ON THE GEOMETRY OF THE SCHMIDT-LEGENDRE TRANSFORMATION

OĞUL ESEN
Department of Mathematics, Gebze Technical University
41400 Çayırova, Gebze, Kocaeli, Turkey

PARTHA GUHA
S.N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake, Kolkata - 700098, India

(Communicated by David Martín de Diego)

ABSTRACT. Tulczyjew’s triples are constructed for the Schmidt-Legendre transformations of both second and third-order Lagrangians. Symplectic diffeomorphisms relating the Ostrogradsky-Legendre and the Schmidt-Legendre transformations are derived. Several examples are presented.

1. Introduction. The dynamics of a mechanical system can either be formulated by a Lagrangian function on the tangent bundle of the configuration space or by a Hamiltonian function on the cotangent bundle \([1, 4]\). For a non-degenerate system, it is immediate to construct the Legendre transformation relating the Lagrangian and the Hamiltonian formalisms. If a system is degenerate or/and possesses various constraints, then defining the Legendre transformation becomes complicated. A geometric framework realizing the Legendre transformation for both degenerate and non-degenerate systems is the Tulczyjew triple \([75, 76, 78]\).

Although in classical mechanics a Lagrangian function is a function of positions and velocities, it is possible to find various theories involving Lagrangian functions depending on higher order derivatives as well. In order to arrive at Hamiltonian formalisms of higher order Lagrangian theories, it is common to employ the Ostrogradsky-Legendre transformation \([61]\). Ostrogradsky’s approach is based on the idea that consecutive time derivatives of initial coordinates form new coordinates. In this way, a higher order Lagrangian can be written in a form of a first order Lagrangian on a proper tangent bundle. There are extensive studies in the literature for degenerate or/and constrained higher order Lagrangian systems and their Hamiltonian analysis, see for example \([6, 12, 15, 21, 41, 65, 47, 50, 57, 60, 64]\). In several papers \([5, 6, 41]\), the Ostrogradsky-Legendre transformation has been achieved by the introductions of some intermediate steps. In particular, the configuration space is considered as a Lie group in some recent works \([19, 32, 33]\).

In an interesting paper \([19]\) higher order systems are described as pre-symplectic Hamiltonian systems, whereas in \([32, 33]\), a variational approach has been followed.

2010 Mathematics Subject Classification. Primary: 70H50; Secondary: 70H45.

Key words and phrases. Ostrogradsky-Legendre transformation, Schmidt-Legendre transformation, second order degenerate Lagrangian systems, Tulczyjew’s triple.

* Corresponding author.
It is also possible to find some literature on the Ostrogradsky-Legendre transformation in the framework of Tulczyjew’s triple [20, 48]. For instance, Tulczyjew’s triple for higher order field theories has been constructed in [38]. Tulczyjew’s triple for the higher order graded bundles has been presented in [10]. Tulczyjew triple for the case of $n$-vectors has been introduced in [40]. We cite [49] for the construction of the Tulczyjew’s triple in the Lie algebroid setting. In some recent studies [17, 43, 54], higher order dynamics has been studied in terms of the Lie algebroids, and in this context, the Tulczyjew’s triple has been constructed in [2, 44].

In the mid nineties, H.J. Schmidt proposed an alternative method for the Legendre transformation of higher order Lagrangian systems [66, 67]. In this approach, the acceleration is defined as a new coordinate instead of the velocity. Although, the Schmidt-Legendre transformation has not been credited as it deserves, it is possible to find some related works in the literature [23, 42, 45]. In [3], a comparison of the Ostrogradsky-Legendre and the Schmidt-Legendre transformations has been presented.

The main result of this manuscript is to study the second and the third order Lagrangian formalisms and their Legendre transformations. The main objectives are to construct Tulczyjew’s triple for the Schmidt-Legendre transformation and present the symplectic transformations relating the Ostrogradsky-Legendre and the Schmidt-Legendre transformations in a a purely geometric language. Novelty in the present work is the constructions of Tulczyjew’s triples proper for the Schmidt-Legendre transformations in the framework of acceleration bundle. We shall propose two Tulczyjew’s triples, one is for the second-order Lagrangian systems and the other is for the third-order Lagrangian systems. One of the important advantages of the Schmidt-Legendre transformation is to relax the Hessian condition in the Ostrogradsky-Legendre transformation. Combining this advantage of the transformation with the beauty of Tulczyjew’s triple, we derive a powerful geometric tool representing the second and the third-order Lagrangian systems as the Lagrangian submanifolds of some certain symplectic manifolds. This enables us to explore the Schmidt-Legendre transformation as a passage between two different generators of the same Lagrangian submanifold. In this picture, the main role is played by the acceleration bundle which permits us to write a higher-order Lagrangian function as a first order Lagrangian function by defining the accelerations as new independent coordinates. In the classical Ostrogradsky’s approach one needs to introduce Lagrange variables to write a higher Lagrangian function in the form of a first order one, on the other hand, if the acceleration bundle, or the Schmidt’s method, is employed, there is no need to introduce Lagrange multipliers to achieve the reduction [3]. Particularly, for regular second order Lagrangians, the Legendre transformation can be defined in the standard form.

Accordingly, in the present work, after the construction of Tulczyjew’s triples we present examples including both non-degenerate and degenerate Lagrangian theories. For the non-degenerate theories, we shall discuss the Hamiltonian analysis of the Pais-Uhlenbeek oscillator [62] which is perennial favorite among quantum theorists. For the case of degenerate systems, we shall address two second-order Lagrangians coming from the theory of topological massive gravity [24, 25], namely Saroğlu-Tekin Lagrangian [68] and Clemént Lagrangian [16]. We are referring a recent study [11] for the Hamiltonian analysis of Saroğlu-Tekin Lagrangian and Clemént Lagrangian in the framework of the Ostrogradsky-Legendre transformation. So that comparisons of the results of this paper, and the results of [11] could be
instructive for discussing the advantages and disadvantages of the Schmidt-Legendre and the Ostrogradsky-Legendre transformations. In [22, Page 130], a trick has been announced to reduce a second order Lagrangian function to a first order one. In this paper, we are presenting a geometrization of this trick in terms of acceleration bundles and the Schmidt-Legendre transformation.

In order to achieve the goals of this paper, we shall start with the following section by reviewing some necessary fundamental materials, namely higher order tangent bundles, higher order Euler-Lagrange equations, symplectic manifolds, Hamilton’s equations, and the pull-back bundle formalism. The third section is reserved for the Tulczyjew’s construction of the Legendre transformation. In this section, basic ingredients of Tulczyjew’s triples will be presented including Morse families, special symplectic structures. Moreover, a Tulczyjew’s triple will be constructed proper for the Ostrogradsky-Legendre transformation. In this section, we shall summarize the Dirac-Bergmann constraint algorithm as well. In the Section 4, acceleration bundle will be introduced. Tangent and cotangent bundles of the acceleration bundle will be presented. We shall construct Tulczyjew’s triple for the case of acceleration bundles. Using this construction, we shall present geometry of the Schmidt-Legendre transformation. The symplectic transformation between the Ostrogradsky-Legendre and the Schmidt-Legendre transformations will be constructed. The last section will be reserved for several examples including the Pais-Uhlenbeck, the Sarıoğlu-Tekin and the Célcant Lagrangians.

We are expecting that the framework presented in this paper can be extended for several applications including Skinner-Rusk unified formalism [69, 70, 71], symmetry analysis, constraint systems, regularity issues, and optimization problems. We list some possible future projects at the conclusion.

2. Fundamentals.

2.1. Higher order Euler-Lagrange equations. Let $Q$ be a manifold. Consider the set $C_q(Q)$ of smooth curves passing through a point $q$ in $Q$. Two curves $\gamma$ and $\gamma'$ are called $k$-equivalent, and denoted by $\gamma \sim^k_q \gamma'$, if they agree up to their $k$-th order derivatives at $q$ that is if

$$D^r(f \circ \gamma)(0) = D^r(f \circ \gamma')(0), \quad r = 0, 1, 2, \ldots, k,$$

(1)

for all real valued functions $f$ defined on $Q$ [50]. Here, we assumed that $\gamma(0) = \gamma'(0) = q$. Under the equivalence relationship $\sim^k_q$, an equivalence class, denoted by $t^k_q \gamma(0)$, is called a $k$-th order tangent vector at $q$. The set of all equivalence classes of curves, that is the set of all $k$-th order tangent vectors at $q$ defines $k$-th order tangent space $T^k_q Q$. The union

$$T^k Q = \bigcup_{q \in Q} T^k_q Q$$

of all $k$-th order tangent spaces is $k$-th order tangent bundle of $Q$.

There exists a projection from the $k$-th order tangent bundle $T^k Q$ to the manifold $Q$ defined as

$$\tau^k_Q : T^k Q \longrightarrow Q : t^k \gamma(0) \longrightarrow \gamma(0).$$

(2)

All possible triples

$$(T^k Q, \tau^k_Q, Q)$$

are fiber bundles with total space $T^k Q$, the projection $\tau^k_Q$, and the base space $Q$. If $k = 1$ then we arrive at the tangent bundle $(TQ, \tau_Q, Q)$. $TQ$ admits a vector
bundle structure on \( Q \) but in general \( T^k Q \) does not necessarily admit a vector bundle structure on \( Q \) for \( k \geq 2 \) \cite{74}. To equip a vector bundle structure on \( T^k Q \), one needs to employ a linear connection on \( Q \) \cite{28,29,73}. In this case, \( T^k Q \) becomes isomorphic to the Whitney sum of \( k \) numbers of \( TQ \).

Assume that \( Q \) be an \( n \)-dimensional manifold with local coordinates \((q^1,\ldots,q^n)\), then the first order tangent bundle \( TQ \) is a \( 2n \)-dimensional manifold with induced coordinates

\[(q,\dot{q}) : TQ \rightarrow \mathbb{R}^{2n} : t \gamma(0) \rightarrow (q(\gamma(0)), D(q \circ \gamma)(0)). \tag{3}\]

The higher order tangent bundle \( T^k Q \) is a \([k+1)n\]-dimensional manifold with induced coordinates

\[(q,\dot{q},\ldots,q^{(k)}) : T^k Q \rightarrow (k+1)n

: t^k \gamma(0) \rightarrow (q(\gamma(0)), D(q \circ \gamma)(0),\ldots,D^k(q \circ \gamma)). \tag{4}\]

Consider the \( k \)-th order tangent bundle \( T^k Q \). There is a natural embedding of \( T^k Q \) into the iterated tangent bundle \( TT^s Q \) of \( T^s Q \) as given by

\[T^k Q \rightarrow TT^s Q : t^k \gamma(0) \rightarrow t^s \tilde{\gamma}(0), \tag{5}\]

where \( \tilde{\gamma} \) is a curve in \( Q \) depending on \( r \), and defined by \( \tilde{\gamma}(r) = \gamma(t + r) \) \cite{50}. In (5), we are assuming that \( s \) is smaller than \( k \). For future reference, let us record here some particular cases of the embedding. First we take \( k = 2 \) and \( s = 1 \), and arrive at

\[T^2 Q \rightarrow TT^1 Q : (q,\dot{q},\ddot{q}) \rightarrow (q,\dot{q},\dot{q},\ddot{q}). \tag{6}\]

Secondly, we consider \( k = 3 \) and \( s = 2 \), and compute the following

\[T^3 Q \rightarrow TT^2 Q : (q,\dot{q},\ddot{q},q^{(3)}) \rightarrow (q,\dot{q},\ddot{q},\dot{q},\ddot{q},q^{(3)}). \tag{7}\]

A \( k \)-th order Lagrangian function \( L = L(q,\dot{q},\ddot{q},\ldots,q^{(k)}) \) is a real-valued function on \( T^k Q \), and \( k \)-th order Euler-Lagrange equations are

\[\sum_{\kappa=0}^{k}(-1)^\kappa \frac{d^\kappa}{dt^\kappa} \left( \frac{\partial L}{\partial q^{(\kappa)}} \right) = 0. \tag{8}\]

If the partial derivatives \( \partial L/\partial q^{(k)} \) depend on \( q^{(k)} \) then the \( k \)-th order Euler-Lagrange equations (8) are a set of differential equation of order \( 2k \). Particularly, for \( k = 1 \), the Euler-Lagrange equations are

\[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{9}\]

For \( k = 2 \), we have the second order Euler-Lagrange equations

\[\frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} + \frac{\partial L}{\partial \dot{q}} = 0, \tag{10}\]

whereas for \( k = 3 \) we have the third order Euler-Lagrange equations

\[\frac{d^3}{dt^3} \frac{\partial L}{\partial q^{(3)}} - \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q}} + \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} + \frac{\partial L}{\partial \dot{q}} = 0. \tag{11}\]
2.2. Symplectic manifolds and Hamilton’s equations. A manifold $P$ is called a symplectic manifold if it is equipped with a closed, non-degenerate two-form $\Omega$ [51]. We are interested in Lagrangian submanifolds of symplectic manifolds. A submanifold $S$ of a symplectic manifold $(P, \Omega)$ is called a Lagrangian submanifold if $\Omega$ vanishes when it is restricted to $S$, and if the dimension of $S$ is half of the dimension of $P$.

A diffeomorphism $\varphi$ from a symplectic manifold $(P_1, \Omega_1)$ to another symplectic manifold $(P_2, \Omega_2)$ is called a symplectic diffeomorphism if it respects to the symplectic two-forms, that is if $\varphi^*\Omega_2 = \Omega_1$. A symplectic diffeomorphism $\varphi$ maps a Lagrangian submanifold $S$ of $P_1$ to a Lagrangian submanifold $\varphi(S)$ of $P_2$. Product of two symplectic manifolds $(P_1, \Omega_1)$ and $(P_2, \Omega_2)$ is a symplectic manifold. There exist natural projections $\text{pr}_1$ from the product space $P_1 \times P_2$ to $P_1$, and $\text{pr}_2$ from $P_1 \times P_2$ to $P_2$. In order to define the symplectic two-form on the product space $P_1 \times P_2$, first pull the symplectic two-forms $\Omega_1$ and $\Omega_2$ back to the product space using the natural projections, then take the difference of the pull-back two-forms

$$\Omega_2 \ominus \Omega_1 = \text{pr}_2^*(\Omega_2) - \text{pr}_1^*(\Omega_1).$$

(12)

The graph of a diffeomorphism $\varphi$ from $P_1$ to $P_2$ is a Lagrangian submanifold of the product symplectic manifold $(P_1 \times P_2, \Omega_2 \ominus \Omega_1)$ if and only if $\varphi$ is a symplectic diffeomorphism. For the proof of this assertion, we refer [79, Page 339].

Let $(P, \Omega)$ be a symplectic manifold. Since the symplectic two-form $\Omega$ is non-degenerate the musical mapping $\Omega^\flat$, from the space $\mathfrak{X}(P)$ of vector fields to the space $\Lambda(P)$ of differential one-forms, $\Omega^\flat : \mathfrak{X}(P) \rightarrow \Lambda(P) : X \rightarrow \iota_X \Omega,$

(13)

is an isomorphism [53]. Here, $\iota$ is the contraction operator. We denote the inverse of $\Omega^\flat$ by $\Omega^\sharp$. Accordingly, the Hamilton’s equation

$$\iota_{X_H} \Omega = -dH$$

(14)

determines a unique vector field $X_H$, called as a Hamiltonian vector field, for a Hamiltonian function $H$ on $P$. By referring to the symplectic two-form $\Omega$, it is possible to define a Poisson bracket on the space $\mathcal{F}(P)$ of real valued functions as follows

$$\{F, H\} = \Omega(X_F, X_H),$$

(15)

where the vector fields $X_H$ and $X_F$ are Hamiltonian vector fields associated with Hamiltonian functions $H$ and $F$, respectively. They are computed through the definition presented in (14).

A cotangent bundle $T^*Q$ admits a canonical one-form $\Theta_Q$ and a canonical symplectic two-form $\Omega_Q = d\Theta_Q$. On a vector field $X : T^*Q \rightarrow TT^*Q$, the canonical one-form $\Theta_Q$ takes the value

$$\langle \Theta_Q, X \rangle = \langle \tau_{T^*Q}(X), T\pi_Q(X) \rangle,$$

(16)

where $\tau_{T^*Q}$ is the tangent bundle projection from $TT^*Q$ to $T^*Q$ whereas $T\pi_Q$ is tangent mapping of the cotangent bundle projection $\pi_Q$. Pairing on the right hand side of (16) is the dualization between the cotangent bundle $T^*Q$ and the tangent bundle $TQ$. Following commutative diagram, called as the tangent rhombic [1, Page
Consider a local coordinate system \((q)\) defined on \(n\)-dimensional manifold \(Q\). In the Darboux’ coordinates \((q, p)\) on the cotangent bundle \(T^* Q\), the canonical forms turn out to be
\[
\Theta_Q = p \cdot dq, \quad \Omega_Q = dp \wedge dq. \tag{18}
\]
whereas the Hamiltonian vector field \(X_H\) in (14) is written as
\[
X_H = \frac{\partial H}{\partial p} \cdot \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \cdot \frac{\partial}{\partial p}. \tag{19}
\]
The canonical Poisson bracket in (15) is computed to be
\[
\{F, H\} = \frac{\partial F}{\partial q} \cdot \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \cdot \frac{\partial H}{\partial q}. \tag{20}
\]

2.3. Pullback bundle. Consider a fibre bundle \((R, \tau, M)\). Assume existence of a differential mapping \(\psi\) from a manifold \(N\) to the base manifold \(M\) of the fiber bundle. Consider the following submanifold
\[
\psi^* R = \{(n, r) \in N \times R : \psi(n) = \pi(r)\}. \tag{21}
\]
There exists a surjective submersion, denoted by \(\psi^*,\) projecting an element \((n, r)\) in \(\psi^* R\) to its first factor \(n\) in \(N\). This make the triple \((\psi^* R, \psi^* \tau, N)\) a fiber bundle, called as the pullback bundle, [59, Page 355]. This definition can be summarized within the following commutative diagram.
\[
\begin{array}{ccc}
\psi^* R & \xrightarrow{\varepsilon} & R \\
\psi^* \tau & \downarrow & \tau \\
N & \xrightarrow{\psi} & M
\end{array} \tag{22}
\]
Here, \(\varepsilon\) is the projection which maps an element in \(\psi^* R\) to its second factor \(R\). If \(N\) is an embedded submanifold of \(M\) in (22), then it is evident that \(\psi^* R\) is an embedded submanifold of \(R\).

