Neutrino oscillation processes with a change of lepton flavor in quantum field-theoretical approach

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Abstract

The oscillating probabilities of lepton flavor changing neutrino oscillation processes, where neutrinos are detected by charged-current and neutral-current interactions, are calculated in a quantum field-theoretical approach to neutrino oscillations based on a modification of the Feynman propagator in the momentum representation. The approach is most similar to the standard Feynman diagram technique in the momentum representation. It is found that the oscillating distance-dependent probabilities of detecting an electron in experiments with neutrino production in the muonic decay of $\pi^+$-meson and the detection of the produced neutrino by charged-current and neutral-current interactions exactly coincide with the corresponding probabilities calculated in the standard approach.

1 Introduction

Neutrino oscillations are an experimentally confirmed phenomenon that is widely discussed in theoretical physics. It is usually interpreted as the transition from a neutrino flavor state to another neutrino flavor state depending on the distance traveled [1, 2, 3]. This interpretation is based on the standard quantum mechanical description of neutrino oscillations, where the neutrino flavor states are assumed to be superpositions of states with definite masses described by plane waves, and it is postulated that it is these flavor states that are produced in weak interactions. However, in local quantum field theory 4-momentum is conserved in any interaction vertex, which leads to different neutrino mass-eigenstate components of a flavor state having different momenta and energies. As a result, there is a problem with violation of energy-momentum conservation, which was extensively discussed in the literature (see, e.g. [4, 5, 6, 7, 8]).

A solution to the problem can be found by considering off-shell neutrinos. The idea to treat the neutrino mass eigenstates as virtual particles and to describe their motion to a detection point by the Feynman propagators was first put forward in paper [4]. Later this approach was developed in papers [5, 6]. In this approach neutrino oscillations occur as a result of interference of the amplitudes of processes due to all the three intermediate virtual neutrino mass eigenstates. However, the calculations of the amplitudes in this approach are essentially different from the standard calculations in the Feynman diagram technique in the momentum representation. This is due to the standard S-matrix formalism of QFT, which is not convenient for describing processes at finite distances and finite time intervals. To describe a localization
of particles or nuclei, which produce and detect neutrinos, one has to use wave packets, which makes the calculations rather complicated.

In paper [9] a modified perturbative S-matrix formalism was put forward, which allows one to consistently describe the neutrino oscillation processes in the framework of quantum field theory using only plane waves. The formalism is based on the Feynman diagram technique in the coordinate representation [11] supplemented by modified rules of passing to the momentum representation. The calculation procedure proper is very similar to the Feynman diagram technique in the momentum representation, where we make use of a modified Feynman propagator. The approach was developed in paper [10], where we have shown explicitly that the suggested formalism exactly reproduces the results of the standard approach in the case, where neutrinos (together with positrons) are produced in the charged-current interaction with nuclei and detected in both neutral-current and charged-current interactions with electrons.

In the present paper we will use the modified S-matrix formalism to calculate probabilities of neutrino oscillation processes non-diagonal in the lepton flavors. Namely, we will consider the processes, where a neutrino is produced in the muonic decay of a $\pi^+$-meson and detected in the neutral-current and charged-current interactions with electrons. We will show that the results of our approach again exactly coincide with what one expects in the standard approach.

2 Oscillations in experiments with neutrino detection by neutral-current and charged-current interactions

The minimal extension of the Standard Model (SM) by the right neutrino singlets is considered. After the diagonalization of the terms sesquilinear in the neutrino fields, the charged-current interaction Lagrangian of leptons takes the form

$$L_{cc} = -\frac{g}{2\sqrt{2}} \left( \sum_{i,k=1}^{3} \bar{l}_i \gamma^\mu (1 - \gamma^5) U_{ik} \nu_k W^-_\mu + h.c. \right),$$

where $l_i$ denotes the field of the charged lepton of the $i$-th generation, $\nu_i$ denotes the field of the neutrino mass eigenstate most strongly coupled to $l_i$ and $U_{ik}$ stands for the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

We are going to consider the process, where a neutrino is produced in the decay of $\pi^+$-meson and is detected by the charged-current and neutral-current interactions with electrons. Due to the structure of the interaction Lagrangian, the process is represented in the lowest order by the following two diagrams:
In diagram (3) all the three virtual neutrino mass eigenstates contribute, so the corresponding amplitude should be summed up over the index \( k = 1, 2, 3 \). At the same time, both diagrams have neutrino mass eigenstate \( \nu_i \) in the final state, thus we should sum the resulting probability over \( i \) to get the probability of registering an electron.

