L^p-EXPANDER GRAPHS

BY

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ABSTRACT

We discuss how graph expansion is related to the behavior of L^p-functions on the covering tree. We show that the non-trivial eigenvalues of the adjacency operator on a (q + 1)-regular graph are bounded by q^{1/p} + q^{(p-1)/p}—the L^p-norm of the operator on the covering tree—if and only if properly averaged lifts of functions from the graph to the tree lie in L^{p+\epsilon} for every \epsilon > 0. We generalize the result to operators on edges and to bipartite graphs.

The work is based on a combinatorial interpretation of representation-theoretic ideas.

1. Introduction

The goal of this paper is to put on record some claims about expander graphs, which characterize them by the properties of L^p-functions on their infinite covering tree. Let us first set some notations.

Fix 2 \leq q \in \mathbb{N}. Let X be a finite, connected, (q + 1)-regular graph. Let T be the (q + 1)-regular tree which is the universal cover of X and let \pi: T \rightarrow X be a covering map. Let v_0 \in V_T be a vertex in T. For every function f: V_X \rightarrow \mathbb{C} (i.e., a function defined on the vertices of X) let \hat{f}: V_T \rightarrow \mathbb{C} be the lift of f to T, i.e., \hat{f} = f \circ \pi. Let

\rho_{v_0}(\hat{f})(v) = \frac{1}{q\rho} \sum_{v'} \hat{f}(v')

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In particular, \(X_2.4\) and that the graph is an eigenvalue of the operator \(f \in L^2(V_X)\) when \(v' \not\sim v\) and \(q_v\) is the size of the sphere (i.e., \(q_{v_0} = 1\) and for \(v \neq v_0\), \(q_v = (q + 1)q^{d(v,v_0)}\)). Let

\[
L^{(2)}_0(X) = \left\{ f : V_X \to \mathbb{C} : \sum_{v \in V_X} f(v) = 0 \right\}
\]

be the subspace orthogonal to the constant function. If \(X\) is not bipartite, let \(L^{(2)}_0(V_X) = L^{(2)}_0(V_X)\), otherwise let \(L^{(2)}_0(V_X)\) be the subspace of \(f \in L^{(2)}_0(V_X)\) for which the sum of \(f\) is 0 over each of the sides of \(X\). We wish to understand the behavior at infinity of \(\rho_{v_0}(\tilde{f})\) for \(\tilde{f} \in L^{(2)}_0(V_X)\). The function \(\tilde{f}\) itself is periodic, and therefore if \(f \neq 0\) it is not in \(L^p(V_T)\) for any \(p < \infty\).

Let \(A : L^2(V_X) \to L^2(V_X)\) be the vertex adjacency operator,

\[
Af(v) = \sum_{v' \sim v} f(v').
\]

The subspace \(L^{(2)}_0(V_X)\) is the space of functions orthogonal to the eigenvectors with trivial eigenvalues \(\pm (q + 1)\) of \(A\). Let \(\lambda(X)\) be the largest absolute value of an eigenvalue of \(A\) on \(L^{(2)}_0(V_X)\). It is standard that \(\lambda(X) < q + 1\) (see Subsection 2.4) and that the graph \(X\) is a good expander if \(\lambda(X)\) is small (see [HLW06]). In particular, \(X\) is called Ramanujan if \(\lambda(X) \leq 2\sqrt{q}\). By the Alon–Boppana theorem ([Nil91]) Ramanujan graphs are considered the best expanders.

Notice that the function \(\hat{\lambda}(p) = q^{1/p} + q^{(p-1)/p}\) for \(p \in [2, \infty]\) is increasing with \(\hat{\lambda}(2) = 2\sqrt{q}\) and \(\hat{\lambda}(\infty) = q + 1\).

**Theorem 1.1:** For \(p \geq 2\), \(\lambda(X) \leq q^{1/p} + q^{(p-1)/p}\) if and only if for every \(f \in L^{(2)}_0(V_X)\) and \(v_0 \in T\), \(\rho_{v_0}(\tilde{f}) \in L^{p+\epsilon}(V_T)\) for every \(\epsilon > 0\).

In particular, \(X\) is a Ramanujan graph (i.e., \(\lambda(X) \leq 2\sqrt{q}\)) if and only if for every \(f \in L^{(2)}_0(V_X)\) and \(v_0 \in T\), \(\rho_{v_0}(\tilde{f}) \in L^{2+\epsilon}(V_T)\) for every \(\epsilon > 0\).

One of the motivations for the definition of a Ramanujan graph is the classical result of Kesten ([Kes59]), stating that the norm of \(A\) on \(L^2(V_T)\) is \(2\sqrt{q}\). This is the case \(p = 2\) of the following theorem:

**Theorem 1.2:** The value \(\hat{\lambda}(p) = q^{1/p} + q^{(p-1)/p}\) is the norm of the adjacency operator \(A\) on \(L^p(V_T)\), \(1 \leq p \leq \infty\).

More precisely, the spectrum of \(A\) on \(L^p(V_T)\), \(1 \leq p \leq \infty\) is

\[
\{ \theta + q\theta^{-1} : \theta \in \mathbb{C}, q^{1/p'} \leq |\theta| \leq q^{(p'-1)/p'} \}, \quad \text{for } p' = \max\{p, p/(p-1)\}.
\]

Theorem 1.1 allows us to define:
**Definition 1.3:** For \( p \geq 2 \), a graph is an \( L^p \)-expander if it satisfies one of the equivalent conditions of Theorem 1.1.

In particular, our definition of an \( L^2 \)-expander graph is the same as a Ramanujan graph. The referee suggested using the term an \( L^p \)-Ramanujan graph instead of an \( L^p \)-expander, which is perhaps a better choice since the definition is based on comparing the graph to the universal cover \( T \). We preferred this notion since \( L^p \)-expanders are simply expanders with an explicit spectral gap.

Theorem 1.1 and Theorem 1.2 are part of a larger theory that we develop in this paper, which include operators acting also on functions on the directed edges \( E_X \) of \( X \) (i.e., each non-oriented edge of \( X \) is counted twice in \( E_X \)). Let us state here the main result without giving all the definitions. See Theorem 3.24 for a precise and extended form.

**Theorem 1.4:** Let \( X \) be a finite, connected, non-bipartite, \((q + 1)\)-regular graph. For \( p \geq 2 \), the following are equivalent:

1. Every eigenvalue \( \lambda \) of \( A \) on \( L^2_0(V_X) \) satisfies \( |\lambda| \leq q^{1/p} + q^{(p-1)/p} \).
2. For every \( f \in L^2_0(V_X) \) and \( v_0 \in T \), \( \rho_{v_0}(\tilde{f}) \in L^{p+\epsilon}(V_T) \) for every \( \epsilon > 0 \).
3. Every eigenvalue \( \lambda \) of Hashimoto’s non-backtracking operator \( h^N_B \) on \( L^2_0(E_X) \) satisfies \( |\lambda| \leq q^{(p-1)/p} \).
4. The eigenvalues of every “Hecke operator” \( h \) on \( L^2_0(V_X) \oplus L^2_0(E_X) \) are contained in the spectrum of \( h \) on \( L^p(V_T) \oplus L^p(E_X) \).
5. Let \( A_k : L^2(V_X) \to L^2(V_X) \) be the “non-backtracking-distance-\( k \) operator”. For every \( k \geq 0 \), every eigenvalue \( \lambda_k \) of \( A_k \) on \( L^2_0(V_X) \) satisfies \( |\lambda_k| \leq (k + 1)q^{k(p-1)/p} \).

The theory was developed as part of a generalization to higher dimensions, which appears in [Kam16]. In this paper we discuss its extension to finite, connected, bipartite, \((q_0 + 1, q_1 + 1)\)-biregular graphs, where \( q_0 < q_1 \). The graph \( X \) has two types of vertices, 0 and 1, and each vertex of type \( i \in \{0, 1\} \) is contained in \( q_i + 1 \) edges. The non-precise form of the theory in this case is summarized in the following theorem. See Subsection 3.5 for full details.

**Theorem 1.5:** Let \( X \) be a finite, connected, bipartite, \((q_0 + 1, q_1 + 1)\)-biregular graph. For \( p \geq 2 \), the following are equivalent:

1. For every \( f \in L^2_{00}(V_X) \) and \( v_0 \in T \), \( \rho_{v_0}(\tilde{f}) \in L^{p+\epsilon}(V_T) \) for every \( \epsilon > 0 \).
2. Every eigenvalue \( \lambda \) of Hashimoto’s non-backtracking operator \( \hat{h}^N_B \) on \( L^2_0(\tilde{E}_X) \) satisfies \( |\lambda| \leq (q_0q_1)^{(p-1)/p} \).
The eigenvalues of every “Hecke operator” $h$ on $L^2_{00}(V_X) \oplus L^2_0(\tilde{E}_X)$ are contained in the spectrum of $h$ on $L^p(V_T) \oplus L^p(\tilde{E}_T)$.

Let $A_k : L^2(V_X) \to L^2(V_X)$ be the “non-backtracking-distance-$k$ operator” on vertices. For every $k \geq 0$, every eigenvalue $\lambda_k$ of $A_k$ on $L^2_{00}(V_X)$ satisfies $|\lambda_k| \leq (k + 1)q_1(q_0q_1)^{(k/2)(p-1)/p}$.

For $p = 2$, the conditions are equivalent to:

- Every eigenvalue $\lambda$ of $A$ on $L^2_{00}(V_X)$ satisfies $\lambda = 0$ or $\sqrt{q_1} - \sqrt{q_0} \leq |\lambda| \leq \sqrt{q_1} + \sqrt{q_0}$, and the multiplicity of $\lambda = 0$ is $|V^0_X| - |V^1_X|$, where $V^i_X$ is the number of vertices of $X$ of type $i$.

Related Results. Theorem 1.4 is connected to some well known results, mostly about the Zeta function of the graph or the non-backtracking operator. The condition on the eigenvalues of $A$ for $p = 2$ in Theorem 1.5 appears (as a definition) in the work of Hashimoto ([Has89, 3.21]), based on similar considerations. The main new idea of this work is the comparison to $L^p$-functions on the tree, with an extension to $p > 2$.

The proof of Alon’s second eigenvalue conjecture ([Fri03, Bor15]) is closely related to our analysis. Theorem 20 in [Bor15] in conjugation with Theorem 1.5 show that a random cover $X'$ of a biregular graph $X$ is an $L^{2+\epsilon}$-expander-cover. That is, we have a natural decomposition

$$L^2(\tilde{E}_X' \oplus V_X') \cong L^2(\tilde{E}_X \oplus V_X) \oplus W_{\text{new}}$$

and $W_{\text{new}}$ satisfies the conditions of Theorem 1.5 for $p = 2 + \epsilon$. In contrast, the covers built in [MSS13] or [HPS18], are not necessarily $L^2$-expander-covers. The construction only promises that the new eigenvalues of $A$ are bounded from above by $\sqrt{q_1} + \sqrt{q_0}$ (see also [HPS18, Question 6.3]).

One application of the $L^p$-theory developed here is to generalize the results of [LP16, Sar18], which concern “almost-diameter”. The results state that for every $\epsilon > 0$ and $\delta > 0$ there exists $N$ such that if $X$ is a $(q + 1)$-regular Ramanujan graph $X$ with $|V_X| > N$, then for every $x \in V_X$, all but $\delta|V_X|$ of the vertices $y \in V_X$ are of distance within $[(1 - \epsilon) \log_q|V_X|, (1 + \epsilon) \log_q|V_X|]$ from $x$, which is an optimal result. One can show that for Cayley expander graphs it is actually enough to assume far weaker conditions, namely that for every $p > 2$ and $\epsilon' > 0$, the number of bad eigenvalues $\lambda$ satisfying $|\lambda| \geq q^{\ell/p} + q^{(p-1)/p}$ is bounded by $C_{\epsilon'}|V_X|^{2/p+\epsilon'}$, $C_{\epsilon'}$ some constant. During the preparation of this paper those results became available at [BL18] and [GK19].
Arbitrary Graphs. The results of this paper do not extend naturally to a general graph $X$, i.e., which is neither regular nor biregular. In particular, we do not know the relations between the eigenvalues of the adjacency operator $A$, the “non-backtracking-distance-$k$ operator” $A_k$ and the non-backtracking operator $h_{NB}$.

One can, however, give some $L^p$-bounds on operators of the covering tree $T$ of $X$. For example, up to $O(k)$, the norm of $A_k$ on $L^p(V_T)$ is bounded by the $(p-1)/p$-th power of $N_k$—the maximal number of vertices on a sphere of radius $k$. We have that

$$N_k^{1/k} \to \text{gr}(T),$$

the growth rate of the tree (see [AFH15]). Similarly, the spectrum of the non-backtracking operator $h_{NB}$ on $L^p(E_X)$ is bounded in absolute value by $\text{gr}(T)^{(p-1)/p}$. See [AFH15] for similar calculations for $p = 2$.

Structure of the article. The work is divided into two sections: Section 2 concerns operators acting on maps on vertices and Section 3 concerns operators acting on maps on directed edges.

Section 2 contains the proofs of Theorem 1.1 and Theorem 1.2. Theorem 1.1 can be proved directly. We will take a slightly longer path, introducing along the way the basic notions of the vertex Hecke algebra $H_0$ and the Satake isomorphism. We define the algebra in Subsection 2.1. We study the Satake isomorphism and the irreducible representations of $H_0$ in Subsection 2.2. In Subsection 2.3 we show that each irreducible representation can be realized on functions of the vertices $V_T$ of the tree $T$. In Subsection 2.4 we prove Theorem 1.1 and finally in Subsection 2.5 we prove Theorem 1.2.

The second section of this paper is devoted to a generalization of the theory from maps on vertices to maps on directed edges. While this generalization is interesting in its own right, its advantage is apparent in [Kam16], where we study high-dimensional $L^p$-expanders (see [Lub14] for an introduction to the subject of high-dimensional expanders). However, since this algebra is more complicated, and in particular not commutative, working with it requires more preliminaries. In Subsection 3.1 we define the Iwahori–Hecke algebra $H_\phi$ of the tree, acting on functions on directed edges. In Subsection 3.2 we study the representations of $H_\phi$. In Subsection 3.3 we study the realizations of the representations on functions of the directed edges $E_T$ of the tree. In Subsection 3.4 we present Theorem 3.24 which combines all the results about regular graphs.
Finally, in Subsection 3.5 we study the corresponding theory for bipartite biregular (but not regular) graphs.

Our analysis is based on constructions from the representation theory of $p$-adic Lie groups, although no prior knowledge of it is assumed. The main contribution of this work is the interpretation of the representation-theoretic statements into simple combinatorial language, as well as an extension of some results to the case $p > 2$.

