The moduli space of parabolic bundles with fixed determinant over a smooth curve of genus greater than one is proved to be rational whenever one of the multiplicities of the quasi-parabolic structure equals one. This gives a new proof that the moduli space of vector bundles of coprime rank and degree is stably rational, a result originally due to Ballico, and the bound on the level is strong enough to deduce rationality in many cases, extending results of Newstead.

1. Introduction

Let $X$ be a smooth complex curve of genus $g \geq 2$, $L$ be a line bundle of degree $d$ over $X$, and $\mathcal{M}_{r,L}$ be the moduli space of semistable bundles $E$ of rank $r$ with determinant $L$.

**Conjecture 1.1.** $\mathcal{M}_{r,L}$ is rational, that is, it is birational to a projective space.

Despite many positive results [13], this is still an open problem, even for $(r,d) = 1$. In this paper, we study a closely related problem, namely the birational classification of moduli spaces of parabolic bundles over $X$. These moduli spaces occur naturally as

(i) unitary representation spaces of Fuchsian groups [10];

(ii) moduli spaces of Yang–Mills connections on $X$ with an orbifold metric [6];

(iii) moduli spaces of certain semistable bundles over an elliptic surface [3].

The theory developed in [8] and extended here shows that their birational type depends only on the quasi-parabolic structure (see Proposition 4.3). The methods of [13] then prove, in many cases, that these moduli spaces are rational. The weaker result, Theorem 6.1, uses only Newstead's theorem, while the stronger one, Theorem 6.2, requires an adaptation of his inductive argument.

Using the theory developed in §4, it then follows from Theorem 6.2 that $\mathcal{M}_{r,L} \times \mathbb{P}^{r-1}$ is rational if $(r,d) = 1$ (see Corollary 6.4). Stable rationality of the moduli spaces had been proved in this case by Ballico [2], and our result is a strengthening of his. For instance, a consequence is that one can conclude Conjecture 1.1 under the assumption $(r,d) = 1$ for most values of the genus, that is, choosing $d' \equiv d \mod (r)$ with $0 < d' < r$, the hypothesis is that either $(d',g) = 1$ or $(r-d',g) = 1$ (see Corollary 6.5).
A number of useful facts are established along the way. One key point is Proposition 3.2, which gives a simple criterion for the existence of a universal bundle of stable parabolic bundles. We also extend the theory developed in [8] in several important ways (Theorems 4.1, 4.2 and 5.4); the first two are standard but necessary for our purposes and the third is completely new. Its proof requires the idea of shifting a parabolic sheaf (Definition 5.1), which also provides a framework for the Hecke correspondence (Figure 3). All of these results play a crucial role in the proofs of Theorems 6.1 and 6.2.

A brief word about the organization of this paper: §2 introduces the notation used in the following sections, §3 discusses the existence of universal families, §4 summarizes and extends the theory of [8], §5 describes shifting and the Hecke correspondence, and §6 contains the proofs of the main results and their corollaries.

Before we begin, we would like to acknowledge a certain debt to the work of Newstead, upon which a number of our arguments depend, and without which this paper would be inconceivable.

2. Notation

Let $X$ be a smooth curve of genus $g \geq 2$ and $D$ be a reduced divisor in $X$. If $E$ is a $\mathbb{C}^*$ bundle over $X$, then a parabolic structure on $E$ with respect to $D$ is just a collection of weighted flags in the fibers of $E$ over each $p \in D$ of the form

$$E_p = F_1(p) \supset F_2(p) \supset \ldots \supset F_r(p) \supset 0$$

and

$$0 \leq \alpha_1(p) < \alpha_2(p) < \ldots < \alpha_r(p) < 1.$$  

Holomorphic bundles $E$ with parabolic structures are called parabolic bundles, and we use the notation $E_\alpha$ to indicate the bundle (or, equivalently, locally free sheaf) $E$ together with a choice of parabolic structure. A morphism $\phi: E_{\alpha_1} \to E_{\alpha_2}$ of parabolic bundles is a bundle map satisfying $\phi(F_i(p)) \subset F_{\alpha_1}(p)$ whenever $\alpha_i(p) > \alpha_j(p)$ for all $p \in D$. We use the tensor product notation $H^0(E_{\alpha} \otimes E_{\beta})$ for these morphisms, where $E_{\alpha}$ denotes the dual parabolic bundle (cf. [18]).

A quasi-parabolic structure on $E$ is what is left after the weights are forgotten; it is determined topologically by its flag type $m$, which specifies multiplicities $m(p) = (m_1(p), \ldots, m_r(p))$ for each $p \in D$ defined by $m_1(p) = \dim F_i(p) - \dim F_{i+1}(p)$.

A subbundle $E'$ inherits a parabolic structure from one on $E$ in a canonical way: the flag in $E'$ is obtained by intersecting with the flag in $E$ and the weights are determined by choosing maximal weights such that the inclusion map from $E'$ to $E$ is parabolic [10, p. 213]. Parabolic structures on quotients have a similar description [10, p. 213].

A parabolic bundle $E_\alpha$ is called stable if every proper holomorphic subbundle $E'$ satisfies $\mu(E'_\alpha) < \mu(E_\alpha)$, where

$$\mu(E_\alpha) = \text{pardeg }E_\alpha/r = \deg E/r + \sum_{p \in D} \sum_{i=1}^r m_i(p) \alpha_i(p)/r.$$  

The parabolic bundle $E_\alpha$ is called semistable if $\mu(E'_\alpha) \leq \mu(E_\alpha)$ for each subbundle $E'_\alpha$. The construction of the moduli space $\mathcal{M}_\alpha$ of semistable parabolic bundles, as a normal, projective variety, is given in [10]. The subspace $\mathcal{M}_\alpha^*$ of stable bundles is smooth and Zariski-open; in particular, if every semistable bundle is stable, then $\mathcal{M}_\alpha^*$ is smooth.

Let $\Delta = \{a_i, \ldots, a_r \mid 0 \leq a_i \leq \ldots \leq a_r < 1\}$ and define $W = \{\alpha: D \to \Delta\}$. Points in $W$ determine both the weights and the multiplicities. Conversely, given a weight $\alpha$
in the sense of (2), the associated point in $W$ is obtained by repeating each $\alpha_i(p)$ according to its multiplicity $m_i(p)$. We abuse notation slightly by referring to points in $W$ as weights. This gives an obvious notion when a weight is compatible with a choice of multiplicities, and for a given $m$, we define the open face of weights compatible with $m$ to be

$$V_m = \left\{ \alpha \in W | \alpha_{i-1}(p) = \alpha_i(p) \iff \sum_{k=1}^{i} m_k(p) < i - 1 < \sum_{k=1}^{i+1} m_k(p) \right\}. $$

A weight in the interior of $W$ specifies full flags at each $p \in D$. For every other choice of $m$, $V_m$ is contained in the boundary of $W$. Now $W$ is a simplicial set, and the face relations give a natural ordering on $\{ V_m \}$ and we write $V_m > V_m'$ if $V_m'$ is a proper face contained in the closure of $V_m$. This agrees with the natural ordering on $m$ obtained by successive refinement.

Weights for which $V_m$ is not necessarily smooth satisfy $\mu(E_\alpha) = \mu(E_\alpha')$ for some proper subbundle $E'$. Letting $E'$ be the quotient, then the short exact sequence of parabolic bundles $E_\alpha \to E_\alpha' \to E_\alpha''$ determines a partition of $(d, r, m)$ in the obvious way: $(d', d'')$, $(r', r'')$ and $(m', m'')$ are the degrees, ranks and multiplicities of $(E', E'')$. We define $m'$ and $m''$ here slightly unconventionally, namely

$$m'_i(p) = \dim(F_i(p) \cap i(E_\alpha')) - \dim(F_i(p) \cap i(E_\alpha''))$$

$$m''_i(p) = \dim(\pi(F_i(p)) \cap E'_\alpha) - \dim(\pi(F_i(p)) \cap E''_\alpha)$$

for $p \in D$ and $1 \leq i \leq \kappa_\alpha$. Notice that $r', r'' > 0$ and $m'_i(p), m''_i(p) \geq 0$. Write $\xi = (d', r', m')$. For fixed $\xi$, the set of weights compatible with $m$ for which $\mu(E_\alpha') = \mu(E_\alpha'')$ is given by those that satisfy the equation

$$\sum_{p \in D} \sum_{i=1}^{\kappa_\alpha} (r'm_i(p) - rm_i(p)) \alpha_i(p) = rd' - r'd. \quad (3)$$

If $m(p) = (1, \ldots, 1)$ for all $p$, this equation determines a hyperplane $H_\xi$ in the interior of $W$. We shall refer to $H_\xi \cap V_m$ as a wall in $V_m$. These walls induce a chamber structure on $V_m$, a chamber being a connected component of $V_m \setminus \bigcup H_\xi$ (it is possible that $V_m \subset H$). We also define the incidence number of a wall to be the number of distinct hyperplanes containing it. This number is finite; indeed, there are only finitely many hyperplanes overall; the above equation puts a bound on $d'$ and all other quantities are already bounded. Weights $\alpha \in W \setminus \bigcup H_\xi$ are called generic, and for these weights, $V_m$ contains a generic weight if and only if the degree $d$ and the set of multiplicities $\{m_i(p)\}$ have greatest common divisor equal to one.