Let us denote the 4-momenta of the particles as it is depicted in the diagram: the momentum of the antimuon is \( q \), the momentum of the virtual neutrinos is \( p_n \), the momentum of the outgoing electron is \( k \), the momentum of the incoming electron is \( k_1 \) and the momentum of the outgoing neutrino is \( k_2 \).

One can write out the amplitude in the coordinate representation corresponding to diagrams (2)-(3) using the standard Feynman rules formulated in textbook [12]. Next, according to the prescription of the S-matrix formalism, in order to pass to the momentum representation one would have to integrate the amplitude with respect to \( x \) and \( y \) over Minkowski space, which means that one considers the process to take place throughout Minkowski space-time and that the resulting probability of the process will be the probability per unit volume and per unit time.

However, such an integration would result in losing the information about the space-time interval between the production event and the detection event, because the experimental situation in neutrino oscillation experiments implies that the distance between the production point and the detection point along the neutrino propagation direction remains fixed. To generalize the standard S-matrix formalism to the case of processes passing at fixed distances, we introduce the delta function \( \delta(\vec{p}(\vec{y} - \vec{x})/|\vec{p}| - L) \) into the integral, \( \vec{p} \) denoting the momentum of the virtual neutrinos and \( L \) denoting the distance between a neutrino source and a detector. In so doing we fix the distance between the production and detection events, and only then we integrate the amplitude with respect to \( x \) and \( y \) over Minkowski space. Thus, just like in the standard S-matrix formalism, we consider the process taking place throughout Minkowski space-time, but the distance between the production and detection events along the momentum of the neutrino beam is now fixed by the delta function. This is equivalent to replacing the standard Feynman fermion propagator in the coordinate representation \( S^c_i(y - x) \) by \( S^c_i(y - x)\delta(\vec{p}(\vec{y} - \vec{x})/|\vec{p}| - L) \).

The Fourier transform of this expression gives us the so-called distance-dependent propagator of the neutrino mass eigenstate \( \nu_i \) in the momentum representation [9, 10]. However, in paper [10] it was argued that this distance-dependent propagator is inconvenient for calculations, because its inverse Fourier transformation cannot be defined, if the momentum of the Fourier transform coincides with the momentum of the virtual neutrinos in the argument of the delta function, which is needed to describe neutrino oscillation processes. To circumvent this problem, we introduce a different delta function \( \delta(y^0 - x^0 - T) \) into the integral, which fixes the time interval between the production and detection events. Later we will be able to express the time interval \( T \) in terms of the distance traveled by the neutrinos in accordance with the formula \( T = Lp^0/|\vec{p}| \), which is often used in describing neutrino oscillation processes.

Now the introduction of the delta function is equivalent to the replacement of the standard Feynman fermion propagator in the coordinate representation \( S^c_i(y - x) \) by \( S^c_i(y - x)\delta(y^0 -
In this case the Fourier transform gives us the so-called time-dependent propagator of the neutrino mass eigenstate $\nu_i$ in the momentum representation, defined by the relation:

$$S^c_i(p_n, T) = \int dx \, e^{ip_n \cdot x} S^c_i(x) \delta(x^0 - T).$$

(4)

This integral can be evaluated exactly [9, 10]:

$$S^c_i(p_n, T) = i \frac{\hat{p}_n - \gamma_0 \left( p_n^0 - \sqrt{\left(p_n^0\right)^2 + m_i^2 - p_n^2} \right) + m_i}{2\sqrt{\left(p_n^0\right)^2 + m_i^2 - p_n^2}} e^{i \left( p_n^0 - \sqrt{\left(p_n^0\right)^2 + m_i^2 - p_n^2} \right) T},$$

(5)

where the standard notation $\hat{p}_n = \gamma_\mu p_n^\mu$ is used. The inverse Fourier transformation of this time-dependent propagator is well defined, which allows us to retain the standard Feynman diagram technique in the momentum representation just by replacing the Feynman propagator by the time-dependent propagator.