3. The Legendre transformation.

3.1. Morse families. Let \((R, \tau, M)\) be a fiber bundle. Vertical bundle \(VR\) over \(R\) is space of vertical vectors lying in the kernel of the tangent mapping \(T\tau\). Conormal bundle \(V^0 R\) consists of covectors in \(T^* R\) annihilating the vectors in \(VR\). A real-valued function \(E\) defined on the total space \(R\) of the fiber bundle is said to be a Morse or a generating family if, for all \(z \in \text{Im} (dE) \cap V^0 R\),
\[
T_z \text{Im} (dE) + T_z V^0 R = T_z T^* R. \tag{23}
\]
Here, $\text{Im}(dE)$ is the image space of exterior derivative $dE$. A Morse family $E$ on a smooth bundle $(R, \tau, M)$ generates an immersed Lagrangian submanifold

$$S = \{ z \in T^*M : T^*\tau(z) = dE(r) \forall r \in R \text{ satisfying } \tau(r) = \pi_{T^*M}(z) \}$$

(24)

of the canonically symplectic space $T^*M$. The Morse family $E$ is called generator of the Lagrangian submanifold $S$. The inverse of this assertion is also true. That is if $S$ is a Lagrangian submanifold of $T^*M$ then there always exists, at least locally, a Morse family $E$ generating $S$. This assertion is known as the generalized Poincaré lemma or Maslov-Hörmander theorem. We refer to the Lecture 5 in [80] and the Section 9.5 of the book [7] for the proofs of the assertions. Here is a diagram summarizing the discussion.

$$\begin{array}{c}
\mathbb{R} \\
E \\
R \\
\downarrow \\
\tau \\
\downarrow \\
T^*M \\
\pi_M \\
\downarrow \\
M \\
\downarrow \\
M \\
\end{array}$$

(25)

We refer [13] for Morse families in the theory of constrained mechanical systems.

Assume that $M$ is equipped with local coordinates $(m)$, and consider the induced local coordinates $(m, \lambda)$ on the total space $R$. In this picture, a function $E$ is called a Morse family if the rank of the matrix

$$\begin{pmatrix}
\frac{\partial^2 E}{\partial m \partial m} & \frac{\partial^2 E}{\partial m \partial \lambda}
\end{pmatrix}$$

(26)

is maximal. In such a case, the Lagrangian submanifold (24) generated by $E$ locally looks like

$$S = \left\{ \left( m, \frac{\partial E}{\partial m}(m, \lambda) \right) \in T^*M : \frac{\partial E}{\partial \lambda}(m, \lambda) = 0 \right\}.$$  

(27)

See that the dimension of $S$ is half of the dimension of $T^*M$, and that the canonical symplectic two-form $\Omega_M$ vanishes on $S$.

For a given Lagrangian submanifold, its Morse family generator is far from being unique. One may find a Morse family with less number of fiber variables generating the same Lagrangian submanifold. This procedure is called reduction of Morse family. We refer [7] for detailed discussions on this subject.

3.2. Special symplectic structures. Let $P$ be a symplectic manifold carrying an exact symplectic two-form $\Omega = d\Theta$. Assume also that, $P$ is the total space of a fibre bundle $(P, \pi, M)$. A special symplectic structure is a quintuple $(P, \pi, M, \Theta, \chi)$ where $\chi$ is a fiber preserving symplectic diffeomorphism from $P$ to the cotangent bundle $T^*M$. Here, $\chi$ can uniquely be characterized by

$$\langle \chi(p), \pi_*X(m) \rangle = \langle \Theta(p), X(p) \rangle$$

(28)

for a vector field $X$ on $P$, for any point $p$ in $P$ where $\pi(p) = m$. Note that, pairing on the left hand side of (28) is the natural pairing between the cotangent space $T^*_mM$ and the tangent space $T_mM$. Pairing on the right hand side of (28) is the one between the cotangent space $T^*_pP$ and the tangent space $T_pP$. We refer [46, 72, 77] for further discussions on special symplectic structures. Here is a diagram exhibiting the special symplectic structure.
The two-tuple \((P, \Omega)\) is called as underlying symplectic manifold of the special symplectic structure \((P, \pi, M, \Theta, \chi)\).

Let \((P, \pi, M, \Theta, \chi)\) be a special symplectic structure. Assume also that \(S_P\) be a Lagrangian submanifold of \(P\). The image \(\chi(S_P)\) of \(S_P\) is a Lagrangian submanifold of \(T^*M\). By referring to the generalized Poincaré lemma presented in the previous subsection (3.1), we argue that the Lagrangian submanifold \(\chi(S_P)\) can locally be generated by a Morse family \(E\) on a fiber bundle \((R, \tau, M)\). Accordingly, we are calling the Morse family \(E\) as a generator of both \(S\) and \(S_P\) since they are the same up to \(\chi\). The following diagram summarizes this discussion by equipping a Morse family to the special symplectic structure (29).

\[
\begin{array}{c}
\mathbb{R} \\
\downarrow E \quad \quad \quad \quad \quad \downarrow R \\
R \\
\downarrow \tau \\
M \\
\downarrow \pi' \\
M' \\
\downarrow \pi \\
T^*M \\
\downarrow \pi' \\
M' \\
\downarrow \pi \\
\chi \\
\pi \\
\chi' \\
\pi' \\
\chi \\
\pi \\
T^*M' \\
\end{array}
\]

3.3. Tulczyjew’s triple. Assume that \((P, \Omega)\) admits two different special symplectic structures, say \((P, \pi, M, \Theta_1, \chi)\) and \((P, \pi', M', \Theta_2, \chi')\). The generalized version of the Tulczyjew’s triple [78] is a diagram combining two special symplectic structures in one picture

\[
\begin{array}{c}
T^*M \\
\downarrow \pi \\
M \\
\downarrow \pi' \\
M' \\
\downarrow \pi \\
\chi \\
\pi \\
\chi' \\
\pi' \\
\chi \\
\pi \\
T^*M' \\
\end{array}
\]

In this case, the symplectic two-form \(\Omega\) have two potential one-forms, namely \(\Theta_1\) and \(\Theta_2\). The difference of these one-forms is closed. So that there exists, at least locally, a function \(\Delta\) on \(P\) satisfying

\[
d\Delta = \Theta_2 - \Theta_1 = (\chi')^*(\Theta_{M'}) - \chi^*(\Theta_M).
\]

Here, \(\Theta_M\) and \(\Theta_{M'}\) are the canonical one-forms on \(T^*M\) and \(T^*M'\), respectively.

Let \(S_P\) be a Lagrangian submanifold of \(P\). Then \(\chi(S_P)\) is a Lagrangian submanifold of \(T^*M\), and \(\chi'(S_P)\) is a Lagrangian submanifold of \(T^*M'\). For the Lagrangian submanifold \(\chi'(S_P)\), in the lights of the discussions done in the previous subsections, we know that there exists a Morse family \(E'\), on a fiber bundle \((R', \tau', M')\), generating \(\chi(S_P)\). Similarly, for the Lagrangian submanifold \(\chi'(S_P)\), there exists a Morse family \(E'\), on a fiber bundle \((R', \tau', M')\), generating \(\chi'(S_P)\). So that, we have two different generators of the Lagrangian submanifold \(S_P\).

Product space \(T^*M \times T^*M'\) is a symplectic manifold. To arrive at the symplectic two-form, by referring to the equation (12), we are pulling the canonical symplectic two-forms \(\Omega_M\) and \(\Omega_{M'}\) back to the product space using the natural projections, then we are taking the difference

\[
\Omega'_{M} \oplus \Omega_{M} = pr_2^*(\Omega_{M'}) - pr_1^*(\Omega_M).
\]

Note that the composition \(\chi' \circ \chi^{-1}\) pictured in (31) is a symplectic diffeomorphism from \(T^*M\) to \(T^*M'\). So that, the graph of the mapping \(\chi' \circ \chi^{-1}\) is a Lagrangian
submanifold of $T^* M \times T^* M'$. The Morse family generating the graph of $\chi' \circ \chi^{-1}$
is the potential function $\Delta$ defined in (32). For a proof this assertion in case of
the classical dynamics we refer the discussion done in the next section (3.4) (espe-
cially the calculations presented in (36)-(39)). We cite [7, 75, 76, 77] for symplectic
diffeomorphisms generating by Morse families. Let us record this in the following
diagram

$\mathbb{R} \xleftarrow{\Delta} P \xrightarrow{\pi \times \pi'} T^*(M \times M')$

$M \times M' \xrightarrow{\pi_M \times \pi_M'} M \times M'$

(33)

Notice that the diagrams (31) and (33) are two different representations of the Le-
gendre transformation in the sense of Tulczyjew. The first one is in terms of special
symplectic structures whereas the latter is in terms of symplectic diffeomorphisms
and Morse families.

3.4. Classical Tulczyjew’s triple. This subsection is a quick review of [78]. So
that for the proofs and for more detailed discussions, we refer directly to [78]. Let $Q$
be a manifold which is assumed to be the configuration space of a physical system.
Tangent bundle $TQ$ is the velocity phase space, the cotangent bundle $T^* Q$
is the
momentum phase space of the dynamics. The tangent bundle $TT^* Q$ of $T^* Q$ carries
a symplectic two-form $\Omega_{TQ}$ that derives from two potential one-forms

$\Theta_1 = i_T \Omega_Q = \dot{p} \cdot dq - \dot{q} \cdot dp$,
$\Theta_2 = d_T \Theta_Q = \dot{p} \cdot dq + p \cdot d\dot{q}$.

(36)

Exterior derivatives of these one-forms are the same and define the symplectic two-
form

$\Omega_{TQ}^T = d\dot{p} \wedge dq + dp \wedge d\dot{q}$.

(37)

Note that, the difference $\Theta_2 - \Theta_1$ is an exact one-form. Actually, it is the exterior
derivative of coupling function

$\Delta(q, p, \dot{q}, \dot{p}) = \dot{q} \cdot p$.

(38)

By assuming $\Delta$ as a function on the Whitney sum $TQ \times_Q T^* Q$, we argue that
it generates a Lagrangian submanifold of $T^* TQ \times_T T^* T^* Q$. This Lagrangian
submanifold is the graph of the symplectic diffeomorphism

$T^* TQ \rightarrow T^* T^* Q: (q, \dot{q}, \dot{p}, \ddot{p}) \rightarrow (q, \dot{p}, p, -\dot{q})$.

(39)
The symplectic manifold \((TT^*Q, \Omega^T_Q)\) admits two special symplectic structures. The first special symplectic structure is due to the musical isomorphism \(\Omega^T_Q\) presented in (13). It is a matter of a direct calculation to show that quintuple
\[
(TT^*Q, \tau_{TT^*Q}, T^*Q, \Theta_1, \Omega^T_Q)
\]
is a special symplectic manifold. Here, \(\tau_{TT^*Q}\) is the tangent bundle projection from \(TT^*Q\) to \(T^*Q\), \(\Theta_1\) is the differential one-form defined in (36).

In order to construct a second special symplectic structure for \((TT^*Q, \Omega^T_Q)\), we introduce the followings. Consider a differential mapping \(\Gamma = \Gamma(t, s)\) from \(\mathbb{R}^2\) to \(Q\). The differential of \(\Gamma\) with respect to \(t\), at \(t = 0\), results with a curve \(\gamma(s)\) lying in the tangent bundle \(TQ\) depending on the free variable \(s\). If one further takes the differential of the curve \(\gamma(s) \subset TQ\) with respect to \(s\), at \(s = 0\), then arrives at a vector in \(TTQ\). The canonical involution \(\kappa_Q\) on \(TQ\) is defined by changing the order of differentiations
\[
\kappa_Q : TTQ \rightarrow TTQ : \frac{d}{dt}
\]
defined, for all \(X\) in \(TT^*Q\) and for all \(Y\) in \(TTQ\), as
\[
\langle \Xi_Q(X), Y \rangle = \langle X, \kappa_Q(Y) \rangle^T. 
\]
Here, pairing on the left hand side is the natural paring between \(T^*TQ\) and \(TTQ\) whereas pairing on the right hand side is the tangent lift of the natural paring between \(T^*Q\) and \(TQ\). Notice that, the mapping \(\Xi_Q\) can be considered as a dual of the canonical involution \(\kappa_Q\). Let us comment on the pairing \(\langle \cdot, \cdot \rangle^T\). See that \(X\) is a vector in \(TT^*Q\) so that there exists a curve \(\xi\) in \(T^*Q\) satisfying that \(X = t\xi(0)\). On the other hand, \(\kappa_Q(Y)\) is a vector in \(TTQ\) so that there exists a curve \(\psi\) in \(T^*Q\) satisfying that \(Y = t\psi(0)\). In this notation, the pairing is defined by
\[
\langle X, \kappa_Q(Y) \rangle^T = \frac{d}{dt}
\]

The mapping \(\Xi_Q\) is a symplectic diffeomorphism, and
\[
(TT^*Q, T\pi_Q, TQ, \Theta_2, \Xi_Q)
\]
is a special symplectic manifold. Here, \(T\pi_Q\) is the tangent mapping of the cotangent bundle projection \(\pi_Q\), \(\Theta_2\) is the differential one-form defined in (36).

As a result, we have derived two special symplectic structures, exhibited in (40) and (45), for the symplectic manifold \((TT^*Q, \Omega^T_Q)\). Tulczyjew’s triple is the combination of these two special symplectic structures in one commutative diagram
\[
T^*TQ \xrightarrow{\Xi_Q} TT^*Q \xrightarrow{\Omega^T_Q} T^*T^*Q. 
\]

Assume a local coordinate system \((q, \dot{q}, \delta q, \dot{\delta} q)\) on \(TTQ\). Canonical involution \(\kappa_Q\) becomes
\[
\kappa_Q(q, \dot{q}, \delta q, \dot{\delta} q) = (q, \delta q, \dot{q}, \dot{\delta} q). 
\]
ON THE GEOMETRY OF THE SCHMIDT-LEGENDRE TRANSFORMATION

In the local coordinates (35) on $TT^*Q$, the symplectic diffeomorphisms turn out to be

$$\Xi_Q(q, p, \dot{q}, \dot{p}) = (q, \dot{q}, \dot{p}, p),$$
$$\Omega_Q(q, p, \dot{q}, \dot{p}) = (q, p, \dot{p}, -\dot{q}).$$

(48)

Let us now define a Lagrangian submanifold of $TT^*Q$ using the left side of Tulczyjew’s triple (46) by starting with a Lagrangian function $L$ on $TQ$. The image space of the exterior derivative of $L$ is a Lagrangian submanifold of $T^*TQ$. The inverse of $\Xi_Q$ maps this Lagrangian submanifold $S_{TT^*Q}$ to a Lagrangian submanifold of $TT^*Q$. In a compact form, the Lagrangian submanifold $S_{TT^*Q}$ is defined by the equation

$$(T\pi_Q)^* dL = \Theta_2,$$

(49)

where $\Theta_2$ is the potential one-form in (36). In terms of the local coordinates (35) on $TT^*Q$, it is computed to be

$$S_{TT^*Q} = \left\{ \left( q, \dot{q}, \partial L / \partial \dot{q} \right) \in TT^*Q : L = L(q, \dot{q}) \right\}.$$

(50)

Dynamics determined by the Lagrangian submanifold $S_{TT^*Q}$ is computed simply by taking the time derivatives of the base coordinates $(q, \partial L / \partial \dot{q})$ and equate them to the fiber coordinates $(\dot{q}, \partial L / \partial q)$ respectively. By this, we arrive at the Euler-Lagrange equations (9). In other words, we say that $S_{TT^*Q}$ is a Lagrangian submanifold realization of the Euler-Lagrange equations (9). Note that, $S_{TT^*Q}$ can be generated from the right side of the Tulczyjew’s triple (46) as well. For this, define the following Morse family

$$E(q, \dot{q}, p) = p \cdot \dot{q} - L(q, \dot{q})$$

(51)

on the Whitney sum $TQ \times Q T^*Q$. To represent this generating family, we are drawing the right side of the Tulczyjew’s triple (46) by equipping it with $TQ \times Q T^*Q$ as follows

$$\begin{align*}
TT^*Q &\xrightarrow{\Omega_Q} T^*T^*Q & TQ \times T^*Q &\xrightarrow{E} \mathbb{R} \\
T_Q^* &\xrightarrow{\tau_{TT^*Q}} T^*T^*Q & T^*Q &\xrightarrow{pr_2} T^*Q \\
\end{align*}$$

(52)

Lagrangian submanifold generated by this Morse family is

$$\left\{ \left( q, p, \frac{\partial E}{\partial \dot{q}} - \frac{\partial E}{\partial q} \right) \in TT^*Q : \frac{\partial E}{\partial \dot{q}} = 0 \right\}.$$

(53)

A direct computation proves that the Lagrangian submanifold (53) and the Lagrangian submanifold $S_{TT^*Q}$ in (50) are the same. See that, this transformation is valid for degenerate Lagrangians as well. If the Lagrangian function is non-degenerate then from the equation

$$\frac{\partial E}{\partial q}(q, \dot{q}, p) = p - \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = 0$$

(54)

one can explicitly determine the velocity $\dot{q}$ in terms of $(q, p)$. In other words, for a non-degenerate Lagrangian function $L = L(q, \dot{q})$ the fiber derivative

$$\mathcal{F}L : TQ \rightarrow T^*Q : (q, \dot{q}) \rightarrow \left( q, \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right)$$

(55)
is a bijective mapping relating the tangent and cotangent bundles. In this case, the Morse family $E$ can be reduced to a well-defined Hamiltonian function

$$H(q, p) = \dot{q}(q, p) \cdot p - L(q, \dot{q}(q, p))$$

on $T^*Q$. The Lagrangian submanifold generated by the Hamiltonian function can compactly be defined by equation

$$-\tau^*_TQdH = \Theta_1,$$

where $\Theta_1$ is the one-form in (36).