### 3. Families of parabolic bundles

In this section, we present Proposition 3.2, which establishes the existence of a universal family of stable parabolic bundles parameterized by $\mathcal{M}_\alpha^*$ whenever $V_m$ contains a generic weight. Although results of this type are well known to experts, the proposition, as well as the proof, is original (cf. [15, Théorème 32]). It is important because, in the case of ordinary bundles, the non-existence of the universal family [14] is the obstruction to proving Conjecture 1.1 by induction and, as shown in §6, the analogous argument works for parabolic bundles precisely because the necessary conditions for the vanishing of this obstruction given by Proposition 3.2 are often satisfied.
Given positive integers \( m_1, \ldots, m_n \) such that \( m_1 + \ldots + m_n = r \), define \( \mathcal{F}_m \) to be the variety of flags of type \( m \). These are simply flags \( C' = F_1 \supset \cdots \supset F_r \supset 0 \) with \( \dim F_1 - \dim F_{r+1} = m_i \). Furthermore, for any bundle \( E \rightarrow S \) of rank \( r \), let \( \mathcal{F}_m(E) \rightarrow S \) be the bundle of flags of type \( m \).

Given a bundle \( U \rightarrow S \times X \), we adopt the notation \( U_i = U|_{(s,x) \times X} \). We also use \( \pi_{i} \) for the projection map \( S \times X \rightarrow S \).

**Definition 3.1.** Fix multiplicities \( m(p) \) for each \( p \in D \).

(i) A family of quasi-parabolic bundles (of type \( m \)) parameterized by a variety \( S \) is a bundle \( U \) over \( S \times X \) together with a section \( \phi_p \) of the flag bundle \( \mathcal{F}_{m(p)}(U|_{S \times X}) \rightarrow S \) for each \( p \in D \).

(ii) Two families \( (U, \phi) \) and \( (U', \phi') \) parameterized by \( S \) are equivalent, written \( (U, \phi) \sim (U', \phi') \), if there exists a line bundle \( L \) over \( S \) and an isomorphism \( U \cong U' \otimes \pi^*_S L \) under which \( \phi \mapsto \phi' \).

Note that the section \( \phi_p \) in (i) is just a choice of a nested chain of subbundles of \( U|_{S \times X} \) whose relative coranks are given by the multiplicities \( m(p) \). A family of parabolic bundles is obtained by associating a fixed set of weights to each chain of subbundles. Let \( U_p = (U, \phi, s) \) be the resulting family of parabolic bundles and \( U_{s,*} = (U_s, \phi(s), x) \) be the parabolic bundle above \( s \in S \). Then \( U_p \) is a family of (semi)stable parabolic bundles if \( U_{s,*} \) is (semi)stable for each \( s \in S \).

It follows from the construction of Mehta and Seshadri [10] that \( \mathcal{M}_e \) is a coarse moduli space. Proposition 1.8 of [12] then gives two conditions which are necessary and sufficient for a coarse moduli space to be fine, that is, to admit a universal family. The second condition is not difficult to verify using an argument similar to that given in [12, Lemma 5.10]. The first condition requires that we construct a family \( \mathcal{W}_e^s \) parameterized by \( \mathcal{M}_e^s \) with the property that \( \mathcal{W}_e^s \) is a parabolic stable bundle isomorphic to \( E_s \) for all \( [E_s] = e \in \mathcal{M}_e^s \).

To construct this family, we need to review the construction of \( \mathcal{M}_e \) [5, 10]. Let \( Q \) be the Hilbert scheme of coherent sheaves over \( X \) which are quotients of \( \mathcal{O}_X^N \) with fixed Hilbert polynomial (that of \( E(k) \) for \( k \gg g \)), where \( N = h^0(E) \). Let \( U \) be the universal family on \( Q \times X \). Define \( \mathcal{R} \) to be the subscheme of \( Q \) of points \( r \in Q \) so that \( U_r \) is a locally free sheaf which is generated by its global sections and \( h^0(U_r) = 0 \). Let \( \mathcal{R} \) be the total space of the universal flag bundle over \( R \) with flag type \( \prod_{p \in D} \mathcal{F}_{m(p)} \), and let \( \tilde{U} \) be the pullback of \( U \) to \( \mathcal{R} \). Then \( \tilde{U} \) is canonically a family of parabolic bundles parameterized by \( \mathcal{R} \) by letting, for each \( p \in D \), \( \phi_p \) be the tautological section and \( \pi(p) \) be the fixed weights. It follows that \( \mathcal{R} \) has the local universal property for parabolic bundles [5, p. 16].

The subsets \( \mathcal{R}^s \) (\( \mathcal{R}^s \)) corresponding to the stable (semistable) parabolic bundles are invariant under the natural action of \( \text{GL}(N) = \text{Aut}(\mathcal{O}_X^N) \), and \( \mathcal{M}_e^s \) is a good quotient of \( \mathcal{R}^s \) (with linearization induced by the weights \( \pi \)), and \( \mathcal{M}_e^s \) is the geometric quotient of \( \mathcal{R}^s \).

The centre of \( \text{GL}(N) \) acts trivially on \( \mathcal{R} \) and \( \mathcal{R} \), but non-trivially on the locally universal bundle \( \tilde{U} \). In fact, \( \lambda(id) \) acts on \( \tilde{U} \) by scalar multiplication by \( \lambda \) in the fibers (this follows from [12, p. 138]). Given a line bundle \( L \) over \( \mathcal{R}^s \) with a natural lift of the \( \text{GL}(N) \) action such that \( \lambda(id) \) acts by multiplication by \( \lambda \), then using \( \tilde{U}^s \) to denote \( \tilde{U}|_{\mathcal{R}^s \times X} \), the quotient of \( \tilde{U}^s \otimes \pi^*_S L^{-1} \), together with the tautological sections and weights \( \{\phi_p, \pi(p)\} \) mentioned above, gives the desired family.
Proposition 3.2. A line bundle $L$ over $\hat{\mathbb{R}}^e$ with natural action of $\text{GL}(N)$ so that central elements $\lambda(\text{id})$ act by multiplication by $\lambda$ exists if either

(i) the elements of the set $\{d, m_i(p)\}_{p \in D, 1 \leq i \leq \kappa_p}$ have greatest common divisor equal to one; or

(ii) the face $V_m$ containing $\alpha$ contains a generic weight.

Moreover, these two conditions are equivalent, and when they are satisfied, the moduli space $\mathcal{M}_n^*$ is fine.

The idea of the proof is to find line bundles $L_k$ for each $k \in \{d, m_i(p)\}$ over $\hat{\mathbb{R}}^e$ with natural actions of $\text{GL}(N)$ such that $\lambda(\text{id})$ acts by scalar multiplication by $\lambda^k$. Then (i) gives the existence of $k_1, \ldots, k_r \in \{d, m_i(p)\}$ and integers $a_1, \ldots, a_r$ so that $a_1k_1 + \ldots + a_rk_r = 1$. The required line bundle is then the tensor product $L = L_{k_1}^{a_1} \otimes \ldots \otimes L_{k_r}^{a_r}$. At the end of the proof, we will show that (i) and (ii) are equivalent.

For the following lemma, let $n$ be the number $|D|$ of parabolic points.

