Cylindrical gravitational waves: radiation and resonance

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ABSTRACT: In the weak field approximation the gravitational wave is approximated as a linear wave, which ignores the nonlinear effect. In this paper, we present an exact general solution of the cylindrical gravitational wave. The exact solution of the cylindrical gravitational wave is far different from the weak field approximation. This solution implies the following conclusions. (1) There exist gravitational monopole radiations in the cylindrical gravitational radiation. (2) The gravitational radiation may generate the resonance in the spacetime. (3) The nonlinearity of the gravity source vanishes after time averaging, so the observed result of a long-time measurement may be linear.

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1 Introduction

General relativity is a nonlinear theory. The quadrupole radiation, the polarization, the spin of the graviton, however, are mostly obtained under the linear approximation [1, 2]. In the linear approximation, it is supposed that the energy and the momentum of the source are invariable. As a result, the monopole and the dipole of the source are invariable and make no contribution to the gravitational radiation [3]. Besides, when the spacetime is regarded as a Minkowski spacetime, the dipole vanishes in the center-of-mass coordinate [2]. Therefore, the leading contribution of the gravitational radiation is made by the variation of the quadrupole [1, 2], i.e., the gravitational quadrupole radiation.

In this paper, we present an exact cylindrical solution which implies a gravitational monopole radiation. The exact nonlinear solution is far different from the linear approximation. By the Birkhoff theorem we know that there is no gravitational wave in the spherical vacuum solution of the Einstein equation [1–3]. The simplest metric involving the gravitational radiation is the Einstein-Rosen metric. The source of an infinite length may exist in the Einstein-Rosen metric. It is convenient to discuss the gravitational radiation with the Einstein-Rosen metric, such as the energy loss of the source by the gravitational radiation [4], the C-energy, the super-energy, and the associated dynamical effect of the cylindrical gravitational wave [5], the energy-momentum of the gravitational wave [6], the superenergy flux of the Einstein-Rosen wave [7], the nonlinear effect such as the Faraday rotation and the time-shift phenomenon of the cylindrical gravitational soliton solution [8], the nonlinear evolution of cylindrical gravitational waves [9], the twisted gravitational wave [10], the scattering of the gravitational waves [11], the gravitational collapse of the energy of
gravitational waves [12], the asymptotic structure of the radiation spacetime [13], the interaction between the gravitational wave and the cosmic string [14, 15], the cosmic censorship hypothesis [16], and the midisuperspace quantization [17–19].

We also discuss the resonance of the gravitational radiation in the spacetime. In literature, the resonance between the gravitational radiation and the matter (especially the gravitational radiation detectors) are considered [20–23]. In this paper, we focus on the resonance of gravitational waves.

In the linear approximation, two gravitational radiations do not interact with each other. In this paper, we show the interaction between two cylindrical gravitational radiations. When two cylindrical gravitational radiations exist simultaneously, the interaction terms arise both in the metric and the energy-momentum tensor. Especially, we show that, though the interaction always exists in nonlinear gravity waves, the nonlinearity of the gravity source vanishes when taking a time average. This implies that the observed result of a long-time measurement may be linear, though the gravity is nonlinear.

In section 2, we present an exact general cylindrical gravitational wave solution. In section 3, we show the existence of the cylindrical gravitational monopole radiation. In section 4, we discuss the resonance of the gravitational radiation. In section 5, we show the interaction of two cylindrical gravitational radiations. The conclusion and outlooks are given in section 6.

2 Cylindrical gravitational wave: general solution

In this section, we present a general vacuum solution of the cylindrical gravitational wave. With the cylindrical gravitational wave solution, we discuss the gravitational monopole radiation, the resonance, and the interaction of the cylindrical gravitational radiations in the latter sections.

The cylindrical gravitational wave is described by the 1+3 dimensional Einstein-Rosen metric [24],

\[ ds^2 = e^{2\gamma(t,\rho)} - 2\psi(t,\rho) (-dt^2 + d\rho^2) + e^{-2\psi(t,\rho)} \rho^2 d\phi^2 + e^{2\psi(t,\rho)} dz^2. \]  

(2.1)

The Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \) in the orthogonal frame [25] with \( \eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1) \) is

\[ G_{00} = G_{11} = e^{2\psi-2\gamma} \left[ \frac{1}{\rho} \frac{\partial \gamma}{\partial \rho} - \left( \frac{\partial \psi}{\partial \rho} \right)^2 - \left( \frac{\partial \psi}{\partial t} \right)^2 \right], \]  

(2.2)

\[ G_{01} = G_{10} = e^{2\psi-2\gamma} \left[ \frac{1}{\rho} \frac{\partial \gamma}{\partial t} - 2 \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial \rho} \right], \]  

(2.3)

\[ G_{22} = e^{2\psi-2\gamma} \left[ \left( \frac{\partial \psi}{\partial \rho} \right)^2 - \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{\partial^2 \gamma}{\partial \rho^2} - \frac{\partial^2 \gamma}{\partial t^2} \right], \]  

(2.4)

\[ G_{33} = e^{2\psi-2\gamma} \left[ \left( \frac{\partial \psi}{\partial \rho} \right)^2 - \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{\partial^2 \gamma}{\partial \rho^2} + \frac{\partial^2 \gamma}{\partial t^2} + 2 \frac{\partial^2 \psi}{\partial \rho^2} - 2 \frac{\partial \psi}{\rho} \frac{\partial \psi}{\partial \rho} \right]. \]  

