Uncertainty orders on the sublinear expectation space

Abstract: In this paper, we introduce some definitions of uncertainty orders for random vectors in a sublinear expectation space. We all know that, under some continuity conditions, each sublinear expectation $\mathbb{E}$ has a robust representation as the supremum of a family of probability measures. We describe uncertainty orders from two different viewpoints. One is from sublinear operator viewpoint. After giving definitions such as monotonic orders, convex orders and increasing convex orders, we use these uncertainty orders to derive characterizations for maximal distributions, $G$-normal distributions and $G$-distributions, which are the most important random vectors in the sublinear expectation space theory. On the other hand, we also establish some uncertainty orders’ characterizations from the viewpoint of probability measures and build some connections with the theory of risk measures.

Keywords: Uncertainty order, Sublinear expectation, Choquet integral, Quantile function, Risk measure

MSC: 62C86, 91B06, 90B50

1 Introduction

Nonlinear expectations have been a prominent topic in mathematical economics ever since the famous Allais paradox. Typical examples are monetary risk measures introduced in [1, 2], Choquet expectations developed in [3], $g$-expectations and $G$-expectations established and studied by [4–10] respectively.

The theory of sublinear expectation, an important and a special nonlinear expectation, is not based on a given (linear) probability space. We all know that, under some mild continuity conditions, each sublinear expectation $\mathbb{E}$ has a robust representation as the supremum of a subset of probability measures. It thus provides a new way to describe the uncertainties.

In a financial context, when facing risks one often uses the stochastic orders to compare the “good or bad” between the portfolios. In this case, a probability measure is given on the set of scenarios, so we can focus on the resulting payoff or loss distributions. The comparison of financial risks plays an important role for both regulators and agents in financial markets. Under the framework of a classical probability space, based on expected utility theory developed by [11], lots of elegant results about the stochastic orders have been obtained. For example, in the stochastic order theory it has been shown that the monotonic order is equivalent to saying that one risk position is preferred over the other by all decision makers who have increasing real functions for which the expectations exist. For more details and properties of stochastic orders, see [12, 13].

Motivated by the classical stochastic orders, the object of this paper is to explore the uncertainty orders for random vectors in a sublinear expectation space. These uncertainty orders will provide useful criterions for describing the comparisons of the uncertainty degrees between two random vectors on the sublinear expectation space.
We establish the uncertainty orders in the sublinear expectation space from two different viewpoints. One is from the sublinear operator viewpoint. We give some definitions of uncertainty orders such as monotonic orders, convex orders and increasing convex orders. Then we use these uncertainty orders to derive the characterizations for maximal distributions, $G$-normal distributions and $G$-distributions, which are the most important random vectors in the theory of sublinear expectation space.

On the other hand, we study the uncertainty orders in the sublinear expectation space by a family of probability measures $\mathcal{P}$ induced by the sublinear expectation $\mathbb{E}$. We can define the capacity space from $\mathcal{P}$, then we get some characterizations with the help of the recent results about capacity orders obtained by [14, 15] in the capacity space. Besides, we also give the characterization of uncertainty orders by distortion functions.

The paper is organized as follows. In Section 2, we first introduce some preliminaries about the sublinear expectation space. Then we give the definitions of uncertainty orders from the sublinear operator viewpoint, and establish some characterizations for maximal distributions, $G$-normal distributions and $G$-distributions. In Section 3, we introduce the other viewpoint of characterizations for uncertainty orders in the sublinear expectation space by using some terminologies of a capacity space. We conclude the paper in Section 4.

2 Characterizations of uncertainty orders from the sublinear operator viewpoint

We first present some preliminaries of sublinear expectation space theory. Then we give the definitions of uncertainty orders in the sublinear expectation space. More details for the first subsection can be found in [6, 8].

2.1 Sublinear expectation spaces

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ such that $c \in \mathcal{H}$ for all constants $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables. We further suppose that if $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$, where $C_{l,lip}(\mathbb{R}^n)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

where $m$ depends only on $\varphi$.

**Definition 2.1** (see [8]). A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(i) **Monotonicity:** $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.

(ii) **Constant preserving:** $\mathbb{E}[c] = c$ for $c \in \mathbb{R}$.

(iii) **Subadditivity:** $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(iv) **Positive homogeneity:** $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

Let $X = (X_1, \cdots, X_n)$, $X_i \in \mathcal{H}$, denoted by $X \in \mathcal{H}^n$, be a given $n$-dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Define a function on $C_{l,lip}(\mathbb{R}^n)$ by

$$\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)], \quad \forall \varphi \in C_{l,lip}(\mathbb{R}^n).$$

The triple $(\mathbb{R}^n, C_{l,lip}(\mathbb{R}^n), \mathbb{F}_X)$ forms a sublinear expectation space. $\mathbb{F}_X$ is called the distribution of $X$. Let $Y$ be another $n$-dimensional random vector on $(\Omega, \mathcal{H}, \mathbb{E})$, we denote $X \preceq Y$, if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ for all $\varphi \in C_{l,lip}(\mathbb{R}^n)$. 
Definition 2.2 (see [6, 8]). Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a sublinear expectation space. A random vector \(Y \in \mathcal{H}^n\) is said to be independent from another random vector \(X \in \mathcal{H}^m\) under \(\mathbb{E}\), if for each test function \(\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^{m+n})\) we have
\[
\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)|x=X]].
\]
And \(Y\) is said to be weakly independent from \(X\) under \(\mathbb{E}\), if the above test functions \(\varphi\) are taken only among, instead of \(\mathcal{C}_{l,Lip}(\mathbb{R}^{m+n})\),
\[
\varphi(x, y) = \psi_0(y) + \psi_1(y)|x| + \psi_2(y)|x|^2, \quad \psi_i \in \mathcal{C}_{l,Lip}(\mathbb{R}^n), i = 0, 1, 2.
\]
Let \(X, \bar{X}\) be two \(n\)-dimensional random vectors on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). \(\bar{X}\) is called an independent copy of \(X\) if \(X \overset{d}{=} \bar{X}\) and \(\bar{X}\) is independent from \(X\).

