FURTHER RESULTS ON STABILIZATION OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAYED FEEDBACK CONTROL UNDER G-EXPECTATION FRAMEWORK

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Abstract. In this paper we establish a comparison approach to study stabilization of stochastic differential equations driven by G-Brownian motion with delayed (G-SDDEs for short) feedback control. This theory also extends to a general range of moment order and brings more choices of \( p \). Finally, a simple example is proposed to demonstrate the applications of our theory.

1. Introduction. The mathematical formulation involving G-Brownian motions ([27, 28, 29]) provide more realistic models to describe measuring finance risk and volatility uncertainty. The volatility is assumed to be a constant in the Black-Scholes model, which brings about a lot of criticisms and doubts. Taking the model uncertainty into consideration, the G-expectation has been proposed and well studied. Peng also constructed Itô’s stochastic calculus with respect to the G-Brownian motion, and initialized the study of stochastic differential equations driven by G-Brownian motion (G-SDEs for short), see e.g. [3, 10, 29] for more details. Further analytical properties of G-SDEs have been studied so far, see e.g. [15, 18, 13, 21, 22, 37]. In particular for the topic of their stability, we refer to [20, 31, 32, 34, 39, 40] and the references therein.

In order to capture the impact of time delay on financial markets and decision making, stochastic models with delay are heavily used within the field of mathematical finance, as indicated by some statistics data of stock prices [35]. Hence, it is of great importance to consider time delay for financial problems. And it is

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interesting to observe that such problem becomes a class of stochastic differential equations with time delay (SDDEs for short). Taking advantage of feedback control, the stabilization problems with time delay were extensively explored, see [1, 30] for determined case, [16, 23] for stochastic case. Since SDDEs can be viewed as a special stochastic functional differential equation, we also refer to the monograph by Mohammed [25] and the literature by Mohammed and Scheutzow [26] for more details. We also remark that there has been some literatures devoted to the study of $G$-SDEs with time delay, see e.g. [4, 5, 6, 8, 7, 9, 19, 36, 38]. Here we further explore some new results on stability of $G$-SDDEs.

The present study is motivated by the recent work by Guo et al. [11] on stability of SDDEs and Hu et al. [16] on stabilization of hybrid SDEs with delayed feedback control. Indeed, it is worthwhile to study the stabilization of $G$-SDEs with delayed feedback control. In this article, we use the method of comparison to obtain the $p$-moment (quasi-sure) exponential stabilization of SDEs with delayed feedback control under $G$-expectation framework. Our methodology consists of two steps. In the first step, we study the $p$-moment exponential stabilization problems of $G$-SDEs through state feedback control (without delay). In particular we make use of the conclusions obtained in [20, 34]. In the second step, we focus on the pathwise comparison of the solution to $G$-SDEs with that to $G$-SDDEs, in terms of the quasi-sure exponentially stable. For example, let $X(t)$ denote the continuous solution to (1), since $X(t - \delta)$ tends to $X(t)$ as $\delta \to 0$, the difference between $X(t)$ and $X(t - \delta)$ can be controlled as long as $\delta$ is reasonably small. Motivated by this idea, we show the controlled $G$-SDEs with delayed feedback control is $p$-moment (quasi-sure) exponentially stable as long as delay constant $\delta$ satisfies $\delta < \delta^*$, where $\delta^*$ is unique determined by equation (29) in terms of the coefficients of the associated $G$-SDEs.

Comparing our work with [16], we would like to point out three main differences. Firstly, the current paper establishes stochastic stabilization of SDEs driven by $G$-Brownian motion, following the numerical simulation approach of [24]. The link between the solution to $G$-SDEs and the $G$-SDDEs is more subtle than in the classical case due to subadditivity and positive homogeneity of $G$-expectation. Second, we explore the link between the stability of $G$-SDDEs and $G$-SDEs based on the $p$-th moment estimate, see Lemmas 3.2, 3.3, 4.6 and 4.7. Thirdly, partial arguments in Section 3.1 can be adopted to deal with the stabilization problem of $G$-SDEs with nonlinear case.

Let us compare our results with conclusions in [33] in the case of the discrete-time observations feedback control. Ren et al. [33] made use of $G$-Lyapunov function, yet their argument was flawed in calculation. And to avoid this point, here the stabilization of $G$-SDEs with delayed feedback control relies on a comparison approach. In summary, the contributions and innovations of this paper are as follows.

- We adopt a more general formulation involving $G$-Brownian motion. For the $G$-SDEs with delayed feedback control, both $p$-th moment and quasi-sure exponential stability of the controlled linear $G$-SDDEs are obtained. Partial arguments in Section 3.1 can be adopted to deal with the stabilization problem of $G$-SDEs with Lipschitzian coefficients.

- A delicate comparison enables us to simplify the problem. Using a comparative method, the stabilization problem of $G$-SDEs with delayed feedback control is converted to the stability problem of some $G$-SDEs without delay, which allows us to use the previous results.
Theoretical efficiency is checked through an example. An example about linear SDE under the \( G \)-expectation framework proves to be stabilizable using delayed feedback control.

This article is organized as follows. Section 2 is devoted to \( G \)-Itô’s stochastic analysis. After introducing useful notations and standing assumptions, Sections 3 and 4 are the central part of this article. An application of the main result concerning stochastic population growth model effected by \( G \)-Brownian motion can be found in Section 5. Section 6 collects some conclusions and discussions.

2. Preliminaries. In this section, we give some frequently used notations of \( G \)-Itô’s stochastic analysis and provide some terminologies, we refer to [2, 27, 28, 29] for more details and historical remarks.

2.1. \( G \)-Brownian motion.

**Definition 2.1.** Let \( \Omega \) be a complete separable metric space and let \( \mathcal{H} \) be a linear space of real-valued bounded functions defined on \( \Omega \) and if \( X_i \in \mathcal{H}, i = 1, \cdots, n \) then \( \varphi(X_1, \cdots, X_n) \in \mathcal{H} \) for \( \varphi \in C_{b,Lip}(R^{d \times n}) \), where \( C_{b,Lip}(R^{d \times n}) \) denotes the space of bounded and Lipschitz functions on \( R^{d \times n} \). A functional \( \mathbb{E} : \mathcal{H} \to R \) is called a sublinear expectation if it satisfies

\[
\begin{align*}
(1) & \quad \mathbb{E}[X] \geq \mathbb{E}[Y], \quad X \geq Y; \\
(2) & \quad \mathbb{E}[c] = c, \quad \forall c \in R; \\
(3) & \quad \mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]; \\
(4) & \quad \mathbb{E}[%X] = \lambda \mathbb{E}[X], \quad \forall \lambda \geq 0.
\end{align*}
\]

The triple \((\Omega, \mathcal{H}, \mathbb{E})\) is called a sublinear expectation space. According to the definition of the sublinear expectation \( \mathbb{E} \), the following properties hold.

**Proposition 1.** For any \( X, Y \in \mathcal{H} \), then we get

\[
\begin{align*}
(1) & \quad \mathbb{E}[\eta X] = \eta^+ \mathbb{E}[X] + \eta^- \mathbb{E}[-X] \text{ for any } \eta \in R. \\
(2) & \quad \text{if } \mathbb{E}[X] = -\mathbb{E}[-X], \text{ then } \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]. \\
(3) & \quad \mathbb{E}[|X|] \leq \mathbb{E}[|X + Y|]. \\
(4) & \quad \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}}, \text{ for any } 1 \leq p, q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} = 1.
\end{align*}
\]

In particular, \( \mathbb{E}||X|| \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} \).

**Definition 2.2.** Let \( X = (X_1, \cdots, X_d) \). We say random vector \( X \) is \( G \)-normally distributed if the function \( u \) defined by

\[
u(t, x) := \mathbb{E}[[\varphi(x + \sqrt{t}X)], \quad (t, x) \in [0, +\infty) \times R^n,
\]

is a viscosity of the following PDE

\[
\partial u_t - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x),
\]

where \( G(r) = \frac{1}{2}(\bar{\sigma}^2 r^+ - \underline{\sigma}^2 r^-) \) for some \( \bar{\sigma} \geq \underline{\sigma} > 0 \).

