Statistics of polymer extensions in turbulent channel flow

Faranggis Bagheri,1 Dhrubaditya Mitra,2 Prasad Perlekar,3 and Luca Brandt1,†

1Linné Flow Centre, KTH Mechanics, SE-104 44 Stockholm, Sweden
2NORDITA, Roslagstullsbacken 23, 106 91 Stockholm, Sweden
3Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

We present direct numerical simulations (DNSs) of turbulent channel flow with passive Lagrangian polymers. To understand the polymer behavior we investigate the behavior of infinitesimal line elements and calculate, for the first time, the PDF of finite-time Lyapunov exponents and from them the corresponding Cramer’s function for the channel flow. We study the statistics of polymer elongation for both the Oldroyd-B model (for Weissenberg number $Wi < 1$) and the FENE model. We use the location of the minima of the Cramer’s function to define the Weissenberg number precisely such that we observe coil-stretch transition at $Wi \approx 1$. For linear polymers (Oldroyd-B model), with $Wi < 1$, we find that the PDF of polymer extensions shows power-law behavior with an exponent consistent with earlier analytical results.1 For $Wi > 1$ (FENE model) the polymers are significantly more stretched near the wall than at the centre of the flow. Furthermore near the wall the polymers show a strong tendency to orient along the stream-wise direction of the flow but near the centerline the statistics of orientation of the polymers is consistent with analogous results obtained recently in homogeneous and isotropic flows.2

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I. INTRODUCTION

Turbulent flows with polymer additives have been an active field of interest since the early experiments of Toms3 which showed the phenomenon of drag reduction on the addition of small amounts (few parts per million) of long-chained polymers to turbulent wall-bounded flows. If disturbed from its equilibrium configuration it relaxes back and for simplest polymer models this relaxation is governed by a single time scale $\tau_{\text{poly}}$. If such a polymer is then placed in a straining flow where the strain can be characterised by inverse of a time scale $\tau_{\text{fluid}}$ the polymer can go from its coiled state to a stretched state if the ratio of the two time scales, the Weissenberg number $Wi > 1$.4 Thus in turbulent flows with strong strain the polymers can go through coil-stretch transition; the stretched polymers can then make significant contributions to the Reynolds stresses and this can result in drag-reduction5 3. Thus to understand drag-reduction we must understand not only the mechanism of coil-stretch transition and but also the statistical properties of the stretched polymers.

On the analytical side, the mechanism of coil-stretch transition in synthetic turbulent flows for simple dumbell models of polymers have been elucidated in Refs.1 7 8. These studies use a flow that is white-in-time and smooth on the smallest scales – the so called Batchelor-Kraichnan flow – and linear and nonlinear models for polymers with a single characteristic relaxation time $\tau_{\text{poly}}$. Further they ignore the backreaction of the polymers to the fluid. For linear polymers below coil-stretch transition Ref.1 predicts a power-law tail for the probability distribution function (PDF) of polymer extension. The exponent of the power-law $\alpha$ can be obtained via the Cramer’s function of the PDF of the finite-time Lyapunov exponent of Lagrangian particles in the flow, v.i.z.,

$$\alpha = S'(\beta + \frac{1}{\tau_{\text{poly}}} - \bar{\mu})$$

(1)

where $\beta$ must be obtained by solving the differential equation,

$$S(\beta + \frac{1}{\tau_{\text{poly}}} - \bar{\mu}) - \beta S'(\beta + \frac{1}{\tau_{\text{poly}}} - \bar{\mu}) = 0$$

(2)

Here $S$ is the Cramer’s function and $\bar{\mu}$ is the location of the minima of $S$. These relationships have been numerically verified for homogeneous and isotropic two dimensional flows9 but not yet for wall bounded flows.

In this paper we confirm the existence of this relationship by Direct Numerical Simulation (DNS) of a channel flow with polymer additives. The polymers are modelled by Lagrangian passive dumbells. We also calculate for the first time, the PDF of finite-time Lyapunov exponents for Lagrangian particles in a channel flow and the Cramer’s function. We further find that the range of scaling shown by the PDF of polymer extensions depends on the wall-normal coordinate but the scaling exponent $\alpha$ is independent of the wall-normal coordinate.

We use the location of the minima of the Cramer’s function, $\bar{\mu}$, as the inverse characteristic time scale of the fluid to define our Weissenberg number as

$$Wi \equiv \bar{\mu} \tau_{\text{poly}}$$

(3)

\[\text{[arXiv:1011.3766v1 [physics.flu-dyn] 16 Nov 2010]}\]
Our simulations show coil-stretch transition for $Wi \approx 1$. For $Wi > 1$ it is not possible to obtain a stationary PDF for linear polymers. In this regime we use the nonlinear FENE (Finitely Extendable Nonlinear Elastic) model. As we use a Lagrangian model for the polymer we are able to use relatively high $Wi$.

