ON THE MINIMAL DRIFT FOR RECURRENCE IN THE FROG MODEL ON $d$-ARY TREES

CHENGKUN GUO, SI TANG, AND NINGXI WEI

Abstract. We study the recurrence of one-per-site frog model $FM(d,p)$ on a $d$-ary tree with drift parameter $p \in [0,1]$, which determines the bias of frogs' random walks. We are interested in the minimal drift $p_d$ so that the frog model is recurrent. Using a coupling argument together with a generating function technique, we prove that for all $d \geq 2$, $p_d \leq 1/3$, which is the optimal universal upper bound.

1. Introduction

Let $T_d$ be a $d$-ary rooted tree where each vertex has $d$ child vertices and one parent vertex except for the root $\emptyset$ which does not have a parent vertex. We study the standard one-per-site frog model on $T_d$ with drift parameter $p$, which we denote by $FM(d,p)$. The model is defined as follows. At time $t = 0$, each vertex of $T_d$ other than the root is occupied by a sleeping frog ("inactive"); the root has a conscious frog ("active") at time $t = 0$. Active frogs perform independent random walks according to the following rules: (a) from vertices other than the root, frogs make "upward" jumps (to the parent of the current vertex) with probability $p$ and "downward" jumps (to child vertices) with probability $(1-p)/d$; and (b) from the root vertex, frogs always jump downward to one of the child vertices, with probability $1/d$ each. Whenever an active frog visits a site with a sleeping frog, the sleeping frog wakes and begins its own independent random walk, following the same rules. Let $V_{d,p}$ denote the total number of visits to the root in $FM(d,p)$. It is known that $P(V_{d,p} = \infty)$ is either 0 or 1 \[KZ17, HJJ17\], marking the two phases of $FM(d,p)$, transience and recurrence, respectively.

The recurrence of the frog model has been studied in various settings. It depends on the graph structure (e.g., the degree distribution of vertices), the distribution of the number of frogs initially at each vertex, and the law of the random walk. Gantert and Schmidt \[GS09\] showed that, on the integer lattice $\mathbb{Z}$, if the numbers of sleeping frogs on the different vertices are i.i.d. copies of a random variable $\eta$ and the random walk has a nonzero drift, then the process is recurrent if and only if $E(\log \eta)_+ = \infty$ no matter how strong the drift is. The asymptotic behavior of the range of the random walks was studied in \[GNR17\] in the transient case. In higher dimensions $d \geq 2$, a similar condition $E(\log \eta)_{+}^{(d+1)/2} = \infty$ was proved to be sufficient for recurrence \[DP14\]. However, unlike the $d = 1$ case, the frog model with one-per-site setting can be either transient or recurrent, where the preference in choosing one direction (e.g., $\pm e_1$) and the net drift in the preferred direction are the key determining factors \[DGH17\].

In \[HJJ17\], Hoffman, Johnson and Junge studied the recurrence of the one-per-site frog model $FM_2$ on the $d$-ary tree $T_d$, where active frogs perform simple, nearest-neighbor random walks. This is the special case of the frog model $FM(d,p)$ where $p = 1/(d+1)$. They showed that $FM_2$ is recurrent for $d = 2$ and transient when $d \geq 5$; it is currently unknown if $FM_3$ and...
FM₄ are recurrent. It was later improved in [HJJ16, JJ16a, JJ16b] that if the “one-per-site” setting in FM₄ is changed to “Ω(d) frogs on each vertex”, then the frog model becomes recurrent for all d.

In this work, we focus on the one-per-site setting. Since FM₄ is transient when d ≥ 5, it is natural to ask what is the minimal drift needed for a frog model on a d-ary tree to be recurrent. Define

\[ p_d := \inf \{ p : \text{FM}(d, p) \text{ is recurrent} \}. \]

It is easy to see that \( p_d \leq 1/2 \), as simple random walk on Z is recurrent. By estimating the expected number of frogs that can reach the root, we see that \( p_d \) must be at least \( 1/(d + 1) \) for all d. Since FM₂ = FM(2, 1/3) is recurrent [HJJ17], implying \( p_2 \leq 1/3 \), we thus can conclude \( p_2 = 1/3 \).

A better universal lower bound for \( p_d \) can be obtained by dominating FM(d, p) with a branching random walk on \( \mathbb{Z}_{>0} \) where particles perform independent \( p \)-biased random walk and split to two whenever moving to the right. This branching random walk is equivalent to FM(∞, p), in which a new active frog is added to the process every time a frog moves away from the root. When \( p < q^* := \frac{1+\sqrt{2}}{4} \), the branching random walk is transient [BFJ+19, HJJ17] and therefore so is the frog model FM(d, p). Thus, \( p_d \geq q^* \) for all \( d \geq 2 \). By refining the branching random walk to approximate two steps of frog model, one can also show that \( p_d - q^* > 0 \) for all \( d \geq 2 \). Since the branching random walk can be considered as a frog model with \( d = \infty \), it is natural to ask if \( p_d \to q^* \) as \( d \to \infty \). A more basic question is whether \( (p_d)_{d \geq 2} \) converges at all. It is natural to expect that \( p_{d+1} \leq p_d \), since it would seem that there would be more active frogs in FM(d + 1, p) than in FM(d, p). But although monotonicity of \( p_d \) is believed to be true for the frog model on homogeneous trees (as conjectured in [LMP05]), it is not generally true for other types of nested graphs [FMS04].

**Conjecture 1.1.** For all \( d \geq 2 \), \( p_{d+1} \leq p_d \).

Direct coupling of two frog models FM(d, p) and FM(d’, p) appears to be difficult, except for some special cases, for example when \( d’ \) is an integer multiple of \( d \) [BFJ+19 Proposition 1.2]. In this case, it was proved that \( p_{kd} \leq p_d \) for all \( d \geq 2, k \geq 1 \), implying the convergence of \( (p_d)_{d \geq 1} \) along certain subsequences. Unfortunately, the coupling can not generalize to other pairs of degrees.

Since \( p_2 = 1/3 \), Conjecture [1.1] if true, would imply that the universal upper bound of \( p_d \) is 1/3. Several improvements to the trivial bound \( p_d \leq 1/2 \) have been made: in [FMS04], it was shown \( p_d \leq (d + 1)/(2d - 2) \), and in [BFJ+19] that \( p_d \leq 0.4155 \). But none of these results is close to the conjectured bound 1/3. Our main contribution here is to prove the sharp upper bound for \( p_d \), i.e.,

**Theorem 1.2.** For all \( d \geq 2 \), \( p_d \leq \frac{1}{3} \).

**Remark 1.3.** Since \( p_2 = 1/3 \), this result implies \( p_3 \leq p_2 \), as predicted by Conjecture [1.1]. More importantly, this inequality is the first such result that compares two frog models where neither tree degree is an integer multiple of the other, bypassing the difficulty of directly coupling two frog models.

### 1.1. Proof Strategy

It suffices to prove that FM(d, 1/3) is recurrent, and we will construct a frog process \( \mathcal{P} \) on \( \mathbb{T}_d \) that is dominated by FM(d, 1/3), so that if \( \mathcal{P} \) is recurrent (i.e., there are infinitely many visits to the root), then FM(d, 1/3) must also be recurrent. This strategy was used in [HJJ17] to prove FM(2, 1/3) is recurrent; there the dominated process \( \mathcal{P} \) was the self-similar frog model SFM(2, 1/3). This process is a modification of the ordinary frog
model in which some active frogs are removed, resulting in a self-similar structure; a precise definition is given in Section 2. In [BFJ+19], when proving a universal upper bound for \( p_d \), the authors made use of a recursive frog model RFM\((d,p)\), where one can directly compare RFM\((d,p)\) with RFM\((d+1,p)\) and the critical drift \( p'_d \) needed for RFM\((d,p)\) to be recurrent appears to be monotone in \( d \). Unfortunately, since too many frogs are removed in RFM, it is very difficult (i.e., a strong drift is needed) for RFM to be recurrent. To this end, the authors made use of a recursive frog model RFM\((d,p)\) as a comparison object. Once Proposition 1.4 has been proved, we will then need to prove the recurrence of SFM\((d,p^\ast)\) for \( p^\ast > p^\ast(d,p) \) with \( p = 1/3 \), that is,

\[
p^\ast(d,1/3) = 1/3 + 1/3d/d(d+1) = 1 - 1/(2d-1).
\]

This will be accomplished by the following proposition.

**Proposition 1.4.** For \( p \leq 1/2 \) and \( d \geq 2 \), if SFM\((d,p^\ast)\) is recurrent, then FM\((d,p)\) is also recurrent, where

\[
p^\ast = p^\ast(d,p) = \frac{p(d-1)}{d-(d+1)p}.
\]

This will be proved by coupling arguments in Section 2. Although Proposition 1.4 holds generally for any \( p \leq 1/2 \) and \( d \geq 2 \), we will only need the result for \( p = 1/3 \).

Once Proposition 1.4 has been proved, we will then need to prove the recurrence of SFM\((d,p^\ast)\) for \( p^\ast = p^\ast(d,p) \) with \( p = 1/3 \), that is,

\[
p^\ast(d,1/3) = 1/3 + 1/3d/d(d+1) = 1 - 1/(2d-1).
\]

This will be accomplished by the following proposition.

