A Lagrangian representation of tangles

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Abstract

We construct a functor from the category of oriented tangles in $\mathbb{R}^3$ to the category of Hermitian modules and Lagrangian relations over $\mathbb{Z}[t, t^{-1}]$. This functor extends the Burau representations of the braid groups and its generalization to string links due to Le Dimet.

Key words: Braids, tangles, string links, Burau representation, Lagrangian relations.

1 Introduction

The aim of this paper is to generalize the classical Burau representation of braid groups to tangles. The Burau representation is a homomorphism from the group of braids on $n$ strands to the group of $(n \times n)$-matrices over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, where $n$ is a positive integer. This representation has been extensively studied by various authors since the foundational work of Burau [2]. In the last 15 years, new important representations of braid groups came to light, specifically those associated with the Jones knot polynomial, $R$-matrices, and ribbon categories. These latter representations do extend to tangles, so it is natural to ask whether the Burau representation has a similar property.

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An extension of the Burau representation to a certain class of tangles was first pointed out by Le Dimet [6]. He considered so-called ‘string links’, which are tangles whose all components are intervals going from the bottom to the top but not necessarily monotonically. The string links on \( n \) strands form a monoid with respect to the usual composition of tangles. Le Dimet’s work yields a homomorphism of this monoid into the group of \((n \times n)\)-matrices over the quotient field of \( \Lambda \). For braids, this gives the Burau representation. The construction of Le Dimet also applies to colored string links, giving a generalization of the Gassner representation of the pure braid group. These representations of Le Dimet were studied by Kirk, Livingston and Wang [5] (see also [7,9]).

To extend the Burau representation to arbitrary oriented tangles, we first observe that oriented tangles do not form a group or a monoid but rather a category \( \text{Tangles} \) whose objects are finite sequences of \( \pm 1 \). An extension of the Burau representation to \( \text{Tangles} \) should be a functor from \( \text{Tangles} \) to some algebraically defined category. We show that the relevant algebraic category is the one of Hermitian \( \Lambda \)-modules and Lagrangian relations. Our principal result is a construction of a functor from \( \text{Tangles} \) to this category. For braids and string links, our constructions are equivalent to those of Burau and Le Dimet.

The appearance of Lagrangian relations rather than homomorphisms is parallel to the following well-known observations concerning cobordisms. Generally speaking, a cobordism \((W, M_-, M_+)\) does not induce a homomorphism from the homology (with any coefficients) of the bottom base \( M_- \) to the homology of the top base \( M_+ \). However, the kernel of the inclusion homomorphism \( H_*(M_-) \oplus H_*(M_+) \to H_*(W) \) can be viewed as a morphism from \( H_*(M_-) \) to \( H_*(M_+) \) determined by \( W \). This kernel is Lagrangian with respect to the usual intersection form in homology. These observations suggest a definition of a Lagrangian category over any integral domain with involution. Applying these ideas to the infinite cyclic covering of the tangle exterior, we obtain our functor from the category of tangles to the category of Lagrangian relations over \( \Lambda \).

Note that recently, a most interesting representation of braid groups due to R. Lawrence was shown to be faithful by S. Bigelow and D. Krammer. We do not know whether this representation extends to tangles.

The organization of the paper is as follows. In Section 2, we introduce the category \( \text{Lagr}_\Lambda \) of Lagrangian relations over the ring \( \Lambda \). In Section 3, we define our functor \( \text{Tangles} \to \text{Lagr}_\Lambda \). Section 4 deals with the proof of three technical lemmas stated in the previous section. In Section 5, we discuss the case of braids and string links. Section 6 concerns technical questions about the Lagrangian relations associated with tangles. In Section 7, we discuss con-
nexions between these Lagrangian relations and the Alexander polynomial of
the link obtained as the closure of the tangle. (These connexions are tradi-
tionally studied in this context.) Finally, Section 8 outlines a multivariable
generalization of the theory as well as a high-dimensional version.

2 Category of Lagrangian relations

Fix throughout this section an integral domain Λ (i.e., a commutative ring
with unit and without zero-divisors) with ring involution Λ → Λ,

2.1 Hermitian modules

A skew-hermitian form on a Λ-module H is a form ω: H × H → Λ such that
for all x, x', y ∈ H and all λ, λ' ∈ Λ,

(1) ω(λx + λ'x', y) = λω(x, y) + λ'ω(x', y),
(2) ω(x, y) = −ω(y, x).

Such a form is called non-degenerate when it satisfies:

(3) If ω(x, y) = 0 for all y ∈ H, then x = 0.

A Hermitian Λ-module is a finitely generated Λ-module H endowed with a
non-degenerate skew-hermitian form ω. The same module H with the opposite
form −ω will be denoted by −H. Note that a Hermitian Λ-module is always
torsion-free.

For a submodule A ⊂ H, denote by Ann(A) the annihilator of A with respect
to ω, that is, the module \{x ∈ H | ω(x, a) = 0 for all a ∈ A\}. Set

\[ \overline{A} = \{ x ∈ H | \lambda x ∈ A \text{ for a non-zero } \lambda ∈ \Lambda \} \]

Clearly A ⊂ A and \overline{\text{Ann}(A)} = \text{Ann}(A) = \text{Ann}(\overline{A}).

We say that a submodule A of H is isotropic if A ⊂ Ann(A). Observe that
then A ⊂ \overline{A} ⊂ Ann(A). We say that a submodule A of H is Lagrangian if
\overline{A} = Ann(A). Note that A is Lagrangian if and only if A is Lagrangian.

**Lemma 1** For any submodule A of a Hermitian Λ-module H,

\[ \text{Ann}(\text{Ann}(A)) = \overline{A} \]
**Proof.** Let $Q = Q(\Lambda)$ denote the field of fractions of $\Lambda$. Given a $\Lambda$-module $F$, denote by $F_Q$ the vector space $F \otimes \Lambda Q$. Note that the kernel of the natural homomorphism $F \to F_Q$ is the $\Lambda$-torsion $\text{Tors}_\Lambda F \subset F$.

The form $\omega$ uniquely extends to a skew-hermitian form $H_Q \times H_Q \to \mathbb{Q}$. Given a $\Lambda$-module $F$, denote by $F_Q$ the vector space $F \otimes \Lambda Q$. Note that the kernel of the natural homomorphism $F \to F_Q$ is the $\Lambda$-torsion $\text{Tors}_\Lambda F \subset F$.

The form $\omega$ uniquely extends to a skew-hermitian form $H_Q \times H_Q \to \mathbb{Q}$. Given a linear subspace $V$ of $H_Q$, let $\text{Ann}_Q(V)$ be the annihilator of $V$ with respect to the latter form. Observe that $\text{Ann}_Q(\text{Ann}_Q(V)) = V$. Indeed, one inclusion is trivial and the other one follows from dimension count, since $\dim(\text{Ann}_Q(V)) = \dim(H_Q) - \dim(V)$.

The inclusion $A \hookrightarrow H$ induces an inclusion $A_Q \hookrightarrow H_Q$. Since $H$ is torsion-free, $H \subset H_Q$ (and $A \subset A_Q$). Clearly, $\overline{A} = A_Q \cap H$ and $\text{Ann}(A)_Q = \text{Ann}_Q(A_Q)$. Replacing in the latter formula $A$ with $\text{Ann}(A)$, we obtain

$$\text{Ann}(\text{Ann}(A))_Q = \text{Ann}_Q(\text{Ann}(A)_Q) = \text{Ann}_Q(\text{Ann}_Q(A_Q)) = A_Q.$$ 

Therefore

$$\overline{A} = A_Q \cap H = \text{Ann}(\text{Ann}(A))_Q \cap H = \overline{\text{Ann}(\text{Ann}(A))} = \text{Ann}(\text{Ann}(A)),$$

and the lemma is proved. □

**Lemma 2** For any submodules $A, B \subset H$, we have

$$\text{Ann}(A + B) = \text{Ann}(A) \cap \text{Ann}(B),$$
$$\text{Ann}(A \cap B) = \text{Ann}(A) + \text{Ann}(B).$$

**Proof.** The first equality is obvious, and implies

$$\text{Ann}(\text{Ann}(A) + \text{Ann}(B)) = \text{Ann}(\text{Ann}(A)) \cap \text{Ann}(\text{Ann}(B))$$
$$= \overline{A} \cap \overline{B} = \overline{A \cap B}.$$ 

Therefore

$$\text{Ann}(A \cap B) = \text{Ann}(\overline{A \cap B}) = \text{Ann}(\text{Ann}(A) + \text{Ann}(B))),$$

which is equal to $\text{Ann}(A) + \text{Ann}(B)$ by Lemma 1. □

**Lemma 3** For any submodules $A \subset B \subset H$, we have $\overline{B}/A = \overline{B}/\overline{A}$. 

PROOF. Consider the canonical projection \( \pi : H \rightarrow H/A \). Clearly,
\[
\pi(B) = \{ \xi \in H/A \mid \lambda \xi \in B/A \text{ for a non-zero } \lambda \in \Lambda \} = B/A.
\]
Also \( \ker(\pi|_B) = \ker(\pi) \cap B = A \cap B = A \). Hence \( B/A = B/A \). \( \square \)

2.2 Lagrangian contractions

The results above in hand, it is easy to develop the theory of Lagrangian contractions and Lagrangian relations over \( \Lambda \) by mimicking the well-known theory over \( \mathbb{R} \) (see, for instance, [10, Section IV.3]).

Let \((H, \omega)\) be a Hermitian \( \Lambda \)-module as above. Let \( A \) be an isotropic submodule of \( H \) such that \( A = \overline{A} \). Denote by \( H|A \) the quotient module \( \text{Ann}(A)/A \) with the skew-hermitian form
\[
(x \mod A, y \mod A) = \omega(x, y).
\]

For a submodule \( L \subset H \), set
\[
L|A = ((L + A) \cap \text{Ann}(A))/A \subset H|A.
\]

We say that \( L|A \) is obtained from \( L \) by \textit{contraction along} \( A \).

**Lemma 4** \( H|A \) is a Hermitian \( \Lambda \)-module. If \( L \) is a Lagrangian submodule of \( H \), then \( L|A \) is a Lagrangian submodule of \( H|A \).

**PROOF.** To check that the form on \( H|A \) is non-degenerate, pick \( x \in \text{Ann}(A) \) such that \( \omega(x, y) = 0 \) for all \( y \in \text{Ann}(A) \). Then, \( x \in \text{Ann}(\text{Ann}(A)) = \overline{A} = A \) so that \( x \mod A = 0 \).

To prove the second claim of the lemma, set \( B = (L + A) \cap \text{Ann}(A) \subset H \). We claim that \( B \) is Lagrangian. Since both \( A \) and \( L \) are isotropic, it is easy to check that \( B \subset \text{Ann}(B) \) and therefore \( \overline{B} \subset \text{Ann}(B) \). Let us verify the opposite inclusion. Lemmas 1 and 2 imply that
\[
\text{Ann}(B) = \text{Ann}((L + A) \cap \text{Ann}(A)) = \overline{\text{Ann}(L + A) + \text{Ann}(\text{Ann}(A))} \subset \overline{\text{Ann}(L)} + \overline{A} = \overline{L + A}.
\]
Since \( A \subset B \), we have \( \text{Ann}(B) \subset \text{Ann}(A) \) and therefore
\[
\text{Ann}(B) \subset \overline{L + A} \cap \text{Ann}(A) = (L + A) \cap \overline{\text{Ann}(A)} = \overline{B}.
\]
Thus \( B \) is Lagrangian. This implies that \( \text{Ann}(B/A) = \overline{B}/A \), which is equal to \( \overline{B}/A \) by Lemma 3. So \( B/A \) is Lagrangian. \( \square \)

### 2.3 Categories of Lagrangian relations

Let \( H_1, H_2 \) be Hermitian \( \Lambda \)-modules. A Lagrangian relation between \( H_1 \) and \( H_2 \) is a Lagrangian submodule of \( (-H_1) \oplus H_2 \) (the latter is a Hermitian \( \Lambda \)-module in the obvious way). For a Lagrangian relation \( N \subset (-H_1) \oplus H_2 \), we shall use the notation \( N : H_1 \Rightarrow H_2 \).

For a Hermitian \( \Lambda \)-module \( H \), the submodule of \( H \oplus H \)

\[
\text{diag}_H = \{ h \oplus h \in (-H) \oplus H \mid h \in H \}
\]

is clearly a Lagrangian relation \( H \Rightarrow H \). It is called the diagonal Lagrangian relation. Given two Lagrangian relations \( N_1 : H_1 \Rightarrow H_2 \) and \( N_2 : H_2 \Rightarrow H_3 \), their composition is the following submodule of \( (-H_1) \oplus H_3 \):

\[
N_2N_1 = \{ h_1 \oplus h_3 \mid h_1 \oplus h_2 \in N_1 \text{ and } h_2 \oplus h_3 \in N_2 \text{ for a certain } h_2 \in H_2 \}.
\]

**Lemma 5** The composition of two Lagrangian relations is a Lagrangian relation.

**Proof.** Given two Lagrangian relations \( N_1 : H_1 \Rightarrow H_2 \) and \( N_2 : H_2 \Rightarrow H_3 \), consider the Hermitian \( \Lambda \)-module \( H = (-H_1) \oplus H_2 \oplus (-H_2) \oplus H_3 \) and its isotropic submodule

\[
A = 0 \oplus \text{diag}_{H_2} \oplus 0 = \{ 0 \oplus h \oplus h \oplus 0 \mid h \in H_2 \}.
\]

Note that \( \overline{A} = A \). It follows from the non-degeneracy of \( H_2 \) that \( \text{Ann}(A) = (-H_1) \oplus \text{diag}_{H_2} \oplus H_3 \). Therefore \( H|A = (-H_1) \oplus H_3 \). Observe that \( N_2N_1 = (N_1 \oplus N_2)|A \). Lemma 4 implies that \( N_2N_1 \) is a Lagrangian submodule of \( (-H_1) \oplus H_3 \). \( \square \)

**Theorem 6** Hermitian \( \Lambda \)-modules, as objects, and Lagrangian relations, as morphisms, form a category.

**Proof.** The composition law is well-defined by Lemma 5 and the associativity follows from the definitions. Finally, the role of the identity morphisms is played by the diagonal Lagrangian relations. \( \square \)
We shall call this category the category of Lagrangian relations over \( \Lambda \). It will be denoted by \( \text{Lagr}_\Lambda \). Let us conclude this section with the definition of another Lagrangian category \( \text{Lagr}_\Lambda \) closely related to the former one. The objects of \( \text{Lagr}_\Lambda \) are Hermitian \( \Lambda \)-modules, and the morphisms are Lagrangian relations \( N \) such that \( \overline{N} = N \). Finally, the composition between two morphisms \( N_1: H_1 \Rightarrow H_2 \) and \( N_2: H_2 \Rightarrow H_3 \) is defined by \( N_2 \circ N_1 = \overline{N_2 N_1}: H_1 \Rightarrow H_3 \).