In addition we also find that the PDF of polymer extensions depends on the wall-normal coordinate, v.i.z, the polymer are more stretched near the wall than at the centre of the flow. We further study the orientation of the polymers with respect to the channel geometry and the local velocity gradient tensor. Our results show that the orientation of the polymers is predominantly determined by the inhomogeneity of the flow, i.e., the wall-normal coordinate as opposed to the local strain tensor. However, for polymers near the centre of the channel we find that the orientation is also influenced by the principal directions of the rate-of-strain tensor, as has been seen in DNS of polymers in homogeneous isotropic flows [2] [10].

The rest of the paper is organised as follows. In the next section, Section II we describe the equations we solve and the details of the numerical algorithm we use. Our results follow next in Section III which is divided into three parts. In Section III A we present our results on the PDF of finite-time-Lyapunov-exponents (FTLEs). The results described in this section are therefore independent of the polymer equation. The statistics of polymer extensions for the two models considered are presented in Section III B and III C. The polymer orientation is characterized by calculating the correlations between the polymer end-to-end vector and fluid vorticity and the rate of strain tensor (Section III D). We show that the polymers are oriented along the direction of the mean flow near the wall whereas near the center of the channel polymer orientation statistics is similar to that of homogeneous, isotropic turbulence [2]. Finally we present our conclusions in Section IV.

II. EQUATIONS AND NUMERICAL METHODS

The fluid is described by the Navier–Stokes equations,

\[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} + \nabla p \]  

with the incompressibility constraint,

\[ \nabla \cdot \mathbf{u} = 0. \]  

Here $\mathbf{u}$ is the fluid velocity, $\nu$ the kinematic viscosity and $p$ the pressure. We use no-slip boundary condition at the walls and periodic boundary condition at all other boundaries. We have chosen our units such that the constant density $\rho = 1$. The $x$ axis of our coordinate system is taken along the stream-wise direction, the $y$ axis along the wall-normal direction and $z$ axis along the span-wise direction. For brevity, we shall often use the common notation, $U \equiv u_x$, $V \equiv u_y$ and $W \equiv u_z$. The $x$, $y$ and $z$ dimensions of our channel are $L_x \times L_y \times L_z = 4\pi \times 2 \times 2\pi$, with resolution $Nx \times Ny \times Nz = 128 \times 128 \times 128$.

The turbulent Reynolds number $Re_t = U_\ast L/\nu = 180$ is defined by the friction velocity $U_\ast = \sqrt{\sigma_w}$ and $L \equiv L_y/2$, the half-channel width, where

\[ \sigma_w \equiv \nu \frac{\partial U}{\partial y}|_{\text{wall}} \]  

is the shear stress at the wall [11]. In the following we non-dimensionalise velocity and distance by $U^+ \equiv U/U_\ast$ and $y^+ \equiv y/y_\ast$ respectively, using the friction length $y_\ast = \nu/U_\ast$. Time will also be measured in the unit of the large-eddy-turnover-time $\tau_L \equiv (U_{\text{center}}/L)^{-1}$, where $U_{\text{center}}$ is average streamwise velocity at the center of the channel. The large-scale Reynolds number defined by $Re = U_0 L/\nu = 4200$ where $U_0$ is the centreline streamwise velocity for the laminar flow of same mass flux.

We solve Eqs. (1) and (2) by using the SIMSON code which uses a pseudo-spectral method in space (Chebychev-Fourier). For time integration a third-order Runge-Kutta method is used for advection and uniform pressure gradient forcing term. The viscous term is discretized using a Crank-Nicolson method [13]. In Fig. 1 we plot $(U^+)$ as a function of $y^+$ at the stationary state of our simulations, where $\langle \cdot \rangle$ denotes averaging over the coordinate directions $x$ and $z$ and also over time. Further details about the code validation can be found in Ref. [14].

![Figure 1](image)

**FIG. 1:** (Colour online) Normalised mean stream-wise velocity $(U^+)$ versus the wall-normal coordinate $y^+$ plotted for one half of the channel.

