Evaluating holonomic quantum computation: beyond adiabatic limitation

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The proposal of the optical scheme for holonomic quantum computation is evaluated based on dynamical resolution to the system beyond adiabatic limitation. The time-dependent Schrödinger equation is exactly solved by virtue of the cranking representation and gauge transformation approach. Besides providing rigorous confirmation to holonomies of the geometrical prediction that holds for the ideally adiabatic situation, the dynamical resolution enables one to evaluate elaborately the amplitude of the nonadiabatic deviation, so that the errors induced to the quantum computation can be explicitly estimated.

The recently proposed holonomic approach to quantum computation\textsuperscript{[1]-[3]} surely predicts a striking contribution to the application of quantum physics. Transcending the traditional dynamical means for quantum computation, the holonomic approach realizes quantum information processing by endowing the quantum code with a non-trivial global topology (a gauge field potential) and the associated holonomies then allow for the universal quantum computing. Specifically, in the scheme of holonomic quantum computation, information is encoded in a degenerate eigenspace of the governing Hamiltonian and the holonomies (abelian as well as non-abelian)\textsuperscript{[4]-[6]} are acquired by driving the system to undergo appropriate loops in the parameter space adiabatically. Besides suggesting an intriguing connection between the gauge fields and the information processing, such a geometrical means for quantum manipulation is believed to have built-in fault-tolerant features\textsuperscript{[7]-[9]} due to its inherent stability against local perturbations. Considerable attention has been addressed to this topic recently and the all-geometrical implementation for universal quantum gates has been proposed by optical schemes, based on laser manipulation of ions confined in a Paul trap\textsuperscript{[10]} or neutral atoms in an optical resonator\textsuperscript{[11]}.

The existent exploration for holonomic quantum computation is based on the analysis by pure geometrical fashion. It is true that in the adiabatic limit the holonomy associated with the evolving loop is determined by the path traced by the time-evolution ray and the curvature of the ray space. This involves the abelian holonomy (the Berry phase) and the non-abelian one merely known as adiabatic connection. Nevertheless, as a whole physical problem, as the dynamics of a system generates a time-dependent physical state, a specified geometrical object (the ray) is generated as well. In such a sense, dynamics determines the holonomy through determining the ray itself and its path. Moreover, in view that the realistic evolution of a physical system could not be ideally adiabatic and the nonadiabaticity shall alter the time-evolution of the ray and thus inevitably induce deviation from the adiabatic consequences. The evaluation of such deviation and the resulting errors in quantum computation is definitely a dynamical problem that goes beyond the geometrical exploration.

In this paper we employ a tractable model of the optical scheme to exploit this subject. For the appropriately chosen loops of the Hamiltonian in the parameter space, the time-dependent Schrödinger equation is exactly solved by virtue of the cranking representation and gauge transformation approach. The derived dynamical evolution of the system recovers the holonomic transformation provided by geometrical consequences, including the simple abelian phase factor and the general non-abelian operation. Thus our results provide further confirmation of the geometrical prediction, and besides, the errors caused by nonadiabatic effects for the holonomic quantum gate operation can be estimated explicitly.

For the proposed optical scheme of holonomy quantum computation\textsuperscript{[12,13]}, the basic idea relies on the adiabatic passage via the dark states since the dynamical evolution restricted to such a space is completely trivial. The system encoding the qubit is realized by a four-level Λ-type trapped ion (or a similar cavity atom). The three ground levels\textsuperscript{[12,13]} (\(|g_i⟩\) \(i = 1, 2, 3\)) are highly degenerate and each couples to the excited state \(|e⟩\) in a tunable way. The states \(|g_1⟩\) and \(|g_2⟩\) stand for the computational bases \(|0⟩\) and \(|1⟩\), respectively, and \(|g_3⟩\) is an ancillary level required for implementation of gate operations. Such a system admits two dark states that have no contribution from the excited state. Through changing the Rabi frequencies and driving the dark states to undergo appropriate cyclic evolutions in an adiabatic fashion, the universal single-bit gate operations \(e^{i\sigma_1}\) and \(e^{i\sigma_x}\) can be generated due to the global geometry of the bundle of the eigenspace of the dark states.

