On the $L^p$-$L^q$ estimates of the gradient of solutions to the Stokes problem

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Abstract - The paper is concerned with estimates of the gradient of the solutions to the Stokes IBVP both in a bounded and in an exterior domain. More precisely, we look for estimates of the kind $|\nabla v(t)|_q \leq g(t)|\nabla v_0|_p$, $q \geq p > 1$, for all $t > 0$ where function $g$ is independent of $v_0$.

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1 Introduction

We consider the Stokes initial boundary value problem in a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, that can be assumed bounded or exterior, whose boundary $\partial \Omega$ is supposed to be smooth:

\begin{equation}
\begin{aligned}
v_t - \Delta v &= -\nabla \pi v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, \\
v &= 0, \quad \text{on } (0, T) \times \partial \Omega, \\
v &= v_0, \quad \text{on } \{0\} \times \Omega.
\end{aligned}
\end{equation}

Several authors (see e.g. [5]-[7], [10]-[12],[15],[16],[20]-[21],[26],[31],[35]-[36]) have contributed to the study of semigroup properties of the Stokes operator associated to problem (1), and of the related $L^p$-$L^q$ estimates of solutions. In particular, for $q \in [p, \infty]$, set $\mu := \frac{n}{2p} \left( \frac{1}{p} - \frac{1}{q} \right)$, the following hold:

\begin{equation}
\begin{aligned}
\|v(t)\|_q &\leq c\|v(s)\|_p(t-s)^{-\mu}, & \text{for all } t-s > 0; \\
\|\nabla v(t)\|_q &\leq c\|v(s)\|_p(t-s)^{-\mu_1}, & \mu_1 := \begin{cases} 
\frac{1}{2} + \mu & \text{if } t-s \in (0, 1], \\
\frac{1}{2} + \mu & \text{if } t-s > 0, q \in [p, n], \\
\frac{1}{2p} & \text{if } t-s > 1, q \geq n;
\end{cases} \\
\|v_t(t)\|_q &\leq c\|v(s)\|_p(t-s)^{-\mu_2}, & \mu_2 := 1 + \mu, \quad \text{for all } t-s > 0; \\
\|D^2v(t)\|_q &\leq c\|v(s)\|_p(t-s)^{-\mu_3}, & \mu_3 := \begin{cases} 
1 + \mu & \text{if } t-s \in (0, 1], \\
1 + \mu & \text{if } t-s > 0, q \in [p, \frac{n}{2}], \\
\frac{n}{2p} & \text{if } t-s > 1, q \geq \frac{n}{2};
\end{cases}
\end{aligned}
\end{equation}

where the constant $c$ is independent of $v$ and, in a suitable sense, the exponents $\mu, \mu_i, i = 1-3$, are sharp (see Lemma 24 below and related references). More recently, also the case of...
the initial data in \( L^\infty(\Omega) \) has been considered by some authors (see [1]-[4],[8],[24],[27],[33]-[34]). In particular for \( n \geq 3 \) the following estimates hold:

\[
\begin{align*}
\|v(t)\|_\infty & \leq c\|v(s)\|_\infty, \quad t-s > 0, \\
\|\nabla v(t)\|_\infty & \leq c\|v(s)\|_\infty (t-s+1)^{-\frac{1}{2}}, \quad t-s > 0, \\
\|v_t(t)\|_\infty & \leq c\|v(s)\|_\infty (t-s)^{-1}, \quad t-s > 0,
\end{align*}
\]

(3)

where the constant \( c \) is independent of \( v_0 \) and again the estimate (3)2 for \( \nabla v \) is sharp (see [24]).

The aim of this paper is to study \( L^p-L^q \)-norm of the gradient of the solutions, that is, we look for estimates of the kind, \( q \geq p \) and \( p \in (1, \infty) \),

\[
\|\nabla v(t)\|_q \leq g(t)\|\nabla v_0\|_p, \quad \text{for all } t > 0,
\]

(4)

where \( g(t) \) is independent of \( v_0 \). As far as we know, the literature related to the previous question is not wide. In the case of the Stokes operator, for any domain which is sufficiently regular, estimate (4) holds for \( p = 2 \) (see e.g. [22, 23]). Moreover, for all \( p \in (1, \infty) \), making use of the representation formula of the solutions, estimate (4) holds in the case of solutions to the Cauchy problem and of the IBVP in the half-space [29], and recently, in the interesting paper [25], the result is achieved for \( p = \infty \) (see Proposition 3.2). Even the heat equation has only few results. In [13, 14], for solutions to the \( p \)-parabolic equation, the authors obtain some special results which are related to some bounded domains. More precisely, in [13] the author considers the heat equation (that is \( p = 2 \)), with homogenous Dirichlet boundary condition or homogeneous Neumann boundary condition, and proves that the function \( e^{\lambda_p t}\|\nabla u(t)\|_p \) is non increasing, for all \( p \in (1, \infty) \). The constant \( \lambda_p \) is the minimal eigenvalue of a suitable boundary value problem associated to \( -\Delta \), where \( \lambda_p \) can be negative (e.g. if \( \Omega \subset \mathbb{R}^2 \) is multiconnected). Finally, estimate (4) is proved with \( g(t) = c \) in the case of \( q = p \in \left[ \frac{3}{3}, \frac{3}{3} \right) \), for a suitable \( p > 2 \). In the paper [14] there is an extension of the results proved in [13] to the solutions to the \( p \)-parabolic heat equation.

Before going into the results of this paper, we point out that, beyond the intrinsic interest related to the Stokes problem, the paper is motivated by the fact that the results allow us to extend to the three-dimensional initial boundary value problem some results of the ones obtained in [28, 29] for the 2D-Navier-Stokes, in particular furnishing weak solutions for non decaying data.

In order to state our chief result we introduce the set of the hydrodynamic test functions \( \mathcal{C}_0(\Omega) \) and, for \( p \in (1, \infty) \), \( J_0^p(\Omega) := \text{completion of} \mathcal{C}_0(\Omega) \) with respect to the seminorm \( \|\nabla \cdot \|_p \) (norm for \( p \in (1, n) \)).

We are able to prove

**Theorem 1.** Let \( \Omega \) be a bounded domain. Let \( v_0 \in J_0^p(\Omega) \). Then there exists a unique solution to problem (1) such that, for all \( T > 0 \),

\[
v \in C([0,T; J_0^p(\Omega)) \), \quad t^\frac{4}{p} v_t, \quad t^\frac{2}{p} D^2 v, \quad t^\frac{2}{p} \nabla \pi_v \in L^\infty(0,T; L^p(\Omega)) \).
\]

(5)
For $q \geq p$, set $\mu := \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)$, the following hold with a constant $c$ independent of $v$:

$$
\| \nabla v(t) \|_q \leq c(t - \sigma)^{-\mu} \exp[-\gamma(t - \sigma)] \| \nabla v(\sigma) \|_p, \text{ for all } t > \sigma \geq 0,
$$

$$
\| v_t(t) \|_q \leq c(t - \sigma)^{-\frac{1}{4} - \mu} \exp[-\gamma(t - \sigma)] \| \nabla v(\sigma) \|_p, \text{ for all } t > \sigma \geq 0.
$$

(6)

**Theorem 2.** Let $\Omega$ be an exterior domain. Let $v_0 \in J^p_0(\Omega)$. Then there exists a unique solution to problem (1) such that, for all $T > 0$,

$$
v \in C([0, T]; J^p_0(\Omega)), \ t^\frac{1}{2} v_t, t^\frac{1}{2} D^2 v, t^\frac{1}{2} \nabla v, v \in L^\infty(0, T; L^p(\Omega)).
$$

(7)

For $q \geq p$, set $\mu := \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)$, the following hold with a constant $c$ independent of $v$:

$$
\| \nabla v(t) \|_q \leq c g_p(t - \sigma) \| \nabla v(\sigma) \|_p, \text{ for all } t > \sigma \geq 0,
$$

(8)

where

$$
g_p(t - \sigma) :=
\begin{cases}
\frac{1}{\varepsilon}, & t - \sigma > 0, \quad \text{if } q = p = 2, \\
(t - \sigma)^{-\mu}, & t - \sigma \in (0, 1), \\
(t - \sigma)^{\frac{1}{2} - \mu}, & t - \sigma > 1, \quad \text{if } p \neq 2, \ n = 2, \\
\log^{\frac{2}{3}}(t - \sigma + e), & t - \sigma \geq 1, \quad \text{if } q = p = n = 3, \\
\log(t - \sigma + e), & t - \sigma \geq 1, \quad \text{if } q > p = n = 3, \\
(t - \sigma)^{\frac{1}{2} - \frac{\delta}{2}}, & t - \sigma \geq 1, \quad \text{if } q \geq p = n > 3,
\end{cases}
$$

(9)

and, for $n \geq 2$, $p \in (1, n)$,

$$
g_p(t - \sigma) :=
\begin{cases}
1, & t - \sigma \geq 1, \quad \text{if } q \geq p = 2, \ n = 2, \\
(t - \sigma)^{-\mu}, & t - \sigma \geq 1, \quad \text{if } q \in [p, n), \ n > 2, \\
(t - \sigma)^{\frac{1}{2} - \frac{\delta}{2}}, & t - \sigma \geq 1, \quad \text{if } q \geq n, \ p \in (1, n), \ n \geq 3, \ p \neq 3, \\
(t - \sigma)^{\frac{1}{2} - \frac{\delta}{2}}, & t - \sigma \geq 1, \quad \text{if } q = n = 3, \ \delta \in (0, \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)].
\end{cases}
$$

(10)

Finally, for $q \geq p > 1$, we have

$$
\| v_t(t) \|_q \leq c(t - \sigma)^{-\frac{1}{2} - \mu} \| \nabla v(\sigma) \|_p, \text{ for all } t > \sigma \geq 0.
$$

(11)

The following result ensures that in a suitable sense the estimates (8) for $t \geq 1$ with $g_p$ defined by (10)$_2$–(10)$_4$ are sharp in suitable sense and that $g_p$ in (9)$_5$–(9)$_6$ for $q > p$ cannot be substituted with a $\xi(t)$ such that $t^{-1} \xi(t) \in L^1(t_0, \infty)$, $t_0 \geq 1$.

**Proposition 1.** For the solutions of Theorem 2, we get

i. for $n > 2$ and $p \in [\frac{2}{3}, n)$, estimate (9) with $g_p$ defined by (10)$_2, 3$ is sharp, in the sense that there is no function $\xi(t)$ such that

$$
\xi(t)^{t^{-1}} \in L^1(t_0, \infty) \text{ and } \| \nabla v(t) \|_{L^q(\Omega \cap S_R)} \leq \xi(t) \| \nabla v_0 \|_p \left( t^\frac{1}{2} - \frac{\delta}{2} \right),
$$

and for $q = n = 3$, $\delta \in (0, \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)]$,

$$
\xi(t)^{t^{-1} - \delta} \in L^1(t_0, \infty) \text{ and } \| \nabla v(t) \|_{L^q(\Omega \cap S_R)} \leq \xi(t) \| \nabla v_0 \|_p t^\frac{1}{2} - \frac{\delta}{2} - \frac{\delta}{2},
$$

where $R > diam(\Omega^c)$ and $\xi$ are independent of $v$. 

ii. for all \( q \geq p \geq n > 2 \) and \( q \geq p > 2, n = 2 \), on the right hand side of the estimate (9) the function \( g_p \) cannot be substituted by a function \( \xi(t) \) such that \( \xi(t) = o(1) \).

We assume \( \Omega \subset \mathbb{R}^n \). Actually in the case of \( \Omega = \mathbb{R}^n \) the result of Theorem 1 is improved by estimate (53) of Lemma 11. An analogous remark holds for the case of the half-space for which we refer to [29] and, for \( q = p = \infty \), to [25].

The paper is developed on the wake of the technique adopted in [31]. Actually the arguments are essentially based on the Green identity (15) (below) related to the Stokes problem. Initially, we consider the solution \( U \) to the Stokes Cauchy problem (52) (below) with data \( v_0 \) extended to zero in \( \mathbb{R}^n - \Omega \). The use of the representation formula simplifies the realization of the task for the field \( U \). Then we study the Stokes initial boundary value problem (59) (below) related to \( u := v - U \) with homogeneous initial data and boundary data \( -U \). This approach leads to estimate the function \( u \) simply by making use of the Green identity (15) (below) written by means of the adjoint problem. The Green identity (15) involves only the boundary value, that is the trace on \( (0, T) \times \partial \Omega \) of the field \( U \), and the trace on \( (0, T) \times \partial \Omega \) of the stress tensor of the adjoint problem that, roughly speaking, obeys the usual \( L^p - L^q \) estimates of solutions.

By Theorem 1, \( \Omega \) bounded domain, we completely realize our aims. Of course, our theorem include the results of paper [13] related to the heat equation with Dirichlet boundary condition.

By Theorem 2, \( \Omega \) exterior domain, if we consider \( p \in (1, n), n > 2 \), the results are, roughly speaking, in line with expectations. In the case of \( n = 2 \) the result is weaker.

For \( p \geq n \), the \( L^p - L^p \) estimates of Theorem 2 furnish just a result of continuous dependence. We are neither able to prove that \( L^p - L^p \) estimates holds with \( g_p \in L^\infty(0, \infty) \) nor that they do not hold for a such \( g_p \). In the case of \( L^p - L^q \) estimates, for \( t \in (0, 1) \) the function \( g_p(t) \) in (8) has the right dimensional balance, for \( t > 1 \), the function \( g_p(t) \) in (8) is growing. However Proposition 1 ensures that no decay is possible\footnote{The present statement ii. of Proposition 1 improves a result previously stated by the author. It has been achieved by the author during a conversation with Prof. G.P. Galdi. Actually, by a comment on the results of the paper G.P. Galdi implicitly makes to realize the proof of the statement.}. Different is the case of \( p \in [\frac{n}{2}, n) \), where the sharpness holds as the one expressed by means of (2)\textsubscript{2} and (3)\textsubscript{2} in the case of the ordinary \( L^p - L^q \) estimates.

