INTERNAL NEIGHBOURHOOD STRUCTURES

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Abstract. The main aim of this paper is to provide a description of neighbourhood operators in finitely complete categories with finite coproducts and a proper factorisation system such that the semilattice of admissible subobjects make a distributive complete lattice. The equivalence between neighbourhoods, Kuratowski interior operators and pseudo-frame sets is proved. Furthermore the categories of internal neighbourhoods is shown to be topological. Regular epimorphisms of categories of neighbourhoods are described and conditions ensuring hereditary regular epimorphisms are probed. It is shown the category of internal neighbourhoods of topological spaces is the category of bitopological spaces, while in the category of locales every locale comes equipped with a natural internal topology.

1. Introduction

The introduction in [DikranjanGiuli1987] and [DikranjanGiuliTholen1989] of categorical closure operators led to systematic study of topological properties in general categories. The theory of categorical closure operators was subsequently developed by many authors, for instance in [ClementinoGiuliTholen1996], [CastelliniGiuli2001], [MMCEGWT2004], [Slapal2005] and the references therein. A concise treatment of this development is available in the self-contained monograph [DikranjanTholen1995], as well as in the later published book [Castellini2003].

Closure operators give rise to the notion of neighbourhood operators on a category. Categorical neighbourhoods have been treated in [Slapal2001], [Slapal2008], [GiuliSlapal2009], [HolgateSlapal2011], [Razafindrakato2012], [Slapal2012], [HolgateIragiRazafindrakatos2016], [HolgateRazafindrakato2017]. Since neighbourhoods are required for the study of convergence, investigation of neighbourhood structures is important in its own right apart from being a consequence of the notion of a closure operator.

The purpose of this paper is to show that the notion of a neighbourhood on an object of a category can be provided with minimal assumptions. In this paper we assume $\mathcal{A}$ to be a finitely complete category with finite coproducts equipped with a proper $(E,M)$-factorisation system, such that for each object $X$, the set $\text{Sub}_M(X)$ of $M$-subobjects of $X$ is a distributive complete lattice (see page 6).

Obviously, one can extract more information when stronger properties of the lattice of admissible subobjects is assumed — for instance, Boolean algebra. However, the usual

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2010 Mathematics Subject Classification. 06D10, 18A40 (Primary), 06D15 (Secondary), 18D99 (Tertiary).

Key words and phrases. Beck-Chevalley condition, filter, frame, interior operator, Kuratowski interior operator, proper factorisation system, regular epimorphism, topological functor.
correspondence between neighbourhoods, Kuratowski interiors, and pseudo-frame sets (see Definition 3.1.6, page 17) can be obtained in this general setup (see Theorem 3.1.5, page 16 and Theorem 3.1.6, page 17 for details). In the remainder of this introductory section we shall highlight the major results that have been obtained in this general context.

The notion of a neighbourhood has three layers. The description of these layers involve a filter on an object. A filter on an object $X$ is a filter in the lattice $\text{Sub}_M(X)$ of admissible subobjects of $X$. As soon as the lattice of admissible subobjects is not atomic, a collection of neighbourhoods of an admissible subobject — which is a filter, becomes an order reversing assignment from the lattice of admissible subobjects of the object to the ordered set of filters on the object. This is the first layer in the definition of a neighbourhood, herein called preneighbourhoods (see Definition 3.1(a), page 11). The second layer in the definition of a neighbourhood is its interpolability — given a neighbourhood $N$ of a subobject $P$, it must be possible to obtain a neighbourhood $N'$ of $P$ such that $N$ is a neighbourhood of $N'$. These neighbourhoods are called weak neighbourhoods (see Definition 3.1(b), page 11). Finally come the neighbourhoods (see Definition 3.1(c), page 11), a collection of which preserve arbitrary meets.

In the special case when the lattice $\text{Sub}_M(X)$ of admissible subobjects of the object $X$ is a frame, neighbourhoods (in the sense of Definition 3.1(c), hereafter) become topologies on $X$. Furthermore, when the lattice of admissible subobjects is atomic, the topologies are provided by prescribing the neighbourhoods of the atoms. In general, a topology on $X$ (see Definition 3.1.7, page 19) is a special collection of neighbourhoods, the set of open sets (see equation (22), page 14) is a frame in the order induced from the lattice $\text{Sub}_M(X)$.

An object along with a preneighbourhood, or a weak neighbourhood, or a neighbourhood, or a topology is said to be an internal preneighbourhood space, or an internal weak neighbourhood space, or an internal neighbourhood space, or an internal topological space, respectively. To define the notion of continuous maps for these spaces, one requires the notion of a preimage. This is achieved from the proper factorisation available on $\mathcal{A}$. A preneighbourhood morphism is a morphism $f$ of $\mathcal{A}$ with the property: if $U$ be a preneighbourhood of an admissible subobject $P$ of the codomain of $f$ then $f^{-1}U$ is a preneighbourhood of $f^{-1}P$ (see Definition 3.2, page 19).

The collection of internal preneighbourhood spaces of a category $\mathcal{A}$ along with the preneighbourhood morphisms make the category $\mathbf{pNbd}[\mathcal{A}]$. The full subcategory of internal weak neighbourhood spaces make the category $\mathbf{wNbd}[\mathcal{A}]$. $\mathbf{wNbd}[\mathcal{A}]$ is bireflective in $\mathbf{pNbd}[\mathcal{A}]$ (see Remark 17, page 22 and Theorem 3.2.1, page 20). Furthermore, both $\mathbf{pNbd}[\mathcal{A}]$ and $\mathbf{wNbd}[\mathcal{A}]$ are topological over $\mathcal{A}$ (see Theorem 4.2((a) & (b)), page 23). The category $\mathbf{pNbd}[\mathbf{Set}]$ of internal preneighbourhood spaces of $\mathbf{Set}$ is equivalent to the category $\mathbf{preTop}$ of pretopological spaces. The category $\mathbf{preTop}$ of pretopological spaces is investigated in [Kent1969], [BentleyHerrlichLowen1991] and [HerrlichLowenSchwarz1991].

The morphisms are restricted in the category of internal neighbourhood spaces. This is suggested from existence of the largest neighbourhood smaller than a weak neighbourhood in Theorem 3.2.2 (see page 21). The internal neighbourhood spaces along with preneighbourhood morphisms $f$ for which the preimage $f^{-1}$ preserve arbitrary joins constitute
the subcategory \( \text{Nbd}[A] \) of internal neighbourhood spaces. Since morphisms of neighbourhoods are restricted to those whose preimage preserve joins, it is topological over \( A_{ppj} \) (see Theorem 4.2(c), page 24), where \( A_{ppj} \) is the non-full subcategory of \( A \) having same objects as \( A \) and precisely those morphisms of \( A \) whose preimages preserve joins. Furthermore, \( \text{Nbd}[A] \) is bireflective in \( w\text{Nbd}[A]_{ppj} \) (see Remark 18, page 22 and Theorem 3.2.2, page 21).

The full subcategory of \( \text{Nbd}[A] \) consisting of internal topological spaces is \( \text{Top}[A] \). \( \text{Top}[A] \) is reflective in \( \text{Nbd}[A] \) if and only if \( \text{Top}[A] \) is topological over \( A_{ppj} \) if and only if each object has a largest internal topology (see Theorem 4.1, page 22). In categories, as in \( \text{Set} \), where each lattice of admissible subobjects is a frame, every object has a largest internal topology. However, internal topologies are interesting in many situations beyond \( \text{Set} \). For instance, \( \text{Loc} \) is a category in which not every lattice of admissible subobjects is a frame and the preimage of every morphism preserve only finite joins. Yet, as shown in the papers [DubeIghedo2016] & [DubeIghedo2016a], the open sublocales do provide a natural way to define an internal topology \( \text{Sub}_{\text{RegMon}}(X)^{\text{op}} \xrightarrow{o_X} \text{Fil}(X) \) (see equation (38), page 35) on each locale \( X \). In fact, as observed in Theorem 6.3 (see page 36), the assignment \( X \mapsto (X, o_X) \) on a locale defines a right inverse to the forgetful functor \( p\text{Nbd}[\text{Loc}] \xrightarrow{U} \text{Loc} \).

The lattice of admissible subobjects play an important role in the development of this paper. Further, apart from the category \( \text{Set} \) of sets and functions there are several categories which satisfy the basic assumption of this paper (see page 6). The following is a list of such instances, apart from \( \text{Set} \):

(i) \( \text{Top} \) satisfy the conditions (see §6.2, page 35). The lattices of admissible subobjects of a topological space is the Boolean algebra of subspaces of the space, and it transpires \( \text{Top}[\text{Top}] = \text{BiTop} \), the category of bitopological spaces and bicontinuous maps.

(ii) \( \text{Loc} \) satisfy the conditions (see §6.3, page 35). As observed earlier, \( \text{Loc} \) is significantly different from \( \text{Set} \) or \( \text{Top} \) but there are interesting internal topologies on a locale, as investigated in the papers [DubeIghedo2016] & [DubeIghedo2016a].

(iii) A category \( A \) is said to be regular if it has finite limits, kernel pairs have coequalisers and regular epimorphisms are pullback stable. Every regular category has a \( (\text{RegEpi}, \text{Mon}) \)-factorisation system. Hence every subobject of an object is admissible.

A regular category \( A \) is said to be coherent or a pre-logos (see [FreydScedrov1990]) if for each object \( X \) the semilattice \( \text{Sub}(X) \) is a lattice and for every morphism \( f \) the preimage \( f^{-1} \) is a lattice homomorphism.

A coherent category \( A \) is a Heyting category or a logos (see [FreydScedrov1990]) or a quasi-category (see [Joyal2008]), if further for every morphism \( f \) the preimage \( f^{-1} \) preserve arbitrary joins. It is well known from [FreydScedrov1990], as well as shown in Corollary 2.3 (see page 11), if each preimage preserve arbitrary
joins then each lattice of admissible subobjects is a frame.

In particular, every Heyting category satisfies the conditions. Since a topos is an example of a Heyting category, every topos satisfies the conditions.

(iv) A category $\mathcal{A}$ is said to be **extensive** if it has finite sums and for objects $X$ and $Y$ of $\mathcal{A}$, the canonical functor $(\mathcal{A} \downarrow X) \times (\mathcal{A} \downarrow Y) \xrightarrow{\tilde{\circ}} (\mathcal{A} \downarrow (X+Y))$ is an equivalence of categories (see [CarboniLackWalters1993]). If further $\mathcal{A}$ is small complete and small cocomplete then it has an $(\text{Epi}, \text{ExtMon})$-factorisation system and for each object $X$ of $\mathcal{A}$ the lattice $\text{Sub}_M(X)$ is a distributive complete lattice.

Any quasitopos with disjoint coproducts is extensive; if further it has a proper factorisation system then it satisfies the conditions.

The categories $\text{Cat}$ of small categories, $\text{CRing}^{\text{op}}$ of affine schemes, and the category $\text{Sch}$ of schemes are all infinitary lextensive with proper factorisation structures. Hence they satisfy the conditions.

(v) If $\mathcal{A}$ has an $(E, M)$-factorisation system then for any object $X$ of $\mathcal{A}$ the category $(\mathcal{A} \downarrow X)$ of bundles over $X$ has $(E_X, M_X)$-factorisation system, where:

$$E_X = \{ (X, x) \xrightarrow{e} (Y, y) : e \in E \}$$

and

$$M_X = \{ (X, x) \xrightarrow{m} (Y, y) : m \in M \}.$$

If the $(E, M)$-factorisation is proper then so also is the $(E_X, M_X)$-factorisation (see [MMCEGWT2004] for details).

Hence, if $\mathcal{A}$ satisfy the conditions of this paper then so does each $(\mathcal{A} \downarrow X)$.

Finally, the regular epimorphisms of internal neighbourhood spaces have been established in §5 (pages 24 - 34). Theorem 5.1 (see page 24) describes the regular epimorphisms of internal preneighbourhood spaces. This is similar to the description of regular epimorphisms of pretopological spaces (see [BentleyHerrlichLowen1991] and also in [Kent1969]). The dissimilarity is a consequence of the pullback stability of epimorphisms of $\text{Set}$, which is not the case in general (see Remark 20 & Remark 21, page 26). The pullback stability of epimorphisms in $\text{Set}$ is also responsible for the regular epimorphisms of pretopological spaces to be hereditary (see [BentleyHerrlichLowen1991] for the hereditary property). In Theorem 5.3.1 (see page 28) the hereditary regular epimorphisms of internal preneighbourhood spaces are described.

The pullback stability of epimorphisms is a weak condition ensuring regular epimorphism of internal preneighbourhood spaces to be hereditary. Theorem 5.3.3 (see page 31) provide five conditions which ensure heridity of regular epimorphisms of internal preneighbourhood spaces. Regular epimorphisms of preneighbourhood spaces are not stable under pullbacks — for instance, regular epimorphisms of pretopological spaces are not closed under products (see [BentleyHerrlichLowen1991] for details).
The regular epimorphisms of internal neighbourhood spaces are similar to the regular epimorphisms of internal preneighbourhood spaces (see Theorem 5.2, page 27). The dissimilarity lies in the replacement of $A$ by $A^{ppj}$ over which $Nbd[A]$ is topological. The hereditary regular epimorphisms of internal neighbourhood spaces is a little more intricate. Theorem 5.3.4 (page 33) provides alternative characterisations for hereditary regular epimorphisms of internal neighbourhood spaces. In the special case of Theorem 5.3.4, a morphism of internal neighbourhood spaces is a regular epimorphism of internal preneighbourhood spaces if and only if it is pseudo-open (see Remark 29, page 34 and also [BentleyHerrlichLowen1991] for comparison with Set).

A summary of the above mentioned connections between different categories of internal neighbourhood spaces appear in Figure 1 (page 37).

The notation and terminology follows [Maclane1997] on categories and [PicadoPultr2012] on frames and ordered algebraic systems.

2. Preliminaries

2.1. Factorisation Systems and Admissible Subobjects. The modern notion of a factorisation system $(E, M)$ was introduced in [FreydKelly1972]. The earlier bicategorical structures of Maclane (see [Maclane1950]) can be seen today as those factorisation systems where every $E$ is an epimorphism and each $M$ is a monomorphism. In this part the necessary facts for factorisation systems are collected from [Janel1997b].

2.1.1. Prefactorisation Systems. Given the morphisms $p$ and $i$ of a category $A$, the symbol $p \downarrow i$ is used to denote the statement: if $v \circ p = i \circ u$, then there exists a unique diagonal morphism $w$ such that the diagram

\[
\begin{array}{ccc}
\cdot & \xrightarrow{w} & \cdot \\
\downarrow{u} & \downarrow{v} & \downarrow{w} \\
\cdot & \xrightarrow{i} & \cdot
\end{array}
\]

commutes.

Let for any set $\mathcal{H}$ of morphisms of $A$:

\[
\mathcal{H}^\uparrow = \{ p : h \in \mathcal{H} \Rightarrow p \downarrow h \} \quad \text{and} \quad \mathcal{H}^\downarrow = \{ p : h \in \mathcal{H} \Rightarrow h \downarrow p \}.
\]

Definition. A prefactorisation system for a category $A$ is a pair $(E, M)$ of sets of morphisms of $A$ such that $E = M^\uparrow$ and $M = E^\downarrow$.