In paper [5] it was shown that virtual particles propagating at large macroscopic distances (or, equivalently, propagating over macroscopic times) are almost on the mass shell, which means that $|p_n^2 - m_i^2|/p_n^2 \ll 1$. Applying also the approximation of small neutrino masses, i.e. keeping the neutrino masses only in the exponential, we can explicitly represent the time-dependent neutrino propagator in the momentum representation in the form [10]

$$S^c_i(p_n, T) = i \frac{\hat{p}_n}{2p_n^0} e^{-i \frac{m_i^2 + p_n^2}{2p_n^0} T}.$$  

(6)

This time-dependent propagator will be used in the calculations below replacing the standard Feynman propagator. Such a technical simplicity is an evident advantage of the discussed approach.

Now we are in a position to write out the amplitudes corresponding to diagrams (2)–(3) in the case, where the time difference $y^0 - x^0$ is fixed and equal to $T$. We assume that the momentum transfer in the production and detection processes is small, so that we can use the approximation of Fermi’s interaction. The pion decay vertex is described in accordance with the formulas in §5 of textbook [13]. The amplitude corresponding to diagram (2) in the momentum representation looks like

$$M^{(i)}_{\nu e} = - \frac{G_F^2}{2p_n^0} \cos \theta_c \, f_\pi \varphi_\pi m_{(\mu)} U^*_{21} e^{-i \frac{m_n^2 - p_n^2}{2p_n^0} T} \bar{\nu}_i(k_2) \gamma^\mu \hat{p}_n \left( 1 + \gamma^5 \right) v(q) \times$$

$$\times \left[ \left( -\frac{1}{2} + \sin^2 \theta_W \right) \bar{u}(k) \gamma_\mu \left( 1 - \gamma^5 \right) u(k_1) + \sin^2 \theta_W \bar{u}(k) \gamma_\mu \left( 1 + \gamma^5 \right) u(k_1) \right],$$

(7)

where $\theta_c$ is the Cabibbo angle, $f_\pi$ is the pion decay constant of the dimension of mass, $\varphi_\pi$ is the (constant) pion wave function, $m_{(\mu)}$ is the muon mass, and we have already applied the 4-momentum conservation condition in the production vertex. Here and below we drop the fermion polarization indices.
Similarly, the amplitude corresponding to diagram (3) summed over the type $k$ of the intermediate virtual neutrino can be written out to be

$$M_{cc}^{(i)} = \frac{G_F^2}{2p_n^0} \cos \theta_c f_\pi^2 \pi^2 m_{(\mu)} U_{1i}^* \sum_{k=1}^3 U_{1k}^* U_{2k}^* e^{-i \frac{m_k^2 - p_n^2}{2p_n^0} T} \times$$

$$\times \bar{v}_i(k_2) \gamma_\mu (1 - \gamma^5) u(k_1) \bar{u}(k) \gamma^\mu \hat{p}_n (1 + \gamma^5) v(q).$$

It is convenient to use the Fierz identity to transpose the spinors $\bar{u}(k)$ and $\bar{v}_i(k_2)$ in the latter amplitude, which makes it look similar to the former one. We also introduce the following notations for the time-dependent factors:

$$A_i = U_{2i}^* e^{-i \frac{m_i^2 - p_n^2}{2p_n^0} T}, \quad B_i = U_{1i}^* \sum_{k=1}^3 U_{1k}^* U_{2k}^* e^{-i \frac{m_k^2 - p_n^2}{2p_n^0} T}.$$  \hspace{1cm} (9)

The total amplitude of the process with the neutrino $\nu_i$ in the final state, which is the sum of the amplitudes $M_{nc}^{(i)}$ and $M_{cc}^{(i)}$, takes the form

$$M_{tot}^{(i)} = \frac{G_F^2}{2p_n^0} \cos \theta_c f_\pi^2 \pi^2 m_{(\mu)} \bar{v}_i(k_2) \gamma^\mu \hat{p}_n (1 + \gamma^5) v(q) \times$$

$$\times \left[ B_i + A_i \left( -\frac{1}{2} + \sin^2 \theta_W \right) \bar{u}(k) \gamma_\mu (1 - \gamma^5) u(k_1) + A_i \sin^2 \theta_W \bar{u}(k) \gamma_\mu (1 + \gamma^5) u(k_1) \right].$$

One can notice that the dimension of this amplitude is not usual. Formally, it corresponds to the process, in which the time difference $y^0 - x^0$ between the production and the detection is exactly equal to $T$. However, in reality, a registration process is not instant, it takes some time $\Delta t$, $\Delta t/T \ll 1$. To find the amplitude of the process with the registration time $\Delta t$ we have to integrate amplitude (10) with respect to $T$ from $T - \Delta t/2$ to $T + \Delta t/2$. After dropping the terms of the order $\Delta t/T$, which are negligibly small, the integration results in the multiplication of the amplitude by $\Delta t$. Hence, expression (10) should be understood as the amplitude per unit time.

