Arithmetic $\mathcal{D}$-modules on Locally Noetherian Formal Schemes

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Abstract

We extend Berthelot’s theory of arithmetic $\mathcal{D}$-modules to a class of morphisms that are not necessarily of finite type. As an application we give a new construction of the category of convergent isocrystals on a separated scheme of finite type over a field, and show that the pullback by Frobenius is an auto-equivalence. This extends results of Berthelot that were proven in the smooth case.

Introduction

The subject of this paper is an extension of Pierre Berthelot’s theory of arithmetic differential operators [3], [4], [6], [7]. The original setting of the theory was that of a formal scheme $\mathcal{X}$ smooth (i.e. formally smooth and of finite type) over a noetherian $p$-adic formal scheme $\mathcal{S}$. The main constructions of the theory are of a sheaf of rings $\mathcal{D}^\dagger_{\mathcal{X}/\mathcal{S}}$ on $\mathcal{X}$ and the category of holonomic $F\mathcal{D}^\dagger$-complexes, together with a set of cohomological operations satisfying Grothendieck’s six-functor formalism. This project is still conjectural in part, but Caro and Tsuzuki have shown [11] that a full subcategory of the category of holonomic $F\mathcal{D}^\dagger$-complexes, Caro’s category of overholonomic complexes is stable under the six operations and contains anything of geometric origin. Nonetheless it can be difficult to show that any particular object is overholonomic if it does not evidently come from geometry, so there is some interest in Berthelot’s original project and in particular in studying the effect of cohomological operations relative to closed immersion.

In this paper we extend Berthelot’s theory to a class of morphisms $\mathcal{X} \to \mathcal{S}$ which are formally smooth but not necessarily of finite type or even adic. We will give a precise definition below; a typical case would be a morphism formally of finite type, i.e. the induced morphism of closed fibers is of finite
Such morphisms are the natural setting for the theory of convergent and overconvergent isocrystals. A simple case also turned up in the author’s study [12] of cohomological operations relative to the inclusion of a point in a smooth curve, and the study of local monodromy in [13]. The results of this paper are meant to prepare the way for work in higher dimensions and codimensions; such work will probably require methods of Clausen and Scholze’s condensed mathematics and for the moment I will say no more of this. On the other hand, the methods of this paper allow one to give an alternate construction of the category of convergent isocrystals on a separated scheme of finite type over a field, and to prove Berthelot’s Frobenius descent theory for this category. The category of overconvergent isocrystals can be handled in the basically same way but involves some additional ideas, so I will not discuss it here. Finally our construction of convergent (and, presumably) overconvergent isocrystals can be extended to the case of any morphism $X \to S$ satisfying the finiteness conditions described below. This too I will leave for later.

After some brief remarks on flat and formally smooth morphisms of formal schemes, §1 our basic finiteness condition: a morphism $X \to S$ of noetherian formal schemes is universally noetherian if for every morphism $T \to S$ with $T$ a noetherian formal scheme, $T \times_S X$ is noetherian. One can make a similar definition by replacing “noetherian” by “locally noetherian” but we will not use this notion in this paper. This category has nice stability properties: it is stable under composition, pullbacks and fiber products. Morphisms of finite type are universally noetherian. The inclusion $\hat{X}_Y \to X$ of the completion of a noetherian formal scheme along a closed subscheme is universally noetherian. Localizations are universally noetherian. We can thus generate a large class of examples of these morphisms.

If $X \to S$ is universally noetherian the ideal of the diagonal $X \to X \times_S X$ is locally finitely generated. Thus if $X/S$ is quasi-smooth, i.e. formally of finite type and universally noetherian, the construction of the usual differential invariants of $X/S$ proceeds without essential changes; this we do in §2. The usual characterization of smooth, étale and unramified morphisms has an analogue in the more general case, as does first-order deformation theory. However when in §3 we turn to the construction of Berthelot’s construction of arithmetic differential operators we run into the following problem. In [4] the ring $D_{\mathcal{X}/S}^{(m)}$ is the $p$-adic completion of $D_{\mathcal{X}/S}^{(m)}$. In the general case the $p$-adic completion is not what we want since $D_{\mathcal{X}/S}^{(m)}/p^nD_{\mathcal{X}/S}^{(m)}$ is not a quasi-coherent sheaf on a scheme (this fact is used in [4], for example to prove Cartan’s theorem A and B for $\hat{D}_{\mathcal{X}/S}^{(m)}$). What we want is to take a comple-
tion with respect to an ideal of definition $J \subset \mathcal{O}_X$, but $\mathcal{D}_X^{(m)}/J^n \mathcal{D}_X^{(m)}$ is not a ring unless the $J$ is carefully chosen: we want the left or right ideal generated by $J$ in $\mathcal{D}_X^{(m)}$ to be a bilateral ideal. Fortunately there is a large supply of such ideals, which we call $m$-bilateralising ideals. A similar problem turns up in section §4 when we want to construct $m$-PD-envelopes of ideals in a formal setting.

In §5 we construct the $\mathcal{O}_X$-algebras that are the formal versions of closed tubes in the analytic space $X^{an}$. The construction of differential operator rings also extends to these tubes. For later purposes I have worked out this theory in somewhat more generality than what is really needed in this paper. These tube algebras are used in the final §6 in our construction of the category of convergent isocrystals. Suppose $X/k$ is a separated $k$-scheme of finite type; let $\mathcal{V}$ be a complete discrete valuation ring with residue field $k$ and fraction field $K$ of characteristic 0. Locally we can realize $X$ as the closed fiber of a formal $\mathcal{V}$-scheme $\mathcal{X}$ that is formally of finite type over $\mathcal{V}$. If $\text{sp} : \mathcal{X}^{an} \to \mathcal{X}$ is the specialization map, the direct image $\mathcal{O}_{\mathcal{X}}^{an} := \text{sp}_* \mathcal{O}_X$ is a sheaf of Fréchet-Stein algebras, and we show that the category of convergent isocrystals is equivalent to the category of coadmissible $\mathcal{O}_{\mathcal{X}}^{an}$-modules with a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}^{\mathbb{Q}}}^+$-module structure (see §6 for the definitions). A suitable extension of Berthelot’s Frobenius descent theorem applies to the category of such objects, and from this it follows that pullback by Frobenius induces an equivalence (theorem 6.3.1).

The reader will probably recognize that the use of formal methods in preference to rigid-analytic ones is a return to Ogus’s point of view on the theory of convergent isocrystals; in fact the tube algebras studied in §5 are generalizations of what are called enlargements in [17]. However the study of what we call the tube algebras in §5 goes far beyond what is needed for this paper. In fact it is not hard to extend the most of the results of §6 to the case of an arbitrary quasi-smooth morphism $\mathcal{X} \to \mathcal{S}$ of noetherian formal $\mathbb{Z}_p$-schemes, and the Frobenius descent theorem should apply if $\mathcal{X}/\mathcal{S}$ is formally of finite type. In a recent book [15] Lazda and Pál have proposed a theory of overconvergent isocrystals and rigid cohomology of this sort when $\mathcal{S} = \text{Spf}(\mathcal{V}[[t]])$ and $\mathcal{V}[[t]]$ has the $p$-adic topology (not the topology defined by the maximal ideal $(p, t)$). However they use the theory of adic spaces rather than formal methods.

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Notation and Conventions. Rings are assumed to have an identity. Terminology and notation regarding commutative algebra and formal schemes generally follows EGA [14]. For example if $A$ is a commutative ring and $M$ is an $A$-module an $f \in A$, $M_f$ is the (algebraic) localization of $M$, and if $M$ is a topological $A$-module $M_{\{f\}}$ is the formal localization, i.e. the completion of $M_f$. A topological ring $R$ is preadic if it has the $J$-adic topology for some ideal $J \subseteq R$, and adic if it is preadic, separated and complete.

Starting with section 2 we will assume, without explicit statement to the contrary that formal schemes are locally noetherian, i.e. locally of the form $\text{Spf}(A)$ with $A$ an adic noetherian ring. Even so, this condition will frequently be stated explicitly for emphasis.

In any category $C$ with fibered products the notation $X_S(r)$ denotes the fibered product of $r + 1$ copies of $X$ over $S$. If $K \subseteq \mathbb{N}$ is a finite subset we write $X_S(K) = X_S(r)$ where $r = \#K$ is the cardinality of $K$, and the factors in the product $X_S(K)$ are indexed by the elements of $K$ in increasing order. Recall the usual notation for the projection maps relating the $X_S(K)$: for $L \subseteq K$ the projection $p_{LK} : X_S(K) \to X_S(L)$ is the product of the projections $p_i : X_S(K) \to X$ for all $i \in L$. When $K = [0, 1, \ldots, r]$ and $L = K \setminus \{i\}$ we will write these projections as $p_L : X_S(r) \to X_S(r - 1)$. For example, if $C$ is the category of schemes and $T$ is any scheme, $p_{02} : X_S(2) \to X_S(1)$ is the morphism inducing $(x, y, z) \mapsto (x, z)$ on $T$-valued points.

We use the same notation for tensor products of rings, or completed tensor products of topological rings; thus if the category $C$ in the last paragraph is the category of schemes and $X = \text{Spec}(A)$, $S = \text{Spec}(R)$ then $X_S(r) = \text{Spec}(A_R(r))$. The tensor product of an abelian group $M$ with $\mathbb{Q}$ will usually be written $M_{\mathbb{Q}}$, as in [3].

When dealing with rings or geometric constructions in an affine setting, completions will usually be denoted by a “hat” which is dropped in purely geometric situations. For example if $A$ is an $R$-algebra, $\Omega^1_{A/R}$ is the usual module of 1-forms, $\hat{\Omega}^1_{A/R}$ is its completion in the natural topology, but the
sheafification of $\Omega^1_{A/R}$ for a morphism $X \to S$ is $\Omega^1_{X/S}$. Exceptions to this convention occur when a formal construction is subject to further completion, as for example the ring $\hat{\mathcal{O}}_{X/S}$, which is a completion of $\mathcal{O}_{X/S}$ (in any case this notation is completely entrenched in the literature). Another example: $\hat{X}_S(r)$ is the formal completion of the product $X_S(r)$ along the diagonal.

In addition to the standard notations for multi-indices, we use the following: for $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $a \in \mathbb{Z}$ we write $K < a$ (resp. $K \leq a$) to mean $k_i < a$ (resp. $k_i \leq a$) for all $i$.

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1 Formal Geometry

1.1 Flatness and formal smoothness. We will use the same definition of formal smoothness for a morphism of formal schemes as for ordinary schemes:

1.1.1 Definition a morphism $X \to S$ of formal schemes is formally smooth (resp. formally unramified, formally étale) if for any commutative square

\[
\begin{array}{ccc}
Z_0 & \rightarrow & X \\
\downarrow & & \downarrow \\
Z & \rightarrow & S
\end{array}
\]

in which $Z$ is an affine scheme and $Z_0 \to Z$ is a nilpotent closed immersion, there exists a morphism $Z \to X$ (resp. there is at most one morphism, there exists a unique morphism) making the extended diagram commutative.

When $X = \text{Spf}(B)$ and $S = \text{Spf}(A)$ are affine $X \to S$ is formally smooth if and only if $B$ is a formally smooth $A$-algebra in the sense of [14, IV, 14.3.1]. The reader may check that most of the elementary properties of formally smooth, formally unramified and formally étale morphisms of ordinary schemes (e.g. [14, IV §17] propositions 17.1.3 and 17.1.4) are also valid in the present context of locally noetherian formal schemes. There is one important exception: with this definition, formal smoothness is not a local property, either on the base or the source. We will return to this question in section 2.2. For now we observe that if $f : X \to S$ is formally smooth (resp. unramified, étale) and $U \subseteq X$, $V \subseteq S$ are open formal subschemes such that $f(U) \subseteq V$, the induced morphisms $U \to V$ is formally smooth (resp. unramified, étale); this follows from the definitions and the fact that $Z$ and $Z_0$ have the same underlying topological space.

1.1.2 Lemma Let $A$ be an adic noetherian ring with ideal of definition $J$, $x \in \text{Spf}(A)$, $m \subset A$ the open prime ideal corresponding to $x$. If $M$ is a coherent $A$-module let $M$ be the sheaf on $\text{Spf}(A)$ corresponding to $M$ and $M_x$ the stalk of $M$ at $x$. Then the natural morphism $M_m \to M_x$ induces an isomorphism $\hat{M}_m \simeq \hat{M}_x$ of the $J$-adic completions.

Proof. We have $M_m = \varinjlim_f M_f$ and $M_x = \varinjlim_f M_{\{f\}}$ where $f$ runs through $A \setminus m$, and the natural map $M_m \to M_x$ is induced by $M_f \to M_{\{f\}}$. It suffices to show that

\[(\varinjlim_f M_f) \otimes_A A/J^n \to (\varinjlim_f M_{\{f\}}) \otimes_A A/J^n\]
is an isomorphism for all $n$. Since inductive limits commute with tensor products, this is the same as

$$\lim_{\underset{f}{\to}} (M_f \otimes_A A/J^n) \to \lim_{\underset{f}{\to}} (M_{\{f\}} \otimes_A A/J^n)$$

and the assertion is clear, since $M_f \otimes_A A/J^n \to M_{\{f\}} \otimes_A A/J^n$ is an isomorphism.

The morphism $M_m \to M_x$ is functorial in $M$, and also in $A$ in the sense that if $A \to B$ is a continuous homomorphism of adic rings yielding $f : \text{Spf}(B) \to \text{Spf}(A)$, and $n \subset B$ is an open prime ideal corresponding to $y \in \text{Spf}(B)$ such that $f(y) = x$, the diagram

$$\begin{array}{c}
(B \otimes_A ^\wedge M)_n \ar[r] & f^*M_y \\
\downarrow & \downarrow \\
M_m \ar[r] & M_x
\end{array}$$

is commutative. It follows that the isomorphisms of completions is functorial in the same sense.

Recall that a morphism $f : X \to S$ of locally ringed spaces, and in particular of locally noetherian formal schemes is flat at a point $x \in X$ if the morphism $O_{f(x)} \to O_x$ of local rings is flat, and $f$ is flat if it is flat at every point of $x$.

1.1.3 Lemma For any morphism $f : X \to S$ of locally noetherian formal schemes, the following are equivalent:

(i) $f$ is flat;

(ii) $f$ is flat at every closed point of $X$;

(iii) for every pair $\text{Spf}(B) \subseteq X$, $\text{Spf}(A) \subseteq S$ of open affines such that $f(\text{Spf}(B)) \subseteq \text{Spf}(A)$, $B$ is a flat $A$-algebra.

Proof. The implications (iii) $\implies$ (i) and (1.1.3)1 $\implies$ (ii) are clear. Suppose now that (ii) holds; we can also assume that $A$ and $B$ in (iii) are noetherian. It suffices to show that for every maximal ideal $n \subset B$ and $m$ equal to the inverse image of $n$ in $A$, $B_n$ is a flat $A_m$-algebra [10 Ch. II §3 no. 4 Prop. 15]. By the faithful flatness of completions (e.g. [13, 0f 7.6.18]) it suffices to show that $\hat{B}_n$ is a flat $\hat{A}_m$-algebra, where the completions are taken with respect to the adic topologies of $A$ and $B$. 

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Since $B$ is adic and noetherian it is a Zariski ring, and every maximal ideal is open. Therefore $\mathfrak{n} \subset B$ corresponds to a closed point $x \in \mathcal{X}$ and $y = f(x)$ corresponds to $\mathfrak{m}$. Now (ii) asserts $\mathcal{O}_x$ is a flat $\mathcal{O}_y$-algebra, and thus $\hat{\mathcal{O}}_x$ is a flat $\hat{\mathcal{O}}_y$-algebra. By lemma 1.1.2 there are isomorphisms $\hat{\mathcal{O}}_x \cong \hat{B}_n$ and $\hat{\mathcal{O}}_y \cong \hat{A}_m$, and by functoriality the map $\hat{A}_m \to \hat{B}_n$ corresponds via these identifications to $\hat{\mathcal{O}}_y \to \hat{\mathcal{O}}_x$.

1.1.4 Proposition A formally smooth morphism $f : \mathcal{X} \to S$ of locally noetherian formal schemes is flat.

Proof. It suffices to show that for all $x \in \mathcal{X}$ and $y = f(x)$ that $\mathcal{O}_x$ is a flat $\mathcal{O}_y$-algebra. Pick open affine neighborhoods $x \in U$, $y \in V$ such that $f(U) \subseteq V$; by the remark just before the proposition, the morphism $U \to V$ induced by $f$ is formally smooth. If $U = \text{Spf}(B)$ and $V = \text{Spf}(A)$, the topological $A$-algebra $B$ is formally smooth. Then $\mathcal{O}_x$ is a formally smooth $\mathcal{O}_y$-algebra for the preadic topologies induced by $B$ and $A$, and therefore formally smooth for the preadic topologies defined by the maximal ideals of $\mathcal{O}_x$ and $\mathcal{O}_y$. The assertion then follows from theorem 19.7.1 of [14, 0 IV].

1.1.5 Corollary If $A$ is an adic noetherian ring and $B$ is an adic noetherian ring and a formally smooth $A$-algebra, $B$ is flat over $A$.

1.2 Universally noetherian morphisms. The basic finiteness condition of this paper is the following:

1.2.1 Definition A morphism $f : \mathcal{X} \to S$ of locally noetherian formal schemes is universally noetherian (resp. universally locally noetherian if for every noetherian formal scheme $S'$ and every morphism $g : S' \to S$ of formal schemes, the fiber product $\mathcal{X} \times_S S'$ is noetherian (resp. locally noetherian).

We will mainly be concerned with universally noetherian morphisms, and leave it to the reader to formulate in what follows the corresponding results for universally locally noetherian morphisms. In the situation of definition 1.2.1 we will also say that $\mathcal{X}$ is a universally noetherian formal $S$-scheme. If $A$ and $B$ are adic noetherian rings and $A \to B$ is a continuous homomorphism, we say that $B$ is a universally noetherian $A$-algebra if $\text{Spf}(B)$ is a universally noetherian formal $\text{Spf}(A)$-scheme. To check that $\mathcal{X} \to S$ is universally noetherian, it suffices to check the condition of 1.2.1 for morphisms $S' \to S$ with $S'$ formally affine and noetherian. In particular
if $A$ is a noetherian ring, an $A$-algebra $B$ is universally noetherian if and only if $A \hat{\otimes}_B C$ is a noetherian ring for any adic noetherian $B$-algebra $C$.

It is immediate from the definition that a universally noetherian morphism is quasicompact, and that if $f : \mathcal{X} \to S$ is universally noetherian and $S$ is noetherian, then so is $\mathcal{X}$.

1.2.2 Lemma If $f : \mathcal{X} \to S$ is universally noetherian and $S' \to S$ is morphism with $S'$ locally noetherian, the fiber product $\mathcal{X} \times_S S'$ is locally noetherian.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of $S'$ by noetherian formal schemes. By definition the $f^{-1}(U_\alpha) = \mathcal{X} \times_S U_\alpha$ are noetherian, and since they cover $\mathcal{X}$, $\mathcal{X}$ is locally noetherian.

Thus fibered products with a universally noetherian morphism do not force us to leave the category of adic locally noetherian schemes. The usual sorites hold for the class of universally noetherian morphisms:

1.2.3 Proposition (i) An immersion is universally noetherian.

(ii) Let $f : \mathcal{X} \to S$ be a morphism of locally noetherian schemes. If $f : \mathcal{X} \to S$ is universally noetherian and $S'$ is any locally noetherian formal $S$-scheme, the base-change $\mathcal{X} \times_S S' \to S'$ is universally noetherian.

(iii) If $f : \mathcal{X} \to S$ and $g : \mathcal{Y} \to \mathcal{X}$ then $f \circ g : \mathcal{Y} \to S$ is universally noetherian.

(iv) If $f : \mathcal{X} \to S$ and $g : \mathcal{Y} \to S$ are universally noetherian morphisms then so is $f \times g : \mathcal{X} \times_S \mathcal{Y} \to S$.

Proof. For (i) it suffices to treat the case of open and closed immersions. Since any base-change of an open (resp. closed) immersion is open (resp. closed), the assertion is clear in the case of closed immersions, and for open immersions it suffices to add that an open immersion of locally noetherian formal schemes is quasi-compact. Assertions (ii) and (iii) follow from the definition and the transitivity of fibered products, while (iv) follows from (i) and (ii).

We will show that (iii) in the proposition has a partial converse (proposition 1.2.5). The property of being universally noetherian is local on the base and, with a suitable restriction, on the source:
1.2.4 Proposition Let \( f : \mathcal{X} \to S \) be a morphism of locally noetherian schemes.

(i) If \( f \) is quasi-compact and \( \{ U_\alpha \}_{\alpha \in I} \) is an open cover of \( \mathcal{X} \), then \( f \) is universally noetherian if and only if each of the induced morphisms \( U_\alpha \to S \) is universally noetherian.

(ii) If \( \{ V_\alpha \}_{\alpha \in I} \) is an open cover of \( S \), then \( f \) is universally noetherian if and only if the morphisms \( f^{-1}(V_\alpha) \to V_\alpha \) are universally noetherian.

Proof. Necessity in assertions (i) and (ii) follows from the sorites. To prove the condition is sufficient in (i) we first observe that for any \( S' \to S \) with \( S' \) noetherian, each of the \( U_\alpha \times_S S' \) are noetherian. Since the \( U_\alpha \times_S S' \) cover \( \mathcal{X} \times_S S' \), the latter is locally noetherian. On the other hand since \( \mathcal{X} \to S \) is quasicompact, so is \( \mathcal{X} \times_S S' \to S' \), from which it follows that \( \mathcal{X} \times_S S' \) is quasicompact, hence noetherian. This proves (i), and sufficiency in (ii) follows from (i), as one sees by taking \( U_\alpha = f^{-1}(V_\alpha) \) and observing, first, that \( f \) is necessarily quasi-compact since all of the \( f^{-1}(V_\alpha) \to V_\alpha \) are, and second that the composite morphisms \( U_\alpha \to V_\alpha \to S \) are universally noetherian.

1.2.5 Proposition If \( \mathcal{X} \to S \) is universally noetherian and \( \mathcal{Y} \) is a locally noetherian formal \( S \)-scheme, any morphism \( \mathcal{X} \to \mathcal{Y} \) is universally noetherian.

Proof. Suppose \( T \) is a noetherian formal scheme and \( T \to S \) is a morphism. By (ii) of the last proposition we may assume that \( \mathcal{Y} \) is affine, in which case \( \mathcal{Y} \to S \) is separated and \( \mathcal{X} \times \mathcal{Y} T \to \mathcal{X} \times_S T \) is a closed immersion. Since by hypothesis \( \mathcal{X} \times_S T \) is noetherian, it follows that \( \mathcal{X} \times \mathcal{Y} T \) is noetherian as well, as required.

1.2.6 Definition A morphism \( f : X \to S \) of locally noetherian schemes is universally noetherian if \( X \times_S S' \) is noetherian for any morphism of schemes \( S' \to S \) with \( S' \) noetherian.

We say that an \( A \)-algebra \( B \) is universally noetherian if \( \text{Spec}(B) \to \text{Spec}(A) \) is universally noetherian in the above sense. Such \( A \)-algebras are sometimes called strongly noetherian but I avoid this terminology since it conflicts with a different notion from the theory of adic spaces.

The condition of definition 1.2.6 is a priori weaker than the condition that \( f \) be universally noetherian in the sense of definition 1.2.1 when \( X \) and
$S$ are regarded as (discrete) formal schemes. In fact the latter condition requires that $X \times_S S' \to S$ be noetherian for any morphism $S' \to S$ with $S'$ a noetherian formal scheme. In fact 1.2.1 and 1.2.6 are equivalent for a morphism of schemes. To see this we will need a general result of topological algebra, which combines [10, Ch. III §2 no. 12 Cor. 2] and [10, Ch. III §2 no. 10 Cor. 5]; see also the general discussion of completions in [10, III §2 no. 12]. In the next proposition and its corollary the “natural” topology of $\hat{R}$ is the topology it has as the completion of the preadic topological ring $(R, I)$: the ideals $(\hat{I}^n)$ are a basis of the neighborhoods of $0$. The completion of an ideal $M \subseteq R$ will be denoted by $\hat{M}$, and identified with an ideal of $\hat{R}$.

1.2.7 Proposition Suppose $R$ is a commutative ring and $I \subseteq R$ is a finitely generated ideal. Then for the $I$-preadic topology of $R$, 

$$(\hat{I}^n) = (\hat{I})^n = I^n \hat{R}.$$ 

(1.2.7.1)

Furthermore the natural topology of $\hat{R}$ is the $\hat{I}$-adic topology and the natural homomorphism $R/I^n \to \hat{R}/\hat{I}^n$ is an isomorphism. Finally if $R/I$ is noetherian then so is $\hat{R}$, and in particular $\hat{R}$ is a Zariski ring.

1.2.8 Corollary Let $(R, J)$ be a preadic ring with $J$ finitely generated. If $R/J$ is noetherian then for any ideal $M \subseteq R$, $M^n = \hat{M}^n$.

Proof. The proposition says that $\hat{R}$ is noetherian and therefore the ideal $\hat{M}^n \subseteq R$ is finitely generated, and thus closed since $\hat{R}$ is a Zariski ring. If $i : R \to \hat{R}$ is the canonical homomorphism, $i(M^n) \subseteq \hat{M}^n \subseteq \hat{M}^n$ and $i(M^n)$ is dense in $\hat{M}^n$. Since both $M^n$ and $\hat{M}^n$ are closed, $M^n = \hat{M}^n$. ■

1.2.9 Lemma A universally noetherian morphism $X \to S$ of locally noetherian schemes is also universally noetherian when $X$ and $S$ are considered as formal schemes.

Proof. It suffices to treat the affine case $X = \text{Spec}(B)$, $S = \text{Spec}(A)$. Let $(C, J)$ be an adic noetherian ring. Then $B \otimes_A J$ is an ideal of finite type in $B \otimes_A C$ and an ideal of definition of $B \otimes_A C$, and by hypothesis the ring $(B \otimes_A C)/(B \otimes_A J) \simeq B \otimes_A (C/J)$ is noetherian. The last proposition then shows that $B \otimes_A C$ is noetherian, as required. ■

1.2.10 Proposition Suppose $f : X \to S$ is a morphism of locally noetherian formal schemes and $f_0 : X \to S$ is the corresponding morphism of reduced closed subschemes. Then $f$ is universally noetherian if and only if $f_0$ is universally noetherian.
Proof. Necessity: if \( f \) is universally noetherian then so is \( \mathcal{X} \times \mathcal{S} \to \mathcal{S} \), and the closed immersion \( X \to \mathcal{X} \times \mathcal{S} \to \mathcal{S} \) is universally noetherian as well.

Sufficiency: it is enough to check the case where \( \mathcal{X} = \text{Spf}(B) \) and \( \mathcal{S} = \text{Spf}(A) \) are formally affine. Then \( A \) and \( B \) are noetherian, and if \( I \subset A \), \( J \subset B \) are maximal ideals of definition, \( X = \text{Spec}(B/J) \) and \( S = \text{Spec}(A/I) \).

Suppose \( B' \) is a noetherian \( A \)-algebra with ideal of definition \( J' \) such that \( IB' \subseteq J' \). By hypothesis the ring \( (B/J) \otimes_{A/I} (B'/J') \) is noetherian. Set \( B'' = B \otimes_A B' \) and \( J'' = J \otimes B' + B \otimes J' \subseteq B'' \). By definition \( B \otimes_A B' \) is the completion \( \hat{B}'' \) for the \( J'' \)-adic topology. Since \( J \subset B \) and \( J' \subset B' \) are finitely generated, so is \( J'' \) and it follows from proposition 1.2.7 that \( \hat{B}'' \), i.e. \( B \hat{\otimes}_A B' \) is noetherian, as required.

Recall that \( f \) is formally of finite type if, in the notation of the last proposition, \( f_0 \) is of finite type.

1.2.11 Corollary A morphism of formal schemes that is formally of finite type is universally noetherian.

Proof. Since a morphism of finite type is universally noetherian this follows from proposition 1.2.10.

If \( \mathcal{X} \) is an locally noetherian formal scheme and \( Y \subset \mathcal{X} \) is a closed subscheme, the completion \( \hat{\mathcal{X}}_Y \) of \( \mathcal{X} \) along \( Y \) is defined in the same way as in the case of ordinary schemes, c.f. [14] I Ch. 10.

1.2.12 Corollary If \( f : \mathcal{X} \) is a locally noetherian formal scheme and \( Y \subset \mathcal{X} \) is a closed subscheme, the canonical morphism \( i_Y : \hat{\mathcal{X}}_Y \to \mathcal{X} \) is universally noetherian.

Proof. The morphism of reduced schemes induced by \( i_Y \) is a closed immersion.

From the corollary and the sorites we conclude that if \( \mathcal{X} \to \mathcal{S} \) is of finite type and \( Y \subset \mathcal{X} \) is closed, \( \hat{\mathcal{X}}_Y \to \mathcal{S} \) is universally noetherian; this will be the main case of interest in section 5.

1.2.13 Corollary Suppose \((A, I)\) is an adic noetherian ring, \( S \subset A \) is a multiplicative system and \( B \) is the completion of \( S^{-1}A \) with respect to the ideal \( S^{-1}I \). Then \( B \) is a universally noetherian \( A \)-algebra.

Proof. We may assume that \( I \) is a maximal ideal of definition of \( A \). Suppose \( C \) is an adic noetherian ring with maximal ideal of definition \( K \). If \( A \to C \) is a continuous homomorphism then \( CI \subseteq K \). The completion \( J \) of \( S^{-1}I \)
is a maximal ideal of definition of $B$, $B/J \cong S^{-1}A/S^{-1}I$ and the isomorphism $(B/J) \otimes_{(A/I)} (C/K) \cong S^{-1}C/S^{-1}K$ shows that $(B/J) \otimes_{(A/I)} (C/K)$ is noetherian. Thus $B/J$ is a universally noetherian $A/I$-algebra, and it follows from proposition 1.2.10 that $A \to B$ is universally noetherian.

Applying the corollary when $S$ is the complement of an open prime ideal, we get:

1.2.14 Corollary Suppose $X$ is a locally noetherian formal scheme and $x$ is a point of $X$. Denote by $\hat{O}_x$ the completion of the local ring of $x$ with respect to an ideal of definition of $X$. Then $\text{Spf}(\hat{O}_x) \to X$ is an adic universally noetherian morphism.

From corollary 1.2.12 we see that $\text{Spf}(\hat{O}_x) \to X$ is also universally noetherian if $\hat{O}_x$ is given the adic topology defined by the maximal ideal, although the morphism $\text{Spf}(\hat{O}_x) \to X$ is not adic in this case.

The following proposition is an easy consequence of the fact that a formal scheme has the same underlying topological space as its reduced closed sub-scheme. Its equivalent properties define the notion of a radial morphism of adic locally noetherian schemes.

1.2.15 Proposition For any universally noetherian morphism $f: Y \to X$, the following are equivalent:

(i) $f$ is universally injective, i.e. for any morphism $X' \to X$ with $X'$ locally noetherian, $Y \times_X X' \to X'$ is injective.

(ii) The morphism induced by $f$ on the reduced closed subschemes of $Y$ and $X$ is radial.

The following result of Vámos [19, Theorem 11] is useful in constructing (counter)examples:

1.2.16 Proposition A field extension extension $L/K$ is finitely generated if and only if $L \otimes_K L$ is noetherian.

Since [19, Theorem 11] the comes at the end of a long discussion of more general topics we give a direct proof for the reader’s convenience. The direct implication is clear since by hypothesis $L$ is a localization of a finitely generated $K$-algebra. Suppose conversely that $L \otimes_K L$ is noetherian; the reverse implication is proven in a sequence of cases:
(i) $L/K$ inseparable algebraic. In this case we show that $L/K$ is finite. Observe first that if $E/F$ is an extension of fields and $E \otimes_F E$ is noetherian then $\Omega_{E/K}^1$ is a finite-dimensional $F$-vector space. Let $p > 0$ be the characteristic of $K$.

(a) We show that there is an $n \geq 0$ such that $L^n \subseteq K$. If not, there is an infinite sequence $\{u_n\}_{n>0}$ of elements of $L$ such that $u_n^p \in K$ and $u_n^{p-1} \notin K$. We claim that the $u_n$ are $p$-independent over $K$, i.e. that the monomials $\prod_{i \in S} u_k^{(n)}$ are linearly independent over $K$, where $S$ runs through finite sets of positive integers and $0 \leq k(n) < p$. Given this, theorem 21.4.5 of [14, 0 IV] shows that the elements $du_n \in \Omega_{L/K}^1$ for all $n > 0$ are linearly independent over $L$. By the preceding observation this implies that $L \otimes_K L$ is not noetherian, so $L^n \subseteq K$ for $n \gg 0$.

We show that $u_1, \ldots, u_n$ are $p$-independent by induction on $n$. When $n = 1$, the assertion is that $u_1, u_2, \ldots, u_1^{p-1}$ are linearly independent over $K$. If not, $u_1$ is separable over $K$, and since $u_1^p \in K$ we find that $u_1 \in K$, a contradiction. Suppose now that $u_1, \ldots, u_{n-1}$ are $p$-independent and let $x = u_n^{p-1}$. A non-trivial $p$-dependence relation among $u_1, \ldots, u_n$ can be written $\sum_{i<p} a_i u_i = 0$ with $a_i \in K(u_1, \ldots, u_{n-1})$ and not all of the $a_i$ are equal to zero since $u_1, \ldots, u_{n-1}$ are $p$-independent. Taking the $p^{n-1}$ power of this relation yields $\sum_{i<p} b_i x^i = 0$ with $b_i = a_i^{p^{n-1}} \in K$ and not all of the $b_i$ are zero. Again this implies that $x$ is separable over $K$, and since $x^p = u_n^p \in K$ we again conclude that $x \in K$, i.e. $u_n^{p-1} \in K$, a contradiction.

(b) We now set $L_n = Kl^n$, and show that $L_n/L_{n+1}$ is finite for all $n$. By part (a) we know $L_n = K$ for $n \gg 0$, so this will show that $L/K$ is finite.

Since $L \otimes_K L$ is noetherian so is $L \otimes_{L_1} L$, and by our earlier observation the $L$-space $\Omega_{L_1/L_1}^1$ has finite dimension. By theorem 21.4.5 of [14, 0 IV] $L/L_1$ has a finite $p$-base, so $L/L_1$ is finite since a $p$-base of $L/L_1$ generates it as an extension. Since Frobenius is an injective homomorphism, $L_p^n$ is a finite extension of $L_1^n = Kp^n L_{p+1}$. We conclude that $L_n = Kl^n$ is a finite extension of $L_{n+1} = Kl^{p+1}$ for all $n \geq 0$.

(ii) $L/K$ algebraic. We again show that $L/K$ is finite. Let $K^s$ be the separable closure of $K$ in $L$, so that $L/K^s$ is purely inseparable. Since
$L \otimes_K L$ is noetherian so is $L \otimes_K L$ and thus $L/K^s$ is finite by the last case. If $K^s/K$ were not finite, $K^s \otimes_K K^s$ would be an infinite direct sum of fields and the underlying topological space of $\text{Spec}(K^s \otimes_K K^s)$ would not be noetherian. Since $\text{Spec}(L \otimes_K L) \to \text{Spec}(K^s \otimes_K K^s)$ is faithfully flat the underlying space of $\text{Spec}(L \otimes_K L)$ would not be noetherian either. Therefore $K^s/K$ is finite, and so is $L/K$.

(iii) $L/K$ purely transcendental. In this case we show that if $L/K$ is not finitely generated then the ring $L \otimes_K L$ has an infinite strictly ascending chain of prime ideals.

Suppose that $x_1, x_2, x_3 \ldots$ is a sequence of algebraically independent elements and let $K_n = K[x_1, \ldots, x_n]$. The projection $f_n : L \otimes_K L \to L \otimes_K K_n$ is surjective and $L \otimes_K K_n$ is a domain, being the localization of a polynomial ring. Therefore $p_n = \ker(f_n)$ is a prime ideal. Clearly $p_n \subseteq p_{n+1}$, and the inclusion is strict since $x_{n+1} \otimes 1 - 1 \otimes x_{n+1}$ is in $p_{n+1}$ but not in $p_n$.

(iv) General case. Let $K \subseteq L_1 \subseteq L$ be an intermediate field with $L_1/K$ purely transcendental and $L/L_1$ algebraic. As before, $L \otimes_{L_1} L$ is noetherian, so $L/L_1$ is finite and it suffices to show that $L_1/K$ is finitely generated. Suppose that it is not and let $p_0 \subset p_1 \subset p_2 \subset \cdots$ be the chain of ideals of $L_1 \otimes_K L_1$ constructed in the last step. Since $L/L_1$ is finite, $L \otimes_K L$ is a finite $L_1 \otimes_K L_1$-algebra and for all $n$ there is a prime $q_n \subset L \otimes_K L$ above $p_n$. Then $q_1 \subset q_2 \subset q_3 \cdots$ is an infinite strictly ascending chain of prime ideals of $L \otimes_K L$, a contradiction.

1.2.17 Corollary Let $L/K$ be an extension of fields. Then $L$ is a universally noetherian $K$-algebra if and only if $L$ is a finitely generated extension of $K$.

Proof. If $L/K$ is finitely generated, $L$ is a localization of a $K$-algebra of finite type, and the conclusion follows from corollaries 1.2.11 and 1.2.3 (iii). Conversely if $L/K$ is universally noetherian, $L \otimes_K L$ is a noetherian ring and the proposition shows that $L/K$ is finitely generated.

1.2.18 Corollary Suppose $f : X \to S$ is universally noetherian, $x \in X$ and $y = f(x)$. The field extension $\kappa(x)/\kappa(y)$ is finitely generated.

Proof. The morphism $\mathcal{X} \times_S \text{Spf}(\kappa(y)) \to \text{Spf}(\kappa(y))$ is also universally noetherian, so we may assume $S = \text{Spf}(\kappa(y))$. The morphism $\text{Spf}(\mathcal{O}_x) \to X$ is universally noetherian, where $\mathcal{O}_x$ is the completion of the local ring at $x$. 

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for the adic topology of $\mathcal{O}_x$. The closed immersion $\text{Spec}(\kappa(x)) \to \text{Spf}(\hat{\mathcal{O}}_X)$ is also universally noetherian. It follows that $\kappa(y) \to \kappa(x)$ is universally noetherian, and the assertion follows from the theorem.

1.2.19 Example Suppose $k$ is a field. If $k[[x]]$ is given the $x$-adic topology, $k[x] \to k[[x]]$ is universally noetherian by corollary 1.2.12. On the other hand if $k[[x]]$ is given the discrete topology $k[x] \to k[[x]]$ is not universally noetherian; if it were $k(x) \to k((x))$ would be universally noetherian as well by corollary 1.2.13. Since $k((x))/k(x)$ is not finitely generated this contradicts corollary 1.2.17.