**Proposition 1.5.** The self-similar frog model SFM\((d,(d-1)/(2d-1))\) is recurrent.

The idea is to compare SFM\((d,(d-1)/(2d-1))\) with SFM\((2,1/3)\), which is known to be recurrent [HJJ17]. To do this, let \( V_{d,p} \) be the total number of visits to the root in SFM\((d,p)\), and consider the probability generating function \( g_{d,p}(x) := \mathbb{E}(x^{V_{d,p}}) \) for \( x \in [0,1) \). We will use the self-similar structure in SFM\((d,p)\) to derive the following self-consistency equation for \( g_{d,p}(x) \):

\[
g_{d,p}(x) = A_{d,p}g_{d,p}(x),
\]

where \( A_{d,p} \) is an operator on the set \( \mathcal{I} = \{ f : [0,1) \to [0,1], \text{nondecreasing} \} \) of functions that will be defined in Section 3. This is the most technical part of the paper: a recursive algorithm to prove (2) for any \( d \) is proposed in Section 3.

When \( d = 2 \) and \( p = 1/3 \), it was shown in [HJJ17] that

\[
g_{2,1/3}(x) = A_{2,1/3}g_{2,1/3}(x) \leq A_{2,1/3}^{n} \to 0,
\]

implying that \( g_{2,1/3}(x) \equiv 1 \) and \( V_{2,1/3}^{*} = \infty \) almost surely. The recurrence of SFM\((2,1/3)\) thus follows. Once we establish (2), we can compare (2) with the \( d = 2 \) case thanks to the recursive algorithm. It turns out that when choosing \( p^\ast = \frac{d-1}{2d-1} \), all operators \( A_{d,d-1}/2d-1 \) are dominated by \( A_{2,1/3} \), yielding \( g_{d,d-1/2}(x) \equiv 1 \) (see Section 4), thus finishing the proof of Theorem 1.2.
2. Proof of Proposition 1.4: the coupling

In this section, we construct couplings among three types of frog processes on rooted $d$-ary tree $T_d$, namely, the classic frog model $FM(d, p)$, the non-backtracking frog model $nbFM(d, p)$, and the self-similar frog model $SFM(d, p)$. There are other auxiliary frog processes involved in the couplings, which we call $P_1, P_2$ and so on. Proposition 1.4 follows from these couplings.

We first give precise descriptions for the non-backtracking frog model and the self-similar frog model.

The **non-backtracking frog model** $nbFM(d, p)$ on the rooted $d$-ary tree $T_d$ with drift parameter $p \in [0, 1]$ evolves according to the same rules as $FM(d, p)$, with two exceptions. First, the paths of active frogs are **non-backtracking**, that is, an active frog never returns to a site it has previously visited. Second, active frogs are killed upon visits to the root vertex. Thus, only one child vertex of the root, which we will henceforth denote by $\emptyset$, can ever be visited in $nbFM(d, p)$: this is the vertex to which the initially active frog at $\emptyset$ jumps on its first step. Any other frog, upon awakening, will begin its journey with either an upward move to the parent vertex with probability $p$ or otherwise a downward move to a uniformly-chosen child vertex. In accordance with the non-backtracking rule, all following steps must satisfy (i) if the last step is upward, the next step will be upward with probability $p/(p + (1 - p)(d - 1)/d)$ or otherwise downward to any one of the child vertices equally likely, unless it is at the root $\emptyset$, in which case it is killed; and (ii) if the last step is downward, the next step will always be downward, equally likely to any child vertex.

The **self-similar frog model** $SFM(d, p)$ on the rooted $d$-ary tree $T_d$ with drift parameter $p \in [0, 1]$ can be constructed by modifying the frog paths in $nbFM(d, p)$ as follows. Let $o_1, \ldots, o_d$ be the child vertices of $\emptyset'$, the vertex chosen by the initially active frog at the root on its first jump. Whenever one of these vertices $o_i$ is visited for a first time (from $\emptyset'$), it becomes lethal to frogs that attempt to jump to it from $\emptyset'$ forever afterward; any such attempt results in the death of the frog attempting the jump. Thus, each of the subtrees $T_d(o_i)$ is entered from the outside at most once, and conditional on the event that there is such an entry, the restriction of $SFM(d, p)$ to this subtree is a (time-shifted) replica of $SFM(d, p)$ in the tree $T_d(\emptyset')$. This “self-similarity” will allow us to write a “recursive distributional equation” or “self-consistency equation” for the generating function of the total number of visits to the root: see Section 3 below.

We next consider a natural embedding of $T_d$ in an unrooted $(d + 1)$-ary homogeneous tree $T_{d+1}^{\text{homo}}$ so that $T_d$ is isomorphic to a subtree $T_d' \subset T_{d+1}^{\text{homo}}$. Here, by an unrooted $(d + 1)$-ary homogeneous tree, we mean an infinite tree in which every vertex is connected to exactly $(d + 1)$ other vertices (see Figure 1). In this embedding, we associate the root vertex $\emptyset$ of $T_d$ with an arbitrary vertex in $T_{d+1}^{\text{homo}}$, and call it $\emptyset$. Fixing an embedding, we assign a level to each vertex in $T_{d+1}^{\text{homo}}$, starting from $\emptyset \in T_{d+1}^{\text{homo}}$. The levels correspond to the “generations” in $T_d$. First, the level of $\emptyset$ is set to 0 and the level of the $d$ neighbors of $\emptyset$ in $T_{d+1}^{\text{homo}}$ corresponding to the $d$ child vertices of $\emptyset \in T_d$ is set to 1. We proceed until finishing assigning levels for all vertices in the subtree $T_d'$. For other vertices in $T_{d+1}^{\text{homo}} \setminus T_d'$, we assign them levels so that each vertex at level $k$ is connected to $d$ “child” vertices in level $(k + 1)$ and one “parent” vertex in level $(k - 1)$. For example, for $\emptyset$, since it is already connected to $d$ vertices in level 1 during the first stage, the only vertex that has not been assigned a level is then marked a level $-1$. Figure 1 illustrates how levels $-1$ to 3 would look like in $T_3^{\text{homo}}$.

Having determined the subtree $T_d' \subset T_{d+1}^{\text{homo}}$ and the levels, we run a frog process on $T_{d+1}^{\text{homo}}$, and call it $P_1 = P_1(T_{d+1}^{\text{homo}}, p)$. At time $t = 0$, a sleeping frog is placed at every vertex of the subtree $T_d'$ (black vertices in Figure 1), and at time $t = 1$ the frog at $\emptyset$ (call it $f_{\emptyset}$)
wakes up and moves to an uniformly chosen child vertex in level 1. After the first step, it performs a $p$-biased random walk on $T_d'$ with a reflecting boundary at $\varnothing$, that is, with probability $p$ it moves to the vertex at one level above, with probability $(1 - p)$ it moves to a uniformly-chosen vertex in the level below, and its path is always reflected at $\varnothing$. As before, whenever an active frog visits a vertex with a sleeping frog, it wakes up that frog. Upon waking up, all frogs (except $f_3$) perform independent $p$-biased random walks on the entire homogeneous tree $T_{d+1}^{\text{homo}}$ starting from the vertices where they originally sleep.

From $P_1$, we may construct a one-per-site frog process $P_2 = P_2(T_d, p)$ on $T_d$. Since there is a natural graph isomorphism between $T_d$ and $T_d' \subset T_{d+1}^{\text{homo}}$, any frog random walk path $\gamma'$ in $P_1$, trimming off the segments spent outside the subgraph $T_d'$, can be translated under the isomorphism to a $\gamma$ on $T_d$ (which might be of finite length). To this end, we can assign to each frog $f$ in $P_2$ the path trimmed and translated from that of the the frog $f'$ in $P_1$ originally placed on the corresponding vertex of $T_d'$. It is not hard to see that $P_2$ is dominated by the frog model $FM(d, p)$ in the sense that at any time $t$, the number of active frogs and the total number of visits to the root vertex in $P_2$ in the long run are no more than the corresponding quantities in $FM(d, p)$. This is because frog paths in $P_2$ can be considered as those in $FM(d, p)$ randomly stopped at the root vertex $\varnothing$, and such modifications can only delay waking up frogs and/or reduce the number of visits to the root vertex $\varnothing$. To this end, if we let $V_P$ be the total number of visits to the root vertex in a frog process $P$ on $T_d$, then $V_{P_2(T_d, p)} \leq V_{FM(d, p)}$.

Now let’s consider another one-per-site frog process $P_3 = P_3(T_d, p)$ on $T_d$ also constructed from $P_1$: for any frog $f'$ in process $P_1$ not starting from $\varnothing$, if its $p$-biased random walk path is $\gamma'_{f'}$, we remove all loops in $\gamma'_{f'}$ first and then terminate the loopless path at the first visit to $\varnothing$. Under the graph isomorphism, the resulting path is mapped to a path $\gamma$ on $T_d$ (possibly of finite length) and assigned to the corresponding frog $f$ in the process $P_3$; (ii) for the frog starting from $\varnothing$, we simply remove all loops in its $p$-biased random walk path, which is then assigned to the corresponding frog in $P_3$ starting from $\varnothing \in T_d$. We have that $V_{P_3(T_d, p)} \leq V_{P_2(T_d, p)}$. To see this, if we drive both $P_2$ and $P_3$ by the same realization of $P_1(T_d^{\text{homo}}, p)$, then the paths of frogs in $P_3$ can always be obtained by further trimming the paths of corresponding frogs in $P_2$. Therefore, under this coupling, if a frog $f_v$ sleeping at some vertex $v \in T_d$ is ever activated in $P_3$, the frog in $P_2$ sleeping at the same vertex of $T_d$ must also be activated. Furthermore, since each activated frog in $P_3$ will visit no more sites than its counterpart in $P_2$, the desired stochastic dominance follows. We further observe that
Lemma 2.1. The frog process $P_3 = P_3(\mathbb{T}_d, p)$ is a nbFM($d, p^*$) with $p^*$ chosen as in (1).

