On the fundamental group of Hom($\mathbb{Z}^k$, $G$)

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Abstract Let $G$ be a compact Lie group. Consider the variety $\text{Hom}(\mathbb{Z}^k, G)$ of representations of $\mathbb{Z}^k$ into $G$. We can see this as a based space by taking as base point the trivial representation $1$. The goal of this paper is to prove that $\pi_1(\text{Hom}(\mathbb{Z}^k, G))$ is naturally isomorphic to $\pi_1(G)^k$.

1 Introduction

Let $G$ be a compact Lie group. The set $\text{Hom}(\mathbb{Z}^k, G)$ can naturally be identified with the subset of $G^k$ consisting of ordered commuting $k$-tuples in $G$. In this way, $\text{Hom}(\mathbb{Z}^k, G)$ can be given a topology as a subspace of $G^k$ making it into a, possibly singular, real analytic variety. Let $1 \in \text{Hom}(\mathbb{Z}^k, G)$ be the trivial representation. Then $\text{Hom}(\mathbb{Z}^k, G)$ can be seen as a based space with base point $1$. As announced in the abstract, the goal of this paper is to prove the following result:

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**Theorem 1.1** Let $G$ be a compact Lie group. Then for every $k \geq 1$ there is a natural isomorphism

$$\pi_1(\text{Hom}(\mathbb{Z}^k, G)) \cong \pi_1(G)^k.$$  

Theorem 1.1 is due to Torres-Giese and Sjerve [10] in the case that $G$ is either SO(3), SU(2) or U(2). In their work, Torres-Giese and Sjerve determine the topological type of $\text{Hom}(\mathbb{Z}^k, G)$ and compute its fundamental group via the Seifert-van Kampen Theorem. Our approach is as follows. Let $G$ be a compact Lie group and denote by $G_0$ the connected component of $G$ containing the unit $1_G$. The natural inclusion $i : G_0 \hookrightarrow G$ gives rise to a map

$$\text{Hom}(\mathbb{Z}^k, G_0) \xrightarrow{i^*} \text{Hom}(\mathbb{Z}^k, G)$$

that induces an isomorphism of fundamental groups. Therefore, we can assume without loss of generality that $G$ is a compact connected Lie group. Observe that in general $\text{Hom}(\mathbb{Z}^k, G)$ is not connected even if $G$ is connected and simply connected. We denote by $\text{Hom}(\mathbb{Z}^k, G)_\mathbb{1}$ the connected component of $\text{Hom}(\mathbb{Z}^k, G)$ containing the trivial representation $\mathbb{1}$. We are thus interested in computing $\pi_1(\text{Hom}(\mathbb{Z}^k, G)_\mathbb{1})$. Fix $T$ a maximal torus in $G$, let $N(T)$ be the normalizer of $T$ in $G$ and $W = N(T)/T$ the associated Weyl group. Following Baird [2], we consider the continuous surjection

$$\sigma_k : G/T \times_W T^k = G \times_{N(T)} T^k \to \text{Hom}(\mathbb{Z}^k, G)_\mathbb{1}$$

$$(g, t_1, \ldots, t_k) \mapsto (gt_1g^{-1}, \ldots, gt_kg^{-1}).$$

When $k = 1$ this map corresponds to the classical map given by conjugation

$$\sigma_1 : G/T \times_W T \to G$$

$$(g, t) \mapsto gtg^{-1}.$$  

If $G^{\text{reg}}$ denotes the subspace of regular elements in $G$, then Weyl’s covering theorem (see [3, Theorem 3.7.2]) asserts that the restriction of $\sigma_1$ to $G/T \times_W (G^{\text{reg}} \cap T)$ is a $G$-equivariant real-analytic diffeomorphism onto $G^{\text{reg}}$. An analogous result is true in general for $k \geq 2$. The map $\sigma_k$ is the main tool that we will use to compute $\pi_1(\text{Hom}(\mathbb{Z}^k, G)_\mathbb{1})$. Using a general position argument we show that the map $\sigma_k$ is $\pi_1$-surjective. Then, under the additional assumption that $G$ is simply connected, we show that every element in a suitable generating set of $\pi_1(G \times_{N(T)} T^k)$ is in the kernel of $\pi_1(\sigma_k)$. At this point we will have proved Theorem 1.1 in the case that $\pi_1(G)$ is trivial. To finish, we reduce the general case to the simply connected case by passing to a suitable cover $\tilde{G}$ of the group $G$ and studying the relation between $\text{Hom}(\mathbb{Z}^k, G)_\mathbb{1}$ and $\text{Hom}(\mathbb{Z}^k, \tilde{G})_\mathbb{1}$.

In the course of the proof of Theorem 1.1 we will need in a key way that $G$ is compact because otherwise the map $\sigma_k$ above will fail to be surjective. However, we would like to mention that the two last authors of this note have proved in [8] that if $G$ is the group of complex points of a connected reductive algebraic group and $K$ is a maximal compact subgroup, then the inclusion of $\text{Hom}(\mathbb{Z}^k, K)$ into $\text{Hom}(\mathbb{Z}^k, G)$ is a homotopy equivalence. Since also $G$ and $K$ are homotopy equivalent, we deduce from Theorem 1.1:

**Corollary 1.2** Let $G$ be the group of complex points of a connected reductive algebraic group. For every $k \geq 1$ there is a natural isomorphism $\pi_1(\text{Hom}(\mathbb{Z}^k, G)) \cong \pi_1(G)^k$.  

