The fate of the bootstrap percolation hybrid critical point in finite dimension

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Bootstrap, or $k$-core, percolation displays in its mean-field Bethe-lattice version a mixed first/second order phase transition with both a discontinuous order parameter and diverging critical fluctuations. Using the $M$-layer construction to go beyond mean-field I show that at all orders in the loop expansion the problem is equivalent to a spinodal with disorder. Thus the hybrid mean-field transition is destroyed in finite dimension by long-wavelength fluctuations. The result is discussed in connection with the theory of supercooled liquids where the same mapping is found within Random-First-Order-Theory for the Mode-Coupling-Theory critical point.

I. INTRODUCTION

Bootstrap, or $k$-core, percolation (BP) is an extension of ordinary percolation in which the occupancy of a given site is constrained by that of its neighbors. The definition of the problem is very simple: first the sites of a given site is constrained by that of its neighbors. The definition of ordinary percolation in which the occupancy of a given field. The problem is not only to compute the critical exponents below the upper critical dimension, first and foremost we have to assess whether the hybrid transition survives or not in finite dimension.

A notable result is that on regular lattices in dimension $d$ the transition is destroyed for $k > d$: the $k$-core needs to be extended in this regime but it does not survives culling as soon as $p$ is smaller than one, meaning that $p_c = 1$. Thus the question remains of what happens on regular lattices with $k \leq d$ and on generic lattices in any dimension. A perturbative expansion around the large dimension limit of [17] suggested a positive answer. It is also known that finite size effects are severe when the $k > d$ [11] implying that one has to be careful with numerical methods. Nevertheless numerical studies on regular lattices show that for $k = d = 4$ the transition becomes definitively first-order with $p_c < 1$ while for $k = d = 3$ the transition is continuous [13].

On the other hand, if the hybrid transition survives in finite dimension, universality should arise because of the diverging correlation length predicted at the mean-field level, thus the answer should not depend on the microscopic details of the problem, i.e. the geometry of the lattice and the value of $k$. Correspondingly a failure to find the hybrid transition on a specific model could be attributed to so-called short-wavelength fluctuations (the microscopic details) thus not providing the universal answer one is seeking. The theory of second-order phase transition [19][22] prescribes that the universal non mean-field values of the critical exponents can be computed, by means of renormalization group arguments, from the loop expansion of the appropriate field theory describing the long-wavelength fluctuations. Until now the main difficulty was the lack of known field-theoretical formulation of bootstrap percolation, at variance with ordinary percolation where the Fortuin-Kastelein map provides the answer.

To overcome this problem, in this work I apply to bootstrap percolation the $M$-layer construction, a tool recently proposed in [29] to generate the loop expansion starting from the Bethe solution of the problem and not from a field theoretical formulation. As in the $1/d$ expansion it turns out at the mixed nature of the transition survives in perturbation theory, but unexpectedly it
is possible to show (without computing them explicitly!) that the Feynman diagrams with any number of loops are the same of a spinodal point in a random field, which is in turn equivalent to a cubic stochastic equation. It follows that the mixed nature of the transition is destroyed in finite dimension due to long wavelength fluctuations and no critical exponents can be defined.

At the technical level this result is possible taking full advantage of a result that as been known for quite a while \cite{10}, namely the existence of two susceptibilities, one diverging in a standard way and the other with a so-called double pole. In the following I will sketch the proof of the mapping which is rather straightforward but requires basic knowledge of i) the field theoretical approach to second-order phase transition at a textbook level \cite{19–22} ii) the M-layer expansion of \cite{23} iii) the replica theory of the random field Ising model and the mapping to stochastic equations leading to Parisi-Sourlas dimensional reduction \cite{24}, an good compact presentation is given in sections 2.2-2.4 of \cite{25}.

The paper is organized as follows. In section (II) I introduce BP on the Bethe lattice, the M-layer construction, the loop expansion an the mapping. In sec. (III) I give my conclusions discussing in particular the connection with Kinetically constrained models of supercooled liquids. An essential analysis of the two-point functions on the Bethe lattice is skipped to the appendix.

