COMPLETE MINORS IN COMPLEMENTS OF NON-SEPARATING
PLANAR GRAPHS

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ABSTRACT. We prove that the complement of any non-separating planar graph of order $2n - 3$ contains a $K_n$ minor, and argue that the order $2n - 3$ is lowest possible with this property. To illustrate the necessity of the non-separating hypothesis, we give an example of a planar graph of order 11 whose complement does not contain a $K_7$ minor. We argue that the complements of planar graphs of order 11 are intrinsically knotted. We compute the Hadwiger numbers of complements of wheel graphs.

1. INTRODUCTION

A planar graph $G$ is non-separating planar if and only if there exists a planar embedding of $G$ in $\mathbb{R}^2$ where for any cycle $C \subseteq G$, the vertices of $G \setminus C$ are not separated between the region $R$ in $\mathbb{R}^2$ enclosed by $C$ and the region $\mathbb{R}^2 \setminus R$. Dehkordi and Farr [3] classified maximal non-separating planar graphs as (1) wheel graphs, (2) elongated triangular prism graphs, or (3) maximal outerplanar graphs. In [10], Pavelescu and Pavelescu proved that the complements of non-separating planar graphs of order ten are intrinsically knotted. In this article, using the classification in [3], we prove that the complements of non-separating planar graphs of order 11 contain $K_7$ as a minor, and we generalize this result by the following theorem:

Theorem 1. Let $G$ be a non-separating planar graph of order $2n - 3$, with $n \geq 7$. Then $\overline{G}$, the complement of $G$, admits a $K_n$ minor.

While this result does not hold for arbitrary planar graphs of order $2n - 3$, we conjecture that the complements of planar graphs of order (at least) 11 are intrinsically knotted. An embedding of a graph in space is a knotted embedding if there exists a cycle which forms a nontrivial knot. A graph $G$ is intrinsically knotted (IK) if all embeddings of $G$ in $\mathbb{R}^3$ are knotted embeddings. The class of knotlessly embeddable (nIK) graphs, consisting of graphs which are not IK, is a minor-closed family of graphs. By the Robertson–Seymour theorem [11], the class of nIK graphs possesses a forbidden minor characterization. While there are over 264 known minor minimal IK (MMIK) graphs [5], a complete list of forbidden minors is not yet available. Among those which are known are $K_7$, shown to be IK by the work of Conway and Gordon [2], and the complete 4-partite graph $K_{3,3,1,1}$, shown to be MMIK by the work of Foisy [4]. This means a graph $G$ which contains $K_7$ or $K_{1,1,3,3}$ as a minor is IK. We investigate graphs of order 11 and show that the complement of non-separating planar graphs of order 11 must contain a $K_7$ minor.

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2. Notation and Definitions

We denote the set of vertices and edges of a simple, undirected, and finite graph \( G \) as \( V(G) \) and \( E(G) \), respectively. For each pair of vertices \( u, v \in V(G) \) we say \((u, v) \in E(G)\) if and only if \( u \) and \( v \) are adjacent in \( G \). If \((u, v) \in E(G)\), we say that the edge \((u, v)\) is incident to both \( u \) and \( v \). The set of all vertices \( u \in V(G) \) satisfying \((u, v) \in E(G)\) for a fixed vertex \( v \) is the open neighbor set \( N_G(v) \), or \( N(v) \) when \( G \) is clear from context. For each graph \( G \), we denote its (edge) complement graph \( \overline{G} = (V(G), E^\neg(G)) \), such that
\[
E^\neg(G) = \{(u, v)|u, v \in V(G)\text{ and } (u, v) \notin E(G)\}.
\]
If a graph \( G' \) is a subgraph of \( G \) we write \( G' \subseteq G \). We say a graph \( H \) is a minor of \( G \), or \( H \preceq G \), if there exists an isomorphism from a subdivision of \( H \) to a subgraph \( G' \subseteq G \). If a graph isomorphic to \( H \) can be obtained by applying a (possibly empty) series of vertex deletions, edge deletions, and edge contractions to \( G \), then also \( H \preceq G \). The minor relation \( \preceq \) is reflexive, anti-symmetric, and transitive. A graph \( G \) is planar when there exists an embedding of \( G \) in \( \mathbb{R}^2 \) in which edges intersect only at points in \( V(G) \).

