A $p$-Spin Interaction Ashkin-Teller Spin-Glass Model

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Abstract

A $p$-spin interaction Ashkin-Teller spin glass, with three independent Gaussian probability distributions for the exchange interactions, is studied by means of the replica method. A simple phase diagram is obtained within the replica-symmetric approximation, presenting an instability of the paramagnetic solution at low temperatures. The replica-symmetry-breaking procedure is implemented and a rich phase diagram is obtained; besides the paramagnetic phase, three distinct spin-glass phases appear. Three first-order critical frontiers are found and they all meet at a triple point; among such lines, two of them present discontinuities in the order parameters, but no latent heat, whereas the other one exhibits both discontinuities in the order parameters and a finite latent heat.

Key words: Spin-glasses, Ashkin-Teller model, first-order phase transitions

1 Introduction

In recent years much progress has been achieved in the understanding of magnetic disordered systems. Among those, one may single out the spin glasses (SGs) [1–3], for which significant advances were obtained, as a result of a

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large effort dedicated to them. Most of the SG studies were carried for two-spin-interaction models, that were at the early stage, intended to explain the physical behavior of some peculiar magnetic compounds. Nowadays, apart from this motivation, such SG models were identified to be closely related to a large diversity of physical problems, like neural networks, protein folding, and optimization problems [2,3]. The mean-field theory of the Ising SG model, considered in terms of an infinite-range-interaction model [4], is nowadays accepted as satisfactorily understood. The simplest solution, based on a single SG order parameter, known as replica-symmetric (RS) solution [4], presented serious difficulties, and it was shown to be unstable at low temperatures [5]. The correct mean-field solution for the Ising SG was proposed by Parisi [6], and it consists in an infinite number of order parameters – a procedure called of replica-symmetry breaking (RSB). However, it is not clear up to the moment, whether the mean-field solution is appropriate for the description of real – short-range-interaction – SG systems [3].

The $p$-spin-interaction ($p > 2$) models were initially introduced as merely theoretical problems [7,8]; however, the identification of a close analogy between the transitions occurring in $p$-spin interaction SG models and those obtained in mode coupling theories of structural glasses [9,10], motivated many studies on such models. A nice feature of infinite-range $p$-spin-interaction SG models is that in the $p \to \infty$ limit the energy levels become independent random variables, yielding an exactly solvable model, known as the random-energy model [7,8]. Also, for $p > 2$ the phase transition scenario of these SG models is quite different from the one found in the corresponding $p = 2$ model. Besides the usual equilibrium transition temperature, there exists another transition temperature presenting a dynamical character. Right below the equilibrium transition, a single step in the RSB procedure is sufficient for stability [11], i.e., the order parameter is properly represented in terms of a one-step Parisi function. For models where there is some kind of competition between different types of interactions, the one-step RSB order parameter changes dramatically the phase diagrams obtained from RS solutions [12,13]. Usually, in the $p \to \infty$ limit, the one-step RSB yields the correct solution, revealing new phases which are not present in the corresponding RS approach.

It should be mentioned that, recently, there has been an increasing interest in the study of $p$-spin-interaction SG models, motivated either by attaining a better understanding of such models, or by the possible applications in other fields of science. The study of infinite-range $p$-spin-interaction SG models, by means of the replica method, has produced many interesting new features as a consequence of the inclusion of magnetic fields [14,15], ferro- [16–18] and antiferromagnetic [19,20] interactions, as well as a competition between quadrupolar and SG orderings [13]. Recent dynamical studies also have been carried [21,22], leading, in particular, to an analysis of the barriers separating metastable states [21]. Besides that, such models have been investigated lately.
through rigorous approaches [23] and in a quantum version [24]; new applications were explored, e.g., connections to the protein folding problem [25], to error correcting codes [26], and to many biological systems [27].

