Rigidity and vanishing theorems for complete translating solitons

Abstract. In this paper, we prove some rigidity theorems for complete translating solitons. Assume that the $L^q$-norm of the trace-free second fundamental form is finite, for some $q \in \mathbb{R}$ and using a Sobolev inequality, we show that a translator must be a hyperspace. Our results can be considered as a generalization of Ma and Miquel (Manuscripta Math 162:115–132, 2020), Wang et al. (Pure Appl Math Q 12(4):603–619, 2016), Xin (Calc Var Partial Differ Equ 54:1995–2016, 2015). We also investigate a vanishing property for translators which states that there are no nontrivial $L^p_f$ $(p \geq 2)$ weighted harmonic 1-forms on $M$ if the $L^n$-norm of the second fundamental form is bounded.

1. Introduction

Let $X_0 : M^n \to \mathbb{R}^{n+m}$ be an $n$-dimensional smooth submanifold immersed in an $(n+m)$-dimensional Euclidean space $\mathbb{R}^{n+m}$. The mean curvature flow with initial value $X_0$ is a family of immersions $X : M \times [0, T) \to \mathbb{R}^{n+m}$ satisfying

$$\begin{cases}
\frac{d}{dt} X(x, t) = H(x, t), \\
X(x, 0) = X_0(x),
\end{cases}$$

(1.1)

for $x \in M$, $t \in [0, T)$, where $H(x, t)$ is the mean curvature vector of $X_t(M) = M_t$ at $X(x, t)$ in $\mathbb{R}^{n+m}$ and $X_t(\cdot) = X(t, \cdot)$.

One of the most important parts in the study of mean curvature flow is the singularity analysis. For several situations, the second fundamental form with respect to the family $M_t$ may develop singularities. For example, if $M$ is compact, the second fundamental form will blow up in a finite time. According to the blow-up rate of the second fundamental form, we divide singularities of the mean curvature flow into two types, Type-I singularities and Type-II singularities. Since the geometry of the solution near the Type-II singularities cannot be controlled well, the study of Type-II singularities is much more complicated than type-I singularities.

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A very important example of Type-II singularities is the translating soliton. A submanifold \( X : M^n \to \mathbb{R}^{n+m} \) is said to be a translating soliton if there exists a unit constant vector \( V \) in \( \mathbb{R}^{n+m} \) such that

\[
H = V^N,
\]

where \( V^N \) denotes the normal component of \( V \) in \( \mathbb{R}^{n+m} \). Let \( V^T \) be the tangent component of \( V \), then

\[
H + V^T = V. \tag{1.3}
\]

Translating solitons often occur as Type-II singularities after rescaling. Also, a translating soliton corresponds to a translating solution \( M_t \) of the mean curvature flow defined by \( M_t = M + tV \). There are very few examples of translating solitons even in the hypersurface case. The first basic examples are translating solitons which are also minimal hypersurfaces. By (1.3) we know that \( V \) must be tangential to the soliton. Consequently, these solitons could have the form of \( \tilde{M} \times L \), where \( L \) is a line parallel to \( V \) and \( \tilde{M} \) is a minimal hypersurface in \( L^\perp \). Some other non-trivial examples are grim reapers and grim reaper cylinders. The grim reaper \( \Gamma \) is a one-dimensional translating soliton in \( \mathbb{R}^2 \) defined as the graph of

\[
y = -\log \cos x, \quad x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
\]

A trivial generalization is the Euclidean product \( \Gamma \times \mathbb{R}^{n-1} \) in \( \mathbb{R}^{n+1} \), which is called grim reaper cylinder or grim hyperplane (see [12]). For further examples of translators, we refer the reader to [12,14,15].

As mentioned above, translators play an important role in the study of mean curvature flow because they arise as blow-up solutions at type II singularities. Huisken and Sinestrari [5] proved that at a type II singularity of a mean curvature flow with a mean convex solution, there exists a blow-up solution which is a convex translating soliton. In [21], Wang proved that when \( n = 2 \), every entire convex translator must be rotationally symmetric. However, in every dimension greater than two, there exist non-rotationally symmetric, entire convex translators. Later, Haslhofer [3] obtained the uniqueness theorem of the bowl soliton in all dimensions under the assumptions of uniformly 2-convexity and a noncollapsing condition. In [22], Xin studied some basic properties of translating solitons: the volume growth, generalized maximum principle, Gauss maps and certain functions related to the Gauss map. He also gave integral estimates for the squared norm of the second fundamental form and used them to show rigidity theorems for translators in the Euclidean space in higher co-dimension. Some of Xin’s results then were extended by Wang, Xu, and Zhao by using integral curvature pinching conditions of the trace-free second fundamental form (see [20]). Recently, in [6,7] Impera and Rimoldi studied rigidity results and topology at infinity of translating solitons of the mean curvature flow. Their approach relies on the theory of \( f \)-minimal hypersurfaces. In particular, they established weighted Sobolev inequalities and used them to show that an \( f \)-stable translator has at most one end. They also investigate some relationship between the space of \( L^2 \)-weighted harmonic 1-forms, cohomology with compact support, and the index of the translator in terms of generalized Morse
index of a stable operator. Using Sobolev inequalities discovered by Impera and Rimoldi, Kunikawa and Sato [9] pointed out that any complete $f$-stable translating soliton admits no codimension one cycle which does not disconnect $M$. As a consequence, any two dimensional complete $f$-stable translator has genus zero. For further discussion on translators, we refer to [2, 3, 5–9, 12, 14, 15, 18, 20–22] and the references therein.

In this paper, motivated by [6, 11, 20, 22], we investigate some rigidity theorems and study the connectedness at infinity of complete translators in Euclidean spaces. Our first main theorem is as follows

**Theorem 1.1.** Let $M^n (n \geq 3)$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+m}$. If the trace-free second fundamental form $\Phi$ of $M$ satisfies

$$\left( \int_M |\Phi|^n \text{d} \mu \right)^{1/n} < K(n, a) \text{ and } \int_M |\Phi|^{2a} e^{(V,X)} < \infty,$$

where

$$1 \leq a < \frac{n + \sqrt{n^2 - 2n}}{2},$$

$$K(n, a) = \frac{(n-2)^2 (a-\frac{1}{2})}{D^2(n) \left[ \frac{(n-2)^2 (a-\frac{1}{2})}{2n(na - \frac{2}{a} - a^2)} + (n-1)^2 \right] a^2},$$

$$t = \begin{cases} 2 & \text{if } m = 1, \\ 4 & \text{if } m \geq 2, \end{cases}$$

and $D(n)$ is the Sobolev constant defined in Lemma 2.2, then $M$ is a linear subspace.

The proof of this theorem relies on a Sobolev inequality on immersed submanifolds which was first verified in [4] and [13]. When $a = \frac{n}{2}$, Theorem 1.1 recovers Theorem 1 in [20]. As noted in [20], the curvature condition in Theorem 1.1 is weaker than that in Theorem 7.1 in [22]. If translators are located in a halfspace, in [6, 7], Impera and Rimoldi proved a weighted Sobolev inequality by using the bijective correspondence found by Smoczyk [18] between translators and minimal hypersurfaces in a suitable warped product. Applying Sobolev inequality, we are able to obtain the following theorem.

**Theorem 1.2.** Let $M^n (n \geq 3)$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ contained in the halfspace $\Pi_{v,a} = \{ y \in \mathbb{R}^{n+1} : \langle y, v \rangle \geq a \}$, where $a \in \mathbb{R}$, $v \in \mathbb{R}^{n+1}$ are fixed. If the second fundamental form $A$ of $M$ satisfies

$$\left( \int_M |A|^n \text{d} \mu \right)^{\frac{1}{n}} < \sqrt{\frac{(n^2 - 2n + 2)(n-2)^2}{n^3 S(n)^2 (n-1)^2}},$$

where $S(n)$ is the Sobolev constant as in Lemma 2.5 and $\rho = e^{(V,X)}$, then $M$ is a hyperplane.
Compared with the results in [20, 22], this result drops the assumption on the smallness of the $L^n$-norm of $|A|$, instead of this, we require the weighed $L^n$-norm of $|A|$ to be small. In fact, in [22], the author supposed that the weighted $L^n$-norm of $|A|$ is finite and the $L^n$-norm is small. Hence, when the weighed $L^n$-norm of $|A|$ is small, this theorem can be considered as a refinement of Theorem 7.1 in [22]. Moreover, using the weighted Sobolev inequality, we obtain a vanishing theorem as follows.

