On the Cramér-Rao Lower Bound for Spatial Correlation Matrices of Doubly Selective Fading Channels for MIMO OFDM Systems

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Abstract—In this paper, the Cramér-Rao lower bound (CRLB) for spatial correlation matrices is derived based on a rigorous model of the doubly selective fading channel for multiple-input multiple-output (MIMO) orthogonal frequency division multiplexing (OFDM) systems. Adopting an orthogonal pilot pattern for multiple transmitting antennas and assuming independent samples along the time, the sample auto-correlation matrix of the channel response is complex Wishart distributed. Then, the maximum likelihood estimator (MLE) and the analytic expression of CRLB are derived by assuming that temporal and frequency correlations are known. Furthermore, lower bounds of total mean squared error (TMSE) and average mean squared error (AvgMSE) are obtained for asymptotically infinite and finite signal-to-noise ratios (SNR’s), respectively. Based on the lower bound of AvgMSE, several factors influencing the accuracy of estimation, including the amount of samples, the order of frequency selectivity, the normalized maximum Doppler spread, the number of pilot tones per antenna and SNR, are further analyzed.

This paper is organized as follows. In Section II, the MIMO OFDM system and channel model are introduced. Then, in Section III, CRLB of the sample spatial correlation matrix is derived and further lower bounded to uncover the essential factors. Numerical results appear in Section IV. Finally, Section V concludes the paper.

Notation: Lowercase and uppercase boldface letters denote column vectors and matrices, respectively. $\cdot^*$, $\cdot^T$, $\cdot^H$, $\cdot^\dagger$, and $\cdot \triangledown F$ denote conjugate, transposition, conjugate transposition, Moore-Penrose pseudo-inverse and Frobenius norm, respectively. $\otimes$ denotes the Kronecker product. $E(\cdot)$ represents expectation. $[A]_{i,j}$ and $[a]_i$ denotes the $(i,j)$-th element of $A$ and the $i$-th element of $a$, respectively. $\text{diag}(\mathbf{a})$ is a diagonal matrix by placing $\mathbf{a}$ on the diagonal.

I. INTRODUCTION

Due to the virtues of orthogonal frequency division multiplexing (OFDM) for converting frequency selective fading channels into flat fading ones and of multiple-input multiple-output (MIMO) techniques for exploiting spatial diversity gain and/or enhancing the system capacity, many current systems combine them two together to achieve a better quality as well as a higher throughput [1].

For MIMO systems, the spatial correlation matrices play very important roles and are widely utilized, for example, to facilitate the transmitting precoding [2] [3], MMSE receiver [4] and multi-user strategies [5]. However, since the true spatial correlation matrices are unknown in real applications, the sample correlation matrices have to be used in stead and are usually obtained through the channel estimation.

In this paper, we study the Cramér-Rao lower bound (CRLB) for the sample spatial correlation matrices of doubly selective fading channels for MIMO OFDM systems. Based on the rigorous doubly fading channel model and assuming invariant pilot sequence along the time, the maximum likelihood estimator (MLE) and the CRLB are derived. Then, the analytic expressions of lower bounds of the total mean squared error (TMSE) and average mean squared error (AvgMSE) are obtained for asymptotically infinite and finite signal-to-noise ratios (SNR’s), respectively. Based on the lower bound of AvgMSE, several factors influencing the accuracy of estimation, including the amount of samples, the order of frequency selectivity, the normalized maximum Doppler spread, the number of pilot tones per antenna and SNR, are further analyzed.

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II. SYSTEM MODEL

Consider an MIMO OFDM system with a bandwidth of $BW = 1/T$ Hz ($T$ is the sampling period). $N$ denotes the total number of tones, and a cyclic prefix (CP) of length $L_{cp}$ is inserted before each symbol to eliminate inter-block interference. Thus the whole symbol duration is $T_s = (N + L_{cp})T$. Then, $n_T$ transmitting antennas at the base station (BS) and $n_R$ receiving antennas at the mobile station (MS) are assumed, respectively.
Between the $i$-th transmitting antenna and $j$-th receiving antenna, the complex baseband model of a linear time-variant mobile channel with $L^{(j,i)}$ paths can be described by [6]

$$h^{(j,i)}(t,\tau) = \sum_{l=1}^{L^{(j,i)}} h_l^{(j,i)}(t)\delta\left(\tau - \tau_l^{(j,i)}\right)$$

(1)

where $(\tau_l^{(j,i)}) \in \mathbb{R}$ is the normalized non-sample-spaced delay of the $i$-th path, and $h_l^{(j,i)}(t)$ is the corresponding complex amplitude with the power $(\sigma_l^{2(j,i)})$.