(2.5)
We consider the gravitational field outside the source, i.e., $\rho \neq 0$, which satisfies $G_{\mu \nu} = 0$. For $G_{\mu \nu} = 0$, eqs. (2.2), (2.3), (2.4), and (2.5) can be simplified as [24]

\[
\frac{\partial \gamma}{\partial \rho} = \rho \left( \frac{\partial \psi}{\partial t} \right)^2 + \rho \left( \frac{\partial \psi}{\partial \rho} \right)^2, \quad (2.6)
\]

\[
\frac{\partial \gamma}{\partial t} = 2\rho \frac{\partial \psi}{\partial t} \frac{\partial \psi}{\partial \rho}. \quad (2.7)
\]

For gravitational wave solutions, we require $\frac{\partial \psi}{\partial t} \neq 0$. Then we obtain the equation of $\psi$ from eqs. (2.6) and (2.7) [24],

\[
- \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = 0. \quad (2.8)
\]

The equation of $\psi$ (2.8) is a linear equation. The general solution of eq. (2.8) is

\[
\psi = \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} d\lambda A(\tau, \lambda) J_0(\lambda \rho) \cos(\lambda (t - \tau) + \alpha(\tau, \lambda)) + \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} d\lambda B(\tau, \lambda) Y_0(\lambda \rho) \cos(\lambda (t - \tau) + \beta(\tau, \lambda)) + \kappa_1 t + \kappa_2 \ln \rho + \kappa_0, \quad (2.9)
\]

where $\alpha(\tau, \lambda), \beta(\tau, \lambda), A(\tau, \lambda),$ and $B(\tau, \lambda)$ are arbitrary functions of $\tau$ and $\lambda$, $J_0$ is the Bessel function of first kind, $Y_0$ is the Bessel function of second kind, and $\kappa_0, \kappa_1,$ and $\kappa_2$ are constants. From eqs. (2.6) and (2.7) we can see that the equations of $\gamma$ are also linear equations. Substituting eq. (2.9) into eqs. (2.6) and (2.7) gives the general solution of $\gamma$:

\[
\gamma = -\frac{\rho}{2} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \int_{0}^{\infty} d\lambda_1 \int_{0}^{\infty} d\lambda_2 (F_+ + F_-) - 2\kappa_1 \rho \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} d\lambda [A(\tau, \lambda) J_1(\lambda \rho) \sin(\lambda (t - \tau) + \alpha(\tau, \lambda)) + B(\tau, \lambda) Y_1(\lambda \rho) \sin(\lambda (t - \tau) + \beta(\tau, \lambda))] - 2\kappa_2 \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} d\lambda [A(\tau, \lambda) J_1(\lambda \rho) \cos(\lambda (t - \tau) + \alpha(\tau, \lambda)) + B(\tau, \lambda) Y_1(\lambda \rho) \cos(\lambda (t - \tau) + \beta(\tau, \lambda))] + 2\kappa_1 \kappa_2 t + \frac{1}{2} \kappa_1^2 \rho^2 + \kappa_2^2 \ln \rho + \kappa_3 \quad (2.10)
\]

with

\[
F_+ = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \left\{ A(\tau_1, \lambda_1) A(\tau_2, \lambda_2) [J_0(\lambda_1 \rho) J_1(\lambda_2 \rho) + J_0(\lambda_2 \rho) J_1(\lambda_1 \rho)] \right. \\
\times \cos(\lambda_1 \tau + \lambda_2 t - \lambda_1 \tau_1 - \lambda_2 \tau_2 + \alpha(\tau_1, \lambda_1) + \alpha(\tau_2, \lambda_2)) + A(\tau_1, \lambda_1) B(\tau_2, \lambda_2) [J_0(\lambda_1 \rho) Y_1(\lambda_2 \rho) + Y_0(\lambda_2 \rho) J_1(\lambda_1 \rho)] \\
\times \cos(\lambda_1 \tau + \lambda_2 t - \lambda_1 \tau_1 - \lambda_2 \tau_2 + \alpha(\tau_1, \lambda_1) + \beta(\tau_2, \lambda_2)) + B(\tau_1, \lambda_1) A(\tau_2, \lambda_2) [Y_0(\lambda_1 \rho) J_1(\lambda_2 \rho) + J_0(\lambda_2 \rho) Y_1(\lambda_1 \rho)] \\
\times \cos(\lambda_1 \tau + \lambda_2 t - \lambda_1 \tau_1 - \lambda_2 \tau_2 + \beta(\tau_1, \lambda_1) + \alpha(\tau_2, \lambda_2)) + B(\tau_1, \lambda_1) B(\tau_2, \lambda_2) [Y_0(\lambda_1 \rho) Y_1(\lambda_2 \rho) + Y_0(\lambda_2 \rho) Y_1(\lambda_1 \rho)] \\
\times \cos(\lambda_1 \tau + \lambda_2 t - \lambda_1 \tau_1 - \lambda_2 \tau_2 + \beta(\tau_1, \lambda_1) + \beta(\tau_2, \lambda_2)) \right\} \quad (2.11)
\]
2.10

