Homotopy groups of ascending unions of infinite-dimensional manifolds

Helge Glöckner

Abstract

Let $M$ be a topological manifold modelled on topological vector spaces, which is the union of an ascending sequence $M_1 \subseteq M_2 \subseteq \cdots$ of such manifolds. We formulate a mild condition ensuring that $\pi_k(M, p) = \lim \pi_k(M_n, p)$ for all $k \in \mathbb{N}_0$ and $p \in M$. This result is useful for Lie theory, because many important examples of infinite-dimensional Lie groups can be expressed as ascending unions of finite- or infinite-dimensional Lie groups (whose homotopy groups may be easier to access). Information on $\pi_0(G) = G/G_0$, $\pi_1(G)$ and $\pi_2(G)$ is needed to understand the Lie group extensions $1 \to A \to \hat{G} \to G \to 1$ of $G$ with abelian kernels. The above conclusion remains valid if $\bigcup_{n \in \mathbb{N}} M_n$ is merely dense in $M$ (under suitable hypotheses). Also, ascending unions can be replaced by (possibly uncountable) directed unions.

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1 Introduction and statement of results

A classical result of Palais sheds light on the homotopy groups of an open subset \( U \) of a locally convex topological vector space \( E \). He considered a dense vector subspace \( E_\infty \) of \( E \), endowed with the “finite topology” (the final topology with respect to the inclusion maps \( F \to E_\infty \) of finite-dimensional vector subspaces \( F \)), and gave \( U_\infty := E_\infty \cap U \) the topology induced by \( E_\infty \). Then the inclusion map \( U_\infty \to U \) is a weak homotopy equivalence, i.e.,

\[
\pi_k(U, p) \cong \pi_k(U_\infty, p) \quad \text{for each } k \in \mathbb{N}_0 \text{ and } p \in U_\infty
\]

(see [50, Theorem 12]; cf. also [54] if \( E \) is a Banach space). Furthermore,

\[
\pi_k(U_\infty, p) = \lim_{\to F \in \mathcal{F}_p} \pi_k(U \cap F, p),
\]

where \( \mathcal{F}_p \) is the set of all finite-dimensional vector subspaces \( F \subseteq E_\infty \) such that \( p \in F \) (see, e.g., [43, Lemma II.9]).

In this article, we prove certain non-linear variants of these facts, in situations where linear spaces have been replaced by topological manifolds (or certain more general topological spaces).

All topological spaces in this article are assumed Hausdorff. Until further notice, let \( M \) be a topological manifold modelled on (not necessarily locally convex) topological vector space \( 1 \); we then simply call \( M \) a manifold \( 2 \). Also, let \( (M_\alpha)_{\alpha \in A} \) be an upward directed family of such manifolds, such that \( M_\infty := \bigcup_{\alpha \in A} M_\alpha \) is dense in \( M \) and all inclusion maps \( M_\alpha \to M \) and \( M_\alpha \to M_\beta \) (for \( \alpha \leq \beta \)) are continuous (but not necessarily embeddings). We describe conditions ensuring that

\[
\pi_k(M, p) = \lim_{\to \alpha \in A_p} \pi_k(M_\alpha, p) \quad (1)
\]

for all \( k \in \mathbb{N}_0 \) and \( p \in M_\infty \), where \( A_p := \{ \alpha \in A : p \in M_\alpha \} \).

If \( M = \bigcup_{\alpha \in A} M_\alpha \) and \( M \) is compactly retractive in the sense that each

1Thus \( M \) is a Hausdorff topological space and for each \( p \in M \), there exists an open neighbourhood \( U \subseteq M \) of \( p \), a topological vector space \( E \) and a homeomorphism \( \phi: U \to V \) (called a “chart”) from \( U \) onto an open subset \( V \subseteq E \).

2Likewise, (possibly infinite-dimensional) Lie groups modelled on locally convex spaces (as in [19], [29] and [17]; cf. also [38]) will simply be called “Lie groups.”
compact set $K \subseteq M$ is a compact subset of some $M_\alpha$, then (1) is quite obvious (Proposition 3.3; compare [25, Remark 3.9] and [42, Lemma I.1] for special cases, as well as many works on homotopy theory or $K$-theory).

Our goal is to get beyond this limited situation. To explain our results, let us assume first that $M = \bigcup_{\alpha \in A} M_\alpha$. In this case, we can prove (1) provided $M$ admits weak direct limit charts, i.e., each point $q \in M$ is contained in the domain $U$ of a so-called weak direct limit chart $\phi: U \to V$.

**Definition 1.1** A weak direct limit chart of $M = \bigcup_{\alpha \in A} M_\alpha$ is a chart $\phi: U \to V$ of $M$ taking $U$ homeomorphically onto an open subset $V$ of a topological vector space $E$, such that there exist $\alpha_0 \in A$, charts $\phi_\alpha: U_\alpha \to V_\alpha \subseteq E_\alpha$ of $M_\alpha$ onto open subsets $V_\alpha \subseteq E_\alpha$ of topological vector spaces $E_\alpha$ for $\alpha \geq \alpha_0$, and continuous linear maps $\lambda_\alpha: E_\alpha \to E$ and $\lambda_{\beta,\alpha}: E_\alpha \to E_\beta$ (if $\beta \geq \alpha \geq \alpha_0$) satisfying the following:

(a) For all $\alpha \geq \alpha_0$, we have $U_\alpha \subseteq U$ and $\phi|_{U_\alpha} = \lambda_\alpha \circ \phi_\alpha$;

(b) For all $\beta \geq \alpha \geq \alpha_0$, we have $U_\alpha \subseteq U_\beta$ and $\phi_\beta|_{U_\alpha} = \lambda_{\beta,\alpha} \circ \phi_\alpha$;

(c) $U = \bigcup_{\alpha \geq \alpha_0} U_\alpha$.

By (a) and (b), each $\lambda_\alpha$ and $\lambda_{\beta,\alpha}$ is injective; after replacing $E_\alpha$ with $\text{im}(\lambda_\alpha)$ (equipped with the topology making $\lambda_\alpha$ a homeomorphism onto its image), we may therefore assume henceforth that $E_\alpha \subseteq E$ for each $\alpha \geq \alpha_0$, $E_\alpha \subseteq E_\beta$ for all $\beta \geq \alpha \geq \alpha_0$, and that $\lambda_\alpha$ and $\lambda_{\beta,\alpha}$ are the inclusion maps. Then $E = \bigcup_{\alpha \geq \alpha_0} E_\alpha$, as a consequence of (c) and (a).

Our results comprise:

**Theorem 1.2** Assume that a manifold $M$ is a directed union $M = \bigcup_{\alpha \in A} M_\alpha$ of manifolds $M_\alpha$, such that all inclusion maps $M_\alpha \to M$ and $M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous. If $M$ admits weak direct limit charts, then

$$\pi_k(M, p) = \lim_{\rightarrow \alpha \in A} \pi_k(M_\alpha, p) \quad \text{for all } k \in \mathbb{N}_0 \text{ and } p \in M.$$  

**Remark 1.3** The concept of a weak direct limit chart was introduced in [26] in the special case of ascending sequences $M_1 \subseteq M_2 \subseteq \cdots$ of manifolds modelled on locally convex spaces. In these studies, a certain strengthened concept of “direct limit chart” provided the key to an understanding of the direct limit properties of ascending unions $G = \bigcup_{n \in \mathbb{N}} G_n$ of Lie groups (all
prominent examples of which admit direct limit charts). Following [26], we might call a weak direct limit chart a direct limit chart if, moreover, $E$ and each $E_\alpha$ is locally convex and $E = \lim \to E_\alpha$ as a locally convex space. However, this additional property is irrelevant for our current ends.

**Remark 1.4** We also mention that if $M$ and each $M_\alpha$ is a $C^r$-manifold with $r \geq 1$ (in the sense of [19] or [47]), $\phi: U \to V$ and each $\phi_\alpha$ a $C^r$-diffeomorphism, and $p \in U$, then there are canonical choices for the spaces $E$ and $E_\alpha$, namely the tangent spaces $E := T_p(M)$ and $E_\alpha = T_p(M_\alpha)$.

**Remark 1.5** In contrast to topological manifolds, the $C^r$-manifolds used in this article are always assumed to be pure manifolds (when $r \geq 1$), i.e., they are modelled on a single locally convex space.

If $M_\infty$ is dense, but not all of $M$, then “well-filled charts” are an appropriate substitute for weak direct limit charts. The following notation will be useful:

**Definition 1.6** If $E$ is a vector space, $Y \subseteq E$ and $n \in \mathbb{N}$ is fixed, we let $\text{conv}_n(Y) \subseteq E$ be the set of all convex combinations of the special form

$$t_1y_1 + \cdots + t_ny_n,$$

where $y_1, \ldots, y_n \in Y$ and $t_1, \ldots, t_n \geq 0$ such that $\sum_{j=1}^n t_j = 1$. Thus $\bigcup_{n \in \mathbb{N}} \text{conv}_n(Y)$ is the convex hull $\text{conv}(Y)$ of $Y$. Given $X, Y \subseteq E$, we set

$$\text{conv}_2(X, Y) := \{tx + (1-t)y : x \in X, y \in Y, t \in [0, 1]\}.$$

Then $\text{conv}_2(X, \text{conv}_n(X)) = \text{conv}_{n+1}(X)$ for all $n \in \mathbb{N}$. (2)

Actually, we can leave the framework of manifolds, and consider more general topological spaces (like manifolds with boundary or manifolds with corners). The following definition captures exactly what we need.

**Definition 1.7** Let $M$ be a topological space and $(M_\alpha)_{\alpha \in A}$ be a directed family of topological spaces such that $M_\infty := \bigcup_{\alpha \in A} M_\alpha$ is dense in $M$ and all inclusion maps $M_\alpha \to M$ and $M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous. We say that a homeomorphism $\phi: U \to V \subseteq E$ from an open subset $U \subseteq M$ onto an arbitrary subset $V$ of a topological vector space $E$ is a well-filled chart of $M$ if there exist $\alpha_0 \in A$, homeomorphisms $\phi_\alpha: U_\alpha \to V_\alpha \subseteq E_\alpha$ from
open subsets $U_{\alpha} \subseteq M_{\alpha}$ onto subsets $V_{\alpha}$ of certain topological vector spaces $E_{\alpha}$ for $\alpha \geq \alpha_0$, and continuous linear maps $\lambda_{\alpha} : E_{\alpha} \to E$, $\lambda_{\beta,\alpha} : E_{\alpha} \to E_{\beta}$ (for $\beta \geq \alpha \geq \alpha_0$) such that (a) and (b) from Definition 1.1 hold as well as the following conditions (d), (e) and (f):

- **(d)** $U_\infty := \bigcup_{\alpha \geq \alpha_0} U_{\alpha} = U \cap M_\infty$.

- **(e)** There exists a non-empty (relatively) open set $V^{(2)} \subseteq V$ such that $\text{conv}_2(V^{(2)}) \subseteq V$ and $\text{conv}_2(V^{(2)}_\infty) \subseteq V_\infty$, where $V_\infty := \bigcup_{\alpha \geq \alpha_0} V_{\alpha}$ and $V^{(2)}_\infty := V^{(2)} \cap V_\infty$.

- **(f)** For each $\alpha \geq \alpha_0$ and compact set $K \subseteq V^{(2)}_{\alpha} := V^{(2)} \cap V_{\alpha}$, there exists $\beta \geq \alpha$ such that $\text{conv}_2(K) \subseteq V_{\beta}$.

Then $U^{(2)} := \phi^{-1}(V^{(2)})$ is an open subset of $U$, called a core of $\phi$. For later use, we set $U^{(2)}_\infty := \phi^{-1}(V^{(2)}_\infty)$; then $U^{(2)}_{\alpha} := U^{(2)}_\infty \cap U_{\alpha} = \phi_\alpha^{-1}(V^{(2)}_{\alpha}) = \phi^{-1}(V^{(2)} \cap V_{\alpha})$ is open in $M_{\alpha}$ and $U^{(2)}_\infty = \bigcup_{\alpha \geq \alpha_0} U^{(2)}_{\alpha}$. Also, we abbreviate $E_\infty := \bigcup_{\alpha \geq \alpha_0} E_{\alpha}$. If cores of well-filled charts cover $M$, then $M$ is said to admit well-filled charts.

**Remark 1.8** We hasten to add that we assumed in (e) and (f) that $E_{\alpha} \subseteq E$ and that $\lambda_{\alpha}$, $\lambda_{\beta,\alpha}$ are the respective inclusion maps (which we always may as explained after Definition 1.1).

**Remark 1.9** Note that $U_\infty$ is dense in $U$ because $U$ is open and $M_\infty$ is dense in $M$. Consequently, $V_\infty$ is dense in $V$.

**Remark 1.10** If $V_{\alpha}$ is open in $E_{\alpha}$ for each $\alpha \geq \alpha_0$, or if each $V_{\alpha}$ is convex, then condition (f) follows from (e) and hence can be omitted.

The reader may find the concept of a well-filled chart somewhat elusive. To make it more tangible, let us consider relevant special cases:

**Example 1.11** If (a), (b) and (d) hold, $V$ is open in $E$, $E_\infty \cap V = V_\infty$ and each $V_{\alpha}$ is convex or open in $E_{\alpha}$ (to ensure (f)), then $\phi$ is a well-filled chart.

[In fact, pick any $v \in V$ and balanced, open 0-neighbourhood $W \subseteq E$ such that $W + W \subseteq V - v$; then $V^{(2)} := v + W$ satisfies (e).] In particular:

- **(i)** Every weak direct limit chart is a well-filled chart.
(ii) $\phi$ is a well-filled chart if (a), (b) and (d) hold, $V$ is open and $E_\alpha \cap V = V_\alpha$ for each $\alpha \geq \alpha_0$.

**Example 1.12** If (a), (b) and (d) hold, $V$ is convex, $E_\infty \cap V = V_\infty$ and each $V_\alpha$ is convex or open in $E_\alpha$, then $\phi$ is a well-filled chart (with $V^{(2)} := V$).

We shall obtain the following far-reaching generalization of Theorem 1.2.

**Theorem 1.13** Consider a topological space $M$ and a directed family $(M_\alpha)_{\alpha \in A}$ of topological spaces whose union $M_\infty := \bigcup_{\alpha \in A} M_\alpha$ is dense in $M$. Assume that all inclusion maps $M_\alpha \rightarrow M$ and $M_\alpha \rightarrow M_\beta$ (for $\alpha \leq \beta$) are continuous. If $M$ admits well-filled charts, then

$$\pi_k(M, p) = \lim_{\to \alpha \in A_p} \pi_k(M_\alpha, p) \quad \text{for all } k \in \mathbb{N}_0 \text{ and } p \in M_\infty.$$  

We shall also see that the inclusion map $M_\infty \rightarrow M$ is a weak homotopy equivalence for suitable topologies on $M_\infty$ (Proposition 6.1). As a very special case, we obtain a generalization of Palais’ original result:

**Corollary 1.14** Let $E$ be a topological vector space (which need not be locally convex) and $U \subseteq E$ be a subset such that

(a) $U$ is open; or:

(b) $U$ is semi-locally convex, i.e., each $p \in U$ has a neighbourhood (relative $U$) which is a convex subset of $E$.

Let $E_\infty$ be a vector subspace of $E$ such that $U_\infty := U \cap E_\infty$ is dense in $U$. Endow $U_\infty$ with the topology $\mathcal{O}$ induced by the finite topology on $E_\infty$. Then the inclusion map $(U_\infty, \mathcal{O}) \rightarrow U$ is a weak homotopy equivalence. Furthermore,

$$\pi_k(U, p) = \lim_{\to F \in F_p} \pi_k(U \cap F, p) \quad \text{for each } k \in \mathbb{N}_0 \text{ and } p \in U_\infty,$$

where $F_p$ is the set of finite-dimensional vector subspaces $F \subseteq E_\infty$ with $p \in F$. So far, generalizations to non-locally convex spaces had been established only for isolated examples [51].

**Results concerning homotopy classes of general maps.** Theorem 1.13 will be deduced from an analogous result (Theorem 5.3) for homotopy classes

\footnote{Compare [50] Theorem 13 and end of p. 1} for indications of related generalizations.
of continuous maps $|\Sigma| \to M$, where $\Sigma$ a finite simplicial complex. This theorem is our main result. Its proof does not cause additional effort.

**Applications in Lie theory.** Once all tools enabling calculations of homotopy groups are established (in Section 6), we apply them to typical examples of infinite-dimensional Lie groups.

In Section 7 we inspect the prime examples of infinite-dimensional Lie groups which are directed unions of Lie groups or manifolds (as compiled in [26]). As we shall see, our methods apply to all of them. Many of the examples are compactly retractive (whence the elementary Proposition 3.3 applies), but not all of them (in which case Theorem 1.2 cannot be avoided). It deserves mention that the existence of a direct limit chart is usually quite obvious, while the proof of compact retractivity may require specialized functional-analytic tools. Therefore Theorem 1.2 is usually easier to apply than Proposition 3.3 (although its proof is much harder).

The main applications of our results are given in Section 8, which is devoted to the calculation of the homotopy groups of prime examples of Lie groups that contain a dense directed union of Lie groups (notably various types of mapping groups and diffeomorphism groups). In particular, we prove a (formerly open) conjecture by Boreck, Czichowski and Rudolph [7] from 1981, concerning the homotopy groups of Lie groups of rapidly decreasing Lie-group valued maps on $\mathbb{R}^d$ (see Remark 8.6).