Our next step is to calculate the squared amplitude, averaged with respect to the polarizations of the incoming particles and summed over the polarizations of the outgoing particles. The operation of averaging and summation will be denoted by angle brackets. Applying again the approximation of small masses of almost real intermediate neutrinos, $p_n^2 = 0$, we find that the squared amplitude factorizes as follows:

$$\left\langle \left| M_{tot}^{(i)} \right|^2 \right\rangle = \left\langle \left| M_1 \right|^2 \right\rangle \left\langle \left| M_2 \right|^2 \right\rangle \frac{1}{4(p_n^0)^2},$$

$$\left\langle \left| M_1 \right|^2 \right\rangle = 4G_F^2 \cos^2 \theta_c f_\pi^2 \pi^2 m_{(\mu)} (p_n q),$$

$$\left\langle \left| M_2 \right|^2 \right\rangle = 64G_F^2 \left[ B_i + A_i \left( -\frac{1}{2} + \sin^2 \theta_W \right) \left( k_1 p_n \right)^2 + \left| A_i \right|^2 \sin^4 \theta_W \left( k p_n \right)^2 - \right.$$

$$\left. - \left( \text{Re}(A_i B_i^*) + \left| A_i \right|^2 \left( -\frac{1}{2} + \sin^2 \theta_W \right) \right) \sin^2 \theta_W m^2 \left( k_2 p_n \right) \right].$$
where $\langle |M_1|^2 \rangle$ is the squared amplitude of the decay process of $\pi^+$-meson into antimuon and a massless fermion, $\langle |M_2^{(i)}|^2 \rangle$ is the squared amplitude of the scattering process of a massless fermion and the initial electron, $m$ standing for the electron mass.

Let us denote the 4-momentum of the decaying pion by $p_\pi$ and the 4-momentum of the neutrinos to be detected by $p$. The experimental setting defines that the momentum $p_\pi$ is directed from a source to a detector and satisfies the momentum conservation condition $p_\pi - q - p = 0$ in the production vertex. In other words, $p$ is a special value of $p_\pi$, which is directed from the source to the detector. Actually, the selection of the single value $p$ of the momenta of the neutrinos to be detected is an approximation, which is applicable, when the distance between the source and the detector is much larger than their sizes. We also recall that we work in the approximation $p^2 = 0$. Following the prescription formulated in paper \cite{10}, in order to find the differential probability of the process one should multiply the amplitude $dW^{(i)}_{tot}$ by the delta function of energy-momentum conservation $(2\pi)^4 \delta(p_\pi + k_1 - q - k - k_2)$ and by the delta function $2\pi \delta(p_\pi - q - p)$, which selects the momentum of the intermediate neutrinos, substitute $p$ instead of $p_\pi$ and integrate the result with respect to the momenta of the final particles, namely antimuon, electron and neutrino, in accordance with the standard rules of probability calculations. The factor $2\pi$ in front of the latter delta function arises after an averaging over the momenta of the neutrinos to be detected, which, because of non-zero sizes of the source and the detector, really lie inside a small cone with the axis along the vector $p$.

Due to the factorization of the squared amplitude the differential probability factorizes as follows:

$$\frac{dW^{(i)}}{d\vec{p}} = \frac{dW_1}{d\vec{p}} W_2^{(i)},$$

$$\frac{dW_1}{d\vec{p}} = \frac{1}{2p_\pi^0} \frac{1}{(2\pi)^3 2p_\pi^0} \int \frac{d^3q}{(2\pi)^3 2q^0} \langle |M_1|^2 \rangle (2\pi)^4 \delta (p_\pi - q - p),$$

$$W_2^{(i)} = \frac{1}{2p_\pi^0 2k_2^0} \int \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} \langle |M_2^{(i)}|^2 \rangle (2\pi)^4 \delta (k_1 + p - k - k_2).$$

Here $\frac{dW_1}{d\vec{p}}$ is the differential probability of the $\pi$-meson decay into an antimuon and a massless fermion with the fixed momentum $\vec{p}$. $W_2^{(i)}$ is the probability of the scattering process of electron and a massless fermion with the production of an electron and neutrino mass eigenstate $\nu_i$.