We use a Lagrangian model for the polymers where we solve one stochastic differential equation (SDE) for each polymer molecule. This model uses several approximations which are as follows [15][16]: (A) The centre-of-mass of a polymer molecule follows the path of a Lagrangian particle. (B) Even when fully stretched the polymer molecule is very small compared to the smallest scales of turbulence. This approximation is well justified [6]. (C) A polymer molecule is modelled by two beads separated by a vector which represents the end-to-end distance of the polymer molecule. (D) The forces acting on the beads...
are stokes drag, restoring force of an overdamped spring with time scale $\tau_{\text{poly}}$, and thermal noise. To be specific, we track $N_p = 2.16 \times 10^5$ Lagrangian passive tracers in the flow by solving

$$\partial_t r_j^i(t|t_0, r_0^i) = v_j^i(t|t_0, r_0^i).$$  \hspace{1cm} (7)

Where $r_j^i(t|t_0, r_0^i)$ is the position of the $j$-th Lagrangian particle which was at position $r_0^i$ at time $t_0$ and $v_j^i(t|t_0, r_0^i)$ is its velocity with $j = 1, \ldots, N_p$. The Lagrangian velocity of a particle, which is generally at an off-grid point, is obtained by tri-linear interpolation from Eulerian velocity at the neighboring grid points.

Equation (7) is integrated by a third order Runge-Kutta scheme. Each of these Lagrangian particles represent a polymer molecule. For $j$-th Lagrangian particle the vector representing the end-to-end distance is denoted by $R_j$ and obeys the following dynamical equation:

$$\partial_t R_j^i(t) = \sigma_{\alpha\beta}^j R_{\alpha}^j + f(R) + \sqrt{\frac{2R_0^2}{3\tau_{\text{poly}}}} B_j^\alpha. \hspace{1cm} (8)$$

Here $\sigma_{\alpha\beta}^j = \partial_x v_j^i(t|t_0, r_0^i)$, $f(R)$ is the restoring force of the polymer, $\tau_{\text{poly}}$ is the characteristic decay time of the polymer and $B_j$ is a Gaussian random noise with $\langle B_j \rangle = 0$ and $\langle B_j(t) B_{\beta}(t') \rangle = \delta_{\alpha\beta} \delta(t - t')$. The prefactor of the random noise is chosen such that in absence of external flow, i.e., $\sigma_{\alpha\beta}^j = 0$, the polymer attains thermal equilibrium, $\langle R_0^2 R_0^j \rangle = \frac{R_0^2 \delta_{\alpha\beta}}{3}$. Here $\langle \cdot \rangle$ denotes averaging over the noise $B$. For the linear Oldroyd-B model $f(R) = -R/\tau_{\text{poly}}$. For the FENE model $f(R) = -R/\tau_{\text{poly}}\{1 - (R/R_{\text{max}})^2\}$. Eq. (8) is also solved by a third order Runge-Kutta scheme except for the noise which is integrated by an Euler-Maruyama method.

To compare with the analytical theory of Ref. [1] we also need to calculate the PDF of finite-time Lyapunov exponents of Lagrangian particles in this flow. For this we need to calculate the rate at which two infinitesimally separated Lagrangian particles diverge as time progresses. For this purpose we also calculate the evolution of an infinitesimal vector in our turbulent flow, given by the equations,

$$\partial_t \delta x^j_{\alpha} = \sigma_{\alpha\beta}^j \delta x^j_{\beta}. \hspace{1cm} (9)$$

Where $\delta x^j$ is a vector carried by the $j$-th Lagrangian particle. This is of course the same equation obeyed by a Lagrangian polymer, Eq. (8), if the restoring force of the polymer and the Brownian noise are omitted.

The correspondence between our Lagrangian description and the Eulerian description of polymeric fluids is that in the latter the dynamical variable for the polymers is the symmetric positive definite (SPD) tensor $C_{\alpha\beta} \equiv \langle R_0 R_0^\beta \rangle$. A DNS of the Eulerian description has certain difficulties [18, 21]. Firstly the numerical schemes used must preserve the SPD nature of $C_{\alpha\beta}$. Secondly for high Weissenberg numbers large gradients of $C_{\alpha\beta}$ can develop which can lead to numerical instability of the code. Stability can generally be restored by employing either shock-capturing schemes [19, 21, 23] or by introducing dissipation in the Eulerian description of the polymer [24, 25]. Lagrangian methods [26, 27] are generally able to avoid such numerical pitfalls and can attain higher Weissenberg number. On the other hand it is quite straightforward to incorporate the back-reaction of the polymer into the flow in the Eulerian model but is tricky in the Lagrangian model [27, 23]. Note finally that more complicated Lagrangian models have also been employed where a single polymer is represented by a chain of beads connected by springs [2, 10].

