$SO(5)$ structure of p-wave superconductivity for spin-dipole interaction model

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Abstract

A closed $SO(5)$ algebraic structure in the mean-field form of the Hamiltonian for pure p-wave superconductivity is found that can help to diagonalized by making use of the Bogoliubov rotation instead of the Balian-Werthamer approach. We point out that the eigenstate is nothing but $SO(5)$-coherent state with fermionic realization. By applying the approach to the Hamiltonian with dipole interaction of Leggett the consistency between the diagonalization and gap equation is proved through the double-time Green function. The relationship between the s-wave and p-wave superconductivities turns out to be recognized through Yangian algebra, a new type of infinite-dimensional algebra.

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The p-wave superconductivity theories and their applications to liquid $^3$He have intensively been studied in many initial literatures, for example, in Refs[1]-[5]. As was pointed out by Anderson and Brinkman[1] that the BW formalism[6] underlies all the further models in the field. The Hamiltonian takes the Aderson reduced form: $H = H_0 + V$

$$H_0 = \sum_{\mathbf{k},\alpha} \epsilon_\mathbf{k} n_{\alpha\mathbf{k}}, \quad V = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\alpha,\beta} V_{\mathbf{k}\mathbf{k}'} a^+_{\mathbf{k}'\alpha} a^{-}_{-\mathbf{k}'\beta} a^{-}_{-\mathbf{k}\beta} a_{\mathbf{k}\alpha}$$

(1)

where $\epsilon_\mathbf{k} = \frac{k^2}{2m} - \mu$, $\alpha, \beta = \uparrow, \downarrow$, and for p-wave $V_{\mathbf{k}\mathbf{k}'} = -3V_1(k, k')n \cdot n'$ ($n = \frac{k}{k}$). To explain the p-pair interaction Leggett[2] introduced an useful algebra, but not closed. Meanwhile, the dipole type of interaction was proposed that naturally distinguishes the energy difference for ABM and BW phases[2] -[4] and gives the correct spin dynamics. All the theories look working perfectly, but with growing interest of the applications of the currently algebraic method there still desirable efforts could be made to yield further understanding. In this paper we would like to show the following results. a) The set obeying by $S_i(\mathbf{k}) = a^+_{\mathbf{k}\alpha}(\sigma_i)_{\alpha\beta} a_{\mathbf{k}\beta}$ and $T_i(\mathbf{k}) = a_{-\mathbf{k}\alpha}(\sigma_i)_{\alpha\beta} a_{\mathbf{k}\beta}$ as well as their conjugates[2] forms $SO(5)$ algebra ($i = 0, 1, 2, 3, \sigma_0 = 1$ and summation over the repeat $\alpha$ and $\beta$) that is larger than the usual $U(1) \otimes SU(\mathbf{L}) \otimes SU(\mathbf{S})$ as shown in Ref.[3]. Equipping with the algebraic structure for the lower pair excitation we then diagonalize the Hamiltonian eq.(1) that together with the invariance $U(1) \otimes SU(\mathbf{L}) \otimes SU(\mathbf{S})$ by using the algebraic average method (AAM, general Bogoliubov rotation) yield all the known results. b) We show that the eigenfunction of eq.(1) is $SO(5)$-coherent state with fermionic realization, hence the coherence property related to eq.(1) origins in the closed $SO(5)$ structure. c) The above calculation can be applied to the dipole interaction Hamiltonian of Leggett. There appears nontrivial consistency between the diagonalization of microscopic-form of $H_D$ by AAM and the gap equation. d) Finally, in difference from the $SO(5)$ unification of Zhang et al[8][9] we attempt to find what price we have to pay in order to form an assumed unification involving both s-wave and p-wave superconductivities (as shown in Ref. [2]), in stead of the claimed transition between s-wave superconductivity and AF phase shown in Refs. [3][4].