In order to find the total differential probability of detecting an electron in the final state we have to sum the differential probability $\frac{dW^{(i)}}{d\vec{p}}$ over $i = 1, 2, 3$. Since $\frac{dW_1}{d\vec{p}}$ does not depend on $i$, we should sum only $W_2^{(i)}$; the result will be denoted by $W_2$. Then the total differential probability of detecting an electron in the final state can be written as

$$\frac{dW}{d\vec{p}} = \frac{dW_1}{d\vec{p}} W_2.$$ 

Since the momentum $p_\pi$ of the intermediate virtual neutrinos is now fixed and equal to $p$, we can substitute $T = Lp^0/|\vec{p}|$ into all the formulas from now on. This substitution is consistent, because the neutrinos are almost on the mass shell, and $|\vec{p}|/p^0$ can be considered as the neutrino speed with a very high accuracy.
Next we observe that the experimental setting fixes only the direction of the neutrino momentum, but not its length $|\vec{p}_\nu| = |\vec{p}|$. Therefore, to find the probability of the process we must also integrate (17) with respect to $|\vec{p}|$ over all the admissible values. The maximal value of $|\vec{p}|$ is determined by the production process and the minimal one is determined by the detection process. Here the production process is a two-body decay, which means that the lengths of the neutrino and antimuon momenta are already fixed by energy-momentum conservation. It results in $\frac{dW_1}{dp}$ being singular, and this singularity is eliminated by the integration. The final result for the probability of the process is as follows:

$$ \frac{dW}{d\Omega} = \int \sum_{i=1}^{3} \frac{dW^{(i)}}{d\vec{p}} |\vec{p}|^2 d|\vec{p}| = \frac{dW_1}{d\Omega} W_2 |\vec{p}| = |\vec{p}|^*, $$

where

$$ \frac{dW_1}{d\Omega} = \frac{G_F^2 \cos^2 \theta_c f^2 \pi m^2_{(\mu)}}{8 (2\pi)^2} \frac{m^2_\pi - m^2_{(\mu)}}{p^0_\pi (p^0_\pi - |\vec{p}_\pi| \cos \theta)^2} $$

is the differential probability of the $\pi$-meson decay into an antimuon and a massless fermion with the fixed direction of the momentum, and

$$ |\vec{p}|^* = \frac{m^2_\pi - m^2_{\mu}}{2 (p^0_\pi - |\vec{p}_\pi| \cos \theta)}; $$

the coordinate system is chosen in such a way that the pion momentum $\vec{p}_\pi$ is directed along the Z-axis, and $\theta$ is the polar angle of $\vec{p}$. After all these transformations the probability (18) can be interpreted as the probability per unit length of the source and per unit length of the detector.

As one can see, differential probability (19) has the maximum at $\theta = 0$, i.e. in the direction of the initial pion momentum. Therefore, it is natural to place the detector in this direction from the source in order to register the maximal possible number of events.

Since the azimuthal angle $\varphi$ is not defined for $\theta = 0$, in order to find the differential probability $\frac{dW_1}{sin \theta d\varphi}$ at $\theta = 0$ first we have to average the differential probability $\frac{dW_1}{d\Omega}$ over the angle $\varphi$ and then to take the limit $\theta \to 0$. As a result, we get the following differential probability of the neutrino production process in the direction of the initial pion momentum:

$$ \frac{dW_1}{sin \theta d\theta} \bigg|_{\theta=0} = \frac{G_F^2 \cos^2 \theta_c f^2 \pi m^2_{(\mu)}}{8 (2\pi)^2} \frac{m^2_\pi - m^2_{(\mu)}}{p^0_\pi (p^0_\pi - |\vec{p}_\pi| \cos \theta)^2}. $$

Let us take a closer look at the registration probability $W_2$. After all the substitutions the absolute values and products of the time-dependent factors $A_i$ and $B_i$ defined in (9) are
expressed in the form:

\[ |A_i|^2 = |U_{2i}|^2, \]

\[ |B_i|^2 = |U_{1i}|^2 \sum_{k,l=1}^{3} \sum_{k<l} -4 \text{Re} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin^2 \left( \frac{m_k^2 - m_l^2}{4|\vec{p}|} L \right) + \]

\[ + 2 \text{Im} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin \left( \frac{m_k^2 - m_l^2}{2|\vec{p}|} L \right) \],

