The metric theory of the pair correlation function for small non-integer powers

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Abstract
For \(0 < \theta < 1\), we show that for almost all \(\alpha\), the pair correlation function of the sequence of fractional parts of \(\{\alpha n^\theta : n \geq 1\}\) is Poissonian.

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1 | INTRODUCTION

The theory of uniform distribution of sequences modulo one has a long history, with many developments since its creation over a century ago, see ref. [6]. Much more recent is the study of “local” statistics of sequences, motivated by problems in quantum chaos and the theory of the Riemann zeta function, see, for example, ref. [12]. A key example of such a local statistic is the pair correlation function \(R_2(x)\) which, for a sequence of points \(\{\theta_n\} \subset \mathbb{R}/\mathbb{Z}\), assigns to every \(s > 0\) the limit

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq x \neq y \leq N : |\theta_x - \theta_y| \leq \frac{s}{N} \right\} = \int_{-s}^{s} R_2(t)dt \quad (1.1)
\]

assuming that the limit exists, which is in itself a major problem. The pair correlation is analytically the most accessible example of a “local” statistic, the easiest to visualize being the nearest neighbour spacing distribution \(P(s)\), which is defined as the limit distribution (assuming it exists) of the gaps between neighbouring elements in the sequence, rescaled so as to have mean value unity. A simple model of what to expect is a “random sequence” (the Poisson model), say taking

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independent uniform random variables. In this case the nearest neighbour spacing distribution is exponential: \( P(s) = \exp(-s) \) and the pair correlation function is \( R_2(t) \equiv 1 \):

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq x \neq y \leq N : |\vartheta_x - \vartheta_y| \leq \frac{S}{N} \right\} = \int_{-s}^{s} 1 \, dx = 2s. \tag{1.2}
\]

One of the few cases where the pair correlation of a fixed (deterministic) sequence is known is the sequence of square roots of integers \( \vartheta_n = \sqrt{n} \). It was shown by Elkies and McMullen [3] that the nearest neighbour spacing distribution of \( \{ \vartheta_n \mod 1 \} \) exists, and is non-Poissonian (this was discovered numerically by Boshernitzan in 1993). In particular, the level spacing distribution is flat near the origin \( P(s) = 6/\pi^2 \), \( 0 \leq s \leq 1/2 \), and is piecewise analytic. Surprisingly, El-Baz, Marklof and Vinogradov [2] showed that the pair correlation of the fractional parts of \( \{ \sqrt{n} : n \neq k^2 \} \) is nonetheless Poissonian, after removing perfect squares \( n = k^2 \) (without this removal the pair correlation blows up). Their work has as its departure point the methods of ref. [3], which use homogeneous dynamics, and does not apply to the more general case of \( \alpha \sqrt{n} \) when \( \alpha^2 \) is irrational.

Our goal in this paper is to show that the pair correlation of \( \alpha \sqrt{n} \) is indeed Poissonian for almost all \( \alpha \), in the sense that the exceptional sense has Lebesgue measure zero. In fact, we show more generally

**Theorem 1.1.** Fix \( \vartheta \in (0,1) \). Then for almost all \( \alpha \), the pair correlation function of the sequence \( \alpha n^\vartheta \) is Poissonian.

The corresponding result for integer \( \vartheta > 1 \) is due to Rudnick and Sarnak [14]. The case of non-integer \( \vartheta > 1 \) was recently solved by Aistleitner, El-Baz and Munsch [1], and we build on some of their work, though their methods fail to cover \( \vartheta < 1 \), see ref. [1, Section 9] for an explanation. Very recently, Lutsko, Sourmelidis and Technau [7] have succeeded in showing a deterministic result, that for \( \vartheta \leq 1/3 \) and all \( \alpha \neq 0 \), the pair correlation of \( \alpha n^\vartheta \) is Poissonian.

## 2 AN OVERVIEW OF THE PROOF

### 2.1 Smoothing

Fix \( \vartheta \in (0,1) \). We study a smooth version of the pair correlation function in which we fix \( h \in C^\infty([1,2]), h \geq 0 \), and an even \( f \in C_c^\infty(\mathbb{R}) \) to define

\[
R_{f,h}(\alpha, N) := \frac{1}{N} \sum_{x \neq y \geq 1} h(\frac{x}{N})h(\frac{y}{N})F_N(\alpha(x^\vartheta - y^\vartheta)), \tag{2.1}
\]

where \( F_N \) denotes the 1-periodized version of \( f \) localized to the scale \( 1/N \), that is

\[
F_N(x) := \sum_{k \in \mathbb{Z}} f(N(x + k)).
\]

It is a standard argument (see, e.g., ref. [8], Sections 8.6.1 and 8.6.3) that a sequence \( x \) has Poissonian pair correlation, that is Equation (1.2) holds, if and only if for any \( h, f \in C_c^\infty(\mathbb{R}) \)

\[
\lim_{N \to \infty} R_{f,h}(\alpha, N) = \int_{-\infty}^{\infty} f(t)dt \left( \int_{-\infty}^{\infty} h(s)ds \right)^2. \tag{2.2}
\]
Hence we want to show that Equation (2.2) holds for almost all $\alpha > 0$. We fix $R > 0$ and assume that $\alpha \in [R, R + 1]$.

Note that choosing $h$ supported in refs. [1,2] rather than [0,1] corresponds to taking $N/2 < x, y \leq N$ in Equation (1.1). We do this for convenience, and this kind of modification is natural in statistical physics (to work “in the bulk”) and one usually does not expect it to change any of the local statistics. A striking exception is for the sequence $\sqrt{n}$ (that is $\theta = 1/2$, $\alpha = 1$), where the limiting level spacing distribution depends on the choice of $c \in (0,1)$ in choosing $cN < x, y < N$, as was found by Elkies and McMullen [3, Section 3.5]; but the pair correlation (after removing squares) does not, as is pointed out by El-Baz, Marklof and Vinogradov [2, remark 1, page 5]. Furthermore, it will be convenient to use the notation

$$\Theta := \frac{1}{1-\theta}.$$  

Note that for $\theta \in (0,1)$, we have $\Theta > 1$, for example, $\theta = 1/2$ gives $\Theta = 2$.

Throughout the paper, we denote $e(\nu) := \exp(2\pi \sqrt{-1} \nu)$, and use a normalized Fourier transform $\hat{f}(x) := \int_{-\infty}^{\infty} f(y) e(-xy) dy$.

