ON THE ZARISKI TOPOLOGY OF AUTOMORPHISM GROUPS OF AFFINE SPACES AND ALGEBRAS

ALEXEI BELOV-KANEL AND JIE-TAI YU

Abstract. We study the Zariski topology of the ind-groups of polynomial and free associative algebras $\text{Aut}(K[x_1, \ldots, x_n])$ (which is equivalent to the automorphism group of the affine space $\text{Aut}(K^n)$) and $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ via Ind-schemes, toric varieties, approximations and singularities.

We obtain some nice properties of $\text{Aut}(\text{Aut}(A))$, where $A$ is polynomial or free associative algebra over a field $K$. We prove that all Ind-scheme automorphisms of $\text{Aut}(K[x_1, \ldots, x_n])$ are inner for $n \geq 3$, and all Ind-scheme automorphisms of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ are semi-inner. We also establish that any effective action of torus $T^n$ on $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ is linearizable provided $K$ is infinity. That is, it is conjugated to a standard one.

As an application, we prove that $\text{Aut}(K[x_1, \ldots, x_n])$ cannot be embedded into $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ induced by the natural abelianization. In other words, the Automorphism Group Lifting Problem has a negative solution.

We explore the close connection between the above results and the Jacobian conjecture, and Kontsevich-Belov conjecture, and formulate the Jacobian conjecture for fields of any characteristic.

Contents

1. Introduction and main results
   1.1. Automorphisms of $K[x_1, \ldots, x_n]$ and $K\langle x_1, \ldots, x_n \rangle$
   1.2. Main results

2. Basic setup and terminologies
   2.1. Elementary and tame automorphisms

2010 Mathematics Subject Classification. Primary 13S10, 16S10. Secondary 13F20, 14R10, 16W20, 16Z05.

Key words and phrases. Ind-Scheme, Approximation, Zariski topology, Singularities, Toric varieties, Affine spaces Automorphisms, polynomial algebras, free associative algebras, lifting problem, tame and wild automorphisms, coordinates, Nagata conjecture, linearization.

The research of Jie-Tai Yu was partially supported by an RGC-GRF Grant.
1. Introduction and main results

1.1. Automorphisms of $K[x_1,\ldots,x_n]$ and $K\langle x_1,\ldots,x_n \rangle$. Let $K$ be an arbitrary field. In this article we study the Zarisky topology of the ind-groups of polynomial and free associative algebras Aut($K[x_1,\ldots,x_n]$) (which is equivalent to the automorphism group of the affine space Aut($K^n$)) and Aut($K\langle x_1,\ldots,x_n \rangle$) via Ind-schemes, toric varieties, approximations an singularities.
Automorphisms of Ind-schemes are closely related with the Jacobian Conjecture and

**Kontsevich-Belev Conjecture KB \[6, 7\]**

Does \( \operatorname{Aut}(W_n) \simeq \text{Sympl}(\mathbb{C}^{2n})? \)

This conjecture is related with the proof of stable equivalence of the Jacobian and Dixmier conjectures saying that \( \operatorname{Aut}(W_n) = \text{End}(W_n) \), where \( W_n \) is the Weil algebra. In order to do it, in the papers [6, 7], some monomorphism \( \operatorname{Aut}(W_n) \to \text{Sympl}(\mathbb{C}^{2n}) \) was constructed, and a natural question whether it is an isomorphism, is raised. It means that the automorphism group remains the same after quantization of standard symplectic structure. This monomorphism was defined by using sufficiently large prime. In the paper [7] it was raised the following

**Question.** Prove that this monomorphism is independent with respect to the choice of sufficiently large prime.

A precise formulation of this question in the paper [7] is follows:

For a finitely generated algebra \( R \) smooth over \( \mathbb{Z} \), does there exist an unique homomorphism

\[
\phi_R : \operatorname{Aut}(W_n)(R) \to \operatorname{Aut}(P_n)(R_\infty)
\]

such that \( \psi_R = Fr_* \circ \phi_R \)? Here \( Fr_* : \operatorname{Aut}(P_n)(R_\infty) \to \operatorname{Aut}(P_n)(R_\infty) \) is the group homomorphism induced by the endomorphism \( Fr : R_\infty \to R_\infty \) of the coefficient ring.

**Question.** In the above formulation, does the image of \( \phi_R \) belong to

\[ \operatorname{Aut}(P_n)(i(R) \otimes \mathbb{Q}) \],

where \( i : R \to R_\infty \) is the topological inclusion? In other words, does there exist a unique homomorphism

\[
\phi_R^{\text{can}} : \operatorname{Aut}(P_n)(R) \to \operatorname{Aut}(P_n)(R \otimes \mathbb{Q})
\]

such that \( \psi_R = Fr_* \circ i_* \circ \phi_R^{\text{can}} \), where \( P_n \) is free poisson algebra?
Comparing the two morphisms $\phi$ and $\varphi$ defined by two different ultra-filters, we get element $\phi\varphi^{-1}$ of $\text{Aut}_{\text{Ind}}(\text{Aut}(W_n))$, (i.e. an automorphism preserving the structure of infinite dimensional algebraic group). Describing this group would provide the solution of this question.

In spirit of the above we propose the following

**Conjecture.** *All automorphisms of $\text{Sympl}({\mathbb{C}}^n)$ as Ind-scheme are inner.*

The same conjecture can be proposed for $\text{Aut}(W_n)$.

We are focused on the investigation of the group of $\text{Aut}(\text{Aut}(K[x_1, \ldots, x_n]))$ and the corresponding noncommutative (free associative algebra) case.

Question regarding the structure of this group was proposed by B.I. Plotkin, motivated by the theory of universal algebraic geometry.

**Wild automorphisms and the lifting problem.** In 2004, the famous Nagata conjecture over a field $K$ of characteristic zero was proved by Shestakov and Umirbaev [27, 28] and a stronger version of the conjecture was proved by Umirbaev and Yu [31]. That is, let $K$ be a field of characteristic zero. Every wild $K[z]$-automorphism (wild $K[z]$-coordinate) of $K[z][x, y]$ is wild viewed as a $K$-automorphism ($K$-coordinate) of $K[x, y, z]$. In particular, the Nagata automorphism $(x - 2y(y^2 + xz) - (y^2 + xz)^2z, y + (y^2 + xz)z, z)$ (Nagata coordinates $x - 2y(y^2 + xz) - (y^2 + xz)^2z$ and $y + (y^2 + xz)z$) is (are) wild. In [31], a related question was raised:

**The lifting problem.** Whether or not a wild automorphism (wild coordinate) of the polynomial algebra $K[x, y, z]$ over a field $K$ can be lifted to an automorphism (coordinate) of the free associative $K\langle x, y, z \rangle$?
In our paper [8], based on the degree estimate [22, 21], it was proved that any wild $z$-automorphism including the Nagata automorphism cannot be lifted as $z$-automorphism (moreover, in [9] we proved every $z$-automorphism of $K\langle x, y, z \rangle$ is stably tame and becomes tame after adding at most one variable). It means, if every automorphism can be lifted, then it provides an obstacle $z'$ to $z$-lifting and the question to estimate such obstacle is naturally raised.

In view of the above, naturally we could ask

**The automorphism group lifting problem.** Whether $\text{Aut}(K[x_1, \ldots, x_n])$ is isomorphic to a subgroup of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ under the natural abelianization?

The following examples shows this problem is interesting and non-trivial.

**Example 1.** There is a surjective homomorphism (taking the absolute value) from $\mathbb{C}^*$ onto $\mathbb{R}^+$. But $\mathbb{R}^+$ is isomorphic to the subgroup $\mathbb{R}^+$ of $\mathbb{C}^*$ under the homomorphism.

**Example 2.** There is a surjective homomorphism (taking the determinant) from $\text{GL}_n(\mathbb{R})$ onto $\mathbb{R}^*$. But obviously $\mathbb{R}^*$ is isomorphic to the subgroup $\mathbb{R}^*I_n$ of $\text{GL}_n(\mathbb{R})$.

In this article we prove that the automorphism group lifting problem has a negative answer.

The lifting problem and the automorphism group lifting problem are closely related to the Kontsevich-Belov Conjecture (see Section 3.1). Consider a symplectomorphism $\varphi : x_i \rightarrow P_i, y_i \rightarrow Q_i$. It can be lifted to some automorphism $\hat{\varphi}$ of quantized algebra $W_{\hbar}[[\hbar]]$:

$$\hat{\varphi} : x_i \rightarrow P_i + P_i^1\hbar + \cdots + P_i^m\hbar^m; \quad y_i \rightarrow Q_i + Q_i^1\hbar + \cdots + Q_i^m\hbar^m.$$
The point is to choose a lifting $\varphi$ in such a way that the degrees of all $P_i^m, Q_i^m$ are bounded. In this case Kontsevich-Belov conjecture follows.

1.2. Main results. The Main results of this paper are

**Theorem 1.1.** Any Ind-scheme automorphism $\varphi$ of $\text{NAut}(K[x_1, \ldots, x_n])$ for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

**Theorem 1.2.** Any Ind-scheme automorphism $\varphi$ of $\text{NAut}(K\langle x_1, \ldots, x_n \rangle)$ for $n \geq 3$ is semi-inner (see definition 1.6).

$\text{NAut}$ means the group of nice automorphisms, i.e. automorphisms which can be approximated by tame ones (see definition 3.1). In characteristic zero case every automorphism is nice.

For the group of automorphisms of a semi-group the similar results on set-theoretical level were obtained previously by A.Belov, R.Lipyaniskii and I.Berzinsh [4, 5]. All these questions (including $\text{Aut}(\text{Aut})$ investigations) are closely related to Universal Algebraic Geometry and were proposed by B.Plotkin. Equivalence of two algebras have same generalized identities and isomorphism of first order means semi-inner properties of automorphisms (see [4, 5] for details).

**Automorphisms of the tame automorphism groups.** Regarding the tame automorphism group, something can be done on the group-theoretical level. In H.Kraft and I.Stampfli [20], the automorphism group of the tame automorphism group of polynomial algebra was brilliantly studied. In that paper, conjugation of elementary automorphisms by translations plays very important role. Our results in the current article are different. We calculate $\text{Aut}(\text{TAut}_0)$ of the tame automorphism group $\text{TAut}_0$ preserving the origin (i.e. taking the augmentation ideal onto an ideal which is a subset of the augmentation ideal). This is technically more difficult, the advantage is that our methodology can be universally and systematically done for both commutative
(polynomial algebra) case and noncommutative (free associative algebra) case. We see some problems in the shift conjugation approach for the noncommutative (free associative algebra) case, as did for commutative case in [20]. Any substitution of a ground field element for a variable can cause zero, for example in Lie polynomial $[[x, y], z]$. Note that calculation of $\text{Aut}(\text{TAut}_0)$ (resp. $\text{Aut} (\text{TAut}_0)$, $\text{Aut} (\text{Aut}_0)$) imply also the same results for $\text{Aut}(\text{TAut})$ (resp. $\text{Aut} (\text{TAut})$, $\text{Aut} (\text{Aut})$) according the approach of this article via stabilization by the torus action.

