AN HDG METHOD FOR LINEAR ELASTICITY WITH STRONG SYMMETRIC STRESSES

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Abstract. This paper presents a new hybridizable discontinuous Galerkin (HDG) method for linear elasticity on general polyhedral meshes, based on a strong symmetric stress formulation. The key feature of this new HDG method is the use of a special form of the numerical trace of the stresses, which makes the error analysis different from the projection-based error analysis used for most other HDG methods. For arbitrary polyhedral elements, we approximate the stress by using polynomials of degree \( k \geq 1 \) and the displacement by using polynomials of degree \( k + 1 \). In contrast, to approximate the numerical trace of the displacement on the faces, we use polynomials of degree \( k \) only. This allows for a very efficient implementation of the method, since the numerical trace of the displacement is the only globally-coupled unknown, but does not degrade the convergence properties of the method. Indeed, we prove optimal orders of convergence for both the stresses and displacements on the elements. These optimal results are possible thanks to a special superconvergence property of the numerical traces of the displacement, and thanks to the use of a crucial elementwise Korn’s inequality.

1. Introduction

In this paper, we introduce a new hybridizable discontinuous Galerkin (HDG) method for the system of linear elasticity

\[
\begin{align*}
A\sigma - \epsilon(u) &= 0 & &\text{in } \Omega \subset \mathbb{R}^3, \\
\nabla \cdot \sigma &= f & &\text{in } \Omega, \\
\n\sigma &= g & &\text{on } \partial \Omega,
\end{align*}
\]

(1.1a)

(1.1b)

(1.1c)

Here, the displacement is denoted by the vector field \( u : \Omega \mapsto \mathbb{R}^3 \). The strain tensor is represented by \( \epsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top) \). The stress tensor is represented by \( \sigma : \Omega \mapsto S \), where \( S \) denotes the set of all symmetric matrices in \( \mathbb{R}^{3 \times 3} \). The compliance tensor \( A \) is assumed to be a bounded, symmetric, positive definite tensor over \( S \). The body force \( f \) lies in \( L^2(\Omega) \), the displacement of the boundary \( g \) is a function in \( H^{1/2}(\partial \Omega) \) and \( \Omega \) is a polyhedral domain.

In general, there are two approaches to design mixed finite element methods for linear elasticity. The first approach is to enforce the symmetry of the stress tensor weakly (\([4, 5, 10, 15, 23, 26, 29, 30, 33]\)). In this category, is included the HDG method considered in \([20]\). The other approach is to exactly enforce the symmetry of the approximate stresses. The methods considered in \([19, 11, 2, 8, 17, 8, 24, 28, 32, 34, 35]\) belong to the second category, and so does the contribution of this paper. In general, the methods in the first category are easier to implement. On the other hand, the methods in the second category preserve the
balance of angular momentum strongly and have less degrees of freedom. Next, we compare our HDG method with several methods of the second category.

In [19], an LDG method using strongly symmetric stresses (for isotropic linear elasticity) was introduced and proved to yield convergence properties that remain unchanged when the material becomes incompressible; simplexes and polynomial approximations or degree k in all variables were used. However, as all LDG methods for second-order elliptic problems, although the displacement converges with order $k + 1$, the strain and pressure converge sub-optimally with order $k$. Also, the method cannot be hybridized. Stress finite elements satisfying both strong symmetry and $H(\text{div})$-conformity are introduced in [11, 2]. The main drawback of these methods is that they have too many degrees of freedom of stress elements and hybridization is not available for them (see detailed description in [26]). In [3, 7, 8, 24, 28, 31, 32, 34, 35], non-conforming methods using symmetric stress elements are introduced. But, methods in [3, 7, 8, 28, 31, 32, 34, 35] use low order finite element spaces only (most of them are restricted to rectangular or cubical meshes except [3, 7]). In [24], a family of simplicial elements (one for each $k \geq 1$) are developed in both two and three dimensions. (The degrees of freedom of $P_{k+1}(S, K)$ were studied in [24] and then used to design the projection operator $\Pi^{(\text{div}, S)}$ in [25]). However, the convergence rate of stress is suboptimal. The first HDG method for linear and nonlinear elasticity was introduced in [31, 32]; see also the related HDG method proposed in [36]. These methods also use simplexes and polynomial approximations of degree $k$ in all variables. For general polyhedral elements, this method was recently analyzed in [24] where it was shown that the method converges optimally in the displacement with order $k + 1$, but with the suboptimal order of $k + 1/2$ for the pressure and the stress. For $k = 1$, these orders of convergence were numerically shown to be sharp for triangular elements. In this paper, we prove that by enriching the local stress space to be polynomials of degree no more than $k + 1$, and by using a modified numerical trace, we are able to obtain optimal order of convergence for all unknowns. In addition, this analysis is valid for general polyhedral meshes. To the best of our knowledge, this is so far the only result which has optimal accuracy with general polyhedral triangulations for linear elasticity problems.

Our HDG method provides approximation to stress and displacement in each element and trace of displacement along interfaces of meshes. The corresponding finite element spaces are $V_h, W_h, M_h$, which are defined to be

$$V_h = \{ v \in L^2(S, \Omega) : v|_K \in P_k(S, K) \quad \forall K \in \mathcal{T}_h \};$$

$$W_h = \{ \omega \in L^2(\Omega) : \omega|_K \in P_{k+1}(K) \quad \forall K \in \mathcal{T}_h \};$$

$$M_h = \{ \mu \in L^2(\mathcal{E}_h) : \mu|_F \in P_k(F) \quad \forall F \in \mathcal{E}_h \}.$$  

Here, $\mathcal{T}_h$ is any conforming polyhedral triangulation of $\Omega$, $\mathcal{E}_h$ is the set of all faces $F$ of all elements $K \in \mathcal{T}_h$, and $k \geq 1$. The space of vector-valued functions defined on $D$ whose entries are polynomials of total degree $k$ is denoted by $P_k(D)$. Similarly, $P_k(S, K)$ denotes the space of symmetric-valued functions defined on $K$ whose entries are polynomials of total degree $k$.

Note the fact that the only globally-coupled degrees of freedom are those of the numerical trace of displacement along $\mathcal{E}_h$, renders the method efficiently implementable. However, the fact that the polynomial degree of the approximate numerical traces of the displacement is one less than that of the approximate displacement inside the elements, might cause a degradation in the approximation properties of the displacement. However, this unpleasant
situation is avoided altogether by taking a special form of the numerical trace of the stresses inspired on the choice taken in \cite{27} in the framework of diffusion problems. This choice allows for a special superconvergence of part of the numerical traces of the stresses which, in turn, guarantees that, for \( k \geq 1 \), the \( L^2 \)-order of convergence for the stress is \( k + 1 \) and that of the displacement \( k + 2 \). So, we obtain optimal convergence for both stress and displacement for general polyhedral elements. Let us mention that the approach of error analysis of our HDG method is different from the traditional projection-based error analysis in \cite{17, 18, 20} in three aspects. First, here, we use simple \( L^2 \)-projections, not the numerical trace-tailored projections typically used for the analysis of other HDG methods. Second, we take the stabilization parameter to be of order \( 1/h \) instead of of order one. And finally, we use an elementwise Korn’s inequality (Lemma \ref{lem:elementwise_korn}) to deal with the symmetry of the stresses.

We notice that mixed methods in \cite{15, 23} and HDG methods in \cite{20} also achieve optimal convergence for stress and superconvergence for displacement by post processing. However, there are two disadvantages regarding of implementation. First, these methods enforce the stress symmetry weakly, which means that they have a much larger space for the stress. In addition, these methods usually need to add matrix bubble functions (\( \delta V \) in \cite{15}) into their stress elements in order to obtain optimal approximations. In fact, the construction of such bubbles on general polyhedral elements is still an open problem. In contrast, our method avoids using matrix bubble functions but only use simple polynomial space of degree \( k, k+1 \).

