Energy of spherically symmetric spacetimes on regularizing teleparallelism

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We calculate the total energy of an exact spherically symmetric solutions, i.e., Schwarzschild and Reissner Nordström, using the gravitational energy-momentum 3-form within the tetrad formulation of general relativity. We explain how the effect of the inertial makes the total energy unphysical! Therefore, we use the covariant teleparallel approach which makes the energy always physical one. We also show that the inertial has no effect on the calculation of momentum.

1. Introduction

The best well known gravitational theory is the Einstein one. This theory up to this day is consistence with observational data. The geometry in which Einstein general relativity (GR) based on is the Riemannian geometry with unique metric and unique connection. However, due to the geometric structure and the equivalence principle of the gravitational theory the problem of energy is not completely solved until now. Using the Lagrange-Noether approach, one can derives the conserved currents that are arise from the invariance of the classical action under transformations of fields. However, in Riemannian geometry one can not find symmetries that can be used to generate the conserved energy-momentum currents. Only one can speak about the energy of asymptotically flat spacetime. Earlier analyses of this problem can be found in details in ([1]∼[5] and references therein) for example.

As is well known, gravitational interaction can be described either by curvature or torsion [6]. According to GR, curvature is used to geometrize spacetime, and in this way a successful
description to the gravitational interaction is carried out. On the other hand teleparallelism, attributes gravitation to torsion. In this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. This means that, in the teleparallel equivalent of general relativity, there are no geodesics, but force equations quite analogous to the Lorentz force equation of electrodynamics [7]. Therefore, Gravitational interaction, can be described either in terms of curvature, as is usually done in GR, or in terms of torsion, in which case we have the teleparallel gravity.

Teleparallel theories are interesting for several reasons: first of all, GR can be viewed as a particular theory of teleparallelism and, thus, teleparallelism could be considered at the very least as a different point of view that can lead to the same results [8]. Second, in this framework, one can define an energy-momentum tensor for the gravitational field that is a true tensor under all general coordinate transformations. This is the reason why teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy-momentum tensor for the gravitational field [9]. The idea was taken over by Pellegrini and Plebański that constructed the general Lagrangian for these theories [10]. The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full Poincaré group) and, thus, they give an alternative interpretation of GR [11, 12].

An important difference between Einstein GR theory and teleparallel theories is that it is possible to distinguish gravitation and inertia [13]. Since inertia is in the realm of the pseudotensor behavior of the usual expressions for the gravitational energy-momentum density, it turns out possible in teleparallel gravity to write down a tensorial expression for such density [14]. With the purpose of getting a deeper insight into the covariant teleparallel formalism, as well as to test how it works Lucas et al. [15] reanalyze the computation of the total energy of two examples. Recently Obukhov et al. [16] computed the energy and momentum transported by exact plane gravitational-wave solutions of Einstein equations using the teleparallel equivalent formulation of Einsteins theory. It is our aim to extend the calculations done by Lucas et al. [15] using Schwarzschild and Reissner Nordström solutions with local Lorentz transformations contain two constants \( c_1 \) and \( c_2 \). Also we show how is inertia related to Weitzenböck connection*.

In §2, we use the language of exterior forms to give an outline of the teleparallel approach. A brief review of the covariant formalism for the gravitational energy-momentum which is described by the pair \( (\vartheta^\alpha, \Gamma^\alpha_{\beta}) \) is also given in §2. In §3, we show by calculations that due to an inconvenient choice of a reference system, the traditional computation of the total energy of Schwarzschild and Reissner Nordström solutions are unphysical! Using the covariant formalism, we show that the Weitzenböck connection acts as a regularizing tool that separates the inertial contribution and provides the physically meaningful result. Final section is devoted for main results and discussion.

We use the Latin indices \( i, j, \cdots \) for local holonomic spacetime coordinates and the Greek indices \( \alpha, \beta, \cdots \) label (co)frame components. Particular frame components are denoted by hats, \( \hat{0}, \hat{1}, \) etc. As usual, the exterior product is denoted by \( \wedge \), while the interior product of a vector \( \xi \) and a p-form \( \Psi \) is denoted by \( \xi \lrcorner \Psi \). The vector basis dual to the frame 1-forms \( \vartheta^\alpha \) is denoted by \( e_\alpha \) and they satisfy \( e_\alpha \lrcorner \vartheta^\alpha = \delta^\alpha_{\alpha} \). Using local coordinates \( x^i \), we have \( \vartheta^\alpha = h^\alpha_i dx^i \) and \( e_\alpha = h^\alpha_i \partial_i \) where \( h^\alpha_i \) and \( h_i^\alpha \) are the covariant and contravariant components of the tetrad.