The Ashkin-Teller model [28] is one of most studied systems in statistical mechanics, due mainly to the richness of critical phenomena revealed by its phase diagrams both in two and three dimensions [29]. This wealth of results steams from the competition between the two- and four-spin interactions present in the model. Herein we consider an infinite-range Ashkin-Teller-like SG with \( p \)-spins interactions in order to investigate the effects of an analogous competition. We present the full solution in the \( p \rightarrow \infty \) limit. The resulting phase diagram shows three distinct phases, similarly to what has been found in other Ashkin-Teller SG models with \( p = 2 \) [30–32]. However, the nature of the present transition lines are changed, and we get a triple point common to the three phases, instead of a multicritical one as found in the previous \( p = 2 \) works [30–32]. In particular, we find that in the limit of two independent Ising-like models (or 4-state clock model) the equilibrium transition occurs at a multiphase point. We also show that, for \( p \rightarrow \infty \), some particular cases of the present model are equivalent to random-energy models with uncorrelated energy levels.

This paper is organized as follows. In the next section we introduce the model and determine the free-energy density functional, obtained through the replica method. In section 3 we determine the phase diagram within the RS solution and consider the stability of such a solution against Gaussian fluctuations in replica space. In section 4 we apply the first stage of Parisi’s Ansatz to determine the phase diagram. Our main conclusions are drawn in section 4. Equivalences with random-energy models, in the limit \( p \rightarrow \infty \), are shown in an appendix.

2 The Model

In the present work we will consider a \( p \)-spin interaction Ashkin-Teller-like SG model, defined by the Hamiltonian

\[
H = - \sum_{1 \leq i_1 < \ldots < i_p \leq N} \left\{ J^{(1)}_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + J^{(2)}_{i_1 \ldots i_p} \tau_{i_1} \ldots \tau_{i_p} + J^{(4)}_{i_1 \ldots i_p} \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_p} \tau_{i_1} \ldots \tau_{i_p} \right\}, \quad (1)
\]

where \( \sigma \) and \( \tau \) (= \( \pm 1 \)) are Ising spin variables. All interactions are infinite-range-like, and similarly to what has been done in previous works [30,31],
herein we will be restricted to the case of independent couplings $J_{i_1 \ldots i_p}^{(\alpha)}$ ($\alpha = 1, 2, 4$), each one following its own Gaussian probability distribution,

$$P(J_{i_1 \ldots i_p}) = \sqrt{\frac{N^{p-1}}{\pi p! J_\alpha^2}} \exp \left\{ -\frac{N^{p-1} J_{i_1 \ldots i_p}^{(\alpha)}}{p! J_\alpha^2} \right\}, \quad (2)$$

The free-energy density is given by

$$\beta f = -\lim_{N \to \infty} \frac{1}{N} \langle \ln Z \rangle \quad (3)$$

where $\beta = \frac{1}{k_B T}$, and $\langle \quad \rangle$ represents an average over the disorder. In the following we will use the replica method to calculate

$$\langle \ln Z \rangle = \lim_{n \to 0} \langle Z^n \rangle - \frac{1}{n} \quad (4)$$

Performing the averages over the random couplings, one gets

$$\langle Z^n \rangle = \text{Tr} \exp \left[ \frac{p!(\beta J_1)^2}{4N^{p-1}} \sum_{1 \leq i_1 < \ldots < i_p \leq N} \left( \sum_{a=1}^n \sigma_a^{i_1} \sigma_a^{i_2} \cdots \sigma_a^{i_p} \right)^2 \right. \right.$$

$$+ \frac{p!(\beta J_2)^2}{4N^{p-1}} \sum_{1 \leq i_1 < \ldots < i_p \leq N} \left( \sum_{a=1}^n \tau_a^{i_1} \tau_a^{i_2} \cdots \tau_a^{i_p} \right)^2 \right.$$

$$+ \frac{p!(\beta J_4)^2}{4N^{p-1}} \sum_{1 \leq i_1 < \ldots < i_p \leq N} \left( \sum_{a=1}^n \sigma_a^{i_1} \tau_a^{i_1} \sigma_a^{i_2} \tau_a^{i_2} \cdots \sigma_a^{i_p} \tau_a^{i_p} \right)^2 \right], \quad (5)$$

