Research Article
The Hermite–Hadamard–Mercer Type Inequalities via Generalized Proportional Fractional Integral Concerning Another Function

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Received 17 December 2021; Accepted 29 January 2022; Published 28 March 2022

Academic Editor: Ram N. Mohapatra

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In order to be able to study cosmic phenomena more accurately and broadly, it was necessary to expand the concept of calculus. In this study, we aim to introduce a new fractional Hermite–Hadamard–Mercer’s inequality and its fractional integral type inequalities. To facilitate that, we use the proportional fractional integral operators of integrable functions with respect to another continuous and strictly increasing function. Moreover, we establish some new fractional weighted \(\varphi\)-proportional fractional integral Hermite–Hadamard–Mercer type inequalities. Furthermore, in this article, we are keen to present some special cases related to our current study compared to the previous work of the inequality under study.

1. Introduction

There is no doubt that a researcher in the field of calculus knows the significant importance that fractional calculus has acquired recently due to its multiple and important uses in many fields in the natural sciences and technology, especially in physics, fluid dynamics, biology, image processing, control theory, computer networking, and signal processing.

Fractional calculus is the generalized form of classical integrals and derivatives for the order is a noninteger, which comes within the framework of mathematicians’ relentless pursuit of developing mathematics to make it more general and useable in most cases that may encounter when studying and analyzing natural phenomena. According to this, we can say fractional calculus has become the focus of a large number of researchers’ attention. As a result, a lot of extensions and generalizations have appeared especially on the classical fractional calculus like the definitions of Riemann–Liouville (RL) and Caputo. Actually, the derivative Riemann–Liouville is the most general concept and the most uniform and natural. In general, there are numerous other definitions of fractional operators such as Erdélyi–Kober, H"{u}ller, Katugampola, Hadamard, and Riesz which are just a few examples to make reference to [1, 2]. It should be noted that there are many modern fractional operators proposed by many researchers and perhaps the most prominent of them is the recently proposed ABC operator by Atangana and Baleanu [3, 4].

**Definition 1.** The function \(g: ([a, z] \subseteq \mathbb{R}) \rightarrow \mathbb{R}\) is said to be a convex function if the inequality

\[
g(\eta r + (1 - \eta)s) \leq \eta g(r) + (1 - \eta)g(s),
\]

holds for all \(r, s \in [a, z]\) and \(\eta \in [0, 1]\). We say that \(g\) is a concave function if inequality (1) is reversed. In general, the real-valued function \(g\) is said to be a convex function on \([a, z]\) if and only if for all \(y_1, y_2, \ldots, y_n \in [a, z]\) and for any \(\eta_i \in [a, z], i = 1, 2, \ldots, n\) with \(\sum_{i=1}^{n} \eta_i = 1\), we have

\[
g \left( \sum_{i=1}^{n} \eta_i y_i \right) \leq \sum_{i=1}^{n} \eta_i g(y_i).
\]
This well-known inequality is called Jensen inequality [5].

Convexity of functions with their features is one of the most useful properties among other categories of functions in the important fields of applied sciences, especially statistics and mathematics, which according to its own useful definition has a geometric interpretation. Furthermore, it is a vital part of inequalities theory and has become the leading point for creating numerous inequalities such as Jensen’s inequality, Hadamard’s inequality with its type inequalities, and Steffensen’s inequality. One of these inequalities that are closely related to the convexity of functions is the Hermite–Hadamard inequality, which has a well-known area in the space of inequalities theory. This inequality was initially proposed by Hermite in 1881, but it did not come into prominence until it was enriched by Hadamard in 1893 [6] as follows:

\[
\frac{g(a + z) + g(z)}{2} \leq \frac{1}{(a - z)} \int_a^z g(y) 
\] 

(3)

where \( g \) is a convex function on \( [a, z] \), which is called the Hermite–Hadamard (H-H) inequality. Significantly, H-H inequality has become the leading definition has a geometric interpretation. Furthermore, it is a vital part of inequalities theory and has become the leading point for creating numerous inequalities such as Jensen’s inequality, Hadamard’s inequality with its type inequalities, and Steffensen’s inequality. One of these inequalities that are closely related to the convexity of functions is the Hermite–Hadamard inequality, which has a well-known area in the space of inequalities theory. This inequality was initially proposed by Hermite in 1881, but it did not come into prominence until it was enriched by Hadamard in 1893 [6] as follows:

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where \( g \) is a convex function on \( [a, z] \), which is called the Hermite–Hadamard (H-H) inequality. Significantly, H-H inequality has become the leading definition has a geometric interpretation. Furtherm
Theorem 4. Let \( g \) : \([a, z] \rightarrow \mathbb{R} \) be a convex differentiable function on \((a, z)\) and \( w : [a, z] \rightarrow \mathbb{R} \) be a nonnegative integrable function. Then, the following inequalities hold:

\[
g \left( a + z - \frac{x + y}{2} \right) \mathcal{F}^\beta_{[a; z-x]} w(a + z - y)
\]

\[
\leq \frac{1}{2} \left\{ \mathcal{F}^\beta_{[a; z-x]} gw(a + z - y) + \mathcal{F}^\beta_{[a; z-x]} gw(a + z - x) \right\}
\]

\[
\leq \frac{1}{2} \left\{ \mathcal{F}^\beta_{[a; z-x]} gw(a + z - x) \right\}.
\]

Isc also, in the same work, gave the following weighted Hermite–Hadamard–Mercer’s inequality.

Theorem 5. Let \( g \) : \([a, z] \rightarrow \mathbb{R} \) be a convex differentiable function on \((a, z)\) and \( w : [a, z] \rightarrow \mathbb{R} \) be a nonnegative integrable function. Then, the following inequalities hold:

\[
g \left( a + z - \frac{x + y}{2} \right) \mathcal{F}^\beta_{[a; z-x]} w(a + z - y)
\]

\[
\leq \frac{1}{2} \left\{ \mathcal{F}^\beta_{[a; z-x]} gw(a + z - y) + \mathcal{F}^\beta_{[a; z-x]} gw(a + z - x) \right\}
\]

\[
\leq \left\{ g(a) + g(z), \mathcal{F}^\beta_{\{x\}} w(y) - \frac{1}{2} \left\{ \mathcal{F}^\beta_{\{x\}} gw(y) + \mathcal{F}^\beta_{\{y\}} gw(x) \right\} \right\}
\]

\[
\leq \left\{ g(a) + g(z) - g \left( \frac{x + y}{2} \right) \right\} \mathcal{F}^\beta_{\{y\}} w(y).
\]

Abdeljawad et al. [21] (2020) established some inequalities of Hermite–Hadamard–Mercer type inequalities employing RL fractional integral. Butt et al. [22] (2020) proved some Hermite–Hadamard–Mercer type inequalities for convex functions by employing the conformable fractional integrals. Chu et al. [23] (2020) presented some generalizations of Hermite–Hadamard–Mercer type inequalities via Katugampola fractional integral. Recently, in 2021, Vivas-Cortez et al. [24] used generalized RL to present some Hermite–Hadamard–Mercer type inequalities involving convex functions. For more recent studies and generalizations of this inequality, please see [25–28].