This paper is organized as follows. In Sect. 2 we prove Theorem 1.1 for simply connected groups. In Sect. 3 we extend this to general compact Lie groups. Finally in Sect. 4 we discuss some examples showing that Theorem 1.1 fails if the base point of $\text{Hom}(\mathbb{Z}^k, G)$ is no longer assumed to be in $\text{Hom}(\mathbb{Z}^k, G)_\mathbb{1}$.
Remark  The space \( \text{Hom}(\mathbb{Z}, G) \) is naturally homeomorphic to \( G \); hence, Theorem 1.1 is trivially satisfied for \( k = 1 \). Therefore, we will assume from now on that \( k \geq 2 \). Also, note that Theorem 1.1 holds trivially for finite groups. Thus we can also assume that \( G \) has rank at least 1.

2 The simply connected case

In this section we prove Theorem 1.1 for the particular case where \( G \) is a simply connected Lie group.

From now on fix a compact connected Lie group \( G \) and \( T \) a maximal torus in \( G \). Let \( N(T) \) be the normalizer of \( T \) in \( G \) and denote by \( W = N(T)/T \) the Weyl group associated to \( T \). Let \( N(T) \) be the normalizer of \( T \) in \( G \) and denote by \( W = N(T)/T \) the Weyl group associated to \( T \).

To \( g \in G \) and \( t_1, \ldots, t_k \in T \) we can associate the representation

\[
\rho(g, t_1, \ldots, t_k) : \mathbb{Z}^k \to G
\]

\[
(n_1, \ldots, n_k) \mapsto g t_1^{n_1} \cdots t_k^{n_k} g^{-1}.
\]

This way we obtain a continuous map

\[
\tilde{\sigma}_k : G \times T^k \to \text{Hom}(\mathbb{Z}^k, G)
\]

\[
(g, t_1, \ldots, t_k) \mapsto \rho(g, t_1, \ldots, t_k)
\]

which is constant along the orbits of the diagonal action of \( N(T) \) on \( G \times T^k \). Thus we have an induced map

\[
\sigma_k : G \times_{N(T)} T^k \to \text{Hom}(\mathbb{Z}^k, G).
\]

Observe that

\[
G \times_{N(T)} T^k = G/T \times_W T^k
\]

is a real-analytic manifold and that the map \( \sigma_k \) is a morphism of real-analytic spaces. Moreover, since \( W \) acts freely on \( G/T \), the projection onto the first factor induces a fibration sequence of the form

\[
T^k \to G/T \times_W T^k \to G/N(T).
\]

Let \( \text{Hom}(\mathbb{Z}^k, G)_1 \) be the connected component of \( \text{Hom}(\mathbb{Z}^k, G) \) containing the trivial representation

\[
1 : \mathbb{Z}^k \to G
\]

\[
(n_1, \ldots, n_k) \mapsto 1_G.
\]

In [2], Baird studied properties of the map \( \sigma_k \). For instance, by [2, Lemma 4.2] the map \( \sigma_k \) is a surjection onto \( \text{Hom}(\mathbb{Z}^k, G)_1 \), and this space is precisely the subspace of \( \text{Hom}(\mathbb{Z}^k, G) \) consisting of commuting \( k \)-tuples contained in some maximal torus of \( G \). Also, by [2, Theorem 4.3], the fibers of \( \sigma_k \) have the cohomology of a point if one has, for example, coefficients over a field of characteristic 0. From this we deduce in particular that the fibers of \( \sigma_k \) are connected. These facts are summarized in the following proposition:

**Proposition 2.1** The space \( \text{Hom}(\mathbb{Z}^k, G)_1 \) is precisely the subspace of \( \text{Hom}(\mathbb{Z}^k, G) \) of commuting \( k \)-tuples contained in some maximal torus of \( G \), and the map \( \sigma_k : G \times_{N(T)} T^k \to \text{Hom}(\mathbb{Z}^k, G)_1 \) is surjective and has connected fibers.
The map $\sigma_k$ is certainly not injective; however, there is a large set on which it has this desirable property. Recall that the action of $N(T)$ on $T$ by conjugation induces the action $W \sim T$. We denote by $(T^k)^* \subset T^k$ consisting of all $k$-tuples $(t_1, \ldots, t_k)$ with the property that the trivial element is the only element in $W$ which fixes $t_i$ for $i = 1, \ldots, k$. Clearly, $T^k \setminus (T^k)^*$ is a compact analytic subset of co-dimension at least $k$ because $(T^k)^*$ contains the subspace of $k$-tuples $(t_1, \ldots, t_k) \in T^k$ for which at least one of the $t_i$’s is regular. Therefore we obtain the following:

**Lemma 2.2** The complement of $G \times_{N(T)} (T^k)^*$ in $G \times_{N(T)} T^k$ is a compact analytic subset of co-dimension at least $k \geq 2$.