In Table \ref{tab:orders_of_convergence} we compare methods which use \( \bar{\mathbf{u}}_h \) for approximating trace of displacement \( \mathbf{u}_h \) on \( \mathcal{E}_h \). There, \( \mathbf{u}_h^\star \) is a post-processed numerical solution of displacement, and \( M \) is the set of all matrices in \( \mathbb{R}^{3 \times 3} \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\text{method} & \text{\( \mathbf{V}(K) \)} & \text{\( \mathbf{W}(K) \)} & \text{\( \| \mathbf{\sigma} - \mathbf{\sigma}_h \|_{\mathcal{T}_h} \)} & \text{\( \| \mathbf{u} - \mathbf{u}_h \|_{\mathcal{T}_h} \)} & \text{\( \| \mathbf{u} - \mathbf{u}_h^\star \|_{\mathcal{T}_h} \)} \\
\hline
AFW \cite{5} & \( P_k(\mathbb{R}^{3 \times 3}, K) \) & \( P_{k-1}(K) \) & \( k \) & \( k \) & \( - \) \\
CGG \cite{15} & \( RT_k(K) + \delta V \) & \( P_{k}(K) \) & \( k + 1 \) & \( k + 1 \) & \( k + 2 \) \\
GC \cite{23} & \( P_k(\mathbb{R}^{3 \times 3}, K) + \delta V \) & \( P_{k-1}(K) \) & \( k + 1 \) & \( k \) & \( k + 1 \) \\
CS \cite{20} & \( P_k(\mathbb{R}^{3 \times 3}, K) + \delta V \) & \( P_{k}(K) \) & \( k + 1 \) & \( k + 1 \) & \( k + 2 \) \\
GG \cite{21} & \( P_{k+1}(S, K) \) & \( P_{k}(K) \) & \( k \) & \( k + 1 \) & \( - \) \\
HDG-S & \( P_{k}(S, K) \) & \( P_{k+1}(K) \) & \( k + 1 \) & \( k + 2 \) & \( - \) \\
\hline
\end{tabular}
\caption{Orders of convergence for methods for which \( \bar{\mathbf{u}}_h \in M(F) = P_k(F), k \geq 1 \), and \( K \) is a tetrahedron.}
\end{table}

The remainder of this paper is organized as follows. In section 2, we introduce our HDG method and present our a priori error estimates. In section 3, we give a characterization of the HDG method and show the global matrix is symmetric and positive definite. In section 4, we give elementwise Korn’s inequality in Lemma \ref{lem:elementwise_korn} then provide a detailed proof of the a priori error estimates.

2. Main results

In this section we first present the method in details and then show the main results for the error estimates.

2.1. The HDG formulation with strong symmetry. Let us begin by introducing some notations and conventions. We adapt to our setting the notation used in \cite{18}. Let \( \mathcal{T}_h \) denote
a conforming triangulation of $\Omega$ made of shape-regular polyhedral elements $K$. We recall that $\partial \mathcal{T}_h := \{ \partial K : K \in \mathcal{T}_h \}$, and $\mathcal{E}_h$ denotes the set of all faces $F$ of all elements. We denote by $\mathcal{F}(K)$ the set of all faces $F$ of the element $K$. We also use the standard notation to denote scalar, vector, and tensor spaces. Thus, if $D(K)$ denotes a space of scalar-valued functions defined on $K$, the corresponding space of vector-valued functions is $D(K) := [D(K)]^d$ and the corresponding space of matrix-valued functions is $\hat{D}(K) := [D(K)]^{d \times d}$. Finally, $\hat{D}(S; K)$ denotes the symmetric subspace of $\hat{D}(K)$.

The methods we consider seek an approximation $(\hat{\sigma}_h, u_h, \hat{u}_h)$ to the exact solution $(\sigma, u, u)$ in the finite dimensional space $\mathbf{V}_h \times \mathbf{W}_h \times M_h \subset L^2(\mathbf{S}; \Omega) \times L^2(\Omega) \times L^2(\mathcal{E}_h)$ given by

$$\begin{align*}
\mathbf{V}_h &= \{ \mathbf{v} \in L^2(\mathbf{S}; \Omega) : \mathbf{v}|_K \in P_k(\mathbf{S}, K) \quad \forall \ K \in \mathcal{T}_h \}, \\
\mathbf{W}_h &= \{ \omega \in L^2(\Omega) : \omega|_K \in P_{k+1}(K) \quad \forall \ K \in \mathcal{T}_h \}, \\
M_h &= \{ \mu \in L^2(\mathcal{E}_h) : \mu|_F \in P_k(F) \quad \forall \ F \in \mathcal{E}_h \}.
\end{align*}$$

(2.1a) (2.1b) (2.1c)

Here $P_k(D)$ denotes the standard space of polynomials of degree no more than $k$ on $D$. Here we require $k \geq 1$.

The numerical approximation $(\hat{\sigma}_h, u_h, \hat{u}_h)$ can now be defined as the solution of the following system:

$$\begin{align*}
(A\hat{\sigma}_h, \mathbf{v})_{\mathcal{T}_h} + (u_h, \nabla \cdot \mathbf{v})_{\mathcal{T}_h} - \langle \hat{u}_h, \mathbf{v} n \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\hat{\sigma}_h, \nabla \omega)_{\mathcal{T}_h} - \langle \hat{\sigma}_h n, \omega \rangle_{\partial \mathcal{T}_h} &= -(f, \omega)_{\mathcal{T}_h}, \\
\langle \hat{\sigma}_h n, \mu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} &= 0, \\
\langle \hat{u}_h, \mu \rangle_{\partial \Omega} &= (g, \mu)_{\partial \Omega}.
\end{align*}$$

(2.2a) (2.2b) (2.2c) (2.2d)

for all $(\mathbf{v}, \omega, \mu) \in \mathbf{V}_h \times \mathbf{W}_h \times M_h$, where

$$\hat{\sigma}_h n = \sigma_h n - \tau (P_M u_h - \hat{u}_h) \quad \text{on } \partial \mathcal{T}_h.$$  

(2.2e)

In fact, in Christoph Lehrenfeld’s thesis, the author defines the numerical flux in this way for diffusion problems (see Remark 1.2.4 in [27]). However, there is no corresponding error analysis for this numerical flux in [27]. Here, $P_M$ denotes the standard $L^2$-orthogonal projection from $L^2(\mathcal{E}_h)$ onto $M_h$. We write $(\eta, \zeta)_{\mathcal{T}_h} := \sum_{i,j=1}^n (\eta_{i,j} : \zeta_{i,j})_{\mathcal{T}_h}$, $(\eta, \zeta)_{\mathcal{T}_h} := \sum_{i=1}^n (\eta_i, \zeta_i)_{\mathcal{T}_h}$, and $(\eta, \zeta)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\eta, \zeta)_K$, where $(\eta, \zeta)_D$ denotes the integral of $\eta \zeta$ over $D \subset \mathbb{R}^n$. Similarly, we write $(\eta, \zeta)_{\partial \mathcal{T}_h} := \sum_{i=1}^n (\eta_i, \zeta_i)_{\partial \mathcal{T}_h}$ and $(\eta, \zeta)_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\eta, \zeta)_{\partial K}$, where $(\eta, \zeta)_D$ denotes the integral of $\eta \zeta$ over $D \subset \mathbb{R}^{n-1}$.

The parameter $\tau$ in (2.2e) is called the stabilization parameter. In this paper, we assume it is a fixed positive number on all faces. It is worth to mention that the numerical trace (2.2e) is defined slightly different from the usual HDG setting, see [13]. Namely, in the definition, we use $P_M u_h$ instead of $u_h$. Indeed, this is a crucial modification in order to get error estimate. An intuitive explanation is that we want to preserve the strong continuity of the flux across the interfaces. Without the projection $P_M$, by (2.2a) the normal component of $\hat{\sigma}_h$ is only weakly continuous across the interfaces.

2.2. A priori error estimates. To state our main result, we need to introduce some notations. We define

$$\| \mathbf{v} \|_{L^2(\mathbf{A}, \Omega)} = \sqrt{(A\mathbf{v}, \mathbf{v})_\Omega}, \quad \forall \mathbf{v} \in L^2(\mathbf{S}, \Omega).$$

We use $\| \cdot \|_{s,D}, | \cdot |_{s,D}$ to denote the usual norm and semi-norm on the Sobolev space $H^s(D)$. We discard the first index $s$ if $s = 0$. A differential operator with a sub-index $h$ means it is
defined on each element \( K \in \mathcal{T}_h \). Similarly, the norm \( \| \cdot \|_{s, \mathcal{T}_h} \) is the discrete norm defined as \( \| \cdot \|_{s, \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \| \cdot \|_{s, K} \). Finally, we need an elliptic regularity assumption stated as follows:

Let \( (\phi, \psi) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \) be the solution of the adjoint problem:

\[
\begin{align*}
A\psi - \epsilon(\phi) &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \psi &= e_u \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

We assume the solution \( (\phi, \psi) \) has the following elliptic regularity property:

\[
\| \psi \|_{1, \Omega} + \| \phi \|_{2, \Omega} \leq C_{\text{reg}} \| e_u \|_{\Omega},
\]

The assumption holds in the case of planar elasticity with scalar coefficients on a convex domain, see [9].

