Breather Solutions to the Cubic Whitham Equation

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We show numerical evidence that breather solutions exist in the cubic Whitham equation which arises as a water-wave model for interfacial waves. The equation combines nonlinearity with the non-local character of the water-wave problem and is non-integrable as suggested by the inelastic interaction of solitary waves. It generalizes the completely-integrable modified KdV (mKdV) equation. In the mKdV equation, breather solutions appear naturally as ground states of invariant integrals, suggesting that such structures may also exist in non-integrable models, at least in an approximate sense.

I. INTRODUCTION

In this work, we are concerned with the numerical approximation of breather solutions in nonlinear dispersive equations. A breather is a localized but oscillatory wave-packet like solution of a dispersive equation which is either stationary or propagates in a similar fashion as a solitary wave. While breathers have been observed experimentally, most if not all theoretical constructions of breathers are based on explicit solutions of a completely integrable nonlinear dispersive differential equation such as the nonlinear Schrödinger (NLS) equation. In the present work, we construct numerical approximations of breather solutions in a non-integrable dispersive equation.

Discussions of breather solutions have featured prominently in the literature on nonlinear dispersive model equations for several decades. Historically, breather solutions have been considered in completely integrable equations such as the sine-Gordon equation \[1, 2\] and the modified Korteweg-de Vries (mKdV) equation \[3, 4\]. So far, the greatest number of exact breather solutions have been found in the NLS equation. Indeed, the first breather-type solution of the NLS equation was found over 40 years ago by Kuznetsov \[5\] and Ma \[6\], followed by the discovery of the Peregrine breather a few years later \[7\]. The authors of \[8, 9\] found a new solution of the NLS equation now called the Akhmediev breather. This solution is temporally localized, and for large negative and positive times approaches a plane-wave solution of the NLS equation. Thus in a sense this solution may be thought of as the nonlinear stage of the modulational instability observed in periodic wave trains in surface water \[10\] and also exhibited by plane-wave solutions of the NLS equation \[11\].

Due to the temporal localization, the Akhmediev breather has been put forward as a possible mechanism for the development of rogue waves, which are unexpectedly large ocean waves that seem to appear out of nowhere. One common definition of a rogue wave is that the amplitude of the wave be about two times higher than the surrounding wavefield. Indeed, one diagnostic for the possible occurrence of rogue waves is the Benjamin-Feir index \[12\] which essentially measures the probability of modulational instability to occur. As the Akhmediev breather provides a path from modulational instability to a singular wave event with large amplitude, it may be used to predict the occurrence of a rogue wave.

One major step forward in the study of breathers and rogue waves was the observation of the Akhmediev breather and associated large wave event in an experimental wave flume \[13\]. Subsequently, the authors of \[14\] used an ingenious method to use a 9 meter wave tank for the creation of a higher-order breather by restarting the experiment seven times using data from the previous experiment as starting data for the next run. The authors also provided convincing comparisons between experimentally obtained wave profiles and the exact breather solutions of the NLS equations to show that the Akhmediev breather had indeed been created in the laboratory.

One feature that all of the theoretical breather solutions mentioned above have in common is that they are exact or closed-form solutions of a completely integrable partial differential equation. On the other hand, as shown in the experiments in \[13, 14\] breathers exist in reality, and there is no reason one would expect that breather solutions only exist in completely integrable models. Approximate breather solutions were also considered in internal wave models. Indeed, the fully-
The cubic Whitham equation features exact breather solutions [15], but it is shown in [16] that when inserted into non-integrable models, the solutions are constantly radiating energy which casts doubt on whether these are truly coherent structures in the non-integrable models. In [17], approximate breather solutions are observed in two non-integrable models: the Dysthe equation [18], essentially a higher-order NLS equation, and the full water-wave problem based on the inviscid and incompressible Euler equations. However, as can be seen in figures 5 and 6 in [17], these numerical solutions are not as regular as the closed-form solutions of the NLS equation such as the Akhmediev breather. Indeed, a slight asymmetry is observed in the solutions provided in [17], and the solutions are also radiating energy. This may be due to the non-integrability of the models used in [17] and may also be connected with wave breaking which was observed in the experiments reported in [17].

As suggested by the short summary above, the study of breather solutions is an active field of research with a wide range of open problems concerning existence, stability and relation to extreme events. For the NLS equation, new breather solutions are found in closed form on a regular basis (see for example [19]), and there are several results showing existence [1], stability [20, 21], or non-existence or instability [22–24]. In the present work we present numerical evidence for the existence of breather solutions in a non-integrable model.

