ABSTRACT

We briefly review a hamiltonian path integral formalism developed earlier by one of us. An important feature of this formalism is that the path integral quantization in arbitrary co-ordinates is set up making use of only classical hamiltonian without addition of adhoc $\hbar^2$ terms. In this paper we use this hamiltonian formalism and show how exact path integration may be done for several potentials.

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1. Introduction

The Feynman path integral formalism has been in use for over forty years and has been successfully applied to a variety of problems\(^1\). An important development in the past fifteen years has been exact treatment of large number of potential problems within the path integral formalism. Duru and Kleinert gave an exact path integral treatment of H-atom Green function using Kustaanheimo-Stiefel transformation\(^3\). The arguments originally used by Duru and Kleinert were formal and the manipulations used lacked mathematical justification. The H atom problem was given a correct treatment by Ho and Inomata\(^4\) soon after Duru and Kleinerts work. It has since been possible to give an exact path integral solution of several problems in quantum mechanics using the new technique scaling of local time. Notable among these are Morse oscillator\(^5\), Rosen-Morse oscillator\(^6\), Poschl-Teller potential\(^7\), Hartmann potential\(^8\), Hulthen potential\(^9\), $\delta$ - function potential\(^10\), square well\(^11\), and many other potential problems problem\(^12,13\). Besides the scaling of local time in combination with change of variables, the techniques of adding new degrees of freedom has been useful in most of the above mentioned applications.

The path integral formalism can be derived from quantum mechanics and this aspect has been a subject of extensive study in the literature. One can arrive at different types of path integral representations of the propagator by working with different bases. Thus, for example, we have configuration space path integrals, phase space path integral or coherent state representation of path integrals. In all these derivations quantum mechanics is assumed.

One of the most important features of the path integral formalism is that it gives an alternative route to quantization. In use of the path integrals as a scheme of quantization a prescription for setting up path integral is assumed and quantum mechanics is derived. The path integral quantization can be set up starting from classical lagrangian or hamiltonian form of the classical action for the system of interest. In the hamiltonian formalism of classical mechanics one can write dynamical equations in any pair of conjugate variables related to a given set by a canonical transformation. However, it appears very difficult to implement general canonical transformations in quantum mechanics even though there are a large number of investigations.
on this subject\textsuperscript{14}. However, the point transformations do not present any
difficulty in quantum mechanics. So it is of interest to ask if path integral
quantization scheme, based on lagrangian or hamiltonian form for the
action, can be formulated in arbitrary co-ordinates. The answer has been
known in the lagrangian formulation known for a long time. The configu-
ration space path integral can be set up in such a way that it works in arbitrary
co-ordinates utilizing the configuration space form of classical action\textsuperscript{15}.

In a quantization scheme developed by one of us\textsuperscript{16,17}, quantization is car-
ried out in arbitrary co-ordinates using the hamiltonian form of path inte-
grals. A unique feature of this scheme is that addition of $O(\hbar^2)$ terms is not
necessary in any co-ordinate system and only the classical hamiltonian is used
to set up hamiltonian path integral quantization in arbitrary co-ordinates.
This formalism makes an essential use of the idea of local scaling of time to
introduce a path integral representation for quantum mechanical propagator.

In ref. 17 the basic property of the canonical path integrals, that only
classical hamiltonian is needed for quantization in arbitrary co-ordinates, was
established; discussion of further properties and also of applications was not
taken up. In ref.18 some important properties of the canonical path integral,
such as scaling of local time, equivalence with lagrangian form of path inte-
gration have been investigated in detail. In this paper possible applications of
the scheme were indicated only briefly. In the past fifteen years many exactly
solvable quantum mechanical problems have been treated by means of path
integration by relating the given problem to another problem for which ex-
act path integration can be done or is already known. In this paper we show
that our scheme offers results which can be used in a very simple manner to
relate different path integrals. Although we, generally, follow the techniques
already available in the literature to give exact treatment, the details of the
treatments are different. There are some new points which require special
and careful treatment. For example, initial condition for the propagator has
to be treated carefully.

In the next section we summarize all the necessary results on hamiltonian
path integration from ref. 18. Some aspects of changes of variables and the
technique of adding new degrees of freedom are discussed in Sec.3 and Sec.4
respectively. Application of the hamiltonian path integral method to exact
solution of the quantum mechanical problems is discussed in the Sec. 5 to 7.

2. Definition and properties of the hamiltonian path integrals

In this section we briefly recall the definition of the hamiltonian path integral\textsuperscript{16,17}. The proofs and detailed discussion of the results can be found in our earlier paper\textsuperscript{18}. We shall assume that all the hamiltonian like functions appearing in this paper are independent of time and are quadratic in momenta.

2.1 Definitions: Given a phase space function $H(q,p)$, we introduce a quantity $(q|t|q_0t_0)$ which stands for the short time approximation and in terms of which a path integral for finite times is constructed. We assume $(q|t|q_0t_0)$ to be of the form

$$
(q|t|q_0t_0) = \frac{1}{\sqrt{\rho(q)\rho(q_0)}} \int d^n p_1 (q|t|p_1t_1)(p_1t_1|q_0t_0) \quad (1)
$$

The definition given below is such that (1) satisfies the semi-group property with respect to the measure $\rho(q)dnq$

$$
\int \rho(q_2)d^n q_2(q_3t_3||q_2t_2)(q_2t_2||q_1t_1) \approx (q_3t_3||q_1t_1) \quad (2)
$$

when the integral is calculated in the stationary phase approximation. The 'mixed short time propagators $(q|t|p_1t_1)$ and $(p_1t_1|q_0t_0)$ are defined below. Let $\gamma_1$ and $\gamma_2$ be two classical trajectories with boundary conditions as indicated below.

$$
\gamma_1 : \tau \rightarrow (\tilde{q}(\tau),\tilde{p}(\tau)), \text{ for } t_0 \leq \tau \leq t_1 \quad (3)
$$

$$
\tilde{q}(t_0) = q_0, \quad \tilde{p}(t_1) = p_1 \quad (4)
$$

$$
\gamma_2 : \tau \rightarrow (\tilde{q}(\tau),\tilde{p}(\tau)), \text{ for } t_1 \leq \tau \leq t \quad (5)
$$

$$
\tilde{p}(t_1) = p_1, \quad \tilde{q}(t) = q \quad (6)
$$

4
We define

\[(q_t | p_{1t_1}) = (2\pi\hbar)^{-n/2} \sqrt{D_{++}} \exp[iS_{++}(q_t, p_{1t_1})/\hbar]\] (7)

\[(p_{1t_1} | q_{0t_0}) = (2\pi\hbar)^{-n/2} \sqrt{D_{--}} \exp[iS_{--}(p_{1t_1}, q_{0t_0})/\hbar]\] (8)

where

\[D_{++} = \det \left( \frac{\partial^2 S_{++}}{\partial q_i^* \partial p_{1j}} \right)\] (9)

\[D_{--} = \det \left( \frac{\partial^2 S_{--}}{\partial \tilde{q}_i^* \partial p_{1j}} \right)\] (10)

and \(S_{++}, S_{--}\) are Legendre transforms of the classical action along the two trajectories,

\[S_{++}(q_t, p_{1t_1}) = p_{1i} \bar{q}_1^i + \int_{\gamma_2} (\tilde{p}_i d \bar{q}^i (\tau) - H(\bar{q}(\tau), \bar{p}(\tau))) d\tau\] (11)

\[S_{--}(q_{0t_0}, p_{1t_1}) = -p_{1i} \bar{q}_1^i + \int_{\gamma_1} (\tilde{p}_i d \bar{q}^i - H(\bar{q}(\tau), \bar{p}(\tau))) d\tau\] (12)

where

\[\bar{q}_1 \equiv \bar{q}(t_1), \quad \bar{q}_1^i \equiv \bar{q}_i(t_1)\] (13)

With the above notation we now introduce the definition of the short time propagator (STP) \((q_t || q_{0t_0})\) by

\[(q_t || q_{0t_0}) = \frac{1}{\sqrt{\rho(q)\rho(q_0)}} \int d^n p_1 (q_t | p_{1t_1}) (p_{1t_1} | q_{0t_0})\] (14)

In (7) to (12) it is understood that the independent variables are those explicitly shown on the left hand side. It is also understood that the right hand sides have been expressed in terms of the independent variables using the equations (4), (6) for the classical trajectories \(\gamma_1\) and \(\gamma_2\). The function \(S_{++}\) is the generator of canonical transformation connecting ‘old’ co-ordinates and momenta at time \(t\) to the ‘new’ ones at time \(t_1\). Similarly, \(S_{--}\) is the generator of canonical transformation connecting the conjugate
variables for the trajectory $\gamma_1$ at time $t_1$ to those at time $t_0$. It must be emphasized that $S_{\pm \pm}$ are need for short times only.

