LARGE NUMBER OF BUBBLE SOLUTIONS FOR A PERTURBED FRACTIONAL LAPLACIAN EQUATION

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Abstract. This paper deals with the following nonlinear perturbed fractional Laplacian equation

\[ (-\Delta)^s u = K(|y'|, y'') u^{\frac{N+2s}{N-2s}} + \epsilon, \quad u > 0, \quad u \in D^{1,s}(\mathbb{R}^N), \]

where \( 0 < s < 1, N \geq 4, (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \epsilon > 0 \) is a small parameter and \( K(y) \) is nonnegative and bounded. By combining a finite reduction argument and local Pohozaev type of identities, we prove that if \( N \geq 4, \max\{\frac{N+1}{4} - \frac{N^2 - 6N + 13}{2}, 3 - \frac{\sqrt{N^2 - 6N} + 13}{2}\} < s < 1 \) and \( K(r, y'') \) has a stable critical point \((r_0, y''_0)\) with \( r_0 > 0 \) and \( K(r_0, y''_0) > 0 \), then the above problem has large number of bubble solutions if \( \epsilon > 0 \) is small enough. Also there exist solutions whose functional energy is in the order \( \epsilon^{\frac{N-2s}{2}} \). Here, instead of estimating directly the derivatives of the reduced functional, we apply some local Pohozaev identities to locate the concentration points of the bubble solutions. Moreover, the concentration points of the bubble solutions include a saddle point of \( K(y) \).

Key words : bubble solutions; fractional Laplacian; Pohozaev identities; finite dimensional reduction method.

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1. Introduction and main result

In this paper, we are interested in the existence of large number of bubble solutions to the following perturbed fractional nonlinear elliptic equation

\[ (-\Delta)^s u = K(|y'|, y'') u^{\frac{N+2s}{N-2s}} + \epsilon, \quad u > 0, \quad u \in D^{1,s}(\mathbb{R}^N), \tag{1.1} \]

where \( 0 < s < 1 \) for \( N \geq 4, (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \epsilon > 0 \) is a small parameter and \( K(y) \) is nonnegative and bounded.

For any \( s \in (0, 1) \), \( (-\Delta)^s \) is the nonlocal operator defined as

\[ (-\Delta)^s f(y) = C_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\epsilon}(y)} \frac{f(y) - f(x)}{|x - y|^{N+2s}} \, dx, \]

where \( P.V. \) stands for the principal value and

\[ C_{N,s} = \pi^{-(2s+N/2)} \frac{\Gamma(N/2 + s)}{\Gamma(-s)}. \]

This operator is well defined in \( C^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap L_s(\mathbb{R}^N) \), where

\[ L_s(\mathbb{R}^N) = \left\{ u \in L^{1}_{\text{loc}}(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2s}} \, dy < +\infty \right\}. \]

For more details about the fractional Laplacian operator, we refer the readers to \([8, 12]\) and the reference therein.
The fractional Laplacian operator appears in many areas including biological modeling, physics and mathematical finances, and can be regarded as the infinitesimal generator of a stable Levy process (see for example[1]). From the view point of mathematics, an important feature of the fractional Laplacian operator is its nonlocal property, which makes it more challenge than the classical Laplacian operator. This nonlocal operator in \( \mathbb{R}^N \) can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half-space \( \mathbb{R}^N_+ = \{(y,t): y \in \mathbb{R}^N, t > 0\} \), we also learn from [7]: given a solution \( u = u(y) \) of \((-\nabla)^s u(y) = 0\) in \( \mathbb{R}^N \), one can equivalently consider the dimensionally extended problem for \( \tilde{u} = \tilde{u}(y,t) \) which solves

\[
\begin{aligned}
& \text{div}(t^{1-2s}\nabla \tilde{u}) = 0, \quad \text{in } \mathbb{R}^{N+1}_+ \\
& -\lim_{t\to 0} d_s t^{1-2s} \partial_t \tilde{u}(y,t) = (-\Delta)^s u(y), \quad \text{on } \partial\mathbb{R}^{N+1}_+,
\end{aligned}
\]  

(1.2)

where \( d_s = 2^{2s-1}\Gamma(s)/\Gamma(1-s) \) is a positive constant. Here, \( \tilde{u}(y,t) \) satisfies

\[
\tilde{u}(y,t) = \mathcal{P}_s[u] := \int_{\mathbb{R}^N} \mathcal{P}_s(y - \xi, t) u(\xi) d\xi, \quad (y,t) \in \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0, \infty),
\]  

(1.3)

where

\[
\mathcal{P}_s(y,t) = \beta(N,s) \frac{t^{2s}}{(|y|^2 + t^2)^{N+2s/2}}
\]  

with constant \( \beta(N,s) \) such that \( \int_{\mathbb{R}^N} \mathcal{P}_s(y,t) dx = 1 \). Moreover, \( \tilde{u} \in L^2(t^{1-2s}, \mathbb{R}^{N+1}_+) \) and \( \tilde{u} \in C^\infty(\mathbb{R}^{N+1}_+) \). Moreover, \( \tilde{u} \) satisfies (see [7])

\[
\|\nabla \tilde{u}\|_{L^2(t^{1-2s}, \mathbb{R}^{N+1}_+)} = N_s \|u\|_{H^s(\mathbb{R}^N)}.
\]  

(1.5)

In recent years, fractional problems have been extensively investigated, see for example [2, 3, 4, 6, 7, 13, 17, 18, 22, 25, 26, 37] and the reference therein.

It is well-known that when \( K(y) \equiv 1 \), the following functions

\[
U_{x,\lambda}(y) = (4^s \gamma)^{-\frac{N-2s}{4s}} \left(\frac{\lambda}{1 + \lambda^2|y-x|^2}\right)^\frac{N-2s}{2}, \quad \lambda > 0, \ x \in \mathbb{R}^N,
\]  

where \( \gamma = \frac{\Gamma(N/2 + 1)}{\Gamma(N/2)} \) are the unique (up to the translation and scaling) solutions for the problem

\[
(-\Delta)^s u = u^{N+2s/2}, \ u > 0 \text{ in } \mathbb{R}^N.
\]  

(1.6)

When \( s = 1 \), in [30], Wang and Wei constructed a single bubble solution to (1.1) provided that \( K(y) \) has a non-degenerate critical point and \( |K(y)| \leq C(1 + |y|^m) \), \( m < 2 \). In [21], assuming that there exist positive constant \( \tau_0 > 0 \), \( \theta \) and \( \iota \) such that

\[
K(r) = K(r_0) - C_0 |r - r_0|^m + O(|r - r_0|^{m+\theta}), \ r \in (r_0 - \iota, r_0 + \iota), \ m \in [2, N-2),
\]

Liu proved that (1.1) has large number of bubble solutions if \( \epsilon > 0 \) small enough. Moreover, he proved there exists solutions whose functional energy is in the order of \( \epsilon^{-\frac{N-2m}{(N-2-m)\overline{\alpha}}} \). Motivated by [27] and [20], we intend in this paper to construct large number of bubbles to (1.1) whose energy is very large if \( \epsilon > 0 \) is small enough under more general assumptions on \( K(y) \). Here we consider the case \( K(y) = K(|y'|, y'') = K(r, y'') \), where \( y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \). We assume that \( K(y) \geq 0 \) is bounded and satisfies the following two conditions which are the same as that of [27]:

\[
\begin{aligned}
& \text{Assumption (i): } \max_{y \in \mathbb{R}^N} K(y) < +\infty, \quad \text{Assumption (ii): } K(y) \leq \frac{C}{(1 + |y|^m)^{m/2}}, \quad m \in [2, N-2),
\end{aligned}
\]
(K_1) $K(r, y'')$ has a stable critical point $(r_0, y_0'')$ in the following sense: $K(r, y'')$ has a critical point $(r_0, y_0'')$ satisfying $r_0 > 0$ and $K(r_0, y_0'') = 1$, and
\[
\deg(\nabla(K(r, y'')), (r_0, y_0'')) \neq 0.
\]

(K_2) $K(r, y'') \in C^3(B_\vartheta((r_0, y_0'')))$, where $\vartheta > 0$ small and
\[
\Delta K(r_0, y_0'') := \frac{\partial^2 K(r_0, y_0'')}{\partial r^2} + \sum_{i=3}^{N} \frac{\partial^2 K(r_0, y_0'')}{\partial y_i^2} < 0. \quad (1.7)
\]

Obviously, the assumption (K_2) contains a saddle point $(r_0, y_0'')$ of $K(y)$. In the sequel, we denote $y_0 = (y_0', y_0'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, where $|y_0'| = r_0$. Since $(r_0, y_0'')$ is a critical point of $K(r, y'')$, (K_2) implies $K(y) \in C^3(B_\vartheta(y_0))$ and
\[
\Delta K(y_0) := \sum_{i=1}^{N} \frac{\partial^2 K(y_0)}{\partial y_i^2} < 0. \quad (1.8)
\]

Our main result is the following

**Theorem 1.1.** Suppose that $K \geq 0$ is bounded and satisfies (K_1) and (K_2). If $N \geq 4, \max\{N+\frac{1}{4}N^2+9, \frac{3}{2}-\frac{6N+13}{6} \} < s < 1$, then there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ problem (1.1) has a solution $u_\epsilon$ whose number to the bubbles is of the order $\epsilon^{-\frac{N+\frac{1}{4}N^2+9}{N-2s}}$ as $\epsilon \to 0$. Particularly, (1.1) has large number of bubble solutions for small $\epsilon > 0$

Remark 1.2. $2s > \tau$ (defined in section 2) and $0 < s < 1$ are equivalent to $\frac{N+\frac{1}{4}N^2+9}{4} < s < 1$, which is needed in (2.25) and (2.27) in Lemma 2.5. $N > 4+2\tau - 2s$ and $0 < s < 1$ are equivalent to $\frac{3}{2}-\frac{6N+13}{6} < s < 1$, which is needed in (2.31) in Lemma 2.5. Both of them are technical assumptions and they can be satisfied automatically when $s = 1$, but here we do not know how to get rid of them.

Remark 1.3. We conjecture that when $N = 3 = 2 + 2s$, a similar result can also be obtained. Since in this case the number of bubbles in construction behaves as a logarithm of the bubbles’ height, one have to make some corresponding modifications. For the details, the readers can refer to [29].

Now we outline the main idea in the proof of Theorem 1.1 and discuss the main difficulties in the proof of such a result.

Throughout the remainder of this paper, we shall prove Theorem 1.1 in detail for the case $2^* - 1 + \epsilon$ since the case $2^* - 1 - \epsilon$ can be obtained by slightly modifying the arguments. We use a Lyapunov-Schmidt reduction argument to prove Theorem 1.5. More precisely, we follow the method in [27] to construct bubble solutions of problem (1.1), where the existence of infinitely many solutions for the prescribed scalar curvature problem is proved. In [27], there is no parameter appearing in their problem. Peng, Wang and Wei used $m$, the number of the bubbles of the solution, as the parameter to construct infinitely many positive bubble solutions. This idea was first introduced by Wei and Yan in [33], which was applied to study other problems, such as [9, 11, 14, 19, 31, 32, 33, 34, 35, 36]. Unlike [27], in our proof, we use $\epsilon$ as the parameter in the construction of bubble solutions, but the number of bubbles depends on the parameter $\epsilon$. This is motivated by [20], where they constructed multiple spikes to a singular perturbed problem and the number of spike depends on the small parameter. Such problems are considered in [21, 30]. Since $(r_0, y_0'')$ may be a saddle point of $K(r, y'')$, we can
not determine the location of the bubbles by using minimization or maximization procedure. Here we will use the Pohozaev identities to find algebraic equations which determine the location of the bubbles. We will discuss this in more details later. This idea was first introduced by Peng, Wang and Yan [28], which was used to deal with other problems such as [15, 16]. Moreover, the application of some local Pohozaev identities can simplify many complicated and tedious computations which were involved in estimating directly the partial derivatives of the reduced functional such as [23, 24].

Define

\[ H_\varepsilon = \left\{ u : u \in D^{1,s}(\mathbb{R}^N), u(y_1, -y_2, y''_3) = u(y_1, y_2, y''_3), u(r \cos \theta, r \sin \theta, y'') = u\left( r \cos \left( \theta + \frac{2\pi j}{m} \right), r \sin \left( \theta + \frac{2\pi j}{m} \right), y'' \right) \right\} \]

Let

\[ x_j = \left( \bar{r} \cos \frac{2(j-1)\pi}{m}, \bar{r} \sin \frac{2(j-1)\pi}{m}, \bar{y}''_j \right), \quad j = 1, \ldots, m, \]

where \( \bar{y}'' \) is a vector in \( \mathbb{R}^{N-2} \).

We will use \( U_{x_j, \lambda} \) as an approximate solution. Let \( \delta > 0 \) be a small constant, such that \( K(r, y'') > 0 \) if \( |(r, y'') - (r_0, y''_0)| \leq 10\delta \).

Denote

\[ Z_{\bar{r}, \bar{y}'', \lambda}(y) = \sum_{j=1}^m U_{x_j, \lambda}(y). \]

By the weak symmetry of \( K(y) \), we observe that \( K(x_j) = K(\bar{r}, \bar{y}''), j = 1, \ldots, m \). In this paper, we always assume that \( m = \left\lceil \varepsilon^{-\frac{N-2s}{(N-2s)^2}} \right\rceil \) is a large integer, \( \lambda \in [L_0 \varepsilon^{-\frac{1}{N-2s}}, L_1 \varepsilon^{-\frac{1}{N-2s}}] \) for some constants \( L_1 > L_0 > 0 \) and

\[ |(\bar{r}, \bar{y}'') - (r_0, y''_0)| \leq \varepsilon \frac{1+\varepsilon}{N-2s}, \quad (1.9) \]

where \( \varepsilon > 0 \) is a small constant.

Remark 1.4. Note that \( \lim_{\varepsilon \to 0} \lambda^{-\frac{N-2s}{2}} = c \), where \( c \) is some positive constant. Then there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) independent of \( \varepsilon \) such that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[ \lambda^{-\frac{N-2s}{2}} \leq C. \quad (1.10) \]

In order to prove Theorem 1.1, we will show the following result.

Theorem 1.5. Under the assumptions of Theorem 1.1, there exists \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \), problem (1.1) has a solution \( u_\varepsilon \) of the form

\[ u_\varepsilon = Z_{\bar{r}, \bar{y}'', \lambda} + \varphi_\varepsilon = \sum_{j=1}^m U_{x_j, \lambda} + \varphi_\varepsilon, \quad (1.11) \]

where \( \varphi_\varepsilon \in H_\varepsilon \). Moreover, as \( \varepsilon \to 0 \), \( \lambda_\varepsilon \in [L_0 \varepsilon^{-\frac{1}{N-2s}}, L_1 \varepsilon^{-\frac{1}{N-2s}}] \), \( (\bar{r}_\varepsilon, \bar{y}'') \to (r_0, y''_0) \), and

\[ \lambda_\varepsilon^{-\frac{N-2s}{2}} \| \varphi_\varepsilon \|_{L^\infty} \to 0. \]

Now we outline some of the main ideas in the proof of such a result. The functional corresponding to problem (1.1) is

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)\tilde{u}|^2 dy - \frac{1}{2^*_s + \varepsilon} \int_{\mathbb{R}^N} K(y)(u)_{2^*_s + \varepsilon} dy. \]
Using the reduction argument, the problem of finding a critical point for $I(u)$ with the form (1.11) can be reduced to that of finding a critical point of the following function

$$F(\bar{r}, \bar{g}', \lambda) := I(Z_{\bar{r}, \bar{g}', \lambda} + \varphi_{\bar{r}, \bar{g}', \lambda}),$$

where $\lambda \in [L_0 e^{-\frac{1}{N}}, L_1 e^{-\frac{1}{N-2}}]$ for some constants $L_1 > L_0 > 0$ and $(\bar{r}, \bar{g}')$ satisfies (1.9).

Instead of estimating the derivatives of the reduced function $F(\bar{r}, \bar{g}', \lambda)$ with respect to $\bar{r}$ and $\bar{g}_k, k = 3, \cdots, N$ directly which involve very complicated calculations, we turn to prove that if $(\bar{r}, \bar{g}', \lambda)$ satisfies the following local Pohozaev identities:

$$
\int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \frac{\partial \bar{u}_\epsilon}{\partial \nu} \frac{\partial \bar{u}_\epsilon}{\partial y_i} dS - \frac{1}{2} \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \nabla |\bar{u}_\epsilon|^2 \nu_i dS
+ \frac{1}{2s + \epsilon} \int_{\partial B_\rho(y_0)} K(y)(u_\epsilon)_{2^*_s + \epsilon}^{2^*_s + \epsilon} dy = 0 \quad (1.12)
$$

and

$$
\int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \langle Y, \nabla \bar{u}_\epsilon \rangle \frac{\partial \bar{u}_\epsilon}{\partial \nu} dS - \frac{1}{2} \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \nabla |\bar{u}_\epsilon|^2 \langle Y, \nu \rangle dS
+ \frac{N - 2s}{2} \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \bar{u}_\epsilon \frac{\partial \bar{u}_\epsilon}{\partial y_i} + \frac{1}{2s + \epsilon} \int_{\partial B_\rho(y_0)} K(r, y')(u_\epsilon)_{2^*_s + \epsilon}^{2^*_s + \epsilon} (y, \nu) dS
- \frac{N}{2s + \epsilon} \int_{\partial B_\rho(y_0)} \langle \nabla K, y \rangle (u_\epsilon)_{2^*_s + \epsilon}^{2^*_s + \epsilon} dy = 0, \quad (1.13)
$$

where $u_\epsilon = Z_{\bar{r}, \bar{g}', \lambda} + \varphi, \tilde{u}_\epsilon$ is the harmonic extension of $u_\epsilon$ (see (1.3)) and satisfies the following equation

$$
\begin{cases}
\text{div}(t^{1-2s} \nabla \tilde{u}(y, t)) = 0, \\
- \lim_{t \to 0} t^{1-2s} \partial_t \tilde{u}(y, t) = K(y)(u)_{2^*_s - 1 + \epsilon}^{2^*_s - 1 + \epsilon}. 
\end{cases} \quad (1.14)
$$

Moreover,

$$
\begin{align*}
\mathcal{B}^+_\rho(y_0) &= \{ Y = (y, t) : |Y - (y_0, 0)| \leq \rho \quad \text{and} \quad t > 0 \} \subset \mathbb{R}^{N+1}_+, \\
\partial^\nu \mathcal{B}^+_\rho(y_0) &= \{ Y = (y, t) : |Y - (y_0, 0)| \leq \rho \quad \text{and} \quad t = 0 \} \subset \mathbb{R}^N, \\
\partial'' \mathcal{B}^+_\rho(y_0) &= \{ Y = (y, t) : |Y - (y_0, 0)| = \rho \quad \text{and} \quad t > 0 \} \subset \mathbb{R}^{N+1}_+, \\
\partial \mathcal{B}_\rho(y_0) &= \{ y : |y - y_0| = \rho \} \subset \mathbb{R}^N,
\end{align*}
$$

where $\rho$ is a small positive constant.

