A Tight Converse to the Spectral Resolution Limit via Convex Programming

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Abstract—It is now well understood that convex programming can be used to estimate the frequency components of a spectrally sparse signal from \( m \) uniform temporal measurements. It is conjectured that a phase transition on the success of the total-variation regularization occurs when the distance between the spectral components of the signal to estimate crosses \( 1/m \). We prove the necessity part of this conjecture by demonstrating that this regularization can fail whenever the spectral distance of the signal of interest is asymptotically equal to \( 1/m \).

I. INTRODUCTION

A. Line spectral estimation

Inferring on the fine scale properties of a signal from its coarse measurements is a common signal processing problem that finds myriad of applications in various areas of applied and experimental sciences. Line spectral estimation is probably one of the most iconic and fundamental instance of this category of problem, with direct application in optics, radar processing, medical imaging and telecommunications. In its most common formulation, it consists in recovering the location of highly informative peaks in the power spectrum. While the line spectral estimation problem aims to estimate the parameters of a sparse measure \( \mu \in \mathcal{M}(\mathbb{T}) \) from the observation \( y \) of a finite number of its uniform samples.

Denote by \( \mathbb{T} = [0, 1) \) the unidimensional torus and let by \( \mathcal{M}(\mathbb{T}) \) the set of integrable complex-valued measures defined over \( \mathbb{T} \). In mathematical terms, the line spectral estimation problem aims to estimate the parameters of a sparse measure \( \mu \in \mathcal{M}(\mathbb{T}) \) of the form

\[
\forall \omega \in \mathbb{T}, \quad \mu(\omega) = \sum_{k=1}^{s} c_k \delta_{x_k}(\omega)
\]

from its projection unto the first \( 2m+1 \) complex trigonometric moments \( y \in \mathbb{C}^{2m+1} \) of the form \( y_k = \langle e^{-i2\pi m\omega}, \mu \rangle \) for \( -m \leq k \leq m \). In the above, the finite subset \( X = \{x_k\}_{k=1}^{s} \subset \mathbb{T} \) classically represents the support of the frequencies to estimate and the subset \( C = \{c_k\}_{k=1}^{s} \subset \mathbb{C} \) represents their associated complex amplitudes. The sparse measure \( \mu \) is assumed to be unknown, meaning that both \( X \), \( C \), and \( s \) are unknown parameters to estimate. As a result, the overall observation vector \( y = [y_{-m}, y_{-m+1}, \ldots, y_m]^T \in \mathbb{C}^{2m+1} \) is given by the integral representation

\[
y = \int_{\mathbb{T}} a(\omega) \, d\mu(\omega),
\]

whereby each “atom” \( a(\cdot) \in \mathbb{C}^{2m+1} \) is the vector defined by

\[
a(\omega) = [e^{-i2\pi m\omega}, e^{-i2\pi (m-1)\omega}, \ldots, e^{i2\pi m\omega}]^T \quad \text{for all } \omega \in \mathbb{T}.
\]

Recovering \( \mu \) from the sole knowledge of \( y \) is obviously an ill-posed problem, since the set of measures \( \mu \in \mathcal{M}(\mathbb{T}) \) leading to the same observation \( y \) forms a affine subspace of \( \mathcal{M}(\mathbb{T}) \) of uncountable dimension. The line spectral estimation problem aims to recover the sparest measure \( \mu \) (the one with minimal support) that is consistent with the measurement \( y \) for the observation model \([1]\). Hence, the optimal estimator can be formulated as the output of the abstract optimization program

\[
\mu_0 = \arg \min_{\mu \in \mathcal{M}(\mathbb{T})} ||\mu||_0, \quad \text{subject to } y = \int_{\mathbb{T}} a(\omega) \, d\mu(\omega), \quad (2)
\]

whereby \( ||\cdot||_0 \) is the pseudo-norm counting the potentially infinite cardinality of a complex measure in \( \mathcal{M}(\mathbb{T}) \).