Theorem (see [Janel1997b]). In any category $A$:

(1) If $f = m \circ e$, $f \downarrow m$ and $e \downarrow m$ then $m$ is an isomorphism.
(2) Given any prefactorisation system $(E, M)$ of $A$:
(a) $E \cap M = \text{Iso}(A)$.
(b) $M$ is closed under compositions.
(c) If $g \circ f \in M$ and $g$ is either a monomorphism or $g \in M$ then $f \in M$.
(d) $M$ is stable under pullbacks.
(e) $M$ is closed under limits, i.e., if $Z \xrightarrow{F} A \xrightarrow{G} B$ with each component of $\alpha$ in $M$ and both the limits $\lim F$ and $\lim G$ exist, then $\lim \alpha \in M$.

2.1.2. Factorisation Systems.

**Definition.** A factorisation system for a category $A$ is a prefactorisation system $(E, M)$, such that any morphism $f$ of $A$, $f = m \circ e$, for some $m \in M$ and $e \in E$.

An $(E, M)$-factorisation for a category $A$ is proper if $E \subseteq \text{Epi}(A)$ and $M \subseteq \text{Mon}(A)$.

Every finitely complete and finitely cocomplete category with all intersections admit a $(\text{Epi}(A), \text{ExtMon}(A))$-factorisation.

Furthermore, in any category $A$ with binary products and coproducts the condition $E \subseteq \text{Epi}(A)$ implies $\text{ExtMon}(A) \subseteq M$ and dually $M \subseteq \text{Mon}(A)$ implies $\text{ExtEpi}(A) \subseteq E$. Hence such a category $A$ has a proper $(E, M)$-factorisation implies $\text{ExtEpi}(A) \subseteq E \subseteq \text{Epi}(A)$ and $\text{ExtMon}(A) \subseteq M \subseteq \text{Mon}(A)$.

2.1.3. Admissible Subobjects. Let $A$ be a finitely complete category with coproducts and a proper factorisation system $(E, M)$.

Given any object $X$ of $A$, the (possibly large) set $\{ M \xrightarrow{m} X : m \in M \}$ is endowed with a natural preorder:

$m \leq n \Leftrightarrow (\exists p)(m = n \circ p)$.

The corresponding quotient set is the poset $\text{Sub}_M(A)$ of all $M$-subobjects or admissible subobjects of $X$. For brevity, an admissible subobject $M \xrightarrow{m} X$ of $X$ shall be simply expressed by the morphism $m$ or even sometimes by $M$.

Since $A$ is finitely complete and $M$ is pullback stable, the poset $\text{Sub}_M(A)$ of admissible subobjects is a meet semilattice with largest element. Furthermore the existence of finite coproducts along with the $(E, M)$-factorisation ensures the existence of finite joins in $\text{Sub}_M(A)$. Hence $\text{Sub}_M(A)$ is a lattice.

Henceforth in this paper the following stipulation is made on $A$:

$A$ is a finitely complete category with finite coproducts and a proper $(E, M)$-factorisation such that for each object $X$, $\text{Sub}_M(X)$ is a distributive complete lattice.

Such categories certainly exist. $\text{Set}$ is the most familiar example. In §6 (page 34) it is shown that $\text{Top}$, $\text{Loc}$ are examples, and in §1 (page 3) other examples are described.

2.1.4. Images and Preimages. From our assumption, for each object $X$ of $A$, $\text{Sub}_M(X)$ is a distributive complete lattice. The smallest admissible subobject in $\text{Sub}_M(X)$ is $\emptyset_X \xrightarrow{\emptyset} X$ and the largest is obviously $1_X$. If $A$ has a strict initial object $\emptyset$ then for each object $X$, $\emptyset_X = \emptyset$ — a situation familiarly seen in $\text{Set}$, $\text{Top}$, $\text{Loc}$ or in any extensive category (see [CarboniLackWalters1993]).
Given a morphism $X \xrightarrow{f} Y$ of $\mathcal{A}$ and an admissible subobject $N \xrightarrow{n} Y$ of $Y$, the pullback $f^{-1} N \xrightarrow{f_n} N$ of $f$ along $n$ exists. Since $\mathcal{M}$ is pullback stable (see Theorem 2.1.1(2d), page 5), $f^{-1} n \in \text{Sub}_{\mathcal{M}}(X)$, and is called the preimage of $n$ under $f$. The morphism $f^{-1} N \xrightarrow{f_n} N$ shall be called the restriction of $f$ to the admissible subobject $N$.

Given a morphism $X \xrightarrow{f} Y$ of $\mathcal{A}$ and an admissible subobject $M \xrightarrow{m} X$ of $X$ the $(E, M)$-factorisation:

$$
\begin{array}{ccc}
M & \xrightarrow{f|_M^{(E)}} & \exists_j M \\
\downarrow m & & \downarrow \exists_j m^{(E)} \\
X & \xrightarrow{f} & Y
\end{array}
$$

of $f$ yields the admissible subobject $\exists_j m$ of $Y$, called the image of $m$ under $f$. The morphism $f|_M$ shall be called the trace of $f$ on $M$.

**Theorem.** Given any morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$, the image $\text{Sub}_{\mathcal{M}}(X) \xrightarrow{\exists_j} \text{Sub}_{\mathcal{M}}(Y)$ and preimage $\text{Sub}_{\mathcal{M}}(Y) \xrightarrow{f^{-1}} \text{Sub}_{\mathcal{M}}(X)$ are order preserving maps between the distributive complete lattices of admissible subobjects with $\exists_j^{-1} f^{-1}$.

**Corollary.** Given any morphism $X \xrightarrow{f} Y$ of $\mathcal{A}$, we have:

(a) For every admissible subobject $M$ of $X$, $M \subseteq f^{-1} \exists_j M$.

(b) For every admissible subobject $N$ of $Y$, $\exists_j f^{-1} N \subseteq N$.

(c) $\exists_j$ preserve all joins and $f^{-1}$ preserve all meets.

(d) For any admissible subobject $N \xrightarrow{n} Y$ of $Y$, $f_n \in E$, if and only if, $\exists_j f^{-1} n = n$.

(e) If $f \in \mathcal{M}$ then $f^{-1} \exists_j M = M$, for every admissible subobject $M$ of $X$.

(f) $f^{-1} Y = X$, $\exists_j \emptyset_X = \emptyset_Y$ and $f \in E$, if and only if, $Y = \exists_j X$.

2.2. Filters on an Object.

2.2.1. The Coherent Frame of Filters. A filter on an object $X$ of $\mathcal{A}$ is just a filter in the distributive lattice$^1$ $\text{Sub}_{\mathcal{M}}(X)$ of admissible subobjects of $X$. The set of all filters on $X$ is $\text{Fil}(X)$ and is ordered by set theoretic inclusion.

**Definition.** Let $X$ be a bounded lattice.

---

$^1$A filter on a meet semilattice $A$ is a subset $F \subseteq A$ which is up-closed (i.e., $x \geq y \in F \Rightarrow x \in F$) and closed under finite meets (i.e., $x, y \in F \Rightarrow x \wedge y \in F$).
(i) An element \( p \) in a lattice \( X \) is said to be \textit{compact}, if for every subset \( S \subseteq X \), there exists a finite subset \( T \subseteq S \) such that \( p \leq \bigvee T \), whenever \( p \leq \bigvee S \).

(ii) A lattice is \textit{compact} if its largest element is compact.

(iii) A lattice is \textit{algebraic} if each of its elements is the supremum of compact elements.

(iv) A frame is \textit{coherent} if it is a compact, algebraic lattice in which the set of compact elements is closed under finite meets.

**Theorem** ([IberkleidMcGovern2009b]). For any object \( X \) of \( \mathbb{A} \), \( \text{Fil}(X) \) is a coherent frame.

The compact elements of the frame \( \text{Fil}(X) \) are precisely \( \uparrow x \ (x \in \text{Sub}_M(X)) \), where:

\[
\uparrow x = \{ u \in \text{Sub}_M(X) : x \leq u \}.
\]

is the \textit{principal} filter on \( X \) containing the admissible subobject \( x \). Clearly:

\[
\uparrow (x \lor y) = (\uparrow x) \cap (\uparrow y) \quad \text{and} \quad \uparrow (x \land y) = (\uparrow x) \lor (\uparrow y).
\]

2.2.2. \textit{Forward and Inverse Filters}. Given a morphism \( X \xrightarrow{f} Y \) of \( \mathbb{A} \), filters \( A \in \text{Fil}(X) \), \( B \in \text{Fil}(Y) \), let:

\[
\text{\( \xrightarrow{\text{f}} A \rightarrow \)} = \{ y \in \text{Sub}_M(Y) : (\exists x \in A)(\exists y \leq x) \} = \{ y \in \text{Sub}_M(Y) : f^{-1}y \in A \},
\]

and

\[
\text{\( \xleftarrow{\text{f}} B \leftarrow \)} = \{ x \in \text{Sub}_M(X) : (\exists y \in B)(f^{-1}y \leq x) \}.
\]

The filter \( \xrightarrow{\text{f}} A \) is the smallest filter in \( \text{Fil}(Y) \) which contains the images \( \exists_i a \ (a \in A) \) and shall be called the \textit{forward filter of} \( A \) \textit{under} \( f \). Similarly, \( \xleftarrow{\text{f}} B \) is the smallest filter in \( \text{Fil}(X) \) which contains the preimages \( f^{-1}b \ (b \in B) \), and shall be called the \textit{inverse filter of} \( B \) \textit{under} \( f \).

**Theorem.** Given any morphism \( X \xrightarrow{f} Y \) of \( \mathbb{A} \), both \( \xleftarrow{\text{f}} \) and \( \xrightarrow{\text{f}} \) preserve principal filters, and there is the Galois connection:

\[
\xleftarrow{\text{f}} \quad \text{Fil}(X) \quad \text{Fil}(Y) \quad \xrightarrow{\text{f}}
\]

If further \( \text{Sub}_M(X) \xrightarrow{\text{\( \exists \)}} \text{Sub}_M(Y) \) preserve finite meets then \( \text{Fil}(X) \xrightarrow{\text{f}} \text{Fil}(Y) \) has a right adjoint \( \text{Sub}_M(Y) \xleftarrow{\text{\( \exists \)}} \text{Sub}_M(X) \) defined by:

\[
\xleftarrow{\text{f}} B = \{ x \in \text{Sub}_M(X) : \exists_i x \in B \}.
\]
Theorem. \( (a) \) If \( A \in \text{Fil}(X) \), \( B \in \text{Fil}(Y) \):

\[
\xymatrix{ f B \subseteq A \iff (\forall b \in B)(f^{-1} b \in A) \\
\iff (\forall b \in B)(\exists a \in A)(a \leq f^{-1} b) \\
\iff (\forall b \in B)(\exists a \in A)(\exists a \leq b) \iff B \subseteq f A,}
\]

proving \( \xymatrix{ f \dashv \tilde{f} } \).

On the other hand, given any family \( (A_i)_{i \in I} \) of filters on \( X \) if \( \exists_f \) preserve finite meets then:

\[
b \in \tilde{f}(\bigvee_{i \in I} A_i) \iff f^{-1} b \in \bigvee_{i \in I} A_i
\]

\[
\iff (\exists n \geq 1)(\exists i_1, i_2, \ldots, i_n \in I)(\exists a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n})(a_1 \land a_2 \ldots \land a_n \leq f^{-1} b)
\]

\[
\iff (\exists n \geq 1)(\exists i_1, i_2, \ldots, i_n \in I)(\exists a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n})(\exists_f (a_1 \land a_2 \ldots \land a_n) \leq b)
\]

\[
\iff (\exists n \geq 1)(\exists i_1, i_2, \ldots, i_n \in I)(\exists a_1 \in A_{i_1}, a_2 \in A_{i_2}, \ldots, a_n \in A_{i_n})(\exists_f a_1 \land \exists_f a_2 \land \ldots \land \exists_f a_n \leq b)
\]

\[
\iff b \in \bigvee_{i \in I} \tilde{f} A_i,
\]

indicate \( \tilde{f} \) preserve all joins and hence must have a right adjoint \( \text{Fil}(Y) \xymatrix{ \tilde{f} \ar[r] & \text{Fil}(X) } \).

Finally, for any admissible subobject \( x \) of \( X \):

\[
x \in \tilde{f} B \iff \uparrow x \subseteq \tilde{f} B \iff \tilde{f}(\uparrow x) \subseteq B \iff \uparrow (\exists_f x) \subseteq B \iff \exists_f x \in B
\]

implying:

\[
\tilde{f} B = \{ x \in \text{Sub}_M(X) : \exists_f x \in B \}.
\]

\( \square \)

2.3. Preimage preserving joins. Sometimes the preimage \( \text{Sub}_M(Y) \xymatrix{ \tilde{f}^{-1} \ar[r] & \text{Sub}_M(X) } \) for a morphism \( X \xymatrix{ \tilde{f} \ar[r] & Y } \) of \( \text{A} \) preserve joins — for instance in \( \text{Set, Top, Meas} \), and in many other concrete categories. However, in \( \text{Loc} \) the preimages usually preserve finite joins only (see [PicadoPultr2012]).

Theorem. \( (a) \) If \( P \xymatrix{ \tilde{P} \ar[r] & X } \) be an admissible subobject of \( X \) then the preimage function \( \text{Sub}_M(X) \xymatrix{ \tilde{P}^{-1} \ar[r] & \text{Sub}_M(P) } \) preserve arbitrary joins, if and only if, for every family \( (m_i)_{i \in I} \) of admissible subobjects of \( X \):

\[
p \land \bigvee_{i \in I} m_i = \bigvee_{i \in I} (p \land m_i).
\]

\( (b) \) The following are equivalent for any morphism \( X \xymatrix{ \tilde{f} \ar[r] & Y } \) of \( \text{A} \):

\( (i) \) \( \text{Sub}_M(Y) \xymatrix{ \tilde{f}^{-1} \ar[r] & \text{Sub}_M(X) } \) has a right adjoint \( \text{Sub}_M(X) \xymatrix{ \forall \ar[r] & \text{Sub}_M(Y) } \).
(ii) $\text{Sub}_{M}(Y) \xleftarrow{f^{-1}} \text{Sub}_{M}(X)$ preserve all arbitrary joins.

(iii) $\text{Fil}(Y) \xrightarrow{f} \text{Fil}(X)$ preserve all arbitrary meets.

(iv) $\text{Fil}(Y) \xrightarrow{f} \text{Fil}(X)$ has a left adjoint $\text{Fil}(X) \xleftarrow{\sqcup} \text{Fil}(Y)$.

**Proof.**

(a) If $\text{Sub}_{M}(X) \xrightarrow{p^{-1}} \text{Sub}_{M}(P)$ preserve arbitrary joins, then for any family $(m_i)_{i \in I}$ of admissible subobjects of $X$:

$$p \land \bigvee_{i \in I} m_i = p \circ p^{-1}(\bigvee_{i \in I} m_i)$$

$$= p \circ \bigvee_{i \in I} p^{-1} m_i = \exists_p(\bigvee_{i \in I} p^{-1} m_i)$$

$$= \bigvee_{i \in I} \exists_p p^{-1} m_i = \bigvee_{i \in I} (p \circ p^{-1} m_i) = \bigvee_{i \in I} (p \land m_i),$$

proving (6).