II. THE LOOP EXPANSION OF BOOTSTRAP PERCOLATION

A. Bootstrap percolation on the Bethe lattice

Bootstrap percolation can be defined on any lattice with edges connecting sites that can be either occupied or empty. The k-core is obtained through the steps described above. The final result does not depend on the sequence by which cull sites, in particular we may first consider the modified problem in which a given site \(i\) is occupied with probability one and no culling is applied to it and then apply an extraction and culling (eac) move to obtain its probability \(P_{\text{site}}\) to be in the k-core. The latter is given by \(p\) times the sum over all configuration of its neighbors \(\{s_j, j \in \partial i\}\) satisfying the constraint of their occupancy probability \(P^{(i)}(s_j, j \in \partial i)\) prior to eac

\[
P^{(i)}(\{s_j : j \in \partial i\}) = \prod_{j \in \partial i} P^{(i)}(s_j) \tag{1}
\]

and we have \(P^{(i)}(s_j) = P_B(s_j)\) where

\[
P_B(s) = \delta_{s,1} P + \delta_{s,0}(1 - P) \tag{2}
\]

where \(P\) satisfies an iterative equation. With the definition \(P_{s,k} = \sum_{i,s} \binom{k}{i} P^i(1 - P)^{k-i}\) we have

\[
P = p P_{k-1,c-1}, \quad P_{\text{site}} = p P_{k,c}. \tag{3}
\]

For all \(k > 2\) a solution with non-zero \(P\) is found at large values of \(p\); the solution disappears at a critical value \(p = p_c\) with a square-root singularity thus exhibiting a mixed first-order/second-order character.

B. The M-layer construction

To develop the M-layer construction we start from a given lattice with connectivity \(c\) and consider \(M\) copies of the lattice so that on each site \(i\) of the original lattice we have \(M\) sites \(s^\alpha_i, \alpha = 1, \ldots, M\). Then for each edge of the original lattice separately we rewrite randomly the corresponding \(M\) edges to obtain a new lattice with the same connectivity \(c\), see fig. \(1\). One can argue that in the limit \(M \to \infty\) the new lattice is actually a Bethe lattice, in the sense that any observable takes the value corresponding to the Bethe lattice with small 1/M corrections. The construction is particularly interesting when applied to a lattice in finite dimension, for instance a regular, triangular or honeycomb lattice in \(d = 2\) or a regular, diamond or close-packing lattice in \(d = 3\). In these cases the M-layer construction at large \(M\) has two main advantages: i) it depletes non-universal short wavelength fluctuations so that the mean-field Bethe result become accurate everywhere except close to the Bethe critical point (the critical region), ii) deviations from mean-field theory in the critical region, as obtained from the 1/M

\[
FIG. 1: The M-layer construction: the original lattice is replicated \(M\) times and then each edge is rewired between different copies (\(M = 3\)) (top). The 1/M expansion is obtained summing over Feynaman diagrams (middle) that are transformed into fat diagrams (bottom).
\]
expansion, are due to universal long-wavelength fluctuations described by a universal loop expansion.

C. The loop expansion and the mapping to the stochastic equation

In the $M$-layer framework, according to [23], the average of a given observable over all the possible rewirings can be expressed as a sum over Feynman diagrams. A given Feynman diagram is transformed into a fat diagram substituting its $I$ lines with one-dimensional chains of given lengths \{\(L_1,\ldots,L_I\)\} and attaching Bethe lines so that the connectivity of each vertex is exactly $c$, see the example of fig. [1]. Within bootstrap percolation the natural observable is the joint probability that two or more sites are all on the $k$-core. In the following I will specialize to two-point correlations and thus to Feynman diagrams with two external vertexes. The derivation of the mapping can be easily generalized to higher orders correlations.