### III. RESULTS

#### A. Finite-time Lyapunov Exponents

How two infinitesimally separated Lagrangian particles diverge in a turbulent flow has been a central topic in turbulence research for a long time. See, e.g., Ref. [29] for a recent review. Below we reproduce the essential points needed to apply such ideas to stretching of polymers in turbulence. For incompressible flows and for long time $T$, much larger than the correlation time of $\sigma_{\alpha\beta}^j$, $|\delta x^j(T)|$ grows exponentially with time. We integrate Eq. (9) for each Lagrangian particle over a finite time interval $T$ and compute the the finite-time Lyapunov exponent (FTLE)

$$\mu_T = \frac{1}{T} \ln \left( \frac{|\delta x^j(t)|}{|\delta x^j(t - T)|} \right). \hspace{1cm} (10)$$

For large $T$, $\mu_T \rightarrow \infty$ the PDF of FTLEs is conjectured to have a Large deviation form [11, 30, 31]

$$P(\mu_T) \sim \exp[-T S(\mu_T)] \hspace{1cm} (11)$$

where $S(\mu_T)$ is called the Cramer’s function or the entropy function. The Cramer’s function has earlier been calculated from direct numerical simulations of two [9]- and three [32] dimensional homogenous isotropic turbulence, turbulence in the presence of homogeneous shear [33], and for hydromagnetic convection [34].

Below we present, for the first time, the analysis of the PDF of FTLEs for channel flows.

Since the channel flow is not homogeneous in the wall-normal direction the statistics can, in principle, depend on $y^+$. Hence we label our particles by their wall-normal coordinate ($y^+$) at the final position, i.e., at time $T$. While integrating the equations for $\delta x^j$, Eq. (9), we store the evolution of $\delta x^j$ and use this to calculate $\mu_T$ for each of $\delta x^j$. To calculate the PDF of $\mu_T$ we gather statistics in two different ways. First we calculate the PDF of $\mu_T$ for all particles at a fixed $y^+$. Furthermore we run our simulations over several $T$ and after each time interval $T$ the particles are redistributed uniformly across the channel and their initial separation vector $\delta x^j(t = 0)$ oriented
randomly. By definition then we generate a $P(\mu_T, y^+)$ which depends on $y^+$.

The PDFs for two different values of $y^+$, one close to the wall, and one near the centerline, are respectively plotted in the left and the right panel of Fig. 2 for several time intervals $T$. The peak and mean of the PDFs are always positive showing that it is more probable for $|\delta a^l|$ to increase exponentially as a function of time. For small $T$ the PDFs near the centre and the PDF near the wall are very different from each other. Significantly larger elongation is found for those elements that are located closer to the wall. However, the two PDFs approach each other for large $T$. This can also be seen by plotting the mean value of the PDFs for three different $y^+$ values as a function of time, Fig. (3). The peak value also shows a similar trend, see inset in Fig. (3). Hence an unique Cramer’s function independent of $y^+$ can be defined for the channel flow for only very large time when the PDFs for different $y^+$ merge with one another. In a channel flow the stress tensor $\sigma_{\alpha\beta}$ depends strongly on the wall-normal coordinate. Thus for short $T$ we can expect that the PDF of $\mu_T$ depend of $y^+$. Conversely, when $T$ becomes much larger than the typical time it takes for a particle to travel from a position near the wall to a position near the centreline, we expect $\mu_T$ to be independent of $y^+$. Let us call this typical time the exit time $T_{exit}$. Surprisingly, we observe from our data that we need to have $T_{exit} \gtrsim 80\tau_L$ for $\mu_T$ to be independent of $T$. An estimate of the time it takes for a particle to travel from the wall to the centre of the channel can be given by the ratio of the half-width of the channel to the friction velocity, $T_{friction} \equiv (L_x/2)/U_s \approx 15$ in our simulations. In units of this time $T_{exit} \gtrsim 5T_{friction}$ which gives a better estimate than $\tau_L$.

From PDF of $\mu_T$ for large $T$ we calculate the Cramer’s function using Eq. (11). We normalise $P(\mu_T)$ such that its integral over the range of $\mu_T$ is unity. For $T > T_{exit}$ the Cramer’s function $S(\mu)$ calculated at different times $T$ are found to be independent of $T$ as it should be. This is shown by the collapse of the Cramer’s function calculated at different times for a fixed $y^+ = 62$ in Fig. 3. Furthermore, the Cramer’s function thus found is independent of $y^+$. The connection between the Cramer’s function and the PDF of end-to-end polymer distance was shown in Ref [1] for linear polymers and in Ref [7] for nonlinear polymers. We discuss such relations in the next section where it will turn out to be useful to have an algebraic expression for the Cramer’s function. To obtain such an expression we fit a polynomial function of the form

$$S(\mu) = a_2(\mu - \bar{\mu})^2 + a_3(\mu - \bar{\mu})^3 + a_4(\mu - \bar{\mu})^4$$