**Preliminaries from Representation Theory.** Let us collect here some basic definitions and facts from representation theory. Note that the first section of this work concerns the commutative vertex Hecke algebra $H_0$ whose analysis does not require most of those facts. The reader is referred to [EGH+11] for basic representation theory.

An algebra $H$ in this work is a vector space over $\mathbb{C}$, with multiplication satisfying the usual properties (including associativity), and a unit $Id = Id_H$. Every subset of elements $S \subset H$ generates a subalgebra $H_S$, which is the intersection of the subalgebras (with the same unit $Id_{H'} = Id_H$) $H' \subset H$ containing $S$. We say that the algebra $H$ is generated by $S$ if $H_S = H$. We say that the algebra $H$ is freely generated by an element $A \in H$ if there is an algebra isomorphism $f: \mathbb{C}[x] \to H$ defined by $f(x) = A$. An algebra representation $(\pi, V)$ of $H$ is an algebra homomorphism $\pi: H \to \text{End}_\mathbb{C}(V)$ such that

$$\pi(Id_H) = Id_V.$$ 

To make the notations simpler we sometimes omit $\pi$ and refer to $V$ directly as the representation.

A homomorphism of two $H$-representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ is a linear map $\varphi: V_1 \to V_2$ such that for all $v \in V_1$, $h \in H$, we have that

$$\pi_2(h)\varphi(v) = \varphi(\pi_1(h)v).$$

If $U \subset V$ is a $\pi(H)$-invariant subspace of $V$, we say that $U$ is a subrepresentation of $(\pi, V)$. If $U$ is a subrepresentation of $V$, there is a natural action of $H$ on the vector space $V/U$, which we call a quotient representation of $(\pi, V)$. If $\varphi: V_1 \to V_2$ is an $H$-algebra homomorphism, then $\ker \varphi \subset V_1$ is a subrepresentation of $V_1$ and $\varphi(V_1) \subset V_2$ is a subrepresentation of $V_2$ isomorphic to the quotient $V_1/\ker \varphi$. In particular, if $\varphi$ is onto then $V_2$ is isomorphic to a quotient of $V_1$. 
If $U_1, U_2, \ldots, U_k$ are subrepresentations of $V$ and $V = \bigoplus_{i=1}^{k} U_i$, we say that the representation $V$ is a direct sum of the representations $U_1, U_2, \ldots, U_k$. An algebra representation is called indecomposable if it is not a direct sum of two proper non-trivial subrepresentations. It is called irreducible if there is no proper non-trivial subrepresentations.

An $H$-representation $(\pi, V)$ is also called a left $H$-module. Similarly, one may define a right $H$-module $(\rho, W)$ by a linear transformation $\rho: H \to \text{End}_C(W)$ such that

$$\rho(h h') = \rho(h') \rho(h).$$

Given a left $H$-module $(\pi, V)$ and a right $H$-module $(\rho, W)$, one may define the vector space $W \otimes_H V$, which is the quotient of the vector space $V \otimes W$ by the vector subspace spanned by

$$\{\rho(h)w \otimes v - w \otimes \pi(h)v = 0 : v \in V, w \in W, h \in H\}.$$

Moreover, if $W$ is an $H'$-representation then $W \otimes_H V$ is also an $H'$-representation.

We call an algebra $H$ a $*$-algebra if it has an involution $*: H \to H$, i.e., a map satisfying for $h_1, h_2 \in H$, $\alpha \in \mathbb{C}$,

$$(h_1 + \alpha h_2)^* = h_1^* + \alpha h_2^* \quad \text{and} \quad (h_1 h_2)^* = h_2^* h_1^*.$$

A representation $(\pi, V)$ of a $*$-algebra $H$ is called unitary if there exists an inner product $\langle \cdot, \cdot \rangle$ on $V$ satisfying

$$\langle \pi(h)v_1, v_2 \rangle = \langle v_1, \pi(h^*) v_2 \rangle$$

for every $v_1, v_2 \in V$ and $h \in H$. We will need the standard claim:

**Proposition 1.6:** Every finite-dimensional unitary representation $(\pi, V)$ of a $*$-algebra $H$ decomposes into a direct sum of irreducible representations.

**Proof.** Assume $\{0\} \neq V' \subset V$ is a proper subrepresentation. Let

$$U = \{u \in V : \forall v \in V' \langle v, u \rangle = 0\}.$$

Since $\langle \cdot, \cdot \rangle$ is an inner product we have $V = V' \oplus U$ as vector spaces. Moreover, if $u \in U$, $h \in H$ then for every $v \in V'$,

$$\langle v, \pi(h)u \rangle = \langle \pi(h^*) v, u \rangle = 0.$$

Therefore $\pi(h)u \in U$ and $U$ is also a subrepresentation. The claim follows by induction. $\blacksquare$
2. Operators on vertices

2.1. The Vertex Hecke Algebra of a Regular Tree. Let $T$ be the $(q+1)$-regular tree and let $V_T$ be its set of vertices. Let $d: V_T \times V_T \to \{0, 1, \ldots\}$ be the natural distance function.

**Definition 2.1:** Let $A_k: \mathbb{C}^{V_T} \to \mathbb{C}^{V_T}$, $k = 0, 1, \ldots$ be the operator $A_k f(v) = \sum_{v': d(v,v') = k} f(v')$.

Notice that $A_0 = I_d$, that $A_1 = A$ is the vertex adjacency operator of $T$, and that for $k \geq 1$, $A_k$ sums $(q+1)q^{k-1} \approx q^k$ different vertices. The **Hecke relations** can be easily verified:

- $A^2 = A_2 + (q+1)A_0,$
- $AA_k = A_{k+1} + qA_{k-1}$ for $k = 2, 3, \ldots$.

**Definition 2.2:** The vertex **Hecke algebra** $H_0$ (sometimes called the spherical Hecke algebra) is the algebra spanned as a vector space by $A_k$, $k \geq 0$.

By the Hecke relations $H_0$ is indeed an algebra. The relations also show that $H_0$ is commutative and freely generated by $A = A_1$.

There is a more abstract definition of the vertex Hecke algebra.

**Definition 2.3:** Let $S$ be a discrete set. We say that a linear operator $h: \mathbb{C}^{S} \to \mathbb{C}^{S}$ is **row and column finite** if it can be written as $hf(x) = \sum_{y \in S} \alpha_{x,y} f(y)$, for some $\alpha: S \times S \to \mathbb{C}$, with $\# \{y : \alpha_{x,y} \neq 0\} < \infty$ and $\# \{y : \alpha_{y,x} \neq 0\} < \infty$ for every $x \in S$.

Notice that every operator $h \in H_0$ is row and column finite since this is true for the spanning vectors $A_k$, $k \geq 0$.

**Proposition 2.4:** Let $\gamma \in \text{Aut}(T)$ be an automorphism of the tree. Then $\gamma$ acts naturally on $\mathbb{C}^{V_T}$ by $\gamma \cdot f(x) = f(\gamma^{-1}x)$. Let $h: \mathbb{C}^{V_T} \to \mathbb{C}^{V_T}$ be a linear operator. Then $h \in H_0$ if and only if $h$ is row and column finite and the action of $h$ on $\mathbb{C}^{V_T}$ commutes with the action of each $\gamma \in \text{Aut}(T)$ on $\mathbb{C}^{V_T}$. 
Proof. Since automorphisms preserve distances in $T$, the only if part follows.

As for the if part, write $h: \mathbb{C}^{V_T} \rightarrow \mathbb{C}^{V_T}$ as $hf(x) = \sum_{y \in V_T} \alpha_{x,y} f(y)$, as in the definition of a row and column finite operator. Assume that $h$ commutes with every $\gamma \in \text{Aut}(T)$. If $x, y, x', y' \in V_T$, $d(x, y) = d(x', y')$, then there exists $\gamma \in \text{Aut}(T)$ such that $\gamma(x') = x, \gamma(y') = y$. Since $h\gamma = \gamma h$ we have $\alpha_{x,y} = \alpha_{x',y'}$. Therefore $\alpha_{x,y}$ depends only on $d(x, y)$ which means that $h \in H_0$.

Remark 2.5: The definition of a Hecke algebra as an algebra of operators commuting with automorphisms appears in a similar context in [Fir16, Kam16]. In both places it is used to extend the definition of Ramanujan graphs to higher-dimensional complexes.

2.2. Representations of $H_0$. The following theorem is called the Satake isomorphism:

Theorem 2.6: The algebra $H_0$ is isomorphic to the subalgebra of $\mathbb{C}[x, x^{-1}]$ which is invariant with respect to the automorphism $x \leftrightarrow x^{-1}$. The isomorphism is given by

\[ A_0 \leftrightarrow 1 \]
\[ A \leftrightarrow \hat{A} = q^{1/2}(x + x^{-1}), \]

and for $k \geq 2$
\[ A_k \leftrightarrow \hat{A}_k = q^{k/2}(x^k + x^{-k} + (1 - q^{-1})(x^{k-2} + x^{k-4} + \cdots + x^{-k+2})) \]
\[ = q^{(k-1)/2}(q^{1/2}x - q^{1/2}x^{-1})^{-1}(x^{k-1}(qx^2 - 1) - x^{-k+1}(qx^{-2} - 1)). \]

Remark 2.7: The algebra $H_0$ is freely generated by $A_1$ and as such is isomorphic to $\mathbb{C}[x]$. The same is true for the invariant subalgebra of $\mathbb{C}[x, x^{-1}]$, which is generated by $x + x^{-1}$. The important statement in the theorem is the explicit description of the isomorphism.

Proof. Since both algebras are freely generated (as algebras) by a single element, $A \rightarrow q^{1/2}(x + x^{-1})$ indeed defines an isomorphism.

The following calculations verify the explicit description:
\[ \hat{A}^2 = q(x + x^{-1})^2 = q(x^2 + x^{-2} + 2) \]
\[ = q + 1 + q(x^2 + x^{-2} + 1 - q^{-1}) \]
\[ = (q + 1)\hat{A}_0 + \hat{A}_2, \]
and for $k \geq 2$
\[
\hat{A} \cdot \hat{A}_k = q^{k/2}(x + x^{-1}) \cdot q^{k/2}(x^k + x^{-k} + (1 - q^{-1})(x^{k-2} + x^{k-4} + \cdots + x^{-k+2})) \\
= q^{(k+1)/2}(x^{k+1} + x^{-k-1} + x^{k-1} + x^{1-k} + (1 - q^{-1})(x^{k-1} + x^{1-k}) \\
+ (2 - 2q^{-1})(x^{k-3} + x^{k-5} + \cdots + x^{3-k})) \\
= q^{(k+1)/2}(x^{k+1} + x^{-k-1} + (1 - q^{-1})(x^{k-1} + x^{k-3} + \cdots + x^{1-k}) \\
+ q \cdot q^{(k-1)/2}(x^{k-1} + x^{1-k} + (1 - q^{-1})(x^{k-3} + x^{k-5} + \cdots + x^{3-k}))) \\
= \hat{A}_{k+1} + q\hat{A}_{k-1}.
\]

Finally, we have
\[
\hat{A}_k = q^{k/2}(x^k + x^{-k} + (1 - q^{-1})(x^{k-2} + x^{k-4} + \cdots + x^{-k+2})) \\
= q^{k/2}(x^k + x^{-k} + (1 - q^{-1})(x - x^{-1})^{-1}(x^{k-1} - x^{-k+1}) \\
= q^{k/2}(x^{k-1}(x + (1 - q^{-1})(x - x^{-1})^{-1} \\
+ x^{-k+1}(x^{-1} - (1 - q^{-1})(x - x^{-1})^{-1})) \\
= q^{k/2}(x - x^{-1})^{-1}(x^{k-1}(x^2 - q^{-1}) - x^{-k+1}(x^2 - q^{-1})) \\
= q^{(k-1)/2}(q^{1/2}x - q^{1/2}x^{-1})^{-1}(x^{k-1}(qx^2 - 1) - x^{-k+1}(qx^{-2} - 1)).
\]

Let us twist the Satake isomorphism by choosing $\theta = q^{1/2}x$. Write
\[
\tilde{\theta} = q\theta^{-1} = q^{1/2}x^{-1}.
\]

**Corollary 2.8:** The algebra $H_0$ is isomorphic to the subalgebra of $\mathbb{C}[\theta, \theta^{-1}]$ which is invariant with respect to the automorphism $\theta \leftrightarrow \tilde{\theta} = q\theta^{-1}$. The isomorphism is given by
\[
A_0 \leftrightarrow 1 \\
A \leftrightarrow A(\theta) = \theta + \tilde{\theta},
\]
and for $k \geq 1$
\[
A_k \leftrightarrow A_k(\theta) = \theta^k + \tilde{\theta}^k + (1 - q^{-1}) \sum_{i=1}^{k-1} \theta^{k-i} \tilde{\theta}^i \\
= (\theta - \tilde{\theta})^{-1}(\theta^{k-1}(\theta^2 - 1) - \tilde{\theta}^{k-1}(\tilde{\theta}^2 - 1)),
\]
where the last equality holds for $\theta \neq \tilde{\theta}$. 

We can now classify the irreducible representations of $H_0$. For $0 \neq \theta \in \mathbb{C}$, write $A_k(\theta)$ as in Corollary 2.8.

**Corollary 2.9:** The linear function $\pi_0^\theta : H_0 \to V_\theta \cong \mathbb{C}$ given by

$$\pi_0^\theta(A_k) = A_k(\theta)$$

defines a representation of $H_0$. The representations $(\pi_0^\theta, V_\theta)$ and $(\pi_0^{\theta'}, V_{\theta'})$ are isomorphic if and only if $\theta' = q\theta^{-1}$ or $\theta' = \theta$.

Each irreducible finite-dimensional representation of $H_0$ is one-dimensional and is isomorphic to one of the representations $V_\theta$, for $0 \neq \theta \in \mathbb{C}$.

**Proof.** Every eigenvector of $\pi(A)$ in a representation $(\pi, V)$ of $H_0$ spans a sub-representation, and therefore each irreducible finite-dimensional representation is one-dimensional.

By the (twisted) Satake isomorphism, $(\pi_0^\theta, V_\theta)$ is indeed a representation of $H_0$.

On the other hand, an irreducible representation $(\pi, V)$ is parameterized by the eigenvalue $\lambda$ of $\pi(A)$. Such a representation is isomorphic to $V_\theta$ if and only if $\theta + q\theta^{-1} = \lambda$. This equation always has one or two solutions $\theta, \theta' \in \mathbb{C}$ satisfying $\theta' = q\theta^{-1}$. 

We say that $\theta$ (or $q\theta^{-1}$) is the **Satake parameter** of the representation $V_\theta$. For a general operator $h \in H_0$, write $h(\theta) \in \mathbb{C}$ for the eigenvalue of $\pi_0^\theta(h)$ on $V_\theta$.