**Lemma 7** Given two submodules \( N_1 \subset H_1 \oplus H_2 \) and \( N_2 \subset H_2 \oplus H_3 \),

\[ \overline{N_2 N_1} = \overline{N_2 N_1}. \]

**Proof.** Consider an element \( h_1 \oplus h_3 \) of \( \overline{N_2 N_1} \). By definition, \( h_1 \oplus h_2 \in \overline{N_1} \) and \( h_2 \oplus h_3 \in \overline{N_2} \) for some \( h_2 \in H_2 \), so \( \lambda_1(h_1 \oplus h_2) \in N_1 \) and \( \lambda_2(h_2 \oplus h_3) \in N_2 \) for some \( \lambda_1, \lambda_2 \neq 0 \). Then \( \lambda_1 \lambda_2(h_1 \oplus h_3) \in N_2 N_1 \), so \( h_1 \oplus h_3 \in \overline{N_2 N_1} \). Hence, \( \overline{N_2 N_1} \subset \overline{N_2 N_1} \). Taking the closure on both sides, we get \( \overline{N_2 N_1} \subset \overline{N_2 N_1} \). The opposite inclusion is obvious. \( \Box \)

**Theorem 8** \( \text{Lagr}_\Lambda \) is a category, and the map \( N \mapsto \overline{N} \) defines a functor \( \text{Lagr}_\Lambda \xrightarrow{j} \text{Lagr}_\Lambda \).

**Proof.** The composition law in \( \text{Lagr}_\Lambda \) is well-defined by Lemma 5; let us check that it is associative. Consider Lagrangian relations \( N_1: H_1 \Rightarrow H_2 \), \( N_2: H_2 \Rightarrow H_3 \), and \( N_3: H_3 \Rightarrow H_4 \) such that \( \overline{N_i} = N_i \) for \( i = 1, 2, 3 \). By Lemma 7,

\[ N_3 \circ (N_2 \circ N_1) = \overline{N_3 N_2 N_1} = \overline{N_3 N_2 N_1} = \overline{N_3 (N_2 N_1)}. \]

Similarly, \( (N_3 \circ N_2) \circ N_1 = (N_3 N_2)N_1 \). The result now follows from the associativity of the composition in \( \text{Lagr}_\Lambda \). The role of the identity morphisms is played by the diagonal Lagrangian relations. Indeed, for any Lagrangian relation \( N: H_1 \Rightarrow H_2 \) such that \( \overline{N} = N \),

\[ \text{diag}_{H_2} \circ N = \overline{\text{diag}_{H_2} N} = \overline{N} = N. \]

Similarly, \( N \circ \text{diag}_{H_1} = N \). Finally, let us check that the map \( N \mapsto \overline{N} \) is functorial. Consider two Lagrangian relations \( N_1: H_1 \Rightarrow H_2 \) and \( N_2: H_2 \Rightarrow H_3 \). By Lemma 7,

\[ \overline{N_2 \circ N_1} = \overline{N_2 N_1} = \overline{N_2 N_1}. \]

This finishes the proof. \( \Box \)
2.4 Lagrangian relations from unitary isomorphisms

By the graph of a homomorphism \( f: A \to B \) of abelian groups, we mean the set

\[
\Gamma_f = \{a \oplus f(a) | a \in A\} \subset A \oplus B.
\]

Let \( H_1, H_2 \) be Hermitian \( \Lambda \)-modules. Consider the Hermitian \( Q \)-modules \( H_1 \otimes Q \) and \( H_2 \otimes Q \), where \( Q = Q(\Lambda) \) is the field of fractions of \( \Lambda \) and \( \otimes = \otimes_\Lambda \). For a unitary \( Q \)-isomorphism \( \varphi: H_1 \otimes Q \to H_2 \otimes Q \), we define its restricted graph \( \Gamma^0_\varphi \) by

\[
\Gamma^0_\varphi = \Gamma_\varphi \cap (H_1 \oplus H_2) = \{h \oplus \varphi(h) | h \in H_1, \varphi(h) \in H_2\} \subset H_1 \oplus H_2.
\]

If \( \varphi \) is induced by a unitary \( \Lambda \)-isomorphism \( f: H_1 \to H_2 \), then clearly \( \Gamma^0_\varphi = \Gamma_f \).

Lemma 9 Given any unitary isomorphism \( \varphi: H_1 \otimes Q \to H_2 \otimes Q \), the restricted graph \( \Gamma^0_\varphi \) is a Lagrangian relation \( H_1 \Rightarrow H_2 \).

**Proof.** Denote by \( \omega_1 \) (resp. \( \omega_2, \omega \)) the skew-hermitian form on \( H_1 \) (resp. \( H_2, -(H_1) \oplus H_2 \)), and pick \( h, h' \in H_1 \) such that \( \varphi(h), \varphi(h') \in H_2 \). Then,

\[
\omega(h \oplus \varphi(h), h' \oplus \varphi(h')) = -\omega_1(h, h') + \omega_2(\varphi(h), \varphi(h')) = 0.
\]

Therefore, \( \Gamma^0_\varphi \) is isotropic. To check that it is Lagrangian, consider an element \( x = x_1 \oplus x_2 \in \text{Ann}(\Gamma^0_\varphi) \subset (-(H_1) \oplus H_2) \). For all \( h \in H_1 \) such that \( \varphi(h) \in H_2 \),

\[
0 = \omega(x, h \oplus \varphi(h)) = -\omega_1(x_1, h) + \omega_2(x_2, \varphi(h))
\]

\[
= -\omega_2(\varphi(x_1), \varphi(h)) + \omega_2(x_2, \varphi(h)) = \omega_2(x_2 - \varphi(x_1), \varphi(h)).
\]

Since \( \varphi \) is an isomorphism, we have \( H_2 \subset \{\varphi(h) | h \in H_1, \varphi(h) \in H_2\} \). Therefore, \( \omega_2(x_2 - \varphi(x_1), h_2) = 0 \) for all \( h_2 \in H_2 \). Since \( \omega_2 \) is non-degenerate, it follows that \( x_2 = \varphi(x_1) \) so \( x = x_1 \oplus \varphi(x_1) \in \Gamma^0_\varphi \) and the lemma is proved. \( \square \)

Therefore, Lagrangian relations can be understood as a generalization of unitary isomorphisms. More precisely, let \( \textbf{U}_\Lambda \) be the category of Hermitian \( \Lambda \)-modules and unitary \( \Lambda \)-isomorphisms. Also, let \( \textbf{U}^0_\Lambda \) be the category of Hermitian \( \Lambda \)-modules, where the morphisms between \( H_1 \) and \( H_2 \) are the unitary \( Q \)-isomorphisms between \( H_1 \otimes Q \) and \( H_2 \otimes Q \).

**Theorem 10** The maps \( f \mapsto f \otimes \text{id}_Q, f \mapsto \Gamma_f \) and \( \varphi \mapsto \Gamma^0_\varphi \) define embeddings...

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of categories $U_\Lambda \subset U_\Lambda^0$, $U_\Lambda \subset \text{Lagr}_\Lambda$ and $U_\Lambda^0 \subset \overline{\text{Lagr}}_\Lambda$ which fit in the commutative diagram

\[
\begin{array}{c}
U_\Lambda \xrightarrow{\varphi} U_\Lambda^0 \\
\downarrow \quad \downarrow \\
\text{Lagr}_\Lambda \xrightarrow{\varphi} \overline{\text{Lagr}}_\Lambda.
\end{array}
\]

**PROOF.** The first embedding is clear. For the second one, note that the graph $\Gamma_f$ of a unitary $\Lambda$-isomorphism $f$ is equal to the restricted graph of the induced unitary $Q$-isomorphism $f \otimes id_Q$. By Lemma 9, $\Gamma_f$ is a Lagrangian relation. If $\Gamma_1$ and $\Gamma_2$ are the graphs of unitary $\Lambda$-isomorphisms $f_1$ and $f_2$, then $\Gamma_2 \Gamma_1$ is clearly the graph of $f_2 \circ f_1$. Finally, two $\Lambda$-isomorphisms with the same graph are equal.

By Lemma 9, $\Gamma_\varphi^0$ is a Lagrangian relation, and it follows from the definition that $\overline{\Gamma_\varphi^0} = \Gamma_\varphi^0$. Also, note that $\Gamma_\varphi^0 \otimes Q = \Gamma_\varphi$. Therefore, given two unitary $Q$-isomorphisms $\varphi_1$ and $\varphi_2$,

\[
\Gamma_{\varphi_2 \varphi_1}^0 = \Gamma_{\varphi_2 \varphi_1} \cap (H_1 \oplus H_3) = \Gamma_{\varphi_2} \varphi_1 \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3) = (\Gamma_{\varphi_2}^0 \varphi_1 Q \cap (H_1 \oplus H_3)
\]

It is clear that a $Q$-isomorphism $\varphi$ is entirely determined by its restricted graph $\Gamma_\varphi^0$. $\blacksquare$

3 The Lagrangian representation

3.1 The category of oriented tangles

Let $D^2$ be the closed unit disk in $\mathbb{R}^2$. Given a positive integer $n$, denote by $x_i$ the point $((2i - n - 1) / n, 0)$ in $D^2$, for $i = 1, \ldots, n$. Let $\varepsilon$ and $\varepsilon'$ be sequences of $\pm 1$ of respective length $n$ and $n'$. An $(\varepsilon, \varepsilon')$-tangle is the pair consisting of the cylinder $D^2 \times [0, 1]$ and its oriented piecewise linear 1-submanifold $\tau$ whose oriented boundary $\partial \tau$ is $\sum_{j=1}^{n'} \varepsilon_j'(x'_j, 1) - \sum_{i=1}^{n} \varepsilon_i(x_i, 0)$. Note that for such a tangle to exist, we must have $\sum_i \varepsilon_i = \sum_j \varepsilon'_j$.

Two $(\varepsilon, \varepsilon')$-tangles $(D^2 \times [0, 1], \tau_1)$ and $(D^2 \times [0, 1], \tau_2)$ are isotopic if there exists an auto-homeomorphism $h$ of $D^2 \times [0, 1]$, keeping $D^2 \times \{0, 1\}$ fixed, such that $h(\tau_1) = \tau_2$ and $h|_{\tau_2}: \tau_1 \sim \tau_2$ is orientation-preserving. We shall denote by
Given an \((\varepsilon, \varepsilon')\)-tangle \(\tau_1\) and an \((\varepsilon', \varepsilon'')\)-tangle \(\tau_2\), their composition is the \((\varepsilon, \varepsilon'')\)-tangle \(\tau_2 \circ \tau_1\) obtained by gluing the two cylinders along the disk corresponding to \(\varepsilon'\) and shrinking the length of the resulting cylinder by a factor 2 (see Figure 1). Clearly, the composition of tangles induces a composition

\[
T(\varepsilon, \varepsilon') \times T(\varepsilon', \varepsilon'') \rightarrow T(\varepsilon, \varepsilon'')
\]
on the isotopy classes of tangles.

The category of oriented tangles \(\text{Tangles}\) is defined as follows: the objects are the finite sequences \(\varepsilon\) of \(\pm 1\), and the morphisms are given by \(\text{Hom}(\varepsilon, \varepsilon') = T(\varepsilon, \varepsilon')\). The composition is clearly associative, and the trivial tangle \(\text{id}_\varepsilon\) plays the role of the identity endomorphism of \(\varepsilon\). The aim of this section is to construct a functor \(\text{Tangles} \rightarrow \text{Lagr}_\Lambda\).

### 3.2 Objects

Denote by \(N(\{x_1, \ldots, x_n\})\) an open tubular neighborhood of \(\{x_1, \ldots, x_n\}\) in \(D^2 \subset \mathbb{R}^2\), and by \(S^2\) the 2-sphere \(\mathbb{R}^2 \cup \{\infty\}\). Given a sequence \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)\) of \(\pm 1\), let \(\ell_\varepsilon\) be the sum \(\sum_{i=1}^n \varepsilon_i\). We shall denote by \(D_\varepsilon\) the compact surface

\[
D_\varepsilon = \begin{cases} 
D^2 \setminus N(\{x_1, \ldots, x_n\}) & \text{if } \ell_\varepsilon \neq 0; \\
S^2 \setminus N(\{x_1, \ldots, x_n\}) & \text{if } \ell_\varepsilon = 0,
\end{cases}
\]

endowed with the counterclockwise orientation, a base point \(z\), and the generating family \(\{e_1, \ldots, e_n\}\) of \(\pi_1(D_\varepsilon, z)\), where \(e_i\) is a simple loop turning once around \(x_i\) counterclockwise if \(\varepsilon_i = +1\), clockwise if \(\varepsilon_i = -1\) (see Figure 2). The same space with the clockwise orientation will be denoted by \(-D_\varepsilon\).
The natural epimorphism $\pi_1(D_\varepsilon) \to \mathbb{Z}$, $e_i \mapsto 1$ gives an infinite cyclic covering $\hat{D}_\varepsilon \to D_\varepsilon$. Choosing a generator $t$ of the group of the covering transformations endows the homology $H_1(\hat{D}_\varepsilon)$ with a structure of module over $\Lambda = \mathbb{Z}[t, t^{-1}]$. If $\ell_\varepsilon \neq 0$, then $D_\varepsilon$ retracts by deformation on the wedge of $n$ circles representing $e_1, \ldots, e_n$, and one easily checks that $H_1(\hat{D}_\varepsilon)$ is a free $\Lambda$-module with basis $v_1 = \hat{e}_1 - \hat{e}_2, \ldots, v_{n-1} = \hat{e}_{n-1} - \hat{e}_n$, where $\hat{e}_i$ is the path in $\hat{D}_\varepsilon$ lifting $e_i$ starting at some fixed lift $\hat{z} \in \hat{D}_\varepsilon$ of $z$. If $\ell_\varepsilon = 0$, then $H_1(\hat{D}_\varepsilon) = \bigoplus_i \Lambda v_i / \Lambda \hat{\gamma}$, where $\hat{\gamma}$ is a lift of $\gamma = e_\varepsilon^1 \cdots e_\varepsilon^n$ to $\hat{D}_\varepsilon$. Note that in any case, $H_1(\hat{D}_\varepsilon)$ is a free $\Lambda$-module.

Let $< , >: H_1(\hat{D}_\varepsilon) \times H_1(\hat{D}_\varepsilon) \to \mathbb{Z}$ be the ($\mathbb{Z}$-bilinear, skew-symmetric) intersection form induced by the orientation of $D_\varepsilon$ lifted to $\hat{D}_\varepsilon$. Consider the pairing $\omega_\varepsilon: H_1(\hat{D}_\varepsilon) \times H_1(\hat{D}_\varepsilon) \to \Lambda$ given by

$$\omega_\varepsilon(x, y) = \sum_k < t^k x, y > t^{-k}.$$ 

Note that this form is well-defined since, for any given $x, y \in H_1(\hat{D}_\varepsilon)$, the intersection $< t^k x, y >$ vanishes for all but a finite number of integers $k$. The multiplication by $t$ being an isometry with respect to the intersection form, it is easy to check that $\omega_\varepsilon$ is skew-hermitian with respect to the involution $\Lambda \to \Lambda$ induced by $t \mapsto t^{-1}$.

