Automorphisms of Partially Commutative Groups III: Inversions and Transvections

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Abstract

The structure of a certain subgroup $S_t$ of the automorphism group of a partially commutative group (RAAG) $G$ is described in detail: namely the subgroup generated by inversions and elementary transvections. We define admissible subsets of the generators of $G$, and show that $S_t$ is the subgroup of automorphisms which fix all subgroups $\langle Y \rangle$ of $G$, for all admissible subsets $Y$. A decomposition of $S_t$ as an iterated tower of semi-direct products is given and the structure of the factors of this decomposition described. The construction allows a presentation of $S_t$ to be computed, from the commutation graph of $G$.

1 Introduction

A partially commutative group (also known as a right-angled Artin group) is a group given by a finite presentation $\langle X \mid R \rangle$, where $R$ is a subset of $\{ [x, y] \mid x, y \in X, x \neq y \}$. (Our convention is that $[x, y] = x^{-1}y^{-1}xy$.) The commutation graph of a partially commutative group is the simple graph $\Gamma$ with vertices $X$ and an edge joining $x$ to $y$ if and only if $[x, y] \in R$. (A simple graph is one without multiple edges or self-incident vertices.) A simple graph $\Gamma$ uniquely determines a presentation $\langle X \mid R \rangle$ of a partially commutative group with commutation graph $\Gamma$, which we denote $G_\Gamma$, and if $\Gamma$ and $\Gamma'$ are simple graphs such that $G_\Gamma \cong G_{\Gamma'}$, then $\Gamma$ and $\Gamma'$ are isomorphic graphs.
The study of isomorphisms between partially commutative groups therefore reduces to the study of automorphisms of groups $G_\Gamma$. For background information on automorphisms of partially commutative groups we refer to [2], [3], [6] and the references therein. In particular, the automorphism group $\text{Aut}(G_\Gamma)$ of $G_\Gamma$ was shown to have a finite generating set by Laurence, building on work of Servatius [7, 9]; a finite presentation for these groups was found by Day [3]; and geometric models of the Outer automorphism group of $\Gamma_\Gamma$ were constructed in [2]. Here we consider the decomposition of $\text{Aut}(G_\Gamma)$ into subgroups corresponding to particular types of the generators found by Laurence and Servatius.

Laurence and Servatius identified four types of elementary automorphism which together generate $\text{Aut}(G_\Gamma)$. These are

- automorphisms which permute the elements of $X$, called graph automorphisms,
- automorphisms which map an element $x \in X$ to $x^{-1}$ and fix all other elements of $X$, called inversions,
- elementary transvections which map an element $x \in X^{\pm 1}$ to $xy^{\pm 1}$, for some element $y \in X$ and fix all elements of $X\setminus\{x\}$, and
- vertex conjugating automorphisms which, for some element $x \in X^{\pm 1}$ and some subset $C \subseteq X$, map $c$ to $cx$ and fix all elements of $X\setminus C$.

Conditions on elements and subsets of $X$ under which elementary transvections and vertex conjugating automorphisms exist are discussed in Section 2 below. The subgroup $\text{Aut}^*(G_\Gamma)$ generated by inversions, elementary transvections and elementary vertex conjugating automorphisms has finite index; and $\text{Aut}(G_\Gamma) = \text{Aut}(\Gamma_\Gamma) \ltimes \text{Aut}^*(G_\Gamma)$, where $\text{Aut}(\Gamma_\Gamma)$ is a subgroup of the group of automorphisms of $G_\Gamma$ which permute $X^{\pm 1}$ (see Section 2.1 below for more detail).

Generalising the notion of vertex conjugating automorphism: an automorphism $\phi \in \text{Aut}(G)$ is called a conjugating automorphism if there exists $g_x \in G$ such that $x\phi = x^{g_x}$, for all $x \in X$. The subgroup of $\text{Aut}(G)$ consisting of all conjugating automorphisms is denoted $\text{Conj}(G)$. Laurence [7] proved that $\text{Conj}(G)$ is the group generated by the vertex conjugating automorphisms and later Toinet [10] constructed a finite presentation for this group (with generators the vertex conjugating automorphisms). Here we give a description of the structure of the subgroup generated by inversions and elementary transvections. We use the methods of [6], where a characterisation of $\text{Aut}^*(G_\Gamma)$ was given, in terms of stabilisers; which we shall now describe.
For $x \in X$, the link, $\text{lk}(x)$, of $x$ is the set of all vertices joined to $x$ by an edge of $\Gamma$. The star, $\text{st}(x)$, of $x$ is $\text{lk}(x) \cup \{x\}$. We define an equivalence relation $\sim$ on $X$ by $x \sim y$ if and only if either $\text{st}(x) = \text{st}(y)$ or $\text{lk}(x) = \text{lk}(y)$; and denote by $[x]$ the $\sim$ equivalence class of $x$. (See Section 2 for more detail.) The admissible set, $a(x)$ of $x$ is
\[
a(x) = \bigcap_{y \in \text{lk}(x)} \text{st}(y),
\]
and we define
\[
\mathcal{K} = \{a(x) \mid x \in X\},
\]
the set of all admissible sets. (See Example 3.1 below.)

For any subset $Y$ of $X$ we denote by $G(Y)$ the subgroup of $G$ generated by $Y$. In $[6]$ we defined
\[
\text{St}(\mathcal{K}) = \{\phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y), \text{ for all } Y \in \mathcal{K}\}
\]
and
\[
\text{St}^{\text{conj}}(\mathcal{K}) = \{\phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y)^{f_Y}, \text{ for some } f_Y \in G, \text{ for all } Y \in \mathcal{K}\},
\]
and proved that $\text{Aut}^*(G_\Gamma) = \text{St}^{\text{conj}}(\mathcal{K})$. As inversions and elementary transvections all belong to $\text{St}(\mathcal{K})$, it follows that $\text{Aut}^*(G)$ is generated by $\text{St}(\mathcal{K})$ and $\text{Conj}(G_\Gamma)$. However, in general these two subgroups intersect non-trivially and it is not the case that $\text{Aut}^*(G_\Gamma) = \text{St}(\mathcal{K}) \cdot \text{Conj}(G_\Gamma)$. Necessary and sufficient conditions on the graph $\Gamma$ under which the latter holds are given in $[6]$.

In this paper we give a decomposition of $\text{St}(\mathcal{K})$ as chain of semi-direct products, of subgroups whose structure we can, to a significant extent, understand. To this end, the height $h(x)$ of an element $x \in X$ is defined to be the largest integer $i$ such that there exists a strictly descending chain $a(x_i) > a(x_{i-1}) > \cdots > a(x_0)$, where $x_i = x$. The $\mathcal{K}$-height of $h_{\mathcal{K}}(G)$ of $\mathcal{K}$ is the maximum of the heights of elements of $X$. Let $h_{\mathcal{K}} = h_{\mathcal{K}}(G)$ and for $0 \leq k \leq h_{\mathcal{K}}$, let the level $k$ vertex set of $X$ be
\[
v(k) = \{y \in X \mid h(y) = k\}.
\]
(See Example 3.1.) We define
\[
\text{St}^*_k(\mathcal{K}) = \{\phi \in \text{St}(\mathcal{K}) \mid y\phi = y, \text{ for all } y \in X \setminus v(k)\}
\]
and
\[
\text{St}^*_x(\mathcal{K}) = \{\phi \in \text{St}(\mathcal{K}) \mid y\phi = y, \text{ for all } y \in X \setminus [x]\}.
\]
Our main results are the following, where we write $\text{St}^*_k$ and $\text{St}^*_x$ for $\text{St}^*_k(\mathcal{K})$ and $\text{St}^*_x(\mathcal{K})$, respectively.
Theorem 1.1. Let $C$ be a transversal for $\sim$ and let $C(k) = \mathcal{V}(k) \cap C$. Then

(i) $\text{St}(\mathcal{K}) = (\cdots (\text{St}_0^x \ltimes \text{St}_1^x) \ltimes \cdots \ltimes \text{St}_{h_{\mathcal{K} - 1}}^x) \ltimes \text{St}_{h_{\mathcal{K}}}^x$ and

(ii) $\text{St}_k^x = \prod_{y \in \mathcal{C}(k)} \text{St}_y^x$, for $k = 0, \ldots, h_{\mathcal{K}}$.

(See Example 3.5.) This leaves the structure of $\text{St}_x^y$ to be determined. There are two cases to consider, which depend on the size of a further set, the closure of an element $x$ of $X$, defined as

$$\text{cl}(x) = \cap_{y \in \text{st}(x)} \text{st}(y).$$

(See Section 2 for details.) As $\text{cl}(x) = \mathcal{A}(x) \cap \text{st}(x)$ we always have $\text{cl}(x) \subseteq \mathcal{A}(x)$. If $\text{cl}(x) = \mathcal{A}(x)$, then $\mathcal{G}([x])$ is a free Abelian group and consequently $\text{St}_x^y$ has the following form, where, for positive integers $a, b$, we denote the group of $a \times b$ integer matrices under addition by $\mathcal{M}(a, b)$.

Theorem 1.2 (cf. Theorem 4.1). Let $x \in X$ such that $\mathcal{A}(x) = \text{cl}(x)$. Assume and that $|\mathcal{A}(x)| = r$ and $|[x]| = s$. Then $s \leq r$ and

$$\text{St}_x^y(\mathcal{K}) \cong \text{GL}(s, \mathbb{Z}) \ltimes \mathcal{M}(s, r - s),$$

where, for $A \in \text{GL}(s, \mathbb{Z})$ and $B \in \mathcal{M}(s, r - s)$, the automorphism $A \theta$ maps $B$ to $A^{-1}B \in \mathcal{M}(s, r - s)$.

On the other hand, if $\text{cl}(x)$ is a proper subset of $\mathcal{A}(x)$ then $\mathcal{G}([x])$ is a free group. In this case we define $\mathcal{A}_{\text{out}}(x) = \mathcal{A}(x) \setminus \text{cl}(x)$ and we have the following decomposition of $\text{St}_x^y$.

Theorem 1.3 (cf. Theorem 4.3). Let $x \in X$ such that $\mathcal{A}(x) \neq \text{cl}(x)$. Assume and that $|\text{cl}(x)| = q$ and $|[x]| = p$. Then $p \leq q$ and $\text{St}_x^y$ has subgroups $\text{St}_{x,l}^y$ and $\text{St}_{x,s}^y$ such that

$$\text{St}_x^y = \text{St}_{x,l}^y \ltimes \text{St}_{x,s}^y,$$

$$\text{St}_{x,l}^y = \{ \phi \in \text{St}_x^y \mid y \phi \in \mathcal{G}([x] \cup \mathcal{A}_{\text{out}}(x)), \forall y \in [x] \}$$

and

$$\text{St}_{x,s}^y \cong \mathcal{M}(p, q - p).$$

Although we do not have a structural decomposition we give a finite presentation of $\text{St}_{x,l}^y$ in Theorem 4.6. Combining these theorems allows us to find generators of $\text{St}(\mathcal{K})$, giving our final result.

Corollary 1.4. The subgroup of $\text{Aut}(\mathcal{G}_T)$ generated by the set of all inversions and elementary transvections is precisely $\text{St}(\mathcal{K})$. Moreover, $\text{St}(\mathcal{K})$ has a finite presentation with these generators.
Indeed, such a finite presentation of St(\(K\)) may be explicitly constructed from the decomposition appearing in the theorems above. (See Examples 3.5, 4.2, 4.4 and 4.7.)

In Section 2 we cover the necessary background on partially commutative groups, admissible sets and closure, and generators of the automorphism group of \(\mathbb{G}_\Gamma\). Section 3 contains the proof of Theorem 1.1. Section 4 is concerned with St\(^v_x\). In Section 4.1 a more detailed version of Theorem 1.2 is stated and proved, namely, Theorem 1.2. In Section 4.2 we prove Theorem 4.3, which is a more detailed version of Theorem 1.3 and then define generators and relations for St\(^v_{x,l}\). The remainder of the paper consists of the proof of Theorem 4.6. For this we use peak reduction, constructing a modification of the process of [3] to work within the given generating set of St\(^v_{x,l}\).

2 Preliminaries

Throughout this article, let \(\mathbb{G} = \mathbb{G}_\Gamma\) be the partially commutative group with commutation graph \(\Gamma\) and presentation \(\langle X \mid R \rangle\), as above. For \(Y \subset X\) the subgroup \(\mathbb{G}(Y)\) of \(\mathbb{G}\) generated by \(Y\) is also a partially commutative group with commutation graph equal to the full subgraph of \(\Gamma\) induced by \(Y\) [1].

For \(w \in \mathbb{G}\) denote by Supp\((w)\) the minimal subset \(Y\) of \(X\) such that \(w \in \mathbb{G}(Y)\). The length \(|w|\) of an element \(w\) of \(\mathbb{G}\) is the minimum of the lengths of words in \(F(X)\) representing \(w \in \mathbb{G}\). If \(u\) is a word of \(F(X)\) of \(F(X)\)-length equal to the length \(|u|\) of \(u\) in \(\mathbb{G}\), then we say \(u\) is a minimal word. If \(u\) and \(v\) are minimal words such that \(|uv| = |u| + |v|\), we write \(uv = u \circ v\).

We extend definitions of star and link from single elements of \(X\) to subsets of \(X\): for \(Y \subset X\) define the star of \(Y\) to be \(\text{st}(Y) = \cap_{x \in Y} \text{st}(y)\). By convention we set \(\text{st}(\emptyset) = X\). We define the closure of \(Y\) to be \(\text{cl}(Y) = \text{st}(\text{st}(Y))\). The closure operator on \(\Gamma\) satisfies, among other things, the properties that \(\text{cl}(Y)\) is a simplex (i.e. the full subgraph on \(\text{cl}(Y)\) is a complete graph) and for \(x \in X\), the closure \(\text{cl}(x)\) is the maximal simplex contained in \(\text{st}(x)\). We set \(\mathcal{L} = \{\text{cl}(x) \mid x \in X\}\).

(See [5, Lemma 2.4] for further details.)

As in [6, Lemma 2.5], we have \(a(x) = \text{cl}(x)\) if and only if \(a(x) \subseteq \text{st}(x)\); from which it follows that \(a(x) = \text{cl}(x)\) if and only if \(a(x)\) is a simplex. The following straightforward lemmas are proved in [6].

**Lemma 2.1.** For all \(x, y, z \in X\), the following hold.

1. If \(y \in a(x)\) then \(a(y) \subseteq a(x)\).
(ii) If \([x, y] = 1\) then \([\mathbb{G}(a(x)), \mathbb{G}(a(y))] = 1\).

(iii) \(a(y) \subseteq a(x)\) if and only if \(lk(x) \subset st(y)\).

(iv) \(a(x) = a(z)\) if and only if \(z \in [x]\).

(v) \([x] = a(x) \setminus (\bigcup\{a(y) | y \in a(x) \text{ and } a(y) \subsetneq a(x)\})\).

Let \(\sim_{st}\) be the relation on \(X\) given by \(x \sim_{st} y\) if and only if \(st(x) = st(y)\) and \(\sim_{lk}\) be the relation given by \(x \sim_{lk} y\) if and only if \(lk(x) = lk(y)\). These are equivalence relations and the equivalence classes of \(x\) under \(\sim_{st}\) and \(\sim_{lk}\) are denoted by \([x]^{st}\) and \([x]^{lk}\), respectively. Moreover \(\sim_{st} = \sim_{st} \cup \sim_{lk}\). In addition (see [6, Lemma 2.7] for details) if \(a(x) = cl(x)\) then \([x] = [x]^{st}\) and otherwise \([x] = [x]^{lk}\).

Let \(L = X \cup X^{-1}\) and for \(x \in L\) let \(v(x) = X \cap \{x, x^{-1}\}\). We extend the notation for stars, links, closures and admissible sets from \(X\) to \(L\) as follows.

- For \(x \in L\), let \(st(x), lk(x), [x], a(x)\) and \(cl(x)\) denote \(st(v(x)), lk(v(x)), [v(x)], a(v(x))\) and \(cl(v(x))\), respectively, and similarly for \([x]^{st}\) and \([x]^{lk}\).

