Non-Linear Massive Gravity with Additional Primary Constraint and Absence of Ghosts

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ABSTRACT: We complete the Hamiltonian analysis of specific model of non-linear massive gravity that was started in arXiv:1112.5267. We identify the primary constraint and corresponding secondary constraint. We show that they are the second class constraints and hence they lead to the elimination of the additional scalar mode. We also find that the remaining constraints are the first class constraints with the structure that corresponds to the manifestly diffeomorphism invariant theory. Finally we determine the number of physical degrees of freedom and we show that it corresponds to the number of physical modes of massive gravity.

KEYWORDS: Massive Gravity, Hamiltonian Formalism
1. Introduction and Summary

Recently de Rham and Gabadadze proposed in [1] an interesting formulation of the massive gravity which is ghost free in the decoupling limit. Then it was shown in [2] that this action that was written in the perturbative form can be resumed into fully non-linear action. The general analysis of the constraints of given theory has been performed in [3]. It was argued there that it is possible to perform such a redefinition of the shift function so that the resulting theory still contains the Hamiltonian constraint. Then it was argued that the presence of this constraint allows to eliminate the scalar mode and hence the resulting theory is the ghost free massive gravity. However this analysis was questioned in [31] where it was argued that it is possible that this constraint is the second class constraint so that the phase space of given theory would be odd dimensional. On the other hand in the paper [32] a very nice analysis of the Hamiltonian formulation of the most general gauge fixed non-linear massive gravity actions was performed with an important conclusion that the Hamiltonian constraints has zero Poisson brackets. Then the requirement of the preservation of this constraint during the time evolution of the system implies an additional constraint. As a result given theory has the right number of constraints for the construction of non-linear massive gravity without additional scalar mode.

The Hamiltonian analysis of the manifestly diffeomorphism invariant non-linear massive gravity with St"uckelberg fields was performed in [29] where corresponding Hamiltonian was found. Then using the observation firstly published in [40] it was shown that this theory possesses one primary constraint. Unfortunately the presence of this constraint makes the calculation of the Poisson brackets between constraints very difficult due to the absence of the inverse of the matrix $V^{AB} = g^{ij} \partial_i \phi^A \partial_j \phi^B$ and we were not able to perform this analysis for the case of four dimensional non-linear massive gravity. On the other hand such an analysis was performed for the case of two dimensional non-linear massive gravity.

\footnote{For related works, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].}

\footnote{Alternative arguments for the existence of an additional constraints were given in [33] even if the Hamiltonian analysis was not complete and the minimal non-linear massive gravity action was considered only.
with conclusions that there are no physical degrees of freedom left with agreement with [30, 44].

In [41] we analyzed the model of the non-linear massive gravity action introduced in [33] written in the Stückelberg formalism. This analysis was then reconsidered in [40] with conclusion that this theory is free from the ghosts.

The goal of this paper is to complete the analysis of the non-linear massive gravity action presented in [29]. We find the Hamiltonian for given theory and identify primary constraints. Then we rewrite the Hamiltonian to such a form where the scalar part of the Hamiltonian constraint will be proportional to the trace of the square root of the regular matrix. Then it would be possible to use the standard formula for the variation of the trace of the square root of the regular matrix and calculate corresponding Poisson brackets. Then we can analyze the requirement of the preservation of the primary constraints during the time evolution of the system and hence identify corresponding secondary constraints. Finally we will check the stability of all constraints during the time evolution of the system. We find that the Hamiltonian and diffeomorphism constraints are still the first class constraints and obey the basic principles of geometrodynamics [36, 37, 38, 39]. On the other hand we show that the additional primary constraint together with corresponding secondary constraint are the second class constraints and that these constrains could eliminate one additional degree of freedom so that the number of physical degrees of freedom correspond to the case of the massive gravity. In other words our results are in full agreement with the conclusion presented in [40]. However we mean that result derived in this paper is non-trivial and should be considered as an independent check of the absence of the ghosts in given theory due to the fact that we analyze theory without additional auxiliary fields so that the Hamiltonian analysis presented here is different from the analysis presented in [40].

