FINITE-TIME BLOWUP OF SOLUTIONS TO SOME ACTIVATOR-INHIBITOR SYSTEMS

Grzegorz Karch
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Kanako Suzuki∗
College of Science, Ibaraki University
2-1-1 Bunkyo, Mito 310-8512, Japan

Jacek Zienkiewicz
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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Abstract. We study a dynamics of solutions to a system of reaction-diffusion equations modeling a biological pattern formation. This model has activator-inhibitor type nonlinearities and we show that it has solutions blowing up in a finite time. More precisely, in the case of absence of a diffusion of an activator, we show that there are solutions which blow up in a finite time at one point, only. This result holds true for the whole range of nonlinearity exponents in the considered activator-inhibitor system. Next, we consider a range of nonlinearities, where some space-homogeneous solutions blow up in a finite time and we show an analogous result for space-inhomogeneous solutions.

1. Introduction. In biology, a pattern formation mechanism is one of the most interesting subjects to understand. Mathematical analysis of such a phenomenon began by the seminal paper of Turing [25], where a notion of diffusion-driven instability was used to explain that a reaction between two chemicals with different diffusion rates may cause a destabilization of a spatially homogeneous state, thus leading to the formation of nontrivial spatial structure. Turing’s idea can be demonstrated by using reaction-diffusion equations for two interacting morphogens \((u, v)\):

\[
\begin{align*}
    u_t &= \varepsilon \Delta u + f(u, v), \\
    v_t &= D \Delta v + g(u, v),
\end{align*}
\]  

(1.1)

where \(\varepsilon, D\) are nonnegative constants. Many such models based on Turing’s idea have been proposed and there are several mathematical results on corresponding reaction-diffusion systems when both diffusion coefficients \(\varepsilon, D\) are positive.

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∗ Corresponding author: Kanako Suzuki.
Recently, however, the diffusion-driven instability has been observed in models describing a coupling of cell-localized processes with a cell-to-cell communication via diffusion. Such models are of a form of systems consisting of a single ordinary differential equation coupled with a reaction-diffusion equation:

\[ u_t = f(u, v), \quad v_t = D\Delta v + g(u, v), \]  

(1.2)
such as in Refs. [8, 12, 13, 18]. There are numerical simulations describing a formation of spatial patterns in such reaction-diffusion-ODE models, but there are few mathematical results on the existence and stability of stationary solutions (and their dependence on kinetics of such systems) and on a dynamics of non-stationary solutions. Our aim in this work is to contribute to the mathematical theory of system (1.2) with particular nonlinearities by showing that it has solutions which become unbounded (blow up) in a finite time.

In this work, we consider both systems (1.1) and (1.2) with the following nonlinearities (kinetics)

\[ f(u, v) = -au + \frac{u^p}{v^q}, \quad g(u, v) = -bv + \gamma \frac{u^r}{v^s}. \]  

(1.3)

Here, \( a, b, \gamma \) are positive constants, and the exponents \( p, q, r, s \) satisfy \( p > 1, q, r > 0 \) and \( s \geq 0 \). This is a typical kinetics based on Turing’s idea which leads to the so-called activator-inhibitor system. Reaction-diffusion equations (1.1) and (1.3) with both non-degenerate diffusion coefficients \( \varepsilon > 0 \) and \( D > 0 \) and under the assumption

\[ 0 < \frac{p - 1}{r} < \frac{q}{s + 1} \]  

(1.4)

were proposed by Gierer and Meinhardt [14] and have been widely used to model various biological pattern formations. In this work, we consider these systems in a bounded domain \( \Omega \subset \mathbb{R}^n \) with a sufficiently smooth boundary \( \partial \Omega \), supplemented with the Neumann boundary conditions and with nonnegative initial data.

First, we deal with equations (1.2)–(1.3), hence there is no-diffusion of the activator \( u(x, t) \). We show in Theorem 2.1 below that, for the whole range of exponents \( p, q, r, s \), there are solutions of the initial-boundary value problem for the considered reaction-diffusion-ODE system where the function \( u(x, t) \) blows up in a finite time and at one point.

Here, let us recall the work [17] which gives a complete description of entire dynamics of the kinetic system of ordinary differential equations associated with (1.1)–(1.3):

\[ \frac{d}{dt} \bar{u} = -a\bar{u} + \frac{\bar{u}^p}{\bar{v}^q}, \quad \frac{d}{dt} \bar{v} = -b\bar{v} + \gamma \frac{\bar{u}^r}{\bar{v}^s}. \]  

(1.5)

It turns out that this dynamics already exhibits various kinds of interesting behaviors including the convergence to the equilibria \((0, 0)\) and \((1, 1)\), periodic solutions, unbounded oscillating global solutions, and a blowup of solutions in finite time. In particular, if inequalities (1.4) and \( p - 1 \leq r \) are satisfied, then solutions of (1.5) are global-in-time, while there are solutions blowing up in finite time under the conditions (1.4) and \( p - 1 > r \). Thus, our Theorem 2.1 shows that the diffusion of the inhibitor described by \( v(x, t) \) induces a blowup of the space-inhomogeneous and non-diffusing activator \( u(x, t) \) – also in the case when space-homogeneous solutions are global-in-time, see Remarks 2.2 and 2.4 below for a discussion of this result.

The idea used in the proof of Theorem 2.1 can be applied also to reaction-diffusion equations (1.1) with nonlinearities (1.3) and with the diffusion coefficients \( \varepsilon \geq 0 \)
and \( D > 0 \). We do it in Theorem 2.3, where we show that the blowup phenomena described in Ref. [17] concerning kinetic system (1.5) hold also true in the case of space-inhomogeneous solutions to the system of the reaction-diffusion equations (1.1) and (1.3), see Remark 2.4 for more comments.