The following conditions are assumed to characterize the correlation property of the channel.

1) Space Correlation: the stochastic MIMO radio channel model [7] is adopted.
2) Frequency Correlation: the wide-sense stationary uncorrelated scattering (WSSUS) [6] is assumed.
3) Time Correlation: the uniform scattering environment introduced by Clarke [8] is assumed.
4) Scattering Function Separability: the channel has degeneracy in all three dimensions [9].

Therefore, the complete spatial correlation matrix of the MIMO radio channel is given by [7]

$$\mathbf{\Xi}_s = \mathbf{\Xi}_{s,T} \otimes \mathbf{\Xi}_{s,R}$$

(2)

where $\mathbf{\Xi}_{s,T} \in \mathbb{C}^{n_T \times n_T}$ and $\mathbf{\Xi}_{s,R} \in \mathbb{C}^{n_R \times n_R}$ are the symmetrical complex correlation matrices of antenna arrays of BS and MS, respectively. Then, the normalized time correlation function of any path is identical, i.e., [8]

$$r_\tau(\Delta t) = J_0(2\pi f_d \Delta t)$$

(3)

where $f_d$ is the maximum Doppler spread, and $J_0(\cdot)$ is the zeroth order Bessel function of the first kind. In addition, $L^{(j,i)} = L$, $(\tau_l^{(j,i)}) = \tau_l$ and $(\sigma_l^{2(j,i)}) = \sigma_l^2$, hence, the frequency correlation matrix is

$$\mathbf{R}_f = \mathbf{F}_f \mathbf{D}_f \mathbf{F}_f^H$$

(4)

where $\mathbf{D} = \text{diag}(\sigma_l^2)$, $l = 1, \ldots, L$ and $\mathbf{F}_f \in \mathbb{C}^{N \times L}$ is the unbalanced Fourier transform matrix, defined as $[\mathbf{F}_f]_{k,l} = e^{-j2\pi kl/N}$. Moreover, the power of the channel between each pair of transmitting and receiving antennas is normalized, i.e., $\text{tr}(\mathbf{D}) = 1$.

Assuming a sufficient CP, i.e., $L_{cp} \geq L$, the discrete signal model in the frequency domain is written as

$$y_f^{(j,i)}(n) = \sum_{i=1}^{n_T} \mathbf{H}_f^{(j,i)}(n) x_f^{(j,i)}(n) + n_f^{(j,i)}(n)$$

(5)

where $x_f^{(j,i)}(n), y_f^{(j,i)}(n), n_f^{(j,i)}(n) \in \mathbb{C}^{N \times 1}$ are the $n$-th transmitted and received signal and additive white Gaussian noise (AWGN) vectors, respectively, and $\mathbf{H}_f^{(j,i)}(n) \in \mathbb{C}^{N \times N}$ is the channel transfer matrix between the $i$-th transmitting and $j$-th receiving antennas with the $(k + \nu, k)$-th element as

$$[\mathbf{H}_f^{(j,i)}(n)]_{k+\nu,k} = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{l=1}^{L} h_l^{(j,i)}(n,m) e^{-j2\pi(\nu m + k\nu)/N}$$

(6)

where $h_l^{(j,i)}(n,m) = h_l^{(j,i)}(nT_s + (L_{cp} + m)T)$ is the sampled complex amplitude of the $l$-th path. $k$ and $\nu$ denote frequency and Doppler spread, respectively.

