On Random Sampling Auctions for Digital Goods †
Saeed Alaei ‡  Azarakhsh Malekian §  Aravind Srinivasan ¶

Abstract
In the context of auctions for digital goods, an interesting random sampling auction has been proposed by Goldberg, Hartline, and Wright [2001]. This auction has been analyzed by Feige, Flaxman, Hartline, and Kleinberg [2005], who have shown that it is 15-competitive in the worst case – which is substantially better than the previously proven constant bounds but still far from the conjectured competitive ratio of 4. In this paper, we prove that the aforementioned random sampling auction is indeed 4-competitive for a large class of instances where the number of bids above (or equal to) the optimal sale price is at least 6. We also show that it is 4.68-competitive for the small class of remaining instances thus leaving a negligible gap between the lower and upper bound. We employ a mix of probabilistic techniques and dynamic programming to compute these bounds.

1 Introduction
In recent years, there has been a considerable amount of work in algorithmic mechanism design. Most of this work can be divided into two categories based on their assumption about prior: (i) Bayesian, and (ii) prior free. Bayesian mechanism design is based on exploiting the knowledge of the prior to optimize the expected performance, whereas prior free mechanism design is aimed at optimizing the worst case performance. Random sampling is perhaps the most popular technique in prior free mechanism design, yet an accurate analysis of its performance has proven difficult even in the simplest applications.

This paper focuses on analyzing the performance of the random sampling auction proposed by Goldberg et al. [2001], known as the “Random Sampling Optimal Price (RSOP)” auction. The basic problem can be described as follows. A seller has unlimited supply of a good (e.g., a digital good) which he is going to sell to unit demand bidders through the following auction: bids are partitioned into two sets uniformly at random; then the optimal (revenue maximizing) sale price is computed for each set, and offered as the sale price to the opposite set. The expected revenue of RSOP is then compared against the optimal revenue of single price sale of at least two copies.

Most of our analysis is based on the following approach: we develop a lower bound on the performance of RSOP that depends on the level of balancedness of the partitions, but independent of the bid values; we then take the expectation of this lower bound over the varying level of balancedness to obtain a general lower bound on the performance of RSOP. That is in contrast to the previous work based on showing that a certain level of balancedness is met with a reasonable probability, which inevitably requires a tradeoff between how strong the balancedness condition is versus how likely it holds.

---

* A preliminary version appeared in the 10th ACM conference on Electronic Commerce, 2009.
† Id: rsop.tex 55 2013-03-10 23:35:24Z saeed
‡ Department of Computer Science Cornell University Ithaca, NY 14853. saeed@cs.cornell.edu. Supported in part by NSF Award CNS-0720528.
§ EECS department, Massachusetts Institute of Technology, Cambridge MA 02139. malekian@mit.edu. Supported in part by NSF Award CCF-0728839.
¶ Department of Computer Science and Institute for Advanced Computer Studies, University of Maryland, College Park, MD 20742. srin@cs.umd.edu. Supported in part by NSF ITR Award CNS-0426683, NSF Award CNS-0626636, and NSF Award CNS 1010789.

† If there is a fixed production cost per copy, the auction can still be used by simply subtracting the production cost from every bid.
Related work. The random sampling optimal price (RSOP) auction has been proposed by Goldberg et al. [2001], but the problems was first studied by Goldberg and Hartline [2001]. The revenue of RSOP has been shown to be close to optimal for many classes of interesting inputs by Segal [2003], and Balcan et al. [2005]. There has also been a fair amount of work analyzing the competitive ratio of RSOP. Goldberg and Hartline [2001] showed that RSOP obtains a constant fraction of the optimal revenue, and conjectured the constant to be $1/4$; note that the conjecture is tight for an instance with 2 bidders with distinct bids. A better analysis was proposed by Feige et al. [2005] which proved the constant to be at least $1/15$.

It is important to prove that RSOP is 4-competitive, because it is a natural and popular mechanism which is easily implementable and adaptable to various settings (e.g., double auctions Baliga and Vohra [2003], online limited-supply auctions Hajiaighayi et al. [2004], combinatorial auctions Balcan et al. [2005], Goldberg and Hartline [2001], and other setting such as Hartline and Roughgarden [2008]). Indeed the results of this paper have been used in analysis of other auctions such as the random sampling based auction of Devanur and Hartline [2009] for limited and online supply.

Results. The following is a summary of our main results.

I. Improved lower bounds: We prove that the ratio of the expected revenue of RSOP to its benchmark is:

- at least 1/4.68 (e.g., Theorem 1 and Theorem 2), improving the previous lower-bound of 1/15 due to Feige et al. [2005];
- at least 1/4, if there are at least 6 bids above (or equal to) the sale price.
- at least 1/3.52, as the number of bids above (or equal to) the sale price approaches infinity.

Our analysis suggests that the worst case performance of RSOP is attained when there are only two bidders with distinct bids.

II. Upper bound: We show that there exist instances where the expected revenue of RSOP is still less than 1/2.65 of its benchmark, even when the number of bids above the optimal sale price approaches infinity.

III. Combinatorial approach: We also present a combinatorial lower bound on the performance of RSOP for a special case when each non-zero bid can take one of only two possible values.

2 Preliminaries

We consider auctioning a digital good to $n$ bidders whose bids are represented by the vector $v = (v_1, \ldots, v_n)$ which is, without loss of generality, sorted in decreasing order.

Definition 1 (RSOP). The random sampling optimal price auction partitions the bids into two sets $A$ and $B$ uniformly at random \footnote{I.e., each bid independently goes to one of $A$ or $B$ with probability $\frac{1}{2}$.}, computes the optimal sale price of each set, and offers it as the sale price to the opposite set.

Definition 2 (OPT). The optimal revenue from single price sale to at least two bidders is

$$\text{OPT} = \max_{j \geq 2} j v_j.$$  \hspace{1cm} (1)

See Goldberg et al. [2006] for motivation of the definition of OPT and why it requires selling to at least two bidders.
Assumptions. We assume \( A \) and \( B \) contain the indices of the bids (as opposed to the actual value of the bids). Without loss of generality we assume there are infinitely many 0 bids, i.e., \( v_j = 0 \) for all \( j > n \); consequently \((A, B)\) is a partitioning of \( N \). The previous assumption allows us to make our analysis independent of \( n \). Also, without loss of generality we assume \( 1 \in B \).\(^3\)

Throughout most of our analysis we ignore the revenue of RSOP from bidders in \( A \) because in the pathological case where \( v_1 \) is too large (e.g., \( v_1 > \text{OPT} \)), the optimal sale price for \( B \) is equal to \( v_1 \) which yields no revenue when offered to \( A \).

Notation. We adopt the convention of using bold letters for vectors, capital roman letters for sets, capital italic letters for single-dimensional random variables, capital bold letters for multi-dimensional random variables (such as sets or vectors), and capital calligraphy letters for events.

We will use \( \mathbf{E}[\text{RSOP}] \) to denote the expected revenue of RSOP on an implicit bid vector \( \mathbf{v} \), where the expectation is taken over all random partitions \((A, B)\); however we sometimes specify an explicit bid vector by writing \( \mathbf{E}[\text{RSOP}(\mathbf{v})] \) or \( \text{OPT}(\mathbf{v}) \).

We use \( \lambda \) to denote the index of the optimal sale price which sells to at least two bidders, i.e.,

\[
\lambda \in \arg \max_{j \geq 2} j v_j
\]

For every \( j \in \mathbb{N} \), we define

\[
S_j = |A \cap \{1 \ldots j\}|,
\]

\[
Z_j = \frac{|B \cap \{1 \ldots j\}| - j}{S_j},
\]

\[
Z = \min \{ \{Z_j\}_{j \in \mathbb{N}, 1} \}.
\]

Note that \( S_j, Z_j \) and \( Z \) are random variables which depend only on how the bids are partitioned, but not on the actual value of the bids.

For every \( T \subset \mathbb{N} \) and \( \alpha, \alpha' \in [0, 1] \), we define the following events:

\[
\mathcal{E}_\alpha^T = \left\{ \max_{j \in T} \frac{S_j}{j} \leq \alpha \right\},
\]

\[
\mathcal{E}_{(\alpha', \alpha]}^T = \left\{ \alpha' < \max_{j \in T} \frac{S_j}{j} \leq \alpha \right\} = \mathcal{E}_\alpha^T \setminus \mathcal{E}_{\alpha'}^T
\]

Figure 1 illustrates an example of \( \mathcal{E}_\alpha^T \) and \( \mathcal{E}_{(\alpha', \alpha]}^T \). We omit \( T \) if \( T = \mathbb{N} \), i.e., \( \mathcal{E}_\alpha = \mathcal{E}_\alpha^\mathbb{N} \) and \( \mathcal{E}_{(\alpha', \alpha]} = \mathcal{E}_{(\alpha', \alpha]}^\mathbb{N} \).

Finally, for any random variable \( X \) and event \( \mathcal{E} \), we use \( \hat{\mathbf{E}}[X \mid \mathcal{E}] \) to denote the expectation of \( X \) conditioned on event \( \mathcal{E} \) normalized by the probability of \( \mathcal{E} \), i.e.,

\[
\hat{\mathbf{E}}[X \mid \mathcal{E}] = \mathbf{E}[X \mid \mathcal{E}] \frac{\Pr[\mathcal{E}]}{\Pr[\mathcal{E}]}.
\]

We will use the following proposition extensively throughout this paper.

