Decoupling Limits in M-Theory

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ABSTRACT

Limits of a system of $N$ D$n$-branes in which the bulk and string degrees of freedom decouple to leave a ‘matter’ theory are investigated and, for $n > 4$, either give a free theory or require taking $N \to \infty$. The decoupled matter theory is described at low energies by the $N \to \infty$ limit of $n + 1$ dimensional super Yang-Mills, and at high energies by a free type II string theory in a curved space-time. Metastable bound states of D6-branes with mass $M$ and D0-branes with mass $m$ are shown to have an energy proportional to $M^{1/3}m^{2/3}$ and decouple, whereas in matrix theory they only decouple in the large $N$ limit.
1. Introduction

The study of field theory limits of M-theory or string theory in the presence of branes has proved very fruitful. In particular, the limit used in matrix theory has been related to the infinite momentum frame or discrete light cone quantization of M-theory [1-17]. Similar limits have led to new non-trivial ‘matter’ theories in 5 + 1 dimensions [9-14] arising from limits of M-theory in which gravity decouples. However, the construction of matrix theories has proved problematic for M-theory compactified on an \( n \)-torus if \( n > 5 \) [16,17] (and perhaps for \( n = 5 \) [18]), and is related to the difficulty in constructing non-trivial theories in greater than 6 dimensions. Here we will consider a different limit from that considered in [1-17], which appears to give rise to a decoupled matter theory in 5+1 or 6+1 dimensions, but which does not seem to be directly related to any kind of light-cone quantization of M-theory. This limit involves taking \( N \rightarrow \infty \) in the underlying \( U(N) \) gauge theory and is closely related to the limits considered by Maldacena et al [19,20].

Consider weakly-coupled type II string theory in the presence of \( N \) coincident Dn-branes. (We will mostly restrict ourselves to \( n < 7 \), due to the problems with having arbitrary numbers of Dn-branes, for \( n \geq 7 \).) This is described by \( n + 1 \) dimensional super Yang-Mills with \( U(N) \) gauge symmetry and coupling constant \( g_{YM} \), coupled to 10-dimensional supergravity with gravitational coupling \( \kappa \), together with stringy corrections characterised by the string mass scale \( m_S = 1/\sqrt{\alpha'} \). The coupling of the bulk degrees of freedom to the fields on the brane are also characterised by \( \kappa \). At strong coupling, the type IIA theory (with \( n \) even) is described by M-theory and the type IIB theory (with \( n \) odd) by the S-dual type IIB theory. The field theory limits that it is natural to consider [2] are ones in which \( \kappa \rightarrow 0 \) so that gravity and the bulk degrees of freedom decouple and \( m_S \rightarrow \infty \) so that the string excitations decouple, while \( g_{YM} \) remains fixed, in order to obtain a non-trivial theory with gauge interactions. Such a limit is possible for \( n < 5 \), and is central to the construction of the matrix theory for M-theory compactified on \( T^n \). (In this construction, the type II theory is taken to be compactified on
the T-dual torus, and the D\textsubscript{n}-branes are wrapped on this torus.) However, the coupling constants are not independent and such a limit is not possible for \( n \geq 5 \).

In particular, for \( n = 5 \), the string mass, after the S-duality transformation, is \( \hat{m}_S = g_{YM}^{-1} \), so that taking \( \hat{m}_S \to \infty \) and decoupling the string modes is possible only if \( g_{YM} \to 0 \). Instead, in [11,13], \( \hat{m}_S = g_{YM}^{-1} \) was kept fixed, resulting in a 5+1 dimensional string theory [11,13]. This is the world-volume theory of \( N \) NS 5-branes and the bulk degrees of freedom do not completely decouple from the throat region of the 5-brane background [18]. For \( n = 6 \), the 11-dimensional Planck mass is \( m_p = g_{YM}^{-2/3} \), so that if gravity is to decouple, again it is necessary to take \( g_{YM} \to 0 \). In [15,16,17], \( m_p = g_{YM}^{-2/3} \) was kept finite, so that gravity does not decouple and the result is M-theory on an ALE \( A_{N-1} \) singularity. (For \( n = 7 \), one would have \( g_{YM}^{-2/3} = m_p^{(10)} \), where \( m_p^{(10)} \) is the 10-dimensional Planck mass, and gravity would again not decouple.)

In this limit for \( n = 6 \), the radius \( R \) of the 11’th dimension in the M-theory description diverges so that the theory decompactifies, and at the same time the BPS states of the bulk theory include D0 branes, which have masses of order \( 1/R \), and so become massless. This decompactification and the presence of an infinite number of massless states are further obstacles to the formulation of a matrix theory for M-theory on \( T^6 \) [16,17].

In this paper, an alternative limit will be considered, in which for \( n = 5,6 \) \( g_{YM} \to 0 \) so that bulk and string modes decouple. If \( N \) is kept fixed, the result is a free theory, which is consistent but not interesting. In order to obtain a non-trivial theory, \( N \) will be taken to infinity at the same time as \( g_{YM} \to 0 \), while keeping the effective coupling

\[
g_N = g_{YM} \sqrt{N}
\]

fixed, giving the ’t Hooft limit [24] of the super Yang-Mills theory, in which only the planar Feynman diagrams survive. In this limit, for large but finite \( N \), the bulk and/or string modes do not decouple and are presumably crucial to the consistency of the theory, but in the infinite \( N \) limit, the bulk and string modes all
appear to decouple, and presumably leave a consistent matter theory in 5+1 or 6+1 dimensions. In addition, $R$ will be kept fixed in the limit, so that the 11'th dimension does not decompactify and there are no extra massless states. Whereas in the usual matrix limit, the characteristic scale of the D0-branes on the dual torus, which is the 11-dimensional Planck scale of the T-dual theory, plays a key role, the limit considered here for $n = 6$ corresponds to keeping the states with energies characterised by the energy scale of the D0-branes on $T^6$ that are becoming light. This can be thought of as performing a Weyl rescaling at the same time as $m_S$ is scaled, in such a way that $R$ is kept constant and the mass of the D0-branes is kept fixed.

Limits in which $R$ is not kept fixed, but scales with $N$ so that $R \to 0$ as $N \to \infty$ will also be considered. For $n = 4$, the D4-brane is described by an M5-brane wrapped on a circle of radius $R$. The world-volume theory of $N$ M5-branes is the $U(N)$ self-dual tensor multiplet theory in 5+1 dimensions, which dimensionally reduces to 4+1 super Yang-Mills with coupling constant $g_{YM}^2 = R$. Taking the 't Hooft limit of the theory corresponds to taking $R \propto 1/N$, so that as $N \to \infty$, the extra dimension disappears, leaving the large $N$ limit of 4+1 super Yang-Mills. For $n = 6$, taking $R \to 0$ completely decouples the D0-branes from the bulk theory, leaving 6+1 super Yang-Mills.