We shall define two path integrals, the first one is without any scaling of time and the second one with scaling of time. The first hamiltonian path integral $K[H, \rho]$ defined as a summation over histories with $(q_0 || q_0)$ inserted for short times:

$$K[H, \rho](q_t; q_0) \equiv \lim_{N \to \infty} \int \prod_{k=1}^{N-1} \rho(q_k) dq_k \int \prod_{k=0}^{N-1} (q_{k+1} || q_k)$$

with $\epsilon = t/N$ and $q_N \equiv q$.

Motivated by the importance of the local scaling of time in the path integral formalism, we now introduce the second hamiltonian path integral which will be described as hamiltonian path integral with scaling of time. The definition makes use of hamiltonian path integral HPI1. A special case of the hamiltonian path integral with scaling was found suitable for quantization in arbitrary coordinates.

Given a hamiltonian function $H(q, p)$, measure $\rho(q)$, and a strictly positive function $\alpha(q)$, we introduce second hamiltonian path integral with local scaling of time, to be denoted by $\mathcal{K}[H, \rho, \alpha]$ as

$$\mathcal{K}[H, \rho, \alpha](q_t; q_0) \equiv \frac{\sqrt{\alpha(q)\alpha(q_0)\int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[\alpha(H-E), \rho](q_\sigma; q_0)\cdots}}{\cdots}$$

The right hand side of (16) has the first hamiltonian path integral with $\alpha(q)(H - E)$ as ‘hamiltonian’ function. The two hamiltonian path integrals $K$ and $\mathcal{K}$ will henceforth be called HPI1 and HPI2 respectively. This completes the definition of HPI2. We shall now give several remarks concerning the definintion of HPI2 which will be helpful in clarifying our approach.

1. Our first remark concerns the definition of HPI2 and use of local scaling of time in this paper. Although the definition of HPI2 is inspired by the expression and formulas appearing in the literature, we do not utilize or rely on any of the other details, mathematical or otherwise, from papers in the existing literature on local scaling of time.
2. The second remark is about the relationship between HPI1 and HPI2. The path integral HPI2 generalizes HPI1 and conversely HPI1 becomes a special case of HPI2 in the sense that for trivial scaling, when the scaling function $\alpha(q)$ is a constant independent of time, HPI2 becomes equal to HPI1. For this reason we shall continue to describe HPI2 as the Hamiltonian path integral with scaling.

3. The third remark is on the appearance of the functions $\alpha(q)$ and $\rho(q)$ in the definition of HPI2. As far as the definition is concerned the HPI2 is defined for arbitrary scaling function $\alpha(q)$ and measure $\rho(q)$ just as the Hamiltonian function $H(q,p)$ is kept arbitrary in the definition of any path integral. Specific values such as those given by $\alpha(q) = \sqrt{g}$ and $\rho(q) = \sqrt{g}$ give rise to the path integral scheme suitable for quantization in arbitrary coordinates (see Sec. 3.2 below).

4. Our last remark concerns the properties and applications of HPI2. The results on scaling developed earlier and others obtained here have been applied to exact path integration by relating the path integral for the problem to be solved to another problem for which answers are already known or can be worked out. When one attempts to do this one has to, in general, deal with other intermediate expressions involving HPI1 or HPI2 with different choices of $\alpha(q)$ and $\rho(q)$. It is for this purpose $\alpha(q)$ and $\rho(q)$ have to be kept arbitrary while defining HPI2.

In the remaining part of this section we shall summarize important properties of the two Hamiltonian path integrals.

2.2 Descretized form: In the limit $\epsilon \to 0$, the only important terms are the terms of order $\epsilon$ and only these need be retained in the definition of STP. Retaining such terms only, the canonical STP is equivalent to

\[
(q\epsilon|q_00) = (\rho(q)\rho(q_0))^{-1/2} \int \frac{dp^n}{(2\pi\hbar)^n} \exp \left[ ip_i(q^i - q_{0i})/\hbar \right] \times 
\]

\[
\exp \left[ -i\epsilon (H(q,p) + H(q_0,p))/\hbar \right] \left[ 1 - \frac{\epsilon}{2} \frac{\partial^2 H(q,p)}{\partial q^i \partial p_i} \right. + \frac{\epsilon}{2} \left. \frac{\partial^2 H(q_0,p)}{\partial q_{0i} \partial p_i} \right] (17)
\]

Inserting (17) in (13)
\[ K[H, \rho](q_t; q_0) = \lim_{N \to \infty} \int \left( \prod_{k=1}^{N-1} \rho(q_k) dq_k \right) \prod_{m=0}^{N-1} (q_{m+1} - q_m) \]  
(18)
gives discrete form for the first hamiltonian path integral. Notice that products over \( k \) and over \( m \) appearing in (18) have different ranges. Hence this expression involves \( N - 1 \) fold \( q \) integrations and \( N - \) fold \( p \) integrations. It is easy to do the \( p \)-integrations in the STP if the hamiltonian is quadratic in momenta. This gives a lagrangian form of STP which can be brought to a standard form making use of the McLaughlin Schulman trick. This makes a connection of our hamiltonian path integrals with lagrangian form of path integral possible.

2.3 Immediate consequences of definition of path integrals: We make some useful observations which follow directly from the definitions and the discrete form for the short time approximation.

(a) Addition of a constant to hamiltonian \( H \) is equivalent to multiplying the HPI1 by a phase

\[ K[H - E, \rho](q_t; q_0) = \exp(iEt/\hbar)K[H, \rho](q_t; q_0). \]  
(19)

(b) The dependence of the HPI1 and HPI2 on the measure \( \rho \) is particularly simple. Thus for example we have

\[ \sqrt{\rho(q)\rho(q_0)}K[H, \rho] = K[H, 1] \]  
(20)

\[ K[H, \rho_1] = \sqrt{(\alpha(q)\alpha(q_0))}K[H, \rho_2] \]  
(21)

where \( \rho_1 \) and \( \rho_2 \) are related by

\[ \rho_2(q) = \alpha(q)\rho_1(q). \]  
(22)

Similar relations are true for HPI2 \( K \) also.

(c) When the scaling function \( \alpha \) is a constant, say \( c \), we have

\[ K[H, \rho, \alpha = c] = K[H, \rho] \]  
(23)

This equation is equivalent to
\[ \int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[c(H - E), \rho](qt; q_00) = \frac{1}{c} K[H, \rho](qt; q_00) \] (24)

(d) In the special case when the function \( H(q, p) \) is independent of \( q^j \) for some \( j \) and depends on \( p_j \) alone, the HPI1 depends only on the difference \( (q - q_0) \), apart from an overall factor of \( \sqrt{\rho(q)\rho(q_0)} \). This is easily seen by doing the corresponding \( p \) integrations after writing the discrete form for HPI1.

2.4 Schrödinger equation for \( K \): If we take

\[ H = \frac{1}{2m} g^{ij} p_i p_j + V(q) \] (25)

the HPI2 \( K[H, \alpha, \rho](qt; q_0t_0) \) as defined in (16) satisfies the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} K[H, \alpha, \rho] = (H_\alpha)_{op} K[H, \alpha, \rho] \] (26)

with

\[ (H_\alpha)_{op} = -\frac{\hbar^2}{2m} \rho^{-1/2} \alpha(q)^{-1/2} \left( \frac{\partial}{\partial q^i} (g^{ij} \alpha) \frac{\partial}{\partial q^j} \right) \rho^{+1/2} \alpha(q)^{-1/2} + V(q) \] (27)

and \( K \) has initial value given by

\[ \lim_{t \to t_0} K[H, \sqrt{g}, \sqrt{g}](qt; q_0t_0) = (1/\rho(q)) \delta(q - q_0) \] (28)

For the special case when \( \alpha = \rho = \sqrt{g} \) the HPI2 \( K[H, \sqrt{g}, \sqrt{g}](qt; q_0t_0) \) with scaling, as defined in (16) satisfies the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} K[H, \sqrt{g}, \sqrt{g}] = \hat{H} K[H, \sqrt{g}, \sqrt{g}] \] (29)

with

\[ \hat{H} = -\frac{\hbar^2}{2m} g^{-1/2} \partial \partial q^i (g^{ij} g^{1/2}) \partial q^j + V(q) \] (30)
and $\mathcal{K}$ has initial value given by

$$
\lim_{t \to t_0} \mathcal{K}[\mathcal{H}, \sqrt{g}, \sqrt{g}](qt; q_0t_0) = g^{-1/2}(q)\delta(q - q_0)
$$

(31)

The relations (26) to (31) are the central results in our scheme. In our scheme $\mathcal{K}[\mathcal{H}, \sqrt{g}, \sqrt{g}]$ is the candidate for the quantum mechanical propagator for a Hamiltonian of type (25) because the right hand side of (30) can be recognized as the Schrodinger operator $-(\bar{\hbar}^2/2m)\nabla^2 + V(q)$.

