On Quantum Mechanics on Noncommutative Quantum Phase Space

A.E.F. Djemai1,* and H. Smail2

1 Abdus Salam International Centre for Theoretical Physics, 34100 Trieste, Italy
2 Département de Physique, Institut d’Hydraulique, Centre Universitaire Mustapha Stambouli, Mascara 29000, Algeria

(Received August 27, 2003)

Abstract In this work, we develop a general framework in which Noncommutative Quantum Mechanics (NCQM), characterized by a space noncommutativity matrix parameter $\theta = \epsilon^{ij}_\beta \delta_{ij}$ and a momentum noncommutativity matrix parameter $\beta_{ij} = \epsilon^{ij}_\beta \delta_{ij}$, is shown to be equivalent to Quantum Mechanics (QM) on a suitable transformed Quantum Phase Space (QPS). Imposing some constraints on this particular transformation, we firstly find that the product of the two parameters $\theta$ and $\beta$ possesses a lower bound in direct relation with Heisenberg incertitude relations, and secondly that the two parameters are equivalent but with opposite sign, up to a dimension factor depending on the physical system under study. This means that noncommutativity is represented by a unique parameter which may play the role of a fundamental constant characterizing the whole NCQPS. Within our framework, we treat some physical systems on NCQPS: free particle, harmonic oscillator, system of two charged particles, Hydrogen atom. Among the obtained results, we discover a new phenomenon which consists of a free particle on NCQPS viewed as equivalent to a harmonic oscillator with Larmor frequency depending on $\beta$, representing the same particle in presence of a magnetic field $\vec{B} = q^{-1} \vec{\beta}$. For the other examples, additional correction terms depending on $\beta$ appear in the expression of the energy spectrum. Finally, in the two-particle system case, we emphasize the fact that for two opposite charges noncommutativity is effectively felt with opposite sign.

PACS numbers: 03.65.Ca, 11.10.Nx

Key words: noncommutative space, quantum mechanics, Moyal product

1 Introduction

Let $x_i$ and $p_i$ be the position and momentum operators which generate the Heisenberg algebra of Quantum Mechanics (QM):

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar \delta_{ij} \mathbf{1}, \quad [p_i, p_j] = 0, \quad (1)$$

where the dynamics is described by the canonical equations,

$$\dot{x}_i = [x_i, H], \quad \dot{p}_i = [p_i, H] \quad (2)$$

with $H$ being the Hamiltonian operator of the quantum system described on Quantum Phase Space (QPS). In Ref. [1], it has been shown that this operator algebra is equivalent to an $\hbar$-star deformation of the Poisson algebra of classical observables equipped with a Weyl–Wigner–Moyal product defined as

$$(f \star g)(u) = \exp \left[ \frac{i\hbar}{2} \omega^{ab} \hat{g}_a^{(1)} \hat{g}_b^{(2)} \right] f(u_1)g(u_2)|_{u_1 = u_2 = u},$$

where $u_a$ are phase space variables, $a = 1, \ldots, 2N$, and $\omega$ is the usual symplectic structure.

Moreover, it has been shown in Ref. [2] that QM can be formulated as a noncommutative symplectic geometry by means of a discrete Weyl–Schwinger realization of the Heisenberg group and by developing a discrete version of the Weyl–Wigner–Moyal formalism. In analogy with the classical case, the noncommutative (quantum) symplectic geometry associated with the matrix algebra $M_N(C)$ generated by Schwinger basis was described. Recently, there has been growing interest on the description of QM on non-commutative spaces (NCQM).[3–6] This was motivated by string theory arguments, which tell us that the low energy effective theory of D–brane in the background of NS–NS $B$ field lives on noncommutative spaces.[7] In fact, in this context, our space–time may be the world–volume of a D–brane, and thus may be noncommutative. Moreover, it has been shown that the study of the dynamics of a quantum system on a noncommutative space is equivalent to the study of the dynamics of this system on a commutative space in presence of a magnetic field $\vec{B}$.[5]

A noncommutative space can be realized by coordinate operators satisfying

$$[x_\mu, x_\nu] = i\hbar \theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is the noncommutativity parameter with the dimension of $(\text{length})^2/\hbar$ represented by an antisymmetric matrix whose entries $\theta_{ij}$ are considered to be vanishing, otherwise the theory is not unitary. Performing explicit loop calculations, it has been proved that, for instance, the noncommutative QED (NCQED) up to one loop is renormalizable. Then, in order to study phenomenological consequences of the noncommutativity of the space, it
is significant to learn more about the QM on noncommutative spaces.

The aim of this work is to study QM on NCQPS. We find that, in addition to the results obtained in various papers where the noncommutativity is only present in the space sector, there are additional terms due to the noncommutativity \( \beta \) present in the momentum sector of QPS since its existence is, in fact, essentially due to the existence of the noncommutativity \( \theta \) on the space. In this work, we construct a general \( \alpha \)-star deformation of the algebra of classical observables that gives rise to a general NCQM. It appears that this \( \alpha \)-star deformation is equivalent to some general transformation on the usual quantum phase space variables \((x_i, p_j)\). Indeed, we show that NCQM is equivalent to QM on a transformed QPS.

This paper is organized as follows. In Sec. 2, we describe a general NCQM. Section 3 is devoted to studying the noncommutative Hamiltonian dynamics of a quantum system. Some simple examples of quantum systems are considered, such as free particle, harmonic oscillator, two-particle system, and as a particular case the Hydrogen atom. Finally, section 4 is devoted to some conclusions and perspectives.

