BLOWUPS OF SURFACES AND MODULI OF
HOLOMORPHIC VECTOR BUNDLES

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Abstract. We examine the moduli of framed holomorphic bundles over the blowup of a complex surface, by studying a filtration induced by the behavior of the bundles on a neighborhood of the exceptional divisor.

1. Introduction

The motivation for this paper comes from the study of moduli spaces of based instantons. In [4], [19], it was shown that the moduli space of based instantons over a connected sum of $q$ copies of $\mathbb{P}^2$ is isomorphic as a real analytic space, to the moduli space of holomorphic bundles over a blow up of $\mathbb{P}^2$ at $q$ points, framed at a rational curve $L_\infty \subset \mathbb{P}^2$. From the study of this moduli space we were led to consider the relationship between the moduli space over an algebraic surface $X$ and the moduli space over its blow up $\tilde{X}$. Bundles on the blow up of a complex surface have been studied in [2], [9], [10], [11]. Our approach is inspired by that in [5]. This paper builds up on results in [20], extending them to the compactification of the moduli space.

1.1. Results. Let $X$ be a smooth algebraic surface and let $C \subset X$ be a curve of positive self-intersection. When $X = \mathbb{P}^2$ we will always take $C$ as a rational curve. Fix an ample divisor $H$ and a polynomial $\bar{\delta}(n)$ with positive coefficients. Let $\mathcal{M}_{k^{ss}}^r(X, H, \bar{\delta})$ denote the moduli space of $(H, \delta)$ semi-stable pairs $(\mathcal{E}, \phi)$ where $\mathcal{E} \to X$ is a rank $r$ coherent sheaf with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$, and $\phi : \mathcal{E} \to \mathcal{O}_{C_\infty}$ is a non-zero homomorphism (the framing). Stability of the pair $(\mathcal{E}, \phi)$ means the following:

Definition 1.1. A pair $(\mathcal{E}, \phi)$ is said to be (semi)stable with respect to $(H, \delta)$ if for all subsheaves $\mathcal{A} \subset \mathcal{E}$ we have

$$\text{rk} \mathcal{E} \left( \chi(\mathcal{A}(n)) - \varepsilon \bar{\delta}(n) \right) (\leq) < \text{rk} \mathcal{A} \left( \chi(\mathcal{E}(n)) - \bar{\delta}(n) \right)$$

where $\varepsilon = 0$ if $\mathcal{A} \subset \ker \phi$ and $\varepsilon = 1$ otherwise.
This is a special case of the construction of moduli of framed sheaves in \([13, 14]\). We will omit reference to \((H, \delta)\) unless the dependance on the polarization is important.

In this paper we will be looking at the subspace \(\mathcal{M}_k^r(X) \subset \mathcal{M}_k^{r,ss}(X)\) of pairs \((E, \phi)\) where \(E \to X\) is a holomorphic rank \(r\) vector bundle, trivial when restricted to \(C\), and \(\phi\) induces a trivialization. Let \(\overline{\mathcal{M}}_k^r(X)\) denote the closure of \(\mathcal{M}_k^r(X)\). The objective of this paper is to prove the theorem

**Theorem 1.2.** Let \(\pi : \tilde{X} \to X\) be the blow up of \(X\) at a point \(x_0 \notin C\). Given a sheaf \(\mathcal{E} \to \tilde{X}\), let \(\pi^\vee \mathcal{E} \to X\) be the sheaf defined by \(\pi^\vee \mathcal{E}(U) = \pi_* \mathcal{E}(U \setminus \{x_0\})\).

1. Let \(S_i \mathcal{M}_k^r(\tilde{X}) = \{(E, \phi) \mid c_2(\pi^\vee \mathcal{E}) = i\}\) Then the map \(\pi^\vee : S_i \mathcal{M}_k^r(\tilde{X}) \to \mathcal{M}_i^r(X)\) is a topologically trivial fibration with fiber \(S_0 \mathcal{M}_{k-i}(\mathbb{P}^2)\);

2. Let \(F_i \mathcal{M}_k^r(X) = \{(E, \phi) \mid c_2(\pi^\vee \mathcal{E}) \leq i\}\) and let \(\overline{S_i \mathcal{M}}_k = F_i \mathcal{M}_k \setminus F_{i-1} \mathcal{M}_k\). Then the map \(\pi^\vee\) extends to a map \(\pi_* : F_i \mathcal{M}_k(\tilde{X}) \to \mathcal{M}_i(X)\). The restriction of this map to \(S_i \mathcal{M}_k(\tilde{X})\) is a fibration with fiber \(S_0 \mathcal{M}_{k-i}(\mathbb{P}^2)\).

The filtration \(F_i \mathcal{M}_k^r\) induces a spectral sequence converging to \(H_*\left(\overline{\mathcal{M}}_k^r\right)\) with \(E^1\) term given by

\[ E^1_{p,q} = H_{p+q}\left( F_p \overline{\mathcal{M}}_k^r(\tilde{X}), F_{p-1} \overline{\mathcal{M}}_k^r(\tilde{X}) \right) \]

As a corollary we will prove

**Corollary 1.3.** Let \(\mathcal{M}_i(X) = \pi_* F_{i-1} \overline{\mathcal{M}}_k^r(\tilde{X}) \subset \mathcal{M}_i(X)\). Then there is a spectral sequence converging to \(H_*\left( F_i \mathcal{M}, F_{i-1} \mathcal{M} \right)\) with \(E^2\) term given by

\[ E^2_{p,q} = H_p\left( \overline{\mathcal{M}}_i(\tilde{X}), \mathcal{M}_i(X) ; H_q(S_0 \mathcal{M}_{k-1}(\mathbb{P}^2)) \right) \]

We conjecture a similar result holds in the non compactified case. Here the triviality of the bundle would lead to a more powerful result:

**Conjecture 1.4.** There is an isomorphism

\[ H_*\left( F_i \mathcal{M}_k^r(\tilde{X}), F_{i-1} \mathcal{M}_k^r(\tilde{X}) \right) \cong H_*\left( \pi_* F_i \mathcal{M}_k^r(\tilde{X}), \mathcal{M}_i(X) \right) \otimes H_*\left( S_0 \mathcal{M}_{k-1}(\mathbb{P}^2) \right) \]
The plan of the paper is as follows: In section 2 we recall the definitions of the moduli space and prove some theorems concerning stability; In section 3 we prove the second part of the theorem 1.2, and corollary 1.3. In section 4 we prove the first part of the theorem 1.2. Finally in section 5 we give a monad description of the space $S_0 \mathfrak{M}_k(\mathbb{P}^2)$.

2. Definition of the moduli space

The objective of this section is to recall the definitions of the moduli spaces we will be considering, and to prove some stability results.

First we introduce a technical restriction on the pairs $(X, C)$ we will be considering in this paper. Then we present two definitions of the moduli space: Following [18], we look at $\mathfrak{M}_k^r$ as the space of holomorphic structures on a fixed topological bundle $E_{top}$. The second definition uses the language of moduli functors, following [15], [14].

2.1. Admissible pairs.

**Definition 2.1.** Let $X$ be an algebraic surface and let $C \subset X$ be a divisor with $C^2 > 0$. We say the pair $(X, C)$ is admissible if there is an ample divisor $H$ such that the set $\{ c_1(A) \cdot H \}_{A \in C}$ is bounded above for all $k, r$. Here $C^r_k$ denotes the set of sheaves $A \to X$ such that

- $c_1(A) \cdot C \leq 0$;
- There is a torsion free rank $r$ sheaf $E \to X$ with $c_1(E) = 0$, $c_2(E) = k$ such that $A \subset E$.

Clearly, if $C$ is ample, $(X, C)$ is admissible.

**Proposition 2.2.** Let $(X, C)$ be an admissible pair with ample divisor $H$ and let $\tilde{X} \to X$ be the blow up of $X$ at a point $x_0 \notin C$. Then the pair $(\tilde{X}, C)$ is admissible with respect to the ample divisor $\tilde{H} = H + C - L$ where $L$ denotes the exceptional divisor.

**Proof.** Let $A \in C^r_k$, $A \subset E$. Then $(\pi_* A)_{\mathbb{P}^2} \subset (\pi_* E)_{\mathbb{P}^2}$. Hence, the proposition will follow if we prove that $-c_1(A) \cdot L \leq 2k$. 
We may assume without loss of generality that \( A, E \) are locally free. Let \( \mathcal{T} = \text{Tor} \mathcal{E} / \mathcal{A} \). Then we have the diagram

\[
\begin{array}{cccc}
0 & 0 & Q \\
& & \\
0 & \mathcal{A} & \mathcal{E} & \mathcal{E} / \mathcal{A} & 0 \\
& & \uparrow & \downarrow & \\
& & \mathcal{K} & \mathcal{T} & \\
& & \uparrow & \downarrow & \\
0 & 0 & \\
\end{array}
\]

for some sheaves \( Q \) and \( \mathcal{K} \). Now observe that

\[
c_1(\mathcal{K}) \cdot H \geq c_1(\mathcal{A}) \cdot H
\]

So we may assume without loss of generality that the quotient \( \mathcal{E} / \mathcal{A} \) is torsion free. Now write

\[
\mathcal{A}|_L = \sum_j \mathcal{O}(a_j), \quad \mathcal{E}|_L = \sum_l \mathcal{O}(b_l)
\]

Then \( a_j = b_l \), so

\[
|c_1(\mathcal{A}) \cdot L| \leq \sum_j |a_j| \leq \sum_l |b_l| \leq 2c_2(\mathcal{E})
\]

which concludes the proof. \( \square \)

2.2. Analytic definition. Let \( E \to X \) be an \( SU(r) \) topological bundle with \( c_2(E) = k \).

Let \( \mathcal{C}(X, E) \) be the space of pairs \((\bar{\partial}, \phi)\) where \( \bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E) \) is a holomorphic structure on \( E \) holomorphically trivial on \( C \) and \( \phi : E|_C \to \mathcal{O}_C^r \) is an isomorphism of holomorphic bundles.

**Proposition 2.3.** The group \( \text{Aut}(E) \) acts freely on \( \mathcal{C}(X, E) \) and the quotient has the structure of a finite dimensional Hausdorff complex analytic space.

