Two-proton radioactivity and three-body decay. III. Integral formulae for decay widths in a simplified semianalytical approach.

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Three-body decays of resonant states are studied using integral formulae for decay widths. Theoretical approach with a simplified Hamiltonian allows semianalytical treatment of the process. The model is applied to decays of the first excited \(3/2^-\) state of \(17\text{Ne}\) and the \(3/2^-\) ground state of \(45\text{Fe}\). The convergence of three-body hyperspherical model calculations to the exact result for widths and energy distributions are studied. The theoretical results for \(17\text{Ne}\) and \(45\text{Fe}\) decays are updated and uncertainties of the derived values are discussed in detail. Correlations for the decay of \(17\text{Ne}\) \(3/2^-\) state are also studied.

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I. INTRODUCTION

The idea of the “true” two-proton radioactivity was proposed about 50 years ago in a classical paper of Goldansky [1]. The word “true” denotes here that we are dealing not with a relatively simple emission of two protons, which becomes possible in every nucleus above two-proton decay threshold, but with a specific situation where one-proton emission is energetically (due to the proton separation energy in the daughter system) or dynamically (due to various reasons) prohibited. Only simultaneous emission of two protons is possible in that case (see Fig. 1 more details on the modes of the three-body decays can be found in Ref. [2]). The dynamics of such decays can not be reduced to a sequence of two-body decays and from theoretical point of view we have to deal with a three-body Coulomb problem in the continuum, which is known to be very complicated.

Progress in this field was quite slow. Only recently a consistent quantum mechanical theory of the process was developed [2, 3, 4], which allows to study the two-proton (three-body) decay phenomenon in a three-body cluster model. It has been applied to a range of light nuclear systems ([12O, 16Ne [5], 6Be, 8Li\textsuperscript{3}, 9Be\textsuperscript{4}, 17Ne\textsuperscript{5}, 19Mg\textsuperscript{6}]). Systematic exploratory studies of heavier prospective \(2p\) emitters [30Ar, \(3^4\)Ca, \(45\text{Fe}, \(48\text{Ni}, \(54\text{Zn}, \(58\text{Ge}, \(62\text{Se}, and \(68\text{Kr} [4, 8]\) have been performed providing predictions of lifetime ranges and possible correlations among fragments.

Experimental studies of the two-proton radioactivity is presently an actively developing field. Since the first experimental identification of \(2p\) radioactivity in \(45\text{Fe} [8][10]\) it was also found in \(54\text{Zn} [11]\). Some fingerprints of the \(48\text{Ni} 2p\) decay were observed and the \(45\text{Fe}\) lifetime and decay energy were measured with improved accuracy [12]. There was an intriguing discovery of the extreme enhancement of the \(2p\) decay mode for the high-spin \(21^+\) isomer of \(94\text{Ag}\), interpreted so far only in terms of the hyperdeformation of this state [13]. New experiments, aimed at more detailed \(2p\) decay studies (e.g. observation of correlations), are under way at GSI (\(19\text{Mg}\)), MSU (\(45\text{Fe}\)), GANIL (\(45\text{Fe}\)), and Jyväskylä (\(94\text{Ag}\)).

Several other theoretical approaches were applied to the problem in the recent years. We should mention the “diproton” model [14, 15], “R-matrix” approach [16, 17, 18, 19], continuum shell model [20], and adiabatic hyperspherical approach of [21]. Some issues of a compatibility between different approaches will be addressed in this work.

Another, possibly very important, field of application of the two-proton decay studies was shown in Refs. [22, 23]. It was demonstrated in [22] that the importance of direct resonant two-proton radiative capture processes was underestimated in earlier treatment of the \(rp\)-process waiting points [24]. The scale of modification of the astrophysical \(2p\) capture rates can be as large as several orders of magnitude in certain temperature ranges. In paper [23] it has been found that nonresonant E1 contributions to three-body (two-proton) capture rates can also be much larger than was expected before. The updated \(2p\) astrophysical capture rate for the \(15\text{O}(2p, \gamma)^{17}\text{Ne}\) reaction appears to be competing with the standard \(15\text{O}(a, \gamma)^{18}\text{Ne}\) breakout reaction for the hot CNO cycle. The improvements of the \(2p\) capture rates obtained in [22, 23] are connected to consistent quantum mechanical treatment.
of the three-body Coulomb continuum in contrast to the essentially quasiclassical approach typically used in astrophysical calculations of three-body capture reactions (e.g. [24, 25]).

The growing quality of the experimental studies of the $2p$ decays and the high precision required for certain astrophysical calculations inspired us to revisit the issues connected with different uncertainties and technical difficulties of our studies. In this work we make the following. (i) Extend the two-body formalism of the integral formulae for width to the three-body case. We perform the relevant derivations for the two-body case to make the relevant approximations and assumptions explicit. (ii) Formulate a simplified three-body model which has many dynamical features similar to the realistic case, but allows the exact semianalytical treatment and thus makes possible a precise calibration of three-body calculations. It is also possible to study in great detail several important dependencies of three-body widths in the frame of this model. (iii) Perform practical studies of some systems of interest and demonstrate a connection between the simplified semianalytical formalism and the realistic three-body calculations.

The unit system $\hbar = c = 1$ is used in the article.

II. INTEGRAL FORMULA FOR WIDTH

Integral formalisms of width calculations for narrow two-body states are known for a long time, e.g. [26, 27]. The prime objective of those studies was $\alpha$-decay widths. An interesting overview of this field can be found in the book [28]. This approach, to our opinion, did not produce novel results as the inherent uncertainties of the method are essentially the same as those of the R-matrix phenomenology, which is technically much simpler (see e.g. a discussion in [29]). An important nontrivial application of the integral formalism was calculation of widths for proton emission off deformed states [30, 31]. There were attempts to extend the integral formalism to the three-body decays, using a formal generalization for the hyperspherical space [2, 32]. Those were shown to be difficult with respect to technical realisation and to be inferior to other methods developed in [2, 3].

Here we develop an integral formalism for the three-body (two-proton) decay width in a different way. However, first we review the standard formalism to define (clearer) the approximations used.

A. Width definition, complex energy WF

For decay studies we consider the wave function (WF) with complex pole energy

$$
\hat{E}_r = \frac{k^2_r}{2M} = E_r - i\Gamma/2 \ , \quad \hat{k}_r \approx k_r - i\Gamma/(2v_r) ,
$$

where $v = \sqrt{2E/M}$. The pole solution for Hamiltonian

$$
(H - \hat{E}_r)\psi_{lm}^{(+)}(r) = (T + V - \hat{E}_r)\psi_{lm}^{(+)}(r) = 0
$$

provides the WF with outgoing asymptotic

$$
\psi_{lm}^{(+)}(r) = \frac{1}{r^{-1}}\psi_l^{(+)}(kr) Y_{lm}(\hat{r}) .
$$

For single channel two-body problem the pole solution is formed only for one selected value of angular momentum $l$. In the asymptotic region

$$
\psi_l^{(+)}(\hat{k}_r r)^{r \geq R} H_l^{(+)}(\hat{k}_r r) = G_l(\hat{k}_r r) + iF_l(\hat{k}_r r) .
$$

The above asymptotic is growing exponentially

$$
\psi_l^{(+)}(\hat{k}_r r)^{r \geq R} \exp[\pm i\hat{k}_r r] \approx \exp[\pm i\hat{k}_r r]\exp[\mp i\Gamma/(2v_r)]
$$

as a function of the radius at pole energy. This unphysical growth is connected to the use of time-independent formalism and could be reliably neglected for typical radioactivity time scale as it has a noticeable effect at very large distances.

Applying Green’s procedure to complex energy WF

$$
\psi^{(+)*} \left[(H - \hat{E}_r)\psi^{(+)}\right] - \left[(H - \hat{E}_r)\psi^{(+)}\right]^\dagger \psi^{(+)} = 0
$$

we get for the partial components at pole energy $\hat{E}_r$

$$
\frac{i\Gamma}{2M} \left[ \psi_l^{(+)*} \left( \frac{d}{dr} \psi_l^{(+)} \right) - \left( \frac{d}{dr} \psi_l^{(+)*} \right) \psi_l^{(+)} \right]_{r = R} = \frac{1}{N_l} j_l .
$$

After radial integration from 0 to $R$ (here and below $R$ denotes the radius sufficiently large that the nuclear interaction disappears) we obtain

$$
\Gamma = \frac{1}{2Mi} \int_0^R \left[ \psi_l^{(+)*} \left( \frac{d}{dr} \psi_l^{(+)} \right) - \left( \frac{d}{dr} \psi_l^{(+)*} \right) \psi_l^{(+)} \right] dr = \frac{j_l}{N_l} ,
$$

which corresponds to a definition of the width as a decay probability (reciprocal of the lifetime):

$$
N = N_0 \exp[-t/\tau] = N_0 \exp[-\Gamma t] .
$$

The width $\Gamma$ is then equal to the outgoing flux $j_l$ through the sphere of sufficiently large radius $R$, divided by number of particles $N_l$ inside the sphere.