Due to the non-localness of the fractional Laplacian operator, we can not built some local Pohozaev identities for problem (1.1). So we need to study the corresponding harmonic extension problem (1.14). Hence, we have to estimate this kind of integrals which do not appear in the local problem. Here we use some similar arguments as [16].

The rest of this paper is organized as follows. In section 2, we will carry out the reduction procedure. Then, we will study the reduced finite dimensional problem and prove Theorem 1.5 in section 3. We put all the technical estimates in Appendices A, B, C and D.
2. Finite-dimensional reduction

In this section, we perform a finite dimensional reduction by using $Z_{r,y'',\lambda}(y)$ as the approximation solution and considering the linearization of the problem (1.1) around the approximation solution. First, we introduce the following norms:

$$
\|u\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N-2s+\tau}} \right)^{-1} \lambda^{-\frac{N-2s}{2}} |u(y)|
$$

and

$$
\|f\|_{**} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N+2s+\tau}} \right)^{-1} \lambda^{-\frac{N+2s}{2}} |f(y)|,
$$

where $\tau = \frac{N-2s-2}{N-2s}$.

Denote

$$
Z_{j,1} = \frac{\partial U_{x_j,\lambda}}{\partial \lambda}, \quad Z_{j,2} = \frac{\partial U_{x_j,\lambda}}{\partial \bar{f}}, \quad Z_{j,k} = \frac{\partial U_{x_j,\lambda}}{\partial y''_k}, \quad k = 3, \ldots, N.
$$

We consider the following problem:

$$
\begin{align*}
(-\Delta)^s \varphi &- (2^*_s - 1 + \epsilon) K(r, y'') Z_{r,y'',\lambda}^{2^*_s-2+\epsilon} \varphi = h + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2^*_s-2} Z_{j,l}, \quad \text{in } \mathbb{R}^N, \\
\varphi &\in H_s, \quad \sum_{j=1}^{m} \int_{\mathbb{R}^N} Z_{x_j,\lambda}^{2^*_s-2} Z_{j,l} \varphi = 0, \quad l = 1, \ldots, N,
\end{align*}
$$

for some real numbers $c_l$.

**Lemma 2.1.** Suppose that $\varphi_\epsilon$ solves (2.3) for $h = h_\epsilon$. If $\|h_\epsilon\|_{**}$ goes to zero as $\epsilon$ goes to zero, so does $\|\varphi_\epsilon\|_*$.

**Proof.** We follow the idea in [34] and proceed the proof by contradiction. Suppose that there exist $\epsilon \to 0$, $\bar{r}_\epsilon \to r_0$, $\bar{y}_''_\epsilon \to \bar{y}_''_0$, $\lambda_\epsilon \in [L_0\epsilon^{-\frac{1}{N-2s}}, L_1\epsilon^{-\frac{1}{N-2s}}]$ and $\varphi_\epsilon$ solving problem (2.3) for $h = h_\epsilon$, $\lambda = \lambda_\epsilon$, $\bar{r} = \bar{r}_\epsilon$, $\bar{y}'' = \bar{y}_''_\epsilon$ with $\|h_\epsilon\|_{**} \to 0$ and $\|\varphi_\epsilon\|_* \geq c > 0$. Without loss of generality, we may assume that $\|\varphi_\epsilon\|_* = 1$.

We have

$$
|\varphi_\epsilon(y)| \leq C \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} Z_{r,y'',\lambda}^{2^*_s-2+\epsilon}(z) |\varphi_\epsilon(z)| dz \\
+ C \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} \left( |h_\epsilon(z)| + \left| \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2^*_s-2}(z) Z_{j,l}(z) \right| \right) dz,
$$

where $C$ defines some positive constant.

For the first term $I_1$, using Lemma B.6, we can prove

$$
I_1 \leq C \|\varphi_\epsilon\|_* \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} Z_{r,y'',\lambda}^{2^*_s-2+\epsilon}(z) \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|z - x_j|)^{\frac{N-2s}{2} + \tau}} dz
$$

$$
\leq C \|\varphi_\epsilon\|_* \lambda^{\frac{N-2s}{2} \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau + \sigma}}.
$$
For the second term $I_2$, applying Lemma B.2, we have

$$I_2 \leq C\|h_{\epsilon}\|_{**} \int_{\mathbb{R}^N} \sum_{j=1}^{m} \frac{\lambda^{N + 2s}}{|y - z|^{N - 2s}(1 + \lambda|y - x_j|)^{\frac{N - 2s}{2} + \tau}} \ dz$$

$$= C\lambda^{N + 2s}\|h_{\epsilon}\|_{**} \sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{1}{|\lambda(z - y)|^{N - 2s}(1 + \lambda|y - x_j|)^{2s + \frac{N - 2s}{2} + \tau}} \ dz$$

$$= C\lambda^{N + 2s}\|h_{\epsilon}\|_{**} \sum_{j=1}^{m} \int_{\mathbb{R}^N} 1 \ dz$$

$$\leq C\|h_{\epsilon}\|_{**} \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N - 2s}{2} + \tau}}. \tag{2.6}$$

Also, by Lemma B.2 we have

$$|I_3| \leq C \sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}U_{x_j, \lambda}^{2s - 2}(z)} \frac{\lambda^{N - 2s + n_l}}{(1 + \lambda|z - x_j|)^{N - 2s}} \ dz$$

$$\leq \lambda^{N - 2s + n_l} C \sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}(1 + \lambda|z - x_j|)^{4s}(1 + \lambda|z - x_j|)^{N - 2s}} \ dz$$

$$\leq \lambda^{N - 2s + n_l} C \sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}(1 + \lambda|z - x_j|)^{N + 2s}} \ dz \tag{2.7}$$

$$\leq C \sum_{l=1}^{N} c_l \left| |c_l| \sum_{j=1}^{m} \frac{\lambda^{N - 2s + n_l}}{(1 + \lambda|y - x_j|)^{\frac{N - 2s}{2} + \tau}} \right|,$$

where $n_1 = -1$, $n_l = 1$, $l = 2, \ldots, N$.

Therefore, we have

$$\left( \sum_{j=1}^{m} \frac{\lambda^{N - 2s}}{(1 + \lambda|y - x_j|)^{\frac{N - 2s}{2} + \tau}} \right)^{-1} |\varphi_{\epsilon}(z)|$$

$$\leq C\|\varphi_{\epsilon}\|_{**} + C\|h_{\epsilon}\|_{**} \leq C\sum_{l=1}^{N} c_l |\lambda|^{n_l}. \tag{2.8}$$

Next, we will estimate $c_l, l = 1, \ldots, N$. Multiplying both sides of (2.3) by $Z_{1,k} (k = 1, \ldots, N)$ and integrating, we get

$$\sum_{l=1}^{N} c_l \sum_{j=1}^{m} \langle U_{x_j, \lambda}^{2s - 2} Z_{l,j}^s, Z_{1,k} \rangle \tag{2.9}$$

$$= \langle (-\Delta)^{s} \varphi_{\epsilon} - (2s^2 - 1 + \epsilon) K(r, y^\nu)Z_{f, \lambda}^{2s - 2 + \nu} \varphi_{\epsilon}, Z_{1,k} \rangle - \langle h_{\epsilon}, Z_{1,k} \rangle.$$
First of all, there exists a constant $\bar{c} > 0$ such that
\[
\sum_{j=1}^{m} \langle U_{x_j,\lambda}^{2s-2} Z_{j,l}, Z_{1,k} \rangle = (\bar{c} + o(1)) \delta_{lk} \lambda^{n_l+n_k}.
\] (2.10)

From Lemma B.1, we obtain that
\[
\left| \langle h_\epsilon(y), Z_{1,k}(y) \rangle \right| \leq C \| h_\epsilon \|_{**} \int_{\mathbb{R}^N} \frac{\lambda^{N-2s+n_k}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{N+2s}}{(1 + \lambda |y - x_j|)^{N+2s+r}} dy
\]
\[
= C \lambda^{nk} \| h_\epsilon \|_{**} \left\{ \int_{\mathbb{R}^N} \frac{\lambda^{N-2s+n_k}}{(1 + \lambda |y - x_1|)^{N-2s}} dy + \sum_{j=2}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{N+2s}}{(1 + \lambda |y - x_j|)^{N+2s+r}} dy \right\}
\]
\[
\leq C \lambda^{nk} \| h_\epsilon \|_{**} \left( C + C \sum_{j=2}^{m} \frac{1}{(\lambda |x_j - x_1|)^{r}} \right)
\]
\[
\leq C \lambda^{nk} \| h_\epsilon \|_{**}.
\] (2.11)

On the other hand, direct calculation gives
\[
\left| \langle (-\Delta)^s \varphi_\epsilon - (2s - 1 + \epsilon) K(r, y''') Z_{r,y''',\lambda}^{2s-2+\epsilon} \varphi_\epsilon, Z_{1,k} \rangle \right| = O\left( \frac{\lambda^{nk} \| \varphi_\epsilon \|_{**}}{\lambda^{1+\epsilon}} \right),
\] (2.12)
whose proof we put in Appendix C.

Combining (2.10), (2.11) and (2.12), we have
\[
|c_i| \leq C \frac{1}{\lambda^{n_i}} \left( \| \varphi_\epsilon \|_{**} + \| h_\epsilon \|_{**} \right).
\] (2.13)

Then by (2.8) and $\| \varphi_\epsilon \|_{**} = 1$, that there is $R > 0$ such that
\[
\| \lambda^{-\frac{N-2s}{2}} \varphi_\epsilon \|_{L^\infty(B_R(x_j))} \geq a > 0,
\] (2.14)
for some $j$. But $\tilde{\varphi}_\epsilon(y) = \lambda^{-\frac{N-2s}{2}} \varphi_\epsilon(y + x_j)$ converges uniformly in any compact set to a solution $u$ of
\[
(-\Delta)^s u - (2s - 1) U_{0,\lambda}^{2s-2} u = 0, \quad \text{in } \mathbb{R}^N,
\] (2.15)
for some $\Lambda \in [\Lambda_1, \Lambda_2]$ and $u$ is perpendicular to the kernel of (2.15). So $u = 0$. This is a contradiction to (2.14).

From Lemma 2.1, applying the same argument as in the proof of Proposition 4.1 in [10], we can prove the following result:

**Lemma 2.2.** There exist $\epsilon_0 > 0$ and a constant $C > 0$ independent of $\epsilon$, such that for $\epsilon \in (0, \epsilon_0)$ and all $h_\epsilon \in L^\infty(\mathbb{R}^N)$, problem (2.3) has a unique solution $\varphi_\epsilon \equiv L_\epsilon(h_\epsilon)$. Moreover,
\[
\| L_\epsilon(h_\epsilon) \|_{**} \leq C \| h_\epsilon \|_{**}, \quad |c_i| \leq C \epsilon^{n_i} \| h_\epsilon \|_{**}.
\] (2.16)

Now we consider
\[
\left\{ \begin{array}{l}
(-\Delta)^s (Z_{r,y''',\lambda} + \varphi) = K(r, y''')(Z_{r,y''',\lambda} + \varphi)^{2s-1+\epsilon} + \sum_{l=1}^{N} \sum_{j=1}^{m} c_l U_{x_j,\lambda}^{2s-2} Z_{j,l}, \quad \text{in } \mathbb{R}^N,
\varphi \in H_\epsilon, \quad \int_{\mathbb{R}^N} \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \varphi = 0, \quad l = 1, 2, \ldots, N.
\end{array} \right.
\] (2.17)
In the rest of this section, we devote ourselves to prove the following proposition by using the contraction mapping theorem.

**Proposition 2.3.** There exist $\epsilon_0 > 0$ and a constant $C > 0$, independent of $\epsilon$, such that for each $\epsilon \in (0, \epsilon_0)$, $\lambda \in [L_0 \epsilon^{-\frac{1}{2s-2}}, L_1 \epsilon^{-\frac{1}{2s-2}}]$, $\bar{r} \in [r_0 - \theta, r_0 + \theta]$, $\bar{y}'' \in B_0(y_0')$, where $\theta > 0$ small, (2.17) has a unique solution $\varphi = \varphi_{\bar{r}, \bar{y}''} \in H_s$ satisfying

$$
\|\varphi\|_s \leq C \epsilon^{\frac{1}{N-2s}}, \quad |c_l| \leq C \epsilon^{\frac{1}{N-2s}},
$$

where $\epsilon > 0$ is a small constant.

We first rewrite (2.17) as

$$
\begin{align*}
\left\{ \begin{array}{l}
(\Delta)^s \varphi - (2_s - 1 + \epsilon)K(r, y'')Z_{\bar{r}, \bar{y}'', \lambda}^2 \varphi \\
= N(\varphi) + l_\epsilon + \sum_{l=1}^N c_l \sum_{j=1}^m U_{x_j, \lambda}^2 Z_{j,l} \varphi, \quad \text{in } \mathbb{R}^N, \\
\varphi \in H_s, \quad \int_{\mathbb{R}^N} \sum_{j=1}^m U_{x_j, \lambda}^2 Z_{j,l} \varphi = 0, \quad l = 1, \cdots, N,
\end{array} \right.
\end{align*}
$$

where

$$
N(\varphi) = K(r, y'') \left[ (Z_{\bar{r}, \bar{y}'', \lambda} + \varphi)^{2_s - 1 + \epsilon} - Z_{\bar{r}, \bar{y}'', \lambda}^{2_s - 1 + \epsilon} - (2_s - 1 + \epsilon)Z_{\bar{r}, \bar{y}'', \lambda}^{2_s - 2 + \epsilon} \varphi \right],
$$

and

$$
l_\epsilon = \left[ K(r, y'')Z_{\bar{r}, \bar{y}'', \lambda}^{2_s - 1 + \epsilon} - \sum_{j=1}^m U_{x_j, \lambda}^{2_s - 1} \right].
$$

In order to apply the contraction mapping theorem to prove Proposition 2.3, we need to estimate $N(\varphi)$ and $l_\epsilon$ respectively.

**Lemma 2.4.** If $N \geq 4$, then

$$
\|N(\varphi)\|_{**} \leq C\|\varphi\|_{!*}^{\min(2_s - 1 + \epsilon, 2)}.
$$

**Proof.** We have

$$
N(\varphi) \leq \left\{ \begin{array}{l}
C|\varphi|_{!*}^{2_s - 1 + \epsilon}, \quad 2_s < 3, \\
|Z_{\bar{r}, \bar{y}'', \lambda}|^{2_s - 3 + \epsilon} \varphi^2 + C|\varphi|_{!*}^{2_s - 1 + \epsilon}, \quad 2_s \geq 3.
\end{array} \right.
$$

First, we consider $2_s < 3$. By the discrete Hölder inequality, we get

$$
\begin{align*}
|N(\varphi)| &\leq C\|\varphi\|_{!*}^{2_s - 1 + \epsilon} \left( \sum_{j=1}^m \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s+2}{2} + \tau}} \right)^{2_s - 1 + \epsilon} \\
&\leq C\|\varphi\|_{!*}^{2_s - 1 + \epsilon} \sum_{j=1}^m \frac{\lambda^{\frac{N-2s+2}{2} + \frac{N-2s}{2} + \epsilon}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2} + \tau}} \\
&\times \left( \sum_{j=1}^m \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2} + \tau}} \right)^{2_s - 1 + \epsilon} \\
&\leq C\|\varphi\|_{!*}^{2_s - 1 + \epsilon} \sum_{j=1}^m \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2} + \tau}},
\end{align*}
$$

where $\lambda > 0$ is a constant.
and we have used Remark 1.4. Therefore,
\[\|N(\varphi)\|_{**} \leq C\|\varphi\|_{2s}^{2s-1+\epsilon}.\]

Similarly, if \(2^s \geq 3\), we have
\[
\|N(\varphi)\| \leq C|Z_{\tilde{r}, \tilde{y}', \lambda}|^{2s-3+\epsilon}\varphi^2 + C|\varphi|^{2s-1+\epsilon}
\]
\[
\leq C\|\varphi\|_2^2 \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right) \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2s-3+\epsilon}
\]
\[
+ C\|\varphi\|_2^{2s-1+\epsilon} \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2s-1+\epsilon}
\]
\[
\leq C\|\varphi\|_2^2 + \|\varphi\|_2^{2s-1+\epsilon} \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2s-1+\epsilon}
\]
\[
\leq C\|\varphi\|_2^2 \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}}.
\]
Hence, we obtain \(\|N(\varphi)\|_{**} \leq C\|\varphi\|_{2s}^{\min(2s-1+\epsilon, 2)}.\]

Next, we will give the estimate of \(l_\epsilon\).

