An explicit Euler method for McKean–Vlasov SDEs
driven by fractional Brownian motion

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Abstract

In this paper, we establish the theory of chaos propagation and propose an Euler-Maruyama scheme
for McKean–Vlasov stochastic differential equations driven by fractional Brownian motion with Hurst
exponent $H \in (0,1)$. Meanwhile, upper bounds for errors in the Euler method is obtained. A numerical
example is demonstrated to verify the theoretical results.

Keywords: Propagation of chaos, Explicit Euler method, McKean–Vlasov, Fractional Brownian
motion, Interacting particle system.

1 Introduction

The pioneering work of McKean–Vlasov stochastic differential equations (SDEs) has been done by McKean
in \cite{19–21} connected with a mathematical foundation of the Boltzmann equation. Due to their widespread
applications in many fields, McKean–Vlasov SDEs have been researched by many scholars. In \cite{17}, existence
and uniqueness are proved for distribution-dependent SDEs with non-degenerate noise under integrability con-
ditions on distribution-dependent coefficients. Some theories about McKean–Vlasov SDEs were investigated,
including ergodicity \cite{11}, Harnack inequality \cite{27}, and the Bismut formula \cite{12, 25}. And the integration by
parts formulae on Wiener space for solutions of the SDEs with general McKean–Vlasov interaction was derived
in \cite{8}. In \cite{6}, Buckdahn et al. characterized the function on the coefficients of the stochastic differential
equation under appropriate regularity conditions as the unique classical solution of a nonlocal partial differential
equation of mean-field type. A complete probabilistic analysis of a large class of stochastic differential games
with mean field interactions was provided in \cite{7}.

It is well known that the explicit solutions to McKean–Vlasov SDEs are difficult to be shown. Hence,
the numerical methods for McKean–Vlasov SDEs driven by standard Brownian motion are studied by many
scholars \cite{1, 2, 4, 5, 9}. Moreover, it should be noted that SDEs driven by fractional Brownian motion (fBm)
have wider applications \cite{3, 23, 24}. On the other hand, the numerical methods for SDEs driven by fBm have

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attracted increasing interest; see [15, 16, 18, 28, 30] for example. Galeati et al. [14] examined the distribution-dependent stochastic differential equations with erratic, potentially distributional drift, driven by an additive fBm of Hurst parameter $H \in (0, 1)$, and they established strong well-posedness under a variety of assumptions on the drift. To our knowledge, the numerical method for McKean–Vlasov SDEs driven by fBm has not been discussed yet. As we know, propagation of chaos plays a key role to approximate the McKean-Vlasov SDEs. This paper aims at establishing the theory about propagation of chaos and the strong convergence rate in $L^p$ sense of EM method for McKean–Vlasov SDEs driven by fBm under the globally Lipschitz condition.

In this paper, we consider the following $d$-dimensional McKean–Vlasov SDEs driven by fBm of the form

$$dX_i = b(X_i, \mathcal{L}(X_i)) \, dt + \sigma(\mathcal{L}(X_i)) \, dB_i^H, \quad (1.1)$$

where the coefficients $b : \mathbb{R}^d \times \mathcal{P}_\theta(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathcal{P}_\theta(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$. Here, the initial value $X_0 \in L^p(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})$ with $p \geq \theta \geq 2$, and $B_i^H$ is a $d$-dimensional fBm with Hurst parameter $H \in (0, 1)$. As we know, the covariance of $B_i^H$ is

$$R_H(t, s) = \mathbb{E}(B_i^H B_i^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad \forall t, s \in [0, T].$$

The fBm $\{B_i^H\}_{t \geq 0}$ corresponds to a standard Brownian motion when $H = 1/2$. Further, the fBm $\{B_i^H\}_{t \geq 0}$ is not a semi-martingale or a Markov process unless $H = 1/2$. Therefore, when working with the fBm $\{B_i^H\}_{t \geq 0}$, many of the powerful features are unavailable. We rewrite (1.1) to the system of noninteracting particles

$$X_i^t = X_i^0 + \int_0^t b(X_i^s, \mathcal{L}_{X_i^s}) \, ds + \int_0^t \sigma(\mathcal{L}_{X_i^s}) \, dB_{s,i}^H, \quad i = 1, \ldots, N, \quad (1.2)$$

where $\mathcal{L}_{X_i^t}$ denotes the law of the process $X_i^t$ at time $t$. Compared with the standard SDEs, McKean–Vlasov SDEs provide an additional complexity, that is, it is required to approximate the law at each time step. Although there are other technologies, the most common one is the so-called interacting particle system

$$X_i^{t,N} = X_i^t + \int_0^t b (X_{s,N}^{i,N}, \mu_{s,N}^{X,N}) \, ds + \int_0^t \sigma(\mu_{s,N}^{X,N}) \, dB_{s,i}^{H,i}, \quad (1.3)$$

where the empirical measures is defined by

$$\mu_{t}^{X,N}(\cdot) := \frac{1}{N} \sum_{j=1}^N \delta_{X_{t,j,N}(\cdot)},$$

and $\delta_x$ denotes the Dirac measure at point $x$.

The structure of this work is as follows. The mathematical preliminaries on the McKean–Vlasov SDEs driven by fBm are presented in Section 2. Section 3 gives the main theorem and its proof. Numerical simulations are provided in Section 4.

## 2 Mathematical Preliminaries

Throughout the article, we will always work on a finite time interval $[0, T]$ and consider an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precisely, $\Omega$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel $\sigma$-algebra and $\mathbb{P}$ is the unique probability measure on $\Omega$ such that the canonical process $\{B_i^H\}_{t \in [0, T]}$ is a $d$-dimensional fBm with Hurst parameter $H \in (0, 1)$. For any $p \geq 1$, let

$$\|V(x)\|_{L^p} := (\mathbb{E}|V(x)|^p)^{1/p}.$$
We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ for the Euclidean norm and inner product, respectively, and let $a \& b := \min(a,b)$ and $a \lor b := \max(a,b)$. The notation $u \otimes v$ for $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ means the tensor product of $u$ and $v$. We will denote the set of all probability measures on $\mathbb{R}^d$ by

$$
P_p(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty \right\}
$$

For $\theta \geq 2$ and any $\mu, \nu \in \mathcal{P}_\theta(\mathbb{R}^d)$, the $\theta$-Wasserstein distance is defined by,

$$W_\theta(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\theta \pi(dx, dy) \right)^{1/\theta}
$$

where $\Pi(\mu, \nu)$ is the set of couplings of $\mu$ and $\nu$, and $\mathcal{P}_\theta(\mathbb{R}^d)$ is a Polish space under the $\theta$-Wasserstein metric.

Let $a, b \in \mathbb{R}$ with $a < b$. For $g \in L^1(a, b)$ and $\alpha > 0$, the left-sided fractional Riemann-Liouville integral of $g$ of order $\alpha$ on $[a, b]$ is defined as

$$I_\alpha^a g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} g(y)dy,
$$

and the right-sided fractional Riemann-Liouville integral of $f$ of order $\alpha$ on $[a, b]$ is defined as

$$I_\alpha^b g(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha - 1} g(y)dy,
$$

where $x \in (a, b)$ and $\Gamma$ denotes the Gamma function. For further details about fractional integral and derivative, we refer the reader to [3, 26].