We define the join \( H = H_1 + H_2 \) of two graphs \( H_1 \) and \( H_2 \) as follows. The set of vertices of \( H \) is \( V(H) = V(H_1) \cup V(H_2) \), the disjoint union of the sets of vertices of \( H_1 \) and \( H_2 \). In addition to existing adjacencies in \( H_1 \) and \( H_2 \), each vertex in \( H \) corresponding to a vertex in \( H_1 \) is adjacent to every vertex arising from \( H_2 \), so that \( E(H) = E(H_1) \cup E(H_2) \cup (V(H_1) \times V(H_2)) \), where \( V(H_1) \times V(H_2) \) is the Cartesian product of the vertex sets. For any \( H' \cong H \), we say \( H' = H_1 + H_2 \).

3. Maximal Non-Separating Planar Graphs

We begin by noting that if \( H \) is a minor of \( G \) of the same order, \( \overline{G} \) is a subgraph of \( \overline{H} \). This allows us to only consider maximal non-separating planar graphs of a given order. We organize the proof of Theorem 3 following the classification of non-separating planar graphs by Dehkordi and Farr [3].

**Theorem 2** (Dehkordi, Farr [3]). A graph \( G \) is maximal non-separating planar if and only if \( G \) belongs to one of the following categories:

1. wheel graphs
2. elongated triangular prism graphs
3. maximal outerplanar graphs

For well-defined classes of graphs, the cardinality of the vertex set will be denoted in subscript. The wheel graph \( W_n \) on \( n \) vertices (for \( n \geq 4 \)) is isomorphic to the join of the cycle graph \( C_{n-1} \) and a vertex; equivalently, \( W_n = K_1 + C_{n-1} \). In what follows, we denote the vertices of \( W_n \) by \( v_1, v_2, \ldots, v_n \), where \( v_1, v_2, \ldots, v_{n-1} \) represent the vertices of \( C_{n-1} \) in clockwise order, and \( v_n \) is adjacent to all \( v_i, i = 1, 2, \ldots, n - 1 \). An elongated triangular prism graph is any graph constructed from the triangular prism graph in Figure 1(a) by consecutively subdividing any or all of the edges \((u_A, w_A), (u_B, w_B), \text{ and } (u_C, w_C)\). An example is given in Figure 1(b). An outerplanar graph is a graph which has a planar embedding in which
all vertices lie on a single face. A graph which is maximal with this property is *maximal outerplanar*. A maximal outerplanar graph can be represented by an \( n \)-cycle with its interior triangulated in the plane. In the next three sections, we look at each type of maximal non-separating planar graphs.

4. **Wheel Graphs**

**Theorem 3.** The complement \( \overline{W_n} \) of the wheel graph \( W_n \) satisfies \( K_{\left\lceil \frac{3(n-1)}{4} \right\rceil} \preceq \overline{W_n} \) for \( n \geq 6 \).

**Proof.** We shall consider the remainder of \( n \) modulo 4.

For \( n = 4t + 1 \), in \( \overline{W_n} \), the vertices \( v_1, v_3, \ldots, v_{4t-1} \) induce a complete subgraph on \( 2t \) vertices. Contracting the edges \((v_{2l}, v_{2l+2})\) for \( 1 \leq l \leq t \), produces a minor of \( \overline{W_n} \) isomorphic to \( K_{3t} \). See Figure 2(c) for the case \( n = 13, t = 3 \).

For \( n = 4t + 2 \), in \( \overline{W_n} \), the vertices \( v_1, v_3, \ldots, v_{4t-1} \) induce a complete subgraph. Contracting the edges \((v_{2l}, v_{2l+2})\) for \( 1 \leq l \leq t \), and contracting any of the edges incident to \( v_{4t+1} \), produces a minor of \( \overline{W_n} \) isomorphic to \( K_{3t} \).
For \( n = 4t + 3 \), in \( W_n \), the vertices \( v_1, v_2, \ldots, v_{4t+1} \) induce a complete subgraph of order \( 2t + 1 \). Contracting the edges \((v_2, v_3, v_{4t+2})\) for \( 1 \leq l \leq t \) and deleting the vertex \( v_{4t+2} \) produces a minor of \( W_n \) isomorphic to \( K_{3t+1} \). See Figure 2(a) for the case \( n = 11, t = 2 \).

For \( n = 4t + 4 \), in \( W_n \), the vertices \( v_1, v_2, \ldots, v_{4t+1} \) induce a complete subgraph of order \( 2t + 1 \). Contracting the edges \((v_2, v_3, v_{4t+2})\) for \( 1 \leq l \leq t \), and \((v_{2t+2}, v_{4t+3})\), produces a minor of \( W_n \) isomorphic to \( K_{3t+2} \). See Figure 2(b) for the case \( n = 12, t = 2 \). \( \square \)

**Corollary 4.** The complement \( W_{2t-3} \) of the wheel graph \( W_{2t-3} \) satisfies \( K_t \leq W_{2t-3} \) for \( t \geq 6 \).