We will need the following estimates for the representation $V_\theta$:

**Lemma 2.10:** Let $0 \neq \theta \in \mathbb{C}$ satisfy $|\theta| \geq q|\theta|^{-1}$. Then,

1. For every $k \geq 0$, $|A_k(\theta)| \leq (k + 1)|\theta|^k$.
2. There exists an infinite number of $k$’s for which $|A_k(\theta)| \geq 2^{-3} \cdot |\theta|^k$.

**Proof.** For (1),

$$|A_k(\theta)| = \left| \theta^k + \tilde{\theta}^k + (1 - q^{-1}) \sum_{i=1}^{k-1} \theta^{k-i}\tilde{\theta}^i \right|$$

$$\leq |\theta|^k + |\tilde{\theta}|^k + (1 - q^{-1}) \sum_{i=1}^{k-1} |\theta^{k-i}\tilde{\theta}^i|$$

$$\leq |\theta|^k + |\theta|^k + (1 - q^{-1}) \sum_{i=1}^{k-1} |\theta|^k \leq (k + 1)|\theta|^k.$$
For (2) consider the following 3 cases.

(a) If $\theta = \tilde{\theta}$, then $\theta = \tilde{\theta} = \pm \sqrt{q}$. Then for every $k$ even, $A_k(\theta)$ is a sum of positive terms and $A_k(\theta) \geq |\theta|^k$.

(b) If $\theta \neq \tilde{\theta}$ and $|\theta| > |\tilde{\theta}|$, then for $k$ large enough

$$|\tilde{\theta}|^k - |\theta|^k - 1 \leq 2^{-1}|\theta|^{k-1}|\theta^2 - 1|,$$

so

$$|A_k(\theta)| \geq 2^{-1}|\theta - \tilde{\theta}|^{-1}|\theta^2 - 1||\theta|^{k-1}. $$

We have that $|\theta - \tilde{\theta}| \leq 2|\theta|$ so $|\theta - \tilde{\theta}|^{-1} \geq 2^{-1}|\theta|^{-1}$. Since $|\theta|^2 \geq q \geq 2$, $|\theta^2 - 1| \geq 2^{-1}|\theta|^2$. Therefore $|A_k(\theta)| \geq 2^{-3}|\theta|^{k}$.

(c) If $\theta \neq \tilde{\theta}$ and $|\theta| = |\tilde{\theta}|$, then $\tilde{\theta} = \tilde{\theta}$ and $|\theta| = q^{1/2}$. Since $q^{-1/2}\theta$ is on the unit circle and is different from $\pm 1$, there exists an infinite number of $k$’s such that the imaginary part of $(\theta^2 - 1)\theta^{k-1}$ is at least half of its absolute value. For such $k$’s

$$|(\theta^2 - 1)\theta^{k-1} - (\tilde{\theta}^2 - 1)\tilde{\theta}^{k-1}| = |(\theta^2 - 1)\theta^{k-1} - (\theta^2 - 1)\theta^{k-1}| \geq 2^{-1}|(\theta^2 - 1)\theta^{k-1}|. $$

Therefore

$$|A_k(\theta)| \geq 2^{-1}|\theta - \tilde{\theta}|^{-1}|\theta^2 - 1||\theta|^{k-1} \geq 2^{-3}|\theta|^{k}.$$ 

The last inequality follows from the same computation as in (b). Combining (a), (b) and (c) gives the explicit constant.

**Remark 2.11:** In number theory texts, it is more common to work with

$$B_k = \sum_{0 \leq l \leq k/2} A_{k-2l}$$

(see, for example, [LPS88, 4.10]). The bounds for $|\theta| = \sqrt{q}$ are well known: the bound $|B_k(\theta)| \leq (k+1)q^{(k-1)/2}$ with a similar derivation appears in Ramanujan’s original conjecture about the $\tau$ function ([Ram16, Section 18]).

### 2.3. Geometric Realization

The construction of $V_\theta$ can be realized as a subrepresentation of the action of $H_0$ on $C^{V_\tau}$ in two ways: the sectorial model and the spherical model.

To describe the sectorial model, fix an infinite ray (i.e., an infinite non-backtracking path) $R = (v_0, v_1, \ldots,)$ on the tree. We define the **relative distance** $c(v) = c_R(v) \in \mathbb{Z}$ of a vertex $v$ to the ray $R$ as follows: let $c(v_k) = -k$ for every vertex $v_k$ on the ray $R$, and for any other vertex $v$, if $v_k$ is the closest vertex to $v$ among all vertices in $R$, define $c(v) = c(v_k) + d(v, v_k)$—see Figure 2.1.
Figure 2.1. The sectorial distance $c(v)$ of a 3-regular tree with a chosen ray $(v_0,v_1,...)$. The value of $c$ is the same for all the vertices in each row of the figure.

Define $\tilde{f}_\theta \in \mathbb{C}^{V_T}$ by $\tilde{f}_\theta(v) = \theta^{-c(v)}$. Notice that the relative distance $c$ has the property that every vertex $v$ has one neighbor $u$ with $c(u) = c(v) - 1$, and $q$ neighbors $u_1,\ldots,u_q$ with $c(u_i) = c(v) + 1$. By this property of $c$, $\tilde{f}_\theta$ is an eigenvector of $A$, with eigenvalue $\theta + q\theta^{-1}$. Therefore, $\tilde{f}_\theta$ spans a representation space of $H_0$ isomorphic to $V_\theta$, which we call the sectorial model of $V_\theta$.

The relative distance to the ray will be used again in Proposition 2.20, and similar considerations can be used to derive the (twisted) Satake isomorphism.
While \( \tilde{f}_\theta \) realizes \( V_\theta \) as a subrepresentation of \( \mathbb{C}^{V_T} \), there exists an infinite number of vertices \( v \in V_T \) with \( \tilde{f}_\theta(v) = 1 \), so the function is not in \( L^p(V_T) \) for any \( p < \infty \). To obtain a representation with functions of controlled \( L^p \) norm, look at the vertex \( v_0 \in V_T \) that is the start of the ray \( R \). Let \( f_\theta = \rho_{v_0}(\tilde{f}_\theta) \), where \( \rho_{v_0} : \mathbb{C}^{V_T} \to \mathbb{C}^{V_T} \) is the spherical average operator of the introduction:

for \( f \in \mathbb{C}^{V_T} \), \( (\rho_{v_0}(f))(v_0) = f(v_0) \), and for \( v \neq v_0 \),

\[
(\rho_{v_0}(f))(v) = \frac{1}{\# \{v' : d(v_0, v) = d(v_0, v') \}} \sum_{v' : d(v_0, v_0) = d(v_0, v')} f(v')
\]

\[
= \frac{1}{(q + 1)q^{d(v_0, v) - 1}} (A_{d(v_0, v)} f)(v_0).
\]

Since \( \tilde{f}_\theta \) spans \( V_\theta \) we have \( (A_k \tilde{f}_\theta)(v_0) = A_k(\theta) \tilde{f}_\theta(v_0) = A_k(\theta) \), so explicitly \( f_\theta(v_0) = 1 \) and for \( v \neq v_0 \),

\[
f_\theta(v) = \frac{1}{(q + 1)q^{d(v_0, v) - 1}} A_{d(v_0, v)}(\theta).
\]

We claim that \( f_\theta \) also spans a representation which is isomorphic to \( V_\theta \), that is, \( A_k \) acts on \( f_\theta \) by \( A_k(\theta) \). We call the resulting representation the \textbf{spherical model} of \( V_\theta \), or the \textbf{geometric realization} of \( V_\theta \). The claim can be proven directly, but also follows from the following interesting lemma:

**Lemma 2.12:** The operator \( \rho_{v_0} \) commutes with the action of \( H_0 \) on \( \mathbb{C}^{V_T} \).

The intuition for the lemma is that \( H_0 \) commutes with automorphisms and \( \rho_{v_0} \) is the “average” of all automorphisms fixing \( v_0 \). Formalizing this intuition is left to the reader.

As for the \( L^p \)-norm of \( f_\theta \), we have

**Proposition 2.13:** Let \( |\theta| \geq |q|^{-1} \). Then \( f_\theta \in L^p(V_T) \) for \( 2 \leq p < \infty \) such that \( |\theta| < q^{(p-1)/p} \), and \( f_\theta \notin L^p(V_T) \) for \( 2 \leq p < \infty \) such that \( |\theta| \geq q^{(p-1)/p} \). For \( p = \infty \), for \( \sqrt{q} \leq |\theta| \leq q \) we have that \( f_\theta \in L^\infty(V_T) \) and for \( |\theta| > |q| \) we have that \( f_\theta \notin L^\infty(V_T) \).

**Proof.** We have \( f_\theta(v) = \frac{1}{(q + 1)q^{d(v_0, v) - 1}} A_{d(v_0, v)}(\theta) \) and there are \( (q + 1)q^{k-1} \) vertices of distance \( k \) from \( v_0 \).

First consider \( p < \infty \). Then \( ||f_\theta||_p^p = 1 + \sum_{k \geq 1} (q + 1)q^{k-1}(\frac{1}{(q + 1)q^{k-1}}|A_k(\theta)|)^p \).

Write

\[
a_k = ((q + 1)q^{k-1})^{1-p}|A_k(\theta)|^p = ((q + 1)q^{-1}q^k)^{1-p}|A_k(\theta)|^p
\]
for the $k$th element of the resulting series. Let $C = ((q+1)q^{-1})^{1-p}$. By Lemma 2.10 we have for $k \geq 1$, $a_k \leq C(k+1)^p q^{k(1-p)}|\theta|^{kp}$, and for an infinite number of $k$’s, $a_k \geq 2^{-3p}C q^{k(1-p)}|\theta|^{kp}$. Therefore, if $|\theta| < q^{(p-1)/p}$ then $\limsup a_k^{1/k} < 1$ and $\|f_\theta\|_p^p < \infty$, and if $|\theta| \geq q^{(p-1)/p}$ then $\|f_\theta\|_p^p = \infty$.

For $p = \infty$, by Lemma 2.10, $|f_\theta(v)|$ is not bounded for $|\theta| > q$, so $f_\theta \notin L^\infty(V_T)$. Since for $p < \infty$, $L^p(V_T) \subset L^\infty(V_T)$, it remains to consider $|\theta| = q$. Then $|A_k(\theta)| \leq A_k(q) = (q + 1)q^{k-1}$, so $|f_\theta(v)| \leq 1$ and $f_\theta \in L^\infty(V_T)$.

The calculations motivate us to define:

**Definition 2.14:** Given a representation $(\pi, V)$ of $H_0$, $u \in V$ and $\phi \in V^*$, we call the linear functional $c_{\phi, u}: H_0 \to \mathbb{C}$,

$$c_{\phi, u}(h) = \langle \phi, \pi(h)u \rangle$$

a **matrix coefficient** of $V$.

For every matrix coefficient we associate a **geometric realization** $f_{\phi, u} = f_{\phi, u} \in \mathbb{C}^{V_T}$ given by

$$f_{\phi, u}(v) = \frac{1}{(q+1)q^{d(v_0,v)-1}} \langle \phi, \pi(A_{d(v_0,v)})u \rangle.$$

We say that $V$ is **$p$-finite** if for every $u \in V$ and $\phi \in V^*$, we have that $f_{\phi, u}(v) \in L^p(V_T)$, or equivalently (for $p < \infty$)

$$1 + \sum_{k \geq 1} ((q+1)q^{k-1})^{1-p} |\langle \phi, \pi(A_k)u \rangle|^p < \infty.$$

We say that $V$ is **$p$-tempered** if it is $p'$-finite for every $p' > p$.

Notice that $\infty$-finite representations are also $\infty$-tempered by this definition.

The representation $V_\theta$ is one-dimensional, so it has only one matrix coefficient up to scale. We can then conclude:

**Corollary 2.15:** The representation $V_\theta$ is $p$-tempered if and only if

$$\max\{|\theta|, q|\theta|^{-1}\} \leq q^{(p-1)/p}.$$

A finite-dimensional representation which is a direct sum of irreducible representations $V = \bigoplus_i V_{\theta_i}$ is $p$-tempered if and only if each $V_{\theta_i}$ is $p$-tempered.

**Remark 2.16:** Matrix coefficients are very standard in representation theory, and in particular in representation theory of $p$-adic algebraic groups. The notion of geometric realization is not standard. The notions of sectorial model and
spherical model are not standard, and correspond to the Poincaré disk model and Poincaré half-plane model of the hyperbolic plane. The notion of a tempered representation usually refers to what we call a 2-tempered representation, and the notion of a \( p \)-tempered representation is not standard.

2.4. Action on Finite Graphs. Let \( X \) be a finite, connected, \((q + 1)\)-regular graph. As \( H_0 \) is freely generated by \( A \), an action of an operator \( A_0 \) on a vector space \( V \) extends to a representation \((\pi, V)\) of \( H_0 \), given by \( \pi(A) = A_0 \). Therefore, the standard action of the vertex adjacency operator \( A_X \) on \( \mathbb{C}^V_X \cong L^2(V_X) \) extends to a representation \((\pi_X, L^2(V_X))\) of \( H_0 \), given by \( \pi_X(A) = A_X \). Moreover, with respect to the standard \( L^2 \)-norm on \( V_X \) the operator \( A_X \) is self-adjoint, so it is diagonalizable and has real eigenvalues. By looking at the maximal value in absolute value of an eigenvector of \( A_X \), each eigenvalue of \( A_X \) is bounded in absolute value by \( q + 1 \), so its spectrum is within the range \([-q - 1, q + 1]\). Therefore the representation \((\pi_X, L^2(V_X))\) of \( H_0 \) is a finite direct sum of one-dimensional representations, and each Satake parameter \( \theta \) of such a representation satisfies that \( \lambda = \theta + q\theta^{-1} \) is real and of absolute value \( \leq q + 1 \).

Solving

\[
\theta, \tilde{\theta} = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2},
\]

we have two options: either \(|\lambda| \leq 2\sqrt{q} \) (the Ramanujan range) in which case \(|\theta| = \sqrt{q} \) and \( V_\theta \) is 2-tempered, or \( 2\sqrt{q} \leq |\lambda| \leq q + 1, \theta \) is real and \( 1 \leq |\theta| \leq q \) (i.e., \(-q \leq \theta \leq -1 \) or \( 1 \leq \theta \leq q \)), and \( V_\theta \) is \( p \)-tempered for \( p \) such that \( \max\{|\theta|, q|\theta|^{-1}\} \leq q^{(p-1)/p} \).