With Lemma 2.1, we establish the hierarchy of stochastic dominance, namely

$$V_{\text{SF}M(d,p^*)} \preceq V_{\text{nbFM}(d,p^*)} = V_{P_3(\mathbb{T}_d,p)} \preceq V_{P_3(\mathbb{T}_d,p^*)} \preceq V_{\text{FM}(d,p)},$$

which implies Proposition 1.4.

Proof of Lemma 2.1. In $P_1$, since $p < 1/2$, every active frog will eventually drift away to level $\infty$. In particular, after removing all loops, the resulting path assigned to the frog originally placed at $\emptyset$ in $P_3$ is an infinite ray in $\mathbb{T}_d$ chosen uniformly from all possible such rays, the same law as that of the frog path in nbFM($d, p^*$) started at the root vertex, because in nbFM($d, p^*$), the frog started at the root will move to a uniformly-chosen child vertex at every step.

Now we consider frogs in $P_3$ that are not placed at the root vertex. Note that for any frog path $\gamma'$ in $P_1$ started at some vertex $v' \in \mathbb{T}_d$ and $v' \neq \emptyset$, removing all its loops and then terminating it at the first visit to $\emptyset$ would then always map to a non-backtracking path $\gamma$ on $\mathbb{T}_d$ that (i) starts at the corresponding non-root vertex $v \in \mathbb{T}_d$, (ii) first leads up toward the root for $k_1$ steps (for some $k_1 \leq |v|$, where $|v|$ denotes the graph distance between $v$ and $\emptyset$), and then (iii) drifts to infinitely far away from the root (when $k_1 < |v|$) or stops at $\emptyset$ (when $k_1 = |v|$). By symmetry, for any such path, the last segment leading directly to infinity from some vertex is an infinite ray chosen uniformly from all rays from the aforementioned vertex to infinity for both $P_3$ and nbFM($d, p^*$). It suffices to show that for all possible values of $k_1$, the probability that a frog in $P_3$ gets assigned a path of such a pattern is the same as the probability that such a pattern is observed in nbFM($d, p^*$) if we choose $p^*$ as in (1). There are three cases.

(a) $k_1 = 0$. In nbFM($d, p^*$), a frog at a non-root vertex immediately moves away from the root upon waking up with probability $(1 - p^*)$. In $P_3$, such a non-backtracking path can be obtained by the loop-erasural procedure from a frog path in $P_1$ with probability

$$\sum_{l=0}^{\infty} \rho(l - 1) \left( \frac{1}{d} \right)^l = \frac{1 - \rho}{1 - \rho/d} = 1 - \frac{\rho(1 - 1/d)}{1 - \rho/d} = 1 - p^*,$$

where $\rho := \frac{p^*}{1 - p^*}$ is the probability that a random walk on $\mathbb{Z}$ starting from 0 with step distribution $p\delta_{-1} + (1 - p)\delta_{+1}$ never visits location $-1$. In the summation, the variable $l$ tracks the furthest distance that a frog in $P_1$ has ever reached above its sleeping level. After that, it must trace backward along the same route, return to the vertex it starts from and then drift to level $\infty$, because only in this way, this loop will be removed and we are left with a loop-erased path $\gamma$ with the desired pattern.

(b) $k_1 = 1, \ldots, |v| - 1$ when $|v| \geq 2$. In nbFM($d, p^*$), the probability that a frog at a non-root vertex $v$ first moves $k_1 < |v|$ steps toward the root and then moves away to infinity is

$$p^* \left( \frac{1}{d^2(1 - p^*) + p^*} \right)^{k_1-1} \left( \frac{d-1}{d}(1 - p^*) \right)^{k_1-1} \left( \frac{d-1}{d}(1 - p^*) + p^* \right).$$
In the frog process $P_3$, such a non-backtracking path can be obtained from a frog path in $P_1$ with probability
\[
\sum_{l=0}^{\infty} \rho^{k_1+l}(1-\rho) \left( \frac{1}{d} \right)^l \left( \frac{d-1}{d} \right) = \frac{\rho^{k_1}(1-\rho)(1-1/d)}{1-\rho/d}.
\]

Similar to the first case, $k_1 + l$ denotes the furthest distance that a frog in $P_1$ has ever reached above its sleeping level. If after removing all loops, there are still $k_1$ upward steps left in the resulting non-backtracking path, then the frog in $P_1$ must have travelled exactly $l$ steps along the same route that had led it $(k_1 + l)$ levels up.

By choosing $p^*$ as in (1), the above two expressions are equal.

(c) $k_1 = |v|$. In nbFM($d, p^*$), the probability that a frog at a non-root vertex moves straight to the root vertex upon waking up is
\[
p^* \left( \frac{d-1}{d}(1-p^*) + p^* \right)^{|v|-1},
\]
whereas a path of the same pattern can be obtained by trimming a frog path in $P_1$ with probability
\[
\sum_{l=0}^{\infty} \rho^{|v|+l}(1-\rho) \sum_{m=0}^{l} \left( \frac{1}{d} \right)^m \left( \frac{d-1}{d} \right).
\]

In the above expression, $l$ marks the number of levels above $\emptyset$ that the frog in $P_1$ has ever reached and $m$ denotes the number of steps that the frog has traced back.

It is again easy to verify that these two expressions are equal when $p^*$ is chosen as in (1). The proof of Lemma 2.1 is complete.

\[\square\]

3. Proof of Proposition 1.5: the self-consistency equation

In this section and the next, we will prove that the self-similar frog model SFM($d, \frac{d-1}{2d-1}$) is recurrent. The proof has two steps. The first step, to which this section is devoted, will be to establish a fixed-point equation
\[
g_{d,p}(x) = \mathcal{A}_{d,p}g_{d,p}(x)
\]
for the probability generating function $g_{d,p}(x) := \mathbb{E}(x^{V^{*}_{d,p}})$ of the number $V^{*}_{d,p}$ of visits to the root vertex $\emptyset$. The second, which will be carried out in Section 4, will be to use the fixed-point equation to show that $g_{d,p}(x) = 0$ for all $x \in (0,1)$; it will then follow that $V^{*}_{d,p} = \infty$ almost surely, proving that SFM($d, \frac{d-1}{2d-1}$) is recurrent. This strategy was used in [HJJ17] to show that SFM($2, 1/3$) is recurrent and in [Ros17] to show that the frog model on a (2, 3)-alternating tree is recurrent.

The fixed-point equation (3) involves a nonlinear operator $\mathcal{A}_{d,p}$ whose domain is the function space
\[\mathcal{I} := \{ f : [0,1) \to [0,1], \text{ nondecreasing} \},\]
which contains the generating function $g_{d,p}(x)$ as an element. This operator is a polynomial in composition operators $Z_k : \mathcal{I} \to \mathcal{I}$ defined for any $p \in (0,1)$ and $k = 1, \cdots, d$ by
\[
Z_k(h) := Z_{k,d,p}(h) = h \circ e_{d,p}^{(k-1)},
\]

where $e_{d,p}^{(k-1)}$ is the $(k-1)$-th power of the composition operator $e_{d,p}$.
Remark which is implied by the fixed point equation (3). The additional terms $k$ where $P$ for
the generating functions $d,p$ where $P$ and only write $d,p$.

Observe that the functions $e^{(k)}_{d,p}$ where $k = 0,1, \ldots, (d-1)$, are linear functions that map
the unit interval monotonically onto subintervals of $[0,1]$. The operator $A_{d,p}$ is defined as follows: for any $h \in I$,

$$A_{d,p}h(x) = \left(px + \frac{1-p}{d}\right) \sum_{k=1}^{d} \left(\frac{d-1}{k-1}\right) P_k(Z_1(h)(x), \ldots, Z_k(h)(x))$$

$$+ \frac{(d-1)(1-p)}{d} \sum_{k=2}^{d} \left(\frac{d-2}{k-2}\right) Q_k(Z_1(h)(x), \ldots, Z_k(h)(x)),$$

where $P_k : \mathbb{R}^d \to \mathbb{R}$ and $Q_k : \mathbb{R}^k \to \mathbb{R}$ are polynomials in $k$ variables defined recursively by

$$P_{k+1}(z_1, \ldots, z_{k+1}) = z_{k+1}^{k+1} - \sum_{l=1}^{k} \left(\frac{k}{l-1}\right) z_{k+1}^{k+1-l} P_l(z_1, \ldots, z_l);$$

$$Q_{k+1}(z_1, \ldots, z_{k+1}) = z_{k+1}^{k+1} - \sum_{l=2}^{k} \left(\frac{k-1}{l-2}\right) z_{k+1}^{k+1-l} Q_l(z_1, \ldots, z_k),$$

starting with $P_1(z_1) = z_1$ and $Q_2(z_1, z_2) = z_2^2$.