There are more direct arguments for this example. Let $A$ be the integral closure of $k[x]$ in $k[[x]]$, i.e. the henselisation of $k[x]$ at $(x)$ and $L$ the fraction field of $A$. As in the proof of the proposition $\text{Spec}(L \otimes_{k(x)} L)$ is not a noetherian space, and since $\text{Spec}(L \otimes_{k(x)} L)$ is open in $\text{Spec}(A \otimes_{k[x]} A)$, $\text{Spec}(A \otimes_{k[x]} A)$ is not a noetherian space either. Finally $k[[x]] \otimes_{k[x]} k[[x]]$ is a faithfully flat $A \otimes_{k[x]} A$-algebra, so $\text{Spec}(k[[x]] \otimes_{k[x]} k[[x]])$ is not a noetherian space.

2 Differentials and Smoothness

From now on all formal schemes are assumed to be locally noetherian. For emphasis we will frequently restate this assumption anyway.

2.1 Differential invariants. As usual we start with the affine case, and then globalize. For the whole of this section we fix a topological ring $R$ and a topological $R$-algebra $A$, preadic with ideal of definition $J \subset A$.

2.1.1 We begin with a review of the topological aspects of the module of relative 1-forms. We denote by $I$ the diagonal ideal $I = \text{Ker}(A \otimes_R A \to A)$, so that the ring of principal parts of order $r$ and the module of relative 1-forms are

$$P^r_{A/R} = (A \otimes_R A)/I^{r+1}, \quad \Omega^1_{A/R} = I/I^2.$$  

We denote by $d_0, d_1 : A \to P^n_{A/R}$ the morphisms $d_0(b) = b \otimes 1$, resp. $d_1(b) = 1 \otimes b$.

Ideals of $A \hat{\otimes}_R A$ will always have the induced topology. We topologize $\Omega^1_{A/R} = I/I^2$ as a subquotient of $A \otimes_R A$ (i.e. as a quotient of $I$ in the induced topology, or as a subobject of $P^2_{A/R}$; these are the same). For $K = A \otimes J + J \otimes A$ this coincides with the $K$-adic topology; this is evident
if \( A \otimes_R A \) is noetherian (Artin-Rees), but in general it follows from the fact that for any ideal \( M \subseteq A \)

\[
I \cap (A \otimes M^2 + M^2 \otimes A) \subseteq MI + I^2
\]

by [14 0IV 20.4.5.1] (c.f. [14 0IV Prop. 20.4.5]). For the \( A \)-module structure of \( \Omega^1_{A/R} \) defined by \( d_0 \) or \( d_1 \), we have

\[
J^n \Omega^1_{A/R} = K^n \Omega^1_{A/R}
\]

and the topology of \( \Omega^1_{A/R} \) is also the \( J \)-adic topology by [14 0IV Prop. 20.4.5] (this is without any assumption that \( \Omega^1_{A/R} \) is finitely generated). As in [14 0IV §20.7] we denote by \( \hat{\Omega}^1_{A/R} \) the completion of \( \Omega^1_{A/R} \) with respect to its subquotient topology. By the previous remarks \( \hat{\Omega}^1_{A/R} \) is also the \( J \)-adic completion of \( \Omega^1_{A/R} \) when the latter is regarded as a \( A \)-module via \( d_0 \) or \( d_1 \).

We denote by \( \hat{P}^n_{A/R} \) the completion of \( P^n_{A/R} \) with respect to its topology as a quotient of \( A \otimes_R A \), i.e. the \( K \)-adic topology.

If \( R \to A \) and \( A \to B \) are continuous homomorphisms of preadic rings, the canonical exact sequence of relative 1-forms for the triple \( R \to A \to B \) induces a sequence

\[
B \hat{\otimes}_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \to \hat{\Omega}^1_{B/A} \to 0 (2.1.1.1)
\]

which is not necessarily exact. It is “nearly exact” in the sense that \( \hat{\Omega}^1_{B/R} \to \hat{\Omega}^1_{B/A} \) is surjective and the image of \( B \hat{\otimes}_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \) is dense in the kernel of \( \hat{\Omega}^1_{B/R} \to \hat{\Omega}^1_{B/A} \), c.f. [14 0IV 20.7.17.3] and the discussion there. If \( A \to B \) is surjective with kernel \( K \), \( \hat{\Omega}^1_{B/A} = 0 \) and there is similar sequence

\[
K/K^2 \to B \hat{\otimes}_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \to 0 (2.1.1.2)
\]

with the same “near exactness” property of (2.1.1.1) c.f. [14 0IV 20.7.20]

The sequence

\[
0 \to I \to A \otimes_R A \to A \to 0
\]

is strict exact; in fact by construction \( I \) has the induced topology, and the image of \( K^n \subseteq A \otimes_R A \) in \( A \) is \( J^n \). Since the completion of a strict exact sequence is strict exact [10 Ch. III §2 no. 12 Lemme 2], the sequence

\[
0 \to \hat{I} \to A \hat{\otimes}_R A \to A \to 0 (2.1.1.3)
\]

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is exact, in which $A \hat{\otimes}_R A \to A$ is induced by $a \otimes b \mapsto ab$. Our first goal is to show that when $A$ is a universally noetherian $R$-algebra we may identify $\hat{\Omega}^1_{A/R} \simeq \hat{I}/\hat{I}^2$ and $\hat{P}^n_{A/R} \simeq A \hat{\otimes}_R A/\hat{I}^{n+1}$, c.f. proposition 2.1.2 below.

2.1.2 Proposition With the hypotheses and notation of §2.1, suppose that $R$ is noetherian, and denote by $I \subset A \otimes_R A$ the diagonal ideal of $R \to A$. If $A$ is a universally noetherian $R$-algebra, there are functorial isomorphisms $\hat{\Omega}^1_{A/R} \simeq \hat{I}/\hat{I}^2$ and $\hat{P}^n_{A/R} \simeq (A \hat{\otimes}_R A)/\hat{I}^{n+1}$.

Proof. If $J \subset A$ is an ideal of definition, $J$ is finitely generated and thus $K = A \otimes J + J \otimes A$ is a finitely generated ideal of $R = A \otimes_R A$. The sequence $0 \to I^2 \to I \to \Omega^1_{A/R} \to 0$ is strict exact by definition of the topologies involved, so its completion $0 \to (I^2)^{\hat{}} \to \hat{I} \to \hat{\Omega}^1_{A/R} \to 0$ is also strict exact. Since $A$ is a universally noetherian $R$-algebra, $A \hat{\otimes}_R A$ is noetherian and corollary 1.2.8 shows that this exact sequence is $0 \to \hat{I}^2 \to \hat{I} \to \hat{\Omega}^1_{A/R} \to 0$ and the first assertion follows. The second is proven in the same way. \[\blacksquare\]

2.1.3 Corollary If $R$ is noetherian and $A$ is an universally noetherian $R$-algebra, $\Omega^1_{A/R}$ is generated as a $A$-module by finitely many elements of the form $1 \otimes x - x \otimes 1$.

Proof. We know $I$ is generated by elements of the form $1 \otimes x - x \otimes 1$ and $\hat{I}$ is the $\hat{K}$-adic completion of $I$, where as before $K = A \otimes_R J + J \otimes_R A$. Then $\hat{I}$ is generated by the $1 \otimes x - x \otimes 1$ for all $x \in A$, and thus by finitely many of them, since $A \hat{\otimes}_R A$ is noetherian. \[\blacksquare\]

For $x \in A$ we will use $dx$ to denote both the image of $1 \otimes x - x \otimes 1$ in $\Omega^1_{A/R}$ and the image of $1 \hat{\otimes} x - x \hat{\otimes} 1$ in $\hat{\Omega}^1_{A/R}$; this should not cause confusion.

2.1.4 Corollary If $R \to A \to B$ are homomorphisms of adic noetherian rings with $R \to A$ and $R \to B$ universally noetherian, the sequence $B \otimes_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \to \hat{\Omega}^1_{B/A} \to 0$
is exact. If \( A \to B \) is surjective with kernel \( K \), the sequence

\[
K/K^2 \to B \otimes_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R} \to 0
\]

is exact.

**Proof.** By proposition \( \text{1.2.5} \) the hypotheses imply that \( A \to B \) is universally noetherian. Therefore \( \Omega^1_{A/R} \) is a finitely generated \( A \)-module and \( \hat{\Omega}^1_{B/A} \) and \( \hat{\Omega}^1_{B/R} \) are finitely generated \( B \)-modules. Thus in \( \text{2.1.1} \) we may replace the completed tensor product by an ordinary one. Since \( B \) is noetherian, any submodule of the finitely generated modules \( \hat{\Omega}^1_{B/R} \) and \( \hat{\Omega}^1_{B/A} \) is closed, and exactness follows from the “near exactness” of the sequence \( \text{2.1.1.1} \). The argument in the case of the second sequence is the same.

Let \( J \) be an ideal of definition \( J \subset A \) and set \( A_n = A/J^{n+1} \). For \( n' \geq n \) there is a natural \( A \)-module homomorphism \( \Omega^1_{A_n/R} \to \Omega^1_{A_n/R} \), and the discussion of \( \text{[14, OIV 20.7.14]} \) shows that their inverse limit is the separated completion of \( \Omega^1_{A/R} \), whence a canonical isomorphism

\[
\hat{\Omega}^1_{A/R} \cong \lim_{\leftarrow n} \Omega^1_{A_n/R} \tag{2.1.4.1}
\]

**2.1.5 Example** Let \( J \) be an ideal of definition of \( R \), and let \( A = R \{ T_1, \ldots, T_d \} \) be the \( J \)-adic completion of the polynomial ring \( R \{ T_1, \ldots, T_d \} \). With \( JA \) is an ideal of definition of \( A \), \( A \) is an \( R \)-algebra that is topologically of finite type, and therefore universally noetherian. Then \( 2.1.4.1 \) shows that \( \hat{\Omega}^1_{A/R} \) is free over \( A \) with basis \( dT_1, \ldots, dT_d \).

**2.1.6 Proposition** Let \( A \) be a universally noetherian \( R \)-algebra. If \( M \subset A \) is an ideal containing an ideal of definition and \( B \) is the \( M \)-adic completion of \( A \), there is a natural and functorial isomorphism

\[
B \otimes_A \hat{\Omega}^1_{A/R} \cong \hat{\Omega}^1_{B/R}
\]

**Proof.** Setting \( K = M^n \) in the second exact sequence of corollary \( \text{2.1.4} \) and \( B_n = A/M^n \) yields exact sequences

\[
M^n/M^{2n} \to (A/M^n) \otimes_A \hat{\Omega}^1_{A/R} \to \Omega^1_{B_n/R} \to 0
\]

for all \( n \geq 0 \). Since the pro-object \( \{ M^n/M^{2n} \}_{n \geq 0} \) is essentially zero, its image in \( \{(A/M^n) \otimes_A \hat{\Omega}^1_{A/R} \}_{n \geq 0} \) is essentially zero and in particular Mittag-Leffler. Therefore the inverse limit over \( n \) is an isomorphism

\[
B \otimes_A \hat{\Omega}^1_{A/R} \cong \lim_{\leftarrow n} \Omega^1_{B_n/R}
\]

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and we may replace the completed tensor product by an ordinary one since \( \hat{\Omega}_A^{1/R} \) is finitely generated. The assertion then follows from 2.1.4.1.

2.1.7 Example If \((R, J)\) is adic noetherian, we saw in the last example that for \(R\)-algebra \( A = R\{X_1, \ldots, X_d\} \) with the \(J\)-adic topology, \( \hat{\Omega}_A^{1/R} \) is finitely generated. The assertion then follows from 2.1.4.1.

2.1.8 Deformations. One more consequence of \([14, 0 \text{ IV} 20.7.14]\) will be useful. Suppose \( A \) is a topological \( R \)-algebra and \( B \) is a discrete topological \( R \)-algebra with an ideal \( I \subset B \) such that \( I^2 = 0 \). If a continuous \( R \)-homomorphism \( u_0 : A \to B \) is given, the set of continuous \( R \)-homomorphisms \( u : A \to B \) having the same composite with \( B \to B/I \) is principal homogenous under the \( A \)-module of continuous derivations \( A \to I \); the argument is the same as the discrete case \([14, 0 \text{ IV} 20.1.1]\). Since \( u_0 \) is continuous and \( B \) is discrete, \( I \) is annihilated by an open ideal of \( A \), and it follows that the \( A \)-module of continuous homomorphisms \( \hat{\Omega}_A^{1/R} \to I \) of \( \hat{A} \)-modules, c.f. \([14, 0 \text{ IV} 20.7.14.4]\). When \( A \) is adic, the topology of \( \hat{\Omega}_A^{1/R} \) is induced by the topology of \( A \), and it follows that any \( A \)-linear \( \hat{\Omega}_A^{1/R} \to I \) is continuous.

2.1.9 Globalization. If \( A \) is a universally noetherian \( R \)-algebra and \( S \subset A \) is a multiplicative system, \( \hat{\Omega}_A^{1/R} \) is a finitely generated \( A \)-module, and therefore the completion of the canonical isomorphism \( \Omega_{S^{-1}A/R} \simeq S^{-1}\Omega_A^{1/R} \) is an isomorphism

\[
\hat{\Omega}_{A^{(S^{-1})}/R} \simeq A\{S^{-1}\} \otimes_A \hat{\Omega}_{A/R}^{1}.
\]

of finitely generated \( A\{S^{-1}\}\)-modules, where as before \( A\{S^{-1}\} \) is the completion of \( S^{-1}A \). From this we see that the construction of \( \hat{\Omega}_A^{1/R} \) globalizes: for any universally noetherian morphism \( f : X \to S \) there is a coherent
\( \mathcal{O}_X \)-module \( \Omega^1_{X/S} \) such that

\[
\Gamma(\text{Spf}(A), \Omega^1_{X/S}) = \hat{\Omega}^1_{A/R} \tag{2.1.9.1}
\]

for all open affines \( \text{Spf}(A) \subseteq X, \text{Spf}(R) \subseteq S \) such that \( f(\text{Spf}(A)) \subseteq \text{Spf}(R) \) and \( A \) is a universally noetherian \( R \)-algebra (note the absence of a “hat,” as per our convention on completions in the introduction). A similar argument shows that for any \( r \geq 0 \) there is a sheaf \( \mathcal{P}^r_{X/S} \) of rings such that

\[
\Gamma(\text{Spf}(A), \mathcal{P}^r_{X/S}) = \hat{P}^r_{A/R}. \tag{2.1.9.2}
\]

Like its affine counterpart \( \mathcal{P}^r_{X/S} \) has two \( \mathcal{O}_X \)-algebra structures, with respect to both of which it is a finite \( \mathcal{O}_X \)-algebra.

In the previous discussion we did not need to assume that \( X \to S \) was separated, but if it is the diagonal \( X \to X \times_S X \) is a closed immersion defined by an ideal \( \mathcal{I} \subset \mathcal{O}_{X \times_S X} \) with the property that in the above affine setting,

\[
\Gamma(\text{Spf}(A), \mathcal{I}) = \hat{I}
\]

where as before \( I \subset A \otimes_R A \) is the kernel of \( A \otimes_R A \to A \). In this case proposition \( \text{[2.1.2]} \) yields canonical and functorial isomorphisms

\[
\Omega^1_{X/S} = \mathcal{I}/\mathcal{I}^2, \quad \mathcal{P}^n_{X/S} = \mathcal{O}_{X \times_S X}/\mathcal{I}^n. \tag{2.1.9.3}
\]

The standard exact sequences globalize immediately: if \( Y \to S \) and \( X \to S \) are universally noetherian and \( f : Y \to X \) is a morphism (necessarily universally noetherian), the sequence

\[
f^* \Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0 \quad \tag{2.1.9.4}
\]

is exact; if in addition \( Y \to X \) is a closed immersion with ideal \( \mathcal{K} \), the sequence

\[
\mathcal{K}/\mathcal{K}^2 \to f^* \Omega^1_{X/S} \to \Omega^1_{Y/S} \to 0 \quad \tag{2.1.9.5}
\]

is exact; these assertions follow immediately for corollary \( \text{[2.1.4]} \). Finally, if \( J \subset \mathcal{O}_X \) is an ideal of definition and \( X_n \) is the closed subscheme of \( X \) defined by \( J^{n+1} \), the isomorphism \( \text{[2.1.4.1]} \) globalizes to

\[
\Omega^1_{X/S} \simeq \varprojlim_n \Omega^1_{X_n/S}. \tag{2.1.9.6}
\]

From proposition \( \text{[2.1.10]} \) we get
2.1.10 Proposition If \( \mathcal{X} \to \mathcal{S} \) is separated and universally noetherian and \( Y \subset \mathcal{X} \) is a closed subscheme, the canonical morphism

\[
i^*_Y \Omega^1_{\mathcal{X}/\mathcal{S}} \to \Omega^1_{\mathcal{X}_Y/\mathcal{S}}
\]
is an isomorphism.

The deformation theory of section 2.1.8 globalizes in the same way. If \( f : \mathcal{X} \to \mathcal{S} \) is universally noetherian, \( Z \) is an affine scheme over \( \mathcal{S}, Z_0 \hookrightarrow Z \) is a closed immersion whose ideal \( I \) is such that \( I^2 = 0 \), and \( g : Z_0 \to \mathcal{X} \) is a \( \mathcal{S} \)-morphism, the sheaf of liftings of \( g \) to a morphism \( u : Z \to \mathcal{X} \) making the diagram

\[
\begin{array}{ccc}
Z_0 & \xrightarrow{g} & \mathcal{X} \\
\downarrow u & & \downarrow \quad \\
Z & \xrightarrow{\quad} & \mathcal{S}
\end{array}
\tag{2.1.10.1}
\]

commutative is a pseudo-torsor under the sheaf \( M = Hom_{\mathcal{O}_{Z_0}}(g^*\Omega^1_{\mathcal{X}/\mathcal{S}}, I) \).

2.2 Smooth and quasi-smooth morphisms. The standard definition of smoothness extends immediately to the setting of formal schemes:

2.2.1 Definition A morphism \( f : \mathcal{X} \to \mathcal{S} \) of locally noetherian formal schemes is smooth (resp. unramified, étale) if it is of finite type and formally smooth (resp. formally unramified, formally étale) in the sense of definition 1.1.1.

This definition is equivalent to the one given by Berthelot [4, 2.1.5] in the case of morphisms of locally noethrian formal schemes over a complete discrete valuation ring.

2.2.2 Definition A morphism \( f : \mathcal{X} \to \mathcal{S} \) of locally noetherian formal schemes is quasi-smooth (resp. quasi-unramified, quasi-étale) if it is universally noetherian and formally smooth (resp. formally unramified, formally étale)

This definition of “quasi-smooth” conflicts with [2, Ch. IV 1.5.1]. As we will not use Berthelot’s notion, this will not be a problem.

It is clear that a smooth morphism is quasi-smooth, and conversely a quasi-smooth morphism of finite type is smooth. By proposition 1.1.4 a quasi-smooth morphism is flat. In particular a quasi-étale morphism is flat and quasi-unramified; I do not know if the converse is true.
The next proposition summarizes the basic properties of quasi-smooth, quasi-unramified and quasi-étale morphisms. They follow from the results on universally noetherian morphisms in section §2.2 and basic properties of formally smooth (resp. formally unramified, formally étale) morphisms whose proofs are entirely parallel to the corresponding assertions for morphisms of schemes, c.f. [14, §17] propositions 17.1.3–5.

2.2.3 Proposition

(i) An immersion is quasi-unramified. An open immersion is quasi-étale.

(ii) If $f : X \to S$ and $g : Y \to X$ are quasi-smooth (resp. quasi-unramified, quasi-étale) then so is $g \circ f : Y \to S$.

(iii) If $X \to S$ is quasi-smooth (resp. quasi-unramified, quasi-étale) and $S' \to S$ is any morphism of locally noetherian schemes, $X \times_S S' \to S'$ is quasi-smooth (resp. quasi-unramified, quasi-étale).

(iv) If $f : X \to S$ and $g : Y \to S$ are quasi-smooth (resp. quasi-unramified, quasi-étale) then so is $f \times g : X \times_S Y \to S$.

(v) If $f : X \to S$ and $g : Y \to X$ are morphisms such that $f \circ g$ is quasi-unramified, then $g$ is quasi-unramified.

(vi) If $f : X \to S$ is quasi-unramified, $g : Y \to X$ a morphism and $f \circ g$ is quasi-smooth (resp. quasi-étale) then $g$ is quasi-smooth (resp. quasi-étale).

(vii) If $f : X \to S$ is quasi-étale and $g : Y \to X$ a morphism then $f \circ g$ is quasi-smooth (resp. quasi-étale) if and only if $g$ is quasi-smooth (resp. quasi-étale).

In statements (v) through (vii) note that $f \circ g$ being quasi-smooth or quasi-étale implies that it is universally noetherian, so that $g$ is universally noetherian by proposition §2.5.

We can now return to a question that was left open in section §1.1:

2.2.4 Proposition Let $f : X \to S$ be a universally noetherian morphism.

(i) If $\{U_\alpha\}$ is an open cover of $X$ and $f_\alpha : U_\alpha \to S$ is the composite of $f$ with the open immersion $U_\alpha \to X$, then $f$ is quasi-smooth (resp. quasi-unramified, quasi-étale) if and only if all the $f_\alpha$ are quasi-smooth (resp. quasi-unramified, quasi-étale).
(ii) If \( \{V_\alpha\} \) is an open cover of \( S \) then \( f \) is quasi-smooth (resp. quasi-unramified, quasi-étale) if and only if the morphisms \( f^{-1}(V_\alpha) \to V_\alpha \) are quasi-smooth (resp. quasi-unramified, quasi-étale).

**Proof.** Assertion (ii) follows from (i), and the quasi-unramified and quasi-étale cases of (i) are formal consequences of the definitions, as in the proof of [14 IV Prop. 17.1.6], together with the basic properties of universally noetherian morphisms. In the quasi-smooth case the main thing is to prove formal smoothness, and we can again follow the argument of loc. cit., the point being that with the assumptions of (i), the set of local liftings \( u \) in the diagram \( \xymatrix{ R \ar[r] & \hat{X}_Y \ar[r] & X_Y \ar[r] & X \ar[l] } \) is a torsor under the sheaf \( M = \text{Hom}_{O_{Z_0}}(h^*\Omega^1_{X/S}, T) \), and \( Z_0 \) being an affine scheme, this torsor is trivial.

**2.2.5 Proposition** A universally noetherian morphism \( X \to S \) is quasi-unramified if and only if \( \Omega^1_{X/S} = 0 \).

**Proof.** We may assume that \( X = \text{Spf}(A) \) and \( S = \text{Spf}(R) \) are formally affine, so that \( A \) is a universally noetherian \( R \)-algebra. It suffices to check that the \( R \)-algebra \( A \) is formally unramified if and only if \( \hat{\Omega}^1_{A/R} = 0 \), which is [14 0 IV Prop. 20.7.4].

**2.2.6 Corollary** If \( Y \subset X \) is a closed subscheme then \( i_Y^* : \hat{X}_Y \to X \) is quasi-étale.

**Proof.** We know that \( i_Y \) is universally noetherian, so by proposition 2.2.5 it suffices to show that \( \Omega^1_{X/Y} = 0 \). We can assume \( X = \text{Spf}(A) \) is formally affine, in which case \( X_Y = \text{Spf}(\hat{A}) \), and then \( \hat{\Omega}^1_{A/A} \simeq \hat{\Omega}^1_{\hat{A}/\hat{A}} = 0 \).

For example, if \( Y \to S \) is quasi-smooth and \( X \) is the completion of \( Y \) along a closed subscheme then \( X \to S \) is quasi-smooth. Note that the corresponding morphism of reduced closed subschemes need not be smooth. The next proposition shows that a standard criterion for smoothness in the case of a morphism of finite type remains true in the general case; as a consequence we get a structure theorem for quasi-smooth morphisms analogous to the usual one for morphisms of finite type.

**2.2.7 Lemma** Suppose \( B \) is adic and noetherian and \( f : M \to N \) is a homomorphism of finitely generated \( B \)-modules.

(i) \( f \) is the inclusion of a direct summand in the category of \( B \)-modules if and only if \( \text{Hom}_B(N, L) \to \text{Hom}_B(M, L) \) is surjective for all discrete \( B \)-modules \( L \) annihilated by an open ideal of \( B \).
(ii) If $N$ is projective and $p \in \text{Spf}(B)$ is such that the $B_p$-module homomorphism $M_p \to N_p$ is the inclusion of a direct summand, there is a $f \in B \setminus p$ such that the $B_f$-module homomorphism $M_f \to N_f$ is the inclusion of a direct summand.

**Proof.** For (i), the condition is clearly necessary, and to show that it is sufficient it suffices to show that $\text{Hom}_B(N, L) \to \text{Hom}_B(M, L)$ is surjective for all finitely generated $B$-modules $L$. If $L$ is finitely generated and $J \subset B$ is an ideal of definition, the hypothesis implies that $\text{Hom}_B(N, L/J^n L) \to \text{Hom}_B(M, L/J^n)$ is surjective; since $M$ and $N$ are finitely generated this says that

$$\text{Hom}_B(N, L) \otimes_B B/J^n \to \text{Hom}_B(M, L) \otimes_B B/J^n$$

is surjective for all $n$, and the assertion follows by the faithful flatness of the $J$-adic completion. Part (ii) is a consequence of [14, Cor. 19.1.12].

The proof of the next proposition uses properties of the functor $\text{Exalcotop}$ defined in [14, 0IV §18.5]. One property of this functor apparently not stated in loc. cit. is the isomorphism

$$\text{Exalcotop}_A(B, L) \simeq \text{Hom}_B(K/K^2, L)$$

which holds when $A \to B$ is a surjective homomorphism of linearly topologized rings with kernel $K$. It is however an immediate consequence of a similar isomorphism [14, 0IV §18.5.1] for the related functor $\text{Exantop}$; the case of discrete rings is [14, 0IV §18.3.8.1] and [14, 0IV §18.4.2.1].

**2.2.8 Proposition** Suppose that $\mathcal{Y}/S$ and $\mathcal{X}/S$ are universally noetherian and $f: \mathcal{Y} \to \mathcal{X}$ is an $S$-morphism.

(i) If $\mathcal{Y} \to S$ is quasi-smooth, then $f$ is quasi-smooth if and only if the $O_{\mathcal{Y}}$-module homomorphism $f^*\Omega^1_{\mathcal{X}/S} \to \Omega^1_{\mathcal{Y}/S}$ is the inclusion of a local direct summand.

(ii) (Jacobian criterion) If $f$ is a closed immersion with ideal $\mathcal{K}$ and $\mathcal{X}/S$ is quasi-smooth, then $\mathcal{Y} \to S$ is quasi-smooth if and only if the $O_{\mathcal{Y}}$-module homomorphism $\mathcal{K}/\mathcal{K}^2 \to f^*\Omega^1_{\mathcal{X}/S}$ is the inclusion of a local direct summand.

**Proof.** The hypotheses imply that $\mathcal{Y} \to \mathcal{X}$ is universally noetherian (prop. 1.2.4 (iii)), so we only have to prove formal smoothness in both cases. We may work locally everywhere, so we may assume that $\mathcal{Y} \to \mathcal{X} \to S$ is the
formal spectrum of $R \to A \to B$. By [14] 0IV 19.4.4, a morphism $A \to B$ of
topological algebras is formally smooth if and only if $\text{Exalcotop}_A(B, L) = 0$
for any discrete $B$-module $L$ annihilated by an open ideal of $B$. In the
situation of the proposition there is an exact sequence

$$0 \to \text{Der}_A(B, L) \to \text{Der}_R(B, L) \to \text{Der}_R(A, L) \to$$

$$\to \text{Exalcotop}_A(B, L) \to \text{Exalcotop}_R(B, L) \to \text{Exalcotop}_R(A, L)$$

for any discrete $B$-module $L$ annihilated by an open ideal of $B$ (c.f. [14]
0IV) Propositions 20.3.5 and 20.3.7).

In case (i) we have $\text{Exalcotop}_R(B, L) = 0$ for all $L$ as above, and thus
$\text{Exalcotop}_A(B, L) = 0$ if and only if $\text{Der}_R(B, L) \to \text{Der}_R(A, L)$ is surjective.
Equivalently, $A \to B$ is formally smooth if and only if $\text{Hom}_B(\hat{\Omega}^1_{B/R}, L) \to$
$\text{Hom}_B(B \otimes_A \hat{\Omega}^1_{A/R}, L)$ is surjective for all discrete $L$ annihilated by an open
ideal of $B$. By (i) of the lemma, this is equivalent to $B \otimes_A \hat{\Omega}^1_{A/R} \to \hat{\Omega}^1_{B/R}$
being the inclusion of a direct summand.

In the case of (ii) we have $\text{Exalcotop}_R(A, L) = 0$ and $\text{Exalcotop}_A(B, L)$
is given by [2.2.7.1]. Therefore $R \to B$ is formally smooth if and only if $\text{Hom}_B(B \otimes_A \hat{\Omega}^1_{A/R}, L) \to \text{Hom}_B(K/K^2, L)$ is surjective for all $L$. As before,
this condition is equivalent to $K/K^2 \to B \otimes_A \hat{\Omega}^1_{A/R}$ being the inclusion of a
direct summand.

2.2.9 Corollary Let $\mathcal{X} \to S$ be a morphism of formal schemes, $x \in \mathcal{X}$ is
a point and $\hat{\mathcal{O}}_x$ is the completion of the local ring $\mathcal{O}_x$ with respect to the
adic topology of $\mathcal{O}_X$. Suppose that some neighborhood $\mathcal{V}$ of $x$ has a closed
embedding $\mathcal{V} \hookrightarrow \mathcal{Y}$ over $S$ into a quasi-smooth $S$-scheme $\mathcal{Y}$. If the composite
morphism $\text{Spf}(\hat{\mathcal{O}}_x) \to S$ is quasi-smooth, there is an open neighborhood $U$
of $x$ such that $U \to S$ is quasi-smooth.

Proof. We can assume $\mathcal{X} = \mathcal{V}$, and denote by $\mathcal{K}$ the ideal of $f : \mathcal{X} \to \mathcal{V}$. Since
$\mathcal{V} \to S$ is universally noetherian, so is $\mathcal{X} \to S$. By (ii) of lemma [2.2.7] there is
an open neighborhood $U \subseteq \mathcal{X}$ of $x$ on which the $\mathcal{O}_X$-module homomorphism
$\mathcal{K}/\mathcal{K}^2 \to f^*\Omega^1_{\mathcal{Y}/S}$ is the inclusion of a (local) direct summand, and the
assertion follows from (ii) of proposition [2.2.8].

The condition that $\mathcal{X} \to S$ is locally embeddable into a quasi-smooth
$\mathcal{Y} \to S$ is satisfied when $\mathcal{X} \to S$ is of finite type, so this condition could be
regarded as a weak finiteness property.

The next proposition is a very special case of a very general (and diffi-
cult) criterion of Grothendieck for a homomorphism of topological rings to
be formally smooth, c.f. loc. cit. Th. 19.5.3 and Cor. 19.5.7, and more particularly [14] 0IV Rem. 19.5.8. Its conclusion could be taken as a definition of differentiably smooth in the formal case:

2.2.10 Proposition Let \( A \to B \to C \) be continuous homomorphisms of topological rings, and suppose that \( A \to B \) formally smooth, \( B \to C \) surjective with finitely generated kernel \( I \), and \( C \) is a Zariski ring. Then \( I/I^2 \) is a projective \( C \)-module and for all \( n \geq 0 \) the natural map \( \text{Sym}^n_C(I/I^2) \to I^n/I^{n+1} \) is an isomorphism. ■

2.2.11 Proposition Suppose \( f : \mathcal{X} \to S \) is a quasi-smooth morphism of locally noetherian formal schemes, and let \( \mathcal{I} \) be the ideal of the diagonal of \( f \). Then

(i) \( \Omega^1_{\mathcal{X}/S} \) is a locally free \( \mathcal{O}_\mathcal{X} \)-module of finite type, and

(ii) the natural morphism

\[
\text{Sym}^n_{\mathcal{O}_\mathcal{X}}(\Omega^1_{\mathcal{X}/S}) \to \mathcal{I}^n/\mathcal{I}^{n+1}
\]

is an isomorphism for all \( n \geq 0 \).

Proof. We may assume that \( \mathcal{X} \to S \) is the formal spectrum of \( A \to B \), so that \( A \) and \( B \) are noetherian and \( A \to B \) is universally noetherian. Then \( B \hat{\otimes}_A B \) is noetherian, and in fact a Zariski ring, so we may apply to previous proposition with \( B, C \) and \( I \) replaced by \( B \hat{\otimes}_A B \), \( B \) and \( \hat{I} = \text{Ker}(B \hat{\otimes}_A B \to B) \) respectively. The conclusion follows since \( \mathcal{I} \) is the sheaf of ideals associated to \( \hat{I} \), and \( \Omega^1_{\mathcal{X}/S} \) is the module associated to \( \hat{I}/\hat{I}^2 \). ■

2.2.12 Corollary If \( f : \mathcal{X} \to S \) is a quasi-smooth morphism of locally noetherian formal schemes, the diagonal \( \mathcal{X} \to \mathcal{X}_S(r) \) is a regular immersion.

Proof. As assertion is local we may assume \( \mathcal{X} = \text{Spf}(A) \) and \( S = \text{Spf}(R) \) are affine, and then \( \mathcal{X}_S(r) = \text{Spf}(A_R(r)) \). Both \( A \) and \( A_R(r) \) are formally smooth \( R \)-algebras, and \( A \) is a Zariski ring. The ideal \( I \subset A_R(r) \) of the diagonal is the kernel of the multiplication map \( A_R(r) \to A \); as this is surjective, proposition [2.2.10] shows that \( I \) is quasi-regular, i.e. locally generated by a quasi-regular sequence \( (f_i) \). Since \( B \) is noetherian it is \( I \)-adically separated, and \( (f_i) \) is regular by [13] 0IV 15.1.9. ■

If \( \mathcal{X}/S \) is quasi-smooth and \( \Omega^1_{\mathcal{X}/S} \) has constant rank, we call this rank the formal relative dimension of \( \mathcal{X}/S \), and denote it by \( \text{fdim}(\mathcal{X}/S) \). If \( \mathcal{X}/S \)
is of finite type, proposition 2.1.10 shows that $\text{fdim}(\mathcal{X}/S)$ is the relative dimension of $\mathcal{X}/S$, but this is not true in general. When $\mathcal{X}/S$ is quasi-smooth we will say that an open affine $U \subseteq \mathcal{X}$ is parallelizable if there are $x_1, \ldots, x_d \in \Gamma(U, \mathcal{O}_\mathcal{X})$ such that $\{dx_1, \ldots, dx_d\}$ is a basis of $\Gamma(U, \Omega^1_{\mathcal{X}/S})$; we will also say that the $x_1, \ldots, x_d$ are local coordinates on $\mathcal{X}/S$. It is clear that when $\mathcal{X}/S$ is quasi-smooth, the topology of $\mathcal{X}$ has a basis consisting of parallelizable open sets.

As an application we get a structure theorem for quasi-smooth morphisms similar to the one that obtains for formally smooth morphisms of finite type:

**2.2.13 Corollary** Suppose $f: \mathcal{X} \to S$ is quasi-smooth and $d = \text{fdim}(\mathcal{X}/S)$. Locally on $\mathcal{X}$ there is a factorisation $f = p \circ g$ where $g: \mathcal{X} \to \mathbb{A}^d_S$ is quasi-étale and $p: \mathbb{A}^d_S \to S$ is the canonical projection. In fact this factorisation exists on any open parallelizable $U \subseteq \mathcal{X}$.

**Proof.** We may assume $\mathcal{X} = \text{Spf}(B)$ and $S = \text{Spf}(R)$, so that $B$ a quasi-smooth $R$-algebra, and by further shrinking $\mathcal{X}$ we can assume $\Omega^1_{B/R}$ is free with basis $dx_1, \ldots, dx_d$. The sections $x_1, \ldots, x_d$ give a factorisation $R \to R[T_1, \ldots, T_d] \to B$ with $T_i \mapsto x_i$, and since $R \to B$ is continuous this extends to a factorisation $R \to R{T_1, \ldots, T_d} \to B$. If $A = R{T_1, \ldots, T_d}$ is given the topology induced by $A$, proposition 2.2.8 and the example after equation 2.1.4.1 show that $\hat{\Omega}^1_{B/A} = 0$. Therefore $A \to B$ is quasi-unramified by proposition 2.2.5 and it suffices to show that $A \to B$ is formally smooth. By proposition 2.2.8 this is so if and only if $B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R}$ is the inclusion of a local direct summand in the category of $B$-modules, but by construction this map is actually an isomorphism.

**2.2.14 Proposition** A morphism of locally noetherian formal schemes that is étale and radicial is an open immersion.

**Proof.** Suppose $f: \mathcal{Y} \to \mathcal{X}$ is étale and radicial, and let $J \subset \mathcal{O}_\mathcal{X}$ be an ideal of definition. Since $f$ is étale it is adic, and $f^*J \subset \mathcal{O}_\mathcal{Y}$ is an ideal of definition. If we set $X_n = V(J^n)$ and $Y_n = V(J^n \mathcal{O}_\mathcal{Y})$ then $X_n$, $Y_n$ are ordinary schemes and for all $n$ the induced map $f_n : Y_n \to X_n$ is étale and radicial. It is therefore an open immersion by [14, IV Th. 17.9.1], and since $f$ is the inductive limit of the $f_n$, it is an open immersion as well.

**2.2.15 Remark** We cannot weaken the hypothesis “étale” to “quasi-étale.” In fact if $\mathcal{X}$ is any formal scheme and $\mathcal{X}_Y$ is the completion of $\mathcal{X}$ along a
proper closed subscheme, the canonical morphism \( i_Y : \overline{X}_Y \rightarrow X \) is quasi-étale and radicial, but not an open immersion.

2.2.16 Proposition Let \( f : X \rightarrow S \) be a quasi-smooth morphism of formal schemes of characteristic \( p \).

(i) The relative \( q \)th power Frobenius \( F_{X/S} : X \rightarrow X^{(q)} \) is flat.

(ii) If \( f \) is formally of finite type, \( F_{X/S} \) is finite.

Proof. We first prove (i) and (ii) in the case when \( f \) is of finite type. Then \( f \) is smooth, and is the inductive limit of a sequence of smooth morphisms \( f_n : X_n \rightarrow S_n \) of schemes; here we choose an ideal of definition \( J \) of \( S \), and have set \( S_n = V(J^n) \) and \( X_n = V(f^*J^n) \). Then \( F_{X_n/S_n} \) is finite and flat for all \( n > 0 \), and for \( n' \geq n \) \( f_{n'} \) is the reduction modulo \( J^n \) of \( f_{n'} \). It follows that \( F_{X/S} \) is finite and flat.