Concerning the existence and boundedness of solutions to initial-boundary value problems for reaction-diffusion equations (1.1) and (1.3) with both \( \varepsilon > 0 \) and \( D > 0 \), there are several results, see, e.g., [22, 11, 7, 23, 3, 26]. In particular, under the assumption \((p-1)/r < 2/(N+2)\), Masuda and Takahashi [11] proved that all solutions exist for all \( t > 0 \) and are uniformly bounded in time in the case when the first equation in (1.1) and (1.3) is supplemented with a nontrivial “basic production term”, see [24] for studies of stability properties of stationary solutions to such equations. The global existence result from [11] was extended by Li, Chen and Qin [7] to the case of all exponents satisfying \( p-1 < r \). Jiang [3] and the author of [23] independently obtained similar results on the existence and the boundedness of solutions to a general activator-inhibitor system including the one in (1.1) and (1.3). Other recent results on the global-in-time existence of solutions to system (1.1) and (1.3) can be found in Ref. [26].

On the other hand, the dynamics of solutions to the activator-inhibitor system (1.1) and (1.3) is far from being understood and one may quote here a few works, only. In [24], a phenomenon called a collapse of patterns has been studied and, in [4], a stability of a periodic solution has been proved. Moreover, a blowup of solutions to the corresponding shadow system (namely, when the second equation in (1.1) and (1.3) is replaced by a nonlocal equation obtained formally in the fast diffusion limit \( D \to \infty \)) has been shown in [6].

Methods developed in this work appeared to be useful to study blowup phenomena in reaction-diffusion-ODE systems (1.2) with other types of nonlinearities. In particular, in our recent work [10], we apply them to another class of equations which appear in mathematical biology.

2. Formulation of results. Now, let us state precisely of our results. We consider the following system of reaction-diffusion equations

\[
\begin{align*}
    u_t &= \varepsilon \Delta u - au + \frac{u^p}{v^q}, & \text{for } x \in \Omega, & t > 0, \\
    v_t &= D \Delta v - bv + \gamma \frac{u^r}{v^s}, & \text{for } x \in \Omega, & t > 0,
\end{align*}
\]

(2.1) \( (2.2) \)

with \( \varepsilon \geq 0 \) and \( D > 0 \), where \( a, b, \gamma \) are nonnegative constants, and the nonlinearity exponents in (2.1)-(2.2) satisfy

\[
p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0.
\]

(2.3)

We consider this system in a bounded domain \( \Omega \subset \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \), supplemented with the initial data \( u_0, v_0 \in C(\overline{\Omega}) \) such that

\[
u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) > 0 \quad \text{for all } x \in \overline{\Omega}.
\]

(2.4)

Thus, we have \( \inf u_0 \equiv \inf_{x \in \Omega} u_0(x) > 0 \) and \( \inf v_0 \equiv \inf_{x \in \Omega} v_0(x) > 0 \).

Moreover, we impose the Neumann boundary conditions

\[
\frac{\partial u}{\partial n} = 0 \quad (\text{if } \varepsilon > 0) \quad \text{and} \quad \frac{\partial v}{\partial n} = 0 \quad \text{for } x \in \partial \Omega, \quad t > 0.
\]

(2.5)
In the following, for simplicity of notation, we use the quantities
\[ f_{0,T} \equiv \inf_{t \in [0,T]} e^{a(1-p+q)t} \] and \[ g_{1,T} \equiv \sup_{t \in [0,T]} e^{b(1-r+s)t}. \] (2.6)

In our first result, we show that the theory on the global-in-time existence of solutions to problem (2.1)-(2.5) with both \( \varepsilon > 0 \) and \( D > 0 \) developed in [11, 7, 23, 3, 26] is no longer valid if \( \varepsilon = 0 \). Thus, in the following, we consider the initial-boundary value problem for the reaction-diffusion-ODE system of the form
\[ u_t = -au + \frac{u^p}{v^q}, \quad \text{for } x \in \Omega, \ t \in [0,T_{\text{max}}) \] (2.7)
\[ v_t = \Delta v - bv + \frac{u^r}{v^s}, \quad \text{for } x \in \Omega, \ t \in [0,T_{\text{max}}) \] (2.8)
\[ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times [0,T_{\text{max}}), \] (2.9)
\[ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0 \quad \text{for } x \in \Omega, \ t \in [0,T_{\text{max}}). \] (2.10)

Here, as before, \( \Omega \subset \mathbb{R}^n \) is an arbitrary bounded domain with a smooth boundary and, without loss of generality, we assume that \( 0 \in \Omega \). Moreover, system (2.7)-(2.8) is rescaled in such a way so that the diffusion coefficient in equation (2.8) is equal to one.

In the following theorem, we prove that if \( u_0 \) is sufficiently well concentrated around an arbitrary point \( x_0 \in \Omega \) (here, for simplicity of notation, we choose \( x_0 = 0 \)), if \( v_0 \) is a constant function, and if \( \gamma > 0 \) is sufficiently small then the corresponding solution to problem (2.7)-(2.10) blows up in a finite time \( T_{\text{max}} > 0 \), without additional restrictions on the exponents in nonlinearities.

