On ERT and MERT-Rings

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ABSTRACT

The main purpose of this paper is to study ERT and MERT rings, in order to study the connection between such rings and II-regular rings.

Keyword: MERT-Rings, ERT-Rings and \(\pi\)-Regular Rings

MERT و ERT حول الحلقات من النوع

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ملخص

الهدف الرئيس من البحث هو دراسة الحلقات من النوع ERT و MERT. لكي ندرس العلاقة بين هذه الحلقات والحلقات المنتظمة من النوع-II

الكلمات المفتاحية: حلقات من النوع ERT, حلقات من النوع MERT, حلقات من النوع \(\pi\)-من النظام.
1- Introduction:
Throughout this paper, $R$ denotes an associative ring with identity, and all modules are unitary right $R$-module. Recall that:
1- An ideal $I$ of the ring $R$ is essential if $I$ has a non-zero intersection with every non-zero ideal of $R$; 2- A ring $R$ is said to be \Pi-regular if for every $a$ in $R$ there exist a positive integer $n$ and $b$ in $R$ such that $a^n = a^n b a^n$ 3- A right $R$-module $M$ is said to be GP- injective if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and any right $R$-homomorphism of $a^n R$ into $M$ extends to one of $R$ into $M$. 4- For any element $a$ in $R$, $r(a), I(a)$ denote the right annihilator of $a$ and the left annihilator of $a$, respectively.

2- ERT-RINGS:
Following [3J, a ring $R$ is said to be ERT-ring if every essential right ideal of $R$ is a two-sided ideal.

Definition 2-1:
A ring $R$ is said to be right weakly regular if for all $a$ in $R$, there exists $b$ in $RaR$ such that $a = ab$, or equivalently every right ideal of $R$ is idempotent.

We begin this section with the following main result:

Theorem 2.2:
If $R$ is ERT-ring with every essential right ideal is idempotent, then $R$ is weakly regular.

Proof:
For any $a \in R$, if $RaR$ not essential, then there exists an ideal $I$, such that $K = RaR \oplus I$ is essential then $K = K^2$.

In order to prove that $R$ is weakly regular, we need to prove $RaR = (RaR)^2$.

For $a \in K$, we have $a \in K^2$, that is $a \in (RaR \oplus I)^2$;

Thus $a = (rar' + i)(ras' + i')$ for some $r, r', s, s' \in R$ and $i, i' \in I$. 29
This implies that \( a = (rar' + i)sas' + (rar' + i) i' \)
\[ \begin{align*}
= rar'sas' + isas' + (rar^1 + i) i' 
\end{align*} \]
but \( isas' \in I \cap RaR = 0 \), also we have \( (rar' + i)i' \in RaR \cap I = 0 \). Therefore \( a = (rar')(sas') \in (RaR)^2 \), this implies that \( RaR \subseteq (RaR)^2 \). Thus \( RaR = (RaR)^2 \), this proves that \( R \) is weakly regular.

Following [2], the singular submodule of \( R \) is
\[ Y(R) = \{ y \in R, \text{r}(y) \text{ is essential right ideal of R} \} \].

**Theorem 2.3:**
Let \( R \) be a semi-prime ERT right GP-injective ring. Then \( R \) is a right non singular.

**Proof:**
Let \( E \) be an essential right ideal of \( R \). Then \( E \) is a two-sided ideal, and hence \( l(E) \) is a two-sided ideal of \( R \).
Now \( (l(E) \cap E)^2 \subseteq (E)E = 0 \).
Since \( R \) is semi-prime, then \( l(E) \cap E = 0 \), whence \( l(E) = 0 \). This proves that \( R \) is right non singular.

**3- MERT-RINGS:**
Following [3], a ring \( R \) is said to be MERT-ring if every maximal essential right ideal of \( R \) is a two-sided ideal.

**Theorem 3.1:**
Let \( R \) be an MERT-ring, if for any maximal right ideal \( A \) of \( R \), and for any \( b \in M \), \( bR/bM \) is GP-injective, then \( R \) is strongly Pi-regular ring.

