Self-organized vortex and antivortex patterns in laser arrays

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Recently it is shown that dissipatively coupled laser arrays simulate the classical XY model. We show that phase-locking of laser arrays can give rise to the spontaneous formation of vortex and antivortex phase patterns that are analogous to topological defects of the XY model. These patterns are stable although their formation is less likely in comparison to the ground state lasing mode. In addition, we show that small ratios of photon to gain lifetime destabilizes vortex and antivortex phase patterns. These findings are important for studying topological effects in optics as well as for designing laser array devices.

I. INTRODUCTION

The two-dimensional XY model consists of a lattice of interacting fixed-length spins that are constrained to rotate in a plane. The classical XY model in two spatial dimensions is governed by the Hamiltonian $H(\phi_1, \phi_2, \cdots, \phi_n)$:

$$H = \sum_{i,j} \kappa_{ij} \cos(\phi_i - \phi_j)$$

(1)

where, $\phi_1, \cdots, \phi_n$ represent the orientation of the $n$ spins, and $\kappa_{ij}$ is the interaction for the pair $i,j$. This model supports non-trivial equilibrium spin configurations — a class of topological defects — known as vortices, which are characterized by the phase of the spins going through a multiple of $2\pi$ as one traces a loop enclosing the vortex, e.g.,

$$\int d\phi = \pm 2\pi$$

(2)

for a single vortex/antivortex. In the 2D XY model, vortices have found applications in various areas of condensed matter physics including superfluid helium-4, superconductivity in thin films, liquid crystals, and the melting of 2D crystals.

![FIG. 1. A schematic of the equilibrium phase patterns of a dissipatively coupled laser array. (a) The ground state. (b) A vortex state. Here, the arrows represent the phase of each element.](image)

Recently, it has been realized that the classical XY model can be optically simulated with an array of coupled optical oscillators. What makes this possible is the random phase of a laser above oscillation thresholds, which emulates a classical spin confined to a two-dimensional plane. In addition, dissipative interaction facilitates synchronization of an array of lasers to a globally-phase-locked state. In this case, one can show that the laser array reaches an equilibrium phase pattern that locally minimizes a cost function that is asymptotically equivalent with the classical XY Hamiltonian. Accordingly, laser arrays have been utilized for simulating interesting phenomena related to spin systems such as geometric frustration.

In this work, we investigate the formation of vortex and antivortex singularities as self-organized phase patterns in laser arrays. As shown schematically in Fig. 1, these patterns are equilibrium states of laser arrays when reaching a globally-phase-locked state. From a nonlinear dynamics point of view, these topological defects are fixed-point solutions of nonlinear dynamical equations governing laser arrays. However, in general, they can be considered as metastable states with finite basins of attraction which restrict their formation to proper initial conditions. In addition, we find that the stability of these states depends critically on the gain level and lifetime. In the following, after introducing a dynamical model governing laser arrays, first we numerically investigate the formation of vortices. Next, we draw a connection between the governing dynamical model and a class of Ginzburg-Landau systems that are known to support vortex patterns. Finally, we investigate the stability of the vortex patterns with respect to the competing time scales of optical cavity and gain decay rates of the lasers.

II. MODEL

To build a dynamical model governing laser arrays, we first consider an array of passive and single-mode optical resonators that dissipatively interact. For the sake of simplicity, we assume all resonators being identical in resonance frequency $\omega_p$ and linewidth $1/\tau_p$. Thus, in the framework of the temporal coupled mode theory the complex modal am-
FIG. 2. Equilibrium phase patterns of an array of dissipatively coupled lasers arranged on a 16 × 16 square lattice. (a-d) The ground state, vortex, antivortex and bound vortex-antivortex states for the case of attractive coupling (κ > 0), associated with a ferromagnetic spin system [first row], and repulsive coupling (κ < 0), associated with an antiferromagnetic spin system [second row]. The XY energy level associated with these phase patterns are shown on top of the panels. Here, \( g_0 = 30, \kappa = -1 \) for the top row and \( \kappa = 1 \) for the bottom row.