We break down our argument into several key steps:

### 2.2 Step 1

Let

$$E_{N,j}(\alpha) = \sum_{y \geq 1} h\left(\frac{y}{N}\right)e(\alpha jy\theta).$$  

(2.3)

By the Fourier expansion of $F_N$ and the decay of $\hat{f}$, write $R_{f,h}(N)(\alpha)$ as

$$S(f, h; N)(\alpha) + \int_{-\infty}^{\infty} f(x)dx \left(\int_{-\infty}^{\infty} h(y)dy\right)^2 - f(0) \int_{-\infty}^{\infty} h(y)^2dy + O(N^{-100})$$

with

$$S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{1}{N} |\hat{f}\left(\frac{j}{N}\right)|^2 |E_{N,j}(\alpha)|^2.$$  

Now use Poisson summation and stationary phase expansion (van der Corput’s “B-process”) to replace $E_{N,j}(\alpha)$ by a shorter sum:

**Proposition 2.1.** For $j > 0$,

$$E_{N,j}(\alpha) = \tilde{E}_{N,j}(\alpha) + O\left(\frac{N^{1-\frac{\theta}{2}}}{j^{1/2}}\right).$$
with
\[ \tilde{E}_{N,j}(\alpha) := c_1 \cdot (\alpha j)_{\Theta}^{\frac{\theta}{2}} \sum_{m = j/\lceil N^{1-\theta} \rceil}^{\infty} \frac{1}{m^{\frac{\theta+1}{2}}} h\left(\frac{(\theta \alpha j)^{\Theta}}{Nm^{\Theta}}\right) e\left(c_2 \frac{(\alpha j)^{\Theta}}{m^{\Theta-1}}\right) \]

where \( c_1 = \frac{\Theta}{\sqrt{1-\theta}} e\left(-\frac{1}{8}\right) \) and \( c_2 = \Theta^{\frac{\theta}{1-\theta}} - \Theta^{\frac{1}{1-\theta}} \).

Note that \( \tilde{E}_{N,j}(\alpha) = 0 \) if \( 0 < j \ll N^{1-\theta} \).

### 2.3 | Step 2

Show that \( \tilde{E}_{N,j}(\alpha) \) is typically of size \( N^{1/2} \) along a polynomially sparse subsequence: We define a probability measure \( d\mu(\alpha) \) on \([R, R + 1]\) by

\[ d\mu(\alpha) = \frac{\Theta \rho(\alpha^{\Theta}) d\alpha}{\alpha} \]

where \( \rho \in C^\infty([R, R + 1]), \rho \geq 0 \), normalized by \( \int_{-\infty}^{\infty} \rho(x) \frac{dx}{x} = 1 \).

**Proposition 2.2.** The estimate

\[ \int_{-\infty}^{\infty} |\tilde{E}_{N,j}(\alpha)|^2 d\mu(\alpha) \ll N \]

holds uniformly for any \( 0 < j < N^2 \).

Let

\[ N_\ell := \ell^{C} \]

where \( C \in \mathbb{N} \) is a large constant. We deduce that on the subsequence

\[ \mathcal{N} = \{N_\ell, \ell \geq 1\} \]

for almost all \( \alpha \in [R, R + 1] \) all frequencies \( j \gg N^{1-\varepsilon} \) (where \( \varepsilon > 0 \) is small) are negligible for \( N \in \mathcal{N} \):

**Corollary 2.3.** For almost all \( \alpha \in [R, R + 1] \) and all \( N_\ell \in \mathcal{N} \), we have that

\[ S(f, h; N_\ell)(\alpha) = \frac{2}{N_\ell^2} \sum_{N_\ell^{1-\varepsilon} \ll j \ll N_\ell^{1+\varepsilon}} \hat{f}\left(\frac{j}{N_\ell}\right) |\tilde{E}_{N_\ell,j}(\alpha)|^2 + O(N_\ell^{-\varepsilon/2}), \]

and

\[ \frac{2}{N_\ell^2} \sum_{N_\ell^{1-\varepsilon} \ll j \ll N_\ell^{1+\varepsilon}} \hat{f}\left(\frac{j}{N_\ell}\right) |\tilde{E}_{N_\ell,j}(\alpha)|^2 \ll N_\ell^{2\varepsilon}. \]
2.4  Step 3

Let

\[ R_{\text{off}}(N)(\alpha) := \frac{2|c_1|^2\alpha^\Theta}{N^2} \sum_{N^{1-\epsilon} < j < N^{1+\epsilon}} \hat{f}\left(\frac{j}{N}\right) j^\Theta \]

\[ \sum_{m \neq n} \frac{1}{(mn)^{\Theta+1/2}} h\left(\frac{(\theta\alpha j)^\Theta}{Nm^\Theta}\right) h\left(\frac{(\theta\alpha j)^\Theta}{Nn^\Theta}\right) e\left(c_2 \cdot (\alpha j)^\Theta \left(\frac{1}{m^{\Theta-1}} - \frac{1}{n^{\Theta-1}}\right)\right) \]

the sum is over \( m \neq n \). We will show

**Proposition 2.4.** For almost all \( \alpha \in [R, R + 1] \) and all \( N_\epsilon \in \mathcal{N} \) we have that

\[ R_{f,h}(N_\epsilon)(\alpha) = \int_{-\infty}^{\infty} f(t)d\mu(\alpha) \left(\int_{-\infty}^{\infty} h(s)ds\right)^2 + R_{\text{off}}(N_\epsilon)(\alpha) + O(N_\epsilon^{-\epsilon/2}). \]

2.5  Step 4

We average over \( \alpha \) and want to show that the second moment of \( R_{\text{off}}(N)(\alpha) \) is small.

**Theorem 2.5.** For any fixed \( \delta \in (0, 1 - \Theta) \), we have

\[ \int_{-\infty}^{\infty} |R_{\text{off}}(N)(\alpha)|^2 d\mu(\alpha) \ll N^{-\delta}. \]

By using Chebychev’s inequality and the Borell–Cantelli lemma, we deduce from Theorem 2.5 that

\[ \lim_{\epsilon \to \infty} R_{\text{off}}(N_\epsilon)(\alpha) = 0 \]

holds for almost all \( \alpha \in [R, R + 1] \) and all \( N_\epsilon \in \mathcal{N} \). Inserting into Proposition 2.4, we obtain

**Theorem 2.6.** For almost all \( \alpha \in [R, R + 1] \), and all \( N_\epsilon \in \mathcal{N} \), we have

\[ \lim_{\epsilon \to \infty} R_{f,h}(N_\epsilon)(\alpha) = \int_{-\infty}^{\infty} f(t)d\mu(\alpha) \left(\int_{-\infty}^{\infty} h(s)ds\right)^2. \]

By employing a standard (deterministic) argument, one deduces Theorem 1.1 from Theorem 2.6. For instance, one can use ref. [15, Lemma 3.1] which give that if Theorem 2.6 holds for all \( f \in C^\infty_c(\mathbb{R}) \), for a strictly increasing sequence of positive integers \( \mathcal{N} = \{N_\epsilon\}_{\epsilon=1}^{\infty} \), with \( N_{\epsilon+1}/N_\epsilon \sim 1 \) then we can pass from the subsequence \( \mathcal{N} \) to the set of all integers, in the sense that Theorem 2.6 holds for all integers \( N \geq 1 \).
3 | APPLYING POISSON SUMMATION