The large $N$ behaviour has recently been considered in [19,20] in the context of the usual limit [2]. The limit considered here is closely related, as will be described in section 4. For energy scales $U$ that are small compared to the scale set by the effective coupling $g_N^2 = g_{YM}^2 N$, the theory is described by the large $N$ limit of super Yang-Mills, while for energies large compared to the effective coupling, the description is in terms of string theory in a particular background, as in [19,20].
2. The Limits of Matrix Theory

Consider the system of coincident $N$ Dn-branes in IIA string theory (for $n$ even) or in type IIB string theory ($n$ odd), with string tension $m_S^2$ and string coupling $g_S$. At low energies and weak coupling, the bulk degrees of freedom are described by type II supergravity and the brane dynamics by super Yang-Mills (with 16 supersymmetries) in $n + 1$ dimensions, with gauge group $U(N)$. The kinetic terms include

$$m_S^8 \int d^{10} x \sqrt{-g} e^{-2\phi} R + m_S^{n-3} \int d^{n+1} \sigma \sqrt{-g} e^{-\phi} F^2$$

giving the following effective gravitational coupling $\kappa$ and 10-dimensional Planck scale $m_p^{(10)}$:

$$m_p^{(10)} \equiv \kappa^{-1/4} = m_s g_S^{-1/4} \quad (2.1)$$

while the Yang-Mills coupling $g_{YM}$ is

$$g_{YM}^2 = \frac{g_S}{m_S^{n-3}} \quad (2.2)$$

Here

$$g_S = e^{\phi_\infty} \quad (2.3)$$

where $\phi_\infty$ is the asymptotic value of the dilaton $\phi$.

For $n$ even, the type IIA theory arises as M-theory compactified on $S^1$ with radius $R$, with

$$R = \frac{g_S}{m_S}, \quad m_p = \frac{m_S}{g_S^{1/3}} \quad (2.4)$$

where $m_p$ is the 11-dimensional Planck mass, and the type IIA description is appropriate only when $g_S$ is small. For $n$ odd, an S-duality transformation takes the
type IIB theory to a dual type IIB theory with string mass scale $\hat{m}_S$ and string coupling $\hat{g}_S$ given by

\begin{align}
\hat{m}_S &= m_S g_S^{-1/2} \\
\hat{g}_S &= g_S^{-1}
\end{align}

so that the strong coupling limit $g_S \to \infty$ is described in the dual theory as the weak coupling limit $\hat{g}_S \to 0$.

The BPS spectrum is most easily considered if the theory is compactified on an $n$-torus $T^n$ with radii $R_i$, with the $N$ D$n$-branes wrapped on the torus. Consider the BPS states of the bulk type II theory compactified on $T^n$. There are momentum states with mass of order $1/R_i$ and string winding states with mass of order $m_S^2 R_i$. There are Dirichlet $p$-branes wrapped on a $p$-cycle for $p \leq n$. Such a $p$-brane, wrapped on the first $p$ cycles of $T^n$ has mass of order:

\begin{equation}
g_S^{-1} m_S^{p+1} \left( \prod_{i=1}^{p} R_i \right)
\end{equation}

For $n \geq 5$ there are NS 5-branes wrapped on five of the circular dimensions, with mass

\begin{equation}
g_S^{-2} m_S^6 \left( \prod_{i=1}^{5} R_i \right)
\end{equation}

In addition, for $n \geq 6$ there are states arising from wrapping Kaluza-Klein monopoles of the type II theory, which should be included as they are T-dual to the NS 5-branes. These depend on 6 radii $R_i$, $i = 1, ..., 6$, one of which ($R_6$, say) is distinguished as the fibre of the Kaluza-Klein monopole solution. These have mass

\begin{equation}
g_S^{-2} m_S^8 \left( \prod_{i=1}^{5} R_i \right) R_6^2
\end{equation}

Some of these BPS states form bound states with the D$n$-branes, in which case the binding energy can alter the mass formulae; this will be discussed in section 6.
If the type II theory is put on an $n$-torus with radii $R_i$ and the $N$ D$n$-branes are wrapped on $T^n$, then a T-duality transformation takes this to $N$ D0-branes in the type IIA theory with string coupling $\tilde{g}_S$ and string mass $\tilde{m}_S$ on the dual torus $\tilde{T}^n$ with radii $\tilde{R}_i$. The parameters $\tilde{g}_S$, $\tilde{m}_S$ and $\tilde{R}_i$ are given by

$$\tilde{m}_S = m_S, \quad \tilde{R}_i = m_S^2 R_i^{-1}$$

$$\tilde{g}_S = g_S / \prod_{i=1}^n (m_S R_i)$$

This is in turn related to an $\tilde{M}$-theory compactified on a space-like circle, and boosting and rescaling and taking a certain limit gives M-theory compactified on a null circle (regarded as the limit of a boosted space-like circle) and hence the matrix theory representation of M-theory [17].

A natural decoupling limit to consider is one in which $m_S \to \infty$ and $g_{YM}$ is kept fixed, and, if the theory is on a torus $T^n$, the radii $R_i$ are also kept fixed. In the T-dual picture, this corresponds to taking the limit $\tilde{g}_S \to 0$ of the T-dual theory, keeping fixed

$$\tilde{m}_p \tilde{R}_i = \tilde{g}_S^{-1/3} \tilde{m}_S \tilde{R}_i, \quad \tilde{m}_p \tilde{R} = \tilde{m}_S \tilde{g}_S^{-1/3}$$

and this is the limit used in the matrix theory construction [1-17]. We will refer here to the limit $m_S \to \infty$ with $g_{YM}$ fixed as the matrix limit; it gives a theory related to $n + 1$ dimensional super Yang-Mills (on $T^n \times \mathbb{R}$). Then

$$g_S = \tilde{g}_S^{-1} = g_{YM}^2 m_S^{n-3}, \quad m_p^{(10)} = g_{YM}^{-1/2} m_S^{(7-n)/4},$$

$$\tilde{m}_s = g_{YM}^{-1} m_S^{(5-n)/2}, \quad R = g_{YM}^2 m_S^{-4}, \quad m_p = g_{YM}^{-2/3} m_S^{(6-n)/3}$$