**Proof.** It follows from

\[
\forall_{k>0} H^0(\text{End} \mathcal{E}_0 \otimes \mathcal{O}_C(-k)) = 0
\]

where \( \mathcal{E}_0 = \mathcal{O}_C^r \) (see [18] theorem 1.1 and lemma 2.6). \( \square \)
Definition 2.4. We define \(\mathcal{M}(X, E)\) as the quotient \(\mathcal{C}(X, E)/\text{Aut}(E)\). We will also use the notation \(\mathcal{M}^r_k(X)\).

2.3. Algebraic definition. Now we present the algebraic definition. For details see [15], [14].

Let \(E_0 = \mathcal{O}_C\). A family of framed sheaves parametrized by a Noetherian scheme \(T\) consists of a pair \((\mathcal{F}, \alpha)\) where \(\mathcal{F}\) is a coherent \(\mathcal{O}_{T \times X}\) module, flat over \(T\), and \(\alpha : \mathcal{F} \to \mathcal{O}_T \otimes E_0\) is a homomorphism with \(\alpha_t \neq 0\).

A homomorphism of families \((\mathcal{F}, \alpha) \to (\mathcal{F}', \alpha')\) is a homomorphism of sheaves \(\mathcal{F} \to \mathcal{F}'\) compatible with \(\alpha, \alpha'\).

The moduli functor \(\mathcal{M}^{ss}(X, E_0)\) is the functor from \((\text{Schemes})\) to \((\text{Sets})\) that to a scheme \(T\) associates the set of isomorphism classes of flat families of semistable pairs parametrized by \(T\) (recall the definition of stability 1.1). In a similar way we define the functor \(\mathcal{M}^s(X, E_0)\) by replacing the word semistable with stable in the definition.

In [14] Huybrechts and Lehn proved

Theorem 2.5. There is a projective scheme \(\mathcal{M}^{r, ss}_k(X)\) which corepresents the functor \(\mathcal{M}^{ss}_k(X)\). Moreover there is an open subscheme \(\mathcal{M}^{r, ss}_k(X)\) which is a fine moduli space representing \(\mathcal{M}^s_k(X)\).

\(\mathcal{M}^{r}_k(X) \subset \mathcal{M}^{r, ss}_k(X)\) is defined as the subspace of pairs \((\mathcal{E}, \phi)\) where \(\mathcal{E} \to X\) is a holomorphic vector bundle trivial when restricted to \(C\), and \(\phi\) induces a trivialization. \(\mathcal{M}^{r}_k(X)\) is defined as the closure of \(\mathcal{M}^{r}_k(X)\).

2.4. Some results about stability. A pair \((\mathcal{E}, \phi)\) is said to be \(\mu\)-stable with respect to \((H, \delta)\) if for every subsheaf \(\mathcal{A} \subset \mathcal{E}\) we have \(\text{rk} \mathcal{E}(c_1(\mathcal{A}) \cdot H - \varepsilon \delta) \leq -\text{rk} \mathcal{A} \delta\) where \(\varepsilon\) is defined as in [1.3]. Then, from Riemann-Roch we get the implications

\[
\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu - \text{semistable}
\]

Proposition 2.6. Let \(H\) be an ample divisor and consider a polarization \((H + MC, \delta' + n\delta)\) with \(M > \delta\). Then, if a pair \((\mathcal{E}, \phi)\) is semistable, \(\mathcal{E}\) is torsion free.

Proof. We apply the semistability condition to \(\mathcal{A} = \text{Tor} \mathcal{E}\). We want to show \(\mathcal{A} = 0\). We divide the proof into two steps:

1. \(\mu\)-semistability implies \(c_1(\mathcal{A}) \cdot (H + MC) \leq \varepsilon \delta\). Suppose \(c_1(\mathcal{A}) \neq 0\). Then \(\varepsilon = 1\) so the restriction of \(\phi\) to \(\mathcal{A}\) is not identically zero.
But then we must have $c_1(A) \cdot C > 0$ so $(H + MC) \cdot c_1(A) \geq M > \delta$ contradicting semistability. Hence $c_1(A) = 0$.

(2) Since $c_1(A) = 0$, $A$ is supported in codimension 2, so $\varepsilon = 0$ and $c_2(A) \leq 0$. On the other hand semistability implies $\chi(A) = -c_2(A) \geq 0$. Hence $c_2(A) = 0$. So $A = 0$.

□

Lemma 2.7. Let $(X, C)$ be an admissible pair with ample divisor $H$. Fix a pair $(E, \phi)$. Then there is an integer $\delta_0 > 0$ such that, for all $\delta > \delta_0$ there is an integer $M_0$ depending on $\delta$ such that for all $M > M_0$ the following holds with respect to the choice of polarization $(H + MC, \delta + \delta n)$:

(1) Let $A \subset E$ be such that $c_1(A) \cdot C < 0$. Then $A$ is not destabilizing.

(2) Let $A \subset E$ be such that $c_1(A) \cdot C \leq 0$ and $\varepsilon = 1$. Then $A$ is not destabilizing.

Proof. For simplicity we assume $c_1(E) = 0$. Let $r_E = \text{rk } E$, $r_A = \text{rk } A$. Since $(X, C)$ is admissible we can choose $\delta_0 > r_E c_1(A) \cdot H$. Now pick any $\delta > \delta_0$. Then choose $M_0$ so that $M_0 > c_1(A) \cdot H + \delta$. Finally pick any $M > M_0$. Let $H_M = H + MC$.

(1) Assume $c_1(A) \cdot C < 0$. Then

$$r_E c_1(A) \cdot H_M \leq r_E (c_1(A) \cdot H - M) < -r_E \delta < -r_A \delta.$$ 

So, we get

$$\frac{c_1(A) \cdot H_M - \varepsilon \delta}{r_A} < \frac{c_1(E) \cdot H_M - \delta}{r_E}$$

(2) Assume $c_1(A) \cdot C \leq 0$. Then

$$r_E (c_1(A) \cdot H_M - \delta) \leq r_E c_1(A) \cdot H - r_E \delta < (1 - r_E) \delta \leq -r_A \delta.$$

Now, since by assumption $\varepsilon = 1$,

$$\frac{c_1(A) \cdot H_M - \varepsilon \delta}{r_A} < \frac{c_1(E) \cdot H_M - \delta}{r_E}$$

□

As a corollary we have

Proposition 2.8. There is an ample divisor $H$ and a polynomial $\bar{\delta}$ such that, for any $M > 0$, the following holds with respect to the polarization $(H + MC, \bar{\delta})$: 


Let $\mathcal{E} \to X$ be a torsion free sheaf, and let $\phi : \mathcal{E} \to \mathcal{E}_0$ induce an isomorphism $\mathcal{E}|_C \to \mathcal{O}_C$. Then the pair $(\mathcal{E}, \phi)$ is stable.

Proof. Let $(H + MC, \delta' + n\delta)$ be such that the conclusions of lemma 2.7 hold. The proof will follow from the following statements:

- If $A \subset \mathcal{E}$ then $C \cdot c_1(A) \leq 0$.
- If $\varepsilon = 0$ then $C \cdot c_1(A) = 0$.

The first statement follows from $A|_C \subset \mathcal{E}|_C \cong \mathcal{O}_C$. To prove the second we observe that $\text{Ker} \phi = \mathcal{E}(-C)$ so if $A \subset \text{Ker} \phi \subset \mathcal{E}$ then $A(C) \subset \mathcal{E}$.

The result follows. □

Now we prove a converse to lemma 2.7:

**Proposition 2.9.** Let $(H, \bar{\delta}(n) = \delta' + n\delta)$ be a polarization for which the conclusions of lemma 2.7 hold. Let $(\mathcal{E}, \phi) \in \mathcal{M}^r_k(X)$. Then, for all subsheaves $A \subset \mathcal{E}$, we have either $c_1(A) \cdot C < 0$ or $c_1(A) \cdot C = 0, \varepsilon = 1$.

Proof. Let $(\mathcal{E}, \phi) \in \mathcal{M}^r_k(X) \subset \mathcal{M}^{ss}(H, \bar{\delta})$ and suppose there was some $A \subset \mathcal{E}$ with either $c_1(A) \cdot C > 0$ or $c_1(A) \cdot C = \varepsilon = 0$. We want to show this is not possible. Consider a new polarization $(H_M, \bar{\delta}_M)$ with $H_M = H + M_H C$, $\delta_M(n) = \delta(n) + nM\delta$, for some constants $M_H > M\delta > 0$.

Then

1. We claim that if $(\mathcal{E}', \phi')$ is any pair semistable with respect to $(H_M, \bar{\delta}_M)$, then $(\mathcal{E}', \phi')$ is stable with respect to $(H, \bar{\delta})$. Let $A' \subset \mathcal{E}'$. By lemma 2.7 we may assume either $c_1(A') \cdot C > 0$ or $c_1(A') \cdot C = \varepsilon = 0$. Then $\mu$-semistability with respect to $(H_M, \bar{\delta}_M)$ implies

   
   \begin{align*}
   0 \geq r_E(c_1(A') \cdot H_M - \varepsilon(\delta + M\delta)) + r_A(\delta + M\delta) \geq
   \geq r_E(c_1(A') \cdot H - \varepsilon\delta) + r_A\delta + M_H r_E c_1(A') \cdot C + M\delta(1 - r_E) >
   > r_E(c_1(A') \cdot H - \varepsilon\delta) + r_A\delta
   \end{align*}

   where $r_E = \text{rk} \mathcal{E}$, $r_A = \text{rk} \mathcal{A}$. So $(\mathcal{E}', \phi,)$ is $\mu$-stable with respect to $(H, \bar{\delta})$.

2. We claim that for $M_H > M\delta \gg 0$, $(\mathcal{E}, \phi)$ is unstable with respect to $(H_M, \bar{\delta}_M)$; We have two cases:

   - If $c_1(A) \cdot C > 0$ then we just have to choose $M_H$ big enough so that $(\mathcal{E}, \phi)$ is $\mu$-unstable;
• If \( c_1(A) \cdot C = \varepsilon = 0 \) then stability questions are not affected by \( M_H \) and making \( M_\delta \) big enough we can make \((E, \phi)\) \( \mu \)-unstable.