The plan of the paper is the following. In sect. 2 we give some notation and preliminary results concerning the trace spaces and the Gagliardo-Nirenberg inequalities, and the solutions to the Stokes problem. In sect. 3 we study the Stokes Cauchy problem assuming \( v_0 \in J_0(\mathbb{R}^n) \). In sect. 4 we study a special auxiliary Stokes IBVP. In sect. 5 we furnish some implications of the results proved in sect. 3 and in sect. 4. Finally in sect. 6 we are able to furnish the proof of Theorem 1, Theorem 2 and of Proposition 1.

## 2 Some notation and preliminary results

The following spaces of completion will be considered: \( J^p(\Omega) := \text{completion of} \mathcal{C}_0(\Omega) \) with respect to the \( L^p \)-norm, \( J^{1,p}(\Omega) := \text{completion of} \mathcal{C}_0(\Omega) \) with respect to the \( W^{1,p}(\Omega) \), and, as in the introduction, \( J^p_0(\Omega) := \text{completion of} \mathcal{C}_0(\Omega) \) with respect to \( |\nabla \cdot | \_p \). We refer the reader to [18] Theorem 6.1 (p.68) for some properties related to the functions belonging to \( J^p_0(\Omega) \).
We adopt the notations: \((u, v) := \int u \cdot v \, dx\) and \((u, v)_{\partial \Omega} := \int u \cdot v \, d\omega\), respectively to mean the integral on the domain \(\Omega\) and the one on the surface \(\partial \Omega\) related to the product of two functions \(u, v\). The normal to the boundary is denoted by \(\nu\).

By the symbol \(< h >_r^\lambda\) we mean the seminorm:

\[
\lambda \in (0, 1), \quad \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|h(x) - h(y)|^r}{|x - y|^{n-1+\lambda r}} \, d\omega_x \, d\omega_y \right)^{\frac{1}{r}},
\]

that for \(\lambda = 1 - \frac{1}{r}\) furnish the classical one of the trace space \(W^{1-\frac{1}{r}}(\partial \Omega)\), that we consider normed by the functional

\[
\|h\|_{1 - \frac{1}{r}, r} := \|h\|_{L^r(\partial \Omega)} + < h >_r^{1 - \frac{1}{r}}.
\]

We recall (see e.g. [17]) that, for all \(\text{Lipchitz domain } D\) such that \(\overline{D} \cap \partial \Omega = \partial \Omega\), we get

\[
\begin{align*}
\|h\|_{L^r(\partial \Omega)} &\leq c_1 \|h\|_{L^r(D)} + \|h\|_{L^r(D)}^{\frac{1}{r}} \|\nabla h\|_{L^r(D)}^{\frac{1}{r}} , \\
\|h\|_{1 - \frac{1}{r}, r} &\leq c_2 \|h\|_{L^r(D)} + \|\nabla h\|_{L^r(D)} ,
\end{align*}
\]

with \(c\) independent of \(h \in W^{1-\frac{1}{r}}(D)\). By the symbol \(W^{-\frac{1}{r}}(\partial \Omega)\) we mean the dual space of \(W^{1-\frac{1}{r}}(\partial \Omega)\).

We denote by \(T(w, \pi_w)\) the newtonian stress tensor for a soleinodal field, and recall that \(\nabla \cdot T(w, \pi_w) = \Delta w - \nabla \pi_w\).

A key tool in our proof is the Green identity. If \((\varphi, \pi_\varphi)\) is a solutions to system \((1)_1\), we define \(\hat{\varphi}(t, x) := \varphi(t - \tau, x)\) and \(\pi_{\hat{\varphi}} := \pi_\varphi(t - \tau, x)\). It is known that \((\hat{\varphi}, \pi_{\hat{\varphi}})\) is a solution to the adjoint problem on \((0, t) \times \Omega\), that is

\[
\begin{align*}
\hat{\varphi}_{\tau} + \Delta \hat{\varphi} - \nabla \hat{\varphi} &= 0, \quad \nabla \cdot \hat{\varphi} = 0, \text{ in } (0, t) \times \Omega, \\
\hat{\varphi} &= 0 \text{ on } (0, t) \times \partial \Omega, \\
\hat{\varphi} &= \varphi_0 \text{ on } \{t\} \times \Omega.
\end{align*}
\]

Multiplying the first equation of \((u, \pi_u)\) by \(\hat{\varphi}\), and after integrating by parts on \((s, t) \times \Omega\), we get the Green identity:

\[
(u(t), \varphi(0)) + \int_{s}^{t} \left( \hat{\varphi}, \nu \cdot T(u, \pi_u) \right)_{\partial \Omega} \, d\tau
\]

\[
= (u(s), \varphi(t - s)) + \int_{s}^{t} \left( u, \nu \cdot T(\hat{\varphi}, \pi_{\hat{\varphi}}) \right)_{\partial \Omega} \, d\tau,
\]

where the symbol \(\nu\) denotes the normal on \(\partial \Omega\) and \(s \in [0, t)\).

We recall some results that will be crucial for our aims.

**Lemma 1.** Let \(\Omega\) be a bounded domain with the cone property. Let \(m \in \mathbb{N}\) and let \(r \in [1, \infty)\) and \(q \in [1, \infty]\). Let \(u \in L^q(\Omega)\) and, for \(|\alpha| = m\), \(D^\alpha u \in L^r(\Omega)\). Then there exists a constant \(c\) independent of \(u\) such that

\[
\|D^\beta u\|_p \leq c \|D^\alpha u\|_r^\alpha \|u\|^\beta_{q} + c_0 \|u\|_q ,
\]

\[(16)\]
provided that for $j := |\beta|$ the following relation holds:

$$\frac{1}{p} = \frac{1}{n} + a \left( \frac{1}{p} - \frac{m}{n} \right) + (1 - a)\frac{1}{q},$$

with $a \in \left[ \frac{1}{m}, 1 \right]$ either if $p = 1$ or if $p > 1$ and $m - j - \frac{n}{r} \notin \mathbb{N} \cup \{0\}$, while $a \in \left[ \frac{1}{m}, 1 \right]$ if $p > 1$ and $m - j - \frac{n}{r} \in \mathbb{N} \cup \{0\}$. Finally, if $u \in W^{m,r}_0(\Omega)$, then we can set $c_0 = 0$ in (16).

**Lemma 2.** Let $\Omega$ be an exterior domain with the cone property. Let $m \in \mathbb{N}$ and let $q, r \in [1, \infty)$. Let $u \in L^q(\Omega)$ and, for $|\alpha| = m$, $D^\alpha u \in L^r(\Omega)$. Then there exists a constant $c$ independent of $u$ such that

$$\|D^\beta u\|_p \leq c\|D^\alpha u\|_q^{|\alpha| - |\beta|},$$

(17)

provided that for $j := |\beta|$ the following relation holds:

$$\frac{1}{p} = \frac{1}{n} + a \left( \frac{1}{p} - \frac{m}{n} \right) + (1 - a)\frac{1}{q},$$

with $a \in \left[ \frac{1}{m}, 1 \right]$ either if $p = 1$ or if $p > 1$ and $m - j - \frac{n}{r} \notin \mathbb{N} \cup \{0\}$, while $a \in \left[ \frac{1}{m}, 1 \right]$ if $p > 1$ and $m - j - \frac{n}{r} \in \mathbb{N} \cup \{0\}$.

The above lemma, proved in [9], gives an interpolation inequality of Gagliardo-Nirenberg’s type (16) with $c_0 = 0$ in exterior domains. The difference with respect to the usual result is the fact that the function $u$ does not belong to a completion space of $C_0^\infty(\Omega)$.

**Lemma 3.** Let $D^2 u \in L^q(\Omega)$ and, for all bounded $\Omega' \subset \Omega$ such that $\partial \Omega \cap \partial (\Omega - \Omega') = \emptyset$, assume that $u \in W^{1,q}(\Omega')$ with zero trace on $\partial \Omega$. Finally, assume that $\nabla \cdot u = 0$ almost everywhere. Then there exists a pressure field $\pi_u$ and a constant $c$ independent of $u$ such that

$$\|D^2 u\|_q + \|\nabla \pi_u\|_q + \|u\|_{W^{1,q}(\Omega')} \leq c(\|P_q \Delta u\|_q + \|u\|_{L^q(\Omega')}).$$

(18)

If $\Omega$ is a bounded domain, then we get

$$\|D^2 u\|_q + \|\nabla \pi_u\|_q + \|u\|_{W^{1,q}(\Omega)} \leq c\|P_q \Delta u\|_q,$$

(19)

with $c$ independent of $u$ and depending on $\Omega$.

**Proof.** For the proof see for example [18] or [30].

Let us consider the Stokes homogeneous problem

$$- \Delta V + \nabla \pi_V = 0, \quad \nabla \cdot V = 0 \text{ on } \Omega, \quad \text{with } V = 0 \text{ on } \partial \Omega.$$  

(20)

**Theorem 3.** Let $\Omega$ be an exterior domain. Let $p \geq n > 2$ or $p > 2$ if $n = 2$. Then problem (20) admits a regular non trivial solution $(V, \pi_V) \in \mathbb{V}_0^p(\Omega) \times L_p(\Omega)$.

**Proof.** See Lemma 5.1 in [18].

**Lemma 4.** Let $\Phi \in C(\overline{\Omega}) \cap L^q(\Omega)$ with $\int_{\partial \Omega} \Phi \cdot \nu d\sigma = 0$. Assume that $\nabla \cdot \Phi = 0$ in weak sense. If the following holds

$$|\langle \Phi, v_0 \rangle| \leq M\|v_0\|_{q'}, \text{ for all } v_0 \in \mathcal{C}_0(\Omega),$$

then there exists a constant $c$ independent of $\Phi$ such that

$$\|\Phi\|_q \leq c(M + \|\gamma_{q'}(\Phi \cdot \nu)\|^{-\frac{1}{q'}}) \|\Phi\|_q.$$  

(21)
Proof. By virtue of the Helmholtz decomposition, for all \( \psi \in C_0(\Omega) \) we get \( \psi = P\psi + \nabla \varphi \) with \( \|P\psi\|_{q'} + \|\nabla \varphi\|_{q'} \leq c\|\psi\|_{q'} \). Hence we get
\[
|\langle \Phi, \psi \rangle| \leq |\langle \Phi, P\psi \rangle| + |\langle \Phi, \nabla \varphi \rangle| =: I_1 + I_2.
\] (22)
By the assumption we deduce \( I_1 \leq M\|\psi\|_{q'} \). Instead for \( I_2 \), via the assumption of zero flux for \( \Phi \), applying the trace theorem, we get
\[
I_2 = |\langle \Phi \cdot \nu, \Pi\psi - \Pi\varphi \rangle_{\partial\Omega}| \leq \|\gamma_{tr}(\Phi \cdot \nu)\|_{\frac{1}{q'},q} \|\Pi\psi - \Pi\varphi\|_{1-q',q'} \leq c\|\gamma_{tr}(\Phi \cdot \nu)\|_{\frac{1}{q'},q} \|\nabla \varphi\|_{q'} \leq c\|\gamma_{tr}(\Phi \cdot \nu)\|_{\frac{1}{q'},q} \|\psi\|_{q'},
\]
where we applied the Poincaré inequality after setting \( \Pi\psi := \frac{1}{|\Omega'|} \int_{\Omega'} \pi_v \, dx \) with \( \Omega' \subset \Omega \) and \( \partial(\Omega' - \Omega') \cap \partial\Omega = \emptyset \). Estimating the right hand side of (22) by means of the estimates deduced for \( I_1 \) and \( I_2 \), since \( \psi \) is arbitrary we easily arrive at (21).

Lemma 5. Let \( n \geq 3 \) and \( \Phi \in L^p(\Omega) \), \( p > \frac{n}{n-1} \). Assume that \( \nabla \cdot \Phi = 0 \) in weak sense and
\[
|\langle \Phi, v_0 \rangle| \leq M\|v_0\|_{q'}, \quad \text{for all } v_0 \in C_0(\Omega),
\]
for some \( q' > p' \), then \( \Phi \in L^q(\Omega) \).

Proof. See Lemma 2.6 p.406 of [31].