**Example 11** Consider $\varepsilon$ of length $2$. If $\varepsilon_1 + \varepsilon_2 = 0$, then $\hat{D}_\varepsilon$ is contractible so $H_1(\hat{D}_\varepsilon) = 0$. If $\varepsilon_1 + \varepsilon_2 \neq 0$, then $H_1(\hat{D}_\varepsilon) = \Lambda v$ with $v = \hat{e}_1 - \hat{e}_2$, and $\omega_\varepsilon(v, v) = \frac{\varepsilon_1 + \varepsilon_2}{2} (t - t^{-1})$, cf. Figure 3.

We shall give a proof of the following result in Section 4.

**Lemma 12** For any $\varepsilon$, the form $\omega_\varepsilon: H_1(\hat{D}_\varepsilon) \times H_1(\hat{D}_\varepsilon) \to \Lambda$ is non-degenerate.
3.3 Morphisms

Given an \((\varepsilon, \varepsilon')\)-tangle \(\tau \subset D^2 \times [0, 1]\), denote by \(N(\tau)\) an open tubular neighborhood of \(\tau\) and by \(X_\tau\) its exterior

\[
X_\tau = \begin{cases} 
(D^2 \times [0, 1]) \setminus N(\tau) & \text{if } \ell_\varepsilon \neq 0; \\
(S^2 \times [0, 1]) \setminus N(\tau) & \text{if } \ell_\varepsilon = 0.
\end{cases}
\]

Note that \(\ell_\varepsilon = \ell_{\varepsilon'}\). We shall orient \(X_\tau\) so that the induced orientation on \(\partial X_\tau\) extends the orientation on \((-D_\varepsilon) \sqcup D_{\varepsilon'}\). If \(\ell_\varepsilon \neq 0\), then the exact sequence of the pair \((D^2 \times [0, 1], X_\tau)\) and the excision isomorphism give

\[
H_1(X_\tau) = H_2(D^2 \times [0, 1], X_\tau) = H_2(N(\tau), N(\tau) \cap X_\tau)
\]

\[
= \bigoplus_{j=1}^{\mu} H_2(N(\tau_j), N(\tau_j) \cap X_\tau),
\]

where \(\tau_1, \ldots, \tau_\mu\) are the connected components of \(\tau\). Since \((N(\tau_j), N(\tau_j) \cap X_\tau)\) is homeomorphic to \((\tau_j \times D^2, \tau_j \times S^1)\), we have \(H_2(N(\tau_j), N(\tau_j) \cap X_\tau) = \mathbb{Z}m_j\), where \(m_j\) is a meridian of \(\tau_j\) oriented so that its linking number with \(\tau_j\) is 1. Hence, \(H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z}m_j\). If \(\ell_\varepsilon = 0\), then \(H_1(X_\tau) = \bigoplus_{j=1}^{\mu} \mathbb{Z}m_j / \sum_{i=1}^{a} \varepsilon_i e_i\).

The composition of the Hurewicz homomorphism and the homomorphism \(H_1(X_\tau) \to \mathbb{Z}, m_j \mapsto 1\) gives an epimorphism \(\pi_1(X_\tau) \to \mathbb{Z}\) which extends the previously defined homomorphisms \(\pi_1(D_\varepsilon) \to \mathbb{Z}\) and \(\pi_1(D_{\varepsilon'}) \to \mathbb{Z}\). As before, it determines an infinite cyclic covering \(\widehat{X_\tau} \to X_\tau\), so the homology of \(\widehat{X_\tau}\) is endowed with a natural structure of module over \(\Lambda = \mathbb{Z}[t, t^{-1}]\).

Let \(i_\varepsilon: H_1(D_\varepsilon) \to H_1(\widehat{X_\tau})\) and \(i_{\varepsilon'}: H_1(D_{\varepsilon'}) \to H_1(\widehat{X_\tau})\) be the homomorphisms induced by the obvious inclusion \(\widehat{D_\varepsilon} \sqcup \widehat{D_{\varepsilon'}} \subset \widehat{X_\tau}\). Denote by \(j_\varepsilon\) the homomor-
phism $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ given by $j_\tau(x,x') = i'_\tau(x') - i_\tau(x)$. Finally, set
$$N(\tau) = \ker(j_\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}).$$

Note that if $\tau$ and $\tau'$ are two isotopic $(\varepsilon, \varepsilon')$-tangles, then $N(\tau) = N(\tau')$.

**Lemma 13** $N(\tau)$ is a Lagrangian submodule of $(-H_1(\widehat{D}_\varepsilon)) \oplus H_1(\widehat{D}_{\varepsilon'})$.

**Lemma 14** If $\tau_1 \in T(\varepsilon, \varepsilon'')$ and $\tau_2 \in T(\varepsilon', \varepsilon''')$, then $N(\tau_2 \circ \tau_1) = N(\tau_2)N(\tau_1)$.

We postpone the proof of these lemmas to the next section, and summarize our results in the following theorem.

**Theorem 15** Given a sequence $\varepsilon$ of $\pm 1$, denote by $\mathcal{F}(\varepsilon)$ the Hermitian $\Lambda$-module $(H_1(\widehat{D}_\varepsilon), \omega_\varepsilon)$. For $\tau \in T(\varepsilon, \varepsilon')$, let $\mathcal{F}(\tau)$ be the Lagrangian relation $N(\tau) : H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_{\varepsilon'})$. Then, $\mathcal{F}$ is a functor Tangles $\to$ Lagr$_\Lambda$.

The usual notions of cobordism and $I$-equivalence for links generalize to tangles in the obvious way. (The surface in $D^2 \times [0, 1] \times [0, 1]$ interpolating between two tangles $\tau_1, \tau_2 \subset D^2 \times [0, 1]$ should be standard on $D^2 \times \{0, 1\} \times [0, 1]$ and homeomorphic to $\tau_1 \times [0, 1]$.) It is easy to see (cf. [5] Theorem 5.1 and the proof of Proposition 5.3) that the Lagrangian relation $N(\tau)$ is an $I$-equivalence invariant of $\tau$.

Finally, note that the usual computation of the Alexander module of a link $L$ from a diagram of $L$ extends to our setting. This gives a computation of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \stackrel{j_\tau}{\to} H_1(\widehat{X}_\tau)$ (cf. [5, Proposition 4.4]). Hence, it is possible to compute $N(\tau)$ from a diagram of $\tau$.

## 4 Proof of the lemmas

The proof of Lemmas 12 and 13 rely on the Blanchfield duality theorem. We recall this fundamental result referring for a proof and further details to [4, Appendix E].

Let $M$ be a piecewise linear compact connected oriented $m$-dimensional manifold possibly with boundary. Consider an epimorphism of $\pi_1(M)$ onto a finitely generated free abelian group $G$. It induces a $G$-covering $\hat{M} \to M$, so the homology modules of $\hat{M}$ are modules over $\Lambda = \mathbb{Z}G$. For any integer $q$, let $<, > : H_q(\hat{M}) \times H_{m-q}(\hat{M}, \partial \hat{M}) \to \mathbb{Z}$ be the $\mathbb{Z}$-bilinear intersection form induced by the orientation of $M$ lifted to $\hat{M}$. The Blanchfield pairing is the form

$$<a,b> = \frac{1}{2\pi} \int_{\hat{M}} (\partial a)(b).$$

The pairing $<, >$ is a dualizing form on $H_q(\hat{M})$, and the Blanchfield duality theorem states that $H_q(\hat{M})$ is self-dual, meaning that $N(\tau)$ is an $I$-equivalence invariant of $\tau$. This allows us to compute $N(\tau)$ from a diagram of $\tau$. The proof of Lemmas 12 and 13 relies on this fundamental result, and we refer the reader to [4, Appendix E] for further details.
Let us now prove the lemmas stated in the previous section.

**Theorem 16 (Blanchfield)** The latter form is non-degenerate.

Let us now prove the lemmas stated in the previous section.

**Proof of Lemma 12.** Consider the Blanchfield pairing

\[ S_{\varepsilon}: H_1(\widehat{D}_\varepsilon) \times H_1(\widehat{D}_\varepsilon, \partial \widehat{D}_\varepsilon) \to \Lambda. \]

It follows from the definitions that \( \omega_{\varepsilon}(x, y) = S_{\varepsilon}(x, j_{\varepsilon}(y)) \), where \( j_{\varepsilon}: H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{D}_\varepsilon, \partial \widehat{D}_\varepsilon) \) is the inclusion homomorphism. Note that \( \partial \widehat{D}_\varepsilon \) consists of a finite number of copies of \( \mathbb{R} \), so \( H_1(\partial \widehat{D}_\varepsilon) = 0 \) and \( j_{\varepsilon} \) is injective. Pick \( y \in H_1(\widehat{D}_\varepsilon) \) and assume that for all \( x \in H_1(\widehat{D}_\varepsilon) \), \( 0 = \omega_{\varepsilon}(x, y) = S_{\varepsilon}(x, j_{\varepsilon}(y)) \). By the Blanchfield duality theorem, \( j_{\varepsilon}(y) \in Tors_\Lambda(H_1(\widehat{D}_\varepsilon, \partial \widehat{D}_\varepsilon)) \), so \( 0 = \lambda j_{\varepsilon}(y) = j_{\varepsilon}(\lambda y) \) for some \( \lambda \in \Lambda \), \( \lambda \neq 0 \). Since \( j_{\varepsilon} \) is injective, \( \lambda y = 0 \). As \( H_1(\widehat{D}_\varepsilon) \) is torsion-free, \( y = 0 \), so \( \omega_{\varepsilon} \) is non-degenerate. \( \square \)

**Proof of Lemma 13.** Let \( H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \xrightarrow{i} H_1(\partial \widehat{X}_\tau) \) be the inclusion homomorphism, and denote by

\[ H_2(\widehat{X}_\tau, \partial \widehat{X}_\tau) \xrightarrow{\partial} H_1(\partial \widehat{X}_\tau) \xrightarrow{j} H_1(\widehat{X}_\tau) \]

the homomorphisms appearing in the exact sequence of the pair \( (\widehat{X}_\tau, \partial \widehat{X}_\tau) \). Also, denote by \( \omega \) the pairing \((-\omega_{\varepsilon}) \oplus \omega_{\varepsilon'} \) on \((-H_1(\widehat{D}_\varepsilon)) \oplus H_1(\widehat{D}_{\varepsilon'})\) and by

\[ S_{\partial \mathcal{X}}: H_1(\partial \widehat{X}_\tau) \times H_1(\partial \widehat{X}_\tau) \to \Lambda, \quad S_{\mathcal{X}}: H_1(\widehat{X}_\tau) \times H_2(\widehat{X}_\tau, \partial \widehat{X}_\tau) \to \Lambda \]

the Blanchfield pairings. Clearly, \( N(\tau) = ((-1)i_{\partial} \oplus i_{\partial'})(L) \), where \( L = \ker(j \circ i) \) and \( i_{\partial} \) (resp. \( i_{\partial'} \)) is the identity endomorphism of \( H_1(\widehat{D}_\varepsilon) \) (resp. \( H_1(\widehat{D}_{\varepsilon'}) \)). Then, \( \text{Ann}(N(\tau)) = ((-1)i_{\partial} \oplus i_{\partial'})\text{Ann}(L) \) and we just need to check that \( L \) is Lagrangian.
First, we check that \( K = \ker(j) = \text{Im}(\partial) \) satisfies \( \text{Ann}_{\partial X}(K) = \overline{K} \), where \( \text{Ann}_{\partial X} \) denotes the annihilator with respect to the form \( S_{\partial X} \). Observe that for any \( x \in H_1(\partial \widehat{X}_r) \) and \( Y \in H_2(\widehat{X}_r, \partial \widehat{X}_r) \), we have \( S_{\partial X}(x, \partial(Y)) = S_X(j(x), Y) \). Therefore

\[
\text{Ann}_{\partial X}(K) = \{x \in H_1(\partial \widehat{X}_r) \mid S_{\partial X}(x, K) = 0\} = \{x \in H_1(\partial \widehat{X}_r) \mid S_X(j(x), H_2(\widehat{X}_r, \partial \widehat{X}_r)) = 0\}.
\]

By the Blanchfield duality, the latter set is just \( j^{-1}(\text{Tors}_{\Lambda}(H_1(\widehat{X}_r))) = \overline{K} \).

Clearly, \( i(L) \subset K \). The exact sequence of the pair \( (\partial \widehat{X}_r, \widehat{D}_\varepsilon \sqcup \widehat{D}_c) \) gives

\[
H_1(\widehat{D}_c) \oplus H_1(\widehat{D}_c) \xrightarrow{i} H_1(\partial \widehat{X}_r) \longrightarrow T,
\]

where \( T \) is a torsion \( \Lambda \)-module. This implies that \( K \subset \overline{i(L)} \) and therefore \( \overline{i(L)} = \overline{K} \). Since the forms \( \omega \) and \( S_{\partial X} \) are compatible under \( i \),

\[
\text{Ann}(L) = i^{-1}(\text{Ann}_{\partial X}(i(L))) = i^{-1}(\text{Ann}_{\partial X}(\overline{i(L)})) = i^{-1}(\text{Ann}_{\partial X}(\overline{K})) = i^{-1}(\overline{K}) = \overline{L},
\]

so \( L \) is Lagrangian and the lemma is proved. □

**Proof of Lemma 14.** Denote by \( \tau \) the composition \( \tau_2 \circ \tau_1 \). Since \( X_r = X_{\tau_1} \cup X_{\tau_2} \) and \( X_{\tau_1} \cap X_{\tau_2} = D_{c'} \), we get the following Mayer-Vietoris exact sequence of \( \Lambda \)-modules:

\[
H_1(\widehat{D}_c') \xrightarrow{\alpha_0} H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}) \xrightarrow{\beta} H_1(\widehat{X}_r) \rightarrow H_0(\widehat{D}_c') \xrightarrow{\alpha_0} H_0(\widehat{X}_{\tau_1}) \oplus H_0(\widehat{X}_{\tau_2}).
\]

The homomorphism \( \alpha_0 \) is clearly injective, so \( \beta \) is onto and we get a short exact sequence which fits in the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H_{c'} & \xrightarrow{i} & H_\varepsilon \oplus H_{c'} \oplus H_{c''} & \overset{\pi}{\longrightarrow} & H_\varepsilon \oplus H_{c''} & \longrightarrow & 0 \\
& & \downarrow^{\alpha} & & \downarrow^{\varphi} & & \downarrow^{j_{c'}} & & \\
0 & \longrightarrow & \ker(\beta) & \longrightarrow & H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}) & \xrightarrow{\beta} & H_1(\widehat{X}_r) & \longrightarrow & 0,
\end{array}
\]

where \( H_\bullet \) denotes \( H_1(\widehat{D}_\bullet) \), \( i \) is the natural inclusion, \( \pi \) the canonical projection, and \( \varphi(x, x', x'') = (j_{\tau_1}(x, x'), j_{\tau_2}(x', x'')) \). Clearly,

\[
\pi(\ker(\varphi)) = \{x \oplus x'' \mid \varphi(x, x', x'') = 0 \text{ for some } x' \in H_{c'}\} = \ker(j_{\tau_2}) \ker(j_{\tau_1}).
\]
Therefore, we just need to check that $\pi(\ker(\varphi)) = \ker(j_\tau)$, which is an easy diagram chasing exercise using the surjectivity of $\alpha: H_{\tau'} \to \ker(\beta)$. □

5 Examples

5.1 Braids

An $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \ldots \cup \tau_n \subset D^2 \times [0, 1]$ is called an oriented braid if every component $\tau_i$ of $\tau$ is strictly increasing or strictly decreasing with respect to the projection to $[0, 1]$. Note that for such an oriented braid to exist, we must have $\sharp\{i \mid \varepsilon_i = 1\} = \sharp\{j \mid \varepsilon_j' = 1\}$ and $\sharp\{i \mid \varepsilon_i = -1\} = \sharp\{j \mid \varepsilon_j' = -1\}$. The finite sequences of $\pm 1$, as objects, and the isotopy classes of oriented braids, as morphisms, form a subcategory Braids of the category of oriented tangles.

