M(ysterious) Patterns in $SO(9)$

Tekparson Pengpan, and Pierre Ramond

Institute for Fundamental Theory,
Department of Physics, University of Florida
Gainesville FL 32611, USA

Abstract

The light-cone little group, $SO(9)$, classifies the massless degrees of freedom of eleven-dimensional supergravity, with a triplet of representations. We observe that this triplet generalizes to four-fold infinite families with the quantum numbers of massless higher spin states. Their mathematical structure stems from the three equivalent ways of embedding $SO(9)$ into the exceptional group $F_4$.

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1 N=1 Supergravity in Eleven Dimensions

It has been recently pointed out that 11-dimensional supergravity is the local limit of a much bigger theory, called M-theory [1], that also contains in different limits all known string theories in ten dimensions. At present, it is still elusive, and only a partial formulation [2] exists in the literature.

Since M-theory lives in eleven dimensions, its massive degrees of freedom must be expressible as multiplets of \(SO(10)\), the Lorentz little group of eleven dimensions and its massless degrees of freedom must form in representations of \(SO(9)\). Among those are the fields of the local supergravity theory which reveal themselves in the local limit. While it is likely that some of the physical objects in M-theory are not local, one still expects that they would be expressible in terms of infinite towers of representations of these little groups.

There is a pervading lore against interacting theories that contain massless states of higher spin. It is based on several no-go theorems, formulated in terms of local field theory [3]. They state that relativistically invariant theories with a finite number of local massless fields of spin higher than two, and with a finite number of derivatives in their interactions, do not exist. It follows that any such theory with an infinite tower of fields, and arbitrarily high derivative couplings escapes the no-go theorems and could conceivably exist. Even with supersymmetry, building such a theory seems like a hopeless task, and the many published attempts have met with partial success. A four-dimensional formulation uses an infinite-dimensional superalgebra, with the interesting feature that it necessarily contains a cosmological constant. Hence it seems that the lore against massless high-spin interacting theories is mainly based on the difficulties associated with their construction rather than on their impossibility. Since M-theory is most likely non-local, it may evade the no-go theorems, and could contain an infinite number of fields. It is therefore interesting to examine the \(SO(9)\) properties of eleven-dimensional supergravity, whose massless states are local limit of M-theory.

In the following, we would like to draw attention to a remarkable mathematical fact, which shows that the supergravity triplet of \(SO(9)\) representations is actually the tip of a mathematical iceberg. We will start by presenting group-theoretical evidence that the supergravity representations are the first of an infinite family of massless states of higher spin. Then we will offer a mathematical resolution in terms of embeddings of \(SO(9)\) into the exceptional group \(F_4\), as well as some generalizations. Since there are no coincidences in the study of these highly constrained theories, it is tempting to muse that these extra higher-spin massless states represent the degrees of freedom of M-theory, even though we have not been able to obtain any dynamical evidence for this conjecture.

2 Group Phenomenology of \(SO(9)\)

The classical Lie group \(SO(9)\) plays an important dual role in the study of theories in ten and eleven dimensions, as the light-cone little group of Lorentz-invariant theories in ten space and one time dimensions, and as the little group of massive representations of theories in nine space and one time dimensions.

The representations of \(SO(9)\) are best described in Dynkin’s language, which Dick Slansky used to great effectiveness in particle physics [4]. As a rank 4 Lie algebra, it takes four positive integers to label its irreducible representations, in the form \([a_1 \ a_2 \ a_3 \ a_4]\). Its four basic representations are:

- Vector, \([1000]\), with 9 components, \(V_i\),
- Adjoint, \([0100]\), with 36 components, \(B_{[ij]}\),
- Three-form, \([0010]\), with 84 components, \(B_{[ijk]}\),
• Spinor, [0001], with 16 components, $\psi_\alpha$.

All representations with odd $a_4$ are spinorial. The irreps of $SO(9)$ are characterized by five generalized Dynkin indices

$$I_p \equiv \sum_{\text{rep}} w^p, \quad p = 0, 2, 4, 6, 8,$$

where $w$ are the weights in the representation. Thus $I_0$ is the dimension of the irrep, and $I_2$ is related to the quadratic Casimir invariant by $C_2 = 36I_2/I_0$.

$N = 1$ supergravity in eleven dimension is a local field theory that contains three different massless fields, two bosonic that describe gravity and a three-form, and one Rarita-Schwinger spinor. Its physical degrees of freedom are classified in terms of the light-cone little group, $SO(9)$,

• Graviton as a symmetric second-rank tensor, [2000], $G_{(ij)}$,

• Third-rank antisymmetric tensor, [0010], $B_{[ijk]}$,

• Rarita-Schwinger spinor-vector, [1001], $\Psi_{\alpha i}$.