3.1 | Applying Poisson summation for the first time

The Fourier expansion of $F_N(x)$ can be written, upon recalling that $f$ is even (and therefore also $\hat{f}$ is even), as

$$F_N(x) = \frac{1}{N} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) e(jx) = \frac{1}{N} \int_{-\infty}^{\infty} f(x) dx + \frac{2}{N} \sum_{j=1}^{\infty} \hat{f} \left( \frac{j}{N} \right) e(jx).$$

Inserting into the definition of $R_{f,h}(N)(\alpha)$ we obtain

$$R_{f,h}(N)(\alpha) = \frac{1}{N^2} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) \sum_{x \neq y \geq 1} h \left( \frac{x}{N} \right) h \left( \frac{y}{N} \right) e(\alpha jx^\theta) e(-\alpha jy^\theta).$$

Next by adding and subtracting the $x = y$ diagonal, the right-hand side is

$$\frac{1}{N^2} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) \left( \sum_{x \geq 1} h \left( \frac{x}{N} \right) e(\alpha jx^\theta) \sum_{y \geq 1} h \left( \frac{y}{N} \right) e(-\alpha jy^\theta) - \sum_{x \geq 1} h \left( \frac{x}{N} \right)^2 \right).$$

By recalling the definition of $\mathcal{E}_{N,j}(\alpha)$, see Equation (2.3), we find that

$$R_{f,h}(N)(\alpha) = \frac{1}{N^2} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) \left| \mathcal{E}_{N,j}(\alpha) \right|^2 - \frac{1}{N^2} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) \sum_{x \geq 1} h \left( \frac{x}{N} \right)^2. \quad (3.1)$$

Using the trivial bound $|\mathcal{E}_{N,j}(\alpha)| \ll N$, we can truncate the frequencies $j > N^{1+\epsilon}$ with negligible error. Further, the term with $j = 0$ contributes

$$\hat{f}(0) \frac{|\mathcal{E}_{N,0}(\alpha)|^2}{N^2} = \hat{f}(0) \left| \frac{1}{N} \sum_{y \geq 1} h \left( \frac{y}{N} \right) \right|^2 = \int_{-\infty}^{\infty} f(x) dx \left( \int_{-\infty}^{\infty} h(y) dy \right)^2 + O(N^{-100})$$

which is the limit that we are aiming for.

The diagonal $x = y$ term equals:

$$\frac{1}{N^2} \sum_{j \in \mathbb{Z}} \hat{f} \left( \frac{j}{N} \right) \sum_{x \geq 1} h \left( \frac{x}{N} \right)^2 = \int_{-\infty}^{\infty} \hat{f}(x) dx \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-100})$$

$$= f(0) \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-100}).$$

We obtain

$$R_{f,h}(N)(\alpha) = S(f,h;N)(\alpha) + \int_{-\infty}^{\infty} f(x) dx \left( \int_{-\infty}^{\infty} h(y) dy \right)^2$$

$$- f(0) \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-100}) \quad (3.2)$$
where
\[
S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{1 \leq j < N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right)|\mathcal{E}_{N,j}(\alpha)|^2.
\]

(3.3)

### 3.2 Applying van der Corput’s B-process

To deal with the terms including the smooth exponential sum \(\mathcal{E}_{N,j}(\alpha)\), we apply Poisson summation and a stationary phase expansion (van der Corput’s “B-process”). Recall
\[
\mathcal{E}_{N,j}(\alpha) = c_1 \cdot (\alpha j)^\theta \sum_{m \approx j/N^{1-\theta}} \frac{1}{m^{\theta+1/2}} h\left(\frac{(\theta \alpha j)^\theta}{Nm^{\theta}}\right)e\left(c_2 \frac{(\alpha j)^\theta}{m^{\theta-1}}\right),
\]

as well as the constants \(c_1\) and \(c_2\) from Proposition 2.1.

**Proposition 3.1.** For \(\alpha \in [R, R+1]\), and \(j > 0\),
\[
\mathcal{E}_{N,j}(\alpha) = \hat{\mathcal{E}}_{N,j}(\alpha) + O\left(\frac{N^{1-\theta}}{j^{1/2}}\right).
\]

Note that for \(1 \leq j \ll N^{1-\theta}\), we just obtain an upper bound since in that case \(\hat{\mathcal{E}}_{N,j}(\alpha) = 0\).

For proving the above proposition, we quote the following version of the smooth B-process (see ref. [4, Equation (8.47)]):

**Theorem 3.2.** Let \(\phi \in C^4[A,B]\) be real valued so that there are \(\Lambda > 0\) and \(\eta \geq 1\) with
\[
\Lambda \leq |\phi^{(2)}(x)| \leq \eta \Lambda, \quad |\phi^{(3)}(x)| < \frac{\eta \Lambda}{B - A}, \quad |\phi^{(4)}(x)| < \frac{\eta \Lambda}{(B - A)^2}
\]

for all \(x \in [A, B]\). Further, assume that \(\phi^{(2)} < 0\) on \([A, B]\). Let \(a = \phi'(A)\), and \(b = \phi'(B)\). Then for all smooth functions \(g\),
\[
\sum_{n \in [A,B]} g(n)e(\phi(n)) = \sum_{m \in [b,a]} \frac{g(x_m)}{\phi''(x_m)^{1/2}} e\left(\phi(x_m) - mx_m - \frac{1}{8}\right)
\]
\[+ O\left(G\Lambda^{-1/2} + G\eta^2 \log(a - b + 1)\right)
\]

where \(x_m\) is the unique solution to \(\phi'(x_m) = m\), and \(G = |g(B)| + \int_B^B |g'(t)| dt\).

**Proof of Proposition 3.1.** We apply this to the sum
\[
\mathcal{E}_{N,j}(\alpha) = \sum_{y \geq 1} h\left(\frac{y}{N}\right)e(\alpha jy^\theta)
\]
where $g(y) = h(y/N)$, so that $[A, B] = [N, 2N]$, $G = O(1)$ and $\phi(x) = \alpha j x^\delta$ for which
\[\phi'(x) = \frac{\theta \alpha j}{x^{1-\delta}}, \quad \phi''(x) = -\frac{\theta(1 - \theta) \alpha j}{x^{2-\delta}}.\]

Therefore we can choose
\[\Lambda \asymp \frac{j}{N^{2-\delta}}, \quad \eta \ll 1,\]
and
\[[b, a] \approx \frac{j}{N^{1-\delta}}[R, 2^{1-\delta}(R + 1)]\]
(also recall that we assume that $\alpha \in \text{supp } \rho = [R, R + 1]$). The critical points are solutions of
\[\frac{\theta \alpha j}{x_m^{1-\delta}} = m \iff x_m = \left(\frac{\theta \alpha j}{m}\right)^\Theta.\]

We find
\[\mathcal{E}_{N,j}(\alpha) = \frac{\theta^{\Theta/2}}{\sqrt{1 - \theta}} (\alpha j)^{\Theta/2} e\left(-\frac{1}{8}\right) \sum_{m \approx j/N^{1-\delta}} \frac{1}{m^{\Theta+1}} h\left(\frac{(\theta \alpha j)^\Theta}{Nm^\Theta}\right) e\left(\frac{(\theta^{\Theta-1} - \theta^\Theta)(\alpha j)^\Theta}{m^{\Theta-1}}\right)\]
\[+ O\left(\frac{N^{1-\delta}}{j^{1/2}} + \log N\right) = \tilde{\mathcal{E}}_{N,j}(\alpha) + O\left(\frac{N^{1-\delta}}{j^{1/2}}\right)\]
as claimed.