Lemma 3.3. Suppose $E_\ast$ is parabolic semistable of degree $d$ and rank $r$ and $H_\ast$ is a parabolic line bundle of degree $h$, then

$$h^i(H_\ast^r \otimes E_\ast) \neq 0 \quad \Rightarrow \quad d \leq r(2g-2+h)+rn.$$ (4)

Proof. Serre duality for parabolic bundles [18, Proposition 3.7] implies that

$$h^i(H_\ast^r \otimes E_\ast) \leq h^0(E_\ast^r \otimes H_\ast \otimes K(D)).$$

(If we had used $h^h(E_\ast^r \otimes \hat{H}_\ast \otimes K(D))$, the circumflex over $H_\ast$ indicating strongly parabolic morphisms, we would get the usual statement of Serre duality with equality, cf. [9, 18].) Suppose that $\phi : E \longrightarrow H \otimes K(D)$ is a non-zero map and let $E'$ be the subbundle generated by $\text{Ker} \phi$. Then

$$\text{deg } E' \geq \text{deg } E - \text{deg } H \otimes K(D) = d - h - (2g-2+n),$$

Considering $E_\ast$ with its canonical parabolic structure as a subbundle of rank $r-1$, the inequality (4) follows easily from this, the semistability of $E_\ast$, and the inequalities pardeg $E_\ast' \geq \text{deg } E'$ and pardeg $E_\ast \geq \text{deg } E + rn$. \hfill \square

Proof of Proposition 3.2. Write the weights $\alpha$ without repetition. Choose $\ell : D \longrightarrow \mathbb{Z}$ with $1 \leq \ell_p \leq \kappa_p + 1$ and set $\beta(p) = \chi_p(\ell_p)$. (Take $\beta(p) > \chi_p$ if $\ell_p = \kappa_p + 1$.) For $h \in \mathbb{Z}$, define

$$\chi(\ell, h) = d + r(1-g-h) - \sum_{p \in D} \sum_{i=1}^{\ell_p} m_i(p).$$

Let $H_\ast$ be a parabolic line bundle with deg $H = h < d/r - rn - (2g-2)$ and with weights $\beta(p)$ at $p \in D$. It follows from Lemma 3.3 that if $E_\ast$ is semistable, then $h^h(H_\ast \otimes E_\ast) = 0$. Thus $h^h(H_\ast \otimes E_\ast) = \chi(\ell, h)$ by Riemann and Roch. Hence $(R^h\pi_\ast)(U^\ast \otimes \pi_\ast^\ast H_\ast)$ is a locally free sheaf of rank $\chi(\ell, h)$ over $\hat{\mathbb{R}}^e$. Let $L(\ell, h)$ be the determinant of the corresponding bundle. By construction, the $\text{GL}(N)$ action on $\hat{U}$ induces one on this bundle (and hence on $L(\ell, h)$); $\lambda(\text{id})$ acts by scalar multiplication by $\lambda$ on the bundle and by $\lambda^h$ on $L(\ell, h)$. It is now a simple exercise in high-school algebra to see that we can choose $h, h', \ell, \ell'$ so that $\lambda(\text{id})$ acts on $L(\ell, h) \otimes L(\ell', h')$ by $\lambda^k$ for any $k \in \{d, m_i(p)\}$.

This proves the conclusion of the proposition assuming (i), and now we show that conditions (i) and (ii) are equivalent. Suppose first that (i) does not hold. Consider $E_\ast$
as a quasi-parabolic bundle without holomorphic structure, which will be specified later. Since the set \(\{d, m_i(p)\}\) is not relatively prime, there exists a prime number \(q\) evenly dividing each element of the set. Clearly \(q\) also divides \(r\). Set \(d' = d/q, r' = r/q\) and \(m'_i(p) = m_i(p)/q\). Consider the quasi-parabolic bundle \(E_*\) with degree \(d'\), rank \(r'\), and multiplicities \(m'\). Any choice of weights \(\alpha\) on \(E_*\) induces (the same!) weights on \(E'_*\), and it follows that since \(g \geq 2\), there is some holomorphic structure for which \(E'_*\) is semistable. Define the holomorphic structure of \(E_*\) by
\[
E_* = E'_* \oplus \cdots \oplus E'_{**}.
\]

It follows that \(E_*\) is semistable but not stable for any choice of compatible weights. This implies that \(V_m\) does not contain a generic weight.

Suppose conversely that \(V_m\) does not contain a generic weight. Since \(V_m\) is affine, \(V_m \subseteq H_\xi\) for some \(\xi = (r', d', m')\). Using (3), we conclude that for all \(\alpha \in V_m\),
\[
\sum_{p=\beta} \sum_{i=1}^{k_p} (rm_i(p) - r'm_i(p)) \alpha_i(p) = rd' - r'd.
\]
(Here, we are still thinking of \(\alpha\) without repetition.) We can vary each \(\alpha_i(p)\) continuously by some small amount, and it follows that
\[
rm_i(p) - r'm_i(p) = 0 = rd' - r'd
\]
for all \(i\) and \(p\). Since \(r' < r\), there exists a prime \(q\) such that \(q^k\) divides \(r\) but not \(r'\). Hence \(q\) divides \(d\) and each element of the set \(\{m_i(p)\}_{p \in D, 1 \leq i \leq k_p}\).

4. The variation and degeneration theorems

In this section, we describe the results of [8]. This allows us to compare the moduli spaces of parabolic bundles \(\mathcal{M}_\alpha\) and \(\mathcal{M}_\beta\) when

(i) \(\alpha, \beta \in V_m\) are generic weights separated by a wall with incidence number one;

(ii) \(\alpha \in V'\) and \(\beta \in V_m\) are generic weights not separated by any hyperplanes and \(V' > V_m\).

Cases (i) and (ii) correspond to [8, Theorem 3.1, Proposition 3.4]. We present slightly stronger versions of those results tailored for our purposes here.

Starting with (i), suppose that \(\alpha, \beta \in V_m\) are generic weights separated by a wall \(H_\xi \cap V_m\) of incidence number one. Choose \(\gamma \in H_\xi \cap V_m\) on the straight line connecting \(\alpha\) to \(\beta\). Then \(\mathcal{M}_\gamma\) is stratified by the Jordan–Hölder type of the underlying bundle, and since \(\gamma\) lies on only one hyperplane, there are exactly two strata: the stable bundles \(\mathcal{M}_\gamma^s\) and the strictly semistable bundles \(\Sigma_\gamma\). If we write \(\xi = (r', d', m')\) for the partition, then it is not hard to see that \(\Sigma_\xi \cong \mathcal{M}_{\gamma'} \times \mathcal{M}_{\gamma''}\), with the obvious definitions for \(\gamma', \gamma''\) coming from the partition \(\xi\).

![Figure 1](image-url)
**Theorem 4.1.** There are natural algebraic maps \( \phi_\alpha \) and \( \phi_\beta \) (see Figure 1) which are generalized blow-downs along projectivizations of vector bundles over \( \Sigma_r \) where the projective fiber dimensions \( e_\alpha \) and \( e_\beta \) satisfy \( e_\alpha + e_\beta + 1 = \text{codim} \Sigma_r \).

**Proof.** The proof is the same as in [8], the only difference being the actual computation of the numbers \( e_\alpha \) and \( e_\beta \) which we discuss now. We assume that \( E_\alpha \sim_h E'_\alpha \oplus E'_\alpha \), where \([E_\alpha] \in \Sigma_r \) and \( \sim_h \) denotes Seshadri equivalence (that is, isomorphic Jordan–Hoelder form). The topological type of the parabolic bundles \( E_\alpha \) and \( E'_\alpha \) does not change as \([E_\alpha] \) varies within \( \Sigma_r \). We use \((r', r''), (d', d''), \) and \((m', m'')\) to denote the ranks, degrees and multiplicities of \((E_\alpha, E'_\alpha)\), written as in §2. The moduli spaces \( \mathcal{M}_\alpha \), \( \mathcal{M}_\beta \) and \( \mathcal{M}_\gamma \) have dimension \[
\text{dim} \mathcal{M}_\gamma = r' r'^{2g-1} - 1 - \sum_{p \in D} m'_i(p) m''_i(p). \]

Now we claim that \( h^0(E'_\alpha \otimes E'_\alpha) = h^0(E''_\alpha \otimes E''_\alpha) \).

This is true for any \( \alpha' \in V_m \), as one of these equations is true for \( \alpha \), and the other is true for \( \beta \), but \( H^* \) is constant as the weights are varied within \( V_m \). Let \( \mathcal{W} \) and \( \mathcal{W}' \) be the families parameterized by \( \Sigma \) obtained by pulling back the universal families \( \mathcal{W} \) and \( \mathcal{W}' \), whose existence follows from Proposition 3.2. Then the vector bundles referred to in the theorem are \( (R^1 \pi_\Sigma \mathcal{W}) \otimes \mathcal{W} \) and \( (R^1 \pi_\Sigma \mathcal{W}) \otimes \mathcal{W}' \).