*We will use the same notation given in Ref. [15]
field. We define the volume 4-form by \( \eta \overset{\text{def.}}{=} \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \). Furthermore, with the help of the interior product we define
\[
\eta_{\alpha} \overset{\text{def.}}{=} e_{\alpha} ] \eta = \frac{1}{3!} \epsilon_{\alpha \beta \gamma \delta} \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta,
\]
where \( \epsilon_{\alpha \beta \gamma \delta} \) completely antisymmetric with \( \epsilon_{0123} = 1 \).
\[
\eta_{\alpha \beta} \overset{\text{def.}}{=} e_{\beta} ] \eta_{\alpha} = \frac{1}{2!} \epsilon_{\alpha \beta \gamma \delta} \vartheta^\gamma \wedge \vartheta^\delta,
\]
\[
\eta_{\alpha \beta \gamma} \overset{\text{def.}}{=} e_{\gamma} ] \eta_{\alpha \beta} = \frac{1}{1!} \epsilon_{\alpha \beta \gamma \delta} \vartheta^\delta,
\]
which are bases for 3-, 2- and 1-forms respectively. Finally,
\[
\eta_{\alpha \beta \mu \nu} \overset{\text{def.}}{=} e_{\nu} ] \eta_{\alpha \beta \mu} = e_{\nu} ] e_{\mu} ] e_{\beta} ] e_{\alpha} ] \eta,
\]
is the Levi-Civita tensor density. The \( \eta \)-forms satisfy the useful identities:
\[
\vartheta^\beta \wedge \eta_{\alpha} \overset{\text{def.}}{=} \delta^\beta_{\alpha} \eta, \\
\vartheta^\beta \wedge \eta_{\mu \nu} \overset{\text{def.}}{=} \delta^\beta_{\mu} \eta_{\nu} - \delta^\beta_{\nu} \eta_{\mu}, \\
\vartheta^\beta \wedge \eta_{\alpha \mu \nu} \overset{\text{def.}}{=} \delta^\beta_{\alpha} \eta_{\mu \nu} + \delta^\beta_{\mu} \eta_{\alpha \nu} + \delta^\beta_{\nu} \eta_{\alpha \mu}, \\
\vartheta^\beta \wedge \eta_{\alpha \beta \mu \nu} \overset{\text{def.}}{=} \delta^\beta_{\nu} \eta_{\alpha \gamma \mu} - \delta^\beta_{\mu} \eta_{\alpha \gamma \nu} + \delta^\beta_{\gamma} \eta_{\alpha \mu \nu} - \delta^\beta_{\alpha} \eta_{\gamma \mu \nu}.
\]

The line element
\[
ds^2 \overset{\text{def.}}{=} g_{\alpha \beta} \vartheta^\alpha \otimes \vartheta^\beta
\]
is defined by the spacetime metric \( g_{\alpha \beta} \).

2. Brief review of teleparallel gravity and energy-momentum conservation

Teleparallel geometry can be viewed as a gauge theory of translation \([11, 12, 17] \sim [20]\). The coframe \( \vartheta^\alpha \) can be viewed as a one-form that plays the role of the gauge translational potential of the gravitational field. Einstein’s general relativity theory can be reformulated as teleparallel equivalent to GR theory. Geometrically, one can view the teleparallel gravity as a special (degenerate) case \([6, 21, 22]\) of the metric-affine gravity in which the coframe \( \vartheta^\alpha \) and the local Lorentz connection \( \Gamma^\alpha_{\beta \delta} \) are subject to the distant parallelism constraint \( R^\alpha_{\beta \delta} = 0 \) \([6]\). The torsion 2-form
\[
 T^\alpha \overset{\text{def.}}{=} d\vartheta^\alpha + \Gamma^\alpha_{\beta \delta} \wedge \vartheta^\beta,
\]
arises as the gravitational gauge field strength and \( \Gamma^\alpha_{\beta \delta} \) being the Weitzenböck connection. The torsion \( T^\alpha \) can be decomposed into three irreducible pieces: the tensor part, the trace, and the axial trace, given respectively by, \([15]\), for example
\[
(1) T^\alpha \overset{\text{def.}}{=} T^\alpha - (2) T^\alpha - (3) T^\alpha,
\]
where
\[
(2) T^\alpha \overset{\text{def.}}{=} \frac{1}{3} \vartheta^\alpha \wedge (e_\beta] T^\beta), \\
(3) T^\alpha \overset{\text{def.}}{=} \frac{1}{3} e_\alpha ] (\vartheta^\beta \wedge T^\beta).
\]
The Lagrangian of the teleparallel equivalent gravity model reads

\[ V = -\frac{1}{2\kappa} T^\alpha \wedge^* \left( T_\alpha - 2 (2) T_\alpha - \frac{1}{2} (3) T_\alpha \right). \] (4)