In particular there are no critical points unless $j \gg N^{1-\delta}$, in which case we just obtain an upper bound.

\[\square\]

### 4 | THE SECOND MOMENT OF $\tilde{\mathcal{E}}_{N,j}$ AND PROOF OF COROLLARY 2.3

Let $\rho \in C^\infty([R, R + 1]), \rho \geq 0$, normalized so that $\int_R^{R+1} \rho(x) \frac{dx}{x} = 1$. We define a smooth measure on $[R, R + 1]$ by
\[d\mu(\alpha) = \frac{\Theta \rho(\alpha^\Theta) d\alpha}{\alpha}\]
which satisfies $\int_{-\infty}^{\infty} d\mu(\alpha) = 1$. Further, we require the following application of the non-stationary phase principle dealing with the oscillatory integrals
\[I(j, m, n, N) := \int_R^{R+1} \alpha^\Theta h\left(\frac{(\theta \alpha j)^\Theta}{Nm^\Theta}\right) h\left(\frac{(\theta \alpha j)^\Theta}{Nn^\Theta}\right) e\left(c_2 \left(\frac{(\alpha j)^\Theta}{m^{\Theta-1}} - \frac{(\alpha j)^\Theta}{n^{\Theta-1}}\right)\right) d\mu(\alpha).\]

To bind the off-diagonal terms, we invoke
Lemma 4.1. For all $K \geq 1$, there is some $C_K = C_{K,h,\varphi} > 0$ so that for all distinct $m, n \asymp j/N^{1-\delta}$, and all $j \gg N^{1-\delta}$, we have that

$$|I(j, m, n, N)| \leq C_K N^{-K}.$$  

Proof. Changing variables $\beta = \alpha^\Theta$, and recalling the definition (2.4) of

$$d\mu(\alpha) = \frac{\Theta \rho(\alpha^\Theta) d\alpha}{\alpha} = \frac{\rho(\beta) d\beta}{\beta},$$

the oscillatory integral $I(j, m, n, N)$ can be written as

$$I(j, m, n, N) = \int_0^\infty e\left(\beta \cdot c_2 j^{\Theta} \left(\frac{1}{m^{\Theta-1}} - \frac{1}{n^{\Theta-1}}\right)\right) h(\beta x_m) h(\beta x_n) \rho(\beta) d\beta$$

with

$$x_m = \left(\frac{\Theta j}{N^{1-\delta} m}\right)^\Theta.$$  

Since $\text{supp}(\rho) \subseteq [R, R+1]$ and $\text{supp}(h) \subseteq [1, 2]$, we must have $\beta \in [R, R+1]$ and $\beta x_m, \beta x_n \in [1, 2]$. Hence for the integral to be non-zero, we must have

$$x_m, x_n \in \left[\frac{1}{R+1}, \frac{2}{R}\right].$$

The integral is the Fourier transform of the function $F \in C_c^\infty(\mathbb{R})$ (which depends on $j, m, n, N$):

$$F(\beta) = \rho(\beta) h(\beta x_m) h(\beta x_n),$$

evaluated at the point $-c_2 j^{\Theta} \left(\frac{1}{m^{\Theta-1}} - \frac{1}{n^{\Theta-1}}\right)$. For any $F \in C_c^\infty(\mathbb{R})$, we can bound the Fourier transform using integration by parts $K$ times:

$$|\hat{F}(y)| \leq \frac{1}{(2\pi |y|)^K} \int_{-\infty}^{\infty} |F^{(K)}(\beta)| d\beta, \quad y \neq 0.$$  

In our case, $|F|_\infty \ll 1$, and the derivatives of $F$ are bounded by

$$|F^{(K)}|_\infty \ll C_{h,\varphi} \left(1 + \max (x^K_m, x^K_n)\right)$$

and since $x_m, x_n = O(1)$ we obtain $|F^{(K)}|_\infty = O(1)$. Therefore if $m \neq n$ then

$$|\hat{F}(y)| \ll_K \frac{1}{|y|^K}.$$  

Finally, note that if $m \neq n$ (but both $m, n \asymp j/N^{1-\delta}$), then the frequency $y$ of the Fourier transform is bounded below by
\[ |y| = \left| c_2 j^\Theta \left( \frac{1}{m^{\Theta-1}} - \frac{1}{n^{\Theta-1}} \right) \right| \]
\[ \gg j^\Theta \left| \frac{1}{n^{\Theta-1}} - \frac{1}{m^{\Theta-1}} \right| \]
\[ \gg j^\Theta \frac{1}{(mn)^{\Theta-1}} (n - m)m^{\Theta-2} \]
\[ \gg \frac{j^\Theta N^{2(\Theta-1)(1-\Theta)}}{j^{2(\Theta-1)}} \cdot 1 \cdot \frac{1}{N^{1-\Theta}} = N^{\Theta(1-\Theta)} = N. \]

Thus \( |I(j, m, n, N)| \ll |\hat{F}(y)| \ll K N^{-K} \).

We proceed to show that \( |\tilde{\mathcal{E}}_{N,j}(\alpha)|^2 \) exhibits, essentially, square-root cancellation on average over \( \alpha \in [R, R+1] \). An important step to this end is:

**Proposition 4.2.** Assume \( N^{1-\Theta} \ll j < N^2 \). Then
\[ \int_{-\infty}^{\infty} |\tilde{\mathcal{E}}_{N,j}(\alpha)|^2 d\mu(\alpha) \ll N. \]

**Proof.** Recall that
\[ \tilde{\mathcal{E}}_{N,j}(\alpha) = c_1(\alpha j)^{\Theta/2} \sum_{m \approx j/N^{1-\Theta}} \frac{1}{m^{\Theta+1}} h \left( \frac{(\partial \alpha j)^{\Theta}}{Nm^{\Theta}} \right) e \left( c_2(\alpha j)^{\Theta} \frac{m^{\Theta-1}}{m^{\Theta-1}} \right). \]

Hence
\[ \int_{R}^{R+1} |\tilde{\mathcal{E}}_{N,j}(\alpha)|^2 d\mu(\alpha) \ll j^\Theta \sum_{m, n \approx j/N^{1-\Theta}} \frac{1}{(mn)^{\Theta+1}} |I(j, m, n, N)| \]
\[ \ll \frac{N^{2-\Theta}}{j} \sum_{m, n \approx j/N^{1-\Theta}} |I(j, m, n, N)|. \] (4.1)