where $a = 1, \ldots, n$ represents the replica index. As usual, the sums over $p$ sites may be reduced to sums over a single site, e.g.,

$$\sum_{1 \leq i_1 < \ldots < i_p \leq N} \left( \sum_{a=1}^n \sigma_a^{i_1} \sigma_a^{i_2} \cdots \sigma_a^{i_p} \right)^2 = \sum_{a,b=1}^n \sigma_a^{i_1} \sigma_a^{i_2} \sigma_b^{i_2} \cdots \sigma_b^{i_p} \sigma_b^{i_p}$$

$$= \frac{1}{p!} \sum_{a,b=1}^n \left( \sum_{i} \sigma_a^i \sigma_b^i \right)^2 + O(N^{p-1})$$

$$= \frac{1}{p!} N^p n + \frac{n}{p!} N^p \sum_{(ab)} \left( \frac{1}{N} \sum_{i} \sigma_a^i \sigma_b^i \right)^p + O(N^{p-1}),$$

where $\sum_{(ab)}$ denotes a sum over distinct pairs of replicas. One may now introduce, for each distinct pair of replicas $(ab)$, the order parameters,
\[ q_{1,ab} = \frac{1}{N} \sum_i \sigma^a_i \sigma^b_i = \langle \sigma^a \sigma^b \rangle, \quad (6.a) \]
\[ q_{2,ab} = \frac{1}{N} \sum_i \tau^a_i \tau^b_i = \langle \tau^a \tau^b \rangle, \quad (6.b) \]
\[ r_{ab} = \frac{1}{N} \sum_i \sigma^a_i \tau^b_i = \langle \sigma^a \tau^b \rangle, \quad (6.c) \]

as well as their respective auxiliary fields (Lagrange's multipliers) \( \gamma_{1,ab}, \gamma_{2,ab}, \) and \( \xi_{ab} \) through standard identities, e.g., for the pair \((q_1, \gamma_1)\),

\[
\left( \frac{N}{2\pi i} \right)^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{(ab)} d\gamma_{1,ab} dq_{1,ab} e^{L(\{\gamma_{1,ab}\};\{q_{1,ab}\})} = 1
\]

where,

\[
L = -\sum_{(ab)} \gamma_{1,ab} \left( Nq_{1,ab} - \sum_i \sigma^a_i \sigma^b_i \right). \quad (11)
\]

Applying a similar procedure for the pairs of matrices \((q_2, \gamma_2)\) and \((r, \xi)\), one may write

\[
\langle Z^n \rangle = \left( \frac{N}{2\pi i} \right)^{3n^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{(ab)} d\gamma_{1,ab} dq_{1,ab} d\gamma_{2,ab} dq_{2,ab} d\xi_{ab} dr_{ab}
\]
\[
\quad \times \exp \left[ -Ng_n(\{\gamma_{1,ab}\}, \{q_{1,ab}\}, \{\gamma_{2,ab}\}, \{q_{2,ab}\}, \{\xi_{ab}\}, \{r_{ab}\}) \right]
\]

where

\[
g_n = \sum_{(a,b)} \left( \gamma_{1,ab}q_{1,ab} + \gamma_{2,ab}q_{2,ab} + \xi_{ab}r_{ab} \right)
\]
\[
-\frac{\beta^2}{2} \sum_{(a,b)} \left( J_1^2 q_{1,ab}^p + J_2^2 q_{2,ab}^p + J_4^2 r_{ab}^p \right)
- \frac{\beta^2 \eta}{4} \left( J_1^2 + J_2^2 + J_4^2 \right)
\]
\[
- \ln \text{Tr} \exp \left\{ \sum_{(ab)} \left[ \gamma_{1,ab} \sigma^a_i \sigma^b_i + \gamma_{2,ab} \tau^a_i \tau^b_i + \xi_{ab} \sigma^a_i \tau^b_i \sigma^b_i \tau^a_i \right] \right\}. \quad (13)
\]

Substituting eq. (12) into eq. (4), we obtain the free-energy density functional

\[
\beta f = \lim_{n \to 0} \frac{1}{n} g_n(\{\gamma_{1,ab}\}, \{q_{1,ab}\}, \{\gamma_{2,ab}\}, \{q_{2,ab}\}, \{\xi_{ab}\}, \{r_{ab}\}) \quad (14)
\]
where $\tilde{g}_n$ stands for the global minimum of $g_n$, taken with respect to the variational parameters.