The eigenvalue \( q + 1 \) for \( A \) is only achieved on constant functions on \( X \). Similarly, the eigenvalue \(-q - 1 \) appears if and only if \( X \) is bipartite and is achieved on functions that are constant on each side of the graph, with one side negative of the other. Ignoring these two cases we seek the decomposition of the \( H_0 \)-representation \((\pi_X, L^2_{00}(V_X))\), as in the introduction. We can now prove Theorem 1.1:

**Proof of Theorem 1.1.** Let \( \lambda \) be the largest absolute value of an eigenvalue of \( A_X \) on \( L^2_{00}(V_X) \). By Corollary 2.15, \( \lambda \leq q^{1/p} + q^{(p-1)/p} \) if and only if \( L^2_{00}(V_X) \) is \( p \)-tempered.

Note that the function \( \rho_{\bar{v}_0}(\tilde{f}) \) as in the introduction is a special case of a geometric realization of \( L^2_{00}(V_X) \). Let \( \tilde{v}_0 \) be a projection of \( v_0 \in V_T \) to \( V_X \), and denote by \( 1_{\tilde{v}_0} \in L^2(V_X) \) the function whose value is 1 on \( \tilde{v}_0 \) and 0 elsewhere.
We have that for \( f \in L^2_{00}(V_X) \),
\[
c_1\tilde{v}_0, f(h) = \langle 1_{\tilde{v}_0} \pi_X(h) f \rangle = (\pi_X(h) f)(\tilde{v}_0)
\]
is a matrix coefficient, and its corresponding geometric realization \( f_{1_{\tilde{v}_0}, f} \) equals \( \rho_{v_0}(\tilde{f}) \) (note that \( 1_{\tilde{v}_0} \notin L^2_{00}(V_X) \), but still defines a linear functional on \( L^2_{00}(V_X) \)). Therefore, if \( \lambda \leq q^{1/p} + q^{(p-1)/p} \) then \( \rho_{v_0}(\tilde{f}) \in L^{p+\epsilon}(V_T) \) for every \( \epsilon > 0 \).

As for the other implication, notice that every matrix coefficient of \( L^2_{00}(V_X) \) is a finite linear sum of matrix coefficients of the form
\[
c_1\tilde{v}_0, \tilde{f}(h) = (\pi_X(h) \tilde{f})(\tilde{v}_0),
\]
for \( f \in L^2_{00}(V_X) \), \( \tilde{v}_0 \in V_X \). Therefore if \( \rho_{v_0}(\tilde{f}) \in L^{p+\epsilon}(V_T) \) for every \( v_0 \in V_T \), \( f \in L^2_{00}(V_X) \), then every geometric realization of \( L^2_{00}(V_X) \) is in \( L^{p+\epsilon}(V_T) \) and \( L^2_{00}(V_X) \) is \( p \)-tempered.

2.5. THE \( L^p \)-SPECTRUM OF HECKE OPERATORS. In this subsection we explain the connection between the notion of \( p \)-temperedness and the spectrum of Hecke operators on \( L^p(V_T) \).

Recall that the norm bounded operator \( h \) on a Banach space \( V \) is
\[
\|h\| = \sup_{\|v\|=1} \|hv\|.
\]
The set of eigenvalues of \( h \) is the set of \( \lambda \in \mathbb{C} \) such that there exists \( 0 \neq v \in V \), with \( hv = \lambda v \). The approximate point spectrum of \( h \) is the set of \( \lambda \in \mathbb{C} \) such that for every \( \epsilon > 0 \) there exists \( 0 \neq v \in V \), with \( \|h - \lambda v\| < \epsilon \|v\| \). The spectrum of \( h \) is the set of \( \lambda \in \mathbb{C} \) such that \( h - \lambda \) has no bounded inverse. The residual spectrum is the complement in the spectrum of the approximate point spectrum. It is well known that the norm of \( h \) bounds the absolute value of every \( \lambda \) in its spectrum.

We will need the following lemmas in our calculations:

**Lemma 2.17:** Let \( x_1, \ldots, x_m \in \mathbb{C} \). Then
\[
\left| \sum_{i=1}^m x_i \right|^p \leq m^{p-1} \sum_{i=1}^m |x_i|^p
\]
for every \( p \geq 1 \), with an equality if all the numbers are equal.

**Proof.** First, \( \left| \sum_{i=1}^m x_i \right|^p \leq (\sum_{i=1}^m |x_i|)^p \). By the convexity of \( f(x) = x^p \) in \( \mathbb{R}_{\geq 0} \) we have \( \frac{1}{m} \sum_{i=1}^m |x_i| \leq \frac{1}{m} \sum_{i=1}^m |x_i|^p \). The equality part is trivial.
Lemma 2.18: Let $X = X_0 \cup X_1$ be a biregular graph, such that every $x \in X_0$ is connected to $K_0$ vertices in $X_1$, and every $y \in X_1$ is connected to $K_1$ vertices in $X_0$.

Let $\tilde{A} : \mathbb{C}^{X_0} \to \mathbb{C}^{X_1}$ be the adjacency operator from $X_0$ to $X_1$, i.e.,

$$\tilde{A} f(y) = \sum_{x \sim y} f(x).$$

Then as an operator $\tilde{A} : L^p(X_0) \to L^p(X_1)$, we have $\|\tilde{A}\|_p \leq K_0^{1/p} K_1^{(p-1)/p}$, with an equality if the graph is finite.

Proof. For $f \in L^p(X_0)$, we have

$$\|\tilde{A}f\|_p^p = \sum_{y \in X_1} |\tilde{A}f(y)|^p = \sum_{y \in X_1} \left| \sum_{x \sim y} f(x) \right|^p \leq \sum_{y \in X_1} K_1^{p-1} \sum_{x \sim y} |f(x)|^p = K_1^{p-1} \sum_{x \in X_0} |f(x)|^p \sum_{y \sim x} 1 = K_1^{p-1} K_0 \|f\|_p^p.$$

The inequality is a result of Lemma 2.17. It is an equality if $f$ is constant, and if the graph is finite such a function is in $L^p(X_0)$.

The following proposition shows that the $L^p$-spectrum of Hecke operators must contain certain elements.

Proposition 2.19: Let $h \in H_0$. If $V_\theta$ is $p$-tempered, then $h(\theta)$ is an eigenvalue of $h$ on $L^{p'}(V_T)$ for every $p' > p$, and $h(\theta)$ is in the approximate point spectrum of $h$ on $L^p(V_T)$.

Proof. The geometric realization of $V_\theta$ provides us with a function

$$f_\theta \in \bigcap_{p' > p} L^{p'}(V_T)$$

which is an eigenvector of $h$ with eigenvalue $h(\theta)$. The first claim follows.

For the second claim, let $\epsilon > 0$ and define $f_\theta^\epsilon \in \mathbb{C}^{V_T}$ by

$$f_\theta^\epsilon(v) = f_\theta(v)(1 - \epsilon)^{d(v,v_0)}.$$

We claim that $f_\theta^\epsilon \in L^p(V_T)$. Using the same arguments as in Proposition 2.13, let $a_k$, $a_k^\epsilon$ be the $k$th elements in the series in the calculations of $\|f_\theta\|_p^p$, $\|f_\theta^\epsilon\|_p^p$. 

Let Proposition 2.20: for the approximate eigenvalue $\lambda$ as approximate eigenvectors.

and the series of $\|f_\ell^\epsilon\|_p^p$ converges.

Let us calculate $\|Af_\ell^\epsilon - A(\theta)f_\ell^\epsilon\|_p$. Assume that $\epsilon < 1/2$. For $v \in V_T$,

$$|(Af_\ell^\epsilon - A(\theta)f_\ell^\epsilon)(v)|^p$$

$$=|(Af_\ell^\epsilon - (1 - \epsilon)d(v,v_0)Af_\ell^\epsilon)(v) + ((1 - \epsilon)d(v,v_0)Af_\ell^\epsilon - A(\theta)f_\ell^\epsilon)(v)|^p$$

$$=|(Af_\ell^\epsilon - (1 - \epsilon)d(v,v_0)Af_\ell^\epsilon)(v) + 0|^p$$

$$=\sum_{v' \sim v} |f_\ell^\epsilon(v') - (1 - \epsilon)d(v,v_0)f_\ell(v')|^p$$

$$=\sum_{v' \sim v} |(1 - \epsilon)d(v',v_0)f_\ell(v') - (1 - \epsilon)d(v,v_0)f_\ell(v')|^p$$

$$\leq(q + 1)^{p-1} \sum_{v' \sim v} |1 - (1 - \epsilon)d(v,v_0) - d(v',v_0)|p|f_\ell^\epsilon(v')|^p$$

$$\leq(q + 1)^{p-1}2\epsilon^p \sum_{v' \sim v} |f_\ell^\epsilon(v')|^p.$$

The first inequality follows from Lemma 2.17. For the second inequality, since $|d(v,v_0) - d(v',v_0)| = 1$ and $\epsilon < 1/2$, we have $|1 - (1 - \epsilon)d(v,v_0) - d(v',v_0)| \leq 2\epsilon$. Summing over all $v \in V_T$, we get

$$\|Af_\ell^\epsilon - A(\theta)f_\ell^\epsilon\|_p^p < C\epsilon^p\|f_\ell^\epsilon\|_p^p,$$

for $C = 2^p(q + 1)^p$. Therefore as $\epsilon \to 0$, the $f_\ell^\epsilon$ are approximate eigenvectors for the approximate eigenvalue $A(\theta)$ of $A$.

Finally, since $A$ generates $H_0$, $h(\theta)$ is an approximate eigenvalue of $h$, with $f_\ell^\epsilon$ as approximate eigenvectors.

The following proposition bounds the $L^p$-norm of Hecke operators:

**Proposition 2.20:** Let $p \geq 2$. The norm (and therefore the absolute value of every element of the spectrum) of $A_k$ on $L^p(V_T)$ is bounded by

$$\|A_k\|_p \leq A_k(q^{(p-1)/p}) \leq (k + 1)q^{k(p-1)/p}.$$ 

In particular, the norm of $A$ is bounded by

$$\|A\|_p \leq A(q^{(p-1)/p}) = q^{1/p} + q^{(p-1)/p}.$$
Proof. Consider an infinite ray $R$ on the tree. Recall from the discussion in Subsection 2.3 that every vertex $v$ has one neighbor $u_0^v$ with relative distance $c(u_0^v) = c(v) - 1$ and $q$ neighbors $u_1^v, \ldots, u_q^v$ with $c(u_i^v) = c(v) + 1$.

Define $h_0, h_1 : \mathbb{C}^{|V_T|} \rightarrow \mathbb{C}^{|V_T|}$ as follows: let $f \in \mathbb{C}^{|V_T|}$. Then

$$h_0 f(v) = \sum_{i=1}^{q} f(u_i^v),$$

i.e., the sum of $f$ on the $q$ vertices that have greater relative distance. Similarly,

$$h_1 f(v) = f(u_0^v)$$

is the value of $f$ on the single neighbor of $v$ that has shorter relative distance.

We first prove the proposition for $A$. By definition, we have $A = h_0 + h_1$. We claim that

$$\|h_0 f\|_p \leq q^{(p-1)/p} \|f\|_p,$$

$$\|h_1 f\|_p = q^{1/p} \|f\|_p.$$

The equality is immediate, since every value in (the series of) $\|h_1 f\|_p^p$ is a value in (the series of) $\|f\|_p^p$, while each value in $\|f\|_p^p$ appears $q$ times in $\|h_1 f\|_p^p$. The inequality follows from Lemma 2.17, since each value in $\|h_0 f\|_p^p$ is a sum of $q$ values in $\|f\|_p^p$, and each value in $\|f\|_p^p$ appears in exactly one such sum.

Therefore $\|A\|_p \leq \|h_0\|_p + \|h_1\|_p \leq q^{1/p} + q^{(p-1)/p}$.

The proof for $A_k$ is a direct generalization: we can write $A_k = h_0 + \cdots + h_k$, where:

- $h_0 f(v)$ is the sum of $f$ on the $q^k$ vertices $u$, with $d(v, u) = k$ and $c(u) - c(v) = k$.
- $h_k f(v)$ is the value of $f$ on the single vertex $u$, with $d(v, u) = k$ and $c(u) - c(v) = -k$.
- $h_i f(v)$, $0 < i < k$ is the sum of $f$ on the $(q-1)^{q^k - i}$, $(k - 1)^{q^k - i}$ vertices $u$ with $d(v, u) = k$ and $c(u) - c(v) = k - 2i$.

Write for simplicity $\theta_p = q^{(p-1)/p}$. Then:

$$\|h_0\|_p \leq q^{k(p-1)/p} = \theta_k^p,$$

$$\|h_k\|_p = q^{k/p} = (q^{1/p})^k,$$

$$\|h_i\|_p \leq (1 - q^{-1})^{(q^k - i)(p-1)/p} q^{i/p} = (1 - q^{-1})^{k}q^{i/p} = (1 - q^{-1})^{k+i}(q^{1/p})^i.$$

The bounds for $h_0$ and $h_k$ are proved similarly to the bounds in the calculations for $A$. Let us prove the bounds for $0 < i < k$: build a bipartite (infinite)
directed graph $G_i$, with $X_0 = V_T \times \{0\}$, $X_1 = V_T \times \{1\}$. Connect $(u, 0)$ to $(v, 1)$ if $d(v, u) = k$ and $c(u) - c(v) = k - 2i$. Then the adjacency operator from $C^{X_0}$ to $C^{X_1}$ acts exactly like the operator $h_i$ acts on $C^{V_T}$. With the notations of Lemma 2.18, have that

$$K_0 = (q - 1)q^{i-1} = (1 - q^{-1})q^i$$

and

$$K_1 = (q - 1)q^{k-i-1} = (1 - q^{-1})q^{k-i}.$$

Now apply Lemma 2.18 and organize to arrive at the given bounds.

Therefore,

$$\|A_k\|_p \leq \|h_0\|_p + \cdots + \|h_k\|_p$$

$$\leq \theta^k + (q\theta^{-1})^k + \sum_{i=1}^{k-1} (1 - q^{-1})\theta^{k-i}(q\theta^{-1})^i$$

$$= A_k(\theta_p) = A_k(q^{(p-1)/p}).$$

We finish by using Lemma 2.10.

We can now prove Theorem 1.2:

**Corollary 2.21:** For $2 \leq p \leq \infty$, the spectrum of $A$ on $L^p(V_T)$ is

$$\{\theta + q\theta^{-1} : \theta \in \mathbb{C}, q^{1/p} \leq |\theta| \leq q^{(p-1)/p}\},$$

and each point of it belongs to the approximate point spectrum. For $p = \infty$ each element of the spectrum is an eigenvalue. For $2 \leq p < \infty$ the set of eigenvalues is the interior $\{\theta + q\theta^{-1} : \theta \in \mathbb{C}, q^{1/p} < |\theta| < q^{(p-1)/p}\}$.