**Theorem 1.3.** Let $x : M^n \to \mathbb{R}^{n+1}$ be a smooth complete translator contained in the halfspace $\Pi_{v, a} = \{ p \in \mathbb{R}^{n+1} : \langle p, v \rangle \geq a \}$, where $v \in \mathbb{R}^{n+1}$ is a given vector, $a \in \mathbb{R}$ is fixed, and $n \geq 3$. Assume that for any $p \geq 2$,

$$\|A\|_{n,f} < \frac{\sqrt{(p - 1)(n - 1)}}{p S(n)},$$

where $S(n)$ is the Sobolev constant as in Lemma 2.5. Then there are no nontrivial $L^p_f$ weighted harmonic 1-forms on $M$.

This paper has four sections. Section 1 is used to derive some rigidity theorems. Then we prove a vanishing result for weighted harmonic forms in Section 3. Finally, we study translators in the Euclidean space with a Sobolev inequality in Section 4 and give another rigidity theorem.

### 2. Rigidity theorems

Let $X : M \to \mathbb{R}^{n+m}$ be an $n$-dimensional translating soliton. $H$, $A$, $\Phi$ denote the mean curvature vector, the second fundamental form, and the trace-free second fundamental form of $M$, respectively. $V$ is the unit vector so that $V N = H$. Let $f = -\langle V, X \rangle$, we define

$$\Delta_f = \Delta + \langle V, \nabla(\cdot) \rangle = e^{-(V, X)} \text{div} \left( e^{(V, X)} \nabla(\cdot) \right) = e^{f} \text{div}(e^{-f} \nabla(\cdot)).$$

The trace-free second fundamental form is given by $\Phi = A - \frac{1}{n} g \otimes H$. It is well-known that

$$|\Phi|^2 = |A|^2 - \frac{1}{n} |H|^2 \quad \text{and} \quad |\nabla \Phi|^2 = |\nabla A|^2 - \frac{1}{n} \nabla |H|^2.$$

In order to prove our theorems, we need the following Simons type identity which has been obtained by Xin [22] (see also [20]).

**Lemma 2.1.** ([6, 20]) On a translating soliton $M^n$ in $\mathbb{R}^{n+m}$, we have

$$\Delta_f |\Phi|^2 \geq 2|\nabla|\Phi||^2 - \nu|\Phi|^4 - \frac{2}{n} |H|^2 |\Phi|^2,$$

where

$$\nu = \begin{cases} 2, & \text{if } m = 1 \\ 4, & \text{if } m \geq 2. \end{cases}$$
Moreover, when $m = 1$, we have
\[
\Delta_f |\Phi|^2 = 2|\nabla \Phi|^2 - 2|A|^2|\Phi|^2.
\]
(2.2)

We now recall that the following Sobolev inequality for submanifolds in the Euclidean space is very helpful to derive our rigidity theorems (see [23]).

**Lemma 2.2.** (Sobolev inequality) Let $M^n$ $(n \geq 3)$ be a complete submanifold in the Euclidean space $\mathbb{R}^{n+m}$. Let $f$ be a nonnegative $C^1$ function with compact support. Then for all $s \in \mathbb{R}^+$, we have
\[
\|f\|^2_{\frac{2a}{n-2}} \leq D^2(n) \left[ \frac{4(n-1)^2(1+s)}{(n-2)^2} \|\nabla f\|^2_2 + \left( \frac{1}{s} \right) \frac{1}{n^2} \|H\|_2^2 \right],
\]
where
\[
D(n) = 2^n (1 + n) \frac{n+1}{n} (n-1)^{-1} \frac{\sigma_n^{-\frac{1}{n}}}{},
\]
and $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$.

For the Proof of Theorem 1.1, let $\rho = e^{(V,X)}$. We can follow the proof in [20], but instead of using the function $f$ defined in Lemma 5 of [20], we use the function $\varphi = |\Phi|^a \rho^{1/2} \eta$, where $a \geq 1$ is a constant to be determined later and $\eta$ is a smooth function with compact support on $M$. For the convenience of the reader, in order to help him/her checking the influence of the constant $a$ in every step, we give all the details of the computations.

**Lemma 2.3.** Assume that $|\Phi| \neq 0$ on $M$. If $\eta$ is a smooth function with compact support on $M$, then
\[
\int_M |\nabla \varphi|^2 = \int_M \nabla \left( |\Phi|^a \eta \right)^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} \eta^2 \varphi + \frac{1}{4} \int_M |\Phi|^{2a} |V^T|^2 \eta^2 \varphi. \tag{2.3}
\]

**Proof.** Integrating by parts, we have
\[
\int_M |\nabla \varphi|^2 = \int_M \nabla \left( |\Phi|^a \eta \right)^2 \varphi + \frac{1}{2} \int_M \nabla \left( |\Phi|^{2a} \eta^2 \right) \nabla \varphi + \int_M |\Phi|^{2a} \eta^2 \nabla \varphi \nabla \eta^2 \|V\|^2_2^2
\]
\[
= \int_M |\nabla \left( |\Phi|^a \eta \right)|^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} \eta^2 \Delta \varphi + \int_M |\Phi|^{2a} \eta^2 \nabla \varphi \nabla \eta^2 \|V\|^2_2^2.
\]

Since $M^n$ is a translating soliton, we have
\[
\nabla \varphi = \nabla e^{(V,X)} = \varphi V^T,
\]
\[
\nabla \varphi \nabla \eta^2 = \frac{1}{2} \varphi^{\frac{1}{2}} \nabla \varphi = \frac{1}{2} \varphi \frac{1}{2} V^T,
\]
and
\[
\Delta \varphi = \sum_i \nabla_i \varphi \langle V, e_i \rangle + \sum_i \varphi \langle V, \nabla e_i \rangle = \varphi \left( |V^T|^2 + |H|^2 \right) = \varphi \left( |V^T|^2 + |V^N|^2 \right) = \varphi.
\]

Using this, we get
\[
\int_M |\nabla \varphi|^2 = \int_M \nabla \left( |\Phi|^a \eta \right)^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} \eta^2 \varphi + \frac{1}{4} \int_M |\Phi|^{2a} |V^T|^2 \eta^2 \varphi.
\]
\[\blacksquare\]
Now, combining the Sobolev’s inequality in Lemma 2.2 and (2.3), we get

\[
\left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left[ \int_M |\nabla \varphi|^2 + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |H|^2 \varphi^2 \right] \right\} \\
= D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left[ \int_M |\nabla (|\Phi|^a \eta)|^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} \eta^2 \varphi \right] + \frac{1}{4} \int_M |\Phi|^{2a} \left| V^T \right|^2 \eta^2 \varphi \right\} + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\Phi|^{2a} |H|^2 \eta^2 \varphi \right\}.
\]

Note that

\[
\left| V^T \right|^2 + |V|^2 = |V^T|^2 + |H|^2 = 1.
\]

Thus, we obtain

\[
\left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq D^2(n) \cdot \left\{ \frac{4(n-1)^2(1+s)}{(n-2)^2} \left[ \int_M |\nabla (|\Phi|^a \eta)|^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} \left| V^T \right|^2 \eta^2 \varphi \right] + \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\Phi|^{2a} |H|^2 \eta^2 \varphi \right\}.
\]

