S_1), \quad (\text{II.8}) \]

\[ \Delta S_1 = (S_0, S_2) + \frac{1}{2} (S_1, S_1) \quad (\text{II.9}) \]

\[ \vdots \quad (\text{II.10}) \]

Note that Equation (II.7) is the CME which we assume to hold. Then, using the CME and the formula

\[ \Delta(f, g) = (f, \Delta g) - (-1)^{gh g} (\Delta f, g), \quad \forall f, g \in \mathcal{O}_{loc}(\mathcal{F}), \]

we get

\[ 0 = \Delta(S_0, S_0) = (S_0, \Delta S_0) = Q(\Delta S_0). \]

Hence \(\Delta S_0\) is closed with respect to the coboundary operator \(Q = (S_0, \cdot)\). Moreover, if we assume that it is also \(Q\)-exact, we get that there is some \(S_1 \in \mathcal{O}_{loc}(\mathcal{F})\) such that \(\Delta S_0 = Q(S_1) = (S_0, S_1)\), which is exactly the statement of Equation (II.8). This will automatically imply that all the higher order equations hold. Indeed, if \(\Delta S_1 = (S_0, S_1)\) for some \(S_1 \in \mathcal{O}_{loc}(\mathcal{F})\), we get

\[ 0 = \Delta(S_0, S_1) = (\Delta S_0, S_1) - (-1)^{gh S_1}(S_0, \Delta S_1) \]

\[ = ((S_0, S_1), S_1) - (S_0, \Delta S_1), \quad (\text{II.11}) \]

where we used \(\Delta^2 = 0\). Using the graded Jacobi formula for the BV bracket, we get

\[ ((S_0, S_1), S_1) = (S_0, (S_1, S_1)) - \]

\[ - (-1)^{(gh S_0-1)(gh S_1-1)}(S_1, (S_0, S_1)). \quad (\text{II.12}) \]
Furthermore, by graded commutativity of the BV bracket we have
\[
((S_0, S_1), S_1) = (-1)^{\text{gh}(S_0, S_1) - 1} \text{gh}(S_1 - 1)(S_1, (S_0, S_1)). \tag{II.13}
\]
Now since
\[
\text{gh}(S_0, S_1) = \text{gh} S_0 + \text{gh} S_1 - \text{gh}( ,)
\]
we get
\[
2((S_0, S_1), S_1) = (S_0, (S_1, S_1)).
\]
Hence, using Equation (II.11), we get
\[
(S_0, \Delta S_1) = \left( S_0, \frac{1}{2} (S_1, S_1) \right).
\]
This will give us
\[
\Delta S_1 = \frac{1}{2} (S_1, S_1) + Q\text{-exact term},
\]
so we can find some \( S_2 \in \mathcal{O}_{\text{loc}}(\mathcal{F}) \) such that the \( Q \)-exact term is given by \( Q(S_2) = (S_0, S_2) \). This implies that Equation (II.9) holds. The higher order equations hold in a similar iterative computation. \( \square \)

**B. Obstruction space on the boundary**

Let us describe the BFV approach for the space of boundary fields. A BFV manifold is a triple
\[
(\mathcal{F}^\partial, \omega^\partial, Q^\partial),
\]
where \( \mathcal{F}^\partial \) is a \( \mathbb{Z} \)-graded supermanifold, \( \omega^\partial \in \Omega^2(\mathcal{F}^\partial) \) an even symplectic form of ghost number 0 and \( Q^\partial \) cohomological and symplectic vector field of degree +1 with odd Hamiltonian function \( S^\partial \in \mathcal{O}_{\text{loc}}(\mathcal{F}^\partial) \) of ghost number +1, i.e. \( \iota_{\omega^\partial} \omega^\partial = \delta S^\partial \), where \( \delta \) denotes the de Rham differential on \( \mathcal{F}^\partial \). Moreover, we want
\[
Q^\partial(S^\partial) = \{ S^\partial, S^\partial \} = 0.
\]
We say that a BFV manifold is *exact*, if there exists a primitive 1-form \( \alpha^\partial \), such that \( \omega^\partial = \delta \alpha^\partial \). A BV-BFV manifold over an exact BFV manifold \( (\mathcal{F}^\partial, \omega^\partial = \delta \alpha^\partial, Q^\partial) \) is a quintuple
\[
(\mathcal{F}, \omega, S, Q, \pi),
\]
where \( \pi : \mathcal{F} \to \mathcal{F}^\partial \) is a surjective submersion such that
- \( \delta \pi Q = Q^\partial \),
- \( \iota_{\omega} \omega = \delta S + \pi^* \alpha^\partial \).