- For \(o\) equal to any one of the operators \(st, lk, [\cdot], a\) or \(cl\) above, let \(o_L(x)\) denote \(o(x) \cup o(x)^{-1}\); so \(st_L(x) = st(x) \cup st(x)^{-1}\), etc..

### 2.1 Generators for \(\text{Aut}(\mathbb{G})\)

First we describe the conditions under which elementary transvections and vertex conjugating automorphisms exist, then we define the subgroup \(\text{Aut}(\Gamma^c)\) and finally we extend the definitions of Laurence and Servatius to give a larger generating set, which is convenient for peak reduction proofs.

For \(x, y \in L\), with \(x \neq y^{\pm 1}\) there exists an elementary transvection in \(\text{Aut}(\mathbb{G})\) mapping \(x\) to \(xy\) if and only if \(lk(x) \subseteq st(y)\) (see for example [9]). Given \(y \in L\) and \(T \subset L \setminus \{y^{\pm 1}\}\) such that \(T \cap T^{-1} = \emptyset\) and \(lk(t) \subseteq st(y)\), for all \(t \in T\); the automorphism \(\tau_{L,y} = \prod_{t \in T} \tau_{t,y}\) is called a transvection.

Let \(x \in L\) and \(C \subseteq X \setminus st(x)\). Then there exists an automorphism of \(\mathbb{G}\) mapping \(c \in C\) to \(x^{-1}cx\), and fixing all other elements of \(X\), if and only if \(C\) is the vertex set of a union of connected components of \(\Gamma \setminus st(x)\); the graph obtained from \(\Gamma\) by removing all vertices of \(st(x)\) and all their incident edges. (see for example [9]). We denote this vertex conjugating automorphism by \(\alpha_{C,x}\). If \(C\) consists of the vertices of a single connected component of \(\Gamma \setminus st(x)\) then \(\alpha_{C,x}\) is called an elementary vertex conjugating automorphism.

For ease of reference we make the following definitions.
Definition 2.2. Denote by

1. $\text{Aut}(\Gamma^\pm)$ the subgroup of automorphisms which permute $L$;
2. $\text{Inv} = \text{Inv}(G)$ the set of inversions;
3. $\text{Tr} = \text{Tr}(G)$ the set of elementary transvections;
4. $\text{LInn} = \text{LInn}(G)$ the set of elementary vertex conjugating automorphisms.

The set of all transvections is denoted $\text{Tr}^+$ and the set of all vertex conjugating automorphisms by $\text{LInn}^+$.

The group $\text{Aut}^C$, mentioned in the introduction depends on the choice of an ordering on each of the sets $[x]$. Choose a total order $<$ on each set $[x] \subseteq X$. Then $\text{Aut}^C$ is defined to be the group of automorphisms of $\mathcal{G}_T$ which permute $L$ and respect the order on $[x]$, for all $x \in X$. That is, an automorphism $\phi$ belongs to $\text{Aut}^C$ if it belongs to $\text{Aut}(\Gamma^\pm)$, and whenever $u, v \in [x]$, with $u < v$, then $u\phi < v\phi$. (See [6] for details.) As $\text{Aut}(G) = \text{Aut}(\Gamma^\pm) \rtimes \text{Aut}^*(G)$ the focus of attention is the subgroup $\langle \text{Inv}, \text{Tr}, \text{LInn} \rangle$.

In the sequel we shall make use of a larger set than the Laurence-Servatius generators, known as Whitehead automorphisms, to generate $\text{Aut}(G)$. These originate in work of Whitehead, and were developed by Rapaport, Higgins and Lyndon, and McCool, to study Automorphisms of Free groups, using peak reduction. Day [3] defined Whitehead automorphisms over partially commutative groups and used them in peak reduction arguments, to construct finite presentations of their automorphism groups.

Definition 2.3. A Whitehead automorphism is an element of $\text{Aut}(G)$ of one of two types.

Type 1. Elements of $\text{Aut}(\Gamma^\pm)$.

Type 2. Elements of the form $\alpha_{C,x}\tau_{T,x}$, where $\alpha_{C,x} \in \text{LInn}^+$, $\tau_{T,x} \in \text{Tr}^+$, and $(C \cup C^{-1}) \cap T = \emptyset$.

In the definition of Type 2 elements we allow $\alpha_{C,x}$ or $\tau_{T,x}$, but not both, to be trivial; so $\text{LInn}^+$ and $\text{Tr}^+$ are sets of Whitehead automorphisms of Type 2.

Notation The Whitehead automorphism $\alpha_{C,x}\tau_{T,x}$ of Type 2 is denoted $(A, x)$, where $A$ is any subset of $L$ such that

1. $x \in A$ and $x^{-1} \notin A$;
2. \( A \setminus \{x\} \) is the disjoint union of the set \( C \cup C^{-1} \), the set \( T \) and a set \( U \cup U^{-1} \), where \( U \) is some subset \( \text{lk}(x) \subset X \), such that \((U \cup U^{-1}) \cap T = \emptyset \). (We always assume \( C \cup T \) is not empty.)

The set of all Whitehead automorphisms is denoted \( \Omega \).

**Remark 2.4.** 1. The notation \((A, x)\) uniquely determines an automorphism \( \phi \) say, although there may be more than one expression of \( \phi \) in terms of Laurence-Servatius generators. For example, if \( \text{lk}(x) \subseteq \text{st}(y) \) then \((\{x, x^{-1}\}, y) = \tau_{x,y} \tau_{x^{-1},y} = \alpha_{\{x, x^{-1}\},y} \).

2. For \( x \in L \) and \( A \subset L \) the pair \((A, x)\) denotes a Whitehead automorphism if and only if \( A = (C \cup U)^{\pm 1} \cup T \cup \{x\} \), for some \( C \cup U \subset X \) and \( T \subset L \) such that \( T \cap (C \cup U)^{\pm 1} = \emptyset \), \( T \cap T^{-1} = \emptyset \), \( C \) is a union of connected components of \( \Gamma \setminus \text{st}(x) \), \( U \subseteq \text{lk}(x) \), and \( T \subseteq a_L(x) \setminus \{x^{\pm 1}\} \).

In this case \((A, x) = \alpha_{C,x} \tau_{T,x} \).

Day [3] defines a Whitehead automorphism \( \phi \) to be

(i) **long range** if either \( \phi \) is of Type 1; or \( \phi \) is of Type 2, \( \phi = (A, a) \) and \( y\phi = y \), for all \( y \in \text{st}(a) \), and

(ii) **short range** if it is of Type 2, \( \phi = (A, a) \) and \( y\phi = y \), for all \( y \in X \setminus \text{st}(a) \).

**Remark 2.5.** In general, if \( \phi = (A, a) \) is of Type 2, and we set \( A_s = A \cap \text{st}(a) \) \( L \) and \( A_l = A \setminus A_s \) then \( \phi_s = (A_s, a) \) is short range, \( \phi_l = (A_l \cup \{a\}, a) \) is long range and \( \phi = \phi_s \phi_l \). Hence every Whitehead automorphism factors uniquely as a product of a short range and a long range automorphism.

**Definition 2.6.** The set of short range automorphisms is denoted \( \Omega_s \) and the set of long range automorphisms is denoted \( \Omega_l \).

As the Laurence-Servatius generators are all either short or long range Whitehead automorphisms it follows that \( \text{Aut}(G) \) is generated by the union \( \Omega_s \cup \Omega_l \) of short and long range Whitehead automorphisms.

Day [3] shows that \( \text{Aut}(G) \) has a finite presentation with generators \( \Omega_s \cup \Omega_l \) and a set of relations \( R \), partitioned into subsets \( R_1 \sim R_7 \), which we shall refer to as \( DR_1 \sim DR_7 \) in the sequel.

**3 The structure of \( \text{St}(K) \)**

The decomposition of \( \text{St}(K) \) reflects the structure of the partial order, by inclusion, on the set \( K \) which we stratify as follows. Let the **level k admissible**...
set of $X$ be

$$\mathcal{A}(k) = \bigcup_{i=0}^{k} \bigcup_{y \in v(i)} a(y).$$

With this notation $\mathcal{A}(h_K) = X$, $v(h_K) = \{y \in X | h(y) = h_K\}$,

$$\mathcal{A}(0) = v(0) = \{y \in X | a(y) = [y]\},$$

and it follows from Lemma 2.1 (v) that

$$\mathcal{A}(k) = \bigcup_{y \in v(k)} \left[ [y] \cup \bigcup_{z \in v(k-1)} a(z) \right] \cup \mathcal{A}(k-1)$$

$$= \bigcup_{y \in v(k)} [y] \cup \mathcal{A}(k-1)$$

$$= v(k) \cup \mathcal{A}(k-1)$$

and $v(k) \cap \mathcal{A}(k-1) = \emptyset$. Moreover, if $C$ is a transversal for $\sim$ (a set of representatives of equivalence classes), then setting $C(k) = C \cap v(k)$, we have $v(k) = \bigcup_{y \in C(k)} [y]$.