Concerning the Stokes problem (1) we recall the following

Lemma 6. Let \( s \in (1, \infty) \). For all \( \varphi_0 \in J^s(\Omega) \) there exist a unique solution to problem (1) such that
\[
\eta > 0, \quad \varphi \in C([0,T),J^s(\Omega)) \cap L^\infty(\eta,T;J^{1-s}(\Omega) \cap W^{2,s}(\Omega)),
\]
\[
\nabla \pi_\varphi, \varphi_1 \in L^\infty(\eta,T;L^s(\Omega)).
\] (23)
Moreover, for \( q \in [s, \infty] \) and \( t > \sigma \geq 0 \), set \( \mu := \frac{n}{2s} \left( \frac{1}{q} - \frac{1}{q'} \right) \), we get
\[
\|\varphi(t)\|_q \leq c\|\varphi(\sigma)\|_s(t-\sigma)^{-\mu}, \quad \text{for all } t - \sigma > 0;
\]
\[
\|
abla \varphi(t)\|_q \leq c\|\varphi(\sigma)\|_s(t-\sigma)^{-\mu_1}, \quad \mu_1 := \begin{cases} \frac{1}{2} + \mu & \text{if } t - \sigma \in (0,1], \\ \frac{1}{2} + \mu & \text{if } t - \sigma > 1, q \leq n, \\ \frac{1}{2} & \text{if } t - \sigma > 1, q \geq n; \end{cases}
\]
\[
\|\varphi_t(t)\|_q \leq c\|\varphi(\sigma)\|_s(t-\sigma)^{-\mu_2}, \quad \mu_2 := 1 + \mu, \quad \text{for all } t - \sigma > 0;
\]
\[
\|D^2 \varphi(t)\|_q \leq c\|\varphi(\sigma)\|_s(t-\sigma)^{-\mu_3}, \quad \mu_3 := \begin{cases} 1 + \mu & \text{if } t - \sigma \in (0,1], \\ 1 + \mu & \text{if } t - \sigma > 1, q < \frac{n}{2}, \\ \frac{1}{2} & \text{if } t - \sigma > 1, q \geq \frac{n}{2}; \end{cases}
\] (24)
where the constant \( c \) is independent of \( \varphi_0 \) and the exponent \( \mu_1 \) is sharp for \( s \geq \frac{n}{2} \), \( n \geq 3 \) in the sense that there is no function \( \xi(t) \) such that
\[
t^{-1}\xi(t) \in L^1(t_0, \infty) \text{ and } \|
abla \varphi(t)\|_{L^q(\Omega \cap S_R)} \leq \xi(t)t^{-\mu}\|\varphi_0\|_s,
\] (25)
where $R > \text{diam}(\Omega^c)$ and $\xi$ are independent $\varphi$. Finally, for all $s \in (1, \infty)$ and $\varphi_0 \in J^s(\Omega)$ the following limit property holds:

$$\lim_{t \to +\infty} \|\varphi(t)\|_s = 0. \quad (26)$$

**Proof.** With exception of (24)$_1$ in the case of $n = 2$ and $q = \infty$, for which we refer to [10, 11], the claims of the lemma are essentially the ones proved in [31]. Estimate (24)$_4$ is contained in [31] but it is not stated in no theorem. However, after remarking that $P\Delta \varphi = \varphi_t$, for the task it is enough to apply estimate (18) and suitably estimates (24)$_{1, 2, 3}$. As well the optimality expressed by (25) is an improvement of the ones given in [31] (see also [10, 11, 19]). We furnish the proof of the optimality stated by means of (25) in Lemma 10 below. \[\Box\]

**Corollary 1.** In the same hypotheses of Lemma 6 and furthermore assuming $\Omega$ bounded domain, then for $t > \sigma \geq 0$ the following holds:

$$(t - \sigma)\|\varphi_t(t)\|_q + (t - \sigma)^\frac{s}{2}\|\nabla \varphi(t)\|_q + \|\varphi(t)\|_q \leq c\|\varphi(\sigma)\|_s e^{-c_1(t - \sigma)}(t - \sigma)^{-\mu}, \quad (27)$$

where $c_1$ is a constant depending on the size of $\Omega$ and constants $c, c_1$ are independent of $\varphi_0$ and of $t, \sigma$.

**Corollary 2.** Let $\varphi_0 \in C_0(\Omega)$. Then, for all $\eta > 0$, the solution of Lemma 6 is such that

$$\varphi \in \bigcap_{s \geq 1} \left[ C([0, T]; J^s(\Omega)) \cap L^\infty(\eta, T; J^{1,s}(\Omega) \cap W^{2,s}(\Omega)) \right],$$

$$\nabla \pi_\varphi, \varphi_t \in \bigcap_{s \geq 1} L^\infty(\eta, T; L^s(\Omega)).$$

**Proof.** For the proof of the Corollary see e.g. [31]. \[\Box\]

**Lemma 7.** Let $(\varphi, \pi_\varphi)$ be the solution of Lemma 6. For $r \geq s$ the following estimates hold:

$$t - \sigma \in (0, 1), \quad \|\nabla \varphi(t)\|_{L^r(\Omega)} \leq c\|\varphi(\sigma)\|_s (t - \sigma)^{-\mu_4},$$

$$t - \sigma > 1, \quad \|\nabla \varphi(t)\|_{L^r(\Omega)} \leq c\|\varphi(\sigma)\|_s (t - \sigma)^{-\mu_5}, \quad (29)$$

where $c$ is a constant independent of $\varphi_0$ and $t - \sigma$, furthermore we have set $\mu_4 := \frac{1}{2} + \mu + \frac{1}{2r}$ and $\mu_5 := \begin{cases} \frac{1}{2} + \mu & \text{if } r \in [s, n], \\ \frac{1}{2s} & \text{if } r \geq n. \end{cases}$

**Proof.** Estimate (29) is an immediate consequence of the trace inequality (13) and of estimates (24)$_{2, 4}$. \[\Box\]

**Lemma 8.** Let $(\varphi, \pi_\varphi)$ be the solution of Lemma 6. Then the pressure field $\pi_\varphi$ enjoys the estimates:

$$\lambda \in (0, 1), \quad |\pi_\varphi|_{L^\lambda(\Omega \cap B_R)} \leq c < \nabla \varphi >_r^\lambda,$$

$$|\nabla \pi_\varphi|_r \leq c < \nabla \varphi >_r^{1 - \frac{d}{n}}, \quad (30)$$

with $c$ independent of $\varphi$. In particular, if $r > n \geq 2$ we get

$$|\pi_\varphi(x)| \leq c (\|\nabla \varphi\|_r + \|\nabla \nabla \varphi\|_r)|x|^{2 - n}, \quad |x| > R,$$

for $n = 2$, $\pi_\varphi - \pi_\infty = o(\|\nabla \varphi\|_r + \|\nabla \nabla \varphi\|_r).$ \[\Box\]
Proof. The estimates (30) are consequence of the results due to Solonnikov in [32] related to the Neumann problem:

\[
\Delta \pi = 0 \text{ in } \Omega, \quad \frac{\partial \pi}{\partial n} = \nu \cdot \Delta \phi \text{ on } \partial \Omega, \\
n = 2, \pi \sim \pi_\infty, n \geq 3, \pi \to 0, \ |x| \to \infty.
\]

The following lemma furnishes the behavior in \(t\) related to a trace-norm of the pressure field \(\pi_\phi\). The behavior depends on the neighborhood of \(t = 0\) and of \(t = \infty\). Of course, our task is to deduce behavior that turns to be the best for our aims.

Lemma 9. Let \((\phi, \pi_\phi)\) be the solution of Lemma 6. Then, for \(r \geq s\), set \(\mu := \frac{d}{2} \left( \frac{1}{r} - \frac{1}{s} \right)\), we get

\[
\begin{align*}
t - \sigma & \in (0, 1), \quad \|\phi_\pi(t)\|_{L^r(\partial \Omega)} \leq c\|\phi(\sigma)\|_s(t - \sigma)^{-\rho_0 - \mu}, \\
t - \sigma > 1, \quad \|\phi_\pi(t)\|_{L^r(\partial \Omega)} \leq c\|\phi(\sigma)\|_s(t - \sigma)^{-\rho_1 - \mu}, \quad (32)
\end{align*}
\]

where \(c\) is a constant independent of \(\phi\) and \(t - \sigma\), and

\[
\begin{align*}
\rho_0 & := \frac{1}{2} + \frac{1 + (\lambda - 1)}{2^{n-1}+r} + \frac{r(1 - \lambda - 1)}{r(n - 2 + r)} + \frac{n - 1 + \lambda r}{2^{n-1}+r} < 1, \\
\rho_1 & := \left\{ \begin{array}{ll}
\frac{1}{2} + \frac{1}{2^{n-1}+r} & \text{if } r \in (1, \frac{n}{2}], \\
\frac{1}{2} + \frac{1}{2^{n-1}+r} \left( \frac{u}{2^{n-1}+r} - 1 \right) & \text{if } r \in [\frac{n}{2}, n], \\
\frac{2^{n-1}+r}{2^{n-1}+r} & \text{if } r > n,
\end{array} \right.
\end{align*}
\]

where \(\lambda \in (0, 1 - \frac{1}{r})\) and \(d := \frac{n - 2 + r}{n - 1 + \lambda r}\).

Proof. We prove estimates (32) for \(\sigma = 0\) and \(s = r\). Subsequently, one deduces estimates (32) in a complete form by means of the semigroup properties of \(\phi\). Assuming in (30) \(\lambda < 1 - \frac{1}{r}\), applying Hölder’s inequality with exponents \((d, \frac{d}{d-1})\), \(d = \frac{n - 2 + r}{n - 1 + \lambda r}\), (we stress that \(1 < d < r\)) we get

\[
\begin{align*}
(\langle \nabla \phi >_r^\beta)^r & = \int_{\partial \Omega} \int_{\partial \Omega} |\nabla \phi(x) - \nabla \phi(y)|^{r(1 - \frac{1}{r})} |\nabla \phi(x) - \nabla \phi(y)|^{\frac{r\beta}{r - \beta}} |x - y|^{\frac{1}{n - 1 + \lambda r}} d\sigma_x d\sigma_y, \\
& \leq c|\nabla \phi|_{L^r(\partial \Omega)}^{r(1 - \frac{1}{r})} (\langle \nabla \phi >_r^\beta)^r
\end{align*}
\] (33)

Employing the trace inequality (13) and estimates (30)-(33), \(1 - \frac{1}{r} = \frac{1}{\mu}\) and \(1 - \frac{1}{s} = \frac{1}{\mu}\) we deduce

\[
\begin{align*}
\|\pi_\phi\|_{L^r(\partial \Omega)} & \leq c\|\pi_\phi\|_{L^r(\partial \Omega \cap S_{\rho})} + \|\pi_\phi\|_{L^r(\partial \Omega \cap S_{\rho})}^{\frac{1}{\mu}} |\nabla \pi_\phi|_{L^r(\partial \Omega \cap S_{\rho})}^{\frac{1}{\mu}} \\
& \leq c(\langle \nabla \phi >_r^\beta) + (\langle \nabla \phi >_r^\beta)^\frac{1}{\mu} (\langle \nabla \phi >_r^\beta)^\frac{1}{\mu} \\
& \leq c\left[ |\nabla \phi|_{L^r(\partial \Omega)}^{\frac{1}{\mu}} (\langle \nabla \phi >_r^\beta) + |\nabla \phi|_{L^r(\partial \Omega)}^{\frac{1}{\mu}} (\langle \nabla \phi >_r^\beta)^\frac{1}{\mu} \right]
\end{align*}
\] (34)

\[
= I_1(r, t) + I_2(r, t).
\]

\[\square\]
Employing again the trace inequality (13)\textsubscript{2}, we get

\[ I_1(r,t) \leq c(\|\nabla \varphi\|_{L^r(\Omega \cap SR)} + \|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|D^2 \varphi\|_{L^r(\Omega)}) \]
\[ \leq c(\|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|D^2 \varphi\|_{L^r(\Omega)}) \]
\[ I_2(r,t) \leq c(\|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|D^2 \varphi\|_{L^r(\Omega)}) \]
\[ \leq c(\|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|\nabla \varphi\|_{L^r(\Omega \cap SR)} \|D^2 \varphi\|_{L^r(\Omega)}) \] (35)
where we have set \( \nu := \frac{1}{d r} + \frac{1}{r} \) and \( \nu_1 := \frac{1}{d r} + \frac{1}{r} . \)

For the right hand side of (35) we look for an estimate in \( t \) and \( \| \varphi_0 \|_r \). We firstly evaluate \( I_1(r,t) \) and \( I_2(r,t) \) for \( t \in (0,1) \). We estimates the terms on the right hand side of (35) by inequalities (24)\textsubscript{2.4}. Since we evaluate for \( t \in (0,1) \), we can limit ourselves to consider the terms on the right hand side of (24)\textsubscript{2.4} which have max exponent. This max exponent is leaded by the last term of \( I_2(r,t) \). Hence we have

\[ I_1(r,t) + I_2(r,t) \leq c(\| \varphi_0 \|_r t^{-\rho_0}, t \in (0,1), \]
where we have set \( \rho_0 := \frac{1}{2} + \frac{1}{2} \lambda \left( \frac{1}{d} - 1 \right) + \frac{1}{2} \left( 1 - \frac{1}{d} \right) + \frac{1}{2d} \). We compute \( \rho_0 \). Recalling that \( r' \), \( d' \) are the coniugate exponents of \( r \), \( d \), we get \( 2 \)

\[ \rho_0 = \frac{1}{2} + \frac{1}{2} \lambda \left( \frac{1}{d} - 1 \right) + \frac{1}{2} \left( 1 - \frac{1}{d} \right) + \frac{1}{2d} . \] (37)

By the definition of \( d \), we have that (37) is equivalent to

\[ \rho_0 = \frac{1}{2} + \frac{1}{2} \left( \frac{r(\lambda - 1)}{2(n-2+r)} + \frac{(1-\lambda)}{2(n-2+r)} + \frac{r-1+\lambda r}{2(n-2+r)} \right) . \]

We are interested to verify that under our assumption on \( \lambda \) and for \( r > 1 \) we get \( \rho_0 < 1 \), that is

\[ \frac{1}{2} + \frac{1}{2} \left( \frac{r(\lambda - 1)}{2(n-2+r)} + \frac{(1-\lambda)}{2(n-2+r)} + \frac{r-1+\lambda r}{2(n-2+r)} \right) < 1 . \] (38)

For \( \varepsilon > 0 \), we set \( r := 1 + \varepsilon \). Hence (38) becomes equivalent to \( 3 \)

\[ \lambda < (1 - \lambda) \varepsilon \Leftrightarrow \lambda < (1 - \lambda)(r - 1) \Leftrightarrow \lambda < 1 - \frac{1}{r} . \] (39)

Since it is \( \lambda < 1 - \frac{1}{r} \), we have verified (38). Now we look for the estimate of \( I_1(r,t) \) and \( I_2(r,t) \) for \( t > 1 \). Since exponent in (24)\textsubscript{2.4} depends on \( r \) we distinguish the cases of \( r \in (1, \frac{n}{2}] \) from the ones of \( r \in \left[ \frac{n}{2}, n \right] \) and \( r > n \). Suppose \( r \in (1, \frac{n}{2}] \). Since \( t > 1 \) we look for exponents minimum. Hence, evaluating the right hand side of (35) via the estimates (24)\textsubscript{2.4}, we get

\[ r \in (1, \frac{n}{2}], \quad I_1(r,t) + I_2(r,t) \leq c(\| \varphi_0 \|_r t^{-\rho_1}, t > 1 \] (40)

\[ \text{2 Actually it holds} \]
\[ \rho_0 = \left( \frac{\lambda r}{2(n-2+r)} \right) + \frac{1}{2r} + \frac{1}{r} = \left( \frac{1}{2} + \frac{1}{2} \right) \left( \frac{1}{d} \right) + \frac{1}{2r} + \frac{1}{r} \]
\[ = \frac{1}{2} \left( 1 + \frac{1}{2} \right) \left( 1 + \frac{1}{d} \right) + \frac{1}{2r} + \frac{1}{r} \]
that leads (37) substituting again \( \frac{1}{d} \) with \( 1 - \frac{1}{r} \).