(12)

to our numerical data for $S(\mu)$ averaged over all values of $y^+$ and extract the coefficients $a_2, a_3$, and $a_4$ above. To estimate the errors in the coefficients $a_k$ we use the same fit to $S(\mu)$ obtained for individual $y^+$ and quote the range of $a_k$ obtained from such fits as the error in $a_k$. The best fit is also plotted in Fig. (4). The coefficients corresponding to the best fit and their errors are given in the caption of Fig. (4).

B. Statistics of polymer extensions: Oldroyd-B model

Before we present detailed results on statistics of polymer extension let us precisely define the Weissenberg number, $Wi$. In simulations the Weissenberg number is defined as the ratio of the characteristic time-scale of the polymer, $\tau_{poly}$ over a characteristic time scale of the fluid. Different definitions of the characteristic time scale has been used in literature to define the Weissenberg number. Refs. [2] 10 [20] use the Kolmogorov time scale $\tau_l$ to define Weissenberg number. We denote this Weissenberg number by $Wi_l = \tau_{poly}/\tau_l$ where $\tau_l$ is the Kolmogorov time scale. In this paper we principally use the following definition for Weissenberg number

$$Wi \equiv \bar{\mu}\tau_{poly}$$

(13)

where $\bar{\mu}$ is the location of the minima of the Cramer’s function $S(\mu)$. Our choice has two principal advantages. Firstly in channel flows the Kolmogorov scale depends on the wall-normal coordinate and hence is not unique. Secondly and more importantly a proper choice of Weissenberg number gives the coil-stretch transition of the polymer at $Wi \approx 1$ which is exactly what we obtain. To compare with earlier simulations, which were all done in homogeneous flows, we also calculate $Wi_{wall}$ and $Wi_{center}$ where we use the Kolmogorov time scale at the wall and at the center of the flow respectively. We typically obtain, $Wi_{wall} \approx 30 Wi$ and $Wi_{center} \approx 5 Wi$. The different values of $Wi$ that we use are given below, in brackets we mention the corresponding values of $Wi_{center}$ for easy comparison with earlier simulations of homogeneous and isotropic turbulence. For the Oldroyd-B model, $Wi(Wi_{center}) \equiv 0.1(0.5), 0.2(1.0), 0.3(1.5), 0.5(2.5)$ and $Wi(Wi_{center}) = 0.1(0.5), 0.3(1.5), 0.5(2.5), 1.5(7.5), 2.5(12.5), 3.5(17.5), 4.5(22.5), 5.5(27.5), 7(35)$, and $10(50)$ for the FENE model. We use $R_0 = 10^{-7}, 10^{-8}$, and $R_{max}/R_0 = 100$ and 1000 for the FENE model.

Let us first present the results for the Oldroyd-B model. Here we expect to see a power-law behavior for the PDF of polymer extensions, $Q(R) \sim R^{1-\alpha}$ [1] for large $R$. In general, the calculation of PDFs from numerical data is plagued by errors originating from the binning of the data to make histograms. Thus it is often a difficult task to extract exponents such as $\alpha$ from such PDFs. A reliable estimate of such an exponent can be obtained by using the rank-order method [23] to calculate the corresponding cumulative probability distribution function,

$$Q^c(R) \equiv \int_0^R Q(\xi)d\xi$$

(14)

If the PDF has a scaling range the cumulative PDF also shows scaling, i.e., $Q^c(R) \sim R^{-\alpha}$. For different values of $Wi$ these the cumulative PDFs for $y^+ = 74$ are plotted in Fig. 5. The cleanest power-law is seen for $Wi = 0.5$. So we choose this Weissenberg number for further detailed investigation. First we show that the exponent of
FIG. 2: (Colour online) (left panel) PDF of $\mu_T$ near the wall ($y^+ \approx 6$) for several values of $T$, v.i.z., $T = 1$, $3$, $5$, $35$, and $100$. (right panel) PDF of $\mu_T$ near the centerline ($y^+ \approx 180$) for several values of $T$, v.i.z., $T = 1$, $3$, $5$, $35$, and $100$. Plots at some other intermediate values of $T$ are consistent with this plot, but are not shown here for clarity. All times are measured in the unit of $\tau_L$.