All of what we mentioned above prompts us to study Hermite–Hadamard–Mercer’s inequality via the recently generalized operators. Here, in this study, we aim to establish Hermite–Hadamard–Mercer’s inequality and its type inequalities for convex functions employing proportional fractional integral operators involving continuous strictly increasing functions. We also aim to present some fractional weighted Hermite–Hadamard–Mercer type inequalities via the current generalized integral operators. Along with this study, we are able to discuss some special cases and some relationships between our current study and previous studies.

The organization of this research paper will be as follows: In Section 2, we will mention some notations, definitions, and preparatory acquaintance which are used in this work. Section 3 is devoted to the first part of our major results which contain Hermite–Hadamard–Mercer’s inequalities. Throughout Section 4, we provide the fractional weighted Hermite–Hadamard–Mercer’s type inequalities.

2. Essential Preliminaries

Here, we characterize some of the basic properties and some definitions of several elementary fractional integral operators which include the final generalized fractional operator we used to obtain and discuss our new results.

Definition 2 (see [1]). Suppose that the function \( g \) is integrable on \([a, z] \) and \( a \geq 0 \). Then, for all \( \beta > 0 \), we have

\[
\mathcal{F}^\beta_{a} g(y) = \frac{1}{\Gamma(\beta)} \int_{a}^{y} (y - u)^{\beta - 1} g(u) du, u > a,
\]

\[
\mathcal{F}^\beta_{y} g(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{z} (u - y)^{\beta - 1} g(u) du, u > z,
\]

where \( \Gamma(\beta) = \int_{0}^{\infty} \exp(-x)x^{\beta - 1} dx \) is the Gamma function and \( \mathcal{F}^\beta_{y} g(y) = \mathcal{F}_{\delta = 1}^\beta g(y) = \mathcal{F}^\beta_{y} g(y) \). The notations \( \mathcal{F}^\beta_{a} g(y) \) and \( \mathcal{F}^\beta_{y} g(y) \) are called, respectively, the left- and right-sided Riemann–Liouville fractional integrals of a function \( g \) for the order \( \beta \).

Definition 3 (see [1, 2]). Suppose that the function \( g \) is integrable on the interval \( \mathbb{I} \), and let \( \varphi \) be an increasing function, where \( \varphi(y) \in C^{1}(\mathbb{I}, \mathbb{R}) \) such that \( \varphi'(y) \neq 0 \) and \( y \in \mathbb{I} \). Then, for all \( \beta > 0 \), we have

\[
\varphi \mathcal{F}^\beta_{a} g(y) = \frac{1}{\Gamma(\beta)} \int_{a}^{y} \varphi'(u) \varphi(y) - \varphi(u) u^{\beta - 1} g(u) du,
\]

\[
\varphi \mathcal{F}^\beta_{y} g(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{z} \varphi'(u) \varphi(u) - \varphi(y) u^{\beta - 1} g(u) du.
\]

The notations \( \varphi \mathcal{F}^\beta_{a} g(y) \) and \( \varphi \mathcal{F}^\beta_{y} g(y) \) are, respectively, called the left- and right-sided \( \varphi \)-Riemann–Liouville fractional integrals of a function \( g \) for the order \( \beta \).

Definition 4 (see [29]). For the function \( g \), let \( \delta > 0 \), and we have for all \( \beta \in \mathbb{C} \) and \( \text{Re}(\beta) \geq 0 \),

\[
\left( D^\beta_{a} g \right)(y) = D^{m \lambda}_{a} g_{m \lambda \beta} g(y)
\]

\[
= \frac{D^{m \lambda}_{a} g(y)}{\delta^{m \lambda} \Gamma(m - \beta)} \int_{a}^{y} \exp \left[ \frac{\delta - 1}{\delta} (y - u) \right]
\]

\[
\cdot (y - u)^{m \lambda - \beta - 1} g(u) du,
\]

\[
\left( D_{y}^{m \lambda \beta} g \right)(y) = D^{m \lambda}_{y} g_{m \lambda \beta} g(y)
\]

\[
= \frac{D^{m \lambda}_{y} g(y)}{\delta^{m \lambda} \Gamma(m - \beta)} \int_{y}^{z} \exp \left[ \frac{\delta - 1}{\delta} (u - y) \right]
\]

\[
\cdot (u - y)^{m \lambda - \beta - 1} g(u) du,
\]
\[
D^{m,\delta} = D^{\delta} D^{\delta} \ldots D^{\delta}, \quad m = \lceil \text{Re}(\beta) \rceil + 1,
\]
(17)

\[
(y D^{\delta} g)(y) = (1 - \delta) g(y) - \delta g'(y), \quad y D^{m,\delta}
\]
(18)

The notations \((D_{a_1} \delta g)(y)\) and \((D_{z_2} \delta g)(y)\) are, respectively, called the left- and right-sided proportional fractional derivatives of a function \(g\) for the order \(\beta\).

**Definition 5** (see [29]). For the integrable function \(g\), let \(\delta > 0\), and we have for all \(\beta \in C\) and \(\text{Re}(\beta) \geq 0\),

\[
(J^{\delta,\beta}_a g)(y) = \frac{1}{\delta^T(\beta)} \int_a^y \exp \left[ \frac{\delta - 1}{\delta} (y - u) \right]
\]
\[
\cdot (y - u)^{\beta - 1} g(u) du,
\]
(19)

\[
(J^{\delta,\beta}_z g)(y) = \frac{1}{\delta^T(\beta)} \int_y^z \exp \left[ \frac{\delta - 1}{\delta} (u - y) \right]
\]
\[
\cdot (u - y)^{\beta - 1} g(u) du.
\]
(20)

The notations \((J^{\delta,\beta}_a g)(y)\) and \((J^{\delta,\beta}_z g)(y)\) are, respectively, called the left- and right-sided proportional fractional integrals of a function \(g\) for the order \(\beta\).