If the D$n$-branes are wrapped on a torus and the radii $R_i$ are kept fixed in the limit, the BPS states surviving in the limit are the momentum modes and wrapped D$p$-branes with $p \leq n - 4$, since the D$p$-brane mass (2.6) can be rewritten as

$$g_{YM}^{-2} m_S^{p-n+4} \prod_i^p R_i$$

For $n < 3$, $g_S \to 0$, so that the D-brane picture with weakly coupled strings is
appropriate, $m_S \rightarrow \infty$ so that the massive string modes decouple and $m_p^{(10)} \rightarrow \infty$, so that supergravity and the bulk degrees of freedom decouple, leaving $n + 1$ dimensional super Yang-Mills. For $n = 3$, $g_S = g_{YM}^2$ remains finite, but the string and bulk degrees of freedom decouple and the limiting theory is four-dimensional super Yang-Mills, with $N = 4$ superconformal symmetry. For $n > 3$, $g_S \rightarrow \infty$ and we need to use the dual strong coupling theory.

For even $n > 3$, we use M-theory with Planck mass $m_p \sim m_S^{(6-n)/3}$ and the radius $R$ of the 11’th dimension $R \sim m_S^{n-4}$, so that for $n = 4$, $m_p \rightarrow \infty$ and gravity and the bulk degrees of freedom decouple, together with the membrane excitations that reduce to string excitations for small $R$, while $R$ remains finite. The M-theory description of the D4-brane is as an M5-brane wrapped around the 11’th dimension, so that the D4-brane wrapped on $T^4$ corresponds to an M5-brane wrapped on $T^5$. The limiting theory for $n = 4$ is the M5-brane world-volume theory, which is a generalisation of the (2,0) supersymmetric self-dual tensor theory with $U(N)$ gauge symmetry [10-14]. The D0-branes have mass proportional to $1/R = g_{YM}^{-2}$ and arise from solitons in the super Yang-Mills theory obtained by lifting instantons in 4 Euclidean dimensions to 4 + 1 dimensions. They are interpreted as Kaluza-Klein modes from the point of view of the 5+1 dimensional theory. The D0 branes with mass $1/R$ combine with the momentum modes with mass $1/R_i$ (all of which form bound states at threshold with the D4-brane) to give the momentum modes on $T^5$ in the M-theory description, transforming as a 5 of $SL(5,\mathbb{Z})$, which is the U-duality group for M-theory compactified on $T^4$ [25].

For $n = 6$, the Planck scale $m_p$ remains finite, but $R \rightarrow \infty$. A single D6-brane corresponds to a Kaluza-Klein monopole of M-theory, and the solution with $N$ D6-branes lifts to a multi-Kaluza-Klein monopole solution of M-theory, in the singular limit in which $N$ monopoles coincide. The matrix limit gives the $R \rightarrow \infty$ limit of this, which is an $A_{N-1}$ ALE singularity, so we obtain M-theory on an ALE

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* The representation of $SL(5,\mathbb{R})$ on $R^5$ induces a representation of $SL(5,\mathbb{Z})$ on the 5-dimensional lattice $\mathbb{Z}^5$, which we will refer to as the 5 representation of $SL(5,\mathbb{Z})$, and similarly for other U-duality groups.
singularity with gravity and the bulk degrees of freedom not decoupling [16,17]. The BPS states considered above whose mass remains finite and non-zero in this limit are the 6 momentum modes, the 15 wrapped D2-branes and the 6 wrapped NS 5-branes; each forms a bound state at threshold with the D6-brane. These combine to form a 27 of $E_6(Z)$, which is the U-duality group for M-theory compactified on $T^6$. In addition, the bulk D0-branes are becoming massless in the limit, leading to an infinite number of massless states; these correspond to gravitons (and their superpartners) moving in the 11’th dimension, which is decompactifying in the limit [16]. In section 6, the metastable bound state formed between the D0 and D6-branes will be considered; these have finite energy for fixed $N$ and so play a role in the world-volume theory, but decouple in the large $N$ limit of [1].

For $n = 5$, the $N$ D5-branes become $N$ Neveu-Schwarz 5-branes in the S-dual type IIB theory, which is free in the limit as $\hat{g}_S \to 0$, with $m_p^{(10)} \to \infty$ and gravity and the bulk decoupling, but the string scale $\hat{m}_S$ remaining finite. For $n = 5$, the marginally bound wrapped D1-strings in the D5-brane have finite mass and become fundamental strings moving in the NS 5-brane after the S-duality; the 5 momentum modes and 5 string winding modes combine to form a 10 of $SO(5,5; Z)$. This limit has been argued to give rise to a 5+1 dimensional ‘non-critical’ string theory with string tension $\hat{m}_S^2$ and $(1,1)$ supersymmetry, whose zero slope limit is $(1,1)$ super Yang-Mills [13,14]. For the NS 5-brane of the type IIA theory, it was argued in [18] that in regimes in which string theory in a near-extremal 5-brane space-time is a good description, there is Hawking radiation with temperature $T_H \propto \hat{m}_S$, so that it appears that there is not a complete decoupling from the bulk in the throat region of the NS 5-brane space-time. The NS 5-brane of the type IIB theory is the same supergravity background as for the type IIA case (the RR gauge fields all vanish) and the arguments of [18] apply in this case also, so that there appears not to be a complete decoupling from the bulk in the throat region.
3. Other Limits

For the usual matrix limit with $n = 6$, there are two problems: $m_p$ remains finite, so that gravity doesn’t decouple, and $R$ becomes infinite, so that the bulk D0-branes become massless. For $n = 5$, $\hat{m}_S$ remains finite, so that one obtains a string theory instead of a field theory, and it appears that there is not a complete decoupling from the bulk, due to Hawking radiation in the throat region. This leads naturally to the search for more general limits in which $m_p \to \infty$ while $R$ remains finite for $n = 6$, and $\hat{m}_S \to \infty$ for $n = 5$. Such limits could lead to new decoupled ‘matter’ theories in 6+1 or 5+1 dimensions, but are unlikely to be related to light cone quantizations of M-theory.