and

\[
F_- = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \{A (\tau_1, \lambda_1) A (\tau_2, \lambda_2) [J_0 (\lambda_1 \rho) J_1 (\lambda_2 \rho) - J_0 (\lambda_2 \rho) J_1 (\lambda_1 \rho)] \\
\times \cos (\lambda_1 t - \lambda_2 t - \lambda_1 \tau_1 + \lambda_2 \tau_2 + \alpha (\tau_1, \lambda_1) - \alpha (\tau_2, \lambda_2)) \\
+ A (\tau_1, \lambda_1) B (\tau_2, \lambda_2) [J_0 (\lambda_1 \rho) Y_1 (\lambda_2 \rho) - Y_0 (\lambda_2 \rho) J_1 (\lambda_1 \rho)] \\
\times \cos (\lambda_1 t - \lambda_2 t - \lambda_1 \tau_1 + \lambda_2 \tau_2 + \alpha (\tau_1, \lambda_1) - \beta (\tau_2, \lambda_2)) \\
+ B (\tau_1, \lambda_1) A (\tau_2, \lambda_2) [Y_0 (\lambda_1 \rho) J_1 (\lambda_2 \rho) - J_0 (\lambda_2 \rho) Y_1 (\lambda_1 \rho)] \\
\times \cos (\lambda_1 t - \lambda_2 t - \lambda_1 \tau_1 + \lambda_2 \tau_2 + \beta (\tau_1, \lambda_1) - \alpha (\tau_2, \lambda_2)) \\
+ B (\tau_1, \lambda_1) B (\tau_2, \lambda_2) [Y_0 (\lambda_1 \rho) Y_1 (\lambda_2 \rho) - Y_0 (\lambda_2 \rho) Y_1 (\lambda_1 \rho)] \\
\times \cos (\lambda_1 t - \lambda_2 t - \lambda_1 \tau_1 + \lambda_2 \tau_2 + \beta (\tau_1, \lambda_1) - \beta (\tau_2, \lambda_2)) \}.
\]

(2.12)

Here \(\kappa_3\) in eq. (2.10) can be eliminated by a coordinate transformation and, without loss of generality, we set \(\kappa_3 = 0\).

The solutions (2.9) and (2.10) are the general solutions of the cylindrical gravitational wave. The solutions given in ref. [24] and the solutions mentioned after are particular cases of the solutions (2.9) and (2.10).

3 Gravitational monopole radiation

In this section, we give an exact cylindrical gravitational monopole radiation solution. The radiation is a wave produced by a source. For example, the plane electromagnetic wave is not an electromagnetic radiation and the electromagnetic wave produced by the antenna is an electromagnetic radiation. The gravitational radiation can be recognized by observing if a wave solution has a time-varying source.

Next (1) we first choose a particular solution and show that the solution is a gravitational wave solution. (2) We separate the monopole radiation solution from this gravitational wave solution.

3.1 Gravitational wave solution

In this section, we present an exact solution, and we show that the solution is a gravitational wave by investigating the Weinberg energy-momentum pseudotensor [1] and the Laudau-Lifscitz energy-momentum pseudotensor [26]. The energy-momentum pseudotensors are defined to describe the energy-momentum of the gravitational field. If the energy-momentum pseudotensor of the gravitational field is time-varying, we say that the solution represents a gravitational wave [26].

Taking

\[
A (\tau, \lambda) = A \delta (\omega - \lambda) \delta (\tau), \\
B (\tau, \lambda) = B \delta (\omega - \lambda) \delta (\tau), \\
\kappa_0 = \kappa_1 = \kappa_2 = 0
\]

(3.1)
in eqs. (2.9) and (2.10), where $\omega \neq 0$ is a constant, we obtain a particular solution
\[
\psi = AJ_0 (\omega \rho) \cos (\omega t + \alpha) + BY_0 (\omega \rho) \cos (\omega t + \beta),
\]  
with $\alpha \equiv \alpha (0, \omega)$, $\beta \equiv \beta (0, \omega)$, and
\[
f (\rho) = \frac{A^2}{4} \omega^2 \rho^2 \left[ J_0^2 (\omega \rho) + 2 J_1^2 (\omega \rho) - J_0 (\omega \rho) J_2 (\omega \rho) \right]
\]
\[+ \frac{B^2}{4} \omega^2 \rho^2 \left[ Y_0^2 (\omega \rho) + 2 Y_1^2 (\omega \rho) - Y_0 (\omega \rho) Y_2 (\omega \rho) \right]
\]
\[+ \frac{AB}{4} \omega^2 \rho^2 \left[ 2 J_0 (\omega \rho) Y_0 (\omega \rho) + 4 J_1 (\omega \rho) Y_1 (\omega \rho) - J_0 (\omega \rho) Y_2 (\omega \rho) - J_2 (\omega \rho) Y_0 (\omega \rho) \right] \cos (\alpha - \beta). \]  

Note here that $\psi$ given by eq. (3.2) has been given in ref. [24]; $\gamma$ given by eq. (3.3) is obtained in the present paper.

In the Cartesian coordinates
\[
x = \rho \cos \phi, \\
y = \rho \sin \phi,
\]  
the Einstein-Rosen metric reads
\[
ds^2 = -e^{2\gamma-2\psi} dt^2 + \frac{e^{-2\psi}}{\rho^2} \left( x^2 e^{2\gamma} + y^2 \right) dx^2 + \frac{e^{-2\psi}}{\rho^2} \left( y^2 e^{2\gamma} + x^2 \right) dy^2
\]
\[+ \frac{2xy}{\rho^2} e^{-2\psi} (e^{2\gamma} - 1) dxdy + e^{2\psi} dz^2.
\]  

It can be checked by the numerical method that the metric (3.6) with $\psi$ given by eq. (3.2) and $\gamma$ given by eq. (3.3) has a time-varying Weinberg energy-momentum pseudotensor [1] and a time-varying Laudau-Lifscitz energy-momentum pseudotensor [26].