**Definition 6** (see [30]). For the integrable function \(g\) and for the strictly increasing continuous function \(\varphi\) on \([a, z]\), let \(\delta \in (0, 1]\), and we have for all \(\beta \in C\) and \(\text{Re}(\beta) \geq 0\),

\[
(\varphi D^{\delta,\beta}_a g)(y) = \varphi D^{m,\delta}_a J^{m-\beta,\delta}_a g(y)
\]
\[
= \frac{\varphi D^{m,\delta}_a}{(\delta - 1)^{m-\beta}(m-\beta)} \int_a^y \exp \left[ \frac{\delta - 1}{\delta} (\varphi(y) - \varphi(u)) \right]
\]
\[
\cdot (\varphi(y) - \varphi(u))^{m-\beta-1} \varphi'(u) g(u) du,
\]
(21)

\[
(\varphi D^{\delta,\beta}_z g)(y) = \varphi D^{m,\delta}_z J^{m-\beta,\delta}_z g(y)
\]
\[
= \frac{\varphi D^{m,\delta}_z}{(\delta - 1)^{m-\beta}(m-\beta)} \int_y^z \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \varphi(y)) \right]
\]
\[
\cdot (\varphi(u) - \varphi(y))^{m-\beta-1} \varphi'(u) g(u) du,
\]
(22)

where

\[
\varphi D^{m,\delta} = \varphi D^{\delta} D^{\delta} \ldots D^{\delta}, \quad m = \lceil \text{Re}(\beta) \rceil + 1,
\]
(23)

\[
\left(\frac{\varphi D^{\delta}_g}{\gamma y} \right)(y) = (1 - \delta) g(y) - \delta g'(y), \quad \frac{\varphi D^{m,\delta}}{y}
\]
(24)

The notations \((\varphi D^{\delta,\beta}_a g)(y)\) and \((\varphi D^{\delta,\beta}_z g)(y)\) are, respectively, called the left- and right-sided proportional fractional derivatives of a function \(g\) with respect to \(\varphi\) for the order \(\beta\).

**Definition 7** (see [30]). For the integrable function \(g\) and for the continuous and strictly increasing function \(\varphi\) on \([a, z]\), let \(\delta \in (0, 1]\), and we have for all \(\beta \in C\) and \(\text{Re}(\beta) \geq 0\),

\[
(\varphi J^{\delta,\beta}_a g)(y) = \frac{1}{\delta^T(\beta)} \int_a^y \exp \left[ \frac{\delta - 1}{\delta} (\varphi(y) - \varphi(u)) \right]
\]
\[
\cdot (\varphi(y) - \varphi(u))^{\beta - 1} \varphi'(u) g(u) du,
\]
(25)

\[
(\varphi J^{\delta,\beta}_z g)(y) = \frac{1}{\delta^T(\beta)} \int_y^z \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \varphi(y)) \right]
\]
\[
\cdot (\varphi(u) - \varphi(y))^{\beta - 1} \varphi'(u) g(u) du.
\]
(26)

The notations \((\varphi J^{\delta,\beta}_a g)(y)\) and \((\varphi J^{\delta,\beta}_z g)(y)\) are, respectively, called the left- and right-sided proportional fractional integrals of a function \(g\) with respect to \(\varphi\) for the order \(\beta\).

**Lemma 2** (see [30]). Let \(\varphi\) be a continuous function on \(y \geq a\). If \(\delta \in (0, 1]\) and \(\text{Re}(\alpha), \text{Re}(\beta) > 0\), we have

\[
\varphi J^{\delta,\beta}_a (\varphi D^{\delta,\beta}_a g)(y) = \varphi J^{\delta,\beta}_a (\varphi D^{\delta,\beta}_z g)(y)
\]
(27)

\[
= (\varphi J^{\delta,\beta}_a g)(y),
\]

\[
\varphi J^{\delta,\beta}_z (\varphi D^{\delta,\beta}_z g)(y) = \varphi J^{\delta,\beta}_z (\varphi D^{\delta,\beta}_a g)(y)
\]
(28)

\[
= (\varphi J^{\delta,\beta}_z g)(y).
\]

**Lemma 3** (see [30]). Let \(\varphi\) be an integrable function defined on \([a, y]\) and \(\delta > a\). If \(0 \leq m < \lceil \text{Re}(\beta) \rceil + 1\), then we have

\[
\varphi D^{m,\delta}_a (\varphi D^{\delta,\beta}_a g)(y) = (\varphi D^{m,\delta}_a g)(y)
\]
(29)

\[
\varphi D^{m,\delta}_z (\varphi D^{\delta,\beta}_z g)(y) = (\varphi D^{m,\delta}_z g)(y).
\]
(30)

In this paper, we need the following identity as in [31]. Let \(\delta \in (0, 1]\), \(\beta \in C\), \(\text{Re}(\beta) \geq 0\), and \(\varphi\) be a strictly increasing continuous function. Then, for any constant \(k\), we have

\[
(\varphi J^{\delta,\beta}_z k)(y) = \frac{\varphi(y) - \varphi(x)^\beta}{\delta^T(\beta + 1)} k.
\]
(31)
3. Fractional Hermite–Hadamard–Mercer Inequalities Involving \( \varphi \)-Proportional Fractional Integrals

This section is the first part of our main contributions. Here, we present basic generalization in Hermite–Hadamard–Mercer’s inequalities which involve convex functions for generalized proportional fractional integral operators concerning another strictly increasing continuous function.

**Theorem 6.** Let \( \varphi: \mathbb{R} \to [a, z] \subseteq \mathbb{R} \), with \( 0 \leq a < z \), be a continuous strictly increasing function and \( g: [a, z] \to \mathbb{R} \) be a convex differentiable function on \((a, z)\) satisfying that \((g' \varphi): \mathbb{R} \to \mathbb{R}\) is an integrable mapping on \( \mathbb{R} \). Then, we have

\[
g\left(a + z - \frac{x + y}{2}\right) \leq g(a) + g(z) - \frac{g(r) + g(s)}{2}
\]

**Proof.** According to the Jensen–Mercer inequality and for \( r, s \in [a, z] \), we have

\[
g\left(a + z - \frac{r + s}{2}\right) \leq g(a) + g(z) - \frac{g(r) + g(s)}{2} \tag{34}
\]

Now, we change the variables \( r \) and \( s \) with \( r = \eta x + (1 - \eta)y \) and \( s = (1 - \eta)x + \eta y \), and we get

\[
g\left(a + z - \frac{\eta x + (1 - \eta)y + (1 - \eta)x + \eta y}{2}\right) \leq g(a) + g(z) - \frac{g(\eta x + (1 - \eta)y) + g(1 - \eta)x + \eta y)}{2} \tag{35}
\]

\[
\int_0^1 g\left(a + z - \frac{x + y}{2}\right) \exp\left[\frac{\delta - 1}{\delta} \eta(y - x)\right] \eta^{\delta - 1} d\eta
\]

\[
\leq \int_0^1 [g(a) + g(z)] \exp\left[\frac{\delta - 1}{\delta} \eta(y - x)\right] \eta^{\delta - 1} d\eta - \int_0^1 \exp\left[\frac{\delta - 1}{\delta} \eta(y - x)\right] \left(g(\eta x + (1 - \eta)y) + g((1 - \eta)x + \eta y)\right) \eta^{\delta - 1} d\eta. \tag{37}
\]

Using identity (31) on both sides of (37), we obtain
\[
\frac{2}{g} \left[ g(a) + g(z) - \frac{1}{2} \int_{0}^{1} \exp \left[ \frac{\delta - 1}{\delta} \eta(y - x) \right] \eta^{\beta - 1} g(\eta x + (1 - \eta)y) d\eta \right]
\]