We next prove (i) in the general case. Since the assertion to be proven is of local nature we may assume \( f \) factors

\[
X \xrightarrow{g} Y := \mathbb{A}^d_S \xrightarrow{p} S
\]

in which \( g \) is quasi-étale and \( p \) is the canonical projection. There is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/S}} & X^{(q)} \\
\downarrow{g} & & \downarrow{p_2} \\
Y \times_{Y^{(q)}} X^{(q)} & \xrightarrow{p_1} & Y^{(q)} \\
\downarrow{g^{(q)}} & & \downarrow{g^{(q)}} \\
F_{Y/S} & & F_{Y/S}
\end{array}
\]

in which the square is Cartesian. Since \( p \) is smooth, \( F_{Y/S} \) is finite and flat, and by base change the same is true for \( p_2 \). Since \( F_{X/S} = h \circ p_2 \) it suffices to show that \( h \) is flat. Again by base change \( g^{(q)} \) and \( p_1 \) are quasi-étale, and in particular quasi-unramified; then since \( g \) is quasi-étale, \( h \) is quasi-étale as well by proposition 2.2.13(vii). In particular \( h \) is quasi-smooth, and therefore flat, as required.

We assume finally that if \( f \) is formally of finite type, and show that \( F_{X/S} \) is finite. Again the assertion is local and we may assume \( f = p \circ g \) with the same diagram as above. By the first part of the argument we know that
If \( F_{Y/S} \) is finite, so by base change \( p_2 \) is also finite. It thus suffices to show that \( h \) is finite. In fact it is formally of finite type since \( g \) is, and its construction shows that it is an adic morphism; the claim then follows by [14 I Prop. 10.13.1].

I do not know if there are quasi-smooth morphisms \( X \to S \) such that \( F_{X/S} \) is not finite.

### 2.3 Ordinary Differential Operators.

In this article we are mainly concerned with arithmetic differential operators, but for the sake of completeness we show that the construction of the usual (Grothendieck) ring of differential operators extends to the case of a quasi-smooth morphism.

When \( X/S \) is quasi-smooth, the rings \( \mathcal{P}^n_{X/S} \) are coherent, locally free \( \mathcal{O}_X \)-modules, as are the \( \mathcal{O}_X \)-modules \( \text{Diff}^n_{X/S} = \text{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S}, \mathcal{O}_X) \) of differential operators of order \( n \). The projection maps for the \( \mathcal{P}^n_{X/S} \) induce injective maps \( \text{Diff}^{n'}_{X/S} \to \text{Diff}^n_{X/S} \) for \( n' \geq 0 \) and the \( \mathcal{O}_X \)-module of differential operators

\[
\mathcal{D}_{X/S} = \varinjlim_n \text{Diff}^n_{X/S}.
\]

is the direct limit in the category of \( \mathcal{O}_X \)-modules. On any parallelizable open affine, \( \mathcal{P}^n_{X/S} \) has a free basis \( \xi^I, |I| \leq n \) where \( \xi = 1 \otimes x - x \otimes 1 + I^{n+1} \) (we use the usual multi-index notation). As usual the basis of \( \text{Diff}^n_{X/S} \) dual to \( (\xi^I)_{|I| \leq n} \) will be denoted by \( (\partial_\xi^I)_{|I| \leq n} \).

The same construction as in the algebraic case (c.f. for example [14 0IV]) gives \( \mathcal{D}_{X/S} \) an \( \mathcal{O}_X \)-ring structure; recall that this is done by dualizing a family of morphisms

\[
\delta_{n,n'} : \mathcal{P}^{n+n'}_{X/S} \to \mathcal{P}^n_{X/S} \otimes \mathcal{P}^{n'}_{X/S} \quad x \otimes y \mapsto x \otimes 1 \otimes 1 \otimes y.
\]

(c.f. [14 IV]). Note that no completed tensor products are involved.

Although the lack of a category of “quasicoherent \( \mathcal{O}_X \)-modules” makes itself felt at this point, the fact that \( \mathcal{D}_{X/S} \) is an inductive limit of coherent \( \mathcal{O}_X \)-modules means that the sections of \( \mathcal{D}_{X/S} \) on any open affine are easily described; in particular if \( U = \text{Spf}(A) \) is a parallelizable open affine in \( X \) mapping to \( \text{Spf}(R) \subseteq S \), the \( A \)-module of sections \( D_{A/R} = \Gamma(U, \mathcal{D}_{U/S}) \) is simply the free \( R \) module with basis \( (\partial^{|I|}) \). Thus local computations in \( \mathcal{D}_{X/S} \) may be done just as in the algebraic case.
If

\[
\begin{array}{ccl}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{g} & S
\end{array}
\]

(2.3.0.1)

is a commutative diagram of locally noetherian formal schemes with \(X \to S\) and \(X' \to S'\) quasi-smooth, the functoriality morphisms

\[ df : \operatorname{Diff}_{X'/S'} \to f^* \operatorname{Diff}_{X/S} \]

are defined as usual; here the \(f^*\) must be understood in the sense of \(\mathcal{O}\)-modules on ringed spaces. The direct limit

\[ f^* D_X \to S = \lim_{\rightarrow} f^* \operatorname{Diff}_{X/S}^n \]

has a \((D_{X'/S'}, f^{-1} D_X)\)-bimodule structure, and the morphism

\[ df : D_{X'/S'} \to f^* D_X \to S \]

can be used to give a construction of the left \(D_{X'/S'}\)-module structure of the inverse image \(f^* M\) of a left \(D_X\)-module \(M\).

2.3.1 Stratifications. Stratifications of an \(\mathcal{O}_X\)-module relative to \(X/S\) are defined in the same way as in the algebraic case; we review this to set notation and terminology; we will use the same notation and terminology for analogous concepts to be discussed later: \(m\)-PD-stratifications, \(m\)-HPD-stratifications, and level \(m\) analytic stratifications.

If \(f : X \to S\) is a noetherian morphism of locally noetherian schemes we recall from the notation section of the introduction that \(X_S(r)\) denotes the \((r+1)\)-fold iterated fiber product of \(X\) over \(S\), and for any ordered subset \(L \subseteq [0, 1, \ldots, r]\) there is a standard projection morphism \(p_L : X_S(r) \to X_S(r')\) (where \(r' = \#L\)). When \(L\) is a singleton we identify \(X_S(0)\) with \(X\). Finally the \((r + 1)\)-fold relative diagonal morphism \(X \to X_S(r)\) is defined by the multiplication map \(m(r) : A_R(r) \to A\) when \(X = \text{Spf}(A)\) and \(S = \text{Spf}(R)\), and by patching in general. We denote by \(I(r) \subset A_R(r)\) the kernel of \(m(r)\) (note that \(A_R(r)\) is the completed tensor product of \(r + 1\) copies of \(A\)).

For \(n \in \mathbb{N}\) we denote by \(X_S^n(r) \subset X_S(r)\) the \(n\)th order infinitesimal neighborhood of the diagonal. When \(X = \text{Spf}(A)\) and \(S = \text{Spf}(R)\), \(X_S^n(r)\) is the affine formal scheme associated \(\hat{P}_{A/R}^n := A_R(r)/I(r)^{n+1}\). For any
\[ L \subseteq [0, 1, \ldots, r] \] with \( \#L = r' \) the projection \( p_L : X_S(r) \to X_S(r') \) induces morphisms
\[
p^L_0 : X^0_S(r) \to X^0_S(r').
\]
In particular when \( \#L = 1 \) the \( r + 1 \) projections
\[
p_i : X^0_S(r) \to X
\]
give \( X^0_S(r) \) \( r + 1 \) structures of a formal \( X \)-scheme, and for \( 0 \leq i \leq r \) the morphism \( p_i \) is finite and formally affine. In fact the rings \( P^m_{A/R}(r) \) of \( (r+1) \)-fold principal parts of order \( n \) sheafify to yield sheaves of rings \( P^n_{X/S}(r) \) on \( X \) with \( r + 1 \) distinct \( O_X \)-algebra structures, which we denote by
\[
d_i : O_X \to P^n_{X/S}(r).
\]
When \( r = 1 \) we drop the \( (1) \), and \( P^n_{X/S} \) is the ring of principal parts of order \( n \) that was defined earlier.

If \( M \) is an \( O_X \)-module, a series of isomorphisms
\[
\chi_n : p^{n*}_1 M \sim \to p^{n*}_0 M
\]
for \( n \geq 0 \) will be said to be compatible if

(i) \( \chi_0 = id_M \),

(ii) for \( n' \geq n \), the restriction of \( \chi_{n'} \) to \( X^0_S(r) \) is \( \chi_n \),

and a stratification of \( M \) relative to \( S \) if the cocycle condition holds as well:

(i) \( p^{n_1*}_0 (\chi_n) \circ p^{n_2*}_0 (\chi_n) = p^{n_2*}_0 (\chi_n) \) for all \( n \geq 0 \).

An \( S \)-stratified \( O_X \)-module is an \( O_X \)-module with an (unspecified) stratification. An \( O_S \)-linear morphism \( (M, \chi_n) \to (M', \chi'_n) \) of \( S \)-stratified \( O_X \)-modules is horizontal if it is compatible with the stratifications.

The data of an \( S \)-stratification \( \chi_n \) of \( M \) is equivalent to a family of morphisms
\[
\theta_n = p^{n*}_{1*}(\chi_n) : M \to p^{n*}_1 p^{0*}_0(M) = M \otimes_{O_X} P^n_{X/S}
\]
for \( n \geq 0 \), \( O_X \)-linear for the right structure of \( P^n_{X/S} \), which are compatible with the canonical morphisms \( P^n_{X'/S} \to P^n_{X/S} \), the identity for \( n = 0 \), and such that the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{\theta_{n+n'}} & M \otimes P^{n+n'}_{X/S} \\
\downarrow{\theta_n} & & \downarrow{\delta_{n+n'}} \\
M \otimes P^n_{X/S} & \xrightarrow{\theta_n \otimes 1} & M \otimes P^n_{X/S} \otimes P^n_{X/S}
\end{array}
\]
commutes for all \( n, n' \geq 0 \).

The same argument as in the algebraic case shows that the category of left \( D_{\mathcal{X}/S} \)-modules is equivalent to the category of \( S \)-stratified \( \mathcal{O}_X \)-modules. The essential point is the commutativity of \ref{dia:2.3.1.6} and the fact that the product in \( D_{\mathcal{X}/S} \) is defined, essentially, by dualizing the morphism \( \delta_{n,n'} \) in the diagram \ref{dia:2.3.1.6} we refer the reader to \cite[§2]{8} for the details. Finally, \( S \)-stratified \( \mathcal{O}_X \)-modules have the following “crystalline” property: if \( M \) is an \( S \)-stratified \( \mathcal{O}_X \)-module and \( f, g : Y \to X \) are \( S \)-morphisms of locally noetherian formal schemes that restrict to the same morphism \( Y \to \mathcal{X} \) on the reduced closed subscheme \( Y \subset Y \), there is a canonical isomorphism \( \chi(f,g) : g^*M \cong f^*M \) of \( \mathcal{O}_Y \)-modules; it is an isomorphism of \( D_{Y/S} \)-modules if \( Y \to S \) is quasi-smooth. The system of maps \( \chi(f,g) \) is transitive in the sense that

\[
\chi(f,g) \circ \chi(g,h) = \chi(f,h)
\]

for any three \( f, g, h : Y \to X \) reducing to the same map in the reduced closed subscheme of \( Y \).

3 Arithmetic Differential Operators

3.1 \( m \)-PD-structures. We are fortunate that the theory of divided power ideals was worked out in \cite{2} and \cite{3} in a very general setting, and few modifications are needed to adapt the theory to the formal case. We will briefly review this theory and how it is used to construct the arithmetic differential operator rings, pointing out the few places where perhaps something needs to be said about the formal case. We will assume the reader is familiar with the terminology and notation of \cite{4} and \cite{6}, but we will summarize some of the main points first.

From now on we fix a prime \( p \), and all formal schemes will be formal schemes over \( \mathbb{Z}_p \) (in addition to being noetherian). All divided powers will be assumed compatible with the canonical divided powers of \( (p) \).

3.1.1 Partial Divided Powers. Let \( R \) be a commutative ring and \( I \subset R \) an ideal. Recall that a partial divided power structure of level \( m \) on \( I \) or an \( m \)-PD-structure on \( I \) is a PD-ideal \((J, \gamma)\) in \( R \) such that

\[
I(p^m) + pI \subseteq J \subseteq I.
\]

We also say that \((I, J, \gamma)\) is an \( m \)-PD-structure on \( R \), that \((R, I, J, \gamma)\) is an \( m \)-PD-ring and that \((I, J, \gamma)\) is an \( m \)-PD-ideal in \( R \). A morphism \((R, I, J, \gamma) \to \)
(R', I', J', γ') of m-PD-rings is a ring homomorphism $f : R \to R'$ such that $f(I) \subseteq I'$ and $f$ induces a morphism $(J, γ) \to (J', γ')$ of PD-ideals.

We will use the following form of the definition of compatibility of $m$-PD-structures, as in [4 §1.2]:

3.1.2 Definition Suppose $R \to A$ is a homomorphism and $(a, b, α), (I, J, γ)$ are $m$-PD-structures on $R$ and $A$ respectively. The $m$-PD-structures $(a, b, α)$ and $(I, J, γ)$ are compatible if the following conditions hold, in which $b_1 = b + pR$:

(i) $b_1A + J$ has a PD-structure inducing the PD-structures $α$, $γ$ and the canonical PD-structure of $(p)$;

(ii) $b_1A \cap I \subseteq b_1A$ is a sub-PD-ideal.

There are a number of ways to reformulate the first condition, c.f. [4 Lemme 1.2.1 and Def. 1.2.2]. The second condition is used in the construction of the $m$-PD-adic filtration, see §3.1.4 below.

We will need the following from [4 1.3.2 and 1.3.4]:

3.1.3 Lemma (i) If $K \subset A$ is an ideal, the $m$-PD-structure $(I, J, γ)$ induces an $m$-PD-structure on $I(A/K)$ such that $A \to A/K$ is an $m$-PD-morphism if and only if $(J + pA) \cap K$ is a sub-PD-ideal of $J + pA$.

(ii) The induced $m$-PD-structure on $A/K$ is compatible with $(a, b, α)$ if and only if

(a) $(J + b_1A) \cap K$ is a sub-PD-ideal of $J + b_1A$, and

(b) $b_1A \cap (I + K)$ is a sub-PD-ideal of $b_1A$

In the situation of (i) we say that the $m$-PD-structure $(I, J, γ)$ descends to $A/K$.

If $(I, J, γ)$ is an $m$-PD-ideal in $R$, the partially divided powers $x^{(k)}_{(m)}$ of an element $x \in I$ are defined by

$$x^{(k)}_{(m)} = x^rγ_q(x^{p^m})$$

where $q$ and $r$ are integers satisfying $k = p^mq + r$ and $0 \leq r < p^m$. It follows from the definition and the equality $q!γ_q(x) = x^q$ that

$$q!x^{(k)}_{(m)} = x^k.$$  

Thus if $p$ is nilpotent in $R$, $I$ is a nilideal. The partially divided powers have a large number of formal properties which we will not bother to state here.
3.1.6 The Regular Case. The case when $I$ is important. The algebras $P$ locally generated by a regular sequence) is particularly nice, and particularly important. There is an $A$-algebra homomorphism $A \to P$ such that $(P^{n}, \sigma^{n}, [\cdot])$ is compatible with $(a, b, \alpha)$ and having the following universal property: for any $A$-algebra $B$ with an $m$-PD-structure compatible with $(a, b, \alpha)$, the structure morphism $A \to B$ factors uniquely through an $m$-PD-morphism $P^{n}(I) \to B$. The ideal $I^{*}$ (resp. $I^{o}$) is called the canonical $m$-PD-ideal (resp. the canonical PD-ideal) of the $m$-PD-envelope $P^{n}(I)$.

The quotient of $P^{n}(I)$ by the $(n + 1)$-st step of the $m$-PD-adic filtration is denoted by $P^{n}(m, I)$; it has a similar universal property with respect to homomorphisms of $A$ to $m$-PD-nilpotent $m$-PD-rings. From 3.1.3.1 we see that the image of $I$ in $P^{n}(m, I)$ is a nilideal.

Let $J_{1} =$ $J + pR$; then for all $n \geq 0$, $I^{n} \cap J_{1}$ is a sub-PD-ideal of $J_{1}$. In particular $I^{n} \cap J$ is a sub-PD-ideal of $J$.

The construction is quite involved and we refer the reader to [4, App.]. An $m$-PD-ideal $(I, J, \gamma)$ is $m$-PD-nilpotent if $I^{n} = 0$ for some $n$.

3.1.5 The $m$-PD-envelope of an ideal. The principal construction of the theory is the $m$-PD-envelope of an ideal. Let $(R, a, b, \alpha)$ be a ring with $m$-PD-structure, $A$ an $R$-algebra and $I \subset A$ an ideal. There is an $m$-PD-ring

$$(P^{n}(m, I), I^{*}, I^{o}, [\cdot])$$

and an $A$-algebra homomorphism $A \to P^{n}(m, I)$ such that $(P^{n}, I^{o}, [\cdot])$ is compatible with $(a, b, \alpha)$ and having the following universal property: for any $A$-algebra $B$ with an $m$-PD-structure compatible with $(a, b, \alpha)$, the structure morphism $A \to B$ factors uniquely through an $m$-PD-morphism $P^{n}(m, I) \to B$. The ideal $I^{*}$ (resp. $I^{o}$) is called the canonical $m$-PD-ideal (resp. the canonical PD-ideal) of the $m$-PD-envelope $P^{n}(m, I)$.

The formation of $P^{n}(m, I)$ commutes with flat base change: if $A \to A'$ is flat, the natural homomorphisms $A' \otimes_{A} P^{n}(m, I) \to P^{n}(A', I)$ and $A' \otimes_{A} P^{n}(m, I) \to P^{n}(A', I)$ are isomorphisms.

3.1.6 The Regular Case. The case when $I \subset A$ is a regular ideal (Zariski-locally generated by a regular sequence) is particularly nice, and particularly important. The algebras $P^{n}(m, I)$ are independent of the $m$-PD-structure of $R$, flat over $R$ and their formation commutes with arbitrary base change $R \to R'$. Suppose furthermore that $I$ is generated by a regular sequence $x_{1}, \ldots, x_{d}$, that $A/I$ is flat over $R$ and that the quotient map $A \to A/I$ has a section $\sigma : A/I \to A$. Then via $\sigma$, $P^{n}(m, I)$ is a free $A/I$-module.
on the $m$-PD-polynomials $x^{(K)}(m)$ for $|K| \leq n$ (in the usual multi-index notation). Furthermore the image of $I^*$ in $P^n_{(m),\alpha}(I)$ is free on the $x^{(K)}(m)$ for $|K| > 0$, and $I^\circ$ is generated by $p P^n_{(m),\alpha}(I)$ and by the $x^{(K)}(m)$ for those $K = (k_1, \ldots, k_d)$ for which at least one entry is $\geq p^m$.

If $p$ is nilpotent in $A$ these assertions hold for the full $m$-PD-envelope $P_{(m),\alpha}(I)$, the only modification being that when $A \to A/I$ has a section and $x_1, \ldots, x_d$ is a regular sequence generating $I$, the $A/I$-module $P_{(m),\alpha}(I)$ is free on the entire set of $x^{(K)}(m)$.

An important example is the $m$-PD-polynomial algebra $R(X_1, \ldots, X_d)$, defined as the $m$-PD-envelope of the regular ideal $(X_1, \ldots, X_d) \subset R[X_1, \ldots, X_d]$. Elements of $R(X_1, \ldots, X_d)$ are called $m$-PD-polynomials; as an example of their use in computation we recall, from the proof of [4, Prop. 4.2.1] that for any natural number $r$ divisible by $p^{m+1}$ there is an $m$-PD-polynomial $\varphi_r^{(m)}(X_1, X_2)$ such that for all $t_1, t_2$ in some $m$-PD-ring $(R, I, J, \gamma)$ such that $t_1 - t_2 \in I$, $t_2^r - t_1^r = p \varphi_r^{(m)}(t_1, t_2)$. (3.1.6.1)

We may work in $\mathbb{Z}_p[X_1]/(X_2 - X_1)$, in which case, writing $r = p^{m+1}q$, we see that

$$X_2^r - X_1^r = ((X_1 + (X_2 - X_1))^{p^{m+1}})^q - X_1^r$$

$$= (X_1^{p^{m+1}} + p(*) + (X_2 - X_1)^{p^{m+1}})^q - X_1^r$$

$$= (X_1^{p^{m+1}} + p(*) + p!(X_2 - X_1)^{(p^{m+1})(m)})^q - X_1^r$$

from which the assertion follows. The identities

$$\varphi_r^{(m)}(X, X) = 0, \quad \varphi_r^{(m)}(X_1, X_2) + \varphi_r^{(m)}(X_2, X_3) = \varphi_r^{(m)}(X_1, X_3)$$

(3.1.6.2)

can be proven by reduction to the case of $\mathbb{Z}_p[X_1]/(X_2 - X_1)$. Since the latter has no $p$-torsion, these identities can be checked after multiplication by $p$, in which case they are obvious consequences of (3.1.6.1).

3.1.7 Application to Formal Schemes. The construction of $m$-PD-envelopes sheafifies on a scheme because it commutes with flat base change, and localization are flat. Thus if $S$ is an $m$-PD-scheme and $X$ is an $S$-scheme, any ideal $I \subset \mathcal{O}_X$ has an $m$-PD-envelope $P_{(m),\alpha}(I)$; it is a quasi-coherent $\mathcal{O}_X$-module satisfying the same universal property as in the affine case. The same holds for the $P^n_{(m),\alpha}(I)$. 

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Suppose now $S$ is an locally noetherian formal scheme with an $m$-PD-structure $(a, b, \alpha)$, and $X$ is a locally noetherian formal $S$-scheme. Since formal localizations are also flat, one might expect that the construction of $m$-PD-envelopes sheafifies in the same way. However the lack of a category of quasi-coherent $O_X$-modules makes itself felt at this point: there is no analogue here of the sheafification procedure that is available in complete generality for schemes. If $\mathcal{I} \subset O_X$ is an ideal one can of course sheafify the presheaf of divided power envelopes of $\mathcal{I}$ on affines; the trouble starts when one tries to prove that the ring of sections of this sheaf over an affine open is a divided power envelope.

The situation is somewhat better for regular ideals, when one is concerned only with the truncated divided power envelopes. Suppose $I \subset O_X$ is a regular ideal, and that the closed immersion $Y \to X$ defined by $I$ has a retraction $X \to Y$ (i.e. the quotient map $O_X \to O_X/\mathcal{I}$ has a section). Let $U = \text{Spf}(A) \subset X$ and $V = \text{Spf}(R) \subset S$ be open affines such that $X \to S$ maps $U$ into $V$, and $U \to V$ is parallelizable. If we set $I = \Gamma(U, \mathcal{I})$, then by hypothesis $A \to A/\mathcal{I}$ has a section $\sigma$, and we know that $P^n_{m, \alpha}(I)$ is a free $A/\mathcal{I}$-module of finite rank via the section $\sigma$. For any $f \in \Gamma(U, O_X)$ the morphism $A \to A \{f\}$ is flat and the canonical morphism

$$A \{f\} \otimes_A P^n_{m, \alpha}(I) \to P^n_{m, \alpha}(IA \{f\})$$

is an isomorphism. Since $P^n_{m, \alpha}(I)$ is finitely generated the tensor product may be replaced by a completed tensor product. Finally, the sections of $O_X \to O_X/\mathcal{I}$ being used all come from a single global section. We conclude that there is a coherent $O_Y$-module $P^n_{m, \alpha}(I)$ with the property that

$$\Gamma(U, P^n_{m, \alpha}(I)) \simeq P^n_{m, \alpha}(I)$$

when $U$ is affine and $I = \Gamma(U, \mathcal{I})$. Since $X$ is locally noetherian, the ideal $\mathcal{I}$ is locally nilpotent in $P^n_{m, \alpha}(I)$, and therefore $P^n_{m, \alpha}(I)$ is supported on the closed formal subscheme of $X$ defined by $\mathcal{I}$.

Like its affine counterpart, the $O_X$-algebra $P^n_{m, \alpha}(I)$ has a universal property, best expressed by introducing the formal scheme

$$X^n_{m, \alpha}(I) = \text{Spf}_{O_X}(P^n_{m, \alpha}(I)).$$

Suppose $f : X' \to X$ is an $S$-morphism of an adic noetherian formal schemes, and $X'$ has an $m$-PD-structure $(I', J', \gamma')$ compatible with $(a, b, \alpha)$ and nilpotent of order $n$. If $f^* \mathcal{I} \subset \mathcal{I}'$, $f$ has a unique factorization

$$X' \xrightarrow{g} X^n_{m, \alpha}(I) \xrightarrow{\Phi} X$$

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for some $m$-PD-morphism $g$, where $p$ is the morphism corresponding to the structure map of the coherent $\mathcal{O}_X$-algebra $\mathcal{P}^n_{(m),\alpha}(\mathcal{I})$.

The case of the full $m$-PD-envelope is more difficult, and requires further hypotheses and no small amount of technicalities. We will deal with it in section 4.1.

3.1.8 Lemma Let $A$ be an noetherian ring with $m$-PD-structure $(I, J, \gamma)$. For any ideal $K \subset A$, the $m$-PD-structure $(I, J, \gamma)$ descends to $A/K^n$ for all sufficiently large $n$.

Proof. By 3.1.3 we need that $K^n \cap (J + pA) \subseteq J + pA$ is a sub-PD-ideal for all $n \gg 0$, i.e. $\gamma_k(x) \in K^n \cap (J + pA)$ for all $x \in K^n \cap (J + pA)$ and $k > 0$, and since $\gamma_1(x) = x$ we may assume $k > 1$. By Artin-Rees there is an integer $c$ such that

$$K^n \cap (J + pA) \subseteq K^{n-c}(K^c \cap (J + pA))$$

for $n > c$. Then for all $x \in K^n \cap (J + pA)$,

$$\gamma_k(x) \in K^{k(n-c)}(J + pA)$$

by the basic PD-identities. We are done if $k(n - c) \geq n$, and since $k \geq 2$ this holds when $n \geq 2c$.

3.1.9 Remark The conclusion of the lemma can be restated as follows: $A$ has an cofinal set of ideals of definition $K$ such that the $m$-PD-structure of $A$ descends to $A/K^n$ for all sufficiently large $n$; it suffices to replace $K$ by $K^N$ for all sufficiently large $N$.

3.2 The ring $\mathcal{D}^{(m)}_{\mathcal{X}/\mathcal{S}}$. Suppose now $\mathcal{X} \to \mathcal{S}$ is quasi-smooth. The considerations of the last paragraph apply to the sheaf of rings $\mathcal{O}_{\mathcal{X}_S(r)}$ and its diagonal ideal $\mathcal{I}(r)$.

3.2.1 Principal parts of level $m$ We denote by $\mathcal{P}^n_{\mathcal{X}/\mathcal{S},(m)}(r)$ the $m$-PD-envelope of order $n$ of $\mathcal{I}(r)$ in $\mathcal{O}_{\mathcal{X}_S(r)}$; it is supported on the diagonal of $\mathcal{X}_S(r)$ and may be regarded as a sheaf of rings on $\mathcal{X}$. It has $r + 1$ $\mathcal{O}_\mathcal{X}$-algebra structures, with respect to each of which it is a coherent locally free $\mathcal{O}_{\mathcal{X}_S(r)}$-algebra. As before we drop the $(r)$ when $r = 1$. We reuse the notation

$$d^n_K : \mathcal{P}^n_{\mathcal{X}/\mathcal{S},(m)}(r) \to \mathcal{P}^n_{\mathcal{X}/\mathcal{S},(m)}(r')$$  (3.2.1.1)
of \([2.3.1]\) for the canonical projections; their existence follows from the universal property of the truncated \(m\)-PD-envelopes. If we define

\[
\mathcal{X}_{S,(m)}^n(r) = \text{Spf} \mathcal{O}_X(\mathcal{P}_{X/S,(m)}^n(r))
\]

(3.2.1.2)

then \(\mathcal{X}_{S,(m)}^n(r)\) is a formally affine formal \(X\)-scheme for each of its \(r+1\) \(\mathcal{O}_X\)-algebra structures. The \(r+1\) structure morphisms are finite and the homomorphisms \([3.2.1.1]\) induce morphisms

\[
p^R_n : \mathcal{X}_{S,(m)}^n(r') \to \mathcal{X}_{S,(m)}^n(r)
\]

(3.2.1.3)

As before when \(r = 1\) we drop the (1).

The \(\mathcal{O}_X\)-module of level \(m\) operators of order \(\leq n\) is defined to be

\[
\text{Diff}^n_{X/S,(m)} = \text{Hom}_{\mathcal{O}_X}(\mathcal{P}_{X/S,(m)}^n, \mathcal{O}_X).
\]

As in \([3, \S 2.2]\) the canonical projections \(\mathcal{P}_{X/S,(m)}^{n'} \to \mathcal{P}_{X/S,(m)}^n\) for \(n' \geq n\) induce injections \(\text{Diff}^n_{X/S,(m)} \to \text{Diff}^{n'}_{X/S,(m)}\) and the ring of arithmetic differential operators of level \(m\) is

\[
\mathcal{D}_{X/S}^{(m)} = \lim_{n} \text{Diff}^n_{X/S,(m)}
\]

where again the limit is to be understood in the sense of \(\mathcal{O}_X\)-modules on the ringed space \((|X|, \mathcal{O}_X)\). When \(X = \text{Spf}(B)\) and \(S = \text{Spf}(A)\) are affine, we define \(\Gamma(X, \text{Diff}^n_{X/S,(m)}) = \text{Diff}^n_{A/R,(m)}\) and \(\Gamma(X, \mathcal{D}_{X/S}) = \mathcal{D}_{B/A}^{(m)}\), and then

\[
\mathcal{D}_{B/A}^{(m)} = \lim_{n} \text{Diff}^n_{A/R,(m)}
\]

when \(X\) is noetherian (so that taking global sections commutes with the inductive limit). When \(X \to S\) is parallelizable with local coordinates \(x_1, \ldots, x_d\), \(\mathcal{P}_{X/S,(m)}^n\) is the free module with basis \((\xi^{(I)}_m)|_{|I| \leq n}\), where as usual \(\xi_i = 1 \otimes x_i - x_i \otimes 1\). The dual basis of \(\text{Diff}^n_{X/S,(m)}\) is denoted by \(\{\partial^{(K)}(m)\}_{|K| \leq n}\).

We make \(\mathcal{D}_{X/S}^{(m)}\) into a ring by means of a map

\[
\delta^{n,n'}_{(m)} : \mathcal{P}_{X/S,(m)}^{n+n'} \to \mathcal{P}_{X/S,(m)}^n \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)}^{n'}
\]

arising from \(\delta^{n,n'}\) via the universal property of \(m\)-PD-envelopes. The formal properties of the ring \(\mathcal{D}_{X/S}^{(m)}\) are the same as in \([4, \S 2.2]\), and are proven in the same way; we will not bother to state them here.
Once again we should point out that in the situation of the diagram 2.3.0.1 the \( (\mathcal{D}^{(m)}_{\mathcal{X}'/S'}, f^{-1}\mathcal{D}^{(m)}_{\mathcal{X}/S}) \)-bimodule \( f^*\mathcal{D}^{(m)}_{\mathcal{X}/S} \) is to be understood in the sense of ringed spaces. This will not be an issue, as we will see later.

For \( m' \geq m \) and all \( n \geq 0 \) there is a canonical \( m \)-PD-morphism

\[
i_{m',m}^n : \mathcal{P}^{n}_{\mathcal{X}/S,(m')} \to \mathcal{P}^{n}_{\mathcal{X}/S,(m)} \tag{3.2.1.4}
\]

arising from the universal property by regarding the canonical \( m' \)-PD-ideal of \( \mathcal{P}^{n}_{\mathcal{X}/S,(m')} \) as an \( m \)-PD-ideal. Since \( \mathcal{P}^{n}_{\mathcal{X}/S,(m)} \) is generated by the \( x^{(k)}(m) \) for \( 0 \leq k < n \), \( \iota_{m',m}^n \) is characterized by the formula

\[
k = qp^m + r = q'p^{m'} + r' \implies \iota_{m',m}^n(x^{(k)}(m)) = \frac{q}{q'} i^{(k)}(m') \tag{3.2.1.5}
\]

which follows from 3.1.3.2 by reduction to the universal case. Dualizing 3.2.1.5 and taking the inductive limit in \( n \) results in a ring homomorphism

\[
\rho_{m',m} : \mathcal{D}^{(m)}_{\mathcal{X}/S} \to \mathcal{D}^{(m')}_{\mathcal{X}/S} \tag{3.2.1.6}
\]

for all \( m' \geq m \). In local coordinates it is given by the formula

\[
\rho_{m',m}(\partial^{(K)(m)}) = \frac{Q!}{Q'!} \partial^{(K)(m')} \tag{3.2.1.7}
\]

where \( Q, Q' \) are defined by

\[
K = p^m Q + R = p^{m'} Q' + R'
\]

with \( 0 \leq R < p^m \) and \( 0 \leq R' < p^{m'} \).

3.2.2 Base change. Let

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{S}' & \longrightarrow & \mathcal{S}
\end{array}
\tag{3.2.2.1}
\]

be a commutative diagram of locally noetherian formal schemes, with \( \mathcal{X}' \to \mathcal{S}' \) and \( \mathcal{X} \to \mathcal{S} \) quasi-smooth and \( \mathcal{S}' \to \mathcal{S} \) an \( m \)-PD-morphism. For any left \( \mathcal{D}^{(m)}_{\mathcal{X}/S} \)-module \( M \) there are, as in [4 2.2.2] and [6 §2.1] two equivalent ways of placing left \( \mathcal{D}^{(m)}_{\mathcal{X}'/S'} \)-module structure on \( f^*M \). One is via the natural homomorphism

\[
d : \mathcal{D}^{(m)}_{\mathcal{X}'/S'} \to \mathcal{D}^{(m)}_{\mathcal{X}/S} = f^*\mathcal{D}^{(m)}_{\mathcal{X}/S} \tag{3.2.2.2}
\]

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which is deduced from the natural homomorphisms

\[ f^* \mathcal{P}_X^n \to \mathcal{P}_{X'/S'}^n \]

by duality and passage to the limit; the induced \((\mathcal{D}_{X'/S'}, f^{-1} \mathcal{D}_{X/S}^{(m)})\)-bimodule structure on \(\mathcal{D}_{X/S}^{(m)}\) yields a left \(\mathcal{D}_{X'/S'}^{(m)}\)-module structure via the canonical isomorphism

\[ \mathcal{D}_{X'/X} \otimes f^{-1} \mathcal{D}_{X/S}^{(m)} \to f^* M \]

(c.f. [6, 2.1.3]). The other method is to use the commutative diagrams

\[
\begin{array}{ccc}
\mathcal{X}'_S^n & \xrightarrow{f \times f} & \mathcal{X}'_S^n \\
\downarrow p_i & & \downarrow p_i \\
S' & \to & S
\end{array}
\]

for \(i = 0, 1\) to show that an \(m\)-PD-stratification of \(M\) relative to \(S\) pulls back to an \(m\)-PD-stratification of \(f^* M\) relative to \(S'\). The latter method is perhaps more convenient for proving the transitivity formula \((fg)^* M \simeq g^* f^* M\), c.f. [6, 2.1.1]. On this point nothing needs to be added to the treatment of [6, 2.1.3] and [6, 2.1.1].

### 3.2.3 \(m\)-PD-stratifications

An \(m\)-PD-stratification relative to \(S\) of an \(\mathcal{O}_X\)-module \(M\) is defined just as before, but with the \(\mathcal{X}'_S^n(r)\) in place of the \(\mathcal{X}'_S^n(r)\): it is a series of isomorphisms

\[ \chi_n : p_1^n(M) \xrightarrow{\sim} p_0^n(M) \]  

(3.2.3.1) satisfying the conditions 2.3.1.4–6 (same conditions on different maps!). More generally, a sequence \(\{\chi_n\}_{n \geq 0}\) is compatible if it satisfies 2.3.1.4–5. The isomorphisms \(\chi_n\) can also be given as a series of isomorphisms

\[ \chi_n : \mathcal{P}_X^n(\mathcal{O}_S) \otimes_{\mathcal{O}_X} M \xrightarrow{\sim} M \otimes_{\mathcal{O}_X} \mathcal{P}_X^n(\mathcal{O}_S(m)) \]  

(3.2.3.2)

of \(\mathcal{O}_X\)-modules with analogous properties, or via the adjunction isomorphism as a series of morphisms

\[ \theta_n : M \to M \otimes_{\mathcal{O}_X} \mathcal{P}_X^n(\mathcal{O}_S(m)) \]  

(3.2.3.3)

that are \(\mathcal{O}_X\)-linear for the right structure of \(\mathcal{P}_X^n(\mathcal{O}_S(m))\), compatible with the canonical morphisms \(\mathcal{P}_X^n(\mathcal{O}_S(m)) \to \mathcal{P}_X^n(\mathcal{O}_S(m))\) for \(n' \geq n\), the identity for \(n = 0\), and making commutative a diagram analogous to 2.3.1.6. The argument that the category of left \(\mathcal{D}_{X/S}^{(m)}\)-modules is equivalent to the category of \(\mathcal{O}_X\)-modules with an \(m\)-PD-stratification relative to \(S\) is the same as the usual one.
3.3 The ring $\hat{D}_X^{(m)}$. In the setting of [4], the next step in the theory is to form the $p$-adic completion $\hat{D}_X^{(m)}$ of $D_X^{(m)}$, and then take the inductive limit of the $\hat{D}_X^{(m)}$ to get the full ring of arithmetic differential operators. We have explained in the introduction why this is not the thing to do here, and the reader will see a case of this in the proof of theorem 3.3.10. Instead we must complete $\hat{D}_X^{(m)}$ with respect to an ideal of definition; this does not obviously result in a sheaf of rings, and what makes this idea workable is the fact that rings like $D_X^{(m)}$ are particularly rich in two-sided ideals, as we see from corollary 3.3.4.

We first recall some definitions and results from [16] (c.f. also [4, §3.2]). An (left or right) ideal $I \subset R$ in a ring is central if it is generated by a set of central elements, and centralising if it is generated by a centralising sequence, i.e. a sequence $x_1, \ldots, x_n \in R$ such that for all $i$ the image of $x_i$ in $R/(x_1, \ldots, x_{i-1})$ lies in the center. A centralising ideal is evidently 2-sided, and when $R$ is noetherian the standard results from commutative algebra concerning $I$-adic topologies and completions extend to the case when $I \subset R$ is centralising. We refer the reader to [16, D III] for proofs of the following assertions, in which $R$ is any left and right noetherian ring with unit and $I$ is a centralising ideal:

- The Artin-Rees lemma holds in the following form: if $M$ is a finitely generated left $R$-module and $N$ is submodule of $M$, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that
  \[ N \cap I^{f(n)} M \subseteq I^n N. \]

- The $I$-adic completion functor $M \mapsto \hat{M}$ is exact on the category of finitely generated left (or right) $R$-modules.