Definition 2.3 (see [8]). Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a sublinear expectation space. A \(n\)-dimensional random vector \(\eta\) on \((\Omega, \mathcal{H}, \mathbb{E})\) is called maximal distributed if
\[
a\eta + b\bar{\eta} \overset{d}{=} (a + b)\eta, \quad \text{for } a, b \geq 0,
\]
where \(\bar{\eta}\) is an independent copy of \(\eta\). In particular, for \(n = 1\), we denote \(\eta \overset{d}{=} N([\underline{\mu}, \bar{\mu}] \times \{0\})\) where \(\underline{\mu} = -\mathbb{E}[-\eta]\) and \(\bar{\mu} = \mathbb{E}[\eta]\).

A \(n\)-dimensional random vector \(X\) on \((\Omega, \mathcal{H}, \mathbb{E})\) is called \(G\)-normal distributed if
\[
aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X, \quad \text{for } a, b \geq 0,
\]
where \(\bar{X}\) is an independent copy of \(X\). In particular, for \(n = 1\), we denote \(X \overset{d}{=} N(\{0\} \times [\underline{\sigma}, \bar{\sigma}^2])\) where \(\underline{\sigma} = -\mathbb{E}[-X^2]\) and \(\bar{\sigma} = \mathbb{E}[X^2]\) and \(\underline{\sigma} \geq \sigma \geq 0\).

A pair of \(n\)-dimensional random vectors \((X, \eta)\) on \((\Omega, \mathcal{H}, \mathbb{E})\) is called \(G\)-distributed if
\[
(aX + b\bar{X}, a^2\eta + b^2\bar{\eta}) \overset{d}{=} (\sqrt{a^2 + b^2}X, (a^2 + b^2)\eta) \quad \text{for } a, b \geq 0,
\]
where \((\bar{X}, \bar{\eta})\) is an independent copy of \((X, \eta)\). For \(n = 1\), we denote \((X, \eta) \overset{d}{=} N([\underline{\mu}, \bar{\mu}] \times [\underline{\sigma}^2, \bar{\sigma}^2])\) where \(\underline{\mu}, \bar{\mu}, \underline{\sigma}^2\) and \(\bar{\sigma}^2\) are defined as above.

### 2.2 Characterizations from the sublinear operator viewpoint

Throughout the following paper, we interpret the risk positions as loss random vectors. Motivated by the definitions of stochastic orders in a probability space, we give the definitions of uncertainty orders in a sublinear expectation space. We then use these uncertainty orders to derive the characterizations for maximal distributions, \(G\)-normal distributions and \(G\)-distributions.

Definition 2.4. Let \((\Omega, \mathcal{H}, \mathbb{E})\) be a sublinear expectation space. Let \(X, Y\) be two \(n\)-dimensional random vectors on \((\Omega, \mathcal{H}, \mathbb{E})\).

(i) \(X\) is said to precede \(Y\) in the monotonic order sense under \(\mathbb{E}\), denoted by \(X \preceq_{\text{mon}} Y\), if for all increasing functions \(\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)\), we have
\[
\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)] \quad \text{and} \quad -\mathbb{E}[-\varphi(X)] \leq -\mathbb{E}[-\varphi(Y)].
\]

(ii) \(X\) is said to precede \(Y\) in the convex order sense under \(\mathbb{E}\), denoted by \(X \preceq_{\text{con}} Y\), if (1) holds for all convex functions \(\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)\).

(iii) \(X\) is said to precede \(Y\) in the increasing convex order sense under \(\mathbb{E}\), denoted by \(X \preceq_{\text{icon}} Y\), if (1) holds for all increasing convex functions \(\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)\).

Remark 2.5. From the definitions of uncertainty orders as above, we can see that the uncertainty orders only involve the distributions of random vectors \(X\) and \(Y\), thus we can also consider the random vectors \(X\) and \(Y\) from the different sublinear expectation spaces.
Remark 2.6. Compared with the stochastic orders, here we impose an extra restriction condition \(-E[-\varphi(X)] \leq -E[-\varphi(Y)]\) on the uncertainty orders, which is a redundant condition in the linear probability space.

Recall Lemma 3.4 (the representation theorem) in Chapter I established in [8], we can see that it is reasonable to define the uncertainty orders as above. In fact, the condition (1) is equivalent to

\[
\max_{\theta \in \Theta} E_{\theta}[\varphi(X)] \leq \max_{\theta \in \Theta} E_{\theta}[\varphi(Y)] \quad \text{and} \quad \min_{\theta \in \Theta} E_{\theta}[\varphi(X)] \leq \min_{\theta \in \Theta} E_{\theta}[\varphi(Y)],
\]

where \(\Theta\) is a family of probability measures on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\). In this sense, we say \(X \leq_{\text{mon}} Y\), i.e., the best and worst expectations of a subset of linear expectations \(\{E_{\theta} : \theta \in \Theta\}\) for loss random variables \(\varphi(X)\) are both less than those of \(\varphi(Y)\) for all increasing functions \(\varphi \in C_{L,Lip}(\mathbb{R}^n)\).

For any loss random variable \(X \in \mathcal{H}\) of a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), we have the following four typical parameters:

\[
\mu = -E[-X], \quad \bar{\mu} = E[X], \quad \bar{\sigma}^2 = -E[-X^2], \quad \sigma^2 = E[X^2].
\]

The intervals \([\mu, \bar{\mu}]\) and \([\sigma^2, \bar{\sigma}^2]\) characterize the mean-uncertainty and the variance-uncertainty of \(X\) respectively.

From Definition 2.4 of the uncertainty orders, we easily obtain that for another loss random variable \(Y\) in \((\Omega, \mathcal{H}, \mathbb{E})\), with the mean-uncertainty interval \([\bar{v}, \bar{v}]\) and the variance-uncertainty interval \([\bar{\rho}^2, \bar{\rho}^2]\),

- If \(X \leq_{\text{mon}} Y\) or \(X \leq_{\text{con}} Y\), then \(\bar{\mu} \leq \bar{v}\) and \(\bar{\sigma}^2 \leq \bar{\rho}^2\).
- If \(X \leq_{\text{con}} Y\), then \(\bar{\mu} = \bar{v}, \mu = \bar{v}, \sigma^2 \leq \rho^2\) and \(\bar{\sigma}^2 \leq \bar{\rho}^2\).

In the following three theorems we show that for some particular distributions the above necessary conditions are also the sufficient conditions. And these distributions are very important in the sublinear expectation space theory.