1) Observing the algebra defined in Ref.[2], if $T_0(\mathbf{k})$ is picked up, it becomes closed. Defining $\bar{S}(\mathbf{k}) = \frac{1}{2}[S(\mathbf{k}) + S(-\mathbf{k})]$, $Q(\mathbf{k}) = \frac{1}{2}(S_0(\mathbf{k}) + S_0(-\mathbf{k}) - 2)$ it can be checked that the set $(Q(\mathbf{k}), \bar{S}(\mathbf{k}), T(\mathbf{k}), T^\dagger(\mathbf{k}))$ forms $SO(5)$ algebra:

$$[I_{ab}(\mathbf{k}), I_{cd}(\mathbf{k}')] = -i\delta(\mathbf{k} - \mathbf{k}') \delta_{ac} I_{bd}(\mathbf{k}) + \delta_{bd} I_{ac}(\mathbf{k}) - \delta_{ad} I_{bc}(\mathbf{k}) - \delta_{bc} I_{ad}(\mathbf{k})$$
where $I_{ab}(\mathbf{k}) = -I_{ba}(\mathbf{k})$ ($a, b = 1, 2, 3, 4, 5$) takes the form:

$$
\begin{pmatrix}
0 \\
-\frac{1}{2}(T^+_x(\mathbf{k}) + T_x(\mathbf{k})) \\
-\frac{1}{2}(T^+_y(\mathbf{k}) + T_y(\mathbf{k})) \\
-\frac{1}{2}(T^+_z(\mathbf{k}) + T_z(\mathbf{k})) \\
Q(\mathbf{k})
\end{pmatrix}
\begin{pmatrix}
0 \\
\bar{S}_z(\mathbf{k}) \\
0 \\
\frac{1}{2i}(T_y(\mathbf{k}) - T_y(\mathbf{k})) \\
\frac{1}{2i}(T_z(\mathbf{k}) - T_z(\mathbf{k}))
\end{pmatrix}
$$
(2)

The Hamiltonian eq.(1) can then be written in the form:

$$
H = \sum_{\mathbf{k}} \epsilon_k(Q(\mathbf{k}) + 1) + \frac{1}{4} \sum_{\mathbf{k}, \mathbf{k}'} V_{\mathbf{k}'\mathbf{k}} T^\dagger(\mathbf{k}) \cdot T(\mathbf{k}')
$$
(3)

Noting that the mean-field approximation is enough to obtain gap equation since we work at the equilibrium state. Using eq.(10)

$$
AB \simeq A\langle B \rangle + \langle A \rangle B - \langle A \rangle \langle B \rangle
$$
eq
(4)

eq.(3) can be linearized with respect to $SO(5)$ generators

$$
H_{mf} = \sum_{\mathbf{k}} \{H(\mathbf{k}) - E_\ast(\mathbf{k})\}
$$
with

$$
H(\mathbf{k}) = \epsilon_k Q(\mathbf{k}) + \Delta(\mathbf{k}) \cdot T^\dagger(\mathbf{k}) + \Delta^\ast(\mathbf{k}) \cdot T(\mathbf{k})
$$
(5)

$$
E_\ast(\mathbf{k}) = \epsilon_k - \Delta(\mathbf{k}) \cdot \langle T^\dagger(\mathbf{k}) \rangle
$$
(6)

where $\Delta(\mathbf{k}) = \frac{1}{4} \sum_{\mathbf{k}'} V_{\mathbf{k}'\mathbf{k}} \langle T(\mathbf{k}) \rangle$ and $\langle ... \rangle$ represents the average over both quantum states and thermodynamics. We emphasized that the set $\Lambda = \{ \frac{1}{\sqrt{2}} T_3(\mathbf{k}), \frac{-1}{\sqrt{2}} T_3(\mathbf{k}), -Q(\mathbf{k}) \}$ i.e. $\{-i\sqrt{2}\pi_2, i\sqrt{2}\pi_2, -Q\}$ in Refs. [5] [6] forms the quasi-spin $\Lambda$ for pairs. $\Lambda$ does not commute with spin operators $\mathbf{S}(\mathbf{k})$ that give rise to $T^\dagger_\pm(\mathbf{k})$ and $T_\pm(\mathbf{k})$ which are beyond two $SU(2)$'s and the total set forms $SO(5)$. In order to make the diagonalization of eq.(4) we introduce the unitary transformation such that $W(\xi_\mathbf{k})H(\mathbf{k})W(\xi_\mathbf{k})$ becomes diagonal for any $\mathbf{k}$. Following the general strategy[7] we introduce the $SO(5)$-coherent operator:

$$
W(\xi_\mathbf{k}) = \exp\{\xi_\mathbf{k} [\mathbf{d}(\mathbf{n}) \cdot T(\mathbf{k})] - h.c.\}
$$
(7)

where $\xi_\mathbf{k} = r_\mathbf{k} e^{i\lambda_\mathbf{k}}, \mathbf{d}(\mathbf{n}) = (\sin \Theta_\mathbf{k} \cos \Phi_\mathbf{k}, \sin \Theta_\mathbf{k} \sin \Phi_\mathbf{k}, \cos \Theta_\mathbf{k})$, $\Theta_\mathbf{k}$ and $\Phi_\mathbf{k}$ are angulars in spin space for a given momentum $\mathbf{k}$. $\lambda_\mathbf{k}$ is a parameter to be determined by the gap equation. Taking the commutation relations for $SO(5)$ into account after lengthy but elementary calculations we derive

$$
W(\xi_\mathbf{k})^{-1} H(\mathbf{k}) W(\xi_\mathbf{k}) = -E_k Q(\mathbf{k}), \quad E_k = \sqrt{\epsilon_k^2 + |\Delta(\mathbf{k})|^2}
$$
where
\[ \tan 2r_k = \frac{2\epsilon_k}{\Delta(k)}, \quad \Delta(k) = \frac{1}{4} \sum_{k'} V_{kk'} \langle T(k') \rangle = -\frac{1}{2} \Delta(k) e^{i\lambda k} d(n) \]  
(8)

The eigenstate is given by
\[ |\xi\rangle = \otimes_k |\xi_k\rangle, \quad |\xi_k\rangle = W(\xi_k)|\text{vac}\rangle \]  
(9)

At temperature \( T = 0 \), the vacuum state is \( |\text{vac}\rangle = |0, 0\rangle_k \equiv |n_\alpha = 0, n_{-\alpha} = 0\rangle \). The expectation value \( \langle T(k) \rangle = \langle \xi_k | T(k) | \xi_k \rangle = \sin 2r_k e^{i\lambda k} d(n) \) yields the well-known gap equation at \( T = 0 \):
\[ \Delta(k) = -\sum_{k'} V_{kk'} \frac{\Delta(k')}{2E_{k'}} \tanh\left(\frac{1}{2} \beta E_{k'}\right) \]  
(10)

To satisfy eq.(10) we simply choose \( \lambda_k = \lambda = \text{constant henceforth}, \) for finite temperature, making use of the double-time Green function we obtain
\[ \langle T(k) \rangle = \langle \xi_k | T(k) | \xi_k \rangle = \sin 2r_k e^{i\lambda} \tanh\left(\frac{1}{2} \beta E_k\right) d(n) \]  
where \( \beta = \frac{1}{kT} \) and \( k \) is the Boltzmann constant. Therefore the gap equation reads
\[ \Delta(k) = -\sum_{k'} V_{kk'} \frac{\Delta(k')}{2E_{k'}} \tanh\left(\frac{1}{2} \beta E_{k'}\right) \]  
(11)

The \( SO(5) \) coherent state \( |\xi_k\rangle \) eq.(9) gives
\[ |\xi_k\rangle = W(\xi_k)|0, 0\rangle = \cos^2 r_k |0, 0\rangle - e^{i2\lambda} \sin^2 r_k |\uparrow\downarrow, \uparrow\downarrow\rangle \]
\[ + \frac{i}{2} e^{i\lambda} \sin 2r_k \{ \cos \Theta_k (|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle) - \sin \Theta_k e^{-i\Phi_k} |\uparrow, \uparrow\rangle + \sin \Theta_k e^{i\Phi_k} |\downarrow, \downarrow\rangle \} \]  
(12)