**Theorem 2.1.** Assume the nonlinearity exponents satisfy (2.3) and let \( T > 0 \) be arbitrary. Suppose that \( 0 \in \Omega \) and

- there exists a number
\[ \alpha \in \left(0, \frac{2(p-1)}{r}\right) \quad \text{if } n \geq 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \quad \text{if } n = 1 \]
such that \( u_0 \in C(\Omega) \) satisfies
\[ 0 < u_0(x) \leq \frac{1}{(u_0(0)^{1-r} + 2|x|^\alpha)^{\frac{1}{p-1}}} \quad \text{for all } x \in \Omega, \] (2.11)

- \( v(x,0) = \bar{v}_0 \) is a constant such that
\[ 0 < \bar{v}_0 < R_0 \equiv \left(T(p-1)f_{0,T}(\inf_{x \in \Omega} u_0(x))^{p-1}\right)^{\frac{1}{2}} \quad \text{for all } x \in \Omega, \] (2.12)

- \( \gamma \in [0,\gamma_0), \) where \( \gamma_0 = \gamma_0(u_0, \bar{v}_0, T, p, q, r, s, n) \) is a certain number determined in the proof.

Then the corresponding solution to problem (2.7)-(2.10) blows up at some \( T_{\text{max}} \leq T \).
Moreover, the following uniform estimates are valid
\[ 0 < u(x,t) < |x|^{-\frac{\alpha}{p-1}} \quad \text{and} \quad 0 < v(x,t) < R_0 \] (2.13)
for all \( (x,t) \in \Omega \times [0,T_{\text{max}}) \).

**Remark 2.2.** (Diffusion-induced blowup) Let us emphasize one application of Theorem 2.1 in the range of exponents
\[ 0 < \frac{p-1}{r} < \min \left\{ \frac{q}{s+1}, 1 \right\}. \]
If an initial datum in (2.10) is constant (i.e. \( x \)-independent), the corresponding solution of problem (2.7)-(2.10) is also \( x \)-independent and global-in-time, see [7, 17]. On the other hand, by Theorem 2.1, there are nonconstant initial conditions, such that the corresponding solution to (2.7)-(2.10) with small \( \gamma > 0 \) blows up at one point in a finite time. This is another example of an initial-boundary value problem for reaction-diffusion equations, where a diffusion in one equation induces a blowup of solutions. First example of one reaction-diffusion equation coupled with one ODE, where some solutions blow up due to a diffusion, appeared in the paper by Morgan [16] and another example can be found in Ref. [2]. The term “diffusion-induced blowup” was introduced by Mizoguchi et al. [15] who proved a blowup of solutions to certain system of reaction-diffusion equations with nonzero diffusion coefficients in both equations for which space homogeneous solutions are global in time. Another system of reaction-diffusion equations with such a property was discovered by Pierre and Schmitt [19, 20]. A discussion of other initial-boundary value problems with a diffusion-induced blowup, as well as several references, can be found in the survey paper [1] as well as in the monograph [21, Ch. 33.2].

In our next theorem, we deal with the initial-boundary value problem for the reaction-diffusion system (2.1)-(2.5) with arbitrary \( \varepsilon \geq 0 \) and \( D > 0 \). We extend blowup results from [7, 17] concerning kinetic system (1.5) to some space-inhomogeneous solutions to problem (2.1)-(2.5), however, for a smaller range of nonlinearity exponents.

**Theorem 2.3.** Assume inequalities (2.3) as well as

\[
\frac{r}{2p} \left( \frac{n}{2} + \frac{1}{p-1} \right) < 1, \quad \varepsilon \geq 0, \quad D > 0, \quad \text{and} \quad \gamma \geq 0
\]

(2.14)
in problem (2.1)-(2.5). Let \( T > 0 \) be an arbitrary fixed number. Suppose that positive initial conditions \( u_0, v_0 \in C(\Omega) \) satisfy

\[
\sup v_0 < R_0 = \left( T(p-1)f_{0,T}(\inf u_0)^{p-1} \right)^{\frac{1}{p}},
\]

(2.15)
where the constant \( f_{0,T} \) is defined in (2.6). Then, there exists a number

\[
C = C(R_0, p, r, \inf v_0, g_{1,T}, |\Omega|, T) > 0 \quad \text{(but independent of } \gamma \text{)}
\]

such that for each (small) \( \gamma \geq 0 \) satisfying

\[
\sup v_0 + \gamma C < R_0
\]

(2.16)
the corresponding solution to problem (2.1)-(2.5) blows up at certain \( T_{\max} \leq T \).

**Remark 2.4.** If \( \varepsilon = 0 \) and \( D > 0 \), Theorem 2.3 is a particular case of Theorem 2.1 (however, with a different proof). By this remark, we would like to emphasize that Theorem 2.3 provides conditions for a blowup of solutions to problem (2.1)-(2.5) with both non degenerate diffusion coefficients \( \varepsilon > 0 \) and \( D > 0 \). This result is related to the one in Ref. [17] on a blowup of solutions to the kinetic system (1.5) with nonlinearity exponents satisfying \( r/(p-1) < 1 \) and with initial data satisfying estimates more-or-less as the one in (2.15). Theorem 2.3 asserts that nonlinearities in system (2.1)-(2.5) cause an analogous blow up of space inhomogeneous solutions of this initial boundary value problem, however, for a smaller range of nonlinearity exponents (i.e. those satisfying first inequality in (2.14)).