Theorem 2.7. Let \(\eta \overset{d}{=} N(\mu, \bar{\mu} \times \{0\})\) and \(\xi \overset{d}{=} N(\bar{v}, \bar{\nu} \times \{0\})\) be two maximal distributions on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). Then we have

(i) \(\eta \leq_{\text{mon}} \xi \iff \eta \leq_{\text{con}} \xi \iff \bar{\mu} \leq \bar{v}\) and \(\bar{\sigma}^2 \leq \bar{\rho}^2\).

(ii) \(\eta \leq_{\text{con}} \xi \iff \bar{\mu} = \bar{v}\) and \(\mu = \bar{v}\) i.e., \(\eta \overset{d}{=} \xi\).

Proof. (i) From the definitions of the monotonic order and the increasing convex order, \(\eta \leq_{\text{mon}} \xi\) implies \(\eta \leq_{\text{con}} \xi\) is obvious.

If \(\eta \leq_{\text{con}} \xi\), choosing an increasing convex function \(\varphi(x) = x\) satisfying \(\varphi \in C_{L,Lip}(\mathbb{R})\), from the definitions of the increasing convex orders and maximal distributions, we have

\[
\bar{\mu} = E[\eta] \leq E[\xi] = \bar{v} \quad \text{and} \quad \mu = -E[-\eta] \leq -E[-\xi] = \bar{v}.
\]

If \(\bar{\mu} \leq \bar{v}\) and \(\mu \leq \bar{v}\), for any increasing function \(\varphi \in C_{L,Lip}(\mathbb{R})\), from the equivalent definition of the maximal distribution Definition 1.1 in Chapter II of [8], we have

\[
E[\varphi(\eta)] = \max_{\mu \leq \varphi \leq \bar{\mu}} \varphi(y) = \varphi(\bar{\mu}) \leq \varphi(\bar{v}) = \max_{\mu \leq \varphi \leq \bar{\nu}} \varphi(y) = E[\varphi(\xi)],
\]

and

\[
-E[-\varphi(\eta)] = \min_{\mu \geq \varphi \geq \bar{\mu}} \varphi(y) = \varphi(\mu) \leq \varphi(\bar{v}) = \min_{\mu \geq \varphi \geq \bar{\nu}} \varphi(y) = -E[-\varphi(\xi)].
\]

Thus we have \(\eta \leq_{\text{mon}} \xi\).

(ii) It is obvious that \(\eta \overset{d}{=} \xi\) implies \(\eta \leq_{\text{con}} \xi\). As for the other direction, choosing the convex functions \(\varphi(x) = x\) and \(\varphi(x) = -x\) respectively, we can easily obtain \(\bar{\mu} = \bar{v}\) and \(\bar{\mu} = \bar{v}\) by the definitions of \(\leq_{\text{con}}\) and maximal distributions.

Theorem 2.8. Let \(X \overset{d}{=} N(\{0\} \times [\sigma^2, \bar{\sigma}^2])\) and \(Y \overset{d}{=} N(\{0\} \times [\rho^2, \bar{\rho}^2])\) be two \(G\)-normal distributions on a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). Then we have

(i) \(X \leq_{\text{mon}} Y \iff \bar{\sigma}^2 \geq \bar{\rho}^2\) and \(\sigma^2 \leq \rho^2\), i.e., \(X \overset{d}{=} Y\).

(ii) \(X \leq_{\text{con}} Y \iff X \leq_{\text{con}} Y \iff \bar{\sigma}^2 \leq \bar{\rho}^2\) and \(\sigma^2 \leq \rho^2\).
Proof. (i) $X \leq Y$ implies $X \leq_{\text{mon}} Y$ is obvious.

Suppose $X \leq_{\text{mon}} Y$. Recall the fact that $\mathbb{E}[\varphi(x)]$ can be explicitly calculated for $G$-normal distributions such that $\varphi \in C_{l,Lip}(\mathbb{R})$ is a convex or concave function in [8], for an increasing convex function $\varphi(x) = x^+$ satisfying $\varphi \in C_{l,Lip}(\mathbb{R})$, thus we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\sigma y) e^{-\frac{y^2}{2}} dy = \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{+\infty} y e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \sigma.$$

Similarly we obtain $\mathbb{E}[\varphi(Y)] = \frac{1}{\sqrt{2\pi}} \tilde{\rho}$. Since $\sigma, \tilde{\rho} \geq 0$, we have $\tilde{\sigma}^2 \leq \tilde{\rho}^2$ by the definition of $\leq_{\text{mon}}$.

On the other hand, we have

$$-\mathbb{E}[-\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\sigma y) e^{-\frac{y^2}{2}} dy = \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{+\infty} y e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \sigma.$$

Similarly we can get $-\mathbb{E}[-\varphi(Y)] = \frac{1}{\sqrt{2\pi}} \tilde{\rho}$. Since $\sigma, \tilde{\rho} \geq 0$, we have $\tilde{\sigma}^2 \leq \tilde{\rho}^2$ from the definition of $\leq_{\text{mon}}$.

Taking an increasing concave function $\varphi(x) = x^-$, we can derive $\tilde{\sigma}^2 \geq \tilde{\rho}^2$ and $\sigma^2 \geq \rho^2$ using the arguments as $\varphi(x) = x^+$. We conclude from the above that $\tilde{\sigma}^2 \geq \tilde{\rho}^2$ and $\sigma^2 \geq \rho^2$, i.e., $X \leq Y$.

(ii) Clearly we have $X \leq_{\text{con}} Y \implies X \leq_{\text{mon}} Y$. Repeating the arguments in the part (i) we have $X \leq_{\text{con}} Y \implies \tilde{\sigma}^2 \leq \tilde{\rho}^2$ and $\sigma^2 \leq \rho^2$ by choosing the increasing convex function $\varphi(x) = x^+$.

It only needs to show that $\tilde{\sigma}^2 \leq \tilde{\rho}^2$ and $\sigma^2 \leq \rho^2 \implies X \leq_{\text{con}} Y$.