As an additional input, our applications in Section 8 require that the test function group $C^\infty_c(M,H)$ is dense in $C^r_c(M,H)$ for each finite-dimensional smooth manifold $M$, Lie group $H$ and $r \in \mathbb{N}_0$. And a similar density result is also needed for certain weighted mapping groups. These more specialized technical tools have been relegated to a separate paper [28]. They are based on results concerning smooth approximations of $C^r$-sections in fibre bundles, which generalize the $C^0$-case discussed in [57].

**Motivation.** In the extension theory of infinite-dimensional Lie groups, the homotopy groups $\pi_0(G) = G/G_0$, $\pi_1(G)$ and $\pi_2(G)$ are needed to see whether a central extension

$$\{0\} \to \mathfrak{a} \to \mathfrak{a} \oplus \omega \mathfrak{g} \to \mathfrak{g} \to \{0\}$$

of topological Lie algebras (with $\mathfrak{g} = L(G)$) gives rise to a central extension

$$\mathbf{1} \to \mathfrak{a}/\Gamma \to \widehat{G} \to G \to \mathbf{1}$$
of Lie groups for some discrete subgroup $\Gamma \subseteq a$ and some Lie group $\hat{G}$ such that $L(\hat{G}) = a \oplus_\omega g$ (where $a$ is a complete locally convex space and $\omega$ an $a$-valued 2-cocycle on $g$). If $G$ is connected (i.e., if $\pi_0(G) = 1$), then such a Lie group extension exists if and only if

- The “period group” $\Pi$ is discrete in $a$ (which is the image of a certain “period homomorphism” $\text{per}_\omega : \pi_2(G) \to a$); and
- A certain “flux homomorphism” $F_\omega : \pi_1(G) \to H^1_c(g, a)$ vanishes [44].

In this case, one can take $\Gamma = \Pi$. Similar results are available for abelian [46] and general extensions [48]. In view of these applications, it is very well motivated to study the homotopy groups of infinite-dimensional Lie groups.

**Related literature.** Much of the literature on the homotopy groups of infinite-dimensional manifolds has concentrated on the case of manifolds modelled on Hilbert, Banach or Fréchet spaces, for which strongest results are available. We recall two landmark results: Every smoothly paracompact smooth manifold modelled on a separable Hilbert space is diffeomorphic to an open subset of modelling spaces [14], and its diffeomorphism type is determined by the homotopy type [10]. Finite-dimensional submanifolds play a vital role in [14]. Frequently, Banach manifolds are homotopy equivalent to an ascending union of finite-dimensional submanifolds (see [14] and [39]).

Various authors have studied the homotopy groups of certain classical Banach-Lie groups of operators of Hilbert spaces (see [49] and [13] for the case of separable Hilbert spaces, [43] for discussions subsuming the non-separable case); also some results on groups of operators of Banach spaces are available [18]. Typically, one shows that the group is homotopy equivalent to a direct limit of classical groups like $\text{GL}_\infty(\mathbb{R}) = \lim \rightarrow \text{GL}_n(\mathbb{R})$, $U_\infty(\mathbb{C}) = \lim \rightarrow U_n(\mathbb{C})$ or $O_\infty(\mathbb{R}) = \lim \rightarrow O_n(\mathbb{R})$. The homotopy groups of these direct limit groups can be calculated using the Bott periodicity theorems [8]. In [42], dense unions of finite-dimensional Lie groups are used to describe the homotopy groups of unit groups of approximately finite $C^*$-algebras.

Some results beyond Banach-Lie groups are established in [45], notably approximation theorems enabling the calculation of the homotopy groups of various types of mapping groups, like $C_0(M, H)$ with $M$ a $\sigma$-compact finite-dimensional smooth manifold and $H$ a Lie group [45, Theorem A.10].
Typical applications of direct limits of finite-dimensional Lie groups (and manifolds) in algebraic topology are described in [35, §47].

2 Preliminaries and notation

In addition to the definitions already given in the introduction, we now compile further notation, conventions and basic facts.

General conventions. As usual, \( \mathbb{R} \) denotes the field of real numbers, \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \mathbb{Z} := \mathbb{N}_0 \cup \{-\mathbb{N}\} \). A subset \( U \) of a vector space \( E \) is called balanced if \( tU \subseteq U \) for all \( t \in \mathbb{R} \) such that \(|t| \leq 1\). If \((X, d)\) is a metric space, \( x \in X \) and \( \varepsilon > 0 \), we write \( B^d_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\} \) and \( \overline{B}^d_\varepsilon(x) := \{y \in X : d(x, y) \leq \varepsilon\} \), or simply \( B_\varepsilon(x) \) and \( \overline{B}_\varepsilon(x) \) if \( X \) and \( d \) are clear from the context. If \((X, \|\cdot\|)\) is a normed space and \( d(x, y) = \|x - y\| \), we also write \( B^X_\varepsilon(x) := B^d_\varepsilon(x) \). By a directed family, we mean a family \((X_\alpha)_{\alpha \in A}\) of sets \( X_\alpha \) indexed by a directed set \((A, \leq)\) such that \( X_\alpha \subseteq X_\beta \) for all \( \alpha, \beta \in A \) such that \( \alpha \leq \beta \). If \( G \) is a topological group, we write \( 1 \) for its neutral element and abbreviate \( \pi_k(G) := \pi_k(G, 1) \) for \( k \in \mathbb{N}_0 \). If \( G(1) \) is the path component of \( 1 \), then \( \pi_0(G) = G/G(1) \), whence \( \pi_0(G) \) is a group in a natural way. The following convention is useful:

2.1 Let \( M \) be a topological space, \( p \in M \) and \( k \in \mathbb{N}_0 \). If \( k \geq 1 \) or \( M \) is a topological group and \( p = 1 \), then \( \pi_k(M, p) \) is considered as a group, “morphism” reads “homomorphism,” and we are working in the category of groups and homomorphisms. Otherwise, \( \pi_0(M) := \pi_0(M, p) \) is a set, “morphism” reads “map,” and we are working in the category of sets and maps.

Given a map \( f : X \to Y \) and \( A \subseteq X \), we write \( f|_A \) for the restriction of \( f \) to \( A \). If \( B \subseteq Y \) is a subset which contains the image \( \text{im}(f) \) of \( f \), we write \( f|_B : X \to B \) for the co-restriction of \( f \) to \( B \). Given a topological space \( X \) and \( p \in X \), we let \( X(p) \) be the path component of \( p \) in \( X \). If \( f : X \to Y \) is a continuous map and \( p \in X \), then \( f \) restricts and co-restricts to a continuous map \( f(p) : X(p) \to Y(f(p)) \). If \((X, d)\) is a metric space and \( A \subseteq X \) a subset, we let \( \text{diam}(A) := \sup\{d(x, y) : x, y \in A\} \in [0, \infty] \) be its diameter.

Simplicial complexes. In this article, we shall only need finite simplicial complexes \( \Sigma \), and we shall always consider these as sets of simplices \( \Delta = \text{conv}\{v_1, \ldots, v_r\} \) in a finite-dimensional vector space \( F \) (where \( v_1, \ldots, v_r \in F \) are affinely independent and \( \text{rk}(\Delta) := r \), not as abstract simplicial
complexes. We write $|\Sigma| := \bigcup_{\Delta \in \Sigma} \Delta$ and call $\sup\{\text{rk}(\Delta) : \Delta \in \Sigma\} \in \mathbb{N}$ the rank of $\Sigma$. Given a simplex $\Delta = \text{conv}\{v_1, \ldots, v_r\}$ as above, we let $\mathcal{V}(\Delta) := \{v_1, \ldots, v_r\}$ be its set of vertices. We define $\mathcal{V}(\Sigma) := \bigcup_{\Delta \in \Sigma} \mathcal{V}(\Delta)$. A simplicial complex $\Sigma'$ is called a refinement of $\Sigma$ if each $|\Sigma'| = |\Sigma|$ and each $\Delta \in \Sigma$ is a union of simplices in $\Sigma'$. A typical example of a refinement is the barycentric subdivision $bsd(\Sigma)$ of $\Sigma$ (see [32, 119-120]), which we may iterate: $bsd^j(\Sigma) := bsd(bsd^{j-1}(\Sigma))$ for $j \in \mathbb{N}$. We recall: If a euclidean norm $\|\cdot\|$ on $F$ is given, $D := \sup\{\text{diam}(\Delta) : \Delta \in \Sigma\}$ and $r := \text{rk}(\Sigma)$, then
\begin{equation}
\text{diam}(\Delta) \leq \frac{r-1}{r} D \quad \text{for each } \Delta \in bsd(\Sigma) \tag{3}
\end{equation}
(cf. [32, p. 120]). Triangulating $|\Sigma|$ by affine simplices ensures that
\begin{equation}
\bigcup\{\Delta \in \Sigma : \Delta \subseteq X\} = X \tag{4}
\end{equation}
if $|\Sigma|$ is a convex set and $X$ a face of $|\Sigma|$ (or a union of faces).

**Basic facts concerning the compact-open topology.** If $X$ and $Y$ are topological spaces, we write $C(X,Y)_{c.o.}$ for the set of continuous functions from $X$ to $Y$, equipped with the compact-open topology. The sets
$$[K,W] := \{\gamma \in C(X,Y) : \gamma(K) \subseteq W\}$$
form a subbasis for this topology, for $K$ ranging through the compact subsets of $X$ and $W$ through the open subsets of $Y$. The following well-known facts (proved, e.g., in [15] and [29]) will be used repeatedly:

**Lemma 2.2** Let $X$, $Y$ and $Z$ be topological spaces and $f : Y \to Z$ be a continuous map. The following holds:

(a) The map $C(X,f) : C(X,Y)_{c.o.} \to C(X,Z)_{c.o.}$, $\gamma \mapsto f \circ \gamma$ is continuous.

(b) The map $C(f,X) : C(Z,X)_{c.o.} \to C(Y,X)_{c.o.}$, $\gamma \mapsto \gamma \circ f$ is continuous.
   In particular, $C(\iota,X) : C(Z,X) \to C(Y,X)$, $\gamma \mapsto \gamma|_Y$ is continuous if $Y \subseteq Z$ is equipped with a topology making the inclusion map $\iota : Y \to Z$, $y \mapsto y$ continuous.

(c) If $\gamma : X \times Y \to Z$ is continuous, then $\gamma^\vee : X \to C(Y,Z)_{c.o.}$, $\gamma^\vee(x) := \gamma(x,\bullet)$ is continuous.
(d) If $\gamma : X \to C(Y,Z)_{c.o.}$ is continuous and $Y$ is locally compact, then the map $\gamma^\wedge : X \times Y \to Z$, $\gamma^\wedge(x,y) := \gamma(x)(y)$ is continuous.

(e) If $X$ is locally compact, then the evaluation map $\varepsilon : C(X,Y)_{c.o.} \times X \to Y$, $\varepsilon(\gamma,x) := \gamma(x)$ is continuous. \hspace{1cm} \Box

Direct limits. We assume that the reader is familiar with the concepts of a direct system $S = ((X_\alpha)_{\alpha \in A}, (\phi_{\beta,\alpha})_{\beta \geq \alpha})$ over a directed set $(A, \leq)$ in a category $C$, the notion of a cone $(X, (\phi_\alpha)_{\alpha \in A})$ over $S$ and that of a direct limit cone and its universal property. It is well known that every direct system $((X_\alpha)_{\alpha \in A}, (\phi_{\beta,\alpha})_{\beta \geq \alpha})$ in the category of sets has a direct limit $(X, (\phi_\alpha)_{\alpha \in A})$. It is also known that

$$X = \bigcup_{\alpha \in A} \phi_\alpha(X_\alpha) \hspace{1cm} (5)$$

and if $\alpha, \beta \in A$ and $x \in X_\alpha$, $y \in X_\beta$, then

$$\phi_\alpha(x) = \phi_\beta(y) \iff (\exists \gamma \geq \alpha, \beta) \quad \phi_{\gamma,\alpha}(x) = \phi_{\gamma,\beta}(y). \hspace{1cm} (6)$$

Likewise, each direct system in the category of groups and homomorphisms has a direct limit. Its underlying set is the direct limit of the given direct system in the category of sets. See, e.g., [21, §2] for these well-known facts.

3 Elementary observations

In this section, we make some simple observations concerning the path components, homotopy groups and homology modules of directed unions of topological spaces, assuming that these are compactly retractive. Special cases of these results are known or part of the folklore, but they are so useful (and apply to so many examples in Lie theory) that they deserve to be recorded in full generality, despite their simplicity. First applications of weak direct limit charts will also be given.

3.1 Throughout this section, we assume that $M$ is a topological space and $M = \bigcup_{\alpha \in A} M_\alpha$ for a directed family $(M_\alpha)_{\alpha \in A}$ of topological spaces $M_\alpha$, such that the inclusion maps $\lambda_\alpha : M_\alpha \to M$ (for $\alpha \in A$) and $\lambda_{\beta,\alpha} : M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous.

Definition 3.2 We say that $M = \bigcup_{\alpha \in A} M_\alpha$ is compactly retractive if every compact subset $K \subseteq M$ is contained in $M_\alpha$ for some $\alpha \in A$ and $M_\alpha$ induces the same topology on $K$ as $M$. 

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Compact reactivity has useful consequences:

**Proposition 3.3** Let $M = \bigcup_{\alpha \in A} M_\alpha$ be compactly retractive, $p \in M$ and $A_p := \{ \alpha \in A : p \in M_\alpha \}$. Then the following holds:

(a) The path component $M(p)$ of $p$ in $M$ is the union $M(p) = \bigcup_{\alpha \in A_p} (M_\alpha)_p$;

(b) $\pi_k(M, p) = \lim_{\alpha \to A_p} \pi_k(M_\alpha, p)$ as a group, for each $k \in \mathbb{N}$;

(c) $\pi_0(M, p) = \lim_{\alpha \to A_p} \pi_0(M_\alpha, p)$ as a set;

(d) If $M$ and each $M_\alpha$ is a topological group and all $\lambda_\alpha$ and $\lambda_{\beta, \alpha}$ are continuous homomorphisms, then $\pi_0(M) = \lim_{\to} \pi_0(M_\alpha)$ as a group;

(e) The singular homology modules of $M$ over $R$ are of the form $H_k(M, R) = \lim H_k(M_\alpha, R)$, for each $k \in \mathbb{N}_0$ and each commutative ring $R$.

**Proof.** (a) By compact reactivity, every path in $M$ is a path in some $M_\alpha$, from which the assertion follows.

(b), (c) and (d): We shall use the conventions of [2.1]. In the situation of (d), we let $p := 1$; in the situation of (c) and (d), we let $k := 0$. We first fix notation which can be re-used later.

3.4 After passing to a cofinal subsystem, we may assume that $p \in M_\alpha$ for each $\alpha \in A$. Since $\lambda_\beta \circ \lambda_{\beta, \alpha} = \lambda_\alpha$ if $\alpha \leq \beta$, we have $(\lambda_\beta)_* \circ (\lambda_{\beta, \alpha})_* = (\lambda_\alpha)_*$, where $(\lambda_\alpha)_*: \pi_k(M_\alpha, p) \to \pi_k(M, p)$ and $(\lambda_{\beta, \alpha})_*: \pi_k(M_\alpha, p) \to \pi_k(M_\beta, p)$.

Hence $(\pi_k(M, p), ((\lambda_\alpha)_*)_{\alpha \in A})$ is a cone over the direct system

$$((\pi_k(M_\alpha, p))_{\alpha \in A}, ((\lambda_{\beta, \alpha})_*)_{\alpha \leq \beta}).$$

By the universal property of the direct limit, there exists a unique morphism $\psi: D := \lim \pi_k(M_\alpha, p) \to \pi_k(M, p)$ such that $\psi \circ \mu_\alpha = (\lambda_\alpha)_*$ for each $\alpha \in A$, where $\mu_\alpha: \pi_k(M_\alpha, p) \to D$ is the limit map.

$\psi$ is surjective. To see this, let $[\gamma] \in \pi_k(M, p)$, where $\gamma: [0, 1]^k \to M$ is a continuous map with $\gamma|_{[0,1]^k} = p$. By compact reactivity, $\gamma$ co-restricts to a continuous map $\eta: [0, 1]^k \to M_\alpha$ for some $\alpha \in A$. Then $[\eta] \in \pi_k(M_\alpha, p)$ and $(\lambda_\alpha)_*([\eta]) = [\gamma]$.

$\psi$ is injective. To see this, let $g_1, g_2 \in D$ such that $\psi(g_1) = \psi(g_2)$. There exist $\alpha \in A$ and $[\gamma_1], [\gamma_2] \in \pi_k(M_\alpha, p)$ such that $g_j = \mu_\alpha([\gamma_j])$ for $j \in \{1, 2\}$. 12
Then $γ_1, γ_2$ are homotopic relative $∂[0, 1]^k$ in $M$, by means of the homotopy $F: [0, 1]^k × [0, 1] → M$, say. By compact retractive, there is $β ≥ α$ such that $F$ co-restricts to a continuous map to $M_β$. Then $(λ_β α)_*(γ_1) = (λ_β α)_*(γ_2)$ and hence $g_1 = μ_α(γ_1) = μ_β((λ_β α)_*(γ_1)) = μ_β((λ_β α)_*(γ_2)) = g_2$.