Both Theorems 2.1 and 2.3 are proved in the following section.
3. Blowup of solutions in a finite time. To show that some solutions to problems (2.1)-(2.5) and (2.7)-(2.10) blow up in a finite time, we first notice that if \((u(x,t),v(x,t))\) is their solution, then the functions \(u(x,t)e^{\alpha t}\) and \(v(x,t)e^{\beta t}\) satisfy the following boundary-value problem

\[
\begin{align*}
  u_t &= \varepsilon \Delta u + \frac{u^p}{v^q} f(t) & \text{for } x \in \Omega, \ t > 0, \quad (3.1) \\
  v_t &= D \Delta v + \gamma \frac{u^r}{v^s} g(t) & \text{for } x \in \Omega, \ t > 0, \quad (3.2) \\
  \frac{\partial u}{\partial \nu} &= 0 \quad \text{if } \varepsilon > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = 0 & \text{for } x \in \partial \Omega, \ t > 0, \quad (3.3) \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x),
\end{align*}
\]

where

\[
f(t) = e^{a(1-p+q)t} \quad \text{and} \quad g(t) = e^{b(1-r+s)t}.
\]

Obviously, it suffices to prove a blowup of solutions to the new problem (3.1)-(3.4).

Preliminary estimates. The proofs of Theorems 2.1 and 2.3 are preceded by auxiliary lemmas. We begin by preliminary estimates of solutions on an maximal interval of their existence.

Lemma 3.1. Assume inequalities (2.3). For every nonnegative \(u_0,v_0 \in C(\overline{\Omega})\), problem (3.1)-(3.4) with \(\varepsilon \geq 0\) and \(D > 0\) has a unique solution on its maximal interval \([0,T_{\max})\). Moreover,

\[
u(x,t) \geq \inf u_0 \quad \text{and} \quad v(x,t) \geq \inf v_0 \quad \text{for all} \quad (x,t) \in \Omega \times [0,T_{\max}). \quad (3.6)
\]

If \(T_{\max} < +\infty\), then \(\sup_{t \in [0,T_{\max})} ||u(\cdot,t)||_{\infty} = +\infty\).

Proof: The proof of the existence and the uniqueness of solutions to problem (3.1)-(3.4) is quite standard. Here, it suffices to apply the theory reported in the monograph [22] and in the work [11].

It is clear that, as long as both functions \(u\) and \(v\) are positive, they satisfy the inequalities \(u_t \geq \varepsilon \Delta u\) and \(v_t \geq D \Delta v\) in the domain \(\Omega\) together with the Neumann boundary conditions. Thus, for \(\varepsilon > 0\) and \(D > 0\), this solution satisfies estimates (3.6) by a comparison principle for parabolic equations. For \(\varepsilon = 0\), we have \(u_t(x,t) \geq 0\) for all \(x \in \Omega\) and \(t \in (0,T_{\max})\), hence \(u(x,t)\) increases on \([0,T_{\max})\) for each fixed \(x \in \overline{\Omega}\). Thus, \(u(x,t)\) satisfies the estimate in (3.6), again. We refer the reader to [9, Lemma 3.4] for a similar reasoning in the case of another reaction-diffusion-ODE system.

Now, we show that an upper bound for \(v(x,t)\) leads to the blowup of \(u(x,t)\) in a finite time.

Lemma 3.2. Let \((u(x,t),v(x,t))\) be a nonnegative solution to (3.1)-(3.4) with \(\varepsilon \geq 0\) and \(D > 0\). Suppose that for some constant \(T > 0\) we have

\[
0 < v(x,t) < R_0 = \left( T(p-1) f_0, T(\inf u_0)^{p-1} \right)^{\frac{1}{p}} \quad \text{for all} \quad (x,t) \in \Omega \times [0,T_{\max}).
\]

Then \(u(x,t)\) blows up at certain \(T_{\max} \leq T\).
Proof. Applying the comparison principle to equation (3.1) either for parabolic equations if \( \epsilon > 0 \) or for ordinary differential equations if \( \epsilon = 0 \), we obtain the estimate
\[
u(x, t) \geq \bar{u}_1(t) \quad \text{for all} \quad (x, t) \in \Omega \times [0, T_{max}), \tag{3.8}
\]
where \( \bar{u}_1 = \bar{u}_1(t) \) is the solution of the Cauchy problem
\[
\frac{d}{dt} \bar{u}_1 = \frac{\bar{u}_1^p}{R_0^{\epsilon}} f_{0,T}, \quad \bar{u}(0) = \inf u_0. \tag{3.9}
\]
The function \( \bar{u}_1 \) may be computed explicitly:
\[
\bar{u}_1(t) = \frac{1}{(\inf u_0)^{1-p} - t(p-1)R_0^{-q} f_{0,T}}. \tag{3.10}
\]
Recalling the definition of the number \( R_0 \) in (3.7), we obtain that \( \bar{u}_1(t) \) blows up at \( t = T \), which due to inequality (3.8) implies that \( T_{max} \leq T \).

Proof of Theorem 2.1. We show a one-point blowup of solutions to the reaction-diffusion-ODE problem (2.7)-(2.10) by considering the following particular case of problem (3.1)-(3.4):
\[
\begin{align*}
u_t &= \frac{\nu^p}{\nu^q} f(t), & x \in \Omega, & t \in [0, T_{max}) \quad \text{(3.11)} \\
v_t &= \Delta v + \gamma \frac{v^r}{v^s} g(t) & x \in \Omega, & t \in [0, T_{max}) \quad \text{(3.12)} \\
\frac{\partial v}{\partial \nu} &= 0 & x \in \partial \Omega \times [0, T_{max}), \quad \text{(3.13)} \\
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & x \in \Omega, & t \in [0, T_{max}), \quad \text{(3.14)}
\end{align*}
\]
where the functions \( f(t) \) and \( g(t) \) are defined in (3.5).

