Parametric bootstrapping in a generalized extreme value regression model for binary response

DIOP Aba∗
E-mail: aba.diop@uadb.edu.sn
Equipe de Recherche en Statistique et Modèles Aléatoires
Université Alioune Diop, Bambey, Sénégal

DEME El Hadji
E-mail: elhadjideumeufsat@gmail.com
Laboratoire d’Etude et de Recherche en Statistique et Développement
Université Gaston Berger, Saint-Louis, Sénégal

Abstract

Generalized extreme value (GEV) regression is often more adapted when we investigate a relationship between a binary response variable Y which represents a rare event and potential predictors X. In particular, we use the quantile function of the GEV distribution as link function. Bootstrapping assigns measures of accuracy (bias, variance, confidence intervals, prediction error, test of hypothesis) to sample estimates. This technique allows estimation of the sampling distribution of almost any statistic using random sampling methods. Bootstrapping estimates the properties of an estimator by measuring those properties when sampling from an approximating distribution. In this paper, we fitted the generalized extreme value regression model, then we performed parametric bootstrap method for testing hypothesis, estimating confidence interval of parameters for generalized extreme value regression model and a real data application.

Keywords: generalized extreme value, parametric bootstrap, confidence interval, test of hypothesis, Dengue.

∗Corresponding author
1 Introduction

Classical methods of statistical inference do not provide correct answers to all the concrete problems the user may have. They are only valid under some specific application conditions (e.g. normal distribution of populations, independence of samples etc.). The estimation of some characteristics such as dispersion measures (variance, standard deviation), confidence intervals, decision tables for hypothesis tests, is also based on the mathematical expressions of the probability laws, as well as approximations of these when the calculation was not feasible.

A good estimate of the nature of the population distribution can lead to powerful results. However, the price to pay is high if the assumption of the distribution is incorrect. It is therefore important to consider other analysis methods (such as non-parametric methods, for which the conditions of application are more restrictive), which are more flexible with the choice of the distribution and based on this, bootstrap methods was introduced.

Bootstrapping is a relatively new, computer-intensive statistical methodology introduced by Efron (1979). The bootstrap method replaces complex analytical procedures by computer intensive empirical analysis. It relies heavily on Monte Carlo Method where several random resamples are drawn from a given original sample. The bootstrap method has been shown to be an effective technique in situations where it is necessary to determine the sampling distribution of (usually) a complex statistic with an unknown probability distribution using these data in a single sample. The bootstrap method has been applied effectively in a variety of situations. Efron and Tibshirani (1994), Shao (1996) and Shao (2010) provide a comprehensive discussion of the bootstrap method. Andronov and Kulynska (2020) discussed statistical properties of the approximations of the "bootstrap-generated" data sets in more detail. Many studies have shown that the bootstrap resampling tech-
nique provides a more accurate estimate of a parameter than the analysis of any one of
the samples (see for example Carpenter and Bithell (2000), Zoubir and Iskander (2004),
Manly (1997), Shao and Tu (1995)).

The bootstrap method is a powerful method to assess statistical accuracy or to esti-
mate distribution from sample’s statistics (Reynolds and Templin (2004), Davison et al.
(2003), Chernick (1998)). In principle there are three different ways of obtaining and
evaluating bootstrap estimates: non-parametric bootstrap which does not assume any dis-
tribution of the population; semi-parametric bootstrap, which partly has an assumption
on the distribution on parameter and whose residuals have no distributional assumption;
and finally parametric bootstrap which assumes a particular distribution for the sample at
hand. Adjeil and Karim (2016) have considered parametric and non-parametric bootstrap
in the case of classical logistic regression model.

In this work we aim to estimate the probability of infection as function of potential
predictors $X = x$ using a generalized extreme value regression model for binary data,
construct confidence intervals and testing hypothesis for the unknown parameters of the
modele using both classical and bootstrap (non-parametric and parametric) methods.

