Anderson localization as a parametric instability of the linear kicked oscillator

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We rigorously analyse the correspondence between the one-dimensional standard Anderson model and a related classical system, the ‘kicked oscillator’ with noisy frequency. We show that the Anderson localization corresponds to a parametric instability of the oscillator, with the localization length determined by an increment of the exponential growth of the energy. Analytical expression for a weak disorder is obtained, which is valid both inside the energy band and at the band edge.

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I. INTRODUCTION

Recently it was shown that quantum one-dimensional tight-binding models with any diagonal site-potential can be formally represented in terms of a two-dimensional Hamiltonian map \[1\]. On the other hand, this classical map is associated with a linear oscillator subjected to a linear force given in the form of time-dependent delta-kicks. In this picture, both the frequency of the unperturbed oscillator and the period of the kicks are determined by the energy of an eigenstate, and the amplitudes of the kicks are defined by the site-potential in the original quantum model. It was shown that by exploring the dynamics of this classical system, one can obtain global characteristics of the eigenstates, such as the localization length defined by the Lyapunov exponent of classical trajectories.

In particular, analytical estimates have been obtained in \[1\] for a specific diagonal site-potential potential with short-range correlations (the so-called dimer model, see \[2\]). Other applications to the case of general correlated diagonal \[3\] and off-diagonal \[4\] disorder have revealed very important peculiarities. One of the most interesting results has been obtained in \[3\] where it was shown how to construct random potentials with specific two-point correlators which result in the emergence of the mobility edge in one-dimensional geometry. Based on these predictions, very recently experimental realization of this effect has been done in one-mode microwave guides \[5\].

In this paper, we perform an analytical study of the standard Anderson model with diagonal uncorrelated disorder, paying main attention to the problem of the mathematical correspondence between the quantum model and its classical representation in the form of a linear kicked oscillator. More specifically, we are interested in the connection between the Anderson localization and the parametric instability of the corresponding classical system. Although the results obtained for the localization length in the weak disorder limit are well-known from other studies, the method we use here is a new one and it may explain the mechanism of the Anderson transition in new terms. Moreover, this approach may be very useful for 2D and 3D cases, for which analytical results for global properties of eigenstates are very restricted.

II. DEFINITION OF THE MODEL

In this paper we study the relation existing between the standard 1D Anderson model and a related physical system, a linear oscillator with noisy frequency. The quantum model is defined by the stationary Schrödinger equation \[6\]

\[
\psi_{n+1} + \psi_{n-1} + \epsilon_n \psi_n = E \psi_n, \tag{1}
\]

where \(\psi_n\) represents the electron wave-function at the \(n\)th lattice site, and the site-energies \(\epsilon_n\) are independent random variables with a common distribution \(p(\epsilon)\). In the standard Anderson model the probability \(p(\epsilon)\) has the form of a box distribution,

\[
p(\epsilon) = \frac{1}{W} \theta \left( \frac{W}{2} - |\epsilon| \right), \tag{2}
\]

whose width \(W\) sets the strength of the disorder. In the following, however, we will not restrict our considerations to the specific form \(6\) of the probability distribution, but simply assume that the variables \(\epsilon_n\) have zero mean \((\langle \epsilon_n \rangle = 0)\) and a finite variance \((\langle \epsilon_n^2 \rangle)\).

The kicked oscillator is a harmonic oscillator that undergoes periodic and instantaneous variations of the momentum (the ‘kicks’). Such a system is defined by the Hamiltonian

\[
H = \omega \left( \frac{p^2}{2} + \frac{x^2}{2} \right) + \frac{x^2}{2} \xi(t), \tag{3}
\]

where

\[
\xi(t) = \sum_{n=-\infty}^{+\infty} A_n \delta(t - nT). \tag{4}
\]