The projectivizations of these bundles have dimensions \[
\begin{align*}
\mathcal{E}_\alpha &= h^1(E'_\alpha \otimes E'_\alpha) - 1 = r' d' - r' d'' + r' r'' (g - 1) + \chi(2) - 1, \\
\mathcal{E}_\beta &= h^1(E''_\alpha \otimes E''_\alpha) - 1 = r'' d'' - r'' d' + r'' r'' (g - 1) + \chi(2') - 1.
\end{align*}
\]

where \( \mathcal{E} \) and \( \mathcal{E}' \) are skyscraper sheaves supported on \( D \) obtained as the quotients \[
\begin{align*}
\Psi \text{ar} \text{Hom}(E'_\alpha, E'_\alpha) &\to \text{Hom}(E'_\alpha, E'_\alpha) \to \mathcal{E}, \\
\Psi \text{ar} \text{Hom}(E'\alpha, E''_\alpha) &\to \text{Hom}(E'_\alpha, E'_\alpha) \to \mathcal{E}'.
\end{align*}
\]

It is a nice exercise to see that \[
\chi(2) + \chi(2') = \sum_{p \in D} \left( r' r'' - \sum_{(i,j) \in S_i(p)} m'_i(p) m''_j(p) \right)
\]

where \( S_i(p) = \{(i,j) \mid \gamma_j(p) = \gamma'_i(p) \} \). This shows that \( e_\alpha + e_\beta + 1 = \text{codim} \Sigma_r \). \( \Box \)

**Theorem 4.2.** Suppose that \( \alpha \in V_\gamma, \beta \in V_m, V_\gamma > V_m \), and that \( \alpha \) and \( \beta \) are generic and are not separated by any hyperplanes. Then there exists a fibration \( \psi : \mathcal{M}_\alpha \to \mathcal{M}_\beta \) with fiber a (possibly twisted) product of flag varieties and this fibration is locally trivial in the Zariski topology. In particular, \( \mathcal{M}_\alpha \) is birational to the product of \( \mathcal{M}_\beta \) with a product of flag varieties.
Proof. The hypothesis \( V_1 > V_n \) just means that the flag structure degenerates as we pass from \( \alpha \) to \( \beta \). By induction, it is enough to prove the above statement when the degeneration of the flag structure is taking place at only one parabolic point. Since \( \alpha \) and \( \beta \) are not separated by any hyperplanes, we can choose \( \alpha \) arbitrarily close to \( \beta \) without changing the stability conditions. Given \( E_\alpha \), a parabolic bundle with multiplicities \( m \) and weights \( \alpha \), let \( E'_\alpha \) be the parabolic bundle with multiplicities \( \ell \) and weights \( \beta \) resulting from forgetting part of the flag structure and interchanging the weights. One easily verifies that if \( E_\alpha \) is \( \alpha \)-stable, then \( E'_\alpha \) is \( \beta \)-stable, and the existence of the morphism \( \psi \) then follows from the universality property of \( \mathcal{M}_p \).

The remaining issues are to identify the fiber and to prove local triviality. For the first issue, notice that there is an inverse procedure to the forgetful map described above. Given a parabolic bundle \( E'_\alpha \) with multiplicities \( \ell \) and weights \( \beta \), consider all parabolic bundles \( E_\alpha \) with weights \( \alpha \) obtained from \( E'_\alpha \) by refining the flag structure to one with multiplicities \( m \) and exchanging the weights. For a given \( E'_\alpha \), the set of all such possible refinements \( E_\alpha \) is parameterized by a flag variety.

A straightforward numerical verification shows that applying this procedure to a \( \beta \)-stable parabolic bundle \( E_\alpha \) yields an \( \alpha \)-stable \( E_\alpha \) for every possible refinement. It is not hard to see that the same procedure, when applied to the universal family \( \mathcal{U}_\alpha \), identifies \( \mathcal{M}_\alpha \) with the total space of the flag bundle of \( \mathcal{U}^0 \) restricted to \( \mathcal{M}_\beta \times \{p\} \) and the map \( \psi \) with the bundle projection. \( \square \)

One might expect from Theorem 4.1 that the birational type of \( \mathcal{M}_\alpha \) depends only on the underlying quasi-parabolic structure. This is the content of the following proposition.

Proposition 4.3. Suppose that \( g \geq 2 \). Then the birational type of \( \mathcal{M}_\alpha \) is independent of the choice of \( \alpha \in V_m \).

Proof. We prove the proposition by showing that \( \mathcal{M}_\alpha \) and \( \mathcal{M}_\beta \) are birational whenever \( \alpha, \beta \in V_m \) are not separated by any walls (although one may lie on a wall which does not contain the other). So assume that \( \alpha \in \bigcap_{i=1}^n H_{\xi_i} \) and \( \beta \in \bigcap_{i=1}^n H_{\xi_i} \), where \( m \geq n \). By Theorem 4.1 [10], \( \mathcal{M}_\alpha \) and \( \mathcal{M}_\beta \) are normal, projective varieties and \( \dim \mathcal{M}_\alpha = \dim \mathcal{M}_\beta \); hence we only need to construct an injective morphism \( \phi : \mathcal{M}_\beta \to \mathcal{M}_\alpha \) to conclude that \( \mathcal{M}_\beta \) is birational to \( \mathcal{M}_\alpha \). One easily verifies that every \( \beta \)-stable bundle is \( \alpha \)-stable, and the existence of \( \phi \) follows from the universality of \( \mathcal{M}_\beta \). \( \square \)

5. Shifting and the Hecke correspondence

In this section, we introduce the notion of a shifted parabolic bundle, which is the result of changing the weights, multiplicities and degree of \( E_\alpha \) in a prescribed way. In some sense, shifting is a symmetry of a larger weight space, one which includes bundles of different degrees. Two applications of shifting are discussed.

Shifting is most naturally described in terms of parabolic sheaves. If \( \mathcal{E} \) is a locally free sheaf on \( X \), then a parabolic structure on \( \mathcal{E} \) consists of a weighted filtration of the form

\[
\mathcal{E} = \mathcal{E}_{\lambda_1} \supset \mathcal{E}_{\lambda_2} \supset \cdots \supset \mathcal{E}_{\lambda_t} \supset \mathcal{E}_{\lambda_{t+1}} = \mathcal{E}(-D) \quad (7)
\]

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_t < \lambda_{t+1} = 1. \quad (8)
\]

We can define \( \mathcal{E}_x \) for \( x \in [0,1] \) by setting \( \mathcal{E}_x = \mathcal{E}_{\lambda_i} \) if \( \lambda_{i-1} < x \leq \lambda_i \), and then extend to
$x \in \mathbb{R}$ by setting $\delta_{x+1} = \delta_x(-D)$. We call the resulting filtered sheaf $\delta_*$ a parabolic sheaf and $\delta = \delta_0$ the underlying sheaf.

We can define parabolic subsheaves, degree and stability for these objects, and there is a categorical equivalence between locally free parabolic sheaves and parabolic bundles. We describe this in case $D = p$, the general case being quite similar [9, 18].

Suppose that $E_*$ is a parabolic bundle given by flags and weights in the fibers as in (1) and (2). Define $\delta_*$ by setting

$$\delta_x = \ker(E \longrightarrow E_p/F_i)$$

for $x_{i-1} < x < x_i$. Thus $\delta_*$ is a parabolic sheaf. Conversely, given a parabolic sheaf $\delta_*$, the quotient $\delta_*/\delta_1 = \delta/\delta(-p)$ is a skyscraper sheaf with support $p$ and fiber that of $\delta$. Defining a flag in this fiber by setting $E_i = (\delta_*/\delta_1)_p$ and associating the weight $x_i$, we obtain a parabolic bundle in the sense of (1) and (2).

The category of parabolic sheaves is developed in [18], where one finds for example the definitions of tensor products $\delta_0 \otimes \delta'_0$ and duals $\delta_0^\vee$. We use this notation freely in the calculations of §6 involving sheaf cohomology and point out that $H(\delta_0) = H(\delta_0^\vee)$.

**Definition 5.1.** Given a parabolic sheaf $\delta_*$ and $\eta \in \mathbb{R}$, we define the shifted parabolic sheaf $\delta_0[\eta]$ by setting $\delta_0[\eta]_i = \delta_{x_i+\eta}$ (see Figure 2).