On the other hand, if (6) holds, then:

$$p \circ p^{-1}(\bigvee_{i \in I} m_i) = p \land \bigvee_{i \in I} m_i = \bigvee_{i \in I} (p \land m_i)$$

$$= \bigvee_{i \in I} \exists_p p^{-1} m_i = \exists_p(\bigvee_{i \in I} p^{-1} m_i) = p \circ \bigvee_{i \in I} p^{-1} m_i,$$

which implies $p^{-1}(\bigvee_{i \in I} m_i) = \bigvee_{i \in I} p^{-1} m_i$, completing the proof for this part.

(b) Obviously, (i) and (ii) are equivalent and so also for the pair (iii) and (iv).

Assuming (ii), given a family $(B_i)_{i \in I}$ of filters on $Y$, $p \in \bigcap_{i \in I} f B_i$, if and only if, for each $i \in I$ there exist a $b_i \in B_i$ such that $f^{-1} b_i \leq p$, which imply

$$\bigvee_{i \in I} f^{-1} b_i = f^{-1} (\bigvee_{i \in I} b_i) \leq p \Rightarrow p \in \bigcap_{i \in I} f B_i$$

which shows (iii) follows.

On the other hand, assuming (iii) and using it on principal filters shows (ii) to follow.

□

**Remark 1.** A morphism $X \xrightarrow{f} Y$ of $\mathcal{A}$ for which the preimage $\text{Sub}_{M}(Y) \xleftarrow{f^{-1}} \text{Sub}_{M}(X)$ preserve arbitrary joins shall be said to have preimage preserve joins property. For any such morphism:

$$\forall x = \bigvee \{ y \in \text{Sub}_{M}(Y) : f^{-1} y \leq x \}, \quad \text{for } x \in \text{Sub}_{M}(X),$$

and

$$\sqcup A = \{ y \in \text{Sub}_{M}(Y) : (\exists a \in A)(\forall a \leq y) \}, \quad \text{for } A \in \text{Fil}(X).$$
In particular, \( f \) preserve principal filters and:

\[
\uplus f \uplus x = \uplus \forall f x, \quad \text{for } x \in \text{Sub}_M(X).
\]

As a consequence of Theorem 2.3(a):

**Corollary.** If every morphism (or, every admissible monomorphism) of \( \mathcal{A} \) has the preimage preserve join property then each \( \text{Sub}_M(X) \) is a frame.

The results in this section are well known and are also seen in [FreydScedrov1990].

3. Preneighbourhoods, Weak Neighbourhoods and Neighbourhoods

In this section we shall define the notion of a *neighbourhood* of an admissible subobject and develop some of their relevant properties.

3.1. Neighbourhoods.

**Definition.** Given an object \( X \) of \( \mathcal{A} \):

(a) An order preserving map \( \text{Sub}_M(X)^{\text{op}} \xrightarrow{\mu} \text{Fil}(X) \) is a preneighbourhood on \( X \) if for each \( m \in \text{Sub}_M(X) \):

\[
n \in \mu(m) \Rightarrow m \leq n.
\]

(b) A preneighbourhood \( \mu \) on \( X \) is a weak neighbourhood on \( X \) if:

\[
\mu(m) \subseteq \bigcup_{p \in \mu(m), x \leq p} \mu(x), \quad \text{for } m \in \text{Sub}_M(X).
\]

(c) A weak neighbourhood \( \mu \) on \( X \) is a neighbourhood on \( X \) if:

\[
\mu(\bigvee G) = \bigcap_{x \in G} \mu(x), \quad \text{for all } G \subseteq \text{Sub}_M(X).
\]

**Remark 2** The set of all preneighbourhoods, weak neighbourhoods or neighbourhoods on and object \( X \) is denoted by the symbols \( \text{pnbd}_X \), \( \text{wnbd}_X \) or \( \text{nbd}_X \), respectively.

If \( \mu \) is a preneighbourhood (respectively, weak neighbourhood, neighbourhood) on and object \( X \) of \( \mathcal{A} \) then the pair \( (X, \mu) \) shall be called an internal preneighbourhood space (respectively, internal weak neighbourhood space, internal neighbourhood space).

**Remark 3** Surely, \( \text{Sub}_M(X)^{\text{op}} \xrightarrow{\uparrow} \text{Fil}(X) \), where \( \uparrow m = \{ p \in \text{Sub}_M(X) : m \leq p \} \) for any \( m \in \text{Sub}_M(X) \), is a neighbourhood on \( X \).

Further, \( \text{Sub}_M(X)^{\text{op}} \xrightarrow{\nabla} \text{Fil}(X) \) defined by:

\[

\nabla(m) = \begin{cases} 
\text{Sub}_M(X), & \text{if } m = \emptyset_X \\
\{1_X\}, & \text{otherwise}
\end{cases}
\]
is also a neighbourhood on $X$.

**Remark 4** The set $\text{pnbd}[X]$ is ordered pointwise, i.e., given preneighbourhoods $\mu$ and $\nu$ on $X$, $\mu \leq \nu$ if for each $m \in \text{Sub}_M(X)$, $\mu(m) \subseteq \nu(m)$.

Consequently, (10) equivalently states $\nabla \leq \mu \leq \nabla$ for every preneighbourhood $\mu$ on $X$, i.e., $\text{pnbd}[X]$ is a bounded poset.

3.1.1. Weak Neighbourhoods are Interpolative Preneighbourhoods.

**Theorem.** A preneighbourhood $\mu$ on an object $X$ of $\mathcal{A}$ is a weak neighbourhood if and only if it is interpolative, i.e., the following equation holds:

\[
\mu(m) = \{ p \in \text{Sub}_M(X) : (\exists q \in \mu(m))(p \in \mu(q)) \}, \quad \text{for all } m \in \text{Sub}_M(X).
\]

**Proof.** Firstly, if $\mu$ be a preneighbourhood on $X$, then for any admissible subobject $p$ of $X$, the set $\{ \mu(x) : x \leq p \}$ of filters on $X$ has $\mu(p)$ as the smallest filter. Hence:

\[
\mu(p) = \bigcap_{x \leq p} \mu(x).
\]

On the other hand, if $p \in \mu(m)$ then using (10), $\mu(p) \subseteq \mu(m)$, and hence:

\[
\mu(m) \supseteq \bigcup_{p \in \mu(m)} \mu(p).
\]

Consequently, $\mu$ is a weak neighbourhood on $X$, if and only if, for each admissible subobject $m$ of $X$:

\[
\mu(m) \subseteq \bigcup_{p \in \mu(m)} \bigcap_{x \leq p} \mu(x) = \bigcup_{p \in \mu(m)} \mu(p) \subseteq \mu(m) \implies \mu(m) = \bigcup_{p \in \mu(m)} \mu(p),
\]

completing the proof. \(\square\)

**Remark 5** For any preneighbourhood $\mu$ on $X$ and any $m \in \text{Sub}_M(X)$, $\bigcup_{p \in \mu(m)} \mu(p)$ is a subset of $\mu(m)$. Theorem 3.1.1 asserts that a preneighbourhood $\mu$ is a weak neighbourhood, if and only if, the subset $\bigcup_{p \in \mu(m)} \mu(p)$ is a filter and:

\[
\mu(m) = \bigvee_{p \in \mu(m)} \mu(p) = \bigcup_{p \in \mu(m)} \mu(p).
\]

3.1.2. Complete Lattices $\text{pnbd}[X], \text{wnbd}[X]$.

**Theorem.** The set $\text{pnbd}[X]$ of all preneighbourhoods on $X$ with the pointwise order in Remark 4 (see page 12) is a complete lattice.

The subset $\text{wnbd}[X]$ of all weak neighbourhoods on $X$ is also a complete lattice with joins computed as in $\text{pnbd}[X]$. 
Proof. As observed in Remark 4 on page 12, \( \text{pnbd}[X] \) is already a bounded poset.

Furthermore, given any set \( T \subseteq \text{pnbd}[X] \) of preneighbourhoods on \( X \), let for each \( m \in \text{Sub}_M(X) \):
\[
(\bigvee T)(m) = \bigvee_{\tau \in T} \tau(m),
\]
and
\[
(\bigwedge T)(m) = \bigcap_{\tau \in T} \tau(m).
\]

Clearly \( \bigvee T, \bigwedge T \in \text{pnbd}[X] \) and \( \bigvee T \) (respectively, \( \bigwedge T \)) is the supremum (respectively, infimum) of \( T \) in \( \text{pnbd}[X] \). Hence \( \text{pnbd}[X] \) is a complete lattice.

Given the subset \( T \subseteq \text{wnbd}[X] \), \( \bigvee T \) is a preneighbourhood on \( X \). If \( p \in (\bigvee T)(m) \), then there exists a natural number \( n \geq 1 \), \( \tau_1, \tau_2, \ldots, \tau_n \in T \), \( p_1 \in \tau_1(m) \), \( p_2 \in \tau_2(m) \), \ldots, \( p_n \in \tau_n(m) \) such that \( p = p_1 \land p_2 \land \cdots \land p_n \). Using Theorem 3.1.1, there exist \( q_1 \in \tau_1(m) \), \( q_2 \in \tau_2(m) \), \ldots, \( q_n \in \tau_n(m) \) such that \( p_1 \in \tau_1(q_1) \), \( p_2 \in \tau_2(q_2) \), \ldots, \( p_n \in \tau_n(q_n) \). Then \( q = q_1 \land q_2 \land \cdots \land q_n \in (\bigvee T)(m) \) and
\[
p = p_1 \land p_2 \land \cdots \land p_n \in \tau_1(q_1) \lor \tau_2(q_2) \lor \cdots \lor \tau_n(q_n) \subseteq \tau_1(q) \lor \tau_2(q) \lor \cdots \lor \tau_n(q) \subseteq (\bigvee T)(q)
\]
shows \( \bigvee T \) to be interpolative. Hence by Theorem 3.1.1, \( \bigvee T \in \text{wnbd}[X] \) and is the supremum of \( T \) in \( \text{wnbd}[X] \).

Since \( \text{wnbd}[X] \) is a bounded poset with every subset having a supremum, it is a complete lattice. \( \square \)

Remark 6 For any preneighbourhood \( \mu \) on \( X \), the largest weak neighbourhood on \( X \) smaller than \( \mu \) is:
\[
\mu_w = \bigvee \{ \nu \in \text{wnbd}[X] : \nu \leq \mu \}.
\]

Remark 7 For any subset \( T \subseteq \text{wnbd}[X] \):
\[
(\bigwedge T) = (\bigcap T)_w.
\]

3.1.3. Open Subobjects and Interiors. Given a preneighbourhood \( \mu \) on \( X \) it is easy to observe for any \( p \in \text{Sub}_M(X) \) the three statements in:
\[
p \in \mu(p),
\]
\[
(\forall m \in \text{Sub}_M(X))(m \leq p \Rightarrow p \in \mu(m)),
\]
and
\[
\mu(p) = \uparrow p
\]
are equivalent: given (19), if \( m \leq p \) then \( p \in \mu(p) \subseteq \mu(m) \) proves (20), given (20), if \( q \geq p \) then \( q \in \mu(p) \) proves (21), and (21) automatically implies (19).

Let:
\begin{align*}
(22) \quad \mathcal{D}_\mu &= \{ p \in \text{Sub}_M(X) : p \in \mu(p) \} \\
&= \{ p \in \text{Sub}_M(X) : m \leq p \iff p \in \mu(m) \} \\
&= \{ p \in \text{Sub}_M(X) : \mu(p) = \uparrow p \}
\end{align*}

and

\( (23) \quad \text{int}_\mu m = \bigvee \{ p \in \mathcal{D}_\mu : p \leq m \}, \quad \text{for } m \in \text{Sub}_M(X). \)

The admissible subobjects in \( \mathcal{D}_\mu \) are called \( \mu \)-open subobjects of \( X \); for any admissible subobject \( m \in \text{Sub}_M(X) \), the admissible subobject \( \text{int}_\mu m \) is \( \mu \)-interior of \( m \).

Observe: for any preneighbourhood \( \mu \) on \( X \), the largest admissible subobject \( 1_X \) is always \( \mu \)-open, and from (12) (page 11), if \( \mu \) is a neighbourhood on \( X \) then the smallest subobject \( \emptyset_X \) is \( \mu \)-open. Using (2) (page 8) it follows that the set \( \mathcal{D}_\mu \) of \( \mu \)-open subobjects is closed under finite meets. Furthermore, for any preneighbourhood \( \mu \) on \( X \), \( \text{Sub}_M(X) \xrightarrow{\text{int}_\mu} \text{Sub}_M(X) \) is an order preserving idempotent function fixing every \( \mu \)-open subobjects such that \( \text{int}_\mu m \leq m \) \( (m \in \text{Sub}_M(X)). \)

**Theorem.** If \( \text{Sub}_M(X)^{\text{op}} \xrightarrow{\mu} \text{Fil}(X) \) be a preneighbourhood on \( X \) then the set \( \mathcal{D}_\mu \) of \( \mu \)-open subobjects is closed under arbitrary joins if and only if for every \( m \in \text{Sub}_M(X) \) its \( \mu \)-interior \( \text{int}_\mu m \) is \( \mu \)-open.

Furthermore, in such a case the following two statements are equivalent:

(a) For any \( m \in \text{Sub}_M(X) \), \( \mu(m) = \bigcup \{ \uparrow q : m \leq q \in \mathcal{D}_\mu \} \).

(b) For any \( m \in \text{Sub}_M(X) \), \( p \in \mu(m) \iff m \leq \text{int}_\mu p \).

**Proof.** The only if part of the first statement is immediate from the definition of \( \mu \)-interior in (23).

Conversely, if for each \( m \in \text{Sub}_M(X) \), the \( \mu \)-interior \( \text{int}_\mu m \) of \( m \) is \( \mu \)-open then for any \( T \subseteq \mathcal{D}_\mu \), \( \text{int}_\mu (\bigvee T) \in \mathcal{D}_\mu \).

Since the elements of \( \mathcal{D}_\mu \) are fixed points of \( \mu \)-interior assignment:

\[ t \in T \Rightarrow t \leq \bigvee T \Rightarrow t = \text{int}_\mu t \leq \text{int}_\mu \bigvee T \Rightarrow \bigvee T \leq \text{int}_\mu \bigvee T \Rightarrow \bigvee T = \text{int}_\mu \bigvee T \in \mathcal{D}_\mu. \]

For the second part of the statement, assume \( \mu \) is a preneighbourhood on \( X \) such that every \( \mu \)-interior is \( \mu \)-open. In this case, using (22):

\[ m \leq \text{int}_\mu p \leq p \Rightarrow p \in \mu(\text{int}_\mu p) \subseteq \mu(m), \]

implies the \( \subseteq \) part of the statement in (b) is true.

The implication of (a) from (b) is trivial.