**Proof.** We apply Theorem 3 with \( n = 2t - 3 \). We have as a result

\[
K_{\left\lfloor \frac{3(2t-4)}{4} \right\rfloor} = K_{\left\lfloor \frac{3n}{2} \right\rfloor - 3} \leq W_{2t-3}.
\]

For \( t \geq 6 \), we have \( \left\lfloor \frac{3t}{2} \right\rfloor - 3 \geq t \) and thus

\[
K_t \leq K_{\left\lfloor \frac{3t}{2} \right\rfloor - 3} \leq W_{2t-3}.
\]

\( \square \)

The following theorem shows that the value \( \left\lfloor \frac{3(n-1)}{4} \right\rfloor \) is the best possible.

**Theorem 5.** The edge complement \( W_n \) of the wheel graph \( W_n \) has no minor isomorphic to \( K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \), equivalently \( K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \not\leq W_n \) for \( n \geq 6 \).

**Proof.** Assume on the contrary, \( K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \leq W_n \). The graph \( W_n \) contains an isolated vertex, \( v_n \). Because \( n \geq 1 \), we have \( K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \leq W_n \iff K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \leq W_n \setminus v_n \).

As the minor is complete, it follows there is a sequence of \( \left\lfloor \frac{n-2}{4} \right\rfloor \) edge contractions applied to \( W_n \setminus v_n \) which results in a graph isomorphic to \( K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \). Let us enumerate the intermediate graphs thus obtained by the sequence \((H_1, H_2, \ldots, H_{\left\lfloor \frac{n-2}{4} \right\rfloor - 1})\) where

\[
W_n \supsetneq W_n \setminus v_n \supsetneq H_1 \supsetneq H_2 \supsetneq \cdots \supsetneq H_{\left\lfloor \frac{n-2}{4} \right\rfloor - 1} \supsetneq K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1}.
\]

In general, denoting \( H_{-1} = W_n \), \( H_0 = W_n \setminus v_n \), and \( H_{\left\lfloor \frac{n-2}{4} \right\rfloor} = K_{\left\lfloor \frac{3(n-1)}{4} \right\rfloor + 1} \), we see \( H_k \) is a graph obtained from \( H_{k-1} \) (for \( 1 \leq k \leq n-1 \)) by applying exactly one edge contraction. We construct bounds on the number \( |E(H_k)| \) of edges in each successive graph.

We have \( \text{deg}_{H_0}(v) = n - 4 \) for all \( v \in V(H_0) \), that is \( H_0 \) is \( (n-4) \)-regular, and \( |E(H_0)| = \frac{1}{2} \sum_{v \in V(H_0)} \text{deg}_{H_0}(v) = \frac{(n-1)(n-4)}{2} \). Moreover, for each edge \((u, v) \in E(H_0)\), the number of shared neighbors of \( u \) and \( v \) is \( |N_{H_0}(u) \cap N_{H_0}(v)| \geq n - 7 \). This implies the minimum number \( L_0 \) of edges lost by performing one edge contraction in \( H_0 \) satisfies \( L_0 = \min_{(u, v) \in E(H_0)} |N_{H_0}(u) \cap N_{H_0}(v)| + 1 \geq n - 6 \). We define \( L_k \) the minimum number of edges lost by performing one edge contraction in \( H_k \). If we define a variable \( \Delta E_k = |E(H_{k-1})| - |E(H_k)| \) for the successive differences in edge set sizes, we see \( \Delta E_k \geq L_k \).
We note that each edge contraction decreases the number of common neighbors (between two vertices) by at most one. In general, for $k \in [1, \lceil \frac{n-2}{4} \rceil]$ we have that if $L_{k-1} \geq \lambda$, then $L_k \geq \lambda - 1$, with $L_0 \geq n - 6$. Then
\[
\Delta E_k = (n - 6) - (k - 1) = (n - 5) - k.
\]
From this, we can evaluate
\[
|E(H_k)| = |E(H_0)| - \sum_{i=1}^{k} \Delta E_i \leq \frac{(n-1)(n-4)}{2} - k(n-5) + \frac{k(k+1)}{2},
\]
for $k \in [1, \lceil \frac{n-2}{4} \rceil]$.