**Assumption 2.1** There exists a positive constant $L$ such that

$$|b(x, \mu) - b(y, \nu)| \leq L (|x - y| + W_\theta(\mu, \nu)),$$

for all $x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_\theta(\mathbb{R}^d)$.

$$|\sigma(\mu) - \sigma(\nu)| \leq LW_\theta(\mu, \nu),$$

for all $\mu, \nu \in \mathcal{P}_\theta(\mathbb{R}^d)$. And for initial experience distribution $\delta_0$,

$$|b(0, \delta_0)| \lor |\sigma(\delta_0)| \leq L.$$

Furthermore, from Assumption 2.1, there exists a positive constant $L$ such that

$$|b(x, \mu)| \leq L (1 + |x| + W_\theta(\mu, \delta_0)),$$

and

$$|\sigma(\mu)| \leq L (1 + W_\theta(\mu, \delta_0)).$$

### 3 Main Result

**Lemma 3.1** [12] When $H \in (1/2, 1)$ and $p > 1/H$ hold, the solution of $X_t$ in (1.1) exists and is unique. Moreover, when $H \in (0, 1/2)$ and $\sigma(\mu)$ is independent of distribution, the solution of $X_t$ in (1.1) exists and is unique.
Lemma 3.2 For two empirical measures $\mu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j^i,N}$ and $\nu_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{y_j^i,N}$. Then,

$$E \left( W_\theta^q (\mu_t^N, \nu_t^N) \right) \leq E \left( \frac{1}{N} \sum_{j=1}^{N} |Z_{i,j}^1 - Z_{i,j}^2|^q \right).$$

This lemma follows from constructing a simple transport plan $\pi(dx, dy) = \frac{1}{N} \sum_{j=1}^{N} \delta_{z_{i,j}^1,N} (dx) \otimes \delta_{z_{i,j}^2,N}(dy)$.

### 3.1 Case $H > 1/2$

**Theorem 3.3** Let Assumption 2.1 holds and $q > p \geq 2$, then

$$\sup_{t \in [0,T]} E \left( |X_t^i|^q \right) + \sup_{t \in [0,T]} E \left( |X_t^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E|X_0|^q \right),$$

where $C_{q,H,T,L}$ is a positive constant dependent on $q, H, T, L$.

**Proof.** From (1.3) and elementary inequality, we get

$$E \left( |X_t^{i,N}|^q \right) \leq 3^{q-1} E \left( |X_i^0|^q \right) + 3^{q-1} E \left( \left| \int_0^t b(X_s^{i,N}, \mu_s^{X,N}) \, ds \right|^q \right) + 3^{q-1} E \left( \left| \int_0^t \sigma(\mu_s^{X,N}) \, dB_s^{H,i} \right|^q \right). \quad (3.1)$$

For the second part at the right side of (3.1), using the Hölder inequality, we obtain

$$E \left( \left| \int_0^t b(X_s^{i,N}, \mu_s^{X,N}) \, ds \right|^q \right) \leq T^{q-1} E \left( \int_0^t |b(X_s^{i,N}, \mu_s^{X,N})|^q \, ds \right).$$

Applying (2.4), we see that

$$E \left( \left| \int_0^t b(X_s^{i,N}, \mu_s^{X,N}) \, ds \right|^q \right) \leq T^{q-1} E \left( \int_0^t L^q \left( 1 + |X_s^{i,N}| + W_0(\mu_s^{X,N}, \delta_0) \right)^q \, ds \right) \quad (3.2)$$

$$\leq C_{T,q} + C_{T,q,L} \int_0^t E \left( |X_s^{i,N}|^q \right) \, ds + C_{T,q} \int_0^t E \left( W_0(\mu_s^{X,N}, \delta_0) \right) \, ds.$$

For the last part of the right end of (3.1), apply Theorem 1.1 in [22], we get

$$E \left( \left| \int_0^t \sigma(\mu_s^{X,N}) \, dB_s^{H,i} \right|^q \right) \leq C_{q,H} \left( \int_0^t |\sigma(\mu_s^{X,N})|^{1/H} \, ds \right)^{qH}. \quad (3.3)$$

Using the Hölder inequality,

$$E \left( \left| \int_0^t \sigma(\mu_s^{X,N}) \, dB_s^{H,i} \right|^q \right) \leq C_{q,H,T} \int_0^t |\sigma(\mu_s^{X,N})|^q \, ds.$$

By (2.5), we have

$$E \left( \left| \int_0^t \sigma(\mu_s^{X,N}) \, dB_s^{H,i} \right|^q \right) \leq C_{q,H,T} \int_0^t L^q E \left( 1 + W_0(\mu_s^{X,N}, \delta_0) \right)^q \, ds \quad (3.4)$$

$$\leq C_{q,H,T,L} + C_{q,H,T,L} \int_0^t E \left( W_0(\mu_s^{X,N}, \delta_0) \right) \, ds.$$

Through a similar proof in [10, Lemma 2.3], for $\theta \geq 2$ one can observe that

$$W_0^\theta(\mu_s^{X,N}, \delta_0) = \frac{1}{N} \sum_{j=1}^{N} |X_j^{i,N}|. \quad (3.5)$$
Combine (3.2) and (3.4) into (3.1) and (3.5), we get

\[
E \left( |X_t^{i,N}|^q \right) \leq C_q E \left( |X_0^i|^q \right) + C_{T,q,L} + C_{T,q,L} \int_0^t E \left( |X_s^{i,N}|^q \right) \, ds \\
+ C_{T,q,L} \int_0^t E \left( \mathcal{W}_t^q (\mu_s^{X,N}, \delta_0) \right) \, ds + C_{q,H,T,L} + C_{q,H,T,L} \int_0^t E \left( \mathcal{W}_t^q (\mu_s^{X,N}, \delta_0) \right) \, ds \\
\leq C_{q,H,T,L} \left( 1 + E |X_0^i|^q \right) + C_{T,q,L} \int_0^t E \left( |X_s^{i,N}|^q \right) \, ds + C_{T,q,L} \left( \frac{1}{N} \sum_{j=1}^N |X_s^{j,N}|^q \right)^{q/\theta}.
\]

For the last term, we apply the Minkowski inequality and since all \( j \) are identically distributed, we have

\[
E \left( \left( \frac{1}{N} \sum_{j=1}^N |X_s^{j,N}|^q \right)^{q/\theta} \right) \leq \left( \frac{1}{N} \sum_{j=1}^N \left( \|X_s^{j,N}\|_{L^q} \right)^q \right)^{q/\theta} = \left( \frac{1}{N} \sum_{j=1}^N (E |X_s^{j,N}|^q)^{\theta/q} \right)^{q/\theta} = E \left( |X_s^{j,N}|^q \right).
\]

Thus, we get

\[
E \left( |X_t^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E |X_0^i|^q \right) + C_{q,H,T,L} \int_0^t E \left( |X_s^{i,N}|^q \right) \, ds.
\]

Then applying the Grönwall inequality, we obtain

\[
E \left( |X_t^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E |X_0^i|^q \right).
\]

Similarly, we can show

\[
E \left( |X_t^i|^q \right) \leq C_{q,H,T,L} \left( 1 + E |X_0^i|^q \right).
\]

The proof is complete. \( \blacksquare \)

**Theorem 3.4 (Propagation of Chaos)** Let Assumption 2.1 be satisfied. If for some \( p \in [\theta, q) \), then it holds that

\[
\sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0,T]} E \left| X_t^i - X_t^{i,N} \right|^p \leq C_{p,T,H,L,\theta} \\
\leq \begin{cases} 
N^{-1/2} + N^{-(q-p)/q}, & \text{if } p > d/2 \text{ and } q \neq 2p, \\
N^{-1/2} \log(1 + N) + N^{-(q-p)/q}, & \text{if } p = d/2 \text{ and } q \neq 2p, \\
N^{-p/d} + N^{-(q-p)/q}, & \text{if } p \in [2, d/2) \text{ and } q \neq d/(d-p),
\end{cases}
\]

where the constant \( C_{p,T,H,L,\theta} > 0 \) depends on \( p, T, H, L \) and \( \theta \) but does not depend on \( N \).