The random coefficients \(A_n\) that appear in the definition of the noise \(6\) represent the intensity of the ‘kicks’, i.e.,
they are proportional to the sudden momentum changes experienced by the oscillator at times \( t = nT \). In other words, the system (3) represents a harmonic oscillator with a mean frequency \( \omega \) perturbed by the noise term \( \xi(t) \). Using the definition (3), one can easily reduct the statistical properties of the noise \( \xi(t) \) to the corresponding properties of the variables \( A_n \); in particular, the mean and the variance of \( \xi(t) \) can be expressed as

\[
\langle \xi(t) \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} \xi(t) \, dt
\]

and

\[
\langle \xi(t)\xi(t+s) \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau}^{\tau} \xi(t)\xi(t+s) \, dt
\]

\[
= \frac{1}{T} \delta(s) \lim_{N \to \infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} A_n^2 = \frac{\langle A_n^2 \rangle}{T} \delta(s) .
\]

The equivalence of the models (1) and (3) has been discussed in [1]. There it was shown how the two-dimensional map

\[
\begin{cases}
  x_{n+1} = x_n \cos(\omega T) + (p_n - A_n x_n) \sin(\omega T) \\
  p_{n+1} = -x_n \sin(\omega T) + (p_n - A_n x_n) \cos(\omega T)
\end{cases}
\]

(5)
can be derived by integrating over a period \( T \) the Hamiltonian equations of motion of the kicked oscillator (3). Note that in the map (3) \( x_n \) and \( p_n \) stand for the coordinate and momentum of the oscillator immediately before the \( n \)-th kick. Eliminating the momentum from equations (3), one eventually obtains the relation

\[
x_{n+1} + x_{n-1} + A_n \cos(\omega T) x_n = 2x_n \cos(\omega T) ,
\]

which coincides with the Anderson equation (3) if one identifies the site amplitude \( \psi_n \) with the coordinate \( x_n \) of the oscillator and if the parameters of the models (3) and (3) are related to each other by the equalities

\[
\epsilon_n = A_n \sin(\omega T) ; \quad E = 2\cos(\omega T) .
\]

In Ref. [1] the classical map (3) was used as a tool to investigate the properties of the model (3); here we focus instead on the direct analysis of the Hamiltonian model (3).

### III. THE OSCILLATOR WITH NOISY FREQUENCY

The dynamics of the kicked oscillator (3) is determined by the Hamiltonian equations of motion:

\[
\begin{aligned}
  \dot{p} &= - (\omega + \xi(t)) x , \\
  \dot{x} &= \omega p .
\end{aligned}
\]

(7)

In order to study the behaviour of the kicked oscillator, it is convenient to substitute the couple of differential equations (7) with the system of stochastic Itô equations,

\[
\begin{aligned}
  dp &= - \omega x \, dt - x \sqrt{A_n^2/T} \, dW(t) \\
  dx &= \omega p \, dt
\end{aligned}
\]

(8)

where \( W(t) \) is a Wiener process with \( \langle dW(t) \rangle = 0 \) and \( \langle dW(t)^2 \rangle = dt \).

The systems (3) and (8) can be considered equivalent inasmuch as the shot noise \( \xi(t) \) is adequately represented by a Wiener process \( W(t) \). That is the case if the strength of the single kicks is weak, i.e., if the condition

\[
\langle A_n^2 \rangle \ll 1
\]

(9)
is fulfilled. That can be understood by considering that the present situation is analogous to the one that occurs in the Brownian motion of a heavy particle suspended in a fluid of light molecules. The instantaneous impacts of the fluid molecules on the massive particle can be successfully described by a continuous Wiener process, provided that each single collision does not produce a significant displacement of the heavy particle. When the mass of the suspended particle is not much bigger than the one of the impinging molecules, the nature of the motion changes and the effect of the molecular collisions can no longer be depicted by a Wiener process.