### III. CRLB of Spatial Correlation Matrices

Usually the correlation matrices of the channel response are obtained through the least squared (LS) channel estimation on pilot tones, that is, only pilot symbols are extracted and used to perform LS channel estimation. Therefore,

$$\mathbf{h}_{f,ls}^{(j,i)}(n) = \mathbf{X}_f^{(j,i)}(n)^{-1} y_f^{(j,i)}(n)$$

(7)

where $\mathbf{X}_f^{(j,i)}(n) = \text{diag}(\mathbf{x}_f^{(j,i)}(n))$ is a diagonal matrix consisting of pilot symbols. To alleviate the interference between multiple transmitting antennas, pilot symbols, i.e., $\mathbf{x}_f^{(j,i)}(n)$, $i = 1, \ldots, n_T$, are designed to be orthogonal, therefore

$$\mathbf{x}_f^{(j,i)}(n)^H \mathbf{x}_f^{(j,i)}(n) = \delta(i_1 - i_2)\mathbf{x}_f^{(j,i)}(n)^2$$

(8)

In this paper, we adopt the frequency division pilot pattern of which the pilot tones allocated to a certain transmitting antenna is exclusive of the others. To be more specifically, it is assumed that the pilot tone set of the $i$-th transmitting antenna is $\mathcal{I}_p^{(i)} = \{i + k \times \theta; k = 0, \ldots, P - 1\}$, where $P$ is the number of pilot tones of each antenna satisfying $L \leq P$, $\theta \geq n_T$ and $P \leq N$. Thus, the $i$-th transmitting antenna transmits non-zero pilots on $\mathcal{I}_p^{(i)}$ while nulls on the rest. Besides, we assume that the non-zero elements of $\mathbf{x}_f^{(j,i)}(n)$ constitute a fixed vector $\mathbf{x}_p \in \mathbb{C}^{P \times 1}$, which is independent of $n$ and of the normalized power so that $\mathbf{X}_p \mathbf{X}_p^H = \mathbf{I}_P$, where $\mathbf{X}_p = \text{diag}(\mathbf{x}_p)$. Then, by discarding the zero elements, (7) is further rewritten into

$$\mathbf{h}_{p,ls}^{(j,i)}(n) = \mathbf{X}_p^{-1} \mathbf{h}_{p,ls}^{(j,i)}(n) \mathbf{x}_p + \mathbf{X}_p^{-1} \mathbf{n}_{p,ls}^{(j,i)}(n)$$

(9)

where the subscript $p$ denotes elements on pilot tones.

According to the orthogonal pilot pattern, the instantaneous channel impulse response (CIR) vector between the $i$-th transmitting antenna and the $j$-th receiving antenna corresponding to the $(i + k \times \theta)$-th sample of the $n$-th OFDM symbol can be denoted as $\mathbf{h}^{(j,i)}(n,i + k \times \theta) = [h_l^{(j,i)}(n,i + k \times \theta), \ldots, h_{L}^{(j,i)}(n,i + k \times \theta)]^T$, $k = 0, \ldots, P - 1$. Then, the $(j,i)$-th CIR matrix for the $n$-th OFDM symbol is formed as $\mathbf{H}_{c}^{(j,i)}(n) = [\mathbf{h}_1^{(j,i)}(n), \ldots, \mathbf{h}_{L}^{(j,i)}(n,i + (P - 1) \times \theta)]$. According to the assumptions of WSSUS and uniform scattering, $\mathbf{H}_{c}^{(j,i)}(n)$ is complex normal, i.e.,

$$\mathbf{H}_{c}^{(j,i)}(n) \sim \mathcal{CN}_{L \times P}(0, \mathbf{\Omega} \otimes \mathbf{D})$$

(10)

where $\mathbf{\Omega} \in \mathbb{C}^{P \times P}$ is a Toeplitz matrix, defined as

$$[\mathbf{\Omega}]_{k_1,k_2} = J_0(2\pi f_d (k_1 - k_2) \theta T)$$

(11)