Their group-theoretical properties are summarized in the following table:

| irrep | [1001] | [2000] | [0010] |
|-------|--------|--------|--------|
| $I_0$ | 128    | 44     | 84     |
| $I_2$ | 256    | 88     | 168    |
| $I_4$ | 640    | 232    | 408    |
| $I_6$ | 1792   | 712    | 1080   |
| $I_8$ | 5248   | 2440   | 3000   |

We note that these indices, except for $I_8$, match between the fermion and the two bosons. As is well known, equality of the bosonic and fermionic dimensions is an indication of supersymmetry. On the light-cone, the supersymmetry algebra reduces to

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta},$$

where the supersymmetric generators transform as the 16 spinor of $SO(9)$. They split into creation and annihilation operators under the decomposition

$$SO(9) \supset SO(6) \times SO(3); \quad 16 = \bar{(4,2)} + (4,2),$$

and we obtain a Clifford algebra

$$\{\bar{Q}, \bar{Q}^\dagger\} = 1,$$

where $\bar{Q}$ transforms as $\bar{(4,2)}$, and $\bar{Q}^\dagger$ as $(4,2)$. The states of the Sugra multiplet are then obtained by successive applications of the $\bar{Q}^\dagger$ on the vacuum state, to yield 128 bosons and 128 fermions

$$\left\{1, (\bar{Q}^\dagger)^2, \ldots, (\bar{Q}^\dagger)^7, (\bar{Q}^\dagger)^8\right\} | 0 >.$$

The equality between the number of bosons and fermions is manifest. All three irreps have the same quadratic Casimir invariant, since they have the same $I_2/I_0$ ratio.

Surprisingly, we have found that some higher spin representations of $SO(9)$ also occur in triples with the same quadratic Casimir invariant, and show remarkable group-theoretical kinships with the supergravity triplet. The higher spin triplets appear in four different types, $S - T - T$, $S - S - S$, $T - T - S$, and $T - S - T$, where $S$ describes fermionic (odd $a_4$), and $T$ bosonic (even $a_4$) degrees of freedom. The largest representation is listed first, and its dimension is equal to the sum of dimensions of the other two irreps of the triplet. Thus only triplet of the $S - T - T$ type display supersymmetry-like properties.
2.1 S-T-T Triples

In Dynkinese, these triples are of the form

\[
[1 + p + 2r, n, p, 1+ 2q + 2r] \oplus [2 + p + 2q + 2r, n, p, 2r] \\
\oplus [p, n, 1 + p + 2r, 2q]
\] (6)

labelled by four integers, \(n, p, q, r\) = 1, 2, \ldots ; the sum of the Dynkin invariants \(I_0\), and \(I_2\), \(I_4\), \(I_6\), over the bosons match those of the fermion representation. All three have the same quadratic Casimir invariant. The simplest of this class is the supergravity multiplet which we have already discussed, and only the lowest of these triples has manifest supersymmetry. The number of fermions and bosons of each triplets are equal, and a multiple of 128, but their construction does not follow that of the supergravity multiplet as polynomials of \(\tilde{Q}^\dagger\) acting on some state. They appear to be supersymmetric without supersymmetry, the simplest being:

- The supertriple, with \(n = 1, p = q = r = 0\), contains

\[
[2100]_b + [0110]_b + [1101]_f 
\]

with group-theoretic numbers given by

| irrep   | [2100] | [0110] | [1101] |
|---------|--------|--------|--------|
| \(I_0\) | 910    | 1650   | 2560   |
| \(I_2\) | 3640   | 6600   | 10240  |
| \(I_4\) | 19864  | 34920  | 54784  |
| \(I_6\) | 130840 | 217320 | 348160 |
| \(I_8\) | 977944 | 1498344| 2466304|

and described by fields of the form

\[
h_{(ijk)l} + A_{(ij)(kl)m} + \Psi_{\alpha(ij)k} .
\]

Their index structure indicates the appearance of higher spin fields. It is not possible to generate this triple by repeated use of the light-cone supersymmetry algebra acting on some field \(|\lambda\rangle\), with dimension equal to 20

\[
\left(1, \tilde{Q}^\dagger, (\tilde{Q}^\dagger)^2, \ldots, (\tilde{Q}^\dagger)^7, (\tilde{Q}^\dagger)^8\right) |\lambda\rangle.
\]

This would imply that \(|\lambda\rangle\) appears twice in the triple, but the triple contains no duplicate representations of \(SO(6) \times SO(3)\) that add up to dimension equal to 20. These fields appear in the Kronecker product of the supergravity triplet with the two-form [0100],

\[
[2000] \otimes [0100] = [2100] \oplus [2000] \oplus \{[1010] \oplus [0100]\} , \quad (10)
\]

\[
[0010] \otimes [0100] = [0110] \oplus [0010] \oplus \{[1002] \oplus [1000] \oplus [0110] \oplus [1100] \oplus [0002]\} , \quad (11)
\]

\[
[1001] \otimes [0100] = [1101] \oplus [1001] \oplus \{[2001] \oplus [1101] \oplus [1001] \oplus [0101] \oplus [0011] \oplus [0001]\} . \quad (12)
\]

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A suitable product can be found that automatically traces out the extra representations (in the curly
brackets), subtracts all traces, and all totally antisymmetric tensors. The states of this triple could
be understood as some sort of bound state between the supergravity states and something having
the Lorentz properties of a 2- or 7-form in the light cone little group. Whether this union can be
consumated through actual dynamics remains to be seen. Although this feature usually associated
with supersymmetry remain, it is not a supermultiplet.