**Proof.** It follows from (1.2) and (1.3) that

\[
X_t^i - X_t^{i,N} = \int_0^t \left( b(X_s^i, \mathcal{L}_X^i) - b(X_s^{i,N}, \mu_s^{X,N}) \right) \, ds + \int_0^t \left( \sigma (\mathcal{L}_X^i) - \sigma (\mu_s^{X,N}) \right) \, dB_s^{H,i}.
\]

Using the elementary inequality, we can show that

\[
E \left( \left| X_t^i - X_t^{i,N} \right|^p \right) \leq 2^{p-1} E \left( \left| \int_0^t \left( b(X_s^i, \mathcal{L}_X^i) - b(X_s^{i,N}, \mu_s^{X,N}) \right) \, ds \right|^p \right) \\
+ 2^{p-1} E \left( \left| \int_0^t \left( \sigma (\mathcal{L}_X^i) - \sigma (\mu_s^{X,N}) \right) \, dB_s^{H,i} \right|^p \right).
\]
For the first part on the right of (3.6), use the Hölder inequality and Assumption 2.1, we get
\[
E \left( \left| \int_0^t b \left( X_s^i, \mathcal{L}_{X_s^i} \right) - b \left( X_s^{i,N}, \mu_s^{X,N} \right) \right|^p \right) \\
\leq t^{p-1} E \left( \left| \int_0^t b \left( X_s^i, \mathcal{L}_{X_s^i} \right) - b \left( X_s^{i,N}, \mu_s^{X,N} \right) \right|^p \right) ds \\
\leq t^{p-1} E \left( \left| \int_0^t \left| X_s^i - X_s^{i,N} \right| + \mathcal{W}_\theta \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right|^p \right) ds \\
\leq C_{p,T,L} E \left( \int_0^t \left| X_s^i - X_s^{i,N} \right|^p ds \right) + C_{p,T,L} E \left( \int_0^t \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) ds \right). \tag{3.7}
\]

For the second part on the right of (3.6), through the same technique as (3.3), we get
\[
E \left( \left| \int_0^t \sigma \left( \mathcal{L}_{X_s^i} \right) - \sigma \left( \mu_s^{X,N} \right) dB_s^{H,\xi} \right|^p \right) \leq C_{p,T,H} E \left( \left( \int_0^t |\sigma \left( \mathcal{L}_{X_s^i} \right) - \sigma \left( \mu_s^{X,N} \right)|^{1/H} ds \right)^p \right). \tag{3.8}
\]

Apply Assumption 2.1, we obtain
\[
E \left( \left| \int_0^t \sigma \left( \mathcal{L}_{X_s^i} \right) - \sigma \left( \mu_s^{X,N} \right) dB_s^{H,\xi} \right|^p \right) \leq C_{p,T,H,L} \int_0^t E \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right) ds.
\]

Combining this with (3.7), then
\[
E \left( \left| X_t^i - X_t^{i,N} \right|^p \right) \leq C_{p,T,L} \int_0^t \left| X_s^i - X_s^{i,N} \right|^p ds + C_{p,T,L} \int_0^t E \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right) ds \\
+ C_{p,T,H,L} \int_0^t \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right) ds \\
\leq C_{p,T,L} E \left( \int_0^t \left| X_s^i - X_s^{i,N} \right|^p ds \right) + C_{p,T,H,L} \int_0^t \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right) ds.
\]

For the part of Wasserstein distance, we note \( \mu_s^X := \frac{1}{N} \sum_{j=1}^N \delta_{x_s^j} \) and we obtain
\[
\mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) = \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right)^{p/\theta} \\
\leq \left( 2^{\frac{p}{\theta}} \mathcal{W}_\theta^p \left( \mu_s^X, \mu_s^{X,N} \right) \right)^{p/\theta} + \left( 2^{\frac{p}{\theta}} \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^X \right) \right)^{p/\theta} \\
\leq C_{p,\theta} \mathcal{W}_\theta^p \left( \mu_s^X, \mu_s^{X,N} \right) + C_{p,\theta} \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^X \right).
\]

By Lemma 3.2, we see
\[
E \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right) \right) \leq C_{p,\theta} E \left( \left( \frac{1}{N} \sum_{j=1}^N \left| X_s^j - X_s^{1,j,N} \right| \right)^{p/\theta} \right) + C_{p,\theta} \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^{X,N} \right).
\]

Thought the fact that \( \mathcal{W}_\theta \left( \mu, \nu \right) \leq \mathcal{W}_p \left( \mu, \nu \right) \) for \( \theta \leq p \), we have
\[
E \left( \left| X_t^i - X_t^{i,N} \right|^p \right) \\
\leq C_{p,T,L} \int_0^t \left| X_s^i - X_s^{i,N} \right|^p ds + C_{p,T,H,L,\theta} \int_0^t \left( \left( \frac{1}{N} \sum_{j=1}^N \left| X_s^j - X_s^{1,j,N} \right| \right)^{p/\theta} \right) ds \\
+ C_{p,T,H,L,\theta} \int_0^t \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^X \right) \right) ds \\
\leq C_{p,T,L} \int_0^t \left| X_s^i - X_s^{i,N} \right|^p ds + C_{p,T,H,L,\theta} \int_0^t \left( \left( \frac{1}{N} \sum_{j=1}^N \left| X_s^j - X_s^{1,j,N} \right| \right)^{p/\theta} \right) ds \\
+ C_{p,T,H,L,\theta} \int_0^t \left( \mathcal{W}_\theta^p \left( \mathcal{L}_{X_s^i}, \mu_s^X \right) \right) ds,
\]
where we use the Minkowski inequality in the last inequality. Then through simple sorting, we have

\[ E \left( \left| X^i_{t_k} - X^{i,N}_{t_k} \right|^p \right) \leq C_{p,T,L} E \left( \int_0^t \left| X^i_s - X^{i,N}_s \right|^p \ ds \right) + C_{p,T,H,L,\theta} \int_0^t E \left( \left| X^i_s - X^{i,N}_s \right|^p \right) \ ds \]

\[ + C_{p,T,H,L,\theta} \int_0^t E \left( \mathcal{W}_p^p(\mathcal{L}_{X^i_s}, \mu^X_s) \right) \ ds \]

\[ \leq C_{p,T,H,L,\theta} \int_0^t E \left( \left| X^i_s - X^{i,N}_s \right|^p \right) \ ds + C_{p,T,H,L,\theta} \int_0^t E \left( \mathcal{W}_p^p(\mathcal{L}_{X^i_s}, \mu^X_s) \right) \ ds. \]

What’s particularly interesting is that \( \mathcal{W}_p^p(\mathcal{L}_{X^i_s}, \mu^X_s) \) is controlled by the Wasserstein distance estimate in [13, Theorem 1]. Therefore,

\[ E \left( \mathcal{W}_p^p(\mathcal{L}_{X^i_s}, \mu^X_s) \right) \leq CM_{q}^{p/q}(\mathcal{L}_{X^i_s}) \]

\[ \times \begin{cases} 
N^{-1/2} + N^{-(q-p)/q}, & \text{if } p > d/2 \text{ and } q \neq 2p, \\
N^{-1/2} \log(1 + N) + N^{-(q-p)/q}, & \text{if } p = d/2 \text{ and } q \neq 2p, \\
N^{-p/d} + N^{-(q-p)/q}, & \text{if } p \in [2, d/2) \text{ and } q \neq d/(d - p),
\end{cases} \]

where

\[ M_q(\mathcal{L}_{X^i_s}) = \int_{\mathbb{R}^d} \left| X^i_s \right|^q \mathcal{L}_{X^i_s}(\text{d}X^i_s). \]

By Theorem 3.3, we note that \( M_q(\mathcal{L}_{X^i_s}) \leq C \). Thus,

\[ E \left( \mathcal{W}_p^p(\mathcal{L}_{X^i_s}, \mu^X_s) \right) \leq C \begin{cases} 
N^{-1/2} + N^{-(q-p)/q}, & \text{if } p > d/2 \text{ and } q \neq 2p, \\
N^{-1/2} \log(1 + N) + N^{-(q-p)/q}, & \text{if } p = d/2 \text{ and } q \neq 2p, \\
N^{-p/d} + N^{-(q-p)/q}, & \text{if } p \in [2, d/2) \text{ and } q \neq d/(d - p).
\end{cases} \]

Then, applying the Grönwall inequality completes the proof.