\( \kappa = 8\pi G/c^3 \), where \( G \) is the Newtonian constant and \( c \) is the speed of light, \( * \) denotes the Hodge duality in the metric \( g_{\alpha\beta} \) which is assumed to be flat Minkowski metric \( g_{\alpha\beta} = o_{\alpha\beta} = \text{diag}(+1, -1, -1, -1) \), that is used to raise and lower local frame (Greek) indices.

The teleparallel field equations are obtained from the variation of the total action with respect to the coframe

\[ DH_\alpha - E_\alpha = \Sigma_\alpha, \] (5)

where \( DH_\alpha = dH_\alpha - \Gamma_\alpha^\beta H_\beta \) denotes the covariant exterior derivative and \( \Sigma_\alpha \) is the canonical energy-momentum current 3-form of matter

\[ \Sigma_\alpha \overset{\text{def}}{=} \frac{\delta L_{\text{matter}}}{\delta \theta^\alpha} \] (6)

as the source. In accordance with the general Lagrange-Noether scheme [18, 23] one derives from (4) the translational momentum 2-form and the canonical energy-momentum 3-form:

\[ H_\alpha \overset{\text{def}}{=} -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} \ast \left( T_\alpha - 2 T_\alpha - \frac{1}{2} (3) T_\alpha \right), \] (7)

\[ E_\alpha \overset{\text{def}}{=} \frac{\partial V}{\partial \theta^\alpha} = e_\alpha]V + \left( e_\alpha]T^\beta \right) \wedge H_\beta. \] (8)

Due to geometric identities [24], the gauge momentum (4) can be recast as

\[ V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \] (9)

The model resulting from the Lagrangian (4) is degenerate from the metric-affine viewpoint, because the variational derivatives of the action with the respect to the metric and connection are trivial. This means that the field equations are satisfied for any \( \Gamma_\alpha^\beta \). The presence of the connection field plays an important regularizing role as shown in [15]. The latter is twofold:

**First**: The teleparallel gravity becomes explicitly covariant under the local Lorentz transformations of the coframe. In particular, the Lagrangian (4) is invariant under the change of variables

\[ \theta'^\alpha = L^\alpha_\beta \theta^\beta, \quad \Gamma'^\alpha_\beta = \left( L^{-1}\right)^\mu_\sigma \Gamma_\mu^\nu L^\beta_\nu + L^\beta_\nu d\left( L^{-1}\right)^\gamma_\sigma, \] (10)

where \( L^\alpha_\beta(x) \in SO(1, 3) \). In the pure tetrad gravity which can be recovered when \( \Gamma_\alpha^\beta = 0 \), the Lagrangian is only quasi-invariantit changes by a total divergence.

The connection \( \Gamma_\alpha^\beta \) can be decomposed into Riemannian and post-Riemannian parts as

\[ \Gamma_\alpha^\beta \overset{\text{def}}{=} \Gamma_\alpha^\beta - K_\alpha^\beta, \] (11)
with $\tilde{\Gamma}^\beta_\alpha$ is the purely Riemannian connection and $K^\mu\nu$ is the contorsion 1-form which is related to the torsion through the relation

$$T^\alpha \stackrel{\text{def.}}{=} K^\alpha_\beta \land \vartheta^\beta.$$ \hfill (12)

The translational momentum (7) can be rewritten as [24]

$$H_\alpha = \frac{1}{2\kappa} K^{\mu\nu} \land \eta_{\alpha\mu\nu}.$$ \hfill (13)

Second: The more important property of the teleparallel framework is that the Weitzenböck connection actually represents inertial effects that arise due to the choice of the reference system [14]. The inertial contributions in many cases yield unphysical results for the total energy of the system, producing either trivial or divergent answers. The teleparallel connection acts as a regularizing tool which helps to subtract the inertial effects without distorting the true gravitational contribution [15].