**Proposition 3.1.** For any $p \leq 1/2$ and $d \geq 2$, the generating function $g_{d,p}(x) := \mathbb{E}(x^{V^*})$ for
the total number of visits to the root in SFM$(d,p)$ satisfies equation (3).

**Remark 3.2.** The self-consistency equation (3) for $g_{d+1,p}(x)$ involves the same polynomials
$P_k, Q_k$ as that for $g_{d,p}$, but evaluated at the variables

$$(z_1, \ldots, z_k) = (Z_{1,d+1,p}(g_{d+1,p}(x)), \ldots, Z_{k,d+1,p}(g_{d+1,p}(x)))$$

$$= (g_{d+1,p} \circ c_{d+1,p}^{(0)}(x), \ldots, g_{d+1,p} \circ c_{d+1,p}^{(k-1)}(x))$$

in place of

$$(z_1, \ldots, z_k) = (Z_{1,d,p}(g_{d,p}(x)), \ldots, Z_{k,d,p}(g_{d,p}(x)))$$

$$= (g_{d,p} \circ c_{d,p}^{(0)}(x), \ldots, g_{d,p} \circ c_{d,p}^{(k-1)}(x))$$

The additional terms $k = d + 1$ in the sums defining $A_{d+1,p}$ involve only the polynomials
$P_{d+1}, Q_{d+1}$, which, by equations (7) and (8), are themselves defined in terms of the polynomials
$P_k, Q_k$ that appear in the definition of $A_{d,p}$. This suggests that the equations (3) for
the generating functions $g_{d,p}$ can be established recursively, given the "base" case $A_{2,p}$.

**Remark 3.3.** The definition of the operator $A_{d,p}$ does not ensure automatically that $A_{d,p}h \in I$
for every $h \in I$. However, we will only need to be able to apply $A_{d,p}$ to $g_{d,p}$ repeatedly,
which is implied by the fixed point equation (3).

### 3.1. The self-similar structure.

We begin by describing the self-similar structure in SFM$(d,p)$, which explains why $Z_{k,d,p}(g_{d,p})$’s, as define in (4), appear in the self-consistency equation. To simplify the notation, when there is no ambiguity we omit the subscript $d,p$ and only write

$$g(x) = g_{d,p}(x) = \mathbb{E}(x^{V^*}) = A_g(x),$$

where $V^* = V_{d,p}^*$. 


Recall that, in SFM, only one of the $d$ child vertices of the root $∅$ could be visited, and we call this vertex $∅'$ and denote its child vertices by $o_1, \ldots, o_d$ as before. The $d$-ary subtree rooted at any vertex $u$ is denoted by $T_d(u)$. For $j = ∅$ or $∅'$ define
\[ V^*_i-j := \text{number of visits to vertex } j \text{ by all frogs originally placed in } T_d(o_i). \quad (9) \]
Denote by $f_v$ the frog initially placed at a vertex $v \in T_d$ and \{ $f_v \rightarrow u$ \} the event that the frog $f_v$ has visited the vertex $u$. Since only one frog is ever allowed to enter $T_d(o_i)$ in SFM, conditional on the event that $T_d(o_i)$ has been visited, $V^*_i-j$ would have the same distribution as $V^*$, which implies
\[ P(V^* = n) = P(V^*_i-j = n|T_d(o_i) \text{ is ever visited}). \quad (10) \]
Since all frogs behave independently, given that $V^*_i-j = n \geq 1$, the number of visits from $o_i$ to the root vertex $∅$ in SFM($d, p$) via $∅'$, $V^*_i-∅$, follows a $\text{Bin}(n, \frac{p}{p+(1-p)^{\frac{d-1}{d}}})$ distribution, and thus the generating function satisfies
\[ E(x^{V^*_i-∅}|V^*_i-∅ = n) = \left( \frac{px}{p+(1-p)^{\frac{d-1}{d}}} + \frac{(1-p)^{-\frac{d-1}{d}}}{p+(1-p)^{\frac{d-1}{d}}} \right)^n. \quad (11) \]
Lemma 3.4 is a generalization of the observation. For any child vertex $o_j$ of $∅'$ and $J \subseteq \{o_1, \ldots, o_d\} \setminus \{o_j\}$, define the event
\[ B_d(o_j; J) := \bigcap_{s \in J} \left\{ \text{frogs initially placed on vertices of } T_d(o_j) \text{ never enter subtrees } T_d(s) \right\} \quad (12) \]
and if $J$ is the empty set, then $B_d(o_j; J) = \Omega$, the entire probability space. We note that, if $|J| = k$, due to symmetry, the probability that frogs in $T_d(o_j)$ never enter any $k$ other subtrees specified by $J$ is the same for all possible choices of $J$.

**Lemma 3.4.** For any $d \geq 2$, $p \in [0, 1/2]$, $j \in [d]$, and $J \subseteq \{o_1, \ldots, o_d\} \setminus \{o_j\}$, we have
\[ E(x^{V^*_j-∅} 1_{B_d(o_j; J) | T_d(o_j) \text{ is ever visited}}) = g \circ c^{(d-1-|J|)}(x), \]
where $c^{(k)}(x) = c^{(k)}_{d,p}(x)$ is as defined in (6), for $k = 0, 1, \ldots, (d-1)$.

**Proof.** We compute the expectation by conditioning on the number of visits from $o_j$ to $∅'$.
\[
E(x^{V^*_j-∅} 1_{B_d(o_j; J) | T_d(o_j) \text{ is ever visited}})
= E[E(x^{V^*_j-∅} 1_{B_d(o_j; J) | V^*_j-∅', T_d(o_j) \text{ is ever visited}} | V^*_j-∅', T_d(o_j) \text{ is ever visited})]
\]
Remark 3.5. When $J$ is the empty set, frogs initially placed at vertices of $T_d(a_j)$ may go to any of the other $(d - 1)$ subtrees (i.e., no constraints) and thus \( 1_{B_d(a_j; J)} = 1 \). We have
\[
\mathbb{E}(x^{V^*} \mid T_d(a_j) \text{ is ever visited}) = g \circ c^{(d-1)}(x).
\]  
(13)

Remark 3.6. The composition of $g$ with $c^{(k-1)}$ is exactly the generating function for the number of visits to the root by frogs in a subtree $T_d(a_j)$ (if the branch is activated) on the event that none of these frogs ever go to some other \( (k-1) \) subtrees.

3.2. The $P$- and $Q$- polynomials. For the clarity of the recursion, we now put back the subscript $d$ but still omit the parameter $p$, since $p$ is unchanged. In this section, we explain what the $P$- and $Q$-polynomials represent in the self-consistency equation \( \mathbf{1} \). We begin by decomposing $g_d(x)$ according to the number of child vertices of $\emptyset'$ that have been ever visited. Write
\[
g_d(x) = \sum_{k=1}^{d} \sum_{\mathcal{S}, |\mathcal{S}| = k, \mathcal{S} \subseteq [d]} \mathbb{E}(x^{V^*} \mathbf{1}_{\cap_{x \in \mathcal{S}} \{T_d(a_j) \text{ is ever visited} \}} \mathbf{1}_{\cap_{x \in \emptyset'} \{T_d(a_j) \text{ is not visited} \}})
\]
We then express

\[ g = \sum_{k=1}^{d} \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{A_d,k}}) \]  

where

\[ A_{d,k} := \bigcap_{i=1}^{k} \{ T_d(o_i) \text{ is visited} \} \cap \bigcap_{j=k+1}^{d} \{ T_d(o_j) \text{ is not visited} \}. \]

The factors \( \binom{d}{k} \) and \( k \) in (14) are due to symmetry: (i) any subset \( S \subseteq [d] \) of size \( k \) would contribute to \( g_d(x) \) in the same way and there are \( \binom{d}{k} \) choices for \( S \), and (ii) on the event that exactly \( T_d(o_1), \ldots, T_d(o_k) \) are ever visited, the frog \( f_\emptyset \) originating from the root vertex is equally like to visit any one of them, resulting in the additional factor \( k \).

We further decompose each summand into two cases, depending on whether or not \( f_{\emptyset'} \) activates a new subtree \( T_d(o_i) \neq T_d(o_1) \). Define events

\[ D_1 := \{ f_{\emptyset'} \to o_1 \} \cup \{ f_{\emptyset'} \to \emptyset \}, \quad D_2 := D_1^c = \bigcup_{i=2}^{d} \{ f_{\emptyset'} \to o_i \}. \]

Note that on \( D_2 \), at least two branches of \( \emptyset' \) are visited, the intersection \( D_2 \cap A_{d,1} \) is empty. We then express \( g_d(x) \) as two summations of \((2d - 1)\) terms in total.

\[ g_d(x) = \sum_{k=1}^{d} \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_1 \cap A_{d,k}}}) + E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_2 \cap A_{d,k}}}) \]

\[ = \sum_{k=1}^{d} \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_1 \cap A_{d,k}}}) \]

\[ + \sum_{k=2}^{d} \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_2 \cap A_{d,k}}}). \]

(15)

On \( D_1 \), since \( f_{\emptyset'} \) goes to either \( o_1 \) (with probability \((1-p)/d)\) or the root vertex \( \emptyset \) (with probability \( p \), which contributes one visit to the root), then \( V_{d,j}^* \) can be expressed as

\[ V_{d,j}^* = 1_{\{f_\emptyset \to \emptyset\}} + \sum_{j=1}^{d} V_{d,j \to \emptyset}. \]