The proof of (27) and (28) is long and can be given in several ways. Here we indicate, in brief, one of the methods used in ref. 18. For this purpose consider the "propagation of functions of $q$ in time" governed by the HPI2 $\mathcal{K}$ and define

$$
\Psi(q, t) = \int \rho(q_0) dq_0 \mathcal{K}[\mathcal{H}, \rho, \alpha](qt; q_0t_0)\Psi(q_0)
$$

(32)

The quantity $\Psi(q, t + \Delta t) - \Psi(q, t)$ is computed up to the first order terms for small $\Delta t$. To do this we by first insert discrete form for the HPI1 $K[\alpha(H - E), \rho]$ in the right hand side of (16) and use the resulting expression for $\mathcal{K}[\mathcal{H}, \rho, \alpha(q)]$ in the right hand side of (32). Because in (16) integral of $K[\alpha(H - E), \rho](q\sigma, q_0\sigma)$ over $\sigma$ is needed, it is not possible to replace $K$ by a single STP and we have to retain full discrete expression for finite $N$. The expression (32) involves $N$ fold $q -$ and $N$ fold $p -$ integrations as well as integrations over $E$ and $\sigma$. The integration over $E$ results in a Dirac delta function implying a linear constraint in $\sigma$ and $\Delta t$ with functions of $q$ appearing as coefficients. The $\sigma -$ integration is easily done using the delta function. The resulting expression is expanded in powers of $\Delta t$ retaining all $O(\Delta t)$ terms. The result of the remaining $N$ fold $q-$ and $p-$integrations in (32) can be expressed as an operator acting on $\Psi$ after long algebraic manipulations. The result can also be proved by making use of the correspondence of the HPI1 with lagrangian form of path integral mentioned earlier at the end of Sec. 2.2.

As a by product of the above result we also get the scaling formula given below.

2.5 Scaling formula: The HPI2 $\mathcal{K}[\mathcal{H}, \rho, \alpha]$ with scaling function $\alpha(q)$ and density $\rho$, as defined by (16) above, is related to the HPI2 with trivial scaling and hence to HPI1 with hamiltonian $H - U(\alpha)$

$$
\mathcal{K}[\mathcal{H}, \rho, \alpha] = \mathcal{K}[H - U(\alpha), \rho, 1] = K[H - U(\alpha), \rho]
$$

(33)
The precise form of $U(\alpha)$ depends on $H$ used. For $H(q, p)$ given by (25) we have

$$U(\alpha) = -\frac{\hbar^2}{8m} \left( g^{ij} \partial_i (\ln \alpha) \partial_j (\ln \alpha) + 2 \partial_i (g^{ij} \partial_j (\ln \alpha)) \right)$$

(34)

Redefining $H$ in the scaling formula (33), it can be put into a useful form as a relation between two HPI1.

$$K[H, \rho](q_t; q_{00}) = K[H + U(\alpha), \rho, \alpha](q_t; q_{00}) = \sqrt{\alpha(q)} \alpha(q_0) \int \frac{dE}{(2\pi\hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[\alpha(H - E + U(\alpha)), \rho](q\sigma; q_{00})$$

(35)

This formula expresses HPI1 without scaling in terms of a HPI1 with scaling. The factor $\sqrt{\alpha(q)} \alpha(q_0)$ can be absorbed into change of measure from $\rho$ to $\rho/\alpha$ for the HPI1 in the right hand side of (35).

$$K[H, \rho](q_t; q_{00}) = \frac{1}{\sqrt{\alpha(q)} \alpha(q_0)} \int_0^\infty dt K[\alpha(H - E), \rho](q_t; q_{00})$$

(36)

Other useful forms of the above relations are obtained by integrating over time or taking inverse Fourier transforms. As an example, we can obtain

$$\int dt K[\alpha(H - E), \rho](q_t; q_{00}) = \frac{1}{\sqrt{\alpha(q)} \alpha(q_0)} \int_0^\infty dt K[(H - E + U(\alpha)), \rho](q_t; q_{00})$$

(37)

by integrating the scaling relation (33) over time.

3. Point transformations and canonical path integration

3.1 Initial condition for canonical path integration: For a path integral to give rise to correct propagator, it is important that not only the correct
Schrodinger equation be reproduced, it must also satisfy boundary conditions appropriate to the co-ordinates chosen. In this section we will at first take an illustrative example and then discuss the general case of initial condition for the canonical path integration. As an example of initial condition on the propagator in arbitrary coordinates, consider the change from the cartesian co-ordinates to polar co-ordinates in a plane.

\[ x = r \cos \theta, \quad y = r \sin \theta \]  
\( (38) \)

The right hand side of (38) is invariant, separately, under the following transformations.

\((i)\) \( r \to r, \quad \theta \to \theta + 2m\pi; \) \( (39) \)

\((ii)\) \( r \to -r, \quad \theta \to \theta + (2m + 1) \) \( (40) \)

Thus we have the relation

\[ \delta(x-x_0)\delta(y-y_0) = \frac{1}{r} \sum_{m=-\infty}^{\infty} (\delta(r-r_0)\delta(\theta-\theta_0+2\pi m) + \delta(r+r_0)\delta[\theta-\theta_0+(2m+1)\pi]) \]  
\( (41) \)

Quantum mechanical propagator, \( \langle \vec{r}t|\vec{r}_0t_0 \rangle \), in two dimensions satisfies the initial condition in cartesian co-ordinates

\[ \lim_{t \to t_0} \langle \vec{r}t|\vec{r}_0t_0 \rangle = \delta(\vec{r} - \vec{r}_0) \]  
\( (42) \)

Therefore, in plane polar co-ordinates the propagator, at equal times, reduces to the right hand side of (41). Conversely for the change of variable from \( r, \theta \) to \( x, y \)

\[ r = (x^2 + y^2)^{1/2}, \quad \theta = \tan^{-1}(y/x), \]  
\( (43) \)

let us assume that a hamiltonian path integral, say HPI1, is constructed to satisfy the initial condition

\[ K[H,r](r\theta t_0; r_0\theta_0 t_0) = \frac{1}{r} \delta(r-r_0)\delta(\theta - \theta_0) \]  
\( (44) \)

Because the right hand side of (43) does not change when \( (x, y) \to (-x, -y) \), therefore, the equation
\[
\frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_0) = \delta(x - x_0) \delta(y - y_0) + \delta(x + x_0) \delta(y + y_0)
\] (45)

correctly gives the boundary condition (44) in the cartesian co-ordinates. In general let \( q, p \to Q, P \) be a point transformation.

\[
Q = Q(q), \quad P = P(q, p)
\] (46)

with the inverse transformations written as

\[
q = q(Q), \quad p = p(Q, P)
\] (47)

Let \( h(q, p) \) and \( H(Q, P) = h(q(Q), p(Q, P)) \) be hamiltonian functions in the two sets of co-ordinates. Let us set up \( \text{HPI}_2 \) in the two sets of co-ordinates \( q \) and \( Q \) directly using the scaled canonical path integrals using the same classical Hamiltonian but expressed in terms of the appropriate canonical variables. The two \( \text{HPI}_2 \) \( \mathcal{K}[h, \sqrt{g}_1, \sqrt{g}_1](qt; q_00) \) and \( \mathcal{K}[H, \sqrt{g}_2, \sqrt{g}_2](Qt; Q_00) \) satisfy the same Schrödinger equation and by construction they obey the following boundary conditions in the limit \( t \to 0 \).

\[
\mathcal{K}[h, \sqrt{g}_1, \sqrt{g}_1](qt = 0; q_00) = g_1^{1/2}(q) \delta(q - q_0)
\] (48)

\[
\mathcal{K}[H, \sqrt{g}_2, \sqrt{g}_2](Qt = 0; Q_00) = g_2^{1/2}(Q) \delta(Q - Q_0)
\] (49)

Whenever the function \( q(Q) \) can be solved for \( Q \) uniquely, \( \mathcal{K}[H, \sqrt{g}_2, \sqrt{g}_2] \) will be obtained from \( \mathcal{K}[h, \sqrt{g}_1, \sqrt{g}_1] \) of (48) by expressing \( q \) in terms of \( Q \). This relationship will not hold whenever the relation \( q = q(Q) \) cannot be inverted to give \( Q \) uniquely in terms of \( q \). This is due to the fact that in such a case the right hand sides of (48) and (49) are not equal. Let \( \Lambda \) be a transformation on \( Q' \)s such that \( q(\Lambda Q) = q(Q) \). The set of all such transformations form a group. Following Moshinsky et al. we call it ambiguity group of the transformation \( q \to Q \), and denote it by \( \mathcal{A} \). Then

\[
\delta(q - q_0) = |J| \sum_{\Lambda \in \mathcal{A}} \delta(Q - \Lambda Q_0)
\] (50)

In the above equation \( Q_0 \) stands for any one solution of the \( q = q(Q) \) and \( J \) is the Jacobian of the transformation \( q \to Q \). The relation between the two
propagators (48) and (49) can be written down
\[ K[h, \sqrt{g_1}, \sqrt{g_1}](qt; q_0) = \sum_{\Lambda \in A} K[H, \sqrt{g_2}, \sqrt{g_2}](Q\Lambda t; Q_00) \]

In the equation (51) it is understood that the variables \( q, q_0 \) are to be expressed in terms of \( Q \) and \( Q_0 \) by using the equations for the point transformation. The result (51) follows from the fact that both sides obey the same Schrodinger equation and the same boundary condition.