- The completion $\hat{R}$ is left and right flat over $R$, and

- The natural map $\hat{R} \otimes_R M \to \hat{M}$ is an isomorphism for any finitely generated left $R$-module $M$.

If $I$ is a central ideal, the function $f$ may be taken to be of the form $f(n) = c + n$ for some constant $c$, and the proof of the Artin-Rees lemma is the same as in the commutative case. The general case is more complicated, c.f. [16, D V]. The remaining statements follow from the first by the usual arguments.

We can relativize the notion of a centralising ideal. Suppose $R \to A$ is a ring homomorphism with $R$ commutative; we say that an ideal $J \subset R$
is centralising in $A$ if it is generated by a sequence whose image in $A$ is centralising. Thus if $J \subset R$ is centralising in $A$, $JA = AJ$ is a centralising ideal of $A$.

3.3.1 Lemma Let $R \to A$ be a ring homomorphism with $R$ commutative and noetherian. Let $I \subset R$ be an ideal such that $IA$ is a central ideal in $A$. Let $J \subset R$ be any ideal and denote by $f : A \to A/IA$ the canonical homomorphism. The ideal $J' = J \cap f^{-1}(Z(A/I))$ is centralising in $A$.

Proof. This is an immediate consequence of the definitions; the noetherian hypothesis is there to ensure that $J'$ is finitely generated, which is implicit in the definition. 

3.3.2 Proposition Suppose $\mathcal{X}/\mathcal{S}$ is quasi-smooth and $m \geq 0$. Any ideal of definition containing the prime $p$ contains an ideal of definition that is centralising in $D_{\mathcal{X}/\mathcal{S}}^{(m)}$.

Proof. Suppose first that $\mathcal{X}$ and $\mathcal{S}$ are affine. Let $J \subset \mathcal{O}_\mathcal{X}$ be an ideal of definition and write $J = (p, f_1, \ldots, f_n)$. The ideal $J'$ constructed in lemma 3.3.1 with $I = (p)$ is topologically nilpotent since it is contained in $J$ and open since it contains $(p, f_1^{p^m+1}, \ldots, f_n^{p^m+1})$, by [4, prop. 2.2.6], c.f. also the remark at the end of [4, §3.2.3]. It is therefore an ideal of definition.

If $J = (p, f_1, \ldots, f_n) = (p, g_1, \ldots, g_r)$ then

$$(p, f_1^{p^m+1}, \ldots, f_n^{p^m+1}) = (p, g_1^{p^m+1}, \ldots, g_r^{p^m+1}),$$

so this construction globalizes. 

3.3.3 Definition Suppose $R$ is a commutative ring and $R \to A$ is an $R$-ring. We will say that an ideal $I \subset R$ is bilateralising in $A$ if $IA = AI$.

If the reference to $A$ is clear we will simply say that $I$ bilateralising. The following assertions are immediate:

- If $I$ is bilateralising in $A$, $IA \subset A$ is a 2-sided ideal.

- Sums and products of bilateralising ideals are bilateralising. In particular, powers of bilateralising ideals are bilateralising.

- A centralising ideal in $A$ is bilateralising.

- If $M$ is a left $A$-module and $I \subset R$ is bilateralising, $M/IM$ is a left $A/IA$-module.
3.3.4 Corollary The set of ideals of definition of $\mathcal{O}_X$ that are bilateralising in $D^{(m)}_{X/S}$ is cofinal in the set of all ideals of definition.

Proof. In fact the lemma says that centralising ideals of definition exist, and if $J$ is one such, $\{J^n\}_{n \geq 0}$ is a cofinal system of bilateralising ideals of definition.

3.3.5 Definition Suppose $S$ is a formal scheme with an $m$-PD-structure and $X \rightarrow S$ is quasi-smooth. An ideal $J \subseteq \mathcal{O}_X$ is $m$-bilateralising if it is bilateralising in $D^{(m)}_{X/S}$.

As before, if $m$ is understood we will simply say that $J$ is bilateralising.

It is easy to characterize the $m$-bilateralising ideals of $\mathcal{O}_X$. We first recall the level $m$ Leibnitz identity

$$\partial^{(K)(m)} f = \sum_{I+J=K} \binom{K}{I} \partial^{(I)(m)}(f)\partial^{(J)(m)}$$

for all $f \in J$ and $K \in \mathbb{N}^d$. Thus if $J$ is $m$-bilateralising and $f \in J$,

$$\partial^{(K)(m)} f = f \partial^{(K)(m)} \in JD^{(m)}$$

for all $K \in \mathbb{N}^d$. Since $D^{(m)}$ is free on the $\partial^{(K)(m)}$, [3.3.5.1] shows that $\partial^{(K)(m)} f \in J$. Conversely if $D^{(m)}J \subseteq J$, the containment $\partial^{(K)(m)} f \subseteq JD^{(m)}$ is an immediate consequence of [3.3.5.1] and the containment $\partial^{(K)(m)} f \subseteq D^{(m)}J$ follows from [3.3.5.1] by induction on $|K|$.

Since locally $D^{(m)}_{X/S}$ is generated locally by the $\partial^{(K)(m)}$ for $|K| \leq p^m$, we deduce:
3.3.7 Corollary If $J \subseteq \mathcal{O}_X$ is $m'$-bilateralising and $m' \geq m$ then $J$ is $m$-bilateralising. □

3.3.8 Corollary If $J \subseteq \mathcal{O}_X$ is $m$-bilateralising then

$$J \mathcal{P}_{X/S,(m)}^n = \mathcal{P}_{X/S,(m)}^n J$$

for all $n \geq 0$.

Proof. For $f \in \mathcal{O}_X$ we set $\delta^n(f) = d^n_1(f) - d^n_0(f)$, and write $\mathcal{P}^n$ for $\mathcal{P}_{X/S,(m)}^n$. It suffices to show that $\delta^n(J) \subseteq \mathcal{P}^n J \cap J \mathcal{P}^n$ for all $n$. The Taylor formula says that

$$\delta^n(f) = \sum_{0 < |K| \leq n} d_0(\partial^{(K)}(m)(f))\xi^{(K)m}$$

which by proposition 3.3.6 yields $\delta^n(J) \subseteq J \mathcal{P}^n$ for all $n$. It also shows that

$$\delta^n(f) = \sum_{0 < |K| \leq n} d_1(\partial^{(K)}(m)(f))\xi^{(K)m} - \sum_{0 < |K| \leq n} \delta^n(\partial^{(K)}(m)(f))\xi^{(K)m}$$

from which we deduce that

$$\delta^n(J) \subseteq \mathcal{P}^n J + I^* \delta^n(J)$$

and the result follows by iteration since $I^*$ is nilpotent in $\mathcal{P}^n$. □

If $J \subseteq \mathcal{O}_X$ is any open ideal we denote by $X_J$ the (ordinary) scheme $(|X|, \mathcal{O}_X/J)$. If $J$ is $m$-bilateralising we set

$$\mathcal{D}_{X,J/S}^{(m)} = \mathcal{D}_{X/S}^{(m)}/J \mathcal{D}_{X/S}^{(m)}$$

(3.3.8.1)

(the notation is purely formal, since $X_J$ is not quasi-smooth over $S$) which may be regarded indifferently as a sheaf of $\mathcal{O}_X$-rings, or of $\mathcal{O}_{X,J}$-rings on $X_J$. Via the latter structure, it is clearly a quasicoherent $\mathcal{O}_{X,J}$-module; in fact on any parallelizable open $U \subseteq X$, $\mathcal{D}_{X,J/S}^{(m)}$ is free $\mathcal{O}_X$-module on the $\partial^{(I)}(m)$, so that $\mathcal{D}_{X,J/S}^{(m)}$ is a free $\mathcal{O}_{X,J}$-module on the images of the $\partial^{(I)}(m)$ (for which we use the same notation).

If $J' \subseteq J$ is another open $m$-bilateralising ideal there is an evident homomorphism

$$\mathcal{D}_{X,J/S}^{(m)} \to \mathcal{D}_{X,J'/(m)}^{(m)}$$

of $\mathcal{O}_X$-rings, inducing an isomorphism

$$\mathcal{D}_{X,J'/S}^{(m)} \otimes_{\mathcal{O}_J} \mathcal{O}_{X_J} \sim \mathcal{D}_{X,J/S}^{(m)}$$

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of $\mathcal{O}_{X_j}$-rings. The $\mathcal{O}_X$-ring $\mathcal{D}^{(m)}_{X/S}$ is the inverse limit

$$\hat{\mathcal{D}}^{(m)}_{X/S} = \varprojlim_J \mathcal{D}^{(m)}_{X/J/S} \quad (3.3.8.2)$$

where $J$ runs through any cofinal set of ideals of definition of $\mathcal{O}_X$ bilateralising in $\mathcal{D}^{(m)}_{X/S}$. On any parallelizable open affine $\text{Spf}(A) = U \subseteq \mathcal{X}$, elements of

$$\hat{\mathcal{D}}^{(m)}_{A/R} = \Gamma(U, \mathcal{D}^{(m)}_{X/S})$$

may be identified with formal series $\sum_{I \in \mathbb{N}} a_I \partial^{(I)}(m)$ with $a_I \to 0$ in the adic topology of $A$.

**3.3.9 Remark** The proof of corollary 3.3.4 shows that the inverse limit in 3.3.8.2 is actually a $J$-adic completion for any centralising ideal of definition of $\mathcal{O}_X$. In particular the properties of such completions summarized in section 3.3 apply in this case.

It follows from corollary 3.3.7 that the natural morphism $\mathcal{D}^{(m)}_{X/S} \to \mathcal{D}^{(m')}_{X/S}$ extends uniquely to a morphism

$$\hat{\rho}_{m',m} : \hat{\mathcal{D}}^{(m)}_{X/S} \to \hat{\mathcal{D}}^{(m')}_{X/S}. \quad (3.3.9.1)$$

In fact in 3.3.8.2 we can use a set of $J$ bilateralising for $\mathcal{D}^{(m')}_{X/S}$ to compute both $\mathcal{D}^{(m)}_{X/S}$ and $\mathcal{D}^{(m')}_{X/S}$. The uniqueness of the extensions shows that this system of morphisms is transitive.

**3.3.10 Theorem** Suppose $\mathcal{X} \to \mathcal{S}$ is a quasi-smooth morphism of adic locally noetherian schemes.

(i) For any open ideal $J \subset \mathcal{O}_X$ bilateralising in $\mathcal{D}^{(m)}_{X/S}$, the ring $\mathcal{D}^{(m)}_{X_J/S}$ is left and right coherent.

(ii) The ring $\hat{\mathcal{D}}^{(m)}_{X/S}$ is left and right coherent.

**Proof.** For (i) it suffices, by [4 Prop. 3.1.3] to show that (a) for the canonical injection $\mathcal{O}_X \to \mathcal{D}^{(m)}_{X_J/S}$, $\mathcal{D}^{(m)}_{X_J/S}$ is quasi-coherent for the $\mathcal{O}_{X_J}$-module structures given by left and right multiplication, and (b) for any open affine $U \subseteq \mathcal{X}$, $\Gamma(U, \mathcal{D}^{(m)}_{X_J/S})$ is a left and right noetherian. We have already seen that (a) is true, and (b) is proven in the same way as in [4 Cor. 2.2.5 (ii)], i.e. by showing the the graded algebra for the filtration by order is finitely

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generated. Since the completion $\hat{D}^{(m)}_{\mathcal{X}/S}$ may be taken to be the $J$-adic completion for some $J \subseteq \mathcal{O}_\mathcal{X}$ centralising in $\mathcal{D}^{(m)}_{\mathcal{X}/S}$, part (ii) follows from the facts about centralising ideals recalled in §3.3, c.f. also [4 §3.2.3] and the last remark in that section.

From [4 Prop. 3.1.3] we also get:

3.3.11 Proposition For $\mathcal{X} \to S$ and $J$ as in theorem 3.3.10, a left (resp. right) $\mathcal{D}^{(m)}_{\mathcal{X}/S}$-module $M$ is coherent if and only if it is quasi-coherent as an $\mathcal{O}_\mathcal{X}$ (resp. $\mathcal{O}_\mathcal{X}/J$)-module, and for every $U \subseteq \mathcal{X}$ belonging to an open cover of $\mathcal{X}$, the left (resp. right) $\Gamma(U, \mathcal{D}^{(m)}_{\mathcal{X}/S})$-module of sections $\Gamma(U, M)$ is of finite type.

The description of coherent left or right $\hat{D}^{(m)}_{\mathcal{X}/S}$-modules is a little more involved, but very little needs to be added to the treatment of [4 §3.3]. For the reader’s convenience we recall some results regarding completions from [4 §3.3], slightly reformulated for the present purposes. In what follows $\mathcal{D}$ is a sheaf of rings on $\mathcal{X}$ endowed with a homomorphism $\mathcal{O}_\mathcal{X} \to \mathcal{D}$, and we assume that $\mathcal{D}$ satisfies the following conditions:

$\mathcal{O}_\mathcal{X}$ has an ideal of definition centralising in $\mathcal{D}$; in particular $\mathcal{O}_\mathcal{X}$ has a fundamental system of ideals of definition that are bilateralising in $\mathcal{D}$.

For any open affine $U \subseteq \mathcal{X}$, $\Gamma(U, \mathcal{D})$ is left noetherian. (3.3.11.2)

As a left $\mathcal{O}_\mathcal{X}$-module, $\mathcal{D}$ is a filtered inductive limit of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{D}_\lambda$ such that for all $\lambda$, $\mathcal{D}_\lambda \simeq \varprojlim_j \mathcal{D}_\lambda/J\mathcal{D}_\lambda$ (where $J$ runs through the set of bilateralising ideals of definition), and for all $\lambda$ and bilateralising open $J \subseteq \mathcal{O}_\mathcal{X}$, $\mathcal{D}_\lambda/J\mathcal{D}_\lambda$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}_J}$-module.

These hypotheses apply in particular to

$$\mathcal{D} = \mathcal{D}^{(m)}_{\mathcal{X}/S}, \quad \mathcal{D}_J = \mathcal{D}^{(m)}_{\mathcal{X}_J/S}, \quad \hat{\mathcal{D}} = \hat{\mathcal{D}}^{(m)}_{\mathcal{X}/S} = \varprojlim_j \mathcal{D}_J$$

and when $\mathcal{X}$ is affine we write

$$\mathcal{D} = \Gamma(\mathcal{X}, \mathcal{D}^{(m)}_{\mathcal{X}/S}), \quad \mathcal{D}_J = \Gamma(\mathcal{X}, \mathcal{D}^{(m)}_{\mathcal{X}_J/S}), \quad \hat{\mathcal{D}} = \Gamma(\mathcal{X}, \hat{\mathcal{D}}^{(m)}_{\mathcal{X}/S}) = \varprojlim_j \mathcal{D}_J.$$
3.3.12 Proposition Suppose $\mathcal{D}$ satisfies conditions 3.3.11.1–3.

(i) For any open affine $U \subseteq \mathcal{X}$, the ring $\Gamma(U, \hat{\mathcal{D}})$ is left noetherian.

(ii) For any pair of open affines $U' \subseteq U$, the homomorphism

$$\Gamma(U, \hat{\mathcal{D}}) \rightarrow \Gamma(U', \hat{\mathcal{D}})$$

is right flat.

When $\mathcal{X}$ is affine we denote by $M \mapsto M^\triangle$ the functor on $\mathcal{D}$-modules defined by

$$M^\triangle = \lim_{\leftarrow J} (M/JM) \rightMo (3.3.12.1)$$

where the tilde denotes sheaf associated to a $O_{\mathcal{X}J}$-module. If we identify $M$ with the constant presheaf with value $M$, there is a natural homomorphism $\mathcal{D} \otimes_{\mathcal{D}} M \rightarrow M^\triangle$. Arguing as in [4, 3.3.7–8] we obtain:

3.3.13 Proposition With the above hypotheses and notation,

(i) The canonical homomorphism $\hat{\mathcal{D}} \rightarrow \hat{\mathcal{D}}^\triangle$ is an isomorphism.

(ii) The functor $M \mapsto M^\triangle$ is exact on the category of $\hat{\mathcal{D}}$-modules of finite type.

(iii) For any $\hat{\mathcal{D}}$-module $M$ of finite type, the canonical homomorphism $M \rightarrow \Gamma(\mathcal{X}, M^\triangle)$ is an isomorphism.

(iv) For all $\hat{\mathcal{D}}$-modules $M$, $N$ of finite type, the canonical homomorphism

$$\text{Hom}_{\hat{\mathcal{D}}}(M, N) \rightarrow \text{Hom}_{\hat{\mathcal{D}}}(M^\triangle, N^\triangle)$$

is an isomorphism.

As in [4, 3.3.8], the essential point is to show that the canonical homomorphism $\Gamma(\mathcal{X}, \mathcal{D}_J) \rightarrow \hat{\mathcal{D}}/J\hat{\mathcal{D}}$ is an isomorphism for all ideals of definition that are bilateralising in $\mathcal{D}$, and here it is important that $\mathcal{D}_J$ is a quasicoherent $O_{\mathcal{X}J}$-module. The argument is basically that of [14, I 10.10.2]. Continuing as in [4, §3.3] and [14, I 10], we obtain the following “theorem A”:

3.3.14 Theorem Suppose $\mathcal{X}$ is affine and $\mathcal{D}$ satisfies conditions 3.3.11.1–3. The following are equivalent, for any $\hat{\mathcal{D}}$-module $M$:
(i) For every ideal of definition $J \subset \mathcal{O}_\mathcal{X}$ bilateralising in $\mathcal{D}$, the $\mathcal{D}_J$-module $\mathcal{M}/JM$ is coherent, and the canonical homomorphism $\mathcal{M} \to \varprojlim_j \mathcal{M}/JM$ is an isomorphism, where the limit is over ideals of definition bilateralising in $\mathcal{D}$.

(ii) There is an isomorphism $\mathcal{M} \sim \varprojlim_j \mathcal{M}_J$ where for all $J$ as before $\mathcal{M}_J$ is a coherent $\mathcal{D}_{\mathcal{X}_J}$-module, and for $J \subseteq K$, the canonical homomorphism $\mathcal{M}_K \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathcal{X}_J} \to \mathcal{M}_J$ is an isomorphism.

(iii) There is a $\hat{\mathcal{D}}$-module $\mathcal{M}$ and an isomorphism $\mathcal{M} \sim \mathcal{M}^\Delta$.

(iv) The $\hat{\mathcal{D}}$-module $\Gamma(\mathcal{X}, \mathcal{M})$ is of finite type and the canonical homomorphism $\hat{\mathcal{D}} \otimes_{\hat{\mathcal{D}}} \Gamma(\mathcal{X}, \mathcal{M}) \to \mathcal{M}$ is an isomorphism.

(v) $\mathcal{M}$ is a coherent $\hat{\mathcal{D}}$-module.

3.3.15 Corollary Suppose $\mathcal{X}$ is affine. With the above notation, the functors

\[ \mathcal{M} \mapsto \Gamma(\mathcal{X}, \mathcal{M}), \quad M \mapsto M^\Delta \]

are inverse equivalences between the category of coherent $\hat{\mathcal{D}}$-modules and the category of $\hat{\mathcal{D}}$-modules of finite type.

We also get “theorem B”:

3.3.16 Theorem Suppose $\mathcal{X}$ is affine. With the above notation

\[ H^q(\mathcal{X}, \mathcal{M}) = 0 \]

for any coherent $\hat{\mathcal{D}}$-module $\mathcal{M}$.

For emphasis we repeat that corollary 3.3.15 and theorem 3.3.16 apply to the ring $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$ for quasi-smooth $\mathcal{X} \to \mathcal{S}$.

3.4 The ring $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}\mathbb{Q}}$.

3.4.1 Note that corollary 3.3.15 and theorem 3.3.16 also apply to the ring

\[ \hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}\mathbb{Q}} := \hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}} \otimes \mathbb{Q} \] (3.4.1.1)

In fact if $\mathcal{X}$ is affine and $M$ is a coherent $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}\mathbb{Q}}$-module there is a coherent $\hat{\mathcal{D}}^{(m)}_{\mathcal{X}/\mathcal{S}}$-module $M_0$ such that $M \simeq M_0 \otimes \mathbb{Q}$ (we will later prove a more general
result of this sort, namely proposition 4.4.1). The analogue of corollary 3.3.15 is an immediate consequence of this observation; for theorem 3.3.16 it suffices to observe that since \( X \) is noetherian, cohomology commutes with inductive limits.

The most important result concerning these rings is:

**3.4.2 Theorem** For all \( m \leq m' \), the canonical homomorphism

\[
\hat{D}_{X/SQ}^{(m)} \rightarrow \hat{D}_{X/SQ}^{(m')}
\]

is left and right flat.

By induction one can assume \( m' = m + 1 \). We will give the proof as a series of lemmas; as the argument is roughly parallel to that of [4, Th. 3.5.3] we will concentrate the few modifications that are needed. It begins with reductions to the case where \( S = \text{Spf}(R) \) is affine and \( \mathcal{X} = \text{Spf}(A) \) is affine and parallelizable relative to \( S \). The next reduction is to the case where \( \mathcal{O}_S \) is \( p \)-torsion free. If \( S' \hookrightarrow S \) is the closed formal subscheme defined by the ideal of \( p \)-torsion elements of \( R \) and \( X' = X \times_S S' \), then \( \mathcal{X}' \rightarrow S' \) is quasi-smooth and therefore flat. Then \( \mathcal{X} \) is flat over \( \mathbb{Z}_p \) since this is the case for \( S' \). Finally if we write \( S' = \text{Spf}(R') \) and \( X' = \text{Spf}(A') \) then the natural morphism \( \hat{D}_{A/R}^{(m)} \rightarrow \hat{D}_{A'/R'}^{(m)} \) induces an isomorphism \( \hat{D}_{A/RQ}^{(m)} \rightarrow \hat{D}_{A'/R'Q}^{(m)} \).

In what follows we denote the \( p \)-adic completion of a \( \mathbb{Z}_p \)-algebra \( A \) by \( \hat{A} \); the \( J \)-adic completion of an \( \mathcal{O}_X \)-ring \( A \) will be denoted by \( \hat{A} \) as before (we assume \( J \) is bilateralising in \( A \)). We abbreviate \( D^{(m)} = D_{A/R}^{(m)} \) and similarly for \( \hat{D}^{(m)} \), \( \hat{D}_{A/R}^{(m)} \) etc. Note that since \( \mathcal{O}_X \) is \( \mathbb{Z}_p \)-flat and \( D^{(m)} \) is free on the \( \partial^{(K)}(m) \), the maps \( \hat{D}^{(m)} \rightarrow \hat{D}^{(m)} \) and \( \hat{D}_{Q}^{(m)} \rightarrow \hat{D}_{Q}^{(m)} \) are injective. We define

\[
D_1 = \text{the subring of } \hat{D}_{Q}^{(m+1)} \text{ generated by } \hat{D}^{(m)} \text{ and } D^{(m+1)},
\]

\[
D_2 = \text{the subring of } \hat{D}_{Q}^{(m+1)} \text{ generated by } \hat{D}^{(m)} \text{ and } D^{(m+1)}
\]

and note that

\[
D_1 \text{ (resp. } D_2 \text{) is generated as a left } \hat{D}^{(m)} \text{-module (resp. left } \hat{D}^{(m)} \text{-module) by the } (\partial^{(p^{m+1})})^K \text{ for all } K \in \mathbb{N}^d. \tag{3.4.2.1}
\]

From this it follows that

\[
D^{(m+1)} \cap JD_2 \subseteq JD^{(m+1)}. \tag{3.4.2.2}
\]
We now consider the injective homomorphisms
\[ D^{(m+1)} \hookrightarrow D_1 \hookrightarrow D_2. \tag{3.4.2.3} \]

**3.4.3 Lemma** (i) For any \( n \geq 0 \),
\[ D_1 = D^{(m+1)} + p^n \hat{D}^{(m)}. \]

(ii) The first map in 3.4.2.3 induces an isomorphism
\[ \hat{D}^{(m+1)} \xrightarrow{\sim} \hat{D}_1. \]

*Proof.* The argument of [4, Th. 3.5.3] can be used without change to prove (i) in the case \( n = 0 \) and to deduce from this that \( D^{(m+1)} \hookrightarrow D_1 \) induces an isomorphism \( \hat{D}^{(m+1)} \xrightarrow{\sim} \hat{D}_1 \) (note that one needs \( A \) to be \( \mathbb{Z}_p \)-flat to show that the map \( D^{(m+1)}/p^i D^{(m+1)} \rightarrow D_1/p^i D_1 \) is injective). Since \( \hat{D}^{(m)} = D^{(m)} + p^n \hat{D}^{(m)} \), the general case of (i) follows from the case \( n = 0 \).

Any open \((m + 1)\)-bilateralising ideal \( J \) contains a power of \( p \), so the \( J \)-adic completions of \( \hat{D}^{(m+1)} \) and \( \hat{D}_1 \) may be identified with \( \hat{D}(m+1) \) and \( \hat{D}_1 \) respectively. The isomorphism \( \hat{D}^{(m+1)} \xrightarrow{\sim} \hat{D}_1 \) thus induces an isomorphism \( \hat{D}^{(m+1)} \xrightarrow{\sim} \hat{D}_1 \).

Note that the inclusion \( D^{(m+1)} + J^n \hat{D}^{(m)} \subset D_2 \) is strict, which is what necessitates the introduction of the ring \( D_1 \).

**3.4.4 Lemma** The second map in 3.4.2.3 induces an isomorphism
\[ \hat{D}_1 \xrightarrow{\sim} \hat{D}_2. \]

*Proof.* It suffices to show that form any open \((m + 1)\)-bilateralising ideal \( J \subset A \), the induced map
\[ D_1/JD_1 \rightarrow D_2/JD_2 \]
is a bijection. That it is surjective follows from 3.4.2.1. By (i) of lemma 3.4.3 injectivity is equivalent to
\[ (D^{(m+1)} + \hat{D}^{(m)}) \cap JD_2 \subset JD^{(m+1)} + J\hat{D}^{(m)}. \]
Suppose \( P \in (\hat{D}^{(m)} + D^{(m+1)}) \cap JD_2 \); again by (i) in lemma 3.4.3 we can write \( P = Q + p^n R \) with \( Q \in D^{(m+1)} \) and \( R \in \hat{D}^{(m)} \) for any given \( n \geq 0 \). If we choose \( n \) so that \( p^n \in J \),
\[ P - Q = p^n R \in J\hat{D}^{(m)} \subset JD_2 \]
and since $P \in JD_2$ this implies $Q \in D^{(m+1)} \cap JD_2$. Then $Q \in JD^{(m+1)}$ by 3.4.2.2 and consequently $P = Q + p^n R \in JD^{(m+1)} + J\hat{D}^{(m)}$, as required.

The last bit of the proof of [4, Th. 3.5.3] can be used without change to prove:

3.4.5 Lemma The ring $D_2$ is left noetherian.

The proof of theorem 3.4.2 is concluded as follows: since $D_2$ is noetherian, $\hat{D}_2$ is a left flat $D_2$-module, and thus $\hat{D}_2Q$ is a left flat $D_2Q$-module. Now lemmas 3.4.3 and 3.4.4 show that $\hat{D}_2 \simeq \hat{D}^{(m+1)}$ and $\hat{D}_2Q \simeq \hat{D}_Q^{(m+1)}$. On the other hand 3.4.2.1 yields $D_2Q = \hat{D}_Q^{(m)}$, and it follows that $\hat{D}_Q^{(m+1)}$ is a left flat $\hat{D}_Q^{(m)}$-module. The assertion for any $m' \geq m$ follows by induction.

3.5 The ring $\mathcal{D}^\dagger_{X/SQ}$. As in [4, §2.5] we define

$$
\mathcal{D}^\dagger_{X/SQ} = \lim_{\longrightarrow} \mathcal{D}^{(m)}_{X/SQ} 
$$

with the inductive limit over the canonical morphisms 3.3.9.1. When $X = \text{Spf}(A)$ and $S = \text{Spf}(R)$ are formally affine we put

$$
\mathcal{D}^\dagger_{A/R} = \Gamma(X, \mathcal{D}^\dagger_{X/S}).
$$

It follows from theorem 3.4.2 by taking inductive limits that the canonical inclusion

$$
\mathcal{D}^{(m)}_{X/SQ} \rightarrow \mathcal{D}^\dagger_{X/SQ}
$$

is also flat.

3.5.1 Theorem If $X/S$ is quasi-smooth, $\mathcal{D}^\dagger_{X/SQ}$ is a coherent sheaf of rings.

As in [4, §3.5], this follows from theorem 3.4.2 by standard arguments. Also standard is the deduction of theorems A and B from the corresponding theorems for the rings $\mathcal{D}^{(m)}_{X/S}$:

3.5.2 Theorem Suppose $X = \text{Spf}(A)$ and $S = \text{Spf}(R)$ are noetherian and formally affine $X \rightarrow S$ is quasi-smooth. The functors

$$
\mathcal{M} \mapsto \Gamma(X, \mathcal{M}), \quad M \mapsto M^\wedge
$$

are exact.
are inverse equivalences between the category of coherent \( \mathcal{D}^\dagger_{X/S} \)-modules and the category of \( \mathcal{D}^\dagger_{A/R} \)-modules of finite type.

### 3.5.3 Theorem

Suppose \( X \rightarrow S \) is quasi-smooth and \( X \) is affine. With the above notation

\[
H^q(X, \mathcal{M}) = 0
\]

for any coherent \( \mathcal{D}^\dagger_{X/S} \)-module \( \mathcal{M} \).

## 4 Stratifications

We can now return to a question that was left open in section 3.1.7, that of the sheafification of the full \( m \)-PD-envelope of a regular ideal.

### 4.1 The ring \( P_{(m)}(I) \)

As always we begin with the affine case, so let \( R \) be an adic noetherian \( \mathbb{Z}_p \)-algebra with \( m \)-PD-structure \( (a, b, \alpha) \) and \( A \) an adic noetherian \( R \)-algebra. By hypothesis any ideal of definition of \( A \) contains a power of \( p \). For \( n \geq 0 \) we set \( R_n = R/p^{n+1}R \) and \( A_n = A/p^{n+1}A \).

### 4.1.1 Definition

An ideal \( I \subset A \) is split-regular if

(i) the ideal \( IA_0 \subset A_0 \) is regular;

(ii) the quotient homomorphism \( \pi : A \rightarrow A/I \) has a section \( \sigma : A/I \rightarrow A \);

(iii) \( A/I \) is a flat \( R \)-algebra.

In what follows it will be convenient to set \( A' = A/I \). If \( (\bar{f}_1, \ldots, \bar{f}_r) \) is a regular sequence generating \( IA_0 \) we pick \( f_1, \ldots, f_r \) in \( I \) lifting \( \bar{f}_1, \ldots, \bar{f}_r \). Then \( I = (f_1, \ldots, f_r) \) and \( (p, f_1, \ldots, f_r) \) are regular, as is \( (p^{n+1}, f_1, \ldots, f_r) \) for all \( n \geq 0 \). It follows that \( IA_n \) is regular for all \( n \geq 0 \).

The \( m \)-PD-structure \( (a, b, \alpha) \) descends to \( R_n \) and we denote by \( P_{(m)}(IA_n) \) the \( (a, b, \alpha) \)-compatible \( m \)-PD-envelope of \( IA_n \subset A_n \) (we will never deal with the full \( m \)-PD-envelope \( P_{(m)}(I) \) of \( I \subset A \)). If we set \( A'_n = A' \otimes A A_n \) then \( P_{(m)}(IA_n) \) with the \( A'_n \)-module structure determined by \( \sigma \) is a free \( A'_n \)-module, with basis the \( m \)-PD-monomials in the regular generators of \( IA_n \).

We note, for use in the arguments that follow that \( p \) is nilpotent in \( A_n \), so that \( P_{(m)}(IA_n) \) has the usual base-change properties.

For any \( n' \geq n \) there is an isomorphism

\[
P_{(m)}(IA_{n'}) \otimes_{R_{n'}} R_n \xrightarrow{\sim} P_{(m)}(IA_n)
\]
since the formation of $P_{(m)}(IA_n)$ is compatible with arbitrary base-change in $R_n$ (c.f. the remark in the last paragraph). Since this merely says that

$$P_{(m)}(IA_n) \simeq P_{(m)}(IA_{n'})/p^n P_{(m)}(IA_{n'})$$

we may rewrite it as an isomorphism

$$P_{(m)}(IA_{n'}) \otimes_{A_{n'}} A_n \xrightarrow{\sim} P_{(m)}(IA_n). \quad (4.1.1.1)$$

Let $J \subset A$ be an ideal of definition and choose $n \geq 0$ such that $p^{n+1} \in J$. Then $A/J$ is an $A_n$-algebra and the isomorphism $4.1.1.1$ shows that

$$P_{J,(m)}(I) := P_{(m)}(IA_n) \otimes_{A_n} (A/J) \simeq P_{(m)}(IA_n)/JP_{(m)}(IA_n) \quad (4.1.1.2)$$

is independent of the choice of $n$ (and justifies the notation). For $m' \geq m$ there is a natural morphism

$$P_{J,(m')}((m)) \rightarrow P_{J,(m)}((m)) \quad (4.1.1.3)$$
arising from the fact that an $m$-PD-structure is automatically an $m'$-PD-structure. Finally, we set

$$\hat{P}_{(m)}(I) = \lim_{\leftarrow} J P_{J,(m)}(I) \quad (4.1.1.4)$$

where $J$ runs through the set of ideals of definition of $A$. We denote by

$$\bar{l}^0 \subseteq \bar{l}^* \subset \hat{P}_{(m)}(I) \quad (4.1.1.5)$$

the closures of $l^0$ and $l^*$ in $\hat{P}_{(m)}(I)$ (or equivalently, the $J$-adic completions). The change-of-level morphisms $4.1.1.3$ induce ring homomorphisms

$$\hat{P}_{J,(m')}((m)) \rightarrow \hat{P}_{J,(m)}((m)) \quad (4.1.1.6)$$

for all $m' \geq m$. By construction the change-of-level formula $3.2.1.5$ holds for any $x \in I$.

**4.1.2 Lemma** Suppose $I$ is split-regular. If $R$ is $\mathbb{Z}_p$-flat, so is $\hat{P}_{(m)}(I)$.

*Proof.* We know that for all $n \neq 0$, $P_{(m)}(IA_n)$ is flat over $R_n$. Tensoring the exact sequence

$$0 \rightarrow p^n R/p^{n+1} R \rightarrow R/p^{n+1} R \xrightarrow{p} R/p^{n+1} R$$

we obtain
with \( P_m(IA_n) \) yields an exact sequence

\[
0 \to P_m(IA_n) \otimes_R p^n R/p^{n+1} R \to P_m(IA_n) \stackrel{p}{\to} P_m(IA_n).
\]

Since the inverse system \( \{p^n R/p^{n+1} R\} \) is essentially null, passing to the inverse limit in \( n \) shows that multiplication by \( p \) is injective.

What is not obvious in this construction is whether the canonical \( m \)-PD-structure of \( P_m(IA_n) \) descends to \( P_J(I) \) when \( p^{n+1} \in J \), or extends to the completion \( \hat{P}_{(m)}(I) \), and if so, whether the extensions are compatible with \( (a, b, \alpha) \). We will restrict our attention to the case where \( J \) satisfies

\[
J P_m(IA_n) = \sigma(J') P_m(IA_n) \text{ for some ideal } J' \subset A' = A/I \text{ and some } n \text{ such that } p^{n+1} \in J'.
\]

When \( p \) is nilpotent in \( A \), \( P_m(I)/I^\bullet \simeq A/I \simeq A' \), and thus \( 4.1.2.1 \) implies that \( J' = p(J) \) where \( \pi : A \to A' \) is the canonical projection, and we see that \( 4.1.2.1 \) is equivalent to

\[
\sigma(\pi(J)) P_m(IA_n) = J P_m(IA_n) \text{ for some } n \text{ such that } p^{n+1} \in J.
\]

For any given \( n \), \( p^{n+1} \in J \) if and only if \( p^{n+1} \in J' \); it follows that the conditions \( 4.1.2.1, 4.1.2.2 \) are independent of the particular value of \( n \).

**4.1.3 Definition** Suppose \( I \subseteq A \) is a split-regular ideal. An ideal \( J \subseteq A \) is adapted to \( I \) if the equivalent conditions \( 4.1.2.1, 4.1.2.2 \) hold.

The following is immediate from either \( 4.1.2.1, 4.1.2.2 \):

**4.1.4 Lemma** Suppose \( I \subseteq A \) is split-regular. If \( J \) is adapted to \( I \) then so is \( J^n \) for any \( n \).

When \( I \) is generated by a regular sequence \((x_1, \ldots, x_d)\) and \( J \subset A' \) satisfies \( 4.1.2.1 \) \( P_{(m)}(I) \) is the free \( A'/J \)-module on the \( m \)-PD-monomials \( x^{(K)}(m) \); this follows from the description of the full \( m \)-PD-envelope in section \( 3.1.6 \).

If \( A \) has an ideal of definition adapted to \( I \) we can give a similar description of the completion \( \hat{P}_{(m)}(I) \): elements of \( \hat{P}_{(m)}(I) \) can be identified with series \( \sum_K a_K x^{(K)}(m) \) with \( a_K \in A' \) and \( a_K \to 0 \) as \( |K| \to \infty \). In fact if \( J \subset A \) is an ideal of definition adapted to \( I \) then \( J^n \) satisfies \( 4.1.2.1 \) as well, and furthermore \( J' = p(J) \) is an ideal of definition of \( A' \). The above description of \( \hat{P}_{(m)}(I) \) follows from the previous description of the \( \hat{P}_{J^n,(m)}(I) \).

In particular \( \hat{P}_{(m)}(I)/I^\bullet \simeq A' \) is a flat \( R \)-algebra.
4.1.5 Proposition Let $R$ be a ring with $m$-PD-structure $(a, b, \alpha)$, $A$ an $R$-algebra. Suppose $I \subset A$ is a split-regular ideal, and $J \subset A$ is adapted to $I$.

(i) For any $n$ such that $p^{n+1} \in J$, the canonical $m$-PD-structure of the $m$-PD-envelope $P_{(m)}(A_n I)$ descends to $P_{J,(m)}(I)$.

(ii) If $b_1 = b + pR$, the $m$-PD-structure on $P_{J,(m)}(I)$ is compatible with $(a, b, \alpha)$ if and only if $J' \cap b_1 A'$ is a sub-PD-ideal of $b_1 A'$.