A similar analogy can be drawn between the present case and the random walk problem. To be exact, let us consider a one-dimensional random walk made by someone that takes steps of length \( l \) at times \( nT \) (with \( n \) integral). At each step the walker is supposed to go to the right or to the left with equal probability. In this model the walker’s position changes with each step much in the way the momentum of the kicked oscillator does under the action of a kick: in both cases the relevant physical quantity is varied in a sudden and random way at regular time intervals. This analogy makes interesting to notice that, by going to the limit

\[
l \to 0 , \quad T \to 0 ,
\]

while holding fixed the ratio

\[
D = \frac{l^2}{T}
\]

the discrete time random walk evolves in a Brownian motion with diffusion constant \( D \) (see, e.g., Ref. [3]). In other words, a Wiener process can be regarded as a limit case of random walk in the limit of very small and fast-spaced steps. In a similar way, the ‘jump process’ \( \xi(t) \) can be described by a ‘diffusion process’ \( W(t) \) when the condition (3) is satisfied, with the ratio

\[
k = \frac{\langle A_n^2 \rangle}{\omega T}
\]

(10)
IV. LYAPUNOV EXPONENT

Once we have established the correspondence of the Anderson model with the stochastic oscillator \( \mathcal{S} \), we can proceed to redefine essential features of the first model in the dynamical language of the second. In particular, we are interested in deriving a formula for the Lyapunov exponent, which gives the inverse localization length for the eigenstates of the equation \( (1) \). Since these eigenstates correspond to trajectories of the stochastic oscillator \( \mathcal{S} \), the Lyapunov exponent is naturally redefined as the exponential divergence rate of neighbouring trajectories, i.e. through the limit

\[
\lambda = \lim_{T \to \infty} \lim_{\delta \to 0} \frac{1}{T} \int_0^T dt \frac{1}{\delta} \log \frac{x(t + \delta)}{x(t)},
\]

which corresponds to the standard expression

\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \log \frac{\psi_{n+1}}{\psi_n}
\]

for the Anderson model \( \mathcal{W} \). By taking the limit \( \delta \to 0 \) first, the expression \((11)\) can be put in the simpler form

\[
\lambda = \langle z \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ z(t),
\]

where the Ricatti variable \( z = \dot{x}/x \) has been introduced and the symbol \( \langle z \rangle \) stands for the (time) average of \( z \).

To compute the Lyapunov exponent, as defined by Eq. \((12)\), it is necessary to analyse the dynamics of the variable \( z = \omega p/x \). The time evolution of this quantity is determined by the Itô stochastic equation

\[
dz = -\lambda (\omega^2 + z^2) dt - \omega \sqrt{\langle A_n^2 \rangle/T} dW(t),
\]

which can be easily derived using \((8)\) and the standard rules of the Itô calculus.

Notice that, while the position and momentum of the oscillator \( \mathcal{S} \) do not evolve independently from each other, the dynamics of their ratio \( z = \omega p/x \) is totally autonomous from that of any other variable. As a consequence, one deals with the single differential equation \((13)\) instead of having to cope with a set of coupled equations like \( \mathcal{S} \). Thus, the introduction of the variable \( z \), which is suggested by the definition \((4)\) of the Lyapunov exponent, turns out to be beneficial also for the study of the stochastic oscillator \( \mathcal{S} \).

As is known, the Itô stochastic differential equation \((13)\) is equivalent to the Fokker-Planck equation \((8)\):

\[
\frac{\partial p}{\partial t}(z,t) = \frac{\partial}{\partial z} \left[ (\omega^2 + z^2) p(z,t) \right] + \frac{\omega^2 \langle A_n^2 \rangle}{2T} \frac{\partial^2 p}{\partial z^2}(z,t),
\]

which gives the time evolution of the probability density \( p(z,t) \) of the stochastic variable \( z \). In other words, the evolution of \( z(t) \) dictated by Eq. \((13)\) is a diffusion process with a deterministic drift coefficient \((\omega^2 + z^2)\) and a noise-induced diffusion coefficient \( \omega^2 \langle A_n^2 \rangle/T \).

