Variable-Length Source Dispersions Differ under Maximum and Average Error Criteria

Yuta Sakai, Member, IEEE, and Vincent Y. F. Tan, Senior Member, IEEE,

Abstract

Variable-length compression without prefix-free constraints and with side-information available at both encoder and decoder is considered. Instead of requiring the code to be error-free, we allow for it to have a non-vanishing error probability. We derive one-shot bounds on the optimal average codeword length by proposing two new information quantities; namely, the conditional and unconditional ε-cutoff entropies. Using these one-shot bounds, we obtain the second-order asymptotics of the problem under two different formalisms—the average and maximum probabilities of error over the realization of the side-information. While the first-order terms in the asymptotic expansions for both formalisms are identical, we find that the source dispersion under the average error formalism is, in most cases, strictly smaller than its maximum counterpart. Applications to a certain class of guessing problems, previously studied by Kuzuoka (2019), are also discussed.

I. INTRODUCTION

In this paper, we are concerned with the problem of variable-length compression without prefix-free constraints. In the simplest version of this problem, a source $X$ is to be compressed to finite-length binary strings. The objective is to ensure that the average codeword length is minimized under the condition that the source code is one-to-one. One-to-one codes have been studied by several researchers (see [1] and references therein). Specifically, Wyner [2] and Alon–Orlitsky [3] derived the following upper and lower bounds:

$$H(X) - \log(H(X) + 1) - \log e \leq L^*(X) \leq H(X),$$

respectively, where $\log$ stands for the base-2 logarithm and $L^*(X)$ stands for the minimum average codeword length of the one-to-one codes for the source $X$. A direct consequence of these bounds is that for a stationary memoryless source $X^n = (X_1, \ldots, X_n)$, one has

$$L^*(X^n) = n H(X) + O(\log n) \quad (\text{as } n \to \infty).$$

The above-mentioned studies and results assume that the code is not allowed to commit any error. In practical latency-constrained applications, occasional errors are often tolerable. Hence, it is worthwhile to study the counterparts to the above zero-error results when one allows the code to have a decoding error probability $\varepsilon$ that is non-vanishing. Towards this end, Kostina–Polyanskiy–Verdú [4] showed that the fundamental limit on the average codeword length $L^*(\varepsilon, X^n)$, again without the prefix-free constraint, admits the following asymptotic expansion:

$$L^*(\varepsilon, X^n) = n (1 - \varepsilon) H(X) - \sqrt{n V(X)} f_0(\varepsilon) + O(\log n) \quad (\text{as } n \to \infty)$$

for every $0 \leq \varepsilon \leq 1$. In this expression, the quantity $V(X)$ denotes the variance of the information density of the source $X$ (often referred to as the varentropy [5]) and the map $f_0 : [0, 1] \to [0, 1/\sqrt{2\pi}]$ is defined as

$$f_0(s) := \begin{cases} \Phi(\Phi^{-1}(s)) & \text{if } 0 < s < 1, \\ 0 & \text{if } s = 0 \text{ or } s = 1, \end{cases}$$

and $\Phi^{-1} : (0, 1) \to \mathbb{R}$ denotes the inverse function of the Gaussian cumulative distribution function

$$\Phi(u) := \int_{-\infty}^{u} \varphi(t) \, dt.$$
$n(1-\varepsilon)H(X)$. This is in contrast to, say, almost lossless fixed-length source coding [6], [7] in which if the tolerable error probability $\varepsilon$ is less than 1/2, the second-order term is positive, which means that the optimal code rate at a finite blocklength $n$ is larger than the first-order term.

A. Main Contributions

In this paper, we extend this setting and result by considering the presence of side-information $Y^n$ at both encoder and decoder. In this case, the notion of the error probability can take one of two different forms. One can consider the maximum error probability in which we would like

$$\mathbb{P}\{X^n \neq \hat{X}^n \mid Y^n\} \leq \varepsilon \quad (a.s.).$$

(7)

Said differently, we require that the reconstructed source $\hat{X}^n$ is equal to the original source sequence $X^n$ with probability at least $1-\varepsilon$ almost surely with respect to the side-information $Y^n$. This is obviously a more stringent criterion than the average error probability criterion in which one simply requires that

$$\mathbb{P}\{X^n \neq \hat{X}^n\} \leq \varepsilon.$$  

(8)

Here, the error probability is averaged over the realizations of $Y^n$. Clearly, the rate of compression under the maximum error criterion is at least as large as the average error criterion. In this paper, we quantify this gap precisely in terms of the second-order asymptotics, i.e., the analogue of the term scaling $\sqrt{n}$ in (3). We show that the first-order terms in the asymptotic expansions are identical and equal to $n(1-\varepsilon)H(X \mid Y)$, but the source dispersion for the maximum error case is smaller than that of its average error counterpart. That is, the backoff from $n(1-\varepsilon)H(X \mid Y)$ is smaller for the former, more stringent, case compared to the latter. In fact, the maximum (resp. average) error source dispersion is the conditional (resp. unconditional) information variance of the conditional information density. By the law of total variance, the conditional information variance is not larger than its unconditional counterpart. It is easy to show that the difference is non-zero for most sources. En route to proving our second-order results, we develop new and novel one-shot bounds for both these error probability formalisms. We introduce two new information measures, namely the unconditional and conditional $\varepsilon$-cutoff entropies; in the $n$-shot setting, these characterize the fundamental compression limits up to a term scaling as $O(\log n)$ and $O(n^{1/6})$, respectively. Finally, we discuss applications of the second-order asymptotic results to guessing problems with a “giving-up” policy; this class of problems was recently introduced by Kuzuoka [12].

B. Related Works

1) Prefix-Free Codes: The problem of variable-length compression allowing errors was initiated by Han [13] who considered the fundamental limits of prefix-free codes with vanishing error probability. Han [13] derived a general formula for the normalized average codeword length. A general formula allowing for non-vanishing error probabilities was derived by Koga–Yamamoto [14]. For a stationary memoryless source $X^n$ on a finite alphabet $\mathcal{X}^n$, Koga–Yamamoto’s general formula can be reduced to the first-order term $n(1-\varepsilon)H(X)$, which coincides with (3) up to a term scaling as $o(n)$. In fact, as mentioned by Kuzuoka–Watanabe [15, Remark 2], the asymptotics of the prefix-free codes and the one-to-one codes are equal up to a constant factor.

2) Guessing Problems: One-to-one codes with side-information are essentially equivalent to strategies for guessing problems [16], [17] via Campbell’s source coding problem [18] without prefix-free constraints (cf. [19]–[21]). Kuzuoka [12] generalized the guessing problem by allowing positive error probabilities. The guesser can also give up guessing at each stage; in this case, an error is declared. Kuzuoka [12] derived general formulas of both Campbell’s source coding problems and guessing problems with non-vanishing error probability by introducing the conditional smooth Rényi entropy and by exploiting its properties.

3) Conditional Rate-Distortion Theory and State-Dependent Channels: A related topic to our present considerations is the conditional rate-distortion problem [22], [23]. Gray considered the problem of lossy compression with common side-information at both encoder and decoder. The duality between source coding and state-dependent channel coding problems with side-information available at both encoder and decoder have been characterized by Cover–Chiang [24] and Pradhan–Chou–Ramchandran [25].

4) Variable-Length Slepian–Wolf Coding: He–Lastras-Montaño–Yang–Jagmohan–Chen [26] investigated fixed- and variable-length Slepian–Wolf coding problems [27] with error probabilities that vanish but not exponentially fast. They derived the second-order coding rates and showed that variable-length Slepian–Wolf coding has a better second-order term compared to its fixed-length counterpart. These are characterized by some forms of the conditional and unconditional information variances, and the superiority of variable-length Slepian–Wolf coding is characterized by these differences. Variable-length Slepian–Wolf coding problems were also investigated by Kimura–Uyematsu [28] and Kuzuoka–Watanabe [15].

1Strictly speaking, Eq. (7) means error probability constraints except on a null set. However, we call it the maximum error probability as usual.
C. Paper Organization

This paper is organized as follows: The problem setting is formulated in Section II. Section II-A presents some definitions and notations of information measures for a correlated source \((X, Y)\). The \(\varepsilon\)-cutoff entropies are defined in Section II-B. Section II-C introduces the variable-length conditional lossless source coding problems. The main results of this study are given in Section III. Specifically, the second-order asymptotics of variable-length compression under maximum and average error formalisms are stated in Theorems 1 and 2 of Section III-A, respectively. Our one-shot coding theorems are stated in Lemmas 1 and 4 of Sections III-B and III-C, respectively; those are used to prove Theorems 1 and 2, respectively. Applications of Theorems 1 and 2 to guessing problems with a “giving-up policy” are discussed in Section IV. Section V concludes this study.

II. Preliminaries

A. Information Measures for Correlated Sources

Assume throughout that the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is rich enough so that all random variables (r.v.’s) are well-defined on the space. Consider a countably infinite alphabet\(^2\) \(X = \{1, 2, \ldots \}\) and an abstract alphabet \(\mathcal{Y}\). Let \(X\) be an \(X\)-valued r.v. and \(Y\) a \(\mathcal{Y}\)-valued r.v. Then, the pair \((X, Y)\) can be thought of as a correlated source pair.

In the conditional source coding, the second source \(Y\) plays the role of the side-information for the first source \(X\). We now introduce several information quantities. Let \(P_{X|Y}(x \mid Y)\) be a version of the conditional probability \(\mathbb{P}(X = x \mid Y)\) for each \(x \in X\). Denote by

\[
i(X \mid Y) = i_{X|Y}(X \mid Y) := \log \frac{1}{P_{X|Y}(X \mid Y)}
\]

the conditional information density of \(X\) given \(Y\). Define three \(\sigma(Y)\)-measurable information measures of \(X\) given \(Y\) as follows:

\[
\begin{align*}
H(P_{X|Y}) & := \mathbb{E}[i(X \mid Y)], \\
V(P_{X|Y}) & := \mathbb{E}[(i(X \mid Y) - H(P_{X|Y}))^2], \\
T(P_{X|Y}) & := \mathbb{E}[(i(X \mid Y) - H(P_{X|Y}))^3],
\end{align*}
\]

where \(\mathbb{E}[Z \mid W]\) stands for the conditional expectation of a real-valued r.v. \(Z\) given a sub-\(\sigma\)-algebra \(\sigma(W)\) generated by a r.v. \(W\). Moreover, we define four information measures of \(X\) given \(Y\) as follows:

\[
\begin{align*}
H(X \mid Y) & := \mathbb{E}[H(P_{X|Y})], \\
V_c(X \mid Y) & := \mathbb{E}[V(P_{X|Y})], \\
V_u(X \mid Y) & := \mathbb{E}[(i(X \mid Y) - H(X \mid Y))^2], \\
T_u(X \mid Y) & := \mathbb{E}[(i(X \mid Y) - H(X \mid Y))^3],
\end{align*}
\]

where \(\mathbb{E}[Z]\) stands for the expectation of a real-valued r.v. \(Z\). The quantity \(H(X \mid Y)\) is the well-known conditional Shannon entropy of \(X\) given \(Y\). In this study, we respectively call \(V_c(X \mid Y)\) and \(V_u(X \mid Y)\) the conditional and unconditional information variances\(^4\) of \(X\) given \(Y\). It follows by the law of total variance that

\[
V_u(X \mid Y) = V_c(X \mid Y) + \mathbb{E}[(H(P_{X|Y}) - H(X \mid Y))^2].
\]

Thus, the unconditional information variance \(V_u(X \mid Y)\) is larger than \(V_c(X \mid Y)\) by the term \(\mathbb{E}[(H(P_{X|Y}) - H(X \mid Y))^2]\), and these variances coincide if and only if \(H(P_{X|Y})\) is almost surely constant.

B. \(\varepsilon\)-Cutoff Entropies

Given a real-valued r.v. \(Z\), define the (unconditional) \(\varepsilon\)-cutoff transformation action of \(Z\) by

\[
\langle Z \rangle_\varepsilon := \begin{cases} 
Z & \text{if } Z < \eta, \\
B Z & \text{if } Z = \eta, \\
0 & \text{if } Z > \eta,
\end{cases}
\]

where \(B\) denotes a Bernoulli r.v. with parameter \((1 - \beta)\) in which the independence \(B \perp Z\) holds, and two parameters \(\eta \in \mathbb{R}\) and \(0 \leq \beta < 1\) are chosen so that

\[
\mathbb{P}(Z > \eta) + \beta \mathbb{P}(Z = \eta) = \varepsilon.
\]

\(^2\)In this study, assume that the \(\sigma\)-algebra on a countable alphabet is always the power set of the alphabet, as usual.

\(^3\)Note that \(P_{X|Y}(\cdot \mid Y)\) is a probability measure on \(X\) almost surely (a.s.) because the conditional probability is \(\sigma\)-additive.

\(^4\)These terminologies are inspired by Polyanskiy’s second-order asymptotic analysis in the channel coding problem [9, Equations (3.97)–(3.100)].
This is the same definition as [4, Equation (13)], and the notation $\langle Z \rangle_{\varepsilon}$ is consistent with that used in [4]. In addition, given a real-valued r.v. $Z$ and an arbitrary r.v. $W$, define the conditional $\varepsilon$-cutoff transformation action of $Z$ given $W$ by

$$\langle Z \mid W \rangle_{\varepsilon} := \begin{cases} Z & \text{if } Z < \eta_W, \\ B_W Z & \text{if } Z = \eta_W, \\ 0 & \text{if } Z > \eta_W, \end{cases}$$

where $B_W$ denotes a Bernoulli r.v. with parameter $(1 - \beta_W)$ in which the conditional independence $B_W \perp \! \! \! \perp Z \mid W$ holds, and two $\sigma(W)$-measurable real-valued r.v.’s $\eta_W \in \mathbb{R}$ and $0 \leq \beta_W < 1$ are chosen so that

$$\mathbb{P}(Z > \eta_W \mid W) + \beta_W \mathbb{P}(Z = \eta_W \mid W) = \varepsilon \quad (\text{a.s.}).$$

Using these cutoff operations, we now define the unconditional and conditional $\varepsilon$-cutoff entropies as follows:

$$\mathcal{C}^{\varepsilon}(X \mid Y) := \mathbb{E}[\langle \ell(X \mid Y) \rangle_{\varepsilon}],$$

$$\mathcal{C}^{\varepsilon}(X \mid Y) := \mathbb{E}[\langle \ell(X \mid Y) \rangle_{\varepsilon}],$$

respectively. Note that these $\varepsilon$-cutoff entropies are not additive in general.

Finally, the following proposition gives some basic properties of the $\varepsilon$-cutoff transformation actions.