Then according to (10) and the pilot pattern, the $(j,i)$-th channel transfer matrix is $\mathbf{H}_{c}^{(j,i)}(n) = \mathbf{F}_f^{(i)} \mathbf{H}_{p}^{(j,i)}(n)$, where $\mathbf{F}_f^{(i)}$ is the submatrix of $\mathbf{F}_f$ by drawing rows from $\mathcal{I}_p^{(i)}$. Thus

$$\mathbf{H}_{p}^{(j,i)}(n) \sim \mathcal{CN}_{P \times P}(0, \mathbf{\Omega} \otimes [\mathbf{F}_f^{(j)} \mathbf{D}(\mathbf{F}_f^{(i)})^H])$$

(12)
Assuming CIR is independent of the thermal noise, with \( \ref{eq:omega} \) and \( \ref{eq:pi} \), we have
\[
\mathbf{l}_{p,i}^{(j)}(n) \sim \mathcal{CN}_p(0, \mathbf{\Upsilon}^{(j,i)})
\]
where the covariance matrix \( \mathbf{\Upsilon}^{(j,i)} \) is defined as
\[
\mathbf{\Upsilon}^{(j,i)} = (\mathbf{x}_p^H \mathbf{\Omega}_p \mathbf{x}_p) (\mathbf{x}_p^H \mathbf{F}_r^{(j)} \mathbf{D}(\mathbf{F}_r^{(j)} \mathbf{\Phi}) + \sigma_n^2 \mathbf{I}_p)
\]
(14)

Apparent, \( \mathbf{\Upsilon}^{(j,i)} \) is irrelevant of the indexes of the receiving antennas, which follows the assumption \( \ref{eq:omega} \). According to the orthogonal pilot pattern, \( \mathbf{F}_r^{(j)} = \mathbf{F}_r^{(0)} \mathbf{\Phi} \), where \( \mathbf{F}_r^{(0)} \in \mathbb{C}^{P \times L} \) is a submatrix of \( \mathbf{F}_r \) by drawing rows from \( \mathcal{I}_{p}^{(0)} = \{ k \times \theta; k = 0, \ldots, P - 1 \} \), and \( \mathbf{\Phi} \) is a diagonal phase-twisted matrix with \( [\mathbf{\Phi}]_{\ell,l} = e^{-j2\pi n_l/N} \). Then, the frequency auto-correlation matrix of \( \mathcal{I}_{p}^{(0)} \) is
\[
\mathbf{R}_p^{(i)} = \mathbf{F}_r^{(i)} \mathbf{D}(\mathbf{F}_r^{(i)})^H = \mathbf{F}_r^{(0)} \mathbf{\Phi} \mathbf{D}(\mathbf{F}_r^{(0)} \mathbf{\Phi}^i)^H = \mathbf{R}_p
\]
where \( \mathbf{R}_p = \mathbf{F}_r^{(0)} \mathbf{D}(\mathbf{F}_r^{(0)})^H \). Furthermore, to simplify the following derivation, we assume that CFR varies slightly within \( n_T \) contiguous tones, so that the frequency cross-correlation matrix between the pilot sets \( \mathcal{I}_{p}^{(1)} \) and \( \mathcal{I}_{p}^{(2)} \) is
\[
\mathbf{R}_p^{(i_1,i_2)} = \mathbf{F}_r^{(i_1)} \mathbf{D}(\mathbf{F}_r^{(i_2)})^H = \mathbf{F}_r^{(0)} \mathbf{\Phi}^{i_1-i_2} \mathbf{D}(\mathbf{F}_r^{(0)})^H \approx \mathbf{R}_p
\]
(16)