A consequence of this definition is
\[
Q(S) = \pi^* (2S^\partial - \iota_{\omega^\partial} \alpha^\partial) \tag{II.14}
\]
which is called the *modified classical master equation* (mCME). Similarly as for BV theories one can ask about the quantization obstruction for a BV-BFV theory, i.e. for a codimension 1 theory. In fact, we get the following theorem.

**Theorem II.2.** Let \( (\mathcal{F}, \omega, S, Q, \pi; \mathcal{F} \to \mathcal{F}^\partial) \) be a BV-BFV manifold over an exact BFV manifold \( (\mathcal{F}^\partial, \omega^\partial = \delta \alpha^\partial, Q^\partial) \). The obstruction space for quantization on the underlying boundary BFV manifold \( \mathcal{F}^\partial \) is given by
\[
H^2_Q(\mathcal{O}_{\text{loc}}(\mathcal{F}^\partial)), \tag{II.15}
\]
where
\[
Q^\partial = \{ S^\partial, \}
\]
with \( \{ \cdot, \cdot \} \) the Poisson bracket induced by the symplectic form \( \omega^\partial \).

**Proof.** Consider a deformation of the BFV action \( S^\partial \), denoted by \( S^\partial_k \), depending on \( h \) and consider its expansion as a formal power series
\[
S^\partial_k := S^\partial_0 + h S^\partial_1 + h^2 S^\partial_2 + O(h^3)
\]
where \( S^\partial_k \in \mathcal{O}_{\text{loc}}(\mathcal{F}^\partial) \) for all \( k \geq 0 \) such that \( S^\partial_0 := S^\partial \). Note that \( gh S^\partial_0 = +1 \) since \( gh S^\partial = +1 \) and the corresponding symplectic form \( \omega^\partial \) is even of ghost number 0. In the BV-BFV construction one assumes a symplectic splitting of the BV space of fields
\[
\mathcal{F} = \mathcal{B} \times \mathcal{Y} \tag{II.17}
\]
where the BV symplectic form \( \omega \) is constant on \( \mathcal{B} \). One should think of \( \mathcal{B} \) as the boundary part and \( \mathcal{Y} \) as the bulk part of the fields. In fact, the space \( \mathcal{B} \) is constructed as the leaf space for a chosen polarization on the space of boundary fields \( \mathcal{F}^\partial \) and \( \mathcal{Y} \) is just a symplectic complement. Using this splitting, we can write the mCME
\[
\delta_{\mathcal{Y}} S = \iota_{Q_{\mathcal{Y}}} \omega, \tag{II.18}
\]
\[
\delta_{\mathcal{B}} S = -\alpha^\partial, \tag{II.19}
\]
where \( Q_{\mathcal{Y}} \) denotes the part of the cohomological vector field \( Q \) on \( \mathcal{Y} \), \( \delta_{\mathcal{Y}} \) and \( \delta_{\mathcal{B}} \) denote the corresponding parts of the de Rham differential \( \delta \) on the BV space of fields \( \mathcal{F} \) according to the splitting (II.17). Note that we have dropped the pullback \( \pi^* \). These two equations together with (II.14) imply
\[
\frac{1}{2} (S, S)_{\mathcal{Y}} = \frac{1}{2} \iota_{Q_{\mathcal{Y}}} \iota_{Q_{\mathcal{Y}}} \omega = S^\partial. \tag{II.20}
\]
Choose Darboux coordinates \((b^i, p_i)\) on \( \mathcal{F}^\partial \) such that \( b^i \) denotes the coordinates on the base \( \mathcal{B} \) and \( p_i \) on the
Finally, considering the ordered standard quantization of the fiber is a Hilbert space (see \[\text{II.18}\] for similar discussions). However, for the quantization we want to perturb around each critical point, thus we only have to use the linear structure. Additionally, we have to assume that the tangent spaces are split and that there is Darboux’s theorem if we work in the Fréchet setting (see \[\text{II.19}\] for discussions about Darboux’s theorem on infinite-dimensional Fréchet manifolds). This allows us to write

\[
\alpha^g = -\sum_i p_i \delta b^i.
\]