Now we give definitions of \( G \)-expectation and \( G \)-Brownian motion.

**Definition 2.3.** For any fixed \( T > 0 \), let \( \Omega_T = \{\omega|[0,T] \ni t \to \omega_t \in R^d \text{ is continuousw}ith \omega(0) = 0\} \) and let \( B(t)(\omega) = \omega(t) \) be the canonical process. \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) denotes the natural filtration generated by the canonical process in \( \Omega_T \). Set

\[
\text{Lip}(\Omega_T) := \{\varphi(B(t_1), \cdots, B(t_n) : n \geq 1, t_1, \cdots, t_n \in [0, T], \varphi \in C_{b,Lip}(R^{d \times n})\}.
\]
Definition 2.4. $G$-expectation on $(\Omega_T, Lip(\Omega_T))$ is a sublinear expectation defined by
\[
E[X] = \widetilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m)],
\]
for all $X \in Lip(\Omega_T)$ with $X = \varphi(B(t_1) - B(t_0), \ldots, B(t_m) - B(t_{m-1}))$ for some $\varphi \in C_b, Lip(\mathbb{R}^{d \times n})$ where $(\xi_m)_{m=1}^\infty$ are identically distributed d-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, H, \tilde{\mathbb{E}})$ such that $\xi_{i+1}$ is independent from $(\xi_1, \ldots, \xi_i)$ for each $i = 1, \ldots, m - 1$. The corresponding canonical process $B(t)$ in the $G$-expectation space $(\Omega_T, Lip(\Omega_T), \mathbb{E})$ is called a $G$-Brownian motion.

2.2. $G$-Itô calculus. We need the following notations before giving $G$-Itô integral with respect to one dimensional $G$-Brownian motion $B(\cdot)$ and its quadratic variation $\langle B \rangle(\cdot)$.

For $\xi \in Lip(\Omega_T)$ and $p \geq 1$, we consider the norm $\|\xi\|_{L^p_G} = (E[\|\xi\|^p])^{1/p}$. Denote by $Lip(\Omega_T)$ the Banach completion of $Lip(\Omega_T)$ under $\|\cdot\|_{L^p_G}$, and
\[
M^p_G([0, T]) := \\{ \eta := \sum_{j=0}^{N-1} \xi_j(t_{j, j+1}) : \xi_j \in L^p_G(\Omega_{t_j}), j = 1, 2, \ldots, N - 1 \}.
\]
For each $p \geq 1$, defining the following norm
\[
\|\eta\|_{M^p_G} = \left( E\left[\int_0^T |\eta_s|^p ds \right] \right)^{1/p}.
\]
Let $M^p_G([0, T])$ be the completion of $M^p_G([0, T])$ under $\|\cdot\|_{M^p_G}$. $H^p_G([0, T])$ denotes the completion of $M^p_G([0, T])$ under norm $\|\eta\|_{H^p_G} := \{ E[|\int_0^T |\eta_s|^2 ds|^{p/2}] \}^{1/p}$.

Definition 2.5. For two processes $\eta(\omega) \in M^2_G([0, T])$ and $\zeta(\omega) \in M^2_G([0, T])$ we define $I, H$ as follows:
\[
I(\eta) = \int_0^T \eta_s dB(s) = \sum_{j=1}^{N-1} \xi_j(B(t_{j+1}) - B(t_j)),
\]
\[
H(\zeta) = \int_0^T \zeta_s d\langle B \rangle(s) = \sum_{j=1}^{N-1} \xi_j(\langle B \rangle(t_{j+1}) - \langle B \rangle(t_j)).
\]

Definition 2.6. (Denis et al. [3]) There exists a weakly compact set $\mathcal{P}$, the set of all probability measures defined on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that
\[
E[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \quad \text{for all } \xi \in L^1_G(\Omega_T).
\]
For this weakly compact set $\mathcal{P}$, we define capacity $\bar{C}(A) := \sup_{P \in \mathcal{P}} P(A)$, $\bar{C}$ defined here is independent of the choice of $\mathcal{P}$.

Definition 2.7. (Denis et al. [3]) A set $A \in \mathcal{B}(\Omega_T)$ is polar set if $\bar{C}(A) = 0$. A property holds quasi-surely (q.s. for short) if it holds outside a polar set.

Lemma 2.8. (Borel-Cantelli lemma) There is a sequence of $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{B}(\Omega)$ such that $\bar{C}(A_n) < \infty$. Then, $\limsup_{n \to \infty} A_n$ is a polar set.

The following lemma is useful for moment estimates.
Lemma 2.9. (Hu et al. [14]) For any $\varphi \in H^q_G([0,T])$ with $q \geq 1$, $p > 0$, the estimate holds
\[
\sigma^p c_p \mathbb{E} \left[ \left( \int_0^T |\varphi(s)|^2 ds \right)^{p/2} \right] \leq \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \varphi(s) dB(s) \right|^p \right) \leq \sigma^p C_p \mathbb{E} \left[ \left( \int_0^T |\varphi(s)|^2 ds \right)^{p/2} \right],
\]
where $0 < c_p < C_p < \infty$ are constants independent of $\varphi$, $\sigma$ and $\bar{\sigma}$.

2.3. Formulation. For a fixed constant $\delta > 0$, let $\ell$ denote the continuous path space $D([-\delta,0]; \mathbb{R}^n)$ endowed with the uniform norm $\|\xi\| = \sup_{-\delta < s < 0} |\xi(s)|$ for $\xi \in \ell$. Denote by $L^p_G(\Omega, D([-\delta,0]; \mathbb{R}^n))$ ($p \geq 4$) the family of $D([-\delta,0]; \mathbb{R}^n)$-valued random variables $\zeta$ with $\mathbb{E}\|\zeta\|^p < \infty$. Our approach generalizes the classical one by allowing for mean-uncertainty perturbation and volatility-uncertainty perturbation.

3. Stabilization of linear G-SDE. We consider the following G-SDEs with control input $u = \beta X(t - \delta)$,
\[
dX(t) = [\alpha X(t) + \beta X(t - \delta)] dt + \gamma X(t) dB(t) + \sigma X(t) dB(t) \tag{1}
\]
with the initial datum $X_{t_0} := \{X(t_0 + s), -\delta \leq s \leq 0\} = \xi(s) \in D([-\delta,0]; \mathbb{R}^n)$, where $\alpha$, $\beta$, $\gamma$, and $\sigma$ are constants. Let us introduce the auxiliary G-SDEs with control input $u = \beta Y(t)$,
\[
dY(t) = [\alpha Y(t) + \beta Y(t)] dt + \gamma Y(t) dB(t) + \sigma Y(t) dB(t) \tag{2}
\]
on $t \geq t_0$ with initial value $Y_{t_0} \in L^p_G(\Omega_{t_0}; \mathbb{R}^n)$ ($p \geq 4$). In the case of linear G-SDEs, the explicit solutions would of course be useful in determining the $p$-th moment (or quasi-sure) exponential stability ($p > 0$). It has been shown that equation (2) has the explicit solution
\[
Y(t) = Y_{t_0} \exp \left( (\alpha + \beta)t + \left( \gamma - \frac{1}{2} \sigma^2 \right) \langle B \rangle(t) + \sigma B(t) \right). \tag{3}
\]
Therefore,
\[
\mathbb{E}|Y(t)|^p = \mathbb{E}|Y_{t_0}|^p \exp \left( p(\alpha + \beta)t + \frac{1}{2} p \left( 2\gamma + (p - 1)\sigma^2 \right) \langle B \rangle(t) \right). \tag{4}
\]

Assumption 3.1. Assume that (4) satisfies
\[
\mathbb{E}|Y(t)|^p = \mathbb{E}|Y_{t_0}|^p e^{-\tilde{\gamma}(t-t_0)}, \text{ for some } \tilde{\gamma} > 0. \tag{5}
\]

Remark 1. Let
\[
\tilde{\gamma} = - \left( p(\alpha + \beta) + \frac{1}{2} p \left( 2\gamma + (p - 1)\sigma^2 \right) + \frac{1}{2} p \left( 2\gamma + (p - 1)\sigma^2 \right) - \frac{1}{2} \sigma^2 \right) > 0.
\]
In view of (3), (4) and Remark 1, one can easily obtain
\[
\mathbb{E}|Y(t)|^p = \mathbb{E}|Y_{t_0}|^p \exp \left( p(\alpha + \beta)t + \left( \gamma - \frac{1}{2} \sigma^2 \right) \langle B \rangle(t) + p\sigma B(t) \right) \leq \mathbb{E}|Y_{t_0}|^p e^{-\tilde{\gamma}(t-t_0)},
\]
which indicates Assumption 3.1 is reasonable.
3.1. Lemmas. We have the following estimates.