**Proposition 1.** Let $Z$ be a nonnegative-valued r.v. $W$ an arbitrary r.v. and $0 \leq \varepsilon \leq 1$ a real number. Then, it holds that

$$\mathbb{E}[\langle Z \rangle_{\varepsilon}] = \min_{\mathbb{E}[\langle Z \rangle] \leq \varepsilon} \mathbb{E}[(1 - \varepsilon)(Z)],$$

$$\mathbb{E}[\langle Z \mid W \rangle_{\varepsilon}] = \min_{E: \mathbb{E}[\langle Z \rangle] \mid W \leq \varepsilon \text{ a.s.}} \mathbb{E}[(1 - E)(Z)],$$

where the minimization in (24) (resp. (25)) is taken over the mappings $\varepsilon : [0, \infty) \to [0, 1]$ (resp. the random maps $E : [0, \infty) \to [0, 1]$) satisfying $\mathbb{E}[(\varepsilon(Z)) \leq \varepsilon$ (resp. $\mathbb{E}[E(Z) \mid W] \leq \varepsilon$ a.s.). Moreover, the following identities hold:

$$\mathbb{E}[\langle Z \rangle_{\varepsilon}] = (1 - \varepsilon) \mathbb{E}[Z] - \int_{\eta}^{\infty} \mathbb{P}(Z > t) \, dt - \varepsilon (\eta - \mathbb{E}[Z]),$$

$$\mathbb{E}[\langle Z \mid W \rangle_{\varepsilon} \mid W] = (1 - \varepsilon) \mathbb{E}[Z \mid W] - \int_{\eta_W}^{\infty} \mathbb{P}(Z > t \mid W) \, dt - \varepsilon (\eta_W - \mathbb{E}[Z \mid W]) \quad (\text{a.s.}),$$

where $\eta$ and $\eta_W$ are given in (19) and (21) respectively. Furthermore, the following inequality holds:

$$\mathbb{E}[\langle Z \rangle_{\varepsilon}] \leq \mathbb{E}[\langle Z \mid W \rangle_{\varepsilon}].$$

**Proof of Proposition 1:** See Appendix A.

Note that (26) and (27) in Proposition 1 are useful in the subsequent second-order asymptotic analysis of the $\varepsilon$-cutoff entropies $\mathcal{C}^{\varepsilon}(X \mid Y)$ and $\mathcal{C}^{\varepsilon}(X \mid Y)$, respectively. The identities (24) and (25) will be used in the proofs of one-shot bounds stated in Lemmas 4 and 1, respectively, of Sections III-B and III-C, respectively. It follows from (28) of Proposition 1 that

$$\mathcal{C}^{\varepsilon}(X \mid Y) \leq \mathcal{C}^{\varepsilon}(X \mid Y).$$

**C. Variable-Length Compression Under Two Error Criteria**

Given an integer $n \geq 1$, denote by $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ $n$ i.i.d. copies of the source pair $(X, Y)$. Then, we may think of $(X^n, Y^n)$ as a sequence of outputs from the stationary memoryless correlated source $(X, Y)$, where $X^n = (X_1, \ldots, X_n)$ and $Y^n = (Y_1, \ldots, Y_n)$. In this subsection, we formalize the variable-length conditional (almost) lossless source coding problems. Let $\{0, 1\}^*$ be the set of finite-length binary strings containing the empty string $\varnothing$. For each $n \geq 1$, consider two random maps $F_n : X^n \times Y^n \to \{0, 1\}^*$ and $G_n : \{0, 1\}^* \times Y^n \to X^n$ in which both $F_n(X^n, Y^n)$ and $G_n(F_n(X^n, Y^n), Y^n)$ are $\mathcal{F}$-measurable. Then, we call the pair $(F_n, G_n)$ a variable-length stochastic code for the source $X^n$ with side-information $Y^n$ available at both encoder $F_n$ and decoder $G_n$.

**Remark 1.** Another way to consider a variable-length stochastic code is to design a $\{0, 1\}^*$-valued r.v. $B_n$ and an $X^n$-valued r.v. $\hat{X}^n$ in which those probability laws are determined by versions of the conditional probabilities $\mathbb{P}(B_n = b \mid X^n, Y^n)$ for $b \in \{0, 1\}^*$ and $\mathbb{P}(\hat{X}^n = x \mid B_n, Y^n)$ for $x \in X^n$, respectively.

Kostina–Polyanskiy–Verdú [4] studied variable-length stochastic codes without side-information $Y^n$ in this manner.

Let $\ell : \{0, 1\}^* \to \mathbb{N} \cup \{0\}$ be the length function of a finite-length binary string; e.g., $\ell(\varnothing) = 0$, $\ell(0) = \ell(1) = 1$, $\ell(00) = \ell(01) = \ell(10) = \ell(11) = 2$, and so on. Given a variable-length stochastic code $(F_n, G_n)$, we are interested in the average

More precisely, the random maps $E : [0, \infty) \to [0, 1]$ are given as a measurable mapping $(\omega, z) \mapsto E(\omega(z))$ from $\Omega \times [0, \infty)$ to $[0, 1]$.}
codeword length $\mathbb{E}[\ell(F_n(X^n, Y^n))]$ to measure the efficiency of the data compressor for the source $X^n$ with side-information $Y^n$.

**Definition 1** (Maximum error criterion). Let $n \geq 1$ be an integer, and $L \geq 0$ and $0 < \varepsilon < 1$ real numbers. Given a source $X$ with side-information $Y$, an $(n, L, \varepsilon)_{\text{max}}$-code is a variable-length stochastic code $(F_n, G_n)$ satisfying

$$\mathbb{E}[\ell(F_n(X^n, Y^n))] \leq L,$$

$$\mathbb{P}\{X^n \neq G_n(F_n(X^n, Y^n), Y^n) \mid Y^n\} \leq \varepsilon \quad \text{(a.s.)}. \quad (30)$$

**Definition 2** (Average error criterion). Let $n \geq 1$ be an integer, and $L \geq 0$ and $0 < \varepsilon < 1$ real numbers. Given a source $X$ with side-information $Y$, an $(n, L, \varepsilon)_{\text{avg}}$-code is a variable-length stochastic code $(F_n, G_n)$ satisfying

$$\mathbb{E}[\ell(F_n(X^n, Y^n))] \leq L,$$

$$\mathbb{P}\{X^n \neq G_n(F_n(X^n, Y^n), Y^n) \mid Y^n\} \leq \varepsilon. \quad (33)$$

Given a probability of error $0 \leq \varepsilon \leq 1$, this study deals with the fundamental limits of the average codeword length under these error criteria. Specifically, we will investigate the following two operational quantities:

$$L^*_\text{max}(n, \varepsilon, X, Y) := \inf\{L > 0 \mid \text{there exists an } (n, L, \varepsilon)_{\text{max}}\text{-code for the source } X \text{ with side-information } Y\}, \quad (34)$$

$$L^*_\text{avg}(n, \varepsilon, X, Y) := \inf\{L > 0 \mid \text{there exists an } (n, L, \varepsilon)_{\text{avg}}\text{-code for the source } X \text{ with side-information } Y\}. \quad (35)$$

### III. SECOND-ORDER ASYMPTOTICS AND ONE-SHOT BOUNDS

**A. Statements of Second-Order Asymptotic Results**

**Theorem 1** (Under maximum error criterion). Suppose that the following two hypotheses hold:

(a) $V(P_{X|Y})$ is bounded away from zero almost surely; and

(b) $T(P_{X|Y})$ is bounded away from infinity almost surely.

Then, it holds that

$$L^*_\text{max}(n, \varepsilon, X, Y) = n (1 - \varepsilon) H(X \mid Y) - \sqrt{n V_0(X \mid Y) f_0(\varepsilon)} + O(n^{1/6}) \quad \text{(as } n \to \infty) \quad (36)$$

for every $0 \leq \varepsilon \leq 1$.

*Proof of Theorem 1:* See Section III-B.

**Theorem 2** (Under average error criterion). Suppose that $T_0(X \mid Y)$ is finite. Then, it holds that

$$L^*_\text{avg}(n, \varepsilon, X, Y) = n (1 - \varepsilon) H(X \mid Y) - \sqrt{n V_0(X \mid Y) f_0(\varepsilon)} + O(\log n) \quad \text{(as } n \to \infty) \quad (37)$$

for every $0 \leq \varepsilon \leq 1$.

*Proof of Theorem 2:* See Section III-C.

Since an $(n, L, \varepsilon)_{\text{max}}$-code is an $(n, L, \varepsilon)_{\text{avg}}$-code, it is clear that

$$L^*_\text{avg}(n, \varepsilon, X, Y) \leq L^*_\text{max}(n, \varepsilon, X, Y). \quad (38)$$

Theorems 1 and 2 state that the first-order optimal coding rates are the same under both the maximum and average error criteria; they are equal to $n (1 - \varepsilon) H(X \mid Y)$. This is somewhat surprising because under the maximum error criterion, we might expect the first-order term to be

$$n (1 - \varepsilon) \inf\{h \geq 0 \mid \mathbb{P}\{H(P_{X|Y}) \leq h\} = 1\}. \quad (39)$$

On the other hand, we see from (17) that unless $H(P_{X|Y})$ is almost surely constant, the optimal second-order coding rates differ under maximum and average error criteria. Since $f_0(\varepsilon) \geq 0$ with equality if and only if either $\varepsilon = 0$ or $\varepsilon = 1$, note in Theorems 1 and 2 that a larger dispersion implies a shorter average codeword length on the $\sqrt{n}$ scale for every fixed $0 < \varepsilon < 1$.

In particular, it follows from (17) that the variable-length source dispersion $V_0(X \mid Y)$ under the average error criterion is larger than that $V_0(X \mid Y)$ under the maximum error criterion by the term $\mathbb{E}[\ell(F_n(X^n, Y^n)) - H(X \mid Y)^2]$.

**Remark 2.** In channel coding, the counterparts of both conditional and unconditional information variances coincide for every capacity-achieving input distribution (cf. [9, Lemma 46] or [10, Lemma 62]). Since the first-order term determines the choice of input distribution (cf. [9, Lemma 48]), the $\varepsilon$-channel dispersion is determined by a capacity-achieving input distribution. Therefore, there is no difference between the conditional and unconditional information variances in channel coding without input cost constraints. On the other hand, the conditional and unconditional information variances are different for the problem at hand as there is no optimization over input distributions. Thus, the variable-length source dispersion under the maximum and average error formalisms are different.
Remark 3. In Theorems 1 and 2, the code is allowed to be stochastic. Namely, an encoder (resp. a decoder) outputs a compressed binary string \( B \) (resp. the reconstructed source \( \hat{X} \)) stochastically according to some probability law given a source \( X \) (resp. a compressed binary string \( B \)) and the side-information \( Y \) (see Remark 1). Since the average codeword length of a stochastic code is nearly equal to that of a deterministic code up to a constant additive term (cf. [4, Section II-A]), our asymptotic analysis is the same as that if we assumed the code is deterministic. See also [12, Remark 6].

Remark 4. In Theorems 1 and 2, prefix-free constraints are not imposed on the codes. However, after some considerations, one can see that our second-order asymptotic results hold for codes with prefix-free constraints. In [15, Proposition 1], Kuzuoka–Watanabe provided a one-shot coding theorem for variable-length conditional lossless source coding with prefix-free constraints under the average error criterion. According to [15, Remark 2], one observes that the unconditional \( \varepsilon \)-cutoff entropy is equal to its conditional quantity \( \hat{H}_\varepsilon(X | Y) \) up to an additive constant. Moreover, relations between \( \hat{H}_\varepsilon(X | Y) \) and Koga–Yamamoto’s quantity \( G_{1,\varepsilon} \) introduced in [14, Theorem 3] are also discussed in [15, Remark 1 and Theorem 1]. These considerations allow us to adapt our analysis of the \( \varepsilon \)-cutoff entropies so that they are also applicable to prefix-free codes.

Even if the side-information alphabet \( \mathcal{Y} \) is countably infinite, there is a correlated source \((X,Y)\) that \( V(P_{X,Y}) \) and \( T(P_{X,Y}) \) are not bounded away from zero and infinity a.s., respectively, but \( T_0(X | Y) \) is finite. Therefore, Hypotheses (a) and (b) in Theorem 1 are stronger than the hypothesis in Theorem 2 in general. On the other hand, these hypotheses can be removed when the source and the side-information alphabets are finite.

Proposition 2. If \( X \) is supported on some finite subalphabet \( \mathcal{A} \subset X \), then Hypothesis (b) in Theorem 1 holds.

Proposition 3. If \( \mathcal{Y} \) is finite, then (36) in Theorem 1 holds without Hypothesis (a).

Proof of Propositions 2 and 3: See Appendix I.

B. Proof of Theorem 1

The following lemma gives us one-shot bounds on the fundamental limit \( L_{\max}^\varepsilon(n, \varepsilon, X, Y) \) defined in (34) in terms of the conditional \( \varepsilon \)-cutoff entropy \( \zeta_{\varepsilon} \) defined in (23).

Lemma 1. For every \( n \geq 1 \) and every \( 0 \leq \varepsilon \leq 1 \), it holds that

\[
0 \leq \zeta_{\varepsilon}(X^n | Y^n) - L_{\max}^\varepsilon(n, \varepsilon, X, Y) \leq \log(n H(X | Y) + 1) + \log \varepsilon. \tag{40}
\]

Proof of Lemma 1: See Appendix B.

Note that Lemma 1 holds without Hypotheses (a) and (b) in Theorem 1. Since \( H(X | Y) < \infty \) if Hypothesis (b) in Theorem 1 holds, Lemma 1 implies that

\[
L_{\max}^\varepsilon(n, \varepsilon, X, Y) = \zeta_{\varepsilon}(X^n | Y^n) + O(\log n) \quad (\text{as } n \to \infty). \tag{41}
\]

Thus, it suffices to provide an appropriate asymptotic estimate on \( \zeta_{\varepsilon}(X^n | Y^n) \).

Lemma 2. Given a fixed \( 0 \leq \varepsilon \leq 1 \), it holds that

\[
\zeta_{\varepsilon}(X^n | Y^n) = n (1 - \varepsilon) H(X | Y) - \mathbb{E} \left[ \sqrt{V(P_{X^n | Y^n})} \right] f_\varepsilon(\varepsilon) + O(1) \quad (\text{as } n \to \infty), \tag{42}
\]

provided that Hypotheses (a) and (b) in Theorem 1 hold.

Proof of Lemma 2: See Appendix C.

Unfortunately, obtaining an exact single-letter expression for the “dispersion” term \( \mathbb{E} \left[ \sqrt{V(P_{X^n | Y^n})} \right] \) that appears in (42) is difficult unless \( V(P_{X^n | Y^n}) \) is almost surely constant. In fact, it can be verified by Jensen’s inequality that

\[
\mathbb{E} \left[ \sqrt{V(P_{X^n | Y^n})} \right] \leq \sqrt{n} V_{\varepsilon}(X | Y). \tag{43}
\]

with equality if and only if \( V(P_{X^n | Y^n}) \) is almost surely constant, because \( V(P_{X^n | Y^n}) \) is a sequence of i.i.d. copies of \( V(P_{X^n | Y^n}) \). Hence, Lemmas 1 and 2 can be readily reduced to Kostina–Polyanski–Verdú’s result [4, Theorem 4] in (3), provided that \( X \) and \( Y \) are independent.

The following lemma provides an asymptotic estimate of \( \mathbb{E} \left[ \sqrt{V(P_{X^n | Y^n})} \right] \).

Lemma 3. If \( \mathbb{E} [V(P_{X^n | Y^n})] < \infty \), then

\[
\mathbb{E} \left[ \sqrt{V(P_{X^n | Y^n})} \right] = \sqrt{n} V_{\varepsilon}(X | Y) + O(n^{1/6}) \quad (\text{as } n \to \infty). \tag{44}
\]

Proof of Lemma 3: See Appendix D.