**Lemma 2.5.** If \(N \geq 4\), \(\max\left\{ \frac{N+1-\sqrt{N^2-2N+9}}{4}, \frac{3-\sqrt{N^2-6N+13}}{2} \right\} < s < 1\), then there is a small constant \(\iota > 0\), such that
\[\|l_\epsilon\|_{**} \leq C\epsilon^{\frac{1+\iota}{N-2s}}.\]

**Proof.** Recall that
\[
l_\epsilon = K(y)Z_{\tilde{r}, \tilde{y}', \lambda}^{2s-1+\epsilon} - \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1}
\]
\[
= K(y) \left( Z_{\tilde{r}, \tilde{y}', \lambda}^{2s-1+\epsilon} - Z_{\tilde{r}, \tilde{y}', \lambda}^{2s-1} \right) + K(y) \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} + (K(y) - 1) \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1}
\]
\[
:= J_1 + J_2 + J_3. \tag{2.22}
\]

By the mean value theorem and the discrete Hölder inequality, it follows from Remark 1.4 that
\[
J_1 \leq C\epsilon \left| \ln Z_{\tilde{r}, \tilde{y}', \lambda} \right|
\]
\[
\leq C\epsilon \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2s-1+\epsilon} \ln \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}}
\]
\[
\leq C\epsilon \ln \lambda^{N-2s} \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \left( \sum_{j=1}^{m} \frac{1}{(N-2s-\frac{(N+2s)\iota}{4(1+\iota)}} \right)^{2s-2+\epsilon}
\]
\[
\leq C\epsilon \ln \frac{1}{\epsilon} \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \left( \sum_{j=1}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N-2s-\frac{(N+2s)\iota}{4(1+\iota)}}} \right)^{2s-2+\epsilon}
\]

\[ \leq C \epsilon \ln \frac{1}{\epsilon} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{N+2s+\tau}} \]

\[ \leq C \epsilon^{\frac{1+\alpha}{2}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{N+2s+\tau}}, \tag{2.23} \]

where \(0 < \kappa < 1\) and we have used the fact that \(N > 2s + 2\) to implies \(\epsilon \ln \frac{1}{\epsilon} \leq C \epsilon^{\frac{1+\alpha}{2}}\).

In order to estimate \(J_{2}\), first we define

\[ \Omega_j = \{ y : y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \left\{ \frac{y'}{|y'|}, \frac{x_j'}{|x_j'|} \right\} \geq \cos \frac{\pi}{m} \}, \quad j = 1, \ldots, m. \]

By symmetry, we can assume that \(y \in \Omega_1\), then \(|y - x_j| \geq |y - x_1|\). Note that

\[ |J_2| \leq C \left( \sum_{j=2}^{m} U_{x_j, \lambda}^{2s-1} U_{x_1, \lambda}^{2s} + \sum_{j=2}^{m} U_{x_j, \lambda}^{2s-1} \right) \]

\[ \leq C \left( \sum_{j=2}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{N-2s}} \right)^{2s-1} + \frac{C\lambda^{2s}}{(1 + \lambda |y - x_1|)^{4s}} \sum_{j=2}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{N-2s}}. \tag{2.24} \]

Note that \(\tau < 2s\) implies that \(\frac{2}{N-2s} \leq \frac{N-2s}{N+2s} \frac{4s}{N-2s} > 1\). As in [34], using Hölder inequality, we can derive

\[ \left( \sum_{j=2}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{N-2s}} \right)^{2s-1} \]

\[ \leq \sum_{j=2}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{N+2s+\tau}} \left( \sum_{j=2}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{N+2s}} \right)^{\frac{4s}{N-2s}} \]

\[ \leq C \left( \frac{m}{\lambda} \right)^{\frac{N+2s}{4s}} \left( \frac{N-2s}{N+2s} \right)^{\frac{4s}{N-2s}} \sum_{j=2}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{N+2s+\tau}}. \tag{2.25} \]

By Lemma B.1, taking \(0 < \alpha \leq \min\{\frac{N+2s}{2}, N-2s\}\), we obtain that for any \(y \in \Omega_1\) and \(j > 1\)

\[ \frac{1}{(1 + \lambda |y - x_1|)^{4s}} \frac{1}{(1 + \lambda |y - x_j|)^{N-2s}} \leq C \frac{1}{(1 + \lambda |y - x_1|)^{N+2s-\alpha}} \frac{1}{|\lambda(x_j - x_1)|^{\alpha}}. \tag{2.26} \]

Since \(\tau < 2s\), we can choose \(\alpha > N-2s\) satisfying \(N+2s - \alpha \geq \frac{N+2s}{2} + \tau\).

Then

\[ \frac{1}{(1 + \lambda |y - x_1|)^{4s}} \sum_{j=2}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{N-2s}} \]

\[ \leq \frac{C}{(1 + \lambda |y - x_1|)^{N+2s-\alpha}} \left( \frac{m}{\lambda} \right)^{\alpha} \leq C \frac{1}{(1 + \lambda |y - x_1|)^{N+2s-\alpha}} \epsilon^{\frac{2s}{N-2s}}. \tag{2.27} \]

\[ \leq C \epsilon^{\frac{1+\alpha}{2}} \frac{1}{(1 + \lambda |y - x_1|)^{\frac{N+2s}{2}+\tau}}. \]
Thus we have proved
\[ \|J_2\|_{**} \leq C\epsilon^{3^+\frac{1}{N-2s}}. \] (2.28)

Next, we will estimate the term \( J_3 \). Using the Taylor expansion, in a neighborhood of \( y_0 \) we can rewrite \( K(y) \) in the following form
\[ K(y) = K(y_0) + \nabla K(y_0) \cdot (y - y_0) + \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (y_i - y_{0i})(y_j - y_{0j}) + o(|y - y_0|^2). \]

In the region \( \tilde{B} := \{(r, y'') - (r_0, y'_0)| \leq \sigma \epsilon^{\frac{1}{N-2s}} \), where \( \sigma > 0 \) is a fixed constant. Then we have
\[ |J_3| = \left| \left( \sum_{i,j=1}^{N} \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (y_i - y_{0i})(y_j - y_{0j}) + o(|y - y_0|^2) \right) \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} \right| \]
\[ \leq C\epsilon^{\frac{1}{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda|y - x_j|)^{N+2s}} \leq C\epsilon^{\frac{1}{N-2s}} \sum_{j=1}^{l} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2}+\frac{1}{\tau}}} \] (2.29)

On the other hand, in the region \( \sigma\epsilon^{\frac{1}{N-2s}} \leq |(r, y'') - (r_0, y'_0)| \leq 2\delta \), we have
\[ |y - x_j| \geq |(r, y'') - (r_0, y'_0)| - |(r_0, y'_0) - (\hat{r}, y'')| \geq \frac{\sigma}{2} \epsilon^{\frac{1}{N-2s}}, \]
which implies that
\[ \frac{1}{1 + \lambda|y - x_j|} \leq C\epsilon^{-\frac{1}{N-2s}}. \] (2.30)

Then we have
\[ |J_3| \leq C \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda|y - x_j|)^{N+2s}} \]
\[ \leq C\epsilon^{\frac{1}{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2}+\frac{1}{\tau}}} \frac{1}{\epsilon^{\frac{1}{N-2s}} (1 + \lambda|y - x_j|)^{\frac{N+2s}{2}+\frac{1}{\tau}}} \]
\[ \leq C\epsilon^{\frac{1}{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2}+\frac{1}{\tau}}} \] (2.31)

since \( \frac{1}{\epsilon^{\frac{1}{N-2s}} (1 + \lambda|y - x_j|)^{\frac{N+2s}{2}+\frac{1}{\tau}}} \geq 0 \).

Combining (2.29) and (2.31), we obtain that
\[ \|J_3\|_{**} \leq C\epsilon^{\frac{1}{N-2s}}. \] (2.32)

As a result, from (2.28) to (2.32), we have
\[ \|I_5\|_{**} \leq C\epsilon^{\frac{1}{N-2s}}. \]
**Proof of Proposition 2.3.** First we recall that \( \lambda \in [L_0 e^{-\frac{1}{N-\epsilon}}, L_1 e^{-\frac{1}{N-2\epsilon}}] \). Set
\[
\mathcal{N} = \left\{ w : w \in C(\mathbb{R}^N) \cap H_s, \|w\|_s \leq \epsilon^{\frac{1}{N-2\epsilon}}, \int_{\mathbb{R}^N} \sum_{j=1}^m U_{x_j, \lambda}^{2s-2} Z_{j,i} w = 0 \right\},
\]
where \( l = 1, \cdots, N \). Then (2.19) is equivalent to
\[
\varphi = A(\varphi) := L_\epsilon(N(\varphi)) + L_\epsilon(l_\epsilon),
\]
where \( L_\epsilon \) is defined in Lemma 2.2. We will prove that \( A \) is a contraction map from \( \mathcal{N} \) to \( \mathcal{N} \).

In fact, since
\[
\|A(\varphi)\|_s \leq C \|L_\epsilon(N(\varphi))\|_s + C \|L_\epsilon(l_\epsilon)\|_s,
\]
we deduce
\[
\|A(\varphi)\|_s \leq C \left[ \|N(\varphi)\|_{**} + \|l_\epsilon\|_{**} \right] \leq C \left[ \|\varphi\|_{\min(2s-1+\epsilon, 2)} + \epsilon^{\frac{1}{N-2\epsilon}} \right] \leq \epsilon^{\frac{1}{N-2\epsilon}},
\]
therefore \( A \) maps \( \mathcal{N} \) to \( \mathcal{N} \).

On the other hand, we have
\[
|N'(t)| \leq \begin{cases} 
C \|t\|_{2s-2+\epsilon}, & 2s < 3, \\
C \left( \|Z_{F,g^{\nu, \lambda}}\|_{2s-3+\epsilon} |t| + |t|^{2s-2+\epsilon} \right), & 2s \geq 3.
\end{cases}
\]

If \( 2s < 3 \), \( \forall \varphi_1, \varphi_2 \in \mathcal{N} \), by the discrete Hölder inequality and Remark 1.4, we have
\[
|N(\varphi_1) - N(\varphi_2)| \\
\leq C(\|\varphi_1\|_{2s-2+\epsilon} + |\varphi_2|_{2s-2+\epsilon})|\varphi_1 - \varphi_2| \\
\leq C(\|\varphi_1\|_{2s-2+\epsilon} + |\varphi_2|_{2s-2+\epsilon})|\varphi_1 - \varphi_2| \left( \sum_{j=1}^m \frac{\lambda^{\frac{N-2s}{2s}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2s} + \tau}} \right)^{2s-1+\epsilon} \\
\leq C(\|\varphi_1\|_{2s-2+\epsilon} + |\varphi_2|_{2s-2+\epsilon})|\varphi_1 - \varphi_2| \sum_{j=1}^m \frac{\lambda^{\frac{N-2s}{2s}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2s} + \tau}}.
\]

Hence
\[
\|A(\varphi_1) - A(\varphi_2)\|_s = \|L_\epsilon(N(\varphi_1)) - L_\epsilon(N(\varphi_2))\|_s \\
\leq C\|N(\varphi_1) - N(\varphi_2)\|_{**} \\
\leq C\epsilon^{\frac{4\alpha}{(N-2s)^2}} |\varphi_1 - \varphi_2|_s \\
\leq \frac{1}{2} |\varphi_1 - \varphi_2|_s.
\]
The case \( 2s \geq 3 \) can be discussed in a similar way.

Hence \( A \) is a contraction map. Now by the contraction mapping theorem, there exists a unique \( \varphi = \varphi_{F,g^{\nu, \lambda}} \in \mathcal{N} \) such that (2.33) holds. Moreover, by Lemma 2.2, Lemma 2.4 and Lemma 2.5, we deduce
\[
\|\varphi\|_s \leq \|L_\epsilon(N(\varphi))\|_s + \|L_\epsilon(l_\epsilon)\|_s \leq C\|N(\varphi)\|_{**} + C\|l_\epsilon\|_{**} \leq C\epsilon^{\frac{1}{N-2\epsilon}}.
\]
Moreover, we get the estimate of \( c_l \) from (2.16). \( \square \)
3. Proof of the main result

Let
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\bar{s}}{2}} u \right|^2 dy - \frac{1}{2^*_s} \int_{\mathbb{R}^N} K(|y'|, y'')(u)^{2^*_s + \epsilon} dy. \]

In this section, we will choose suitable \((\bar{r}, \bar{y}'', \lambda)\) so that \(Z_{\bar{r}, \bar{y}'', \lambda} + \varphi_{\bar{r}, \bar{y}'', \lambda}\) is a solution of problem (1.1). For this purpose, we need the following result.

**Proposition 3.1.** Suppose that \((\bar{r}, \bar{y}'', \lambda)\) satisfies

\[
\int_{\partial^* B^+_1(y_0)} t^{1-2s} (Y, \nabla \tilde{u}_\epsilon) \frac{\partial \tilde{u}_\epsilon}{\partial \nu} dS - \frac{1}{2} \int_{\partial^* B^+_1(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 dS + \frac{N - 2s}{2} \int_{B^+_1(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 dy + \frac{1}{2^*_s} \int_{\partial B(y_0)} K(r, y'')(u\epsilon)^{2^*_s + \epsilon} \langle y, \nu \rangle ds - \frac{N}{2^*_s + \epsilon} \int_{B(y_0)} K(r, y'')(u\epsilon)^{2^*_s + \epsilon} dy = 0, \tag{3.1}
\]

and

\[
\int_{\partial^* B^+_1(y_0)} t^{1-2s} \frac{\partial \tilde{u}_\epsilon}{\partial \nu} dS + \frac{1}{2} \int_{\partial^* B^+_1(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 \nu_i dS + \frac{1}{2} \int_{\partial B(y_0)} \frac{\partial K(r, y'')(u\epsilon)^{2^*_s + \epsilon}}{\partial \nu_i} \nu_i dS = 0. \tag{3.2}
\]

where \(u\epsilon = Z_{\bar{r}, \bar{y}'', \lambda} + \varphi_{\bar{r}, \bar{y}'', \lambda}\) and \(B(y_0) = \{(r, y'') : \| (r, y'') - (r_0, y_0') \| \leq \rho \} \), \(\rho\) is a small positive constant. Then \(c_l = 0, l = 1, \ldots, N\).

**Proof.** If (3.1) and (3.2) hold, then it follows from (A.2) and (A.6) that

\[
\int_{B(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} \langle y, \nabla u\epsilon \rangle dy = 0, \tag{3.4}
\]

and

\[
\int_{B(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} \frac{\partial u\epsilon}{\partial \nu_i} dy = 0, (i = 3, \ldots, N). \tag{3.5}
\]

By direct computations, we can check that

\[
\int_{B(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} \langle y, \nabla u\epsilon \rangle dy = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m|c_1|), \tag{3.6}
\]

and

\[
\int_{B(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} \frac{\partial u\epsilon}{\partial \nu_i} dy = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m|c_1|), (i = 3, \ldots, N), \tag{3.7}
\]

whose proofs we put in Appendix D.
Also if (3.3) holds, then by (2.17) we have
\[
\int_{\mathbb{R}^N} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2_{*}-2} Z_{j,l} \frac{\partial Z_{\gamma,\gamma',\lambda}}{\partial \lambda} dy = 0. \tag{3.8}
\]
From (3.4) to (3.8), we get
\[
\sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2_{*}-2} Z_{j,l} v dy = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m|c_1|) \tag{3.9}
\]
for \( v = \langle y, \nabla u_i \rangle, v = \frac{\partial u_i}{\partial y_i}, (i = 3, \ldots, N) \) and \( v = \frac{\partial Z_{\gamma,\gamma',\lambda}}{\partial \lambda} \).

By direct computations, it is easy to obtain that
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,2} \langle y', \nabla y' Z_{\gamma,\gamma',\lambda} \rangle dy = m(a_1 + o(1))\lambda^2, \tag{3.10}
\]
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,i} \frac{\partial Z_{\gamma,\gamma',\lambda}}{\partial y_i} dy = m(a_2 + o(1))\lambda^2, \quad i = 3, \ldots, N, \tag{3.11}
\]
and
\[
\sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,1} \frac{\partial Z_{\gamma,\gamma',\lambda}}{\partial \lambda} dy = m(a_3 + o(1)), \tag{3.12}
\]
for some constants \( a_1 \neq 0, a_2 \neq 0 \) and \( a_3 > 0 \).

For any functions \( f, \varphi \in H^1(\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} f \frac{\partial \varphi}{\partial y_i} dy = - \int_{\mathbb{R}^N} \varphi \frac{\partial f}{\partial y_i} dy. \tag{3.13}
\]
Using (3.13), we can prove from (2.18) that
\[
\sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,l} v dy = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m|c_1|), \tag{3.14}
\]
holds for \( v = \langle y, \nabla \varphi_{\gamma,\gamma',\lambda} \rangle \) and \( v = \frac{\partial \varphi_{\gamma,\gamma',\lambda}}{\partial y_i} \). Therefore, from (3.4), we obtain
\[
\sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,l} v dy = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m|c_1|), \tag{3.15}
\]
holds for \( v = \langle y, \nabla Z_{\gamma,\gamma',\lambda} \rangle \) and \( v = \frac{\partial Z_{\gamma,\gamma',\lambda}}{\partial y_i} \).

From
\[
\langle y, \nabla Z_{\gamma,\gamma',\lambda} \rangle = \langle y', \nabla y' Z_{\gamma,\gamma',\lambda} \rangle + \langle y'', \nabla y'' Z_{\gamma,\gamma',\lambda} \rangle,
\]
we find
\[
\sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,l} \langle y, \nabla Z_{\gamma,\gamma',\lambda} \rangle dy
\]
\[
= c_2 \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2_{*}-2} Z_{j,2} \langle y', \nabla y' Z_{\gamma,\gamma',\lambda} \rangle dy + o(m\lambda^2) \sum_{l=3}^{N} |c_l| + o(m|c_1|), \tag{3.16}
\]

and

\[
\sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,l} \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial y_i} dy
\]

\[
= c_i \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,i} \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial y_i} dy + o(m\lambda^2) \sum_{t\neq 1, i} |c_t| + o(m|c_1|), \quad i = 3, \cdots, N.
\]

Combining (3.15), (3.16) and (3.17), we are led to

\[
c_2 \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,2} \langle y', \nabla' Z_{\tilde{r},\tilde{y}'',\lambda} \rangle dy = o(m\lambda^2) \sum_{t=3}^{N} |c_t| + o(m|c_1|),
\]

and

\[
c_i \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,i} \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial y_i} dy = o(m\lambda^2) \sum_{t\neq 1, i} |c_t| + o(m|c_1|), \quad i = 3, \cdots, N,
\]

which, together with (3.10) and (3.11), imply

\[
c_i = o\left(\frac{1}{\lambda^2}\right) c_1, \quad i = 2, \cdots, N.
\]

Now we have

\[
0 = \sum_{l=1}^{N} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,l} \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial \lambda} dy
\]

\[
= c_i \sum_{j=1}^{m} \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2^*-2} Z_{j,1} \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial \lambda} dy
\]

\[
= m \frac{a_3}{\lambda^2} c_1 + o\left(\frac{m}{\lambda^2}\right) c_1.
\]

So \(c_1 = 0\). We also have \(c_i = 0\). □

Next, we will estimate (3.1), (3.2) and (3.3). Moreover, we will choose suitable \((\tilde{r}, \tilde{y}'', \lambda)\) so that (3.1), (3.2) and (3.3) holds. First of all, the following lemma gives the estimate of (3.3).