\[ \text{3 Estimate (38) is equivalent to} \]
\[ \frac{1}{2} \left( \frac{r(\lambda - 1)}{2(n-2+r)} + \frac{(1-\lambda)}{2(n-2+r)} + \frac{r-1+\lambda r}{2(n-2+r)} \right) < \frac{1}{2} \]
which is equivalent to \( 1 + r(\lambda - 1) + 2r^2(1-\lambda) - 2r < -r^2 + (1-\lambda)r^3 \). Introducing \( r := 1 + \varepsilon \) we obtain the first of (39).
where we have set $\rho_1 := \frac{1}{2} + \frac{1}{2s}$. In the case of $r \in \left[\frac{n}{2}, n\right]$, for the right hand side of (35), we obtain the exponents:

$$I_1(r, t) \leq c\|\varphi_0\|_r (t^{-\rho_{11}} + t^{-\rho_{12}}), \quad t > 1,$$

with $\rho_{11} := \frac{1}{2} + \frac{1}{2s} \left(\frac{n}{r} - 1\right)$ and $\rho_{12} := \rho_{11} + \frac{1}{2s} \left(\frac{n}{r} - 1\right) \left(1 - \frac{1}{d}\right)$, as well

$$I_2(r, t) \leq c\|\varphi_0\|_r (t^{-\rho_{21}} + t^{-\rho_{22}}), \quad t > 1,$$

with $\rho_{21} := \rho_{12}$ and $\rho_{22} := \rho_{11} + \frac{1}{r} \left(\frac{n}{r} - 1\right) \left(1 - \frac{1}{d}\right)$. Hence choosing the minimum exponent we get

$$r \in \left[\frac{n}{2}, n\right], \quad I_1(r, t) + I_2(r, t) \leq c\|\varphi_0\|_r t^{-\rho_1}, \quad t > 1, \quad (41)$$

where we have set $\rho_2 := \rho_{11}$. Now we consider the case of $r > n$. Summing the exponents of terms on the right hand side of (35) we get $\frac{1}{2s}$ as minimum exponent. Hence we get

$$r > n, \quad I_1(r, t) + I_2(r, t) \leq c\|\varphi_0\|_r t^{-\rho_1}, \quad t > 1. \quad (42)$$

Finally, via estimates (34)-(36) and (34) with (40)-(42), we get

$$\|\pi_{\varphi}(t)\|_{L^r(\partial\Omega)} \leq c\|\varphi_0\|_r t^{-\rho_0}, \quad \forall t \in (0, 1) \quad \text{and for all} \quad r > 1,$$

$$\|\pi_{\varphi}(t)\|_{L^r(\partial\Omega)} \leq c\|\varphi_0\|_r t^{-\rho_1}, \quad \forall t \geq 1. \quad (43)$$

**Remark 1.** We point out that $\rho_0 > \rho_1$. Moreover comparing estimates (29) with the estimates (32), related to $\nabla \varphi$ and to $\pi_{\varphi}$ respectively, we note that, in a neighborhood of $t = 0$, $\rho_0 + \mu > \frac{1}{2} + \mu$ for all $r \geq s$, as well, in neighborhood of infinity, if $r < n$ then $\rho_1 + \mu > \frac{1}{2} + \mu$, and if $r \geq n$ then $\rho_1 + \mu = \frac{1}{2s}$.

The following result holds:

**Corollary 3.** Let $(\varphi, \pi_{\varphi})$ be the solution furnished in Lemma 6, then for $r \geq s$, set $\mu := \frac{s}{2} \left(\frac{1}{s} - \frac{1}{r}\right)$, we get

$$\|T(\varphi, \pi_{\varphi})(t)\|_{L^r(\partial\Omega)} \leq c\|\varphi(\sigma)\|_s \begin{cases} (t-\sigma)^{-\rho_0-\mu} & \text{if } t-\sigma \in (0, 1), \\ (t-\sigma)^{-\frac{\mu}{2}} & \text{if } t-\sigma > 1 \quad \text{and } r \leq n, \\ (t-\sigma)^{-\frac{\mu}{2s}} & \text{if } t-\sigma > 1 \quad \text{and } r > n. \end{cases} \quad (44)$$

Finally, we also have

$$\|T(\varphi, \pi_{\varphi})(t)\|_{L^r(\partial\Omega)} \leq c\|\varphi(\sigma)\|_s \begin{cases} (t-\sigma)^{-\rho_0-\mu-1} & \text{if } t-\sigma \in (0, 1), \\ (t-\sigma)^{-\frac{\mu}{2}} & \text{if } t-\sigma > 1 \quad \text{and } r \leq n, \\ (t-\sigma)^{-\frac{\mu}{2s}} & \text{if } t-\sigma > 1 \quad \text{and } r > n. \end{cases} \quad (45)$$

**Proof.** The proof is an immediate consequence of Lemma 8, Lemma 9 and Remark 1. \qed

**Lemma 10.** Let $(\varphi, \pi_{\varphi})$ be a solution to problem (1) given by Corollary 2. Then, for $s \geq \frac{n}{2}$, there is no function $\xi(t)$ such that

$$t^{-1} \xi(t) \in L^1(t_0, \infty) \quad \text{and} \quad \|\nabla \varphi(t)\|_{L^s(\Omega \cap S_t)} \leq \xi(t) t^{-\mu_1} \|\varphi_0\|_s, \quad t > 0, \quad (46)$$

where $R > \text{diam}(\Omega^c)$ and $\xi$ are independent of $\varphi$ and $\mu_1$ is given in (24)\textsubscript{2}. \quad
Proof. Let \( \varphi_0 \in \mathcal{C}_0(\Omega) \) and let \( (\varphi, \pi_\varphi) \) be the solution ensured by Corollary 2. Employing (13) and subsequently (18), for all \( R > \text{diam}(\Omega) \), we have

\[
\| \nabla \varphi(t) \|_{L^q(\partial \Omega)} \leq c(\| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} + \| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} ) \\
\leq c(\| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} + \| P \Delta \varphi(t) \|_{q} + \| \varphi(t) \|_{L^q(\Omega \cap S_R)} ) \\
\leq c(\| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} + \| \varphi(t) \|_{L^q(\Omega \cap S_R)} ) .
\]

(47)

We point out that

- estimating the penultimate row, for the second term, we take into account that the equation \( (1)_1 \) furnishes \( P \Delta \varphi = \varphi_t \),

- since in \( (18) \) \( \Omega' \) is bounded with \( \partial \Omega \cap \partial \Omega' = \partial \Omega \) we can choose \( \Omega' \equiv \Omega \cap S_R \), and, for the third term, we employed the Poincaré inequality.

Estimate (34) ensures

\[
\| \pi_\varphi(t) \|_{q} \leq I_1(t) + I_2(t) , \text{ for all } t > 0 .
\]

Recalling estimate (35), computing the exponents and by making use of Young inequality, we get

\[
I_1(q, t) + I_2(q, t) \leq c(\| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} + \| D^2 \varphi(t) \|_{q}) .
\]

By the same arguments employed in the previous computation for the \( D^2 \varphi \) we obtain

\[
\| \pi_\varphi(t) \|_{q} \leq I_1(q, t) + I_2(q, t) \leq c(\| \nabla \varphi(t) \|_{L^q(\Omega \cap S_R)} + \| \varphi(t) \|_{q}) , \text{ for all } t > 0 .
\]

(48)

Now we are in a position to prove the lemma. We adapt the idea already employed in [31]. Let consider the exterior problem

\[
\Delta \Phi = \nabla P , \quad \nabla \cdot \Phi = 0 \text{ in } \Omega , \quad \Phi = a \text{ on } \partial \Omega , \quad \Phi \to 0 \text{ for } |x| \to \infty .
\]

(49)

It is well known that assuming \( a \in C^2(\partial \Omega) \) there exists a solution such that \( \Phi = O(|x|^{-n+2}) \) at infinity. Hence \( \Phi \in L^q(\Omega) \) with arbitrary \( q > \frac{n}{n-2} \). Our task is to prove that if (46) is true, then the following holds:

\[
| (\Phi, \psi_0) | \leq c | \psi_0 \|_{2} , \quad \psi_0 \in \mathcal{C}_0(\Omega) .
\]

(50)

By virtue of Lemma 5 this last implies that \( \Phi \in L^{\frac{n}{n-2}}(\Omega) \) for arbitrary boundary data \( a \in C^2(\Omega) \), which is impossible. Hence (46) can not be true. Now our task is to prove (50) via the assumption (46). Assuming that estimate holds for the solution \( (\varphi, \pi_\varphi) \) to problem (1) with initial data \( \varphi_0 \in \mathcal{C}_0(\Omega) \), multiplying equation (49), by \( \varphi \) and integrating on \( (0, T) \times \Omega \), we obtain:

\[
(\Phi, \varphi_0) = (\Phi, \varphi(t)) - \int_0^t \int_{\Omega} [ (a, \nu \cdot \nabla \varphi) d\Omega - (a \cdot \nu, \pi_\varphi) d\Omega ] d\tau \\
+ \int_0^t \int_{\Omega} [ (a, \nu \cdot \nabla \varphi) d\Omega - (a \cdot \nu, \pi_\varphi) d\Omega ] d\tau .
\]
Applying Hölder’s inequality, for $\eta > \frac{n}{\sigma - r}$, we get

$$
|\langle \Phi, \varphi_0 \rangle| \leq \|\Phi\|_{\eta} \|\varphi(t)\|_{\eta} + c\|a\|_{\infty} \int_{0}^{t} \left[ \|\nabla \varphi\|_{L_{\infty}(\partial \Omega)} + \|\pi_{\varphi}\|_{L_{\infty}(\partial \Omega)} \right] d\tau
$$

$$
+ c\|a\|_{\infty} \int_{t_0}^{t} \left[ \|\nabla \varphi\|_{L_{\infty}(\partial \Omega)} + \|\pi_{\varphi}\|_{L_{\infty}(\partial \Omega)} \right] d\tau,
$$

(51)

here, assuming $\varphi_0 \in \mathcal{C}_0(\Omega)$, we have tacitly considered that (23) holds. Applying to the right hand side of (51) for $(\nabla \phi, \pi_\phi)$ estimates (29) and (32) for $t \in (0, t_0)$, and estimates (47)-(48) for $t > t_0$, by virtue of assumption (46), we get

$$
|\langle \Phi, \psi_0 \rangle| \leq \|\Phi\|_{\eta} \|\psi(t)\|_{\eta} + c\|a\|_{\infty} \|\psi_0\|_{\frac{n}{\sigma}},
$$

which, letting $t \to \infty$ and employing (26), implies (50).

\[ \square \]

3 The Stokes Cauchy problem

Let us consider the Stokes Cauchy problem:

$$
U_t - \Delta U + \nabla \pi U = 0, \quad \nabla \cdot U = 0, \text{ in } (0, T) \times \mathbb{R}^n,
$$

$$
U = U_0 \text{ on } \{0\} \times \mathbb{R}^n.
$$

Lemma 11. For all $U_0 \in J^0_0(\mathbb{R}^n)$ there exits, up to a function of $t$ for the pressure field, a unique solution $U$ to problem (52) such that $\nabla U \in C((0, T); J^0_0(\mathbb{R}^n))$ and, for all $r \geq s \geq 1$ and $t > \sigma \geq 0$, set $\mu := \frac{n}{2} \left( \frac{1}{r} - \frac{1}{s} \right)$, we get

$$
\|\nabla U(t)\|_r \leq c\|\nabla U(\sigma)\|_s (t - \sigma)^{-\mu}, \text{ for all } t - \sigma > 0,
$$

$$
\|D^2 U(t)\|_r + \|U_t(t)\|_r \leq c\|\nabla U(\sigma)\|_s (t - \sigma)^{-\frac{1}{s} - \mu}, \text{ for all } t - \sigma > 0,
$$

(53)

where the constant $c$ is independent of $U_0$.

Proof. By means of the representation of the solution by heat kernel, after integrating by parts, and via the Young theorem we get (53)$_{1,2}$.

In the following corollary we make the special assumption of $U_0 \in \mathcal{C}_0(\Omega) \subset \mathcal{C}_0(\mathbb{R}^n)$ (we mean that $U_0$ has a trivial extension on $\mathbb{R}^n$), and we study the behavior of the solutions corresponding to these special initial data in neighborhood of $t = 0$ and of $t = \infty$. Of course, the special data influence the quoted behavior. They are special in such a way that they are useful for our subsequent tasks.