Besides, there is a special case of the solutions (3.2) and (3.3) satisfying the out-going wave condition: when $t - r = \text{const}$, $\psi$ and $\gamma$ remain unchanged at $\rho \to \infty$. When
\[
A = B,
\]
\[
\beta = \alpha - \frac{\pi}{2},
\]  
the approximation of $\psi$ and $\gamma$ at $\rho \to \infty$ are
\[
\psi = A \sqrt{\frac{2}{\pi \omega \rho}} \cos \left( \omega t - \omega \rho + \alpha + \frac{1}{4} \right),
\]
\[
\gamma = A^2 \frac{\pi}{\pi} \left[ \cos (2\omega t - 2\omega \rho + 2\alpha) - 2 (\omega t - \omega \rho) \right]. \]  

In addition, it can be found that when the out-going wave condition is satisfied, the curvature of the spacetime vanishes at $\rho \to \infty$. That is, the spacetime is asymptotically flat in spite of that the metric is not asymptotic to the Minkowski metric.
3.2 Gravitational radiation

In this section, we show that the solutions (3.2) and (3.3) contain a gravitational radiation. As mentioned above, the gravitational radiation is a gravitational wave solution with a source.

The solutions (3.2) and (3.3) have two special cases.

$A = 0$:

$$
\psi_{\text{rad}} = BY_0 (\omega \rho) \cos (\omega t + \beta), \quad (3.9)
$$

$$
\gamma_{\text{rad}} = \frac{B^2}{4} \omega^2 \rho^2 \left[ Y_0^2 (\omega \rho) + 2Y_1^2 (\omega \rho) - Y_0 (\omega \rho) Y_2 (\omega \rho) \right] - \frac{B^2}{2} \omega \rho Y_0 (\omega \rho) Y_1 (\omega \rho) \cos (2\omega t + 2\beta), \quad (3.10)
$$

$B = 0$:

$$
\psi_{\text{nrad}} = AJ_0 (\omega \rho) \cos (\omega t + \alpha), \quad (3.11)
$$

$$
\gamma_{\text{nrad}} = \frac{A^2}{4} \omega^2 \rho^2 \left[ J_0^2 (\omega \rho) + 2J_1^2 (\omega \rho) - J_0 (\omega \rho) J_2 (\omega \rho) \right] - \frac{A^2}{2} \omega \rho J_0 (\omega \rho) J_1 (\omega \rho) \cos (2\omega t + 2\alpha). \quad (3.12)
$$

These two cases, eqs. (3.9), (3.10), (3.11), and (3.12), have been given in ref. [24].

In this paper, we point out that $\psi_{\text{rad}}$ and $\gamma_{\text{rad}}$ reprent radiations, but $\psi_{\text{nrad}}$ and $\gamma_{\text{nrad}}$ do not reprent radiations. Here we use the subscripts "rad" to denote the gravitational radiation and use "nrad" to denote the nonradiation gravitational wave.

Next we show that $\psi_{\text{rad}}$ and $\gamma_{\text{rad}}$ describe a gravitational monopole radiation, which has a time-varying energy density or a monopole, and $\psi_{\text{nrad}}$ and $\gamma_{\text{nrad}}$ are pure gravitational waves without sources.

The solutions (3.2) and (3.3) represent a gravitational wave in the vacuum for $\rho > 0$. Nevertheless, when $\rho = 0$, there may exist a source. The singularity in the solutions (3.2) and (3.3), according to ref. [24], might be interpreted as a matter presented along the $z$ axis. Just as the Coulomb potential $1/r$, when $r \neq 0$, the solution is a vacuum solution; nevertheless, there is a point charge at $r = 0$. We use a standard mathematical analysis method to calculate the energy-momentum tensor at $\rho = 0$. More details of this method can be found in our previous work [27]. Replacing $\rho$ by $\sqrt{\rho^2 + \epsilon^2}$ in eqs. (3.2) and (3.3) and substituting the metric (2.1) into eqs. (2.2), (2.3), (2.4) and (2.5), by the Einstein equation $G_{\mu \nu} = 8\pi T_{\mu \nu}$, we arrive at

$$
T_{00} (\omega, \epsilon) = T_{11} (\omega, \epsilon) = \frac{e^{2\psi - 2\gamma}}{8\pi} \frac{\omega^2 \epsilon^2}{\rho^2 + \epsilon^2} \left[ AJ_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos (\omega t + \alpha) + BY_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos (\omega t + \beta) \right]^2, \quad (3.13)
$$

$$
T_{01} (\omega, \epsilon) = \frac{e^{2\psi - 2\gamma}}{8\pi} \frac{2\omega^2 \epsilon^2}{\rho \sqrt{\rho^2 + \epsilon^2}} \left[ AJ_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \sin (\omega t + \alpha) + BY_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \sin (\omega t + \beta) \right] \times \left[ AJ_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos (\omega t + \alpha) + BY_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos (\omega t + \beta) \right], \quad (3.14)
$$