Using the fact \( \mathcal{M}^{ss} \) corepresents the moduli functor, property (1) implies there is a map \( \mathcal{M}^{ss}(H_M, \bar{\delta}_M) \to \mathcal{M}^s(H, \bar{\delta}) \). Let \( D \subset \mathcal{M}_k^s(X) \subset \mathcal{M}^{ss}(H, \bar{\delta}) \) be a one dimensional disk and suppose the origin \( 0 \in D \) corresponds to \((E, \phi)\) and \( D \setminus 0 \) is contained in \( \mathcal{M}_k^s(X) \). Then, by proposition 2.8, the restriction of the universal family to \( D \setminus 0 \) gives a family of \((H_M, \bar{\delta}_M)\)-stable pairs. Hence we have a map \( D \setminus 0 \to \mathcal{M}^{ss}(H_M, \bar{\delta}_M) \). Since \( \mathcal{M}^{ss}(H_M, \bar{\delta}_M) \) is projective this map extends to a map \( D \to \mathcal{M}^{ss}(H_M, \bar{\delta}_M) \). We get a commutative diagram

\[
\begin{array}{ccc}
D & \longrightarrow & \mathcal{M}^{ss}(H_M, \bar{\delta}_M) \\
\downarrow & & \downarrow \\
& \mathcal{M}^{ss}(H, \bar{\delta})
\end{array}
\]

But this contradicts the fact that \((E, \phi) \notin \mathcal{M}^{ss}(H_M, \bar{\delta}_M)\). This concludes the proof. \( \square \)

As a corollary we have

**Corollary 2.10.** For a convenient choice of polarization, \( \mathcal{M}_k^s(X) \subset \mathcal{M}_k^{r,s} \).

### 3. The map \( \pi_\blacksquare \)

In this section we prove the second part of theorem 1.2. Recall that

\[
\begin{align*}
F_i \mathcal{M}_k(X) & = \{(E, \phi) \mid c_2(\pi^*_\blacksquare \mathcal{E}) \leq i \} \\
S_i \mathcal{M}_k(X) & = \{(E, \phi) \mid c_2(\pi^*_\blacksquare \mathcal{E}) = i \}
\end{align*}
\]

First we want to show the existence of a map \( \pi_\blacksquare : F_i \mathcal{M}_k(X) \to \overline{\mathcal{M}_i(X)} \) extending \( \pi^*_\blacksquare \). We will define this map locally. Let \( T \subset F_i \mathcal{M}_k(X) \) be an open subset and consider the universal family \((\mathcal{F}, \alpha)\), \( \mathcal{F} \to T \times \tilde{X} \), \( \alpha : \mathcal{F} \to p^* \mathcal{E}_0 \). Let \( L \) be the exceptional divisor.

**Proposition 3.1.** For each \( N \in \mathbb{Z} \) define the family \( \pi^N\mathcal{F} \to T \times X \) by

\[
\pi^N\mathcal{F}(W) = (\Xi \times \pi)_* \mathcal{F}(-NL)(W \setminus (T \times x_0))
\]

For each \( t = [E, \phi] \in T \) let \( \iota_t : \{t\} \times X \to T \times X \) be the inclusion. Then we have, for \( N \gg 0 \),
(1) $(\pi_* F, \alpha)$ is a flat family of stable pairs over $X$, hence it induces a map $\pi_\ast : T \to \text{M}_k(X)$ given by $t \mapsto (\iota_t^* \pi_* F, \alpha|_t)$.

(2) If $t \in T \cap S \text{M}_k$, then $\iota_t^* \pi_* F = (\pi_* E)^\vee$. Hence $\pi_* F$ extends $\pi_* E$.

**Proof.** We begin by showing part (1). We have to show flatness and stability.

- We prove flatness in two steps. First we show that $\pi_\ast F'$ is constant with $t$ (see [16], proposition 2.1.2). This follows from the Grothendieck-Riemann-Roch theorem applied to $\iota_t^*(1 \times \pi)_* F(-NL) = \pi_* \iota_t^* F(-NL)$ (for this equality see the appendix) plus the vanishing of the higher direct image sheaves $R^i \pi_\ast$ for $N \gg 0$.

Now we want to show $\pi_* F$ is flat. It is enough to check for points $(t_0, x_0)$ for some $t_0 \in T$. Fix an ideal $I \subset \mathcal{O}_{t_0, T}$. Let $\sum_i a_i \otimes m_i \in I \otimes \pi_* F_{t_0, x_0}$ and assume $\sum_i a_i m_i = 0 \in \pi_* F_{t_0, x_0}$. We want to show $\sum_i a_i \otimes m_i = 0$.

Pick an open set $W \subset T \times X$ such that the following holds:
- For all $i$, $a_i \in \mathcal{O}_T(p(W))$ and $m_i \in \mathcal{F}'(W \setminus (T \times x_0))$;
- There are generators $f_1, \ldots, f_j$ of $I$ such that $f_j \in \mathcal{O}_T(p(W))$.

Then $\sum_i a_i m_i = 0 \in \mathcal{F}'(W \setminus (T \times x_0))$ hence $\sum_i a_i m_i = 0 \in \mathcal{F}'_{t, x}$ for any $x \neq x_0$ and any $t$.

Define the sheaf $\mathcal{I}$ as the sheaf of ideals $\mathcal{I}(V) = \langle I \cap \mathcal{O}_T(p(W)) \rangle \subset \mathcal{O}_T(V)$. Then $\mathcal{I}_{t_0} = I$. Then by flatness of $\mathcal{F}'$ it follows that $\sum_i a_i \otimes m_i = 0 \in \mathcal{I}_t \otimes \mathcal{F}'_{t, x}$. Hence $\sum_i a_i \otimes m_i = 0 \in (\mathcal{I} \otimes \mathcal{F}')(W \setminus (T \times x_0))$. Now

$$(\mathcal{I} \otimes \mathcal{F}')(W \setminus (T \times x_0)) = \mathcal{I}(p(W)) \otimes \pi_* \mathcal{F}(W)$$

The flatness of $\pi_* F$ follows.

- Now we prove stability. We want to show that, for every $t \in T$, $\iota_t^* \pi_* F$ is stable. Let $\mathcal{E} \to X$ be a sheaf defined by $\mathcal{E}(V) = \iota_t^* \pi_* F(V \setminus \{x_0\})$. Then stability of $\mathcal{E}$ is equivalent to stability of $\iota_t^* \pi_* F$. Let $\mathcal{A} \subset \mathcal{E}$. We may assume $\mathcal{A}$ is locally free at $x_0$.

We claim that, for $M \gg 0$, $\pi^* \mathcal{A}(-ML) \subset \iota_t^* F$. To see this notice that $\mathcal{E} = \pi_* \iota_t^* F(ML)$ hence we have the inclusion map $\pi^* \mathcal{A} \to \pi^* \pi_* \iota_t^* F(ML) \to \iota_t^* F(ML)$.

Now notice that $\mathcal{A}$ and $\pi^* \mathcal{A}(-ML)$ are isomorphic on $X \setminus x_0$. Hence, since $\iota_t^* \mathcal{E}$ is stable, it follows by proposition 2.9 that either $c_1(\mathcal{A}) \cdot C < 0$ or $c_1(\mathcal{A}) \cdot C = 0$, $\varepsilon = 1$. But then, by lemma 2.7, $\mathcal{A}$ is not destabilizing.
Now we prove statement (2): \( \pi_* \) restricted to \( S_i \mathcal{M}_k^r(X) \) is given by \( (\mathcal{E}, \phi) \mapsto ((\pi_* \mathcal{E})^{\vee \vee}, \phi) \). That is, if \( t = [\mathcal{E}, \phi] \in S_i \mathcal{M}_k^r(X) \) then \( i^* \pi_* \mathcal{F} = (\pi_* \mathcal{E})^{\vee \vee} \).

First observe that, for \( t = [\mathcal{E}, \phi] \in S_i \mathcal{M}_k^r \cap T \), \( i^* \pi_* \mathcal{F} = \pi_* \mathcal{E} \) on \( X \setminus \{x_0\} \) hence \( (i^* \pi_* \mathcal{F})^{\vee \vee} = (\pi_* \mathcal{E})^{\vee \vee} \). In particular \( c_2((i^* \pi_* \mathcal{F})^{\vee \vee}) = i \). Now flatness implies \( c_2(i^* \pi_* \mathcal{F}) \) is constant with \( t \) so it is enough to show that for some \( t \in S_i \mathcal{M}_k^r \cap T \), \( i^* \pi_* \mathcal{F} \) is locally free.

To prove this last statement observe that in \( (S_i \mathcal{M}_k^r \cap T) \times X \) we have \( \pi_* \mathcal{F} = ((1 \times \pi)_* \mathcal{F})^{\vee \vee} \). Hence its singularities lie in codimension three. This implies the desired result. \( \Box \)

Now we want to show that the restriction of \( \pi_* \) to \( S_i \mathcal{M}_k^r(X) \) is a fibration with fiber \( S_0 \mathcal{M}_{k-1}(\mathbb{P}^2) \).

Let \( \mathcal{M}_i^r(X) \) be the subspace of pairs \( (\mathcal{E}, \phi) \) such that \( \mathcal{E} \) is locally free at \( x_0 \).

**Proposition 3.2.** \( \pi_*^{-1} \mathcal{M}_i^r(X) = S_i \mathcal{M}_k^r(X) \)

**Proof.** We begin by remarking that, for any sheaf \( \mathcal{E} \to \mathcal{X} \), \( \pi_* \mathcal{E} \) and \( \pi_*^{\vee \vee} \mathcal{E} \) coincide over \( X \setminus \{x_0\} \). So \( (\mathcal{E}, \phi) \in \pi_*^{-1} \mathcal{M}_i^r(X) \) if and only if \( \pi_* \mathcal{E} = \pi_*^{\vee \vee} \mathcal{E} \). Now

- Suppose \( \pi_* \mathcal{E} = \pi_*^{\vee \vee} \mathcal{E} \). Then, by definition of \( S_i \), \( (\mathcal{E}, \phi) \in S_i \mathcal{M}_k^r(X) \).
- If \( (\mathcal{E}, \phi) \in S_i \mathcal{M}_k^r(X) \) then \( c_2(\pi_*^{\vee \vee} \mathcal{E}) = c_2(\pi_* \mathcal{E}) \) hence \( \pi_* \mathcal{E} = \pi_*^{\vee \vee} \mathcal{E} \).

