The Coulomb static gauge

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Abstract

The existence of gauge conditions involving second-order derivatives of potentials is not well known in classical electrodynamics. We introduce one of these gauges, the Coulomb static gauge, in which the scalar potential is given by the Coulomb static potential. We obtain an explicit expression for the associated vector potential and show how the scalar and vector potentials in this gauge yield the retarded electric and magnetic fields. We note the close relation between the proposed gauge and the temporal gauge.
I. INTRODUCTION

The topic of electromagnetic gauges has recently received renewed attention, motivated in part by a paper by Jackson and Okun, in which the history that led to the conclusion that potentials in different gauges describe the same retarded electric and magnetic fields was reviewed. In a subsequent paper, Jackson developed a simple approach for finding novel expressions for the vector potential in the Coulomb, velocity, and temporal gauges and demonstrated how the potentials in these gauges yield the same retarded electric and magnetic fields. Jackson’s approach has been used to express the retarded fields in terms of the instantaneous fields of a Galilean-invariant electromagnetic theory and to discuss the Kirchhoff gauge in which the scalar potential “propagates” with the imaginary speed \(ic\). The present author has proposed an alternative method for showing how the equations of potentials in various gauges yield the same retarded electric and magnetic fields.

The Lorenz, Coulomb, and Kirchhoff gauges are members of a family of gauges known generically as the velocity gauge. The members of this family are characterized by gauge conditions defined by first-order derivatives of potentials. For example, the Coulomb gauge condition, \(\nabla \cdot A = 0\). The existence of gauges defined by gauge conditions involving second-order derivatives of potentials, which may explicitly involve the sources of the fields, does not seem to have been reported before.

In this paper we introduce the Coulomb static gauge defined by the gauge condition

\[
\frac{\partial}{\partial t}(\nabla \cdot A) = 4\pi c (\rho_s - \rho),
\]  

(1)

where \(\rho_s = \rho(x, t_0)\) is the static charge density, which follows from evaluating the time-dependent charge density \(\rho(x, t)\) at an arbitrarily chosen time \(t = t_0\). We use Gaussian units and consider confined sources in vacuum. The gauge condition (1) is defined by second-order derivatives of the vector potential and explicitly involves the charge density, one of the sources of the retarded fields.

In Sec. II we show how a vector potential can always be found to satisfy Eq. (1). We then apply Jackson’s approach to find an explicit expression for the vector potential in the Coulomb static gauge. In Sec. III we calculate the retarded electric and magnetic fields from the explicit expressions for potentials in the Coulomb static gauge. In Sec. IV we show that
an equivalent gauge condition for the temporal gauge is given by
\[ \frac{\partial}{\partial t}(\nabla \cdot A) = -4\pi c \rho, \]
which can be considered as a special case of Eq. (1), namely, when \( \rho_s = 0 \). We also emphasize the close relation between the Coulomb static gauge and the temporal gauge. In Sec. V we present our conclusions.

II. THE COULOMB STATIC GAUGE

It is well known that the electric and magnetic fields \( E \) and \( B \) due to confined sources in vacuum are determined from the scalar and vector potentials \( \Phi \) and \( A \) according to
\[ E = -\nabla \Phi - \frac{1}{c} \frac{\partial A}{\partial t}, \]
\[ B = \nabla \times A. \]
The fields \( E \) and \( B \) are invariant under the gauge transformations
\[ \Phi' = \Phi - \frac{1}{c} \frac{\partial \chi}{\partial t}, \]
\[ A' = A + \nabla \chi, \]
where \( \chi(\mathbf{x}, t) \) is an arbitrary function. The inhomogeneous Maxwell equations together with Eq. (3) lead to the coupled equations
\[ \nabla^2 \Phi + \frac{1}{c} \frac{\partial}{\partial t}(\nabla \cdot A) = -4\pi \rho, \]
\[ \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right) = -\frac{4\pi}{c} \mathbf{J}. \]
The arbitrariness of the gauge function \( \chi \) in Eq. (4) allows us to choose a gauge condition. For the case of the gauge condition in Eq. (1), Eq. (5) becomes
\[ \nabla^2 \Phi_s = -4\pi \rho_s, \]
\[ \nabla \times (\nabla \times A_s) + \frac{1}{c^2} \frac{\partial^2 A_s}{\partial t^2} = \frac{4\pi}{c} \mathbf{J}, \]
where \( \Phi_s \) and \( A_s \) denote the scalar and vector potentials in the Coulomb static gauge. The novelty of this new gauge is that it is defined by a gauge condition possessing second order derivatives of the vector potential in contrast to the Coulomb gauge which is defined by a
gauge condition involving first order derivatives of the vector potential: \( \nabla \cdot \mathbf{A} = 0 \). The solution of Eq. (6a) is given by the Coulomb static potential