Proof. If $p^{n+1} \in J$ we can replace $A$ and $I$ by $A_n$ and $IA_n$, which is a regular ideal in $A_n$; furthermore $A_n/IA_n$ is a flat $R_n$-algebra and the section $\sigma$ induces a section of $A_n \to A_n/IA_n$. We may therefore assume that $p$ is nilpotent in $A$ and set $P = P_{(m)}(I)$; from lemma 3.1.3 (c.f. also [4, 1.3.4]) we see that the conditions to be checked are that the following are sub-PD-ideals:

$$\begin{align*}
(I^o + pP) \cap J &\subseteq I^o + pP & (4.1.5.1) \\
(I^o + b_1 P) \cap J &\subseteq I^o + b_1 P & (4.1.5.2) \\
b_1 P + (I^o + J) &\subseteq b_1 P & (4.1.5.3)
\end{align*}$$

where 4.1.5.1 guarantees that the $m$-PD-structure of $P$ descends to $P/J$, and 4.1.5.2 and 4.1.5.3 guarantee that it is compatible with $(a, b, \alpha)$. The question is Zariski-local so we may assume that $I$ is generated by a regular sequence $(x_1, \ldots, x_n)$. Then $P$ is a free $A'$-module on the $m$-PD-monomials $x^{(K)}(m)$ and the ideals in 4.1.5.1–4.1.5.3 have the following descriptions, where $x = \sum_K a_K x^{(K)}(m)$ and $K < p^m$ means that every entry of $K$ is less than $p^m$:

$$\begin{align*}
x \in (I^o + pP) \cap J &\iff a_K \in J', \text{ and } K < p^m \implies a_K \in pA'
\end{align*}$$

$$\begin{align*}
x \in (I^o + b_1 P) \cap J &\iff a_K \in J', \text{ and } K < p^m \implies a_K \in b_1 A'
\end{align*}$$

$$\begin{align*}
x \in b_1 P \cap (I^o + J) &\iff a_0 \in J', \text{ and } a_K \in b_1 A'.
\end{align*}$$

Thus 4.1.5.1 is a sub-PD-ideal because $pA' \cap J'$ is a sub-PD-ideal of $pA'$. If 4.1.5.3 is a sub-PD-ideal then $b_1 A' \cap J'$ is a sub-PD-ideal of $b_1 A'$, and conversely this implies that 4.1.5.2 and 4.1.5.3 are sub-PD-ideals.

With the hypotheses of the lemma, $(a, b, \alpha)$ extends to $A'$, and the condition in (ii) is equivalent to the assertion that the $m$-PD-structure $(aA', bA', \alpha)$ descends to $A'/J'$. Note that this is automatic if $b_1$ is principal, or if $A'/J'$ is a flat $R$-algebra.

If $J$ is adapted to $I$, $P_{J,(m)}(I)$ has an $m$-PD-adic filtration, and we denote by $P^n_{J,(m)}(I)$ the quotient of $P^n_{J,(m)}(I)$ by the $n + 1$-st step of that filtration.
On the other hand the truncations $P_{(m)}^n(I)$ of the full $m$-PD-envelope of the diagonal commutes with arbitrary base change in $R$, and in particular with the base change $R \to R_m$. From the construction we see that $P_{(m)}^n(I)$ is isomorphic to $P_{J_{(m)}(I)}^n(I)/JP_{J_{(m)}(I)}^n(I)$.

4.1.6 Corollary With the assumptions of proposition 4.1.5, for any $J \subset A$ adapted to $I$ the canonical $m$-PD-structure of $P_{J_{(m)}(I)}(I)$ is compatible with the $m$-PD-structure $(a, b, \alpha)$ of $R$ for all sufficiently large $n$.

Proof. With the notation of the proposition and its proof, it suffices to show that $(J')^n \cap b_1 A'$ is a sub-PD-ideal of $b_1 A'$ for $n \gg 0$, but this follows from lemma 3.1.8.

4.1.7 Theorem Suppose $I \subset A$ is a split-regular ideal and $A$ has an ideal of definition adapted to $I$. The canonical $m$-PD-structure of $P_{(m)}(I)$ extends to an $m$-PD-structure $(\hat{I}^\bullet, \hat{J}^\circ, \hat{\gamma})$ on $\hat{P}_{(m)}(I)$ with $\hat{I}^\bullet$ and $\hat{J}^\circ$ as in 4.1.1.3 and this $m$-PD-structure is compatible with $(a, b, \alpha)$.

Proof. Let $J$ be an ideal of definition adapted to $I$ and denote by $(\hat{I}^\bullet_n, \hat{J}^\circ_n, \gamma_n)$ the quotient $m$-PD-structure of $P_{J_{(n)}(I)}(I)$. By construction

$$\hat{I}^\bullet = \lim_{\longrightarrow} \hat{I}^\bullet_n, \quad \hat{J}^\circ = \lim_{\longrightarrow} \hat{J}^\circ_n$$

and the containments $(I^\bullet_n)^{(p^n)} + p\hat{I}^\bullet_n \subseteq \hat{I}^\circ_n$ for all $n$ show that $(\hat{I}^\bullet)^{(p^n)} + p\hat{I}^\bullet \subseteq \hat{I}^\circ$. If $\gamma_n = \{\gamma_{n,k}\}_{k \geq 0}$, the functions $\hat{\gamma}_k = \lim_{\longrightarrow} \gamma_{n,k}$ define a PD-structure on $\hat{I}^\circ$, and $(\hat{I}^\bullet, \hat{J}^\circ, \hat{\gamma})$ an $m$-PD-structure on $\hat{P}_{(m)}(I)$. We must show that $(\hat{I}^\bullet, \hat{J}^\circ, \hat{\gamma})$ is compatible with $(a, b, \alpha)$; this means that $b_1 \hat{P}_{(m)}(I) + \hat{I}^\circ$ has a PD-structure extending the PD-structures $\bar{\alpha}$ of $b_1$ and $\hat{\alpha}$ of $\hat{I}^\circ$, and that $b_1 \hat{P}_{(m)}(I) \cap \hat{I}^\bullet$ is a sub-PD-ideal of $b_1 \hat{P}_{(m)}(I)$. By construction $b_1 P_{(m)}(I) + J^\circ$ has a PD-structure $\{\delta_k\}_{k > 0}$ extending $\bar{\alpha}$ and $\gamma$; on the other hand corollary 4.1.6 says that $(I^\bullet_n, \hat{I}^\circ_n, \gamma_n)$ of is compatible with $(a, b, \alpha)$ for all $n \gg 0$, which implies that for all $k > 0$, $\delta_k$ is $J$-adically continuous on $b_1 P_{(m)}(I) + J^\circ$. Therefore $\delta$ extends by continuity to the closure of $b_1 P_{(m)}(I) + J^\circ$, and in particular to $b_1 \hat{P}_{(m)}(I) + \hat{I}^\circ$. Finally we observed earlier that $\hat{P}_{(m)}(I)/\hat{I}^\bullet \simeq A'$ is a flat $R$-algebra, which implies that $b_1 \hat{P}_{(m)}(I) \cap \hat{I}^\bullet = b_1 \hat{I}^\bullet$ is a sub-PD-ideal of $b_1 \hat{P}_{(m)}(I)$.

Passing to the limit over $J$ in 4.1.6 yields ring homomorphisms

$$\hat{P}_{(m)}(I) \to \hat{P}_{(m)}(I)$$

(4.1.7.1)
for all $m' \geq m$. By construction the change-of-level formula \[3.2.1.5\] holds for any $x \in I$.

The universal properties of these rings are as follows:

**4.1.8 Proposition** Suppose that $A$ has an ideal of definition adapted to $I$.

(i) Let $A'$ be a discrete topological $R$-algebra with an $m$-PD-structure $(I', J', \gamma)$ compatible with $(a, b, \alpha)$, and suppose $f : A \to A'$ is a continuous $R$-algebra homomorphism such that $f(I) \subseteq I'$. For any $m$-bilateralising ideal of definition $K \subset A$ such that $f(K) = 0$, $f$ has a unique factorisation

$$A \to P_{K,(m)}(I) \xrightarrow{f_K} A'$$

in which $f_K$ is an $m$-PD-homomorphism over $R$.

(ii) Suppose $A'$ is an adic noetherian $R$-algebra with an $m$-PD-structure $(I', J', \gamma)$ compatible with $(a, b, \alpha)$. Any continuous $R$-algebra homomorphism $f : A \to A'$ such that $f(I) \subseteq I'$ has a unique factorisation

$$A \to \hat{P}_{(m)}(I) \xrightarrow{g} A'$$

in which $g$ is an some $m$-PD-homomorphism $g$ over $R$.

**Proof.** (i) Pick $n$ such that $p^{n+1} \in J$; then $f$ factors through a morphism $f_n : A_n \to A'$, and the $m$-PD-structure $(I'A_n, J'A_n, \tilde{\gamma})$ descends to an $m$-PD-structure $f_n$ factors through a unique $m$-PD-morphism $f' : P_{(m),\alpha}(IA_n) \to A'$, and since $JA' = 0$, $f'$ factors through an $m$-PD-morphism $f' : P_{(m),\alpha}(I) \to A'$ which is unique since $f'$ is.

(ii) The same argument as before shows that for all $n \geq 0$ the reduction $f_n : A_n \to A_n'$ of $f$ factors uniquely through an $m$-PD-morphism $P_{(m),\alpha}(IA_n) \to A_n'$ for all $n \geq 0$. By lemma \[3.1.8\] we know that $A'$ has an ideal of definition $K'$ such that the $m$-PD-structure of $A'$ descends to $A/(K')^n$ for all $n$ (c.f. the remark after the lemma). We can then find an ideal of definition $K \subset A$ such that $f(K) \subseteq K'$ and the $m$-PD-structure of $A$ descends to $A/K^n$ for all $n$. For any particular $n$ we can choose an $n'$ such that $p^{n'+1} \in K^n$; then the morphism $f_{n'} : P_{(m)}(IA_{n'}) \to A_{n'}'$ induces a morphism $g_n : P_{(m),K^n}(I) \to A'/((K')^n)$. The latter morphism is necessarily an $m$-PD-morphism since $f_{n'}$ is, and since $P_{(m)}(IA_{n'}) \to P_{(m),K^n}(IA_{n'})$ is an $m$-PD-morphism. Since $A'$ is $K'$-adically complete, the inverse limit of
the $g_n$ is an $m$-PD-morphism $\hat{g} : \hat{P}_{(m)}(I) \to A'$, and the construction shows that it is the unique morphism that factors $f$.

We now globalize these constructions. Let $X \to S$ be a universally noetherian morphism of locally noetherian adic formal $\mathbb{Z}_p$-schemes and $(a, b, \alpha)$ is an $m$-PD-structure on $S$.

4.1.9 Definition An ideal $I \subseteq O_X$ is split-regular if

(i) The image of $I$ in $O_X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is regular ideal;

(ii) If $Y = V(I)$, the canonical closed immersion $Y \to X$ locally has a retraction $X \to Y$ over $S$.

(iii) $Y \to S$ is flat.

If $I \subseteq O_X$ is split-regular, and ideal $J \subseteq O_X$ is adapted to $I$ if it locally satisfies the equivalent conditions [4.1.2.1] [4.1.2.2] in which $\sigma$ is the homomorphism associated to the (local) retraction $X \to Y$.

Fix a split-regular ideal $I \subseteq O_X$ and and ideal $J \subseteq O_X$ adapted to $I$. To explain how the preceding constructions patch together we can assume that $S = \text{Spf}(R)$ is affine. Let $U = \text{Spf}(A) \subseteq X$ be an affine open and set $I = \Gamma(U, I)$, $J = \Gamma(U, J)$. For any $f \in A$ the $A_n$-algebra $(A_n)_f$ is flat, and the natural morphism

$$P_{(m)}(IA_n)_f \to P_{(m)}(I(A_n)_f)$$

is an isomorphism. Thus if $J$ is an open ideal such that $p^{n+1} \in J$, the natural morphism

$$A_f \otimes_A P_{J,(m)}(I) \to P_{J,(m)}(IA_f)$$

is an isomorphism. It follows that there is a quasicoherent sheaf of $O_{X_J}$-algebras $P_{J,(m)}(I)$ with an $m$-PD-structure such that for affine opens $U = \text{Spf}(A) \subseteq X$, $V = \text{Spf}(R) \subseteq S$ such that $X \to S$ sends $U \to V$,

$$\Gamma(U, P_{J,(m)}(I)) = P_{J,(m)}(I)$$

where $I = \Gamma(U, I)$ and $J = \Gamma(U, J)$. The $O_{X_J}$-algebra $P_{J,(m)}(I)$ gives us an affine morphism $X_{S,(m)}^J(I) \to X$ of formal schemes, where

$$X_{S,(m)}^J(I) := \text{Spec}_{O_{X_J}}(P_{J,(m)}(I)).$$

(4.1.9.1)
Suppose now \( \mathcal{O}_X \) has an ideal of definition adapted to \( \mathcal{I} \). By lemma 4.1.4 it has a cofinal set of ideals of definition adapted to \( \mathcal{I} \), and we define
\[
\mathcal{P}_{(m)}(\mathcal{I}) = \lim_{\leftarrow \mathcal{J}} \mathcal{P}_{\mathcal{J},(m)}(\mathcal{I})
\]
(4.1.9.2)
where \( \mathcal{J} \) runs through the set of ideals of definition adapted to \( \mathcal{I} \). If \( U = \text{Spf}(A) \subseteq X \) is an open affine mapping to an open affine \( \text{Spf}(A) \subseteq S \) then
\[
\Gamma(U, \mathcal{P}_{(m)}(\mathcal{I})) \simeq \hat{P}_{(m)}(\mathcal{I})
\]
where as before \( \mathcal{I} = \Gamma(U, \mathcal{I}) \). By definition \( \mathcal{P}_{(m)}(\mathcal{I}) \) is a sheaf of \( \mathcal{O}_X \)-algebras whose reduction modulo \( J \subset \mathcal{O}_X \) for any \( J \) adapted to \( \mathcal{I} \) is the sheaf \( \mathcal{P}_{\mathcal{J},(m)}(\mathcal{I}) \). We will not attach a formal scheme to \( \mathcal{P}_{(m)}(\mathcal{I}) \) since this would take us out of the category of locally noetherian formal schemes. As before there is a change-of-level morphism
\[
\mathcal{P}_{(m')}(\mathcal{I}) \to \mathcal{P}_{(m)}(\mathcal{I})
\]
(4.1.9.3)
for \( m' \geq m \).

We can now state the universal properties of \( X^{J, S}_{(m)} \to X \) and \( \mathcal{O}_X \to \mathcal{P}_{(m)}(\mathcal{I}) \) when \( J \) is adapted to \( \mathcal{I} \). Suppose, first, that \( X' \) is an \( S \)-scheme with an \( m \)-PD-structure \((I', J', \gamma')\) compatible with \((a, b, \alpha)\), and \( f : X' \to X \) is an \( S \)-morphism such that \( f^* \mathcal{I} \subseteq \mathcal{I}' \). There is a cofinal set of ideals of definition \( J \subset \mathcal{O}_X \) satisfying 4.1.2.1 such that \( f \) factors
\[
X' \to X^{J, S}_{(m)}(\mathcal{I}) \xrightarrow{f_J} X
\]
for some unique \( m \)-PD-morphism \( f_J \). Suppose, on the other hand that \( X' \) is a formal \( S \)-scheme with an \( m \)-PD-structure \((I', J', \gamma')\) compatible with \((a, b, \alpha)\), and \( f : X' \to X \) is an \( S \)-morphism such that \( f^* \mathcal{I} \subseteq \mathcal{I}' \). Then the canonical morphism \( f^* \mathcal{O}_X \to \mathcal{O}_X \) has a unique factorization
\[
f^* \mathcal{O}_X \to f^* \mathcal{P}_{(m)}(\mathcal{I}) \xrightarrow{g} \mathcal{O}_{X'}
\]
where \( g \) is an \( m \)-PD-morphism (here the \( f^* \) is understood in the sense of ringed spaces).

### 4.2 The sheaf \( \mathcal{P}_{X/S,(m)}(r) \)

When \( R \to A \) is a quasi-smooth homomorphism of adic noetherian \( \mathbb{Z}_p \)-algebras we can apply the preceding constructions to the diagonal ideal \( I(r) \) of the completed tensor product \( \hat{A}_R(r) \) of \( r + 1 \) copies of \( A \) over \( R \). Recall \( I(r) \) is the kernel of the multiplication map

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$A_R(r) \to A$, which has $r + 1$ sections, namely the maps $d_i : A \to A_R(r)$ for $0 \leq i \leq r$. Set $R_n = R/p^{n+1}R$ and $A_n = A/p^{n+1}$ as before, and set $I_n = I(A_n)R_n(r)$. Since $R_0 \to A_0 = A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is quasi-smooth, $A_n$ is a flat $R_n$-algebra and $I_n$ is a regular ideal. It follows that the ideal $I(r) \subseteq \hat{A}_R(r)$ is split-regular.

If $J \subseteq A$ is any ideal of definition, $J(r) = \sum_{0 \leq i \leq r} d_i(J)A_R(r)$ (4.2.0.1)

is an ideal of definition of $A_R(r)$; that it is adapted to $I(r)$ with $\sigma = d_i$ for any $i$ follows from:

**4.2.1 Lemma** Suppose $R \to A$ is quasi-smooth and $(a, b, \alpha)$ is an $m$-PD-structure on $R$. If $J \subseteq A$ is an $m$-bilateralising ideal, $J(r)P_{A/R,(m)} = d_i(J)P_{A/R,(m)}$ for $0 \leq i \leq r$ and any $n$ such that $p^n \in J$.

Proof. As before we reduce to the case where $p^n = 0$ in $A$. The canonical isomorphisms

$$P_{A/R,(m)}(r) \otimes_{A} P_{A/R,(m)}(r') \simeq P_{A/R,(m)}(r + r')$$

show that it suffices to treat the case $r = 1$, which may be rephrased as an equality

$$JP_{A/R,(m)} = P_{A/R,(m)}J$$

(4.2.1.1) for all $m$-bilateralising $J$. Set $P = P_{A/R,(m)}$; in the notation of corollary 3.3.8 we must show that $\delta(J) \subseteq JP \cap PJ$, and the same Taylor series argument shows that $\delta(J) \subseteq JP$. By Zariski localization we may assume that $A$ has local coordinates relative to $R$, which we may use to get a basis $\{\xi^{(K)}(m}\} K \geq 0$ of $P$ as a $B$-module via $d_1 : B \to P$. The corollary tells us that the image of $\delta(J)$ under the natural projection $P \to P^n = P_{A/R,(m)}$ is contained in $P^nJ$ for all $n$. Thus if $x \in \delta(J)$ is $\sum K d_1(a_K)\xi^{(I)}(m)$ in terms of the basis we have $a_K \in J$ for all $K$, and thus $x \in PJ$. \hfill \blacksquare

**4.2.2 Remark** The equality 4.2.1.1 looks like the definition of “bilateralising” but has nothing to do with it, since in fact $P_{B/A,(m)}$ is a commutative ring. The ideals $JP_{B/A,(m)}$, $P_{B/A,(m)}J$ are the ideals generated by the image of $J$ under the two ring homomorphisms $d_0, d_1 : B \to B_{B/A,(m)}$. 62
Suppose now \( S \) is an adic locally noetherian formal \( \mathbb{Z}_p \)-scheme with \( m \)-PD-structure \((a, b, \alpha)\) and \( \mathcal{X} \to S \) is quasi-smooth. We may apply the results of §4.1, with the result that for any bilateralising \( J \subset \mathcal{O}_X \) there is a sheaf \( \mathcal{P}_{X/S, J, (m)}(r) \) with \( r + 1 \) \( \mathcal{O}_X \)-module structures, quasi-coherent for any one of them. The sheaf \( \mathcal{P}_{X/S, (m)}(r) \) is the inverse limit

\[
\mathcal{P}_{X/S, (m)}(r) = \lim_{\leftarrow J} \mathcal{P}_{X/S, J, (m)}(r)
\]

(4.2.2.1)

where \( J \) runs through all bilateralising ideals of definition (this is another case of dropping the hat in a geometric context). By construction it is an algebra over the structure sheaf of the completion \( \hat{X}_S(r) \) of the \( r + 1 \)-fold fiber product \( X_S(1) \) with respect to the diagonal ideal \( I(r) \).

When \( U = \text{Spf}(A) \subseteq \mathcal{X} \) is an open affine lying over \( \text{Spf}(R) \subseteq S \) we have

\[
\Gamma(U, \mathcal{P}_{X/S, (m)}(r)) \simeq \hat{P}_{B/A, (m)}(r)
\]

(4.2.2.2)

As before we omit the \( (r) \) when \( r = 1 \).

The scheme

\[
\mathcal{X}_{S,(m)}^J = \text{Spec}_{\mathcal{O}_X} (\mathcal{P}_{X/S, J, (m)})
\]

is relatively affine over \((X_J)_S(1)\). We denote by \( q_0, q_1 : \mathcal{X}_{S,(m)}^J \to X_J \) the composites of the structure morphism \( \mathcal{X}_{S,(m)}^J \to (X_J)_S(1) \) with the natural projections \( p_0, p_1 : (X_J)_S(1) \to X_J \). The inductive system of \( \mathcal{X}_{S,(m)}^J \) has the following universal property. Let \( Y \) be a scheme over \( S \) such that the \( m \)-PD-structure \((a, b, \alpha)\) of \( S \) extends to \( Y \), and let \( f_0, f_1 : Y \to \mathcal{X} \) be \( S \)-morphisms congruent modulo \( a \) in the sense that if \( Y_0 \subset Y \) is the closed subscheme defined by \( a \mathcal{O}_Y \), the two composite morphisms

\[
Y_0 \xrightarrow{f_0} Y \xrightarrow{f_1} \mathcal{X}
\]

are equal. There is a cofinal set of \( J \subset \mathcal{O}_X \) such that there is a unique morphism \( g_J : Y \to \mathcal{X}_{S,(m)}^J \) such that the morphism \((f_0, f_1) : Y \to \mathcal{X} \times_S \mathcal{X} \) factors

\[
Y \xrightarrow{g_J} \mathcal{X}_{S,(m)}^J \to X_J \times_S X_J \to \mathcal{X} \times_S \mathcal{X}.
\]

4.3 \textit{m-HPD-stratifications.} We have already observed that a left \( \mathcal{D}_{X/S}^{(m)} \) module structure on an \( \mathcal{O}_X \)-module \( M \) is equivalent to an \( m \)-PD-stratification of \( M \) relative to \( S \). When \( J \subset \mathcal{O}_X \) is bilateralising this construction can be restricted to the case of \( \mathcal{O}_{X_J} \)-modules, yielding an equivalence of the
category of left $\mathcal{D}_{X,J/S}^{(m)}$-modules with the category of $\mathcal{O}_{X,J}$-modules endowed with an $\mathcal{S}$-stratification in the previous sense.

Denote by $p_0, p_1 : \mathcal{X}_S(1) \rightarrow \mathcal{X}$ the natural projections.

4.3.1 Definition An $m$-HPD-stratification of an $\mathcal{O}_{X,J}$-module $M$ relative to $\mathcal{S}$ is an isomorphism

$$\chi : \mathcal{P}_{X/S,J,(m)} \otimes_{\mathcal{O}_{X_S(1)}} p_1^*M \cong p_0^*M \otimes_{\mathcal{O}_{X_S(1)}} \mathcal{P}_{X/S,J,(m)}$$

restricting to the identity on the diagonal and satisfying the cocycle condition.

Note that the order of the tensor products is immaterial and the isomorphism 4.3.1.1 is written this way to preserve consistency with other well-known notations.

If $\chi$ is an $m$-HPD-stratification, extending scalars by $\mathcal{P}_{X/S,J,(m)} \rightarrow \mathcal{P}_{X,J,S,(m)}$ for all $n$ results in an $m$-PD-stratification of $M$ relative to $\mathcal{S}$, and $\chi$ is determined by this $m$-PD-stratification. We may then say that a left $\mathcal{D}_{X,J/S}^{(m)}$-module $M$ is quasi-nilpotent if its associated $m$-PD-stratification extends to an $m$-HPD-stratification. The argument of [4, Prop. 2.3.7] with $\mathcal{P}_{X/S,J,(m)}$ in place of $\mathcal{P}_{X/S,J,(m)}$ then shows:

4.3.2 Proposition A left $\mathcal{D}_{X,J/S}^{(m)}$-module $M$ is quasi-nilpotent if and only if for every local section $x$ of $M$ and some system of local coordinates (defined in the same neighborhood as $m$), $\partial^{[I]}(x) = 0$ for $|I| \gg 0$. If this is so, then in fact $\partial^{[I]}(x) = 0$ for any system of local coordinates and $|I| \gg 0$.

4.3.3 Corollary If $M$ is a quasi-nilpotent left $\mathcal{D}_{X,J/S}^{(m)}$-module, then so is any submodule or quotient module of $M$, and conversely if $M$ is a left $\mathcal{D}_{X,J/S}^{(m)}$-module and $N \subseteq M$ is a submodule such that $N$ and $M/N$ are quasi-nilpotent, then $M$ is quasi-nilpotent. If $M$ and $N$ are quasi-nilpotent left $\mathcal{D}_{X,J/S}^{(m)}$-modules then so are $M \otimes_{\mathcal{O}_X} N$ and $\text{Hom}_{\mathcal{O}_X}(M, N)$.

For example, if $J \subseteq \mathcal{O}_X$ is $m$-bilateralising then the standard $\mathcal{D}_{X/S}^{(m)}$-module structure of $\mathcal{O}_X$ induces a quasi-nilpotent $\mathcal{D}_{X,J/S}^{(m)}$-module structure $\mathcal{O}_X/J$. Applying this to the $\mathcal{D}_{X,J/S,J}^{(m)}$-module $\mathcal{O}_X/J^n$, we see from the corollary that $J^n/J^{n+1}$ is a quasi-nilpotent $\mathcal{D}_{X,J/S}^{(m)}$-module for all $n \geq 0$.

It might make sense at this point to say that a left $\mathcal{D}_{X,S}^{(m)}$-module $M$ is topologically quasi-nilpotent if $M/JM$ is quasi-nilpotent for some $m$-bilateralising ideal if definition of $\mathcal{O}_X$. However we will want to use this
terminology in situations where for example $M/JM = 0$ even thought $M \neq 0$. We therefore place restrictions on $M$:

For every ideal of definition $J \subseteq \mathcal{O}_X$; $M/JM$ is a quasi-coherent sheaf of $O_{X,J}$-modules;

$$M \simeq \varprojlim J M/JM$$

(4.3.3.2)

4.3.4 Definition An $\mathcal{D}_{X/S}^{(m)}$-module $M$ satisfying [4.3.3.1] and [4.3.3.2] is topologically quasi-nilpotent for every bilateralising ideal of definition $J \subseteq \mathcal{O}_X$ the reduction $M/JM$ is a quasi-nilpotent $\mathcal{D}_{X/J}^{(m)}$-module.

4.3.5 Proposition A left $\mathcal{D}_{X/S}^{(m)}$-module $M$ satisfying [4.3.3.1] and [4.3.3.2] is topologically quasi-nilpotent if and only if $M/JM$ is a quasi-nilpotent $\mathcal{D}_{X/J}^{(m)}$-module for some $m$-bilateralising ideal of definition $J \subseteq \mathcal{O}_X$.

Proof. The condition is evidently necessary, and for the converse it suffices to show that if $M/JM$ is quasi-nilpotent for $J$ in the proposition then $M/J^nM$ is a quasi-nilpotent $\mathcal{D}_{X/J}^{(m)}$-module for all $n > 0$. We have seen that $J^k/J^{k+1}$ is a quasi-nilpotent $\mathcal{D}_{X/J}^{(m)}$-module for all $k$, so by the hypothesis and corollary [4.3.3] the same holds for the tensor product $J^k/J^{k+1} \otimes \mathcal{O}_X$ $M/JM$ and for its quotient $J^kM/J^{k+1}M$. Then for $k < n$, $J^kM/J^{k+1}M$ is quasi-nilpotent as a $\mathcal{D}_{X/J}^{(m)}$-module, and corollary [4.3.3] shows that $M/J^n$ is a quasi-nilpotent $\mathcal{D}_{X/J}^{(m)}$-module as well.

4.3.6 Corollary If $M$ is a topologically quasi-nilpotent left $\mathcal{D}_{X}^{(m)}$-module, so is any submodule or quotient module of $M$. If $M$ is a left $\mathcal{D}_{X}^{(m)}$-module and $N \subseteq M$ is a submodule such that $N$ and $M/N$ are topologically quasi-nilpotent, then so is $M$. If $M$ and $N$ are topologically quasi-nilpotent $\mathcal{D}_{X}^{(m)}$ then so are $M \otimes_{\mathcal{O}_X} N$ and $\text{Hom}_{\mathcal{O}_X}(M,N)$.

Proof. This follows from proposition [4.3.5] and corollary [4.3.3].

4.3.7 Corollary If $M$ is a left $\mathcal{D}_{X/S}^{(m+1)}$-module satisfying [4.3.3.1] and [4.3.3.2] the $\mathcal{D}_{X/S}^{(m)}$-module induced by restriction is topologically quasi-nilpotent.
Proof. Topological quasi-nilpotence can be detected locally, so we can work in local coordinates. It suffices to observe that in the ring \( D^{(m+1)}_{X/S} \), the differential operators \( \partial_i^{(p^m)(m+1)} \) are multiples of \( p \); since \( p \) is topologically nilpotent in \( O_X \) we can apply proposition 4.3.2.

A quasi-nilpotent \( D^{(m)}_{X/S} \)-module \( M \) gives rise to an isomorphism

\[ \chi : \mathcal{P}_{X/S,(m)} \otimes_{O_X} M \xrightarrow{\sim} M \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,(m)} \]  

(4.3.7.1)

reducing to the identity on the diagonal and satisfying the cocycle condition; this defines the notion of an \( m \)-HPD-structure on the \( O_X \)-module \( M \). Suppose, in fact that \( M \) is a \( D^{(m)}_{X/S} \)-module such that \( M/JM \) arises from an \( m \)-PD-stratification \( \chi_J \) for all \( J \). Since a left \( D^{(m)}_{X/S} \)-module structure arises from at most one \( m \)-HPD-stratification on an \( O_{X_J} \)-module, the various \( \chi_J \) of the \( M/JM \) for variable \( J \) must all be compatible, and then \( \chi = \lim_J \chi_J \) has the required properties. Conversely an isomorphism \[ \text{4.3.7.1} \] induces \( m \)-HPD-stratifications of \( M/JM \) for all \( J \), all of which correspond to the same left \( D^{(m)}_{X/S} \)-module structure.

4.3.8 Proposition Suppose \( X \to S \) is quasi-smooth and \( (a, b, \alpha) \) is an \( m \)-PD-structure on \( S \), and \( M \) is a left \( D^{(m)}_{X/S} \)-module. Suppose \( X' \) is a formal \( S \)-scheme such that \( (a, b, \alpha) \) extends to \( X' \), and let \( f, f' : X' \to X \) be two \( S \)-morphisms having the same restriction to the closed formal subscheme \( X'_0 \subset X' \) defined by \( aO_{X'} \).

(i) If the \( m \)-PD-ideal \( (a, b, \alpha) \) is \( m \)-PD-nilpotent, there is a canonical isomorphism

\[ \tau_{f, f'} : (f')^* M \xrightarrow{\sim} f^* M \]

of left \( O_{X'} \)-modules, such that \( \tau_{f, f} = \text{id}_{f^* M} \), and the system of \( \tau_{f, f'} \) is transitive in the sense that if \( f'' : X' \to X \) is a third such morphism then \( \tau_{f, f'} \circ \tau_{f', f''} = \tau_{f, f''} \).

(ii) If \( (a, b, \alpha) \) is not assumed to be \( m \)-PD-nilpotent, but if \( M \) is a topologically quasi-nilpotent left \( D^{(m)}_{X/S} \)-module, the system of \( \tau_{f, f'} \) also exists and has the same properties.

(iii) In either case, in the situation of the diagram \[ \text{4.3.2.1} \] when \( X'/S' \) is quasi-smooth, the morphism \( \tau_{f, f'} \) is \( D^{(m)}_{X'/S'} \)-linear.
The argument is the same as in [6, Prop. 2.1.5], and is a direct consequence of the universal property of the system of $X^J_{S,(m)}$. In (iii), the proof of $D_{m}$-linearity uses the compatibility of the canonical ideal of $X^n_{S,(m)}$ with the $m$-PD-structure of $a$, which relies on the structure theorem [4, Prop. 1.5.3] for $m$-PD-envelopes of a regular ideal.

4.3.9 Coefficient Rings. As in [4, 2.3.4], a left $D_{m}$-$X/S$-module structure on a commutative $O_X$-algebra $B$ is compatible with its algebra structure if the given $O_X$-module structure of $B$ coincides with the one derived from its left $D_{m}$-$X/S$-module structure, and if the isomorphisms [5.2.3.2] are isomorphisms of $P_{(m)}$-$X/S$-algebras. An equivalent condition is that the multiplication map $B \otimes_{O_X} B \to B$ is $D_{m}$-$X/S$-linear. In local coordinates, this is equivalent to the level $m$ Leibnitz rule

$$\partial(K_{(m)}(ab)) = \sum_{I+J=K} \left\{ \begin{array}{c} K \\ I \\ J \end{array} \right\}_{(m)} \partial(I_{(m)}(a))\partial(J_{(m)}(b))$$ (4.3.9.1)

holding for all local sections $a, b$ of $B$.

When $B$ is an $O_X$-algebra with a compatible left $D_{m}$-$X/S$-module structure, the $O_X$-module $B \otimes_{O_X} D_{m}$-$X/S$ has a unique ring structure such that the canonical homomorphisms

$$B \to B \otimes_{O_X} D_{m}$-X/S ; \ b \mapsto b \otimes 1$$

$$D_{m}$-X/S \to B \otimes_{O_X} D_{m}$-X/S ; \ P \mapsto 1 \otimes P$$

are ring homomorphisms. The product is defined as follows: with the identification

$$B \otimes_{O_X} \text{Diff}^{n'}_{X/S,(m)} \simeq \text{Hom}_{O_X}(P^n_{X/S,(m)}, B)$$ (4.3.9.2)

the product of local sections $P \in B \otimes_{O_X} \text{Diff}^{n'}_{X/S,(m)}; Q \in B \otimes_{O_X} \text{Diff}^{n'}_{X/S,(m)}$ is

$$\begin{array}{c}
\delta ^{n,n'} : P^{n'}_{X/S,(m)} \oplus P^n_{X/S,(m)} \\
\begin{array}{c}
\begin{array}{c}
\text{Id} \\
\otimes Q \\
\chi _{n',n} \\
\delta _{n,n'} \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
P^n_{X/S,(m)} \otimes B \\
B \otimes P^n_{X/S,(m)}
\end{array}$$ (4.3.9.3)

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where $\chi$ is the stratification of $\mathcal{B}$.

Since on occasion we will be considering several $\mathcal{B}$ at once we will use the notation
\[ D^{(m)}_{\mathcal{B}/S} = \mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}}} D^{(m)}_{\mathcal{X}/S} \]  
(4.3.9.4)
in preference to that of [4]. For any open affine $U \subseteq \mathcal{X}$ the canonical homomorphism
\[ \Gamma(U, \mathcal{B}) \otimes_{\Gamma(U, \mathcal{O}_{\mathcal{X}})} \Gamma(U, D^{(m)}_{\mathcal{X}/S}) \to \Gamma(U, D^{(m)}_{\mathcal{B}/S}) \]  
(4.3.9.5)
is an isomorphism; argument is the same as that of [4, Prop. 2.3.6] and depends mainly on the fact that $D^{(m)}_{\mathcal{X}/S}$ is an inductive limit of locally free $\mathcal{O}_{\mathcal{X}}$-modules.

From now on the following conditions will be imposed in $\mathcal{B}$, without explicit mention to the contrary:

for every open affine $U \subseteq \mathcal{X}$, $\Gamma(U, \mathcal{B})$ is noetherian;  
(4.3.9.6)
for every ideal of definition $J \subseteq \mathcal{O}_{\mathcal{X}}$; $\mathcal{B}/J\mathcal{B}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}_J}$-algebra;  
(4.3.9.7)
$\mathcal{B} \simeq \varprojlim J(\mathcal{B}/J\mathcal{B})$ where the inverse limit is over ideals of definition of $\mathcal{X}$.  
(4.3.9.8)

We define
\[ \hat{D}^{(m)}_{\mathcal{B}/S} = \varprojlim J(\mathcal{B}/J\mathcal{B}) \otimes_{\mathcal{O}_{\mathcal{X}}} D^{(m)}_{\mathcal{X}/S} \]  
(4.3.9.9)
where the inverse limit is over $m$-bilateralising ideals of definition. Then $\hat{D}^{(m)}_{\mathcal{B}/S}$ is a ring, and the previous discussion shows that a left $\hat{D}^{(m)}_{\mathcal{B}/S}$-module is the same as $\mathcal{B}$-module with a compatible left $\hat{D}^{(m)}_{\mathcal{X}/S}$-module structure in the previous sense. With the hypotheses 4.3.9.6 4.3.9.8 on $\mathcal{B}$, theorem 3.3.10 3.3.14 and its corollary, theorem 3.3.16 and propositions 3.3.11 3.3.13 hold for $\hat{D}^{(m)}_{\mathcal{B}/S}$ without modification; we will not bother to restate them.

For homomorphisms of rings satsifying 4.3.9.7 and 4.3.9.8 we can define pullbacks as in rigid geometry, i.e. by reducing modulo powers of an ideal of definition and passing to a limit. The same applies to tensor products; we will explicitly note completion of the tensor products unless one factor is coherent, in which case completion is unnecessary. We can then reformulate the condition that an $\mathcal{O}_{\mathcal{X}}$-algebra $\mathcal{B}$ has an $m$-PD-stratification of an $\mathcal{O}_{\mathcal{X}}$ compatible with its algebra structure. Suppose $\mathcal{P}^{n}_{\mathcal{B}/S,(m)}$ is an $\mathcal{O}_{\mathcal{P}^{n}_{\mathcal{X}/S,(m)}}$-algebra and

\[ \alpha^n_0 : p_0^*(\mathcal{B}) \otimes_{\mathcal{O}_{\mathcal{X}(1)}} \mathcal{P}^{n}_{\mathcal{X}/S,(m)} \simeq Q \]  
\[ \alpha^n_1 : \mathcal{P}^{n}_{\mathcal{X}/S,(m)} \otimes_{\mathcal{O}_{\mathcal{X}(1)}} p_1^*(\mathcal{B}) \simeq Q \]  
(4.3.9.10)
are isomorphisms of $P^m_{X/S(m)}$-algebras. We will say that $\{\alpha^n_0\}_{n \geq 0}$ and $\{\alpha^n_1\}_{n \geq 0}$ are compatible if for $n' \geq n$ they define the same morphism $P^m_{B/S(m)} \to P^n_{B/S(m)}$ and for $n = 0$ they yield the same identification $P^0_{B/S(m)} \simeq B$. If $\{\alpha^n_0\}_{n \geq 0}$ and $\{\alpha^n_1\}_{n \geq 0}$ are compatible, the isomorphisms

$$\chi_n = (\alpha^n_0)^{-1} \circ \alpha^n_1 : P^n_{X/S(m)} \otimes \mathcal{O}_{\hat{X}(1)} B \to B \otimes \mathcal{O}_{\hat{X}(1)} P^n_{X/S(m)}$$

are compatible in the previous sense. Conversely if $\chi_n$ is given we set $P^n_{B/S(m)} = B \otimes \mathcal{O}_X P^n_{X/S(m)}$; then $\alpha^n_0 = id$ and $\alpha^n_1 = \chi_n$ are compatible. When $\alpha^n_0$ and $\alpha^n_1$ are compatible we will take the isomorphisms (4.3.9.10) to define $P^n_{B/S(m)} = Q$.

