Multiscale Talbot effects in Fibonacci geometry

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Abstract

This article investigates the Talbot effects in Fibonacci geometry by introducing the cut-and-projection construction, which allows for capturing the entire infinite Fibonacci structure in a single computational cell. Theoretical and numerical calculations demonstrate the Talbot foci of Fibonacci geometry at distances that are multiples of the Talbot distance. Here \((p, q)\) are coprime integers, \(\mu\) is an integer, \(\tau\) is the golden mean, and \(F_p\) and \(L_p\) are Fibonacci and Lucas numbers, respectively. The image of a single Talbot focus exhibits a multiscale-interval pattern due to the self-similarity of the scaling Fourier spectrum.

Keywords: Talbot effect, Fibonacci geometry, cut and projection method, multiscale

1. Introduction

The Talbot effect, also referred to as self-imaging or has lensless imaging, is of the phenomena manifested by a running repetition of lateral periodic fields \(u(x)\) along the wave propagation \(z\) in Fresnel diffraction. The simplicity and beauty of the effect has triggered diverse research related to coherent optical signal models [1, 2] and has resulted in numerous interesting and original applications [3–5], providing competitive solutions to scientific and technological problems [6, 7]. Moreover, this the concept has been also transferred to a wide variety of problems which portray laws that are equivalent to the physics of diffraction of periodic wave fields, including electron microscopy [8], plasmonics [9], matter–wave interactions [10], and quantum mechanics [11, 12].

At the same time, Montgomery [13] demonstrated that lateral periodicity is a sufficient but not necessary condition for self-imaging, and hence has encouraged unexplored avenues to relevant fundamental optical sciences and device technology. For instance, aperiodic optical media generated by mathematical rules recently attracted significant attention in optics communities due to their unconventional nature [14, 15] and full compatibility with current materials deposition and device technologies [16–18]. Several recent research efforts have also leveraged aperiodicity as an efficient strategy to engineer optical manipulation, devices, and functionalities [19, 20]. In particular, Fibonacci geometries have been proved to demonstrate various novel optical properties [21–26].

This work proposes to transfer the concept of self-imaging to lateral-aperiodic objects [27, 28] and mainly investigates the Talbot effects for Fibonacci geometry. Considering that the Talbot effects are distinct only in the paraxial approximation and when the illuminated geometry tends to infinity, this work introduces the cut-and-projection construction techniques [14, 29], which capture the entire infinite Fibonacci structure in a single two-dimensional computational cell. Since the cut-and-projection procedure illustrates the fact that the Fourier transform of a projection operation is equivalent to a cut operation and vice versa, an analytical equation for the Talbot images of the Fibonacci slice embedded in 2D hyper-lattices is derived straightforwardly. We further demonstrate the approximate solution for the Talbot images of common 1D Fibonacci gratings at a propagation distance \(z_0 < z_T\). Here \(z_T\) is the primary Talbot distance. Numerical results show that the image of a single...
Talbot focus at \( z_0 < z_T \) exhibits a multiscale-interval pattern due to the self-similarity of the scaling Fourier spectrum.

2. Formulae and results

By introducing the cut-and-projection procedure in the field of the quasi-crystalline \[ 14 \], the Fibonacci structure \( u(x) \) along the dimension \( x \) can be constructed by an irrational cut of a periodic array of strips in the two-dimensional space \((x_1, x_2)\) (see figure 1(a)). This leads to a result of the Fourier spectrum \( \tilde{u}_{mn,k} \), becoming a function of the projection of the two-dimensional reciprocal lattice \( m\vec{k}_{x1} + n\vec{k}_{x2} \) on the \( k_x \) axis, as in figure 1(b). As shown in figure 1, the rectangular strips with width \( w \) (along the \( x \) direction) and length \( \Delta = \Lambda[\cos(\alpha) + \sin(\alpha)] \) (along the \( x \) direction) are rotated by an angle \( \alpha = \tan^{-1}[1/r] \) with respect to the basis \((x_1, x_2)\) of the square lattice having periodicity \( \Lambda \). \( \vec{k}_{x1} \) and \( \vec{k}_{x2} \) are the basis vectors in the reciprocal space, and \( \tau = (1 + \sqrt{5})/2 \) is the golden mean.