Hypothesis (b) in Theorem 1 implies that \( \mathbb{E} [V(P_{X^n | Y^n})] < \infty \); therefore, Lemmas 1–3 yield Theorem 1, as desired.
C. Proof of Theorem 2

Similar to Lemma 1, the following lemma states one-shot bounds on the fundamental limit \( L^*_{\text{avg}}(n, \varepsilon, X, Y) \) defined in (35) in terms of the unconditional \( \varepsilon \)-cutoff entropy \( \xi^*_n \) defined in (22).

**Lemma 4.** It holds that

\[
0 \leq \xi^*_n(X^n | Y^n) - L^*_{\text{avg}}(n, \varepsilon, X, Y) \leq \log(n H(X | Y) + 1) + \log e. \tag{45}
\]

**Proof of Lemma 4:** See Appendix E.

Note that Lemma 4 holds without Hypotheses (a) and (b) in Theorem 2. Since \( H(X | Y) < \infty \) if \( T_0(X | Y) < \infty \), Lemma 4 tells us that

\[
L^*_{\text{avg}}(n, \varepsilon, X, Y) = \xi^*_n(X^n | Y^n) + O(\log n) \quad \text{as } n \to \infty.
\]

Thus, it suffices to provide an appropriate asymptotic estimate on \( \xi^*_n(X^n | Y^n) \).

**Lemma 5.** Given a fixed \( 0 \leq \varepsilon \leq 1 \), it holds that

\[
\xi^*_n(X^n | Y^n) = n (1 - \varepsilon) H(X | Y) - \sqrt{n V_0(X | Y)} f_0(\varepsilon) + O(1) \quad \text{as } n \to \infty,
\]

provided that \( T_0(X | Y) < \infty \).

**Proof of Lemma 5:** Since \( \xi(X^n | Y^n) = \iota(X_1 | Y_1) + \iota(X_2 | Y_2) + \cdots + \iota(X_n | Y_n) \), and since \( \iota(X_1 | Y_1), \iota(X_2 | Y_2), \ldots, \iota(X_1 | Y_1) \) are i.i.d. real-valued r.v.’s, a naïve application of (19) readily proves Lemma 5. For the readers’ convenience, we now only give a sketch of the proof as follows: If \( V_c(X | Y) = 0 \), then we readily see that

\[
\xi^*_n(X^n | Y^n) = n (1 - \varepsilon) H(X | Y).
\]

Thus, it suffices to consider the case where \( V_c(X | Y) > 0 \). It follows from (26) of Proposition 1 that

\[
\xi^*_n(X^n | Y^n) = n (1 - \varepsilon) H(X | Y) - \int_{\eta_n}^{\infty} P\{\iota(X^n | Y^n) > t\} \, dt - \varepsilon (\eta_n - n H(X | Y)),
\]

where \( \eta_n > 0 \) is given so that

\[
P\{\iota(X^n | Y^n) > \eta_n\} + \beta_n P\{\iota(X^n | Y^n) = \eta_n\} = \varepsilon
\]

with an appropriate \( 0 \leq \beta_n < 1 \). Then, the uniform Berry–Esseen bound (cf. (186) in Appendix H) shows that

\[
\eta_n = n H(X | Y) + \sqrt{n V_0(X | Y)} \Phi^{-1}(1 - \varepsilon) + O(1) \quad \text{as } n \to \infty,
\]

provided that \( T_0(X | Y) < \infty \). On the other hand, it can be verified by the non-uniform Berry–Esseen bound (cf. Lemma 10 in Appendix H) that

\[
\int_{\eta_n}^{\infty} P\{\iota(X^n | Y^n) > t\} \, dt = \sqrt{n V_0(X | Y)} \left(f_0(\varepsilon) - \varepsilon \Phi^{-1}(1 - \varepsilon)\right) + O(1) \quad \text{as } n \to \infty,
\]

provided that \( T_0(X | Y) < \infty \). Therefore, Lemma 5 can be proven by combining (49), (51), and (52).

The proof of Theorem 2 is finally completed by combining Lemmas 4 and 5.

**IV. GUESSING PROBLEM**

Following [12, Section III], we now introduce the guessing problem with a “giving-up” policy. Let \( n \geq 1 \) be an integer. Consider the stationary memoryless correlated source \( (X^n, Y^n) \) as in Section II-C. A guessing function \( g_n : X^n \times Y^n \to X \) is a deterministic map in which \( g_n(x, y) : X^n \to X \) is bijective for each \( y \in Y^n \). This function induces the following strategy: the guesser asks “Is \( X^n = x^n? \)” if \( g_n(x_1, y_1) = 1 \) at time 1; if not, the guesser asks “Is \( X^n = x_2^n? \)” if \( g_n(x_2, y_2) = 2 \) at time 2; if not again, the guesser asks “Is \( X^n = x_3^n? \)” if \( g_n(x_3, y_3) = 3 \) at time 3, and so on. By introducing a certain error probability for guessing, the guesser can stochastically give up at each time. For each \( (k, y) \in X \times Y^n \), let \( 0 \leq \pi_n(k | y) \leq 1 \) be the real number that plays the role of a giving-up policy: just before starting on the \( k \)-th guess, the guesser can give up his task with probability \( \pi_n(k | Y^n) \). We call the pair \( (g_n, \pi_n(\cdot | \cdot)) \) a guessing strategy with a giving-up policy. Formally, for each \( k \geq 1 \), the guesser declares an error just before starting on the \( k \)-th guess if \( E_{n,1} = E_{n,2} = \cdots = E_{n,k-1} = 0 \) and \( E_{n,k} = 1 \), where \( \{E_{n,k}\}_{k=1}^{\infty} \) denotes a sequence of conditionally (and mutually) independent Bernoulli r.v.’s given \( Y^n \) in which

\[
P\{E_{n,k} = 1 \mid Y^n\} = \pi_n(k | Y^n) \quad \text{a.s.}
\]

for each \( k \geq 1 \). Then, the giving-up guessing function \( G_n : X^n \times Y^n \to X \cup \{c_e\} \) is a random map given as

\[
G_n(X^n, Y^n) := \begin{cases} 
  g_n(X^n, Y^n) & \text{if } E_{n,l} = 0 \text{ for all } 1 \leq l \leq n, \\
  c_e & \text{otherwise},
\end{cases}
\]

(54)
where \( c_\varepsilon > 0 \) denotes the cost of marking an error.\(^6\) While Kuzuoka investigated the fundamental limits of the \( \rho \)-th moment \( \mathbb{E}[G_n(X^n, Y^n)] \) with a fixed real \( \rho > 0 \) to evaluate the guessing cost (see [12, Equation (33)]), we are now interested in the fundamental limits of \( \mathbb{E}[\log G_n(X^n, Y^n)] \). In fact, if \( \mathbb{E}[G_n(X^n, Y^n)] \) is finite for some \( \rho > 0 \), then it follows by l'Hôpital's rule and the dominated convergence theorem that

\[
\lim_{\rho \to 0} \frac{1}{\rho} \log \mathbb{E}[G_n(X^n, Y^n)] = \mathbb{E}[\log G_n(X^n, Y^n)].
\]

so our study in this section can be thought of as a limiting case of that in [12]. Noting that errors are declared if and only if \( G_n(X^n, Y^n) \neq g_n(X^n, Y^n) \), we define two error formalisms as follows:

**Definition 3** (Maximum error criterion). Given a source \( X \) with side-information \( Y \), an \( (n, N, \varepsilon)_{\text{max}} \)-guessing strategy is a guessing strategy \( (G_n, \pi_n(\cdot | \cdot)) \) satisfying

\[
\mathbb{E}[\log G_n(X^n, Y^n)] \leq N,
\]

\[
\mathbb{P}(G_n(X^n, Y^n) \neq g_n(X^n, Y^n) | Y^n) \leq \varepsilon \quad \text{(a.s.)}.
\]

**Definition 4** (Average error criterion). Given a source \( X \) with side-information \( Y \), an \( (n, N, \varepsilon)_{\text{avg}} \)-guessing strategy is a guessing strategy \( (G_n, \pi_n(\cdot | \cdot)) \) satisfying

\[
\mathbb{E}[\log G_n(X^n, Y^n)] \leq N,
\]

\[
\mathbb{P}(G_n(X^n, Y^n) \neq g_n(X^n, Y^n)) \leq \varepsilon.
\]

Given a probability of error \( 0 \leq \varepsilon \leq 1 \), we investigate the following two operational quantities:

\[
N_{\text{max}}^*(n, \varepsilon, X, Y) := \inf \{N > 0 \mid \text{there exists an } (n, N, \varepsilon)_{\text{max}} \text{-guessing strategy for the source } X \text{ with side-information } Y \},
\]

\[
N_{\text{avg}}^*(n, \varepsilon, X, Y) := \inf \{N > 0 \mid \text{there exists an } (n, N, \varepsilon)_{\text{avg}} \text{-guessing strategy for the source } X \text{ with side-information } Y \}.
\]

**Corollary 1** (Under maximum error criterion). Suppose Hypotheses (a) and (b) in Theorem 1. Then, it holds that

\[
N_{\text{max}}^*(n, \varepsilon, X, Y) = n(1 - \varepsilon) H(X | Y) - \sqrt{n} V_\varepsilon(X | Y) f_\varepsilon(\varepsilon) + O(n^{1/6}) \quad \text{(as } n \to \infty)\]

for every \( 0 \leq \varepsilon \leq 1 \).

**Corollary 2** (Under average error criterion). Suppose that \( T_\varepsilon(X | Y) \) is finite. Then, it holds that

\[
N_{\text{avg}}^*(n, \varepsilon, X, Y) = n(1 - \varepsilon) H(X | Y) - \sqrt{n} V_\varepsilon(X | Y) f_\varepsilon(\varepsilon) + O(\log n) \quad \text{(as } n \to \infty)\]

for every \( 0 \leq \varepsilon \leq 1 \).

**Proof of Corollaries 1 and 2:** Relying on Theorems 1 and 2, it suffices to prove the following lemma:

**Lemma 6.** For every \( n \geq 1 \), every \( 0 \leq \varepsilon \leq 1 \), and every correlated source \( (X, Y) \), it holds that

\[
|N_{\text{max}}^*(n, \varepsilon, X, Y) - L_{\text{max}}^*(n, \varepsilon, X, Y)| \leq 1 + |\log c_\varepsilon|,
\]

\[
|N_{\text{avg}}^*(n, \varepsilon, X, Y) - L_{\text{avg}}^*(n, \varepsilon, X, Y)| \leq 1 + |\log c_\varepsilon|.
\]

Lemma 6 is proven in Appendix F, completing the proof of Corollaries 1 and 2.\(\blacksquare\)

From Lemma 6, it is worth pointing out that the asymptotic results of Corollaries 1 and 2 still hold even if the error cost \( c_\varepsilon \) grows polynomially in \( n \).

\[\]

**V. Concluding Remarks**

We considered two variable-length conditional lossless source coding problems in this paper. We derived one-shot coding theorems and the second-order asymptotic results under two error formalisms: the maximum and the average probabilities of error. The one-shot bounds of Lemmas 1 and 4 are stated in terms of the \( \varepsilon \)-cutoff entropies \( \mathbb{E}^\varepsilon(X | Y) \) and \( \mathbb{C}^\varepsilon(X | Y) \), respectively. These one-shot bounds are generalizations of Kostina–Polyanskiy–Verdú’s one-shot coding theorem [4, Theorem 2] to the case in which side-information \( Y \) is available at both encoder and decoder. While Kostina–Polyanskiy–Verdú proved the one-shot coding theorem by showing how to construct an optimal stochastic code (see [4, Section II-A]), we have provided the converse bounds of our one-shot bounds explicitly in our proofs (see Appendices G and J). The variable-length source dispersions under the maximum and average error criteria were derived by proving asymptotic estimates on the \( \varepsilon \)-cutoff entropies \( \mathbb{E}^\varepsilon(X^n | Y^n) \) and \( \mathbb{C}^\varepsilon(X^n | Y^n) \) in Lemmas 2 and 5, respectively. In Section IV, we showed that our results can be applicable to Kuzuoka’s guessing problem [12, Section III].

\[\]

\[\]

For simplicity of our analysis, we assume that \( c_\varepsilon \) is not an integer. This assumption simplifies the guessing error event \( \{G_n(X^n, Y^n) \neq g_n(X^n, Y^n)\} \), and does not affect the results in [12] and Corollaries 1 and 2 under a valid definition of the error event.
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APPENDIX A

PROOF OF PROPOSITION 1

A. Proof of (24)

The identity (24) is stated in [4, Equation (38)], and can be thought of as a special case of (25) in which \( \sigma(W) \) is the trivial \( \sigma \)-algebra \( \{\emptyset, \Omega\} \), where \( \emptyset \) stands for the empty set. We leave the proof to the next subsection.

B. Proof of (25)

Let \( E^* : [0, \infty) \rightarrow [0, 1] \) be a random map given as

\[
E^*(z) = \begin{cases} 
0 & \text{if } z < \eta_w, \\
\beta_w & \text{if } z = \eta_w, \\
1 & \text{if } z > \eta_w
\end{cases}
\] (66)

for each \( z \geq 0 \), where the \( \sigma(W) \)-measurable r.v.’s \( \eta_w \geq 0 \) and \( 0 \leq \beta_w < 1 \) are given in (21). It is clear from (21) and (66) that

\[
\mathbb{E}[E^*(Z) \mid W] = \varepsilon \quad (\text{a.s.}).
\] (67)

After some algebra, we get

\[
\mathbb{P}\{\langle Z \mid W \rangle_e > t \mid W\} = \mathbb{P}\{(1 - E^*(Z))Z > t \mid W\} = \begin{cases} 
\mathbb{P}\{Z > t \mid W\} & \text{if } t < 0, \\
\mathbb{P}\{Z > t \mid W\} - \varepsilon & \text{if } 0 \leq t < \eta, \\
0 & \text{if } t \geq \eta,
\end{cases} \quad (\text{a.s.}).
\] (68)

Therefore, two r.v.’s \( \langle Z \mid W \rangle_e \) and \( (1 - E^*(Z))Z \) are equal in distribution, which implies that

\[
\mathbb{E}[\langle Z \mid W \rangle_e] = \mathbb{E}[(1 - E^*(Z))Z].
\] (69)

Consider an arbitrary random map \( E : [0, \infty) \rightarrow [0, 1] \) satisfying

\[
\mathbb{E}[E(Z) \mid W] \leq \varepsilon \quad (\text{a.s.}).
\] (70)

Denoting by \( \mathbf{1}_A \) the indicator function of \( A \subset \Omega \), a direct calculation shows

\[
\mathbb{E}[(E(Z) - E^*(Z))Z \mid W] = \mathbb{E}[(E(Z) - E^*(Z))Z(\mathbf{1}_{z < \eta_w} + \mathbf{1}_{z = \eta_w} + \mathbf{1}_{z > \eta_w})] \mid W]
\]
\[
\overset{(a)}{=} \mathbb{E}[E(Z) \mathbf{1}_{z < \eta_w} \mid W] + \mathbb{E}[(E(Z) - \beta_w)Z \mathbf{1}_{z = \eta_w} \mid W] + \mathbb{E}[(E(Z) - 1)Z \mathbf{1}_{z > \eta_w} \mid W]
\]
\[
\overset{(b)}{=} \eta_w \left[ \mathbb{E}[E(Z) \mathbf{1}_{z < \eta_w} \mid W] + \mathbb{E}[(E(Z) - \beta_w) \mathbf{1}_{z = \eta_w} \mid W] + \mathbb{E}[(E(Z) - 1) \mathbf{1}_{z > \eta_w} \mid W] \right]
\]
\[
\overset{(c)}{=} \eta_w \mathbb{E}[E(Z) - E^*(Z) \mid W]
\]
\[
\overset{(d)}{\leq} 0 \quad (\text{a.s.}), \tag{71}
\]

where

\[
\begin{align*}
(a) & \text{ and (c) follow from the definition of } E^* \text{ stated in (66)}; \\
(b) & \text{ follows from the fact that } \eta_w \text{ is } \sigma(W) \text{-measurable; and} \\
(d) & \text{ follows from (67) and (70) and the fact that } \eta_w \geq 0.
\end{align*}
\]

Combining (67) and (69)–(71), we obtain (25) of Proposition 1, as desired.