- 6 -
\[ T_{22}(\omega, \epsilon) = \frac{e^{2\psi-2\gamma}}{8\pi} \frac{\omega^2 \epsilon^2}{\rho^2 + \epsilon^2} \left[ A J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \alpha) + B Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \beta) \right]^2 \\
- \frac{e^{2\psi-2\gamma}}{8\pi} \frac{2\omega^3 \epsilon^2}{\sqrt{\rho^2 + \epsilon^2}} \left[ A^2 J_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + 2\alpha) \\
+ B^2 Y_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + 2\beta) \right] \\
- \frac{e^{2\psi-2\gamma}}{8\pi} \frac{2\omega^3 \epsilon^2}{\sqrt{\rho^2 + \epsilon^2}} A B [J_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \\
+ J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + \alpha + \beta) \right], \quad (3.15) \]

\[ T_{33}(\omega, \epsilon) = \frac{e^{2\psi-2\gamma}}{8\pi} \frac{\omega^2 \epsilon^2}{\rho^2 + \epsilon^2} \left[ A J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \alpha) + B Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \beta) \right]^2 \\
+ \frac{e^{2\psi-2\gamma}}{8\pi} \frac{2\omega^2 \epsilon^2}{\rho^2 + \epsilon^2} \left[ A J_2 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \alpha) + B Y_2 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(\omega t + \beta) \right] \\
- 2 \frac{e^{2\psi-2\gamma}}{8\pi} \frac{\omega^3 \epsilon^2}{\sqrt{\rho^2 + \epsilon^2}} \left[ A^2 J_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + 2\alpha) \\
+ B^2 Y_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + 2\beta) \right] \\
- 2 \frac{e^{2\psi-2\gamma}}{8\pi} \frac{\omega^3 \epsilon^2}{\sqrt{\rho^2 + \epsilon^2}} A B [J_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \\
+ J_1 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) Y_0 \left( \omega \sqrt{\rho^2 + \epsilon^2} \right) \cos(2\omega t + \alpha + \beta) \right]. \quad (3.16) \]

The energy-momentum tensor is given by

\[ T_{\mu\nu}(\omega) = \lim_{\epsilon \to 0} T_{\mu\nu}(\omega, \epsilon). \quad (3.17) \]

When \( \rho \neq 0 \), we have

\[ T_{\mu\nu}(\omega) = \lim_{\epsilon \to 0} T_{\mu\nu}(\omega, \epsilon) = 0. \quad (3.18) \]

When \( \rho = 0 \), using

\[ \lim_{z \to 0} J_0(z) = 1, \quad \lim_{z \to 0} Y_0(z) = \frac{2}{\pi} \ln z, \]
\[ \lim_{z \to 0} J_\nu(z) = \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu, \quad \nu \neq 0, \]
\[ \lim_{z \to 0} Y_\nu(z) = -\frac{\Gamma(\nu)}{\pi} \left( \frac{z}{2} \right)^{-\nu}, \quad \nu \neq 0, \quad (3.19) \]
we have

$$T_{00} (\omega) = \lim_{\epsilon \to 0} T_{00} (\omega, \epsilon) = T_{11} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \cos^2 (\omega t + \beta) \lim_{\epsilon \to 0} \frac{\epsilon^2}{(\rho^2 + \epsilon^2)^2},$$

$$T_{01} (\omega) = \lim_{\epsilon \to 0} T_{01} (\omega, \epsilon) = -e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \sin (2\omega t + 2\beta) \lim_{\epsilon \to 0} \frac{\omega \epsilon^2}{\rho (\rho^2 + \epsilon^2)} \ln \left( \omega \sqrt{\rho^2 + \epsilon^2} \right),$$

$$T_{22} (\omega) = \lim_{\epsilon \to 0} T_{22} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \cos^2 (\omega t + \beta) \lim_{\epsilon \to 0} \frac{\epsilon^2}{(\rho^2 + \epsilon^2)^2},$$

$$T_{33} (\omega) = \lim_{\epsilon \to 0} T_{33} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \left[ \cos^2 (\omega t + \beta) - \frac{2\pi}{B} \cos (\omega t + \beta) \right] \lim_{\epsilon \to 0} \frac{\epsilon^2}{(\rho^2 + \epsilon^2)^2}.$$

By use of [28]

$$\lim_{\epsilon \to 0} \frac{\epsilon^2}{(x^2 + y^2 + \epsilon^2)^2} = \pi \delta (x) \delta (y),$$

we have

$$T_{00} (\omega) = T_{11} (\omega, \epsilon) = \lim_{\epsilon \to 0} T_{00} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \delta (x) \delta (y) \cos^2 (\omega t + \beta), \quad (3.21)$$

$$T_{22} (\omega) = \lim_{\epsilon \to 0} T_{22} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^2} \delta (x) \delta (y) \cos^2 (\omega t + \beta), \quad (3.22)$$

$$T_{33} (\omega) = \lim_{\epsilon \to 0} T_{33} (\omega, \epsilon) = e^{2\psi - 2\gamma} \frac{B^2}{2\pi^3} \delta (x) \delta (y) \left[ \cos^2 (\omega t + \beta) - \frac{2\pi}{B} \cos (\omega t + \beta) \right]. \quad (3.23)$$

Because

$$\lim_{\epsilon \to 0} \frac{T_{01} (\omega, \epsilon)}{T_{00} (\omega, \epsilon)} = 0,$$

$$T_{01} (\omega)$$ should be regraded as zero:

$$T_{01} (\omega) = 0. \quad (3.24)$$

$$T_{01} (\omega) = 0$$ can be explained from a different way which we will show below.