The stationary solution of Eq. \((14)\) is

\[
p(z) = \left[ C_1 + C_2 \int_{-\infty}^z dx \exp\left\{\Phi(x/\omega)\right\}\exp\left\{-\Phi(z/\omega)\right\} \right],
\]

where \( C_1 \) and \( C_2 \) are integration constants and the function \( \Phi(x) \) is given by the relation

\[
\Phi(x) = \frac{2}{k} \left( x + \frac{x^3}{3} \right),
\]
which contains the parameter $k$ defined by Eq. (10). Since $p(z)$ is a probability distribution, it must be integrable and therefore the constant $C_1$ must vanish. The residual constant $C_2$ is determined by the normalization condition $\int_{-\infty}^{\infty} p(z) \, dz = 1$. The resulting distribution is:

$$p(z) = \frac{1}{N \omega^2} \int_{-\infty}^{\infty} dx \, \exp \{ \Phi(x/\omega) - \Phi(z/\omega) \}$$  \hspace{1cm} (16)

with

$$N = \frac{\sqrt{\pi k}}{2} \int_{0}^{\infty} dx \, \frac{1}{\sqrt{x}} \exp \left[ -\frac{2}{k} (x + x^3/12) \right] .$$ \hspace{1cm} (17)

Once the steady-state probability distribution (16) is known, one can use it to compute the average of $z$ which represents an equivalent form of the system (7).

The problem is defined by the continuous Schrödinger equation for the kicked oscillator

$$\ddot{x}(t) + (\omega^2 + \omega \xi(t)) \, x(t) = 0 ,$$

which represents an equivalent form of the system (7).

\[ \] 

V. WEAK DISORDER EXPANSION

The weak disorder case is defined by the condition (4). This condition implies that, except that at the band edge (i.e., for $\omega T \to 0$), the parameter $|k| \ll 1$. In this section we analyse therefore the expansion of the Lyapunov exponent (19) in the limit $k \to 0$. This corresponds to studying the behaviour of the localization length inside the energy band for the Anderson model (8) with a weak disorder.

Making use of expressions (19) and (17), it is easy to verify that for $k \to 0$ the Lyapunov exponent can be written in the form

$$\lambda = \frac{\langle A_n^2 \rangle}{4T} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(3n + 1/2)}{n!} \frac{k^2}{48} \cdot 1 \hspace{1cm} (21)$$

To the lowest order in $k$ this expression reduces to

$$\lambda = \frac{\langle A_n^2 \rangle}{8T}, \hspace{1cm} (22)$$

which represents the basic approximation for the inverse localization length in the weak disorder case.

Taking into account the relations (1) between the parameters of the Anderson model (8) and those of the stochastic oscillators (9) and (8), the variance $\langle A_n^2 \rangle$ that appears in formula (22) can be expressed as

$$\langle A_n^2 \rangle = \frac{\langle \epsilon_n^2 \rangle}{1 - E^2/4} .$$

When the distribution for the random site-energies $\epsilon_n$ is the box distribution (4), one can further write $\langle \epsilon_n^2 \rangle = W^2/12$; as a consequence, expression (22) takes the form

$$\lambda = \frac{1}{T} \frac{W^2}{96 (1 - E^2/4)} , \hspace{1cm} (23)$$

which, for $T = 1$, coincides with the well-known standard formula for the inverse localization length in the Anderson model (12).

The extra-factor $1/T$ stems from definition (12) of the Lyapunov exponent, which implies that $\lambda = \langle x/t \rangle$ has the dimension of an inverse time. In order to have the correct physical dimension, therefore, $\lambda$ must be inversely proportional to a time parameter which, on the other hand, must be a specific feature of the noise (4), since that is the physical origin of the orbit instability. This
The expression \( \langle A^2_n \rangle / \langle A^2 \rangle \) corresponds to the result derived by Thouless using standard perturbation methods \([12]\). As such, formula \( \langle A^2_n \rangle / \langle A^2 \rangle \) fails to reproduce the correct behaviour of the Lyapunov exponent at the band centre, where the second order perturbation theory of Thouless breaks down and an anomaly appears which was first explained in Ref. \([13]\). This deviation of the inverse localization length from the behaviour predicted by formula \( \langle A^2_n \rangle / \langle A^2 \rangle \) is a resonance phenomenon, which can be conveniently understood by considering the dynamics of the kicked oscillator \([10]\). In fact, the band centre corresponds to the case \( \omega T = \pi/2 \) and this equality can be interpreted as the condition that the frequency \( 1/T \) of the kicks be exactly four times the frequency \( \omega/2\pi \) of the unperturbed oscillator. This generates a resonance effect that manifests itself in a small but clear increase of the localization length with respect to the value predicted by formula \( \langle A^2_n \rangle / \langle A^2 \rangle \) for \( E = 0 \). Once the origin of the anomaly at the band centre is explained in these terms, it is not surprising that the model \([3]\) fails to reproduce this feature, because it is obvious that the Wiener noise \( W(t) \) cannot conveniently mimic the regularly time-spaced character of the shot noise \([1]\). One might then worry that the model \([3]\) provides an inadequate description of the Anderson model whenever the period of the unperturbed oscillator and that of the kicks stand in any rational ratio. This is not the case, however, because the resonance effect at the band centre is the only one that affects the localization length at the second order of the perturbation theory \([14]\) and is thus of interest for the present work. For all the other ‘rational’ values of the energy \( E = 2 \cos \pi \alpha \) with \( \alpha \) rational, the effect of the resonance on the Lyapunov exponent can be seen only by going beyond the second order approximation in the weak disorder expansion (see details in Ref. \([11]\)).