Next,

\[
\frac{\delta}{2(y - x)^{\beta}} \left\{ \varphi \mathcal{F}^{\beta, \delta} \left[ \varphi^{-1} \right] \left( g \varphi \right) \left( \varphi^{-1} \right) + \varphi \mathcal{F}^{\beta, \delta} \left[ \varphi^{-1} \right] \left( g \varphi \right) \left( \varphi^{-1} \right) \right\}
\]

Putting \( \varphi(u) = \eta x + (1 - \eta)y \) and \( \varphi(v) = (1 - \eta)x + \eta y \), we get

\[
g(a) + g(z) = \frac{\delta}{2(y - x)^{\beta}} \left\{ \varphi \mathcal{F}^{\beta, \delta} \left[ \varphi^{-1} \right] \left( g \varphi \right) \left( \varphi^{-1} \right) + \varphi \mathcal{F}^{\beta, \delta} \left[ \varphi^{-1} \right] \left( g \varphi \right) \left( \varphi^{-1} \right) \right\}
\]

This proves the first inequality in (32). To prove the second inequality and by using the convexity of \( g \), we can be certain that

\[
g \left( \varphi(u) + \varphi(v) \right) \leq \frac{g \varphi(u) + (g \varphi)(v)}{2}
\]

Then, for \( \varphi(u) = \eta x + (1 - \eta)y \) and \( \varphi(v) = (1 - \eta)x + \eta y \), we have

\[
g \left( \frac{x + y}{2} \right) \leq \frac{g(\eta x + (1 - \eta)y) + g((1 - \eta)x + \eta y)}{2}, \quad \eta \in [0, 1].
\]
\[ g\left(\frac{x + y}{2}\right) \geq \frac{\beta}{2} \left[ \int_0^1 \exp \left( \frac{\delta - 1}{\delta} \eta (y - x) \right) \eta^{\beta - 1} g(\eta x + (1 - \eta)y) d\eta \right] + \frac{1}{0} \exp \left[ \frac{\delta - 1}{\delta} \eta (y - x) \right] \eta^{\beta - 1} g((1 - \eta)x + \eta y) d\eta \]

Therefore, we have

\[ -g\left(\frac{x + y}{2}\right) \geq \frac{\beta}{2} \left[ \int_0^1 \exp \left( \frac{\delta - 1}{\delta} \eta (y - x) \right) \eta^{\beta - 1} g[\eta (a + z - x) + (1 - \eta)(a + z - y)] + (1 - \eta)(a + z - y)] d\eta \]

On both sides of inequality (44), adding \( g(a) + g(z) \), we get the second inequality in (32). Hence, desired inequality (32) is thus proved. We now give the proof of inequalities in (33). We have, according to the convexity of the function \( g \), for all \( r, s \in [a, z] \), that

\[ g\left(a + z - \frac{r + s}{2}\right) = g\left(\frac{(a + z - r + a + z - s)}{2}\right) \leq \frac{1}{2} \left[ g(a + z - r) + g(a + z - s) \right]. \tag{45} \]

On both sides of (46), taking product by \( \exp[\delta - 1/\delta \eta (y - x)] \eta^{\beta - 1} \) and then integrating the estimating inequality with respect to \( \eta \) over \([0, 1]\), we obtain

\[ g\left(a + z - \frac{x + y}{2}\right) \leq \frac{1}{2} \left[ g(\eta (a + z - x) + (1 - \eta)(a + z - y)) + g(\eta (a + z - y) + (1 - \eta)(a + z - x)) \right]. \tag{46} \]
Putting \( \varphi(u) = \eta(a + z - y) + (1 - \eta)(a + z - x) \) and 
\( \varphi(v) = \eta(a + z - x) + (1 - \eta)(a + z - y) \), we get

\[
\frac{\eta^\beta(\beta + 1)}{2(\gamma - \eta)} \left\{ \begin{array}{l}
\eta^\beta \left[ \varphi^\beta \delta \frac{\partial^\beta}{\partial \eta^\beta} \right] \eta^\beta \varphi \left[ \varphi^{-1} \delta \frac{\partial^\beta}{\partial \eta^{-1}} \right] \\
+ \int_0^1 \exp \left[ \frac{\delta - 1}{\eta} \right] \eta^\beta \varphi \left[ \varphi^{-1} \delta \frac{\partial^\beta}{\partial \eta^{-1}} \right] d\eta
\end{array} \right.
\]

This completes the proof of the first inequality in (33). To prove the second inequality and by using the convexity of \( g \), we can be certain that

\[
g[\eta(a + z - x) + (1 - \eta)(a + z - y)] \leq \eta g(a + z - x) + (1 - \eta)g(a + z - y),
\]

\[
g[\eta(a + z - y) + (1 - \eta)(a + z - x)] \leq \eta g(a + z - y) + (1 - \eta)g(a + z - x).
\]

Adding inequalities (50) and (51), we obtain

\[
g[\eta(a + z - x) + (1 - \eta)(a + z - y)] + g[\eta(a + z - y) + (1 - \eta)(a + z - x)]
\leq g(a + z - y) + g(a + z - x) \leq 2[g(a) + g(z)] - [g(x) + g(y)].
\]

On both sides of (52), taking product by \( \exp[\delta - 1/\delta \eta(\gamma - \eta)] \eta^\beta \) and then integrating the estimating inequality with respect to \( \eta \) over \([0, 1]\), we obtain

\[
\left\{ \begin{array}{l}
\int_0^1 \exp \left[ \frac{\delta - 1}{\eta} \right] \eta^\beta g \left[ \eta(a + z - x) + (1 - \eta)(a + z - y) \right] d\eta \\
+ \int_0^1 \exp \left[ \frac{\delta - 1}{\eta} \right] \eta^\beta g \left[ \eta(a + z - y) + (1 - \eta)(a + z - x) \right] d\eta
\end{array} \right.
\]

\[
\leq \frac{1}{\beta} g(a + z - y) + \frac{1}{\beta} g(a + z - x)
\]

\[
\leq \frac{2}{\beta} [g(a) + g(z)] - \frac{1}{\beta} [g(x) + g(y)].
\]

On the left-hand side in (53), applying the same arguments as above, we obtain
\[
\frac{\delta^\beta \Gamma(\beta + 1)}{2(y - x)\beta} \left\{ \varphi \mathcal{P}^{\beta, \delta}_{\left[ y - x \right]} \delta^\beta \left[ \varphi^{-1}(a + z - x) \right] + \varphi \mathcal{P}^{\beta, \delta}_{\left[ y - x \right]} \delta^\beta \left[ \varphi^{-1}(a + z - y) \right] \right\}
\leq \frac{g(a + z - y) + g(a + z - x)}{2} \leq g(a) + g(z) - \frac{g(x) + g(y)}{2},
\]

which is the second and third inequalities in (33). Hence, the desired inequalities in (33) are thus proved. \[\Box\]

**Remark 1**

The proportional fractional integral version of Theorem 6 was provided by K. Yildirim and S. Yildirim in [32].