The diagonal terms \( m = n \) contribute
\[ \ll \frac{N^{2-\Theta}}{j} \sum_{m \approx j/N^{1-\Theta}} 1 \ll \frac{N^{2-\Theta}}{j} \frac{j}{N^{1-\Theta}} = N. \]

We conclude the proof of Proposition 4.2 by deducing that the total contribution of all the off-diagonal terms to Equation (4.1) is bounded by
\[ \frac{N^{2-\Theta}}{j} \sum_{m, n \approx j/N^{1-\Theta}} |I(j, m, n, N)| \ll \frac{N^{2-\Theta}}{j} \left( \frac{j}{N^{1-\Theta}} \right)^2 N^{-100} \ll jN^{-99} \ll N^{-90} \]
since we assumed that \( j \ll N^2 \).
We can now proceed to the:

**Proof of Corollary 2.3.** Recall $N_{\varepsilon} = \lceil \varepsilon^C \rceil$ as well as, see Equation (3.3), that

$$S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{1 \leq j < N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right) |E_{N,j}(\alpha)|^2.$$  

Now fix $\tau \in \{1 - \varepsilon, 1 + \varepsilon\}$. We observe, by using Proposition 3.1, that

$$\frac{2}{N^2} \left| \sum_{1 \leq j < N^{\tau+\varepsilon}} \hat{f}\left(\frac{j}{N}\right) |E_{N,j}(\alpha)|^2 \right| \ll \frac{1}{N^2} \sum_{1 \leq j < N^{\tau}} \left( |\tilde{E}_{N,j}(\alpha)|^2 + \frac{N^{2-\theta}}{j} \right) \ll \frac{1}{N^2} \left( \sum_{1 \leq j < N^{\tau}} |\tilde{E}_{N,j}(\alpha)|^2 \right) + \frac{1}{N^\varepsilon} \quad (4.2)$$

Next, we consider

$$Y_{N, \tau}(\alpha) := \frac{1}{N^{1+\tau}} \sum_{j < N^\tau} |\tilde{E}_{N,j}(\alpha)|^2,$$

and infer from Proposition 4.2 that

$$\int_{-\infty}^{\infty} Y_{N, \tau}(\alpha) d\mu(\alpha) \ll 1.$$

By the Borel–Cantelli lemma and Markov’s inequality,

$$Y_{N_{\varepsilon}, \tau}(\alpha) \ll N_{\varepsilon}^\varepsilon$$

almost all $\alpha \in [R, R + 1]$. Using this bound in Equation (4.2) completes the proof. \qed

## 5 PROOF OF PROPOSITION 2.4

**Proof.** Recall Equation (3.2):

$$R_{f, h}(N)(\alpha) = S(f, h; N)(\alpha) + \int_{-\infty}^{\infty} f(x) dx \left( \int_{-\infty}^{\infty} h(y) dy \right)^2 - f(0) \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-100}) \quad (5.1)$$

with

$$S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{1 \leq j < N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right) |E_{N,j}(\alpha)|^2. \quad (5.2)$$
Inserting Proposition 3.1 and Corollary 2.3 in Equation (5.2) gives
\[
S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{N^{1-\epsilon} \leq j \ll N^{1+\epsilon}} \hat{f} \left( \frac{j}{N} \right) |\tilde{\varepsilon}_{N,j}(\alpha)|^2 + O(N^{-\epsilon/2})
\]
\[
= \frac{2}{N^2} \sum_{1 \leq j \ll N^{1+\epsilon}} \hat{f} \left( \frac{j}{N} \right) |\tilde{\varepsilon}_{N,j}(\alpha)|^2
\]
\[
+ O \left( \frac{N^{1-\theta/2}}{N^2} \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{|\hat{f}(\frac{j}{N})|}{j^{1/2}} |\tilde{\varepsilon}_{N,j}(\alpha)| + N^{-\theta} \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{|\hat{f}(\frac{j}{N})|}{j} \right)
\]

The second term in the O-symbol can be dispensed with by
\[
\sum_{1 \leq j \ll N^{1+\epsilon}} \frac{|\hat{f}(\frac{j}{N})|}{j} \ll \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{1}{j} \ll \log N.
\]

We proceed to bound the first term in the O-symbol. By using the estimate just above and $|\hat{f}(\frac{j}{N})| = O(1)$, the Cauchy–Schwarz inequality implies
\[
\frac{N^{1-\theta/2}}{N^2} \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{|\hat{f}(\frac{j}{N})|}{j^{1/2}} |\tilde{\varepsilon}_{N,j}(\alpha)| \ll N^{-1/2-\theta/2+\epsilon(1)} \left( \sum_{1 \leq j \ll N^{1+\epsilon}} |\tilde{\varepsilon}_{N,j}(\alpha)|^2 \right)^{1/2}.
\]

Hence we infer from Equation (2.5) that for almost all $\alpha \in [1, 2]$ the bound
\[
\frac{N^{1-\theta/2}}{N^2} \sum_{1 \leq j \ll N^{1+\epsilon}} \frac{|\hat{f}(\frac{j}{N})|}{j^{1/2}} |\tilde{\varepsilon}_{N,j}(\alpha)| \ll N^{-\theta/2+2\epsilon}
\]
holds. The upshot is that, for almost all $\alpha$ and all $N = N_\epsilon \in \mathcal{N}$, we have
\[
S(f, h; N_\epsilon)(\alpha) = \frac{2}{N_\epsilon^2} \sum_{N_\epsilon^{1-\epsilon} \leq j \ll N_\epsilon^{1+\epsilon}} \hat{f} \left( \frac{j}{N_\epsilon} \right) |\tilde{\varepsilon}_{N,j}(\alpha)|^2 + O(N_\epsilon^{-\theta/2+O(\epsilon)}).
\]

We for ease of notation, we write now $N$ in place of $N_\epsilon$. Thus we find that for almost all $\alpha$, and all $N \in \mathcal{N}$,
\[
S(f, h; N)(\alpha) = \tilde{S}(f, h; N)(\alpha) + O(N^{-\epsilon/2})
\]
(5.3)

where
\[
S(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{N^{1-\epsilon} \leq j \ll N^{1+\epsilon}} \hat{f} \left( \frac{j}{N} \right) |\tilde{\varepsilon}_{N,j}(\alpha)|^2.
\]
We write out

\[ \tilde{S}(f, h; N)(\alpha) = \frac{2}{N^2} \sum_{N^{1-\varepsilon} \leq j \leq N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right)|\hat{c}_{N, j}(\alpha)|^2 \]

\[ = \frac{2|c_1|^2 \alpha^\Theta}{N^2} \sum_{N^{1-\varepsilon} \leq j \leq N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right)j^\Theta \]

\[ \sum_{m, n \approx j/N^{1-\varepsilon}} \frac{1}{(mn)^{\Theta+1}} h\left(\frac{(\partial_x j)^\Theta}{Nm^\Theta}\right) h\left(\frac{(\partial_x j)^\Theta}{Nn^\Theta}\right) e\left(c_2 (\alpha j)^\Theta\left(\frac{1}{m^\Theta-1} - \frac{1}{n^\Theta-1}\right)\right). \]