2 Description of General Noncommutative Quantum Mechanics

Let us consider a general \( \alpha \)-star deformation on the Poisson algebra of the classical observables as

\[
(f \star _{\alpha} g)(u) = \exp \left[ \frac{1}{2} \alpha \delta \theta (\theta_1) \delta \beta (\beta_2) \right] f(u_1)g(u_2) |_{u_1 = u_2 = u},
\]

where \( \alpha \) generalizes the usual classical symplectic structure \( \omega \) and its general form may be given by

\[
\alpha = \begin{pmatrix}
\theta & 1 + \sigma \\
-1 - \sigma & \beta
\end{pmatrix},
\]

where \( \theta \) and \( \beta \) are antisymmetric \( 3 \times 3 \) matrices and \( \sigma \) a symmetric \( 3 \times 3 \) matrix, such that

\[
\theta_{ij} = \epsilon_{ijk} \theta_k, \quad \beta_{ij} = \epsilon_{ijk} \beta_k.
\]

The \( \alpha \)-star deformed Poisson algebra is given by

\[
\{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij},
\]

\[
\{p_i, p_j\}_\alpha = \beta_{ij}.
\]

It is easy to see that the classical limit is guaranteed by the condition \( \theta = \beta = \sigma = 0 \). Considering the noncommutativity as a small perturbation on the structure of the phase space, the real parameters \( \theta_i \) and \( \beta_i \) are taken very small and our calculations are taken up to the first order in \( \theta \) and \( \beta \). We will see that \( \sigma \) can be ignored since it is of the second order.

The above star-product permits us to deduce the star deformed Heisenberg algebra which defines a general NCQM,

\[
[x_i, x_j]_\alpha = i \hbar \delta_{ij} 1, \quad [x_i, p_j]_\alpha = i \hbar (\delta_{ij} + \sigma_{ij}) 1, \quad [p_i, p_j]_\alpha = i \hbar \beta_{ij} 1,
\]

via a generalized Dirac quantization,

\[
\{,\}_\alpha \rightarrow \frac{1}{i\hbar} [\cdot, \cdot]_\alpha
\]

and with a noncommutative quantum dynamics governed by the motion equations,

\[
\dot{x}_i = [x_i, H]_\alpha, \quad \dot{p}_i = [p_i, H]_\alpha.
\]

Here, we argue that introducing a noncommutativity parametrized by \( \theta \) on the space sector of the QPS, we automatically introduce a noncommutativity parametrized by another parameter, say \( \beta \), on the momentum sector of QPS, since they are linked by the famous Heisenberg incertitude relations, and then, the parameter \( \sigma \), which appears in the star-commutator of a space operator with a momentum operator must be tied to the two parameters \( \theta \) and \( \beta \). Hereunder, we will give a more precise explanation to this point.

Let us consider a general linear transformation on the usual quantum phase space variables,

\[
\begin{pmatrix}
x'_i \\
p'_i
\end{pmatrix} = \begin{pmatrix}
a_{ik} x_k + b_{ik} p_k \\
c_{ik} x_k + d_{ik} p_k
\end{pmatrix}.
\]

At this stage, we remark that for the following particular choice of the matrix parameters \( a, b, c, \) and \( d \),

\[
a = d = 1, \quad c = \frac{1}{2} \beta, \quad b = \frac{1}{2} \theta,
\]

the new quantum variables \( x'_i \) and \( p'_i \)

\[
\begin{pmatrix}
x'_i \\
p'_i
\end{pmatrix} = \begin{pmatrix}
\delta_{ik} - \beta_{ik}/2 \\
\beta_{ik}/2 \delta_{ik}
\end{pmatrix} \begin{pmatrix}
x_k \\
p_k
\end{pmatrix} = T_{ik} \begin{pmatrix}
x_k \\
p_k
\end{pmatrix}.
\]

Or, in another manner,

\[
\begin{pmatrix}
x'_i \\
p'_i
\end{pmatrix} = \begin{pmatrix}
\bar{x}_i - \frac{1}{2} \bar{p} \wedge \bar{\theta} = \bar{x} - \frac{1}{2} \theta \bar{p}, \\
\bar{p}'_i = \bar{p} + \frac{1}{2} \bar{x} \wedge \bar{\beta} + \bar{p} + \frac{1}{2} \beta \bar{x},
\end{pmatrix}
\]

satisfy commutation relations that look like Eq. (4),

\[
[x'_i, x'_j] = i \hbar \theta_{ij} 1, \quad [x'_i, p'_j] = i \hbar (\delta_{ij} + \sigma_{ij}) 1, \quad [p'_i, p'_j] = i \hbar \beta_{ij} 1,
\]

where \( \sigma \) must satisfy the following condition

\[
\sigma = -\frac{1}{8} (\theta \beta + \beta \theta),
\]

and so, it can be ignored since we are only interested in first order terms in \( \theta \) and \( \beta \). Then, instead of using quantum variables with \( \star_{\alpha} \)-product (see Ref. (4)), it is more suitable to use the transformed quantum variables \( (x'_i, p'_i) \).
with the usual operator product. Investigating the transformation (6), we see that its Jacobian is given by

\[ J = \det(T) = \left(1 + \frac{1}{8}\rho\right)^2 \neq 0, \]

where

\[ \rho = \text{tr}(\theta \cdot \beta) = \text{tr}(\beta \cdot \theta) = -2\theta \cdot \beta. \]