Lemma 3.2. Let \( \xi \in L^p_{\mathcal{C}}(\Omega_{t_0}, D([-\delta, 0]; \mathbb{R}^n)) \) for \( p \geq 4 \) or \( \xi \in L^p_{\mathcal{C}}(\Omega_{t_0}, D([-\delta, 0]; \mathbb{R}^n)) \) for \( 0 < p < 4 \) and write \( X(t; t_0, \xi) = X(t) \). Then we have for any \( t_0 \geq 0 \) and \( T \geq 0 \)

\[
\sup_{t_0 \leq t \leq t_0 + T + \delta} \mathbb{E}|X(t)|^p \leq L_1 \mathbb{E}\|\xi\|^p, \tag{6}
\]

\[
\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + T + \delta} |X(t)|^p \right) \leq L_2 \mathbb{E}\|\xi\|^p, \tag{7}
\]

\[
\sup_{t_0 \leq t \leq t_0 + T} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t + r) - X(t)|^p \right) \leq L_3 \mathbb{E}\|\xi\|^p, \tag{8}
\]

where \( L_1 = L_1(p, \bar{T}, \delta) \)

\[
= (1 + |\beta|\delta)^{(1 + \frac{2}{p})} \exp \left[ \left| p|\alpha| + p|\beta| + p\alpha^2 |\gamma| + \frac{1}{2} \beta^2 p(p - 1) \right| \right],
\]

\[
L_2 = L_2(p, \bar{T}, \delta) = \begin{cases} 
5^{p-1} \left( (1 + \delta)|\beta|^p (T + \delta)^{p-1} \right) (\bar{T} + \delta)^p (|\alpha|^p + |\beta|^p + \alpha^2 |\gamma|^p) \\
\quad + C_3 |\beta|^2 (T + \delta)^2 |\sigma|^4 L_1(p, \bar{T}, \delta), \quad p \geq 4,
\end{cases}
\]

\[
L_3 = L_3(p, \bar{T}, \delta) = \begin{cases} 
4^{p-1} \left( (|\alpha|^p \delta^p + |\gamma|^p \alpha^2 \delta^p + C_4 |\sigma|^p \delta^2) L_1(p, \bar{T}, \delta) \\
\quad + |\beta|^p \delta^p (1 + L_1(p, \bar{T}, \delta)), \quad p \geq 4,
\end{cases}
\]

and

\[
L_3 = L_3(p, \bar{T}, \delta) = \begin{cases} 
4^{p-1} \left( (|\alpha|^4 \delta^4 + |\gamma|^4 \delta^4 + C_4 |\sigma|^4 \delta^2) L_1(p, \bar{T}, \delta) \\
\quad + |\beta|^4 \delta^4 (1 + L_1(p, \bar{T}, \delta)), \quad 0 < p < 4.
\end{cases}
\]

Proof. The proof is separated into two cases. Firstly, we consider the case \( p \geq 4 \). Applying G-Itô’s formula, for \( t_0 \leq t \leq t_0 + \bar{T} + \delta \), we obtain that

\[
\text{d}|X(t)|^p = \langle p|X(t)|^{p-2}X(t), \alpha X(t) \rangle \text{d}t + \langle p|X(t)|^{p-2}X(t), \beta X(t - \delta) \rangle \text{d}t \\
\quad + \langle p|X(t)|^{p-2}X(t), \gamma X(t) \rangle \text{d}(B)(t) + \frac{1}{2} |p|X(t)|^{p-2} \text{d}(B)(t) \\
\quad + \frac{1}{2} p(p - 2)|X(t)|^{p-4} \sigma X(t) X(t) \text{d}(B)(t) \\
\quad + \langle p|X(t)|^{p-2}X(t), \sigma X(t) \rangle \text{d}B(t) \\
\leq p|X(t)|^{p-1} (|\alpha X(t)| + |\beta X(t - \delta)|) \text{d}t \\
\quad + \left( p|\gamma| |X(t)|^p + \frac{1}{2} p(p - 1) |\sigma|^2 |X(t)|^p \right) \text{d}(B)(t) \\
\quad + \langle p|X(t)|^{p-2}X(t), \sigma X(t) \rangle \text{d}B(t). \tag{9}
\]
Taking $G$-expectation on both sides of (9), we get
\[
E|X(t)|^p \leq E|X(t_0)|^p + E \int_{t_0}^t \left( p|\alpha||X(s)|^p + p|\beta||X(s)|^{p-1}|X(s-\delta)| \right) ds \\
+ \left( p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2 \right) E \int_{t_0}^t |X(s)|^p ds \\
\leq E|X(t_0)|^p + \left( p|\alpha| + (p-1)|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2 \right) \\
\times E \int_{t_0}^t |X(s)|^p ds + |\beta| E \int_{t_0}^t |X(s-\delta)|^p ds \\
\leq (1 + |\beta\delta|)E\|\xi\|^p + \left( p|\alpha| + p|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2 \right) \\
\times \int_{t_0}^t \left( \sup_{t_0 \leq r \leq s} E|X(r)|^p \right) ds. \tag{10}
\]
Noting that $\int_{t_0}^t \left( \sup_{t_0 \leq r \leq s} E|X(r)|^p \right) ds$ is increasing in $t$, we have
\[
\sup_{t_0 \leq r \leq t} E|X(r)|^p \leq (1 + |\beta\delta|)E\|\xi\|^p + \left( p|\alpha| + p|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2 \right) \\
\times \int_{t_0}^t \left( \sup_{t_0 \leq r \leq s} E|X(r)|^p \right) ds. \tag{11}
\]
Then the Gronwall inequality leads to
\[
\sup_{t_0 \leq r \leq t} E|X(r)|^p \leq (1 + |\beta\delta|)e^{(p|\alpha| + p|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2)(t-t_0)}E\|\xi\|^p. \tag{12}
\]
In particular,
\[
E|X(t)|^p \leq (1 + |\beta\delta|)e^{(p|\alpha| + p|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2)(t-t_0)}E\|\xi\|^p. \tag{13}
\]
Then for $t_0 \leq t \leq t_0 + T + \delta$, we have
\[
\sup_{t_0 \leq t \leq t_0 + T + \delta} E|X(t)|^p \leq (1 + |\beta\delta|)e^{(p|\alpha| + p|\beta| + p\sigma^2|\gamma| + \frac{1}{2}\sigma^2(p-1)|\sigma|^2)(T+\delta)}E\|\xi\|^p. \tag{14}
\]
Secondly, when $0 < p < 4$, invoking $G$-Itô’s formula to $|X(t)|^4$ and conditional $G$-expectation, this leads to
\[
E(|X(t)|^4|\mathcal{F}_{t_0}) \leq (1 + |\beta\delta|)|\xi|^4 e^{(4|\alpha|+4|\beta|+4\sigma^2|\gamma|+6\sigma^2|\sigma|^2)(T+\delta)} \tag{15}
\]
for any $t \leq t_0 + T + \delta$. Due to the Hölder inequality, we have
\[
E(|X(t)|^p|\mathcal{F}_{t_0}) \leq E(|X(t)|^4|\mathcal{F}_{t_0})^{\frac{p}{4}} \leq (1 + |\beta\delta|)^{\frac{p}{4}} e^{(p|\alpha|+p|\beta|+p\sigma^2|\gamma|+\frac{3}{2}p\sigma^2|\sigma|^2)(T+\delta)}||\xi||^p. \tag{16}
\]
Taking $G$-expectation on both sides of (16), we have
\[
E(|X(t)|^p) \leq (1 + |\beta\delta|)^{\frac{p}{4}} e^{(p|\alpha|+p|\beta|+p\sigma^2|\gamma|+\frac{3}{2}p\sigma^2|\sigma|^2)(T+\delta)}E||\xi||^p. \tag{17}
\]
As a consequence, (6) follows from (14) as well as (17).
Analogously, we prove the second assertion for $p \geq 4$. By Burkholder-Davis-Gundy inequality, it follows from (1) that