(e) Let $c = ∑ r_σ σ$ be a singular chain in $M$, where $r_σ ∈ R$ and $F := \{σ: r_σ ≠ 0\}$ is finite. Then there exists $α ∈ A$ such that each $σ ∈ F$ co-restricts to a continuous map to $M_α$. Thus $c$ can be considered as a singular chain in $M_α$. The assertion now follows as in the proof of (b).

In the presence of weak direct limit charts, compact reactivity can be checked on the level of modelling spaces.

**Proposition 3.5** Let $M$ be a topological manifold which is a directed union $M = ∪_{α ∈ A} M_α$ of topological manifolds.

(a) If $M$ is covered by the domains of weak direct limit charts $φ: M ⊇ U → V ⊆ E$ (as in Definition 3.4) such that $E = ∪_{α ≥ α_0} E_α$ is compactly retractive, then $M$ is compactly retractive.

(b) If $M$ is compactly retractive and $φ = ∪_{α ≥ α_0} φ_α: M ⊇ U → V ⊆ E = ∪_{α ≥ α_0} E_α$ a weak direct limit chart with charts $φ_α: M_α ⊇ U_α → V_α ⊆ E_α$, then $E = ∪_{α ≥ α_0} E_α$ is compactly retractive.

**Proof.** (a) Let $K ⊆ M$ be compact. Given $x ∈ K$, let $φ: U → V$ be a weak direct limit chart as described in (a), with $x ∈ U$. There exists a compact neighbourhood $K_x ⊆ K ∩ U$ of $x$ in $K$. Now the compact reactivity of $E$ shows that $φ(K_x)$ is a compact subset of $E_β_x$ for some $β_x ≥ α_0$. Since $(V_α ∩ φ(K_x))_{α ≥ β_x}$ is a directed family of sets and an open cover of the compact set $φ(K_x)$, after increasing $β_x$ if necessary we may assume that $φ(K_x)$ is a compact subset of $V_β_x$. Then $K_x$ is a compact subset of $U_β_x$. There exists a finite subset $F ⊆ K$ such that $K = ∪_{x ∈ F} K_x$, and $α ≥ α_0$ such that $α ≥ β_x$ for all $x ∈ F$. Then $K = ∪_{x ∈ F} K_x$ is a compact subset of $M_α$.

(b) We may assume that $0 ∈ V$. Let $K ⊆ E$ be compact. Given $x ∈ K$, there exists a compact neighbourhood $K_x$ of $0$ in $(K − x) ∩ V$. By compact reactivity of $M$, there exists $β_x ≥ α_0$ such that $φ^{-1}(K_x)$ is a compact subset of $M_β_x$. Since $(U_α ∩ φ^{-1}(K_x))_{α ≥ β_x}$ is a directed family of sets and an open cover of the compact set $φ^{-1}(K_x)$, after increasing $β_x$ if necessary we may assume that $φ^{-1}(K_x)$ is a compact subset of $U_β_x$. Then $K_x$ is a compact subset of $V_β_x$ and hence of $E_β_x$. There exists a finite subset $F ⊆ K$ such that
\[ K = \bigcup_{x \in F} (x + K_x), \text{ and } \alpha \geq \alpha_0 \text{ such that } F \subseteq E_\alpha \text{ and } \alpha \geq \beta_x \text{ for all } x \in F. \]
Then \( K = \bigcup_{x \in F} (x + K_x) \) is a compact subset of \( E_\alpha \). \( \square \)

The following corollary refers to Lie groups modelled on locally convex spaces, smooth maps and \( C^1 \)-maps as in [19], [29] and [47] (cf. also [38] for the case of sequentially complete modelling spaces). The tangent space of a Lie group at the identity element will be denoted by \( L(G) := T_1(G) \). If \( E \) and each \( E_\alpha \) is a locally convex space in the definition of a weak direct limit chart and \( \phi \) and each \( \phi_\alpha \) is a \( C^1 \)-diffeomorphism, then we speak of a weak direct limit chart of class \( \mathcal{C}^1 \). Since all translates of a weak direct limit chart of a Lie group are weak direct limit charts, Proposition 3.5 (and Remark 1.4) imply:

**Corollary 3.6** Assume that a Lie group \( G \) is a directed union \( G = \bigcup_{\alpha \in A} G_\alpha \) of Lie groups \( G_\alpha \), such that all inclusion maps are smooth homomorphisms. If \( G \) admits a weak direct limit chart of class \( \mathcal{C}^1 \) around \( 1 \), then \( G \) is compactly retractive if and only if \( L(G) = \bigcup_{\alpha \in A} L(G_\alpha) \) is compactly retractive. \( \square \)

In the case of Lie groups, other simple hypotheses lead to conclusions similar to the preceding ones. We write \( G_0 \) for the connected component of the identity element \( 1 \) in a topological group \( G \). If \( G \) is a Lie group, then \( G_0 = G(1) \) coincides with the path component.

**Lemma 3.7** Consider a Lie group \( G = \bigcup_{\alpha \in A} G_\alpha \) which is a directed union of Lie groups (such that each inclusion map is a smooth homomorphism). Assume that \( G \) and each \( G_\alpha \) has an exponential map, \( L(G) = \bigcup_{\alpha} L(G_\alpha) \), and that \( \exp_G(L(G)) \) is an identity neighbourhood in \( G \). Then the identity component of \( G \) is the union \( G_0 = \bigcup_{\alpha \in A} (G_\alpha)_0 \).

**Proof.** \( S := \bigcup_{\alpha \in A} (G_\alpha)_0 \) is a subgroup of \( G_0 \). Each \( v \in L(G) \) belongs to \( L(G_\alpha) \) for some \( \alpha \). Then \( \exp_G(v) = \exp_{G_\alpha}(v) \in (G_\alpha)_0 \subseteq S \), by naturality of \( \exp \). Thus \( \exp_G(L(G)) \subseteq S \), whence \( G_0 \subseteq S \) and therefore \( G_0 = S \). \( \square \)

### 4 Technical preparations

We now prove several preparatory lemmas, which will be used in the next section to establish our main result (Theorem 5.3). The first lemma yields extensions of continuous maps from the boundary \( \partial \Delta \) of a simplex to all of \( \Delta \).
We start with the following setting: Let $E$ be a topological vector space, $F$ be a finite-dimensional vector space, $v_1, \ldots, v_r \in F$ be affinely independent points, $\Delta := \text{conv}\{v_1, \ldots, v_r\}$ and $b := \sum_{j=1}^{r} \frac{1}{r} v_j$ be the barycentre of $\Delta$. We pick (and fix) any point $x_{\Delta} \in \partial \Delta$. To $\gamma \in C(\partial \Delta, E)$, we associate a function $\Phi(\gamma): \Delta \rightarrow E$ as follows: Given $x \in \Delta$, there exists a proper face $X$ of $\Delta$ such that $x \in \text{conv}(X \cup \{b\})$. Then $X = \text{conv}(J)$ for a proper subset $J \subset \{v_1, \ldots, v_r\}$ and

$$x = tb + \sum_{j \in J} t_j v_j$$

with uniquely determined non-negative real numbers $t$ and $t_j$ for $j \in J$ such that $t + \sum_{j \in J} t_j = 1$. We define

$$\Phi(\gamma)(x) := \begin{cases} t \gamma(x_{\Delta}) + (1-t) \gamma \left( \frac{\sum_{j \in J} t_j v_j}{1-t} \right) & \text{if } t < 1; \\ \gamma(x_{\Delta}) & \text{if } t = 1. \end{cases}$$

This definition is independent of the choice of $X$, as follows from the following consideration: If also $x \in X' = \text{conv}(J')$ and $x = t'b + \sum_{j \in J'} t_j' v_j$, then $x \in X \cap X' = \text{conv}(J \cap J')$ and thus $t_j = 0$ for all $j \in J \setminus J'$ as well as $t_j' = 0$ for all $j \in J' \setminus J$. Now $t_j = t_j'$ for all $j \in J \cap J'$, by uniqueness.

**Lemma 4.1 (Filling Lemma)** In the preceding situation, we have:

(a) For each $\gamma \in C(\partial \Delta, E)$, the function $\Phi(\gamma): \Delta \rightarrow E$ is continuous, and $\Phi(\gamma)|_{\partial \Delta} = \gamma$.

(b) $\Phi: C(\partial \Delta, E)_{c.o.} \rightarrow C(\Delta, E)_{c.o.}$, $\gamma \mapsto \Phi(\gamma)$ is continuous and linear.

(c) If $\gamma$ is constant, taking the value $y$, then also $\Phi(\gamma)(x) = y$ for all $x \in \Delta$.

**Proof.** We first note that $\Phi(\gamma)|_{\partial \Delta} = \gamma$, by construction. Next, we claim that the map

$$\Phi^\wedge: C(\partial \Delta, E) \times \Delta \rightarrow E, \quad \Phi^\wedge(\gamma, x) := \Phi(\gamma)(x)$$

is continuous. If this is true, then $\Phi(\gamma) = \Phi^\wedge(\gamma, \bullet)$ is continuous, proving (a). Moreover, $\Phi = (\Phi^\wedge)^\vee$ will be continuous, by Lemma 2.2(c). Since $\Phi$ is linear by definition, this gives (b). Property (c) holds by construction.

The sets $C(\partial \Delta, E) \times \text{conv}(J \cup \{b\})$ form a finite cover of $C(\partial \Delta, E) \times \Delta$ by closed sets, if $J$ ranges through the proper subsets of $\{v_1, \ldots, v_r\}$. Hence $\Phi^\wedge$
will be continuous if its restriction to each set \( C(\partial\Delta, E) \times \text{conv}(J \cup \{b\}) \) is continuous (by the Glueing Lemma, [52 Satz 3.7]). To verify this property, let \((\gamma_\alpha, x_\alpha)\) be a convergent net in \( C(\partial\Delta, E) \times \text{conv}(J \cup \{b\})\), with limit \((\gamma, x)\). We write \( x_\alpha = t_\alpha b + \sum_{j \in J} t_{j,\alpha} v_j \) and \( x = tb + \sum_{j \in J} t_j v_j \) as above. Then \( t_\alpha \to t \) and \( t_{j,\alpha} \to t_j \).

Case 1: If \( t < 1 \), then \( t_\alpha < 1 \) eventually and

\[
\Phi^\wedge(\gamma_\alpha, x_\alpha) = t_\alpha \gamma_\alpha(x_\Delta) + (1 - t_\alpha) \gamma_\alpha \left( \frac{\sum_{j \in J} t_{j,\alpha} v_j}{1 - t_\alpha} \right)
\]

\[
= t_\gamma(x_\Delta) + (1 - t) \gamma \left( \frac{\sum_{j \in J} t_j v_j}{1 - t} \right) = \Phi^\wedge(\gamma, x),
\]

exploiting that the evaluation map \( C(\partial\Delta, E) \times \partial\Delta \to E, (\eta, y) \mapsto \eta(y) \) is continuous because \( \partial\Delta \) is compact (see Lemma 2.2(e)).

Case 2: If \( t = 1 \), then

\[
\Phi^\wedge(\gamma_\alpha, x_\alpha) - \Phi^\wedge(\gamma, x) = \gamma_\alpha(x_\Delta) - \gamma(x_\Delta) + R_\alpha
\]

where \( R_\alpha = 0 \) if \( t_\alpha = 1 \) while

\[
R_\alpha = (t_\alpha - 1)\gamma_\alpha(x_\Delta) + (1 - t_\alpha) \gamma_\alpha \left( \frac{\sum_{j \in J} t_{j,\alpha} v_j}{1 - t_\alpha} \right)
\]

if \( t_\alpha < 1 \). Since \( \gamma_\alpha(x_\Delta) - \gamma(x_\Delta) \to 0 \) by continuity of evaluation (see Lemma 2.2(e)), it only remains to show that \( R_\alpha \to 0 \). To verify this, let \( U \subseteq E \) be a balanced 0-neighbourhood. Pick a balanced open 0-neighbourhood \( V \subseteq E \) such that \( V + V + V + V \subseteq U \). Since \( \gamma(\partial\Delta) \) is compact and hence bounded, there exists \( \rho > 0 \) such that \( \gamma(\partial\Delta) \subseteq \rho V \). Then \( \gamma(\partial\Delta) + \rho V \) is an open neighbourhood of \( \gamma(\partial\Delta) \) and hence \( \Omega := [\partial\Delta, \gamma(\partial\Delta) + \rho V] \) is a neighbourhood of \( \gamma \) in \( C(\partial\Delta, E)_{c.o.} \). For \( \alpha \) sufficiently large, we have \( \gamma_\alpha \in \Omega \) and \( 1 - t_\alpha < \rho^{-1} \). If \( t_\alpha = 1 \), then \( R_\alpha = 0 \in U \). If \( t_\alpha < 1 \), then \( R_\alpha \in (1 - t_\alpha)(\gamma_\alpha(\partial\Delta) - \gamma_\alpha(x_\Delta)) \subseteq (1 - t_\alpha)(\gamma(\partial\Delta) + \rho V - \gamma(\partial\Delta) - \rho V) \subseteq (1 - t_\alpha)(\rho V + \rho V + \rho V + \rho V) \subseteq (1 - t_\alpha)\rho U \subseteq U \) as well. Thus \( R_\alpha \to 0 \). \( \square \)

\[4.2\] The next lemmas refer to a setting already encountered in Theorem 1.13: \( M \) is a topological space and \((M_\alpha)_{\alpha \in A}\) a directed family of topological spaces whose union \( M_\infty := \bigcup_{\alpha \in A} M_\alpha \) is dense in \( M \). We assume that all inclusion maps \( M_\alpha \to M \) and \( M_\alpha \to M_\beta \) (for \( \alpha \leq \beta \)) are continuous. Furthermore, we assume that \( M \) admits well-filled charts.
Lemma 4.3 In the situation of 4.2, let \( \alpha \in A \) and \( K \subseteq M_\alpha \) be a compact set such that \( K \subseteq U^{(2)} \) for a core \( U^{(2)} \) of a well-filled chart \( \phi: U \to V \) (as in Definition 1.7). Then there exists \( \beta \geq \alpha_0 \) such that \( K \subseteq U^{(2)}_\beta \).

Proof. We may assume that \( \alpha \geq \alpha_0 \), where \( \alpha_0 \) is as in Definition 1.7. The sets \( K \cap U^{(2)}_\beta \) (for \( \beta \geq \alpha \)) form an open cover of \( K \), and a directed family of sets. Since \( K \) is compact, there is \( \beta \geq \alpha \) such that \( K \subseteq K \cap U^{(2)}_\beta \). \( \square \)

If \( M \) admits well-filled charts, then there are well-filled charts with arbitrarily small domain around each point. More precisely:

Lemma 4.4 In the situation of 4.2, let \( q \in M \) and \( W \) be a neighbourhood of \( q \) in \( M \). Then there exists a well-filled chart \( \phi: U \to V \) and a core \( U^{(2)} \) of \( \phi \) such that \( q \in U^{(2)} \subseteq U \subseteq W \).

Proof. By hypothesis, there exists a well-filled chart \( \bar{\phi}: U \to V \subseteq E \) and a core \( \overline{U}^{(2)} \) thereof such that \( q \in \overline{U}^{(2)} \). Let \( \alpha_0 \) and the homeomorphism \( \bar{\phi}_\alpha: M_\alpha \supseteq \overline{U}_\alpha \to \overline{V}_\alpha \) for \( \alpha \geq \alpha_0 \) be as in Definition 1.7 and \( \overline{U}_\infty := \bigcup_{\alpha \geq \alpha_0} \overline{U}_\alpha \). There exists a balanced, open \( 0 \)-neighbourhood \( Q \subseteq E \) such that \( V := (\bar{\phi}(q) + Q + Q) \cap V \subseteq \overline{\phi}(W \cap \overline{U}) \). Set \( V^{(2)} := (\bar{\phi}(q) + Q) \cap \overline{V}^{(2)} \), \( U := \bar{\phi}^{-1}(V) \), \( U^{(2)} := \bar{\phi}^{-1}(V^{(2)}) \), \( \overline{U}_\alpha := \overline{U}_\alpha \cap U \), \( V_\alpha := \phi(U_\alpha) = \overline{V}_\alpha \cap V \), \( U_\infty := \bigcup_{\alpha \geq \alpha_0} U_\alpha \) and \( \phi_\alpha := \bar{\phi}_\alpha|_{U_\alpha} \). Then \( q \in U^{(2)} \subseteq U \subseteq W \). Furthermore, \( \phi := \bar{\phi}|_U: U \to V \) is a well-filled chart. In fact, (a) and (b) required in Definition 1.7 hold by construction. Since \( U \cap M_\infty = U \cap \overline{U} \cap M_\infty = U \cap \overline{U}_\infty = U \cap \bigcup_{\alpha \geq \alpha_0} \overline{U}_\alpha = U \cap \overline{U}_\infty = \bigcup_{\alpha \geq \alpha_0} U_\alpha = U_\infty \), also (d) holds. Next, observe that \( \text{conv}_2(V^{(2)}) \subseteq (\bar{\phi}(q) + Q + Q) \cap \overline{V} = V \). Moreover, \( V_\infty := \bigcup_{\alpha \geq \alpha_0} V_\alpha = V \cap \overline{V}_\infty \) and \( V^{(2)} := V^{(2)} \cap V_\infty \) satisfy \( \text{conv}_2(V_\infty^{(2)}) \subseteq V \cap \text{conv}_2(\overline{V}_\infty^{(2)}) \subseteq V \cap \overline{V}_\infty = V_\infty \). Hence (e) holds. To verify (f), let \( \alpha \geq \alpha_0 \) and \( K \subseteq V_\alpha^{(2)} := V^{(2)} \cap V_\alpha \) be a compact set. Then \( \text{conv}_2(K) \subseteq \overline{V}_\beta \) for some \( \beta \geq \alpha \). Since also \( \text{conv}_2(K) \subseteq (\phi(q) + Q + Q) \cap \overline{V} = V \cap \overline{V}_\beta = V_\beta \), we deduce that \( \text{conv}_2(K) \subseteq (\phi(q) + Q + Q) \cap \overline{V}_\beta = V \cap \overline{V}_\beta = V_\beta \). \( \square \)

Lemma 4.5 In the situation of 4.2, let \( q \in M \), \( \phi: U \to V \) be a well-filled chart and \( U^{(2)} \subseteq U \) be a core of \( \phi \) such that \( q \in U^{(2)} \). Then there exists an open neighbourhood \( U^{(4)} \subseteq U^{(2)} \) of \( q \) such that \( V^{(4)} := \phi(U^{(4)}) \) satisfies \( \text{conv}_2(V^{(4)}) \subseteq V^{(2)} \).
Proof. Apply the construction from the proof of Lemma 4.4 to $\phi := \phi$ and $W := U^{(2)}$.