**Example 3.1.** Let $\Gamma$ be the graph:

```
  a -- e -- f
  |   |   |
  b ---- d -- g
  |   |
  h ---- i
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Then $[a] = \{a, b\}$, $[f] = \{f, g\}$, all other equivalence classes contain a single element, $a(a) = a(b) = \{a, b, d, h\}$, $a(c) = \{c, d, e\}$, $a(d) = \{d\}$, $a(e) = \{d, e\}$, $a(f) = a(g) = \{d, e, f, g\}$, $a(h) = \{h\}$ and $a(i) = \{c, d, e, h, i\}$; with inclusions as shown in the following diagram.
We take a transversal $C = \{a, c, d, e, f, h, i\}$ for $\sim$. Thus $h_K(G) = 3 = h(i), \quad \nu(3) = C(3) = \{i\}, \quad \nu(2) = \{c, f, g\}, \quad C(2) = \{c, f\}, \quad \nu(1) = \{a, b, e\}, \quad C(1) = \{a, e\}$ and $\nu(0) = C(0) = \{d, h\}$. Finally $A(3) = a(i) \cup a(f) \cup a(a) = X, \quad A(2) = a(c) \cup a(f) \cup a(a) = X \backslash \{i\}, \quad A(1) = a(a) \cup a(e) = \{a, b, d, e, h\}$ and $A(0) = a(d) \cup a(h) = \{d, h\}$.

Let $k \in \mathbb{N}$ such that $0 \leq k \leq h(x)$. Define the level $k$ stabiliser of $K$ to be

$$St_k(K) = \{\phi \in St(K) | y\phi = y, \text{ for all } y \in X \backslash A(k)\}.$$

With this definition, for $x \in X$ of height $k$, we have

$$St_k^x(K) \subseteq St_k^y(K) \subseteq St_k(K).$$

We shall abbreviate $St(K)$, $St_k(K)$, $St_k^x(K)$ and $St_k^y(K)$ to $St$, $St_k$, $St^x$ and $St^y$, respectively, when no ambiguity arises.

**Definition 3.2.** Let $h_K = h_K(G)$ and let $\phi \in St(K)$. For each $k$ such that $0 \leq k \leq h_K$ define the level $k$ restriction $\phi_k$ of $\phi$ to be the map given by

$$z\phi_k = \begin{cases} z, & \text{if } z \notin A(k) \\ z\phi, & \text{if } z \in A(k), \end{cases}$$

for all $z \in X$.

**Lemma 3.3.** For $\phi \in St(K)$ and $0 \leq k \leq h_K$ the map $\phi_k$ extends uniquely to an element of $St(K)$ (also denoted $\phi_k$). Moreover the map $s_k$, such that $\phi \in St(K)$ is mapped to $\phi_k$ is a retraction of $St(K)$ onto $St_k(K)$.

**Proof.** First we show that if $\phi \in St(K)$ then $\phi_k$ extends to an endomorphism of $G$, which is necessarily unique as the images of generators are determined. Suppose that $a, b \in X$ such that $[a, b] = 1$. It suffices to show that $[a\phi_k, b\phi_k] = 1$. If either $\{a, b\} \subseteq A(k)$ or $\{a, b\} \cap A(k) = \emptyset$ then clearly $[a\phi_k, b\phi_k] = 1$. This leaves the case $a \notin A(k), b \in A(k)$. As $\phi \in St(K)$ we have $b\phi \in G(a(b))$ and so, from Lemma 2.1 (ii), $[a\phi_k, b\phi_k] = [a, b\phi] = 1$. Therefore $\phi_k$ is an endomorphism of $G$.

To see that $\phi_k$ is an automorphism suppose first that $\phi, \psi \in St(K)$. If $z \notin A(k)$ then $z\phi_k\psi_k = z$, while for $z \in A(k)$, $z\phi_k\psi_k = (z\phi)\psi_k$. By definition, $z \in A(k)$ implies $z \in a(y)$, for some $y \in X$, with $h(y) \leq k$. Since $\phi \in St(K)$ we have $z\phi \in G(a(y))$, so $z\phi \in G(A(k))$ and $\text{Supp}(z\phi) \subseteq A(k)$. Therefore $z\phi_k\psi_k = (z\phi)\psi$. In particular, $z\phi_k(\phi^{-1})_k = z\phi^{-1} = z = (\phi^{-1})_k\phi_k$, so $\phi_k$ has inverse $(\phi^{-1})_k$ and is therefore an automorphism. Moreover, for $z \notin A(k)$ we have $z\phi_k\psi_k = z = (z\phi)\psi_k$ and for $z \in A(k)$ we have shown that $z\phi_k\psi_k = z(\phi\psi) = (z\phi\psi)_k$; so the map $s_k : \phi \mapsto \phi_k$ is an endomorphism of $St(K)$. By definition $\phi_k \in St_k(K)$. If $\alpha \in St_k(K)$ then $\alpha s_k = \alpha k = \alpha$, so $s_k$ has image $St_k(K)$ which is a retract of $St(K)$, as claimed. \(\square\)
Corollary 3.4. Let \( 1 \leq k \leq h_K \). The restriction of \( s_{k-1} : St \rightarrow St_{k-1} \) to \( St_k \) is a retraction onto \( St_{k-1} \) with kernel \( St_k^\gamma \). Therefore \( St_k = St_k^\gamma \times St_{k-1} \).

Proof. As \( \mathfrak{A}(k-1) \subseteq \mathfrak{A}(k) \), we have \( St_{k-1} \leq St_k \leq St \) and as \( s_{k-1} \) is a retraction of \( St \) onto \( St_{k-1} \), the restriction of \( s_{k-1} \) to \( St_k \) is surjective, and so also a retraction. Denote this restriction by \( s_{k-1}^k \). Since \( \mathfrak{A}(k) = v(k) \sqcup \mathfrak{A}(k-1) \), from the definition, \( \ker(s_{k-1}^k) = St_k^\gamma \).

Proof of Theorem 1.1. (i) follows from Corollary 3.4. To prove (ii) first note that if \( y, z \in C(k) \) with \( y \neq z \) then \( [y] \cap [z] = \emptyset \), so \( St^\gamma_y \cap St^\gamma_z = 1 \). We claim next that, in this case, \( [St^\gamma_y, St^\gamma_z] = 1 \). Suppose that \( \phi_y \in St^\gamma_y \) and \( \phi_z \in St^\gamma_z \). Then for \( u \in [y] \) there is \( w \in G(\mathfrak{a}(y)) \) such that \( \phi_y = w \). We have, from Lemma 2.1 (i) and (iv), \( [z] \cap \mathfrak{a}(y) = \emptyset \), so \( \phi_z = v \), for all \( v \in \text{Supp}(w) \), and thus \( \phi_z = w \) and \( \phi_y \phi_z = w \). On the other hand \( u \phi_z \phi_y = u \phi_y = w \). Similarly, for \( u \in [z] \) we have \( u \phi_z \phi_y = u \phi_y \phi_z \). For all other \( u \in X \) both \( u \phi_y = u \) and \( u \phi_z = u \), so \( [\phi_y, \phi_z] = 1 \), as claimed. Therefore \( \prod_{y \in C(k)} St^\gamma_y \leq St_k^\gamma \).

Now let \( \phi \in St_k^\gamma \) and let \( y \in C(k) \). Define a map \( \phi_y \) from \( X \) to \( G \) by

\[
z \phi_y = \begin{cases} z \phi, & \text{if } z \in [y] \\ z, & \text{otherwise.} \end{cases}
\]

As in the case of \( \phi_k \) in the proof of Lemma 3.3, to see that \( \phi_y \) is a homomorphism, we need only check that if \( a, b \in X \) such that \( [a, b] = 1 \), \( a \in [y] \) and \( b \notin [y] \) then \( [a \phi_y, b] = 1 \). In this situation, since \( a \in [y] \) implies \( \mathfrak{a}(a) = \mathfrak{a}(y) \), we have \( G(\mathfrak{a}(y)), b] = 1 \), and since \( \phi \in St_k^\gamma \) we have \( a \phi_y = a \phi \in G(\mathfrak{a}(y)) \), so \( [a \phi_y, b] = 1 \), as required. Moreover, if \( \phi, \psi \in St_k^\gamma \) and \( a \in [y] \), then \( a \phi_y \psi_y = (a \phi) \psi_y \). For \( u \in [y] \) we have \( u \psi_y = u \psi \), while for \( u \in \mathfrak{a}(y) \setminus [y] \), since \( h(u) < h(y) \), we have \( u \psi = u = u \psi_y \). Hence for all \( u \in \mathfrak{a}(y) \) we have \( u \psi_y = u \psi \) and therefore \( (a \phi) \psi_y = a \phi \psi \). It follows, as in the proof of Lemma 3.3, that \( \phi_y \) is an automorphism of \( G \), and by definition \( \phi_y \in St^\gamma_y \).

Define \( \phi' = \prod_{y \in C(k)} \phi_y \) and for \( z \in v(k) \) let \( \bar{z} \) denote the unique element of \( C(k) \) such that \( z \sim \bar{z} \). From the remark above Example 3.1, \( v(k) = \bigcup_{y \in C(k)} [y] \) so, for all \( z \in X \),

\[ z \phi' = \begin{cases} z \phi_z = z \phi, & \text{if } z \in v(k) \\ z, & \text{otherwise.} \end{cases} \]

Since \( \phi \in St_k^\gamma \) it follows that \( \phi = \phi' \). Therefore \( St_k^\gamma = \prod_{y \in C(k)} St^\gamma_y \).

Example 3.5. Continuing Example 3.1, we have

\[ St(K) = St^\gamma_1 \times (St^\gamma_2 \times (St^\gamma_1 \times St^\gamma_0)) \]

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where

\[ \text{St}^y_3 = \text{St}^y_1, \]
\[ \text{St}^y_2 = \text{St}^y_c \times \text{St}^y_f, \]
\[ \text{St}^y_1 = \text{St}^y_a \times \text{St}^y_e \quad \text{and} \]
\[ \text{St}^y_0 = \text{St}^y_d \times \text{St}^y_h. \]

4 The structure of \( \text{St}^y_x \)

As pointed out in the introduction, the structure of \( \text{St}^y_x \) depends on the difference between \( \text{cl}(x) \) and \( \text{a}(x) \).

4.1 \( \text{cl}(x) = \text{a}(x) \)

In case \( \text{cl}(x) = \text{a}(x) \) a description of the structure of \( \text{St}^y_x \) may be obtained by applying the results of [5], where the analogue of \( \text{St}(\mathcal{K}) \), for sets \( \text{cl}(x) \) instead of \( \text{a}(x) \) is investigated. In more detail, in [5], the subgroups

\[ \text{St}(\mathcal{L}) = \{ \phi \in \text{Aut}(\mathcal{G}) \mid \mathcal{G}(\text{cl}(x))\phi = \mathcal{G}(\text{cl}(x)), \text{ for all } x \in X \} \]

and

\[ \text{St}^\text{conj}(\mathcal{L}) = \{ \phi \in \text{Aut}(\mathcal{G}) \mid \forall x \in X, \exists g_x \in \mathcal{G}, \text{ such that } \mathcal{G}(\text{cl}(x))\phi = \mathcal{G}(\text{cl}(x))g_x \} \]

of \( \text{Aut}(\mathcal{G}) \) are defined and

- it is shown that \( \text{St}^\text{conj}(\mathcal{L}) = \text{St}(\mathcal{L}) \ltimes \text{Conj}(\mathcal{G}) \) [5, Theorem 2.20]; and

- in Section 2.6, that \( \text{St}(\mathcal{L}) \) is isomorphic to a subgroup of \( \text{GL}(|X|, \mathbb{Z}) \) generated by upper block-triangular matrices, with diagonal blocks corresponding to the equivalence classes \([x]\), of elements of \( x \in X \), together with a subgroup of the unipotent upper triangular matrices \( U(|X|, \mathbb{Z}) \), of nilpotency class equal to the centraliser dimension of \( \mathcal{G} \).

- For \( x \in X \) and \( \phi \in \text{St}(\mathcal{L}) \), the restriction of \( \phi \) to \( \mathcal{G}(\text{cl}(x)) \) is an automorphism of \( \mathcal{G}(\text{cl}(x)) \) denoted \( \phi_x \). Define the subgroup

\[ \text{St}_x(\mathcal{L}) = \{ \phi_x \mid \phi \in \text{St}(\mathcal{L}) \} \]

of \( \text{Aut}(\mathcal{G}(\text{cl}(x))) \). Then the map \( \rho_x : \text{St}(\mathcal{L}) \to \text{St}_x(\mathcal{L}) \) sending \( \phi \) to \( \phi_x \), is a surjective homomorphism [5, Lemma 2.15].
As \( \text{cl}(x) \) is a simplex, \( \mathbb{G}(\text{cl}(x)) \) is finitely generated free Abelian of rank \( |\text{cl}(x)| \); and for all \( y \in \text{cl}(x) \), we have \( [y] = [y]^\ast \). Let \( \text{cl}(x) = \{x_1, \ldots, x_r\} \) and \( [x] = \{x_i \mid 1 \leq i \leq s\} \), where \( s \leq r \). If \( \phi \in \text{St}_x(\mathcal{L}) \) then, for \( 1 \leq i \leq r \), we have \( x_i = x_1^{a_{i1}} \cdots x_r^{a_{ir}} \), for some integers \( a_{ij} \). Therefore \( \phi \) corresponds to the \( r \times r \) integer matrix \( [\phi] = (a_{ij}) \), when matrices act on the right on row vectors. Moreover, as shown in [5], \([\phi]\) is an upper block-triangular matrix, with diagonal blocks corresponding to the equivalence classes of elements of \( \text{cl}(x) \). More precisely, let \( \{y_i \mid 1 \leq i \leq m\} \) be a transversal for the equivalence relation \( \sim \) restricted to \( \text{cl}(x) \) (and assume \( y_1 = x \)). Then \( \text{cl}(x) = \cup_{i=1}^{m}[y_i] \) and \( [\phi] \) has

(i) \( m \) diagonal blocks \( A_1, \ldots, A_m \), where \( A_i \in \text{GL}(|[y_i]|, \mathbb{Z}) \) and

(ii) \( a_{ij} = 0 \) if \( i > j \) and \( a_{ij} \) is not in the \( i \)th block \( A_i \) of \([\phi]\).

Let \( \mathcal{S}_x \) denote the set of matrices satisfying the two conditions above. From [5, Lemma 2.15], the map \( \pi_x \) such that \( \phi \pi_x = [\phi] \) is a homomorphism from \( \text{St}_x(\mathcal{L}) \) to the subgroup \( \mathcal{S}_x \) of \( \text{GL}(r, \mathbb{Z}) \).

Also, writing \( A = [\phi] \) and \( A_D \) for the block-diagonal matrix which has diagonal blocks \( A_1, \ldots, A_m \) and zeros elsewhere, we have \( A_D \in \prod_{i=1}^{m} \text{GL}(|[y_i]|, \mathbb{Z}) \) and \( A_D^{-1}A \) is a unipotent upper triangular matrix \( A_U \): that is an element of \( \text{U}(r, \mathbb{Z}) \), satisfying (i) and (ii) above, but with \( A_i \) equal to the identity matrix, for all \( i \). It follows that \( (A_U - I)^m = 0 \), so the subgroup \( \mathcal{S}_U = \{A_U \mid A = [\phi], \phi \in \text{St}_x\} \) is a nilpotent subgroup of \( \text{GL}(r, \mathbb{Z}) \) of class \( m - 1 \). Furthermore ([5, Lemma 2.18]) setting \( \mathcal{S}_D = \{A_D \mid A = [\phi], \phi \in \text{St}_x\} \), we have \( \mathcal{S}_x = \mathcal{S}_D \times \mathcal{S}_U \) with \( \mathcal{S}_D = \prod_{i=1}^{m} \text{GL}(|[y_i]|, \mathbb{Z}) \). ([Errata: In [5, Lemma 2.18], the equality for \( D_Y \) should be \( D_Y = \prod_{i=1}^{m} \text{GL}(|[v_i]|, \mathbb{Z}) \) in both statement and proof.)

Note that, for all \( z \in X \), \( y \in a(z) \) implies \( y \in \text{cl}(y) \subseteq a(y) \subseteq a(z) \), so \( a(z) = \cup_{y \in a(z)} \text{cl}(y) \), and it follows that \( \text{St}(\mathcal{L}) \subseteq \text{St}(\mathcal{K}) \).

Returning to \( x \) such that \( \text{cl}(x) = a(x) \), we claim that in this case \( \text{St}_x^*(\mathcal{K}) \) is a subgroup of \( \text{St}(\mathcal{L}) \). To see this, suppose that \( \phi \in \text{St}_x^*(\mathcal{K}) \) and \( y \in X \). If \( y \in a(x) \) then \( a(y) \subseteq a(x) \), which is a simplex, from which it follows that \( a(y) = \text{cl}(y) \).

Hence, as \( \phi \in \text{St}(\mathcal{K}) \), we have \( \mathcal{G}(\text{cl}(y)) \phi = \mathcal{G}(a(y)) \phi = \mathcal{G}(a(y)) \mathcal{G}(\text{cl}(y)) \).

On the other hand if \( y \notin a(x) \) let \( u \in \text{cl}(y) \). If \( u \notin [x] \) then \( u \phi = u \in \text{cl}(y) \). If \( u \in [x] \) then \( [x] \subseteq \text{cl}(y) \) so \( u \in a(x) = \text{cl}(x) \subseteq \text{cl}(y) \).