3.5. Dirac-Bergmann constraint algorithm. The fiber derivative $\mathcal{F}L$ of a Lagrangian function $L$, defined in (55), is a mapping from the tangent bundle $TQ$ to the cotangent bundle $T^*Q$. If the Lagrangian is non-degenerate then $\mathcal{F}L$ becomes a bijection. This permits us to write the velocity variables in terms of the momenta. If the Lagrangian is degenerate, then $\mathcal{F}L$ is not bijective. The image $\mathcal{F}L(TQ)$ is only a submanifold, called as the primary constraint submanifold, of the cotangent bundle $T^*Q$. The graph

$$\Gamma_L = \left\{(q, p, \dot{q}) \in TQ \times Q : p - \frac{\partial L}{\partial \dot{q}} = 0 \right\}$$

of $\mathcal{F}L$ is a submanifold of the Whitney sum $TQ \times Q$. Note that, restriction $E|_{\Gamma_L}$ of the Morse family $E$, in (51), to the graph $\Gamma_L$ also generates the Lagrangian submanifold (53). This is a Morse family reduction. $E|_{\Gamma_L}$ is free of velocity variables ($\dot{q}$) since

$$\frac{\partial E|_{\Gamma_L}}{\partial \dot{q}} = \left(p - \frac{\partial L}{\partial \dot{q}}\right)|_{\Gamma_L} = 0.$$ 

This shows that there exists a function $H$ defined on an open cover of the image space $\mathcal{F}L(TQ)$ of the fiber derivative depending on the position and momenta $(q, p)$ such that

$$H(q, p) = E(q, p, \dot{q}), \quad (q, p, \dot{q}) \in \Gamma_L.$$ 

Notice that, $H$ satisfying (60) is far from being unique. Let us assume that there exists constraint functions $\Phi_\alpha = 0$ determining $\mathcal{F}L(TQ)$. By adding any linear combination of the constraint functions $\Phi_\alpha$, to $H$, one may arrive at some other functions satisfying (60). This observation leads to a more general expression called the total Hamiltonian function

$$H_T = H + u^\alpha \Phi_\alpha,$$

on $T^*Q$. Here, $u^\alpha$’s are Lagrange multipliers. The requirement that the solutions of Euler-Lagrange equations remain on constraint submanifold is described by the weak equality

$$\dot{\Phi}_\beta = \{\Phi_\beta, H\} + u^\alpha \{\Phi_\beta, \Phi_\alpha\} = 0, \quad \beta = 1, ..., 2n - r,$$

modulo primary constraints. Brackets on the right hand side are the canonical Poisson bracket (20) on the cotangent bundle $T^*Q$. The consistency conditions in (62) may lead to determination of the Lagrange multipliers or may result on a new set of constraints. If the consistency check results with a new set of constraints one should repeat the consistency checks by enlarging the constraint set with the introductions of secondary constraints. The process should be stopped when no more new constraints appear. At the end, one arrives at a well defined Hamiltonian function on a submanifold, called as the final constraint submanifold, of $\mathcal{F}L(TQ)$. 
This is the Dirac-Bergmann constraint algorithm [26, 27]. For a geometric version of the algorithm, we refer [34, 35, 36, 37].

3.6. Tulczyjew’s triple for the Ostrogradsky-Legendre transformation.

Traditional framework to obtain Hamiltonian formulation of a higher order Lagrangian formalism is due to Ostrogradsky [61]. Ostrogradsky’s approach is based on the idea that consecutive time derivatives of initial coordinates form new coordinates, hence a higher order Lagrangian function can be written in a form of a first order Lagrangian function on an iterated tangent bundle. In this subsection, we are exhibiting geometry of the Ostrogradsky-Legendre transformation by constructing a proper Tulczyjew’s triple [20, 48].

Let us first introduce an alternative notation for local coordinates of $T^kQ$

$$(q_0, q_1, ..., q_{k}) := (q, \dot{q}, ..., q^{(k)})$$

which is especially useful while working on the iterated tangent bundle $TT^kQ$. Accordingly, we denote the induced coordinates on $TT^kQ$ as

$$(q_0, q_1, ..., q_{k}) : TT^kQ \rightarrow \mathbb{R}^{2(k+1)n}.$$ 

In terms of the coordinates (63), the embedding in (5) reads that

$$T^{k+1}Q \rightarrow TT^kQ : (q_0, q_1, ..., q_{k+1})$$

$$\mapsto (q_{0}, q_{1}, ..., q_{k}(k), q_{1(1)}, q_{2}, ..., q_{k(k)}).$$

By replacing the configuration manifold $Q$ in the triple (46) with the $k$-th order tangent bundle $T^kQ$, we draw the following Tulczyjew’s triple

$$T^*TT^kQ \xrightarrow{\Xi_{T^kQ}} TT^*T^kQ \xrightarrow{\Omega_{T^*Q}^k} T^*T^*TQ.$$ 

Here, $\pi_{TT^kQ}$ is the cotangent bundle projection, $T\pi_{T^kQ}$ is the tangent lift of $\pi_{T^kQ}$, $\tau_{T^kT^*Q}$ is the tangent bundle projection, and $\pi_{T^*T^kQ}$ is the cotangent bundle projection. Being a cotangent bundle, $T^*T^kQ$ is a symplectic manifold with the canonical symplectic two-form $\Omega_{TT^kQ} = d\Theta_{TT^kQ}$. Darboux’s coordinates on $T^*T^kQ$ are

$$(q_0, q_1, ..., q_{k}, \pi^{(0)}, \pi^{(1)}, ..., \pi^{(k)}) : T^*T^kQ \rightarrow \mathbb{R}^{2(k+1)n}. $$

In this coordinate frame, the canonical forms are computed to be

$$\Theta_{TT^kQ} = \sum_{k=0}^{k} \pi^{(k)} \cdot dq^{(k)} \quad \Omega_{TT^kQ} = \sum_{k=0}^{k} d\pi^{(k)} \wedge dq^{(k)}.$$ 

On $TT^*T^kQ$, introduce the following local coordinate system

$$(q^{(k)}, \pi^{(k)}, \dot{q}^{(k)}), \tilde{T}^*T^kQ ~ \rightarrow \mathbb{R}^{4(k+1)n},$$ 

where $k$ runs from 0 to $k$. Tangent bundle of a symplectic manifold is a symplectic manifold with lifted symplectic two-form whose local description is available in (37). Under the light of this fact, we argue that $TT^*T^kQ$ is a symplectic manifold with lifted symplectic two-form $\Omega_{TT^kQ}$. In terms of the coordinates in (67), $\Omega_{TT^kQ}$ can be written as

$$\Omega_{TT^kQ}^{T^kQ} = \sum_{k=0}^{k} d\tilde{\pi}^{(k)} \wedge dq^{(k)} + \sum_{k=0}^{k} d\pi^{(k)} \wedge d\dot{q}^{(k)}.$$ 

Definitions of the symplectic diffeomorphism $\Xi_{T^*Q}$ and $\Omega^p_{T^*Q}$ are direct adaptations of the definitions of the symplectic diffeomorphism $\Xi_Q$ and $\Omega^p_Q$ presented in (48). Accordingly, they are computed as

$$\Xi_{T^*Q}(q_{(\kappa)}, \pi^{(\kappa)}, \dot{q}_{(\kappa)}, \dot{\pi}^{(\kappa)}) = (q_{(\kappa)}, \dot{q}_{(\kappa)}, \dot{\pi}^{(\kappa)}, \pi^{(\kappa)}),$$

(71)

$$\Omega^p_{T^*Q}(q_{(\kappa)}, \pi^{(\kappa)}, \dot{q}_{(\kappa)}, \dot{\pi}^{(\kappa)}) = (q_{(\kappa)}, \pi^{(\kappa)}, \dot{\pi}^{(\kappa)}, -\dot{q}_{(\kappa)}).$$

(72)

We define the following fiber bundle structure

$$(TT^kQ \times \mathbb{R}^{k \times n}, pr_1, TT^kQ)$$

(73)

where $pr_1$ is the projection form the product space to $TT^kQ$. Fibers of the bundle are $(k \times n)$-dimensional Euclidean spaces $\mathbb{R}^{k \times n}$ equipped with global coordinates $(\lambda^0, ..., \lambda^{k-1})$ where each of $\lambda^{(\delta)}$ are belonging to $\mathbb{R}^n$ for $\delta = 0, 1, ..., k - 1$. Let us start with a Lagrangian function $L$ on $T^{k+1}Q$. By referring to the natural embedding of $T^{k+1}Q$ into $TT^kQ$, defined in (65), we introduce the following Morse family

$$L_O = L(q_0, q_{(1)}, ..., q_{(k)}, \dot{q}_{(k)}) + \sum_{\delta=0}^{k-1} \lambda^{(\delta)} \cdot (\dot{q}_{(\delta)} - q_{(\delta+1)})$$

(74)

on the total space of the fiber bundle (73). Note that, $\dot{q}_{(k)}$ refers to the $(k+1)$-th order derivative of the position $q$. The Morse family $L_O$ generates a Lagrangian submanifold of the symplectic manifold $(TT^kQ, \Omega^p_{T^*Q})$. In order to exhibit this, we redraw the left side of the Tulczyjew’s triple (66) equipped with the fiber bundle (73).

\[
\begin{array}{ccc}
\mathbb{R} & \xleftarrow{L_O} & TT^kQ \times \mathbb{R}^{k \times n} \\
\downarrow{pr_1} & & \downarrow{\pi^*_{TT^kQ}} \\
TT^kQ & \xrightarrow{T^*TT^kQ} & TT^*T^kQ
\end{array}
\]

(75)

Following the general description in (50), in terms of the induced coordinates (69) on $T^*TT^kQ$, the Lagrangian submanifold generated by $L_O$ is computed to be

$$\{(q_0, ..., q_{(k)}, \lambda^0, ..., \lambda^{(k-1)}, \frac{\partial L}{\partial q_{(k)}}, \dot{q}_0, ..., \dot{q}_k) : \frac{\partial L}{\partial q_{(k)}} - \lambda^0, ..., \frac{\partial L}{\partial q_k} - \lambda^{(k-1)} \}$$

(76)

equipped with the constraints

$$\dot{q}_{(\delta)} = q_{(\delta+1)}, \quad \delta = 0, 1, ..., k-1.$$  

(77)

The dynamical equations defined by this Lagrangian submanifold are

$$\frac{d}{dt}q_{(\kappa)} = \dot{q}_{(\kappa)}, \quad \kappa = 0, 1, ..., k,$$

(78)

$$\frac{d}{dt}\lambda^{(\delta)} = \frac{\partial L}{\partial q_{(\delta)}} - \lambda^{(\delta-1)}, \quad \delta = 0, 1, ..., k-1,$$

(79)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{(k)}} = \frac{\partial L}{\partial q_{(k)}} - \lambda^{(k-1)},$$

(80)

where $\lambda^{(-1)}$ is assumed to be 0. It is easy now to show that this system is equivalent to the Euler-Lagrange equations generated by the $(k+1)$-th order Lagrangian
function $L$. In other words, we say that (76) with (77) is a Lagrangian submanifold realization of the $(k + 1)$-th order Euler-Lagrange equations.

In Tulczyjew’s approach, the Legendre transformation of the $(k + 1)$-th order Euler-Lagrange equations is to regenerate the Lagrangian submanifold (76)-(77) by referring to the right side of the Tulczyjew’s triple (66). To this end, we are now considering the following fiber bundle structure

$$(TT^kQ \times T^*Q, TT^kQ \times \mathbb{R}^{(k \times n)}, pr_1, T^*T^kQ).$$

We take the following local coordinates on the total space of the fiber bundle considering the following fiber bundle structure $E$

$$\kappa \in (83)$$

generates a Lagrangian submanifold of $E$.

Referring to (53), it is a matter of a direct calculation to show that the Morse family

$$E = \sum_{\kappa = 0}^k q_{(\kappa)} \cdot \pi^{(\kappa)} - L(q_{(0)}, q_{(1)}, \ldots, q_{(k)}), \dot{q}_{(\delta)} - \sum_{\kappa = 0}^{k-1} \lambda^{(\delta)} \cdot (q_{(\delta)} - q_{(\delta + 1)}),$$

(83)

We redraw the right side of the Tulczyjew’s triple (66) by adding the fibration (81)

$$TT^*Q \xrightarrow{\Omega_{T^*Q}} T^*T^kQ \xrightarrow{\pi_{T^*Q}} T^*T^kQ \xrightarrow{pr_1} T^*T^kQ.$$  

Referring to (83), it is a matter of a direct calculation to show that the Morse family $E$ in (83) generates a Lagrangian submanifold of $TT^*T^kQ$ given by

$$\{q_{(0)}, \ldots, q_{(1)}, \ldots, q_{(k)}, \pi^{(0)}, \ldots, \pi^{(k)}, \dot{q}_{(0)}, \ldots, \dot{q}_{(k)}, \frac{\partial L}{\partial q_{(0)}}, \ldots, \frac{\partial L}{\partial q_{(k-1)}}, \lambda^{(0)}, \ldots, \lambda^{(k)}, \frac{\partial L}{\partial q_{(k)}}\}$$

equipped with the constraints

$$\pi^{(k)} = \frac{\partial L}{\partial q_{(k)}}, \quad \dot{q}_{(\delta)} = q_{(\delta + 1)}, \quad \lambda^{(\delta)} = \pi^{(\delta)}, \quad \delta = 0, \ldots, k - 1.$$  

(85)

Note that, this Lagrangian submanifold is exactly the same with the one presented in (76). Substitutions of the Lagrange multipliers $\lambda^{(\delta)}$ into the Morse family $E$ in (83) result with a reduction of the family given by

$$E = \pi^{(k)} \cdot \dot{q}_{(k)} + \sum_{\delta = 0}^{k-1} \pi^{(\delta)} \cdot q_{(\delta + 1)} - L(q_{(0)}, q_{(1)}, \ldots, q_{(k)}), \dot{q}_{(k)}).$$

(87)

Note that, this Morse family is defined on the Whitney sum $T^*T^kQ \times T^*Q TT^kQ$. Further reduction is possible if $\dot{q}_{(k)}$ can be written in terms of the positions and the momenta. This is possible if the matrix $[\partial^2 L/\partial \dot{q}_{(k)}^2]$ is non-degenerate. If this Hessian condition holds then we arrive at a well-defined Hamiltonian function on $T^*T^kQ$. If not, one may use the Dirac-Bergmann algorithm, summarized in the subsection (3.5), in order to arrive at a well-defined Hamiltonian function on a proper submanifold of $T^*T^kQ$. Let us finish this subsection by exhibition of the Ostrogradsky momenta

$$\pi^{(\kappa)} = \sum_{j=\kappa}^k \left(-\frac{d}{dt}\right)^{j-\kappa} \frac{\partial L}{\partial q^{(\kappa + 1)}}$$

(88)
where \( \kappa \) runs from 0 to \( k \).

4. The Schmidt-Legendre transformation.

4.1. Acceleration bundle. Let \( Q \) be a manifold. Consider the set \( C_q(Q) \) of smooth curves passing through the point \( q \) in \( Q \). We define a subset \( K_q(Q) \) of \( C_q(Q) \) by considering only the curves whose first derivatives are vanishing at \( q \). More formally,

\[
K_q(Q) = \{ \gamma \in C_q(Q) : D(f \circ \gamma)(0) = 0, \ \forall f : Q \to \mathbb{R} \}.
\]

(89)

It is worthless to say that since vanishing of the first derivative is asked only at a single point, the curve \( \gamma \) in \( K_q(Q) \) does not have to be a constant curve. We are now defining an equivalence relation on \( K_q(Q) \). We call two curves \( \gamma \) and \( \gamma' \) in \( K_q(Q) \) equivalent if the second derivatives of \( \gamma \) and \( \gamma' \) are equal at \( q \), that is if

\[
D^2(f \circ \gamma)(0) = D^2(f \circ \gamma')(0),
\]

for all real valued functions \( f \) on \( Q \). Here, it is assumed that \( \gamma(0) = \gamma'(0) = q \). An equivalence class is denoted by \( a_\gamma(0) \). The set of all equivalence classes is called acceleration space \( A_q Q \) at \( q \in Q \). If \( Q \) is an \( n \)-dimensional manifold with local coordinates \((q)\), then union of all acceleration spaces

\[
AQ = \bigsqcup_{q \in Q} A_q Q.
\]

is a 2\( n \)-dimensional manifold with induced local coordinates

\[
(q, a) : AQ \to \mathbb{R}^{2n} : a \gamma(0) \to (q \circ \gamma(0), D^2(q \circ \gamma)(0)).
\]

(90)

Assume that \((q)\) and \((x)\) be two compatible charts around a point \( q \) in \( Q \). Then, the induced local charts on \( AQ \), given by \((q, a)\) and \((x, b)\), are also compatible. Transformations relating these two local pictures are computed to be

\[
x = x(q), \quad b = \left( a \cdot \frac{\partial}{\partial q} \right) x.
\]

(91)

These coordinate transformations suggest a vector bundle structure of \( AQ \) over \( Q \) with projection

\[
\alpha_Q : AQ \to Q : a \gamma(0) \to \gamma(0).
\]

(92)

We call \( AQ \) the acceleration bundle of \( Q \).

One can understand \( AQ \) as a subbundle of \( T^2 Q \to Q \) as follows

\[
AQ = \{ X \in T^2 Q : \tau_Q^2(X) = 0 \in TQ \},
\]

(93)

where \( \tau_Q^{12} \) is projection from \( T^2 Q \) to \( TQ \) defined as

\[
\tau_Q^{12} : T^2 Q \to TQ : t^2 \gamma(0) \to t^1 \gamma(0).
\]

(94)

In terms of the local coordinates, the embedding of \( AQ \) into \( T^2 Q \) is computed to be

\[
\iota : AQ \to T^2 Q : (q, a) \to (q, 0, a).
\]

(95)

If a linear connection is defined on \( Q \) then \( T^2 Q \) becomes a vector bundle over \( Q \) \([28, 73]\). In this case, we can consider the following short exact sequence of vector bundles

\[
0 \to AQ \to T^2 Q \to TQ \to 0
\]

where the second mapping \( AQ \to T^2 Q \) is the one presented in (93), and the third mapping \( T^2 Q \to TQ \) is the projection \( \tau_Q^{12} \) defined in (94).
Alternatively, the acceleration bundle \( AQ \) can be defined as a subbundle of iterated tangent bundle \( TTQ \) as follows

\[
AQ = \{ X \in TTQ : T\tau_Q(X) = \tau_{TQ}(X) = 0 \},
\]

(96)

where \( T\tau_Q \) is the tangent lift of projection \( \tau_Q : TQ \rightarrow Q \), and \( \tau_{TQ} \) is the tangent bundle projection \( TTQ \rightarrow TQ \). Let us establish a relationship between the two definitions of \( AQ \) given in (93) and (96). Recall the embedding of the second order tangent bundle \( T^2Q \) into the iterated bundle \( TTQ \) defined in (6). An element of \( X \in TTQ \) is in the image space of this embedding if it satisfies the equality

\[
T\tau_Q(X) = \tau_{TQ}(X) = 0.
\]

(97)

In the literature, a vector field \( X \) satisfying (97) is called a second order vector field. Notice that, a second order vector is a fixed point of the involution map \( \kappa_Q \) in (41). Accordingly, acceleration bundle can be identified with intersection

\[
AQ = VTQ \cap \epsilon(T^2Q),
\]

(98)

where \( VTQ \) is the vertical bundle consisting of vectors on \( TQ \) projecting to the zero vectors on \( Q \) via the mapping \( T\tau_Q \) whereas \( \epsilon(T^2Q) \) is the space of second order vectors in \( TTQ \).