B. The spectral resolution limit

By opposition to the “classic” finite-dimensional inverse problem framework, one seeks, in the studied settings, to reconstruct continuously a subset \( X \) over \( \mathbb{T} \), instead of assuming that \( X \) belongs to a predefined finite subset of atoms. As a result, the notion of restricted isometric property (RIP) or incoherence commonly used in order to guarantee a robust inversion cannot be translated in the presented problem. In particular, two atoms \( a(\omega) \) and \( a(\omega + \Delta \omega) \) will become more that more coherent as \( \Delta \omega \) tends to zero, and inferring on their joined presence in the support set \( X \) will become a harder and harder task \([1]\). Hence, one can intuit that the reconstruction performances of the support set \( X \) are driven by its minimal warp-around distance over the torus \( \Delta_T(X) \), defined as

\[
\forall X \subseteq \mathbb{T}, \quad \Delta_T(X) \triangleq \inf_{x,x' \in X, p} \min_{x \neq x'} |x - x' + p|.
\]

It was recently proven \([2]\) that the line spectral estimation problem is untractable whenever \( \Delta_T(X) < \frac{1}{m} \) in the sense one can always find another discrete support set \( X' \subset \mathbb{T} \) that can explain the observations \( y \) within exponentially small noise levels with respect to the number of measurements \( m \). Hence, under this critical resolution limit, one cannot statistically distinguish between \( X \) and \( X' \) in the limit where \( m \rightarrow \infty \), no matter the estimation algorithm. This result is explained by the presence of a phase transition on the behaviors of the extremal singular values of Vandermonde matrices with collapsing nodes around the unit circle. The interested reader may refer to \([3]\) for discussion and extensions of this phenomenon. Moreover, it is particularly relevant to study those results under the light of the early work of Slepian \([4]\), who showed that no discrete
time signal of length \( m \) can asymptotically concentrate its energy in a spectral bandwidth narrower than \( \frac{1}{m} \).

C. Reconstruction via convex optimization

There is a vast literature in signal processing on spectral deconvolution methods. The multiple signal classification algorithm (MUSIC) is probably one of the most commonly used subspace based method for spectral spike deconvolution, with well understood theoretical performances \([5]\).

In the recent years, a growing enthusiasm has been placed in tackling the line spectral estimation problem though the lens of convex optimization after the pioneer work \([6]\) demonstrated that convex programming could recover any sparse measure having a support verifying \( \Delta_T(X) \geq \frac{2}{m} \) in absence of noise and for sufficiently large \( m \). The authors’ original idea consists in swapping the cardinality counting pseudo-norm in \([2]\) by the total mass \( |\mu(T)| \) of the measure defined by \( |\mu| (T) = \int_T d |\mu| \) for every measure \( \mu \in \mathcal{M}(T) \). The total mass can be easily interpreted as an extension of the classic \( L_1 \)-norm to the set of measures. The so-called total-variation (TV) regularization of the combinatorial Program \((2)\) reads in noiseless settings

\[
\mu_{TV} = \arg\min_{\mu \in \mathcal{M}(T)} |\mu| (T) \text{ subject to } y = \int_T a(\omega) d \mu (\omega),
\]

which is a well-defined convex program over \( \mathcal{M}(T) \).

The sufficient separation limit was later enhanced to \( \frac{26}{m} \) \([7]\). As suggested by simulation results \([8]\), the convex approach is conjectured to work asymptotically in the regime \( \Delta_T(X) > \frac{1}{m} \). Performance guarantees and stability of the reconstruction under white Gaussian noise have been derived in \([9]\), \([10]\).

Line spectral estimation is a canonical example of sparse inverse problems defined over the set of measures, we refer the interested reader to \([11]\)–\([13]\) for more generic aspects and extensions of this theory.

II. MAIN RESULTS

A. Spectral resolution limit of TV-regularization

This work focuses on tightening the necessary minimal separation \( \Delta_T(X) \) for the success of the TV-regularization. The best bound up-to-date was derived in \([11]\), showing that Program \((3)\) can fail whenever \( \Delta_T(X) < \frac{5}{26m} \). The proof relies on an argument on the decay rate of trigonometric polynomials around their supremal values \([12]\).

