MORE NONCOMMUTATIVE 4-SPHERES

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Abstract. New examples of noncommutative 4-spheres are introduced.

1. INTRODUCTION

Recently some new examples of noncommutative algebras, which are deformations of spheres (3 and 4-dimensional) were presented [4, 6]. The significant part of the construction was the deformation of the 3-sphere, then the 4-sphere in both cases is obtained by a commutative suspension.

The deformation [4], which, on the algebraic level is a twist of the algebra of functions relative to the action of two $U(1)$ groups, preserves much of the geometry of the sphere. In particular, the spectral triple is also deformed with the Dirac operator unchanged (similarly as for the noncommutative torus), which explains the name of an isospectral deformation [4].

In this Letter we propose another family of noncommutative 4-spheres constructed using the method of “twisting”. Let us stress, however, that such deformed algebras will probably not be ‘noncommutative spin manifolds’ as described by spectral triples (see [2, 3] for details) and their geometry seems to be much different from the commutative four-sphere.

2. NONCOMMUTATIVE 2-SPHERE

Let us remind here the construction of the Podlés quantum sphere [4]. Let $q$ be a real number $0 < q < 1$ and $S_q$ be an algebra generated by operators $a$, $a^*$ and $b = b^*$, which satisfy the following relations:

\begin{align}
ba &= qab, \\
a^*a + b^2 &= 1, \\
\frac{1}{q^2}b^2 &= 1.
\end{align}

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1 We have chosen here a particular quantum sphere.
This algebra admits the action of the $U_q(su(2))$ quantum group and in the $q = 1$ limit corresponds to the continuous functions on the two-dimensional sphere.

3. Noncommutative 4-spheres

Let us consider an algebra generated by $\alpha, \alpha^*, \beta = \beta^*$ and $U, U^*$ with the following contraints:

$$
\begin{align*}
\beta \alpha &= q \alpha \beta, \\
\alpha^* \alpha + \beta^2 + U^* U &= 1, \\
U \alpha &= e^{2\pi i \theta} \alpha U, \\
U \beta &= \beta U, \\
U^* &= U U^*. 
\end{align*}
$$

(2)

This defines a family of algebras $S^4_{q, \theta}$. By taking the supremum of operator norm over all admissible representations, taking out the ideal of zero-norm elements and completing the quotient we obtain a $C^*$ algebra structure.

Let us make here some remarks.

Remark 1. The center of the algebra $S^4_{q, \theta}$ (for $0 < q < 1$ and irrational $\theta$) is generated by a non-negative operator $U U^*$. The space of characters, as one can easily verify, contains all following maps $\chi$:

$$
\begin{align*}
\chi(\alpha) &\in S^1, \chi(\beta) = 0, \chi(U) = 0, \\
or \\
\chi(\alpha) &= 0, \chi(\beta) = 0, \chi(U) \in S^1,
\end{align*}
$$

(3)

so that the 'classical' space underlying the noncommutative construction is a union of two disjoint circles.

Remark 2. In the $\theta = 0$ limit $S^4_{q, 0}$ is the complex (2-dimensional) suspension of the Podleś sphere. Note that this is not equivalent to the real (1-dimensional) suspension of $SU_q(2)$ constructed in [6].

Remark 3. A map $\rho : S^4_{q, \theta} \to S^3_{\theta}$ defined by $\rho(\beta) = 0$ and $\rho(\alpha) = \tilde{\alpha}$, $\rho(U) = \tilde{\beta}$ (for definitions of the generators of $S^3_{\theta}$ see [4]) is a star algebra morphism.

Remark 4. In the $q = 1$ limit $S^4_{1, \theta}$ is exactly the sphere $S^4_{\theta}$ of [4] (obtained through the suspension od $S^3_{\theta}$).
Remark 5. The sphere \( S^4_{q,\theta} \) arises as a subalgebra of the crossproduct algebra from nontrivial action of \( \mathbb{Z} \) on the one-dimensional suspension of the Podleś sphere.

\[
\begin{align*}
n \triangleright a &= e^{2\pi i \theta} a, \\
n \triangleright a^* &= e^{-2\pi i \theta} a^*, \\
n \triangleright b &= b, \\
n \triangleright t &= t,
\end{align*}
\]

(4)

where \( t \) is a central selfadjoint suspension generator. The crossproduct algebra of \( \mathbb{C}Z \) is generated by \( a, a^*, b, t \) and a unitary generator \( V \) of \( \mathbb{C}Z \). It is easy to observe that the subalgebra generated by \( a, a^*, b \) and \( Vt = tv \) is isomorphic to \( S^4_{q,\theta} \).