II. THE CUBIC WHITHAM EQUATION

The Korteweg-de Vries (KdV) equation is a simplified model equation for waves at the surface of an inviscid incompressible fluid. The equation includes the essential effects of nonlinearity and dispersion, and balancing these two effects is the basic mechanism behind the existence of both solitary-wave solutions and periodic travelling waves.

The KdV equation can also be used as a model for waves at the interface in a two-fluid system although it has been noted that the so-called extended KdV (eKdV) equation also known as the Gardner equation is more advantageous for internal waves as it more closely describes the typically broader solitary waves [25]. In the two-layer case, the unknown function $\eta(x, t)$ is the deflection of the interface from its rest position.

Following an idea of G.B. Whitham [26], one may use the full representation of the the dispersion relation for a two-fluid system. In particular for a lower layer of depth $h_1$ and density $\rho_1$ and an upper layer of depth $h_2$ and density $\rho_2$, the equation will have the form

$$\eta_t + \alpha_1 \eta_{xx} + \alpha_2 \eta^2 \eta_x + K_I * \eta_x = 0,$$  

where the coefficients are given in terms of $c_0^2 = gh_1 h_2 \rho_1 \rho_2 / (\rho_1 h_2 + \rho_2 h_1)$ by $\alpha_1 = \frac{3}{2} c_0 \rho_1 h_2 - \rho_2 h_1$ and $\alpha_2 = \frac{3}{2} c_0 \rho_1 h_2 + \rho_2 h_1 / (\rho_1 h_2 + \rho_2 h_1)$. The convolution operator is given by the kernel

$$K_I = F^{-1} \left( \frac{\sqrt{\rho_1 \rho_2 \tanh(k h_1) \tanh(k h_2)}}{\rho_1 \tanh(k h_2) + \rho_2 \tanh(k h_1)} \right),$$

where $F^{-1}$ is the inverse Fourier transform, defined by

$$F^{-1} U(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(k) e^{ikx} dk.$$  

(see [25, 27, 29]). In the case when $\rho_1 h_2^2 \sim \rho_2 h_1^2$, the quadratic coefficient vanishes, and the pure cubic equation (without the quadratic term) appears as the most appropriate model. The equation can be normalized by taking the depth of the lower fluid $h_1$ as the unit of distance, and $h_1/c_0$ as a unit of time.

For the purpose of this article, we use the the so-called cubic Whitham equation, which is a simplified version of (1) given by

$$\eta_t + \eta^2 \eta_x + K * \eta_x = 0$$  

with

$$K = F^{-1} \sqrt{\tanh k}.$$  

FIG. 1. Interaction of solitary waves of amplitude 0.67 and 0.34 in the the cubic Whitham equation (top panel) and in the modified KdV (mKdV) equation (bottom panel). Solitary waves are propagating to the right. Pre-interaction (left), interaction point (center) and post-interaction (right) are shown. A dispersive tail is visible in the post-interaction plot in the case of the cubic Whitham equation, but not in the mKdV equation. Note that the position of the pre-and post-interaction plots on the x-axis is not to scale. Nevertheless, it can be seen that solitary waves of the same height are much wider in the mKdV equation than in the cubic Whitham equation.
FIG. 2. Evolution of a breather in the mKdV equation (5). The breather propagates to the left. The right-most profile is taken the initial data, a breather with $\alpha = 0.3$ and $\beta = 0.4$. The wave then evolves in the mKdV equation, forming a negative shape (center waveform), then evolving further until it reaches the original shape, but translated left.

Equation (5) has the advantage of being closer to the original quadratic Whitham equation which has seen great attention lately (see [30–33] and references therein), and it can also be more easily compared with the modified KdV equation

$$\eta_t + \eta^2 \eta_x + \eta_{xxx} = 0. \quad (5)$$

It is well known that the mKdV equation is an integrable model equation, and solutions may be found with the help of the inverse scattering transform method, or using the Miura transform which maps solutions of the KdV equation into solutions of the mKdV equation [34]. On the other hand, the cubic Whitham equation (3) is not integrable as can be seen explicitly by examining the interaction of two solitary waves. The existence of traveling-wave solutions and the existence and stability of solitary wave solutions to the cubic Whitham equation (3) was studied in [35–39]. More recently the formation of shocks has been proved in [40]. Approximate solitary-wave solutions can be found numerically using a manual cleaning process such as explained in [41]. As shown in Figure 1, the interaction between two solitary waves in the cubic Whitham equation creates a small dispersive tail after the interaction, while the same code applied to two solitary waves of the mKdV equation shows a clean interaction. The appearance of small dispersive ripples after the interaction is usually a strong indicator of non-integrability.