$$
\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + \bar{T} + \delta} |X(t)|^p \right) \\
\leq 5^{p-1} \left( \mathbb{E}|X(t_0)|^p + (\bar{T} + \delta)^{p-1}|\alpha|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds \right) \\
+ (\bar{T} + \delta)^{p-1}|\beta|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s - \delta)|^p ds \\
+ (\bar{T} + \delta)^{p-1}\bar{\sigma}^2|\gamma|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds \\
+ C_p\bar{\sigma}^p(\bar{T} + \delta)^{\frac{2}{p}-1}|\sigma|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds \\
\leq 5^{p-1} \left( 1 + \tau |\beta|^p (\bar{T} + \delta)^{p-1} \mathbb{E}\|\xi\|^p \right) \\
+ 5^{p-1} \left( (\bar{T} + \delta)^p \left( |\alpha|^p + |\beta|^p + \bar{\sigma}^2|\gamma|^p \right) \right) \\
+ C_p\bar{\sigma}^p(\bar{T} + \delta)^{\frac{2}{p}-1}|\sigma|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds. \quad (18)
$$

Recalling (13), it holds that

$$
\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + \bar{T} + \delta} |X(t)|^p \right) \\
\leq 5^{p-1} \left( 1 + \tau |\beta|^p (\bar{T} + \delta)^{p-1} \mathbb{E}\|\xi\|^p \right) \\
+ 5^{p-1} \left( (\bar{T} + \delta)^p \left( |\alpha|^p + |\beta|^p + \bar{\sigma}^2|\gamma|^p \right) \right) \\
+ C_p\bar{\sigma}^p(\bar{T} + \delta)^{\frac{2}{p}-1}|\sigma|^p \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds. \quad (19)
$$

The second case is when $0 < p < 4$, thus the corresponding assertion

$$
\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + \bar{T} + \delta} |X(t)|^p |\mathcal{F}_{t_0} \right) \\
\leq \left[ \mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + \bar{T} + \delta} |X(t)|^4 |\mathcal{F}_{t_0} \right) \right]^{\frac{p}{4}} \\
\leq 5^{\frac{3p}{4}} \left( 1 + \delta |\beta|^4 (\bar{T} + \delta)^3 \right) \\
+ (\bar{T} + \delta)^4 \left( |\alpha|^4 + |\beta|^4 + \bar{\sigma}^8|\gamma|^4 \right) \\
+ C_4\bar{\sigma}^4(\bar{T} + \delta)^2|\sigma|^4 \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E}|X(s)|^p ds \quad (20)
$$

Taking $G$-expectation on both sides of (20) and combining (19) give the required result (7).
To proceed, let us show the last assertion. When \( p \geq 4 \), then
\[
\mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t + r) - X(t)|^p \right) 
\leq 4^{p-1} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} \int_t^{t+r} \alpha X(s) ds \right)^p 
\leq 4^{p-1} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} \int_t^{t+r} \beta X(s - \delta) ds \right)^p 
+ 4^{p-1} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} \int_t^{t+r} \gamma X(s) dB(s) \right)^p 
+ 4^{p-1} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} \int_t^{t+r} \sigma X(s) dB(s) \right)^p 
\leq 4^{p-1} \left( |\alpha|^p \delta^p - 1 + |\gamma|^p \bar{\sigma}^2 \delta^p - 1 + C_p |\sigma|^p \bar{\sigma}^2 \delta^2 - 1 \right) 
\times \int_t^{t+\delta} \mathbb{E} |X(s)|^p ds + |\beta|^p \delta^p \int_t^{t+\delta} \mathbb{E} |X(s - \delta)|^p ds 
\leq 4^{p-1} \left( |\alpha|^p \delta^p - 1 + |\gamma|^p \bar{\sigma}^2 \delta^p - 1 + C_p |\sigma|^p \bar{\sigma}^2 \delta^2 - 1 \right) 
\times \int_t^{t+\delta} \mathbb{E} |X(s)|^p ds + |\beta|^p \delta^p \int_t^{t+\delta} \mathbb{E} |X(s - \delta)|^p ds. 
\tag{21}
\]

Since
\[
\sup_{t_0 - \delta \leq r \leq t} \mathbb{E} |X(r)|^p \leq \mathbb{E} \|\xi\|^p \sup_{t_0 \leq r \leq t} \mathbb{E} |X(r)|^p. 
\tag{22}
\]

Hence, the combination of (12) and (22) results in
\[
\sup_{t_0 - \delta \leq r \leq t} \mathbb{E} |X(r)|^p \leq (1 + L_1(p, T, \delta)) \mathbb{E} \|\xi\|^p 
\tag{23}
\]

for \( t_0 \leq t \leq t_0 + T \). By (21) and (23), we further derive that
\[
\mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t + r) - X(t)|^p \right) 
\leq 4^{p-1} \left[ \left( |\alpha|^p \delta^p - 1 + |\gamma|^p \bar{\sigma}^2 \delta^p - 1 + C_p |\sigma|^p \bar{\sigma}^2 \delta^2 - 1 \right) L_1(p, T, \delta) 
\right. 
+ |\beta|^p \delta^p (1 + L_1(p, T, \delta)) \left. \right] \mathbb{E} \|\xi\|^p. 
\tag{24}
\]

For the case of \( 0 < p < 4 \), we have
\[
\mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t + r) - X(t)|^p |\mathcal{F}_{t_0} \right) 
\leq \left[ \mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t + r) - X(t)|^4 |\mathcal{F}_{t_0} \right) \right]^\frac{p}{4} 
\leq 4^{p-4} \left[ \left( |\alpha|^4 \delta^4 + |\gamma|^4 \bar{\sigma}^8 \delta^4 - 1 + C_4 |\sigma|^4 \bar{\sigma}^4 \delta^2 \right) L_1(4, T, \delta) 
\right.
+ |\beta|^4 \delta^4 (1 + L_1(4, T, \delta)) \left. \right] \left[ \mathbb{E} \|\xi\|^p \right]^\frac{p}{4}. 
\tag{25}
\]
Taking $G$-expectation on both sides of (25) and combining (24) we get the third conclusion of Lemma 3.2.

**Lemma 3.3.** Let $\xi \in L^p_G(\Omega_{t_0}, D([-\delta, 0]; R^n))$ for $p \geq 4$ or $\xi \in L^2_G(\Omega_{t_0}, D([-\delta, 0]; R^n))$ for $0 < p < 4$ and write $Y(t; t_0 + \delta, X(t_0 + \delta)) = Y(t)$ for $t_0 \geq 0$ and $T \geq 0$. Then we have for any $t \geq t_0 + \delta$,  

$$E|X(t) - Y(t)|^p \leq L_4 \|\xi\|^p,$$

where

$$L_4 = L_4(p, \bar{T}, \delta) = \left\{ \begin{array}{ll}
|\beta|L_3(p, \bar{T}, \delta)\bar{T}e^{(p|\alpha| + (2p-1)|\beta| + p\sigma^2|\gamma| + \frac{1}{2}p(p-1)\sigma^2|\sigma|^2)^2}, & p \geq 4,
|\beta|L_3(4, \bar{T}, \delta)\bar{T}e^{(4|\alpha| + 7|\beta| + 4\sigma^2|\gamma| + 6\sigma^2|\sigma|^2)^2}, & 0 < p < 4.
\end{array} \right.$$