Then, the complete LS-estimated channel transfer matrix is constructed as
\[
\mathbb{H}_{p,ls}(n) = \begin{pmatrix}
\mathbf{h}_{p,ls}^{(1,1)}(n) & \cdots & \mathbf{h}_{p,ls}^{(1,n_T)}(n) \\
\vdots & \ddots & \vdots \\
\mathbf{h}_{p,ls}^{(n_R,1)}(n) & \cdots & \mathbf{h}_{p,ls}^{(n_R,n_T)}(n)
\end{pmatrix}_{n_R \times n_T}
\]
and it is complex normal, i.e.,
\[
\mathbb{H}_{p,ls}(n) \sim \mathcal{CN}_{n_R \times n_T}(0, \mathbf{\Sigma})
\]
where \( \mathbf{\Sigma} \in \mathbb{C}^{n_R \times n_T \times n_T} \). According to the assumptions \( \ref{eq:omega} \), its \( ((i_1 - 1)n_R + j_1, (i_2 - 1)n_R + j_2) \)-th submatrix is \( \mathbb{H}_{p,ls}(n) \), shown at bottom of the next page. Then, with \( \ref{eq:omega} \) and \( \ref{eq:pi} \), \( \mathbf{\Sigma} \) is expressed as
\[
\mathbf{\Sigma} = \mathbf{\Xi} \otimes (\mathbf{\Omega} \mathbf{A}) + \sigma_n^2 \mathbf{I}_{n_T n_R P}
\]
where \( \mathbf{\Xi} \) is
\[
\mathbf{\Xi} = \begin{pmatrix}
\mathbf{\Xi}_p & \cdots & \mathbf{\Xi}_p
\end{pmatrix}
\]
(20)

When \( N_t \) complete LS estimated channel transfer matrices are available, the sample auto-correlation matrix is formed as
\[
\hat{\mathbf{\Sigma}} = \frac{1}{N_t} \sum_{n=1}^{N_t} \text{vec}(\mathbb{H}_{p,ls}(n)) (\text{vec}(\mathbb{H}_{p,ls}(n)))^H
\]
(21)
where \( N_t \) is the number of samples. To derive the probability density function (PDF) of the sample auto-correlation matrix, we assume that samples are independent of each other, which may be a strict constraint. However, when the spacing between two contiguous pilot symbols is sufficiently large, the correlation between them is rather low, which alleviates the effect of model mismatch. Then, \( \hat{\mathbf{\Sigma}} \) has the complex central Wishart distribution with \( N_t \) degrees of freedom and covariance matrix \( \mathbf{\Sigma'} = \mathbf{\Sigma}/N_t \) [10], denoted as
\[
\hat{\mathbf{\Sigma}} \sim \mathcal{CW}_{n_T n_R P}(N_t, \mathbf{\Sigma'})
\]
and its PDF is
\[
f(\hat{\mathbf{\Sigma}}) = \frac{\text{etr}(-\hat{\mathbf{\Sigma}}^{-1} \hat{\mathbf{\Sigma}}) [\det(\hat{\mathbf{\Sigma}})]^{N_t-n_T n_R P}}{\pi^{n_T n_R P(n_T n_R P-1)/2} \prod_{k=1}^{N_t} \Gamma(N_t - k + 1)}
\]
(23)

Then, according to \( \ref{eq:omega} \), the likelihood function with respect to the parameter matrix \( \mathbf{\Xi} \) can be written as
\[
L(\mathbf{\Xi}) = \text{tr}(-\mathbf{\Sigma'}^{-1} \hat{\mathbf{\Sigma}}) + (N_t - n_T n_R P) \ln(\det(\hat{\mathbf{\Sigma}})) - \ln(\det(\mathbf{\Sigma'}))
\]
(24)

Therefore, the score function [11] is
\[
\frac{\partial \text{vec}(\mathbf{\Sigma'})^T}{\partial \text{vec}(\mathbf{\Xi})} = \frac{\omega \partial \text{vec}(\mathbf{\Xi} \otimes \mathbf{A})^T}{N_t} \frac{\partial \text{vec}(\mathbf{A})}{\partial \text{vec}(\mathbf{\Xi})}
\]
(26)

and the second term is
\[
\frac{\partial L(\mathbf{\Xi})}{\partial \text{vec}(\mathbf{\Sigma'})} = \text{vec}([\mathbf{\Sigma'}^{-1} \hat{\mathbf{\Sigma}} \mathbf{\Sigma'}^{-1} - N_t \mathbf{\Sigma'}^{-1}]^T)
\]
(27)