4.1.1. Acceleration rhombic. Let \( \phi \) be a differential mapping from a manifold \( Q \) to another manifold \( M \). Consider an element \( a_\gamma(0) \) in \( A_\gamma Q \). The mapping \( \phi \) maps the curve \( \gamma \) to a curve \( \phi \circ \gamma \) in \( M \). Notice that, tangent vector to the curve \( \phi \circ \gamma \) at \( \phi(q) \) is zero. That is, \( \phi \circ \gamma \) is an element of \( K_{\phi(q)}(M) \), and \( a(\phi \circ \gamma)(0) \) is an element of the acceleration space \( A_{\phi(q)}M \). Hence, we define the acceleration lift of a differentiable mapping \( \phi \) by

\[
A\phi : AQ \rightarrow AM : a_\gamma(0) \rightarrow a(\phi \circ \gamma)(0).
\]

(99)

This definition also satisfies the following commutative diagram.

\[
\begin{array}{ccc}
AQ & \xrightarrow{A\phi} & AM \\
\downarrow{\alpha_Q} & & \downarrow{\alpha_M} \\
Q & \xrightarrow{\phi} & M
\end{array}
\]

(100)

Here, \( \alpha_Q \) and \( \alpha_M \) are the acceleration bundle projections defined in (92).

A tangent vector can be represented by a curve on the base manifold. Accordingly, we can represent a vector \( X \) in the iterated tangent bundle \( TAQ \) by a differential mapping \( \Gamma \) from \( \mathbb{R}^2 \) to the manifold \( Q \) as follows. By taking the differential of \( \Gamma = \Gamma(t,s) \) with respect to \( s \) at \( s = 0 \) we arrive at a curve in \( AQ \) depending on the free parameter \( t \). Denote this curve by \( \Upsilon(t) \subset AQ \). By taking the differential of \( \Upsilon(t) \) with respect to the parameter \( t \) at \( t = 0 \), we arrive at

\[
X = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \Gamma(t,s).
\]

(101)

Using this representation, we define the following projections

\[
\tau_{AQ} : TAQ \rightarrow AQ : \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \Gamma(t,s) \rightarrow \left. \frac{d}{ds} \right|_{s=0} \Gamma(0,s)
\]

(102)

\[
T\alpha_{AQ} : TAQ \rightarrow TQ : \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \Gamma(t,s) \rightarrow \left. \frac{d}{dt} \right|_{t=0} \Gamma(t,0).
\]

(103)
Here, $T\alpha_Q$ is the tangent lift of acceleration bundle projection $\alpha_Q$. These two fibrations determine a double vector structure on $TAQ$ summarized in the following diagram.

\[
\begin{array}{c}
TAQ \\
\tau_{AQ} \swarrow \searrow T\alpha_Q \\
AQ \\
\alpha_Q \swarrow \searrow \tau_Q \\
Q
\end{array}
\]

We call this commutative diagram an acceleration rhombic motivating by the tangent rhombic presented in (17).

Assuming a local coordinate system $(q, a)$ on $AQ$, we introduce a local coordinate system on the tangent bundle $TAQ$ of $AQ$ as follows

\[(q, a, \dot{q}, \dot{a}) : TAQ \rightarrow \mathbb{R}^{4n}. \quad (104)\]

In terms of the coordinates (104), the local picture of the mapping $\Gamma$ in (101) is

\[
\Gamma(t, s) = q(X) + \dot{q}(X)t + a(X)\frac{s^2}{2} + \dot{a}(X)\frac{ts^2}{2}. \quad (105)
\]

whereas the projections presented in (102) and (103) are computed to be

\[
\tau_{AQ} : TAQ \rightarrow AQ : (q, a, \dot{q}, \dot{a}) \rightarrow (q, a), \quad T\alpha_Q : TAQ \rightarrow TQ : (q, a, \dot{q}, \dot{a}) \rightarrow (q, \dot{q}),
\]

respectively.

An element $Y$ in $ATQ$ can be represented by a differential mapping $\Sigma$ from $\mathbb{R}^2$ to the manifold $Q$. By taking the derivative of $\Sigma = \Sigma(t, s)$ with respect to $t$ at $t = 0$ we arrive at a curve in $TQ$ depending on the free parameter $s$. Denote this curve by $Y(s)$. By taking the derivative of $Y(s)$ with respect to the parameter $s$ at $s = 0$, we arrive at

\[
Y = \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} \Sigma(t, s). \quad (106)
\]

Using this representation, two fibrations of $ATQ$ can be defined as follows

\[
\alpha_{TQ} : ATQ \rightarrow TQ : \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} \Sigma(t, s) \rightarrow \left. \frac{d}{dt} \right|_{t=0} \Sigma(t, 0) \quad (107)
\]

\[
A\tau_Q : ATQ \rightarrow AQ : \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} \Sigma(t, s) \rightarrow \left. \frac{d}{ds} \right|_{s=0} \Sigma(0, s), \quad (108)
\]

respectively. Here, $\alpha_{TQ}$ is the acceleration bundle projection whereas $A\tau_Q$ is the acceleration lift of tangent bundle projection $\tau_Q$. Here is the diagram

\[
\begin{array}{c}
ATQ \\
\alpha_{TQ} \swarrow \searrow A\tau_Q \\
TQ \\
\tau_Q \swarrow \searrow \tau_Q \\
AQ \\
\alpha_Q \swarrow \searrow \tau_Q \\
Q
\end{array}
\]

relating two projections manifesting the double vector bundle structure of $ATQ$. 


Consider the following coordinates
\[(q, \dot{q}, a_q, \ddot{a}_q) : ATQ \rightarrow \mathbb{R}^{4n} \quad (109)\]
on acceleration bundle \(ATQ\) of \(TQ\). In terms of the coordinates (109), local picture of \(\Sigma\) in (106) is
\[\Sigma(t, s) = q(Y) + \dot{q}(Y)t + a_q(Y)\frac{s^2}{2} + \ddot{a}_q(Y)\frac{ts^2}{2}. \quad (110)\]
The projections (107) and (108) are
\[\alpha_{TQ} : ATQ \rightarrow TQ : (q, \dot{q}, a_q, \ddot{a}_q) \rightarrow (q, \dot{q}) \quad (111)\]
\[A\tau : ATQ \rightarrow AQ : (q, \dot{q}, a_q, \ddot{a}_q) \rightarrow (q, a_q), \quad \text{respectively.} \]

Recall the differential mapping \(\Gamma\) presented in (105). By changing the order of differentiations, we define an isomorphism from \(TAQ\) to \(ATQ\) as follows
\[\Phi : TAQ \rightarrow ATQ : \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \Gamma(t, s) \rightarrow \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \Gamma(t, s). \quad (112)\]

In terms of the local pictures, the isomorphism \(\Phi\) can be computed as
\[\Phi : TAQ \rightarrow ATQ : (q, a, \dot{a}, \ddot{a}) \rightarrow (q, \dot{q}, a, \ddot{a}). \quad (112)\]

Let us now exhibit relationship between the third order tangent bundle \(T^3Q\) and the tangent bundle of the acceleration bundle \(TAQ\). Recall the embedding
\[f : T^3Q \rightarrow TT^2Q : (q, \dot{q}, \ddot{q}, \dot{q}^{(3)}) \rightarrow (q, \dot{q}, \ddot{q}, \dddot{q}, \dddot{q}^{(3)}). \quad (113)\]
The image \(f(T^3Q)\) is a 4n-dimensional submanifold of \(TT^2Q\). \((f(T^3Q), \tau, T^2Q)\) is a subbundle of the tangent bundle \(TT^2Q\). Here, \(\tau\) denotes restriction of the tangent bundle projection \(\tau_{TT^2Q}\) to \(f(T^3Q)\). Recall the pullback bundle definition summarized in the diagram (22), and consider the following particular case.
\[
\begin{array}{ccc}
\iota^* f(T^3Q) & \xrightarrow{\varepsilon} & f(T^3Q) \\
\iota^* \tau & \downarrow & \downarrow \tau \\
AQ & \xrightarrow{\iota} & T^2Q
\end{array}
\quad (114)
\]

Here, \(\iota\) is the inclusion in (95). According to (21), the total space of the pullback bundle \((\iota^* f(T^3Q), \iota^* \tau, AQ)\) is a 4n-dimensional manifold with local coordinates
\[(q, \dot{q}, \ddot{q}, \dot{q}^{(3)} : \iota^* f(T^3Q) \rightarrow \mathbb{R}^{4n}. \quad (115)\]

In this local picture, the projection \(\iota^* \tau\) maps the four tuple in (115) to the first two components. The local representation of total space in (115) suggests a local diffeomorphism from \(\iota^* f(T^3Q)\) to the tangent bundle \(TAQ\) of the acceleration bundle. To have this, start with an element \(Z\) in \(\iota^* f(T^3Q)\), and define a mapping \(\Lambda\) from \(\mathbb{R}^2\) to \(Q\) given by
\[\Lambda(t, s) = q(Z) + \dot{q}(Z)t + \ddot{q}(Z)\frac{s^2}{2} + \dddot{q}^{(3)}(Z)\frac{ts^2}{2}. \quad (116)\]
This mapping represents an element in \(TAQ\) which is locally in form \((q, \dot{q}, \ddot{q}, \dot{q}^{(3)}\). Accordingly, we have the following local diffeomorphism
\[\varepsilon : \iota^* f(T^3Q) \simeq TAQ \rightarrow f(T^3Q) \simeq T^3Q : (q, \dot{q}, \ddot{q}, \dot{q}^{(3)}) \rightarrow (q, \dot{q}, \ddot{q}, \dddot{q}, \dot{q}^{(3)}). \quad (117)\]
4.2. **Gauge symmetry of second order Lagrangian formalisms.** Consider a second order Lagrangian function

\[ L = L(q, \dot{q}, \ddot{q}) \quad (118) \]

on \( T^2Q \). Add the total derivative of an arbitrary function \( F = F(q, \dot{q}, \ddot{q}) \) to the Lagrangian function in order to arrive at

\[
\dot{L}(q, \dot{q}, \ddot{q}, q^{(3)}) = L(q, \dot{q}, \ddot{q}) + \frac{d}{dt}F(q, \dot{q}, \ddot{q})
\]

\[
= L(q, \dot{q}, \ddot{q}) + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial \dot{q}} \cdot \ddot{q} + \frac{\partial F}{\partial \ddot{q}} \cdot q^{(3)}.
\quad (119)
\]

Note that, \( \dot{L} \) is a third order Lagrangian function defined on \( T^3Q \) whereas both \( L \) and \( F \) are second order Lagrangians defined on \( T^2Q \). A straightforward computation proves that, the third order Euler-Lagrange equations generated by \( \dot{L} \) are the same with the second order Euler-Lagrange equations generated by \( L \). To see this directly, let us compute the third order Euler-Lagrange equations

\[
\frac{d^3}{dt^3} \frac{\partial \dot{L}}{\partial q^{(3)}} - \frac{d^2}{dt^2} \frac{\partial \ddot{L}}{\partial q} + \frac{d}{dt} \frac{\partial \dot{L}}{\partial q} - \frac{\partial L}{\partial q} = 0
\]

for the Lagrangian function \( \dot{L} \) in \( (119) \). So that the left hand side is

\[
\frac{d^3}{dt^3} \frac{\partial F}{\partial q} - \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial \dot{q} \partial q} \cdot \dot{q} + \frac{\partial^2 F}{\partial \ddot{q} \partial q} \cdot \ddot{q} + \frac{\partial^2 F}{\partial \dot{q} \partial \ddot{q}} \cdot q^{(3)} \right)
\]

\[
+ \frac{d}{dt} \left( \frac{\partial L}{\partial q} + \frac{\partial F}{\partial q} \right) + \frac{d}{dt} \left( \frac{\partial^2 F}{\partial \dot{q} \partial q} \cdot \dot{q} + \frac{\partial^2 F}{\partial \ddot{q} \partial q} \cdot \ddot{q} + \frac{\partial^2 F}{\partial \dot{q} \partial \ddot{q}} \cdot q^{(3)} \right)
\]

\[
- \left( \frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial \ddot{q} \partial q} \cdot \ddot{q} + \frac{\partial^2 F}{\partial \dot{q} \partial \ddot{q}} \cdot q^{(3)} \right)
\]

\[
= \frac{d^3}{dt^3} \frac{\partial F}{\partial q} - \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial q} + \frac{d}{dt} \left( \frac{\partial F}{\partial q} \right) \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial F}{\partial q} \right) - \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial F}{\partial q} \right)
\]

\[
= \frac{d^3}{dt^3} \frac{\partial F}{\partial q} - \frac{d^2}{dt^2} \frac{\partial L}{\partial q} - \frac{d^2}{dt^3} \left( \frac{\partial F}{\partial q} \right) - \frac{d^2}{dt^2} \frac{\partial F}{\partial q} + \frac{d}{dt} \frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial F}{\partial q} + \frac{d}{dt} \frac{\partial F}{\partial q}
\]

\[
- \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial F}{\partial q} \right)
\]

\[
= - \frac{d^2}{dt^2} \frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial L}{\partial q} - \frac{\partial L}{\partial q}
\]

This is a gauge invariance of the second order Euler-Lagrange equations. One of the major role in the Schmidt-Legendre transformation is played by this gauge invariance property [3, 66, 67]. Accordingly, in this section, we are discussing this invariance in the framework of acceleration bundle.

By recalling the mapping \( \varepsilon \) presented in Eq.\((117)\), we pull the Lagrangian \( \dot{L} \) in Eq.\((119)\) back to the tangent bundle \( TAQ \). This results with a first order Lagrangian function

\[
L_S : TAQ \rightarrow \mathbb{R} : (q, a, \dot{q}, \dot{a}) \rightarrow L(q, \dot{q}, a) + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial \dot{q}} \cdot \ddot{q} + \frac{\partial F}{\partial a} \cdot \dot{a}
\quad (120)
\]
defined on the first order tangent bundle $TAQ$. Euler-Lagrange equations generated by $L_S$ in (120) are computed to be
\[
\frac{\partial L_S}{\partial q} - \frac{d}{dt} \frac{\partial L_S}{\partial \dot{q}} = 0, \quad \frac{\partial L_S}{\partial a} - \frac{d}{dt} \frac{\partial L_S}{\partial \dot{a}} = 0.
\] (121)

The second set of equations in (121) is computed to be
\[
\left( \frac{\partial L}{\partial a} + \frac{\partial F}{\partial \dot{q}} \right) + \frac{\partial^2 F}{\partial q \partial a} \cdot (a - \ddot{q}) = 0.
\] (122)

Here and in the forthcoming equations we are using abbreviation
\[
\frac{\partial^2 F}{\partial q \partial a} \cdot a := \frac{\partial^2 F}{\partial q^i \partial a^j} a^j,
\] (123)

where on the right hand side summation is understood for the repeated indices.

Assume particularly that $L$ is a non-degenerate Lagrangian that is the rank of the Hessian matrix $[\frac{\partial^2 L}{\partial a^2}]$ is full, and assume also that the auxiliary function $F$ satisfies
\[
\frac{\partial L}{\partial a} + \frac{\partial F}{\partial \dot{q}} = 0.
\] (124)

The non-degeneracy of $[\frac{\partial^2 L}{\partial a^2}]$ forces non-degeneracy of the matrix $[\frac{\partial^2 F}{\partial a \partial \dot{q}}]$. So that, the equations (122) reduce to set of constraints $a - \ddot{q} = 0$. In this case, the first set in (121) result with the same Euler-Lagrange equations generated by $L$ in (118). Let us take the partial derivative of (124) with respect to $\dot{q}$. This brings an integrability condition on $F$ as follows. The symmetry of the matrix $[\frac{\partial^2 F}{\partial q \partial a}]$ enforces the symmetry of the matrix $[\frac{\partial^2 L}{\partial a \partial \dot{q}}]$. In other words, to define an auxiliary function satisfying (124), the matrix $[\frac{\partial^2 L}{\partial a \partial \dot{q}}]$ has to be symmetric, that is
\[
[\frac{\partial^2 L}{\partial a \partial \dot{q}}] = [\frac{\partial^2 L}{\partial \dot{q} \partial a}].
\] (125)

We shall discuss degenerate second order Lagrangians in the Subsection (4.5).