![Figure 2. The parabolic sheaf $\delta_0$ shifted by $\eta$ with $x_i < \eta < x_i$](image)

**Remark 5.2.** The above operation can be refined in case $D = p_1 + \ldots + p_n$. If $\eta = (\eta_1, \ldots, \eta_n)$, then one can shift $\delta_0$ by $\eta_j$ at each $p_j \in D$ [8, 18].

It is not difficult to verify that $\delta_0[\eta]$ is (semi)stable if and only if $\delta_0$ is (semi)stable, and it follows that this defines an isomorphism between the associated moduli spaces of parabolic bundles.

We can easily describe the parabolic structure on the shifted bundle $\delta_0' = \delta_0[\eta]$ in case $0 < \eta \leq 1$ and $D = p$. Let $E'_*$ denote the parabolic bundle associated to $\delta_0'$. If $i$ is the integer with $x_i < \eta \leq x_{i+1}$, then the weights of $E'_*$ are given by

$$x'_j = \begin{cases} x_{j+i} - \eta & \text{for } j = 1, \ldots, r-i \\ 1 + x_{j+i} - \eta & \text{for } j = r-i+1, \ldots, r. \end{cases}$$

(9)

The quasi-parabolic structure of $E'_*$ has multiplicities $m'$ given by a cyclic permutation
of \( m \), that is, \( m' = (m_{-1}, \ldots, m_0, m_1, \ldots, m_l) \). Although \( \mathcal{E}' \) is a subsheaf of \( \mathcal{E} \), \( E' \) is not a subbundle of \( E \), so one must appeal to sheaf theory in order to define the flag in \( E' \).

This is a simple exercise in tracing through the equivalence between locally free parabolic sheaves and parabolic bundles given above.

We now discuss two interesting applications of shifting. The first is the Hecke correspondence. Using \( \mathcal{M}_{r,d} \) to denote the moduli space of semistable bundles of rank \( r \) and degree \( d \), the Hecke correspondence gives a means of comparing \( \mathcal{M}_{r,d} \) and \( \mathcal{M}_{r,a} \) through the use of parabolic bundles. For \( r = 2 \), this was observed in a remark at the end of [10].

To start, define \( \varepsilon_1(d,r) \), \( \varepsilon_2(d,r) \) and \( \varepsilon(d,r) \) for \( d, r \in \mathbb{Z} \) with \( r > 0 \) by

\[
\varepsilon_1(d,r) = \inf \left\{ \pm \left( \frac{d}{r} - \frac{d'}{r'} \right) \middle| \begin{array}{c}
d', r' \in \mathbb{Z}, 1 \leq r' < r, \\
\text{and } \pm \left( \frac{d}{r} - \frac{d'}{r'} \right) > 0
\end{array} \right\}
\]

\[
\varepsilon(d,r) = \min \{ \varepsilon_2(d,k) \mid k = 1, \ldots, r \}.
\]

It is easy to see that \( \varepsilon_2(d,k) > 0 \) for all \( k \); thus \( \varepsilon(d,r) > 0 \) as well.

Suppose that \( E \) is a bundle over \( X \) of degree \( d \) and rank \( r \) and suppose further that \( E' \) is a proper subbundle. If \( \mu(E') < \mu(E) \), then \( \mu(E) - \mu(E') \geq \varepsilon_1(d,r) \). Similarly, if \( \mu(E') > \mu(E) \), then \( \mu(E') - \mu(E) \geq \varepsilon_2(d,r) \).

**Proposition 5.3.** Suppose that \( E_\ast \) satisfies \( \sum_{p \in D} \sum_{i \in \mathbb{Z}} m_i(p) \eta_i(p) < \varepsilon(d,r)/2 \).

(i) If \( E \) is stable as a regular bundle, then \( E_\ast \) is parabolic stable.

(ii) If \( E_\ast \) is parabolic stable, then \( E \) is semistable as a regular bundle.

**Proof.** (i) If \( E_\ast \) is a proper parabolic subbundle of \( E_\ast \), then

\[
\mu(E_\ast) \leq \mu(E') + \varepsilon(d,r)/2 < \mu(E') + \varepsilon_1(d,r) \leq \mu(E) < \mu(E_\ast).
\]

Thus \( E_\ast \) is parabolic stable.

(ii) If \( E' \) is a subbundle of \( E \), then

\[
\mu(E') \leq \mu(E_\ast) \leq \mu(E) + \varepsilon(d,r)/2 < \mu(E) + \varepsilon_2(d,r).
\]

Hence \( \mu(E') \leq \mu(E) \) and \( E \) is semistable.

We thus get a morphism \( \mathcal{M}_r \longrightarrow \mathcal{M}_{r,d} \) which is the map of Theorem 4.2 in case \( (r,d) = 1 \). By choosing the weights and quasi-parabolic structure correctly, we can fit \( \mathcal{M}_{r,d} \) and \( \mathcal{M}_{r,d-1} \) into a chain diagram of maps as follows. Let \( D = p \) and \( m = (1, \ldots, 1) \), and choose weights \( \xi = (\xi_1, \ldots, \xi_r) \) with \( \xi_1 + \ldots + \xi_r < \varepsilon(r,d)/2 \). Suppose that \( \xi_1 < \eta < \xi_2 \) and set \( E_\ast \) to be the parabolic bundle \( E_\ast \) shifted by \( \eta \). Notice that \( E_\ast \) has degree \( d - 1 \), multiplicities \( m' = (1, \ldots, 1) \), and weights \( \xi' = (\xi_2 - \eta, \ldots, \xi_r - \eta, 1 - \eta + \xi_r) \). If \( \beta \in V_{\xi'} \) is generic with \( \beta_1 + \ldots + \beta_r < \varepsilon(r,d)/2 \), then we can connect \( \xi' \) to \( \beta \) in \( V_{\xi'} \) by a line passing through a finite number of hyperplanes \( H_{\xi''}, \ldots, H_{\xi''} \), all of the form to which Theorem 4.1 applies. Choose weights \( \xi'' \) in the intermediate chambers and \( \xi'' \in H_{\xi'} \) for \( i = 1, \ldots, n \) with \( \xi'' = \beta' \). Applying Theorem 4.1 each time we cross a hyperplane, we get the diagram shown in Figure 3, where, by Proposition

![Figure 3](image-url)
5.3, the vertical maps $\psi$ and $\psi'$ have fibers the (full) flag variety over $\mathcal{M}_{r,d}$ and $\mathcal{M}_{r,d-1}$, respectively. By Theorem 4.2, $\psi$ is a fibration which is locally trivial in the Zariski topology provided that $(r,d) = 1$, and the same follows for $\psi'$ if $(r,d-1) = 1$.

The second application of shifting is to extend the results of [8] to a case which is natural from the point of view of representations of Fuchsian groups but less natural from the point of view of parabolic bundles. Assume for simplicity that $\mu(E_\alpha) = 0$ and $D = p$. Thus, $\deg E = -k$ for some $0 \leq k < r$, and the relevant weight space is

$$W_k = \{ (x_1, \ldots, x_r) \in \Delta^r \mid x_1 + \ldots + x_r = k \}.$$ 

Denote by $\bar{W}_k$ the closure of $W_k$ in $\mathbb{R}^r$. Consider the union $\bar{W} = \bigcup_{k=0}^{r-1} \bar{W}_k$, where we identify

$$\partial_0 W_k = \{ \gamma \in W_k \mid \gamma_1 = 0 \}$$

with its companion set

$$\partial_1 W_{k+1} = \{ \gamma \in W_{k+1} \mid \gamma_r = 1 \}$$

via the identification

$$\partial_0 W_k \ni \gamma = (0, \gamma_2, \ldots, \gamma_r) \sim (\gamma_2, \ldots, \gamma_r, 1) = \gamma \in \partial_1 W_{k+1}. \quad (10)$$

One can think of this set $\bar{W}$ as the space of all weights modulo shifting (because every bundle can be shifted so that $\mu(E_\alpha) = 0$), which in this case is just the quotient $\mathcal{U}(r)/Ad$ and which can be naturally identified with the standard $(r-1)$-simplex. From this point of view $\partial_0 W_k$ is an interior hyperplane of $\bar{W}$ because it satisfies condition (3).