For any convex function $\varphi \in C_{l,Lip}(\mathbb{R})$. Consider the following $G$-heat equation for $X$

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 u}{\partial x^2} = 0, \\ u|_{t=0} = \varphi. \end{cases}$$

(3)

We have that $u(t,x) := \mathbb{E}[\varphi(x + \sqrt{T}X)]$, $(t,x) \in [0, +\infty) \times \mathbb{R}$, is the unique viscosity solution of (3). Furthermore, we can check that $u(t,x)$ is convex in $x$. Thus the above $G$-heat equation (3) becomes

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} = 0, \\ u|_{t=0} = \varphi. \end{cases}$$

(4)

Similarly we obtain $v(t,x) := \mathbb{E}[\varphi(x + \sqrt{T}Y)]$, $(t,x) \in [0, +\infty) \times \mathbb{R}$, is the unique viscosity solution of the following $G$-heat equation

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{1}{2} \tilde{\rho}^2 \frac{\partial^2 v}{\partial x^2} = 0, \\ v|_{t=0} = \varphi. \end{cases}$$

(5)

Since $\tilde{\sigma}^2 \leq \tilde{\rho}^2$, then by the comparison theorem for the viscosity solutions of (4) and (5) (For example, see [8], Theorem 2.6 in Appendix C), we derive that

$$u(t,x) \leq v(t,x), \ (t,x) \in [0, +\infty) \times \mathbb{R}.$$

In particular, taking $(t,x) = (1,0)$, we have

$$\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)].$$

(6)

Since $-\varphi$ is a concave function, we can similarly show that $m(t,x) := \mathbb{E}[-\varphi(x + \sqrt{T}X)]$ and $n(t,x) := \mathbb{E}[-\varphi(x + \sqrt{T}Y)]$, $(t,x) \in [0, +\infty) \times \mathbb{R}$, are the unique viscosity solutions of the following $G$-heat equations respectively

$$\begin{cases} \frac{\partial m}{\partial t} + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 m}{\partial x^2} = 0, \\ m|_{t=0} = -\varphi. \end{cases}$$

$$\begin{cases} \frac{\partial n}{\partial t} + \frac{1}{2} \tilde{\rho}^2 \frac{\partial^2 n}{\partial x^2} = 0, \\ n|_{t=0} = -\varphi. \end{cases}$$

(7)

Due to the facts that $\tilde{\sigma}^2 \leq \tilde{\rho}^2$ and the comparison theorem for the viscosity solutions of (7), setting $(t,x) = (1,0)$, we have

$$-\mathbb{E}[-\varphi(X)] \leq -\mathbb{E}[-\varphi(Y)].$$

(8)

By combining (6) with (8), we get $X \leq_{\text{con}} Y$. The proof is complete. \qed
Theorem 2.9. Let \((\eta, X) \overset{d}{=} N(\langle \mu, \bar{\mu} \rangle \times [\sigma^2, \tilde{\sigma}^2])\) and \((\xi, Y) \overset{d}{=} N(\langle \nu, \bar{\nu} \rangle \times [\rho^2, \tilde{\rho}^2])\) be two \(G\)-distributions on a sublinear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\). Moreover, suppose \(\eta\) is weakly independent from \(X\) and \(\xi\) is weakly independent from \(Y\) respectively. Then we have

(i) \((\eta, X) \overset{\mathcal{E}}{\leq}_{\text{mon}} (\xi, Y)\), if and only if \(\bar{\mu} \leq \bar{\nu}, \underline{\mu} \leq \underline{\nu}, \bar{\sigma}^2 \leq \bar{\rho}^2\) and \(\tilde{\sigma}^2 \leq \tilde{\rho}^2\).

(ii) \((\eta, X) \overset{\mathcal{E}}{\leq}_{\text{con}} (\xi, Y)\), if and only if \(\bar{\mu} = \bar{\nu}, \underline{\mu} = \underline{\nu}, \bar{\sigma}^2 \leq \bar{\rho}^2\) and \(\tilde{\sigma}^2 \leq \tilde{\rho}^2\).

(iii) \((\eta, X) \overset{\mathcal{E}}{\leq}_{\text{icon}} (\xi, Y)\), if and only if \(\bar{\mu} \leq \bar{\nu}, \underline{\mu} \leq \underline{\nu}, \bar{\sigma}^2 \leq \bar{\rho}^2\) and \(\tilde{\sigma}^2 \leq \tilde{\rho}^2\).

Proof. The “only if” parts are the combinations of the results of Theorem 2.7 and Theorem 2.8. In fact, for example, if \((\eta, X) \overset{\mathcal{E}}{\leq}_{\text{mon}} (\xi, Y)\), then we can derive that \(\eta \overset{\mathcal{E}}{\leq}_{\text{mon}} \xi\) and \(X \overset{\mathcal{E}}{\leq}_{\text{mon}} Y\). Thus from Theorem 2.7 and Theorem 2.8 we get the results.

For the proof of the converse implications, the key ideas are both the applications of the comparison theorem of the viscosity solutions to \(G\)-equations. We only show the case (iii). Cases (i) and (ii) are verified by an analogous argument.

Assume \(\bar{\mu} \leq \bar{\nu}, \underline{\mu} \leq \underline{\nu}, \bar{\sigma}^2 \leq \bar{\rho}^2\) and \(\tilde{\sigma}^2 \leq \tilde{\rho}^2\). For any increasing convex function \(\varphi \in C_{l,lip}(\mathbb{R}^{2})\), by Proposition 1.10 in Chapter II of [8], we have that \(u(t, x, y) := \mathbb{E}[\varphi(x + \sqrt{t}X, y + t\eta)]\), \((t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}\), is the unique viscosity solution of the following \(G\)-equation for \((\eta, X)\)

\[
\begin{cases}
\frac{\partial u}{\partial t} - G\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0, \\
u|_{t=0} = \psi.
\end{cases}
\]

where \(G(p, a) = \mathbb{E}[\frac{1}{2}aX^2 + p\eta]\).