The amplitude of the electric field in the \( i \)th resonator is governed by:

\[
\dot{a}_i(t) = -a_i - \gamma a_i - \sum_{j \neq i} \kappa_{ij} a_j, \tag{3}
\]

where, the equations are written within the gauge \( a_i \to a_i e^{-i\omega_0 t} \) and time is normalized to the photon lifetime \( \tau_p \). In this relation, \( \kappa_{ij} \) represents the rate of dissipative coupling between the \( i \)th and \( j \)th resonators, \( \gamma = \sum_{j} |\kappa_{ij}| \) is the external loss of the \( i \)th resonator due to its coupling with other resonators as demanded by conservation relations. Here, all coupling coefficients \( \kappa_{ij} \) are normalized to the photon decay rate \( 1/\tau_p \), while the choice of normalization for the complex amplitudes depends on the gain as discussed next.

By incorporating a saturable gain mechanism, equations (3) can be modified to support self-sustained finite stationary solutions of the field amplitudes. Here, we consider the gain being a dynamical variable as in the so-called class-B laser model. In this model, the gain of a laser oscillator is driven at a finite pump rate, while it decays linearly for small field intensities and nonlinearly when the field intensity grows. The normalized rate equations for the \( i \)th oscillator can then be written as:

\[
\dot{a}_i(t) = [g_i(t) - 1]a_i - \gamma a_i - \sum_{j \neq i} \kappa_{ij} a_j, \tag{4a}
\]
\[
\dot{g}_i(t) = (\tau_p/\tau_g)[g_0 - (1 + |a_i|^2)g_i], \tag{4b}
\]

where \( g_i \) represents the gain of the \( i \)th oscillator, \( g_0 \) is the pump parameter, and \( 1/\tau_g \) is the gain decay rate. In these relations both the field amplitude and gain are dimensionless and the time is normalized to the photon lifetime \( \tau_p \). This model has been applied to solid-state lasers. In addition, it can be generalized to model semiconductor lasers by incorporating the linewidth enhancement factor which plays an important role in the dynamics.

Equations (4) can be greatly simplified when the gain decay rate is much larger than the photon decay rate, i.e., \( 1/\tau_g \gg 1/\tau_p \). In this case, the gain almost instantaneously follows
the dynamics of the field. Thus, one can adiabatically eliminate the dynamics of the gain, i.e., $g_i(t) \approx 0$, to reach at an instantaneous nonlinear gain $g_i(|a_i|) = g_0/(1 + |a_i|^2)$. In this manner, one reaches at a reduced model that is suitable for a so-called class-A laser. Here, we use a polynomial gain term $g_i(a_i) = g_0(1 - |a_i|^2)$, which is enough to guarantee the bounded oscillations of the laser. Thus, we reach at the following reduced model,

$$
\dot{a}_i = [g_0(1 - |a_i|^2) - 1]a_i - \gamma a_i - \sum_{j \neq i} \kappa_{ij} a_j,
$$

which is accurate as long as the photon decay rate is smaller than the decay rates of the atomic degrees of freedom that give result to the gain. In this work, first we focus on simulating the dynamics of the field. Thus, we adiabatically eliminated the effect of the gain dynamics.

The dynamical model of Eq. (5) admits a Lyapunov function $F$ such that $\dot{a}_i = -\partial F / \partial a_i^*$, where

$$
F = \sum_i -[g_0(1 - |a_i|^2) - 1]a_i - \gamma a_i + \frac{1}{2} \sum_{j \neq i} \kappa_{ij} a_j.
$$

The fixed point solutions of the dynamical system of Eq. (5) are local minima of this Lyapunov function. In a recent work we showed that the diagonal term in this cost function behaves like a penalty term that tends to force all oscillators to a constant amplitude in the large gain limit, $g_0 \gg 1$. Thus, considering the stationary state solution of the oscillators as $a_i = \sqrt{T_i e^{i\theta_i}}$, by enforcing the condition of $J_i = I_0$, the cost function of Eq. (6) reduces to the XY Hamiltonian of Eq. (1).