Using Eq. (2) the flux in the asymptotic region could be rewritten for $\hat{k}_r \sim k_r$ in terms of a Wronskian

$$
\frac{j_l}{2Mi} = \frac{1}{2Mi} \left[ \psi_l^{(+)*} \left( \frac{d}{dr} \psi_l^{(+)} \right) - \left( \frac{d}{dr} \psi_l^{(+)*} \right) \psi_l^{(+)} \right]_{r = R} = \langle k_r/M \rangle \ W(F_l(k_r R), G_l(k_r R)) = v_r ,
$$

where the Wronskian for real energy functions $F_l, G_l$ is

$$
W(F_l, G_l) = G_l F_l' - G_l' F_l \equiv 1 .
$$
The effect of the complex energy is easy to estimate (actually without loss of a generality) in a small energy approximation
\[ F_i(kr) \sim C_i(kr)^{l+1}, \quad G_i(kr) \sim \frac{(kr)^{-l}}{(2l+1)C_i}, \quad (5) \]
where \( C_i \) is a Coulomb coefficient (defined e.g. in Ref. [33]). The flux is then
\[
\frac{\partial}{\partial t} = \frac{\tilde{k}_r H_i^{-\dagger}(\tilde{k}_r r) H_i^{i}(\tilde{k}_r r) - \tilde{k}_F H_i^{-\dagger}(\tilde{k}_r r) H_i^{i}(\tilde{k}_r r)}{2iM} = v_r \left( 1 - \frac{2l(l+1)}{k^2_r} \left( \frac{\Gamma}{2v_r} \right)^2 + l \times o[\Gamma^3] \right).
\]
So, the equality (4) is always valid for \( l = 0 \) and for \( l \neq 0 \) we get
\[
\Gamma \ll \left( \frac{8}{l(l+1)} \right)^{1/2} E_r.
\]

**B. Two-body case, real energy WF**

Now we need a WF as real energy \( E = \frac{k^2}{2M} \) solution of Schrödinger equation
\[
(H - E)\Psi_k(r) = (T + V^{\text{nuc}} + V^{\text{coul}} - E)\Psi_k(r) = 0,
\]
\[
\Psi_k(r) = 4\pi \sum_i^{l} i^{l} (kr)^{-1} \psi_i(kr) \sum_m Y^{*}_{lm}(\hat{k}) Y_{lm}(\hat{r}),
\]
in S-matrix representation, which means that for \( r > R \)
\[
\psi_i(kr) = \frac{i}{2} \left[ (G_i(kr) - iF_i(kr)) - \tilde{S}_i(G_i(kr) + iF_i(kr)) \right].
\]
At resonance energy \( E_r \)
\[
S_i(E_r) = e^{2i\delta_i(E_r)} = e^{2i\pi/2} = -1
\]
and in asymptotic region, defined by the maximal size of nuclear interaction \( R \),
\[
\psi_i(kr, r \geq R) \sim i G_i(kr, r).
\]
At resonance energy we can define a “quasibound” WF \( \psi \) as matching the irregular solution \( G_i \) and normalized to unity for the integration in the internal region limited by radius \( R \):
\[
\tilde{\psi}_i(kr, r) = \frac{(-i)}{(\int_0^R |\psi_i(kr, x)|^2 dx)^{1/2}} \psi_i(kr, r) = -i \frac{\psi_i(kr, r)}{N_i^{1/2}}.
\]

Now we introduce an auxiliary Hamiltonian \( \tilde{H} \) with different short range nuclear interaction \( V^{\text{nuc}} \),
\[
(\tilde{H} - E)\Phi_k(r) = (T + \tilde{V}^{\text{nuc}} + V^{\text{coul}} - E)\Phi_k(r) = 0,
\]
and also construct other WF in S-matrix representation
\[
\Phi_k(r) = 4\pi \sum_i^{l} i^{l} (kr)^{-1} \varphi_i(kr) \sum_m Y^{*}_{lm}(\hat{k}) Y_{lm}(\hat{r}),
\]
\[
\varphi_i(kr) = \frac{i}{2} \left[ (G_i(kr) - iF_i(kr)) - \tilde{S}_i(G_i(kr) + iF_i(kr)) \right],
\]
for \( r > R \). Or in equivalent form:
\[
\varphi_i(kr) = \exp(i\delta_i) \left[ F_i(kr) \cos(\delta_i) + G_i(kr) \sin(\delta_i) \right].
\]
The Hamiltonian \( \tilde{H} \) should provide the WF \( \Phi_k(r) \) which at energy \( E_r \) is sufficiently far from being a resonance WF and for this WF \( \delta_i(E_r) \sim 0 \).

For real energy WFs \( \Psi_k(r) \) and \( \Phi_k(r) \) we can write:
\[
\Phi_k(r) \approx \left[ (H - E)\Psi_k(r) \right] - \left[ (\tilde{H} - E)\Phi_k(r) \right] \Psi_k(r) = 0,
\]
\[
\varphi_i^*(V - \tilde{V}) \psi_i = \frac{1}{2M} \left[ \varphi_i^* \left( \frac{d^2}{dr^2} \psi_i \right) - \left( \frac{d^2}{dr^2} \varphi_i^* \right) \psi_i \right].
\]
For WFs taken at resonance energy \( E_r \) this expression provides
\[
2M \int_0^R \varphi_i^*(V - \tilde{V}) \psi_i dr = 2MiN_i^{1/2} \int_0^R \varphi_i^*(V - \tilde{V}) \tilde{\psi}_i dr = 0,
\]
\[
N_i^{1/2} = \frac{-i}{2M} \int_0^R \varphi_i^*(V - \tilde{V}) \tilde{\psi}_i dr.
\]
From Eqs. (3), (4), (6) and the approximation \( \psi_i^{(+) \approx} \psi_i \) it follows that
\[
\Gamma = \frac{v_r}{\int_0^R |\psi_i^{(+) \approx}|^2 dr} \approx \frac{v_r}{\int_0^R |\psi_i|^2 dr} = \frac{v_r}{N_i^{1/2}},
\]
\[
\Gamma = \frac{4}{v_r \cos^2(\delta_i)} \left( \int_0^R \varphi_i^*(V - \tilde{V}) \tilde{\psi}_i dr \right)^2.
\]
So, the idea of the integral method is to define the internal normalizations for the WF with resonant boundary conditions (this is equivalent to determination of the outgoing flux for normalized “quasibound” WF) by the help of the eigenfunction of the auxiliary Hamiltonian, which has the same long-range behaviour and differs only in the compact region.

**III. ALTERNATIVE DERIVATION**

Let us reformulate the derivation of Eq. (10) in a more general way, so that the detailed knowledge of the WF structure for \( \psi_i \) and \( \psi_i^{(+) \approx} \) is not required. It would allow a straightforward extension of the formalism to the three-body case. We start from Schrödinger equation in continuum with solution \( \Psi^{(+) \approx} \) at the pole energy \( \tilde{E}_r = E_r + i\Gamma/2 \):
\[
\left( H - \tilde{E}_r \right) \Psi^{(+) \approx} = \left( T + V - \tilde{E}_r \right) \Psi^{(+) \approx} = 0.
\]
Then we rewrite it identically via the auxiliary Hamiltonian \( H = T + V \)

\[
(H + V - V - E_r) \Psi(+) = (V - V) \Psi(+) \\
(H - E_r) \Psi(+) = (V - V + i \Gamma/2) \Psi(+) .
\]

(12)

Thus we can use the real-energy Green’s function \( G_{E_r} \) of auxiliary Hamiltonian \( H \) to “regenerate” the WF with outgoing asymptotic

\[
\hat{\Psi}(+) = G_{E_r}(V - V + i \Gamma/2) \Psi(+) .
\]

(13)

At this point in Eq. (13) \( \Psi(+) \equiv \Psi(+) \) and the bar in the notation for “corrected” WF \( \bar{\Psi}(+) \) is introduced for later use to distinguish it from the “initial” WF \( \Psi(+) \) [the one before application of Eq. (13)]. Further assumptions we should consider separately in two-body and three-body cases.

A. Two-body case

To define the width \( \Gamma \) by Eq. (3) we need to know the complex-energy solution \( \Psi(+) \) at pole energy. For narrow states \( \Gamma \ll E_r \) this solution can be obtained in a simplified way using the following approximations.

(i) For narrow states we can always choose the auxiliary Hamiltonian in such a way that \( \Gamma \ll V - V \), and we can assume \( \Gamma \to 0 \) in the Eq. (13).

(ii) Instead of complex-energy solution \( \Psi(+) \) in the right-hand side of (13) we can use the normalized real-energy quasi-bound solution \( \hat{\Psi} \) defined for one real resonant value of energy \( E_r = k_x^2/2M \)

\[
N_i = \int d\Omega \int_0^R dr \; r^2 \left| \hat{\Psi}_{lm}(r) \right|^2 = 1 .
\]

So, the Eq. (13) is used in the form

\[
\hat{\Psi}_{lm}(r) = G_{E_r}(V - V) \hat{\Psi}_{lm} .
\]

(14)

The solution \( \hat{\Psi}(+) \) is matched to function

\[
h^{(+)\text{barrier}}(kr) = G_l(kr) + iF_l(kr) ,
\]

while the solution \( \hat{\Psi} \) is matched to function \( G_l \). For deep subbarrier energies it is reasonable to expect that in the internal region \( r \leq R \)

\[
G_l \gg F_l \rightarrow \left\| \text{Re}[\hat{\Psi}(+)] \right\| \approx \left\| \hat{\Psi} \right\| \gg \left\| \text{Im}[\hat{\Psi}(+)] \right\| .
\]

In the single channel case it can be shown by direct calculation that an approximate equality

\[
\frac{MR^2\Gamma}{\pi} \left\| \text{Re}[\hat{\Psi}(+)] \right\| \gg \left\| \text{Im}[\hat{\Psi}(+)] \right\|.
\]

holds in the internal region and thus for narrow states \( \Gamma \ll E_r \) the approximation (13) \( \rightarrow \) (14) should be very reliable.