Since \(\eta\) is weakly independent from \(X\), we have

\[
G(p, a) = \mathbb{E}[\frac{1}{2}aX^2 + p\eta] = \mathbb{E}[\frac{1}{2}aX^2] + \mathbb{E}[p\eta] = \frac{1}{2}(\bar{\sigma}^2 a^+ - \tilde{\sigma}^2 a^-) + (\bar{\mu} p^+ - \underline{\mu} p^-).
\]

On the other hand, since \(\varphi\) is an increasing convex function in \(C_{l,lip}(\mathbb{R}^{2})\), we can get \(u(t, x, y)\) is convex in \(x\) and increasing in \(y\). Thus (9) becomes

\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{1}{2} \bar{\sigma}^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^+ + \bar{\mu} \left(\frac{\partial u}{\partial x}\right)^+ = 0, \\
u|_{t=0} = \psi.
\end{cases}
\]

Similarly we obtain \(v(t, x, y) := \mathbb{E}[\varphi(x + \sqrt{t}Y, y + t\xi)]\), \((t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}\), is the unique viscosity solution of the following \(G\)-equation

\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{1}{2} \tilde{\rho}^2 \left(\frac{\partial^2 u}{\partial y^2}\right)^+ + \tilde{\nu} \left(\frac{\partial u}{\partial y}\right)^+ = 0, \\
u|_{t=0} = \psi.
\end{cases}
\]

Since \(\bar{\mu} \leq \bar{\nu}\) and \(\bar{\sigma}^2 \leq \tilde{\rho}^2\), then by the comparison theorem for the viscosity solutions of (10) and (11), setting \((t, x, y) = (1, 0, 0)\), we have

\[
\mathbb{E}[\varphi(X, \eta)] \leq \mathbb{E}[\varphi(Y, \xi)].
\]

Since \(\varphi\) is an increasing convex function, we can similarly obtain that \(m(t, x, y) := \mathbb{E}[-\varphi(x + \sqrt{t}X, y + t\eta)]\) and \(n(t, x, y) := \mathbb{E}[-\varphi(x + \sqrt{t}Y, y + t\xi)]\), \((t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}\), are the unique viscosity solutions of the following \(G\)-equations respectively

\[
\begin{cases}
\frac{\partial m}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \left(\frac{\partial^2 m}{\partial x^2}\right)^- + \bar{\mu} \left(\frac{\partial m}{\partial x}\right)^- = 0, \\
m|_{t=0} = -\psi.
\end{cases}
\]

and

\[
\begin{cases}
\frac{\partial n}{\partial t} + \frac{1}{2} \tilde{\rho}^2 \left(\frac{\partial^2 n}{\partial y^2}\right)^- + \tilde{\nu} \left(\frac{\partial n}{\partial y}\right)^- = 0, \\
n|_{t=0} = -\psi.
\end{cases}
\]

Due to the facts that \(\bar{\mu} \leq \bar{\nu}\) and \(\bar{\sigma}^2 \leq \tilde{\rho}^2\), by the comparison theorem for the viscosity solutions of (13), setting \((t, x, y) = (1, 0, 0)\), we have

\[
-\mathbb{E}[-\varphi(X, \eta)] \leq -\mathbb{E}[-\varphi(Y, \xi)].
\]

By combining (12) with (14), we obtain \((\eta, X) \overset{\mathcal{E}}{\leq}_{\text{icon}} (\xi, Y)\). The proof is complete.
Remark 2.10. In the classical linear expectation space, for the stochastic orders' results to the normal distributions, the reader can refer to [12] and [16]. We list the results as follows. Let \( X \overset{d}{\sim} N(\mu, \sigma^2) \) and \( Y \overset{d}{\sim} N(v, \rho^2) \) be the two normal distributions on the probability space \((\Omega, \mathcal{F}, P)\), we have

- \( X \preceq_{\text{mon}} Y \) if and only if \( \mu \leq v, \sigma^2 = \rho^2 \),
- \( X \preceq_{\text{con}} Y \) if and only if \( \mu = v, \sigma^2 \leq \rho^2 \),
- \( X \preceq_{\text{icon}} Y \) if and only if \( \mu \leq v, \sigma^2 \leq \rho^2 \).

Hence, our results generalize the classical results.

Remark 2.11. Theorem 2.9 looks like just combining stochastic orders' results of two normal distributions \( X \overset{d}{\sim} N(\bar{\mu}, \bar{\sigma}^2) \) and \( Y \overset{d}{\sim} N(\bar{v}, \bar{\rho}^2) \) with \( X' \overset{d}{\sim} N(\mu, \sigma^2) \) and \( Y' \overset{d}{\sim} N(v, \rho^2) \) together. However, we can not understand it in this way, because \( G \)-distribution is not a simple collection of a family of normal distributions, see [8].

In this subsection, we introduce some uncertainty orders in the sublinear expectation space. Here we only consider some random variables with special distributions. It is not easy to characterize other distributions. For more properties or computations, the readers can refer to [17, 18].

### 3 Characterizations of uncertainty orders from the probability measures viewpoint

In this section, we first list some properties of a capacity and quantile functions. We then introduce the recent results of uncertainty orders in the capacity space introduced by [14, 15]. We also establish some characterizations by distortion functions. Finally, we derive the characterizations for uncertainty orders by capacity space theory, induced by sublinear expectation space.

#### 3.1 Quantile functions and risk measures

Let \((\Omega, \mathcal{F})\) be a measurable space, and for simplicity we only consider the situation bounded random variables. Let \( L^\infty = L^\infty(\Omega, \mathcal{F}) \) be the space of bounded \( \mathcal{F} \)-measurable functions, endowed with the supremum norm \( \| \cdot \| \).

Firstly, we introduce some properties of set functions \( \mu : \mathcal{F} \rightarrow [0, 1] \) by [3]:

- Monotonicity: if \( A, B \in \mathcal{F} \) and \( A \subseteq B \), then \( \mu(A) \leq \mu(B) \);
- Normalization: if \( \mu(\emptyset) = 0 \) and \( \mu(\Omega) = 1 \);
- Continuous from below: if \( A_n, A \in \mathcal{F} \), and \( A_n \uparrow A \), then \( \mu(A_n) \uparrow \mu(A) \);
- Continuous from above: if \( A_n, A \in \mathcal{F} \), and \( A_n \downarrow A \), then \( \mu(A_n) \downarrow \mu(A) \).

Now we introduce the definitions of capacity and Choquet integral (see, for instance, [19], [3]).

**Definition 3.1.** A set function \( \mu : \mathcal{F} \rightarrow [0, 1] \) is called a capacity if it is monotonic, normalized and continuous from below and continuous from above.

**Definition 3.2.** Let \( \mu \) be a capacity, \( X \in L^\infty \), and denote \( \mu[X] \) by

\[
\mu(X) := \int_0^\infty (\mu(X > t) - 1)dt + \int_0^\infty \mu(X > t)dt.
\]

We call \( \mu(X) \) the Choquet integral of \( X \) with respective to the capacity \( \mu \).