If we put \(\delta = 1\), in Theorem 6, we obtain its \(\varphi\)-Riemann–Liouville fractional integral version, which was proved by Butt et al. in [33]. Fixing \(\delta = 1\) and \(\varphi(x) = x\) in Theorem 6 for all \(x \in [a, z]\), it gives

This was given by Kian and Moslehian in [34].

**Theorem 7.** Let \(\varphi: \mathbb{R} \rightarrow [a, z] \subseteq \mathbb{R}\), with \(0 \leq a < z\), be a continuous strictly increasing function and \(g: [a, z] \rightarrow \mathbb{R}\) be a convex differentiable function on \((a, z)\) satisfying that \((g^\prime \varphi): \mathbb{R} \rightarrow \mathbb{R}\) is an integrable mapping on \(\mathbb{R}\). Then, we have

\[
g\left( a + z - \frac{x + y}{2} \right) \leq g(a) + g(z) - \frac{1}{y - x} \int_x^y g(a + z - \eta) d\eta
\]

\[
\leq g(a) + g(z) - \frac{g(x) + g(y)}{2}.
\]
Proof. According to the convexity of the function \( g \) for all \( r, s \in [a, z] \), we have
\[
g\left( a + z - \frac{r + s}{2} \right) = g\left( \frac{a + z - r + a + z - s}{2} \right)
\leq \frac{1}{2} g(a + z - r) + g(a + z - s).
\]
Putting \( r = \eta/2x + 2 - \eta/2y \) and \( s = 2 - \eta/2x + \eta/2y \), it follows, for all \( r, s \in [a, z] \) and \( \eta \in [0, 1] \), that
\[
g\left( a + z - \frac{x + y}{2} \right) \leq \frac{1}{2} \left\{ g\left( a + z - \frac{\eta (x + 2 - \eta y)}{2} \right) + g\left( a + z - \frac{2 - \eta x + \eta y}{2} \right) \right\}.
\]
On both sides of (61), taking product by \( \exp[\delta - 1/\delta\eta/2(y - x)](\eta/2)^{\beta - 1} \) and then integrating the estimating inequality with respect to \( \eta \) over \([0, 1]\), we obtain
\[
\frac{1}{\beta} \left\{ \exp\left[ \frac{\delta - 1}{\delta} \left( y - x \right) \right] \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot \exp\left[ \frac{\delta - 1}{\delta} \left( y - x \right) \right] \left( \frac{\eta}{2} \right)^{\beta - 1} \right\}
\leq \int_0^1 \frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{\eta (x + 2 - \eta y)}{2} \right) \right) \cdot d\eta
\]
\[
+ \int_0^1 \frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{2 - \eta x + \eta y}{2} \right) \right) \cdot d\eta.
\]
Next,
\[
\leq \int_0^1 \frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{\eta (x + 2 - \eta y)}{2} \right) \right) \cdot d\eta
\]
\[
+ \int_0^1 \frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{2 - \eta x + \eta y}{2} \right) \right) \cdot d\eta.
\]
Putting \( \varphi(\eta) = a + z - (\eta/2x + 2 - \eta/2y) \) and \( \varphi(\eta) = a + z - (2 - \eta/2x + \eta/2y) \), we obtain
\[
\frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{\eta (x + 2 - \eta y)}{2} \right) \right) \cdot d\eta
\]
\[
+ \int_0^1 \frac{\delta - 1}{\delta} \eta \cdot (y - x) \left( \frac{\eta}{2} \right)^{\beta - 1} \cdot g\left( a + z - \left( \frac{2 - \eta x + \eta y}{2} \right) \right) \cdot d\eta.
\]
This proves the first inequality in (59). To prove the second inequality and by using the Jensen–Mercer inequality, we can be certain that

\[
g(a + z - \left( \frac{\eta}{2} x + \frac{2 - \eta}{2} y \right)) \leq g(a) + g(z) - \left( \frac{\eta}{2} g(x) + \frac{2 - \eta}{2} g(y) \right). \tag{65}
\]

Adding inequalities (65) and (66), we get

\[
g(a + z - \left( \frac{\eta}{2} x + \frac{2 - \eta}{2} y \right)) + g(a + z - \left( \frac{2 - \eta}{2} x + \frac{\eta}{2} y \right)) \leq 2[g(a) + g(z)] - [g(x) + g(y)]. \tag{67}
\]

By comparing the left-hand side of inequality (64) with the left-hand side of inequality (68), we can deduce

\[
\frac{\delta^{\beta} (\beta + 1)}{2(y - x)^{\beta}} \left[ \phi, \varphi^{\beta \delta} \left[ \phi^{-1}(a + z - x + y/2) \right] \right] \left( g \varphi' \phi^{-1}(a + z - y) \right) + \frac{\delta^{\beta} (\beta + 1)}{2(y - x)^{\beta}} \left[ \phi, \varphi^{\beta \delta} \left[ \phi^{-1}(a + z - x + y/2) \right] \right] \left( g \varphi' \phi^{-1}(a + z - x) \right) \leq g(a) + g(z) - \frac{g(x) + g(y)}{2}, \tag{69}
\]

which is the second inequality in (59). The proof is thus completed. \( \square \)

4. Weighted Fractional

Hermite–Hadamard–Mercer Inequalities

Involving \( \phi \)-Proportional

Fractional Integrals

This section is the second part of our main contributions, within which we give the fractional weighted

Hermite–Hadamar–Mercer’s inequalities which involve convex functions for generalized proportional fractional integral operators concerning another strictly increasing continuous function.

Theorem 8. Let \( \phi: \mathbb{R} \rightarrow [a, z] \subseteq \mathbb{R} \), with \( 0 \leq a < z \), be a continuous strictly increasing function, \( g: [a, z] \rightarrow \mathbb{R} \) be a convex differentiable function on \( (a, z) \), and \( w: [a, z] \rightarrow \mathbb{R} \) be a nonnegative integrable function satisfying that \( (g \varphi'), (w \varphi'): \mathbb{R} \rightarrow \mathbb{R} \) are integrable mappings on \( \mathbb{R} \). Then, we have
\[ g \left( a + z - \frac{x + y}{2} \right) \frac{\partial^\delta}{\partial \{\varphi^{-1} (a + z - x)\}} \left( (w') \varphi^{-1} (a + z - y) \right) \]
\[ \leq \frac{1}{2} \left\{ \varphi \frac{\partial^\delta}{\partial \{\varphi^{-1} (a + z - x)\}} \left( (w') \varphi^{-1} (a + z - y) \right) \right\} \]
\[ + \varphi \frac{\partial^\delta}{\partial \{\varphi^{-1} (a + z - y)\}} \left( (w') \varphi^{-1} (a + z - x) \right) \]
\[ \leq \frac{1}{2} \left\{ (a + z - x) + g(a + z - y) \right\} \frac{\partial^\delta}{\partial \{\varphi^{-1} (a + z - x)\}} \left( (w') \varphi^{-1} (a + z - y) \right) \]
\[ \leq \left\{ g(a) + g(z) - \frac{g(x) + g(y)}{2} \right\} \frac{\partial^\delta}{\partial \{\varphi^{-1} (a + z - x)\}} \left( (w') \varphi^{-1} (a + z - y) \right). \]