We are now ready to state our main result.

**Theorem 2.1.** If the meshes are quasi-uniform and \( \tau = O(\frac{1}{h}) \), then we have

\[
\| \sigma - \sigma_h \|_{L^2(\mathcal{A}, \Omega)} \leq C h^{s}(\| u \|_{s+1, \Omega} + \| \sigma \|_{s, \Omega}),
\]

for all \( 1 \leq s \leq k + 1 \). Moreover, if the elliptic regularity property (2.4) holds, then we have

\[
\| u - u_h \|_{\Omega} \leq C h^{s+1}(\| u \|_{s+1, \Omega} + \| \sigma \|_{s, \Omega}),
\]

for all \( 1 \leq s \leq k + 1 \). Here the constant \( C \) depends on the upper bound of compliance tensor \( \mathcal{A} \) but it is independent of the mesh size \( h \).

This result shows that the numerical errors for both unknowns \((u, \sigma)\) are optimal. In addition, since the only globally-coupled unknown, \( \tilde{u}_h \), stays in \( P_k(\mathcal{E}_h) \), the order of convergence for the displacement remains optimal only because of a key superconvergence property, see the remark right after Corollary 4.2. In addition, we restrict our result on quasi-uniform meshes to make the proof simple and clear. This result holds for shape-regular meshes also.

2.3. **Numerical approximation for nearly incompressible materials.** Here, we consider the numerical approximation of stress for isotropic nearly incompressible materials.

We define isotropic materials to be those whose compliance tensor satisfying the following Assumption 2.1.

**Assumption 2.1.**

\[
\mathcal{A}\tau = P_D \tau_D + P_T \frac{\text{tr}(\tau)}{3} I_3
\]

where \( \tau_D = \tau - \frac{\text{tr}(\tau)}{3} I_3 \),

for any \( \tau \in \mathbb{R}^{3 \times 3} \), and \( P_D \) and \( P_T \) are two positive constants.

An isotropic material is nearly incompressible if \( P_T \) is close to zero.

**Theorem 2.2.** If the material is isotropic (whose compliance tensor satisfies Assumption 2.1), \( P_T \) is positive, the boundary data \( g = 0 \), the meshes are quasi-uniform and \( \tau = O(\frac{1}{h}) \), then we have

\[
\| \sigma - \sigma_h \|_{L^2(\Omega)} \leq C h^{s-1}(\| u \|_{s+1, \Omega} + \| \sigma \|_{s, \Omega}),
\]

for all \( 1 \leq s \leq k + 1 \). Here, the constant \( C \) is independent of \( P_T^{-1} \).
This result shows that the HDG method (2.2) is locking-free for nearly incompressible materials. We emphasize that the convergence rate of stress for nearly incompressible materials is the same as [5, 24] with the same finite element space for numerical trace of displacement.

3. A characterization of the HDG method

In this section we show how to eliminate elementwise the unknowns \( \sigma_h \) and \( u_h \) from the equations (2.2) and rewrite the original system solely in terms of the unknown \( \tilde{u}_h \), see also [32]. Via this elimination, we do not have to deal with the large indefinite linear system generated by (2.2), but with the inversion of a sparser symmetric positive definite matrix of remarkably smaller size.

3.1. The local problems. The result on the above mentioned elimination can be described using additional “local” operators defined as follows:

On each element \( K \), for any \( \lambda \in M_h|\partial K \), we denote \((Q\lambda, U\lambda) \in V(K) \times W(K)\) to be the unique solution of the local problem:

\[
\begin{align*}
\langle AQ\lambda, v \rangle_K + \langle U\lambda, \nabla \cdot v \rangle_K &= \langle \lambda, \tau \cdot v \rangle_{\partial K}, \\
-(\nabla \cdot Q\lambda, \omega)_{\partial K} + \langle \tau P_M U\lambda, \omega \rangle_{\partial K} &= \langle \tau \lambda, \omega \rangle_{\partial K},
\end{align*}
\]

(3.1)

for all \((v, \omega) \in V(K) \times W(K)\).

On each element \( K \), we also denote \((Q_S\lambda, U_S\lambda) \in V(K) \times W(K)\) to be the unique solution of the local problem:

\[
\begin{align*}
\langle AQ_S f, v \rangle_K + \langle U_S f, \nabla \cdot v \rangle_K &= 0, \\
-(\nabla \cdot Q_S f, \omega)_{\partial K} + \langle \tau P_M U_S f, \omega \rangle_{\partial K} &= -(f, \omega)_K,
\end{align*}
\]

(3.2)

for all \((v, \omega) \in V(K) \times W(K)\). Here, \( V(K) = V_h|_K \) and \( W(K) = W_h|_K \).

It is easy to show the two local problems are well-posed. In addition, due to the linearity of the global system (2.2), the numerical solution \((\sigma_h, u_h, \tilde{u}_h)\) satisfies

\[
\sigma_h = Q\tilde{u}_h + Q_S f, \quad u_h = U\tilde{u}_h + U_S f.
\]

(3.3)

3.2. The global problem. For the sake of simplicity, we assume the boundary data \( g = 0 \). Then, the HDG method (2.2) is to find \((\sigma_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M^0_h\) satisfying

\[
\begin{align*}
\langle A\sigma_h, v \rangle_T + \langle u_h, \nabla \cdot v \rangle_T - \langle \tilde{u}_h, \tau v \rangle_{\partial T} &= 0, \\
-(\nabla \cdot \sigma_h, \omega)_T + \langle \tau (P_M u_h - \tilde{u}_h), \omega \rangle_{\partial T} &= -(f, \omega)_T, \\
\langle \sigma_h n - \tau (P_M u_h - \tilde{u}_h), \mu \rangle_{\partial T \setminus \partial \Omega} &= 0,
\end{align*}
\]

(3.4)

for all \((v, \omega, \mu) \in V_h \times W_h \times M^0_h\), where \( M^0_h = \{ \mu \in M_h : \mu|_{\partial \Omega} = 0 \} \).

Combining (3.4) with (3.3), we have that for all \( \mu \in M^0_h \),

\[
\langle (Q\tilde{u}_h) n - \tau (P_M U\tilde{u}_h - \tilde{u}_h), \mu \rangle_{\partial T} = \langle (Q_S f) n - \tau P_M U_S f, \mu \rangle_{\partial T}.
\]

(3.5)

Up to now we can see that we only need to solve the reduced global linear system (3.5) first, then recover \((\sigma_h, u_h)\) by (3.3) element by element. Next we show that the global system (3.5) is in fact symmetric positive definite.
3.3. A characterization of the approximate solution. The above results suggest the following characterization of the numerical solution of the HDG method.

**Theorem 3.1.** The numerical solution of the HDG method \((2.2)\) satisfies

\[
\boldsymbol{\sigma}_h = Q\hat{\mathbf{u}}_h + Q_s f, \quad \mathbf{u}_h = U\hat{\mathbf{u}}_h + U_s f.
\]

If we assume the boundary data \(g = 0\), then \(\hat{\mathbf{u}}_h \in M_h^0\) is the solution of

\[
a_h(\hat{\mathbf{u}}_h, \mu) = \langle (Q_s f)n - \tau P_M U_s f, \mu \rangle_{\partial T_h}, \quad \forall \mu \in M_h^0,
\]

where

\[
a_h(\hat{\mathbf{u}}_h, \mu) = (A Q \hat{\mathbf{u}}_h, Q \mu)_{T_h} + \langle \tau (P_M U \hat{\mathbf{u}}_h - \hat{\mathbf{u}}_h), P_M \mu - \mu \rangle_{T_h}.
\]

In addition, the bilinear operator \(a_h(\lambda, \lambda)\) is positive definite.