In the Maxwell-type the field equation (5) can be rewritten in the form

$$DH_\alpha = E_\alpha + \Sigma_\alpha.$$ \hfill (14)

The Maxwell 2-form $F = dA$ represents the gauge field strength of the electromagnetic potential 1-form $A$. Using the Lagrangian $V(F)$, the 2-form of the electromagnetic excitations is defined by $H = -\frac{\partial V}{\partial F}$. The field equations has the form $dH = J$ where $J$ is the 3-form of the electric current density of matter. In view of the nilpotency of the exterior differential, $dd = 0$, the Maxwell equation yields the conservation law of the electric current, $dJ = 0$.

Similarly to electrodynamics, gravity is a self-interacting field, and the gauge field potential 1-form $\vartheta^\alpha$ carries an “internal” index $\alpha$. The gauge field strength 2-form $T^\alpha = D\vartheta^\alpha$ is now defined by the covariant derivative of the potential (compare with $F = dA$). The gravitational field excitation 2-form $H_\alpha$ introduced by (7), in a direct analogy to the Maxwell theory ($H = -\frac{\partial V}{\partial F}$). Finally, we observe that as compared to the Maxwell field equation $dH = J$, the gravitational field equation (14) contains now the covariant derivative $D$, and in addition, the right-hand side is represented by a modified current 3-form, $E_\alpha + \Sigma_\alpha$. The last term is the energy-momentum of matter, and we naturally conclude that the 3-form $E_\alpha$ describes the energy-momentum current of the gravitational field. Its presence in the right-hand side of the field equation (14) reflects the self-interacting nature of the gravitational field, and such contribution is absent in the linear electromagnetic theory.

Comparison with electrodynamics can be completed by deriving the corresponding conservation law. Indeed, since $DD = 0$ for the teleparallel connection, (14) tells us that the sum of the energy-momentum currents of gravity and matter, $E_\alpha + \Sigma_\alpha$, is covariantly conserved [14],

$$D(E_\alpha + \Sigma_\alpha) = 0.$$ \hfill (15)

This law is consistent with the covariant transformation properties of the currents $E_\alpha$ and $\Sigma_\alpha$.

One can rewrite the conservation of energy-momentum in terms of the ordinary derivatives. Using the explicit expression $DH_\alpha = dH_\alpha - \tilde{\Gamma}^\alpha_\beta \land H_\beta$, the field equation (5) and (14)
can be recast in an alternative form

\[ d (\varepsilon_\alpha + \Sigma_\alpha) = 0. \]  

(16)

The 3-form \( E_\alpha \) describes the gravitational energy-momentum in a covariant way, whereas the 3-form \( \varepsilon_\alpha \) is a noncovariant object. In terms of components, it gives rise to the energy-momentum pseudotensor. It is worthwhile to note that \( H_\alpha \) plays the role of energy-momentum superpotential both for the covariant energy-momentum current \( (E_\alpha + \Sigma_\alpha) \) and for the total (including inertia) non-covariant current \( (\varepsilon_\alpha + \Sigma_\alpha) \).

The \( \eta \)-forms defined in Eq. (1) serve as the basis of the spaces of forms of different rank, and when we expand the above objects with respect to the \( \eta \)-forms, the usual tensor formulation is recovered. Explicitly,

\[ H_\alpha = \frac{1}{\kappa} S_\alpha^{\mu \nu} \eta_{\mu \nu}, \]  

(17)

with \( S_\alpha^{\mu \nu} = -S_\alpha^{\nu \mu} \) has the form [6]

\[ S_\rho^{\mu \nu} \overset{\text{def}}{=} \frac{1}{4} (T_\rho^{\mu \nu} + T_\rho^{\nu \mu} - T_\rho^{\nu \mu}) - \frac{1}{2} \left( \delta_\rho^{\nu \mu} T_\theta^{\mu \theta} - \delta_\rho^{\mu \nu} T_\theta^{\nu \theta} \right). \]  

(18)

Similarly, the explicit form of the gravitational energy-momentum

\[ E_\alpha = \frac{1}{2} \left[ (e_\alpha \parallel T^\beta) \wedge H_\beta - T^\beta \wedge (e_\alpha \parallel H_\beta) \right]. \]  

(19)

Using (1), (17) and \( T^\alpha = T_\mu^{\alpha \nu} \partial^\mu \wedge \partial^\nu = 2T_{[\mu \nu]}^{\alpha} \partial^\mu \wedge \partial^\nu \) in (18) one can find [15]

\[ E_\alpha = t_\alpha^{\beta \gamma} \eta_{\beta \gamma}, \quad t_\alpha^{\beta \gamma} = \frac{1}{2\kappa} \left( 4T_{\alpha \nu}^{\lambda} S_{\lambda}^{\beta \nu} - T_{\mu \nu}^{\lambda} S_{\lambda}^{\mu \nu} \delta_\alpha^{\beta} + 4\Gamma_{\nu \lambda}^{\lambda} S_{\lambda}^{\beta \nu} \right). \]  