The next lemma is the technical backbone of this article. Given a continuous map $\gamma_0: |\Sigma| \to M$, it ensures that any $\gamma$ close to $\gamma_0$ can be approximated by a continuous map $\eta_\gamma: |\Sigma| \to M$ nearby, which has specific additional properties. In our later applications, we shall only need the approximation $\eta_{\gamma_0}$ to $\gamma_0$. However, the inductive proof makes it necessary to formulate and prove the lemma in the stated form.

Lemma 4.6 (Simultaneous Approximations) In the setting of 4.2 let $\Sigma$ be a finite simplicial complex, $\gamma_0: |\Sigma| \to M$ be a continuous function and $Q \subseteq C(|\Sigma|, M)_{c.o.}$ be a neighbourhood of $\gamma_0$. Let $\mathcal{E} \subseteq |\Sigma|$ be a subset such that $\mathcal{E} = \bigcup \{ \Delta \in \Sigma: \Delta \subseteq \mathcal{E}\}$. Then there exist a finite subset $S \subseteq |\Sigma|$ containing $V(\Sigma)$, an open neighbourhood $P$ of $\gamma_0$ in $C(|\Sigma|, M)_{c.o.}$, and a continuous map $\Theta: P \times |\Sigma| \times [0,1] \to M$ with the following properties:

(a) $\Theta(\gamma, \bullet, 0) = \gamma$, for each $\gamma \in P$;

(b) $\Theta(\gamma, \bullet, t) \in Q$, for each $\gamma \in P$ and $t \in [0,1]$;

(c) For each $\gamma \in P$, the map $\eta_\gamma := \Theta(\gamma, \bullet, 1): |\Sigma| \to M$ only depends on $\gamma|_{S \cup \mathcal{E}}$. Also, for each $\Delta \in \Sigma$, the restriction $\eta_\gamma|_\Delta$ only depends on $\gamma|_{(S \cup \mathcal{E}) \cap \Delta}$;

(d) Let $\gamma \in P$ such that $\gamma(S \cup \mathcal{E}) \subseteq M_\alpha$ for some $\alpha \in A$, and $\gamma|_\mathcal{E}: \mathcal{E} \to M_\alpha$ is continuous. Then there exists $\beta \geq \alpha$ such that $\eta_\gamma$ takes its values in $M_\beta$ and is continuous as a map to $M_\beta$;

(e) $F_\gamma := \Theta(\gamma, \bullet, 1): |\Sigma| \times [0,1] \to M$ is a homotopy from $\gamma$ to $\eta_\gamma$, for each $\gamma \in P$;

(f) If $\gamma \in P$ is such that $\text{im}(\gamma) \subseteq M_\alpha$ for some $\alpha \in A$ and $\gamma|_{M_\alpha}: |\Sigma| \to M_\alpha$ is continuous, then there exists $\beta \geq \alpha$ such that $\text{im}(F_\gamma) \subseteq M_\beta$ and $F_\gamma: |\Sigma| \times [0,1] \to M_\beta$ is continuous;

(g) If $\gamma \in P$ and $\Delta \in \Sigma$ are such that $\gamma|_\Delta$ is a constant function, taking the value $y \in M$, say, then $F_\gamma(x, t) = y$ for all $x \in \Delta$ and $t \in [0,1]$;

(h) $F_\gamma(x, t) = \gamma(x)$ for all $\gamma \in P$, $x \in S \cup \mathcal{E}$ and $t \in [0,1]$.  

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Proof. The proof is by induction on the rank $r$ of $\Sigma$. We assume that $r = 1$ first, in which case $S := |\Sigma|$ is a finite subset of a finite-dimensional vector space $F$. Then $P := Q$ and $\Theta : P \times |\Sigma| \times [0, 1] \to M, \Theta(\gamma, x, t) := \gamma(x)$ have the asserted properties.

To perform the induction step, let $\Sigma$ be a simplicial complex of rank $r \geq 2$ and assume that the assertion holds for complexes of rank $r - 1$.

By definition of the compact-open topology, there exist $\ell \in \mathbb{N}$, compact subsets $K_j \subseteq |\Sigma|$ for $j \in \{1, \ldots, \ell\}$ and open sets $W_j \subseteq M$ such that

$$\gamma_0 \in \bigcap_{j=1}^\ell [K_j, W_j] \subseteq Q. \quad (9)$$

Our first objective is to make more intelligent choices of the sets $K_j$ and $W_j$. We shall improve them in several steps.

Since $[K_j, W_j] = \bigcap_{\Delta \in |\Sigma|} [K_j \cap \Delta, W_j]$, we may assume without loss of generality that each $K_j$ is a subset of some $\Delta_j \in \Sigma$. Since $\gamma_0^{-1}(W_j) \cap \Delta_j$ is an open neighbourhood of $K_j$ in $\Delta_j$, there exists $m_j \in \mathbb{N}$ such that $\gamma_0(\Delta') \subseteq W_j$ for all $\Delta' \in \text{bsd}^{m_j}((\Delta_j))$ such that $\Delta' \cap K_j \neq \emptyset$ (cf. (4)). Let $m$ be the maximum of the $m_j$ for $j \in \{1, \ldots, \ell\}$. After replacing $K_j$ by all $\Delta' \in \text{bsd}^m(\Delta_j)$ such that $\Delta' \cap K_j \neq \emptyset$, and after replacing $\Sigma$ with $\text{bsd}^m(\Sigma)$, we may assume without loss of generality that $K_j \in \Sigma$ for each $j$. Given $\Delta \in \Sigma$, define $W_\Delta$ as the intersection of the $W_j$, for all $j \in \{1, \ldots, \ell\}$ such that $K_j = \Delta$ (with the convention that $\bigcap \emptyset := M$). Improving (9), we now have

$$\gamma_0 \in \bigcap_{\Delta \in \Sigma} [\Delta, W_\Delta] \subseteq Q. \quad (10)$$

In the next step, we replace some $W_\Delta$ by cores of well-filled charts.

Recall that $|\Sigma| \subseteq F$ for some finite-dimensional vector space $F$; we choose any norm $\|\cdot\|$ on $F$ and let $d$ be the metric on $|\Sigma|$ arising from $\|\cdot\|$. Given $\Delta' \in \Sigma$ and $x \in \Delta'$, there exists a well-filled chart $\phi_{\Delta', x} : U_{\Delta', x} \to V_{\Delta', x}$ of $M$ such that $U_{\Delta', x} \subseteq W_{\Delta'}$ and $\gamma_0(x) \in U^{(2)}_{\Delta', x}$ for some core $U^{(2)}_{\Delta', x}$ of $\phi_{\Delta', x}$, by Lemma 4.4. Let $U^{(4)}_{\Delta', x} \subseteq U^{(2)}_{\Delta', x}$ be a neighbourhood of $\gamma_0(x)$ as in Lemma 4.5. Since $\gamma_0$ is continuous, $x$ has an open neighbourhood $Y_{\Delta', x}$ in $\Delta'$ such that $\gamma_0(Y_{\Delta', x}) \subseteq U^{(4)}_{\Delta', x}$. Choose $\delta > 0$ such that $\delta$ is a Lebesgue number for the open cover $(Y_{\Delta', x})_{x \in \Delta'}$ of $\Delta'$, for each $\Delta' \in \Sigma$. There exists $m \in \mathbb{N}$ such
that \( \text{diam}(\Delta) < \delta \) for each \( \Delta \in \bsd^m(\Sigma) \) (cf. \( \square \)). Given \( \Delta \in \bsd^m(\Sigma) \), there exists a unique \( \Delta' \in \Sigma \) such that \( \Delta \in \bsd^m(\Delta') \) but \( \Delta \notin \bsd^m(\Delta'') \) for each proper face \( \Delta'' \) of \( \Delta' \). We pick \( x \in \Delta' \) such that \( \Delta \subseteq U_{\Delta,x} \) and set 
\[
\phi_\Delta := \phi_{\Delta,x}, \quad U_\Delta := U_{\Delta,x}, \quad V_\Delta := V_{\Delta,x}, \quad U_\Delta^{(2)} := U_{\Delta,x}^{(2)} \quad \text{and} \quad U_\Delta^{(4)} := U_{\Delta,x}^{(4)}.
\]
We let \( E_\Delta \) be the topological vector space with \( V_\Delta \subseteq E_\Delta \). Then \( \gamma_0 \in [\Delta, U_\Delta^{(4)}] \).

As before, replace \( \Sigma \) with \( \bsd^m(\Sigma) \) for simplicity of notation. Let \( \Sigma^* \) be the simplicial complex formed by all simplices \( \Delta \in \Sigma \) of rank at most \( r - 1 \).

Given \( \Delta' \in \Sigma^* \), let \( Z_{\Delta'} := \bigcap_\Delta U_\Delta^{(4)} \), where \( \Delta \) ranges through all \( \Delta \in \Sigma \) such that \( \Delta' \subseteq \Delta \). We have achieved the following:

(i) \( U_\Delta \) is the domain of a well-filled chart of \( M \), for each simplex \( \Delta \in \Sigma \) of rank \( r \);

(ii) If \( \Delta, \Delta' \in \Sigma \) such that \( \text{rk}(\Delta) = r \) and \( \Delta' \) is a proper subset of \( \Delta \), then
\[
Z_{\Delta'} \subseteq U_{\Delta}^{(4)};
\]

(iii) \( \gamma_0 \in R := \bigcap_{\Delta \in \Sigma^*} [\Delta, Z_\Delta] \cap \bigcap_{\Delta \in \Sigma^*} [\Delta, U_\Delta^{(4)}] \subseteq \bigcap_{\Delta \in \Sigma} [\Delta, U_\Delta] \subseteq Q \).

Define
\[
Q^* := \bigcap_{\Delta \in \Sigma^*} [\Delta, Z_\Delta] \subseteq C(|\Sigma^*|, M).
\]

By induction, there exists an open neighbourhood \( P^* \subseteq C(|\Sigma^*|, M)_{c.o.} \) of \( \gamma_0|_{|\Sigma^*|} \), a continuous map \( \Theta^*: P^* \times [\Sigma^*] \times [0, 1] \to M \) and a finite subset \( S \subseteq [\Sigma^*] \) with \( \mathcal{V}(\Sigma) = \mathcal{V}(\Sigma^*) \subseteq S \) satisfying analogues of (a)–(h), with \( \Sigma \) replaced by \( \Sigma^* \), \( P \) by \( P^* \), \( Q \) by \( Q^* \), \( \Theta \) by \( \Theta^* \), and \( \mathcal{E} \) by \( \mathcal{E}^* := \mathcal{E} \cap [\Sigma^*] \). We let
\[
P := \{ \gamma \in R: \gamma|_{|\Sigma^*|} \in P^* \};
\]
by Lemma \( \square \)(b), this is an open neighbourhood of \( \gamma_0 \) in \( C(|\Sigma|, M)_{c.o.} \).

To enable a piecewise definition of \( \Theta \), let \( \Delta \in \Sigma \) be a simplex of rank \( r \) (which we fix for the moment). The well-filled chart \( \phi_\Delta : U_\Delta \to V_\Delta \subseteq E_\Delta := E \) from above goes along with \( \alpha_0 \in A \), homeomorphisms \( \phi_{\Delta,a} : M_a \supseteq U_{\Delta,a} \to V_{\Delta,a} \subseteq E_{\Delta,a} \) and sets \( U_{\Delta,\infty}, V_{\Delta,\infty}, E_{\Delta,\infty} \subseteq E, V_{\Delta}^{(2)}, V_{\Delta,\infty}^{(2)} \) (etc.) as in Definition \( \square \).

Then
\[
\Theta^*(\gamma, x, t) \in U_{\Delta}^{(4)} \quad \text{for all} \quad \gamma \in P^*, \ x \in \partial \Delta \quad \text{and} \quad t \in [0, 1]. \quad (10)
\]
In fact, given \( x \in \partial \Delta \), there exists a proper face \( \Delta' \) of \( \Delta \) such that \( x \in \Delta' \). Now \( \Theta^*(\gamma, x, t) \in Z_{\Delta'} \subseteq U_{\Delta'}^{(4)} \), by definition of \( Q^* \) and \( Z_{\Delta'} \). The preceding enables us to define a map \( \Xi_\Delta : P^* \times [0, 1] \to C(\partial \Delta, E) \) via
\[
\Xi_\Delta(\gamma, t) := \phi_\Delta \circ \Theta^*(\gamma, \bullet, t)|_{\partial \Delta}.
\]
As a consequence of (10), the map \( \Xi \) has image in \( C(\partial \Delta, V^{(4)}_\Delta) \); and by Lemma 2.2(a)–(c), \( \Xi \) is continuous. We pick \( x_\Delta \in S \cap \partial \Delta \) and let

\[
\Phi_\Delta := \Phi : C(\partial \Delta, E) \to C(\Delta, E)
\]

be as in Lemma 4.1. By (8), the values of \( \Phi_\Delta(\gamma) \) lie in \( \text{conv}_2(\text{im}(\gamma)) \). Hence

\[
\Phi_\Delta(\gamma)(\Delta) \subseteq V^{(2)}_\Delta \quad \text{for each } \gamma \in [\partial \Delta, V^{(4)}_\Delta] \subseteq C(\partial \Delta, E).
\]

(11)

Because \( \gamma(x) \in U^{(4)}_\Delta \) and thus \( \phi_\Delta(\gamma(x)) \in V^{(4)}_\Delta \) for all \( \gamma \in P \subseteq R \) and \( x \in \Delta \), we can define a map \( \Theta_\Delta : P \times \Delta \times [0,1] \to U_\Delta \subseteq M \) via

\[
\Theta_\Delta(\gamma, x, t) := \left\{ \begin{array}{ll}
\phi_\Delta^{-1}((1 - 2t) \phi_\Delta(\gamma(x)) + 2t \Phi_\Delta(\phi_\Delta \circ \gamma|_{\partial \Delta})(x)) & \text{if } t \in [0, \frac{1}{2}]; \\
\phi_\Delta^{-1}(\Phi_\Delta(\Xi(\gamma|_{\Sigma^*}), 2t - 1))(x)) & \text{if } t \in [\frac{1}{2}, 1].
\end{array} \right.
\]

This map is continuous as a consequence of Lemma 2.2(a), (b) and (e). We now define a map \( \Theta : P \times |\Sigma| \times [0,1] \to M \), as follows: If \( x \in \Delta \) for some \( \Delta \in \Sigma \) such that \( \text{rk}(\Delta) = r \) and \( \Delta \not\subseteq \mathcal{E} \), we set

\[
\Theta(\gamma, x, t) := \Theta_\Delta(\gamma, x, t).
\]

If \( x \in \Delta \) for some \( \Delta \in \Sigma \) such that \( \text{rk}(\Delta) = r \) and \( \Delta \subseteq \mathcal{E} \), we set

\[
\Theta(\gamma, x, t) := \gamma(x).
\]

If \( x \in |\Sigma^*| \), we define

\[
\Theta(\gamma, x, t) := \left\{ \begin{array}{ll}
\gamma(x) & \text{if } t \in [0, \frac{1}{2}]; \\
\Theta^*(\gamma|_{\Sigma^*}, x, 2t - 1) & \text{if } t \in [\frac{1}{2}, 1].
\end{array} \right.
\]

If \( \Delta \in \Sigma \) is a simplex of rank \( r \), then \( \Theta_\Delta(\gamma, x, t) = \gamma(x) \) for all \( t \in [0, \frac{1}{2}] \) and \( x \in \partial \Delta \). Therefore \( \Theta \) is well defined. By the Glueing Lemma, \( \Theta \) is continuous. It remains to show that \( \Theta \) satisfies all of (a)–(h).

(a) Let \( \gamma \in P \) and \( x \in |\Sigma| \). If \( x \in |\Sigma^*| \), then \( \Theta(\gamma, x, 0) = \gamma(x) \) by definition of \( \Theta \). Otherwise, \( x \in \Delta \) for some \( \Delta \in \Sigma \) of rank \( r \). If \( \Delta \subseteq \mathcal{E} \), then \( \Theta(\gamma, x, 0) = \gamma(x) \) by definition of \( \Theta \). If \( \Delta \not\subseteq \mathcal{E} \), then \( \Theta(\gamma, x, 0) = \Theta_\Delta(\gamma, x, 0) = \gamma(x) \) by definition of \( \Theta_\Delta \).