**Lemma 3.2.** We have

\[
\int_{\mathbb{R}^N} ((-\Delta)^s u_\epsilon - K(r, y'')(u_\epsilon)_+^{2^*_+ - 1 + \epsilon}) \frac{\partial Z_{\tilde{r},\tilde{y}'',\lambda}}{\partial \lambda} dy
\]

\[
= m\left( - \frac{B_1}{\lambda^3} + \frac{B_2}{\lambda^{N-2s+1}} |x_1 - x_j|^{|N-2s|} + O\left(\epsilon^\frac{3s}{N-2s}\right) \right)
\]

\[
= m\left( - \frac{B_1}{\lambda^3} + \frac{B_3 m^{N-2s}}{\lambda^{N-2s+1}} + O\left(\epsilon^\frac{3s}{N-2s}\right) \right),
\]

where \(B_i > 0, i = 1, 2, 3\).
Proof. We have
\[
\int_{\mathbb{R}^N} ((-\Delta)^{s} u_\epsilon - K(r, y'') (u_\epsilon)^{2_s - 1 + \epsilon}) \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \, dy
= \left\langle I' (Z_{F\cdot g', \lambda}), \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right\rangle + m \left\langle (-\Delta)^{s} \varphi - (2_s - 1 + \epsilon) K(r, y'') Z_{F\cdot g', \lambda}^{2_s - 2 + \epsilon}, \frac{\partial Z_{x_1, \lambda}}{\partial \lambda} \right\rangle
- \int_{\mathbb{R}^N} K(r, y'') ((Z_{F\cdot g', \lambda} + \varphi)^{2_s - 1 + \epsilon} - Z_{F\cdot g', \lambda}^{2_s - 1 + \epsilon} - (2_s - 1 + \epsilon) Z_{F\cdot g', \lambda}^{2_s - 2 + \epsilon}) \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \, dy
= \left\langle I' (Z_{F\cdot g', \lambda}), \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right\rangle + mK_1 - K_2.
\]
Using (2.12), we have
\[
K_1 = O \left( \frac{\|\varphi\|_{L^{2^{*}+\epsilon}}}{{\lambda}^{2^{*}+\epsilon}} \right) = O \left( \epsilon \frac{3^{+}}{N^{-2s}} \right). \tag{3.24}
\]
Note that
\[
|(1 + t)\zeta - 1 - \zeta| \leq \begin{cases} C t^2, & 1 < \zeta \leq 2, \\ C t^2 + |t|\zeta, & \zeta > 2. \end{cases} \tag{3.25}
\]
Suppose that $2_s^* < 3$. Then it follows from Remark 1.4 that
\[
|K_2| \leq C \int_{\mathbb{R}^N} Z_{F\cdot g', \lambda}^{2_s^* - 3 + \epsilon} \varphi^2 \left| \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right| \, dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{m} U_{x_j, \lambda} \right)^{2_s^* - 2 + \epsilon} \varphi^2 \, dy
\]
\[
\leq \frac{C\|\varphi\|^2}{\lambda} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2_s^* - 2 + \epsilon} \left( \sum_{i=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_i|)^{N-2s}} \right)^{-\epsilon} \, dy
= C m\|\varphi\|^2 = O \left( m\epsilon \frac{3^{+}}{N^{-2s}} \right). \tag{3.26}
\]
Similarly, for $2_s^* \geq 3$, we have
\[
|K_2| \leq \int_{\mathbb{R}^N} \left( Z_{F\cdot g', \lambda}^{2_s^* - 3 + \epsilon} + |\varphi|^{2_s^* - 1 + \epsilon} \left| \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right| \right) \, dy = O \left( m\epsilon \frac{3^{+}}{N^{-2s}} \right). \tag{3.27}
\]
So, we have proved
\[
\left\langle I' (Z_{F\cdot g', \lambda} + \varphi), \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right\rangle = \left\langle I' (Z_{F\cdot g', \lambda}), \frac{\partial Z_{F\cdot g', \lambda}}{\partial \lambda} \right\rangle + O \left( m\epsilon \frac{3^{+}}{N^{-2s}} \right). \tag{3.28}
\]
Using Lemma B.7 in Appendix B, we obtain the result. \qed

Next, we will estimate (3.1) and (3.2). Let us point out that (3.1) is the local Pohozaev identity generating from scaling, while (3.2) is the local Pohozaev identities generating from translations.

Noting that $\tilde{u}_\epsilon$ satisfies the following equation
\[
\begin{cases} 
\text{div}(t^{-1-2s} \nabla \tilde{u}_\epsilon) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
- \lim_{t \to 0} t^{1-2s} \partial_t \tilde{u}_\epsilon (x, t) = K(r, y'')(u_\epsilon)^{2_s^* - 1 + \epsilon} + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2_s^* - 2} Z_{j, l}, & \text{in } \mathbb{R}^N. \tag{3.29}
\end{cases}
\]
Then, we can get that

\[
0 = \int_{B_r^+(y_0)} \text{div}(t^{1-2s} \nabla \tilde{u}_e) \tilde{u}_e \, dy \, dt
= \int_{\partial B_r^+(y_0)} t^{1-2s} \nabla \tilde{u}_e \tilde{u}_e \, dS - \int_{B_r^+(y_0)} t^{1-2s} \nabla \tilde{u}_e \nabla \tilde{u}_e \, dy \, dt
= \int_{\partial B_r^+(y_0)} t^{1-2s} \nabla \tilde{u}_e \tilde{u}_e \, dS + \int_{\partial B_r^+(y_0)} t^{1-2s} \nabla \tilde{u}_e \tilde{u}_e \, dS - \int_{B_r^+(y_0)} t^{1-2s} |\nabla \tilde{u}_e|^2 \, dy \, dt
= -\int_{\partial B_r^+(y_0)} t^{1-2s} \partial_t \tilde{u}_e \, dS + \int_{\partial B_r^+(y_0)} t^{1-2s} \nabla \tilde{u}_e \tilde{u}_e \, dS - \int_{B_r^+(y_0)} t^{1-2s} |\nabla \tilde{u}_e|^2 \, dy \, dt
= \int_{B_r^+(y_0)} \left[K(r, y'')(u_e)_{+}^{2^*_s + 1 + \epsilon} + \sum_{l=1}^{N} \sum_{j=1}^{m} c_l U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} u_e\right] \, dy
+ \int_{\partial B_r^+(y_0)} t^{1-2s} \frac{\partial \tilde{u}_e}{\partial \nu} \tilde{u}_e \, dS.
\]

Therefore, we get

\[
\int_{B_r^+(y_0)} t^{1-2s} |\nabla \tilde{u}_e|^2 \, dy \, dt = \int_{B_r^+(y_0)} \left[K(r, y'')(u_e)_{+}^{2^*_s + 1 + \epsilon} + \sum_{l=1}^{N} \sum_{j=1}^{m} c_l U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} u_e\right] \, dy
+ \int_{\partial B_r^+(y_0)} t^{1-2s} \frac{\partial \tilde{u}_e}{\partial \nu} \tilde{u}_e \, dS. \tag{3.31}
\]

Since

\[
\int_{\mathbb{R}^N} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} \varphi \, dy = 0,
\]

then

\[
\int_{\mathbb{R}^N} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} u_e \, dy = \int_{\mathbb{R}^N} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} (Z_{r, g''} + \varphi) \, dy
= \int_{\mathbb{R}^N} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} Z_{r, g''} \, dy.
\]

Therefore (3.1) is equivalent to

\[
\frac{1}{2^*_s + \epsilon} \int_{B_r^+(y_0)} \langle \nabla K, y''(u_e) \rangle_{+}^{2^*_s + \epsilon} \, dy
= \left(\frac{N - 2s}{2} - \frac{N}{2^*_s + \epsilon}\right) \int_{B_r^+(y_0)} K(r, y'')(u_e)_{+}^{2^*_s + \epsilon} \, dy
+ \frac{1}{2^*_s + \epsilon} \int_{\partial B_r^+(y_0)} K(r, y'')(u_e)_{+}^{2^*_s + \epsilon} \, dy \, ds
+ \frac{N - 2s}{2} \int_{\partial B_r^+(y_0)} t^{1-2s} \tilde{u}_e \frac{\partial \tilde{u}_e}{\partial \nu} \, dS + \int_{\partial B_r^+(y_0)} t^{1-2s} \langle Y, \nabla \tilde{u}_e \rangle \frac{\partial \tilde{u}_e}{\partial \nu} \, dS + \frac{N - 2s}{2} \int_{\mathbb{R}^N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2^*_s - 2} Z_{j,l} Z_{r, g''} \, dy. \tag{3.32}
\]
\[-\frac{N - 2s}{2} \int_{B^c(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} \phi dy.\]

**Lemma 3.3.** (3.1) and (3.2) are equivalent to

\[
\int_{B^c(y_0)} \langle \nabla K, y \rangle (u_\epsilon)^{2\epsilon + i} dy = O(m \epsilon^{\frac{1 + \epsilon}{2s}}) \tag{3.33}
\]

and

\[
\int_{B^c(y_0)} \frac{\partial K(y)}{\partial y_i} (u_\epsilon)^{2\epsilon + i} dy = O(m \epsilon^{\frac{1 + \epsilon}{2s}}) \quad i = 3, \cdots, N. \tag{3.34}
\]

**Proof.** Here we only prove (3.33) since the proof of (3.34) is similar.

First, we have

\[
\left( \frac{N - 2s}{2} - \frac{N}{2s} + \epsilon \right) \int_{B^c(y_0)} K(r, y') (u_\epsilon)^{2\epsilon + \epsilon} dy = o(\epsilon) = o(m \epsilon^{\frac{1 + \epsilon}{(N - 2s)^2}}) = o(m \epsilon^{\frac{1 + \epsilon}{2s}}). \tag{3.35}
\]

Noting that $K(r, y')$ is bounded, we have

\[
\frac{1}{2s} + \epsilon \int_{\partial B^c(y_0)} K(r, y') (u_\epsilon)^{2\epsilon + \epsilon} (y, \nu) ds \tag{3.36}
\]

\[
\leq C \int_{\partial B^c(y_0)} |\phi_\epsilon|^{2s} ds + C \int_{\partial B^c(y_0)} Z_{x, y, \lambda}^{2s} ds
\]

\[
\leq C\|\phi\|^{2s} \int_{\partial B^c(y_0)} \left( \sum_{j=1}^{m} \lambda^{\frac{N - 2s}{2}} (1 + \lambda |y - x_j|)^{\frac{N - 2s}{2}} \right)^{2s} ds
\]

\[
+ C \int_{\partial B^c(y_0)} \left( \sum_{j=1}^{m} \lambda^{\frac{N - 2s}{2}} (1 + \lambda |y - x_j|)^{N - 2s} \right)^{2s} ds
\]

\[
\leq C m^{2s} \|\phi\|^{2s} + C \frac{m^{2s}}{\lambda^{2s}} \leq C m \epsilon^{\frac{1 + \epsilon}{2s}}. \tag{3.37}
\]

Note that

\[
\int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{u}_\epsilon, Y \rangle \frac{\partial \bar{u}_\epsilon}{\partial \nu} dS
\]

\[
= \int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{Z}_{x, y', \lambda}, Y \rangle \frac{\partial \bar{Z}_{x, y', \lambda}}{\partial \nu} dS + \int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{\varphi}, Y \rangle \frac{\partial \bar{\varphi}}{\partial \nu} dS \tag{3.38}
\]

\[
+ \int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{Z}_{x, y', \lambda}, Y \rangle \frac{\partial \bar{\varphi}}{\partial \nu} dS + \int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{\varphi}, Y \rangle \frac{\partial \bar{Z}_{x, y', \lambda}}{\partial \nu} dS.
\]

Next, we will estimates the terms in (3.38) one by one.

By Lemma B.3, we have

\[
\left| \int_{\partial^c B^c(y_0)} t^{1-2s} \langle \nabla \bar{Z}_{x, y', \lambda}, Y \rangle \frac{\partial \bar{Z}_{x, y', \lambda}}{\partial \nu} dS \right|
\]

\[
\leq C \int_{\partial^c B^c(y_0)} t^{1-2s} | \nabla \bar{Z}_{x, y', \lambda} |^2 dS.
\]
\[
\begin{align*}
\int_{\partial''B^+_p(y_0)} t^{1-2s} \left( \sum_{j=1}^{m} \frac{1}{(1 + |y - x_j|)^{N-2s+1}} \right)^2 dS & \leq C \frac{m^2}{\lambda^N} \int_{\partial''B^+_p(y_0)} t^{1-2s} \left( \frac{1}{(1 + |y - x_j|)^{2N-4s+2}} \right) dS \\
& \leq C \frac{m^2}{\lambda^N} \leq C m \epsilon^{\frac{1+\sigma}{N-2s}}.
\end{align*}
\]

By Lemma B.4, we have
\[
\left| \int_{\partial''B^+_p(y_0)} t^{1-2s} \langle \nabla \tilde{\varphi}, Y \rangle \frac{\partial \tilde{\varphi}}{\partial \nu} dS \right| \leq C \int_{\partial''B^+_p(y_0)} t^{1-2s} |\nabla \tilde{\varphi}|^2 dS \leq C m \|\varphi\|_2^2 \leq C m \epsilon^{\frac{1+\sigma}{\lambda^\tau}}.
\]

By (3.39) and (3.40), we have
\[
\begin{align*}
\left| \int_{\partial''B^+_p(y_0)} t^{1-2s} \langle \nabla \tilde{Z}_{f, g''}, \lambda \rangle \frac{\partial \tilde{\varphi}}{\partial \nu} dS \right| & \leq C \int_{\partial''B^+_p(y_0)} t^{1-2s} |\nabla \tilde{\varphi}|^2 dS + C \int_{\partial''B^+_p(y_0)} t^{1-2s} |\nabla \tilde{\varphi}|^2 dS \\
& \leq C \frac{m^2}{\lambda^N} \leq C m \epsilon^{\frac{1+\sigma}{N-2s}}.
\end{align*}
\]

Similar to (3.41), we have
\[
\begin{align*}
\left| \int_{\partial''B^+_p(y_0)} t^{1-2s} \langle \nabla \tilde{Z}_{f, g''}, Y \rangle \frac{\partial \tilde{u}_\epsilon}{\partial \nu} dS \right| & \leq C \int_{\partial''B^+_p(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 dS \leq C m \epsilon^{\frac{1+\sigma}{N-2s}}.
\end{align*}
\]

From (3.38) to (3.42), we have
\[
\left| \int_{\partial''B^+_p(y_0)} t^{1-2s} \langle \nabla \tilde{u}_\epsilon, Y \rangle \frac{\partial \tilde{u}_\epsilon}{\partial \nu} dS \right| \leq C m \epsilon^{\frac{1+\sigma}{N-2s}}.
\]

Just by the same argument as that of (3.43), we can prove
\[
\left| \int_{\partial''B^+_p(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 dS \right| \leq C m \epsilon^{\frac{1+\sigma}{N-2s}}.
\]
Similarly to (3.39), by Lemma B.3 we have

\[
\left| \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{\partial \tilde{Z}_{f,g',\lambda}}{\partial \nu} \tilde{Z}_{f,g',\lambda} dS \right|
\leq C \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \left| \nabla \tilde{Z}_{f,g',\lambda} \right| \left| \tilde{Z}_{f,g',\lambda} \right| dS
\leq \frac{C}{\lambda^{N-2s}} \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \sum_{j=1}^{m} \left( \frac{1}{1 + |y - x_j|^{N-2s+1}} \right) \sum_{j=1}^{m} \frac{1}{1 + |y - x_j|^{N-2s}} dS \quad (3.45)
\leq \frac{C m^2}{\lambda^{N-2s}} \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{1}{(1 + |y - x_j|^{2N-4s+1})} dS
\leq \frac{C m^2}{\lambda^{N-2s}} \leq C m \epsilon^{N-2s}.
\]

By Lemma B.5, we have

\[
\left| \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} |\tilde{\varphi}|^2 dS \right| \leq C \frac{\|\varphi\|^2}{\lambda^{2r}} \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \left( \sum_{j=1}^{m} \frac{1}{1 + |y - x_j|^{N-2s+r}} \right)^2 dS
\leq C m \frac{\|\varphi\|^2}{\lambda^{2r}} \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{1}{(1 + |y - x_j|^{N-2s+2r})} dS \quad (3.46)
\leq C m \epsilon^{N-2s}.
\]

It follows from (3.40) and (3.46) that

\[
\left| \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{\partial \tilde{\varphi}}{\partial \nu} \tilde{\varphi} dS \right| \leq C \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} |\nabla \tilde{\varphi}| |\tilde{\varphi}| dS
\leq C \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} |\nabla \tilde{\varphi}|^2 dS + C \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} |\tilde{\varphi}|^2 dS \quad (3.47)
\leq C m \epsilon^{N-2s}.
\]

Similarly, by (3.39) and (3.46), we have

\[
\left| \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{\partial \tilde{Z}_{f,g',\lambda}}{\partial \nu} \tilde{Z}_{f,g',\lambda} dS \right| \leq C m \epsilon^{N-2s}. \quad (3.48)
\]

Similar to (3.41), by (3.40) and Lemma B.3 we can also prove

\[
\left| \int_{\partial^r B^+_\rho(y_0)} t^{1-2s} \frac{\partial \tilde{Z}_{f,g',\lambda}}{\partial \nu} \tilde{Z}_{f,g',\lambda} dS \right| \leq C m \epsilon^{N-2s}. \quad (3.49)
\]
Hence, from (3.45) to (3.49) we have

\[
\left| \int_{\partial \Gamma_{+}^\lambda(y_0)} t^{1-2s} \frac{\partial \bar{\nu}_t}{\partial \nu} dS \right| \\
= \left| \int_{\partial \Gamma_{+}^\lambda(y_0)} t^{1-2s} \frac{\partial \bar{Z}_{\varphi, g''}, \lambda}{\partial \nu} \bar{Z}_{\varphi, g''}, \lambda dS + \int_{\partial \Gamma_{+}^\lambda(y_0)} t^{1-2s} \frac{\partial \tilde{\varphi}}{\partial \nu} \tilde{\varphi} dS \right| \\
+ \int_{\partial \Gamma_{+}^{\lambda}(y_0)} t^{1-2s} \frac{\partial \bar{Z}_{\varphi, g''}, \lambda}{\partial \nu} \bar{Z}_{\varphi, g''}, \lambda dS \right| \\
\leq C \epsilon^{\frac{1+ n}{N-2s}}. 
\] (3.50)

By (2.18), we know

\[ |c_l| \leq C \epsilon^{\frac{1+ n}{N-2s}}. \]

Note that

\[
\int_{B_{\rho}(x_0)} \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} Z_{\varphi, g''}, \lambda = \sum_{j=1}^{m} \int_{B_{\rho}(x_0)} U_{x_j, \lambda}^{2s-1} Z_{j,l} + \sum_{j=1}^{m} \int_{B_{\rho}(x_0)} \sum_{l \neq j} U_{x_j, \lambda}^{2s-2} Z_{j,l} U_{x_j, \lambda} dy \\
= O\left( m \lambda^{n_l} \right). 
\] (3.51)

Therefore, we have

\[
\sum_{l=1}^{N} \int_{B_{\rho}(x_0)} \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} Z_{\varphi, g''}, \lambda \leq C m \epsilon^{\frac{1+ n}{N-2s}}. 
\] (3.52)

Similarly, we can prove that

\[
\sum_{l=1}^{N} \int_{B_{\rho}(x_0)} \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} \varphi \leq C m \epsilon^{\frac{1+ n}{N-2s}}. 
\] (3.53)

Combining (3.32), (3.35), (3.36), (3.43), (3.44), (3.50), (3.52) and (3.53), we can prove that (3.33) holds.