We shall now investigate the restriction of the functor $\mathcal{F}$ to this subcategory.

Consider an oriented braid $\beta = \beta_1 \cup \ldots \cup \beta_n \subset D^2 \times [0, 1]$. Clearly, there exists an isotopy $H_\beta: D^2 \times [0, 1] \to D^2 \times [0, 1]$ with $H_\beta(x, t) = (x, t)$ for $(x, t) \in (D^2 \times \{0\}) \cup (\partial D^2 \times [0, 1])$, such that $t \mapsto H_\beta(x_i, t)$ is a homeomorphism of $[0, 1]$ onto the arc $\beta_i$ for $i = 1, \ldots, n$. Let $h_\beta: D_\varepsilon \to D_{\varepsilon'}$ be the homeomorphism given by $x \mapsto H_\beta(x, 1)$, and by the identity on $S^2 \setminus D^2$ if $\varepsilon_1 + \ldots + \varepsilon_n = 0$. It is a standard result that the isotopy class $(\text{rel } \partial D^2)$ of $h_\beta$ only depends on the isotopy class of $\beta$. Consider the lift $\hat{h}_\beta: \hat{D}_\varepsilon \to \hat{D}_{\varepsilon'}$ of $h_\beta$ fixing $\partial\hat{D}^2$ pointwise, and denote by $f_\beta$ the induced unitary isomorphism $(\hat{h}_\beta)_*: H_1(\hat{D}_\varepsilon) \to H_1(\hat{D}_{\varepsilon'})$.

The isotopy $H_\beta$ provides a deformation retraction of $X_\beta$ to $D_{\varepsilon'}$: let us identify $H_1(\hat{X}_\beta)$ and $H_1(\hat{D}_{\varepsilon'})$ via this deformation. Clearly, the homomorphism $j_\beta: H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_{\varepsilon'}) \to H_1(\hat{X}_\beta)$ is given by $j_\beta(x, y) = y - f_\beta(x)$. Therefore,

$$N(\beta) = \ker(j_\beta) = \{x \oplus f_\beta(x) \mid x \in H_1(\hat{D}_\varepsilon)\} = \Gamma_{f_\beta},$$

the graph of the unitary isomorphism $f_\beta$. We have proved:

**Proposition 17** The restriction of $\mathcal{F}$ to the subcategory of oriented braids gives a functor $\text{Braids} \to U_{\Lambda}$.

Consider an $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \ldots \cup \tau_n \subset D^2 \times [0, 1]$ such that every component $\tau_i$ of $\tau$ is strictly increasing with respect to the projection to $[0, 1]$. Here, $\varepsilon = \varepsilon' = (1, \ldots, 1)$. We will simply call $\tau$ a braid, or an $n$-strand braid. As usual, we will denote by $B_n$ the group of isotopy classes of $n$-strand braids, and by $\sigma_1, \ldots, \sigma_{n-1}$ its standard set of generators (see Figure 5). Recall that
the **Burau representation** $B_n \rightarrow GL_n(\Lambda)$ maps the generator $\sigma_i$ to the matrix

$$I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

where $I_k$ denotes the identity $(k \times k)$-matrix. This representation is reducible: it splits into the direct sum of an $(n-1)$-dimensional representation $\rho$ and the trivial one-dimensional representation (see e.g. [1]). Using the Artin presentation of $B_n$, one easily checks that the map $\sigma_i \mapsto \rho(\sigma_i)^T$, where $(\_)^T$ denotes the transposition, also defines a representation $\rho^T: B_n \rightarrow GL_{n-1}(\Lambda)$.

**Proposition 18** The restriction of the functor $\mathcal{F}$ to $B_n$ gives a linear anti-representation $B_n \rightarrow GL_{n-1}(\Lambda)$ which is the dual of $\rho^T$.

**Proof.** Consider two braids $\alpha, \beta \in B_n$. By Proposition 17, $N(\alpha)$ (resp. $N(\beta)$, $N(\alpha\beta)$) is the graph of a unitary automorphism $f_\alpha$ (resp. $f_\beta$, $f_{\alpha\beta}$) of $H_1(\hat{D}_\varepsilon)$. Note that the product $\alpha\beta \in B_n$ represents the composition $\beta \circ \alpha$ in the category of tangles. Clearly, $f_{\alpha\beta} = f_\beta \circ f_\alpha$. Therefore, $\mathcal{F}$ restricted to $B_n$ is an anti-representation. In order to check that it corresponds to the dual of $\rho^T$, we just need to verify that these anti-representations coincide on the generators $\sigma_i$ of $B_n$.

Denote by $f_i$ the unitary isomorphism corresponding to $\sigma_i$. We shall now compute the matrix of $f_i$ with respect to the basis $v_1, \ldots, v_{n-1}$ of $H_1(\hat{D}_\varepsilon)$. Consider the homeomorphism $h_i$ of $D_\varepsilon$ associated with $\sigma_i$. As shown in Figure 4, its action on the loops $e_j$ is given by

$$h_i(e_j) = \begin{cases} e_i e_{i+1} e_i^{-1} & \text{if } j = i; \\
e_i & \text{if } j = i + 1; \\
e_j & \text{else.} \end{cases}$$

Fig. 4. The action of $h_i$ on the loops $e_{i-1}, \ldots, e_{i+2}$. 

PROOF. Consider two braids $\alpha, \beta \in B_n$. By Proposition 17, $N(\alpha)$ (resp. $N(\beta)$, $N(\alpha\beta)$) is the graph of a unitary automorphism $f_\alpha$ (resp. $f_\beta$, $f_{\alpha\beta}$) of $H_1(\hat{D}_\varepsilon)$. Note that the product $\alpha\beta \in B_n$ represents the composition $\beta \circ \alpha$ in the category of tangles. Clearly, $f_{\alpha\beta} = f_\beta \circ f_\alpha$. Therefore, $\mathcal{F}$ restricted to $B_n$ is an anti-representation. In order to check that it corresponds to the dual of $\rho^T$, we just need to verify that these anti-representations coincide on the generators $\sigma_i$ of $B_n$.

Denote by $f_i$ the unitary isomorphism corresponding to $\sigma_i$. We shall now compute the matrix of $f_i$ with respect to the basis $v_1, \ldots, v_{n-1}$ of $H_1(\hat{D}_\varepsilon)$. Consider the homeomorphism $h_i$ of $D_\varepsilon$ associated with $\sigma_i$. As shown in Figure 4, its action on the loops $e_j$ is given by

$$h_i(e_j) = \begin{cases} e_i e_{i+1} e_i^{-1} & \text{if } j = i; \\
e_i & \text{if } j = i + 1; \\
e_j & \text{else.} \end{cases}$$
Therefore, the lift $\hat{h}_i$ of $h_i$ satisfies

$$\hat{h}_i(\hat{e}_j) = \begin{cases} 
\hat{e}_i - t(\hat{e}_i - \hat{e}_{i+1}) & \text{if } j = i; \\
\hat{e}_i & \text{if } j = i + 1; \\
\hat{e}_j & \text{else},
\end{cases}$$

and the matrix of $f_i = (\hat{h}_i)_*$ with respect to the basis $v_j = \hat{e}_j - \hat{e}_{j+1}$ is

$$M_{f_1} = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix},$$

$$M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leq i \leq n-2.$$  

This is exactly $\rho(\sigma_i)$ (see, for instance, [1, p.121]). □

5.2 String links

An $(\varepsilon, \varepsilon')$-tangle $\tau = \tau_1 \cup \ldots \cup \tau_n \subset D^2 \times [0,1]$ is called an oriented string link if every component $\tau_j$ of $\tau$ joins $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Oriented string links clearly form a category $\text{Strings}$ which satisfies

$$\text{Braids} \subset \text{Strings} \subset \text{Tangles},$$

where all the inclusions denote embeddings of categories. Recall the functor $\mathcal{J}: \text{Lagr}_A \to \text{Lagr}_A$ from Section 2.3.

**Proposition 19** The restriction of $\mathcal{J} \circ \mathcal{F}$ to the subcategory of oriented string links gives a functor $\text{Strings} \to U_0^A$.

**PROOF.** Since $\tau$ is an oriented string link, the inclusions $D_\varepsilon \subset X_\tau$ and $D_{\varepsilon'} \subset X_\tau$ induce isomorphisms in integral homology. Therefore, the induced homomorphisms $H_1(\hat{D}_\varepsilon; Q) \xrightarrow{i_*} H_1(\hat{X}_\tau; Q)$ and $H_1(\hat{D}_{\varepsilon'}; Q) \xrightarrow{i'_{*}} H_1(\hat{X}_\tau; Q)$ are isomorphisms (see e.g. [5, Proposition 2.3]). Since $Q = Q(\Lambda)$ is a flat $\Lambda$-module, $N(\tau) \otimes Q$ is the kernel of

$$H_1(\hat{D}_\varepsilon; Q) \oplus H_1(\hat{D}_{\varepsilon'}; Q) \xrightarrow{\iota_*-i'_{*}} H_1(\hat{X}_\tau; Q).$$

Hence,

$$\mathcal{J} \circ \mathcal{F}(\tau) = \overline{N(\tau)} = (N(\tau) \otimes Q) \cap (H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_{\varepsilon'})) = \Gamma^0_{\varphi},$$
Fig. 5. The elementary tangles.

If all the components of an oriented string link \( \tau \) are oriented from bottom to top, we will simply speak of \( \tau \) as a string link. By Proposition 19, the restriction of \( \mathcal{F} \) to the category of string links gives a functor to the category \( U_\Lambda^0 \). This functor is due to Le Dimet [6] and was studied further in [5].

5.3 Elementary tangles

Every tangle \( \tau \in T(\varepsilon, \varepsilon') \) can be expressed as a composition of the elementary tangles given in Figure 5, where the orientation of the strands is determined by the signs \( \varepsilon \) and \( \varepsilon' \). We shall now compute explicitly the functor \( \mathcal{F} \) on these tangles, assuming that \( \ell_\varepsilon \neq 0 \).

Let us start with the tangle \( u \in T(\varepsilon, \varepsilon') \). Here, \( H_1(\bar{D}_\varepsilon) = \bigoplus_{i=1}^{n-3} \Lambda v_i \) and \( H_1(\bar{D}_{\varepsilon'}) = \bigoplus_{i=1}^{n-1} \Lambda v'_i \) where \( v_i = \hat{e}_i - \hat{e}_{i+1} \) and \( v'_i = \hat{e}'_i - \hat{e}'_{i+1} \). Moreover, \( X_u \) is homeomorphic to the exterior of the trivial \((\varepsilon'', \varepsilon''')\)-tangle, where \( \varepsilon'' = (-\varepsilon'_1, \varepsilon_1, \ldots, \varepsilon_{n-2}) = (\varepsilon'_2, \ldots, \varepsilon'_n) \). Hence, \( H_1(\bar{X}_u) = \bigoplus_{i=1}^{n-2} \Lambda v''_i \) with \( v''_i = \hat{e}''_i - \hat{e}''_{i+1} \) and the homomorphism \( j_u: H_1(\bar{D}_\varepsilon) \oplus H_1(\bar{D}_{\varepsilon'}) \to H_1(\bar{X}_u) \) is given by \( j_u(v_i) = -v''_{i+1} \) for \( i = 1, \ldots, n-3 \), \( j_u(v'_1) = 0 \) and \( j_u(v'_i) = v''_{i-1} \) for \( i = 2, \ldots, n-1 \). Therefore,

\[
N(u) = \ker(j_u) = \Lambda v'_1 \oplus \bigoplus_{i=1}^{n-3} \Lambda(v_i \oplus v'_{i+2}).
\]

Similarly, we easily compute

\[
N(\eta) = \Lambda v_1 \oplus \bigoplus_{i=1}^{n-3} \Lambda(v_{i+2} \oplus v'_i).
\]
Now, consider the oriented braid $\sigma_i \in T(\varepsilon, \varepsilon')$ given in Figure 5. Then, $N(\sigma_i)$ is equal to the graph $\Gamma_{f_i}$ of a unitary isomorphism $f_i: H_1(\hat{D}_\varepsilon) \rightarrow H_1(\hat{D}_{\varepsilon'})$. As in the proof of Proposition 18, we can compute the matrix $M_{f_i}$ of $f_i$ with respect to the bases $v_1, \ldots, v_{n-1}$ of $H_1(\hat{D}_\varepsilon)$ and $v'_1, \ldots, v'_{n-1}$ of $H_1(\hat{D}_{\varepsilon'})$:

$$M_{f_1} = \begin{pmatrix} -t^{\varepsilon_2} & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3},$$

$$M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t^{\varepsilon_n} & -t^{\varepsilon_n} \end{pmatrix},$$

$$M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t^{\varepsilon_{i+1}} & -t^{\varepsilon_{i+1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \text{ for } 2 \leq i \leq n-2.$$ Finally, consider the tangle $\sigma_i^{-1}$ given in Figure 5. Since it is an oriented braid, $N(\sigma_i^{-1})$ is equal to the graph of a unitary isomorphism $g_i: H_1(\hat{D}_{\varepsilon'}) \rightarrow H_1(\hat{D}_\varepsilon)$. Furthermore, we have

$$\text{diag}_{H_1(\hat{D}_\varepsilon)} = N(id_\varepsilon) = N(\sigma_i^{-1} \circ \sigma_i) = N(\sigma_i^{-1})N(\sigma_i) = \Gamma_{g_i} \Gamma_{f_i} = \Gamma_{g_i \circ f_i}.$$ Therefore, $g_i \circ f_i$ is the identity endomorphism of $H_1(\hat{D}_\varepsilon)$, so the matrix of $g_i$ with respect to the basis given above is equal to $M_{g_i} = M_{f_i}^{-1}$.