To evaluate the gate operation \(e^{i\sigma_1}\) from a dynamical viewpoint, let us explore the state evolution generated by the periodic Hamiltonian\textsuperscript{[12,13]}

\[ H(t) = \Omega \sin \theta (\sigma_2 e^{i\varphi} + \sigma_2 e^{-i\varphi}) + \Omega \cos \theta (\sigma_3 e^{i\varphi} + \sigma_3 e^{-i\varphi}), \]

(1)

where \(\theta\) is a fixed parameter and \(\varphi\) is assumed to rotate at a constant frequency \(\gamma\) for convenience. The equation of motion for the system is...
\[ i \frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \] (2)

It is known that in the adiabatic limit, the geometrical exploration shows that the dark state of the system, \( |D(t)\rangle = \cos \theta |g_2\rangle - \sin \theta e^{i\gamma t} |g_3\rangle \), shall acquire a net Berry phase \([\hat{\Phi}]\) after a period \( T = 2\pi/\gamma \); \(|D(T)\rangle = e^{i\Phi}|D(0)\rangle\) with \( \phi = 4\pi \sin^2 \theta \). To give a dynamical resolution to the system beyond adiabatic limitation, we use the crank phase \([5,6]\) after a period \( T = 2\pi/\gamma \) and the unitary transformation \( e^{i\gamma t}\sigma_{33} \) can be obtained by solving the instantaneous eigensolutions of \( H(t) \) and the eigenvalues of \( I \). Consequently, the dynamical invariant of the system can be shown as an element of the SU(3) group. Hence, the Hamiltonian \([4]\) can be regarded as a cranked one

\[ H(t) = e^{i\gamma t}\sigma_{33}H_0e^{-i\gamma t}\sigma_{33}, \] (3)

where \( H_0 = \Omega \sin \theta (\sigma_{2c} + \sigma_{c2}) + \Omega \cos \theta (\sigma_{3c} + \sigma_{c3}) \),

and the unitary transformation \( e^{i\gamma t}\sigma_{33} \) can be viewed as an element of the SU(3) group. Consequently, the dynamical invariant of the system can be shown as

\[ I(t) = e^{i\gamma t}\sigma_{33}(H_0 + \gamma \sigma_{33})e^{-i\gamma t}\sigma_{33} = H(t) + \gamma \sigma_{33}, \] (5)

which satisfies \([4]\)

\[ \frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} - i[I(t), H(t)] = 0. \] (6)

The second term of \([3]\) accounts for an extra gauge potential since \( H(t) \) depends on time explicitly. Now the recurrent basis \( |\psi(t)\rangle \) of the system [differs from the basic solution \( |\Psi(t)\rangle \) of the Schrödinger equation \([2]\)] only by a phase factor can be obtained by solving the instantaneous eigensolutions of \( I(t) \). It turns out, one only needs to solve the characteristic equation

\[ x^3 - (\gamma/\Omega)x^2 - x + (\gamma/\Omega)\sin^2 \theta = 0 \] (7)

and the eigenvalues of \( I(t) \) are given by \( E_i(\frac{\gamma}{\Omega}) = \Omega x_i(\frac{\gamma}{\Omega}) \) \( (i = -1, 0, 1) \). It is straightforward to show that the recurrent basis \( |\psi_0(t)\rangle \) represented by the middle number \( E_0 \) approaches to the dark state \( |D(t)\rangle \) in the adiabatic limit \( \gamma/\Omega \rightarrow 0 \). Now the leakage error induced by the nonadiabatic effect can be conveniently estimated by the overlap [see Fig. 1(a)]

\[ \eta(\theta, \frac{\gamma}{\Omega}) = |\langle \psi_0(0)|D(0)\rangle|^2 = |\langle \psi_0(T)|D(T)\rangle|^2. \] (8)

Besides the leakage, the nonadiabatic evolution shall result in deviation to the desired phase factor. It follows, instead of the net Berry phase, the cyclic evolution here induces a total phase (the so-called Lewis-Riesenfeld phase)

\[ \Phi = \int_0^T \langle \psi_0(t)|i\frac{\partial}{\partial t} - H(t)|\psi_0(t)\rangle dt = E_0 \frac{2\pi}{\gamma}. \] (9)

The detailed depiction of the deviation for the phase factor is shown in Fig. 1(b). Noting that in the adiabatic limit, the total phase \([13]\)

\[ \Phi = \lim_{\gamma/\Omega \rightarrow 0} \frac{2\pi}{\gamma/\Omega} [\frac{\pi}{2} - \frac{\pi}{2} e^{i\gamma t}\sigma_{33}^{-1}(\sigma_{12} - \sigma_{21})] = 4\pi \sin^2 \theta, \] (10)

the geometrical consequence is thus recovered.