- The second tower of triples is obtained by multiplying all its representations by [1010], and
  performing suitable subtractions. This representation appears in the antisymmetric product of two
second-rank antisymmetric tensor fields. It could therefore be generated by applying two two-forms
on the supergravity multiplet, resulting in a bound state between supergravity and two branes. The
simplest with \( p = 1, n = q = r = 0 \), contains

\[
[3010]_b + [1020]_b + [2011]_f ,
\]  

with group theory table

| irrep | 3010 | 1020 | 2011 |
|-------|------|------|------|
| \( I_0 \) | 7700 | 12012 | 19712 |
| \( I_2 \) | 46200 | 72072 | 118272 |
| \( I_4 \) | 384360 | 585624 | 969984 |
| \( I_6 \) | 3938760 | 5748792 | 9687552 |
| \( I_8 \) | 46646664 | 64127736 | 110529792 |

- The third tower is more complicated, multiplying the fermion and the second boson by [0002], and
  the first boson by [2000]. The simplest in this series, with \( q = 1 n = p = r = 0 \), and contains

\[
[4000]_b + [0012]_b + [1003]_f
\]

with group theory table

| irrep | 4000 | 0012 | 1003 |
|-------|------|------|------|
| \( I_0 \) | 450 | 4158 | 4608 |
| \( I_2 \) | 2200 | 20328 | 22528 |
| \( I_4 \) | 15160 | 131784 | 146944 |
| \( I_6 \) | 130360 | 1016520 | 1146880 |
| \( I_8 \) | 1325944 | 8839560 | 10103296 |

- The fourth infinite tower is also twisted. The simplest of this series has \( r = 1 \), with content

\[
[4002]_b + [0030]_b + [3003]_f
\]

and group-theory mugshot

| irrep | 4002 | 0030 | 3003 |
|-------|------|------|------|
| \( I_0 \) | 32725 | 23595 | 56320 |
| \( I_2 \) | 261800 | 188760 | 450560 |
| \( I_4 \) | 2938280 | 2055768 | 4994048 |
| \( I_6 \) | 41127080 | 27239256 | 68366336 |
| \( I_8 \) | 673801256 | 414212568 | 1084279808 |
2.2 S-S-S Triples

Here all three representations are spinors

\[
[2 + p + 2r, n, p, 3 + 2q + 2r] \oplus [4 + p + 2q + 2r, n, p, 1 + 2r] \\
\oplus [p, n, 2 + p + 2r, 1 + 2q],
\]

and the dimension of the first is the sum of the other two. The simplest example is

\[
[2003] \oplus [4001] \oplus [0021].
\]

Its group-theory mugshot is

| irrep | [2003] | [4001] | [0021] |
|-------|--------|--------|--------|
| \(I_0\) | 18480 | 5280 | 13200 |
| \(I_2\) | 117040 | 33440 | 83600 |
| \(I_4\) | 1010992 | 297632 | 713360 |
| \(I_6\) | 10640944 | 3303584 | 7337360 |
| \(I_8\) | 128166448 | 43030688 | 85922192 |

2.3 T-T-S Triples

In this class, the dimension of the largest boson (listed first) is equal to that of the spinor and the second boson,

\[
[1 + p + 2r, n, p, 2 + 2q + 2r] \oplus [3 + p + 2q + 2r, n, p, 2r] \\
\oplus [p, n, 1 + p + 2r, 1 + 2q].
\]

The lowest member of this class is

\[
[1002] \oplus [3000] \oplus [0011],
\]

with mugshot

| irrep | [1002] | [3000] | [0011] |
|-------|--------|--------|--------|
| \(I_0\) | 924 | 156 | 768 |
| \(I_2\) | 3080 | 520 | 2560 |
| \(I_4\) | 13400 | 2392 | 11008 |
| \(I_6\) | 68216 | 13432 | 54784 |
| \(I_8\) | 382328 | 87544 | 299776 |

2.4 T-S-T Triples

The last class contains the representations

\[
[2 + p + 2r, n, p, 2 + 2q + 2r] \oplus [3 + p + 2q + 2r, n, p, 1 + 2r] \\
\oplus [p, n, 2 + p + 2r, 2q].
\]

Its lowest-lying member is

\[
[2002] \oplus [3001] \oplus [0020],
\]

with mugshot
| irrep | \([0002]\) | \([3001]\) | \([0020]\) |
|-------|------------|------------|------------|
| \(I_0\) | 3900       | 1920       | 1980       |
| \(I_2\) | 18200      | 8960       | 9240       |
| \(I_4\) | 114920     | 57728      | 57192      |
| \(I_6\) | 875720     | 455936     | 419784     |
| \(I_8\) | 7549064    | 4148096    | 3453384    |

There are several triples which only match dimensions and quadratic Casimir invariants; we found one made entirely of spinors

\[
[1033] \oplus [7001] \oplus [0305] ; \quad [7122] \oplus [6008] \oplus [4018],
\]

with the dimension of the first equal to the sum of the other two, and all with the same quadratic Casimir, both their \(I_{4,6}\) do not match.