### 3.1.1 EM Method for Interacting Particle System

Now, we define a uniform mesh \( T_N : 0 = t_0 < t_1 < \cdots < t_N = T \) with \( t_k = k\Delta \), where \( \Delta = T/N \) for \( N \in \mathbb{N} \). The numerical solutions are then generated by the EM method

\[ Y^{i,N}_{t_{k+1}} = Y^{i,N}_{t_k} + b \left( Y^{i,N}_{t_k}, \mu^{Y,N}_{t_k} \right) \Delta + \sigma \left( \mu^{Y,N}_{t_k} \right) \Delta B^{H,i}_{t_k}, \tag{3.9} \]

where the empirical measures \( \mu^{Y,N}_{t_k}(\cdot) := \frac{1}{N} \sum_{j=1}^{N} \delta_{Y^{j,N}_{t_k}}(\cdot) \) and \( \Delta B^{H,i}_{t_k} = B^{H,i}_{t_{k+1}} - B^{H,i}_{t_k} \). We show two versions of extension of the numerical solution at the discrete time points to \( t \geq 0 \). The first is the piecewise constant extension given by

\[ Y^{i,N}_{t} = Y^{i,N}_{t_k}, \quad t_k \leq t < t_{k+1}, \tag{3.10} \]

and the second is the continuous extension of the EM method defined by

\[ Y^{i,N}_{t} = Y^{i,N}_{t_k} + \int_{t_k}^{t} b \left( Y^{i,N}_{s}, \mu^{Y,N}_{s} \right) \ ds + \int_{t_k}^{t} \sigma \left( \mu^{Y,N}_{s} \right) \ dB^{H,i}_{s}. \tag{3.11} \]

Here, \( \mu^{Y,N}_{t}(\cdot) := \frac{1}{N} \sum_{j=1}^{N} \delta_{Y^{j,N}_{t}}(\cdot) \). From (3.11), for all \( t \in [0, T] \), we have

\[ Y^{i,N}_{t} = X_0^{i} + \int_{0}^{t} b \left( Y^{i,N}_{s}, \mu^{Y,N}_{s} \right) \ ds + \int_{0}^{t} \sigma \left( \mu^{Y,N}_{s} \right) \ dB^{H,i}_{s}. \tag{3.12} \]
**Theorem 3.5** Let Assumption 2.1 holds. For some $q > p$, then

$$\sup_{t \in [0,T]} E \left( |Y_t^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E|X_0^i|^q \right),$$

where $C_{q,H,T,L}$ is a positive constant dependent on $q, H, T, L$ but independent of $\Delta$.

**Proof.** Similar to the proof of Theorem 3.3, we can show

$$E \left( |Y_t^{i,N}|^q \right) \leq 3^{q-1} E \left( |X_0^i|^q \right) + 3^{q-1} E \left( \left| \int_0^t b \left( \bar{Y}_s^{i,N}, \bar{\mu}_s^{Y,N} \right) \, ds \right|^q \right) + 3^{q-1} E \left( \left| \int_0^t \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_s^{H,\bar{i}} \right|^q \right)$$

$$\leq C_{q,H,T,L} \left( 1 + E|X_0^i|^q \right) + C_{q,H,T,L} \int_0^t E \left( |Y_s^{i,N}|^q \right) \, ds$$

$$\leq C_{q,H,T,L} \left( 1 + E|X_0^i|^q \right) + C_{q,H,T,L} \int_0^t \sup_{0 \leq s \leq \tau} E \left( |Y_s^{i,N}|^q \right) \, ds.$$

Therefore, for $0 \leq t \leq T$, we have

$$\sup_{0 \leq \tau \leq t} E \left( |Y_{\tau}^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E|X_0^i|^q \right) + C_{q,H,T,L} \int_0^t \sup_{0 \leq s \leq \tau} E \left( |Y_s^{i,N}|^q \right) \, ds.$$

By the Grönwall inequality, we see

$$\sup_{0 \leq \tau \leq t} E \left( |Y_{\tau}^{i,N}|^q \right) \leq C_{q,H,T,L} \left( 1 + E|X_0^i|^q \right), \quad \forall t \in [0,T].$$

Therefore, the assertion holds. 

**Lemma 3.6** Assume (2.4) and (2.5) hold, for a constant $\kappa \in (1 - H, 1 - 1/p)$, then

$$E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^p \right) \leq C_{p,H,p,L} \Delta^p \Delta^H,$$

where $C_{p,H,p,L}$ is a positive constant dependent on $\kappa, H, p, L$ but independent of $\Delta$.

**Proof.** From (3.11), we separate the left hand side of (3.13) into two parts

$$E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^p \right) = E \left( \left| \int_{t_k}^t b \left( \tilde{Y}_s^{i,N}, \bar{\mu}_s^{Y,N} \right) \, ds + \int_{t_k}^t \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_s^{H,\bar{i}} \right|^p \right)$$

$$\leq 2^{p-1} E \left( \left| \int_{t_k}^t b \left( \tilde{Y}_s^{i,N}, \bar{\mu}_s^{Y,N} \right) \, ds \right|^p \right) + 2^{p-1} E \left( \left| \int_{t_k}^t \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_s^{H,\bar{i}} \right|^p \right).$$

Let us first consider the first part on the right of (3.14), by using the Hölder inequality and (2.4), we obtain

$$E \left( \left| \int_{t_k}^t b \left( \tilde{Y}_s^{i,N}, \bar{\mu}_s^{Y,N} \right) \, ds \right|^p \right)$$

$$\leq \Delta^{p-1} E \left( \int_{t_k}^t |b \left( \tilde{Y}_s^{i,N}, \bar{\mu}_s^{Y,N} \right)|^p \, ds \right)$$

$$\leq \Delta^{p-1} E \left( \int_{t_k}^t L^p \left( 1 + |\tilde{Y}_s^{i,N}| + \sup_{0 \leq \theta \leq s} \left| \bar{\mu}_s^{Y,N} \right| \right)^p \, ds \right)$$