Proposition 1. For any random variable \( X \) and any two events \( \mathcal{E}, \mathcal{E}' \),

- if \( \mathcal{E}' \subseteq \mathcal{E} \), then \( \hat{\mathbf{E}}[X \mid \mathcal{E} \setminus \mathcal{E}'] = \hat{\mathbf{E}}[X \mid \mathcal{E}] - \hat{\mathbf{E}}[X \mid \mathcal{E}'] \);

- if \( \mathcal{E} \cap \mathcal{E}' = \emptyset \), then \( \hat{\mathbf{E}}[X \mid \mathcal{E} \cup \mathcal{E}'] = \hat{\mathbf{E}}[X \mid \mathcal{E}] + \hat{\mathbf{E}}[X \mid \mathcal{E}'] \).

The following lemmas will be useful throughout the rest of this paper.

Lemma 1. For any \( T, T' \subset \mathbb{N} \) and \( \alpha \in [0, 1] \), the two events \( \mathcal{E}_\alpha^T \) and \( \mathcal{E}_{\alpha'}^{T'} \) are positively correlated, i.e.,

\[
\Pr[\mathcal{E}_\alpha^T \cap \mathcal{E}_{\alpha'}^{T'}] \geq \Pr[\mathcal{E}_\alpha^T] \Pr[\mathcal{E}_{\alpha'}^{T'}] \quad \text{(alternatively } \Pr[\mathcal{E}_\alpha^T \cap \mathcal{E}_{\alpha'}^{T'}] \leq \Pr[\mathcal{E}_\alpha^T] \Pr[\mathcal{E}_{\alpha'}^{T'}] \text{)}.
\]

\(^3\)Otherwise we can swap \( A \) and \( B \).
Lemma 2. For any \( \alpha \in (0, 1) \) and \( j \in \mathbb{N} \),

\[
\text{if } \alpha \geq 0.5, \text{ then } \Pr\left[E_{\alpha}^{(j)}\right] \geq 1 - (r_{\alpha})^{j}, \quad \text{where} \quad r_{\alpha} = \frac{1}{2\alpha^{\alpha}(1 - \alpha)^{1 - \alpha}},
\]

\[
\text{if } \alpha \leq 0.5 - 1/j, \text{ then } \Pr\left[E_{\alpha}^{(j)}\right] \leq (r_{(\alpha + 1/j)})^{j - 1} \quad \text{where } r_{\alpha} \text{ is the same as above.}
\]

Proof. The claim follows from a direct application of Chernoff-Hoeffding bound and can be found in Appendix B.

3 The Basic Lower Bound

In this section we prove that RSOP is indeed 4-competitive for a large class of input instances (i.e., when \( \lambda > 10 \)). In the next section, we improve this result for \( \lambda \leq 10 \) using a more sophisticated analysis, but based on the same ideas. The following theorem summarizes the main result of this section.

Theorem 1. \( \mathbb{E}[\text{RSOP}] \geq \frac{1}{4} \text{OPT} \) for all \( \lambda > 10 \). Furthermore, \( \mathbb{E}[\text{RSOP}] \geq \frac{1}{3.52} \text{OPT} \) for all \( \lambda > 5000 \).

Table 1 lists the actual lower bounds obtained for various choices of \( \lambda \).

The outline of this section is as follows. First, we present a lower bound on \( \mathbb{E}[\text{RSOP}] \) as a function of \( \lambda \). Recall that expectation is taken over all random partitions \((A, B)\) for a fixed set of bids (and thus a fixed \( \lambda \)). Our proposed lower bound depends only on \( \lambda \) and not the actual value of the bids. We present a dynamic program for numerically computing the lower bound for any fixed \( \lambda \). By computing the lower bound on \( \mathbb{E}[\text{RSOP}] \) for all \( \lambda \in \{11 \cdots 5000\} \) we confirm that it is indeed greater than \( \frac{1}{4} \text{OPT} \). We then prove a lower bound of \( \frac{1}{3.52} \text{OPT} \) on \( \mathbb{E}[\text{RSOP}] \) for all \( \lambda > 5000 \).

The following lemma provides a lower bound on \( \mathbb{E}[\text{RSOP}] \) as a function of \( \lambda \).

Lemma 3. \( \mathbb{E}[\text{RSOP}] \geq \mathbb{E}[\frac{S}{\lambda} Z] \text{OPT} \).

Proof. Let \( v_{\lambda, A} \) be the optimal price for \( A \) which RSOP offers to bidders in \( B \); observe that \( S_{\lambda, A} v_{\lambda, A} \geq S_{j} v_{j} \) for all \( j \in \mathbb{N} \). The revenue of RSOP is at least the revenue it obtains from \( B \), therefore

\[
\text{RSOP} \geq (\lambda_{A} - S_{\lambda, A}) v_{\lambda, A} = Z_{\lambda, A} S_{\lambda, A} v_{\lambda, A} \geq Z S_{\lambda} v_{\lambda} = Z \frac{S}{\lambda} \text{OPT}
\]

because at least \( \lambda_{A} - S_{\lambda, A} \) bids in \( B \) are above or equal to \( v_{\lambda, A} \), by definition of \( Z_{\lambda, A} \) in \ref{eq:bound}, \( Z \leq Z_{\lambda, A} \) and \( S_{\lambda} v_{\lambda} \leq S_{\lambda, A} v_{\lambda, A} \), because OPT = \( \lambda v_{\lambda} \).

Consequently, \( \mathbb{E}[\text{RSOP}] \geq \mathbb{E}[\frac{S}{\lambda} Z] \text{OPT} \) which proves the claim.

It is crucial that the lower bound provided by the above lemma only depends on \( \lambda \) and not on the exact value of the bids. Recall that \( \lambda \) depends only on the value of the bids and not on how the bids are partitioned.

3.1 Small \( \lambda \)

We start by proving the first part of Theorem 1, i.e., that \( \mathbb{E}[\text{RSOP}] \geq \frac{1}{4} \text{OPT} \) for all \( 10 < \lambda \leq 5000 \).

Recall that \( \mathbb{E}[\text{RSOP}] \geq \mathbb{E}[\frac{S}{\lambda} Z] \text{OPT} \) by Lemma 3. Ideally, we would like to approximate \( \mathbb{E}[\frac{S}{\lambda} Z] \) by \( \mathbb{E}[\frac{S}{\lambda}] \mathbb{E}[Z] \), however \( \frac{S}{\lambda} \) and \( Z \) are negatively correlated. To work around this obstacle we will decompose \( \mathbb{E}[\frac{S}{\lambda} Z] \) over a set of small and disjoint events such that, conditioned on each such event, \( Z \) can be approximated closely by a constant. The events are defined as follows. We partition the interval \([0, 1]\) to small disjoint intervals by picking \( m \) points \( 0.5 < \alpha_{1} < \cdots < \alpha_{m} < 1 \). For each interval \((\alpha_{i-1}, \alpha_{i}]\) we consider the event \( E_{(\alpha_{i-1}, \alpha_{i}]} \). Recall that \( E_{(\alpha_{i-1}, \alpha_{i}]} \) is the event that \((\max_{x} \frac{S}{\lambda}) \in (\alpha_{i-1}, \alpha_{i}] \) (see Figure 1). Conditioned on \( E_{(\alpha_{i-1}, \alpha_{i}]} \), it is easy to see that \( Z \in [1 - \alpha_{i}, 1 - \alpha_{i-1}] \), and therefore we can obtain a good lower bound by substituting \( Z \) with \( \frac{1}{\alpha_{i}} \). Notice that there is no use in picking \( \alpha_{i} \) from \([0, 0.5]\) because for any \( \alpha \in [0, 0.5] \), \( \Pr[E_{\alpha}^{(j)}] = 0 \) and therefore, for any bounded random variable \( X \), we get \( \mathbb{E}[X \mid E_{\alpha}^{(j)}] = \mathbb{E}[X \mid E_{\alpha}^{(j)}] \Pr[E_{\alpha}^{(j)}] = 0 \).

Also notice that there is no use in considering the event \( E_{(\alpha_{i-1}, \alpha_{i}]} \) because we can only guarantee a trivial lower bound of \( 0 \) for \( Z \) under \( E_{(\alpha_{m}, 1]} \).
Lemma 4. Given an increasing sequence $\alpha_1, \ldots, \alpha_m \in (0, 1)$, the following inequality holds for any non-negative random variable $X$.

$$E[XZ] \geq \sum_{i=1}^{m} \left( \frac{1}{\alpha_i} - \frac{1}{\alpha_{i+1}} \right) E[X \mid \mathcal{E}_{\alpha_i}].$$  \hfill (11)

Assume $\alpha_{m+1} = 1$.

Proof. Let $\alpha_0 = 0$. We decompose $E[XZ]$ over the set of disjoint events $\mathcal{E}_{(\alpha_0, \alpha_1)}, \ldots, \mathcal{E}_{(\alpha_{m-1}, \alpha_m)}$ as follows.

$$E[XZ] \geq \sum_{i=1}^{m} \mathbb{E}[XZ \mid \mathcal{E}_{(\alpha_{i-1}, \alpha_i)}]$$  \hfill by law of total expectation

$$\geq \sum_{i=1}^{m} \mathbb{E}\left[ X \left( \frac{1 - \alpha_i}{\alpha_i} \right) \mathcal{E}_{(\alpha_{i-1}, \alpha_i)} \right]$$  \hfill because $Z \geq \frac{1 - \alpha_i}{\alpha_i}$ conditioned on $\mathcal{E}_{(\alpha_{i-1}, \alpha_i)}$

$$= \sum_{i=1}^{m} \frac{1 - \alpha_i}{\alpha_i} \left( \mathbb{E}[X \mid \mathcal{E}_{\alpha_i}] - \mathbb{E}[X \mid \mathcal{E}_{\alpha_{i-1}}] \right)$$  \hfill by Proposition 1 given that $\mathcal{E}_{(\alpha_{i-1}, \alpha_i)} = \mathcal{E}_{\alpha_i} \setminus \mathcal{E}_{\alpha_{i-1}}$

$$= \sum_{i=1}^{m} \left( \frac{1}{\alpha_i} - \frac{1}{\alpha_{i+1}} \right) \mathbb{E}[X \mid \mathcal{E}_{\alpha_i}]$$  \hfill by rearranging the terms.