Theorem \([1]\) proposes an improvement of this result by showing the existence of measures having a minimal separation asymptotically close to \( \frac{1}{m} \) for which the convex approach fails. This tight result validates one side of the conjecture on the decay rate of trigonometric polynomials and highlights their relationships with the existence of dual certificates. Theorem \([6]\) states that such families can’t exist if the support set is not enough separated.

B. Impact of the second order term

Figure \(1\) presents sufficient values of the parameter \( M_\delta \) defined in Theorem \([1]\) for different choices of the second order term \( \delta \). Those results are a by-product of the analysis \((12)\) in the proof of Theorem \([6]\) and are presented for illustration purposes. However, the present curve has a priori no reason to act as a sharp bound on the minimal achievable value of \( M_\delta \).

C. Notations

Through this paper, \([s]\) denotes the sequence \([1, \cdots, s]\) for any \( s \in \mathbb{N} \). The set a complex trigonometric polynomials of degree \( m \) is denoted \( \mathcal{T}_m \). Trigonometric polynomials are assumed to be 1-periodic, so that any element \( Q \in \mathcal{T}_m \) writes

\[
\forall \omega \in \mathbb{T}, \quad Q(\omega) = \sum_{k=-m}^{m} q_k e^{i2\pi k \omega},
\]

for some vector \( q \in \mathbb{C}^{2m+1} \). The supremal norm over \( \mathbb{T} \) is denoted \( \|\cdot\|_{l_\infty} \). For any \( z \in \mathbb{C}^* \), the complex sign of \( z \) is defined by \( \text{sign}(z) = \frac{z}{|z|} \), and we let \( \mathbb{U} = \{ z \in \mathbb{C} : |z| = 1 \} \).

III. PROOF OF THEOREM\([1]\)

A. Dual certifiability

It is now well understood that the success of TV-regularization methods over the set of measures is conditioned by the existence of a so called dual certificate \([11]\), \([13]\): A function representing the values of the dual optimal Lagrange variables of Problem \((3)\) and satisfying some extremal interpolation properties. We recall from \([6]\) the following result as a starting point of our analysis.

Proposition \([2]\) (Dual certificate). The output of the convex optimization program \((3)\) is equal to the ground truth measure.

Figure 1. Upper bound on the minimal number of observations \( M_\delta \) requested by Theorem \([7]\) against the second order term \( \delta \). The curve admits a vertical asymptote of equation \( \log_2 (M_\delta) = \Theta \left((\delta - 2)^{-1}\right) \) at \( \delta \to 2 \).
\[ \mu = \sum_{k=1}^{s} c_k \delta_{x_k} \text{ if and if only there exists a complex trigonometric polynomial } Q \in T_m \text{ satisfying } \\
Q(x_k) = \text{sign}(c_k), \quad \forall k \in [s] \\
|Q(\omega)| < 1, \quad \forall \omega \notin X. \quad (4) \]

We aim to demonstrate Theorem 1 by constructing a sequence of well-separated measures \( \{\mu_m\}_{m \in \mathbb{N}} \) for which there is no element of \( T_m \) satisfying the extremal interpolation properties \( \{4\} \).

B. Diagonalizing families

In this subsection, we introduce the notion of diagonalizing families over \( T_m \). Lemma \( 3 \) draws an important connection between the existence of dual certificate for a measure \( \mu \) and the existence of a stable diagonalizing family on its support.