$$\leq L^p \Delta^p \Delta^{p-1} \int_{t_k}^t E \left( |\tilde{Y}_s^{i,N}|^p \right) \, ds + C_{p,L} \Delta^{p-1} \int_{t_k}^t \left( \sup_{0 \leq \theta \leq s} \left| \bar{\mu}_s^{Y,N} \right| \right) \, ds.$$
Thanks to Stochastic Fubini Theorem for the Wiener Integrals with regard to fBm [23, Theorem 1.13.1], we obtain
\[
\int_{t_k}^{t} \left( \int_{s}^{t} (t-r)^{-\kappa} (r-s)^{\kappa-1} \, dr \right) \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} = \int_{t_k}^{t} (t-r)^{-\kappa} \left( \int_{0}^{r} (r-s)^{\kappa-1} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right) \, dr.
\]
Therefore, by the Hölder inequality, we get
\[
E \left( \left| \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \right) \leq C_{\kappa}^{-p} E \left( \left( \int_{t_k}^{t} (t-r)^{-p\kappa/(p-1)} \, dr \right)^{p-1} \int_{t_k}^{t} \left| \int_{0}^{r} (r-s)^{\kappa-1} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \, dr \right) \leq \frac{C_{\kappa}^{-p}(p-1)}{(p-1-p\kappa)^{p-1}} \Delta^{p-1-p\kappa} E \left( \int_{t_k}^{t} \left| \int_{0}^{r} (r-s)^{\kappa-1} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \, dr \right).
\]
Applying the Theorem 1.1 in [22], we see
\[
E \left( \left| \int_{0}^{r} (r-s)^{\kappa-1} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \right) \leq C_{H,p} \left( \int_{0}^{r} (r-s)^{(\kappa-1)/H} \left| \sigma \left( \bar{\mu}_s^{Y,N} \right) \right|^{1/H} \, ds \right)^{pH}.
\]
Thus, we obtain
\[
E \left( \left| \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \right) \leq C_{H,p} \Delta^{p-1-p\kappa} \int_{t_k}^{t} \left( \int_{0}^{r} (r-s)^{(\kappa-1)/H} \left| \sigma \left( \bar{\mu}_s^{Y,N} \right) \right|^{1/H} \, ds \right)^{pH} \, dr.
\]
By [12, Lemma 3.2] with \( \bar{\eta} = pH \), \( \alpha = \frac{H-1-\kappa}{H} \) and \( \bar{\nu} = \frac{pH}{p(\kappa+H-1)+1} \), we have
\[
E \left( \left| \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \right) \leq C_{H,p} \Delta^{p-1-p\kappa} \int_{t_k}^{t} \left( \int_{0}^{r} \left| \sigma \left( \bar{\mu}_s^{Y,N} \right) \right|^{p/[p(\kappa+H-1)+1]} \, ds \right)^{p(\kappa+H-1)+1} \leq C_{H,p} \Delta^{p-1-p\kappa} \Delta^{p(\kappa+H-1)} \int_{t_k}^{t} \left( \sigma \left( \bar{\mu}_s^{Y,N} \right) \right)^{p} \, ds,
\]
where we use the Hölder inequality in the last inequality. Then by (2.5), we know that
\[
E \left( \left| \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, dB_{s}^{H,i} \right|^p \right) \leq C_{H,p} \Delta^{pH-1} \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, ds \leq C_{H,p} \Delta^{pH-1} \int_{t_k}^{t} L^p \sigma \left( \bar{\mu}_s^{Y,N} \right) \left( 1 + \mathcal{W}_0 \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right)^p \, ds \leq C_{H,p} \Delta^{pH} + C_{H,p,L} \Delta^{pH-1} \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, ds (3.16).
\]
Combining (3.15) and (3.16) into (3.14) yields
\[
E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^p \right) \leq L^p \Delta^p + C_{p,L} \Delta^{p-1} \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, ds + C_{p,L} \Delta^{p-1} \int_{t_k}^{t} \sigma \left( \bar{\mu}_s^{Y,N} \right) \, ds \leq C_{H,p,L} \Delta^{pH} + C_{H,p,L} \Delta^{pH-1} \int_{t_k}^{t} E \left( \mathcal{W}_0^p \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right) \, ds.
\]
By Lemma 3.2 and the Minkowski inequality, we note that
\[
E \left( \mathcal{W}_0^p \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right) = E \left( \mathcal{W}_0^p \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right)^{p/\theta} \leq E \left( \left( \frac{1}{N} \sum_{j=1}^{N} \left| \bar{Y}_s^{j,N} \right|^{\theta} \right)^{p/\theta} \right) = E \left( \left| \bar{Y}_s^{j,N} \right|^{p} \right) (3.18).
\]
Substituting (3.18) into (3.17), we get

\[
E \left( \left| Y_{t}^{i,N} - \tilde{Y}_{t}^{i,N} \right|^{p} \right) \leq L^{p} \Delta^{p} + C_{p,L} \Delta^{p-1} \int_{t_{k}}^{t} E \left( \left| \tilde{Y}_{s}^{i,N} \right|^{p} \right) \, ds + C_{p,L} \Delta^{p-1} \int_{t_{k}}^{t} E \left( \left| \tilde{Y}_{s}^{i,N} \right|^{p} \right) \, ds
\]

\[
+ C_{\kappa,H,p,L} \Delta^{pH} + C_{\kappa,H,p,L} \Delta^{pH-1} \int_{t_{k}}^{t} E \left( \left| \tilde{Y}_{s}^{i,N} \right|^{p} \right) \, ds
\]

\[
\leq C_{\kappa,H,p,L} \Delta^{pH} + C_{\kappa,H,p,L} \Delta^{pH-1} \int_{t_{k}}^{t} E \left( \left| \tilde{Y}_{s}^{i,N} \right|^{p} \right) \, ds.
\]

Applying the Theorem 3.5 to above inequality gives that

\[
E \left( \left| Y_{t}^{i,N} - \tilde{Y}_{t}^{i,N} \right|^{p} \right) \leq C_{\kappa,H,p,L} \Delta^{pH}.
\]

Thus, we obtain the desired result. □

**Theorem 3.7** Let Assumption 2.1 holds, for \( p \geq \theta \), then

\[
E \left( \left| X_{t}^{i,N} - Y_{t}^{i,N} \right|^{p} \right) \leq C_{p,T,H,L,\kappa} \Delta^{pH},
\]

where \( C_{p,T,H,L,\kappa} \) is a positive constant independent of \( \Delta \).

**Proof.** It follows from (1.3) and (3.12) that

\[
E \left( \left| X_{t}^{i,N} - Y_{t}^{i,N} \right|^{p} \right)
= E \left( \left| \int_{0}^{t} \left( b \left( X_{s}^{i,N}, \mu_{s}^{X,N} \right) - b \left( \tilde{Y}_{s}^{i,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds + \int_{0}^{t} \left( \sigma \left( \mu_{s}^{X,N} \right) - \sigma \left( \bar{\mu}_{s}^{Y,N} \right) \right) \, dB_{s}^{H,i} \right|^{p} \right)
\]

\[
\leq 2^{p-1} E \left( \left| \int_{0}^{t} \left( b \left( X_{s}^{i,N}, \mu_{s}^{X,N} \right) - b \left( \tilde{Y}_{s}^{i,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds \right|^{p} \right)
\]

\[
+ 2^{p-1} E \left( \left| \int_{0}^{t} \left( \sigma \left( \mu_{s}^{X,N} \right) - \sigma \left( \bar{\mu}_{s}^{Y,N} \right) \right) \, dB_{s}^{H,i} \right|^{p} \right).
\]

Thanks to the Hölder inequality and Assumption 2.1 yield

\[
E \left( \left| \int_{0}^{t} \left( b \left( X_{s}^{i,N}, \mu_{s}^{X,N} \right) - b \left( \tilde{Y}_{s}^{i,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds \right|^{p} \right)
\]

\[
\leq t^{p-1} E \left( \left| \int_{0}^{t} \left| b \left( X_{s}^{i,N}, \mu_{s}^{X,N} \right) - b \left( \tilde{Y}_{s}^{i,N}, \bar{\mu}_{s}^{Y,N} \right) \right|^{p} \, ds \right) \right)
\]

\[
\leq T^{p-1} E \left( \int_{0}^{t} L^{p} \left( \left| X_{s}^{i,N} - \tilde{Y}_{s}^{i,N} \right| \right) + \mathcal{W}_{\theta} \left( \mu_{s}^{X,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds \right)
\]

\[
\leq C_{p,T,L} E \left( \int_{0}^{t} \left| X_{s}^{i,N} - \tilde{Y}_{s}^{i,N} \right| \, ds \right) + C_{p,T,L} E \left( \int_{0}^{t} \left( \mathcal{W}_{\theta}^{p} \left( \mu_{s}^{X,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds \right).
\]

Following a very similar approach used for (3.8), we can show

\[
E \left( \left| \int_{0}^{t} \left( \sigma \left( \mu_{s}^{X,N} \right) - \sigma \left( \bar{\mu}_{s}^{Y,N} \right) \right) \, dB_{s}^{H,i} \right|^{p} \right) \leq C_{p,T,H,L} \int_{0}^{t} E \left( \mathcal{W}_{\theta}^{p} \left( \mu_{s}^{X,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds.
\]