Since
\[
\text{vec}(\mathbf{\Xi} \otimes \mathbf{A}) = \mathbf{K}_{\otimes} [\text{vec}(\mathbf{\Xi}) \otimes \text{vec}(\mathbf{A})]
\]
where \( \mathbf{K}_{\otimes} \) is
\[
\mathbf{K}_{\otimes} = \mathbf{I}_{n_T n_R} \otimes \mathbf{K}_{\otimes}(n_T n_R P) \otimes \mathbf{I}_p
\]
(29)

where \( \mathbf{K}_{\otimes}(n_T n_R P) \in \mathbb{R}^{(n_T n_R P) \times (n_T n_R P)} \) is a transpose matrix satisfying \( \text{vec}(\mathbf{B}^T) = \mathbf{K}_{\otimes}(n_T n_R P) \text{vec}(\mathbf{B}) \), where \( \mathbf{B} \in \mathbb{C}^{n_T n_R P} \). Hence, \( \ref{eq:omega} \) is rewritten into
\[
\frac{\partial \text{vec}(\mathbf{\Sigma'})^T}{\partial \text{vec}(\mathbf{\Xi})} = \frac{\omega}{N_t} \left[ \text{vec}(\mathbf{A})^T \mathbf{K}_{\otimes} \right]^T
\]
(30)

By letting the score function equal zero, we know that the maximum likelihood estimator (MLE) of \( \hat{\mathbf{\Sigma}} \) is \( \overline{\hat{\mathbf{\Sigma}}} \), so, with \( \ref{eq:omega} \), its \( (k_1, k_2) \)-th submatrix, denoted as \( \{ \mathbf{\Sigma} \}_{k_1,k_2}^{(P \times P)} \in \mathbb{C}^{P \times P} \), can be used to estimate \( \mathbf{\Xi}_{k_1,k_2} \) by
\[
\mathbf{\Xi}_{k_1,k_2} \times (\omega \mathbf{A}_{k_1,k_2}) + \sigma_n^2 \delta(k_1 - k_2) \mathbf{I}_p = \{ \mathbf{\Sigma} \}_{k_1,k_2}^{(P \times P)}
\]
where \( k_1 = (i_1 - 1)n_R + j_1, k_2 = (i_2 - 1)n_R + j_2 \), and
\[
\mathbf{A}_{k_1,k_2} = \mathbf{X}_p^H \mathbf{R}_p^{(i_1,i_2)} \mathbf{X}_p
\]
(32)
Then, with (37), according to [12], (36) is rewritten into
\[ \begin{align*}
\omega \Xi k_{k_1},k_2 A_{k_1},k_2 + \sigma_n^2 \delta(k_1 - k_2) I_p = U_{k_1},k_2 (\Sigma k_{k_1},k_2 \times V_{k_1},k_2)
\end{align*} \]
where \( A_{k_1},k_2 \) is a diagonal matrix with singular values of \( A_{k_1},k_2 \) on the diagonal. Then, since rank\( (A_{k_1},k_2) = \) rank\( (R_p) = L \), the MLE of \( \Xi k_{k_1},k_2 \) is
\[ MLE(\Xi k_{k_1},k_2) = \frac{1}{L} \sum_{l=1}^{L} \left[ U_{k_1},k_2 (\Sigma k_{k_1},k_2 \times V_{k_1},k_2) \right]_{l,l} \]
where \( c_l \)'s are normalized non-negative weight coefficients, i.e., \( c_l \geq 0 \) and \( \sum_{l=1}^{L} c_l = 1 \).