For $1 \leq p < 2$ the spectrum is the same as for $p/(p-1)$. For $1 < p \leq 2$ the interior $\{\theta + q\theta^{-1} : \theta \in \mathbb{C}, q^{(p-1)/p} < |\theta| < q^{1/p}\}$ is in the residual spectrum, and the boundary $\{\theta + q\theta^{-1} : \theta \in \mathbb{C}, |\theta| = q^{(p-1)/p}\}$ belongs to the approximate point spectrum but is not an eigenvalue. For $p = 1$ the entire spectrum belongs to the residual spectrum.

**Proof.** Assume $p \geq 2$. By Corollary 2.15 and Proposition 2.19, every point in the interior is an eigenvalue and every point in the boundary is in the approximate point spectrum.

For $2 \leq p < \infty$ points in the boundary are not eigenvalues, since if $f \in L^p(V_T)$ is an eigenvector with eigenvalue $A(\theta)$, and $f(v_0) \neq 0$, then also $\rho_{v_0}f \in L^p(V_T)$ and $\rho_{v_0}f(v_0) \neq 0$. But then necessarily $\rho_{v_0}f = f(v_0)f_{\theta}$ and $f_{\theta} \notin L^p(V_T)$ for $\theta$.
in the boundary by Corollary 2.15. If \( A(\theta) \in \text{Spec}_{L^p(V_T)} A \), then by the Satake isomorphism \( A_k(\theta) \in \text{Spec}_{L^p(V_T)} A_k \) for every \( k \geq 1 \) and, in particular, the norm of \( A_k \) on \( L^p(V_T) \) is at least \( |A_k(\theta)| \). Assume \( |\theta| \notin [q^{1/p}, q^{(p-1)/p}] \) and \( |\theta| \leq q|\theta|^{-1} \), then \( |\theta| > q^{(p-1)/p} \). By Lemma 2.10(b) there exist infinitely many \( k > 0 \), with \( \|A_k\|_p \geq |A_k(\theta)| \geq 0.001|\theta|^k \). By Proposition 2.20, \( \|A_k\|_p \leq (k + 1)(q^{(p-1)/p})^k \), and since \( |\theta| > q^{(p-1)/p} \) we have a contradiction.

For \( 1 \leq p < 2 \), note that the action of \( A \) on \( L^p/(p-1)(V_T) \) is the dual of the action of \( A \) on \( L^p(V_T) \) (this is also true for \( p = 1 \), \( p/(p - 1) = \infty \)). There are no eigenvalues since \( L^p(V_T) \subset L^2(V_T) \) and there are no eigenvectors for \( L^2(V_T) \). By basic facts of spectral theory, the spectrum of an operator is equal to the spectrum of its dual. Moreover, for reflexive Banach spaces, the discrete spectrum (i.e., the eigenvalues union the residual spectrum) and the continuous spectrum (i.e., the approximate point spectrum without the eigenvalues) of dual operators agree. As there are no eigenvalues, for \( 1 < p < 2 \) the interior is in the residual spectrum and the boundary is in the continuous spectrum. For \( p = 1 \) one uses the fact that the residual spectrum of an operator without eigenvalues is equal to the set of eigenvalues of its dual.

Remark 2.22: For \( p = 2 \), Proposition 2.19 and Proposition 2.20 are versions of Theorem 1 and Theorem 2 of [CHH88]. The proof of Proposition 2.20 is based on the proof of Theorem 2 in [CHH88]. A similar combinatorial proof for \( p = 2 \) is given in [AFH15] Theorem 4.2.

3. Operators on edges

3.1. The Iwahori–Hecke Algebra. We wish to extend the \( L^p \)-theory to operators acting on the directed edges of the tree or the graph. The theory here is slightly more complicated, since the algebra is not commutative and the operators are not self-adjoint. Since the proofs are very similar to the vertex case, some of them are omitted. In any case, a generalized full treatment is given in [Kam16].

We denote by \( E_T \) the directed edges of the tree and by \( E_X \) the directed edges of the (finite, non-oriented) graph \( X \) from the introduction. Each non-oriented edge is counted twice in \( E_T \) and \( E_X \).
Definition 3.1: Let $h_{s_0}, h_{s_1}, h_\tau, h_{NB} : \mathbb{C}^{ET} \to \mathbb{C}^{ET}$ be the following operators:

\[
\begin{align*}
&h_{s_0} \tilde{f}(x, y) = \sum_{y' \sim x, y' \neq y} \tilde{f}(x, y'), \\
&h_{s_1} \tilde{f}(x, y) = \sum_{x' \sim y, x' \neq y} \tilde{f}(x', y), \\
&h_\tau \tilde{f}(x, y) = \tilde{f}(y, x), \\
&h_{NB} \tilde{f}(x, y) = h_\tau h_{s_0} \tilde{f} = h_{s_1} h_\tau \tilde{f} = \sum_{x' \sim y, x' \neq x} \tilde{f}(y, x').
\end{align*}
\]

The Iwahori–Hecke algebra $H_\phi$, or the directed edge Hecke algebra, is the algebra of operators acting on $\mathbb{C}^{ET}$ generated by the operators $h_{s_0}, h_{s_1}$ and $h_\tau$.

We will show in the beginning of Subsection 3.2 that there is a natural representation $(\pi_X, L^2(E_X))$ of $H_\phi$. For now we analyze the action of $H_\phi$ on $\mathbb{C}^{ET}$. Notice, however, that the operator $\pi_X(h_{NB})$ is Hashimoto’s non-backtracking operator (see [Has89]). The non-backtracking operator is used in the theory of the graph Zeta function, which is defined as

\[
\zeta_X(u) = \frac{1}{\det(1 - u \pi_X(h_{NB}))}.
\]

Our discussion here is indeed similar to the discussion of Hashimoto on the Zeta function in [Has89].

Definition 3.2: Let $(\hat{W}, S)$ be the extended Coxeter group

$$
\hat{W} = \langle s_0, s_1, \tau | s_0^2 = s_1^2 = \tau^2 = 1, \tau s_0 = s_1 \tau \rangle,
$$

with its set of generators $S = \{ s_0, s_1, \tau \}$.

Let $w_{NB} \in \hat{W}$ be the element $\tau s_0 = s_1 \tau$.

Lemma 3.3: Each $w \in \hat{W}$ can be written uniquely as $w = \tau^{\delta_\tau} w_{NB}^{m} s_1^{\delta_1}$, for $\delta_\tau, \delta_1 \in \{0, 1\}$ and $m \geq 0$.

Proof. By the relations involving $\tau$, every $w \in \hat{W}$ may be written uniquely as $w = \tau^{\delta'} w'$ where $\delta' \in \{0, 1\}$ and $w'$ is a product of $s_0$ and $s_1$ only. Since $s_0^2 = s_1^2 = 1$, $w'$ may be written uniquely as $w' = s_0^{\alpha_0} (s_1 s_0)^{m'} s_1^{\delta_1}$,
with $m' \geq 0$, $\alpha_0, \delta_1 \in \{0, 1\}$. Since $w_{NB}^2 = s_1 r s_0 = s_1 s_0$,
\[ w = \tau^{\delta'} w' = \tau^{\delta'} s_0^{\alpha_0} (s_1 s_0)^{m'} s_1^{\delta_1} \]
\[ = \tau^{\delta'} \tau^{\alpha_0} w_{NB}^{2m'} + \alpha_0 s_1^{\delta_1} = \tau^{\delta'} w_{NB}^m s_1^{\delta_1}, \]
with $\delta = \delta' + \alpha_0 \mod 2$, $m = 2m' + \alpha_0$.

As one may recover $m', \alpha_0, \delta'$ from $m, \delta$, it also proves uniqueness.

**Definition 3.4:** The Coxeter length function $l : \hat{W} \rightarrow \mathbb{N}$ is defined by
\[ l(\tau^{\delta'} w_{NB}^m s_1^{\delta_1}) = m + \delta_1. \]

For $w = \tau^{\delta'} w_{NB}^m s_1^{\delta_1} \in \hat{W}$ we denote
\[ h_w = h_{\tau^{\delta'}} h_{NB}^m h_{s_1}^{\delta_1}. \]

Notice that our two different notations for $h_{s_0}, h_{\tau}, h_{s_1}$ agree with each other and that $h_{w_{NB}} = h_{NB}$.

For $e \in E_T$, we denote by $1_e \in C^{ET}$ the function whose value is 1 on $e$ and 0 elsewhere.

**Lemma 3.5:** Let $e_0, e_1 \in E_T$. Then:

1. The function $h_w 1_{e_1}$ is non-zero on $q^{l(w)}$ edges.
2. There exists a unique $w \in \hat{W}$ such that $h_w 1_{e_1}$ is non-zero on $e_0$, and then $h_{w^{-1}} 1_{e_0}$ is non-zero on $e_1$.

**Proof.** Proved easily by the decomposition
\[ w = \tau^{\delta'} w_{NB}^m s_1^{\delta_1} \]
and induction on $l(w)$.

**Definition 3.6:** For $e_0, e_1 \in E_T$, the **distance** $d(e_0, e_1) \in \hat{W}$ is the unique $w \in \hat{W}$ such that $h_w 1_{e_1}$ is supported on $e_0$.

Notice that by Lemma 3.5, if $d(e_0, e_1) = w$ then $d(e_1, e_0) = w^{-1}$.

While this definition is not standard in combinatorics, it is standard when treating the tree $T$ as a building. The abstract reason for this definition is the following lemma, which is left for the reader.

**Lemma 3.7:** Let $e_0, e_1, e_0', e_1' \in E_T$. Then $d(e_0, e_1) = d(e_0', e_1')$ if and only if there exists a tree automorphism $\gamma \in \text{Aut}(T)$ such that $\gamma(e_0) = e_0'$ and $\gamma(e_1) = e_1'$. 

As a result of Lemma 3.5 and the definition of \( d(e_0, e_1) \), we may write for \( w \in \hat{W} \) and \( f \in \mathbb{C}^{\mathcal{E}_T} \)

\[
h_w f(e_0) = \sum_{e_1 : d(e_0, e_1) = w} f(e_1).
\]

We can now describe \( H_\phi \) as follows:

**Lemma 3.8:** The algebra \( H_\phi \) is isomorphic to the algebra defined abstractly by the generating operators \( h_{s_0}, h_{s_1}, h_\tau \), and the relations

\[
\begin{align*}
h_{s_0}^2 &= q \cdot \text{Id} + (q - 1) h_{s_0}, \\
h_{s_1}^2 &= q \cdot \text{Id} + (q - 1) h_{s_1}, \\
h_\tau^2 &= \text{Id}, \\
h_\tau h_{s_0} &= h_{s_1} h_\tau.
\end{align*}
\]

The algebra \( H_\phi \) is also isomorphic to the algebra which is the linear span of the basis operators \( h_w, w \in \hat{W} \), with the relations above and the relation for \( w \in \hat{W} \) and \( s \in S \)

\[
h_w h_s = h_{ws} \quad \text{if } s = \tau \text{ or } l(ws) = l(w) + 1.
\]

**Remark 3.9:** The relations of Lemma 3.8 are called the **Iwahori–Hecke relations**. It easily follows from them that for \( w, w' \in \hat{W} \) and \( s \in S \),

\[
\begin{align*}
h_{ww'} &= h_w h_{w'}, & \text{if } l(ww') = l(w) + l(w'), \\
h_w h_s &= q h_w + (q - 1) h_{ws}, & \text{if } l(ws) = l(w) - 1.
\end{align*}
\]

Some of the relations are actually redundant, as \( H_\phi \) is generated by \( h_{s_0}, h_\tau \) and the two relations \( h_{s_0}^2 = q \cdot \text{Id} + (q - 1) h_{s_0} \) and \( h_\tau^2 = \text{Id} \).

**Proof.** The fact that the two abstract descriptions are isomorphic is standard and left to the reader (see [Mac96] for the general case). So it is enough to prove that \( H_\phi \) is isomorphic to the second description, using a linear basis of the algebra. By Lemma 3.5, the \( h_w \in H_\phi, w \in \hat{W} \) are indeed linearly independent. The first 4 relations may be verified directly. The last relation follows from Lemma 3.5.

**Lemma 3.10:** The algebra \( H_\phi \) has an involution (or adjunction) \( * : H_\phi \to H_\phi \) sending each \( \alpha \cdot h_w, \alpha \in \mathbb{C}, w \in \hat{W} \) to \( \bar{\alpha} \cdot h^{-1}_w \).
Proof. Consider the natural inner product on $L^2(E_T)$. By Lemma 3.5, for each $e_0, e_1 \in E_T$ and $w \in \hat{W}$ we have

$$\langle h_w 1_{e_0}, 1_{e_1} \rangle = \langle 1_{e_0}, h_{w^{-1}} 1_{e_1} \rangle.$$ 

It follows that $h_{w^{-1}}$ is the adjoint of $h_w$ relatively to the natural inner product on $L^2(E_T)$. As taking an adjoint is an involution, the claim follows. 

The following proposition, analogous to Proposition 2.4, gives an abstract definition of $H_\phi$.

**Proposition 3.11:** Let $\gamma \in \text{Aut}(T)$ be an automorphism of the tree. Then $\gamma$ acts naturally on $C^{E_T}$ by

$$\gamma \cdot f(x, y) = f(\gamma^{-1} x, \gamma^{-1} y).$$

The algebra $H_\phi$ is the algebra of row and column finite operators acting on $C^{E_T}$ and commuting with tree automorphisms.

**Proof.** As in Proposition 2.4, making use of Lemma 3.7. 

It is natural to study $H_0$ and $H_\phi$ together. We can do it by defining a larger algebra containing them both. Following Proposition 2.4 and Proposition 3.11, one can define:

**Definition 3.12:** The **full graph Hecke algebra** $H$ is the algebra of row and column finite operators acting on $C^{E_T} \oplus C^{V_T}$ and commuting with tree automorphisms.

Consider the composition: $C^{E_T} \oplus C^{V_T} \xrightarrow{p} C^{V_T} \xrightarrow{A_m} C^{V_T} \xrightarrow{i} C^{E_T} \oplus C^{V_T}$, where $p$ is the projection and $i$ is the extension by zeros. Using it, $A_m$ extends to an operator acting on $C^{E_T} \oplus C^{V_T}$, which we will denote by abuse of notations $A_m$ again. This extension belongs to the full graph Hecke algebra $H$. Similarly, and again by abuse of notations, we may extend each operator $h \in H_\phi$ to an operator $h \in H$ to an operator $h \in H$. In other words, the algebras $H_0, H_\phi$ occur as subalgebras (with different units) of $H$ and $\text{Id}_H = \text{Id}_{H_0} + \text{Id}_{H_1}$.