VI. THE NEIGHBOURHOOD OF THE BAND EDGE

Besides the \( k \to 0 \) limit considered in the previous section, one can also study the behaviour of the Lyapunov exponent \([19]\) in the complementary case \( k \to \infty \). Physically speaking, the limit \( k \to \infty \) can be interpreted in different ways, depending on the reference model. If one bears in mind the kicked oscillator \([4]\), then taking the limit \( k \to \infty \) is tantamount to studying the case of a very strong noise. More precisely, the condition \( k \gg 1 \) implies that the kicks play a predominant role in the oscillator dynamics with respect to the elastic force. Notice that this is not in contrast with the requirement that each single kick be weak, as established by the condition \([4]\). In fact, regardless of how weak the individual kicks are, their collective effect can be arbitrarily enhanced by making the interval \( T \) between two successive kicks sufficiently shorter than the fraction \( \langle A^2_n \rangle / \omega \) of the period of the unperturbed oscillator.

From the point of view of the Anderson model \([1]\), the analysis of the case \( k \gg 1 \) is equivalent to the study of the localization length in a neighbourhood of the band edge. To understand this point, one should consider that in the weak disorder case, as defined by the relation \([4]\), the only way to fulfill the condition \( k = \langle A^2_n \rangle / (\omega T) \to \infty \) is to have \( \omega T \to 0 \). Correspondingly, the energy \( E = 2 \cos(\omega T) \) must approach the limit \( E \to 2^− \), i.e. the edge of the band. Using relations \([4]\), one can also express the condition \( k \gg 1 \) in the significant form

\[
2 - E \ll \langle A^2_n \rangle / \omega T \to \infty \,
\]

which shows that the investigation of the case \( k \gg 1 \) corresponds to studying the behaviour of the inverse localization length for energy values which are close to the band edge on a distance scale set by the fluctuations of the random site-potential.

With the physical meaning of the limit \( k \to \infty \) clear in mind, we can proceed to verify that Eq. \([19]\) reproduces the correct behaviour of the Lyapunov exponent in a neighbourhood of the band edge. For this, it suffices to notice that the substitution

\[
k = \frac{\langle A^2_n \rangle}{\omega T \sin^2(\omega T)} \to k' = \frac{\langle A^2_n \rangle}{(\omega T)^3}
\]

transforms formula \([19]\) in the expression originally obtained by Derrida and Gardner for the Lyapunov exponent at the band edge \([10]\). This implies that Derrida and Gardner’s expression coincides with our own for \( \omega T \to 0 \), since in this limit the difference between parameters \( k \) and \( k' \) vanishes. The limit \( \omega T \to 0 \), on the other hand, identifies the band edge case: this proves that formula \([19]\) is correct not only inside the energy band (except that for \( E = 0 \)), but also for \( E \to 2 \). The extended validity range of expression \([19]\) is a relevant and novel feature; indeed, to the best of our knowledge, no other formula encompassing the whole energy band has been previously found for the Lyapunov exponent in the Anderson model.