**Proof.** Similar to Lemma 3.2, we can also obtain the associated assertions for the following two cases. We first consider the case $p \geq 4$. By the $G$-Itô formula, we have for any $t_0 + \delta \leq t \leq t_0 + \delta + \bar{T}$,

$$d (|X(t) - Y(t)|^p) = p|X(t) - Y(t)|^{p-2}(X(t) - Y(t))^\top d\langle X(t) - Y(t) \rangle dt$$
$$+ p|X(t) - Y(t)|^{p-2}(X(t) - Y(t))^\top \beta(X(t) - Y(t)) dt$$
$$+ p|X(t) - Y(t)|^{p-2}(X(t) - Y(t))^\top \gamma(X(t) - Y(t)) d\langle B(t) \rangle$$
$$+ \frac{1}{2} p|X(t) - Y(t)|^{p-2}\sigma(X(t) - Y(t))^2 d\langle B(t) \rangle$$
$$\times |X(t) - Y(t)|^{p-4}(|X(t) - Y(t)|^2 \sigma(X(t) - Y(t))^2 d\langle B(t) \rangle$$
$$\leq p|X(t) - Y(t)|^{p-4}(|X(t) - Y(t)|^{p-1}|X(t) - Y(t)|^2 d|X(t) - Y(t)|^2$$
$$+ \left( p|X(t) - Y(t)|^{p-4} + \frac{1}{2} p(p-1)|X(t) - Y(t)|^2 \right) d\langle B(t) \rangle$$
$$+ p|X(t) - Y(t)|^{p-2}(X(t) - Y(t))^\top \sigma(X(t) - Y(t)) d\langle B(t) \rangle.$$

Taking $G$-expectation on the equality above, we have

$$E|X(t) - Y(t)|^p$$
$$\leq \left( p|\alpha| + p\sigma^2|\gamma| + \frac{1}{2} p(p-1)\sigma^2|\sigma|^2 \right) \int_{t_0 + \delta}^{t} E|X(s) - Y(s)|^p ds$$
$$+ p|\beta| \int_{t_0 + \delta}^{t} E|X(s) - Y(s)|^{p-1}|X(s) - Y(s)| ds$$
$$\leq \left( p|\alpha| + (2p-1)|\beta| + p\sigma^2|\gamma| + \frac{1}{2} p(p-1)\sigma^2|\sigma|^2 \right) \int_{t_0 + \delta}^{t} E|X(s) - Y(s)|^p ds$$
$$+ |\beta| \int_{t_0 + \delta}^{t} E|X(s) - Y(s)|^p ds$$
$$\leq \left( p|\alpha| + (2p-1)|\beta| + p\sigma^2|\gamma| + \frac{1}{2} p(p-1)\sigma^2|\sigma|^2 \right) \int_{t_0 + \delta}^{t} E|X(s) - Y(s)|^p ds$$
$$+ |\beta|L_2(p, \bar{T}, \delta)\bar{T}E\|\xi\|^p.$$  \hfill (26)

By the Gronwall inequality, then

$$E|X(t) - Y(t)|^p \leq |\beta|L_2(p, \bar{T}, \delta)\bar{T}e^{(p|\alpha| + (2p-1)|\beta| + p\sigma^2|\gamma| + \frac{1}{2} p(p-1)\sigma^2|\sigma|^2)^2} E\|\xi\|^p.$$  \hfill (27)
We now switch to case $0 < p < 4$, we note that
\[
\mathbb{E}(|X(t) - Y(t)|^p |\mathcal{F}_t) \leq |\beta| L_2(4, \bar{T}, \delta) \bar{T} \mathbb{E} \left(\frac{4|\alpha| + 7|\beta| + 4\sigma^2 |\gamma| + 6\sigma^2 |\sigma|^2 \bar{T}^2}{|\xi|^p}\right). \tag{28}
\]
To complete the proof of the second assertion of Lemma 3.3 it remains to apply the H"older inequality and the method of $G$-expectation for (28). The proof is complete.

The main result of this section is stated as follows.

**Theorem 3.4.** Let $\xi \in L_G^{\bar{p}}(\Omega_{t_0}, D([-\delta, 0]; \mathbb{R}^n))$ for $p \geq 4$ or $\xi \in L_G^\bar{p}(\Omega_{t_0}, D([-\delta, 0]; \mathbb{R}^n))$ for $0 < p < 4$. Let $\delta^*$ be the unique root to the following
\[
\theta + 2\bar{p} L_4(p, \bar{T}, \delta) + 2\bar{p} L_3(p, \bar{T}, \delta) = 1, \tag{29}
\]
where $\bar{p} = 0 \lor (p - 1)$. Then (1) satisfies
\[
\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}|X(t)|^p) < 0 \tag{30}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t} \log(|X(t)|) < 0, \text{ q.s.} \tag{31}
\]
Namely, the trivial solution of system (1) is $p$-th moment (quasi-surely) exponentially stable if $\delta < \delta^*$.

**Proof.** Now let us analyze the value $\delta^*$. Choose a constant $\theta \in (0, 1)$ and let
\[
H(\delta) = \theta + 2\bar{p} L_4(p, \bar{T}, \delta) + 2\bar{p} L_3(p, \bar{T}, \delta) - 1.
\]
We observe that $H(\delta)$ is an increasing continuous function about $\delta$, $H(0) = \theta - 1 < 0$ and $H(+\infty) = +\infty$. According to the existence theorem of roots, $\delta^*$ is determined uniquely by equation (29) and $\delta^* > 0$. In addition, the value of $\delta^*$ depends on the choice of $\theta$, namely, $\delta^* = \delta^*(\theta)$. Let
\[
\bar{T} = \frac{1}{\bar{\gamma}} \log \left( \frac{2\bar{p}}{\theta} \right), \tag{32}
\]
namely, $2\bar{p} e^{-\bar{\gamma}\bar{T}} = \theta$. For notation brevity, we set $X(t) = X(t; 0, \xi)$ for $t \geq 0$. Let us consider $X(t)$ on $t \in [\delta, 2\delta + \bar{T}]$ at time $t = \delta$. We write $Y(\delta + \bar{T}) = Y(\delta + T; \delta, X(\delta))$. It is easy to see from Assumption 3.1 that
\[
\mathbb{E}|Y(\delta + \bar{T})|^p \leq \mathbb{E}|X(\delta)|^p e^{-\bar{\gamma}\bar{T}}. \tag{33}
\]
Let $\bar{p} = (p - 1) \lor 0$. An application of the triangle inequality yields
\[
\mathbb{E}|X(\delta + \bar{T})|^p \leq 2\bar{p} \mathbb{E}|Y(\delta + \bar{T})|^p + 2\bar{p} \mathbb{E}|X(\delta + \bar{T}) - Y(\delta + \bar{T})|^p. \tag{34}
\]
A combination of (33) and Lemma 3.3 give
\[
\mathbb{E}|X(\delta + \bar{T})|^p \leq 2\bar{p} \left(e^{-\bar{\gamma}\bar{T}} \mathbb{E}|X(\delta)|^p + L_4(p, \delta, \bar{T}) \mathbb{E}\|X_\delta\|^p\right)
\leq 2\bar{p} \left(e^{-\bar{\gamma}\bar{T}} + L_4(p, \delta, \bar{T})\right) \mathbb{E}\|X_\delta\|^p.
\]
Further, due to Lemma 3.2, we have
\[
\mathbb{E}\|X_{2\delta + \bar{T}}\|^p \leq \mathbb{E} \left(\sup_{0 \leq r \leq \delta} |X(\delta + \bar{T} + r)|^p\right)
\leq 2\bar{p} \mathbb{E}|X(\delta + \bar{T})|^p + 2\bar{p} \mathbb{E} \left(\sup_{0 \leq r \leq \delta} |X(\delta + \bar{T} + r) - X(\delta + \bar{T})|^p\right)
\leq 2\bar{p} \left(2\bar{p} \left(e^{-\bar{\gamma}\bar{T}} + L_4(p, \bar{T}, \delta)\right) + L_3(p, \bar{T}, \delta)\right) \mathbb{E}\|X_\delta\|^p. \tag{35}
\]
It is clear from (32) that
\[ 2^{2\overline{p}}e^{-\overline{\gamma}\overline{T}} = \theta. \]