Letting \( J \) be equal to 1, the parameter \( \rho \) is constrained to fulfill the following condition,

\[ \rho = -16 \implies \theta \cdot \beta = 8. \tag{8} \]

From the relations (7), we see that the dimension of \( \theta_i \) is the same as \((\Delta x_i)^2/\hbar\) and the dimension of \( \beta_j \) as \((\Delta p_j)^2/\hbar\) and then the above condition looks like the well-known Heisenberg uncertainty relations,

\[ \Delta x_i \Delta p_i \geq \hbar. \]

It follows that the noncommutative perturbation along the space-like and momentum-like sectors must verify the same kind of uncertainty relations as do the position and the momentum operators. The right side of Eq. (8) gives in fact the lower bound of the product of the two parameters \( \theta \) and \( \beta \).

Furthermore, letting the matrix transformation \( T \) be orthogonal,

\[ TT^\dagger = T^\dagger T = 1, \]

we deduce that the deformation parameters \( \theta_i \) in the space-like sector of the quantum phase space are of the same “magnitude”, up to a dimension factor depending on the physical system under study, as the deformation parameters \( \beta_j \) in the momentum-like sector,

\[ \theta_i \sim -\beta_j. \]

Effectively, this special orthogonal transformation on the quantum phase space expresses that, if one introduces a noncommutativity as a perturbation on the space like sector of QPS, this leads automatically to introduce a noncommutativity as an opposite equivalent perturbation on the momentum-like sector. In fact, \( T \) looks like an infinitesimal transformation on QPS:

\[ T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & \theta \\ -\beta & 0 \end{pmatrix}, \]

Hereunder, we will show that noncommutative effects are always tied to the angular momentum of the system under study and so, they emphasize the presence of some kind of rotations relatively to the axis defined by \( \vec{\theta} \).

3 Noncommutative Quantum Dynamics

In this section we investigate some examples of dynamical systems on NCQPS within the framework presented in the previous section. In particular, we propose to treat the cases of a free particle, a harmonic oscillator, two-particle system and in particular, Hydrogen atom.

Let us begin to investigate the general case of a quantum system with a Hamiltonian operator,

\[ H(\vec{x}, \vec{p}) = \frac{\vec{p}^2}{2m} + V(\vec{x}), \]

where \( \vec{x} \) and \( \vec{p} \) satisfy Eq. (1). Its quantum dynamics is governed by the motion equation (2). We have shown that the study of this quantum system on an NCQPS, where the position and momentum operators obey the nontrivial commutation relations (4), is equivalent to studying it on the usual QPS subject to the transformation (6), where the new position and momentum operators \( \vec{x}' \) and \( \vec{p}' \) obey the commutation relations (7).

In this context, the Hamiltonian operator varies as

\[ \Delta H = H(\vec{x}', \vec{p}') - H(\vec{x}, \vec{p}) = -\frac{1}{2m} \beta_{jk} x_j p_k + [V(\vec{x}') - V(\vec{x})] = -\frac{1}{2m} \vec{L} \cdot \vec{\beta} + \Delta V, \tag{9} \]

where \( \vec{L} = \vec{x} \wedge \vec{p} \) is the angular momentum of the quantum system. It is clear that there exists a correction term \( \Delta E_{NC} \) to the energy spectrum of our quantum dynamical system due to the presence of noncommutativity in the QPS reflected by the two additional terms in the right side of Eq. (9).

3.1 Free Particle

In the case of a quantum free particle \( V(\vec{x}) = 0 \), the Hamiltonian transforms as (see Eq. (9)):

\[ H'(\vec{x}, \vec{p}) = \frac{\vec{p}'^2}{2m} = \frac{\vec{p}^2}{2m} - \frac{1}{2m} \beta_{jk} x_j p_k = \vec{p}'^2 - \frac{1}{2m} \vec{L} \cdot \vec{\beta}, \]

where

\[ \vec{p}' = \vec{p} + \frac{1}{2}(\vec{x} \wedge \vec{\beta}). \tag{10} \]

This shows that the study of the dynamics of a particle on an NCQPS is equivalent to the study of the dynamics of this particle of charge \( q \) on the usual QPS in presence of a magnetic field. This interpretation is justified by the identification of the additional term in Eq. (10) as a vector potential \( \vec{A} \) associated with a magnetic field \( \vec{B} \),

\[ q\vec{A} = -\frac{1}{2} \vec{x} \wedge \vec{\beta}, \quad \vec{\beta} = q\vec{B}. \]

Considering the simple case \( \beta_1 = \beta_2 = 0 \) and \( \beta_3 = \beta = qB \), and defining

\[ z = \sqrt{\frac{|\beta|}{2\hbar}} (x + iy), \]

and the annihilation and creation operators

\[ a = \partial_z + \frac{1}{2} z, \quad a^\dagger = -\partial_z + \frac{1}{2} \bar{z}, \]

the Hamiltonian becomes

\[ H' = \frac{a^\dagger a}{2} + \frac{1}{2} m \vec{L} \cdot \vec{\beta} + \frac{1}{2} m \vec{L} \cdot \vec{\beta}^\dagger - \frac{1}{2} m \vec{L} \cdot \vec{\beta} \cdot \vec{\beta}^\dagger = a^\dagger a, \]

where

\[ E = \frac{a^\dagger a}{2} + \frac{1}{2} m \vec{L} \cdot \vec{\beta} + \frac{1}{2} m \vec{L} \cdot \vec{\beta}^\dagger - \frac{1}{2} m \vec{L} \cdot \vec{\beta} \cdot \vec{\beta}^\dagger = a^\dagger a. \]