B. Simplified model for three-body case

In papers [2, 3] the widths for three-body decays were defined by the following procedure. We solve numerically the problem

\[
(H - E_{3r}) \hat{\Psi} = 0
\]

FIG. 2: Single particle coordinate systems: (a) “V” system typical for a shell model. In the Jacobi “T” system (b), “diproton” and core are explicitly in configurations with definite angular momenta \( l_t \) and \( l_y \). For a heavy core the Jacobi “Y” system (c) is close to the single particle system (a).
with some box boundary conditions (e.g. zero or quasi-sibound in diagonal channels at large distances) getting the WF $\Psi$ normalized in the finite domain and the value of the real resonant energy $E_{3\pi}$. Thereupon we search for the outgoing solution $\Psi^{(+)}$ of the equation

$$ (H - E_{3\pi}) \Psi^{(+)} = -i\Gamma/2 \dot{\Psi} $$

with approximate boundary conditions of three-body Coulomb problem (see Ref. [2] for details) and arbitrary $\Gamma$. The width is then defined as the flux through the hypersphere of the large radius divided by normalization within this radius:

$$ \Gamma = \frac{\int d\Omega_5 \Psi^{(+)*} \rho^{5/2} \frac{d}{dp} \rho^{5/2} \Psi^{(+)} \bigg|_{p=p_{\text{max}}}^1}{M \int d\Omega_5 \int_0^{p_{\text{max}}} \rho^{5/2} d\rho \left| \Psi^{(+)} \right|^2} \quad (18) $$

The 3-body WF with outgoing asymptotic is

$$ \Psi^{(+)}_{JM}(\rho, \Omega_5) = \rho^{-5/2} \sum_{K\gamma} J_{K\gamma}^{JM}(\rho) \Phi_{K\gamma}^{JM}(\Omega_5) \quad (19) $$

where the definitions of the hyperspherical variables $\rho$, $\Omega_5$ and hyperspherical harmonics $J_{K\gamma}^{JM}$ can be found in Ref. [4].

Here we formulate the simplified three-body model in the way which, on one hand, keeps the important dynamical features of the three-body decays (typical sizes of the nuclear potentials, typical energies in the subsystems, correct ratios of masses, etc.), and, on the other hand, allows a semi-analytical treatment of the problem. Two types of approximations are made here.

The three-body Coulomb interaction is

$$ V_{coul}^{\text{core}} = \frac{Z_1 Z_2 \alpha}{X} + \frac{Z_1 Z_3 \alpha}{Y + \frac{A_2}{A_1 + A_2}} + \frac{Z_2 Z_3 \alpha}{Y - \frac{A_1}{A_1 + A_2}} \quad (20) $$

where $\alpha$ is the fine structure constant. By convention, see e.g. Fig. [2] in the “T” Jacobi system the core is particle number 3 and in “Y” system it is particle number 2. We assume that the above potential can be approximated by Coulomb terms which depend on Jacobi variables $X$ and $Y$ only:

$$ V^{coul}_{x}(X) = \frac{Z_x \alpha}{X} \quad , \quad V^{coul}_{y}(Y) = \frac{Z_y \alpha}{Y} \quad (21) $$

(in reality for the small $X$ and $Y$ values the Coulomb formfactors of the homogeneously charged sphere with radius $r_{\text{sph}}$ are always used). The effective charges $Z_x$ and $Z_y$ could be considered in two ways.

1. We can neglect one of the Coulomb interactions. This approximation is consistent with physical situation of heavy core and treatment of two final state interactions. Such a situation presumes that Jacobi “Y” system is preferable and there is a symmetry in the treatment of the $X$ and $Y$ coordinates, which are close to shell-model single particle coordinates.

$$ Z_x = Z_1 Z_{\text{core}} \quad , \quad Z_y = Z_2 Z_{\text{core}} \quad (22) $$

Further we refer this approximation as “no p-p Coulomb” case, as typically the proton-proton Coulomb interaction is neglected compared to Coulomb interaction of a proton with heavy core.

2. We can also consider two particles on the $X$ coordinate as one single particle. The Coulomb interaction in $p-p$ channel is thus somehow taken into account effectively via a modification of the $Z_y$ charge:

$$ Z_x = Z_1 Z_{\text{core}} \quad , \quad Z_y = Z_2 (Z_{\text{core}} + Z_1) \quad (23) $$

Below we call this situation as “effective p-p Coulomb” case.

For nuclear interactions we can assume that

1. There is only one nuclear pairwise interaction and

$$ H = T + V_3(\rho) + V^{coul}_x(X) + V^{\text{nuc}}_x(X) + V^{coul}_y(Y) + V^{\text{nuc}}_y(Y), \quad \Delta V(X, Y) = V^{\text{nuc}}_x(Y) - V_3(\rho) \quad (24) $$

This approximation is good for methodological purposes as it allows to focus on one degree of freedom and isolate it from the others. From physical point of view it could be reasonable if only one FSI is strong [12], or we have reasons to think that decay mechanism associated with this particular FSI is dominating. Potential $V^{\text{nuc}}_y(Y)$ in the auxiliary Hamiltonian (27) is “unphysical” in that case and can be put zero [13]. We further refer this model as “one final state interaction” (OFSI).

2. We can consider two final state interactions (TFSI). Simple form of the Green’s function in that case can be preserved only if the core mass is considered as infinite (the $X$ and $Y$ coordinates in the Jacobi “Y” system coincide with single-particle core-$p$ coordinates). In that case both pairwise interactions $V^{\text{nuc}}_x(X)$ and $V^{\text{nuc}}_y(Y)$ are treated as “physical”, that means that they are both present in the initial and in the auxiliary Hamiltonians. Thus only three-body potential “survive” the $V - V$ subtraction:

$$ H = T + V_3(\rho) + V^{coul}_x(X) + V^{\text{nuc}}_x(X) + V^{coul}_y(Y) + V^{\text{nuc}}_y(Y), \quad \Delta V(X, Y) = -V_3(\rho) \quad (25) $$

The three-body potential is used in this work in Woods-Saxon form

$$ V_3(\rho) = V_3^0 \left(1 + \exp \left[\frac{(\rho - \rho_0)}{a_0}\right]\right)^{-1} \quad (26) $$

with $\rho_0 = 5$ fm for $^{17}\text{Ne}$, $\rho_0 = 6$ fm for $^{45}\text{Fe}$ [14], and a small value of diffuseness parameter $a_0 = 0.4$ fm. Use of such three-body potential is an important difference
from our previous calculations, where it was utilized in the form
\[ V_3(\rho) = V_0^0 \left(1 + (\rho/\rho_0)^3\right)^{-1}, \quad (26) \]
which provides the long-range behaviour \( \sim \rho^{-3} \). Such an asymptotic in \( \rho \) variable is produced by short-range pairwise nuclear interactions and thus the interpretation of three-body potential \((26)\) is phenomenological taking into account those components of pairwise interactions which were omitted for some reasons in calculations. In this work the aim of the potential \( V_3 \) is different. On one hand we would like to keep the three-body energy fixed while the properties (and number) of pairwise interactions are varied. On the other hand we do not want to change the properties of the Coulomb barriers beyond the typical nuclear distance (this is achieved by the small diffuseness of the potential). Thus this potential is phenomenological taking into account interactions that act only when both valence nucleons are close to the core (both move in the mean field of the nucleus).

The auxiliary Hamiltonian is taken in the form that allows a separate treatment of \( X \) and \( Y \) variables
\[ \hat{H} = T + V_{x}^{\text{conf}}(X) + V_{y}^{\text{nucl}}(X) + V_{y}^{\text{conf}}(Y) + V_{y}^{\text{nucl}}(Y) \quad (27) \]
In this formulation of the model the Coulomb potentials \([V_{x}^{\text{nucl}}(X) \text{ if present}] \) define the position of the state in the \( X \) subsystem. The three-body potential \( V_3(\rho) \) defines the position of the three-body state, which is found using \([2, 4]\). After that a new WF with outgoing asymptotic is generated by means of the three-body Green’s function which can be written for \((27)\) in a factorized form (without paying attention to the angular coupling)
\[ G_{E_{3\rho}}^{(+)}(XY, X'Y') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE_x G_{E_x}^{(+)}(X, X') G_{E_y}^{(+)}(Y, Y'), \]
where \( E_{3\rho} = E_x + E_y \) (\( E_x, E_y \) are energies of subsystems). The two-body Green’s functions in the expressions above are defined as in \([10]\) via eigenfunctions of the subhamiltonians
\[ \begin{cases} \hat{H}_x - E_x = T_x + V_{x}^{\text{conf}}(X) + V_{x}^{\text{nucl}}(X) - E_x \\ \hat{H}_y - E_y = T_y + V_{y}^{\text{conf}}(Y) + V_{y}^{\text{nucl}}(Y) - E_y \end{cases}. \]
In the OFSI case the nuclear potential in the “\( Y \)” subsystem should be put \( V_{y}^{\text{nucl}}(Y) \equiv 0 \). The “corrected” continuum WF \( \Psi_{JM}^{(+)} \) is
\[ \Psi_{JM}^{(+)}(X, Y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE_x G_{E_x}^{(+)}(X, X') \times G_{E_y}^{(+)}(Y, Y') \Delta V(X', Y') \Psi^{(+)}(X'Y'). \]
The “initial” solution \( \Psi^{(+)} \) of Eq. \([10]\) rewritten in the coordinates \( X \) and \( Y \) is
\[ \Psi_{JM}^{(+)}(X, Y) = \frac{\bar{\varphi}_{L_y} i_{y} S(X, Y)}{XY} [I_{y} \otimes L_{y}]_{L} \otimes S \] \[ J_{M} \quad (28) \]