In any case the isomorphisms (4.3.9.10) give $P^n_{B/S(m)}$ a $(p^*_0 B, p^*_1 B)$-bimodule structure, the left (resp. right) one arising from $\alpha^n_0$ (resp. $\alpha^n_1$). These structures are exchanged by the isomorphism $\chi_n = (\alpha^n_0)^{-1} \circ \alpha^n_1$, and by construction $\chi_n$ is $P^n_{X/S(m)}$-linear.

Given (4.3.9.10) one can define two ring homomorphisms

$$p^0_{ij} P^{n+n'}_{B/S(m)} \to p^*_{ij} P^n_{B/S(m)} \otimes_{p^*_1 B} p^*_{12} P^{n'}_{B/S(m)}$$

where $p_{ij} : \hat{X}(2) \to \hat{X}(1)$ and $p_1 : \hat{X}(2) \to X$ are the usual projections. We first remark that we can identify

$$p^*_{01} P^n_{X/S(m)} \otimes \mathcal{O}_{\hat{X}(2)} p^*_1 B \otimes \mathcal{O}_{\hat{X}(2)} p^*_{12} P^{n'}_{X/S(m)} \xrightarrow{\sim} (p^*_{01} P^n_{X/S(m)} \otimes \mathcal{O}_{\hat{X}(2)} p^*_1 B) \otimes_{p^*_1 B} (p^*_{12} P^{n'}_{X/S(m)})$$

$$\xrightarrow{\alpha^n_0 \otimes \alpha^n_1} (p^*_{01} P_{B/S(m)}^n) \otimes_{p^*_1 B} (p^*_{12} P_{B/S(m)}^{n'}).$$

The first homomorphism is the composite

$$\delta^{n,n'}_{0,1} : p^*_{02} P^{n+n'}_{B/S(m)} \xrightarrow{(\alpha^n_0)^{-1}} p^*_{01} B \otimes \mathcal{O}_{\hat{X}(2)} p^*_{02} P^{n+n'}_{X/S(m)}$$

$$\xrightarrow{\delta^{n,n'} \otimes 1} p^*_{01} B \otimes \mathcal{O}_{\hat{X}(2)} p^*_{01} P^n_{X/S(m)} \otimes_{p^*_1 B} p^*_{12} P^{n'}_{X/S(m)}$$

$$\xrightarrow{\chi_n \otimes 1} (p^*_{01} P^n_{B/S(m)}) \otimes_{p^*_1 B} (p^*_{12} P^{n'}_{B/S(m)}).$$

$$\simeq (p^*_{01} P^n_{B/S(m)}) \otimes_{p^*_1 B} (p^*_{12} P^{n'}_{B/S(m)}).$$

(4.3.9.12)
and the second is the composite

$$
\delta_{B,1}^{n,n'} : p_0^* \mathcal{P}_B^{n,n'} \times_{\mathcal{O}_{\hat{X}/S,(m)}} \mathcal{P}_B^{n,n'} \rightarrow p_0^* \mathcal{P}_{\hat{X}/S,(m)} \otimes_{\mathcal{O}_{\hat{X}/S,(m)}} p_2^* \mathcal{B}
$$

(4.3.10.1)

Note that $\mathcal{P}_B^{n,n'} \otimes_{\mathcal{B}} \mathcal{P}_B^{n,n'}$ has a $(p_0^* \mathcal{B}, p_2^* \mathcal{B})$-bimodule structure, coming from $d_0 \otimes 1$ on the left and and $1 \otimes d_1$ on the right. By construction, $\delta_{B,0}^{n,n'}$ is B-linear for the left structure and $\delta_{B,1}^{n,n'}$ is B-linear for the right structure.

**4.3.10 Proposition** For any sheaf $\mathcal{B}$ of $\mathcal{O}_X$-algebras $\mathcal{B}$ satisfying conditions [4.3.9.6](#) [4.3.9.8](#) and for any compatible system of isomorphisms [4.3.9.10](#) the following are equivalent:

(i) The isomorphisms [4.3.9.11](#) define an $m$-PD-stratification of $\mathcal{B}$ compatible with its $O_X$-algebra structure.

(ii) For all $n, n' \geq 0$, $\delta_{B,0}^{n,n'} = \delta_{B,1}^{n,n'}$.

(iii) There is a ring homomorphism

$$
\delta_{B}^{n,n'} : p_0^* \mathcal{P}_B^{n,n'} \rightarrow (p_0^* \mathcal{P}_B^{n,n'}) \otimes_{\mathcal{B}} (p_1^* \mathcal{P}_B^{n,n'})
$$

(4.3.10.1)

that is a homomorphism of $(p_0^* \mathcal{B}, p_2^* \mathcal{B})$-bimodules and semilinear for the homomorphism $\delta_{B}^{n,n'} : p_0^* \mathcal{P}_B^{n,n'} \rightarrow p_0^* \mathcal{P}_B^{n,n'} \otimes_{\mathcal{O}_X} p_1^* \mathcal{P}_B^{n,n'}$.

**Proof.** In the following calculations we drop the $p^*ij, p_1^*$, $(m)$ and some of the the tensor product subscripts. Consider the diagram

$$
\begin{array}{ccccccc}
\mathcal{P}_{\hat{X}/S}^{n,n'} \otimes \mathcal{B} & \xrightarrow{1 \otimes \chi_{n,n'}} & \mathcal{P}_{\hat{X}/S}^{n,n'} \otimes \mathcal{B} & \xrightarrow{\chi_{n,n'}} & \mathcal{B} \otimes \mathcal{P}_{\hat{X}/S}^{n,n'} \\
\delta_{B}^{n,n'} \otimes 1 & & 1 \otimes \delta_{B}^{n,n'} & & 1 \otimes \delta_{B}^{n,n'} \\
\mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{P}_{\hat{X}/S}^{n'} \otimes \mathcal{B} & & \mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{P}_{\hat{X}/S}^{n'} \otimes \mathcal{B} & & \mathcal{B} \otimes \mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{P}_{\hat{X}/S}^{n'} \\
1 \otimes \chi_{n,n} & & 1 \otimes \chi_{n,n} & & 1 \otimes \chi_{n,n} \\
\mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{B} \otimes \mathcal{P}_{\hat{X}/S}^{n} & & (\mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{B}) \otimes (\mathcal{B} \otimes \mathcal{P}_{\hat{X}/S}^{n}) & & \mathcal{P}_{\hat{X}/S}^{n} \otimes \mathcal{B} \otimes \mathcal{P}_{\hat{X}/S}^{n} \\
\end{array}
$$

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where the equalities denote the appropriate canonical isomorphisms. By [4 Prop. 2.3.2] the cocycle condition for \( \chi_n \) is equivalent to the commutativity of the top rectangle. The equality \( \delta_{B,0}^{n,n'} = \delta_{B,1}^{n,n'} \) is equivalent to the commutativity of the outside square. Since all morphisms in lower part of the diagram are isomorphisms, (i) and (ii) are equivalent.

Since \( \chi_n \) (resp. \( \chi_{n'} \)) is \( \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \)-linear (resp. \( \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \)-linear) the morphisms \( \delta_{B,0}^{n,n'} \) and \( \delta_{B,1}^{n,n'} \) are semilinear for \( \delta^{n,n'} \). Since \( \delta_{B,0}^{n,n'} \) (resp. \( \delta_{B,1}^{n,n'} \)) \( \mathcal{B} \)-linear for the left (resp. right) structure, (ii) implies (i) with \( \delta_{B,0}^{n,n'} = \delta_{B,1}^{n,n'} = \delta_{B,1}^{n,n'} \). Suppose conversely that (iii) holds. Since \( \mathcal{P}_{B/\mathcal{S},(m)} \) is generated as a \( \mathcal{B} \)-module (for either structure) by the image of \( \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \to \mathcal{P}_{B/\mathcal{S},(m)} \), \( \delta_{B,0}^{n,n'} \) (resp. \( \delta_{B,1}^{n,n'} \)) is the unique morphism that is semilinear for \( \delta^{n,n'} \) and \( \mathcal{B} \)-linear for the left (resp. right) \( \mathcal{B} \)-structure. Thus (iii) implies (ii) ■

We will let the reader check that the definition of the product given by [4.3.9.2] and [4.3.9.3] can be rephrased in terms of \( \mathcal{P}_{B/\mathcal{S},(m)} \) as follows. The isomorphism [4.3.9.2] may be rewritten

\[
\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}/\mathcal{S},(m)} \simeq \text{Hom}_{\mathcal{B}}(\mathcal{P}_{B/\mathcal{S},(m)}, \mathcal{B}) \tag{4.3.10.2}
\]

and with this identification the product of \( P \in \text{Hom}_{\mathcal{B}}(\mathcal{P}_{B/\mathcal{S},(m)}, \mathcal{B}) \) and \( Q \in \text{Hom}_{\mathcal{B}}(\mathcal{P}_{B/\mathcal{S},(m)}, \mathcal{B}) \) is

\[
\mathcal{P}_{B/\mathcal{S},(m)} \xrightarrow{\delta_{B,1}^{n,n'}} \mathcal{P}_{B/\mathcal{S},(m)} \otimes_{\mathcal{B}} \mathcal{P}_{B/\mathcal{S},(m)} \\
\xrightarrow{1 \otimes Q} \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \xrightarrow{P} \mathcal{B} \tag{4.3.10.3}
\]

We can treat \( m \)-HPD-stratified \( \mathcal{O}_{\mathcal{X}} \)-algebras in the same way. A pair of ring isomorphisms

\[
\alpha_0 : p_0^* \mathcal{B} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \xrightarrow{\sim} Q \\
\alpha_1 : \mathcal{P}_{\mathcal{X}/\mathcal{S},(m)} \otimes_{\mathcal{O}_{\mathcal{X}}} p_1^* \mathcal{B} \xrightarrow{\sim} Q \tag{4.3.10.4}
\]

is compatible if they induce the same surjective homomorphism \( \mathcal{P}_{B/\mathcal{S},(m)} := Q \to \mathcal{B} \). Note the use of completed tensor products in place of ordinary ones. As before they induce a \( (p_0^* \mathcal{B}, p_1^* \mathcal{B}) \)-module structure on \( \mathcal{P}_{B/\mathcal{S},(m)} \) via \( d_0 \) and \( d_1 \), and on \( (p_0^* \mathcal{P}_{B/\mathcal{S},(m)}) \otimes p_1^* \mathcal{B} \) via \( d_0 \otimes 1 \) and \( 1 \otimes d_1 \). As before, one shows:
4.3.11 Proposition A pair of compatible ring isomorphisms \[4.3.10.4\] defines an \(m\)-HPD-stratification of \(\mathcal{B}\) compatible with its ring structure if and only there is a ring homomorphism

\[
\delta_{\mathcal{B},(m)} : p_0^* \mathcal{P}_{\mathcal{B}/S,(m)} \to (p_0^* \mathcal{P}_{\mathcal{B}/S,(m)}) \circ (p_1^* \mathcal{P}_{\mathcal{B}/S,(m)})
\] (4.3.11.1)

that is \((p_0^* \mathcal{B}, p_1^* \mathcal{B})\)-bilinear and semilinear for the homomorphism

\[
\delta : p_0^* \mathcal{P}_{\mathcal{X}/S,(m)} \to (p_0^* \mathcal{P}_{\mathcal{X}/S,(m)}) \circ (p_1^* \mathcal{P}_{\mathcal{X}/S,(m)}).
\] (4.3.11.4)

In this connection one should recall the canonical isomorphism

\[
p_0^* \mathcal{P}_{\mathcal{X}/S,(m)} \circ (p_1^* \mathcal{P}_{\mathcal{X}/S,(m)}) \simeq \mathcal{P}_{\mathcal{X}/S,(m)}(2)
\] (4.3.11.2)

so that it would make sense to define

\[
p_0^* \mathcal{P}_{\mathcal{B}/S,(m)} \circ (p_1^* \mathcal{P}_{\mathcal{B}/S,(m)}) \simeq \mathcal{P}_{\mathcal{B}/S,(m)}(2).
\] (4.3.11.3)

Then \[4.3.11.1\] corresponds to a homomorphism

\[
d_{02}^B : p_0^* \mathcal{P}_{\mathcal{B}/S,(m)} \to \mathcal{P}_{\mathcal{B}/S,(m)}(2).
\] (4.3.11.3)

On the other hand there are homomorphisms

\[
d_{01}^B, d_{12}^B : \mathcal{P}_{\mathcal{B}/S,(m)} \to \mathcal{P}_{\mathcal{B}/S,(m)}(2)
\] (4.3.11.4)

(we will start dropping the \(p^*\) again) which via \[4.3.11.1\] correspond to the morphisms \(x \mapsto x \otimes 1\) and \(x \mapsto 1 \otimes x\) for \(x\) in \(\mathcal{P}_{\mathcal{B}/S,(m)}\). These homomorphisms satisfy the simplicial identities

\[
d_0^B d_{02}^B = d_0^B d_{01}^B, \quad d_1^B d_{02}^B = d_1^B d_{12}^B, \quad d_0^B d_{12}^B = d_1^B d_{01}^B
\] (4.3.11.5)

and the diagrams

\[
\begin{array}{ccc}
\mathcal{P}_{\mathcal{X}/S,(m)} & \xrightarrow{d_{ij}} & \mathcal{P}_{\mathcal{B}/S,(m)}(2) \\
\downarrow & & \downarrow \\
\mathcal{P}_{\mathcal{B}/S,(m)} & \xrightarrow{d_{ij}} & \mathcal{P}_{\mathcal{B}/S,(m)}(2)
\end{array}
\] (4.3.11.6)

commute for \((ij) = (01), (02)\) and \((12)\).
4.3.12 Modules over coefficient rings. Suppose \( M \) is a \( \mathcal{B} \)-module satisfying 4.3.3.1, 4.3.3.2 and \( \mathcal{B} \) has a compatible left \( \mathcal{D}^{(m)}_{X/S} \)-module structure. An \( m \)-PD-stratification on \( M \) can be viewed as a set of isomorphisms

\[
\chi_n : (\mathcal{P}^n_{X/(S,(m))} \otimes p^*_1 \mathcal{B}) \otimes p^*_1 \mathcal{B} M \sim \rightarrow M \otimes p^*_1 \mathcal{B} (p^*_1 \mathcal{B} \otimes \mathcal{P}^n_{X/(S,(m))})
\] (4.3.12.1)

and we say that the \( m \)-PD-stratification of \( M \) is compatible with the \( \mathcal{B} \)-module structure if the isomorphism 4.3.12.1 is semilinear with respect to the stratification

\[
\mathcal{P}^n_{X/(S,(m))} \otimes \mathcal{B} \sim \rightarrow \mathcal{B} \otimes \mathcal{P}^n_{X/(S,(m))}
\]
of \( \mathcal{B} \). A left \( \mathcal{D}^{(m)}_{X/S} \)-module structure on \( M \) is compatible with the \( \mathcal{B} \)-module structure if this is the case for the corresponding \( m \)-PD-stratification. An equivalent condition is that the map \( \mathcal{B} \otimes_{\mathcal{O}_X} M \rightarrow M \) defining the \( \mathcal{B} \)-module structure is compatible with the \( m \)-PD-stratifications, or in other words is \( \mathcal{D}^{(m)}_{X/S} \)-linear. In local coordinates, this condition says that

\[
\partial^{(K)(m)}(ax) = \sum_{I+J=K} \begin{pmatrix} K \\ I \end{pmatrix}_{(m)} \partial^{(I)(m)}(a) \partial^{(J)(m)}(x)
\] (4.3.12.2)

for local sections \( a \) of \( \mathcal{B} \) and \( x \) of \( M \).

A \( \mathcal{B} \)-module with a compatible left \( \mathcal{D}^{(m)}_{X/S} \)-module structure evidently gives rise to a left \( \mathcal{D}^{(m)}_{B/S} \)-module structure. Conversely a left \( \mathcal{D}^{(m)}_{B/S} \)-module \( M \) gets compatible \( \mathcal{B} \)-module and \( \mathcal{D}^{(m)}_{X/S} \)-module structures from the canonical inclusions of \( \mathcal{B} \) and \( \mathcal{D}^{(m)}_{X/S} \) into \( \mathcal{D}^{(m)}_{B/S} \).

If we are given a compatible pair of isomorphisms 4.3.9.10 the isomorphisms 4.3.12.1 defining the stratification may be rewritten

\[
\chi^\mathcal{B}_n : \mathcal{P}^n_{B/(S,(m))} \otimes p^*_1 \mathcal{B} p^*_1 M \sim \rightarrow p^1_0 M \otimes p^*_0 \mathcal{B} \mathcal{P}^n_{B/(S,(m))}
\] (4.3.12.3)

and 4.3.12.1 is semilinear for the stratification of \( \mathcal{B} \) if and only if the isomorphisms 4.3.12.3 are \( \mathcal{P}^n_{B/S} \)-linear. The compatibility of 4.3.9.10 guarantees the compatibility of the \( \chi^\mathcal{B}_n \), while the cocycle condition can be expressed in various ways. One is to introduce the morphisms

\[
\partial^\mathcal{B}_n : p^*_1 M \rightarrow p^*_0 M \otimes p^*_0 \mathcal{B} \mathcal{P}^n_{B/(S,(m))}
\] (4.3.12.4)

induced by 4.3.12.3 which are \( \mathcal{B} \)-linear for the right structure of \( \mathcal{P}^n_{B/(S,(m))} \); they are compatible in an obvious sense, and \( \chi_n \) satisfies the cocycle condi-
tion if and only if diagram

\[
\begin{array}{c}
M \xrightarrow{\theta_{n+n'}^B} M \otimes \mathcal{P}_{B/S,(m)}^{n+n'} \\
\downarrow \theta_{n'}^B \downarrow \quad \downarrow \theta_{n+n'}^B \downarrow \\
M \otimes \mathcal{P}_{B/S,(m)}^{n'} \otimes \mathcal{P}_{B/S,(m)}^n \xrightarrow{\delta_{B,(m)}} \mathcal{P}_{B/S,(m)}^{n+n'}
\end{array}
\]  

(4.3.12.5)

commutes for all \( n, n' \geq 0 \). On the other hand the commutativity of (4.3.12.5) shows that the \( \theta_{n}^B \) give \( M \) the structure of a left \( \mathcal{D}_{B/S}^{(m)} \)-module, and thus yields another way of understanding the equivalence of the category of left \( \mathcal{D}_{B/S}^{(m)} \)-modules with the category of \( B \)-modules endowed with a left \( \mathcal{D}_{X/S}^{(m)} \)-module structure compatible with the \( B \)-algebra structure.

When \( B \) has a quasi-nilpotent left \( \mathcal{D}_{X/S}^{(m)} \)-module structure compatible with its algebra structure, the same picture holds for quasi-nilpotent left \( \mathcal{D}_{X/S}^{(m)} \)-modules endowed with a compatible \( B \)-module structure. The \( m \)-HPD-stratification

\[
\chi : \mathcal{P}_{X/S,(m)} \otimes p_1^* M \xrightarrow{\cong} p_0^* M \otimes \mathcal{P}_{X/S,(m)}
\]

can be rewritten

\[
\chi^B : \mathcal{P}_{B/S,(m)} \otimes p_1^B M \xrightarrow{\cong} p_0^B M \otimes \mathcal{P}_{B/S,(m)}
\]

(4.3.12.6)

as before, or as a morphism

\[
\theta^B : p_1^B M \rightarrow p_0^B M \otimes \mathcal{P}_{B/S,(m)}
\]

(4.3.12.7)

linear for the right \( \mathcal{P}_{B/S,(m)} \)-structure. The analogue of (4.3.12.5) is the commutative diagram

\[
\begin{array}{c}
M \xrightarrow{\theta^B} \mathcal{P}_{B/S,(m)} \\
\downarrow \theta^B \downarrow \quad \downarrow \delta_{B,(m)} \\
M \otimes \mathcal{P}_{B/S,(m)} \otimes \mathcal{P}_{B/S,(m)} \xrightarrow{\delta_{B,(m)}} \mathcal{P}_{B/S,(m)} \otimes \mathcal{P}_{B/S,(m)}
\end{array}
\]  

(4.3.12.8)

where \( \delta_{B,(m)} \) is (4.3.11.1). But we can also linearize (4.3.12.8) and use the isomorphism (4.3.11.2) and the maps \( d_{ij}^B \); then the cocycle condition takes the usual form of an equality

\[
(d_{02}^B)_{*}(\chi^B) = (d_{01}^B)_{*}(\chi^B) \circ (d_{12}^B)_{*}(\chi^B)
\]

(4.3.12.9)

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of $\mathcal{P}_{B/S,(m)}(2)$-modules, where $(d^B_{ij}, \chi^B)$ denotes the extension of scalars of $\chi^B$ by $d^B_{ij}$ (corresponding to the pullbacks by the projections for the nonexistent formal schemes corresponding the the $\mathcal{O}_X$-algebras $\mathcal{P}_{B/S,(m)}$ and $\mathcal{P}_{B/S,(m)}(2)$).

The following proposition is proven in exactly the same way in [3, Prop. 3.1.3].

**4.3.13 Proposition** Suppose $M$ is a coherent $D^{(m)}_{B/S}$-module, that is coherent as a $B$-module. Then $M$ is coherent as a $\hat{D}^{(m)}_{B/S}$-module, and the canonical homomorphism

$$M \to \hat{D}^{(m)}_{B/S} \otimes_{D^{(m)}_{B/S}} M$$

is an isomorphism.

**4.3.14 Change of coefficient ring.** Suppose $C$ is a second $\mathcal{O}_X$-algebra with a left $D^{(m)}_{X/S}$-module structure compatible with its $\mathcal{O}_X$-algebra structure. If $B \to C$ is an $\mathcal{O}_X$-algebra homomorphism linear for the $D^{(m)}_{X/S}$-module structures, there is an obvious ring homomorphism $D^{(m)}_{B/S} \to D^{(m)}_{C/S}$ inducing a canonical $D^{(m)}_{X/S}$-linear and $C$-linear isomorphism

$$C \otimes_B D^{(m)}_{B/S} \to D^{(m)}_{C/S}. \quad (4.3.14.1)$$

If $M$ is a left $D^{(m)}_{B/S}$-module, the transitivity of tensor shows that there is an isomorphism

$$C \otimes_B M \cong D^{(m)}_{C/S} \otimes_{D^{(m)}_{B/S}} M. \quad (4.3.14.2)$$

In particular, if $M$ is a coherent $D^{(m)}_{B/S}$-module then $C \otimes_B M$ is a coherent $D^{(m)}_{C/S}$-module.

If $B$ and $C$ satisfy conditions [4.3.9.6–4.3.9.8], the same argument can be applied to $M/JM$ for any $m$-bilateralising ideal $J \subset \mathcal{O}_X$, and passing to the inverse limit yields an isomorphism

$$C \hat{\otimes}_B D^{(m)}_{B/S} \to \hat{D}^{(m)}_{C/S}. \quad (4.3.14.3)$$

and, for any coherent left $\hat{D}^{(m)}_{B/S}$-module $M$, an isomorphism

$$C \hat{\otimes}_B M \cong \hat{D}^{(m)}_{C/S} \otimes_{\hat{D}^{(m)}_{B/S}} M. \quad (4.3.14.4)$$
Thus if $M$ is a coherent $\hat{D}^{(m)}_{B/S}$-module then $\mathcal{C} \otimes_B M$ is a coherent $\hat{D}^{(m)}_{\mathcal{C}/S}$-module.

4.3.15 Proposition Suppose $\mathcal{B}$ (resp. $\mathcal{C}$) is an $\mathcal{O}_X$-algebra with a compatible $\hat{D}^{(m')}_{\mathcal{X}/S}$-module structure (resp. a compatible $\hat{D}^{(m)}_{\mathcal{X}/S}$-module structure), $m' > m$ and $\mathcal{B} \to \mathcal{C}$ is a $\hat{D}^{(m)}_{\mathcal{X}/S}$-linear homomorphism of $\mathcal{O}_X$-algebras. If $M$ is a coherent $\hat{D}^{(m')}_{B/S}$-module and $\mathcal{C}$ is topologically quasi-nilpotent as a $\hat{D}^{(m)}_{\mathcal{X}/S}$-module, $\mathcal{C} \hat{\otimes}_B M$ is topologically quasi-nilpotent for its induced $\hat{D}^{(m)}_{\mathcal{X}/S}$-module structure.

Proof. We may work locally, so suppose $M$ is generated as a $\hat{D}^{(m')}_{B/S}$-module by $a_1, \ldots, a_n$. Since $m' > m$, the formula 3.2.1.7 shows that $\partial^{(K)}(a_i) \to 0$ for $|K| \to \infty$. Since $\mathcal{C}$ is a quasi-nilpotent $\hat{D}^{(m)}_{\mathcal{X}/S}$-module the Leibnitz formula 4.3.12.2 shows that $\partial^{(K)(m)}(x) \to 0$ for any local section of $\mathcal{C} \hat{\otimes}_B M$, and we are done.

4.4 The isogeny category. To extend these results to coherent $\mathcal{D}^{(m)}_{B/S\mathbb{Q}}$-modules we use the following theorem of Ogus, which was proven by him when $\mathcal{B} = \mathcal{O}_X$ and $m = 0$, but the argument in the general case is the same; c.f. also [3, Prop. 3.1.2] for the case $\mathcal{B} = \mathcal{O}_X$ and $m \geq 0$.

4.4.1 Proposition Suppose $\mathcal{B}$ is an $\mathcal{O}_X$-algebra with a compatible $\hat{D}^{(m)}_{B/S}$-module structure satisfying conditions 4.3.9.5–7. If $M$ is a $\mathcal{B}_\mathbb{Q}$-coherent left $\hat{D}^{(m)}_{B/S\mathbb{Q}}$-module, then locally on $X$ there is a $\mathcal{B}$-coherent left $\hat{D}^{(m)}_{B/S}$-module $M^0$ such that $(M^0)_\mathbb{Q} \simeq M$.

Proof. Let $M'$ be any coherent $\mathcal{B}$-module such that $M'_\mathbb{Q} = M$. Pick an open affine on which $\mathcal{X}/S$ is parallelizable and choose local coordinates $x_1, \ldots, x_d$; then $\mathcal{P}^n_{\mathcal{X}/S,(m)}$ is free on the corresponding $\xi^{(K)}(m)$. For $n \geq 0$ the map

$$\theta_n : M \to M \otimes_{\mathcal{O}_X} \mathcal{P}^n_{\mathcal{X}/S,(m)}$$

arising from the $m$-PD-stratification of $M$ is

$$\theta_n(x) = \sum_{|K| \leq n} \partial^{(K)(m)}(x) \otimes \xi^{(K)}(m).$$

We will show that

$$M^0 = \bigcap_{n \geq 0} \theta_n^{-1}(M' \otimes_{\mathcal{O}_X} \mathcal{P}^n_{\mathcal{X}/S,(m)})$$

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satisfies the conditions of the proposition. From the definition we see that \( x \in M^0 \) if and only if \( \partial^{(K)}(x) \in M' \) for all \( K \), and thus \( M^0 \) is preserved by the \( \partial^{(K)}(m) \). Since the \( m \)-PD-stratification of \( M \) is semilinear with respect to the \( m \)-PD-stratification of \( B \), \( M^0 \) is a \( B \)-submodule of \( M' \); this is most easily seen from condition 1.3.12.2. Clearly \( M^0_0 \simeq M \), and thus \( M^0 \) is preserved by the \( \partial^{(K)}(m) \). Since the \( m \)-PD-stratification of \( M \) is semilinear with respect to the \( m \)-PD-stratification of \( B \), \( M^0 \) is a \( B \)-submodule of \( M' \); this is most easily seen from condition 4.3.12.2. Clearly \( M^0_0 \simeq M \), and thus \( M^0 \) is preserved by the \( \partial^{(K)}(m) \).

Finally, since the \( B \)-module and \( \hat{D}^{(m)}_{B/S} \)-module structures of \( M \) are compatible this will also be the case for \( M^0 \), and thus \( M^0 \) is a \( \hat{D}^{(m)}_{B/S} \)-module.

If \( X \) is quasicompact, the \( M \) in the proposition can be chosen globally, but we will not need this refinement. An immediate consequence of Ogus’s theorem is a version of 4.3.13 for \( \hat{D}^{(m)}_{B/S} \)-modules:

**4.4.2 Proposition** Suppose \( M \) is a left \( \hat{D}^{(m)}_{B/S} \)-module that is coherent as a \( B \)-module. Then \( M \) is coherent as a left \( \hat{D}^{(m)}_{B/S} \)-module, and the canonical homomorphism

\[
M \to \hat{D}^{(m)}_{B/S} \otimes_{\hat{D}^{(m)}_{B/S}} M
\]

is an isomorphism.

The following is also useful when going back and forth between \( \hat{D}^{(m)}_{B/S} \)-modules and \( \hat{D}^{(m)}_{B/S} \)-modules:

**4.4.3 Lemma** Let \( A \) be a noetherian topological \( \mathbb{Z}_p \)-algebra (not necessarily commutative) and \( A \to B \) a ring homomorphism. Suppose \( A \) has a ideal \( J \) centralising in both \( A \) and \( B \), and such that \( A \) and \( B \) both have the \( J \)-adic topology. Let \( f : M \to M' \) be a homomorphism of finitely generated left \( A \)-modules. If the kernel and cokernel of \( f \) are \( p \)-torsion, \( f \) induces an isomorphism \( (B \otimes A M)_\mathbb{Q} \sim (B \otimes A M')_\mathbb{Q} \).

**Proof.** It suffices to treat the cases in which \( f \) is surjective or injective. If \( f \) is injective we define \( M'' \) by the exactness of

\[
0 \to M \xrightarrow{f} M' \to M'' \to 0
\]

and there is an exact sequence

\[
0 \to K \to B \otimes_A M \to B \otimes_A M' \to B \otimes_A M'' \to 0
\]
where $K$ is a quotient of $\text{Tor}_A^1(B, M'')$. Since $M''$ is $p$-torsion and a finitely generated $A$-module, $K$ and $B \otimes_A M''$ are $p$-torsion finitely generated $B$-modules, and thus annihilated by some power of $p$. By the hypotheses on the topologies of $A$ and $B$, the completion $0 \to \hat{K} \to B \otimes_A M \to B \otimes_A M' \to B \otimes_A M'' \to 0$

of the previous exact sequence is exact. Furthermore $\hat{K}$ and $B \otimes_A M''$ are annihilated by some power of $p$, and the assertion follows. The case where $f$ is surjective is similar.

The following lemma is evident:

**4.4.4 Lemma** Suppose $B$ is an $\mathcal{O}_X$-algebra with a compatible $m$-HPD-stratification and $M$ is a coherent left $\hat{D}_{B/S}^{(m)}$-module. The following are equivalent:

(i) Any finitely generated $\hat{D}_{B/S}^{(m)}$-submodule of $M$ is topologically quasi-nilpotent.

(ii) There is a topologically quasi-nilpotent left $\hat{D}_{B/S}^{(m)}$-module $M^0$ and an isomorphism $M^0_\mathbb{Q} \simeq M$ of $\hat{D}_{B/S}^{(m)}$-modules.

**4.4.5 Definition** Suppose $B$ is as in lemma 4.4.4. A coherent left $\hat{D}_{B/S}^{(m)}$-module $M$ is topologically quasi-nilpotent if it satisfies the equivalent conditions of the lemma.

If $M$ is a coherent $\hat{D}_{B/S}^{(m)}$-module we define the $C \hat{\otimes}_B M$ as follows: choose locally a coherent $\hat{D}_{B/S}^{(m)}$-submodule $M^0$ of $M$ such that $M^0_\mathbb{Q} \simeq M$; then

$$C \hat{\otimes}_B M := (C \hat{\otimes}_B M^0)_\mathbb{Q} \quad (4.4.5.1)$$

where in the right hand side we have the usual completed tensor product. Lemma 4.4.3 shows that the left hand side of 4.4.5.1 is independent of the choice of $M^0$, so the definition is justified. Note that if $M$ is coherent as a $B_\mathbb{Q}$-module we may choose an $M^0$ that is coherent as a $B$-module. In any case the resulting $\hat{D}_{C/S}^{(m)}$-module is coherent.

From 4.3.15, lemma 4.4.4 and definition 4.4.5 we find:
4.4.6 Corollary Let $\mathcal{B}$ and $\mathcal{C}$ be as in proposition 4.3.15 and let $M$ be a coherent left $\mathcal{D}_{B/S_Q}^{(m')}$-module. If $m' > m$, $\mathcal{C} \otimes_B M$ is a quasi-nilpotent left $\mathcal{D}_{C/S_Q}^{(m)}$-module.

In particular:

4.4.7 Corollary Suppose $B$ is an $\mathcal{O}_X$ with a compatible $m$-HPD-stratification satisfying 4.3.9.6-4.3.9.8 and $M$ is a coherent left $\mathcal{D}_{B/S_Q}^{(m+1)}$-module. Then $M$ with the induced $\mathcal{D}_{B/S_Q}^{(m)}$-module structure is topologically quasi-nilpotent.

4.5 Descent by Frobenius. Suppose now $S$ has the $m$-PD-structure $(a,b,\alpha)$ and $p \in a$. As before we set $S_0 = V(a)$, set $q = p^s$ and denote by $F_{S_0}$ the $q$th power Frobenius of $S_0$. For any formal $S$-scheme $\mathcal{X}$ we again set $X_0 = V(a\mathcal{O}_X)$ and denote by $F_{X_0}$ the $q$th power Frobenius; then the relative Frobenius $F_{X_0/S_0} : X_0 \to X_0(q)$ is defined, and we denote by $W_{X_0/S_0} : X_0(q) \to X_0$ the canonical projection. Suppose $F_{X_0/S_0}$ lifts to a morphism $F : \mathcal{X} \to \mathcal{X}'$. The argument of [6, Prop. 2.2.2] can be used as is to show:

4.5.1 Proposition For any left $\mathcal{D}_{\mathcal{X}'/S}^{(m)}$-module $M$, $F^* M$ has a canonical and functorial left $\mathcal{D}_{\mathcal{X}/S}^{(m+s)}$-module structure restricting to the $\mathcal{D}_{\mathcal{X}/S}^{(m)}$-module structure derived from base change. If $M$ is a topologically quasi-nilpotent $\mathcal{D}_{\mathcal{X}/S}^{(m)}$-module, $F^* M$ is a topologically quasi-nilpotent $\mathcal{D}_{\mathcal{X}/S}^{(m+s)}$-module. 

An important point in the proof is that $F$ is flat, a fact used in the proofs of [6, Lemme 2.3.2] and [6, Lemme 2.3.3]. This is true in the present case, since $F$ is a lifting of $F_{X_0/S_0} : X_0 \to X_0(q)$ which is flat by proposition 2.2.16.

To prove the descent theorem we need to make the further assumption that $F$ is finite, a fact needed in the argument of [6, Théorème 2.3.6]. This will be true if $F_{X_0/S_0}$ is finite, but so far we only know this when $X_0 \to S_0$ is formally of finite type, again by proposition 2.2.16. We therefore state the Frobenius descent theorem for $\mathcal{D}^{(m)}$-modules as follows; no changes are needed in adapting the argument of [6, §2.3].

4.5.2 Theorem Suppose $f : \mathcal{X} \to S$ is quasi-smooth and the relative Frobenius $F_{X_0/S_0}$ is finite. The functor $F^*$ induces an equivalence of the category of left $\mathcal{D}_{\mathcal{X}/S}^{(m)}$-modules with the category of left $\mathcal{D}_{\mathcal{X}/S}^{(m+s)}$-modules. 

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The Frobenius descent theorem for $\hat{D}^{(m)}$-modules is an immediate consequence. If $J' \subseteq \mathcal{O}_{X'}$ is open and $\mathcal{D}^{(m)}_{X'}/S$-bilateralising, $J = F^*J' \subseteq \mathcal{O}_X$ is open and $\mathcal{D}^{(m)}_{X}/S$-bilateralising. From the equivalence of categories in the last paragraph we deduce, again with the assumption that $F$ is finite and flat, that the category of left $\mathcal{D}^{(m)}_{X'/S}$-modules is equivalent to the category of left $\mathcal{D}^{(m)}_{X'/S,J}$-modules. If $J'$ is an ideal of definition, so is $J$; this is because $F_{X_0/S_0}$ is a homeomorphism. Furthermore, as $J' \subseteq \mathcal{O}_{X'}$ runs through a cofinal set of $\mathcal{D}^{(m)}_{X'/S}$-bilateralising ideals of definition, $F^*J$ runs through a cofinal set of $\mathcal{D}^{(m+s)}_{X'/S}$-bilateralising ideals of definition. From this and the previous discussion we get the following:

4.5.3 Theorem Suppose $X \to S$ is quasi-smooth and the relative Frobenius $F_{X_0/S_0}$ lifts to an $S$-morphism $F : X \to X'$. For any left $\hat{D}^{(m)}_{X'/S}$-module $M$, $F^*M$ has a canonical left $\hat{D}^{(m+s)}_{X'/S}$-module structure. If $F_{X_0/S_0}$ is finite, this induces an equivalence of the category of left $\hat{D}^{(m)}_{X'/S}$-modules with the category of left $\hat{D}^{(m+s)}_{X'/S}$-modules.

Suppose $F' : X \to X'$ is a second lifting of $F_{X_0/S_0}$. If the $m$-PD-structure $(a, b, \alpha)$ of $S$ is $m$-PD-nilpotent, the isomorphism $\tau_{F,F'}$ of proposition 4.3.3 is $\mathcal{D}^{(m+s)}_{X'/S}$-linear, the argument being the same as [4, Prop. 2.2.5].