Note that in the last step we have used the fact that $\mathbb{E}[X \mid \mathcal{E}_{\alpha_0}] = 0$ (because $Pr[\mathcal{E}_{\alpha_0}] = 0$).

The choice of $m$ and $\alpha_1, \ldots, \alpha_m$ in Lemma 4 greatly affects the value of the lower bound. Generally speaking, increasing $m$ improves the lower bound but at the cost of more computation.

In order to use Lemma 4 effectively, we need to be able to compute $E[S_{\alpha_i} \mid \mathcal{E}_{\alpha_i}]$ for each $\alpha_i$. However, the events $\mathcal{E}_{\alpha_i}$ are hard to deal with computationally. The next two lemmas show that $E[S_{\alpha_i} \mid \mathcal{E}_{\alpha_i}]$ can be bounded below and thus approximated by $E[S_{\alpha_i} \mid \mathcal{E}_{\alpha_i}^{(1:\ell)}] - \epsilon$ where $\epsilon$ approaches 0 exponentially fast as a function of $\ell$. 

Figure 1
Lemma 5. For any random variable $X \in [0, 1]$, any $\alpha \in (0.5, 1]$, and $\ell \in \mathbb{N}$ the following holds:

$$
\hat{E} \left[ X \mid \mathcal{E}_\alpha \right] \geq \hat{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \right] - \epsilon
$$

where $\epsilon = \Pr \left[ \mathcal{E}_\alpha^{(1-\ell)} \right] \left( 1 - \Pr \left[ \mathcal{E}_\alpha^{(1-1\cdots)} \right] \right)$ \quad (12)

Proof. Observe that $\mathcal{E}_\alpha = \mathcal{E}_\alpha^{(1-\ell)} \setminus (\mathcal{E}_\alpha^{(1-\ell)} \cap \mathcal{E}_\alpha^{(1-1\cdots)})$ \[ \square \] therefore

$$
\hat{E} \left[ X \mid \mathcal{E}_\alpha \right] = \hat{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \right] - \hat{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \cap \mathcal{E}_\alpha^{(1-1\cdots)} \right]
$$

by Proposition 1

$$
\geq \hat{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \right] - \Pr \left[ \mathcal{E}_\alpha^{(1-\ell)} \cap \mathcal{E}_\alpha^{(1-1\cdots)} \right]
$$

because $X \in [0, 1]

$$
\geq \hat{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \right] - \Pr \left[ \mathcal{E}_\alpha^{(1-\ell)} \right] \left( 1 - \Pr \left[ \mathcal{E}_\alpha^{(1-1\cdots)} \right] \right)
$$

by Lemma 1

\[ \square \]

The following lemma allows us to compute an upper bound on the $\epsilon$ of the previous lemma.

Lemma 6. For any $\alpha \in (0.5, 1]$ and any $\ell, \ell' \in \mathbb{N}$ such that $\ell \leq \ell'$, the following holds:

$$
\Pr \left[ \mathcal{E}_\alpha^{(\ell+1\cdots)} \right] \geq \left( 1 - \left( \frac{(r_\alpha)^{\ell+1}}{1 - r_\alpha} \right) \right) \prod_{j=\ell+1}^{\ell'} \left( 1 - (r_\alpha)^j \right)
$$

where $r_\alpha$ is defined in \[ \square \] (13)

Proof.

$$
\Pr \left[ \mathcal{E}_\alpha^{(\ell+1\cdots)} \right] = \Pr \left[ \bigcap_{j=\ell+1}^{\infty} \mathcal{E}_\alpha^{(j)} \right] \geq \Pr \left[ \bigcap_{j=\ell+1}^{\infty} \mathcal{E}_\alpha^{(j)} \right] \prod_{j=\ell+1}^{\ell'} \Pr \left[ \mathcal{E}_\alpha^{(j)} \right]
$$

by Lemma 1

$$
\geq \left( 1 - \sum_{j=\ell+1}^{\infty} \Pr \left[ \mathcal{E}_\alpha^{(j)} \right] \right) \prod_{j=\ell+1}^{\ell'} \Pr \left[ \mathcal{E}_\alpha^{(j)} \right]
$$

by union bound

$$
\geq \left( 1 - \left( \frac{(r_\alpha)^{\ell+1}}{1 - r_\alpha} \right) \right) \prod_{j=\ell+1}^{\ell'} \left( 1 - (r_\alpha)^j \right)
$$

by Lemma 2

\[ \square \]

Observe that in the special case of the above lemma in which $\ell = \ell'$, the right hand side of (13) approaches 1 exponentially fast as a function of $\ell$ which implies that $\epsilon$ in (12) approaches 0 exponentially fast as a function of $\ell$. Choosing $\ell' > \ell$ only improves the bound.

The next lemma provides a recurrence relation which can be used to compute the exact value of $\mathbb{E} \left[ X \mid \mathcal{E}_\alpha^{(1-\ell)} \right]$ and $\Pr \left[ \mathcal{E}_\alpha^{(1-\ell)} \right]$ in time $O(\ell^2)$.
Lemma 7. For any $\ell \in \mathbb{N}$ and $\alpha \in [0, 1]$, the exact value of $\mathbb{E}[S_\alpha | \mathcal{E}_\alpha^{(1:\ell)}]$ and $\text{Pr}[\mathcal{E}_\alpha^{(1:\ell)}]$ can be computed using the following recurrence in which $\mathcal{E}_\alpha^{(1:\ell)} = \mathcal{E}_\alpha \cap \{S_\ell = k\}$ is the event that $\mathcal{E}_\alpha$ happens and $S_\ell = k$.

$$
\text{Pr}[\mathcal{E}_{\alpha,k}^{(1:\ell)}] = \begin{cases} 
\frac{1}{2} \text{Pr}[\mathcal{E}_{\alpha,k-1}^{(1:\ell-1)}] + \frac{1}{2} \text{Pr}[\mathcal{E}_{\alpha,k}^{(1:\ell-1)}] & \ell > 1, k \leq \alpha \ell \\
1 & \ell = 1, k = 0 \\
0 & \text{otherwise}
\end{cases}
$$

(14)

$$
\mathbb{E}\left[\frac{S_\alpha}{\lambda} | \mathcal{E}_{\alpha,k}^{(1:\ell)}\right] = \begin{cases} 
\frac{1}{2} \mathbb{E}\left[\frac{S_\alpha}{\lambda} | \mathcal{E}_{\alpha,k-1}^{(1:\ell-1)}\right] + \frac{1}{2} \mathbb{E}\left[\frac{S_\alpha}{\lambda} | \mathcal{E}_{\alpha,k}^{(1:\ell-1)}\right] & \ell > \lambda, k \leq \alpha \ell \\
\frac{1}{\lambda} \text{Pr}[\mathcal{E}_{\alpha,k}^{(1:\ell)}] & \ell = \lambda \\
0 & \text{otherwise}
\end{cases}
$$

(15)

$$
\text{Pr}[\mathcal{E}_\alpha^{(1:\ell)}] = \sum_{\ell=0}^{k} \text{Pr}[\mathcal{E}_{\alpha,k}^{(1:\ell)}]
$$

(16)

$$
\mathbb{E}\left[\frac{S_\alpha}{\lambda} | \mathcal{E}_\alpha^{(1:\ell)}\right] = \sum_{k=0}^{\ell} \mathbb{E}\left[\frac{S_\alpha}{\lambda} | \mathcal{E}_{\alpha,k}^{(1:\ell)}\right]
$$

(17)

Proof. Let $\mathcal{A}_k$ denote the event that $\ell \in \mathcal{A}$. First consider (14): if $\ell > 1$ and $k \leq \alpha \ell$, then $\mathcal{E}_{\alpha,k}^{(1:\ell)}$ can be decomposed as two disjoint events $\mathcal{E}_{\alpha,k}^{(1:\ell-1)} \cap \mathcal{A}_k$ and $\mathcal{E}_{\alpha,k}^{(1:\ell-1)} \cap \overline{\mathcal{A}_k}$, therefore its probability is the sum of the probabilities of those two event; note that $\mathcal{E}_{\alpha,k}^{(1:\ell-1)}$ and $\mathcal{A}_k$ are independent for any $\ell$ and $k$ and $\text{Pr}[\mathcal{A}_k] = \frac{1}{2}$; furthermore the base of the recursion is $\text{Pr}[\mathcal{E}_{\alpha,0}^{(1)}] = 1$ because by our assumption $\mathcal{A}_1 = 0$ (i.e., the highest bid is always in $\mathcal{B}$). The same argument implies the correctness of (15) for the case of $\ell > \lambda$. Furthermore, $\mathcal{E}_\alpha^{(1:\lambda)}$ by its definition implies $S_\lambda = k$ which implies the correctness of (15) for the case of $\ell = \lambda$. Finally (16) and (17) follow trivially from the law of total probability and the law of total expectation.