Here the quantities \( V_{d,j \to \emptyset} \) are the same as \( V_{j \to \emptyset} \) in (9) except that we now put back the subscript \( d \) to emphasize that we are working with the SFM\((d,p)\) case. By independence of \( f_\emptyset, f_{\emptyset'} \) and frogs in \( T_d(o_i) \), each summand in (15) can be written as

\[ \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_1 \cap A_{d,k}}}) \]

\[ = \binom{d}{k} k P(f_\emptyset \to o_1) E(x^{\sum_{j \in S} V_{d,j}^*} 1_{D_1}) E \left( x^{\sum_{j=1}^{k} V_{d,j}^*} 1_{A_{d,k}} \mid f_\emptyset \to o_1, D_1 \right) \]

\[ = \binom{d}{k} k \left( \frac{p}{d} + \frac{1-p}{d} \right) \left( d - 1 \right) E \left( x^{\sum_{j=1}^{k} V_{d,j}^*} 1_{A_{d,k}} \mid f_\emptyset \to o_1, D_1 \right). \]

Similarly, each summand in (16) can be written as

\[ \binom{d}{k} k E(x^{\sum_{j \in S} V_{d,j}^*} 1_{\{f_\emptyset \to o_1\} 1_{D_2 \cap A_{d,k}}}) = \frac{(d-1)(1-p)}{d} \binom{d}{k} \left( d - 2 \right) E \left( x^{\sum_{j=1}^{k} V_{d,j}^*} 1_{A_{d,k}} \mid f_\emptyset \to o_1, f_{\emptyset'} \to o_2 \right). \]
where we used fact that on $A_{d,k}$, if $f_{\emptyset}$ activates a new branch other than $T_d(o_1)$, it is equally likely to go to any of the other $(k - 1)$ vertices $o_2, \ldots, o_k$. Denote these conditional expectations by

$$P_{d,k}(x) := \mathbb{E} \left( x^{\sum_{j=1}^{k} V'_{j,i} \rightarrow \emptyset} 1_{A_{d,k}} \mid f_{\emptyset} \rightarrow o_1, D_1 \right)$$

and

$$Q_{d,k}(x) := \mathbb{E} \left( x^{\sum_{j=1}^{k} V'_{j,i} \rightarrow \emptyset} 1_{A_{d,k}} \mid f_{\emptyset} \rightarrow o_1 \right.$$ \hspace{1cm} \left. f_{\emptyset'} \rightarrow o_2 \right). \tag{18}$$

and we have

$$g_d(x) = \left( px + \frac{1 - p}{d} \right) \sum_{k=1}^{d} \frac{(d-1)(k-1)}{d} P_{d,k}(x) + \frac{(d-2)}{d} \sum_{k=2}^{d} (\frac{d-2}{k-2}) Q_{d,k}(x). \tag{19}$$

Comparing (19) with (6), to show that $g_d$ is a fixed point of $A_{d,p}$, we require that the polynomials $P_k: \mathbb{R}^k \to \mathbb{R}$ and $Q_k: \mathbb{R}^k \to \mathbb{R}$, constructed recursively through equations (7) and (8), to satisfy

$$P_{d,k}(x) = P_k(g_d \circ c_d^{(0)}(x), \ldots, g_d \circ c_d^{(k-1)}(x)), \tag{20}$$

$$Q_{d,k}(x) = Q_k(g_d \circ c_d^{(k)}(x), \ldots, g_d \circ c_d^{(k-1)}(x)). \tag{21}$$

We point out that these $P$- and $Q$-polynomials in the $z$ variables would not depend on the degree $d$ of the tree, and the dependence of the conditional expectations (17) and (18) on $d$ is reflected only when plugging in $z_i = g_d \circ c_d^{(i-1)}(x)$. Consequently, the first $(2d - 1)$ terms in the self-consistency equation for $g_{d+1}(x)$ would share the same “structures” as the $(2d - 1)$ terms for $g_d(x)$.

**Lemma 3.7.** For each $d \geq 2$ and $k \leq d$, $P_{d,k}(x)$ and $Q_{d,k}(x)$, as defined in (17) and (18), respectively, are polynomials of $g_d \circ c_d^{(0)}(x), \ldots, g_d \circ c_d^{(k-1)}(x)$, where $g_d(x) := \mathbb{E}(x^{V'_d})$ and $c_d^{(i)}: [0,1] \to [0,1]$ are linear functions defined in (5). Moreover, if there are polynomials $P_k: \mathbb{R}^k \to \mathbb{R}$ and $Q_k: \mathbb{R}^k \to \mathbb{R}$ such that (20) and (21) hold for some $d \geq k$, then

$$P_{d+1,k}(x) = P_k(g_{d+1} \circ c_{d+1}^{(0)}(x), \ldots, g_{d+1} \circ c_{d+1}^{(k-1)}(x))$$

$$Q_{d+1,k}(x) = Q_k(g_{d+1} \circ c_{d+1}^{(0)}(x), \ldots, g_{d+1} \circ c_{d+1}^{(k-1)}(x)).$$

We defer the proof of Lemma 3.7 to the end of the section.

### 3.3. The base case

We explain in detail how we derive the self-consistency equation in the SFM(2, $p$) case. We then give an example to show how to construct the self-consistency equation in the SFM(3, $p$) case using Lemma 3.7 and equations (7) and (8).

According to (19), the generating function $g_2(x)$ can be written as

$$g_2(x) = \left( px + \frac{1 - p}{2} \right) [P_{2,1}(x) + P_{2,2}(x)] + \frac{(1 - p)}{2} Q_{2,2}(x). \tag{22}$$

By definition (17), the first term

$$P_{2,1}(x) = \mathbb{E} \left( x^{V'_{2,i} \rightarrow \emptyset} 1_{A_{2,1}} \mid f_{\emptyset} \rightarrow o_1, D_1 \right)$$

$$= \mathbb{E} \left( x^{V'_{2,i} \rightarrow \emptyset} 1_{B_2(o_1;\{o_2\})} \mid f_{\emptyset} \rightarrow o_1, D_1 \right)$$

$$= \mathbb{E} \left( x^{V'_{2,i} \rightarrow \emptyset} 1_{B_2(o_1;\{o_2\})} \mid T_2(o_1) \text{ is visited} \right)$$

$$= g_2 \circ c_2^{(0)}(x), \quad \text{by Lemma 3.4}.$$
The second equality above is because on the event \( \{f_\emptyset \to o_1 \} \cap D_1 \), the event \( A_2,1 \) is equivalent to that “no frogs in \( T_2(o_1) \) ever enter the subtree \( T_2(o_2) \)”, which is exactly \( B_2(o_1; \{o_2\}) \), defined in (12). The third equality is due to the independence of frogs in the subtree \( T_2(o_1) \) and the frog \( f_\emptyset \), that is, the number of visits to the root vertex by frogs in \( T_2(o_1) \) only depends on whether or not \( T_2(o_1) \) is ever visited but not where \( f_\emptyset \) goes.

For \( Q_{2,2}(x) \), on the event \( \{f_\emptyset \to o_1 \} \cap \{f_{\emptyset'} \to o_2 \} \), we have \( 1_{A_{2,2}} = 1 \). By independence of frogs initially placed in subtree \( T_2(o_1) \) and those in \( T_2(o_2) \),

\[
Q_{2,2}(x) = E \left( x^{V_{z_1} \to a + V_{z_2} \to a} 1_{A_{2,2}} | f_\emptyset \to o_1, f_{\emptyset'} \to o_2 \right) \\
= E \left( x^{V_{z_1} \to a} | T_2(o_1) \text{ is visited} \right) E \left( x^{V_{z_2} \to a} | T_2(o_2) \text{ is visited} \right) \\
= [g_2 \circ c_2^{(1)}(x)]^2, \text{ by Lemma 3.4}
\]

Finally, conditional on \( \{f_\emptyset \to o_1 \} \cap D_1 \), the event \( A_{2,2} \) is the same as that “some frog initially placed in the subtree \( T_2(o_1) \) has entered \( T_2(o_2) \)”. Denote this event by \( G \), and we have \( G = B_2(o_1; \{o_2\}) \) on the event \( \{f_\emptyset \to o_1 \} \cap D_1 \). Knowing the subtree \( T_2(o_2) \) has been visited, the number of visits to the root vertex by frogs in \( T_2(o_2) \) does not depend on the behavior of the frogs in \( T_2(o_1) \), \( f_\emptyset \) or \( f_{\emptyset'} \). This gives

\[
P_{2,2}(x) = E \left( x^{V_{z_1} \to a + V_{z_2} \to a} 1_{A_{2,2}} | f_\emptyset \to o_1, D_1 \right) \\
= E \left( x^{V_{z_1} \to a + V_{z_2} \to a} 1_G | f_\emptyset \to o_1 \right) \\
= E \left( x^{V_{z_1} \to a} 1_G | f_\emptyset \to o_1 \right) E \left( x^{V_{z_2} \to a} | G \right) \\
= E \left( x^{V_{z_1} \to a} (1 - 1_{B_2(o_1;\{o_2\})}) | f_\emptyset \to o_1 \right) E \left( x^{V_{z_2} \to a} | T_2(o_2) \text{ is activated} \right) \\
= g_2 \circ c_2^{(1)}(x) \left( g_2 \circ c_2^{(0)}(x) - g_2 \circ c_2^{(0)}(x) \right), \text{ by Lemma 3.4}
\]