In the T-dual description in terms of $\tilde{D}0$-branes on $\tilde{T}^n$, one can consider replacing the usual limit $\tilde{g}_S \to 0$ keeping (2.10) fixed with the limit $\tilde{g}_S \to 0$ keeping $g_{S}^{-A}\hat{m}_S\tilde{R}_i, \quad \hat{m}_S\tilde{g}_S^B$ fixed for some constants $A, B$ (generalising the usual limit (2.10) in which $A = B = 1/3$). The new limits that will be proposed correspond to choosing

\[
A = B = \frac{1}{n-1}
\]

(3.2)

for $n > 1$, so that for $n = 4$, the limit considered here is the same as the usual one. Whereas the usual limit [1,2] focuses on scales of order the Planck scale $\hat{m}_p$, the new limit for $n = 6$ focuses on the scale associated with the branes that are becoming massless (the D0-branes on $T^6$, or the $\tilde{D}6$-branes wrapped on $\tilde{T}^6$ in the T-dual description). This can be thought of as performing a $\tilde{g}_S$-dependent Weyl-rescaling at the same time as scaling $\tilde{g}_S$ in such a way that $R$ and the mass of the bulk D0-branes is kept fixed. It will be more convenient to present the results in terms of the theory on $T^n$, however.

For $n = 6$, in order for the bulk modes to decouple, we clearly need a limit in which $m_p \to \infty$, but for $n = 6$, $g_{YM}^2 = m_p^{-3}$, so that this would require $g_{YM} \to 0$. 

10
Similarly, for a decoupled theory with \( n = 5 \), it seems a limit in which \( \hat{m}_S \to \infty \) is needed, but for \( n = 5 \), \( g_{YM} = \hat{m}_S^{-1} \), so that again this would require \( g_{YM} \to 0 \). In either case, such a limit leads to decoupling of bulk modes, but if \( N \) is kept fixed, the result is a free super Yang-Mills theory. If, on the other hand, \( N \) is taken to infinity at the same time, while keeping \( g^2_N \equiv g^2_{YM} N \) fixed, then the result is the 't Hooft limit of \( n + 1 \) dimensional super Yang-Mills, for \( n = 5, 6 \). If in addition a rescaling is chosen so that \( R \) is kept fixed in the limit, then the bulk D0-branes will remain of finite non-zero mass in the limit. We will subsequently consider more general limits in which \( R \) is not kept fixed, and which give rise for \( n = 4 \) to the 't Hooft limit of \( 4+1 \) super Yang-Mills.

We now propose a limit satisfying both of these criteria, and which generalises to all \( n \), although the meaning is unclear for \( n > 6 \). In this limit we take \( m_S \to \infty \) and keep fixed \( R \equiv g_S/m_S \) both for even and for odd \( n \); for even \( n \), \( R \) is the radius of the 11’th dimension. Then

\[
\begin{align*}
g_S &= 1/\hat{g}_S = m_S R, \\
m_p^{(10)} &= m_S^{3/4} R^{-1/4}, \\
\hat{m}_S &= m_S^{1/2} R^{-1/2}, \\
g^2_{YM} &= R m_S^{4-n}\end{align*}
\]  

(3.3)

Thus in the limit \( m_S \to \infty \), the mass scales \( m_S, m_p^{(10)}, m_p, \hat{m}_s \) all diverge, so that bulk modes and string modes decouple for all \( n \). The string coupling \( g_S \to \infty \) in this limit, so that the dual strong coupling description is appropriate. For the type IIA theory (\( n \) even), this gives M-theory with \( m_p \to \infty \) and a finite size \( R \) of the compact 11’th dimension, while for type IIB (\( n \) odd) this gives the weak coupling limit \( \hat{g}_S \to 0 \) of the S-dual type IIB theory, with \( \hat{m}_S, m_p^{(10)} \to \infty \).

When on a torus \( T^n \), whose radii \( R_i \) are kept fixed in the limit, most of the bulk BPS states considered above decouple. The masses of the winding strings, the wrapped Dp-branes for \( p > 0 \) and the wrapped NS 5-branes all diverge and these
BPS states are absent in the limit. The momentum modes and the D0 branes (which are present for $n$ even) have finite mass, and the mass of the D0-branes is now finite and given by $1/R$ (for all even $n$), whereas in the matrix limit the D0-branes became massless for $n = 6$.

The Yang-Mills coupling constant diverges in the limit for $n < 4$, is finite for $n = 4$ and tends to zero for $n > 4$. For $n = 4$, the limit is exactly the same as the usual matrix limit. For $n = 3$, the type IIB S-duality leads to a super Yang-Mills S-duality transforming $g_{YM} \rightarrow \tilde{g}_{YM} \equiv 1/g_{YM}$, and $\tilde{g}_{YM} \rightarrow 0$ in the limit. For $n = 3$ and $n > 4$, taking this limit with $N$ fixed gives a free theory, while taking the limit with $N \rightarrow \infty$ and $g_N = g_{YM}^2 N$ fixed gives a non-trivial limit.

For $n > 4$, keeping $g_N$ and $R$ fixed implies

$$m_S = \alpha N \frac{1}{n-4}, \quad \alpha = \left( \frac{g_N^2}{R} \right) \frac{1}{n-4}$$  \hspace{1cm} (3.5)$$

and for $n = 3$, keeping $\tilde{g}_N^2 = \tilde{g}_{YM}^2 N = N/g_{YM}^2$ fixed implies

$$m_S = \beta N, \quad \beta = \frac{1}{R \tilde{g}_N^2}$$  \hspace{1cm} (3.6)$$

giving the rate at which $m_S$ diverges with $N$.

For all $n$ these limits are ones in which in which bulk and string modes decouple, and which has a description in terms of super Yang-Mills in $n + 1$ dimensions. If $N$ is kept fixed, the result is a free theory, while if $N$ is taken to $\infty$ for $n \geq 3$, we obtain an 't Hooft limit. For $n = 3, 5$, we obtain in this way the large $N$ limit of $n + 1$ super Yang-Mills. For $n = 6$, we obtain $6 + 1$ super Yang-Mills, either as a free theory or a large $N$ limit, together with bulk D0 branes with mass of order $1/R$. If we now take $R \rightarrow 0$, these D0-branes decouple leaving pure super Yang-Mills on the brane. For finite $R$, there could be extra degrees of freedom corresponding to the D0-branes in the 6+1 dimensional theory, but these could be consistently decoupled by taking $R \rightarrow 0$; this will be discussed further in sections 5.6.
For $n = 4$ with fixed $N$, we obtain the usual limit consisting of the $(2,0)$ tensor theory in $5 + 1$ dimensions with $U(N)$ symmetry, on a circle of radius $R$. The dimensional reduction to $4 + 1$ dimensions gives super Yang-Mills with $g_{YM}^2 = R$; this is non-renormalizable, and the extra degrees of freedom needed at short distances are provided by the Kaluza-Klein modes. If we take $N \to \infty$ in this theory while keeping $g_{YM}^2 N$ fixed, then $R \sim 1/N \to 0$, and the resulting theory is the large $N$ limit of $4+1$ super Yang-Mills, with the extra dimension disappearing.