The computation of the occupation probability on a given fat diagram can be done it two steps: first we study the case in which the vertexes of the Feynman diagram are occupied and perform eac on the internal lines and then we apply eac to the vertexes.

Let us discuss the properties of a chain of length $L$ in the Bethe lattice. At variance with the single site case, knowledge of the configuration of sites $s_1$ and $s_{L-1}$ prior to eac on sites $s_0$ and $s_L$ is not enough to perform eac on $s_0$ and $s_L$: one has to take into account the possibility that eac on $s_0$ ($s_L$) changes the value of $s_{L-1}$ ($s_1$). This happens if and only if site $s_0$ is emptied by eac and every site $\{s_1,\ldots,s_{L-1}\}$ had exactly $c$ occupied neighbors before eac on $s_0$. Following standard terminology we say that sites $\{s_1,\ldots,s_{L-1}\}$ belong to the corona cluster before eac on $s_0$ and $s_L$. It is convenient to introduce an additional variable $\chi$ that is equal to one if every site on the chain is on the corona and vanishes otherwise. If $\chi = 0$ eac on $s_0$ ($s_L$) depends only on $s_1$ ($s_{L-1}$) before eac; if $\chi = 1$ eac on $s_0$ and $s_L$ cannot be done independently but is easily performed: essentially the whole line can be replaced by a single edge connecting site $s_0$ and $s_L$ directly.

Let us consider the difference between the probability distribution $P_L(s_1,s_{L-1},\chi)$ and its $L \to \infty$ limit $P_B(s_1)P_B(s_{L-1})\delta_{\chi,0}$:

\[
\Delta P_L(\sigma,\tau,\chi) = P_L(\sigma,\tau,\chi) - P_B(\sigma)P_B(\tau)\delta_{\chi,0}.
\]

One can write an iterative equation for $P_L(\sigma,\tau,\chi)$ that leads to the following large distance behavior (see the appendix):

\[
\Delta P_L(\sigma,\tau,\chi) = \delta_{\chi,0}[g(\sigma)g(\tau)\Delta L \lambda^{L-1} + O(\lambda^L)] + \delta_{\chi,1} \delta_{\sigma,1} \delta_{\tau,1} \lambda^{L-1}
\]

where $g(s) = \delta_{s,1} - \delta_{s,0}$ (leading notably to $\sum_\chi g(s) = 0$) and $\Delta$ is a positive constant. Note that the corona probability decays exactly as $\lambda^{L-1}$ but there is a larger $O(L\lambda^L)$ off-corona contribution. The parameter $\lambda$ has the following behavior close to the Bethe critical point $p = p_c$:

\[
\lambda \approx \frac{1}{c-1}(1 - \sigma^{1/2})
\]

where $\sigma$ is a positive quantity vanishing linearly at the critical point $\sigma \propto |p - p_c|$. The configuration of the vertexes of any Feynman diagram after eac on them is completely determined by: i) the configuration of the (Feynman) vertexes themselves after extraction but before culling, ii) the configuration of their neighbors before culling on the Feynman vertexes but after eac on the lines iii) the values of the $\chi$’s of the lines.

More specifically, for any Feynman diagram $G$, we may define an indicator function $I_{12}$ that is equal to one if an only if site $s_1$ and $s_2$ (the external vertexes of the Feynman diagram) are both occupied after eac on all the vertexes:

\[
I_{12}(G,s_1,s_2) = 1_{s_1,s_2 \text{ occupied}}(1_{\chi_1,\chi_2 \text{ occupied}})
\]

where $V \geq 2$ is the total number of vertexes of $G$, $V' = V + cV$ is the number of sites including the $c$ sites at the frontier of each of each vertex and $I$ is the number of lines.