With these elementary tangles, we can sketch an alternative proof of Lemma 13 which does not make use of the Blanchfield duality. Indeed, any tangle $\tau \in T(\varepsilon, \varepsilon')$ can be written as a composition of $\sigma_i$, $\sigma_i^{-1}$, $u$ and $\eta$. By Lemmas 5 and 14, we just need to check that $N(\sigma_i)$, $N(\sigma_i^{-1})$, $N(u)$ and $N(\eta)$ are Lagrangian. For $N(\sigma_i)$ and $N(\sigma_i^{-1})$, this follows from Proposition 17, Lemma 12 and Lemma 9. For $N(u)$ and $N(\eta)$, it can be verified by a direct computation of $\omega_\varepsilon$.

6 The module $N(\tau)$

6.1 Freeness of $N(\tau)$ and $\overline{N}(\tau)$

In this section, we deal with the following technical question: Given a tangle $\tau \in T(\varepsilon, \varepsilon')$, are the modules $N(\tau)$ and $\overline{N}(\tau)$ free? Clearly, these modules are contained in the free module $H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_{\varepsilon'})$. But since the ring $\mathbb{Z}[t, t^{-1}]$ is not a principal ideal domain, this is not sufficient to conclude that $N(\tau)$ and $\overline{N}(\tau)$ are free. Nevertheless, we have the following result.

Let us say that a tangle $\tau \in T(\varepsilon, \varepsilon')$ is straight if it has no closed components, and if at least one strand of $\tau$ joins $D_\varepsilon$ with $D_{\varepsilon'}$. 

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Proposition 20  Given any tangle $\tau$, the $\mathbb{Z}[t, t^{-1}]$-module $N(\tau)$ is free. If $\tau$ is a straight tangle, then $N(\tau)$ is also free.

We shall need several notions of homological algebra, that we recall now. Let $\Lambda$ be a commutative ring with unit. The projective dimension $pd(A)$ of a $\Lambda$-module $A$ is the minimum integer $n$ (if it exists) such that there is a projective resolution of length $n$ of $A$, that is, an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

where all the $P_i$’s are projective modules. It is a well-known fact that if $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ is any resolution of $A$ with $pd(A) \leq n$ and all the $P_i$’s projective, then $K_n$ is projective as well (see, for instance, [11, Lemma 4.1.6]). The global dimension of a ring $\Lambda$ is the (possibly $\infty$) number

$$\text{sup}\{pd(A) \mid A \text{ is a } \Lambda\text{-module}\}.$$ 

For example, the global dimension of $\Lambda$ is zero if $\Lambda$ is a field, and at most one if $\Lambda$ is a principal ideal domain.

Note that the ring $\mathbb{Z}[t, t^{-1}]$ has global dimension 2 (see e.g. [11, Theorem 4.3.7]). We shall also need the fact that all projective $\mathbb{Z}[t, t^{-1}]$-modules are free ([8, Chapter 3.3]). From now one, set $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Lemma 21  Consider an exact sequence of $\Lambda$-modules

$$0 \to K \to P \to F,$$

where $P$ and $F$ are free $\Lambda$-modules. Then $K$ is free.

**PROOF.** Let $A$ be the image of the homomorphism $P \to F$. We claim that the projective dimension of $A$ is at most 1. Indeed, since the global dimension of $\Lambda$ is at most two, there is a projective resolution $0 \to P_2 \xrightarrow{\partial} P_1 \to P_0 \to A \to 0$ of $A$. Splicing this resolution with the exact sequence $0 \to A \hookrightarrow F \to F/A \to 0$, we get a resolution of $F/A$

$$0 \to P_1/\partial P_2 \to P_0 \to F \to F/A \to 0,$$

where $P_0$ and $F$ are projective. Since the global dimension of $\Lambda$ is 2, we have $pd(F/A) \leq 2$. Hence, $P_1/\partial P_2$ is projective as well. Therefore, the resolution of $A$

$$0 \to P_1/\partial P_2 \to P_0 \to A \to 0$$

is projective, so $pd(A) \leq 1$. Now, the exact sequence $0 \to K \to P \to A \to$
0 together with the fact that $P$ is free and $pd(A) \leq 1$, implies that $K$ is projective. Therefore, it is free. □

**Lemma 22** Let $H$, $H'$ and $H''$ be finitely generated free $\Lambda$-modules. Consider free submodules $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$ such that $(N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0) = 0$. Then $N_2 N_1$ is a free submodule of $H \oplus H''$.

**PROOF.** Denote by $f_1$ (resp. $f'_1$) the homomorphism $N_1 \subset H \oplus H' \xrightarrow{\pi} H$ (resp. $N_1 \subset H \oplus H' \xrightarrow{\pi'} H'$), where $\pi$ and $\pi'$ are the canonical projections. Similarly, denote by $f'_2$ and $f''_2$ the homomorphisms $N_2 \subset H' \oplus H'' \rightarrow H'$ and $N_2 \subset H' \oplus H'' \rightarrow H''$. Let $K$ be the kernel of $(-f'_1) \oplus f''_2: N_1 \oplus N_2 \rightarrow H'$. Our assumptions and Lemma 21 imply that $K$ is free. We have an exact sequence

$$0 \rightarrow (N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0) \rightarrow K \xrightarrow{f_1 \oplus f'_2} N_2 N_1 \rightarrow 0.$$ 

Therefore, if $(N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0) = 0$, then $N_2 N_1 = K$ is free. □

**Lemma 23** Consider tangles $\tau_1 \in T(\varepsilon, \varepsilon')$, $\tau_2 \in T(\varepsilon', \varepsilon'')$ such that $\tau_2 \circ \tau_1$ is straight. Then

$$(N(\tau_1) \oplus N(\tau_2)) \cap (0 \oplus \text{diag}_{H_1(\hat{D}_\varepsilon)} \oplus 0) = 0.$$ 

**PROOF.** Denote by $\tau$ the tangle $\tau_2 \circ \tau_1$. We claim that $H_2(X_\tau) = 0$. Let us first assume that $\ell_\varepsilon \neq 0$. By excision,

$$H_2(X_\tau) = H_3(D^2 \times [0, 1], X_\tau) = H_3(\tau \times D^2, \tau \times S^1) = 0$$

since $\tau$ has no closed components. If $\ell_\varepsilon = 0$, consider the Mayer-Vietoris exact sequence associated with the decomposition $X_\tau = ((D^2 \times [0, 1]) \setminus N(\tau)) \cup (D^2 \times [0, 1])$:

$$0 \rightarrow H_2(X_\tau) \rightarrow \mathbb{Z} \gamma \xrightarrow{i} H_1((D^2 \times [0, 1]) \setminus N(\tau)),$$

where $\gamma$ is a 1-cycle parametrizing $\partial D^2$. Since one strand of $\tau$ joins $D_\varepsilon$ with $D_{\varepsilon''}$, we have $i(\gamma) \neq 0 \in H_1((D^2 \times [0, 1]) \setminus N(\tau)) = \mathbb{Z}^\mu$, where $\mu$ is the number of components of $\tau$. Therefore, $i$ is injective, so $H_2(X_\tau) = 0$ and the claim is proved.

Since $X_\tau$ has the homotopy type of a 2-dimensional $CW$-complex and $H_2(X_\tau) =$
0, we have \( H_2(\hat{X}_\tau) = 0 \). The decomposition \( X_\tau = X_{\tau_1} \cup X_{\tau_2} \) gives the Mayer-Vietoris exact sequence

\[
H_2(\hat{X}_\tau) = 0 \rightarrow H_1(\hat{D}_\epsilon') \xrightarrow{j} H_1(\hat{X}_{\tau_1}) \oplus H_1(\hat{X}_{\tau_2}).
\]

Therefore,

\[
0 = \ker(j) = \{ x \in H_1(\hat{D}_\epsilon') \mid j_{\tau_1}(0 \oplus x) = j_{\tau_2}(x \oplus 0) = 0 \}
\]

\[
\cong (\ker(j_{\tau_1}) \oplus \ker(j_{\tau_2})) \cap (0 \oplus \text{diag}_{H_1(\hat{D}_\epsilon')} \oplus 0)
\]

and the lemma is proved. \( \square \)

**Lemma 24** Let \( \tau \) be an elementary tangle, as described in Figure 5. Then, \( \hat{N}(\tau) \) is a free \( \Lambda \)-module.

**PROOF.** We already checked this statement by a direct computation when \( \ell_\epsilon \neq 0 \). In any case, note that \( X_\tau \) has the homotopy type of a 1-dimensional connected \( CW \)-complex \( Y_\tau \) (unless \( \tau \) is one of the 1-strand tangles \( u \) and \( \eta \), in which case the lemma is obvious). Therefore, \( H_1(\hat{X}_\tau) \) is the kernel of \( \partial: C_1(\hat{Y}_\tau) \rightarrow C_0(\hat{Y}_\tau) \). Since the latter two modules are free, Lemma 21 implies that \( H_1(\hat{X}_\tau) \) is free. Now, consider the exact sequence

\[
0 \rightarrow \hat{N}(\tau) \hookrightarrow H_1(\hat{D}_\epsilon) \oplus H_1(\hat{D}_\epsilon') \xrightarrow{j} H_1(\hat{X}_\tau).
\]

Since \( H_1(\hat{D}_\epsilon) \oplus H_1(\hat{D}_\epsilon') \) and \( H_1(\hat{X}_\tau) \) are free, the conclusion follows from Lemma 21. \( \square \)

**Proof of Proposition 20.** Consider the exact sequence

\[
0 \rightarrow \overline{\hat{N}(\tau)} \hookrightarrow H_1(\hat{D}_\epsilon) \oplus H_1(\hat{D}_\epsilon') \rightarrow (H_1(\hat{D}_\epsilon) \oplus H_1(\hat{D}_\epsilon'))/\hat{N}(\tau) \rightarrow 0.
\]

Clearly, the latter module is finitely generated and torsion free. Since \( \Lambda \) is a noetherian ring, such a module embeds in a free \( \Lambda \)-module \( F \), giving an exact sequence

\[
0 \rightarrow \overline{\hat{N}(\tau)} \hookrightarrow H_1(\hat{D}_\epsilon) \oplus H_1(\hat{D}_\epsilon') \rightarrow F.
\]

By Lemma 21, \( \overline{\hat{N}(\tau)} \) is free. The second statement follows from Lemmas 14, 22, 23 and 24. \( \square \)
Recall that for a $\Lambda$-module $N$, its rank $\text{rk}_\Lambda N$ is defined by $\text{rk}_\Lambda N = \dim_Q(N \otimes_{\Lambda} Q)$.

**Proposition 25** Consider $\tau \in T(\varepsilon, \varepsilon')$ with $\varepsilon$ of length $n$ and $\varepsilon'$ of length $n'$. Then, the rank of $N(\tau)$ is given by

$$
\text{rk}_\Lambda N(\tau) = \begin{cases} 
0 & \text{if } n = n' = 0; \\
n + n' - 1 & \text{if } \ell_\varepsilon \neq 0 \text{ or } nn' = 0 \text{ and } (n, n') \neq (0, 0); \\
n + n' - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } nn' > 0.
\end{cases}
$$

**PROOF.** Since $N(\tau)$ is a Lagrangian submodule of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$, we have $\text{rk}_\Lambda N(\tau) = \frac{1}{2} \text{rk}_\Lambda (H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}))$. If $\varepsilon$ has length $n$, we know from Section 3.2 that

$$
\text{rk}_\Lambda H_1(\widehat{D}_\varepsilon) = \begin{cases} 
0 & \text{if } n = 0; \\
n - 1 & \text{if } \ell_\varepsilon \neq 0; \\
n - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } n > 0.
\end{cases}
$$

The result follows. $\Box$

6.2 Recursive computation of $N(\tau)$

Consider two finitely generated free $\Lambda$-modules $H$ and $H'$ with fixed basis. A homomorphism of $\Lambda$-modules $f: H \to H'$ is canonically described by its matrix $M_f$, and the composition of homomorphisms corresponds to the product of matrices. What about morphisms in the Lagrangian category? A free submodule $N$ of $H \oplus H'$ is determined by a matrix of the inclusion $N \subset H \oplus H'$ with respect to a basis of $N$. We will say that $N \subset H \oplus H'$ is encoded by this matrix. For example, the graph $\Gamma_f$ of an isomorphism $f: H \to H'$ is encoded by the matrix $\left(\begin{smallmatrix} I & M_f \end{smallmatrix}\right)$, where $I$ is the identity matrix.

Let $H$, $H'$, $H''$ be finitely generated free $\Lambda$-modules with fixed basis. Consider free submodules $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$. A choice of a basis for $N_1$ and $N_2$ determines matrices $\left(\begin{smallmatrix} M_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} M_2' \end{smallmatrix}\right)$ of the inclusions $N_1 \subset H \oplus H'$ and $N_2 \subset H' \oplus H''$. By Lemma 22, if $(N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0) = 0$, then $N_2N_1$ is free. A natural question is: how can we compute a matrix of the inclusion $N_2N_1 \subset H \oplus H'$ from the matrices $\left(\begin{smallmatrix} M_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} M_2' \end{smallmatrix}\right)$?

**Lemma 26** If $(N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0) = 0$, then the inclusion of $N_2N_1$ in $H \oplus H''$ is encoded by the matrix $\left(\begin{smallmatrix} M_1W_1 \\ M_2'W_2 \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right)$ is a matrix of the inclusion of $K = \{x \in N_1 \oplus N_2 \mid (-M_1', M_2') \cdot x = 0\}$ in $N_1 \oplus N_2$. 

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PROOF. We will assume the notation of the proof of Lemma 22. By definition, $M_1$, $M'_1$, $M''_1$ and $M''_2$ are the matrices of $f_1$, $f'_1$, $f'_2$ and $f''_2$ with respect to the bases of $N_1$, $N_2$, $H$, $H'$ and $H''$. Furthermore, we saw in the proof of Lemma 22 that $K = \ker((f'_1) \oplus f'_2)$ is free. Let $\begin{pmatrix} W_1' \\ W_2' \end{pmatrix}$ be a matrix of the inclusion $K \subset N_1 \oplus N_2$ with respect to a basis of $K$ and the fixed basis of $N_1 \oplus N_2$. By definition, $N_2 = (f_1 \oplus f''_2)(K)$. Clearly, $\ker(f_1 \oplus f''_2) \cap K = (N_1 \oplus N_2) \cap (0 \oplus \text{diag}_{H'} \oplus 0)$. Since the latter module is assumed to be trivial, $f_1 \oplus f''_2$ restricted to $K$ gives an isomorphism onto $N_2$. The lemma follows easily. □

Lemma 26 gives the following recursive method for the computation of $N(\tau)$, where $\tau$ is a straight tangle.