One can easily obtain
\[ E\|X_{2\delta + \overline{T}}\|^p \leq \left( \theta + 2^{2\overline{p}}L_4(p, \overline{T}, \delta) + 2^{\overline{p}}L_3(p, \overline{T}, \delta) \right) E\|X_\delta\|^p. \] (36)

Since \( \delta < \delta^* \), it follows from (29) that
\[ \theta + 2^{2\overline{p}}L_4(p, \overline{T}, \delta) + 2^{\overline{p}}L_3(p, \overline{T}, \delta) < 1. \] (37)

Since (37), we have for some \( \bar{\gamma} > 0 \)
\[ \theta + 2^{2\overline{p}}L_4(p, \overline{T}, \delta) + 2^{\overline{p}}L_3(p, \overline{T}, \delta) = e^{-\bar{\gamma}(\overline{T} + \delta)}, \] (38)

which together with (35) yields
\[ E\|X_{2\delta + \overline{T}}\|^p \leq e^{-\bar{\gamma}(\overline{T} + \delta)} E\|X_\delta\|^p. \]

Similarly, we can prove that for \( 2\delta + \overline{T} \leq t \leq \delta + 2(\delta + \overline{T}) \),
\[ E\|X_{\delta + 2(\delta + \overline{T})}\|^p \leq e^{-\bar{\gamma}(\delta + \overline{T})} E\|X_{2\delta + \overline{T}}\|^p. \]

Combining the above estimate, we have
\[ E\|X_{\delta + m(2\delta + \overline{T})}\|^p \leq e^{-m\bar{\gamma}(\delta + \overline{T})} E\|X_\delta\|^p \] (39)

holds for any \( m = 0, 1, \cdots \). By (7) and (39), we get
\[ E \left( \sup_{\delta + m(\overline{T} + \delta) \leq t \leq \delta + (m + 1)(\overline{T} + \delta)} |X(t)|^p \right) \leq L_2 E\|X_{\delta + m(\overline{T} + \delta)}\|^p \leq L_2 e^{-m\bar{\gamma}(\overline{T} + \delta)} E\|X_\delta\|^p. \] (40)

For \( \delta + m(\overline{T} + \delta) \leq t \leq \delta + (m + 1)(\overline{T} + \delta) \), we have
\[ \frac{1}{t} \log(E\|X(t)\|^p) \leq \frac{-m\bar{\gamma}(\overline{T} + \delta)}{\delta + m(\overline{T} + \delta)} + \frac{\log(L_2(p, \overline{T} + \delta) E\|X_\delta\|^p)}{\delta + m(\overline{T} + \delta)}. \] (41)

Passing to the limit as \( t \to \infty \), which leads to
\[ \limsup_{t \to \infty} \frac{1}{t} \log(E\|X(t)\|^p) < -\bar{\gamma}. \]

Hence, (30) holds.

We proceed to show the second result. Using Markov’s inequality and (40), it follows that
\[ \bar{C} \left( \sup_{\delta + m(\overline{T} + 2\delta) \leq t \leq \delta + (m + 1)(\overline{T} + \overline{T})} |X(t)|^p \geq e^{-\frac{1}{2}m\bar{\gamma}(\overline{T} + \delta)} \right) \]
\[ \leq L_2(p, \overline{T}, \delta) e^{-\frac{1}{2}m\bar{\gamma}(\delta + \overline{T})} E\|X_\delta\|^p. \]

From Borel-Cantelli lemma, we see that for q.s. every \( \omega \), there exists \( m_0 := m_0(\omega) \), such that for \( m \geq m_0(\omega) \) and \( t \in [\delta + m(\overline{T} + 2\delta), \delta + (m + 1)(\overline{T} + 2\delta)] \),
\[ \sup_{\delta + m(\overline{T} + 2\delta) \leq t \leq \delta + (m + 1)(\overline{T} + 2\delta)} |X(t)|^p \leq e^{-\frac{1}{2}m\bar{\gamma}(\overline{T} + \delta)}, \]
which implies
\[
\limsup_{t \to \infty} \frac{1}{t} \log(|X(t)|) \leq -\frac{\gamma}{2p}, \quad \text{q.s.}
\]
As a consequence, (31) holds. We henceforth complete the proof.

4. **Stabilization of Lipschitz G-SDE: Nonlinear case.** In order to focus on the main ideas, in this section we consider the G-SDEs with delayed feedback control under global Lipschitz conditions, and leave more general situation for future studies. Here we only discuss the stabilization in the mean square exponential sense. For the other case of \( p \), the proof of these moment estimates are analogous to that of case of linear, we omit details. Concretely speaking, we consider the equation

\[
dX(t) = \left[ f(t, X(t)) + u(t, X(t - \delta)) \right] dt + g(t, X(t)) d\langle B \rangle(t) + h(t, X(t)) dB(t)
\]

with the initial datum \( X_{t_0}(s) = \xi(s) \) for \( s \in [-\delta, 0] \), where \( X_t \in \ell \) is defined by \( X_t(s) = X(t + s) \) for \( s \in [-\delta, 0] \). \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \), \( u : [0, T] \times \mathbb{R} \to \mathbb{R}^n \), \( g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) and \( h : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) satisfy the following hypotheses:

**Assumption 4.1.** For the deterministic functions \( f(\cdot, x), u(\cdot, x), g(\cdot, x) \) and \( h(\cdot, x) \in M^2(\mathcal{G}([0, T]; \mathbb{R}^n)) \). Suppose that

\[
|f(t, x) - f(t, y)| \leq K_1|x - y|,
|u(t, x) - u(t, y)| \leq K_2|x - y|,
|g(t, x) - g(t, y)| \leq K_3|x - y|,
|h(t, x) - h(t, y)| \leq K_4|x - y|
\]

for any \( x, y \in \mathbb{R}^n \).

**Assumption 4.2.** Assume that

\[
f(t, 0) \equiv 0, \quad u(t, 0) \equiv 0, \quad g(t, 0) \equiv 0, \quad h(t, 0) \equiv 0.
\]

It follows from Assumptions above that

\[
|f(t, x)| \leq K_1|x|, \quad |u(t, x)| \leq K_2|x|, \quad |g(t, x)| \leq K_3|x|, \quad |h(t, x)| \leq K_4|x|.
\]

**Remark 2.** According to ([12], Lemma 2.1), under Assumptions 4.1 and 4.2, the equation (42) has a unique global solution denoted by \( X(t, t_0, \xi) \).

Let us consider the following controlled G-SDEs

\[
dY(t) = \left[ f(t, Y(t)) + u(t, Y(t)) \right] dt + g(t, Y(t)) d\langle B \rangle(t) + h(t, Y(t)) dB(t),
\]

with initial value \( Y(t_0) \). According to Gao [10] and Peng [29], a unique global solution \( Y(t) \), with \( \mathbb{E}|Y(t)|^2 < \infty \), to the G-SDEs (43) is ensured by Assumptions 4.1 and 4.2.

**Assumption 4.3.** There exist a pair of constants \( \eta \geq 1 \) and \( \tilde{\gamma} > 0 \) such that the trivial solution of reference equation (43) satisfies

\[
\mathbb{E}|Y(t)|^2 \leq \eta \mathbb{E}|Y(t_0)|^2 e^{-\tilde{\gamma}(t-t_0)}, \quad \text{for } t > t_0,
\]

for \( Y_{t_0} \in L^2(\mathcal{G}(\Omega_{t_0}; \mathbb{R}^n)) \).

**Assumption 4.4.** There exists a constant \( c_1 > 0 \) such that

\[
Y^T [f(t, Y) + u(t, Y)] + G \left( 2Y^T g(t, Y) + |h(t, Y)|^2 \right) \leq -c_1 |Y|^2.
\]

**Theorem 4.5.** Let Assumptions 4.1, 4.2 and 4.4 hold. Then (44) holds.
Proof. The proof of (44) follows by applying the G-Itô formula to $e^{\gamma t}|Y|^2$. For more details, the readers are referred to Hu et al. ([15, 17], Lemma 3.2 and Lemma 4.1).