In order to obtain the probability that both sites $s_1$ and $s_2$ survive culling we have to sum the function $I_{12}$ over all possible configurations prior to culling with the corresponding probabilities. As usual it is convenient to specify for each edge $l$ of the Feynman diagram a direction and this allows to define a function $i_n(l)$ (out(l)) that yields the index $j$ of the site at the beginning (at the end) of edge $l$. We also define $S_e$ as the set of frontier sites that are on the edges of the Feynman diagram, i.e. the $2I$ sites $s_j$ such that $j = in(l)$, out(l) for $l = 1,\ldots,I$. We then have:

\[
\langle s_1s_2 \rangle = \sum_{s_1,\ldots,s_{V'},\chi_1,\ldots,\chi_I} I_{12} \left[ \prod_{i=1}^V \left( p(s_i) \prod_{j \in \partial i,j \notin S_e} P_B(s_j) \right) \right] \left[ \prod_{l=1}^I P_{l_i}(s_{in(l)},s_{out(l)},\chi_l) \right]
\]

where $L_l$ is the length of edge $l$ and we have omitted the explicit dependence of $I_{12}$ from its arguments. Actually the correct objects to be computed are the so-called line-connected observables for which a general expression was
provided in [23]. In this context this means that we have just to replace $P_L(\sigma, \tau, \chi)$ with $\Delta P_L(\sigma, \tau, \chi)$:

$$\langle s_1 s_2 \rangle_{I_2} = \sum_{s_1, \ldots, s_L, \ldots, s_L} I_{12} \left[ \prod_{j \in \partial i, j \notin S_0} P(s_i) \prod_{j \in \partial 2} P_B(s_j) \right] \left[ \prod_{j = 1}^L \Delta P_{i_j} (s_{in}(t), s_{out}(t), \chi(t)) \right]. \quad (9)$$

The function $I_{12}$ depends only on the topology of the Feynman graph and not on the actual lengths of the lines. As discussed in [24] in the critical region we must consider the large $L$ limit, therefore we should select the leading $L \lambda^L$ off-corona contribution in $\Delta P_L$ for all lines. However such a contribution has a zero prefactor. This can be seen noticing that if $\chi_i = 0$ on all $l$ the function $I_{12}$ depends only on the frontier of $s_1$ and $s_2$. In particular it does not depend on any of the neighbors of all the other vertexes. As a consequence we can sum on any of the neighboring sites independently and this yields a term $\sum_g g(s) = 0$ for each of them, thus showing that the total contribution is actually zero. Thus when we take the leading contribution for a given line we must be careful not to get a vanishing contribution. Let us start from the beginning and concentrate on a single line $l$ for which we select that leading term $\chi_1 = 0$. We consider two configurations that differ only by the state (occupied or empty) of one of the sites of $l$, say $in(l)$. If the function $I_{12}$ does not change on the two configurations then again the term $\sum_{s_{in}(l)} g(s_{in}(l)) = 0$ produces a vanishing contribution. Thus a necessary condition for having a non-vanishing contribution is that the configuration of the other sites and of the $\chi$’s must be such that the internal vertex to which $s_{in}(l)$ is connected is on the same corona cluster as either $s_1$ or $s_2$ when $s_{in}(l)$ is occupied. This is the only case in which $I_{12}$ can change when $s_{in}(l)$ is emptied (unless $I_{12}$ is zero on both configurations because the other site, respectively $s_2$ or $s_1$, is empty). Similarly if we assign the values of $\chi$ on each edge of the graph and study the effect of summing over the configurations of the sites on the edges with $\chi = 0$, we can easily see that the necessary condition to have a non-vanishing contribution is that when all sites on those edges that are occupied all sites connected to them must be on the same corona cluster of either $s_1$ or $s_2$. The previous observations imply that in order to maximize the number of off-corona edges to have the largest possible contribution while having a non-vanishing prefactor one has to arrange the corona edges on the graph on trees in such a way that each internal vertex is connected to either $s_1$ and $s_2$ by a path of corona edges. Once this condition is fulfilled we can set all other $\chi$’s to zero thus selecting the leading contribution. In fig. 2 we show one possible non-zero leading assignment of the $\chi$’s on a given Feynman diagram. We have represented subleading corona contributions with simple lines and leading off-corona contributions with crossed lines as in the random-field Ising model literature [24] [25]. The only configurations of corona edges and of the vertexes that survive the sum over the off-corona edges vertexes are those such that all vertexes including $s_1$ and $s_2$ are on the corona when all sites on off-corona edges are occupied, so that $I_{12} = 1$ on the latter and $I_{12} = 0$ as soon as any of the off-corona edges is emptied. The total weight of those configurations is obtained multiplying for each vertex a factor