\( \Box \)

**Proposition 3.3.** The restriction of \( \pi_* \) to \( \pi_*^{-1}(\mathcal{M}_i^r(X)) \) is a fibration with fiber \( S_0 \mathcal{M}_{k-1}(\mathbb{P}^2) \).

**Proof.** Let \( T_X \subset \mathcal{M}_i^r(X) \) be an open set. We want to build an isomorphism \( T_X \times S_0 \mathcal{M}(\mathbb{P}^2) \cong \pi_*^{-1}(T_X) \). Let \( T_P \subset \mathcal{M}_k^r(\mathbb{P}^2) \) be an open set. Consider the universal families \( (\mathcal{F}_X, \alpha_X) \) over \( T_X \) and \( (\mathcal{F}_P, \alpha_P) \) over \( T_P \). The next step is to build trivializations \( \psi_X, \psi_P \) of \( \mathcal{F}_X, \mathcal{F}_P \):

- For \( T_X \) small enough we can choose a neighborhood of \( x_0, U \subset X \), such that \( \mathcal{F}_X \) is free on \( T_X \times U \). Fix a trivialization \( \psi: \mathcal{F} \mid_{T_X \times U} \to \mathcal{O}_{T_X \times U} \);
• By definition of $S_0\overline{\mathcal{M}}_{k-1}(\mathbb{P}^2)$, for any $t \in T_P$ the sheaves $i_t^*F_P|_{\widetilde{\mathbb{P}}^2 \setminus L}$ are free. It follows that, for $T_P$ small enough, $F_P|_{\widetilde{\mathbb{P}}^2 \setminus L}$ is a free sheaf. Then Hartog’s theorem implies there is a unique isomorphism $\psi_P : F_P|_{\widetilde{\mathbb{P}}^2 \setminus L} \to \mathcal{O}_{\widetilde{\mathbb{P}}^2 \setminus L}$ which is compatible with $\alpha_P$.

Now consider the sheaf $F_{XP} \to \tilde{X} \times T_X \times T_P$ given by

$$F_{XP} = p_X^*F_X|_{X \setminus \{x_0\}} \bigcup_{\psi^{-1}\psi_X} p_P^*F_P|_{\widetilde{\mathbb{P}}^2 \setminus E}$$

where $p_X : T_X \times T_P \to T_X$, $p_P : T_X \times T_P \to T_P$ are the projections, and let $\alpha_{XP} = \alpha_{XPX}$. We claim $(F_{XP}, \alpha_{XP})$ is a flat family of stable framed sheaves. Flatness follows since both $p_X^*F_X$ and $p_P^*F_P$ are flat (see \cite{12}, proposition III.9.2). To prove stability let $(t_X, t_P) \in T_X \times T_P$ and let $A \subset i_t^*F_{XP}$. Then $(\pi_s A)^{\vee \vee} \subset i_t^*F_X$. Now we repeat the argument used in the proof of proposition \cite{3.1}

The family $(F_{XP}, \alpha_{XP})$ induces a map $g_\psi : T_X \times T_P \to \overline{\mathcal{M}}_k(\tilde{X})$ such that $\pi_\bullet g_\psi = p_X$. We want to build an inverse to this map, $\pi_\bullet \times g_\tilde{\psi}$, where

$$g_\tilde{\psi} : \pi_\bullet^{-1}(T_X) \to S_0\overline{\mathcal{M}}_{k-1}(\mathbb{P}^2)$$

Let $T \subset \pi_\bullet^{-1}(T_X)$ and consider the universal family $(\mathcal{F}, \alpha)$ over $T$. We have an isomorphism

$$\tilde{\psi} : F|_{T \times U \setminus L} = \pi_\bullet F|_{T \times U \setminus \{x_0\}} \cong (\mathbb{1} \times \pi_\bullet)^*F_X|_{T \times U \setminus \{x_0\}} \overset{(\mathbb{1} \times \pi_\bullet)^*\psi}{\rightarrow} \mathcal{O}_{T \times U \setminus \{x_0\}}$$

Then we define the sheaf over $\mathbb{P}^2 \times T$

$$F_p = F|_{\tilde{U}} \bigcup_{\tilde{\psi}} \mathcal{O}_{\mathbb{P}^2 \setminus \{x_0\}}$$

As above, this is a flat family of framed sheaves inducing the desired map $g_\tilde{\psi}$.

Now it is a direct verification to check that $\pi_\bullet \times g_\tilde{\psi}$ is the inverse map of $g_\psi$.

Now we turn to the proof of corollary \cite{13}. We will need the lemma:

**Lemma 3.4.** Let $f : X \to Y$ be a proper map between metric spaces $X,Y$. Let $C \subset Y$ be a closed subspace. Then, for any neighborhood $W$ of $f^{-1}(C)$, there is a neighborhood $V$ of $C$ such that $f^{-1}(V) \subset W$.

**Proof.** We claim that, for any $y \in C$ we can build a neighborhood $V_y$ of $y$ such that $f^{-1}(V_y) \subset W$: if not we could build a sequence $x_n \notin W$
with $f(x_\alpha) \to y$. Then properness of $f$ leads to a contradiction. Now, just take $V = \bigcup_{y \in C} V_y$. □

Proof of Corollary 1.3. To simplify notation we will write $F_i = F_i\mathcal{M}(X)$, $\mathcal{M}_i = \mathcal{M}_i(X)$. Recall that $\mathcal{M}_i = \pi_*F_{i-1}$. We begin by building open sets $V_0, V_1 \subset \mathcal{M}_i^r(X)$ and $W_0, W_1 \subset F_i\mathcal{M}_k^r(X)$ such that

1. $\pi_*^{-1}V_1 \subset W_1 \subset \pi_*^{-1}V_0 \subset W_0$;
2. $\mathcal{M}_i \subset V_j, j = 0, 1$ are strong deformation retracts and $V_1 \setminus \hat{\mathcal{M}}_i \to V_0 \setminus \hat{\mathcal{M}}_i$ is a homotopy equivalence;
3. $F_{i-1} \subset W_n, n = 0, 1$ are strong deformation retracts and $W_1 \setminus F_{i-1} \to W_0 \setminus F_{i-1}$ is a homotopy equivalence.

The existence of such neighborhoods follows from lemma 3.4 and [13]. Now it follows from the five lemma applied to the homotopy exact sequence coming from the fibrations $\pi_*^{-1}V_j \setminus F_{i-1} \to V_j \setminus \hat{\mathcal{M}}_i, j = 1, 2$, that the inclusion $\pi_*^{-1}V_1 \setminus F_{i-1} \to \pi_*^{-1}V_0 \setminus F_{i-1}$ is a homotopy equivalence. Now, using the inclusion maps in (1) we see that the spaces $\pi_*^{-1}V_1 \setminus F_{i-1}$ and $W_1 \setminus F_{i-1}$ are homotopy equivalent. Now, by excision,

$$H_\ast(F_i, F_{i-1}) \cong H_\ast(S_i, W_1 \setminus F_{i-1}) \cong H_\ast(S_i, \pi_*^{-1}V_1 \setminus F_{i-1})$$

where $S_i = S_i\mathcal{M}_k^r(X)$. To conclude the proof we apply the relative Leray-Serre spectral sequence to the pair of fibrations $S_i \to \mathcal{M}_i \setminus \hat{\mathcal{M}}_i$ and $\pi_*^{-1}V_1 \setminus F_{i-1} \to V_1 \setminus \hat{\mathcal{M}}_i$. The $E^2$ term is

$$E^2_{p,q} = H_p\left(\mathcal{M}_i \setminus \hat{\mathcal{M}}_i, V_1 \setminus \hat{\mathcal{M}}_i; H_q(S_0\mathcal{M}_k^{-i}(\mathbb{P}^2))\right)$$

To finish the proof we apply excision. □

4. Stratification of $\mathcal{M}_k^r(\tilde{X})$

The results in this section first appeared in [20]. Recall that

$$S_i\mathcal{M}_k^r(\tilde{X}) = \left\{(\mathcal{E}, \phi) \in \mathcal{M}_k^r(\tilde{X}) \mid c_2((\pi_*\mathcal{E})^{\vee\vee}) = i\right\}$$

The objective of this section is to prove part one of theorem 1.2.

**Theorem 4.1.** The map

$$\pi_* : S_i\mathcal{M}_k^r(\tilde{X}) \to \mathcal{M}_i^r(X)$$

is a trivial fibration with fiber $S_0\mathcal{M}(\mathbb{P}^2)$.
The proof will be done using the analytic definition of the moduli space. Fix $SU(r)$ bundles $E \to \tilde{X}$, $E_X \to X$, $E_P \to \mathbb{P}^2$ with $c_2(E) = k$, $c_2(E_X) = i$ and $c_2(E_P) = k - i$. We will use the notation $\mathcal{M}(\tilde{X}, E)$, $\mathcal{M}(X, E_X)$, $\mathcal{M}(\mathbb{P}^2, E_P)$ for the moduli spaces.