**Theorem 1.3.** Any automorphism $\varphi$ of $\text{TAut}_0(K[x_1, \ldots, x_n])$ (in group theoretical sense) for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

**Theorem 1.4.** The group $\text{TAut}_0(K[x_1, \ldots, x_n])$ is generated by automorphisms $x_1 \to x_2x_3, x_i \to x_i, i \neq 1$ and linear ones if $\text{Char}(K) \neq 2$ or $n > 3$.

Let $G_n \subset \text{TAut}(K[x_1, \ldots, x_n]), E_n \subset \text{TAut}(K\langle x_1, \ldots, x_n \rangle)$ be tame automorphism groups preserving the augmentation ideal.

**Theorem 1.5.** Any automorphism $\varphi$ of $G_n$ (in the group theoretical sense) for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

**Definition 1.6.** An anti-automorphism $\Psi$ of $K$-algebra $B$ is an automorphism $B$ as a vector space such that $\Psi(ab) = \Psi(b)\Psi(a)$. e.g. Transposition of matrices is an anti-automorphism. An anti-automorphism of free associative algebra $A$ is **mirror** if it sends $x_ix_j$ to $x_jx_i$ for some fixed $i$ and $j$. An automorphism of $\text{Aut}(A)$ is **semi-inner** if it can be obtained as a composition of an inner automorphism and conjugation by a mirror anti-automorphism.

**Theorem 1.7.** a) Any group automorphism of the group $\varphi$ of $\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle)$ or the group $\text{TAut}(K\langle x_1, \ldots, x_n \rangle)$ (in the group theoretical sense) for
n ≥ 4 is semi-inner, i.e. is a conjugation via some automorphism and/or mirror anti-automorphism.

b) The same is true for $E_n$, $n ≥ 4$.

The case of $\text{TAut}(K\langle x, y, z \rangle)$ is much more difficult. We can treat it only on the $\text{Ind}$-scheme level, but even then it is the most technical part of this paper (see section 5.2).

**Theorem 1.8.** a) Let $\text{Char}(K) \neq 2$. Then $\text{Aut}_\text{Ind}(\text{TAut}(K\langle x, y, z \rangle))$ (resp. $\text{Aut}_\text{Ind}(\text{TAut}_0(K\langle x, y, z \rangle))$) is generated by conjugations on automorphisms or mirror anti-automorphisms.

b) The same is true for $\text{Aut}_\text{Ind}(E_3)$.

By $\text{TAut}$ we mean the tame automorphism group, $\text{Aut}_\text{Ind}$ is the group of $\text{Ind}$-scheme automorphisms (see section 2.2).

Approximation allows us to formulate the famous Jacobian conjecture for any characteristic.

**Lifting of the automorphism groups.** In this article we prove that the automorphism group of polynomial algebra over an arbitrary field $K$ cannot be embedded into the automorphism group of free associative algebra induced by the natural abelianization.

**Theorem 1.9.** Let $K$ be an arbitrary field, $G = \text{Aut}_0(K[x_1, \ldots, x_n])$ and $n > 2$. Then $G$ cannot be isomorphic to any subgroup $H$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ induced by the natural abelianization. The same is true for $\text{NAut}(K[x_1, \ldots, x_n])$.

2. **Basic setup and terminologies**

2.1. **Elementary and tame automorphisms.** Let $P$ be a polynomial independent respect $x_i$. Automorphism $x_i \to x_i + P, x_j \to x_j$ for $i \not= j$ is called elementary. The group generated by linear automorphisms and elementary ones for all possible $P$ are called tame
automorphism group TAut and elements of TAut are tame automorphisms.

2.2. Ind-schemes and Ind-groups.

Definition 2.1. An Ind-variety $M$ is the direct limit of algebraic varieties $M = \varprojlim M_i \subseteq M_2 \ldots$. An Ind-scheme is an Ind-variety which is a group such that group inversion is a morphism $M_i \rightarrow M_{j(i)}$, and the group multiplication induces a morphism from $M_i \times M_j$ to $M_{k(i,j)}$. A map $\varphi$ is a morphism of Ind-variety $M$ to Ind-variety $N$, if $\varphi(M_i) \subseteq N_{j(i)}$ and restriction $\varphi$ on $M_i$ is morphism for all $i$. Monomorphism, epimorphism and isomorphism can be defined similarly in the natural way.

Example. Let $M$ be the group of automorphisms of an affine space, and $M_j$ be set of all automorphisms in $M$ with degree $\leq j$.

There is an interesting

Question. Investigating the growth function on Ind-variety. For example, the dimension of varieties of polynomial automorphisms of degree $\leq n$.

Note that coincidence of the growth functions for $\text{Aut}(W_n)$ and $\text{Sympl}(\mathbb{C}^{2n})$ would imply Kontsevich-Belov conjecture [7].

Definition 2.2. The ideal $I$ generated by variables $x_i$ is the augmentation ideal. The augmentation subgroup $H_n$ is the group of all automorphisms $\varphi$ such that $\varphi(x_i) \equiv x_i \mod I^n$. The set $G_n \supset H_n$ is a group of automorphisms whose linear part is scalar, and $\varphi(x_i) \equiv \lambda x_i \mod I^n$ ($\lambda$ does not dependant on $i$).

3. The Jacobian Conjecture in any Characteristic, Kontsevich-Belov Conjecture and Approximation

3.1. Approximation problems and Kontsevich-Belov Conjecture. First let us remind the Kontsevich-Belov Conjecture:
**KB**$_n$: Does Aut($W_n$) $\simeq$ Sympl($\mathbb{C}^{2n}$).

A similar conjecture can be stated for endomorphisms

**KB**$_n$: Does End($W_n$) $\simeq$ Sympl End($\mathbb{C}^{2n}$).

If the Jacobian conjecture $JC_{2n}$ is true, then these two conjectures are equivalent. $W_n = \mathbb{C}[x_1, \ldots, x_n; \partial_1, \ldots, \partial_n]$ is the *Weil algebra* of differential operators.

It is natural to approximate automorphisms by tame ones. There exists such approximation up to terms of any order not only in the situation of polynomial automorphisms, but also for automorphisms of Weil algebra, symplectomorphisms etc. However, naive approach fails.

It is known that Aut($W_1$) $\equiv$ Aut$_1(K[x, y])$ where Aut$_1$ means the Jacobian determinant is one. However, considerations from [25] shows that Lie algebra of the first group is derivations of $W_1$ and hence has no identities apart ones which have free Lie algebra, another coincidence of the vector fields which divergents to zero, and has polynomial identities. They cannot be isomorphic [6, 7]. In other words, this group has two coordinate system non-smooth with respect to each other (but integral with respect to each other). One system provided by coefficients of differential operators, another with coefficients of polynomials, which are images of $\tilde{x}_i, \tilde{y}_i$. The group Aut($W_n$) can be embedded into Sympl($\mathbb{C}^{2n}$), for any $n$. But the Lie algebra Der($W_n$) has no polynomial identities apart from ones which have free Lie algebras, another coincidence of the vector fields preserving symplectic form and has polynomial identities.

In the paper [25] functionals on $\mathfrak{m}/\mathfrak{m}^2$ were considered in order to define the Lie algebra structure. In the spirit of that we have the following

**Conjecture.** The natural limit of $\mathfrak{m}/\mathfrak{m}^2$ is zero.
It means that the definition of the Lie algebra has some functoriality problem and it depends on the presentation of (reducible) Ind-scheme.

In his remarkable paper, Yu.Bodnarchuck [14] established a result similar to our Theorem 1.1 by using the Shafarevich results for tame automorphism group and for case when automorphism of Ind-scheme is regular in following sense: sent polynomials on coordinate functions (coordinate – coefficient before corresponding monomial) to coordinate functions. In this case tame approximation works (as well as for the symplectic case as well). For this case his method is similar to ours, but we display it here for self-contain-ness, and convenience of readers, and also to treat the noncommutative (free associative algebra) case. But in general, for regular functions, if the approximation via the Shafarevich approach is correct, then the Kontsevich-Belov conjecture (for isomorphism between Aut($W_n$) and Sympl($K^n$)) would follow directly, which would be absurd.

We would like to mention also the very recent paper of H. Kraft and I. Stampfli [20]. They show brilliantly that every automorphism of the group $\mathcal{G}_n := \text{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of complex affine $n$-space $\mathbb{A}^n = \mathbb{C}^n$ is inner up to some field automorphisms when restricted to the subgroup $T\mathcal{G}_n$ of the tame automorphisms. They play on conjugation with translation. This generalizes a result of J.Deserti [15] who proved this for dimension two where all automorphisms are tame: $T\mathcal{G}_2 = \mathcal{G}_2$. Our method is slightly different. We calculate automorphism of tame automorphism group preserving the origin (i.e. taking the augmentation ideal onto a subset of the augmentation ideal). In this case we cannot play on translations. One advantage of our approach is that we also established the same results for the noncommutative (free associative algebra) case, which could not be treated by the approaches of Bodnarchuck and that of Kraft and Stampfli. We always treat dimension more than two.
In the sequel, we do not assume regularity in the sense of [14] but only assume that the restriction of a morphism on any subvariety is a morphism again. Note that morphisms of Ind-schemes $\text{Aut}(W_n) \to \text{Sympl}(\mathbb{C}^{2n})$ has this property, but not regular in the sense of Bodnar-chuk [14].

In order to make approximation work, we use the idea of singularity which allows us to prove the augmentation subgroup structure preserving, so approximation works in the case (not in all situations, in a much more complicated way).

Consider the isomorphism $\text{Aut}(W_1) \cong \text{Aut}_1(K[x, y])$. It has some strange property. Let us add a small parameter $t$. Then an element arbitrary close to zero with respect to $t^k$ does not go to zero arbitrarily, so it is impossible to make tame limit! There is a sequence of convergent product of elementary automorphisms, which is not convergent under this isomorphism. Exactly same situation happens for $W_n$. These effects cause problems in the quantum field theory.

3.2. The Jacobian Conjecture for any characteristic. Naive formulation is not good because of example of mapping $x \to x - x^p$ in characteristic $p$. Approximation provides a way to formulate a question generalizing the Jacobian conjecture for any characteristic and put it into framework of other questions.

According to Anick [1], any automorphism of $K[x_1, \ldots, x_n]$ if $\text{Char}(K) = 0$ can be approximated by tame ones with respect to augmentation subgroups $H_n$.

**Definition 3.1.** An endomorphism $\varphi \in \text{End}(K[x_1, \ldots, x_n])$ is good if for any $m$ there exist $\psi_m \in \text{End}(K[x_1, \ldots, x_n])$ and $\phi_m \in \text{Aut}(K[x_1, \ldots, x_n])$ such that

- $\varphi = \psi_m \phi_m$
- $\psi(x_i) = x_i + P_i; P_i \in \text{Id}(x_1, \ldots, x_n)^m$. 