Assuming (a):

\[ p \in \mu(m) \iff (\exists q \in \mathcal{D}_\mu)(m \leq q \leq p) \Rightarrow m \leq \text{int}_\mu p, \]
(b) follows, completing the proof. □

Remark 8 Observe, for any preneighbourhood $\mu$ on $X$, since $\mu$-interior have $\mu$-open subobjects as fixed points:

$$\forall p \in \mathcal{O}_\mu \left( p \leq m \iff p \leq \text{int}_\mu m \right).$$

(24)

In the special case when every $\mu$-interior is $\mu$-open, (24) provides the familiar meaning: $\text{int}_\mu m$ is the largest $\mu$-open subobject contained in $m$.

Remark 9 If for a preneighbourhood $\mu$ every $\mu$-interior is $\mu$-open then for admissible subobjects $m$ and $n$ of an object $X$, $m \land n \geq (\text{int}_\mu m \land \text{int}_\mu n) \in \mathcal{O}_\mu \Rightarrow \text{int}_\mu (m \land n) \geq (\text{int}_\mu m \land \text{int}_\mu n)$.

The order preserving property of interior returns:

$$\text{int}_\mu (m \land n) = \text{int}_\mu m \land \text{int}_\mu n.$$

This leads to:

Corollary. The interior operation $\text{int}_\mu$ of a preneighbourhood $\mu$ for which the $\mu$-interiors are $\mu$-open is a Kuratowski interior operation.

Remark 10 Any preneighbourhood on $X$ which satisfies the condition in Theorem 3.1.3(a) is interpolative and hence a weak neighbourhood.

Remark 11 The preneighbourhoods $\text{Sub}_M(X)^\text{op} \xrightarrow{\mu} \text{Fil}(X)$ satisfying the condition in Theorem 3.1.3(a) are a very special kind of weak neighbourhoods — the ones which are determined completely by the $\mu$-open subobjects. Using Theorem 3.1.4 these are in between the weak neighbourhoods and neighbourhoods.

3.1.4. interiors for neighbourhoods.

Theorem. If $\text{Sub}_M(X)^\text{op} \xrightarrow{\mu} \text{Fil}(X)$ be a neighbourhood on $X$ then every $\mu$-interior is $\mu$-open and

$$\mu(m) = \bigcup \{ \uparrow q : m \leq q \in \mathcal{O}_\mu \}.$$

Proof. If $T \subseteq \mathcal{O}_\mu$ then $\mu(\bigvee T) = \bigcap_{t \in T} \mu(t) = \bigcap_{t \in T} \uparrow t = \uparrow (\bigvee T)$, shows the set $\mathcal{O}_\mu$ of all $\mu$-open subobjects closed under arbitrary joins. Using Theorem 3.1.3 (page 14) completes the proof of the first statement.

Choose and fix a $p \in \mu(m)$. Let:

$$T_p = \{ u \in \text{Sub}_M(X) : p \in \mu(u) \} \quad \text{and} \quad p_0 = \bigvee T_p.$$

By definition of $T_p$, $m \in T_p$. Since $\mu$ is a neighbourhood:

$$\mu(p_0) = \bigcap_{u \in T_p} \mu(u) \Rightarrow p \in \mu(p_0) \Rightarrow p_0 \in T_p.$$
Since \(\mu\) is a weak neighbourhood on \(X\), from Theorem 3.1.1 (page 12) it is interpolative. Hence, if \(u \in T_p\), there exists a \(v \in \mu(u)\) such that \(p \in \mu(v)\). Consequently, \(v \in T_p\) and \(p_0 \geq v \in \mu(u) \Rightarrow p_0 \in \mu(u)\). Thus \(p_0 \in \mu(p_0)\), yielding:

\[ p \in \mu(m) \implies (\exists q \in \mathcal{O}_\mu)(m \leq q \leq p). \]

Since \(\mu\) is a weak neighbourhood:

\[ \mu(m) \subseteq \bigcup \{ \uparrow q : m \leq q \in \mathcal{O}_\mu \} = \bigcup \{ \mu(q) : m \leq q \in \mathcal{O}_\mu \} \subseteq \bigcup \{ \mu(t) : t \in \mu(m) \} = \mu(m) \]

completing the proof. \(\square\)

3.1.5. Kuratowski Interiors and Neighbourhoods. Let \(\mathcal{K}_X\) be the set of all Kuratowski operations on an object \(X\). This set can be ordered pointwise, producing a partially ordered set. Using Remark 3 (see page 11), Theorem 3.1.4 (page 15) and Corollary in 3.1.3 (page 15), \(\text{int}_\triangledown, \text{int}_\uparrow \in \mathcal{K}_X\).

Let \(\mathcal{P}_X\) be the subset of all preneighbourhoods \(\mu\) on \(X\) for which the \(\mu\)-interiors are \(\mu\)-open. The order from \(\text{pndb}[X]\) restricts to provide another partially ordered set.

**Theorem.** The interior operation \(\mathcal{P}_X \xrightarrow{\text{int}} \mathcal{K}_X\) is a split epimorphism of bounded posets having a left adjoint which restricts to an isomorphism precisely on \(\text{nbd}[X]\).

Furthermore, for each Kuratowski interior operation \(i\) on \(X\) the fibre \(\text{int}^{-1} i\) has exactly one neighbourhood on \(X\), which is the smallest element of the fibre.

**Proof.** Since \(\mathcal{O}_\triangledown = \{ \emptyset_X, X \}\) (see (13), page 11) and \(\mathcal{O}_\uparrow = \text{Sub}_M(X)\) one obtains:

\[ \text{int}_\triangledown m = \begin{cases} \emptyset_X, & \text{if } m \neq 1_X \\ X, & \text{otherwise} \end{cases}, \quad \text{and} \quad \text{int}_\uparrow m = m, \]

it follows that int preserve the bounds.

If \(\mu, \nu \in \mathcal{P}_X\) with \(\mu \leq \nu\) then \(p \in \mathcal{O}_\mu \iff p \in \mu(p) \subseteq \nu(p) \Rightarrow p \in \mathcal{O}_\nu\) implying \(\mathcal{O}_\mu \subseteq \mathcal{O}_\nu\).

Hence \(\text{int}_\mu m = \bigvee \{ p \in \mathcal{O}_\mu : p \leq m \} \leq \bigvee \{ p \in \mathcal{O}_\nu : p \leq m \} = \text{int}_\nu m\), showing \(\mathcal{P}_X \xrightarrow{\text{int}} \mathcal{K}_X\) to be a morphism of bounded posets.

Let \(\text{Sub}_M(X) \xrightarrow{i} \text{Sub}_M(X)\) be a Kuratowski interior operation on \(X\) and let:

\[ (25) \quad p_i(m) = \{ p \in \text{Sub}_M(X) : m \leq i(p) \}, \quad \text{for } m \in \text{Sub}_M(X). \]

Clearly: \(p_i(\emptyset_X) = \text{Sub}_M(X), p_i(1_X) = \{ 1_X \}, m \leq n \Rightarrow p_i(n) \subseteq p_i(m), q \geq p \in p_i(m) \Rightarrow q \in p_i(m)\) and since \(i\) preserve finite meets, \(p, q \in p_i(m) \Rightarrow p \land q \in p_i(m)\).

Hence, for each \(m \in \text{Sub}_M(X), p_i(m)\) is a filter on \(X\), and \(\text{Sub}_M(X)^{op} \xrightarrow{p_i} \text{Fil}(X)\) is a preneighbourhood on \(X\).

Further:

\[ p \in p_i(p) \iff p \leq i(p) \iff i(p) = p \Rightarrow \mathcal{O}_{p_i} = \{ p \in \text{Sub}_M(X) : i(p) = p \} \]
shows for any \( T \subseteq \mathcal{O}_p \), \( t \in T \Rightarrow t \leq \bigvee T \Rightarrow t = i(t) \leq i(\bigvee T) \Rightarrow \bigvee T \leq i(\bigvee T) \), and hence \( \mathcal{O}_p \) is closed under arbitrary joins.

Moreover, for any \( m \in \text{Sub}_M(X) \):
\[
\text{int}_p m = \bigvee \{ p \in \mathcal{O}_p : p \leq m \} = \bigvee \{ p \in \text{Sub}_M(X) : i(p) = p \leq m \} = i(m),
\]
\( p \in \text{int}_p(m) \Leftrightarrow p \leq i(m) \Leftrightarrow p \leq \text{int}_p m, \)
and for any \( S \subseteq \text{Sub}_M(X) \)
\[
p \in \bigcap_{x \in S} p_i(x) \Leftrightarrow (\forall x \in S)(x \leq i(p)) \Leftrightarrow \bigvee S \leq i(p) \Leftrightarrow p \in p_i(\bigvee S).
\]

Hence \( p_i \in \text{nbd}[X] \subseteq \mathcal{P} \). Furthermore, for any \( \mu \in \mathcal{P}_X \), if \( i \leq \text{int}_\mu \) then:
\[
p \in p_i(m) \Rightarrow m \leq i(p) \leq \text{int}_\mu p \leq p \Rightarrow p \in \mu(m),
\]
shows that the assignment \( i \mapsto p_i \) extends to an order preserving map \( K_X \overset{p}{\rightarrow} \mathcal{P}_X \) such that \( p \dashv \text{int}_\mu \) with \( \text{int} \circ p = 1_{K_X} \).

Clearly, from the adjunction the fibre \( \text{int}^{-1} i \) of any \( i \in K_X \) has the neighbourhood \( p_i \) as the smallest element.

Finally, if \( \mu \in \mathcal{P}_X \) be a neighbourhood on \( X \) then from Theorem 3.1.4:
\[
p \in \mu(m) \Leftrightarrow m \leq \text{int}_\mu p \Leftrightarrow p \in p_{\text{int}_\mu}(m),
\]
yields along with the observation \( p_i \) for each \( i \in K_X \) is a neighbourhood that \( \mu \in \mathcal{P}_X \) is a neighbourhood if and only if \( \mu = p_{\text{int}_\mu} \), completing the proof. \( \square \)

3.1.6. Neighbourhoods and Pseudo-frame subsets.

Definition. A set \( \mathcal{O} \subseteq \text{Sub}_M(X) \) of admissible subobjects of \( X \) is said to be a pseudo-frame set if it is closed under finite meets and arbitrary joins.

We denote the set of all pseudo-frame sets by \( \text{Pfs}[X] \) and is ordered by usual set inclusion.

Remark 12 Clearly \( \{0, 1_X\} \) is the smallest and \( \text{Sub}_M(X) \) is the largest element of \( \text{Pfs}[X] \).

Remark 13 Since any intersection of pseudo-frame sets is again a pseudo-frame set, it follows that \( \text{Pfs}[X] \) is a complete lattice with intersection being the arbitrary meet.

Hence for every subset \( \mathcal{T} \subseteq \text{Pfs}[X] \) the supremum \( \bigvee \mathcal{T} \) exists, but a simple intrinsic description may be difficult to obtain. However, in case when \( \text{Sub}_M(X) \) is itself a frame it has a simple description — \( \bigvee \mathcal{T} \) is the set of all arbitrary joins of finite meets of elements of \( \bigcup \mathcal{T} \).

Theorem. Given \( \mathcal{O} \subseteq \text{Pfs}[X] \) let:
\[
(26) \quad \mu_\mathcal{O}(m) = \bigcup \{ \uparrow q : m \leq q \in \mathcal{O} \}, m \in \text{Sub}_M(X).
\]

The assignment \( \mathcal{O} \mapsto \mu_\mathcal{O} \) is an isomorphism of the complete lattices \( \text{Pfs}[X] \) and \( \text{nbd}[X] \).
Proof. Since:

- \( \mu_\mathcal{O}(0) = \text{Sub}_M(X), \mu_\mathcal{O}(1_X) = \{1_X\} \),
- \( p \in \mu_\mathcal{O}(m) \Rightarrow m \leq p \),
- \( m \leq n \Rightarrow \{q \in \mathcal{O} : q \geq n\} \subseteq \{q \in \mathcal{O} : q \geq m\} \Rightarrow \mu_\mathcal{O}(m) \geq \mu_\mathcal{O}(n) \),
- \( p' \geq p \in \mu_\mathcal{O}(m) \iff (\exists q \in \mathcal{O})(m \leq q \leq p \leq p') \Rightarrow p' \in \mu_\mathcal{O}(m) \),
- \( p, p' \in \mu_\mathcal{O}(m) \iff (\exists q, q' \in \mathcal{O})(m \leq q \leq p \text{ and } m \leq q' \leq p') \Rightarrow m \leq q \land q' \leq p \land p' \),

and since \( \mathcal{O} \) is closed under finite meets, \( p \land p' \in \mu_\mathcal{O}(m) \),

it follows that \( \text{Sub}_M(X)^{op} \xrightarrow{\mu_\mathcal{O}} \text{Fil}(X) \) is indeed a preneighbourhood on \( X \).

Further, \( p \in \mu_\mathcal{O}(p) \iff (\exists q \in \mathcal{O})(p \leq q \leq p) \iff p \in \mathcal{O} \), it follows that \( \mathcal{O}_{\mu_\mathcal{O}} = \mathcal{O} \), implying \( \mu_\mathcal{O} \in \mathcal{P}_X \) by Theorem 3.1.3 (page 14). In particular, from Theorem 3.1.5 (page 16), \( \text{int}_{\mu_\mathcal{O}} \) is a Kuratowski interior operation.

Using (26) on the equivalence between (a) \& (b) in Theorem 3.1.3 (see page 14) indicates \( \mu_\mathcal{O} = \mathcal{P}_{\text{int}_{\mu_\mathcal{O}}} \), and hence from Theorem 3.1.5 again, \( \mu_\mathcal{O} \in \text{nbd}[X] \).

If \( \mathcal{O}, \mathcal{O}' \in \text{Pfs}[X], \mathcal{O} \subseteq \mathcal{O}' \) then:

\[
p \in \mu_\mathcal{O}(m) \iff (\exists q \in \mathcal{O})(m \leq q \leq p) \iff (\exists q \in \mathcal{O}')(m \leq q \leq p) \iff p \in \mu_{\mathcal{O}'}(m)
\]

implies \( \mu_\mathcal{O} \leq \mu_{\mathcal{O}'} \); conversely, if \( \mu_\mathcal{O} \leq \mu_{\mathcal{O}'} \) then using \( \mathcal{O} = \mathcal{O}_{\mu_\mathcal{O}} \) one obtains:

\[
p \in \mathcal{O} \iff p \in \mu_\mathcal{O}(p) \leq \mu_{\mathcal{O}'}(p) \Rightarrow p \in \mathcal{O}'.
\]

Hence \( \mathcal{O} \subseteq \mathcal{O}' \iff \mu_\mathcal{O} \leq \mu_{\mathcal{O}'} \) with \( \mathcal{O} = \mathcal{O}' \iff \mu_\mathcal{O} = \mu_{\mathcal{O}'} \).

Thus, the function \( \text{Pfs}[X] \xrightarrow{P} \text{nbd}[X] \) defined by \( P(\mathcal{O}) = \mu_\mathcal{O} \) is an order preserving bijection with \( P(\{0, 1_X\}) = \nabla \) and \( P(\text{Sub}_M(X)) = \uparrow \).