C. Proof of (26)

The identity (26) can be shown in the same manner as [4, Equations (155)–(156)], and can be thought of as a special case of (27) in which \( \sigma(W) \) is the trivial \( \sigma \)-algebra. We leave the proof to the next subsection.
D. Proof of (27)

It can be verified that the following conditional version of [4, Equation (157)] holds:  
\[ \mathbb{E}[Z \mathbf{1}_{\{Z > z\}} | W] = \int_{z}^{\infty} \mathbb{P}\{Z > t | W\} \, dt + z \mathbb{P}\{Z > z | W\} \quad \text{(a.s.)} \]  
where the minimization in the left-hand side (resp. the right-hand side) is taken over the mappings \( \mathbb{P}\{Z > t | W\} \) or \( z \mathbb{P}\{Z > z | W\} \) for every real number \( z \geq 0 \).

We have
\[ \mathbb{E}[\langle Z | W\rangle_{\tau} | W] \overset{(a)}{=} \mathbb{E}[Z \mathbf{1}_{\{Z < \eta W\}} | W] + \eta W (1 - \beta W) \mathbb{P}\{Z = \eta W | W\} \\
\quad = \mathbb{E}[Z | W] - \mathbb{E}[Z \mathbf{1}_{\{Z > \eta W\}} | W] - \eta W \beta W \mathbb{P}\{Z = \eta W | W\} \\
\overset{(b)}{=} \mathbb{E}[Z | W] - \int_{\eta W}^{\infty} \mathbb{P}\{Z > t\} \, dt - \eta W \mathbb{P}\{Z > \eta W | W\} \\
\overset{(c)}{=} \mathbb{E}[Z | W] - \int_{\eta W}^{\infty} \mathbb{P}\{Z > t | W\} \, dt - \eta W \mathbb{P}\{Z = \eta W | W\} \quad \text{(a.s.)}, \]

where
- (a) follows from the definition of \( \langle \cdot | \cdot \rangle_{\tau} \) stated in (20);
- (b) follows from (72) by noting that \( \eta W \) is \( \sigma(W) \)-measurable; and
- (c) follows from (21).

This completes the proof of (27) of Proposition 1. \( \blacksquare \)

E. Proof of (28)

Since the functional \( \epsilon \mapsto \mathbb{E}[(1 - \epsilon(Z)) Z] \) of a mapping \( \epsilon : [0, \infty) \to [0, 1] \) is linear, we readily see that
\[ \min_{\epsilon : \mathbb{E}[\epsilon(Z)] = \epsilon} \mathbb{E}[(1 - \epsilon(Z)) Z] = \min_{E : \mathbb{E}[E(Z)] = \epsilon} \mathbb{E}[(1 - E(Z)) Z], \]

where the minimization in the left-hand side (resp. the right-hand side) is taken over the mappings \( \epsilon : [0, \infty) \to [0, 1] \) (resp. the random maps \( E : [0, \infty) \to [0, 1] \)) satisfying \( \mathbb{E}[\epsilon(Z)] = \epsilon \) (resp. \( \mathbb{E}[E(Z)] = \epsilon \)). Therefore, the proof is completed by the identities (24) and (25) of Proposition 1, and the fact that \( \mathbb{E}[E(Z)] = \epsilon \) if \( \mathbb{P}\{\mathbb{E}[E(Z) | W] = \epsilon\} = 1 \).

\( \blacksquare \)

Appendix B

Proof of Lemma 1

It suffices to consider the case in which \( n = 1 \), i.e., it suffices to prove that
\[ 0 \leq \xi_\mathcal{S}^* (X | Y) - L_\max^* (\epsilon, X, Y) \leq \log(H(X | Y) + 1) + \log \epsilon, \]

where \( L_\max^* (\epsilon, X, Y) := L_\max^* (1, \epsilon, X, Y) \). The following lemma gives us a formula of the fundamental limit \( L_\max^* (\epsilon, X, Y) \).

Lemma 7. Given \( 0 \leq \epsilon \leq 1 \), it holds that
\[ L_\max^* (\epsilon, X, Y) = \mathbb{E}[(\log \mathcal{S}^{-1}_Y (X) | Y)_{\epsilon}], \]

where \( \mathcal{S}_Y \) stands for a random permutation on \( X \) satisfying
\[ P_{X|Y}(\mathcal{S}_Y(1) | Y) \geq P_{X|Y}(\mathcal{S}_Y(2) | Y) \geq P_{X|Y}(\mathcal{S}_Y(3) | Y) \geq \cdots \quad \text{(a.s.)}, \]

which rearranges the probability masses in \( P_{X|Y}(\cdot | Y) \) in non-increasing order.

Proof of Lemma 7: See Appendix G.

We are now ready to prove (75) which immediately ensures Lemma 1. Define the event
\[ C_k := \left\{ \frac{1}{\mathbb{P}_{X|Y}(\mathcal{S}_Y(x) | Y)} \log x \leq \log \frac{1}{\mathbb{P}_{X|Y}(\mathcal{S}_Y(x) | Y)} \text{ for all } 1 \leq x \leq k \right\} \]

for each integer \( k \geq 1 \). Since \( \mathcal{S}_Y \) rearranges the probability masses in \( P_{X|Y}(\cdot | Y) \) in non-increasing order (see (77)), similar to [5, Theorem 2], it can be verified by induction that
\[ \mathbb{P}(C_k) = 1 \]

7While [4, Equation (157)] can be verified by applying Tonelli’s theorem only once, one can show (72) by applying Tonelli’s theorem twice.
for every $k \geq 1$. Hence, the monotonicity $C_1 \supset C_2 \supset C_3 \supset \cdots$ implies that
\[
P \left\{ \log \sigma_Y^{-1}(x) \right\} \leq \log \frac{1}{P(x | Y)} \quad \text{for all } x \in X \right\} = 1, \tag{80}
\]
yielding
\[
\mathbb{E}[\langle \log \sigma_Y^{-1}(X) \rangle | Y] \leq \mathcal{C}_e(X | Y). \tag{81}
\]
Thus, it follows from Lemma 7 that the left-hand inequality of (75) holds.

On the other hand, we observe that
\[
\mathbb{E}[\langle \log \sigma_Y^{-1}(X) \rangle | Y] = \mathcal{C}_e(X | Y) \tag{82}
\]
where
- (a) follows from (25) of Proposition 1;
- (b) follows from (80);
- (c) follows from the fact that the same argument as [3] proves
\[
\mathbb{E}[\langle \log \sigma_Y^{-1}(X) \rangle | Y] \geq H(P_{X|Y}) - \log(H(P_{X|Y}) + 1) - \log e \quad \text{(a.s.)}, \tag{83}
\]
which leads together with Jensen’s inequality that
\[
\mathbb{E}[\langle \log \sigma_Y^{-1}(X) \rangle | Y] \geq H(X | Y) - \log(H(X | Y) + 1) - \log e; \tag{84}
\]
and
- (d) follows as in (a).

Therefore, it follows from Lemma 7 that the right-hand side of (75) holds. This completes the proof of Lemma 1. \hfill \blacksquare

APPENDIX C

PROOF OF Lemma 2

It is clear from the definition of $\mathcal{C}_e$ stated in (23) that
\[
\varepsilon = 0 \implies \mathcal{C}_e(X^n | Y^n) = nH(X | Y), \quad \varepsilon = 1 \implies \mathcal{C}_e(X^n | Y^n) = 0. \tag{85, 86}
\]

Hence, it suffices to consider the case where $0 < \varepsilon < 1$. It follows from (27) of Proposition 1 that
\[
\mathbb{E}[\langle \log Y^n \rangle | Y^n] = \int_{t}^{n} \mathbb{P}\{t(X^n | Y^n) > \eta_Y^n \} \, dt - \varepsilon \left( \eta_Y^n - H(P_{X^n|Y^n}) \right) \quad \text{(a.s.)} \tag{87}
\]
for every $n \geq 1$, where $\sigma(Y^n)$-measurable r.v.’s $\eta_Y^n \geq 0$ and $0 \leq \beta_Y^n < 1$ are given so that
\[
\mathbb{P}\{t(X^n | Y^n) > \eta_Y^n | Y^n\} + \beta_Y^n \mathbb{P}\{t(X^n | Y^n) = \eta_Y^n | Y^n\} = \varepsilon \quad \text{(a.s.)}. \tag{88}
\]
Similar to [4, Equations (159)–(165)], we see that
\[
\int_{\eta_{Y_n}}^{\infty} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > t \mid Y^n \right\} \, dt 
\]
\[
\quad \stackrel{(a)}{=} \int_{\eta_{Y_n}}^{\infty} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > H(P_{X^n|Y^n}) + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) + t \mid Y^n \right\} \, dt 
\]
\[
\quad \stackrel{(b)}{=} \int_{0}^{\infty} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > H(P_{X^n|Y^n}) + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) + t \mid Y^n \right\} \, dt - B_{Y^n} 
\]
\[
\quad \stackrel{(c)}{=} \sqrt{V(P_{X^n|Y^n})} \int_{0}^{\infty} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > H(P_{X^n|Y^n}) + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) + r \mid Y^n \right\} \, dr - B_{Y^n} 
\]
\[
\quad \stackrel{(d)}{=} \sqrt{V(P_{X^n|Y^n})} \int_{0}^{\infty} (1 - \Phi(r)) \, dr - B_{Y^n} + D_{Y^n} 
\]
\[
\quad \stackrel{(e)}{=} \sqrt{V(P_{X^n|Y^n})} \int_{0}^{\infty} r \varphi(r) \, dr - \varepsilon \Phi^{-1}(1-\varepsilon) - B_{Y^n} + D_{Y^n} 
\]
\[
\quad \stackrel{(f)}{=} \sqrt{V(P_{X^n|Y^n})} \left( f_{\tilde{G}}(1-\varepsilon) - \varepsilon \Phi^{-1}(1-\varepsilon) \right) - B_{Y^n} + D_{Y^n} \quad \text{a.s.} \tag{89}
\]
for every \( n \geq 1 \), where
- (a) follows by the definition
  \[
  b_{Y^n} := \eta_{Y^n} - H(P_{X^n|Y^n}) - \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon); \tag{90}
  \]
- (b) follows by the definition
  \[
  B_{Y^n} := \text{sgn}(b_{Y^n}) \int_{\min(0, b_{Y^n})}^{\max(0, b_{Y^n})} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > H(P_{X^n|Y^n}) + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) + t \mid Y^n \right\} \, dt \tag{91}
  \]
  with the sign function \( \text{sgn} : \mathbb{R} \to \{-1, 0, 1\} \) defined by
  \[
  \text{sgn}(u) := \begin{cases} 
  -1 & \text{if } u < 0, \\
  0 & \text{if } u = 0, \\
  1 & \text{if } u > 0;
  \end{cases} \tag{92}
  \]
- (c) follows by the substitution rule for integrals with
  \[
  t = r \sqrt{V(P_{X^n|Y^n})}; \tag{93}
  \]
- (d) follows by the definition
  \[
  D_{Y^n} := \sqrt{V(P_{X^n|Y^n})} \int_{\Phi^{-1}(1-\varepsilon)}^{\infty} \mathbb{P}\left\{ \tilde{t}(X^n | Y^n) > H(P_{X^n|Y^n}) + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) - \Phi(r) \mid Y^n \right\} \, dr; \tag{94}
  \]
- (e) follows from (72) with the trivial \( \sigma \)-algebra \( \sigma(W) = \{\emptyset, \Omega\} \); and
- (f) follows by the definition of \( f_{\tilde{G}} : [0, 1] \to [0, 1/\sqrt{2\pi}] \) stated in (4).

Substituting (89) into (87), we obtain
\[
E[\tilde{t}(X^n | Y^n) | Y^n] = (1-\varepsilon) H(X^n | Y^n) - \sqrt{V(P_{X^n|Y^n})} \left( f_{\tilde{G}}(1-\varepsilon) - \varepsilon \Phi^{-1}(1-\varepsilon) \right)
\]
\[
+ B_{Y^n} - D_{Y^n} - \varepsilon \left( \eta_{Y^n} - H(P_{X^n|Y^n}) \right)
\]
\[
\stackrel{(a)}{=} (1-\varepsilon) H(P_{X^n|Y^n}) - \sqrt{V(P_{X^n|Y^n})} \left( f_{\tilde{G}}(1-\varepsilon) - \varepsilon \Phi^{-1}(1-\varepsilon) \right)
\]
\[
+ B_{Y^n} - D_{Y^n} - \varepsilon \left( b_{Y^n} + \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1-\varepsilon) \right)
\]
\[
= (1-\varepsilon) H(P_{X^n|Y^n}) - \sqrt{V(P_{X^n|Y^n})} f_{\tilde{G}}(1-\varepsilon) + B_{Y^n} - D_{Y^n} - \varepsilon b_{Y^n} \tag{95}
\]
where (a) follows by the definition of \( b_{Y^n} \) stated in (90). Taking expectations in both sides of (95), we have
\[
E[\tilde{t}(X^n | Y^n)] = n (1-\varepsilon) H(X | Y) - E \left[ \sqrt{V(P_{X^n|Y^n})} f_{\tilde{G}}(1-\varepsilon) + E[B_{Y^n}] + E[D_{Y^n}] - \varepsilon E[b_{Y^n}] \right]. \tag{96}
\]
Finally, we shall prove that the last three terms in (96) can be scaled as +O(1) as $n \to \infty$. By Hypotheses (a) and (b) in Theorem 1, there exist two positive constants $V_{\text{inf}}$ and $T_{\text{sup}}$ satisfying

$$
V(P_{X|Y}) \geq V_{\text{inf}} \quad (\text{a.s.}),
$$

$$
T(P_{X|Y}) \leq T_{\text{sup}} \quad (\text{a.s.}),
$$

respectively. Using those constants, we state the following lemma.

**Lemma 8.** Suppose that Hypotheses (a) and (b) in Theorem 1 hold. Given $0 < \varepsilon < 1$, it holds that

$$
\left| \mathbb{E}[b_{Y^{n}}] + \mathbb{E}[D_{Y^{n}}] - \varepsilon \mathbb{E}[b_{Y^{n}}] \right| \leq \frac{A(1 + \varepsilon)T_{\text{sup}}^{4/3}}{cV_{\text{inf}}^{3/2}} + \frac{3AT_{\text{sup}}}{V_{\text{inf}}}
$$

for every $n \geq n_0$, where $A > 0$ is an absolute constant, $c = c(\varepsilon) > 0$ is a constant depending only on $\varepsilon$, and $n_0 = n_0(\varepsilon, V_{\text{inf}}, T_{\text{sup}}) \geq 1$ is a constant depending on $\varepsilon$, $V_{\text{inf}}$, and $T_{\text{max}}$.