FIG. 3: (Color online) $\langle \mu_T \rangle$ for different values of $T$ for three different positions, in the channel, near the wall (●), near centerline (○) and at $y^+ = 84$ (△). All times are measured in the units of $\tau_L$.

the power-law ($Wi = 0.5$) $\alpha = 0.81 \pm 0.02$ does not depend on the $y^+$ although the range over which scaling is obtained does, Fig. 4. The exponent $\alpha$ is obtained by fitting a power-law for five different values of $y^+$. The mean is reported as the exponent above and the standard deviation from the mean is reported as the error.

This exponent $\alpha$ can be obtained from the Cramer’s function $S(\mu)$ using the set of couple equations Eq. (1) and (2) which we rewrite below,

$$\alpha = S'(\beta + \frac{1}{\tau_{poly}} - \bar{\mu})$$  \hspace{1cm} (15)

where $\beta$ must be obtained by solving the differential equation,

$$S(\beta + \frac{1}{\tau_{poly}} - \bar{\mu}) - \beta S'(\beta + \frac{1}{\tau_{poly}} - \bar{\mu}) = 0$$  \hspace{1cm} (16)

Had the Cramer’s function been well approximate by a parabola of the form $S(\mu) = (\mu - \bar{\mu})^2/\Delta$, Eq. (1) would simplify to $\alpha = (2/\Delta)(1/\tau_{poly} - \bar{\mu})$. We have checked that this quadratic approximation does not give accurate result for $\alpha$ in our case. Using the algebraic expression for $S$ given in Eq. (12) with $\bar{\mu} = 0.165[0.088 \ldots 0.13]$ and $a_2 = 3.55[3.09 \ldots 4.35]$, $a_3 = -12.60[-27.48 \ldots 4.29]$ and $a_4 = 39.64[3.84 \ldots 90.07]$ equation,
error bars. We note here that the $\alpha$ we calculate using the Cramer’s function has large margin of error because the $\alpha$ depends sensitively on the coefficients $a_k$ in Eq. [12]. To find these coefficients accurately we need to know the Cramer’s function accurately for a large range of its argument not just the location of its minima. Numerically this is a difficult task and would require collecting data over very long times.

Finally let us comment on the possible experimental determination of the exponent $\alpha$. In practice no polymers are linear and in most cases the ratio of $R_{\text{max}}$ (maximum possible extension of the polymer) to $R_0$ (the equilibrium length) ranges between 100 to 1000. To see the effect of a maximum extension, we first select one of the cumulative PDFs plotted in Fig. [6], say for $y^+ = 74$. From this cumulative PDF we remove all the polymers for which $R$ is so large that $R/R_0 > R_{\text{cutoff}}$ where we choose $R_{\text{cutoff}} = 100$ and 1000. The resultant cumulative PDFs are plotted in Fig. [7] where the original cumulative PDF is also plotted for comparison. It can be seen that the scaling behavior, although present, is valid over a much smaller range. In the same figure we have also plotted the cumulative PDF for the FENE model with $R_{\text{max}}/R_0 = 1000$. This also shows scaling with a reduced range. Thus we expect that in experiments reminiscence of this scaling law should be visible although it may be difficult to detect because of reduced range of scaling.

C. Statistics of polymer extensions: FENE model

Till now we have described the polymer statistics for $Wi \leq 0.5$. As we increase the Wi and make it close to unity no stationary statistics of the polymers is obtained. We interpret this by noting that we are close to the coil-stretch transition. A stationary state can be obtained by either including the feedback for the polymers into the fluid or by using nonlinear polymers e.g., the FENE model. We choose the second option. In the FENE model we have used $R_{\text{max}}/R_0 = 100$ and 1000. Our results as reported below does not depend on this parameter.

First we show the coil-stretch transition by plotting the PDF of polymer extensions for three different Weissenberg numbers $Wi = 0.5$, 1.5 and 10 in Fig. [8]. The sequence of these figures illustrate the coil-stretch transition clearly. In the first plot the PDF is strongly peaked near zero which corresponds to the coiled state. The associated cumulative PDF is displayed in Fig. [7] and shows power-law behavior over a limited range of extensions.
The data for Wi = 1.5 are shown in the second figure: although the peak of the PDF is still close to zero the PDF is well spread over the whole range. The PDF for polymers with Wi = 10 has peak near \( R_{\text{max}} \) which is the stretched state of the polymer. In these three figures the PDF is plotted for \( y^+ = 74 \) but the PDF at other wall-normal coordinates in the channel shows the same qualitative nature as was seen in the case of the Oldroyd-B model in Fig. (6).