By the Cauchy inequality, for \( \delta > 0 \) we have

\[
\left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2} \left\{ (1 + \delta) a^2 \int_M |\nabla \varphi|^2 \cdot |\Phi|^{2a-2} \eta^2 \varphi \right\} + \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^{2a} \left| \nabla \eta \right|^2 \varphi - \frac{1}{4} \int_M |\Phi|^{2a} \left| V^T \right|^2 \eta^2 \varphi - \frac{1}{2} \int_M |\Phi|^{2a} |H|^2 \eta^2 \varphi \right\} + D^2(n) \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\Phi|^{2a} |H|^2 \eta^2 \varphi \right\}.
\]

(2.4)
In order to estimate the term $\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\mathcal{Q}$, we multiply $|\Phi|^{2a-2}\eta^2$ on both sides of (2.1) and integrating by parts with respect to the measure $\vartheta d\mu$ on $M$ gives

$$0 \geq 2 \int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\mathcal{Q} - t \int_M |\Phi|^{2a+2}\eta^2\mathcal{Q} - \frac{2}{n} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q}$$

$$- \int_M |\Phi|^{2a-2}\eta^2 \Delta f |\Phi|^2\mathcal{Q}. \quad (2.6)$$

Since $\eta$ has compact support on $M$, by the Stokes theorem, we obtain

$$- \int_M |\Phi|^{2a-2}\eta^2 \Delta f |\Phi|^2\mathcal{Q}$$

$$= - \int_M |\Phi|^{2a-2}\eta^2 \text{div} \left( \vartheta \cdot |\nabla|\Phi|^2 \right)$$

$$= 2 \int_M \vartheta |\Phi| |\nabla|\Phi| \cdot \nabla \left( |\Phi|^{2a-2}\eta^2 \right) \quad (2.7)$$

$$= 4(a-1) \int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\mathcal{Q} + 4 \int_M (\nabla|\Phi| \cdot \nabla \eta)|\Phi|^{2a-1}\eta\mathcal{Q}.$$ 

Combining (2.6) and (2.7), we get

$$0 \geq 4 \left( a - \frac{1}{2} \right) \int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\mathcal{Q} - t \int_M |\Phi|^{2a+2}\eta^2\mathcal{Q} - \frac{2}{n} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q}$$

$$+ 4 \int_M (\nabla|\Phi| \cdot \nabla \eta)|\Phi|^{2a-1}\eta\mathcal{Q}. \quad (2.8)$$

By the Cauchy inequality, for $0 < \varepsilon < a - \frac{1}{2}$, we have

$$t \int_M |\Phi|^{2a+2}\eta^2\mathcal{Q} + \frac{2}{n} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q} + \frac{1}{\varepsilon} \int_M |\Phi|^{2a}|\nabla \eta|^2\mathcal{Q}$$

$$\geq 4 \left( a - \frac{1}{2} - \varepsilon \right) \int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\mathcal{Q}. \quad (2.8)$$

Substituting (2.8) into (2.5), we get

$$\left( \int_M |\vartheta|^\frac{2n}{n-2} \right)^{\frac{n-2}{n}} \leq \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2} \left\{ \frac{a^2(1+\delta)}{4(a - \frac{1}{2} - \varepsilon)} \left( t \int_M |\Phi|^{2a+2}\eta^2\mathcal{Q} \right) \right.$$

$$+ \frac{2}{n} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q} + \frac{1}{\varepsilon} \int_M |\Phi|^{2a}|\nabla \eta|^2\mathcal{Q} \right\}$$

$$+ \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^{2a}|\nabla \eta|^2\mathcal{Q} - \frac{1}{2} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q}$$

$$+ D^2(n) \left( 1 + \frac{1}{s} \right) \cdot \frac{1}{n^2} \int_M |\Phi|^{2a}|H|^2\eta^2\mathcal{Q}. \quad (2.9)$$
We want to get rid of the term $\int_M |\Phi|^{2a}|H|^{2}\eta^2 \, \mathcal{Q}$ by choosing $\delta > 0$ appropriately. Put

$$
\delta = \delta(n, \varepsilon, a) = \frac{(2(n - 1)^2n^2s - (n - 2)^2)(a - \frac{1}{2} - \varepsilon)}{2(n - 1)^2a^2ns} - 1.
$$

We would require $\delta > 0$, this occurs only if $s$ satisfies

$$
(2.10)\quad s > \frac{(n - 2)^2(a - \frac{1}{2} - \varepsilon)}{2(n - 1)^2n(na - \frac{n}{2} - a^2 - n\varepsilon)}
$$

for some $\varepsilon \in (0, a - \frac{1}{2} - \frac{a^2}{n})$ defined later and also, we need

$$
1 \leq a < \frac{n + \sqrt{n^2 - 2n}}{2}.
$$

Consequently, we have

$$
\begin{align*}
\kappa^{-1} \left( \int_M |\varphi|^{\frac{2a}{n-2}} \right)^{\frac{n-2}{n}} &\leq \frac{a^2(1 + s)(1 + \delta)}{4(a - \frac{1}{2} - \varepsilon)} \left( \int_M |\Phi|^{2a+2}\eta^2 \, \mathcal{Q} + \frac{1}{\varepsilon} \int_M |\Phi|^{2a} |\nabla \eta|^2 \, \mathcal{Q} \right) \\
&+ (1 + s) \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^{2a} |\nabla \eta|^2 \, \mathcal{Q} \\
&= \frac{(1 + s)(2sn^2(n - 1)^2 - (n - 2)^2)}{8sn(n - 1)^2} \int_M |\Phi|^{2a+2}\eta^2 \, \mathcal{Q} \\
&+ C(s, \varepsilon, n, a) \int_M |\Phi|^{2a} |\nabla \eta|^2 \, \mathcal{Q},
\end{align*}
$$

(2.11)

where $C(s, \varepsilon, n, a)$ is an explicit positive constant depending on $s, \varepsilon, n, a$ and

$$
\kappa = \frac{4D^2(n(n - 1)^2)}{(n - 2)^2}.
$$

By Hölder’s inequality we have

$$
\begin{align*}
\int_M |\Phi|^{2a+2}\eta^2 \, \mathcal{Q} &\leq \left( \int_M |\Phi|^{\frac{2a}{n}} \right)^{\frac{2}{n}} \cdot \left( \int_M \left( |\Phi|^{2a}\eta^2 \, \mathcal{Q} \right)^{\frac{n}{n-2}} \right)^{\frac{n-2}{n}} \\
&= \left( \int_M |\Phi|^{\frac{2}{n}} \right)^{\frac{2}{n}} \cdot \left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}.
\end{align*}
$$

(2.12)
Applying this to (2.11), we get
\[ \kappa^{-1} \left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \]
\[ \leq \frac{(1 + \varepsilon) t \left[ 2s (n - 1)^2 - (n - 2)^2 \right]}{8s n (n - 1)^2} \left( \int_M |\Phi|^{2a} \right)^{\frac{2}{n}} \left( \int_M |\varphi|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \]
\[ + C(s, \varepsilon, n, a) \int_M |\Phi|^{2a} |\nabla \eta|^2. \]
(2.13)

Put
\[ K(n, s) = \sqrt{\frac{8s n (n - 1)^2}{(1 + \varepsilon) t \left[ 2s (n - 1)^2 - (n - 2)^2 \right] \kappa}}. \]

By condition (2.10) we can choose
\[ s = \frac{(n - 2)^2 (a - \frac{1}{2})}{2(n - 1)^2 n \left( na - \frac{n}{2} - a^2 - n \varepsilon \right)}. \]

Hence, substituting \( s \) into \( K(n, s) \), we have
\[ K(n, a, \varepsilon) = K(n, s(a, \varepsilon)) = \frac{(n - 2)^2 (a - \frac{1}{2})}{D^2(n) \left[ \frac{(n-2)^2 (a-\frac{1}{2})}{2n(na-\frac{n}{2} - a^2 - n \varepsilon)} + (n - 1)^2 \right] t(n \varepsilon + a^2)}. \]
(2.14)