4.3. Tulczyjew’s triple for the acceleration bundle. In this section, we build Tulczyjew’s triple over an acceleration bundle. Let us first draw the diagram then start to comment on it.

\[
\begin{array}{ccc}
T^*TAQ & \xleftarrow{\Xi_{AQ}} & TT^*AQ \\
\pi_{T^*AQ} & & \Omega^*_{AQ} \\
TAQ & \xrightarrow{T\pi_{AQ}} & T^*AQ \\
\end{array}
\] (126)

Assume the following coordinates
\[
(q, a, p_q, p_a, \dot{q}, \dot{a}, \dot{p}_q, \dot{p}_a) : TT^*AQ \to \mathbb{R}^{8n}
\] (127)
on the tangent bundle $TT^*AQ$. See that $(TT^*AQ, \Omega^T_{AQ})$ is a symplectic manifold with the lifted symplectic two-form $\Omega^T_{AQ}$. Referring to the local definition in (37), $\Omega^T_{AQ}$ is computed be
\[
\Omega^T_{AQ} = d\dot{p}_q \wedge dq + \dot{p}_a \cdot da + dp_q \wedge d\dot{q} + dp_a \wedge d\dot{a}.
\]

$\Omega^T_{AQ}$ admits two potential one-forms $\vartheta_1$ and $\vartheta_2$ given by
\[
\vartheta_1 = \dot{p}_q \cdot dq + \dot{p}_a \cdot da - \dot{q} \cdot dp_q - \dot{a} \cdot dp_a,
\] (128)
\[
\vartheta_2 = \dot{p}_q \cdot dq + \dot{p}_a \cdot da + p_q \cdot d\dot{q} + p_a \cdot d\dot{a}.
\] (129)
Note that, the difference of potential one-forms is an exact one-form
\[ \vartheta_2 - \vartheta_1 = d(p_\dot{q} \cdot \dot{q} + p_\dot{a} \cdot \dot{a}) \]
on \(T^*A^Q\). Referring to the local definitions in (48), we arrive at the symplectic diffeomorphisms
\[
\Xi_{AQ}(q, a, p_q, p_a, \dot{q}, \dot{a}, \dot{p}_q, \dot{p}_a) = (q, a, \dot{q}, \dot{a}, \dot{p}_q, \dot{p}_a, p_q, p_a), \quad (130)
\]
\[
\Omega^*_{AQ}(q, a, p_q, p_a, \dot{q}, \dot{a}, \dot{p}_q, \dot{p}_a) = (q, a, p_q, p_a, \dot{p}_q, \dot{p}_a, -\dot{q}, -\dot{a}) \]. \quad (131)

4.4. The Schmidt-Legendre transformation for the second order Lagrangians. Recall the Lagrangian function \(L_S\) defined in (120). The image of \(dL_S\) is a Lagrangian submanifold of the cotangent bundle \(T^*TAQ\). Inverse of the symplectic diffeomorphism \(\Xi_{AQ}\) maps this Lagrangian submanifold to a Lagrangian submanifold of \(T^*A^Q\) given, compactly, by
\[
(T\pi_{AQ})^*dL_S = \vartheta_2. \quad (132)
\]
Here, \(\vartheta_2\) is the potential form in (129), and \(T\pi_{AQ}\) is tangent mapping of the projection \(\pi_{AQ}\). Referring to (50), according to the local coordinates (127), a local representation of the Lagrangian submanifold of \(T^*A^Q\) generated by \(L_S\) is computed to be
\[
p_q = \frac{\partial L_S}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot \dot{q} + \frac{\partial F}{\partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial q \partial a} \cdot \dot{a}, \quad (133)
\]
\[
p_a = \frac{\partial L_S}{\partial \dot{a}} = \frac{\partial L}{\partial \dot{a}} + \frac{\partial^2 F}{\partial q \partial \dot{a}} \cdot \dot{q} + \frac{\partial F}{\partial \dot{a}} \cdot a + \frac{\partial^2 F}{\partial q \partial a} \cdot \dot{a}, \quad (134)
\]
\[
\dot{p}_q = \frac{\partial L_S}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot \dot{q} + \frac{\partial F}{\partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial q \partial a} \cdot \dot{a}, \quad (135)
\]
\[
\dot{p}_a = \frac{\partial L_S}{\partial a} = \frac{\partial L}{\partial a} + \frac{\partial^2 F}{\partial q \partial \dot{a}} \cdot \dot{q} + \frac{\partial F}{\partial \dot{a}} \cdot a + \frac{\partial^2 F}{\partial q \partial a} \cdot \dot{a}, \quad (136)
\]
where \((q, a, \dot{q}, \dot{a})\) being free.

Notice that, while arriving at the Lagrangian submanifold (133)-(136), we have referred to the left side of the Tulczyjew’s triple in (126). To generate the Lagrangian submanifold once more from the right side of the triple, motivating from (51), we define the following Morse family
\[
E(q, a, p_q, p_a, \dot{q}, \dot{a}) = p_q \cdot \dot{q} + p_a \cdot \dot{a} - L_S(q, a, \dot{q}, \dot{a}) \quad (137)
\]
\[
= p_q \cdot \dot{q} + p_a \cdot \dot{a} - L(q, \dot{q}, a) + \frac{\partial F}{\partial q} \cdot \dot{q} - \frac{\partial F}{\partial a} \cdot a - \frac{\partial F}{\partial \dot{a}} \cdot \dot{a}
\]
on the Whitney sum \(TAQ \times_{AQ} T^*AQ\). The right side of Tulczyjew’s triple in (126) and the Morse family \(E\) are summarized in the diagram
\[
\begin{align*}
T^*A^Q & \xrightarrow{T^*T^*A^Q} T^*A^Q \xrightarrow{E} \mathbb{R}. \quad (138)
\end{align*}
\]
cotangent bundle 
\[ \mathcal{T}^* T^* AQ \longrightarrow \mathbb{R}^{8n}. \]

In terms of this coordinate system, and under the light of the local description presented in Eq. (24), Lagrangian submanifold of \( T^* T^* AQ \) generated by the Morse family \( E \) is computed to be
\[
\Pi_q = \partial E_{L \rightarrow H}^{\text{LS}} \frac{\partial}{\partial q}, \quad \Pi_a = \partial E_{L \rightarrow H}^{\text{LS}} \frac{\partial}{\partial a}, \quad \phi_q = \partial E_{L \rightarrow H}^{\text{LS}} \frac{\partial}{\partial p_q}, \quad \phi_a = \partial E_{L \rightarrow H}^{\text{LS}} \frac{\partial}{\partial p_a}, \quad \Pi_q = \partial F \frac{\partial}{\partial p_q}, \quad \Pi_a = \partial F \frac{\partial}{\partial p_a}, \quad \phi_q = \partial F \frac{\partial}{\partial a}, \quad \phi_a = \partial F \frac{\partial}{\partial a}. \quad (139)
\]

Inverse musical isomorphism \( \Omega^*_{AQ} \) maps this Lagrangian submanifold to the Lagrangian submanifold presented in (133-136). Alternatively, for a direct passage to the Lagrangian submanifold (133-136) from the Morse family \( E \) in (137), one may apply the definition in (53).

Two equations in the last line (141) define canonical momenta
\[
p_q = \frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial q} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot \dot{q} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial q \partial a} \cdot \dot{a}, \quad (142)
\]
\[
p_a = \frac{\partial F}{\partial a} (q, \dot{q}, a). \quad (143)
\]

We substitute the momenta (143) into the definition of the Morse family (137). This substitution makes the Morse family free of \( \dot{a} \). This is an example of a Morse family reduction. Eventually, we have the following reduced Morse family
\[
E(q, a, p_q, p_a, \dot{q}) = p_q \cdot \dot{q} - L(q, \dot{q}, a) - \frac{\partial F}{\partial q} \cdot \dot{q} - \frac{\partial F}{\partial q} \cdot a. \quad (144)
\]

defined on the Whitney sum \( T^* AQ \times_Q TQ \) over \( Q \). A further reduction on the Morse family is possible. For this, recall the assumption that the matrix \( [\partial^2 F / \partial a \partial \dot{q}] \) is non-degenerate. So that we can, at least locally, solve \( \dot{q} \) in terms of the momenta from the equation (143). Let us write this solution as
\[
\dot{q} = z(q, a, p_a). \quad (145)
\]

Substitution of the solution (145) into the Morse family (144) results with a well-define Hamiltonian function
\[
H(q, a, p_q, p_a) = p_q \cdot \dot{q} - L(q, \dot{q}, a) - \frac{\partial F}{\partial q} \cdot \dot{q} - \frac{\partial F}{\partial q} \cdot a \quad (146)
\]
on \( T^* AQ \). Now, the Lagrangian submanifold (133)-(136) can be written directly by means of the Hamilton’s equations
\[
- (\tau_{T^* AQ})^* dH = \vartheta_1, \quad (147)
\]
where \( \vartheta_1 \) is the one-form in (128). Locally, the Hamilton’s equations are
\[
\dot{q} = \frac{\partial H}{\partial p_q}, \quad \dot{a} = \frac{\partial H}{\partial p_a}, \quad \dot{p}_q = - \frac{\partial H}{\partial q}, \quad \dot{p}_a = - \frac{\partial H}{\partial a}. \quad (148)
\]
The first set of equations in (148) is the definition of the velocity variable in (145). The second set of equations is identically satisfied after the substitutions of the
transformations (142) and (143). The third and the fourth sets of equations are the same with the second order Euler-Lagrange equations.

Let us now concentrate on a particular case where the auxiliary function $F$ in the definition of the Lagrangian $L_S$ in (120) has the following simple form

$$F = F(a, \dot{q}) = a \cdot \dot{q}. \quad (149)$$

In this case, $L_S$ turns out to be

$$L_S(q, a, \dot{q}, \dot{a}) = L(q, \dot{q}, a) + a \cdot \dot{a}. \quad (150)$$

As in the general case, the Lagrangian function $L_S$ is defined on the tangent bundle $T^2Q$, and generates a Lagrangian submanifold of $TT^*AQ$. The local expression of this Lagrangian submanifold can be computed as

$$p_q = \frac{\partial L}{\partial \dot{q}} + \dot{a}, \quad p_a = \dot{q}, \quad \dot{p}_q = \frac{\partial L}{\partial q}, \quad \dot{p}_a = \frac{\partial L}{\partial a} + a. \quad (151)$$

where $(q, a, \dot{q}, \dot{a})$ being free. In this case, the equation (145) defining the relationship between the velocity and momenta turns out to be $\dot{q} = p_a$ hence the Hamiltonian function (146) reduces to

$$H(q, a, p_q, p_a) = p_q \cdot p_a - L(q, p_a, a) - a \cdot a. \quad (152)$$

4.4.1. Symplectic relation with the Ostrogradsky-Legendre transformation. For a second order non-degenerate Lagrangian function on $T^2Q$, the Ostrogradsky-Legendre transformation results with a Hamiltonian function $H$ on $T^*TQ$, whereas the Schmidt-Legendre transformation results with a Hamiltonian function on $T^*AQ$.

In this subsection, we establish a symplectic transformation relating Hamiltonian formulations on $T^*TQ$ and $T^*AQ$.

Let us first recall the coordinates $(q(0), q(1), \pi(0), \pi(1))$ on $T^*TQ$, and $(q, a, p_q, p_a)$ on $T^*AQ$. In terms of these local coordinates, the canonical one-forms on the cotangent bundles $T^*TQ$ and $T^*AQ$ are given by

$$\Theta_{TQ} = \pi(0) \cdot dq(0) + \pi(1) \cdot dq(1), \quad \Theta_{AQ} = p_q \cdot dq + p_a \cdot da, \quad (153)$$

respectively. We introduce a projection from the cotangent bundle $T^*AQ$ to the tangent bundle $TQ$

$$T^*AQ \to TQ : (q, a, p_q, p_a) \to (q, z(q, a, p_a)), \quad (154)$$

where $z$ is the function in (145). Using this projection, define a Whitney sum

$$W = T^*AQ \times_{TQ} T^*TQ \quad (155)$$

over $TQ$. Here, the projection $T^*AQ \to TQ$ in the one in Eq.(154) and $T^*TQ \to TQ$ is the cotangent bundle projection $\pi_{TQ}$. By pulling the canonical forms $\Theta_{TQ}$ and $\Theta_{AQ}$ in (153) back to the Whitney sum $W$, we arrive at the following one form

$$\vartheta_W = \Theta_{AQ} \ominus \Theta_{TQ} = p_q \cdot dq + p_a \cdot da - \pi(0) \cdot dq - \pi(1) \cdot dz$$

on $W$. Exterior derivative $d\vartheta_W$ of $\vartheta_W$ is a symplectic two-form on $W$. Locally, $T^2Q$ can be identified with Whitney sum $TQ \times_Q AQ$. So that, It is possible, locally, to identify $W$ with the cotangent bundle $T^*T^2Q$ of the second order tangent bundle.

In this case, exterior derivative of the auxiliary function $F = F(q, z, a)$ on $T^2Q$ defines a Lagrangian submanifold of $W$. To arrive at the Legendre transformation, we equate the one-form $\vartheta_W$ to the exterior derivative $dF$ so that

$$p_q \cdot dq + p_a \cdot da - \pi(0) \cdot dq - \pi(1) \cdot dz = \frac{\partial F}{\partial q} \cdot dq + \frac{\partial F}{\partial z} \cdot dz + \frac{\partial F}{\partial a} \cdot da.$$
From this, we have the following relations
\[
\begin{align*}
\pi_{a} - \pi^{(0)} &= 0, \\
\pi_{a} - \frac{\partial F}{\partial a} &= 0, \\
\pi^{(1)} + \frac{\partial F}{\partial z} &= 0
\end{align*}
\] (156)
where we choose \( \dot{q} = z (q, a, p_{a}) \). These equations determine the Legendre transformation relating the Schmidt-Legendre and the Ostrogradsky-Legendre transformations \( [3] \). Note that, the symplectic diffeomorphism is generated by the auxiliary function \( F \). Explicitly, the symplectic diffeomorphism is given by
\[
\begin{align*}
T^{*}AQ & \rightarrow T^{*}TQ : \\
(q, a, p_{q}, p_{a}) & \rightarrow \left( q, z (q, a, p_{a}), p_{q} - \frac{\partial F}{\partial q} (q, z (q, a, p_{a}), a), -\frac{\partial F}{\partial z} \right).
\end{align*}
\]
4.5. The Schmidt-Legendre transformation for the third order Lagrangians. We have assumed that the configuration manifold \( Q \) is an \( n \)-dimensional manifold. Introduce now another \( n \)-dimensional manifold \( M \) equipped with the local coordinates \( (m) = (m^{1}, ..., m^{n}) \). Tangent bundle of \( M \) is a \( 2n \)-dimensional manifold \( TM \) with induced coordinates \( (m, \dot{m}) \). Recall the Tulczyjew’s triple presented in (46), and replacing the configuration space \( Q \) with the product manifold \( AQ \times M \), we introduce the following Tulczyjew’s triple
\[
\begin{align*}
T^{*}T(AQ \times M) & \xrightarrow{\Xi_{(AQ \times M)}} T^{*}T^{*}(AQ \times M) \\
\pi_{T(AQ \times M)} & \xrightarrow{\pi_{T^{*}(AQ \times M)}} T^{*}T^{*}(AQ \times M) \\
\pi_{T^{*}(AQ \times M)} & \xrightarrow{\pi_{T^{*}(AQ \times M)}} T^{*}(AQ \times M)
\end{align*}
\] (157)
Definitions of the projections as well as local descriptions of the symplectic diffeomorphisms can easily be deduced by making proper modifications to the definitions exhibited in the subsection (3.4). More explicitly, one should replace the local coordinates \( (q) \) with \( (q, a, m) \). Then induced local coordinates of fibers of the tangent bundle \( T^{*}(AQ \times M) \) and the cotangent bundle \( T^{*}(AQ \times M) \) become \( (\dot{q}, \dot{a}, \ddot{m}) \) and \( (p_{q}, p_{a}, p_{m}) \), respectively. Accordingly, induced coordinates on the iterated tangent bundle \( T^{*}T^{*}(AQ \times M) \) are
\[
(q, a, m, p_{q}, p_{a}, p_{m}, \dot{q}, \dot{a}, \ddot{m}, \dot{p}_{q}, \dot{p}_{a}, \dot{p}_{m}) : T^{*}T^{*}(AQ \times M) \rightarrow \mathbb{R}^{12n}.
\] (158)
Let us start with a third order Lagrangian function \( L(q, \dot{q}, \ddot{q}, q^{(3)}) \) defined on \( T^{*}Q \). Recalling the local diffeomorphism in (117), we pull the Lagrangian function \( L \) back to the tangent bundle \( TAQ \) of the acceleration bundle. By this we arrive at a first order Lagrangian function \( L = L(q, a, \dot{a}) \). Note that, by the abuse of notation, we continue to use the letter \( L \) for the first order Lagrangian function as well. Let us now define a Lagrangian function
\[
L_{3}(q, a, \dot{a}, \ddot{m}) = L(q, a, \dot{a}) + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial a} \cdot \dot{a} + \frac{\partial F}{\partial \dot{m}} \cdot \ddot{m}.
\] (159)
on the tangent bundle \( T(AQ \times M) \). In this case, the auxiliary function is \( F = F(q, \dot{q}, a, m) \). Euler Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L_{3}}{\partial \dot{q}} \right) - \frac{\partial L_{3}}{\partial q} = 0, \\
\frac{d}{dt} \left( \frac{\partial L_{3}}{\partial \dot{a}} \right) - \frac{\partial L_{3}}{\partial a} = 0, \\
\frac{d}{dt} \left( \frac{\partial L_{3}}{\partial \ddot{m}} \right) - \frac{\partial L_{3}}{\partial \dot{m}} = 0,
\] (160)
generated by the first order Lagrangian function \( L_{3} \) equal to the Euler-Lagrange equations (10) generated by the third order Lagrangian function \( L \) if the single
requirement
\[ \det[\partial^2 F/\partial \dot{q} \partial m] \neq 0 \] (161)
is employed. Let us start to write explicit calculations of the sets of equations in (160) one by one. The third set of equations in (160) can be computed to be
\[
\frac{d}{dt} \left( \frac{\partial F}{\partial m} \right) - \frac{\partial^2 F}{\partial m \partial q} \cdot \dot{q} - \frac{\partial^2 F}{\partial m \partial a} \cdot a - \frac{\partial^2 F}{\partial m \partial \dot{m}} \cdot \ddot{m} = 0
\]
\[ \frac{\partial^2 F}{\partial q \partial m} \cdot (\ddot{q} - a) = 0. \] (162)
If the condition in (161) is assumed then \( \ddot{q} = a \). By considering \( \ddot{q} = a \), the second set of equations in (160) turns out to be
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{a}} \right) - \frac{\partial L}{\partial a} - \frac{\partial F}{\partial q} = 0,
\] (163)
whereas the first set of equations in (160) results with
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial \dot{q}^2} = 0. \] (164)
To arrive at the third order Euler Lagrange equations (11) generated by the third order Lagrangian \( L \), one needs to solve \( \partial F/\partial q \) from equation (163) and substitute the solution into 164.