Proof of Theorem 1 for small $\lambda$ (i.e., $10 < \lambda \leq 5000$). We show how to numerically compute a lower bound on $\mathbb{E}[\text{RSOP}]$ for any fixed $\lambda$. Let $m = 100$ and $\alpha_i = 0.5 + \frac{1}{m+i}$ for each $i \in [m]$. Observe that

$$
\mathbb{E}[\text{RSOP}] \geq \mathbb{E}\left[\frac{S_\lambda}{\lambda}Z\right] \text{OPT}
$$

by Lemma 3

$$
\geq \sum_{i=1}^{m} \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i+1}}\right) \mathbb{E}\left[\frac{S_\lambda}{\lambda} | \mathcal{E}_{\alpha_i}\right] \text{OPT}
$$

by Lemma 4

We then compute a lower bound for each $\mathbb{E}\left[\frac{S_\lambda}{\lambda} | \mathcal{E}_{\alpha_i}\right]$ (using Lemma 5, Lemma 6, and Lemma 7 with $\ell = 5000$ and $\ell' = 100000$), and substitute them in the last inequality above to obtain a lower bound on $\mathbb{E}[\text{RSOP}]$. We have confirmed that $\mathbb{E}[\text{RSOP}] \geq \frac{1}{2} \text{OPT}$ for all $\lambda \in \{11 \cdots 5000\}$ by numerically computing the lower bound for each choice of $\lambda$ in that range. The computed numerical values of our lower bound are listed in Table 1 for various choices of $\lambda$.

3.2 Large $\lambda$

We now prove the second part of Theorem 1, i.e., $\mathbb{E}[\text{RSOP}] \geq \frac{1}{\sqrt{2\lambda}} \text{OPT}$ for all $\lambda > 5000$.

Recall that $\mathbb{E}[\text{RSOP}] \geq \mathbb{E}\left[\frac{S_\lambda}{\lambda}Z\right] \text{OPT}$ by Lemma 3 Also recall that $\frac{S_\lambda}{\lambda}$ and $Z$ are negatively correlated, thus $\mathbb{E}[\frac{S_\lambda}{\lambda}] \mathbb{E}[Z]$ does not yield a lower bound on $\mathbb{E}\left[\frac{S_\lambda}{\lambda}Z\right]$. Nevertheless, the correlation decreases as $\lambda$ increases which suggests that for sufficiently large $\lambda$ we can separate the two terms. In other words, when $\lambda$ is large (i.e., $\lambda > 5000$), the two random variables $\frac{S_\lambda}{\lambda}$ and $Z$ are almost independent and so the expected value of their product is very close to the product of their expected values. Also for a large $\lambda$ the value of $\frac{S_\lambda}{\lambda}$ is very close to $\frac{1}{\lambda}$ so $\mathbb{E}\left[\frac{S_\lambda}{\lambda}Z\right]$ is close to $\frac{1}{\lambda} \mathbb{E}[Z]$. We formalize this argument in the following lemma.

Lemma 8. For any $\alpha \in [0, 1]$:

$$
\mathbb{E}\left[\frac{S_\lambda}{\lambda}Z\right] \geq \alpha \left(\mathbb{E}[Z] - \text{Pr}[\mathcal{E}_{\alpha}^{(\lambda)}]\right)
$$

(18)
Proof.

\[
\mathbb{E}\left[\frac{S_\alpha Z}{\lambda}\right] = \mathbb{E}\left[\frac{S_\alpha Z}{\lambda} \mid \mathcal{E}_\alpha^{(\lambda)}\right] + \mathbb{E}\left[\frac{S_\alpha Z}{\lambda} \mid \mathcal{E}_\alpha^{(\lambda)}\right]
\]
\[
\geq \alpha \mathbb{E}\left[Z \mid \mathcal{E}_\alpha^{(\lambda)}\right]
\]
\[
= \alpha \left(\mathbb{E}\left[Z\right] - \mathbb{E}\left[Z \mid \mathcal{E}_\alpha^{(\lambda)}\right]\right)
\]
\[
\geq \alpha \left(\mathbb{E}\left[Z\right] - \mathbb{P}\left[\mathcal{E}_\alpha^{(\lambda)}\right]\right)
\]
because $\frac{S_\alpha}{\lambda} > \alpha$ conditioned on $\mathcal{E}_\alpha^{(\lambda)}$.

Recall that we can compute an upper bound on $\mathbb{P}[\mathcal{E}_\alpha^{(\lambda)}]$ using Lemma 2. Also observe that, for any fixed $\alpha \in (0, 0.5)$, $\mathbb{P}[\mathcal{E}_\alpha^{(\lambda)}]$ approaches 0 exponentially fast as a function of $\lambda$ as $\lambda \to \infty$. The only remaining task is to compute a good lower bound on $\mathbb{E}[Z]$.

**Lemma 9.** $\mathbb{E}[Z] \geq 0.61$.

Proof. Let $\ell = 60000$, $m = 100$, and $\alpha_i = 0.5 + \frac{1}{m+i}$ for each $i \in [m]$. By applying Lemma 4 and plugging $X = 1$ we get
\[
\mathbb{E}[Z] \geq \sum_{i=1}^{m} \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i+1}}\right) \mathbb{P}[\mathcal{E}_{\alpha_i}^{(1:\ell)}]
\]
\[
\geq \sum_{i=1}^{m} \left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i+1}}\right) \mathbb{P}[\mathcal{E}_{\alpha_i}^{(1:\ell)}] \mathbb{P}[\mathcal{E}_{\alpha_i}^{(\ell+1:\infty)}]
\]
by Lemma 1.

We then use Lemma 7 to compute $\mathbb{P}[\mathcal{E}_{\alpha_i}^{(1:\ell)}]$, and use Lemma 6 with $\ell' = 100000$ to compute a lower bound on $\mathbb{P}[\mathcal{E}_{\alpha_i}^{(\ell+1:\infty)}]$; by substituting the computed numerical values in the above inequality we get $\mathbb{E}[Z] \geq 0.61$.

It is worth mentioning that by using a similar method we have computed an upper bound of $\mathbb{E}[Z] \leq 0.63$ which indicates that our analysis is almost tight.

Proof of Theorem 4 for large $\lambda$ (i.e., $\lambda > 5000$). Let $\alpha = 0.48$. Then
\[
\mathbb{E}[RSOP] \geq \mathbb{E}\left[\frac{S_\alpha}{\lambda} Z\right] \text{OPT}
\]
\[
\geq \alpha \left(\mathbb{E}\left[Z\right] - \mathbb{P}\left[\mathcal{E}_\alpha^{(\lambda)}\right]\right) \text{OPT}
\]
by Lemma 8.

Using Lemma 2 we get $\mathbb{P}[\mathcal{E}_\alpha^{(\lambda)}] \leq 0.0183$ for all $\lambda > 5000$; furthermore $\mathbb{E}[Z] \geq 0.61$ by Lemma 9; substituting them in the above inequality we get $\mathbb{E}[RSOP] \geq 0.284$ which is equivalent to a competitive ratio of 3.52.

## 4 The Exhaustive Search Lower-Bound

In this section we propose an exhaustive search approach which yields an improved lower bound for RSOP for small choices of $\lambda$ (i.e., $\lambda \leq 10$). The following theorem summarizes the main result of this section.

**Theorem 2.** $\mathbb{E}[RSOP] \geq \frac{1}{10.03} \text{OPT}$ for $\lambda \geq 2$ and $\mathbb{E}[RSOP] \geq \frac{1}{4} \text{OPT}$ for $\lambda \geq 6$. Table 2 lists the actual lower bounds obtained for various choices of $\lambda$.

\[ \text{Note that } 1 - 1/e \simeq 0.6321 \text{ which is slightly greater than the upper bound of } \mathbb{E}[Z] \leq 0.63. \]
The basic lower bound of $E[RSOP] \geq E[\frac{S_{\lambda}}{Z}]OPT$ which we used in Section 3 does not yield a good enough bound when $\lambda$ is small, mainly because

(I) $\frac{S_{\lambda}}{Z}$ and $Z$ are negatively correlated and their correlation is much stronger when $\lambda$ is small, also

(II) the highest bid is always in $B$, so $E[\frac{S_{\lambda}}{Z}]$ approaches $\frac{1}{4}$ as $\lambda$ goes down to 2.

Therefore, for $\lambda = 2$, $E[\frac{S_{\lambda}}{Z}] < E[\frac{S_{\lambda}}{Z}]E[Z] < 0.25 \times 0.63 \approx \frac{1}{6.33}$. In fact, the lower bounds of Table 1 are quite close to the exact value of $E[\frac{S_{\lambda}}{Z}]$, which suggests for small values of $\lambda$ we need a different approach.

We now provide a high level description of the approach of this section. Without loss of generality we assume $OPT = 1$. In addition to fixing the index of the optimal price, $\lambda$, we fix the index of the second optimal price of a higher index, $\lambda'$, and also fix its corresponding revenue, $OPT'$, i.e.,

$$\lambda' = \arg \max_{j > \lambda} jv_j$$

$$OPT' = \max_{j > \lambda} jv_j$$

We then try all possible values for $OPT'$ and $v_1, \ldots, v_d$ (for an appropriate choice of $d$), and apply the techniques from Section 3 to the remaining bids; however instead of fixing the exact value of each bid and of $OPT'$ — which would require checking infinitely many instances — we restrict each bid and $OPT'$ to an interval, i.e., $OPT' \in [OPT', OPT']$ and $v_j \in [v_j, \tau_j]$ for every $j \in [d]$. We then try various configurations of such intervals to cover all possible scenarios. For each configuration we compute a lower bound on $E[RSOP]$ for each $\lambda' \leq 5000$ as a function of both $\lambda$ and $\lambda'$, and another lower bound as a function of only $\lambda$ assuming a reasonably large $\lambda'$ (e.g., $\lambda' > 5000$). We then take the minimum lower bound among all configurations and all $\lambda'$ to obtain a global lower bound on $E[RSOP]$ for each $\lambda \in \{2 \cdots 10\}$; the computed lower bounds are listed in Table 3.