Set
\[ K(n, a) = \sup_{\varepsilon \in \left( 0, \frac{a^2}{n-1} \right)} K(n, a, \varepsilon) = \frac{(n - 2)^2 (a - \frac{1}{2})}{D^2(n) \left[ \frac{(n-2)^2 (a-\frac{1}{2})}{2n(na-\frac{n}{2} - a^2)} + (n - 1)^2 \right] t a^2}. \]

We now can give a Proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since we made the assumption \( \left( \int_M |\Phi|^n d\mu \right)^{1/n} < K(n, a) \), there exists a positive constant \( \tilde{K} \) such that
\[ \left( \int_M |\Phi|^n d\mu \right)^{1/n} < \tilde{K} < K(n, a). \]
(2.15)

Thus, there exists \( \varepsilon = \varepsilon_0 > 0 \) such that
\[ \tilde{K} < K(n, a, \varepsilon_0) < K(n, a). \]
Using this and combining (2.13), (2.15), there exists $0 < \epsilon < 1$ such that

$$
\kappa^{-1} \left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} 
\leq \kappa^{-1} \cdot K (n, a, \epsilon_0)^{-2} \cdot \tilde{K}^2 \left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \tilde{C} (n, a, \epsilon_0) \int_M |\Phi|^{2a} |\nabla \eta|^2 \varrho 
\leq \frac{1 - \epsilon}{\kappa} \left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \tilde{C} (n, a, \epsilon_0) \int_M |\Phi|^{2a} |\nabla \eta|^2 \varrho
$$

(2.16)

or equivalently

$$
\frac{\epsilon}{\kappa} \left( \int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \tilde{C} (n, a, \epsilon_0) \int_M |\Phi|^{2a} |\nabla \eta|^2 \varrho.
$$

(2.17)

Let

$$
\eta(X) = \eta_r (X) = \phi \left( \frac{|X|}{r} \right)
$$

for any $r > 0$, where $\phi$ is a non-negative smooth function on $[0, +\infty)$ satisfying

$$
\phi(x) = \begin{cases} 
1, & \text{if } x \in [0, 1) \\
0, & \text{if } x \in [2, +\infty) 
\end{cases}
$$

(2.18)

and $|\phi'| \leq C$ for some absolute constant. Since $\int_M |\Phi|^q \varrho$ and $\tilde{C}(n, a, \epsilon_0)$ are bounded the right hand side of (2.17) approaches zero as $r \to \infty$, which implies the left hand side to be equal to zero i.e. $|\Phi| \equiv 0$.

Finally, using the assertion that $|\Phi| = 0$, it was confirmed in [20] that $M$ is a linear subspace. In the rest of the proof, we give a detail of argument which is inspired by Impera and Rimoldi in [6]. We argue as follows. Since $|\Phi| = 0$, it turns out that $|A|^2 = \frac{1}{n} H^2$. Moreover, we note that $|\nabla \Phi|^2 = |\nabla |\Phi||^2 = 0$. This implies

$$
0 = |\nabla \Phi|^2 = |\nabla A|^2 - \frac{1}{n} |\nabla H|^2.
$$

Therefore, we get

$$
|\nabla |A||^2 = \frac{1}{n} |\nabla H|^2 = \frac{2}{n} |H||\nabla H| = \frac{2}{n} \left( \sqrt{n} |A| \right) \left( \sqrt{n} |\nabla A| \right) = 2 |A||\nabla A|.
$$

As a consequence, $|\nabla A| = |\nabla |A||$. Therefore, we can apply the argument in the proof of Theorem A in [6] to conclude that $M$ is a linear subspace. The proof is complete.

Observe that $[1, n - 1] \subset \left[ 1, \frac{n + \sqrt{n^2 - 2n}}{2} \right)$, the weighted $L^{2a}$ norm of $|\Phi|$ in our theorem is wider than those in [20]. Moreover, when $a = \frac{n}{2}$, our theorem recovers the following rigidity property which was obtained by Wang, Xu and Zhao in [20].
Theorem 2.4. Let $M^n (n \geq 3)$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+m}$. If the trace-free second fundamental form $\Phi$ of $M$ satisfies
\[ \left( \int_M |\Phi|^n \, d\mu \right)^{1/n} < K(n) \quad \text{and} \quad \int_M |\Phi|^n \, e^{(V,X)} < \infty \]
where $K(n)$ is defined as above, then $M$ is a linear subspace.

It is worth mentioning that the above condition is weaker than that in the rigidity theorem of Xin [22]. Now, to derive another rigidity result, we can use the following version of the Sobolev inequality.

Lemma 2.5. ([6]) Let $X: M^n \to \mathbb{R}^{n+1}$ be a translator contained in the halfspace $\Pi_{v,a} = \{ p \in \mathbb{R}^{n+1} : \langle p, v \rangle \geq a \}$ for some $a \in \mathbb{R}$ and $n \geq 3$. Let $u$ be a non-negative compactly supported $C^1$ function on $M$. Then
\[ \left( \int_M u^{2n/(n-2)} \, d\mu \right)^{n-2/n} \leq \left( \frac{2(n-1)S(n)}{n-2} \right)^2 \int_M |\nabla u|^2 \, d\mu \] (2.19)
where $S(n)$ is the Sobolev constant given in Lemma 4.2 in [6].

Repeating the same computation as above, we can give a verification of Theorem 1.2 as follows.

Proof of Theorem 1.2. Applying the Sobolev’s inequalities (2.19) to $u = |\Phi|^n \eta$ and using the Cauchy inequality, we have
\[ \left( \int_M (|\Phi|^n \eta)^{2n/(n-2)} \, d\mu \right)^{n-2/n} \leq \left( \frac{2S(n)(n-1)}{n-2} \right)^2 \int_M |\nabla (|\Phi|^n \eta)|^2 \, d\mu \]
\[ = \left( \frac{2S(n)(n-1)}{n-2} \right)^2 \left( \int_M \frac{n^2}{4} |\nabla |\Phi||^2 |\Phi|^{n-2} \eta^2 Q \right) \]
\[ + \int_M n |\Phi|^{n-1} \eta |\nabla |\Phi| \cdot \nabla \eta Q + \int_M |\Phi|^n |\nabla \eta|^2 Q \]
\[ \leq \left( \frac{2S(n)(n-1)}{n-2} \right)^2 \left( 1 + \delta \right) \int_M \frac{n^2}{4} |\nabla |\Phi||^2 |\Phi|^{n-2} \eta^2 Q \]
\[ + \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^n |\nabla \eta|^2 Q \].

Applying (2.8) and notice that $\iota = 2$, we have, for $0 < \varepsilon < \frac{n}{2} - \frac{1}{2}$
\[ \kappa_2^{-1} \left[ \int_M (|\Phi|^n \eta)^{2n/(n-2)} \, d\mu \right]^{n-2/n} \leq \left\{ \frac{n^2}{4} \left( 1 + \delta \right) \left( 2 \int_M |\Phi|^{n+2} \eta^2 Q \right) \right. \]
\[ + \frac{2}{n} \int_M |\Phi|^n |H|^2 \eta^2 Q + \frac{1}{\varepsilon} \int_M |\Phi|^n |\nabla \eta|^2 Q \left. \right) + \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^n |\nabla \eta|^2 Q \right\}, \] (2.20)
where $\kappa_2 = \left( \frac{2S(n)(n-1)}{n-2} \right)^2$. By the fact that $|A|^2 = |\Phi|^2 + \frac{1}{n}|H|^2$, we can rewrite (2.20) as

$$
\kappa_2^{-1} \left[ \int_M \left( |\Phi|^{\frac{2}{n}} \right)^{\frac{n-2}{n}} \rho \right]^{\frac{n-2}{n}} \leq \frac{n^2 (1 + \delta)}{8 \left( \frac{n}{2} - 1 - \epsilon \right)} \left[ \int_M |\Phi|^n |A|^2 \rho^{\frac{2n}{n-2}} \right]^{\frac{n-2}{n}} \left[ \int_M |\Phi|^{\frac{2}{n}} |\nabla \eta|^2 \right]^{\frac{n-2}{n}} + \hat{C}(n, \delta, \epsilon) \int_M |\Phi|^n |\nabla \eta|^2 \rho,
$$