This result is consistent with the known quantum mechanics of the magnetic monopole, and it also shows that the magnetic field is “false” in the sense that it is related to the magnetic field of a monopole noncommutative QPS.
one finds that our quantum particle looks like a harmonic oscillator system:

\[ [a, a] = a^\dagger a = 0, \quad [a, a^\dagger] = 1 \]

with Hamiltonian operator

\[ H(a, a^\dagger) = \hbar \omega_\beta (a^\dagger a + \frac{1}{2}) , \]

giving us a spectrum formed by infinite degenerate Landau levels,

\[ E_l = \hbar \omega_\beta (l + \frac{1}{2}), \quad l \in \mathbb{Z}, \]

where

\[ \omega_\beta = \frac{|\beta|}{m} = \frac{|qB|}{m} \]

denotes the Larmor frequency.

Finally, we conclude that introducing a noncommutativity on the QPS, the free quantum particle behaves as a harmonic oscillator with a (very small) frequency depending on the noncommutative perturbation \( \beta \) on the momentum sector.

### 3.2 Harmonic Oscillator

Let us consider now the example of a quantum harmonic oscillator of charge \( q \) and with a potential,

\[ V(\vec{x}) = \frac{1}{2} \kappa \vec{x}^2, \]

and let us set

\[ \vec{\mu} = [\vec{\beta} + m \kappa \vec{\theta}] . \]

Then, the modified Hamiltonian becomes (see Eq. (9))

\[ H'(\vec{x}, \vec{p}) = H(\vec{x}, \vec{p}) - \frac{1}{2m} \vec{L} \cdot \vec{\mu} = \frac{\vec{P}^2}{2m} + V(\vec{x}) , \]

where

\[ \vec{P} = \vec{p} + \frac{1}{2} \vec{x} \land \vec{\mu} . \]

The resulting system is then always a quantum harmonic oscillator but with a shifted energy spectrum. In fact, it looks also like the ordinary harmonic oscillator in presence of a magnetic field,

\[ \vec{B} = q^{-1} \vec{\mu} = q^{-1} [\vec{\beta} + m \kappa \vec{\theta}] , \]

which is the sum of two magnetic fields \( \vec{B}_1 = q^{-1} \vec{\beta} \) and \( \vec{B}_2 = q^{-1} m \kappa \vec{\theta} \) due to the presence of noncommutativity in the momentum and space sectors of QPS respectively.

The new Larmor frequency characterizing the correction term in the system spectrum is

\[ \omega_\mu = \frac{q}{m} = \frac{|qB|}{m} . \]

### 3.3 Two-Particle System

Let us treat now a system of two distinct quantum particles with respective masses and charges \((m_a, q_a)\) and \((m_b, q_b)\), defined on NCQPS. Moreover, since the dynamical states of two different particles belong to two different Hilbert spaces, one considers that their position and momentum operators commute under any product, stared or not. This leads to the following set of non-trivial commutation relations,

\[ [\hat{x}^{(a)}_i, \hat{\mathbf{p}}^{(a)}_j] = i \hbar \theta^{(a)}_{ij} \mathbf{1} , \]

\[ [\hat{x}^{(b)}_i, \hat{\mathbf{p}}^{(b)}_j] = i \hbar \theta^{(b)}_{ij} \mathbf{1} , \]  

\[ [\hat{x}^{(a)}_i, \hat{x}^{(a)}_j] = [\hat{\mathbf{p}}^{(a)}_i, \hat{\mathbf{p}}^{(a)}_j] = 0 . \]

Within our framework, the study of this two-particle system on NCQPS is equivalent to studying it on the usual QPS on which we perform the transformation (6), where \( \sigma \) can be ignored since it is of second order. This means that we have the new variables,

\[ \mathbf{x}^{(a)}_i = \mathbf{x}^{(a)}_i - \frac{1}{2} \theta^{(a)}_{ij} \mathbf{p}^{(a)}_j , \quad \mathbf{p}^{(a)}_i = \mathbf{p}^{(a)}_i + \frac{1}{2} \beta^{(a)}_{ij} \mathbf{x}^{(a)}_j , \]

\[ \mathbf{x}^{(b)}_i = \mathbf{x}^{(b)}_i - \frac{1}{2} \theta^{(b)}_{ij} \mathbf{p}^{(b)}_j , \quad \mathbf{p}^{(b)}_i = \mathbf{p}^{(b)}_i + \frac{1}{2} \beta^{(b)}_{ij} \mathbf{x}^{(b)}_j , \]

where the primed variables obey the same commutation relations (12), (13), and (14) as the noncommutative variables do, but without \( \ast \)-product. The non-primed variables \( \mathbf{x}^{(a,b)}_i \) and \( \mathbf{p}^{(a,b)}_i \) generate usual Heisenberg algebras for each particle.