Using Equation \(\text{II.19}\), we get

\[
\overline{\frac{\delta}{\delta b^i}}S = p_i, \quad \forall i.
\]

Denote by \(\Delta_\mathcal{Y}\) the BV Laplacian restricted to \(\mathcal{Y}\). We will assume that \(\Delta_\mathcal{Y} S = 0\). For the closed case this means that we assume that \(S\) solves both, the CME and the QME. For the case with boundary, the BV Laplacian anyway only makes sense on \(\mathcal{Y}\), so \(\Delta = \Delta_\mathcal{Y}\). Next, we can obtain

\[
\Delta_\mathcal{Y} \exp (iS/h) = \left( \frac{i}{h} \right)^2 \frac{1}{2} (\mathcal{S}, \mathcal{S})_\mathcal{Y} \exp (iS/h)
\]

and by Equation \(\text{II.20}\), we get

\[
-\hbar^2 \Delta_\mathcal{Y} \exp (iS/h) = S^g \exp (iS/h).
\]

Now consider the standard quantization \(\hat{\pi}_i := -i\hbar \overline{\frac{\delta}{\delta b^i}}\). If \(\hat{\pi}_i\) acts on a function \(S\) on \(\mathcal{B}\) parametrized by \(\mathcal{Y}\), we get

\[
\hat{\pi}_i S = -i\hbar \pi_i, \quad \pi_i \in \mathcal{Y}.
\]

Finally, considering the ordered standard quantization of \(S^g\) given by

\[
\mathcal{S}^g := S^g \left( b^i, -i\hbar \overline{\frac{\delta}{\delta b^i}} \right),
\]

where all the derivatives are placed to the right, and using Equation \(\text{II.21}\), we get the modified quantum master equation \(\text{(mQME)}\) \[\text{[13]}\]

\[
\left( \hbar^2 \Delta_\mathcal{Y} + \mathcal{S}^g \right) \exp (iS/h) = 0.
\]

In order to get a well-defined cohomology theory, we require that

\[
\left( \hbar^2 \Delta + \mathcal{S}^g \right)^2 = 0.
\]

Since \(\Delta^2 = 0\) and obviously the commutator \(\left[ \Delta, \mathcal{S}^g \right]\) vanishes, we have to assume that \(\left( \mathcal{S}^g \right)^2 = 0\). This clearly follows if

\[
S^g_h * S^g_h = 0,
\]

\(\text{II.23}\)

where

\[
*: \mathcal{O}_{loc}(\mathcal{F}^0)[h] \times \mathcal{O}_{loc}(\mathcal{F}^0)[h] \rightarrow \mathcal{O}_{loc}(\mathcal{F}^0)[h]
\]

denotes the star product (deformation quantization) induced by the BFV form \(\omega^g\) and the standard ordering as mentioned above. Actually, the construction with the star product does not require the notion of a BV-BFV manifold and thus can be also considered independently for the BFV case. Moreover, the deformed boundary action \(S^g_h\) satisfying \(\text{II.23}\) might spoil the mQME \(\text{II.22}\). Note that we can endow the deformed algebra \(\mathcal{O}_{loc}(\mathcal{F}^0)[h]\) with a dg structure by considering the differential given by

\[
Q^g_h := S^g_h * -.
\]

\(\text{II.24}\)

Then we have

\[
Q^g_h (S^g_h) = S^g_h * S^g_h = S^g_h S^g_h + \sum_{k \geq 1} \hbar^k B_k (S^g_h, S^g_h)
\]

\(= S^g_h S^g_h + \hbar \{S^g_h, S^g_h\} + h^2 B_2 (S^g_h, S^g_h) + O(h^3), \quad \text{II.25}\)

where \(B_k\) denotes some bidifferential operator for all \(k \geq 1\) with \(B_1 := \{ , \}\). Moreover, note that we have

\[
\{S^g_h, S^g_h\} = \{S^g_0, S^g_0\} + \hbar \{S^g_0, S^g_1\} + h^2 \{S^g_1, S^g_0\} + h^2 \{S^g_1, S^g_1\} + h^2 \{S^g_2, S^g_0\} + O(h^3).
\]