where $\hat{C}(n, \delta, \epsilon)$ is explicit positive constant depending on $n, \delta, \epsilon$. Applying Hölder inequality, we have

$$
\kappa_2^{-1} \left[ \int_M \left( |\Phi|^{\frac{2}{n}} \right)^{\frac{n-2}{n}} \rho \right]^{\frac{n-2}{n}} \leq \frac{n^2 (1 + \delta)}{8 \left( \frac{n}{2} - 1 - \epsilon \right)} \left( \int_M |A|^2 \rho^{\frac{2n}{n-2}} \right)^{\frac{n}{n-2}} \left( \int_M |\Phi|^{\frac{2}{n}} |\nabla \eta|^2 \right)^{\frac{n-2}{n}} + \hat{C}(n, \delta, \epsilon) \int_M |\Phi|^n |\nabla \eta|^2 \rho,
$$

here we used $|\Phi| \leq |A|$ in the last inequality. Put

$$
K_2(n, \delta, \epsilon) = \sqrt{\frac{8 \left( \frac{n}{2} - 1 - \epsilon \right)}{n^2 (1 + \delta) \kappa_2}},
$$

and

$$
K_2(n) = \sup_{\delta > 0, 0 < \epsilon < \frac{n-1}{n}} K_2(n, \delta, \epsilon) = \sqrt{\frac{(n-2)^2}{S(n)^2 (n-1)n^2}}.
$$

By the assumption

$$
\left( \int_M |A|^n \rho \right)^{\frac{1}{n}} < K_2(n)
$$

and using the same argument as Theorem 1.1, we complete the proof. \qed

Now, as mentioned in [6], an application of the maximum principle and the weighted version of a result in [1] give that translators with mean curvature that does not change sign are either $f$-stable (generalizing in particular Theorem 1.2.5 in [16], and Theorem 2.5 in [17]) or they split as the product of a line parallel to the translating direction and a minimal hypersurface in the orthogonal complement of the line. Note that, in this latter case, by Fubini’s theorem, the condition $|A| \in L^p(M_f)$ for some $p > 0$ is met if and only if $|A| \equiv 0$ (i.e. $M$ is a translator hyperplane). Moreover, to adapt the ideas in [19] for minimal surface, Ma and Miquel proved in [11] a refined Kato inequality on translating solitons as follows.
Lemma 2.6. ([11]) Let $M^n$ be a hypersurface immersed in $\mathbb{R}^{n+1}$ satisfying
\[ |\nabla A| \leq \frac{3n+1}{2n} |\nabla H|, \]
then we have
\[ |\nabla \Phi|^2 \geq \frac{n+1}{n} |\nabla \Phi|^2. \]

Note that on the translating soliton $M$, we have $|\nabla H| = \langle \nabla v, v \rangle = A(\cdot, v)$, so the condition becomes $|\nabla A| \leq \frac{3n+1}{2n} |A(\cdot, v)|$. Now, under these assumptions, we obtain the following result which can be considered as an improvement of Theorem 6 in [11].

Theorem 2.7. Let $x : M^{n \geq 2} \to \mathbb{R}^{n+1}$ be a translator with mean curvature which does not change sign. Suppose that
\[ |\nabla A| \leq \frac{3n+1}{2n} |\nabla H| \]
and the traceless second fundamental form of the immersion satisfies $|\Phi| \in L^p (\text{M}_f)$ for $p \in \left( 2 - \frac{2}{\sqrt{n}}, 2 + \frac{2}{\sqrt{n}} \right)$. Then $M$ is a hyperplane.

**Proof.** Since the curvature does not change sign, we may assume that $M$ is $f$-stable. Otherwise, $|A| \equiv 0$, so $M$ is a hyperplane. From the definition of the $f$-Laplacian operator and the Eq. (2.2), we have
\[ |\Phi| \Delta_f |\Phi| = |\nabla \Phi|^2 - |\nabla ||\Phi|^2 - |A|^2 |\Phi|^2. \]

By the Kato-type inequality in Lemma 2.6, this implies
\[ |\Phi| \Delta_f |\Phi| \geq \frac{1}{n} |\nabla |\Phi||^2 - |A|^2 |\Phi|^2. \tag{2.21} \]

Now, let $\eta$ be a smooth compactly supported function on $M$. For any $a > 1$, multiplying $|\Phi|^{a-1} \eta^2$ both sides of the (2.21) and integrating by parts with respect to the measure $e^{-f} d\mu$ on $M$ yield
\[ \int_M \eta^2 |\Phi|^{a} \Delta_f |\Phi| e^{-f} d\mu \geq \frac{1}{n} \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu - \int_M |A|^2 \eta^2 |\Phi|^{a+1} e^{-f} d\mu. \tag{2.22} \]

Since $\eta$ has compact support on $M$, by the Stokes theorem, it shows that
\[ \int_M \eta^2 |\Phi|^{a} \Delta_f |\Phi| e^{-f} d\mu = -\int_M \left( \nabla \left( \eta^2 |\Phi|^a \right), \nabla |\Phi| \right) e^{-f} d\mu \]
\[ = -\int_M \left( 2\eta |\Phi|^a \nabla \eta + a\eta^2 |\Phi|^{a-1} \nabla |\Phi|, \nabla |\Phi| \right) e^{-f} d\mu \]
\[ = -2 \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu - a \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu. \]
Substituting the above identity into (2.22), we obtain
\[-2 \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu - a \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu \]
\[\geq \frac{1}{n} \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu - \int_M |A|^2 \eta^2 |\Phi|^{a+1} e^{-f} d\mu, \]
or equivalently
\[(a + \frac{1}{n}) \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu \]
\[\leq 2 \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu + \int_M |A|^2 \eta^2 |\Phi|^{a+1} e^{-f} d\mu. \tag{2.23} \]
On the other hand, since $M$ satisfies the stability inequality, we have
\[\int_M |A|^2 \psi^2 e^{-f} d\mu \leq \int_M |\nabla \psi|^2 e^{-f} d\mu. \]
Replacing $\psi$ by $\eta|\Phi|^{a+1}$ in the above inequality gives
\[\int_M |A|^2 \eta^2 |\Phi|^{a+1} e^{-f} d\mu \leq \int_M |\nabla (\eta|\Phi|^{a+1})|^2 e^{-f} d\mu \]
\[= \int_M |\Phi|^{a+1} |\nabla \eta|^2 e^{-f} d\mu + (a + 1) \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu \]
\[+ \frac{(a + 1)^2}{4} \int_M |\Phi|^{a-1} |\nabla |\Phi||^2 \eta^2 e^{-f} d\mu. \tag{2.24} \]
Combining (2.23) and (2.24), we have
\[\left(a + \frac{1}{n} \right) \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu \leq 2 \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu + \int_M |\Phi|^{a+1} |\nabla \eta|^2 e^{-f} d\mu \]
\[+ (a + 1) \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu \]
\[+ \frac{(a + 1)^2}{4} \int_M |\Phi|^{a-1} |\nabla |\Phi||^2 \eta^2 e^{-f} d\mu. \]
Hence,
\[\left[a + \frac{1}{n} - \frac{(a + 1)^2}{4}\right] \int_M \eta^2 |\Phi|^{a-1} |\nabla |\Phi||^2 e^{-f} d\mu \leq \int_M |\Phi|^{a+1} |\nabla \eta|^2 e^{-f} d\mu \]
\[+ (a + 3) \int_M |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta e^{-f} d\mu. \tag{2.25} \]
From the Cauchy-Schwarz inequality and the inequality $xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2$ for all $\varepsilon > 0$, we see that
\[(a + 3) |\Phi|^a \langle \nabla \eta, \nabla |\Phi| \rangle \eta \leq |a + 3||\Phi|^a |\nabla \eta|| |\nabla |\Phi|| |\eta| \]
\[= |a + 3| \left(|\Phi|^{a-1} |\nabla |\Phi|| |\eta| \right) \left(|\Phi|^{a+1} |\nabla \eta| \right) \]
\[\leq \varepsilon |\Phi|^{a-1} |\nabla |\Phi||^2 \eta^2 + \frac{(a + 3)^2}{4\varepsilon} |\Phi|^{a+1} |\nabla \eta|^2. \tag{2.26} \]
Substituting (2.26) into (2.25), we get
\[ \left[ a + \frac{1}{n} - \frac{(a + 1)^2}{4} - \varepsilon \right] \int_M |\Phi|^{a-1} |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu \leq \left[ 1 + \frac{(a + 3)^2}{4\varepsilon} \right] \int_M |\Phi|^{a+1} |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu. \]