(20)

By the same method one can have [15]

\[ \varepsilon_\alpha = j_\alpha^{\beta \gamma} \eta_{\beta \gamma}, \quad j_\alpha^{\beta \gamma} = \frac{1}{2\kappa} \left( 4T_{\alpha \nu}^{\lambda} S_{\lambda}^{\beta \nu} - T_{\mu \nu}^{\lambda} S_{\lambda}^{\mu \nu} \delta_\alpha^{\beta} + 4\Gamma_{\nu \lambda}^{\lambda} S_{\lambda}^{\beta \nu} \right). \]  

(21)

Now \( t_\alpha^{\beta \gamma} \) is understood to be a true tensor since it depends explicitly on the Weitzenbök connection \( \Gamma_{\nu \lambda}^{\lambda} \) the current \( j_\alpha^{\beta \gamma} \) is a pseudotensor. Since the Weitzenbök connection \( \Gamma_{\nu \lambda}^{\lambda} \) represents the inertial effects related to the choice of the frame, we see clearly that the origin of the pseudotensor behavior of the usual energy-momentum densities is that they include those inertial effects [14].

Taking into account the analogous expansion of the matter energy-momentum, \( \Sigma_\alpha = \Sigma_\alpha^{\beta \gamma} \eta_{\beta \gamma} \), which introduces the energy-momentum tensor \( \Sigma_\alpha^{\beta \gamma} \), and using (17) and (20), we easily recover the field equation in tensor language (used, for example, in [24]). Note that
the conservation laws (15) and (16) coincide when we put $\Gamma^\alpha_{\beta\gamma} = 0$. The last term in (21) then disappears, whereas torsion reduces to the anholonomity 2-form, $T^\alpha = F^\alpha = d\vartheta^\alpha$. We denote the corresponding energy-momentum and superpotential with a tilde:

$$\tilde{E}_\alpha = E_\alpha|_{\Gamma^\alpha_{\beta\gamma}=0}, \quad \tilde{H}_\alpha = H_\alpha|_{\Gamma^\alpha_{\beta\gamma}=0}. \tag{22}$$

The properties of these quantities and their use for the computation of the total energy of the exact solutions was discussed in [22, 25]. Explicitly, one can have [15]

$$\tilde{H}_\alpha = \frac{1}{2\kappa} \bar{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \tag{23}$$

$$\tilde{E}_\alpha = \frac{1}{2} \left[ (e_{\alpha}] \vartheta^\delta) \wedge \tilde{H}_\beta - d\vartheta^\beta \wedge (e_{\alpha}]\tilde{H}_\beta) \right]. \tag{24}$$

3. Total energy of Schwarzschild and Reissner Nordström solutions

Lucas et al. [15] have calculated the energy of three different solutions that reproduce the same metric which gives the Schwarzschild metric. Also they have calculated the total energy of Kerr metric. Here we are going to generalized this calculation for another solutions that give the Schwarzschild and Reissner Nordström metrics. We will calculate the total energy using the tensorial expression of the energy-momentum.

3.1 Schwarzschild metric

Using the spherical local coordinates $(t, r, \theta, \phi)$, Schwarzschild solution is described by the coframe components:

$$\vartheta^\alpha = (\Lambda^\alpha_\gamma) (\Lambda'^\gamma_\delta) \vartheta^\delta, \tag{25}$$

where the coframe $\vartheta^\delta$ has the form

$$\vartheta^0 = \frac{1}{\alpha} c dt, \quad \vartheta^1 = \alpha dr, \quad \vartheta^2 = rd\theta, \quad \vartheta^3 = r \sin \theta d\phi, \text{ where}$$

$$\alpha = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}, \text{ and} \quad m = GM/c^2, \tag{26}$$

the matrices $\Lambda^\alpha_\gamma$ and $\Lambda'^\gamma_\delta$ are defined as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ 0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ 0 & \cos \theta & -\sin \theta & 0 \end{pmatrix}. \tag{27}$$
which is the global Lorentz transformation and the local Lorentz transformation \((\Lambda^\gamma_\delta)\) has the form