\(\text{II.26}\)

Using \(\text{II.16}\) and \(\text{II.26}\), we get

\[
S^g_h * S^g_h = (S^g_0 + h S^g_1 + h^2 S^g_2 + O(h^3)) \times (S^g_0 + h S^g_1 + h^2 S^g_2 + O(h^3)) + \hbar \{S^g_0, S^g_0\} + \hbar \{S^g_0, S^g_1\} + h \{S^g_1, S^g_0\} + h^2 \{S^g_1, S^g_1\} + h^2 \{S^g_1, S^g_2\} + O(h^3) + h^2 B_2 (S^g_0, S^g_0) + O(h^3)
\]

\[
= S^g_0 S^g_0 + h (S^g_1 S^g_0 + S^g_0 S^g_1 + \{S^g_0, S^g_0\}) + h^2 (S^g_1 S^g_1 + S^g_0 S^g_1 + \{S^g_0, S^g_1\}) + B_2 (S^g_0, S^g_0) + O(h^3)
\]

\[
= \hbar \{S^g_0, S^g_0\} + h^2 \{S^g_1, S^g_0\} + B_2 (S^g_0, S^g_0) + O(h^3), \quad \text{II.27}\)

were we have used the graded commutativity relation

\[
\{f, g\} = -(-1)^{(gb f + 1)(gb g + 1)} \{g, f\},
\]

the fact that \(\{ , \}\) is even of ghost number 0 and that each \(S^g_h\) is odd of ghost number +1 for all \(k \geq 0\). Note that by the CME for \(S^g_h\) the first term in \(\text{II.27}\) vanishes. Moreover, by the associativity of the star product we get

\[
\{S^g_0, B_2 (S^g_0, S^g_0)\} = Q^g (B_2 (S^g_0, S^g_0)) = 0,
\]

\(\text{II.28}\)
and thus $B_2(S^0, S^0)$ is closed under the coboundary operator $Q^0 = \{S^0, \}$, If we assume that $B_2(S^0, S^0)$ is also $Q^0$-exact, there exists some $S^1 \in \mathcal{O}_{loc}(\mathcal{F}^0)$, such that

$$B_2(S^0, S^0) = -\{S^0, S^1\} = -Q^0(S^1).$$

Thus the coefficients in degree 2 vanish and one can check that by the construction of the star product all the higher coefficients will also vanish using a similar iterative procedure as we have seen before.

More general, in the quantum BV-BFV construction [13] one can construct a geometric quantization [22] on the space of boundary fields $\mathcal{F}^0$ using the symplectic form $\omega^0$ and the chosen polarization. This will give a vector space $\mathcal{H}_\partial$ (actually a chain complex $(\mathcal{H}_\partial, h^2 \Delta + \mathcal{S}^\partial)$) associated to the source boundary. We call $\mathcal{H}_\partial$ the space of boundary states. In order to deal with high energy parts of the bulk fields, we choose a gauge-constructing Lagrangian submanifold $\mathcal{L} \subset \mathcal{Y}$ of $\mathcal{Y}$, a boundary state manifold. Moreover, in [13] it was shown that there is always a quantization $\mathcal{S}^\partial$ of $\mathcal{S}^\partial$ that squares to zero and satisfies (II.32). It is fully described by integrals over the boundary of suitable configuration spaces determined by the underlying Feynman graphs.