Further, according to the score function, the Fisher Information matrix with respect to \( \Xi k_{k_1},k_2 \) [11] is
\[ J(\Xi k_{k_1},k_2) = E \left[ \left( \frac{\partial \Xi}{\partial \Xi} \right) \left( \frac{\partial \Xi}{\partial \Xi} \right)^T \right] \]
with (25) (30) (27), (34) is rewritten into (35), shown at the bottom of the next page, where \( B \sim C W N (N_t, \Sigma_{\parallel}^{-1}) \). Notice that
\[ B \sim C W N (N_t, \Sigma_{\parallel}^{-1}) \]
therefore
\[ E \{ \Xi (B - N_t \Sigma_{\parallel}^{-1} B)^T \} = \text{Var}(\text{vec}(B^T)) \]
According to [12], (36) is
\[ \text{Var}(\text{vec}(B^T)) = N_t (\Sigma_{\parallel}^{-H} \otimes \Sigma_{\parallel}^{-T}) \]
Then, with (37), \( J(\Xi k_{k_1},k_2) \) is
\[ J(\Xi k_{k_1},k_2) = \frac{\omega^2}{N_t} \text{vec}(A_{k_1},k_2)^T K_{\Xi} (\Sigma_{\parallel}^{-H} \otimes \Sigma_{\parallel}^{-T}) K_{\Xi}^T \text{vec}(A_{k_1},k_2)^* \]
Therefore, the CRLB of \( \Xi k_{k_1},k_2 \) is [11]
\[ CRLB(\Xi k_{k_1},k_2) = J^{-1}(\Xi k_{k_1},k_2) \]
Now we consider the case that the SNR is asymptotically infinite, or, equivalently, the power of noise is zero. According to (20), then, \( \Sigma_{\parallel} \) is reduced to
\[ \Sigma_{\parallel} = \frac{\omega}{N_t} \Xi \otimes A \]
Since \( A \) is rank deficient if \( L < P \), \( \Sigma_{\parallel}^{-1} \) should be replaced by \( \Sigma_{\parallel}^{-1} \). Then
\[ \Sigma_{\parallel}^{-1} = \frac{\omega^2}{N_t} \Xi \otimes A \]
and, correspondingly, (39) is rewritten into
\[ CRLB(\Xi) = \frac{1}{\alpha N_t} (\Xi^H \otimes \Xi^T) \]
In fact, \( L_s \) represents the order of frequency selectivity, that is, the number of equivalent independent parallel transmission branches of the multipath channels for OFDM systems. When the SNR is finite, \( \Sigma_{\parallel} \) is approximated as
\[ \Sigma_{\parallel} = \frac{\omega}{N_t} \Xi \otimes (A + \frac{\sigma_n^2}{\omega} I_p) \]
and, correspondingly, (39) is rewritten into
\[ CRLB(\Xi) = \frac{1}{\beta N_t} (\Xi^H \otimes \Xi^T) \]
and (46) is rewritten into
\[ \text{AvgMSE}_{LB}(\Xi) = \frac{1}{\beta N_t} (\Xi^H \otimes \Xi^T) \]

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where
\[
\beta = \text{vec}(A)^T[(A + \frac{\sigma_n^2}{\omega}I_p)^{-H} \otimes (A + \frac{\sigma_n^2}{\omega}I_p)^{-T}] \text{vec}(A)^*
\]
\[
= \sum_{l=1}^{L} \frac{1}{1 + (\omega \rho_l)^{-1}} |^2 \tag{52}
\]
where \( \rho_l = \frac{\lambda_l}{\sigma_n^2} \) and \( \lambda_l \) is the \( l \)-th eigenvalue of \( A \). Moreover, since
\[
\omega = x_p^H \Omega x_p = \|x_p\|^2 = \frac{x_p^H \Omega x_p}{x_p^H x_p} = \|x_p\|^2 \frac{R(\Omega)}{x_p^H x_p} \tag{53}
\]
where \( \|x_p\|^2 \) is the power of pilot symbol, and \( R(\Omega) \) is the Rayleigh quotient of \( \Omega \) [13]. Due to the normalized power, \( \|x_p\|^2 = P \). Besides, it is straightforward that \( R(\Omega) \leq \lambda_{\text{max}}(\Omega) \), where \( \lambda_{\text{max}}(\Omega) \) denotes the maximum eigenvalue of \( \Omega \). According to [14], when \( \theta_d T_s \ll 0.35 \), \( \lambda_{\text{max}}(\Omega) \) can be well approximated by
\[
\lambda_{\text{max}}(\Omega) \approx PJ_0(2\pi c \theta_d f_d T_s) \tag{54}
\]
where \( c = 0.35 \). Therefore, \( \beta \) is upper bounded by
\[
\beta \leq \sum_{l=1}^{L} \frac{1}{1 + (P^2 J_0(2\pi c \theta_d f_d T_s) \rho_l)^{-1}} \leq \beta_{\text{max}} \tag{55}
\]
Then, (51) is further lower bounded by
\[
\text{AvgMSE}_{\text{LB}}(\Xi_s) = \frac{1}{\beta_{\text{max}} N_t} \tag{56}
\]
Note that \( \omega \rho_l \) can be regarded as the effective SNR on the \( l \)-th subchannel. When \( \omega \rho_l \) is too small, say, below 0 dB, it should not be used in MLE [47], which would reduce the effective order of frequency selectivity. According to (55), therefore, the number of pilot tones, SNR and maximum Doppler spread together influence the effective order of frequency selectivity and, further, the accuracy of estimation.