We should also stress that our treatment has its own limitation due to the fact that we restrict to the case of one specific model of non-linear massive gravity action. It turns out that the extension of given analysis to the more general form of the non-linear massive gravity actions is very difficult due to the complicated relation between canonical momenta and time derivatives of the scalar fields. Unfortunately we were not able to find an inverse mapping that would allow us to write the Hamiltonian as a function canonical variables in these cases. It would be very interesting to find such Hamiltonian formulation and corresponding primary constraints between Stückelberg fields even for the most general form of the non-linear massive gravity action. We hope to return to this problem in future.

The structure of given note is as follows. In the next section (2) we review some basic facts about the non-linear massive gravity action in the formulation presented in [29]. Then in section (3) which is the main body of this paper we perform corresponding Hamiltonian analysis. We also identify primary and the secondary constraints and determine the number of the physical degrees of freedom.
2. Non-Linear Massive Gravity

Our goal is to study non-linear massive gravity action in the following form
\[ S = M_p^2 \int d^3x dt N \sqrt{g} \left[ K_{ij} \mathcal{G}^{ijkl} K_{kl} + (3)R - m^2 \text{Tr}_A \sqrt{\mathbf{A}} \right] , \] (2.1)
where we used 3 + 1 notation \[34]\ and write the four dimensional metric components as
\[
\hat{g}_{00} = -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}^{ij} = g^{ij} - \frac{N_i N^j}{N^2}.
\] (2.2)

Note that \((3)R\) is three-dimensional spatial curvature, \(K_{ij}\) is extrinsic curvature defined as
\[
K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i),
\] (2.3)
where \(\nabla_i\) is covariant derivative built from the metric components \(g_{ij}\) and \(\mathcal{G}^{ijkl}\) is de Witt metric defined as
\[
\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl}
\] (2.4)
with inverse
\[
\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{2}g_{ij} g_{kl}, \quad \mathcal{G}_{ijkl} \mathcal{G}_{klmn} = \frac{1}{2}(\delta^m_i \delta^n_j + \delta^m_j \delta^n_i). (2.5)
\]
Finally note that the matrix \(A^A_B\) is defined as
\[
A^A_B = -\nabla_n \phi^A \nabla_n \phi_B + g^{ij} \partial_i \phi^A \partial_j \phi_B, \quad \nabla_n \phi^A = \frac{1}{N}(\partial_t \phi^A - N^i \partial_i \phi^A),
\] (2.6)
and the trace defined in (2.1) is the trace over Lorentz indices \(A, B, C, \ldots = 0, 1, 2, 3\).

We see that the action contains the potential term which is the square root of the matrix which can be defined as
\[
(\sqrt{\mathbf{A}})^B_A (\sqrt{\mathbf{A}})^C_B = A^A_C .
\] (2.7)
For further purposes it is crucial to presume that \(A\) is regular matrix. Then when we perform the variation of this expression and multiply by \((\sqrt{\mathbf{A}})^{-1}\) from the right we obtain
\[
\delta(\sqrt{\mathbf{A}})^A_B + (\sqrt{\mathbf{A}})^A_C \delta(\sqrt{\mathbf{A}})^C_D ((\sqrt{\mathbf{A}})^{-1})^D_B = \delta(\mathbf{A})^A_C ((\sqrt{\mathbf{A}})^{-1})^C_B .
\] (2.8)
Taking the trace the equation (2.8) we immediately obtain
\[
\delta \text{Tr}_L \sqrt{\mathbf{A}} = \frac{1}{2} \delta A^A_B \left( (\sqrt{\mathbf{A}})^{-1} \right)^B_A .
\] (2.12)
This is the key formula that is used in the calculation of the Poisson brackets as we will see in the next section.

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\(^3\)We use notations introduced in the paper \[2\].

\(^4\)For review, see \[3\].