Referring to the left side of the Tulczyjew’s triple (157), the first order Lagrangian function \( L_3 \) generates a Lagrangian submanifold of \( T^*T(AQ \times M) \). Following the definition in (50), in terms of the local coordinates (158) of \( T^*T(AQ \times M) \), this Lagrangian submanifold is computed to be
\[
\begin{align*}
 p_q &= \frac{\partial L_3}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial \dot{q}} \cdot \dot{q} + \frac{\partial^2 F}{\partial \dot{q}^2} \cdot \ddot{q} + \frac{\partial^2 F}{\partial \dot{q} \partial a} \cdot a + \frac{\partial^2 F}{\partial \dot{q} \partial \dot{m}} \cdot \ddot{m}, \\
p_a &= \frac{\partial L_3}{\partial \dot{a}} = \frac{\partial L}{\partial \dot{a}} + \frac{\partial F}{\partial \dot{a}}, \\
p_m &= \frac{\partial L_3}{\partial \dot{m}} = \frac{\partial L}{\partial \dot{m}}, \\
pq &= \frac{\partial L_3}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial F}{\partial q} \cdot q + \frac{\partial^2 F}{\partial q \partial a} \cdot a + \frac{\partial^2 F}{\partial q \partial \dot{m}} \cdot \ddot{m}, \\
pa &= \frac{\partial L_3}{\partial a} = \frac{\partial L}{\partial a} + \frac{\partial F}{\partial a} \cdot a + \frac{\partial^2 F}{\partial a \partial \dot{m}} \cdot \ddot{m}, \\
pm &= \frac{\partial L_3}{\partial m} = \frac{\partial L}{\partial m} \cdot \ddot{m}.
\end{align*}
\] (165)
To derive the equations of the motion governed by this Lagrangian submanifold, we simply take the time derivative of the first three components and equate them with the last three one. So that we have three sets of equations. More explicitly, we take the time derivative of (165) and equate it to (168), take the time derivative of (166) and equate it to (169), and finally take the time derivative of (167) and equate it to (170). The last one results with \( a = \ddot{q} \) under the assumption that the matrix \( \partial^2 F/\partial \dot{q} \partial m \) is non-degenerate. The second set of equations determine the equality
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{a}} = \frac{\partial L}{\partial a} + \frac{\partial F}{\partial q},
\] (171)
In the lights of these, the first set of equations reads
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial F}{\partial \dot{q}} + \frac{\partial L}{\partial a} = 0. \] (172)

By solving \( \partial F/\partial \dot{q} \) from the equation (171) and substituting it into (172), we arrive at the third order Euler-Lagrange equations (11). From this observation, we see that (165)-(170) is a Lagrangian submanifold realization of the third order Euler-Lagrange equations (11).

In the present picture, the Legendre transformation of the third order Euler-Lagrange equations (11) is to generate the Lagrangian submanifold (165)-(170) from the right side of Tulczyjew’s triple (157). To this end, introduce the following Morse family
\[ E = p_q \cdot \dot{q} + p_a \cdot \dot{a} + p_m \cdot \dot{m} - L - \frac{\partial F}{\partial q} \cdot \dot{q} - \frac{\partial F}{\partial a} \cdot \dot{a} - \frac{\partial F}{\partial m} \cdot \dot{m} \] (173)
on the Whitney sum \( T(AQ \times M) \times T^*(AQ \times M) \). The following diagram
\[ TT^*(AQ \times M) \xrightarrow{\alpha^*(AQ \times M)} T^*(AQ \times M) \xrightarrow{\pi_T} T^*(AQ \times M) \]
shows the right side of Tulczyjew triple (157) equipped with the fiber bundle where the Morse family (173) is defined. It is a matter of direct calculation to show that the Morse family (173) generates the Lagrangian submanifold (165)-(170).

In order to perform some possible reductions of the Morse family (173) we consider the first three equations (165)-(167) defining the conjugate momenta. If the velocities \( (\dot{q}, \dot{a}, \dot{m}) \) can be solved in terms of the positions and momenta, then we can reduce the Morse family (173) to the base manifold \( T^*(AQ \times M) \), and the Morse family becomes a well-defined Hamiltonian function. But this is not true for degenerate Lagrangian functions. Let us start to reduce the Morse family by starting with a Lagrangian function which is not necessarily non-degenerate. At first, notice that substitution the momenta \( p_m \) in (167) into the Morse family results with the absence of \( \dot{m} \) in the family. Secondly observe that non-degeneracy of \[ [\partial^2 F/\partial \dot{q} \partial \dot{m}] \] makes it is possible to solve \( \dot{q} \) in terms of \( (q, a, m, p_m) \) from the third equation (167). Let us write this solution as
\[ \dot{q} = v(q, a, m, p_m). \] (175)

By substituting (175) in to the Morse family (173), and by replacing \( \dot{a} \) in (173) with a set \( \lambda \) of Lagrange multipliers, we arrive at a total Hamiltonian computed as
\[ H_T = p_q \cdot v(q, a, m, p_m) - L - \frac{\partial F}{\partial q} \cdot v(q, a, m, p_m) \] (176)
Note that, we have a set of constraints \( \Phi_1 = p_a - \partial F/\partial a = 0 \). A consistency check should be performed for \( \Phi_1 \) according to the Dirac-Bergmann constraint algorithm summarized in (3.5). Instead of applying the Dirac-Bergmann algorithm to the present system, we are preferring to apply the algorithm to a particular case given in the following paragraphs.
We now particularly choose that the Lagrangian function $L$ in (159) is a second order Lagrangian function defined on $T^2Q$, that is $L$ does not depend on $\dot{a}$. This interesting case is a revisit to the Legendre transformation of the second order Lagrangian systems presented in the previous subsection (4.4). Apart of the discussion done in (4.4), in this case we are not asking that $L$ necessarily be non-degenerate. So this case is applicable both for non-degenerate and degenerate second order Lagrangian theories. Moreover, we are not asking the integrability condition (125).

In the definition of $L_3$ given in (159), we choose $L = L(q, \dot{q}, a)$, and consider an auxiliary function $F = F(q, \dot{q}, m)$. So that

$$L_3(q, a, m, \dot{q}, \dot{a}, \dot{m}) = L(q, \dot{q}, a) + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial F}{\partial \dot{q}} \cdot a + \frac{\partial F}{\partial \dot{m}} \cdot \dot{m},$$

(177) defined on the tangent bundle $T(AQ \times M)$. For the auxiliary function we are asking the non-degeneracy of the matrix $[\partial^2 F/\partial q \partial m]$. In this case, the Lagrangian submanifold in (165)-(170) reduces to

$$p_q = \frac{\partial L_3}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial q} \cdot \dot{q} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial q \partial \dot{m}} \cdot \dot{m},$$

(178)

$$p_a = \frac{\partial L_3}{\partial \dot{a}} = 0,$$

(179)

$$p_m = \frac{\partial L_3}{\partial \dot{m}} = \frac{\partial F}{\partial \dot{m}},$$

(180)

$$\dot{p}_q = \frac{\partial L_3}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial q \partial q} \cdot \dot{q} + \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial q \partial \dot{m}} \cdot \dot{m},$$

(181)

$$\dot{p}_a = \frac{\partial L_3}{\partial a} = \frac{\partial L}{\partial a} + \frac{\partial F}{\partial q},$$

(182)

$$\dot{p}_m = \frac{\partial L_3}{\partial \dot{r}} = \frac{\partial^2 F}{\partial q \partial \dot{q}} \cdot \dot{q} + \frac{\partial^2 F}{\partial m \partial \dot{q}} \cdot a + \frac{\partial^2 F}{\partial m \partial \dot{m}} \cdot \dot{m}.$$  

(183)

To derive the equations of the motion governed by the Lagrangian submanifold, we simply take the time derivative of the first three components and equate them with the last three one. Explicitly, take the time derivative of (178) and equate it to (181), take the time derivative of (179) and equate it to (182), and finally take the time derivative of (180) and equate it to (183). The last one results with that $a = \dot{q}$ under the assumption that the matrix $[\partial^2 F/\partial q \partial m]$ is non-degenerate. The second set of the equations determine the equality in (124). In the lights of these two sets of equations, the first equation reads the second order Euler-Lagrange equations (10). Eventually, we have two different Lagrangian submanifold realization of the second order Euler-Lagrange equations, the present one in (178)-(183) and the one in (133)-(136).

Let us now perform the Legendre transformation of the system (178)-(183). Notice that, this time, the solutions $v$ in (175) depend on $(q, m, p_m)$ by being free of $(a)$. The total Hamiltonian function in (176) takes the particular form

$$H_T = p_q \cdot \dot{q} - L - \frac{\partial F}{\partial q} \cdot \dot{q} - \frac{\partial F}{\partial \dot{q}} \cdot a + \lambda \cdot p_a,$$

(184)

with primary constraint $\Phi_1 = p_a = 0$. To check the consistency of $\Phi_1$, we take the Poisson bracket of $\Phi_1$ and $H_T$ which gives

$$\Phi_2 = \{\Phi_1, H_T\} = \frac{\partial L}{\partial a} + \frac{\partial F}{\partial \dot{q}} = 0.$$
Note that, this is nothing but Eq. (124). To check the consistency of \( \Phi_2 \), we compute
\[
\dot{\Phi}_2 = \left\{ \Phi_2, H_T \right\} = \frac{\partial^2 L}{\partial a \partial a} \cdot \lambda + \frac{\partial^2 L}{\partial q \partial a} \cdot \ddot{q} + \frac{\partial^2 L}{\partial q \partial a} \cdot a + p_q - \frac{\partial F}{\partial q} = 0.
\]
If the Lagrangian is assumed to be non-degenerate, that is if the rank of the matrix \( \left[ \frac{\partial^2 L}{\partial a^2} \right] \) is full then this step determines the Lagrange multipliers \( \lambda \), and the constraint algorithm is finished up. If the Lagrangian is degenerate further steps may be needed to determine the Lagrange multipliers and/or to finish the algorithm. Number of possible further steps in the algorithm are determined by degeneracy level of the Lagrangian function \( L \). We refer [3] for the present and further discussions on this.

4.5.1. A trick to reduce a second order Lagrangian to a first order one. In [22, Page 130], a trick has been introduced which reduces a second order Lagrangian to the first order one. In this subsection, we shall first summarize this trick, then exhibit its geometry in terms of acceleration bundle and the Schmidt-Legendre transformation. Recall the notation in (64), and define the local coordinates on geometry in terms of acceleration bundle and the Schmidt-Legendre transformation.

We start with a second order Lagrangian function \( L = L(q, \dot{q}, \ddot{q}) \), by identifying \( q = q(0) \) and \( \dot{q} = q(2) \), introduce following action integral
\[
\int L(q(0), q(2)) + k \cdot (q(2) - \ddot{q}(0)) dt,
\]
where \( k = (k^1, ..., k^n) \) are a set of Lagrange multipliers. Applying by-parts technique to the integral (186), we arrive at the following Lagrangian function
\[
L_D = L(q(0), q(0), q(2)) + k \cdot q(2) + \dot{k} \cdot \dot{q}(0)
\]
on the product space \( T^2T^2Q \). Here, \( T^2K \) is a \( 2n \)-dimensional tangent bundle with coordinates \((k, \dot{k})\). Notice that \( L_D \) is a first order Lagrangian function defined on the first order tangent bundle \( T(T^2Q \times K) \). \( L_D \) does not depend on the variables \( q(1) \) or \( \dot{q}(1) \). This observation reads that the domain of the Lagrangian function \( L_D \) lies in the image space of the embedding \( \varepsilon \) presented in (117). So that, we can pull the Lagrangian function \( L_D \) back to the tangent bundle \( TAQ \) and have the following Lagrangian function
\[
L_D = L(q, \dot{q}, a) + k \cdot a + \dot{k} \cdot \dot{a}
\]
on the tangent bundle \( T(AQ \times K) \). It is a now matter of direct calculation to observe that Euler-Lagrange equations generated by the first order Lagrangian function \( L_D \) in (188) is equivalent to Euler-Lagrange equations (10) generated by the second order Lagrangian function \( L \).

Let us exhibit the relationship between \( L_2 \) in (177) and \( L_D \) in (188). In the definition of the Lagrangian function \( L_2 \), choose the auxiliary function as \( F = m \cdot q \). This auxiliary function is the simplest one satisfying the non-degeneracy condition of the matrix \( \left[ \frac{\partial^2 F}{\partial q \partial \dot{m}} \right] \). In this case the Lagrangian function \( L_2 \) reduces to
\[
L_2(q, a, m, \dot{m}, \ddot{m}) = L(q, \dot{q}, a) + m \cdot a + \dot{q} \cdot \dot{m}
\]
on the product manifold \( T(AQ \times M) \). It is immediate now to see that the Lagrangian function in (188) and the Lagrangian function in (189) are the same if we identify the manifolds \( K \) and \( M \) so that \( m = k \) and \( \dot{m} = \dot{k} \). This is an important subcase of the Schmidt-Legendre transformation, since many of the physical systems fit such
kind of geometry. Lagrangian submanifold generated by the Lagrangian function (189), in terms of the coordinates (158) of \( TT^*(AQ \times M) \), is computed to be

\[
p_q = \frac{\partial L}{\partial \dot{q}} + m, \quad p_a = 0, \quad p_m = \dot{q}, \quad (190)
\]

\[
\dot{p}_q = \frac{\partial L}{\partial q}, \quad \dot{p}_a = \frac{\partial L}{\partial a} + m, \quad \dot{p}_m = a, \quad (191)
\]

where the coordinates \((q, a, \dot{q}, \dot{a}, m, \dot{m})\) are free.

Let us present Hamiltonian formalism of the Lagrangian submanifold (190)-(191). In this case, Morse family (176) generating the Lagrangian submanifold by means of the right side of Tulczyjew’s triple turns out to be

\[
E = p_q \cdot \dot{q} + p_a \cdot \dot{a} + p_m \cdot \dot{m} - L - m \cdot a - \dot{q} \cdot \dot{m} \quad (192)
\]

After the substitutions of the velocity variables \(\dot{q}\) and \(\dot{m}\), and considering the set of primary constraints \(\Phi_1 = p_a = 0\), we arrive at the following total Hamiltonian function

\[
H_T = p_q \cdot p_m - L - r \cdot a + \lambda \cdot p_a, \quad (193)
\]

where \(\lambda = \dot{q}\) is a set of Lagrange multipliers. One needs to ask compatibility of \(\Phi_1\).

This results with a new set of constraints computed to be

\[
\Phi_2 = \{\Phi_1, H_T\} = \frac{\partial L}{\partial a} + m = 0. \quad (194)
\]

The compatibility condition for the constraint \(\Phi_2\) leads to the following computation

\[
\{\Phi_2, H_T\} = \frac{\partial^2 L}{\partial a \partial a} \cdot \lambda + \frac{\partial^2 L}{\partial q \partial a} \cdot \dot{q} + \frac{\partial^2 L}{\partial q \partial a} \cdot a + p_q - \frac{\partial L}{\partial \dot{q}} = 0.
\]

As in the previous subsection, the determination of the Lagrange multipliers are related with degeneracy level of the Lagrangian \(L\).

5. Examples.

5.1. Example 1. Let \(Q = \mathbb{R}\) and consider the non-degenerate second order Lagrangian function given by

\[
L = \frac{1}{2} q^2 + 5q^2 \dot{q}^2 + q^6 \quad (195)
\]

on \(T^2Q\) with coordinates \((q, \dot{q}, \ddot{q})\). Euler-Lagrange equations yield the following fourth-order differential equation

\[
q^{(iv)} - 10q^2 \ddot{q} - 10q \dot{q}^2 + 6q^5 = 0. \quad (196)
\]

Introduce the coordinates \((q, a, \dot{q}, \dot{a})\) on \(TAQ\). By integrating the condition (124), we arrive at the auxiliary function

\[
F(q, a, \dot{q}, \dot{a}) = -\dot{q}a + G(q, a),
\]

where \(G\) being an arbitrary function depending on \(q\) and \(a\). The first order Lagrangian \(L_S\) in (120) takes the particular form

\[
L_S(q, a, \dot{q}, \dot{a}) = -\frac{1}{2} a^2 + 5q^2 \dot{q}^2 + q^6 - \dot{q}a + \frac{d}{dt} G(q, a) \quad (197)
\]

It is obvious that the total derivative term may be neglected. Following (142) and (143), we compute the conjugate momenta

\[
p_q = 10q^2 \dot{q} - \dot{a}, \quad p_a = -\dot{q}.
\]
The Hamiltonian function (146) on $T^*A Q$ is computed to be
\[ H = \frac{1}{2} \dot{a}^2 - p_q p_a - 5q^2 p_a^2 - q^6, \tag{198} \]
whereas the Hamilton’s equations (148) are
\[ \dot{q} = p_a, \quad \dot{a} = p_q - 10q^2 p_a, \quad \dot{p}_q = -10q p_a^2 - 6q^5, \quad \dot{p}_a = a. \]

Let us now apply the Ostrogradsky-Legendre transformation, presented in the Section (3.6), to the second order Lagrangian function $L$ in (195). We take the coordinates $(q(0), q(1), \pi(0), \pi(1))$ on the cotangent bundle $T^*TQ$, and compute the Ostrogradsky momenta (88) as
\[ \pi(0) = 10q^2 q(0), \quad \pi(1) = \frac{d}{dt} q(1), \quad \pi(1) = \dot{q}(1). \]
Since, the velocity $\dot{q}(1)$ can be written in terms of the momenta, the Morse family (87) reduces to a Hamiltonian function
\[ H = \frac{1}{2} \left( \pi(1) \right)^2 + \pi(0) q(1) - 5q(0)^2 q(1) - q(0)^6 \tag{199} \]
on $T^*TQ$. Recall the symplectic transformation (156) relating the Schmidt-Legendre and the Ostrogradsky-Legendre transformations. In the present case, the transformation is generated by the auxiliary function $F = -\dot{q}a$. Explicitly we have that
\[ q = q(0), \quad a = -\pi(1), \quad p_q = \pi(0), \quad p_a = -q(1). \tag{200} \]
Notice that, substitution of these equations into the Hamiltonian function in (199) results with the Hamiltonian function in (198).