**Proof.** According to the convexity of the function \( g \) on \([a, z] \), we have

\[ g \left( a + z - \frac{x + y}{2} \right) = g \left( \frac{1}{2} (a + z - (\eta x + (1 - \eta) y) + a + z - (\eta y + (1 - \eta) x)) \right) \]
\[ \leq \frac{g(a + z - (\eta x + (1 - \eta) y)) + g(a + z - (\eta y + (1 - \eta) x))}{2} \]

On both sides of (71), taking product by \( h(a + z - (\eta x + (1 - \eta) y)) \) and then integrating the estimating inequality with respect to \( \eta \) over \([0, 1] \), we obtain

\[ g \left( a + z - \frac{x + y}{2} \right) \int_0^1 h(a + z - (\eta x + (1 - \eta) y)) d\eta \]
\[ \leq \frac{1}{2} \left\{ \int_0^1 g(a + z - (\eta x + (1 - \eta) y)) h(a + z - (\eta x + (1 - \eta) y)) d\eta \right\} \]
\[ + \int_0^1 g(a + z - (\eta y + (1 - \eta) x)) h(a + z - (\eta y + (1 - \eta) x)) d\eta \].

Applying the change of variables \( \varphi(u) = a + z - (\eta x + (1 - \eta) y) \), to each of the left-hand side and the first integration in (72) and applying the change of variables \( \varphi(u) = a + z - (\eta y + (1 - \eta) x) \), to the second integration on the right-hand side, we get

\[ g \left( a + z - \frac{x + y}{2} \right) \int_{\varphi^{-1}(a + z - x)}^{\varphi^{-1}(a + z - y)} \left( h') \varphi(u) \frac{\varphi'(u)}{y - x} \right) du \]
\[ \leq \frac{1}{2} \left\{ \int_{\varphi^{-1}(a + z - x)}^{\varphi^{-1}(a + z - y)} \left( g' \varphi(u) (h') \varphi(u) \right) \frac{\varphi'(u)}{y - x} du \right\} \]
\[ + \int_{\varphi^{-1}(a + z - y)}^{\varphi^{-1}(a + z - x)} \left( g' \varphi(u) h(2(a + z) - (x + y + \varphi(u))) \right) \frac{\varphi'(u)}{y - x} du \].
Choosing
\[ (h\varphi)(u) = \frac{1}{\delta \Gamma(\beta)} \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \{a + z - y\}) \right] \]
(74)

\[ \cdot (\varphi(u) - \{a + z - y\})^{\beta - 1}(w^\varphi)(u), \]

then (73) leads to

\[ \frac{1}{y - x} g(a + z - \frac{x + y}{2}) \mathcal{F}_{\varphi^{-1}(a+z-x)}(\varphi^{-1}(a + z - y)) \]
\[ \leq \frac{1}{2(y - x)} \left\{ \varphi \mathcal{F}_{\varphi^{-1}(a+z-x)}(g') \varphi(w^\varphi)(\varphi^{-1}(a + z - y)) \right\} \]
\[ + \varphi \mathcal{F}_{\varphi^{-1}(a+z-x)}(g' \varphi)(w^\varphi)(\varphi^{-1}(a + z - x)), \]

which proves the first inequality in (70). To prove the second inequality, by using the convexity of \( g \), we have

\[ g(a + z - (\eta x + (1 - \eta)y)) = g(\eta(a + z - x) + (1 - \eta)(a + z - y)) \]
\[ \leq \eta g(a + z - x) + (1 - \eta)g(a + z - y), \]

(76)

\[ g(a + z - (\eta y + (1 - \eta)x)) = g(\eta(a + z - y) + (1 - \eta)(a + z - x)) \]
\[ \leq \eta g(a + z - y) + (1 - \eta)g(a + z - x). \]

(77)

Adding inequalities (76) and (77), we obtain

\[ g(a + z - (\eta x + (1 - \eta)y)) + g(a + z - (\eta y + (1 - \eta)x)) \]
\[ \leq g(a + z - x) + g(a + z - y). \]

(78)

\[ \frac{1}{2} \left\{ \int_0^1 g(a + z - (\eta x + (1 - \eta)y))h(a + z - (\eta x + (1 - \eta)y))d\eta \right\} \]
\[ + \int_0^1 g(a + z - (\eta y + (1 - \eta)x))h(a + z - (\eta x + (1 - \eta)y))d\eta \]
\[ \leq \frac{1}{2} \left\{ \int_0^1 [g(a + z - x) + g(a + z - y)]h(a + z - (\eta x + (1 - \eta)y))d\eta \right\}. \]

(79)

Applying the change of variables \( \varphi(u) = a + z - (\eta x + (1 - \eta)y) \), to the second integration in the left-hand side of (79) and applying the change of variables \( \varphi(u) = a + z - (\eta y + (1 - \eta)x) \), to the first integration on left-hand side and to the right-hand side, then we obtain
and the second inequality is thus proved. To prove the third inequality in (70), by using the convexity of \( g \) and Lemma 1, we have

\[
g(a + z - x) \leq g(a) + g(z) - g(x), \quad (81)
\]

\[
g(a + z - y) \leq g(a) + g(z) - g(y). \quad (82)
\]

Adding inequalities (81) and (82), we get

\[
g(a + z - x) + g(a + z - y) \leq 2[g(a) + g(z)] - g(x) - g(y).
\]

On both sides of (83), taking product by 1/2h(a + z - (\eta x + (1 - \eta)y)), integrating the estimating inequality with respect to \( \eta \) over [0, 1], and then applying the changing of variables \( \varphi(u) = a + z - (\eta x + (1 - \eta)y) \), we get

\[
g(a + z - x) + g(a + z - y) \leq \frac{1}{2} \int_{\varphi^{-1}(a + z - x)}^{\varphi^{-1}(a + z - y)} \left[ (h^* \varphi)(u) \frac{\varphi'(u)}{y - x} \right] du
\]

\[
\leq \frac{1}{2} \left\{ g(a) + g(z) - \frac{g(x) + g(y)}{2} \right\} \int_{\varphi^{-1}(a + z - y)}^{\varphi^{-1}(a + z - x)} \left( h^* \varphi(u) \frac{\varphi'(u)}{y - x} \right) du.
\]

(84)

By choosing

\[
(\tilde{h}^* \varphi)(u) = \frac{1}{\delta^\beta(\beta)} \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \varphi(u)\beta^\beta - 1 (w^* \varphi)(u)) \right]
\]

we can reach the third desired inequality in (70). Hence, the proof is thus completed. \( \square \)