**Proposition 27** Let $\tau_1 \in T(\varepsilon, \varepsilon')$ and $\tau_2 \in T(\varepsilon', \varepsilon'')$ be tangles such that $\tau_2 \circ \tau_1$ is straight. Then, $N(\tau_1)$, $N(\tau_2)$ and $N(\tau_2 \circ \tau_1)$ are free. Furthermore, if the inclusions $N(\tau_1) \subset H_1(\hat{D}_{\varepsilon}) \oplus H_1(\hat{D}_{\varepsilon'})$ and $N(\tau_2) \subset H_1(\hat{D}_{\varepsilon'}) \oplus H_1(\hat{D}_{\varepsilon''})$ are encoded by matrices $\begin{pmatrix} M_1 \\ M'_1 \end{pmatrix}$ and $\begin{pmatrix} M'_2 \\ M''_2 \end{pmatrix}$, then $N(\tau_2 \circ \tau_1) \subset H_1(\hat{D}_{\varepsilon}) \oplus H_1(\hat{D}_{\varepsilon''})$ is encoded by the matrix $\begin{pmatrix} M_1 W_1 \\ M'_2 W_2 \end{pmatrix}$, where $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ is a matrix of the inclusion $\{ x \in N(\tau_1) \oplus N(\tau_2) \mid (M_1 M'_2) \cdot x = 0 \} \subset N(\tau_1) \oplus N(\tau_2)$.

In order to compute $N(\tau)$ for any straight tangle $\tau$, we only need to understand how $N(\tau_2 \circ \tau_1)$ is obtained from $N(\tau_1)$ for any elementary tangle $\tau_2$. Assuming the notation of Section 5.2, we easily get the following result.

**Proposition 28** Let $\tau_1 \in T(\varepsilon, \varepsilon')$ be a straight tangle with $\ell_\varepsilon \neq 0$. If the inclusion $N(\tau_1) \subset H_1(\hat{D}_{\varepsilon}) \oplus H_1(\hat{D}_{\varepsilon'})$ is encoded by $\begin{pmatrix} M_1 \\ M'_1 \end{pmatrix}$ and if $\tau_2 \circ \tau_1$ is straight, then $N(\tau_2 \circ \tau_1)$ is encoded by $\begin{pmatrix} M \\ M'' \end{pmatrix}$, with

- $M = (0 \ M_1)$ and $M'' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus M'_1$ if $\tau_2 = u$;
- $M = M_1$ and $M'' = M'_1 \ M'_1$ if $\tau_2 = \sigma_\varepsilon^\epsilon$, for $\varepsilon = \pm 1$;
- $M = M_1 W$ and $M'' = \hat{M}'_1 W$ if $\tau_2 = \eta$, where $\hat{M}'_1$ denotes $M'_1$ without its first two lines $\ell_1, \ell_2$, and $W$ encodes the solutions of the equation $\ell_2 \cdot x = 0$.

7 The Alexander polynomial

Let $\tau \subset D^2 \times [0, 1]$ be an $(\varepsilon, \varepsilon)$-tangle, with $\varepsilon$ of length $n$. The closure of $\tau$ is the oriented link $\hat{\tau} \subset S^3$ obtained from $\tau$ by adding $n$ oriented parallel strands in $S^3 \setminus (D^2 \times [0, 1])$ as indicated in Figure 6. The orientation of these strands is determined by $\varepsilon$ in order to obtain a well-defined oriented link $\hat{\tau}$.

In this section, we show how the Alexander polynomial $\Delta_\varepsilon$ of $\hat{\tau}$ is related to
the Lagrangian module $N(\tau) \subset H_1(\hat{D}_\varepsilon) \oplus H_1(\tilde{D}_\varepsilon)$.

7.1 Basics

Let $\Lambda$ be a unique factorization domain. Consider a finite presentation $\Lambda^r \xrightarrow{f} \Lambda^g \rightarrow M \rightarrow 0$ of a $\Lambda$-module $M$. We will denote by $\Delta(M)$ the greatest common divisor of the $(g \times g)$-minors of the matrix of $f$. It is well-known that, up to multiplication by units of $\Lambda$, the element $\Delta(M)$ of $\Lambda$ only depends on the isomorphism class of $M$. Furthermore, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of $\Lambda$-modules, then $\Delta(B) \doteq \Delta(A)\Delta(C)$, where $\doteq$ denotes the equality up to multiplication by units of $\Lambda$.

We briefly recall the definition of the 1-variable Alexander polynomial of an oriented link $L \subset S^3$. Denote by $X_L$ the exterior of $L$ in $S^3$, and consider the epimorphism $\pi_1(X_L) \rightarrow \mathbb{Z}$ given by the total linking number with $L$. It induces an infinite cyclic covering $\hat{X}_L \rightarrow X_L$. The $\mathbb{Z}[t, t^{-1}]$-module $H_1(\hat{X}_L)$ is called the Alexander module of $L$ and the Laurent polynomial $\Delta_L(t) = \Delta(H_1(\hat{X}_L))$ is the Alexander polynomial of $L$. It is defined up to multiplication by $\pm t^\nu$, with $\nu \in \mathbb{Z}$.

7.2 A factorization of the Alexander polynomial

We use throughout this section the notation of Section 3.

Lemma 29 For $\tau \in T(\varepsilon, \varepsilon)$ with $\ell_\varepsilon \neq 0$,

$$(t^{\ell_\varepsilon} - 1)\Delta_\tau(t) \doteq (t - 1)\Delta(A),$$

where $A$ is the cokernel of $i_\tau^\prime - i_\tau : H_1(\hat{D}_\varepsilon) \rightarrow H_1(\hat{X}_\tau)$.

Proof. Consider the compact manifold $Y_\tau$ obtained by pasting $X_\tau$ and $X_{id_\varepsilon}$ along $D_\varepsilon \sqcup D_\varepsilon$. The epimorphisms $\pi_1(X_\tau) \rightarrow \mathbb{Z}$ and $\pi_1(X_{id_\varepsilon}) \rightarrow \mathbb{Z}$ extend
to an epimorphism \( \pi_1(Y_\tau) \to \mathbb{Z} \) which defines a \( \mathbb{Z} \)-covering \( \hat{Y}_\tau \to Y_\tau \). Hence, we have the Mayer-Vietoris exact sequence

\[
H_1(D_\varepsilon) \oplus H_1(D_\varepsilon) \xrightarrow{\partial} H_1(D_\varepsilon) \oplus H_1(\hat{X}_\varepsilon) \xrightarrow{\beta} H_1(\hat{Y}_\tau) \xrightarrow{\alpha} H_0(D_\varepsilon) \oplus H_0(D_\varepsilon)
\]

where \( \alpha_1(x, y) = (x + y, i_\tau(x) + i'_\tau(y)) \). Since \( H_0(D_\varepsilon) = H_0(\hat{X}_\varepsilon) = \Lambda/(t-1) \), the module \( \text{Im}(\partial) = \ker(\alpha_0) \) is equal to \( \Lambda/(t-1) \). This and the equality \( A = \text{Im}(\beta) \) lead to the exact sequence

\[
0 \to A = \text{Im}(\beta) \to H_1(\hat{Y}_\tau) \to \Lambda/(t-1) \to 0.
\]

Hence, \( \Delta(H_1(\hat{Y}_\tau)) = (t-1)\Delta(A) \).

Clearly, \( X_\hat{\tau} \) is the union of \( Y_\tau \) and \( D^2 \times S^1 \) along a torus \( T \subset \partial Y_\tau \). The epimorphism \( \pi_1(X_\hat{\tau}) \to \mathbb{Z} \) given by the total linking number with \( \hat{\tau} \) extends the previously defined epimorphism \( \pi_1(Y_\tau) \to \mathbb{Z} \). Therefore, the exact sequence of the pair \( (\hat{X}_\tau, \hat{Y}_\tau) \) gives

\[
0 \to H_2(\hat{Y}_\tau) \to H_2(\hat{X}_\tau) \to \Lambda/(t^{l_\varepsilon} - 1) \to H_1(\hat{Y}_\tau) \to H_1(\hat{X}_\tau) \to 0.
\]

Note that both \( H_2(\hat{Y}_\tau) \) and \( H_2(\hat{X}_\tau) \) are free \( \Lambda \)-modules. (This follows from the fact that \( X_\hat{\tau} \) and \( Y_\tau \) have the homotopy type of a 2-dimensional CW-complex, and from Lemma 21.) If \( H_2(\hat{X}_\tau) = 0 \), then \( \Delta(H_1(\hat{Y}_\tau)) = (t^{l_\varepsilon} - 1)\Delta(\hat{X}_\tau) = (t^{l_\varepsilon} - 1)\Delta(\hat{\tau})(t) \) and the lemma holds. If \( H_2(\hat{X}_\tau) \neq 0 \), then \( H_2(\hat{Y}_\tau) \neq 0 \) so both modules have positive rank. By an Euler characteristic argument, the rank of \( H_1(\hat{X}_\tau) \) and \( H_1(\hat{Y}_\tau) \) is also positive. Therefore, \( \Delta(H_1(\hat{X}_\tau)) = \Delta(H_1(\hat{Y}_\tau)) = 0 \), and the lemma is proved. \( \Box \)

**Theorem 30** Let \( \tau \in T(\varepsilon, \varepsilon) \) be a tangle with \( l_\varepsilon \neq 0 \), such that \( N(\tau) \) is free. Then,

\[
\frac{t^{l_\varepsilon} - 1}{t - 1} \Delta_{\hat{\tau}}(t) \geq \det(M' - M) \Delta(\text{coker}(j_\tau))
\]

where \((M, M')\) is a matrix of the inclusion \( N(\tau) \subset H_1(D_\varepsilon) \oplus H_1(D_\varepsilon) \).

**PROOF.** Since \( N(\tau) = \ker(j_\tau) \), we have the exact sequence

\[
0 \to N(\tau) \to H_1(D_\varepsilon) \oplus H_1(D_\varepsilon) \xrightarrow{\partial} H_1(\hat{X}_\tau) \xrightarrow{\alpha} \text{coker}(j_\tau) \to 0.
\]
The module $A$ defined by the exact sequence $H_1(\hat{D}_\varepsilon) \xrightarrow{i' - i} H_1(\hat{X}_\tau) \xrightarrow{p} A \rightarrow 0$ fits in the sequence

$$N(\tau) \xrightarrow{\alpha} H_1(\hat{D}_\varepsilon) \xrightarrow{\beta} A \xrightarrow{j_\tau} \text{coker}(j_\tau) \rightarrow 0,$$

where $\alpha(x, y) = y - x$ for $x, y \in H_1(\hat{D}_\varepsilon)$, $\beta = p \circ i_\tau = p \circ i'_\tau$, and $\gamma(\zeta) = \pi(z)$ for $\zeta = p(z) \in A$, $z \in H_1(\hat{X}_\tau)$. We leave to the reader the proof that this sequence is exact. It then splits into two exact sequences

$$N(\tau) \xrightarrow{\alpha} H_1(\hat{D}_\varepsilon) \xrightarrow{\beta} \text{Im}(\beta) \rightarrow 0$$

and

$$0 \rightarrow \text{Im}(\beta) \xhookrightarrow{f_\beta} A \xrightarrow{j_\tau} \text{coker}(j_\tau) \rightarrow 0.$$

The latter sequence implies that $\Delta(A) = \Delta(\text{coker}(j_\tau))\Delta(\text{Im}(\beta))$. By Lemma 29, we get $(t^{\ell_\varepsilon} - 1)\Delta_\beta(t) = \Delta(\text{Im}(\beta))\Delta(\text{coker}(j_\tau))$. The former sequence is nothing but a finite presentation of the module $\text{Im}(\beta)$. Furthermore, if a matrix of the inclusion $N(\tau) \subset H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_\varepsilon)$ is given by $\left( \begin{array}{c} M \\ M' \end{array} \right)$, then a matrix of $\alpha$ is given by $(M' - M)$. Since $N(\tau)$ is a Lagrangian submodule of $H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_\varepsilon)$, its rank is equal to the rank of $H_1(\hat{D}_\varepsilon)$. Therefore, $M$ and $M'$ are square matrices and $\Delta(\text{Im}(\beta)) = \det(M' - M)$. \[\square\]

We have the following generalization of [1, Theorem 3.11]. (There, all the strands of the braid must be oriented in the same direction.)

**Corollary 31** If $\beta \in T(\varepsilon, \varepsilon)$ is an oriented braid with $\ell_\varepsilon \neq 0$, then

$$\frac{t^{\ell_\varepsilon} - 1}{t - 1} \Delta_\beta(t) = \det(M_{f_\beta} - I),$$

where $M_{f_\beta}$ is a matrix of $f_\beta: H_1(\hat{D}_\varepsilon) \rightarrow H_1(\hat{D}_\varepsilon)$ (cf. Section 5.1) and $I$ is the identity matrix.

**PROOF.** Since $N(\beta) = \Gamma_{f_\beta}$, its inclusion in $H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_\varepsilon)$ is given by the matrix $\left( \begin{array}{c} I \\ M_{f_\beta} \end{array} \right)$. Furthermore, $\hat{D}_\varepsilon$ is a deformation retract of $\hat{X}_\beta$, so the homomorphism $j_\beta$ is onto. The equality then follows from Theorem 30. \[\square\]

A tangle $\tau \in T(\varepsilon, \varepsilon')$ is said to be *topologically trivial* if the oriented pair $(D^2 \times [0, 1], \tau)$ is homeomorphic to the oriented pair $(D^2 \times [0, 1], \text{id}_{\varepsilon''})$ for some
$\varepsilon''$. For instance, the oriented braids are topologically trivial, as well as the elementary tangles described in Figure 5. Note that a topologically tangle with $\ell_\varepsilon \neq 0$ is always straight. Therefore, $N(\tau)$ is a free module if $\ell_\varepsilon \neq 0$.

**Corollary 32** Consider a topologically trivial tangle $\tau \in T(\varepsilon, \varepsilon)$ with $\ell_\varepsilon \neq 0$. Given a matrix $\left(\begin{array}{cc} M \\ M' \end{array}\right)$ of the inclusion $N(\tau) \subset H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_\varepsilon)$, we have

$$\delta \Delta_\varepsilon(t) \doteq \det(M' - M),$$

where $\delta$ is a divisor of $\frac{t^{\ell_\varepsilon - 1}}{t - 1}$ in $\Lambda$.