To conclude our discussion of the \( k \to \infty \) limit, we observe that in this case it may be appropriate to expand the integrals that appear in expressions \([19]\) and \([7]\) in series of the inverse powers of \( k \). One thus obtains

\[
\lambda = \left( \frac{3}{4k^2} \right)^{1/3} \frac{\langle A^2_n \rangle}{T} \left( \sum_{n=0}^{\infty} \frac{(-2 \sqrt{6})^n}{n!} \Gamma \left( \frac{2n + 3}{6} \right) (k^{-1})^{2n/3} \right)
\]

which is the counterpart of the expansion \([21]\) of the preceding section. To the lowest order in \( k^{-1} \) this expression reduces to
\[ \lambda = \frac{1}{2} \frac{\sqrt{6}}{\Gamma(1/6)} \left( \frac{\omega T}{\sin(\omega T)} \right)^{2/3} \langle \epsilon_n^2 \rangle^{1/3}, \]

so that for \( E = 2 \) (i.e., for \( \omega T = 0 \)), the Lyapunov exponent turns out to be

\[ \lambda = \frac{1}{2} \frac{\sqrt{6}}{\Gamma(1/6)} \langle \epsilon_n^2 \rangle^{1/3}, \]

in perfect agreement with the result originally found in \( \text{(10)} \).

**VII. DYNAMICS OF ACTION-ANGLE VARIABLES**

To gain further insight in the dynamics of the stochastic oscillator \( \text{(3)} \), it is useful to consider its description in terms of action-angle variables. To this end, we introduce the polar coordinates \((r, \theta)\) through the standard relations

\[ r = \sqrt{x^2 + p^2}, \quad \theta = \arctan(x/p) \tag{25} \]

Starting from this definition and Eqs. \( \text{(3)} \), it is straightforward to obtain the following pair of stochastic Itô equations

\[
\begin{align*}
\frac{dr}{dt} &= \frac{(A_n^2)}{2T} \sin^4 \theta \ dt - \sqrt{\frac{(A_n^2)}{T}} \sin \theta \ cos \theta \ dW(t) \\
\frac{d\theta}{dt} &= \left( \omega + \frac{(A_n^2)}{T} \cos \theta \sin^3 \theta \right) dt + \sqrt{\frac{(A_n^2)}{T}} \sin^2 \theta \ dW(t) \tag{27}
\end{align*}
\]

A simple examination of Eqs. \( \text{(26)} \) and \( \text{(27)} \) reveals that, while the action variable is coupled to the angular one, the latter evolves in an independent way. Therefore, one can determine the probability distribution for the variable \( \theta \) by solving the Fokker-Planck equation associated to the Itô equation \( \text{(27)} \), \( \text{(3)} \), i.e.,

\[
\frac{\partial \rho}{\partial t}(\theta, t) = -\frac{\partial}{\partial \theta} \left[ \left( \omega + \frac{(A_n^2)}{T} \cos \theta \sin^3 \theta \right) \rho(\theta, t) \right] + \frac{(A_n^2)}{2T} \frac{\partial^2}{\partial \theta^2} \left[ (\sin^4 \theta) \rho(\theta, t) \right]. \tag{28}
\]

In geometrical terms, the introduction of the angular variable \( \theta \) is equivalent to the projection of the point \((x(t), p(t))\), representative of the stochastic oscillator \( \text{(3)} \), in the phase space, onto the unit circle by the relation \( \text{(25)} \). A glance at the Fokker-Planck equation \( \text{(28)} \) shows that the ensuing motion of the projected point \( \theta(t) \) is the combination of a drift with a noise-induced diffusion. In the Fokker-Planck equation \( \text{(28)} \) the diffusion term vanishes for \( \theta = 0 \) and \( \theta = \pi \), but the drift coefficient is not zero at these points (unless \( \omega \to 0 \)), so that it is reasonable to expect the invariant measure \( \rho(\theta) \) to be finite and non-vanishing at any point of the interval \([0 : 2\pi]\). Things are different when \( \omega \to 0 \), i.e., at the band edge, for in this case both the drift and the diffusion coefficient tend to vanish in a neighbourhood of \( \theta = 0 \) and \( \theta = \pi \). Therefore, for \( \omega \to 0 \), the projection of the representative point onto the unit circle tends to stick at the critical points \( \theta = 0 \) and \( \theta = \pi \); as a consequence, one can expect the invariant measure \( \rho(\theta) \) to be sharply peaked around these two values of \( \theta \) and almost zero everywhere else. This peculiar behaviour for \( \omega \to 0 \) gives an intuitive feeling for the origin of the anomalous scaling of the Lyapunov exponent with the noise strength at the band edge.