**Proof of Lemma 8:** See Appendix H.

The proof of Lemma 2 is completed by applying Lemma 8 to (96).

**APPENDIX D**

**PROOF OF Lemma 3**

Since Lemma 3 is obviously satisfied if $V_c(X \mid Y) = 0$, it suffices to consider the case where $V_c(X \mid Y) > 0$. Define two events

$$
\mathcal{A}_n(\delta) \coloneqq \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} V(P_{X_i|Y_i}) - V_c(X \mid Y) \right| \leq \delta \right\},
$$

$$
\mathcal{B}_n(\delta) \coloneqq \left\{ \left| \sqrt{n} \sum_{i=1}^{n} V(P_{X_i|Y_i}) - \sqrt{V_c(X \mid Y)} \right| \leq \delta \right\}
$$

for each $\delta > 0$ and each $n \geq 1$. Noting that

$$
b > 0 \quad \text{and} \quad |a - b| \leq \delta \quad \Rightarrow \quad |\sqrt{a} - \sqrt{b}| \leq \frac{\delta}{\sqrt{b}}
$$

for every $a \geq 0$ and $\delta > 0$, we observe that

$$
\mathcal{A}_n(\delta) \subset \mathcal{B}_n(\delta/\sqrt{V_c(X \mid Y)}).
$$

Hence, we have

$$
\mathbb{E}\left[ \sqrt{V(P_{X^n|Y^n})} \right] \geq \mathbb{E}\left[ \sqrt{V(P_{X^n|Y^n})} \mathbf{1}_{\mathcal{A}_n(\delta)} \right]
\geq \mathbb{E}\left[ \left( \sqrt{n} V_c(X \mid Y) - \frac{\sqrt{n} \delta}{\sqrt{V_c(X \mid Y)}} \right) \mathbf{1}_{\mathcal{A}_n(\delta)} \right]
\geq \mathbb{E}\left[ \left( \sqrt{n} V_c(X \mid Y) - \frac{\sqrt{n} \delta}{\sqrt{V_c(X \mid Y)}} \right) \mathbf{1}_{\mathcal{A}_n(\delta)} \right]
\geq \mathbb{E}\left[ \left( \sqrt{n} V_c(X \mid Y) - \frac{\sqrt{n} \delta}{\sqrt{V_c(X \mid Y)}} \right) \right]
\geq \mathbb{E}\left[ \frac{\mathbb{E}[V(P_{X|Y})^2] - V_c(X \mid Y)^2}{n \delta^2} \right],
\tag{104}
$$

where note that we have assumed that $\mathbb{E}[V(P_{X|Y})^2] < \infty$, which implies that $V_c(X \mid Y)^2 < \infty$ as well; and

- (a) follows from (103) and the definition of $\mathcal{B}_n(\delta)$; and
- (b) follows from Chebyshev’s inequality.

By taking $\delta = \delta_n \coloneqq n^{-1/3}$, we obtain (44) from (104) and the right-hand side of (43), completing the proof.
APPENDIX E
PROOF OF LEMMA 4

Similar to Appendix B, it suffices to consider the case in which \( n = 1 \), i.e., it suffices to prove that

\[
0 \leq \mathbb{E}_0^{\varepsilon}(X \mid Y) - L^*_{\text{avg}}(\varepsilon, X, Y) \leq \log(H(X \mid Y) + 1) + \log e,
\]

(105)

where \( L^*_{\text{avg}}(\varepsilon, X, Y) := L^*_{\text{avg}}(1, \varepsilon, X, Y) \). The following lemma gives us a formula of the fundamental limit \( L^*_{\text{avg}}(\varepsilon, X, Y) \).

**Lemma 9.** Given \( 0 \leq \varepsilon \leq 1 \), it holds that

\[
L^*_{\text{avg}}(\varepsilon, X, Y) = \mathbb{E}[\langle \log \xi_Y^{-1}(X) \rangle]_{\varepsilon},
\]

(106)

where \( \xi_Y \) is defined in (77).

**Proof of Lemma 9:** See Appendix J.

We are now ready to prove Lemma 4.

**Proof of Lemma 4:** Employing (24) of Proposition 1, (80), (84), and Lemma 9, we can prove Lemma 4 by the same manner as the proof of Lemma 1, see Appendix B for details. This completes the proof of Lemma 4.

\[
\text{APPENDIX F}
\]

**PROOF OF LEMA 6**

A. **Proof of (64)**

By Lemma 7 in Appendix B, it suffices to show that

\[
-|\log c_\varepsilon| \leq N^*_n(\varepsilon, X, Y) - \mathbb{E}[\langle \log \xi_Y^{-1}(X) \rangle | Y]_{\varepsilon} \leq 1 + |\log c_\varepsilon|,
\]

(107)

where \( N^*_n(\varepsilon, X, Y) := N^*_n(1, \varepsilon, X, Y) \). Throughout Appendix F, we consider one-shot (\( n = 1 \)) guessing strategies as defined in Section IV (by taking \( n \) therein to be 1). Specifically, we now consider an \((N, \varepsilon)_\text{max}-guessing strategy} \((g, \pi(\cdot | \cdot))\) satisfying

\[
\mathbb{E}[\log G(X, Y)] \leq N,
\]

(108)

\[
\mathbb{P}\{G(X, Y) \neq g(X, Y) | Y\} \leq \varepsilon \quad \text{(a.s.)}.
\]

(109)

It is clear that

\[
|\mathbb{E}[\langle \log G(X, Y) \rangle 1_{\{G(X, Y) = g(X, Y)\}}]| \leq |\log c_\varepsilon|.
\]

(110)

It follows from (109) that

\[
1 - \varepsilon \leq 1 - \mathbb{P}\{G(X, Y) \neq g(X, Y) | Y\} = \mathbb{P}\{G(X, Y) = g(X, Y) | Y\} = \sum_{k=1}^{\infty} \mathbb{P}\{G(X, Y) = k | Y\} \quad \text{(a.s.),}
\]

(111)

where the last equality follows from the fact that \( G(X, Y) = k \) only if \( g(X, Y) = k \). Based on (111), define the \( \sigma(Y) \)-measurable real-valued r.v.’s \( \nu_Y \) and \( \gamma_Y \) as follows:

\[
\nu_Y := \sup \left\{ k \geq 1 \mid \sum_{x=1}^{k} \mathbb{P}\{G(X, Y) = k | Y\} \leq 1 - \varepsilon \right\},
\]

(112)

\[
\gamma_Y := 1 - \varepsilon - \sum_{x=1}^{\nu_Y} \mathbb{P}\{G(X, Y) = k | Y\},
\]

(113)

respectively. In addition, define the \( \sigma(Y) \)-measurable r.v.’s \( \kappa_Y \) and \( \gamma_Y \) so that

\[
\kappa_Y := \sup \left\{ k \geq 0 \mid \sum_{x=1}^{k} P_{X|Y}(\xi_Y(x) | Y) \leq 1 - \varepsilon \right\},
\]

(114)

\[
\gamma_Y := 1 - \varepsilon - \sum_{x=1}^{\kappa_Y} P_{X|Y}(\xi_Y(x) | Y),
\]

(115)
respectively, where $\zeta_Y$ is given in (77). Furthermore, define

$$
p_1(k \mid Y) := \begin{cases} 
P(G(X, Y) = k \mid Y) & \text{if } 1 \leq k \leq \nu_Y, \\
\nu_Y & \text{if } k = \nu_Y + 1, \\
0 & \text{if } \nu_Y + 2 \leq k < \infty,
\end{cases}
$$

(116)

$$
p_2(x \mid Y) := \begin{cases} 
P_{X \mid Y}(\zeta_Y(x) \mid Y) & \text{if } 1 \leq x \leq \nu_Y, \\
\gamma_Y & \text{if } x = \nu_Y + 1, \\
0 & \text{if } \nu_Y + 2 \leq x < \infty,
\end{cases}
$$

(117)

Then, a direct calculation shows

$$
\mathbb{E}[(\log G(X, Y))_e \mid G(X, Y) = g(X, Y)] \geq \sum_{k=1}^{\infty} \log k \ p_1(k \mid Y) = \sum_{j=0}^{2^{j+1}-1} \sum_{k=2^j}^{2^{j+1}-1} \sum_{x=1}^{\infty} p_1(k \mid Y) \left( \sum_{x=1}^{\infty} p_2(x \mid Y) \right) \text{ (a.s.),}
$$

(118)

$$
= \mathbb{E}[(\log \zeta_Y^{-1}(X)) \mid Y] \mathbb{E}[(\log G(X, Y))_e] \geq \sum_{x=1}^{\infty} \log x \ p_2(x \mid Y) = \sum_{x=1}^{\infty} \sum_{j=1}^{2^{j+1}-1} \sum_{k=2^j}^{2^{j+1}-1} \sum_{x=1}^{\infty} p_2(x \mid Y) \text{ (a.s.),}
$$

(119)

respectively, where the second equalities in (118) and (119) follow from the fact that $\log k = j$ if and only if $2^j \leq k < 2^{j+1}$ for every $k \geq 1$. On the other hand, since $\zeta_Y$ rearranges the probability masses in $P_{X \mid Y}(\cdot \mid Y)$ in non-increasing order (see (77)), it can be verified that $p_1(\cdot \mid Y)$ is majorized by $p_2(\cdot \mid Y)$ a.s., i.e., it follows that

$$
\sum_{k=1}^{l} p_1(k \mid Y) \leq \sum_{x=1}^{l} p_2(x \mid Y) \text{ (a.s.)}
$$

(120)

for every $l \geq 1$, and

$$
\sum_{k=1}^{\infty} p_1(k \mid Y) \geq \sum_{x=1}^{\infty} p_2(x \mid Y) = 1 - \epsilon \text{ (a.s.).}
$$

(121)

Combining (108)–(110) and (118)–(121), the existence of an $(N, \epsilon)_{\text{max}}$-guessing strategy implies that

$$
N + |\log \epsilon| \geq \mathbb{E}[(\log \zeta_Y^{-1}(X)) \mid Y]_e
$$

(122)

which corresponds to the left-hand inequality of (107).

Finally, considering the guessing strategy $(g^*, \pi_{\text{max}}^*(\cdot \mid \cdot))$ given by

$$
g^*(x, Y) = \zeta_Y^{-1}(x) \text{ (a.s.),}
$$

(123)

$$
\pi_{\text{max}}^*(x \mid Y) = \begin{cases} 
0 & \text{if } 1 \leq x \leq \nu_Y, \\
1 - \frac{\gamma_Y}{P_{X \mid Y}(\zeta_Y(x) \mid Y)} & \text{if } x = \nu_Y + 1, \\
1 & \text{if } \nu_Y + 2 \leq x < \infty,
\end{cases}
$$

(124)

and denoting by $G_{\text{max}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ the giving-up guessing function induced by the strategy $(g^*, \pi_{\text{max}}^*(\cdot \mid \cdot))$, we obtain after some algebra that

$$
P\{G_{\text{max}}^*(X, Y) \neq g^*(X, Y) \mid Y\} = \epsilon \text{ (a.s.)}
$$

(125)

and

$$
\mathbb{E}[(\log G_{\text{max}}^*(X, Y))_e] = \mathbb{E}[(\log G_{\text{max}}^*(X, Y))_e] + 1 = \mathbb{E}[(\log \zeta_Y^{-1}(X)) \mid Y]_e + 1,
$$

(126)

which, together with (110) and (118)–(121), imply the right-hand inequality of (107). This completes the proof of (64).

\textbf{B. Proof of (65)}

By Lemma 9 in Appendix E, it suffices to show that

$$
-|\log \epsilon| \leq N_{\text{avg}}^*(\epsilon, X, Y) - \mathbb{E}[(\log \zeta_Y^{-1}(X)) \mid Y]_e \leq 1 - |\log \epsilon|,
$$

(127)

where $N_{\text{avg}}^*(\epsilon, X, Y) := N_{\text{avg}}^*(1, \epsilon, X, Y)$. Consider an $(N, \epsilon)_{\text{avg}}$-guessing strategy $(g, \pi(\cdot \mid \cdot))$ satisfying

$$
\mathbb{E}[\log G(X, Y)] \leq N,
$$

(128)

$$
P\{G(X, Y) \neq g(X, Y)\} \leq \epsilon,
$$

(129)
where \( G : X \times Y \to X \) is the giving-up guessing function induced by the strategy \((g, \pi(\cdot \mid \cdot))\). Similar to (111), one has

\[
1 - \varepsilon \leq \sum_{k=1}^{\infty} \mathbb{P}\{G(X, Y) = k\}. \tag{130}
\]

Based on (130), define two parameters \( \nu \) and \( \nu \) by

\[
\nu := \sup \left\{ k \geq 1 \left| \sum_{x=1}^{\nu} \mathbb{P}\{G(X, Y) = k\} \leq 1 - \varepsilon \right. \right\}, \tag{131}
\]

\[
\nu := 1 - \varepsilon - \sum_{x=1}^{\nu} \mathbb{P}\{G(X, Y) = k\}, \tag{132}
\]

respectively. In addition, define two parameters \( \kappa \) and \( \gamma \) so that

\[
\kappa := \sup \left\{ k \geq 0 \left| \sum_{x=1}^{\kappa} \mathbb{P}\{X = \varsigma_Y(x)\} \leq 1 - \varepsilon \right. \right\}, \tag{133}
\]

\[
\gamma := 1 - \varepsilon - \sum_{x=1}^{\nu} \mathbb{P}\{X = \varsigma_Y(x)\}. \tag{134}
\]

respectively, where \( \varsigma_Y \) is given in (77). Furthermore, define

\[
q_1(k) := \begin{cases} \mathbb{P}\{G(X, Y) = k\} & \text{if } 1 \leq k \leq \nu, \\ \nu & \text{if } k = \nu + 1, \\ 0 & \text{if } \nu + 2 \leq k < \infty, \end{cases} \tag{135}
\]

\[
q_2(x) := \begin{cases} \mathbb{P}\{X = \varsigma_Y(x)\} & \text{if } 1 \leq x \leq \kappa, \\ \gamma & \text{if } x = \kappa + 1, \\ 0 & \text{if } \kappa + 2 \leq x < \infty. \end{cases} \tag{136}
\]

Similar to (118) and (119), we observe that

\[
\mathbb{E}[\log G(X, Y) \mathbb{1}_{\{G(X, Y) = g_0(X, Y)\}}] \geq \sum_{j=1}^{\infty} \sum_{k=2^j}^{\infty} q_1(k), \tag{137}
\]

\[
\mathbb{E}[\log \varsigma_Y^{-1}(X)]_{v} = \sum_{j=1}^{\infty} \sum_{x=2^j}^{\infty} q_2(x). \tag{138}
\]

Since \( \varsigma_Y \) defined in (77) ensures that

\[
\mathbb{P}\{X = \varsigma_Y(1)\} \geq \mathbb{P}\{X = \varsigma_Y(2)\} \geq \mathbb{P}\{X = \varsigma_Y(3)\} \geq \cdots, \tag{139}
\]

it can be verified that \( q_1(\cdot) \) is majorized by \( q_2(\cdot) \), i.e., it follows that

\[
\sum_{k=1}^{l} q_1(k) \leq \sum_{x=1}^{l} q_2(x) \quad \text{(a.s.)} \tag{140}
\]

for every \( l \geq 1 \), and

\[
\sum_{k=1}^{\infty} q_1(k) = \sum_{x=1}^{\infty} q_2(x) = 1 - \varepsilon \quad \text{(a.s.)}. \tag{141}
\]

Combining (110), (128), (129), and (137)–(141), the existence of an \( (N, \varepsilon)_{\text{avg}} \)-guessing strategy implies that

\[
N + \left| \log c_x \right| \geq \mathbb{E}[\log \varsigma_Y^{-1}(X)]_{v}, \tag{142}
\]

which corresponds to the left-hand inequality of (127).