Let \( \mu \) be a capacity and \( X \in L^\infty \). Put

\[
G_{\mu, X}(x) := \mu(X > x).
\]
We call $G_{\mu,X}$ the decreasing distribution function of $X$ with respect to $\mu$. Taking into account the continuity property from below of the capacity $\mu$, we derive that $G_{\mu,X}$ is right continuous. We introduce the definition of quantile functions of $X$ with respect to $\mu$ by [3].

**Definition 3.3** ([3]). Let $\mu$ be a capacity and $X \in L^\infty$. Then we say that $\tilde{G}_{\mu,X}$ is a quantile function of $X$ under the capacity $\mu$ if for all $\lambda \in (0, 1)$,

$$
\tilde{G}_{\mu,X}^+(\lambda) := \sup\{x \mid G_{\mu,X}(x) \geq \lambda\} \geq \tilde{G}_{\mu,X}(\lambda) \geq \sup\{x \mid G_{\mu,X}(x) > \lambda\} =: \tilde{G}_{\mu,X}^-(\lambda).
$$

And $\tilde{G}_{\mu,X}^+(\lambda)$ and $\tilde{G}_{\mu,X}^-(\lambda)$ are called the upper and lower $(1 - \lambda)$-quantile of $X$ with respect to $\mu$.

**Remark 3.4.** It is easy to verify that the lower and upper $(1 - \lambda)$-quantile of the distribution of $X$ with respect to $\mu$ can also be represented as

$$
\begin{align*}
\tilde{G}_{\mu,X}^- (\lambda) &= \inf\{x \mid \mu(X > x) \leq \lambda\}, \\
\tilde{G}_{\mu,X}^+ (\lambda) &= \inf\{x \mid \mu(X > x) < \lambda\}.
\end{align*}
$$

Any two quantile functions coincide for all levels $\lambda$, except on at most a countable set. We also have the following properties about the quantile functions of $X$ under the capacity $\mu$ (see Chapter 1 and Chapter 4 of [3]). Note that (iv) of Lemma 3.5 holds here because the capacity is continuous from below and above, the reader can obtain a similar proof of Lemma A.23 in [12], see also Remark 2.4 in [15].

**Lemma 3.5.** Let $\mu$ be a capacity and $X \in L^\infty$, we have

(i) $\tilde{G}_{\mu,X}(\cdot)$ is a decreasing function;

(ii) If $v$ is another capacity and $Y \in L^\infty$, such that $G_{v,Y} \leq G_{\mu,X}$, then $\tilde{G}_{v,Y} \leq \tilde{G}_{\mu,X}$ except on at most a countable set;

(iii) $\mu[X] = \int_0^1 \tilde{G}_{\mu,X}(t) dt$;

(iv) If $u$ is an increasing function, then $\tilde{G}_{\mu,u(X)} = u \circ \tilde{G}_{\mu,X}$ except on at most a countable set.

Note that VaR of a financial position $X \in L^\infty$ under a given probability measure $P$ on $(\Omega, \mathcal{F})$ is the quantile function of the distribution of $X$. We give the following definitions, which generalize the definitions of VaR and AVaR under a priori probability measure. These definitions can also be found in [14, 15], where Grigorova used the different notations but the economic implications are the same.

**Definition 3.6.** Let $\mu$ be a capacity and a loss random variable $X \in L^\infty$. For $\lambda \in (0, 1)$, we define the Value at Risk with a confidence level $\lambda \in (0, 1)$ of $X$ under the capacity $\mu$ as

$$
\text{VaR}_{\mu,\lambda}(X) := \tilde{G}_{\mu,X}^- (\lambda) = \inf\{x \mid \mu(X > x) \leq \lambda\}.
$$

**Definition 3.7.** The Average Value at Risk under a capacity $\mu$ at $\lambda \in (0, 1)$ of a loss position $X \in L^\infty$ is given by

$$
\text{AVaR}_{\mu,\lambda}(X) := \frac{1}{\lambda} \int_0^\lambda \text{VaR}_{\mu,t}(X) dt.
$$

**Remark 3.8.** For any $X \in L^\infty$, Definition 3.6 can also be used to define $\text{VaR}_{\mu,0}(X)$ and $\text{VaR}_{\mu,1}(X)$. We set that $\text{VaR}_{\mu,0}(X) = -\infty$ and $\text{VaR}_{\mu,1}(X) = \sup X$. In the sequel, we will often use the following equivalence relation which holds for all $x \in \mathbb{R}$ and $\lambda \in [0, 1]$:

$$
\text{VaR}_{\mu,\lambda}(X) \leq x \iff G_{\mu,X}(x) \leq \lambda. 
$$

(16)

Since any two quantiles coincide for all levels $\lambda$, except on at most a countable set, then the AVaR under a capacity $\mu$ at $\lambda \in (0, 1]$ of a financial position $X \in L^\infty$ can be defined by any quantile function of $X$ under $\mu$, i.e.,

$$
\text{AVaR}_{\mu,\lambda}(X) = \frac{1}{\lambda} \int_0^\lambda \tilde{G}_{\mu,X}(t) dt.
$$
We list some concepts of uncertainty orders under a capacity introduced by Grigorova. See Definition 2.7 and Definition 3.1 in [15].

**Definition 3.9.** Let $\mu$ be a capacity. Let $X$ and $Y$ be two losses positions in $L^\infty$.
(i) $X$ is said to precede $Y$ in the increasing convex ordering under $\mu$, denoted by $X \lesssim_{ic} Y$, if for all increasing and convex function $\phi : \mathbb{R} \to \mathbb{R}$
$$\mu(\phi(X)) \leq \mu(\phi(Y)).$$

(ii) $X$ is said to precede $Y$ in monotone ordering under $\mu$, denoted by $X \lesssim_{mon} Y$, if for all increasing function $\phi : \mathbb{R} \to \mathbb{R}$
$$\mu(\phi(X)) \leq \mu(\phi(Y)).$$

Some characterizations about these uncertainty orders were considered in [14, 15], see Propositions 2.6-2.8 and Proposition 3.1 in [15].