\[
(\Lambda^\gamma_\delta) = \begin{pmatrix}
\beta & \beta_1 \sin \theta \cos \phi & \beta_1 \sin \theta \sin \phi & \beta_1 \cos \theta \\
-\beta_1 \sin \theta \cos \phi & -1 + (1 - \beta) \sin^2 \theta \cos^2 \phi & (1 - \beta) \sin^2 \theta \sin \phi \cos \phi & (1 - \beta) \sin \theta \cos \theta \cos \phi \\
-\beta_1 \sin \theta \sin \phi & (1 - \beta) \sin^2 \theta \sin \phi \cos \phi & -1 + (1 - \beta) \sin^2 \theta \sin^2 \phi & (1 - \beta) \sin \theta \cos \theta \sin \phi \\
-\beta_1 \cos \theta & (1 - \beta) \sin \theta \cos \theta \cos \phi & (1 - \beta) \sin \theta \cos \theta \sin \phi & -1 + (1 - \beta) \cos^2 \theta
\end{pmatrix},
\]

where \(\beta\) and \(\beta_1\) have the form

\[
\beta = \frac{1}{\sqrt{1 - \frac{2c_1}{r}}}, \quad \beta_1 = \sqrt{\frac{2c_1}{1 - \frac{2c_1}{r}}}. \tag{29}
\]

where \(c_1\) is a constant. If we take tetrad (26), as well as the trivial Weitzenböck connection \(\Gamma^\alpha_\beta = 0\) and substitute into (22) we finally get

\[
\tilde{H}_0 = \frac{r \sin \theta}{8\pi} \left[ \left( 1 - \sin^2 \theta \cos^2 \phi - \cos \theta \sin \phi \right) \left( 1 + \sin \theta \cos \phi \right) \right] \left( \beta - 1 \right) - \frac{2\beta}{\alpha} \left( d\theta \wedge d\phi \right). \tag{30}
\]

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface \(\partial S = \{ r = R, \theta, \phi \} \) we obtain

\[
\tilde{E} = \int_{\partial S} \tilde{H}_0 = \frac{R}{3} \left\{ 2 + \beta \left( 1 - \frac{3}{\alpha} \right) \right\}.
\]

This case is similar to the freely falling discussed in [15, 25] when \(c_1 \neq 0\), i.e., \(E \neq M\) but here, the acceleration is not vanishing. If we put the constant \(c_1 = 0\) in Eq. (31), then \(\beta = 1\) and the energy will be the ADM, i.e., \(E = M\) which is the case of the proper tetrad [15]. This is due to the fact that local Lorentz transformation will be Minkowski metric, \(\delta_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)\).

Using the regularization framework which is based on the covariance property, i.e., we will take into account the Weitzenböck connection \(\Gamma^\alpha_\beta \neq 0\) in Eq. (17) and calculate the necessary components we finally get the superpotential

\[
H_0 = \frac{r \beta \sin \theta \left( 1 - \frac{1}{\alpha} \right)}{4\pi} \left( d\theta \wedge d\phi \right). \tag{32}
\]

The total energy of (31) thus has the form

\[
E = \int_{\partial S} H_0 = R\beta \left( 1 - \frac{1}{\alpha} \right) \approx M\beta = M + O \left( \frac{1}{R} \right). \tag{33}
\]
The non vanishing components needed to calculate the spatial momentum have the form
\[ \vec{H}_\alpha = H_\alpha, \ \hat{\alpha} = 1, 2, 3 \] have the form
\[
\begin{align*}
\vec{H}_1 &= H_1 \approx \frac{r \beta_1 (1 - \frac{1}{\alpha}) \sin \theta \left[ \sin \phi \cos \phi \left[ 1 - \sin \theta \cos \theta \right] - \cos^2 \phi \sin^2 \phi \right]}{4\pi} (d\theta \wedge d\phi), \\
\vec{H}_2 &= H_2 \approx \frac{r \beta_1 (1 - \frac{1}{\alpha}) \sin \theta \left[ \cos \theta \cos \phi \left[ \sin \theta \cos \phi - 1 \right] - \cos \phi \sin \phi \sin^2 \theta - \cos \theta \sin \theta \right]}{4\pi} (d\theta \wedge d\phi), \\
\vec{H}_3 &= H_3 \approx \frac{r \beta_1 (1 - \frac{1}{\alpha}) \sin^2 \theta \left[ \cos \theta \cos \phi - \sin \theta \sin \phi \right]}{4\pi} (d\theta \wedge d\phi). 
\end{align*}
\] (34)

Using Eqs. (34) we finally get the spatial momentum in the form
\[
P_1 = \int_{\partial S} H_1 = R \beta_1 \left( 1 - \frac{1}{\alpha} \right) \approx M \beta_1 = O \left( \frac{1}{R} \right), \quad P_2 = P_3 = 0. \tag{35}
\]