Thus,

\[
E \left( \left| X_{t}^{i,N} - Y_{t}^{i,N} \right|^{p} \right) \leq C_{p,T,L} E \left( \int_{0}^{t} \left| X_{s}^{i,N} - \tilde{Y}_{s}^{i,N} \right| \, ds \right) + C_{p,T,H,L} \int_{0}^{t} E \left( \mathcal{W}_{\theta}^{p} \left( \mu_{s}^{X,N}, \bar{\mu}_{s}^{Y,N} \right) \right) \, ds.
\]
Due to Lemma 3.2 and the Minkowski inequality, we observe

\[ E \left( W_\theta^p(\mu_s^{X,N}, \bar{\mu}_s^{Y,N}) \right) = E \left( (W_\theta^p(\mu_s^{X,N}, \bar{\mu}_s^{Y,N}))^{p/q} \right) \]

\[ \leq E \left( \frac{1}{N} \sum_{j=1}^N |X_s^{j,N} - \bar{Y}_s^{j,N}|^\theta \right)^{p/q} \]

\[ \leq \left( \frac{1}{N} \sum_{j=1}^N \left\| X_s^{j,N} - \bar{Y}_s^{j,N} \right\|_{L^{p/q}}^q \right)^{p/q} \]

\[ = E \left( |X_s^{i,N} - \bar{Y}_s^{i,N}|^p \right). \tag{3.21} \]

Substituting (3.21) into (3.20), we get

\[ E \left( |X_t^{i,N} - Y_t^{i,N}|^p \right) \leq C_{p,T,L} \int_0^t E \left( |X_s^{i,N} - \bar{Y}_s^{i,N}|^p \right) ds + C_{p,T,H,L} \int_0^t E \left( |X_s^{i,N} - \bar{Y}_s^{i,N}|^p \right) ds \]

\[ \leq C_{p,T,H,L} \int_0^t E \left( |Y_s^{i,N} - \bar{Y}_s^{i,N}|^p \right) + C_{p,T,H,L} \int_0^t E \left( |Y_s^{i,N} - \bar{Y}_s^{i,N}|^p \right) ds. \]

We derive from Lemma 3.6 that

\[ E \left( |X_t^{i,N} - Y_t^{i,N}|^p \right) \leq C_{p,T,H,L} \int_0^t E \left( |X_t^{i,N} - Y_t^{i,N}|^p \right) ds + C_{p,T,H,L,\kappa} \Delta^{pH}. \]

By the Grönwall inequality, we derive that

\[ E \left( |X_t^{i,N} - Y_t^{i,N}|^p \right) \leq C_{p,T,H,L,\kappa} \Delta^{pH}. \]

Therefore, the proof is complete. \qed

**Theorem 3.8** Let Assumption 2.1 be satisfied. If \( \theta \leq p < q \), then it holds that

\[
\sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} E \left( |X_t^i - Y_t^i|^p \right) \leq C_{p,T,H,L,\theta,\kappa} \times \begin{cases} 
N^{-1/2} + N^{-(q-p)/q} + \Delta^{pH}, & \text{if } p > d/2 \text{ and } q \neq 2p, \\
N^{-1/2} \log(1 + N) + N^{-(q-p)/q} + \Delta^{pH}, & \text{if } p = d/2 \text{ and } q \neq 2p, \\
N^{-p/d} + N^{-(q-p)/q} + \Delta^{pH}, & \text{if } p \in [2, d/2) \text{ and } q \neq d/(d - p),
\end{cases}
\]

where the constant \( C_{p,T,H,L,\theta,\kappa} > 0 \) does not depend on \( N \) and \( \Delta \).

We can easily get this theorem through trigonometric inequality, Theorem 3.4 and Theorem 3.7.

### 3.2 Case \( H < 1/2 \)

For the case of \( H \in (0, 1/2) \), due to the Theorem 3.1(II) in [12], \( \sigma(\mu) \) is independent of distribution and then the solution of \( X_t \) in (1.1) exists and is unique. In other words, the coefficient of diffusion \( \sigma(\mu_s^{X,N}) = \xi \), where \( \xi \) is a constant. Here, we only consider the case of \( p = 2 \) (imply \( \theta = 2 \)).

**Theorem 3.9** Let Assumption 2.1 holds, for \( q > 2 \) then

\[
\sup_{t \in [0, T]} E \left( |X_t^i|^q \right) + \sup_{t \in [0, T]} E \left( |X_t^{i,N}|^q \right) \leq C_{q,T,L,H,\xi} \left( 1 + E |X_0^{i}|^\theta \right),
\]

where \( C_{q,T,L,H,\xi} \) is a positive constant dependent on \( q, T, L, H, \xi \) but does not depend on \( N \).
Proof. From (1.3), (3.2) and Theorem 2.1 in [29], we get
\[ E \left( |X_t^{i,N}|^q \right) \leq 3^{q-1} E \left( |X_0^{i}|^q \right) + 3^{q-1} E \left( \left| \int_0^t b \left( X_s^{i,N}, \mu_s^{X,N} \right) ds \right|^2 \right) + 3^{q-1} E \left( \left| \int_0^t \xi dB_s^{H,i} \right|^q \right) \]
\[ \leq C_q E \left( |X_0^{i}|^q \right) + C_{q,T,L} \int_0^t E \left( |X_s^{i,N}|^2 \right) ds + C_{q,T,L} \int_0^t E \left( \mathcal{W}_s^2(\mu_s^{X,N}, \delta_0) \right) ds + C_q \xi^q T^q H \]
\[ \leq C_{q,T,L,H,\xi} \left( 1 + E |X_0^{i}|^q \right) + C_{T,L} \int_0^t E \left( |X_s^{i,N}|^q \right) ds. \]

Then applying the Grönwall inequality, we obtain
\[ E \left( |X_t^{i,N}|^q \right) \leq C_{q,T,L,H,\xi} \left( 1 + E |X_0^{i}|^q \right). \]

Similarly, we can show
\[ E \left( |X_t^{i}|^q \right) \leq C_{q,T,L,H,\xi} \left( 1 + E |X_0^{i}|^q \right). \]

The proof is complete. □

**Theorem 3.10 (Propagation of Chaos)** Let Assumption 2.1 be satisfied. If for some \( q > 2 \), then it holds that

\[ \sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0,T]} E \left| X_t^{i} - X_t^{i,N} \right|^2 \leq C_{T,H,L} \]
\[ \times \begin{cases} N^{-1/2} + N^{-(q-2)/q}, & \text{if } d < 4, \\ N^{-1/2} \log(1 + N) + N^{-(q-2)/q}, & \text{if } d = 4, \\ N^{-2/d} + N^{-(q-2)/q}, & \text{if } d > 4, \end{cases} \]

where the constant \( C_{T,H,L} > 0 \) depends on \( T \), \( H \) and \( L \) but does not depend on \( N \).

The proof of this lemma is similar to that of Theorem 3.4, we put it in the Appendix A.

### 3.2.1 EM Method for Interacting Particle System

The numerical solutions are generated by the EM method
\[ Y_{t_{k+1}}^{i,N} = Y_{t_k}^{i,N} + b \left( Y_{t_k}^{i,N}, \bar{\mu}_{t_k}^{Y,N} \right) \Delta \xi B_{t_k}^{H,i}, \tag{3.22} \]
and the continuous extension of the EM method defined by
\[ Y_t^{i,N} = Y_{t_k}^{i,N} + \int_{t_k}^t b \left( Y_s^{i,N}, \bar{\mu}_s^{Y,N} \right) ds + \int_{t_k}^t \xi dB_s^{H,i}. \tag{3.23} \]

From (3.23), for all \( t \in [0,T] \), we have
\[ Y_t^{i,N} = X_t^{i} + \int_0^t b \left( Y_s^{i,N}, \bar{\mu}_s^{Y,N} \right) ds + \int_0^t \xi dB_s^{H,i}. \tag{3.24} \]