An automorphism \( \varphi \in \text{Aut}(K[x_1, \ldots, x_n]) \) is nice if for any \( m \) there exist \( \psi_m \in \text{Aut}(K[x_1, \ldots, x_n]) \) and \( \phi_m \in \text{TAut}(K[x_1, \ldots, x_n]) \) such that

- \( \varphi = \psi_m \phi_m \)
- \( \psi(x_i) = x_i + P_i; \) \( P_i \in \text{Id}(x_1, \ldots, x_n)^m \), i.e. \( \Psi \in H_m \).

D. Anick \[1\] shown that if \( \text{Char}(K) = 0 \) any automorphism is nice. However, this is unclear in positive characteristic.

**Question.** Is any automorphism over arbitrary field nice?

**The Jacobian conjecture for any characteristic** Is any good endomorphism over arbitrary field an automorphism?

Each good automorphism has Jacobian 1, and all such automorphisms are good (even nice) when \( \text{Char}(K) = 0 \).

Similar notions can be formulated for the free associative algebra.

**Question.** Is any automorphism of free associative algebra over arbitrary field nice?

Now we came to generalization of the Jacobian conjecture to arbitrary characteristic:

**Question.** Is any good endomorphism of free associative algebra over arbitrary field an automorphism?

### 3.3. Approximation for the automorphism group of affine spaces.

The approximation is the most important method of the current paper.
In order to do it, we have to prove that \( \varphi \in \text{Aut}_{\text{ind}}(\text{Auto}_0(K[x_1, \ldots x_n])) \) preserves the structure of the augmentation subgroup. We treat here only the affine case. For symplectomorphisms for example, the situation is more complicate and we can treat just the general automorphism group.
Theorem 3.2. \( \varphi(H_n) \subseteq H_n \) where \( \varphi \in \text{Aut} \ (\text{Aut}_0(K[x_1, \ldots x_n])) \), \( H_n \) is a subgroup of elements identity modulo ideal \((x_1, \ldots ,x_k)^n\).

Theorem 3.3. \( \varphi(H_n) \subseteq H_n \) where \( H_n \) is subgroup of elements identity modulo ideal \((x_1, \ldots ,x_k)^n\) also for free associative case.

Corollary 3.4. \( \varphi = \text{Id} \).

Proof. Every automorphism can be approximated via the tame ones. i.e. for any \( \psi \) and any \( n \) there exists a tame automorphism \( \psi'_n \) such that \( \psi \psi'_n^{-1} \in H_n \).

In fact this theorem implies \( \text{Aut}(K[x_1, \ldots ,x_n]) \) cannot be embedding into \( \text{Aut}(K\langle x_1, \ldots ,x_n \rangle) \) under the natural abelianization, because elementary actions determine coordinates but we have approximations.

So the main point is why \( \varphi(H_n) \subseteq H_n \).

Proof of Theorem 3.2. Consider matrix \( A(t) \) with a parameter \( t \) such that eigenvalues are \( t^{n_i} \) and \( n_i k \leq n_j \). \( \varphi(A(t)) = A(t) \), because \( \varphi \) preserves the linear transformations.

This follows from the next two lemmas.

Lemma 3.5. Let \( M \) be an automorphism of the polynomial algebra. Then \( A(t)MA(t)^{-1} \) has no singularities. i.e. It is an affine curve for \( t = 0 \) for any \( A(t) \) with the properties that

\[
A(t) \text{ dependent on parameter } t \text{ such that eigenvalues are } t^{n_i} \text{ and } n_i k \leq n_j. \quad (*)
\]

if and only if \( M \in \hat{H}_n \) where \( \hat{H}_n \) is homothety modulo the augmentation ideal.

Proof. The ‘If’ part is obvious, because the sum \( \sum_{j=1}^{k} n_{ij} \) is greater then \( n_m \) and the homothety commutes with linear map hence conjugation of the homothety via the linear map is itself.

We have to prove that if the linear part of \( \varphi \) does not satisfy the condition \((*)\), then \( A(t)MA(t)^{-1} \) has a singularity at \( t = 0 \).
Case 1. The linear part $M$ of $M$ is not a scalar matrix. Then after basis change it is not a diagonal matrix and has a non-zero coefficient in $i,j$ position $E_{ij}$. Consider diagonal matrix $A(t) = D(t)$ such that on all position on the main diagonal except $j$-th it has $t^{n_i}$ and on $j$-th position $t^{n_j}$. Then $D(t)M^{-1}(t)$ has $(i,j)$ entry with the coefficient $\lambda t^{n_i-n_j}$ and if $n_j > n_i$ it has a singularity at $t = 0$.

Let also $n_i < 2n_j$. Then the non-linear part of $M$ does not produce singularities and cannot compensate the singularity of the linear part. So we are done.

Case 2. The linear part $M$ of $M$ is a scalar matrix. Then conjugation of linear part can not produce singularities and we are interested just in the smallest non-linear term. Let $\phi \in H^k \setminus H^{k+1}$. Due to a linear base change we can assume that $\phi(x_1) = \lambda \cdot x_1 + \delta x_2^k + S$, where $S$ is the sum of monomials of degree $\geq k$ different from $x_2^k$ with coefficients in $K$.

Let $A(t) = D(t)$ be a diagonal matrix of the form $(t^{n_1}, t^{n_2}, \ldots, t^{n_1})$. Let $(k+1) \cdot n_2 > n_1 > k \cdot n_2$. Then in $A^{-1}MA$ term $\delta x_2^k$ will be transformed in $\delta x_2^k t^{kn_2-n_1}$ all other terms produce power $t^{ln_2+sn_1-n_1}$ such that $(l,s) \neq (1,0), l, s > 0$. In this case $ln_2 + sn_1 - n_1 > 0$ and we are done with the proof of Lemma 3.5.

The next lemma can be proved by concrete calculations:

Lemma 3.6. a) $[G_n,G_n] \subset H_n, n > 2$. There exist elements $\varphi \in H_n \setminus H_{n+k-1}, \psi_1 \in G_k, \psi_2 \in G_n$, such that $\varphi = [\psi_1, \psi_2]$.

b) $[H_n,H_k] \subset H_{n+k-1}$.

c) Let $\varphi \in G_n \setminus H_n, \psi \in H_k \setminus H_{k+1}, k > n$. Then $[\varphi, \psi] \in H_k \setminus H_{k+1}$.

Proof. a) Let $\psi_1 : x \rightarrow x + y^{k}, y \rightarrow y, \psi_2 : x \rightarrow x, y \rightarrow y + x^n$, $\psi_1, i = 1, 2$ fixing other variables and $\varphi = [\psi_1, \psi_2] = \psi_1^{-1} \psi_2^{-1} \psi_1 \psi_2$.

Then

$$
\varphi : x \rightarrow x + (y + x^n)^k - (y + x^n - (x + (y + x^n)^k)^n)^k, \\
y \rightarrow y + x^n - (x + (y + x^n)^k)^n.
$$
It is easy to see that if either \( k \) or \( n \) relatively prime with \( \text{Char}(K) \), then all the terms of degree \( k+n-1 \) does not cancel and \( \varphi \in H_{n+k-1} \setminus H_{n+k} \).

Now suppose that \( \text{Char}(K) \nmid n \), then obviously \( n-1 \) is relatively prime with \( \text{Char}(K) \). Consider mappings \( \psi_1 : x \to x + y^k, y \to y, \psi_2 : x \to x, y \to y + x^{n-1}z, \psi_i, i = 1,2 \) stable on other variables. Let \( \varphi' = [\psi_1, \psi_2] = \psi_1^{-1} \psi_2^{-1} \psi_1 \psi_2 \). Then

\[
\varphi' : \begin{cases} 
  x \to x + (y + zx^{n-1})^k - (y + zx^{n-1} - (x + z(y + x^n)^k)^{n-1})^k, \\
  y \to y + zx^{n-1} - z(x + (y + zx^{n-1})^k)^{n-1} = y + (n-1)y^kx^{n-1} + o.
\end{cases}
\]

\( o \) means sum of terms of degree \( \geq n+k \). We see that \( \varphi \in H_{n+k-1} \setminus H_{n+k} \).

**Corollary 3.7.** Let \( \Psi \in \text{NAut} \ (\text{Aut}(K[x_1, \ldots, x_n])) \). Then \( \Psi(G_n) = G_n, \Psi(H_n) = H_n \).

Corollary 3.7 and Proposition 4.3 imply Theorem 3.2 because every nice automorphism can be approximated via tame ones. Note that in zero characteristics every automorphism is nice.

### 3.4. Lifting of automorphism groups.

#### 3.4.1. Lifting of automorphisms from \( \text{Aut}(K[x_1, \ldots, x_n]) \) to \( \text{Aut}(K\langle x_1, \ldots, x_n \rangle) \).

**Theorem 3.8.** Any effective action of torus \( \mathbb{T}^n \) on \( K\langle x_1, \ldots, x_n \rangle \) is linearizable. That is, it is conjugated to a standard one.

**Proof.** Similar to the proof of Theorem 4.1

As a consequence of the above theorem, we get
Proposition 3.9. Let $T^n$ be standard torus action. Let $\hat{T}^n$ its lifting to automorphism group of the free algebra. Then $\hat{T}^n$ is also standard torus action.

Proof. Consider the roots $\hat{x}_i$ of this action. They are liftings of the coordinates $x_i$. We have to prove that they generate the whole associative algebra.

According to the reducibility of this action, all elements are product of eigenvalues of this action. Hence it is enough to prove that eigenvalues of this action can be presented as a linear combination of this action. This can be done as did by Byalickii-Birula [13]. Note that all propositions of previous section hold for free associative algebras. Proof of the Theorem 3.3 is similar. Hence we have the following

Theorem 3.10. Any Ind-scheme automorphism $\varphi$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

Hence the group lifting (under the sense of isomorphism induced by the natural abelianization) implies the analogue of Theorem 3.2.

This also implies that an automorphism group lifting, if exists, satisfies the approximation properties.

Proposition 3.11. Let $H = \text{Aut}(K[x_1, \ldots, x_n]), G = \text{Aut}(K\langle z_1, \ldots, z_n \rangle)$. Suppose $\Psi : H \rightarrow G$ be a group homomorphism such that its composition with natural projection is the identity map. Then

1. After some coordinate change $\psi$ provide correspondence between standard torus actions $x_i \rightarrow \lambda_i x_i$ and $z_i \rightarrow \lambda_i z_i$.
2. Images of elementary automorphisms
   
   $x_j \rightarrow x_j, j \neq i, \ x_i \rightarrow x_i + f(x_1, \ldots, \hat{x}_i, \ldots, x_n)$

   are elementary automorphisms of the form
   
   $z_j \rightarrow z_j, j \neq i, \ z_i \rightarrow z_i + f(z_1, \ldots, \hat{z}_i, \ldots, z_n)$.

   (Hence image of tame automorphism is tame automorphism).
(3) $\psi(H_n) = G_n$. Hence $\psi$ induces map between completion of the groups of $H$ and $G$ respect to augmentation subgroup structure.