Definition 3 (Diagonalizing family). Let \( X = \{x_k\}_{k=1}^{s} \) be a finite subset of \( \mathbb{T} \). A first order diagonalizing family of \( X \) over \( T_m \) is a set of \( s \) elements \( P_X = \{P_l\}_{l=1}^{s} \) of \( T_m \) satisfying

\[ \forall l \in [s], \quad \begin{cases} P_l(x_k) = \delta_{l=k}, & \forall k \in [s] \\
P_l^*(x_k) = 0, & \forall k \in [s]. \end{cases} \quad (5) \]

Definition 4 (Stable diagonalizing family). A first order diagonalizing family \( P_X = \{P_l\}_{l=1}^{s} \) of \( X \) is said to be stable if and only if \( \|P_l\|_{L_{\infty}} = 1 \) for all \( l \in [s] \).

Lemma 5. Let \( X = \{x_k\}_{k=1}^{s} \) be a discrete subset of \( \mathbb{T} \) of cardinality \( s \leq m \). Suppose that for every \( u \in \mathbb{U}^s \), there exists \( Q_u \in T_m \) such that

\[ \begin{align*}
Q_u(x_k) &= u_k, & \forall k \in [s] \\
\|Q_u(x)\| < 1, & \forall x \notin X,
\end{align*} \quad (6) \]

then \( X \) has a first order stable diagonalizing family over \( T_m \).

Proof: Denote by \( U = \{u^{(k)}\}_{k=1}^{s} \) the set of vectors defined by \( u^{(k)} = [1, e^{2\pi(k-1)}, \ldots, e^{2\pi(k-1)(s-1)}]^T \in \mathbb{U}^s \) for all \( k \in [s] \). By assumption there exist \( s \) polynomials \( Q = \{Q_u^{(k)}\}_{k=1}^{s} \) satisfying Property \( \{6\} \). Since \( s \leq m \), the set of vectors \( U \) forms a basis of \( \mathbb{C}^s \). Hence, a classic interpolation theory argument ensures that the set of trigonometric polynomials \( Q \) constitutes a free family of \( T_m \). Consequently, \( Q \) spans a sub-vectorial space of \( T_m \) of dimension \( s \).

We aim to build a stable diagonalizing family \( P_X \) of \( X \) lying in the span of the family \( Q \). Namely, we construct

\[ \forall l \in [s], \quad P_l = \sum_{k=1}^{s} \alpha_k^{(l)} Q_u^{(k)}, \]

where \( \{\alpha_k^{(l)}\}_{k=1}^{s} \subset \mathbb{C}^s \) are coefficients to be determined. Each vector \( \alpha_k^{(l)} \) is the unique solution of the linear system

\[ \forall k \in [s], \quad \delta_{l=k} = P_l(x_k) = \sum_{k=1}^{s} \alpha_k^{(l)} Q_u^{(k)}(x_k) = \sum_{k=1}^{s} \alpha_k^{(l)} e^{2\pi(k-1)(l-1)}, \]

that reformulates for every \( l \in [s] \) under the matrix form \( F_s \alpha^{(l)} = e_l \), whereby \( F_s \in M_s(\mathbb{C}) \) is the discrete Fourier transform matrix in dimension \( s \), and \( e_l \) denotes the \( l \)th vector of the canonical basis of \( \mathbb{C}^s \). \( F_s \) is invertible with inverse \( F_s^{-1} = \frac{1}{s} F_s^* \), and consequently each polynomial \( P_l \) reads

\[ \forall l \in [s], \quad P_l = \frac{1}{s} \sum_{k=1}^{s} e^{i2\pi(k-1)(l-1)} Q_u^{(k)}, \quad (7) \]

and \( P_X \) verifies by construction the first condition of \( \{5\} \).

Next, since \( |Q_u^{(k)}(x_k)| = |u_k| = 1 \) and \( |Q_u(\omega)| < 1 \) for every element \( \omega \) lying in a small open ball centered on \( x_k \), one may conclude that \( Q_u^{(k)}(x_k) = 0 \) for all \( k \in [s] \). Hence, by linearity \( P_l \) also satisfies \( P_l^{(k)}(x_k) = 0 \) for all \( k \in [s] \). The second condition of \( \{5\} \) is verified and \( P_X \) is a first order diagonalizing family for \( X \) over \( T_m \).