5.2. Example 2. The Pais-Uhlenbeck oscillator, see [62],
\[ q^{(iv)} + (\omega_1^2 + \omega_2^2)q = 0, \tag{201} \]
is a quantum mechanical prototype of a field theory containing both second and fourth order derivative terms. In view of its fundamental importance it provides a good reference point on which to build higher order equations which have possible physical relevance, see [9, 52, 55, 56, 58]. The fourth order equation (201) is generated by the following non-degenerate second order Lagrangian
\[ L = \frac{1}{2} \dot{a}^2 - \frac{1}{2}(\omega_1^2 + \omega_2^2) \dot{q}^2 + \frac{1}{2} \omega_1^2 \omega_2^2 q^2, \tag{202} \]
If we substitute $L_{PU}$ in (120), then the Schmidt Lagrangian takes the particular form
\[ L_S = -\frac{1}{2} \alpha^2 - \frac{1}{2}(\omega_1^2 + \omega_2^2) \dot{q}^2 - \dot{a} \dot{a} + \frac{1}{2} w_1^2 w_2^2 q^2, \tag{203} \]
where we choose the auxiliary function $F = -\dot{q}a$ by solving the defining equation (124). The canonical momenta in Eqs. (12) and (13) become
\[ p_q = -w_1^2 + w_2^2), \quad p_a = -\dot{q}, \]
whereas the canonical Hamiltonian function (146) turns out to be
\[ H = -p_q p_a + \frac{1}{2} \alpha^2 + \frac{1}{2}(\omega_1^2 + \omega_2^2) p_a^2 - \frac{1}{2} \omega_1^2 \omega_2^2 q^2. \tag{204} \]
In this case the motion is determined by the Hamilton’s equations
\[ \dot{q} = -p_a, \quad \dot{a} = -p_q + (w_1^2 + w_2^2) p_a, \quad \dot{p}_q = w_1^2 w_2^2 q, \quad \dot{p}_a = -a. \tag{205} \]
It is interesting at this juncture to compare these equations with those following from Ostrogradsky’s formulation. For the latter the new coordinates are chosen to be \( q(0) = q \) and \( q(1) = \dot{q} \) while the corresponding Ostrogradsky momenta ((88)) are
\[
\pi^{(0)} = -(w_1^2 + w_2^2)q - \frac{d}{dt}q, \quad \pi^{(1)} = \dot{q}.
\]
Since, the velocity \( \dot{q} \) can be written in terms of the momenta, the Morse family (87) reduces to a Hamiltonian function
\[
H = \frac{1}{2} \left( \pi^{(1)} \right)^2 + \pi^{(0)}q(1) + \frac{1}{2}(w_1^2 + w_2^2)q^2(1) - \frac{1}{2}w_1^2w_2^2q^2(0)
\]
and leads to the following equations:
\[
\frac{d}{dt}q^{(0)} = q^{(1)}, \quad \frac{d}{dt}q^{(1)} = \pi^{(1)}, \quad \frac{d}{dt}\pi^{(1)} = w_1^2w_2^2q^{(0)}.
\]

The symplectic transformation relating the Hamiltonian formalisms in (204) and (206) is the same with the one in (200).

5.3. Example 3. Let \( Q = \mathbb{R}^2 \) equipped with the coordinates \((x, y)\) and consider the non-degenerate second order Lagrangian function
\[
L = \frac{1}{2}(\dot{x}\ddot{y}^2 + \dot{y}\ddot{x}^2).
\]
on \( T^2\mathbb{R}^2 \). Note that this Lagrangian function is not satisfying the integrability condition in (124). Indeed, for this present case, we compute
\[
\frac{\partial F}{\partial \dot{x}} = -\ddot{x}, \quad \frac{\partial F}{\partial \dot{y}} = -\ddot{y}.
\]
Notice that partial derivatives of \( F \) with respect to \( \dot{x} \) and \( \dot{y} \) do not commute. That is \( \partial^2 F / \partial \dot{x} \partial \dot{y} \) is not equal to \( \partial^2 F / \partial \dot{y} \partial \dot{x} \). Due to this, we cannot apply the Schmidt-Legendre transformation presented in subsection (4.4) to the Lagrangian function (208) but we can use the Schmidt-Legendre transformation presented in subsection (4.5). Even though the method in (4.5) is proper for the third order Lagrangians, as mentioned in the text, it is also proper for the degenerate second order Lagrangians and the ones not satisfying the integrability criterion in (125). Since the Lagrangian 208 does not satisfy (125), we are now addressing the method in (4.5).

Let \( Q = \mathbb{R}^2 \), and consider local coordinates \((x, y, a, b)\) on \( AQ = A\mathbb{R}^2 \), and local coordinates \((r, s)\) on \( M = \mathbb{R}^2 \). Induced coordinates on the tangent and cotangent bundles are
\[
(x, y, a, b, r, s, \dot{x}, \dot{y}, \dot{a}, \dot{b}, \dot{r}, \dot{s}) : T(AQ \times M) \rightarrow \mathbb{R}^{12},
\]
\[
(x, y, a, b, r, s, p_x, p_y, p_a, p_b, p_r, p_s) : T^*(AQ \times M) \rightarrow \mathbb{R}^{12}.
\]
In these coordinates, obeying the condition (161), we introduce the auxiliary function
\[
F = \dot{x}r + bs.
\]
Conjugate momenta exhibited in (178)-(180) take the particular form
\[
p_x = b^2/2 + \dot{r}, \quad p_y = a^2/2 + \dot{s}, \quad p_a = p_b = 0, \quad p_r = \dot{x}, \quad p_s = \dot{y}.
\]
Accordingly, the total Hamiltonian \( H_T \) in (193) becomes
\[
H_T = p_xp_r + p_yp_s - (a^2/2)p_s - (b^2/2)p_r - ra - sb + \lambda_1p_a + \lambda_2p_b.
\]
where \((\lambda_1, \lambda_2)\) are Lagrange multipliers. Consistency check for the primary constraints \(\Phi_1^a = p_a = 0\) and \(\Phi_1^b = p_b = 0\) result with the secondary constraints
\[
\Phi_2^a = \{p_a, H_T\} = -ap_s - r = 0, \quad \Phi_2^b = \{p_b, H_T\} = -bp_r - s = 0.
\]
Consistency checks of the secondary constraints
\[
\{ap_s + r, H_T\} = 0, \quad \{bp_r + s, H_T\} = 0,
\]
modulo primary constraints, determine the Lagrangian multipliers
\[
\lambda_1 = \frac{1}{p_s} \left( \frac{b^2}{2} - p_x - ab \right), \quad \lambda_2 = \frac{1}{p_r} \left( \frac{a^2}{2} - p_y - ab \right).
\]
By the substitutions of the Lagrange multipliers into the Hamiltonian function (210), we achieve to write a Hamiltonian formalism for the Lagrangian system (208).

5.4. Example 4. Take \(Q = \mathbb{R}^3 \times \mathbb{R}^3\) with coordinates given by a two tuple \((x, y)\) both of which are three dimensional vectors. We consider the following degenerate second order Lagrangian function
\[
L = \frac{c}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{\mu} \dot{y} \cdot \dot{x} - \frac{m^2}{2} (x^2 + y^2) \tag{211}
\]
introduced in [68]. Here \(c, \mu\) and \(m\) are constants. The Euler-Lagrange equations (10) generated by the Lagrangian function (211) are computed as
\[
m^2 \dot{x} + c \ddot{x} = \frac{1}{\mu} y^{(3)}, \quad m^2 \dot{y} + c \ddot{y} = -\frac{1}{\mu} x^{(3)}. \tag{212}
\]
Recall geometry of the Schmidt-Legendre transformation presented in the subsection (4.5). In the present case, acceleration bundle \(AQ = \mathbb{R}^{12}\) is equipped with induced coordinates \((x, y, a, b)\). \(M\) is \(\mathbb{R}^6\) with coordinates \((r, s)\). Consider auxiliary function
\[
F = \dot{x} \cdot r + \dot{y} \cdot s
\]
obeying the condition (161). The Lagrangian function (189) becomes
\[
L_2 = \frac{c}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{\mu} \dot{y} \cdot a - \frac{m^2}{2} (x^2 + y^2) + \dot{x} \cdot \dot{r} + \dot{y} \cdot \dot{s} + a \cdot r + b \cdot s
\]
on the tangent bundle \(T(AQ \times M)\). Conjugate momenta are computed to be
\[
p_x = c \dot{x} + \dot{r}, \quad p_y = c \dot{y} + \dot{s} + \frac{1}{\mu} s, \quad p_a = p_b = 0, \quad p_r = \dot{x}, \quad p_s = \dot{y},
\]
where \((p_x, p_y, p_a, p_b, p_r, p_s)\) are the fiber coordinates of the bundle \(T^*(AQ \times M)\). Accordingly, the primary constraints are \(p_a = 0\) and \(p_b = 0\). In this case, the total Hamiltonian function (193) takes the particular form
\[
H_T = p_x \cdot p_r + p_s \cdot (p_y - \frac{1}{\mu} a) - \frac{c}{2}((p_r)^2 + (p_s)^2) + \frac{m^2}{2}(x^2 + y^2) - r \cdot a - s \cdot b + u \cdot p_a + v \cdot p_b \tag{213}
\]
where we have six numbers of Lagrange multipliers denoted by \((u, v)\). Consistency checks of the primary constraints lead to secondary constraint functions
\[
\{p_a, H_T\} = \frac{1}{\mu} p_s + r, \quad \{p_b, H_T\} = s. \tag{214}
\]
We check the compatibility of the secondary constraints. These give the tertiary constraint functions

\[
\begin{align*}
\{ \frac{1}{\mu} p_s + r, H_T \} &= \frac{1}{\mu} b + p_x - c p_r, \\
\{ s, H_T \} &= p_y - \frac{1}{\mu} a - c p_s.
\end{align*}
\tag{215}
\]

Note that, the secondary and tertiary constraints in (214) and (215) enable to write the auxiliary variables \((r, s)\) and their conjugate momenta \((p_r, p_s)\) in terms of the coordinates of \(T^*AQ\). Continuing in this way, we compute the fourtiery constraints

\[
\begin{align*}
\left\{ \frac{1}{\mu} b + p_x - c p_r, H_T \right\} &= \frac{1}{\mu} v - m^2 x - ca = 0, \\
\left\{ p_y - \frac{1}{\mu} a - c p_s, H_T \right\} &= -m^2 y - \frac{1}{\mu} u - cb = 0.
\end{align*}
\tag{216}
\tag{217}
\]

Notice that, (216) and (217) determine the Lagrange multipliers

\[
u = -\mu(m^2 y + cb), \quad v = \mu(m^2 x + ca).
\]

So that this is the final step of the Dirac-Bergmann algorithm. Substitutions of the Lagrange multipliers \(u, v\) into the total Hamiltonian (213), one arrives at the final form

\[
H_T = p_x \cdot p_r + p_s \cdot (p_y - \frac{1}{\mu} a) - \frac{c}{2} ((p_r)^2 + (p_s)^2) + \frac{m^2}{2} (x^2 + y^2) - r \cdot a - s \cdot b - \mu(m^2 y + cb) \cdot p_a + \mu(m^2 x + ca) \cdot p_b.
\tag{218}
\]

Using this total Hamiltonian function, Hamilton’s equations of motion are

\[
\begin{align*}
x & = p_r, \quad y = p_s, \quad \dot{a} = -\mu(m^2 y + cb), \quad \dot{b} = \mu(m^2 x + ca), \quad \dot{r} = p_x - c p_r, \\
\dot{s} & = p_y - \frac{1}{\mu} a - c p_s, \quad \dot{p}_x = -m^2 x, \quad \dot{p}_y = -m^2 y, \quad \dot{p}_a = \frac{1}{\mu} p_s \cdot r, \quad \dot{p}_b = a,
\end{align*}
\tag{219}
\tag{220}
\]

It is easy to check that Hamiltonian system in (219)-(220) is equivalent to the Euler-Lagrange equations (212).

Notice that the Lagrangian (211) is a second order Lagrangian with respect to the variable \(x\) whereas it is a first order Lagrangian with respect to the variable \(y\). So that, instead of considering the acceleration bundle \(AQ\) of \(Q = \mathbb{R}^3 \times \mathbb{R}^3\), one can only consider \(AR^3 \times \mathbb{R}\), where \(AR^3\) is the acceleration bundle of \(\mathbb{R}^3\). Local coordinates on \(AR^3 \times \mathbb{R}\) are \((x, a, y, r)\). See that no acceleration is assigned to the variable \((y)\). In this case, the auxiliary manifold \(M\) in the theory reduces to \(\mathbb{R}^3\) with coordinates \((r)\). So that the configuration space is the product manifold \(AR^3 \times \mathbb{R}^3 \times \mathbb{R}^3\) with coordinates \((x, a, y, r)\). Coordinates on the tangent bundle \(T(AR^3 \times \mathbb{R}^3 \times \mathbb{R}^3)\) and the cotangent bundle \(T^*(AR^3 \times \mathbb{R}^3 \times \mathbb{R}^3)\) are assumed to be

\[
\begin{align*}
(x, a, y, r, \dot{x}, \dot{a}, \dot{y}, \dot{r}) : T(AR^3 \times \mathbb{R}^3 \times \mathbb{R}^3) & \rightarrow \mathbb{R}^{24}, \\
(x, a, y, r, p_x, p_a, p_y, p_r) : T^*(AR^3 \times \mathbb{R}^3 \times \mathbb{R}^3) & \rightarrow \mathbb{R}^{24}.
\end{align*}
\tag{222}
\tag{223}
\]

Accordingly, we take auxiliary function as \(F = \dot{x} \cdot r\). By adding the total time derivative of the auxiliary function \(F\) to (211), define the following Lagrangian
function
\[ L_2 = \frac{c}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{\mu} \dot{r} \cdot a - \frac{m^2}{2} (x^2 + y^2) + \dot{x} \cdot \dot{r} + a \cdot r \]  
\hspace{2cm} (224)

on the tangent bundle \( T(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3) \). Conjugate momenta defined by this Lagrangian function are computed to be
\[ p_x = c \dot{x} + \dot{r}, \quad p_y = c \dot{y} + \frac{1}{\mu} a, \quad p_a = 0, \quad p_r = \dot{x}. \]  
\hspace{2cm} (225)

From these equations, it is possible to solve \( \dot{x}, \dot{y} \) and \( \dot{r} \) in terms of the momentum variables. The equations \( p_a \approx 0 \) are assumed to be primary constraints. Here is the total Hamiltonian function
\[ H_T = p_x \cdot \dot{r} + \frac{1}{2c} (p_y - \frac{1}{\mu} a)^2 - \frac{c}{2} (p_r)^2 + \frac{m^2}{2} (x^2 + y^2) - r \cdot a + u \cdot p_a \]  
\hspace{2cm} (226)

where \( u \) is a set of Lagrange multipliers. Let us check the consistencies of the primary constraints \( p_a = 0 \). Computation results with a set of secondary constraint functions
\[ \{ p_a, H_T \} = \frac{1}{c\mu}(p_y - \frac{1}{\mu} a) + r. \]  
\hspace{2cm} (227)

Compatibilities of the secondary constraints give
\[ \left\{ \frac{1}{c\mu}(p_y - \frac{1}{\mu} a) + r, H_T \right\} = \frac{1}{c\mu}(-m^2 y - \frac{1}{\mu} u) + p_x - a p_r = 0. \]  
\hspace{2cm} (228)

These three equations determine the Lagrange multipliers \( u \), so that constraint algorithm is finished up. Substitutions of the Lagrange multipliers into the Hamiltonian function (226) result with
\[ H_T = p_x \cdot \dot{r} + \frac{1}{2c} (p_y - \frac{1}{\mu} a)^2 - \frac{c}{2} (p_r)^2 + \frac{m^2}{2} (x^2 + y^2) - r \cdot a + (c\mu^2 (p_x - a p_r) - \mu m^2 y) \cdot p_a. \]  
\hspace{2cm} (229)

Hamilton’s equations are
\[ \dot{x} = p_r, \quad \dot{y} = c (p_y - \frac{1}{\mu} a), \quad \dot{a} = -\mu m^2 y + c \mu^2 (p_x - a p_r), \quad \dot{r} = p_x - c p_r, \]  
\hspace{2cm} (230)

\[ \dot{p}_x = -m^2 x, \quad \dot{p}_y = -m^2 y, \quad \dot{p}_a = \frac{1}{c\mu}(p_y - \frac{1}{\mu} a) + r, \quad \dot{p}_r = a. \]  
\hspace{2cm} (231)

5.5. Example 5. In [16], while makings investigations of particle-like solutions to the theory of topological massive gravity, Clément’s introduced the following second order degenerate Lagrangian function
\[ L = -\frac{m}{2} \zeta \dot{x}^2 - \frac{2m \Lambda}{\zeta} + \frac{\zeta^2}{2\mu m} x \cdot (\dot{x} \times x) \]  
\hspace{2cm} (232)

on the second order tangent bundle \( T^2 Q \). Here, \( Q \) is a three dimensional space equipped with a Lorentzian metric \( x^2 = T^2 - x^2 - y^2 \). In (232), we have considered the following set \( (x, \dot{x}, \ddot{x}) \) as a coordinate chart on \( T^2 Q \). The triple product \( x \cdot (\dot{x} \times \ddot{x}) = \epsilon_{ijk} x^i \dot{x}^j \ddot{x}^k \) is defined by a completely antisymmetric tensor of rank three \( \epsilon_{ijk} \). \( \zeta = \zeta(t) \) is a function which allows arbitrary reparametrization of the variable \( t \). \( \Lambda \) and \( 1/2m \) are cosmological and Einstein gravitational constants, respectively.
The Euler-Lagrange equations generated by the Clément’s Lagrangian (232) are computed to be

\[ 2m^2\mu \ddot{x} + 3\dot{x} \times \ddot{x} + 2\dot{x} \times x^{(3)} = 0. \tag{233} \]

Recall geometry of the Schmidt-Legendre transformation presented in the subsection (4.5). For this example, \( AQ = \mathbb{R}^6 \) with coordinates \((x, a)\) and \( M = \mathbb{R}^3 \) with coordinates \((r)\). Tangent and cotangent bundles of the product manifold \( AQ \times M \) are

\[(x, a, r, \dot{x}, \dot{a}, \dot{r}) : T(AQ \times M) \longrightarrow \mathbb{R}^{18} \]
\[(x, a, r, p_x, p_a, p_r) : T^*(AQ \times M) \longrightarrow \mathbb{R}^{18}. \]

By considering auxiliary function \( F = \dot{x} \cdot r \) define the following Lagrangian function

\[ L_2 = -m\zeta^2 \dot{x}^2 + \frac{\zeta^2}{2\mu m} x \cdot (\dot{x} \times a) + \dot{x} \cdot \dot{r} + r \cdot A \]
on \( T(AQ \times M) \). The conjugate momenta are computed to be

\[ p_x = -m\zeta \dot{x} + \frac{\zeta^2}{2\mu m} a \times x + \dot{r}, \quad p_a = 0, \quad p_r = \dot{x}. \]

The velocities \( \dot{x} \) and \( \dot{r} \) can be solved from these equations, and \( \Phi_1 = p_a = 0 \) turns out to be a set of primary constraints for the system. Accordingly, total Hamiltonian function can be written as

\[ H_T = p_x \cdot p_r + \frac{m\zeta}{2} (p_r)^2 - \frac{\zeta^2}{2\mu m} x \cdot (p_r \times a) - r \cdot a + w \cdot p_a, \tag{234} \]

where \( w \) are Lagrange multipliers. Consistency checks of the primary constraints \( \Phi_1 \) result with a set of secondary constraint functions

\[ \Phi_2 = \{p_a, H_T\} = \frac{\zeta^2}{2\mu m} x \times p_r + r. \tag{235} \]