\(^5\)Note also that due to the matrix nature of objects \(A\) and \(B\) the following relation is not valid
\[
\sqrt{AB} = \sqrt{A} \sqrt{B}
\] (2.9)
3. Hamiltonian Analysis

In this section we perform the Hamiltonian analysis of the action (2.1). For the General Relativity part of the action the procedure is standard. Explicitly, the momenta conjugate to $N, N^i$ are the primary constraints of the theory

$$\pi_N(x) \approx 0, \quad \pi_i(x) \approx 0$$

while the Hamiltonian takes the form

$$H^{GR} = \int d^3x (N\mathcal{H}^{GR}_T + N^i\mathcal{H}^{GR}_i),$$

$$\mathcal{H}^{GR}_T = \frac{1}{\sqrt{g}M_p^2} \pi^{ij}g_{ijkl}\pi^{kl} - M_p^2 \sqrt{g}^{(3)}R,$$

$$\mathcal{H}^{GR}_i = -2g_{ik}\nabla_l \pi^{kl},$$

where $\pi^{ij}$ are momenta conjugate to $g^{ij}$ with following non-zero Poisson brackets

$$\{g_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2} \left( \delta^{k}_l \delta^i_j + \delta^{l}_i \delta^k_j \right) \delta(x - y).$$

Note also that $\pi_N, \pi_i$ have following Poisson brackets with $N, N^i$

$$\{N(x), \pi_N(y)\} = \delta(x - y), \quad \{N^i(x), \pi_j(y)\} = \delta^i_j \delta(x - y).$$

Now we proceed to the Hamiltonian analysis of the scalar field part of the action. Note that in 3 + 1 formalism the matrix $A^A_B$ takes the form

$$A^A_B = -\nabla_n \phi^A \nabla_n \phi_B + g^{ij} \partial_i \phi^A \partial_j \phi_B \equiv K^A_B + V^A_B,$$

where

$$K^A_B = -\nabla_n \phi^A \nabla_n \phi_B, \quad K_{AB} = \eta_{AC} K^C_B = K_{BA},$$

$$V^A_B = g^{ij} \partial_i \phi^A \partial_j \phi_B, \quad V^{AB} = V^A_C \eta^{CB} = V^{BA}.$$  

Then the conjugate momenta $p_A$ are equal to

$$p_A = -\frac{M_p^2 m^2}{2} \sqrt{g} \frac{\delta A^C_B}{\delta \partial_l \phi^A} (A^{-1/2})^D_C =$$

$$= \frac{M_p^2 m^2}{2} \sqrt{g} (\nabla_n \phi_C (A^{-1/2})^C_A + \eta_{AC} (A^{-1/2})^C_B \nabla_n \phi_B), \quad A^{-1/2} = (\sqrt{A})^{-1}. $$

unless $A$ and $B$ commute. On the other hand since obviously $A$ and $A^{-1}$ commute the equation (2.9) gives

$$\sqrt{A} \sqrt{A^{-1}} = I$$

which implies following important relation

$$(\sqrt{A})^{-1} = \sqrt{A^{-1}}.$$
Note that using the symmetry of $A_{AB} = A_{BA}$ we can write (3.7) in simpler form

$$p_A = M_p^2 m^2 \sqrt{g} (A^{-1/2})_{AB} \nabla_n \phi^B .$$

Using this expression we derive following relation

$$\frac{1}{g M_p^4 m^4} p_{AB} = (A^{-1/2})_{AC} (\nabla_n \phi^C \nabla_n \phi^D) (A^{-1/2})_{DB} = (A^{-1/2})_{AC} (V^{CD} - A^{CD}) (A^{-1/2})_{DB}$$

(3.9)

which implies

$$\Pi_{AB} = (A^{-1/2})_{AC} V^{CD} (A^{-1/2})_{DB} ,$$

(3.10)

where we introduced the matrix $\Pi_{AB}$ defined as

$$\Pi_{AB} = \frac{1}{g m^4 M_p^4} p_{AB} + \eta_{AB} .$$

(3.11)

Note that when we multiply (3.10) by $V$ from the right we obtain (we use matrix notation)

$$\Pi V = (A^{-1/2} V) (A^{-1/2} V)$$

(3.12)

which implies

$$A^{-1/2} V = \sqrt{\Pi V} .$$

(3.13)

This relation will be important below. The crucial point for the Hamiltonian analysis of the non-linear massive gravity is the fact that $V^{AB}$ has the rank 3 as was firstly explicitly stressed in [40]. In fact, if we introduce the $4 \times 3$ matrix $W^A_i = \partial_i \phi^A$ and its transpose matrix $(W^T)_j^A = \partial_j \phi^A$ which is $3 \times 4$ matrix we can write

$$V^{AB} = W^A_i g^{ij} (W^T)_j^B .$$

(3.14)

Then since $W^A_i, g^{ij}$ have the rank 3 we obtain that $V^{AB}$ has the rank 3 as well. As a result $\det V = 0$. In other words $V$ is not invertible matrix.