Define the operators $u: C^{V_T} \to C^{E_T}, d: C^{E_T} \to C^{V_T}$ by

$$uf(x, y) = f(x),$$

$$df(x) = \sum_{y \sim x} f(x, y),$$
and extend them similarly to operators acting on $\mathbb{C}^{E_T} \oplus \mathbb{C}^{V_T}$. We have that $u, d \in H$, and the following relations hold:

\[
ud = hs_0 + Id_{H_\phi},
\]

\[
du = (q + 1)Id_{H_0},
\]

\[
A = dh_\tau u, \quad A_m = dh_\tau h^{m-1}_N u.
\]

One can give a complete description of $H$, either in terms of generators of an algebra or in terms of a linear basis. We will only give the description in terms of generators and relations.

**Proposition 3.13:** The algebra $H$ is isomorphic to the algebra defined abstractly by the generators $Id_{H_\phi}, h_{s_0}, h_{s_1}, h_\tau, d$ and $u$, the generating relations of $H_\phi$ (with $Id_{H_\phi}$ instead of $Id$, including the relations saying that $Id_{H_\phi}$ is the identity of $H_\phi$), and the relations

\[
du = hs_0 + id_{H_\phi},
\]

\[
u^2 = d^2 = uh = 0,
\]

for any $h \in H_\phi$.

**Proof.** Left to the reader. □

**Remark 3.14:** This subsection essentially proves that the $(q + 1)$-regular tree is a building. See [Ron09] for an introduction to buildings.

The group $W = \langle s_0, s_1 | s_0^2 = s_1^2 = 1 \rangle$ is the infinite dihedral group, which is an affine Coxeter group, or an affine Weyl group. Specifying the generators $S = \{s_0, s_1\}$ makes $(W, S)$ into a Coxeter system. Adding an automorphism $\tau$ which acts on $W$ by $\tau s_0 = s_1 \tau$ makes $\hat{W}$, with the extra data of $\{s_1, s_2\}$ and $\tau$, an extended affine Coxeter group, which is more commonly called an extended affine Weyl group. As it turns out, $\hat{W}$ is generated by $s_0, \tau$ and is isomorphic as a group to $W$ with generators $s_0, s_1$, but they are not isomorphic as Coxeter systems.

Similarly, the Iwahori–Hecke algebra we defined, which is common in representation theory, is an extended version of the standard Iwahori–Hecke algebra, which does not include $h_\tau$. While the name an “extended Iwahori–Hecke algebra” is appropriate for it, it is called in the literature either an Iwahori–Hecke algebra, a Hecke algebra, or an affine Hecke algebra. Standard references for Iwahori–Hecke algebras and Coxeter groups are [Mac96, Lus03].
In his fundamental paper [Has89], Hashimoto studied the graph Zeta function using the Iwahori–Hecke algebra and its representation theory. However, he used its non-extended version, which does not include $h_{NB}$.

Readers familiar with the Bernstein–Lusztig presentation of the Iwahori–Hecke algebra may note that $h_{NB}$ plays a crucial role in this presentation, as it corresponds (in the notations of [Mac96]) to the operator $Y^\lambda$, $\lambda$ a fundamental coweight.

The full graph Hecke algebra $H$ is not standard in representation theory.

3.2. The Representation Theory of the Iwahori–Hecke Algebra.

Recall from Lemma 3.10 that $H_\phi$ has an involution $*: H_\phi \to H_\phi$. Also recall from the Preliminaries that a representation $(\pi, V)$ of $H_\phi$ is called unitary if there exists an inner product on $V$ satisfying $\langle \pi(h)v_1, v_2 \rangle = \langle v_1, \pi(h^*)v_2 \rangle$ for every $v_1, v_2 \in V$ and $h \in H$.

Let $X$ be a $(q + 1)$-regular graph. As $T$ is the universal cover of $X$ (see [HLW06, Chapter 6]), we may consider $X$ as a quotient of $T$ by a discrete cocompact torsion free group $\Gamma \subset \text{Aut}(T)$. Since the action of $H_\phi$ on $\mathbb{C}^{E_T}$ commutes with automorphisms, we have an action of $H_\phi$ on the $\Gamma$-invariant vectors of $\mathbb{C}^{E_T}$, which we identify with $\mathbb{C}^{E_X} \cong L^2(E_X)$, i.e., functions on the directed edges of the finite graph. Moreover, this representation $(\pi_X, L^2(E_X))$ is unitary with respect to the usual inner product on $L^2(E_X)$.

3.2.1. Classification of Irreducible Representations. The following theorem is the Iwahori–Hecke analog of the Satake isomorphism.

**Theorem 3.15:** There exists an embedding $\Phi: H_\phi \to M_{2 \times 2}(\mathbb{C}[[\theta, \theta^{-1}]])$ ($\theta$ indeterminate), given by $(\tilde{\theta} = q\theta^{-1})$:

\[
\Phi(h_\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi(h_{s_0}) = \begin{pmatrix} 0 & \tilde{\theta} \\ \theta & q - 1 \end{pmatrix}, \quad \Phi(h_{s_1}) = \begin{pmatrix} q - 1 & \theta \\ \tilde{\theta} & 0 \end{pmatrix},
\]

\[
\Phi(h_{NB}^k) = \begin{pmatrix} \theta^k & (q - 1)(\theta^{-1} + \theta^{-2}\tilde{\theta} + \cdots + \tilde{\theta}^{-1}) \\ 0 & \tilde{\theta}^k \end{pmatrix},
\]

and for $w = \tau^\delta w_{NB}^m s_1^\delta$,

\[
\Phi(h_w) = \Phi(h_\tau)^\delta \Phi(h_{NB}^m) \Phi(h_{s_1})^\delta.
\]

For any specific $0 \neq \theta \in \mathbb{C}$, the evaluation map

\[
\tilde{\pi}_\theta: M_{2 \times 2}(\mathbb{C}[[\theta, \theta^{-1}]]) \to M_{2 \times 2}(\mathbb{C})
\]
defines a 2-dimensional representation \((\pi_\theta^\phi, U_\theta)\) of \(H_\phi\), with \(\pi_\theta^\phi = \pi_\theta \circ \Phi\) and \(U_\theta = \mathbb{C}^2\). Every irreducible finite-dimensional representation is isomorphic to a quotient of such a representation.

**Proof.** By Lemma 3.8, one should prove that the set \(\{\Phi(h_w) : w \in \hat{W}\}\) is linearly independent (over \(\mathbb{C}\)) and that the Iwahori–Hecke relations hold. We leave the verification to the reader.

To prove the statement about every irreducible representation, let \((\pi, V)\) be a finite-dimensional representation of \(H_\phi\), and let \(\theta\) be an eigenvalue of \(\pi(h_{NB})\) with eigenvector \(0 \neq v_0 \in V\). Let \(u_0, u_1 = \pi_\theta(\tau)u_0\) be the standard basis of \(U_\theta\). Define a linear transformation \(\varphi: U_\theta \to V\) by \(\varphi(u_0) = v_0\) and \(\varphi(u_1) = v_1 = \pi(\tau)v_0\). The fact that \(v_0\) is an eigenvector of \(\pi(h_{NB})\) with eigenvalue \(\theta\) and the Iwahori–Hecke relations says that the following relations hold:

\[
\begin{align*}
\pi(h_\tau)v_0 &= v_1, & \pi(h_\tau)v_1 &= v_0, \\
\pi(h_{s_1})v_1 &= \pi(h_{NB})v_0 = \theta v_0, \\
\pi(h_{s_1})v_0 &= \pi(h_{s_1}^2)\theta^{-1}v_1 = (q \cdot Id + (q - 1)\pi(h_{s_1}))\theta^{-1}v_1 = (q - 1)v_0 + \tilde{\theta}v_1, \\
\pi(h_{s_0})v_1 &= \pi(h_{s_0}h_\tau)v_0 = \pi(h_\tau h_{s_1})v_0 = (q - 1)v_1 + \tilde{\theta}v_0, \\
\pi(h_{s_0})v_0 &= \pi(h_\tau h_{s_1})v_1 = \theta v_1.
\end{align*}
\]

The relations show that \(\varphi\) is a homomorphism of representations of \(H_\phi\). Therefore, if \(V\) is irreducible it is equal to the image of \(\varphi\), and therefore \(V\) is isomorphic to a quotient of \(U_\theta\) (see the Preliminaries).

**Corollary 3.16:** For every \(\theta \neq \pm 1, \pm q\) the representation \((\pi_\theta^\phi, U_\theta)\) is irreducible, and is isomorphic to the representation \((\pi_{\theta^{-1}}^\phi, U_{\theta^{-1}})\).

There are four one-dimensional representations which occur as quotients of \((\pi_{\pm 1}^\phi, U_{\pm 1}^\phi), (\pi_{\pm q}^\phi, U_{\pm q}^\phi)\):

Two trivial representations \((\pi_T^\phi, U_T^\phi)\), where \(\tau\) acts by \(\pm 1\) and \(h_{s_0}, h_{s_1}\) act by \(q\);

Two Steinberg (or special) representations \((\pi_S^\phi, U_S^\phi)\), where \(\tau\) acts by \(\pm 1\) and \(h_{s_0}, h_{s_1}\) act by \(-1\).

**Proof.** The only possible irreducible representations \((\pi, V)\) that are not \(U_\theta\) are of dimension 1. In this case \(\pi(h_\tau)\) acts by \(\alpha_\tau = \pm 1\), and \(\pi(h_{s_0})\) acts by multiplication by a scalar \(\alpha_0\). By the Iwahori–Hecke relation \((h_{s_0} + 1)(h_{s_0} - q) = 0\),
we have $\alpha_0 = -1$ or $\alpha_0 = q$. Since $h_{s_1} = h_\tau h_{s_0} h_\tau$, the operator $\pi(h_{s_1})$ also acts by $\alpha_0$. On the other hand, each choice of $\alpha_\tau \in \{ \pm 1 \}$ and $\alpha_0 \in \{ -1, q \}$ defines a one-dimensional representation, as an easy verification of the Iwahori–Hecke relations shows. This gives us the four representations $U_S^\pm, U_{\overline{S}}^\pm$.

Let us explain the connection between the $H_0$-representation $(\pi_0^0, V_\theta)$ and the $H_{\phi}$-representation $(\pi_{\phi}^0, U_\theta)$. Recall that we defined in Definition 3.12 a larger algebra $H$ containing both $H_0$ and $H_{\phi}$ as subalgebras (with different units).

Now, $H \cdot \text{Id}_{H_{\phi}}$ is a right $H_{\phi}$-module and a left $H_0$-representation, so (see the Preliminaries) given a representation $(\pi, V)$ of $H_{\phi}$, $H \cdot \text{Id}_{H_{\phi}} \otimes_{H_{\phi}} V$ is an $H_0$-representation. To simplify notations we write it as $H \otimes_{H_{\phi}} V$, which is well defined if we extend the tensor notation to modules in which $\text{Id}_{H_{\phi}}$ does not act as the identity.

**Proposition 3.17:** We can induce an $H_{\phi}$-representation $(\pi, U)$ to an $H_0$-representation $(\text{ind}^H_{H_{\phi}} \pi_0, \text{ind}^H_{H_{\phi}} U_\theta)$ by choosing $\text{ind}^H_{H_{\phi}} U = H \otimes_{H_{\phi}} U$. We can restrict an $H_0$-representation $(\pi, W)$ to an $H_{\phi}$-representation $(\text{res}^H_{H_{\phi}} \pi_0, \text{res}^H_{H_{\phi}} W)$ by choosing $\text{res}^H_{H_{\phi}} W = \pi(\text{Id}_{H_{\phi}}) W$. Induction and restriction define a bijection between isomorphism classes of irreducible finite-dimensional $H_{\phi}$-representations and $H_0$-representations.

**Proof.** To simplify notations in the proof, we let $H$ and $H_{\phi}$ act directly on the representation spaces without mentioning $\pi$.

The idea behind the proof is the decomposition

$$H = H_{\phi} \oplus H_{\phi} u \oplus dH_{\phi} \oplus dH_{\phi} u,$$

where $H_{\phi} u = \{ hu : h \in H_{\phi} \}$, and similarly for $dH_{\phi}, dH_{\phi} u$. This decomposition follows immediately from the relations stated in Proposition 3.13.

We start by showing that $U$ and $\text{res}^H_{H_{\phi}} \text{ind}^H_{H_{\phi}} U$ are isomorphic $H_{\phi}$-representations, and $W$ and $\text{ind}^H_{H_{\phi}} \text{res}^H_{H_{\phi}} W$ are isomorphic $H_0$-representations. It proves that induction and restriction define a bijection between isomorphism classes of $H_{\phi}$-representations and $H_0$-representations.

From Decomposition (3.1), we have that $H \cdot \text{Id}_{H_{\phi}} = H_{\phi} \oplus dH_{\phi}$, so

$$\text{ind}^H_{H_{\phi}} U = H \otimes_{H_{\phi}} U = \{ d \otimes v : v \in U \} \oplus \{ \text{Id}_{H_{\phi}} \otimes v : v \in U \}$$

$$= (d \otimes U) \oplus (\text{Id}_{H_{\phi}} \otimes U).$$
On the first factor $\text{Id}_{H_{\phi}}$ acts by 0, so
\[
\text{res}^H_{H_{\phi}} \text{ind}^H_{H_{\phi}} U = \text{Id}_{H_{\phi}} \cdot H \otimes_{H_{\phi}} U \\
= (\text{Id}_{H_{\phi}} d \otimes U) \oplus (\text{Id}_{H_{\phi}} \otimes U) \\
= \text{Id}_{H_{\phi}} \otimes U.
\]

It is immediate that $v \rightarrow \text{Id}_{H_{\phi}} \otimes v$ is an isomorphism of the $H_{\phi}$-representations $U$ and $\text{res}^H_{H_{\phi}} \text{ind}^H_{H_{\phi}} U = \text{Id}_{H_{\phi}} \otimes U$.

Given an $H$-representation $W$, from $\text{Id}_H = \text{Id}_{H_{\phi}} + \text{Id}_{H_0} = \text{Id}_{H_{\phi}} + (q+1)^{-1} d u$, we have
\[
W = \text{Id}_H W = \text{Id}_{H_{\phi}} W \oplus \text{Id}_{H_0} W = \text{Id}_{H_{\phi}} W \oplus d u W.
\]
We have $u W = \text{Id}_{H_{\phi}} W$, so $d u W = d \text{Id}_{H_{\phi}} W = d W$, so
\[
W = \text{Id}_{H_{\phi}} W \oplus d W.
\]

Therefore,
\[
\text{ind}^H_{H_{\phi}} \text{res}^H_{H_{\phi}} W = H \otimes_{H_{\phi}} \text{Id}_{H_{\phi}} W \\
= (\text{Id}_{H_{\phi}} \otimes_{H_{\phi}} \text{Id}_{H_{\phi}} W) \oplus (d \otimes_{H_{\phi}} \text{Id}_{H_{\phi}} W).
\]
The vector spaces $\text{Id}_{H_{\phi}} W$ and $\text{Id}_{H_{\phi}} \otimes_{H_{\phi}} \text{Id}_{H_{\phi}} W$ are naturally isomorphic as $H_{\phi}$-representations. Similarly, $d V = d \text{Id}_{H_{\phi}} W$ and $d \otimes_{H_{\phi}} \text{Id}_{H_{\phi}} W$ are naturally isomorphic as vector spaces. Therefore, $W$ and $\text{ind}^H_{H_{\phi}} \text{res}^H_{H_{\phi}} W$ are isomorphic as vector spaces. It is also easy to see that the actions of $d, u$ and the elements of $H_{\phi}$ on the two spaces agree, so by Proposition 3.13, $W$ and $\text{ind}^H_{H_{\phi}} \text{res}^H_{H_{\phi}} W$ are isomorphic $H$-representations.