Finally, considering the guessing strategy \((g^*, \pi(\cdot \mid \cdot))\) so that \( g^* \) is given as (123) and

\[
\pi^{\text{avg}}_\kappa(x | Y) = \begin{cases} 0 & \text{if } 1 \leq x \leq \kappa, \\ \frac{\gamma}{\mathbb{P}\{X = \varsigma_Y(x)\}} & \text{if } x = \kappa + 1, \\ 1 & \text{if } \kappa + 2 \leq x < \infty, \end{cases} \tag{143}
\]
and denoting by $\mathbf{G}^*_{\text{avg}} : X \times \mathcal{Y} \to X$ the giving-up guessing function induced by the strategy $(g^*, \pi^*_\text{avg}(\cdot | \cdot))$, we obtain after some algebra that
\begin{equation}
\mathbb{P}(\mathbf{G}^*_{\text{avg}}(X, Y) \neq g^*(X, Y)) = \varepsilon \tag{144}
\end{equation}
and
\begin{equation}
\mathbb{E}[(\log \mathbf{G}^*_{\text{avg}}(X, Y)) \mathbf{1}_{\{|(g^*(X, Y))\}}] \leq \mathbb{E}[(\log \mathbf{G}^*_{\text{avg}}(X, Y)) \mathbf{1}_{\{|g^*(X, Y)\}}] + 1
\end{equation}
\begin{equation}
= \mathbb{E}[(\log \mathbf{c}^{-1}_Y(X))_{\varepsilon}] + 1, \tag{145}
\end{equation}
which, together with (110) and (137)–(141), imply the right-hand inequality of (127). This completes the proof of (65).

**Appendix G**

**Proof of Lemma 7**

Throughout Appendix G, we consider one-shot $(n = 1)$ variable-length stochastic codes $(F, G)$ as defined in Section II-C. We say that a decoder $G$ is deterministic if $G(F(X, Y), X)$ is $\sigma(F(X, Y), X)$-measurable. To specify the determinism, we use the lower case $g$ to denote a deterministic decoder.

Consider an $(L, \varepsilon)_{\text{max}}$-code $(F, G)$ satisfying
\begin{equation}
\mathbb{E}[\ell(F(X, Y))] \leq L, \tag{146}
\end{equation}
\begin{equation}
\mathbb{P}\{X \neq G(F(X, Y), Y) \mid Y\} \leq \varepsilon \quad \text{(a.s.)}. \tag{147}
\end{equation}
It can be verified that there exists a deterministic decoder $g_0$ satisfying\(^8\)
\begin{equation}
\mathbb{P}\{X \neq g_0(F(X, Y), Y) \mid Y\} \leq \mathbb{P}\{X \neq G(F(X, Y), Y) \mid Y\} \quad \text{(a.s.)}. \tag{148}
\end{equation}
In addition, for each $(x, y) \in X \times \mathcal{Y}$, construct another stochastic encoder $F_0$ as
\begin{equation}
F_0(x, y) := \begin{cases}
\emptyset & \text{if } x \neq g_0(F(x, y), y), \\
F(x, y) & \text{otherwise}.
\end{cases} \tag{149}
\end{equation}
As shown later, the new code $(F_0, g_0)$ has a better performance than that of the initial code $(F, G)$. It is clear that
\begin{equation}
F_0(X, Y) \neq \emptyset \quad \implies \quad X = g_0(F_0(X, Y), Y). \tag{150}
\end{equation}

Now, generate a random collection $\mathcal{B}(Y)$ of subsets of $\{0, 1\}^*$ as
\begin{equation}
\mathcal{B}(Y) := \{\mathcal{B}(x \mid Y) \mid x \in X\} \setminus \{\emptyset\}, \tag{151}
\end{equation}
where the random subset $\mathcal{B}(x \mid Y)$ of $\{0, 1\}^*$ is defined by
\begin{equation}
\mathcal{B}(x \mid Y) := \begin{cases}
\{\emptyset\} & \text{if } x = g_0(\emptyset, Y), \\
\{b \in \{0, 1\}^* \setminus \{\emptyset\} \mid \mathbb{P}\{F_0(x, Y) = b \mid Y\} > 0\} & \text{if } x \neq g_0(\emptyset, Y)
\end{cases} \tag{152}
\end{equation}
for each $x \in X$. We shall prove the disjointness of the sets $\mathcal{B}(x \mid Y), x \in X,$ in $\mathcal{B}(Y)$ as follows: Choose $b^* \in \{0, 1\}^* \setminus \{\emptyset\}$ and $x_1, x_2 \in X$ so that $\mathbb{P}(\mathcal{E}_i \cap \mathcal{E}_j) > 0$, where the two events $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by
\begin{equation}
\mathcal{E}_1 = \{\mathbb{P}(F_0(x_1, Y) = b^* \mid Y) > 0\}, \tag{153}
\end{equation}
\begin{equation}
\mathcal{E}_2 = \{\mathbb{P}(F_0(x_2, Y) = b^* \mid Y) > 0\}, \tag{154}
\end{equation}
respectively. It is clear that $\mathcal{E}_i \in \sigma(Y)$ for each $i = 1, 2$. Moreover, it follows from (150) that $g_0(b^*, Y) = x_i$ on the event $\mathcal{E}_i$ for each $i = 1, 2$. Thus, since $g_0(b^*, Y)$ is $\sigma(Y)$-measurable, we observe that $x_1 = x_2$ whenever $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) > 0$. Therefore, the random collection $\mathcal{B}(Y)$ is disjoint a.s., i.e.,
\begin{equation}
\mathbb{P}\{\mathcal{B}(x_1, Y) \cap \mathcal{B}(x_2, Y) = \emptyset\} \text{ for all } x_1 \neq x_2 = 1. \tag{155}
\end{equation}
By the disjointness of (155), one can find an index set $I(Y) = \{1, 2, \ldots, |\mathcal{B}(Y)|\}$ of the collection $\mathcal{B}(Y)$ so that
\begin{equation}
\mathcal{B}(Y) = \{\mathcal{B}_i(Y) \mid i \in I(Y)\} \tag{156}
\end{equation}
and
\begin{equation}
\mathbb{P}\{\text{for all } i < j, \text{ there exists } b \in \mathcal{B}_i(Y) \text{ s.t. } b < b^* \text{ for all } \tilde{b} \in \mathcal{B}_j(Y)\} = 1, \tag{157}
\end{equation}
\(^8\)Note that in general, the determinism of decoders is not a necessary condition to be optimal.
where the binary relation $<$ on $\{0, 1\}^*$ represents the lexicographical order in $\{0, 1\}^*$. Let $\{b_i\}_{i=1}^\infty$ be the lexicographical ordering of the strings in $\{0, 1\}^*$ so that $b_i < b_j$ whenever $i < j$; e.g., $b_1 = \emptyset$, $b_2 = 0$, $b_3 = 1$, $b_4 = 00$, $b_5 = 01$, $b_6 = 10$, $b_7 = 11$, $b_8 = 000$, and so on. It is trivial that
\[
P\{B_1(Y) = \emptyset\} = 1;
\] (158)
consequently, it follows from (155) that
\[
P\{\emptyset \notin B_i(Y) \text{ for all } i \in I(Y) \setminus \{1\}\} = 1.
\] (159)

Now, define the event
\[
A_k := \{\ell(b_i) \leq \ell(b) \text{ for all } 1 \leq i \leq \min\{k, |B(Y)|\} \text{ and } b \in B_i(Y)\}
\] (160)
for each integer $k \geq 1$. It can be verified from (157) and (158) by induction that
\[
P(A_k) = 1
\] (161)
for every $k \geq 1$. Hence, it follows from the monotonicity $A_1 \supset A_2 \supset A_3 \supset \cdots$ that
\[
P\{\ell(b_i) \leq \ell(b) \text{ for all } i \in I(Y) \text{ and } b \in B_i(Y)\} = 1.
\] (162)

Based on the previous paragraph, define the random map $\Phi_Y : \{0, 1\}^* \to \{0, 1\}^*$ so that
\[
\Phi_Y(b) := \begin{cases} \emptyset & \text{if } b \notin B_i(Y) \text{ for all } i \in I(Y), \\ b_i & \text{if } b \in B_i(Y) \text{ for some } i \in I(Y). \end{cases}
\] (163)

Moreover, the disjointness of (155) ensures the existence of a random map $\Psi_Y : X \to X \cup \{0\}$ satisfying
\[
\Psi_Y(i) = \begin{cases} x & \text{if } B_i(Y) = B(x \mid Y) \text{ for some } x \in X, \\ 0 & \text{otherwise}. \end{cases}
\] (164)

Note that $\Phi_Y(n)$ and $\Psi_Y(i)$ are $\sigma(Y)$-measurable for each $b \in \{0, 1\}^*$ and $i \in X$, respectively. Then, construct another variable-length stochastic code $(F_1, g_1)$ so that
\[
F_1(x, Y) := \Phi_Y(F_0(x, Y)),
\] (165)
\[
g_1(b, Y) := \Psi_Y(i) \text{ if } b = b_i \text{ for some } i \geq 1.
\] (166)

Now, we shall evaluate the average codeword length of the encoder $F_1$. A direct calculation shows
\[
\mathbb{E}[\ell(F_1(X, Y))] \leq \begin{aligned}
&\mathbb{E}\left[\sum_{i=2}^{B(Y)} \ell(b_i) 1_{\{F_0(X, Y) \in B_i(Y)\}}\right] \\
&\leq \mathbb{E}\left[\sum_{i=2}^{B(Y)} \ell(F_0(X, Y)) 1_{\{F_0(X, Y) \in B_i(Y)\}}\right] \\
&\leq \mathbb{E}[\ell(F_0(X, Y)) 1_{\{x \neq g_0(\emptyset, Y)\}}] \\
&\leq \mathbb{E}[\ell(F(X, Y)) \mid \{x = g_0(F(X, Y), Y)\} \cap \{x \neq g_0(\emptyset, Y)\}] \\
&\leq \mathbb{E}[\ell(F(X, Y))] \\
&\leq L,
\end{aligned}
\] (167)
where
\begin{itemize}
\item (a) follows from the definitions of $B(Y)$ and $\phi$ stated in (156) and (163), respectively, and the fact that $\ell(\emptyset) = 0$;
\item (b) follows from (162);
\item (c) follows from the disjointness of $B(Y)$ stated in (155) and the fact that
\[
P\{F_0(X, Y) \in B_i(Y) \text{ for some } 1 \leq i \leq |B(Y)|\} = 1;
\] (168)
that is,
\[
\sum_{i=2}^{B(Y)} 1_{\{F_0(X, Y) \in B_i(Y)\}} = 1_{\{F_0(X, Y) \in B_i(Y) \text{ for some } 2 \leq i \leq |B(Y)|\}} = 1_{\{F_0(X, Y) \neq \emptyset\}} = 1_{\{X \neq g_0(\emptyset, Y)\} \cap \{X = g_0(F_0(X, Y), Y)\}} \leq 1_{\{X \neq g_0(\emptyset, Y)\}}
\] (169)
\end{itemize}
a.s.; and
- (d) follows by the definition of $F_0$ stated in (149); and
- (e) follows from (146).

Namely, the average codeword length of the encoder $F_1$ is shorter than or equal to that of the initial encoder $F$.

Next, we shall evaluate the error probability of the code $(F_1, g_1)$. We observe that

$$
\mathbb{P}\{X \neq g_1(F_1(X, Y), Y) \mid Y\} = \sum_{i=1}^{\lfloor |B(Y)| \rfloor} \mathbb{P}\{F_1(X, Y) = b_i \text{ and } X \neq g_1(b_i, Y) \mid Y\}
$$

$$
= (a) \sum_{i=1}^{\lfloor |B(Y)| \rfloor} \mathbb{P}\{\Psi_Y(F_0(X, Y)) = b_i \text{ and } X \neq \Psi_Y(i) \mid Y\}
$$

$$
= (b) \sum_{i=1}^{\lfloor |B(Y)| \rfloor} \mathbb{P}\{F_0(X, Y) \in B_i(Y) \text{ and } X \neq \Psi_Y(i) \mid Y\}
$$

$$
= (c) \sum_{i=1}^{\lfloor |B(Y)| \rfloor} \mathbb{P}\{F_0(X, Y) \in B_i(Y) \text{ and } B(X, Y) \neq B_i(Y) \mid Y\}
$$

$$
= (d) \mathbb{P}\{F_0(X, Y) \in B_1(Y) \text{ and } B(X, Y) \neq B_1(Y) \mid Y\}
$$

$$
= (e) \mathbb{P}\{F_0(X, Y) = \emptyset \text{ and } X \neq g_0(\emptyset, Y) \mid Y\}
$$

$$
= (f) \mathbb{P}\{X \neq g_0(F_0(X, Y), Y) \text{ and } X \neq g_0(\emptyset, Y) \mid Y\}
$$

$$
= (g) \mathbb{P}\{X \neq g_0(F(X, Y), Y) \text{ and } X \neq g_0(\emptyset, Y) \mid Y\}
$$

$$
\leq \mathbb{P}\{X \neq g_0(F(X, Y), Y) \mid Y\}
$$

$$
\leq \varepsilon \quad \text{(a.s.)},
$$

where

- (a) follows by the definition of $(F_1, g_1)$ stated in (165) and (166);
- (b) follows by the definition of the random map $\Psi_Y(\cdot)$ stated in (163);
- (c) follows by the definition of the random map $\Psi_Y(\cdot)$ stated in (164);
- (d) follows from the fact that $F_0(X, Y) \in B_i(Y)$ only if $B(X, Y) = B_i(Y)$ for each $2 \leq i \leq |B(Y)|$ a.s.;
- (e) follows from (158);
- (f) follows from the fact that $F_0(X, Y) = \emptyset$ if and only if $X = g(\emptyset, Y)$ or $X \neq g_0(F_0(X, Y), Y)$;
- (g) follows by the definition of $F_0$ stated in (149); and
- (h) follows from (147) and (148).

In other words, the maximum probability of error for the code $(F_1, g_1)$ is smaller than or equal to that for the initial code $(F, G)$.