(b) Let \( \gamma \in P \) and \( t \in [0, \frac{1}{2}] \). Let \( \Delta \in \Sigma \). If \( \text{rk}(\Delta) < r \), then \( x \in |\Sigma^*| \) for each \( x \in \Delta \) and thus \( \Theta(\gamma, x, t) = \gamma(x) \in Z_\Delta \subseteq U_\Delta \) (since \( P \subseteq R \)), i.e.,

\[
\Theta(\gamma, *, t) \in [\Delta, U_\Delta]. \quad (12)
\]
Now assume that \( \Delta \) has rank \( r \). If \( \Delta \subseteq \mathcal{E} \), then \( \Theta(\gamma, x, t) = \gamma(x) \in U_{\Delta} \). If \( \Delta \not\subseteq \mathcal{E} \), then \( \Theta(\gamma, x, t) = \Theta_{\Delta}(\gamma, x, t) \in U_{\Delta} \) for each \( x \in \Delta \), whence again (12) holds. Thus

\[
\Theta(\gamma, \bullet, t) \in \bigcap_{\Delta \in \Sigma} [\Delta, U_{\Delta}] \subseteq Q. \tag{13}
\]

Now let \( \gamma \in P \) and \( t \in [\frac{1}{2}, 1] \). Let \( \Delta \subseteq \Sigma \). If \( \text{rk}(\Delta) = r \), we see as before that (12) holds. If \( \text{rk}(\Delta) < r \), we exploit that \( \gamma|_{[\Sigma^{\ast}|, 2t - 1) \in \mathcal{E} \) holds. Thus (13) is established.

(c) It suffices to prove the second assertion. To this end, let \( \Delta \subseteq \Sigma \) and \( x \in \Delta \). If \( \Delta \subseteq \Sigma^{\ast} \), we have \( \Theta(\gamma, x, 1) = \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, x, 1) \), which only depends on \( \gamma|_{[\Sigma^{\ast}|} \cap \Delta \) by the inductive hypothesis. If \( \text{rk}(\Delta) = r \) and \( \Delta \not\subseteq \mathcal{E} \), then \( \Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1) = \phi_{\Delta} \circ \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{\partial \Delta} \) only depends on \( \gamma|_{[\Sigma^{\ast}|} \cap \Delta \) (because \( \partial \Delta \) is a union of proper faces \( \Delta' \) of \( \Delta \), and \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{\partial \Delta} \) only depends on \( \gamma|_{[\Sigma^{\ast}|} \cap \Delta \)). Hence also \( \Theta(\gamma, x, 1) = \Theta_{\Delta}(\gamma, x, 1) = \phi_{\Delta}^{-1}(\Phi_{\Delta}(\Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1))(x)) \) only depends on \( \gamma|_{[\Sigma^{\ast}|} \cap \Delta \). Finally, assume \( \text{rk}(\Delta) = r \) and \( \Delta \subseteq \mathcal{E} \). Then\( \eta_{\gamma}|_{\Delta} = \gamma|_{[\Sigma^{\ast}|} \cap \Delta \) only depends on \( \gamma|_{[\Sigma^{\ast}|} \cap \Delta \).

(d) It suffices to show that for each \( \Delta \subseteq \Sigma \), there exists \( \beta \geq \alpha \) such that \( \eta_{\gamma}(\Delta) \subseteq M_{\beta} \) and \( \eta_{\gamma}|_{\Delta}^{M_{\beta}} : \Delta \to M_{\beta} \) is continuous. If \( \text{rk}(\Delta) = r \) and \( \Delta \subseteq \mathcal{E} \), then the latter holds by hypothesis (with \( \beta := \alpha \)), since \( \eta_{\gamma}|_{\Delta} = \gamma|_{\Delta} \).

To tackle the remaining cases, we exploit that there is \( \tau \geq \alpha \) such that \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, [\Sigma^{\ast}|, 1) \subseteq M_{\tau} \) and \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{[\Sigma^{\ast}|} \cap \Delta \) is continuous, by the inductive hypothesis.

If \( \text{rk}(\Delta) < r \), then \( \eta_{\gamma}(\Delta) \subseteq \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, [\Sigma^{\ast}|, 1) \subseteq M_{\tau} \) and \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{[\Sigma^{\ast}|} \cap \Delta = \eta_{\gamma}|_{\Delta}^{M_{\tau}} \) is continuous, whence \( \beta := \tau \) satisfies our needs.

Now assume \( \text{rk}(\Delta) = r \) and \( \Delta \not\subseteq \mathcal{E} \). Since \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{\partial \Delta} \) has image in \( U_{\Delta}^{(2)} \cap M_{\tau} \) and is continuous as a map to \( M_{\tau} \), using Lemma 113 we find \( \sigma \geq \tau \) such that \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \partial \Delta, 1) \subseteq U_{\Delta}^{(2)} \) and \( \Theta^{\ast}(\gamma|_{[\Sigma^{\ast}|}, \bullet, 1)|_{\partial \Delta} \subseteq M_{\tau} \) is continuous as a map to \( U_{\Delta}^{(2)} \). As a consequence, \( \Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1) \subseteq C(\partial \Delta, V_{\Delta}^{(2)}) \). Now \( \Phi_{\Delta} \), restricted to \( C(\partial \Delta, E_{\Delta, \sigma}) \), is a map to \( C(\Delta, E_{\Delta, \sigma}) \) by Lemma 114(a) (applied with \( E_{\Delta, \sigma} \) rather than \( E = E_{\Delta} \)). Hence \( \phi_{\Delta} \circ \eta_{\gamma}|_{\Delta} = \Phi_{\Delta}(\Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1)) \in C(\Delta, E_{\Delta, \sigma}) \). The image \( K \) of \( \Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1) \) is a compact subset of \( V_{\Delta, \beta}^{(2)} \). Hence, by Definition 115(f), there exists \( \beta \geq \sigma \) such that \( \text{conv}_{\beta}(K) \subseteq V_{\Delta, \beta}^{(2)} \). As a consequence, \( \eta_{\gamma}(x) = \phi_{\Delta, \beta}^{-1}(\Phi_{\Delta}(\Xi_{\Delta}(\gamma|_{[\Sigma^{\ast}|}, 1))(x)) \in U_{\Delta, \beta}^{(2)} \) for each \( x \in \Delta \) and
\(\eta\gamma|\Delta\) is continuous as a map to \(U_{\Delta,\beta}\).

(e) \(F_\gamma\) is continuous and hence is a homotopy from \(F_\gamma(\bullet, 0) = \gamma\) (see (a)) to \(F_\gamma(\bullet, 1) = \eta\gamma\).

(f) Define \(F_\gamma^* := \Theta^*(\gamma|_{[\Sigma^*], \bullet})\): \([\Sigma^*] \times [0, 1] \to M\). It suffices to show that for each \(\Delta \in \Sigma\), there exists \(\beta \geq \alpha\) such that \(F_\gamma(\Delta \times [0, 1]) \subseteq M_\beta\) and \(F_\gamma|_{\Delta \times [0, 1]}: \Delta \times [0, 1] \to M_\beta\) is continuous. If \(\text{rk}(\Delta) = r\) and \(\Delta \nsubseteq \mathcal{E}\), then \(F_\gamma(x, t) = \gamma(x) \in M_\alpha\); since \(\gamma|_{M_\alpha}\) is continuous by hypothesis, the desired property is satisfied with \(\beta := \alpha\). To tackle the remaining cases, we shall exploit that there exists \(\tau \geq \alpha\) such that \(F_\gamma^*([\Sigma^*] \times [0, 1]) \subseteq M_\tau\) and \(F_\gamma^*\) is continuous as a map to \(M_\tau\), by the inductive hypothesis.

If \(\text{rk}(\Delta) < r\), then \(F_\gamma(\Delta \times [0, 1]) \subseteq F_\gamma^*([\Sigma^*] \times [0, 1]) \subseteq M_\tau\) holds and \(F_\gamma|_{\Delta \times [0, 1]}\) is continuous as \(F_\gamma(\bullet, t)|_\Delta = \gamma^M_\tau\) if \(t \in [0, \frac{1}{2}]\) and \(F_\gamma(\bullet, t)|_\Delta = F_\gamma^*(\bullet, 2t - 1)|^M_\Delta\) if \(t \in [\frac{1}{2}, 1]\). Hence \(\tau := t\) does the job.

Now assume \(\text{rk}(\Delta) = r\) and \(\Delta \nsubseteq \mathcal{E}\). Since \(F_\gamma^*|_{\partial \Delta \times [0, 1]} = \Theta^*(\gamma|_{[\Sigma^*], \bullet})|_{\partial \Delta \times [0, 1]}\) has image in \(U^{(2)}_\Delta\) (by (11)) and is continuous as a map to \(M_\tau\), using Lemma 4.3 we find \(\sigma \geq \tau\) such that \(F_\gamma^*(\partial \Delta \times [0, 1]) \subseteq U^{(2)}_{\Delta, \sigma}\) and \(F_\gamma^*|_{\partial \Delta \times [0, 1]}\) is continuous as a map to \(U^{(2)}_{\Delta, \sigma}\). Thus \(\Xi_\Delta(\{\gamma|_{[\Sigma^*]}\} \times [0, 1]) \subseteq C(\partial \Delta, V^{(2)}_{\Delta, \sigma})\). The mapping \(\partial \Delta \times [0, 1] \to V^{(2)}_{\Delta, \sigma}, (x, t) \mapsto \phi_\Delta(F_\gamma^*(x, t)) = \Xi_\Delta(\gamma|_{[\Sigma^*]}, t)(x)\) is continuous and has compact image \(K \subseteq V^{(2)}_{\Delta, \sigma}\). By Definition 1.7(f), there is \(\beta \geq \sigma\) such that \(\text{conv}_2(K) \subseteq V_{\Delta, \beta}\). Now \(\Phi_\Delta\), restricted to \(C(\partial \Delta, E_{\Delta, \sigma})\), is continuous as a map to \(C(\Delta, E_{\Delta, \sigma})\) c.o. by Lemma 4.1(a) (applied with \(E_{\Delta, \sigma}\) rather than \(E = E_\Delta\)), and hence also as a map to \(C(\Delta, E_{\Delta, \beta})\) c.o. Furthermore, \(\Phi_\Delta(\Xi_\Delta(\gamma|_{[\Sigma^*]}, t))(x) \in \text{conv}_2(K) \subseteq V_{\Delta, \beta}\) for each \(t \in [0, 1]\) and \(x \in \Delta\), by choice of \(\beta\). As a consequence, \(F_\gamma: \Delta \times [\frac{1}{2}, 1] \to M_\beta, (x, t) \mapsto \phi^{-1}_{\Delta, \beta}(\Phi_\Delta(\Xi_\Delta(\gamma|_{[\Sigma^*]}, 2t - 1))(x))\) is a continuous map to \(U_{\Delta, \beta} \subseteq M_\beta\).

To tackle also the case \(t \in [0, \frac{1}{2}]\), note that we may assume that \(U^1_{\Delta, \gamma}\) has been obtained by applying the construction from the proof of Lemma 4.4 to \(\overline{\phi} := \phi_\Delta\) and \(W := U^{(2)}_\Delta\). Hence, we may assume that the restriction of \(\phi_\Delta\) to some subset of \(U^{(2)}_\Delta\) is a well-filled chart with core \(U^{(4)}_\Delta\), and \(V^{(4)}_\Delta := \phi_\Delta(U^{(4)}_\Delta) = \bigcap_{\theta \geq 0} V^{(4)}_{\Delta, \theta}\). Since \(\gamma(\Delta) \subseteq U^{(4)}_\Delta\) and \(\gamma|\Delta\) is a continuous map to \(M_\alpha\), after increasing \(\sigma\) (and \(\beta\)) we may assume that \(\gamma(\Delta)\) is a compact subset of \(U^{(4)}_{\Delta, \sigma} := \phi^{-1}_{\Delta, \sigma}(V^{(4)}_{\Delta, \sigma})\), by Lemma 4.3. Thus \(L := \phi_\Delta(\gamma(\Delta))\) is a compact subset of \(V^{(4)}_{\Delta, \sigma}\). Since \(V^{(4)}_{\Delta, \sigma} \subseteq V^{(2)}_{\Delta, \sigma}\) (by the construction in the proof of Lemma 4.4), after increasing \(\beta\) if necessary we may assume that
conv}_2(L) \subseteq V_{\Delta,\beta} \) (by Definition 1.7(f)). Moreover, \( \text{conv}_2(L) \subseteq \text{conv}_2(V_\Delta^{(4)}) \subseteq \phi_\Delta(W) = V_\Delta^{(2)} \). Thus \( \phi_\Delta^{-1}(\text{conv}_2(L)) \) is a compact subset of \( M_\beta \) which is contained in \( U_\Delta^{(2)} \). In view of Lemma 4.3 after increasing \( \beta \) we may assume that \( \phi_\Delta^{-1}(\text{conv}_2(L)) \) is a compact subset of \( U_\Delta^{(2)} \), and thus \( \text{conv}_2(L) \subseteq V_\Delta^{(2)} \).

Hence \( \Phi_\Delta(\phi_\Delta \circ \gamma|_{\partial \Delta}) \in C(\Delta, \text{conv}_2(L)) \subseteq C(\Delta, V_\Delta^{(2)}) \) (exploiting that \( \Phi_\Delta \) takes \( C(\partial \Delta, E_{\Delta,\beta}) \) to \( C(\Delta, E_{\Delta,\beta}) \)). Since \( \phi_\Delta \) is a well-filled chart, using Definition 1.7(f) we see that \( \text{conv}_2(\text{conv}_2(L)) \subseteq V_{\Delta,\beta} \) may be assumed after increasing \( \beta \) further. Thus \( \text{conv}_2(L, \text{conv}_2(L)) \subseteq V_{\Delta,\beta} \) in particular and we obtain a continuous map \( \Delta \times [0,\frac{1}{2}] \rightarrow V_{\Delta,\beta} \subseteq E_{\Delta,\beta}, \)

\[
(x, t) \mapsto (1 - 2t) \phi_\Delta(\gamma(x)) + 2t \Phi_\Delta(\phi_\Delta \circ \gamma|_{\partial \Delta})(x).
\]

Hence \( F_\gamma(x, t) = \Theta_\Delta(\gamma(x, t) = \phi_\Delta^{-1}((1 - 2t) \phi_\Delta(\gamma(x)) + 2t \Phi_\Delta(\phi_\Delta \circ \gamma|_{\partial \Delta})(x)) \in U_{\Delta,\beta} \) holds, and \( F_\gamma: \Delta \times [0,\frac{1}{2}] \rightarrow U_{\Delta,\beta} \) is continuous.

(g) Define \( F_\gamma^* \) as in the proof of (f). If \( \Delta \in \Sigma^* \) in the situation of (g), then \( F_\gamma(x, t) = \gamma(x) = y \) for each \( x \in \Delta \) if \( t \in [0,\frac{1}{2}] \), while \( F_\gamma(x, y) = F_\gamma^*(x, 2t - 1) = y \) if \( t \in \left[\frac{1}{2},1 \right] \), by the inductive hypothesis.

Now assume that \( \text{rk}(\Delta) = r \). If \( \Delta \subseteq \mathcal{E} \), then \( F_\gamma(t, x) = \gamma(x) = y \) for each \( x \in \Delta \) and \( t \in [0,1] \). If \( \Delta \not\subseteq \mathcal{E} \) and \( t \in [0,\frac{1}{2}] \), given \( x \in \Delta \) we have that \( \phi_\Delta(F_\gamma(x, t)) \) is a convex combination of the vectors \( \phi_\Delta(\gamma(x')) = \phi_\Delta(y) \) for several \( x' \in \Delta \), and thus \( F_\gamma(x, t) = y \). If \( t \in \left[\frac{1}{2},1 \right] \) and \( x \in \Delta \), then \( F_\gamma(x, t) \) is the image under \( \phi_\Delta^{-1} \) of a convex combination of the vectors \( \phi_\Delta(\gamma(x', t)) = \phi_\Delta(y) \) with \( x' \in \partial \Delta \). Since any such convex combination is \( \phi_\Delta(y) \), it follows that \( F_\gamma(x, t) = y \).

(h) Let \( \gamma \in P, x \in \mathcal{S} \) and \( t \in [0,1] \). Then \( x \in |\Sigma^*| \). Hence \( F_\gamma(x, t) = \gamma(x) \) if \( t \in [0,\frac{1}{2}] \), while \( F_\gamma(x, t) = F_\gamma^*(x, 2t - 1) = \gamma(x) \) if \( t \in \left[\frac{1}{2},1 \right] \), by induction.

Now take \( x \in \mathcal{E} \). Then \( x \in \Delta \) for some \( \Delta \in \Sigma \) such that \( \Delta \subseteq \mathcal{E} \). If \( \text{rk}(\Delta) = r \), then \( F_\gamma(x, t) = \gamma(x) \) by definition of \( \Theta \). If \( \text{rk}(\Delta) < r \), then \( \Delta \subseteq \Sigma^* \) and we see as in the case \( x \in \mathcal{S} \) that \( F_\gamma(x, t) = \gamma(x) \). \( \square \)

For a single map \( \gamma_0: |\Sigma| \rightarrow M \), we can deduce stronger conclusions.