### 2.5 Basic Operations

It is possible to understand these different triples in terms of four basic operations, which starting from the supergravity multiplet, generate all triples:

- \(\Delta_1\): Increase the Dynkin labels all three irreps within a triple by \([0100]\).
- \(\Delta_2\): Increase the Dynkin labels of all three irreps within a triple by \([1010]\).
- \(\Delta_3\): Increase the Dynkin labels of the first and third irreps by \([0001]\), the second by \([1000]\).
- \(\Delta_4\): Increase the Dynkin labels of the first and second irreps by \([1001]\), the third by \([0010]\)

The \(\Delta_{1,2}\) operations may be simplest to understand as they can be generated by applying representations that appear either as the light-cone 2-form \([0100]\), or in its twice-antisymmetrized product, since

\[
([0100] \otimes [0100])_A = [0100] \oplus [1010].
\]

A light-cone 2-form may indicate a brane state, and these triples could then be understood as bound states of the supergravity fields with these branes. The third and fourth operations are more complicated as they treat the different members differently. However, starting from the supergravity multiplet, they generate all other triples, as shown in the diagram below, where the upward arrow denotes \(\Delta_4\), and the downward arrow denotes \(\Delta_3\):
It is clear that the supergravity multiplet sits at the beginning of a very intricate and beautiful complex of irreps of $SO(9)$. Limited by the two dimensions of the paper, we have not shown the effect of the $\Delta_{1,2}$ operations which act uniformly on any of the triples in the picture. The whole pattern is summarized by the general form of the triples

$$
[1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4] \oplus [2 + a_2 + a_3 + a_4, a_1, a_2, a_3] \oplus [a_2, a_1, 1 + a_2 + a_3, a_4].
$$

where $a_i$ are non-negative integers.

3 Mathematical Origin of the Triples

So far we have only offered numerical evidence for the remarkable structure of the $SO(9)$ representations. A recent paper \[7\] by B. Gross, B. Kostant, S. Sternberg and one of us (PR), unveils its mathematical origin. The following is a watered-down version of its contents. It points to a construction of a more general character, but does not (yet) seem to shed light on its physical interpretation.

The triples stem from the triality of $SO(8)$, which is explicitly realized in $F_4$. That very triality is already familiar to particle physicists: the three equivalent ways to embed $SU(2) \times U(1)$ in $SU(3)$, called I-spin, U-spin, and V-spin \[9\].

The general idea behind the mathematical construction goes as follows. Let $F$ and $B$ be two Lie algebras of equal rank such that $F \supset B$. The Weyl group of $F$, $W(F)$, is bigger than that of $B$, $W(B)$, with $r$-times as many operations. The fundamental Weyl chamber of $F$, is the sliver of weight space that contains all weights with positive or zero Dynkin labels; it is $r$ times smaller than the fundamental Weyl chamber of the subgroup $B$.

Let $\lambda$ be the highest weight of an irrep of $F$; it lies either inside or at the boundary of the Weyl chamber. We can choose $r$ operations of $W(F)$ not in $W(B)$ which map the fundamental Weyl
chamber of F into that of B. When applied to this highest weight, they produce \( r \) copies inside the chamber of B (unless the weight is at the chamber boundary). In order to make sure it is inside the chamber, we add to it the weight \( \rho = [1,1,1,\ldots,1] \), the Weyl vector (or half sum of positive roots). Then we are sure the Weyl group will act on this weight non-trivially. We now construct the \( r \) weights

\[
\lambda_i \equiv w_i(\lambda + \rho_F) - \rho_B, \quad i = 1,2\ldots r,
\]

where \( w_i \) are the operations \( W(F) \) not in \( W(B) \). They all lie on the fundamental Weyl chamber of B, and on its boundary, and therefore describe an \( r \)-plet of irreps of B.

Apply this reasoning to the case \( F_4 \supset SO(9) \). The Weyl group of \( F_4 \) has dimension 1152, that of \( SO(9) \), 384 so that \( r = 3 \). Starting from any representation of \( F_4 \) this construction generates a triplet of representations of \( SO(9) \). There remains to identify those three elements of the Weyl group, the reason for the relations among their invariants and the emergence of supersymmetry in the construction.

To understand the origin of the triality in \( F_4 \), the octonion language is convenient since the adjoint of \( F_4 \) is generated by antihermitian traceless \( 3 \times 3 \) matrices over the octonions, supplemented by their automorphism group, \( G_2 \). Triality is then related to the three inequivalent ways of picking out one of the matrix’s off-diagonal elements, and this construction generalizes to the Lie algebras of the magic square.