Combining all computations above, we get

\[
g_2(x) = \left[p x + \frac{1-p}{2}\right] \left\{ g_2 \circ c_2^{(0)}(x) + g_2 \circ c_2^{(1)}(x) \left( g_2 \circ c_2^{(1)}(x) - g_2 \circ c_2^{(0)}(x) \right) \right\} + \frac{1-p}{2} [g_2 \circ c_2^{(1)}(x)]^2.
\]

Thus, if we define polynomials

\[
P_1(z_1) := z_1 \\
P_2(z_1, z_2) := z_2(z_2 - z_1) \\
Q_2(z_1, z_2) := z_2^2,
\]

then we obtain

\[
P_{2,1}(x) = P_1(g_2 \circ c_2^{(0)}(x)) \\
P_{2,2}(x) = P_2(g_2 \circ c_2^{(0)}(x), g_2 \circ c_2^{(1)}(x)) \\
Q_{2,2}(x) = Q_2(g_2 \circ c_2^{(0)}(x), g_2 \circ c_2^{(1)}(x)).
\]

**Remark 3.8.** With \( p = 1/3 \), we have \( \frac{1-p}{2} = \frac{1}{3} \); \( px + \frac{1-p}{2} = \frac{x+1}{3} \), \( c_2^{(0)}(x) = x/2 \) and \( c_2^{(1)}(x) = (x+1)/2 \), and the above recovers Equation (1) in [111,117] derived for SFM(2,1/3):

\[
g_2(x) = \frac{x+2}{3} g_2 \left( \frac{x+1}{2} \right)^2 + \frac{x+1}{3} g_2 \left( \frac{x}{2} \right) \left[ 1 - g_2 \left( \frac{x+1}{2} \right) \right]. \tag{23}
\]
Example. Find the self-consistency equation for \( g_3(x) = \mathbb{E}(x^{V_3}) \) for SFM(3, p).

First, according to (19), \( g_3(x) \) can be written as

\[
g_3(x) = \left( px + \frac{1-p}{3} \right) \sum_{k=1}^{3} \left( \frac{3-1}{k-1} \right) P_{3,k}(x) + \frac{2(1-p)}{3} \sum_{k=2}^{3} \left( \frac{3-2}{k-2} \right) Q_{3,k}(x)
\]

\[
= \left( px + \frac{1-p}{3} \right) [P_{3,1}(x) + 2P_{3,2}(x) + P_{3,3}(x)] + \frac{2(1-p)}{3} [Q_{3,2}(x) + Q_{3,3}(x)],
\]

(24)
in which each term is related to a \( P \)- or \( Q \)-polynomial via equations (20) and (21).

Secondly, the computation of the SFM(2, p) case gives the \( P \)- and \( Q \)-polynomials

\[
P_1(z_1) = z_1; \quad P_2(z_1, z_2) = z_2(z_2 - z_1); \quad Q_2(z_1, z_2) = z_2^2,
\]

whereas the polynomials \( P_3 \) and \( Q_3 \) can be easily deduced from (7) and (8), i.e.,

\[
P_3(z_1, z_2, z_3) = z_3^3 - z_3^2 P_1(z_1) - 2z_3 P_2(z_1, z_2) = z_3^3 - z_3^2 z_1 - 2z_3 z_2(z_2 - z_1);
\]

\[
Q_3(z_1, z_2, z_3) = z_3^3 - z_3 Q_2(z_1, z_2) = z_3^3 - z_3 z_2^2.
\]

Finally, combining (20), (21) and (24), we have

\[
g_3(x) = \left( px + \frac{1-p}{3} \right) \left\{ z_3^3 + (1 - z_3^2)z_1 + 2z_2(z_2 - z_1)(1 - z_3) \right\}
\]

\[
+ \frac{2(1-p)}{3} \left[ z_3^3 + (1 - z_3^2)z_2^2 \right],
\]

where \( z_k = g_3 \circ c_3^{(k-1)}(x) = g_3 \left( \frac{3px + (k-1)(1-p)}{p+2} \right) \) for \( k = 1, 2, 3 \). This is the desired self-consistency equation for \( g_3 \).

Remark 3.9. The corresponding operator \( A_3 = A_{3,p} \) in this case can be defined according to (6): for any \( h \in \mathcal{I} \), \( A_3 \) maps \( h \) to a function \( A_3 h \) whose value at any \( x \in [0, 1) \) is given by

\[
(A_3 h)(x) = \left( px + \frac{1-p}{3} \right) \sum_{k=1}^{3} \left( \frac{3-1}{k-1} \right) P_k(Z_1(h)(x), \ldots, Z_k(h)(x))
\]

\[
+ \frac{2(1-p)}{3} \sum_{k=2}^{3} \left( \frac{3-2}{k-2} \right) Q_k(Z_1(h)(x), \ldots, Z_k(h)(x)),
\]

where, as defined in (4) and (5),

\[
Z_k(h)(x) = h \circ c_{3,p}^{(k-1)}(x) = h \left( \frac{px + (k-1)/3}{p + 2(1-p)/3} \right), \quad \text{for } k = 1, 2, 3.
\]

3.4. Proof of Proposition 3.1 It suffices to prove Lemma 3.7 and the recursive relations that the \( P \)- and \( Q \)-polynomials satisfy.

Proof of Lemma 3.7. Starting from the definition of \( P_{d,k}(x) \) in (17), we will first show that

\[
P_{d,k}(x) := \mathbb{E} \left( x^{\sum_{j=1}^{d} V_{a,j}^{(1)} \mathbb{1}_{A_{d,k}} | I_0 \to o_1, D_1} \right)
\]

is a polynomial of degree \( k \).

Proof of Lemma 3.7. Starting from the definition of \( P_{d,k}(x) \) in (17), we will first show that

\[
P_{d,k}(x) := \mathbb{E} \left( x^{\sum_{j=1}^{d} V_{a,j}^{(1)} \mathbb{1}_{A_{d,k}} | I_0 \to o_1, D_1} \right)
\]

is a polynomial of degree \( k \).
can be written as a polynomial of \((g_d \circ c_d^{(0)}(x), \ldots, g_d \circ c_d^{(k-1)}(x))\). For any \(o_i \in \{o_1, \ldots, o_d\}\), define \(J(o_i)\) to be the subset
\[
J(o_i) := \{o_j \neq o_i : \text{some frog initially placed in } T_d(o_i) \text{ has visited } o_j\} \subset \{o_1, \ldots, o_d\}.
\]
By slightly abusing the notation, we write \(J(A) = \cup_{o_i \in A} J(o_i)\) for any \(A \subset \{o_1, \ldots, o_d\}\). At the beginning, only subtree \(T_d(o_1)\) is activated. After that, some frogs in \(T_d(o_1)\) may enter some other subtrees \(T_d(o_j)\) with \(j \neq 1\) and wake up frogs there, and the newly-wakened frogs may continue to explore and activate new branches. Eventually, if event \(A_{d,k}\) occurs, we have, exactly the subtrees \(T_d(o_1), \ldots, T_d(o_k)\) are visited but none of the subtrees \(T_d(o_{k+1}), \ldots, T_d(o_d)\) are ever visited. Thus on the event \(\{f_\emptyset \rightarrow o_1\} \cap D_1, A_{d,k}\) is equivalent to
\[
\bigcup_{m=0}^{k-1} J^{(m)}(o_1) = \{o_1, \ldots, o_k\}, \quad (25)
\]
where \(J^{(m)}\) denotes the \(m\)-fold composition of \(J\) (the convention is \(J^{(0)}(A) = A\)). Note that we only need to do at most \((k-1)\) compositions here because if \(J^{(m)}(o_1)\) does not contain any new vertex that did not appear previously in \(J^{(0)}(o_1), \ldots, J^{(m-1)}(o_1)\), then no more new subtrees can be activated thereafter (i.e., by applying the function \(J\) even more times).

We can thus write \(P_{d,k}\) as
\[
P_{d,k}(x) = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \mathbb{E} \left[ x^{V_{d,m} \rightarrow 0} \mathbf{1}_{J(o_m) = J_m} \mid T_d(o_m) \text{ is ever visited} \right], \quad (26)
\]
where the summation \((\ast)\) is over all choices of \(J_1, \ldots, J_k \subset \{o_1, \ldots, o_k\}\) such that if \(J(o_m) = J_m\) for \(m = 1, 2, \ldots, k\), then \((25)\) is satisfied. Each choice of \(J_1, \ldots, J_k\) defines a pattern of frog flows among the \(d\) branches attached to \(\emptyset'\). The above expression is because the exact number of visits to the root vertex \(\emptyset\) by frogs in \(T_d(o_m)\) only depends on two events: (a) whether the subtree \(T_d(o_m)\) is activated and (b) whether some frogs in \(T_d(o_m)\) have entered some other subtrees (thus these frogs can not visit the root \(\emptyset\) due to non-backtracking). For each term in \((26)\), we re-write the event \(\{J(o_m) = J_m\}\) in terms of the events \(B_d(o_m; J)\), whose definition is in \((12)\). We can immediately see the following equivalence: for any set \(J \subset \{o_1, \ldots, o_d\} \setminus \{o_m\}\),
\[
B_d(o_m; J) := \left\{ \text{frogs originally placed on vertices of } T_d(o_m) \text{ never enter subtrees } T_d(o_j), o_j \notin J \right\} = \{J(o_m) \subseteq \{o_1, \ldots, o_{m-1}, o_{m+1}, \ldots, o_d\} \setminus J\}.
\]
Now for any \(J_m \subset \{o_1, \ldots, o_d\}\), let \(J_{m,d} := \{o_1, \ldots, o_{m-1}, o_{m+1}, \ldots, o_d\} \setminus J_m\), and we write the subscript \(d\) to emphasize that the compliment is taken in the case \(\text{SFM}(d,p)\). Then,
\[
\{J(o_m) = J_m\} = \left\{ \text{frogs in } T_d(o_m) \text{ have never visited any of } (T_d(o_j))_{o_j \in J_{m,d}} \right\} \cap \left\{ \text{frogs in } T_d(o_m) \text{ have visited every tree in } (T_d(o_j))_{o_j \in J_m} \right\}
= B_d(o_m; J_{m,d}) \cap \left\{ \text{frogs in } T_d(o_m) \text{ have not visited at least one tree in } (T_d(o_j))_{o_j \in J_m} \right\}
= B_d(o_m; J_{m,d}) \setminus \left( \bigcup_{a \in J_m} B_d(o_m; \{a\}) \right)
\]
Therefore, (26) can be expressed as