**Proof of Theorem 1.9**

Any automorphism, including the Nagata automorphism can be approximated via product of elementary automorphisms with respect to augmentation topology. In the case of the Nagata automorphism corresponding to

$$\text{Aut}(K\langle x_1, \ldots, x_n \rangle),$$

all such elementary automorphisms fix all coordinates except $x_1, x_2$. Due to (2) and (3) of Proposition 3.11, the lifted automorphism would be an automorphism induced by automorphism of $K\langle x_1, x_2, x_3 \rangle$ fixing $z_3$. However, it is impossible to lift the Nagata automorphism to such an automorphism due to the main result of [8]. Therefore, Theorem 1.9 is proved.

4. **Automorphisms of the polynomial algebra and the approach of Bodnarchuk-Kraft-Rips**

Let $\Psi \in \text{Aut(} \text{Aut}(K[x_1, \ldots, x_n]))$ (resp. in $\text{Aut(TAut}(K[x_1, \ldots, x_n]))$, $\text{Aut(TAut}_0(K[x_1, \ldots, x_n]))$, $\text{Aut(Aut}_0(K[x_1, \ldots, x_n]))$).

4.1. **Reduction to the case when $\Psi$ is identical on $\text{SL}_n$.** We follow [20] and [14] using the classical theorem of Byalickii-Birula [12, 13]:

**Theorem 4.1** (Byalickii-Birula). *Any effective action of torus $\mathbb{T}^n$ on $\mathbb{C}^n$ is linearizable. That is, it is conjugated to a standard one.*

**Remark.** An effective action of $\mathbb{T}^{n-1}$ on $\mathbb{C}^n$ is linearizable [13, 12]. There is a conjecture whether an action of $\mathbb{T}^{n-2}$ on $\mathbb{C}^n$ is linearizable, established for $n = 3$. For codimension more than 2, counterexamples were constructed [2].
Remark. H. Kraft and I. Stampfli [20] proved (considering periodic elements in \( T \)) that this action is not just abstract group action but also if \( \Psi \in \text{Aut}(\text{Aut}) \) its image of \( T \) is an algebraic group. In fact their proof is also applicable for free associative algebra. (It based on consideration of elements of finite order.) We use this result.

Consider the standard action of torus \( T^n \) on \( C^n \): \( x_i \to \lambda_i x_i \), let \( H \) be the image of \( T^n \) under \( \varphi \). Then by Theorem 4.1 \( H \) is conjugated to the standard torus via some automorphism \( \psi \). Composing \( \varphi \) with conjugation with respect to \( \psi \), we come to the case when \( \varphi \) is the identity on the maximal torus. Then we have the following

**Corollary 4.2.** Without loss of generality, it is enough to prove Theorem 1.1 for the case when \( \varphi|_T = \text{Id} \).

Now we are in the situation when \( \varphi \) preserves all linear mappings \( x_i \to \lambda_i x_i \). We have to prove that it is identity.

**Proposition 4.3** (E. Rips, private communication). Let \( n > 2 \) and let \( \varphi \) preserves the standard torus action for a polynomial algebra. Then \( \varphi \) preserves all elementary transformations.

**Corollary 4.4.** Let \( \varphi \) satisfies the conditions of Proposition 4.3. Then \( \varphi \) preserves all tame automorphisms.

**Proof of Proposition 4.3.** We need several lemmas. First of all, we need to see compositions of given automorphisms with action of maximal torus.

**Lemma 4.5.** Consider the diagonal \( T^1 \) automorphisms: \( \alpha : x_i \to \alpha_i x_i \), \( \beta : x_i \to \beta_i x_i \). Let \( \psi : x_i \to \sum_{i,J} a_{i,J} x^J, i = 1, \ldots, n; J = (j_1, \ldots, j_n) \) - multi-index, \( x^J = x_1^{j_1} \cdots x_n^{j_n} \). Then

\[
\alpha \circ \psi \circ \beta : x_i \to \sum_{i,J} \alpha_i a_{i,J} x^J \beta^J,
\]
In particular,

\[ \alpha \circ \psi \circ \alpha^{-1} : x_i \to \sum_{i,j} \alpha_i a_i,j x^j \alpha^{-j}. \]

Applying Lemma 4.5 and comparing the coefficients we get the following

**Lemma 4.6.** Consider the diagonal \( \mathbb{T}^1 \) action: \( x_i \to \lambda x_i \). Then the set of automorphisms commuting with this action is exactly the linear automorphisms.

Similarly (using Lemma 4.5) we obtain Lemmas 4.7, 4.9, 4.10:

**Lemma 4.7.** a) Consider the following \( \mathbb{T}^2 \) action: \( x_1 \to \lambda \delta x_1, x_2 \to \lambda x_2, x_3 \to \delta x_3, x_i \to x_i, i > 3 \). Then the set \( S \) of automorphisms commuting with this action generated with the following automorphisms

\[ x_1 \to x_1 + \beta \cdot x_2 x_3, x_i \to \varepsilon_i x_i, i > 1, (\alpha, \beta, \varepsilon \in K). \]

b) Consider the following \( \mathbb{T}^2 \) action: \( x_1 \to \lambda' x_1, x_j \to \lambda_j x_j, j > 1 \). Then the set \( S \) of automorphisms commuting with this action generated with following automorphisms

\[ x_1 \to x_1 + \beta \cdot \prod_{j=2}^{n} x_j^{i_j}, (\lambda = (\lambda_2, \ldots, \lambda_n), \beta, \lambda_j \in K). \]

**Remark.** The similar statement for the noncommutative (free associative algebra) case is true, but one has to consider the set \( \hat{S} \) of automorphisms \( x_1 \to x_1 + H, x_i \to \varepsilon_i x_i, i > 1, (\varepsilon \in K, \text{the polynomial } H \in K\langle x_2, \ldots, x_n \rangle \text{ has multi-degree } J, \text{in non-commutative case it is not just monomial anymore}).

**Corollary 4.8.** Let \( \varphi \in \text{Aut}(TAut(K(x_1, \ldots, x_n))) \) stabilizing all elements from \( \mathbb{T} \). Then \( \varphi(S) = S \).

**Lemma 4.9.** Consider the following \( \mathbb{T}^1 \) action: \( x_1 \to \lambda^2 x_1, x_2 \to \lambda x_2, x_i \to x_i, i \neq 1, 2 \). Then the set \( S \) of automorphisms commuting with this
action generated with following automorphisms \( x_1 \rightarrow x_1 + \beta \cdot x_2, x_i \rightarrow \lambda_i x_i, i > 2, (\beta, \lambda_i \in K) \).

**Lemma 4.10.** Consider the set \( S \) defined in the previous lemma. Then 
\([S,S] = \{uvu^{-1}v^{-1}\}\) consists of the following automorphisms \( x_1 \rightarrow x_1 + \beta \cdot x_2x_3, x_2 \rightarrow x_2, x_3 \rightarrow x_3, (\beta \in K) \).

**Lemma 4.11.** Let \( n \geq 3 \). Consider the following set of automorphisms 
\( \psi_i : x_i \rightarrow x_i + \beta_i x_{i+1}x_{i+2}, \beta_i \neq 0, x_k = x_k, k \neq i \) for \( i = 1, \ldots, n-1 \). (Numeration is cyclic, so for example \( x_{n+1} = x_1 \)). Let \( \beta_i \neq 0 \) for all \( i \). Then all of \( \psi_i \) simultaneously conjugated by torus action to \( \psi'_i : x_i \rightarrow x_i + x_{i+1}x_{i+2}, x_k = x_k, k \neq i \) for \( i = 1, \ldots, n \) in a unique way.

**Proof.** Let \( \alpha : x_i \rightarrow \alpha_i x_i \), then by Lemma 4.5 we obtain
\[
\alpha \circ \psi_i \circ \alpha^{-1} : x_i \rightarrow x_i + \beta_i x_{i+1}x_{i+2} \alpha_i^{-1} \alpha_i^{-1} \alpha_i^{-1} \alpha_i^{-1}
\]
and
\[
\alpha \circ \psi_i \circ \alpha^{-1} : x_k \rightarrow x_k
\]
for \( k \neq i \).

Comparing the coefficients of the quadratic terms, we see that it is sufficient to solve the system:
\[
\beta_i \alpha_i^{-1} \alpha_i^{-1} \alpha_i^{-1} \alpha_i^{-1} = 1, i = 1, \ldots, n-1.
\]
because \( \beta_i \neq 0 \) for all \( i \), this system has unique solution.

**Remark.** In the free associative algebra case, instead of \( \beta x_2x_3 \) one has to consider \( \beta x_2x_3 + \gamma x_3x_2 \).

4.2. The lemma of Rips.

**Lemma 4.12** (Rips). Let \( \text{Char}(K) \neq 2, |K| = \infty \). Then linear transformations and \( \psi'_i \) generate the tame automorphism group of the polynomial algebra.
Proposition 4.3 follows from Lemmas 4.6, 4.7, 4.9, 4.10, 4.11, 4.12. Note that we have proved an analogue of Theorem 1.1 for tame automorphisms.

**Proof of Lemma 4.12.** Let $G$ be group generated by elementary transformations as in Lemma 4.11. We have to prove that $G = \text{TAut}_0$, tame automorphism group fixing the augmentation ideal. We need some preliminaries.

**Lemma 4.13.** The linear transformations and $\psi : x \to x, y \to y, z \to z + xy$ generate all the mappings of the following form
\[ \phi_n^b(x, y, z) : x \to x, y \to y, z \to z + bx^n, \quad b \in K. \]

**Proof of Lemma 4.13.** We proceed by induction. Suppose we have automorphism
\[ \phi_{n-1}^b(x, y, z) : x \to x, y \to y, z \to z + bx^{n-1}. \]
Conjugating by the linear transformation $(z \to y, y \to z, x \to x)$, we obtain the automorphism
\[ \phi_{n-1}^b(x, z, y) : x \to x, y \to y + bx^{n-1}, z \to z. \]
Composing this with $\psi$ from the right side, we get the automorphism
\[ \varphi(x, y, z) : x \to x, y \to y + bx^{n-1}, z \to z + yx + x^n. \]
Note that
\[ \phi_{n-1}(x, y, z)^{-1} \circ \varphi(x, y, z) : x \to x, y \to y, z \to z + xy + bx^n. \]
Now we see that
\[ \psi^{-1} \phi_{n-1}^b(x, y, z)^{-1} \circ \varphi(x, y, z) = \phi_n^b \]
and the lemma is proved.

**Corollary 4.14.** Let $\text{Char}(K) \nmid n$ (in particular, $\text{Char}(K) \neq 0$) and $|K| = \infty$. Then $G$ contains all the transformations $z \to z + bx^ky^l, y \to y, x \to x$ such that $k + l = n$. 
Proof. For any invertible linear transformation

\[ \varphi : x \to a_{11}x + a_{12}y; y \to a_{21}x + a_{22}y, z \to z; a_{ij} \in K \]

we have

\[ \varphi^{-1}\phi^b_n\varphi : x \to x, y \to y, z \to z + b(a_{11}x + a_{12}y)^n. \]

Note that sums of such expressions contains all the terms of the form \( bx^ky^l \). Corollary is proven.