Although consistent perturbative quantum super Yang-Mills theories do not exist in 5,6 or 7 dimensions with finite gauge groups, we have seen that the large $N$ limit of 5,6 or 7 dimensional $U(N)$ theories arise as limits of M-theory in which the bulk modes appear to decouple, and so could be consistent matter theories in these dimensions. We will return to this in section 7.

Note that in these limits for the theory on $T^n$, only the geometric $SL(n, \mathbb{Z})$ subgroup of the full U-duality group will survive, as the Kaluza-Klein modes for $n = 4$, the string winding modes for $n = 5$ and the membrane wrapping modes and NS 5-brane wrapping modes for $n = 6$ all decouple. (Note that for $n = 6$, a limit in which the NS 5-branes survived and the membranes did not could lead to a theory with duality group $SL(6; \mathbb{Z}) \times SL(2; \mathbb{Z})$, which is a maximal subgroup of $E_6(\mathbb{Z})$. This $SL(2, \mathbb{Z})$ S-duality identified in [23] would interchange the six 5-branes with the six momentum modes, and acts on $g_{YM}$ and the generalised $\theta$ angle of [23], which combine into a complexified coupling constant.)
4. The Supergravity Solutions

The type II solution for $N$ coincident D$n$-branes is, for $n < 7$, [27,20,28]

$$ds^2 = f_n^{-1/2}(-dt^2 + dx_1^2 + \ldots + dx_n^2) + f_n^{1/2}(dx_{n+1}^2 + \ldots + dx_9^2),$$

$$e^\phi = e^{\phi_\infty} f_n^{-(n-3)/4},$$

$$A_0...n = -\frac{1}{2}(f_n^{-1} - 1),$$

where $f_n$ is a harmonic function of the transverse coordinates $x_{n+1}, \ldots, x_9$, which, for $N$ coincident branes at the origin $r = 0$ (where the radial coordinate $r = (x_{n+1}^2 + \ldots + x_9^2)^{1/2}$) is

$$f_n = 1 + \left(\frac{a}{r}\right)^{7-n}$$

where

$$a^{7-n} = g_S N d_n m_S^{7-n} = R N d_n m_S^{6-n} = \gamma^2 m_S^{2(n-5)}$$

Here $d_n$ is a constant given in [20], and

$$\gamma = \sqrt{d_n g_N} = g_{YM} \sqrt{d_n N}$$

In the asymptotic region $r/a \gg 1$, the solution approaches flat space with $\phi \to \phi_\infty$, giving string theory in flat space with coupling constant $g_S$, while in the near region $r/a \ll 1$, it is approximately described by the geometry

$$ds^2 = \left(\frac{a}{r}\right)^{(7-n)/2} dx_\parallel^2 + \left(\frac{a}{r}\right)^{(7-n)/2} (dr^2 + r^2 d\Omega_{8-n}^2)$$

and is singular as $r \to 0$ (for $n \neq 3$). The dilaton is given by

$$\phi = \phi_\infty + \frac{(7-n)(n-3)}{4} \log (r/a)$$

so that as $r \to 0$, $e^\phi \to 0$ for $3 < n < 7$ and $e^\phi \to \infty$ for $n < 3$. 

14
In the limits considered here, if $N$ is kept fixed then $a \to 0$ for $n < 6$, and is finite ($a = d_6RN$) for $n = 6$, while if $N \to \infty$ keeping $g_{YM}^2N$ fixed, $a \to \infty$. If $r$ is rescaled at the same time, the limiting form will be the asymptotic flat geometry if $a/r$ remains small, but will be the near geometry (4.5) if $a/r$ becomes large. The limiting forms of these geometries in the matrix limit have been considered in [19,20], in the limit in which the radial coordinate is also scaled so that

$$U = m_S^2r$$

(4.7)

is held fixed; this corresponds to keeping fixed the energy of a string joining D-branes separated by a distance $r$ in the D-brane description, and $U$ is the energy scale of the super Yang-Mills description [19,20]. In terms of $U$,

$$f_n = 1 + \left( \frac{b}{U} \right)^{7-n}$$

(4.8)

where

$$b^{7-n} = g_SN d_n m_S^{7-n} = R N d_n m_S^{8-n} = \gamma^2 m_S^4$$

(4.9)

Then in any limit in which $m_S \to \infty$ while $U$ and $g_N = g_{YM} \sqrt{N}$ are kept fixed, the solution becomes [20]

$$ds^2 = m_S^{-2} \left( \frac{U(7-n)/2}{\gamma} dx_\parallel^2 + \frac{\gamma}{U(7-n)/2} dU^2 + \gamma U^{(n-3)/2} d\Omega_{8-n}^2 \right),$$

$$e^\phi = g_{YM}^2 \left( \frac{g_N^2}{U^{7-n}} \right)^{\frac{3-n}{4}} \sim \frac{g_{eff}^{(7-n)/2}}{N}$$

(4.10)

where $g_{eff}^2 = g_N^2 U^{n-3}$ is the effective coupling at energy scale $U$. This is the geometry of the near region (4.5), rescaled by a factor $m_S^2$ which diverges in the limit. It is the remarkable fact that the metric only depends on $g_{YM}$ and $N$ in the combination $g_{YM}^2N$ that leads to the structure of the large $N$ theory [20].
In particular, if \( g_{YM} \rightarrow 0, N \rightarrow \infty \), as in the case here, we have the metric (4.10) but \( e^\phi \rightarrow 0 \). Note that in this limit, although \( g_S = e^{\phi \infty} \rightarrow \infty \) for \( n > 3 \), \( e^\phi = e^{\phi \infty} f_{n}^{-(n-3)/4} \rightarrow 0 \). In terms of \( g_{eff} \) the curvature \( \mathcal{R} \) associated with the metric is [20]

\[
m_{S}^{-2} \mathcal{R} \approx \frac{1}{g_{eff}} \sim \sqrt{\frac{U^{3-n}}{g_{YM}^2 N}}
\]

(4.11)

This will be small if \( g_{eff} \) is large, and this is a necessary condition for the supergravity solution to be reliable. As the position-dependent string coupling \( e^\phi \) becomes zero in this limit, the weakly-coupled type II picture is reliable. Then if \( g_{eff} >> 1 \), i.e. at energy scales \( U \) such that

\[
U^{n-3} >> g_N^2
\]

(4.12)

the limiting theory is described by free type II superstrings in the near-geometry background (4.10), while for energies

\[
U^{n-3} << g_N^2
\]

(4.13)

the effective coupling \( g_{eff} \) is small, and the theory is described by the large \( N \) limit of super Yang-Mills for \( n = 5, 6 \).