Corollary 4. Let $U_0 \in \mathcal{C}_0(\Omega)$. Then the solution of Lemma 11 is such that

$$
\|U(t)\|_{\infty} \leq c\|\nabla U_0\|_p t^{-\frac{1}{p} - \frac{n}{2p}}, \text{ if } p \in (1, n),
$$

$$
\|U(t)\|_{L^\infty(\Omega \cap B_R)} \leq c\|\nabla v_0\|_p \zeta_\delta(t)
$$

(54)
Employing \( \text{inequality, for some } p > n \) any case we get
\[
\|U(t)\|_{L^\infty(\partial \Omega)} + \|U(t) \cdot \nu\|_{L^{\frac{1}{2-n}}} \leq c\|\nabla U_0\|_p \zeta(t),
\]
\[
\|U(t)\|_{L^p(\partial \Omega)} \leq c\|\nabla U_0\|_p t^\frac{1}{2},
\]
\[
\|U_t(t) \cdot \nu\|_{L^{\frac{1}{2-n}}} \leq c\|\nabla U_0\|_p t^{-\frac{1}{2} - \mu},
\]
with \( \zeta_d := \begin{cases} 
1 + \frac{1}{2} \tau & \text{if } p = n, \\
1 + \frac{1}{2} \tau - \frac{1}{d} & \text{if } p \neq n,
\end{cases} \) and \( \zeta_0(t) := \begin{cases} 
t^{\frac{1}{2} - \frac{1}{d}} & \text{if } p \neq n, \\
\log(t + c) & \text{if } p = n, \end{cases} \) where \( c \) is a constant independent of \( U_0 \) and all the estimates hold uniformly in \( t > 0 \). There exists a constant \( c \) such that for all \( U_0 \in \mathcal{C}_0(\mathbb{R}^n) \)
\[
\|U_t(t)\|_p + \|D^2 U(t)\|_p \leq c\|D^2 U_0\|_p, \text{ for all } t > 0.
\]

\textbf{Proof.} In the case of \( p \in (1, n) \) estimate \((54)_1\) is an immediate consequence of \((53)_2\) and of the fact that
\[
\|U(t)\|_\infty \leq c \int t \|U_\tau(\tau)\|_\infty d\tau \text{ for all } t > 0.
\]

Analogously for \( p > n \), employing again \((53)_2\), we get
\[
\|U(t)\|_{L^\infty(\partial \Omega \cap B_R)} \leq \|U_0\|_{L^\infty(\partial \Omega \cap B_R)} + \int_{0}^{t} \|U_\tau(\tau)\|_\infty d\tau
\]
\[
\leq (c(R) + t^{\frac{1}{2} - \frac{1}{d}})\|\nabla U_0\|_p,
\]
where in the last step we estimate the \( \|U_0\|_{L^\infty(\partial \Omega \cap B_R)} \) by \( \|U_0\|_{W^{1,p}(\Omega \cap B_R)} \), and, taking into account that \( U_0 = 0 \) on \( \mathbb{R}^n - \Omega \), by applying the Poincaré inequality to \( \|U_0\|_{L^p(\partial \Omega \cap B_R)} \).
The case of \( p = n \) is a bit different. Initially we estimate \( \|U(1)\|_{L^\infty(\partial \Omega \cap B_R)} \). By Sobolev inequality, for some \( p > n \),
\[
\|U(t)\|_\infty \leq c\|U(t)\|_{W^{2,n}(\partial \Omega \cap B_R)}.
\]

Employing \((53)_1,2\), we get
\[
\|\nabla U(t)\|_n + \|D^2 U(t)\|_n \leq c(1 + t^{-\frac{n}{2}})\|\nabla U_0\|_n,
\]
and
\[
\|U(t)\|_{L^n(\partial \Omega \cap B_R)} \leq c\|U_0\|_{L^n(\partial \Omega \cap B_R)} + \int_{0}^{t} \|U_\tau(\tau)\|_n d\tau \leq c\|\nabla U_0\|_n(1 + t^{\frac{1}{2}}),
\]
where in the last step we employ the Poincaré inequality again. Therefore we can claim that \( \|U(1)\|_{L^\infty(\partial \Omega \cap B_R)} \leq c\|\nabla U_0\|_n \). Finally, in order to obtain \((54)_2\) in complete form, it is enough to consider a path with end point 1 for \( t < 1 \) and initial point 1 for \( t > 1 \), in any case we get
\[
\|U(t)\|_{L^\infty(\partial \Omega \cap B_R)} \leq \|U(1)\|_\infty + \int_{1}^{t} \|U_\tau(\tau)\|_\infty d\tau \leq \|\nabla U_0\|_n(1 + t\int_{1}^{t} \log \tau d\tau).
\]
In order to prove (55)₁, we remark that for \( s < n \) and \( r = \infty \) estimate (57) holds, hence we have the thesis. In the case of \( s > n \), we remark that \( U = 0 \) on \( \{0\} \times \partial \Omega \), hence we can compute in the following way:

\[
\|U(t)\|_{L^\infty(\partial \Omega)} \leq \int_0^t \|U(\tau)\|_{\infty} d\tau ,
\]

that, via (53)₂, implies the thesis. Finally, we consider the case of \( s = n \). We repeat the same arguments of the previous case but working in \( \Omega^c := \mathbb{R}^n - \overline{\Omega} \). First of all we note that for all \( t > 0 \) it holds

\[
\|U(t)\|_{C(\Omega^c)} \leq c \|D^2 U(t)\|_{L^1(\mathbb{R}^n - \overline{\Omega})} + c_0 \|U(t)\|_{L^n(\mathbb{R}^n - \overline{\Omega})} ,
\]

with \( c \) independent of \( t \) and \( U \). Moreover, since \( \|U_0\|_{L^n(\mathbb{R}^n - \Omega)} = 0 \), employing (53)₂, we have

\[
\|U(t)\|_{L^n(\mathbb{R}^n - \overline{\Omega})} \leq \int_0^t \frac{d}{d\tau} \|U(\tau)\|_{L^n(\mathbb{R}^n - \overline{\Omega})} d\tau \leq c \\|\nabla U_0\|_n t^\frac{1}{2} , \quad t > 0 .
\]

Therefore, we obtain

\[
\|U(t)\|_{L^\infty(\partial \Omega)} \leq \|U(t)\|_{C(\Omega^c)} \leq c \|\nabla U_0\|_n + c_0 c \\|\nabla U_0\|_n t^\frac{1}{2} , \quad t > 0 .
\]

Considering the following inequality:

\[
\|U(t)\|_{L^\infty(\partial \Omega)} \leq \|U(1)\|_{L^\infty(\partial \Omega)} + \int_1^t \|U(\tau)\|_{L^\infty(\partial \Omega)} d\tau ,
\]

via (53)₂ for the integral term, and via estimate (58) for \( \|U(1)\|_{L^\infty(\partial \Omega)} \), we complete the proof. In order to prove (55)₂ we consider the following formula:

\[
\|U(t)\|_{L^p(\partial \Omega)} \leq \int_0^t \|U(\tau)\|_{L^p(\partial \Omega)} d\tau ,
\]

that by means of (53)₂ gives

\[
\|U(t)\|_{L^p(\partial \Omega)} \leq c \|\nabla U_0\|_p t^\frac{1}{2} .
\]

In order to complete the estimates of the lemma we have to prove the one related to \( W^{-\frac{1}{q},q}(\partial \Omega) \) norm. To this end, it is enough to observe that, by the regularity of \( U(t) \) and \( \partial \Omega \) bounded, we get \( \|U(t) \cdot \nu\|_{W^{-\frac{1}{q},q}(\partial \Omega)} \leq c \|U(t)\|_\infty \). The same holds in the case of \( U_t \). Hence the estimates are a consequence of (54)₁ and (55)₁, and of (53)₂, respectively. Finally, estimate (56) is a consequence of the regularity of \( U_0 \) and the representation formula. \( \square \)
4 A special auxiliary Stokes initial boundary value problem

Let \( v_0 \in \mathscr{C}_0(\Omega) \). Denoted by \((v, \pi_v)\) and by \((U, c)\) the solutions to problems (1) and (52) both with initial data \( v_0 \), whose existence are ensured by Lemma 6 and by Lemma 11, respectively. The pair \((u, \pi_u)\) with \( u := v - U \) and \( \pi_u := \pi_v - c(t) \) is a solution to the problem

\[
\begin{align*}
    u_t - \Delta u &= -\nabla \pi_u, \quad \nabla \cdot u = 0, & \text{ in } (0, T) \times \Omega, \\
    u &= -U & \text{ on } (0, T) \times \partial \Omega, \\
    u &= 0 & \text{ on } \{0\} \times \Omega.
\end{align*}
\]

(59)

Trivially, we get

\[
\int_{\partial \Omega} u \cdot \nu d\sigma = 0 \quad \text{and} \quad \int_{\partial \Omega} u_t \cdot \nu d\sigma = 0 \quad \text{for all } t > 0.
\]

(60)

Lemma 12. Let \( p \in (1, \infty) \), and \( q \geq p \). Set \( \mu := \frac{1}{p} \left( \frac{1}{p} - \frac{1}{q} \right) \), for solution \( u \) to problem (59) the following estimates hold:

\[
\begin{align*}
    \|\nabla u(t)\|_q &\leq ct^{-\mu} \|\nabla v_0\|_p, & \text{for } t \in (0, 1), \\
    \|u_t(t)\|_q &\leq ct^{-\frac{1}{2} - \mu} \|\nabla v_0\|_p, & \text{for } t \in (0, 1), \\
    \|u(t)\|_q &\leq ct^{\frac{1}{2} - \mu} \|\nabla v_0\|_p, & \text{for } t \in (0, 1),
\end{align*}
\]

(61)

with \( c \) independent of \( v_0 \).

Proof. We set \( \tilde{\varphi}(\tau, x) := \varphi(t - \tau, x) \) for all \( \tau \in [0, t] \), where \( t > 0 \) is fixed, and \( \varphi_0 \in \mathscr{C}_0(\Omega) \). Taking into account problem (59), in the case of the pair \((u_t, \pi_{u_t})\) and \((\tilde{\varphi}, \pi_{\tilde{\varphi}})\) the Green identity (15) becomes:

\[
(u_t(t), \varphi_0) = (u_t(s), \varphi(t - s)) + \int_s^t (U_\tau, \nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}})) d\tau.
\]

(62)

Applying the divergence theorem, recalling that \( \tilde{\varphi} \) is solution to the adjoint problem on \((0, t) \times \Omega\), we get

\[
(u_t(t), \varphi_0) = (u_t(s), \varphi(t - s)) + \int_s^t (\nabla U_\tau, \nabla \tilde{\varphi}) d\tau - \int_s^t (U_\tau, \tilde{\varphi}_\tau) d\tau.
\]

A further integration by parts furnishes

\[
(u_t(t), \varphi_0) = (u_t(s), \varphi(t - s)) - \int_s^t (\Delta U_\tau, \tilde{\varphi}) d\tau - \int_s^t (U_\tau, \tilde{\varphi}_\tau) d\tau.
\]

Hence via (52), integrating by parts with respect to the time, we get

\[
(u_t(t), \varphi_0) = (u_s(s), \varphi(t - s)) - (U_t(t), \varphi_0) + (U_s(s), \varphi(t - s)).
\]

(63)
Since \( u_s(s) = P \Delta u(s) \), an integration by parts furnishes
\[
(u_s(s), \varphi(t-s)) = (U(s), \nu \cdot \nabla \varphi(t-s))_{\partial \Omega} + (u(s), \Delta \varphi(t-s)).
\]
Letting \( s \to 0 \), we have \( u(s, x) \to 0 \) in \( L^p(\Omega) \), as well, recalling (55)_2, we have \( U(s) \to 0 \) in \( L^p(\partial \Omega) \), then, we get
\[
\lim_{s \to 0} (u_s(s), \varphi(t-s)) = 0.
\]  
(64)
Therefore from (63) and \( U_s(s) = \Delta U(s) \), an integration by parts allows us to deduce
\[
(u_t(t), \varphi_0) = (u_s(s), \varphi(t-s)) - (U_t(t), \varphi_0) + (U_s(s), \varphi(t-s))
\]
\[
= (u_s(s), \varphi(t-s)) - (U_t(t), \varphi_0) - (\nabla U(s), \nabla \varphi(t-s)),
\]
and letting \( s \to 0 \), we get
\[
(u_t(t), \varphi_0) = -(U_t(t), \varphi_0) - (\nabla v_0, \nabla \varphi(t)), \text{ for } t > 0.
\]  
(65)
Applying Holder’s inequality, via estimates (24)_2 for \( \varphi \) and (53)_2 for \( U \), we obtain
\[
|(u_t(t), \varphi_0)| \leq \|U_t(t)\|_q \|\varphi_0\|_{q'} + \|\nabla v_0\|_p \|\nabla \varphi(t)\|_{p'}
\]
\[
\leq c \|\nabla v_0\|_p t^{-\frac{1}{2}-\mu} \|\varphi_0\|_{p'}, \text{ for } t \in (0,1).
\]  
(66)
Recalling (60), by means of estimate (21) we also obtain
\[
\|u_t(t)\|_q \leq c(\|\nabla v_0\|_p t^{-\frac{1}{2}-\mu} + \|u_t \cdot \nu\|_{\frac{1}{4},q}) \leq c(\|\nabla v_0\|_p t^{-\frac{1}{2}-\mu} + \|U_t \cdot \nu\|_{\infty})
\]
which implies (61)_2 after applying (55)_3 for \( U_t \). As a consequence we also prove that
\[
\|u(t)\|_p \leq \int_0^t \|u_\tau(\tau)\|_p d\tau \leq c t^{\frac{1}{4}} \|\nabla v_0\|_p, \text{ for } t \in (0,1),
\]  
(67)
which proves (61)_3 for \( q = p \). Since for all \( t > 0 \), \( \|v(t)\|_q \leq c \|v_0\|_p t^{-\mu} \), via (54), we obtain that for all \( t > 0 \) and \( R > 0 \) the estimate \( \|u(t)\|_{L^q(\Omega \cap B_R)} < \infty \) holds. Fixing \( t \) in (1), by Lemma 3 we get
\[
\|D^2v\|_q \leq c(\|v_t\|_q + \|v\|_{L^q(\Omega \cap B_R)}), \text{ for } t \in (0,1).
\]
Hence the following holds
\[
\|D^2u\|_q \leq c(\|v_t\|_q + \|U_t\|_q + \|u\|_{L^q(\Omega \cap B_R)} + \|U\|_{L^q(\Omega \cap B_R)}), \text{ for } t \in (0,1).
\]
Since (16) for all \( \delta \in (0,1) \) furnishes \( \|u\|_q \leq \delta \|D^2u\|_q + c(\delta)\|u\|_p \), by virtue of estimates (53)_2 and (67), and by virtue of estimate (61)_2, for a suitable \( \delta \), we get
\[
\|D^2u\|_q \leq c(\|v_t\|_q + \|U_t\|_q + \|u\|_{L^p(\Omega \cap B_R)} + \|U\|_{L^q(\Omega \cap B_R)})
\]
\[
\leq c t^{-\frac{1}{2}-\mu} \|\nabla v_0\|_p, \text{ for } t \in (0,1),
\]  
(68)
where the constant \( c \) is independent of \( v_0 \) and \( t \). Employing estimate (17) of Lemma 2 we deduce
\[
\|\nabla u(t)\|_q \leq c \|D^2u(t)\|_q \|u(t)\|_{\frac{1}{2}-a} \text{ for } t \in (0,1).
\]
Hence estimate (61)_1 follows by means of (67)-(68). Employing again estimate (17), for all \( q > p \) we deduce that \( \|u(t)\|_q \leq c \|\nabla u(t)\|_p \|u(t)\|_{\frac{1}{2}-a} \), which completes the proof via (61)_1 and (67). □
Lemma 13. Let \( \Omega \) be an exterior domain and \( p \in (1, \infty) \). For \( q \geq p \) we set \( \mu = \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \). Then for the time derivative of solution \( u \) to problem (59) the following estimate holds:

\[
\| u_t(t) \|_q \leq c t^{-\frac{1}{2} - \mu} \| \nabla v_0 \|_p, \quad \text{for } t > 1,
\]

where \( c \) is a constant independent of \( v_0 \).