The open set $G \times_{N(T)} (T^k)^*$ will be important to us because it is homeomorphic to a very large subset of $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$:

**Lemma 2.3** The restriction of the map $\sigma_k$ to $G \times_{N(T)} (T^k)^*$ is a homeomorphism onto its image.

**Proof** Note that $G \times_{N(T)} T^k$ is a compact space and $\sigma_k : G \times_{N(T)} T^k \to \text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$ is a continuous map. In particular, $\sigma_k$ is a closed map. This shows that the restriction of $\sigma_k$ to the open subspace $G \times_{N(T)} (T^k)^*$ is a continuous, closed and surjective map onto its image. Therefore it suffices to see that restriction of $\sigma_k$ to $G \times_{N(T)} (T^k)^*$ is injective. This is clear by definition of $(T^k)^*$.

**Definition 1** Define $\mathcal{H}^\ast$ to be the image of $G \times_{N(T)} (T^k)^*$ under the map $\sigma_k$. We will refer to $\mathcal{H}^\ast$ as the regular part of $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$. Also define $\mathcal{H}^\circ = \text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}} \setminus \mathcal{H}^\ast$, the complement of the regular part in $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$. We will refer to $\mathcal{H}^\circ$ as the singular part of $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$.

**Remark** The subspace $\mathcal{H}^\circ$ is precisely the set of all representations $\rho : \mathbb{Z}^k \to G$ whose image has a maximal torus as its Zariski closure; we will not need this fact.

**Lemma 2.4** The singular part $\mathcal{H}^\circ$ is nowhere dense and does not disconnect connected open subsets of $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$.

**Proof** The fact that $\mathcal{H}^\circ$ is nowhere dense follows from the fact that $\sigma_k$ is surjective and that the preimage $G \times_{N(T)} (T^k)^*$ of its complement $\mathcal{H}^\ast = \text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}} \setminus \mathcal{H}^\circ$ is dense in $G \times_{N(T)} T^k$.

We prove now that $\mathcal{H}^\circ$ does not separate any connected open set $U \subset \text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$. Suppose that we have such a set $U$; if $U \cap \mathcal{H}^\circ = \emptyset$ then there is nothing to prove, so also suppose that this is not the case. Then the connectivity of the fibers of $\sigma_k$ implies that the preimage $\sigma_k^{-1}(U)$ of $U$ under the surjective map $\sigma_k$ is connected as well. On the other hand,

$$\sigma_k^{-1}(\mathcal{H}^\circ) = (G \times T^k)/N(T) \setminus (G \times (T^k)^*)/N(T)$$

has co-dimension at least $k \geq 2$ by Lemma 2.2. A set of co-dimension at least 2 in a manifold does not disconnect connected open sets, and hence $\sigma_k^{-1}(U \setminus \mathcal{H}^\circ)$ is connected. As $\sigma_k^{-1}(U \setminus \mathcal{H}^\circ)$ and $U \setminus \mathcal{H}^\circ$ are homeomorphic, by Lemma 2.3, we have that $\mathcal{H}^\circ$ does not disconnect connected open sets.

Recall that $\text{Hom} (\mathbb{Z}^k, G)_{\mathbb{I}}$ is real analytic and that, as the image of the compact analytic set

$$(G \times_{N(T)} T^k) \setminus (G \times_{N(T)} (T^k)^*)$$
under the analytic map $\sigma_k$, the subset $\mathcal{H}^s$ is closed and analytic. In particular, by the Whitney stratification theorem \cite{9,11}, $\text{Hom}(\mathbb{Z}^k, G)_1$ admits the structure of a simplicial complex in such a way that $\mathcal{H}^s$ is a subcomplex. The following lemma gives the reason we proved Lemma 2.4 at all:

**Lemma 2.5** Let $X$ be a compact simplicial complex and $Y \subset X$ a subcomplex. Suppose that $X \setminus Y$ is dense, and that $Y$ does not separate any connected open set in $X$. If $x_0 \in X \setminus Y$ is the basepoint, then $\pi_1(X \setminus Y, x_0)$ surjects onto $\pi_1(X, x_0)$.

**Proof** Suppose that we have a loop $\gamma$ in $X$ based at $x_0$; we want to homotope $\gamma$ away from $Y$. Note that, as $Y$ is a nowhere dense subcomplex, we can homotope $\gamma$ so that it meets $Y$ in a finite number of points; suppose that $\gamma$ has been chosen to minimize this number.

Seeking a contradiction, assume that $\gamma$ meets $Y$ at some point $p$, and let $U \subset X$ be a small open contractible neighborhood of $p$. Let also $J \subset U$ be the proper subarc of $\gamma$ containing $p$ and let $p_\pm \notin Y$ be the endpoints of $J$. Since $U \setminus Y$ is connected, we can connect $p_\pm$ inside $U \setminus Y$ by some arc $I$. Since $U$ is contractible, both $I$ and $J$ are homotopic to each other in $U$ while fixing $p_\pm$. It follows that we can replace the curve $\gamma$ by a homotopic curve which meets $Y$ in a point less than $\gamma$ did. This is not possible by the choice of $\gamma$, so the lemma follows. \hfill \Box