Let us distinguish two cases: In the BW phase, to satisfy the gap equation it holds \( d(n) = n, \ \Theta_k = \theta_k \) and \( \Phi_k = \psi_k \) that correspond intuitively to Cooper pair with total angular momentum \( J = 0 \) and has an isotropic gap, \( |\Delta(k)| e^{i\lambda} = c\)-number. The wave function of the BW solution in the conventional notation reads
\[ |\xi_k\rangle = \frac{E_k + \epsilon_k}{2E_k} |0, 0\rangle - e^{i2\lambda} \frac{E_k - \epsilon_k}{2E_k} |\uparrow\downarrow, \uparrow\downarrow\rangle \]
\[ - e^{i\lambda} \frac{i|\Delta(k)|}{2E_k} \left[ \frac{8\pi}{3} \{ Y_{11} |\downarrow, \downarrow\rangle - \frac{1}{\sqrt{2}} Y_{10} (|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle) + Y_{01} |\uparrow, \uparrow\rangle \} \right] \]  
(13)

In the AM case, there is another solution of the gap equation by taking \( |\Delta(k)| e^{i(\lambda + \frac{\pi}{2})} = Y_{11} \) and \( \sin \Theta_k = 0 \), it is non-ESP state:
\[ |\xi_k\rangle = \frac{E_k + \epsilon_k}{2E_k} |0, 0\rangle + e^{i2\lambda} \frac{E_k - \epsilon_k}{2E_k} |\uparrow\downarrow, \uparrow\downarrow\rangle \]
\[ + \frac{Y_{11}}{2E_k} (| \uparrow, \downarrow \rangle + | \downarrow, \uparrow \rangle) \]  

However, the solution for \( \cos \Theta_k = 0 \) appears only under an applied magnetic field. For instance, when \( \mathbf{B} = \mu \mathbf{B} e_z \), the Hamiltonian becomes \( H_B = \sum_k H(k) - \mu B \sum_k (a_k^+ a_k - a_k^+ a_k^+) \) and \( \mathcal{W} H B \mathcal{W} = \frac{1}{2} E_k (1 + \frac{\mu B}{\epsilon_k}) (n_{k\uparrow} + n_{k\downarrow}) + \frac{1}{2} E_k (1 - \frac{\mu B}{\epsilon_k}) (n_{k\uparrow} + n_{k\downarrow}) - E_k \). Through the double-time Green-function the non-vanishing components of \( \Delta \) are \( \Delta_{1\uparrow}(k) = \frac{i}{4} \sum_k' V_{kk'} | \Delta(k') | e^{i \lambda} e^{i \Phi_k} \frac{\tanh[\frac{1}{2} \beta E_{k'} (1 - \frac{\mu B}{\epsilon_{k'}})]}{\frac{1}{2} T_{\pm}(k') \frac{1}{2} T_{\pm}(k') \frac{1}{2} T_{\pm}(k')} \) and \( \Delta_{1\downarrow}(k) \) obtained by changing in \( \Delta_{1\uparrow}(k) \) by \( \pi + \lambda, - \Phi_{k'} \) and \( -B \), then times (-1). The eigenstate is \( \sim Y_{11} (| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle) \). If we study s-superconductivity by AAM, then coherence comes from the \( SU(2) \) coherent operators in Ref.[7]. The AAM is exactly the usual Bogoliubov transformation.