With the help of these results it is easy to determine corresponding Hamiltonian

$$\mathcal{H}^{sc} = \partial_t \phi^A p_A - \mathcal{L}_{sc} = M_p^2 m^2 \sqrt{g} N V^{AB} (A^{-1/2})_{BA} + N^i p_A \partial_i \phi^A = N M_p^2 m^2 \sqrt{g} \text{Tr}_L \sqrt{\Pi V} + N^i p_A \partial_i \phi^A \equiv N \mathcal{H}^{sc}_T + N^i \mathcal{H}^{sc}_i$$

(3.15)

using (3.13) and using an obvious relation $\text{Tr}_L \sqrt{\Pi V} = \text{Tr}_L \sqrt{\Pi V}$. With the help of these results we find the Hamiltonian for the action (2.1) in the form

$$H = \int d^3 x (N \mathcal{H}_T + N^i \mathcal{H}_i + v^T \pi_i + v^N \pi_N + v_c C) ,$$

(3.16)

where

$$\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^{sc} \quad \mathcal{H}_i = \mathcal{H}_i^{GR} + \mathcal{H}_i^{sc} ,$$

(3.17)
and where $\pi_i \approx 0, \pi_N \approx 0$ are the primary constraints of the theory. Note also that the Hamiltonian (3.16) contains primary constraint $C$ whose explicit form follows from (3.10) when we calculate the determinant of the matrix $\Pi_{AB}$. Using

$$\det \Pi_{AB} = - \left( 1 + \frac{1}{gM_p^2m^4}p_A p_A \right)$$

(3.18)

and using (3.11) together with the fact that $\det V = 0$ we derive primary constraint $C$ in the form

$$C : 1 + \frac{1}{gM_p^2m^4}p_A p_A \approx 0 .$$

(3.19)

It is also important to stress that using the definition of $\Pi_{AB}$ and the existence of the constraint $C$ we obtain an important relation

$$p^A \Pi_{AB} = \left( \frac{1}{gM_p^2m^4}p_A + 1 \right) p_B = C p_B \approx 0 .$$

(3.20)

Now we analyze the requirement of the preservation of the primary constraints. As usual the requirement of the preservation of the primary constraints $\pi_N \approx 0, \pi_i \approx 0$ implies an existence of the secondary constraints

$$\mathcal{H}_T(x) \approx 0 , \quad \mathcal{H}_i(x) \approx 0 .$$

(3.21)

For further purposes we introduce the smeared form of these constraints (3.21)

$$T_T(N) = \int d^3x N \mathcal{H}_T , \quad T_S(N^i) = \int d^3x N^i \mathcal{H}_i .$$

(3.22)

It is not easy to determine the time evolution of the constraint $C$ due to the fact that $\Pi V$ is singular matrix. To proceed let us express the trace of the square root of the matrix as power series in the form

$$\text{Tr}_L \sqrt{\Pi V} = \sum_n c_n \text{Tr}_L (\Pi V)^n .$$

(3.23)

Now we can write

$$\text{Tr}_L \Pi V = \Pi^{AB} \partial_i \phi^B g^{ij} \partial_j \phi_A = \partial_j \phi_A \Pi^{AB} \partial_i \phi_B g^{ij} \equiv \tilde{\Pi}_j^i \equiv \text{Tr}_s \tilde{\Pi} ,$$

$$\text{Tr}_L \Pi V = (\partial_i \phi_A \Pi^{AB} \partial_j \phi_B g^{ij}) (\partial_k \phi_C \Pi^{CD} \partial_i \phi_D g^{li}) = \tilde{\Pi}_i^k \tilde{\Pi}_k^i \equiv \text{Tr}_s \tilde{\Pi}^2 ,$$

$$\vdots$$

(3.24)

where the trace $\text{Tr}_s$ is the trace over spatial indices $i, j, k \ldots = 1, 2, 3$. Then with the help of (3.24) it is easy to see that

$$\text{Tr}_L \sqrt{\Pi V} = \text{Tr}_s \sqrt{\Pi} .$$

(3.25)
Now $\tilde{\Pi}$ is $3 \times 3$ matrix with the rank equal to 3 which implies an existence of the inverse matrix $\tilde{\Pi}^{-1}$. As a result we can easily determine the variation of the trace of the square root of given matrix

$$\delta \text{Tr}_x \sqrt{\tilde{\Pi}} = \frac{1}{2} \text{Tr}_x \delta \tilde{\Pi} \sqrt{\tilde{\Pi}^{-1}}.$$ (3.26)