Octonions, together with real numbers, \( R \), complex numbers, \( C \), quaternions, \( Q \), are the four Hurwitz (division) algebras. \( 3 \times 3 \) matrices with elements belonging to these algebras generate interesting mathematical structures.

- For real numbers, these matrices generate the Lie algebra \( SO(3) \). Its maximal subgroup is \( SO(2) \).
- For complex numbers, they generate the Lie algebra \( A_2 \sim SU(3) \), and singling out one of the three off-diagonal elements picks out the subgroup \( SU(2) \times U(1) \sim SO(3) \times SO(2) \).
- For quaternions, together with their automorphism group \( A_1 \sim SU(2) \), they generate \( C_3 \sim Sp(6) \). Two off-diagonal elements are treated equally by the subgroup \( Sp(4) \times Sp(2) \sim SO(5) \times SO(3) \).
- With octonions, and their automorphism group \( G_2 \), they generate the exceptional group \( F_4 \). Its subgroup \( B_4 \sim SO(9) \) naturally picks out one of the three off-diagonal elements.

Under \( F_4 \supset SO(9) \), its adjoint breaks up as \( 52 = 36 + 16 \), where \( 36 \) is the adjoint of \( SO(9) \) and \( 16 \) its spinor representation. Another way to look at this embedding is to say that it generates a 16-dimensional coset space acted on by the orthogonal group \( SO(16) \). It yields the anomalous embedding \( SO(16) \supset SO(9) \) according to which the spinor of \( SO(9) \) fits in the vector of \( SO(16) \).

4 The Magic Square

Starting from the four Hurwitz algebras, it is possible to construct the so-called composition algebras, which include all the exceptional groups, except \( G_2 \), the automorphism group of the octonions. That construction relies on the triality of both \( SO(8) \) and on the structure of \( 3 \times 3 \) matrices. Start from \( 3 \times 3 \) antihermitian traceless matrices with elements over the product of any two of the four Hurwitz algebras, with three off-diagonal elements, and two diagonal elements which are pure imaginary. They are acted on by the automorphism groups of the algebras, and each of their parameters generate a Lie algebra transformation, to produce one of the ten Lie algebras in the magic square.
In particular, the exceptional group $F_4$ is generated by $3 \times 3$ traceless antihermitian matrices over the octonions, together with $G_2$, the automorphism group of the octonion multiplication table. This produces the $3 \times 8 + 2 \times 7 + 14 = 52$ parameters of the algebra.

Each of the algebras appearing in the magic square have subalgebras of equal rank, and we can apply our mathematical construction to each. The results are summarized in table below, first for the exceptional groups,

| Group | Subgroup | Coset Dimension | $r$ |
|-------|----------|----------------|-----|
| $E_8$ | $SO(16)$ | 128            | 135 |
| $E_7$ | $SO(12) \times SO(3)$ | 64             | 63  |
| $E_6$ | $SO(10) \times SO(2)$ | 32             | 27  |
| $F_4$ | $SO(9)$  | 16             | 3   |

and then for the non-exceptional groups in the square

| Group | Subgroup | Coset Dimension | $r$ |
|-------|----------|----------------|-----|
| $SO(12)$ | $SO(8) \times SO(4)$ | 32 | 30 |
| $SU(6)$ | $SO(6) \times SO(3) \times SO(2)$ | 16 | 15 |
| $Sp(6)$ | $Sp(4) \times Sp(2)$ | 8  | 3  |
| $SU(3) \times SU(3)$ | $SO(3) \times SO(3) \times SO(2) \times SO(2)$ | 16 | 3  |
| $SU(3)$ | $SO(3) \times SO(2)$ | 4  | 3  |

The connection with supersymmetry occurs through the generation of representations of the subgroup in terms of polynomials in Clifford charges.

4.1 $SU(3)$ Triples

By studying the simplest non-trivial example, of the embedding of $SU(2) \times U(1)$ into $SU(3)$, the emergence of supersymmetry in our general construction should become clear. We begin with some well known facts about $SU(3)$, the algebra generated by $3 \times 3$ antihermitian traceless matrices over the complex numbers. Its maximal subgroup is

$$SU(3) \supset SU(2) \times U(1) \sim SO(3) \times SO(2).$$

(26)

There are three equivalent ways to imbed this subalgebra in $SU(3)$, corresponding, in particle physics language, to $I$-spin, $U$-spin and $V$-spin. These three embeddings can be understood in terms of the Weyl group. The Weyl group of $SU(3)$ is $S_3$, the permutation group on three objects. It is three times as big as the Weyl group of its maximal subgroup $SO(3) \times SO(2)$.