We begin by introducing the enlarged moduli spaces:

**Definition 4.2.** let $U \subset X$ be an open topological ball around $x_0$ intersecting $C$ in a non-empty disk and let $\tilde{U} = \pi^{-1}(U)$. Then we define

1. $\mathcal{M}^U(X, E_X)$ is the quotient by $\text{Aut}(E_X)$ of the space of triples $(\tilde{\partial}_X, \phi_X, \psi_X)$ where $\tilde{\partial}_X$ is a holomorphic structure on $E_X$, $\phi_X : E_X|_C \to \mathcal{E}_0$ is an isomorphism and $\psi_X$ is a holomorphic trivialization of $E_X|_U$ that agrees with $\phi_X$ in $U \cap C$;
2. $S_1 \mathcal{M}^{\tilde{U} \setminus C}(\tilde{X}, E)$ is the quotient by $\text{Aut}(E)$ of the space of triples $(\tilde{\partial}, \phi, \psi)$ where $\tilde{\partial}$ is a holomorphic structure on $E$ such that $c_2(\pi_* E^{\vee \vee}) = i$, $\phi : E|_C \to \mathcal{E}_0$ is an isomorphism and $\psi$ is a holomorphic trivialization of $E|_{\tilde{U} \setminus C}$ (this bundle is always trivial) that agrees with $\phi$ in $\tilde{U} \cap C$;
3. $S_0 \mathcal{M}^{\mathbb{P}^2 \setminus C}(\mathbb{P}^2, E_P)$ is the quotient by $\text{Aut}(E_P)$ of the space of triples $(\tilde{\partial}_P, \phi_P, \psi_P)$ where $\tilde{\partial}_P$ is a holomorphic structure on $E_P$ such that $c_2(\pi_* E_P^{\vee}) = 0$, $\phi_P : E_P|_C \to \mathcal{E}_0$ is an isomorphism and $\psi_P$ is a holomorphic trivialization of $E_P|_{\mathbb{P}^2 \setminus C}$ (trivial since $c_2(\pi_* E_P^{\vee}) = 0$) that agrees with $\phi_P$ in $C_\infty$.

Before we proceed we introduce the useful result:

**Lemma 4.3.** Let $B$ be a 4 dimensional ball and consider two holomorphic vector bundles $E_1, E_2 \to B$. Let $\phi : E_1|_{B \setminus 0} \to E_2|_{B \setminus 0}$ be an isomorphism. Then $\phi$ extends to an isomorphism $\tilde{\phi} : E_1 \to E_2$.

**Proof.** $\phi$ is equivalent to a map $\phi : B \setminus 0 \to Gl(r, \mathbb{C}) \subset \mathbb{C}^{r^2}$. By Hartog’s theorem this map extends to a map $\phi : B \to \mathbb{C}^{r^2}$. Composing with the determinant we get a map $\text{det} \circ \phi : B \to \mathbb{C}$ which can only vanish at $0 \in B$, hence it never vanishes. We conclude that the image of $\phi$ lies in $Gl(r, \mathbb{C})$. \hfill $\Box$

**Proposition 4.4.** The spaces $\mathcal{M}^U(X, E_X) \times S_0 \mathcal{M}^{\mathbb{P}^2 \setminus C}(\mathbb{P}^2, E_P)$ and $S_1 \mathcal{M}^{\tilde{U} \setminus C}(\tilde{X}, E)$ are homeomorphic.

**Proof.** We divide the proof into four steps:
The first goal is to define a map
\[ g : \mathcal{M}^U(X, E_X) \times S_0\mathcal{M}_{\mathbb{P}^2}(\mathbb{P}^2, E_P) \to \mathcal{M}_{\tilde{U}}(\tilde{X}, E). \]

Let \([\bar{\partial}_X, \phi_X, \psi_X] \in \mathcal{M}^U(X, E_X), [\bar{\partial}_P, \phi_P, \psi_P] \in S_0\mathcal{M}_{\mathbb{P}^2}(\mathbb{P}^2, E_P).\n
Let
\[ E_{XP} = E_X|_{X\setminus\{x_0\}} \bigcup_{\psi^{-1}_P \psi_X} E_P|_{\tilde{U}} \]

We claim \( E_{XP} \) is isomorphic to \( E \) as topological vector bundles. It is enough to show that \( c_2(E_{XP}) = k \). To prove this fix trivializations \( h_X \) of \( E_X \) on \( X \setminus \{x_0\} \) and \( h_P \) of \( E_P \) on \( \tilde{U} \). Then,
\[ E_X \cong \mathbb{C}_{X\setminus\{x_0\}} \bigcup_{\psi_X h^{-1}_X} \mathbb{C}_U, \quad E_P \cong \mathbb{C}_{\mathbb{P}^2} \bigcup_{h_P \psi^{-1}_P} \mathbb{C}_{\tilde{U}} \]

Hence, seen as maps \( S^3 \to \text{Gl}(r, \mathbb{C}) \), \( \psi_X h^{-1}_X \) and \( h_P \psi^{-1}_P \) have degree \( i \) and \( k - i \) respectively. Now
\[ E_{XP} \cong \mathbb{C}_{X\setminus\{x_0\}} \bigcup_{\psi_X h^{-1}_X h_P \psi^{-1}_P} \mathbb{C}_{\tilde{U}} \]

hence \( c_2(E_{XP}) = k \).

Now we define the map \( g \). \( \bar{\partial}_X \) and \( \bar{\partial}_P \) induce a holomorphic structure \( \bar{\partial}_{XP} \) on \( E_{XP} \) and we have \([\bar{\partial}_{XP}, \phi_X, \psi_X|_{\tilde{U}}] \in \mathcal{M}_{\tilde{U}}(\tilde{X}, E_{XP}) \).

Choose an isomorphism \( f : E_{XP} \to E \). Then \( f \) induces a map \( f_\# : \mathcal{M}_{\tilde{U}}(\tilde{X}, E_{XP}) \to \mathcal{M}_{\tilde{U}}(\tilde{X}, E) \). We define
\[ g([\bar{\partial}_X, \phi_X, \psi_X], [\bar{\partial}_P, \phi_P, \psi_P]) = f_\#([\bar{\partial}_{XP}, \phi_X, \psi_X|_{\tilde{U}}]) \]

This map does not depend on the choice of isomorphism \( f \).

Now we show that the image of \( g \) lies in \( S_1\mathcal{M}_{\tilde{U}}(\tilde{X}, E) \). Let \( \mathcal{E} \to \tilde{X} \) be the bundle \( E_{XP} \) with holomorphic structure induced by \( \bar{\partial}_X \) and \( \bar{\partial}_P \). We claim that \((\pi_*\mathcal{E})^{\vee \vee}\) is biholomorphic to \( E_X \) with holomorphic structure \( \bar{\partial}_X \). This is a consequence of lemma 4.3 since the restriction of the bundles to \( X \setminus \{x_0\} \) are biholomorphic. Hence the image of \( g \) lies in \( S_1\mathcal{M}_{\tilde{U}}(\tilde{X}, E) \).

Now we show \( g \) is continuous. Fix \( q_X = (\bar{\partial}_X, \phi_X, \psi_X) \in \mathbb{C}^U(X, E_X), q_P = (\bar{\partial}_P, \phi_P, \psi_P) \in \mathbb{C}_{\mathbb{P}^2}(\mathbb{P}^2, E_P). \)

Fix balls \( B_r(x_0) \subset B_R(x_0) \subset U \) and let \( K = B_R(x_0) \setminus B_r(x_0) \). Choose \( \varepsilon > 0 \) such that \( W = B_r(\mathbb{1}) \subset \text{Gl}(r, \mathbb{C}) \) Then \((K,W)\) defines an open neighborhood of \( \mathbb{1} \) in \( C^\infty(U, \text{Gl}(r, \mathbb{C})) \).
For each pair \( q = ([\bar{\partial}_X, \phi_X, \psi_X], [\bar{\partial}_P, \phi_P, \psi_P]) \) let
\[
E_q = E_X|_{X \setminus \{x_0 \}} \bigcup_{\psi_P^{-1}\psi_X} E_P|_U
\]
Continuity of \( g \) will follow if we construct a continuous family of isomorphisms \( f_q : E_q \to E \), or, what amounts to the same, a continuous family \( f_q : E_q \to E_{q_0} \) for fixed \( q_0 = ([\bar{\partial}_{X_0}, \phi_{X_0}, \psi_{X_0}], [\bar{\partial}_{P_0}, \phi_{P_0}, \psi_{P_0}]) \)

A map \( f_q : E_q \to E_{q_0} \) is equivalent to the diagram
\[
\begin{array}{ccc}
E_X|_{X \setminus \{x_0 \}} & \xrightarrow{f_X} & E_X|_{X \setminus \{x_0 \}} \\
\downarrow \psi_X & & \downarrow \psi_{X_0} \\
U \setminus \{x_0 \} \times \mathbb{C}^r & \xrightarrow{f_U} & U \setminus \{x_0 \} \times \mathbb{C}^r \\
\downarrow \psi_P & & \downarrow \psi_{P_0} \\
E_P|_U & \xrightarrow{f_P} & E_P|_U
\end{array}
\]
Assume \( \psi_{X_0}^{-1} \psi_X \) and \( \psi_{P_0}^{-1} \psi_P \) are in a \((K, W)\) neighborhood of \( q_0 \). Fix a monotonous \( C^\infty \) function \( \eta \) such that \( \eta(\rho) = 0 \) for \( \rho < r \) and \( \eta(\rho) = 1 \) for \( \rho > R \). \( \eta \) induces a map \( \tilde{\eta} : K \to \mathbb{R}, \ x \mapsto \eta(|x|) \). We define
\[
\begin{align*}
\hat{f}_X &= \mathbb{1} & |x| > R \\
\hat{f}_U &= \tilde{\eta}(x_0) \psi_X^{-1} + (1 - \tilde{\eta}(x_0)) \psi_{P_0} \psi_P^{-1} & r < |x| < R \\
\hat{f}_P &= \mathbb{1} & |x| < r
\end{align*}
\]
This completes the proof of continuity of \( g \).

