ON THE CONVERGENCE RATE OF BERGMAN METRICS

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Abstract. We study the convergence rate of Bergman metrics on the class of polarized pointed Kähler n-manifolds \((M, L, g, x)\) with \(\text{Vol}(B_1(x)) > v\) and \(|\sec| \leq K\) on \(M\). Relying on Tian’s peak section method \[21\], we show that the \(C^{1,\alpha}\) convergence of Bergman metrics is uniform. In the end, we discuss the sharpness of our estimates.

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1. Introduction

Let \((M, g)\) be an \(n\)-dimensional complete Kähler manifold, \(L\) be a positive on \(M\) equipped with a hermitian metric \(h\) whose curvature form is \(2\pi\omega\). The \(L^2\) orthonormal basis of \(H^0(M, L^m)\) will induce canonical embeddings \(\varphi_m\) of \(M\) into \(\mathbb{C}P^{N_m-1}\), where \(N_m = \dim H^0(M, L^m)\). The pullbacks of the \(\frac{1}{m}\)–multiple of Fubini-Study metrics \(g_m = \frac{1}{m\varphi_m}g_{FS}\) are usually called Bergman metrics.

A natural question is to compare the Bergman metrics with the original Kähler metrics. In the pioneering work \[21\], Tian used his peak section method to prove that Bergman metrics converge to the original polarized metric in the \(C^2\)-topology. By the similar method, Ruan \[20\] proved that this convergence is \(C^\infty\). Later, Zelditch \[22\], also Catlin \[5\] independently, used the Szegő kernel to obtain an alternative proof of the \(C^\infty\)-convergence of Bergman metrics and they gave the asymptotic expansion of Bergman kernel, which is the potential of Bergman metric. This expansion can be also obtained by Tian’s peak section method (see \[15\]) and is often called Tian-Yau-Zelditch expansion. By using the heat kernel, Dai-Liu-Ma \[7\] gave another proof of the Tian-Yau-Zelditch expansion, and moreover, they also considered the asymptotic behavior of Bergman kernels on symplectic manifolds and Kähler orbifolds (see also Ma-Marinescu’s book \[17\]). There are many important applications of using Bergman metrics to approximate a given Kähler metric, for example, see \[9\].
In this paper, we study the problem on dependence of the convergence rate. We focus on Bergman metrics in this paper. For Bergman kernel, please view the discussion in [23]. Our first result is an estimate of the $C^1$-convergence rate of Bergman metrics stated as follows.

**Theorem 1.1.** Let $(M, g)$ be a polarized Kähler manifold. Assume that there are constants $K, v > 0$ such that $|\text{sec}| \leq K$ on $M$ and $\text{Vol}(B_1(x_0)) > v$, for $x_0 \in M$. Then we have constants $m_0 = m_0(K, v) \in \mathbb{N}$ and $C = C(K, v) > 0$ such that

$$\sqrt{m} \| \nabla g_m(x_0) \| + m \| g_m(x_0) - g(x_0) \| \leq C, \forall m > m_0,$$

where $\| \cdot \|$ is the norm of tensors which induced by $g$.

**Remark.** We emphasize that the constants in the estimate are uniform for all $(M, g)$ satisfying the assumptions there. For a fixed $(M, g)$, Ruan ([20]) proved that $\| g_m - g \|_{C^\infty} = O \left( \frac{1}{m} \right)$. But in the uniform sense, even if we consider the $C^1$ case, we cannot prove that the difference between metrics is $O \left( \frac{1}{m} \right)$. In fact, the $O \left( \frac{1}{m} \right)$ rate in Theorem 1.1 is sharp. See example 7.4.

Our second main result is about $C^{1,\alpha}$-convergence of Bergman metrics:

**Theorem 1.2.** Let $(M, g)$ be a polarized Kähler manifold. Assume that there are constants $K, v > 0$ such that $|\text{sec}| \leq K$ on $M$, and $\text{Vol}(B_1(x_0)) > v$, for $x_0 \in M$. Then there are constants $r$ and $C$ depend only on $K$ and $v$, such that there exists a holomorphic chart $(z_1, \cdots, z_n)$ containing $B_r(x_0)$ with

$$e^{-C} g_{C^n} \leq g \leq e^{C} g_{C^n},$$

and for each $\alpha \in (0, 1)$, we have a constant $m_0 = m_0(K, v, \alpha)$, such that

$$\| g_{ij,m} - g_{ij} \|_{C^{1,\alpha}} \leq C m \| \log(m) \|^{\alpha}, \forall m > m_0,$$

for $i, j = 1, \cdots, n$. Here $g_{C^n}$ is the normal flat metric on $\mathbb{C}^n$, $\| \cdot \|_{C^{1,\alpha}}$ is the $C^{1,\alpha}$-norm on the chart, $g_{ij,m} = g_m \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)$, and $g_{ij} = g \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)$.

As a byproduct, we also obtain an estimate for the $W^{2,p}$ convergence. In addition, uniform $W^{2,1}$ convergence is not generally true (see Section 7).

For the proof of Theorem 1.1 and 1.2, we will use the holomorphic version of the Cheeger-Gromov convergence theory and Tian’s peak section method. By Taylor expansion, we use the peak sections to approximate the holomorphic section with the largest norm at a given point. Since the second-order expansion is not uniform, we also need to estimate some special Fourier coefficients on distinguish boundaries of polydiscs.

This paper is organized as follows. In Section 2, we collect some preliminary results. We will give the method of constructing local coordinates in Section 3. Then we will make some estimates about Tian’s peak sections in Section 4. We prove Theorem 1.1 and 1.2 in Section 5 and Section 6. In Section 7, we will give some examples about surfaces to illustrate what happens if some conditions are removed or if we consider faster convergence. For convenience, we save some details of it in Appendix A. We also put some ODE estimates in Appendix B.

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2. Preliminaries

First, recall some basic notations in Kähler geometry. Let $\pi : L \to M$ be a holomorphic line bundle on the $n$-dimensional Kähler manifold $M$, and let $h$ be a hermitian metric on $L$. Then the curvature of $(L, h)$ is the $(1, 1)$ form $Ric(h) = -\sqrt{-1}\partial\bar{\partial}\log h$. For each ample line bundle $L$ on $M$, we can find a hermitian metric $h$ on $L$ such that $\omega = \frac{1}{2\pi}Ric(h) > 0$. Then we say that $(M, \omega)$ is a polarized Kähler manifold with polarized Kähler metric $\omega$. The $L^2$ orthonormal basis of $H^0(M, L^m)$ will induce canonical embeddings $\varphi_m$ of $M$ into $\mathbb{C}P^{N_m-1}$, where $N_m = \dim H^0(M, L^m)$. The pullbacks of the $\frac{1}{m}$-multiple of Fubini-Study metrics $\omega_m = \frac{1}{m}\varphi^*_m\omega_{FS}$ are usually called Bergman metrics. Let $\{\tilde{S}^m_j\}_{j=0}^{N_m-1}$ be a $L^2$ orthonormal basis of $H^0(M, L^m)$. Then it’s obvious that

$$\omega_m = \frac{\sqrt{-1}}{2\pi m}\partial\bar{\partial}\log \left( \sum_{j=0}^{N_m-1} |\tilde{S}^m_j(x)|^2 \right).$$

We introduce the holomorphic version of Cheeger-Gromov $C^{m,\alpha}$-norm for Kähler manifolds now.

**Definition 2.1 (Holomorphic Norms).** Let $(M, g, x)$ be a pointed Kähler manifold. We say that the holomorphic $C^{m,\alpha}$-norm on the scale of $r$ at $x$:

$$\|((M, g, x))\|_{C^{m,\alpha},r}^{\text{holo}} \leq Q,$$

provided there exists a biholomorphic chart $\phi : (B_r(0), 0) \subset \mathbb{C}^n \to (U, x) \subset M$ such that

1. $|D\phi| \leq e^Q$ on $B_r(0)$ and $|D\phi^{-1}| \leq e^Q$ on $U$.
2. For all multi-indices $I$ with $0 \leq |I| \leq m$,

$$r^{|I|+\alpha}\|D^I g_{i\bar{j}}\|_{\alpha} \leq Q.$$

Globally we define

$$\|((M, g))\|_{C^{m,\alpha},r}^{\text{holo}} = \sup_{x \in M} \|((M, g, x))\|_{C^{m,\alpha},r}^{\text{holo}}.$$

Then we state the Hömander’s $L^2$ theory:

**Proposition 2.2.** Let $(M, \omega)$ be a connected but not necessarily complete Kähler manifold with $\dim M = n$. Assume that $M$ is Stein if it isn’t compact. Let $(L, h)$ be a hermitian holomorphic line bundle, and let $\psi \in L^1_{\text{loc}}(M)$ be a weight function on $M$. Suppose that

$$\sqrt{-1}\partial\bar{\partial}\psi + Ric(\omega) + Ric(h) \geq \gamma g$$

for some positive continuous function $\gamma$ on $M$. Then for any $L$-valued $(0,1)$-form $\zeta \in L^2$ on $M$ with $\bar{\partial}\zeta = 0$ and $\int_M ||\zeta||^2 e^{-\psi} dV_g$ finite, then there exists an $L$-valued function $u \in L^2$ such that $\bar{\partial}u = \zeta$ and

$$\int_M ||u||^2 e^{-\psi} dV_g \leq \int_M \gamma^{-1}||\zeta||^2 e^{-\psi} dV_g.$$

The proof can be found in [8]. By the theory of elliptic equations, we can choose the solution $u \in C^{k+2,\alpha}$ (resp. $W^{k+2,p}$), if $\zeta \in C^{k+1,\alpha}$ (resp. $W^{k+1,p}$), for $k \geq 0$.

In this paper, the notation $(M, g, J, x)$ means that a pointed Kähler manifold with Kähler metric $g$, complex structure $J$, and $x \in M$. For simplicity, we denote it by $(M, g, x)$ if we don’t emphasize the complex structure.
3. Construct Uniform Holomorphic Charts

Let $M$ be an $n$-dimensional algebraic manifold with a polarization $L$, and let $g$ be a polarized Kähler metric with respect to $L$, i.e., $\omega_g \in \mathcal{C}_1(L) \in H^2(M, \mathbb{Z})$. Then there exists a hermitian metric $h$ on $L$ such that $\frac{1}{2\pi} \text{Ric}(h) = \omega_g$. In order to use the Schauder interior estimates, we introduce some norms here.

**Definition 3.1** (Interior norms). Let $U \subset \mathbb{R}^n$ be a domain, for each function $f : U \to \mathbb{C}$, we define

$$[f]_{k,0;U} = \sup_{x \in U} \text{dist}(x, \partial U) |\nabla^k f(x)|,$$

$$[f]_{k,\alpha;U} = \sup_{x \neq y \in U} \min \{\text{dist}(x, \partial U), \text{dist}(y, \partial U)\}^\alpha \frac{|\nabla^k f(x) - \nabla^k f(y)|}{|x - y|^\alpha},$$

$$\|f\|_{k,\alpha;U} = \sum_{j=0}^k [f]_{j,\alpha;U}^* + [f]_{k,\alpha;U}^*,$$

where $k \in \mathbb{Z}_+$, and $\alpha \in (0, 1)$.

We construct holomorphic charts with uniform size now.

**Proposition 3.2.** Let $(M, g, L, h)$ be given as at above, $x \in M$. Suppose that $r \in (0, 1]$, $Q > 0$ and $\|(M, g, x)\|_{C^{1,\alpha}}^{\text{holo}} \leq Q$. Then there exists a holomorphic chart $\phi : B_{3r}(0) \to B_r(x) \subset M$, which satisfies the conditions in Definition 2.1 with constant $2Q$, and we can find a holomorphic frame $e_L$ of $L$ on $\phi(B_{3r}(0))$ satisfying that

$$g_{ij}(0) = \delta_{ij},$$

$$d g_{ij}(0) = 0,$$

$$a(0) = 1,$$

$$\frac{\partial^I a}{\partial z^I}(0) = 0,$$

$$\|a\|_{3,\alpha;B_{3r}(0)} \leq C r^2,$$

for each multi-index $I$ with $|I| \leq 3$, where $g_{ij} = g(\phi(\frac{\partial}{\partial z_i}), \phi(\frac{\partial}{\partial z_j}))$, $a = h(e_L, e_L)$, $\delta$ and $C$ are positive constants depend only on $n, Q$, and $\|\|_{k,\alpha;U}$ is the interior norm on the domain $U$.

**Proof.** Without loss of generality, we can assume that $r = 1$. By the definition of $\|(M, g)\|_{C^{1,\alpha}}^{\text{holo}}$, we can find a holomorphic chart

$$\phi_0 : B_r(0) \to B_r(x)$$

which satisfies the conditions in Definition 2.1 with constant $2Q > 0$. Replacing the ball $B_r(0)$ by a polydisc

$$U = D_{n^{-1} \epsilon}(0) \times ... \times D_{n^{-1} \epsilon}(0) = D_{n^{-1} \epsilon}(0)^n,$$

then $H^1(U, \mathcal{O}^*) = 0$ shows that $L|_U$ is a trivial bundle, and hence we can choose a holomorphic frame $e_0$ on a smaller ball $B_{n^{-2} \epsilon}(0) \subset U$. Without loss of generality, we can assume that $L|_{B_r(0)}$ is trivial, and $e_0 \in H^0(B_r(0)), L$. If necessary, we will appropriately shrink $\epsilon$. 