Here, we shall prove the converse result of Lemma 7, i.e., we shall show that the average codeword length of $(F, G)$ satisfying (147) is always bounded from below by the right-hand side of (76). We see that

$$
1 - \varepsilon \leq \mathbb{P}\{X = g_1(F_1(X, Y), Y) \mid Y\}
$$

$$
= \mathbb{P}\{X = g_1(\emptyset, Y) \text{ and } F_1(X, Y) = \emptyset \mid Y\} + \mathbb{P}\{X = g_1(F_1(X, Y), Y) \text{ and } F_1(X, Y) \neq \emptyset \mid Y\}
$$

$$
= (a) \mathbb{P}\{X = \Psi_Y(1) \mid Y\} + \mathbb{P}\{X = g_1(F_1(X, Y), Y) \text{ and } F_1(X, Y) \neq \emptyset \mid Y\}
$$

$$
= \mathbb{P}\{X = \Psi_Y(1) \mid Y\} + \mathbb{P}\{X = g_1(b_i, Y) \text{ and } F_1(X, Y) = b_i \text{ for some } i \geq 2 \mid Y\}
$$

$$
= (a) \mathbb{P}\{X = \Psi_Y(1) \mid Y\} + \mathbb{P}\{X = \Psi_Y(i) \text{ for some } 2 \leq i \leq |B(Y)| \mid Y\}
$$

$$
= (d) \sum_{i=1}^{\lfloor |B(Y)| \rfloor} P_X(\Psi_Y(i) \mid Y) \quad \text{(a.s.)},
$$

where

- (a) follows from (170);
- (b) and (c) follows from the definition of $(F_1, g_1)$ stated in (165) and (166);
- (d) follows from the fact that $\Psi_Y(i) \neq \Psi_Y(j)$ if $1 \leq i < j \leq |B(Y)|$ a.s.
Now, Define the $\sigma(Y)$-measurable r.v.’s $\xi_Y$ and $\zeta_Y$ so that
\begin{equation}
\xi_Y := \sup \left\{ k \geq 0 \mid \sum_{i=1}^{k} P_{X|Y}(\Psi_Y(i) \mid Y) \leq 1 - \varepsilon \right\},
\end{equation}
\begin{equation}
\zeta_Y := 1 - \varepsilon - \sum_{i=1}^{\xi_Y} P_{X|Y}(\Psi_Y(i) \mid Y),
\end{equation}
respectively. Moreover, define
\begin{equation}
p_3(i \mid Y) := \begin{cases} P_{X|Y}(\Psi_Y(i) \mid Y) & \text{if } 1 \leq i \leq \xi_Y, \\ \zeta_Y & \text{if } i = \xi_Y + 1, \\ 0 & \text{if } \xi_Y + 2 \leq i < \infty. \end{cases}
\end{equation}

Similar to (118) in Appendix F-A, we get
\begin{equation}
\mathbb{E}[\ell(F_1(X,Y)) \mid Y] \geq \sum_{j=1}^\infty \sum_{i=2j}^\infty p_3(i \mid Y) \quad \text{(a.s.),}
\end{equation}
Moreover, in the same way as (120) and (121) in Appendix F-A, it can be verified that $p_3(\cdot \mid Y)$ is majorized by $p_2(\cdot \mid Y)$ a.s. Therefore, it follows from (119) and (175) that
\begin{equation}
\mathbb{E}[\ell(F_1(X,Y))] \geq \mathbb{E}[\langle \log \varsigma_Y^{-1}(X) \rangle \mid Y].
\end{equation}

Combining (167) and (176), we observe that the existence of an $(L, \varepsilon)$-max-code $(F, G)$ implies that
\begin{equation}
L \geq \mathbb{E}[\langle \log \varsigma_Y^{-1}(X) \rangle \mid Y],
\end{equation}
which corresponds to the converse bound of Lemma 7.

Finally, we shall show the existence of an $(L, \varepsilon)_{\text{max}}$-code meeting the equality in (177). In fact, constructing a variable-length stochastic code $(F_{\sup}, g^*)$ so that
\begin{equation}
F_{\sup}(x,Y) := \begin{cases} b_{\varsigma_Y^{-1}(x)} & \text{if } 1 \leq \varsigma_Y^{-1}(x) \leq \kappa_Y, \\ B_{\sup} & \text{if } \varsigma_Y^{-1}(x) = \kappa_Y + 1, \\ \emptyset & \text{if } \kappa_Y < \varsigma_Y^{-1}(x) < \infty, \end{cases} \quad g^*(b,Y) := x \quad \text{if } b = b_{\varsigma_Y^{-1}(x)} \text{ for some } x \in X,
\end{equation}
where $B_{\sup}$ denotes a $\{0, 1\}^*$-valued r.v. satisfying the conditional independence $B_{\sup} \perp X \mid Y$ and
\begin{equation}
\mathbb{P}(B_{\sup} = \emptyset \mid Y) = 1 - \mathbb{P}(B_{\sup} = b_{\kappa_Y + 1} \mid Y) = 1 - \gamma_Y \quad \text{(a.s.),}
\end{equation}
we readily see that
\begin{equation}
\mathbb{E}[\ell(F_{\sup}(X,Y))] = \mathbb{E}[\langle \log \varsigma_Y^{-1}(X) \rangle \mid Y],
\end{equation}
\begin{equation}
\mathbb{P}(X \neq g^*(F_{\sup}(X,Y), Y) \mid Y) = \varepsilon \quad \text{(a.s.).}
\end{equation}

This completes the proof of Lemma 7.

\section*{Appendix H

\textbf{Proof of Lemma 8}

We shall use the following non-uniform strengthened Berry–Esseen bound.

\textbf{Lemma 10 (non-uniform Berry–Esseen bound [29])}. Let $n \geq 1$ be an integer, and $Z_1, Z_2, \ldots, Z_n$ independent, but not necessarily identically distributed, real-valued r.v.’s. Define the following two quantities:
\begin{equation}
V_n := \sum_{i=1}^n \mathbb{E}[(Z_i - \mathbb{E}[Z_i])^2],
\end{equation}
\begin{equation}
T_n := \sum_{i=1}^n \mathbb{E}|Z_i - \mathbb{E}[Z_i]|^3.
\end{equation}
\footnote{Note that $\xi_Y = 0$ if $P_{X|Y}(\psi(1,Y) \mid Y) \geq 1 - \varepsilon$; and $\xi_Y = \infty$ if $\sum_{i=1}^\infty P_{X|Y}(\psi(i,Y) \mid Y) = 1 - \varepsilon$ and $P_{X|Y}(\psi(i,Y) \mid Y) > 0$ for all $i \geq 1.$}
Then, it holds that

\[ \Pr \left( \sum_{i=1}^{n} (Z_i - \mathbb{E}[Z_i]) \leq z \sqrt{V_n} \right) - \Phi(z) \leq \frac{A T_n}{(1 + |z|^3) V_n^{3/2}} \]  (185)

for every \( z \in \mathbb{R} \), provided that \( V_n > 0 \) and \( T_n < \infty \), where \( A > 0 \) is an absolute constant.

Note that Lemma 10 can be readily reduced to the uniform Berry–Esseen bound:

\[ \sup_{z \in \mathbb{R}} \left| \Pr \left( \sum_{i=1}^{n} (Z_i - \mathbb{E}[Z_i]) \leq z \sqrt{V_n} \right) - \Phi(z) \right| \leq \frac{A T_n}{V_n^{3/2}}. \]  (186)

Since \( (X^n \mid Y^n) \) is a real-valued r.v., we see that \( \Pr \{ \epsilon(X^n \mid Y^n) \leq r \mid Y^n \} \) forms a cumulative distribution function of \( r \in \mathbb{R} \) a.s. (see, e.g., [30, Theorem 10.2.2]). Thus, noting that \( \eta_Y = \text{given in (88)} \) is \( \sigma(Y^n) \)-measurable, it follows from the Berry–Esseen bound stated in (186) with an absolute constant \( A > 0 \) that

\[ \Pr \{ \epsilon(X^n \mid Y^n) \leq \eta_Y \mid Y^n \} \leq \Phi \left( \frac{\eta_Y - H(P_{X^n|Y^n})}{\sqrt{V(P_{X^n|Y^n})}} + \frac{A T(P_{X^n|Y^n})}{V(P_{X^n|Y^n})^{3/2}} \right) \]  (a.s.),

\[ \Pr \{ \epsilon(X^n \mid Y^n) < \eta_Y \mid Y^n \} \geq \Phi \left( \frac{\eta_Y - H(P_{X^n|Y^n})}{\sqrt{V(P_{X^n|Y^n})}} - \frac{A T(P_{X^n|Y^n})}{V(P_{X^n|Y^n})^{3/2}} \right) \]  (a.s.).

It is clear that there exists an \( n_0 = n_0(\epsilon, V_{\inf}, T_{\sup}) \geq 1 \) satisfying

\[ \frac{A T_{\sup}}{\sqrt{n} V_{\inf}^{3/2}} < \frac{\min\{1 - \epsilon, \epsilon\} + A T_{\sup}}{2} \]  (189)

for every \( n \geq n_0 \), where note that we have assumed that \( \epsilon > 0 \). Since

\[ \frac{A T(P_{X^n|Y^n})}{V(P_{X^n|Y^n})^{3/2}} = \frac{\sum_{i=1}^{n} T(P_{X_i|Y_i})}{(\sum_{i=1}^{n} V(P_{X_i|Y_i}))^{3/2}} \leq \frac{A T_{\sup}}{\sqrt{n} V_{\inf}^{3/2}} \]  (a.s.),

substituting (187) and (188) into (88), we get

\[ \Phi^{-1} \left( 1 - \frac{A T_{\sup}}{\sqrt{n} V_{\inf}^{3/2}} \right) \leq \Phi^{-1} \left( 1 - \frac{\eta_Y - H(P_{X^n|Y^n})}{\sqrt{V(P_{X^n|Y^n})}} \right) \leq \Phi^{-1} \left( 1 - \frac{A T_{\sup}}{\sqrt{n} V_{\inf}^{3/2}} \frac{\eta_Y - H(P_{X^n|Y^n})}{\sqrt{V(P_{X^n|Y^n})}} \right) \]  (a.s.)

for every \( n \geq n_0 \). In addition, it follows by Taylor’s theorem (and the inverse function theorem) that

\[ \Phi^{-1}(t + u) - \Phi^{-1}(t) = \frac{u}{f(s)} \]  (192)

for every \( 0 < t < 1 \), every \( u \in (-t, 1 - t) \), and some \( s \in [t, u + t] \). Applying (192) to (191), we have

\[ |b_Y| = \left| \eta_Y - H(P_{X^n|Y^n}) - \sqrt{V(P_{X^n|Y^n})} \Phi^{-1}(1 - \epsilon) \right| \leq \frac{A T_{\sup}}{c \sqrt{n} V_{\inf}^{3/2}} \]  (a.s.)

for every \( n \geq n_0 \), where the constant \( c > 0 \) is given as

\[ c = c(\epsilon) := \begin{cases} f_s(\epsilon) & \text{if } 0 < \epsilon \leq \frac{1}{2}, \\ f_s \left( \frac{1 + \epsilon}{2} \right) & \text{if } \frac{1}{2} < \epsilon < 1; \end{cases} \]  (194)

and the last inequality follows from the fact that\(^{10} \)

\[ V(P_{X^n|Y^n}) = \sum_{i=1}^{n} V(P_{X_i|Y_i}) \leq \sum_{i=1}^{n} T(P_{X_i|Y_i}) \leq n T_{\sup}^{3/2} \]  (a.s.).

We readily see that \( F_n \) is bounded away from zero for sufficiently large \( n \).

Now, it follows by the definition of \( B_{Y^n} \) stated in (91) that

\[ \mathbb{E}[|B_{Y^n}|] \leq \mathbb{E} \left[ \max_{\min \{0, b_Y \}} \right] \leq \mathbb{E}[|b_{Y^n}|] \]  (196)

\(^{10}\) The first inequality in (195) can be verified by \( \mathbb{E}[|Z|^p]^{1/p} \leq \mathbb{E}[|Z|^q]^{1/q} \) for \( 1 \leq p < q \).
for every \( n \geq 1 \). Moreover, it follows from (193) that
\[
\mathbb{E}[|b_{Y^n}|] \leq \frac{AT^{4/3}}{c\sqrt[3]{V_{n0}}}
\]  
(197)
for every \( n \geq n_0 \). Furthermore, it follows by the definition of \( D_{Y^n} \) stated in (94) that
\[
\mathbb{E}[D_{Y^n}] = \mathbb{E} \left[ \int_{-\infty}^{\infty} \sqrt{V(P_{X^n|Y^n})} \, dr \right] 
\]
\[
\leq \frac{AT_{sup}}{V_{inf}} \int_{-\infty}^{\infty} \frac{dr}{1 + |r|^3}
\]
\[
\leq \frac{2AT_{sup}}{V_{inf}} \left( \int_{0}^{1} \frac{dr}{1 + r^3} + \int_{1}^{\infty} \frac{dr}{r} \right)
\]
\[
= \frac{3AT_{sup}}{V_{inf}} \left( 1 + \frac{1}{2} \right)
\]
(198)
for every \( n \geq 1 \), where
- (a) follows by the non-uniform Berry–Esseen theorem stated in Lemma 10 with an absolute constant \( A > 0 \); and
- (b) follows from (97) and (98).

Analogously, we may see that
\[
\mathbb{E}[D_{Y^n}] \geq -\frac{3AT_{sup}}{V_{inf}}
\]  
(199)
for every \( n \geq 1 \). Combining (196)–(199), we obtain Lemma 8, as desired.

\section*{Appendix I}
\textbf{Relaxations of Hypotheses in Theorem 1}

\subsection*{A. Proof of Proposition 2}
Suppose that \( P(X \in \mathcal{A}) = 1 \) for some finite \( \mathcal{A} \subset X \). Then, we readily see that
\[
\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) = 1 \quad \text{(a.s.)}.
\]
(200)
Hence, the well-known upper bound on the Shannon entropy shows that
\[
\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)} = \sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)} \leq \log |\mathcal{A}| \quad \text{(a.s.)}.
\]
(201)
Therefore, it can be verified by the dominated convergence theorem for the conditional expectation that
\[
H(P_{X|Y}) \leq \log |\mathcal{A}| \quad \text{(a.s.)}.
\]
(202)
On the other hand, we get
\[
\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)} - H(P_{X|Y})^3 \leq \sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log \frac{1}{P_{X|Y}(x \mid Y)} + H(P_{X|Y})^3.
\]
(203)
Since the mapping \( u \mapsto u \ln^3(1/u) \) on \([0,1]\) is maximized at \( u = e^{-3} \), it follows from (200) and (203) that
\[
\sum_{x \in \mathcal{A}} P_{X|Y}(x \mid Y) \log^3 \frac{1}{P_{X|Y}(x \mid Y)} \leq \frac{27|\mathcal{A}|}{(e \ln 2)^3} \quad \text{(a.s.)},
\]
(204)
where $\ln$ stands for the natural logarithm. Combining (202)–(204), we can obtain from the dominated convergence theorem that

$$T(P_{X|Y}) \leq \frac{27 |\mathcal{A}|}{(6 \ln 2)^3} + \log^3 |\mathcal{A}| \quad (\text{a.s.}),$$

which implies that $T(P_{X|Y})$ is bounded away from infinity a.s. This completes the proof of Proposition 2.

B. Proof of Proposition 3

Assume without loss of generality that $P_\gamma(y) := \mathbb{P}(Y = y)$ is positive for each $y \in \mathcal{Y}$. Hypothesis (a) in Theorem 1 is used only to ensure Lemma 2; thus, it suffices to prove Lemma 2 without Hypothesis (a) in Theorem 1.