The asymptotic form of the “corrected” continuum WF \( \Psi_{JM}^{(+)} \) is
\[ \Psi_{JM}^{(+)}(X, Y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE_x \frac{4}{v_x (\epsilon) v_y (\epsilon)} A(\epsilon) \times e^{i k_x (\epsilon) x + i k_y (\epsilon) y} \left[I_{y} \otimes L_{y}\right]_{L} \otimes S \] \[ J_{M} \quad (29) \]
The “corrected” outgoing flux \( j_c \) can be calculated on the sphere of the large radius for any of two Jacobi variables. E.g. for \( X \) coordinate we have \([13]\)
\[ j_c(E_{3\rho}) = \Im \left[X^2 \int d\Omega_x \int d\Omega_y \left( \frac{\bar{\varphi}^{(+)} S(X, Y)}{M_x} \right) \right]_{x \rightarrow -\infty} \]
\[ = E_{3\rho}^2 \int_{0}^{1} d\epsilon A^*(\epsilon) \frac{4}{2\pi v_x v_y} \int_{0}^{1} d\epsilon' \frac{k_x (\epsilon) k_y (\epsilon)}{M_x} \frac{2\pi}{2\pi} \]
\[ \times 4 \] \[ \frac{4}{v_x v_y} 2\pi \delta(k_y (\epsilon') - k_y (\epsilon)) \] \[ \] \[ J_{c}(E_{3\rho}) = \frac{8}{\pi} E_{3\rho} \int_{0}^{1} d\epsilon \frac{1}{v_x (\epsilon) v_y (\epsilon)} |A(\epsilon)|^2. \quad (31) \]
In principle as we have seen above that the widths obtained with both fluxes Eqs. \([18]\) and \([31]\) should be equal
\[ \Gamma = \frac{j_c}{N} \equiv \Gamma_c = \frac{j_c}{N}. \quad (32) \]
This is the idea of calibration procedure for the simplified three-body model. The convergence of the HH method (for WF $\Psi_{JI\nu}$) is expected to be fast in the internal region and much slower in the distant subbarrier region. This should be true for the width $\Gamma$ calculated in the HH method. However, the procedure for calculation of the "corrected" width $\Gamma_c$ is exact under the barrier and it is sensitive only to HH convergence in the internal region, which is achieved easily. Below we demonstrate this in particular calculations.

IV. DECAYS OF THE $^{17}\text{Ne}$ $3/2^-$ AND $^{45}\text{Fe}$ $3/2^-$ STATES IN A SIMPLIFIED MODEL

In this Section when we refer widths of $^{17}\text{Ne}$ and $^{45}\text{Fe}$ we always mean the $^{17}\text{Ne}$ $3/2^-$ state ($E_{3s_r} = 0.344\text{ MeV}$) and the $^{45}\text{Fe}$ $3/2^-$ ground state ($E_{3s_r} = 1.154\text{ MeV}$) calculated in a very simple models. We expect that important regularities found for these models should be true also in realistic calculations. However, particular values obtained in realistic models may differ significantly, and this issue is considered specially in the Section V.

To keep only the most significant features of the systems we assume pure $sd$ structure ($l_x = 0, l_y = 2$) for $^{17}\text{Ne}$ and pure $p^2$ structure ($l_x = 1, l_y = 1$) for $^{45}\text{Fe}$ in "Y" Jacobi system (see Fig. 2). Spin dependencies of the interactions are neglected. The Gaussian formfactor

$$V_i^{\text{nuc}}(r) = V_{i0}\exp\left[-(r/r_0)^2\right],$$

where $i = \{x, y\}$, is taken for $^{17}\text{Ne}$ (see Table I), and a standard Woods-Saxon formfactor is used for $^{45}\text{Fe}$ (see Table II).

$$V_i^{\text{nuc}}(r) = V_{i0}\left[1 + \exp[(r - r_0)/a]\right]^{-1}.$$  \hspace{1cm} (33)

The simplistic structure models can be expected to overestimate the widths. There should be a considerable weight of $d^2$ component ($l_x = 2, l_y = 2$) in $^{17}\text{Ne}$ and $f^2$ component ($l_x = 3, l_y = 3$) in $^{45}\text{Fe}$. Also the spin-angular coupling should lead to splitting of the single-particle strength and corresponding reduction of the width estimates (e.g. we assume one s-wave state at 0.535 keV in the "X" subsystem of $^{17}\text{Ne}$ while in reality there are two s-wave states in $^{16}\text{F}$: 0$^-$ at 0.535 MeV and 1$^-$ at 0.728 MeV). Thus the results of the simplified model should most likely be regarded as upper limits for widths.

A. One final state interaction — core-$p$ channel

First we take into account only the 0.535 MeV s-wave two-body resonance in the $^{16}\text{F}$ subsystem (this is the experimental energy of the first state in $^{16}\text{F}$). Convergence of the $^{17}\text{Ne}$ width in a simplified model for Jacobi "Y" system is shown in Fig. 4. The convergence of the corrected width $\Gamma_c$ as a function of $K_{\text{max}}$ is very fast: $K_{\text{max}} > 8$ for the width is stable within $\sim 1\%$. For maximal achieved in the fully dynamic calculation $K_{\text{max}} = 24$ the three-body width $\Gamma$ is calculated within 30% precision. Further increase of the effective basis size is possible within the adiabatic procedure based on the so called Feschbach reduction (FR).

Feschbach reduction is a procedure, which eliminates from the total WF $\Psi = \Psi_p + \Psi_q$ an arbitrary subspace $q$ using the Green’s function of this subspace:

$$H_p = T_p + V_p + V_{pq}G_qV_{pq}$$

In a certain adiabatic approximation we can assume that the radial part of kinetic energy is small and constant un-
under the centrifugal barrier in the channels with so high centrifugal barrier that it is much higher than any other interaction. In this approximation the reduction procedure becomes trivial as it is reduced to construction of effective three-body interactions $V^{eff}_{K\gamma,K'\gamma'}$ by matrix operations

$$G^{−1}_{K\gamma,K'\gamma'} = (H − E)^{−1}_{K\gamma,K'\gamma'} = V^{K\gamma,K'\gamma'}_{K\gamma,K'\gamma'} + \left( E_f − E + \frac{(K + 3/2)(K + 5/2)}{2Mρ^2} \right) \delta_{K\gamma,K'\gamma'}.$$

Summation over indexes with bar is made for eliminated channels. No strong sensitivity to the exact value of the “Feshbach energy” $E_f$ is found and we take it as $E_f = E$ in our calculations. More detailed account of the procedure applied within HH method can be found in Ref. 36.

It can be seen in Fig. 7 (solid line) that Feshbach reduction procedure drastically improves the convergence. However, the calculation converges to a width value, which is somewhat smaller than the corrected width value (that should be exact). The reason for this effect can be understood if we make a reduction to a smaller “dynamic” basis size ($K_{\max} = 12$, gray line). The calculation in this case also converges, but even to a smaller width value. We can conclude that FR procedure allows anyhow to approach the real width value, but provides a good result only for sufficiently large size of the dynamic sector of the basis.

The next issue to be discussed is a convergence of the width in calculations with different positions $E_{2\nu}$ of two-body resonance in the core+p subsystem. It is demonstrated for several energies $E_{2\nu}$ in Fig. 4. When the resonance in the subsystem is absent (or located relatively high) the convergence of the width value to the exact result is very fast both in the pure three-body and in the “corrected” calculation (in that case, however, much faster). Here even FR is not required as the

![FIG. 6: Convergence of energy distribution for $^{17}$Ne in the “Y” Jacobi system.](image)

![FIG. 7: Energy distributions for $^{17}$Ne in the “Y” Jacobi system for different two-body resonance positions $E_{2\nu}$. The three-body decay energy is $E_{3\nu} = 0.344$ MeV. The distributions are normalized to have unity value on maximum of three-body components. The values near the peaks show the fraction of the total intensity concentrated within the peak. Note the change of the scale at vertical axis.](image)

**TABLE I:** Parameters for $^{17}$Ne calculations. Potential parameters for $^{15}$O+p channel in s-wave ($V_{x0}$ in MeV, $r_0 = 3.53$ fm) and $^{16}$F+p channel in d-wave ($V_{x0}$ in MeV). Radius of the charged sphere is $r_{sph} = 3.904$ fm. Widths $\Gamma_i$ of the state in the subsystem and experimental width values $\Gamma_{exp}$ for really existing at these energies states are given in keV. The corrected three-body width $\Gamma_c$ is given in the units $10^{-14}$ MeV. TFSI calculations with d-wave state at 1.2 MeV are made with s-wave state at 0.728 MeV.

| $E_{2\nu}$ (MeV) | $I_x$ (I_y) | $V_{x0}$ (MeV) | $\Gamma_x (\Gamma_y)$ | $\Gamma_{exp}$ (MeV) | $\Gamma_c (\text{keV})$ |
|-----------------|-------------|----------------|-----------------------|---------------------|------------------------|
| 0.258           | 0           | −14.4          | 0.221                 | 144                 |
| 0.275           | 0           | −14.35         | 0.355                 | 16.6                |
| 0.292           | 0           | −14.3          | 0.544                 | 7.75                |
| 0.360           | 0           | −14.1          | 2.09                  | 2.34                |
| 0.535           | 0           | −13.55         | 17.9                  | 25(5) [34]          |
| 0.728           | 0           | −12.89         | 72.0                  | 70(5)              |
| 1.0             | 0           | −12.0          | 252                   | 0.093               |
| 2.0             | 0           | −9.0           | ~1500                 | 0.021               |

This is TFSI calculation with “$p$-p” Coulomb, $r_0 = 2.75$ fm.