Hence \( u \phi \in \mathcal{G}(a(x)) \subseteq \mathcal{G}(\text{cl}(y)) \). In both cases \( u \phi \in \mathcal{G}(\text{cl}(y)) \).

The same arguments apply to \( \phi^{-1} \), and it follows that \( \mathcal{G}(\text{cl}(y)) \phi = \mathcal{G}(\text{cl}(y)) \), completing the proof of the claim.

As \( \text{St}_x^*(\mathcal{K}) \subseteq \text{St}(\mathcal{L}) \) we may consider the restriction of the homomorphism \( \rho_x \) above to \( \text{St}_x^* \).

This restriction maps \( \text{St}_x^* \) isomorphically to its image in \( \text{St}_x \). Indeed, if \( \phi, \phi' \in \text{St}_x^* \) are such that \( \phi \rho_x = \phi' \rho_x \) then \( y \phi \rho_x = y \phi' \rho_x \), for all \( y \in \text{cl}(x) \), so \( y \phi \rho_x = y \phi' \rho_x \), for all \( y \in [x] \), and it follows that \( \phi = \phi' \).
Therefore, the composition $\rho_x \pi_x$ maps $\text{St}_x^v$ isomorphically to its image in $\mathcal{S}_x$, which we call $\mathcal{S}_x^v$. Now let $\phi \in \text{St}_x^v$, let $[\phi] = \phi \rho_x \pi_x$ and write $[\phi] = (a_{i,j})_{i,j=1}^r$.

Then $(a_{i,j})_{i,j=1}^r$ satisfies (i) and (ii) above. Moreover as $\phi \in \text{St}_x^v$, for $i > s$, we have $x_i \phi = x_i$, so $a_{i,i} = 1$ and $a_{i,j} = 0$, for $i \neq j$. Hence $A_i$ is the identity matrix for $1 < i \leq m$. Thus $(a_{i,j})_{i,j=1}^r$ satisfies

(iii) $A_1 = (a_{i,j})_{i,j=1}^r$ is in $\text{GL}([x], \mathbb{Z})$;

(iv) $a_{i,i} = 1$, for $i > s$, and

(v) $a_{i,j} = 0$ if $i \neq j$ and $i > s$.

Conversely, any matrix $A$ satisfying (iii), (iv) and (v) determines a unique element $A \pi_x^{-1} \rho_x^{-1}$ of $\text{St}_x^v$. If $A \in \mathcal{S}_x^v$ then the matrix $A_D$ obtained from $A$ by setting $a_{i,j} = 0$, for $(i, j)$ such that $1 \leq i \leq s$ and $j > s$, is uniquely determined by the block $A_1$, while $A_U = A_D^{-1}A$ satisfies $(A_U - I)^2 = 0$. This gives the following theorem, in which for $1 \leq i, j \leq n$ and $i \neq j$,

- $E^n_{i,j}$ is an $n \times n$ square matrix, with 1’s on the leading diagonal, a 1 in position $i, j$, and zeros elsewhere; and

- $O^n_i$ is an $n \times n$ square diagonal matrix with 1’s on the leading diagonal except for row $i$ which has diagonal entry $-1$ (and zeros off the leading diagonal).

- For $m \leq n$, $\mathcal{M}(m, n - m)$ is the group of $m \times (n - m)$ integer matrices under addition, and $Z_{i,j} \in \mathcal{M}(m, n - m)$ is the matrix with every coefficient equal to 0, except the $(i, j)$ coefficient which is equal to 1.

Since the Whitehead automorphisms of Type 2 involved here are all transvections we use the Laurence-Servatius notation for generators in this case: that is, in the terminology of Section 2.1 we use $\tau_{x,y}$ rather than ($\{x,y\}, y$), to denote the transvection mapping $x$ to $xy$.

**Theorem 4.1.** Let $x \in X$ such that $a(x) = \text{cl}(x)$. Assume and that $a(x) = \{x_i \mid 1 \leq i \leq r\}$ and $[x] = \{x_i \mid 1 \leq i \leq s\}$, where $s \leq r$. Then

$$\text{St}_x^v(K) = \text{St}_x^v \times \text{St}_x^v \cong \text{GL}(s, \mathbb{Z}) \times \mathcal{M}(s, r - s),$$

where,

(i) $\text{St}_x^v \times \text{St}_x^v$ is free Abelian of rank $s(r - s)$, freely generated by the set of automorphisms $\{\tau_{x_i, x_j} \mid 1 \leq i \leq s, s + 1 \leq j \leq r\}$, and is isomorphic to $\mathcal{M}(s, r - s)$ by an isomorphism taking $\tau_{x_i, x_j}$ to $Z_{i,j-s}$;
(ii) $\text{St}_{x,D}^v$ is generated by $\{\tau_{x,i,j}, \tau_x | 1 \leq i, j \leq s, i \neq j\}$ and the map $\tau_{x,i,j} \mapsto E_{i,j}^s$, $\tau_x \mapsto O_i^s$, where $E_{i,j}^s$ and $O_i^s$ are the $s \times s$ matrices above, extends to an isomorphism $\text{St}_{x,D}^v \to \text{GL}(s, \mathbb{Z})$; and

(iii) for $A \in \text{GL}(s, \mathbb{Z})$ and $B \in \mathcal{M}(s, r-s)$, the automorphism $A\theta$ maps $B$ to $A^{-1}B \in \mathcal{M}(s, r-s)$.

**Proof.** We have established, using results of [5], that $\text{St}_x^v \cong \mathcal{S}_x^v \leq \text{GL}(r, \mathbb{Z})$ via the isomorphism $\rho_x \pi_x$, such that $\phi \mapsto [\phi]$, for $\phi \in \text{St}_x^v$. The subgroup $\mathcal{S}_x^v$ consists of matrices satisfying (iii), (iv) and (v) above, so if $A \in \mathcal{S}_x^v$, then, as above, we may write $A = A_D A_U$, where

$$A_U = \begin{bmatrix} I_s & A'_U \\ 0 & I_{r-s} \end{bmatrix}, \quad A_D = \begin{bmatrix} A'_D & 0 \\ 0 & I_{r-s} \end{bmatrix},$$

with $A'_U \in \mathcal{M}(s, r-s)$, $A'_D \in \text{GL}(s, \mathbb{Z})$, and $I_s$ and $I_{r-s}$ the identity matrices of the appropriate dimensions.

Define subgroups $\mathcal{S}_{x,D}^v = \{A_D | A \in \mathcal{S}_x^v\}$ and $\mathcal{S}_{x,U}^v = \{A_U | A \in \mathcal{S}_x^v\}$. If $U, V \in \mathcal{S}_{x,U}^v$ and $W \in \mathcal{S}_{x,D}^v$ with

$$U = \begin{bmatrix} I_s & U' \\ 0 & I_{r-s} \end{bmatrix}, \quad V = \begin{bmatrix} I_s & V' \\ 0 & I_{r-s} \end{bmatrix}, \quad W = \begin{bmatrix} W' & 0 \\ 0 & I_{r-s} \end{bmatrix},$$

then

$$UV = \begin{bmatrix} I_s & U' + V' \\ 0 & I_{r-s} \end{bmatrix} = VU \quad \text{and} \quad W^{-1}UW = \begin{bmatrix} I_s & W'^{-1}U' \\ 0 & I_{r-s} \end{bmatrix} \in \mathcal{S}_{x,U}^v$$

and we deduce that $\mathcal{S}_{x,U}^v$ is free Abelian and $\mathcal{S}_x^v = \mathcal{S}_{x,D}^v \rtimes \mathcal{S}_{x,U}^v$. Also the map $\pi_U$, sending $U$ above to $U' \in \mathcal{M}(s, r-s)$, and the map $\pi_D$, sending $W$ above to $W'$ in $\text{GL}(s, \mathbb{Z})$ are isomorphisms from $\mathcal{S}_{x,U}^v$ to $\mathcal{M}(s, r-s)$ and from $\mathcal{S}_{x,D}^v$ to $\text{GL}(s, \mathbb{Z})$, respectively.

Let $\text{Tr}_{x,U}$ denote the set of transvections $\{\tau_{x,i,j} | 1 \leq i \leq s, s+1 \leq j \leq r\}$ and define $\text{St}_{x,U}^v$ to be the subgroup of $\text{St}_x^v$ generated by $\text{Tr}_{x,U}$. Elements of $\text{St}_{x,U}^v$ fix the set $\{x_i | 1 \leq i \leq s\}$ point-wise, and the map $\rho_x \pi_x \pi_U$ maps $\tau_{x,i,j}$ to $Z_{i,j-s} \in \mathcal{M}(s, r-s)$, for all $\tau_{x,i,j} \in \text{Tr}_{x,U}$. It follows that $\rho_x \pi_x \pi_U$ maps $\text{St}_{x,U}^v$ isomorphically to $\mathcal{M}(s, r-s)$, and as the latter is freely generated by the $Z_{i,j}$, $1 \leq i \leq s$, $1 \leq j \leq r-s$, this proves (i). Similarly, $\rho_x \pi_x \pi_D$ sends $\tau_{x,i,j}$ to $E_{i,j}^s$ and $\tau_x$ to $O_i^s$, for $1 \leq i, j \leq s$, and $i \neq j$; so determines an isomorphism from $\text{St}_{x,D}^v$ to $\text{GL}(s, \mathbb{Z})$. As the latter is generated by the matrices $E_{i,j}^s$ and $O_i^s$, (ii) follows. Finally, (iii) follows from the identity for $W^{-1}UW$ above.

**Example 4.2.** Continuing Example 3.1; we have $a(x) = \text{cl}(x)$ for $x = d, e, f$ and $h$. 

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1. \( a(f) = \text{cl}(f) = \{d, e, f, g\} \) and \( [f] = \{f, g\} \), so \( r = 4 \) and \( s = 2 \).

\( \text{St}^x_{f,U} = \langle \tau_{f,d}, \tau_{f,e}, \tau_{g,d}, \tau_{g,e} \rangle \) and is isomorphic to \( \mathcal{M}(2, 2) \) via the map sending \( \tau_{f,d}, \tau_{f,e}, \tau_{g,d} \) and \( \tau_{g,e} \) to \( Z_{1,1}, Z_{1,2}, Z_{2,1} \) and \( Z_{2,2} \), respectively.

\( \text{St}^x_{f,D} = \langle \tau_{f,g}, \tau_{g,f}, \tau_{f}, \tau_{g} \rangle \) and is isomorphic to \( GL(2, Z) \) via the map sending \( \tau_{f,g}, \tau_{g,f}, \tau_{f} \) and \( \tau_{g} \) to \( E_{1,2}^d, E_{2,1}^d, O_1^2 \) and \( O_2^2 \).

Combining these two isomorphisms, \( \text{St}^x \) is isomorphic to \( GL(2, Z) \times \mathcal{M}(2, 2) \), where, for \( A \in GL(2, Z) \), the automorphism \( A \theta \) of \( \mathcal{M}(2, 2) \) is the map sending \( B \) to \( A^{-1}B \), for \( B \in \mathcal{M}(2, 2) \).

2. In the same way we see that \( \text{St}^v = \langle \tau_e | \tau^2_e \rangle < \langle \tau^2_e \rangle \cong D_\infty \), the infinite dihedral group.

3. Similar considerations show that \( \text{St}^h = \langle \tau_h | \tau^2_h \rangle \cong Z_2 \) and \( \text{St}^s = \langle \tau_s | \tau^2_s \rangle \cong Z_2 \).

4.2 \( \text{cl}(x) \neq a(x) \)

In this case \( \text{cl}(x) \) is a proper subset of \( a(x) \) and we define \( a_{\text{out}}(x) = a(x) \setminus \text{cl}(x) \) and \( a_s(x) = \text{cl}(x) \setminus [x] \). Then we have a disjoint union \( a(x) = [x] \cup a_s(x) \cup a_{\text{out}}(x) \), with \( a_s(x) \subseteq \text{lk}(x) \) and \( a_{\text{out}}(x) \cap \text{st}(x) = \emptyset \). Let \( a(x) = \{x_i | 1 \leq i \leq r\} \), where for some \( p \leq q < r \) we have \( [x] = \{x_i | 1 \leq i \leq p\} \) and \( a_s(x) = \{x_i | p+1 \leq i \leq q\} \). Then \( G(a(x)) = G(a_s(x)) \times [G([x]) \ast G(a_{\text{out}}(x))] \), where \( G([x]) \) is free of rank \( p \) and \( G(a_s(x)) \) is isomorphic to \( Z^{p-q} \) and \( G(a_{\text{out}}(x)) \) is a partially commutative group on a graph of \( r - q \) vertices.

**Theorem 4.3.** \( \text{St}^x = \text{St}^x_{f,l} \times \text{St}^x_{f,s} \) where

\[
\text{St}^x_{f,l} = \{ \phi \in \text{St}^x_f \mid \forall \phi \in G([x] \cup a_{\text{out}}(x)), \phi \in [x] \}
\]

and

\[
\text{St}^x_{f,s} = \{ \phi \in \text{St}^x_f \mid \forall y \in [x], \exists w_y \in G(a_s(x)) \text{ such that } \phi y = yw_y \}
\]

Moreover, with the above notation, \( \text{St}^x_{f,s} \) is a free Abelian group of rank \( p(q - p) \), freely generated by \( \{\tau_{x_i,x_j} \mid 1 \leq i \leq p, p + 1 \leq j \leq q\} \), and is isomorphic to \( \mathcal{M}(p,q - p) \) by an isomorphism taking \( \tau_{x_i,x_j} \) to \( Z_{i,j-p} \) (cf. Theorem 4.1).

**Proof.** Let \( \phi \in \text{St}^x_f \), so \( y\phi = y \) unless \( y \in \{x_1, \ldots, x_p\} \). For \( 1 \leq i \leq p \), there exists \( w_i \in G([x] \cup a_{\text{out}}(x)) \) such that

\[
x_i\phi = w_i x_{p+1}^{a_i,p+1} \cdots x_q^{a_i,q},
\]
as \( G(a_s(x)) \) is the centre of \( G(a(x)) \). Let
\[
\phi_1 = \prod_{i=1}^{p} \tau_{x_i,x_{i+1}}^{a_i + 1} \cdots \tau_{x_q,x_1}^{a_q},
\]
so \( x_i^\phi_1 = x_i^{a_i + 1} \cdots x_q^{a_q} \), for \( i = 1, \ldots, p \). Then, as \( \phi_1 \in St_{x,i}^y \), so is \( \phi_0 = \phi_1^{-1} \phi \) and \( x_i^\phi_0 = w_i \in G([x] \cup a_{out}(x)) \), for \( x_i \in [x] \). Therefore \( \phi = \phi_1 \phi_0 \) with \( \phi_0 \in St_{x,i}^y \) and \( \phi_1 \in St_{x,s}^y \). Moreover, as in the previous subsection, \( St_{x,s}^y \) is a free Abelian group generated by \( \{ \tau_{x_i,x_j} \mid 1 \leq i \leq p, \, p+1 \leq j \leq q \} \), isomorphic to \( M(p, q - p) \) via the map sending \( \tau_{x_i,x_j} \) to \( Z_{p,j-p} \). From the definitions \( St_{x,s}^y \cap St_{x,l}^y = \{ 1 \} \). To see that \( St_{x,s}^y \) is normal in \( St_{x,l}^y \), let \( \tau_{x_k,x_j} \) be a generator of \( St_{x,l}^y \) and \( \phi \in St_{x,l}^y \). Then, for \( x_i \in [x] \) there exists \( w_i \in G([x] \cup a_{out}(x)) \) such that \( x_i^\phi = w_i \). For each \( i \), let \( s(i,k) \) be the exponent sum of \( x_k \) in \( w_i \). As \( x_k \in [x] \) and \( x_j \in a_s(x) \), we have \( x_i^\phi \tau_{x_k,x_j} \phi = w_i \tau_{x_k,x_j} \phi = w_i x_j^{s(i,k)} \phi = x_i^\phi x_j^{s(i,k)} = x_i x_j^{s(i,k)} \). Therefore \( \phi^{-1} \tau_{x_k,x_j} \phi \in St_{x,s}^y \), from which it follows that \( St_{x,s}^y \) is normal in \( St_{x,l}^y \).

**Example 4.4.** Continuing Example 3.1; \( a(x) \neq cl(x) \) when \( x = a, b, c \) or \( i \). We have \( [a] = \{ a, b \} \) and \( a_s(a) = \{ d \} \) so \( St_{a,s}^y = \langle \tau_{a,d}, \tau_{b,d} \rangle \) is free Abelian of rank 2. As \( [c] = \{ c \} \) and \( a_s(c) = \{ d \} \), we have \( St_{c,s}^y = \langle \tau_{c,d} \rangle \) and similarly \( St_{c,s}^y = \langle \tau_{i,h} \rangle \), both infinite cyclic.

This lemma allows us to reduce determination of the structure of \( St_{x,i}^y \) to that of \( St_{x,l}^y \). For this purpose it is convenient to consider the set of Whitehead automorphisms \( \Omega \) as in Definition 2.3, as the generating set for \( Aut(G) \). Our candidate generating set for \( St_{x,i}^y \) is given in the next definition.

**Definition 4.5.** Given \( x \in X \), let \( \Omega_x = \Omega \cap St_{x,l}^y \).

From the definitions we have
\[
\Omega_x = \{ \sigma \in Aut(\Gamma^+) \mid y\sigma = y, \forall y \in X \setminus [x] \}
\cup \{ (A, a) \in \Omega \mid A \setminus \{ a \} \subseteq [x]_L, a \in [x]_L \cup a_{out,L}(x) \}.
\]
As the full graph on \([x]\) is a null graph in the current case, the set of Type 1 automorphisms in \( \Omega_x \) is \( \{ \sigma \in Aut(\Gamma^+) \mid y\sigma = y, \forall y \in X \setminus [x] \} \), which is the set of permutations \( \sigma \) of \([x]_L \) such that \( x^{-1}\sigma = (x\sigma)^{-1} \).