### III. FORMATION OF TOPOLOGICAL DEFECTS

In the following, we investigate self-organization of topological defects by numerically simulating the dynamical model of Eqs. (5). We first consider an array of lasers arranged on a square lattice with uniform nearest-neighbor coupling of strength $\kappa$. Here, the gain is assumed to be large ($g_0 \gg 1$), so that steady-state amplitudes become nearly uniform and the phase pattern obeys the XY Hamiltonian. Figure 2 depicts the equilibrium phase patterns obtained by simulating a square lattice arrangement of $16 \times 16$ oscillators for both scenarios of attractive ($\kappa < 0$) and repulsive ($\kappa > 0$) coupling, which are respectively associated with ferromagnetic and antiferromagnetic cases. In both cases, different stable patterns are observed, including the ground state, isolated vortex and antivortex states, and paired vortex-antivortex states. The XY energy levels associated with these equilibrium phase patterns are listed in Fig. 2, which shows higher energy levels for the topological defects. The difference between the XY energy of the vortex (Fig. 2(b)) and the ground state (Fig. 2(a)) is comparable with the approximate formula $\Delta E = \pi \kappa \ln L$, where $L$ is the lattice length.

The stability of these fixed point solutions is directly evaluated through the Jacobian matrix of the dynamical system of Eqs. (5):

$$
J = g_0 \begin{pmatrix}
\text{diag}(1 - 2a \odot \bar{a}) & -I - Q \\
-\text{diag}(\bar{a} \odot \bar{a}) & \text{diag}(1 - 2a \odot \bar{a}) - I - Q
\end{pmatrix}.
$$

In this relation, $\bar{a} = (\bar{a}_1, \cdots, \bar{a}_n)^T$ is the stationary state, $I$ represents the identity matrix, $Q$ is the coupling matrix, where $q_{ij} = \kappa_{ij}$ and $q_{ii} = \sum_j |\kappa_{ij}|$, $\odot$ represents the entry-wise product, and diag creates a diagonal matrix of a given vector. The Jacobian matrix turns out to be a negative semi-definite matrix for all cases shown in Fig. 2.

**FIG. 5.** The XY energy distribution associated with the equilibrium phase patterns of a triangular lattice arrangement of lasers with (a) ferromagnetic ($\kappa < 0$), (b) anti-ferromagnetic ($\kappa > 0$) coupling. The lattice size and array parameters are the same as in Fig. 4.
FIG. 6. The transient dynamics of a $16 \times 16$ laser array. (a) Snapshots of the phase pattern at intermediate times. (b) Amplitudes and and phases of the array elements. In part (b), the dashed lines show the time instants associated with the snapshots in part (a). All parameters are the same as in Fig. 2.

IV. DISCRETE GINZBURG LANDAU EQUATION

In order to provide insight to the formation vortices in laser arrays, we draw a connection between the lattice model of Eq. (5) with its continuum counterpart, which turns out to be the well-known Ginzburg-Landau Equation (GLE). To understand this analogy, we focus our attention to the case of a 2D square lattice arrangement of lasers with uniform nearest neighbor coupling $\kappa$, and consider the ferromagnetic case ($\kappa < 0$). By considering a pair of integer indices $(m,n)$ for describing the horizontal and vertical coordinates of a square lattice, one can rewrite Eqs. (5) as:

$$a_{m,n}(t) = [g_0(1 - |a_{m,n}|^2) - 1]|a_{m,n} - \kappa L a_{m,n}$$

where $L$ is the discrete Laplacian operator on a square lattice graph that acts as $L a_{m,n} = a_{m-1,n} + a_{m+1,n} + a_{m,n-1} + a_{m,n+1} - 4a_{m,n}$. This relation is clearly in a finite difference form. To construct the continuum counterpart of this relation, we use $(m,n) \rightarrow (x,y)$, $a_{m,n}(t) \rightarrow \psi(x,y,t)$, and $L \rightarrow \nabla^2$, which results in the Ginzburg-Landau Equation (GLE):

$$\psi(x,y,t) = [g_0(1 - |\psi|^2) - 1]|\psi - \kappa \nabla^2 \psi.$$

Given that all coefficients are real-valued, despite the fact that $\psi$ is complex, this equation is often referred to as the real Ginzburg-Landau equation.