3.2 Quantization in arbitrary co-ordinates: We are now in a position to write the path integral for quantization in arbitrary co-ordinates needed for later applications. We are interested in the potential problems of quantum mechanics. The classical hamiltonian, \( h_{cl} \), for these problems is of the form
\[ h_{cl} = \vec{p}^2 / 2m + V(\vec{r}) \]
where \( \vec{r} \) denotes cartesian co-ordinates and \( \vec{p} \) are the conjugate momenta. Let \( Q \) be any other set of co-ordinates related to \( \vec{r} \) by a point transformation. Then the quantum mechanical propagator \( \langle \vec{r}t | \vec{r}_0 \rangle \) satisfying the Schrodinger equation and the correct boundary condition
\[ \lim_{t \to 0} \langle \vec{r}t | \vec{r}_0 \rangle = \delta(\vec{r} - \vec{r}_0) \]
can be written in terms of HPI2 set up in \( Q, P \) variables as
\[ \langle \vec{r}t | \vec{r}_0 \rangle = \sum_{\Lambda \in A} \int dE \left( \frac{2\pi}{\bar{\hbar}} \right)^{1/4} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_E, \sqrt{g}](\Lambda Q\sigma; Q_00) \]

where \( H_E = \sqrt{g}(H_{cl} - E) \). The corresponding energy dependent Green function is defined by
\[ G(\vec{r}t, \vec{r}_0|E) = \int_0^\infty dt \exp(iE_+t/\hbar) \langle \vec{r}t | \vec{r}_0 \rangle, \]
where $E_+ = E + i\eta, \eta > 0$ and $\lim \eta \to 0$ is taken at the end. For the energy dependent Green function we have the following path integral representation in arbitrary co-ordinates.

$$G(\vec{r}, \vec{r}_0|E) = (\frac{g(Q)}{\sqrt{g(H_{cl} - E)}})_{\Lambda \in A} \sum_{\Lambda \in A} \int_0^\infty d\sigma K[\sqrt{g}(H_{cl} - E), \sqrt{g}](\Lambda Q, Q_0)$$

$$= \sum_{\Lambda \in A} \int_0^\infty d\sigma K[\sqrt{g}(H_{cl} - E), \rho = 1](\Lambda Q, Q_0).$$

In the above $H_{cl}$ is the classical hamiltonian written in terms of $Q$ and $P$. The expressions (56) and (57) define, respectively, the scheme for obtaining the quantum mechanical propagator and the energy dependent Green function. They involve the classical Hamiltonian only which is to be expressed in terms of appropriate co-ordinates and conjugate momenta, no ad hoc addition of $O(h^2)$ terms is necessary in any co-ordinate system.

4. Relating path integrals

In many of the exact evaluations of the path integrals in literature, the results are be obtained by relating the desired propagators to one for which answers are known. There are some notable investigations in which the authors actually compute the multi-dimensional integrations to complete path integration. Besides the use of local rescaling of time, the technique of adding new degrees of freedom with known, preferably trivial, dynamics has proved to be useful in relating path integral representations for quantum mechanical propagators of different quantum mechanical problems. Adding new degrees of freedom is like inverse of separating the variables in the differential equation approach. We shall briefly recall this technique here.

Let $H_{ext}(q, p, P)$ be independent of $Q$ and $H(q, p)$ be such that for $P = \text{constant}$, say $\nu$, we have

$$H_{ext}(q, p, P)|_{p=\nu} = H(q, p)$$

We will say that $H_{ext}$ is an extension of $H$ by addition of a new degree freedom $Q, P$ with trivial dynamics. Let $K$ and $K^{ext}$ be the two HPII using the unit measure and with $H$ and $H_{ext}$ as hamiltonians. Then
\[ K[H, 1](qt; q_00) = \int d(\Delta Q) \exp(i\nu\Delta Q) K[H_{\text{ext}}, 1](qQt; q_0Q_0) \] 

(59)

Notice that \( H_{\text{ext}} \) is assumed to be independent of \( Q \). It, therefore, follows that \( K[H_{\text{ext}}, 1] \) depends only on \( Q - Q_0(= \Delta Q) \). The limits of integration over \( \Delta Q \) are usually from \(-\infty\) to \(+\infty\). However, if \( K[H_{\text{ext}}, 1] \) is periodic in \( \Delta Q \) the region of integration can be restricted to one period for the case \( \nu = \text{integer} \). Similar results have been used in literature and one can easily arrived at the result by considering the differential equation satisfied by the the two path integrals appearing in (59).

In this paper almost all the examples of path integrations to be discussed, with the exception of the hydrogen atom in the parabolic co-ordinates, the desired propagator or the energy dependent Green function will be obtained by relating the corresponding path integral representation to another path integral which is known. A careful distinction has to be maintained between the path integrals which are related to quantum mechanical answers for propagator of some problem and other path integrals which appear only in the intermediate steps. For any quantum mechanical problem the propagator is given by (a) the corresponding HPI1 in cartesian co-ordinates or an equivalent HPI2 expression with trivial scaling, or (b) expression (54) appropriate to the non-cartesian co-ordinate system selected, or (c) the expression in the line just before (54).

Apart from the addition of new degrees of freedom, properties about of HPI1 and HPI2 when \( \rho \) or \( \alpha \) are changed will be used in obtaining relations between different path integrals.

A few remarks about the notation to be followed are given in the following. We shall always use the bold face symbols \( K_0(qt|q_0t_0) \) and \( G_0(q, q_0|E) \), with a zero subscript, to denote the full quantum mechanical solution for the propagator and the energy dependent Green function, respectively. Similarly, the Hamiltonian for the problem to be solved will have a subscript 0.

Different superscripts or subscripts on roman, script or bold face symbols will be used to denote other intermediate or related expressions. Many of the exact treatments in this paper begin with a series of observations and from these observations we “zero in” to one of the two required functions \( K_0(qt|q_0t_0) \) or \( G_0(q, q_0|E) \).

In the following sections the symbol \( k_1(xt|x_00) \) will be used to denote harmonic oscillator propagator in one dimension for mass \( \mu \) and frequency \( \omega \).
\[ k^{osc}_i \langle xt|x_00 \rangle = \frac{\mu \omega}{2\pi i \sin \omega t} \exp \left( i \frac{\mu \omega}{2 \sin \omega t} [(x^2 + x_0^2) \cos \omega t - 2xx_0] \right) \] (60)

The propagators for harmonic oscillator in \( n \) dimension will be product of \( n \) such factors and will be denoted by \( k^{osc}_n \). Also in the following we shall use same symbol to denote the classical hamiltonian in different co-ordinates.

5. Potential problems in one dimension

5.1 The \( ax^2 + b/x^2 \) potential on half line: The solution for this problem will be obtained by relating it, following Duru\(^{18}\), to two dimensional oscillator problem in a half plane. The derivation is written as a sequence of steps consisting of several small intermediate results and observations.

Step 1: The hamiltonian for the problem given by

\[ H_0 = \frac{p^2}{2m} + ax^2 + b/x^2. \] (61)

The required propagator \( K_0 \langle xt|x_0t_0 \rangle \) has initial value

\[ \lim_{t \to t_0} K_0 \langle xt|x_0t_0 \rangle = \delta(x - x_0) \] (62)

The propagator \( K_0 \), in terms of the hamiltonian path integral representation, is given by

\[ K_0 \langle xt|x_00 \rangle = K[H_0, \rho = 1, \alpha = 1](xt; x_0t) = K[H_0, \rho = 1](xt; x_00) \] (63)

This is due to the fact that HPII in (63), the last expression, is the right answer in the Cartesian co-ordinates. It obeys the correct Schrodinger equation and the correct initial condition at \( t = 0 \).

Step 2: The classical hamiltonian \( H_0 \) looks like the hamiltonian \( H_1 \)

\[ H_1 = \frac{p_x^2}{2m} + \frac{p_\theta^2}{2mx^2} + ax^2 \] (64)

with \( p_\theta \) taken as suitable constant. Under a point transformation
\[ x_1 = x \cosh \theta, \quad x_2 = x \sinh \theta \]  
\[ dx_1 dx_2 = x dx d\theta, \quad \sqrt{g} = x \] (65)

the hamiltonian \( H_1 \) takes the form of harmonic oscillator hamiltonian in two dimensions

\[ H_1 = \frac{p_1^2}{2m} - \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1^2 - x_2^2) \] (66)

for which the the exact quantum mechanical propagator is known.