Combining the Hörmander $L^2$-estimate and the zig-zag argument (see Proposition 8.5 in [3]), we can find a real function $f \in C^\infty (B_\epsilon (0))$ such that

$$\int_{B_\epsilon (0)} |f|^2 e^{-|z|^2} dV_{\omega} \leq C_0 \epsilon e^4 \int_{B_\epsilon (0)} e^{-|z|^2} dV_{\omega},$$

where $C_0 = C_0 (n)$ is a constant, and $\frac{\sum_{j=1}^{n} \partial \bar{\partial} f}{2\pi} = \omega_g$. By the $C^0$ estimate of solution of Poisson’s equation, we can assume that

$$\sup_{B_\epsilon (0)} |f| \leq C_1 \epsilon^{3+\alpha},$$

where $C_1 = C_1 (n)$ is a constant.

It is clear that $\partial \bar{\partial} f = \partial \bar{\partial} \log (a_0)$, where $a_0 = h (e_0, e_0)$. Then $f - \log (a_0)$ is a pluriharmonic function, and hence we have a holomorphic function $\psi$ which satisfies that $Re (\psi) = f - \log (a_0)$. Let $e_1 = e^z e_0$, we have $\log (h(e_1, e_1)) = f$.

Next, Schauder’s interior estimate([10], Theorem 6.2) implies that

$$\|f\|^*_5 \leq C_2 \epsilon^2 \left\| \Delta f\right\|^*_1 + C_3 \sup_{B_\epsilon (0)} \|f\| \leq C_4 \epsilon^2,$$

where $C_2, C_3, C_4 > 0$ are constants depending only on $n, Q$.

By the Kähler conditions $dw_g = 0$, we can make $B_\epsilon (0)$ satisfy the equation $dg_{ij} (0) = 0$ through a biholomorphic mapping. Likewise, we can assume that $a$ satisfies the vanishing properties at $0 \in B_\epsilon (0)$. \qed

**Remark.** If $\|(M, g, x)\|^{\text{holo}}_{C^{3, \alpha, r}} \leq Q$ for some $k \in \mathbb{N}$, then we can assume that $\|a\|^*_k \leq C_5$ and $\frac{\partial (\psi)}{\partial z_I} (0) = 0$, for each multi-index $I$ with $|I| \leq k + 2$.

4. Estimate Tian’s peak sections

In this section, we will make some estimates about Tian’s peak sections.

Let $(M, g)$ be an $n$-dimensional algebraic manifold with Kähler metric $g$ and a polarization $(L, h)$ such that $\frac{1}{2\pi} Ric (h) = \omega_g$. Fix a local coordinate $(z_1, \ldots, z_n)$ defined on an open neighborhood $U$ around $x_0 \in M$. Define $|z| = \sqrt{\sum_{j=1}^{n} |z_j|^2}$ for $z \in U$.

We assume that $\|(M, g, x_0)\|^{\text{holo}}_{C^{3, \alpha, r}} \leq Q$ for some $r, Q > 0$, $Ric (g) \geq t g$ for some $t \leq 0$, and the local coordinate is the one we constructed in Proposition 3.2. Now we can construct the peak sections.

**Lemma 4.1** ([21], Lemma 1.2). For an $n$-tuple of integers $P = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}_+^n$ and an integer $p' > p = \sum_{j=1}^{n} p_j$, then we can find constant $m_0$ which depends on $n, p, p', Q$, and there exists another constant $C_0$ which depends on $n, p, p', Q$, such that for each $m > \max \left\{m_0, \frac{1 + \log r}{r^2} \right\}$, there are sequences $a_m$ and $b_m$, smooth $L$-valued sections $\varphi_m$, and holomorphic global sections $S_m$ in $H^0 (M, L^m)$ satisfying

$$\int_{M} \|\varphi_m\|^2_{h_m} dV_g \leq \frac{C_0}{m^{8p + 2n}},$$

$$\int_{M} \|S_m\|^2_{h_m} dV_g = 1,$$

$$\int_{M \setminus \{|z| \leq \frac{\log m}{m^2}\}} \|S_m\|^2_{h_m} dV_g \leq \frac{C_0}{m^{2p}}.$$
and locally at $x_0$,

\begin{equation}
S_m(z) = \lambda_{(p_1,p_2,...,p_n)} \left( 1 + \frac{a_m}{m^{2p}} \right) \left( \bar{z}_1^{p_1} \cdots \bar{z}_n^{p_n} + \varphi_m \right) e^{mz},
\end{equation}

where $|| \cdot ||_{h^m}$ is the norm on $L^m$ given by $h^m$, $|a_m| \leq C_0$, $\varphi_m$ is holomorphic on $\left\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \right\}$, and $||\varphi_m||_{h^m} \leq b_m |z|^{2p'}$ on $U$, moreover

\begin{equation}
\lambda_{(p_1,p_2,...,p_n)}^{-2} = \int \left\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \right\} |z_1^{p_1} \cdots z_n^{p_n}|^2 a^m \, dV_g,
\end{equation}

where $dV_g = \left( \frac{m}{2\pi} \right)^n \det(g_{ij}) \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ is the volume form.

**Proof.** Through the proof of Lemma 1.2 in [21], combined with Proposition 3.2, this lemma can be proved. \hfill $\square$

So the estimation of Tian’s Peak sections is reduced to the estimation of $\lambda_{(p_1,p_2,...,p_n)}$. Now we need to control some special Fourier coefficients on distinguish boundaries of polydiscs.

**Lemma 4.2.** Let $f$ be a smooth real function on a domain that contains the closed polydisc $U = \overline{D}(0,\delta)^n$, where $\delta > 0$ be a constant.

(i). If $k \in \mathbb{Z}_+$, $f(0) = \nabla f(0) = 0$, $|f| \leq K_1$ for some $K_1 > 0$, and $|\partial \bar{\partial} f| \leq K_2$ for some $K_2 > 0$, then we have a constant $C > 0$ which depends only on $\delta$, $n$, $k$, $K_1$ and $K_2$, such that

$$
\int \prod_{z,j=1}^n \partial D(0,r_j) f(z) \cos k \theta_1 \partial \theta_1 \wedge \cdots \wedge \partial \theta_n \leq C r^2 |\log(r)| \delta_{k,2} + C r^2,
$$

when $r = \sqrt{\sum_{j=1}^n r_j^2} < \frac{\delta}{\sqrt{n}}$, where $z_j = r_j e^{\theta_j \sqrt{-1}}$, and $\delta_{i,j}$ is the Kronecker symbol. In addition, when $k = 0$, we can assume that $C = C(n,K_2)$.

(ii). If $k \in \mathbb{Z}_+$, $f(0) = \nabla f(0) = \nabla^2 f(0) = \nabla^3 f(0) = 0$, $|f| \leq K_1$ for some $K_1 > 0$, and

$$
\left| \frac{\partial^4 f}{\partial z_1 \partial z_j \partial z_i, \partial z_t} \right| \leq K_2 \text{ for some } K_2 > 0, \\forall i,j,s,t,
$$

then we have a constant $C > 0$ which depends only on $\delta$, $n$, $k$, and $K$, such that

$$
\int \prod_{z,j=1}^n \partial D(0,r_j) f(z) \cos k \theta_1 \partial \theta_1 \wedge \cdots \wedge \partial \theta_n \leq C \left( \delta_{k,2} + \delta_{k,4} \right) r^4 |\log(r)| + C r^4,
$$

when $r = \sqrt{\sum_{j=1}^n r_j^2} < \frac{\delta}{\sqrt{n}}$, where $z_j = r_j e^{\theta_j \sqrt{-1}}$, and $\delta_{i,j}$ is the Kronecker symbol. In addition, when $k = 0$, we can assume that $C = C(n,K_2)$.

**Proof.** See Appendix B. \hfill $\square$

Assume that $m > \max\left\{ m_0, \frac{\log(m)}{r^2} \right\}$ from now. We begin to estimate $\lambda_{(p_1,...,p_n)}^{-2}$ on $M$.

**Lemma 4.3.** Under the notations and assumptions of Lemma 4.1 with the additional condition $|\text{sec}| \leq K$ on $B_1(x_0)$, we have the following estimates:

\begin{equation}
\left| \frac{n^{n+p}}{p!} \lambda_{(p_1,...,p_n)}^{-2} - \frac{1}{\pi^p} \right| \leq C r^{-2} m^{-1},
\end{equation}

where $C = C(Q,n,p,K,\alpha,C_0) > 0$ is a constant.
Proof. Recall the definition

\[ \lambda_{(p_1, \ldots, p_n)}^{-2} = \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 a^m dV_g, \]

we have

\[
| \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 a^m dV_g \\
- \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 e^{-\pi m|z|^2} dV_{C^n} \\
\leq \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 |a^m - e^{-\pi m|z|^2}| \left| \log (g_{ij}) - 1 \right| dV_{C^n} \\
+ \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 |a^m - |z_1| \cdots |z_n|^2 e^{-\pi m|z|^2} dV_{C^n} \\
+ \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 \left( \log (g_{ij}) - 1 \right) e^{-\pi m|z|^2} dV_{C^n}.
\]

Proposition 3.2 shows that:

\[
\int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 |a^m - e^{-\pi m|z|^2}| \left| \log (g_{ij}) - 1 \right| dV_{C^n} \\
\leq C_1 r^{-4-2\alpha} \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z|^{2p+4+2\alpha} dV_{C^n},
\]

where \( C_1 \) is a positive constant depending only on \( Q, \alpha, \) and \( p \).

Since

\[ R_{kk} = -\frac{\partial^2 g_{ij}}{\partial z_k \partial \bar{z}_k} - g^{st} \frac{\partial g_{kj}}{\partial z_k} \frac{\partial g_{is}}{\partial \bar{z}_k}, \]

Lemma 4.2 now implies that

\[
\left| \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z_1^{p_1} \cdots z_n^{p_n}|^2 \left( \log (g_{ij}) - 1 \right) e^{-\pi m|z|^2} dV_{C^n} \right| \\
\leq C_2 r^{-2} \int_{\{|z| \leq \frac{\log(m)}{\sqrt{m}}\}} |z|^{2p+2} e^{-\pi m|z|^2} dV_{C^n},
\]

where \( C_2 \) is a positive constant depending only on \( Q, n, p, \) and \( K \).

By the definition of \( a \), we have

\[
\frac{1}{2\pi} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log (a) = -g_{ij},
\]
then we can apply Lemma 4.2 to $a - e^{-\pi|z|^2}$. It gives a constant $C_3$ depending only on $Q, n, p$, and $K$, such that

$$\left| \int_{\{|z| \leq \log(m)/\sqrt{m} \}} |z_1^{p_1} \cdots z_n^{p_n}|^2 \left[ a^m - e^{-\pi m|z|^2} \right] dV_{C^n} \right|$$

$$\leq m C_3 r^{-2} \int_{\{|z| \leq \log(m)/\sqrt{m} \}} |z|^{2p+4} e^{-\pi m|z|^2} dV_{C^n}$$

$$+ m^2 C_3 r^{-3} \int_{\{|z| \leq \log(m)/\sqrt{m} \}} |z|^{2p+7} e^{-\pi m|z|^2} dV_{C^n}.$$

Then a straightforward calculation shows that

$$\int_{\mathbb{C}^n} |z|^{2p+4} e^{-\pi m|z|^2} + m^3 |z|^{2p+7} e^{-\pi m|z|^2} dV_{C^n} \leq C_4 m^{-n-p-2},$$

where $C_4 = C_4(n)$ is a constant.

By direct computation, we have

$$\left| \int_{\{|z| \leq \log(m)/\sqrt{m} \}} |z_1^{p_1} \cdots z_n^{p_n}|^2 e^{-\pi m|z|^2} dV_{C^n} - \frac{P!}{\pi^p m^{n+p}} \right| \leq C_5 m^{-n-p-2},$$

where $C_5 = C_5(p, n) > 0$ is a constant. We thus get the estimate (8).

\[\square\]

Remark. When $P = (0, \cdots, 0)$, we can replace the condition $|\sec| \leq K$ by $|Ric| \leq K$, and the proof is similar to the above.

We now estimate the inner product between peak sections with some other sections.

Lemma 4.4. Let $S_m$ be the sections we have constructed in Lemma 4.1, and let $T$ be another section of $L^m$ with $\int_M |T|^2_{h_m} dV_g = 1$, which contain no term $z_1^{p_1} \cdots z_n^{p_n}$ in its Taylor expansion at $x_0$. Then

$$\left| \int_M \langle S_m, T_{h_m} \rangle dV_g \right| \leq Cr^{-1-\alpha} m^{-\frac{1+\alpha}{2}},$$

where $\langle , \rangle$ is the inner product on the linear space $H^0(M, L^m)$ induced by the metric $h^m$, and $C = C(Q, n, p, \alpha)$ is a constant.