All the states spanned by this algebra live in a 2-dimensional lattice. In the Dynkin basis, the weights are labelled by two integers $[a_1, a_2]$. The action of the Weyl group is simplest acting on an orthonormal basis $\{e_i\}$ in which the roots have components which are half-integers between $-2$ and 2.
The simple roots of $SU(3)$ are given in terms of the three vectors $e_a$ which span a three-dimensional
the Euclidean space
\[ \alpha_1 = e_1 - e_2 ; \quad \alpha_2 = e_2 - e_3 \, . \] (27)
A weight is labelled in the orthonormal basis by three numbers, $(b_1, b_2, b_3)$, such that $b_1 + b_2 + b_3 = 0$.
It can also be expressed in the Dynkin basis as
\[ w = a_1 \omega_1 + a_2 \omega_2 \, , \] (28)
where the fundamental vectors $\omega_{1,2}$ are determined through
\[ 2 \frac{(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij} \, . \] (29)
It follows that the same weight has the orthonormal basis components
\[ b_1 = \frac{1}{3}(2a_1 + a_2) \, , \quad b_2 = \frac{1}{3}(a_2 - a_1) \, , \quad b_3 = -\frac{1}{3}(a_1 + 2a_2) \, . \] (30)
The action of the Weyl group on the orthonormal components is just the permutation group on the
three $b_i$’s. Furthermore, the fundamental Weyl chamber is described by the inequalities $b_1 > b_2 > b_3$.
We will use the Weyl vector, defined as half the sum of the positive roots:
\[ \rho_{SU(3)} = (1, 0, -1) \sim [1, 1] \, , \] (31)
where the square brackets in indicate the Dynkin basis and the curved brackets the orthonormal
basis.

Start from an irrep of $SU(3)$, labelled by its maximum Dynkin weight $[a_1, a_2]$, or orthonormal
components $(b_1, b_2, b_3)$, and add the Weyl vector, so that the resulting weight is inside the chamber.
Since the Weyl group of $SU(3)$ is three times as large as that of $SU(2)$, the fundamental Weyl
chamber of $SU(2)$, is three times as large as that of $SU(3)$. Identify the three Weyl transformations
(permutations) that produce a weight in the fundamental Weyl chamber of $SU(2)$, defined by $b'_1 \geq b'_2$.
These yield three weights
\[ (b_1 + 1, b_2, b_3 - 1) \quad (b_1 + 1, b_3 - 1, b_2) \quad (b_2, b_3 - 1, b_1 + 1) \, . \] (32)
Next we subtract the $SU(2)$ Weyl vector, $\rho_{SU(2)} = (1/2, -1/2) \sim [1]$, and revert to the Dynkin basis.
This construction yields the $SU(3)$-generated triples, with the $U(1) \sim SO(2)$ charge indicated as a
subscript
\[ SU(3) : \quad [a_1] \frac{1}{2}(a_1 + 1) \oplus [a_1 + a_2 + 1] \frac{1}{2}(a_1 - a_2) \oplus [a_2] \frac{1}{2}(a_2 + 1) \, . \] (33)
The three Weyl operations are best identified by their action on the fundamental irrep of $SU(3)$,
$u, d, s$:
\[ w_1 : (u, d, s) \rightarrow (u, d, s) \, , \] (34)
\[ w_2 : (u, d, s) \rightarrow (u, s, d) \, , \] (35)
\[ w_3 : (u, d, s) \rightarrow (s, u, d) \, . \] (36)
Starting from the singlet of $SU(3)$, with $a_1 = a_2 = 0$, we get the simplest triple
\[ [0] \frac{1}{2} \oplus [1]_0 \oplus [0] \frac{1}{2} \, . \] (37)
This triplet of representations forms a supersymmetric multiplet, generated by a charge that is a doublet under the \( SU(2) \) and has \( U(1) \) value of \(-1/2\). To understand its origin, note that the group acting on the four-dimensional coset \( SU(3)/SU(2) \times U(1) \) is \( SO(4) \equiv SO(3) \times SO(3) \), defining an embedding of \( SO(4) \supset SU(2) \times U(1) \). The states of this lowest triple correspond to the decomposition of the \( SO(4) \) spinor.

An obvious interpretation is to view \( SO(3) \times SO(2) \) as the compact subgroup as the light cone little group of either \( SO(4,1) \times SO(2) \), or \( SO(3) \times SO(3,1) \), which lead to theories in \( d = 5 \) and \( d = 4 \) dimensions, respectively.

Two fundamental operations generate the higher triples, in one to one correspondence with the rank of the mother algebra \( SU(3) \). The operation that increases \( a_1 \) by two, produces on \([0]_\frac{1}{2}\) the infinite chain

\[ [0]_\frac{1}{2} \oplus [2]_1 \oplus [4]_2 \oplus \cdots \]  

It is amusing that these states fit in the spin zero singleton (Rac) representation of \( SO(3,2) \). The same operation on \([1]_0\) yields the Di representation. The \( SO(2) \) charge is the energy and the \( SO(3) \) representation determines the spin.