Then we can determine following Poisson brackets

$$\{ p_A(x), \text{Tr} \sqrt{\tilde{\Pi}}(y) \} = -\frac{1}{2} \frac{\delta \text{Tr}_x \sqrt{\tilde{\Pi}(y)}}{\delta \phi^A(x)} = -\frac{1}{2} \frac{\delta \tilde{\Pi}^i_j(y)}{\delta \phi^A(x)} \sqrt{\tilde{\Pi}^{-1}(y)}$$

$$= -\frac{1}{2} (\partial_y \delta(x-y) \eta_{AC} \Pi^{CD} \partial_y \phi_D g^{kj} + \partial_y \phi_C \Pi^{CD} \eta_{DA} \partial_y \delta(x-y) g^{kj}(y) \sqrt{\tilde{\Pi}^{-1}(y)})$$

$$\{ \phi^A(x), \text{Tr} \sqrt{\tilde{\Pi}}(y) \} = \frac{\delta \text{Tr}_x \sqrt{\tilde{\Pi}(y)}}{\delta p_A(x)} = \frac{1}{2} \frac{\delta \tilde{\Pi}^i_j(y)}{\delta p_A(x)} \sqrt{\tilde{\Pi}^{-1}(y)}$$

$$= \frac{1}{2g m^4 \tilde{\Pi}} (\partial_i \phi^A p_k \partial_k \phi^K + \partial_i \phi^K p_k \partial_k \phi^A) g^{kj}(\sqrt{\tilde{\Pi}^{-1}(y)}) \delta(x-y).$$ (3.27)

Using these results we find

$$\{ T_S(N^i), \mathcal{C}(x) \} = -N^i \partial_0 \mathcal{C}(x)$$ (3.28)

and also

$$\{ T_T(N), \mathcal{C}(x) \} = -\frac{1}{g m^2} p_A \left( \partial_i [N \sqrt{\eta_{AC} \Pi^{CD} \partial_k \phi_D g^{kj} \sqrt{\tilde{\Pi}^{-1}}} + \partial_k [N \partial_i (\phi^{CD} \eta_{DA} \partial_k \delta(x-y)) g^{kj}(\sqrt{\tilde{\Pi}^{-1}}))] \right)$$

$$= \frac{2N}{g m^2} p_A g_{ij} \sqrt{\Pi} \frac{N \sqrt{\eta_{AC} \Pi^{CD} \partial_k \phi_D g^{kj} \sqrt{\tilde{\Pi}^{-1}}}}{g m^2}$$

$$\{ H, \mathcal{C}(x) \} \approx \{ T_T(N), \mathcal{C}(x) \} = \frac{N}{g m^2} C^{II} \approx 0.$$ (3.29)

where we used (3.31) and where we defined $C^{II}$ as

$$C^{II} = p_A \partial_i (\Pi^{AB} \sqrt{\eta} \partial_j \phi_B) \left( \sqrt{\Pi^{-1}} \right)^{ji} - \frac{2}{g m^2} \sqrt{g} p_A g_{ij} \pi^{ji},$$ (3.30)

where we defined $\sqrt{\Pi^{-1}}^{ij} = \left( \sqrt{\Pi^{-1}} \right)^{ji} = \sqrt{\Pi^{-1}}^{ik} g^{kj}$. Now it is easy to see that the requirement of the preservation of the constraint $\mathcal{C}$ during the time evolution of the system implies following secondary constraint

$$\partial_t \mathcal{C} = \{ H, \mathcal{C} \} \approx \{ T_T(N), \mathcal{C} \} = \frac{N}{g m^2} C^{II} \approx 0.$$ (3.31)

In summary, the theory possesses following collection of the primary constraints $\pi_N \approx 0$, $\pi_i \approx 0$, $\mathcal{C} \approx 0$ and secondary constraints $\mathcal{H}_N \approx 0$, $\mathcal{H}_i \approx 0$, $\mathcal{C}^{II} \approx 0$. As a result the total Hamiltonian has the form

$$H_T = \int d^3x (N \mathcal{H}_N + N^i \mathcal{H}_i + v_N \pi_N + v_i \pi_i + v_C \mathcal{C} + \Gamma^i \mathcal{H}_i + \Gamma_C \mathcal{C}^{II}),$$ (3.32)
where \( v_N, v^i, u_C, \Gamma^i, \Gamma_C \) are corresponding Lagrange multipliers.