$$p_v = p \left( \frac{c - d_v}{k - d_v} \right) p^{k - d_v} (1 - P)^{c - k} \quad (10)$$

where $d_v$ is the connectivity (also called the degree) of the vertex in the Feynman diagram. In this context the relevant contributions are obtained considering cubic Feynman diagrams (like the one in fig. 2) with $d_v = 3$ for all internal vertexes, while external vertexes have $d_v = 1$. More importantly we have a factor $\lambda^{L_{i} - 1}$ and $\Delta L_{i} \lambda^{L_{i} - 1}$ respectively for each corona and off-corona line.

According to [23] the diagram must then be embedded on the original lattice (with $M = 1$) assigning the positions of the vertexes and the incoming and outgoing

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**FIG. 2:** Top-Left: A cubic 13-loops Feynman diagram contributing to the two-points correlation. Top-Right: one of the possible leading non-vanishing assignments of the subleading $\chi = 1$ corona contributions (simple lines) and of the leading $\chi = 0$ off-corona contributions (crossed lines). Bottom: Graphical representation of the mapping between the random field spinodal and quadratic stochastic equations (see text).
directions of each line. For each line we have to multiply a factor $N_{L_1}(x, y, \mu, \mu')$ that yields the number of non-backtracking paths of length $L_1$ starting at lattice point $x$ in the direction $\mu$ and ending at lattice point $y$ from direction $\mu'$. In the interesting regime $L_1 \gg 1$ one can argue that $N_{L_1}(x, y, \mu, \mu')$ is independent of the directions $\mu$ and $\mu'$ and can be replaced by the total number of paths between $x$ and $y$ divided by the connectivity squared $N_{L_1}(x, y)/c^2$. Then the sum over different directions yields for each vertex a factor

$$n_v = d_v! \left( \frac{c}{d_v} \right).$$

(11)

It is convenient to study the Fourier transform of the two-point function so that we have to sum over the positions of all vertexes, including the external ones. In the relevant region where the vertexes are far apart the number of paths of length $L$ between two lattice points at $x$ and $y$ can be approximated by

$$N_L(x, y) \approx \frac{c(c-1)^{L-1}}{\rho} G(x - y)$$

(12)

where $G(x)$ is a Gaussian distribution and $c(c-1)^{L-1}$ is the total number of paths starting from a given lattice point. Furthermore since the integrand varies slowly the sum over the lattice points can be replaced by an integral: $\sum \to \rho \int d^d x$. Performing also the internal integrals in Fourier space we end up with a momentum conservation condition (and a factor $(2\pi)^{d-d_c/2}$) at each internal vertex and a factor $c\rho$ for each vertex. Furthermore for each line we have a factor

$$\frac{1}{c^2} N_L(k, k') \approx \delta(k + k') \frac{(c-1)^{L-1}}{c\rho} \exp[-L \sum_{ij} a_{ij} k_i k_j]$$

(13)

where the factor $c^2$ is to get the contribution at fixed incoming and outgoing directions. The constants $a_{ij}$ depend on the lattice, in the following for simplicity we assume that the lattice is isotropic at large distances so that $a_{ij} = a \delta_{ij}$.