4.3. Generators of the tame automorphism group.

**Theorem 4.15.** If \( \text{Char}(K) \neq 2 \) and \(|K| = \infty\), then the linear transformations and \( \psi : x \to x, y \to y, z \to z + xy \) generate all the mappings of the following form \( \alpha_n^b(x, y, z) : x \to x, y \to y, z \to z + byx^n, b \in K. \)

**Proof of Theorem 4.15.** Note that \( \alpha = \beta \circ \phi_n^b(x, z, y) : x \to x + by^n, y \to y + x + by^n, z \to z \) where \( \beta : x \to x, y \to x + y, z \to z. \) Then \( \gamma : \alpha^{-1}\psi\alpha : x \to x, y \to y, z \to z + xy + 2bxy^n + by^{2n}. \) Composing with \( \psi^{-1} \) and \( \psi_{2n-2b} \) we get needed \( \alpha_n^b(x, y, z) : x \to x, y \to y, z \to z + 2byx^n, b \in K. \)

**Corollary 4.16.** Let \( \text{Char}(K) \nmid n \) and \(|K| = \infty\). Then \( G \) contains all the transformations \( z \to z + bx^ky^l, y \to y, x \to x \) such that \( k = n + 1. \)

The proof is similar to the proof of Corollary 4.14. Note that either \( n \) or \( n + 1 \) is not a multiple of \( p \) so we have

**Lemma 4.17.** If \( \text{Char}(K) \neq 2 \) then the linear transformations and \( \psi : x \to x, y \to y, z \to z + xy \) generate all the mappings of the following form \( \alpha_P : x \to x, y \to y, z \to z + P(x, y), P(x, y) \in K[x, y]. \)

We have proved 4.12 for the three variable case. In order to treat the case when \( n \geq 4 \) we need one more lemma.
Lemma 4.18. Let $M(\vec{x}) = a \prod x_i^{k_i}, a \in K, |K| = \infty, \text{Char}(K) \nmid k_i$ for at least one of $k_i$. Consider the linear transformations
\[ f : x_i \to y_i = \sum a_{ij} x_j, \det(a_{ij}) \neq 0, \]
and monomials $M_f = M(\vec{y})$. Then linear span of $M_f$ for different $f$ contains all homogenous polynomials of degree $k = \sum k_i$ in $K[x_1, \ldots, x_n]$.

Proof. It is a direct consequence of the following fact. Let $S$ be a homogenous subspace of $K[x_1, \ldots, x_n]$, which is an invariant with respect to $GL_n$ of degree $m$. Then $S = S^{p^k}_{m/p^k}, p = \text{Char}(K)$, $S_l$ is the space of all polynomials of degree $l$.

Lemma 4.12 follows from Lemma 4.18 same as in the proofs of Corollaries 4.14 and 4.16.

4.4. Aut(TAut) for general case. Now we consider the case when \text{Char}(K) is arbitrary, i.e. consider remaining case $\text{Char}(K) = 2$. Still $|K| = \infty$. Although we are unable to prove the analogue of Proposition 4.3, we can still play on the relations.

Let $M = a \prod x_i^{k_i}$ be monomial, $a \in K$. For polynomial $P(x,y) \in K[x,y]$ we define the elementary automorphism
\[ \psi_P : x_i \to x_i, i = 1, \ldots, n - 1, x_n \to x_n + P(x_1, \ldots, x_{n-1}). \]
We have $P = \sum M_j$ and $\psi_P$ naturally decomposes as a product of commuting $\psi_{M_j}$. Let $\Psi \in \text{Aut}(\text{TAut}(K[x,y,z]))$ stabilizing linear mappings and $\phi$ (Automorphism $\phi$ defined in the lemma 4.13). Then according to the corollary 4.8 $\Psi(\psi_P) = \prod \Psi(\psi_{M_j})$. If $M = ax^n$ then due to Lemma 4.13
\[ \Psi(\psi_M) = \psi_M. \]
We have to prove the same for other type of monomials:

**Lemma 4.19.** Let $M$ be a monomial. Then

$$\Psi(\psi_M) = \psi_M.$$

**Proof.** Let $M = a \prod_{i=1}^{n-1} x_i^{k_i}$. Consider the automorphism $\alpha : x_i \rightarrow x_i + x_1, i = 2, \ldots, n - 1; x_1 \rightarrow x_1, x_n \rightarrow x_n$. Then

$$\alpha^{-1} \psi_M \alpha = \psi_{x_1^{k_1}} \prod_{i=2}^{n-1} (x_i + x_1)^{k_i} = \psi_Q \psi_{\sum_{i=2}^{n-1} k_i}$$

Here the polynomial

$$Q = x_1^{k_1} \left( \prod_{i=2}^{n-1} (x_i + x_1)^{k_i} - ax_1^{\sum_{i=2}^{n-1} k_i} \right)$$

It has the following form

$$Q = \sum_{i=2}^{n-1} N_i,$$

where $N_i$ are monomials such that none of them is proportional to a power of $x_1$.

According to Corollary 4.18 $\Psi(\psi_M) = \psi_{bM}$ for some $b \in K$. We need only to prove that $b = 1$. Suppose the contrary, $b \neq 1$. Then

$$\Psi(\alpha^{-1} \psi_M \alpha) = \left( \prod_{[N_i,x_1] \neq 0} \Psi(\psi_{N_i}) \right) \circ \Psi(\psi_{\sum_{i=2}^{n-1} k_i}) = \left( \prod_{[N_i,x_1] \neq 0} \psi_{bN_i} \right) \circ \psi_{\sum_{i=2}^{n-1} k_i}$$

for some $b_i \in K$.

From the other hand

$$\Psi(\alpha^{-1} \psi_M \alpha) = \alpha^{-1} \Psi(\psi_M) \alpha = \alpha^{-1} \psi_{bM} \alpha = \left( \prod_{[N_i,x_1] \neq 0} \psi_{bN_i} \right) \circ \psi_{\sum_{i=2}^{n-1} k_i}$$

Comparing the factors $\psi_{\sum_{i=2}^{n-1} k_i}$ and $\psi_{\sum_{i=2}^{n-1} k_i}$ in the last two products we get $b = 1$. Lemma 4.19 and hence the proposition 4.3 are proved.
5. The Automorphism group of $\text{TAut}(K\langle x_1, \ldots, x_n \rangle)$ ($n > 2$)

Now consider the noncommutative case. We treat the case $n > 3$ on the group-theoretical level and case $n = 3$ on Ind-scheme level. Note that if $n = 2$ then $\text{Aut}_0(K[x, y]) = \text{TAut}_0(K[x, y]) \simeq \text{TAut}_0(K\langle x, y \rangle) = \text{Aut}_0(K\langle x, y \rangle)$ and description of automorphism group of such object is known due to J.Deserty.

5.1. The automorphism group of the tame automorphisms group of $K\langle x_1, \ldots, x_n \rangle$, $n \geq 4$.

**Proposition 5.1** (E.Rips, private communication). Let $n > 3$ and let $\varphi$ preserves the standard torus action for a free associative algebra. Then $\varphi$ preserves all elementary transformations.

**Corollary 5.2.** Let $\varphi$ satisfies the conditions of the proposition. Then $\varphi$ preserves all tame automorphisms.

For free associative algebras, we note that any automorphism preserving torus action preserves also symmetric

$$x_1 \rightarrow x_1 + \beta(x_2x_3 + x_3x_2), x_i \rightarrow x_i, i > 1$$

and the skew symmetric

$$x_1 \rightarrow x_1 + \beta(x_2x_3 - x_3x_2), x_i \rightarrow x_i, i > 1$$

elementary automorphisms. The first property follows from Lemma 4.9. The second follows from the fact that the skew symmetric automorphisms commute with automorphisms of the following type

$$x_2 \rightarrow x_2 + x_3^2, x_i \rightarrow x_i, i \neq 2$$

and this property distinguish them from elementary automorphisms of the type

$$x_1 \rightarrow x_1 + \beta x_2x_3 + \gamma x_3x_2, x_i \rightarrow x_i, i > 1.$$
Theorem 1.2 follows from the fact that only forms $\beta x_2 x_3 + \gamma x_3 x_2$ corresponding to multiplication preserving the associative law when either $\beta = 0$ or $\gamma = 0$ and the approximation issue (see section 3.3).

**Proposition 5.3.** The group $G$ containing all linear transformations and mappings $x \to x, y \to y, z \to z + xy, t \to t$ contains also all the transformations of form $x \to x, y \to y, z \to z + P(x, y), t \to t$.

**Proof.** It is enough to prove that $G$ contains all transformations of the following form $x \to x, y \to y, z \to z + aM, t \to t; a \in K, M$ is monomial.

**Step 1.** Let

$$M = a \prod_{i=1}^{n} x^{k_i} y^{l_i} \text{ or } M = a \prod_{i=1}^{n} y^{l_i} x^{k_i} y^{l_i}$$

or

$$M = a \prod_{i=1}^{n} x^{k_i} y^{l_i} \text{ or } M = a \prod_{i=1}^{n} x^{k_i} y^{l_i} x^{k_{n+1}}.$$ 

$H(M) - \text{Height of } M$ is number of different powers of variables needed to compose $M$. (For example, $H(a \prod_{i=1}^{n} x^{k_i} y^{l_i} x^{k_{n+1}}) = 2n + 1$.) Using induction on $H$ one can reduce situation to the case when $M = x^k y$. Let $M = M' x^k$ such that $H(M') < H(M)$. (Case when $M = M' y^l$ is similar.) Let

$$\phi : x \to x, y \to y, z \to z + M', t \to t.$$ 

$$\alpha : x \to x, y \to y, z \to z, t \to t + zx^k.$$ 

Then

$$\phi^{-1} \circ \alpha \circ \phi : x \to x, y \to y, z \to z, t \to t + M + zx^k.$$ 

Automorphism $\phi^{-1} \circ \alpha \circ \phi$ is the composition of automorphisms $\beta : x \to x, y \to y, z \to z, t \to t + M$ and $\gamma : x \to x, y \to y, z \to z, t \to t + zx^k$. $\beta$ is conjugated to the automorphism $\beta' : x \to x, y \to y, z \to z + M, t \to t$ by the linear automorphism $x \to x, y \to y, z \to t, t \to z$. 
similarly \( \gamma \) is conjugated to automorphism \( \gamma' : x \to x, y \to y, z \to z + yx^k, t \to t \). We have reduced to the case when \( M = x^k \) or \( M = yx^k \).

**Step 2.** Consider automorphisms \( \alpha : x \to x, y \to y + x^k, z \to z, t \to t \) and \( \beta : x \to x, y \to y, z \to z, t \to t + azy \). Then
\[
\alpha^{-1} \circ \beta \circ \alpha : x \to x, y \to y, z \to z, t \to t + azx^k + azy.
\]
It is composition of automorphism \( \gamma : x \to x, y \to y, z \to z, t \to t + axy, t \to t \) and then to the automorphism \( \delta : x \to x, y \to y, z \to z + x, t \to t \) (using similarities). We reduced the problem to proving inclusion \( G \ni \psi_M, M = x^k \) for all \( k \).