As the final step we have to analyze the preservation of all constraints. Note that in case of the General Relativity part of the constraints we have following Poisson brackets

\[
\{ \mathcal{H}^{GR}_T(x), \mathcal{H}^{GR}_T(y) \} = - \left[ \mathcal{H}^{GR}_T(x) \frac{\partial}{\partial x^i} \delta(x - y) - \mathcal{H}^{GR}_T(y) \frac{\partial}{\partial y^i} \delta(x - y) \right],
\]

\[
\{ \mathcal{H}^{GR}_T(x), \mathcal{H}^{GR}_T(y) \} = \mathcal{H}^{GR}_T(y) \frac{\partial}{\partial x^i} \delta(x - y),
\]

\[
\{ \mathcal{H}^{GR}_T(x), \mathcal{H}^{GR}_T(y) \} = \mathcal{H}^{GR}_T(x) \frac{\partial}{\partial y^i} \delta(x - y) \left[ \frac{\partial}{\partial x^i} \delta(x - y) - \mathcal{H}^{\text{GR}}(y) \frac{\partial}{\partial y^i} \delta(x - y) \right].
\]

(3.33)

The calculation of the Poisson brackets that contains scalar phase space degrees of freedom is more involved. However it is easy to find the Poisson bracket between generators of spatial diffeomorphisms

\[
\{ \mathcal{H}^{sc}_i(x), \mathcal{H}^{sc}_j(y) \} = \left[ \mathcal{H}^{sc}_j(x) \frac{\partial}{\partial x^i} \delta(x - y) - \mathcal{H}^{sc}_i(y) \frac{\partial}{\partial y^i} \delta(x - y) \right]
\]

(3.34)

that together with the Poisson bracket on the third line in (3.33) implies following form of Poisson bracket between smeared form of the diffeomorphism constraints

\[
\{ T_S(N^i), T_S(M^j) \} = T_S(N^i \partial_j M^i - M^i \partial_j N^i).
\]

(3.35)

It is also easy to see that

\[
\{ T_S(N^i), T_T(N) \} = T_T(\partial_k N N^k).
\]

(3.36)

Now we proceed to the calculation of the Poisson bracket \( \{ T_T(N), T_T(M) \} \). By definition we have

\[
\{ T_T^{sc}(N), T_T^{sc}(M) \} = \int d^3 x d^3 y N(x) \{ \mathcal{H}^{sc}_T(x), \mathcal{H}^{sc}_T(y) \} M(y) =
\]

\[
= -M_p^4 \int d^3 x d^3 y d^3 z N(x) M(y) \left( \sqrt{g(x)} \frac{\delta(\sqrt{\Pi})^i_j(x)}{\delta p_X(z)} \frac{\delta(\sqrt{\Pi})^j_i(y)}{\delta \phi^X(z)} \sqrt{g(y)} - \sqrt{g(y)} \frac{\delta(\sqrt{\Pi})^i_j(y)}{\delta p_X(z)} \frac{\delta(\sqrt{\Pi})^j_i(x)}{\delta \phi^X(z)} \sqrt{g(x)} \right)
\]

\[
= T_T^{sc}(\{ N \partial_j M - M \partial_j N \} g^{ji}),
\]

(3.37)

where we used (3.27). This result together with (3.33) implies \(^6\)

\[
\{ T_T(N), T_T(M) \} = T_S((N \partial_j M - M \partial_j N) g^{ji}).
\]

(3.38)