**Proof.** In order to show \((3.6)\) is true, we only need to show that for all \(\lambda, \mu \in M_h^0\), then

\[
a_h(\lambda, \mu) = \langle (Q \lambda)n - \tau (P_M U \lambda - \lambda), \mu \rangle_{\partial T_h}.
\]

According to \((3.1)\), we have

\[
\begin{align}
(A Q \mathbf{m}, \mathbf{v})_{T_h} + (U \mathbf{m}, \nabla \cdot \mathbf{v})_{T_h} &= \langle \mathbf{m}, \mathbf{v} \cdot n \rangle_{\partial T_h}, \quad (3.7a) \\
(\nabla \cdot Q \mathbf{m}, \omega)_{T_h} &= \langle \tau (P_M U \mathbf{m} - \mathbf{m}), \omega \rangle_{\partial T_h}, \quad (3.7b)
\end{align}
\]

for all \((\mathbf{m}, \omega) \in \mathbf{V}_h \times W_h, \mathbf{m} \in M_h^0\). Then, we have

\[
\begin{align}
\langle (Q \lambda)n - \tau (P_M U \lambda - \lambda), \mu \rangle_{\partial T_h} &= \langle \mu, (Q \lambda)n \rangle_{\partial T_h} - \langle \tau (P_M U \lambda - \lambda), \mu \rangle_{\partial T_h} \\
&= (A Q \mu, Q \lambda)_{T_h} + (U \mu, \nabla \cdot Q \lambda)_{T_h} - \langle \tau (P_M U \lambda - \lambda), \mu \rangle_{\partial T_h} \\
&= (A Q \mu, Q \lambda)_{T_h} + (\nabla \cdot Q \lambda, U \mu)_{T_h} - \langle \tau (P_M U \lambda - \lambda), \mu \rangle_{\partial T_h} \\
&= (A Q \mu, Q \lambda)_{T_h} + \langle \tau (P_M U \lambda - \lambda), P_M \mu - \mu \rangle_{\partial T_h} \quad \text{by \((3.7a)\)} \\
&= a_h(\lambda, \mu).
\end{align}
\]

So, we can conclude that \((3.6)\) holds. We end the proof by showing the bilinear operator \(a_h(\cdot, \cdot)\) is positive definite.

If \(a_h(\lambda, \lambda) = 0\) for some \(\lambda \in M_h^0\), from the previous result we have

\[
Q \lambda = 0, \quad P_M U \lambda - \lambda|_{\partial T_h} = 0.
\]

We apply integration by parts on \((3.1)\), we have

\[
\langle \xi(U \lambda), \mathbf{v} \rangle_{\partial K} = 0, \quad \forall \quad \mathbf{v} \in \mathbf{V}(K).
\]

This implies that \(\xi(U \lambda)|_K = 0\) for all \(K \in T_h\). So, for any \(K \in T_h\), there are \(a_K, b_K \in \mathbb{R}^3\) such that \(U \lambda|_K = a_K \times x + b_K\). Since \(k \geq 1\), we have \(P_M U \lambda = U \lambda\). Combining this result with the fact that \(P_M U \lambda - \lambda|_{\partial T_h} = 0\) and \(\lambda|_{\partial \Omega} = 0\), we can conclude that \(U \lambda \in C^0(\Omega)\) and \(U \lambda|_{\partial \Omega} = 0\).

Finally, let us consider two adjacent element \(K_1, K_2\) with the interface \(F = K_1 \cap K_2\). In addition, we assume that on \(K_i, U \lambda\) can be expressed as

\[
U \lambda = a_i \times x + b_i, \quad i = 1, 2.
\]
We claim that \( a_1 = a_2 \) and \( b_1 = b_2 \). This fact can be shown by considering the continuity of the function on the interface \( F \). We omit the detailed proof since it only involves elementary linear algebra.

From this result we conclude that there exist \( a, b \in \mathbb{R}^3 \) such that \( U\lambda = a \times x + b \) in \( \Omega \). By the fact that \( U\lambda|_{\partial\Omega} = 0 \), we can conclude that \( U\lambda = 0 \), hence \( \lambda = 0 \). This completes the proof.

\[ \square \]

**Remark 3.2.** In Theorem 3.1 we assume the boundary data \( g = 0 \). Actually, if \( g \) is not zero, we can still obtain the same linear system as \( a_h \) in Theorem 3.1 by the same treatment of boundary data in [14].

4. Error Analysis

In this section we provide detailed proofs for our a priori error estimates - Theorem 2.1 and Theorem 2.2. We use elementwise Korn’s inequality (Lemma 4.1), which is novel and crucial in error analysis. We use \( \Pi_V, \Pi_W \) to denote the standard \( L^2 \)-orthogonal projection onto \( \mathcal{V}_h, \mathcal{W}_h \) respectively. In addition, we denote

\[
eq \Pi_V \sigma - \sigma_h, \quad e_u = \Pi_W u - u_h, \quad e_\hat{u} = P_M u - \hat{u}_h,
\]

In the analysis, we are going to use the following classical results:

\[
\begin{align*}
\| u - \Pi_W u \|_{\Omega} &\leq Ch^s \| u \|_{s, \Omega}, & 0 \leq s \leq k + 2, \\
\| \sigma - \Pi_V \sigma \|_{t, \Omega} &\leq Ch^t \| \sigma \|_{t, \Omega}, & 0 \leq t \leq k + 1, \\
\| u - P_M u \|_{\hat{e}_h} &\leq Ch^{s - \frac{1}{2}} \| u \|_{s, \Omega}, & 1 \leq s \leq k + 2, \\
\| u - \Pi_W u \|_{\partial K} &\leq Ch^{s - \frac{1}{2}} \| u \|_{s, K}, & 1 \leq s \leq k + 2, \\
\| \sigma n - \Pi_V \sigma n \|_{\partial K} &\leq Ch^{t - \frac{1}{2}} \| \sigma \|_{t, K}, & 1 \leq t \leq k + 1, \\
\| v \|_{\partial K} &\leq Ch^{-\frac{1}{2}} \| v \|_{K}, & \forall v \in P_s(K), \\
\| \sigma n - P_M(\sigma n) \|_{\partial K} &\leq Ch^{t - \frac{1}{2}} \| \sigma \|_{t, K}, & 1 \leq t \leq k + 1.
\end{align*}
\]

The above results are due to standard approximation theory of polynomials, trace inequality.

Let \( \mathcal{E}_h \) denote the discrete symmetric gradient operator, such that for any \( K \in \mathcal{T}_h, \mathcal{E}_h|_K = \mathcal{E}|_K \). It is well known (see Theorem 2.2 in [12]) the kernel of the operator \( \mathcal{E}(\cdot)|_h \) is:

\[
\ker(\mathcal{E}_h) = Y_h := \{ \Lambda \in L^2(\Omega), \Lambda|_K = B_K x + b_K, \forall B_K \in \mathcal{A}, b_K \in \mathbb{R}^3, K \in \mathcal{T}_h \}.
\]

Here, \( \mathcal{A} \) denotes the set of all anti-symmetric matrices in \( \mathbb{R}^{3 \times 3} \).

In the analysis, we need the following elementwise Korn’s inequality:

**Lemma 4.1.** Let \( K \in \mathcal{T}_h \) be a generic element with size \( h_K \) and \( Y(K) := Y_h|_K \). Then for any function \( v \in \mathcal{W}_h|_K \), we have

\[
\inf_{\Lambda \in Y(K)} \| \nabla(v + \Lambda) \|_K \leq C \| \mathcal{E}(v) \|_K,
\]

Here \( C \) is independent of the size \( h_K \). In addition, if \( K \) is a tetrahedron, the above inequality holds for any \( v \in H^1(K) \).