**Lemma 10.** Let $RSOP_{\text{MINEXPECT}}(\lambda, \{[v_j, \tau_j]\}_{j \in [d]}, [OPT', OPT'])$ denote the minimum expected revenue of $E[RSOP]$ over all instances where $OPT' \in [OPT', OPT']$ and $v_j \in [v_j, \tau_j]$ for all $j \in [d]$ and given that $\lambda$ is the index of the optimal price. Then for any $\theta, \theta' \in \mathbb{N}$,

$$E[RSOP] \geq \min_{(i_1, \ldots, i_d) \in [\theta]^d} \min_{i' \in [\theta']} RSOP_{\text{MINEXPECT}}\left(\lambda, \left\{\left[\frac{i_j - 1}{\theta}, \frac{i_j}{\theta}, \frac{1}{\theta'}\right] \cdot \theta \right\}_{j \in [d]}, \left[\frac{i' - 1}{\theta'}, \frac{i'}{\theta'}, \frac{1}{\theta'}\right]\right).$$

**Proof.** The claim follows because the minimum is taken over all possible combinations of intervals and that any bid vector is covered by at least one of the combinations.

Note that some combinations of intervals in (21) might be inconsistent/infeasible; for example it is infeasible to have both $[v_2, \tau_2] = [\frac{0}{10}, \frac{1}{10}]$ and $[v_3, \tau_3] = [\frac{4}{15}, \frac{5}{15}]$ because that would imply $v_3 > v_2$; we define $RSOP_{\text{MINEXPECT}}$ to be $\infty$ if a configuration of intervals is infeasible.

**Computing a lower bound on $RSOP_{\text{MINEXPECT}}$.** In the rest of this section we show how to compute a lower bound on $E[RSOP]$ given the assumption that $OPT' \in [OPT', OPT']$ and $v_j \in [v_j, \tau_j]$ for all $j \in [d]$, where these intervals are specified exogenously. The high level idea is to enumerate all possible partitions of the first $d$ bids, define an event for each such partition and decompose $E[RSOP]$ over those events, and compute a lower bound conditioned on each such event.

We start with a few definitions. For every $T \subset [d]$, we define the following event

$$\mathcal{A}_T = \{A \cap \{1 \cdots d\} = T\}$$

Intuitively, $\mathcal{A}_T$ is the event that, among the first $d$ bids, the subset of bids that fall in $A$ is exactly $T$. Observe that under the event $\mathcal{A}_T$, both $S_j$ and $Z_j$ are constants (for every $j \in [d]$); we will denote those constants respectively by

$$s_j^T = (S_j | \mathcal{A}_T) = |\{1 \cdots j\} \cap T|$$

$$z_j^T = (Z_j | \mathcal{A}_T) = \frac{|\{1 \cdots j\} \setminus T|}{|\{1 \cdots j\} \cap T|}$$
Our approach is to decompose \( E[\text{RSOP}] \) over the set of disjoint events \( \{A_T\}_{T \subseteq \{2 \cdots d\}} \) and then decompose \( \hat{E}[\text{RSOP} | A_T] \) further over the set of disjoint events \( \{E^{(d+1\cdots \infty)}_{\alpha_{i-1},\alpha_i}\}_{i \in [m]} \) for some choice of \( 0.5 < \alpha_1 < \cdots < \alpha_m < 1 \) (the second decomposition is similar to Section 3); formally,

\[
E[\text{RSOP}] = \sum_{T \subseteq \{2 \cdots d\}} \sum_{i=1}^{m} \hat{E}[\text{RSOP} | A_T \cap E^{(d+1\cdots \infty)}_{\alpha_{i-1},\alpha_i}]
\]

Next we show how to compute a lower bound on \( \hat{E}[\text{RSOP} | A_T \cap E^{(d+1\cdots \infty)}_{\alpha_{i-1},\alpha_i}] \) that does not depend on the exact value of the bids.

**Lemma 11.** For any \( d \geq 2 \), any \( T \subseteq \{2 \cdots d\} \), and any \( 0.5 < \alpha' < \alpha \leq 1 \),

\[
\hat{E}[\text{RSOP} | A_T \cap E^{(d+1\cdots \infty)}_{\alpha',\alpha}] \geq \hat{E}\left[ \max\left( r^T, \frac{S_{\lambda'}}{\lambda'} \text{OPT}' \right) \rho^T_{\alpha} \mid A_T \cap E^{(d+1\cdots \infty)}_{\alpha',\alpha} \right]
\]

where \( r^T \) and \( \rho^T_{\alpha} \) are constants defined as

\[
r^T = \max_{j \in [d]} s^T_j v_j,
\]

\[
\lambda^T = \{ j \in T \mid s^T_j v_j \geq r^T \},
\]

\[
\rho^T_{\alpha} = \begin{cases} 
\min \left\{ s^T_j \mid j \in \lambda^T \right\} & \text{if } r^T > \alpha \text{OPT}' \\
\min \left\{ s^T_j \mid j \in \lambda^T \right\}, \frac{1-\alpha}{\alpha} & \text{otherwise}
\end{cases}
\]

**Proof.** Let \( v_{\lambda_A} \) be the optimal price for \( A \) which RSOP offers to bidders in \( B \); observe that

\[
\text{RSOP} \geq (\lambda_A - S_{\lambda_A}) v_{\lambda_A} = S_{\lambda_A} v_{\lambda_A} Z_{\lambda_A}
\]

Under event \( A_T \cap E^{(d+1\cdots \infty)}_{\alpha',\alpha} \), we show that \( S_{\lambda_A} v_{\lambda_A} \geq \max\left( r^T, \frac{S_{\lambda'}}{\lambda'} \text{OPT}' \right) \) and \( Z_{\lambda_A} \geq \rho^T_{\alpha} \), which combined with the above inequality imply the statement of the lemma.

- \( S_{\lambda_A} v_{\lambda_A} \geq \max\left( r^T, \frac{S_{\lambda'}}{\lambda'} \text{OPT}' \right) \). Notice that \( S_{\lambda_A} v_{\lambda_A} \) is the optimal revenue of \( A \) which must be at least \( r^T \); furthermore, the optimal revenue of \( A \) is no less than the revenue of selling to \( A \) at price \( v_{\lambda} \), which is at least \( \frac{S_{\lambda'}}{\lambda'} \text{OPT}' \).

- \( Z_{\lambda_A} \geq \rho^T_{\alpha} \). The inequality follows immediately by considering the following two possibilities:

  (I) \( \lambda_A \leq d \). In this case \( \lambda_A \) must be in \( \lambda^T \), because for any \( j \in \{1 \cdots d\} \setminus \lambda^T \), selling to \( A \) at price \( v_j \) generates a revenue which is less than \( r^T \), therefore \( v_j \) cannot be the optimal price for \( A \).

  (II) \( \lambda_A > d \). First we claim that this case cannot happen if \( r^T > \alpha \text{OPT}' \), because otherwise the revenue of selling to \( A \) at price \( v_{\lambda_A} \) is less than \( r^T \) which contradicts its optimality.\footnote{The revenue of selling to \( A \) at price \( v_{\lambda_A} \) is \( S_{\lambda_A} v_{\lambda_A} \) which is at most \( \alpha \lambda_A v_{\lambda_A} = \alpha \text{OPT}' \) under event \( E^{(d+1\cdots \infty)}_{\alpha',\alpha} \).}

If indeed \( \lambda_A > d \), then \( Z_{\lambda_A} \geq \frac{1-\alpha}{\alpha} \) under event \( E^{(d+1\cdots \infty)}_{\alpha',\alpha} \).

\[\square\]

**Lemma 12.** For any increasing sequence \( \alpha_1, \ldots, \alpha_m \in (0.5, 1) \) the following inequality holds (assume \( \alpha_{m+1} = 1 \)).

\[
E[\text{RSOP}] \geq \sum_{T \subseteq \{2 \cdots d\}} \sum_{i=1}^{m} \left( \rho^T_{\alpha_i} - \rho^T_{\alpha_{i+1}} \right) \hat{E}\left[ \max\left( r^T, \frac{S_{\lambda'}}{\lambda'} \text{OPT}' \right) \mid A_T \cap E^{(d+1\cdots \infty)}_{\alpha_i} \right]
\]
Proof. The claim follows by applying Lemma 11 to equation (25), then decomposing each event $E_{\alpha_i}^{\{d+1,\ldots,\infty\}}$ as $E_{\alpha_i}^{\{d+1,\ldots,\infty\}} \setminus E_{\alpha_{i-1}}^{\{d+1,\ldots,\infty\}}$ and applying Proposition 1 and then rearranging the terms. □

Next we sketch the proof of the main theorem of this section.