5. The Role of the D0-branes for \( n = 6 \).

We now return to the role of the D0-branes for \( n = 6 \). For \( n = 6 \), the free limit or large \( N \) limit of super Yang-Mills was obtained by taking \( R \rightarrow 0 \) after the limits described above. If \( R \) is kept finite, there are D0-branes of finite mass surviving in the bulk and which might lead to extra degrees of freedom in the world-volume theory. The interaction between D0 branes and D6 branes is not completely understood, but D0-branes do not form a stable BPS bound state with the D6-branes, and there is a repulsive force. If a D0-brane is put on a D6-brane,
it would tend to shrink to zero size and then escape from the brane into the bulk. However, there are non-supersymmetric bound states that are quadratically stable [26], so that D0-brane states might be expected to play a role in the world-volume theory for finite \( R \), and this need not be supersymmetric. In the free theory arising from the limit \( m_S \to \infty \) keeping \( N \) fixed, the D0-brane could correspond to an extra free scalar moving in a compact dimension, corresponding to the fibre of the multi-Taub-NUT space.

In [29], it was argued that the supersymmetric world-volume dynamics of the M-theory Kaluza-Klein monopole \( \mathbb{R}^{6,1} \times (\text{Taub} - \text{NUT}) \) should be described by the super Yang-Mills multiplet in 6+1 dimensions. In a covariant approach, the 6+1 world-volume theory should include a vector field and 11 scalars \( X^M \), of which 6+1 scalars can be identified with the world-volume coordinates by choosing a static gauge, leaving 4 scalars taking values in the Taub-NUT space. The vector multiplet in 7 dimensions has only 3 scalars however, so that we obtain a vector multiplet and an extra scalar, taking values in the fibre of the Taub-NUT, which is a circle of radius \( R \). Translations in this fibre direction do not constitute true deformations, and are pure gauge as they constitute an isometry of the space-time; in particular, there is no modulus associated with them [30]. In [31], it was proposed that the effective description of the 4 scalars should be a sigma-model whose target space is the Taub-NUT space, in which the fibre isometry is gauged, leaving a theory of 3 scalars, which fit into the vector multiplet. In [31], a kinetic term was proposed for the bosonic degrees of freedom, and shown to have appropriate properties, and in particular the reduction to 5+1 dimensions gives the D6-brane action. The general form of the Wess-Zumino term for gauged sigma-models in any dimension was given in [32], and it is straightforward to add such a term to the kinetic term of [31] to obtain an action that reduces to the full D6-brane action; Wess-Zumino terms for such actions have also been considered in [33].

The D0-brane degree of freedom clearly corresponds to the scalar taking values in the Taub-NUT fibre, and its zero-mode drops out because of the isometry symmetry of the target space. The scalar can be decoupled from the Taub-NUT
sigma-model by gauging, as in [31], or by taking $R \to 0$. It cannot fit into a 7-dimensional supermultiplet with the vector and other scalars.

To discuss whether the D0-brane survives in the limit $N \to \infty$, we need to know how the energy of the metastable D0-D6 bound state depends on $N$ and $m_S$. In the next section, we will show that, at least in regimes in which supergravity is reliable, the bound state of $k$ D0-branes to the $N$ D6-branes does not decouple for matrix theory with fixed $N$, but does in the $N \to \infty$ limit of [1]. For the limits proposed here with $m_S \to \infty$ and $R, R_i$ fixed, these decouple both for the free limit with $N$ fixed and in the large $N$ limit.

6. D-Brane Bound States

The D$n$-D$n - p$ brane bound state has been extensively studied for $p = 2, 4$. For $p = 2$, there is a true bound state preserving half the supersymmetries, and the D$n$-D$n - 2$ brane bound state wrapped on an $n$-torus gives a state of mass $M$ satisfying

\[ M^2 = Q^2 + P^2 \]  

(6.1)

For $p = 4$, there is a marginally bound state preserving $1/4$ of the supersymmetry, and the wrapped D$n$-D$n - 4$ brane bound state has mass

\[ M = |Q| + |P| \]  

(6.2)

Here $P$ is the mass of $N$ D$n$-branes,

\[ P = g_S^{-1} m_S^{n+1} \left( \prod_{i=1}^{n} R_i \right) \]  

(6.3)

and $Q$ is the mass of $k$ D$n - p$ branes

\[ Q = k g_S^{-1} m_S^{n-p+1} \left( \prod_{i=1}^{n-p} R_i \right) \]  

(6.4)

The energy $E = M - |P|$ is then $E = |Q|$ for the marginally bound states with
\[ p = 4, \text{ while for } p = 2, \]
\[ E \sim \frac{Q^2}{2|P|} \]  

(6.5)

if \( Q/P << 1 \), as will be the case for large \( N \), so that \( E \sim 1/N \). The formulae are also the mass formulae for the black hole solution obtained by dimensionally reducing the brane solutions to \( 10 - n \) dimensions (or less). The black hole has two electric charges \( Q, P \) with respect to two different gauge fields and preserves \( 1/2 \) (\( p = 2 \)) or \( 1/4 \) (\( p = 4 \)) of the supersymmetry.