**Step 3**: We set up the path integral quantization scheme for oscillator problem in the two sets of co-ordinate systems, viz., the plane polar co-ordinates \((x, \theta)\) and cartesian co-ordinates \(\vec{x} = (x_1, x_2)\) and define

\[ K_1 \equiv K[H_1, 1, 1](\vec{x}, t; \vec{x}_0, 0) = K[H_1, 1](\vec{x}, t; \vec{x}_0) \] (67)

and

\[ K_2 \equiv K[H_1, \sqrt{g}, \sqrt{g}](x, \theta t; x_0 \theta_0) \]

\[ = \int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) \int d\sigma K[H_2, \sqrt{g}](x, \theta \sigma; x_0 \theta_0) \] (70)

where

\[ H_2 = x(H_1 - E) = x(\frac{p_x^2}{2m} + \frac{p_\theta^2}{2mx^2} + ax^2 - E) \] (71)

The two functions \( K_1 \) and \( K_2 \) become equal when expressed in terms of either co-ordinates because the mapping \( x_1, x_2 \) to \( x, \theta \) is one to one. They are also equal to the quantum mechanical propagator in two dimensions for (67).

\[ K_1 = K_2 = k_1^{osc}(x_1 t | x_{10})k_1^{osc}(x_2 t | x_{20}) \] (72)

\( k_1^{osc}(xt|x_{00}) \) appearing in the above equation is the propagator for one dimensional oscillator with mass \( m \) and frequency \( \omega \) given by \( \sqrt{2a/m} \).

**Step 4**: Although \( H_1 \) reduces to \( H_0 \) for \( p_\theta = \text{const.} \) but \( x \) dependences of the normalizations of \( K_2 \) and \( K_0 \) do not match:

\[ K_0(x, t = 0; x_0, 0) = \delta(x - x_0) \] (73)
\[ K_2(x, \theta, t = 0; x_0, \theta_0, 0) = \frac{1}{x} \delta(x - x_0) \delta(\theta - \theta_0) \]  

(74)

The steps to relate these two have to take into account of this difference in the initial conditions at \( t = 0 \). In order to have correct initial value for the two dimensional problem, we define

\[ \tilde{K}(xt|x_0t_0) = 1 \frac{1}{\sqrt{xx_0}} K_0(xt|x_0t_0) \]  

(75)

which obeys

\[ \tilde{K}(xt|x_0t_0)|_{t=t_0} = \frac{1}{x} \delta(x - x_0) \]  

(76)

Step 5: Next we insert the HPI1 of (63) for \( K_0(xt;x_0) \) in the right hand side of (75) and use the formula (37). Dividing (35) by \( \sqrt{\alpha(q)\alpha(q_0)} \) and using it we get

\[ \tilde{K}(xt|x_00) \]  

\[ = \frac{1}{\sqrt{xx_0}} K[H_0, \rho = 1] \]  

\[ = \int dE \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_3, \rho = 1](x, \sigma; x_00) \]  

(77)

where

\[ H_3 = x(H_0 + U(x) - E) \]  

\[ = x \left\{ \frac{p^2}{2m} + \left( b + \frac{\hbar^2}{2m} \right) \frac{1}{x^2} + ax^2 - E \right\} \]

Step 6: This problem of obtaining HPI1 for \( H_3 \) in (76) is related to the HPI1 for the extended hamiltonian \( H_2 \)

\[ H_2 = x \left( \frac{p^2_x}{2m} + \frac{p^2_\theta}{2m} + \frac{\hbar^2}{2m} \frac{1}{x^2} + ax^2 - E \right) \]  

(78)

for \( p^2_\theta = (2mb/\hbar + 1/4) \). Thus using (4.2) we get the relation

\[ K[H_3, \rho = 1](x, \sigma; x_00) = \int d(\Delta\theta) \exp(i\nu \Delta\theta) K[H_2, \rho = \sqrt{g}](x\theta\sigma; x_0\theta_00) \]  

(79)
where $\Delta \theta = \theta - \theta_0$ and

$$\nu^2 = \left( \frac{2mb}{\hbar^2} + \frac{1}{4} \right)$$  \hspace{1cm} (80)

**Step 7**: From (75) $K_0$ is given in terms of $\tilde{K}_0$ which can be expressed in terms of $K[H_2, 1]$ using (77) and (79). Thus we obtain

$$K_0 \langle xt | x_0, 0 \rangle = \sqrt{xx_0} \tilde{K} \langle xt | x_00 \rangle = \sqrt{xx_0} \int dE \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_3, \rho = 1](x, \sigma; x_00)$$

$$= \int d(\Delta \sigma) \exp(i\nu \theta) \int dE \exp(-iEt/\hbar) \int d\sigma K[H_2, x](x \theta \sigma; x_0 \theta 0)$$

$$= \int d(\Delta \sigma) \exp(i\nu \theta) \kappa_{osc}^2$$  \hspace{1cm} (81)

Here the last step comes from equality (72) already noted and $\kappa_{osc}^2$ is the propagator for the two dimensional problem (67). We get the desired result by and following ref. 19 for the remaining steps and integrations. Omitting the details, the final answer is

$$K_0 \langle xt | x_0, 0 \rangle = \frac{m\omega \sqrt{xx_0}}{\hbar \sin \omega t} I_\nu \left( \frac{m\omega \sqrt{xx_0}}{i \sin \omega t} \right) \exp \left[ \left( \frac{im\omega}{2\hbar} \right) \cot \omega t (x^2 + x_0^2) \right]$$  \hspace{1cm} (82)

5.2 Morse Oscillator : We shall briefly indicate how the propagator for this problem is related to that for $ax^2 + b/x^2$ potential on half line. The Morse oscillator hamiltonian

$$H_0 = \frac{p^2}{2m} + A \exp(-2ax) - B \exp(-ax)$$  \hspace{1cm} (83)

after a point transformation to a new coordinate $r = \exp(-ax/2)$ becomes

$$H_0 = \frac{1}{2m} a^2 r^2 p_r^2 + (Ar^4 - Br^2)$$  \hspace{1cm} (84)

Quantization of the Morse oscillator in the new variable $r$ requires us to set up

$$\mathcal{K}[H_0, \sqrt{g}, \sqrt{g}](rt; r_00)$$
\[ \equiv \sqrt{\alpha(q)\alpha(q_0)} \int \frac{dE}{(2\pi\hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_E, \rho](r\sigma; r_0) \]  

(85)

where \( \alpha = \sqrt{g} = (2/ar) \) and \( H_E \) is given by

\[
H_E = (2/ar) (H_0 - E) 
= (ar/2) \left( \frac{p_r^2}{2m} + \frac{4}{a^2}(Ar^2 - B) - \frac{4E}{a^2r^2} \right) 
\equiv (ar/2)H_2
\]  

(86)

where \( H_2 \) is the function in the brackets in the second line of equation (86).

The HPI1 in r.h.s. of (85) can be related to HPI1 for \( H_2 \) using the scaling formula with a \( U(ar/2) \) appearing in the Hamiltonian. The HPI1 for the resulting hamiltonian \( H_2 - U(ar/2) \) is just the HPI1 for the radial oscillator and is known from the results in the previous subsection.

6. Coulomb problem in two dimensions

The Coulomb problem in two dimensions is solved easily by our method in the parabolic co-ordinates as it gets related to harmonic oscillator in two dimensions. The steps leading to the solution are as follows.

Step 1: The classical hamiltonian for the Coulomb problem in two dimensions is

\[
H_0 = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} 
\]

\( \vec{p} = (p_x, p_y), \vec{r} = (x, y) \)  

(87)

In the ” 2-dimensional parabolic coordinates ” \( u_1, u_2 \) defined

\[
x = u_1^2 - u_2^2; \quad y = 2u_1u_2
\]

(89)

the classical hamiltonian takes the form

\[
H_0 = \frac{\vec{p}_u^2}{8m\vec{u}^2} - \frac{e^2}{\vec{u}^2}
\]

\( dx dy = 4\vec{u}^2 du_1 du_2, \quad g^{1/2} = 4\vec{u}^2 \)

(90)

(91)

Step 2 : For quantization of the Coulomb system in the cartesian, recall for cartesian co-ordinates the desired propagator \( K_0(\vec{r}|\vec{r_0}) \) is just the HPI2
with a trivial scaling, which is also equal to HPI1 for the same hamiltonian. Thus we have

\[ K_0(\vec{r}|\vec{r}0) = \mathcal{K}[H_0, 1, 1] = K[H_0, 1] \]  (92)

**Step 3:** Quantization in parabolic coordinates \( u_1, u_2 \) will proceed via path integral HPI2 with hamiltonian \( H_0 \), scaling function \( \alpha = 4u^2 \). Thus we should set up \( K[H_0, 4u^2, 4u^2](\vec{u}t; \vec{u}00) \)

\[ K[H_0, 4u^2, 4u^2](\vec{u}t; \vec{u}00) = \int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_E, 1](\vec{u}\sigma; \vec{u}00) \]  (93)

with

\[ H_E = 4u^2(H_0 - E) = \left( \frac{\vec{p}_u^2}{2m} - 4e^2 - 4E\vec{u}^2 \right) \]  (94)

Notice that apart from an additive constant, \( H_E \) is just the harmonic oscillator hamiltonian in two dimensions. Thus we see that the HPI1 in (93) is related to the propagator for the two dimensional oscillator and the relation is given by

\[ K[H_E, \rho = 1](\vec{u}\sigma; \vec{u}00) = \exp(i4e^2\sigma/\hbar)k^{osc}(u_1\sigma|u_{10}0)k^{osc}(u_2\sigma|u_{20}) \]  (95)

with \( k^{osc} \) is the one dimensional oscillator propagator for mass \( \mu = m \) and frequency \( \omega^2 = -8E/M \).