### 3.2 Reissner Nordström metric

Using the spherical local coordinates \((t, r, \theta, \phi)\), Reissner Nordström solution is described by the coframe components:
\[
^{R} \vartheta^\alpha = \Lambda^\alpha_\gamma \Lambda^\gamma_\delta \vartheta^\delta, \tag{36}
\]
where the coframe \(\vartheta^\delta\) has the form
\[
\vartheta^0 = \frac{1}{\alpha_1} dt, \quad \vartheta^1 = \alpha_1 dr, \quad \vartheta^2 = r d\theta, \quad \vartheta^3 = r \sin \theta d\phi, \quad \text{where} \quad \alpha_1 = \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-\frac{1}{2}},
\] (37)

\((\Lambda^\alpha_\gamma)\) is given by Eq. (27) and \((\Lambda^\gamma_\delta)\) is defined as
\[
(\Lambda^\gamma_\delta) = \begin{pmatrix}
\beta_2 & \beta_3 \sin \theta \cos \phi & \beta_3 \sin \theta \sin \phi & \beta_3 \cos \theta \\
-\beta_3 \sin \theta \cos \phi & -1 + (1 - \beta_2) \sin^2 \theta \cos^2 \phi & (1 - \beta_2) \sin^2 \theta \sin \phi \cos \phi & (1 - \beta_2) \sin \theta \cos \theta \cos \phi \\
-\beta_3 \sin \theta \sin \phi & (1 - \beta_2) \sin^2 \theta \sin \phi \cos \phi & -1 + (1 - \beta_2) \sin^2 \theta \sin^2 \phi & (1 - \beta_2) \sin \theta \cos \theta \sin \phi \\
-\beta_3 \cos \theta & (1 - \beta_2) \sin \theta \cos \theta \cos \phi & (1 - \beta_2) \sin \theta \cos \theta \sin \phi & -1 + (1 - \beta_2) \cos^2 \theta
\end{pmatrix},
\] (38)

where \(\beta_2\) and \(\beta_3\) have the form
\[
\beta_2 = \frac{1}{\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}}, \quad \beta_3 = \sqrt{\frac{\frac{2m}{r} + \frac{Q^2}{r^2}}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}}, \tag{39}
\]

with \(c_2\) is another constant. Take tetrad (36), as well as the trivial Weitzenböck connection \(\Gamma^\alpha_\beta = 0\) and substitute into (22) we finally get
\[
\vec{H}_0 \frac{r \sin \theta}{8\pi} \left[ \left( 1 - \sin^2 \theta \cos^2 \phi - \cos \theta \sin \phi \left[ 1 + \sin \theta \cos \phi \right] \right) \left( \beta_2 - 1 \right) \right] (d\theta \wedge d\phi). \tag{40}
\]
In general the acceleration of solution (36) is not vanishing. If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface \( \partial S = \{ r = R, \theta, \phi \} \), we obtain

\[
\tilde{E} = \int_{\partial S} \tilde{H}_0 = \frac{R}{3} \left\{ 2 + \beta_2 \left( 1 - \frac{3}{\alpha_1} \right) \right\}.
\]

(41)

When we put the parameter \( c_1 = 0 \) and \( c_2 = 0 \) in Eq. (39), the energy will be identical to Reissner Nordström [26] because \( \beta_2 = 1 \).

Using the regularization framework which is based on the covariance property, i.e., we will take into account the Weitzenböck connection \( \Gamma^{\alpha}_{\beta \gamma} \neq 0 \) in Eq. (17) and calculate the necessary components we finally get the superpotential

\[
\tilde{H}_0 = \frac{r \beta_2 \sin \theta}{4\pi} \left( 1 - \frac{1}{\alpha_1} \right)(d\theta \wedge d\phi).
\]

(42)

The total energy of (42) thus has the form

\[
E = \int_{\partial S} \tilde{H}_0 = R\beta_2 \left( 1 - \frac{1}{\alpha_1} \right) = M - \frac{Q^2 - 2Mc_1}{2R} + O \left( \frac{1}{R^2} \right).
\]

(43)