The validity of the above evaluation is based on a presumption that the initial state \( |D(0)\rangle = \cos \theta |g_2\rangle - \sin \theta e^{i\gamma t} |g_3\rangle \) can be generated from the computational basis \( |g_2\rangle \) and so the inverse process. Explicitly, such processes can be accomplished by the driven Hamiltonian \([4]\) through changing the parameter \( \theta \) adiabatically. Conventionally, the nonadiabatic effect here shall lead to an additional error for the quantum computation. However, such an error can be in principle avoided through appending a matching interaction to compensate the gauge potential term induced to the system. Specifically, one can use the following Hamiltonian (setting \( \varphi = 0 \))

\[ H_{ad}(t) = H(t) + H_{ad}(t), H_{ad}(t) = i\theta(t)(\sigma_{23} - \sigma_{32}). \] (11)

It follows that the dynamical invariant of the system \( H_{ad}(t) \) now has a form \( I(t) = H(t) \), thus the above state transformation can be processed exactly. Physically, the interaction \( H_{ad}(t) \) can be realized by a microwave coupling to the two degenerate levels \( |g_2\rangle \) and \( |g_3\rangle \), with its intensity accurately controlled through a derivative feedback process.

Now we investigate the gate operation \( e^{i\varphi t}\sigma_y \) achieved by the holonomic means. The corresponding evolution is generated by the Hamiltonian

\[ H(t) = \Omega \sin \theta \cos \varphi (\sigma_{1c} + \sigma_{c1}) + \Omega \sin \theta \sin \varphi (\sigma_{2c} + \sigma_{c2}) + \Omega \cos \theta (\sigma_{3c} + \sigma_{c3}), \] (12)

where the parameter \( \varphi = \gamma t \). As is known, the adiabatic cyclic evolution of the Hamiltonian generates a non-abelian holonomy due to its degeneracy structure of the dark states. It can be easily worked out, from the formula of Ref. \([9]\), that the holonomic transformation

\[ u_C = e^{i2\pi \cos \theta D_y}, \] (13)

where \( D_y = i(|D_2\rangle\langle D_1| - |D_1\rangle\langle D_2|), \) and the two dark states, \( |D_1\rangle = \cos \theta |g_1\rangle - \sin \theta |g_3\rangle \) and \( |D_2\rangle = |g_2\rangle \), span the degenerate space of the starting (ending) Hamiltonian. Note that the Hamiltonian \([12]\) possesses an su(4) Lie algebraic structure and dynamical resolution to the system is usually very complicated. Surprisingly, as we shall show in the following, this system can be exactly solved by the gauge transformation approach \([16,17]\), and its dynamical evolution analytically manifested thus leads to a complete understanding of the adiabatic and nonadiabatic properties for the time-dependent Hamiltonian system.

Similar to the cranking method used above, we introduce the unitary gauge transformation