However, Theorem 4.1 does not obviously carry over to this case because points in $W_k$ and $W_{k+1}$ are weights on parabolic bundles of different degrees. Given a parabolic bundle of degree $-k$, what is needed is a canonical procedure to construct a parabolic bundle of degree $-(k+1)$. This is precisely what is provided by the shifting operation. Thought of in terms of $\bar{W}$, the following theorem extends Theorem 4.1 to the case where $H_1 = \partial_0 W$.

We use the notation $\mathcal{M}_=(k,m)$ for the moduli space when $E_\alpha$ has degree $-k$, multiplicities $m$, and weights $\alpha$. The assumption that $D = p$ is not necessary for the following theorem, we only need that $\mu(E_\alpha) = 0$.

**Theorem 5.4.** Suppose that $\gamma \in \partial_0 W_k \cap V_m$ lies on no hyperplane other than $\partial_0 W_k$ and that $\alpha \in W_k \cap V_m$ is a generic weight near $\gamma$. This implies that $m_1 = 1$. Choose $\eta \in \mathbb{R}$ with $0 = \gamma_1(p) < \eta < \gamma_h(p)$ and define $\tilde{\gamma} \in \partial_0 W_k$ as in (10). Let $E'_\alpha$ be $E_\alpha$ shifted by $\eta$ at $p$, as in Remark 5.2, and denote the multiplicities of $E'_\alpha$ by $m'$. Set $k' = -\deg E' = k + 1$. Let $\beta \in W_k \cap V_{m'}$ be generic near $\tilde{\gamma}$. Then there are projective algebraic maps $\phi_\alpha$ and $\phi_\beta$ (see Figure 4) satisfying the conclusion of Theorem 4.1.

![Figure 4](image)

**Proof.** By the choice of $\alpha$, $\beta$ and $\eta$, we see that $\alpha_{m_1} < \eta < \alpha_{m_1+1}$, $\eta < \beta_1$ and $\eta < \gamma_{m_1+1}$. Consequently, the shifting operation defines the following isomorphisms:
where \( \alpha', \beta', \gamma' \in V_m \) are defined from \( \alpha, \beta, \gamma \) as in (9). The hypothesis that \( \gamma \) lies on no other hyperplanes implies that \( \gamma' \) lies on a wall of incidence number one. Now Theorem 4.1 applies to the shifted moduli spaces to prove the theorem. One can calculate \( e_\alpha \) and \( e_\beta \) by applying formulas (5) and (6) to \( \alpha', \beta' \) and \( \gamma' \).

**Remark 5.5.** Theorem 5.4 solves the problem mentioned at the end of [8] and extends the wall-crossing formula for knot invariants introduced in [7].

### 6. Rationality of moduli spaces of parabolic bundles

Let \( L \) be a holomorphic line bundle over a curve \( X \) of genus \( g \geq 2 \). Denote

(i) by \( \mathcal{M}_{s,L} \) the moduli space of semistable bundles \( E \) of rank \( r \) with \( \det E = L \);

(ii) by \( \mathcal{M}_{s,L} \) the moduli space of parabolic bundles \( E_\alpha \), with weights \( \alpha \) and \( \det E = L \).

The main results of §4 hold for the moduli spaces with fixed determinant with no essential difference. In view of Theorem 4.2, the goal is therefore to prove rationality with the coarsest possible choice of flag structure. At one extreme, we have the trivial flag, whose moduli space is exactly \( \mathcal{M}_{s,L} \). Proposition 2 of [13] implies that \( \mathcal{M}_{s,L} \) is rational if \( \deg L \equiv \pm 1 \mod (r) \), and then Theorem 4.2 and Proposition 4.3 imply that \( \mathcal{M}_{s,L} \) is also rational for any \( \alpha \in V_m \) provided that \( \deg L \equiv \pm 1 \mod (r) \).

**Theorem 6.1.** If \( m(p) = (1, \ldots, 1) \) for some \( p \in D \), then \( \mathcal{M}_{s,L} \) is rational for all \( \alpha \in V_m \).

**Proof.** By applying Propositions 4.3 and 3.2, we can arrange that the hypotheses of Theorem 4.2 are satisfied. This theorem then allows us to reduce to the case \( D = p \) by forgetting all the other flag structures. If \( E_\alpha \) denotes the bundle obtained by shifting \( E_\alpha \) by some \( \eta \) with \( \alpha < \eta < \alpha_2 \), then \( \det E' = L' = L(-p) \). It follows that shifting by \( \eta \) defines an isomorphism from \( \mathcal{M}_{s,L} \) to \( \mathcal{M}_{s,L} \). Repeated application of shifting puts us in the case \( \deg L \equiv 1 \mod (r) \), and then Newstead’s theorem and Theorem 4.2 imply that \( \mathcal{M}_{s,L} \) is rational.

The above argument works in slightly more generality. We can always shift our bundle to be any of the \( \delta_i \) appearing in the filtration (7) and illustrated in Figure 2. Thus, whenever one of these terms in the filtration is of a degree to which Newstead’s theorem applies, the corresponding moduli space of parabolic bundles is rational.

The next theorem is a considerable strengthening of the previous one.

**Theorem 6.2.** If \( m_i(p) = 1 \) for some \( p \in D \) and some \( 1 \leq i \leq \kappa_p \), then \( \mathcal{M}_{s,L} \) is rational for all \( \alpha \in V_m \).

Before delving into the proof of this theorem, we mention some interesting consequences. Recall first the following definition.

**Definition 6.3.** A variety \( V \) is stably rational of level \( k \) if \( V \times \mathbb{P}^k \) is rational. The level is the smallest integer \( k \) with this property.
The following result, with a weaker bound on the level, was proved in [2].

**Corollary 6.4.** For \((r, d) = 1\), \(\mathcal{M}_{g,L}\) is stably rational with level \(k \leq r - 1\).

**Proof.** Theorem 6.2 implies that \(\mathcal{M}_{g,L}\) is rational, where \(m(p) = (r - 1, 1)\), and Theorem 4.2 shows that \(\mathcal{M}_{g,L}\) is birational to \(\mathcal{M}_{g,L} \times \mathbb{P}^{r-1}\), which proves the corollary.

We now apply this last result to Conjecture 1.1.

**Corollary 6.5.** Suppose that \((r, d) = 1\). By tensoring with a line bundle, we can assume that \(0 < d < r\). If either \((g, d) = 1\) or \((g, r - d) = 1\), then \(\mathcal{M}_{g,L}\) is rational.

**Proof.** Suppose first that \((g, r-d) = 1\). Let \(L\) be a line bundle of degree \(r(g-1)+d\). Then Newstead's construction applies and proves that \(\mathcal{M}_{g,L}\) is birational to \(\mathcal{M}_{d-1,L} \times \mathbb{P}^{2}\), where \(\chi = (g-1)(r^2 - (r-d)^2)\). But Corollary 6.4 implies that \(\mathcal{M}_{g,L} \times \mathbb{P}^{2}\) is stably rational with level \(k \leq r - d - 1 \leq \chi\); hence \(\mathcal{M}_{g,L}\) is rational.

The case \((g, d) = 1\) follows by the same argument after applying duality, which interchanges \((r, d)\) and \((r, r-d)\).

**Remark 6.6.** Conjecture 1.1 was previously known [13] in a number of cases, including

(i) \(d \equiv \pm 1 \mod (r)\);
(ii) \((r, d) = 1\) and \(g\) a prime power;
(iii) \((r, d) = 1\) and the two smallest distinct primes factors of \(g\) have sum greater than \(r\).

Newstead's method applies more generally, and the 'up-and-down' argument of Ballico also proves rationality sometimes. Corollary 6.5 proves rationality more generally. The simplest case where it applies and previous methods do not is \(g = 6, r = 7, d \equiv 2 \mod (7)\). In fact, if \((r, d) = 1\), it is easy to list those \(g\) for which the conjecture remains open. For example, if \(r = 110\) and \(d = 43\), then Corollary 6.5 applies as long as \(g\) is not a multiple of \(d \cdot (r-d) = 43 \cdot 67 = 2881\). The simplest case where no known method proves rationality is \(g = 6, r = 5, d \equiv 2 \mod (5)\).