(5.4)

Restricting the sum over \( m, n \) to the diagonal \( m = n \) gives a term of the form

\[ \sum_{n \geq 1} \frac{W}{n^\Theta} h(\frac{W}{n^\Theta})^2 = \frac{1 - \Theta}{W} \int_{-\infty}^{\infty} h(y)^2 dy + O(W^{-100}), \]

with

\[ W = \frac{(\partial_x j)^\Theta}{N} \gg N^{\Theta-1-\varepsilon} \to \infty. \]

Thus the contribution of the diagonal \( m = n \) to Equation (5.4) is

\[ \text{terms with } (m = n) + |c_1|^2 \frac{1 - \Theta}{\Theta} \frac{1}{N} \sum_{j \geq 1} \hat{f}\left(\frac{j}{N}\right) \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-10}) \]

\[ = f(0) \int_{-\infty}^{\infty} h(y)^2 dy + O(N^{-10}) \]

which exactly cancels off the diagonal term in Equation (3.1). Thus

\[ \tilde{S}(f, h; N)(\alpha) = f(0) \int_{-\infty}^{\infty} h(y)^2 dy + R_{\text{off}}(N)(\alpha) + O(N^{-10}) \]

(5.5)

where the off-diagonal terms are \( R_{\text{off}}(N)(\alpha) \), that is

\[ \frac{2|c_1|^2 \alpha^\Theta}{N^2} \sum_{N^{1-\varepsilon} \leq j \leq N^{1+\varepsilon}} \hat{f}\left(\frac{j}{N}\right)j^\Theta \]

\[ \sum_{m \neq n} \frac{1}{(mn)^{\Theta+1}} h\left(\frac{(\partial_x j)^\Theta}{Nm^\Theta}\right) h\left(\frac{(\partial_x j)^\Theta}{Nn^\Theta}\right) e\left(c_2 (\alpha j)^\Theta\left(\frac{1}{m^\Theta-1} - \frac{1}{n^\Theta-1}\right)\right). \]

(5.6)

Inserting Equations (5.3) and (5.5) into Equation (5.1) we obtain

\[ R_{f, h}(N)(\alpha) = \int_{-\infty}^{\infty} f(t) dt \left( \int_{-\infty}^{\infty} h(s) ds \right)^2 + R_{\text{off}}(N)(\alpha) + O(N^{-\varepsilon/2}) \]

for a.e. \( \alpha \in [R, R + 1] \) and all \( N \in \mathcal{N} \), which is Proposition 2.4.
6 | THE SECOND MOMENT OF $R_{\text{off}}(N)(\alpha)$

We now average $|R_{\text{off}}(N)(\alpha)|^2$ over $\alpha$, by using the measure Equation (2.4). The goal is to show Theorem 2.5, namely

$$
\int_{-\infty}^{\infty} |R_{\text{off}}(N)(\alpha)|^2 d\mu(\alpha) \ll N^{-\delta}
$$

(6.1)

for $\delta \in (0, 1 - \Theta)$. To this end, we require a simple lemma on oscillatory integrals. By arguing as in the proof of Lemma 4.1 we find:

**Lemma 6.1.** With $z_1 = m_1^{1-\Theta} - n_1^{1-\Theta}$ and $z_2 = m_2^{1-\Theta} - n_2^{1-\Theta}$, we define

$$
I(\tilde{j}, \tilde{m}, \tilde{n}, N) = \int_{-\infty}^{\infty} e(c_2 \cdot \alpha^{\Theta}(j_1^{\Theta}z_1 - j_2^{\Theta}z_2)) \prod_{r=1}^{2} h\left(\frac{\theta \alpha j^{\Theta}}{Nm_r^{\Theta}}\right) h\left(\frac{\theta \alpha j^{\Theta}}{Nn_r^{\Theta}}\right) \alpha^{2\Theta} d\mu(\alpha).
$$

For all $K \geq 1$, there is some $C = C_{K, h, \phi}$ so that for all $m, n, j$,

$$
|I(\tilde{j}, \tilde{m}, \tilde{n}, N)| \leq C \min\left(1, \frac{1}{|j_1^{\Theta}z_1 - j_2^{\Theta}z_2|^K}\right).
$$

Moreover, $I(\tilde{j}, \tilde{m}, \tilde{n}, N) = 0$ unless

$$
m_r, n_r \asymp j_r / N^{1-\Theta} \quad (r = 1, 2).
$$

(6.2)

Next, we relate the variance of $R_{\text{off}}$ to counting the solutions of Diophantine inequalities:

**Proposition 6.2.** Let $U = (1 + \epsilon) \log N$, $Q = (\Theta + \epsilon) \log N$. Then

$$
\int_{-\infty}^{\infty} |R_{\text{off}}(N)(\alpha)|^2 d\mu(\alpha) \ll \frac{(\log N)^2}{N^{2\Theta}} \sum_{(1-\epsilon)\log N < u \leq U} \sum_{0 \leq q \leq Q} e^{-2u} D_{u, q}
$$

(6.3)

where

$$
D_{u, q} = \# \left\{ e^u \leq j_1, j_2 < e^{u+1}, m_1, m_2, n_1, n_2 \asymp \frac{e^u}{N^{1-\Theta}} : 
\begin{align*}
|j_1 (\frac{1}{m_1^{\Theta-1}} - \frac{1}{n_1^{\Theta-1}}) - j_2 (\frac{1}{m_2^{\Theta-1}} - \frac{1}{n_2^{\Theta-1}})| &\leq N^\epsilon 
\end{align*}
\right\}.
$$