First, we recall a classical result on the H"older continuity of solutions to the inhomogeneous heat equation.

**Lemma 3.3.** Let \( f \in L^\infty([0, T], L^\ell(\Omega)) \) with some \( \ell > \frac{n}{2} \) and \( T > 0 \). Denote
\[
w(x, t) = \int_0^t e^{(t-\tau)D\Delta} f(x, \tau) \, d\tau,
\]
where \( \{e^{tD\Delta}\}_{t \geq 0} \) is the semigroup of linear operators on \( L^\ell(\Omega) \) generated by \( D\Delta \) in a bounded domain with a smooth boundary, and supplemented with the Neumann boundary conditions. There exist numbers \( \beta \in (0, 1) \) and \( C = C > 0 \) depending on \( \sup_{0 \leq t \leq T} \|f(t)\|_\ell \) such that
\[
|w(x, t) - w(y, t)| \leq C|x - y|^{\beta} \quad \text{for all} \quad x, y \in \Omega. \tag{3.15}
\]

**Proof.** Note that the function \( w(x, t) \) is the solution of the problem
\[
w_t = D\Delta w + f, \quad w(x, 0) = 0
\]
supplemented with the Neumann boundary conditions. Hence, estimate (3.15) is a classical and well-known result on the Hölder continuity of solutions to linear parabolic equations, see e.g. [5, Ch. III, §10].

From now on, we deal with problem (3.11)-(3.14). It follows from Lemma 3.2 that it suffices to estimate the function \( v(x, t) \) from above to obtain a blowup of \( u(x, t) \) in a finite time. Now, we prove that such an upper bound for \( v(x, t) \) is a consequence of a certain a priori estimate imposed on \( u(x, t) \).
Lemma 3.4. Let $u(x, t)$ and $v(x, t)$ be a solution to problem (3.11)-(3.14). Suppose that there is a number
\[
\alpha \in \left(0, \frac{2(p-1)}{r}\right) \text{ if } n \geq 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \text{ if } n = 1
\]
such that, a priori, the following inequality holds true
\[
0 < u(x, t) < |x|^{-\frac{n}{p-1}} \quad \text{for all } (x, t) \in \Omega \times [0, T_{\max}).
\]
Then, there is an explicit number $C_0 > 0$ (see equation (3.23) below) such that for all $\gamma \geq 0$ we have
\[
\|v(t)\|_{\infty} \leq \|v_0\|_{\infty} + \gamma C_0 \quad \text{for all } (x, t) \in \Omega \times [0, T_{\max}).
\]
Proof. We use the following integral formulation of equation (3.12)
\[
v(t) = e^{t\Delta} v_0 + \gamma \int_0^t e^{(t-\tau)\Delta} \left( \frac{u^r(\tau)}{v^s(\tau)} g(\tau) \right) \, d\tau.
\]
Here, we recall the following well-known estimates for the heat semigroup which are valid for all $t > 0$, $D > 0$, and all $w_0 \in L^\infty(\Omega)$:
\[
\|e^{t\Delta} w_0\|_{\infty} \leq \|w_0\|_{\infty}
\]
and
\[
\|e^{t\Delta} w_0\|_{\infty} \leq C_\ell \left(1 + t^{-\frac{1}{2\ell}}\right) \|w_0\|_{\ell}
\]
for each $\ell \in [1, \infty]$, with a constant $C_\ell = C(\ell, n, D, \Omega)$ independent of $w_0$ and of $t$, see e.g. [22, p. 25].

Now, we compute the $L^\infty$-norm of equation (3.19). Using inequalities (3.20) and (3.21), the lower bound of $v$ in (3.6) as well as the a priori assumption on $u$ in (3.17) we obtain the estimate
\[
\|v(t)\|_{\infty} \leq \|v_0\|_{\infty} + \gamma \int_0^t \left\| e^{(t-\tau)\Delta} \left( \frac{u^r(\tau)}{v^s(\tau)} g(\tau) \right) \right\|_{\infty} \, d\tau
\]
\[
\leq \|v_0\|_{\infty} + \gamma C_\ell \left( \inf v_0 \right)^{-s} g_{1, T} \int_0^t (1 + (t-\tau)^{-\frac{1}{2\ell}}) \left\| |x|^{-\frac{n}{p-1}} \right\|_{\ell} \, d\tau,
\]
where the constant $g_{1, T}$ is defined in (2.6). Here, we choose $\ell > n/2$ to have $n/(2\ell) < 1$, which leads to the equality
\[
\int_0^t (1 + (t-\tau)^{-\frac{1}{2\ell}}) \, d\tau = t + \left(1 - \frac{n}{2\ell}\right)^{-1} t^{1-\frac{1}{2\ell}}.
\]
Moreover, we assure that $\ell < n(p-1)/(\alpha r)$ or, equivalently, that $\alpha r/(p-1) < n$ to have $|x|^{-\frac{n}{p-1}} \in L^r(\Omega)$. Such a choice of $\ell \in [1, \infty)$ is always possible because max$\{1, n/2\} < n(p-1)/(\alpha r)$ under our assumptions on $\alpha$ in (3.16).