\[ J = B_d(\alpha_m; J_{m,d}^*) \setminus \left( \bigcup_{a \in J_m} B_d(\alpha_m; J_{m,d}^* \cup \{a\}) \right), \]

where we used the fact that

\[ B_d(\alpha_m, A) \cap B_d(\alpha_m, B) = B_d(\alpha_m, A \cup B). \]

The union on the right can be re-written using the intersection of these B-sets according to the inclusion-exclusion principle, and we thus have

\begin{align*}
1_{\{J(\alpha_m) = J_m\}} &= 1_{B_d(\alpha_m; J_{m,d}^*)} - \sum_{a_1 \in J_m} 1_{B_d(\alpha_m; J_{m,d}^* \cup \{a_1\})} + \sum_{a_1, a_2 \in J_m, a_1 \neq a_2} 1_{B_d(\alpha_m; J_{m,d}^* \cup \{a_1, a_2\})} \\
&\quad - \sum_{a_1, a_2, a_3 \in J_m \text{ distinct}} 1_{B_d(\alpha_m; J_{m,d}^* \cup \{a_1, a_2, a_3\})} + \cdots + (-1)^{|J_m|} 1_{B_d(\alpha_m; J_{m,d}^* \cup J_m)} \\
&= 1_{B_d(\alpha_m; J_{m,d}^*)} + \sum_{S \subseteq J_m \text{ nonempty}} (-1)^{|S|} 1_{B_d(\alpha_m; J_{m,d}^* \cup S)} = \sum_{S \subseteq J_m} (-1)^{|S|} 1_{B_d(\alpha_m; J_{m,d}^* \cup S)}.
\end{align*}

Therefore, (26) can be expressed as

\[ P_{d,k}(x) = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \sum_{S_m \subseteq J_m} (-1)^{|S_m|} E \left( x^{V_{d,m} \rightarrow x} 1_{B_d(\alpha_m; J_{m,d}^* \cup S_m)} \mid T_d(\alpha_m) \text{ is ever visited} \right) \]

\[ = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \sum_{S_m \subseteq J_m} (-1)^{|S_m|} g_d \circ c_d^{(d-1-|J_{m,d}^* \cup S_m|)}(x), \quad \text{(by Lemma 3.4)}. \]

Since \(|J_{m,d}^*| + |J_m| = d - 1\) by definition and \(J_{m,d}^*\) and \(S_m\) are disjoint, we have

\[ d - 1 - |J_{m,d}^* \cup S_m| = d - 1 - |J_{m,d}^*| - |S_m| = |J_m| - |S_m| \]

and it then follows

\[ P_{d,k}(x) = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \sum_{S_m \subseteq J_m} (-1)^{|S_m|} g_d \circ c_d^{(|J_m| - |S_m|)}(x), \]

Since \(J_m \subseteq \{\alpha_1, \ldots, \alpha_k\} \setminus \{\alpha_m\}\) then we have \(0 \leq |J_m|, |J_m| - |S_m| \leq k - 1\). The above is a polynomial of \(g_d \circ c_d^{(i)}(x)\) for \(i = 0, 1, \ldots, k - 1\). Now, following exactly the same argument, we can also get

\[ P_{d+1,k}(x) = E \left( \sum_{j=1}^d V_{d+1,j} \rightarrow x 1_{A_{d+1,k} \mid f_x \rightarrow 0_1, D_1} \right) \]

\[ = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \sum_{S_m \subseteq J_m} (-1)^{|S_m|} g_{d+1} \circ c_{d+1}^{(d+1-1-|J_{m,d+1} \cup S_m|)}(x) \]

\[ = \sum_{J_1, \ldots, J_k} \prod_{m=1}^k \sum_{S_m \subseteq J_m} (-1)^{|S_m|} g_{d+1} \circ c_{d+1}^{(|J_m| - |S_m|)}(x) \]

where the condition for \((J_1, \ldots, J_k)\) to satisfy in the summation (**) is the same as that in (26); this is because on \(A_{d+1,k}\) when only subtrees \(T_d(\alpha_1), \ldots, T_d(\alpha_k)\) are activated, the
possible choices for \( J_1, \ldots, J_k \) are still restricted to those subsets of \( \{q_1, \ldots, q_k \} \) such that \((25)\) is satisfied. The only difference is that, for the SFM\( (d + 1, p) \) case, the complement for each \( J_m \) needs to be taken with respect to the set \( \{q_1, \ldots, q_{m-1}, q_{m+1}, \ldots, q_{d+1} \} \), that is
\[
J_{m,d+1}^c = \{q_1, \ldots, q_{m-1}, q_m, q_{m+1}, \ldots, q_{d+1} \} \setminus J_m,
\]
and so now we have \( |J_{m,d+1}^c| + |J_m| = d \). This suggests that \( P_{d+1,k}(x) \) can be written as a polynomial of \( g_{d+1} \circ c_d^{(i)}(x) \) for \( 0 \leq i \leq k - 1 \).

We call this process a re-activated self-similar frog model (rSFM) and denote by \( \tilde{T}_{k+1,j}(o_i) \)
the number of visits to the root vertex \( \varnothing \) made by frogs initiated in the subtree \( T_{k+1}(o_i) \)

\[
\text{Proof of Proposition 3.7} \quad \text{Having shown that } P_{d,k}(x) \text{ and } Q_{d,k}(x) \text{ are polynomials of } g_d \circ c_d^{(i)}(x), i = 0, \ldots, (k - 1), \text{ it suffices to prove that these polynomials } P_k : \mathbb{R}^k \to \mathbb{R} \text{ and } Q_k : \mathbb{R}^k \to \mathbb{R}, \text{ which determine } P_{d,k}(x), Q_{d,k}(x) \text{ via } (20), (21), \text{ satisfy the recursive relations } (7) \text{ and } (8). \text{ Again, since the arguments for proving these two equations are essentially the same, we only provide the details for the proof of } (7). \text{ In view of Lemma 3.7 it suffices to show that for } d = k + 1, \text{ we have}
\]
\[
P_{k+1,k+1}(x) = [g_{k+1} \circ c_{k+1}^{(k)}(x)]^{k+1} - \sum_{l=1}^{k} \binom{k}{l-1} [g_{k+1} \circ c_{k+1}^{(k)}(x)]^{k+1-l} P_{k+1,l}(x). \quad (27)
\]
Thus if \( P_{k+1} \) is defined according to (7), then for all \( d \geq k + 1, \)
\[
P_{d,k+1}(x) = P_{k+1}(g_d \circ c_d^{(0)}(x), \ldots, g_d \circ c_d^{(k)}(x)).
\]
To do this, we would like to add to the self-similar frog model SFM\( (k + 1, p) \) a second stage of frog re-activation.

- **Stage I:** Run an ordinary SFM\( (k + 1, p) \), starting from one activated frog placed at the root vertex. For \( i = 1, 2, \ldots, \) let \( \tau_i \) be the first time that \( o_i \) is visited by some active frog. Set \( \tau_i = \infty \) if \( o_i \) is never visited in this stage.

- **Stage II:** For each \( i = 1, 2, \ldots, k + 1, \) if \( \tau_i = \infty, \) introduce another activated frog at \( \varnothing', \) have it move to \( o_i, \) and wake up the sleeping frog there. These activated frogs then perform independent non-backtracking random walks and activate sleeping frogs according to the rules of SFM\( (k + 1, p), \).
via \( \emptyset' \). Let \( \mathcal{L} \subseteq \{ o_1, \ldots, o_{k+1} \} \) be the branches activated in Stage I and set \( L := |\mathcal{L}| \), that is
\[
L = \sum_{i=1}^{d} 1\{ \tau_i < \infty \}.
\]
Since the first stage is an ordinary SFM\((k+1, p)\), we have
\[
P_{k+1,k+1}(x) = E \left( x \sum_{i=1}^{k+1} V_{k+1,i}^* \mathbf{1}_{A_{k+1,k+1}} | f_\emptyset \to o_1, D_1 \right)
\]
\. By Proposition 1.4, this implies FM\((d, r)\) is recurrent, which finishes the proof of Theorem 1.2. The special case that SFM\((2, 1/3)\) is recurrent was proved in [HJJ17], and our goal is to show that for any \( d \geq 3 \), SFM\((d, d-1)\) is at least “as recurrent” as SFM\((2, 1/3)\). Throughout this section, the drift parameter is always set to \( p = p^*(d, 1/3) = \frac{d-1}{2d-1} \). Since the model
SFM\((d, \frac{d-1}{2d-1})\) is parametrized by \(d\) only, we abbreviate the notation \(X_{d, \frac{d-1}{2d-1}}\) as \(X_d\), for \(X = g, A, V^*\) or \(c^{(k)}\).