Proof. We divide the integral into three parts:

$$\int_M \langle S_m, T \rangle_{h_m} dV_g = \int_M \left\{ |z| \leq \log(m)/\sqrt{m} \right\} \langle S_m, T \rangle_{h_m} dV_g$$

$$+ \left( 1 + \frac{a_m}{m^{2p}} \right) \lambda_{(p_1, p_2, \ldots, p_n)} \int_{\{|z| \leq \log(m)/\sqrt{m} \}} \langle \varphi_m, T \rangle_{h_m} dV_g$$

$$+ \left( 1 + \frac{a_m}{m^{2p}} \right) \lambda_{(p_1, p_2, \ldots, p_n)} \int_{\{|z| \leq \log(m)/\sqrt{m} \}} \langle z_1^{p_1} \cdots z_n^{p_n} e^m, T \rangle_{h_m} dV_g.$$
Lemma 4.1 shows that
\[
\left| \int_{M \setminus \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} \langle S_m, T \rangle_{h_m} dV_g \right|
\leq \left( \int_{M \setminus \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} \| S_m \|_{h_m}^2 dV_g \right)^{\frac{1}{2}} \left( \int_{M \setminus \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} \| T \|_{h_m}^2 dV_g \right)^{\frac{1}{2}}
\leq C_1 m^{-p'},
\]
and the similar argument gives
\[
\left| \left( 1 + \frac{a_m}{m^{2p'}} \right) \lambda \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \} \int_{\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} \langle \varphi_m, T \rangle_{h_m} dV_g \right| \leq C_2 m^{-p'},
\]
where \( C_1, C_2 \) are constants depending only on \( Q, n, p, a \).

It is sufficient to estimate the last term of (9) now.

We assume that \( T = f_T e_{\omega_L} \) on \( \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \} \), then \( f_T \) is holomorphic on \( \{ |z| \leq \frac{\log(m)}{\sqrt{m}} \} \) and contains no term \( z_1^{p_1} \cdots z_n^{p_n} \) in the Taylor expansion at \( z = 0 \). It follows that
\[
\int_{\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} z_1^{p_1} \cdots z_n^{p_n} \int_{
\prod_{j=1}^n \partial D(0, r_j)} f_T e^{-\pi m |z|^2} dV_{\mathbb{C}^n} = 0,
\]
because for each \( P \neq 0, \)
\[
\int_{\prod_{j=1}^n \partial D(0, r_j)} z_1^{p_1} \cdots z_n^{p_n} d\theta_1 \wedge \cdots \wedge d\theta_n = 0.
\]

By Schwarz inequality we have
\[
\left| \left( \int_{\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} \langle z_1^{p_1} \cdots z_n^{p_n} e_{\omega_L}, T \rangle_{h_m} dV_g \right) \right|
\leq \left( \int_{\{ |z| \leq \frac{\log(m)}{\sqrt{m}} \}} |z|^{2p} \left[ a^m \det (g_{ij}) - e^{-\pi m |z|^2} \right]^2 a^{-m} \det (g_{ij})^{-1} dV_{\mathbb{C}^n} \right)^{\frac{1}{2}}.
\]

It follows from Lemma 3.2 that
\[
\left[ a^m \det (g_{ij}) - e^{-\pi m |z|^2} \right]^2 a^{-m} \det (g_{ij})^{-1}
\leq C_3 \left[ a^m \left( \det (g_{ij}) - 1 \right) + \left( a^m - e^{-\pi m |z|^2} \right) \right]^2 a^{-m}
\leq C_4 \left[ r^{-2-2\alpha} |z|^{2+2\alpha} e^{-m |z|^2} + mr^{-1-\alpha} |z|^{3+3\alpha} e^{-m |z|^2} \left( 1 - \left( \frac{e^{-\pi |z|^2}}{a} \right)^{m} \right) \right]
\leq C_5 e^{-m |z|^2} \left( r^{-2-2\alpha} |z|^{2+2\alpha} + m^2 r^{-2-2\alpha} |z|^{6+2\alpha} \right),
\]
where $C_3, C_4, C_5$ are positive constants depend only on $Q$, $n$, $p$, $\alpha$. Then we can conclude that

$$\left| \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} (z_1^{p_1} \cdots z_n^{p_n} e_\zeta^m, T)_{h_m} dV_g \right|$$

$$\leq \left( \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z \right|^{2p} \left[ a^m \det (g_{ij}) - e^{-2\pi m |z|^2} \right]^2 a^{-m} \det (g_{ij})^{-1} dV_{C_n} \right)$$

$$\leq \left( C_5 r^{-2-2\alpha} \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z \right|^{2p+2+2\alpha} e^{-m |z|^2} (1 + m^2 |z|^4) dV_{C_n} \right)$$

$$\leq C_6 r^{-1-\alpha} m^{-\frac{p+4+\alpha}{2}}$$

for some constant $C_6 = C_6(Q, n, p, \alpha) > 0$.

From (8), we see that $\lambda_{(p_1, p_2, \ldots, p_n)} \leq C_7 m^{\frac{p_1}{2}}$ for some constant $C_7 = C_7(Q, p, n, \alpha)$. This gives

$$\left| \int_M \langle S_m, T \rangle_{h_m} dV_g \right| \leq C r^{-1-\alpha} m^{-\frac{1+\alpha}{2}},$$

where $C = C(Q, n, p, \alpha)$ is a constant.

Now we focus on the adjacent peak sections.

**Lemma 4.5.** Let $S_m$ be the peak sections we have constructed in Lemma 4.1 for $P = (p_1, \ldots, p_n)$, and let $T_m$ be the peak sections for $P' = (p_1 + k, \ldots, p_n)$ for some $k \in \mathbb{N}$. We assume that $| \sec | \leq K$ for some $K > 0$, then we can find a constant $C = C(n, p, Q, k, K)$, s.t.

$$\left| \int_M \langle S_m, T_m \rangle_{h_m} dV_g \right| \leq C m^{-1} + C m^{-1} \log(m) (\delta_{k, 2} + \delta_{k, 4}),$$

for each $m > m_0$.

**Proof.** It is sufficient to show that

$$\left| \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 z_1^k a^m dV_g \right| \leq C m^{-n-p-1-\frac{1}{2}} (1 + \delta_{k, 2} \log(m) + \delta_{k, 4} \log(m)).$$

Let $\psi = \log(a) + \pi |z|^2$, $\varphi = \det (g_{ij}) - 1$. Then we have

$$\int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 z_1^k a^m dV_g$$

$$= \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 z_1^k e^{-\pi m |z|^2} (e^{m \psi} - 1 - m \psi) (1 + \varphi) dV_{C_n}$$

$$+ \int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 z_1^k e^{-\pi m |z|^2} (m \psi + \varphi + m \psi \varphi) dV_{C_n}.$$

Since there exists a constant $C_1 = C_1(n, p, Q, k, K)$ such that $|\psi| \leq C_1 |z|^3$, $|\varphi| \leq C_1 |z|$, and $|e^{m \psi} - 1 - m \psi| \leq C_1 m^2 |z|^7$, we can conclude that

$$\int \left\{ \left| z \right| \leq \frac{\log(m)}{\sqrt{m}} \right\} \left| z_1^{p_1} \cdots z_n^{p_n} \right|^2 z_1^k a^m dV_g$$

$$\leq C m^{-n-p-1-\frac{1}{2}} (1 + \delta_{k, 2} \log(m) + \delta_{k, 4} \log(m)).$$
By an orthogonal transformation we may further assume that
\[
\int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{p_1} \cdots z^{p_n}|^2 z_1^k a^n dV_g
\]
\[
\leq C_2 \int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} e^{-\pi m|z|^2} (m|z|^4 + |z|^2 + m|z|^5 + m^2|z|^7) dV_C^n
\]
\[
+ C_2 \int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} e^{-\pi m|z|^2} (m|z|^4 + |z|^2) |\log(|z|)| (\delta_{k,2} + \delta_{k,4}) dV_C^n.
\]

Then a straightforward computation gives the following inequality:
\[
\int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} e^{-\pi m|z|^2} (m|z|^4 + |z|^2 + m|z|^5 + m^2|z|^7) dV_C^n \leq C_3 m^{-n-p-1-\frac{5}{2}},
\]
where \( C_3 = C_3(n, p, k) \) is a constant.

Clearly, \( \sqrt{|z|} |\log(|z|)| \geq -2e^{-1}, \) and hence
\[
\int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} e^{-\pi m|z|^2} (m|z|^4 + |z|^2) |\log(|z|)| dV_C^n
\]
\[
\leq \int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} e^{-2\pi m|z|^2} (m|z|^4 + |z|^2) |\log(m^2)| dV_C^n
\]
\[
+ \int_{\{|z| \leq \frac{\log(m)}{m}\}} |z^{2p+k} (m|z|^4 + |z|^2) |\log(|z|)| dV_C^n
\]
\[
\leq 2C_3 m^{-n-p-1-\frac{5}{2}} |\log(m)| + C_4 m^{-2n-4p-2k-3},
\]
where \( C_4 \) is a constant depends only on \( n, p, Q, k, K. \)

5. Pointwise Estimates

In this section, we will prove Theorem 1.1.

Choosing an \( L^2 \) orthonormal basis \( \{S_j^m\}_{j=0}^{N_m-1} \) of \( H^0(M, L^m) \), where \( N_m = \dim H^0(M, L^m) \).
Since \( L|_U \) is a trivial bundle, we can find holomorphic functions \( f_j^m \in \mathcal{O}(U) \) s.t. \( S_j^m = f_j^m e_L \) on \( U \).
By an orthogonal transformation we may further assume that
\[
f_j^m(0) = 0, \quad \text{for } j \geq 1,
\]
\[
\partial f_j^m(0) = 0, \quad \text{for } j \geq k + 1, \quad j = 1, 2, \cdots, n,
\]
\[
\partial^2 f_j^m(0) = 0, \quad \text{for } j \geq n + 2.
\]
Lemma 5.1. Under the conditions stated above, for each given \( k \in \mathbb{N}, \alpha \in (0, 1) \), there exists a positive constant \( C \) depends only on \( t, Q, r, n, k, \alpha, K \), such that

\[
\begin{align*}
(13) & \quad \left| \sqrt{\frac{1}{m^n}} f_0^m(t) - 1 \right| \leq C m^{-1}, \\
(14) & \quad \left| \sqrt{\frac{1}{m^{n+1}}} \frac{\partial f_1^m}{\partial z_1}(0) - \sqrt{\pi} \right| \leq C m^{-1}, \\
(15) & \quad \left| \sqrt{\frac{1}{2m^{n+2}}} \frac{\partial f_{n+1}^m}{\partial z_1}(0) - \pi \right| \leq C m^{-1}, \\
(16) & \quad m^{-\frac{n+k-2}{2}} \left| \frac{\partial f_1^m}{\partial z_1}(0) \right| + m^{-\frac{n+k-1}{2}} \left| \frac{\partial^2 f_1^m}{\partial z_1^2}(0) \right| \leq C + C \log(m) (\delta_{k,2} + \delta_{k,4}),
\end{align*}
\]

and

\[
(17) \quad m^{-\frac{n+k}{2}} \left| \frac{\partial^2 f_j^m}{\partial z_1^2}(0) \right| \leq C, \ 1 \leq j \leq n.
\]

Proof. Let \( T_0, T_1, \cdots, T_{n+1} \) be peak sections of \( L^m \) for \( P = (0, \cdots, 0), (1, 0, \cdots, 0), \cdots, (0, \cdots, 1) \), and \( (2, 0, \cdots, 0) \), respectively. We can find constants \( \beta_{ij} \), satisfying that \( T_i = \sum_{j=0}^{N_m-1} \beta_{ij} S_j^m \), for \( j = 1, 2, \cdots, n+1 \). By Lemma 4.4,

\[
\begin{align*}
\left| \int_M \left( T_0, \sum_{j=1}^{N_m-1} \beta_{0j} S_j^m \right) dV_g \right| & \leq C_1 m^{-\frac{1+n}{2}} \left( \int_M \left( \sum_{j=1}^{N_m-1} \beta_{0j} S_j^m, \sum_{j=1}^{N_m-1} \beta_{0j} S_j^m \right) dV_g \right)^{\frac{1}{2}} \\
& \leq C_1 m^{-\frac{1+n}{2}},
\end{align*}
\]

and hence

\[
\sum_{j=1}^{N_m-1} |\beta_{0j}|^2 \leq C_1 m^{-\frac{1+\alpha}{2}},
\]

where \( C_1 = C_1(t, Q, r, n, p, \alpha) \) is a constant. Since \( \int_M \|T_0\|_{h_m}^2 dV_g = 1 \), it follows that

\[
\sum_{j=1}^{N_m-1} |\beta_{0j}|^2 = \left| \int_M \left( T_0, \sum_{j=1}^{N_m-1} \beta_{0j} S_j^m \right) dV_g \right| \leq C_1 m^{-\frac{1+n}{2}} \left( \sum_{j=1}^{N_m-1} |\beta_{0j}|^2 \right) \leq C_1^2 m^{-1-\alpha}.
\]
Thus we have
\[
\left| \sqrt{\frac{1}{m^n}} |f_0^m(0)| - 1 \right| \leq \left| \sqrt{\frac{1}{m^n}} |\beta_0^m|^{-1} \lambda(0,\ldots,0) - 1 \right| + C_2 m^{-1}
\]
\[
\leq C_3 m^{-1},
\]
where \( C_2 = C_2(t, Q, r, n, p, \alpha, K) \), \( C_3 = C_3(t, Q, r, n, p, \alpha, K) \) are constants.