As one can see, the family of 4-spheres we have just introduced, shares both the properties of the \( q \)-deformation as well as that of the twisted deformation. Notice that for \( \theta = 0 \) (as well as for the suspension of \( SU_q(2) \) in \([1]\)) \( U_q (su(2)) \) is still a symmetry group. This does not hold, however, for (irrational) \( \theta \neq 0 \).

4. Representation of \( S^4_{q,\theta} \)

To obtain the Hilbert space representation of \( S^4_{q,\theta} \) we proceed as follows. First, let \( \mathcal{H}_q \) be the Hilbert space representation of the Podleś sphere and \( \rho \) the corresponding representation map (see \([3]\) for details on the representations of \( S^2_q \)). Then for every \( 0 \leq \phi < 2\pi \) we construct the following representation \( \rho_\phi \) on \( \mathcal{H}_q \otimes l^2(\mathbb{Z}) \):

\[
\begin{align*}
\rho_\phi(\alpha) (v \otimes |n\rangle) &= (\cos \phi) e^{-2\pi i \phi} (\rho(\alpha)(v) \otimes |n\rangle), \\
\rho_\phi(\beta) (v \otimes |n\rangle) &= (\cos \phi) (\rho(\beta)(v) \otimes |n\rangle), \\
\rho_\phi(U) (v \otimes |n\rangle) &= (\sin \phi) (v \otimes |n+1\rangle)
\end{align*}
\]

(5)

Given this example, let us finish this section with a remark that using various representations of \( S^2_q \) (which may correspond to nontrivial projective modules over the quantum sphere) one might obtain many interesting models for field theory.

5. The projector.

A question, which is interesting in itself is the construction of noncommutative vector bundles over our noncommutative space. This is expressed algebraically by construction of finitely generated projective modules over an algebra \( \mathcal{A} \), defined by idempotents in the algebra \( M_n(\mathcal{A}) \). We shall provide here an example of an idempotent in \( M_4(\mathcal{A}) \)
and calculate its first Chern class in order to make contact with the construction of [4, 6].

Let us take:

\[
\begin{pmatrix}
1 + \beta & 0 & U & \alpha^* \\
0 & 1 + \frac{1}{q} \beta & \alpha & -e^{2\pi i \theta} U^*
\end{pmatrix},
\]

(6)

We may easily calculate its Chern-Connes character in the reduced $(b, B)$ standard complex of the algebra $S^4_{q,\theta}$ using the Dennis trace map [7]. Clearly, $ch_0(e) = 0$ and:

\[
ch_1(e) \sim \left(1 - \frac{1}{q}\right) \left(\beta \otimes (U \otimes U^* - U^* \otimes U) + U \otimes U^* \beta - \beta \otimes U^* + U^* \otimes (\beta \otimes U - U \otimes \beta)\right).
\]

(7)

Notice that here, $ch_1(e)$ does not depend on $\theta$ and vanishes for $q = 1$, exactly as in [3]. In fact, the formula (6) is almost the same as for the suspension of $SU_q(2)$. In the commutative limit $q = 1$ and $\theta = 0$ we recover the projector of the instanton bundle. Note that $ch_2(e)$ does not vanish and contains 222 terms.

Of course, this is not the only projector one can find, in particular, using the monopole projector for the quantum sphere we define, similarly as in [6]:

\[
\begin{pmatrix}
1 + x & 0 & \beta & \alpha^* \\
0 & 1 + x & \alpha & -\frac{1}{q} \beta \\
\beta & \alpha^* & 1 - x & 0 \\
\alpha & -\frac{1}{q} \beta & 0 & 1 - x
\end{pmatrix},
\]

(8)

where $x = \sqrt{U^* U}$ is a central element. However, this projector lives effectively on the subalgebra of the $S^4_{q,\theta}$, which could be identified as a deformation of a 3-sphere and therefore it would correspond to a trivial bundle over $S^3$ [5]. In fact, it is easy to verify that all its Chern classes vanish.

6. Conclusions

In this letter we have shown that there exists a large family of objects, which could be called noncommutative 4-spheres: this includes the $\theta$-deformation ("isospectral deformation") defined in [4] and extends the family of objects presented in [6].
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All of them have the same commutative limit and all are deformations, however, only in the case of the “isospectral” deformation \((q = 1)\) the lower chern characters of the deformed instanton bundle projector vanish.

Such objects should be studied in greater details: the classes and examples of projective modules over the algebra, their cyclic cohomology etc. It would be interesting to verify whether these deformations could be understood as noncommutative manifolds in the sense of [3]. Let us remark that despite some negative results like dimension change in cyclic cohomology [8] or no-go theorems for some differential calculi [11] the main problem of understanding such deformations as noncommutative manifolds might be algebraic and related with the choice of the dense subalgebra of \(C^\infty\) functions on our deformations (see Rieffel’s "Question 19" in [10]).

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