Thus, for the constant
\[
C_0 = C_\ell \left( \inf v_0 \right)^{-s} \left\| |x|^{-\frac{n}{p-1}} \right\|_{\ell} \left( T_{\max} + \left(1 - \frac{n}{2\ell}\right)^{-1} T_{\max}^{1-\frac{1}{2\ell}} \right),
\]
inequality (3.22) implies the upper bound (3.18). \qed

Next, we apply Lemma 3.3 to show the Hölder continuity of $v(x, t)$.
Lemma 3.5. Let \((u(x,t), v(x,t))\) be a solution to problem (3.11)-(3.14) such that \(u(x,t)\) satisfies a priori estimate (3.17) and \(v(x,t)\) corresponds to the constant initial datum \(v(x,0) = \bar{v}_0\). There exists a constant \(\alpha \in (0,1)\) satisfying also (3.16) and a number \(C > 0\), the both independent of \(\gamma > 0\), such that
\[
|v(x,t) - v(y,t)| \leq \gamma C|x - y|^{\alpha}
\]
for all \((x,t) \in \Omega \times [0,T_{\text{max}})\).

Proof. As in the proof of Lemma 3.4, we use the following integral representation
\[
v(x,t) = \bar{v}_0 + \gamma \int_0^t e^{(t-\tau)\Delta} \left( \frac{u(\tau)}{v^s(\tau)} g(\tau) \right) ds,
\]
because \(e^{t\Delta} \bar{v}_0 = \bar{v}_0\) for every constant \(\bar{v}_0\).

First, suppose that \(\alpha \in (0,1)\) is an arbitrary number satisfying (3.16). Since, by assumption (3.17) and inequality (3.6), we have
\[
\left| \frac{u^\tau(\tau)}{v^s(\tau)} g(\tau) \right| \leq |x|^{-\alpha r/(p-1)} (\inf v_0)^{-s} g_{1,T}
\]
we obtain \(u^\tau v^{-s} g \in L^\infty((0,T_{\text{max}}), L^1(\Omega))\) for some \(\ell > n/2\), see the proof of Lemma 3.4. By Lemma 3.3, there exist constants \(C > 0\) and \(\beta \in (0,1)\), independent of \(\gamma\), such that \(|v(x,t) - v(y,t)| \leq \gamma C|x - y|^{\beta}\) for all \(x, y \in \Omega\) and \(t \in [0,T_{\text{max}})\).

If \(\beta < \alpha\), we replace \(\alpha\) by \(\beta\) in the following way. First, instead of (3.17), we impose the a priori estimate \(0 \leq u(x,t) < \varepsilon |x|^{-\beta/(p-1)}\) for all \(x \in \Omega\) and \(t \in [0,T_{\text{max}}]\). Thus, for each \(\alpha \geq \beta\), there exists a constant \(C = C(\alpha, \beta, p, \Omega) > 0\) such that
\[
0 \leq u(x,t) < \varepsilon |x|^{-\beta/(p-1)} = \varepsilon |x|^{-\beta/(p-1)} |x|^\frac{\alpha}{\alpha - \beta} \leq C \varepsilon |x|^{-\beta/(p-1)}.
\]
Thus, repeating the reasoning in the beginning of this proof, we obtain the estimate \(|v(x,t) - v(y,t)| \leq \varepsilon^p C|x - y|^{\beta}\) for all \(x, y \in \Omega\) and \(t \in [0,T_{\text{max}})\) with a modified constant \(C > 0\), but still independent of \(\varepsilon > 0\).

We are ready to prove a result on the one-point blowup of solutions to the reaction-diffusion-ODE problem (3.11)-(3.14).

Proof of Theorem 2.1. Let \((u(x,t), v(x,t))\) be a solution to the modified problem (3.11)-(3.14). By Lemmas 3.2 and 3.4, it suffices to show the a priori estimate
\[
0 < u(x,t) < |x|^{-\frac{\alpha}{\alpha - \beta}}
\]
under the assumption that \(\gamma > 0\) is sufficiently small. Let \(T > 0\) be a number such that inequality (2.12) holds true.

By assumption (2.11), we have \(0 < u_0(x) < |x|^{-\frac{\alpha}{\alpha - \beta}}\) for all \(x \in \Omega\), hence, by a continuity argument, inequality (3.24) is satisfied on a certain initial time interval. Suppose a contrario that there exists \(T_1 \in (0,\min\{T_{\text{max}}, T\})\) such the solution of problem (3.11)-(3.14) exists on the interval \([0,T_1]\) and satisfies
\[
\sup_{x \in \Omega} |x|^{-\frac{\alpha}{\alpha - \beta}} u(x,t) = 1 \quad \text{for} \quad t = T_1
\]

Now, for a given function \(v(x,t)\), we solve equation (3.11) with respect to \(u(x,t)\) to obtain
\[
u(x,t) = \frac{1}{\left(\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{j(\tau)}{v(z,\tau)^{p-1}} d\tau\right)^{-\frac{\alpha}{\alpha - \beta}}}.
\]
We are going to use this explicit formula for \( u(x,t) \) and the Hölder regularity of \( v(x,t) \) from Lemma 3.3 to obtain a contradiction with equality (3.26).