\[ U_g(t) = e^{-\gamma t(\sigma_{12} - \sigma_{21})}, \] (14)
to the equation of motion for the system, from which a
covariant Schrödinger equation is stemmed
\[ |\Psi_g(t)\rangle = U_g^{-1}(t)|\Psi(t)\rangle, \]
\[ i\frac{\partial}{\partial t}|\Psi_g(t)\rangle = H_g|\Psi_g(t)\rangle, \] (15)
with the gauged Hamiltonian
\[ H_g = U_g^{-1}HU_g - iU_g^{-1}\frac{\partial U_g}{\partial t} \]
\[ = \Omega \sin\theta(\sigma_{1e} + \sigma_{e1}) + \Omega \cos\theta(\sigma_{3e} + \sigma_{e3}) + i\gamma(\sigma_{12} - \sigma_{21}). \] (16)
In view that the above Hamiltonian is time independent,
the basic solutions \( |\Psi^0_n(t)\rangle \) to the covariant equation (14) can be easily obtained and the corresponding eigenvalues are as follows
\[ E_{1,2} = \pm \frac{\sqrt{2}}{2} \Omega \left[ 1 - \sqrt{1 - 4(\frac{\gamma}{\Omega})^2 \cos^2 \theta} \right]^{1/2}, \]
\[ E_{3,4} = \pm \frac{\sqrt{2}}{2} \Omega \left[ 1 + \sqrt{1 - 4(\frac{\gamma}{\Omega})^2 \cos^2 \theta} \right]^{1/2}, \] (17)
where
\[ \Omega = \Omega \sqrt{1 + (\gamma/\Omega)^2}, \quad \cos \theta = \frac{\cos\theta}{1 + (\gamma/\Omega)^2}. \] (18)
The dynamical basis of the system (12) can be directly obtained as \( |\Psi^0_n(t)\rangle = U_g|\Psi^0_n(t)\rangle \), from which one can see that \( E_n \) has the natural implication related to the total phase. Now the time evolution operator generated by the Hamiltonian (12) can be given
\[ U_C(T) = \sum_{n=1}^{4} |\Psi_n(T)\rangle \langle \Psi_n(0)| \]
\[ = \sum_{n=1}^{4} e^{-iE_n \frac{T}{\Omega}} |\Psi_n(0)\rangle \langle \Psi_n(0)|. \] (19)
Considering the asymptotic behavior of the evolution in the adiabatic limit, it follows that \( \lim_{\gamma/\Omega \to 0} \frac{E_{1,2}}{\gamma} = \pm \cos \theta \), and the phase-equipped dynamical bases \( |\Psi_1(t)\rangle \) and \( |\Psi_2(t)\rangle \) have the form
\[ |\Psi_1(t)\rangle = \frac{\sqrt{2}}{2} e^{-i\gamma t \cos \theta} \left[ (\cos\theta \cos \gamma t + i \sin \gamma t)|g_1\rangle + (\cos\theta \sin \gamma t - i \cos \gamma t)|g_2\rangle - \sin \theta|g_3\rangle \right], \]
\[ |\Psi_2(t)\rangle = \frac{\sqrt{2}}{2} e^{i\gamma t \cos \theta} \left[ (\cos\theta \cos \gamma t - i \sin \gamma t)|g_1\rangle + (\cos\theta \sin \gamma t + i \cos \gamma t)|g_2\rangle - \sin \theta|g_3\rangle \right]. \] (20)
One can verify that they are the instantaneous eigenstates of the Hamiltonian (12) with a two-degeneracy eigenvalue 0, and the equipped phases are just the Berry phases accordingly. Thus the cyclic evolution restricted to the space spanned by these two states is purely geometrical and can be denoted as
\[ u(T) = e^{-i2\pi \cos \theta}|\Psi_1(0)\rangle \langle \Psi_1(0)| + e^{i2\pi \cos \theta}|\Psi_2(0)\rangle \langle \Psi_2(0)| \] (21)
with \( |\Psi_1(0)\rangle = (|D_1\rangle - i|D_2\rangle)/\sqrt{2} \) and \( |\Psi_2(0)\rangle = (|D_1\rangle + i|D_2\rangle)/\sqrt{2} \). It can be easily recognized that the operator (21) is just the non-abelian holonomy (13), thus the geometrical nature is verified again.
The above dynamical resolution to the system is important. Besides offering a vivid verification to the remarkable formula of non-abelian holonomy [3], which holds for the ideally adiabatic situation, it enables one to evaluate elaborately the amplitude of the nonadiabaticity deviation and the resulting errors to the holonomic gate operation \( e^{i\theta_\sigma} \). In detail, the population transfer from the initial state \( |\Psi(0)\rangle = |g_2\rangle \) is pictured in Fig. 2. The leakage out of the computational space can be estimated by the projection (see also Fig. 2)
\[ \eta(\theta, \frac{\sqrt{2}}{\Omega}) = \sum_{i=1}^{2} |\langle D_i|U(T)|\Psi(0)\rangle|^2. \] (22)

Similar to the former case, to transform the computational basis \( |g_1\rangle \) into the dark state \( |D_1\rangle = \cos\theta|g_1\rangle - \sin \theta|g_3\rangle \) and to invert the process successfully, one needs to use the Hamiltonian (12) (with \( \theta \) tunable and \( \varphi = 0 \)) along with a matching interaction
\[ H_{ad}(t) = H(t) + H_{ad}(t), \quad H_{ad}(t) = i\dot{\theta}(t)|\sigma_{13} - \sigma_{31}|. \] (23)