Integrating $T_{00} (\omega, \epsilon)$ over the whole space

$$E (\omega, \epsilon) \equiv \int \sqrt{g} \rho d\theta d\phi T_{00} (\omega, \epsilon)$$

$$= \frac{\omega^2 e^2 L_z}{8} \left[ A^2 (J_0^2 (\omega \epsilon) + J_1^2 (\omega \epsilon)) \cos^2 (\omega t + \alpha) + B^2 (Y_0^2 (\omega \epsilon) + Y_1^2 (\omega \epsilon)) \cos^2 (\omega t + \beta) + 2AB (J_0 (\omega \epsilon) Y_0 (\omega \epsilon) + J_1 (\omega \epsilon) Y_1 (\omega \epsilon)) \cos (\omega t + \alpha) \cos (\omega t + \beta) \right] \quad (3.25)$$

and taking the limit $\epsilon \to 0$ give

$$E (\omega) \equiv \lim_{\epsilon \to 0} E (\omega, \epsilon) = \frac{B^2 L_z}{2\pi^2} \cos^2 (\omega t + \beta) \quad (3.26)$$
with \( L_z \equiv \int_{-\infty}^{\infty} dz \). Again, integrating \( T_{01} (\epsilon) \) over the whole space

\[
P_\rho (\omega, \epsilon) = \int \sqrt{\gamma} dp d\theta d\phi T_{01} (\omega, \epsilon) \]

\[
= \frac{\omega^2 \epsilon^2 L_z}{2} \left[ A^2 (\omega J_0^2 (\omega) + \omega J_1^2 (\omega) - J_0 (\omega) J_1 (\omega) ) \sin (\omega t + \alpha) \cos (\omega t + \alpha) \\
+ B^2 (\omega Y_0^2 (\omega) + \omega Y_1^2 (\omega) - Y_0 (\omega) Y_1 (\omega) ) \sin (\omega t + \beta) \cos (\omega t + \beta) \\
+ AB (\omega J_0 (\omega) Y_0 (\omega) + \omega J_1 (\omega) Y_1 (\omega) - J_0 (\omega) Y_0 (\omega) ) \cos (\omega t + \alpha) \sin (\omega t + \beta) \\
+ AB (\omega J_0 (\omega) Y_0 (\omega) + \omega J_1 (\omega) Y_1 (\omega) - J_0 (\omega) Y_1 (\omega) ) \sin (\omega t + \alpha) \cos (\omega t + \beta) \right]
\]

and taking the limit \( \epsilon \to 0 \) give

\[
P_\rho (\omega) \equiv \lim_{\epsilon \to 0} P_\rho (\omega, \epsilon) = \lim_{\epsilon \to 0} \frac{2B^2 L_z \omega^2}{\pi^2} \epsilon \ln \epsilon = 0.
\]

Then the energy-momentum tensor of the metric (2.1) with \( \psi \) given by eq. (3.2) and \( \gamma \) given by eq. (3.3) is

\[
T_{\mu\nu} = e^{2\psi - 2\gamma} B^2 \frac{\pi^2}{2} \delta (x) \delta (y) \\
\times \text{diag} \left( \cos^2 (\omega t + \beta), \cos^2 (\omega t + \beta), \cos^2 (\omega t + \beta), \cos^2 (\omega t + \beta) - \frac{2\pi}{B} \cos (\omega t + \beta) \right)
\]

(3.29)

with \( x = \rho \cos \phi \) and \( y = \rho \sin \phi \).

The energy-momentum tensor corresponding to \( \psi_{\text{rad}} \) and \( \gamma_{\text{rad}} \) in eqs. (3.11) and (3.12) is zero and the energy-momentum tensor corresponding to \( \psi_{\text{rad}} \) and \( \gamma_{\text{rad}} \), eq. (3.29), is not zero. In other words, the solutions with \( \psi_{\text{rad}} \) and \( \gamma_{\text{rad}} \) in eqs. (3.9) and (3.10) have a time-varying energy density. In general relativity, the total energy itself is the monopole of the source. The solution with \( \psi_{\text{rad}} \) and \( \gamma_{\text{rad}} \) describes a gravitational monopole radiation with the time-varying total energy of the source (3.29). The solution with \( \psi_{\text{rad}} \) and \( \gamma_{\text{rad}} \) is a wave solution without sources, which has no singularity and is just like a plane electromagnetic wave.

There is a problem in the gravitational quadrupole radiation. If insisting that the energy of the source is invariable in a process of gravitational radiations, where the energy of the gravitational radiation comes form. If the energy of the gravitational radiation comes from the source, then the total energy of the source should decrease with time instead of being invariable. In general relativity, the charge of the gravitational field is the energy density, just like the charge of the electromagnetic field is the electric charge density. The monopole in general relativity is the integration of the energy density, i.e., the total energy. The contribution of the gravitational radiation should be made by the variation of the monopole.

4 Resonance of cylindrical radiation

In the following we show that the solution given above contains resonance structures.
To see this, we take
\[ A(\tau, \lambda) = A\delta(\omega_1 - \lambda) \delta(\tau), \]
\[ B(\tau, \lambda) = B\delta(\omega_2 - \lambda) \delta(\tau), \]
\[ \kappa_1 = \kappa_2 = \kappa_3 = 0 \] (4.1)
in the general solution (2.9) and (2.10):
\[ \psi = A_1 J_0(\omega_1 \rho) \cos(\omega_1 t + \alpha_1) + B_2 Y_0(\omega_2 \rho) \cos(\omega_2 t + \beta_2), \] (4.2)
\[ \gamma = -\frac{A_1^2}{2} \omega_1 \rho J_0(\omega_1 \rho) J_1(\omega_1 \rho) \cos(2\omega_1 t + 2\alpha_1) - \frac{B_2^2}{2} \omega_2 \rho Y_0(\omega_2 \rho) Y_1(\omega_2 \rho) \cos(2\omega_2 t + 2\beta_2) \]
\[ - A_1 B_2 \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \rho [J_0(\omega_1 \rho) Y_1(\omega_2 \rho) + Y_0(\omega_2 \rho) J_1(\omega_1 \rho)] \cos(\omega_1 t + \omega_2 t + \alpha_2 + \beta_2) \]
\[ + \frac{A_1^2}{4} \frac{\omega_1^2 \rho^2}{\omega_1} J_0^2(\omega_1 \rho) + 2J_1^2(\omega_1 \rho) - J_0(\omega_1 \rho) J_2(\omega_1 \rho) \]
\[ + \frac{B_2^2}{4} \frac{\omega_2^2 \rho^2}{\omega_2} Y_0^2(\omega_2 \rho) + 2Y_1^2(\omega_2 \rho) - Y_0(\omega_2 \rho) Y_2(\omega_2 \rho) \]
\[ - A_1 B_2 \frac{\omega_1 \omega_2}{\omega_1 - \omega_2} \rho [J_0(\omega_1 \rho) Y_1(\omega_2 \rho) - Y_0(\omega_2 \rho) J_1(\omega_1 \rho)] \cos(\omega_1 t - \omega_2 t + \alpha_1 - \beta_2), \] (4.3)
where \( A_1 = A(0, \omega_1), B_2 = B(0, \omega_2), \alpha_1 = \alpha(0, \omega_1), \) and \( \beta_2 = \beta(0, \omega_2). \)