Let $\eta(y) \geq 0$ be a real number satisfying

$$\mathbb{P}\{u(X | Y) > \eta(y) | Y = y\} + \beta(y) \mathbb{P}\{u(X | Y) = \eta(y) | Y = y\} = \varepsilon$$

for some $0 \leq \beta(y) < 1$. Define

$$\eta_{\max} = \max_{y \in \mathcal{Y}} \eta(y).$$

In addition, define

$$H(P_{X|Y=y}) := \sum_{x \in \mathcal{X}} P_{X|Y=y}(x | y) \log \frac{1}{P_{X|Y=x}(x | y)},$$

$$V(P_{X|Y=y}) := \sum_{x \in \mathcal{X}} P_{X|Y=y}(x | y) \left( \log \frac{1}{P_{X|Y=x}(x | y)} - H(P_{X|Y=y}) \right),$$

$$T(P_{X|Y=y}) := \sum_{x \in \mathcal{X}} P_{X|Y=y}(x | y) \left( \log \frac{1}{P_{X|Y=x}(x | y)} - H(P_{X|Y=y}) \right)$$

for each $y \in \mathcal{Y}$, where $P_{X|Y=y}(x | y) := \mathbb{P}\{X = x | Y = y\}$ stands for the conditional probability given the event $\{Y = y\}$ for each $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Since we now do not assume Hypothesis (a) in Theorem 1, there may exist a $y \in \mathcal{Y}$ satisfying $V(P_{X|Y=y}) = 0$. If $V(P_{X|Y=y}) = 0$ for every $y \in \mathcal{Y}$, then we readily see that

$$\mathbb{E}(\langle u(X^n | Y^n) | Y^n = y \rangle) = (1 - \varepsilon) \sum_{i=1}^{n} H(P_{X|Y=y_i})$$

for every $n \geq 1$ and $y \in \mathcal{Y}^n$. Therefore, Lemma 2 also holds even if $V(P_{X|Y=y}) = 0$ for every $y \in \mathcal{Y}$.

In the following, we assume that there exists a $y \in \mathcal{Y}$ satisfying $V(P_{X|Y=y}) > 0$. Define

$$H_{\max} := \max_{y \in \mathcal{Y}} H(P_{X|Y=y}),$$

$$V_{\max} := \max_{y \in \mathcal{Y}} V(P_{X|Y=y}),$$

$$V_{\min} := \min_{y \in \mathcal{Y} : V(P_{X|Y=y}) > 0} V(P_{X|Y=y}),$$

$$T_{\max} := \max_{y \in \mathcal{Y}} V(P_{X|Y=y}).$$

Note that the three numbers $H_{\max}$, $V_{\max}$, and $T_{\max}$ are finite since Hypothesis (b) in Theorem 1 is satisfied. To prove Lemma 2 without Hypothesis (a) in Theorem 1, it suffices to prove an analog of Lemma 8 stated in Appendix C without Hypothesis (a) in Theorem 1. More precisely, we shall prove the following lemma.

Lemma 11. Recall that $b_{\gamma^n}$, $B_{\gamma^n}$, and $D_{\gamma^n}$ are defined in (90), (91), and (94), respectively (see Appendix C). Suppose that the following hold: (i) $\mathcal{Y}$ is a finite alphabet; (ii) there exists a $y \in \mathcal{Y}$ satisfying $V(P_{X|Y=y}) > 0$; and (iii) Hypothesis (b) in Theorem 1 holds. Given $0 < \varepsilon < 1$, it holds that

$$\left| \mathbb{E}[B_{\gamma^n}] + \mathbb{E}[D_{\gamma^n}] - \varepsilon \mathbb{E}[b_{\gamma^n}] \right| \leq (1 + \varepsilon) \left( \frac{A T_{\max}^{d/3}}{c V_{\min}^{1/2}} + K (\eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon)) \right) + \frac{3 A T_{\max}}{V_{\min}}$$

for every $n \geq K$, where $A > 0$ is an absolute constant, $c = c(\varepsilon) > 0$ is a constant depending only on $\varepsilon$, and $K = K(\varepsilon, V_{\min}, T_{\max}) > 0$ is a constant depending on $\varepsilon$, $V_{\min}$, and $T_{\max}$.

Proof of Lemma 11: See Appendix K.

Lemma 2 without Hypothesis (a) in Theorem 1 is now ensured by applying Lemma 11 to (96). This completes the proof of Proposition 3.
Consider an \((L, \varepsilon)_{\text{avg}}\)-code \((F, G)\) satisfying
\[
\mathbb{E}[\ell(F(X,Y))] \leq L, \tag{217}
\]
\[
\mathbb{P}(X \neq G(F(X,Y), Y)) \leq \varepsilon \quad \text{(a.s.)}. \tag{218}
\]

As similarly done in (165) and (166) of Appendix G, construct another code \((F_1, g_1)\) from the initial code \((F, G)\) via the random maps \(\Phi_Y\) and \(\Psi_Y\) defined in (163) and (164), respectively. Obviously, the same derivations as (167) and (171) yield
\[
\mathbb{E}[\ell(F_1(X,Y))] \leq L, \tag{219}
\]
\[
1 - \varepsilon \leq \mathbb{E} \left[ \sum_{i=1}^{[\mathcal{B}(Y)]} \mathbb{P}(X = \Psi_Y(i) | Y) \right] = \sum_{i=1}^{\infty} \mathbb{P}(X = \Psi_Y(i)), \tag{220}
\]
respectively, where the last equality follows from the fact that \(\Psi_Y(i) = 0\) whenever \(i > [\mathcal{B}(Y)]\) a.s. (see (164)). Now, defining two parameters \(\alpha\) and \(\beta\) so that\(^{11}\)
\[
\xi := \sup \left\{ k \geq 0 \left| \sum_{i=1}^{k} \mathbb{P}(X = \Psi_Y(i)) < 1 - \varepsilon \right. \right\}, \tag{221}
\]
\[
\zeta := 1 - \varepsilon - \sum_{i=1}^{\xi} \mathbb{P}(X = \Psi_Y(i)), \tag{222}
\]
respectively. Moreover, define
\[
q_3(i) := \begin{cases} 
\mathbb{P}(X = \Psi_Y(i)) & \text{if } 1 \leq i \leq \xi, \\
\zeta & \text{if } i = \xi + 1, \\
0 & \text{if } \xi + 2 \leq i < \infty.
\end{cases} \tag{223}
\]

Similar to (137) in Appendix F-B, one sees that
\[
\mathbb{E}[\ell(F_1(X,Y))] \geq \sum_{j=1}^{\infty} \sum_{i=2^j}^{\infty} q_3(i). \tag{224}
\]

Moreover, in the same way as (140) and (141) in Appendix F-B, it can be verified that \(q_3(\cdot)\) is majorized by \(q_2(\cdot)\). Therefore, it follows from (138) and (224) that
\[
\mathbb{E}[\ell(F_1(X,Y))] \geq \mathbb{E}[(\log \zeta_Y^{-1}(X))]_{\varepsilon}. \tag{225}
\]

Combining (219) and (225), we observe that the existence of an \((L, \varepsilon)_{\text{avg}}\)-code \((F, G)\) implies that
\[
L \geq \mathbb{E}[(\log \zeta_Y^{-1}(X))]_{\varepsilon}, \tag{226}
\]
which corresponds to the converse bound of Lemma 9.

Finally, we shall show the existence of an \((L, \varepsilon)_{\text{avg}}\)-code meeting the equality in (226). In fact, constructing a variable-length stochastic code \((F^{\star}_{\text{avg}}, g^\star)\) so that
\[
F^{\star}_{\text{avg}}(x, Y) := \begin{cases} 
\mathbf{b} \quad &\text{if } 1 \leq \zeta_Y^{-1}(x) \leq \kappa, \\
B_{\text{avg}} &\text{if } \zeta_Y^{-1}(x) = \kappa + 1, \\
\varnothing &\text{if } \kappa < \zeta_Y^{-1}(x) < \infty,
\end{cases} \tag{227}
\]
\[
g^\star(b, Y) := x \quad \text{if } b = \mathbf{b}_{\zeta_Y^{-1}(x)} \text{ for some } x \in \mathcal{X}, \tag{228}
\]
where \(B_{\text{avg}}\) denotes a \(\varnothing, \mathbf{b}_{\kappa+1}\)-valued r.v. satisfying the independence \(B_{\text{avg}} \perp (X, Y)\), and
\[
B_{\text{avg}} = \begin{cases} 
\mathbf{b}_{\kappa+1} &\text{with probability } \gamma, \\
\varnothing &\text{with probability } 1 - \gamma,
\end{cases} \tag{229}
\]
we readily see that
\[
\mathbb{E}[\ell(F^{\star}_{\text{avg}}(X,Y))] = \mathbb{E}[(\log \zeta_Y^{-1}(X))]_{\varepsilon}, \tag{230}
\]
\[
\mathbb{P}(X \neq g^\star(F^{\star}_{\text{avg}}(X,Y), Y)) = \varepsilon. \tag{231}
\]

This completes the proof of Lemma 9.

\(^{11}\)Note that \(\alpha = 0\) if \(\mathbb{P}(X = \psi(1, Y)) \geq 1 - \varepsilon\); and \(\alpha = \infty\) if \(\sum_{i=1}^{\infty} \mathbb{P}(X = \psi(i, Y)) = 1 - \varepsilon\) and \(\mathbb{P}(X = \psi(i, Y)) > 0\) for all \(i \geq 1\).
**APPENDIX K**

**PROOF OF LEMMA 11**

Fix an infinite sequence \( y = (y_1, y_2, \ldots) \in \mathcal{Y}^\mathbb{N} \) arbitrarily. For each \( n \geq 1 \), denote by \( y^{(n)} = (y_1, \ldots, y_n) \) the \( n \)-length prefix of \( y \). For each \( n \geq 1 \), consider two parameters \( \eta_n \geq 0 \) and \( 0 \leq \beta_n < 1 \) given so that

\[
P \{ \eta(X^n | Y^n) > \eta_n | Y^n = y^{(n)} \} + \beta_n \mathbb{P} \{ \eta(X^n | Y^n) = \eta_n | Y^n = y^{(n)} \} = \varepsilon.
\]  

(232)

Let \( \mathcal{K} \subset \mathbb{N} \) be the subset in which for all \( k \in \mathcal{K} \), there exists a finite \( \mathcal{A}_k \subset X \) satisfying \( P_{X|Y} (x | y_k) = 1/|\mathcal{A}_k| \) for each \( x \in \mathcal{A}_k \). Define \( k(n) := \{ 1 \leq k \leq n \mid k \not\in \mathcal{K} \} \) for each \( n \geq 1 \). Moreover, let \( n_1 \geq 1 \) be chosen so that

\[
n_1 := \sup \left\{ n \geq 1 \left| \frac{2 \sum_{i=1}^{n} H(P_{X^n|Y^n=y^{(n)}}) V(P_{X^n|Y^n=y^{(n)}})^{3/2}}{\sqrt{\mathbb{E}[|k(n) + 1| V_{\min}]}} \right. \geq \frac{\min\{1 - \varepsilon, \varepsilon\}}{2} \right\}.
\]  

(233)

Since \( V(P_{X^n|Y^n=y^{(n)}}) = T(P_{X^n|Y^n=y^{(n)}}) = 0 \) for every \( k \in \mathcal{K} \), and since \( V(P_{X^n|Y^n=y^{(n)}})T(P_{X^n|Y^n=y^{(n)}}) > 0 \) for every \( i \in \mathbb{N} \setminus \mathcal{K} \), we observe that

\[
\frac{A T(P_{X^n|Y^n=y^{(n)}})}{V(P_{X^n|Y^n=y^{(n)}})^{3/2}} = \frac{A \sum_{i=1}^{n} H(P_{X^n|Y^n=y^{(n)}}) V(P_{X^n|Y^n=y^{(n)}})^{1/2}}{\sqrt{k(n) V_{\min}}} \leq \frac{A T_{\max}}{\sqrt{k(n) V_{\min}}}.
\]  

(234)

therefore, it can be shown by the same way as (193) that

\[
|\eta_n - H(P_{X^n|Y^n=y^{(n)}}) - \sqrt{V(P_{X^n|Y^n=y^{(n)}})} \Phi^{-1}(1 - \varepsilon) | \leq \frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}}
\]  

(235)

for every \( n \geq n_1 \), provided that \( n_1 < \infty \), where \( A > 0 \) is an absolute constant that appears in Lemma 10, and \( c = c(\varepsilon) > 0 \) a constant is given in (194).

Now, consider the case where \( n_1 = \infty \). Note that \( n_1 = \infty \) if and only if

\[
\lim_{n \to \infty} k(n) \leq \left( \frac{2 A T_{\max}}{V_{\min} \min\{1 - \varepsilon, \varepsilon\}} \right)^2 - 1 =: K(\varepsilon, V_{\min}, T_{\max})
\]  

(236)

It is clear from the definition of \( \mathcal{K} \) that

\[
H(P_{X^n|Y^n=y^{(n)}}) = \sum_{i=1}^{n} H(P_{X^n|Y^n=y^{(n)}}) + \sum_{j=1}^{n} \log |\mathcal{A}_j|
\]  

(237)

for each \( n \geq 1 \). If \( 1 \in \mathcal{K} \), then

\[
P \{ \eta(X_1 | Y_1) > t | Y_1 = y_1 \} = \begin{cases} 0 & \text{if } t \leq \log |\mathcal{A}_1|, \\ 1 & \text{if } t > \log |\mathcal{A}_1|, \end{cases}
\]  

(238)

implying that

\[
\eta_1 = \log |\mathcal{A}_1|.
\]  

(239)

Moreover, for each \( k \in \mathcal{K} \) satisfying \( k \geq 2 \), we see that

\[
P \{ \eta(X^k | Y^k) > \eta_k | Y^k = y^{(n)} \} = P \{ \eta(X^{k-1} | Y^{k-1}) > \eta_{k-1} + \log |\mathcal{A}_k| | Y^k = y^{(n)} \}
\]  

for every \( t \in \mathbb{R} \), implying that

\[
\eta_k = \eta_{k-1} + \log |\mathcal{A}_k|.
\]  

(241)

Therefore, since \( V(P_{X^n|Y^n=y^{(n)}}) = 0 \) for each \( k \in \mathcal{K} \), it follows from (236), (237), (239), and (240) that

\[
|\eta_n - H(P_{X^n|Y^n=y^{(n)}}) - \sqrt{V(P_{X^n|Y^n=y^{(n)}})} \Phi^{-1}(1 - \varepsilon) | \leq K(\varepsilon, V_{\min}, T_{\max}) \left( \eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right)
\]  

(242)

for every \( n \geq 1 \), provided that \( n_1 = \infty \).

Combining (235) and (242), we obtain

\[
|\eta_n - H(P_{X^n|Y^n=y^{(n)}}) - \sqrt{V(P_{X^n|Y^n=y^{(n)}})} \Phi^{-1}(1 - \varepsilon) | \leq \frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}} + K(\varepsilon, V_{\min}, T_{\max}) \left( \eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right)
\]  

(243)

for every \( n \geq K(\varepsilon, V_{\min}, T_{\max}) \). Therefore, since the infinite sequence \( y = (y_1, y_2, \ldots) \) is arbitrary, we have

\[
\mathbb{E}[|b_{\eta_y}|] \leq \frac{A T_{\max}^{4/3}}{c V_{\min}^{3/2}} + K(\varepsilon, V_{\min}, T_{\max}) \left( \eta_{\max} + H_{\max} + V_{\max} \Phi^{-1}(1 - \varepsilon) \right)
\]  

(244)

for every \( n \geq K(\varepsilon, V_{\min}, T_{\max}) \). The proof of Lemma 11 is finally completed by combining (196), (198), (199), and (244).
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