(23)

\[ \text{Re} \left( A_i B_i^* \right) = \text{Re} \left( U_{1i} U_{2i}^* \sum_{k=1}^{3} U_{1k} U_{2k} e^{-i\frac{m_k^2 - m_l^2}{2|\vec{p}|} L} \right). \]

(24)

Substituting these expressions into (13) summed over \( i \) gives:

\[ \sum_{i=1}^{3} \left( |M_i^{(i)}|^2 \right) = 64G_F^2 \left\{ \right. \\

\begin{align*}
2 \sin^2 \theta_W & \sum_{k,l=1}^{3} \left[ -4 \text{Re} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin^2 \left( \frac{m_k^2 - m_l^2}{4|\vec{p}|} L \right) + \right. \\
+ 2 \text{Im} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin \left( \frac{m_k^2 - m_l^2}{2|\vec{p}|} L \right) \right] \left( \frac{1}{2} + \sin^2 \theta_W \right)^2 \left( k_1 p \right)^2 + \\
+ \sin^4 \theta_W (k p)^2 & - \left[ -4 \text{Re} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin^2 \left( \frac{m_k^2 - m_l^2}{4|\vec{p}|} L \right) + \right. \\
+ 2 \text{Im} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin \left( \frac{m_k^2 - m_l^2}{2|\vec{p}|} L \right) \right] \sin^2 \theta_W m^2 \left( k_2 p \right) \left. \right\}. \]

(25)

Now one should substitute this expression into (16) summed over \( i \). Using the formulas for neutrino-electron scattering kinematics presented in §16 of textbook [13], evaluating the integral and substituting \( |\vec{p}| = |\vec{p}|^* \) defined in eq. (20), we get the following result:

\[ W_2 = \frac{G_F^2 m}{2 \pi} \frac{dW_1}{d\theta_1} \left[ 1 - 2 \sin^2 \theta_W \left( 1 + \frac{2 |\vec{p}|^*}{2 |\vec{p}|^* + m} \right) + 4 \sin^4 \theta_W \left( 1 + \frac{1}{3} \left( \frac{2 |\vec{p}|^*}{2 |\vec{p}|^* + m} \right)^2 \right) + \right. \\
+ 4 \sin^2 \theta_W \left( 1 + \frac{2 |\vec{p}|^*}{2 |\vec{p}|^* + m} \right) \right\} - 4 \sum_{k,l=1}^{3} \left[ \text{Re} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin^2 \left( \frac{m_k^2 - m_l^2}{4|\vec{p}|} L \right) \right] + \right. \\
+ 2 \sum_{k,l=1}^{3} \left[ \text{Im} \left( U_{1k} U_{1l}^* U_{2k} U_{2l} \right) \sin \left( \frac{m_k^2 - m_l^2}{2|\vec{p}|} L \right) \right] \right\}. \]

(26)

In the approximation of massless neutrinos \( \frac{dW_1}{d\theta_1} \) coincides with the neutrino probability flux and \( W_2 \) coincides with the cross section of the scattering process of a massless fermion on an
electron, which can be expressed as \( P_{\mu e}(L)\sigma_{\nu e} + (1 - P_{\mu e}(L))\sigma_{\nu_{\mu} e} \), where

\[
P_{\mu e}(L) = -4 \sum_{\substack{k,l=1 \atop k > l}}^{3} \left[ \text{Re} (U_{1k}U_{1l}^{*}U_{2k}^{*}U_{2l}) \sin^{2} \left( \frac{m_{k}^{2} - m_{l}^{2}}{4 |\vec{p}|} L \right) \right] + \\
+ 2 \sum_{\substack{k,l=1 \atop k > l}}^{3} \left[ \text{Im} (U_{1k}U_{1l}^{*}U_{2k}^{*}U_{2l}) \sin \left( \frac{m_{k}^{2} - m_{l}^{2}}{2 |\vec{p}|} L \right) \right]
\]

denotes the distance-dependent probability of the transition \( \nu_{\mu} \rightarrow \nu_{e} \). Thus, we have obtained that the probability of detecting an electron is equal to the probability of the production, in the source, of neutrino with the momentum aimed in direction of the detector multiplied by the probability of the neutrino interaction in the detector, which is expressed in terms of the muon and electron neutrino interaction cross sections and the standard distance-dependent \( \nu_{\mu} \rightarrow \nu_{e} \) transition probability, i.e. we have actually exactly reproduced the result of the standard approach to neutrino oscillations in the framework of QFT without making use of the neutrino flavor states and difficulties associated with applying of wave packets.