**Lemma 4.7 (Individual Approximations)** In the setting of 4.2, let \( \Sigma \) be a finite simplicial complex, \( \gamma_0: |\Sigma| \rightarrow M \) be a continuous function and \( Q \subseteq C(|\Sigma|, M)_{c.o.} \) be a neighbourhood of \( \gamma_0 \). Let \( \mathcal{E} \subseteq |\Sigma| \) be a subset such that \( \mathcal{E} = \bigcup \{ \Delta \in \Sigma: \Delta \subseteq \mathcal{E} \} \). Assume that there exists \( \alpha \in A \) such that \( \gamma_0(\mathcal{E}) \subseteq M_\alpha \) and \( \gamma_0|_{\mathcal{E}} \) is continuous as a map to \( M_\alpha \). Then there exists
\[ \beta \geq \alpha \text{ and a continuous map } \eta : |\Sigma| \to M_\beta \text{ such that } \eta|_E = \gamma_0|_E \text{ and } \eta \in Q. \]
Moreover, there exists a homotopy \( H : |\Sigma| \times [0, 1] \to M \) relative \( E \) from \( \gamma_0 \) to \( \eta \), such that \( H(\bullet, t) \in Q \) for each \( t \in [0, 1] \).

**Proof.** Let \( P, S, \Theta \) and further notation be as in Lemma 4.6 applied to the given data \( \gamma_0, Q \) and \( E \). Abbreviate \( X := |\Sigma| \). We claim:

There exists a homotopy \( G : X \times [0, 1] \to M \) relative \( E \) from \( \gamma_0 \) to some \( \gamma : X \to M \) such that \( \gamma(x) \in M_\infty \) for all \( x \in S \) and \( G(\bullet, t) \in P \) for \( t \in [0, 1] \).

If this is true, then \( \eta := \eta_\gamma \) (from Lemma 4.6(c)) is a continuous map from \( X \) to some \( M_\beta \), by Lemma 4.6(d) \footnote{Using that \( \gamma|_E = \gamma_0|_E \) is a continuous map to \( M_\alpha \).} Furthermore, the map

\[ H : X \times [0, 1] \to M, \quad H(x, t) := \Theta(G(\bullet, t), x, t) \]

is a homotopy from \( \gamma_0 \) to \( \eta \) (cf. Lemma 2.2(c) for the continuity of \( H \)), and in fact a homotopy relative \( E \), because \( G(\bullet, t)|_E = \gamma|_E \) and hence \( \Theta(G(\bullet, t), x, t) = G(x, t) = \gamma(x) \) for each \( x \in E \) and \( t \in [0, 1] \), by Lemma 4.6(h).

**Proof of the claim.** There exist \( \ell \in \mathbb{N} \), compact sets \( K_1, \ldots, K_\ell \subseteq X \) and open subsets \( W_1, \ldots, W_\ell \subseteq M \) such that \( \gamma_0 \in \bigcap_{j=1}^\ell (K_j \setminus W_j) \subseteq P \). Given \( x \in S \), let \( I_x \) be the set of all \( j \in \{1, \ldots, \ell\} \) such that \( x \in K_j \), define \( J_x := \{1, \ldots, \ell\} \setminus I_x \), and \( G_x := X \setminus \bigcup_{j \in J_x} K_j \). By Lemma 4.4, for each \( x \in S \) there is a well-filled chart \( \phi_x : U_x \to V_x \) such that \( \gamma_0(x) \in U_x^{(2)} \) for some core \( U_x^{(2)} \) of \( \phi_x \), and \( U_x \subseteq \bigcap_{j \in I_x} W_j \) (if \( I_x = \emptyset \), we define the preceding intersection as \( M \)). We choose \( U_x^{(4)} \subseteq U_x^{(2)} \) with \( \gamma_0(x) \in U_x^{(4)} \) as in Lemma 4.5 and set \( V_x^{(4)} := \phi_x(U_x^{(4)}) \). Pick a metric \( d \) on \( X \) defining its topology. There exists \( \varepsilon > 0 \) such that the closed \( d \)-balls \( \overline{B}_\varepsilon(x) \subseteq X \) for \( x \in S \) are pairwise disjoint, \( \overline{B}_\varepsilon(x) \subseteq G_x \), and \( \gamma_0(\overline{B}_\varepsilon(x)) \subseteq U_x^{(4)} \). Set \( T := \{x \in S : \gamma_0(x) \notin M_\infty\} \). Since \( \gamma_0(E) \subseteq M_\infty \), we then have \( T \subseteq X \setminus E \). Hence, after shrinking \( \varepsilon \) further if necessary, we may assume that

\[ E \cap \bigcup_{x \in T} \overline{B}_\varepsilon(x) = \emptyset. \quad (14) \]

Given \( x \in S \), pick \( v_x \in V_x^{(4)} \cap V_{x, \infty} \), where \( V_{x, \infty} = \bigcup_{\alpha \geq \alpha_0} V_{x, \alpha} \) is as in Definition 4.7(e). We define \( G : X \times [0, 1] \to M \) for \( z \in X \) and \( t \in [0, 1] \) as follows: If \( z \in X \setminus \bigcup_{x \in T} B_\varepsilon(x) \), we set

\[ G(z, t) := \gamma_0(z). \]
If \( z \in B(x) \) for some \( x \in T \), we set
\[
G(z,t) := \phi_x^{-1} \left( t \left( \frac{d(z,x)}{\varepsilon} \right) v_x + \frac{d(z,x)}{\varepsilon} \phi_x(\gamma_0(z)) + (1-t)\phi_x(\gamma_0(z)) \right).
\]
Then \( G \) is continuous, and \( \gamma := G(\bullet,1) \) satisfies \( \gamma(x) \in M_\infty \) for all \( x \in S \).
In fact: \( \gamma(x) = \gamma_0(x) \in M_\infty \) if \( x \in S \setminus T \), while \( \gamma(x) = \phi_x^{-1}(v_x) \in M_\infty \) if \( x \in T \). If \( z \in E \) and \( t \in [0,1] \), then \( G(z,t) = \gamma_0(z) \), by (14) and definition of \( G \). Hence \( G \) is a homotopy relative \( E \) from \( \gamma_0 \) to \( \gamma \). Finally, we have \( \zeta := G(\bullet,t) \in P \) for each \( t \in [0,1] \). To see this, let \( j \in \{1,\ldots,\ell\} \) and \( z \in K_j \).

5 The main result and first consequences

We shall deduce Theorem 1.13 from a more general theorem dealing with sets \([(X,C),(M,p)]\) of homotopy classes.

5.1 If \( X \) and \( Y \) are topological spaces, \( C \subseteq X \) a closed set and \( p \in Y \), let
\[
[(X,C),(Y,p)]
\]
be the set of all equivalence classes \([\gamma]\) of continuous mappings \( \gamma: X \to Y \) such that \( \gamma|_C = p \), using homotopy relative \( C \) as the equivalence relation. If also \( Z \) is a topological space, \( q \in Z \) and \( f: Y \to Z \) is a continuous map such that \( f(p) = q \), we obtain a map
\[
[(X,C),f]: [(X,C),(Y,p)] \to [(X,C),(Z,q)] , \quad [\gamma] \mapsto [f \circ \gamma].
\]
We simply write \( f_* := [(X,C),f] \) if the meaning is clear from the context. If \( C = \emptyset \), it is customary to write \([X,Y]\) instead of \([(X,C),(Y,p)]\).
5.2 Now let $F$ be a finite-dimensional vector space, $\Sigma$ be a finite simplicial complex of simplices in $F$ and $X := |\Sigma| \subseteq F$. Let $C \subseteq X$ be a subset which is a union of simplices, i.e., $C = \bigcup \{\Delta \in \Sigma : \Delta \subseteq C\}$.

**Theorem 5.3** Let $X = |\Sigma|$ and $C \subseteq X$ be as in 5.2. $M$ a topological space and $(M_\alpha)_{\alpha \in A}$ be a directed family of topological spaces whose union $M_\infty := \bigcup_{\alpha \in A} M_\alpha$ is dense in $M$. Assume that all inclusion maps $\lambda_\alpha : M_\alpha \to M$ and $\lambda_{\beta,\alpha} : M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous. For $p \in M_\infty$, abbreviate $A_p := \{\alpha \in A : p \in M_\alpha\}$. If $M$ admits well-filled charts, then

$$[(X, C), (M, p)] = \lim_{\rightarrow_{\alpha \in A_p}} [(X, C), (M_\alpha, p)]$$

as a set, for each $p \in M_\infty$.

**Proof.** We may assume that $p \in M_\alpha$ for each $\alpha \in A$. The sets $[(X, C), (M, p)]$ form a direct system $\mathcal{S}$ of sets, with the bonding maps $(\lambda_{\beta,\alpha})_* := [(X, C), \lambda_{\beta,\alpha}]$. We let $D := \lim_{\rightarrow_{\alpha \in A}} [(X, C), (M_\alpha, p)]$ be the direct limit in the category of sets, with limit maps $\mu_\alpha : [(X, C), (M_\alpha, p)] \to D$. Since the maps $(\lambda_\alpha)_* : [(X, C), (M_\alpha, p)] \to [(X, C), (M, p)]$ form a cone over $\mathcal{S}$, there is a unique map $\psi : D \to [(X, C), (M_\alpha, p)]$ with $\psi \circ \mu_\alpha = (\lambda_\alpha)_*$ for all $\alpha \in A$.

$\psi$ is surjective. Let $[\gamma_0] \in [(X, C), (M, p)]$ be the equivalence class of a continuous map $\gamma_0 : X \to M$ with $\gamma_0|_C = p$. Applying Lemma 4.7 with $Q := C(X, M)$ and $\mathcal{E} := C$, we obtain $\beta \in A$ and a homotopy $H : X \times [0, 1] \to M$ relative $C$ from $\gamma_0$ to some continuous map $\eta : X \to M_\beta$. Then $[\gamma_0] = [\eta] = (\lambda_\beta)_*([\eta]) = \psi(\mu_\alpha([\eta])) \in \im(\psi)$. Thus $\psi$ is surjective.

$\psi$ is injective. To see this, let $g, h \in D$ with $\psi(g) = \psi(h)$. There is $\alpha \in A$ such that $g = \mu_\alpha([\sigma])$ and $h = \mu_\alpha([\tau])$ for certain $[\sigma], [\tau] \in [(X, C), (M_\alpha, p)]$ with continuous maps $\sigma, \tau : X \to M_\alpha$. Then $(\lambda_\alpha)_*([\sigma]) = (\lambda_\alpha)_*([\tau])$, whence there is a homotopy $\gamma_0 : X \times [0, 1] \to M$ relative $C$ from $\sigma$ to $\tau$, considered as maps to $M$. Choose a triangulation $\Sigma'$ of $X \times [0, 1] \subseteq F \times \mathbb{R}$ such that

$$C \times [0, 1] = \bigcup \{\Delta \in \Sigma' : \Delta \subseteq C \times [0, 1]\}$$

and

$$X \times \{0, 1\} = \bigcup \{\Delta \in \Sigma' : \Delta \subseteq X \times \{0, 1\}\}$$

(this is always possible, by standard arguments). Applying Lemma 4.6 to $\Sigma'$, $\gamma_0$, $Q := C(X \times [0, 1], M)$ and

$$\mathcal{E} := (C \times [0, 1]) \cup (X \times \{0, 1\}),$$

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we obtain $\beta \geq \alpha$, a continuous map $\eta: X \times [0,1] \to M_\beta$ and a homotopy $H: (X \times [0,1]) \times [0,1] \to M$ relative $\mathcal{E}$ from $\gamma_0$ to $\eta$. Because $H$ is a homotopy relative $\mathcal{E}$, we have

$$\eta(x,0) = \gamma_0(x,0) = \sigma(x)$$

and $\eta(x,1) = \gamma_0(x,1) = \tau(x)$ for all $x \in X$, and furthermore

$$\eta(x,t) = \gamma_0(x,t) = p \text{ for all } x \in C \text{ and } t \in [0,1].$$

Hence $\eta$ is a homotopy relative $C$ from $\sigma$ to $\tau$, considered as maps to $M_\beta$. Consequently, $[\sigma] = [\tau]$ in $[(X,C),(M,P)]$ and thus $g = (\lambda_\beta)_*([\sigma]) = (\lambda_\beta)_*([\tau]) = h.$

Remark 5.4 Theorem 5.3 and its proof easily extend to sets $[(X,C),(M,P)]$ of homotopy classes of mappings between space pairs, where $X, C$ and $M$ are as before and $P \subseteq M$ is a subset such that $P \subseteq M_\theta$ for some $\theta \in A$ and both $M$ and $M_\theta$ induce the same topology on $P$.

Proof of Theorem 1.13. Let $D := \lim \pi_k(M_\alpha,p)$ and $\psi: D \to \pi_k(M,p)$ be as in 3.4. If $k \geq 1$ or if $M$ and each $\bar{M}_\alpha$ is a topological group and each $\lambda_\alpha$ and $\lambda_{\beta,\alpha}$ a homomorphism, then also $\psi$ is a homomorphism of groups. Since $\psi$ is a bijection by Theorem 5.3 (and hence an isomorphism of groups in the cases just described), Theorem 1.13 is established.

We record another simple consequence. It mainly is of interest if a manifold $M$ is a directed union of manifolds admitting weak direct limit charts.

Corollary 5.5 Let $M$ be a topological space and $(M_\alpha)_{\alpha \in A}$ be a directed family of topological spaces such that $M = \bigcup_{\alpha \in A} M_\alpha$. Assume that all inclusion maps $M_\alpha \to M$ and $M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous, and that $M$ admits well-filled charts. Then the path components of $M$ are the unions of those of the steps:

$$M(p) = \bigcup_{\alpha \in A_p} (M_\alpha)_p \text{ for all } p \in M. \quad (15)$$

Proof. Let $p \in M$, say $p \in M_\alpha$. It is clear that $\bigcup_{\beta \geq \alpha} (M_\beta)_p \subseteq M(p)$. To prove the converse inclusion, let $q \in M(p)$. There exists $\beta \geq \alpha$ such that $q \in M_\beta$. Since

$$(\lambda_\alpha)_*((M_\alpha)_p) = M(p) = M(q) = (\lambda_\beta)_*((M_\beta)_q)$$

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and \( \pi_0(M) = \lim \pi_0(M_\gamma) \) by Theorem 1.13 (applied with \( k = 0 \)), there exists \( \gamma \geq \alpha, \beta \) such that

\[
(\lambda_{\gamma,\alpha})_*((M_\alpha)(p)) = (\lambda_{\gamma,\beta})_*((M_\beta)(q))
\]

(see (16)), where we use the natural mappings \((\lambda_\alpha)_*: \pi_0(M_\alpha) \to \pi_0(M)\), \((\lambda_\beta)_*: \pi_0(M_\beta) \to \pi_0(M)\), \((\lambda_{\gamma,\alpha})_*: \pi_0(M_\alpha) \to \pi_0(M_\gamma)\) as well as \((\lambda_{\gamma,\beta})_*: \pi_0(M_\beta) \to \pi_0(M_\gamma)\). Thus \((M_\gamma)(p) = (M_\gamma)(q)\) and thus \( q \in (M_\gamma)(p) \), entailing that equality holds in (15).

If \( M_\infty := \bigcup_{\alpha \in A} M_\alpha \) is merely dense in \( M \), the same argument shows that \((M_\infty)(p) = \bigcup_{\alpha \in A_p} (M_\alpha)(p)\) for each \( p \in M_\infty \).

6 When the inclusion map is a weak homotopy equivalence

We now extend Palais’ result recalled in the introduction: under suitable hypotheses, the inclusion map \( M_\infty \to M \) is a weak homotopy equivalence.

**Proposition 6.1** Assume that \( M \) admits well-filled charts in the situation of Theorem 1.13 and that \( \mathcal{O} \) is a topology on \( M_\infty \) with the following properties:

(a) All of the inclusion maps \( \sigma_\alpha: M_\alpha \to (M_\infty, \mathcal{O}) \) (for \( \alpha \in A \)) as well as \( \sigma: (M_\infty, \mathcal{O}) \to M \) are continuous;

(b) \( (M_\infty, \mathcal{O}) = \bigcup_{\alpha \in A} M_\alpha \) is compactly retractive.

Then \( \sigma \) is a weak homotopy equivalence.

**Proof.** We shall re-use notation from 3.4. Let \( k \in \mathbb{N}_0 \) and \( p \in M_\infty \); equip \( M_\infty \) with the topology \( \mathcal{O} \). We have to show that \( \sigma_*: \pi_k(M_\infty, p) \to \pi_k(M, p) \) is a bijection.

\( \sigma_* \) is surjective. If \( g \in \pi_k(M, p) \), then \( g = (\lambda_\alpha)_*(h) \) for some \( \alpha \in A_p \) and \( h \in \pi_k(M_\alpha, p) \), by Theorem 1.13. Since \( \lambda_\alpha = \sigma \circ \sigma_\alpha \), it follows that \( g = \sigma_*((\lambda_\alpha)_*(h)) \) is in the image of \( \sigma_* \).