For \( p = 6 \), the D6-D0 brane configuration is only metastable and is not supersymmetric. Its dimensional reduction should give a dyonic black hole in 4 dimensions carrying an electric charge \( Q \) and a magnetic charge \( P \) with respect to the same gauge field, which is the vector field obtained from the dimensional reduction of the Ramond-Ramond one-form. Such dyonic black holes have been considered in this context in [34-39]. The relevant black hole solution is that of [40,41,42], and, for the extreme black hole, the mass \( M \) satisfies

\[ M^2 = 4 (Q^2 + P^2) - \Sigma^2 \]  

(6.6)

where \( \Sigma \) is the dilaton charge defined by

\[ \frac{1}{6} \Sigma = \frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M} \]  

(6.7)

For the magnetically charged solution with \( Q = 0 \),

\[ P = M, \quad \Sigma = -\sqrt{3}M, \quad Q = 0 \]  

(6.8)

while for the electrically charged solution with \( P = 0 \),

\[ Q = M, \quad \Sigma = \sqrt{3}M, \quad P = 0 \]  

(6.9)

The dilaton charge can be eliminated from these formulae (using e.g. the
parameterisation of [40]) to give

\[ M = (Q^{2/3} + P^{2/3})^{3/2} \]  \hspace{1cm} (6.10)

Then if \(|Q| \ll |P|\),

\[ E = M - P \sim \frac{3}{2} P^{1/3} Q^{2/3} \]  \hspace{1cm} (6.11)

plus corrections of order \(|Q|^{4/3} |P|^{-1/3}\)

Defining the volume of the 6-torus in string units,

\[ \hat{V}_6 = m_6^S V_6 = m_6^S \left( \prod_{i=1}^{6} R_i \right) \]  \hspace{1cm} (6.12)

the energy can be written as

\[ E \sim \frac{3}{2} N^{1/3} k^{2/3} \frac{1}{R} \hat{V}_6^{1/3} \]
\[ = \frac{3}{2} N^{1/3} k^{2/3} \frac{1}{g_{YM}^2} V_6^{1/3} \]  \hspace{1cm} (6.13)
\[ = \frac{3}{2} N^{4/3} k^{2/3} \frac{1}{g_N^2} V_6^{1/3} \]

This formula will apply whenever \( Q \ll P \); as \(|Q/P| \sim k/(\hat{N}_6^V)\), we have that \(|Q/P| \rightarrow 0\) if \( m_S \rightarrow \infty \) with \( R_i \) fixed, as is the case for the matrix and decoupling limits discussed here. Then the higher order corrections, which are suppressed by a factor of \(|Q/P|^{2/3}\), vanish in the limit and the formula (6.13) is exact. In the usual matrix limit in which \( g_{YM}, N, V_6 \) are kept fixed, this gives \( E \propto N^{1/3} m_p^3 \) and this remains finite, so that the finite \( N \) matrix theory should include these bound states, but if \( N \) is taken to infinity (keeping \( g_{YM} \) fixed, as in [1,2]) these states decouple. For the limits proposed here, \( V_6 \) is kept fixed and \( g_{YM} \rightarrow 0 \), so that these states decouple either if \( N \) is kept fixed or if \( N \rightarrow \infty \), \( g_N \) fixed, in which case \( E \sim N^{4/3} \). If \( R \) is kept fixed, the states in which D0-branes bind to the D6-branes
decouple, even though the bulk D0-branes remain at finite mass, while if $R \to 0$, the bulk D0-branes decouple also. (Note that if $R$ and the string-scale volume $\hat{V}_6$ were kept fixed, these states would remain for finite $N$ but decouple as $N \to \infty$, with $E \sim N^{1/3}$.)

7. Discussion

We have seen that for $n = 3, 4, 5, 6$, there is a limit of the theory of $N$ Dn-branes that gives the $N \to \infty$ limit of $n + 1$ dimensional super Yang-Mills, for which the Planck mass and string mass both tend to infinity, so that the bulk and stringy degrees of freedom should all decouple. For $n = 4$, this involves taking $R \to 0$, so that the extra dimension disappears, while for $n = 3, 5$ this involves $\hat{g}_S \to 0, \hat{m}_S \to \infty$, so that the stringy degrees of freedom decouple. The Dn-brane action has higher-derivative terms such as those from the expansion of the Dirac-Born-Infeld kinetic term, but these are suppressed by powers of the string scale and disappear in the limit. If there is a complete decoupling of all other degrees of freedom, then this limit should give a consistent quantum theory, and the simplest way in which this might occur would be if the large $N$ limits of these super Yang-Mills theories were finite. As the theories with finite $N$ are not finite or renormalizable for $n > 3$, this would be quite remarkable.

If one considers the situation with $N$ large but finite, the theory should make sense (assuming M-theory does) and there are extra degrees of freedom that become important at the short distance scale $l_{YM} = g_{YM}^{2/(n-3)}$, with $g_{YM}$ small but finite. For $n = 4$, these are Kaluza-Klein modes associated with a circle of radius $R = l_{YM}$, for $n = 5$ these include string excitations of non-critical strings with string mass $\hat{m}_S = 1/l_{YM}$, and for $n = 6$ these include bulk modes coupling at the Planck scale $m_p = 1/l_{YM}$. However, taking $N \to \infty$ takes the effective cut-off $l_{YM} \to 0$, and it appears that the extra short-distance degrees of freedom decouple. Some evidence for this is given in [43], where it is shown that for certain processes and in certain kinematical regimes, certain non-planar super Yang-Mills processes
correspond to supergravity effects that are suppressed by powers of the Planck scale.

For finite $N$, the divergence structure of super Yang-Mills in $D = n + 1$ dimensions has been explicitly calculated to two loops by Marcus and Sagnotti [44], and their results have been confirmed in [45], while the implications of non-renormalisation theorems were investigated in [46]. For $D = 8, 10$, there is a divergence occurring at one-loop, for $D = 7, 9$ there are divergences at 2-loops, calculated in [44]. For $D = 5, 6$, the theory is finite to two-loop order [44], but there are supersymmetric counterterms at 3-loops for $D = 6$ and at 4-loops for $D = 5$ that are allowed by $N = 2$ non-renormalisation theorems [46]. In [45], it was argued that for $D = 5$, the 4-loop and 5-loop $D = 5$ divergences should be absent also, and that the first ultra-violet divergence should occur at 6-loops (or higher). Whenever counterterms are not forbidden, they are expected to actually occur.

The 2-loop counterterm in 7 dimensions is of the form $C_{abcd} F^a F^b D F^c D F^d + ....$ (with certain contractions of the space-time indices, suppressed here) and there are two possible group-theory factors $C_{abcd}$, one of which is associated with planar graphs and the other with non-planar graphs [44,45]. The actual counterterm is a linear combination of the two, so that in the large $N$ limit the non-planar one is suppressed, but the planar one survives. Thus for $D = 7$, the theory defined by taking the large $N$ limit while keeping the ultra-violet cut-off fixed remains ultra-violet divergent, i.e. there are still planar terms which diverge as the cut-off is taken to infinity. For $D = 5, 6$ the allowed counterterms can also have planar or non-planar group-theory factors, and there is no reason to expect that the coefficients of the planar ones should vanish, although it would be interesting to have explicit confirmation of this.