**Theorem 3.11** Let Assumption 2.1 holds, then

\[ \sup_{t \in [0,T]} E \left( Y_t^{i,N} \right)^2 \leq C_{T,L,H,\xi} \left( 1 + E |X_0^{i}|^2 \right), \]

where \( C_{T,L,H,\xi} \) is a positive constant dependent on \( T, L, H, \xi \) but independent of \( \Delta \).
Proof. By (3.2) and (3.24), we can show
\[ E \left( \left| Y_t^{i,N} \right|^2 \right) \leq 3E \left( \left| X_0^{i} \right|^2 \right) + 3E \left( \left| \int_0^t b \left( \tilde{Y}_s^{i,N}, \tilde{\mu}_s^{Y,N} \right) ds \right|^2 \right) \]
\[ \leq 3E \left( \left| X_0^{i} \right|^2 \right) + C_{T,L} \int_0^t E \left( \left| \tilde{Y}_s^{i,N} \right|^2 \right) ds + C_{T,L} \int_0^t E \left( W_2^2 \left( \tilde{\mu}_s^{Y,N}, \delta_0 \right) \right) ds + 3\xi^2 t^{2H} \]
\[ \leq C_{T,L,H,\xi} \left( 1 + E \left| X_0^{i} \right|^2 \right) + C_{T,L} \int_0^t E \left( \left| Y_s^{i,N} \right|^2 \right) ds \]
\[ \leq C_{T,L,H,\xi} \left( 1 + E \left| X_0^{i} \right|^2 \right) + C_{T,L} \int_0^t \sup_{0 \leq \tau \leq s} E \left( \left| Y_s^{i,N} \right|^2 \right) ds. \]
Therefore, for \( 0 \leq t \leq T \), we have
\[ \sup_{0 \leq \tau \leq t} E \left( \left| Y_{\tau}^{i,N} \right|^2 \right) \leq C_{T,L,H,\xi} \left( 1 + E \left| X_0^{i} \right|^2 \right) + C_{T,L} \int_0^t \sup_{0 \leq \tau \leq s} E \left( \left| Y_s^{i,N} \right|^2 \right) ds. \]
By the Grönwall inequality, the assertion holds. \( \square \)

Lemma 3.12 Assume (2.4) and (2.5) hold, then
\[ E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2 \right) \leq C_L \Delta^{2H}, \quad (3.25) \]
where \( C_L \) is a positive constant dependent on \( L \) but independent of \( \Delta \).

Proof. From (3.23), we get
\[ E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2 \right) = E \left( \left| \int_0^t b \left( \bar{Y}_s^{i,N}, \tilde{\mu}_s^{Y,N} \right) ds + \int_{t_k}^t \xi \ dB^{H,i}_s \right|^2 \right) \]
\[ \leq 2E \left( \left| \int_{t_k}^t b \left( \bar{Y}_s^{i,N}, \tilde{\mu}_s^{Y,N} \right) ds \right|^2 \right) + 2E \left( \left| \int_{t_k}^t \xi \ dB^{H,i}_s \right|^2 \right). \quad (3.26) \]
By (3.15) and the fact that \( E \left( \left| B_t^{H} - B_s^{H} \right|^2 \right) = \left| t - s \right|^{2H} \), we obtain
\[ E \left( \left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2 \right) \leq C_L \Delta^{2} + C_L \Delta \int_{t_k}^t E \left( \left| \bar{Y}_s^{i,N} \right|^2 \right) ds + C_L \Delta \int_{t_k}^t E \left( W_2^2 \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right) ds \]
\[ + C_L \Delta^{2H} + C_L \Delta^{2H-1} \int_{t_k}^t E \left( W_2^2 \left( \bar{\mu}_s^{Y,N}, \delta_0 \right) \right) ds. \]
\[ \leq C_L \Delta^{2} + C_L \Delta \int_{t_k}^t E \left( \left| \bar{Y}_s^{i,N} \right|^2 \right) ds + C_L \Delta \int_{t_k}^t E \left( \left| \bar{Y}_s^{i,N} \right|^2 \right) ds \]
\[ + C_L \Delta^{2H} + C_L \Delta^{2H-1} \int_{t_k}^t E \left( \left| \bar{Y}_s^{i,N} \right|^2 \right) ds \]
\[ \leq C_L \Delta^{2H} + C_L \Delta^{2H-1} \int_{t_k}^t E \left( \left| \bar{Y}_s^{i,N} \right|^2 \right) ds. \]
Applying Theorem 3.11 to above inequality and then getting the desired result. \( \square \)

Theorem 3.13 Let Assumption 2.1 holds, then
\[ E \left( \left| X_t^{i,N} - Y_t^{i,N} \right|^2 \right) \leq C_{T,L} \Delta^{2H}, \]
where \( C_{T,L} \) is a positive constant independent of \( \Delta \).
Proof. It follows from (1.3) and (3.24) that
\[
E \left( \left| X_t^{i,N} - Y_t^{i,N} \right|^2 \right) = E \left( \left\| \int_0^t \left( b \left( X^{i,N}_s, \mu^{X,N}_s \right) - b \left( Y^{i,N}_s, \mu^{Y,N}_s \right) \right) \, ds \right\|^2 \right).
\]
By (3.19), we have
\[
E \left( \left| X_t^{i,N} - Y_t^{i,N} \right|^2 \right) \leq C_{T,L} E \left( \int_0^t \left| X_s^{i,N} - Y_s^{i,N} \right|^2 \, ds \right) + C_{T,L} E \left( \int_0^t W_2^2 \left( \mu^{X,N}_s, \mu^{Y,N}_s \right) \, ds \right).
\]
By (3.21) and Lemma 3.12, we get
\[
E \left( \left| X_t^{i,N} - Y_t^{i,N} \right|^2 \right) \leq C_{T,L} \int_0^t E \left( \left| X_s^{i,N} - Y_s^{i,N} \right|^2 \right) \, ds + C_{T,L} \int_0^t E \left( \left| Y_s^{i,N} - Y_t^{i,N} \right|^2 \right) \, ds
\leq C_{T,L} \int_0^t E \left( \left| X_s^{i,N} - Y_t^{i,N} \right|^2 \right) \, ds + C_{T,L} \Delta^{2H}.
\]
Then applying the Grönwall inequality, we obtain the desired result. 

**Theorem 3.14** Let Assumption 2.1 be satisfied. If \( q > 2 \), then it holds that
\[
\sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0,T]} E \left( \left| X_t^i - Y_t^{i,N} \right|^2 \right) \leq C_{T,H,L} \times \left\{ \begin{array}{ll}
N^{-1/2} + N^{-(q-2)/q} + \Delta^{2H}, & \text{if } d < 4 \\
N^{-1/2} \log(1 + N) + N^{-(q-2)/q} + \Delta^{2H}, & \text{if } d = 4 \\
N^{-2/d} + N^{-(q-2)/q} + \Delta^{2H}, & \text{if } d > 4,
\end{array} \right.
\]
where the constant \( C_{T,H,L} > 0 \) does not depend on \( N \) and \( \Delta \).

We can easily get this theorem through trigonometric inequality, Theorem 3.10 and Theorem 3.13.

## 4 Numerical Example

**Example 4.1** Consider the following McKean–Vlasov SDEs driven by fBm
\[
\begin{align*}
\text{d}X_t = & \left( X_t + \int_{\mathbb{R}} (X_t - y) \mu(dy) \right) \, dt + \left( \int_{\mathbb{R}} (X_t - y) \mu(dy) \right) \, dB_t^H,
\end{align*}
\]
where initial value \( X_0 \) is a constant.

In Figure 1, we draw the terminal mean square error at \( T = 1 \) with four time step sizes (\( \Delta = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8} \)) and use 100 sample paths to obtain the convergence order of \( H = 0.6, H = 0.7, H = 0.8 \) and \( H = 0.9 \) respectively. We use the numerical solution with smaller step size \( 2^{12} \) to replace the exact solution. For the four subfigures in Figure 1, the red dotted reference line has a slope of 1, this observation is in line with our theoretical result.