\( \sigma_* \) is injective. To see this, let \( [\gamma_1], [\gamma_2] \in \pi_k(M_\infty, p) \) such that \( \sigma_*([\gamma_1]) = \sigma_*([\gamma_2]) \). By compact reactivity of \( M_\infty = \bigcup_{\alpha \in A} M_\alpha \), there exists \( \alpha \in A \) such that both \( \gamma_1 \) and \( \gamma_2 \) have image in \( M_\alpha \) and their corestrictions \( \eta_i := \ldots \)
\[ \gamma_j |_{M_\alpha} \text{ are continuous for } j \in \{1, 2\}. \] Then \([\gamma_j] = (\sigma_\alpha)_*([\eta_j])\) and hence
\[ \psi(\mu_\alpha([\eta_j])) = (\lambda_\alpha)_*([\eta_j]) = \sigma_*((\sigma_\alpha)_*([\eta_j])) = \sigma_*([\gamma_j]), \] implying \(\psi(\mu_\alpha([\eta_1])) = \psi(\mu_\alpha([\eta_2]))\).

Since \(\psi\) is bijective, it follows that \(\mu_\alpha([\eta_1]) = \mu_\alpha([\eta_2])\) and thus \((\lambda_\beta, \alpha)_*([\eta_1]) = (\lambda_\beta, \alpha)_*([\eta_2])\) for some \(\beta \geq \alpha\) (see (6)). Because \(\sigma_\alpha = \sigma_\beta \circ \lambda_\beta, \alpha\) and hence \([\gamma_j] = (\sigma_\alpha)_*([\eta_j]) = (\sigma_\beta)_*((\lambda_\beta, \alpha)_*([\eta_j]))\) for \(j \in \{1, 2\}\), we deduce that \([\gamma_1] = [\gamma_2]\).

Now Corollary 1.14 (and slightly more) readily follows.

**Proof of Corollary 1.14**

Let \(A := \mathcal{F}\) be the set of all finite-dimensional vector subspaces \(F\) of \(E_\infty\). If \(U\) is open, then \(\phi := \text{id}_U : U \to U \subseteq E\) is a well-filled chart of \(U\) such that each \(q \in U\) is contained in some core of \(\phi\) (see Example 1.11(ii)), with \(U_F := V_F := U \cap F\) and \(\phi_F := \text{id}_{U_F}\). If \(U\) is semi-locally convex, then each \(q \in U\) has a convex relatively open neighbourhood \(W \subseteq U\). Then \(W \cap E_\infty\) is dense in \(W\), and \(\phi := \text{id}_W : W \to W \subseteq E\) is a well-filled chart such that \(q\) is contained in some core of \(\phi\), by Example 1.12 with \(U_F := V_F := W \cap F\) and \(\phi_F := \text{id}_{U_F}\). We are therefore in the situation of Theorem 1.13. Let \(\mathcal{T}\) be the topology \(\mathcal{O}\) on \(U_\infty\) described in Corollary 1.14.

Or, more generally, let \(\mathcal{T}\) be any topology on \(U_\infty\) which is coarser than the direct limit topology on \(\lim U \cap F\) but finer than the topology induced by the finest vector topology on \(E_\infty\) (if \(E\) is locally convex, one can also use the finest locally convex vector topology as a lower bound). Then \(U_\infty = \bigcup_{F \in \mathcal{F}} (U \cap F)\) is compactly retractive because so is \(E_\infty = \bigcup_{F \in \mathcal{F}} F\) with the finest locally convex topology (see, e.g., [33, Proposition 7.25(iv)]). Thus Proposition 6.1 applies: The inclusion map \((U_\infty, \mathcal{T}) \to U\) is a weak homotopy equivalence.

**Remark 6.2** Many criteria for compact retractive are known.

(a) For example, the direct limit topology on the union \(M = \bigcup_{n \in \mathbb{N}} M_n\) of an ascending sequence \(M_1 \subseteq M_2 \subseteq \cdots\) of Hausdorff topological spaces is compactly retractive if the direct sequence is strict in the sense that each inclusion map \(M_n \to M_{n+1}\) is a topological embedding (e.g., by [25, Lemma 1.7(d)] combined with [41, Lemma A.5]; cf. also [31]).

Further conditions (beyond strictness) arise from the reduction to modelling spaces performed in Proposition 3.5(a). On the level of locally convex spaces,
various criteria for compact retractivity are known. One such criterion was already encountered in preceding proof. Here are further ones:

(b) The locally convex direct limit topology on $E = \bigcup_{n \in \mathbb{N}} E_n$ is compactly retractive for each strict ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of complete locally convex topological vector spaces (cf. Proposition 9 (i) and (ii) in [9, Ch. II, §4, no. 6] and Proposition 6 in [9, Ch. III, §1, no. 4]).

(c) The locally convex direct limit topology on $E = \bigcup_{n \in \mathbb{N}} E_n$ is compactly retractive for each ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that all inclusion maps $E_n \to E_{n+1}$ are compact operators (see Proposition 7 in [9, Ch. III, §1, no. 4], or [16]). In this situation, $E$ is called a Silva space (or also a DFS-space).

(d) For (LF)-spaces, a quite concrete characterization of compact retraction is given in [56, Theorem 6.4]: Let $E_1 \subseteq E_2 \subseteq \cdots$ be Fréchet spaces, with continuous linear inclusion maps. Equip $E = \bigcup_{n \in \mathbb{N}} E_n$ with the locally convex direct limit topology. Then $E = \bigcup_{n \in \mathbb{N}} E_n$ is compactly retractive if and only if for each $n \in \mathbb{N}$, there exists $m \geq n$ such that for all $k \geq m$, there is a 0-neighbourhood $U$ in $E_n$ on which $E_k$ and $E_m$ induce the same topology. In this case, $E$ is also regular and complete [56, Corollary to Theorem 6.4].

Further criteria and references to the research literature can be found in [8].

7 Applications to typical Lie groups that are directed unions of Lie groups or manifolds

In this section, we show that our techniques apply to all major classes of examples of Lie groups $G$ which are an ascending union $G = \bigcup_{n \in \mathbb{N}} G_n$ of Lie groups or manifolds $G_n$ (as compiled in [26]).

In Examples 7.1–7.6 we shall see that $G = \bigcup_{n \in \mathbb{N}} G_n$ has a weak direct limit chart and $L(G) = \bigcup_{n \in \mathbb{N}} L(G_n)$ is compactly retractive, whence $G = \bigcup_{n \in \mathbb{N}} G_n$ is compactly retractive (by Proposition 3.5(a)). Hence Proposition 3.3 gives information both concerning the homotopy groups and the singular homology groups of $G$. In Example 7.7 the same reasoning applies to certain Lie groups $G$ which can be written as a union $G = \bigcup_{n \in \mathbb{N}} M_n$ of Banach manifolds. In Example 7.8 compact retractivity can be violated, but the group still has
a direct limit chart and thus Theorem 1.2 provides information concerning the homotopy groups.

**Example 7.1** (*Direct limits of finite-dimensional Lie groups*). Consider an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional Lie groups, such that the inclusion maps $G_n \to G_{n+1}$ are smooth homomorphisms. Give $G = \bigcup_n G_n$ the Lie group structure making it the direct limit Lie group $\lim_{\to} G_n$ (see [25, Theorem 4.3], or also [40] and [21] in special cases). Then $G$ has a direct limit chart by construction and $L(G) = \lim_{\to} L(G_n)$ is compactly retractive (see Remark 6.2 (a) or (b)).

**Example 7.2** (*Groups of compactly supported diffeomorphisms*). If $M$ is a $\sigma$-compact, finite-dimensional smooth manifold, there exists a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of $M$ such that $M = \bigcup_{n \in \mathbb{N}} K_n$ and $K_n \subseteq K_{n+1}^0$ (the interior in $M$) for each $n \in \mathbb{N}$. Then $(K_n)_{n \in \mathbb{N}}$ is a cofinal subsequence of the directed set $\mathcal{K}$ of all compact subsets of $M$. Let $\text{Diff}_c(M)$ be the Lie group of all $C^\infty$-diffeomorphisms $\gamma : M \to M$ such that the closure of $\{x \in M : \gamma(x) \neq x\}$ (the support of $\gamma$) is compact; this Lie group is modelled on the LF-space $\mathcal{V}_c(M)$ of compactly supported smooth vector fields on $M$. Given $K \in \mathcal{K}$, let $\text{Diff}_K(M)$ be the Lie group of all $\gamma \in \text{Diff}_c(M)$ supported in $K$, modelled on the Fréchet space $\mathcal{V}_K(M)$ of smooth vector fields supported in $K$ (cf. [30], [27] and [29] for the Lie group structures on these groups). Then

$$\text{Diff}_c(M) = \bigcup_{K \in \mathcal{K}} \text{Diff}_K(M)$$

and $\text{Diff}_c(M)$ admits a direct limit chart (cf. [26, §5.1]). Moreover, $\mathcal{V}_c(M) = \bigcup_{K \in \mathcal{K}} \mathcal{V}_K(M)$ is compactly retractive (see Remark 6.2(b)).

**Example 7.3** (involving a mere weak direct limit chart). We mention that $\text{Diff}_c(M)$ can also be made a Lie group modelled on the space $\mathcal{V}_c(M)$ of compactly supported smooth vector fields, equipped with the (usually properly coarser) topology making it the projective limit

$$\bigcap_{k \in \mathbb{N}_0} \mathcal{V}^k_c(M) = \lim_{\leftarrow k \in \mathbb{N}_0} \mathcal{V}^k_c(M)$$

of the LB-spaces of compactly supported $C^k$-vector fields (see [27], where this Lie group is denoted $\text{Diff}_c(M)$). Then $\text{Diff}_c(M) = \bigcup_{K \in \mathcal{K}} \text{Diff}_K(M)$.
and the chart of \( \text{Diff}_c(M) \) around \( \text{id}_M \) described in [26, §5.1] is a weak direct limit chart (albeit not a direct limit chart). It is not hard to see (with Remark 6.2 (b)) that \( \lim_{k \to \infty} V_k(M) = \bigcup_{K \in \mathcal{K}} V_K(M) \) is compactly retractive. Hence \( \text{Diff}_c(M) = \bigcup_{K \in \mathcal{K}} \text{Diff}_K(M) \) is compactly retractive (by Proposition 3.5 (a)).

**Example 7.4 (Test function groups).** Let \( M \) and \( \mathcal{K} \) be as in Example 7.2, \( H \) be a Lie group modelled on a locally convex space, and \( r \in \mathbb{N}_0 \cup \{\infty\} \). Consider the “test function group” \( C^r_c(M, H) \) of \( C^r \)-maps \( \gamma: M \to H \) such that the closure of \( \{x \in M : \gamma(x) \neq 1\} \) (the support of \( \gamma \)) is compact. Given \( K \in \mathcal{K} \), let \( C^r_K(M, H) \) be the subgroup of functions supported in \( K \). Then \( C^r_K(M, H) \) is a Lie group modelled on \( C^r(M, L(H)) = \lim_{\to} C^r_K(M, L(H)) \) (see [20]; cf. [41] for special cases, also [26, §7.1]). Then

\[
C^r_c(M, H) = \bigcup_{K \in \mathcal{K}} C^r_K(M, H)
\]

admits a direct limit chart (cf. [26, §7.1]). Furthermore, \( C^r_c(M, L(H)) = \bigcup_{K \in \mathcal{K}} C^r_K(M, L(H)) \) is compactly retractive as a consequence of Remark 6.2 (b).

**Example 7.5 (Weak direct products of Lie groups).** Given a sequence \( (H_n)_{n \in \mathbb{N}} \) of Lie groups, its weak direct product \( G := \prod_{n \in \mathbb{N}} H_n \) is defined as the group of all \( (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} H_n \) such that \( x_n = 1 \) for all but finitely many \( n \); it has a natural Lie group structure [22, §7]. Then \( G = \bigcup_{n \in \mathbb{N}} G_n \), identifying the partial product \( G_n := \prod_{k=1}^n H_k \) with a subgroup of \( G \). By construction, \( G = \bigcup_{n \in \mathbb{N}} G_n \) has a direct limit chart. Moreover, \( L(G) = \bigoplus_{n \in \mathbb{N}} L(H_n) = \lim_{\to} L(G_n) \) is compactly retractive, as locally convex direct sums are regular [9, Ch. 3, §1, no. 4, Proposition 5] and induce the given topology on each finite partial product (cf. Propositions 7 or 8 (i) in [9, Ch. 2, §4, no. 5]).

**Example 7.6 (Lie groups of germs of analytic mappings).** Let \( H \) be a complex Banach-Lie group, \( \| . \| \) be a norm on \( L(H) \) defining its topology, \( X \) be a complex metrizable locally convex space and \( K \subseteq X \) be a non-empty compact set. Then the set \( \text{Germ}(K, H) \) of germs around \( K \) of \( H \)-valued complex analytic functions on open neighbourhoods of \( K \) can be made a Lie group modelled on the locally convex direct limit

\[
\text{Germ}(K, L(H)) = \lim_{\to} \text{Hol}_b(W_n, L(H))
\]

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of the Banach spaces $g_n := \text{Hol}_b(W_n, L(H))$ of bounded $L(H)$-valued complex analytic functions on $W_n$ (with the supremum norm), where $W_1 \supseteq W_2 \supseteq \cdots$ is a fundamental sequence of open neighbourhoods of $K$ in $X$ such that each connected component of $W_n$ meets $K$ (see [23]). The group operation arises from pointwise multiplication of representatives of germs. The identity component $\text{Germ}(K, H)_0$ is the union

$$\text{Germ}(K, H)_0 = \bigcup_{n \in \mathbb{N}} G_n$$

of the Banach-Lie groups $G_n := \langle \{\exp_H \circ \gamma \} : \gamma \in g_n \rangle$, and $\text{Germ}(K, H)_0 = \bigcup_{n \in \mathbb{N}} G_n$ admits a direct limit chart [26, §10.4]. Moreover, Wengenroth’s result recalled in Remark 6.2(d) implies that $\text{Germ}(K, L(H)) = \bigcup_{n \in \mathbb{N}} g_n$ is compactly retractive (see [12], §4) and thus also $\text{Germ}(K, H)_0 = \bigcup_{n \in \mathbb{N}} G_n$.

**Example 7.7 (Lie groups of germs of analytic diffeomorphisms).** If $X$ is a complex Banach space and $K \subseteq X$ a non-empty compact subset, let $\text{GermDiff}(K)$ be the set of germs around $K$ of complex analytic diffeomorphisms $\gamma : U \to V$ between open neighbourhoods $U$ and $V$ of $K$ (which may depend on $\gamma$), such that $\gamma|_K = \text{id}_K$. Then $\text{GermDiff}(K)$ can be made a Lie group modelled on the locally convex direct limit

$$\text{GermDiff}(K) = \bigcup_{n \in \mathbb{N}} M_n$$

where the $W_n$ and $\text{Hol}_b(W_n, X)_K := \{\zeta \in \text{Hol}_b(W_n, X) : \zeta|_K = 0\}$ (see [26, §15] for the special case $\dim(X) < \infty$, and [11] for the general result). The group operation arises from composition of representatives of germs. Now the set $M_n$ of all elements of $\text{GermDiff}(K)$ having a representative in $\text{Hol}_b(W_n, X)_K$ is a Banach manifold, and

$$\text{GermDiff}(K) = \bigcup_{n \in \mathbb{N}} M_n$$

has a direct limit chart (see [11]; cf. [26, Lemma 14.5 and §15]). Again, Wengenroth’s characterization implies that $\text{Germ}(K, X)_K = \bigcup_{n \in \mathbb{N}} \text{Hol}_b(W_n, X)_K$ is compactly retractive (see [11]), and hence also $\text{GermDiff}(K) = \bigcup_{n \in \mathbb{N}} M_n$.

---

6If $X$ and $H$ are finite-dimensional and $W_{n+1}$ is relatively compact in $W_n$, then the restriction maps $\text{Hol}_b(W_n, L(H)) \to \text{Hol}_b(W_{n+1}, L(H))$ are compact operators [26, §10.5], whence $\text{Germ}(K, L(H)) = \bigcup_{n \in \mathbb{N}} g_n$ is compactly retractive by the simpler Remark 6.2(c).

7Or simply Remark 6.2(c), if $\dim(X) < \infty$.  

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Example 7.8 (Unit groups of ascending unions of Banach algebras). Let $A_1 \subseteq A_2 \subseteq \cdots$ be unital complex Banach algebras (such that all inclusion maps are continuous homomorphisms of unital algebras). Give $A := \bigcup_{n \in \mathbb{N}} A_n$ the locally convex direct limit topology. Then $A^\times$ is open in $A$ and if $A$ is Hausdorff (which we assume now), then $A^\times$ is a complex Lie group [26, Proposition 12.1]. Moreover, $A^\times = \bigcup_{n \in \mathbb{N}} A_n^\times$ and the identity map $id_{A^\times}$ is a direct limit chart.

If each inclusion map $A_n \to A_{n+1}$ is a topological embedding or each a compact operator, then $A = \bigcup_{n \in \mathbb{N}} A_n$ and hence also $A^\times = \bigcup_{n \in \mathbb{N}} A_n^\times$ is compactly retractive (and thus Proposition 3.3 applies). However, for more general choices of the steps, $A = \bigcup_{n \in \mathbb{N}} A_n$ is not compactly retractive.