(6.4)
Proof. We rewrite Equation (5.6) as
\[
R_{off}(N)(\alpha) = \frac{2|c_1|^2\alpha^\Theta}{N^2} \sum_{0 \leq u \leq U} \sum_{0 \leq q \leq Q} \sum_{e^u \leq j \leq e^{u+1}} \hat{f}\left(\frac{j}{N}\right) j^\Theta
\]
\[
\sum_{m,n,e^u/N^N_1 = e^u/m \leq m < e^{u+1}} \frac{1}{(mn)^{\Theta+1/2}} \left(\frac{(\Theta \alpha j)^\Theta}{Nm^\Theta}\right) h\left(\frac{\Theta \alpha j}{Nn^\Theta}\right) e\left(c_2 \cdot (\alpha j)^\Theta \left(\frac{1}{m^\Theta - 1} - \frac{1}{n^\Theta - 1}\right)\right).
\]
Integrating over \(\alpha\) and applying Cauchy-Schwarz gives that the left hand side of Equation (6.3) is at most a constant time
\[
\frac{UQ}{N^4} \sum_{0 \leq q \leq Q} \sum_{e^u \leq j_1, j_2 < e^{u+1}} \frac{|I(j_1, m, n, N)|}{(m_1 n_1 m_2 n_2)^{1+\Theta}}.
\]
We see from Lemma 6.1 that the oscillatory integral \(I(j_1, m, n, N)\) vanishes unless Equation (6.2) holds, and is negligible unless
\[
|j_1^\Theta z_1 - j_2^\Theta z_2| \leq N^\epsilon.
\]
Hence, noting \(j_\gamma^\Theta / (m_\gamma n_\gamma) \approx e^{-\gamma} N^{2-\gamma}\), we infer \(\int_{-\infty}^\infty |R_{off}(N)(\alpha)|^2 d\mu(\alpha)\) is at most a constant times
\[
\frac{(\log N)^2}{N^4} \sum_{0 \leq q \leq Q} \sum_{e^u \leq j_1, j_2 < e^{u+1}} e^{-2uN^{4-2\Theta}} \left\{ |j_1^\Theta z_1 - j_2^\Theta z_2| \leq N^\epsilon \right\}
\]
\[
\ll \frac{(\log N)^2}{N^{2\Theta}} \sum_{0 \leq q \leq Q} \sum_{e^u \leq j_1, j_2 < e^{u+1}} e^{-2u} \left\{ |j_1^\Theta z_1 - j_2^\Theta z_2| \leq N^\epsilon \right\}
\]
which is the statement of Proposition 6.2.

\section{Decoupling and Using the Robert-Sargos Theorem}

We now bound \(D_{u,q}\) by using an integral representation, a modification of that used by Aistleitner, El-Baz and Munsch [1], and then invoke classical results about Dirichlet polynomials.

Recall that \(D_{u,q}\) is given by Equation (6.4):
\[
D_{u,q} := \# \{ e^u \leq j_1, j_2 < e^{u+1}, z_1, z_2 \in Z_{u,q} : |j_1^\Theta z_1 - j_2^\Theta z_2| < N^\epsilon \}
\]
where \( N^{1-\varepsilon} < e^u < N^{1+\varepsilon} \) and

\[
\mathcal{Z}_{u,q} = \left\{ z = \frac{1}{m^{\Theta-1}} - \frac{1}{n^{\Theta-1}} : m, n \asymp \frac{e^u}{N^{1-\Theta}}, \quad e^q \leq n - m < e^{q+1} \right\}
\]

(we consider it as a multi-set, that is the elements are counted with their multiplicities, if any).

Note that \( z \in \mathcal{Z}_{u,q} \) satisfies \( z \asymp e^q N / e^{\Theta u} \) and \( 1 \leq e^q \leq e^u / N^{1-\Theta} \). Further,

\[
\# \mathcal{Z}_{u,q} \asymp \frac{e^u}{N^{1-\Theta}} e^q.
\]

**Theorem 7.1.** If \( N^{1-\varepsilon} \ll e^u \ll N^{1+\varepsilon} \), then

\[
D_{u,q} \ll N^{1+3\Theta+o(1)}
\]

Together with Proposition 6.2, this will give the required bound Equation (6.1) on the variance, that is prove Theorem 2.5.

### 7.1 Estimating \( D_{u,q} \) by an integral

We define (following Regavim [10])

\[
P_{u,q}(t) := \sum_{z \in \mathcal{Z}_{u,q}} z^{2\pi it}.
\]

Also let \( D_u(t) \) be the Dirichlet polynomial

\[
D_u(t) := \sum_{e^u \leq j < e^{u+1}} j^{2\pi it},
\]

and set

\[
T = \frac{e^q N}{N^\varepsilon}.
\]

Note that \( T \asymp j^{\Theta} z / N^\varepsilon \) when \( j \asymp e^u \) and \( z \in \mathcal{Z}_{u,q} \), and that \( T \ll N^{1+\Theta} \).

The following lemma produces useful cut-off functions, by using Beurling–Selberg functions, see Appendix A.

**Lemma 7.2.** There is a smooth function \( \Phi : \mathbb{R} \to \mathbb{R} \) satisfying

1. \( \Phi \) is supported in \([-1, 1]\),
2. \( \Phi \geq 0 \) is non-negative,
3. the Fourier transform \( \hat{\Phi} \geq 0 \) is non-negative,
4. \( \hat{\Phi}(x) \geq 1 \) for \(|x| \leq 1\).
We first show:

**Lemma 7.3.** We can bound $D_{u,q}$ by the twisted second moment

$$D_{u,q} \ll \frac{1}{T} \int_{-\infty}^{\infty} |D_u(\Theta t)|^2 |P_{u,q}(t)|^2 \Phi \left( \frac{t}{T} \right) dt.$$  \hspace{1cm} (7.1)

**Proof.** We expand the integral

$$\int_{-\infty}^{\infty} |D_u(\Theta t)|^2 |P_{u,q}(t)|^2 \Phi \left( \frac{t}{T} \right) dt = \sum_{e^u \leq j_1, j_2 < e^{u+1}} \int_{-\infty}^{\infty} e^{-2\pi it \log \frac{j_1 z_1}{j_2 z_2}} \Phi \left( \frac{t}{T} \right) \frac{dt}{T}$$

$$= \sum_{e^u \leq j_1, j_2 < e^{u+1}} \hat{\Phi} \left( T \log \frac{j_1 z_1}{j_2 z_2} \right).$$

Since $\hat{\Phi} \geq 0$, we decrease the right-hand side if we drop all terms except those where $|j_1^\Theta z_1 - j_2^\Theta z_2| < N^\varepsilon$. For these, we have

$$| \log \frac{j_1^\Theta z_1}{j_2^\Theta z_2} | = | \log \left( 1 + \frac{j_1^\Theta z_1 - j_2^\Theta z_2}{j_2^\Theta z_2} \right) | \sim \frac{|j_1^\Theta z_1 - j_2^\Theta z_2|}{j_2^\Theta z_2} < \frac{N^\varepsilon}{e^{\Theta u} e^{\Theta u}} = \frac{1}{T}$$

and therefore if $|j_1^\Theta z_1 - j_2^\Theta z_2| < N^\varepsilon$ then $|T \log \frac{j_1^\Theta z_1}{j_2^\Theta z_2}| < 1$ so that

$$\hat{\Phi} \left( T \log \frac{j_1^\Theta z_1}{j_2^\Theta z_2} \right) \geq 1.$$ 

Therefore

$$\frac{1}{T} \int_{-\infty}^{\infty} |D_u(\Theta t)|^2 |P_{u,q}(t)|^2 \Phi \left( \frac{t}{T} \right) dt \geq \sum_{e^u \leq j_1, j_2 < e^{u+1}} 1 =: D_{u,q}.$$ 

\[ \square \]