Since the incoming \( \pi \)-mesons always have a momentum distribution, the total neutrino probability flux can be obtained by performing the average of \( \frac{dW}{d\Omega} \) over the momentum distribution of the incoming \( \pi \)-mesons. In this case the magnitude of the momentum of the virtual neutrinos is not fixed, which results in the blurring of the interference pattern and gives rise to the corresponding coherence length. The number of events in the detector per unit time can be found by integrating the corresponding probability and the densities of \( \pi \)-mesons and electrons over the volumes of the neutrino source and detector.

### 3 Oscillations in experiments with neutrino detection by charged-current interactions only

Let us consider the process, where a neutrino is produced in the muonic decay of \( \pi^{+} \)-meson, as in the previous case, but it is detected only by the charged-current interaction with a nucleus. The process is described in the lowest order by the diagram:

\[
\begin{align*}
\pi^{+} & \quad \text{x} \quad \mu^{+}(q) \\
\nu_{i}(p_{n}) & \quad \text{y} \quad e^{-}(k) \\
& \quad W^{+} \quad \text{X}
\end{align*}
\]

which should be summed over the type \( i = 1, 2, 3 \) of the intermediate neutrino mass eigenstate. The filled circle stands for the matrix element \( j_{\mu} \) of the charged weak hadron current. Since the neutrino energy in the muonic decay of pion is of the order of 30 MeV, the interaction of the virtual neutrinos with a nucleus can result in the disintegration of the latter. To be specific, we will consider first only the two body final state and suppose that an initial nucleus \( \frac{A}{Z} \text{X} \) absorbs
$W^+$-boson and turns into the final nucleus $\frac{A}{2}+1X$, thus
\[ j_\mu = \langle \frac{A}{2}+1X | j_\mu^{(0)} | \frac{A}{2}X \rangle. \]

Using again the approximation of Fermi’s interaction one can write out the amplitude in the momentum representation corresponding to diagram (27) summed over all three neutrino mass eigenstates in the case, where the time difference $y^0 - x^0$ between the production and detection points is fixed and equal to $T$:
\[ M = -i \frac{G_F^2}{2p_n^0} \cos \theta_c f_\pi \varphi \pi m_\mu \sum_{i=1}^3 U_{1i}U_{2i}^* e^{-\frac{i m_i^2 - p_i^2}{2p_n^0} T} j_\mu \bar{u}(k) \gamma^\mu \vec{p}_n \left(1 + \gamma^5\right) v(q). \] (28)

Here the particle 4-momenta are defined similarly to the previous section, as it is shown in the diagram.

The squared amplitude averaged with respect to the incoming particles polarizations and summed over the outgoing particles polarizations factorizes as follows:
\[ \langle |M|^2 \rangle = \langle |M_1|^2 \rangle \langle |M_2|^2 \rangle \frac{1}{4 (p_n^0)^2} \times \]
\[ \times \sum_{i<k=1}^3 \left[ -4 \text{Re} (U_{1i}U_{1k}^*U_{2i}^*U_{2k}) \sin^2 \left(\frac{m_i^2 - m_k^2}{4p_n^0} T\right) + 2 \text{Im} (U_{1i}U_{1k}^*U_{2i}^*U_{2k}) \sin \left(\frac{m_i^2 - m_k^2}{2p_n^0} T\right) \right], \] (29)

where $\langle |M_1|^2 \rangle$ is the squared amplitude of the pion decay into antimuon and a massless fermion, given by formula (12), and
\[ \langle |M_2|^2 \rangle = 4G_F^2 \left[ k^\mu p_n^\nu + k^\nu p_n^\mu - (p_n k) g^{\mu\nu} + i \varepsilon^{\mu\nu\alpha\beta} k_\alpha p_{n\beta} \right] (W_{\mu\nu}^{(S)} + iW_{\mu\nu}^{(A)}) \] (30)

is the squared amplitude of the scattering process of the initial nucleus and a massless fermion resulting in the production of the final nucleus and an electron. Here the nuclear tensor $W_{\mu\nu} = W_{\mu\nu}^{(S)} + iW_{\mu\nu}^{(A)} = \langle j_\mu j_\nu^+ \rangle$ characterizes the interaction of the nucleus with a virtual $W^+$-boson, its symmetric part $W_{\mu\nu}^{(S)}$ being real and antisymmetric part $iW_{\mu\nu}^{(A)}$ being imaginary.