Proof. By virtue of the semigroup property (24)_1 for \( v_t \), that is \( \| v_t(t) \|_q \leq c(t-s)^{-\mu} \| v_0(s) \|_p \), and by virtue of (53)_2 for \( u_t \), since \( v = u + U \) we can limit ourselves to consider the proof for \( q = p \), that is \( \mu = 0 \). We distinguish the cases: \( p \in (1, \frac{1}{\mu}) \) and \( p \geq \frac{1}{\mu} \). In the latter case we have \( p' \in (1, n] \). We can deduce estimate (66) again, hence we get

\[
|(u_t(t), \varphi_0)| \leq |U_1(t)|_p |\varphi_0|_{p'} + \| \nabla v_0 \|_p \| \nabla \varphi(t) \|_{p'} \\
\leq c \| \nabla v_0 \|_p |\varphi_0|_{p'} t^{-\frac{1}{2}}, \quad \text{for } t > 1.
\]

Recalling (60), applying Lemma 4 we arrive at

\[
\| u_t(t) \|_p \leq c (\| \nabla v_0 \|_p t^{-\frac{1}{2}} + \| U_1 \cdot \nu \|_{p', p} \leq c (\| \nabla v_0 \|_p t^{-\frac{1}{2}} + \| U_1 \cdot \nu \|_{\infty})
\]

which, after applying (55)_3 for \( U_t \), implies (69)_2. Now we consider \( p \in (1, \frac{1}{\mu}) \). Hence we have \( p' \in (n, \infty) \). Since integrating by parts we get

\[
(u_t(s), \varphi(t-s)) = (\Delta u(s), \varphi(t-s)) = (U(s), \nu \cdot \nabla \varphi(t-s))_{\partial \Omega} + (u(s), \Delta \varphi(t-s))
\]

recalling that for \( s \to 0 \) we get both \( u(s, x) = v(x, s) - U(x, s) \to 0 \) in \( L^p(\Omega) \) and, by virtue of (55)_2, \( U(s) \to 0 \) in \( L^p(\partial \Omega) \), via the Green Identity (62), we deduce

\[
(u_t(t), \varphi_0) = \int_0^t (U_1, \nu \cdot T(\varphi, \pi))_{\partial \Omega} d\tau = I_1 + I_2, \quad \text{for } t > 1,
\]

where we have set

\[
I_1 := \int_0^t (U_1, \nu \cdot T(\varphi, \pi))_{\partial \Omega} d\tau \quad \text{and} \quad I_2 := \int_0^t (U_1, \nu \cdot T(\varphi, \pi))_{\partial \Omega} d\tau.
\]

Integrating by parts, recalling that, letting \( s \to 0 \), (55)_2 gives \( U(s, x) \to 0 \) in \( L^p(\partial \Omega) \), we obtain

\[
I_1 = (U_1(t), \nu \cdot T(\varphi, \pi)(\varphi))_{\partial \Omega} - \int_0^t (U_1, \nu \cdot T(\varphi, \pi))_{\partial \Omega} d\tau := I_{11} + I_{12}.
\]

\[\text{Actually we have}\frac{1}{2}
\]

\[
\| u_t(t) \|_q \leq |v_0(t)|_q + |U_1(t)|_q \leq c t^{-\mu} |v_3(\frac{1}{2})|_p + c t^{-\frac{1}{2} - \mu} |\nabla v_0|_p \\
\leq c t^{-\mu} \left[ |u_t(\frac{1}{2})|_p + |U_1(\frac{1}{2})|_p \right] + c t^{-\frac{1}{2} - \mu} |\nabla v_0|_p \leq c t^{-\mu} |u_t(\frac{1}{2})|_p + c t^{-\frac{1}{2} - \mu} |\nabla v_0|_p,
\]

where we have employed (53)_2 for \( U_t \) and (24)_1 for \( v_t \).
Applying Hölder’s inequality, employing (54)\textsubscript{1} for \( U \) and (44)\textsubscript{3} for the stress tensor, we get

\[
|I_{11}| \leq c\|U(x, t)\|_{\infty}\|T(\varphi, \pi_\varphi)(x, t)\|_{L^{p'}(\partial\Omega)} \leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}t^{\frac{1}{2} - \frac{2}{p'}}.
\]

Applying Hölder’s inequality, we get

\[
|I_{12}| \leq \int_0^t \|U\|_{L^p(\partial\Omega)}\|T(\hat{\varphi}, \pi_\varphi)\|_{L^{p'}(\partial\Omega)}d\tau + \int_0^t \|U\|_{\infty}\|T(\hat{\varphi}, \pi_\varphi)\|_{L^{p'}(\partial\Omega)}d\tau.
\]

Employing estimates (55)\textsubscript{2} and (54)\textsubscript{1} for \( U \) in the first and for the second integral, respectively, and (45)\textsubscript{3} for the stress tensor, we get

\[
|I_{12}| \leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}\left[t^{-1 - \frac{2}{p'}} + \int_0^t \tau^{\frac{1}{2} - \frac{2}{p'}}(t - \tau)^{-1 - \frac{2}{p'}}d\tau\right]
\]

\[
\leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}t^{-\frac{1}{2}}, \quad \text{for } t > 1,
\]

where we have taken into account that \( t > 1 \). Moreover for \( I_2 \) we obtain

\[
|I_2| \leq c\int_0^t \|U\|_{\infty}\|T(\hat{\varphi}, \pi_\varphi)\|_{p'}d\tau
\]

\[
\leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}\int_0^t \tau^{-\frac{1}{2}(1 + \frac{2}{p'})}\zeta(t - \tau)d\tau \leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}t^{-\frac{1}{2}}, \quad \text{for } t > 1,
\]

where, employing estimates (44), we considered \( \zeta(\sigma) = \begin{cases} \sigma^{\frac{-m}{2}} & \text{if } \sigma \in (0, 1), \\ \sigma^{-\frac{m}{2}} & \text{if } \sigma > 1 \text{ and } p' \geq m. \end{cases} \)

Increasing the right hand side of (71) by means of the estimates related to \( I_1 \) and the one relative to \( I_2 \), we get

\[
|\langle u_s(t), \varphi_0 \rangle| \leq \|U_s(t)\|_p\|\varphi_0\|_{p'} + \|\nabla v_0\|_p\|\nabla \varphi(t)\|_{p'}
\]

\[
\leq c\|\nabla v_0\|_p\|\varphi_0\|_{p'}t^{-\frac{1}{2}}, \quad \text{for } t > 1.
\]

Recalling (60), via Lemma 4 we obtain the estimate (70), that applying (53)\textsubscript{2} furnishes estimate (69) for \( q = p \).

**Lemma 14.** Assume that the Green identity (15) holds for the pairs \( (u, \pi_u) \) and \( (\hat{\varphi}, \pi_\varphi) \) solutions respectively to problem (59) and to problem (14) with \( \varphi_0 \in C_0(\Omega) \). Then, we get

\[
(u(t), \varphi_0) = \int_0^t \langle \nu \cdot T(\hat{\varphi}, \pi\varphi), U \rangle_{\partial\Omega}d\tau.
\]

**Proof.** Recalling (61)\textsubscript{3} and (55)\textsubscript{2}, letting \( s \to 0 \), then \( u(s) = v(s) - U(s) \to 0 \) in \( L^{p}(\Omega) \), as well, in \( L^{p}(\partial\Omega) \) follow, respectively. Hence, letting \( s \to 0 \) in the Green identity (15), we arrive at (72).
Lemma 15. Let $\Omega$ be an exterior domain and $p \in (1, n)$, $n \geq 2$. Then, for all $q \geq p$, the solution $u$ to problem (59) enjoys the following estimates:

$$\|u(t)\|_q \leq c\|\nabla v_0\|_p \begin{cases} t^{\frac{n}{2}-\mu} & \text{if } t > 1, n = 2, \\ t^{-\frac{n}{2}}(t - \tau)^{-\frac{\mu}{2}} & \text{if } t > 1, q \in [p, n), n \neq 2, \\ t^{-\frac{n}{2} + \frac{\mu}{2}} & \text{if } t > 1, q = n > 3, \\ t^{-\frac{n}{q} + \frac{\mu}{q}} & \text{if } t > 1, q > n \geq 3, \\ t^{-\frac{n}{q} + \frac{\mu}{q}}, & \text{if } t > 1, q = n = 3, \end{cases}$$

(73)

where constant $c$ is independent of $v_0$.

Proof. We recall (72):

$$(u(t), \varphi_0) = \int_0^t (\nu \cdot T(\hat{\varphi}, \pi_\varphi), U)_{\partial \Omega} d\tau.$$ 

In order to discuss the last integral we have to distinguish the cases $n = 2, 3$ and $n > 3$.

**n=2.** Applying H"older’s inequality, we get

$$|(u(t), \varphi_0)| \leq c\int_0^t \|\nabla v_0\|_p \rho_0 \|\varphi_0\|_q \int_0^t (t - \tau)^{-\mu_0} d\tau + c\int_0^t \|\nabla v_0\|_p \rho_0 \|\varphi_0\|_q \int_0^t (t - \tau)^{\frac{n}{2} - \frac{\mu}{2}} d\tau.$$ 

(74)

We recall (54)\textsubscript{1} for $U$, and, remarking that the best bound for the latter integral is for $r > 2$, via (44)\textsubscript{1,3} for $T(\hat{\varphi}, \pi_\varphi)$, we obtain:

$$|(u(t), \varphi_0)| \leq c\|\nabla v_0\|_p \|\varphi_0\|_q \int_0^t (t - \tau)^{\frac{n}{2} - \frac{\mu}{2}} (t - \tau)^{-\mu_0} d\tau + c\|\nabla v_0\|_p \|\varphi_0\|_q \int_0^t (t - \tau)^{\frac{n}{2} - \frac{\mu}{2}} (t - \tau)^{-\mu_0} d\tau.$$ 

(75)

**n=3.** Partially the argument is the same of the case $n = 2$:

$$(u(t), \varphi_0) = \int_0^t (\nu \cdot T(\hat{\varphi}, \pi_\varphi), U)_{\partial \Omega} d\tau + \int_0^t (\nu \cdot T(\hat{\varphi}, \pi_\varphi), U)_{\partial \Omega} d\tau, \quad t > 1.$$ 

Applying H"older’s inequality, for all $r > q'$, we get

$$|(u(t), \varphi_0)| \leq c\int_0^t \|\nabla v_0\|_p \rho_0 \|\varphi_0\|_{q'} \int_0^t (t - \tau)^{-\mu_0} d\tau + c\int_0^t \|\nabla v_0\|_p \rho_0 \|\varphi_0\|_{q'} \int_0^t (t - \tau)^{\frac{n}{2} - \frac{\mu}{2}} d\tau.$$ 

Recalling estimates (44)\textsubscript{1,2} for the stress tensor, and estimate (54)\textsubscript{1} for $U$, we obtain

$$|(u(t), \varphi_0)| \leq c\|\nabla v_0\|_p \|\varphi_0\|_{q'} \int_0^t (t - \tau)^{-\mu_0} d\tau + c\|\nabla v_0\|_p \|\varphi_0\|_{q'} \int_0^t (t - \tau)^{\frac{n}{2} - \frac{\mu}{2}} d\tau.$$ 

(76)
Applying Hölder’s inequality, we get
\begin{equation}
(u(t), \varphi_0) = \int_0^t (\nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}}), U)_{\partial \Omega} d\tau =: I_1 + I_2 + I_3, \ t > 1.
\end{equation}

By virtue of estimates (44)_{3} for the stress tensor, and estimate (55)_{2} for \( U \), since \( q' \leq p' \) we get
\begin{align*}
|I_1| & \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} \int_0^t (t - \tau)^{-\frac{1}{2}(1 - \delta)} d\tau \\
& \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} t^{-\frac{1}{2}(\frac{1}{q'} - \frac{1}{q})}, \ t > 1,
\end{align*}
where we have employed the assumption \( t > 1 \). Applying Hölder’s inequality, for the term \( I_2 \) we get
\begin{align*}
|I_2(t)| & = \int_0^t (\nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}}), U)_{\partial \Omega} d\tau \leq c \int_0^t \|U(t)\|_{L^p(\partial \Omega)} \|\nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}})\|_{L^{q'}(\partial \Omega)} d\tau.
\end{align*}

By virtue of estimates (44)_{3} for the stress tensor, and estimate (54)_{1} for \( U \), we get
\begin{align*}
|I_2| & \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} \int_0^t \tau^{\frac{1}{2}(1 - \frac{1}{q'})} (t - \tau)^{-\frac{1}{2}(1 - \frac{1}{q})} d\tau \\
& \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} t^{-\frac{1}{2}(\frac{1}{q'} - \frac{1}{q})}, \ t > 1.
\end{align*}

Finally, applying Hölder’s inequality, we get
\begin{align*}
|I_3| & = \int_0^t (\nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}}), U)_{\partial \Omega} d\tau \leq c \int_0^t \|U\|_{L_\infty(\partial \Omega)} \|\nu \cdot T(\tilde{\varphi}, \pi_{\tilde{\varphi}})\|_{q'} d\tau, \ t > 1.
\end{align*}