At this point we perform the summation over $L$ on each line and we see that at the critical point since $\lambda$ obeys eq. (10) the exponential decay is very slow and leads to a single pole contribution for a corona edge and a double pole contribution for an off-corona edge:

$$\frac{1}{c \rho} \frac{1}{\sigma^{1/2} + a k^2} + \frac{1}{c \rho} \Delta \left( \frac{1}{\sigma^{1/2} + a k^2} \right)^2.$$  

(14)

Finally according to [23] each diagram must be divided by the symmetry factor of the Feynman diagram $S(G)$ and multiplied by a factor $1/M^{n+L-1}$ where $n$ is the order of the observable (the number of external vertexes of the diagram) and $L$ is the number of loops. If we rescale the distances in order to have $ak^2 \to k^2$ we have additional factors for each internal line and for each internal vertex. We have thus constant factors $p^{in}$ associated to internal vertexes, $p_{ext}$ to external vertexes, $i_{in}$ to internal lines and $i_{ext}$ to external lines. For cubic diagrams internal vertexes and internal lines have expressions $V_{in} = 2(L-1) + n$ and $I_{in} = 3(L-1) + n$ thus if we also redefine the observable of order $n$ multiplying it by an appropriate constant $b^n$ all factors (except that $(2\pi)^{d-d_3/2}$ at each internal vertex) can be reabsorbed into a rescaling of the factor $M$:

$$\frac{1}{M^{n+L-1}} b^n i_{in}^{in} i_{ext}^{in} V_{in}^{in} i_{ext}^{ext} P_{ext}^{ext} = \frac{1}{M^{n+L-1}}.$$  

(15)

And thus we arrive at the final result that the (rescaled) observable of order $n$ is obtained summing over all cubic Feynman diagrams with a factor $1/(M^{n+L-1}S(G))$. For each diagram we have to consider all possible valid arrangements of corona and off-corona edges and multiply the propagators [14] without the $(cp)^{-1}$ factors and with $a = 1$. Then we have to perform the integrals over the internal momenta with the momentum conservation condition and a factor $(2\pi)^{d-d_3/2}$ at each internal vertex. The above Feynman rules are exactly the same of a spinodal in a random field. Let us recall the definition of the corresponding replicated action in terms of fields $m_a(x)$ where $a = 1, \ldots n$ and $n \to 0$ at the end:

$$\int \prod_{a,x} dm_a(x) \exp \left[-\tilde{M}\mathcal{L}(m)\right]$$

(16)

The loop expansion around the stable mean-field solution

$$m_a(x) = -\sigma^{1/2}$$

is controlled by the propagator:

$$\langle m_a(k)m_{a'}(k') \rangle_c = \frac{\delta_{ab}}{\sigma^{1/2} + k^2} + \Delta \left( \frac{1}{\sigma^{1/2} + k^2} \right)^2.$$  

(18)
Note that because of the $n \to 0$ limit simple scaling badly fails and the propagator displays a double pole. Following the literature we associate a continuous line to the single-pole contribution and a crossed line to the double-pole contribution. Given a Feynman diagram we would like to select the most diverging contribution but care must be taken because of the replica summations. If we select the crossed term on each line around a given vertex the free summation over the vertex replica indexes yields $\sum a = n = 0$. A standard analysis, see e.g. \cite{25} sec. 2.2-2.4, tells us that the leading non vanishing contributions are obtained arranging the simple and crossed propagators in such a way that all vertexes are on trees of simple propagators connected to the external vertexes precisely as in fig. (2). Thus we see that the Feynman rules for the most diverging diagrams are exactly the same that we have obtained above for bootstrap percolation (the factor $(2\pi)^{d-d/2}$ for the internal vertexes appears when we write the cubic vertex contribution in Fourier space). One can also show that the most diverging diagrams are those generated by the solution of a stochastic equation. Through a Hubbard-Stratonovich transformation we rewrite the term proportional to $\Delta$ in \cite{16} as:
\[
\int \left[ \prod dh(x) \right] e^{-\int d^d x \frac{h^2(x)}{2M} + \beta \int d^d x \frac{h(x)}{M^{1/2}} \sum_a m_a(x)} .
\]
\[
\text{(19)}
\]
Different replicas are now decoupled and a loop expansion of each replica separately before the integration over the random fields generates all possible diagrams with simple poles and sources $h(x)$. The ensuing average over the fields generates the double poles through the merging of two sources $h(x)$. Since as we saw above the relevant diagrams after the average over the random sources are those where the simple poles are arranged on trees we can select these tree diagrams before the average. A classic result is that tree diagrams are generated by the solution of the mean-field equation:
\[
\frac{\sigma}{2} + \frac{h(x)}{M} - \nabla^2 m(x) - \frac{1}{2} m^2(x) = 0 \quad \text{(20)}
\]
and therefore we can write for an $N$-point function in the replicated system
\[
\langle m_a(x_1) \ldots m_a(x_N) \rangle \approx [m(x_1) \ldots m(x_N)] \quad \text{(21)}
\]
where $m(x)$ is the solution of the above stochastic equation and the square brackets mean average over the solution of the above stochastic equation with Gaussian random fields $h(x)$
\[
[h(x)] = 0, \quad [h(x)h(y)] = \Delta \delta(x - y) \quad \text{(22)}
\]
Note that in principle the averages over the random fields have a weight $Z^n$ and are not white, but they become white averages at $n = 0$. The mapping to the stochastic equation is represented graphically in fig. (2). Before average the loop expansion is given by Feynman diagrams with single poles and random sources (represented by arrows), after average different sources are paired to give the crossed lines of the replica treatment. The leading diagrams after average are associated to tree diagrams before average whose sum is given by the stochastic equation.