The transmittance function \( u(x, z = 0) \) for a Fibonacci structure at \( z = 0 \) can be given by \[ 14 \]

\[
\begin{align*}
    u(x, z = 0) &= \sum_{m,n=-\infty}^{\infty} \tilde{u}_{mn,k} \exp(i k_{mn,x} \cdot x) \\
    \tilde{u}_{mn,k} &= \frac{2}{\pi} \frac{\sin\left(\frac{k_{mn,w}}{2}\right) \sin\left(\frac{k_{mn,\Delta}}{2}\right)}{k_{mn,x} \cdot k_{mn,\tau}\Delta} \\
    k_{mn,x} &= \frac{2\pi (m + n\tau)}{\Lambda \sqrt{2 + \tau}}; \\
    k_{mn,xc} &= \frac{-2\pi (n - m\tau)}{\Lambda \sqrt{2 + \tau}}
\end{align*}
\]

It is easy to verify that the distribution of \((L,S)\) intervals of the function \( u(x, z = 0) \) in the \( x \) direction in equation (1) obeys a Fibonacci sequence \( LLSLLLSLLSLLSLLS \ldots \), in which \( L = \Lambda \cos(\alpha) \) and \( S = \Lambda \sin(\alpha) \) indicate the longer and shorter intervals between successive rectangular strips along the \( x \) direction, respectively. This can also be seen in figure 1(c). It is noted that the relation \( \Lambda^2 = L^2 + S^2 \) holds. Other properties of the Fibonacci Fourier transform are that,
with the relation $\tau^{\mu+1} = \tau^{\mu} + \tau^{-1}$, the sequence of Fourier vectors defined by equation (3) can be shown to be invariant when multiplied by any power of $\tau$ [14]. Moreover, in the limit of infinitesimal strip width $w \ll \Lambda$, the Fourier spectrum $\tilde{u}_{mn,k}$ in equation (2) remains invariant after scaling $k_{mn,x}$ by $\tau^{\mu}$, since $k_{mn,k} \to \tau^{\mu}k_{mn,x}$ gives $\tilde{u}_{mn,k}(k_{mn,x}) \to \tilde{u}_{mn,k}(\tau^{\mu}k_{mn,x})$. The Fourier spectrum, however, is rescaled linearly along the $k_x$ axis after scaling $k_{mn,x}$ by $\tau^{\mu}$, since $k_{mn,k} \to \tau^{\mu}k_{mn,x}$ leads to the approximation $\tilde{u}_{mn,k} \to \tilde{u}_{mn,k}(\tau^{\mu}k_{mn,x})$, mathematically. Note that $\tau$, $m$, and $n$ should remain finite in the paraxial approximation. Figure 2 shows numerically the self-similarity of the Fourier spectrum amplitude with different $k_{mn,x}$ or $k_{mn,xc}$ scalings.

We can calculate the propagation field $u(x, z_0)$ with wavelength $\lambda$ at distance $z = z_0$ by the method of the angular spectrum of a plane wave [30]. In Fresnel diffraction, it is given that

$$u_{2d}(x, z_0) = \sum_{m,n=-\infty}^{\infty} \tilde{u}_{mn,k} \exp(ik_{mn,x} \cdot x) \cdot \exp\left[-i\left(\frac{k_{mn,x}^2 + k_{mn,xc}^2}{2}z_0/(4\pi)\right)\right]$$

$$u_{1d}(x, z_0) = \sum_{m,n=-\infty}^{\infty} \tilde{u}_{mn,k} \exp(ik_{mn,x} \cdot x) \cdot \exp\left[-i\left(\frac{k_{mn,x}^2}{2}z_0/(4\pi)\right)\right]$$

Here the field $u_{2d}(x, z_0)$ represents the measurement of an $x$–cut–line of the two-dimensional illumination in figure 1(a), and the field $u_{1d}(x, z_0)$ is for the measurement of a one-dimensional grating in figure 1(c). It is straightforward to define the primary Talbot distance $z_T = 2A^2/\lambda$ by the relation