### 4.2 \( Sp(6) \) Triples

We now apply the same construction to the next entry in the magic square. The Lie algebra \( Sp(6) \equiv C_3 \) is generated by the \( 3 \times 3 \) traceless antihermitian matrices over quaternions, augmented by \( SU(2) \), the automorphism group of the quaternions. By picking out one of its off-diagonal element, we obtain the embedding

\[
Sp(6) \supset Sp(4) \times Sp(2) \sim SO(5) \times SO(3)
\]

with an eight-dimensional coset space acted on by \( SO(8) \). \( Sp(6) \) has three simple roots, given by

\[
\alpha_1 = e_1 - e_2 , \; \alpha_2 = e_2 - e_3 , \; \alpha_3 = 2e_3 .
\]

The same weight, written in Dynkin and orthonormal bases, has components \([a_1, a_2, a_3]\) and \([b_1, b_2, b_3]\), respectively with the relations

\[
b_1 = a_1 + a_2 + a_3 , \; b_2 = a_2 + a_3 , \; b_3 = a_3 ,
\]

derived from Equation (29), where now \( i,j = 1,2,3 \), after carefully accounting for the long root. Its Weyl vector is given by

\[
\rho_{C_3} = (3,2,1) \sim [1,1,1]
\]

The Weyl group of \( C_3 \) contains \( S_3 \), which permutes the three \( b_j \)’s. As in the previous case, three of its operations relate the three equivalent ways to embed \( SO(5) \times SO(3) \).

Add to the highest weight of an irrep of \( C_3 \), \([a_1, a_2, a_3]\) the Weyl vector, to put it inside the fundamental chamber of \( Sp(6) \). There are three elements of the Weyl group which map this weight into the fundamental chamber of the subgroup \( SO(5) \times SO(3) \), defined by \( b_1 \geq b_2 \geq 0 \), and \( b_3 \geq 0 \): the identity element, the parity \( P_{23} \), which interchanges \( b_2 \) and \( b_3 \), and the cyclic element \( C_{231} \) which effects \((b_1, b_2, b_3) \to (b_2, b_3, b_1)\). After subtracting the Weyl vector of the subgroup \( \rho = (2,1;1) \), where the last entry is for the \( SO(3) \) subgroup, we obtain the three weights

\[
(b_1 + 1, b_2 + 1; b_3) ; \quad (b_1 + 1, b_3; b_2 + 1) ; \quad (b_2, b_3; b_1 + 2) .
\]
To rewrite these in the Dynkin basis, we use the expression for the simple roots of $SO(5)$ in the orthonormal basis

$$\alpha_1 = e_1 - e_2 , \quad \alpha_2 = e_2 ,$$

which yield the $Sp(6)$-generated triple

$$[a_1, a_2 + a_3 + 1 : a_3] \oplus [a_1 + a_2 + 1, a_3; a_2 + a_3 + 1]$$

$$\oplus [a_2, a_3; a_1 + a_2 + a_3 + 2]$$

(45)

The scalar irrep of $Sp(6)$ yields the simplest triplet

$$[0, 1; 0] \oplus [1, 0; 1] \oplus [0, 0; 2],$$

or in terms of the dimensions of the representations of $SO(5) \times SO(3)$,

$$\mathbf{(5, 1)} \oplus \mathbf{(4, 2)} \oplus \mathbf{(1, 3)},$$

(47)

the representation content of super-Yang-Mills in ten dimensions. The orthogonal group $SO(8)$ acts on the eight-dimensional coset space, defining an embedding of the spinor of $SO(5) \times SO(3)$ into the vector irrep of $SO(8)$: $8_V = (4, 2)$. The triple is just the decomposition of the spinors of $SO(8)$: $8_S = (4, 2), 8'_S = (5, 1) + (1, 3)$. The subgroup can be interpreted as the light-cone little group of either $SO(6, 1) \times SO(3)$, or $SO(5) \times SO(4, 1)$, which imply Lorentz-invariant theories in either $d = 7$ or $d = 5$ space-time dimensions.

### 4.3 $F_4$ Triples

The $F_4$ simple roots are given by

$$\alpha_1 = e_2 - e_3 , \quad \alpha_2 = e_3 - e_4 ,$$

(48)

$$\alpha_3 = e_4 , \quad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

(49)

Through the use of Equation (29), we find the relation between the components of any weight in the Dynkin basis $[a_1, a_2, a_3, a_4]$ and the orthonormal basis $(b_1, b_2, b_3, b_4)$

$$b_1 = a_1 + 2a_2 + \frac{3}{2}a_3 + a_4 , \quad b_2 = a_1 + a_2 + \frac{1}{2}a_3 ,$$

(50)

$$b_3 = a_2 + \frac{1}{2}a_3 , \quad b_4 = \frac{1}{2}a_3 .$$

(51)

The Weyl vector is given in the orthonormal and Dynkin bases by

$$\rho_{F_4} = \frac{1}{2}(11, 5, 3, 1) \sim [1, 1, 1, 1].$$

(52)