The cubic Whitham equation (3) admits the following conserved quantities:

$$I_1 = \int \eta \, dx, \quad (6a)$$

$$I_2 = \int \eta^2 \, dx, \quad (6b)$$

$$I_3 = \int \left( \eta K_1 \eta + \frac{1}{4} \eta^4 \right) \, dx, \quad (6c)$$

where the integration is over the real line, or over one wavelength depending on the situation. Since the equation is non-integrable, it is unlikely that further conserved integrals can be found.

### III. BREATHER SOLUTIONS

One of the most prominent examples of a breather solution is the Peregrine breather which is an exact solution of the NLS equation [7]. The Peregrine breather can be obtained by perturbing the exponential Stokes-wave solution of the NLS equation with specially chosen functions of polynomial decay.

In most of the models mentioned above, in particular the mKdV and Gardner equations, breather solutions share common features such as having two free parameters which can be interpreted as the amplitude and the rate of the inner oscillation of the wave packet. In particular, the mKdV equation (5) has a closed-form breather solution given in terms of the phase velocity $\gamma = 3\alpha^2 - \beta^2$ and the crest velocity $\delta = \alpha^2 - 3\beta^2$. The evolution of a typical mKdV breather is shown in Figure 2. The formula is represented as

$$\eta(x, t) = 2\sqrt{6} \beta \text{sech} (x + \gamma t) \left[ \frac{\cos(\alpha(x + \delta t)) - (\beta/\alpha) \sin(\alpha(x - \delta t)) \tanh(\beta((x + \gamma t)))}{1 + (\beta/\alpha)^2 \sin^2(\alpha(x + \delta t)) \text{sech}^2(\beta(x + \gamma t))} \right]. \quad (7)$$

However, in recent work, breathers have been characterized as solutions of a fourth-order ordinary differential equations which implies that breather solutions are critical points of a second-order Lyapunov functional, composed of a linear combination of low order conserved quantities [4, 42, 43]. This work essentially raises the question whether breathers exist more generally as stable structures in non-integrable evolution equations.

For the cubic Whitham equation (3) (as for the quadratic Whitham equation) no closed-form solutions
In (3) can be represented exactly. The discretization
advantage that the symbol of the linear operator appears
Fourier-collocation method is used. This method has the
breathers appear naturally.

As we consider the evolution of initial data in the form
may admit breather solutions. In order to find a breather, one
numerical simulations which strongly suggest that (3) does
are known. In particular, it is not known whether (3)
are shown in the upper and middle panel, the breather shown
in the middle panel appears slightly more oscillatory, but
has a much larger total waveheight (defined here as the
elevation of the highest crest minus the elevation of low-
est trough in the wave packet) than the breather in the
upper panel. On the other hand, the breather in the
lowest panel has a similar waveheight to the one in the
middle panel, but features many more oscillations. This
suggests that there is probably a two-parameter family of
breather solutions, in analogy with the mKdV equation.
Since no radiation was detected with the numerical code
used here, our numerical results suggest the mathemati-
cal existence of true breather solutions.

The question of the existence of breather solutions for
non-integrable equations was raised in [44, 45]. These
authors provided an asymptotic development suggesting
the existence of a breather solution in the so-called Klein-

IV. DISCUSSION

The three breathers shown in Figure 5 suggest that
a rich variety of different breathers exist in the cubic
Whitham equation (3). Indeed, comparing the breather
shown in the upper and middle panel, the breather shown
in the middle panel appears slightly more oscillatory, but
has a much larger total waveheight (defined here as the
elevation of the highest crest minus the elevation of low-
est trough in the wave packet) than the breather in the
upper panel. On the other hand, the breather in the
lowest panel has a similar waveheight to the one in the
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FIG. 3. Evolution of a breather in the quadratic (proper)
Whitham equation \( \eta_t + \eta \eta_x + K * \eta_x = 0 \). The light blue
curve shows the initial data. The dark blue curve shows the
numerical solution at \( t = 172 \). As in the previous figure,
the initial data is given by a breather solution for the mKdV
equation with \( \alpha = 0.3 \) and \( \beta = 0.4 \). The solution develops
into a positive solitary wave, an emerging negative solitary wave, an
oscillatory breather-like structure and dispersive noise, shown
at \( t = 1750 \). A large domain is necessary in order to allow the
various structures to separate.