$$(k_{mn,x}^2 + k_{mn,xc}^2) = 4\pi^2(m^2 + n^2)/\lambda^2$$

for the field $u_{2d}$. Since the dominant spots of $\tilde{u}_{mn,k}$ occur with $k_{mn,xc}$ having small $n - m\tau$ (see equations (2) and (3)), the function $u_{1d}$ can be treated as a perturbation case of the function $u_{2d}$ with a phase deviation $\exp\left[i\frac{k_{mn,xc}^2}{2}z_0/(4\pi)\right]$ at small $z_0 < z_T$. The
Applying equation (4), the Talbot self-imaging for Fibonacci geometry is calculated and shown in figure 3. It can be seen that, aside from the conventional fractional Talbot foci at 
\[ z_0 = \frac{p}{q} \cdot T_0 \] 
figure 3 shows the irrational Talbot effects as well. Here \((p, q)\) are coprime integers. This also can be understood by the self-similarity of the scaling Fourier spectrum in the Fibonacci structure. Consider the scaling of \(k_{mn,x}\) by \(\tau^g\), the scaling of \(k_{mn,xc}\) by \(\tau^h\) \((g, h\) are integers), or the linear combination of different scaling terms, e.g., \(k_{mn,xc}^+\) by \(\tau^{2g+1} = \tau^{2g+2} - \tau^{2g}\) in equation (4). Without loss of generality, the propagation phase term 
\[ \exp \left[ -i \left( k_{mn,x}^2 + k_{mn,xc}^2 \right) \lambda z_0 / (4\pi) \right] \equiv \exp \left[ -i \Phi \right] \] 
can be given by
\[
\Phi = \frac{\pi \lambda z_0}{\lambda^2} \left[ \frac{(m + nt)^2}{\tau + 2} \tau^g + \frac{(n - mt)^2}{\tau + 2} \tau^h \right]
\]
properties of \(u_{1d}\) will be discussed in the last paragraph. Applying equation (4), the Talbot self-imaging for Fibonacci geometry is calculated and shown in figure 3.

It can be seen that, aside from the conventional fractional Talbot foci at \(z_0 = z_T \cdot p/q\) [30], figure 3 shows the irrational Talbot effects as well. Here \((p, q)\) are coprime integers. This also can be understood by the self-similarity of the scaling Fourier spectrum in the Fibonacci structure. Consider the scaling of \(k_{mn,x}\) by \(\tau^g\), the scaling of \(k_{mn,xc}\) by \(\tau^h\) \((g, h\) are integers), or the linear combination of different scaling terms, e.g., \(k_{mn,xc}^+\) by \(\tau^{2g+1} = \tau^{2g+2} - \tau^{2g}\) in equation (4). Without loss of generality, the propagation phase term 
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can be given by
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\Phi = \frac{\pi \lambda z_0}{\lambda^2} \left[ \frac{(m + nt)^2}{\tau + 2} \tau^g + \frac{(n - mt)^2}{\tau + 2} \tau^h \right]
\]
\[
\begin{align*}
F_{g-h} &= (-1)^g \left( F_g L_h - L_g F_h \right)/2 \\
L_{g-h} &= (-1)^h \left( L_g L_h - 5F_g F_h \right)/2
\end{align*}
\]

Here, \(F_g\) [31] and \(L_g\) [32] are Fibonacci and Lucas numbers, respectively. From the constraint \(\Phi = \pi \ell\) (\(\ell\) is an integer), the distance of the foci for the irrational Talbot image can be decided by equations (6) and (7):

\[
z_f = (2 + \tau) \left[ L_\mu + \tau L_{\mu+1} \right]^{-1} \left( \frac{z_T}{2} \right)
\quad \text{for even } \frac{g-h}{2}
\]

\[
z_f = (2 + \tau) \left[ F_\mu + \tau F_{\mu+1} \right]^{-1} \left( \frac{z_T}{2} \right)
\quad \text{for odd } \frac{g-h}{2}
\]

where the following relations have been used [31, 32]:

\[
F_{g+1} = \left( F_g + L_g \right)/2,
\]

\[
F_{g-1} = \left( L_g - F_g \right)/2
\]

\[
F_{g-h} = \left( -1 \right)^h F_{g+h} = F_h L_g
\]

\[
L_{g+h} = \left( -1 \right)^h L_{g-h} = L_h F_g
\]

\[
\text{Here } \mu = (g + h)/2 \text{ is an integer. It is easy to show that } z_f \text{ is equal to the conventional Talbot distance } z_T/2 \text{ [33] for } g = h = 0. \text{ Note that the coefficients in the first set of square brackets in equations (6) and (7) give an even integer, which can be proved by equation (8). Consequently an extra factor}
\]

Figure 6. Comparison of Talbot images between \(u_{1d}\) (lower red curves) and \(u_{2d}\) (upper blue curves) at (a) \(z_0 = 0\), (b) \(z_0 = z_T\), (c) \(z_0 = z_f,111\), (d) \(z_0 = z_f,311\), (e) \(z_0 = z_f,012\), and (f) \(z_0 = z_f,113\). Numerical results show similar multiscale-interval patterns except varying diffraction amplitudes.
1/2 entered equations (13) and (14). Together with the extension of the fractional Talbot effect [30] for Fibonacci geometry, the foci of the function \( u_{2d} \) can be decided for the distance

\[
z_{f,\rho \phi q} = \frac{z_T}{2} (2 + \rho)(L_\mu + \tau L_{\mu+1})^{-1} p q^{-1}
\]

for even \((g-h)/2\) \( (15) \)

\[
z_{f,\rho \phi q} = \frac{z_T}{2} (2 + \rho)(F_\mu + \tau F_{\mu+1})^{-1} p q^{-1}
\]

for odd \((g-h)/2\) \( (16) \)

with the additional conditions \(|u_{2d}(x, z_{f,\rho \phi q} + \ell z_T)|^2 = |u_{2d}(x, z_{f,\rho \phi q})|^2 \) and \(|u_{2d}(x, z_T - z_{f,\rho \phi q})|^2 = |u_{2d}(x, z_{f,\rho \phi q})|^2 \).