**Proof of Theorem 6.2.** Set \(d = \deg L\). The theorem is clearly true for \(r = 1\) and follows from Theorem 6.1 for \(r = 2\), so assume that \(r > 2\). Notice that by tensoring with a line bundle, we can suppose that

\[ r(g-1) < d \leq rg. \]

By Theorem 4.2, we can again assume that \(D = p\), and by shifting and another application of Theorem 4.2, if necessary, we can arrange it so that \(m(p) = (r-1, 1)\). Write

\[ \alpha = \alpha(p) = (\alpha_1, \ldots, \alpha_1, \alpha_2). \]

Proposition 3.2. implies that \(V_{\alpha}\) contains a generic weight and that \(\mathcal{M}_{\alpha,L}\) parameterizes a universal family \(\mathcal{M}_g^\alpha\). By Proposition 4.3, the birational type of \(\mathcal{M}_{g,L}\) is independent
of choice of compatible weights, so we can assume that the weights are small enough to satisfy the hypothesis of Proposition 5.3 (this comes up at various technical points in the argument, for example the proof of Claim 6.7).

Consider the following two cases.

Case 1: \( d = rg \). Choose \( \eta \) with \( \alpha < \eta < \alpha_2 \), and let \( E'_* = E_*[\eta]_* \). Denote the weights of \( E'_* \) by \( \alpha' \) as in (9). If \( \det E = L \), then \( \det E' = L' = L(-r-1-p) \) has degree \( d' = d - (r-1) \). Since \( d' \equiv 1 \mod (r) \), [13, Proposition 2] implies that \( \mathcal{M}_{L'} \) is rational, and Theorem 4.2 then implies that \( \mathcal{M}'_{L'} \) is also rational. Rationality of \( \mathcal{M}_{L} \) now follows from the isomorphism of the moduli spaces \( \mathcal{M}_{L} \cong \mathcal{M}'_{L} \) defined by shifting by \( \eta \).

Case 2: \( r(g-1) < d < rg \). The idea is to use induction to construct a non-empty, Zariski-open subset \( \mathcal{M} \) of affine space of dimension \( r^2-1)(g-1)+r-1 = \dim \mathcal{M}_{L} \) and a family of stable parabolic bundles \( \mathcal{U}_* \) parameterized by \( \mathcal{M} \) with \( \det \mathcal{U}_* = L \) for all \( \xi \in \mathcal{M} \). The universal property of \( \mathcal{U}_* \) then gives a map \( \psi_{\xi*}: \mathcal{M} \rightarrow \mathcal{M}_{L*} \). If, in addition, we have \( \mathcal{M}_{L*} \cong \mathcal{M}_{\xi*} \Rightarrow \xi_1 = \xi_2 \), then \( \psi_{\xi*} \) is injective and the rationality of \( \mathcal{M}_{L*} \) follows from that of \( \mathcal{M} \) and the dimension condition.

Set \( r' = rg - d \), \( r'' = r - r' \) and

\[ r' = \left( \begin{array}{c} r-1 \\ \alpha' = (\alpha_1, \ldots, \alpha_r) \end{array} \right) \]

Assume that both \( \alpha \) and \( \alpha' \) are generic. Let \( \mathcal{U}_* \) be the universal family parameterized by \( \mathcal{M}_{L*} \) and let \( I_* = \mathcal{E}_*[\alpha_1]_* \) be the trivial parabolic line bundle with weight \( \alpha_1 \). If \( e' = [E'_*] \in \mathcal{M}_{L*} \), then because \( E'_* \otimes I_* \) is a stable parabolic bundle of negative parabolic degree, \( h^0(E'_* \otimes I_*) = 0 \)

is independent of \( e' \). Since \( \mathcal{U}_* \cong E'_* \), it follows that

\[ (R^1 \pi_{L*})((\mathcal{U}_*^*)^* \otimes \pi_*^* (I_*)) \]

is locally free. The associated vector bundle \( V \xrightarrow{\pi} \mathcal{M}_{L*} \) has rank \( n \) and fiber over \( e' \) naturally isomorphic to \( H^0(\mathcal{U}_*^*) \otimes I_* \).

Let \( \mathcal{U}_* = (\pi \times 1)_* (\mathcal{U}_*^*) \) be the pullback family and \( \mathcal{F}_* = \pi_*^* (I_*^*) \) be the trivial family, where \( \pi': V^\oplus r' \rightarrow \mathcal{M}_{L*} \). There is an extension

\[ 0 \rightarrow \mathcal{F}_* \rightarrow \mathcal{U}_* \rightarrow \mathcal{U}_* ightarrow 0 \]

of families over \( V^\oplus r' \times X \), such that, for \( \xi \in V^\oplus r' \), \( \mathcal{U}_* \) is the parabolic bundle \( E_*^\xi \) described as the short exact sequence

\[ 0 \rightarrow I_*^\oplus r' \rightarrow E_*^\xi \rightarrow E_*^\xi \rightarrow 0 \]

corresponding to the extension class \( \xi \in H^1(E_*^\xi \otimes I_*^\oplus r') \).

Using the stability of \( E_*^\xi \) and the triviality of \( I_*^\oplus r' \), it follows that

\[ \text{Aut}(E_*^\xi) \times \text{Aut}(I_*^\oplus r') \cong \mathbb{C}^* \times \text{GL}(r^*, \mathbb{C}). \]
This group acts naturally as fiber-preserving maps on the bundle $V^{\otimes r'}$ since

$$V^{\otimes r'} \cong H^1(E_5^{\otimes} \otimes I_5^{\otimes r'}) = H^1(E_5^{\otimes} \otimes I_a^{\otimes r'})$$

and two extension classes $\xi_1$ and $\xi_2$ in the same orbit have associated bundles $E^{\xi_1}$ and $E^{\xi_2}$ which are isomorphic. We can ignore the $C^*$ action here because $(z, 1) \cdot \xi = (z, z) \cdot \xi$ for $z \in C^*$ and $\xi \in V^{\otimes r}$.

Using the inductive hypothesis and the local triviality of $V$, we can choose a non-empty Zariski-open subset $\mathcal{M}$ of $\mathcal{M}_{a, l}$ isomorphic to a Zariski-open subset of affine space of dimension $(r^2 - 1)(g - 1) + r' - 1$ such that $V|_M \cong \mathcal{M} \times H^1(E_5^{\otimes} \otimes I_a^{\otimes r'})$ ($E_5^{\otimes r'}$ is fixed). Lemma 2 of [13] applies here and produces a Zariski-open subspace $\mathcal{M} \times W$ of $V^{\otimes r'}|_M$ invariant under the group action, and an affine subspace $A \subset W$ so that every orbit in $W$ intersects $A$ precisely once. In fact, $A$ can be chosen as a Zariski-open subset of the Grassmannian $G(r', n)$. In any case, it should be clear that $A$ has dimension $r'(n - r')$. Using (11) and the fact that $r' + r'' = r$, we see that $\mathcal{M} \times A$ is a Zariski-open subset of affine space of dimension

$$\dim \mathcal{M} \times A = (r^2 - 1)(g - 1) + r' - 1 + r''(n - r'') = (r^2 - 1)(g - 1) + r' - 1 + r''((2r' + r'')(g - 1) + 1) = (r^2 - 1)(g - 1) + r - 1.$$

Let $\mathcal{M}$ be the subset of $V^{\otimes r'}$ defined by

$$\mathcal{M} = \{ \xi \in \mathcal{M} \times A \mid H^1(\mathcal{M}, \mathcal{U}_a^{\xi}) = 0 \}$$

and consider the bundle $\mathcal{M}_a$ restricted to $\mathcal{M}$, which we continue to denote $\mathcal{M}_a$. For $\xi \in V^{\otimes r'}$, let $E_5^{\xi} = \mathcal{M}_a^{\xi}$. Clearly, $E_5^{\xi}$ is a parabolic bundle with weights $\alpha$ and determinant $L$; thus $\mathcal{M}$ parameterizes a family of parabolic bundles. By the upper semi-continuity theorem, $\mathcal{M}$ is Zariski-open in $\mathcal{M} \times A$.