Writing \( T_i = f_{T_i} e_i \) locally, then we have \( f_{T_i}(0) = 0, \forall i > 0 \), and \( \frac{\partial f_{T_i}}{\partial z_j}(0) = 0 \), when \( i \neq j \). Then we have \( \beta_{ij} = 0 \), if \( i > j \). By a similar argument, Lemma 4.4 now shows that
\[
\sum_{j=i}^{N_m-1} |\beta_{ij}|^2 = 1,
\]
and hence
\[
\left| |\beta_i| - 1 \right| \leq C_4 m^{-1-\alpha},
\]
for some constant \( C_4 = C_4(t, Q, r, n, p, \alpha) \).

It follows that there exists a constant \( C_5 = C_5(t, Q, r, n, p, \alpha, K) > 0 \) such that
\[
\left| \sqrt{\frac{1}{m^{n+1}}} \frac{\partial f_1^m}{\partial z_i}(0) - \sqrt{\pi} \right| \leq C_5 m^{-1},
\]
\[
\left| \sqrt{\frac{1}{2m^{n+2}}} \frac{\partial f_{m+1}}{\partial z_i^2}(0) - \pi \right| \leq C_5 m^{-1}.
\]

Next we come to (16) and (17).

Let \( B = (\beta_{ij})_{0 \leq i, j \leq n+1} \) be a matrix, and let \( X = (f_i^m)_{0 \leq i \leq n+1} \), \( Y = (f_{T_i})_{0 \leq i \leq n+1} \) be row vectors with function elements. Then \( T_i = \sum_{j=0}^{N_m-1} \beta_{ij} S_j \) gives
\[
\frac{\partial^2 Y}{\partial z_i^2} = \frac{\partial^2 X}{\partial z_i^2} B, \quad \frac{\partial Y}{\partial z_j} = \frac{\partial X}{\partial z_j} B, \quad 1 \leq j \leq n,
\]
and thus
\[
\frac{\partial^2 X}{\partial z_i^2} = \frac{\partial^2 Y}{\partial z_i^2} B^{-1}, \quad \frac{\partial X}{\partial z_j} = \frac{\partial Y}{\partial z_j} B^{-1}, \quad 1 \leq j \leq n.
\]

Since \( f_{T_i}(0) = 0, \forall i > 0 \), and \( \frac{\partial f_{T_i}}{\partial z_j}(0) = 0 \), when \( i \neq j \), then \( \sum_{j=0}^{N_m-1} |\beta_{ij}|^2 \leq C_4 m^{-1-\alpha} \) shows that \( ||B^{-1} - I_{n+2}|| \leq (1 + C_4)(2 + n)^n m^{-1-\alpha} \), where \( I_{n+2} \) is the identity matrix. Then Lemma 4.3 implies (17).

Apply the Lemma 4.5 to the peak sections \( T_0 \) and \( T_1 \), one can see that
\[
|\beta_{01} \bar{\beta}_{11} + \sum_{j=2}^{N_m-1} \beta_{01} \bar{\beta}_{1j}| \leq C_6 m^{-1},
\]
where \( C_6 = C_6(t, Q, r, n, p, K) \) is a constant, and the Cauchy-Schwartz inequality shows that
\[
\left| \sum_{j=2}^{N_m-1} \beta_{0j} \bar{\beta}_{1j} \right| \leq C_4 m^{-1},
\]
hence we have \( |\beta_{01}\bar{\beta}_{11}| \leq (C_4 + C_6)m^{-1} \), and \( |\beta_{01}| \leq (1 + C_4)(C_4 + C_6)m^{-1} \).

Recall the definition of peak section, \( \partial f_0/\partial z_1(0) = 0 \), and we can rewrite it as
\[
\beta_{00} \partial f_0^m/\partial z_1(0) + \beta_{01} \partial f_1^m/\partial z_1(0) = 0.
\]

By the argument above,
\[
\left| \partial f_0^m/\partial z_1(0) \right| = \left| \beta_{00} \partial f_1^m/\partial z_1(0) \right| \leq (1 + C_4)^2(C_4 + C_6)m^{-1} \cdot (1 + C_5)m^{n/2}.
\]

Similarly, we can find a constant \( C_7 = C_7(t, Q, r, n, p, K) \) satisfying that \( |\beta_{1n+1}| \leq C_7 m^{-1} \), and
\[
\left| \partial f_1^m/\partial z_1(0) \right| \leq |\beta_{11} \beta_{1n+1} \partial f_{n+1}^m/\partial z_1| + C_7 m^{-1} \cdot m^{n/2} \leq (1 + C_4) \cdot (1 + C_7)m^{-1} \cdot (1 + 2C_5)m^{n/2}.
\]

It gives (16) when \( k = 1 \). Likewise, the Lemma 4.5 can obtain (16) in general, if we consider the peak sections of \( L^m \) for \( P = (1, 0, \ldots, 0), \ldots, (k, 0, \ldots, 0) \). The proof is almost the same as that of the case \( k = 1 \), so we omit it. \( \square \)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1:** The proof is completed by showing the following inequalities:
\[
\left| \frac{1}{\pi m} \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) - 1 \right| (x_0) \leq C m^{-1},
\]

and
\[
\left| \frac{1}{m} \frac{\partial^3}{\partial z_1^2 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) \right| (x_0) \leq C m^{-\frac{3}{2}}.
\]

Since \( f_j^m \) are holomorphic functions satisfying the assumptions (10) and (11), we can conclude that
\[
\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0) = \frac{\partial}{\partial z_1} \left( \frac{\sum_{j=0}^{N_m-1} f_j^m \partial f_j^m/\partial \bar{z}_1}{\sum_{j=0}^{N_m-1} |f_j^m|^2} \right) (x_0)
\]
\[
= \frac{|\partial f_0^m/\partial z_1|^2}{|f_0^m|^2} (x_0),
\]

and similarly,

\[
\frac{\partial^3}{\partial z_1^2 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0) = \frac{\partial^2 f_1^m / \partial z_1^2 \cdot \partial f_1^m / \partial \bar{z}_1}{|f_0^m|^2} - \frac{2 \bar{f}_0^m \partial f_1^m / \partial z_1 \cdot \partial f_1^m / \partial \bar{z}_1 \cdot \partial \bar{f}_1^m / \partial \bar{z}_1}{|f_0^m|^4}.
\]

Combining Corollary A.8 with the lemmas in this section, we can assert that

\[
\left| \frac{|\partial f_1^m / \partial z_1|^2}{2\pi m |f_0^m|^2} - 1 \right| \leq \frac{(m + n + 1)!/m!}{m \cdot (m + n)!/m!} - 1 + C_1 m^{-1}
\]

\[
= \frac{n + 1}{m} + C_1 m^{-1},
\]

where \( C_1 = C_1(K, v) > 0 \) is a constant.

Likewise, we can find a constant \( C_2 = C_2(K, v) > 0 \) satisfies that

\[
\left| \frac{\partial^3}{\partial z_1^2 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0) \right| \leq C_2 m^{\frac{1}{2}}.
\]

This theorem follows. \( \square \)

6. Hölder Estimates

In this section, we will show that the sequence of Bergman metrics, \( \{g_m\}_{m=m_0}^\infty \), converges to \( g \) in the \( C^{1,\alpha} \)-topology, and we will try to control the \( C^2 \)-norm and \( L^2 \)-norm of \( g_m \) now. Although \( g_m \) cannot converge into \( g \) in the Sobolev space \( W^{2,1} \) (see section 7), we can still make some estimates.

Fix a uniform holomorphic chart \((z_1, \cdots, z_n)\) as in the previous section, then we follow the assumptions and notations here.

**Lemma 6.1.** Under the conditions stated in the previous section, there exists a positive constant \( C \), \( m_0 \) depends only on \( t, Q, r, n, K \), such that

\[
\left| \frac{\partial^2 g_{k\bar{l},m}}{\partial z_i \partial \bar{z}_j} \right| \leq C \log(m),
\]

when \( m > m_0 \), where \( g_{k\bar{l},m} = g_m \left( \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l} \right) \).

**Proof.** It’s sufficient to show that

\[
\left| \frac{\partial^4}{\partial z_1^3 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0) \right| \leq C m \log(m).
\]
By a straightforward computation, we obtain
\[
\frac{\partial^4}{\partial z_1^3 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0)
= \frac{\partial^3 f_1^m / \partial z_1^3 \cdot \partial f_1^m / \partial \bar{z}_1}{|f_0^m|^2} + \sum_{i=1}^{3} \sum_{j=0}^{1} \frac{\tilde{f}_0^m \partial^i f_0^m / \partial z_1^i}{|f_0^m|^4} \cdot \partial^2 f_j^m / \partial z_1^i \cdot \partial^2 f_j^m / \partial \bar{z}_1^i + 6 \left( \sum_{j=0}^{1} |\partial f_j^m / \partial z_1|^2 \right) \frac{(\sum_{j=0}^{m} |\partial f_j^m / \partial z_1|^2)^2}{|f_0^m|^6}.
\]
Then this lemma follows from Lemma 5.1.

Similarly, we have:

**Lemma 6.2.** Under the conditions stated in the previous section, there exists a positive constant $C$, $m_0$ depends only on $t$, $Q$, $r$, $n$, $K$, such that

\[
| \frac{\partial^2 g_{kl,m}}{\partial z_i \partial \bar{z}_j} | \leq C,
\]
when $m > m_0$, where $g_{kl,m} = g_m \left( \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l} \right)$.

**Proof.** It’s sufficient to show that

\[
| \frac{\partial^4}{\partial z_1^3 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0) | \leq C m.
\]

By a straightforward computation, we obtain
\[
\frac{\partial^4}{\partial z_1^3 \partial \bar{z}_1} \log \left( \sum_{j=0}^{N_m-1} |f_j^m|^2 \right) (x_0)
= \frac{\partial^2 f_1^m / \partial z_1^2 \cdot \partial f_1^m / \partial \bar{z}_1}{|f_0^m|^4} \cdot \partial^2 f_1^m / \partial z_1 \cdot \partial f_1^m / \partial \bar{z}_1 + 4 \frac{|\partial f_1^m / \partial z_1|^2 |\partial f_1^m / \partial \bar{z}_1|^2}{|f_0^m|^4}
- 4 \text{Re} \left( \frac{\partial^2 f_1^m / \partial z_1^2 \cdot \partial f_1^m / \partial \bar{z}_1 \cdot \partial f_1^m / \partial z_1 \cdot \partial f_1^m / \partial \bar{z}_1}{|f_0^m|^4} \right).
\]

By Lemma 5.1,

\[
\sum_{i=1}^{n+1} |\partial^2 f_1^m / \partial z_1^2|^2 \leq C_1 \frac{m^{n+\frac{1}{2}}}{m^n} = C_1 m^{\frac{1}{2}},
\]

\[
|\partial f_1^m / \partial z_1|^2 |\partial f_1^m / \partial \bar{z}_1|^2 \leq C_1 \frac{m^{\frac{1}{2}} \cdot m^{\frac{1}{2}}}{m^{2n}} = C_1,
\]
and
\[
|\partial^2 f_1^m / \partial z_1^2 \cdot \partial f_1^m / \partial \bar{z}_1 \cdot \partial f_1^m / \partial z_1 \cdot \partial f_1^m / \partial \bar{z}_1 | \leq C_1 \frac{m^{\frac{1}{2}} \cdot m^{\frac{1}{2}} \cdot m^{\frac{1}{2}} \cdot m^{\frac{1}{2}}}{m^{2n}} = C_1,
\]
where $C_1 = C_1(t, Q, r, n, K)$ is a constant.

Apply Lemma 5.1 again, we can conclude that

$$
\left| \frac{\partial^2 f_{m+1}^m / \partial z_1^2}{f_0^m} - \frac{2 \partial f_0^m / \partial z_1^4}{f_0^m} \right| 
\leq \frac{2m^{2n+2}}{m^{2n}}(1 + C_2 m^{-1}) - \frac{2m^{2n+2}}{m^{2n}}(1 - C_2 m^{-1})
= 4C_2m,
$$

where $C_2 = C_2(t, Q, r, n, K)$ is a constant.

This proves the Lemma. \hfill \Box

Using the classical $L^p$-estimates of elliptic partial differential equations, we can give the following result as a corollary:

**Corollary 6.3.** Let $(M, g)$ be a polarized Kähler manifold. Assume that there are constants $K, \nu > 0$ and $t \geq 0$ s.t. $\text{Ric} \geq -\nu g$ on $M$, $|\sec| \leq K$ on $B_1(x_0)$, and $\text{Vol}B_1(x_0) > \nu$, for $x_0 \in M$. Then for each $q > 1$, we have constants $\epsilon = \epsilon(K, \nu, t)$, $m_0 = m_0(K, \nu, t) \in \mathbb{N}$ and $C = C(K, \nu, t, q) > 0$, such that

$$
||\nabla^2 g_m||_{L^q B(x_0, \epsilon)} \leq C, \forall m > m_0.
$$

where $|| \cdot ||_{L^q B(x_0, \epsilon)}$ is the $L^q$ norm of tensors on $B(x_0, \epsilon)$ which is induced by $g$.