### III. HIGHER CODIMENSION

#### A. Higher codimension gauge theories: BV-BF$^k$V theories

Since the BV-BFV construction is a codimension 1 formalization, we have an action of a dg algebra of observables, coming from the deformation quantization construction, to a chain complex (or vector space) associated to the boundary via geometric quantization with respect to the symplectic manifold $(\mathcal{F}^0, \omega^0)$. This corresponds to the action of the operator $S^\partial \in \text{End}(\mathcal{H}_\partial)$ on $\Psi \in \mathcal{H}_\partial$. Classical BV-BFV theories can be extended to higher codimension manifolds [12]. One can define an exact BF$^k$V manifold to be a triple $(\mathcal{F}^{\bar{k}}, \omega^{\bar{k}} = \delta \alpha^{\bar{k}}, Q^{\bar{k}})$ where $\mathcal{F}^{\bar{k}}$ is a $\mathbb{Z}$-graded supermanifold, $\omega^{\bar{k}} \in \Omega^2(\mathcal{F}^{\bar{k}})$ is an exact symplectic form of ghost number $k - 1$ with primitive 1-form $\alpha^{\bar{k}}$, and $Q^{\bar{k}} \in \mathcal{X}(\mathcal{F}^{\bar{k}})$ is a cohomological, symplectic vector field with Hamiltonian function $S^{\bar{k}}$ of ghost number $k$. A BV-BF$^k$V manifold over an exact BF$^k$V manifold $(\mathcal{F}^k, \omega^k = \delta \alpha^k, Q^k)$ is a quintuple

$$(\mathcal{F}^{\bar{k}-1}, \omega^{\bar{k}-1}, S^{\bar{k}-1}, Q^{\bar{k}-1}, \pi : \mathcal{F}^{\bar{k}-1} \rightarrow \mathcal{F}^{\bar{k}})$$

such that $\pi$ is a surjective submersion and

- $\delta \pi Q^{\bar{k}-1} = Q^{\bar{k}}$,
- $\iota_{Q^{\bar{k}-1}} \omega^{\bar{k}-1} = \delta S^{\bar{k}-1} + \pi^* \alpha^{\bar{k}}$.

Again, this will lead to a higher codimension version of the mCME

$$Q^{\bar{k}-1}(S^{\bar{k}-1}) = \pi^* \left(2S^{\bar{k}} - \iota_{Q^{\bar{k}}} \alpha^{\bar{k}}\right). \quad (III.1)$$

The quantum extension is more difficult and requires certain algebraic constructions. Following the codimension 1 construction, one can try to formulate a similar procedure by considering a deformation quantization of Poisson structures with higher shifts. However, we will not consider a coupling of higher codimension theories here, but rather describe the idea for quantization of the according BF$^k$V theory for a codimension $k$ stratum. We expect that a coupling on each codimension is possible.

#### B. Algebraic structure for the quantization in higher codimension

Let us denote by $\mathbb{E}_k$ the topological operad of little $k$-dimensional disks and let $\mathbb{P}_k$ denote the operad controlling $(1 - k)$-shifted (unbounded) Poisson dg algebras
It is known that deformation quantization of $\mathbb{P}_1$-algebras corresponds to $\mathbb{E}_1$-algebras [23], which is the same as an $A_{\infty}$-algebra, and for $k$-shifted Poisson structures ($\mathbb{P}_k$-algebras) to $\mathbb{E}_k$-algebras through methods of factorization algebras [9, 15]. This is due to the fact that the $\mathbb{E}_k$ operad is formal [20, 23, 35], i.e. equivalent to its homology, in the category of chain complexes, hence it is equivalent to a $\mathbb{P}_k$-algebra for all $k \geq 2$. Thus for all $k \geq 2$, there exists a deformation quantization for a $\mathbb{P}_k$-algebra and there is a canonical Lie bracket $[,]_{E_k}$ for any $\mathbb{E}_k$-algebra which corresponds to a $(1-k)$-shifted Poisson structure through the equivalence. One can then view $\mathcal{O}_{loc}(\mathcal{F}^k)[[h]]$ as an $\mathbb{E}_k$-algebra endowed with a dg structure induced by the differential

$$Q_h^{(k)} := \left[ S_h^{(k)}, \right]_{E_k}. \quad (\text{III.2})$$

The higher shifted analogue of the quantum master equation is then given by

$$\left[ S_h^{(k)}, S_h^{(k)} \right]_{E_k} = 0. \quad (\text{III.3})$$

This construction is also consistent with the $k$-dimensional version of the Swiss-cheese operad [23, 37] $\mathcal{S}\mathcal{C}_k$ which somehow couples the $E_k$ operad to the $E_{k-1}$ operad [27]. Describe it as an operad of sets. This colored operad has two colors: points may be in the bulk or on the boundary. The set of colors is a poset, that is a category, rather than a set, and there are only operations compatible with this structure. This operad is important when dealing with the coupling in each codimension. The corresponding geometric quantization picture for $k$-shifted symplectic structures uses the notion of higher categories. In particular, it corresponds to the notion of a "dg $k$-category". One can define such an object to be a $k$-category $\mathcal{C}$ [2] for which each set of morphisms $\text{Hom}(X, Y)$ between two objects $X, Y \in \mathcal{C}$ forms a dg module, i.e. it is given by a direct sum