**PROOF.** Let $h$ be the homeomorphism between $(D^2 \times [0, 1], \tau)$ and $(D^2 \times [0, 1], id_{\varepsilon''})$. The induced isomorphism $h_\sharp: \pi_1(X_\tau) \to \pi_1(X_{id_{\varepsilon''}})$ is compatible with the epimorphisms $\pi_1(X_\tau) \to \mathbb{Z}$ and $\pi_1(X_{id_{\varepsilon''}}) \to \mathbb{Z}$. Therefore, $h$ lifts to a homeomorphism $\hat{h}: X_\tau \to X_{id_{\varepsilon''}}$.

Denote by $B_\varepsilon$ the compact surface $(\partial D^2 \times [0, 1]) \cup (D_\varepsilon \times \{0, 1\})$. Since $X_{id_{\varepsilon''}}$ retracts by deformation on $\hat{D}_{\varepsilon''} \subset \hat{B}_{\varepsilon''}$, the manifold $X_\tau$ retracts by deformation on $\hat{C} = \hat{h}^{-1}(\hat{D}_{\varepsilon''}) \subset \hat{B}_\varepsilon$. This leads to the following commutative diagram of inclusion homomorphisms,

$$
\begin{array}{ccc}
H_1(\hat{D}_\varepsilon) & \oplus & H_1(\hat{D}_\varepsilon) \\
\downarrow{i} & & \downarrow{j} \\
H_1(\hat{B}_\varepsilon) & \xrightarrow{k} & H_1(\hat{C})
\end{array}
$$

where $j \circ k$ is an isomorphism. Let $\pi: H_1(\hat{X}_\tau) \to \text{coker}(j \circ i)$ and $\pi': H_1(\hat{B}_\varepsilon) \to \text{coker}(i)$ be the canonical projections. Consider the homomorphism $\varphi: \text{coker}(j \circ i) \to \text{coker}(i)$ given by $\varphi(\pi(x)) = \pi' \circ k \circ (j \circ k)^{-1}(x)$ for $x \in H_1(\hat{X}_\tau)$.

We easily check that $\varphi$ is a well-defined injective homomorphism. Therefore, $\Delta(\text{coker}(j \circ i)) = \Delta(\text{coker}(j \circ i))$ divides $\Delta(\text{coker}(i))$. The exact sequence of the pair $(\hat{B}_\varepsilon, \hat{D}_\varepsilon \sqcup \hat{D}_\varepsilon)$ gives

$$H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_\varepsilon) \xrightarrow{i} H_1(\hat{B}_\varepsilon) \to \Lambda/(t^{\ell_\varepsilon} - 1) \to \Lambda/(t - 1) \to 0.$$

Therefore, $\Delta(\text{coker}(i)) \doteq (t^{\ell_\varepsilon} - 1)/(t - 1)$. The result now follows from Theorem 30. $\square$
7.3 Examples

Given a topologically trivial tangle \( \tau \in T(\varepsilon, \varepsilon) \) with \( \ell_\varepsilon \neq 0 \), Propositions 27, 28 and Corollary 32 provide a method for the computation of the Alexander polynomial \( \Delta_\tau \). We now give several examples of such computations.

**Rational links.** For integers \( a_1, \ldots, a_n \), denote by \( \sigma(a_1, \ldots, a_n) \) the following unoriented 3-strand braid:

\[
\sigma(a_1, \ldots, a_n) = \begin{cases} 
\sigma_2^{a_1} \sigma_1^{-a_2} \sigma_2^{a_3} \cdots \sigma_n^{a_n} & \text{if } n \text{ is odd;} \\
\sigma_2^{a_1} \sigma_1^{-a_2} \sigma_2^{a_3} \cdots \sigma_1^{-a_n} & \text{if } n \text{ is even.}
\end{cases}
\]

Consider the unoriented 3-strand tangle \( \tau(a_1, \ldots, a_n) = \tau_n \circ \sigma(a_1, \ldots, a_n) \), where

\[
\tau_n = \begin{cases} 
\mu \circ \eta & \text{if } n \text{ is odd;} \\
\mu \circ \eta \circ \sigma_2 \circ \sigma_1 & \text{if } n \text{ is even.}
\end{cases}
\]

(Recall Figure 5 for the definition of the tangles \( \mu, \eta \) and \( \sigma_i \).) Finally, denote by \( C(a_1, \ldots, a_n) \) the unoriented link given by the closure of \( \tau(a_1, \ldots, a_n) \). Such a link is called a **rational link** or a 2-bridge link (see [3] and Figure 7 for examples).

Consider the oriented link \( L \) obtained by endowing \( C(a_1, \ldots, a_n) \) with an orientation. (Note that there is no canonical way to do so: \( L \) is not uniquely determined by the integers \( a_1, \ldots, a_n \).) This turns \( \sigma(a_1, \ldots, a_n) \) into an oriented braid \( \beta \), and one easily compute the associated matrix \( M_\beta = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \), where \( m_{ij} \in \Lambda \) for \( i, j = 1, 2 \).

**Proposition 33** The Alexander polynomial of \( L \) is given by

\[
\Delta_L(t) = \begin{cases} 
m_{21} & \text{if } n \text{ is odd;} \\
m_{11} & \text{if } n \text{ is even.}
\end{cases}
\]
PROOF. Let us first assume that $n$ is odd. Consider the decomposition $\tau = \tau_n \circ \beta$. In the canonical bases $v_1, v_2$ of $H_1(\tilde{D}_\epsilon)$ and $v'_1, v'_2$ of $H_1(\tilde{D}_{\epsilon'})$, the inclusion $N(\tau_n) \subset H_1(\tilde{D}_{\epsilon'}) \oplus H_1(\tilde{D}_\epsilon)$ is encoded by the matrix $(M')_M$, with $M' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Furthermore, the inclusion $N(\beta) \subset H_1(\tilde{D}_{\epsilon'}) \oplus H_1(\tilde{D}_\epsilon)$ is encoded by the matrix $(M_{f_{\beta}})_M$. Since $M_{f_{\beta}}$ is invertible, the solutions of the system $(-M_{f_{\beta}}M') \cdot x = 0$ are given by $(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}) = \begin{pmatrix} M_{f_{\beta}}^{-1}M' \\ I \end{pmatrix}$. By Proposition 27, $N(\tau)$ is encoded by $(M_{f_{\beta}}^{-1}M')_M$. By Corollary 32,

$$\Delta_L(t) \doteq \det(M - M_{f_{\beta}}^{-1}M') \doteq \det(M_{f_{\beta}}M - M')$$

$$= \det \left( \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \doteq m_{21}. $$

If $n$ is even, we have $M' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This leads to $\Delta_L(t) \doteq m_{11}$. □

For example, consider an oriented knot $K$ obtained by orienting the knot $C(3, 2)$ described in Figure 7. The corresponding oriented braid $\beta$ is the composition of 5 elementary braids, leading to

$$M_{f_{\beta}} = \begin{pmatrix} -t^\epsilon & 1 \\ 0 & 1 \end{pmatrix}^{-2} \begin{pmatrix} 1 & 0 \\ t^\epsilon & -t^\epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^\epsilon & -t^\epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^\epsilon & -t^\epsilon \end{pmatrix}$$

$$= \begin{pmatrix} 2t^{-2\epsilon} - 3t^{-\epsilon} + 2 & t^{-\epsilon} - 1 \\ 2t^\epsilon - 1 & -t^\epsilon \end{pmatrix},$$

where $\epsilon$ is $\pm 1$ according to the orientation of $K$. By Proposition 33, we have $\Delta_K(t) \doteq 2t - 3 + 2t^{-1}$.

Let $L$ be an oriented link obtained by orienting $C(2, 2, 2)$ so that the linking number of the components is $+2$. Here, we get

$$M_{f_{\beta}} = \begin{pmatrix} 1 & 0 \\ t^\epsilon & -t^\epsilon \end{pmatrix}^2 \begin{pmatrix} -t^\epsilon & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -t^{-\epsilon} & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ t^\epsilon & -t^\epsilon \end{pmatrix}^2$$

$$= \begin{pmatrix} (t^\epsilon - t^{-2\epsilon})^2 (t^\epsilon t^{-2\epsilon} - t^\epsilon + 1) & t^\epsilon - t^{-2\epsilon} \\ 2(t^\epsilon - t^{-2\epsilon}) (2t^\epsilon - t^\epsilon + 1) & 2t^\epsilon - 2t^{-2\epsilon} + t^\epsilon \end{pmatrix},$$

where $\epsilon = \pm 1$ depends on the global orientation of $L$. Therefore, $\Delta_L(t) \doteq 2(t - 1)(t - 1 + t^{-1})$. Finally, if we orient $C(2, 2, 2)$ so that the linking number of the components is $-2$, the resulting oriented link $L'$ has Alexander polynomial $\Delta_{L'} \doteq (t - 1)(t - 4 + t^{-1})$. 

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2-strand tangles. In this section, we use the techniques introduced above to define an invariant of \((2, 2)\)-tangles formed by two arcs and having no closed components. This invariant is a pair of elements of \(\Lambda\) defined up to simultaneous multiplication by a unit of \(\Lambda\). We study the behaviour of this invariant under the basic transformations of \((2, 2)\)-tangles introduced by Conway [3].

Consider a tangle \(\tau \in T(\varepsilon, \varepsilon')\) with no closed components, where \(\varepsilon\) and \(\varepsilon'\) are sequences of length 2. By bending \(\tau\), we get a tangle \(\tau^b \in T(\emptyset, \mu)\) where \(\emptyset\) is the empty sequence and \(\mu = (\varepsilon'_1, \varepsilon'_2, -\varepsilon_2, -\varepsilon_1)\).

Lemma 34 The submodule \(N(\tau^b)\) of \(H_1(\hat{D}_\mu)\) is free of rank one.

PROOF. One can write \(\tau^b\) as a composition \(\tau^b = \tau' \circ u\), where \(u \in T(\emptyset, \bar{\varepsilon})\) is the elementary 1-strand ‘cup’ tangle and \(\tau' \in T(\bar{\varepsilon}, \varepsilon)\) is a straight tangle. Since \(H_1(\hat{D}_\emptyset) = H_1(\hat{D}_{\varepsilon'}) = 0\), we have \(N(u) = 0\). By Lemma 14, \(N(\tau^b) = N(\tau')\) which is free by Proposition 20. Its rank is one by Proposition 25. \(\square\)

Recall from Section 3.2 that \(H_1(\hat{D}_\mu) = (\Lambda v_1 \oplus \Lambda v_2 \oplus \Lambda v_3) / \Lambda \gamma\), where \(v_i = \hat{e}_i - \hat{e}_{i+1}\) and \(\gamma = e_1^{\varepsilon_1} \cdots e_4^{\varepsilon_4}\). Therefore, \(H_1(\hat{D}_\mu)\) is free with basis \(v_1, v_2\). Using this fact and Lemma 34, the inclusion \(N(\tau^b) \subset H_1(\hat{D}_\mu)\) is given by a matrix
\[
\begin{pmatrix}
m_1 \\
m_2
\end{pmatrix}
\] with \(m_1, m_2 \in \Lambda\), unique up to multiplication by \(\pm \nu\) with \(\nu \in \mathbb{Z}\). Let us denote this by \(\tau \sim (m_1, m_2)\).

For concreteness, we shall assume throughout the rest of the discussion that \(\varepsilon = \varepsilon' = (-1, +1)\) as for the tangle \(\tau\) in Figure 8. (The other five cases can be treated similarly.) Consider the tangles \(\tau_1, \tau_2, \tau_3\) and \(\tau_4\) shown in Figure 8: \(\tau_1\) is obtained from \(\tau\) by a horizontal reflection, \(\tau_2\) by a rotation to the angle \(\pi/2\), \(\tau_3\) by addition of a twist to the right, and \(\tau_4\) by addition of a twist to the top.
Proposition 35 If $\tau \sim (m_1, m_2)$, then $\tau_1 \sim (m_1, -m_2)$, $\tau_2 \sim (m_2, -m_1)$, $\tau_3 \sim (tm_1, m_1 - m_2)$ and $\tau_4 \sim (m_2 - tm_1, m_2)$.

**PROOF.** We have $\tau^b \in T(\emptyset, \mu)$ with $\mu = (-1, +1, -1, +1)$, while $\tau_1^b, \tau_2^b \in T(\emptyset, \mu')$ where $\mu' = (+1, -1, +1, -1)$. Hence, $H_1(\tilde{D}_\mu) = (\Lambda v_1' \oplus \Lambda v_2' \oplus \Lambda v_3')/\Lambda(v_1' + v_3')$. The horizontal reflexion induces an isomorphism $H_1(\tilde{D}_\mu) \to H_1(\tilde{D}_\mu')$ given by $v_1 \mapsto -v_3' = v'_1$ and $v_2 \mapsto -v_2'$. Hence, $\tau_1 \sim (m_1', m_2')$ with \[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ -m_2 \end{pmatrix}.
\]
Similarily, the rotation to the angle $\pi/2$ induces an isomorphism $H_1(\tilde{D}_\mu) \to H_1(\tilde{D}_{\mu'})$ given by the matrix \[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Thus, $\tau_2 \sim (m_2, -m_1)$. Note that $\tau_3^b \in T(\emptyset, \mu'')$ where $\mu'' = (-1, -1, +1, +1)$. The transformation from $\tau^b$ to $\tau_3^b$ can be understood as a composition $\tau_3^b = \sigma \circ \tau^b$, where $\sigma$ is a spherical braid. By the results of Section 5.3, the isomorphism $H_1(\tilde{D}_\mu) \to H_1(\tilde{D}_{\mu''})$ corresponding to $\sigma$ is given by $v_1 \mapsto v_1'' + t^{-1}v_2''$ and $v_2 \mapsto -t^{-1}v_3''$. Therefore, $\tau_3 \sim (m_1, t^{-1}(m_1 - m_2))$, which is equivalent to $(tm_1, m_1 - m_2)$. The case of $\tau_4$ is similar. $\square$

Proposition 36 If $\tau$ is topologically trivial and $\tau \sim (m_1, m_2)$, then the oriented links $\tau^D$ and $\tau^N$ described in Figure 8 have the Alexander module
\[
H_1(\tilde{X}_\tau^D) = \Lambda/(m_1) \quad \text{and} \quad H_1(\tilde{X}_\tau^N) = \Lambda/(m_2).
\]
In particular, $\Delta_{\tau^D}(t) \doteq m_1$ and $\Delta_{\tau^N}(t) \doteq m_2$.