**Proof.** It follows the $L^p$-estimates about the solutions of elliptic partial differential equations. For example, see Theorem 9.13 in [10]. \hfill \Box

**Remark.** Actually, we can obtain a $C^{1,\alpha}$ estimate by combining Corollary 6.3 and Lemma 1.1. But this estimate is weaker than the estimate we will make below.

Now we need some estimates of Newtonian potential here. Let $\Gamma(x)$ be the fundamental solution of Laplace’s equation on $\mathbb{R}^n$, i.e.,

$$
\Gamma(x) = \Gamma(|x|) = \begin{cases} \frac{1}{n(2-n)|B_1|}|x|^{2-n}, & \text{if } n > 2, \\
\frac{1}{n}\log |x|, & \text{if } n = 2,
\end{cases}
$$

where $|B_1|$ is the volume of unit ball in $\mathbb{R}^n$. Fix $R > 0$. For each $f \in L^\infty(B_R(0))$, let $\omega$ be the Newtonian potential of $f$, then $\omega(x) = \int_{B_R(0)} \Gamma(x - y)f(y)dy$, $\forall x \in \mathbb{R}^n$. The classical theory of Newtonian potential gives $D_1\omega(x) = \int_{B_R(0)} D_1\Gamma(x - y)f(y)dy$, where $D_i = \frac{\partial}{\partial x_i}$. By the definition of $\Gamma$, we have $D_1\Gamma(x - y) = \frac{1}{n|B_1|} \cdot \frac{x_i - y_i}{|x - y|^n}$. Then we will estimate the difference of of $\nabla \omega$ locally.

**Lemma 6.4.** There are constant $C = C(n)$, $\epsilon = \epsilon(n, R)$, such that

$$
||\nabla \omega(x) - \nabla \omega(0)|| \leq C||f||_{L^\infty} \cdot |x| \cdot |\log(|x|)|,
$$

when $|x| \leq \epsilon$, where $|| \cdot ||_{L^\infty}$ is the $L^\infty$-norm.

**Proof.** Since $D_1\Gamma(x) = \frac{1}{n|B_1|} \cdot \frac{x_i}{|x|^n}$, $i = 1, \cdots, n$, a direct computation shows that

$$
|\nabla^2 \Gamma(x)| \leq \frac{2n^2}{|B_1|} |x|^{-n}.
$$
Let $r \leq \min \left\{ \frac{1}{10R}, \frac{R}{16} \right\}$. If $|x| \leq \frac{r}{4}$, we have

$$D_i \omega(x) - D_i \omega(0) = \int_{B_r(0)} (D_i \Gamma(x - y) - D_i \Gamma(-y)) f(y)dy$$

$$+ \int_{B_r(0) - B_r(0)} (D_i \Gamma(x - y) - D_i \Gamma(-y)) f(y)dy.$$ 

When $|y| \geq r$, we can find $t \in (0, 1)$, such that

$$|D_i \Gamma(x - y) - D_i \Gamma(-y)| \leq |x| \cdot |\nabla^2 \Gamma(tx - y)|.$$ 

Without loss of generality, we can assume that $||f||_{L^\infty} = 1$. Then

$$\left| \int_{B_r(0) - B_r(0)} (D_i \Gamma(x - y) - D_i \Gamma(-y)) f(y)dy \right| \leq |x| \int_{B_r(0) - B_r(0)} |\nabla^2 \Gamma(tx - y)| dy$$

$$= 2^{n+1} n^3 |x| (\log(R) - \log(r)) \leq 2^{n+2} n^3 |x| |\log(r)|.$$ 

On the small ball $B_r(0)$, we can obtain another estimate as follows.

$$\left| \int_{B_r(0)} (D_i \Gamma(x - y) - D_i \Gamma(-y)) f(y)dy \right| \leq 2 \int_{B_r(0)} |D_i \Gamma(-y)| dy$$

$$\leq 2 \int_{B_r(0)} \frac{|y|^{1-n}}{n |B_1|} dy = 4r.$$ 

Combine the estimation of this two integrals, we have

$$|\nabla \omega(x) - \nabla \omega(0)| \leq 4r + 2^{n+1} n^3 |x| |\log(r)|.$$ 

Fix $\epsilon = \min \left\{ \frac{1}{100R}, \frac{R}{200} \right\}$. If $|x| \leq \epsilon$, then we can choose $r = -|x| \log(|x|)$. It is easy to check that $r \leq \min \left\{ \frac{1}{100R}, \frac{R}{200} \right\}$, $r > 5|x|$, and thus

$$|\nabla \omega(x) - \nabla \omega(0)| \leq -4|x| \log(|x|) + 2^{n+1} n^3 |x| |\log(r)|$$

$$\leq 2^{n+3} n^3 |x| \log(|x|).$$ 

This completes the proof. 

Since $\Delta \omega = f$, we can obtain a similar estimate about the solution of Poisson’s equation as following:

**Lemma 6.5.** If $f \in C^\infty(B_R(0))$, and $|f| + |\Delta f| \leq K$, then we can find constant $C = C(n, K)$, $\epsilon = \epsilon(n, R, K)$, such that for each $|x| \leq \epsilon$,

$$|\nabla f(x) - \nabla f(0)| \leq C|x| |\log(|x|)|.$$ 

**Proof.** Let $\psi$ be the Newtonian potential of $\Delta f$, then we have $\Delta(\psi - f) = 0$, and it is immediate that there exists a constant $C_1 = C_1(n, R, K) > 0$ such that $|\psi| \leq C_1$.

By the interior derivative estimates for harmonic functions, there is a constant $C_2 = C_2(n, R, K)$, satisfies that $|\nabla^2(\psi - f)| \leq C_2$ on $B_{\frac{R}{2}}(0)$. By Lemma 6.4, $|\nabla \psi(x) - \nabla \psi(0)| \leq C_3 |x| |\log(|x|)|$, when $|x| \leq \epsilon$, where $\epsilon$ and $C_3$ are constants depend only on $n, R, K$. 


Hence we have
\[ |\nabla f(x) - \nabla f(0)| \leq |\nabla \psi(x) - \nabla \psi(0)| + |\nabla (\psi - f)(x) - \nabla (\psi - f)(0)| \leq C_3 |x| \log(|x|) + C_2 |x|, \]
and the lemma follows.

Now we will consider the $C^{1,\alpha}$-convergence of Bergman metrics. Recall that
\[ R_{kkij} = -\frac{\partial^2 g_{ij}}{\partial z_k \partial \bar{z}_k} - g^{st} \frac{\partial g_{ij}}{\partial z_k} \frac{\partial g_{st}}{\partial \bar{z}_k}, \]
then we can apply Lemma 6.5 to $g_{ij}$:

**Lemma 6.6.** Under the conditions stated in the previous section, there exists a positive constant $\epsilon$, $C$, $m_0$ depends only on $t$, $Q$, $r$, $n$, $K$, such that for each $|x - y| \leq \epsilon$,
\[ |\nabla g_{ij}(x) - \nabla g_{ij}(y)| \leq C|x - y| |\log(|x - y|)|, \]
when $m > m_0$, where $g_{ij} = g \left( \frac{\partial g_{ij}}{\partial z_i}, \frac{\partial g_{ij}}{\partial \bar{z}_j} \right)$.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2:** It is sufficient to show that
\[ |\nabla g_{ij,m}(z) - \nabla g_{ij,m}(0) - \nabla g_{ij}(z) + \nabla g_{ij}(0)| \leq C|z|^\alpha m^{\frac{1-\alpha}{2}} |\log(m)|^\alpha, \]
on the fixed uniform holomorphic chart $(z_1, \ldots, z_n)$.

Clearly, Theorem 1.1 implies that
\[ |\nabla g_{ij,m}(z) - \nabla g_{ij,m}(0) - \nabla g_{ij}(z) + \nabla g_{ij}(0)| \leq C_1 m^{-\frac{1}{4}}, \]
where $C_1 = C_1(K, v, t)$ is a constant.

By Lemma 6.1, Lemma 6.2 and Lemma 6.6, we can conclude that there are constants $\epsilon = \epsilon(K, v, t)$ and $C_2 = C_2(K, v, t)$, such that if $|z| \leq \epsilon$
\[ |\nabla g_{ij,m}(z) - \nabla g_{ij,m}(0) - \nabla g_{ij}(z) + \nabla g_{ij}(0)| \leq C_2 |z| \log(m) + C_2 |z| |\log(|z|)|. \]

Let $\delta = m^{-\frac{1}{4}} |\log(m)|^{-1}$. If $|z| \geq \delta$, an easy computation shows that
\[ |\nabla g_{ij,m}(z) - \nabla g_{ij,m}(0) - \nabla g_{ij}(z) + \nabla g_{ij}(0)| \leq |z|^\alpha m^{\frac{1-\alpha}{2}} |\log(m)|^\alpha \cdot C_1 m^{-\frac{1}{4}} \]
\[ \leq C_1 |z|^\alpha m^{\frac{1-\alpha}{2}} |\log(m)|^\alpha. \]

Since $\lim_{m \to \infty} m^{-\frac{1}{4}} |\log(m)|^{-1} = 0$, we can find $m_1 = m_1(\epsilon) > 0$ such that $\delta \leq \epsilon$, $\forall m > m_1$. By above argument, if $|z| \leq \delta$ and $m > m_1$, we have
\[ |\nabla g_{ij,m}(z) - \nabla g_{ij,m}(0) - \nabla g_{ij}(z) + \nabla g_{ij}(0)| \leq C_2 |z|^\alpha \delta^{1-\alpha} \log(m) + C_2 |z|^\alpha |z|^{1-\alpha} |\log(|z|)| \]
\[ \leq C_2 |z|^\alpha \left( m^{\frac{1-\alpha}{2}} |\log(m)|^\alpha + |z|^{1-\alpha} |\log(|z|)| \right). \]
A direct computation shows that
\[ \delta^{1-\alpha} |\log (\delta)| \leq 2m^{-\frac{1-\alpha}{2}} \log (m). \]
We thus get
\[ |\nabla g_{i,j,m}(z) - \nabla g_{i,j,m}(0) - \nabla g_{i,j}(z) + \nabla g_{i,j}(0)| \]
\[ \leq C_2 |z|^\alpha \left( m^{\frac{1-\alpha}{2}} |\log (m)|^\alpha + |z|^{1-\alpha} |\log (|z|)| \right) \]
\[ \leq 2C_2 |z|^\alpha m^{-\frac{1-\alpha}{2}} |\log (m)|^\alpha, \]
when \( |z| \leq \delta \), and \( m > m_1 + m_2 \), which proves the theorem. \( \square \)

7. Examples

Examples 7.1-7.3 show that we cannot control the \( C^0 \)-convergence rate of Bergman metrics if we drop only one of the conditions \( \sec \geq K \), \( \sec \leq K \) or \( \Vol B_1(x_0) > v \) in Theorem 1.1.

Example 7.4 provides a \( C^{1,1} \) polarized pointed Kähler manifold \((M,g,L,x_0)\) that satisfies
\[ \liminf_{m \to \infty} \sqrt{m} \|\nabla g_m\| > 0. \]

Example 7.5 demonstrates that the conditions in Theorem 1.1 are not sufficient to control the convergence rate of \( \nabla^2 g_m \) in \( L^1 \) norm.

Example 7.1. Let \( M = \mathbb{C}P^1, L = \mathcal{O}(1), \theta \in (0,1) \), then we have two open sets
\[ U_0 = \{ [1, w] \in \mathbb{C}P^1 \}, \]
\[ U_1 = \{ [z, 1] \in \mathbb{C}P^1 : |z| < 1 \} \]
in \( M \), such that \( U_0 \cup U_1 = M \). Choose a radial cut-off function \( \eta \in C_c^\infty (B_1(0)) \subset C_c^\infty (\mathbb{C}) \) such that \( 0 \leq \eta \leq 1 \) and \( \eta = 1 \) on \( B_{\frac{1}{2}}(0) \). Then we construct a sequence of \( C^{1,1} \) functions \( f_n \in C^{1,1} (\mathbb{C}) \),
\[ f_n(z) = \begin{cases} \frac{1}{8} (\theta - 1) \left( e^{2n}|z|^2 - 1 - 2n \right), & \text{if } |z| \leq e^{-n}, \\ \eta(|z|)(\theta - 1) \log(|z|) - (1 - \eta(|z|)) \log (1 + |z|^2), & \text{if } |z| > e^{-n}. \end{cases} \]

Clearly, \( \Delta f_n \leq e^{-n} \leq 0 \). Then we can find a sequence of functions \( u_n = u_n(|z|) \in C^\infty(\mathbb{C}) \) such that \( |u_n - f_n| \leq 1 \), \( u_n = f_n \) on \( \mathbb{C} - B_{\frac{1}{2n}}(0) \), and \( \Delta u_n \leq 0 \) on \( B_{\frac{1}{2n}}(0) \). Then there is a Kähler metric \( g'_n \) on \( M \) such that \( g'_{n,ij} = e^{-2u_n} \) on \( U_0 \), and \( g'_{n,\bar{j}i} = \frac{1}{(1+|z|^2)^2} \) on \( U_1 \). Let \( g_n = \int_M g'_n \), it’s clear that \( \omega_{g_n} \in c_1(L) \), and there exists a constant \( K > 0 \) such that \( \sec > -K \), \( \forall n \).