**Step 4:** The HPI2 \( \mathcal{K} \) of (93) is thus known. We use analysis of Sec. 3 to relate HPI2 for \( H_0 \) in the cartesian and the parabolic coordinates. For this we notice that \( x, y \) defined by (89) do not change if we substitute \( u_1 \to -u_1, \) and \( u_2 \to -u_2 \).

Therefore, we have

\[ \delta(\vec{r} - \vec{r}_0) = g^{-1/2} [\delta(\vec{u} - \vec{u}_0) + \delta(\vec{u} + \vec{u}_0)] \]  (97)

and

\[ \mathcal{K}[H_0, 1, 1](\vec{r}t; \vec{r}00) = [\mathcal{K}[H_0, 4u^2, 4u^2](\vec{u}t; \vec{u}00) + \mathcal{K}[H_0, 4u^2, 4u^2](\vec{u}t; -\vec{u}00)] \]  (98)
Step 5: Thus the desired quantum mechanical propagator $K_0$ is obtained from (92) and (98) using the expressions (93) and (95). Writing the answer in terms of the energy dependent Green function we get

$$G_0(\vec{r},\vec{r}_0|E) = \int_0^\infty \exp(4ie\frac{\sigma}{\hbar})[k_1^{osc}\langle u_1|u_{10}\rangle k_1^{osc}\langle u_2|u_{20}\rangle$$

$$+ k_1^{osc}\langle u_1| - u_{10}\rangle k_1^{osc}\langle u_2|- u_{20}\rangle]$$

This result agrees with equation (28)-(30) of ref. 20 after the integration variable $\sigma$ is replaced by $\tau = 4\sigma$.

7. H-Atom in three dimensions

7.1. K-S Transformation: The H-atom in three dimensions is related to isotropic harmonic oscillator in 4-dimensions via K-S transformation. Local time transformations in the path integral formalism and K-S transformation have been used to relate the Coulomb problem Green function to the Green function for the oscillator problem in 4-dimensions. This relation is derived in our canonical formalism of path integrations follows.

Step 1: Consider the 4-dimensional oscillator problem defined by

$$H_4^{osc} = \frac{\vec{p}^2}{2M} + \frac{1}{2}M\Omega^2\vec{u}^2$$

where

$$\vec{u} = (u_1, u_2, u_3, u_4)$$
$$\vec{p}_u = (p_{u_1}, p_{u_2}, p_{u_3}, p_{u_4})$$

We introduce the new co-ordinates $(x_1, x_2, x_3, x_4)$, in two steps. In the first step the transformation equations

$$u_1 = \sqrt{r} \sin \frac{\theta}{2} \cos(\frac{\alpha + \phi}{2}) ; \quad u_2 = \sqrt{r} \cos \frac{\theta}{2} \sin(\frac{\alpha - \phi}{2})$$
$$u_3 = \sqrt{r} \cos \frac{\theta}{2} \cos(\frac{\alpha - \phi}{2}) ; \quad u_4 = \sqrt{r} \sin \frac{\theta}{2} \sin(\frac{\alpha + \phi}{2})$$

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define the co-ordinates \( r, \theta, \phi, \alpha \). In the next step we identify \( x_4 = \alpha \) and the equations relating \( r, \theta, \phi \) to \((x_1, x_2, x_3) \equiv \vec{x}\) are taken to be the same as those giving the relation between the spherical polar to cartesian co-ordinates in three dimensions. Thus

\[
x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta, \quad x_4 = \alpha \quad (104)
\]

The classical hamiltonian \((100)\) when expressed in terms of the co-ordinates \((\vec{x}, \alpha)\) and conjugate momenta \((\vec{p}, p_\alpha)\) becomes

\[
H_{\text{osc}}^4 = 4r \left\{ \frac{\vec{p}^2}{2M} + \frac{(x_1p_2 - x_2p_1)zp_\alpha}{Mr(x_1^2 + x_2^2)} \right\} + \frac{1}{2}M\Omega^2r \quad (105)
\]

Also

\[
d^4u = 16r \sin \theta \, dr \, d\theta \, d\phi \, d\alpha \quad (106)
\]

\[
g^{1/2} = (16/r) \equiv J \quad (107)
\]

**Step 2:** Next we set up path integrals for the 4-dimensional oscillator problem in two sets of co-ordinates systems. In the cartesian co-ordinates \(\vec{u}\) the relevant HPI2 is equal to the HPI1 for the same hamiltonian

\[
K[H_{\text{osc}}^4, J, J](\vec{x}, \alpha, t; \vec{x}_0, \alpha_0, 0) \quad (108)
\]

The path integrals in \((108)\) at \(t = 0\) become equal to \(\delta^4(\vec{u} - \vec{u}_0)\). Thus two HPI’s in \((108)\) coincide with the exact quantum mechanical propagator, \(K_{\text{osc}}^4\) for the four dimensional oscillator problem with mass \(M\) and frequency \(\Omega\).

To quantize the oscillator \((100)\) in coordinates \((\vec{x}, \alpha)\) we set up the HPI2 \(K[H_{\text{osc}}^4, J](\vec{x}, \alpha, t; \vec{x}_0, \alpha_0, 0)\) which is given by

\[
K[H_{\text{osc}}^4, J](\vec{x}, \alpha, t; \vec{x}_0, \alpha_0, 0)
\]

\[
= 16 \int \frac{d\mathcal{E}}{(2\pi \hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_{\mathcal{E}}, \rho = 1](\vec{x}, \alpha, \sigma; \vec{x}, \alpha_0) \quad (109)
\]
Taking inverse Fourier transform this relation is written as

\[
\int_0^{\infty} d\sigma K[H_\sigma, \rho = 1](\vec{x}, \alpha, \sigma; \vec{x}_0, \alpha_0)
= \frac{1}{16} \int dt \exp(i\mathcal{E}t/\hbar) \mathcal{K}[H_{osc}^4, J, J](\vec{x}, \alpha, t; \vec{x}_0, \alpha_0, 0)
\]

(110)

The hamiltonian function appearing in the HPI1 of r.h.s of (109) is given by

\[
H_\sigma = J(H_{osc}^4 - \mathcal{E})/16
\]

(111)

\[
= 4 \left\{ \frac{\vec{p}^2}{2M} + \frac{(x_1p_2 - x_2p_1)z\alpha}{Mr(x_1^2 + x_2^2)} + \frac{1}{2}M\Omega^2 - \mathcal{E}/r \right\}
\]

(112)

A factor of 16 appears in (109) to (111) because a trivial scaling of time has been included in (109).

**Step 3:** Next we note that two HPI2 set up for the oscillator problem, appearing in (108) and (109), are normalized differently. Their values at \( t = 0 \) are related by

\[
(1/J)\delta^4(\vec{x} - \vec{x}_0)\delta(\alpha - \alpha_0) = [\delta^4(\vec{u} - \vec{u}_0) + \delta^4(\vec{u} + \vec{u}_0)]
\]

(113)

Thus we must have

\[
\mathcal{K}[H_{osc}^4, J, J](\vec{x}, \alpha, t; \vec{x}_0, \alpha_0, 0)
= \mathcal{K}[H_{osc}^4, 1, 1](\vec{u}t; \vec{u}_00) + \mathcal{K}[H_{osc}^4, 1, 1](-\vec{u}t; -\vec{u}_00)
= \mathbf{k}_{osc}^4(\vec{u}t|\vec{u}_00) + \mathbf{k}_{osc}^4(\vec{u}t| -\vec{u}_00)
\]

(114)

where the oscillator propagator, \( \mathbf{k}_{osc}^4 \) appearing in the above equation is the four dimensional oscillator propagator.