**Proof:**
Let \( b \) be a non-zero element in \( R \), we claim that \( b^nr + r(b^n) = R \).
If \( b^nr + r(b^n) \neq R \), let \( M \) be a maximal right ideal containing \( b^nr + r(b^n) \). Then \( M \) is essential right ideal of \( R \).
If $bR = bM$, then $b = bc$, for some $c$ in $M$, this implies $(1-c) \in r(b) \subset r(b^n) \subset M$, therefore $I \in M$, this contradics $M \not= R$.

Now, since $R/ M \cong bR/bM$. Then $R / M$ is GP-injective.

Now, define $f : b^n R \rightarrow R / M$ by $f(b^n r) = r + M$, note that $f$ is a well-defined $R$-homomorphism.

Since $R/M$ is GP-injective, then there exists $c \in R$, such that: $1+M=f(b^n)=cb^n+M$ and so $(1-c b^n) \in M$, since $b^n \in M$, and $R$ is MERT-ring, this implies that $M$ is a two-sided ideal, and hence $\in c b^n \in M$.

Thus $I \in M$, a contradiction.

Therefore $b^n R + r(b^n) = R$.

In particular $l=b^n u+v; v \in r(b^n), u \in R$.

Thus $b^n = b^{2n} u$ and therefore $R$ is strongly $\prod$-regular ring.

**Theorem 3.2:**

If $R$ is MERT-ring with every simple singular right ideal is GP-injective, then $Y(R) = 0$.

**Proof:**

If $Y(R) \not= 0$, by Lemma (7) of [6], there exists $0 \not= y \in Y(R)$ with $y^2 = 0$. Let $L$ be a maximal right ideal of $R$, set $L = y R + r(y)$, we claim that $L$ is essential right ideal of $R$. Suppose this is not true, then there exists a non-zero ideal $T$ of $R$ such that $L \cap T = (0)$. Then $yRT \subseteq LT \subseteq L \cap T = 0$ implies $T \subseteq r(y) \subseteq L$, so $L \cap T = (0)$. This contradiction proves that $L$ is an essential right ideal, that is $R/L$ is simple singular and hence $R/L$ is GP-injective.

Now; Let $f: yR \rightarrow R/L$ be defined by $f(yr) = r + L$, then $f$ is a well-defined $R$-homomorphism.

Since $R/L$ is GP-injective, so $\exists c \in R$, such that $l+L=f(y)=cy+L$.

Hence $l+L=cy+L$, implies that $1-cy \in L$. 

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Since $R$ is MERT, then $\text{eye } L$ and thus $I \in L$, a contradiction. Therefore $Y(R)=[0]$.

Following [1], a ring $R$ is zero insertive (briefly ZI) if for $a, b \in R, ab=0$ implies $aRb=0$.

**Theorem 3.3:**

Let $R$ be a ZT ring. If every simple singular rights-modules is GP-injective which is left self-injective, then $R$ is strongly H-regular ring.

**Proof:**

Since $R$ is simple singular GP-injective, then $R$ is semi-prime, by Lemma (4) of [5].

Thus for any left ideal $I$, $L(I) \cap I = 0$.

Since $R$ is simple singular GP-injective and ZI, then $R$ is reduced and hence $r(a)=l(a)$ for any element $a$ in $R$.

Thus $l(r(a)) \cap l(a)=l(l(a)) \cap l(a)=0$.

Since $R$ is left self-injective ring, then $aR$ is a right annihilator, by Proposition (4) of [4].

Since $r(a) \subseteq r(a^n)$, then $a^nR = r(a^n)$.

Now, since $R= r(l(r(a))) + r(l(l(a)))$ then we have $R = r(l(r(a^n))) + r(l(l(a^n))) = r(a^n) + a^nR$

In particular, for some $b$ in $R$, and $d$ in $r(a^n)$.

Thus $a^n = a^n b$.

Therefore $R$ is strongly $\Pi$-regular.
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