Finally, one proves the stability of the family \( P_X \) by applying the triangular inequality to Equation \( \{7\} \)

\[ \forall \omega \in \mathbb{T}, \quad |P_l(\omega)| \leq \frac{1}{s} \sum_{k=1}^{s} |Q_u^{(k)}(\omega)| \leq 1, \]

which ensures that \( \|P_l\|_{L_{\infty}} \leq 1 \). Furthermore, since \( |P_l(x_k)| = 1 \), one has as well \( \|P_l\|_{L_{\infty}} \geq 1 \) for all \( l \in [s] \). Hence \( \|P_l\|_{L_{\infty}} = 1 \), and the stability property of \( P_X \) follows.

C. Existence of stable diagonalizing families

It is worth noticing that, by a classic linear algebra argument, any set \( X \subset \mathbb{T} \) with cardinality \( s \leq m \) admits infinitely many diagonalizing families. However the existence of a stable one is not necessary guaranteed. Theorem \( 6 \) states that there exist sets with asymptotic minimal distance \( \frac{1}{m} \) that do not admit a stable diagonalization family over \( T_m \). Its demonstration is delayed to Section \( IV \) for readability.

Theorem 6. For every real \( \delta > 2 \), there exists \( M_\delta \in \mathbb{N} \), such that for every \( m \geq M_\delta \), there exists a set \( X_m = \{x_k\}_{k=1}^{s_m} \subset \mathbb{T} \) such that \( \Delta_T(X_m) \geq \frac{1}{m} - \frac{\delta}{m^2} \) and there is no stable diagonalization family for \( X \) over \( T_m \).

D. Conclusion on Theorem 7

We now have all the elements to complete the proof of Theorem 1. First of all, by Theorem 6 for \( \delta > 2 \) and \( m \in \mathbb{N} \) sufficiently large, there exists a subset \( X_m = \{x_k\}_{k=1}^{s_m} \subset \mathbb{T} \) of cardinality \( s_m \) such that \( \Delta_T(X_m) \geq \frac{1}{m} - \frac{\delta}{m^2} \), and that does not admit any stable diagonalization basis over \( T_m \).

Using the contraposition of Lemma 5 on \( X_m \), one can check the existence of a set \( \{x_k\}_{k=1}^{s_m} \subset \mathbb{T} \) of strictly positive reals. One has sign \( \{\tau_k u_k\} \subset \mathbb{R} \), and we conclude using the negation of Proposition 2 that the measure \( \mu_m \) is not solution of Program \( \{3\} \).
IV. PROOF OF THEOREM

Let \( m \in \mathbb{N} \), and let \( X = \{x_k\}_{k=1}^K \) be a subset of \( \mathbb{T} \) of cardinality \( s \). First of all, if \( P_l \in \mathbb{T}_m \) is the \( l \)th element of a diagonalizing family \( \mathcal{P}_\gamma \) over the set \( X, P_l \) and its derivative \( P'_l \) both cancel at every point \( x_k \) for \( k \neq l \). Consequently \( P_l \) has roots with multiplicity two at each of those locations, and \( P_l \) belongs to the ideal generated by the minimal vanishing trigonometric polynomial \( Z_{\gamma,l}^m \in \mathbb{T}_m \) defined by

\[
\forall \omega \in \mathbb{T}, \quad Z_{\gamma,l}^m(\omega) \triangleq \prod_{1 \leq k \leq l} \frac{\sin^2(\pi(\omega - x_k))}{\sin^2(\pi(x_k - x_l))}. \tag{8}
\]

Hence, there exists a factorization of \( P_l \) under the form

\[
\forall \omega \in \mathbb{T}, \quad P_l(\omega) = Z_{\gamma,l}^m(\omega) R_l(\omega), \tag{9}
\]

where \( R_l \in \mathbb{T}_{m-s+1} \). Noticing from Equation (14) that

\[
Z_{\gamma,l}^m(x_l) = 1, \quad Z_{\gamma,l}^m(x_l) = \sum_{1 \leq k \leq l} \frac{\cot(\pi(x_l - x_k))}{\sin(\pi(x_l - x_k))} = \eta_l,
\]

and using the Assumption on \( P_l, R_l \) must obey the two interpolation conditions \( R_l(x_1) = 1 \) and \( R_l'(x_1) = -\eta_l \).