Proof of Theorem 3 We use Lemma 10 with $d = 11, \theta = 3$ and $\theta' = 100$ together with Lemma 12 with $m = 100$ and $\alpha_i = \frac{0.5 + \lambda}{m+1}$ for each $i \in [m]$. To compute an accurate approximation (lower bound) on each term $E[\max\left(r^T, \frac{S^C}{T}\right) \mid \mathcal{A}_T \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}]$, we use a combination of dynamic programming and tail bounds similar to those of Lemma 3, Lemma 6, Lemma 7, Lemma 8, and Lemma 9 (observe that $S_{\lambda}$ is the only random variable in this term). However doing so naively requires computing a lower bound on as many as $\theta^{d-1}\theta'2^{d-1}m$ such terms. Instead, we pre-compute $E[\max(c, S_{\lambda}c') \mid \{S_d = a\} \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}]$ for all $c, c' \in \{0, \frac{S_{\lambda}}{\theta'}, \ldots, \frac{S_{\lambda}}{\theta'}\}$, all $a \in \{0, \ldots, d\}$, and all $\alpha \in \{\alpha_1, \ldots, \alpha_m\}$; and then we approximate $E[\max\left(r^T, \frac{S^C}{T}\right) \mid \mathcal{A}_T \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}] = E[\max(c, S_{\lambda}c') \mid \{S_d = a\} \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}]$ where $c, c'$ are the result of rounding $r^T$ and $\frac{S^C}{T}$ down to the nearest integer multiples of $\frac{1}{\lambda}$ respectively and $a = |T|$ and $\alpha = \alpha_i$. Notice that we only need to pre-compute $(\theta' + 1)^2dm$. Table 2 lists the lower bound obtained for each $\lambda \in \{1, \ldots, 10\}$. As a last note, we should mention that we refine each configuration of intervals by cutting off infeasible regions of each interval prior to any further computation. □

5 An Upper Bound on The Performance of RSOP

It has been previously shown that there exist instances of bids for which $E[RSOP]$ is as low as $\frac{1}{4}OPT$ (e.g., Feige et al. [2005], Goldberg and Hartline [2001]). However, all such instances have $\lambda = 2$. That raises the question of whether the performance of RSOP approaches optimality asymptotically as $\lambda \to \infty$. In this section, we exhibit a family of instances for which $E[RSOP]$ is no more than $\frac{5}{7}\alpha OPT$ as $\lambda \to \infty$, which proves that the asymptotic competitive ratio of RSOP is no better than 2.65.

Theorem 3. For any $\lambda \geq 2$ there exists an input instance where there are $\lambda$ bids above or equal to the optimal sale price and such that $E[RSOP] < \frac{1}{2.65}OPT$.

Next, we define a family of instances which are used in the proof of the above theorem.

Definition 3 (Equal Revenue Instance). An instance of bids is called an equal revenue instance if choosing any of the bids as the sale price yields the same revenue. The equal revenue instance with $n$ non-zero distinct bids is unique (up to scaling) and given by the bid vector $q^{(n)} = (q_1^{(n)}, q_2^{(n)}, \ldots)$, where

$q_j^{(n)} = \begin{cases} \frac{1}{j} & j \leq n \\ 0 & \text{otherwise} \end{cases}$

Proposition 2. For any equal revenue instance, RSOP offers the worst price to each of the sets $A$ and $B$. In other words, the optimal price of each set generates the least revenue when offered to the opposite set (i.e., less revenue than offering any of the other non-zero bids as the sale price).

Proof. It follows immediately from the fact that offering any of the bids as the sale price for both sets generates a total revenue that is equal to OPT. So the price that generates highest revenue for $A$ also generates lowest revenue for $B$ and vice versa. □

Proposition 2 suggests that, for any given $\lambda$, an equal revenue instance might actually be the worst case instance for RSOP among all instances with the same $\lambda$; however based on computer simulation that seems not to be true at least for small values of $\lambda$.

---

7Because there are $\theta^{d-1}\theta'$ possible combinations of intervals in (21) and $2^{d-1}$ events of the form $\mathcal{A}_T$ and $m$ events of the form $E_{\alpha_i}^{\{d+1,\ldots,\infty\}}$.

8It is easy to see that $\hat{E}[\max(c, \frac{S_{\lambda}}{T}c') \mid \mathcal{A}_T \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}] = \hat{E}[\max(c, \frac{S_{\lambda}}{T}c') \mid \{S_d = |T|\} \cap E_{\alpha_i}^{\{d+1,\ldots,\infty\}}]$. For example if $\frac{q_j}{\lambda} < \frac{q_{j+1}}{\lambda}$, we set $\frac{q_j}{\lambda} \leftarrow \frac{q_{j+1}}{\lambda}$.
To prove Theorem 3, we need to show the expected revenue of RSOP is no more than $\frac{1}{2^{65}}$ OPT for any equal revenue instance with distinct bids. However a direct analysis of the performance of RSOP for all such instances is not easy. Instead we define a modified variant of RSOP whose performance is easy to analyze, and whose revenue is close to the revenue of RSOP (e.g., asymptotically equal as $\lambda \to \infty$).

**Definition 4** (RSOP*). The modified random sampling optimal price auction behaves exactly the same way as RSOP (see Definition 1), except if all of the non-zero bids fall in the same set, it offers them the lowest non-zero bid as the sale price (instead of 0).

Note that RSOP* is not a truthful auction, however it is only used to aid the analysis. Next we show that the revenue of RSOP is asymptotically equal to the revenue of RSOP*.

**Lemma 13.** $E[\text{RSOP}] \leq E[\text{RSOP}^*] \leq E[\text{RSOP}] + (\lambda)^{n-1} \text{OPT}$, with the second inequality being met with equality for equal revenue instances.

**Proof.** Recall that RSOP* behaves exactly like RSOP except when all the $n$ bids fall in the same set which happens with probability $(\lambda)^{n-1}$, in which case RSOP* still generates a revenue of at most OPT (exactly OPT if it is an equal revenue instance), while RSOP generates zero revenue.

**Lemma 14.** $E[\text{RSOP}^*(q(n))]$ is a decreasing function of $n$.

**Proof.** Let $\text{Rev}(v, A,p)$ and $\text{Rev}(v, B,p)$ denote the revenue obtained by offering price $p$ to bidders respectively in $A$ and $B$ with the vector of bids $v$. Also let $\text{Rev}^*(v,A)$ and $\text{Rev}^*(v,B)$ denote the revenue RSOP* obtains respectively from each of $A$ and $B$ under partition $(A,B)$. Observe that $\text{Rev}^*(v,A) = 0$, because the price that is offered to $A$ is always 1. So it is enough to show that $E[\text{Rev}^*(q(n),B)]$ is a decreasing function of $n$.

Let $v = q(n)$ and $v' = q'(n)$. We now prove that $\text{Rev}^*(v,B) \leq \text{Rev}^*(v',B)$ which implies the claim of the lemma. Let $v_{\text{A}}$ and $v_{\text{B}}'$ denote the prices offered to $B$ by RSOP* respectively on $v$ and $v'$, i.e., $\lambda \in \arg \max_{j \in A} \text{Rev}(v,A,v_j)$, and $\lambda' \in \arg \max_{j \in A} \text{Rev}(v',A,v_j')$. There are four possible scenarios:

(I) $n \in A$ and $\{1 \cdots n - 1\} \subset B$. In this case $\text{Rev}^*(v,B) = 1 - \frac{1}{n} < 1 = \text{Rev}^*(v',B)$\(^{10}\)

(II) $n \in A$ and $\{1 \cdots n - 1\} \not\subset B$. In this case either

(a) $\lambda = \lambda' < n$ and so $\text{Rev}^*(v,B) = \text{Rev}^*(v',B)$, or

(b) $\lambda < \lambda' = n$, but that means $\text{Rev}(v,A,v_{\lambda}) \geq \text{Rev}(v',A,v'_{\lambda'})$, therefore it must be $\text{Rev}^*(v,B) = \text{Rev}(v,B,v_{\lambda}) \geq \text{Rev}(v',B,v'_{\lambda'}) = \text{Rev}^*(v',B)$.\(^{11}\)

(III) $n \in B$ and $\{1 \cdots n - 1\} \subset B$. In this case $\text{Rev}^*(v,B) = \text{Rev}^*(v',B) = 1$.

(IV) $n \in B$ and $\{1 \cdots n - 1\} \not\subset B$. In this case $\lambda = \lambda' = n$ and $\text{Rev}(v,A,v_{\lambda}) > v_n$ so $v_n$ does not affect the revenue, therefore $\text{Rev}^*(v,B) = \text{Rev}^*(v',B)$.

\(\square\)

The following is obtained by direct calculation using a computer.

**Proposition 3.** $E[\text{RSOP}^*(q^{(400)})] = 0.377208 \pm 10^{-6}$.

We now prove the main theorem of this section.

\(^{10}\)Recall that OPT$(q^{(n)}) = 1$.

\(^{11}\)That is because both $v$ and $v'$ are equal revenue instances, therefore $\text{Rev}(v,B,v_{\lambda}) + \text{Rev}(v,A,v_{\lambda}) = \text{Rev}(v',B,v'_{\lambda'}) + \text{Rev}(v',A,v'_{\lambda'}) = 1$. 