**Proof.** Let \( \hat{K} \) denote the reference tetrahedron element and \( v \in H^1(K) \). The mapping from \( \hat{K} \) to \( K \) is \( x = A_K \hat{x} + c_K \) where \( A_K \) is a non-singular matrix and \( c_K \in \mathbb{R}^3 \).
We define \( \hat{\mathbf{v}} \), which is the pull back of \( \mathbf{v} \) on \( \hat{K} \), by
\[
\mathbf{A}_K^{-T} \hat{\mathbf{v}}(\hat{x}) = \mathbf{v}(x) \quad \forall \mathbf{x} \in \hat{K}.
\]
Obviously, we have
\[
\mathbf{A}_K^{-T} \hat{\mathbf{v}}(\hat{x}) \mathbf{A}_K^{-1} = \nabla \mathbf{v}(x), \quad \mathbf{A}_K^{-T} \hat{\mathbf{v}}(\hat{x}) \mathbf{A}_K^{-1} = \mathbf{\epsilon}(\mathbf{v}).
\]
(4.2)

According to Theorem 2.3 in [12], the following inequality holds:
\[
\inf_{\hat{\Lambda} \in \Upsilon(\hat{K})} \| \hat{\mathbf{v}} + \hat{\Lambda} \|_{1,\hat{K}} \leq C \| \hat{\mathbf{\epsilon}}(\hat{\mathbf{v}}) \|_{0,\hat{K}}.
\]

So, there is \( \hat{\Lambda} = \mathbf{B}_K \mathbf{x} + \mathbf{b}_K \) with \( \mathbf{B}_K \in \mathbf{A} \) and \( \mathbf{b}_K \in \mathbb{R}^3, K \in \mathcal{T}_h, \) such that
\[
\| \hat{\mathbf{v}}(\hat{x} + \hat{\Lambda}) \|_{0,\hat{K}} \leq C \| \hat{\mathbf{\epsilon}}(\hat{\mathbf{v}}) \|_{0,\hat{K}}.
\]
(4.3)

We define
\[
\Lambda(x) = \mathbf{A}_K^{-T} \hat{\Lambda}(\hat{x}) \quad \forall x \in K.
\]

It is easy to see that
\[
\nabla \Lambda = \mathbf{A}_K^{-T} \hat{\mathbf{v}} \mathbf{A}_K^{-1} = \mathbf{A}_K^{-T} \mathbf{B}_K \mathbf{A}_K^{-1} \in \mathbf{A}.
\]

So, \( \Lambda \in \Upsilon(K) \). Then, by standard scaling argument with (4.2, 4.3) and the shape regularity of the meshes, we can conclude that the proof for arbitrary tetrahedron element is complete.

Now, we consider the case of arbitrary shape regular element \( K \), which can be hexahedron, prism or pyramid. Let \( \mathbf{v} = (v_1, v_2, v_3)^\top \in \mathcal{W}_h|_K \). It is well known that for any \( 1 \leq i, j, k \leq 3 \),
\[
\partial_j(\partial_k v_i) = \partial_j(\epsilon_{ik}(\mathbf{v})) + \partial_k(\epsilon_{ij}(\mathbf{v})) - \partial_i(\epsilon_{jk}(\mathbf{v})).
\]

Here, \( \epsilon_{ik}(\mathbf{v}) = (\mathbf{\epsilon}(\mathbf{v}))_{ik} \). Consequently, we have
\[
\| \nabla(\partial_j v_i - \partial_i v_j) \|_{0,K} \leq C \| \nabla \mathbf{\epsilon}(\mathbf{v}) \|_{0,K} \leq C h_K^{-3} \| \mathbf{\epsilon}(\mathbf{v}) \|_{0,K}.
\]

We define an anti-symmetric matrix \( \mathbf{B}_K \) by
\[
(\mathbf{B}_K)_{ij} = \frac{1}{|K|} \int_K (\partial_j v_i - \partial_i v_j) d\mathbf{x} \quad \forall 1 \leq i, j \leq 3.
\]

We take \( \Lambda = \mathbf{B}_K \mathbf{x} \), which is obviously in \( \Upsilon(K) \). Then, we have
\[
\int_K (\nabla (\mathbf{v} - \Lambda) - \mathbf{\epsilon}(\mathbf{v} - \Lambda)) d\mathbf{x} = \int_K (\nabla \mathbf{v} - \mathbf{\epsilon}(\mathbf{v})) d\mathbf{x} - \mathbf{B}_K \int_K 1 d\mathbf{x} = 0.
\]

By the Poincaré inequality, we have
\[
\| \nabla (\mathbf{v} - \Lambda) - \mathbf{\epsilon}(\mathbf{v} - \Lambda) \|_{0,K} \leq C h_K \Sigma_{1 \leq i,j \leq 3} \| \nabla (\partial_j v_i - \partial_i v_j) \|_{0,K} \leq C \| \mathbf{\epsilon}(\mathbf{v}) \|_{0,K}.
\]

On the other hand, \( \mathbf{\epsilon}(\mathbf{v} - \Lambda) = \mathbf{\epsilon}(\mathbf{v}) \). We immediately have that
\[
\| \nabla (\mathbf{v} - \Lambda) \|_{0,K} \leq C \| \mathbf{\epsilon}(\mathbf{v}) \|_{0,K}.
\]

This completes the proof. \( \square \)

**Step 1: The error equation.** We first present the error equation for the analysis.
Lemma 4.2. Let \((u, \sigma), (u_h, \sigma_h, \tilde{u}_h)\) solve (1.1) and (2.2) respectively, we have

\[
\begin{align*}
(\mathcal{A}e_\sigma, v)_{\Omega_h} + (e_u, \nabla \cdot v)_{\Omega_h} - \langle e_\tilde{u}, vn \rangle_{\partial \Omega_h} &= (\mathcal{A}(\Pi_V \sigma - \sigma), v)_{\Omega_h}, \tag{4.4a} \\
(e_\sigma, \nabla \omega)_{\Omega_h} - \langle \sigma n - \tilde{\sigma}_h n, \omega \rangle_{\partial \Omega_h} &= 0, \tag{4.4b} \\
(\sigma n - \tilde{\sigma}_h n, \mu)_{\partial \Omega_h \setminus \partial \Omega} &= 0, \tag{4.4c} \\
(\langle e_\tilde{u}, \mu \rangle_{\partial \Omega} &= 0, \tag{4.4d}
\end{align*}
\]

for all \((v, \omega, \mu) \in V_h \times W_h \times M_h\).

Proof. We notice that the exact solution \((u, \sigma, u|_{\Omega_h})\) also satisfies the equation (2.2). Hence, after simple algebraic manipulations, we get that

\[
\begin{align*}
(\mathcal{A}\Pi_V \sigma, v)_{\Omega_h} + (\Pi_W u, \nabla \cdot v)_{\Omega_h} - \langle P_M u, \nabla \cdot v \rangle_{\Omega_h} &= \tag{4.4e} \\
- (A(\sigma - \Pi_V \sigma), v)_{\Omega_h} + (u - \Pi_W u, \nabla \cdot v)_{\Omega_h} \tag{4.4f} \\
(\Pi_V \sigma, \nabla \omega)_{\Omega_h} - \langle \sigma n, \omega \rangle_{\partial \Omega_h} &= -(f, \omega)_{\Omega_h} \tag{4.4g} \\
(\langle \sigma n, \mu \rangle_{\partial \Omega_h \setminus \partial \Omega} &= 0, \tag{4.4h} \\
\langle P_M u, \mu \rangle_{\partial \Omega} &= -(u - \Pi_W u, \mu)_{\partial \Omega}, \tag{4.4i}
\end{align*}
\]

for all \((v, w, \mu) \in V_h \times W_h \times M_h\). Notice that the local spaces satisfy the following inclusion property:

\[
\nabla \cdot V(K) \subset W(K), \quad \varepsilon(W(K)) \subset V(K), \quad V(K)n|_F \subset M(F).
\]

Hence by the property of the \(L^2\)-projection, the above system can be simplified as:

\[
\begin{align*}
(\mathcal{A}\Pi_V \sigma, v)_{\Omega_h} + (\Pi_W u, \nabla \cdot v)_{\Omega_h} - \langle P_M u, \nabla \cdot v \rangle_{\Omega_h} &= -(A(\sigma - \Pi_V \sigma), v)_{\Omega_h}, \tag{4.4e} \\
(\Pi_V \sigma, \nabla \omega)_{\Omega_h} - \langle \sigma n, \omega \rangle_{\partial \Omega_h} &= -(f, \omega)_{\Omega_h}, \tag{4.4f} \\
(\langle \sigma n, \mu \rangle_{\partial \Omega_h \setminus \partial \Omega} &= 0, \tag{4.4g} \\
\langle P_M u, \mu \rangle_{\partial \Omega} &= 0, \tag{4.4h}
\end{align*}
\]

for all \((v, w, \mu) \in V_h \times W_h \times M_h\). Here we applied the fact that \((\sigma - \Pi_V \sigma, \nabla \omega)_{\Omega_h} = (\sigma - \Pi_V \sigma, \varepsilon(\omega)_{\Omega_h}) = 0\). If we now subtract the equations (2.2), we obtain the result. This completes the proof. \(\square\)

