Signal and Noise in Financial Correlation Matrices

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Abstract

Using Random Matrix Theory one can derive exact relations between the eigenvalue spectrum of the covariance matrix and the eigenvalue spectrum of its estimator (experimentally measured correlation matrix). These relations will be used to analyze a particular case of the correlations in financial series and to show that contrary to earlier claims, correlations can be measured also in the “random” part of the spectrum. Implications for the portfolio optimization are briefly discussed.

Key words: random matrix theory, correlation matrix, eigenvalue spectrum

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1 Introduction

Empirically determined correlation matrices appear in many research areas. In financial analysis they became one of the cornerstones of the financial risk analysis through the idea of the optimal portfolio by Markowitz [1]. The practical way to obtain the correlation matrix is to perform a (large) number \( T \) of measurements of a (large) number \( N \) of experimental quantities (price fluctuations) \( x_{it}, i = 1,\ldots,N, t = 1,\ldots,T \) and to estimate the correlation matrix by

\[
c_{ij} = \frac{1}{T} \sum_{it} x_{it} x_{jt},
\]

(1)

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where we assumed that \( \langle x_i \rangle = 0 \). The relation of the estimated correlation matrix \( c_{ij} \) to the true correlation matrix \( C_{ij} \), in particular the relation of the eigenvalue spectra of the two matrices was addressed in the literature (cf. e.g. [2,3,4,11]), assuming the experimentally measured quantities come from some correlated random process. We shall address this point in the next section, assuming these distributions come from the correlated Gaussian ensemble.

In the context of financial analysis, the correlation matrices were discussed in [5,6,7,8,9,10]. The general conclusion, based on results of [6] was rather pessimistic. Let us quote here from the paper [8]: \( \ldots \)covariance matrices determined from empirical financial time series appear to contain such a high amount of noise that their structure can essentially be regarded as random. This seems, however, to be in contradiction with the fundamental role played by covariance matrices in finance, which constitute the pillars of modern investment theory and have also gained industry-wide applications in risk management. Let us recall the arguments which led to this conclusion. In the paper [6] the authors analyzed the S&P500 price fluctuations of \( N = 406 \) assets in the period of \( T = 1309 \) days between 1991-1996. Price fluctuations were normalized to the standard deviation \( \sigma_i = 1 \) and the correlation matrix was constructed, following (1). The spectrum of the \( 406 \times 406 \) correlation matrix contains a number of large eigenvalues, the largest of which was interpreted as a “market”. The spectrum, where the nine largest eigenvalues are missing is presented on figure 1. The lower part of the spectrum was interpreted as a dis-

![Fig. 1. Eigenvalue distribution of the normalized correlation matrix following [6]. Nine largest eigenvalues are missing. The line represents a RMM distribution with a single eigenvalue.](image)

tribution of eigenvalues of a random uncorrelated Gaussian matrix, therefore lacking any information about the correlations. In the following we will try to dissolve this pessimism. In the next section we will present a simple model of correlated Gaussian fluctuations and discuss the properties of the resolvent of this model. We will use the Random Matrix Theory solution [11] to construct
explicitly the eigenvalue density function \( \rho(\lambda) \) in case the spectrum of the correlation matrix is known. We shall also discuss the problem of obtaining this spectrum from the experimentally measured eigenvalue distribution. We will apply this method to reanalyze the financial time series discussed above. Summary and conclusions will be given in the last section.

2 The correlated Gaussian fluctuations.

Let us consider a Gaussian model, generating a matrix \( X \) of fluctuations \( x_{it}, i = 1, \ldots N, t = 1, \ldots T \) with a matrix measure

\[
P(X)DX = \frac{1}{Z} \exp(-\frac{1}{2} \text{Tr}X^T C^{-1}X)
\]

where \( Z \) is a normalization factor and \( C \) is a real symmetric positive correlation matrix, representing the genuine two-point correlations in the system. A single matrix \( X \) generated from this ensemble can be used to estimate the spectrum of \( C \) by (1). Matrix \( C \) can always be diagonalized and we assume the set of it’s eigenvalues \( \{ \mu_i \} \) to be known. We shall define two resolvents \( G(Z) \) and \( g(z) \) defined by

\[
G(Z) = \frac{1}{N} \text{Tr}(Z - C)^{-1}, \quad g(z) = \frac{1}{N} \langle \text{Tr}(z - c)^{-1} \rangle,
\]

where the averaging is made with the measure (2). In (3) \( Z \) and \( z \) are complex variables. In the limit \( N \to \infty, T \to \infty N/T = r < 1, r = \text{const} \) we can use the diagrammatic technique [11] to find the exact relation between the two resolvents. To take the limit for \( G(Z) \) we assume that eigenvalues appear in blocs with degeneracy \( k_j = p_j N, \quad j = 1, \ldots M \) and the number of blocs \( M \) stays finite. Function \( G(Z) \) is an analytic function of \( Z \) on a complex plane. It has poles at \( Z = \mu_j \) with residues \( p_j \). Similarly \( g(z) \) is an analytic function on the complex \( z \) plane with one or more cuts along the real axis. Discontinuities across these cuts are related to the eigenvalue density \( \rho(\lambda) \) by

\[
\rho(\lambda) = \frac{1}{\pi} \text{Im} g(\lambda + i0^+).
\]