12
Proof of Theorem 3. To prove the theorem for any \( \lambda \) we exhibit a bid vector \( v \) with \( \lambda \) bids above the optimal sale price such that \( E[RSOP(v)] < \frac{1}{2.65} \text{OPT}(v) \). Let \( n = \max(\lambda, 400) \), then

\[
E \left[ RSOP(q^{(n)}) \right] \leq E \left[ RSOP^*(q^{(n)}) \right] \leq E \left[ RSOP^*(q^{(n)}) \right] \leq 0.377209 \text{OPT}(q^{(n)}) \leq \frac{1}{2.65} \text{OPT}(q^{(n)})
\]

because \( \text{OPT}(q^{(n)}) = 1 \)

Observe that the optimal sale price for \( q^{(n)} \) is not unique. Let \( v \) be the same as \( q^{(n)} \) everywhere except \( v_\lambda = q_\lambda^{(n)} + \epsilon \) for a small \( \epsilon \in (0, \frac{1}{h}) \). Observe that \( v_\lambda \) is now the unique optimal sale price for \( v \). It is easy to see that \( \lim_{\epsilon \to 0} E[RSOP(v)] = E[RSOP(q^{(n)})] \) and \( \lim_{\epsilon \to 0} \text{OPT}(v) = \text{OPT}(q^{(n)}) = 1 \), so for a small enough \( \epsilon \), we get \( E[RSOP(v)] < \frac{1}{2.65} \text{OPT}(v) \) which completes the proof.

6 A Combinatorial Lower Bound

In this section we present a combinatorial approach for obtaining a lower bound on the expected revenue of RSOP for equal revenue instance where each non-zero bid is either \( h \) or 1 (for some fixed \( h \in \mathbb{N} \)). We hope the ideas we present in the section help develop a more general combinatorial approach in the future for proving lower bounds on mechanisms based on random sampling.

Observe that in an equal revenue instance where non-zero bids are either \( h \) or 1, if there are \( k \) bids of value \( h \), there must be \( k(h-1) \) bids of value 1. Throughout the rest of this section we assume \( h \) is an implicit constant. The following theorem summarizes the main result of this section.

**Theorem 4.** For any equal revenue instance where each non-zero bid is either \( h \) or 1,

\[
E[\text{RSOP}] \geq \left( \frac{1}{2} + \frac{1}{2h} - \frac{1}{2^{kh-1}} \right) \text{OPT}
\]

where \( k \) is the number of bids of value \( h \).

Observe that in the above theorem the worst case of the lower bound is when \( k = 1 \) and \( h = 2 \) for which the lower bound becomes \( \text{OPT}/4 \). Notice that the lower bound approaches \( \text{OPT}/2 \) quickly as either \( k \) or \( h \) increases.

**Definition 5.** \( Q^{(k)} \) denotes the multi-set of bid corresponding to an equal revenue instance with \( k \) bids of value \( h \) and \( k(h-1) \) bids of value 1.

For the rest of this section we assume that \( A \) and \( B \) are multi-sets containing the actual bids in each side of the partition, as opposed to the previous sections where we assumed \( A \) and \( B \) contained the indices of those bids. Furthermore, for any multi-set of bids such as \( I \), we use the notation \( E[\text{RSOP}(I)] \), \( E[\text{RSOP}^*(I)] \) and \( \text{OPT}(I) \) to denote the respective quantity being computed on bids explicitly specified by \( I \). We also make no assumption about which of \( A \) or \( B \) gets the highest bid, unless explicitly stated otherwise.

We start by proving a lower bound on the expected revenue of \( \text{RSOP}^* \) (see Definition 4) on equal revenue instances where each non-zero bid is either \( h \) or 1. We then extend the lower bound to \( \text{RSOP} \). Recall that \( \text{RSOP}^* \) behaves exactly the same way as \( \text{RSOP} \), except if all non-zero bids fall in the same set, \( \text{RSOP}^* \) offers them the lowest non-zero bid as the sale price (instead of 0).

**Lemma 15.** For any \( k \in \mathbb{N} \),

\[
E[\text{RSOP}^*(Q^{(k)})] \geq \frac{k(h+1)}{2} = \frac{1}{2} \text{OPT}(Q^{(k)}) + \frac{k}{2}.
\]
Proof. We prove the claim by induction on \( k \).

We first prove the base case which is \( k = 1 \). The single bid of value \( h \) is the highest bid. Without loss of generality assume that the \( h \) bid is in \( B \). Observe that the optimal price of \( B \) is \( h \) which is also the price offered to \( A \), so no revenue is obtained from \( A \). Furthermore the optimal price of \( A \) is 1 which is also the price offered to \( B \). Each bid of value 1 falls in \( B \) with probability 1/2, so \( E[RSOP^*(Q^{(1)})] = 1 + \frac{h-1}{2} \) which proves the base of the induction.

We now prove the induction step. For any two multi-sets of bids such as \( T \) and \( U \), let \( Rev^*(T, U) \) denote the revenue obtained from \( T \) by computing the optimal sale price for \( U \) (let the optimal price be 1 if \( U = \emptyset \) and offering that price to \( T \); also let \( Rev(T, p) \) denote the revenue obtained by offering price \( p \) to \( T \). Let \((A, B)\) be a random partition of \( Q^{(1)} \), and let \((A', B')\) be a random partition of \( Q^{(k-1)} \). Observe that \((A \cup A', B \cup B')\) is a random partition of \( Q^{(k)} \). The induction step follows from the following inequities.

\[
E \left[ RSOP^*(Q^{(k)}) \right] = E \left[ Rev^*(A \cup A', B \cup B') + Rev^*(B \cup B', A \cup A') \right]
\]

\[
\geq E \left[ Rev^*(A, B) + Rev^*(A', B') + Rev^*(B, A) + Rev^*(B', A') \right] \quad \text{to be proven}
\]

\[
= E \left[ RSOP^*(Q^{(1)}) \right] + E \left[ RSOP^*(Q^{(k-1)}) \right]
\]

\[
\geq \frac{h+1}{2} + \frac{(k-1)(h+1)}{2} = \frac{k(h+1)}{2} \quad \text{by the induction hypothesis}
\]

We shall prove \( Rev^*(B \cup B', A \cup A') \geq Rev^*(A, B) + Rev^*(B', A') \) and by symmetry we can argue \( Rev^*(A \cup A', B \cup B') \geq Rev^*(A, B) + Rev^*(A', B') \) which completes the proof. Let \( p, p' \) and \( p'' \) denote the optimal price of \( A, A' \) and \( A \cup A' \) respectively as computed by \( RSOP^* \) (i.e., the optimal price for an empty set would be 1). We argue that

\[
Rev^*(B \cup B', A \cup A') = Rev(B, p'') + Rev(B', p''') \geq Rev(B, p) + Rev(B', p') \quad \text{explained below}
\]

\[
= Rev^*(B, A) + Rev^*(B', A').
\]

Observe that both \( A \cup B \) and \( A' \cup B' \) are equal revenue instances and by Proposition 2 in any equal revenue instance the price that is optimal for one side generates the least revenue for the opposite side so \( Rev(B, p'') \geq Rev(B, p) \) and \( Rev(B', p''') \geq Rev(B', p') \).

Proof of Theorem 4. Recall that the only situation where \( RSOP^* \) and \( RSOP \) behave differently is when either \( A \) or \( B \) is empty which happens with probability \( 1/2^{kh-1} \), therefore

\[
E \left[ RSOP(Q^{(k)}) \right] = E \left[ RSOP^*(Q^{(k)}) \right] - \frac{1}{2^{kh-1}} \text{OPT}
\]

\[
\geq \frac{k(h+1)}{2} - \frac{1}{2^{kh-1}} \text{OPT} \quad \text{by Lemma 15}
\]

\[
= \left( \frac{1}{2} + \frac{1}{2^{kh}} - \frac{1}{2^{kh-1}} \right) \text{OPT} \quad \text{because OPT} = kh
\]

That completes the proof.

7 Acknowledgment

We would like to thank Jason Hartline for several valuable discussions. We also thank the anonymous referees for their helpful and detailed comments.
References

M.-F. Balcan, A. Blum, J. D. Hartline, and Y. Mansour. Mechanism design via machine learning. In FOCS, pages 605–614, 2005.

S. Baliga and R. Vohra. Market research and market design. Advances in Theoretical Economics, 3(1):1059–1059, 2003.

N. R. Devanur and J. D. Hartline. Limited and online supply and the bayesian foundations of prior-free mechanism design. In ACM Conference on Electronic Commerce, pages 41–50, 2009.

U. Feige, A. Flaxman, J. D. Hartline, and R. D. Kleinberg. On the competitive ratio of the random sampling auction. In WINE, pages 878–886, 2005.

C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. Communications in Mathematical Physics, 22(2):89–103, 1971.

A. V. Goldberg and J. D. Hartline. Competitive auctions for multiple digital goods. In ESA ’01: Proceedings of the 9th Annual European Symposium on Algorithms, pages 416–427, London, UK, 2001. Springer-Verlag. ISBN 3-540-42493-8.

A. V. Goldberg, J. D. Hartline, and A. Wright. Competitive auctions and digital goods. In SODA ’01: Proceedings of the twelfth annual ACM-SIAM symposium on Discrete algorithms, pages 735–744, Philadelphia, PA, USA, 2001. Society for Industrial and Applied Mathematics. ISBN 0-89871-490-7.

A. V. Goldberg, J. D. Hartline, A. R. Karlin, M. Saks, and A. Wright. Competitive auctions. Games and Economic Behavior, 55(2):242–269, May 2006.

M. T. Hajiaghayi, R. D. Kleinberg, and D. C. Parkes. Adaptive limited-supply online auctions. In ACM Conference on Electronic Commerce, pages 71–80, 2004.

J. D. Hartline and T. Roughgarden. Optimal mechanisms design and money burning. CoRR, abs/0804.2097, 2008.

W. Hoeffding. Probability inequalities for sums of bounded random variables. American Statistical Association Journal, 58:13–30, 1963.