The resonance occurs when \( \omega_1 = \omega_2. \) When \( \omega_1 \rightarrow \omega_2, \gamma \) diverges due to the existence of the factor \( \frac{1}{\omega_1 - \omega_2}. \) Taking \( \epsilon = \omega_1 - \omega_2 \rightarrow 0, \) we have
\[ \lim_{\omega_2 \rightarrow \omega_1} \gamma = \lim_{\epsilon \rightarrow 0} \gamma \sim -\frac{2A_1 B_2}{\pi} \omega_1 \cos(\alpha_1 - \beta_2) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}. \] (4.4)

The resonance also appears in eq. (3.3). In eqs. (3.2) and (3.3), the radiation part and the nonradiation part have the same frequency \( \omega. \) When \( A \neq 0 \) and \( B \neq 0, \gamma \) in eq. (3.3) has an aperiodic term being proportional to \( t. \) This is also a resonance which means the impact stores up and increase with the time \( t. \)

The solutions (4.2) and (4.3) show that the radiation part (with the frequency \( \omega_2 \)) resonates with the nonradiation part (with the frequency \( \omega_1 \)).

5 Interaction between radiations

In this section, we consider the interaction between the radiations. The theory of gravity is a nonlinear theory, so gravitational radiations in principle interact with each other.

5.1 Interaction term in metric

The factor \( \psi \) in the metric (2.1) satisfies a linear equation, eq. (2.8), so
\[ \psi = \psi_1 + \psi_2 \] (5.1)
is also a solution when \( \psi_1 \) and \( \psi_2 \) are the solutions of eq. (2.8). For convenience, we take \( \psi_1 \) and \( \psi_2 \) given by eq. (3.9) as an example,
\[ \psi_1 = B_1 Y_0(\omega_1 \rho) \cos(\omega_1 t + \beta_1), \]
\[ \psi_2 = B_2 Y_0(\omega_2 \rho) \cos(\omega_2 t + \beta_2). \] (5.2)
These two radiations have the frequencies $\omega_1$ and $\omega_2$, respectively. Substituting eq. (5.1) into eqs. (2.6) and (2.7), we have

$$\gamma = \gamma_1 + \gamma_2 - \gamma_{\text{int}},$$

where $\gamma_1$ and $\gamma_2$ are given by eq. (3.10) with frequencies $\omega_1$ and $\omega_2$ and

$$\gamma_{\text{int}} = B_1 B_2 \rho \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \left[ Y_0 (\omega_1 \rho) Y_1 (\omega_2 \rho) + Y_1 (\omega_1 \rho) Y_0 (\omega_2 \rho) \right] \cos (\omega_1 t + \omega_2 t + \beta_1 + \beta_2)$$

$$+ B_1 B_2 \rho \frac{\omega_1 \omega_2}{\omega_1 - \omega_2} \left[ Y_0 (\omega_1 \rho) Y_1 (\omega_2 \rho) - Y_0 (\omega_2 \rho) Y_1 (\omega_1 \rho) \right] \cos (\omega_1 t - \omega_2 t + \beta_1 - \beta_2).$$

$\gamma_{\text{int}}$ can be understood as an interaction between two gravitational radiations.

It should be emphasized that in this case when $\omega_1 = \omega_2 = \omega$

$$\gamma_{\text{int}} = B_1 B_2 \rho Y_0 (\omega \rho) Y (\omega \rho) \cos (2\omega t + \beta_1 + \beta_2)$$

$$- \frac{B_1 B_2}{2} \omega^2 \rho^2 \left[ Y_0^2 (\omega \rho) + 2 Y_1^2 (\omega \rho) - Y_0 (\omega \rho) Y_2 (\omega \rho) \right] \cos (\beta_1 - \beta_2)$$

does not diverge. Two radiations do not resonate. The resonance occurs only between the radiation part and the non-radiation part.

5.2 Interaction term in energy-momentum tensor

Now we calculate the energy-momentum tensor of the radiations (5.1) and (5.3).