In particular, this implies $|E(H_{\lceil \frac{n-2}{4} \rceil})| \leq \frac{n^2 - 5n + 4}{2} - \frac{(2n-11-\lceil \frac{n-2}{4} \rceil)(\lceil \frac{n-2}{4} \rceil)}{2}$, where $H_{\lceil \frac{n-2}{4} \rceil}$ is the graph resulting from the $\lceil \frac{n-2}{4} \rceil$th operation on $H_0$ and $|V(H_{\lceil \frac{n-2}{4} \rceil})| = \lceil \frac{3(n-1)}{4} \rceil + 1$. However, we have $|E(K_{\lceil \frac{3(n-1)}{4} \rceil+1})| = (\lceil \frac{3(n-1)}{4} \rceil + 1)^2/2$. By the calculations done in Table 1 addressing all $n$ (mod 4), we see that
\[
|E(H_{\lceil \frac{n-2}{4} \rceil})| < |E(K_{\lceil \frac{3(n-1)}{4} \rceil+1})|.
\]
This implies $H_{\lceil \frac{n-2}{4} \rceil} \subset K_{\lceil \frac{3(n-1)}{4} \rceil+1}$, and we conclude that $K_{\lceil \frac{3(n-1)}{4} \rceil+1} \not\cong \overline{W}_n$.

| $n$ | $\lceil \frac{n-2}{4} \rceil$ | $\lceil \frac{3(n-1)}{4} \rceil$ | $|E(H_{\lceil \frac{n-2}{4} \rceil})|$ | $|E(K_{\lceil \frac{3(n-1)}{4} \rceil+1})|$ |
|-----|-----------------|-----------------|----------------|----------------|
| $4s$ | $s - 1$ | $3s - 1$ | $\frac{9s^2 - 3s - 6}{2}$ | $\frac{9s^2 - 3s}{2}$ |
| $4s + 1$ | $s - 1$ | $3s$ | $\frac{9s^2 + 3s - 8}{2}$ | $\frac{9s^2 + 3s}{2}$ |
| $4s + 2$ | $s$ | $3s$ | $\frac{9s^2 + 3s - 2}{2}$ | $\frac{9s^2 + 3s}{2}$ |
| $4s + 3$ | $s$ | $3s + 1$ | $\frac{9s^2 + 9s - 2}{2}$ | $\frac{9s^2 + 9s + 2}{2}$ |

Table 1. Calculations show that $|E(H_{\lceil \frac{n-2}{4} \rceil})| < |E(K_{\lceil \frac{3(n-1)}{4} \rceil+1})|$ for all $n$ in the range.

The Hadwiger number of a graph, introduced by Hadwiger in 1943 [6], is defined to be the order of the largest complete minor of the graph. The following corollary of Theorems 3 and 5 gives the Hadwiger number of complements of wheel graphs.

**Corollary 6.** For $n \geq 6$, the Hadwiger number of $\overline{W}_n$ is $\lceil \frac{3(n-1)}{4} \rceil$.

### 5. Elongated Triangular Prisms

An elongated triangular prism graph $G$ is constructed from the triangular prism graph in Figure 1(a) by subdividing any or all of the edges $(u_A, w_A)$, $(u_B, w_B)$, and $(u_C, w_C)$. Note that in $G$, there exists exactly one path $P_i$ from $u_i$ to $w_i$ that does not contain an edge in either triangle $u_Au_Bu_C$ or $w_Aw_Bw_C$, for $i \in \{A, B, C\}$. Moreover, the path $P_i$ is an induced subgraph of $G$. 
Lemma 7. For a graph $G$ of order 11, if $G$ is an elongated triangular prism, then $K_7 \preceq G$.

Proof. The graph $G$ is obtained by subdividing the non-triangular edges of the prism graph five times. The 5 vertices are contained in the path $P_i$ for exactly one $i \in \{A,B,C\}$. In this way, the distinct graphs $G$ up to isomorphism correspond to partitions of five vertices into three indistinguishable, disjoint sets. This can be done in five ways, so there are five elongated triangular prism graphs $G$ of order 11 up to isomorphism. We name these $G_{5,0,0}, G_{4,1,0}, G_{3,2,0}, G_{3,1,1}, G_{2,2,1}$ where the subscript corresponds to the partition which gives rise to the respective graph. See Figure 3.

Figure 3. The five elongated triangular prisms of order 11 up to isomorphism. The dotted edges belong to the complement graph.

A sequence of four edge contractions performed on $\overline{G}$, as specified by the red, dotted edges in the figures, produces $K_7$ minors of $\overline{G}$ in all cases. □

Theorem 8. Every elongated triangular prism graph $G$ on $|V(G)| = 2n - 3$ vertices satisfies $K_n \preceq \overline{G}$ for $n \geq 7$.