As a candidate set of relations \( R_x \) for \( St_{x,l}^y \) we take those relations of the presentation for \( Aut(G) \) in [3] which apply to words in the free group on \( \Omega_x \), augmented by relations required to make peak reduction arguments possible within the set \( \Omega_x \) (namely \( R_3^x \) and \( R_4^x \)). More precisely we define the set \( R_x \) to consist of all relations defined by \( R_1^x-R_7^x, R_3^x \) and \( R_4^x \) below. In
these relations, $A + B$ denotes $A \cup B$, when $A \cap B = \emptyset$, and $B - A$ denotes $A \setminus B$, when $A \subseteq B$. By $A - a$ and $A + a$ we mean $A - \{a\}$ and $A + \{a\}$ respectively.

R1$_x$ $(A, a)^{-1} = (A - a + a^{-1}, a^{-1})$, for $(A, a) \in \Omega_x$.

R2$_x$ $(A, a)(B, a) = (A \cup B, a)$, for $(A, a), (B, a) \in \Omega_x$, such that $A \cap B = \{a\}$.

R3$_x$ $(B, b)^{-1}(A, a)(B, b) = (A, a)$, for $(A, a), (B, b) \in \Omega_x$, such that $a^{-1} \notin B$, $b^{-1} \notin A$ and either

(a) $A \cap B = \emptyset$ or
(b) $a \in \text{lk}_L(b)$.

R3$_x^*$ $(B, b)^{-1}(A, a)(B, b) = (A, a)$, for $(A, a), (B, b) \in \Omega_x$, such that $a^{-1} \in B$, $b \notin A$ and $A \subseteq B$.

R4$_x$ $(B, b)^{-1}(A, a)(B, b) = (A + B - b, a)$, for $(A, a), (B, b) \in \Omega_x$, such that $a^{-1} \notin B$, $b^{-1} \in A$, and $A \cap B = \emptyset$.

R4$_x^*$ $(B, b)^{-1}(A, a)(B, b) = (B - A + b^{-1}, a^{-1})$, for $(A, a), (B, b) \in \Omega_x$, such that $a^{-1} \in B$, $b \in A$, and $A \subseteq B$.

R5$_x$ $(A, a)(A - a + a^{-1}, b) = \sigma_{a,b}(A - b + b^{-1}, a)$, for $(A, a) \in \Omega_x$, $a, b \in [x]_L$, $a \neq b$, $b \in A$, $b^{-1} \notin A$ and $\sigma_{a,b}$ the Type 1 Whitehead automorphism permuting $[x]_L$ by the cycle $(a, b^{-1}, a^{-1}, b)$.

R6$_x$ $\sigma^{-1}(A, a)\sigma = (A\sigma, a\sigma)$, where $(A, a)$ is in $\Omega_x$, of Type 2, and $\sigma \in \Omega_x$ of Type 1.

R7$_x$ The multiplication table of the subgroup of Type 1 automorphisms in $\Omega_x$.

We denote by $R_x$ the set of of relations given by $R1_x$–$R7_x$, $R3_x^*$ and $R4_x^*$. In the remainder of this section we shall prove the following theorem.

**Theorem 4.6.** $\text{St}^x_{x,l}$ has a presentation $\langle \Omega_x | R_x \rangle$.

We shall use the peak reduction theorem of [3], and its analogue for $\text{St}^x_{x,l}$, to prove this theorem, and introduce the necessary terminology in the next sub-section. First we prove Corollary 1.4 and given an example.
Proof of Corollary 1.4. It must be shown that \( \text{St}(K) = \langle \text{Inv}, \text{Tr} \rangle \); and that there is a finite presentation with these generators there is a Every inversion and elementary transvection belongs to \( \text{St}(K) \), by the fundamental results of Laurence and Servatius. On the other hand, it follows from Theorems 1.1, 4.1, 4.3 and 4.6 that \( \text{St}(K) \) is generated by inversions and transvections. Moreover, from the constructions appearing in these theorems a finite presentation may be built. \( \Box \)

**Example 4.7.** Continuing from Example 4.4 we find presentations for \( \text{St}^x_{x, l} \), for \( x = a, c, i \).

1. We have \( [i] = \{i\} \), \( a_i(i) = \{h\} \) and \( a_l(i) = \{c, d, e\} \). Then
   \[
   \Omega_i = \{i, ((i^c, s), s), (i, i^{-1}, s), s : \varepsilon \in \{\pm 1\}, s \in \{c, d, e\}^{\pm 1}\}.
   \]
   The set \( R_i \) consists of relations of types \( R_{1_i}, R_{2_i}, R_{3_i}, R_{6_i} \) and \( R_{7_i} \).
   There are no relations of types \( R_{3_i}^*, R_{4_i}, R_{4_i}^* \) or \( R_{5_i} \). Relations \( R_{1_i} \) and \( R_{2_i} \) allow Tietze transformations to be applied to remove generators \( \{i^c, s^{-1}\}, s \in \{c, d, e\} \), and generators \( \{i, i^{-1}, s\}, s \), where \( s \in \{c, d, e\}^{\pm 1} \). This leaves a presentation with generating set
   \[
   \Omega_i' = \{i, ((i^c, s), s) : \varepsilon \in \{\pm 1\}, s \in \{c, d, e\}\},
   \]
   and relations
   \[
   \begin{align*}
   R_{2_i} + R_{3_i}(a). & \quad \{i, s\}(i^{-1}, s', s') = \{i^{-1}, s', s'(i, s), s, s' \in \{c, d, e\}\}. \\
   R_{3_i}(b). & \quad \{i^c, d\}(i^c, s, s) = \{i^c, s, s(i^c, d), s \in \{c, e\}, \varepsilon \in \{\pm 1\}\}. \\
   R_{6_i}. & \quad i_s(i, s) i_t = (i^{-1}, s), s \in \{c, d, e\}. \\
   R_{7_i}. & \quad i_i^2 = 1.
   \end{align*}
   \]
   For each \( \varepsilon = 1 \) and \( -1 \) we have a subgroup of \( \text{St}^x_{x, l} \) generated by \( \{(i^c, s), s \} : s \in \{c, d, e\}\) which is isomorphic to \( F_2 \times \mathbb{Z} \) (the central \( \mathbb{Z} \) generated by \( \{(i^c, d), d\}\)). The conjugation action of \( i_s \) on this subgroup maps \( (i^c, s), s \) to \( (i^{-c}, s), s \), for all \( s \), so
   \[
   \text{St}^x_{x, l} \cong (F_2 \times \mathbb{Z})^2 \rtimes \phi, \mathbb{Z}_2,
   \]
   where \( \phi \) maps \( 1 \in \mathbb{Z}_2 \) to the automorphism of \( (F_2 \times \mathbb{Z})^2 \) taking \( (a, b) \) to \( (b, a) \), for \( a, b \in F_2 \times \mathbb{Z} \).

2. We have \( [c] = \{c\} \), \( a_c(c) = \{d\} \) and \( a_l(c) = \{e\} \). As before, applying Tietze transformations to the presentation obtained from Theorem 4.6 gives a presentation with generating set
   \[
   \Omega_c' = \{i_c, ((c^c, e), e) : \varepsilon = \pm 1\},
   \]
   and relations
\[ R_2c. \ (\{c^{-1}, e\}, e)(\{c, e\}, e) = (\{c, e\}, e)(\{c^{-1}, e\}, e). \]

\[ R_6c. \ \iota_c(\{c, e\}, e)\iota_c = (\{c^{-1}, e\}, e). \]

\[ R_7c. \ \iota_c^2 = 1. \]

Hence \( \text{St}_{c,l}^v \cong \mathbb{Z}^2 \rtimes \phi_c \mathbb{Z}_2 \), where \( \phi_c \) maps \( 1 \) to the automorphism of \( (\mathbb{Z})^2 \) taking \( (a, b) \) to \( (b, a) \), for \( a, b \in \mathbb{Z} \).

3. We have \([a] = \{a, b\}, a_s(a) = \{d\}\) and \(a_l(a) = \{h\}\). Let \( \Pi_a \) denote the set of permutations of \( \{a^{\pm 1}, b^{\pm 1}\} \) inducing automorphisms of \( G(a, b) \).

We use the more concise notation \( \tau_{x,y} \) for the automorphism \( (\{x, y\}, y) \) here. After applying Tietze transformations as in the previous cases we obtain a presentation for \( \text{St}_{a,l}^v \) with generating set

\[ \Omega_a' = \Pi_a \cup \{\tau_{s,t} : s \in \{a, b\}^{\pm 1}, t \in \{a, b, h\}, s \neq t^{\pm 1}\}, \]

and relations \( R_a' \) as follows.

\[ R_2a. \quad \tau_{s,t}\tau_{s^{-1},t} = \tau_{s^{-1},t}\tau_{s,t}, \]
where \( s \in \{a, b\}, t \in \{a, b, h\}, s \neq t^{\pm 1} \).

\[ R_3^*a. \quad \tau_{s,t}\tau_{s,h}\tau_{t,h}\tau_{t^{-1},h}\tau_{s,t} = \tau_{s,h}\tau_{t,h}\tau_{t^{-1},h}\tau_{s,t}, \]
where \( s \in \{a, b\}^{\pm 1}, t \in \{a, b\}, s \neq t^{\pm 1} \).

\[ R_4^*a. \quad \tau_{s,t}\tau_{t,s}\tau_{t^{-1},s} = \tau_{t,s}\tau_{t^{-1},s}\tau_{s^{-1},t}^{-1}, \]
where \( s, t \in \{a, b\}, s \neq t \).

\[ R_5a. \quad \tau_{s,t}\tau_{s^{-\varepsilon},t} = \sigma_{s,t}\tau_{a^{-\varepsilon},b}, \]
where \( \varepsilon \in \{\pm 1\}, s, t \in \{a, b\}, s \neq t^{\pm 1}, \) and \( \sigma_{s,t} \) is the permutation with cycle \( (s^\varepsilon, t, s^{-\varepsilon}, t^{-1}) \).

\[ R_6a. \quad \text{For all } \sigma \in \Pi_a, \text{ and all } \tau_{s,t}, \]
\[ \sigma^{-1}\tau_{s,t}\sigma = \tau_{\sigma s, t\sigma}. \]

\[ R_7a. \quad \text{A set of defining relations for } \langle \Pi_a \rangle. \]
From Examples 3.5, 4.2 and 4.4,

\[ \text{St}(K) \cong \left[ \mathbb{Z} \times \text{St}_{v}^{\gamma} \right] \]
\[ \times \left\{ \left[ \left( \mathbb{Z} \times \text{St}_{v}^{\gamma} \right) \times \left( \mathcal{M}(2, 2) \rtimes \text{GL}(2, \mathbb{Z}) \right) \right] \right\} \]
\[ \times \left\{ \left[ \left( \mathbb{Z}^{2} \times \text{St}_{v}^{\gamma} \right) \times D_{\infty} \right] \right\} \]

and combining with the current example we have

\[ \text{St}(K) \cong \left[ \mathbb{Z} \times \left( (\mathbb{F}_{2} \times \mathbb{Z})^{2} \times \mathbb{Z}_{2} \right) \right] \]
\[ \times \left\{ \left[ \left( \mathbb{Z} \times (\mathbb{Z}^{2} \times \mathbb{Z}_{2}) \right) \times \left( \mathcal{M}(2, 2) \rtimes \text{GL}(2, \mathbb{Z}) \right) \right] \right\} \]
\[ \times \left\{ \left[ (\mathbb{Z}^{2} \times \langle \Omega'_{a} \mid R'_{a} \rangle) \times D_{\infty} \right] \right\} \]
\[ \times \left\{ [\mathbb{Z}_{2} \times \mathbb{Z}_{2}] \right\} \].

From this decomposition we could construct a presentation of \( \text{St}(cK) \), with generators \text{Inv} and \text{Tr}, as in Corollary 1.4.

Remark 4.8. In the case when \( [x] = \{x\} \) and \( a(x) \neq \text{cl}(x) \) the group \( \text{Aut}(\mathbb{G}([x])) \) is cyclic of order 2, generated by the inversion \( \iota_{x} \) which permutes the elements of \( \{x^{\pm 1}\} \). Every element \( \phi \) of \( \text{St}_{v}^{\gamma} \) maps \( x \) to a word \( w_{1}x^{\varepsilon}w_{2} \), where \( w_{i} \in \mathbb{G}(a_{\text{out}}(x)), and \varepsilon = \pm 1 \). It follows that \( \text{St}_{v}^{\gamma} \) is isomorphic to the wreath product \( C_{2} \wr \mathbb{G}(a_{\text{out}}(x)) \). However, when \( \|x\| \geq 2 \) although, similarly, \( \text{St}_{v}^{\gamma} \) contains an subgroup \( H \) isomorphic to the wreath product \( \text{Sym}(L) \wr \mathbb{G}(a_{\text{out}}(x)) \), it also contains elements outside \( H \); for example \( \tau_{x,y} \tau_{x,a} \), where \( x, y \in [x] \) and \( a \in \alpha_{\text{out}}(x) \).

4.3 Peak reduction in \( \text{Aut}(\mathbb{G}) \)

The length of a conjugacy class \( c \) of \( \mathbb{G} \) is the minimum of the lengths of words representing elements of \( c \), denoted \( |c|_{\sim} \). The length of a \( k \)-tuple \( C = (c_{1}, \ldots, c_{k}) \) of conjugacy classes is \( |C|_{\sim} = \sum_{i=1}^{k} |c_{i}|_{\sim} \). If \( \alpha \in \text{Aut}(\mathbb{G}) \) and \( c \) is a conjugacy class in \( \mathbb{G} \) then by \( \alpha \) we mean the conjugacy class of \( w_{\alpha} \), where \( w \) is an element of \( c \). If \( C = (c_{1}, \ldots, c_{k}) \) is a \( k \)-tuple of conjugacy classes of \( \mathbb{G} \) and \( \alpha \) is an automorphism then we write \( C\alpha = (c_{1}\alpha, \ldots, c_{k}\alpha) \).

Definition 4.9. Let \( \alpha, \beta \in \Omega \) and let \( C \) be a \( k \)-tuple of conjugacy classes. The composition \( \alpha \beta \) is a peak with respect to \( C \) if

\[ |C\alpha|_{\sim} \geq |C|_{\sim} \text{ and } |C\alpha_{\sim} \geq |C\alpha|_{\sim} \]
and at least one of these inequalities is strict. Let $\Omega'$ be a subset of $\Omega$ and let $\alpha \beta$ be a peak with respect to $C$. A peak lowering of $\alpha \beta$ for $C$, in $\Omega'$, is a factorisation $\alpha \beta = \delta_1 \cdots \delta_s$, such that $\delta_i \in \Omega'$ and

$$|C\delta_1 \cdots \delta_i|_\sim < |C\alpha|_\sim,$$

for $1 \leq i \leq s - 1$.

Let $\phi \in \text{Aut}(G)$ have factorisation $\phi = \alpha_1 \cdots \alpha_m$, where $\alpha_i \in \Omega$. For $1 \leq i \leq m - 1$, this factorisation is said to have a peak with respect to $C$, at $i$, if $\alpha_i \alpha_{i+1}$ is a peak with respect to $C\alpha_1 \cdots \alpha_{i-1}$. If the factorisation has no peak with respect to $C$ it is said to be peak reduced with respect to $C$.

Day proves [3, Lemma 3.18] that if $C$ is a $k$-tuple of conjugacy classes of $G$ and $\alpha, \beta \in \Omega_l$ such that $\alpha \beta$ is a peak with respect to $C$ then there is a peak lowering of $\alpha \beta$ for $C$, in $\Omega_l$. We shall first use the peak lowering theorem of [3] to show that $\text{St}^v_{x,l}$ is generated by $\Omega_x$. Then we shall establish that peak lowering can be carried out in the set $\Omega_x$, and use this to prove Theorem 4.6.

### 4.4 Generators for $\text{St}^v_x$

Let $W = (w_1, \ldots, w_k)$ be a $k$-tuple of elements of $G$. The stabiliser $\text{stab}(W)$ of $W$ in $\text{Aut}(G)$ is the set consisting of elements $\alpha$ such that $w_i \alpha = w_i$, for $1 \leq i \leq k$, whereas the stabiliser up to conjugacy $\text{stab}_\sim(W)$ of $W$ is the set of elements $\alpha$ such that $w_i \alpha$ is conjugate to $w_i$, for $1 \leq i \leq k$, (the stabiliser of $W$ as a tuple of conjugacy classes).

**Theorem 4.10.** Let $x \in X$. Then $\text{St}^v_x = \text{stab}(X \setminus \{x\})$ and $\text{St}^v_x$ is generated by $\text{stab}(X \setminus \{x\}) \cap (\Omega_l \cup \Omega_s)$. In particular $\text{St}^v_{x,s}$ is generated by $\text{stab}(X \setminus \{x\}) \cap \Omega_s$ and $\text{St}^v_{x,l}$ is generated by $\text{stab}(X \setminus \{x\}) \cap \Omega_l = \Omega_x$.

To prove this we use the analogue of [3, Corollary 4.5] for $\text{stab}(W)$ instead of $\text{stab}_\sim(W)$.

**Proposition 4.11** (cf. [3, Corollary 4.5]). Let $W = (w_1, \ldots, w_k)$ be a $k$-tuple of elements of $L$, where $k \geq 2$ and $\nu(w_i) \neq \nu(w_j)$, if $1 \leq i < j \leq k$. Then the subgroup $\text{stab}(W)$ is generated by $(\Omega_l \cup \Omega_s) \cap \text{stab}(W)$.

**Proof.** Let $\alpha \in \text{stab}(W)$. From [3, Corollary 4.5], $\alpha \in (\Omega_l \cup \Omega_s) \cap \text{stab}_\sim(W))$. In fact, from the proof of [3, Proposition C] there is a factorisation, which is peak reduced with respect to $W$,

$$\alpha = \phi_1 \cdots \phi_r \sigma^{-1} \delta_1 \cdots \delta_m,$$