**Proposition 3.10.** ([15]) Let $\mu$ be a capacity, $X, Y \in L^\infty$. The following statements hold.
(i) $Y \lesssim_{ic} X$ if and only if $\mu((X - b)^+) \geq \mu((Y - b)^+)$ for any $b \in \mathbb{R}$.

(ii) $Y \lesssim_{ic} X$ if and only if $\text{AVaR}_{\mu, \lambda}(X) \geq \text{AVaR}_{\mu, \lambda}(Y)$, for any $\lambda \in (0, 1)$.

(iii) $Y \lesssim_{mon} X$ if and only if $G_{\mu, X}(x) \geq G_{\mu, Y}(x)$, for all $x \in \mathbb{R}$ and $\text{VaR}_{\mu, \lambda}(X) \geq \text{VaR}_{\mu, \lambda}(Y)$, for all $\lambda \in (0, 1)$.

**Remark 3.11.** In Proposition 3.10, we use our notations. In fact, $\gamma_X(t)$, in the Proposition 2.6 of [15], is equal to our $\hat{G}_{\mu, X} (1-t)$, for all $t \in (0, 1)$, thus (ii) and (iii) of our claims hold by simple transformation.

Here, we give an another characterization of the uncertainty orders $\lesssim_{mon}$ and $\lesssim_{ic}$ by distortion functions. Distortion functions play an important role in the field of mathematical finance. We refer to [20] for decision choice under risk, [21] for insurance premiums and [12] for risk measures.

Motivated by [22], we give the characterizations of the uncertainty orders $\lesssim_{mon}$ and $\lesssim_{ic}$ in terms of distortion functions. We first list the definition of distortion functions, which can be found in any literature referred above.

**Definition 3.12.** A distortion function is defined as a non-decreasing function $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

The distortion risk measure associated with distortion function $g$ and capacity $\mu$ is denoted by $g \circ \mu(\cdot)$ and is defined as
$$g \circ \mu(X) = \int_{-\infty}^{0} (g[\mu(X > x)] - 1)dx + \int_{0}^{+\infty} g[\mu(X > x)]dx, \forall X \in L^\infty.$$

**Proposition 3.13.** Let $\mu$ be a capacity. For two losses $X, Y \in L^\infty$,

(i) $Y \lesssim_{mon} X$ if and only if $g \circ \mu(X) \geq g \circ \mu(Y)$ for all distortion functions $g$.

(ii) $Y \lesssim_{ic} X$ if and only if $g \circ \mu(X) \geq g \circ \mu(Y)$ for all concave distortion functions $g$.

**Proof.** (i) The “$\Rightarrow$” implication follows immediately from $G_{\mu, X}(x) \geq G_{\mu, Y}(x)$ and the non-decreasing property of any distortion function.

“$\Leftarrow$” For $\lambda \in (0, 1)$, let distortion function $g$ be defined by $g(x) = I_{x > \lambda}, x \in [0, 1]$. By the translation invariance of $\text{VaR}_{\mu, \lambda}(\cdot)$ and $g \circ \mu(\cdot)$, we may assume without loss of generality that $X \geq 0$, then by (16), we have
$$g[\mu(X > x)] = \begin{cases} 1, & \text{if } \text{VaR}_{\mu, \lambda}(X) > x, \\ 0, & \text{if } \text{VaR}_{\mu, \lambda}(X) \leq x. \end{cases}$$

Hence, $g \circ \mu(X) = \text{VaR}_{\mu, \lambda}(X)$. Then by Proposition 3.10(iii), we have $Y \lesssim_{mon} X$. 

(ii) “$\Rightarrow$” For any continuous distortion function $g$, by Fubini’s theorem, we get that
\[
g \circ \mu(X) = \int_{0}^{1} VaR_{\mu,t}(X) dg(t).
\]
For $\lambda \in (0, 1]$, taking
\[
g(x) = \min\left(\frac{x}{\lambda}, 1\right), \quad x \in [0, 1],
\]
we see that $g$ is a continuous concave distortion function. Furthermore, we find that
\[
g \circ \mu(x) = \frac{1}{\lambda} \int_{0}^{1} VaR_{\mu,t}(X) dt = AVaR_{\mu,\lambda}(X).
\]
Then by Proposition 3.10, we have $Y \preceq_{icon} X$.

“$\Leftarrow$” Any concave distortion functions (the concave distortion function may be not continuous at the point 0). Without loss of generality, we only need to show $g \circ \mu(X) \geq g \circ \mu(Y)$ for all continuous concave distortion functions $g$.

First we prove it holds for continuous concave piecewise linear distortion functions. As [22] constructed, any such function can be written as
\[
g(x) = \sum_{i=1}^{n} a_i (\beta_i - \beta_{i+1}) \min(x/\alpha_i, 1),
\]
where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = 1$ and $\beta_1 > \beta_2 > \cdots > \beta_n > \beta_{n+1} = 0$. By (17) and (18), we have that $g \circ \mu(\cdot)$ can be represented by
\[
g \circ \mu(\cdot) = \sum_{i=1}^{n} a_i (\beta_i - \beta_{i+1}) AVaR_{\mu,\alpha_i}(\cdot).
\]
By Proposition 3.10, we thus obtain $g \circ \mu(X) \geq g \circ \mu(Y)$ for all continuous concave piecewise linear distortion functions $g$.

For any continuous concave distortion function $g$, there exists an increasing sequence of continuous concave piecewise linear distortion functions $g_n$ such that $\lim_{n \to \infty} g_n(x) = g(x)$ for all $x \in [0, 1]$. Since $g_n \circ \mu(X) \geq g_n \circ \mu(Y)$ for all $n$, by monotone convergence theorem, we have $g \circ \mu(X) \geq g \circ \mu(Y)$. The proof is complete.

**Proposition 3.14.** If $\mu$ is a sub modular capacity, then for any concave distortion function $g$, $g \circ \mu(\cdot)$ is a coherent risk measure on $L^\infty$. In particular, for $\lambda \in (0, 1]$, $AVaR_{\mu,\lambda}(\cdot)$ is a coherent risk measure.

**Proof.** It is known that, for any sub modular capacity $\nu$, $\nu(\cdot)$ is a coherent risk measure on $L^\infty$ (see [12], Theorem 4.88). So the point is to show that
\[
\nu(A) := g(\mu(A)), \quad \forall A \in \mathcal{F},
\]
is a sub modular capacity.