Proof. Using Lemma 7 as our base case, we proceed by induction on $n$. We assume any elongated triangular prism graph $G_0$ with $|V(G_0)| = 2k - 3$ satisfies $K_k \preceq \overline{G_0}$ and show an arbitrary elongated triangular prism graph $G$ with $|V(G)| = 2k - 1$ satisfies $K_{k+1} \preceq \overline{G}$. 
There are two cases to consider: either all subdivisions in $G$ of the triangular prism graph occur on one path $P_i$, or there are at least two paths in $G$ with subdivisions, namely we have $|V(P_i)|, |V(P_j)| \geq 3$ for some distinct $i, j \in \{A, B, C\}$.

To consider the first case, let us say without loss of generality $|V(P_A)| = 2k - 5$ so that $(u_B, w_B), (u_C, w_C) \in E(G)$. We label $V(P_A) = \{u_A, v_1, v_2, \ldots, v_{2k-7}, w_A\}$ such that the vertices $v_1, v_2, \ldots, v_{2k-7}$ occur in this order along $P_A$ from $u_A$ to $w_A$. Since $k \geq 7$, we know $(v_1, v_{2k-7}) \notin E(G)$, so $(v_1, v_{2k-7}) \in E(G)$. Let $H \leq \overline{G}$ be the minor obtained from $\overline{G}$ by contracting the edge $(v_1, v_{2k-7})$, and call the new vertex formed by this edge contraction $v_0$. Because $N_G(v_1) \cap N_G(v_{2k-7}) = \emptyset$, we have $N_H(v_0) = V(H) \setminus \{v_0\}$. Since $v_0$ is adjacent to every other vertex in $H$, we now consider $H \setminus v_0$. The complement $\overline{H \setminus v_0}$ of the graph is a spanning subgraph of some elongated prism graph $G_0$ with $|V(G_0)| = 2k - 3$. The assumption $K_k \leq \overline{G_0}$ implies $K_k \leq \overline{(H \setminus v_0)} \cong H \setminus v_0$. It follows

$$\overline{G} \geq H \geq K_1 + K_k \cong K_{k+1}.$$ 

For the second case, assume without loss of generality $|V(P_A)|, |V(P_B)| \geq 3$. Let $v_a \in V(P_A)$ where $v_a \notin \{u_A, w_A\}$ and $v_b \in V(P_B)$ where $v_b \notin \{u_B, w_B\}$. We have $(v_a, v_b) \in E(G)$ and $N_G(v_a) \cap N_G(v_b) = \emptyset$. As before, let $H \leq \overline{G}$ be the minor obtained from $\overline{G}$ by contracting $(v_a, v_b)$, and call the resulting vertex $v_0$. Then $N_H(v_0) = V(H) \setminus \{v_0\}$ and the complement $\overline{H \setminus v_0}$ of the graph is a spanning subgraph of some elongated prism graph $G_0$ with $|V(G_0)| = 2k - 3$. As before, $K_k \leq \overline{G_0}$ implies $K_k \leq H \setminus v_0$ and it follows that

$$\overline{G} \geq H \geq K_1 + K_k \cong K_{k+1}.$$ 

In both cases, the inductive step follows and completes the proof. \hfill \Box

6. Outerplanar Graphs

For a maximal outerplanar graph $G$ on $n$ vertices consider the planar embedding in $\mathbb{R}^2$ which consists of a triangulated convex regular $n$-gon $C_n \subseteq G$. See Figure 6(a). The graph $G$ is uniquely determined up to isomorphism by this embedding, and we do not distinguish between the two. Label the vertices of $G$ by $v_1, v_2, \ldots, v_n$, in the order they appear in $C_n$. The endpoints of any edge of $G$ which is not an edge of $C_n$ determine two paths along $C_n$. If the length of the shorter of the two paths is $i$, the edge is an $i$-chord in $G$.

Lemma 9. For $G$ a maximal outerplanar graph of order 11, $K_7 \leq \overline{G}$.

Proof. Consider $G$ an outerplanar graph labeled as above. The chords of $G$ can be $i$-chords, for $i = 2, 3, 4, 5$. If $G$ has no 5-chord, up to a rotation, the graph $G$ contains a subgraph isomorphic to the graph $H$ pictured in Figure 6(a). Contracting the edges $(v_1, v_7), (v_2, v_8), (v_3, v_9)$, and $(v_5, v_{11})$ in $\overline{G}$ creates a $K_7$ minor.