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Figure 1: Error in final solution at time $T = 1$ as the mean step size decreases for the truncated EM method applied to (4.1) with parameter set initial value $X_0 = 1$ and particle number $U = 1000$.

A Proof of Theorem 3.10

Proof. It follows from (1.2) and (1.3) that

$$X^i_t - X^i_t,N = \int_0^t (b(X^i_s, \mathcal{L}X^i_s) - b(X^i_s,N, \mu^{X,N}_s)) \, ds.$$  

By (3.7), we obtain

$$E \left( \left| X^i_t - X^i_t,N \right|^2 \right) \leq C_{T,L} \int_0^t E \left( \left| X^i_s - X^i_s,N \right|^2 \right) \, ds + C_{T,L} \int_0^t E \left( W^2_2(\mathcal{L}X^i_s, \mu^{X,N}_s) \right) \, ds$$

$$\leq C_{T,L} \int_0^t E \left( \left| X^i_s - X^i_s,N \right|^2 \right) \, ds + C_{T,L} \int_0^t E \left( W^2_2(\mathcal{L}X^i_s, \mu^X_s) \right) \, ds.$$

We know that $W^2_2(\mathcal{L}X^i_s, \mu^X_s)$ is controlled by the Wasserstein distance estimate in [13, Theorem 1]. Therefore,

$$E \left( W^2_2(\mathcal{L}X^i_s, \mu^X_s) \right) \leq CM^2_q(\mathcal{L}X^i_s)$$

$$\times \begin{cases} 
N^{-1/2} + N^{-(q-2)/2}, & \text{if } d < 4, \\
N^{-1/2} \log(1 + N) + N^{-(q-2)/2}, & \text{if } d = 4, \\
N^{-2/d} + N^{-(q-2)/2}, & \text{if } d > 4, 
\end{cases}$$

where

$$M_q(\mathcal{L}X^i_s) = \int_{\mathbb{R}^d} |X^i_s|^q \mathcal{L}X^i_s(\,dX^i_s).$$

By Theorem 3.9, we note that $M_q(\mathcal{L}X^i_s) \leq C$. Thus,

$$E \left( W^2_2(\mathcal{L}X^i_s, \mu^X_s) \right) \leq C \begin{cases} 
N^{-1/2} + N^{-(q-2)/2}, & \text{if } d < 4, \\
N^{-1/2} \log(1 + N) + N^{-(q-2)/2}, & \text{if } d = 4, \\
N^{-2/d} + N^{-(q-2)/2}, & \text{if } d > 4, 
\end{cases}$$

Then, applying the Gr"onwall inequality completes the proof. □
References

[1] F. Antonelli and A. Kohatsu-Higa. Rate of convergence of a particle method to the solution of the McKean-Vlasov equation. *Ann. Appl. Probab.*, 12(2):423–476, 2002.

[2] J. Bao, C. Reisinger, P. Ren, and W. Stockinger. First-order convergence of Milstein schemes for McKean-Vlasov equations and interacting particle systems. *Proc. Roy. Soc. London Ser. A*, 477(2245):20200258, 2021.

[3] F. Biagini, Y. Hu, B. Oksendal, and T. Zhang. *Stochastic calculus for fractional Brownian motion and applications*. Springer Science and Business Media, 2008.

[4] M. Bossy and D. Talay. Convergence rate for the approximation of the limit law of weakly interacting particles: application to the Burgers equation. *Ann. Appl. Probab.*, 6(3):818–861, 1996.

[5] M. Bossy and D. Talay. A stochastic particle method for the McKean-Vlasov and the Burgers equation. *Math. Comp.*, 66(217):157–192, 1997.

[6] R. Buckdahn, J. Li, S. Peng, and C. Rainer. Mean-field stochastic differential equations and associated PDEs. *Ann. Probab.*, 45(2):824–878, 2017.

[7] R. Carmona and F. Delarue. Probabilistic analysis of mean-field games. *SIAM J. Control Optim.*, 51(4):2705–2734, 2013.

[8] D. Crisan and E. McMurray. Smoothing properties of McKean-Vlasov SDEs. *Probab. Theory Related Fields*, 171(1):97–148, 2018.

[9] G. dos Reis, S. Engelhardt, and G. Smith. Simulation of McKean-Vlasov SDEs with super-linear growth. *IMA J. Numer. Anal.*, 42(1):874–922, 2022.

[10] G. Dos Reis, W. Salkeld, and J. Tugaut. Freidlin-Wentzell LDP in path space for McKean-Vlasov equations and the functional iterated logarithm law. *Ann. Appl. Probab.*, 29(3):1487–1540, 2019.

[11] A. Eberle, A. Guillin, and R. Zimmer. Quantitative Harris-type theorems for diffusions and McKean-Vlasov processes. *Trans. Amer. Math. Soc.*, 371(10):7135–7173, 2019.

[12] X. Fan, X. Huang, Y. Suo, and C. Yuan. Distribution dependent sdes driven by fractional Brownian motions. *Stochastic Process. Appl.*, 2022.

[13] N. Fournier and A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields*, 162(3):707–738, 2015.

[14] L. Galeati, F.A. Harang, and A. Mayorcas. Distribution dependent sdes driven by additive fractional brownian motion. *Probab. Theory Related Fields*, pages 1–59, 2022.

[15] J. Hong, C. Huang, M. Kamrani, and X. Wang. Optimal strong convergence rate of a backward Euler type scheme for the Cox-Ingersoll-Ross model driven by fractional Brownian motion. *Stochastic Process. Appl.*, 130(5):2675–2692, 2020.
[16] Y. Hu, Y. Liu, and D. Nualart. Crank-Nicolson scheme for stochastic differential equations driven by fractional Brownian motions. *Ann. Appl. Probab.*, 31(1):39–83, 2021.

[17] X. Huang and F. Wang. Distribution dependent SDEs with singular coefficients. *Stochastic Process. Appl.*, 129(11):4747–4770, 2019.

[18] M. Li, Y. Hu, C. Huang, and X. Wang. Mean square stability of stochastic theta method for stochastic differential equations driven by fractional Brownian motion. *arXiv preprint arXiv:2109.09009*, 2021.

[19] H.P. McKean. Propagation of chaos for a class of non-linear parabolic equations. *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57, 1967.

[20] H.P. McKean. Fluctuations in the kinetic theory of gases. *Comm. Pure Appl. Math.*, 28(4):435–455, 1975.

[21] H.P. McKean Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Sciences*, 56(6):1907–1911, 1966.

[22] J. Mémin, Y. Mishura, and E. Valkeila. Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. *Statist. Probab. Lett.*, 51(2):197–206, 2001.

[23] I.S. Mishura, IUS Mishura, J.S. Mišura, Y. Mishura, and Ú.S. Mišura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929. Springer Science & Business Media, 2008.

[24] I. Norros. On the use of fractional Brownian motion in the theory of connectionless networks. *IEEE J. Sel. Areas Commun*, 13(6):953–962, 1995.

[25] P. Ren and F. Wang. Bismut formula for lions derivative of distribution dependent SDEs and applications. *J. Differential Equations*, 267(8):4745–4777, 2019.

[26] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives*, volume 1. Gordon and breach science publishers, Yverdon Yverdon-les-Bains, Switzerland, 1993.

[27] F. Wang. Distribution dependent SDEs for landau type equations. *Stochastic Process. Appl.*, 128(2):595–621, 2018.

[28] M. Wang, X. Dai, and A. Xiao. Optimal Convergence Rate of B-Maruyama Method for Stochastic Volterra Integro-Differential Equations with Riemann-Liouville Fractional Brownian Motion. *Adv. Appl. Math. Mech.*, 2022.

[29] P. Yaskov. A maximal inequality for fractional brownian motions. *J. Math. Anal. Appl.*, 472(1):11–21, 2019.

[30] S. Zhang and C. Yuan. Stochastic differential equations driven by fractional Brownian motion with locally Lipschitz drift and their implicit Euler approximation. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(4):1278–1304, 2021.