(4) Now we construct maps
\[
\begin{align*}
g_X : S_i \mathcal{M}^{\bar{\partial}_X \setminus L}(\tilde{X}, E) & \to \mathcal{M}^{\bar{\partial}_X \setminus L}(X, E) \\
g_P : S_i \mathcal{M}^{\bar{\partial}_P \setminus L}(\tilde{X}, E) & \to S_0 \mathcal{M}^{\bar{\partial}_P \setminus L}(\tilde{P}_2, E_P)
\end{align*}
\]
Let \([\bar{\partial}_X, \phi, \psi] \in S_i \mathcal{M}^{\bar{\partial}_X \setminus L}(\tilde{X}, E)\). Define the bundles
\[
\begin{align*}
\tilde{E}_X &= E|_{\tilde{X} \setminus L} \bigcup_{\psi} U \times \mathbb{C}^r \\
\tilde{E}_P &= E|_{\tilde{U}} \bigcup_{\psi} \tilde{P}_2 \setminus L \times \mathbb{C}^r
\end{align*}
\]
Then \( \bar{\partial}_X \) induces holomorphic structures \( \bar{\partial}_X \) on \( \tilde{E}_X \) and \( \bar{\partial}_P \) on \( \tilde{E}_P \). Proceeding as above we choose isomorphisms \( f_X : \tilde{E}_X \to \)
then we define

\[ g_X([\mathcal{E}, \phi, \psi]) = (f_X)_\#([\hat{\mathcal{E}}_X, \phi_X, 1]) \]

\[ g_P([\mathcal{E}, \phi, \psi]) = (f_P)_\#([\hat{\mathcal{E}}_P, 1, 1]) \]

The proof that \( g_X, g_P \) are well defined continuous maps proceeds as the corresponding proof for \( g \). To conclude the proof we observe that \( g \circ (g_X \times g_P) = 1 \) and \( (g_X \times g_P) \circ g = 1 \).

Now we look more closely at the map \( g_X : S_\mathcal{M}_U \setminus L(\tilde{X}, E) \to \mathcal{M}^U(X, E_X) \) introduced in the proof of the previous proposition.

**Proposition 4.5.** Consider the projection maps

\[ pr_{\tilde{X}} : S_\mathcal{M}^{\tilde{U} \setminus L}(\tilde{X}, E) \to S_\mathcal{M}(\tilde{X}, E) \]

\[ pr_X : \mathcal{M}^U(X, E_X) \to \mathcal{M}(X, E_X) \]

\( g_X \) preserves the orbits of \( pr_{\tilde{X}} \) and \( pr_X \), hence it induces a map \( \hat{g}_X : S_\mathcal{M}(\tilde{X}, E) \to \mathcal{M}(X, E_X) \). Furthermore, \( \hat{g}_X = \pi \circ \pi \) and is determined by the following property:

- Let \([\mathcal{E}, \phi] \in \mathcal{M}(\tilde{X}, \mathcal{E})\). Then there is a unique holomorphic bundle \( \mathcal{E}_X \to X \) such that \( \mathcal{E}|_{\tilde{X}\setminus L} = \mathcal{E}_X|_{X\setminus \{x_0\}} \) and we have \( \hat{g}_X([\mathcal{E}, \phi]) = [\mathcal{E}_X, \phi] \).

**Proof.** We begin by proving the last statement. Let \([\mathcal{E}, \phi] \in \mathcal{M}(\tilde{X}, \mathcal{E})\) and define \( \mathcal{E}_X = \pi_X^* \mathcal{E}^{\vee \vee} \). Then clearly \( \mathcal{E}|_{\tilde{X}\setminus L} = \mathcal{E}_X|_{X\setminus \{x_0\}} \). Uniqueness of \( \mathcal{E}_X \) then follows from lemma 4.3.

Now, from this uniqueness property and by definition of \( g_X \) it follows that \( pr_X \circ g_X([\mathcal{E}, \phi, \psi]) = [\mathcal{E}_X, \phi] \). This shows \( g_X \) preserves the fibers of \( pr_{\tilde{X}} \) hence \( \hat{g}_X \) is well defined.

Now it follows that \( \hat{g}_X = \pi_* \).

**Proposition 4.6.** Consider the projection maps

\[ pr_{\tilde{X}} : \mathcal{M}^{\tilde{U} \setminus L}(\tilde{X}, E) \to \mathcal{M}(\tilde{X}, E) \]

\[ pr_X : \mathcal{M}^U_k(X) \to \mathcal{M}_k(X) \]

\[ pr_P : \mathcal{M}^{\tilde{P} \setminus L}(\tilde{P}^2) \to \mathcal{M}_k(\tilde{P}^2) \]

These maps are principal bundle maps with contractible fiber.
Proof. We divide the proof into three steps:

(1) For an open set $A$ we define

$$G(A) = \{ \eta \in \text{Hol}(A, Gl(r, \mathbb{C})) : \eta|_{A \cap C} = 1 \}$$

Then, for $A = \tilde{U} \setminus L$, $U$, $\tilde{P} \setminus L$, $G(A)$ acts on $\mathcal{M}^4(Y)$ (where $Y = \tilde{X}, X, \tilde{P}$ respectively) by

$$\eta[\mathcal{E}, \phi, \psi] = [\mathcal{E}, \phi, \eta \psi]$$

This action is free and its orbits are the fibers of $pr_{X}, pr_{P}$ respectively.

(2) $G(U)$ is clearly contractible. From lemma 4.3 it follows that $G(\tilde{U} \setminus L) = G(U)$ and $G(\tilde{P} \setminus L) = G(\mathbb{P}^2) = \{ 1 \}$.

(3) Finally we need to show the existence of local sections. We will first build a section $s_X$ of $pr_{X}$. As in the proof of proposition 3.3, for $T_X \subset \mathcal{M}(X, E_X)$ we choose a trivialization $\psi_X : F|_{T_X \times U} \rightarrow O_{T_X \times U}$. We can choose $\psi_X$ compatible with $\alpha_X$ on $C \cap U$. Now, for $t = [\mathcal{E}, \phi] \in T_X$, $\psi_X$ gives a map $\psi_X : \mathcal{E}|_U = \tilde{\iota}^* \mathcal{E} \rightarrow \mathcal{O}_U$. Then $s_X(\mathcal{E}, \phi) = (\mathcal{E}, \phi, \psi_X)$.

Sections of $pr_{\tilde{X}}$ are built in a similar way: for $T \subset \mathcal{M}(\tilde{X}, E)$ and $T_P \subset \mathcal{M}(\tilde{P}, E_P)$ we build trivializations $\psi_X$ and $\psi_P$ following the same procedure as in the proof of proposition 3.3.

□

Corollary 4.7. The spaces $\mathcal{M}_i(X, E_X) \times S_0 \mathcal{M}_{k-i}(\mathbb{P}^2, E_P)$ and $S_i \mathcal{M}_{k}(\tilde{X}, E)$ are homotopically equivalent.

We are ready to prove theorem 4.1

Proof. Fix a global section $s : \mathcal{M}(X, E_X) \rightarrow \mathcal{M}^U(X, E_X)$ of $pr_{X}$. Define $f_s : \mathcal{M}(X, E_X) \times S_0 \mathcal{M}(\mathbb{P}^2, E_P) \rightarrow S_i \mathcal{M}(\tilde{X}, E)$ by $f_s = pr \circ g \circ (s \times 1)$. It is easy to check that $f_s$ is a bijection. We need to show that its inverse is continuous. To that end we construct a local inverse as follows: let $S \subset \mathcal{M}(\tilde{X}, E)$ be affine. Then $s$ induces a map $s_X : S \rightarrow \mathcal{M}^U(\tilde{X}, E)$ given by the restriction of $s \circ \tilde{\iota} \tilde{g}_X$ to $\tilde{U} \setminus L$ (see the construction of a trivialization $\tilde{\psi}$ in the proof of proposition 3.3). We define $F_s = (pr \times 1) \circ (\tilde{g}_X \times g_P) \circ s_X$. Then $F_s$ is the inverse of $f_s$ which concludes the proof.

□
4.1. **Rank Stabilization.** Direct sum with a trivial bundle induces maps $j : \mathcal{M}_r^k(X, E) \to \mathcal{M}_r' (X, E \oplus O_X^{-r})$ for $r' > r$. We define the rank stable moduli space by

$$\mathcal{M}_k^\infty(X) = \lim_{\to r} \mathcal{M}_r^k$$

The definitions of $S_i, F_i$ carry through to the rank stable situation. We want to extend the previous results to these moduli spaces:

**Theorem 4.8.** The spaces $S_i \mathcal{M}_k^\infty(\tilde{X})$ and $\mathcal{M}_i^\infty(X) \times S_0 \mathcal{M}_{k-i}(\mathbb{P}^2)$ are homotopically equivalent.

**Proof.** We will show that the maps $f_s, F_s$ introduced in the last section can be defined on the direct limits and they are homotopic inverse of each other.

We first observe that the rank stable enlarged moduli spaces $\mathcal{M}_{k,A}^\infty(X)$ ($A = U, \tilde{U} \setminus L, \tilde{C} \setminus L$) can be defined in the same way Consider then $f_s = pr \circ g \circ (s \times 1)$. $pr$, $g$ commute with the inclusion $j$. So we only need to show that the diagram

$$\begin{array}{ccc}
\mathcal{M}_k^r(X) & \xrightarrow{s_r} & \mathcal{M}_k^{r,U}(X) \\
\downarrow j & & \downarrow j \\
\mathcal{M}_k'(X) & \xrightarrow{s_{r'}} & \mathcal{M}_k^{r',U}(X)
\end{array}$$

is homotopy commutative. First observe that $pr \circ j \circ s_r = pr \circ s_{r'} \circ j = j$. Hence, from proposition 4.6 we can find, for each $t = [E, \phi] \in \mathcal{M}_k(X)$, a map $h_t \in \text{Hol}(U, Gl(r', \mathbb{C}))$ such that $h(j \circ s_1(t)) = s_2 \circ j(t)$. This defines a map $H : \mathcal{M}_k^r(X) \to \text{Hol}(U, Gl(r', \mathbb{C}))$. Since the space of such maps is connected, $j \circ s_r$ is homotopic to $s_{r'} \circ j$. We conclude that the map $f_s$ can be defined in the direct limit. In the same way we define the inverse map $F_s$ in the direct limit. This concludes the proof. □

5. **Characterization of $S_0 \mathcal{M}_k^r(\mathbb{P}^2)$ using Monads**

The objective of this section is to give a characterization of points in $S_0 \mathcal{M}_k^r(\mathbb{P}^2)$ in terms of a monad description. This result also appears in [20] and [4]. We also give an explicit description of the map $S_0 \mathcal{M}(\mathbb{P}^2) \to S_0 \mathcal{M}(\mathbb{P}^2 \setminus L(\mathbb{P}^2))$ from section 4.