Now let \( T \subseteq \text{nbd}[X], \mathcal{O} = \sup\{\mathcal{O}_\tau : \tau \in T\} \) and \( \varnothing = \bigcap_{\tau \in T} \mathcal{O}_\tau \); since \( \text{Pfs}[X] \) is a complete lattice \( \mathcal{O}, \mathcal{O} \in \text{Pfs}[X] \). Then:

- \( \tau \in T \Rightarrow \mu_\varnothing \leq \tau \leq \mu_{\mathcal{O}} \).
- If \( \nu, \mu \in \text{nbd}[X] \) such that \( \tau \in T \Rightarrow \nu \leq \tau \leq \mu \) then \( \tau \in T \Rightarrow \mathcal{O}_\nu \subseteq \mathcal{O}_\tau \subseteq \mathcal{O}_\mu \).

Hence using the definition of the supremum and infimum in \( \text{Pfs}[X], \mathcal{O}_\nu \subseteq \varnothing \Leftrightarrow \nu \leq \mu_\varnothing \) and \( \mathcal{O} \subseteq \mathcal{O}_\mu \Leftrightarrow \mu_\mathcal{O} \leq \mu \).

Hence \( \mu_\varnothing = \inf T \) and \( \mu_{\mathcal{O}} = \sup T \) in \( \text{nbd}[X] \), completing the proof. \( \square \)

Remark 14 The proof also provides the route for computing the suprema or infima in \( \text{nbd}[X] \). Given a \( T \subseteq \text{nbd}[X] \) to obtain the suprema (respectively, infima) in \( \text{nbd}[X] \):

(a) compute the suprema (respectively, infima) \( \mu = \bigvee T \) (respectively, \( \mu = \bigwedge T = (\bigcap T)_w \) in \( \text{unbd}[X] \),

(b) compute the set \( \mathcal{O} \) of \( \mu \)-open sets,
Table 1. Conversion between Facets of Neighbourhood

| From \(\text{nbd}[X]\) (\(\nu\)) | To \(\text{Pfs}[X]\) | \(K_X\) |
|---|---|---|
| \(\nu\) | \(\bigcup q : m \leq q \in \mathbb{D}\) | \(\mathbb{D}_t = \{ p \in \text{Sub}_M(X) : i(p) = p \}\) |
| \(\mu_0(m) = \bigcup \{ q : m \leq q \in \mathbb{D}\} \) | \(\mathbb{D}\) | \(\mu_{\mathcal{O}} = \bigvee \{ p \in \mathbb{D}_v : p \leq m \}\) |
| \(\mu_i(m) = \{ p \in \text{Sub}_M(X) : m \leq i(p) \}\) | \(\mathbb{D}_i = \{ p \in \text{Sub}_M(X) : i(p) = p \}\) | \(\mu_{\mathcal{O}} m = \bigvee \{ p \in \mathbb{D} : p \leq m \}\) |

(c) the candidate for the suprema (respectively, infima) is then \(\mu_{\mathcal{O}}\).

Remark 15 Thus there are three ways to identify a neighbourhood on an object \(X\), which is summarised in Table 1.

3.1.7. Internal Topological Spaces.

Definition. A neighbourhood \(\mu\) is a topology on \(X\) if \(\mathcal{O}_\mu\) is a frame in the partial order of \(\text{Sub}_M(X)\).

The set of all internal topologies on \(X\) is denoted by \(\text{top}[X]\).

Clearly, \(\nabla\) is an internal topology. However, \(\uparrow\) is an internal topology if and only if \(\text{Sub}_M(X)\) is itself a frame.

Moreover, the order isomorphism between \(\text{nbd}[X]\) and \(\text{Pfs}[X]\) indicate:

\[
\mu \leq \nu \in \text{top}[X] \Rightarrow \mu \in \text{top}[X].
\]

Thus \(\text{top}[X]\) being a down set of \(\text{nbd}[X]\) is a complete meet subsemilattice of \(\text{nbd}[X]\). The following is immediate:

Theorem. \(\text{top}[X]\) is a complete sublattice of \(\text{nbd}[X]\) if and only if there exists a largest topology on \(X\).

3.2. Morphisms of Neighbourhoods.

Definition. Given the internal preneighbourhood spaces \((X, \mu)\) and \((Y, \phi)\), a morphism \(X \xrightarrow{f} Y\) of \(\mathcal{A}\) is a preneighbourhood morphism if for every admissible subobject \(n\) of \(Y\):

\[
p \in \phi(n) \Rightarrow f^{-1}p \in \mu(f^{-1}n).
\]

The symbol \((X, \mu) \xrightarrow{f} (Y, \phi)\) denotes \(f\) is a preneighbourhood morphism.

Clearly the adjunctions in Theorem 2.1.4 (page 7) and Theorem 2.2.2 (page 8) easily suggest the following equivalent formulations of a preneighbourhood morphism.

Theorem ([HolgateSlapal2011]). Given the internal preneighbourhood spaces \((X, \mu)\) and \((Y, \phi)\) the following are equivalent for any morphism \(X \xrightarrow{f} Y\) of \(\mathcal{A}\):

(a) \(f\) is a preneighbourhood morphism.

(b) \(f \phi(n) \subseteq \mu(f^{-1}n)\), for every admissible subobject \(n \in \text{Sub}_M(Y)\).
(c) \( \phi(n) \subseteq \overleftarrow{f \mu}(f^{-1}n) \), for every admissible subobject \( n \in \text{Sub}_M(Y) \).

(d) \( \overleftarrow{f \phi}(\exists_f m) \subseteq \mu(m) \), for every admissible subobject \( m \in \text{Sub}_M(X) \).

Remark 16 Seen via diagrams, \((X, \mu) \xrightarrow{f} (Y, \phi)\) is a preneighbourhood morphism if and only if the square below denotes a natural transformation from the composites of order preserving maps on the right to the composites of order preserving maps on the left, where the order preserving maps are considered as functors:

\[
\begin{array}{ccc}
\text{Sub}_M(X)^{\text{op}} & \xrightarrow{f^{-1}} & \text{Sub}_M(Y)^{\text{op}} \\
\downarrow & & \downarrow \\
\text{Fil}(X) & \xrightarrow{\perp} & \text{Fil}(Y) \\
\downarrow & & \downarrow \\
\mu & \subseteq & \phi \\
\end{array}
\]

3.2.1. Universal Weak Neighbourhoods.

**Theorem.** Given a preneighbourhood morphism \((X, \mu) \xrightarrow{f} (Y, \phi)\) from the internal preneighbourhood space \((X, \mu)\) to the internal weak neighbourhood space \((Y, \phi)\), the preneighbourhood morphism \((X, \mu_w) \xrightarrow{f} (Y, \phi)\) is the unique morphism between internal weak neighbourhood spaces such that the diagram \((X, \mu) \xrightarrow{1_X} (X, \mu_w)\) commutes.

**Proof.** It is enough to show that \(\text{Sub}_M(X)^{\text{op}} \xrightarrow{\overleftarrow{f \phi}(\exists_f)} \text{Fil}(X)\) is a weak neighbourhood structure on \(X\).

Having shown the assertion above, since \(f\) is already a preneighbourhood morphism, \(\overleftarrow{f \phi}(\exists_f) \leq \mu\) (using Theorem 3.2(d)), and \(\mu_w\) being the largest weak neighbourhood smaller than \(\mu\) would then immediately yield \(\overleftarrow{f \phi}(\exists_f) \leq \mu_w\). Hence, \((X, \mu_w) \xrightarrow{f} (Y, \phi)\) would become a preneighbourhood morphism, completing the proof.

Towards the proof of the assertion: since \(\phi\) is a weak neighbourhood structure, using Theorem 3.1.1 (page 12), **Remark 5** (page 12) and the adjunction \(\overleftarrow{f} \dashv \overrightarrow{f}\) by Theorem 2.2.2 (page 8), for each admissible subobject \(m \in \text{Sub}_M(X)\):

\(\overleftarrow{f \phi}(\exists_fm) = \overleftarrow{f}(\bigcup\{\phi(a) : a \in \phi(\exists_fm)\})\)
\[
= \bigvee \{ \tilde{f}\phi(a) : a \in \phi(\exists_j^1 m) \} = \bigcup \{ \tilde{f}\phi(a) : a \in \phi(\exists_j^1 m) \}.
\]

Hence for each \( p \in \tilde{f}\phi(\exists_j^1 m) \) there exists an \( a \in \phi(\exists_j^1 m) \) such that \( p \in \tilde{f}\phi(a) \).

Since \( \phi(a) \subseteq \phi(\exists_j f^{-1} a) \Rightarrow \tilde{f}\phi(a) \subseteq \tilde{f}\phi(\exists_j f^{-1} a) \) and \( a \in \phi(\exists_j m) \Rightarrow f^{-1} a \in f \phi(\exists_j m) \),
the statements \( f^{-1} a \in f \phi(\exists_j m) \) and \( p \in \tilde{f}\phi(\exists_j f^{-1} a) \) follow, showing \( \tilde{f}\phi(\exists_j) \) is inter-
polative, completing the proof using Theorem 3.1.1 (page 12).

3.2.2. Universal Neighbourhoods. Since the set \( \mathbf{bd}[X] \) of all neighbourhood structures on \( X \) is a complete lattice (Theorem 3.1.6, page 17), given any weak neighbourhood structure \( \mu \) on \( X \) one has the largest neighbourhood structure
\( \mu_n = \bigvee \{ \nu \in \mathbf{bd}[X] : \nu \leq \mu \} \) on \( X \) smaller than \( \mu \). If \( (X, \mu) \) is an internal weak neighbourhood space, \( (Y, \phi) \) is an internal
neighbourhood space, \( (X, \mu) \xrightarrow{f} (Y, \phi) \) is a preneighbourhood morphism such that \( f \) has preimage preserve join property then for any \( S \subseteq \mathbf{Sub}_M(X) \):

\[
\tilde{f}\phi(\exists_j (\bigvee s \in S \exists_j s)) = \tilde{f}\phi(\bigvee s \in S \phi(\exists_j s)) = \bigcup_{s \in S} \tilde{f}\phi(\exists_j s)
\]

shows \( \tilde{f}\phi(\exists_j) \) to be a neighbourhood structure on \( X \). Since \( f \) is a preneighbourhood
morphism, \( \tilde{f}\phi(\exists_j) \leq \mu \) implies \( \tilde{f}\phi(\exists_j) \leq \mu_n \), yielding:

**Theorem.** Given a preneighbourhood morphism \( (X, \mu) \xrightarrow{f} (Y, \phi) \) from the internal weak
neighbourhood space \( (X, \mu) \) to the internal neighbourhood space \( (Y, \phi) \) where \( f \) has the
preimage preserve join property, the preneighbourhood morphism \( (X, \mu_n) \xrightarrow{f} (Y, \phi) \) is the
unique morphism between internal neighbourhood spaces such that the diagram:

\[
\begin{array}{ccc}
(X, \mu) & \xrightarrow{f} & (X, \mu_n) \\
\downarrow f & & \downarrow f \\
(Y, \phi) & \xrightarrow{f} & (Y, \phi)
\end{array}
\]

commutes.

4. Categories of Neighbourhood Structures

**Definition.** The following categories are now stipulated.

(a) \( \text{pre}[\mathcal{A}] \) is the category of internal preneighbourhood spaces \( (X, \mu) \) and preneigh-
bourhood morphisms \( (X, \mu) \xrightarrow{f} (Y, \phi) \).

(b) \( \text{wNbd}[\mathcal{A}] \) is the full subcategory of \( \text{pre}[\mathcal{A}] \) consisting of all internal weak neigh-
bourhood spaces.

(c) \( \text{Nbd}[\mathcal{A}] \) is the subcategory of \( \text{wNbd}[\mathcal{A}] \) consisting of internal neighbourhood
spaces \( (X, \mu) \) and preneighbourhood morphisms \( (X, \mu) \xrightarrow{f} (Y, \phi) \) between internal
neighbourhood spaces where the morphism $X \xrightarrow{f} Y$ of $A$ has the preimage preserve joins property.

(d) $\text{Top}[A]$ is the full subcategory of $\text{Nbd}[A]$ consisting of internal topological spaces.

4.1. Reflective Subcategories.

Remark 17 Theorem 3.2.1 (page 20) exactly shows $\text{wNbd}[A]$ to be a bireflective full subcategory of $\text{pre}[A]$.

Remark 18 Let $\text{wNbd}[A]_{ppj}$ be the non-full subcategory of $\text{wNbd}[A]$ with internal weak neighbourhood spaces as objects and proneighbourhood morphisms $(X, \mu) \xrightarrow{f} (Y, \phi)$ for which the morphism $X \xrightarrow{f} Y$ of $A$ have the preimage preserve join property.

Theorem 3.2.2 (page 21) shows the category $\text{Nbd}[A]$ is a bireflective full subcategory of $\text{wNbd}[A]_{ppj}$.

**Theorem.** Let $A_{ppj}$ be the subcategory of $A$ consisting of all objects of $A$ and morphisms $X \xrightarrow{f} Y$ of $A$ which have the preimage preserve joins property.

Then, the following are equivalent:

(a) For every object $X$, there exists a largest internal topological structure on $X$.
(b) $\text{Top}[A]$ is a full bireflective subcategory of $\text{Nbd}[A]$.
(c) $\text{Top}[A]$ is topological over $A_{ppj}$.

**Proof.** (a) implies (b): Choose and fix an internal neighbourhood space $(X, \mu)$.

Since from the assumption $\text{top}[X]$ is a complete sublattice of $\text{nbd}[X]$ (Theorem 3.1.7, page 19), $\mu_t = \bigvee \{\nu \in \text{top}[X] : \nu \leq \mu\}$ is the largest internal topology on $X$ smaller than $\mu$. Hence, $(X, \mu) \xrightarrow{1_X} (X, \mu_t)$ is a bimorphism of $\text{Nbd}[A]$.

If $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a morphism of $\text{Nbd}[A]$ from the internal neighbourhood space $(X, \mu)$ to the internal topological space $(Y, \phi)$ then arguments similar to the one just before the statement of Theorem 3.2.2 (page 21) show $\xrightarrow{f \phi(\exists_j)}$ to be a neighbourhood structure on $X$. Furthermore using $\exists_j \dashv f^{-1} \dashv \forall_j$ (Theorem 2.1.4 (page 7) & Theorem 2.3(b) (page 9)):

$$p \in f \phi(\exists_j p) \iff (\exists q \in \phi(\exists_j p))(f^{-1} q \leq p) \iff (\exists q \in \phi(\exists_j p))(q \leq \forall_j p) \iff \forall_j p \in \phi(\exists_j p)$$
$$\iff (\exists u \in \mathcal{D}_\phi)(\exists_j p \leq u \leq \forall_j p) \iff (\exists u \in \mathcal{D}_\phi)(p \leq f^{-1} u \leq p) \iff (\exists u \in \mathcal{D}_\phi)(p = f^{-1} u),$$

shows:

$$\mathcal{D}_{f \phi(\exists_j)} = \{ f^{-1} u : u \in \mathcal{D}_\phi \}.$$
Since $O$ is a frame and $f^{-1}$ preserves all joins and meets, $\mathcal{O} \leftarrow f(\exists \gamma)$ is also a frame. Hence $f(\exists \gamma)$ is an internal topology on $X$, smaller than $\mu$, implying $f(\exists \gamma) \leq \mu t$.