Next, we prove

**Lemma 3.4.** For any $C^1$ bounded function $g(r, y'')$, it holds

\[
\int_{B_{\rho}(y_0)} g(r, y'') |u_\epsilon|^{2s+\epsilon} dy = m\left( g(\bar{r}, \bar{y}'') \int_{\mathbb{R}^N} U_{0, 1}^{2s+\epsilon} dy + o\left( \epsilon^{\frac{1+ n}{N-2s}} \right) \right), 
\] (3.54)

where $o(1)$ denotes a quantity that goes to zero when $\epsilon$ goes to zero.

**Proof.** Since $u_\epsilon = Z_{\varphi, g''}, \mu + \varphi$, we have

\[
\int_{B_{\rho}(y_0)} g(r, y'') |u_\epsilon|^{2s+\epsilon} dy = \int_{B_{\rho}(y_0)} g(r, y'') |Z_{\varphi, g''}, \lambda + \varphi|^{2s+\epsilon} dy \\
= \int_{B_{\rho}(y_0)} g(r, y'') |Z_{\varphi, g''}, \lambda|^{2s+\epsilon} dy + \int_{B_{\rho}(y_0)} g(r, y'') |\varphi|^{2s+\epsilon} dy \\
+ O\left( \int_{B_{\rho}(y_0)} |Z_{\varphi, g''}, \lambda|^{2s+\epsilon-1} \varphi dy + \int_{B_{\rho}(y_0)} |Z_{\varphi, g''}, \lambda|^{2s+\epsilon-1} |\varphi| dy \right). 
\] (3.55)
By the discrete Hölder inequality and Remark 1.4, we can check that

\[
\int_{B_{\rho}(y_0)} |Z_{\tilde{\mathcal{F}},y'''}_\lambda| |\varphi|^{2s+\epsilon-1} dy \\
\leq C \|\varphi\|_{2s+\epsilon-1} \int_{\mathbb{R}^N} \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda |y - x_j|)^{N-2s}} \left( \sum_{i=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda |y - x_i|)^{N-2s}} + \sum_{j \neq i} \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N+2s}} + \frac{1}{(1 + \lambda |y - x_i|)^{N+2s}} \right) dy \\
\leq C \|\varphi\|_{2s+\epsilon-1} \int_{\mathbb{R}^N} \sum_{j \neq i} \frac{1}{(\lambda |x_j - x_i|)^{\tau}} \left( \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N+2s}} + \frac{\lambda^{N}}{(1 + \lambda |y - x_i|)^{N+2s}} \right) dy \\
\leq C \epsilon^{\frac{1}{N-2s}}. \quad (3.56)
\]

and

\[
\int_{B_{\rho}(y_0)} |Z_{\tilde{\mathcal{F}},y'''}_\lambda|^{2s+\epsilon-1} |\varphi| dy \\
\leq C \|\varphi\| \int_{\mathbb{R}^N} \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda |y - x_j|)^{N+2s}} \sum_{i=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda |y - x_i|)^{N+2s}} dy \\
\leq C \|\varphi\| \int_{\mathbb{R}^N} \sum_{j \neq i} \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N+2s}} + \frac{\lambda^{N}}{(1 + \lambda |y - x_i|)^{N+2s}} dy \\
\leq C \epsilon^{\frac{1}{N-2s}} + C \epsilon^{\frac{1}{N-2s}} \int_{\mathbb{R}^N} \sum_{j \neq i} \frac{1}{(\lambda |x_j - x_i|)^{\tau}} \left( \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N+2s}} + \frac{\lambda^{N}}{(1 + \lambda |y - x_i|)^{N+2s}} \right) dy \\
\leq C \epsilon^{\frac{1}{N-2s}}. \quad (3.57)
\]

So from (3.55) and (3.57), we obtain the following estimate

\[
\int_{B_{\rho}(y_0)} g(r, y') |u_k|^{2s+\epsilon} dy = \int_{B_{\rho}(y_0)} g(r, y') Z_{\tilde{\mathcal{F}},y'''}_\lambda dy + o(\epsilon^{\frac{1}{N-2s}}). \quad (3.58)
\]

Since

\[
\int_{B_{\rho}(y_0)} g(r, y'') U_{x_j,\lambda}^{2s+\epsilon} dy = \int_{B_{\rho}(y_0)} g(\tilde{\bar{r}}, \bar{y}'') U_{x_j,\lambda}^{2s+\epsilon} dy + \int_{B_{\rho}(y_0)} [g(r, y'') - g(\tilde{\bar{r}}, \bar{y}'')] U_{x_j,\lambda}^{2s+\epsilon} dy \\
= \int_{\mathbb{R}^N} g(\tilde{\bar{r}}, \bar{y}'') U_{x_j,\lambda}^{2s+\epsilon} + \int_{B_{\rho}(y_0)} g(\tilde{\bar{r}}, \bar{y}'') U_{x_j,\lambda}^{2s+\epsilon} + \int_{B_{\rho}(y_0)} [g(r, y'') - g(\tilde{\bar{r}}, \bar{y}'')] U_{x_j,\lambda}^{2s+\epsilon} dy \\
= g(\tilde{\bar{r}}, \bar{y}'') \int_{\mathbb{R}^N} U_{x_j,\lambda}^{2s+\epsilon} dy + o(\epsilon^{\frac{1}{N-2s}}), \quad (3.59)
\]
and
\[
\sum_{i \neq j} \int_{B_{\rho}(y_0)} g(r, y'') u_{x_i, \lambda} U_{x_j, \lambda}^{2^* + \epsilon - 1} \, dy \leq C \sum_{i \neq j} \int_{\mathbb{R}^N} U_{x_i, \lambda} U_{x_j, \lambda}^{2^* + \epsilon - 1} \, dy
\]
\[
\leq C \sum_{i \neq j} \int_{\mathbb{R}^N} \frac{\lambda_{N+N-2s}}{(1 + \lambda |y - x_i|)^N (1 + \lambda |y - x_j|)^{N+2s}} \, dy
\]
\[
\leq \sum_{i \neq j} \frac{1}{(\lambda |x_i - x_j|)^{N-2s}} \int_{\mathbb{R}^N} \left[ \frac{\lambda_{N}}{(1 + \lambda |y - x_i|)^{N+2s}} + \frac{\lambda_{N}}{(1 + \lambda |y - x_j|)^{N+2s}} \right] \, dy
\]
\[
\leq C \sum_{i \neq j} \left( \frac{m}{\lambda} \right)^{N-2s} = O \left( \frac{m}{\lambda^{N-2s}} \right) = o \left( m \epsilon^{-\frac{1}{N-2s}} \right).
\]

As a result,
\[
\int_{B_{\rho}(y_0)} g(r, y'') |u_i|^{2^* + \epsilon} \, dy = m \left( g(\bar{r}, \bar{y}'') \int_{\mathbb{R}^N} U_{0,1}^{2^* + \epsilon} \, dy + o \left( \epsilon^{\frac{1}{N-2s}} \right) \right).
\]

(3.61)

Now it follows from Lemma 3.4, (3.33) and (3.34) that
\[
m \left( \frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{r} \right) \int_{\mathbb{R}^N} U_{0,1}^{2^* + \epsilon} \, dy + o \left( \epsilon^{\frac{1}{N-2s}} \right) = o \left( m \epsilon^{\frac{1}{N-N-2s}} \right),
\]

(3.62)

and
\[
m \left( \frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{y}_i \right) \int_{\mathbb{R}^N} U_{0,1}^{2^* + \epsilon} \, dy + o \left( \epsilon^{\frac{1}{N-2s}} \right) = o \left( m \epsilon^{\frac{1}{N-N-2s}} \right), \quad i = 3, \ldots, N.
\]

(3.63)

Therefore, the equations to determine \((\bar{r}, \bar{y}'')\) are
\[
\frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{r} = o \left( \epsilon^{\frac{1}{N-2s}} \right),
\]

(3.64)

and
\[
\frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{y}_i = o \left( \epsilon^{\frac{1}{N-2s}} \right), \quad i = 3, \ldots, N.
\]

(3.65)

**Proof of Theorem 1.5.** We have proved that (3.1), (3.2) and (3.3) are equivalent to
\[
\frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{r} = o \left( \epsilon^{\frac{1}{N-2s}} \right),
\]

(3.66)

\[
\frac{\partial K(\bar{r}, \bar{y}'')} \partial \bar{y}_i = o \left( \epsilon^{\frac{1}{N-2s}} \right), \quad i = 3, \ldots, N
\]

(3.67)

and
\[
- \frac{B_1}{\lambda^3} + \frac{B_3 m^{N-2s}}{\lambda^{N-2s+1}} = O \left( \epsilon^{\frac{1}{N-2s}} \right).
\]

(3.68)

Let \(\lambda = t m^{\frac{N-2s}{N-2s+1}}\), then \(t \in [L_0, L_1]\) since \(\lambda \in [L_0 m^{\frac{N-2s}{N-2s+1}}, L_1 m^{\frac{N-2s}{N-2s+1}}]\). Then, from (3.68), we get
\[
- \frac{B_1}{t^3} + \frac{B_3}{t^{N-2s+1}} = o(1), \quad t \in [L_0, L_1].
\]

(3.69)

Let
\[
F(t, \bar{r}, \bar{y}'') = (\nabla_{\bar{r}, \bar{y}''}(K(\bar{r}, \bar{y}''))) - \frac{B_1}{t^3} + \frac{B_3}{t^{N-2s+1}}.
\]
Then

\[ \deg(F(t, \bar{r}, \bar{y}''))[L_0, L_1] \times B_\theta((r_0, y''_0))) = -\deg(\nabla_{\bar{r}, \bar{y}''}(K(\bar{r}, \bar{y}''))), B_\theta((r_0, y''_0))) \neq 0. \]

So, (3.66), (3.67) and (3.69) have a solution \( t_m \in [L_0, L_1], (\bar{r}_m, \bar{y}''_m) \in B_\theta((r_0, y''_0)) \).

\[ \square \]

### Appendix A. Pohozaev Identities

For the readers’ convenient, we give the detailed proof of some local Pohozaev identities. Note that if \( u_\varepsilon \) satisfies (2.17), which is equivalent to \( \bar{u}_\varepsilon \) satisfies

\[
\begin{align*}
\begin{cases}
\text{div}(t^{1-2s}\nabla \bar{u}_\varepsilon) = 0 & \text{in } \mathbb{R}^{N+1}, \\
-\lim_{t \to 0} t^{1-2s} \partial_t \bar{u}_\varepsilon(x, t) = K(r, y'')(u_\varepsilon)_+^{2s-1+\varepsilon} + \sum_{l=1}^N c_l \sum_{j=1}^m U_{x_j, \lambda}^{2s-2} Z_{j,l}, & \text{in } \mathbb{R}^N.
\end{cases}
\end{align*}
\]

(A.1)

First we have the following local Pohozaev identities by translations.

**Lemma A.1.** If \( \bar{u}_\varepsilon \) satisfies (A.1), then there holds

\[
\begin{align*}
&\int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \nabla \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial y_i} dS + \frac{1}{2s + \varepsilon} \int_{\partial B_\rho(y_0)} K(y)(u_\varepsilon)^{2s+\varepsilon} \nu_i dS \\
&- \frac{1}{2s + \varepsilon} \int_{B_\rho(y_0)} \frac{\partial K(y)}{\partial y_i} (u_\varepsilon)^{2s+\varepsilon} dy - \frac{1}{2} \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \nabla |\nabla \bar{u}_\varepsilon|^2 \nu_i dS \\
&+ \int_{B_\rho(y_0)} \sum_{l=1}^N c_l \sum_{j=1}^m U_{x_j, \lambda}^{2s-2} Z_{j,l} \frac{\partial u_\varepsilon}{\partial y_i} dy = 0, \quad i = 3, \ldots, N.
\end{align*}
\]

(A.2)

**Proof.** Noting that \( \bar{u}_\varepsilon \) satisfies (A.1), then we have

\[
\begin{align*}
0 &= \int_{B^+_\rho(y_0)} \text{div}(t^{1-2s}\nabla \bar{u}_\varepsilon) \frac{\partial \bar{u}_\varepsilon}{\partial y_i} dy dt \\
&= \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \nabla \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial y_i} dS - \int_{B^+_\rho(y_0)} t^{1-2s} \nabla \bar{u}_\varepsilon \nabla \left( \frac{\partial \bar{u}_\varepsilon}{\partial y_i} \right) dy dt \\
&= \int_{\partial^\nu B^+_\rho(y_0)} t^{1-2s} \frac{\partial \bar{u}_\varepsilon}{\partial y_i} \frac{\partial \bar{u}_\varepsilon}{\partial \nu} dS + \int_{\partial B^+_\rho(y_0)} t^{1-2s} \nabla \bar{u}_\varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial y_i} \nu dS \\
&- \int_{B^+_\rho(y_0)} t^{1-2s} \nabla \bar{u}_\varepsilon \nabla \left( \frac{\partial \bar{u}_\varepsilon}{\partial y_i} \right) dy dt.
\end{align*}
\]

(A.3)
and

\[
\int_{\partial B_\rho^+(y_0)} t^{1-2s} \nabla \tilde{u}_\epsilon \frac{\partial \tilde{u}_\epsilon}{\partial y_i} \nu dS \\
= - \int_{\partial B_\rho^+(y_0)} t^{1-2s} \partial \tilde{u}_\epsilon \frac{\partial \tilde{u}_\epsilon}{\partial y_i} dS \\
= \int_{B_\rho(y_0)} [K(r, y') (u_\epsilon)^{2s+1}_+ + \sum_{l=1}^N c_l \sum_{j=1}^m U_{x_j, i}^{2s-2} Z_{j, l}] \frac{\partial u_\epsilon}{\partial y_i} dy \\
= \frac{1}{2s + \epsilon} \int_{\partial B_\rho(y_0)} K(y) (u_\epsilon)^{2s+\epsilon}_+ \nu_i ds - \frac{1}{2s + \epsilon} \int_{B_\rho(y_0)} \frac{\partial K}{\partial y_i} (u_\epsilon)^{2s+\epsilon}_+ dy \\
+ \sum_{l=1}^N \sum_{j=1}^m c_l U_{x_j, i}^{2s-2} Z_{j, l} \frac{\partial u_\epsilon}{\partial y_i} dy.
\]

Moreover we have

\[
- \int_{B_\rho^+(y_0)} t^{1-2s} \nabla \tilde{u}_\epsilon \left( \frac{\partial \tilde{u}_\epsilon}{\partial y_i} \right) dy = - \frac{1}{2} \int_{B_\rho^+(y_0)} t^{1-2s} \frac{\partial}{\partial y_i} |\tilde{u}_\epsilon|^2 dy dt \\
= - \frac{1}{2} \int_{\partial B_\rho^+(y_0)} t^{1-2s} |\tilde{u}_\epsilon|^2 \nu_i dS \\
= - \frac{1}{2} \int_{\partial B_\rho^+(y_0)} t^{1-2s} |\tilde{u}_\epsilon|^2 \nu_i dS - \frac{1}{2} \int_{\partial B_\rho^+(y_0)} t^{1-2s} |\tilde{u}_\epsilon|^2 \nu_i dS, \quad i = 1, \ldots, n \\
= - \frac{1}{2} \int_{\partial B_\rho^+(y_0)} t^{1-2s} |\tilde{u}_\epsilon|^2 \nu_i dS,
\]

where we use the fact that on $\partial B_\rho^+(y_0)$, $\nu_i = 0$, so the first term equals to 0. Combining all the equations above, we can get (A.2). \qed

Next, we will obtain a local Pohozaev identity by scaling.