Next, we construct a well-separated subset of \( \mathbb{T} \), and show that no polynomial of the form (9) is stable in the sense of Definition. For convenience, we restrict our analysis to odd number of observations \( m = 2K + 1 \), and claim that the result is extendable for even \( m \). Let \( \alpha_m \triangleq \frac{1}{m} - \frac{\delta}{m^2} \), and consider the set \( X_{m,\delta} = \{x_k(m,\delta)\}_{k=-K}^K \) of \( m \) elements of the form

\[
\forall k \in [-K,K], \quad x_k(m,\delta) \triangleq \frac{k \alpha_m}{m} = k \left( \frac{1}{m} - \frac{\delta}{m^2} \right).
\]

Suppose that \( m \) is large enough so that \( 0 < \alpha_m < 1 \). A direct computation gives

\[
\|x_k(m,\delta) - x_l(m,\delta)\|_\mathbb{T} = 1 - \alpha_m + \frac{\alpha_m}{m} \geq \frac{\alpha_m}{m},
\]

and the minimal distance \( \Delta_T(X_{m,\delta}) \) reads

\[
\Delta_T(X_{m,\delta}) = \|x_1(m,\delta) - x_0(m,\delta)\|_\mathbb{T} = \alpha_m/m = 1 - \frac{\delta}{m^2}.
\]

Let \( P_0 \in \mathbb{T}_m \) be a diagonalizing polynomial of \( X_{m,\delta} \) for the element \( x_0(m,\delta) = 0 \). \( P_0 \) can be factorized under the form \( P_0 = Z_{\delta}^m \times R_0 \), where \( Z_{\delta}^m \triangleq Z_{\delta}^m, \forall 0 \in \mathbb{T}_{m-1} \) is the minimal polynomial vanishing on \( X_{m,\delta} \setminus \{0\} \). The factor \( R_0 \) has maximal degree 1 and must have a vanishing derivative at point 0, since \( \eta_0 = 0 \) by symmetry of \( X_{m,\delta} \). Hence, there exists \( \gamma \in \mathbb{C} \) such that \( R_0 \approx \gamma \), whereby

\[
\forall \gamma \in \mathbb{C}, \forall \omega \in \mathbb{T}, \quad R_0(\omega) \triangleq (1 - \gamma) + \gamma \cos(2\pi \omega), \tag{10}
\]

and \( P_0 \) writes \( P_0 = P_{m,\delta,\gamma} \triangleq Z_{\delta}^m \times R_0 \) for some \( \gamma \in \mathbb{C} \).

It remains to show that if \( \delta \) is sufficiently large \((\alpha_m \) sufficiently small), every polynomial of the form \( P_{m,\delta,\gamma} \) verifies \( \|P_{m,\delta,\gamma}\|_{L_\infty} > 1 \). We introduce the quantity

\[
L(m,\delta) \triangleq \inf_{\gamma \in \mathbb{C}} \|P_{m,\delta,\gamma}\|_{L_\infty} = \inf_{\gamma \in \mathbb{C}} \sup_{\omega \in \mathbb{T}} |P_{m,\delta,\gamma}(\omega)|,
\]

and aim to bound it away from 1 for small enough \( \alpha_m \). Intuitively, we expect \( Z_{\delta}^m \) to be large far away from its roots, at \( \omega \approx \frac{1}{2} \), and expect that the restrictive structure \( \mathbb{T}_m \) on \( R_0 \) will not suffice the drag the product \( Z_{\delta}^m \) down. For ease of calculation, we introduce the translated polynomials \( \tilde{Z}_{\delta}^m(\omega) = Z_{\delta}^m(\omega - \frac{1}{2}) \) and \( \tilde{R}_0(\omega) = R_0(\omega - \frac{1}{2}) \) for all \( \omega \in \mathbb{T} \), and let \( \Omega_m = [-\frac{\alpha_m}{m}, \frac{\alpha_m}{m}] \subset \mathbb{T} \). The two following key lemmas, demonstrated in Section IV, provide lower bounds on \( \tilde{Z}_{\delta}^m \) and \( \tilde{R}_0 \) over the set \( \Omega_m \).