Consistency checks of these secondary constraints lead to the tertiary constraint functions

\[ \Phi_3 = \left\{ \frac{\zeta^2}{2\mu m} x \times p_r + r, H_T \right\} = \frac{\zeta^2}{\mu m} x \times a + m\zeta p_r + p_x. \]

We continue by checking the compatibility of \( \Phi_3 \). These give the fourtiery constraints

\[ \Phi_4 = \left\{ \frac{\zeta^2}{\mu m} x \times a + m\zeta p_r + p_x, H_T \right\} = \frac{\zeta^2}{\mu m} x \times w + \frac{3\zeta^2}{2\mu m} p_r \times a + m\zeta a = 0. \tag{236} \]

There are three equations in (236) but only two of them are linearly independent, so that two components of the Lagrange multiplier \( w \) can be determined depending on the third one. By taking the dot product of (236) with \( x \) we arrive at a new constraint

\[ \frac{3\zeta^2}{2\mu m} x \cdot p_r \times a + m\zeta a \cdot x = 0. \]

The consistency condition for this scalar constraint

\[ \left\{ \frac{3\zeta^2}{2\mu m} x \cdot p_r \times a + m\zeta a \cdot x, H_T \right\} = \frac{3\zeta^2}{2\mu m} p_r \times w \cdot x + m\zeta w \cdot x + m\zeta a \cdot p_r = 0 \]
determines the third and last component of the Lagrange multiplier $w$. Substitution of this component into (236) leads to the determination of all of the components of $w$’s as follows

$$w = \frac{1}{x^2} \left( \mu \left( \frac{3\zeta^2}{2\mu m} x \times p_r + m\zeta x \right) \times \left( \frac{3\zeta^2}{2\mu m} p_r \times a + m\zeta a \right) - m\zeta (a \cdot p_r)x \right).$$

(237)

Substitutions of the Lagrange multipliers $w$ in (237) result with the explicit determination of the total Hamiltonian function (234). Using this total Hamiltonian, the equations of motion are

$$\dot{x} = p_r, \quad \dot{p}_r = \frac{\zeta^2}{2\mu m} p_r \times a,$$

(238)

$$\dot{a} = \frac{1}{x^2} \left( \mu \left( \frac{3\zeta^2}{2\mu m} x \times p_r + m\zeta x \right) \times \left( \frac{3\zeta^2}{2\mu m} p_r \times A + m\zeta a \right) - m\zeta x(a \cdot p_r) \right),$$

(239)

$$\dot{p}_a = \frac{\zeta^2}{2\mu m} x \times p_r + x, \quad \dot{p}_r = a.$$

(240)

6. Conclusions & future work. In this paper, we have constructed Tulczyjew’s triple for acceleration bundles. This allowed us to study second and third-order Lagrangian formalisms in terms of Lagrangian submanifolds and symplectic diffeomorphisms. We have presented the symplectic transformation between the Ostrogradsky-Legendre and the Schmidt-Legendre transformations. Several examples both from the degenerate and the non-degenerate Lagrangians cases have been presented in this paper.

Some possible future research problems are given as follows:

- To construct Skinner-Rusk unified formalism [69, 70, 71] in terms of the acceleration bundle and the Schmidt-Legendre transformation. This was done for the case of the Ostrogradsky-Legendre transformation in [63], and for the Hamilton-Jacobi theory in [18].
- To study the Schmidt-Legendre transformation when the configuration space is a Lie group under the existence of some symmetries. We cite [19, 32, 33] for the Ostrogradsky-Legendre transformation on Lie groups, and for Ostrogradsky-Lie-Poisson reduction. We refer [31, 30, 39] for Tulczyjew’s triple in the framework of Lie groups.
- To discuss the regularity issues studied in [20] in the realm of the acceleration bundle and the Schmidt-Legendre transformation. In [20], this is done in case of the Ostrogradsky-Legendre transformation.
- To work on Lagrangian systems with higher order constraints [14] in terms of the acceleration bundle.
- To apply the present framework to the optimization problems. A relationship between second order Lagrangian systems and the optimization problems has been established in [8].
- To generalize the present geometry to the case of Lagrangian functions defined on higher order tangent bundles $T^kQ$ for $k > 3$.
- To extend the present framework for the higher order dynamics in the Lie algebroid setting in the light of some recent studies [2, 44].
Acknowledgments. We are greatly thankful to two anonymous referees for their valuable remarks and for their constructive comments which helped us to improve the manuscript. Possible applications, and the related citations, of the presented framework to constrained systems, Lie algebroid settings, optimization problems, regularity issues are referees recommendations. One of us (OE) is grateful to Hasan Gümrü for pointing out the examples of Sarıoğlu-Tekin and Clément Lagrangians, and for many discussions on Tulczyjew triple. (OE) is also grateful to Filiz Çağatay Uğun especially for her corrections and help on the examples 4 and 5.

REFERENCES

[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, Reading, Massachusetts, Benjamin/Cummings Publishing Company, 1978.
[2] L. Abrunheiro and L. Colombo, Lagrangian Lie subalgebroids generating dynamics for second-order mechanical systems on Lie algebroids, Mediterranean Journal of Mathematics, 15 (2018), Art. 57, 19 pp.
[3] K. Andrzejewski, J. Gonera, P. Machalski and P. Maślanka, Modified Hamiltonian formalism for higher-derivative theories, Physical Review D, 82 (2010), 045008.
[4] V. I. Arnol’d, Mathematical Methods of Classical Mechanics, Vol. 60, Springer-Verlag, New York, 1989.
[5] C. Battle, J. Gomis, J. M. Pons and N. Roman-Roy, Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems, Journal of Mathematical Physics, 27 (1986), 2953–2962.
[6] C. Battle, J. Gomis, J. M. Pons and N. Roman-Roy, Lagrangian and Hamiltonian constraints for second-order singular Lagrangians, Journal of Physics A: Mathematical and General, 21 (1988), 2693–2703.
[7] S. Benenti, Hamiltonian Structures and Generating Families, Springer Science & Business Media, 2011.
[8] A. M. Bloch and P. E. Crouch, On the equivalence of higher order variational problems and optimal control problems. In Decision and Control, 1996., Proceedings of the 35th IEEE Conference on, IEEE, 2 (1996), 1648–1653.
[9] K. Bolonek and P. Kosinski, Hamiltonian structures for pais–uhlenbeck oscillator, Acta Physica Polonica B, 36 (2005), 2115.
[10] A. J. Bruce, K. Grabowska and J. Grabowski, Higher order mechanics on graded bundles, Journal of Physics A: Mathematical and Theoretical, 48 (2015), 205203, 32pp.
[11] F. Çağatay Uğun, O. Esen and H. Gümrü, Reductions of topologically massive gravity I: Hamiltonian analysis of second order degenerate Lagrangians, Journal of Mathematical Physics, 59 (2018), 013510, 16pp.
[12] C. M. Campos, M. de León, D.M. de Diego and J. Vankerschaver, Unambiguous formalism for higher order Lagrangian field theories, Journal of Physics A: Mathematical and Theoretical, 42 (2009), 475207, 24pp.
[13] F. Cardin, Morse families and constrained mechanical systems, Generalized hyperelastic materials. Meccanica, 26 (1991), 161–167.
[14] H. Cendra and S. D. Grillo, Lagrangian systems with higher order constraints, Journal of Mathematical Physics, 48 (2007), 052904, 35pp.
[15] T. J. Chen, M. Fasiello, E. A. Lim and A. J. Tolley, Higher derivative theories with constraints: Exorcising Ostrogradski’s ghost, Journal of Cosmology and Astroparticle Physics, 2 (2013), 042, front matter+17 pp.
[16] G. Clément, Particle-like solutions to topologically massive gravity, Classical and Quantum Gravity, 11 (1994), L115–L120.
[17] L. Colombo, Second-order constrained variational problems on Lie algebroids: Applications to optimal control, Journal of Geometric Mechanics, 9 (2017), 1–45.
[18] L. Colombo, M. de León, P. D. Prieto-Martínez and N. Román-Roy, Unified formalism for the generalized kth-order Hamilton–Jacobi problem, International Journal of Geometric Methods in Modern Physics, 11 (2014), 1460037, 9pp.
[19] L. Colombo and D. M. de Diego, Higher-order variational problems on Lie groups and optimal control applications, Journal Geometric Mechanics, 6 (2014), 451–478.
[20] L. Colombo and P. D. Prieto-Martínez, Regularity properties of fiber derivatives associated with higher-order mechanical systems, *Journal of Mathematical Physics*, 57 (2016), 082901, 25pp.

[21] M. Crampin, W. Sarlet and F. Cantrijn, Higher-order differential equations and higher-order Lagrangian mechanics. In *Mathematical Proceedings of the Cambridge Philosophical Society*, 99 (1986), 565–587.

[22] A. Deriglazov, *Classical Mechanics*, Springer International Publishing, 2017.

[23] N. Deruelle, Y. Sendouda and A. Youssef, Various Hamiltonian formulations of $f(R)$ gravity and their canonical relationships, *Physical Review D*, 80 (2009), 084032, 11pp.

[24] S. Deser, R. Jackiw and S. Templeton, Topologically massive gauge theories, *Annals of Physics*, 140 (1982), 372–411.

[25] S. Deser, R. Jackiw and S. Templeton, Three-dimensional massive gauge theories, *Physical Review Letters*, 48 (1982), 975–978.

[26] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monograph Series, Yeshiva University, New York, 1967.

[27] P. A. M. Dirac, Generalized hamiltonian dynamics, In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 246 (1958), 326–332.

[28] C. T. J. Dodson and M. S. Radivoiovici, Tangent and frame bundles of order two, *An. Stiint. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. (N.S.)*, 28 (1982), 63–71.

[29] C. T. Dodson, G. Galanis and E. Vassiliou, *Geometry in a Fréchet Context: A Projective Limit Approach*, Cambridge University Press, 2016.

[30] O. Esen and H. Gümral, Tulczyjew’s triplet for Lie groups I: Trivializations and reductions, *Journal of Lie Theory*, 24 (2014), 1115–1160.

[31] O. Esen and H. Gümral, Tulczyjew’s triplet for Lie groups II: Dynamics, *Journal of Lie Theory*, 27 (2017), 329–356.

[32] F. Gay-Balmaz, D. D. Holm, D. M. Meier, T. S. Ratiu and F.-X. Vialard, Invariant higher-order variational problems, *Communications in Mathematical Physics*, 309 (2012), 413–458.

[33] F. Gay-Balmaz, D. D. Holm and T. S. Ratiu, Higher order Lagrange-Poincaré and Hamilton-Poincaré reductions, *Bulletin of the Brazilian Mathematical Society*, 42 (2011), 579–606.

[34] M. J. Gotay and J. M. Nester, Presymplectic Lagrangian systems. I: The constraint algorithm and the equivalence theorem, *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 30 (1979), 129–142.

[35] M. J. Gotay and J. M. Nester, Generalized constraint algorithm and special presymplectic manifolds, *Geometric Methods in Mathematical Physics* (Proc. NSF-CBMS Conf., Univ. Lowell, Lowell, Mass., 1979), Lecture Notes in Math., 775, Springer, Berlin, (1980), 78–104.

[36] M. J. Gotay and J. M. Nester, Apartheid in the Dirac theory of constraints, *Journal of Physics A: Mathematical and General*, 17 (1984), 3063–3066.

[37] M. J. Gotay, J. M. Nester and G. Hinds, Presymplectic manifolds and the Dirac Bergmann theory of constraints, *Journal of Mathematical Physics*, 19 (1978), 2388–2399.

[38] K. Grabowska and L. Vitagliano, Tulczyjew triple in higher derivative field theory, *Journal of Geometric Mechanics*, 7 (2015), 1–33.

[39] K. Grabowska and M. Zajac, The Tulczyjew triple in mechanics on a Lie group, *Journal of Geometric Mechanics*, 8 (2016), 413–435.

[40] J. Grabowski, K. Grabowska and P. Urbański, Geometry of Lagrangian and Hamiltonian formalisms in the dynamics of strings, *Journal of Geometric Mechanics*, 6 (2014), 503–526.

[41] X. Grácia, J. M. Pons and N. Román-Roy, Higher-order Lagrangian systems: Geometric structures, dynamics, and constraints, *Journal of mathematical physics*, 32 (1991), 2744–2763.

[42] S. W. Hawking and T. Hertog, Living with ghosts, *Physical Review D*, 65 (2002), 103515, 8pp.

[43] M. Jóźwikowski and M. Rotkiewicz, Models for higher algebroids, *Journal of Geometric Mechanics*, 7 (2015), 317–359.

[44] M. Jóźwikowski, Prolongations vs. Tulczyjew triples in Geometric Mechanics, arXiv preprint, arXiv:1712.09858, (2017).

[45] U. Kasper, Finding the Hamiltonian for cosmological models in fourth-order gravity theories without resorting to the Ostrogradski or Dirac formalism, *General Relativity and Gravitation*, 29 (1997), 221–233.

[46] B. Lawruk, J. Śniatycki and W. M. Tulczyjew, Special symplectic spaces, *Journal of Differential Equations*, 17 (1975), 477–497.
[47] M. de León and D. M. de Diego, Symmetries and constants of the motion for higher order Lagrangian systems, *Journal of Mathematical Physics*, 36 (1995), 4138–4161.

[48] M. de León and E. A. Lacomba, Lagrangian submanifolds and higher-order mechanical systems, *Journal of Physics A: Mathematical and General*, 22 (1989), 3809–3820.

[49] M. de León, J. C. Marrero and E. Martínez, Lagrangian submanifolds and dynamics on Lie algebroids, *Journal of Physics A: Mathematical and General*, 38 (2005), R241–R308.

[50] M. de León and P. R. Rodrigues, *Generalized Classical Mechanics and Field Theory: A Geometrical Approach of Lagrangian and Hamiltonian Formalisms Involving Higher Order Derivatives*, North-Holland Publishing Co., Amsterdam, 1985.

[51] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publishing Co., Dordrecht, 1987.

[52] P. D. Mannheim and A. Davidson, Dirac quantization of the Pais-Uhlenbeck fourth order oscillator, *Physical Review A*, 71 (2005), 042110, 9pp.

[53] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, Second edition. Texts in Applied Mathematics, 17, Springer-Verlag, New York, 1999.

[54] E. Martínez, Higher-order variational calculus on Lie algebroids, *Journal of Geometric Mechanics*, 7 (2015), 81–108.

[55] I. Masterov, An alternative Hamiltonian formulation for the Pais-Uhlenbeck oscillator, *Nuclear Physics B*, 902 (2016), 95–114.

[56] I. Masterov, The odd-order Pais-Uhlenbeck oscillator, *Nuclear Physics B*, 907 (2016), 495–508.

[57] R. Miron, *The Geometry of Higher-Order Lagrange Spaces: Applications to Mechanics and Physics*, Kluwer Academic Publishers Group, Dordrecht, 1997.

[58] A. Mostafazadeh, A Hamiltonian formulation of the Pais–Uhlenbeck oscillator that yields a stable and unitary quantum system, *Physics Letters A*, 375 (2010), 93–98.

[59] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics, Bristol, 2003.

[60] N. V. Nesterenko, Singular Lagrangians with higher derivatives, *Journal of Physics A: Mathematical and General*, 22 (1989), 1673–1687.

[61] A. Pais and G. E. Uhlenbeck, On field theories with non-localized action, *Physical Review D*, 49 (1994), 6354–6366.

[62] R. Skinner, First order equations of motion for classical mechanics, *Journal of Mathematical Physics*, 24 (1983), 2581–2588.

[63] R. Skinner and R. Rusk, Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$, *Journal of Mathematical Physics*, 24 (1983), 2589–2594.

[64] J. M. Pons, Ostrogradski’s theorem for higher-order singular Lagrangians, *Lett. Math. Phys.*, 17 (1989), 181–189.

[65] M. S. Rashid and S. S. Khalil, Hamiltonian description of higher order Lagrangians, *International Journal of Modern Physics A*, 11 (1996), 4551–4559.

[66] H. J. Schmidt, Stability and Hamiltonian formulation of higher derivative theories, *Physical Review D*, 49 (1994), 6354–6366.

[67] H. J. Schmidt, An alternate Hamiltonian formulation of fourth-order theories and its application to cosmology, arXiv preprint, arXiv:9501019.

[68] Ū Saroğlu and B. Tekin, Topologically massive gravity as a Pais-Uhlenbeck oscillator, *Classical and Quantum Gravity*, 23 (2006), 7541–7549.

[69] R. Skinner, First order equations of motion for classical mechanics, *Journal of Mathematical Physics*, 24 (1983), 2581–2588.

[70] R. Skinner and R. Rusk, Generalized Hamiltonian dynamics. II. Gauge transformations, *Journal of Mathematical Physics*, 24 (1983), 2595–2601.

[71] J. Śniatycki and W. M. Tulczyjew, Generating forms of Lagrangian submanifolds, *Indiana Univ. Math. J.*, 22 (1972/73), 267–275.

[72] A. Suri, Geometry of the double tangent bundles of Banach manifolds, *Journal of Geometry and Physics*, 74 (2013), 91–100.

[73] A. Suri, Higher order tangent bundles, *Mediterr. J. Math.*, 14 (2017), Art. 5, 17pp.

[74] W. M. Tulczyjew, A symplectic formulation of relativistic particle dynamics, *Acta Physica Polonica B*, 8 (1977), 431–447.
[76] W. M. Tulczyjew, The Legendre transformation, *Ann. Inst. Henri Poincaré Sec. A: Phys. Théor.*, 27 (1977), 101–114.
[77] W. M. Tulczyjew, A symplectic formulation of particle dynamics, *Differential Geometric Methods in Mathematical Physics*, Lect. Notes in Math., 570 (1977), 457–463.
[78] W. M. Tulczyjew and P. Urbanski, A slow and careful legendre transformation for singular lagrangians, *Acta Physica Polonica. Series B*, 30 (1999), 2909–2978.
[79] A. Weinstein, Symplectic manifolds and their Lagrangian submanifolds, *Adv. in Math.*, 6 (1971), 329–346.
[80] A. Weinstein, *Lectures on Symplectic Manifolds*, Exp. lec. from the CBMS, Regional Conference Series in Mathematics, 29, A.M.S., Providence, 1977.

Received February 2017; revised August 2018.

E-mail address: oesen@gtu.edu.tr
E-mail address: partha@bose.res.in