We begin by sketching the monad description of the spaces $\mathcal{M}_k^r(\mathbb{P}^2)$ and $\mathcal{M}_k'(\mathbb{P}^2)$. We follow [17]. See also [3].
Let $L_\infty \subset \mathbb{P}^2$ be a rational curve and let $L$ be the exceptional divisor. Let $W, W_0, W_1$ be $k$ dimensional complex vector spaces. Choose sections $x_1, x_2, x_3$ spanning $H^0(\mathcal{O}(L_\infty))$ and $y_1, y_2$ spanning $H^0(\mathcal{O}(L_\infty - L))$ so that $x_3$ vanishes on $L_\infty$ and $x_1y_1 + x_2y_2$ spans the kernel of $H^0(\mathcal{O}(L_\infty)) \otimes H^0(\mathcal{O}(L_\infty - L)) \to H^0(\mathcal{O}(2L_\infty - L))$.

5.1. The moduli space over $\mathbb{P}^2$. Let $\mathcal{R}$ be the space of 4-tuples $m = (a_1, a_2, b, c)$ with $a_i \in \text{End}(W)$, $b \in \text{Hom}(\mathbb{C}^r, W)$, $c \in \text{Hom}(W, \mathbb{C}^r)$, obeying the integrability condition $[a_1, a_2] + bc = 0$. For each $m = (a_1, a_2, b, c) \in \mathcal{R}$ we define maps $A_m, B_m$ by

$$A_m = \begin{bmatrix} x_1 - a_1x_3 \\ x_2 - a_2x_3 \\ cx_3 \end{bmatrix}, \quad B_m = \begin{bmatrix} -x_2 + a_2x_3 \\ x_1 - a_1x_3 \\ bx_3 \end{bmatrix}$$

Then $B_mA_m = 0$. The assignment $m \mapsto \mathcal{E}_m = \text{Ker } B_m/\text{Im } A_m$ induces a map $f : \mathcal{R} \to \mathcal{M}_r^k(\mathbb{P}^2)$.

A point $m \in \mathcal{R}$ is called non-degenerate if $A_m$ and $B_m$ have maximal rank at every point in $\mathbb{P}^2$. Then $\mathcal{E}_m$ is locally free.

**Theorem 5.1.** Let $\mathcal{M}_r^r(\mathbb{P}^2)$ denote the quotient of the space of non-degenerate points in $\mathcal{R}$ by the action of $\text{Gl}(W)$: $g \cdot (a_1, a_2, b, c) = (g^{-1}a_1g, g^{-1}a_2g, g^{-1}b, cg)$

Then the map $f$ induces an isomorphism $\mathcal{M}_r^r(\mathbb{P}^2) \to \mathcal{M}_r^r(\mathbb{P}^2)$. For a proof see [4], proposition 1.

**Theorem 5.2.** Let $\overline{\mathcal{M}}_k^r(\mathbb{P}^2)$ be the algebraic quotient $\mathcal{R}/\text{Gl}(W)$. This space is isomorphic to the Donaldson-Uhlenbeck completion of the moduli space of instantons over $S^4$.

For a proof see [8], sections 3.3, 3.4, 3.4.4.

For future reference we sketch here how the map from $\mathcal{R}/\text{Gl}(W)$ to the Donaldson-Uhlenbeck completion of the moduli space of instantons is constructed (see [17] for details):

Let $m = (a_1, a_2, b, c) \in \mathcal{R}$. A subspace $V \subset W$ is called $b$-special with respect to $m$ if

$$a_i(V) \subset V \quad (i = 1, 2) \quad \text{and } \text{Im } b \subset V$$

(1)
A subspace \( V \subset W \) is called \( c \)-special with respect to \( m \) if
\[
(2) \quad a_i(V) \subset V \ (i = 1, 2) \quad \text{and} \quad V \subset \text{Ker} \ c
\]
m is called completely reducible if for every \( V \subset W \) which is \( b \)-special or \( c \)-special, there is a complement \( V' \subset W \) which is \( c \)-special or \( b \)-special respectively.

**Proposition 5.3.** Let \( m = (a_1, a_2, b, c) \in \mathcal{R} \).

1. \( m \) is non-degenerate if and only if the only \( b \)-special subspace is \( W \) and the only \( c \)-special subspace is 0;
2. For every \( m \), the orbit of \( m \) under \( \text{Gl}(W) \) contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;
3. If \( m \) is completely reducible then, after acting with some \( g \in \text{Gl}(W) \) we can write
\[
a_i = \begin{bmatrix} a_{i,\text{red}} & 0 \\ 0 & a_{i}^\Delta \end{bmatrix}, \quad b = \begin{bmatrix} b_{\text{red}} \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} c_{\text{red}} & 0 \end{bmatrix}
\]
where \((a_{1,\text{red}}, a_{2,\text{red}}, b_{\text{red}}, c_{\text{red}})\) is non-degenerate and the matrices \(a_1^\Delta, a_2^\Delta\) can be simultaneously diagonalized. Such a configuration is equivalent to the following data:
- An irreducible integrable configuration \((a_{1,\text{red}}^\Delta, a_{2,\text{red}}^\Delta, b_{\text{red}}, c_{\text{red}})\) corresponding to a bundle with \( c_2 = l \leq k \);
- \( k - l \) points in \( \mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty \) given by the eigenvalue pairs of \(a_1^\Delta, a_2^\Delta\).

This is precisely the Donaldson-Uhlenbeck completion.

### 5.2. The moduli space over \( \mathbb{P}^2 \).

Let \( \mathcal{R} \) be the space of 5-tuples \( \tilde{m} = (a_1, a_2, d, b, c) \) with \( a_i \in \text{Hom}(W_1, W_0), \ d \in \text{Hom}(W_0, W_1), \ b \in \text{Hom}(\mathbb{C}^r, W_0), \ c \in \text{Hom}(W_1, \mathbb{C}^r) \), such that \( a_1(W_1) + a_2(W_1) + b(\mathbb{C}^r) = W_0 \), obeying the integrability condition \( a_1 da_2 - a_2 da_1 + bc = 0 \). For each \( \tilde{m} = (a_1, a_2, d, b, c) \in \mathcal{R} \) we define maps \( A_{\tilde{m}}, B_{\tilde{m}} \)

\[
W_1(-L_\infty) \oplus W_0(L - L_\infty) \xrightarrow{A_{\tilde{m}}} (W_0 \oplus W_1)^{\oplus 2} \oplus \mathbb{C}^n \xrightarrow{B_{\tilde{m}}} \rightarrow W_0(L_\infty) \oplus W_1(L_\infty - L)
\]

by

\[
A_{\tilde{m}} = \begin{bmatrix} a_1 x_3 & -y_2 \\ x_1 - da_1 x_3 & 0 \\ a_2 x_3 & y_1 \\ x_2 - da_2 x_3 & 0 \\ c x_3 & 0 \end{bmatrix}, \quad B_{\tilde{m}} = \begin{bmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b x_3 \\ dy_1 & y_1 & dy_2 & y_2 & 0 \end{bmatrix}
\]
Then $B_{\tilde{m}}A_{\tilde{m}} = 0$. The assignment $\tilde{m} \mapsto E_{\tilde{m}} = \text{Ker} B_{\tilde{m}} / \text{Im} A_{\tilde{m}}$ induces a map $\tilde{f} : \tilde{R} \to \overline{M_k^r(\tilde{P}^2)}$.

A point $\tilde{m} \in \tilde{R}$ is called non-degenerate if $A_{\tilde{m}}$ and $B_{\tilde{m}}$ have maximal rank at every point in $\tilde{P}^2$.

**Theorem 5.4.** Let $M_k^r(\tilde{P}^2)$ denote the quotient of the space of non-degenerate points in $\tilde{R}$ by the action of $\text{Gl}(W_0) \times \text{Gl}(W_1)$:

$$(g_0, g_1) \cdot (a_1, a_2, b, c, d) = (g_0^{-1}a_1g_1, g_0^{-1}a_2g_1, g_0^{-1}b, cg_1, g_1^{-1}dg_0)$$

Then the map $\tilde{f}$ induces an isomorphism $M_k^r(\tilde{P}^2) \to \overline{M_k^r(\tilde{P}^2)}$.

**5.3. The theorem and its proof.** We are now in conditions to state the theorem:

**Theorem 5.5.** Let $\tilde{m} = (a_1, a_2, d, b, c) \in \tilde{R}$. Then $E_{\tilde{m}} \in S_0\overline{M_k^r(\tilde{P}^2)}$ if and only if $da_1, da_2$ are nilpotent and, for any sequence $i_1, \ldots, i_n \in \{1, 2\}$, we have

$$c \left( \prod_{j=1}^n da_{i_j} \right) db = 0$$

We divide the proof into several propositions.

**Proposition 5.6.** Let $\pi_\# : \tilde{R} \to R$ be given by $\pi_\#(a_1, a_2, d, b, c) = (da_1, da_2, db, c)$. Let $m = \pi_\#\tilde{m}$. Then $E_{\tilde{m}}|_{\tilde{X} \setminus L}$ is isomorphic to $E_m|_{X \setminus \{x_0\}}$.

**Proof.** Let $m = \pi_\#\tilde{m}$. Fix an isomorphism $W \cong W_1$. Let $p$ be the projection $p : W_{\oplus \tilde{P}^2} \oplus \mathbb{C}^r \to (W_0 \oplus W_1)_{\oplus \tilde{P}^2} \oplus \mathbb{C}^r$ with kernel $W_0^{\oplus \tilde{P}^2}$. After restricting to $\tilde{X} \setminus L$ we can rescale the sections so that $y_2 = -x_1$, $y_1 = x_2$. Then a direct verification shows that, for any $\tilde{m}$, $p$ induces maps $\text{Ker} B_{\tilde{m}} \to \text{Ker} B_m$ and $\text{Im} A_{\tilde{m}} \to \text{Im} A_m$. Hence we get a map $E_{\tilde{m}} \to E_m$. It is a direct computation to check that this map is an isomorphism. □

It follows from this proposition and theorem 5.2 that, for $\tilde{m} \in \tilde{R}$, $E_{\tilde{m}} \in S_0\overline{M_k^r(\tilde{P}^2)}$ if and only if $E_{\pi_\#\tilde{m}}$ is a sheaf whose whole charge is concentrated at $[0, 0, 1] \in \tilde{P}^2$, that is, $\ell \left( E_{\pi_\#\tilde{m}}^{\vee \vee} / E_{\pi_\#\tilde{m}} \right) = k$. We proceed to study this situation:

**Lemma 5.7.** Let $m = (a_1, a_2, b, c)$ be such that $E_m$ has its whose whole charge concentrated at $[0, 0, 1]$. Then, after a change of basis we can
write
\[ a_i = \begin{bmatrix} J & * \\ 0 & J \end{bmatrix}, \quad b = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & * \end{bmatrix} \]

where \( J \) represents any nilpotent matrix in the Jordan canonical form.