**Remark** Lemma 2.5 can be proved in greater generality, but we will only need the version presented here.

From Lemmas 2.4 and 2.5, it follows that $\pi_1(\mathcal{H}^r)$ surjects onto $\pi_1(\text{Hom}(\mathbb{Z}^k, G)_1)$ if we take as base point some element $x_0 \in \mathcal{H}^r$. On the other hand, $\mathcal{H}^r$ is the homeomorphic image of $G \times_{N(T)} (T^k)^*$ under the map $\sigma_k$ by Lemma 2.3. Since the fundamental group of a connected space does not depend, up to isomorphism, on the chosen base point we deduce the following:

**Corollary 2.6** If $G$ is a compact connected Lie group, the map

$$\sigma_k : G \times_{N(T)} T^k \to \text{Hom}(\mathbb{Z}^k, G)_1$$

is $\pi_1$-surjective. \hfill \Box

Our next goal is to prove that the homomorphism

$$\pi_1(\sigma_k) : \pi_1(G \times_{N(T)} T^k) \to \pi_1(\text{Hom}(\mathbb{Z}^k, G)_1)$$

is trivial if we further assume that $G$ is simply connected. Recall that $\text{Hom}(\mathbb{Z}^k, G)_1$ is a based space, with base point $1$. We can also view $G \times_{N(T)} T^k$ as a based space by taking as base point the class representing the element $(1_G, \ldots, 1_G) \in G \times T^k$. With this choice of base points, the map $\sigma_k$ is a based map.

We will show that $\pi_1(\sigma_k)$ is the trivial map by showing that a suitable set of generators of $\pi_1(G \times_{N(T)} T^k)$ is in the kernel of $\pi_1(\sigma_k)$. In order to describe such a set of generators, recall that projection onto the first factor $p_1 : G \times_{N(T)} T^k \to G / N(T)$ induces a fibration sequence of the form

$$T^k \to G \times_{N(T)} T^k \xrightarrow{p_1} G / N(T). \quad (2.1)$$

The tail end of the associated homotopy long exact sequence is the following exact sequence:

$$\pi_1(T^k) \to \pi_1(G \times_{N(T)} T^k) \to \pi_1(G / N(T)) \to 1. \quad (2.2)$$
Observe that the map $p_1$ admits a section
\[ s : G/N(T) \to G \times_{N(T)} (T^k) \]
where $1_G$ is the unit element in $G$, and $[\cdot]$ denotes the class of the corresponding element in $G/N(T)$ and $G \times_{N(T)} T^k$, respectively. This section gives a splitting of the sequence (2.2). We deduce:

**Lemma 2.7** $\pi_1(G \times_{N(T)} T^k)$ is generated by $\pi_1([1_G] \times T^k)$ and by $\pi_1(s(G/N(T)))$. $\square$

At this point we would like to notice that the composition of the section $s$ with the map $\sigma_k$ is the constant map; the image is namely the trivial representation $1$. It follows that
\[ \pi_1(s(G/N(T))) \subset \text{Ker}(\pi_1(\sigma_k)). \]

In particular, by Lemma 2.7, in order to show that $\pi_1(\sigma_k)$ is trivial it suffices to show that the restriction of the map $\sigma_k$ to the fiber $[1_G] \times T^k$ is trivial in $\pi_1$. We do this next. Identifying
\[ \pi_1(T^k) = \pi_1(T) \times \cdots \times \pi_1(T) \]
we see that $\pi_1(T^k)$ is generated by loops which are constant on each component but one. More concretely, for every $1 \leq a \leq k$ let
\[ i_a : T \to T \times \cdots \times T \]
\[ x \mapsto (1_G, \ldots, x, \ldots, 1_G). \]
be the natural inclusion of $T$ into the $a$-th factor of $T \times \cdots \times T$. Then $\pi_1(T^k)$ is generated by loops of the form $\eta(t) = i_a(\gamma(t))$, where $\gamma : [0, 1] \to T$ is a loop in $T$ based at $1_G$. Note that the image of a loop of the form $i_a(\gamma)$ under $\sigma_k$ is a loop $(\rho_t)$ in $\text{Hom}(\mathbb{Z}^k, G)_1$ where each $\rho_t = \sigma_k(\eta(t))$ is given by
\[ \rho_t(n_1, \ldots, n_k) = i_a(\gamma(t)^n_a), \]
where here by abuse of notation we also denote by
\[ i_a : G \to \text{Hom}(%001\frac{Z^k}{G}, G) \]
\[ g \mapsto (1_G, \ldots, g, \ldots, 1_G) \]
the inclusion of $G$ into the $a$-th factor of $\text{Hom}(\mathbb{Z}^k, G) \subset G^k$. By assumption, and this is the first and only time that we use this assumption, $\pi_1(G)$ is trivial. Hence, the loop $\gamma(t)$ can be contracted in $G$ to the trivial loop. Let
\[ [0, 1] \times [0, 1] \to G \]
\[ (s, t) \mapsto \gamma^s(t) \]
be such a homotopy with $\gamma^0(t) = \gamma(t)$ and with $\gamma^1(t) = 1_G$ for all $t$. Consider the homotopy
\[ [0, 1] \times [0, 1] \to \text{Hom}(\mathbb{Z}^k, G)_1, \]
\[ (t, s) \mapsto \rho_t^s \]
where
\[ \rho_t^s(n_1, \ldots, n_k) = i_a(\gamma^s(t)^{n_a}). \]
This homotopy begins with the loop \((\rho_t) = \sigma_k(\eta)\) and ends with the constant curve with image the trivial representation \(1\). We have proved that the restriction of \(\sigma_k\) to the fiber \(\{1_G\} \times T^k\) of the fibration