4.1. Lemmas. We have the following estimates.

Lemma 4.6. Let Assumptions 4.1 and 4.2 hold. Let $\xi \in L_0^2(\Omega_{t_0}, D([-\delta, 0]; R^n))$ and write $X(t; t_0, \xi) = X(t)$. Then we have for any $t_0 \geq 0$ and $T \geq 0$

$$\sup_{t_0 \leq t \leq t_0+T+\delta} \mathbb{E}|X(t)|^2 \leq \bar{L}_1 \mathbb{E}\|\xi\|^2,$$

(45)

$$\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0+T+\delta} |X(t)|^2 \right) \leq \bar{L}_2 \mathbb{E}\|\xi\|^2,$$

(46)

$$\sup_{t_0 \leq t \leq t_0+\bar{L}} \mathbb{E} \left( \sup_{0 \leq r \leq \delta} |X(t+r) - X(t)|^2 \right) \leq \bar{L}_3 \mathbb{E}\|\xi\|^2,$$

(47)

where

$$\bar{L}_1 = \bar{L}_1(\bar{T}, \delta) = (1 + K_2 \delta) \exp \left[ \left( -2K_1 + 2K_2 + 2\bar{\sigma}^2K_3 + K_4^2\bar{\sigma}^2 \right) (\bar{T} + \delta) \right],$$

$$\bar{L}_2 = \bar{L}_2(\bar{T}, \delta) = 5 \left( 1 + \delta K_2 K_3 (\bar{T} + \delta) + (T + \delta)^2 \left( K_1^2 + K_2^2 + \bar{\sigma}^2K_3^2 \right) + 4(\bar{T} + \delta)\bar{\sigma}^2K_4^2 \right) \bar{L}_1(\bar{T}, \delta),$$

and

$$\bar{L}_3 = \bar{L}_3(\bar{T}, \delta) = 4 \left[ (K_4^2\bar{\sigma}^2 + K_3^2\bar{\sigma}^2\delta^2 + 4K_4^2\bar{\sigma}^2\delta) \bar{L}_1(\bar{T}, \delta) + K_3^2\delta^2(1 + L_1(\bar{T}, \delta)) \right].$$

Proof. Applying G-Itô’s formula, for $t_0 \leq t \leq t_0 + \bar{T} + \delta$, we have

$$d|X(t)|^2 = \langle X(t), f(t, X(t)) \rangle dt + \langle X(t), u(t, X(t) - \delta) \rangle dt$$

$$+ \langle X(t), g(t, X(t)) \rangle d\langle B \rangle(t) + \langle X(t) \rangle \bar{h}(t, X(t)) d\langle B \rangle(t)$$

$$+ \langle \bar{h}(t, X(t)) \rangle dB(t),$$

(48)

Taking G-expectation on both sides of (48) yields that

$$\mathbb{E}|X(t)|^2 \leq \mathbb{E}|X(t_0)|^2 + (2K_1 + K_2 + 2\bar{\sigma}^2K_3 + K_4^2\bar{\sigma}^2) \mathbb{E} \int_{t_0}^{t} |X(s)|^2 ds$$

$$+ K_2 \mathbb{E} \int_{t_0}^{t} |X(s - \delta)|^2 ds.$$

Since

$$\mathbb{E} \int_{t_0}^{t} |X(s - \delta)|^2 ds = \delta \mathbb{E}\|\xi\|^2 + \int_{t_0}^{t} \left( \sup_{t_0 \leq r \leq s} \mathbb{E}|X(r)|^2 \right) ds,$$

noting that $\int_{t_0}^{t} (\sup_{t_0 \leq r \leq s} \mathbb{E}|X(r)|^2) ds$ is increasing in $t$, we have

$$\sup_{t_0 \leq r \leq t} \mathbb{E}|X(r)|^2 \leq (1 + K_2 \delta) \mathbb{E}\|\xi\|^2$$

$$+ (2K_1 + 2K_2 + 2\bar{\sigma}^2K_3 + K_4^2\bar{\sigma}^2) \int_{t_0}^{t} \left( \sup_{t_0 \leq r \leq s} \mathbb{E}|X(r)|^2 \right) ds.$$
The assertion (45) follows from the Gronwall inequality.

Next we estimate second assertion, it follows from (42) that
\[
\mathbb{E} \left( \sup_{t_0 \leq t \leq t_0 + \bar{T} + \delta} \|X(t)\| \right) 
\leq \ 5 \left( 1 + \tau |\beta|^2 (\bar{T} + \delta) \right) \mathbb{E} \|\xi\|^2 
+ 5 (\bar{T} + \delta) \left( |\alpha|^2 + |\beta|^2 + \sigma^4 |\gamma|^2 \right) 
\times \int_{t_0}^{t_0 + \bar{T} + \delta} \mathbb{E} \|X(s)\|^2 ds 
\leq \ 5 \left( 1 + \tau |\beta|^2 (\bar{T} + \delta) \right) + (\bar{T} + \delta)^2 \left( |\alpha|^2 + |\beta|^2 + \sigma^4 |\gamma|^2 \right) 
+ 4(\bar{T} + \delta) \sigma^2 |\sigma|^2 \bar{L}_1(\bar{T}, \delta) \mathbb{E} \|\xi\|^2.
\]

The remaining estimates can be checked similarly and we complete the proof.

Lemma 4.7. Let Assumptions 4.1 and 4.2 hold. Let \( \xi \in L^2_\nu(\Omega_\infty, D([-\delta, 0]; \mathbb{R}^n)) \) and denote by \( Y(t) = Y(t; t_0 + \delta) \) for \( t_0 + \delta \geq 0 \) and \( \bar{T} \geq 0 \). Then we have for any \( t_0 + \delta \leq t \leq t_0 + \delta + \bar{T} \),
\[
\mathbb{E} \|X(t) - Y(t)\|^2 \leq \bar{L}_4 \mathbb{E} \|\xi\|^2,
\]
where
\[
\bar{L}_4(\delta, \bar{T}, \delta) = K_2 \bar{L}_3(\delta, \bar{T}, \delta) \bar{L}_0(2K_1 + 3K_2 + 2K_3 + \sigma^2 K_4) \bar{T}^3.
\]

4.2. Main results. The main result of this section is stated as follows.

Theorem 4.8. Under Assumptions 4.1 and 4.2, and 4.3. Let \( \delta^* \) be the unique root to the following
\[
\theta = 4 \bar{L}_4(\bar{T}, \delta) + 2 \bar{L}_3(\bar{T}, \delta) = 1.
\]

Then (42) satisfies
\[
\lim_{t \to \infty} \sup \frac{1}{t} \log(\mathbb{E} \|X(t)\|^2) < 0, 
\]
and
\[
\lim_{t \to \infty} \sup \frac{1}{t} \log(\|X(t)\|) < 0, \text{ q.s.}
\]
Namely, the trivial solution of system (42) is \( p \)-th moment (quasi-surely) exponentially stable if \( \delta < \delta^* \).

Proof. We only sketch the proof since the argument used here resembles that in the proof of Theorem 3.4. Choose a constant \( \theta \in (0, 1) \) and let
\[
\bar{T} = \frac{1}{\gamma} \log \left( \frac{4\eta}{\theta} \right),
\]
namely, \( 4\eta e^{-\gamma \bar{T}} = \theta \). For notation brevity, we set \( X(t) = X(t; t_0, \xi) \) for \( t \geq t_0 \) and \( Y(\delta + \bar{T}) = Y(\delta + \bar{T}; \delta, X(\delta)) \). It is easy to see from Assumption 4.3 that
\[
\mathbb{E} \|Y(\delta + \bar{T})\|^2 \leq \eta \mathbb{E} \|X(\delta)\|^2 e^{-\gamma \bar{T}}.
\]
An application of the triangle inequality yields
\[
\mathbb{E} \|X(\delta + \bar{T})\|^2 \leq 2 \mathbb{E} \|Y(\delta + \bar{T})\|^2 + 2 \mathbb{E} \|X(\delta + \bar{T}) - Y(\delta + \bar{T})\|^2.
\]
A combination of (53) and Lemma 4.7 gives
\[ E|X(\delta + \bar{T})|^2 \leq 2 \left( \eta e^{-\gamma \bar{T}} + \bar{L}_4(\bar{T}, \delta) \right) E\|X_\delta\|^2. \] (55)