where $\phi_i \in \Omega_s$, $\sigma \in \text{Aut}(\Gamma^\pm)$ and $\delta_i \in \Omega_i$, with $W\phi_i = W_i$; and so $|W\sigma^{-1}\delta_1 \cdots \delta_i|_\sim = |W|_\sim$, for $1 \leq i \leq m$. If $\delta_i$ is of Type 1 then so is $\delta_i^\sigma$ and if $\delta_i$ is of Type 2 then so is $\delta_i^\sigma$; and (from DR6)

$$\sigma^{-1}\delta_1 \cdots \delta_m = \delta_1^\sigma \cdots \delta_m^\sigma \sigma^{-1}.$$  

Moreover, as elements of $\text{Aut}(\Gamma^\pm)$ do not affect length, $|W|_\sim = |W\delta_1|_\sim = |W\delta_i|_\sim$ and similarly $|W\sigma^{-1}\delta_1 \cdots \delta_i|_\sim = |W\delta_1^\sigma \cdots \delta_i^\sigma|_\sim$, for $1 \leq i \leq m$. Hence we may replace the above factorisation with $\alpha = \phi_1 \cdots \phi_r \delta_1^\sigma \cdots \delta_m^\sigma \sigma^{-1}$, which is also peak reduced with respect to $W$.

Continuing in this way we may move any of the $\delta_i$ which are of Type 1 to the right hand end of the factorisation, until we have, after renaming, a peak reduced factorisation

$$\alpha = \phi_1 \cdots \phi_r \delta_1 \cdots \delta_m \sigma^{-1},$$

satisfying $\phi_i \in \Omega_s \cap \text{stab}(W)$, $\sigma \in \text{Aut}(\Gamma^\pm)$, $\delta_i \in \Omega_i$ of Type 2, and $|W\delta_1 \cdots \delta_i|_\sim = |W|_\sim$, for $1 \leq i \leq m$.

This means that $|W\delta_1|_\sim = |W|_\sim$ and $|w_i\delta_1|_\sim \geq 1$, with equality only if either $w_i\delta_1 = w_i$ or $w_i\delta_1 = w_i^a$, for some $a \in L$. It follows that $w_i\delta_1 = w_i$ or $w_i^a$, $a \in L$, for $1 \leq i \leq k$. That is, writing $\delta_i = (A_i, a_i)$, where $A_i \cap \text{lk}_L(a_i) = \emptyset$, and partitioning $A_i \setminus \{a_i\}$ as the disjoint union $A_{i,0} \cup A_{i,1}$, where $A_{i,0} = A_i^{-1}$ and $A_{i,1} \cap A_{i,1}^{-1} = \emptyset$, as in Remark 2.4.2, we have $\{w_1, \ldots, w_k\} \cap A_i \setminus \{a_i\} = \{w_1, \ldots, w_k\} \cap A_{i,0}$ and $\{w_1, \ldots, w_k\} \cap A_{i,1} = \emptyset$. Hence, $w_i$ is a minimal length representative of the conjugacy class of $w_i\delta_1$, for $1 \leq i \leq k$.

Assume now that, for some $j \geq 1$, $w_i$ is a minimal length representative of the conjugacy class of $w_i\delta_1 \cdots \delta_j$, for $1 \leq i \leq k$. Then $w_i\delta_{j+1}$ is a representative of the conjugacy class of $w_i\delta_1 \cdots \delta_j \delta_{j+1}$ and, if it has length 2 it is conjugate to no shorter element. Therefore $w_i\delta_{j+1}$ is again equal to $w_i$ or $w_i^{\sigma_{j+1}}$,

$$\{w_1, \ldots, w_k\} \cap A_j \setminus \{a_j\} \subseteq A_{j,0} \text{ and } \{w_1, \ldots, w_k\} \cap A_{j,1} = \emptyset,$$

and $w_i$ is a representative of the conjugacy class of $w_i\delta_1 \cdots \delta_j \delta_{j+1}$, for $1 \leq i \leq k$.

Therefore, there exist $g_i \in G$ such that $w_i\delta_1 \cdots \delta_m = g_i^{-1} \circ w_i \circ g_i$. As $W = W\delta_1 \cdots \delta_m \sigma^{-1}$, this implies that $g_i^{-1} \circ w_i \circ g_i = w_i \sigma \in L$. Hence $g_i = 1$ and $w_i \sigma = w_i$, for $1 \leq i \leq k$. Thus $\sigma \in \text{Aut}(\Gamma^\pm) \cap \text{stab}(W)$ and $W = W\delta_1 \cdots \delta_m$, where $\delta_i \in \Omega_i$ is of Type 2, for $i = 1, \ldots, m$.

If $V = (v_1, \ldots, v_k)$ and $W = (w_1, \ldots, w_l)$ are tuples of elements (or conjugacy classes) of $G$, let $VW$ denote the concatenation $(v_1, \ldots, v_k, w_1, \ldots, w_l)$
of \( V \) and \( W \). Inductively define \( W^n = W^{n-1}W \), for \( n \geq 2 \). Given \( W = (w_1, \ldots, w_k) \) let \( V_i = (w_i^2, w_iw_{i+1}, \ldots, w_iw_k) \), for \( 1 \leq i \leq k \), let
\[
V = V_1 \cdots V_k \text{ and } Z = W^kV,
\]
so \( |Z|_\sim = k(k+1)^2 \), and \( Z \) is fixed point wise by \( \alpha, \phi_i, \delta_1 \cdots \delta_m \) and \( \sigma \).

Let \( \alpha_1 = \delta_1 \cdots \delta_m \). Applying [3, Lemma 3.18], we may choose a factorisation
\[
\alpha_1 = \beta_1 \cdots \beta_t,
\]
which is peak reduced with respect to \( Z \), where \( \beta_j \in \Omega_t \), for all \( j \). Let \( j \) be minimal such that there exists some \( i \) with \( w_i \beta_1 \cdots \beta_j \) not equal to a conjugate of an element of \( L \). Then \( |w_i \beta_1 \cdots \beta_{s-1}|_\sim = 1 \), and \( w_i \beta_1 \cdots \beta_{j-1} = v_i \), for some \( v_i \in L \), for all \( i, s \) such that \( 1 \leq i \leq k \) and \( 1 \leq s \leq j \); while there exists \( i \) such that \( v_i \beta_j = v_i \alpha \) or \( \alpha v_i \), for some \( \alpha \in L \). Assume that there are precisely \( r \) elements of \( i \in \{1, \ldots, k\} \) such that \( v_i \beta_j \) is not conjugate to an element of \( L \); so \( |W \beta_1 \cdots \beta_j|_\sim = k + r > k \). As each element of the tuple \( V \beta_1 \cdots \beta_j \) is conjugate to an element of length at least 1, we have \( |V \beta_1 \cdots \beta_j|_\sim \geq k(k+1)/2 \). Therefore
\[
|Z \beta_1 \cdots \beta_j|_\sim \geq (k + r) (k(k+1)) + \frac{k(k+1)}{2} > k(k+1)^2 = |Z|_\sim.
\]
As \( |Z \beta_1 \cdots \beta_t| = |Z| \) and \( \beta_1 \cdots \beta_t \) is peak reduced with respect to \( Z \), this cannot occur, there is no such \( j \), and \( w_i \beta_1 \cdots \beta_j \) is conjugate to an element of \( L \), for all \( i, j \).

Next we shall move all \( \beta_j \)'s of Type 1 to the right hand end of the factorisation of \( \alpha_1 \). We may assume no two consecutive \( \beta_j \)'s are of Type 1. Let \( j \) be minimal such that \( \beta_j \) is of Type 1 and assume that \( j < t \). Assume first that \( \beta_s \) is of Type 2, for all \( s > j \). Let \( Z' = Z \beta_1 \cdots \beta_{j-1} \), so writing \( \tau = \beta_j^{-1} \),
\[
Z \beta_1 \cdots \beta_t = Z' \beta_j \cdots \beta_t = Z' \beta_{j+1}^r \cdots \beta_t \beta_j.
\]
Also, for \( j + 1 \leq s \leq t \),
\[
|Z|_\sim = |Z'|_\sim = |Z' \beta_j \beta_{j+1} \cdots \beta_s|_\sim = |Z' \tau \beta_{j+1} \cdots \beta_s \tau|_\sim = |Z' \beta_{j+1}^r \cdots \beta_s^r|_\sim,
\]
so
\[
\alpha_1 = \beta_1 \cdots \beta_{j-1} \beta_{j+1}^r \cdots \beta_t \beta_j
\]
is also peak reduced with respect to \( Z \). In the case where \( \beta_s \) is also of Type 1, for some \( s \) such that \( j < s \leq t \) we set \( \beta'_s = \beta_j \beta_s \), so \( \beta'_s \) is of Type 1, and the same argument shows that the factorisation
\[
\alpha_1 = \beta_1 \cdots \beta_{j-1} \beta_{j+1}^r \cdots \beta_t \beta'_s \beta_{s+1}^r \cdots \beta_t
\]

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is peak reduced with respect to \( Z \). In all cases the new factorisation has fewer elements of Type 1 to the left of elements of Type 2 than the original factorisation. Thus, continuing in this way we may assume that we have a factorisation \( \alpha_1 = \beta_1 \cdots \beta_t \sigma_1 \) which is peak reduced with respect to \( Z \), such that \( \beta_1 \) is of Type 2 in \( \Omega_t \) and \( \sigma_1 \) is of Type 1. Moreover, from the above, for \( 1 \leq i \leq k \) and \( 1 \leq j \leq t \) we have \( w_i \beta_1 \cdots \beta_j \) conjugate to an element of \( L \); so we have \( w_i \beta_1 \cdots \beta_j \) conjugate to \( w_i \), for all \( i, j \). Also as above, since \( Z \alpha_1 = Z \) it follows that \( w_i \sigma_1 = w_i \) and \( w_i \beta_1 \cdots \beta_t = w_i \), for all \( i \).

Given a factorisation \( \alpha_1 = \beta_1 \cdots \beta_i \sigma_1 \) with the properties above, let \( \beta_j = (B_j, b_j) \), where \( B_j \cap \text{st}(b_j) = \{ b_j \} \), for all \( j \). We claim that, for all \( j \), either

1. \( \{ w_1, \ldots, w_k \} \subseteq B_j \cup \text{st}(b_j) \) or
2. \( \{ w_1, \ldots, w_k \} \subseteq (L - B_j) \cup \text{st}(b_j) \).

To prove this claim, assume that \( j \) is minimal such that the claim does not hold. Note that, as \( w_i \beta_1 \cdots \beta_j \) is conjugate to \( w_i \), we have \( w_i \in B_j \setminus \{ b_j \} \) if and only if \( w_i^{-1} \in B_j \setminus \{ b_j \} \). Suppose that, for some \( p, q \) we have \( w_p \in B_j \), \( w_p \neq b_j^{+1} \) and \( w_q \in L - (B_j \cup \text{st}(b_j)) \). Then \( [w_p, b_j] \neq 1 \) and \( w_p \beta_j = \beta_j^{-1} w_p \circ \beta_j \) while \( [w_q, b_j] \neq 1 \) and \( w_q \beta_j = w_q \). Hence \( w_p w_q \beta_j = b_j^{-1} w_p \circ b_j \circ w_q \); a minimal length representative of its conjugacy class in \( G \).

By assumption the claim holds for \( 1, \ldots, j - 1 \), so there exists \( g \in G \) such that \( w_i \beta_1 \cdots \beta_{j-1} = w_q^g \). Thus \( w_p w_q \beta_1 \cdots \beta_j \) is conjugate to \( w_p w_q \beta_j \) and therefore \( |w_p w_q \beta_1 \cdots \beta_j| \geq 4 \). It follows that \( |w_q w_p \beta_1 \cdots \beta_j| \geq 4 \) and either \( w_q w_p \) or \( w_p w_q \) occurs in \( V \). For all other \( w_r w_s \) in \( V \) we have \( |w_r w_s \beta_1 \cdots \beta_j| \geq 2 \) and \( |W \beta_1 \cdots \beta_j| = |W|_{\sim} \). Hence

\[
|Z \beta_1 \cdots \beta_j|_{\sim} \geq |Z|_{\sim} + 2,
\]

a contradiction. Therefore no such \( j \) exists, and the claim holds.

Now let \( j \) be minimal such that 1 above holds. Define

\[
\beta_j' = (L - B_j - \text{lk}(b_j)) + b_j^{-1}, b_j^{-1}.
\]

Then \( \beta_j' \in \Omega_t \), \( \beta_j = \beta_j' \gamma_{b_j} \), where \( \gamma_{b_j} \) is conjugation by \( b_j \), and

\[
\beta_1 \cdots \beta_t = \beta_1 \cdots \beta_{j-1} \beta_j' \gamma_{b_j} \beta_{j+1} \cdots \beta_t = \beta_1 \cdots \beta_{j-1} \beta_j' \beta_{j+1} \cdots \beta_t \gamma',
\]

for some \( \gamma' \in \text{Inn}(G) \). As \( \{ w_1, \ldots, w_k \} \subseteq B_j \cup \text{st}(b_j) \), for the latter factorisation, \( j \) satisfies condition 2 above. We have thereby reduced the number of indices \( j \) for which 1 holds. Continuing this way we may assume \( \alpha_1 = \beta_1 \cdots \beta_j \gamma \sigma_1 \), where \( \beta_j \) is of Type 2 and satisfies 2 above, for \( j = 1, \ldots, t \). \( \gamma \in \text{Inn}(G) \) and \( \sigma_1 \in \Omega_t \) of Type 1, such that \( W \sigma_1 = W \). In this case
\( W = W\beta_1 \cdots \beta_t = W\gamma \), so \( \gamma \) is conjugation by \( g \in G \) such that \( g \in C_G(w_i) \), for \( 1 \leq i \leq k \). Every element of \( \text{Inn}(G) \) is a product of elements \( \Omega_l \) and it follows that \( \gamma \) is a product of elements of \( \Omega_l \cap \text{stab}(W) \). Hence we have a factorisation of \( \alpha \) as a product of elements of \( \text{stab}(W) \cap \Omega_l \). As \( \phi_i \in \Omega_l \cap \text{stab}(W) \) and \( \sigma \in \text{Aut}(\Gamma^{\pm 1}) \cap \text{stab}(W) \), it follows that \( \alpha \) belongs to the subgroup generated by \( (\Omega_l \cup \Omega_s) \cap \text{stab}(W) \), as required. \( \square \)

**Proof of Theorem 4.10.** The final statement follows from the first, in view of Theorem 4.3 and Definition 4.5. By definition, \( \text{St}_x^y \subseteq \text{stab}(X\setminus [x]) \).

For the opposite inclusion, first consider the case \( |X\setminus [x]| \leq 1 \). If \( |X\setminus [x]| = 0 \), then \( \text{stab}(X\setminus [x]) = \text{Aut}(G) = \text{St}_x^y \) and the Theorem follows from the results of Laurence and Servatius. If \( |X\setminus [x]| = 1 \) then \( \mathbb{G}(\Gamma) = \langle a \rangle \times \mathbb{G}([x]) \), where \( \langle a \rangle \) is infinite cyclic generated by the element \( a \) of \( X\setminus [x] \) and \( \mathbb{G}([x]) \) is the free group on \([x]\). If \( \phi \in \text{stab}(X\setminus [x]) \) then \( a\phi = a \) and, for all \( x \in X \), \( x\phi = w_x a^n \), where \( w_x \in \mathbb{G}([x]) \) and \( n_x \in \mathbb{Z} \). Hence \( \phi \in \text{St}_x^y \). Let \( \tau = \prod_{x \in X} \tau_{x,a}^{-n_x} \) and let \( \theta = \tau^{-1}\phi \). Then \( a\theta = a \) and \( x\theta = (xa^{-n_x})\phi = w_x \in \mathbb{G}([x]) \), for all \( x \in X \). Therefore \( \theta \) restricts to an element of \( \text{Aut}(\mathbb{G}([x])) \); which can be written as a product of Whitehead automorphisms of \( \mathbb{G}([x]) \), and these may all be regarded as Whitehead automorphisms of \( G \), fixing \( a \), and necessarily in \( \Omega_l \), hence in \( \Omega_l \cap \text{stab}(X\setminus [x]) \). Moreover \( \tau \) is a product of elements of \( \Omega_s \), which are also in \( \text{stab}(X\setminus [x]) \). Therefore \( \phi \) is in \( \text{St}_x^y \) and is a product of elements of \( (\Omega_l \cup \Omega_s) \cap \text{stab}(X\setminus [x]) \), as required.