This is TFSI calculation with “effective” Coulomb, $r_0 = 3.2$ fm.
The dependence of the three-body width follows well the analytical expression
\[ \Gamma \sim (E_{3r}/2 - E_{2r})^{-2} \] (34)

The reasons of such a behaviour will be clarified in the forthcoming paper [37]. The deviations from this dependence can be found in the decay window (close to “transition regime”) and at higher energies. This dependence is quite universal; e.g. for \( ^{45}\text{Fe} \) it is demonstrated in Fig. 14, where it follows the calculation results even with higher precision.

Another important issue is a convergence of energy distributions in the HH calculations, demonstrated in Fig. 9 for calculations with \( E_{2r} = 535 \text{ keV} \). The distribution is calculated in “Y” Jacobi subsystem, thus \( E_{2r} \) is the energy between the core and one proton. The energy distribution convergence is fast: the distribution is stable at \( K_{\text{max}} = 10 - 14 \) and does not change visibly with further increase of the basis. There remain a visible disagreement with exact (“corrected”) results, which give more narrow energy distribution. We think that this effect was understood in our work [4]. The three-body calculations are typically done for \( \rho_{\text{max}} \sim 500 - 2000 \text{ fm} \) (\( \rho_{\text{max}} \sim 1000 \text{ fm} \) everywhere in this work). It was demonstrated in Ref. [4] by construction of classical trajectories that we should expect a complete stabilization of the energy distribution in core+p subsystem at \( \rho_{\text{max}} \sim 30000 - 50000 \text{ fm} \) and the effect on the width of the energy distribution should be comparable to one observed in Fig. 9.

The evolution of the energy distribution in core+p subsystem with variation of \( E_{2r} \) is shown in Fig. 7. When we decrease the energy \( E_{2r} \) the distribution is very stable until the two-body resonance enters the three-body decay energy window. After that the peak at about \( \varepsilon \sim 0.5 \) first drifts to higher energy and then for \( E_{2r} \sim 0.85E_{3r} \) the noticeable second narrow peak for sequential decay is formed. At \( E_{2r} \sim 0.7E_{3r} \) the sequential peak becomes so high that the three-body component of the spectrum is practically disappeared in the background.

The result concerning the transition region obtained in this model is consistent with conclusion of the paper [8] (where much simpler model was used for estimates).
only when the sequential decay is energy prohibited as $E_{2\pi} > E_{3\pi}$. Also the three-body approach is valid when the sequential decay is formally allowed (because $E_{2\pi} < E_{3\pi}$) but is not taking place in reality due to Coulomb suppression at $E_{2\pi} \gtrsim 0.8E_{3\pi}$.

Geometric characters of potentials can play an important role in the width convergence. To test this aspect of the convergence we have also made the calculations for potential with repulsive core. This class of potentials was employed in studies of $^{17}$Ne and $^{19}$Mg in Ref. [7]. A comparison of the convergence of HH calculations with s-wave $^{15}$O+p potential from [7] and Gaussian potential is given in Fig. 8. The width convergence in the case of the “complicated” potential with a repulsive core is drastically worse than in the “easy” case of Gaussian potential. For typical dynamic calculations with $K_{\text{max}} = 20 - 24$ the HH calculations provide only $20 - 25\%$ of the width for potential with a repulsive core. On the other hand the calculations with both potentials provide practically the same widths $I_c$ [10] and FR provides practically the same and very well converged result in both cases.

B. One final state interaction — p-p channel

As far as two-proton decay is often interpreted as “diproton” decay we should also consider this case and study how important this channel could be. For this calculation we use a simple $s$-wave Gaussian p-p potential, providing a good low-energy $p$-$p$ phase shifts,

$$V(r) = -31 \exp[-(r/1.8)^2]. \quad (35)$$

Calculations with this potential are shown in Fig. 9 (see also Table [7]). First of all the penetrability enhancement provided by p-p FSI is much less than the enhancement provided by core-p FSI (the widths differs more than two orders of magnitude, see Fig. 9). This is the feature, which has been already outlined in our works. The p-p interaction may boost the penetrability strongly, but only in the situation, when protons occupy predominantly orbitals with high orbital momenta. In such a situation the p-p interaction allows transitions to configurations with smaller orbital momenta in the subbarrier region, which provide a large increase of the penetrability. In our simple model for $^{17}$Ne $3/2^-$ state, we have already assumed the population of orbitals with minimal possible angular momenta and thus no strong effect of the p-p interaction is expected.

Also a very slow convergence of the decay width should be noted in this case. For core-p interaction the $K_{\text{max}} \sim 10 - 40$ were sufficient to obtain a reasonable result. In the case of the p-p interaction the $K_{\text{max}} \sim 100$ is required.

Energy distributions between two protons obtained in this model are shown in Fig. 10. Important feature of these distributions is a strong focusing of protons at small $p$-p energies. This feature is connected, however, not with attractive p-p FSI, but with dominating Coulomb repulsion in the core-p channel. This is demonstrated by the calculation with nuclear FSI turned off, which provides practically the same energy distributions. Similarly to the case of the core-p FSI, very small $K_{\text{max}} > 10$ is sufficient to provide the converged energy distribution. The converged HH distribution is very close to the exact (“corrected”) one but it is, again, somewhat broader.

So far the diproton model has been treated by us as a reliable upper limit for three-body width [8]. With some technical improvements this model was used for the two-proton widths calculations in Refs. [16, 17, 18, 19]. It is important therefore to try to understand qualitatively the reason of the small width values obtained in this form of OFSI model, which evidently represents appropriately formulated diproton model [17]. In Fig. 11 we compared the results of the OFSI calculations for $^{45}$Fe in the “T” system (between two protons).

![FIG. 10: Energy distributions for $^{17}$Ne in “T” Jacobi system (between two protons).](image)

![FIG. 11: Comparison of the OFSI calculations (solid lines) for $^{45}$Fe in the “T” system with diproton model Eq. (35) (dashed lines). Effective equivalent channel radius $r_{ch}(dp)$ for “diproton emission” (a) as a function of radius $\rho_0$ of the three-body potential (25), the value $\rho_0/\sqrt{2}$ should be comparable with typical nuclear sizes. (b) as a function of the position of the peak $Y_{peak}$ in the three-body WF $\Psi^{(+)}$ in Y coordinate. The dash lines are given to guide the eye.](image)
system with diproton width estimated by expression

$$
\Gamma_{dp} = \frac{1}{M_{red} r_{ch}(dp)} P_{l=0}(0.95E_{3\gamma}, r_{ch}(dp), 2Z_{core})
$$

(36)

where $M_{red}$ is the reduced mass for $^{43}$Cr-$pp$ motion and $r_{ch}(dp)$ is channel radius for diproton emission. The energy for the relative $^{43}$Cr-$pp$ motion is taken $0.95E_{3\gamma}$ basing on the energy distribution in the $p-p$ channel (see Fig. 10 for example). In Fig. 11a we show the effective equivalent channel radii for diproton emission obtained by fulfilling condition $\Gamma_{dp} \equiv \Gamma_c$ for OFSI model calculations with different radii $\rho_0$ of the three-body potential Eq. (45). It is easy to see that for realistic values of these radii ($\rho_0 \sim 6$ fm for $^{45}$Fe) the equivalent diproton model radii should be very small ($\sim 1.5$ fm). This happens presumably because the “diproton” is too large to be considered as emitted from nuclear surface of such small $\rho_0$ radius. Technically it can be seen as the nonlinearity of the $r_{ch}(dp)$-$\rho_0$ dependence, with linear region achieved at $\rho_0 \sim 15-20$ fm. Only at such unrealistically large $\rho_0$ values the typical nuclear radius (when it becomes comparable with the “size” of the diproton) can be reasonably interpreted as the surface, off which the “diproton” is emitted. It is interesting to note that in the nonlinearity region for Fig. 11a there exists practically exact correspondence between the $Y$ coordinate of the WF peak in the internal region and the channel radius for diproton emission (Fig. 11b). This fact is reasonable to interpret in such a way that the diproton is actually emitted not from nuclear surface (as it is presumed by the existing systematics of diproton calculations) but from the interior region, where the WF is mostly concentrated.

C. Two final state interactions

As we have already mentioned the situation of one final state interaction is comfortable for studies, but rarely realized in practice. An exception is the case of the E1 transitions to continuum in the three-body systems, considered in our previous work [22]. For narrow states in typical nuclear system of the interest there are at least two comparable final state interactions (in the core-p channel). For systems with heavy core this situation can be treated reasonably well as the $Y$ coordinate (in “$Y$” Jacobi system) for such systems practically coincides with the core-p coordinate. Below we treat in this way $^{17}$Ne (for which this approximation could be not very consistent) and $^{45}$Fe (for which this approximation should be good). In the case of $^{17}$Ne we are thus interested in the scale of the effect, rather in the precise width value.

For calculations with two FSI for $^{17}$Ne we used Gaussian $d$-wave potential (see Table I), in addition to the s-wave potential used in Section IV.A. This potential provides a $d$-wave state at 0.96 MeV ($\Gamma = 13.5$ keV), which corresponds to the experimental position of the first $d$-wave state in $^{16}$O. The convergence of the $^{17}$Ne decay width is shown in Fig. 12. Comparing with Fig. 8 one can see that the absolute value of the width has changed significantly (2–3 times) but not extremely and the convergence is practically the same. Interesting new feature is a kind of the convergence curve “staggering” for odd and even values of $K/2$. Also the convergence of the corrected calculations requires now a considerable $K_{max} \sim 12 – 14$.