**Step 4:** We next set up path integral quantization of Coulomb hamiltonian in cartesian co-ordinates. The required energy dependent Green function is given by (57) in arbitrary co-ordinates. In cartesian co-ordinates we have

\[
\mathbf{G}_0(\vec{x}, \vec{x}_0|E) = \int_0^{\infty} d\sigma K[H_0 - E, \rho = 1](\vec{x}\sigma; \vec{x}_00)
\]

(115)
where $H_0$ is the Coulomb Hamiltonian

$$H_0 = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} \quad (116)$$

**Step 5:** We can now assemble the above results to obtain the H-atom Green function. It should be noticed that for $p_\alpha = 0$ the function $H_E$ is the Coulomb Hamiltonian apart from an additive constant $\frac{1}{2} M \Omega^2$. Therefore, the integral of $K[H^E,1](\vec{x}, \alpha t; \vec{x}_0, \alpha_0 0)$ over $\Delta \alpha (\equiv \alpha - \alpha_0)$ is related to the HPI1 for the function $H_0 - E$

$$K[H_0 - E, \rho = 1](\vec{x}_t; \vec{x}_0) = \int_0^{4\pi} d(\Delta \alpha) K[H_E,1](\vec{x}, \alpha t; \vec{x}_0, \alpha_0 0) \quad (117)$$

where we have identified $m = M/4, \mathcal{E} = -M\Omega^2/2$ and $^2 = \mathcal{E}$. The integral of the left hand side of (117) over $t$ is just the energy dependent Green function (115) for the Coulomb problem. Therefore using (110) and (112) we get,

$$G_0(\vec{x}, \vec{x}_0 | E) = \frac{1}{16} \int d\sigma \exp(i e^2 \sigma) \int_0^{4\pi} d(\Delta \alpha) K[H_4^{osc}, J, J](\vec{x}, \alpha \sigma; \vec{x}_0, \alpha_0 0) \quad (118)$$

which on using (114) gives

$$G_0(\vec{x}, \vec{x}_0 | E) = \frac{1}{16} \int_0^{\infty} d\sigma \exp(i e^2 \sigma) \int_0^{4\pi} d(\Delta \alpha) [k_4^{osc}(u\sigma, -u_0 0) + k_4^{osc}(\bar{u}\sigma, -\bar{u}_0 0)] \quad (119)$$

This is the desired connection between the solutions for the four dimensional oscillator and the Coulomb problems. The above result (119) is in agreement with the equation (108) of ref 19.

7.2 Parabolic Coordinates: In this subsection we shall show how exact path integration can be done for H-atom in parabolic coordinates within the canonical formalism. In this case we proceed directly to set up the path integral representation for the propagator and evaluate it. The Coulomb Hamiltonian
\[ H_{\text{coul}} = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} \]  \hspace{1cm} (120)

takes the form
\[ H_{\text{coul}} = \frac{p^2_\xi}{2m(\xi + \eta)} + \frac{p^2_\eta}{2m(\xi + \eta)} + \frac{p^2_\phi}{2m\xi\eta} - \frac{2e^2}{\xi^2 + \eta^2} \]  \hspace{1cm} (121)
in the parabolic coordinates \( \xi, \eta, \phi \) defined by
\[
  \begin{align*}
    x &= \xi \eta \cos \phi \\
    y &= \xi \eta \sin \phi \\
    z &= \frac{1}{2}(\xi^2 - \eta^2)
  \end{align*}
\]  \hspace{1cm} (122)

Also
\[ dx \, dy \, dz = \xi \eta(\xi^2 + \eta^2)d\xi \, d\eta \, d\phi \]  \hspace{1cm} (123)

We, therefore, set-up the HPI1 \( \mathcal{K}[H_E, \sqrt{g}] \) in the parabolic co-ordinates with
\[
  H_E = g^{1/2}(H_{\text{coul}} - E) = \xi \eta \left\{ \frac{p^2_\xi}{2m} + \frac{p^2_\eta}{2m} + \frac{1}{2m} \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) p^2_\phi - E(\xi^2 + \eta^2) - 2e^2 \right\} \]  \hspace{1cm} (124)

and use it to define HPI2 \( \mathcal{K} \) as follows
\[
  \mathcal{K}[H, \sqrt{g}, \sqrt{g}](\xi \eta \phi, t; \xi_0 \eta_0 \phi_0, 0) = \int \frac{dE}{(2\pi\hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma \mathcal{K}[H_E, 1](\xi \eta \phi, t; \xi_0 \eta_0 \phi_0, 0) \]  \hspace{1cm} (125)

\( \mathcal{K} \) satisfies the Schrodinger equation appropriate to the Coulomb potential but has the initial value
\[
  \lim_{t \to 0} \mathcal{K}[H, \sqrt{g}, \sqrt{g}](\xi \eta \phi, t; \xi_0 \eta_0 \phi_0, 0) = \frac{1}{\xi \eta(\xi^2 + \eta^2)} \delta(\xi - \xi_0)\delta(\eta - \eta_0)\delta(\phi - \phi_0) \]  \hspace{1cm} (126)

whereas the Coulomb propagator should be normalized to
\[ \delta(\vec{r} - \vec{r}_0) = \frac{1}{\xi \eta (\xi^2 + \eta^2)} \delta(\xi - \xi_0) \delta(\eta - \eta_0) \sum_{n=-\infty}^{\infty} \delta(\phi - \phi_0 + 2\pi n) \quad (127) \]

Taking (127) it into account the Coulomb propagator will be given by

\[ K_0\langle \vec{r}_t | \vec{r}_0 \rangle = \sum_{n=-\infty}^{\infty} K[H_0, \sqrt{g}, \sqrt{g}] (\xi \eta \phi + 2\pi nt; \xi_0 \eta_0 \phi_0, 0) \quad (128) \]

\[ = \int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma \sum_{n=-\infty}^{\infty} K[H_E, 1](\xi \eta, \phi + 2\pi n, \sigma; \xi_0 \eta_0 \phi_0, 0) \quad (129) \]

Using time integrated version of (128) and expressing change in scaling as change in potential for HP1 we get, with \( \alpha = \xi \eta \),

\[ \int_0^\infty d\sigma K[H_E, 1](\xi \eta \phi \sigma; \xi_0 \eta_0 \phi_0 0) = \int_0^\infty K[h_E, 1](\xi \eta \phi \sigma; \xi_0 \eta_0 \phi_0 0) d\sigma \quad (130) \]

where

\[ h_E = \frac{p_\xi^2}{2m} + \frac{p_\eta^2}{2m} + \frac{p_\phi^2}{2m} \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) - E(\xi^2 + \eta^2) - 2e^2 - \frac{\hbar^2}{8m} \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) \quad (131) \]

Thus we have

\[ K_0\langle \vec{r}_t | \vec{r}_0 \rangle = \int \frac{dE}{(2\pi \hbar)} \exp(-iEt/\hbar) G_0(\vec{r}, \vec{r}_0 | E) \quad (132) \]

with

\[ G_0(\vec{r}, \vec{r}_0 | E) = \int_0^\infty d\sigma \sum_{n=-\infty}^{\infty} K[h_E, 1](\xi, \eta, \phi + 2\pi n, \sigma; \xi_0, \eta_0, \phi_0, 0) \quad (133) \]

We shall now write the canonical path integral \( K[h_E, 1] \) in the discrete from and do the \( \phi \) and \( p_\phi \) path integration explicitly. For this purpose we write \( K[h_E, 1] \) of (130) as

28
\[
\int \prod_{j=1}^{N-1} (d\xi_j d\eta_j) \left( \prod_{j=0}^{N-1} dp_j^\xi dp_j^\eta \right) \exp \left[ \sum_k ip_k^\xi (\xi_{k+1} - \xi_k) + \sum_k ip_k^\eta (\eta_{k+1} - \eta_k) \right]
\]
(134)

\[
\exp \left[ i \sum_k \frac{p_k^\xi^2}{2m} + \frac{p_k^\eta^2}{2m} - E(\xi_k^2 + \eta_k^2) - \frac{\hbar^2}{8m} \left( \frac{1}{\xi_k^2} + \frac{1}{\eta_k^2} \right) \right] K_\phi
\]
(135)

where

\[
K_\phi = \int \left( \prod_{k} d\phi_k \right) \left( \prod_{k=0}^{N-1} dp_k^\phi \right) \exp \left[ \sum_{j=0}^{N-1} \left\{ p_j^\phi (\phi_{j+1} - \phi_j) + \frac{p_j^\phi^2}{2m} \left( \frac{1}{\xi_j^2} + \frac{1}{\eta_j^2} \right) \right\} \right]
\]
(136)

The \( \phi_k \) integral may first be done giving rise to the product of delta functions \( \delta(p_k^\phi - p_{k+1}^\phi) \) which may in turn be used to do all except one \( p_k^\phi \) integration. The last remaining \( p^\phi \) integral is Gaussian and is easily written down. This gives

\[
K_\phi = (\hbar I/2\pi i T)^{1/2} \exp \left( \frac{2iI(\phi - \phi_0)^2}{2T\hbar} \right)
\]
(137)

where

\[
I = \frac{1}{2m} \sum_{n=-\infty}^{\infty} \left( \frac{1}{\xi_k^2} + \frac{1}{\eta_k^2} \right) \Delta t
\]
(138)