Consequently, $(X, \mu t) \rightarrow (Y, \phi)$ is the unique morphism of internal topological spaces such that the diagram $(X, \mu) \xrightarrow{1_X} (X, \mu t)$ commutes in $Nbd[A]$.

(a) implies (c): Choose and fix a family $((X_i, \mu_i))_{i \in I}$ of internal topological spaces and a family $(X \xrightarrow{f_i} X_i)_{i \in I}$ of morphisms from $A_{\mathrm{ppj}}$.

Since for each $i \in I$, $f_i \mu_i(\exists \gamma_i) \in \text{top}[X]$ and from our assumption $\text{top}[X]$ is a complete lattice, $\mu = \bigvee_{i \in I} f_i \mu_i(\exists \gamma_i) \in \text{top}[X]$. Hence $(X, \mu) \rightarrow (X_i, \mu_i)$, for each $i \in I$, is a morphism of $\text{Top}[A]$.

If $(Z, \zeta)$ be an internal topological space and $Z \xrightarrow{g} X$ be a morphism of $A_{\mathrm{ppj}}$ such that for each $i \in I$, $(Z, \zeta) \xrightarrow{f_i \circ g} (X_i, \mu_i)$ is a morphism of $\text{Top}[A]$, then for each $z \in \text{Sub}_M(Z)$, $\zeta(z) \supseteq (f_i \circ g) \mu_i(\exists \gamma_i z) = \xi(g f_i \mu_i(\exists \gamma_i(\exists \gamma z))) \Rightarrow g \zeta(z) \supseteq f_i \mu_i(\exists \gamma_i(\exists \gamma z))$.

Hence, for each $z \in \text{Sub}_M(Z)$, $\mu(\exists \gamma z) \subseteq g \zeta(z) \iff g \mu(\exists \gamma z) \subseteq \zeta(z)$, implying $(Z, \zeta) \xrightarrow{g} (X, \mu)$ to be a morphism of internal topological spaces, and the unique one making each $(X, \mu) \xrightarrow{f_i} (X_i, \mu_i)$ $(i \in I)$ to commute, proving (c).

(b) implies (a): Assuming (b), given any internal topology $\mu$ on $X$, one has the diagram $(X, \uparrow) \xrightarrow{1_X} (X_i, \uparrow)$ to commute uniquely, yielding $\mu \leq \uparrow$, proving (a).

(c) implies (a): Assuming (c), consider the object $X$, the family $((X, \mu))_{\mu \in \text{top}[X]}$ of internal topological objects and the family $(1_X)_{\mu \in \text{top}[X]}$ of morphisms of $A_{\mathrm{ppj}}$.

From hypothesis, there exists a unique internal topological structure $\lambda X$ on $X$ such that for each $\mu \in \text{top}[X]$, $(X, \lambda X) \xrightarrow{1_X} (X, \mu)$ is a morphism of $\text{top}[X]$, implying $\mu \leq \lambda X$, for each $\mu \in \text{top}[X]$, proving (a).

\[ \square \]

4.2. Results on Topologicity.
Theorem.  
(a) The category $pNbd[A]$ of internal preneighbourhood spaces is topological over $A$.
(b) The category $wNbd[A]$ of internal preneighbourhood spaces is topological over $A$.
(c) The category $Nbd[A]$ of internal neighbourhood spaces is topological over $A_{ppj}$.

Proof. The proof follows from the facts:

- if $X \xrightarrow{f} Y$ is a morphism of $A$ and $(Y, \phi)$ is an internal preneighbourhood space (respectively, an internal weak neighbourhood space), then from the proof of Theorem 3.2.1 (page 20) $f\phi(\exists_j)$ is a preneighbourhood (respectively, weak neighbourhood) structure on $X$, 
- if $X \xrightarrow{f} Y$ is a morphism from $A_{ppj}$ and $(Y, \phi)$ is an internal neighbourhood space then using arguments just before Theorem 3.2.2 (page 21) $\xleftarrow{f} \phi(\exists_j)$ is a neighbourhood structure on $X$, and
- the sets $pnb[x], wnb[x]$ and $nb[x]$ of preneighbourhood structures, weak neighbourhood structures and neighbourhood structures on $X$ make a complete lattice — Theorem 3.1.2 (page 12), Theorem 3.1.6 (page 17) and Remark 14 (page 18).

All of this leads to the diagram in Figure 1 (page 37), which summarises the results obtained so far. While the general situation appears in Figure 1A, the picture is simplified when every morphism has preimage preserve join property (see Figure 1B). As a consequence of Corollary 2.3 (page 11) every lattice of admissible subobjects is a frame. This is the situation for $A = \text{Set}$, in particular.

5. Regular Epimorphisms of Internal Neighbourhood Spaces

5.1. Regular Epimorphisms of $pre[A]$.

Theorem. A morphism $(X, \gamma) \xrightarrow{f} (Y, \phi)$ of $pre[A]$ is a regular epimorphism if and only if the morphism $X \xrightarrow{f} Y$ is a regular epimorphism of $A$ and:

\[
\phi(y) = \{ u \in \text{Sub}_M(Y) : y \leq u \text{ and } f^{-1}u \in \gamma(f^{-1}y) \}.
\]

Proof.  

if part: Since $X \xrightarrow{f} Y$ is a regular epimorphism of $A$, it is the coequaliser of its kernel pair 

\[
\begin{array}{c}
\xymatrix{
\text{Kerp}[f] \ar[r]^-{p_2} & X \\
X \ar[u]^-{p_1} \ar[r]_-{f} & Y
}
\end{array}
\]
Since the forgetful functor $\mathbf{pre}[\mathbb{A}] \xrightarrow{U} \mathbb{A}$ creates kernel pairs, there exists a unique pre-neighbourhood $\kappa$ on $\text{Kerp}[f]$ such that $(\text{Kerp}[f], \kappa) \xrightarrow{p_2} (X, \gamma)$ is the kernel pair of $(X, \gamma) \xrightarrow{f} (Y, \phi)$.

Let $(X, \gamma) \xrightarrow{g} (Z, \zeta)$ be a pretopological morphism such that $g \circ p_1 = g \circ p_2$. Then:

- From the coequaliser in $\mathbb{A}$:

  \[
  \begin{array}{ccc}
  \text{Kerp}[f] & \xrightarrow{p_1} & X \\
  \downarrow{g} & \xrightarrow{f} & \downarrow{h} \\
  & \phantom{x} & Y
  \end{array}
  \]

  there exists the unique morphism $Y \xrightarrow{h} Z$ such that $g = h \circ f$.

- Choose and fix admissible subobjects $u, z$ of $Z$ with $u \in \zeta(z)$.

  Since $(X, \gamma) \xrightarrow{g} (Z, \zeta)$ is a pre-neighbourhood morphism, $g^{-1} u \in \gamma(g^{-1} z)$.

  But $u \in \zeta(z) \Rightarrow z \leq u \Rightarrow h^{-1} z \leq h^{-1} u$ and:

  \[
  g^{-1} u \in \gamma(g^{-1} z) \Leftrightarrow (h \circ f)^{-1} u \in \gamma((h \circ f)^{-1} z) \Leftrightarrow f^{-1}(h^{-1} u) \in \gamma(f^{-1}(h^{-1} z)),
  \]

  so that:

  \[
  u \in \zeta(z) \Rightarrow h^{-1} z \leq h^{-1} u \text{ and } f^{-1}(h^{-1} u) \in \gamma(f^{-1}(h^{-1} z)) \Rightarrow h^{-1} u \in \phi(h^{-1} z),
  \]

  proving $(Y, \phi) \xrightarrow{h} (Z, \zeta)$ to be a pre-neighbourhood morphism.

- Since $U$ is faithful, $(Y, \phi) \xrightarrow{h} (Z, \zeta)$ is the unique pretopological morphism which makes the diagram:

  \[
  \begin{array}{ccc}
  (\text{Kerp}[f], \kappa) & \xrightarrow{p_1} & (X, \gamma) \\
  \downarrow{g} & \xrightarrow{f} & \downarrow{h} \\
  & \phantom{x} & (Z, \zeta)
  \end{array}
  \]

  to commute in $\mathbf{pre}[\mathbb{A}]$.

Hence $(X, \gamma) \xrightarrow{f} (Y, \phi)$ is a regular epimorphism in $\mathbf{pre}[\mathbb{A}]$.

only if part: Since the forgetful functor $\mathbf{pre}[\mathbb{A}] \xrightarrow{U} \mathbb{A}$ preserve coequalisers, the coequaliser diagram:

\[
\begin{array}{ccc}
(Z, \zeta) & \xrightarrow{p} & (X, \gamma) \\
\downarrow{q} & \xrightarrow{f} & \downarrow{h} \\
& \phantom{x} & (Y, \phi)
\end{array}
\]

in $\mathbf{pre}[\mathbb{A}]$ is mapped to the coequaliser diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & X \\
\downarrow{q} & \xrightarrow{f} & \downarrow{h} \\
& \phantom{x} & Y
\end{array}
\]

in $\mathbb{A}$. In particular, $f$ is a regular epimorphism of $\mathbb{A}$, and since $(E, M)$-factorisation is proper, $\text{RegEpi}(\mathbb{A}) \subseteq \text{ExtEpi}(\mathbb{A}) \subseteq E$, implying $f \in E$. 
Define:

\[ \psi(y) = \left\{ v \in \text{Sub}_M(Y) : y \leq v \text{ and } f^{-1}v \in \gamma(f^{-1}y) \right\} \text{, for all } y \in \text{Sub}_M(Y). \]

Then:

- \( u \geq v \in \psi(y) \) implies \( u \geq v \geq y \) and \( f^{-1}u \geq f^{-1}v \in \gamma(f^{-1}y) \), since \( \gamma(f^{-1}y) \) is a filter.
  - Hence \( u \in \psi(y) \), showing \( \psi(y) \) is an upset.
- \( u, v \in \psi(y) \) implies \( u, v \geq y \) and \( f^{-1}u, f^{-1}v \in \gamma(f^{-1}y) \), implying \( u \land v \geq y \) and \( f^{-1}(u \land v) = f^{-1}u \land f^{-1}v \in \gamma(f^{-1}y) \). Hence \( u \land v \in \psi(y) \), showing \( \psi(y) \) to be closed under finite intersections.
- \( y \geq z \) and \( v \in \psi(y) \) implies \( v \geq y \geq z \) and \( f^{-1}v \in \gamma(f^{-1}y) \subseteq \gamma(f^{-1}z) \), so that in particular, \( v \geq z \) and \( f^{-1}v \in \gamma(f^{-1}z) \), i.e., \( v \in \psi(z) \). Hence \( \psi(y) \subseteq \psi(z) \), showing the assignment \( y \mapsto \psi(y) \) to be an order preserving map \( \text{Sub}_M(Y)^{\text{op}} \xrightarrow{\psi} \text{Fil}(Y) \).
- Since \( v \in \psi(y) \) implies \( v \geq y \), it follows that \( \psi \) is a pre-neighbourhood on \( Y \).
- Since \( f \) is a pretopological morphism, if \( u \in \phi(y) \) then \( f^{-1}u \in \gamma(f^{-1}y) \). Hence, \( u \geq y \) and \( f^{-1}u \in \gamma(f^{-1}y) \) implies \( u \in \psi(y) \). Hence \( \phi(y) \subseteq \psi(y) \), as filters and hence \( \phi \leq \psi \), as pre-neighbourhoods on \( Y \).
- From the very definition of \( \psi \), for any \( y \in \text{Sub}_M(Y) \) and a \( v \in \psi(y) \), \( f^{-1}v \) is in \( \gamma(f^{-1}y) \), so that \( (X, \gamma) \xrightarrow{f} (Y, \psi) \) is also a pretopological morphism.
- Since \( (X, \gamma) \xrightarrow{f} (Y, \psi) \) is a regular epimorphism in \( \text{pre} [\mathbb{A}] \), there exists a unique pretopological morphism \( (Y, \phi) \xrightarrow{h} (Y, \psi) \) making the diagram:

\[
\begin{array}{ccc}
(Z, \zeta) & \xrightarrow{p} & (X, \gamma) \xrightarrow{f} (Y, \phi) \\
\downarrow{q} & \downarrow{f} & \downarrow{h} \\
(Y, \psi) & \xrightarrow{t} & (Y, \psi)
\end{array}
\]

to commute in \( \text{pre} [\mathbb{A}] \).
- Since \( f \in \text{RegEpi}(\mathbb{E}) \), \( h \circ f = f = 1_Y \circ f \) in \( \mathbb{A} \) implies \( h = 1_Y \).
- Hence \( (Y, \phi) \xrightarrow{1_Y} (Y, \psi) \) is a pre-neighbourhood morphism, entailing \( \psi \leq \phi \).
  - Hence \( \phi = \psi \), completing the proof.

\[ \Box \]

Remark 19 The proof only requires the forgetful functor \( \text{pre} [\mathbb{A}] \xrightarrow{\phi} \mathbb{A} \) to create kernel pairs and preserve coequalisers. Theorem 4.2(a) provides much more than just this requirement.

Remark 20 If \( f^{-1}u \in \gamma(f^{-1}y) \) then \( f^{-1}y \leq f^{-1}u \Leftrightarrow \exists f^{-1}y \leq u \), and hence the extra restriction in (28) on page 24 is to ensure the description of regular epimorphisms in \( \text{pre} [\mathbb{A}] \).

Remark 21 The condition of a regular epimorphism \( f \) being stably in \( \mathbb{E} \) is necessary to ensure simpler description of the regular epimorphism \( (X, \gamma) \xrightarrow{f} (Y, \phi) \):

\[ u \in \phi(y) \Leftrightarrow f^{-1}u \in \gamma(f^{-1}y). \]
This is exactly the situation in case when $A = \text{Set}$.

5.2. **Regular Epimorphisms of $\text{Nbd}[A]$**. Given a morphism $(X, \gamma) \xrightarrow{f} (Y, \phi)$ of preneighbourhoods, the proof of Theorem 5.1 suggests:

$$\phi(y) = \{ u \in \text{Sub}_M(Y) : y \leq u \text{ and } f^{-1}u \in \gamma(f^{-1}y) \},$$

is actually a preneighbourhood on $Y$, $\phi \leq \psi$ and $(X, \gamma) \xrightarrow{f} (Y, \psi)$ is a morphism of preneighbourhoods.