**Step 2: Estimate of \(e_\sigma\).** We are now ready to obtain our first estimate.

**Proposition 4.1.** We have

\[
(\mathcal{A}e_\sigma, e_\sigma)_{\Omega_h} + \langle \tau(P_M e_u - e_\tilde{u}), P_M e_u - e_\tilde{u} \rangle_{\partial \Omega_h} = -(\mathcal{A}(\sigma - \Pi_V \sigma), e_\sigma)_{\Omega_h} + T_1 - T_2,
\]

where \(T_1, T_2\) are defined as:

\[
T_1 := (e_u - e_\tilde{u}, \sigma n - (\Pi_V \sigma) n)_{\partial \Omega_h}, \quad T_2 := (e_u - e_\tilde{u}, \tau(P_M (u - \Pi_W u)))_{\partial \Omega_h}.
\]

Proof. By the error equation (4.4d) we know that \(e_\tilde{u} = 0\) on \(\partial \Omega\). This implies that

\[
\langle e_\tilde{u}, \sigma n - \tilde{\sigma}_h n \rangle_{\partial \Omega} = 0.
\]

Now taking \((v, w, \mu) = (e_\sigma, e_u, e_\tilde{u})\) in error equations (4.4a) - (4.4d) and adding these equations together with the above identity, we obtain, after some algebraic manipulation,

\[
(\mathcal{A}e_\sigma, e_\sigma)_{\Omega_h} + (e_u - e_\tilde{u}, e_\sigma n - (\sigma n - \tilde{\sigma}_h n))_{\partial \Omega_h} = -(\mathcal{A}(\sigma - \Pi_V \sigma), e_\sigma)_{\Omega_h}.
\]

(4.5)
Now we work with the second term on the left hand side,
\[
e_n - (\sigma n - \tilde{\sigma}_h n) = \Pi_V \sigma n - \sigma_h n - \sigma n + \tilde{\sigma}_h n
\]
by the definition of the numerical trace (2.2e),
\[
= - (\sigma - \Pi_V \sigma) n - \tau (P_M u_h - \hat{u}_h),
\]
the last step is by the definition of \( e_u, e_\tilde{u} \). Inserting the above identity into (4.5), moving terms around, we have
\[
(A e_\sigma, e_\sigma)_{\partial \Omega} + \langle e_u - e_\tilde{u}, \tau (P_M e_u - e_\tilde{u}) \rangle_{\partial \Omega} + \langle \tau (P_M (u - \Pi_W u)) \rangle_{\partial \Omega} = -(A (\sigma - \Pi_V \sigma), e_\sigma)_{\partial \Omega} + T_1 - T_2.
\]
Finally, notice that on each \( F \in \partial \Omega \), \( \tau (P_M e_u - e_\tilde{u}) | F \in M(F) \), so we have
\[
\langle e_u - e_\tilde{u}, \tau (P_M e_u - e_\tilde{u}) \rangle_{\partial \Omega} = \langle P_M e_u - e_\tilde{u}, \tau (P_M e_u - e_\tilde{u}) \rangle_{\partial \Omega}.
\]
This completes the proof. \( \square \)

From the above energy argument we can see that we need to bound \( T_1, T_2 \) in order to have an estimate for \( e_\sigma \). Next we present the estimates for these two terms:

**Lemma 4.3.** If the parameter \( \tau = \mathcal{O}(h^{-1}) \), we have
\[
T_1 \leq Ch^t \| \sigma \|_{t, \Omega} \left( \| \sigma \|_{t, \Omega} + \| e_u \|_{\partial \Omega} \right)
\]
\[
T_2 \leq Ch^{s-1} \| u \|_{s, \Omega} \| \tau \|_{\partial \Omega} \| u - \Pi_W u \|_{\partial \Omega},
\]
for all \( 1 \leq t \leq k+1, 1 \leq s \leq k+2 \).

**Proof.** We first bound \( T_2 \). We have
\[
T_2 = \langle P_M e_u - e_\tilde{u}, \tau (P_M (u - \Pi_W u)) \rangle_{\partial \Omega} = \langle P_M e_u - e_\tilde{u}, \tau (P_M (u - \Pi_W u)) \rangle_{\partial \Omega}
\]
\[
\leq \| \tau \|_{\partial \Omega} \| u - \Pi_W u \|_{\partial \Omega}
\]
\[
\leq Ch^t (\tau h^{-1}) \| u - \Pi_W u \|_{\partial \Omega},
\]
for all \( 1 \leq s \leq k+2 \). The last step we applied the inequality (4.1d).

The estimate for \( T_1 \) is much more sophisticated. We first split \( T_1 \) into two parts:
\[
T_1 = T_{11} + T_{12},
\]
where
\[
T_{11} := \langle P_M e_u - e_\tilde{u}, \sigma n - (\Pi_V \sigma) n \rangle_{\partial \Omega},
\]
\[
T_{12} := \langle e_u - P_M e_u, \sigma n - (\Pi_V \sigma) n \rangle_{\partial \Omega}.
\]
For \( T_{11} \), we simply apply the Cauchy-Schwarz inequality,
\[
T_{11} \leq \| \tau \|_{\partial \Omega} \| u - \Pi_W u \|_{\partial \Omega},
\]
for all \( 1 \leq t \leq k+1 \). Here we used the inequality (4.1e).
Now we work on $T_{12}$. Using the $L^2$-orthogonal property of the projection $P_M$, we can write
\[
T_{12} = \langle e_u - P_M e_u, \sigma n - (\Pi_V \sigma)n \rangle_{\partial T_h} \\
= \langle e_u - P_M e_u, \sigma n - P_M (\sigma n) \rangle_{\partial T_h},
\]
by the fact $\Pi_V \sigma n |_F, P_M (\sigma n) |_F \in M(F)$ for all $F \in \partial T_h$, 
\[
= \langle e_u, \sigma n - P_M (\sigma n) \rangle_{\partial T_h}, \quad \text{since } P_M e_u |_F \in M(F), \forall F \in \partial T_h,
\]
\[
= \langle e_u + \Lambda, \sigma n - P_M (\sigma n) \rangle_{\partial T_h},
\]
here $\Lambda \in L^2(\Omega)$ is any vector-valued function in $\Upsilon_h$. Notice here the last step holds only if $\Upsilon_h |_F \in M(F)$, $\forall F \in \partial T_h$. This is true if $k \geq 1$. Next, on each $K \in T_h$, if we denote $\bar{u}$ to be the average of $u$ over $K$, then we have
\[
\langle e_u + \Lambda, \sigma n - P_M (\sigma n) \rangle_{\partial K} = \langle e_u + \Lambda - \bar{e}_u - \Lambda, \sigma n - P_M (\sigma n) \rangle_{\partial K} \\
\leq \|e_u + \Lambda - \bar{e}_u - \Lambda\|_{\partial K} \|\sigma n - P_M (\sigma n)\|_{\partial K},
\]
by the standard inequalities (4.1f), (4.1g),
\[
\leq Ch^{t-1} \|\sigma\|_{t,K} \|e_u + \Lambda - \bar{e}_u - \Lambda\|_{K} \\
\leq Ch^{t} \|\sigma\|_{t,K} \|\nabla (e_u + \Lambda)\|_{K},
\]
for all $1 \leq t \leq k + 1$. The last step is by the Poincaré inequality. Notice that the constant $C$ in above inequality is independent of $\Lambda \in \Upsilon_h$. Now applying the Lemma 4.1 yields,
\[
\leq Ch^{t} \|\sigma\|_{t,K} \|e(e_u)\|_{K},
\]
Sum over all $K \in T_h$, we have
\[
T_{12} \leq Ch^{t} \|\sigma\|_{t,\Omega} \|e(e_u)\|_{T_h},
\]
for all $1 \leq t \leq k + 1$. We complete the proof by combining the estimates for $T_2, T_{11}, T_{12}$. □

Combining Lemma 4.3 and Proposition 4.1 we obtain the following estimate.