\[
\Phi_s(x, t_0) = \int d^3x' \frac{1}{R} \rho(x', t_0),
\]

(7)

where \( R = |x - x'| \). The potential \( \Phi_s(x, t_0) \) produces the electrostatic field \( \mathbf{E}(x, t_0) = -\nabla \Phi_s(x, t_0) \) which satisfies \( \nabla \times \mathbf{E}(x, t_0) = 0 \). It is clear that the term \( -\nabla \Phi_s \) does not satisfy the properties of causality and propagation at the speed of light, which are characteristic of the retarded electric field. Therefore the explicit presence of the static term \( -\nabla \Phi_s \) in the expression for the retarded electric field \( \mathbf{E} = -\nabla \Phi_s - (1/c) \partial \mathbf{A}_s / \partial t \) seems to suggest some kind of inconsistency. Before discussing this point, we will show that a vector potential can always be found to satisfy Eq. (1).

Suppose that the original potential \( \mathbf{A} \) satisfies Eq. (5) but does not satisfy Eq. (1), that is, \( \partial (\nabla \cdot \mathbf{A}) / \partial t - 4\pi c (\rho_s - \rho) = g \), where \( g = g(x, t) \neq 0 \). We make a gauge transformation to the potential \( \mathbf{A}' \) and require that it satisfies the Coulomb static condition

\[
\frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}') - 4\pi c (\rho_s - \rho) = 0 = g + \nabla^2 \left( \frac{\partial \chi}{\partial t} \right).
\]

(8)

Therefore, if a gauge function can be found to satisfy

\[
\nabla^2 \left( \frac{\partial \chi}{\partial t} \right) = -g,
\]

(9)

the new potential \( \mathbf{A}' \) will satisfy the Coulomb static condition (1) and Eq. (6). We note that Eq. (9) is an instantaneous Poisson equation, which can be solved by assuming that \( \partial \chi / \partial t \) vanishes sufficiently rapidly at spatial infinity. The associated solution \( \partial \chi / \partial t = f \) can be integrated over time. Therefore, the existence of a gauge function leading to the Coulomb static condition is generally guaranteed.

To find the solution of Eq. (6b) we can apply the approach used in Ref. 1. In Eq. (4) we identify the potentials \( \Phi' \) and \( \mathbf{A}' \) with the Coulomb static gauge potentials \( \Phi_s \) and \( \mathbf{A}_s \), and the potentials \( \Phi \) and \( \mathbf{A} \) with the Lorenz gauge potentials \( \Phi_L \) and \( \mathbf{A}_L \),

\[
\Phi_s = \Phi_L - \frac{1}{c} \frac{\partial \chi_s}{\partial t},
\]

(10a)

\[
\mathbf{A}_s = \mathbf{A}_L + \nabla \chi_s,
\]

(10b)

where \( \chi_s(x, t) \) is the gauge function that transforms the potentials \( \Phi_L \) and \( \mathbf{A}_L \) into the potentials \( \Phi_s \) and \( \mathbf{A}_s \). From \( \Phi_s \) and \( \Phi_L \) we obtain \( \chi_s \) using Eq. (10a). Hence we can find \( \nabla \chi_s \). Then \( \mathbf{A}_s \) can be obtained from Eq. (10b) because \( \mathbf{A}_L \) and \( \nabla \chi_s \) are known.
The Lorenz gauge potentials are

\[ \Phi_L(x, t) = \int d^3x' \frac{1}{R} \rho(x', t - R/c), \]  
\[ A_L(x, t) = \frac{1}{c} \int d^3x' \frac{1}{R} J(x', t - R/c), \]  