Up to now, we have investigated the single-qubit holonomic operations of the optical scheme and evaluated the nonadiabaticity-causing errors for quantum computation. It deserves to point out that, the proposal [12] of geometrical implementation for the controlled two-qubit phase shift gate, \( e^{i\theta_\sigma} [11]^{11} \), which is sufficient for the universal quantum computation along with the two single-qubit gates, can be explored in a similar way. The scheme is realized by two-color laser manipulation on the trapped ions [18]. Briefly, the transition from the ground states \( |g_2\rangle \) and \( |g_3\rangle \) to the excited state \( |e\rangle \) is driven by two different bi-chromatic laser beams with their amplitudes and phases of the Rabi frequencies controllable. The frequencies of the laser fields are tuned so that the two-photon process, exciting pair ions, is resonant and the single-photon excitation is off-resonant. Hence the system can be described by an effective Hamiltonian in the notation of Ref. [12]
\[ H_{\text{eff}} \propto -|\Omega_1|^2 e^{i2\varphi} \langle ee|g_2g_2\rangle + \text{h.c.} \]
\[ + |\Omega_2|^2 e^{i2\varphi} \langle ee|g_3g_3\rangle + \text{h.c.}, \] (24)
where the relative intensity of the Rabi frequencies \( \tan \theta = -|\Omega_1|^2/|\Omega_2|^2 \) and the phase difference \( \varphi/2 = \]
\( \varphi_1 - \varphi_2 \) are tunable. One can see that the bases \(|g_1g_1\rangle\), \(|g_2g_2\rangle\), and \(|g_2g_1\rangle\) are decoupled from the evolution, and the component \(|g_3g_2\rangle\) serving as the code [11] evolved in an enclosed space spanned by \(|g_2g_2\rangle, |g_3g_3\rangle, |ee\rangle\). Introducing the \( su(3) \) generators explicitly

\[
\begin{align*}
A_{e_2} &= e^{i2\varphi_1} |ee\rangle\langle g_2g_2|,
A_{e_3} &= e^{i2\varphi_1} |ee\rangle\langle g_3g_3|,
A_{g_2} &= |g_2g_2\rangle\langle g_3g_3|, \quad A_{\mu\nu}^\dagger = A_{\nu\mu},
\end{align*}
\]

the Hamiltonian (24) can be rewritten as

\[
H_{\text{eff}} = g \sin \theta (A_{e_2} + A_{e_3}) + g \cos \theta (A_{e_2} e^{i\varphi_1} + A_{e_3} e^{-i\varphi_1}).
\]

Obviously this Hamiltonian possesses an \( su(3) \) algebraic structure isomorphic to that of system (\( \mathcal{F} \)), thus all the discussions therein also hold for the present system.

It should be noted that, the effective Hamiltonian (24), respecting a second-order process of the interaction, is quite a rough description of the model. Specifically, it ignores the same second-order process induced by virtual photons excitation in the self-transitions of the states: \(|g_{2(3)}\rangle \rightarrow |g_{2(3)}\rangle \) and \(|e\rangle \rightarrow |e\rangle\). It can be anticipated that such self transitions shall dress the energy levels of the ions and lift the degeneracy of the ground states, which in turn affects the desired gate operation. Detailed exploration of this point shall be presented in a future report.

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Caption of Fig. 1:
Deviation induced by nonadiabaticity for abelian holonomy. (a) The overlap \( \eta \) for the parameters \( \theta \in (0,\pi/2) \) and \( \gamma/\Omega \in [0,1] \); (b) The total phase \( \Phi \) for \( \theta \in [0,\pi] \) and \( \gamma/\Omega \in [0,1] \).

Caption of Fig. 2:
Deviation induced by nonadiabaticity for non-abelian holonomy. The initial state is prepared in \( |D_2\rangle \). The two solid curves show the results for the population of the target state on \( |D_1\rangle \) and \( |D_2\rangle \), as a function of \( 1 - \cos \theta \), respectively. The dashed curve depicts the total population \( \eta \) on the computational space. Figures (a), (b), (c) and (d) correspond to \( \gamma/\Omega = 0.01, 0.2, 0.5 \) and 0.8, respectively.
Fig. 2