Lemma A.2. If $u_\epsilon$ satisfies (A.1), then there holds

\[
\int_{\partial B_\rho^+(y_0)} t^{1-2s} \langle Y, \nabla \tilde{u}_\epsilon \rangle \frac{\partial \tilde{u}_\epsilon}{\partial \nu} dS - \frac{1}{2} \int_{\partial B_\rho^+(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 \langle Y, \nu \rangle dS \\
+ \frac{1}{2s + \epsilon} \int_{\partial B_\rho(y_0)} K(r, y') (u_\epsilon)^{2s+\epsilon}_+ \langle y, \nu \rangle ds - \frac{N}{2s + \epsilon} \int_{B_\rho(y_0)} K(r, y') (u_\epsilon)^{2s+\epsilon}_+ dy \\
- \frac{1}{2s + \epsilon} \int_{B_\rho(y_0)} \langle \nabla K, y \rangle (u_\epsilon)^{2s+\epsilon}_+ dy + \frac{N - 2s}{2} \int_{B_\rho(y_0)} t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 dS \\
+ \sum_{l=1}^N \sum_{j=1}^m c_l U_{x_j, i}^{2s-2} Z_{j, l} \langle y, \nabla u_\epsilon \rangle dy = 0.
\]

Proof. Since

\[
\text{div}(t^{1-2s} \nabla \tilde{u}_\epsilon) \langle Y, \nabla \tilde{u}_\epsilon \rangle = \text{div} \left( t^{1-2s} \nabla \tilde{u}_\epsilon \langle Y, \nabla \tilde{u}_\epsilon \rangle - t^{1-2s} Y |\nabla \tilde{u}_\epsilon|^2 \right) \\
- t^{1-2s} |\nabla \tilde{u}_\epsilon|^2 + \frac{\text{div}(t^{1-2s} Y)}{2} |\nabla \tilde{u}_\epsilon|^2,
\]

(A.7)
we have
\[0 = \int_{B^+_1(y_0)} \text{div}(t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle) dy dt\]
\[= \int_{B^+_1(y_0)} \text{div}(t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle - t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2}) dy dt\]
\[- \int_{B^+_1(y_0)} \left[ - t^{1-2s} |\nabla \tilde{u}_e|^2 + \frac{\text{div}(t^{1-2s}Y)}{2} |\nabla \tilde{u}_e|^2 \right] dy dt.\]

It is easy to derive that
\[\int_{\partial B^+_1(y_0)} \text{div}(t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle - t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2}) dy dt\]
\[= \int_{\partial B^+_1(y_0)} [t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle \nu - t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2} \nu] dS\]
\[= \int_{\partial B^+_1(y_0)} [t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle \nu - t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2} \nu] dS\]
\[+ \int_{\partial B^+_1(y_0)} [t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle \nu - t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2} \nu] dS\]
\[:= A_1 + A_2.\]

Noting that
\[\partial' B^+_1(y_0) = \{ Y = (y, t) : |Y - (y_0, 0)| \leq \rho \quad \text{and} \quad t = 0 \} \in \mathbb{R}^N,\]
and $-t^{1-2s}Y \frac{\nabla \tilde{u}_e}{2} \nu$ is odd about $y$, we have
\[A_2 = \int_{\partial B^+_1(y_0)} t^{1-2s}\nabla \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle \nu dS = \int_{\partial B^+_1(y_0)} -t^{1-2s}\partial_t \tilde{u}_e \langle Y, \nabla \tilde{u}_e \rangle dS\]
\[= \int_{B^+_1(y_0)} \left[ K(r, y') (u_e)_+^{2s-1+\epsilon} + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j}^{2s-2} Z_{j,l} \right] \langle y, \nabla u_e \rangle dy\]
\[= \frac{1}{2s + \epsilon} \int_{B^+_1(y_0)} K(r, y') (u_e)_+^{2s+\epsilon} \langle y, \nabla u_e \rangle dy - \frac{1}{2s + \epsilon} \int_{B^+_1(y_0)} \langle y, \nabla K \rangle (u_e)_+^{2s+\epsilon} dy\]
\[- \frac{N}{2s + \epsilon} \int_{B^+_1(y_0)} K(r, y') (u_e)_+^{2s+\epsilon} dy + \int_{B^+_1(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j}^{2s-2} Z_{j,l} \langle y, \nabla u_e \rangle dy.\]

Direct calculation will give that
\[\frac{\text{div}(t^{1-2s}Y)}{2} |\nabla \tilde{u}_e|^2 = \frac{t^{1-2s} + t^{1-2s} + \cdots + (2 - 2s) t^{1-2s}}{2} |\nabla \tilde{u}_e|^2\]
\[= \frac{N + 2 - 2s}{2} t^{1-2s} |\nabla \tilde{u}_e|^2. \quad (A.10)\]
Then it follows from all the equations above that
\[
\int_{\partial^r B^+(y_0)} t^{1-2s} \nabla u_{\epsilon}(Y, \nabla \tilde{u}_{\epsilon}) \nu dS - \frac{1}{2} \int_{\partial^r B^+(y_0)} t^{1-2s} |\nabla \tilde{u}_{\epsilon}|^2 (Y, \nu) dS \\
+ \frac{1}{2_s + \epsilon} \int_{\partial B^+(y_0)} K(r, y''(u_{\epsilon}))^{2s+\epsilon} \langle y, \nu \rangle ds - \frac{N}{2_s + \epsilon} \int_{B^+(y_0)} K(r, y''(u_{\epsilon}))^{2s+\epsilon} dy \\
- \frac{1}{2_s + \epsilon} \int_{B^+(y_0)} \langle \nabla K, y \rangle (u_{\epsilon})^{2s+\epsilon} dy + \frac{N - 2s}{2} \int_{B^+(y_0)} t^{1-2s} |\nabla \tilde{u}_{\epsilon}|^2 dydt \\
+ \int_{B^+(y_0)} \sum_{l=1}^{N} \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \langle y, \nabla u_{\epsilon} \rangle dy = 0.
\]

\[\square\]

**Appendix B. Basic Estimates**

For each fixed \(k\) and \(j, k \neq j\), we consider the following function
\[
g_{k,j}(y) = \frac{1}{1 + |y - x_j|^\alpha} \frac{1}{1 + |y - x_k|^\beta}, \tag{B.1}
\]
where \(\alpha \geq 1\) and \(\beta \geq 1\) are two constants.

**Lemma B.1.** (Lemma B.1, [34]) For any constants \(0 < \delta \leq \min\{\alpha, \beta\}\), there is a constant \(C > 0\), such that
\[
g_{k,j}(y) \leq \frac{C}{|x_k - x_j|^\delta} \left( \frac{1}{1 + |y - x_k|^{\alpha + \beta - \delta}} + \frac{1}{1 + |y - x_j|^{\alpha + \beta - \delta}} \right).
\]

**Lemma B.2.** (Lemma 2.1, [17]) For any constant \(0 < \delta < N - 2s\), there is a constant \(C > 0\), such that
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} \frac{1}{(1 + |z|)^{2s+\delta}} dz \leq \frac{C}{(1 + |y|)^\delta}.
\]

**Lemma B.3.** (Lemma A.5, [16]) Suppose that \((y - x)^2 + t^2 = \rho^2, t > 0\). Then there exists a constant \(C > 0\) such that
\[
|\tilde{U}_{x_j,\lambda}| \leq \frac{C}{\lambda^{N-2s}(1 + |y - x_j|)^{N-2s}} \text{ and } |\nabla \tilde{U}_{x_j,\lambda}| \leq \frac{C}{\lambda^{N-2s}(1 + |y - x_j|)^{N-2s+1}}.
\]

**Lemma B.4.** (Lemma A.6, [16]) For any \(\delta > 0\), there is a \(\rho = \rho(\delta) \in (2\delta, 5\delta)\) such that
\[
\int_{\partial^r B^+(y)} t^{1-2s} |\nabla \tilde{\varphi}|^2 dydt \leq C \frac{m}{\lambda^s} ||\varphi||^2_s,
\]
where \(C > 0\) is a constant, depending on \(\delta\).

From the proof of Lemma A.6 in [16], we have

**Lemma B.5.** There exists a positive constant \(C\) such that
\[
|\tilde{\varphi}(y, t)| \leq \frac{C||\varphi||}{\lambda^s} \sum_{j=1}^{m} \frac{1}{(1 + |y - x_j|)^{N-2s+\tau}}. \tag{B.2}
\]
Let us recall that
\[
Z_{r,g^{\infty },\lambda }(y) = \sum_{j=1}^{m} U_{x_j,\lambda } = (4^{\gamma }\gamma )^{\frac{N-2s}{4s}} \sum_{j=1}^{m} \left( \frac{\lambda}{1 + \lambda^{2}|y - x_j|^{2}} \right)^{\frac{N-2s}{2}}.
\]

**Lemma B.6.** There is a small constant \(\sigma > 0\), such that
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} Z_{r,g^{\infty },\lambda }^{4s+\varepsilon + \epsilon} (z) \sum_{j=1}^{m} \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau}} dz \leq \sum_{j=1}^{m} \frac{C}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau + \sigma}}.
\]

**Proof.** Here we prove it by some different arguments from Lemma B.3 of [34] and Lemma 2.2 of [17]. By direct computations and Remark 1.4, we have
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} Z_{r,g^{\infty },\lambda }^{4s+\varepsilon + \epsilon} (z) \sum_{j=1}^{m} \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau}} dz \leq C \int_{\mathbb{R}^N} \frac{\lambda^{2s}}{|y - z|^{N-2s}} (\sum_{i=1}^{m} \frac{1}{(1 + \lambda |z - x_i|)^{\frac{N-2s}{2} + \tau + (N-2s)\epsilon}})
\]
\[
+ \sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{C\lambda^{2s}}{|y - z|^{N-2s}} (\sum_{j \neq i, j=1}^{m} \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau + (N-2s)\epsilon}} \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau}}) dz
\]
\[
\leq \sum_{j=1}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau + \sigma}}.
\]

since it follow from Lemma B.2 that
\[
\sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{2s}}{|y - z|^{N-2s}} \frac{1}{(1 + \lambda |z - x_i|)^{\frac{N-2s}{2} + \tau + (N-2s)\epsilon}} dz
\]
\[
\leq \sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{2s}}{|y - z|^{N-2s}} \frac{1}{(1 + \lambda |z - x_i|)^{2s + (\frac{N-2s}{2} + \tau + 2s - (N-2s)\epsilon)}} dz
\]
\[
\leq \sum_{j=1}^{m} \frac{1}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau + \sigma}},
\]

and
\[
\sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{2s}}{|y - z|^{N-2s}} \left( \sum_{j \neq i, j=1}^{m} \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau + (N-2s)\epsilon}} \right) \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau}} dz
\]
\[
\leq \sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{2s}}{|y - z|^{N-2s}} \sum_{j \neq i, j=1}^{m} \frac{1}{(1 + \lambda |z - x_i|)^{2s}} \left( \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau + (N-2s)\epsilon}} \right) \frac{1}{(1 + \lambda |z - x_j|)^{\frac{N-2s}{2} + \tau + 2s - (N-2s)\epsilon}} dz
\]
\[
+ \frac{1}{(1 + \lambda |z - x_j|)^{N-2s + (\frac{N-2s}{2} + \tau + (N-2s)\epsilon)}} dz
\]
where we used $2s - \tau \frac{4s}{N-2s} > 0$.

**Lemma B.7.** If $N \geq 4$ and $0 < s < 1$, then

$$\frac{\partial I(Z_{f, g''}, \lambda)}{\partial \lambda} = m \left( -\frac{B_1}{\lambda^3} + \sum_{j=2}^{m} \frac{B_2}{\lambda^{N-2s+1}|x_1 - x_j|^{N-2s}} + O \left( \epsilon^{\frac{N}{N-2s}} \right) \right),$$

where $B_j$, $j = 1, 2$ are some positive constants.

**Proof.** Direct calculations show that

$$\frac{\partial I(Z_{f, g''}, \lambda)}{\partial \lambda} = \int_{\mathbb{R}^N} \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} \frac{\partial Z_{f, g''}}{\partial \lambda} dy = \int_{\mathbb{R}^N} \sum_{j=1}^{m} \left[ (Z_{f, g''})^{2s-1} - \frac{m}{\lambda^{N-2s+1}|x_1 - x_j|^{N-2s}} \right] \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$= - \int_{\mathbb{R}^N} \left[ (Z_{f, g''})^{2s-1} - \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} \right] \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$+ \int_{\mathbb{R}^N} \left[ 1 - K(y) \right] (Z_{f, g''})^{2s-1+\epsilon} \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$- \int_{\mathbb{R}^N} \left[ (Z_{f, g''})^{2s-1+\epsilon} - (Z_{f, g''})^{2s-1} \right] \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$:= - F_1 + F_2 - F_3.$$

First, we have

$$F_1 = \int_{\mathbb{R}^N} \left[ (Z_{f, g''})^{2s-1} - \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} \right] \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$= m \int_{\Omega_1} \left[ (Z_{f, g''})^{2s-1} - \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-1} \right] \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$= m \left( \int_{\Omega_1} (2s-1)U_{x_1, \lambda}^{2s-2} \sum_{j=2}^{m} U_{x_j, \lambda} \frac{\partial U_{x_1, \lambda}}{\partial \lambda} dy + O \left( \epsilon^{\frac{N}{N-2s}} \right) \right)$$

$$= m \left( - \sum_{j=2}^{m} \frac{B_2}{\lambda^{N-2s+1}|x_1 - x_j|^{N-2s}} + O \left( \epsilon^{\frac{N}{N-2s}} \right) \right)$$

for some constant $B_2 > 0$.

Noting that $K(y_0) = 1$, it is easy to check that

$$F_2 = \int_{\mathbb{R}^N} (1 - K(y))(Z_{f, g''})^{2s-1+\epsilon} \frac{\partial Z_{f, g''}}{\partial \lambda} dy$$

$$= m \left( \int_{\mathbb{R}^N} (1 - K(y))U_{x_1, \lambda}^{2s-1+\epsilon} \frac{\partial U_{x_1, \lambda}}{\partial \lambda} dy + O \left( \frac{1}{\lambda} \int_{\mathbb{R}^N} U_{x_1, \lambda}^{2s-1+\epsilon} \sum_{j=2}^{m} U_{x_j, \lambda} dy \right) \right).$$
\begin{align*}
&= m \left( - \int_B \mathcal{L}^{\lambda,\epsilon}(y) \left( \sum_{i,j=1}^N \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (y_i - y_{0i})(y_j - y_{0j}) 
+ \sum_{i,j,k=1}^N \frac{1}{6} \frac{\partial^3 K(y_0 + \vartheta(y - y_0))}{\partial y_i \partial y_j \partial y_k} (y_i - y_{0i})(y_j - y_{0j})(y_k - y_{0k}) \right) \frac{1}{2s} + \epsilon \frac{\partial U_{x1,\lambda}^{2s+\epsilon}}{\partial \lambda} dy 
+ \int_{B_0^C} \mathcal{L}^{\lambda,\epsilon}(y) \left( \sum_{i,j=1}^N \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (y_i - y_{0i})(y_j - y_{0j}) \right) \frac{1}{2s} + \epsilon \frac{\partial U_{x1,\lambda}^{2s+\epsilon}}{\partial \lambda} dy 
+ O \left( \int_{B_0^C} \frac{U_{x1,\lambda}^{2s+\epsilon}}{\lambda} dy \right) + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) \right) 
= m \left( - \int_B \mathcal{L}^{\lambda,\epsilon}(y) \left( \sum_{i,j=1}^N \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (z_i + x_{1i} - y_{0i})(z_j + x_{1j} - y_{0j}) 
+ O \left( \|z + x_1 - y_0\|^3 \right) \right) \frac{1}{2s} + \epsilon \frac{\partial U_{0,\lambda}^{2s+\epsilon}}{\partial \lambda} dz + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) 
= m \left( - \frac{1}{2s} + \epsilon \frac{\partial}{\partial \lambda} \int_{B_0^C} \mathcal{L}^{\lambda,\epsilon}(y) \left( \sum_{i,j=1}^N \frac{1}{2} \frac{\partial^2 K(y_0)}{\partial y_i \partial y_j} (z_i + x_{1i} - y_{0i})(z_j + x_{1j} - y_{0j}) \right) \frac{1}{2s} + \epsilon \frac{\partial U_{0,\lambda}^{2s+\epsilon}}{\partial \lambda} dz 
+ \int O \left( \|z + x_1 - y_0\|^3 \right) \frac{U_{0,1}^{2s+\epsilon}}{\lambda} + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) 
= l \left( - \frac{1}{2s} + \epsilon \frac{\partial}{\partial \lambda} \int \frac{1}{2s} + \epsilon \frac{\partial^2 K(y_0)}{\partial y_i^2} \frac{z_i^2 U_{0,1}^{2s+\epsilon}}{\lambda} dz + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) 
= l \left( \frac{1}{2s} + \epsilon \frac{1}{\lambda^3} \int \frac{\partial^2 K(y_0)}{\partial y_i^2} z_i^2 U_{0,1}^{2s+\epsilon} dz + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) 
= l \left( \frac{1}{2s} + \epsilon \frac{1}{\lambda^3} \int \frac{\Delta K(y_0)}{N} z_i^2 U_{0,1}^{2s+\epsilon} dz + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right) = l \left( - \frac{B_1}{\lambda^3} + O \left( \epsilon^{\frac{3+1}{N-2s}} \right) \right),
\end{align*}

where we have used (1.9) and (1.8).

Finally similar to (2.23), since \( N > 2s + 2 \), we have

\[ F_3 = \int_{\mathbb{R}^N} \left[ (Z_{\alpha,\epsilon}^{2s+\epsilon} - (Z_{\alpha,\epsilon})^{2s+\epsilon}) \frac{\partial Z_{\alpha,\epsilon}}{\partial \lambda} \right] dy \]
\[
\begin{align*}
\leq C\epsilon \ln \frac{1}{\epsilon} \int_{\mathbb{R}^N} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N+2s}{2} + \tau}} \lambda(1 + \lambda |y - x_j|)^{N-2s} \ dy \\
\leq C\epsilon^{\frac{2s+4}{N-2s}} \int_{\mathbb{R}^N} \sum_{j=1}^{m} \frac{\lambda^{\frac{N+2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N+2s}{2} + \tau}} (1 + \lambda |y - x_j|)^{N-2s} \ dy \\
\leq C\epsilon^{\frac{3+2s}{N-2s}}.
\end{align*}
\]

So, we obtain
\[
\frac{\partial I(Z_{\bar{y}',y'',\lambda})}{\partial \lambda} = m \left(- \frac{B_1}{\lambda^3} + \sum_{j=2}^{m} \frac{B_2}{\lambda^{N-2s+1}|x_1 - x_j|^N + O\left(\epsilon^{\frac{3+2s}{N-2s}}\right)} \right).
\]

\[
\square
\]

**APPENDIX C. PROOF OF (2.12)**

In this section, we mainly prove (2.12).