FIG. 4. Appearance of a breather in the cubic Whitham
equation (3). In this case, the initial data is given by a
slightly perturbed breather solution for the mKdV equation
with \( \alpha = 0.3 \) and \( \beta = 0.4 \). The solution develops into a
positive solitary wave, an emerging negative solitary wave, an
oscillatory breather-like structure and dispersive noise, shown
at \( t = 1750 \). A large domain is necessary in order to allow the
various structures to separate.

For the purpose of approximating solutions of (3), a
Fourier-collocation method is used. This method has the
advantage that the symbol of the linear operator appearing
in (3) can be represented exactly. The discretiza-
tion and numerical implementation are similar to meth-
ods used in [37]. The code used here is versatile enough
so that it can be used for the KdV, Whitham and mKdV
and cubic Whitham equations by simply changing coeffi-
cients.

In order to find coherent structures for the Whitham
equation, we insert an mKdV breather solution as ini-
tial data into the fully discrete scheme. As a test, we
first solve the mKdV equation itself and observe that the
breather propagates as it should (see Figure 2). Since
this solution has an exact form, this procedure was also
used to test the implementation of the numerical code.

We then insert a perturbed breather into the quadratic
and cubic Whitham equation. In the quadratic Whitham
equation, a solitary wave and a dispersive shock wave are
formed (see Figure 3). In the cubic Whitham equation,
the initial data evolve into a positive solitary wave, a
negative solitary wave, a breather-like structure and a
dispersive tail (see Figure 4).

In order to extract numerical approximations to a
breather, a cleaning approach needs to be used such as
explained in [41]). In other words, at some intervals, the
dispersive ripples are simply zeroed out by hand, and
the code is restarted. If the coherent structure (solitary
wave of breather) is stable, and the small oscillations are
propagating at a different speed than the main structure,
then this process eventually leads to a close numerical
approximation of the target structure, with small rip-
bles essentially absent almost to machine precision. The
time evolution of three different breathers in the cubic
Whitham equation (3) is depicted in Figure 5. Figure 6
shows the first breather from Figure 5 after evolving for
10 periods, and no difference can be discerned visually
between the starting waveform and the evolved wave.
FIG. 5. Time development of breathers in the cubic Whitham equation (3). The breather in the upper panel has a (highest) crest to (lowest) trough waveheight of 0.53 and a non-dimensional speed of 0.9. The breather in the middle panel has a crest-trough waveheight of 0.30 and a non-dimensional speed of 0.96. The breather in the lower panel has a crest-trough waveheight of 0.34 and a non-dimensional speed of 0.69.

Gordon $\phi^4$ equation. However, it was ultimately shown that this expansion did not converge, and the purported breather solution actually features a tiny amount of radiation, though the rate of radiation is so small that it can only be detected using asymptotics beyond all orders \[46, 47\]. The solutions constructed here are also different from the sine-Gordon breather in the sense that they are propagating, and have a well defined phase and crest velocity, similar to the two-parameter family of mKdV breathers. In contrast, the sine-Gordon breather is a one-parameter family of stationary breathers.

Since the present work is numerical, it is possible that the breathers found here also feature very small radiation which may be below machine precision, and in this case the breathers found here would also be approximate in the mathematical sense, i.e. decaying after a very long time. On the other hand, the question whether the breather is an exact solution or a slowly radiating metastable state may not be of much importance in practice. Indeed while it has been shown for example that the Peregrine breather is unstable to virtually all perturbations \[24, 48\], breathers are nevertheless observable in the laboratory \[49\]. Moreover asymptotic model equations such as the KdV and Whitham equations are generally valid physical models only on a time scale inversely proportional to the amplitude \[31, 50–52\].

We should also mention that there are some works discussing mathematical existence of structures more similar to the sine-Gordon breathers. In \[53\], \[54\] and \[55\] non-explicit real-valued, time-periodic and spatially localized solutions are constructed for nonlinear non-integrable wave equations with periodic potentials using variational methods and spatial dynamics and center manifold reductions. These solutions are of breather type, but are stationary, and not propagating, so they are more similar to the breather of the sine-Gordon equation. It will be interesting to see whether breather solutions are possible in other fully dispersive equations such as the fully dispersive Gardner equation \[56\] and the fully dispersive NLS equation \[57\].

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