To deal with the two-particle system, let us consider the following more convenient set of operators,

\[ \mathbf{X}_i = \mathbf{x}^{(a)}_i - \mathbf{x}^{(b)}_i , \]

(\text{relative coordinate operators}),

\[ \mathbf{Y}_i = \frac{m_a \mathbf{x}^{(a)}_i + m_b \mathbf{x}^{(b)}_i}{m_a + m_b} , \]

(\text{center of mass coordinate operators}),

\[ \mathbf{P}_i = \mathbf{p}^{(a)}_i + \mathbf{p}^{(b)}_i , \]

(\text{total momentum operators}),

\[ \mathbf{Q}_i = \frac{m_a \mathbf{p}^{(a)}_i - m_b \mathbf{p}^{(b)}_i}{m_a + m_b} , \]

(\text{relative momentum operators}),

and let us introduce

\[ M = m_a + m_b \quad (\text{total mass}), \]

\[ \mu = \frac{m_a m_b}{m_a + m_b} \quad (\text{reduced mass}) . \]
These new two-particle system operators verify the following set of commutation relations,
\[ [X_i, Q_j] = [Y_i, P_j] = i \hbar \delta_{ij} 1, \]
and all the others are vanishing.

The primed corresponding operators satisfy
\[ [X'_i, X'_j] = \hbar (\theta^{(a)}_{ij} + \theta^{(b)}_{ij}) 1, \]
\[ [X'_i, Y'_j] = \hbar (m_a \theta^{(a)}_{ij} - m_b \theta^{(b)}_{ij}) 1, \]
\[ [X'_i, P'_j] = 0, \]
\[ [X'_i, Q'_j] = i \hbar \delta_{ij} 1, \]
\[ [Y'_i, Y'_j] = \hbar (\frac{m_a^2}{M} \theta^{(a)}_{ij} + \frac{m_b^2}{M} \theta^{(b)}_{ij}) 1, \]
\[ [Y'_i, P'_j] = 0. \]

Under our transformation, the Hamiltonian of this two-particle system
\[ H(\vec{x}^{(a)}, \vec{x}^{(b)}, \vec{p}^{(a)}, \vec{p}^{(b)}) \]
\[ = \frac{P_i^{(a)} P_i^{(a)}}{2m_a} + \frac{P_i^{(b)} P_i^{(b)}}{2m_b} + V(\vec{x}^{(a)}, \vec{x}^{(b)}), \]
or, in terms of the new (non-primed) variables
\[ H(\vec{X}, \vec{Y}, \vec{P}, \vec{Q}) = \frac{P_i P_i}{2M} + \frac{Q_i Q_i}{2\mu} + V(\vec{X}, \vec{Y}), \]
will change to
\[ H(\vec{x}^{(a)}, \vec{x}^{(b)}, \vec{p}^{(a)}, \vec{p}^{(b)}) = \frac{P_i^{(a)} P_i^{(a)}}{2m_a} + \frac{P_i^{(b)} P_i^{(b)}}{2m_b} + V(\vec{x}^{(a)}, \vec{x}^{(b)}), \]
or equivalently
\[ H(\vec{X}', \vec{Y}', \vec{P}', \vec{Q}') = \frac{P_i P_i}{2M} + \frac{Q_i Q_i}{2\mu} + V(\vec{X}', \vec{Y}'), \]
where
\[ X'_i = X_i - \frac{1}{2} \theta^{(a)}_{ij} P_j + \frac{1}{2} \theta^{(b)}_{ij} P_j, \]
\[ Y'_i = Y_i - \frac{m_a}{2M} \theta^{(a)}_{ij} P_j - \frac{m_b}{2M} \theta^{(b)}_{ij} P_j, \]
\[ P'_i = P_i + \frac{1}{2} \theta^{(a)}_{ij} x_j + \frac{1}{2} \theta^{(b)}_{ij} x_j, \]
\[ Q'_i = Q_i + \frac{m_a}{2M} \theta^{(a)}_{ij} x_j - \frac{m_b}{2M} \theta^{(b)}_{ij} x_j. \]

At this level, we recall our assumption presented in the precedent section that considers that up to a dimension factor depending on the physical system under study, we have for the same particle that
\[ \theta^{(a)}_{ij} = -\theta^{(a)}_{ij}, \quad \theta^{(b)}_{ij} = -\theta^{(b)}_{ij}. \]

Furthermore, considering the particular and very interesting case of two-particle system whose charges \(q_a\) and \(q_b\) have opposite signs, and knowing that noncommutativity means the existence of a magnetic field on the QPS in presence of which charges of opposite signs have opposite motions, we can also argue that each of our two particles is sensible the same noncommutativity but with opposite sign,\(^8\)
\[ \theta^{(a)}_{ij} = -\theta^{(b)}_{ij} = \theta_{ij}, \quad \theta^{(b)}_{ij} = -\theta^{(a)}_{ij} = \theta_{ij}, \]
and so, finally, we are dealing with only one noncommutativity parameter which characterizes the whole NCQPS,
\[ \theta_{ij} = \theta^{(a)}_{ij} = -\theta^{(b)}_{ij} = \beta_{ij} = -\beta_{ij}. \]