Lemma 7. There exists a constant \( C(\delta) > 0 \) such that

\[
\forall m \in \mathbb{N}, \forall \omega \in \Omega_m, \quad \tilde{Z}_{\delta}^m(\omega) \geq C(\delta) m^{2(\delta - 1)}.
\]

Lemma 8. Let \( \tilde{R}_0 \in \mathbb{T}_1 \) be in (10), then

\[
k_m \triangleq \inf_{\gamma \in \mathbb{C}} \sup_{\omega \in \Omega_m} |\tilde{R}_0(\omega)| \geq \frac{\pi^2 \alpha_m^2}{3m^2}. \tag{11}
\]

One may lower bound the quantity \( L(m,\delta) \) by spitting the infimum of \( |P_{m,\delta,\gamma}| \), and apply Lemma 7 and Lemma 8.

\[
L(m,\delta) = \inf_{\gamma \in \mathbb{C}} \sup_{\omega \in \mathbb{T}_m} |\tilde{Z}_{\delta}^m(\omega) R_0(\omega)|
\]

\[
= \inf_{\gamma \in \mathbb{C}} \sup_{\omega \in \mathbb{T}_m} |\tilde{Z}_{\delta}^m(\omega) \tilde{R}_0(\omega)|
\]

\[
\geq \inf_{\gamma \in \mathbb{C}} \inf_{\omega \in \Omega_m} |\tilde{Z}_{\delta}^m(\omega) \tilde{R}_0(\omega)|
\]

\[
\geq C(\delta) \frac{\pi^2 \alpha_m^2 m^{2(\delta - 2)}}{3} = \Theta(m^{2(\delta - 2)}).
\]

Hence, if \( \delta > 2 \), \( L(m,\delta) \) diverges and there exists \( M_\delta > 0 \) such that for all \( m \geq M_\delta \) there is no stable diagonalization family of \( X_{m,\delta} \) over \( \mathbb{T}_m \).

V. PROOFS OF THE AUXILIARY LEMMAS

A. Proof of Lemma 7. Lower bound on \( Z_0^m(\omega) \)

The roots \( \{\tilde{x}_k(m,\delta)\}_{k=1}^K \) of \( \tilde{Z}_{\delta}^m \) are linked to the ones of \( Z_{\delta}^m \) through the relation \( \tilde{x}_k(m,\delta) = \frac{1}{2} - x_{K-k+1}(m,\delta) \), and a direct calculation yields

\[
\tilde{x}_1 = \frac{1}{2} - \frac{K \alpha_m}{m} = \frac{1 - \alpha_m}{2} + \frac{\alpha_m}{2m}.
\]

Consequently \( \{\tilde{x}_k(m,\delta)\}_{1 \leq k \leq K} \) whereby

\[
\beta_m \triangleq m \tilde{x}_1 - \alpha_m = \frac{m(1 - \alpha_m) - \alpha_m}{2} = \frac{\delta (1 + \frac{1}{m}) - 1}{2}.
\]

Using Expression 88, one may rearrange \( \tilde{Z}_{\delta}^m \) as follows

\[
\forall \omega \in \mathbb{T}, \quad \tilde{Z}_{\delta}^m(\omega) = \prod_{k=1}^K \frac{\sin^2\left(\pi \left(\frac{\beta_m + k \alpha_m}{m} - \omega\right)\right)}{\sin^2\left(\pi \left(\frac{\beta_m + k \alpha_m}{m} + \omega\right)\right)}.
\]