The non vanishing components needed to calculate the spatial momentum have of the form \( \tilde{H}_\alpha = H_\alpha \), \( \alpha = 1, 2, 3 \) have the form

\[
\tilde{H}_1 = H_1 \cong \frac{r \beta_3 (1 - \frac{1}{\alpha_1}) \sin \theta \left[ \sin \phi \cos \phi \{ 1 - \sin \theta \cos \theta \} - \cos^2 \phi \sin^2 \phi \right]}{4\pi} (d\theta \wedge d\phi),
\]

\[
\tilde{H}_2 = H_2 \cong \frac{r \beta_3 (1 - \frac{1}{\alpha_1}) \sin \theta \left[ \cos \theta \cos \phi \{ \sin \theta \cos \phi - 1 \} - \cos \phi \sin \phi \sin^2 \theta - \cos \theta \sin \theta \right]}{4\pi} (d\theta \wedge d\phi),
\]

\[
\tilde{H}_3 = H_3 \cong \frac{r \beta_3 (1 - \frac{1}{\alpha_1}) \cos^2 \theta \left[ \cos \theta \cos \phi - \sin \theta \sin \phi \right]}{4\pi} (d\theta \wedge d\phi).
\]

(44)

Using Eqs. (44) we finally get the spatial momentum in the form

\[
P_1 = \int_{\partial S} H_1 = R\beta_3 \left( 1 - \frac{1}{\alpha_1} \right) \cong M\beta_3 = O \left( \frac{1}{\sqrt{R}} \right), \quad P_2 = P_3 = 0.
\]

(45)

### 4. Discussion and conclusion

Teleparallel theory is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [27] ~ [33] or metric-affine gravity [23]. Physics relevant to geometry may be related to the teleparallel description of gravity [12, 34]. Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques ([35] and references therein). Teleparallel approach is used for positive-gravitational-energy proof [36]. It is shown that one of the main differences between general relativity and teleparallel theory is that the
Weitzenböck connection which represents only inertial effects related to the frame [15]. Therefore, one can separate gravitation from inertial effects. A tensorial expression for the energy-momentum density of gravity is obtained in [15]. A covariant teleparallel approach naturally yields regularized solutions for the energy and momentum due to the fact that the frame-related inertial contribution to the conserved quantities is always properly subtracted by the Weitzenböck connection.

In this study we show that in general, the total conserved energy-momentum \( \tilde{P}_\alpha \) corresponding to \( \tilde{H}_\alpha \) does not transform covariantly under a change of frame. However, for local Lorentz transformations which become global at spatial infinity, the total energy-momentum transforms covariantly as a Lorentz vector. This is clear from the two examples we have studied with two constants. These examples reproduce Schwarzschild and Reissner Nordström metrics. The constants \( c_1 \) and \( c_2 \) of these solutions plays the role of inertial as shown in Eqs. (31) and (41) which makes the total energy always unphysics. The only choice which makes the energy always physics is that \( \beta = 1 \) for the first example, i.e., Schwarzschild solution and \( \beta_2 = 1 \) for the second example, i.e., Reissner Nordström solution. The choice \( \beta = 1 \) leads to \( c_1 = 0 \) which makes the local Lorentz transformation (28) has the form of Minkowski metric \( o_{\alpha\beta} = diag(+1, -1, -1, -1) \). When these condition is satisfied we reproduce the case of proper tetrad discussed in [15].

Therefore, we use the tensorial expression for the energy-momentum density of gravity and calculate the total energy associated with the Schwarzschild and Reissner Nordström solutions. We show by calculations that the Weitzenböck connection acts as a regularizing tool which separates the inertial energy-momentum density, leaving the tensorial, physical energy-momentum density of the system untouched. This is clear from Eq. (33) in which the energy is given by \( E \equiv M\beta \approx M(1 + \frac{c_1}{R}) \approx M \). This shows that the terms that will contributes to the energy is of order \( O(1/R) \). In this case, i.e., Schwarzschild, we do not need terms of \( O(1/R) \). On the other hand, for the Reissner Nordström solution the energy is given by Eq. (43), i.e, \( E \equiv \beta_2(M - \frac{Q^2}{2R}) \approx (1 + \frac{c_1}{R} - \frac{c_2^2}{2R})(M - \frac{Q^2}{2R}) \approx M - \frac{Q^2 - 2Mc_1}{2R} + O(1/R^2) \). In this case, the constant \( c_1 \) contributes the total energy. When this constant, i.e., \( c_1 = M/2 \) the value of energy will be consistent [37]. Finally we show that the components needed to calculate the spatial momentum have the same form either we put the Weitzenböck connection trivial or non trivial. As is clear from Eqs (35) and (45) the components of spatial momentum associated with Schwarzschild and Reissner Nordström solutions are in agreement with the previous results [22, 37]. Finally we show that the inertial has no effect on the calculation of the spatial components as explained in the both examples studied in paper.
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