In this case, the relations (17) reduce to
\[ X'_i = X_i - \frac{1}{2} \theta_{ij} P_j, \]
\[ Y'_i = Y_i - \frac{1}{2} \theta_{ij} \left[ Q_j + \left( \frac{m_a - m_b}{M} \right) P_j \right], \]
\[ P'_i = P_i - \frac{1}{2} \theta_{ij} x_j, \]
\[ Q'_i = Q_i - \frac{1}{2} \theta_{ij} \left[ Y_j - \left( \frac{m_a - m_b}{M} \right) X_j \right], \]
and the relations (15) become
\[ [X'_i, Y'_j] = -[P'_i, Q'_j] = i \hbar \delta_{ij} 1, \]
\[ [X'_i, Q'_j] = [Y'_i, P'_j] = i \hbar \delta_{ij} 1, \]
\[ [Y'_i, Y'_j] = [Q'_i, Q'_j] = i \hbar \left( \frac{m_a - m_b}{M} \right) \theta_{ij} 1, \]
\[ [X'_i, X'_j] = [X'_i, P'_j] = [Y'_i, Q'_j] = [P'_i, P'_j] = 0. \]

With these simplifications, the transformed Hamiltonian (16) will differ from the non-transformed one by
\[ \Delta H = \left[ \frac{1}{2M} \theta_{ij} X_j P_i - \frac{1}{2\mu} \theta_{ij} Q_j Y_j - \left( \frac{m_b - m_a}{2m_a m_b} \right) \theta_{ij} Q_j X_j \right], \]
\[ + V(\vec{X}', \vec{Y}') - V(\vec{X}, \vec{Y}) \]
\[ = \left[ \frac{1}{2M} (\vec{X} \wedge \vec{P}) + \frac{1}{2\mu} (\vec{Y} \wedge \vec{Q}) \right] \cdot \vec{b} + \Delta V, \]
which reduces to
\[ \Delta H = \left[ \frac{1}{2m_a} \vec{L}^{(a)} - \frac{1}{2m_b} \vec{L}^{(b)} \right] \cdot \vec{b} + \Delta V, \]
where
\[ \vec{L}^{(a)} = \vec{x}^{(a)} \wedge \vec{p}^{(a)}, \quad \vec{L}^{(b)} = \vec{x}^{(b)} \wedge \vec{p}^{(b)} \]
are the usual momentum operators of the two particles \(a\) and \(b.\)
Let us discuss now the dynamical states of the two-particle system represented by a collective wave function solution of the NC Schrödinger equation,[3,5]

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{x}^{(a)}, \vec{x}^{(b)}, t) = \left[ -\frac{\hbar^2}{2m_a} \Delta^{(a)} - \frac{\hbar^2}{2m_b} \Delta^{(b)} + V(\vec{x}^{(a)}, \vec{x}^{(b)}) \right] \Psi(\vec{x}^{(a)}, \vec{x}^{(b)}, t), \]

where the generalized ⋆-product is defined by

\[ (f \star g)(\vec{x}^{(a)}, \vec{x}^{(b)}) = \exp\left\{ \frac{i}{\hbar} \left( \phi_i^{(a)} \frac{\partial}{\partial y_i^{(a)}} + \phi_j^{(b)} \frac{\partial}{\partial z_j^{(b)}} \right) \right\} \times f(\vec{y}^{(a)}, \vec{z}^{(b)}) g(\vec{z}^{(a)}, \vec{x}^{(b)}) \]

\[ \left\{ \begin{array}{c}
\vec{y}^{(a)} = \vec{x}^{(a)} = \vec{x}^{(a)} \\
\vec{y}^{(b)} = \vec{z}^{(b)} = \vec{z}^{(b)}
\end{array} \right. \]

Let us recall that the above relations concern “hated” variables. Instead of using these variables and following our approach, it is more suitable to use our primed variables that verify the same commutation relations as the first ones do but without ⋆-product. It follows that the NC Schrödinger equation reads now as

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{x}^{(a)}, \vec{x}^{(b)}, t) = \left[ -\frac{\hbar^2}{2m_a} \Delta^{(a)} - \frac{\hbar^2}{2m_b} \Delta^{(b)} + V(\vec{x}^{(a)}, \vec{x}^{(b)}) \right] \Psi(\vec{x}^{(a)}, \vec{x}^{(b)}, t), \]

or equivalently,

\[ i\hbar \frac{\partial}{\partial t} \Psi(\vec{X}', \vec{Y}', t) = \left[ -\frac{\hbar^2}{2\mu} \Delta_{X'} - \frac{\hbar^2}{2M} \Delta_{Y'} + V(\vec{X}', \vec{Y}') \right] \Psi(\vec{X}', \vec{Y}', t). \]

To solve this equation, we use the technique of separation of variables,

\[ \Psi(\vec{X}', \vec{Y}', t) = \varphi(\vec{X}') \phi(\vec{Y}') \exp\left(-\frac{i E' t}{\hbar}\right). \]

It follows two separated differential equations concerning the functions \( \varphi(\vec{X}') \) and \( \phi(\vec{Y}') \). Solving the second one, one obtains

\[ \phi(\vec{Y}') = \exp(i \vec{K}' \cdot \vec{Y}'), \]

where \( \vec{K} \) plays the role of a wave vector associated to the collective wave function relatively to the center of mass and \( \vec{K}' \) is its transformed. The second equation reads

\[ \Delta_{X'} \varphi(\vec{X}') + \frac{2\mu}{\hbar^2} \left[ E' - V(\vec{X}', \vec{Y}') - \frac{\hbar^2}{2M} | \vec{K}' |^2 \right] \varphi(\vec{X}') = 0, \]

where

\[ \vec{X}' = \vec{X} + \frac{1}{2} \vec{P} \wedge \vec{q}, \quad \vec{Y}' = \vec{Y} + \frac{1}{2} \vec{q} \wedge \vec{P} + \left( \frac{m_a - m_b}{M} \right) \vec{q} \wedge \vec{P}, \]

\[ E' = E + \Delta E_{NC}, \quad \Delta V = V(\vec{X}', \vec{Y}') - V(\vec{X}, \vec{Y}). \]