For each given \( m \in \mathbb{N} \), there is a basis \( \left\{ e_{0,1}^{z_j} \right\}_{j=0}^{m} \) of \( H^0 (M, L^m) \). Since \( f_n(z) \leq (\theta - 1) \log(|z|) \) on \( B_{\frac{1}{2}}(0) \) and \( |z|^{1-\theta} \in L^2 (B_1(0)) \), we have a constant \( c > 0 \) satisfying \( c < \int_M \omega_{g_n} < \frac{1}{c}, \forall n \in \mathbb{N} \).

Let \( h_n = e^{\frac{c}{|z|^2 + |z|^2}} \) be the unique hermitian metric on \( L \) such that
\[ \text{Ric}(h_n) = \omega_{g_n} = -\sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \varphi_n + \omega_{FS}, \]
and \( \int_M \varphi_n \omega_{FS} = 0 \). Apply the boundedness of the Green operator to \((M, \omega_{FS})\), we have a constant \( t > 0 \) such that
\[ \| \varphi_n \|_{L^2(M)} = \| G_{FS} \Delta_{FS} \varphi_n \|_{L^2(M)} \leq t \| \Delta_{FS} \varphi_n \|_{L^2(M)}, \forall n \in \mathbb{N}. \]
Now we have a constant \( C > 0 \) such that
\[ \sup_M |\varphi_n| \leq C \left( \| \omega_g - \omega_{FS} \|_{L^2(M)} + \| \varphi_n \|_{L^2(M)} \right) \leq C, \forall n \in \mathbb{N}. \]
Since $h_n$ is unique, $\partial \bar{\partial} \varphi_n(e^{ix} w) = \partial \bar{\partial} \varphi_n(w)$, we have $\varphi_n(w) = \varphi_n(|w|)$ on $U_0$, and hence we have

$$
\int_M \left< \frac{j^{m-j}}{2\pi} \right| h_n^m = 0, \quad \text{if } j \neq k,
\int_M \left< \frac{j^{m-j}}{2\pi} \right| h_n^m = a_{m,n}.
$$

Now we can find a constant $M > 0$ such that $a_{m,n} \in \left( \frac{1}{M}, M \right)$, $\forall n \in \mathbb{N}$. Then the Bergman metrics $\frac{1}{2\pi} \partial \bar{\partial} \log \left( \sum_{j=1}^{m+1} a_{m,n} |z_0|^{2j} |z_1|^{2m-2j} \right) \leq C' \omega_{FS}$ for some constant $C' > 0$. But

$$
limit_{n \to \infty} \frac{g_{n,11}([1,0])}{g_{FS,11}([1,0])} = \infty,
$$

it shows that the distances between $g_n$ and the Bergman metrics $g_{n,m}$ satisfying

$$
limit_{n \to \infty} \sup_M \|g_{n,m} - g_n\| \geq 1,
$$

for each given $m \in \mathbb{N}$.

So we cannot control the rate of convergence of Bergman metrics if we only assume that $\sec \geq -K$ and $Vol(B_1(x)) > v$, $\forall x \in M$.

**Example 7.2.** Let $M = \mathbb{C}P^1$, $L = \mathcal{O}(1)$, choose an open covering $U_0, U_1$ of $M$, where

$$
U_0 = \{ [1, w] \in \mathbb{C}P^1 : |w| < 2 \},
U_1 = \{ [z, 1] \in \mathbb{C}P^1 : |z| < 2 \}.
$$

Pick a radial cut-off function $\eta = \eta(|z|) \in C_0^\infty(B_2(0)) \subset C_0^\infty(\mathbb{C})$, s.t. $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_\frac{3}{2}(0)$. For each $n \in \mathbb{N}$, we define

$$f_n(z) = \eta(z) + (1 - \eta(z)) \eta \left( \frac{z}{e^{zn}} \right) \frac{1}{|z||\log(1)|} + (1 - \eta(z)) \left( 1 - \eta \left( \frac{z}{e^{zn}} \right) \right) \frac{1}{e^n |z|}.$$

It’s obvious that there is a sequence of Kähler metrics $g_n'$ on $M$ such that

$$g_{n,11}' = \begin{cases} e^{4\eta} f_n^2 (e^{zn} w) & \text{on } U_0, \\ e^{4\eta} f_n^2 (e^{zn} z) & \text{on } U_1. \end{cases}$$

Through a direct calculation, we get $2\pi < Vol(M, g_n') < 20\pi$, $\forall n \in \mathbb{N}$, and the sectional curvature of $g_n'$ satisfies that $|\sec| \leq 100\pi$.

Let $g_n = \frac{1}{\sqrt{Vol(M, g_n')}} g_n'$, then $Vol(M, g_n) = 1$ shows that the Kähler form $\omega_n$ associated with $g_n$ belongs to $c_1(L)$, $\forall n \in \mathbb{N}$. Now we can choose a hermitian metric $h_n$ on $L$ for each $n \in \mathbb{N}$, such that $Ric(h_n) = \omega_n$. Since $g_n$ is invariant under the following $S^1$-action:

$$S^1 \times M \rightarrow M, \quad \left( e^{\theta} [z, w] \right) \mapsto \left[ e^{\theta} z, w \right],$$

$\partial \bar{\partial} \log (h_n) = -2\pi \omega_n$ shows that $h_n$ can be represented as radial functions on $U_0$ and $U_1$, if we choose the standard trivialization. It follows that $\left\{ z^j w^{m-j} \right\}_{j=0}^m$ becomes an $L^2$-orthogonal basis of $H^0(M, L^m), \forall n \in \mathbb{N}$. Write $a_{n,m,j} = \left( \int_M \|z^j w^{m-j}\|_{h_n}^2 dV_{g_n} \right)^{-\frac{1}{2}}$, then $a_{n,m,j} = a_{n,m,m-j}$, and the
$m$-th Bergman metric of $g_n$, which is $g_{n,m}$, satisfies that

$$g_{n,m,11} = \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \sum_{j=0}^{m} |a_{n,m,j} z^j|^2 \right)$$

$$= \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \sum_{j=0}^{m} |a_{n,m,j} z^{j-1}|^2 \right) - \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \sum_{j=0}^{m} |a_{n,m,j} z^j|^2 \right),$$

on $U_1$. Setting $z = 1$ and $m = 2l + 1$ one obtains

$$g_{n,m,11}([1,1]) = \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \sum_{j=0}^{m} |a_{n,m,j}|^2 \right) - \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \sum_{j=0}^{m} |a_{n,m,j}|^2 \right)$$

$$\geq \frac{m^2 |a_{n,m,0}|^4 + \sum_{1 \leq j < k \leq m} (j-k)^2 |a_{n,m,j}|^2 |a_{n,m,k}|^2}{2\pi (m+1)^2 \left( \sum_{j=0}^{m} |a_{n,m,j}|^4 \right)}$$

$$\geq \frac{1}{4\pi (m+1)^2},$$

when $l > m_0 + 2$. Since $\lim_{n \to \infty} g_{n,11}([1,1]) = 0$, we get

$$\lim \inf_{n \to \infty} \sup_M \|g_{n,2m+1} - g_n\| \geq 1,$$

for each given $m > m_0 + 2$.

**Example 7.3.** Let $U = \mathbb{R}^2$ be a riemannian manifold with metric

$$g_U = dr^2 + \psi^2(r)d\theta^2,$$

where $\psi$ is a non-negative smooth function on $\mathbb{R}$, such that $\psi(r) = r$ on $[0, 10]$, $\psi > 0$ on $(0, \infty)$, and $\psi(r) = e^{-r}$ when $r > 20$.

When $k > 30$, we can construct a Riemannian manifold $(kT^2, h)$ such that $\text{Vol}_h(kT^2) = 10$, where $kT^2 = T^2 \# \cdots \# T^2$ ($k$ times). By gluing $B_{\log(k)}(0) \subset U$ and $kT^2$, we can get a Riemannian manifold $(M, g)$ that satisfies $\text{Vol}_g(M) = 400$, $\text{Vol}_B_{1}(p) = \pi$, and $\text{sec} = 0$ on $B_1(p)$, where $p = 0 \in U$. Since $\dim M = 2$, there exist a complex structure on $M$ such that $(M, g)$ becomes a Kähler manifold. Let $L = O_{400p}$. It’s clear that $\omega_L \in c_1(L)$.

For each $m \in \mathbb{N}$, choose $k > 30 + 400m$, then we have $H^0(M, L^m) \cong \mathbb{C}$. If not, the Riemann-Hurwitz formula shows that $\chi_M = -2\deg f + \sum_{p \in M} (e_p - 1)$ for each meromorphic function on $M$, where $e_p$ is the ramification index under $f$, and hence $2k - 2 = -\chi_M \leq \sum_{p \in M} (e_p - 1) \leq 400m$.

It shows that if we drop the condition $\sec > -t$, then although the local geometric structure can still be controlled, $m_0$ may become too large.

**Example 7.4.** Let $M = \mathbb{C}P^1$, $L = O(1)$, then we consider the dense open subset

$$U = \{ [z, 1] \in \mathbb{C}P^1 : z \in \mathbb{C} \}$$

of $M$ and the classical frame $e_L$ of $L$ over $U$. Choose a radial cut-off function $\eta \in C_0^\infty(B_1(0))$, such that $0 \leq \eta \leq 1$, $|\nabla \eta| \leq 3$, $|\nabla^2 \eta| \leq 30$ and $\eta = 1$ on $B_{2}(0)$. Of course,

$$h(e_L, e_L) = \frac{1}{1 + |z|^2} e_{-\eta |z| |z|^{2} (z + \bar{z})}$$
is a $C^{3,1}$ hermitian metric on $L$, and, in consequence,
\[ \omega = Ric(h) = \omega_{FS} - \frac{\sqrt{1}}{800 \pi} \partial \bar{\partial} \left( \eta(z)|z|^{3}(z + \bar{z}) \right) \]
is a $C^{1,1}$ polarized Kähler metric with bounded sectional curvature, where $\omega_{FS}$ is the Fubini-Study metric. For each given $k \in \mathbb{Z}_{+}$, a trivial verification shows that
\[ T_{k}^{m} = \left( \int_{M} |z|^{2k} \langle e_{L}, e_{L} \rangle_{h^{m}} \omega \right)^{-\frac{1}{2}} z^{k} e_{L} \]
are peak sections of the point $x_{0} = [0, 1] \in U$. Write $a = h(e_{L}, e_{L})$, then we can conclude that
\[ \int_{M} |z|^{2k} \langle e_{L}, e_{L} \rangle_{h^{m}} \omega = \frac{k !}{m^{k+1}} + O \left( \frac{1}{m^{k+2}} \right), \]
and similarly,
\[ \int_{M} |z|^{2k} \langle e_{L}, e_{L} \rangle_{h^{m}} \omega = \frac{4k + 1}{800} \cdot \sqrt{\pi (2k + 3)!!} + O \left( \frac{1}{m^{k+3}} \right), \]
for each given $k \in \mathbb{Z}_{+}$, where $O \left( \frac{1}{m^{k}} \right)$ denotes a quantity dominated by $\frac{C}{m^{k}}$ with the constant $C$ depending only on $k$.