To get an example for this pathology, let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence of Banach spaces whose locally convex direct limit $E := \bigcup_{n \in \mathbb{N}} E_n$ is not regular (for example, a suitable ascending sequence of weighted function spaces as in [1, Remark 1.5]). Then $E = \bigcup_{n \in \mathbb{N}} E_n$ is not compactly retractive (e.g., by Wengenroth’s result recalled in Remark 6.2(d)). Consider $A_n := \mathbb{C} \times E_n$ as a unital complex Banach algebra with associative multiplication $(z_1, x_1) \cdot (z_2, x_2) := (z_1 z_2, z_1 x_2 + z_2 x_1)$. Since $A := \lim_{\longrightarrow} A_n = \mathbb{C} \times \lim_{\longrightarrow} E_n = \mathbb{C} \times E$ as a locally convex space, $A = \bigcup_{n \in \mathbb{N}} A_n^\times = \mathbb{C} \times (\bigcup_{n \in \mathbb{N}} E_n^\times)$ is not compactly retractive (nor is $A^\times = \bigcup_{n \in \mathbb{N}} A_n^\times$, in view of Corollary 3.6). Of course, the homotopy groups $\pi_k(A^\times) \cong \pi_k(\mathbb{C}^\times) \times \pi_k(E) \cong \pi_k(E^\times)$ (which are infinite cyclic if $k = 1$ and trivial otherwise) can be calculated directly in this example.

8 Applications to typical Lie groups that contain a dense union of Lie groups

We now describe typical examples of Lie groups which contain a dense directed union of Lie groups, and verify that Theorem 1.13 applies.

To test the applicability of Theorem 1.13, it is helpful to have a simple criterion for the existence of well-filled charts. The following lemma serves this purpose. It even applies to certain topological groups.

Lemma 8.1 Let $M$ be a topological group that contains a directed union $M_\infty := \bigcup_{\alpha \in A} M_\alpha$ of topological groups as a dense subset. Assume that all inclusion maps $M_\alpha \to M$ and $M_\alpha \to M_\beta$ (for $\alpha \leq \beta$) are continuous homo-
morphisms. If there exists a well-filled chart $\phi: M \supseteq U \to V \subseteq E$ and a core $U^{(2)}$ of $\phi$ such that $1 \in U^{(2)}$, then $M$ admits well-filled charts.

Proof. We re-use the notation from the introduction. If $g \in M_\infty$, define

$$\psi: gU \to V, \quad x \mapsto \phi(g^{-1}x).$$

After increasing $\alpha_0$, we may assume that $g \in M_{\alpha_0}$. Then $\psi$ is a well-filled chart with core $gU^{(2)}$ (containing $g$), together with the charts $\psi_\alpha: gU_\alpha \to V_\alpha$, $x \mapsto \phi_\alpha(g^{-1}x)$. In fact, the conditions (a) and (b) from Definition 1.1 hold by construction. Since $gU \cap M_\infty = gU \cap gM_\infty = g(U \cap M_\infty) = gU_\infty = \bigcup_{\alpha \geq \alpha_0} gU_\alpha$, condition (d) from Definition 1.7 holds. Also (e) and (f) hold with $\psi(gU^{(2)}) = V^{(2)}$, as $V$ and $V_\alpha$ are unchanged and $\phi$ is a well-filled chart.

Now $M = M_\infty U^{(2)}$ by density of $M_\infty$ in $M$ (cf. [53, Lemma 3.17]). Hence $M = \bigcup_{g \in M_\infty} gU^{(2)}$ is covered by cores of well-filled charts.

We now prepare the discussion of weighted mapping groups. If $(X, ||.||)$ is a normed space, $Y$ a locally convex space, $q$ a continuous seminorm on $Y$ and $p: X \to Y$ a continuous homogeneous polynomial, we set

$$||p||_q := \sup \{q(p(x)) : x \in \overline{B}_1(0)\}. \quad (16)$$

If $Y$ is a normed space and $q$ its norm, we also write $||p|| := ||p||_q$.

8.2 Let $X = \mathbb{R}^d$, equipped with some norm, $Y$ be a locally convex space, $\Omega \subseteq X$ be open, $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathcal{W}$ be a set of smooth functions $f: \Omega \to \mathbb{R}$ such that the constant function 1 belongs to $\mathcal{W}$ and the following conditions are satisfied:

(a) $f(x) \geq 0$ for all $f \in \mathcal{W}$ and $x \in \Omega$;

(b) For each $x \in \Omega$, there exists $f \in \mathcal{W}$ such that $f(x) > 0$;

(c) For all $N \in \mathbb{N}$, $f_1, \ldots, f_N \in \mathcal{W}$ and $k_1, \ldots, k_N \in \mathbb{N}_0$ with $k_1, \ldots, k_N \leq r$, there exist $C > 0$ and $f \in \mathcal{W}$ such that\footnote{Here $\delta^k_x f: X \to \mathbb{R}$ denotes the $k$-th Gateaux differential of $f$ at $x \in \Omega$, defined via $\delta^k_x f(y) := \frac{d^k}{dt^k} f(x + ty)$.}

$$||\delta^{k_1}_x f_1 \cdots \delta^{k_N}_x f_N|| \leq C f(x) \quad \text{for all } x \in \Omega.$$
Let $C^r_W(\Omega, Y)$ be the set of all $C^r$-maps $\gamma: \Omega \to Y$ such that

$$\|\gamma\|_{f,k,q} := \sup_{x \in \Omega} f(x) \|\delta^k_x \gamma\|_q < \infty$$

for each $f \in W$, $k \in \mathbb{N}_0$ such that $k \leq r$, and continuous seminorm $q$ on $Y$. Let $C^r_W(\Omega, Y)^\bullet$ be the set of all $\gamma \in C^r_W(\Omega, Y)$ such that moreover

$$f(x) \|\delta^k_x \gamma\|_q \to 0 \quad \text{as} \quad x \to \infty$$

in the Alexandroff compactification $\Omega \cup \{\infty\}$ of $\Omega$. Then $C^r_W(\Omega, Y)$ and $C^r_W(\Omega, Y)^\bullet$ are vector spaces and the seminorms $\|\cdot\|_{f,k,q}$ turn them into locally convex spaces which are complete if $Y$ is complete (cf. [55]). If $Q \subseteq Y$ is open, then $C^r_W(\Omega, Q)^\bullet := \{\gamma \in C^r_W(\Omega, Y)^\bullet : \gamma(\Omega) \subseteq Q\}$ is open in $C^r_W(\Omega, Y)^\bullet$.

The conditions (a)-(c) imposed on $W$ imply a crucial property:

**Lemma 8.3** $C^\infty_c(\Omega, Y)$ is dense in $C^r_W(\Omega, Y)^\bullet$.

**Proof** (sketch). If $Y$ is finite-dimensional, the assertion is immediate from the scalar-valued case treated in [17, V.7 a), p. 224]. For the general case, one first replaces $Y$ with a completion $\tilde{Y}$ and reworks the proof of [17, V.7 a), p. 224], with minor modifications. Then, in the last line of [17, p. 226], one replaces $(T_{m_1,m_2}f)(x^{(m_i)}) \in \tilde{Y}$ by a nearby element in $Y$. 

**Example 8.4** (Groups of rapidly decreasing Lie group-valued maps). Given a Lie group $H$, let $C^r_W(\Omega, H)^\bullet$ be the set of all $C^r$-maps $\gamma: \Omega \to H$ for which there exists a chart $\kappa: P \to Q \subseteq L(H)$ of $H$ around 1 with $\kappa(1) = 0$, a compact set $K \subseteq \Omega$ such that $\gamma(\Omega \setminus K) \subseteq P$, and a compactly supported, smooth function $h: \Omega \to [0,1]$ taking the constant value 1 on some neighbourhood of $K$, such that $(1-h) \cdot (\kappa \circ \gamma|_{\Omega \setminus K}) \in C^r_W(\Omega \setminus K, L(H))^\bullet$ (cf. [55, Definition 6.4.1], where this set is denoted by $C^r_W(\Omega, H)^{\bullet \max}$). Define

$$C^r_W(\Omega, P)^\bullet := \{\gamma \in C^r_W(\Omega, H)^\bullet : \gamma(\Omega) \subseteq P\}.$$ 

Then $C^r_W(\Omega, H)^\bullet$ can be made a Lie group modelled on $C^r_W(\Omega, L(H))^\bullet$ in a natural way, such that, for some chart $\kappa: P \to Q$ as just described,

$$\phi: U := C^r_W(\Omega, P)^\bullet \to C^r_W(\Omega, Q)^\bullet =: V, \quad \gamma \mapsto \kappa \circ \gamma.$$ 

The completeness of $\tilde{Y}$ ensures that the relevant vector-valued (weak) integrals exist. As one continuous seminorm $q$ on $Y$ suffices to describe a typical neighbourhood of a given function in $C^r_W(\Omega, Y)^\bullet$, the proof goes through if we replace the absolute value $|\cdot|$ by $q$.
is a chart of $C^r_W(\Omega, H)^\bullet$ around 1 (see [55]; cf. [7] for special cases).\footnote{The Lie groups $S(\mathbb{R}^d, H; \mathcal{W})$ constructed in [7] coincide with the groups $C^r_W(\mathbb{R}^d, H)^\bullet$ by [55, Lemma 6.4.23].}

To get some information on the homotopy groups of $C^r_W(\Omega, H)^\bullet$, let $K$ be the set of compact subsets of $\Omega$, directed by inclusion. In [28], it is shown that $C^\infty_c(\Omega, H) = \bigcup_{K \in K} C^\infty_c(\Omega, H)$ is dense in $C^r_W(\Omega, H)^\bullet$. The restriction of $\phi$ to the map

$$\phi_K := C^\infty_K(\Omega, \kappa) : C^\infty_K(\Omega, P) \to C^\infty_K(\Omega, Q)$$

from an open subset of $C^\infty_K(\Omega, H)$ to an open subset of $C^\infty_K(\Omega, L(H))$ is a chart of $C^\infty_K(\Omega, H)$ (see [29]; cf. [20, §3]). Since $C^\infty_K(\Omega, P) = U \cap C^\infty_K(\Omega, H)$ and $C^\infty_K(\Omega, Q) = V \cap C^\infty_K(\Omega, L(H))$, we are in the situation of Example 1.11 (ii). Thus $\phi$ is a well-filled chart admitting cores around each $\gamma \in U$, notably around $1 \in U$. Hence $C^r_W(\Omega, H)^\bullet$ admits well-filled charts (by Lemma 8.1) and thus

$$\pi_k(C^r_W(\Omega, H)^\bullet) \cong \lim_{\to} \pi_k(C^\infty_K(\Omega, H)) = \pi_k(C^\infty_c(\Omega, H)),$$

using Theorem 1.13 for the first equality and Example 7.4 for the second.

If $\Omega = X$, then the homotopy groups can be calculated more explicitly.

**Theorem 8.5** If $\Omega = X = \mathbb{R}^d$ in the preceding situation, then

$$\pi_k(C^r_W(\mathbb{R}^d, H)^\bullet) \cong \pi_{k+d}(H) \quad \text{for all } k \in \mathbb{N}_0.$$

**Proof.** Let $S_d \subseteq \mathbb{R}^{d+1}$ be the $d$-dimensional sphere, $* \in S_d$ be a point and $C_*(S_d, H)$ be the group of $H$-valued continuous maps on $S_d$ taking $*$ to 1 (equipped with the topology of uniform convergence). Then

$$\pi_k(C^r_W(\mathbb{R}^d, H)^\bullet) \cong \pi_k(C^\infty_c(\mathbb{R}^d, H)) \cong \pi_k(C_0(\mathbb{R}^d, H)) \cong \pi_k(C_*(S_d, H)) \cong \pi_{k+d}(H),$$

using (17) for the first isomorphism, [45, Theorem A.10] for the second, and standard facts from homotopy theory for the last. \hfill $\square$

**Remark 8.6** Define $f_m : \mathbb{R}^d \to \mathbb{R}$ via $f_m(x) := (1 + \|x\|^2)^m$ for $m \in \mathbb{N}_0$ (where $\|\cdot\|$ is a euclidean norm on $\mathbb{R}^d$), and $\mathcal{W} := \{f_m : m \in \mathbb{N}_0\}$. Then
$C^\infty_W(\mathbb{R}^d, L(H))^\bullet$ is the Schwartz space $S(\mathbb{R}^d, L(H))$ of rapidly decreasing smooth $L(H)$-valued maps on $\mathbb{R}^d$. As a special case of Theorem 8.5, the group $S(\mathbb{R}^d, H) := C^\infty_W(\mathbb{R}^d, H)^\bullet$ satisfies

$$\pi_k(S(\mathbb{R}^d, H)) = \pi_{k+d}(H) \text{ for all } k \in \mathbb{N}_0.$$  

This had been conjectured in [7, p. 130], and was open since 1981.

**Example 8.7 (Weighted diffeomorphism groups).** Let $\Omega = X = Y$ and $r := \infty$ in 8.2 and let $\text{Diff}_W(X)^\bullet$ be the set of all $C^\infty$-diffeomorphisms $\gamma: X \to X$ with $\gamma - \text{id}_X \in C^\infty_W(X, X)^\bullet$ and $\gamma^{-1} - \text{id}_X \in C^\infty_W(X, X)^\bullet$ (in [55], this set is denoted $\text{Diff}_W(X)^0$). Then

$$V := \{ \gamma \in C^\infty_W(X, X)^\bullet : \gamma + \text{id}_X \in \text{Diff}_W(X)^\bullet \}$$

is open in $C^\infty_W(X, X)^\bullet$ and $\phi: U := \text{Diff}_W(X)^\bullet \to V$, $\gamma \mapsto \gamma - \text{id}_X$ a global chart for $\text{Diff}_W(X)^\bullet$, making it a Lie group (see [53]). Because $C^\infty_c(X, X)$ is dense in $C^\infty_W(X, X)^\bullet$, it follows that $\text{Diff}_c(X) = \bigcup_{K \in \mathcal{K}} \text{Diff}_K(X)$ is dense in $\text{Diff}_W(X)^\bullet$, where $\mathcal{K}$ is the set of compact subsets of $X$ and $\text{Diff}_c(X)$ as well as $\text{Diff}_K(X)$ are as in Example 7.2. Since, for each $K \in \mathcal{K}$, the restriction of $\phi$ to a map $\text{Diff}_K(X) \to V \cap C^\infty_K(X, X)$

is a chart of $\text{Diff}_K(X)$, we are in the situation of Example 1.11(ii) and thus

$$\pi_k(\text{Diff}_W(X)^\bullet) = \lim_{\to K \in \mathcal{K}} \pi_k(\text{Diff}_K(X)) \text{ for each } k \in \mathbb{N}_0, \text{ by Theorem 1.13}.$$  

**Remark 8.8** We mention that (unlike Example 8.4) the preceding example can also be deduced from Palais’ classical theorem. To this end, let $\mathcal{F}$ be the set of finite-dimensional vector subspaces of $C^\infty_c(X, X)$, and $V_\infty := V \cap C^\infty_c(X, X)$. Because $C^\infty_c(X, X)$ is dense in $C^\infty_W(X, X)^\bullet$, using Palais’ Theorem twice we see that

$$\pi_k(\text{Diff}_W(X)^\bullet) \cong \pi_k(V) \cong \lim_{\to F \in \mathcal{F}} \pi_k(V \cap F) = \lim_{\to F \in \mathcal{F}} \pi_k(V_\infty \cap F) \cong \pi_k(V_\infty) \cong \pi_k(\text{Diff}_c(X)).$$

Hence $\pi_k(\text{Diff}_W(X)^\bullet) = \lim \pi_k(\text{Diff}_K(X))$ (see Example 7.2). Notably, the inclusion map $\text{Diff}_c(X) \to \text{Diff}_W(X)^\bullet$ is a weak homotopy equivalence.  

11We mention that special cases of such groups have been used by physicists [30]. The weighted diffeomorphism group of $\mathbb{R}$ modelled on $S(\mathbb{R}, \mathbb{R})$ has also been treated in [37].
Example 8.9 Let $M$ be a $\sigma$-compact, finite-dimensional smooth manifold, $r \in \mathbb{N}_0$ and $H$ be a Lie group. Then the inclusion map
\[ C_c^\infty(M, H) \to C_c^r(M, H) \]
is a weak homotopy equivalence.

To see this, let $K$ be the set of compact subsets of $M$. By [23], $C_c^\infty(M, H) = \bigcup_{K \in K} C_c^\infty(M, H)$ is dense in $C_c^r(M, H)$. Let $\kappa: P \to Q$ be a chart of $H$ around 1 such that $P = P^{-1}$, $\kappa(1) = 0$ and $\kappa$ extends to a chart with domain $R$, such that $PP \subseteq R$. Then
\[ \phi := C_c^r(M, \kappa): C_c^r(M, P) \to C_c^r(M, Q), \quad \gamma \mapsto \kappa \circ \gamma \]
is a chart of $C_c^r(M, H)$ and
\[ \phi_K := C_K^\infty(M, \kappa): C_K^\infty(M, P) \to C_K^\infty(M, Q) \]
is a chart of $C_K^\infty(M, H)$, for each compact subset $K \subseteq M$ (see [20]). It is clear that all conditions described in Example 1.11 (ii) are satisfied, and thus $\phi$ is a well-filled chart admitting a core around 1. Hence $C_c^r(M, H)$ admits well-filled charts (by Lemma 8.1), and thus
\[ \pi_k(C_c^r(M, H)) \cong \lim_{\longrightarrow} \pi_k(C_K^\infty(M, H)) \cong \pi_k(C_c^\infty(M, H)) \quad \text{for each } k \in \mathbb{N}_0, \]
by Theorem 1.13 and Example 7.4. The assertion follows.

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Helge Glöckner, Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany. E-Mail: glockner@math.uni-paderborn.de