Knowing that the usual two-particle Schrödinger equation reads

\[ \Delta_{X} \varphi(\vec{X}) + \frac{2\mu}{\hbar^2} \left[ E - V(\vec{X}, \vec{Y}) - \frac{\hbar^2}{2M} | \vec{K} |^2 \right] \varphi(\vec{X}) = 0, \]

and using

\[ \Delta_{X'} \varphi(\vec{X}') = (\Delta_{X'} + \Delta_{\nu'}) \varphi(\vec{X}') = (\Delta_{X} + \Delta_{\nu}) \varphi(\vec{X} + \frac{1}{2} \vec{q} \wedge \vec{P}) \approx \Delta_{X} \varphi(\vec{X} + \frac{1}{2} \vec{q} \wedge \vec{P}), \]

and considering the following separation of variables

\[ \varphi(\vec{X}') = \varphi(\vec{X}) F(\vec{q} \wedge \vec{P}), \]

we find that the noncommutative correction to the energy spectrum of this two-particle system is given by

\[ \Delta E_{NC} = \Delta V - \frac{\hbar^2}{2M} | \vec{K}' |^2 - | \vec{K} |^2. \quad (19) \]

In fact, it is easy to see that this relation is another equivalent version of the relation (18).

### 3.4 Hydrogen Atom

This example can be treated in two different ways. The first consists of considering this system as a one-particle system (electron) in an external Coulomb potential (nucleus). In this approach, we can directly use the relation (9) to deduce the NC correction of the Hamiltonian by
injecting the potential,
\[ V(\vec{x}) = -\frac{Ze^2}{\sqrt{x^2 + x^2}} \]
The second method consists of considering it as a two-particle system, whose treatment is developed in the previous subsection.

Using the first method, we find that the variation of the potential reads
\[ \Delta V = -\frac{Ze^2}{2x^3} \vec{L} \cdot \vec{r} = -\frac{e}{2}(\vec{\theta} \wedge \vec{p}) \cdot \left( -\frac{Ze^2}{x^3} \right), \]
where \( x = \sqrt{x^2 + x^2} \) and then, the NC correction of the Hamiltonian is given by
\[ \Delta H = \frac{1}{2m} (\vec{\beta} \wedge \vec{p}) \cdot \vec{x} - \frac{e}{2} (\vec{\theta} \wedge \vec{p}) \cdot \left( -\frac{Ze^2}{x^3} \right) = -\frac{1}{2m} \vec{L} \cdot \vec{\eta}, \]
where
\[ \vec{\eta} = \vec{\beta} + mZe^2 \frac{x^3}{x^3} \vec{\theta}. \] (20)
The first term in \( \Delta H \) corresponds to the existence of a magnetic field as already obtained in the precedent cases. It comes as a kinetic correction term. The second one that comes as a potential correction term looks like a spin–orbit coupling term where the noncommutativity induced spin momentum \( \vec{S} \) is given by
\[ \vec{S} = \frac{\hbar}{2\lambda}\vec{\theta}, \]
where \( \lambda \) is the usual Compton wave length of the electron.\[3\]

Following the second approach, we have first to calculate \( \Delta V \) (see Eq. (18)),
\[ \Delta V = -\frac{e}{2} (\vec{\theta} \wedge \vec{P}) \cdot \left( -\frac{Ze^2}{X^3} \right) = \frac{Ze^2}{2X^3} \vec{L} \cdot \vec{\theta}, \]
where \( X = \sqrt{X^2 + X^2} \) and \( \vec{L} = \vec{X} \wedge \vec{P} \).

Then, the NC correction to the Hamiltonian is now given by
\[ \Delta H = \left[ \frac{1}{2m_b} \vec{L}^{(b)} - \frac{1}{2m_a} \vec{L}^{(a)} \right] \cdot \vec{\theta} - \frac{Ze^2}{2X^3} \vec{L} \cdot \vec{\theta}. \]
It is easy to see that this result reduces to the precedent one, if one considers the nucleus (particle \( b \)) as localized at the origin and possessing an infinite mass.

Finally, in addition to the energy level shift at tree level for hydrogen atom obtained in Ref. [3], there exists another term which takes into account the noncommutativity in the momentum sector of NCQPS in such a way that
\[ \Delta E_{\text{NC}}^{\text{H-atom}} = -\frac{1}{2m} \langle \vec{L}, \vec{\eta} \rangle = -\frac{1}{2m} \left[ \langle \vec{L}, \vec{\beta} \rangle + \frac{Ze^2}{2X^3} \langle \vec{L}, \vec{\theta} \rangle \right]. \]

4 Conclusion

In Ref. [4], it is emphasized that there is no noncommutative corrections at tree level for Hydrogen atom. Solving the NC two-body Schrödinger equation and considering that \( \theta_i \) changes the sign under charge conjugation,\[6\] the authors contradict the result obtained (and confirmed) in Ref. [3].