**Step 3.** Obtaining the automorphism \( x \to x, y \to y + x^n, z \to z, t \to t \). Similar to the commutative case of \( k[x_1, \ldots, x_n] \) (see section 4).

Proposition 5.3 is proved.

 Let us formulate the Remark after Lemma 4.7 as follows:

**Lemma 5.4.** Consider the following \( \mathbb{T}^2 \) action: \( x_1 \to \lambda^1 x_1, x_j \to \lambda_j x_j, j > 1 \). Then the set \( S \) of automorphisms commuting with this action generated with following automorphisms \( x_1 \to x_1 + H, x_i \to x_i; i > 1, H \) is homogenous polynomial of the same degree as \( \prod_{j=2}^{n} x_{j}^{i_{j}} \) (\( \lambda = (\lambda_2, \ldots, \lambda_n), \beta, \lambda_j \in K \)).

Proposition 5.3 and Lemma 5.4 imply

**Corollary 5.5.** Let \( \Psi \in \text{Aut}_0(\text{TAut}(K(x_1, \ldots, x_n))) \) stabilizing all elements of torus and linear automorphisms,
\[
\phi_P : x_n \to x_n + P(x_1, \ldots, x_{n-1}), x_i \to x_i, i = 1, \ldots, n - 1.
\]
Let \( P = \sum_{I} P_I, P_I - \text{homogenous component of } P \text{ of multi-degree } I \). Then
\[\text{a)} \ \Psi(\phi_P) : x_n \to x_n + P^\Psi(x_1, \ldots, x_{n-1}), x_i \to x_i, i = 1, \ldots, n - 1.\]
b) \( P^\Psi = \sum_I P_I^\Psi \); where \( P_I^\Psi \) – homogenous of multi-degree \( I \).

c) If \( I \) has positive degree respect to one or two variables, then \( P_I^\Psi = P_I \).

Let \( \Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle)) \) stabilizing all elements of torus and linear automorphisms,

\[
\phi : x_n \rightarrow x_n + P(x_1, \ldots, x_{n-1}), x_i \rightarrow x_i, i = 1, \ldots, n-1.
\]

Let \( \varphi_Q : x_1 \rightarrow x_1, x_2 \rightarrow x_2, x_i \rightarrow x_i + Q_i(x_1, x_2), i = 3, \ldots, n-1, x_n \rightarrow x_n; Q = (Q_3, \ldots, Q_{n-1}) \). Then \( \Psi(\varphi_Q) = \varphi_Q \) due to Proposition 5.3.

**Lemma 5.6.**

a) \( \varphi_Q^{-1} \circ \phi_P \circ \varphi_Q = \phi_P \), where

\[
P_Q(x_1, \ldots, x_{n-1}) = P(x_1, x_2, x_3 + Q_3(x_1, x_2), \ldots, x_{n-1} + Q_{n-1}(x_1, x_2)).
\]

b) Let \( P_Q = P_Q^{(1)} + P_Q^{(2)}, P_Q^{(1)} \) consists of all terms containing one of the variables \( x_3, \ldots, x_{n-1} \), \( P_Q^{(1)} \) consists of all terms containing just variables \( x_1, x_2 \). Then \( P_Q^\Psi = P_Q^\Psi = P_Q^{(1)} + P_Q^{(2)} + P_Q^{(1)} + P_Q^{(2)}. \)

**Lemma 5.7.** If \( P_Q^{(2)} = R_Q^{(2)} \) for all \( Q \) then \( P = R \).

**Proof.** It is enough to prove that if \( P \neq 0 \) then \( P_Q^{(2)} \neq 0 \) for appropriate \( Q = (Q_3, \ldots, Q_{n-1}) \). Let \( m = \text{deg}(P), Q_i = x_1^{2i+1}x_2^{2i+1}m \). Let \( \hat{P} \) be the highest degree component of \( P \), then \( \hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \) is the highest component of \( P_Q^{(2)} \). It is enough to prove that

\[
\hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \neq 0.
\]

Let \( x_1 \prec x_2 \prec x_2 \prec \cdots \prec x_{n-1} \). Consider the lexicographically minimal term \( M \) of \( \hat{P} \). It is easy to see that the term

\[
M|_{Q_i \rightarrow x_i}, i = 3, n-1
\]

can not be cancel with any other term

\[
N|_{Q_i \rightarrow x_i}, i = 3, n-1
\]

of \( \hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \). Hence \( \hat{P}(x_1, x_2, Q_3, \ldots, Q_{n-1}) \neq 0. \)

Lemmas 5.6 and 5.7 imply
Corollary 5.8. Let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilizing all elements of torus and linear automorphisms. Then $P^\Psi = P$ and $\Psi$ stabilizes all elementary automorphisms, hence $\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle)$.

We get the following

Proposition 5.9. Let $n \geq 4$. Let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilizing all elements of torus and linear automorphisms and automorphism $EL : x_i \to x_i; i = 1, \ldots, x_n \to x_n + x_1 x_2$. Then $\Psi = \text{Id}$.

Let $n \geq 4$. Let $\Psi \in \text{Aut}(\text{TAut}_0(K\langle x_1, \ldots, x_n \rangle))$ stabilizing all elements of torus and linear automorphisms. We have to prove that $\Psi(EL) = EL$ or $\Psi(EL) : x_i \to x_i; i = 1, \ldots, x_n \to x_n + x_2 x_1$. In the last case $\Psi$ is the conjugation with mirror anti-automorphism of $K\langle x_1, \ldots, x_n \rangle$. In any case

$\Psi(EL) : x_i \to x_i; i = 1, \ldots, x_n \to x_n + x_1 * x_2,$

where $x * y = axy + byx; a, b \in K$.

The next lemma can be obtained by direct computation:

Lemma 5.10. The operation $*$ is associative if and only if $a = 0$ or $b = 0$. Moreover, $x^2 * y = (x * (x * y))$ if and only if $ab = 0$.

It mean that $*$ is either associative or non-alternative operation.

Now we are ready to prove Proposition 5.9. Consider the automorphisms

$\alpha : x \to x, y \to y, z \to z + xy, t \to t$

$\beta : x \to x, y \to y, z \to z, t \to t + xz,$

$h : x \to x, y \to y, z \to z, t \to t - xz.$

Then

$\gamma = h\alpha^{-1}\beta\alpha : (x \to x, y \to y, z \to z, t \to t + xxy)$

and

$\Psi(\gamma) : x \to x, y \to y, z \to z, t \to t + x * (x * y).$
Let $\delta : x \to x, y \to y, z \to z + x^2, t \to t$,
$$\epsilon : x \to x, y \to y, z \to z, t \to t + zy.$$ Let $\gamma = \epsilon^{-1}\delta^{-1}\epsilon\delta$. Then
$$\gamma : x \to x, y \to y, z \to z, t \to t + x^2y.$$ On the other hand we have $\varepsilon = \Psi(\epsilon^{-1}\delta^{-1}\epsilon\delta) : x \to x, y \to y, z \to z, t \to t + (x^2) \ast y$. We also have $\varepsilon = \gamma$. Equality $\Psi(\varepsilon) = \Psi(\gamma)$ is equivalent to the equality $x \ast (x \ast y) = x^2y$. We conclude.

5.2. The group $\text{Aut}_{\text{Ind}}(\text{TAut}(K\langle x, y, z \rangle))$. This is the most technical part of this article. We are unable to treat this situation on the group theoretical level. In this section we shall determine just $\text{Aut}_0(\text{TAut}(K\langle x, y, z \rangle))$, i.e. Ind-scheme automorphisms and prove Theorem 1.8. We use the approximation results of Section 3.3. In the sequel, we suppose that $\text{Char}(K) \neq 2$. \{a, b, c\}$_*$ denotes associator of $a, b, c$ respect to operation $\ast$, i.e.
$$\{a, b, c\} = (a \ast b) \ast c - a \ast (b \ast c).$$ Let $\Psi \in \text{TAut}_0(K\langle x, y, z \rangle)$ be an Ind-scheme automorphism, stabilizing linear automorphisms.

In this section, we work only on the Ind-scheme level.

**Proposition 5.11.** Let $\Psi \in \text{Aut}_{\text{Ind}}(\text{TAut}_0(k\langle x, y, z \rangle))$ stabilizing linear automorphisms. Let $\phi : x \to x, y \to y, z \to z + xy$. Then $\Psi(\phi) : x \to x, y \to y, z \to z + axy$ or $\Psi(\phi) : x \to x, y \to y, z \to z + ayx$.

**Proof.** Consider the automorphism $t : x \to x, y \to y, z \to z + xy$. Then $\Psi(t) : x \to x, y \to y, z \to z + x \ast y, x \ast y = axy + byx$. Due to conjugation on the mirror anti-automorphism and coordinate exchange one can suppose that $x \ast y = xy + \lambda yx$. We have to prove that $\lambda = 0$. In that case $\Psi = \text{Id}$. 
Lemma 5.12. Let \( A = K\langle x, y, z \rangle \). Let \( a \ast b = ab + \lambda ba \). Then \( \{a, b, c\}_\ast = \lambda[b, [a, c]] \).

In particular \( \{a, b, a\}_\ast = 0 \), \( a \ast (a \ast b) - (a \ast a) \ast b = -\{a, a, b\}_\ast = \lambda[a, [a, b]] \), \( (b \ast a) \ast a - b \ast (a \ast a) = \{b, a, a\}_\ast = \lambda[a, [a, b]] \).

**Proof.** \( \{a, b, c\}_\ast = (a \ast b) \ast c - a \ast (b \ast c) = (ab + \lambda ba)c + \lambda c(ab + \lambda ba) = \lambda(bac + cab - acb - bca) = \lambda([b, ac] + [ca, b]) = \lambda[b, [ac]] \).

Lemma 5.13. Let \( \varphi_1 : x \to x + yz, y \to y, z \to z, \varphi_2 : x \to x, y \to y, z \to z + yx \), \( \varphi = \varphi_2^{-1} \varphi_1^{-1} \varphi_2 \varphi_1 \). Then modulo terms of order \( \geq 4 \) we have:

\[
\varphi : x \to x - y^2x, y \to y, z \to z + y^2z \quad \text{and} \quad \Psi(\varphi) : x \to x - y^*(yx), y \to y, z \to z + y^*(y \ast z).
\]

**Proof.** Direct computation.

Lemma 5.14. a) Let \( \phi_l : x \to x, y \to y, z \to z + y^2x \). Then \( \Psi(\phi) : x \to x, y \to y, z \to z + y^2z \) and \( \Psi(\varphi) : x \to x - y^*(yx), y \to y, z \to z + y^*(y \ast z) \).

b) Let \( \phi_r : x \to x, y \to y, z \to z + xy^2 \). Then \( \Psi(\phi) : x \to x, y \to y, z \to z + (x \ast y) \ast y \).

**Proof.** According to the results of the previous section we have \( \Psi(\phi_l) : x \to x, y \to y, z \to z + P(y, x) \) where \( P(y, x) \) is homogenous of degree 2 respect to \( y \) and degree 1 respect to \( x \). We have to prove that \( H(y, x) = P(y, x) - y^*(yx) = 0 \).