By virtue of estimates (44)_{1} for the stress tensor, and estimate (54)_{1} for \( U \), we get
\begin{align*}
|I_3| & \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} \int_0^t \tau^{\frac{1}{2}(1 - \frac{1}{q'})} (t - \tau)^{-m_0} d\tau \\
& \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} \begin{cases} t^{-\frac{1}{2}(\frac{1}{q'} - \frac{1}{q})}, & \text{if } q \leq n, \\ t^\frac{1}{2} - \frac{n}{2q'}, & \text{if } q > n. \end{cases}
\end{align*}

Collecting the estimates related to \( I_1, I_2, I_3 \), via the Green formula (77), we obtain
\begin{equation}
|(u(t), \varphi_0)| \leq c \|Dv_0\|_p \|\varphi_0\|_{q'} \begin{cases} t^{-\frac{1}{2}(\frac{1}{q'} - \frac{1}{q})}, & \text{if } q \leq n, \\ t^\frac{1}{2} - \frac{n}{2q'}, & \text{if } q > n, \end{cases} \text{ for all } t > 1.
\end{equation}
We are in a position to prove (73). Since estimates (74), (76) for \( q \neq 3 \), and (78) hold for all \( \varphi_0 \in \mathcal{C}_0(\Omega) \), recalling (60) for \( u \), via Lemma 4 and estimates (55), for the norm \( \|u \cdot \nu\|_{\tilde{\mathbb{L}}^q} \equiv \|U \cdot \nu\|_{\tilde{\mathbb{L}}^q} \), one proves (73) in all the cases with exclusion of (73)5. For this last, employing (73)2 and (73)4 for \( n = 3 \), we prove (73)5 interpolating \( \|u(t)\|_3 \) between \( q_1 > 3 \) and \( q_2 = 3 - \eta > p \). Hence we get
\[
\|u(t)\|_3 \leq \|u(t)\|_{\tilde{\mathbb{L}}_q}^{a} \|u(t)\|_{\tilde{\mathbb{L}}_q}^{1-a} \leq c\|\nabla v_0\|_{p}t^{-\vartheta}, \quad t > 1,
\]
where we set \( \vartheta := \frac{3}{2p} - \frac{a}{2} - \frac{1}{2} \) with \( a := \frac{\eta q_1}{3(q_1 - 3 + \eta)} \), that proves the result for \( q_1 \) sufficiently large and \( \eta \) sufficiently small. The lemma is completely proved. 

Lemma 16. Let \( \Omega \) be an exterior domain and \( q \geq p \geq n \), \( n \geq 2 \). Then for the solution \( u \) to problem (59) the following estimates hold:
\[
\|u(t)\|_q \leq c\|\nabla v_0\|_{p} \Gamma(t), \quad t > 1, \tag{79}
\]
where we have set \( \Gamma(t) := \begin{cases} \frac{t^{\frac{1}{2}}}{p} & \text{if } q \geq p \geq 2, \quad n = 2, \\ \frac{\log(t+e)}{q} & \text{if } q \geq p \geq n \geq 2, \\ \log(t+e) & \text{if } q \geq p = n, \quad n > 3, \\ \log^2(t+e) & \text{if } q = p = n = 3. \end{cases} \)

Proof. We start from the Green identity (72) for \( u \):
\[
(u(t), \varphi_0) = \int_{\partial \Omega} (u(T(\tilde{\varphi}, \pi \tilde{\varphi})), U_{\partial \Omega}) d\tau = I_1 + I_2, \quad t > 1, \tag{80}
\]
where
\[
I_1 := \int_{0}^{t-1} (u(T(\tilde{\varphi}, \pi \tilde{\varphi})), U_{\partial \Omega}) d\tau, \\
I_2 := \int_{t-1}^{t} (u(T(\tilde{\varphi}, \pi \tilde{\varphi})), U_{\partial \Omega}) d\tau.
\]

Applying Hölder’s inequality, for the term \( I_1 \) we get:
\[
|I_1| \leq c \int_{0}^{t-1} \|T(\tilde{\varphi}, \pi \tilde{\varphi})\|_{L^{\infty}(\partial \Omega)} \|U\|_{L^{\infty}(\partial \Omega)} d\tau.
\]

Employing (55)1 for \( U \), and, since \( q' \leq p' \leq n' \), employing (44)3 for the stress tensor, we obtain
\[
|I_1| \leq c\|\nabla v_0\|_{p} \|\varphi_0\|_{q'} \int_{0}^{t-1} \zeta_0(\tau)(t-\tau)^{-\frac{q}{2}(1-\frac{1}{q'})} d\tau \leq c\|\nabla v_0\|_{p} \|\varphi_0\|_{q'} \Gamma(t).
\]
Finally, we estimate $I_2$. Applying Hölder’s inequality, employing \((55)_1\) for $U$ and \((44)_1\) for the stress tensor, recalling Remark 1, we get

$$
|I_2| \leq c \int_{t-1}^{t} \|T(\bar{\varphi}, \pi \bar{\varphi})\|_{q'} \|U\|_{L^\infty(\partial \Omega)} d\tau \leq c \|\nabla v_0\|_p \|\varphi_0\|_{q'} \int_{t-1}^{t} \zeta_b(\tau) (t-\tau)^{-\beta_0} d\tau
$$

$$
\leq c \|\nabla v_0\|_p \|\varphi_0\|_{q'} \Gamma(t).
$$

Hence we deduce

$$
|(u(t), \varphi_0)| \leq c \|\nabla v_0\|_p \|\varphi_0\|_{q'} \Gamma(t), \quad \text{for all } \varphi \in \mathcal{C}_0(\Omega) \text{ and } t > 1.
$$

Recalling \((60)\) for $u$, via estimate \((21)\) and estimate \((55)_1\) one completes the proof. \(\square\)

**Lemma 17.** Let $\Omega$ be an exterior domain and $n \geq 2$. Then for the solution $u$ to problem \((59)\), for $q \geq p$, the following estimate holds:

$$
\|\nabla u(t)\|_q \leq c \|\nabla v_0\|_p \widehat{g}_p(t),
$$

where the constant $c$ is independent of $u$ and $\widehat{g}_p(t)$ is defined by

$$
\widehat{g}_p(t) :=
\begin{cases}
t^{-\mu}, & t \in (0, 1), \quad \text{if } q \geq p > 1, \\
(t^2 - \mu^-)^+ t > 1, & \text{if } n = 2, \ p \neq 2, \\
\log^+(t + e), & t \geq 1, \quad \text{if } q = p = n = 3, \\
\log(t + e), & t \geq 1, \quad \text{if } q \geq p = n = 3, \\
(t^2 - \mu^-)^+ t > 0, & \text{if } q \in [p, n), n > 2 \\
t^{-\mu^-}, & t \geq 1, \quad \text{if } q \geq p \neq n > 3.
\end{cases}
$$

**Proof.** The estimates for $t \in (0, 1)$ are contained in \((61)_1\). Hence we limit ourselves to look for the estimates for $t \geq 1$. Since equation \((1)_1\) ensures $v_t = P\Delta v$, by virtue of Lemma 3 we get

$$
\|D^2 v(t)\|_q \leq c (\|v(t)\|_q + \|v(t)\|_{L^\infty(\Omega')}).
$$

Since $v = U + u$, and $\Omega'$ is bounded, for all $r > q$, we deduce

$$
\|D^2 u(t)\|_q \leq \|D^2 U(t)\|_q + \|v(t)\|_q + \|u(t)\|_{L^\infty(\Omega' \cap B_R)} + \|u(t)\|_{L^\infty(\Omega' \cap B_R)}).
$$

Hence via Lemma 11 and Corollary 4 for $U$, Lemma 13 for $u_t$, Lemma 15 - Lemma 16 for $u$, for $q \geq p \neq 2$ and $n = 2$, we get

$$
\|D^2 u(t)\|_q \leq c \|\nabla v_0\|_p t^{\frac{2}{p} + \mu^- + \frac{1}{q}}, \quad \mu > 0, \ t > 1,
$$

and, for $q \geq p$ and $n \neq 2$, we get

$$
\|D^2 u(t)\|_q \leq c \|\nabla v_0\|_p t^{\frac{1}{p} - \frac{n}{q}}\left\{
\begin{array}{ll}
t^\mu, & \text{if } q < p, \\
\log(t + e), & \text{if } q = n, \\
(t^2 - \mu^-)^+ & \text{if } q > n,
\end{array}\right.
$$

Employing Lemma 2, for all $q \in (1, \infty)$, we obtain

$$
\|\nabla u(t)\|_q \leq c \|D^2 u(t)\|_q^\frac{1}{q} \|u(t)\|_q^\frac{1}{q}, \ t > 1.
$$

Hence, via estimate \((86)\) (resp. \((85)\) for $n = 2$) for $D^2 u$, and via estimates \((73)\) and \((79)\) for $u$, we get \((81)\) with $\widehat{g}_p$ given by \((82)\). \(\square\)
5 Some consequences of the results of Section 3 and Section 4

In this section, we assume \( v_0 \in \mathcal{C}_0(\Omega) \) and we establish some properties of the solutions to problem (1) whose existence is ensured by Corollary 2.

**Lemma 18.** Let \( v_0 \in \mathcal{C}_0(\Omega) \) and \((v, \pi_v)\) be the solution of Corollary 2, then \( \nabla v \in C([0, T); L^p(\Omega)) \) holds with \( \lim_{t \to 0} \|\nabla v(t) - \nabla v_0\|_p = 0. \)

**Proof.** By virtue of Lemma 11, \( \nabla U \in C([0, T); L^p(\mathbb{R}^n)) \) with \( \lim_{t \to 0} \|\nabla U(t) - \nabla v_0\|_p = 0. \)

Since \( v = U + u \), the result is achieved if we are able to prove that \( \nabla u \in C([0, T); L^p(\Omega)) \) and \( \lim_{t \to 0} \|\nabla u(t)\|_p = 0. \) From formula (63), applying Hölder’s inequality, via estimate (56), we get

\[
|(u_t(t), \varphi_0)| \leq |(v_s, \varphi(t-s))| + c\|D^2v_0\|_p(\|\varphi_0\|_{p'} + \|\varphi(t-s)\|_{p'}), \ t > 0.
\]

Then the limit property (64) for \( U_i \) and estimate (24)\(_1\) for \( \varphi \) furnish

\[
\|u_t(t)\|_p \leq c\|D^2U_0\|_p, \text{ for all } t > 0.
\]

Via the Minkowski inequality, employing again (56), easily it holds

\[
\|v_t\|_p \leq \|U_t\|_p + \|u_t\|_p \leq c\|D^2v_0\|_p \text{ for all } t > 0.
\]

Since equation (1)\(_1\) ensures \( v_t = P\Delta v \), by virtue of Lemma 3 we get

\[
\|D^2v(t)\|_q \leq c(\|v_t(t)\|_q + \|v(t)\|_{L^q(\Omega)}).
\]

Hence, via (24)\(_1\) for \( \|v(t)\|_p \), the following estimate holds

\[
\|D^2v(t)\|_p \leq c(\|D^2v_0\|_p + \|v_0\|_p).
\]

As well, applying the Minkowski inequality and again (56), we get

\[
\|D^2u(t)\|_p \leq c(\|D^2v_0\|_p + \|v_0\|_p) \text{ for all } t > 0. \tag{87}
\]

Via inequality (17), for all \( t \) and \( s \) we obtain

\[
\|\nabla v(t) - \nabla v(s)\|_p \leq c\|D^2v(t) - D^2v(s)\|_p^{\frac{1}{2}} \|v(t) - v(s)\|_p^{\frac{1}{2}} \leq c(\|D^2v_0\|_p + \|v_0\|_p)^{\frac{1}{2}} \|v(t) - v(s)\|_p^{\frac{1}{2}},
\]

that, via Corollary 2, furnishes the continuity, and as well the one of \( \nabla u(t) \) holds. Applying inequality (17), we obtain

\[
\|\nabla u(t)\|_p \leq c\|D^2u(t)\|_p^{\frac{1}{2}} \|u(t)\|_p^{\frac{1}{2}}, \text{ for all } t > 0.
\]

Hence the limit property for \( \|\nabla u(t)\|_p \) follows from (87) and (61)\(_3\) for \( \|u(t)\|_p. \)

**Lemma 19.** Let \((v, \pi_v)\) be the solution of Corollary 2, then the following estimates hold:

\[
\|v_t(t)\|_q \leq c\|\nabla v_0\|_p^{1 - \frac{q}{2} + \frac{q}{p}(\frac{1}{p} - \frac{1}{q})}, \ t > 0, \tag{88}
\]

where \( c \) is a constant independent of \( v \).
Proof. Estimate (88) is an immediate consequence of estimates (69) and (55)2.

Lemma 20. Let $p \in (1, \infty)$, and $q \geq p$. Set $\mu := \frac{p}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, for solution $(v, \pi_v)$ furnished by Corollary 2 enjoys the following estimates hold:
\[
\|\nabla v(t)\|_q \leq c t^{-\mu}\|\nabla v_0\|_p, \quad \text{for } t \in (0, 1), \tag{89}
\]
with $c$ independent of $v$.

Proof. We consider $v = U + u$. Hence the result is a consequence of Lemma 11 for $\nabla U$ and Lemma 12 for $\nabla u$.

Lemma 21. Let $\Omega$ be an exterior domain and $n \geq 2$. Then for the solution $(v, \pi_v)$ furnished by Corollary 2 enjoys the following estimates hold:
\[
\|\nabla v(t)\|_q \leq c\|\nabla v_0\|_p \hat{g}_p(t), \quad \text{for } t > 1, \quad \text{and } q \geq p, \tag{90}
\]
where the constant $c$ is independent of $v$ and $\hat{g}_p(t)$ is defined by
\[
\hat{g}_p(t) := \begin{cases} 
    t^{-\mu}, & t \in (0, 1), \quad \text{if } q \geq p > 1, \\
    t^{\frac{n}{2} - \mu}, & t > 1, \quad \text{if } n = 2, \quad p \neq 2, \\
    \log^2(t + e), & t \geq 1, \quad \text{if } q = p = n = 3, \\
    \log(t + e), & t \geq 1, \quad \text{if } q > p = n = 3, \\
    t^{-\mu}, & t > 0, \quad \text{if } q \in [p, n), \quad n > 2, \\
    t^{-\frac{2}{2} - \frac{4}{p}}, & t \geq 1, \quad \text{if } q \geq p \neq n > 3.
\end{cases} \tag{91}
\]

Proof. We consider $v = U + u$. Hence the result is a consequence of Lemma 11 for $\nabla U$ and Lemma 17 for $\nabla u$.