\[ G \times_{N(T)} T^k \to G/N(T) \]

is trivial in \(\pi_1\). Combining this fact with our earlier observations, we deduce that \(\pi_1(\sigma_k)\) is the trivial homomorphism. On the other, by Lemma 2.6, the map \(\pi_1(\sigma_k)\) is surjective. This proves that

\[ \pi_1\left(\text{Hom}(\mathbb{Z}^k, G)_1\right) = 1. \]

In conclusion, we have proved the following theorem:

**Theorem 2.8** Let \(G\) be a simply connected compact Lie group. If \(\text{Hom}(\mathbb{Z}^k, G)\) has base point \(1\), then

\[ \pi_1(\text{Hom}(\mathbb{Z}^k, G)) = 1. \]

This is precisely Theorem 1.1 in the case where \(G\) is simply connected.

### 3 The general case

In this section we prove Theorem 1.1 for any compact Lie group \(G\).

To begin with, suppose that \(G\) is a compact Lie group. Denote by \(G_0\) the connected component of \(G\) containing \(1_G\). As mentioned in the introduction, the natural inclusion \(i : G_0 \hookrightarrow G\) gives rise to a map

\[ \text{Hom}(\mathbb{Z}^k, G_0) \overset{i_*}{\to} \text{Hom}(\mathbb{Z}^k, G) \]

that induces an isomorphism of \(\pi_1\) for any \(k\). Because of this we only need to consider the case where \(G\) is a compact connected Lie group. Suppose then that \(G\) is such a Lie group. By [5, Theorem 6.19] we can write \(\tilde{G} = \tilde{G}/K\), where \(K\) is a finite subgroup in the center of \(\tilde{G}\), and where

\[ \tilde{G} = (S^1)^r \times G_1 \times \cdots \times G_s \]

for some compact simply connected and simple Lie groups \(G_1, \ldots, G_s\). If we write

\[ H = G_1 \times \cdots \times G_s \]

then \(\tilde{G} = (S^1)^r \times H\) and \(H\) is a compact and simply connected Lie group. Notice that the projection map

\[ p : \tilde{G} \to G \]

is both a homomorphism and a covering map, with covering group \(K\); in particular, it is a local isomorphism. In [4, Lemma 2.2], Goldman showed that if \(\pi\) is a finitely generated group and \(p : G' \to G\) is a local isomorphism, then composition with \(p\) defines a continuous map

\[ p_* : \text{Hom}(\pi, G') \to \text{Hom}(\pi, G), \]

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such that the image of $p_*$ is a union of connected components of $\text{Hom}(\pi, G)$. Moreover, if $Q$ is a connected component in the image of $p_*$, then the restriction of $p_*$

$$(p_*)|_{p_*^{-1}(Q)} : p_*^{-1}(Q) \to Q$$

is a covering map, with covering group $\text{Hom}(\pi, K)$. We can apply this to the particular case of $\pi = \mathbb{Z}^k$ and $Q = \text{Hom}(\mathbb{Z}^k, G)$. Thus we obtain a covering map

$$p^{-1}(\text{Hom}(\mathbb{Z}^k, G)_1) \to \text{Hom}(\mathbb{Z}^k, G)_1$$

with covering group $K^k = \text{Hom}(\mathbb{Z}^k, K)$. For this covering map, the action of $K^k$ on $p^{-1}(\text{Hom}(\mathbb{Z}^k, G)_1)$ corresponds to left component-wise multiplication. By Lemma 2.1, the space $\text{Hom}(\mathbb{Z}^k, \tilde{G})_1$ is precisely the subspace of $\text{Hom}(\mathbb{Z}^k, \tilde{G})$ of commuting $k$-tuples contained in some maximal torus of $\tilde{G}$. Using this and the fact that in any compact Lie group the center is contained in any maximal torus (see for example [7, Corollary 4.47]), it follows that

$$p^{-1}(\text{Hom}(\mathbb{Z}^k, G)_1) = \text{Hom}(\mathbb{Z}^k, \tilde{G})_1.$$

This shows that we have a covering sequence

$$K^k \xrightarrow{i_*} \text{Hom}(\mathbb{Z}^k, \tilde{G})_1 \xrightarrow{p_*} \text{Hom}(\mathbb{Z}^k, G)_1.$$

The long exact sequence in homotopy associated to this covering map shows that there is a short exact sequence

$$1 \to \pi_1(\text{Hom}(\mathbb{Z}^k, \tilde{G})_1) \xrightarrow{p_*} \pi_1(\text{Hom}(\mathbb{Z}^k, G)_1) \xrightarrow{\delta} K^k \to 1. \quad (3.1)$$