**Corollary 1.** Let \( \varphi : \mathbb{R} \rightarrow [a, z] \subseteq \mathbb{R} \), with \( 0 \leq a < z \), be a continuous strictly increasing function, \( g : [a, z] \rightarrow \mathbb{R} \) be a convex differentiable function on \( (a, z) \), and \( w : [a, z] \rightarrow \mathbb{R} \) be a nonnegative integrable function satisfying that \( (g^\prime \varphi), (w^* \varphi) : \mathbb{R} \rightarrow \mathbb{R} \) are integrable mappings on \( \mathbb{R} \). Then, we have

\[
(\tilde{h}^* \varphi)(u) = \frac{1}{\delta^\beta(\beta)} \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \varphi(u)\beta^\beta - 1 (w^* \varphi)(u)) \right]
\]

(87)

**Proof.** This corollary can be easily demonstrated by following the proof of Theorem 8, taking

\[
(\tilde{h}^* \varphi)(u) = \frac{1}{\delta^\beta(\beta)} \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - \varphi(u)\beta^\beta - 1 (w^* \varphi)(u)) \right]
\]

(87)
Remark 2. By adding inequalities (70) and (86), we can derive the following inequality:

\[
g\left(a + z - \frac{x + y}{2}\right) \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(\nu^\delta_{\{x\}}(a + z - y)\right)
+ \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - y)\right) \right\}
\leq \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - x)\right) \right\}
\leq 1 \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right)
+ \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - x)\right) \right\}
\leq \frac{1}{2} \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right)
\leq \frac{1}{2} \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right),
\]

Remark 3

Putting \( \delta = 1 \), we obtain the proportional fractional integral version of each of Theorem 8, Corollary 1, and inequality (88) with respect to the positive increasing function \( \phi \).

By taking \( \phi(x) = x \), for all \( x \in [a, z] \), we obtain the proportional fractional integral version for each of Theorem 8, Corollary 1, and inequality (88).

If we put \( \delta = 1 \) and \( \phi(x) = x \), for all \( x \in [a, z] \), we obtain inequality (9), for Riemann–Liouville fractional integral introduced by Iscan [35].

Putting \( \delta = 1 \) and \( \phi(x) = x \), for all \( x \in [a, z] \), and if we choose \( x = a \) and \( y = z \), we get the following inequality:

\[
g\left(a + z - \frac{x + y}{2}\right) \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(\nu^\delta_{\{x\}}(a + z - y)\right)
+ \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - y)\right) \right\}
\leq \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - x)\right) \right\}
\leq 1 \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right)
+ \frac{1}{2} \left\{ \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(g^\delta_{\{x\}}(w^\delta_{\{x\}})(a + z - x)\right) \right\}
\leq \frac{1}{2} \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right)
\leq \frac{1}{2} \left\{ g(a) + g(z) \right\} \frac{\nu^\beta_{\{x\}}}{\nu^\beta_{\{x\}}} \cdot \left(w^\delta_{\{x\}}(a + z - y)\right),
\]

which was introduced by Iscan [35].

The next result is as follows.

Theorem 9. Let \( \phi: \mathbb{R} \rightarrow [a, z] \subseteq \mathbb{R} \), with \( 0 < a < z \), be a continuous strictly increasing function, \( g: [a, z] \rightarrow \mathbb{R} \) be a convex differentiable function on \( (a, z) \), and \( w: [a, z] \rightarrow \mathbb{R} \) be a nonnegative integrable function satisfying that \( (g')^\delta_{\{x\}}(w^\delta_{\{x\}}): \mathbb{R} \rightarrow \mathbb{R} \) are integrable mappings on \( \mathbb{R} \). Then, we have
Proof. The first inequality in (90) is already proved in Theorem 8. To prove the second inequality, we have by Lemma (1) according to the convexity of the function $g$ on $[a, z]$ what follows

\[ g(a + z - (\eta x + \eta y + (1 - \eta) y)) \leq g(a) + g(z) - g(\eta x + (1 - \eta) y), \]  

(91)

\[ g(a + z - (\eta y + (1 - \eta) x)) \leq g(a) + g(z) - g(\eta y + (1 - \eta) x). \]  

(92)

Adding inequalities (91) and (92), we get

\[
\frac{1}{2} \left\{ \int_{\varphi^{-1}(a+z)}^{\varphi^{-1}(a+y)} (g' \varphi) (u) (h' \varphi) (u) \frac{\varphi'(u)}{y-x} \, du \right. \\
+ \left. \int_{\varphi^{-1}(a+y)}^{\varphi^{-1}(a+x)} (g' \varphi) (u) h(2a+z)-(x+y+\varphi(u)) \frac{\varphi'(u)}{y-x} \, du \right\} \\
\leq (g(a) + g(z)) \int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} h(a+z-\varphi(u)) \frac{\varphi'(u)}{y-x} \, du \\
- \frac{1}{2} \left\{ \int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} (g' \varphi) (u) h(2a+z-\varphi(u)) \frac{\varphi'(u)}{y-x} \, du \right. \\
+ \left. \int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} (g' \varphi) (u) h(a+z-x-y+\varphi(u)) \frac{\varphi'(u)}{y-x} \, du \right\}. \]  

(94)

Choosing

\[ (h' \varphi) (u) = \frac{1}{\delta^\beta} \exp \left[ \frac{\delta - 1}{\delta} (\varphi(u) - [a + z - y]) \right] \cdot (\varphi(u) - [a + z - y])^{\beta-1} (w' \varphi) (u), \]  

(95)

then (94) leads to

\[
\frac{1}{2(y-x)} \left\{ \phi_{\varphi}^{\beta, \delta} \left[ \varphi^{-1}(a+z-x) \right] \cdot (g' \varphi) (w' \varphi) \left( \varphi^{-1}(a+z-y) \right) \right. \\
+ \phi_{\varphi}^{\beta, \delta} \left[ \varphi^{-1}(a+z-y) \right] \cdot (g' \varphi) (w' \varphi) \left( \varphi^{-1}(a+z-x) \right) \right\} \\
\leq \frac{(g(a) + g(z))}{y-x} \phi_{\varphi}^{\beta, \delta} \left[ \varphi^{-1}(x) \right] \cdot (w' \varphi) \left( \varphi^{-1}(y) \right) \\
- \frac{1}{2} \frac{1}{2(y-x)} \left\{ \phi_{\varphi}^{\beta, \delta} \left[ \varphi^{-1}(x) \right] \cdot (g' \varphi) (w' \varphi) \left( \varphi^{-1}(x) \right) \right. \\
+ \left. \phi_{\varphi}^{\beta, \delta} \left[ \varphi^{-1}(y) \right] \cdot (g' \varphi) (w' \varphi) \left( \varphi^{-1}(y) \right) \right\}. \]  