The Frobenius descent theorem extends to the case of coefficient rings. Let $\mathcal{B}$ be an $\mathcal{O}_{X'}$-algebra with a compatible left $\hat{D}^{(m)}_{X'/S}$-module structure. Then $F^*\mathcal{B}$ has a left $\hat{D}^{(m+s)}_{X'/S}$-module structure compatible with its $\mathcal{O}_{X'}$-algebra structure. Furthermore if $\mathcal{B}$ satisfies the assumptions 4.3.9.3–5 relative to $X'$, the $\mathcal{O}_{X'}$-algebra $F^*\mathcal{B}$ satisfies these same conditions relative to $X$. In these cases the Frobenius descent theorem can be stated as follows:

4.5.4 Theorem Let $\mathcal{B}$ be an $\mathcal{O}_{X'}$-algebra with a left $\mathcal{D}^{(m)}_{X'/S}$-module structure compatible with its $\mathcal{O}_{X'}$-algebra structure. For any left $\mathcal{D}^{(m)}_{\mathcal{B}/S}$-module $M$, $F^*$ has a canonical left $\mathcal{D}^{(m+s)}_{F^*\mathcal{B}/S}$-module structure, and if $F_{X_0/S_0}$ is finite this construction yields an equivalence of the category of left $\mathcal{D}^{(m)}_{\mathcal{B}/S}$-modules with the category of left $\mathcal{D}^{(m+s)}_{F^*\mathcal{B}/S}$-modules. If in addition $\mathcal{B}$ satisfies the conditions 4.3.9.3–5, $F^*$ induces an equivalence of the category of left $\hat{D}^{(m)}_{\mathcal{B}/S}$-modules with the category of left $\hat{D}^{(m+s)}_{F^*\mathcal{B}/S}$-modules.
5 Tubes

Let \( \mathcal{V} \) be a complete discrete valuation ring of mixed characteristic \( p \) with residue field \( k \) and fraction field \( K \). The goal of this section and the next is to reconstruct in purely formal terms the categories of convergent isocrystals on a separated \( k \)-scheme \( X \) of finite type relative to \( K \). This will allow us to apply the Frobenius descent theorem to show that the Frobenius pullback is an autoequivalence of the categories of convergent and overconvergent isocrystals on \( X/K \).

The key to doing this is a formal version of Berthelot’s construction of the tube of \( X \) relative to an embedding over \( \mathcal{V} \) of \( X \) into a smooth formal \( \mathcal{V} \)-scheme \( \mathcal{P} \). If \( \mathcal{X} \) is the completion of \( \mathcal{P} \) along \( X \subset \mathcal{P} \), the tube \( |X[\mathcal{P}] \) is the analytic space \( \mathcal{X}^{an} \) associated to the formal \( \mathcal{V} \)-scheme \( \mathcal{X} \) by the procedure of [5, 0.2.6], which we now recall. Fix an ideal of definition of \( \mathcal{X} \), and assume for the moment that \( \mathcal{X} = \text{Spf}(A) \) is affine, with ideal of definition \( J = (f_1, \ldots, f_r) \). The \( \mathcal{V} \)-algebra

\[
A_n = A\{T_1, \ldots, T_r\}/(\pi T_i - f_i^n, 1 \leq i \leq nj)
\]

is topologically of finite type, and \( A_n \otimes \mathbb{Q} \) is a Tate algebra. For \( n' \geq n \) there are natural continuous homomorphisms \( A_n' \otimes \mathbb{Q} \to A_n \otimes \mathbb{Q} \), and \( \mathcal{X}^{an} \) is the direct limit of the rigid analytic spaces \( \text{Max}(A_n \otimes \mathbb{Q}) \). The general case is handled by patching together the \( A_n \) for various affine opens of \( \mathcal{X} \).

In the next few sections we revisit this construction in a more general setting. We will then be able to construct all of the analytic data involved in the definition of a convergent isocrystal on \( X/K \) in terms of the formal scheme \( \mathcal{X} \) and modules over various differential operator rings. The construction can in fact be done in much greater generality, as we shall soon see.

5.1 Admissible Blowups. In this section we review the definition and basic properties of admissible blowups, for which our basic reference are the books of Abbes [1] and Bosch [9, §8.2]. If \( \mathcal{X} \) be any locally noetherian formal scheme it is idyllic in the sense of [1]. Fix an open ideal \( I \subset O_{\mathcal{X}} \), and let \( \mathcal{B}_I \) be the graded \( O_{\mathcal{X}} \)-algebra

\[
\mathcal{B}_I = \bigoplus_{\ell \in \mathbb{N}} I^\ell.
\]
The formal scheme $\text{Proj}(\mathcal{B}_f)$ associated to $\mathcal{B}_f$ is the direct limit
\[
\text{Proj}(\mathcal{B}_f) = \varprojlim_n \text{Proj}(\mathcal{B}_f \otimes_{\mathcal{O}_X} \mathcal{O}_X/J^{n+1}).
\] (5.1.0.2)
\[
= \varprojlim_n \text{Proj} \left( \bigoplus_{\ell \geq 0} I^\ell/J^{n+1}I^\ell \right).
\] (5.1.0.3)

Since $I$ is an ideal of finite type, the adic formal $\mathcal{X}$-scheme $\text{Proj}(\mathcal{B}_f) \to \mathcal{X}$ is of finite type. The formal $\mathcal{X}$-scheme $\text{Proj}(\mathcal{B}_f)$ has the same universal property as in the algebraic case: it is a final object of the category of formal $\mathcal{X}$-schemes $f : \mathcal{Y} \to \mathcal{X}$ for which $f^*I$ is an invertible sheaf on $\mathcal{Y}$.

For $x \in I$ we denote by $\bar{x}$ the element $x$ viewed as a homogenous element of $\mathcal{B}_f$ of degree 1. More generally we denote by $\bar{x}[n]$ the element $x$ viewed as a homogenous element of degree $n$; thus $\bar{x}[1] = \bar{x}$. For any subset $S \subseteq I$ we denote by $\hat{S}$ the ideal of $\mathcal{B}_f$ generated by the $\bar{x}$ for all $x \in S$. Note that $\hat{S} \subseteq (\mathcal{B}_f)_+$, with equality if and only if $S$ generates $I$.

For ideals $I' \subseteq I \subseteq \mathcal{O}_X$ the natural functoriality induces a morphism
\[
\phi_{I,I'} : G(\phi) \to \text{Proj}(\mathcal{B}_{I'})
\] (5.1.0.4)
where $G(\phi) \subseteq \mathcal{B}_f$ is the complement of $V(\bar{I'})$. When $I' = I^n$ this map is none other than the isomorphism [14, II, Prop. 2.4.7 (i)] in the case of $\mathcal{B}_f$, which is in particular defined on all of $\text{Proj}(\mathcal{B}_{I'})$:
\[
\phi_{I,I^n} : \text{Proj}(\mathcal{B}_f) \sim \text{Proj}(\mathcal{B}_{I^n}).
\] (5.1.0.5)

We will need the following generalization, which is surely well known but for which I do not know of a convenient reference.

5.1.1 Proposition For any ideals $I, I'$ of $\mathcal{O}_X$ such that $I^n \subseteq I' \subseteq I$, the morphism [5.1.0.4] is an isomorphism
\[
\phi_{I,I'} : \text{Proj}(\mathcal{B}_f) \sim \text{Proj}(\mathcal{B}_{I'}).
\]

Proof. The argument is basically that of [14, II]. We can reduced to the case of an affine scheme $\text{Spec}(A)$ and ideals $I' \subseteq I \subseteq A$. The hypothesis yields injective ring homomorphisms $B_{I^n} \to B_{I'} \to B_I$. If $P \subset B_{I^n}$ is a homogenous ideal we set $p'_\ell = p_\ell = P_{\ell/n}$ if $\ell$ is a multiple of $n$; otherwise we let $p'_\ell$ (resp. $p_\ell$) be the set of $x \in B_{I'}$ (resp. $x \in B_I$) such that $x^n \in P_{n\ell}$. It follows from the criterion [14, II 2.1.9] that $p' \subset B_{I'}$ and $p \subset B_I$ are homogenous ideals representing points of $\text{Proj}(B_{I'})$ and $\text{Proj}(B_I)$
respectively, and by construction we have \( P = B_{I^n} \cap p' = B_{I^n} \cap p \) and \( p' = B_I \cap p \). As in [14, II] loc. cit. it follows that there are morphisms of schemes

\[
\text{Proj}(B_I) \rightarrow \text{Proj}(B_{I^n}) \rightarrow \text{Proj}(B_{I^{n+1}})
\]

and the morphisms \( \text{Proj}(B_I) \rightarrow \text{Proj}(B_{I^n}), \text{Proj}(B_{I^n}) \rightarrow \text{Proj}(B_{I^{n+1}}) \) are isomorphisms. Therefore that \( \phi_{I,I'} \) has both a left and a right inverse, and is thus an isomorphism. \( \blacksquare \)

5.2 The Tube of an Open Ideal. Suppose now \( S \) is a locally noetherian formal \( \mathbb{Z}_p \)-scheme and \( \mathcal{X} \) is a universally noetherian \( S \)-scheme. We fix ideals \( c \subseteq O_S \) and \( I \subseteq O_X \). In fact much of the following discussion holds if \( c \) is any ideal of \( O_X \), but this is the setting in which we will use this construction, and at some points we will need \( c \subseteq O_S \).

5.2.1 Tubes. Denote by \( \bar{c} \) the graded ideal of \( B_c O_{X+I} \) generated by the \( \bar{x} \) for all \( x \in c \) (recall \( \bar{x} \) is \( x \) viewed as an element of degree one of \( B_c O_{X+I} \)), and set

\[
\mathcal{X}[I/c] = \text{Proj}(B_c O_{X+I}) \setminus V(\bar{c}). \tag{5.2.1.1}
\]

If \( S \) is affine and \( c = (c_1, \ldots, c_r) \) then \( \mathcal{X}[I/c] \) is the union of the open sets \( D_+(c_i) \subseteq \text{Proj}(B_c O_{X+I}) \), and is thus an adic formal \( \mathcal{X} \)-scheme of finite type. For any \( c \subseteq c \), \( D_+(c) \) is an affine formal \( \mathcal{X} \)-scheme, and in particular if \( c \) is principal, \( \mathcal{X}[I/c] \) is an affine formal \( \mathcal{X} \)-scheme. More generally if \( c \) is locally principal, the natural projection morphism \( \mathcal{X}[I/c] \rightarrow \mathcal{X} \) is affine.

If \( I, I' \) are open ideals of \( O_X \) such that \( cO_X + I = cO_X + I' \) then \( \mathcal{X}[I/c] \simeq \mathcal{X}[I'/c] \) by construction, so there is nothing lost by assuming \( cO_X \subseteq I \).

If \( c \subseteq \mathfrak{d} \) then \( c \subseteq \mathfrak{d} \) and \( V(\mathfrak{d}) \subseteq V(\bar{c}) \), so that \( \mathcal{X}[I/c] \subseteq \mathcal{X}[I/\mathfrak{d}] \). The next proposition, which could thought of as a formal version of the maximum modulus principle shows the sense in which this construction is functorial:

5.2.2 Proposition Suppose \( f : \mathcal{X} \rightarrow \mathcal{X}' \) is a morphism of locally noetherian formal \( S \)-schemes and \( I \subseteq O_X, I' \subseteq O_{X'} \) are open ideals such that \( I'O_X \subseteq I \). There is a unique morphism \( f_c : \mathcal{X}[I/c] \rightarrow \mathcal{X}'[I'/c] \) over \( f \).

Proof. As remarked before we may assume that \( cO_X \subseteq I \) and \( cO_{X'} \subseteq I' \). Let \( g \) be the composition \( \mathcal{X}[I/c] \rightarrow \mathcal{X} \rightarrow \mathcal{X}' \). Since \( cO_{X'} \subseteq I' \) we have \( \bar{c} \subseteq I'O_X \), \( V(\mathcal{I}'O_X) \subseteq V(\bar{c}) \) and thus

\[
\mathcal{X}[I/c] \subseteq \text{Proj}(B_I) \setminus V(\mathcal{I}'O_X)
\]

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(note that for this to make sense we need \(I'O_X \subseteq I\)). Therefore \(g^*I'\) is an invertible sheaf on \(X[I/c]\) and \(g\) factors uniquely through a morphism \(h : X[I/c] \to \text{Proj}(B_I)\). Since \(h^{-1}(c)\) is empty it actually factors through a morphism \(X[I/c] \to X'[I'/c]\) which by construction covers \(f\).

5.2.3 Proposition If \(f : Y \to X\) is a flat morphism of noetherian formal \(S\)-schemes there is a canonical isomorphism

\[
Y \times_X X[I/c] \xrightarrow{\sim} Y[I\mathcal{O}_Y/c]
\]

natural in \(f : Y \to X\) and transitive for pairs of composable morphisms \(f : Y \to X, g : Z \to Y\).

Proof. We start with the canonical isomorphism

\[
Y \times_X \text{Proj}(B_I) \xrightarrow{\sim} \text{Proj}(f^*B_I)
\]

and we may assume that \(X = \text{Spf}(A)\) and \(Y = \text{Spf}(B)\) are affine. Now \(f^*B_I\) is the graded \(A\)-algebra whose degree \(\ell\) component is \(B \otimes_A I^\ell\), and since \(B\) is a flat \(A\)-algebra, \(B \otimes_A I^\ell \simeq BI^\ell\). This shows that that there is a canonical isomorphism

\[
Y \times_X X[I] \xrightarrow{\sim} Y[I\mathcal{O}_Y]
\]

and 5.2.3.1 follows from this.

5.2.4 Lemma Suppose \(X\) is a formal \(S\)-scheme and \((b, \alpha)\) is a PD-structure on \(S\). For any ideal \(I \subseteq \mathcal{O}_X\) the PD-structure \((b, \alpha)\) extends uniquely to \(X[I/b]\).

Proof. The argument is that of [2, Prop. 2.1.1] where the hypothesis is that \(b\) is locally principle. In the present case the point is that \(I\mathcal{O}_X[I/b]\) is locally generated by a section of \(b\). Thus if the extension \(\tilde{\alpha}\) exists and \(a\) is a local section of \(I\mathcal{O}_X[I/b]\) we can write \(a = xt\) with \(t \in b\), and then \(\tilde{\alpha}_n(a) = x^n\tilde{\alpha}_n(t)\). This shows uniqueness. If \(xt = yt\) then

\[
x^n\alpha_n(t) - y^n\alpha_n(t) = (x^{n-1} + \cdots + y^{n-1})(x - y)\alpha_n(t).
\]

On the other hand \(\alpha_n(t) = zt\) for some local section \(z\) of \(I\mathcal{O}_X[I/b]\), so

\[
(x - y)\alpha_n(t) = (x - y)zt = 0.
\]

Thus setting \(\tilde{\alpha}_n(a) = x^n\tilde{\alpha}_n(t)\) when \(a = xt\) and \(t \in b\) defines \(\alpha_n\) locally, and by uniqueness the local extensions patch together to yield a global one.
5.2.5 Tube Algebras. In this paper we are mainly concerned with the case when \( c \) is locally principal, and we remarked after equation 5.2.1.1 that in this case the canonical projection \( \pi : \mathcal{X}[I/c] \to \mathcal{X} \) is affine. In this case we define the tube algebra of \( \mathcal{X} \) about \( I \) of radius \( c \) to be the direct image

\[
\mathcal{O}_X[I/c] := \pi_*\mathcal{O}_X[I/c].
\] (5.2.5.1)

The conditions 4.3.9.6, 4.3.9.7 and 4.3.9.8 are easily checked, so the results of section 4.3.9 apply to \( \mathcal{O}_X[I/c] \).

Recall from proposition 5.2.2 that if \( f : \mathcal{X} \to \mathcal{X}' \) is a morphism and \( I \subset \mathcal{O}_X, I' \subset \mathcal{O}_{X'} \) are ideals such that \( I'\mathcal{O}_X \subseteq I \) there is a morphism \( f_c : \mathcal{X}[I/c] \to \mathcal{X}'[I'/c] \) such that

\[
\begin{array}{ccc}
\mathcal{X}[I/c] & \xrightarrow{f_c} & \mathcal{X}'[I'/c] \\
\pi & & \pi' \\
\mathcal{X} & \xrightarrow{f} & \mathcal{X}'
\end{array}
\]

commutes. The natural morphism \( \mathcal{O}_{X'}[I'/c] \to f_*\mathcal{O}_X[I/c] \) then induces

\[
\pi'_*\mathcal{O}_{X'}[I'/c] \to \pi'_*f_*\mathcal{O}_X[I/c] \xrightarrow{\sim} f_*\pi_*\mathcal{O}_X[I/c]
\]

or in other words a morphism

\[
\mathcal{O}_{X'}[I'/c] \to f_*\mathcal{O}_X[I/c]
\] (5.2.5.2)

in the above notation.

If \( c \) is not locally principal there is a useful variant of this construction. Suppose \( j \subset \mathcal{O}_S \) is an open ideal and \( I \subset \mathcal{O}_X \) is an open ideal containing the image of \( j \) in \( \mathcal{O}_X \). We can apply proposition 5.2.2 with \( \mathcal{X}' = \mathcal{X} \) and \( I' = j\mathcal{O}_X \), obtaining a morphism

\[
\pi : \mathcal{X}[I/c] \to \mathcal{X}[j\mathcal{O}_X/c]
\]

The description of the basic open sets at the beginning of section 5.2.1 shows that \( \pi \) is an affine morphism. In this case we define

\[
\mathcal{O}_X[I/c] := \pi_*\mathcal{O}_X[I/c]
\] (5.2.5.3)

and again observe that the conditions 4.3.9.6, 4.3.9.7 and 4.3.9.8 apply to this \( \mathcal{O}_X[I/c] \), but relative to the formal scheme \( \mathcal{X}[j/c] \) in place of \( \mathcal{X} \). We will make no further use of 5.2.5.3 so the confusion of notation with 5.2.5.1 will not be a problem here.
5.2.6 Affine algebras in blowups. In section 5.3 we will need an explicit
description of the affine pieces of $X[I/c]$ which resembles the usual
description of the blowup of a regular ideal. Suppose $S = \text{Spf}(R)$ is noetherian
and formally affine. For any $c \in \mathfrak{c}$ the morphism $D_+(\overline{c}) \to X$
induced by the canonical projection $X[J/c] \to X$ is formally affine. If in addition
$X = \text{Spf}(A)$ is formally affine, the affine algebra of $D_+(\overline{c})$ has a well known
presentation by generators and relations, for which see e.g. [1 §3.1.6]. For
later use we rework this description in a slightly more intrinsic way.

If $N \subseteq A$ is any subset and $c \in \mathfrak{c}$ we define

$$A[N/c] = A[T_f, f \in N]/(cT_f - f, f \in N)$$  \hspace{1cm} (5.2.6.1)

and

$$A[N/c] = A[N/c]/(c\text{-torsion}).$$  \hspace{1cm} (5.2.6.2)

For $f \in N$ we use the same symbol to denote the polynomial variable $T_f$
in the right hand side of (5.2.6.1) and its image in $A[N/c]$. With this notation,

$$T_f + T_g = T_{f+g}, \quad fT_g = T_{fg} = gT_f$$  \hspace{1cm} (5.2.6.3)

in $A[N/c]$, since in both cases the difference of both sides in $A[N/c]$ is
annihilated by $c$.

The $A$-algebra $A[N/c]$ has the following universal property: if $B$ is a $c$-
torsion free $A$-algebra such that structure map $A \to B$ maps every element of
$N$ to $cB$ then $A \to B$ factors uniquely through an $A$-algebra homomorphism
$A[N/c] \to B$. From this it is clear (just in case it isn’t from 5.2.6.1) that
$A[N/c]$ depends functorially on $A$ and $N$. Note that $A[N/c] \simeq A$ if $N \subseteq cA$
(and in particular if $N$ is empty).

5.2.7 Lemma If $I$ is the ideal generated by a subset $N \subseteq A$, the natural
morphism $A[N/c] \to A[I/c]$ is an isomorphism.

Proof. This follows from the universal property: if $B$ is a $c$-torsion free
$A$-algebra and $A \to B$ maps $N$ into $cB$ then it will do the same for $I$. Thus
$A \to A[N/c]$ factors through a homomorphism $A[I/c] \to A[N/c]$ which is
an inverse to the natural morphism $A[N/c] \to A[I/c]$.

5.2.8 Lemma The kernel of the structure map $A \to A[N/c]$ is the ideal of
$c$-torsion elements of $A$.

Proof. It suffices to show that if $A$ is a $c$-torsion free $A$-algebra the structure
morphism is injective. For such $A$ natural morphism $A[c^{-1}] \to A[N/c][c^{-1}]$
is an isomorphism, and that $A \to A[N/c]$ is injective follows from the commutative diagram

\[
\begin{array}{ccc}
A & \to & A[N/c] \\
\downarrow & & \downarrow \\
A[c^{-1}] & \sim & A[N/c][c^{-1}] \\
\end{array}
\]

For any ring homomorphism $f : A \to A'$ the universal property implies the existence of a canonical homomorphism

\[ A' \otimes_A A[N/c] \to A'[f(N/c)] \quad (5.2.8.1) \]

with $c$-torsion kernel. It is evidently induced by an isomorphism

\[ A' \otimes_A A[N/c] \simrightarrow A'[f(N/c)] \quad (5.2.8.2) \]

from which we see that $5.2.8.1$ is surjective.

5.2.9 Lemma If $A'$ is a flat $A$-algebra, the natural map $5.2.8.1$ is an isomorphism.

Proof. By the universal property it suffices to show that $A' \otimes_A A[N]$ is $p$-torsion-free; this follows from the flatness of $A \to A'$.

Suppose now that $A$ is a noetherian adic ring and $J$ is an ideal of definition. We denote by $\hat{A}[N/c]$ the completion of $A[N]$ with respect to the ideal $JA[N/c]$. Since $A[N]$ is noetherian so is $\hat{A}[N/c]$, and $\hat{A}[N/c]$ is a flat $A[N]$-algebra. In particular since $A[N/c]$ is $c$-torsion-free, so is $\hat{A}[N/c]$. Finally the $A$-algebra $\hat{A}[N/c]$ has a universal property similar to that of $A[N]$: if $B$ is any $J$-adically complete $p$-torsion free $A$-algebra with the property that any element of $N$ becomes divisible by $p$ in $B$, the structure homomorphism $A \to B$ factors uniquely through an $A$-algebra homomorphism $\hat{A}[N/c] \to B$.

It follows from lemma 5.2.9 that the formation of $\hat{A}[N/c]$ is compatible with localization on $\text{Spf}(A)$. If $N$ is a sheaf of subsets of $\mathcal{O}_X$ We denote by $\mathcal{X}[N/c]$ the formal scheme obtained by patching together the formal schemes $\hat{A}[N/c]$.

Suppose now $\mathcal{X}$ is a formal $\mathcal{S}$-scheme, $c \subseteq \mathcal{O}_S$ and $J \subset \mathcal{O}_X$ are ideals and $c$ is a section of $c$. Let $\pi : D_+(\bar{c}) \to \mathcal{X}$ be the restriction of the canonical projection $\mathcal{X}[J/c] \to \mathcal{X}$. Since a local section $f$ of $J$ becomes divisible by $c$ in $D_+(\bar{c})$ the universal property yields a unique homomorphism

\[ \mathcal{O}_{\mathcal{X}[J/c]} \to \pi_*\mathcal{O}_{D_+(\bar{c})} \quad (5.2.9.1) \]
factorizing the canonical $\mathcal{O}_X \to \pi_0\mathcal{O}_{D_+}(e)$. If $X = \text{Spf}(A)$ is affine and $f \in J$, it sends $T_f \mapsto f/c$.

5.2.10 Proposition The homomorphism 5.2.9.1 is an isomorphism.

Proof. We can work locally, so let $X = \text{Spf}(A)$. The affine algebra of $\pi_0\mathcal{O}_{D_+}(e)$ is the $J$-adic completion of $(B_J)(\bar{c})$, the the degree 0 part of the homogenous localization of $B_J$ by $\bar{c}$ (in the notation of section 5.1). The map 5.2.9.1 is the $J$-adic completion of $A[J/c] \to (B_J)(\bar{c})$ so it suffices to show that the latter is an isomorphism. It is clearly surjective since $B_J$ is generated by elements of graded degree one.

Suppose $h \in A[J/c]$ is an element the kernel of 5.2.10.1 By construction any element of $A[J/c]$ can be written as a polynomial in the elements $T_f$ with coefficients in $A$ (not necessarily uniquely). Therefore there is an $N \in \mathbb{N}$ such that $c^N h$ is in the image of the structure morphism $A \to A[J/c]$. Now 5.2.10.1 is an $A$-algebra homomorphism and in both cases the structure map consists of $c$-torsion elements of $A$. It follows that $c^N h = 0$, and thus $h = 0$ since $A[J/c]$ is $c$-torsion free. Consequently 5.2.10.1 is injective.

5.3 The $\hat{\mathcal{D}}_{X/S}^{(m)}$-module structure on a tube algebra. We now fix the following situation: $S$ is a locally noetherian formal $\mathbb{Z}_p$-scheme, $X \to S$ is universally noetherian, $(a, b, \alpha)$ is an $m$-PD-structure on $S$ such that $p \in b$; finally $I \subset \mathcal{O}_X$ is $m$-bilaterising. The aim of this section is to show that for any locally principal $c \subset \mathcal{O}_S$, $\mathcal{O}_X[I/c]$ has a natural $\hat{\mathcal{D}}_{X/S}^{(m)}$-module structure compatible with its $\mathcal{O}_X$-algebra structure and for which it is quasi-nilpotent. In fact we will show, without the hypothesis on $c$ the structure sheaf $\mathcal{O}_X[I/c]$ has a natural $\pi^{-1}\hat{\mathcal{D}}_{X/S}^{(m)}$-module structure.

We first observe that if $I$ is $m$-bilaterising then so is $I^\ell$ for all $\ell \in \mathbb{N}$, so $I^\ell$ has a $m$-HPD-stratification induced by the canonical $m$-HPD-stratification of $\mathcal{O}_X$. The same is true for the $\mathcal{O}_X$-algebra

$$\mathcal{B}_I = \bigoplus_\ell I^\ell$$

and we denote the corresponding $m$-PD-stratification by

$$\theta_n : \mathcal{B}_I \to \mathcal{B}_I \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}^{(m)}$$

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Since this is induced by the canonical $m$-HPD-stratification of $\mathcal{O}_X$, this $m$-HPD-stratification is compatible with the $\mathcal{O}_X$-algebra structure of $\mathcal{B}_I$.

For any local section $b \in c$ we get an $m$-PD-stratification on the degree zero part $(\mathcal{B}_I)_0$ of the homogenous localization of $\mathcal{B}_I$ by setting

$$\theta_n(x/b^k) = \frac{\theta_n(x)}{b^k}$$

for $x \in I^k$. That this works can be seen as follows: if $x/b^k = y/b^j$ for $y \in I^j$ then $b^{j+q}x = b^{k+q}y$ for some $q \in \mathbb{N}$, whence

$$b^{j+k}\theta_n(x) = b^{j+k}\theta_n(x) = b^{k+q}\theta_n(y) b^k$$

and thus

$$\frac{\theta_n(x)}{b^k} = \frac{\theta_n(y)}{b^j}$$

in $\mathcal{B}_I \otimes_{\mathcal{O}_X} \mathcal{P}^n_{\mathcal{X}/S,(m)}$. The same calculation shows that this definition is compatible with the restriction maps $(\mathcal{B}_I)_0 \rightarrow (\mathcal{B}_I)_{bc}$ for $b, c \in c$, so $(\theta_n)$ is defined on all of $\mathcal{O}_X[I/c] = \mathcal{O}_X[I/c]$. It extends to an $m$-HPD-stratification since it came from an $m$-HPD-stratification of $\mathcal{B}_I$.

An important special case is $I = J(p^n)$, which is $m$-bilateralising if $n > m$. Since the canonical $m$-HPD-stratification of $\mathcal{O}_X$ is $\theta(x) = 1 \otimes x$, the corresponding $m$-HPD-stratification of $\pi_*\mathcal{O}_X[I/c]$ is determined by

$$\theta \left( \frac{f^p}{b} \right) = \frac{f^p}{b} \otimes 1 + \left( 1 \otimes \frac{f^p}{b} - \frac{f^p}{b} \otimes 1 \right)$$

(5.3.0.2)

where $\varphi^{(m)}_{p^n}$ is the $m$-PD-polynomial defined in section 3.1.6. In the notation of section 5.2.6, the restriction of the $m$-HPD-stratification to $D_+ (\bar{p})$ is

$$\theta(T_{f^p}) = T_{f^p} + \varphi^{(m)}_{p^n} (f \otimes 1, 1 \otimes f) \otimes 1.$$
5.4.1 Comparison maps. In what follows we set $\mathcal{X}[I, m] = \mathcal{X}[f^{(m)}]/b$ and $\mathcal{O}_\mathcal{X}[I, m] = \mathcal{O}_\mathcal{X}[f^{(m)}]/b$ for readability. By lemma 5.2.4 we know that $(b, \alpha)$ extends to an PD-structure $\bar{\alpha}$ on $f^{(m)}$. We now set

$$I_0 = f^{(m)}\mathcal{O}_\mathcal{X}[I, m], \quad I_1 = f\mathcal{O}_\mathcal{X}[I, m] + b\mathcal{O}_\mathcal{X}[I, m]$$

and claim that $(I_1, I_0, \bar{\alpha})$ is an $m$-PD-structure on $\mathcal{X}[I, m]$ compatible with $(a, b, \alpha)$. It suffices to treat the case when $S = \text{Spf}(R)$ and $\mathcal{X} = \text{Spf}(A)$ are formally affine. Since $p \in b$, $pI_1 \subseteq I_0$. Pick $c \in b$ and suppose $f$ is the section of $f^{(m)}\mathcal{O}_\mathcal{X}[I, m]$ on the affine $D_+(\bar{c})$. Then $f^{(m)}$ is divisible by $c$ and so lies in $b_1\mathcal{O}_\mathcal{X}[I, m]$. We conclude that $I_0^{(m)} \subseteq I_1$. The first condition of section 3.1 holds by construction. The second says $I_1 \cap b\mathcal{O}_\mathcal{X}[I, m]$ is a sub-PD-ideal, which by [4, lemme 1.2.1] is true since $I_0$ is locally principal.

Because of our hypotheses on $I$ the $m$-PD-envelope $\mathcal{P}_{(m)}(I)$ is defined, and its universal property, proposition 4.1.8 shows that the structure morphism $\mathcal{O}_\mathcal{X} \to \pi^*\mathcal{O}_\mathcal{X}[I, m] = \mathcal{O}_\mathcal{X}[I, m]$ factors through a morphism

$$\alpha_m : \mathcal{P}_{(m)}(I) \to \mathcal{O}_\mathcal{X}[I, m]. \quad (5.4.1.1)$$

Since this is an $m$-PD-morphism it is characterized by the fact that for any local section $x \in I$ and $c \in b$,

$$\alpha_m(x^{(k)}(m))|D_+(\bar{c}) = x^r\left(\frac{x^{p^m}}{c}\right)^q \bar{\alpha}_q(c) \quad (5.4.1.2)$$

where as usual $k = p^mq + r$, $0 \leq r < p^m$. Note that for $c$ and $d \in b$,

$$\left(\frac{x^{p^m}}{c}\right)^q \bar{\alpha}_q(c) = \left(\frac{x^{p^m}}{cd}\right)^q d^q \bar{\alpha}_q(c) = \left(\frac{x^{p^m}}{cd}\right)^q \bar{\alpha}_q(cd)$$

and thus

$$\left(\frac{x^{p^m}}{c}\right)^q \bar{\alpha}_q(c) = \left(\frac{x^{p^m}}{d}\right)^q \bar{\alpha}_q(d)$$

in $D_+(cd) = D_+(c) \cap D_+(d)$, so $5.4.1.2$ in fact defines a section of $\mathcal{O}_\mathcal{X}[I, m]$.

We next construct a morphism

$$\beta_m : \mathcal{O}_\mathcal{X}[I, m + 1] \to \mathcal{P}_{(m)}(I) \quad (5.4.1.3)$$

in the other direction. We can assume $\mathcal{X} = \text{Spf}(A)$ is affine. In principle $5.4.1.3$ arises because $f^{(m+1)}$ is a multiple of $p$ in $\mathcal{P}_{(m)}(I)$, but since $\mathcal{P}_{(m)}(I)$ is not a geometric object we cannot appeal to the universal property of blowups, as in e.g. [I Prop. 3.1.9]. Instead we use the construction of
Let $\tau : O_{D_+} \to X$ be the restriction of $\pi$ to the open subset $D_+$. By proposition 5.2.10 we can identify $\tau_* O_{D_+}$ with $A[I^{(p^{m+1})}/p]$, and lemma 5.2.7 says that we can identify $A[I^{(p^{m+1})}/p]$ with $A[N/p]$ where $N$ is the set of $f^{p^{m+1}}$ for all $f \in I$. There is a map

$$\gamma_m : A[N/p] \to \hat{P}(m)(I)$$

induced from

$$A[T_f, f \in N] \to \hat{P}(m)(I)$$

$$T_{f^{p^{m+1}}} \mapsto (p-1)! f^{p^{m+1}}(m)$$

since the latter sends

$$p T_{f^{p^{m+1}}} - f^{p^{m+1}} \mapsto p! f^{p^{m+1}}(m) - f^{p^{m+1}} = 0.$$ 

We define 5.4.1.3 to be the composition of $\gamma_m$ with the restriction map $\tau_* O_{D_+}$.

5.4.2 Proposition (i) The composition

$$\beta_m \alpha_{m+1} : \mathcal{P}(m+1)(I) \to \mathcal{O}_X[I, m+1] \to \mathcal{P}(m)(I)$$

is the change-of-level map 4.1.9.3.

(ii) The composition

$$\alpha_m \beta_m : \mathcal{O}_X[I, m+1] \to \mathcal{P}(m)(I) \to \mathcal{O}_X[I, m]$$

is the natural map induced by the inclusion $I^{(p^{m+1})} \subset I^{(p^m)}$.

Proof. As usual we may work in an affine setting. (i) If $x \in I$, $c \in \mathfrak{c}$ and $k = p^{m+1}q + r$,

$$\alpha_{m+1}(x^{(k)}(m+1))|D_+(\mathfrak{c}) = x^r \left( \frac{x^{p^{m+1}}}{c} \right)^q \bar{\alpha}_q(c)$$

and thus

$$\beta_m \alpha_{m+1}(x^{(k)}(m+1)) = \beta_m(x^r \left( \frac{x^{p^{m+1}}}{p} \right)^q)$$

$$= x^r p^{[q]} \beta_m(T_{x^{p^{m+1}}})$$

$$= x^r p^{[q]} ((p-1)! x^{p^{m+1}}(m))$$

$$= x^r p^{[q]} ((p-1)! (x^{p^m})^{-p}).$$
From the change-of-level formula \[3.2.1.5\] we see that we must check that if \(k = p^mq' + r'\),

\[x^r p^{[q]}((p - 1)! (x^{p^n})^{[q]})q = \frac{q!}{q!} x^{r'} (x^{p^n})^{[q']}\].

This is a PD-identity which can be checked by reduction to the universal case. There it may be checked after multiplication by \(q!\), but the identity \(q! x^{[q]} = x^q\) shows that both sides are \(x^k\).

For (ii) note that \(A[I^{(p^{m+1})}] = A[N]\) where \(N \subset I\) is the set of \(p^{m+1}\)-powers of elements of \(I\). By construction

\[
\beta_m(T_{f^{p^m+1}}) = (p - 1)! f^{(p^{m+1})}_m = (p - 1)! (f^{p^m})^p
\]

and since \(pT_{f^{p^m}} = f^{p^m}\) in \(A[I^{(p^m)}]\),

\[
\alpha_m((p - 1)! (pT_{f^{p^m}})^{[p]}) = (p - 1)! (p)^{[p]} T_{f^{p^m}}^p = p^{p-1} T_{f^{p^m}} = (f^{p^m})^{p-1} T_{f^{p^m}} = f^{p^{m+1}} - p^m T_{f^{p^m}} = T_{f^{p^m+1}}
\]

where the last equality is from \(5.2.6.3\) \(\blacksquare\)

Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \rightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{S}' & \rightarrow & \mathcal{S}
\end{array}
\]

(5.4.2.3)

in which \(\mathcal{S}\) and \(\mathcal{S}'\) are locally noetherian \(\mathbb{Z}_p\)-schemes with \(m\)-PD-structures \((a, b, \alpha)\) and \((a', b', \alpha')\), \(g\) is an \(m\)-PD-morphism, \(\mathcal{X}\) (resp. \(\mathcal{X}'\)) is a quasi-smooth formal \(\mathcal{S}\)-scheme (resp \(\mathcal{S}'\)-scheme), and \(f\) is a morphism of locally noetherian formal schemes such that \(f^* I \subseteq I'\). We assume that conditions \(4.1.1-3\) hold both for \(I \subset O_{\mathcal{X}}\) and \(I' \subset \mathcal{X}'\) relative to \(\mathcal{S}\) and \(\mathcal{S}'\) respectively, and that both \(\mathcal{X}\) and \(\mathcal{X}'\) have ideals of definition locally satisfying condition \(4.1.2.1\). The formulas \(5.4.1.1\) and \(5.4.1.3\) and the fact that \(f\) is an \(m\)-PD-morphism shows that the diagram

\[
\begin{array}{ccc}
O_{\mathcal{X}}[I, m + 1] & \xrightarrow{\beta_m} & \mathcal{P}_{(m)\alpha}(I) \\
\downarrow & & \downarrow \\
f_* O_{\mathcal{X}}'[I', m + 1] & \xrightarrow{\beta_m} & f_* \mathcal{P}_{(m)\alpha'}(I')
\end{array}
\]

\[
\begin{array}{ccc}
\rightarrow & & \rightarrow \\
\alpha_m & & \alpha_m \\
\downarrow & & \downarrow \\
O_{\mathcal{X}}[I, m] & \xrightarrow{\alpha_m} & f_* O_{\mathcal{X}}'[I', m]
\end{array}
\]

(5.4.2.4)

is commutative.

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5.5 Analytic stratifications. The close relation between tubes and \(m\)-PD-envelopes suggests that we work out the corresponding notion of stratification. The setting of section 5.4 remains in force: \(S\) is a locally noetherian formal \(\mathbb{Z}_p\)-scheme, \(\mathcal{X} \rightarrow S\) is universally noetherian, \((a, b, \alpha)\) is an \(m\)-PD-structure on \(S\) such that \(p \in b\). We assume, in addition that \(b\) is locally principal, so that the preceding sections apply to the tubes \(\mathcal{X}[I/b]\).