It is also obvious that if $W$ is a finite-dimensional $H$-representation, then $\text{res}^H_{H_{\phi}} W$ is finite-dimensional, and if $U$ is a finite-dimensional $H_{\phi}$-representation, then $\text{ind}^H_{H_{\phi}} U = (d \otimes U) \oplus (\text{Id}_{H_{\phi}} \otimes U)$ is finite-dimensional.

It also follows from our explicit description above that if $U_1 \subset U$ is a proper non-trivial $H_{\phi}$-subrepresentation, then $\text{ind}^H_{H_{\phi}} U_1 \subset \text{ind}^H_{H_{\phi}} U$ is a proper non-trivial $H$-representation. Similarly, if $W_1 \subset W$ is a proper non-trivial $H$-representation, then $\text{res}^H_{H_{\phi}} W_1 \subset \text{res}^H_{H_{\phi}} W$ is a proper non-trivial $H_{\phi}$-representation.

Finally, if $U$ is irreducible, then $\text{ind}^H_{H_{\phi}} U$ is irreducible, otherwise it has a proper subrepresentation, whose restriction is a proper subrepresentation of $U$. Similarly, if $W$ is irreducible then $\text{res}^H_{H_{\phi}} W$ is irreducible. It concludes the proof. \hfill \blacksquare
A generalized version of Proposition 3.17 appears in [Kam16, Section 10]. To make Proposition 3.17 more explicit, we extend the embedding

$$\Phi: H_\phi \to M_{2 \times 2}(\mathbb{C}[\theta, \theta^{-1}])$$

to an embedding

$$\Phi': H \to M_{3 \times 3}(\mathbb{C}[\theta, \theta^{-1}]),$$

satisfying

$$\Phi'(h_w) = \begin{pmatrix} \Phi(h_w) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for $w \in \hat{W}$, and

$$\Phi'(u) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \theta \\ 0 & 0 & 0 \end{pmatrix}, \quad \Phi'(d) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \tilde{\theta} & 0 \end{pmatrix}.$$

Once again, this embedding can be derived from the presentation of the algebra using generators and relations.

Notice that using this description,

$$\Phi'(A) = \Phi'(dh_T u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \tilde{\theta} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta + \tilde{\theta} \end{pmatrix}.$$

The reader may verify by calculating

$$\Phi'(A_m) = \Phi'(dh_T h_{NB}^{m-1} u)$$

that one recovers the Satake isomorphism of Subsection 2.2.

**Proposition 3.18:** We denote by $(\pi_\theta, W_\theta), \theta \neq \pm 1, \pm q, (\pi_\pm^T, W_\pm^T), (\pi_\pm^S, W_\pm^S)$ the irreducible representations of $H$.

When restricted to an $H_0$-representation by $V = Id_{H_0} \cdot W$, the corresponding representations are:

1. For $(\pi_\theta, W_\theta), \theta \neq \pm 1, \pm q$: $(\pi_0^\theta, V_\theta)$.
2. For $(\pi_\pm^T, W_\pm^T)$: $(\pi_0^{\pm q}, V_{\pm q}) = (\pi_0^{\pm 1}, V_{\pm 1})$.
3. For $(\pi_\pm^S, W_\pm^S)$: the 0-representation.

Note that we recovered all the irreducible $H_0$-representation.
Remark 3.19: The theory presented here is a very simple case of the general representation theory of affine Iwahori–Hecke algebras. See [Mac96].

The names of the trivial representation and the Steinberg representation come from corresponding representations of the automorphism group of the tree.

3.2.2. Unitary Representations. We want to identify the irreducible unitary representations of $H_{\phi}$. It is immediate that the one-dimensional representations are unitary so we look at $(\pi_{\theta}^{\phi}, U_{\theta})$. Unitary representations $(\pi, U)$ satisfy the following:

1. The adjoint of $\pi(h_{NB}) = \pi(h_{s0})$ is

$$\pi(h_{NB}^*) = \pi(h_{s0\tau}),$$

and in unitary representation they have complex conjugate eigenvalues. The eigenvalues of $h_{NB} = h_\tau h_{s0}$ on $U_{\theta}$ are $\theta$ and $q\theta^{-1}$. The eigenvalues of its adjoint $h_{NB}^* = h_{s0\tau}$ are also $\theta$ and $q\theta^{-1}$. Therefore either $\theta = \bar{\theta}$, i.e., $\theta$ is real, or $\theta = q\theta^{-1}$, i.e., $|\theta|^2 = q$.

2. The eigenvalues of $\pi(h_{NB})$ are of absolute value $\leq q$, since $h_{NB} = h_\tau h_{s0}$ and the eigenvalues and therefore the norms of $\pi(h_\tau), \pi(h_{s0})$ are bounded by $1, q$. This condition bounds $\theta$ to $1 \leq |\theta| \leq q$, and since $U_{\pm1}, U_{\pm q}$ are reducible the actual bound is $1 < |\theta| < q$.

Summarizing, we proved one direction of the following proposition.

**Proposition 3.20:** The unitary irreducible representations of $H_{\phi}$ are the following representations:

1. $(\pi_{\theta}^{\phi}, U_{\theta})$, for $|\theta| = q^{1/2}$.
2. $(\pi_{\theta}^{\phi}, U_{\theta})$, for $\theta$ real $1 < |\theta| < q$.
3. The one-dimensional representations: $(\pi_{T}^{\phi, \pm}, U_{T}^{\pm}), (\pi_{S}^{\phi, \pm}, U_{S}^{\pm})$.

The proposition says that the algebraic definition of a unitary $H_{\phi}$-representation captures the combinatoric bounds we found in Subsection 2.4 on the Satake parameter. This is a very simple case of a general result of Barbasch and Moy ([BM93]). To complete the proof we need to prove that $(\pi_{\theta}^{\phi}, U_{\theta})$ for $\theta$ as in the proposition is indeed unitary. Since we will not use this part and it is slightly technical, we skip it and refer the reader to [Sav04, Section 9], where a similar claim is proven in the context of the representation theory of $p$-adic groups.
3.3. **Geometric Realization and the $L^p$-Spectrum.** Similarly to the vertex Hecke algebra, representations of the Iwahori–Hecke algebra can be realized on the tree. We will only describe the spherical model, although there also exists a sectorial model of an irreducible representation.

Recall that by Lemma 3.5 and Definition 3.5 we have a distance $d: E_T \times E_T \rightarrow \hat{W}$ and that for a given $e_0 \in E_T$ the number of $e \in E_T$ with $d(e_0, e) = w$ is $q^{l(w)}$.

**Definition 3.21:** Given a representation $(\pi, U)$ of $H_\phi$, $u \in U$ and $\varphi \in U^*$ we call the function $c_{\varphi, u}: H_\phi \rightarrow \mathbb{C}$, $c_{\varphi, u}(h) = \langle \varphi, \pi(h)u \rangle$ a **matrix coefficient** of $U$.

Fix $e_0 \in E_T$. For every matrix coefficient we associate a geometric realization $f_{\varphi, u}^{e_0} = f_{\varphi, u} \in \mathbb{C}^{V_T}$ given by

$$f_{\varphi, u}(e) = \frac{1}{q^{l(d(e_0, e))}} \langle \varphi, \pi(h_{d(e_0, e)})u \rangle.$$  

We say that $(\pi, U)$ is **$p$-finite** if for every $u \in U$, $\varphi \in U^*$, $f_{\varphi, u}(v) \in L^p(E_T)$, or equivalently

$$\sum_{w \in \hat{W}} q^{l(w)(1-p)} |\langle \varphi, \pi(h_w)u \rangle|^p < \infty.$$  

We say that $(\pi, U)$ is **$p$-tempered** if it is $p'$-finite for every $p' > p$.

A nice feature of working with the Iwahori–Hecke algebra is that $p$-temperedness is directly related to the eigenvalues of $h_{NB}$.

**Proposition 3.22:** Let $(\pi, U)$ be a finite-dimensional representation of $H_\phi$, and let $\rho_U(h_{NB})$ be the largest absolute value of an eigenvalue of $\pi(h_{NB})$. Then $(\pi, U)$ is $p$-tempered if and only if $\rho_U(h_{NB}) \leq q^{(p-1)/p}$.

Therefore, $(\pi_\theta^\phi, U_\theta)$ is $p$-tempered if and only if $\max\{|\theta|, q|\theta|^{-1}\} \leq q^{(p-1)/p}$, $(\pi_\phi^{S, \pm}, U_\phi^S)$ is $1$-tempered and $(\pi_{S, \pm}^{T, \pm}, U_T^\pm)$ is not $p$-finite for any $p < \infty$.

**Proof.** Every $w \in \hat{W}$ can be written uniquely as $w = \tau^{\delta_\tau} w_{NB}^m \delta_1^{\delta_1}$, for $\delta_\tau, \delta_1 \in \{0, 1\}$ and $m \geq 0$. Thus, $p$-finiteness is equivalent to

$$\sum_{\delta_\tau, \delta_1 \in \{0, 1\}} q^{\delta_1(1-p)} \sum_{m \geq 0} q^{(1-p)m} |\langle h_{\tau^{\delta_\tau}}^\delta \varphi, \pi(h_{NB}^m h_{\delta_1}^1 u) \rangle|^p < \infty$$  

for every $u \in U$, $\varphi \in U^*$.
This is reduced to the convergence of
\[ \sum_{m \geq 0} q^{(1-p)m} |\langle \varphi, \pi(h_{NB})^m u \rangle|^p \]
for every \( u \in U, \varphi \in U^* \). If \( u \) is an eigenvector of \( \pi(h_{NB}) \) with eigenvalue \( \theta \) with
\[ |\theta| \geq q^{(p-1)/p} \] and \( \langle \varphi, u \rangle \neq 0 \), then the series diverges. For the other direction, the theory of matrix norms says that for every \( u \in U \) and \( \varphi \in U^* \),
\[ \limsup_{m} |\langle \varphi, \pi(h_{NB})^m u \rangle|^{1/m} \leq \rho_U(h_{NB}), \]
which shows that if \( \rho_U(h_{NB}) < q^{(p-1)/p} \) the series converges. \( \blacksquare \)

As with representations of \( H_0 \), geometric realizations allow us to consider every irreducible representation of \( H_0 \) as a subrepresentation of \( C^E_T \).

The definition can be extended to \( H \) and agrees with the corresponding definition of \( p \)-temperedness of \( H_0 \)-representations for irreducible \( H \)-representations whose restriction to \( H_0 \) is non-zero.

The arguments of Subsection 2.5 can be extended to the following theorem:

**Theorem 3.23:** Let \( V \) be a finite-dimensional representation of \( H \). Let \( p \geq 2 \). The following are equivalent:

1. The eigenvalues of every \( h \in H \) are contained in the spectrum of \( h \) on \( L^p(V_T \oplus E_T) \).
2. \( V \) is \( p \)-tempered.

Specifically, the eigenvalues of \( h_{NB} \) on \( L^p(E_T) \) are
\[ \{ \pm 1 \} \cup \{ \theta \in \mathbb{C} \setminus \{ 0 \} : \max\{ |\theta|, q|\theta|^{-1} \} < q^{(p-1)/p} \} \]
and the approximate point spectrum of \( h_{NB} \) on \( L^p(E_T) \) is
\[ \{ \pm 1 \} \cup \{ \theta \in \mathbb{C} \setminus \{ 0 \} : \max\{ |\theta|, q|\theta|^{-1} \} \leq q^{(p-1)/p} \} \].

**3.4. The \( L^p \)-Expander Theorem.** We summarize the discussion above by the following theorem. For simplicity, we look at a finite non-bipartite graph \( X \), and denote
\[ L^2_0(V_X) = \left\{ f \in L^2(V_X) : \sum_{v \in V_X} f(v) = 0 \right\} \]
and
\[ L^2_0(E_X) = \left\{ f \in L^2(E_X) : \sum_{e \in E_X} f(e) = 0 \right\}. \]
Theorem 3.24: Let $X$ be a finite, connected, non-bipartite, $(q + 1)$-regular graph. For $p \geq 2$, the following are equivalent:

1. Every eigenvalue $\lambda$ of $A_X$ on $L^2_0(V_X)$ satisfies $|\lambda| \leq q^{1/p} + q^{(p-1)/p}$.
2. The only representations appearing in the decomposition of the $H_0$-action on $L^2(V_X)$ are $(\pi^0_\theta, V_\theta)$ with $\max\{|\theta|, q|\theta|^{-1}\} \leq q^{(p-1)/p}$ and $(\pi^0_q, V_q)$.
3. The only representations appearing in the decomposition of the $H_\phi$-action on $L^2(E_X)$ are $(\pi^\phi_\theta, U_\theta)$ with $\max\{|\theta|, q|\theta|^{-1}\} \leq q^{(p-1)/p}$, $(\pi^\phi_{\pm}, U^\pm_S)$ and $(\pi^\phi_{\mp}, U_T)$.
4. The $H_0$-representation $(\pi_X, L^2_0(V_X))$ is $p$-tempered.
5. The eigenvalues of $\pi_X(h)$ for every $h \in H$ on $L^2_0(V_X) \oplus L^2_0(E_X)$ are contained in the spectrum of $h$ on $L^p(V_T \oplus E_T)$.
6. For every $k$, every eigenvalue $\lambda_k$ of $\pi_X(A_k)$ on $L^2_0(V_X)$ satisfies
   $$|\lambda_k| \leq A_k(q^{(p-1)/p}) \leq (k + 1)q^{k(p-1)/p}.$$  
7. Every eigenvalue $\lambda$ of Hashimoto’s non-backtracking operator $\pi_X(h_{NB})$ on $L^2_0(E_X)$ satisfies $|\lambda| \leq q^{(p-1)/p}$.

3.5. Bipartite Biregular Graphs. In this subsection we show how to extend the previous results to biregular graphs.

Let $\tilde{T}$ be a biregular tree, i.e., each vertex $v$ is colored by $t(v) \in \{0, 1\}$, each edge contains one vertex of type 0 and one vertex of type 1, and each vertex of type $i \in \{0, 1\}$ is contained in $q_i + 1$ edges. As the case $q_0 = q_1$ was covered above, we assume $q_1 > q_0 \geq 1$.