**PROOF.** Since $\tau$ is topologically trivial, $H_1(\tilde{X}_\tau^D) = H_1(\tilde{X}_\tau) = \Lambda$ and the inclusion homomorphism $j: H_1(\tilde{D}_\mu) = \Lambda v_1 \oplus \Lambda v_2 \to H_1(\tilde{X}_\tau)$ is onto (cf. the proof of Corollary 32). Therefore, the greatest common divisor of $j(v_1)$ and $j(v_2)$ is 1. Hence, the kernel $N(\tau^b)$ of $j$ is generated by $j(v_2)v_1 - j(v_1)v_2$, so $m_1 = j(v_2)$ and $m_2 = -j(v_1)$. Since the exterior of $\tau^D$ in $S^3$ can be written $X_{\tau^D} = X_\tau \cup X_{id}$, we have the Mayer-Vietoris exact sequence
\[
H_1(\tilde{D}_\mu) \xrightarrow{\varphi} H_1(\tilde{X}_\tau) \oplus H_1(\tilde{X}_{id}) \to H_1(\tilde{X}_{\tau^D}) \to 0.
\]
Clearly, $H_1(\tilde{X}_{id}) = \Lambda v_1$ and a matrix of $\varphi$ is given by \[
\begin{pmatrix} j(v_1) & j(v_2) \\ 1 & 0 \end{pmatrix}.
\]
It is equivalent to $(j(v_2)) = (m_1)$, so $H_1(\tilde{X}_{\tau^D}) = \Lambda/(j(v_2)) = \Lambda/(m_1)$. With the notation of Figure 8, we have $\tau^N = (\tau_2)^D$. Hence, the formula for $\tau^N$ follows from the formula for $\tau^D$ and Proposition 35. $\square$

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8 Generalizations

8.1 The category of $m$-colored tangles

Fix throughout this section a positive integer $m$. An $m$-colored tangle is an oriented tangle $\tau$ together with a map $c$ assigning to each component $\tau_j$ of $\tau$ a color $c(j) \in \{1, \ldots, m\}$. The composition of two $m$-colored tangles is defined if and only if it is compatible with the coloring of each component. Finally, we say that an $m$-colored tangle is an oriented $m$-colored braid if the underlying tangle is a braid.

More formally, $m$-colored tangles can be understood as morphisms of a category in the following way. Consider two maps $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$ and $\varphi': \{1, \ldots, n'\} \to \{\pm 1, \ldots, \pm m\}$, where $n$ and $n'$ are non-negative integers. We will say that an $m$-colored tangle $(\tau, c)$ is a $(\varphi, \varphi')$-tangle if the following conditions hold:

- $\tau$ is an $(\varepsilon, \varepsilon')$-tangle, where $\varepsilon = \varphi/|\varphi|$ and $\varepsilon' = \varphi'/|\varphi'|$;
- if $x_i \in D^2 \times \{0\}$ (resp. $x'_i \in D^2 \times \{1\}$) is an endpoint of a component $\tau_j$ of $\tau$, then $|\varphi(i)| = c(j)$ (resp. $|\varphi'(i)| = c(j)$).

Two $(\varphi, \varphi')$-tangles are isotopic if they are isotopic as $(\varepsilon, \varepsilon')$-tangles under an isotopy that respects the color of each component. We denote by $T(\varphi, \varphi')$ the set of isotopy classes of $(\varphi, \varphi')$-tangles. The composition of oriented tangles induces a composition $T(\varphi, \varphi') \times T(\varphi', \varphi'') \to T(\varphi, \varphi'')$ for any $\varphi, \varphi'$ and $\varphi''$.

This allows us to define the category of $m$-colored tangles $\text{Tangles}_m$. Its objects are the maps $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$ with $n \geq 0$, and its morphisms are given by $\text{Hom}(\varphi, \varphi') = T(\varphi, \varphi')$. Clearly, oriented $m$-colored braids and oriented $m$-colored string links form categories $\text{Braids}_m$ and $\text{Strings}_m$ such that

$$\text{Braids}_m \subset \text{Strings}_m \subset \text{Tangles}_m.$$  

8.2 The multivariable Lagrangian representation

We now define a functor $F_m: \text{Tangles}_m \to \text{Lagr}_{\Lambda_m}$, where $\Lambda_m$ denotes the ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. This construction generalizes the functor of Theorem 15, which corresponds to the case $m = 1$. It also extends the works of Gassner for pure braids and Le Dimet for pure string links.

Consider an object of $\text{Tangles}_m$, that is, a map $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$
with $n \geq 0$. Set $\ell_{\varphi} = (\ell_{\varphi}^{(1)}, \ldots, \ell_{\varphi}^{(m)}) \in \mathbb{Z}^m$, where $\ell_{\varphi}^{(j)} = \sum_{i | \varphi(i) = \pm j} \text{sign}(\varphi(i))$ for $j = 1, \ldots, m$. Using the notation of Section 3.2, we define

$$D_{\varphi} = \begin{cases} D^2 \setminus N(\{x_1, \ldots, x_n\}) & \text{if } \ell_{\varphi} \neq (0, \ldots, 0); \\ S^2 \setminus N(\{x_1, \ldots, x_n\}) & \text{if } \ell_{\varphi} = (0, \ldots, 0). \end{cases}$$

As in the case of oriented tangles, we endow $D_{\varphi}$ with the counterclockwise orientation, a base point $z$, and a coloring of $\pi_1(D_{\varphi}, z)$. Consider the homomorphism from $\pi_1(D_{\varphi})$ to the free abelian group $G \cong \mathbb{Z}^m$ with basis $t_1, \ldots, t_m$ given by $e_i \mapsto t_{|\varphi(i)|}$. It defines a regular $G$-covering $\widehat{D_{\varphi}} \to D_{\varphi}$, so the homology $H_1(\widehat{D_{\varphi}})$ is a module over $\mathbb{Z}G = \Lambda_m$. Finally, let $\omega_{\varphi}: H_1(\widehat{D_{\varphi}}) \times H_1(\widehat{D_{\varphi}}) \to \Lambda_m$ be the skew-hermitian pairing given by

$$\omega_{\varphi}(x, y) = \sum_{g \in G} < gx, y > g^{-1},$$

where $<, >: H_1(\widehat{D_{\varphi}}) \times H_1(\widehat{D_{\varphi}}) \to \mathbb{Z}$ is the intersection form induced by the orientation of $D_{\varphi}$ lifted to $\widehat{D_{\varphi}}$.

Consider now a $(\varphi, \varphi')$-tangle $(\tau, c)$. Note that $\ell_{\varphi} = \ell_{\varphi'}$. Let $X_{\tau}$ be the compact manifold

$$X_{\tau} = \begin{cases} (D^2 \times [0, 1]) \setminus N(\tau) & \text{if } \ell_{\varphi} \neq (0, \ldots, 0); \\ (S^2 \times [0, 1]) \setminus N(\tau) & \text{if } \ell_{\varphi} = (0, \ldots, 0), \end{cases}$$

oriented so that the induced orientation on $\partial X_{\tau}$ extends the orientation on $(-D_{\varphi}) \cup D_{\varphi'}$. We know from Section 3.3 that $H_1(X_{\tau}) = \bigoplus_{j=1}^{m} \mathbb{Z}m_j$ if $\ell_{\varphi} \neq (0, \ldots, 0)$, and $H_1(X_{\tau}) = \bigoplus_{j=1}^{m} \mathbb{Z}m_j / \sum_{j=1}^{m} \text{sign}(\varphi(i))e_i$ otherwise. Hence, the coloring of $\tau$ defines a homomorphism $H_1(X_{\tau}) \to G$, $m_j \mapsto t_{c(j)}$ which induces a homomorphism $\pi_1(X_{\tau}) \to G$ extending the homomorphisms $\pi_1(D_{\varphi}) \to G$ and $\pi_1(D_{\varphi'}) \to G$. It gives a $G$-covering $\widehat{X_{\tau}} \to X_{\tau}$.

Consider the inclusion homomorphisms $i_{\tau}: H_1(\widehat{D_{\varphi}}) \to H_1(\widehat{X_{\tau}})$ and $i'_{\tau}: H_1(\widehat{D_{\varphi'}}) \to H_1(\widehat{X_{\tau}})$. Denote by $j_{\tau}$ the homomorphism $H_1(\widehat{D_{\varphi}}) \oplus H_1(\widehat{D_{\varphi'}}) \to H_1(\widehat{X_{\tau}})$ given by $j_{\tau}(x, x') = i'_{\tau}(x') - i_{\tau}(x)$. Set

$$\mathcal{F}_m(\tau) = \ker(j_{\tau}) \subset H_1(\widehat{D_{\varphi}}) \oplus H_1(\widehat{D_{\varphi'}}).$$

**Theorem 37** Let $\mathcal{F}_m$ assign to each map $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$ the pair $(H_1(\widehat{D_{\varphi}}), \omega_{\varphi})$ and to each $\tau \in T(\varphi, \varphi')$ the submodule $\mathcal{F}_m(\tau)$ of $H_1(\widehat{D_{\varphi}}) \oplus H_1(\widehat{D_{\varphi'}})$. Then, $\mathcal{F}_m$ is a functor $\text{Tangles}_m \to \text{Lagr}_{\Lambda_m}$ which fits
in the diagrams

$$
\begin{align*}
\text{Tangles}_m \xrightarrow{\mathcal{F}_m} \text{Lagr}_{\Lambda_m} \\
\text{Braids}_m \longrightarrow U_{\Lambda_m}
\end{align*}
$$

and

$$
\begin{align*}
\text{Tangles}_m \xrightarrow{\mathcal{J} \circ \mathcal{F}_m} \text{Lagr}_{\Lambda_m} \\
\text{Strings}_m \longrightarrow U^0_{\Lambda_m}
\end{align*}
$$

where the vertical arrows denote embeddings of categories.

**PROOF.** Lemmas 12, 13, 14, Proposition 17, Proposition 19 and their proofs extend to our setting with obvious changes. The only ‘topological’ facts required are the following:

1. \(H_1(\partial \widehat{D}_\varphi) = 0\),
2. the \(\Lambda_m\)-module \(H_1(\widehat{D}_\varphi)\) is torsion-free,
3. \(H_1(\partial \widehat{X}_\tau, \widehat{D}_\varphi \sqcup \widehat{D}_\varphi')\) is a torsion \(\Lambda_m\)-module.

The definition of \(\widehat{D}_\varphi\) easily implies that \(\partial \widehat{D}_\varphi\) consists of copies of \(\mathbb{R}\), so the first claim is checked. Since \(D_\varphi\) has the homotopy type of a 1-dimensional CW-complex \(Y_\varphi\), the \(\Lambda_m\)-module \(H_1(\widehat{D}_\varphi) = H_1(\widehat{Y}_\varphi) = \mathbb{Z}(\widehat{Y}_\varphi)\) is a submodule of the free \(\Lambda_m\)-module \(C_1(\widehat{Y}_\varphi)\). Therefore, \(H_1(\widehat{D}_\varphi)\) is torsion-free. Finally, the third claim follows easily from the definitions and the excision theorem. \(\square\)

### 8.3 High-dimensional Lagrangian representations

The Lagrangian representation of Theorem 15 can be generalized in another direction by considering high-dimensional manifolds. We conclude the paper with a brief sketch of this construction.

Fix throughout this section an integer \(n \geq 1\). In the sequel, all the manifolds are assumed piecewise linear, compact and oriented. Consider a homology \(2n\)-sphere \(D\). To this manifold, we associate a category \(\mathcal{G}_D\) as follows. Its objects are codimension-2 submanifolds \(M\) of \(D\) such that \(H_n(M) = 0\). The morphisms between \(M \subset D\) and \(M' \subset D\) are given by properly embedded codimension-2 submanifolds \(T\) of \(D \times [0, 1]\) such that the oriented boundary \(\partial T\) of \(T\) satisfies \(\partial T \cap (D \times \{0\}) = -M\) and \(\partial T \cap (D \times \{1\}) = M'\), where \(-M\) denotes \(M\) with the opposite orientation. The composition is defined in the obvious way.

If \(D_M\) is the complement of an open tubular neighborhood of \(M\) in \(D\), we easily check that \(H_1(D_M) \cong H_0(M)\). Therefore, the epimorphism \(H_0(M) \rightarrow \mathbb{Z}\) which sends every generator to 1 determines a \(\mathbb{Z}\)-covering \(\widehat{D}_M \rightarrow D_M\). The
lift of the orientation of \( D_M \) to \( \widetilde{D}_M \) defines a \( \mathbb{Z} \)-bilinear intersection form on \( H_n(\widetilde{D}_M) \). This gives a \( \Lambda \)-sesquilinear form on \( H_n(\widetilde{D}_M) \), which in turn induces a \( \Lambda \)-sesquilinear form \( \omega_M \) on \( BH_n(\widetilde{D}_M) \), where \( BH = H/T_{\text{ors}} \Lambda H \) for a \( \Lambda \)-module \( H \). (Note that \( \omega_M \) is skew-hermitian if \( n \) is odd, and Hermitian if \( n \) is even.) Using the fact that \( H_n(M) = 0 \), the proof of Lemma 12 can be applied to this setting, showing that \( \omega_M \) is non-degenerate. Let \( \mathcal{F}_D(M) \) denote the \( \Lambda \)-module \( BH_n(\widetilde{D}_M) \) endowed with the non-degenerate \( \Lambda \)-sesquilinear form \( \omega_M \).

Given a codimension-2 submanifold \( T \) of \( D \times [0,1] \), denote by \( X_T \) the complement of an open tubular neighborhood of \( T \) in \( D \times [0,1] \). Since \( H_1(X_T) \approx H_0(T) \), we have a \( \mathbb{Z} \)-covering \( \widetilde{X}_T \to X_T \) given by the homomorphism \( H_0(T) \to \mathbb{Z} \) which sends every generator to 1. There are obvious inclusions \( \widetilde{D}_M \subset \widetilde{X}_T \) and \( \widetilde{D}_M' \subset \widetilde{X}_T \) which induce homomorphisms \( i \) and \( i' \) in \( n \)-dimensional homology. Let \( j: H_n(\widetilde{D}_M) \oplus H_n(\widetilde{D}_M') \to H_n(\widetilde{X}_T) \) be the homomorphism given by \( j(x, x') = i'(x') - i(x) \). It induces a homomorphism

\[
BH_n(\widetilde{D}_M) \oplus BH_n(\widetilde{D}_M') \xrightarrow{j_T} BH_n(\widetilde{X}_T).
\]

Set \( \mathcal{F}_D(T) = \ker(j_T) \). The proof of Lemma 13 can be applied to check that \( \mathcal{F}_D(T) \) is a Lagrangian submodule of \( (-BH_n(\widetilde{D}_M)) \oplus BH_n(\widetilde{D}_M') \). Lemma 14 can also be adapted to our setting to show that \( \mathcal{F}_D(T_2 \circ T_1) = \mathcal{F}_D(T_2) \circ \mathcal{F}_D(T_1) \). Therefore, \( \mathcal{F}_D \) is a functor from \( \mathcal{C}_D \) to the Lagrangian category \( \mathcal{Lagr}_\Lambda \) amended as follows: the non-degenerate form is Hermitian if \( n \) is even, skew-hermitian if \( n \) is odd.

\section*{Acknowledgements}

A part of this paper was done while the authors visited the Department of Mathematics of the Aarhus University whose hospitality the authors thankfully acknowledge. The first author also wishes to thank the Institut de Recherche Mathématique Avancée (Strasbourg) and the Institut de Mathématiques de Bourgogne (Dijon) for hospitality.

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