IV. NUMERICAL RESULTS

The OFDM system in simulations is of \( BW = 1.25 \text{ MHz} \) \((T = 1/BW = 800 \text{ ns})\), \( N = 128 \), and \( L_{cp} = 16 \). Two 3GPP E-UTRA channel models are adopted: Extended Vehicular A model (EVA) and Extended Typical Urban model (ETU) [15]. The excess tap delay of EVA is \([0, 30, 150, 310, 370, 710, 1090, 1730, 2510] \text{ ns}\), and its relative power is \([0.0, -1.5, -1.4, -3.6, -0.6, -9.1, -7.0, -12.0, -16.9] \text{ dB}\). For ETU, they are \([0, 50, 120, 200, 230, 500, 1600, 2300, 5000] \text{ ns}\) and \([-1.0, -1.0, -1.0, 0.0, 0.0, 0.0, -3.0, -5.0, -7.0] \text{ dB}\), respectively. The classic Doppler spectrum, i.e., Jakes’ spectrum [6], is applied to generate the Rayleigh fading channel. The MIMO configuration is \(4 \times 4\), and the correlation matrices of transmitting and receiving antennas are shown at the bottom of the next page, respectively [7]. Besides, the number of pilot tones per transmitting antenna is 16, the pilot spacing is 8, and the pilot symbols are separated far enough from others to reduce the correlation among them.

In Fig. 1, we compare the analytic results (48) and the numerical results over a range of \( N_t \)’s for EVA and ETU channels, when the SNR is asymptotically infinite and \( f_d = 100 \text{ Hz}\). The pilot sequences are QPSK modulated and randomly chosen. Besides, different \( L_s \)’s are tested to demonstrate that (43) is a tight lower bound of (47). Apparently, the analytic results meet the numerical ones quite well.

In Fig. 2, different SNR’s and maximum Doppler spreads are tested to demonstrate that (56) is a tight lower bound of (47) when the pilot sequence, \( x_p \), is properly selected, specifically, the eigenvector of \( \Omega \) associated with the maximum eigenvalue. Since the number of pilot tones, SNR and normalized maximum Doppler spread together influence the effective order of frequency selectivity, the value of \( L_s \) varies for different cases, which is reflected by varying AvgMSE. Further, it is obvious that SNR has a more significant impact on \( L_s \) than the maximum Doppler spread, when the number of pilot tones are large enough, e.g., over 16.

\[
J(\Xi_s) = \frac{\omega^2}{N_t} [I_{(n_{RNT})}]_2 \otimes \text{vec}(A)^T K_\Theta E \{ \text{vec}[(B - N_s \Sigma^{-1})^T] \text{vec}[(B - N_s \Sigma^{-1})^H] \} K_\Theta [I_{(n_{RNT})}]_2 \otimes \text{vec}(A)^*] \tag{35}
\]
V. Conclusion

In this paper, we derive the CRLB of spatial correlation matrices based on a rigorous model of the doubly selective fading channel for MIMO OFDM systems. With several necessary assumptions, the sample auto-correlation matrix of the channel response is complex Wishart distributed. Then, the maximum likelihood estimator is obtained, and the analytic expressions of CRLB as well as lower bounds of TMSE and AvgMSE are deduced for asymptotically infinite and finite SNR’s, respectively. According to the lower bound of AvgMSE, the amount of samples and the order of frequency selectivity influence the accuracy of estimation dominantly. Besides, the number of pilot tones, SNR and maximum Doppler spread have effects on the effective order of frequency selectivity.

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