### 7.2 Using a theorem of Robert–Sargos

The diagonal sub-system

$$\mathcal{Z}_{u,q,\text{diag}} = \{ z_1, z_2 \in \mathcal{Z}_{u,q} : |z_1 - z_2| < N^{2\varepsilon} e^{-\Theta u} \}$$

of $\mathcal{Z}_{u,q}$ plays an important role in the following. Our current goal is to show that it has essentially only diagonal solutions $z_1 = z_2$. To this end, the next result of Robert and Sargos [11, Theorem 2] is crucial:
Theorem 7.4. Suppose $1 - \Theta \in \mathbb{R} \setminus \{0, 1\}$, and $\varepsilon > 0$. Then there exists a $C_{\varepsilon, \Theta} > 0$ so that

$$
\# \{(x_1, y_1, x_2, y_2) \in [1, M]^4 \cap \mathbb{Z}^4 : |x_1^{1-\Theta} - y_1^{1-\Theta} + x_2^{1-\Theta} - y_2^{1-\Theta}| < \gamma M^{1-\Theta}\}
$$

is at most $C_{\varepsilon, \Theta} M^\varepsilon (M^2 + \gamma M^4)$ for any $\gamma > 0$ and any $M \geq 1$.

Now we are ready to bound $\# Z_{u, q, \text{diag}}$:

Lemma 7.5. We have $\# Z_{u, q, \text{diag}} \ll N^{2\Theta + O(\varepsilon)}$.

Proof. First of all note that $Z_{u, q, \text{diag}}$ equals

$$
\{m_1, n_1, m_2, n_2 \ll M : |m_1^{1-\Theta} - n_1^{1-\Theta} - m_2^{1-\Theta} + n_2^{1-\Theta}| < \gamma M^{(1-\Theta)}\}
$$

where

$$
M := e^u / N^{1-\Theta}, \quad \gamma := N^{2\varepsilon} e^{-\Theta u} M^{\Theta-1}.
$$

We compute that $\gamma = N^{2\varepsilon} / (e^u N^\Theta)$. Therefore,

$$
\gamma M^2 \ll N^{O(\varepsilon)} N^{2\Theta} N^{-(1+\Theta)} \ll 1.
$$

Applying Theorem 7.4 completes the proof. \[\square\]

Now we complete the proof Theorem 7.1 by proving:

Lemma 7.6. The right-hand side of Equation (7.1) is $O(e^u N^{3\Theta + O(\varepsilon)})$.

Proof. As $\Phi(t/T) = 0$ if $|t| > T$, we first note that

$$
\int_{-\infty}^{\infty} |D_u(\Theta t)|^2 |P_{u, q}(t)|^2 \Phi\left(\frac{t}{T}\right) dt = 2 \int_0^T |D_u(\Theta t)|^2 |P_{u, q}(t)|^2 \Phi\left(\frac{t}{T}\right) dt.
$$

Denote by $I_{\text{low}}$ (resp. by $I_{\text{high}}$) the contribution of $t \in [0, e^u]$ (resp. of $t \in [e^u, T]$) to the right-hand side. To bound $I_{\text{low}}$, we use the trivial bound

$$
|P_{u, q}(t)| \leq P_{u, q}(0) = \# Z_{u, q} \ll N^{\Theta + O(\varepsilon)} e^q
$$

combined with the bound (see ref. [9, Equation (34) on page 141])

$$
|D_u(t)| \ll \frac{e^u}{\sqrt{1 + t^2}} \quad \text{for } t \in [0, e^u].
$$

Thus,

$$
I_{\text{low}} \ll \int_0^{e^u} \frac{e^{2u}}{1 + t^2} N^{4\Theta + O(\varepsilon)} dt \ll e^{2u} N^{2\Theta + O(\varepsilon)} e^{2q}.
$$
To estimate $I_{\text{high}}$, we use, see ref. [9, Equation (34) on page 141], that

$$|D_u(t)| \ll t^{1/2}, \quad \text{for } t \in [e^u, e^{2u}];$$

by combining this with Lemma 7.6, we infer that

$$I_{\text{high}} \ll \int_0^T |t^{1/2}| |P_{u,q}(t)|^2 \Phi \left( \frac{t}{T} \right) dt \ll TN^{2\theta+O(\epsilon)}.$$

All in all, upon bearing $T \gg e^q N^{1+O(\epsilon)}$ in mind, we conclude

$$\frac{1}{T} \int_{-\infty}^{\infty} |D_u(\Theta t)|^2 |P_{u,q}(t)|^2 \Phi \left( \frac{t}{T} \right) dt \ll \frac{I_{\text{low}} + I_{\text{high}}}{T} \ll N^{O(\epsilon)}(e^u N^{2\theta} e^q + N^{2\theta}).$$

Using $e^q = O(N^\theta+O(\epsilon))$ completes the proof. □

**APPENDIX A: CONSTRUCTION OF $\Phi$**

Here we prove the existence of the well-behaved cut-off function $\Phi$.

**Proof of Lemma 7.2.** We start with a function $\Psi_+$, supported in $[-1, 1]$, so that $\hat{\Psi}_+ \geq 1_{[-1,1]}$ is a majorant for the indicator function of the interval $[-1,1]$, meaning that $\hat{\Psi}_+ \geq 0$, and $\hat{\Psi}_+(x) \geq 1$ for $|x| \leq 1$. Such functions were constructed by Beurling and Selberg, cf., ref. [16], starting from Beurling’s function

$$B(x) = \left( \frac{\sin \pi x}{\pi} \right)^2 \left( \frac{2}{x} + \sum_{n=0}^{\infty} \frac{1}{(x-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} \right)$$

which is a smooth majorant for the sign function, with Fourier transform $\hat{B}$ supported on $[-1, 1]$, and taking

$$\hat{\Psi}_+(x) = \frac{1}{2} (B(1-x) + B(1+x)),$$

see Figure A1. By construction, both $\Psi_+$ and $\hat{\Psi}_+$ are even and real valued.

Now take $\Phi = |\Psi_+|^2 = \Psi_+^2$, which is positive and has the same support as $\Psi_+$. The Fourier transform $\hat{\Phi}$ is the convolution

$$\hat{\Phi}(x) = \hat{\Psi}_+ * \hat{\Psi}_+(x) = \int_{-\infty}^{\infty} \hat{\Psi}_+(y) \hat{\Psi}_+(x-y) dy.$$

Since $\hat{\Psi}_+ \geq 1_{[-1,1]} \geq 0$, we have $\hat{\Phi}(x) = \hat{\Psi}_+ * \hat{\Psi}_+(x) \geq 1_{[-1,1]} * 1_{[-1,1]}(x)$. But $1_{[-1,1]} * 1_{[-1,1]}(x) = \max(2 - |x|, 0)$ is a tent function, and in particular also majorises the indicator function $1_{[-1,1]}$ so that

as required. □
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