Further:

$$\mathcal{O}_\psi = \{ u \in \text{Sub}_M(Y) : f^{-1}u \in \mathcal{O}_\gamma \},$$

is closed under arbitrary joins, if $\gamma$ preserve arbitrary meets. Hence, if $\gamma$ is a preneighbourhood which preserve arbitrary meets then $\text{int}_\psi$ is a Kuratowski operator. Consequently the smallest preneighbourhood $\hat{\psi}$ in the fibre $\text{int}_\psi$ of $\mathcal{O}_\gamma$ (Remark 18, page 22) yield similarly as in Theorem 5.1:

**Theorem.** A morphism $(X, \gamma) \xrightarrow{f} (Y, \phi)$ of $\text{Nbd}[A]$ is a regular epimorphism if and only if the morphism $X \xrightarrow{f} Y$ is a regular epimorphism of $\text{App}$ and:

$$\phi(y) = \{ u \in \text{Sub}_M(Y) : y \leq u \text{ and } f^{-1}u \in \gamma(f^{-1}y) \}.$$  

**Remark 22** In case where $\gamma$ is a neighbourhood and $f^{-1}$ preserve arbitrary meets then for any $S \subseteq \text{Sub}_M(Y)$:

$$u \in \psi(\bigvee S) \iff u \geq \bigvee S \text{ and } f^{-1}u \in \gamma\left(\bigvee_{s \in S} f^{-1}s\right) = \gamma(\bigvee_{s \in S} f^{-1}s),$$

$$\iff \bigvee S \leq u \text{ and } f^{-1}u \in \bigwedge_{s \in S} \gamma(f^{-1}s),$$

$$\iff (\forall s \in S)(s \leq u \text{ and } f^{-1}u \in \gamma(s))$$

$$\iff (\forall s \in S)(u \in \psi(s)) \iff u \in \bigcap_{s \in S} \psi(s),$$

shows $\psi$ to preserve meets. However, this does not guarantee whether $\psi$ is a neighbourhood.

5.3. **Hereditary Regular Epimorphisms**. Given any preneighbourhood $\gamma$ of an object $X$ of $A$ and an admissible subobject $P \xrightarrow{p} X$ of $X$ there exists from topologicity of the forgetful functor $\text{pre}[A] \xrightarrow{U} A$ a unique smallest preneighbourhood $\gamma_p$ on $P$ such that $(P, \gamma_p) \xrightarrow{p} (X, \gamma)$ is a preneighbourhood morphism. Indeed:

$$\gamma_p(m) = \{ u \in \text{Sub}_M(P) : (\exists w \in \gamma(p \circ m))(p \wedge w \leq p \circ u) \},$$

for all $m \in \text{Sub}_M(U)$, and is the preneighbourhood induced from $\gamma$.  


Theorem. Let $P \xrightarrow{\gamma} X$ be an admissible subobject of $X$.

(a) If $(X, \gamma)$ be an internal weak neighbourhood space of $\mathbb{A}$ then so also is $(P, \gamma_p)$.

(b) If $(X, \gamma)$ be an internal neighbourhood space of $\mathbb{A}$ and $\text{Sub}_M(X) \xrightarrow{P^{-1}} \text{Sub}_M(P)$ preserve joins the $(P, \gamma_p)$ is also an internal neighbourhood space of $\mathbb{A}$.

Proof. (a) Given any $m \in \text{Sub}_M(P)$, $u \in \gamma_p(m) \iff (\exists v \in \gamma(p \circ m))(p \land w \leq p \circ u)$. Since $\gamma$ is a weak neighbourhood, from Theorem 3.1.1 (page 12), there exists a $v \in \gamma(p \circ m)$ such that $w \in \gamma(v)$. Hence $p^{-1}v \in \gamma_p(m)$ and $p^{-1}w \in \gamma_p(p^{-1}v)$. Since $p^{-1}w \leq u, u \in \gamma_p(p^{-1}v)$, showing $\gamma_p$ to be interpolative.

Hence $\gamma_p$ is a weak neighbourhood on $P$.

(b) From (a) $\gamma_p$ is a weak neighbourhood on $P$. It remains to show that $\gamma_p$ preserves meets. Since for any family $(m_i)_{i \in I}$ of admissible subobjects of $P$, $\gamma_p(\bigvee_{i \in I} m_i) \subseteq \bigcap_{i \in I} \gamma_p(m_i)$, it is enough to show the other inequality.

If $u \in \bigcap_{i \in I} \gamma_p(m_i)$ then for each $i \in I$, there exists a $w_i \in \gamma(p \circ m_i)$ such that $p \land w_i \leq p \circ u$.

Let $w = \bigvee_{i \in I} w_i$. Since $p^{-1}$ preserve joins, using Theorem 2.3(a) (page 9) $p \land w = \bigvee_{i \in I} (p \land w_i) \leq p \circ u$. Hence:

$w \in \bigcap_{i \in I} \gamma(p \circ m_i) = \bigcap_{i \in I} \gamma(\exists_i m_i) = \gamma(\bigvee_{i \in I} m_i) = \gamma(\exists m_i (\bigvee_{i \in I} m_i)) = \gamma(p \circ (\bigvee_{i \in I} m_i))$,

implies $u \in \gamma_p(\bigvee_{i \in I} m_i)$.

Hence $\gamma_p$ is a neighbourhood on $P$.

As expected, a regular epimorphism of a category $\mathbb{X}$ would be hereditary if its restriction to every subobject of the codomain in $\mathbb{X}$ is also a regular epimorphism of $\mathbb{X}$. Explicit conditions for $\mathbb{X} = \text{pre}[\mathbb{A}], \text{Nbd}[\mathbb{A}]$ are obtained.

5.3.1. Hereditary Regular Epimorphisms of $\text{pre}[\mathbb{A}]$.

Definition. A regular epimorphism $(X, \gamma) \xrightarrow{f} (Y, \phi)$ of $\text{pre}[\mathbb{A}]$ is said to be hereditary if for every admissible subobject $P \xrightarrow{\gamma} Y$ of $Y$ the restriction $f_p$ of $f$ to $p$ in the pullback

$f^{-1}P \xrightarrow{f_p} P \xrightarrow{f} Y \xrightarrow{p} P$ is a regular epimorphism $(f^{-1}P, \gamma_{f^{-1}P}) \xrightarrow{f_p} (P, \phi_p)$ of $\text{pre}[\mathbb{A}]$.

Theorem. A pre-neighbourhood morphism $(X, \gamma) \xrightarrow{f} (Y, \phi)$ is a hereditary regular epimorphism if and only if for each $t \in \text{Sub}_M(Y)$ $f^{-1}T \xrightarrow{f_t} T$ is a regular epimorphism of $\mathbb{A}$.
and for any $u, v \in \text{Sub}_M(T)$:

\begin{equation}
(32) \quad (\exists p \in \gamma(f^{-1}(t \circ u)))(\exists v \in f^{-1}t \wedge p) \leq t \circ v \Rightarrow (\exists q \in \phi(t \circ u))(t \wedge q \leq t \circ v).
\end{equation}

Proof. Since $f^{-1}t \circ u = f^{-1}t \circ f_t^{-1}u, p \in \gamma(f^{-1}(t \circ u)) = \gamma((f^{-1}t) \circ f_t^{-1}u), \exists v \in \gamma(f^{-1}t \wedge p) \leq t \circ v \Leftrightarrow f^{-1}t \wedge p \leq (f^{-1}t) \circ (f_t^{-1}v)$. Hence the hypothesis of (32) in view of (31) is equivalent to $f_t^{-1}v \in \gamma f^{-1}t(f_t^{-1}u)$.

In view of (31) the consequent of (32) is equivalent to $v \in \phi_t(u)$.

Since $(f^{-1}T, \gamma f^{-1}_t) \xrightarrow{f_t} (T, \phi_t)$ is a pre-neighbourhood morphism, (32) is equivalent to stating:

$$v \in \phi_t(u) \Leftrightarrow f_t^{-1}v \in \gamma f^{-1}_t(f_t^{-1}u).$$

Further, if $X \xrightarrow{f} Y$ is hereditarily in $E$ then $f^{-1}v \in \gamma(f^{-1}u) \Rightarrow f^{-1}u \leq f^{-1}v \Leftrightarrow \exists v, f^{-1}u = u \leq v$.

The equivalence now follows from the description of regular epimorphisms of $\text{pre}[A]$ in Theorem 5.1 (page 24). \qed

5.3.2. Conditions ensuring hereditary regular epimorphisms of $\text{pre}[A]$. The regular epimorphisms of $\text{pre}[\text{Set}]$ are hereditary (see [BentleyHerrlichLowen1991]). The purpose of this and the next subsection is to explain this phenomena.

Consider the diagram in Figure 2 (page 38) for the admissible subobjects $p \in \text{Sub}_M(X)$ and $t \in \text{Sub}_M(Y)$. With respect to Figure 2:

- the blue arrows indicate morphisms from $M$ while the orange arrows indicate morphisms from $E$,
- the front, right and left hand vertical squares are pullback squares,
- the bottom horizontal square is the $(E, M)$-factorisation of $f \circ p$,
- hence the vertical squares on the right and left are completely in $M$, and
- the diagonal on the top horizontal square is the $(E, M)$-factorisation of $f \circ ((f^{-1}t)^{-1}p)$.

Since: $t \circ f_t \circ (f^{-1}t)^{-1}p = f \circ f^{-1}t \circ (f^{-1}t)^{-1}p = f \circ p \circ (f^{-1}t) = \exists p \circ f \mid p \circ (f^{-1}t)_p$, it follows from the right hand vertical pullback square the existence of a unique morphism $(f^{-1}t)^{-1} \xrightarrow{u} t^{-1} \exists \gamma \gamma P$ making the top horizontal and hind vertical squares to commute.

Since the vertical diagonal with vertices $((f^{-1}t)^{-1}P)-T-Y-P$ is a composite of the front and left vertical pullback squares, it is a pullback square; since this is also a composite of the hind and right hand vertical squares, and the right hand vertical square is a pullback, it follows that the hind vertical square is also a pullback square.
Further from the commutative square \((f^{-1}t)^{-1}P \xrightarrow{f_t\mid (f^{-1}t)^{-1}P} \exists_f (f^{-1}t)^{-1}P\), since
\[
\begin{array}{c}
\text{w} \\
\downarrow \downarrow \downarrow \\
\exists_f (f^{-1}t)^{-1}P
\end{array}
\begin{array}{c}
\text{t}^{-1}\exists_f P \\
\downarrow \downarrow \downarrow \\
\text{T}
\end{array}
\]
\[t^{-1}\exists_f P \xrightarrow{r} t^{-1}\exists_f P\]
making the whole diagram to commute.

Hence, the top left triangle on the top horizontal square yields a \((E, M)\)-factorisation of \(w\), entailing:

\[(33)\]
\[\exists_f (p \land f^{-1}t) = t \circ \exists_f (f^{-1}t)^{-1}P\quad \text{and} \quad f_{(p \cap f^{-1}T)} = f_t\mid (f^{-1}t)^{-1}P,\]
and from the existence of \(r\):

\[(34)\]
\[\exists_f (f^{-1}t)^{-1}P \leq t^{-1}\exists_f P \Leftrightarrow t \circ \exists_f (f^{-1}t)^{-1}P \leq \exists_f P \Leftrightarrow \exists_f (p \land f^{-1}t) \leq t \land \exists_f p.\]

**Definition.** An adjunction \(X \xrightarrow{f} Y\) between partially ordered sets is said to be a Frobenius pair if:

\[(35)\]
\[f(g(y) \land x) = y \land f(x), \quad \text{for all } x \in X, y \in Y.\]

If for a given morphism \(P \xrightarrow{f} Q\) of \(A\) the adjunction \(\exists_f \dashv f^{-1}\) is a Frobenius pair then \(f\) is a Frobenius morphism.

In case of the category \(\text{Set}\) of sets and functions every function is a Frobenius morphism. The discussion preceding the definition above produces equivalent formulations for Frobenius morphisms in categories with a proper factorisation system.

**Theorem.** The following are equivalent for any morphism \(X \xrightarrow{f} Y\) of \(A\):

\(a\) \(f\) is a Frobenius morphism.
(b) For every admissible subobject \( T \rightarrow Y \) of \( Y \) the diagram:

\[
\begin{array}{ccc}
\text{Sub}_M(X) & \xrightarrow{f} & \text{Sub}_M(Y) \\
(f^{-1}t)^{-1} \downarrow & & \downarrow t^{-1} \\
\text{Sub}_M(f^{-1}T) & \xrightarrow{f_i} & \text{Sub}_M(T)
\end{array}
\]

of order preserving maps commute.

(c) For every admissible subobject \( T \rightarrow Y \) of \( Y \) and \( P \rightarrow X \) of \( X \) the unique morphism \((f^{-1}t)^{-1} P \rightarrow t^{-1} \exists f, P \) in Figure 2 (see page 38) is in \( E \).

Remark 23 The \((E, M)\)-factorisation system is said to satisfy Beck-Chevalley condition if for every pullback diagram \( X \times_Z Y \xrightarrow{fg} Y \) the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow g_f & & \downarrow g \\
\text{Sub}_M(X) & \xrightarrow{f} & \text{Sub}_M(Z) \\
\downarrow g_f^{-1} & & \downarrow g^{-1} \\
\text{Sub}_M(X \times_Z Y) & \xrightarrow{\exists f, P} & \text{Sub}_M(Y)
\end{array}
\]

of order preserving maps commute.

The diagram in (b) of the Theorem is a special case for \( g \) an admissible subobject. Hence \( f \) is a Frobenius morphism if and only if the \((E, M)\)-factorisation system satisfies Beck-Chevalley condition for admissible subobjects of codomain of \( f \).

5.3.3. Five Conditions for Heredity of Regular Epimorphisms of \( \text{pre}[\mathbb{A}] \). The case for regular epimorphisms for \( \text{pre}[\text{Set}] \) to be hereditary is a consequence of every function being a Frobenius morphism. The heredity of regular epimorphisms in \( \text{pre}[\mathbb{A}] \) holds in fact with weaker conditions as the following theorem establishes.

Theorem. Given the statements:

(a) The set \( E \) is stable under pullbacks.
(b) The set \( E \) is hereditary.
(c) Every morphism of \( \mathbb{A} \) is Frobenius.
(d) Every morphism in \( E \) is Frobenius.
(e) Every regular epimorphism is Frobenius.
(f) Every regular epimorphism of \( \text{pre}[\mathbb{A}] \) is hereditary.
The following implications hold good:

\[(a) \implies (b) \]

\[\uparrow\]

\[(c) \implies (d) \implies (e) \implies (f)\]

**Proof.** Follows immediately from the diagram in Figure 2 (page 38) and Theorem 5.3.2 (page 30). The equivalence of (b) and (c) is obvious. \qed

**Remark 24** The obvious equivalence of (b) and (c) in Theorem 5.3.3 was also observed in [ClementinoGiuliTholen1996].

**Remark 25** It is known from [JanelTholen1994] that the condition (a) in Theorem 5.3.3 is equivalent to the \((E,M)\)-factorisation system satisfying the Beck-Chevalley condition.

In the case of \(\text{Set}\), since \(E = \text{Epi}\) is pullback stable, every regular epimorphism of the category \(\text{pre}[\text{Set}]\) of pretopological spaces is hereditary.

5.3.4. **Hereditary Regular Epimorphisms of \(\text{Nbd}[\mathbb{A}]\).** Since the morphisms of internal neighbourhood spaces have the preimage preserve join property, in view of Theorem 5.3(b) (page 28) it is best to restrict to the case when every morphism of \(\mathbb{A}\) has preimage preserve join property. Hence, from Corollary 2.3 (page 11), every lattice of admissible subobjects is a frame.