To generate the triples, we start with an irrep of $F_4$, $[a_1, a_2, a_3, a_4]$, and add to it the Weyl vector to put it inside the fundamental chamber. Express it in the orthonormal basis. The three elements of the $F_4$ Weyl group which map weights inside the fundamental chamber of $SO(9)$, are the identity, a parity and an anticyclic permutation. These are most easily expressed by their action on the fundamental of $F_4$, which is a $3 \times 3$ hermitian traceless octonionic matrix. The parity interchanges
two off-diagonal octonion elements, and the cyclic permutes the three off-diagonal octonion elements. The first is the original weight, the second and third are obtained in terms of permutations on the simple roots of \( D_4 \), for which we have

\[
\alpha_1 = e_1 - e_2 , \quad \alpha_2 = e_2 - e_3 , \quad \alpha_3 = e_3 - e_4 , \quad \alpha_4 = e_3 + e_4 .
\]

(53)

In this numbering, \( \alpha_2 \) is at the center of the Dynkin diagram, and does not move under the permutations. The two permutations that produce the required weights are

\[
S_{14} : \quad \alpha_1 \mapsto \alpha_4 , \quad \alpha_4 \mapsto \alpha_1 , \quad \alpha_3 \mapsto \alpha_3 ,
\]

(55)

\[
C_{143} : \quad \alpha_1 \mapsto \alpha_4 \mapsto \alpha_3 \mapsto \alpha_1 .
\]

(56)

Weights in the chamber of \( B_4 \) are determined by the inequalities \( b_1 \geq b_2 \geq b_3 \geq b_4 \geq 0 \). Next, to get in the closure of the Weyl chamber, we subtract the Weyl vector of \( SO(9) \),

\[
\rho_{B_4} = \frac{1}{2}(7, 5, 3, 1) \sim [1, 1, 1, 1] ,
\]

(57)

and convert in the Dynkin basis of \( SO(9) \), using the \( B_4 \) relations

\[
\alpha_1 = e_1 - e_2 , \quad \alpha_2 = e_2 - e_3 , \quad \alpha_3 = e_3 - e_4 , \quad \alpha_4 = e_4 ,
\]

(58)

(59)

from which

\[
a_1 = b_1 - b_2 , \quad a_2 = b_2 - b_3 , \quad a_3 = b_3 - b_4 , \quad a_4 = 2b_4 .
\]

(60)

(61)

This leads back to the \( F_4 \) generated triples in the Dynkin basis

\[
[1 + a_2 + a_3, a_1, a_2, 1+] \quad a_3 + a_4] \oplus [2 + a_2 + a_3 + a_4, a_1, a_2, a_3] \oplus [2 + a_2 + a_3 + a_4, a_1, a_2, a_3]
\]

(62)

\[
[ a_2, a_1, 1 + a_2 + a_3, a_4] .
\]

This formula corresponds to that previously derived empirically. For the simplest case, the triple is the supergravity multiplet

\[
[1, 0, 0, 1] \oplus [2, 0, 0, 0] \oplus [0, 0; 1, 0] .
\]

(63)

The group \( SO(16) \) acts on the sixteen-dimensional coset, providing the embedding of the spinor of \( SO(9) \) into the vector of \( SO(16) \). Then the lowest order triple is the decomposition of the two spinors of \( SO(16) \) into \( SO(9) \), which explains the supersymmetry.

### 4.4 \( E_{6,7,8} \)-Multiples

As indicated by the decompositions of the \( E \)-like exceptional groups in the magic square, we can generate higher order Clifford algebras.
• The embedding $E_6 \supset SO(10) \times SO(2)$ produces a 32-dimensional coset space. The lowest multiplet contains 27 terms which are decomposition of the two spinor irreps of $SO(32)$ in terms of $SO(10) \times SO(2)$. These are generated by a Clifford algebra with $2^{16}$ elements, half fermions, half bosons. The $SO(32)$ Clifford is also generated by the magic square embedding $SO(12) \supset SO(8) \times SO(4)$.

• With $E_7 \supset SO(12) \times SO(3)$, a 64-dimensional coset space is generated. The two spinor irreps of $SO(64)$ break into 63 irreps of the subgroup, with $2^{32}$ states generated by a Clifford algebra, again half of them fermions.

• Finally, $E_8 \supset SO(16)$ produces $2^{64}$ states describing the two spinors of $SO(128)$ in terms of 135 irreps of $SO(16)$!

In all these cases, the lowest multiplet that is generated contains states of spin higher than two, which means that any theory based on these structures probably have no local limit, except perhaps by compactification.

Finally we note that the magic square embedding $SU(6) \supset SO(6) \times SO(3) \times SO(2)$ yields in the lowest multiplet the states generated by the $SO(16)$-Clifford, corresponding to the dimensional reduction of the supergravity multiplet, and it could be viewed as coming from $d = 8-, 5-, 4-$dimensional theories or even on $AdS_4 \times S_8$. For the reader interested in proofs concerning the properties of these multiplets, we refer the reader to [7], where a novel formula between representations of Lie algebras of the same rank is derived.

In conclusion, we have described a very rich mathematical structure, which contains as special cases, the supermultiplets of eleven-dimensional supergravity and ten-dimensional super-Yang-Mills. While there is at present no clear interpretation of these structures, it is tempting to believe that they may shed some light on the eventual structure of $M$-theory.

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