**Proof.** Note that
\[
\begin{align*}
&\langle (-\Delta)^s \varphi_\epsilon - (2^*_s - 1 + \epsilon)K(r, y'')Z_{\bar{y}',y'',\lambda}^{2^*_s-2+\epsilon} \varphi_\epsilon, Z_{1,k}\rangle \\
= &\langle (-\Delta)^s Z_{1,k}, \varphi_\epsilon \rangle - (2^*_s - 1 + \epsilon)\langle K(r, y'')Z_{\bar{y}',y'',\lambda}^{2^*_s-2+\epsilon} \varphi_\epsilon, Z_{1,k}\rangle \\
= &\langle (2^*_s - 1)\langle U_{2^*_s,1,\lambda}^{2^*_s-2} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2} Z_{1,k}, \varphi_\epsilon \rangle - (2^*_s - 1 + \epsilon)\langle K(r, y'')Z_{\bar{y}',y'',\lambda}^{2^*_s-2+\epsilon} \varphi_\epsilon, Z_{1,k}\rangle \\
= &\langle (2^*_s - 1)\langle U_{2^*_s,1,\lambda}^{2^*_s-2} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2} Z_{1,k}, \varphi_\epsilon \rangle - (2^*_s - 1)\langle K(r, y'') - 1\rangle Z_{\bar{y}',y'',\lambda}^{2^*_s-2} Z_{1,k}, \varphi_\epsilon \rangle \\
&\quad - (2^*_s - 1)\langle K(r, y'')(Z_{\bar{y}',y'',\lambda}^{2^*_s-2+\epsilon} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2}) Z_{1,k}, \varphi_\epsilon \rangle - \epsilon\langle K(r, y'')Z_{\bar{y}',y'',\lambda}^{2^*_s-2+\epsilon} Z_{1,k}, \varphi_\epsilon \rangle \\
:= &M_1 - M_2 - M_3 - M_4. \tag{C.1}
\end{align*}
\]

First, we can rewrite $M_1$ as following
\[
M_1 = (2^*_s - 1) \int_{\mathbb{R}^N} \langle U_{2^*_s,1,\lambda}^{2^*_s-2} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2} \rangle Z_{1,k} \varphi_\epsilon \ dy
\]
\[
= (2^*_s - 1) \int_{\Omega_1} \langle U_{2^*_s,1,\lambda}^{2^*_s-2} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2} \rangle Z_{1,k} \varphi_\epsilon \ dy + (2^*_s - 1) \int_{\Omega_1^c} \langle U_{2^*_s,1,\lambda}^{2^*_s-2} - Z_{\bar{y}',y'',\lambda}^{2^*_s-2} \rangle Z_{1,k} \varphi_\epsilon \ dy
\]
\[
:= M_{11} + M_{12}.
\]

Next, we will estimate $M_1$ in the following two cases.

If $2^*_s > 3$, then $\frac{4s}{N-2s} > 1$ and
\[
\begin{align*}
M_{11} &\leq C\|\varphi_\epsilon\|_s \int_{\Omega_1} \left[ \left( \sum_{i=2}^{m} \frac{U_{x_i,\lambda}^{2^*_s-2} + U_{x_1,\lambda}^{2^*_s-3}}{\lambda^{\frac{N-2s}{2}}} \right) Z_{1,k} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \right] \\
&\leq C\|\varphi_\epsilon\|_s \lambda^n \int_{\Omega_1} \left( \sum_{i=2}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2^*_s-2} \lambda^N \\
&\quad + C\|\varphi_\epsilon\|_s \lambda^n \int_{\Omega_1} \left( \sum_{i=2}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2^*_s-2} \lambda^N \\
&\quad + C\|\varphi_\epsilon\|_s \lambda^n \int_{\Omega_1} \left( \sum_{i=2}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2^*_s-2} \lambda^N
\end{align*}
\]
In the above, we have used the fact that $N > 2 + 2s$.
Noting that $N > 2 + 2s$, $\frac{4s}{N-2s} > 1$ and $\tau < 2s$, by Lemma B.1 and the discrete Hölder inequality we have

$$
\int_{\Omega_1} \left( \sum_{i=2}^{m} U_{x_i, \lambda} \right)^{2s-2} |Z_{1,k}| \sum_{j=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda |y - x_j|)^{N-2s + \tau}} \\
\leq C\lambda^n \int_{\Omega_1} \left( \sum_{i=2}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2s-2} \frac{\lambda^{N}}{(1 + \lambda |y - x_1|)^{N-2s + \tau}} \sum_{j=1}^{m} \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N-2s + \tau}} \\
\leq C\lambda^n \int_{\Omega_1} \frac{1}{(1 + \lambda |y - x_1|)^{N-2s + \tau}} \frac{\lambda^{N}}{(1 + \lambda |y - x_1|)^{N-2s}} \\
+ C\lambda^n \int_{\Omega_1} \left( \sum_{i=3}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2s-2} \frac{\lambda^{N}}{(1 + \lambda |y - x_1|)^{N-2s + \tau}} \sum_{j=1,j\neq 2}^{m} \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N-2s + \tau}} \\
+ C\lambda^n \int_{\Omega_1} \left( \sum_{i=3}^{m} \frac{1}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2s-2} \frac{\lambda^{N}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1,j\neq 2}^{m} \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N-2s + \tau}} \\
\leq C\lambda^n \int_{\Omega_1} \frac{1}{\lambda(x_1 - x_2)^{N-2s + \tau}} \left( \frac{\lambda^{N}}{(1 + \lambda |y - x_1|)^{N-2s + \tau}} + \frac{\lambda^{N}}{(1 + \lambda |y - x_2|)^{N+2s}} \right) \\
+ C\lambda^n \int_{\Omega_1} \sum_{j=1,j\neq 2}^{m} \frac{1}{\lambda(x_j - x_2)^{N-2s + \tau}} \left( \frac{\lambda^{N}}{(1 + \lambda |y - x_j|)^{N+2s}} + \frac{\lambda^{N}}{(1 + \lambda |y - x_2|)^{N+2s}} \right)
$$
\[ + C\lambda^{n_k} \int_{\Omega_1} \sum_{i=3}^{m} \frac{1}{|\lambda(x_i - x_2)|^{\frac{N-2s}{2} + \tau}} \left( \frac{\lambda^{N}}{(1 + \lambda|y - x_i|)^{N + 2s - \sigma}} + \frac{\lambda^{N}}{(1 + \lambda|y - x_2|)^{N + 2s - \sigma}} \right) \]

\[ + C\lambda^{n_k} \int_{\Omega_1} \sum_{i=3}^{m} \frac{1}{|\lambda(x_i - x_2)|^{\frac{N-2s}{2} + \tau}} \sum_{j=1,j \neq 2}^{m} \frac{1}{|\lambda(x_j - x_2)|^{\tau}} \left( \frac{\lambda^{N}}{(1 + \lambda|y - x_i|)^{N + 2s - \sigma}} + \frac{\lambda^{N}}{(1 + \lambda|y - x_2|)^{N + 2s - \sigma}} \right) \]

\[ \leq C\lambda^{n_k} \left( \frac{m}{\lambda} \right)^{\frac{N-2s}{2} + \tau} \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}} \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}}, \quad (C.3) \]

where \( \tilde{\sigma} \) is a small positive constant. Similar to (C.3), we can prove

\[ \int_{\Omega_1} U_{x_1,\lambda}^{2s-3} \sum_{i=2}^{m} U_{x_1,\lambda} |Z_{1,k}| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}}. \quad (C.4) \]

From (C.2) to (C.4), when \( 2^*_s > 3 \), we have

\[ |M_1| \leq C \|\varphi_\epsilon\|_\infty \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}}. \quad (C.5) \]

When \( 2^*_s \leq 3 \), we can estimate \( M_1 \) similarly. First of all, we have

\[ |M_{11}| \leq C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \left[ Z_{x_1,\lambda}^{2s-2} - U_{x_1,\lambda}^{2s-2} \right] |U_{x_1,\lambda}| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]

\[ \leq C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \left[ \left( U_{x_1,\lambda} + \sum_{j \neq 2} \frac{U_{x_j,\lambda}}{U_{x_1,\lambda}} \right)^{2s-2} - U_{x_1,\lambda}^{2s-2} \right] |U_{x_1,\lambda}| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]

\[ = C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \left[ U_{x_1,\lambda}^{2s-2} \left( 1 + \sum_{j \neq 2} \frac{U_{x_j,\lambda}}{U_{x_1,\lambda}} \right)^{2s-2} - U_{x_1,\lambda}^{2s-2} \right] |U_{x_1,\lambda}| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]

\[ \leq C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} U_{x_1,\lambda}^{2s-3} \sum_{j=2}^{m} U_{x_j,\lambda} |U_{x_1,\lambda}| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]

\[ \leq C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \lambda^{N} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N - 2s}} \]

\[ + C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \lambda^{N} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N - 2s}} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]

\[ \leq C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \sum_{j=2}^{m} \frac{1}{(\lambda|x_1 - x_j|)^{\frac{N-2s}{2}}} \left[ \lambda^{N} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N+2s}} + \lambda^{N} \sum_{j=2}^{m} \frac{1}{(\lambda|x_1 - x_j|)^\tau} \right] \]

\[ + C \|\varphi_\epsilon\|_\infty \lambda^{n_k} \int_{\Omega_1} \lambda^{N} \sum_{j=2}^{m} \frac{1}{(\lambda|x_1 - x_j|)^{\frac{N-2s}{2} + 4s}} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{N - 2s}} \sum_{j=2}^{m} \frac{1}{(\lambda|x_1 - x_j|)^\tau} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \left( \frac{m}{\lambda} \right)^{\frac{N-2s}{2}} \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}} \| \varphi \|_\ast. \quad (C.6) \]

Noting that \( \tau > 2s \) and \( |y - x_1| \geq |y - x_i|, y \in \Omega^i, \forall i \neq 1 \), we also have
\[ |M_{12}| \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{\Omega^i_1} \sum_{i=2}^{m} U_{x_i}^{2s-2} U_{x_1} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \left[ \int_{\Omega^i_1} \sum_{i=2}^{m} \frac{\lambda^{N}}{1} \frac{1}{(1 + \lambda|y - x_i|)^{4s} (1 + \lambda|y - x_1|)^{\frac{3(N-2s)}{2} + \tau}} \right. \]
\[ + \left. \int_{\Omega^i_1} \sum_{i=2}^{m} \frac{\lambda^{N}}{(1 + \lambda|y - x_i|)^{4s} (1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=2}^{m} \frac{1}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \right] \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{\Omega^i_1} \left[ \sum_{i=2}^{m} \frac{1}{\lambda(\lambda(\lambda - 1))^{\frac{N-2s}{2} + \tau} (1 + \lambda|y - x_1|)^{N+2s}} \right. \]
\[ + \left. \sum_{i=2}^{m} \frac{1}{\lambda(\lambda(\lambda - 1))^{\frac{N-2s}{2} + \tau} (1 + \lambda|y - x_1|)^{N+2s}} \right] \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \lambda^{\frac{N}{1+\epsilon}}. \quad (C.7) \]

From (C.6) and (C.7), when \( 2^*_s \leq 3 \), we have
\[ |M_1| \leq C \| \varphi \|_\ast \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}}. \quad (C.8) \]

Similar to (2.29) and (2.31), by the discrete Hölder inequality we have
\[ |M_2| \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{R^N} |K(y) - 1| \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \]
\[ \times \left( \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{N-2s}} \right)^{2^* - 2} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{R^N} |K(y) - 1| \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \left( \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \right)^{2^* - 1} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{R^N} |K(y) - 1| \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \int_{B} |K(y) - 1| \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} dy \]
\[ + C \| \varphi \|_\ast \lambda^{n_k} \int_{(B)^c} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \lambda^{\frac{N-2s}{1+\epsilon}} \int_{B} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \| \varphi \|_\ast \lambda^{n_k} \lambda^{\frac{N-2s}{1+\epsilon}} \int_{B} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ + C \| \varphi_* \|_\star \lambda^{n_k} \int_{(B)^c} \frac{\lambda^N}{(1 + \lambda |y - x_1|)^{N-2s + \tau + N-2s}} \]
\[ + C \| \varphi_* \|_\star \lambda^{n_k} \int_{(B)^c} \frac{\lambda^N}{(1 + \lambda |y - x_1|)^{N-2s + \tau + N-2s}} \]
\[ + C \| \varphi_* \|_\star \lambda^{n_k} \int_{(B)^c} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=2}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N+2s}{2} + \tau}} \]
\[ \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}} \| \varphi_* \|_\star , \]  
(C.9)

where \( \bar{B} \) is the same as that of Lemma 2.5.

Similar to (2.23), by direct computations we have

\[ |M_3| \leq C \epsilon \| \varphi_* \| \int_{\mathbb{R}^N} Z^{2^*_r - 2 + \kappa \epsilon}_{\varphi, y^\prime, \lambda} \| Z_{1,k} \| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} dy \]
\[ \leq C \epsilon \ln \frac{1}{\epsilon} \lambda^{n_k} \| \varphi_* \| \int_{\mathbb{R}^N} \left( \sum_{i=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2^*_r - 2 + \kappa \epsilon} \]
\[ \times \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}} \| \varphi_* \| \int_{\mathbb{R}^N} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \frac{\lambda^{n_k}}{\lambda^{1+\epsilon}} \| \varphi_* \| , \]  
(C.10)

where \( 0 < \kappa < 1 \).

Finally, similar to (C.3) we estimate \( M_4 \) as follows

\[ |M_4| \leq C \epsilon \| \varphi_* \| \int_{\mathbb{R}^N} Z^{2^*_r - 2 + \epsilon}_{\varphi, y^\prime, \lambda} \| Z_{1,k} \| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} dy \]
\[ \leq C \epsilon \| \varphi_* \| \star \lambda^{n_k} \int_{\mathbb{R}^N} \left( \sum_{i=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_i|)^{N-2s}} \right)^{2^*_r - 2 + \epsilon} \]
\[ \times \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \epsilon \lambda^{n_k} \| \varphi_* \| \int_{\mathbb{R}^N} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
\[ \leq C \epsilon \lambda^{n_k} \| \varphi_* \| \int_{\mathbb{R}^N} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_1|)^{N-2s}} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2s}{2}}}{(1 + \lambda |y - x_j|)^{\frac{N-2s}{2} + \tau}} \]
Observe that

\[ \text{Proof.} \]

Next we estimate \( H \)

Noting that \( B \)

Using Lemma \( \text{(C.11)} \)

It follows from \((\text{C.1})\) to \((\text{C.11})\) that \((2.12)\) holds.

\[ \square \]

**Appendix D. Proofs of (3.6) and (3.7)**

Now first we prove \((3.6)\).

**Proof.** Observe that

\[ \int_{B_n(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \left< y, \nabla u_\varepsilon \right> dy \]

\[ = \int_{B_n(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \sum_{i=1}^{N} y_i \frac{\partial u_\varepsilon}{\partial y_i} dy \]

\[ = \int_{B_n(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \sum_{i=1}^{N} y_i \left( \frac{\partial Z_{i,y',\lambda}}{\partial y_i} + \frac{\partial \varphi}{\partial y_i} \right) dy \]

\[ := H_1 + H_2. \]

Next we estimate \( H_1 \) and \( H_2 \) respectively.

First, we will give the estimates of \( H_1 \).

\[ H_1 = \int_{B_{\rho}^{c}(y_0)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} U_{x_j,\lambda}^{2s-2} Z_{j,l} \sum_{i=1}^{N} \sum_{k=1}^{m} (y_i - x_{k,i}) \frac{\partial U_{x_{k,\lambda}}}{\partial y_i} + x_{k,i} \frac{\partial U_{x_{k,\lambda}}}{\partial y_i} dy \]

\[ := H_{11} + H_{12}. \]

Noting that \( B_{\rho}^{c}(y_0) \subset B_{\frac{3}{2}}^{c}(x_j) \), by direct computations, we have

\[ |H_{11}| \leq C \int_{B_{\frac{3}{2}}^{c}(x_j)} \sum_{l=2}^{N} |c_l| \sum_{j=1}^{m} \lambda U_{x_j,\lambda}^{2s-1} \sum_{k=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_k|)^{N-2s}} dy \]

\[ + C \int_{B_{\frac{3}{2}}^{c}(x_j)} |c_l| \sum_{j=1}^{m} \frac{U_{x_j,\lambda}^{2s-1}}{\lambda} \sum_{k=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_k|)^{N-2s}} dy \]

\[ := C(H_{111} + H_{112}). \]

Using Lemma \( B.1 \), it is easy to obtain that

\[ |H_{111}| \leq C \int_{B_{\frac{3}{2}}^{c}(x_j)} \sum_{l=2}^{N} |c_l| \sum_{j=1}^{m} \lambda U_{x_j,\lambda}^{2s-1} \sum_{k=1}^{m} \frac{\lambda^{N-2s}}{(1 + \lambda|y - x_k|)^{N-2s}} dy \]

\[ \leq C \sum_{l=2}^{N} |c_l|m \int_{B_{\frac{3}{2}}^{c}(x_j)} \left[ \frac{\lambda^{N+1}}{(1 + \lambda|y - x_j|)^{2N}} \right] \]

\[ \leq C \sum_{l=2}^{N} |c_l|m \int_{B_{\frac{3}{2}}^{c}(x_j)} \left[ \frac{\lambda^{N+1}}{(1 + \lambda|y - x_j|)^{2N}} \right]. \]
\[ + \sum_{k \neq j} \frac{\lambda}{(\lambda|x_k - x_j|)^{N-2s}} \left( \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s}} + \frac{\lambda^N}{(1 + \lambda|y - x_k|)^{N+2s}} \right) dy \]

\[ \leq C \sum_{l=2}^{N} |c_l| \frac{m}{\lambda^{N-1}} + C \sum_{l=2}^{N} \frac{|c_l|m}{\lambda} = o(m\lambda^2) \sum_{l=2}^{N} |c_l|, \quad \text{(D.4)} \]

and

\[ |H_{112}| \leq C|c_1|m \int_{B^c_\frac{\lambda}{2} (x_j)} \left[ \frac{\lambda^{N-1}}{(1 + \lambda|y - x_j|)^{2N}} \right] dy \]

\[ + \sum_{k \neq j} \frac{\lambda^{-1}}{(\lambda|x_k - x_j|)^{N-2s}} \left( \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s}} + \frac{\lambda^N}{(1 + \lambda|y - x_k|)^{N+2s}} \right) dy \quad \text{(D.5)} \]

\[ \leq C|c_1| \frac{m}{\lambda^{N+1}} + C|c_1| \frac{m}{\lambda^3} = o(m)|c_1|. \]