Before we begin the proof we observe that this lemma implies one direction of theorem 5.5.

Proof. We begin by proving by induction on the charge that, after acting with an element \( g \in GL(W) \), we can write.
\[ a_i = \begin{bmatrix} a_{iu} & * \\ 0 & a_{id} \end{bmatrix}, \quad b = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & * \end{bmatrix} \]

Clearly the configuration \((a_1, a_2, b, c)\) cannot be non-degenerate hence there is a subspace \( V \) which is either \( b \)-special or \( c \)-special (proposition 5.3, 1). We consider both cases:

(1) If \( m \) is \( b \)-special, after a change of basis we can write
\[ a_i = \begin{bmatrix} a_1' & * \\ 0 & f_i \end{bmatrix}, \quad b = \begin{bmatrix} b' \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} c' & * \end{bmatrix} \]

So the point \( \left( \begin{bmatrix} a_1' & 0 \\ 0 & f_i \end{bmatrix}, [b'], [c' 0] \right) \) is in the closure of the orbit of \( m \). It follows then from proposition 5.3 that \( m' = (a'_1, a'_2, b', c') \) corresponds to an ideal bundle with charge concentrated at \([0, 0, 1]\). Hence we can apply the induction hypothesis to \( m' \).

We get
\[ a_i = \begin{bmatrix} a_{iu}' & * & * \\ 0 & a_{id}' & * \\ 0 & 0 & f_i \end{bmatrix}, \quad b = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & * & * \end{bmatrix} \]

which is in the desired form.

(2) If \( m \) is \( c \)-special, after a change of basis we can write
\[ a_i = \begin{bmatrix} f_i & * \\ 0 & a_i' \end{bmatrix}, \quad b = \begin{bmatrix} * \\ b' \end{bmatrix}, \quad c = \begin{bmatrix} 0 & c' \end{bmatrix} \]

Applying induction hypothesis to \((a'_1, a'_2, b', c')\) as in the previous case, we can write
\[ a_i = \begin{bmatrix} f_i & * & * \\ 0 & a_{iu}' & * \\ 0 & 0 & a_{id}' \end{bmatrix}, \quad b = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & * \end{bmatrix} \]

This is in the desired form.
Now the condition \([a_1, a_2] + bc = 0\) implies \([a_{1u}, a_{2u}] = [a_{1d}, a_{2d}] = 0\).
So, after a change of basis we can put all these matrices in the Jordan canonical form. Since all charge is concentrated at \([0, 0, 1]\), proposition 5.3 implies the eigenvalues of these matrices are all 0. □

The other direction of theorem 5.5 follows directly from the proposition

**Proposition 5.8.** Let \(m \in \mathcal{R}\) be such that \(a_1, a_2\) are nilpotent and, for any \(n_1, n_2\) we have

\[
ca_1^{n_1}a_2^{n_2}b = 0
\]

Then \(\mathcal{E}_m|_{\mathbb{P}^2 \setminus [0,0,1]}\) is free.

**Proof.** We will build an explicit trivialization. First we need to introduce some notation. Let \(\mathbb{P}^2 = \{(x_1, x_2, x_3)\}\) and define an open cover of \(\mathbb{P}^2 \setminus [0,0,1]\) by \(U_1 = \{x_1 \neq 0\}\), \(U_2 = \{x_2 \neq 0\}\). Let \(e_1, \ldots, e_r\) be the canonical basis of \(\mathbb{C}^r\). Choose coordinates \((\alpha_2, \alpha_3) \mapsto [1, \alpha_2, \alpha_3]\) in \(U_1\) and \((\beta_1, \beta_3) \mapsto [\beta_1, 1, \beta_3]\) in \(U_2\). Then define functions \(s_i^j : U_j \to \text{Ker } B_m\) by

\[
s_i^1 = (0, -\alpha_3(1 - \alpha_3a_1)^{-1}be_i, e_i)
\]

\[
s_i^2 = (\beta_3(1 - \beta_3a_2)^{-1}be_i, 0, e_i)
\]

(since \(a_1, a_2\) are nilpotent, \(1 - \lambda a_i\) is invertible for any \(\lambda\)). It is a direct verification that indeed the image of \(s_i^1, s_i^2\) lie in \(\text{Ker } B_m\). We want to show that \(s_i^1, s_i^2\) induce a trivialization of \(\mathcal{E}_m\). We will have to show that \(S_i^1 - s_i^2 \in \text{Im } A_m\) in \(U_1 \cap U_2\). Then \(s_i^1, s_i^2\) induce a section of \(\mathcal{E}_m\). We will show these sections are linearly independent.

1. We begin by looking at the space \(\text{Im } A + \text{Im } s_i^1 + \ldots + \text{Im } s_i^1\).

We can represent the image of \(s_i^1\) in terms of column vectors in matrix form. Then, joining this matrix with \(A_m\) we have

\[
\mathfrak{A}_1 = \begin{bmatrix}
1 - \alpha_3a_1 & 0 \\
\alpha_2 - \alpha_3a_2 & -\alpha_3(1 - \alpha_3a_1)^{-1}b \\
\alpha_3c & 1
\end{bmatrix}
\]

This matrix clearly has maximum rank since \(1 - \alpha_3a_1\) is non-singular. Hence, for dimensional reasons its columns form a basis for \(\text{Ker } B\). In particular \(s_i^1\) are linearly independent.

2. Now we repeat the argument for \(s_i^2\). In \(U_2\) we have a similar matrix:

\[
\mathfrak{A}_2 = \begin{bmatrix}
\beta_1 - \beta_3a_1 & \beta_3(1 - \beta_3a_2)^{-1}b \\
1 - \beta_3a_2 & 0 \\
\beta_3c & 1
\end{bmatrix}
\]
which has also clearly maximum rank. Its columns form a basis for \( \ker B \). In particular \( s_2^i \) are linearly independent.

(3) Now we will show \( s_1^i - s_2^i \in \text{Im} A \) in \( U_1 \cap U_2 \). We have, with \( \beta_1 = \alpha_2^{-1} \) and \( \beta_3 = \alpha_2^{-1} \alpha_3 \),

\[
s_2^i(\alpha_2^{-1}, \alpha_2^{-1} \alpha_3) = \begin{bmatrix} \alpha_3(\alpha_2 - \alpha_3 a_2)^{-1} b \\ 0 \\ 1 \end{bmatrix}
\]

Since \( s_2^i \) is in the kernel of \( B \), it must be in the image of the surjective matrix \( A_1 \). So we can solve

\[
\begin{bmatrix} 1 - \alpha_3 a_1 \\ \alpha_2 - \alpha_3 a_2 \\ \alpha_3 c \\ 1 \end{bmatrix}
\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \alpha_3(\alpha_2 - \alpha_3 a_2)^{-1} b \\ 0 \\ 1 \end{bmatrix}
\]

We obtain immediately \( \xi_1 = \alpha_3(1 - \alpha_3 a_1)^{-1}(\alpha_2 - \alpha_3 a_2)^{-1} b \). Now equation \( 3 \) implies \( c \xi_1 = 0 \). From here it follows immediately that \( \xi_2 = 1 \) hence \( s_1^i - s_2^i = A_m \xi_1 \).

This completes the proof. \( \square \)

Notice that this proof gives an explicit description of the map \( S_0 \mathcal{M}(\mathbb{P}^2) \to S_0 \mathcal{M}_{\mathbb{P}^2 \setminus L}(\mathbb{P}^2) \).

**Appendix A. Direct Image**

Let \( S \) be a scheme and let \( s \in S \). Let \( \iota_s : X \to S \times X \) and \( \tilde{i}_s : \tilde{X} \to S \times \tilde{X} \) be the inclusions \( x \mapsto (s, x) \).

**Lemma A.1.** For any sheaf \( \mathcal{F} \) over \( S \times \tilde{X} \) we have

\[
i_s^*(1 \times \pi)_* \mathcal{F} \cong \pi_* \iota_s^* \mathcal{F}
\]

**Proof.** Let \( V \subset X \). Then

\[
i_s^*(1 \times \pi)_* \mathcal{F}(V) = \left( \lim_{s \times V \subset W} \mathcal{F}((1 \times \pi)^{-1} W) \right) \bigotimes_{\lim_{s \times V \subset W} \mathcal{O}_{S \times X}(W)} \mathcal{O}_X(V)
\]

\[
\pi_* \iota_s^* \mathcal{F}(V) = \left( \lim_{s \times \pi^{-1}(V) \subset U} \mathcal{F}(U) \right) \bigotimes_{\lim_{s \times \pi^{-1}(V) \subset U} \mathcal{O}_{S \times \tilde{X}}(U)} \mathcal{O}_X(\pi^{-1}(V))
\]
which we can rewrite as

\[
\begin{align*}
\tau^*_s(1 \times \pi)_* \mathcal{F}(V) &= \lim_{U_1 \in S_1} \mathcal{F}(U_1) \boxtimes \mathcal{O}_X(V) \\
\pi_* \tau^*_s \mathcal{F}(V) &= \lim_{U_2 \in S_2} \mathcal{F}(U_2) \boxtimes \mathcal{O}_X(V)
\end{align*}
\]

where

\[
S_1 = \{ (1 \times \pi)^{-1} W \mid s \times V \subset W \} \\
S_2 = \{ U \mid (1 \times \pi)^{-1}(s \times V) \subset U \}
\]

We claim that \( S_1 = S_2 \): Just observe that if \( U_2 \in S_2 \) then \( U_2 = (1 \times \pi)^{-1}(1 \times \pi)(U_2) \). This concludes the proof. \( \square \)

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