If $G$ has a 5-chord, say $(v_1, v_6)$, then the complement graph $\overline{G}$ contains as a subgraph a complete bipartite graph $K_{4,5}$ with vertex partitions $A = \{v_2, v_3, v_4, v_5\}$ and $B = \{v_7, v_8, v_9, v_{10}, v_{11}\}$. We observe that at least one of the edges $(v_2, v_4)$ and $(v_3, v_5)$ belongs to $\overline{G}$. Assume $(v_2, v_4) \in E(\overline{G})$. In $\overline{G}$, contracting edges $(v_3, v_k)$ and $(v_5, v_j)$, where $k, j \in \{7, 8, 9, 10, 11\}, k \neq j$, and
deleting the vertices $v_1$ and $v_6$ yields a minor isomorphic to $K_7$ minus a triangle rooted in the vertices in $B$. The missing triangle is $v_pv_qv_r$ with $\{p, q, r\} = \{7, 8, 9, 10, 11\} \setminus \{k, j\}$. To show that $\overline{G}$ contains a $K_7$ minor, it suffices to show that within the subgraph induced by \{v_1, v_6, v_7, v_8, v_9, v_{10}, v_{11}\} there either exists a triangle with vertex set in $B$ or edges incident to $v_1$ or $v_6$ can be contracted to give such a triangle. The chord $(v_1, v_6)$ is part of a triangle with the third vertex in $B$. Up to symmetry, we distinguish three cases.

Case 1: $(v_1, v_7) \in E(G)$. See Figure 4(b). Then $(v_6, v_8), (v_6, v_9), (v_6, v_{10}), (v_6, v_{11}) \in E(\overline{G})$. At least one of the two edges $(v_9, v_{11})$ and $(v_8, v_{10})$ belongs to $E(\overline{G})$. If $(v_8, v_{10}) \in E(\overline{G})$, contract $(v_6, v_9)$ to obtain the triangle $v_8v_9v_{10}$. If $(v_9, v_{11}) \in E(\overline{G})$, contract $(v_6, v_{10})$ to obtain the triangle $v_6v_9v_{10}$.

Case 2: $(v_1, v_8), (v_6, v_8) \in E(G)$. See Figure 4(c). Then $(v_6, v_9), (v_6, v_{10}), (v_7, v_9), (v_7, v_{10}) \in E(\overline{G})$. Contract the edge $(v_6, v_{10})$ to obtain the triangle $v_7v_9v_{10}$.

Case 3: $(v_1, v_9), (v_6, v_9) \in E(G)$. See Figure 4(d). Then $(v_6, v_{10}), (v_6, v_{11}), (v_7, v_{10}), (v_7, v_{11}) \in E(\overline{G})$. Contract the edge $(v_6, v_{10})$ to obtain the triangle $v_7v_{10}v_{11}$.

To tackle the general case, we consider the relative position of chords in maximal outerplanar graphs in Lemma 10. We say two edges $e_1, e_2 \in E(G)$ are independent if and only if $V(e_1) \cap V(e_2) = \emptyset$. We start by remarking that every maximal outerplanar graph with $n \geq 4$ has at least one 2-chord.

**Lemma 10.** For a maximal outerplanar graph $G$ of order $n \geq 7$, exactly one of the following is true:

1. there exist a pair of independent edges $e_1, e_2 \in E(G)$ such that both $e_1$ and $e_2$ are 2-chords of $G$.
2. $G$ is isomorphic to $K_1 + P_{n-1}$.
Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( G \) in the order in which they appear in \( C_n \), the boundary of the common face. Let \( k \) be the maximal length of all the chords of \( G \), and let \((v_i, v_{i+k})\) be a \( k \)-chord of \( G \). (a) If there exist a chord \((v_j, v_{j+l})\) independent of \((v_i, v_{i+k})\), we may assume the vertices \( v_i, v_{i+k}, v_j, v_{j+l} \) appear in this order along \( C_n \). See Figure 5(a). The subgraph \( H_1 \) of \( G \) induced by \( \{v_i, v_{i+1}, \ldots, v_{i+k}\} \) has a 2-chord. If this edge is a 2-chord of \( G \), call it \( e_1 \). Else, we may assume this chord is \((v_i, v_{i+k-1})\). The subgraph \( H_2 \) induced by \( \{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \) has a 2-chord. If this edge is a 2-chord of \( G \), call it \( e_1 \). Else, consider \( H_3 \) the subgraph obtained from \( H_2 \) by deleting the vertex \( v \in \{v_i, v_{i+k-1}\} \), which is not an endpoint of the 2-chord. Continue the process until a 2-chord \( e_1 \) of \( G \) is found. The latest this chord could be found is in a subgraph induced by four consecutive vertices of \( H_1 \). Similarly, find a 2-chord in the subgraph of \( G \) induced by \( \{v_j, v_{j+1}, \ldots, v_{j+l}\} \) and call it \( e_2 \). Then, \( e_1 \) and \( e_2 \) are independent 2-chords of \( G \). (b) If every chord of \( G \) shares an endpoint with \((v_i, v_{i+k})\), let \( H_1 \) be a subgraph of \( G \) induced by \( \{v_i, v_{i+1}, \ldots, v_{i+k}\} \). Without loss of generality, we may assume that there exists \( t \) such that \((v_i, v_t) \in E(H_1) \). See Figure 5(b). Let \( H_2 \) be the subgraph of \( G \) induced by \( V(G) \setminus \{v_{i+1}, v_{i+2}, \ldots, v_{i+k-1}\} \). If there exists a chord \((v_{i+k}, v_s) \in E(H_2) \), then \((v_i, v_t) \) and \((v_{i+k}, v_s) \) are independent chords. By part (a), \( G \) has two independent 2-chords. If such a chord does not exist, all chords of \( H_2 \) meet at \( v_i \). If all chords of \( H_1 \) meet at \( v_i \), then \( G \cong K_1 + P_{n-1} \). Else, a chord of \( H_2 \) incident to \( v_i \) is independent of a chord of \( H_1 \) incident to \( v_{i+k} \) and by part (a), \( G \) has two independent 2-chords.