In Fig. 9 we show how the mean polymer extension \( \langle R \rangle_{xz} \), where the averaging is over the stream-wise and the span-wise direction, changes with Wi across the channel for \( R_{\text{max}}/R_0 = 100 \). For a given Wi the average polymer extension is small near the wall, increases to a maximum around \( y^+ \approx 10 \) (this corresponds to the region of maximum strain), and then decreases towards the center of the domain where the flow is close to homogeneous turbulence. A similar trend is also seen for \( R_{\text{max}}/R_0 = 1000 \). This trend has been seen in earlier DNS of polymeric turbulence in channel flows, see, e.g., Ref. [36] and the references therein. But for larger values of Wi the average polymer extension becomes almost uniform across the channel (except very near the wall where it is always small). This is because the polymers that are stretched close to the wall on reaching the centerline are not able to relax fast enough as the polymer relaxation time scales are much larger than the fluid time scales.

**FIG. 8:** (Color online) The PDF of polymer extensions \( Q(R) \) as a function of \( R \) for different Wi showing the coil-stretch transition. The line with \( (\times) \) symbols is for \( Wi = 0.5 \) (\( \tau_{\text{poly}} = 5 \)), the continuous line is for \( Wi = 10 \) (\( \tau_{\text{poly}} = 100 \)) and the inset is for \( Wi = 1.5 \) (\( \tau_{\text{poly}} = 15 \)). The PDF for \( Wi = 10 \) is multiplied by 2 to make it clear in the same scale. The dashed line shows power-law scaling with exponent \( \alpha = 1.48 \).

**FIG. 9:** (Color online) The average polymer extensions \( \langle R \rangle_{xz} \) as a function of the wall-normal coordinate \( y^+ \) for different Weissenberg numbers, v.i.z., Wi = 1.5(\( \times \)), 2.5(■), 3.5(□), 4.5(▲), 5.5(△), 7(•), 10(◦). The maximum occurs at \( y^+ \approx 10 \).

**FIG. 10:** (Color online) The maxima of the average polymer extensions \( \langle R \rangle_{xz} \) across the channel as a function of Wi in semilog scale.

### D. Statistics of polymer orientation

In this section we present the results related to the orientation of the polymers. First let us discuss the orientation of the polymers with respect to the geometry of the channel. Let us denote the unit vector along \( R \) to be \( e \). The PDF of the three components of \( e, e_x, e_y \) and \( e_z \) (i.e., three direction cosines of \( R \)) are plotted in the left panel of Fig. (11) for polymers close to the wall \( (y^+ \approx 7) \) and for three different values of Wi. For Wi < 1 , i.e., below the coil-stretch transition the polymers are almost equally probable to point in any direction or in other words as the polymers are coiled as a
sphere no preferential direction is picked up. Above the coil-stretch transition polymers close to the wall have a high probability of being oriented along the x axis, which is the stream-wise direction. A similar plot for polymers close to the centreline ($y^+ \approx 180$) is given in the right panel of Fig. (11). For small $Wi$ all directions are equally probable. But as $Wi$ increases here too the polymers get preferentially oriented along the stream-wise direction although the trend is much weaker than what is seen near the wall.

We have also investigated the orientation of the polymers with respect to the three principal directions of the rate of strain tensor. For this purpose we first determine the three real eigenvalues of the symmetric rate of strain tensor and order them such that $\lambda_1 > \lambda_2 > \lambda_3$. We denote the components of the unit vector $e$ (which is the unit vector along $R$) along these three perpendicular directions by $e_1$, $e_2$, and $e_3$, these are merely the cosines of the angles between $R$ and the three principal directions of the strain tensor. The PDFs of $e_1$, $e_2$, and $e_3$ are plotted in the left panel of Fig. (12) for polymers close to the wall ($y^+ \approx 7$) and for three different values of $Wi$. The peak seen in Fig. (12) corresponds to the polymers orientating along the stream-wise direction as shown already in Fig. (11). Interestingly the polymers are not preferentially oriented along the strongest direction of strain $\lambda_1$ but along the stream-wise direction. This has an angle of about 45 degrees with respect to the x-axis since the main component of the strain rate is because of the wall-normal shear $\partial U/\partial y$. Close to the centerline however the PDFs look quite different [right panel of Fig. (12)]. For small $Wi$ there is no preferential orientation but as $Wi$ increases the polymers develops a trend of orienting parallel to the direction of either $\lambda_1$ or $\lambda_2$ and shows anti-alignment to $\lambda_3$.