The rest of this paper is organized as follows. In Section 2 we describe the problem
of GEV regression model and parametric bootstrapping method. Section 3 present the
obtained results. A discussion and some perspectives are given in Section 4.
2 Method

2.1 Generalized extreme value regression model

Generalized is used to model rare and extreme event (see Coles (2001). In the case where
the dependent variable $Y$ represents a rare event, the logistic regression model (obviously
used for this category of data) shows relevant drawbacks. We suggest the quantile function
of the GEV distribution as link function to investigate the relationship between the
binary response variable $Y$ and the potential predictors $X$ (see Wang and Dey (2010)
and Calabrese and Osmetti (2013) for more details). We use Bootstrapping method as
a tool to estimate parameters and standard errors for generalized extreme value regression
model. For a binary response variable $Y_i$ and the vector of explanatory variables $x_i$, let
$\pi(x_i) = P(Y_i = 1 | X_i = x_i)$ the conditional probability of infection. Since we consider the
class of Generalized Linear Models, we suggest the GEV cumulative distribution function
proposed by Calabrese and Osmetti (2013) as the response curve given by

$$\pi(x_i) = 1 - \exp\left\{\left[1 - \tau(\beta_1 + \beta_2 x_{i2} + \cdots + \beta_p x_{ip})\right]^{1/\tau}\right\}$$

$$= 1 - GEV(-x_i'\beta; \tau)$$

where $\beta = (\beta_1, \ldots, \beta_p)' \in \mathbb{R}^p$ is an unknown regression parameter measuring the association
between potential predictors and the risk of infection and $GEV(x; \tau)$ represents the
cumulative probability at $x$ for the GEV distribution with a location parameter $\mu = 0$, a
scale parameter $\sigma = 1$, an unknown shape parameter $\tau$.

For $\tau \to 0$, the previous model \(\Pi\) becomes the response curve of the log-log model, for
$\tau > 0$ and $\tau < 0$ it becomes the Frechet and Weibull response curve respectively, a particular case of the GEV one.
The link function of the GEV model is given by

\[
\frac{1 - \log(1 - \pi(x_i))}{\tau} = x_i'\beta = \eta(x_i)
\]  

(2)

The unknown vector parameter \( \beta \) will be estimated with \((1 - \alpha)\%\) confidence intervals \((\alpha \in [0; 1])\) and a test of hypothesis \( H_0: \beta_j = 0 \) by both classical approach of GEV regression model and bootstrap methods.

2.2 Parametric Bootstrapping

Parametric bootstraps resample a known distribution function, whose parameters are estimated from your sample. A parametric model is fitted using parameters estimated from the distribution of the bootstrap estimates, from which confidence limits are obtained analytically. In applications where the standard asymptotic theory does not hold, the null reference distribution can be obtained through parametric bootstrapping (see Reynolds and Templin (2004)). Maximum likelihood estimators are commonly used for parametric bootstrapping despite the fact that this criterion is nearly always based upon their large sample behaviour.

2.2.1 Parametric bootstrap confidence interval

Using an algorithm by Zoubir and Iskander (2004) or Carpenter and Bithell (2000) a parametric bootstrap confidence interval is obtained as follows:
1. Draw $B$ bootstrap samples $\{(Y_i^{(b)}, X_i^{(b)}), i = 1, \ldots, n\}$ ($b = 1, \ldots, B$) from the original data sample, and for each bootstrap sample, estimate $\beta$ and $\pi_i$ by its maximum likelihood estimators $\hat{\beta}_n^{(b)}$ in the model (1).