The improved experimental data for $2p$ decay of $^{45}$Fe is published recently in Ref. [22], $E_{3\gamma} = 1.154(16)$ MeV, $\Gamma_{2p} = 2.85^{+0.65}_{-0.68} \times 10^{-10}$ MeV [T_{1/2}(2p) = 1.6^{+0.5}_{-0.3}$ ms] for two-proton branching ratio $Br(2p) = 0.57$. Below we use the resonance energy from this work.

The convergence of the $^{45}$Fe width is shown in Fig. 13. The character of this convergence is very similar to that
in the 17Ne case, except the “staggering” feature is more expressed.

The dependence of the 45Fe width on the two-body resonance energy $E_{2\text{r}}$ is shown in Fig. 13. Potential parameters for these 45Fe calculations are given in Table II. The result calculated for $E_{3\text{r}} = 1.154$ MeV and $E_{2\text{r}} = 1.48$ MeV in paper [4] for pure $[p^2]$ configuration is $\Gamma = 2.85 \times 10^{-19}$ MeV. The value $K_{\text{max}} = 20$ was used in these calculations. If we take the HH width value from Fig. 13 at $K_{\text{max}} = 20$ it provides $\Gamma = 2.62 \times 10^{-19}$ MeV, which is in a good agreement with a full HH three-body model of Ref. [4]. However, from Fig. 13 we can conclude that the calculations of [4] the width was about 35% underestimated. Thus the value of about $\Gamma = 6.3 \times 10^{-19}$ MeV should be expected in these calculations. On the other hand much larger uncertainty could be inferred from Fig. 14 due to uncertain energy of the 44Mn ground state. If we assume a variation $E_{2\text{r}} = 1.1 - 1.6$ MeV the

![Graph](https://via.placeholder.com/150)

**FIG. 14:** The 45Fe g.s. width as a function of the two-body resonance position $E_{2\text{r}}$. Dashed, dotted and solid lines show cases of a pure HH calculation with $K_{\text{max}} = 24$, the same but with Feshbach reduction from $K_{\text{max}} = 100$, and the corrected width $\Gamma_c$. Gray area shows the transition region from three-body to two-body decay regime. The gray curve shows simple analytical dependence of Eq. (34).

The calculated width values are extrapolated using the convergence curves obtained in TFSI model (Figs. 15 for 17Ne and 26Mn or 45Fe). We have no proof that the width convergence in the realistic three-body case is absolutely the same as in the TFSI case. However, the TFSI model takes into account main dynamic features of the system causing a slow convergence, and we are expecting that the convergence should be nearly the same in both cases.

### V. THREE-BODY CALCULATIONS

Having in mind the experience of the convergence studies we have performed large-basis calculations for 45Fe and 17Ne. They are made with dynamical $K_{\text{max}} = 16 - 18$ (including Feshbach reduction from $K_{\text{max}} = 30 - 40$ for 17Ne and $K_{\text{max}} = 22$ (FR from $K_{\text{max}} = 40$) for 45Fe. The calculated width values are extrapolated using the convergence curves obtained in TFSI model (Figs. 15 for 17Ne and 26Mn or 45Fe). We have no proof that the width convergence in the realistic three-body case is absolutely the same as in the TFSI case. However, the TFSI model takes into account main dynamic features of the system causing a slow convergence, and we are expecting that the convergence should be nearly the same in both cases.

### A. Widths and correlations in 17Ne

The potentials used in the realistic calculations are the same as used for 17Ne studies in Refs. [1, 38]. The GPT potential [39] is used in the $p-p$ channel. The core-$p$ potentials are referred in [38] as “GMZ” (potential in-

![Graph](https://via.placeholder.com/150)

**FIG. 15:** Interpolation of 17Ne decay width obtained in full three-body calculations by means of TFSI convergence curves (see Fig. 9). Upper curves correspond to TFSI case with Gaussian potential in $s$-wave and compatible SI case for full three-body model. Lower curves correspond to TFSI case with repulsive core potential in $s$-wave and compatible GMZ case for full three-body model.

| $E_{2\text{r}}$ | $V_{50}$ | $\Gamma_x$ | $V_{50}$ | $\Gamma_y$ | $\Gamma_c$ |
|-----------------|--------|-----------|--------|-----------|-----------|
| 1.0             | $24.350$ | $4.3 \times 10^{-3}$ | $24.54$ | $2.1 \times 10^{-3}$ | 26.5 |
| 1.2             | $24.03$ | 0.032     | $24.224$ | 0.018     | 11.8 |
| 1.48            | $23.58$ | 0.26      | $23.78$ | 0.15      | 5.6 |
| 2.0             | $22.7$  | 3.6       | $22.93$ | 2.3       | 2.3 |
| 3.0             | $20.93$ | 58        | $21.49$ | 44        | 0.84 |

**TABLE II:** Parameters for 45Fe calculations. Potential parameters for $p$-wave interactions [29] in $^{43}$Cr+p channel ($V_{50}$ in MeV, $r_0 = 4.236$ fm, $r_{spK} = 5.486$ fm) and $^{44}$Mn+p ($V_{50}$ in MeV, $r_0 = 4.268$ fm, $r_{spK} = 5.527$ fm), $a = 0.65$ fm. Calculations are made with “effective Coulomb” of Eq. (22). Widths $\Gamma_x$, $\Gamma_y$ of the states in the subsystems are given in keV. Corrected three-body widths are given in the units $10^{-19}$ MeV.
produced in [5] and “high s” (with centroid of d-wave states is shifted upward which is providing a higher content of $s^2$ components in the $^{17}$Ne g.s. WF). Both potentials provide correct low-lying spectrum of $^{16}$F and differ only for d-wave continuum above 3 MeV (see Table III). The core-p nuclear potentials, including central, $ss$ and $ls$ terms, are taken as

$$V(r) = \frac{V_s^l + (s_1 \cdot s_2)V_s^l}{1 + \exp[(r - r_0^s)/a]} - (1 \cdot s)\frac{2.0153 V_s^l}{a r} \times \exp[(r - r_0^l)/a] (1 + \exp[(r - r_0^l)/a])^{-2},$$

with parameters: $a = 0.65$ fm, $r_0^s = 3.014$ fm, $r_0^{ls} = 2.94$ fm, $V_0^s = -26.381$ MeV, $V_1^s = -9$ MeV, $V_0^l = -57.6$ ($-51.48$) MeV, $V_2^l = -9$ MeV, $V_0^s_s = 0.885$ MeV, $V_2^s_s = 4.5$ (12.66) MeV, $V_1^s = 4.4$ (13.5) MeV (the values in brackets are for “high s” case). There are also repulsive cores for $s$- and $p$-waves described by $a = 0.4$ fm, $r_0^s = 0.89$ fm, $V_{core} = 200$ MeV. These potentials are used together with Coulomb potential obtained for Gaussian charge distribution reproducing the charge radius of $^{15}$O.

To have extra confidence in the results, the width of the $^{17}$Ne $3/2^-$ state is calculated in several models of growing complexity (Tables IV-VI). One can see from those Tables that improvements introduced on each step provide quite smooth transition from the very simple to the most sophisticated model.

In Table V we compare approximations of a different kind: those connected with choice of the Jacobi coordinate system in the simplified model. First we compare the “pure Coulomb” case: all pairwise nuclear interactions are off and the existence of the resonance is provided solely by the three-body potential ([25]). This model provides some hint what should be the width of the system without nuclear pairwise interactions. Then the models are compared with the nuclear FSIs added. The addition of nuclear FSI drastically increase width in all cases. It is the most “efficient” (in the sense of width increase) in the case of TFSI model in the “Y” system. Choice of this model provides the largest widths and can be used for the upper limit estimates.

In Table VI full three-body models are compared. The simplistic S1 and S2 interactions correspond to calculations with simplified spectra of the $^{16}$F subsystem. For S1 case it includes one s-wave state at 0.535 MeV ($\Gamma = 18.8$ keV) and one d-wave state at 0.96 MeV ($\Gamma = 3.5$ keV). These are two lower s- and d-wave states known experimentally. In the S2 case we use instead the experimental positions of the higher component of the s- and d-wave doublets: s-wave at 0.72 MeV ($\Gamma = 73.4$ keV) and d-wave state at 4.6 MeV ($\Gamma = 25(5)$ keV) and the S2 case we use instead the experimental positions of the higher component of the s- and d-wave doublets: s-wave at 0.72 MeV ($\Gamma = 73.4$ keV) and d-wave state at 4.6 MeV ($\Gamma = 25(5)$ keV).

**TABLE III: Low-lying states of $^{16}$F obtained in the “GMZ” and “high s” core-p potentials. The potential is diagonal in the representation with definite total spin of core and proton S, which is given in the third column.**

| Case | GMZ | high s | Exp. |
|------|-----|--------|------|
| $J^P$ | $l$ | $S$ | $E_{2\pi}$ (MeV) | $\Gamma$ (keV) | $E_{2\pi}$ (MeV) | $\Gamma$ (keV) | $\Gamma$ (keV) |
| 0$^-$ 0 0 | 0.535 | 18.8 | 0.535 | 18.8 | 25(5) | [34] |
| 1$^-$ 0 1 | 0.728 | 73.4 | 0.728 | 73.4 | 70(5) | [34] |
| 2$^-$ 2 0 | 0.96 | 3.5 | 0.96 | 3.5 | 6(3) | [34] |
| 3$^-$ 2 1 | 1.2 | 9.9 | 1.2 | 10.5 | < 15 | [35] |
| 2$^-$ 2 1 | 3.2 | 430 | 7.6 | 3000 | |
| 1$^-$ 2 1 | 4.6 | 1350 | 15 | 6000 | |
wave at 1.2 MeV ($\Gamma = 10$ keV). Parameters of the core-\(p\) potentials can be found in Table \ref{table:parameters} Simple Gaussian \(p-p\) potential \cite{55} is used. The variation of the results between these models is moderate ($\sim 30\%$). The calculations with GMZ potential provide the width for \(^{17}\text{Ne} \ 3/2^-\) state which comfortably rests in between the results obtained in the simplified S1 and S2 models. The structure of the WF is also obtained quite close to these calculations. The structure in the “high \(s\)” case is obtained with a strong domination of the \(sd\) component. The width in the “high \(s\)” case is obtained somewhat larger ($\sim 11\%$) than in GMZ case, but this increase is consistent with the increase of the \(sd\) WF component, ($\sim 15\%$) which is expected to be more preferable for decay than \(d^2\) component.