Corollary 4.1. If the parameter $\tau = O(h^{-1})$, then we have
\[
\|e_\sigma\|_{L^2(\Lambda,\Omega)}^2 + \|\tau^{\frac{s}{2}} (P_M e_u - e_\bar{u})\|_{\partial T_h}^2 \\
\leq C \left( h^{2t} \|\sigma\|_{t,\Omega} + h^{2(s-1)} \|u\|_{s,\Omega} + h^{t} \|\sigma\|_{t,\Omega} \|e(e_u)\|_{T_h} \right),
\]
for all $1 \leq s \leq k + 2, 1 \leq t \leq k + 1$, the constant $C$ is independent of $h$ and exact solution.

The proof is omitted. One can obtain the above result by Cauchy-Schwarz inequality and weighted Young’s inequality. Finally, we can finish the estimate for $e_\sigma$ by the following estimate for $e(e_u)$:

Lemma 4.4. Under the same assumption as Theorem 4.1, we have
\[
\|e(e_u)\|_{T_h} \leq C \left( h^{t} \|\sigma\|_{t,\Omega} + \|e_\sigma\|_{L^2(\Lambda,\Omega)} + \|\tau^{\frac{s}{2}} (P_M e_u - e_\bar{u})\|_{\partial T_h} \right),
\]
for all $0 \leq t \leq k + 1$. 

Corollary 4.2. Under the same assumption as in Theorem 4.1, we have
\[
\|e_\sigma\|_{L^2(A,\Omega)} + \|\tau^{\frac{1}{2}}(P_M e_u - e_\bar{u})\|_{\partial T_h} + \|e(u)\|_{T_h} \leq C(h^{t} \|\sigma\|_{t,\Omega} + h^{s-1}\|u\|_{s,\Omega}),
\]
for all \(1 \leq t \leq k+1, 1 \leq s \leq k+2\), the constant \(C\) is independent of \(h\) and exact solution.

One can see that by taking \(t = k+1, s = k+2\), both of the error \(e_\sigma, e(u)\) obtain optimal convergence rate. Moreover, if we take \(t = 1/h\), we readily obtain the superconvergence property
\[
\|h^{\frac{1}{2}}(P_M e_u - e_\bar{u})\|_{\partial T_h} \leq C h^{k+2},
\]
for smooth solutions. It is this superconvergence property the one which allows to obtain the optimal convergence in the stress and, as we are going to see next, in the displacement.

Step 3: Estimate of \(e_u\). Next we use a standard duality argument to get an estimate for \(e_u\). First we present an important identity.

Proposition 4.2. Assume that \((\phi, \psi) \in H^2(\Omega) \times H^1(\Omega)\) is the solution of the adjoint problem (2.3a), we have
\[
\|e_u\|^2_{\Omega} = (e_u, \nabla \cdot \psi)_{T_h} + (e_\sigma, \nabla \phi)_{T_h} - (e_\sigma, \Pi_\sigma \phi)_{T_h} - (e_\sigma, \Pi_\sigma \phi)_{T_h} + (e_u - e_\bar{u}, (\psi - \Pi_\sigma \phi)_{T_h},
\]

Proof. By the dual equation (2.3), we can write
\[
\|e_u\|^2_{\Omega} = (e_u, \nabla \cdot \psi)_{T_h} + (e_\sigma, A\psi - e(\phi))_{T_h} = (e_u, \nabla \cdot \psi)_{T_h} + (Ae_\sigma, \psi)_{T_h} - (e_\sigma, \nabla \phi)_{T_h} = (e_u, \nabla \cdot (\Pi_\sigma \psi))_{T_h} + (Ae_\sigma, \Pi_\sigma \psi)_{T_h} - (e_\sigma, \nabla \Pi_\sigma \phi)_{T_h} + (Ae_\sigma, \psi - \Pi_\sigma \psi)_{T_h} + (e_u, \nabla (\psi - \Pi_\sigma \psi))_{T_h} - (e_\sigma, \nabla (\phi - \Pi_\sigma \phi))_{T_h}.
\]
integrating by parts for the last two terms, applying the property of the $L^2$-projections, yields,

$$
\|e_u\|_{\Omega}^2 = (e_u, \nabla \cdot \Pi_V \psi)_{\Omega_h} + (Ae_\sigma, \Pi_V \psi)_{\Omega_h} - (e_\sigma, \nabla \Pi_W \phi)_{t_h}
+ (Ae_\sigma, \psi - \Pi_V \psi)_{\Omega_h} + (e_u, (\psi - \Pi_V \psi)n)_{\partial \Omega_h} - (e_\sigma n, \phi - \Pi_W \phi)_{\partial \Omega_h}.
$$

Taking $\mathbf{u} := \Pi_V \mathbf{u}$ and $\omega := \Pi_W \phi$ in the error equations (4.4a) and (4.4b), respectively, inserting these two equations into above identity, we obtain

$$
\|e_u\|_{\Omega}^2 = (e_u, \Pi_V \psi n)_{\partial \Omega_h} + (A(\sigma - \Pi_V \sigma), \Pi_V \psi)_{\Omega_h} - (\sigma n - \hat{\sigma}_h n, \Pi_W \phi)_{\partial \Omega_h}
+ (Ae_\sigma, \psi - \Pi_V \psi)_{\Omega_h} + (e_u, (\psi - \Pi_V \psi)n)_{\partial \Omega_h} - (e_\sigma n, \phi - \Pi_W \phi)_{\partial \Omega_h}.
$$

Next, note that by the regularity assumption, $(\psi, \phi) \in H^2(\Omega) \times H^1(\Omega)$, so the normal component of $\psi$ and $\phi$ are continuous across each face $F \in \mathcal{E}_h$. By the equation (2.2d), the normal component of $\hat{\sigma}_h$ is also strongly continuous across each face $F \in \mathcal{E}_h$. This implies that

$$
- (e_\hat{u}, \psi n)_{\partial \Omega} = - (e_\hat{u}, \psi n)_{\partial \Omega} = 0, \quad \text{by (4.4d)},
\quad (\sigma n - \hat{\sigma}_h n, \phi)_{\partial \Omega} = (\sigma n - \hat{\sigma}_h n, \phi)_{\partial \Omega} = 0 \quad \text{by (2.3c)}.
$$

Adding these two zero terms into the previous equation, rearranging the terms, we obtain the expression as presented in the proposition.

As a consequence of the result just proved, we can obtain our estimate of $e_u$.

**Corollary 4.3.** Under the same assumption as in Theorem 4.1, in addition, if the elliptic regularity property (2.4) holds, then we have

$$
\|e_u\|_{\Omega} \leq C(h^{t+1}\|\sigma\|_{t, \Omega} + h^s\|u\|_{s, \Omega}),
$$

for $1 \leq t \leq k + 1$, $1 \leq s \leq k + 2$.

**Proof.** We will estimate each of the terms on the right hand side of the identity in Proposition 4.2

$$
(Ae_\sigma, \psi - \Pi_V \psi)_{\Omega_h} \leq Ch\|e_\sigma\|_{L^2(A, \Omega)}\|\psi\|_{1, \Omega} \leq Ch\|e_\sigma\|_{L^2(A, \Omega)}\|e_u\|_{\Omega},
$$

by the projection property (4.1b) and the regularity assumption (2.4).

$$
(A(\sigma - \Pi_V \sigma), \Pi_V \psi)_{\Omega_h} = (A(\sigma - \Pi_V \sigma), \psi)_{\Omega_h} - (A(\sigma - \Pi_V \sigma), \psi - \Pi_V \psi)_{\Omega_h}
= (\sigma - \Pi_V \sigma, A\psi - A\psi)_{\Omega_h} - (A(\sigma - \Pi_V \sigma), \psi - \Pi_V \psi)_{\Omega_h}
\leq Ch\|\sigma - \Pi_V \sigma\|_{\Omega}\|\psi\|_{1, \Omega}
\leq Ch\|\sigma - \Pi_V \sigma\|_{\Omega}\|e_u\|_{\Omega}
\leq Ch^{t+1}\|\sigma\|_{t, \Omega}\|e_u\|_{\Omega},
$$

for all $0 \leq t \leq k + 1$. Here we applied the Galerkin orthogonal property of the local $L^2$-projection $\Pi_V$ and the regularity assumption (2.4).
For the second term, we apply the same argument for the estimate of

\[ \langle (\sigma - \Pi_V \sigma) n, \phi - \Pi_W \phi \rangle_{\partial \Omega_h} \leq Ch^{\frac{s}{2}} \left\| (\sigma - \Pi_V \sigma) n \right\|_1_{\partial \Omega_h} \langle \phi - \Pi_W \phi \rangle_{\partial \Omega_h} \]

by the standard inequalities, \([1.1d], [1.1e]\),

\[ \leq Ch^{s+1} \left\| (\sigma - \Pi_V \sigma) n \right\|_1 \left\| e_u \right\|_{\partial \Omega_h}, \]

for all \(1 \leq s \leq k + 1\). The last step is due to the regularity assumption \([2.4]\).