From Eqs. (7), (10a), and (11a) we obtain

\[ \frac{1}{c} \frac{\partial \chi_s(x, t)}{\partial t} = \int d^3x' \frac{1}{R} \rho(x', t - R/c) - \int d^3x' \frac{1}{R} \rho(x', t_0). \]  

We integrate both sides with respect to \( ct \) to obtain

\[ \chi_s(x, t) = \int d^3x' \frac{c}{R} \int_{t_0}^{\tau - R/c} dt' \rho(x', t') - (t - t_0) \int d^3x' \frac{c}{R} \rho(x', t_0). \]  

If we change variables in the first term from \( t \) to \( \tau = t - t' \), then Eq. (13) takes the form

\[ \chi_s(x, t) = \int d^3x' \frac{c}{R} \int_{R/c}^{t - t_0} d\tau \rho(x', t - \tau) - (t - t_0) \int d^3x' \frac{c}{R} \rho(x', t_0). \]  

From Eq. (14) it follows that we obtain

\[ \nabla \chi_s(x, t) = - \int d^3x' \frac{\hat{R}}{R} \left( \rho(x', t - R/c) + \frac{c}{R} \int_{R/c}^{t - t_0} d\tau \rho(x', t - \tau) \right) \]
\[ + c(t - t_0) \int d^3x' \frac{\hat{R}}{R^2} \rho(x', t_0), \]  

where \( \hat{R} = R/R \) with \( R = x - x' \). If we substitute Eqs. (11b) and (15) in Eq. (10b), then we obtain an explicit expression for the vector potential in the Coulomb static gauge,

\[ A_s(x, t) = \frac{1}{c} \int d^3x' \left( \frac{1}{R} \left[ J(x', t') - \hat{R} \rho(x', t') \right] \right)_{\text{ret}} - \frac{c^2 \hat{R}}{R^2} \int_{R/c}^{t - t_0} d\tau \rho(x', t - \tau) \]
\[ + c(t - t_0) \int d^3x' \frac{\hat{R}}{R^2} \rho(x', t_0), \]  

where the square brackets \( [ \] \) \(_{\text{ret}} \) indicate that the enclosed quantity is to be evaluated at the retarded time \( t' = t - R/c \).

Equations (7) and (16) are explicit expressions for the scalar and vector potentials in the Coulomb static gauge. The presence of the last term in Eq. (16) is crucial to canceling the electrostatic field produced by the scalar field in the Coulomb static gauge. Also note that the first volume integral in Eq. (16) constitutes the vector potential in the temporal gauge.\(^1\) This result anticipates the existence of a close relation between the Coulomb static gauge and the temporal gauge, which we will discuss in Sec. IV.
We now show that the fields calculated from the potentials in the Coulomb static gauge are the familiar retarded electric and magnetic fields, that is, we will show that

\[ E = -\nabla \Phi_s - \frac{1}{c} \frac{\partial A_s}{\partial t}, \tag{17a} \]

\[ B = \nabla \times A_s. \tag{17b} \]

lead to the retarded form of the fields \( E \) and \( B \). We first calculate \(-\nabla \Phi_s\) using Eq. (7),

\[ -\nabla \Phi_s(x, t_0) = \int d^3x' \frac{\hat{R}}{R^2} \rho(x', t_0), \tag{18} \]

We calculate \(-\frac{1}{c} \frac{\partial A_s}{\partial t}\) using Eq. (16),

\[ -\frac{1}{c} \frac{\partial A_s(x, t)}{\partial t} = \frac{1}{c^2} \int d^3x' \left( \frac{1}{R} \left[ \frac{\partial J(x', t')}{\partial t'} - c \frac{\hat{R}}{R^2} \frac{\partial \rho(x', t')}{\partial t'} \right]_{\text{ret}} \right. \]

\[ \left. - \frac{c^2 \hat{R}}{R^2} \frac{\partial}{\partial t} \left\{ \int_{R/c}^{t-t_0} \rho(x', t - \tau) \right\} \right) \int d^3x' \frac{\hat{R}}{R^2} \rho(x', t_0), \tag{19} \]