![Figure 5](image)

**Figure 5.** (a) \( G \) has independent chords \((v_i, v_{i+k})\) and \((v_j, v_{j+l})\); (b) all the chords of \( G \) share a vertex with \((v_i, v_{i+k})\).

Theorem 11. Every maximal outerplanar graph \( G \) of order \(|V(G)| = 2n-3\), \( n \geq 7 \), satisfies \( K_n \not\leq \overline{G} \).

Proof. We use induction on the number of vertices \( n \geq 7 \). The base case, \( n = 7 \) is provided by Lemma 9. Let \( G \) denote an outerplanar graph with \( 2(n+1) - 3 = 2n - 1 \) vertices. By Lemma 10 we need to consider two cases.
(1) There exists a pair of independent 2-chords \((v_{i-1}, v_{i+1})\) and \((v_{j-1}, v_{j+1})\). Then \(N_G(v_i) \cap N_G(v_j) = \emptyset\) and \(H := G - \{v_i, v_j\}\) is a maximal outerplanar graph of order \(2n - 3\). In turn \(N_G(v_i) \cup N_G(v_j) = V(G)\). By the induction hypothesis, \(\overline{H}\) contains a complete minor of order \(n\). Contracting the edge \((v_i, v_j)\) in \(G\) yields a minor isomorphic to \(H + K_1\) which has a complete minor of order \(n + 1\).

(2) If \(G\) is isomorphic to \(K_1 + P_{2n-4}\), then \(G\) is a spanning subgraph of \(W_{2n-3}\). By Corollary \(4\), \(K_n \preceq W_{2n-3}\), and thus \(K_n \preceq G\).

Complements of maximal outerplanar graphs with \(2n - 4\) vertices do not necessarily have a \(K_n\) minor, as the following example illustrates.

**Example 12.** There exists a maximal outerplanar graph \(M\) of order 10 for which \(K_7 \not\preceq M\).

**Proof.** Consider the graph \(M\) and its complement \(\overline{M}\) as shown and labeled in Figure 6. We see that in \(\overline{M}\) the vertices \(v_1, v_2, v_3, v_6, v_7, v_8\) all have degree at most 5. This implies \(K_7 \not\subseteq \overline{M}\). If \(\overline{M}\) contains a \(K_7\) minor, this minor is obtained from \(\overline{M}\) by three edge contractions. Under an edge contraction, the degree of a vertex not incident to the contracted edge is non-increasing. The number of vertices of degree at least six increases by at most 1 after each edge contraction, and this occurs only if (but not necessarily) the endpoints of the contracted edge both have degree less than six. The vertices \(v_1, v_2, v_3, v_6, v_7, v_8\) of \(G\) all have degree less than six. To achieve a \(K_7\) minor, all endpoints of the three contracted edges must belong to this set, and the only possibility is contracting the edges \((v_1, v_6), (v_2, v_8),\) and \((v_3, v_7)\). By inspection, the graph obtained by contracting these three edges is not complete and so \(\overline{M} \not\cong K_7\). □