Similarly, we have

\[ |H_{12}| \leq C \int_{B^c_\frac{\lambda}{2} (x_j)} \sum_{l=2}^{N} |c_l| \sum_{j=1}^{m} \lambda \mu_{x_j, \lambda}^{l-1} \sum_{k=1}^{m} \frac{\lambda^{N-2s+1}}{(1 + \lambda|y - x_k|)^{N-2s+1}} dy \]

\[ + C \int_{B^c_\frac{\lambda}{2} (x_j)} |c_l| \sum_{j=1}^{m} \mu_{x_j, \lambda}^{l-1} \sum_{k=1}^{m} \frac{\lambda^{N-2s+1}}{(1 + \lambda|y - x_k|)^{N-2s+1}} dy \quad \text{(D.6)} \]

\[ := C(H_{121} + H_{122}), \]

\[ |H_{121}| \leq C \sum_{l=2}^{N} |c_l| \frac{m}{\lambda^{N-1}} \int_{B^c_\frac{\lambda}{2} (x_j)} \left[ \frac{\lambda^{N+2}}{(1 + \lambda|y - x_j|)^{2N+1}} \right] dy \]

\[ + \sum_{k \neq j} \frac{\lambda^2}{(\lambda|x_k - x_j|)^{N-2s}} \left( \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+1+2s}} + \frac{\lambda^N}{(1 + \lambda|y - x_k|)^{N+1+2s}} \right) dy \]

\[ \leq C \sum_{l=2}^{N} |c_l| \frac{m}{\lambda^{N-1}} + C \sum_{l=2}^{N} |c_l|m\lambda^2 \sum_{l=2}^{N} |c_l|, \quad \text{(D.7)} \]

and

\[ |H_{122}| \leq C|c_1|m \int_{B^c_\frac{\lambda}{2} (x_j)} \left[ \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{2N+1}} \right] dy \]

\[ + \sum_{k \neq j} \frac{1}{(\lambda|x_k - x_j|)^{N-2s}} \left( \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+1+2s}} + \frac{\lambda^N}{(1 + \lambda|y - x_k|)^{N+1+2s}} \right) dy \]

\[ \leq C|c_1| \frac{m}{\lambda^{N+1}} + C|c_1| \frac{m}{\lambda^3} = o(m)|c_1|. \quad \text{(D.8)} \]

From (D.2) to (D.8), we have

\[ |H_1| = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \quad \text{(D.9)} \]
Noting that $\partial B^c_\rho(y_0) \subset B^c_\frac{\rho}{2}(x_j)$ and $B^c_\rho(y_0) \subset B^c_\frac{\rho}{2}(x_j)$, applying integrating by parts, by Proposition 2.3 and Lemma B.1 we have

\[
H_2 = \int_{\partial B^c_\rho(y_0)} \sum_{t=1}^N c_t \sum_{j=1}^m \frac{u^{2s-2}_{x,j,\lambda}}{Z_{j,t}} \varphi y \cdot \nu ds \\
- \int_{B^c_\rho(y_0)} \sum_{t=1}^N c_t \sum_{j=1}^m \sum_{i=1}^N \frac{\partial}{\partial y_i} \left( \frac{u^{2s-2}_{x,j,\lambda}}{Z_{j,t}} \varphi y_i \right) dy \\
= \int_{\partial B^c_\rho(y_0)} \sum_{t=1}^N c_t \sum_{j=1}^m \frac{u^{2s-2}_{x,j,\lambda}}{Z_{j,t}} \varphi y \cdot \nu ds \\
- \int_{B^c_\rho(y_0)} \sum_{t=1}^N c_t \sum_{j=1}^m \sum_{i=1}^N \left[ \frac{\partial u^{2s-2}_{x,j,\lambda}}{\partial y_i} Z_{j,t} y_i + \frac{\partial Z_{j,t}}{\partial y_i} y_i + \frac{d y_i}{d y_i} u^{2s-2}_{x,j,\lambda} Z_{j,t} \right] \varphi dy \\
:= H_{21} - H_{22} - H_{23} - H_{24}. 
\]

(D.10)

In the following, we will estimate the terms one by one.

For $H_{21}$, we have

\[
|H_{21}| \leq C|c_1| ||\varphi||_{B^c_\rho(x_j)} \sum_{j=1}^m \frac{\lambda^{N-1}}{(1 + \lambda|y - x_j|)^{N+2s}} \sum_{k=1}^m \frac{1}{(1 + \lambda|y - x_k|)^{\frac{N-2s}{\sigma} + \tau}} \\
+ C \sum_{l=2}^N |c_l| ||\varphi||_{B^c_\rho(x_j)} \sum_{j=1}^m \frac{\lambda^{N+1}}{(1 + \lambda|y - x_j|)^{N+2s}} \sum_{k=1}^m \frac{1}{(1 + \lambda|y - x_k|)^{\frac{N-2s}{\sigma} + \tau}} \\
\leq C|c_1| ||\varphi||_{L^\infty} m \int_{B^c_\rho(x_j)} \left[ \frac{\lambda^{N-1}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2} + \tau}} \\
+ \sum_{k \neq j} \frac{\lambda^{N-1}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \frac{(1 + \lambda|y - x_j|)^{N+2s}}{(1 + \lambda|y - x_k|)^{N+2s}} \\
+ C \sum_{l=2}^N |c_l| ||\varphi||_{L^\infty} \sum_{j=1}^m \frac{\lambda^{N+1}}{(1 + \lambda|y - x_j|)^{\frac{N+2s}{2} + \tau}} \\
+ \sum_{k \neq j} \frac{\lambda^{N}}{(1 + \lambda|y - x_j|)^{\frac{N-2s}{2} + \tau}} \frac{(1 + \lambda|y - x_j|)^{N+2s}}{(1 + \lambda|y - x_k|)^{N+2s}} \right] \\
= o(m\lambda^2) \sum_{l=2}^N |c_l| + o(m)|c_1|. 
\]

(D.11)

From directly computations, we obtain

\[
H_{22} = \int_{B^c_\rho(y_0)} \sum_{t=1}^N c_t \sum_{j=1}^m \sum_{i=1}^N \frac{\partial u^{2s-2}_{x,j,\lambda}}{\partial y_i} Z_{j,t} \left[ (y_i - x_{j,i}) + x_{j,i} \right] \varphi dy 
\]
\[ \leq C\|\varphi\| \sum_{l=1}^{N} |c_l| m \int_{B_{2}^{n}(x_j)} \frac{\lambda^{n+1}}{(1 + \lambda|y - x_j|)^{N+2s+1}} \sum_{k=1}^{m} \frac{1}{(1 + \lambda|y - x_k|)^{N-2s+\gamma}} \]
\[ + C\|\varphi\| \sum_{l=1}^{N} |c_l| m \int_{B_{2}^{n}(x_j)} \frac{\lambda^{n+1}}{(1 + \lambda|y - x_j|)^{N+2s+1}} \sum_{k=1}^{m} \frac{1}{(1 + \lambda|y - x_k|)^{N-2s+\gamma}} \]
\[ \leq C|c_1|\|\varphi\| m \int_{B_{2}^{n}(x_j)} \left[ \frac{\lambda^{N+1}}{(1 + \lambda|y - x_j|)^{N+2s+1}} \right] \]
\[ + \frac{1}{(1 + \lambda|y - x_j|)^{N+2s+1}} \lambda^{N+2} \sum_{k\neq j} \frac{1}{(1 + \lambda|y - x_k|)^{N+2s+1}} \]
\[ + C|c_1|\|\varphi\| m \int_{B_{2}^{n}(x_j)} \left[ \frac{\lambda^{N+2}}{(1 + \lambda|y - x_j|)^{N+2s+1}} \right] \]
\[ + \frac{1}{(1 + \lambda|y - x_j|)^{N+2s+1}} \lambda^{N+2} \sum_{k\neq j} \frac{1}{(1 + \lambda|y - x_k|)^{N+2s+1}} \]
\[ = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \]  

(D.12)

Similar to (D.12), we can also check that

\[ H_{23} = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \]  

(D.13)

Finally, we estimate \(H_{24}\) as follows

\[ H_{24} \leq C\|\varphi\| \sum_{l=1}^{N} |c_l| m \int_{B_{2}^{n}(x_j)} \frac{\lambda^{n+1}}{(1 + \lambda|y - x_j|)^{N+2s}} \sum_{k=1}^{m} \frac{1}{(1 + \lambda|y - x_k|)^{N-2s+\gamma}} \]
\[ = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \]  

(D.14)

It follows from all the estimates above that (3.6) holds. \(\square\)

Then we prove (3.7).
Proof. Note that

\[
\int_{B_{\rho}(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} \frac{\partial u_\tau}{\partial y_i} \, dy \\
= \int_{B_{\rho}(y_0)} \sum_{l=1}^{N} c_l \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} \left( \frac{\partial Z_{F, y''}}{\partial y_i} + \frac{\partial \varphi}{\partial y_i} \right) \\
:= G_1 + G_2.
\]

By direct computations, from Proposition 2.3 and Lemma B.1 we have

\[
|G_1| \leq C \int_{B_{\rho}(x_j)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} \lambda^{n_l} U_{x_j, \lambda}^{2s-1} \left| \frac{\partial Z_{F, y''}}{\partial y_i} \right| \\
\leq C \int_{B_{\rho}(x_j)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} \lambda^{n_l+1} U_{x_j, \lambda}^{2s-1} \sum_{k=1}^{m} \lambda^{N-2s} (1 + \lambda |y - x_k|)^{N-2s+1} \\
\leq C|c_1| m \int_{B_{\rho}(x_j)} \left[ \frac{\lambda^N}{(1 + \lambda |y - x_j|)^{2N+1}} + \frac{\lambda^{N+1}}{(1 + \lambda |y - x_j|)^{2N+1}} \right] \\
+ C \sum_{l=2}^{N} |c_l| m \int_{B_{\rho}(x_j)} \left[ \frac{\lambda^{N+2}}{(1 + \lambda |y - x_j|)^{2N+1}} + \frac{\lambda^{N+1}}{(1 + \lambda |y - x_j|)^{2N+1}} \right] \\
\leq C|c_1| \frac{m}{\lambda^N} + C|c_1| \frac{m}{\lambda^2} + C \sum_{l=2}^{N} |c_l| \frac{m}{\lambda^{N-3}} + Cm \sum_{l=2}^{N} |c_l| \\
= o(m)|c_1| + o(m\lambda^2) \sum_{l=2}^{N} |c_l|. \quad (D.16)
\]

Applying integrating by parts, we have

\[
G_2 = \int_{\partial B_{\rho}(y_0)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} U_{x_j, \lambda}^{2s-2} Z_{j,l} \varphi_i \, ds - \int_{B_{\rho}(y_0)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} \frac{\partial}{\partial y_i} (U_{x_j, \lambda}^{2s-2} Z_{j,l}) \varphi \, dy \\
= \int_{\partial B_{\rho}(y_0)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} \lambda^{n_l} U_{x_j, \lambda}^{2s-1} \varphi_i \, ds \\
- \int_{\partial B_{\rho}(y_0)} \sum_{l=1}^{N} |c_l| \sum_{j=1}^{m} \left( \frac{\partial U_{x_j, \lambda}^{2s-2}}{\partial y_i} Z_{j,l} + \frac{\partial Z_{j,l}}{\partial y_i} U_{x_j, \lambda}^{2s-2} \right) \varphi \, dy \\
:= G_{21} - G_{22} - G_{23}. \quad (D.17)
\]
Similar to (3.6), we can check that
\[ |G_{21}| = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \] (D.18)

Noting that \( \partial B_{\rho}^c(y_0) \subset B_{\frac{\rho}{2}}^c(x_j) \) and \( B_{\rho}^c(y_0) \subset B_{\frac{\rho}{2}}^c(x_j) \), applying integrating by parts, by Proposition 2.3 and Lemma B.1 we have
\[ |G_{22}| \leq C |c_1| \|\varphi\|_{m} \int_{B_{\frac{\rho}{2}}^c(x_j)} \left[ \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s+\tau}} + \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s}} \right] \]
\[ + C \sum_{l=2}^{N} |c_l| \|\varphi\|_{m} \int_{B_{\frac{\rho}{2}}^c(x_j)} \left[ \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s+\tau}} \right] \]
\[ + \sum_{k \neq j} \frac{\lambda^2}{(\lambda|x_k - x_j|)^N} \left[ \frac{\lambda^N}{(1 + \lambda|y - x_j|)^{N+2s+\tau}} + \frac{\lambda^N}{(1 + \lambda|y - x_k|)^{N+2s}} \right] \]
\[ = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \] (D.19)

Just by the same argument as (D.19), we can prove
\[ |G_{23}| = o(m\lambda^2) \sum_{l=2}^{N} |c_l| + o(m)|c_1|. \] (D.20)

By all the estimates above, we know that (3.7) holds. \( \square \)

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**References**

[1] D. Applebaum, Levy processes and stochastic calculus, Second edition, Cambridge Studies in Advanced Mathematics, 116, Cambridge University Press, Cambridge, 2009.

[2] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator. J. Differential Equations 252 (2012), 6133-6162.

[3] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian. Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), 39-71.

[4] B. Barrios, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities. Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), 875-900.

[5] H. Brezis, Y.Y. Li, Some nonlinear elliptic equations have only constant solutions. J. Partial Differ. Equ. 19 (2006), 208-217.

[6] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian. Adv. Math. 224 (2010), 2052-2093.

[7] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32 (2007), 1245-1260.

[8] W. Chen, Y. Li, P. Ma, the fractional Laplacian, World Scientific Publishing Co Pte Ltd, Singapore, 2019.
[9] W. Chen, J. Wei, S. Yan, Infinitely many positive solutions for the Schrödinger equations in \( \mathbb{R}^N \) with critical growth. J. Differential Equations 252 (2012), 2425-2447.

[10] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron’s problem. Calc. Var. Partial Differential Equations 16 (2003), 113-145.

[11] Y. Deng, C.-S. Lin, S. Yan, On the prescribed scalar curvature problem in \( \mathbb{R}^N \), local uniqueness and periodicity. J. Math. Pures Appl. 104 (2015), 1013-1044.

[12] E. Di Nezzaa, G. Palatuccia, E. Valdinocia, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Svi. Math. 136 (2012), 521-573.

[13] P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. Proc. Roy. Soc. Edinburgh Sect. A 142 (2012), 1237-1262.

[14] Y. Guo, S. Peng, S. Yan, Local uniqueness and periodicity induced by concentration. Proc. London Math. Soc. (3) 114 (2017), 1005-1043.

[15] Y. Guo, T. Liu, J. Nie, Construction of solutions for the polyharmonic equation via local Pohozaev identities, Calc. Var. Partial Differential Equations (2019) 58:123.

[16] Y. Guo, T. Liu, J. Nie, Solutions for fractional operator problem via local Pohozaev identities, arXiv:1904.08316, 2019.

[17] Y. Guo, J. Nie, Infinitely many non-radial solutions for the prescribed curvature problem of fractional operator. Discrete Contin. Dyn. Syst. 36 (2016), 6873-6898.

[18] J. Jin, Y.Y. Li, J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. J. Eur. Math. Soc. 16 (2014), 1111-1171.

[19] Y. Y. Li, J. Wei, H. Xu, Multi-bump solutions of \((-\Delta)^su = K(x)u^{\frac{N+2}{N-2}}\) on lattices in \( \mathbb{R}^N \). J. Reine Angew. Math. 743(2018), 163-211.

[20] F. Lin, W.-M. Ni, J. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem. Comm. Pure Appl. Math. 60 (2007), 252-281.

[21] Z. Liu, Large number of bubble solutions for the equation \(\Delta u + K(y)u^{\frac{N+2}{N-2}+\varepsilon} = 0\) on \( \mathbb{R}^N \). Sci. China Math. 59 (2016), 459-478.

[22] M. Niu, Z. Tang, L. Wang, Solutions for conformally invariant fractional Laplacian equations with multi-bumps centered in lattices. J. Differential Equations 266 (2019), 1756-1831.

[23] O. Rey, The role of the Green’s function in a nonlinear elliptic problem involving the critical Sobolev exponent. J. Funct. Anal. 89 (1990), 1-52.

[24] O. Rey, Boundary effect for an elliptic Neumann problem with critical nonlinearity, Commun. in Partial Differential Equations, 22 (1997), 1055-1139.

[25] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian. Calc. Var. Partial Differential Equations 42 (2011), 21-41.

[26] J. Tan, J. Xiong, A Harnack inequality for fractional Laplace equations with lower order terms. Discrete Contin. Dyn. Syst. 31 (2011), 975-983.

[27] S. Peng, C. Wang, S. Wei, Constructing solutions for the prescribed scalar curvature problem via local Pohozaev identities. J. Differential Equations 267 (2019), 2503-2530.

[28] S. Peng, C. Wang, S. Yan, Construction of solutions via local Pohozaev identities. J. Funct. Anal. 274 (2018), 2606-2633.

[29] J. Vátois, S. Wang, Infinitely many solutions for cubic nonlinear Schrödinger equations in dimension four. (English summary) Adv. Nonlinear Anal. 8 (2019), 715-724.

[30] X. Wang, J. Wei, On the equation \(\Delta u + K(x)u^{\frac{N+2}{N-2}+\varepsilon} = 0\) in \( \mathbb{R}^N \). Rend. Circ. Mat. Palermo (2) 44 (1995), 365-400.

[31] L. Wang, J. Wei, S. Yan, A Neumann problem with critical exponent in nonconvex domains and Lin-Ni’s conjecture. Trans. Amer. Math. Soc. 362 (2010), 4581-4615.

[32] L. Wang, J. Wei, S. Yan, On Lin-Ni’s conjecture in convex domains. Proc. Lond. Math. Soc. (3) 102 (2011), 1099-1126.

[33] J. Wei, S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in \( \mathbb{R}^N \). Calc. Var. Partial Differential Equations 37 (2010), 423-439.

[34] J. Wei, S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on \( S^N \). J. Funct. Anal. 258 (2010), 3048-3081.
[35] J. Wei, S. Yan, On a stronger Lazer-McKenna conjecture for Ambrosetti-Prodi type problems. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), 423-457.

[36] J. Wei, S. Yan, Infinitely many positive solutions for an elliptic problem with critical or supercritical growth. J. Math. Pures Appl. (9) 96 (2011), 307-333.

[37] S. Yan, J. Yang, X. Yu, Equations involving fractional Laplacian operator: compactness and application. J. Funct. Anal. 269 (2015), 47-79.

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