First, notice that assumption (2.11) can be written as \( u_0(x)^{1-p} \geq 2|x|^\alpha + u_0(0)^{1-p} \). Thus, we may estimate the denominator of the fraction in (3.27) using this assumption as follows

\[
\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(x,\tau)^q} d\tau 
\geq 2|x|^\alpha + \frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(0,\tau)^q} d\tau \tag{3.28}
\]

\[
+ (p-1) \int_0^t \left( \frac{1}{v(0,\tau)^q} - \frac{1}{v(x,\tau)^q} \right) f(\tau) d\tau.
\]

By the definition of \( T_{\text{max}} \) and due to formula (3.27), we immediately obtain

\[
\frac{1}{u_0(0)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(0,\tau)^q} d\tau > 0 \quad \text{for all} \quad t \in [0, T_{\text{max}}).
\tag{3.29}
\]

Next, we use our hypotheses (3.25) and (3.26) implying estimate (3.18) and the Hölder continuity of \( v(x,t) \) established in Lemma 3.5, as well as the lower bound of \( v(x,t) \) in (3.6), to find constants \( C > 0 \) and \( \alpha \in (0,1) \), satisfying also (3.16), such that

\[
\left| \frac{1}{v(0,\tau)^q} - \frac{1}{v(x,\tau)^q} \right| \leq C \frac{|v(0,\tau) - v(x,\tau)|}{v(0,\tau)^q v(x,\tau)^q} \leq \gamma C |x|^\alpha
\]

for all \((x,\tau) \in \Omega \times [0, T_1]\). Hence, since \( T_1 \leq T \), we obtain the following lower bound

\[
(p-1) \int_0^t \left| \frac{1}{v(0,\tau)^q} - \frac{1}{v(x,\tau)^q} \right| f(\tau) d\tau \leq \gamma C(T)|x|^\alpha
\tag{3.30}
\]

for all \((x,\tau) \in \Omega \times [0, T_1] \). Consequently, applying inequalities (3.29) and (3.30) in (3.28) we obtain the following lower bound for the denominator in (3.27)

\[
\frac{1}{u_0(x)^{p-1}} - (p-1) \int_0^t \frac{f(\tau)}{v(x,\tau)^q} d\tau \geq (2 - \gamma C)|x|^\alpha
\tag{3.31}
\]

for all \((x,\tau) \in \Omega \times [0, T_1] \). Finally, we choose \( \gamma > 0 \) so small that \( \gamma C < 1 \) (hence \( 2 - \gamma C > 1 \)) and we substitute estimate (3.31) into equation (3.10) to obtain

\[
0 < u(x,t) \leq \frac{1}{\left( (2 - \gamma C)|x|^\alpha \right)^{\frac{1}{p-1}}} < \frac{1}{|x|^{\frac{\alpha}{p-1}}} \quad \text{for all} \quad (x,t) \in \Omega \times [0, T_1].
\]

This inequality for \( t = T_1 \) contradicts our hypothesis (3.26).

Thus, estimate (3.24) holds true on the whole interval \([0, \min\{T_{\text{max}}, T\}]\). Then, by Lemma 3.4, the function \( v(x,t) \) is bounded from above by a constant \( \tilde{v}_0 + \gamma C_0 < R_0 \), provided \( \gamma > 0 \) is sufficiently small. Finally, Lemma 3.2 implies that \( u(x,t) \) blows up at certain \( T_{\text{max}} \leq T \). \( \square \)

**Proof of Theorem 2.3.** Let us finally prove our second blowup result.

Let \((u(x,t), v(x,t))\) be a solution to the modified problem (3.1)-(3.4) with \( \varepsilon \geq 0 \) and \( D > 0 \). In view of Lemma 3.2, it suffices to show inequality (3.7) and our proof is by contradiction. Notice that \( \|v(\cdot,0)\|_\infty = \|v_0\|_\infty < R_0 \) by assumption (2.15),
hence, there is \( \delta > 0 \) such that \( 0 < v(x, t) < R_0 \) for all \( (x, t) \in \Omega \times [0, \delta]. \) Suppose that for some \( T_0 \in (0, \min\{T, T_{\max}\}) \) we have \( \|v(t)\|_\infty < R_0 \) for all \( t \in [0, T_0] \), but
\[
\|v(T_0)\|_\infty = R_0.
\] (3.32)
The following calculations are performed for all \( t \in (0, T_0) \) and they lead to a  
contraction with equality (3.32).
Since \( \|v(t)\|_\infty \leq R_0 \) for all \( t \in (0, T_0) \), we obtain the inequality
\[
 u_t \geq \varepsilon \Delta u + u^p B_0 \quad \text{with} \quad B_0 = \frac{f_0}{R_0^p} \quad (3.33)
\] for all \( x \in \Omega \) and \( t \in (0, T_0) \). Integrating this differential inequality over \( \Omega \) and using the Neumann boundary conditions as well as the Hölder inequality, we have
\[
\frac{d}{dt} \int_\Omega u(x, t) \, dx \geq B_0 \int_\Omega u^p(x, t) \, dx \geq B_0|\Omega|^{1-p} \left( \int_\Omega u(x, t) \, dx \right)^p, \quad (3.34)
\] where \( |\Omega| \) is a Lebesgue measure of the domain \( \Omega \). Hence, for all \( t \in (0, T_0) \), we obtain the estimate
\[
\frac{1}{1-p} \frac{d}{dt} \left( \int_\Omega u(x, t) \, dx \right)^{1-p} \geq B_0|\Omega|^{1-p}
\] which after integrating over the interval \([t, T_0]\) leads to
\[
\frac{1}{1-p} \left( \int_\Omega u(x, T_0) \, dx \right)^{1-p} = \frac{1}{(p-1)B_0|\Omega|^{1-p}(T_0-t)} \quad \text{for all} \quad t \in (0, T_0). \] (3.35)
Thus, integrating again inequality (3.33) with arbitrary \( 0 < b < a < T_0 \) and using the upper-bound (3.35) we obtain
\[
\int_{T_0-a}^{T_0-b} \int_\Omega u^p(x, t) \, dx \, dt \leq B_0^{-1} \int_\Omega u(x, T_0 - b) \, dx - B_0^{-1} \int_\Omega u(x, T_0 - a) \, dx
\leq B_0^{-1} \int_\Omega u(x, T_0 - b) \, dx \leq \frac{b^{-\frac{1}{p-1}}}{B_0((p-1)B_0|\Omega|^{1-p})^{\frac{1}{p-1}}}. \] (3.36)
In the next step, we obtain an upper estimate for \( v = v(x, t) \) applying inequality (3.36) in the usual integral formulation of equation (3.2)
\[
v(t) = e^{tD\Delta}v_0 + \gamma \int_0^t e^{(t-\tau)D\Delta} \left( \frac{u^p(\tau)}{v^q(\tau)} \right) g(\tau) \, d\tau. \quad (3.37)
\] By second inequality in (3.6) and estimates (3.20)–(3.21) with \( \ell = \frac{p}{q} > 1 \) we obtain for all \( t \in [0, T_0] \) that
\[
\|v(t)\|_\infty \leq \|v_0\|_\infty + \gamma \frac{g_1(T)}{\inf v_0} C(p/r, n, D) \int_0^t \left( 1 + (t-\tau)^{-\frac{p}{p-1}} \right) \|u(\tau)\|_p^p \, d\tau. \quad (3.38)
\]
Our goal now is to show that the integral
\[
\int_0^{T_0} (1 + (T_0 - \tau)^{-\frac{2}{r} \frac{p}{r + 1}}) \|u(\tau)\|_p^r d\tau
\]
(3.39)
is bounded from above by a constant independent of \(T_0 \in (0, T]\). To do it, we combining the Hölder inequality with estimate (3.36), we obtain first the estimate
\[
\int_0^t \|u(\tau)\|_p^r d\tau \leq t^{1 - \frac{r}{p}} \left(\int_0^t \|u(\tau)\|_p^p d\tau\right)^{r/p} \leq t^{1 - \frac{r}{p}} \left(\frac{t^{-\frac{r}{p - r}}}{B_0((p - 1)B_0[|\Omega|^{1 - p})^{-\frac{r}{p - r}})}\right)^{r/p}
\]
(3.40)
for all \(t \in [0, T_0]\). Hence, under the assumption
\[
\left(1 - \frac{r}{p}\right) - \left(\frac{1}{p - 1}\right) \cdot \frac{r}{p} > 0 \iff \frac{r}{p - 1} < 1,
\]
we find that
\[
\int_0^{T_0} \|u(\tau)\|_p^r d\tau \leq T^{(1 - \frac{r}{p} - (\frac{1}{p - 1}) \frac{r}{p})} \left(B_0((p - 1)B_0[|\Omega|^{1 - p})^{-\frac{r}{p - r}})\right)^{r/p}
\]
for each \(T_0 \in [0, T]\).