The stationary solutions of \( \text{(28)} \) are periodic in \( \theta \) with period \( \pi \). In the interval \([0 : \pi]\) the solution \( \rho(\theta) \) which is bounded as \( \theta \to \pi \) and is normalized with the condition \( \int_0^{2\pi} \rho(\theta) \, d\theta = 1 \) can be expressed in the integral form

\[
\rho(\theta) = \frac{1}{2N \sin^2 \theta} \int_\theta^\pi \frac{dx}{\sin^2 x} \exp \left\{ \Phi(\cot x) - \Phi(\cot \theta) \right\}, \tag{29}
\]

where \( \Phi \) is the function \( \text{(15)} \) and \( N \) is the normalization constant \( \text{(17)} \); outside the interval \([0 : \pi]\), the probability \( \rho(\theta) \) is defined through the periodicity condition \( \rho(\theta + k\pi) = \rho(\theta) \) with \( k \) integer.

Notice that at the points \( \theta = 0 \) and \( \theta = \pi \) the invariant measure \( \rho(\theta) \) takes the value \( \rho(0) = \rho(\pi) = k/4N \). Expanding expression \( \text{(17)} \) in power series of \( k \) and \( k^{-1} \), \( \text{(26)} \) it is possible to show that in the opposite limits \( k \to 0 \) and \( k \to \infty \) one has

\[
\begin{align*}
\rho(0) &= \rho(\pi) = \frac{k}{4N(k)} = \left\{ \begin{array}{ll}
\frac{1}{2\pi} & \text{for } k \to 0 \\
\frac{3}{5} & \text{for } \frac{1}{k^{2/3}} \to 0
\end{array} \right.
\end{align*}
\]

These expansions show that, upon increasing \( k \), the probability distribution \( \text{(24)} \) does become more and more sharply peaked at the points \( \theta = 0 \) and \( \theta = \pi \); the analysis of the function \( \rho(\theta) \) thus confirms the singular behaviour of the invariant measure that was suggested from physical considerations on the dynamics of the stochastic variable \( \theta(t) \).

Using \( \text{(29)} \) one can easily compute the Lyapunov exponent as defined by the expression \( \text{(13)} \). To this end, it suffices to notice that in polar coordinates the Ricatti variable takes the form \( z = \omega \cot \theta \) one can therefore evaluate the Lyapunov exponent as

\[
\lambda = \omega \langle \cot \theta \rangle = 2\omega \int_0^\pi \, d\theta \, \rho(\theta) \cot \theta = \frac{\omega}{2N} \int_0^{\pi} \frac{d\theta}{\sin^2 \theta} \cot \theta \exp \left\{ -\Phi(\cot \theta) \right\} \cdot \int_\theta^{\pi} \frac{dx}{\sin^2 x} \exp \left\{ \Phi(\cot x) \right\}.
\]
With a change of variables this expression can be reduced to the form (18), so that one eventually recovers the result (19) for the Lyapunov exponent. This is not surprising, since the distributions (16) and (29) are strictly related; indeed one has

$$\rho(\theta) = \frac{1}{2} p(z(\theta)) \left| \frac{dz}{d\theta} \right|,$$

where the factor $1/2$ derives from the normalization conditions chosen for the two distributions. From a geometrical point of view, the passage from the process $z(t) < +\infty$ onto the unit circle $0 < \theta(t) < \pi$ via the relation $\theta(t) = \cot^{-1}(z/\omega)$.

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