Let us denote the 4-momentum of $\pi^+$-meson again by $p_\pi$ and the 4-momenta of the initial and final nuclei by $P = (E, \vec{P})$, $P' = (E', \vec{P}')$, respectively. Following the outlined recipe, in order to find the probability of the process one has to multiply the amplitude $\langle |M|^2 \rangle$ by the delta function of the energy-momentum conservation $(2\pi)^4 \delta(p_\pi + P - q - k - P')$ and by the delta function $2\pi \delta(p_n - q - p)$, which fixes the momentum of the intermediate neutrinos, to substitute $p$ instead of $p_n$ and to integrate with respect to the momentum of the final particles. We may also replace the time interval $T$ by $L p_n^0 / |\vec{p}|$ to pass from the time-dependent factor to the distance-dependent factor, because the momentum $p_n$ is now selected to be equal to $p$. The result has to be integrated with respect to $|\vec{p}|$, and this integration is performed using the additional delta function. As a result of all these transformations we have:
\[ \frac{dW}{d\Omega} = \int \frac{dW}{d\vec{p}} |\vec{p}|^2 d |\vec{p}| = \frac{dW_1}{d\Omega} W_2 |\vec{p}| = |\vec{p}| \times \]
\[ \times \sum_{i<k=1}^3 \left[ -4 \text{Re} (U_{1i}U_{1k}^*U_{2i}^*U_{2k}) \sin^2 \left(\frac{m_i^2 - m_k^2}{4|\vec{p}|^*} L\right) + 2 \text{Im} (U_{1i}U_{1k}^*U_{2i}^*U_{2k}) \sin \left(\frac{m_i^2 - m_k^2}{2|\vec{p}|^*} L\right) \right], \] (31)
where $|p|^*$ is given by (20), $\frac{dW_1}{d\Omega}$ stands for the differential probability of the $\pi$-meson decay into an antimuon and a massless fermion with the fixed direction of the momentum, given by (19), and

$$W_2 = \frac{1}{2p^0 2E} \int \frac{d^3 k}{(2\pi)^3 2k^0} \frac{d^3 P'}{(2\pi)^3 2E'} \langle |M_2|^2 \rangle (2\pi)^4 \delta (P + p - P' - k)$$

(32)

is the probability of the scattering process of a nucleus and a massless fermion resulting in the production of the final nucleus and electron. In fact, this probability should be replaced by the probability of the inclusive scattering process, where only the final electron is detected. However, this does not influence the result that the factor in formula (31) exactly coincides with the one we expect for the $\nu_\mu \rightarrow \nu_e$ transition probability in the conventional approach. The number of events in the detector can be found exactly in the same way, as it was explained in the end of the previous section.

4 Conclusion

In the present paper we have shown that the lepton flavor changing neutrino oscillation processes can be consistently described in quantum field theory using only plane wave states of the involved particles. In the framework of the Standard Model minimally extended by the right neutrino singlets we have used the modified perturbative formalism put forward in paper [9] and developed in paper [10]. It is based on the conventional S-matrix approach supplemented by the modified rules of passing from the coordinate representation to the momentum representation. These rules allow us to construct the modified Feynman propagator in the momentum representation corresponding to the experimental situation at hand, which we call the time-dependent propagator. Unlike the standard S-matrix formalism, our approach is adequate for describing the processes passing at finite distances and finite time intervals. The calculations are simple and very similar to those in the standard perturbative S-matrix formalism in the momentum representation. The modified S-matrix formalism is physically transparent and has the advantage of not violating energy-momentum conservation. It is important to note that we do not make use of the neutrino flavor states in the model, working only with the neutrino mass eigenstates.

This technique has been used for calculating the oscillating probabilities of the processes, where neutrinos are produced in the muonic decay of $\pi^+$-meson and detected in the neutral-current and charged-current interactions with electrons or just the charged-current interaction with nuclei. It was explicitly shown that the approach exactly reproduces the results of the standard formalism.

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