Suppose that $g$ is concave. Take $A, B \in \mathcal{F}$ with $\mu(A) \leq \mu(B)$. We show that
\[
g(\mu(A)) - g(\mu(A \cap B)) \leq g(\mu(A \cup B)) - g(\mu(B)).
\]
It is trivial if $r = 0$, where
\[
\mu(A) - \mu(A \cap B) \geq \mu(A \cup B) - \mu(B) := r.
\]
For $r = 0$, the concavity of $g$ yields that
\[
\frac{\nu(A) - \nu(A \cap B)}{\mu(A) - \mu(A \cap B)} \geq \frac{\nu(A \cup B) - \nu(B)}{\mu(A \cup B) - \mu(B)}.
\]
Multiplying both sides with $(\mu(A) - \mu(A \cap B))$ derives the result.

Let $g$ be defined by (17), we then find that $AVaR_{\mu,\lambda}(\cdot) = g \circ \mu(\cdot)$ for a continuous concave function. Hence $AVaR_{\mu,\lambda}(\cdot)$ is a coherent risk measure. The proof is complete.
3.2 Characterizations from the probability measures viewpoint

Given a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\). In this subsection, let \(\Omega\) be a complete separable metric space equipped with the distance \(d\), \(B(\Omega)\) the Borel \(\sigma\)-algebra of \(\Omega\) and \(\mathcal{M}\) the collection of all probability measures on \((\Omega, B(\Omega))\). Let \(B_p(\Omega)\) be the all bounded functions of \(B(\Omega)\)-measurable real functions, and we assume that \(\mathcal{H} = B_p(\Omega)\). For the sublinear operator \(\mathbb{E}\), we assume that there exists a family of probability measures \(P \subseteq \mathcal{P}\) such that

\[
\mathbb{E}[X] = \sup_{Q \in \mathcal{P}} E_Q[X], \quad \forall X \in \mathcal{H}.
\]

And we assume that the induced set function \(\mu = \mathbb{E}[I_A] = \sup_{Q \in \mathcal{P}} E_Q[I_A] = \sup_{Q \in \mathcal{P}} Q(A), \quad \forall A \in B(\Omega),\)

is a capacity. Thus \((\Omega, B(\Omega), \mu)\) become a capacity space.

Kervarec defined the robust VaR and AVaR using a family of probability measures in [23]. [24] also introduced the similar notations under a family of absolutely continuous probability measures. For any \(X \in \mathcal{H}, \lambda \in (0, 1)\), we define

\[
\text{VaR}_{P,\lambda}(X) := \sup_{Q \in \mathcal{P}} \text{VaR}_Q(X), \quad \text{AVaR}_{P,\lambda}(X) := \sup_{Q \in \mathcal{P}} \text{AVaR}_Q(X),
\]

where \(\text{VaR}_Q(X) = \inf\{x \mid Q(X > x) \leq \lambda\}\) and \(\text{AVaR}_Q(X) = \frac{1}{\lambda} \int_0^\lambda \text{VaR}_Q(t) dt\) are the classical definitions (see Pages 177-179 in [12]).

**Remark 3.15.** It can be verified that if \(\mu\) is the supremum of a family of probability measures, then for all \(X \in \mathcal{H}\) and \(\lambda \in (0, 1)\) the following

\[
\text{VaR}_{\mathcal{P},\lambda}(X) = \text{VaR}_{\mu,\lambda}(X), \quad \text{AVaR}_{\mathcal{P},\lambda}(X) = \text{AVaR}_{\mu,\lambda}(X)
\]

hold.

In the case of AVaR under a given probability measure \(P\), we have known that \(\text{AVaR}_{\mathcal{P},\lambda}(-)\) is a coherent risk measure (see [12], Theorem 4.47). For coherent risk measures, we refer to [1] and [2]. From Remark 3.15, we can see that \(\text{AVaR}_{\mu,\lambda}(-)\) is coherent risk measure. And from Theorem 4.47 in [12], we can obtain that for any \(\lambda \in (0, 1), X \in \mathcal{H},\)

\[
\text{AVaR}_{\mathcal{P},\lambda}(X) = \sup_{Q \in \mathcal{P}} \max_{R \in \mathcal{Q}_\lambda} E_R[X],
\]

where

\[
\mathcal{Q}_\lambda = \{R \in \mathcal{M} | R \ll Q \text{ and } \frac{dR}{dQ} \leq \frac{1}{\lambda}, Q \text{ - a.s.}\}.
\]

Finally, from the characterizations results of uncertainty orders in Propositions 3.10-3.13, we conclude the following theorems.

**Theorem 3.16.** Let \(\mu\) be a capacity induced by a family of probability measures \(\mathcal{P}\), which is determined by the sublinear expectation \(\mathbb{E}\). Let \(X, Y \in \mathcal{H}\). The following statements are equivalent.

(i) \(\mu(\phi(X)) \geq \mu(\phi(Y))\) for all increasing and convex function \(\phi : \mathbb{R} \to \mathbb{R}\).

(ii) \(\mu((X-b)^+) \geq \mu((Y-b)^+)\) for any \(b \in \mathbb{R}\).

(iii) \(\text{AVaR}_{\mathcal{P},\lambda}(X) \geq \text{AVaR}_{\mathcal{P},\lambda}(Y)\), for any \(\lambda \in (0, 1)\).

(iv) \(g \circ \mu(X) \geq g \circ \mu(Y)\) for all concave distortion functions \(g\).

**Theorem 3.17.** Let \(\mu\) be a capacity induced by a family of probability measures \(\mathcal{P}\), which is determined by the sublinear expectation \(\mathbb{E}\). Let \(X, Y \in \mathcal{H}\). The following statements are equivalent.
4 Conclusions

This paper considers the uncertainty orders on the sublinear expectation space from two different viewpoints. It is worth noting that the sublinear expectation does not equal to the Choquet integral generally, where the capacity is induced by sublinear expectation. The readers can refer to [25] and [26] for more details. We only consider some special distributions in the first viewpoint, and other plausible formulation leaves to a future study.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgement: The authors are very grateful to the editor and an anonymous referee for several constructive and insightful comments on how to improve the paper. This work was supported by the National Natural Science Foundation of China (11371362), the Natural Science Foundation of Jiangsu Province (BK20150167).

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