With \(p = \frac{d-1}{2d-1}\), the linear functions \(c^{(k)}_{d}(x)\) become

\[
c^{(k)}_{d}(x) = c^{(k)}_{d, \frac{d-1}{2d-1}}(x) = \frac{(d-1)x + k}{2d-1} + \frac{d-1}{2d-1} = \frac{x}{2} + \frac{k}{2(d-1)} \quad \text{for } k = 0, 1, \ldots, (d-1).
\] (28)

The operator \(A_d\) on \(I = \{ f : [0, 1] \rightarrow [0, 1], \text{ nondecreasing} \}\) can be defined according to (6): \(\forall h \in I\), \(A_d = A_d, \frac{d-1}{2d-1}\) maps \(h\) to

\[
A_d h(x) := \frac{(d-1)x + 1}{2d-1} \sum_{k=1}^{d-1} \binom{d-1}{k-1} P_k \left( h\left( \frac{x}{2} \right), \ldots, h\left( \frac{x}{2} + \frac{k-1}{2(d-1)} \right) \right) + \sum_{k=2}^{d-1} \binom{d-2}{k-2} Q_k \left( h\left( \frac{x}{2} \right), \ldots, h\left( \frac{x}{2} + \frac{k-1}{2(d-1)} \right) \right)
\] (29)

and \(g_d \in I\) is a fixed point of \(A_d\) by Proposition [3.1].

In the base case \(d = 2\) and \(\frac{d-1}{2d-1} = \frac{1}{4}\), the operator \(A_2\) maps any function \(h \in I\) to \(A_2 h\), defined as

\[
A_2 h(x) := \frac{x + 2}{3} h\left( \frac{x + 1}{2} \right)^2 + \frac{x + 1}{3} h\left( \frac{x}{2} \right) \left[ 1 - h\left( \frac{x + 1}{2} \right) \right],
\]

and \(g_2(x) := \mathbb{E}(x^{V^*_d}) \in I\) is a fixed point of \(A_2\). It was shown in [HJJ17] that this operator \(A_2\) exhibits a few nice properties:

- (Closed) For any \(h \in I\), \(A_2 h \in I\);
- (Monotone) For any \(h_1, h_2 \in I\) with \(h_1 \leq h_2\), \(A_2 h_1 \leq A_2 h_2\);
- (Vanishing) \(\lim_{n \to \infty} A_2^n 1 = 0\).

It follows that \(g_2 = A_2^2 g_2 \leq A_2^3 1 \to 0\), meaning \(g_2 \equiv 0\) and \(V^*_2 = \infty\) almost surely. This proves that SFM\((2, 1/3)\) is recurrent.

**Proposition 4.1.** For any \(d \geq 2\), let \(g_d(x) = \mathbb{E}(x^{V_d^*})\) be the probability generating function of \(V_d^*\), the number of visits to the root in SFM\((d, \frac{d-1}{2d-1})\). Then \(A_d g_d \leq A_d g_d\), where \(A_d\) is an operator, whose domain is \(I = \{ f : [0, 1] \rightarrow [0, 1], \text{ nondecreasing} \}\), defined by (29).

Proposition 4.1 indicates that all self-similar frog models SFM\((d, \frac{d-1}{2d-1})\) are at least as recurrent as SFM\((2, 1/3)\). To see this, we make use of the fact that \(g_d\) is a fixed point of \(A_d\) and the three properties of \(A_2\) repeatedly:

\[
g_d = A_d g_d \leq A_2(A_d g_d) \leq A_2 A_2 g_d \leq \cdots \leq A_2^n g_d \leq A_2^n 1 \to 0,
\]

which implies \(g_d \equiv 0\) and \(V_d^* = \infty\) almost surely.

**Proof of Proposition 4.1.** Fix \(d \geq 2\) and consider the last conditional expectations \(P_{d,d}\) and \(Q_{d,d}\) in the self-consistency equation of \(g_d\). To simplify the notation, we fixed an arbitrary \(x \in [0, 1]\) and write

\[
z_k = g_d \circ c^{(k-1)}_{d}(x) = g_d \left( \frac{x}{2} + \frac{k-1}{2(d-1)} \right), \quad k = 1, 2, \ldots, d.
\]

Then \(z_{\{k\}}\) is an increasing sequence in \([0, 1]\). By Proposition 3.1 and Lemma 3.7

\[
0 \leq \mathbb{E}\left( x \sum_{j=1}^{d} \mathbb{P}(V_j^* < z_{j}) 1_{A_{d,d}}|f_0 \to a_1, D_1 \right) = P_{d,d}(x) = P_d(z_1, \ldots, z_d)
\]
\[ z_d = \sum_{l=1}^{d-1} \frac{d-1}{l-1} z_d^{d-1-l} P_l(z_1, \ldots, z_l) \]

\[ = z_d \left[ z_d^{d-1} - \sum_{l=1}^{d-1} \frac{d-1}{l-1} z_d^{d-1-l} P_l(z_1, \ldots, z_l) \right] \]

\[ \leq z_d^{d-1} - \sum_{l=1}^{d-1} \frac{d-1}{l-1} z_d^{d-1-l} P_l(z_1, \ldots, z_l) \] (replace the factor \( z_d \) outside by 1)

\[ = z_d^{d-1} - \sum_{l=1}^{d-2} \frac{d-1}{l-1} z_d^{d-1-l} P_l(z_1, \ldots, z_l) - \left( \frac{d-1}{d-2} \right) P_{d-1}(z_1, \ldots, z_{d-1}) \]

\[ \leq \cdots \leq z_d^2 - z_d z_1 - \sum_{l=2}^{d-1} \frac{d-1}{l-1} P_l(z_1, \ldots, z_l), \]

where we keep plugging out the common factor \( z_d \) and replacing it by 1. This means

\[ \sum_{l=1}^{d} \frac{d-1}{l-1} P_l(z_1, \ldots, z_l) = z_1 + \sum_{l=2}^{d-1} \frac{d-1}{l-1} P_l(z_1, \ldots, z_l) + P_d(z_1, \ldots, z_d) \leq z_1 + z_d^2 - z_d z_1. \]

We can get similar result for the \( Q \)-polynomials following the same strategy:

\[ \sum_{k=2}^{d} \frac{d-2}{k-2} Q_l(z_1, \ldots, z_l) \leq z_d^2. \]

Combining the above two inequalities, we get

\[ A_d g_d(x) = \frac{(d-1)x + 1}{2d-1} \sum_{k=1}^{d} \frac{d-1}{k-1} P_k(z_1, \ldots, z_k) + \frac{d-1}{2d-1} \sum_{k=2}^{d} \frac{d-2}{k-2} Q_k(z_1, \ldots, z_k) \]

\[ \leq \frac{(d-1)x + 1}{2d-1} (z_1 + z_d^2 - z_d z_1) + \frac{d-1}{2d-1} z_d^2. \]

Noticing that for \( x \in [0, 1) \) and \( d \geq 2 \)

\[ \frac{(d-1)x + 1}{2d-1} - \frac{x + 1}{3} = \frac{(3d-3)x + 3 - (2d-1)x - (2d-1)}{3(2d-1)} = \frac{(d-2)(x-2)}{3(2d-1)} \leq \frac{-(d-2)}{3(2d-1)} = \frac{1}{3} - \frac{d-1}{2d-1}, \]

then

\[ A_d g_d(x) \leq \frac{x + 1}{3} (z_1 + z_d^2 - z_d z_1) + \frac{1}{3} z_d^2 - \frac{d-2}{3(2d-1)} (z_1 + z_d^2 - z_d z_1 - z_d^2) \]

\[ \leq \frac{x + 1}{3} (z_1 + z_d^2 - z_d z_1) + \frac{1}{3} z_d^2. \]
Observe that for all $d \geq 2$, the exact forms of the first and last linear functions $c_d^{(0)}$ and $c_d^{(d-1)}$ are the same for all $d \geq 2$; see equation (28):

$$c_d^{(0)}(x) = \frac{x}{2} \quad \text{and} \quad c_d^{(d-1)}(x) = \frac{x+1}{2}.$$ 

This gives

$$z_1 = g_d\left(\frac{x}{2}\right) \quad \text{and} \quad z_d = g_d\left(\frac{x+1}{2}\right),$$

and thus

$$A_d g_d(x) \leq \frac{x+2}{3} g_d\left(\frac{x+1}{2}\right)^2 + \frac{x+1}{3} g_d\left(\frac{x}{2}\right) \left[1 - g_d\left(\frac{x+1}{2}\right)\right] = A_2 g_d(x).$$

The above holds for any $x \in [0,1)$, and thus the proof of Proposition 4.1 is complete. □

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E-mail address: chg217@lehigh.edu

E-mail address: sit218@lehigh.edu

E-mail address: niw318@lehigh.edu

Department of Mathematics, Lehigh University