Lemma 22. Let $\Omega$ be an exterior domain and $n = 2$. Then, for $q \geq 2$ the solution $(v, \pi_v)$ of Corollary 2 is such that
\[
\|\nabla v(t)\|_q \leq \tilde{g}_2(t)\|\nabla v_0\|_2, \quad t > 1, \quad \tilde{g}_2(t) := \begin{cases} 
    1 & \text{if } q = 2, \\
    c > 1 & \text{if } q > 2, \tag{92}
\end{cases}
\]
where $c$ is a constant independent of $v$.

Proof. In order to prove (92), by virtue of Lemma 3, employing the Poincaré inequality, we easily get
\[
\|D^2v(t)\|_2 \leq c(\|v(t)\|_2 + \|v(t)\|_{L^2(\Omega)}) \leq c(\|v(t)\|_2 + \|\nabla v(t)\|_2).
\]
Since the $L^2$-theory ensures $\|\nabla v(t)\|_2 \leq \|\nabla v_0\|_2, \quad t > 0$, for all $v_0 \in C_0(\Omega)$, that is (94) with $\hat{g}_2 = 1$, employing (88), we arrive at
\[
\|D^2v(t)\|_2 \leq c\|\nabla v_0\|_2(1 + t^{-\frac{2}{2}}), \quad t > 1. \tag{93}
\]
By virtue of Lemma 2 we get
\[
\|\nabla v(t)\|_q \leq c\|D^2v(t)\|_2\|\nabla v(t)\|^{1-a}_2, \quad t > 1, \quad a := \frac{q-2}{q}.
\]
Employing (93), and employing again $\|\nabla v(t)\|_2 \leq \|\nabla v_0\|_2$, we conclude the proof with $\tilde{g}_2 \equiv c$. \qed
Lemma 23. Let \( \Omega \) be an exterior domain and \( n > 2 \). In (1) assume \( v_0 \in \mathcal{C}_0(\Omega) \). Then, for \( q \geq n, p \in (1, n) \) the solution \( (v, \pi_v) \) of Corollary 2 is such that

\[
\| \nabla v(t) \|_q \leq c \tilde{g}_p(t) \| \nabla v_0 \|_p, \quad t \geq 1, \quad \tilde{g}_p(t) := \begin{cases} t^{\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}, & \text{if } q > 3, \\ \frac{t^{\frac{n}{2}}}{\frac{n}{2} - \frac{1}{2}}, & \text{if } q = 3, \end{cases}
\]

(94)

where \( c \) is a constant independent of \( v \).

Proof. We look for an estimate for \( \| \nabla u(t) \|_q \). In our hypotheses we can deduce again (84), that is, for all \( r > q > p \),

\[
\| D^2 u(t) \|_q \leq \| D^2 U(t) \|_q + c(\| v_I(t) \|_q + \| U(t) \|_{L^\infty(\Omega)} + \| u(t) \|_{L^r(\Omega)}) .
\]

By virtue of estimates (88), (54)_1 and (73)_4, we get

\[
\| D^2 u(t) \|_p \leq c \| \nabla v_0 \|_{\frac{n}{2} - \frac{1}{p}}, \quad t > 1.
\]

(96)

By estimate (2), furnishes \( \| \nabla u(t) \|_q \leq c \| D^2 u(t) \|_q^{\frac{2}{q}} \| u(t) \|_q^{\frac{2}{r}} \), applying (96) and (73)_4 for \( q = n > 3 \) and (73)_5 for \( q = n = 3 \), we get

\[
\| \nabla u(t) \|_p \leq c \| \nabla v_0 \|_{\frac{n}{2} - \frac{1}{p}}, \quad \text{if } q = n > 3, \quad t > 1,
\]

\[
\| \nabla u(t) \|_p \leq c \| \nabla v_0 \|_{\frac{n}{2} - \frac{1}{2}}, \quad \text{if } q = n = 3, \quad t > 1,
\]

moreover, applying (96) and (73)_4, for \( q > n \), we get

\[
\| \nabla u(t) \|_q \leq c \| \nabla v_0 \|_{\frac{n}{2} - \frac{1}{p}}, \quad \text{if } n \geq 3, \quad t > 1.
\]

Since \( v = U + u \), recalling (53)_1, we deduce the result (94) via the Minkowski inequality.

\[ \square \]

6 Proof of Theorem 1, Theorem 2 and Proposition 1

6.1 Proof of Theorem 1.

Let \( v_0 \in J^p_0(\Omega) \). Since \( \Omega \) is bounded, \( v_0 \in J^{1,p}(\Omega) \). Moreover, there exists a sequence \( \{ v_0^k \} \subset \mathcal{C}_0(\Omega) \) which converges to \( v_0 \) in \( J^{1,p}(\Omega) \). We denote by \( (v_k, \pi_{v_k}) \) the sequence of solutions ensured by Lemma 6 and enjoying the estimates (27). We consider the decomposition \( v^k := U^k + u^k \). Using the linearity of the Stokes problem, by virtue of estimate (53)_1 for \( \{ U^k \} \) and estimate (61)_1 for \( \{ u^k \} \), we get \( \| \nabla v^k(t) - \nabla v^m(t) \|_q \leq c t^{-\mu} \| v_0^k - v_0^m \|_p \), for \( t \in (0, 1) \). For \( t \geq 1 \) we employ estimates (27), hence \( \| \nabla v^k(t) - \nabla v^m(t) \|_q \leq c t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})} \exp[-\gamma t \| v_0^k - v_0^m \|_p] \), that we increase via the Poincaré inequality. An analogous argument is developed in the case of \( \{ v_k \} \). We employ (53)_2 for \( \{ U^k \} \) and estimate (61)_2 for \( \{ u^k \} \), provided that \( t \in (0, 1) \). In the case of \( t \geq 1 \), we employ (27) for \( \{ u_k \} \). Hence for the linearity of the problem (1) we get that \( \| v^k_t - v^m_t \|_q \leq c t^{-\frac{n}{2} - \mu} \| v_0^k - v_0^m \|_p \). For \( \{ D^2 v^k \} \) and \( \{ \pi_{v_k} \} \) we employ Lemma 3. Then, for all \( t > 0 \), the sequence \( \{ v^k \} \) enjoys the Cauchy condition. We denote by \( (v, \pi_v) \) the limit. Since for \( q = p \) the above Cauchy conditions for \( \{ \nabla v^k \} \) are uniform with respect to \( t \), and by
virtue of Lemma 18 \( \{v^k\} \subset C([0, T); J^p_0(\Omega)) \) with \( \lim_{t \to 0} \|\nabla v^k(t) - \nabla v_0^k\|_p = 0 \), we get that the limit \( v \in C([0, T); J^p_0(\Omega)) \) and \( \lim_{t \to 0} \|\nabla v(t) - \nabla v_0\|_p = 0 \). Therefore the limit \( (v, \pi_v) \) of the sequence \( \{v^k\} \) enjoys the estimates (5)-(6). The uniqueness holds as in the case of the usual \( L^p \)-theory.

\[ \square \]

6.2 Proof of Theorem 2.

Existence. Let \( v_0 \in J^p_0(\Omega) \). We denote by \( \{v^k_0\} \subset \mathcal{C}_0(\Omega) \) a sequence converging to \( v_0 \) in \( J^p_0(\Omega) \). By virtue of Corollary 2, we denote by \( \{(v^k, \pi_v^k)\} \) the sequence of solutions to problem (1). We also set \( v^k := U^k + u^k \) and \( \pi^k_v := \pi^k_{u^k} \). Hence, by virtue of the linearity of problem (1), employing Lemma 20, Lemma 21, Lemma 22 and Lemma 23, we get

\[ \|\nabla v^k(t) - \nabla v^m(t)\|_q \leq g_p(t)\|\nabla v^k_0 - \nabla v^m_0\|_p, \quad \text{for all} \quad k, m \in \mathbb{N} \quad \text{and} \quad t > 0, \]

where, collecting the estimates given in Lemma 20, Lemma 21, Lemma 22 and Lemma 23, we tacitly defined \( g_p(t) \) as made in the statement of Theorem 1. Moreover, by virtue of Lemma 19, for \( \mu := \frac{q}{p} (\frac{1}{p} - \frac{1}{q}) \), we get

\[ \|v^k(t) - v^m(t)\|_q \leq c t^{-\frac{1}{q} - \mu} \|\nabla v^k_0 - \nabla v^m_0\|_p, \quad \text{for all} \quad k, m \in \mathbb{N} \quad \text{and} \quad t > 0. \]

Finally, employing Lemma 3, via the above estimates for \( v^k - v^m \) and the Poincaré inequality, we get

\[ \|D^2 v^k(t) - D^2 v^m(t)\|_q + \|\nabla \pi_{v^k} - \nabla \pi_{v^m}\|_q \leq c g_p(t)\|\nabla v^k_0 - \nabla v^m_0\|_p, \quad \text{for all} \quad k, m \in \mathbb{N} \quad \text{and} \quad t > 0. \]

Since the right hand side of the above estimates satisfies the Cauchy condition in \( J^p_0(\Omega) \), we get the existence of strong limit \( (v, \pi_v) \) solutions to problems (1). Since for \( q = p \) the above Cauchy conditions for \( \{\nabla v^k\} \) are uniform with respect to \( t \) on any compact interval \([0, T] \), as proved in the case of \( \Omega \) bounded, we get that the limit \( v \in C([0, T); J^p_0(\Omega)) \) and \( v(t, x) \) assume the initial data \( v_0(x) \) by continuity in the norm of \( J^p_0(\Omega) \). The pair \( (v, \pi_v) \) is a solution to problem (1) and enjoys property (7)-(8), the proof of the existence is completed.

Uniqueness. We prove that in the class of existence for \( v_0 \) the unique solution is identically equal to 0. Since \( \nabla v \in C([0, T); J^p_0(\Omega)) \), for all \( t > 0 \) and \( R > 0 \), via the Poincaré inequality, we get

\[ v \in C([0, T); L^p(\Omega \cap B_R)). \]

Since \( v_t \in L^1(0, T; L^p(\Omega)) \), \( v = 0 \) in \( t = 0 \), for all \( t > 0 \) the following also holds:

\[ \|v(t)\|_{L^p(\Omega \cap B_R)} \leq \int_0^t \|v_\tau(\tau)\|_{L^p(\Omega \cap B_R)} d\tau \leq \int_0^t \|v_\tau(\tau)\|_p d\tau. \]

The last inequality holds uniformly in \( R > 0 \). Hence letting \( R \to \infty \), we get \( v \in L^\infty(0, T; L^p(\Omega)) \). Now, the uniqueness follows by the one of the usual \( L^p \)-theory.

\[ \square \]
6.3 Proof of Proposition 1.

We start proving point i. We can assume \( v_0 \in \mathcal{C}_0(\Omega) \). We employ the optimality already known for \( \mu_1 \) in (24). That is, we verify that if (12) holds, then (46) also is true. Hence we arrive at a contradiction. In the case of (10), assume \( q \geq n \) and \( p \in \left[ \frac{3}{2}, n \right) \). Then, under assumption (12)1, recalling (24)2, we get

\[
\| \nabla v(t) \|_p \leq \xi \left( \frac{1}{2} \right)^{\frac{1}{p} - \frac{1}{q}} \| \nabla v(t) \|_p \leq c \xi \left( \frac{1}{2} \right)^{- \frac{1}{p} - \mu} \| v_0 \| \frac{1}{2} = c \xi \left( \frac{1}{2} \right)^{t - 1} \| v_0 \| \frac{1}{2}, \quad t > \max \{ 2, t_0 \},
\]

that is (46). Now let us consider the case of \( q \in [p, n) \). The argument is similar. Assume that (12)1 holds for \( q \in [p, n) \), \( p \geq \frac{n}{2} \). Then, for \( r \geq n \), via (10)3 and (24)2, we also get

\[
\| \nabla v(t) \|_r \leq c t^{\frac{1}{2} - \frac{1}{r}} \| \nabla v(t) \|_r \leq c \xi \left( \frac{1}{2} \right)^{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}} \| v_0 \| \frac{1}{2} \leq c \xi \left( \frac{1}{2} \right)^{- 1} \| v_0 \| \frac{1}{2}, \quad t > \max \{ 4, t_0 \},
\]

that is (97), which is false. Considering the properties (24) of \( v \) in the case of \( n = 3 \), one achieves the proof of (11)4 by repeating the same arguments. Actually, assuming that (12)2 holds for some \( p \in \left[ \frac{3}{2}, n \right) \), and \( v_0 \in \mathcal{C}_0(\Omega) \), by virtue of (24)2, we get the following estimate:

\[
\| \nabla v(t) \|_3 \leq \xi \left( \frac{1}{2} \right)^{\frac{1}{2} - \frac{1}{3}} \| \nabla v(t) \|_3 \leq \xi \left( \frac{1}{2} \right)^{- \frac{1}{2} + \frac{1}{2} - \frac{1}{3}} \| v_0 \| \frac{1}{3} \leq \xi \left( \frac{1}{2} \right)^{- 1 + \frac{1}{2}} \| v_0 \| \frac{1}{3},
\]

where in the last step we set \( \theta = \frac{3}{2} \left( \frac{1}{p} - \frac{1}{3} \right) - \delta > 0 \). So that, by the assumption (12)2, for all \( \delta \in (0, \frac{3}{2} \left( \frac{1}{p} - \frac{1}{3} \right) ) \) we find \( \xi(t) := \xi \left( \frac{1}{2} \right)^{t - \delta} \) such that (46) holds for \( q = 3 \) and \( s = \frac{n}{2} \), that is an absurdum. To prove point ii. of the proposition it is enough to consider a solution ensured by Theorem 3. Such a solution solves problem (1) and satisfies estimate (8) with \( g_0(t) \) constant \( \geq 1 \).

\[ \square \]

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