Next, we deal with the second part of the integral in (3.39), where we use the dyadic decomposition of the range of integration \([0, T_0]\). First, we observe that
\[
\int_0^{T_0} \int_0^{T_0(1 - 2^{-j-1})} (T_0 - \tau)^{-\frac{2}{r} \frac{p}{r + 1}} \|u(\tau)\|_p^r d\tau \leq (T_0 2^{-j-1})^{-\frac{2}{r} \frac{p}{r + 1}} \int_0^{T_0(1 - 2^{-j-1})} \|u(\tau)\|_p^r d\tau
\]
Then, we use the Hölder inequality and estimate (3.36) in an analogous way as in (3.40) to obtain
\[
\int_0^{T_0} (T_0 - \tau)^{-\frac{2}{r} \frac{p}{r + 1}} \|u(\tau)\|_p^r d\tau = \sum_{j=0}^{\infty} \int_0^{T_0(1 - 2^{-j-1})} (T_0 - \tau)^{-\frac{2}{r} \frac{p}{r + 1}} \|u(\tau)\|_p^r d\tau
\]
\[
\leq \sum_{j=0}^{\infty} (T_0 2^{-j-1})^{-\frac{2}{r} \frac{p}{r + 1}} (T_0 2^{-j-1})^{1 - \frac{r}{p}} \left(\frac{(T_0 2^{-j-1})^{-\frac{r}{p - r}}}{B_0((p - 1)B_0[|\Omega|^{1 - p})^{-\frac{r}{p - r}})}\right)^{\frac{r}{p - 1}}
\]
(3.41)
Thus, for the exponent satisfying
\[
-\frac{n}{2p} + 1 - \frac{r}{p} - \frac{1}{p - 1} \cdot \frac{r}{p} > 0 \iff \frac{r}{2p} + \frac{1}{p - 1} < 1,
\]
(3.42)
the series on the right-hand side of (3.41) is convergent and bounded from above by an explicit constant independent of \(T_0 \in [0, T]\).

Consequently, coming back to inequality (3.38), assuming (3.42), we may find a constant \(C = C(R_0, p, r, \inf v_0, g_{1,T}, |\Omega|, T) > 0\) but independent of \(\gamma\), such that
\[
\|v(T_0)\|_\infty \leq \|v_0\|_\infty + \gamma C.
\]
Finally, under assumption (2.16) we obtain the inequality \(\|v(T_0)\|_\infty < R_0\) which contradicts equality (3.32). Hence, estimate (3.7) holds true and the function \(u(x, t)\) blows up at some \(T_{\max} \leq T\) by Lemma 3.2.
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E-mail address: grzegorz.karch@math.uni.wroc.pl
E-mail address: kanako.suzuki.sci2@vc.ibaraki.ac.jp
E-mail address: jacek.zienkiewicz@math.uni.wroc.pl