By the same procedure in section 2, by eqs. (3.21), (3.22), (3.23), we have

$$T_{00} (\omega_1, \omega_2) = T_{11} (\omega_1, \omega_2) = T_{00} (\omega_1) + T_{00} (\omega_2) + T_{\text{int}} (\omega_1, \omega_2),$$

$$T_{01} (\omega_1, \omega_2) = T_{10} (\omega_1, \omega_2) = 0,$$

$$T_{22} (\omega_1, \omega_2) = T_{22} (\omega_1) + T_{22} (\omega_2) + T_{\text{int}} (\omega_1, \omega_2),$$

$$T_{33} (\omega_1, \omega_2) = T_{33} (\omega_1) + T_{33} (\omega_2) + T_{\text{int}} (\omega_1, \omega_2)$$

with the interaction term

$$T_{\text{int}} (\omega_1, \omega_2) = e^{2\psi - 2\gamma} \frac{B_1 B_2}{\pi^2} \delta (x) \delta (y) \cos (\omega_1 t + \beta_1) \cos (\omega_2 t + \beta_2).$$

It can be seen that $T_{\mu\nu} (\omega_1, \omega_2)$ which involves both $\omega_1$ and $\omega_2$ can be written in three parts: $T_{\mu\nu} (\omega_1)$ which involves only $\omega_1$, $T_{\mu\nu} (\omega_2)$ which involves only $\omega_2$, and $T_{\text{int}} (\omega_1, \omega_2)$ which involves both $\omega_1$ and $\omega_2$. Here $T_{\text{int}} (\omega_1, \omega_2)$ is an interaction term due to the nonlinearity of the gravity.

Nevertheless, though the nonlinearity of the gravity leads to the existence of the interaction term $T_{\text{int}} (\omega_1, \omega_2)$, the time average of $T_{\text{int}}$ vanishes:

$$\langle T_{\text{int}} \rangle = \int_0^\infty dt T_{\text{int}} (\omega_1, \omega_2) = 0.$$

The vanishing of the time average of interaction term implies that the result of a long-time measurement may be linear. That is, when the typical time of a detector is more larger
than the period of a gravitational wave, the source of radiations observed may be a linear one though the gravitational wave is nonlinear.

As an analogy, we consider a simple example of an energy superposition of two plane electromagnetic waves. When two plane electromagnetic waves $E_1 = A_1 \cos(\omega_1 t + \alpha)$ and $E_2 = A_2 \cos(\omega_2 t + \beta)$ superpose together, the electric field is

$$E = E_1 + E_2 = A_1 \cos(\omega_1 t + \alpha) + A_2 \cos(\omega_2 t + \beta),$$

where $E$ is the electric field and $A$ is the amplitude. The energy density of the electric field is

$$\varepsilon = E^2 = (E_1 + E_2)^2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_{\text{int}},$$

(5.9)

where $\varepsilon_1 = A_1^2 \cos^2(\omega_1 t + \alpha)$, $\varepsilon_2 = A_2^2 \cos^2(\omega_2 t + \beta)$, and the interference term

$$\varepsilon_{\text{int}} = 2A_1 A_2 \cos(\omega_1 t + \alpha) \cos(\omega_2 t + \beta).$$

(5.10)

The time average of $\varepsilon_{\text{int}}$ vanishes:

$$\langle \varepsilon_{\text{int}} \rangle = \int_0^\infty dt \varepsilon_{\text{int}} = 0.$$  

(5.11)

The interaction behavior of the energy-momentum tensor (5.6) is similar to the interference behavior of the energy density (5.9).

6 Conclusion and outlook

In linear approximation, the gravitational radiation is a quadrupole radiation, two radiations superpose linearly and do not interact with each other. Only the resonance between the radiation and the detector is considered as the probing scheme of the radiation.

In this paper, we discuss the gravitational radiation based on the exact cylindrical gravitational wave solutions rather than the linear approximation. (1) We present a cylindrical gravitational monopole radiation solution which indicates that the leading contribution of the gravitational radiation may be the monopole rather than a quadrupole. (2) We consider a new kind of the resonance between gravitational radiations. Expect the gravity, the radiation and the spacetime are two separate concepts. Nevertheless, in general relativity, the concepts of the radiation and the spacetime are mixed up. The radiation and the spacetime are both described by the metric. We attempt to separate the radiation and the spacetime in this paper. Following this idea, we regard the metric without the radiation as a spacetime background. That is, $\psi_{\text{rad}}$ and $\gamma_{\text{rad}}$ in eqs. (3.11) and (3.12) are regarded as the spacetime background. This idea works well. The spacetime is also a kind of matter. $\psi_{\text{rad}}$ and $\gamma_{\text{rad}}$ are periodic with the intrinsic frequency $\omega$. When the gravitational radiation $\psi_{\text{rad}}$ ($\gamma_{\text{rad}}$) with the same frequency act on the periodic spacetime background, the resonance occurs. That makes the resonance in the spacetime consistent with the resonance in Newton mechanics or other physical theories expect the gravity. We suppose that the
resonance between the gravitational radiation and the spacetime background exists in spite of the symmetry of the system. In recent years, the gravitational wave detection makes rapid progress. It can be expected that the resonance between the gravitational radiation and the spacetime background can be found. (3) We investigate the interaction between the cylindrical gravitational radiations. The interaction arises both in the metric and the energy-momentum tensor. Nevertheless, the time average of the interaction term in the energy-momentum tensor vanishes, which indicates that the energy-momentum tensor do not interact with each other directly in the time-averaging level.

In future works, based on gravitational monopole radiation solutions obtained in this paper, we may figure out if the gravitational dipole radiation solution exists or not. Besides, the gravitational radiation will lead to the energy loss of the source. With the conservation law of the energy, we may define the energy of the cylindrical gravitational radiation in our framework. We can also consider that the matter wave resonates with the gravitational radiation based on the preceding work on scattering [29–31].

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