It is important for us that the results obtained in the three-body models with considerably varying spectra of the two-body subsystems and different convergence systematics appear to be quite close: $\Gamma \sim (5 - 8) \times 10^{-15}$ MeV. Thus we have not found a factor which could lead to a considerable variation of the three-body width, given the ingredients of the model are reasonably realistic.

The decomposition of the \(^{17}\text{Ne} \ WF \) obtained with GMZ potential is provided in Table \ref{table:decomposition} in terms of partial internal normalizations and partial widths. The correspondence between the components with large weights and large partial widths is typically good. However, there are several components giving large contribution to the width in spite of negligible presence in the interior.

Complete correlation information for three-body decay of a resonant state can be described by two variables (with omission of spin degrees of freedom). We use the energy distribution parameter $\varepsilon = E_x / E_{3r}$ and the angle $\cos(\theta_{3r}) = \left| \langle k_x, k_y \rangle \right| / \left( |k_x| |k_y| \right)$ between the Jacobi momenta. The complete correlation information is provided in Fig. \ref{fig:correlations} for realistic \(^{17}\text{Ne} \ 3/2^-\) decay calculations. We can see that the profile of the energy distribution is characterized by formation of the double-hump structure, expected so far for $p^2$ configurations (see, e.g. \cite{4}). This structure can be seen both in “\(T\)” system (in energy distribution) and in “\(Y\)” system (in angular distribution). In the calculations of ground states of the \(s-d\) shell nuclei we were getting such distributions to be quite smooth. It can be found that the profile of this distribution is defined by the \(sd/d^2\) components ratio. For example in the calculations with “high \(s\)” potential the total domination of the \(sd\) configuration leads to washing out of the double-hump profile.

The correlations in the \(^{17}\text{Ne}\) (shown in Fig. \ref{fig:correlations}) are strongly influenced by the nuclear FSIs. Calculations for only Coulomb pairwise FSIs left in the Hamiltonian are

![Fig. 17: Correlations for \(^{17}\text{Ne} \) decay in “\(T\)” and “\(Y\)” Jacobi systems. Three-body calculations with Coulomb FSIs only (all nuclear pairwise potentials are turned off).](image)

| \(3\)-body | \(2\)-body | \(1\)-body | \(0\)-body |
|---|---|---|---|
| \(\varepsilon\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) |
| \(l\) | \(l\) | \(l\) | \(l\) |
| \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) |
| \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) |
| \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) | \(\langle \sigma \rangle\) |

*Small repulsion ($\sim 0.5$ MeV) was added in that case in the \(p\)-wave core-\(p\) channel to split the states with \(sd\) and \(p^2\) structure which appear practically degenerated and strongly mixed in this model.

*TABLE IV: Comparison of widths for \(^{17}\text{Ne}\) (in \(10^{-14}\) MeV units) obtained in simplified model in “\(Y\)” Jacobi system and in full three-body model with correspondingly truncated Hamiltonian. Structure information is provided for the three-body model. In the simplified model the weight of the \([sd]\) configuration is 100\% by construction. “\(p-p\)” column shows the case where Coulomb interaction in \(p-p\) channel is switched off (see, \cite{21}). “Eff.” column corresponds to the effective treatment (see, \cite{22}) of Coulomb interaction in the \(p-p\) channel in the simplified model, but to the exact treatment in full three-body model.*
shown in Fig. 17. The strong peak at small p-p energy is largely dissolved and the most prominent feature of the correlation density in that case is a rise of the distribution for $\cos(\theta_k) \to 1$ in the “Y” Jacobi system. This kinematical region corresponds to motion of protons in the opposite directions from the core and is qualitatively understandable feature of the three-body Coulomb interaction (the p-p Coulomb interaction is minimal along such a trajectory).

The distributions calculated in the simplified (OFSI) model are shown in Fig. 18 on the same $\{\epsilon, \cos(\theta_k)\}$ plane as in Figs. 16 and 17. It should be noted that here the calculations in “T” and “Y” Jacobi systems represent different calculations (with p-p FSI only and with core-p FSI only). In Figs. 16 and 17 two panels show different representations of the same result. Providing reasonable (within factor 2$^{-4}$) approximation to the full three-body model in the sense of the decay width, the simplified model is very deficient in the sense of correlations. The only feature of the realistic correlations which is even qualitatively correctly described in the simplified model is the energy distribution in the “Y” system. The “diproton” model (OFSI model with p-p interaction) fails especially strongly, which is certainly relevant to the very small width provided by this calculation.

B. Width of $^{45}$Fe

The calculation strategy is the same as in [4]. We start with interactions in the core-p channel which give a resonance in p-wave at fixed energy $E_{2r}$. Such a calculation provides $^{45}$Fe with practically pure $p^2$ structure. Then we gradually increase the interaction in the f-wave, until it replaces the p-wave resonance at fixed $E_{2r}$, and then we gradually move the p-wave resonance to high energy. Thus we generate a set of WFs with different $p^2/f^2$ mixing ratios.

The results of the improved calculations with the same settings as in [4] (the $^{44}$Mn g.s. is fixed to have $E_{2r} = 1.48$ MeV) are shown in Fig. 19 (see also Table V) together with updated experimental data [12]. The basis size used in [4] was sufficient to provide stable correlation pictures (as we have found in this work) and they are not updated.

| TABLE V: Comparison of widths calculated for $^{17}$Ne ($10^{-14}$ MeV units) and $^{45}$Fe ($10^{-19}$ MeV units) with pure Coulomb FSIs and for nuclear plus Coulomb FSIs. Simplified OFSI model in “T”, TFSI in “Y” Jacobi systems (“effective” Coulomb is used in both cases) and full three-body calculations. |
| --- |
|  | pure Coulomb | Nuclear+Coulomb |
|  | “T” | “Y” | 3-body | “T” | “Y” | 3-body |
| $^{17}$Ne | 0.0011 | 0.0032 | 0.0041 | 0.0077 | 1.37 | 0.76$^a$ |
| [$sdf$] | 100 | 100 | 99.3 | 100 | 100 | 73.1 |
| $[p^2]$ | 0 | 0 | 0.6 | 0 | 0 | 1.8 |
| $[d^2]$ | 0 | 0 | 0 | 0 | 24.2 |
| $^{45}$Fe | 0.0053 | 0.0167 | 0.26 | 0.034 | 4.94 | 6.3$^b$ |

$^a$This is a calculation with S1 Hamiltonian.
$^b$This is a calculation providing pure $p^2$ structure.

| TABLE VI: Width (in $10^{-14}$ MeV units) and structure of $^{17}$Ne 3/2$^-$ state calculated in a full three-body model with different three-body Hamiltonians. |
| --- |
|  | S1 | S2 | GMZ | high s |
| $K_{max} = 18$ | 0.35 | 0.27 | 0.14 | 0.16 |
| Extrapolated | 0.76 | 0.56 | 0.69 | 0.76 |
| [$sdf$] | 73.1 | 71.7 | 80.2 | 95.1 |
| $[p^2]$ | 1.8 | 1.8 | 2.0 | 1.3 |
| $[d^2]$ | 24.2 | 25.7 | 16.8 | 3.1 |
The sensitivity of the obtained results to the experimentally unknown energy of $^{44}$Mn can be easily studied by means of Eq. (34). The results are shown in Fig. 20 in terms of the regions consistent with experimental data on the $(E_{2r}, W(p^2))$ plane [$W(p^2)$ is the weight of $p^2$ configuration in $^{45}$Fe WF]. It is evident from this plot that our current experimental knowledge is not sufficient to draw definite conclusions. However, it is also clear that with increased precision of the lifetime and energy measurements for $^{45}$Fe and the appearance of more detailed information on $^{44}$Mn subsystem the restrictions on the theoretical models should become strong enough to provide the important structure information.

### VI. DISCUSSION

General trends of the model calculations can be well understood from Tables IV-VI. For the pure Coulomb case the simplified model calculations (in the “Y” and “T” systems) and three-body calculations provide reasonably consistent results. The simplified calculations in the “Y” system always give larger widths than those in the “T” system. From decay dynamics point of view this leads to understanding of the contradictory fact that the sequential decay path is preferable even if no even virtual sequential decay is possible (as the nuclear interactions are totally absent in this case).

The calculations with attractive nuclear FSIs rather expectedly provide larger widths than the corresponding calculations with Coulomb interaction only. The core-proton FSI is much more efficient for width enhancement than $p-p$ FSI. This fact is correlated with the observation of the previous point and is a very simple and strong indication that the wide-spread perception of the two-proton decay as “diproton” decay is to some extent misleading. As it has already been mentioned the $p-p$ FSI influences the penetration strongly in the very special case when the decay occurs from high-$l$ orbitals (e.g. $f^2$ in the case of $^{49}$Fe). Thus we should consider as not fully consistent the attempts to explain two-proton decay results only by the FSI in the $p-p$ channel (e.g. Ref. [19]) as much stronger decay mechanism is neglected in these studies.