Let \( \tau : x \to z, y \to y, z \to x; \tau = \tau^{-1}, \phi' = \tau \phi_l \tau^{-1} : x \to x + y^2z, y \to y, z \to z \). Then \( \Psi(\phi_l) : x \to x + P(y, z), y \to y, z \to z \).

Let \( \phi''_l = \phi_l \phi'_l : x \to x + P(y, z), y \to y, z \to z + P(y, x) \) modulo terms of degree \( \geq 4 \).

Let \( t : x \to x - z, y \to y, z \to z, \varphi_2, \varphi \) be the automorphisms described in Lemma 5.13.

Then \( T = t^{-1} \phi_l^{-1} t \phi''_l : x \to x, y \to y, z \to z \) modulo terms of order \( \geq 4 \).
On the other hand \( \Psi(T) : x \to x + H(y, z) - H(y, x), y \to y, z \to z + P \) modulo terms of order \( \geq 4 \). Because \( \deg_y(H(y, x)) = 2, \deg_x(H(y, x)) = 1 \) we get \( H = 0 \).

Proof of b) is similar.

**Lemma 5.15.** a) Let

\[
\psi_1 : x \to x + y^2, y \to y, z \to z, \psi_2 : x \to x, y \to y, z \to z + x^2.
\]

Then

\[
[\psi_1, \psi_2] = \psi_2^{-1}\psi_1^{-1}\psi_2\psi_1 : x \to x, y \to y, z \to z + y^2x + xy^2,
\]

\[
\Psi([\psi_1, \psi_2]) : x \to x, y \to y, z \to z + (y \ast y) \ast x + x \ast (y \ast y).
\]

b)

\[
\phi_l^{-1}\phi_r^{-1}[\psi_1, \psi_2] : x \to x, y \to y, z \to z
\]

modulo terms of order \( \geq 4 \) but

\[
\Psi(\phi_l^{-1}\phi_r^{-1}[\psi_1, \psi_2]) : x \to x, y \to y, z \to z + (y \ast y) \ast x + x \ast (y \ast y) - (x \ast y)y \ast y(\ast y \ast x) =
\]

\[
= z + 4\lambda[x[x, y]]
\]

modulo terms of order \( \geq 4 \).

**Proof.** a) can be obtained by direct computation. b) follows from a) and Lemma 5.12.

Proposition 5.11 follows from Lemma 5.15.

We need some auxiliary lemmas. The first is an analogue of the hiking procedure from [19, 3].

**Lemma 5.16.** Let \( K \) be algebraically closed, \( n_1, \ldots, n_m \) positive integers. Then there exist \( k_1, \ldots, k_s \in \mathbb{Z} \) and \( \lambda_1, \ldots, \lambda_s \in K \) such that

- \( \sum k_i = 1 \) modulo \( \text{Char}(K) \) (if \( \text{Char}(K) = 0 \) then \( \sum k_i = 1 \)).
- \( \sum_i k_i^{n_j} \lambda_i = 0 \) for all \( j = 1, \ldots, m \).

For \( \lambda \in K \) we define an automorphism \( \psi_\lambda : x \to x, y \to y, z \to \lambda z \).
The next lemma provides some translation between language of polynomials and group action language. It is similar to the hiking process \([3, 19]\).

**Lemma 5.17.** Let \(\varphi \in K\langle x, y, z \rangle\). Let \(\varphi(x) = x, \varphi(y) = y + \sum_i R_i + R', \varphi(z) = z + Q\). Let \(\deg(R_i) = N\), degree of all monomials of \(R'\) strictly greater then \(N\), degree of all monomials of \(Q\) greater equal \(N\), \(\deg_z(R_i) = i\), \(z\)-degree of all monomials of \(R_1\) strictly greater then 0.

Then

a) \(\psi^{-1}_\lambda \varphi \psi_{\lambda} : x \to x, y \to y + \sum_i \lambda^i R_i + R'' , z \to z + Q'. \) And degree of all monomials of \(R'\) strictly greater then \(N\), degree of all monomials of \(Q\) grater equal \(N\).

b) Let \(\phi = \prod (\psi^{-1}_{\lambda_i} \varphi \psi_{\lambda_i})^{k_i}\). Then

\[
\phi : x \to x, y \to y + \sum_i R_i \lambda_i^{k_i} + S, z \to z + T
\]
degree of all monomials of \(S\) strictly grater then \(N\), degree of all monomials of \(T\) grater equal \(N\).

**Proof.** a) Direct calculation, b) It follows from a).

**Remark.** In the case of characteristic zero, the condition of \(K\) to be algebraically closed can be released. After hiking of several steps, we need to prove just

**Lemma 5.18.** Let \(\text{Char}(K) = 0, n \) is a positive integer. Then there exist \(k_1, \ldots, k_s \in \mathbb{Z}\) and \(\lambda_1, \ldots, \lambda_s \in K\) such that

- \(\sum k_i = 1\).
- \(\sum_i k_i^n \lambda_i = 0\).

Using this lemma we can cancel all terms in the product in the lemma \([3.17]\) but the constant. The proof of Lemma \([5.18]\) for any field of zero characteristics can be obtained based on the following observation:
Lemma 5.19.  
\[
\left(\sum_{i=1}^{n} \lambda_i\right)^n - \sum_{j} (\lambda_1 + \cdots + \lambda_i + \cdots + \lambda_n)^n + \cdots + 
\]
\[
+(-1)^{n-k} \sum_{i_1<\cdots<i_k} (x_{i_1} + \cdots + x_{i_k})^n + \cdots + (-1)^{n-1}(x_1^n + \cdots + x_n^n) = n! \prod_{i=1}^{n} x_i
\]

and if \(m < n\) then
\[
\left(\sum_{i=1}^{n} \lambda_i\right)^m - \sum_{j} (\lambda_1 + \cdots + \lambda_i + \cdots + \lambda_n)^m + \cdots + 
\]
\[
+(-1)^{n-k} \sum_{i_1<\cdots<i_k} (x_{i_1} + \cdots + x_{i_k})^m + \cdots + (-1)^{n-1}(x_1^m + \cdots + x_n^m) = 0.
\]

Lemma 5.19 allows us to replace \(n\)-th powers by product of different constant, then statement of Lemma 5.18 became easy.

Lemma 5.20. Let \(\varphi : x \rightarrow x + R_1, y \rightarrow y + R_2, z \rightarrow z'\), degree of all monomials of \(R_1, R_2\) greater equal \(N\). Then \(\Psi(\varphi) : x \rightarrow x + R_1', y \rightarrow y + R_2', z \rightarrow z''\), degree of all monomials of \(R_1', R_2'\) greater equal \(N\).

Proof. Similar to the proof of Theorem 3.2

Lemmas 5.20, 5.17, 5.16 imply the following statement.

Lemma 5.21. Let \(\varphi_j \in \text{Aut}_0(K\langle x, y, z \rangle), j = 1, 2\). Let
\[
\varphi_j(x) = x, \quad \varphi_j(y) = y + \sum_i R_i^j + R_j', \varphi_j(z) = z + Q_j.
\]

Let \(\deg(R_i^j) = N\), degree of all monomials of \(R_j\) strictly grater then \(N\), degree of all monomials of \(Q\) grater equal \(N\), \(\deg_z(R_i) = i\), \(z\)-degree of all monomials of \(R_i\) strictly greater then 0. Let \(R_0^1 = 0, R_0^2 \neq 0\).

Then \(\Psi(\varphi_1) \neq \varphi_2\).

Consider the automorphism
\(\phi : x \rightarrow x, y \rightarrow y, z \rightarrow z + P(x, y)\). Let \(\Psi \in \text{TAut}_0(k\langle x, y, z \rangle)\) which is identical on the standard torus action. Then
\[
\Psi(\phi) : x \rightarrow x, y \rightarrow y, z \rightarrow z + Q(x, y).
\]
We denote
\[ \Psi(P) = Q. \]

Our goal is to prove that \( \Psi(P) = P \) for all \( P \) if \( \Psi \) stabilizes the linear automorphisms and \( \Psi(xy) = xy \).

**Lemma 5.22.**
\[ \Psi(x^k y^l) = x^k y^l \]

**Proof.** Let \( \phi : x \to x, y \to y, z \to z + x^k y^l, \)
\[ \varphi_1 : x \to x + y^l, y \to y, z \to z, \]
\[ \varphi_2 : x \to x, y \to y + x^k, z \to z, \]
\[ \varphi_3 : x \to x, y \to y, z \to z + xy, \]
\[ h : x \to x, y \to y, z \to z - x^{k+1}. \]

Then
\[ g = h\varphi_3^{-1}\varphi_2^{-1}\varphi_3\varphi_2 : x \to x, y \to y, z \to z + x^k \cdot y^l + N; \]

\( N \) is the sum of terms of degree strictly greater than \( k + l \). It means that \( g = \phi \circ L, L \in H_{k+l+1} \). We shall use theorem 3.2. Applying \( \Psi \) we get the result because \( \Psi(\varphi_i) = \varphi_i, i = 1, 2, 3 \) and \( \varphi(H_n) \subseteq H_n \) for all \( n \). The lemma is proved.

Let
\[ M_{k_1,\ldots,k_s} = x^{k_1} y^{k_2} \cdots y^{k_s} \]
for even \( s \) and
\[ M_{k_1,\ldots,k_s} = x^{k_1} y^{k_2} \cdots x^{k_s} \]
for odd \( s \), \( k = \sum_{i=1}^{n} k_i \). Then
\[ M_{k_1,\ldots,k_s} = M_{k_1,\ldots,k_{s-1}} y^{k_s} \]
for even \( s \) and
\[ M_{k_1,\ldots,k_s} = M_{k_1,\ldots,k_{s-1}} x^{k_s} \]
for odd \( s \).
We have to prove that \( \bar{\Psi}(M_{k_1,\ldots,k_s}) = M_{k_1,\ldots,k_s} \). By induction we may assume that \( \bar{\Psi}(M_{k_1,\ldots,k_{s-1}}) = M_{k_1,\ldots,k_{s-1}} \).

For any monomial \( M = M(x,y) \) we shall define an automorphism \( \varphi_M : x \rightarrow x, y \rightarrow y, z \rightarrow z \).

We also define the automorphisms \( \phi_k^e : x \rightarrow x, y \rightarrow y + zx^k, z \rightarrow z \) and \( \phi_k^o : x \rightarrow x + zy^k, y \rightarrow y, z \rightarrow z \). We shall treat case of even \( s \). Odd case is similar.

Let \( D^e_{zx^k} \) be the derivation of \( K(x,y,z) \) such that \( D^e_{zx^k}(x) = 0, D^e_{zx^k}(y) = zx^k, D^e_{zx^k}(z) = 0 \). Similarly \( D^o_{zy^k} \) be derivation of \( k(x,y,z) \) such that \( D^o_{zy^k}(y) = 0, D^o_{zy^k}(x) = zy^k, D^o_{zy^k}(z) = 0 \).