2) The dipole interaction was proposed to describe the spin dynamics for liquid \( ^3H_e \). The computation in Ref.[2] is based on \( \langle T(k) \rangle = \sum_{i=1}^3 T_i n_i \). However, in the microscopic-form of dipole interaction Hamiltonian

\[ H_D = \frac{2\pi \gamma^2}{3} \sum_{kk'} V_{kk'} (T^i(k) \cdot T(k') - 3 \hat{q} \cdot T^i(k) \hat{q} \cdot T(k')) \]  

where \( \hat{q} \) is a unit vector along \( \mathbf{n} - \mathbf{n}' \). The average formula shown in Ref.[2] can no longer be used, since the operator cannot be expended by \( n_i \). However, the \( SO(5) \) AAM procedure works for eq.(13) with the redefinition \( \Delta_i = \sum_{j=1}^3 (\delta_{ij} - 3 \hat{q}_i \hat{q}_j) \langle T_j \rangle \). Repeating the process in 1) we find that the \( \Delta(k) \) satisfies the same gap equation eq.(11) and the eigenstates take the same form for given \( \Delta(k) \) satisfying eq.(11), i.e., eq.(13) can be diagonalized with the following relation instead of eq.(8):

\[
\begin{pmatrix}
\Delta_{1\uparrow}(k) \\
\sqrt{2} \Delta_{1\downarrow}(k) \\
\Delta_{1\downarrow}(k)
\end{pmatrix}
= \frac{4\pi \gamma^2}{3} \sum_{kk'} V_{kk'} \left( \frac{1}{2} I \right)

+ \frac{3}{2} D^{ij=1}(\alpha = \psi_{kk'}, \beta = 2\omega_{kk'}, \gamma = \pi - \psi_{kk'}) \begin{pmatrix}
\frac{1}{2} T_{-}(k') \\
\frac{1}{\sqrt{2}} T_{\pm}(k') \\
\frac{1}{\sqrt{2}} T_{+}(k')
\end{pmatrix}
\]  

where the \( D^{ij=1}(\alpha, \beta, \gamma) \) is the Wigner rotation function with the Euler angles \( \alpha, \beta \) and \( \gamma \) and \( \hat{q} = q(\sin \omega_{kk'} \cos \psi_{kk'}, \sin \omega_{kk'} \sin \psi_{kk'}, \cos \omega_{kk'}) \). The relations for \( \omega_{kk'}, \psi_{kk'} \) and \( \alpha, \beta, \gamma \) have been indicated in eq.(10). The \( H_D \) works well in spin dynamics, but the consistency condition for the diagonalization of \( H_D \) and gap equation, in our knowledge, have not been proved before. Now the eq.(8) is replaced by eq.(13) for the Hamiltonian eq.(13). It means that all the discussion in 1) can be transplanted literally for \( \Delta(k) \).
3) It seems that the $T_{\pm}$ (or $\sim \pi_{\pm}$ in Refs.[8][9]) in $SO(5)$ may give rise to the transition between superconductivity and AF state based on the argument in Refs.[8][9]. However, it is not the case in the present model. This is not only because there is not $SO(5)$ invariance for $H$ or $H_D$, but also for deeper reason. Observing the gap equation for $V_{kk'} \sim P_0$ (constant) and $V_{kk'} \sim P_1 \sim n \cdot n'$, the corresponding wave function $\Psi_0 \sim Y_{00} \chi_{00}$ and $\Psi_1 \sim $ eq.(13) where $\chi_{00}$ is spin singlet. In our case, the generators of $SO(5)$ work only within p-supeconductivity, i.e. nothing with s-superconductivity. If we assume there is transition between $\Psi_0$ and $\Psi_1$, i.e. the form of gap equation is preserved, but with the different potentials $\sim P_0$ and $P_1$, the relationship occurs between two states:

$$\Psi_0 \sim \frac{Y_{00}}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Psi_1 \sim \frac{1}{\sqrt{3}} \left( \begin{pmatrix} Y_{11} \\ -\frac{1}{\sqrt{2}} Y_{10} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} Y_{11} \\ -\frac{1}{2} Y_{10} \end{pmatrix} \right) = \frac{1}{\sqrt{8\pi}} \begin{pmatrix} \hat{k}_+ \hat{k}_- \\ -\hat{k}_z \end{pmatrix}$$

(17)