2. Calculate the bootstrap mean and standard error of $\hat{\beta}_n$ as follows:

\[ \hat{\beta}_n^* = \frac{1}{B} \sum_{b=1}^{B} \hat{\beta}_n^{(b)} \quad \text{and} \quad \hat{s}_n = \sqrt{\frac{1}{B-1} \sum_{b=1}^{B} (\hat{\beta}_n^{(b)} - \hat{\beta}_n^*)^2} \]

3. Calculate $(1 - \alpha)100\%$ bootstrap confidence interval by finding quantile of bootstrap replicates

\[ \left[ \hat{\beta}_{n,L}, \hat{\beta}_{n,U} \right] = \left[ \hat{\beta}_n^{(b), 1-\alpha}, \hat{\beta}_n^{(b), \alpha} \right] \]

### 2.2.2 Parametric Bootstrap for Test of Hypothesis

We use here the algorithm for parametric test of hypothesis given by Fox (2015) defined as follows:
1. Estimates parameters $\hat{\beta}_{n,j}$ of model (I) using the observed data and calculate observed statistic test $t_{\beta_j}^{obs} = \hat{\beta}_{n,j} / \hat{s}_{\beta_{n,j}}$. Let $\hat{\theta} = (\hat{\beta}_{n,j}, t_{\beta_j}^{obs})$

2. Draw $B$ bootstrap samples $\{(Y_i^{(b)}, X_i^{(b)}), i = 1, \ldots, n\} (b = 1, \ldots, B)$ from the original data sample, and for each bootstrap sample, estimate $\beta$ and $t_{\beta_j}^{obs}$ by its maximum likelihood estimators $\hat{\beta}_n^{(b)}$ and $t_{\beta_j}^{obs}$ in the model (I).

3. Calculate bootstrap p-value by

$$p-value = \frac{\# \left\{ |t_{\beta_j}^{obs}^{(b)}| > |t_{\beta_j}^{obs}| \right\}}{B}$$

3 Real data application

In this section, we consider a study of dengue fever, which is a mosquito-borne viral human disease. We consider here a database of size $n = 515$ (with 15.5% of 1’s), which was constituted with individuals recruited in Cambodia, Vietnam, French Guiana, and Brazil (?). Each individual $i$ was diagnosed for dengue infection and coded as $Y_i = 1$ if infection was present and 0 otherwise. We aim at estimating: i) the risk of infection for those individuals, based on this data set which also includes the following covariate: Weight (continuous bounded covariates), ii) testing the impact of weight on infection status by running a generalized extreme value regression analysis of the model defined as follows:

$$\pi_i = \Pr(Y_i = 1|Weight) = 1 - \text{GEV}[-(\beta_1 + \beta_2 \times Weight); \tau].$$

Results obtained from the GEV regression model (I) by parametric bootstrap are shown
Table 1: Parameter estimates of GEV regression model.

| Parameter       | Estimate | SE   | 95% C.I           | P-value |
|-----------------|----------|------|-------------------|---------|
| Intercept ($\beta_1$) | 0.9947   | 0.0739 | [0.8731,1.1162]   |         |
| Weight ($\beta_2$)  | -0.0456  | 0.0012 | [-0.0475,-0.0436] | < 0.0001|

Note: SE: standard error. C.I: confidence interval

Table 2: Confidence intervals and p-value by parametric bootstrap.

| Parameter       | Estimate | SE   | 95% C.I           | P-value |
|-----------------|----------|------|-------------------|---------|
| Intercept ($\beta_1$) | 1.3873   | 0.0926 | [-1.0480,6.7853]  |         |
| Weight ($\beta_2$)  | -0.0522  | 0.0015 | [-0.1567,-0.0005] | < 0.0001|

Note: SE: standard error. C.I: confidence interval

in Table 2. This results lead to similar conclusion from classical method. It can be observed that the parameter estimates are very close. The standard errors of estimates for parametric bootstrap were slightly bigger compared to that of classical approach. It is observed that in both situations of classical approach and parametric bootstrap method, the effect of weight is highly significant.