Choosing an $L^{2}$ orthonormal basis $\{ S_{j}^{m} \}_{j=0}^{m}$ of $H^{0}(M, L^{m})$ such that the local representation $S_{j}^{m} = f_{j}^{m} e_{L}$ satisfies that $\beta_{ii} > 0$, and
\[ \frac{\partial^{i} f_{j}^{m}}{\partial z^{i}} (x_{0}) = 0, \forall i < j. \]
It is easy to check that there are constants $\beta_{ij}^{m}$ such that $T_{i}^{m} = \sum_{j=0}^{m} \beta_{ij}^{m} S_{j}^{m}$, and the above conditions implies that $\beta_{ij} = 0, \forall i < j$. It follows that $\beta_{ii}^{m} = 1 + O \left( \frac{1}{m^{2}} \right)$, $\sum_{j>i} | \beta_{ij}^{m} |^{2} = O \left( \frac{1}{m^{2}} \right)$ and $\sum_{j>i+1} | \beta_{ij}^{m} |^{2} = O \left( \frac{1}{m^{3}} \right)$. Consider the $L^{2}$ inner product of $H^{0}(M, L^{m})$, we have
\[ \beta_{0}^{m} \tilde{\beta}_{11}^{m} + \sum_{j=2}^{m} \beta_{0}^{m} \tilde{\beta}_{j1}^{m} = \int_{M} \langle T_{0}^{m}, T_{1}^{m} \rangle_{h^{m}} \omega \]
\[ = \frac{3\sqrt{\pi}}{6400m} + O \left( \frac{1}{m^{\frac{3}{2}}} \right), \]
hence we can conclude that
\[ \beta_{0}^{m} = \frac{3\sqrt{\pi}}{6400m} + O \left( \frac{1}{m^{\frac{3}{2}}} \right). \]
Similar arguments apply to $T_{i}^{m}$ and $T_{2}^{m}$, one obtains
\[ \beta_{12}^{m} = \frac{3\sqrt{\pi}}{512m^{\sqrt{2}}} + O \left( \frac{1}{m^{3/2}} \right). \]
Since $T_{0}^{m} = \sum_{j=0}^{m} \beta_{0j}^{m} S_{j}^{m}$, we have
\[ \beta_{0}^{m} \frac{\partial f_{0}^{m}}{\partial z} (x_{0}) + \beta_{01}^{m} \frac{\partial f_{1}^{m}}{\partial z} (x_{0}) = 0, \]
hence that
\[ \frac{\partial f_{0}^{m}}{\partial z} (x_{0}) = - \frac{3\sqrt{\pi}}{6400} + O \left( \frac{1}{m^{\frac{3}{2}}} \right). \]
Likewise,

\[
\frac{\partial^2 f^m}{\partial z^2}(x_0) = -\frac{3\sqrt{m}^2}{512} + O(1).
\]

Then we have

\[
\frac{\partial g_{m,1\bar{1}}}{\partial z}(x_0) = \frac{1}{2\pi m} \frac{\partial^2 f^m}{\partial z^2} \cdot \frac{\partial f^m}{\partial \bar{z}} - \frac{1}{\pi m} \frac{f^m_0}{|f_0^m|^2} \frac{\partial f^m_0/\partial z \cdot \partial f^m_0/\partial \bar{z}} = -\frac{369}{128000\sqrt{m^2}} + O\left(\frac{1}{m}\right).
\]

**Remark.** Choose a sequence of smooth polarized pointed Kähler manifold \((M, g_k, L, x_0)\) such that \(\frac{\partial g_{k,1\bar{1}}}{\partial z}(x_0) = 0\), and \(g_k \to g\) in \(C^{1,1}\)-topology as \(k \to \infty\). It is clear that

\[
\liminf_{m \to \infty} \lim_{k \to \infty} \pi m \left| \frac{\partial g_{k,m,1\bar{1}}}{\partial z}(x_0) \right| > 0.
\]

**Example 7.5.** Let \(M = \mathbb{C}P^1, L = O(1)\), then we have two open sets

\[
U_0 = \{[1, w] \in \mathbb{C}P^1\}, \quad U_1 = \{[z, 1] \in \mathbb{C}P^1 : |z| < 1\}
\]

in \(M\), such that \(U_0 \cup U_1 = M\). Choose a radial cut-off function \(\eta \in C_0^\infty(B_1(0)) \subset C_0^\infty(\mathbb{C})\), s.t. \(0 \leq \eta \leq 1\) and \(\eta = 1\) on \(B_{\frac{1}{2}}(0)\). For each \(k \in \mathbb{N}\), we define

\[
\varphi_k = k^{-4} \sin(kz + \bar{kz}) \sin(\sqrt{-1}kz - \sqrt{-1}k\bar{z})\eta(z)
\]

on \(U_1\). Then \(h_k = e^{\varphi_k} h_0\) gives a Hermitian metric on \(L\), where \(h_0\) be the normal metric on \(L\), i.e. \(h_0 = \frac{1}{|w|^2+1}\) on \(U_0\), and \(h_0 = \frac{1}{|z|^2+1}\) on \(U_1\). Clearly, \(Ric(h_0) = \omega_{FS}\) on \(M\), and hence

\[
Ric(h_k) = \omega_{FS} - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi_k.
\]

For sufficiently large \(k\), \(Ric(h_k)\) is also a Kähler form. Let \(g_k\) be the Kähler metric corresponds to \(Ric(h_k)\). Recall that

\[
R_{1\bar{1}1\bar{1}} = -\frac{\partial^2 g_{1\bar{1}}}{\partial z \partial \bar{z}} - g^{1\bar{1}} \frac{\partial g_{1\bar{1}}}{\partial z} \frac{\partial g_{1\bar{1}}}{\partial \bar{z}},
\]

we can find a constant \(C > 0\) such that on \(U_1\),

\[
\left| R_{k,1\bar{1}1\bar{1}} + \frac{64k^4}{\pi} \varphi_k - R_{FS,1\bar{1}1\bar{1}} \right| \leq \frac{C}{k^4}
\]

where \(R_{FS,1\bar{1}1\bar{1}}\) is the curvature of Fubini-Study metric on \(U_1\), and \(R_{k,1\bar{1}1\bar{1}}\) is the curvature of \(g_k\) on \(U_1\).

It is easy to check that there is a constant \(\delta > 0\) satisfies that

\[
\liminf_{k \to \infty} \|k^4 \varphi_k - f\|_{L^1(M)} > \delta,
\]

for each given \(f \in L^1(M)\).

If \(\nabla^2 g_{m,k}\) converges to \(\nabla^2 g_k\) in \(L^1\) norm uniformly as \(m \to \infty\), then \(R_{m,1\bar{1}1\bar{1}}\) also converges to \(R_{k,1\bar{1}1\bar{1}}\) in \(L^1\) sense uniformly, where \(R_{m,1\bar{1}1\bar{1}}\) is the curvature of \(g_{m,k}\) on \(U_1\). But for each fixed
m, it’s clear that the sequence of Bergman metrics $g_{m,k}$ must be convergence in $C^\infty$-topology as $k \to \infty$. Hence $R^m_{k,1111}$ is also convergence for each given $m$, then combining (18) and (19) we deduce that

$$0 = \liminf_{k \to \infty} \left\| R^m_{k,1111} + \frac{64k^4}{\pi} \varphi_k - R_{FS,1111} \right\|_{L^1(M)}$$

$$= \liminf_{m,k \to \infty} \left\| R^m_{k,1111} + \frac{64k^4}{\pi} \varphi_k - R_{FS,1111} \right\|_{L^1(M)} \geq \frac{64\delta}{\pi},$$

contradiction.

If there is a uniform sequence $b_m$ such that $\lim_{m \to \infty} b_m = 0$, and

$$\left\| m^{1-n} \rho_{\omega_k,m} - m - a_1 \right\|_{L^1} \leq b_m,$$

for any sufficiently large $k$ and $m$, then it is obvious that $a_1 = \frac{scal}{2}$, where $scal$ is the scalar curvature. Since $g_k$ converges to $g$ in $C^\infty$-topology as $k \to \infty$, $m^{1-n} \rho_{\omega_k,m} - m$ is also convergence in $C^\infty$-topology as $k \to \infty$, for each given $m$. It follows that the sequence of scalar curvatures of $g_k$ is convergence in $L^1$ norm as $k \to \infty$, contrary to (19).

**Appendix A. Holomorphic Norms of Kähler Manifolds**

The results in this appendix are essentially obtained by M. Anderson, J. Cheeger, and M. Gromov ([1], [2], [6], [12]). In fact, this is just a holomorphic version of harmonic norm. The details of classical $C^{m,\alpha}$-norm and $C^{m,\alpha}$-convergence theory can also be found in Chapter 11 of [19]. Now we will give some basic properties of holomorphic norms. The following result may be proved in much the same way as Proposition 11.3.2 in [19], where $\| \cdot \|_{\text{holo}}^{C^{m,\alpha}}$ is defined in Definition 2.1.

**Proposition A.1.** Given $(M, g, p)$, $m \geq 0$, $\alpha \in (0, 1]$ we have:

(20) $\| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}} \leq \| (M, \lambda^2 g, p) \|_{\text{holo}}^{C^{m,\alpha}}$ for all $\lambda > 0$.

(21) The function $r \mapsto \| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}}$ is increasing, continuous, and converges to 0 as $r \to 0$.

(22) If $\| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}} < Q$, then for all $x_1, x_2 \in B_r(0)$ we have

$$e^{-Q} \min \left\{ \left| x_1 - x_2 \right|, 2r \right\} \leq \left| \psi(x_1) - \psi(x_2) \right| \leq e^Q \left| x_1 - x_2 \right|.$$

(23) The norm $\| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}}$ is realized by a $C^{m+1,\alpha}$-chart.

(24) If $M$ is compact, then $\| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}} = \| (M, g, p) \|_{\text{holo}}^{C^{m,\alpha}}$ for some $p \in M$.

Now we will introduce the $C^{m,\alpha}$-convergence concept for this norm. The classic definition of $C^{m,\alpha}$-convergence in [19] is not directly applicable to Kähler geometry, because we have to consider the complex structure on the Kähler manifold $M$ at the same time. The following example can explain this observation.

**Example A.2.** Pick a cut-functions $\eta = \eta(|z|) \in C_c^\infty(B_1(0))$, such that $\eta(0) = 1$ and $0 \leq \eta \leq 1$. Let $(M_i, g_i, p_i)$ be the constant sequence $(\mathbb{C}, g, 0)$, where the Kähler metric $g_{11} = 1 + \eta(4|z|) + \eta(6|z - \sqrt{-1}|) + \eta(8|z - 1 - 2\sqrt{-1}|)$.

Clearly, $(M_i, g_i, p_i)$ converges to $(\mathbb{C}, g, 0)$ in the classical pointed $C^{m,\alpha}$-topology, $\forall m \geq 0$, but the pointed Kähler manifold $(\mathbb{C}, g, 0)$ doesn’t isomorphic to $(\mathbb{C}, g, 0)$. 
This example tells us that the definition of $C^{m,\alpha}$-convergence needs to be modified when the manifolds are Kähler manifolds.

**Definition A.3 (Holomorphic $C^{m,\alpha}$-convergence).** A sequence of pointed complete Kähler manifolds is said to converge in the pointed $C^{m,\alpha}$-topology, $(M_i, g_i, J_i, p_i) \to (M, g, J, p)$, if for every $R > 0$ we can find a domain $\Omega \supset B_R(p) \subset M$ and $C^{m+1,\alpha}$-embeddings $F_i : \Omega \to M_i$ for large $i$ such that $F_i(p) = p_i$, $F_i(\Omega) \supset B_R(p_i)$, $F_i^* g_i \to g$ and $F_i^* J_i \to J$ in the $C^{m,\alpha}$-topology.

We can now state and prove the Arzela-Ascoli type theorem on the Kähler manifolds.

**Theorem A.4.** For given $Q > 0$, $n \geq 1$, $m \geq 0$, $\alpha \in (0, 1]$, and $r > 0$ consider the class $\mathcal{M}_{\text{holo}}^{m,\alpha}(n, Q, r)$ of complete, pointed Kähler $n$-manifolds $(M, g, p)$ with $\|(M, g)\|_{C^{m,\alpha}, r} \leq Q$. The class $\mathcal{M}_{\text{holo}}^{m,\alpha}(n, Q, r)$ is compact in the pointed $C^{m,\beta}$-topology for all $\beta < \alpha$.

**Proof.** This theorem can be proved by modifying the proof of Theorem 11.3.6 in [19], since the limit of a sequence of holomorphic functions is also holomorphic. \[\square\]

**Remark.** We can also define the holomorphic $W^{m,p}$-norm of Kähler manifolds. When $mp > n$, we can obtain similar propositions about $W^{m,p}$-norm. See [18] for details.

Then we will prove a result about the continuity of holomorphic $C^{m,\alpha}$-norm.

**Proposition A.5.** Suppose $(M_i, g_i, J_i, p_i) \to (M, g, J, p)$ in $C^{m,\alpha}$, $m \geq 1$, $\alpha > 0$, and the Ricci curvature $\text{Ric}(g_i) \geq -c_i$, for some constant $c > 0$ independent of $i$. Then

$$\|(M_i, g_i, p_i)\|_{C^{m,\alpha}, r}^{\text{holo}} \to \|(M, g, p)\|_{C^{m,\alpha}, r}^{\text{holo}} \text{ for all } r > 0.$$  

Moreover, when all the manifolds have uniformly bounded diameter

$$\|(M_i, g_i)\|_{C^{m,\alpha}, r}^{\text{holo}} \to \|(M, g)\|_{C^{m,\alpha}, r}^{\text{holo}} \text{ for all } r > 0.$$

**Proof.** First, we show the easy part:

$$\liminf_{i \to \infty} \|(M_i, g_i, p_i)\|_{C^{m,\alpha}, r}^{\text{holo}} \geq \|(M, g, p)\|_{C^{m,\alpha}, r}^{\text{holo}}.$$  

For each $Q > \liminf \|(M_i, g_i, p_i)\|_{C^{m,\alpha}, r}^{\text{holo}}$, we can find holomorphic charts $\phi_i : B_r(0) \to M_i$ with the requisite properties for sufficiently large $i > 0$. After passing to a subsequence, we can make these charts converge to a holomorphic chart

$$\phi = \lim F_i^{-1} \circ \phi_i : B_r(0) \to M.$$  

Since the metrics and complex structure converge in $C^{m,\alpha}$, $\phi$ must be a holomorphic chart, and it follows that $\|(M, g, p)\|_{C^{m,\alpha}, r}^{\text{holo}} \leq Q$.