### 7. Future Explorations

It is worth noting that the main result of this article is specific to non-separating planar graphs. Consider the maximal planar graph \(G\) with eleven vertices in Figure 7(a) and its complement \(\overline{G}\) in Figure 7(b). The graph \(\overline{G}\) has edge set \(\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_6),\)
We show \( G \) does not have a \( K_7 \)-minor. Since \( \deg_G(v_2) = 4 \), to obtain a \( K_7 \) minor of \( G \), an edge incident to the vertex \( v_2 \) needs to be contracted. Since the edges \((v_1,v_4),(v_4,v_7)\) and \((v_1,v_7)\) are contained in the neighborhood of \( v_2 \), it follows that \( G \) has a \( K_7 \) minor if and only if the graph \( H \) obtained by contracting the edge \((v_2,v_{11})\) of \( G \) has a \( K_7 \) minor. See Figure 8.

\[
\begin{align*}
(v_1,v_7), (v_1,v_8), (v_2,v_4), (v_2,v_7), (v_2,v_{11}), (v_3,v_6), (v_3,v_{10}), (v_3,v_{11}), (v_4,v_6), (v_4,v_7), (v_4,v_8), \\
(v_5,v_8), (v_5,v_9), (v_5,v_{11}), (v_6,v_9), (v_6,v_{10}), (v_6,v_{11}), (v_7,v_9), (v_7,v_{10}), (v_7,v_{11}), (v_8,v_9), (v_8,v_{10}), \\
(v_8,v_{11}) \}. \end{align*}
\]

Figure 7. (a) A maximal planar graph \( G \) of order 11 (b) The graph \( \overline{G} \), the complement of a maximal planar graph of order 11.

Figure 8. (a) The graph \( \overline{H} \) obtained by contracting the edge \((v_2,v_{11})\) of \( \overline{G} \) (b) The graph induced in \( \overline{H} \) by \( \{v_3,v_4,v_5,v_7,v_9,v_{10}\}\).

Note that in \( \overline{H} \), the vertices \( v_3,v_4,v_5,v_7,v_9,v_{10} \) each have degree less than 6. If a \( K_7 \) minor of \( \overline{H} \) exists, it is obtained by contracting three edges of \( \overline{H} \). Vertices \( v_3,v_4,v_5,v_7,v_9,\)
\(v_{10}\) all must be incident to the contracted edges. However, since the subgraph of \(\overline{H}\) induced by \(\{v_3, v_4, v_5, v_7, v_9, v_{10}\}\) is a tree with leaves \(v_3, v_4, v_5\), it must be that the edges to be contracted are \((v_3, v_{10})\), \((v_4, v_7)\), and \((v_5, v_9)\). See Figure 8(b). Contracting these edges, however, produces a minor of order 7 and size 19, thus not isomorphic to \(K_7\).

On the other hand, contracting the edges \((v_2, v_{11})\), \((v_3, v_{10})\), and \((v_5, v_9)\) in \(G\), yields a minor isomorphic to \(K_{3,3,1,1}\), which is a minor minimal intrinsically knotted graph, by the work of Foisy [4]. It follows that the graph \(G\) is intrinsically knotted.

More can be said about the complements of planar graphs of order 11: they are all intrinsically knotted. Independently, Foisy [4] and Taniyama and Yasuhara [12] provided a sufficient condition for a graph to be intrinsically knotted. Namely, if every embedding of the graph contains a double-linked \(D_4\)-minor (a graph of order 4 with a set of 4 double edges, which form a set of two pairs of linked cycles), then the graph is intrinsically linked. Based on these results, Miller and Naimi [9] developed and algorithm which checks whether a given graph has a double-linked \(D_4\)-minor in every embedding. Naimi implemented the algorithm into a Mathematica program. We used this program to confirm that each of the complements of the 1249 maximal planar graphs of order 11 is intrinsically knotted. Thus:

**Proposition 13.** The complement of a planar graph of order 11 is intrinsically knotted.

From a topological perspective, Proposition 13 is the natural continuation of the work of Battle, Harary, and Kodama [1], and Tutte [13], who proved that the complement of a planar graph of order 9 is not planar, and the work of Lovász and Shrijver [8], and Kotlov, Lovász and Vempala [7], which implies that the complement of a planar graph of order 10 is intrinsically linked. As always, a proof of Proposition 13 without computer assistance might provide extra insight into the structure of the complements of planar graphs.

Theorem 1 shows the complement of a non-separating planar graph of order \(2n - 3\) contains a \(K_n\) minor. It would be interesting to investigate, using techniques similar to those in this article, the connection between the order of a (non-separating) planar graph and the existence of a \(K_{3,3} + K_t\) minor for the complement of the graph.

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