4 Discussion and perspectives

Confidence intervals are good indicators of practical significance, unlike p-values and they also provide more information than p values. Unfortunately, confidence intervals are rarely
reported in academic papers. This is because computing confidence intervals are not prac-
tical and not possible for some statistics. This is why bootstraps methods, which are
resampling techniques for assessing uncertainty, have become popular.
In this study we have performed bootstrapping parametric method on a GEV regression
model. Moreover bootstrapped method was compared with classical approch while calcul-
lating the parameters of GEV regression model. The bootstrap technique used for esti-
mation and testing produced flexible results. Several question can be asked: what is the
appropriate value of $B$ for confidence intervalles and for test of hypothesis ? For example
Efron and Tibshirani (1994) suggest that $B$ should be between 1000 and 2000 for 90-95
percent confidence intervals. Non-parametric bootstrap method can also be used for this
study. With the help of statistical software today it is easy to compute confidence intervals
and test of hypothesis for almost any statistics of interest.

References

Agresti A., Kateri M., 2011. Categorical Data Analysis. Springer, Berlin.

Andronov I.L., Kulynska V.P., 2020. Computer modeling of irregularly spaced signals.
Statistical properties of the wavelet approximation using a compact weight function,
Annales Astronomiae Novae, vol. 1, pp. 167-178.

Ariffin S.B., Midi H., 2012. Robust Bootstrap Methods in Logistic Regression Model. 2012
International Conference on Statistics in Science, Business and Engineering (ICSSBE),
Langkawi, 10-12, 1-6.

Adjei1 I.A., Karim R., 2016. An Application of Bootstrapping in Logistic Regression Model.
Open Access Library Journal, 3: e3049.
Chernick M.R., 1999. Bootstrap methods: a practitioner’s guide. New York: Wiley.

Coles S.G., 2001. An Introduction to Statistical Modeling of Extreme Values. Springer, New York.

Calabrese R., Osmetti S., 2013. Modelling SME Loan Defaults as Rare Events: an Application to Credit Defaults. Journal of Applied Statistics 40 (6), 1172-1188.

Carpenter J., Bithell J., 2000. Bootstrap Confidence Intervals: When, Which, What? A Practical Guide for Medical Statisticians. Statistics in Medicine, 19, 1141-1164.

Davison A.C., Hinkley D.V., Young, G.A., 2003. Recent Developments in Bootstrap Methodology. Statistical Science, 18, 141-157.

Davison A.C., Hinkley D.V, 1997. Bootstrap Methods and Their Application, Cambridge University Press.

Efron B., 1979. Bootstrap Methods: Another Look at the Jacknife. Annals of Statistics, vol. 7, N. 1; pp: 1-26.

Efron B., Tibshirani R.J., 1994. An Introduction to the Bootstrap. Chapman and Hall/CRC, UK.

Fox J., 2015. Applied Regression Analysis and Generalized Linear Models. Sage Publications, Thousand Oaks, California.

Hall P., 1992. The bootstrap and Edgeworth expansion. New York: Springer

Léger C., Politis D.H.N., Romano J.P., 1992. Bootstrap technology and applications. Technometrics 34, pp.378–398.
Manly B.F.J., 1997. Randomization, bootstrap and Monte Carlo methods in biology. New York: Chapman and Hall, 399 p.

Reynolds J.H., Templin W.D., 2004. Comparing Mixture Estimates by Parametric Bootstraping Likelihood Ratios. Journal of Agricultural, Biological, and Environmental Statistics, 9, 57-74.

Shao X., 2010. The Dependent Wild Bootstrap, Journal of the American Statistical Association Volume 105, 2010 - Issue 489, Pages 218-235.

Shao J., 1996. Bootstrap model selection. Journal of the American Statistical Association Vol. 91, No. 434, pp. 655-665.

Shao J., Tu D. The Jackknife and Bootstrap. Springer.

Wang X., Dey D.K., 2010. Generalized extreme value regression for binary response data: An application to B2B electronic payments system adoption. Ann. Appl. Stat. 4, 2000-2023.

Zoubir A.M., Iskander D.R., 2004. Bootstrap Techniques for Signal Processing. Cambridge University Press, Cambridge.