Finally we look at the relative orientation between the polymer end-to-end vector $R$ and the vorticity vector $\omega$. Close to the wall we find that PDF of the cosine of the angle $\psi$ between $R$ and $\omega$ has a peak at zero, see left panel of Fig. (13). This implies that the polymers show a weak tendency to lie in the plane perpendicular to $\omega$. However this trend is reversed near the centerline [right panel of Fig. (13)] where the polymers orient along the vorticity vector.

To summarise the polymers near the wall shows the cleanest trend in their orientation. They show a strong tendency to line along the stream-wise directions. Weaker trends are seen near the center. The statistics of orientation of polymers near the center of our flow is very similar to the statistics of orientation of polymers obtained in homogeneous and isotropic flows [2]. But note that the orientation effects we see near the wall are much stronger compared to those near the centerline.

IV. CONCLUSIONS

We have presented in this paper an extensive study of the passive Lagrangian polymers in turbulent channel flow by Direct Numerical Simulations. We have used both linear (Oldroyd-B) and nonlinear (FENE) polymers. To understand the statistics of polymer end-to-end vector it is necessary to know the statistics of the Finite Time Lyapunov Exponents. For this purpose in addition to the polymers we have solved the equation of evolution of infinitesimal line elements in the turbulent flow and for the first time calculated the FTLEs for an inhomogeneous flow. We find that at very large time the PDF of FTLEs admit a large deviation description and a corresponding Cramer’s function can be found. In addition we use the location of the minima of the Cramer’s function to define our Weissenberg number. Consequently for the FENE model we observe coil-stretch transition at $Wi \approx 1$. For the Oldroyd-B model we find that the PDF of polymer extension shows power-law behavior for $Wi < 1$. We calculate the exponent of this power-law using the rank-order method. We also calculate the same exponent from the Cramer’s function using the theory of Ref. [1]. These two different calculations match within error, validating the theory of Ref [1]. This shows that the idealizations used in Ref. [1], in particular the assumption that in Lagrangian coordinates the rate-of-strain tensor $\sigma_{\alpha\beta}$ is delta-correlated in time is a reasonable approximation at least for linear polymers below the coil-stretch transition even in the case of a realistic flow. For the FENE model we cannot meaningfully calculate the PDF of polymer extension from the Cramer’s function using the theory of Ref. [7] because our numerically calculated Cramer’s function is not accurate enough for this exercise. For the FENE model we find that the polymers are more extended near the wall, but the difference decreases as Weissenberg number increases far beyond the coil-stretch transition. We further find that near the centre of the channel the orientational statistics of the polymers show similarity to orientational statistics obtained for homogeneous and isotropic flows [2], i.e., they align along either of the two largest directions of strain and tend to orient orthogonal to the third principal direction of strain. But a much stronger orientational trend is seen near the wall where the orientations of the polymers are along by the stream-wise direction.

Although our DNSs involve passive polymers it is possible to have insights on polymeric drag reductions from these simulations. We can calculate the polymeric stress from our simulations and add this to the Reynolds stresses to see how they change the Reynolds equations of the flow. It would be interesting to see how much of drag-reduction can be described by this simple approach. Such results will be presented in a future publication.
FIG. 11: (Color online) PDF of the three direction cosines of polymer end-to-end separation vector $R$ for polymers near the wall (left panel) and for polymers at the center of the channel (right channel). Three different values of $Wi$ are used. Namely, $Wi = 0.1(\circ), 1.5(\triangle), 4.5(\blacksquare)$. The data for $Wi = 1.5$ and 4.5 coincide on each other. The PDFs of $e_x$ and $e_y$ are respectively plotted using continuous line with symbols ($P_x$) and dashed lines with symbols ($P_y$). The inset shows the PDF of $e_z$, $P_z$.

FIG. 12: (Color online) PDF of $e_1$, $e_2$ and $e_3$, components of the unit vector along $R$ along the three principal directions of strain, for polymers near the wall (left panel), and for polymers near the centerline (right panel). Three different $Wi$ are used. Namely, $Wi = 0.1(\circ), 1.5(\triangle), 4.5(\blacksquare)$. The data for $Wi = 1.5$ and 4.5 coincide on each other. The PDFs of $e_1$ and $e_2$ are respectively plotted using continuous line with symbols ($P_1$) and dashed line with symbols ($P_2$). The inset shows the PDF of $e_3$, $P_3$.

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FIG. 13: (Color online) PDF of $\cos(\psi)$, where $\psi$ is the angle between the polymer end-to-end vector $\mathbf{R}$ and vorticity, for polymers near the wall (left panel) and for polymers near the centerline (right panel). Three different values of Wi are plotted. Namely $Wi = 0.1(\circ), 1.5(\triangle), 4.5(\blacksquare)$.