Now let \( p = a + 1 \). Then, the above inequality becomes
\[ \left[ p - 1 - \frac{p^2}{4} + \frac{1}{n} - \varepsilon \right] \int_M |\Phi|^{p-2} |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu \leq \left[ 1 + \frac{(p + 2)^2}{4\varepsilon} \right] \int_M |\Phi|^p |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu. \]

Next, we choose the number \( p \) to be
\[ p - 1 - \frac{p^2}{4} + \frac{1}{n} > 0, \]
or equivalently
\[ 2 - \frac{2}{\sqrt{n}} < p < 2 + \frac{2}{\sqrt{n}} = 2 \left( 1 + \sqrt{\frac{1}{n}} \right). \]

Hence, for \( 2 - \frac{2}{\sqrt{n}} < p < 2 + \frac{2}{\sqrt{n}} \), we can choose \( \varepsilon > 0 \) small such that there is a constant \( C > 0 \) depending on \( n, p \) such that
\[ \int_M |\Phi|^{p-2} |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu \leq C \int_M |\Phi|^p |\nabla \Phi|^2 \eta^2 e^{-f} \, d\mu. \]

Now, for some fixed point \( o \in M \) and \( R > 0 \), we choose a test function \( \eta \) satisfying \( \eta \in C^\infty(M), 0 \leq \eta \leq 1, \eta = 1 \text{ in } B_o(R), \eta = 0 \text{ outside } B_o(2R), \) and \( |\nabla \eta| \leq \frac{2}{R} \). Plugging \( \eta \) into the above inequality then letting \( R \) tend to infinity, we conclude that \( |\nabla |\Phi|| = 0 \), since \( |\Phi| \in L^p(M_f) \). Therefore, \( |\Phi| \) is constant. Note that a translating soliton is of Euclidean volume growth ([22]), this implies \( \Phi = 0 \) because \( |\Phi| \in L^p(M_f) \). Now, we apply the argument as in the Proof of Theorem 1.1 to conclude that \( M \) is a hyperplane.

As a consequence of this theorem, for \( p = 2 \) we obtain the following corollary which can be considered as an improvement of Theorem 6 by Ma and Miquel in [11].

**Corollary 2.8.** Let \( x : M^{n \geq 2} \to \mathbb{R}^{n+1} \) be a translator with mean curvature which does not change sign and
\[ |\nabla A| \leq \frac{3n + 1}{2n} |\nabla H|. \]
Suppose that the traceless second fundamental form of the immersion satisfies \( |\Phi| \in L^2(M_f) \). Then \( M \) is a hyperplane.
3. Vanishing result for weighted harmonic forms

In this section we give a Proof of Theorem 1.3.

Proof the Theorem 1.3. Let $\omega$ be $L^p_f$ harmonic 1-form on $M$, i.e.,

$$\Delta_f \omega = 0, \int_M |\omega|^p e^{-f} d\mu < \infty.$$  

We denote the dual vector field of $\omega$ by $\omega^\#$. Applying the extended Bochner formula for a $L^p_f$ harmonic 1-form, we get

$$\Delta_f |\omega|^2 = 2|\nabla \omega|^2 + 2\langle \Delta_f \omega, \omega \rangle + 2\text{Ric}_f (\omega^\#, \omega^\#) = 2|\nabla \omega|^2 + 2\text{Ric}_f (\omega^\#, \omega^\#).$$  

(3.1)

Note that

$$\Delta_f |\omega|^2 = 2|\omega| \Delta_f |\omega| + 2|\nabla |\omega||^2.$$  

and the Bakry-Emery Ricci tensor of $M$, satisfies

$$\text{Ric}_f (\omega^\#, \omega^\#) = -\left\langle A^2 \omega^\#, \omega^\# \right\rangle.$$  

This implies

$$|\omega| \Delta_f |\omega| = |\nabla \omega|^2 - |\nabla |\omega||^2 - \left\langle A^2 \omega^\#, \omega^\# \right\rangle.$$  

Consequently, by Kato’s inequality, we have

$$|\omega| \Delta_f |\omega| \geq -\left\langle A^2 \omega^\#, \omega^\# \right\rangle \geq -|A^2 \omega^\#| |\omega^\#| \geq -|A||\omega|^2.$$  

Now, let $\eta$ be a smooth compactly supported function on $M$. By multiplying both sides of the above inequality by $\eta^2|\omega|^{p-2}$ and then integrating the obtained result, we arrive at

$$\int_M \eta^2|\omega|^{p-1} \Delta_f |\omega| e^{-f} d\mu \geq -\int_M |A|^2 |\omega|^p \eta^2 e^{-f} d\mu.$$  

(3.2)

Since $\eta$ has compact support on $M$, by the Stokes theorem, we see that

$$\int_M \eta^2|\omega|^{p-1} \Delta_f |\omega| e^{-f} d\mu$$

$$= -\int_M \left\langle \nabla \left( \eta^2|\omega|^{p-1} \right), \nabla |\omega| \right\rangle e^{-f} d\mu$$

$$= -2\int_M |\omega|^{p-1} \langle \nabla \eta, \nabla |\omega| \rangle \eta e^{-f} d\mu - (p - 1) \int_M \eta^2|\omega|^{p-2} |\nabla |\omega||^2 e^{-f} d\mu.$$
This inequality and (3.2) implies

\[
(p - 1) \int_M \eta^2 |\omega|^{p-2} |\nabla |\omega||^2 e^{-f} d\mu \\
\leq -2 \int_M |\omega|^{p-1} (\nabla \eta, \nabla |\omega|) \eta e^{-f} d\mu + \int_M |A|^2 |\omega|^p \eta^2 e^{-f} d\mu. 
\]

(3.3)

By Hölder’s inequality and the weighted Sobolev inequality, we have

\[
\int_M |A|^2 |\omega|^p \eta^2 e^{-f} d\mu \\
\leq \left( \frac{2(C(n))}{n-1} \right)^2 \|A\|_{n,f}^2 \int_M \left| \nabla (\eta |\omega|^{\frac{p}{2}}) \right|^2 e^{-f} d\mu \\
= D_n \|A\|_{n,f}^2 \int_M \left| \omega |\nabla \eta|^2 + |\omega|^{p-1} (\nabla |\omega|, \nabla \eta) \eta + \frac{p^2}{4} |\omega|^{p-2} \eta^2 |\nabla |\omega||^2 \right| e^{-f} d\mu, 
\]

(3.4)

where \( D_n = \left( \frac{2(C(n))}{n-1} \right)^2 \). Using the Cauchy-Schwarz inequality and the inequality \( xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2 \) for any \( \varepsilon > 0 \), we see that

\[
p|\omega|^{p-1} (\nabla |\omega|, \nabla \eta) \eta \leq p |\omega|^{p-1} |\nabla |\omega|| |\nabla \eta|| \eta| \\
= \frac{|\nabla \eta|^2 |\omega|^p}{\varepsilon} + \frac{\varepsilon p^2}{4} |\omega|^{p-2} \eta^2 |\nabla |\omega||^2.
\]