The exact relation between the resolvents has a form of a duality relation

\[
zg(z) = ZG(Z),
\]

where the complex arguments \( z \) and \( Z \) are related by the conformal map.
\[ Z = z(1 - r + rzG(z)), \quad z = \frac{Z}{1 - r + rZG(Z)}. \] (6)

On the “physical” sheet on the \( z \) plane \( z \to \infty \) for \( Z \to \infty \) and as follows from (6) the poles at \( Z = \mu_j \) are mapped to \( z = \infty \) on other Riemann sheets of \( g(z) \).

The relation between \( g(z) \) and \( G(Z) \) can be used to find the eigenvalue distribution \( \rho(\lambda) \) for any set \( \{\mu_j, p_j\} \). Using (6) we find the position of the cuts on the \( z \) plane by relating them to the map of the \( \text{Im} z = 0^\pm \) lines on the \( Z \) plane. We then find the imaginary part of the resolvent \( g(z) \) using (5).

It is more difficult to obtain \( \{\mu_j, p_j\} \) from the measure distribution \( \rho(\lambda) \). The eigenvalue distribution is known only approximately, from one realization of the estimator \( c_{ij} \). It is nevertheless possible to extract more information about the spectrum of the correlation matrix, than suggested by the pessimistic approach quoted above. Both resolvents contain information about the moments of the correlation matrices. Expanding around \( Z = \infty \) (resp. \( z = \infty \)) we have

\[ G(Z) = \sum_{i=0}^{\infty} \frac{1}{Z^{i+1}} \frac{1}{N} \text{Tr} C^i, \quad g(z) = \sum_{i=0}^{\infty} \frac{1}{z^{i+1}} \frac{1}{N} \langle \text{Tr} C^i \rangle. \] (7)

For \( r < 1 \) we have similar expansions around \( Z = 0 \) (resp. \( z = 0 \)). Duality relation can be viewed as relation between the moments

\[ M_i = \sum_{j=1}^{M} p_j \mu_j^i \quad \text{and} \quad m_i = \int d\lambda \rho(\lambda) \lambda^i. \] (8)

For the first few moments we have:

\[ M_1 = m_1 \]
\[ M_2 = m_2 - rm_1^2 \]
\[ M_3 = m_3 - 3rm_1m_2 + 2r^2m_1^3 \]
\[ \cdots \]

and

\[ M_{-1} = (1 - r)m_{-1} \]
\[ M_{-2} = (1 - r)^2m_{-2} - r(1 - r)m_{-1}^2 \]
\[ M_{-3} = (1 - r)^3m_{-3} - r(1 - r)^2m_{-2}m_{-1} - r^2(1 - r)m_{-1}^2 \]
\[ \cdots \] (9)
These relations can be used to determine the moments of the correlation matrix $C$ from the moments of its estimator $c$. It is clear that high moments will have larger errors.

As an example let us reconsider the analysis of the S&P500 data presented in the Introduction. In the analysis we assumed six eigenvalue blocs in the correlation matrix, which gave the optimal fit of the experimental distribution.

![Fig. 2. Analysis of the eigenvalue distribution presented above. We assumed six eigenvalues, which gave the optimal fit of the distribution. The determined eigenvalues are represented by blue lines and their relative height corresponds to the probability $p_j$. The red line is the fit of the distribution with six eigenvalues, to be compared with the green line, corresponding to the fit presented before.](image)

3 Discussion and summary

Experimental estimator of the correlation matrix, although containing statistical noise, can be nevertheless used to extract much more information about the spectrum of the true correlation matrix than is usually believed. This information can be very important when the optimized portfolio is constructed. In case the matrix $C$ is diagonal, one can derive exact relations between the volatility $\sigma_0$ of the true optimal portfolio, obtained from the correlation matrix $C$, volatility $\sigma_R$ of the portfolio based on the estimate $c$ and the volatility $\sigma$ predicted from this estimate. This relation depends on $r$

$$\sigma_R = \frac{\sigma_0}{\sqrt{1 - r}} = \frac{\sigma}{1 - r}$$

(11)

and shows that, particularly for $r$ close to 1, the error can be quite substantial.
If the spectrum of the correlation matrix $C$ is known, we can always predict the shape of the measured eigenvalue distribution $\rho(\lambda)$ of it’s estimator $c$. Although the formulas presented above are valid in principle only for infinitely large matrices, in practice the predicted spectra agree very well even for the matrix sizes of the order 100 and below.

It is also possible to invert the problem and to determine the spectrum of $C$ from that of the estimator $c$, although this step can be done only approximately.

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