**Definition.** A regular epimorphism \((X, \gamma) \xrightarrow{f} (Y, \phi)\) of \(\text{Nbd}[\mathbb{A}]\) is said to be **hereditary** if for every admissible subobject \(P \xrightarrow{p} Y\) of \(Y\) the restriction \(f_p\) of \(f\) to \(p\) in the pullback diagram

\[
f^{-1}P \xrightarrow{f_p} P \quad \text{is a regular epimorphism } (f^{-1}P, \gamma f^{-1}P) \xrightarrow{f_p} (P, \phi_p) \text{ of } \text{pre}[\mathbb{A}].
\]

**Remark 26** If \((X, \gamma) \xrightarrow{f} (Y, \phi)\) is a hereditary regular epimorphism of \(\text{Nbd}[\mathbb{A}]\), then each restriction \((f^{-1}P, \gamma f^{-1}P) \xrightarrow{f_p} (Y, \phi_p)\) is a regular epimorphism of \(\text{Nbd}[\mathbb{A}]\). Hence each \(f_p \in \text{RegEpi}(\mathbb{A}) \subseteq E\) and each \(f_p\) preserves arbitrary joins.

Furthermore, for each \(p \in \text{Sub}_M(Y)\), \(\exists f^{-1}p = p\).

**Remark 27** For each \(p, q \in \text{Sub}_M(Y)\):

\[f^{-1}p \leq f^{-1}q \iff \exists f^{-1}p \leq q \implies p \leq q.\]

Hence:

\[p \leq q \iff f^{-1}p \leq f^{-1}q.\]

Since for any \(u \leq p \in \text{Sub}_M(Y)\), \(f^{-1}p \circ f_p^{-1}u = f^{-1}(p \circ u)\), the same holds for each \(f_p\).
Remark 28. Consequently, for each $p \in \text{Sub}_M(Y)$:

$$u \in \phi_p(m) \iff f_p^{-1}u \in \gamma_{f_p^{-1}}(f_p^{-1}m).$$

**Theorem.** Assume every morphism of $\mathcal{A}$ has the preimage preserve join property and $(X, \gamma) \xrightarrow{f} (Y, \phi)$ is a morphism of $\text{Nbd}[_\mathcal{A}]$.

(a) If $f$ is a hereditary regular epimorphism of $\text{Nbd}[_\mathcal{A}]$ then it is regular epimorphism of $\mathcal{A}$ such that for each $p \in \text{Sub}_M(Y)$, $f_p \in \text{RegEpi}(_\mathcal{A})$.

(b) If $f$ is a Frobenius morphism, $f_p \in \text{RegEpi}(_\mathcal{A})$ for each $p \in \text{Sub}_M(Y)$ and a regular epimorphism of $\mathcal{A}$ then it is a hereditary regular epimorphism of $\text{Nbd}[_\mathcal{A}]$.

(c) If $f$ is a regular epimorphism of $\mathcal{A}$ with the property that for each $p \in \text{Sub}_M(Y)$, $f_p \in \mathcal{E}$, then $f$ is a regular epimorphism of $\mathcal{A}$, if and only if, for every $y \in \text{Sub}_M(Y)$:

$$u \in \gamma(f^{-1}y) \Rightarrow \exists u \in \phi(y).$$

**Proof.**  
(a) Follows from Remark 27, Theorem 5.2 (page 27) & Theorem 5.1 (page 24).

(b) It is required to show for any $p \in \text{Sub}_M(Y)$ the morphism $(X, \gamma_{f^{-1}}) \xrightarrow{f_p} (Y, \phi_p)$ is a regular epimorphism of $\text{Nbd}[_\mathcal{A}]$.

Given $u, m, p \in \text{Sub}_M(Y)$, $u, m \leq p$, in light of Remark 28, it is enough to show:

$$f_p^{-1}u \in \gamma_{f_p^{-1}}(f_p^{-1}m) \Rightarrow u \in \phi_p(m).$$

Clearly, from equation (31) (page 27):

$$f_p^{-1}u \in \gamma_{f_p^{-1}}(f_p^{-1}m)$$

$$\iff \left( \exists t \in \gamma((f^{-1}p) \circ (f_p^{-1}m)) \right) (f_p^{-1}m \leq (f^{-1}p) \circ (f_p^{-1}m))$$

$$\iff \left( \exists t \in \gamma(f^{-1}(p \circ m)) \right) (f_p^{-1}m \leq f^{-1}(p \circ u))$$

$$\iff \left( \exists t \in \gamma(f^{-1}(p \circ m)) \right) (\exists t (f^{-1}m \leq p \circ u))$$

$$\iff \left( \exists t \in \gamma(f^{-1}(p \circ m)) \right) (p \leq \exists t \leq p \circ u) \quad \text{(since $f$ is Frobenius)}$$

$$\Rightarrow \exists t \in \text{Sub}_M(X) \left( f^{-1}(t \in \gamma(f^{-1}(p \circ m)) \text{ and } p \leq \exists t \leq p \circ u) \right) \quad \text{(since $\exists t \leq f^{-1}$)}$$

$$\Rightarrow \exists t \in \text{Sub}_M(X) \left( \exists t \in \phi(p \circ m) \text{ and } p \leq \exists t \leq p \circ u \right)$$

$$\Rightarrow u \in \phi_p(m),$$

completing the proof of (b).
(c) If \((X, \gamma) \xrightarrow{f} (Y, \phi)\) is a morphism of \(\text{Nbd}[A]\) then \(\exists t \vdash f^{-1} \vdash \forall \gamma\), where for any \(t \in \text{Sub}_M(X)\):

\[
\exists t = \bigwedge \{ p \in \text{Sub}_M(Y) : t \leq f^{-1} p \} \quad \text{and} \quad \forall t = \bigvee \{ q \in \text{Sub}_M(Y) : f^{-1} q \leq t \}.
\]

Hence for any \(t \in \text{Sub}_M(X)\):

\[
\forall t \leq \exists t \iff (\forall p, q \in \text{Sub}_M(Y)) \big( f^{-1} q \leq t \leq f^{-1} p \Rightarrow q \leq p \big).
\]

Since for each \(p \in \text{Sub}_M(Y)\), \(f_p \in E\), and Remark 27 shows the statement on the right hand side of (37) holds, and hence for all \(t \in \text{Sub}_M(X)\), \(\forall t \leq \exists t\).

Assume now for each \(y \in \text{Sub}_M(Y)\):

\[
u \in \gamma(f^{-1} y) \Rightarrow \exists u \in \phi(y).
\]

If \(f^{-1} p \in \gamma f^{-1} y\) then \(p = \exists f^{-1} p \in \phi(y)\), showing \(f\) to be a regular epimorphism of \(\text{pre}[A]\) using Remark 28 (page 33) and Theorem 5.1 (page 24).

Conversely if \(f\) be a regular epimorphism of \(\text{pre}[A]\) then:

\[
u \in \gamma(f^{-1} y) \Rightarrow f^{-1} y \leq u \iff y \leq \forall u \Rightarrow y \leq \forall u \leq \exists u,
\]

and \(u \leq f^{-1} \exists u \Rightarrow f^{-1} \exists u \in \gamma(f^{-1} y)\) implies \(\exists u \in \phi(y)\).

\[\square\]

Remark 29 In the case of \(\text{Set}\), neighbourhood morphisms \((X, \gamma) \xrightarrow{f} (Y, \phi)\) satisfying (36) are called pseudo open.

We could call a morphism of \(\text{Nbd}[A]\) pseudo open if it satisfies (36).

Hence: a morphism of \(\text{Nbd}[A]\) is a regular epimorphism of \(\text{pre}[A]\) if and only if it is a pseudo open regular epimorphism of \(A\) with each of its restrictions in \(E\).

6. Concluding Remarks

6.1. The category \(\text{Set}\). The concepts studied in this paper are well known for the category \(\text{Set}\) of sets and functions.

\(\text{Set}\) comes equipped with its usual \((Epi, Mono)-factorisation\) system. The lattice \(\text{Sub}_{Mono}(X)\) of admissible subobjects is a complete atomic Boolean algebra. Hence, \(\text{Top}[\text{Set}] = \text{Top} = \text{Nbd}[\text{Set}]\) is the category of topological spaces. The category \(\text{pre}[\text{Set}]\) is isomorphic to the category \(\text{preTop}\) of pretopological spaces. The category \(\text{preTop}\) is investigated in [BentleyHerrlichLowen1991] & [HerrlichLowenSchwarz1991].

Since epimorphisms in \(\text{Set}\) are pullback stable the regular epimorphisms of pretopological spaces are hereditary (see [BentleyHerrlichLowen1991] and compare Theorem 5.3.3, page 31). However, it is also known from [BentleyHerrlichLowen1991], that the regular epimorphisms of pretopological spaces are not in general pullback stable.
6.1.1. A Weak Neighbourhood which is not a Neighbourhood. Neighbourhoods in \textbf{Set} can be obtained by just specifying the filters for each point, since the subobject lattices are atomic. This is not true of weak neighbourhoods or preneighbourhoods. 

Given a set \( X \) and a topology \( \Theta \) on \( X \) let \( \Theta^c \) be the set of closed subsets of the topological space \((X, \Theta)\). Define:

\[
\mu(M) = \{ V \subseteq X : (\exists C \in \Theta^c)(M \subseteq C \subseteq V) \}.
\]

Clearly, \( \mu \) defines a preneighbourhood on \( X \) such that \( \mathcal{O}_\mu = \Theta^c \), and:

\[
\text{int}_\mu M = \bigcup \{ C \in \Theta^c : C \subseteq M \}, \\
\mu(M) = \bigcup \{ \uparrow C : C \in \Theta^c \text{ and } M \subseteq C \}.
\]

Hence \( \mu \) is a weak neighbourhood, and a neighbourhood if and only if \( \Theta^c \) is closed under arbitrary joins. Incidentally, under the same condition \( \text{int}_\mu \) becomes a Kurastowski interior.

6.2. The category \textbf{Top}. The category \textbf{Top} of topological spaces comes equipped with its usual \((\text{Epi}, \text{ExtMon})\)-factorisation system. The lattice \( \text{Sub}_{\text{ExtMon}}(X) \) is precisely the set of all subsets of \( X \) equipped with the subspace topology and hence again is a complete atomic Boolean algebra.

A preneighbourhood \( \text{Sub}_{\text{ExtMon}}(X)^{\text{op}} \xrightarrow{\mathcal{F}} \text{Fil}(X) \) on a topological space \( X \) is given on specifying for each \( T \subseteq X \) a filter \( \mathcal{F}_T \) of subspaces of \( X \) such that \( S \in \mathcal{F}_T \Rightarrow T \subseteq S \). Thus, for instance, taking all open sets (or, closed sets) containing \( T \) provides instances of two preneighbourhood structures on \( X \).

Since neighbourhoods are meet preserving and the subobject lattices are atomic, it is enough to specify the neighbourhoods of each \( x \in X \). Thus, neighbourhoods on \( X \) correspond to specifying a second topology on \( X \). Consequently, \( \text{Nbd[Top]} \) is isomorphic to the category \( \text{BiTop} \) of bitopological spaces and functions which are continuous with respect to both the topologies on \( X \).

6.3. The category \textbf{Loc}. The category \textbf{Loc} of locales comes equipped with its usual \((\text{Epi}, \text{RegMon})\)-factorisation system. The lattice \( \text{Sub}_{\text{RegMon}}(X) \) is a distributive complete lattice in which finite joins distribute over arbitrary meets, i.e., is a coframe. For any localic map \( X \xleftarrow{f} Y \) the preimage is \( \text{Sub}_{\text{RegMon}}(Y)^{\text{op}} \xrightarrow{\mathcal{F}} \text{Fil}(X) \), defined as the largest sublocale of \( X \) which is contained inside the subset \( f^{-1} S \) \((S \in \text{Sub}_{\text{RegMon}}(Y))\) of \( X \) (see [PicadoPultr2012]). It is known for every localic map \( f \) its preimage \( f^{-1} \) preserve, apart from arbitrary meets, finite joins (see [PicadoPultr2012]).

Recently in [DubeIghedo2016] & [DubeIghedo2016a] neighbourhoods have been effectively used. The neighbourhood used is \( \text{Sub}_{\text{RegMon}}(X)^{\text{op}} \xrightarrow{\alpha_X} \text{Fil}(X) \):

\[
\alpha_X(S) = \{ T \in \text{Sub}_{\text{RegMon}}(X) : (\exists a \in X)(S \subseteq \alpha(a) \subseteq T) \}.
\]
Since $X \xrightarrow{o} \text{Sub}_{\text{RegMon}}(X)$ preserves finite meets and arbitrary joins, for any $b \in X$ and any family $(a_i)_{i \in I}$ of elements of $X$:

$$o(b) \cap \bigvee_{i \in I} o(a_i) = o(b) \cap o\left(\bigvee_{i \in I} a_i\right) = o\left(b \land \bigvee_{i \in I} a_i\right)$$

$$= o\left(\bigvee_{i \in I} (b \land a_i)\right) = \bigvee_{i \in I} o(b \land a_i) = \bigvee_{i \in I} o\left(b \land o(a_i)\right),$$

shows $\text{OpenSub}(X)$ the set of open sublocales of $X$ is a frame. Further, since $o(a) \leq o(b) \iff a \leq b$, $X \xrightarrow{o} \text{OpenSub}(X)$ is an isomorphism of frames.

Since $\mathcal{O}_{oX} = \text{OpenSub}(X)$, it follows $(X, o)$ is actually an internal topological space of $\text{Loc}$.

Furthermore, for any frame homomorphism $X \xrightarrow{f} Y$, if $S \in \text{Sub}_{\text{RegMon}}(Y)$ is a sublocale of $Y$ and $T \in o_Y(S)$, then there exists a $b \in Y$ such that $S \subseteq o(b) \subseteq T$. Hence:

$$f^{-1}S \subseteq f^{-1}o(b) = o(f^*(b)) \subseteq f^{-1}T,$$

where $Y \xrightarrow{f^*} X$ is the left adjoint of $f$, which is a frame homomorphism. This implies $f^{-1}T \in o_X(f^{-1}S)$, yielding:

**Theorem.** The functor $\text{Loc} \xrightarrow{O} \text{pNbd}[\text{Loc}]$ defined by $O(X) = (X, o_X)$ is a right inverse to the forgetful functor $\text{pNbd}[\text{Loc}] \xrightarrow{U} \text{Loc}$.

### 6.4. Acknowledgments

I am indebted to:

1. T. Dube for supporting my research through NRF Funds for Research Chair here at Unisa.
2. M. Korostenski-Davies for painstakingly going through the draft version of this document and suggesting several editorial changes.
3. Z. Janelidze for his stimulating ideas during our talks on several occasions.
4. A. Razafindrakato, D. Holgate and their students for sharing their experiences.
(A) Categories of Neighbourhood Structures: the *forgetful* functors $U$, $V$, $W$ and $T$ (modulo its existence) are topological. The dotted lines are used to highlight extra necessary conditions.

(B) Categories of Neighbourhood Structures when every morphism has preimage preserve join property, as a consequence of which every lattice of admissible subobjects is a frame.

**Figure 1.** Summarising Categories of Neighbourhood Structures
Figure 2. Frobenius morphisms
INTERNAL NEIGHBOURHOOD STRUCTURES

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