The dynamical equation (9) can be written in terms of a variation of a functional $\mathcal{F}[\psi]$, as $\psi = -\delta \mathcal{F}/\delta \psi^*$. The functional $\mathcal{F}$ is found to be:

$$\mathcal{F} = \int d\xi d\eta \left[-(g_0 - 1)|\psi|^2 + \frac{g_0}{2} |\psi|^4 + |\nabla \psi|^2\right],$$

which, can be considered as a counterpart of the Lyapunov function of Eq. (6). In an amplitude and phase representation...
ψ(r, φ, t) = R(r, φ, t) exp[iΦ(r, φ, t)], Eq. (9) can be rewritten as:

\[ \dot{R}(r, \phi, t) = [g_0(1 - R^2) - 1]R - \kappa(\nabla^2 - |\nabla \Phi|^2)R, \quad (11a) \]
\[ \dot{R}\Phi(r, \phi, t) = -\kappa(2\nabla R \cdot \nabla \Phi + R\nabla^2 \Phi). \quad (11b) \]

These equations are a special class of a reaction-diffusion system, named \( \lambda - \omega \) system.\(^{22}\) It is shown that these equations support stable single-arm spiral wave solutions of the form \( R(r, \phi, t) = R(r) \) and \( \Phi(r, \phi, t) = \phi \), while multi-arm spiral waves are unstable.\(^{22}\) Driven from this intuition, one would expect discrete counterparts of the spiral patterns in lattices of coupled lasers. An obvious deviation of the lattice model from the continuum case is the finite amplitude of the field at the center of the vortex. In the continuum scenario, the vanishing amplitude at the phase singularity guarantees a well-defined field. In the case of the lattice model, however, there is no such requirement. This is clearly irrelevant in the case of the lattice model, although the amplitudes still tend to be lower at the lattice sites that are located near the vortex center. Nevertheless, the amplitudes become uniform in the large gain limit \((g_0 \to \infty)\), where the laser system approaches the XY model.

V. VORTICES IN THE DYNAMIC GAIN REGIME

The results presented in Figs. 2 are based on the complex amplitude model with an instantaneous nonlinear saturable gain described by Eqs. (5). Here, we consider the more realistic scenario that involves gain as a dynamical variable as described through Eqs. (4). The gain lifetime \( \tau_p \), which in reality is dictated by the laser gain material, is found to play a critical role in the dynamics and in formation of stationary topological defects. As mentioned earlier, when the ratio of the photon lifetime over the gain lifetime is large \((\tau_p/\tau_g \gg 1)\), the gain dynamics can be effectively disregarded. Thus, in this regime, stable vortex and antivortex patterns are expected. It remains to investigate the stability of these topological defects when the photon to gain lifetime ratio becomes small \((\tau_p/\tau_g \sim 1) \text{ or } \tau_p/\tau_g \ll 1\).

To explore this aspect, we first simulated the dynamics of a small array of lasers for different values of \( \tau_p/\tau_g \) while using identical initial conditions. The results are depicted in Figs. 7. As these exemplary simulations indicate, by decreasing the ratio of \( \tau_p/\tau_g \), the transient dynamics becomes more complicated, involving effects like self-pulsations that are known processes in class-B lasers. On the other hand, these figures show that the vortex state does not form for smaller ratios of \( \tau_p/\tau_g \) (Figs. 7(b,c)).