The likely persistence of ultra-violet divergences in the large $N$ limit at first seems disappointing. It could mean that there is not a complete decoupling in this limit, but it is hard to see how this could be the case, since the constants
1/m_S, 1/m_p, etc governing these couplings have all gone to zero. Alternatively, it has been proposed that the large $N$ limit can be used to give a finite formulation of perturbatively non-renormalisable theories (there is an extensive literature, but see, for example, [47]), and something similar may play a role here. However, finiteness would have meant that large $N$ super Yang-Mills provides a good description at all energy scales, which would be hard to reconcile with the proposed high energy behaviour of [19,20] and section 4. The picture that seems to fit best is that the large $N$ limit of the theory of $N$ D-branes proposed here gives a limit in which the bulk and string modes decouple, leaving a theory that is consistent (since it is derived from M-theory), is described at low energies by the ’t Hooft limit of super Yang-Mills, and the extra degrees of freedom that become relevant at short distances are described by the curved space string theories proposed in [19,20], as we shall now argue.

Consider the case of $n = 4$, which is the one that is best understood. For finite $R = g_{YM}^2$ and finite $N$, the theory is described by the (2,0) U(N) tensor theory in 6 dimensions, compactified on a circle of radius $R$. This is believed to be a consistent finite superconformally invariant field theory in 6 dimensions [10-14]. The large $N$ limit has been argued to exist, and to be described at high energies by the type IIA string theory on the limiting form of the D4-brane background [20]. Taking $R \to 0$ while keeping $N$ fixed, the Kaluza-Klein modes decouple, leaving free super Yang-Mills theory in 4+1 dimensions, with $g_{YM} \to 0$. Taking the ’t Hooft limit of the $D = 6$ theory, keeping $NR = N g_{YM}^2$ fixed, should give a consistent theory. As $U(N)$ super Yang-Mills is expected to be a good effective description at length scales that are large compared to $R$, the $N \to \infty, R \to 0$ limit might be expected to correspond to the large $N$ limit of 4+1 super Yang-Mills. However, the large $N$ limit of the (2,0) theory should still be finite, while the large $N$ limit of the 4+1 super Yang-Mills may still be divergent. If so, this would suggest that the two theories are not the same, and that the large $N$ super Yang-Mills is only an effective description. The only scale surviving in the limit is that set by $g_N^2 = RN$, which is kept fixed in the limit, so it seems reasonable to assume
that this is the effective cut-off. The resulting theory is effectively described by super Yang-Mills at length scales large compared with $g_N^2$, but extra degrees of freedom must become important at distances small compared to this. However, at distances small compared to $RN$, we can use the description proposed in [20] and the extra degrees of freedom that enter are provided by type IIA string theory on the limiting form of the D4-brane background. (For large enough $N$, the other regime of [20] described by M-theory on an M5-brane background is absent.)

This suggests the following picture. The $D = 6 \ (2,0)$ theory gives a finite theory for all $N, R$. If $R > 0$ and $N$ is small, Kaluza-Klein modes are important at length scales small compared to $R$ and provide the extra degrees of freedom needed at short distances in the non-renormalisable $D = 5$ super Yang-Mills. In the ’t Hooft limit, $R \rightarrow 0$ so that the Kaluza-Klein modes decouple, but it appears that the resulting $D = 5$ theory (the ’t Hooft limit of $D = 5$ super Yang-Mills) remains non-renormalisable. The effective cut-off must be $RN = g_N^2$, and the extra degrees of freedom needed to regulate the theory at shorter distance scales are provided by IIA string theory in the near-region of a D4-brane background.

A similar picture should then apply for the non-renormalisable super Yang-Mills theories in $D = 6, 7$; for finite $N, g_{YM}$, the extra degrees of freedom needed at distances small compared with $l_{YM}$ are string modes in $D = 6$ and certain bulk modes in $D = 7$. For the ’t Hooft limit in which $l_{YM} \rightarrow 0$, the resulting theory is again presumably not finite, even though the usual effective cut-off $l_{YM}$ has been removed, and the only surviving scale in the theory is that set by $g_N$, which should be the effective cut-off. The planar limit of the super Yang-Mills is a good effective description at distance scales large compared with this, and the type II string theory in the near region limit of the Dn-brane background is a good effective description at distance scales small compared with the $g_N$ scale. This generalises the duality between $D = 4$ super Yang-Mills and type IIB string theory in anti-de Sitter space [19,21]: at weak coupling $g_{eff} << 1$ (where $g_{eff}^2 = N g_{YM}^2 U^{n-3}$ is the dimensionless effective coupling at energy $U$) there is a super Yang-Mills description, while for $g_{eff} >> 1$ there is a description in terms of free type II
string theory in the near region limit of a D-brane background.

To summarise, the theory of $N$ D$n$-branes in the large $N$ limit described here gives a theory in which gravity and the bulk degrees of freedom decouple, along with the string excitations, to leave a ‘matter theory’. If M-theory is consistent, these theories should be also. For $n \leq 4$, such a decoupling can be obtained for finite $N$, but for $n > 4$, one either obtains a free theory or one must take $N \to \infty$. For weak effective coupling $g_{eff}$ the theory is the large $N$ limit of $U(N)$ super Yang-Mills, while for strong coupling $g_{eff} \gg 1$ it is described by the free type II string theory in the near region limit of the D$n$-brane space-time proposed in [19,20]. For $n = 3$, the super Yang-Mills theory is perturbatively finite, and the strong coupling formulation in terms of free type IIB strings in anti-de Sitter space and the weak coupling formulation in terms of large $N$ super Yang-Mills are dual formulations of the same theory [19,21]. For $n > 3$, there are again two dual formulations, but here the energy scale $U$ plays a role, so that the super Yang-Mills description gives the behaviour at energies low compared to $U_N = g_N^{-2/(n-3)}$ while the high energy behaviour is that of the free string theories of [19,20]. There are thus two scales that enter into the D-brane theory: $g_{YM}$ and $g_N$, and extra short distance degrees of freedom enter at whichever scale is the larger. In the limits in which $N$ is kept fixed, extra degrees of freedom enter at the energy scale set by $g_{YM}$ which are Kaluza-Klein modes for $n = 4$, string degrees of freedom for $n = 5$ and bulk degrees of freedom for $n = 6$. In the large $N$ limits considered here in which $g_{YM} \to 0$, these degrees of freedom associated with $g_{YM}$ decouple, but the new degrees of freedom that enter at the energy scale set by $g_N$ are associated with free string theories in D-brane space-times.

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