The average over solutions in the r.h.s. of eq. \cite{21} is clearly ill-defined because the stochastic equation does not admit a real solution for certain (negative) values of $h(x)$. The equation is well-defined instead if the random fields are imaginary and yields the universality class of branched polymers \cite{26}.

\section{III. CONCLUSIONS}

The mapping to an ill-defined stochastic equation implies that the hybrid transition cannot exist in finite dimension. The nature of the actual BP transition is not universal, depending on the model, it may be continuous, first-order with $p_c = 1$ or first order with $p_c < 1$. In the last case one could still observe a behavior consistent with a mixed critical point with mean-field exponents that is eventually avoided by a genuine first-order transition as observed in \cite{18}. The possible occurrence of this pseudo-critical scenario is model-dependent and could be relevant in applications.

The result is particularly intriguing given the precise connection existing between BP and Fredrikson-Andersen (FA) models, a class of kinetically constrained models of super-cooled liquids. Indeed a mapping to the very same quadratic stochastic equations have been obtained in recent years for the Mode-Coupling-Theory (MCT) critical point in the context of the Random-First-Order-Theory \cite{27} motivating interest in other random critical points \cite{28,29} and in zero-temperature disordered spinodals \cite{30}. Later a mapping to a well-defined dynamical stochastic equation, called stochastic beta relaxation (SBR), has been obtained through a full-fledged dynamical treatment \cite{31,32} and provides a complete description of how the sharp transition of MCT is transformed into a dynamical crossover. It is likely that SBR is also valid for FA models, but at present such an analysis is hampered by the lack of a Bethe lattice solution of the dynamics, i.e. we are not even able to show analytically that FA models obey critical MCT equations, as shown numerically in \cite{12,14,15}. Be as it may, the result of \cite{27} has already been applied to FA models \cite{33,34} because, even if the quadratic stochastic equation is ill-defined, one can argue that the scaling exponents controlling critical finite-size corrections on the Bethe lattice are valid. Similarly, it is natural to expect that the correct dynamical scaling functions in finite dimension are given by SBR as confirmed in recent numerical simulations in the $M$-layer framework \cite{35}.
Appendix: Two-point correlations on the Bethe lattice