On the other hand there is a natural homeomorphism

$$\text{Hom}(\mathbb{Z}^k, \tilde{G})_1 \cong \text{Hom}(\mathbb{Z}^k, (S^1)^r) \times \text{Hom}(\mathbb{Z}^k, H)_1.$$

In particular

$$\pi_1(\text{Hom}(\mathbb{Z}^k, \tilde{G})_1) \cong \pi_1(((S^1)^r)^k) \times \pi_1(\text{Hom}(\mathbb{Z}^k, H)_1).$$

As $H$ is a compact and simply connected Lie group, by Theorem 2.8 we have $\pi_1(\text{Hom}(\mathbb{Z}^k, H)_1) = 1$. Thus

$$\pi_1(\text{Hom}(\mathbb{Z}^k, \tilde{G})_1) \cong \pi_1(((S^1)^r)^k) \cong (\mathbb{Z}^r)^k,$$

with an isomorphism induced by the inclusion map

$$(S^1)^r \to (S^1)^r \times H = \tilde{G}\]

$$x \mapsto (x, 1).$$

This shows that (3.1) is a short exact sequence of the form

$$1 \to (\mathbb{Z}^r)^k \xrightarrow{p_*} \pi_1(\text{Hom}(\mathbb{Z}^k, G)_1) \xrightarrow{\delta} K^k \to 1. \quad (3.2)$$

On the other hand, we also have a covering space $K \xrightarrow{i} \tilde{G} \xrightarrow{p} G$ and the long exact sequence in homotopy associated to this sequence gives a short exact sequence

$$1 \to \pi_1(\tilde{G}) \xrightarrow{p_*} \pi_1(G) \xrightarrow{\delta} K \to 1.$$
By taking the direct sum $k$-copies of this sequence we obtain a short exact sequence

$$1 \rightarrow (\mathbb{Z}^r)^k \xrightarrow{(p_x)^k} (\pi_1(G))^k \xrightarrow{(\delta)^k} K^k \rightarrow 1.$$  (3.3)

We claim that we can find a natural homomorphism $h_G$ making the following diagram commuting:

$$
\begin{array}{cccccc}
1 & \rightarrow & (\mathbb{Z}^r)^k & \xrightarrow{(p_x)^k} & (\pi_1(G))^k & \xrightarrow{(\delta)^k} K^k & \rightarrow & 1 \\
\downarrow{id} & & \downarrow{h_G} & & \downarrow{id} & & \downarrow{id} \\
1 & \rightarrow & (\mathbb{Z}^r)^k & \xrightarrow{p_*} & \pi_1(\text{Hom}(\mathbb{Z}^k, G)_1) & \xrightarrow{\delta} K^k & \rightarrow & 1.
\end{array}
$$  (3.4)

Then by the five lemma it follows that

$$h_G : (\pi_1(G))^k \rightarrow \pi_1(\text{Hom}(\mathbb{Z}^k, G)_1)$$

is an isomorphism, hence proving Theorem 1.1.

To construct the homomorphism $h_G$, define for every $1 \leq a \leq k$

$$j_a : \pi_1(G) \rightarrow (\pi_1(G))^k$$

$$[\alpha] \mapsto (1, \ldots, [\alpha], \ldots, 1).$$

In other words, $j_a$ is the inclusion of $\pi_1(G)$ into the $a$-th factor of $(\pi_1(G))^k$. Notice that the elements in the image of $j_1, \ldots, j_k$ generate $(\pi_1(G))^k$, and thus it suffices to define $h_G$ on elements of the form $j_a([\alpha])$ for some $1 \leq a \leq k$ and some loop $\alpha : [0, 1] \rightarrow G$ based at $1_G$. For such elements define

$$h_G(j_a([\alpha])) = [i_a(\alpha)] \in \pi_1(\text{Hom}(\mathbb{Z}^k, G)),$$

where as before

$$i_a : G \rightarrow \text{Hom}(\mathbb{Z}^k, G)$$

$$g \mapsto (1_G, \ldots, g, \ldots, 1_G)$$

the inclusion of $G$ into the $a$-th factor of $\text{Hom}(\mathbb{Z}^k, G) \subset G^k$. In this way we obtain a well-defined homomorphism

$$h_G : (\pi_1(G))^k \rightarrow \pi_1(\text{Hom}(\mathbb{Z}^k, G)).$$

From the definition it follows at once that $h_G$ is a natural map. To see that diagram (3.4) commutes note that for every $1 \leq a \leq k$ we have a morphism of fibration sequences

$$
\begin{array}{cccccc}
K & \xrightarrow{i} & \tilde{G} & \xrightarrow{p} & G \\
\downarrow{i_a} & & \downarrow{i_a} & & \downarrow{i_a} \\
K^k & \xrightarrow{i_*} & \text{Hom}(\mathbb{Z}^k, \tilde{G})_1 & \xrightarrow{p_*} & \text{Hom}(\mathbb{Z}^k, G)_1.
\end{array}
$$

The naturality of the long exact sequence in homotopy shows that the corresponding diagram in homotopy groups commutes. This diagram is precisely the restriction of (3.4) onto the $a$-th factor. This proves the commutativity of (3.4).
4 Examples and general remarks

In this section we explore the situation in which the base point of $\text{Hom}(\mathbb{Z}^k, G)$ is no longer assumed to be in the path-connected component $\text{Hom}(\mathbb{Z}^k, G)_1$. For instance, our second example below shows that even if $G$ is simply connected, $\text{Hom}(\mathbb{Z}^k, G)$ may have connected components with non-trivial $\pi_1$.