We claim that $\mathcal{M}$ is non-empty. Fix $e' = [E_5^{\xi}] \in \mathcal{M}$ and consider the set

$$N = \{ \xi \in H^1(E_5^{\otimes} \otimes I_5^{\otimes r'}) \mid h^1(E_5^{\xi}) = 0 \}.$$

If $N \cap A = \emptyset$, then $\mathcal{M}$ is non-empty. Clearly, $N$ is invariant under the action of $GL(r', \mathbb{C})$, so it is enough to show that $N \cap W \neq \emptyset$. There is a natural map

$$\delta: H^1(E_5^{\otimes} \otimes I_5^{\otimes r'}) \times H^0(E_5^{\xi}) \longrightarrow H^1(I_5^{\otimes r'})$$

with $\delta_1 = \delta(\xi, \cdot): H^0(E_5^{\xi}) \longrightarrow H^1(I_5^{\otimes r'})$ the coboundary map of the long exact sequence in homology of (13). Now $H^0(E_5^{\xi}) = H^0(E')$, and since $x_1 + (r' - 1)x_2 < (r, d)/2$, by Proposition 5.3, $E'$ is semistable as a non-parabolic bundle. Serre duality implies that $h^0(E') = h^0(E' \otimes K)$, and we compute that

$$\deg(E' \otimes K) = -d + r'(2g - 2) = (r' - r')(g - 1) - r'' < r'(g - 1).$$

Thus $h^1(E') = 0$ generically, and Riemann–Roch implies that $h^0(E_5^{\xi}) = r'g$. Because $h^1(I_5^{\otimes r'}) = r'g$, we see that

$$\xi \in N \iff h^1(E_5^{\xi}) = 0 \iff \delta_1 \text{ is an isomorphism.}$$

But $\delta$ is obviously onto and $\dim(\ker \delta) = r'n$. The set $N$ has complement

$$N^c = \{ \xi \in H^1(E_5^{\otimes} \otimes I_5^{\otimes r'}) \mid \delta_1(\xi, s) = 0 \text{ for some } 0 \neq s \in H^0(I_5^{\otimes r'}) \}.$$
But \( \delta(\xi, s) = 0 \Rightarrow \delta(\xi, zs) = 0 \) for all \( z \in \mathbb{C} \), which shows that the map \( \ker \delta \rightarrow N^\vee \) has fibers of dimension \( \geq 1 \). Hence \( \dim N^\vee \leq \dim(\ker \delta) - 1 < r'n \), and we see that \( N \) is non-empty and Zariski-open. Thus \( N \cap W \neq \emptyset \) and it follows that \( \mathcal{M} \) is non-empty.

We now prove that \( \mathcal{M} \) parameterizes a family of stable parabolic bundles, using again the inequality \( (r - 1) \alpha + \alpha_2 < \alpha(r, d)/2 \) and Proposition 5.3.

**Claim 6.7.** (i) \( E^*_\alpha \) is stable for all \( \xi \in \mathcal{M} \).
(ii) \( E^*_\alpha \cong E^*_\beta \Leftrightarrow \text{GL}(r^\vee, \mathbb{C}) \cdot \xi_1 = \text{GL}(r^\vee, \mathbb{C}) \cdot \xi_2 \) for all \( \xi_1, \xi_2 \in \mathcal{M} \).

**Proof.** (i) Suppose to the contrary that \( E^*_\alpha \) is not parabolic stable for some \( \xi \in \mathcal{M} \). Let \( G_\alpha \) be a rank \( s \) parabolic subbundle of \( E^*_\alpha \) with \( \mu(G_\alpha) \geq \mu(E^*_\alpha) \).

Then \( \mu(G) \geq \mu(E^*_\alpha) \), since otherwise

\[
\mu(G_\alpha) < \mu(G) + \alpha(d, r)/2 < \mu(E^*_\alpha) < \mu(E^*_\alpha).
\]

As in the argument of Newstead [12, Lemma 6], the map \( G \rightarrow G^1 \rightarrow G^2 \rightarrow E^\prime \) and the arguments there give the following inequalities:

\[
\text{deg}(G^2) \geq \text{deg}(G) \geq \frac{sd}{r} \quad (14)
\]

\[
\text{rank}(G^2) \leq \text{rank}(G) - h^0(G) \leq \frac{sr'}{r}. \quad (15)
\]

These imply that \( \mu(G^2) - \mu(E^\prime) \geq 0 \). But \( E^*_\alpha \) is parabolic stable, so by Proposition 5.3, \( E^\prime \) is semistable and \( \mu(G^2) = \mu(E^\prime) \). Thus, we must have equalities in (14) and (15), in particular \( \mu(G) = \mu(E^\prime) \). But since \( \mu(G_\alpha) \geq \mu(E^*_\alpha) \), we see that \( G_\alpha \) must inherit the weight \( \alpha_2 \), which implies that \( G_\alpha^2 \) also inherits \( \alpha_2 \), and it now follows that

\[
\mu(G_\alpha^2) - \mu(E^\prime) = \frac{(s_2 - 1) \alpha_1 + \alpha_2}{s_2} - \frac{(r' - 1) \alpha_1 + \alpha_2}{r'} > 0
\]

where \( s_2 = \text{rank } G^2 < r' \). This contradicts the parabolic stability of \( E^*_\alpha \) and completes the proof of part (i).

(ii) Since \( \Rightarrow \) is true independent of the vanishing of \( H^1 \), we only prove \( \Rightarrow \). Suppose that \( E^*_\beta \cong E^*_\alpha \) and set \( \pi_\gamma(E^*_\gamma) = \xi_\gamma = [E^*_\gamma] \in \mathcal{M}_{\alpha, \nu} \). Notice that \( h^1(E^*_\beta) = 0 \), and so \( h^0(E^*_\gamma) = \chi(E^*_\gamma) = r'' \). It follows that every holomorphic section of \( E^*_\alpha \) has its image contained in \( f^\circ \nu \). Hence any isomorphism \( \psi: E^*_\gamma \rightarrow E^*_\alpha \) defines a commutative diagram (see Figure 5) where both \( \psi' \) and \( \psi'' \) are isomorphisms, and so \( \xi_\gamma = (\psi' \circ \psi'') \cdot \xi_1 \).

\[
\begin{array}{ccc}
0 & \rightarrow & f^\circ \nu \rightarrow E^*_\gamma \rightarrow E^*_\alpha \rightarrow 0 \\
| \psi'' & \downarrow \psi & \downarrow \psi' \\
0 & \rightarrow & f^\circ \nu \rightarrow E^*_\gamma \rightarrow E^*_\alpha \rightarrow 0
\end{array}
\]

**Figure 5.**

Part (i) of Claim 6.7 and the universal property of \( \mathcal{M}_\alpha \) give a map \( \mathcal{M} \rightarrow \mathcal{M}_{\alpha, \nu} \), which is injective by part (ii). Since \( \mathcal{M} \) is non-empty, \( \dim \mathcal{M} = \dim \mathcal{M}_{\alpha, \nu} \), so the rationality of \( \mathcal{M}_{\alpha, \nu} \) follows from that of \( \mathcal{M} \). This concludes the proof in Case 2. \( \square \)
Remark 6.8. Notice that one can deduce rationality of \( \mathcal{M}_{\alpha},L \) whenever \((r,d) = 1\) from Corollary 6.4 because Proposition 4.3 allows us to choose generic weights \( \alpha \) compatible with \( V_e \), but arbitrarily small and Theorem 4.2 implies that \( \mathcal{M}_{\alpha},L \) is birational to \( \mathcal{M}_{r,v} \times \mathbb{P}^N \). But flag varieties are themselves rational and an easy dimension count yields \( N \geq r-1 \).

More generally, whenever one of the bundles \( E \), appearing in the filtration (7) has degree coprime to the rank, rationality of \( \mathcal{M}_{\alpha},L \), follows by shifting and applying the same argument. Such parabolic bundles necessarily satisfy the hypothesis of Proposition 3.2, but the converse is not true. In fact, a direct proof of rationality in the case of generic weights fails. To see why, consider the case \( D = p \). By tensoring with a line bundle and shifting, we can assume that

\[
r(g-1) < d \leq r(g-1) + m_1.
\]

Hence, the subbundle split off in the induction is again a sum of parabolic line bundles with the same weights. The difficulty is in proving that the quotient \( E' \) has generic weights \( \alpha' \). Proposition 3.2 implies that \( E' \) admits a generic weight if and only if the elements of the set \( \{d,m_i(p)\} \) have greatest common divisor equal to one. The statement

\[
(d,m_1,\ldots,m_n) = 1 \Rightarrow (d,m'_1,\ldots,m'_n) = 1
\]

which is what is needed here, is false (notice that \( m'_1 = m_1 - d + r(g-1) \) and \( m'_n = m_n \) otherwise).

Acknowledgements. Both authors would like to express their gratitude to the Max-Planck-Institut für Mathematik for providing financial support. The first author is also grateful to the Institut des Hautes Études Scientifique for partial support. We would also like to thank I. Dolgachev and L. Göttche for helpful discussions.

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