Now consider the case \( |X\setminus [x]| \geq 2 \). From Proposition 4.11, it suffices to show that every element of \( \text{stab}(X\setminus [x]) \cap (\Omega_l \cup \Omega_s) \) belongs to \( \text{St}_x^y \). If \( \sigma \) is a Type 1 element of \( \text{stab}(X\setminus [x]) \cap (\Omega_l \cup \Omega_s) \) then \( \sigma \) permutes elements of \([x]_L \) and fixes all other elements of \( L \), so belongs to \( \text{St}_x^y \). If \( (A,a) \) is of Type 2 in \( \text{stab}(X\setminus [x]) \cap (\Omega_l \cup \Omega_s) \) then, by definition of \( \text{stab}(X\setminus [x]) \), \( A - a \subseteq [x]_L \). It remains to show that \( a \in a(x) \pm 1 \). If \( a \notin a(x)^\pm 1 \) then there is \( y \in \text{lk}(x) \) such that \( a \notin \text{st}(y) \). In this case \( \text{lk}(x) \notin \text{st}(a) \) and so \( A - a \) consists of the vertices of a union of connected components of \( \Gamma \setminus \text{st}(a) \), and their inverses. Since \( A - a \) contains some element of \([x]_L \) (as we assume \((A,a)\) is non-trivial) this means that \( y \in A \), a contradiction. Therefore \( (A,a) \in \text{St}_x^y \). \( \square \)

### 4.5 Peak reduction for \( \text{St}_x^y \)

First note that the elements of \( \Omega_x \) are all, by definition, long range and in fact \( \Omega_x = \text{St}_x^y \cap \Omega_l \). We shall need the following Lemma in the proof of Lemma 4.13.

**Lemma 4.12.** Let \( \alpha = (A,a) \) and \( \beta = (B,b) \) be Type 2 elements in \( \Omega_l \) and let \( C \) be a \( k \)-tuple of conjugacy classes of \( G \), such that \( A \subseteq B \) and \( \alpha^{-1}\beta \) is a peak for \( C \). In this case
(i) if $a^{-1} \in B$ then $|C\beta|_{\sim} < |C\alpha^{-1}|_{\sim}$ and

(ii) if $b \notin A$ then $|C(B - A + a, b)|_{\sim} < |C\alpha^{-1}|_{\sim}$.

Proof. Let $\beta^* = (L - B - \text{lk}_L(b), b^{-1})$. Then $\beta = \beta^*\gamma_b$, where $\gamma_b = (L - b^{-1}, b)$ is conjugation by $b$. As $A \subseteq B$ we have $A \cap (L - B - \text{lk}_L(b)) = \emptyset$.

(i) As $a^{-1} \in B$ we have $a^{-1} \notin L - B - \text{lk}_L(b)$. Also $|C\alpha^{-1}\beta|_{\sim} = |C\alpha^{-1}\beta^*\gamma_b|_{\sim} = |C\alpha^{-1}\beta^*|_{\sim}$, as $\gamma_b$ is inner, and $\alpha^{-1}\beta$ is a peak for $C$, so $\alpha^{-1}\beta^*$ is a peak for $C$. From [3, Lemma 3.21], we have $|C\beta^*|_{\sim} < |C\alpha^{-1}|_{\sim}$, and again $|C\beta|_{\sim} = |C\beta^*|_{\sim}$.

(ii) As before, since $\alpha^{-1}\beta$ is a peak for $C$, so is $\alpha^{-1}\beta^*$, and so $(\beta^*)^{-1}\alpha$ is a peak for $C^* = C\alpha^{-1}\beta^*$. As $b \notin A$, from [3, Lemma 3.21], we have

$$|C\alpha^{-1}\beta \alpha|_{\sim} = |C\alpha^{-1}\beta^*\alpha|_{\sim} < |C\alpha^{-1}\beta^*(\beta^*)^{-1}|_{\sim} = |C\alpha^{-1}|_{\sim}.$$

Write $B^* = L - B - \text{lk}_L(b)$. From DR4, we have $\alpha^{-1}\beta^*\alpha = (A + B^* - a, b^{-1})$ so

$$\alpha^{-1}\beta\alpha = \alpha^{-1}\beta^*\gamma_b\alpha = \alpha^{-1}\beta^*\alpha\gamma_b = (A + B^* - a, b^{-1})\gamma_b = (B - A + a, b).$$

Lemma 4.13 (cf. [3, Lemma 3.18]). Let $C$ be a $k$-tuple of conjugacy classes of $\Gamma$ and $\alpha, \beta \in \Omega_\ast$ such that $\alpha^{-1}\beta$ is a peak with respect to $C$. Then

PL1. there is a peak lowering $\alpha^{-1}\beta = \delta_1 \cdots \delta_s$ with respect to $C$, in $\Omega_\ast$, and

PL2. the relation $\alpha^{-1}\beta = \delta_1 \cdots \delta_s$ follows from $R1_\ast - R6_\ast, R3^\ast_\ast$ and $R4^\ast_\ast$ above.

Proof. We may assume that $C = (c_1, \ldots, c_k)$ where $c_i$ is a minimal representative of its conjugacy class; so $c_i$ and all its cyclic permutations are minimal words. From [3, Lemma 3.18] there is a peak lowering of $\alpha^{-1}\beta$ in $\Omega_\ast$. We work through Cases 1 to 4 of the proof of Lemma 3.18 in [3] to show that in the case in hand we may find such a factorisation satisfying PL1 and PL2. Cases 1 to 3 go through in the same way as they do in [3]. To cope with Case 4, without using automorphisms from outside $\Omega_\ast$, we extend the treatment of Case 3, following McCool [8]. (Note that in [3] automorphisms act on the left, whereas here automorphisms act on the right.)

Case 1. The case where $\alpha \in \text{Aut}(\Gamma^\pm)$: that is $\alpha$ is in $\Omega_\ast$ and of Type 1. The peak lowering factorisation in [3] is

$$\alpha^{-1}\beta = \beta^\prime\alpha^{-1},$$

(4.1)
where $\beta = (B, b)$ and $\beta' = (B\alpha, b\alpha)$. As $\alpha$ and $\beta$ are in $\Omega_x$, we have $b\alpha = b$ or $b\alpha \in [x]_L$ and $(B-b)\alpha \subseteq [x]_L$; so $\beta' \in \Omega_x$. Moreover, relation (4.1) follows from $R6_x$.

From now on we assume $\alpha$ and $\beta$ are of Type 2, $\alpha = (A, a)$ and $\beta = (B, b)$. Moreover, for elements $a, b \in L$, with $a \neq b^{\pm 1}$ we define $\sigma_{a,b}$ to be the element of $\text{Aut}(\Gamma^\pm)$ which fixes all elements of $L$ not equal to $a^{\pm 1}$ or $b^{\pm 1}$, maps $a$ to $b^{-1}$ and $b$ to $a$ (as in $R5_x$).

**Case 2.** The case $a \in \text{lk}_L(b)$. From $R3_x$, that $\alpha^{-1}\beta = \beta\alpha^{-1}$, so, as in [3], both PL1 and PL2 hold.

**Case 3.** The conditions of this case are that $A \cap B = \emptyset$ and $a \notin \text{lk}_L(b)$. The case is broken (in [3]) into three sub-cases, a, b and c. As noted in [3] these sub-cases exhausts all possibilities in Case 3.

**Sub-case 3a.** In this sub-case $\nu(a) = \nu(b)$. From [3] we have a peak lowering

$$\alpha^{-1}\beta = (A + B + a^{-1}, a^{-1}) = (A - a + a^{-1}, a^{-1})(B, a^{-1}),$$

which follows from $R1_x$ and $R2_x$, since $(A + B + a^{-1}, a^{-1}) \in \Omega_x$.

**Sub-case 3b.** In this sub-case $a^{-1} \notin B$. If $b^{-1} \notin A$ then $b^{\pm 1} \notin A$ and so from $R3_x$, we have $\alpha^{-1}\beta = \beta\alpha^{-1}$. As in [3], this factorisation is peak lowering.

If $b^{-1} \in A$ then, from $R4_x$, we have

$$\alpha^{-1}\beta = \beta(A + B - a - b + a^{-1}, a^{-1}),$$

and, as in [3], this factorisation is peak lowering.

**Sub-case 3c.** In this sub-case $\nu(a) \neq \nu(b)$, $a^{-1} \in B$ and $b^{-1} \in A$. The conditions that $a^{-1} \in B$ and $\nu(a) \neq \nu(b)$ imply that $a^{-1} \in B - b \subseteq [x]_L$. Similarly, $b^{-1} \in [x]_L$, so $a^{\pm 1}, b^{\pm 1} \in [x]_L$. It follows that $\alpha' = (A, b^{-1}), \beta' = (B, a^{-1})$ and $(B - a^{-1} + a - b + b^{-1}, a)$ are in $\Omega_x$, as is the element $\sigma_{a,b}$ of $\text{Aut}(\Gamma^\pm)$. From $R5_x$,

$$(\beta')^{-1}\beta = \sigma_{a,b}(B - a^{-1} + a - b + b^{-1}, a)$$

and from $R2_x$,

$$\alpha^{-1}\beta' = (A + B - a, a^{-1}) \in \Omega_x.$$

Hence

$$\alpha^{-1}\beta = \alpha^{-1}\beta'(\beta')^{-1}\beta = (A + B - a, a^{-1})\sigma_{a,b}(B - a^{-1} + a - b + b^{-1}, a)$$

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and as in [3] this factorisation is peak lowering; so PL1 and PL2 hold in this case.

This concludes Case 3. Some instances of peak lowering in Case 4, in [3], require \( \alpha \) to be replaced by \( \alpha^* = (L - A - \text{lk}_L(a), a^{-1}) \), and if \( \alpha \in \Omega_x \) then \( \alpha^* \) is not. To avoid this replacement we consider the analogue of Case 3 in which we assume \( A \subseteq B \) instead of \( A \cap B = \emptyset \). (In the usual treatment of peak lowering these cases follow, after switching \( \alpha \) and \( \alpha^* \), or making a similar switch for \( \beta \).)

**Case 3\'**. Assume \( A \subseteq B \) and \( a \notin \text{lk}_L(b) \). We break the case into three sub-cases.

**Sub-case 3\'a.** In this sub-case \( v(a) = v(b) \). As \( A \subseteq B \) this implies \( a = b \). Then

\[
\alpha^{-1}\beta = (A - a + a^{-1}, a^{-1})(A, a)(B - A + a, a) = (B - A + a, a),
\]

is a peak lowering factorisation and, as \( (B - A + a, a) \in \Omega_x \), this relation follows from R2\(_x\) and R1\(_x\), so PL1 and PL2 hold.

**Sub-case 3\'b.** Assume \( a^{-1} \in B \). As \( a \in B \) this implies that \( v(a) \neq v(b) \). If \( b \notin A \) then, from R3\(_x^\ast\), \( \alpha^{-1}\beta = \beta\alpha^{-1} \).

If \( b \in A \) then \( (B - A + b^{-1}, a^{-1}) \in \Omega_x \) and, from R4\(_x^\ast\), \( \alpha^{-1}\beta = \beta(B - A + b^{-1}, a^{-1})^{-1} \). In both cases it follows from Lemma 4.12 that these factorisations are peak lowering.

**Sub-case 3\'c** In this sub-case \( v(a) \neq v(b) \) and \( a^{-1} \notin B \). If \( b \notin A \) then \((B - A + a, b) \in \Omega_x\) so from R4\(_x\),

\[
\beta = (A - a + a^{-1}, a^{-1})^{-1}(B - A + a, b)(A - a + a^{-1}, a^{-1}).
\]

This gives a factorisation \( \alpha^{-1}\beta = (B - A + a, b)\alpha^{-1} \), which Lemma 4.12 (ii) implies is peak lowering.

If \( b \in A \) then \( a, b \in [\cdot]_L \), so \( a \sim b \) and \( \alpha_b = (A, b) \) and \( \beta_a = (B, a) \) are defined and in \( \Omega_x \). Also, as in [3], (using an adjacency counter argument and [3, Lemma 3.17]) either \( \alpha_b \) or \( \beta_a \) reduces \( \lvert C\alpha^{-1}\rvert_{\sim} \). Assume first that \( \lvert C\alpha^{-1}\beta_a\rvert_{\sim} < \lvert C\alpha^{-1}\rvert_{\sim} \). We have, from R1\(_x\) and R2\(_x\), that \( \alpha^{-1}\beta_a = (B - A + a, a) \) and so

\[
\alpha^{-1}\beta = (B - A + a, a)\beta_a^{-1}\beta
= (B - A + a, a)\sigma_{a^{-1},b}(B + a^{-1} - a + b^{-1} - b, a^{-1}),
\]

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using $R5_x$. As $\sigma_{a^{-1},b}$ preserves lengths and $|C(B - A + a, a)|_\sim < |C\alpha^{-1}|_\sim$, this factorisation is peak lowering.

On the other hand, if $|C\alpha^{-1}a|_\sim < |C\alpha^{-1}|_\sim$ then, from $R2_x$ and $R6_x$,

$$\beta = \alpha_b(B - A + b, b)$$

so

$$\alpha^{-1}\beta = \alpha^{-1}\alpha_b(B - A + b, b)\sigma^{-1}_{\alpha,b} = (A - b + b^{-1}, a),$$

using $R5_x$ and, since $|C(A - a + a^{-1}, b)\sigma^{-1}_{\alpha,b}|_\sim < |C\alpha^{-1}|_\sim$, the factorisation

$$\alpha^{-1}\beta = (A - a + a^{-1}, b)\sigma^{-1}_{\alpha,b}(B - A + b, b)$$

is peak lowering.

**Case 4.** In the light of Cases 1 to 3 and 3* above, and since we may interchange $\alpha$ and $\beta$, we may now assume $a \notin \text{lk}_L(b)$ and $A \cap B$, $A \cap B^c$ and $A^c \cap B$ are all non-empty.

**Sub-case 4a.** In this case we assume $\text{lk}_L(a) = \text{lk}_L(b)$ and $v(a) \neq v(b)$; so $a \sim b$. As in [8] (using the form of adjacency counting defined in [3]) after interchanging $\alpha$ and $\beta$ if necessary, we may assume one of $\alpha_1 = (A \cap B, a)$, $\alpha_2 = (A \cap B^c, a)$, $\alpha_3 = (A^c \cap B, a^{-1})$ or $\alpha_4' = (A^c \cap B^c, a^{-1})$ is a well-defined Whitehead automorphism and reduces $|C\alpha^{-1}|_\sim$. If $\alpha_4'$ is defined and reduces $|C\alpha^{-1}|\sim$, then by composing with the inner automorphism $\gamma_a$, we see that $\alpha_4 = (A \cup B, a)$ is also defined and also reduces $|C\alpha^{-1}|_\sim$. In addition, if it is defined, $\alpha_i \in \Omega_x$, for $i = 1, \ldots, 4$. (In fact $\alpha_1$ is defined if $a \in B$, $\alpha_2$ if $a \notin B$, $\alpha_3$ if $a^{-1} \in B$ and $\alpha_4$ if $a^{-1} \notin B$.) Now, for $i = 1, \ldots, 4$, define $\hat{\alpha}_i = \alpha^{-1}\alpha_i$. Using $R1_x$ and $R2_x$ we have, when the map in question is defined,

$$\hat{\alpha}_1 = (A \cap B^c + a, a)^{-1},$$

$$\hat{\alpha}_2 = (A \cap B + a, a)^{-1},$$

$$\hat{\alpha}_3 = (A \cup B - a^{-1}, a)^{-1}$$

and

$$\hat{\alpha}_4 = (A^c \cap B + a, a).$$

Assume then $1 \leq i \leq 4$ and that $\alpha_i$ is defined and shortens $|C\alpha^{-1}|_\sim$. Then $|C\hat{\alpha}_i|_\sim = |C\alpha^{-1}\alpha_i|_\sim < |C\alpha^{-1}|_\sim$, so $\alpha_i^{-1}\beta$ is a peak with respect to $C\hat{\alpha}_i$. If $i = 1, 2$ or $4$ then Case 3* gives a peak-lowering of $\alpha_i^{-1}\beta$, with respect to $C\hat{\alpha}_i$, in $\Omega_x$. If $i = 3$, then Case 3 gives a peak-lowering of $\alpha_i^{-1}\beta$, with respect to $C\hat{\alpha}_i$, in $\Omega_x$. In all cases we have a peak lowering factorisation

$$\alpha_i^{-1}\beta = \delta_1 \ldots \delta_k.$$
with $\delta_i \in \Omega_x$. Therefore, as $\hat{\alpha}_i \in \Omega_x$,

$$\alpha^{-1} \beta = \hat{\alpha}_i \delta_1 \ldots \delta_k,$$

is a peak-lowering factorisation of $\alpha^{-1} \beta$, in $\Omega_x$. Moreover this factorisation follows from the relations $R_x$.

**Sub-case 4b.** In this case we assume that $v(a) = v(b)$ or $lk_L(a) \neq lk_L(b)$.

We break this sub-case into two further sub-cases: either $a \in B$ or $b \in A$; or $a \notin B$ and $b \notin A$.