This can be explained through the dynamics of the gain. According to Figs. 7(b,c), by increasing \( \tau_g \), the gain increases slowly and during this process, the laser array finds the time to escape from patterns associated with higher XY energy levels and to stabilize into the ground state phase pattern. This is in agreement with our recent study based on the reduced model, while the gain was adiabatically increased to avoid trapping into local minima\(^{11}\). Here, the dynamics allows the gain to automatically increase over a time scale governed by \( \tau_g \). To systematically explore this aspect, the laser array was simulated with an ensemble of random initial conditions and for three different time scale ratios. In addition, both cases of rectangular and triangular lattices were considered. The XY energy distributions of the associated steady-state phase patterns are shown in Fig. 8. As expected, for \( \tau_p/\tau_g \gg 1 \), the equilibrium energy distribution is similar to the case of instantaneous gain (Figs. 3(b) and 5(b)). As this ratio increases, however, the occurrence of higher energy patterns drops significantly.

These results clearly suggest that smaller ratios of \( \tau_p/\tau_g \) destabilize the vortex and antivortex patterns. However, it is quite interesting to note that these patterns pass the linear stability test in the both regimes of small and large time scale ratios. To show this, first we consider the fixed point solutions of Eqs. (4), i.e., \((\bar{a}_i(t), \bar{g}_i(t)) = (\bar{a}_i, \bar{g}_i)\). The fixed points are...
FIG. 8. The XY energy distribution associated with the equilibrium phase pattern of laser arrays governed by the dynamical equations [4] for different photon to gain lifetime ratios. Top: The square lattice for (a) $\tau_p/\tau_g = 10$, (b) 1.0, and (c) 0.01. Bottom: The triangular lattice for (a) $\tau_p/\tau_g = 10$, (b) 1.0, and (c) 0.01. Here, $\kappa = -1$ and $g_0 = 30$, while the lattice size is the same as in Figs. 3 and 4 for the top and bottom rows, respectively.

FIG. 9. The real part of the Jacobian matrix eigenvalues versus the photon to gain lifetime ratio $\tau_p/\tau_g$. These eigenvalues represent the linear stability test for a vortex pattern (shown as inset) in a 6 × 6 rectangular lattice array. The largest eigenvalue is highlighted in red.

VI. CONCLUSION

In summary, we investigated the formation of vortex and antivortex phase patterns in laser arrays. We showed that for large gains these patterns exist in direct analogy with topological defects of the XY model. However, their stability depends critically on the ratio of photon to gain lifetime $\tau_p/\tau_g$. An important finding was that large gain lifetimes ($\tau_p/\tau_g$) destabilize topological defects and force the system into the ground state.

The presence of self-organized vortex and antivortex phase patterns in laser arrays could have practical applications given that these patterns can emit optical vortex beams. It is worth noting that both solid-state and semiconductor lasers are considered class-B lasers given that in these systems the fluorescence lifetime is by several orders of magnitude larger than the photon lifetime ($\tau_p/\tau_g < 1$). A potential route for enforcing such laser arrays to create vortex phase patterns is to fix the phase of the boundary elements which could be done by seeding through a master laser. In this manner, vortex patterns can be enforced by fixing the topological charge of the array through boundary elements. However, it remains to investigate this aspect given that pinning the topological charge in the nonlinear dynamical systems discussed here can result in wandering of a vortex and potentially other instabilities.

Finally, it is worth mentioning that the formation of bound vortex-antivortex states suggests an analogy with the Berezinskii–Kosterlitz–Thouless (BKT) phase transition. In two spatial dimensions (2D), the theorem by Mermin and Wagner precludes conventional long-range order at any finite temperature in systems with continuous symmetry and short-range interactions[24]. Nevertheless, certain 2D systems can exhibit signs of a BKT transition from a high-temperature disordered state, with short-range correlations, to a quasi-ordered state, with algebraic correlations, below a critical temperature[24,25]. When viewed in terms of the vortices, the BKT transition is a vortex-unbinding transition. These topological objects, while tightly bound in pairs in the low-temperature state, unbind and freely proliferate above the critical temperature, causing the system to melt its quasi-order. While the present work sug-
gests such a transition, a rigorous investigation of this aspect requires a non-equilibrium thermodynamic treatment of the laser array which could be the subject of future studies.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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