To start, let $H$ be a compact connected Lie group. As pointed out above, the space $\text{Hom}(\mathbb{Z}^k, H)$ is not necessarily connected. This can be explained as follows. Suppose first that $H$ is not simply connected. Then $H$ can be written in the form $H = G/K$, where $G$ is the universal cover of $H$ and $K \subset G$ is a closed central subgroup. Let

$$p : G \to G/K = H$$

be the natural projection. Given a commuting sequence $(x_1, \ldots, x_k)$ in $H$ we can find a lifting $\tilde{x}_i$ of $x_i$ in $G$ for all $1 \leq i \leq k$. The sequence $(\tilde{x}_1, \ldots, \tilde{x}_k) \in G^k$ is not necessarily a commuting sequence. Instead, $[\tilde{x}_i, \tilde{x}_j] \in K = \text{Ker}(p) \subset Z(G)$. We call such a sequence a $K$-almost commuting sequence in $G$. Following [1], given a Lie group $G$ and a closed subgroup $K \subset Z(G)$, we denote by $B_k(G, K)$ the set of $K$-almost commuting $k$-tupels; that is, the set of sequences $(x_1, \ldots, x_k)$ such that $[x_i, x_j] \in K$ for all $1 \leq i, j \leq k$. The set $B_k(G, K)$ is given the subspace topology under the natural inclusion $B_k(G, K) \subset G^k$. It is easy to see that projection map $p : G \to G/K$ induces a $K^k$-principal bundle

$$p_* : B_k(G, K) \to \text{Hom}(\mathbb{Z}^k, G/K).$$

This shows that we can understand the space of commuting elements in $G/K$ by studying the space of $K$-almost commuting elements in $G$. For example, by keeping track of the different commutators of sequences in $B_k(G, K)$ this space can be broken down into a disjoint union of subspaces that are both open and closed in $B_k(G, K)$, hence a union of path-connected components. Moreover, the image of these components under the map $p_*$ provides different path-connected components of the space $\text{Hom}(\mathbb{Z}^k, G/K)$.

\textbf{Example 1} Given an integer $m \geq 1$ and any prime number $p$, consider $\text{SU}(p)^m$, the product of $m$ copies of $\text{SU}(p)$. Let $\Delta(\mathbb{Z}/p)$ be the diagonal inclusion of $\mathbb{Z}/p$ into the center of $\text{SU}(p)^m$. Define

$$G_{m,p} := \text{SU}(p)^m/\Delta(\mathbb{Z}/p).$$

Thus $G_{m,p}$ is the $m$-fold central product of $\text{SU}(p)$. The space of commuting elements in $G_{m,p}$ can be understood by studying the space of almost commuting elements in $\text{SU}(p)^m$. Indeed, let $E_p \subset \text{SU}(p)$ be the quaternion group $Q_8$ of order eight when $p = 2$ and the extraspecial $p$–group of order $p^3$ and exponent $p$ when $p > 2$. In [1] it was proved that for any $k \geq 1$ the space $\text{Hom}(\mathbb{Z}^k, G_{m,p})$ has

$$N(k, m, p) = \frac{p^{(m-1)(k-2)} (p^k - 1) (p^{k-1} - 1)}{p^2 - 1} + 1$$

path–connected components. One of these path-connected components is $\text{Hom}(\mathbb{Z}^k, G_{m,p})_1$ and all others are homeomorphic to

$$A_{m,p} := \text{SU}(p)^m/((\mathbb{Z}/p)^{m-1} \times E_p).$$

The path-connected components of $\text{Hom}(\mathbb{Z}^k, G_{m,p})$ that are homeomorphic to $A_{m,p}$ have the additional property that the centralizer in $G_{m,p}$ of any sequence in them is a finite group.
Let \( x \in \operatorname{Hom}(\mathbb{Z}^k, G_{m,p}) \) be a point which is taken as the base point of \( \operatorname{Hom}(\mathbb{Z}^k, G_{m,p}) \). Using Theorem 1.1 it follows that

\[
\pi_1(\operatorname{Hom}(\mathbb{Z}^k, G_{m,p}), x) \cong \pi_1(G_{m,p})^k \cong (\mathbb{Z}/p)^k
\]

whenever \( x \) lies in \( \operatorname{Hom}(\mathbb{Z}^k, G_{m,k})_1 \). On the other hand, if \( x \) lies in a path-connected component of \( \operatorname{Hom}(\mathbb{Z}^k, G_{m,p}) \) that is homeomorphic to \( A_{m,p} \), then since \( SU(p)^m \) is simply connected we have that

\[
\pi_1(\operatorname{Hom}(\mathbb{Z}^k, G_{m,p}), x) \cong (\mathbb{Z}/p)^{m-1} \times E_p.
\]

Note in particular that \( \pi_1(\operatorname{Hom}(\mathbb{Z}^k, G_{m,p}), x) \) is independent of \( k \) in this case.