This together with (3.4) implies

\[
\int_M |A|^2 |\omega|^p \eta^2 e^{-f} d\mu \\
\leq D_n \|A\|_{n,f}^2 \left[ \left( 1 + \frac{1}{\varepsilon} \right) \int_M |\omega|^{p} |\nabla \eta|^2 e^{-f} d\mu + \frac{(1+\varepsilon)p^2}{4} \int_M |\omega|^{p-2} \eta^2 |\nabla |\omega||^2 e^{-f} d\mu \right] \\
= \left( 1 + \frac{1}{\varepsilon} \right) D_n \|A\|_{n,f}^2 \int_M |\omega|^p |\nabla \eta|^2 e^{-f} d\mu \\
+ \frac{(1+\varepsilon)p^2}{4} D_n \|A\|_{n,f}^2 \int_M |\omega|^{p-2} \eta^2 |\nabla |\omega||^2 e^{-f} d\mu. 
\]

(3.5)

On the other hand, for any \( \varepsilon > 0 \), we have

\[
-2 \int_M |\omega|^{p-1} (\nabla \eta, \nabla |\omega|) \eta e^{-f} d\mu \leq 2 \int_M |\omega|^{p-1} |(\nabla \eta, \nabla |\omega|)|| \eta| e^{-f} d\mu \\
\leq \frac{1}{\varepsilon} \int_M |\nabla \eta|^2 |\omega|^p e^{-f} d\mu + \varepsilon \int_M |\omega|^{p-2} |\nabla |\omega||^2 \eta^2 e^{-f} d\mu. 
\]

(3.6)
Combining (3.3), (3.5), and (3.6), we get
\[
\begin{align*}
\left[ p - 1 - \frac{p^2}{4} D_n \| A \|^2_{n,f} - \frac{\varepsilon p^2}{4} D_n \| A \|^2_{n+1,f} + \varepsilon \right] \int_M \eta^2 |\omega|^{p-2} |\nabla| |\omega|^{2e^{-f}} d\mu & \\
\leq \left[ \left( 1 + \frac{1}{\varepsilon} \right) D_n \| A \|^2_{n,f} + \frac{1}{\varepsilon} \right] \int_M |\omega|^p |\nabla\eta|^2 e^{-f} d\mu.
\end{align*}
\]

For a sufficiently small \( \varepsilon > 0 \), the above inequality implies that there is a constant \( C > 0 \) such that
\[
\int_M |\omega|^{p-2} |\nabla| |\omega|^{2e^{-f}} d\mu \leq C \int_M |\omega|^p |\nabla\eta|^2 e^{-f} d\mu,
\]
provided that
\[
p - 1 - \frac{p^2}{4} D_n \| A \|^2_{n,f} > 0,
\]
or equivalently
\[
\| A \|^2_{n,f} < \frac{4 (p - 1)}{p^2 D_n} = \frac{(p - 1) (n - 1)^2}{p^2 C^2 (n)}.
\]

Let \( o \in M \) be a fixed point and let \( B_r(o) \) be the geodesic ball centered at \( o \) with radius \( R \). We choose \( \eta \) to be a smooth function on \( M \) such that \( 0 \leq \eta \leq 1 \).

Moreover, \( \eta \) satisfies:

(i) \( \eta = 1 \) on \( B_{R/2}(o) \) and \( \eta = 0 \) outside \( B_R(o) \);

(ii) \( |\nabla\eta| \leq \frac{2}{R} \).

Applying this test function \( \eta \) to (3.7), we get
\[
\int_{B_R(o)} |\omega|^{p-2} |\nabla| |\omega|^{2e^{-f}} d\mu \leq \frac{4C}{R^2} \int_{B_R(o)} |\omega|^p e^{-f} d\mu.
\]

Letting \( R \) tend to \( \infty \) in the above inequality and noting that \( \omega \in L^f_p \), we conclude that \( |\nabla| |\omega| = 0 \), which shows that \( |\omega| \) is a constant. Moreover, since \( \int_M |\omega|^p e^{-f} d\mu < \infty \) and the weighted volume of \( M \) is infinite, we finally get \( \omega = 0 \). The proof is complete. \( \square \)

Now, we note that if a Sobolev inequality holds true on \( M \) every end of \( M \) is non-\( f \)-parabolic, for example see [6]. Therefore, we have the following corollary.

**Corollary 3.1.** Let \( x : M^n \to \mathbb{R}^{n+1} \) be a smooth complete translator contained in the halfspace \( \Pi_{v,a} = \{ y \in \mathbb{R}^{n+1} : \langle y, v \rangle \geq a \} \), where \( a \in \mathbb{R} \), \( v \in \mathbb{R}^{n+1} \) are fixed and \( n \geq 3 \). Furthermore, assume that
\[
\| A \|_{n,f} \leq \frac{n - 1}{2S(n)},
\]
where \( S(n) \) is the constant as in Lemma 2.5. Then there are no nontrivial \( L^2_f \) harmonic 1-forms on \( M \). In particular, \( M \) has only one end.
Proof. Since every end of $M$ is non-$f$-parabolic, we can argue by contradiction to assume that $M$ has at least two ends. Then by Li-Tam [10], there exists a non-constant $f$-harmonic function $u$ such that $\omega := du$ satisfying $|\omega| \in L^2_f$. An application of Theorem 1.3 implies that $\omega = 0$ or $u$ is constant. This is a contradiction. The proof is complete. \qed

4. Translators with a Sobolev inequality

Suppose that $M$ satisfies the following Sobolev inequality

$$\left[ \int_M u^{\frac{2(n+1)}{n-1}} \rho d\mu \right]^{\frac{n-1}{n+1}} \leq \left( \frac{2C(n)n}{n-1} \right)^2 \int_M |\nabla u|^2 \rho d\mu$$

(4.1)

for any $u$ that is a non-negative compactly supported $C^1$ function on $M$ and $C(n)$ is the Sobolev constant. In fact, the above inequality was proved in [6]. However, the authors pointed out in [7] that there is a gap in their proof of this inequality. Here, we assume that this inequality holds true. The Sobolev inequality (4.1) was used by Kunikawa and Saito in [9] to study the injectivity of the natural map between the first de Rham cohomology group with compact support, the reduced $L^2_f$ cohomology, and the space of $L^2_f$-$f$-harmonic 1-forms. They proved that if $M$ supports the Sobolev inequality (4.1) and admits a codimension one cycle which does not disconnect $M$ then the space of $L^2_f$-$f$-harmonic 1-forms is non-trivial.

Now, we apply the above Sobolev’s inequality above to $u = |\Phi|^a \eta$. Then we have

$$\left[ \int_M (|\Phi|^a \eta)^{\frac{2(n+1)}{n-1}} \rho \right]^{\frac{n-1}{n+1}} \leq \left( \frac{2C(n)n}{n-1} \right)^2 \int_M \left| \nabla (|\Phi|^a \eta) \right|^2 \rho$$

$$= \left( \frac{2C(n)n}{n-1} \right)^2 \left( \int_M a^2 |\nabla |\Phi||^2 |\Phi|^{2a-2} \eta^2 \mathcal{Q} \right)$$

$$+ \int_M 2a |\Phi|^{2a-1} \eta \nabla |\Phi| \cdot \nabla \eta \mathcal{Q} + \int_M |\Phi|^{2a} |\nabla \eta|^2 \mathcal{Q}$$

(4.2)

By the Cauchy inequality, we obtain

$$\left[ \int_M (|\Phi|^a \eta)^{\frac{2(n+1)}{n-1}} \rho \right]^{\frac{n-1}{n+1}} \leq \left( \frac{2C(n)n}{n-1} \right)^2 \left( \int_M a^2 |\nabla |\Phi||^2 |\Phi|^{2a-2} \eta^2 \mathcal{Q} \right)$$

$$+ \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^{2a} |\nabla \eta|^2 \mathcal{Q}.$$  

(4.3)
Apply (2.8) and keep in mind that right now \( \iota = 2 \). For \( 0 < \varepsilon < a - \frac{1}{2} \), we have

\[
\kappa_1^{-1} \left[ \int_M (|\Phi|^{\iota} \eta)^{\frac{2(n+1)}{n+1-1}} \rho \right]^{\frac{n-1}{n+1}} \leq \left\{ \begin{array}{l}
\frac{a^2(1+\delta)}{4(a - \frac{1}{2} - \varepsilon)} \left( 2 \int_M |\Phi|^{2a+2} \eta^2 \right) \\
\frac{2}{n} \int_M |\Phi|^{2a} |H|^2 \eta^2 + \frac{1}{\varepsilon} \int_M |\Phi|^{2a} |\nabla \eta|^2 \\
+ \left( 1 + \frac{1}{\delta} \right) \int_M |\Phi|^{2a} |\nabla \eta|^2 \end{array} \right. 
\]

where \( \kappa_1 = \left( \frac{2C(n)n}{n-1} \right)^2 \).