Similarly, we apply the Cauchy-Schwarz inequality and \([1.1d]\) for the other terms:

\[ \langle \tau(P_M e_u - e_{\hat{u}}), \phi - \Pi_W \phi \rangle_{\partial \Omega_h} \leq C\tau^{\frac{s}{2}} \left\| (\tau(P_M e_u - e_{\hat{u}})) \right\|_1 \langle \phi - \Pi_W \phi \rangle_{\partial \Omega_h} \]

\[ \leq C\tau^{\frac{s}{2}} h^{k+1} \left\| (\tau(P_M e_u - e_{\hat{u}})) \right\|_1 \left\| e_u \right\|_{\partial \Omega_h}, \]

for all \(1 \leq s \leq k + 2\).

Finally, for the last term in Proposition \([4.2]\) we can write:

\[ \langle e_u - e_{\hat{u}}, (\psi - \Pi_V \psi) n \rangle_{\partial \Omega_h} = \langle P_M e_u - e_{\hat{u}}, (\psi - \Pi_V \psi) n \rangle_{\partial \Omega_h} \]

\[ + \langle e_u - P_M e_u, \psi - \Pi_V \psi \rangle_{\partial \Omega_h}. \]

For the first term, we can apply a similar argument as in the previous steps to obtain:

\[ \langle P_M e_u - e_{\hat{u}}, (\psi - \Pi_V \psi) n \rangle_{\partial \Omega_h} \leq C\tau^{\frac{s}{2}} h^{k+1} \left\| (\tau(P_M e_u - e_{\hat{u}})) \right\|_1 \left\| e_u \right\|_{\partial \Omega_h}. \]

For the second term, we apply the same argument for the estimate of \(T_{12}\) in the proof of Lemma \([4.3]\) and obtain:

\[ \langle e_u - P_M e_u, \psi - \Pi_V \psi \rangle_{\partial \Omega_h} \leq Ch \left\| (\psi - \Pi_V \psi) n \right\|_1 \left\| e_u \right\|_{\partial \Omega_h} \]

\[ \leq Ch \left\| e_{\hat{u}} \right\|_{\partial \Omega_h} \left\| e_u \right\|_{\partial \Omega_h}. \]

Finally if we take \(\tau = O(h^{-1})\), combine all the above estimates and Theorem \([4.2]\) we obtain the estimate in the Theorem \([4.3]\). \( \square \)
As a consequence of Theorem 4.2 and Theorem 4.3, we can obtain Theorem 2.1 by a simple triangle inequality and the approximation property of the projections $\Pi_W, \Pi_V$ (4.1a), (4.1b).

We can now give the proof of Theorem 2.2.

Proof. (Proof of Theorem 2.2) By the assumption $g = 0$, taking $v = I_3$ in the first equation of (2.2), we have that $\int_{\Omega} \text{tr}(A\sigma_h) dx = 0$. According to the Assumption 2.1 and the fact that $P_T > 0$, we have $\int_{\Omega} \text{tr}\sigma_h dx = 0$. Similarly, we have

$$\int_{\Omega} \text{tr}(\sigma - \sigma_h) dx = 0. \quad (4.6)$$

Similar to the proof of [11, Prop. 9.1.1], it is easy to obtain

$$\|\text{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)} \leq C \left( \|\nabla \cdot (\sigma - \sigma_h)\|_{H^{-1}(\Omega)}^2 + \|\sigma - \sigma_h\|_{D}^2 \right)^{1/2}$$

due to (4.6). Here, $[\tau \cdot n]_F$ denotes the jump of the normal component of $\tau$ on $F \in \mathcal{E}_h \setminus \partial \Omega$, and $(\sigma - \sigma_h)_D = (\sigma - \sigma_h) - \frac{1}{2} \text{tr}(\sigma - \sigma_h) I_3$.

In what follows, we assume that $s$ is some arbitrary real number in $[1, k + 1]$ and $C$ is a positive constant independent of $P_T$ and $s$. We recall that $e_\sigma = \Pi_V \sigma - \sigma_h$, $e_u = \Pi_W u - u_h$, $\hat{e}_u = P_M u - \hat{u}_h$. According to Corollary 4.2, we have

$$\|\tau^{1/2}(P_M e_u - e_{\hat{u}})\|_{\partial \Gamma_h} \leq C h^s (\|u\|_{s+1, \Omega} + \|\sigma\|_{s, \Omega}). \quad (4.8)$$

By the definition of $e_{\hat{u}}$ and $e_u$, we have

$$\|\tau^{1/2}(P_M u_h - \hat{u}_h)\|_{\partial \Omega}^2 \leq 2\|\tau^{1/2}(P_M e_u - e_{\hat{u}})\|_{\partial \Gamma_h}^2 + 2\|\tau^{1/2}P_M (u - \Pi_W u)\|_{\partial \Gamma_h}^2 \leq 2\|\tau^{1/2}(P_M e_u - e_{\hat{u}})\|_{\partial \Gamma_h}^2 + 2\|\tau^{1/2}(u - \Pi_W u)\|_{\partial \Gamma_h}^2.$$

now applying Young’s inequality and (4.8), (4.1a) we obtain:

$$\|\tau^{1/2}(P_M u_h - \hat{u}_h)\|_{\partial \Gamma_h} \leq C h^s (\|u\|_{s+1, \Omega} + \|\sigma\|_{s, \Omega}). \quad (4.9)$$

Take $\omega = \nabla \cdot e_\sigma$ in the error equation (4.4b), we have

$$\|\nabla \cdot e_\sigma\|_{T_h}^2 = \langle e_\sigma \cdot e_\sigma, \nabla \cdot (\sigma - \sigma_h) \rangle_{\partial \Gamma_h} = -\langle (\sigma - \Pi_V \sigma) \cdot \nabla \cdot e_\sigma, \nabla \cdot e_\sigma \rangle_{\partial \Gamma_h} + \langle \tau (P_M u_h - \hat{u}_h), \nabla \cdot e_\sigma \rangle_{\partial \Gamma_h}.$$

Applying Cauchy-Schwarz inequality on the right hand side and by (4.1), (4.9) and $\tau = \mathcal{O}(\frac{1}{h})$, we obtain:

$$\|\nabla \cdot (\sigma - \sigma_h)\|_{T_h} \leq C h^{s-1} (\|u\|_{s+1, \Omega} + \|\sigma\|_{s, \Omega}).$$

Taking $\mu_F = [e_\sigma \cdot n]_F$ for all $F \in \mathcal{E}_h \setminus \partial \Omega$ in the error equation (4.4c), with a similar argument as the above estimate we can get

$$\left( \Sigma_{F \in \mathcal{E}_h \setminus \partial \Omega} h^{-1/2}_F \|\nabla \cdot (\sigma - \sigma_h) \cdot n\|_{L^2(F)}^2 \right)^{1/2} \leq C h^{s-1} (\|u\|_{s+1, \Omega} + \|\sigma\|_{s, \Omega}).$$

Finally, by the Assumption 2.1 and Theorem 2.1, we have

$$\|P_D^{1/2} (\sigma - \sigma_h)\|_{L^2(\Omega)} \leq \|\sigma - \sigma_h\|_{L^2(A, \Omega)} \leq C h^{s} (\|u\|_{s+1, \Omega} + \|\sigma\|_{s, \Omega}).$$
Combining the above three estimates into (4.7) and with a simple triangle inequality, we get
\[
\|\sigma - \sigma_h\|_{L^2(\Omega)} \leq C\left(\|\sigma - \sigma_h\|_{D}\|_{L^2(\Omega)} + \|\text{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)}\right)
\leq Ch^{s-1}\left(\|u\|_{s+1,\Omega} + \|\sigma\|_{s,\Omega}\right),
\]
for all $1 \leq s \leq k + 1$. 

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