where we have used the fact that \( \partial[ \_ ]_{\text{ret}} / \partial t = [\partial / \partial t']_{\text{ret}} \). After performing the time derivative in the third term on the right-hand side of Eq. (19) and using \( \partial \rho / \partial t = -\partial \rho / \partial \tau \), we obtain

\[ -\frac{1}{c} \frac{\partial A_s(x, t)}{\partial t} = \int d^3x' \left[ \frac{\hat{R}}{R^2} \rho(x', t') + \frac{\hat{R}}{Rc} \frac{\partial \rho(x', t')}{\partial t'} - \frac{1}{Rc^2} \frac{\partial J(x', t')}{{\partial t'}_{\text{ret}}} \right] - \int d^3x' \frac{\hat{R}}{R^2} \rho(x', t_0), \tag{20} \]

The last term on the right-hand side of Eq. (20) cancels the term in Eq. (18). Therefore, if Eqs. (18) and (20) are used in Eq. (17a), we obtain the retarded electric field in the form given by Jefimenko\(^9\)

\[ E(x, t) = \int d^3x' \left[ \frac{\hat{R}}{R^2} \rho(x', t') + \frac{\hat{R}}{Rc} \frac{\partial \rho(x', t')}{\partial t'} - \frac{1}{Rc^2} \frac{\partial J(x', t')}{{\partial t'}_{\text{ret}}} \right]_{\text{ret}}. \tag{21} \]

Equations (16) and (17b) give the usual expression for the retarded magnetic field\(^10\)

\[ B(x, t) = \frac{1}{c} \int d^3x' \frac{1}{R} \left[ \nabla' \times J(x', t') \right]_{\text{ret}}. \tag{22} \]

Therefore, we have shown that the potentials in the Coulomb static gauge lead to the retarded fields.

A recently proposed four-step approach\(^3\) can alternatively be used for showing that the potentials \( \Phi_s \) and \( A_s \) lead to the retarded fields. By applying this alternative approach, the reader can obtain the expression

\[ -\frac{1}{c} \frac{\partial A_s}{\partial t} = \int d^3x' \left[ -\nabla' \rho - \frac{1}{c^2} \frac{\partial J}{\partial t'} \right]_{\text{ret}} + \nabla \Phi_s, \tag{23} \]
which is equivalent to Eq. (20). Equation (23) states that the term $-\frac{1}{c^2} \frac{\partial A_s}{\partial t}$ always contains the static component $\nabla \Phi_s$. Therefore, the explicit presence of the time-independent term $-\nabla \Phi_s$ in the retarded electric field (17a) is irrelevant because such a term is always canceled by the component $\nabla \Phi_s$ of the remaining term $-\frac{1}{c^2} \frac{\partial A_s}{\partial t}$. The static field $-\nabla \Phi_s$ is undetectable and can be interpreted as a spurious field that exists only mathematically. From Eqs. (17a) and (23) we obtain the usual retarded form of the electric field.\(^\text{10}\) The four-step approach leads also to the usual retarded form of the magnetic field in Eq. (22).\(^\text{10}\)

IV. THE COULOMB STATIC GAUGE AND THE TEMPORAL GAUGE

In this section we will discuss the close relation that exists between the Coulomb static gauge and the temporal gauge. Traditionally, the temporal gauge is defined to be one in which the scalar potential is identically zero,\(^1\)

$$\Phi_T = 0. \quad (24)$$

In this gauge the electric and magnetic fields are given only by the vector potential

$$E = -\frac{1}{c} \frac{\partial A_T}{\partial t}, \quad (25a)$$

$$B = \nabla \times A_T, \quad (25b)$$

where we have used the notation $A_T$ to specify that the vector potential is in the temporal gauge. Note the nonexistence of the scalar potential in the temporal gauge, despite a nonzero charge density. This nonexistence of the scalar potential is feasible because the values of the charge density do not necessarily lead to a scalar potential in all gauges. The existence of a scalar potential generally depends on the adopted gauge. The retarded values of the charge density always contribute physically to the electric field, but they do not lead to a scalar potential in the temporal gauge.