Following Proposition 2.4 and Proposition 3.11, we define:

Definition 3.25: The vertex Hecke algebra $\tilde{H}_0$ is the algebra of row and column finite operators acting on $\mathbb{C}^{V_{\tilde{T}}}$ and commuting with automorphisms of $\tilde{T}$.

Since we have two types of vertices and an automorphism never sends a vertex of one type to the other (since $q_1 > q_0$), the algebra $\tilde{H}_0$ is slightly more complicated than the algebra $H_0$ of the regular case. In particular, $\tilde{H}_0$ is not generated by the adjacency operator $A$ and is not commutative. However, the algebra still contains the operators $A_m: \mathbb{C}^{V_{\tilde{T}}} \rightarrow \mathbb{C}^{V_{\tilde{T}}}$. Each operator $A_m$ sums a function on a sphere of radius $m$ on $\tilde{T}$ around each vertex, which is of approximate size $(\sqrt{q_0q_1})^m$ (the exact size depends on the type of the vertex).
As for operators acting on edges, since each edge has a natural “direction” from vertices of type 0 to vertices of type 1, it is simpler to consider the set \( \tilde{E}_T \) of non-directed edges. Give each non-directed edge \( e = \{x, y\} \in \tilde{E}_T \) a direction from its 0-vertex \( x \) to its 1-vertex \( y \). We therefore write \( e = (x, y) \).

Biregular trees also have an Iwahori–Hecke algebra attached to them, which was used by Hashimoto in [Has89]. This time it is natural to use the standard Iwahori–Hecke algebra and not the extended one (see Remark 3.14). The shortest way to define this algebra is as follows:

**Definition 3.26:** The **Iwahori–Hecke** (or **edge-Hecke**) algebra \( \tilde{H}_\phi \) is the algebra of row and column finite operators acting on \( C\tilde{E}_T \) and commuting with automorphisms of \( \tilde{T} \).

The description of \( \tilde{H}_\phi \) is very similar to the regular case. In particular, we have the operators \( h_{s_0} \) and \( h_{s_1} \) (but not \( h_\tau \)), defined as in Subsection 3.1. The basis of \( \tilde{H}_\phi \) consists of the operators \( h_w, w \in W \), where \( W \) is the (non-extended) Coxeter group \( W = \langle s_0, s_1 : s_0^2 = s_1^2 = 1 \rangle \) (see Remark 3.14). The Iwahori–Hecke relations are

\[
\begin{align*}
    h_{s_0}^2 &= (q_0 - 1)h_{s_0} + q_0 Id, \\
    h_{s_1}^2 &= (q_1 - 1)h_{s_1} + q_1 Id, \\
    h_w h_s &= h_{ws} \quad \text{if } l(ws) > l(w).
\end{align*}
\]

Hashimoto’s non-backtracking operator \( h_{NB} \) is not part of our algebra. However, we do have the non-backtracking operator \( \tilde{h}_{NB} = h_{s_1s_0} = h_{s_1}h_{s_0} \) (which corresponds to \( h_{NB}^2 \) in the regular graph case).

**Definition 3.27:** The **full graph Hecke algebra** \( \tilde{H} \) is the algebra of row and column finite operators acting on \( C\tilde{E}_T \oplus C^V_T \) and commuting with tree automorphisms.

In this case it is useful to define the **raising operators** \( u_0, u_1 : C^V_T \to C\tilde{E}_T \), and the **lowering operators** \( d_0, d_1 : C\tilde{E}_T \to C^V_T \) by

\[
\begin{align*}
    u_0 f(x, y) &= f(x), \quad u_1(x, y) = f(y) \\
    d_0 f(x) &= \begin{cases} 
        \sum_{y \sim x} f(x, y) & x \text{ of type 0,} \\
        0 & \text{otherwise,}
    \end{cases} \quad d_1 f(y) = \begin{cases} 
        \sum_{x \sim y} f(x, y) & y \text{ of type 1,} \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
\]
We extend \( u_0, u_1, d_0, d_1 \) as usual to operators in \( \tilde{H} \). The relations satisfied are

\[
\begin{align*}
    u_0 d_0 &= h_{s_0} + \text{Id}_{\tilde{H}_\phi}, \\
    u_1 d_1 &= h_{s_1} + \text{Id}_{\tilde{H}_\phi}, \\
    A &= d_0 u_1 + d_1 u_0.
\end{align*}
\]

The algebra \( \tilde{H} \) and the raising and lowering operators allow us to transfer results between \( \tilde{H}_0 \) and \( \tilde{H}_\phi \). In addition, the restriction operator

\[
\tilde{W} \to \tilde{U} = \text{Id}_{\tilde{H}_\phi} \tilde{W}
\]

from an \( \tilde{H} \)-representation \( \tilde{W} \) to an \( \tilde{H}_\phi \)-representation \( \tilde{U} \) defines a bijection between equivalence classes of irreducible representations of \( \tilde{H} \) and \( \tilde{H}_\phi \), as in Proposition 3.17.

The \( L^p \)-theory remains essentially the same. It is summarized in the following theorem:

**Theorem 3.28:** Let \( (\pi, \tilde{W}) \) be a finite-dimensional representation of \( \tilde{H} \). Let \( p \geq 2 \). The following are equivalent:

1. Each eigenvalue \( \lambda \) of \( \pi(h) \) for every \( h \in \tilde{H} \) on \( \tilde{W} \) is contained in the approximate point spectrum of \( h \) on \( L^p(V_{\tilde{T}} \oplus \tilde{E}_{\tilde{T}}) \).
2. The representation \( (\pi, \tilde{W}) \) is \( p \)-tempered, i.e., all of its geometric realizations are in \( L^{p+\epsilon}(V_{\tilde{T}} \oplus \tilde{E}_{\tilde{T}}) \) for every \( \epsilon > 0 \).
3. Each eigenvalue \( \theta' \) of \( \pi(\tilde{h}_{NB}) \) on \( \text{Id}_{\tilde{H}_\phi} \tilde{W} \) satisfies \( |\theta'| \leq |q_0 q_1|^{(p-1)/p} \).

Since this theorem is a special case of the generalized theory in [Kam16] (specifically, Theorem 1.6 and Corollary 1.12), we only give its sketch. The fact that (2) derives (1) is as in Proposition 2.19. The equivalence between (2) and (3) is as in Proposition 3.22. Finally, to prove that (1) derives (3) one generalizes Proposition 2.20. It also gives a more qualitative part, given as follows:

**Proposition 3.29:** The norm of \( A_m \in \tilde{H}_0 \) on \( L^p(V_{\tilde{T}}) \) (and therefore on every \( p \)-tempered unitary representation) is bounded by \( (m + 1)q_1(q_0 q_1)^{(m/2)(p-1)/p} \).

The proposition states that up to \( O(m) \), \( A_m \) is bounded by the \( (p-1)/p \)-th power of the number of vertices it sums.

**Definition 3.30:** A \((q_0 + 1, q_1 + 1)\)-biregular graph \( X \) is \( L^p \)-**expander** if every non-trivial \( \tilde{H} \)-subrepresentation of \( (\pi_X, L^2(\tilde{E}_X \oplus V_X)) \) is \( p \)-tempered.
The theorems above are general and do not require the classification of $\tilde{H}$-representations. However, understanding the exact connection between the eigenvalues of the adjacency operator $A$ and $p$-temperedness does require the classification. To classify irreducible representations, we embed $\tilde{H}$ in $M_{4 \times 4}(\mathbb{C}[\theta', \theta'-1])$ ($\theta'$ indeterminate) by

$$h_{s_0} \rightarrow \begin{pmatrix} 0 & q_0 & 0 & 0 \\ 1 & q_0 - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_{s_1} \rightarrow \begin{pmatrix} q_1 - 1 & \theta' & 0 & 0 \\ \theta'^{-1}q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{h}_{NB} \rightarrow \begin{pmatrix} \theta'^{-1} & q_0(q_1 - 1) + \theta'(q_0 - 1) & 0 & 0 \\ 0 & \theta'^{-1}q_0q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_0 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & q_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_0 \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_1 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ q_1 & \theta' & 0 & 0 \end{pmatrix}, \quad U_1 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \theta'^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $0 \neq \theta' \in \mathbb{C}$, we denote the resulting 4-dimensional $\tilde{H}$-representation by $(\tilde{\pi}_{\theta'}, \tilde{W}_{\theta'})$.

Notice that $\theta'$ corresponds to an eigenvalue of $\tilde{h}_{NB}$, so $\tilde{W}_{\theta'}$ will be similar to the representation $W_{\theta'^{-1}2}$ of Subsection 3.2.

The proposition below gives the classification of unitary $\tilde{H}$-representations which can be found similarly to Subsection 3.2. For simplicity we omit $\pi$ from the notations. The edge dimension of a representation $\tilde{W}$ is the dimension of $Id_{\tilde{H}_{\phi}}\tilde{W}$. The vertex dimension is the dimension of $Id_{\tilde{H}_0}\tilde{W}$. The eigenvalues of $\tilde{h}_{NB}$ are calculated on $Id_{\tilde{H}_{\phi}}\tilde{W}$ and the eigenvalues of $A$ are calculated on $Id_{\tilde{H}_0}\tilde{W}$. A similar classification can be found in [Has89].
Proposition 3.31: The unitary representations of $\tilde{H}$ are the following representations:

1. The representation $\tilde{W}_{\theta'}$, for (a) $|\theta'| = \sqrt{q_0 q_1}$ or (b) $\theta'$ real with $1 < \theta' < q_0 q_1$ or (c) $\theta'$ real with $-q_1 < \theta' < -q_0$. This representation is of dimension 4: vertex dimension 2 and edge dimension 2. The eigenvalues of $\tilde{h}_{NB}$ are $\theta', q_0 q_1 \theta'^{-1}$ and it is $p$-tempered if and only if $\max\{|\theta'|, q_0 q_1 |\theta'|^{-1}\} \leq (q_0 q_1)^{(p-1)/p}$.

Write $\theta' = \sqrt{q_0 q_1} \tilde{\theta}$. The eigenvalues of $A$ are

$$\pm \sqrt{(1 + \theta'^{-1} q_0)(1 + \theta')} = \pm \sqrt{(\tilde{\theta}^{1/2} \sqrt{q_1} + \tilde{\theta}^{-1/2} \sqrt{q_0})(\tilde{\theta}^{1/2} \sqrt{q_0} + \tilde{\theta}^{-1/2} \sqrt{q_1})}.$$  

(a) If $|\theta'| = \sqrt{q_0 q_1}$, the representation is 2-tempered and the eigenvalues $\lambda_\pm$ of $A$ are $\lambda_\pm = \pm |\tilde{\theta}^{1/2} \sqrt{q_1} + \tilde{\theta}^{-1/2} \sqrt{q_0}|$, and it holds that $\sqrt{q_1} - \sqrt{q_0} \leq |\lambda_\pm| \leq \sqrt{q_1} + \sqrt{q_0}$.

(b) If $|\theta'| \neq \sqrt{q_0 q_1}$, the representation is not 2-tempered. (b) For $1 < \theta' < q_0 q_1$ the eigenvalues $\lambda_\pm$ of $A$ satisfy

$$\sqrt{q_1} + \sqrt{q_0} < |\lambda| < \sqrt{(1 + q_0)(1 + q_1)}.$$  

c) For $-q_1 < \theta' < -q_0$ the eigenvalues $\lambda_\pm$ of $A$ satisfy

$$0 < |\lambda| < \sqrt{q_1} - \sqrt{q_0}.$$  

2. The Steinberg representation $\tilde{W}_S$: $h_0, h_1$ act by $-1$. This representation is of dimension 1: vertex dimension 0 and edge dimension 1. The eigenvalue of $\tilde{h}_{NB}$ is 1. There are no eigenvalues for $A$ since the vertex dimension is 0. The representation is 1-tempered.

3. The trivial representation $\tilde{W}_T$: $h_0, h_1$ act by $q$. This representation is of dimension 3: vertex dimension 2 and edge dimension 1. The eigenvalue of $\tilde{h}_{NB}$ is $q_0 q_1$. The eigenvalues of $A$ are $\pm \sqrt{(1 + q_0)(1 + q_1)}$. The representation is $\infty$-tempered.

4. The representations $\tilde{W}^0$: $h_0$ acts by $q$, $h_1$ acts by $-1$. This representation is of total dimension 2: vertex dimension 1 and edge dimension 1. The eigenvalue of $\tilde{h}_{NB}$ is $-q_0$. The eigenvalue of $A$ is 0, with an eigenvector supported on vertices of type 0. The representation is 2-finite.

5. The representations $\tilde{W}^1$: $h_0$ acts by $-1$, $h_1$ acts by $q_1$. This representation is of total dimension 2: vertex dimension 1 and edge dimension 1. The eigenvalue of $\tilde{h}_{NB}$ is $-q_1$. The eigenvalue of $A$ is 0, with an eigenvector supported on vertices of type 1. The representation is not 2-tempered.
Let us apply the classification to graphs, i.e., to the decomposition of $L^2(\tilde{E}_X \oplus V_X)$ as a unitary $\tilde{H}$-representation.

The trivial representation $\tilde{W}_T$ appears once: it is the subrepresentation consisting of functions having constant value on every type of face. By a dimension argument, the Steinberg representation appears $\chi(X) + 1 = |\tilde{E}_X| - |V_X| + 1$ times in the decomposition. The rest of the representations are either $\tilde{W}_{\theta'}$ or $\tilde{W}_0^1$ or $\tilde{W}_1^1$. Counting dimensions again, we know that the difference between the number of appearances of $\tilde{W}_0^0$ and $\tilde{W}_1^1$ is $|V_X^0| - |V_X^1|$.

To make a graph an $L^2$-expander, we need that:

1. For each $\tilde{W}_{\theta'}$ appearing in the decomposition, $\theta'$ will satisfy $|\theta'| = \sqrt{q_0q_1}$.
2. The representation $\tilde{W}_1^1$ will not appear in the decomposition.

The classification proves the final part of Theorem 1.5:

**Theorem 3.32:** A bipartite $(q_0 + 1, q_1 + 1)$-biregular graph is an $L^2$-expander (i.e., Ramanujan) if and only if the eigenvalues of $A$ are

$$\lambda = 0, \pm \sqrt{(1 + q_0)(1 + q_1)}$$

or satisfy

$$\sqrt{q_1} - \sqrt{q_0} \leq |\lambda| \leq \sqrt{q_1} + \sqrt{q_0},$$

and the multiplicity of $\lambda = 0$ is $|V_X^0| - |V_X^1|$.

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