I. Segal. Optimal pricing mechanisms with unknown demand. American Economic Review, 93(3):509–529, June 2003.
A Results

| λ  | E[RSOP]/OPT | Competitive-Ratio |
|----|-------------|-------------------|
| 2  | 0.2138      | 4.68              |
| 3  | 0.2178      | 4.59              |
| 4  | 0.2338      | 4.20              |
| 5  | 0.2433      | 4.11              |
| 6  | 0.2503      | 3.99              |
| 7  | 0.2545      | 3.93              |
| 8  | 0.2602      | 3.84              |
| 9  | 0.2627      | 3.81              |
| 10 | 0.2669      | 3.76              |

Table 1: Computed numerical values for the basic lower-bound of $\mathbb{E}[\text{RSOP}] \geq \mathbb{E}[\frac{\lambda}{\lambda + Z}] \mathbb{OPT}$.

| λ  | E[RSOP]/OPT | Competitive-Ratio |
|----|-------------|-------------------|
| 2  | 0.2138      | 4.68              |
| 3  | 0.2178      | 4.59              |
| 4  | 0.2338      | 4.20              |
| 5  | 0.2433      | 4.11              |
| 6  | 0.2503      | 3.99              |
| 7  | 0.2545      | 3.93              |
| 8  | 0.2602      | 3.84              |
| 9  | 0.2627      | 3.81              |
| 10 | 0.2669      | 3.76              |

Table 2: Computed numerical values for the exhaustive-search lower-bound

B Proofs

**Theorem 5** (Chernoff-Hoeffding [1963]). For (i.i.d.) random variables $X_1, X_2, \ldots, X_\ell \in \{0, 1\}$ with $\mathbb{E}[X_i] = p$, the following inequality holds for all $\varepsilon \in (0, 1 - p)$:

$$\Pr\left[\frac{1}{\ell} \sum X_i \geq p + \varepsilon\right] \leq \left(\frac{p}{p + \varepsilon}\right)^p \left(\frac{1 - p}{1 - p - \varepsilon}\right)^{1-p-\varepsilon} \ell$$  (33)

**Lemma 2.** For any $\alpha \in (0, 1)$ and $j \in \mathbb{N}$,

- if $\alpha \geq 0.5$, then $\Pr[E_{\alpha}^{(j)}] \geq 1 - (r_{\alpha})^j$,
- if $\alpha \leq 0.5 - 1/j$, then $\Pr[E_{\alpha}^{(j)}] \leq (r_{\alpha+1/j})^{j-1}$

where $r_{\alpha} = \frac{1}{2\alpha^\alpha(1-\alpha)^{1-\alpha}}$ and $r_{\alpha}$ is the same as above.

**Proof.** Let $A_j$ be an indicator random variable which is 1 if $j \in \mathbb{A}$, and 0 otherwise.

The first inequality of the lemma follows immediately from **Theorem 5** by setting $X_j = A_j$, $\ell = j$, $p = 0.5$, and $\varepsilon = \alpha - 0.5$ which yields an upper bound on $\Pr[E_{\alpha}^{(j)}]$ and thus a lower bound on $\Pr[E_{\alpha}^{(j)}]$). Note that $A_1 = 0$ with probability 1, however that only decreases the probability on the left hand side of (33) so it still holds.
To prove the second inequality, we proceed as follows.

\[
\Pr [\mathcal{E}_\alpha^{(j)}] = \Pr \left[ \frac{S_j}{j} \leq \alpha \right] = \Pr \left[ \frac{\sum_{k=1}^{j} A_k}{j} > 1 - \alpha \right] = \Pr \left[ \frac{\sum_{k=2}^{j} A_k}{j} > 1 - \alpha - \frac{1}{j} \right] \quad \text{because } \mathcal{A}_1 = 1 \text{ always.}
\]

The second inequality of the lemma now follows immediately from Theorem 5 by setting \( X_j = \mathcal{A}_{j-1}, \ell = j-1, p = 0.5, \) and \( \epsilon = 0.5 - \alpha - \frac{1}{j}. \) Note that \( r_{1-\alpha-\frac{1}{j}} = r_{\alpha+\frac{1}{j}}. \)

**Theorem 6** (Fortuin, Kasteleyn, and Ginibre [1971]). Let \( L \) be a finite distributive lattice, and \( \mu : L \rightarrow \mathbb{R}_+ \) be a function that satisfies

\[
\mu(x \land y)\mu(x \lor y) \geq \mu(x)\mu(y), \quad \text{for all } x, y \in L.
\]

Then for any two functions \( f, g : L \rightarrow \mathbb{R}_+ \) which are either both increasing, or both decreasing, the following inequality holds.

\[
\left( \sum_{x \in L} f(x)g(x)\mu(x) \right) \left( \sum_{x \in L} \mu(x) \right) \geq \left( \sum_{x \in L} f(x)\mu(x) \right) \left( \sum_{x \in L} g(x)\mu(x) \right)
\]

**Lemma [1].** For any \( T, T' \subset \mathbb{N} \) and \( \alpha \in [0, 1], \) the two events \( \mathcal{E}_\alpha^T \) and \( \mathcal{E}_\alpha^{T'} \) are positively correlated, i.e.,

\[
\Pr[\mathcal{E}_\alpha^T \cap \mathcal{E}_\alpha^{T'}] \geq \Pr[\mathcal{E}_\alpha^T] \Pr[\mathcal{E}_\alpha^{T'}].
\]

**Proof.** For every \( n \in \mathbb{N}, \) define \( T_n = T \cap \{1 \cdots n\} \); similarly define \( T'_n, A_n, B_n, \) etc.

We start by proving \( \Pr[\mathcal{E}_\alpha^{T_n} \cap \mathcal{E}_\alpha^{T'_n}] \geq \Pr[\mathcal{E}_\alpha^{T_n}] \Pr[\mathcal{E}_\alpha^{T'_n}] \) for every \( n \in \mathbb{N}. \) Let \( L_n \) be a distributive lattice whose elements are the subsets of \( \{2 \cdots n\} \) and whose meet/join operators correspond to taking intersection/union. For all \( A \in L_n \) let \( \mu(A) = 1/2^{n-1}. \) Define \( \mathcal{E}_\alpha^{T_n}(A) \) to be an indicator function which is defined for each \( A \in L_n \) as

\[
\mathcal{E}_\alpha^{T_n}(A) = \begin{cases} 
1 & \text{if } |A \cap \{1 \cdots j\}| \leq \alpha j \text{ for all } j \in T_n \\
0 & \text{otherwise}
\end{cases}
\]

By invoking Theorem 6 on lattice \( L_n \) and substituting \( f(x) \) and \( g(x) \) with \( \mathcal{E}_\alpha^{T_n}(A) \) and \( \mathcal{E}_\alpha^{T'_n}(A) \) respectively we get the following inequality.

\[
\left( \sum_{A \subseteq \{2 \cdots n\}} \frac{\mathcal{E}_\alpha^{T_n}(A)\mathcal{E}_\alpha^{T'_n}(A)}{2^{n-1}} \right) \geq \left( \sum_{A \subseteq \{2 \cdots n\}} \frac{\mathcal{E}_\alpha^{T_n}(A)}{2^{n-1}} \right) \left( \sum_{A \subseteq \{2 \cdots n\}} \frac{\mathcal{E}_\alpha^{T'_n}(A)}{2^{n-1}} \right)
\]

Observe that the left hand side of the above inequality is exactly \( \mathbf{E}_A[\mathcal{E}_\alpha^{T_n}(A)\mathcal{E}_\alpha^{T'_n}(A)] = \Pr[\mathcal{E}_\alpha^{T_n} \cap \mathcal{E}_\alpha^{T'_n}] \) while its right hand side is exactly \( \mathbf{E}_A[\mathcal{E}_\alpha^{T_n}(A)] \mathbf{E}_A[\mathcal{E}_\alpha^{T'_n}(A)] = \Pr[\mathcal{E}_\alpha^{T_n}] \Pr[\mathcal{E}_\alpha^{T'_n}], \) so we have proved that

\[
\Pr[\mathcal{E}_\alpha^{T_n} \cap \mathcal{E}_\alpha^{T'_n}] \geq \Pr[\mathcal{E}_\alpha^{T_n}] \Pr[\mathcal{E}_\alpha^{T'_n}] \text{ for every } n \in \mathbb{N}.
\]

We now prove the infinite case. For every \( n \in \mathbb{N}, \) define \( \ell_n = \Pr[\mathcal{E}_\alpha^{T_n} \cap \mathcal{E}_\alpha^{T'_n}], \) \( r_n = \Pr[\mathcal{E}_\alpha^{T'_n}] \Pr[\mathcal{E}_\alpha^{T_n}], \) and \( d_n = \ell_n - r_n. \) Observe that \( d_n \) is an infinite sequence which is bounded in \([0, 1], \) so by invoking BolzanoWeierstrass theorem we argue that it has an infinite converging subsequence, i.e., there exists an infinite sequence of indices \( n_1 < n_2 < \cdots \) and \( d^* \in [0, 1] \) such that \( \lim_{j \to \infty} d_{n_j} = d^*. \) On the other hand both \( \ell_{n_j} \) and \( r_{n_j} \) are decreasing sequences which are bounded below by 0 so they both converge, therefore

\[
\Pr \left[ \mathcal{E}_\alpha^T \cap \mathcal{E}_\alpha^{T'} \right] - \Pr \left[ \mathcal{E}_\alpha^T \right] \Pr \left[ \mathcal{E}_\alpha^{T'} \right] = \lim_{j \to \infty} \ell_{n_j} - \lim_{j \to \infty} r_{n_j} = \lim_{j \to \infty} d_{n_j} = d^* \geq 0
\]

which proves the claim of the lemma. \( \square \)