$$\text{Hom}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(X, Y),$$

endowed with a differential

$$d^{(X,Y)}_C : \text{Hom}_n(X, Y) \to \text{Hom}_{n+1}(X, Y).$$

Composition of morphisms is given by maps of dg modules

$$\text{Hom}(X,Y) \otimes \text{Hom}(Y,Z) \to \text{Hom}(X,Z), \quad \forall X, Y, Z \in \mathcal{C}$$

satisfying some additional relations [17, 36]. One should think of $\text{Hom}$ as the space of 1-morphisms. Denote by $\text{Hom}^{(k)}$ the space of $k$-morphisms, which again forms a dg module. A 2-morphism $\alpha$ is usually denoted as below.

$$\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
\downarrow^{\alpha} & & \downarrow_{g} \\
Y & \overset{g}{\longrightarrow} & Z
\end{array}$$

We require that the they satisfy the same conditions as $\text{Hom} = \text{Hom}^{(1)}$, i.e. for two $(k-1)$-morphisms $f, g$, we want that the space of $k$-morphisms between them is given by a direct sum

$$\text{Hom}^{(k)}(f, g) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_n^{(k)}(f, g),$$

endowed with a differential

$$d^{(f,g)}_C : \text{Hom}_n^{(k)}(f, g) \to \text{Hom}_{n+1}^{(k)}(f, g).$$

The composition of $k$-morphism should then, similarly as for higher categories, satisfy some Stasheff pentagon identity [33, 34]. The diagram below illustrates the quantization for higher codimensions where we have denoted by $\mathcal{A}_{\mathbb{E}_k}$ the category of $\mathbb{E}_k$-algebras, by $\mathcal{D}_{\mathbb{C}_k}$ the category of dg $k$-categories and by $\mathcal{C}_{\mathbb{E}_k}$ the category of chain complexes. One should think of the horizontal arrows as passing to higher codimension and not as a functor in particular. The quantization on the level of deformation quantization focuses on the algebraic structure induced by the space of observables, whereas the quantization on the level of geometric quantization focuses on the geometric structure induced by the space of boundary fields, namely its symplectic manifold structure.

$$\begin{array}{cccc}
\text{DefQuant} & \overset{\mathcal{A}_{\mathbb{E}_k}}{\longrightarrow} & \mathcal{A}_{\mathbb{E}_1} \overset{\mathcal{A}_{\mathbb{E}_0}}{\longrightarrow} & 0 \\
\text{GeomQuant} & \overset{\mathcal{D}_{\mathbb{C}_k}}{\longrightarrow} & \mathcal{D}_{\mathbb{C}_1} \overset{\mathcal{D}_{\mathbb{C}_0}}{\longrightarrow} & \mathcal{C}
\end{array}$$

C. Obstruction spaces

Recall that for codimension 0 theories the quantum obstruction space was given by the first cohomology group with respect to the cohomological vector field in the bulk
(Theorem II.1) and for coboundary 1 theories it was given by the second cohomology group with respect to the cohomological vector field on the boundary (Theorem II.2). A natural question is whether the obstruction space for the quantization of codimension $k$ theories is given by

$$H^{k+1}_{Q^h}(\mathcal{O}_{\text{loc}}(\mathcal{F}^h)),$$

This is not clear at the moment. We plan to consider this more carefully in the future.

ACKNOWLEDGMENTS

The author would like to thank Alberto Cattaneo, Pavel Mnev and Nicola Capacci for several discussions on this topic. Moreover, he would like to thank Pavel Safronov for discussions and ideas about the algebraic construction for higher codimensions. We hope to describe the algebraic ideas of Section III B, especially the bulk-boundary coupling, in another paper by a precise mathematical formulation. This research was supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation, and by the SNF grant No. 200020_192080.

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