Remember that the scaling of \( k_{mn,\ell} (x^\rho \tau^\mu) \) makes the sequence of Fourier vectors and the Fourier spectrum \( \tilde{a}_{mn,k} \), invariant, and so does the term \( \tilde{a}_{mn,k} \exp(ik_{mn,\ell} \cdot x) \) in equation (4). The scaling of \( k_{mn,\ell} (x^\rho \tau^\mu) \), however, signifies a rescale of \( \tilde{a}_{mn,k} \) along the \( k_\ell \) axis (see figure 2) and equivalently suggests an inverse scaling of \( u_{2d}(x, z_0) \) along the \( x \) axis by

\[
\tilde{a}_{[\tau^\nu n]m} [\nu n]_k \exp(ik_{mn,\ell} \cdot x') \rightarrow \tilde{a}_{mn,k}
\]

\[
\exp(ik_{mn,\ell} \cdot x') \equiv \tilde{a}_{mn,k} \exp(ik_{mn,\ell} \cdot x) \text{ in equation (4).}
\]

Since \( \mu = (g + h)/2 \), the function \( z_{f,\rho \phi q} \) thus determines the Talbot foci of Fibonacci geometry having multiscale segments of \((L/q)\tau^{-\nu} \) or \((S/q)\tau^{-\nu} = (L/q)\tau^{-\nu-1} \) with \( \nu \in \{0, 1, \ldots, |\mu|\} \), roughly. Figure 4 depicts the numerical results for several \( z_{f,\rho \phi q} \) and presents multiscale Talbot effects for Fibonacci geometry. Figures 4(a) and (b) show the origin and the repetition of the Fibonacci structure at \( z_0 = 0 \) and \( z_0 = z_T \), respectively. Figure 4(c) is the focus image at \( z_0 = z_{f,111} = 0.3090z_T \) having a dominant \( \tau^{-1} \) and \( \tau^{-2} \) scaling pattern. Figure 4(d) is the focus image at \( z_0 = z_{f,311} = 0.2639z_T \) showing the distinct scaling pattern up to the order about \( \tau^{-5} \). Figure 4(e) is the fractional image at \( z_0 = z_{f,002} = 0.25z_T \) illustrating the half-length Fibonacci segments \( L/q = L/2 \) and \( S/2 \). Figure 4(f) is the focus image at \( z_0 = z_{f,113} = 0.1030z_T \) depicting the 1/3-shrinking as well as the \( \tau \)-scaling configurations.

Figure 5 shows the Talbot self-imaging of Fibonacci structure \( u_{1d} \) using the same parameters for figure 3. This image of \( u_{1d} \) presents properties similar to those of \( u_{2d} \) (see figure 3) at \( z_0 < z_T \) except for small perturbations from the term \exp\left(ik_{mn,\ell}L_\rho x/(4\pi)\right) \) by comparing equations (4) and (5). Figure 6 shows a detailed comparison of Talbot images between \( u_{2d} \) (upper blue curves) and \( u_{1d} \) (lower red curves) for several \( z_0 \leq z_T \) values. Numerical results indicate that the Talbot images of \( u_{1d} \) indeed present similar multiscale diffraction patterns to those of \( u_{2d} \) at \( z_0 < z_T \) except for small intensity deviations. With the increasing propagation distance \( z_0 \geq z_T \), however, the phase deviation is amplified and presents a significant departure from the prediction by equations (15) and (16). In fact, the ergodicity of phase of \( u_{1d} \) at \( z_0 \gg z_T \) will not be settled until further studies are done.

3. Conclusion

This article investigates the Talbot properties for Fibonacci geometry constructed by the cut-and-projection method. Analytical formulae for Talbot foci are deduced and suggest two feasible experiments. Theoretical and numerical calculations demonstrate fractional and irrational Talbot images and exhibit a multiscale Talbot effect due to the self-similarity of the scaling Fourier spectrum in the Fibonacci structure.

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