Using the fact that \(|A|^2 = |\Phi|^2 + \frac{1}{n} |H|^2\), we can rewrite (4.4) as

\[
\kappa_1^{-1} \left[ \int_M (|\Phi|^{\iota} \eta)^{\frac{2(n+1)}{n+1-1}} \rho \right]^{\frac{n-1}{n+1}} \leq \frac{a^2(1+\delta)}{2(a - \frac{1}{2} - \varepsilon)} \int_M |\Phi|^{2a} |A|^2 \eta^2 \\
+ \tilde{C}(n, a, \delta, \varepsilon) \int_M |\Phi|^{2a} |\nabla \eta|^2 \eta, 
\]

where \( \tilde{C}(n, a, \delta, \varepsilon) \) is an explicit positive constant depending on \( n, a, \delta, \varepsilon \).

By the Hölder’s inequality, we have that

\[
\int_M |\Phi|^{2a} |A|^2 \eta^2 \leq \left( \int_M |A|^2 \frac{n+1}{n+1-1} \eta \right)^\frac{n+1}{n+1-1} \left( \int_M (|\Phi|^{2a} \eta)^\frac{n+1}{n+1-1} \right)^\frac{n+1-1}{n+1} 
\]

\[
= \left( \int_M |A|^{n+1} \right)^\frac{n+1}{n+1-1} \left( \int_M (|\Phi|^{2a} \eta)^\frac{2(n+1)}{n+1} \right)^\frac{n+1-1}{n+1} .
\]

Our goal is to decrease the number of conditions in Theorem 1.1, only one condition instead of two as in Theorem 1.1, so we should choose \( a = \frac{n+1}{2} \). For that reason, combining (4.5) and (4.6), we have

\[
\kappa_1^{-1} \left[ \int_M (|\Phi|^{\iota} \eta)^{\frac{2(n+1)}{n+1-1}} \rho \right]^{\frac{n-1}{n+1}} \leq \frac{(n+1)^2(1+\delta)}{8\left(\frac{n+1}{2} - \frac{1}{2} - \varepsilon\right)} \left( \int_M |A|^{n+1} \right)^\frac{n+1}{n+1-1} \left( \int_M (|\Phi|^{\iota} \eta)^{\frac{2(n+1)}{n+1}} \right)^\frac{n+1-1}{n+1} \\
+ \tilde{C}(n, \delta, \varepsilon) \int_M |\Phi|^{n+1} |\nabla \eta|^2 \eta.
\]

Put

\[
K_1(n, \varepsilon, \delta) = \sqrt{\frac{8\left(\frac{n+1}{2} - \frac{1}{2} - \varepsilon\right)}{(n+1)^2(1+\delta)\kappa_1}},
\]

and

\[
K_1(n) = \sup_{\delta > 0, 0 < \varepsilon < a - \frac{1}{2}} K_1(n, \varepsilon, \delta) = \sqrt{\frac{(n-1)^2}{C(n)^2(n+1)^2n^2}}.
\]

Applying the argument as in the Proof of Theorem 1.1, we have the following result.
Theorem 4.1. Let $M^n (n \geq 3)$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ with Sobolev inequality (4.1). If the second fundamental form $A$ of $M$ satisfies

$$\left( \int_M |A|^{n+1} q \right)^{\frac{1}{n+1}} < K_1(n),$$

where $K_1(n)$ is defined as above, then $M$ is a hyperplane.

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Data availability All data generated or analysed during this study are included in this article.

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Fischer-Colbrie, D., Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. Comm. Pure Appl. Math. 33(2), 199–211 (1980)
[2] Guang, Q.: Volume growth, entropy and stability for translating solitons. Comm. Anal. Geom. 27(1), 47–72 (2019)
[3] Haslhofer, R.: Uniqueness of the bowl soliton. Geom. Topol. 19(4), 2393–2406 (2015)
[4] Hoffman, D., Spruck, J.: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Comm. Pure Appl. Math. 27, 715–727 (1974)
[5] Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math. 183(1), 45–70 (1999)
[6] Impera, D., Rimoldi, M.: Rigidity results and topology at infinity of translating solitons of the mean curvature flow. Commun. Contemp. Math. 19(6), 1750002 (2017)
[7] Impera, D., Rimoldi, M.: Quantitative index bounds for translators via topology. Math. Zeits. 292(1–2), 513–527 (2019)
[8] Kim, D.H., Pyo, J.C.: Translating solitons foliated by spheres. Internat. J. Math. 28(1), 1750006 (2017)
[9] Kunikawa, K., Saito, S.: Remarks on topology of stable translating solitons. Geom. Dedicata 202, 1–8 (2019)
[10] Li, P., Tam, L.F.: Harmonic functions and the structure of complete manifolds. J. Diff. Geom. 35(2), 359–383 (1992)
[11] Ma, L., Miquel, V.: Bernstein theorem for translating solitons of hypersurfaces. Manuscripta Math. 162, 115–132 (2020)
[12] Martin, F., Savas-Halilaj, A., Smoczyk, R.: On the topology of translating solitons of the mean curvature flow. Calc. Var. 54, 2853–2882 (2015)
[13] Michael, J.H., Simon, L.M.: Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. Comm. Pure Appl. Math. 26, 361–379 (1973)
[14] Nguyen, X.H.: Doubly periodic self-translating surfaces for the mean curvature flow. Geom. Dedicata 174, 177–185 (2015)
[15] Nguyen, X.H.: Complete embedded self-translating surfaces under mean curvature flow. J. Geom. Anal. 23, 1379–1426 (2013)
[16] Shahriyari, L.: Translating Graphs by Mean Curvature Flow (ProQuest LLC, Ann Arbor, MI, 2013); Ph.D. thesis, The Johns Hopkins University
[17] Shahriyari, L.: Translating graphs by mean curvature flow. Geometriae Dedicata 175(1), 57–64 (2014)
[18] Smoczyk, K.: A relation between mean curvature flow solitons and minimal submanifolds. Math. Nachr. 229, 175–186 (2001)
[19] Schoen, R., Simon, L., Yau, S.Y.: Curvature estimates for minimal hypersurfaces. Acta Math. 134, 275–288 (1975)
[20] Wang, H.J., Xu, H.W., Zhao, E.T.: A global pinching theorem for complete translating solitons of mean curvature flow. Pure Appl. Math. Q. 12(4), 603–619 (2016)
[21] Wang, X.J.: Convex solutions to the mean curvature flow. Ann. Math. (2) 173(3), 1185–1239 (2011)
[22] Xin, Y.L.: Translating solitons of the mean curvature flow. Calc. Var. Partial Differ. Equ. 54, 1995–2016 (2015)
[23] Xu, H.W., Gu, J.R.: A general gap theorem for submanifolds with parallel mean curvature in $\mathbb{R}^{n+m}$. Comm. Anal. Geom. 15, 175–194 (2007)

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