In the temporal gauge, Eq. (5) become

$$\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot A_T) = -4\pi \rho, \quad (26a)$$

$$\nabla \times (\nabla \times A_T) + \frac{1}{c^2} \frac{\partial^2 A_T}{\partial t^2} = \frac{4\pi}{c} J. \quad (26b)$$

Jackson\(^1\) has recently derived an expression for the potential $A_T$,

$$A_T(x, t) = \frac{1}{c} \int d^3 x' \left( \frac{1}{R} \left[ J(x', t') - \mathbf{\hat{R}} c \rho(x', t') \right] \right)_{\text{ret}} - \frac{c^2 \mathbf{\hat{R}}}{R^2} \int_{t/c}^{t-t_0} d\tau \rho(x', t - \tau). \quad (27)$$
We can alternatively define the temporal gauge to be one in which the vector potential satisfies the gauge condition (2). By adopting this condition, Eq. (5) become

\[ \nabla^2 \Phi_T = 0, \]

\[ \nabla \times (\nabla \times A_T) + \frac{1}{c^2} \frac{\partial^2 A_T}{\partial t^2} = \frac{4\pi}{c} J + \frac{1}{c} \nabla \frac{\partial \Phi_T}{\partial t}. \]

(28b)

Because of the usual assumption that \( \Phi_T \) vanishes at infinity, Eq. (28a) implies that \( \Phi_T = 0 \), which is the traditional form of the temporal gauge condition. Because \( \Phi_T = 0 \), Eq. (28b) reduces to the usual form given in Eq. (26b). In other words, if we impose Eq. (2) as a gauge condition, then the associated potentials are seen to satisfy Eqs. (24) and (28b). Therefore, the temporal gauge can alternatively be defined by either Eq. (24) or Eq. (2).

From Eqs. (1) and (2) it follows that the temporal gauge can be considered as a particular case of the Coulomb static gauge, namely, the case \( \rho_s = 0 \). Furthermore, from Eqs. (16) and (27) we obtain the connection between the vector potentials \( A_s \) and \( A_T \),

\[ A_s(x,t) = A_T(x,t) + c(t-t_0)E(x,t_0), \]

(29)

where \( E(x,t_0) = -\nabla \Phi_s(x,t_0) \). If \( \rho_s = 0 \), then \( E(x,t_0) = 0 \) and hence \( A_s = A_T \). If we use Eq. (29) then we can obtain the relations

\[ -\nabla \Phi_s - \frac{1}{c} \frac{\partial A_s}{\partial t} = -\frac{1}{c} \frac{\partial A_T}{\partial t}, \]

(30a)

\[ \nabla \times A_s = \nabla \times A_T, \]

(30b)

which show that both gauges yield the same retarded electric and magnetic fields.

V. CONCLUSION

In this paper we have introduced the Coulomb static gauge in which the scalar potential is given by the Coulomb static potential, Eq. (7). We have defined the associated gauge condition, Eq. (1), derived an expression for the associated vector potential, Eq. (16), and demonstrated that the potentials in this new gauge lead to the usual retarded electric and magnetic fields. We have presented an equivalent gauge condition for the temporal gauge and discussed the connection of this gauge condition with the Coulomb static gauge condition. We have also derived the relation between the vector potential in the Coulomb static gauge and the vector potential in the temporal gauge, Eq. (29).
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7 We define confined (localized) sources as those sources that vanish outside the surface of a finite region of space. We could also consider sources that vanish at infinity, that is, sources of order $O(|x|^{-2-\delta})$ where $\delta > 0$ as $|x|$ goes to infinity.

8 Equation (13) contains an additive integration term $\chi_0$ which is a priori a function of $x$, but not of time. We can show that this term is at most a constant. From Eq. (10) and the fact that the Coulomb static gauge scalar field is time-independent, we obtain the wave equation $\square^2 \chi_s = \nabla \cdot \mathbf{A}_s$. The terms on the right-hand side of Eq. (13) constitute a particular integral of this wave equation. Therefore, the term $\chi_0$ is a solution to the associated homogeneous wave equation. But this solution can consist only of time-dependent plane waves. A time-independent $\chi_0$, a solution of Laplace’s equation, can at most be a constant if we demand finiteness at infinity.

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