(i) If $b \in A$ but $a \notin B$ then interchanging $\alpha$ and $\beta$ we obtain $a \in B$. Hence we may assume that $a \in B$. In this case either $a = b$ or $v(a) \neq v(b)$. If $a = b$ then $a^{-1} = b^{-1} \notin A \cup B$. If $v(a) \neq v(b)$ then $a \in B \setminus \{b\} \subseteq [x]_L$ and, as $lk_L(a) \neq lk_L(b)$, we have $b \notin [x]_L$, so $b^{\pm 1} \notin A$. In both cases $b^{-1} \notin A$ and $a \notin B^\ast$. If $A \cap B^\ast = \emptyset$ then $A \subseteq B \cup lk_L(b)$ and, as $\alpha, \beta \in \Omega_x$, we have $A \setminus \{a\} \subseteq [x]_L$ and $b \in [x]_L \cup a_{\text{out},L}(x)$, so $A \setminus \{a\} \cap lk_L(b) = \emptyset$. As $a \in B$ this implies $A \subseteq B$, a contradiction. Hence, $A \cap B^\ast \neq \emptyset$ and, as $\alpha^{-1} \beta$ is a peak for $C$ so is $\alpha^{-1} \beta^\ast$. As in [3, Sub-case 4b] both $(A^\ast \cap B^\ast, b^{-1})$ and $(A \cap (B^\ast)^{\ast}, a) = (A \cap B, a)$ are defined, and one or other reduces the conjugacy length of $C\alpha^{-1}$. If $(A \cap B, a)$ shortens $C\alpha^{-1}$ then, as $(A \cap B, a) \in \Omega_x$, we may construct a peak lowering as in the case when $i = 2$ of Sub-case 4a above; via elements of $\Omega_x$ and following from the relations $R_x$. If $(A^\ast \cap B^\ast, b^{-1})$ shortens the conjugacy length of $C\alpha^{-1}$ then so does $(A \cup (B^\ast)^{\ast}, b) = (A \cup B, b)$. In this case, after interchanging $\alpha$ and $\beta$ we construct a peak lowering, with the required properties, as in the case when $i = 4$ of Sub-case 4a above.

(ii) If $a \notin B$ and $b \notin A$ then we are in the same situation as Sub-case 4b in [3]. In this case both $(A \cap B^\ast, a)$ and $(A^\ast \cap B, b)$ are defined, necessarily in $\Omega_x$, and one or other of them reduces the conjugacy length of $C\alpha^{-1}$. If this conjugacy length is reduced by $(A \cap B^\ast, a)$ then we may construct a peak lowering, with the required properties, as in Sub-case 4a above, where $i = 2$. For the remaining case we first interchange $\alpha$ and $\beta$ and then proceed as before.

\[\blacksquare\]

### 4.6 Proof of Theorem 4.6

Let $P$ be the group with presentation $\langle \Omega_x \mid R_x \rangle$. The canonical map from $P$ to $\text{St}_{x,I}^\ast$, taking $\alpha \in \Omega_x$ to its realisation as an automorphism of $\mathbb{G}$, induces a surjective homomorphism, in view of Theorem 4.10 and the fact that all the
relations of $R_x$ hold in $St_{x,l}^x$. It remains to show that this homomorphism is also injective.

For the duration of this section $C_2$ denotes a fixed tuple of words of $G$ of length 2, such that $C_2$ contains precisely one representative of each conjugacy class of $G([x])$ of length 2. Note that, if $y, z \in [x]_L$, with $y \neq z^{\pm 1}$ it follows that either $yz$ or $zy$ is in $C_2$.

**Lemma 4.14.** If $\beta \in \Omega_x$ and $|C_2\beta| \sim |C_2|$ then either

(i) $\beta$ is of Type 1, or

(ii) $\beta = (B, b)$, where $b \in [x]_L$ and $B = [x]_L - b^{-1}$, or

(iii) $\beta = (B, b)$, where $b \in a_{\text{out},L}(x)$ and $B = [x]_L$.

In (i) $|C_2\beta| = |C_2|$ and in all cases $|C_2\beta| \sim |C_2|$.

**Proof.** In the case where $|[x]| = 1$, without loss of generality we may assume that $C_2 = (x^2, x^{-2})$. If $\beta$ does not map $x$ to $x^{\pm 1}$ or to a conjugate of $x$ then evidently $|C_2\beta| \sim > |C_2|$.. Therefore the result holds in this case.

Assume then that $|[x]| \geq 2$. If $\beta$ is of Type 1, then the claims of the Lemma follow from the definition of $\Omega_x$. Assume then that $\beta$ is not of Type 1, so $\beta = (B, b)$, where $B - b \subseteq [x]_L$. If $b \in [x]_L$ then $\beta|_{G([x])}$ is an automorphism of the free group $G([x])$. From, for example, [3][Theorem 5.2], in this case the restriction of $\beta$ is an inner automorphism of this group. Hence $B = [x]_L - b^{-1}$, as claimed.

On the other hand, if $b \notin [x]_L$ then $b \in a_{\text{out},L}(x)$. In this case, if $y$ and $z$ are elements of $[x]_L$, with $y \neq z^{\pm 1}$, $y \in B$ and $z \notin B$ then $yz\beta = ybz$ or $yz\beta = b^{-1}ybz$, in both cases a cyclically minimal word of length at least 3. We may assume $yz \in C_2$, and as $b$ is not in $[x]_L$, no conjugacy class in $C_2$ has its length reduced by $\beta$. Hence $|C_2\beta| \sim > |C_2|$., a contradiction. It follows, since the identity map is of Type 1, that $B = [x]_L$. In both cases (ii) and (iii) the map $\beta$ acts by conjugation on $[x]_L$, so the final statement of the Lemma holds.

We shall call elements of $\Omega_x$ of the form occurring in (i), (ii) and (iii) generators of Type $1_x, 2a_x$ and $2b_x$, respectively. Now

- let $\alpha \in F(\Omega_x)$, say $\alpha = \phi_1 \cdots \phi_n$, where $\phi_j \in \Omega_x (= \Omega_x^{-1})$, and this word is reduced.

We may assume that the length of the word $\phi_1 \cdots \phi_n$ cannot be reduced by application of relations $R1_x$ or $R7_x$.  

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Lemma 4.15. Assume $\phi_j$ is of Type $1_x$, $2a_x$ or $2b_x$, for $j = 1, \ldots, n$, and $\phi_1 \cdots \phi_n$ is peak reduced with respect to $C_2$. Then there exist elements $\alpha_i, \beta_i$ and $\sigma$ of $\Omega_x$ such that $\alpha_i$ is of type $2a_x$, $\beta_i$ is of Type $2b_x$, $\sigma$ is of Type $1_x$ and

(i) in the group $P$ the element $\alpha$ is equal to $\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \sigma$ and

(ii) in $\text{St}^{\sigma}_{x,i}$ the factorisation $\alpha = \alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \sigma$ is peak reduced with respect to $C_2$.

Proof. Let $i$ be maximal such that $\phi_i$ is of Type $2a_x$ or $2b_x$ and $\phi_{i-1}$ is of Type $1_x$. Then $\phi_i = (A, a)$. Let $\phi_i' = (A\phi_i^{-1} \alpha, a\phi_i^{-1})$. From the definitions, $\phi_i'$ is of the same Type as $\phi_i$ and relations $R6_x$ imply that $\phi_i - 1 \phi_i = \phi_i' \phi_i - 1$ in $P$.

Let $C_{2,j} = C_2 \phi_1 \cdots \phi_j$, for $j = 1, \ldots, n$, and $C_{2,0} = C_2$. As $\phi_1 \cdots \phi_n$ is peak reduced with respect to $C_2$ it follows, from Lemma 4.14, that $|C_{2,j}|_\sigma = |C_{2,i}|_\sigma$ for all $j$. As $\phi_{i-1}$ does not alter lengths, and $\phi_i$ and $\phi_i'$ preserve conjugacy lengths of elements of $G([x])$, we have

$$|C_{2,i-2}|_\sigma = |C_{2,i-2}\phi_i|_\sigma = |C_{2,i-2}\phi_i'|_\sigma,$$

from which it follows (as $\phi_i' \phi_{i-1} = \phi_{i-1} \phi_i$ in $\text{Aut}(G)$) that the factorisation $\alpha = \phi_1 \cdots \phi_i' \phi_i - 1 \cdots \phi_n$ is peak reduced with respect to $C_2$. Continuing this way we may move all the $\phi_i$ of Type $1_x$ to the right hand side, and use relations $R7_x$, to obtain $\alpha = \gamma_1 \cdots \gamma_m \sigma$ in $P$, where $\gamma_i$ is of Type $2a_x$ or $2b_x$, $\sigma$ is of Type $1_x$ and the factorisation is peak reduced with respect to $C_2$.

Now let $i$ be maximal such that $\gamma_i$ is of Type $2a_x$ and $\gamma_{i-1}$ is of Type $2b_x$. By definition of Types $2a_x$ and $2b_x$, relations $R3_x^\sigma$ imply that $\gamma_{i-1} \gamma_i = \gamma_{i-1} \gamma_i$. As in the previous case, since $\gamma_1 \cdots \gamma_m \sigma$ is peak reduced with respect to $C_2$, so is $\gamma_1 \cdots \gamma_{i-1} \gamma_{i-1} \cdots \gamma_m \sigma$. Continuing this way gives the required result. 

Lemma 4.16. Let $y$ be a word of length 2 in $G([x])$ and let $\alpha = \alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \sigma$ be a factorisation of $\alpha$, satisfying conditions of the conclusion of Lemma 4.15. Then $y\alpha = z^w$, where $z$ is a word of length 2 in $G([x])$, $u \in G([x])$ and $v \in G(a_{out}(x))$.

Proof. First assume that $\sigma = 1$. If $r+s = 0$ there is nothing to prove. Assume next that $s > 0$ and inductively that $\beta_s = (B, b)$, $y\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_{s-1} = z^{uv}$, for some word $z$ of length 2 in $G([x])$, $u \in G([x])$ and $v \in G(a_{out}(x))$. Then $y\alpha = (z^w)\beta_s = z^{uv}v$, where $v' \in G(a_{out}(x))$, so $y\alpha$ is of the required form. If $s = 0$ and $r > 0$ then again $y\alpha = (z^w)\alpha_r = z^{u'}v$, where $u' = ua$, so $y\alpha$ is of the required form. Finally, if $\sigma \neq 1$ then, since $\sigma$ fixes $a_{out}(x)$ point-wise and permutes the elements of $[x]_L$, the result follows.
Lemma 4.17. Let $\alpha = \alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \sigma$ be a factorisation of $\alpha$, as in the conclusion of Lemma 4.15. Assume $\alpha = 1$ in $\text{St}^\nu_{x,t}$. Then $\alpha = 1$ in $P$.

Proof. Let $C_2 = (y_1, \ldots, y_k)$. Then, from Lemma 4.16, we have $C_2 \alpha = (z_1^{u_1 v_1}, \ldots, z_k^{u_k v_k})$, where $z_i$ is of length 2 in $\mathbb{G}([x])$, $u_i \in \mathbb{G}([x])$ and $v_i \in \mathbb{G}(a_{\text{out}}(x))$. As $\alpha = 1$ in $\text{St}^\nu_{x,t}$, we have $C_2 \alpha = C_2$ so $z_i^{u_1 v_1} = y_i$, for $i = 1, \ldots, k$. No letter of $a_{\text{out}}(x)$ commutes with any element of $[x]$, so this implies that $v_i = 1$, for all $i$. Let $\beta_j = ([x]_L, b_j)$, for $j = 1, \ldots, s$, so $\beta_i = b_s \cdots b_1 = 1$, for all $i$. Therefore there exist $1 \leq p < q \leq s$ such that $b_p = b_q^{-1}$ and $b_j \in \text{lk}_L(b_p)$, for $p < j < q$. From R3$_x$ (b) and R1$_x$ it follows that, in $P$,

$$
\beta_p \beta_{p+1} \cdots \beta_{q-1} \beta_q = \beta_{p+1} \cdots \beta_{q-1} \beta_p \beta_q = \beta_{p+1} \cdots \beta_{q-1},
$$

so

$$
\beta_1 \cdots \beta_s = \beta_1 \cdots \beta_{p-1} \beta_{p+1} \cdots \beta_{q-1} \beta_q \beta_{q+1} \cdots \beta_s.
$$

Continuing this process we eventually obtain $\alpha = \alpha_1 \cdots \alpha_r \sigma$ in $P$. Then $y_i \alpha = x_i^{u_i} = y_i$, where $z_i$ and $u_i$ are as before, for $i = 1, \ldots, k$.

If $[x] = \{x\}$, then there are no automorphisms of Type 2$\alpha_x$ and so $\alpha = \sigma$ is of Type 1$_x$. As $\alpha = 1$ in $\text{St}^\nu_{x,t}$, this implies that $\sigma$ is the identity permutation of $[x]_L$, so $\sigma = 1$ in $P$, as required. Assume then that $|[x]| \geq 2$. For $1 \leq j \leq r$ let $\alpha_j = ([x]_L - a_j^{-1}, a_j)$, where $a_j \in [x]_L$. Then, for $1 \leq i \leq k$ we have $y_i = y_i \alpha = y_i^{a_i \cdots a_1} \sigma$. In particular, if $x_1$ and $x_2$ are distinct elements of $[x]$ then $x_1^2$ and $x_2^2$ both appear in $C_2$,

$$
x_1^2 \sigma^{-1} = (x_1^2)^{a_1 \cdots a_1} \text{ and } x_2^2 \sigma^{-1} = (x_2^2)^{a_1 \cdots a_1}.
$$

This occurs only if $x_i \sigma = x_i$, for $i = 1, 2$. Furthermore, from $[1]$, $x_i^2 = (x_i^2)^{a_1 \cdots a_1}$ only if $(a_1 \cdots a_1) \in C_{\mathbb{G}}(x_i)$, for $i = 1, 2$. As $(a_1 \cdots a_1) \in \mathbb{G}([x])$ and $\mathbb{G}([x]) \cap C_{\mathbb{G}}(x_1) \cap C_{\mathbb{G}}(x_2) = \{1\}$, we have $a_1 \cdots a_1 = 1$ in $\mathbb{G}([x])$. As $\mathbb{G}([x])$ is free there must therefore exist $j$ such that $a_j = a_j^{-1}$, so $a_j \cdots a_j = 1$ in $P$, using R1$_x$. Continuing this process we again obtain $\alpha = \sigma$ and as $\alpha = 1$ in $\text{St}^\nu_{x,t}$ we now have $\alpha = 1$ in $P$. \hfill \Box

Proof of Theorem 4.4.6. As observed above it is necessary only to show that the canonical homomorphism from $P$ to $\text{St}^\nu_{x,t}$ is injective. Let $\alpha \in F(\Omega_x)$ and assume that $\alpha = 1$ in $\text{St}^\nu_{x,t}$. Write $\alpha = \phi_1 \cdots \phi_n$, where $\phi_i \in \Omega_x$ and define $C_{\alpha,j} = C_2 \phi_1 \cdots \phi_j$, for $1 \leq j \leq n$, and $C_{\alpha,0} = C_2$. If $\phi_1 \cdots \phi_n$ is not peak reduced with respect to $C_2$ then we say that $\phi_{j+1}$ is a peak of height $m$ (for $C_2$) if $\phi_j \phi_{j+1}$ is a peak for $C_{\alpha,j+1}$ and $|C_{\alpha,j} \sim = m$. Let $m$ be the maximum of the heights of peaks for $C_2$ and let $p$ be minimal such that $\phi_p \phi_{p+1}$ is a peak of height $m$. This implies that $|C_{\alpha,p-1} \sim < |C_{\alpha,p} \sim = m$. Also, define the peak length (with respect to $C_2$) of the factorisation to be the number of
indices \( j \) such that \( |C_j|_\sim = m \); and assume the peak length of \( \phi_1 \cdots \phi_n \) is \( M \).

From Lemma 4.13, there exist \( \delta_1, \ldots, \delta_s \in \Omega_x \) such that \( \phi_p \phi_{p+1} = \delta_1 \cdots \delta_s \) in \( P \) and \( |C_{2,p-1} \delta_1 \cdots \delta_t|_\sim < |C_{2,p}|_\sim \), for \( 1 \leq t \leq s - 1 \). Therefore we have \( \alpha = \phi_1 \cdots \phi_{p-1} \delta_1 \cdots \delta_s \phi_{p+1} \cdots \phi_n \) in \( P \) and in this factorisation, either the maximum height of peaks for \( C_2 \) is less than \( m \), or the peak length is less than \( M \). This process may therefore be repeated until we obtain a factorisation of \( \alpha \), in \( P \), which is peak reduced with respect to \( C_2 \).

Assume then that \( \alpha = \phi_1 \cdots \phi_n \) in \( P \), where \( \phi_i \in \Omega_x \) and that this factorisation is peak reduced with respect to \( C_2 \). From Lemma 4.14 and the fact that \( \alpha = 1 \) in \( \text{St}^e_j \), we have \( |C_{2,j}|_\sim = |C_j|_\sim \); so \( \phi_j \) is of type 1\( _x \), 2\( _a \), or 2\( _b \), for \( j = 1, \ldots, n \). From Lemma 4.15 there is, in \( P \), a peak reduced factorisation \( \alpha = \alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s \sigma \), where \( \alpha_j \) is of Type 2\( _a \), \( \beta_j \) is of Type 2\( _b \) and \( \sigma \) is of Type 1\( _x \). Lemma 4.17 then implies that \( \alpha = 1 \) in \( P \), as required.

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