The next lemma can be obtained via direct computation:

**Lemma 5.23.** Let

\[
    u = \phi_k^e -1 \varphi(M_{k_1,\ldots,k_{s-1}})^{-1} \phi_k^e \varphi(M_{k_1,\ldots,k_{s-1}})
\]

for even \( s \) and

\[
    u = \phi_k^o -1 \varphi(M_{k_1,\ldots,k_{s-1}})^{-1} \phi_k^o \varphi(M_{k_1,\ldots,k_{s-1}})
\]

for odd \( s \). Then

\[
    u : x \rightarrow x, y \rightarrow y + M_{k_1,\ldots,k_s} + N', z \rightarrow z + D^e_{zx^k}(M_{k_1,\ldots,k_{s-1}}) + N
\]

for even \( s \) and

\[
    u : x \rightarrow x + M_{k_1,\ldots,k_s} + N', y \rightarrow y, z \rightarrow z + D^o_{zy^k}(M_{k_1,\ldots,k_{s-1}}) + N
\]

for odd \( s \).

where \( N, N' \) are the sums of terms of degree strictly more then \( k = \sum_{i=1}^s k_i \).

Let \( \psi(M_{k_1,\ldots,k_s}) : x \rightarrow x, y \rightarrow y, z \rightarrow z + M_{k_1,\ldots,k_s} \),

\[
    \alpha_e : x \rightarrow x, y \rightarrow y - z, z \rightarrow z, \alpha_o : x \rightarrow x - z, y \rightarrow y, z \rightarrow z,
\]

Let \( P_M = \Psi(M) - M \). Our goal is to prove that \( P_M = 0 \).
Let
\[ v = \psi(M_{k_1, \ldots, k_s})^{-1} \alpha \psi(M_{k_1, \ldots, k_s}) u \alpha^{-1} \]
for even \( s \) and
\[ v = \psi(M_{k_1, \ldots, k_s})^{-1} \alpha \psi(M_{k_1, \ldots, k_s}) u \alpha^{-1} \]
for odd \( s \).

The next lemma can also be obtained by direct computation:

**Lemma 5.24.** a)
\[ v : x \to x, y \to y + H, z \to z + H_1 + H_2 \]
for even \( s \) and
\[ v : x \to x + H, y \to y, z \to z + H_1 + H_2 \]
for odd \( s \)

b)
\[ \Psi(v) : x \to x, y \to y + P_{M_{k_1, \ldots, k_s}} + \tilde{H}, z \to z + \tilde{H}_1 + \tilde{H}_2 \]
for even \( s \) and
\[ \Psi(v) : x \to x + P_{M_{k_1, \ldots, k_s}} + \tilde{H}, y \to y, z \to z + \tilde{H}_1 + \tilde{H}_2 \]
for odd \( s \)

where \( H_2, \tilde{H}_2 \) are sums of terms of degree strictly greater than \( k = \sum_{i=1}^{s} k_i \), \( H, \tilde{H} \) are the sums of terms of degree \( \geq k \) and positive degree with respect to \( z \), \( H_1, \tilde{H}_1 \) are sums of terms of degree \( k \) and positive degree respect to \( z \).

**Proof of Theorem 1.8.** Part b) follows from part a), in order to prove a) we need to prove that \( \bar{\Psi}(M) = M \) for any monomial \( M(x, y) \) and for any \( \Psi \in \text{Aut} \ T_{\text{Aut}}(\langle x, y, z \rangle) \) stabilizing standard torus \( T^3 \) and \( \phi \). Automorphism \( \Psi(\Phi_M) \) has a form, described in the lemma 5.24. But in this case Lemma 5.21 implies that \( \bar{\Psi}(M) = M = 0 \).
6. SOME OPEN QUESTIONS CONCERNING THE TAME AUTOMORPHISM GROUP

As the conclusion of this article, we would like to raise the following questions.

(1) Is it true that any automorphism $\varphi$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ (in the group theoretical sense) for $n = 3$ is semi-inner, i.e. is a conjugation via some automorphism or mirror anti-automorphism.

(2) Is it true that $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ is generated by affine automorphisms and automorphism $x_n \rightarrow x_n + x_1x_2, x_i \rightarrow x_i, i \neq n$? For $n = 3$ answer is negative, see Umirbaev [30], see also Drensky and Yu [16]. For $n \geq 4$ we think the answer is positive.

(3) Is it true that $\text{Aut}(K[x_1, \ldots, x_n])$ is generated by the linear automorphisms and automorphism $x_n \rightarrow x_n + x_1x_2, x_i \rightarrow x_i, i \neq n$? For $n = 3$ the answer is negative, see the proof of the Nataga conjecture [27, 28, 31]. For $n \geq 4$ it is plausible that the answer is positive.

(4) Is any automorphism $\varphi$ of $\text{Aut}(K\langle x_1, \ldots, x_n \rangle)$ (in the group theoretical sense) for $n = 3$ is semi-inner?

(5) Is it true that the conjugation in Theorems 1.3 and 1.7 can be done by some tame automorphism? Suppose $\psi^{-1}\varphi\psi$ is tame automorphism for any tame $\varphi$. Does it follow that $\psi$ is tame?

(6) Prove Theorem 1.8 for $\text{Char}(k) = 2$. Does it hold on the set theoretical level, i.e. $\text{Aut}(\text{TAut}(K\langle x, y, z \rangle))$ are generated by conjugations on automorphism or mirror anti-automorphism?

The similar questions can be proposed for nice automorphisms.

7. ACKNOWLEDGEMENTS

The authors would like to thank to J.P. Furter, T. Kambayashi, H.Kraft, R.Lipyanski, Boris Plotkin, Eugeny Plotkin, Andrey Regeta,
and M. Zaidenberg for stimulating discussion. We are grateful to Eliahu Rips as he kindly agrees to include and use his crucial results.

The authors also thank Shanghai University, Shanghai, Jilin University, Changchun and South University of Science and Technology of China, Shenzhen for warm hospitality and stimulating atmosphere during their visits, when part of this project was carried out.

References

[1] D.J. Anick, *Limits of tame automorphisms of $k[x_1,\ldots,x_N]$*, J. Algebra 82 (1983) 459-468.

[2] T. Asanuma, *Non-linearizable $k^*$-actions in affine space*, Invent. Math. 138 (1999) 281-306.

[3] A. Belov, *Local finite basis property and local representability of varieties of associative rings*, Izv. Rus. Acad. Sci. 74 (2010) 3-134. English transl. Izvestiya: Mathematics 74(2010) 1-126.

[4] A. Belov and R. Lipyanskiii, *Automorphisms of Endomorphism group of free associative-commutative algebra over an arbitrary field*, J. Algebra 333 (2010) 40-54.

[5] A. Belov, A. Berzins, and R. Lipiansky, *Automorphisms of endomorphism group of free associative algebra*, Int. J. Algebra Comp. 17 (2007) 923-939.

[6] A. Belov and M.L. Kontsevich, *Jacobian and Dixmier Conjectures are stably equivalent*. Moscow Math. J. 7 (2007)209-218 (A special volume dedicated to the 60-th anniversary of A.G. Khovanskii).

[7] A. Belov and M.L. Kontsevich, *Automorphisms of Weyl algebras*, Lett. Math. Phys. 74 (2005) 34-41 (A special volume dedicated to the memory of F.A. Berezin).

[8] A. Belov-Kanel and J.-T. Yu, *On the lifting of the Nagata automorphism*, 17 (2011) Selecta Math. (N.S.) 935-945.

[9] A. Belov-Kanel and J.-T. Yu, *Stable tameness of automorphisms of $F(x,y,z)$ fixing $z$*, 18 (2012) Selecta Math. (N.S) 799-802.

[10] G.M. Bergman, *Conjugates and nth roots in Hahn-Laurent group rings*, Bull. Malay. Math. Soc. 1 (1978) 29-41.

[11] G.M. Bergman, *Historical addendum to: “Conjugates and nth roots in Hahn-Laurent group rings”*, Bull. Malay. Math. Soc. 1 (1978) 29-41; 2 (1979) 41-42.
[12] A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on $k^n$ II*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. **15** (1967) 123-125.

[13] A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on $k^n$*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astro. Phys. **14** (1966) 177-181.

[14] Yu. Bodnarchuk, *Every regular automorphism of the affine Cremona group is inner*, J. Pure Appl. Algebra 157 (2001) 115-119.

[15] J. Deserti, *Sur le groupe des automorphismes polynomiaux du plan affine*, J. Algebra **297** (2006) 584-599.

[16] V. Drensky and J.-T. Yu, *The strong Anick conjecture is true*, J. Eur. Math. Soc. (JEMS) **9** (2007) 659-679.

[17] T. Kambayashi, *Inverse limits of polynomial rings*, Osaka J. Math. **41** (2004) 617-624.

[18] T. Kambayashi, *Some basic results on pro-affine algebras and ind-affine schemes*, Osaka J. Math. **40** (2003) 621-638.

[19] A. Kanel-Belov, L.H. Rowen and U. Vishne, *Full exposition of Specht’s problem*, Serdica Math. J. **38** (2012) 313-370.

[20] H. Kraft and I. Stampfli, *On Automorphisms of the Affine Cremona Group*, arXiv:1105.3739v1.

[21] Y.-C. Li and J.-T. Yu, *Degree estimate for subalgebras*, J. Algebra **362** (2012) 92-98.

[22] L. Makar-Limanov and J.-T. Yu, *Degree estimate for subalgebras generated by two elements*, J. Eur. Math. Soc. (JEMS) **10** (2008) 533-541.

[23] L. Makar-Limanov, *Algebraically closed skew fields*, J. Algebra **93** (1985) 117-135.

[24] B. Plotkin, Algebra with the same (algebraic geometry), in *Proc. of the Steklov Institut of Mathematics* **242** (2003) 176-207.

[25] I. R. Shafarevich, *On some infinitely dimensional groups II*, Izv. Akad. Sch. Ser. Math. **2** (1981) 214-216.

[26] M. K. Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra **58** (1989) 209–212.

[27] I. P. Shestakov and U. U. Umirbaev, *Degree estimate and two-generated subalgebras of rings of polynomials*, J. Amer. Math. Soc. **17** (2004) 181-196.

[28] I. P. Shestakov and U. U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. **17** (2004) 197-220.
[29] M. Suzuki, Proprits topologiques des polynomes de deux variables complexes, et automorphismes algebriques de l’espace $C^2$ (French), J. Math. Soc. Japan 26 (1974) 241-257.

[30] U. U. Umirbaev, The Anick automorphism of free associative algebras, J. angew. Math. (Crelles Journal) 605 (2007) 165-178.

[31] U. U. Umirbaev and J.-T. Yu, The strong Nagata conjecture, Proc. Natl. Acad. Sci. USA 101 (2004) 4352-4355.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel
E-mail address: beloval@cs.biu.ac.il, kanelster@gmail.com

Department of Mathematics, The University of Hong Kong, Hong Kong SAR, China
E-mail address: yujt@hku.hk, yujietai@yahoo.com