From technical point of view the states considered in this work belong to the most complicated cases. The complication is due to the ratio between the decay energy and the strength of the Coulomb interaction (it defines the subbarrier penetration range to be considered dynamically). Thus the convergence effects demonstrated in this work for $^{17}$Ne have the strongest character among the systems studied in our previous works. Because of the relatively small $K_{\text{max}} = 12$ used in the previous works we have found an order of the magnitude underestimation of the $^{17}$Ne(3/2−) width. For systems like $^{48}$Ni ~ $^{66}$Kr the underestimation of widths in our previous calculations is expected to be about factor of 2. A much smaller effect is expected for lighter systems.

It was demonstrated in [22, 23] that the capture rate for the $^{15}$O(2p,γ)$^{17}$Ne reaction depends strongly on the two-proton width of the first excited 3/2− state in $^{17}$Ne. This width was calculated in Ref. [7] as $4.1 \times 10^{-16}$ MeV (some confusion can be connected with misprint in Table

### TABLE VII: Partial widths $\Gamma_{K \gamma}$ of different components of $^{17}$Ne 3/2− WF calculated in “T” Jacobi systems. Partial weights are given in “T” (value $N_{K \gamma}^{(T)}$) and in “Y” (value $N_{K \gamma}^{(Y)}$) Jacobi systems. $S_z$ is the total spin of two protons.

| $K$ | $L$ | $l_x$ | $l_y$ | $S_z$ | $N_{K \gamma}^{(Y)}$ | $N_{K \gamma}^{(T)}$ | $\Gamma_{K \gamma}$ |
|-----|-----|------|------|------|-----------------|-----------------|-----------------|
| 2   | 2   | 0    | 2    | 0    | 23.88           | 33.87           | 44.93           |
| 2   | 2   | 0    | 0    | 0    | 24.97           | 16.52           | 13.29           |
| 2   | 1   | 1    | 1    | 0    | 0.28            | 7.39            | 3.59            |
| 2   | 1   | 1    | 0    | 1    | 1.54            |                 |                 |
| 2   | 0   | 2    | 1    | 0    | 3.68            |                 |                 |
| 2   | 2   | 0    | 1    | 3.68  |                 |                 |                 |
| 2   | 2   | 0    | 1    | 8.97  | 20.04           | 3.19            |
| 2   | 2   | 0    | 0    | 8.68  | 13.57           | 5.57            |
| 2   | 2   | 2    | 0    | 15.49 | 0.32            | 18.80           |
| 2   | 1   | 3    | 1    | 0.03  | 2.18            | 0.95            |
| 4   | 2   | 3    | 1    | 1.89  | 0.63            |                 |
| 4   | 1   | 2    | 2    | 1.02  |                 |                 |
| 4   | 2   | 0    | 2    | 1.99  |                 |                 |
| 4   | 2   | 0    | 0    | 2.07  |                 |                 |
| 6   | 2   | 2    | 4    | 0.14  | 0.77            | 3.57            |
| 6   | 2   | 4    | 2    | 0.14  | 0.77            | 0.78            |
| 6   | 2   | 0    | 2    | 0.50  | 0.09            | 0.69            |
| 6   | 2   | 4    | 4    | 0.02  | 0.003           | 1.58            |

![FIG. 19: The lifetime of $^{45}$Fe as a function of the 2p decay energy $E_{3r}$. The plot is analogue of Fig. 6a from [4] with updated experimental data [12] and improved theoretical results. Solid curves shows the cases of practically pure $p^2$ and $f^2$ configurations, dashed curves stand for different mixed $p^2/f^2$ cases. The numerical labels on the curves show the weights of the $s^2$ and $p^2$ configurations in percents.](image-url)
with realistic calculations confined to the narrow range $\Gamma \sim 5 \times 10^{-14}$ MeV (see Table VII) was obtained in a TFSI calculation neglecting $p$-$p$ Coulomb interaction. The other models systematically produce smaller values, with realistic calculations confined to the narrow range $\Gamma \sim (5 - 8) \times 10^{-15}$ MeV (Table VII). Thus the value $\Gamma \sim 4 \times 10^{-12}$ MeV obtained in paper [21] is very likely to be erroneous. That result is possibly connected with a simplistic quasiclassical procedure for width calculations employed in this work.

III of Ref. [3], see erratum). However, in the subsequent work [21], providing very similar to [21] properties of the $^{17}$Ne WFs for the ground and the lowest excited states, the width of the $3/2^-$ state was found to be $3.6 \times 10^{-12}$ MeV. It was supposed in [21] that such a strong disagreement is connected with poor subbarrier convergence of the HH method in [7] compared to Adiabatic Faddeev HH method of [21]. This point was further reiterated in Ref. [41]. We can see now that this statement has a certain ground. However, the convergence problems of the HH method are far insufficient to explain the huge disagreement: the width increase found in this work is only one order of magnitude. The most conservative upper limit $\Gamma \sim 5 \times 10^{-14}$ MeV (see Table IV) was obtained in a TFSI calculation neglecting $p$-$p$ Coulomb interaction. The other models systematically produce smaller values, with realistic calculations confined to the narrow range $\Gamma \sim (5 - 8) \times 10^{-15}$ MeV (Table VI). Thus the value $\Gamma \sim 4 \times 10^{-12}$ MeV obtained in paper [21] is very likely to be erroneous. That result is possibly connected with a simplistic quasiclassical procedure for width calculations employed in this work.

VII. CONCLUSION.

In this work we derive the integral formula for the widths of the resonances decaying into the three-body channel for simplified Hamiltonians and discuss various aspects of its practical application. The basic idea of the derivation is not new, but for our specific purpose (precision solution of the multichannel problem) several important features of the scheme have not been discussed.

We can draw the following conclusions from our studies:

(i) We presume that HH convergence in realistic calculations should be largely the same as in the simplified calculations as they imitate the most important dynamic aspects of the realistic situation. The width values were somewhat underestimated in our previous calculations. The typical underestimation ranges from few percent to tens of percent for “simple” potential and from tens of percent to an order of magnitude in “complicated” cases (potentials with repulsive core).

(ii) Convergence of the width calculations in the three-body HH model can be drastically improved by a simple adiabatic version of the Feshbach reduction procedure. For a sufficiently large dynamic sector of the basis the calculation with effective FR potential converges from below and practically up to the exact value of the width. For a small dynamic basis the FR calculation converges towards a width value smaller than the exact value, but still improves considerably the result.

(iii) The energy distributions obtained in the HH calculations are quite close to the exact ones. Convergence with respect to basis size is achieved at relatively small $K_{\text{max}}$ values. The disagreement with exact distributions is not very significant and is likely to be connected not with basis size convergence but, with radial extent of the calculations [4].

(iv) Contributions of different decay mechanisms were evaluated in the simplified models. We have found that the “diproton” decay path is much less efficient than the “sequential” decay path. This is true even in the model calculations without nuclear FSIs (no specific dynamics), which means that the “sequential” decay path is somehow kinematically preferable.

(v) The value of the width for $^{17}$Ne $3/2^-$ state was underestimated in our previous works by around an order of magnitude. A very conservative upper limit is obtained in this work as $\Gamma \sim 5 \times 10^{-14}$ MeV, while typical values for realistic calculations are within the $(5 - 8) \times 10^{-15}$ MeV range. Thus the value $\Gamma \sim 4 \times 10^{-12}$ MeV obtained in papers [21, 41] is likely to be erroneous.

From this paper it is clear that the convergence issue is sufficiently serious, and in some cases were underestimated in our previous works. However, from practical point of view, the convergence issue is not a principle problem. For example the uncertain structure issues and subsystem properties impose typically much larger uncertainties for width values. For heavy two-proton emitters (e.g. $^{45}$Fe) the positions of resonances in the subsystems are experimentally quite uncertain. For a moment this is the issue most limiting the precision of theoretical predictions. We have demonstrated that with increased precision the experimental data impose strong restrictions on theoretical calculations allowing to extract an important structure information.
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tinuum considered in Ref. [23]. This case is relevant to the low energy radiative capture reactions, important for astrophysics, but deal with nonresonant continuum only.

[43] Interesting numerical stability test is a variation of the “unphysical” (for OFSI approximation) potential $V_{nuc}(Y)$ in the auxiliary Hamiltonian (27). It can be used for numerical tests of the procedure as it should not influence the width. Really, for variation of this potential from weak attraction (we should not allow an unphysical resonance into decay window) to strong repulsion (scale of the variation is tens of MeV for potential with some typical radius) the width is varied only within couple of percents. This shows high numerical stability of the procedure.

[44] These values can be evaluated as typical nuclear radius for the system multiplied by $\sqrt{2}$: $3.53\sqrt{2} \approx 5$ and $4.29\sqrt{2} \approx 6$.

[45] The derivation of the flux here is given in a schematic form. The complete proof is quite bulky to be provided in the limited space. We would mention only that it is easy to check directly that the derived expression for flux preserves the continuum normalization.

[46] We demonstrate in paper [37] that a three-body width should depend linearly on two-body widths of the subsystems and only very weakly on various geometrical factors. This is confirmed very well by direct calculations.

[47] The assumed nuclear structure is very simple, but the diproton penetration process is treated exactly — without assumptions about the emission of diproton from some nuclear surface, which should be made in “R-matrix” approach.