In the following, by Lemma 4.6, we obtain that
\[ E\|X_{2\delta + \bar{T}}\|^2 \leq E \left( \sup_{0 \leq r \leq \delta} |X(\delta + \bar{T} + r)|^2 \right) \]
\[ \leq 2E|X(\delta + \bar{T})|^2 + 2E \left( \sup_{0 \leq r \leq \delta} |X(\delta + \bar{T} + r) - X(\delta + \bar{T})| \right) \]
\[ \leq 2 \left( 2\eta e^{-\gamma \bar{T}} + 2\bar{L}_4(\delta, \bar{T}) + \bar{L}_3(\bar{T}, \delta) \right) E\|X_\delta\|^2 \]
\[ \leq \left( \theta + 4\bar{L}_4(\bar{T}, \delta) + 2\bar{L}_3(\bar{T}, \delta) \right) E\|X_\delta\|^2. \] (56)

Since \( \delta < \delta^* \), it follows from (49) that
\[ \theta + 4\bar{L}_4(\bar{T}, \delta) + 2\bar{L}_3(\bar{T}, \delta) < 1. \]

Then we may write
\[ \theta + 4\bar{L}_4(\bar{T}, \delta) + 2\bar{L}_3(\bar{T}, \delta) = e^{-\tilde{\gamma}(\delta + \bar{T})} \]
for some \( \tilde{\gamma} > 0 \). From (56), we get
\[ E\|X_{2\delta + \bar{T}}\|^2 \leq e^{-\tilde{\gamma}(\delta + \bar{T})} E\|X_\delta\|^2. \]

Similarly, we can prove that for \( 2\delta + \bar{T} \leq t \leq \delta + 2(\delta + \bar{T}) \),
\[ E\|X_{\delta + 2(\delta + \bar{T})}\|^p \leq e^{-\tilde{\gamma}(\bar{T} + \delta)} E\|X_{2\delta + \bar{T}}\|^p. \]

Combining the above estimate, we have
\[ E\|X_{\delta + m(2\delta + \bar{T})}\|^p \leq e^{-m\tilde{\gamma}(\bar{T} + \delta)} E\|X_{2\delta + \bar{T}}\|^p. \]

By the mathematical induction, and it is straightforward to see that
\[ E\|X_{\delta + m(2\delta + \bar{T})}\|^p \leq e^{-m\tilde{\gamma}(\bar{T} + \delta)} E\|X_{2\delta + \bar{T}}\|^p \]
holds for any \( m = 0, 1, \cdots \). For the following detailed steps, along similar lines of that of Theorem 3.4, the remainder of the proof of Theorem carries over directly. Hence, it is easy to verify that (50) and (51) are satisfied.

5. Example. Now we use Theorem 3.4 to design a linear control of the form \( u = KX(t - \delta) \) to stabilize an unstable system. Moreover, the Theorem 3.4 allows us to study the stability of system (2). There are some existing results and techniques in previous works (see [20], [34]).

Example 5.1. We consider the stochastic population growth model affected by mean-uncertainty perturbation \( \langle B(t) \rangle \) and volatility-uncertainty perturbation \( B(t) \),
\[ X(t) = X(t_0) + \int_{t_0}^t \mu X(s)ds + \int_{t_0}^t \nu X(s)d\langle B \rangle (s) + \int_{t_0}^t \phi X(s)dB(s), \] (57)
where \( X(t_0) = X_0, B(t) \sim N \left( \{0\} \times [0.4t, 0.5t] \right) \). According to Theorem 1.2 in V-1 of Peng [29], the solution to (57) has an explicit expression,
\[ X(t) = X_0 \exp \left( \mu t + \left( \nu - \frac{1}{2} \phi^2 \right) \langle B \rangle (t) + \phi B(t) \right). \]
Let us give several unstable cases according to 
\[ \nu - \frac{1}{2} \phi^2 = 0, \nu - \frac{1}{2} \phi^2 > 0 \text{ and } \nu - \frac{1}{2} \phi^2 < 0. \]

**Case I.** Taking \( \mu = 2, \nu = 2, \phi = 2 \), the sample Lyapunov exponent of system (57) is

\[
\lim_{t \to \infty} \frac{1}{t} \log |X(t)| = \lim_{t \to \infty} \frac{1}{t} \log(|X_0 \exp(\mu t + \phi B(t))|) = 2 + \lim_{t \to \infty} \frac{2B(t)}{t} = 2, \quad (58)
\]

where in the last inequality we have used Corollary 5.9 in Li et al. [20].

**Case II.** Taking \( \mu = -\frac{1}{2}, \nu = 2, \phi = 1 \), the sample Lyapunov exponent of system (57) is

\[
\lim_{t \to \infty} \frac{1}{t} \log |X(t)| \geq \lim_{t \to \infty} \frac{1}{t} \log \left( |X_0 \exp(\mu t + (\nu - \frac{1}{2} \phi^2) \sigma^2 t + \phi B(t))| \right)
\]

\[
= \frac{7}{20} + \lim_{t \to \infty} \frac{2B(t)}{t} = \frac{7}{20}, \quad (59)
\]

**Case III.** Taking \( \mu = 2, \nu = 1, \phi = 2 \), the sample Lyapunov exponent of system (57) is

\[
\lim_{t \to \infty} \frac{1}{t} \log |X(t)| \geq \lim_{t \to \infty} \frac{1}{t} \log \left( |X_0 \exp(\mu t + (\nu - \frac{1}{2} \phi^2) \bar{\sigma}^2 t + \phi B(t))| \right)
\]

\[
\geq \frac{3}{2} + \lim_{t \to \infty} \frac{2B(t)}{t} = \frac{3}{2}, \quad (60)
\]

Firstly, let us design \( u(t, X) = KX(t) \) which satisfies Assumptions 3.1. In this situation, it is a positive contribution to use stabilization of \( B(t) \) when we choose \( 0 < p < 1 \) in sense of quasi-sure exponential stability. We only discuss the case III, since the cases I and II follows the same method. Taking \( K = -\frac{3}{2}, p = \frac{1}{2} \), then \( \tilde{\gamma} = \frac{1}{4} \). Thus, the G-SDE

\[ X(t) = X_0 + \int_0^t [\mu X(s) + KX(s)]ds + \int_0^t \nu X(s)d\langle B \rangle(s) + \int_0^t \phi X(s)dB(s), \]

with control function \( KX(t) \) is quasi-surely exponentially stable. Then choose a constant \( \theta \in (0, 1) \) and obtain

\[ \bar{T} = \frac{1}{\tilde{\gamma}} \log \left( \frac{2}{\theta} \right). \]

Then (29) becomes

\[ \theta + L_4 \left( \frac{1}{2}, \bar{T}, \delta \right) + L_3 \left( \frac{1}{2}, \bar{T}, \delta \right) = 1. \]

(61)

When \( \delta \) satisfies (61), by Theorem 3.4, then the G-SDE with delayed control function \( KX(t - \delta) \):

\[ X(t) = X_0 + \int_0^t [\mu X(s) + KX(s - \delta)]ds + \int_0^t \nu X(s)d\langle B \rangle(s) + \int_0^t \phi X(s)dB(s), \]

is quasi-surely exponentially stable.
6. Conclusion and further discussions. The paper is concerned with recent advances in stability problems of $G$-SDDE. A sufficient condition is proposed to guarantee the stabilization of $G$-SDEs with delayed feedback control. Instead of constructing some $G$-Lyapunov function, our approach takes advantage of a comparison between $G$-SDEs with delayed feedback control with the controlled $G$-SDEs. Firstly we deal with the moment estimate of $G$-SDDEs and $G$-SDEs for linear coefficients, then for equations with Lipschitz coefficients. Next we show the relationship between the stability of these two types of equations. Finally, we take an example to illustrate the applications of the above theoretical results.

Despite the success of stabilization of linear $G$-SDEs with delayed feedback control, a general criteria on stability $G$-SDDEs under global Lipschitz condition have not been established due to limitations of conditional $G$-expectation and $G$-Itô's formula, which are all left for our future exploration.

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