Now let’s turn our attention to the case of a Lie group \( G \) that is assumed to be simply connected. Even in this situation the space \( \operatorname{Hom}(\mathbb{Z}^k, G) \) is not necessarily path-connected. If fact by [2, Theorem 4.1], if \( G \) is a compact simple Lie group such that \( \operatorname{Hom}(\mathbb{Z}^k, G) \) is path-connected for every \( k \geq 1 \), then \( G \) is either \( SU(m) \) or \( Sp(m) \) for some \( m \geq 1 \). Thus in general, the space \( \operatorname{Hom}(\mathbb{Z}^k, G) \) has many path-connected components. This can be seen as follows. Let \( x := (x_1, \ldots, x_k) \in \operatorname{Hom}(\mathbb{Z}^k, G) \) and consider the centralizer \( Z_G(x) \) of the \( k \)-tuple \( (x_1, \ldots, x_k) \) in \( G \). Let \( S \) be a maximal torus in \( Z_G(x) \). Proposition 2.1 shows that \( x \) lies in \( \operatorname{Hom}(\mathbb{Z}^k, G) \) if and only if \( S \) is a maximal torus in \( G \). Therefore the space \( \operatorname{Hom}(\mathbb{Z}^k, G) \) is not path-connected precisely when we can find a commuting \( k \)-tuple \( x \) such that \( Z_G(x) \) does not contain a maximal torus in \( G \). The following example, first studied by Kac and Smilga in [6], illustrates this possibility.

**Example 2** The space \( \operatorname{Hom}(\mathbb{Z}^3, \operatorname{Spin}(7)) \) has two path-connected components. One of these components is \( \operatorname{Hom}(\mathbb{Z}^3, \operatorname{Spin}(7))_1 \) the other component we denote by \( B_3 \). In [6], it was proved directly that in \( \operatorname{Spin}(7) \) there is a commuting triple \( (x_1, x_2, x_3) \), unique up to conjugation, such that any maximal torus in \( Z_{\operatorname{Spin}(7)}(x_1, x_2, x_3) \) has rank 0, thus explaining the existence of \( B_3 \). This can also be seen in the following way. As explained in [6], we can find an element \( x_1 \) in \( \operatorname{Spin}(7) \) such that

\[
Z_{\operatorname{Spin}(7)}(x_1) = (\operatorname{SU}(2))^3 / \Delta(\mathbb{Z}/2) = G_{3,2}.
\]

By the previous example, the space \( \operatorname{Hom}(\mathbb{Z}^2, G_{3,2}) \) has two different path-connected components. In particular, we can choose \( (x_2, x_3) \in \operatorname{Hom}(\mathbb{Z}^2, G_{3,2}) \) outside the path-connected component containing the trivial representation \( 1 \). As pointed out above, elements in this component have the additional property that \( Z_{G_{3,2}}(x_2, x_3) \) is a finite group. This shows that any maximal torus in \( Z_{\operatorname{Spin}(7)}(x_1, x_2, x_3) \) has rank 0, as any maximal torus in \( Z_{G_{3,2}}(x_2, x_3) \) already has rank 0, hence explaining the existence of an exotic path-connected component in \( \operatorname{Hom}(\mathbb{Z}^3, \operatorname{Spin}(7)) \). Moreover, the triple \( (x_1, x_2, x_3) \) is unique up to conjugation in \( \operatorname{Spin}(7) \). This shows that the conjugation action of \( \operatorname{Spin}(7) \) on \( B_3 \) is transitive; in particular there is a homeomorphism

\[
B_3 \cong \operatorname{Spin}(7) / Z_{\operatorname{Spin}(7)}(x_1, x_2, x_3).
\]

Using the work in [1] it is easy to see that

\[
Z_{\operatorname{Spin}(7)}(x_1, x_2, x_3) = Z_{G_{3,2}}(x_2, x_3) \cong (\mathbb{Z}/2)^4.
\]

This shows that

\[
B_3 \cong \operatorname{Spin}(7)/(\mathbb{Z}/2)^4 \cong \operatorname{SO}(7)/(\mathbb{Z}/2)^3,
\]

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for some embedding \((\mathbb{Z}/2)^3 \hookrightarrow \text{SO}(7)\). Using this and Theorem 1.1, we see that if \(y \in \text{Hom}(\mathbb{Z}^3, \text{Spin}(7))\) is taken as the base point, then

\[
\pi_1(\text{Hom}(\mathbb{Z}^3, \text{Spin}(7)), y) = 1
\]

whenever \(y \in \text{Hom}(\mathbb{Z}^3, \text{Spin}(7))_{\mathbb{L}}\). In contrast, if \(y \in B_3\) then by (4.1)

\[
\pi_1(\text{Hom}(\mathbb{Z}^3, \text{Spin}(7)), y) = (\mathbb{Z}/2)^4.
\]

Examples 1 and 2 show that Theorem 1.1 may not hold if the base point of \(\text{Hom}(\mathbb{Z}^k, G)\) is no longer assumed to be in \(\text{Hom}(\mathbb{Z}^k, G)_{\mathbb{L}}\).

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