Let us derive the two-point function on the Bethe lattice. The probability that every site on the chain is 

\[
\left( \frac{P_{L+1}(1, s_1, 0)}{P_{L+1}(0, s_1, 0)} \right) = \left( \frac{p P_{k-2,c-2}}{1 - p P_{k-2,c-2}} \right) \left( \frac{P_L(1, s_1, 0)}{P_L(0, s_1, 0)} \right) + \ldots
\]

where the dots represent the corona contribution to be discussed later. There matrix has two eigenvalues, the Perron-Frobenius eigenvalue \( \lambda_{PF} = 1 \) with right eigenvector \( (P, 1 - P) \) and left eigenvector \( (1, 1) \) and an eigenvalue equal to \( \lambda \) controlling the corona probability with right eigenvector \( (1, -1) \) (orthogonal to left PF eigenvector \( (1, 1) \)). The additional contribution to \( P_{L+1}(s_L, s_0, 0) \) comes from those configurations that were on the corona prior to each on \( s_L \) but are no longer on the corona after. This can either happen if site \( s_L \) after each is still occupied with more than \( m \) occupied neighbors or if it is emptied. The first case contributes to \( P_{L+1}(1, 1, 0) \), the second to \( P_L(0, 0, 0) \) and the sum of the two contributions must be equal to the probability that a corona configuration is no longer a corona configuration after cutting i.e. \( \lambda L^{-1}(1 - \lambda) \).

Now let us write \( P_L(s_{L-1}, s_L, 0) \) in full generality in terms of the eigenvectors of the matrix, we will have:

\[
P_L(\sigma, \tau, 0) = a_{11}^{(L)} P_B(\sigma) P_B(\tau) + a_{12}^{(L)} P_B(\sigma) g(\tau) + a_{21}^{(L)} g(\sigma) P_B(\tau) + a_{22}^{(L)} g(\sigma) g(\tau)
\]

Note that from the normalization condition implies \( a_{11}^{(L)} = 1 - \lambda L^{-1} \) while symmetry of the function with respect to exchange of its arguments implies \( a_{12}^{(L)} = a_{21}^{(L)} \). The iterative equation translates then into an equation for the components \( a_{12}^{(L)} \). The application of the linear matrix breaks the symmetry with respect to the exchange

\[
\lambda = p \left( \frac{c - 2}{k - 2} \right) P^{k-2}(1 - P)^{c-k}
\]

Therefore we have \( \sum_{s_1, s_{L-1}} P_L(s_1, s_{L-1}, 0) = 1 - \lambda L^{-1} \). Next we can write an iterative equation for \( P_{L+1}(s_L, s_1, \chi) \) in terms of \( P_L(s_{L-1}, s_1, \chi) \). We have to perform each on \( s_L \) while keeping \( s_{L+1} \) occupied. Unless every site \( \{s_1, \ldots, s_{L-1}\} \) was on the corona each on \( s_L \) will not affect \( s_1 \) this case will contribute a term while all other terms in the equation are symmetric therefore we must have \( a_{12}^{(L)} = a_{21}^{(L)} = 0 \). For the same reason the contribution coming from the corona cannot have a mixed term of the form \( P_B(\sigma) g(\tau) + P_B(\sigma) g(\sigma) \), besides it is concentrated on \( (1, 1) \) and \( (0, 0) \) and zero otherwise it must be equal to:

\[
\lambda L^{-1}(1 - \lambda) (P_B(\sigma) P_B(\tau) + P(1 - P) g(\sigma) g(\tau))
\]

This leads to the following equation for \( a_{22}^{(L+1)} \):

\[
a_{22}^{(L+1)} = \lambda a_{22}^{(L)} + \lambda^2 \Delta
\]

where

\[
\Delta \equiv \frac{1 - \lambda}{\lambda} P(1 - P).
\]

The above equation is valid for \( L \geq 2 \) with \( a_{22}^{(2)} = \lambda \Delta \) and the exact solution is then

\[
a_{22}^{(L)} = (L - 1) \lambda^{L-1} \Delta
\]

leading to the expression for the propagator used before. For the correlation on the Bethe lattice we have at leading order:

\[
\langle s_0 s_L \rangle = \frac{p^2}{\lambda} L \lambda^L + O(\lambda^L)
\]

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