Using the identity

\[
\sum_{n=-\infty}^{\infty} \left( \frac{I\hbar}{2\pi i T} \right)^{1/2} \exp \left( \frac{iI(\phi - \phi_0 + 2\pi n)^2}{2T\hbar} \right)
\]

\[= \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \right) \exp \left( -i n^2 T/2\hbar I \right) \exp[\text{in}(\phi - \phi_0)]
\]
(139)

We get the Coulomb Green function as
\[ \mathbf{G}_0(\vec{r}, \vec{r}_0| E) = \int_0^\infty \sum_{n=\infty}^{-\infty} K[H_n, 1](\xi, \eta, \sigma; \xi_0, \eta_0, 0) \exp[im(\phi - \phi_0)]d\sigma \] (140)

where \( H_n \) in (140) is the hamiltonian function

\[ H_n = \frac{\vec{p}_\xi^2}{2m} + \frac{\vec{p}_\eta^2}{2m} - E(\xi^2 + \eta^2) + \frac{\hbar^2}{2m}(n^2 - \frac{1}{4}) \left( \frac{1}{\xi^2} + \frac{1}{\eta^2} \right) \] (141)

\[ \equiv \tilde{H}(\xi, p_\xi) + \tilde{H}(\eta, p_\eta) \] (142)

As \( H_n \) is a sum of two hamiltonians each depending only on one set of conjugate variables, we have

\[ K[H_n, 1](\xi, \eta, \sigma; \xi_0, \eta_0, 0) = K[\tilde{H}, 1](\xi_\sigma; \xi_0)K[\tilde{H}, 1](\eta_\sigma; \eta_0) \] (143)

The HPII for \( \tilde{H} \) is known from Sec. 5.1 and hence we get

\[ K[H_n, 1] = \left( \frac{m\omega}{\hbar \sin \omega t} \right) \left( \frac{m\omega}{\hbar \sin \omega t} \right) J_n \left( \frac{m\omega}{\hbar \sin \omega t} \right) \left( \frac{m\omega}{\hbar \sin \omega t} \right) \exp\left\{ \frac{i\omega}{2\hbar} \cot \omega t (\xi^2 + \xi_0^2 + \eta^2 + \eta_0^2) \right\} \] (144)

\[ \exp\left\{ -\frac{2ie^2}{\hbar} \right\} \sum_{n=-\infty}^{\infty} \exp[im(\phi - \phi_0)]K[H_n, 1](\xi, \eta, t; \xi_0, \eta_0, 0) \] (145)

The required Coulomb Green function is then given by

\[ \mathbf{G}_0(\xi, \eta, \phi; \xi_0, \eta_0, \phi_0| E) \]

\[ = \int_0^\infty d\sigma \exp(-2ie^2\sigma/\hbar) \sum_{n=\infty}^{-\infty} \left( \frac{m\omega}{i\hbar \sin \omega \sigma} \right)^2 \exp[im(\phi - \phi_0)] \exp\left\{ -\frac{2ie^2}{\hbar} \right\} \exp\left\{ \frac{i\omega}{2\hbar} \cot \omega \sigma [\xi^2 + \xi_0^2 + \eta^2 + \eta_0^2] \right\} J_n \left( \frac{m\omega\sqrt{\xi_0}}{\hbar \sin \omega \sigma} \right) \left( \frac{m\omega\sqrt{\eta_0}}{\hbar \sin \omega \sigma} \right) \] (146)
Using the identity
\[ \sum_{n=-\infty}^{\infty} \exp(in\alpha)J_n(r)J_n(\rho) = J_0(R) \] (148)
where
\[ R^2 = r^2 + \rho^2 + 2r\rho \cos\alpha \] (149)
the Green function (147) becomes
\[ G_0(\xi, \eta, \phi; \xi_0, \eta_0, \phi_0|E) = \int_{0}^{\infty} d\sigma \exp(2ie^{2}\sigma/\hbar) \left( \frac{m\omega}{i\hbar \sin \omega \sigma} \right)^2 J_0 \left( \frac{m\omega}{\hbar \sin \omega \sigma} R \right) \exp \left\{ \frac{im\omega}{2\hbar} \cot \omega t(\xi^2 + \xi_0^2 + \eta^2 + \eta_0^2) \right\} \] (151)
with \( R \) given by
\[ R^2 = \xi \xi_0 + \eta \eta_0 + 2\xi \xi_0 \eta \eta_0 \cos(\phi - \phi_0) = rr_0 \cos(\psi/2) \] (152)
and
\[ \cos \psi = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0). \] (153)
Defining \( k = (m\omega/i\hbar) \) the above result is seen to coincide with that obtained by Ho and Inomata in ref 13.

7.3 Radial propagator for Hydrogen atom: The hamiltonian for the radial hydrogen atom problem is
\[ H = \frac{p^2}{2m} - \frac{e^2}{r} + \frac{\hbar^2(l+1)}{2mr^2} \] (154)
Note that \( 1/r^2 \) term in the hamiltonian can be thought of as coming from a \( p_0^2/2mr^2 \) term in hamiltonian for corresponding two dimensional problem. Therefore, this path integral solution of this problem can be obtained by relating it to the Coulomb problem in two dimensions. This relation can be derived in a way parallel to the derivation in Sec.5.
8. Concluding Remarks

We have summarized the main properties of the hamiltonian path integral formalism of ref 16, 17, and 18. We have also discussed application to exact path integration. Here, we wish to highlight the following points.

(a) For all other existing literature on path integration, local scaling of time is a technique used in exact solution, whereas for us it is an essential ingredient of quantization scheme itself. In some cases, such as in the example of H-atom in Sec. 6, no further local scaling of time is necessary to complete exact solution.

(b) The best feature of this approach is that path integral quantization is formulated in terms of the classical hamiltonian in arbitrary co-ordinates. Performing changes of variables becomes quite trivial in the path integral formalism in our scheme, as one can do this at the classical level itself and use the classical hamiltonian in arbitrary coordinates to set up the canonical path integral quantization. The boundary conditions are correctly reproduced when initial condition for the propagator is correctly taken into account as discussed in Sec 3. In this respect our approach is new and more direct than all other existing treatments.

(c) The canonical approach offers more general possibilities for local scaling of time, than those exploited here. It should be possible to generalize the scaling relation to the case when \( f(q) \) is replaced by a function of \( p \) or by a function of \( p \) and \( q \). It is not known whether this can be used to do exact path integration of new problems.

(d) A class of potentials, such as Scarf potential, Rosen Morse potential etc., can be related to a free particle on sphere. Though we are not able to do exact path integration for free particle on a sphere, our scheme can be used to obtain inter-relations of propagators for the potentials within the class. A possible approach to solve particle on sphere problem exactly may be to formulate it as a path integral for a constrained systems. For this a discretized form of path integral for the constrained systems is needed.

(e) In ref. 18 correspondence between the canonical and lagrangian forms of path integrals was established. For most of the problems, not taken up for discussion in this paper one can, in principle, utilize correspondence between the hamiltonian and the lagrangian forms of path integration and follow the existing literature to complete the path integration. However, this is not very interesting.
(f) Doing a close analysis of discussion in sections 5, 6 and 7, we see that with exception of the path integral in Sec. 7.2, hardly any path integral, is actually computed. All the details of computing and manipulating path integrals are buried in the results summarized in Sec. 2 which were derived in ref. 18. The present paper thus isolates and identifies a set of properties of Hamiltonian path integrals useful for path integration.

(g) The aim of path integration formalism is not merely to solve those quantum mechanical problems exactly for which the exact solution can already be obtained by several methods. This activity is useful only if it generates new techniques or gives new insights into the path integral formalism. It would be far more interesting and useful to explore the generalizations and advantages of the hamiltonian approach for path integration.

(h) For the constrained hamiltonian systems, in a most direct but a formal approach, one sets up path integral over the reduced phase in terms of an independent (reduced) set of conjugate variables \((q^*, p^*)\) which, in general, are functions of an initial set \((q, p)\). By means of a formal canonical transformation, when the path integral is expressed as path integral over the initial set \((q, p)\) over full phase space, Dirac delta functions enforcing constraints and determinants appear in the integrand. This path integral requires a precise lattice definition. It then appears that, any attempt to precisely define a discrete version of path integration for the constrained systems must solve the question of incorporating canonical transformation as change of variables in the phase space path integral approach. This merits further investigation.

**Acknowledgment:** One of the authors (A.K.K.) thanks Prof. H.S. Mani for hospitality extended to him at the Mehta Research Institute. We thank Ramandip Singh Bhaalla for help in preparation of the Latex version of the manuscript.
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