In our work, following our approach, we have found that the kinetic term in the Hamiltonian gives an additional correction term to the result obtained in Ref. [3], (See Eq. (20)). Following our assumption which consists to say that the noncommutativity \( \beta \) in the momentum sector of QPS is of the same magnitude as the noncommutativity \( \theta \) in the space sector, up to a dimension factor depending on the physical system under study, we are tempted to deduce that effectively there is no noncommutative effects. We are motivated by the fact that \( \beta \sim -\theta \) and by the presence of a coefficient of \( \theta \) in Eq. (20) (as well as in Eq. (11)), which is specific to the considered physical system. It seems that this may ensure the annihilation of the two terms, in such a way that the noncommutative effects do not really appear. The proof that noncommutativity exists really is already given by the simple example of free particle, which behaves like a harmonic oscillator with a (very small) Larmor frequency tied to \( \beta \) (and so to \( \theta \)). It seems that it is the variation of the potential when it exists which contributes to the annihilation of this NC effect.

In another hand, it appears that noncommutativity is deeply tied to the presence of some magnetic sources in the space at scales near the planck one, and it is supposed to be a quantum effect of gravity.\[9\] However, we think that this problem needs a deep analysis to be well understood. Furthermore, in Ref. [10] an analog approach has been followed that consists to construct an “isotropic representation” representing general transformations on NCQPS. Nevertheless, in this approach the authors have not used one important requirement on these transformations that leads to the natural condition \( \theta = -\beta \) we have obtained just by requiring an orthogonality property of these transformations. In fact, their intention being to work in full generality, they assumed in general \( \theta + \beta \neq 0 \). In future works, we plan to treat within our framework some other quantum examples like NCQED, Bohm–Aharonov, Lamb shift , Stark–Zeeman, and Hall effects, etc.

Acknowledgments

The authors would like to thank Abdus Salam ICTP, where this work was performed, for hospitality. A.E.F. Djemaï would also like to address his special thanks to Arab Fund and the Associateship scheme of ICTP for their support and help. The authors are also very indebted to A. Smailagic and E. Spallucci for useful discussions and for having shown us their work, Ref. [10].
References

[1] J.E. Moyal, Proceedings of the Cambridge Philosophical Society 45 (1949) 99; J. Vey, Commentarii Mathematici Helvetici 50 (1975) 421; M. Flato, et al, Compositio Mathematica 31 (195) 47; M. Flato, et al., J. Math. Phys. 17 (1976) 1754; F. Bayen, et al., Lett. Math. Phys. 1 (1977) 521; F. Bayen, et al., Ann. 111 Phys. 111 (1978) 61, 111.

[2] A.E.F. Djemaï and H. Smail, Int. J. Theor. Phys. 35 (1996) 519.

[3] M. Chaichian, et al., Phys. Rev. Lett. 86 (2001) 2716, hep-th/0010175, hep-th/0212259.

[4] P.M. Ho and H.C. Kao, hep-th/0110191.

[5] D. Kochan and M. Demetrian, hep-th/0102050.

[6] C. Duval and P.A. Horvathy, Phys. Lett B479 (2000) 284, hep-th/002233; J. Gamboa, et al., Phys. Rev. D64 (2001) 267, hep-th/0010220; V.P. Nair and A.P. Polychronakos, Phys. Lett. B505 (2001) 267, hep-th/0011172; B. Morariu and A.P. Polychronakos, Nucl. Phys. B610 (2001) 531, hep-th/0102157; A. Jellal, hep-th/0105303; C. Duval and P.A. Horvathy, J. Phys. A34 (2001) 10097, hep-th/0106089; J. Gamboa, et al., hep-th/0106125; S. Bellucci, et al., Phys. Lett. B522 (2001) 345, hep-th/0106138; H.R. Christiansen and F.A. Schaposnik, hep-th/0106181; R. Banerjee, Mod. Phys. Lett. A17 (2002) 631, hep-th/0106280; C. Acatrinei, JHEP 0109 (2001) 007, hep-th/0107078. P.A. Horvathy, Ann. Phys. 299 (2002) 128, hep-th/0201007; S. Bellucci and A. Nersessian, Phys. Lett. B542 (2002) 295, hep-th/020524; A. Smailagic and E. Spallucci, hep-th/0307217, hep-th/0308193.

[7] A. Connes, et al., JHEP 9802 (1998) 003, hep-th/9711162; M.R. Douglas and C.M. Hull, JHEP 9802 (1998) 008, hep-th/9711165; C. Chu and P. Ho, Nucl. Phys. B550 (1999) 151, hep-th/9812219; Nucl. Phys. B568 (2000) 447, hep-th/9906192; V. Schomerus, JHEP 9906 (1999) 030, hep-th/9903205; N. Seiberg and E. Witten, JHEP 9909 (1999) 032, hep-th/9908142 and references therein.

[8] M.M. Sheikh-Jabbari, Phys. Rev. Lett. 84 (2000) 5265, hep-th/0001167.

[9] H.S. Snyder, Phys. Rev. 71 (1946) 38; C.N. Yang, Phys. Rev. 72 (1947) 874.

[10] A. Smailagic and E. Spallucci, Phys. Rev. D65 (2002) 107701, hep-th/0108216, J. Phys. A35 (2002) L363, hep-th/0205242.