For the reverse inequality

$$\limsup \|(M_i, g_i, p_i)\|_{C^{m,\alpha}, r}^{\text{holo}} \leq \|(M, g, p)\|_{C^{m,\alpha}, r}^{\text{holo}},$$  

select $Q > \|(M, g, p)\|_{C^{m,\alpha}, r}^{\text{holo}}$. Then we can find $\varepsilon > 0$ such that also $\|(M, g, p)\|_{C^{m,\alpha}, r+2\varepsilon}^{\text{holo}} < Q$. Choose $\varphi : B_{r+3\varepsilon}(0) \to U \in M$ satisfying the usual conditions, and let

$$U_i = F_i(\varphi(B_{r+\varepsilon}(0))),$$  

$$V_i = F_i(\varphi(B_{r+\varepsilon}(0))).$$
Obviously, we can assume that \(U_i, V_i\) are domains with smooth boundaries
\[
\partial U_i = F_i(\varphi(\partial B_{\gamma_i}(0))) ,
\]
\[
\partial V_i = F_i(\varphi(\partial B_{\tau_i}(0))) .
\]
for each \(i \in \mathbb{N}\).

Consider the coordinates \(z_1, \ldots, z_n\) of \(B_{\gamma_i+3\epsilon}(0) \subset \mathbb{C}^n\), and denote the function \(z_k \circ \varphi^{-1} \circ F_i^{-1}\) on \(F_i(\varphi(B_{\gamma_i+2\epsilon}(0)))\) as \(\tilde{z}_{i,k}\).

Since \(2\partial \tilde{z}_{i,k} = (J_i + \sqrt{-1}) dz_{i,k}\) and \(\partial \tilde{\partial} = 0\) on \(M_i\), we can conclude that
\[
2\partial \partial \left( \sum_{k=1}^{n} |z_{i,k}|^2 \right) = \sum_{k=1}^{n} d \left[ z_{i,k} (J_i + \sqrt{-1}) d\tilde{z}_{i,k} + \bar{z}_{i,k} (J_i + \sqrt{-1}) dz_{i,k} \right]
\]
\[
= \sum_{k=1}^{n} d z_{i,k} \wedge (J_i + \sqrt{-1}) d\tilde{z}_{i,k} + \sum_{k=1}^{n} z_{i,k} d (J_i d\tilde{z}_{i,k})
\]
\[
+ \sum_{k=1}^{n} d\tilde{z}_{i,k} \wedge (J_i + \sqrt{-1}) dz_{i,k} + \sum_{k=1}^{n} \bar{z}_{i,k} d (J_i d z_{i,k}) .
\]

Note that the above formula only contains the first-order derivative of \(J_i\) at most. By the definition of holomorphic \(C^{m,\alpha}\)-convergence, we have
\[
\lim_{i \to \infty} \| \varphi^* [F_i^* \partial \partial \left( \sum_{k=1}^{n} |z_{i,k}|^2 \right)] - \sum_{k=1}^{n} \partial z_{i,k} \wedge \partial \bar{z}_{i,k} \|_{C^{m-1,\alpha}(B_{\gamma_i+2\epsilon}(0))} = 0 .
\]

It shows that \(U_i\) must be Stein manifold for sufficiently large \(i > 0\). We now plan to use the \(L^2\) method to find coordinates \(w_{i,k}\) on \(U_i\) that are close to \(z_{i,k}\).

Apply an argument similar as above, we can obtain
\[
\lim_{i \to \infty} \| \partial \tilde{z}_{i,k} \|_{C^{m,\alpha}(U_i)} = 0 .
\]

Combining \(Ric(g_i) \geq -c \omega_{g_i}\) with (25), we can assert that
\[
10e^Q \partial \partial \left( \sum_{k=1}^{n} |z_{i,k}|^2 \right) - Ric(g_i) \geq \omega_{g_i},
\]
for sufficiently large \(i > 0\).

Pick weight function \(\psi_i = 10e^Q \sum_{k=1}^{n} |z_{i,k}|^2\) on such \(U_i\). Proposition 2.2 now gives a \(C^{m+1,\alpha}\) function \(u_{i,k}\) such that \(\partial (u_{i,k} - z_{i,k}) = 0\), and
\[
\int_{U_i} |u_{i,k}|^2 e^{-\psi_i} dV_{g_i} \leq \int_{U_i} |\partial \tilde{z}_{i,k}|^2 e^{-\psi_i} dV_{g_i} .
\]

Let \(w_{i,k} = z_{i,k} - u_{i,k}\). The proof is completed by showing that
\[
\lim_{i \to \infty} \| u_{i,k} \circ F_i \|_{C^{m+1,\alpha}(B(0,\tau_i))} = 0 .
\]

Using the Kähler conditions on \((M_i, g_i, J_i)\), we can conclude that
\[
\lim_{i \to \infty} \| \Delta_{g_i} z_{i,k} \|_{C^{m-1,\alpha}(U_i)} \leq \lim_{i \to \infty} \| \partial \tilde{z}_{i,k} \|_{C^{m,\alpha}(U_i)} = 0 .
\]
By Moser iteration, \( \lim_{i \to \infty} \|z_{i,k}\|_{C^0(V_i)} = 0 \). According to the Schauder’s elliptic estimate, we have
\[
\lim_{i \to \infty} \|z_{i,k}\|_{C^{m+1,\alpha}(V_i)} \leq \lim_{i \to \infty} C_1 \|\Delta g_i z_{i,k}\|_{C^{m-1,\alpha}(V_i)} + \lim_{i \to \infty} C_2 \|z_{i,k}\|_{C^0(V_i)} = 0,
\]
where \( C_1, C_2 \) are constants independent of \( i \), which completes the proof.

As an application, we will construct \( C^{1,\alpha} \)-bounded holomorphic chart here.

**Lemma A.6.** Let \( \alpha \in (0, 1), n \geq 1, K > 0 \) and \( R > 0 \) be given. For every \( Q > 0 \), there is an \( r = r(n, \alpha, K, R) > 0 \) such that if the Kähler \( n \)-manifold \((M,g)\) satisfies
\[
\sup_{B_1(x)} |\text{Ric}| \leq K,
\]
\[
\text{inj}(p) \geq R, \forall p \in B_1(x),
\]
then \( \|(M, g, x)\|_{C^{1,\alpha}, r}^{\text{holo}} \leq Q \).

**Proof.** Relacing harmonic norms by holomorphic norms in Lemma 2.2 in [1], we can obtain the proof of this lemma.

Then we state the Cheeger’s lemma.

**Lemma A.7** (Cheeger, 1967). Given \( n \geq 2, v, K > 0 \), and an \( n \)-dimensional Riemannian manifold \((M, g)\) with
\[
\sup_{B_1(x)} |\text{sec}| \leq K,
\]
\[
\text{Vol}B_1(x) \geq v,
\]
then \( \text{inj}(x) \geq R \), where \( R \) depends only on \( n, K, \) and \( v \).

**Proof.** The proof can be found on page 34 of [6].

Combining those two lemmas above:

**Corollary A.8.** Given \( n \geq 1 \) and \( \alpha \in (0, 1), v, K > 0 \), one can find \( r(n, \alpha, K, R) > 0 \) for each \( Q > 0 \) such that if the Kähler \( n \)-manifold \((M,g)\) satisfies
\[
\sup_{B_1(x)} |\text{sec}| \leq K,
\]
\[
\text{Vol}B_1(x) \geq v,
\]
then \( \|(M, g, x)\|_{C^{1,\alpha}, r}^{\text{holo}} \leq Q \).

**APPENDIX B. PROOF OF LEMMA 4.2**

We consider the case \( n = 1 \) at first. In this case, we have \( z = re^{\theta\sqrt{-1}} \), and \( U = D_\delta(0) \). Let \( h(r) = \int_{\partial D_r(0)} f \cos k\theta d\theta \). Then \( \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \) shows that
\[
(26) \quad h'' + \frac{1}{r} h' - \frac{k^2 h}{r^2} = \int_{\partial D_r(0)} \Delta f \cos k\theta d\theta - \frac{1}{r^2} \int_{\partial D_r(0)} \frac{\partial^2 f}{\partial \theta^2} \cos k\theta d\theta - \frac{k^2 h}{r^2} = \int_{\partial D_r(0)} \Delta f \cos k\theta d\theta.
\]
Likewise,  
\[ h''' + \frac{2}{r} h'' - \frac{2k^2 + 1}{r^2} h' + \frac{2k^2 + 1}{r^3} h + \frac{k^2(k^2 - 4)}{r^4} h = \int_{\partial D_r(0)} \Delta^2 f \cos k\theta d\theta. \]

Define \( \varphi(t) = h(e^t) \), we can rewrite (26) as  
\[ \varphi'' - k^2 \varphi = e^{2t} \int_{\partial D_r(0)} \Delta f \cos k\theta d\theta, \]
and hence  
\[ |\varphi'' - k^2 \varphi| \leq 16n^2 K e^{2t}. \]

Then \( (e^{kt} \varphi' - ke^{kt} \varphi)' = e^{kt} (\varphi'' - k^2 \varphi) \) shows that we can find \( a \in \mathbb{R} \) such that  
\[ |\varphi' - k\varphi - ae^{-kt}| \leq 16n^2 K e^{2t}, \]
when \( t \leq \log(\delta) \). Similarly, we can find \( b \in \mathbb{R} \) such that  
\[ |\varphi - \frac{a}{k+\delta_k,0} e^{-kt} - a_\delta_k,0 t - be^{kt}| \leq 16n^2 K e^{2t} + 16n^2 K t e^{2t} \delta_k,2, \]
when \( t \leq \log(\delta) \).  
Assume that \( f(0) = \nabla f(0) = 0 \) now, then we have \( a = b = 0 \) when \( k \neq 2 \). If \( k = 2 \), we can obtain \( a = 0 \), and hence  
\[ |h| \leq 16n^2 C_1 r^2 + 16n^2 K r^2 \log(r) \delta_k,2, \quad \forall r \in [0, \delta], \]
where \( C_1 = C_1(n, K, \delta) > 0 \) be a constant. This gives (i).  
Now we will prove (ii) in the case \( n = 1 \). Rewrite (27) as  
\[ \varphi''' - 4\varphi'' - 2(k^2 - 2)\varphi' + 4k^2 \varphi' + k^2(k^2 - 4) \varphi = e^{4t} \int_{\partial D_{e^t}(0)} \Delta^2 f \cos k\theta d\theta. \]

The classical Schauder’s estimates shows that \( |f| + |\nabla f| + |\nabla^2 f| + |\nabla^3 f| \leq C_2 \) on \( \prod_{j=1}^n D(0, \frac{\delta}{2n}) \), where \( C_2 \) is a constant depending on \( n, K, \) and \( \delta \). Recall that \( \varphi(t) = h(e^t) \), we can rewrite (27) as  
\[ \varphi''' - 4\varphi'' - 2(k^2 - 2)\varphi' + 4k^2 \varphi' + k^2(k^2 - 4) \varphi = e^{4t} \int_{\partial D_{e^t}(0)} \Delta^2 f \cos k\theta d\theta, \]
then we can conclude that  
\[ |\varphi''' - 4\varphi'' + 2\varphi' + 4\varphi - \varphi| \leq 16n^2 K e^{4t}. \]

Argument similar to above implies that  
\[ |\varphi(t)| \leq C_5 e^{4t} + C_3 |t| e^{4t} (\delta_k,2 + \delta_k,4), \]
when \( t \leq \log(\delta) - \log(2n) \). It gives (ii) in the case \( n = 1 \).
When \( n > 1 \), a similar argument gives

\[
\left| \int_{\partial D_{r_j} (0)} \prod_{j=1}^{n-1} \partial f (z) \cos k \theta_1 \wedge \cdots \wedge \partial \theta_n \right|
\leq 2 \pi \left| \int_{\partial D_{r_j} (0)} \prod_{j=1}^{n-1} f (z', 0) \cos k \theta_1 \wedge \cdots \wedge \partial \theta_n \right|
+ 2 \pi r_n^2 \left| \prod_{j=1}^{n-1} \frac{\partial^2 f}{\partial z_n \partial \bar{z}_n} (z', 0) \cos k \theta_1 \wedge \cdots \wedge \partial \theta_n \right|
+ (2 \pi)^{n+2} C_1 r_1^4,
\]

where \( z = (z', z_n) \in U \).

By induction on \( n \), we can reduce \((i)\) to the special case \( n = 1 \), which we have proved.

Apply the same argument again, we can conclude that

\[
\left| \int_{\partial D_{r_j} (0)} \prod_{j=1}^{n-1} \partial f (z) \cos k \theta_1 \wedge \cdots \wedge \partial \theta_n \right|
\leq 2 \pi \left| \int_{\partial D_{r_j} (0)} \prod_{j=1}^{n-1} f (z', 0) \cos k \theta_1 \wedge \cdots \wedge \partial \theta_n \right|
+ (2 \pi)^{n+2} C_1 r_1^4.
\]

Combine this and the case \( n = 1 \), we obtain \((ii)\) by induction on \( n \).

\[ \square \]

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