(96)
which proves the second inequality in (90). To prove the last inequality, by using the convexity of \( g \), we have, for all \( x, y \in [a, z] \),

\[
g \left( \frac{x + y}{2} \right) = g \left( \frac{(\eta x + (1 - \eta)y) + (\eta y + (1 - \eta)x)}{2} \right)
\]

\[
\leq \frac{1}{2} g(\eta x + (1 - \eta)y) + g(\eta y + (1 - \eta)x),
\]

which can be rewritten as

\[
g(a) + g(z) - \frac{1}{2} \left[ g(\eta x + (1 - \eta)y) + g(\eta y + (1 - \eta)x) \right]
\]

\[
\leq g(a) + g(z) - \frac{1}{2} g \left( \frac{x + y}{2} \right),
\]

(97)

On both sides of (98), taking product by \( h(a + z - (\eta x + (1 - \eta)y)) \) and integrating the estimating inequality with respect to \( \eta \) over \([0, 1]\),

\[
[g(a) + g(z)] \int_0^1 h(a + z - (\eta x + (1 - \eta)y)) d\eta
\]

\[
- \frac{1}{2} \left\{ \int_0^1 g(\eta x + (1 - \eta)y) h(a + z - (\eta x + (1 - \eta)y)) d\eta \right\}
\]

\[
+ \int_0^1 g(\eta y + (1 - \eta)x) h(a + z - (\eta x + (1 - \eta)y)) d\eta
\]

\[
\leq \left\{ g(a) + g(z) - \frac{1}{2} g \left( \frac{x + y}{2} \right) \right\} \int_0^1 h(a + z - (\eta x + (1 - \eta)y)) d\eta.
\]

(99)

Applying the change of variables \( \varphi(u) = a + z - (\eta x + (1 - \eta)y) \), to the right-hand side and applying the same process to the left-hand side as above, we get

\[
\frac{(g(a) + g(z))^{\varphi^{-1}}}{y - x} \cdot \left( w^* \varphi \right)(\varphi^{-1}(y))
\]

\[
- \frac{1}{2} \left\{ \varphi \cdot \varphi^{\beta, \delta} \left[ \varphi^{-1}(y) \right] \cdot \left( g^* \varphi \right)(\varphi^{-1}(y)) \right\}
\]

\[
+ \varphi \cdot \varphi^{\beta, \delta} \left[ \varphi^{-1}(y) \right] \cdot \left( g^* \varphi \right)(\varphi^{-1}(x))
\]

\[
\leq \left\{ g(a) + g(z) - \frac{1}{2} g \left( \frac{x + y}{2} \right) \right\} \cdot \left( w^* \varphi \right)(\varphi^{-1}(y)),
\]

(100)

which proves the last inequality in (90). Hence, the proof is thus completed.

Corollary 2. Let \( \varphi: [a, z] \subseteq \mathbb{R} \), with \( 0 \leq a < z \), be a continuous strictly increasing function, \( g: [a, z] \rightarrow \mathbb{R} \) be a
convex differentiable function on \((a, z)\), and \(w: [a, z] \rightarrow \mathbb{R}\) be a nonnegative integrable function satisfying that \((g^\ast \varphi), (w^\ast \varphi): \mathbb{R} \rightarrow \mathbb{R}\) are integrable mappings on \(\mathbb{R}\). Then, we have

\[
g\left(a + z - \frac{x + y}{2}\right)^{\varphi^\beta_{\delta}}_{\left[\varphi^{-1}(a + z - y)\right]} (w^\ast \varphi)(\varphi^{-1}(a + z - x))
\]

\[
\leq \frac{1}{2} \left\{ \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (g^\ast \varphi)(\varphi^{-1}(a + z - y)) + \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - x) \right] (g^\ast \varphi)(\varphi^{-1}(a + z - x)) \right\}
\]

\[
\leq (g(a) + g(z)) \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (w^\ast \varphi)(\varphi^{-1}(x)) - \frac{1}{2} \left\{ \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (g^\ast \varphi)(\varphi^{-1}(y)) + \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - x) \right] (g^\ast \varphi)(\varphi^{-1}(x)) \right\}
\]

Remark 5.

Putting \(\delta = 1\), we obtain the proportional fractional integral version of each of Theorem 9, Corollary 2, and inequality (103) with respect to the positive increasing function \(\varphi\)

Proof. This corollary can be easily demonstrated by following the proof of Theorem 9, taking

\[
(h^\ast \varphi)(u) = \frac{1}{\delta^\beta_{\gamma}(\beta)} \exp \left[ \frac{\delta - 1}{\delta} (\{a + z - x\} - \varphi(u)) \right] (a + z - x) - \varphi(u))^\beta_{\gamma}(\varphi^\ast)(u).
\]

Remark 4. By adding inequalities (90) and (101), we can derive the following inequality:

\[
g\left(a + z - \frac{x + y}{2}\right)^{\varphi^\beta_{\delta}}_{\left[\varphi^{-1}(a + z - y)\right]} (w^\ast \varphi)(\varphi^{-1}(a + z - x))
\]

\[
+ \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - x) \right] (w^\ast \varphi)(\varphi^{-1}(a + z - x)) \leq \left\{ \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - x) \right] (g^\ast \varphi)(\varphi^{-1}(y)) + \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (g^\ast \varphi)(\varphi^{-1}(x)) \right\}
\]

\[
\leq (g(a) + g(z)) \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (w^\ast \varphi)(\varphi^{-1}(x)) - \frac{1}{2} \left\{ \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - y) \right] (g^\ast \varphi)(\varphi^{-1}(y)) + \varphi^\beta_{\delta} \left[ \varphi^{-1}(a + z - x) \right] (g^\ast \varphi)(\varphi^{-1}(x)) \right\}
\]

Remark 5.

By taking \(\varphi(x) = x\), for all \(x \in [a, z]\), we obtain the proportional fractional integral version for each of Theorem 9, Corollary 2, and inequality (103).

If we put \(\delta = 1\) and \(\varphi(x) = x\), for all \(x \in [a, z]\), we obtain inequality (10), for Riemann–Liouville fractional integral introduced by Iscan [20].
5. Conclusion

In view of the significant importance recently achieved by fractional calculus and its very important applications in the interpretation and modeling of natural phenomena, it has become necessary to develop and refine our capabilities to generalize some of the recent results related to this topic. We achieved our goals of introducing a new fractional Hermite–Hadamard–Mercer’s inequality and its fractional integral type inequalities by employing the proportional fractional operators of integrable functions with respect to another continuous and strictly increasing function. We enhanced our work by establishing some new fractional weighted ϕ-proportional fractional integral Hermite–Hadamard–Mercer type inequalities. Also, in this article, we were keen to present some special cases related to our current study compared to the previous work of the inequality under study. In future work, we recommend researchers study the current inequality via recent fractional operators such as the Atangana + Baleanu operator or Caputo + Fabrizio operator.

Data Availability

The data analysis in this article is all theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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