The connection should be beyond Lie algebra. We find that such a "transition" may be performed through Yangian[11][13]. Actually,

$$\hat{k}_+ J_\pm \Psi_0 = (\mu_2 - \mu_1 + \frac{h}{2}) Y_{1\pm 1} \chi_{1\mp 1}$$

$$\hat{k}_z J_z \Psi_0 = -\frac{1}{2} (\mu_2 - \mu_1 + \frac{h}{2}) Y_{10} \chi_{10}$$

(18)

where $J_\alpha = \mu_1 S_\alpha \otimes 1 + \mu_2 1 \otimes S_\alpha - \frac{ih}{4} \epsilon_{\alpha\beta\gamma} (S_\beta \otimes S_\gamma - S_\gamma \otimes S_\beta)$ ($\alpha, \beta, \gamma = 1, 2, 3$) and $S_\alpha$ the spin operators. $\mu_1$ and $\mu_2$ are arbitrary constants allowed by Yangian representation theory and play the crucial role in the representation of Yangian[14]. $[S_\alpha, J_\beta] = i \epsilon_{\alpha\beta\gamma} J_\gamma$ and $J_\gamma$’s obey the nonlinear commutation relations[11][13]. The set $\{S_\alpha, J_\beta\}$ forms Yangian associated with $SU(2)$ denoted by $Y(SU(2))$. Noting that $J_\alpha$ act on quantum tensor space only. If the set $\{S, J\}$ satisfies Yangian, so does $J + \eta S$ with arbitrary constant $\eta$ that is called translation of Yangian. By taking an appropriate translation constant we have

$$(\hat{k} \cdot J) \Psi_0 = \frac{\sqrt{3}}{2} (\mu_2 - \mu_1 + \frac{h}{2}) \Psi_1, \quad (\hat{k} \cdot J) \Psi_1 = 0.$$  

(19)

Yangian is an infinte algebra. Therefore any attempt to unify the superconductivity with different l-waves should be through infinite algebra. For the simply physical realization of $Y(SU(2))$, see Ref.[13].

4) In conclusion we believe that the AAM provides an useful approach to discuss physics concerning pair-particles, especially for the nature of coherence and consistency between the diagonalization of given Hamiltonian and gap equation through double-time Green function. Further, this algebraic method may be extended to Yangian algebra that is natural to describe the transition between different condensates.
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References

[1] P.W. Anderson, W.F. Brinkman, Theory of Anisotropic Superfluidity in $^3H_e$, PartII, Chaper3, edited by K.H. Bennemann and J.B. Ketterson, Wiley, New York (1978).

[2] A.J. Leggett, Annals of Physics, 85 11-55 (1974); Rev. Mod. Phys. 47 331-413 (1975).

[3] G.E. Volovik, Exotic properties of superfluid $^3H_e$, world scientific, 1992, Singapore.

[4] D.M. Lee, Rev. Mod. Phys. 69 645-664 (1997).

[5] R. Balian, and N.R. Werthamer, Phys. Rev. 131 1553-1564 (1963).

[6] P.W. Anderson, Phys. Rev. 112, 1900 (1958).

[7] W. Zhang, D. Feng, and R. Gilmore, Rev. Mod. Phys. 62 867 (1990).

[8] S.C. Zhang, Science 275 1089 (1997).

[9] D.J. Scalapinp, S.C. Zhang and W. Hanke, Phys. Rev. B 58 443 (1998).

[10] A.I. Solomon, Y. Feng and V. Penna, Phys.Rev. B 60 3044 (1999).

[11] V. Drinfeld, Sov.Math.Dokl. 32 254 (1985).

[12] V. Drinfeld, Sov.Math.Dokl. 36 212 (1985).

[13] V. Drinfeld, Quantum Group, PICM, Berkely, P269-291 (1986).

[14] V. Chari and A. Pressley, Yangian and R-matrix, L’Enseignement matne-matique, 36 267-302 (1990), A Guide to Quantum Groups, Cambrige University Press 1994.

[15] M.L. Ge, K. Xue and Y.M. Cho, Phys. Lett. A 249 358 (1998).