\(^6\)It is important to stress that \( \{ T_T^{GR}(N), T_T^{sc}(M) \} + \{ T_T^{sc}(M), T_T^{GR}(N) \} = 0 \) due to the fact that \( T_T^{sc} \) does not depend on the spatial derivatives of \( g_{ij} \).
Now we have to determine whether all constraints are preserved during the time evolution of the system. Let us now start with the primary constraints $\pi^N, \pi_i, C \approx 0$. The case of the constraints $\pi_N \approx 0, \pi_i \approx 0$ is trivial. Further, the requirement of the preservation of the constraint $C$ gives

$$\partial_t C(x) = \{H_T, C(x)\} \approx \int d^3y \{N(y)H_T(y), C(x)\} + \Gamma_C(y) \{C^{\prime\prime}(y), C(x)\} =$$

$$= \frac{2N}{M_p^2 m^2 g} C^{\prime\prime}(x) + \int d^3y \Gamma_C(y) \{C^{\prime\prime}(y), C(x)\} \approx$$

$$\approx \int d^3y \Gamma_C(y) \{C^{\prime\prime}(y), C(x)\} = 0$$

(3.39)

that has the solution $\Gamma_C = 0$ using the fact that

$$\{C^{\prime\prime}(x), C(y)\} = 2p_A(x)\partial_i\Pi^{AB}(x)\sqrt{g}(x) \left(\sqrt{\Pi^{-1}}\right)^{ji}(x)\partial_j(x - y)\frac{p_B(y)}{M_p^4 m^4 g(y)} + \ldots$$

(3.40)

does not vanish on the constraint surface. Note that $\ldots$ mean additional terms that arise from the explicit calculations of given Poisson brackets. In other words, $C$ and $C^{\prime\prime}$ are the second class constraints.

Now we come to the requirement of the preservation of the secondary constraints. Let us begin with the diffeomorphism constrains $\mathcal{H}_i$ or their smeared forms. Since $C^{\prime\prime} \approx 0$ is manifestly diffeomorphism invariant we have $\{T_S(N^i), C^{\prime\prime}\} \approx 0$ and also using (3.28) together with (3.35) and (3.36) we find that $\mathcal{H}_i$ is preserved during the time evolution of the system. In case of $\mathcal{H}_T$ we find that its time development is governed by the equation

$$\partial_t \mathcal{H}_T(x) = \{H_T, \mathcal{H}_T(x)\} \approx \int d^3y \{v_C(y), \mathcal{H}_T(x)\} + \Gamma_C \{C^{\prime\prime}(y), \mathcal{H}_T(x)\} =$$

$$= \int d^3y \left\{v_C(y), \mathcal{H}_T(x)\right\} = \frac{2v_C(x)}{M_p^2 m^2 g} C^{\prime\prime}(x) \approx 0$$

(3.41)

using (3.37) and also using the fact that $\Gamma_C = 0$. In other words $\mathcal{H}_T$ is also preserved during the time evolution of the system without any restriction on the lapse function $N$.

Finally the requirement of the preservation of the constraint $C^{\prime\prime}$ has the form

$$\partial_t C^{\prime\prime}(x) = \{H_T, C^{\prime\prime}(x)\} =$$

$$= \int d^3y \left\{\{N \mathcal{H}_T(y), C^{\prime\prime}(x)\} + v_C(y) \{C(y), C^{\prime\prime}(x)\}\right\} = 0$$

(3.42)

using the fact that $\Gamma_C = 0$. Then with the help of the equation (3.40) we can argue that this solution can be solved for $v_C$ at least in principle.

Let us outline our results. We have following first class constrains $\pi_N \approx 0, \pi_i \approx 0, \mathcal{H}_i \approx 0, \mathcal{H}_T \approx 0$ together with the second class constrains $C \approx 0, C^{\prime\prime} \approx 0$. Then we
also have 10 metric components $g_{ij}$, $N$, $N^i$ and corresponding conjugate momenta $\pi^{ij}$, $\pi_N$, $\pi_i$ and 4 scalar fields $\phi^A$ with conjugate momenta $p_A$. In general we have $D = 28$ phase space degrees of freedom. On the other hand we have $S = 2$ second class constraints and $F = 8$ first class constraints. As a result the number of physical degrees of freedom is equal to

$$N_{p.d} = (D - S - 2F) = 10 \quad (3.43)$$

which is the correct number of the physical degrees of freedom of the massive gravity. In other words $C$ and $C^{II}$ eliminate the ghost field and corresponding conjugate momenta at least in principle.

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