5.5.1 Recall that \(\mathcal{X}_S(\cdot)\) is the simplicial formal scheme for which \(\mathcal{X}_S(r)\) is the fibered product of \(r + 1\) copies of \(\mathcal{X}\) over \(S\). We denote by \(I(r)\) the ideal of the diagonal \(\mathcal{X} \rightarrow \mathcal{X}_S(r)\) and by \(\hat{\mathcal{X}}_S(r)\) the formal completion of \(\mathcal{X}_S(r)\) with respect to \(I(r)\). The result is a simplicial scheme \(\hat{\mathcal{X}}_S(\cdot)\) which in every degree has the same underlying space as \(\mathcal{X}\). We denote by \(p_0, p_1 : \hat{\mathcal{X}}_S(1) \rightarrow \mathcal{X}\) the morphisms induced by the corresponding projections \(\mathcal{X}_S(1) \rightarrow \mathcal{X}\). Since \(\mathcal{X} \rightarrow S\) is quasi-smooth, \(I(r)\) is split-regular.

We now denote by \(\mathcal{X}_S[I,m,r]\) the tube \(\hat{\mathcal{X}}_S(r)[I,m] = \hat{\mathcal{X}}_S(r)[I(r)^{\mu^r}] / b\) and by \(\tau : \hat{\mathcal{X}}_S[I,m] \rightarrow \hat{\mathcal{X}}_S(r)\) the canonical projection (note that \(r\) does not appear in the notation here). Finally we denote by \(q_0, q_1 : \mathcal{X}_S[I,m,1] \rightarrow \hat{\mathcal{X}}_S\) the composites \(q_i = p_i \tau\) for \(i = 0, 1\). An analytic stratification of level \(m\) of an \(O_\mathcal{X}\)-module \(M\) is an isomorphism

\[
\chi : q_1^*(M) \congto q_0^*(M) \tag{5.5.1.1}
\]

on \(\mathcal{X}_S[I,m,1]\) which restricts to the identity on the diagonal and satisfies the usual cocycle condition on \(\mathcal{X}_S[I,m,2]\). A morphism \((M, \chi) \rightarrow (M', \chi')\) of \(O_\mathcal{X}\)-modules with an analytic stratification is a morphism \(M \rightarrow M'\) of \(O_\mathcal{X}\)-modules making

\[
\begin{array}{ccc}
q_1^*(M) & \xrightarrow{\chi} & q_0^*(M) \\
| & | & |
q_1^*(M') & \xrightarrow{\chi'} & q_0^*(M')
\end{array}
\tag{5.5.1.2}
\]

commutative.

To put this notion in a closer relation to \(m\)-HPD-stratifications we use the relations \(q_i = p_i \tau\) and the adjunction to rewrite 5.5.1.1 as

\[
p_1^*(M) \rightarrow \tau_* \tau^* p_0^*(M) \cong \tau_*(\tau^* p_0^*(M) \otimes O_{\mathcal{X}_S[I,m,1]}) \\
\cong p_0^*(M) \otimes O_{\hat{\mathcal{X}}_S(1)} \tau_*(O_{\mathcal{X}_S[I,m,1]}). \tag{5.5.1.3}
\]

To emphasize the analogy with \(m\)-PD-envelopes we define

\[
\mathcal{P}^an_{\mathcal{X}/S,(m)}(r) = \tau_* O_{\mathcal{X}[I,m,r]} \tag{5.5.1.4}
\]

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where as before \( \tau : \mathcal{X}[I, m, r] \to \hat{\mathcal{X}}_s(r) \) is the canonical projection. For variable \( r \) the projections and diagonals give \( \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \) the structure of a simplicial ring on \( \mathcal{X} \). When \( r = 1 \) we write \( \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(1) = \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \). With this notation the map \( 5.5.1.3 \) is
\[
p_1^\prime(M) \to p_0^\prime(M) \otimes \mathcal{O}_{\hat{\mathcal{X}}_s(1)} \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}.
\]
This is \( \mathcal{O}_{\hat{\mathcal{X}}_s(1)} \)-linear and its linearization with respect to \( \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \) is a morphism
\[
\chi : \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \otimes \mathcal{O}_{\hat{\mathcal{X}}_s(1)} p_1^\prime(M) \to p_0^\prime(M) \otimes \mathcal{O}_{\hat{\mathcal{X}}_s(1)} \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}.
\]
on \( \hat{\mathcal{X}}_s(1) \). Note that we use \( \chi \) for both \( 5.5.1.1 \) and \( 5.5.1.6 \) and in fact these two morphisms determine each other. The usual argument shows that \( 5.5.1.6 \) is an isomorphism, restricts to the identity on the diagonal and satisfies a suitable cocycle condition.

As is the case for the \( m \)-PD-envelopes \( \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(r) \), there is a canonical isomorphism
\[
\mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(r) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(r') \xrightarrow{\sim} \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(r + r')
\]
for all \( r, r' \geq 0 \). The case \( r = r' = 1 \) is important since it means that the morphism \( d_{02} : \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \to \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}(2) \) can be identified with a morphism
\[
\delta_{(m)} : \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \to \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \otimes \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}.
\]
Suppose now that \( \mathcal{B} \) is an \( \mathcal{O}_\mathcal{X} \)-module satisfying the conditions \( 4.3.9.6 \) and \( 4.3.9.8 \) and endowed with an analytic stratification
\[
\chi : \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \otimes \mathcal{O}_{\hat{\mathcal{X}}_s(1)} p_1^\prime(\mathcal{B}) \to p_0^\prime(\mathcal{B}) \otimes \mathcal{O}_{\hat{\mathcal{X}}_s(1)} \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}
\]
of level \( m \). The arguments of section \( 4.3.9 \) apply \textit{mutatis mutandis} to yield a ring \( \mathcal{P}_{\mathcal{B}/\mathcal{S},(m)} \) and an analogue of proposition \( 4.3.11 \) the ring structure of \( \mathcal{B} \) is compatible with its analytic stratification if and only if there is a ring homomorphism
\[
\delta_{\mathcal{B},(m)} : p_{02}^\prime \mathcal{P}_{\mathcal{B}/\mathcal{S},(m)} \to (p_{01}^\prime \mathcal{P}_{\mathcal{B}/\mathcal{S},(m)}) \otimes_{\mathcal{O}_{\hat{\mathcal{X}}(1)}} (p_{12}^\prime \mathcal{P}_{\mathcal{B}/\mathcal{S},(m)})
\]
that is \( (p_0^\prime \mathcal{B}, p_2^\prime \mathcal{B}) \)-bilinear and semilinear for the homomorphism
\[
\delta : p_{02}^\prime \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)} \to (p_{01}^\prime \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}) \otimes_{\mathcal{O}_{\hat{\mathcal{X}}(2)}} (p_{12}^\prime \mathcal{P}_{\hat{\mathcal{X}}/\mathcal{S},(m)}).
\]
Finally the constructions of section 4.3.12 extend in the same way to analytic stratifications. Suppose $B$ has an analytic stratification of level $m$ and $M$ is a $B$-module satisfying the conditions 4.3.3.1 and 4.3.3.2. An analytic stratification of level $m$ of $M$ is compatible with its $B$-module structure if and only if the analytic stratification induces an isomorphism

$$\chi^B : P_{B/S,(m)} \otimes_{p_1^*B} p_1^*M \cong p_0^*M \otimes_{p_0^*B} P_{B/S,(m)}$$

(5.5.1.8)
satisfying the usual conditions. As before, the existence of a $\chi^B$ with these properties is equivalent to the existence of a morphism

$$\theta^B : p_1^*M \to p_0^*M \otimes_{p_0^*B} P_{B/S,(m)}$$

(5.5.1.9)

making commutative a diagram analogous to 4.3.12.8.

5.5.2 Stratifications on tube algebras. The analytic stratifications we will use arise from HPD-stratifications. With the hypotheses in force, the comparison morphisms 5.4.1.3, 5.4.1.1 yield morphisms

$$\alpha_m : P_{X/S,(m)}(\cdot) \to P_{X/S,(m)}(\cdot)$$

$$\beta_m : P_{X/S,(m+1)}(\cdot) \to P_{X/S,(m)}(\cdot)$$

(5.5.2.1)
of simplicial rings. In the situation of the commutative square 5.4.2.3 there is a commutative diagram analogous to 5.4.2.4. The morphisms $\alpha_m$ allow us to derive an analytic stratification of level $m$ from an $m$-HPD-stratification, and likewise the $\beta_m$ produce an $m$-HPD-stratification from an analytic stratification of level $m + 1$.

Suppose for example $J \subseteq O_X$ is an open ideal. In section 5.3 we showed that if $J$ is $m$-bilateralising the tube algebra $O_X[J/b]$ had a canonical left $\hat{D}^{(m)}_{X/S} \otimes_{\mathbb{S}^{(m)}_S} \cdot$-module structure. Recall however from cor. 4.3.7 that if $J$ is $(m+1)$-bilateralising the left $\hat{D}^{(m+1)}_{X/S} \otimes_{\mathbb{S}^{(m+1)}_S} \cdot$-module structure of $O_X[J/b]$ will induce a topologically quasi-nilpotent left $\hat{D}^{(m)}_{X/S} \otimes_{\mathbb{S}^{(m)}_S} \cdot$-module. Thus if $J$ is $(m+1)$-bilateralising the tube algebra $O_X[J/b]$ has a canonical $m$-HPD-stratification

$$P_{X/S,(m)} \otimes p_1^*(O_X[J/b]) \cong p_0^*(O_X[J/c]) \otimes P_{X/S,(m)}.$$

The comparison morphism $\alpha_m$ then induces an analytic stratification

$$P_{X/S,(m)} \otimes p_1^*(O_X[J/b]) \cong p_0^*(O_X[J/c]) \otimes P_{X/S,(m)}$$

(5.5.2.2)
of level $m$. 95
5.5.3 The isogeny category. I do not know if there is an analogue in this setting of Ogus’s integrality theorem (prop. 4.4.1) in these situations. Nonetheless if \( B \) is a \( \mathcal{O}_X \)-algebra with an analytic stratification of level \( m \) compatible with its ring structure, and \( M \) is a coherent \( B \)-module we can still define an analytic stratifications by analogy with 5.5.1.1 and 5.5.1.6: an analytic stratification of \( M \) of level \( m \) is an isomorphism as in 5.5.1.1 or equivalently as an isomorphism

\[
\chi: \mathcal{P}^{an}_{X/\mathbb{Q}_p}(m) \otimes \mathcal{O}_{X_S(1)} \to \mathcal{P}^{an}_{X/\mathbb{Q}_p}(m),
\]

and one can even omit the tensor products with \( \mathbb{Q} \) in \( \mathcal{O}_{X_S(1)} \) and \( \mathcal{P}^{an}_{X/\mathbb{Q}_p}(m) \). The point is that since \( M \) is coherent, the pullbacks can be defined as in rigid geometry and are still coherent on their base, so that completing the tensor products is unnecessary. Furthermore \( m \)-HPD-stratifications on coherent \( B \)-modules can be defined in the analogous way, and proposition 4.4.1 says that they describe coherent \( B \)-modules with a quasi-nilpotent left \( \mathbb{D}(m) \)-module (i.e. they arise from a \( B \)-coherent quasi-nilpotent left \( \mathbb{D}(m) \)-modules by extension of scalars to \( \mathbb{Q} \)). Finally the constructions of section 5.4.1 show how to go back and forth between \( m \)-HPD-stratifications and analytic stratifications of level \( m \) for coherent \( B \)-modules.

5.5.4 An Example. Here is what this mass of formalism amounts to in the one case we will actually use. Let \( V \) be a complete discrete valuation ring of mixed characteristic \( p > 0 \) with fraction field \( K \) and take \( S = \text{Spf}(V) \). Fix an \( m_0 \in \mathbb{N} \) such that \( (m, (p), [ ]) \) is an \( m_0 \)-PD-structure on \( S \), where \( m \subset V \) is the maximal ideal. Let \( X \) be a formal \( V \)-scheme formally of finite type over \( S \), and fix an ideal of definition \( J \subset \mathcal{O}_X \). We can assume \( X \) is formally affine for this discussion. If \( f_1, \ldots, f_r \) generate \( J \), the analytification \( X_m := X(j^{(p^{m+1})}/(p))^{an} \) of the tube is the locus of

\[
|f_1| \leq |p|^{p-(m+1)}, \ldots, |f_r| \leq |p|^{p-(m+1)}
\]

in the analytic space \( X^{an} \). Similarly if \( x_1, \ldots, x_d \) are local parameters on \( X \) and \( I \) is the diagonal ideal of \( X \times_S X \), \( X_S[I, m, 1]^{an} \) defined by

\[
|1 \otimes x_r - x_r \otimes 1| \leq |p|^{p-(m+1)} \quad 1 \leq r \leq d
\]

If following tradition we write \([X]_{p-(m+1)} \) and \([\Delta]_{p-(m+1)} \) for the tubes 5.5.4.1 and 5.5.4.2 then an isomorphism 5.5.1.1 for some coherent \( \mathcal{O}_{X_m} \)-module is an isomorphism \( p^*_M \sim p^*_0 M \) on of the pullbacks of \( M \) to \([\Delta]_{p-(m+1)} \) by the projections \( p^*_r: [\Delta]_{p-(m+1)} \to [X]_{p-(m+1)} \).
If \( m \geq m_0 \) the results of section 5.5.2 apply to the tube algebra \( B = \mathcal{O}_X[J(p^{n+1})/(p)] \). If \( M \) is a coherent \( B \)-algebra, \( M_\mathbb{Q} \) can be identified with a coherent sheaf \( \mathcal{M} \) on the analytic space \( X_m \). Furthermore an \( m \)-HPD-stratification on \( M \) induces a level \( m \) analytic stratification of \( M \), whence an isomorphism \( p^*_1 M \cong p^*_0 M \) as in the last paragraph.

6 Convergent Isocrystals

Fix a complete discrete valuation ring \( V \) of mixed characteristic \( p \) with fraction field \( K \) and residue field \( k \), and let \( X \) be a separated \( k \)-scheme of finite type. In this final section we apply the results of the preceding sections to give an alternate construction of Berthelot’s category \( \mathrm{Isoc}(X/K) \) of convergent isocrystals on \( X \) with coefficients in \( K \). As an application we are able to generalize Berthelot’s theorem on Frobenius descent to the case of any separated \( X/k \) of finite type (without assuming \( X/k \) is smooth).

6.1 Some Fréchet-Stein algebras. When dealing with coherent sheaves on Stein spaces (and in many other situations) the category of coadmissible modules replaces that of coherent modules. We recall the definitions.

6.1.1 Fréchet-Stein algebras. Let \( A \) be a \( K \)-algebra endowed with a Frechét topology, i.e. the underlying \( K \)-vector space is a Frechét space. The \( K \)-algebra \( A \) is is a \( K \)-Fréchet algebra if its ring structure is compatible with the topology, i.e. if multiplication is continuous. If \( q \) is a continuous seminorm on \( A \), the completion \( A_q \) of \( A \) for the topology induced by \( q \) is a \( K \)-Banach algebra. If \( q' \) is another seminorm on \( A \) such that \( q' \geq q \) there is a natural homomorphism \( A_q \to A_{q'} \) of \( K \)-Banach algebras with norm at most 1 and dense image. We recall, after Schneider and Teitelbaum [18] that \( A \) is left \( K \)-Fréchet-Stein if there is an increasing sequence

\[
q_0 \leq q_1 \leq \cdots
\]

(6.1.1.1)

of continuous seminorms on \( A \) such that for all \( n \geq 0 \),

(i) \( A_{q_n} \) is left noetherian, and

(ii) \( A_{q_{n+1}} \to A_{q_n} \) is right flat.

and we require that the natural map

\[
A \xrightarrow{\sim} \lim_n A_{q_n}
\]

(6.1.1.2)
is a topological isomorphism. Similarly for right or bilateral Fréchet-Stein algebras; in the paper we will only be concerned with the commutative case, and speak simply of Fréchet-Stein algebras. The archetypal example is that of a Stein space $X$ over $K$, where $U_0 \subseteq U_1 \subseteq \cdots$ is an increasing and exhaustive sequence of affinoid domains in $X$ and $q_n$ is the affinoid norm of $\Gamma(U_n, \mathcal{O}_X)$. In fact the Fréchet-Stein algebras in this paper will be of this sort. Note that in any case the topology of $A$ is independent of the choice of seminorms. Finally the canonical map $A \to A_n$ from 6.1.1.2 is flat for all $n$.

6.1.2 Coadmissible Modules. Fix a $K$-Frechet-Stein algebra $A$ and a sequence of seminorms 6.1.1.1 defining its topology and satisfying the above conditions; for the rest of section 6.1.1 we write $A_n$ for $A_{q_n}$. As in [18], a coherent $A$-module is a sequence $\{M_n\}_{n \geq 0}$ in which $M_n$ is a finitely generated $A_n$-module and maps $M_{n+1} \to M_n$ for $n \geq 0$ that are linear with respect to $A_{n+1} \to A_n$ and such that the canonical map

$$A_n \otimes_{A_{n+1}} M_{n+1} \xrightarrow{\sim} M_n$$

is an isomorphism. Morphisms $(M.) \to (N.)$ are morphisms of graded modules such that $M_n \to N_n$ is $A_n$-linear for all $n \geq 0$. Then

$$\Gamma(M.) = \varprojlim_n M_n$$

is a topological $A$-module, and Cartan’s theorems A and B hold for $M$:

- for all $n$ the projection $M \to M_n$ has dense image, and
- $\mathcal{R}^p \varprojlim_n M_n = 0$ for $p > 0$.

Finally, an $A$-module $M$ is coadmissible if there is a coherent $A$-module such that $M \simeq \Gamma(M.)$. Morphisms of coadmissible modules are morphisms induced by morphisms of coherent $A$-modules. However

$$M_n \simeq A_n \otimes_A M$$

for all $n$, and morphisms of coadmissible $A$-modules are simply continuous $A$-module homomorphisms. We denote by Coad$(A)$ the category of coadmissible $A$-modules, which is naturally a full subcategory of the category of $A$-modules. Schneider and Teitelbaum show that Coad$(A)$ is an abelian and the inclusion functor Coad$(A) \to$ Mod$(A)$ is exact. The category Coad$(A)$ has many agreeable stability properties; for example if two out of three terms
of a short exact sequence are coadmissible then the third is also. A closed submodule of a coadmissible \( A \)-module is coadmissible, and any \( A \)-module of finite presentation is coadmissible. For all these facts and more see \cite{18, §3}.

6.1.3 Globalization. We will want to globalize this construction in the following setting. Let \( \mathcal{V}, K \) and \( k \) be as before and let \( \mathcal{X} \) be a quasi-smooth formal \( \mathcal{V} \)-scheme that is formally of finite type. We denote by \( X \) the closed fiber \( \mathcal{X} \times \mathcal{V} k \) and by \( \mathcal{X}^{an} \) the rigid-analytic space associated to \( \mathcal{X} \); we could take this to be the adic space that is the generic fiber of \( \mathcal{X} \) in the sense of adic spaces, but since \( \mathcal{X}/\mathcal{V} \) is formally of finite type we can use Berthelot’s explicit construction from \cite{5}. Fix a decreasing sequence  
\[
\cdots \subseteq J_2 \subseteq J_1 \subseteq J_0 \subseteq \mathcal{O}_{\mathcal{X}^{an}}
\]
(6.1.3.1) of ideals of definition of \( \mathcal{O}_{\mathcal{X}} \) that is cofinal in the set of all ideals of definition, and let  
\[
A_n = \mathcal{O}_X[J_n/(p)]
\]
(6.1.3.2) be the corresponding tube algebras\footnote{5.2.5.1} The inclusions \( J_{n+1} \to J_n \) induce homomorphisms \( A_{n+1} \to A_n \) for all \( n \).

6.1.4 Proposition Suppose \( U \subseteq \mathcal{X} \) is an open formal affinoid and set  
\[
A_n = \Gamma(U, A_n)\mathbb{Q}
\]
for all \( n \geq 0 \). Then each \( A_n \) is an affinoid algebra over \( K \), and \( A = \varprojlim A_n \) is a Fréchet-Stein algebra when each \( A_n \) is given its affinoid norm. Finally \( A = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}^{an}}) \) where \( \mathcal{U} \subseteq \mathcal{X}^{an} \) is the tube of \( \mathcal{U} \) in \( \mathcal{X}^{an} \).

Proof. That \( A_n \) is an affinoid algebra follows from the explicit construction of the tube algebras in section \footnote{5.2.6} and in particular from lemma \footnote{5.2.7} in which we take \( A = \Gamma(\mathcal{U}, \mathcal{O}_X) \), \( I = \Gamma(\mathcal{U}, J_n) \) and \( N \) to be any finite set of generators of \( I \). From the same lemma the \( A_n \) are noetherian since \( \mathcal{X} \) is noetherian, and the flatness of \( A_{n+1} \to A_n \) is classical. Finally the isomorphism  
\[
A = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}^{an}}) \simeq \varprojlim A_n
\]
is clear: \( \mathcal{O}_X \) is a sheaf. \hfill \( \blacksquare \)

The proposition shows that \( \mathcal{U} \mapsto \varprojlim \Gamma(\mathcal{U}, A_n)\mathbb{Q} \) defines a presheaf \( \mathcal{O}_{\mathcal{X}^{an}} \) of algebras on \( \mathcal{X} \) which is easily seen to be a sheaf, in fact a sheaf of topological \( \mathcal{O}_{\mathcal{X}} \)-algebras. The construction shows that \( \mathcal{O}_{\mathcal{X}^{an}} \) is the direct image by the specialization map of the structure sheaf of \( \mathcal{X}^{an} \).
6.1.5 Definition A sheaf $\mathcal{M}$ of $\mathcal{O}_X^{an}$-modules is coadmissible if for every open affine $\mathcal{U} \subseteq \mathcal{X}$, $\Gamma(\mathcal{U}, \mathcal{M})$ is a coadmissible $\Gamma(\mathcal{U}, \mathcal{O}_X^{an})$-module.

By theorems A and B we have $\mathcal{O}_X^{an} \simeq \lim_{\leftarrow n} A_n$, and the maps $A_{n+1} \to A_n$, $A \to A_n$ are flat for all $n \geq 0$. A coadmissible sheaf $\mathcal{M}$ can be identified with a sequence $\{M_n\}_{n \geq 0}$ where each $M_n$ is a coherent $\mathcal{O}_X$-module and the maps $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ are isomorphisms for all $n$. Then $\mathcal{M} \simeq \lim_{\leftarrow n} M_n$ and $M_n \simeq A_n \otimes_A \mathcal{M}$.

We denote by $\text{Coad}(\mathcal{O}_X^{an})$ the category of coadmissible $\mathcal{O}_X^{an}$-modules; it is an abelian category and the inclusion functor $\text{Coad}(\mathcal{O}_X^{an}) \to \text{Mod}(\mathcal{O}_X^{an})$ is exact.

6.2 A Comparison Theorem. We recall that $\mathcal{V}$ is a complete discrete valuation ring of mixed characteristic $p > 0$ with maximal ideal $m$ and residue field $k$, and $\mathcal{X}$ is a formal $\mathcal{V}$-scheme formally of finite type over $\mathcal{V}$. We choose a natural number $m_0$ such that $(m, (p), [\cdot])$ is an $m$-PD-structure on $\mathcal{V}$, where $\pi$ is a uniformizer of $\mathcal{V}$.

6.2.1 $\mathcal{O}_X^{an}$ has a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger}$-module structure. We fix an ideal of definition $J \subset \mathcal{O}_X$ and set $J_n = J^{(p^m+1)}$. The $J_n$ for $n \geq 0$ are a cofinal subset of ideals of definition of $\mathcal{O}_X$, so we can apply the constructions of section 6.1.3. The point of this choice is that for $n > m \geq m_0$ the tube algebra $\mathcal{O}_X[J_n/(p)]$ has a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger(m)}$-module structure compatible with its $\mathcal{O}_X$-algebra structure, c.f. section 5.3. Then $A_{n\mathcal{Q}} = \mathcal{O}_X[J_n/(p)]_{\mathcal{Q}}$ has a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger(m)}$-module for all $n \geq m$. For $n' \geq n \geq m$ the map $A_{n'\mathcal{Q}} \to A_{n\mathcal{Q}}$ is $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger(m)}$-linear, so $\mathcal{O}_X^{an} = \lim_{\leftarrow n} A_{n\mathcal{Q}}$ has a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger}$-module structure.

6.2.2 Definition The category $\text{Coad}^{\dagger}(\mathcal{O}_X^{an})$ is the category whose objects are coadmissible $\mathcal{O}_X^{an}$-modules endowed with a left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger}$-module structure compatible with the left $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger}$-module structure of $\mathcal{O}_X^{an}$, and morphisms are $\mathcal{D}_{\mathcal{X}/\mathcal{V}}^{\dagger}$-linear morphisms of coadmissible $\mathcal{O}_X^{an}$-modules.
As Coad($\mathcal{O}_{X}^{an}$) is an abelian category, it is easily checked that Coad$^\dagger(\mathcal{O}_{X}^{an})$ is also abelian.

The comparison theorem to be proven later will use two alternative constructions of $\text{Isoc}(X/K)$ and Coad$^\dagger(\mathcal{O}_{X}^{an})$. If we set

$$J_m = J^{(pm+2)}(p), \quad X_m = X[j_m/(p)], \quad \mathcal{O}^{(m)} = \mathcal{O}_X[j_m/(p)]$$

then $\mathcal{O}^{(m)}$ has a left $\mathcal{D}^{(m+1)}_{X/V}$-module structure, so the induced left $\mathcal{D}^{(m)}_{X/V}$-module structure is quasi-nilpotent, i.e. $\mathcal{O}^{(m)}$ has a $m$-HPD-stratification $P_{X/S}(m)$.

We denote by $\text{CStrat}(X/K)$ the following category: objects are sequences of triples $(M^{(m)}, \chi_m, \phi_m)_{m \geq m_0}$ where $M^{(m)}$ is a coherent $\mathcal{O}^{(m)}_{\hat{Q}}$-module, $\chi_m$ is a level $m$ analytic stratification of $M^{(m)}$ compatible with the analytic stratification $6.2.2.2$ of $\mathcal{O}^{(m)}$, and $\phi_m$ is an isomorphism

$$\phi_m : \mathcal{O}^{(m)}_{\hat{Q}} \otimes_{\mathcal{O}^{(m+1)}_{\hat{Q}}} M^{(m+1)} \to M^{(m)}.$$ 

We require $\phi_m$ and $\chi_m$ to be compatible in the sense that

$$\mathcal{O}^{(m)} \otimes_{\mathcal{O}^{(m+1)}} \chi_{m+1} = \chi_m$$

if we use the identifications $\phi_m : \mathcal{O}^{(m)} \otimes_{\mathcal{O}^{(m+1)}} M^{(m+1)} \simeq M^{(m)}$ and

$$\mathcal{O}^{(m)} \otimes_{\mathcal{O}^{(m+1)}} \mathcal{P}^{an}_{X/V_Q}(m+1) \simeq \mathcal{P}^{an}_{X/V_Q}(m).$$

A morphism

$$(M^{(m)}, \chi_m, \phi_m)_m \to (M^{(m)}, \chi'_m, \phi'_m)_m$$

is an sequence of morphisms of $\mathcal{O}^{(m)}_{\hat{Q}}$-modules with analytic stratifications, compatible with the $\phi_m$ for all $m$.

The category $\mathcal{O}\text{Coh}(\hat{\mathcal{D}}^{(1)})$ is defined similarly except that we use $\hat{\mathcal{D}}^{(m)}_{X}$-module structures in place of analytic stratifications. In the following we
will write $\hat{\mathcal{D}}_{O(m)Q}^{(m)}$ for $\hat{\mathcal{D}}_{O_{\hat{\mathcal{X}}}^{m}}^{(m)}$. An object of $\mathcal{O}Coh(\mathcal{O}_{\hat{\mathcal{X}}}^{an})$ is a sequence of pairs $(M^{(m)}, \phi_{m})$ where $M^{(m)}$ is a coherent $\mathcal{O}_{\hat{\mathcal{X}}}^{(m)}$-module with the structure of a topologically quasi-nilpotent left $\hat{\mathcal{D}}_{O(m)Q}^{(m)}$-module, and

$$\phi_{m} : \hat{\mathcal{D}}_{O(m)Q}^{(m)} \otimes _{\hat{\mathcal{D}}_{O(m+1)Q}^{(m)}} M^{(m+1)} \simeq M^{(m)}$$

is an isomorphism of left $\hat{\mathcal{D}}_{O(m)Q}^{(m)}$-modules. A morphism $(M^{(m)}, \phi_{m})_{m} \to (M'^{(m)}, \phi'^{m})_{m}$ is a set of $\hat{\mathcal{D}}_{O(m)Q}^{(m)}$-linear maps $M^{(m)} \to M'^{(m)}$ compatible with $\phi_{m}$.

6.2.3 Lemma The categories $\text{Isoc}(X/K)$ and $\text{CStrat}(X/K)$ are equivalent.

Proof. This follows from the example worked out in section 5.5.4 whose notation we resume. If in 5.5.4.1 and 5.5.4.2 we replace $m$ by $m + 1$, the former is the local description of $\mathcal{X}^{an}_{m}$, i.e. the closed tube $[\mathcal{X}]_{p^{-(m+2)}} \subset \mathcal{X}^{an}$ and the latter is the closed tube $[\Delta]_{p^{-(m+2)}} \subset \mathcal{X}^{an}_{X \times \mathcal{X}}$ of the diagonal. The tubes $[\mathcal{X}]_{p^{-(m+2)}}$ (resp. $[\Delta]_{p^{-(m+2)}}$) for $m \gg 0$ form an admissible covering of the open tubes $|X|_{X}$ (resp. $|\Delta|_{X \times \mathcal{X}}$) for $m \gg 0$. Thus if $(M^{(m)}, \chi_{m})$ is an object of $\text{CStrat}(X/K)$, the $\chi_{m}$ and their compatibility describe a locally sheaf on $|\mathcal{X}|_{X}$ with a convergent connection. Conversely if a locally free sheaf $(\mathcal{M}, \nabla)$ with convergent connection is given we set $M^{(m)} = \Gamma(\mathcal{X}^{an}_{m}, M^{(m)})$ and take $\chi_{m}$ to be the level $m$ analytic stratification induced by $\nabla$. 

6.2.4 Lemma The categories $\text{Coad}^{\dag}(\mathcal{O}_{\hat{\mathcal{X}}}^{an})$ and $\mathcal{O}Coh(\hat{\mathcal{D}}_{\mathcal{X}}^{(j)})$ are equivalent.

Proof. Suppose first that $\mathcal{M}$ is an object of $\text{Coad}^{\dag}(\mathcal{O}_{\hat{\mathcal{X}}}^{an})$; since $\mathcal{M}$ is a coadmissible $\mathcal{O}_{\hat{\mathcal{X}}}^{an}$-module,

$$M^{(m)} = \mathcal{O}^{(m)}_{\hat{\mathcal{X}}} \otimes _{\mathcal{O}_{\hat{\mathcal{X}}}^{an}} \mathcal{M}$$

is a coherent $\mathcal{O}_{\hat{\mathcal{X}}}^{(m)}$-module for all $m \geq m_{0}$. The left $\hat{\mathcal{D}}_{\mathcal{X}/V}^{(j)}$-module structure of $\mathcal{M}$ induces a left $\hat{\mathcal{D}}_{\mathcal{X}/V}^{(m)}$-module structure on $M^{(m)}$, and in fact a left $\hat{\mathcal{D}}_{\hat{\mathcal{O}}_{\hat{\mathcal{X}}}^{(m)}Q}^{(m)}$-module structure since the $\hat{\mathcal{D}}_{\mathcal{X}/V}^{(m)}$-module structure of $M^{(m)}$ is compatible with the $\mathcal{O}_{\hat{\mathcal{X}}}^{(m)}$-module structure. We claim that it is a topologically quasi-nilpotent $\hat{\mathcal{D}}_{\hat{\mathcal{O}}_{\hat{\mathcal{X}}}^{(m)}Q}^{(m)}$-module: from the definition of the $M^{(m)}$ we have

$$M^{(m)} \simeq \mathcal{O}^{(m)}_{\hat{\mathcal{X}}} \otimes _{\mathcal{O}_{\hat{\mathcal{X}}}^{an}} M^{(m+1)}$$

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and the $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-module structure of $M^{(m)}$ is induced by the $\hat{D}^{(m)}_{\mathcal{O}(m+1)\mathbb{Q}}$-module structure of $M^{(m+1)}$, which by corollary 4.4.7 is quasi-nilpotent. The map $\phi_m$ for $m \geq m_0$ is the one induced by base-change, and is clearly a $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-linear isomorphism. The resulting sequence $(M^{(m)}, \phi_m)$ is then an object of $\mathcal{O}\text{Coh}(\hat{D}^{(i)}_{X})$.

Suppose conversely that $(M^{(m)}, \phi_m)$ is an object of $\mathcal{O}\text{Coh}(\hat{D}^{(i)}_{X})$. Since the $M^{(m)}$ are coherent $\mathcal{O}^{an}_{\mathcal{X}}$-modules and the $\phi_m$ are isomorphisms, $\mathcal{M} = \lim_{\leftarrow m} M^{(m)}$ is a coadmissible $\mathcal{O}^{an}_{\mathcal{X}}$-module. Since $M^{(m')}$ has a left $\hat{D}^{(m')}_{\mathcal{O}(m)\mathbb{Q}}$-module structure for all $m' \geq m$, $\mathcal{M}$ has a left $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-module structure for all $m$, and thus a left $\mathcal{D}^{\dagger}_{\mathcal{X}/\mathcal{V}}$-module structure. Then $(M^{(m)}, \phi_m)$ is an object of $\text{Coad}^{\dagger}(\mathcal{O}^{an}_{\mathcal{X}})$.

6.2.5 Theorem Suppose $\mathcal{X}$, $\mathcal{V}$ and its $m$-PD-structure are as in the beginning of section 6.2 and $X/k$ is the closed fiber of $\mathcal{X}$. The category $\text{Coad}^{\dagger}(\mathcal{O}^{an}_{\mathcal{X}})$ is equivalent to the category $\text{Isoc}(X/K)$ of convergent isocrystals on $X$ with coefficients in $K$.

Proof. By the last two lemmas it suffices to produce an equivalence between $\text{CStrat}(X/K)$ and $\mathcal{O}\text{Coh}(\hat{D}^{(i)}_{X})$. Suppose first that $(M^{(m)}, \phi_m)_m$ is an object of $\mathcal{O}\text{Coh}(\hat{D}^{(i)}_{X})$. Since each $M^{(m)}$ is a topologically quasi-nilpotent $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-module we get an $m$-HPD-stratification $\chi_m$ of $M^{(m)}$. We apply to the $\chi_m$ the construction of section 5.5.2, i.e. the base-change of the $m$-HPD-stratification by the $\alpha_m$ of that section; the result is a level $m$ analytic stratification of $M^{(m)}$. That these stratifications are compatible with the $\phi_m$ follows from the $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-linearity of the base change map. Therefore $(M^{(m)}, \chi_m, \phi_m)_m$ is an object of $\text{CStrat}(X/K)$.

Suppose conversely that $(M^{(m)}, \chi_{m+1}, \phi_{m+1})_m$ is an object of $\text{CStrat}(X/K)$. We now apply the base change by $\beta_m$ of section 5.5.2 obtaining an $m$-HPD-stratification of $M^{(m)}$, whose compatibility with $\phi_m$ is proven in the same way as before. This gives $M^{(m)}$ the structure of a topologically nilpotent $\hat{D}^{(m)}_{\mathcal{O}(m)\mathbb{Q}}$-module. As these structures are compatible with the $\phi_m$, we have an object of $\mathcal{O}\text{Coh}(\hat{D}^{(i)}_{X})$.

The functoriality of these constructions is easily verified. It remains to be checked that these constructions are inverse to each other. This follows from proposition 5.4.2: the composites $\alpha_m \beta_m$ and $\beta_m \alpha_{m+1}$ induce the canonical change-of-level maps for the respective stratifications.
It follows from the theorem that \( \text{Coad}^\dagger(\mathcal{O}^\text{an}_X) \) is independent up to equivalence of the choice of the ideal of definition \( J \subset \mathcal{O}_X \). In fact it is not hard to show directly that the ring \( \mathcal{O}^\text{an}_X \) and its left \( \mathcal{D}^b \) module structure is independent of this choice; we will leave this to the reader.

### 6.3 Frobenius descent.

Suppose now that \( \mathcal{V} \) is a complete discrete valuation of mixed characteristic \( p > 0 \), fraction field \( K \) and residue field. Suppose also that \( X/k \) is a separated \( k \)-scheme of finite type. Denote by \( F_{X/k} : X \rightarrow X^{(q)} \) the relative \( q \)th power Frobenius of \( X \). We will use the comparison theorem to prove the following:

#### 6.3.1 Theorem

The pullback functor

\[
\text{Isoc}(X^{(q)}/K) \rightarrow \text{Isoc}(X/K)
\]

is an equivalence.

**Proof.** Since both categories are of local nature we can assume, first that \( X \) is the closed fiber of a formal \( \mathcal{V} \)-scheme formally of finite type over \( \mathcal{V} \), and that that there is a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
F_{X/k} \downarrow & & \downarrow F \\
X^{(q)} & \longrightarrow & X''
\end{array}
\]

for some formal \( \mathcal{V} \)-scheme formally of finite type over \( \mathcal{V} \), where the horizontal maps are the inclusions of the closed fibers and \( F \) is finite and flat. By theorem [6.2.5] and lemma [6.2.4] it will suffice to show that the pullback functor

\[
\mathcal{O}\text{Coh}(\mathcal{O}^\text{an}_{X'}) \rightarrow \mathcal{O}\text{Coh}(\mathcal{O}^\text{an}_X)
\]

is an equivalence. Choose an ideal of definition \( J \subset \mathcal{O}_X \) and note that \( J' := F^*J \subset \mathcal{O}_{X'} \) is also an ideal of definition. Let \( J_m \subset \mathcal{O}_X \) and \( X_m \) be as in [6.2.2] but we will write \( \mathcal{O}_{X'}^{(m)} \) for \( \mathcal{O}_{X'}[J_m/(p)] \) since \( X' \) is also in the picture. Note that

\[
J'_m := (J')^{(p^{m+2})} + (p) = F^*J_m
\]

and

\[
\mathcal{O}_{X'}^{(m)} := \mathcal{O}_{X'}[J'_m/(p)] \simeq F^*\mathcal{O}^{(m)}.
\]
The theorem 4.5.4 on Frobenius descent with coefficients shows that $F^*$ induces an equivalence of the category of left $\hat{D}_{X'/V}^{(m)}$-modules with the category of left $\hat{D}_{X'/V}^{(m+s)}$-modules. Furthermore $F$ is faithfully flat since it is finite and flat, so an $O_{X'}^{(m)}$-module $M$ is coherent if and only if $F^*M$ is a coherent $O_{X'}^{(m+s)}$-module. It follows that $F^*$ induces an equivalence of $O\text{Coh}(O_{X'}^{an})$ with $O\text{Coh}(O_{X'}^{an})$, as required.

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