(14)
Since $\delta > 1$, $\beta_m$ is positive through $[13]$ and $\tilde{Z}_{m,\delta}$ has no roots over the set $\Omega_m$. Consequently, the logarithm $\tilde{z}_{m,\delta}$ of $|\tilde{Z}_{m,\delta}|$ is well defined over $\Omega_m$, and it yields

$$
\forall \omega \in \Omega_m, \quad \tilde{z}_{m,\delta}(\omega) = \sum_{k=1}^{K} \ln \left( \frac{\beta_m + k\alpha_m}{m} - \omega \right) + \ln \left( \frac{\beta_m + k\alpha_m + \omega}{m} \right) - 2\ln \left( \frac{k\alpha_m}{m} \right).
$$

We derive a lower bound on $\tilde{z}_{m,\delta}$ over $\Omega_m$ by using the two following results, whose elementary proofs have been skipped.

**Fact 9.** For any $t, h \in \mathbb{R}^+$ such that $t + h \leq \frac{\pi}{2}$, we have that

$$
\ln \sin(t + h) - \ln \sin(t) \geq h \cot(t) - \frac{h^2}{2} \csc^2(t).
$$

**Fact 10.** For all $m \in \mathbb{N}$ and all $\frac{2}{\pi} < \alpha < 1$, the following inequalities hold.

$$
\sum_{k=1}^{K} \cot \left( \frac{\pi k\alpha_m}{m} \right) \geq \frac{m}{\pi \alpha_m} \ln \left( \frac{m}{\pi \alpha_m} \right) - \frac{2\beta_m}{\alpha_m} \ln \left( \frac{m}{\pi \alpha_m} \right) - \frac{2\beta_m}{\alpha_m} \ln \left( \frac{m}{\pi \alpha_m} \right) - 2 \left( \frac{2m^2}{\alpha_m} \right) \geq \frac{2\beta_m}{\alpha_m} \ln \left( \frac{m}{\pi \alpha_m} \right). 
$$

Moreover, for a fixed value of $\gamma$, the symmetry of the function $\hat{R}_\gamma(\omega)$ and its monotonic behaviors over $[0, \omega_{\text{max}}]$ imply that the supremum is reached either on $0$ or on $\omega_{\text{max}}$, leading to

$$
\sup_{\omega \in \Omega} \left| \hat{R}_\gamma(\omega) \right| = \max \left\{ \hat{R}_\gamma(0), \hat{R}_\gamma(\omega_{\text{max}}) \right\} 
$$

Defining the auxiliary function $y$ over $\mathbb{R}^+$ as $y(\gamma) = (1 - 2\gamma)^2 - (1 - 2\gamma)^2$, $y(\gamma)$ is positive whenever the maximum ($17$) is reached at $0$ and negative whenever it is reached at $\omega_{\text{max}}$. The auxiliary function is parabolic on $\gamma$ and we have

$$
\left\{ \begin{array}{l}
(1 + c)\gamma - (1 - c) \gamma \\
\text{for } \gamma \geq \frac{1}{1+c},
\end{array} \right.
$$

which takes positive values for $\gamma \geq \frac{1}{1+c}$. Hence

$$
\sup_{\omega \in \Omega} \left| \hat{R}_\gamma(\omega) \right| = \left\{ \begin{array}{l}
(1 - 2\gamma) \gamma \\
(1 - 2\gamma) \gamma \\
\text{if } \gamma \geq \frac{1}{1+c},
\end{array} \right.
$$

is a piecewise monotonic function in $\gamma$. By similar argument,

$$
\kappa(\omega_{\text{max}}) = \min \left\{ \frac{1}{1 - 2\gamma} \gamma \right\} = \frac{1 - c}{1 + c} \frac{\sin^2(\pi \omega_{\text{max}})}{1 + \cos^2(\pi \omega_{\text{max}})} \geq \frac{\pi^2 \omega_{\text{max}}^2}{3}.
$$

And one concludes on the lemma by letting $\omega_{\text{max}} = \frac{\alpha_m}{m}$.

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