One must now look for stable solutions of the saddle-point equations. The simplest Ansatz consists in considering all replicas in an equal footing, and thus assume the RS solution. In the next section we will analyse this solution, and it will be shown that the only stable RS solution is the paramagnetic one.

### 3 Replica-Symmetric Solution

The RS solution is obtained by considering

\begin{align}
q_{1,ab} &= q_1, & q_{2,ab} &= q_2, & r_{ab} &= r, & \forall (ab), \quad (12.a)
\end{align}

as well as

\begin{align}
\gamma_{1,ab} &= \gamma_1, & \gamma_{2,ab} &= \gamma_2, & \xi_{ab} &= \xi, & \forall (ab). \quad (12.b)
\end{align}

Performing standard simplifications through Gaussian identities, we get the free-energy density functional as

\begin{align}
f &= \beta \frac{1}{4} \left[ J_1^2(q_1^p - 1) + J_2^2(q_2^p - 1) + J_4^2(r^p - 1) \right] \\
&\quad - \frac{1}{2\beta} \left[ \gamma_1(q_1 - 1) + \gamma_2(q_2 - 1) + \xi(r - 1) \right] - \frac{1}{\beta} \langle \ln \Xi(x, y, z) \rangle_{x, y, z}, \quad (16)
\end{align}

where

\begin{align}
\Xi(x, y, z) &= 4 \left[ \cosh(\sqrt{\gamma_1}x) \cosh(\sqrt{\gamma_2}y) \cosh(\sqrt{\xi}z) \\
&\quad + \sinh(\sqrt{\gamma_1}x) \sinh(\sqrt{\gamma_2}y) \sinh(\sqrt{\xi}z) \right]. \quad (17)
\end{align}

In Eq. (16) the double brackets $\langle \langle \rangle \rangle_{x, y, z}$ stand for Gaussian averages with respect to the set of variables $(x, y, z)$, e.g., for an arbitrary function $\varphi(x, y, z)$, one has

\begin{align}
\langle \langle \varphi(x, y, z) \rangle \rangle_{x, y, z} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz \frac{1}{(2\pi)^{\frac{3}{2}}} e^{\frac{1}{2}(x^2+y^2+z^2)} \varphi(x, y, z).
\end{align}
The parameters $q_1$, $q_2$, $r$, $\gamma_1$, $\gamma_2$ and $\xi$ in Eq. (16) may be determined from the saddle-point conditions. Thus, the equations relating the auxiliary fields ($\gamma_1, \gamma_2, \xi$) to the order parameters ($q_1, q_2, r$) become

$$
\gamma_1 = \frac{1}{2} p (\beta J_1)^2 q_1^{p-1}, \quad \gamma_2 = \frac{1}{2} p (\beta J_2)^2 q_2^{p-1}, \quad \xi = \frac{1}{2} p (\beta J_4)^2 r^{p-1},
$$

(18)

whereas the order parameters are self-consistently given by

$$
q_1 = \langle \langle \varphi_1^2(x, y, z) \rangle \rangle_{x,y,z}, \quad q_2 = \langle \langle \varphi_2^2(x, y, z) \rangle \rangle_{x,y,z}, \quad r = \langle \langle \varphi_3^2(x, y, z) \rangle \rangle_{x,y,z},
$$

(19)

with

$$
\varphi_1(x, y, z) = \frac{1}{D} \left[ \tanh(\sqrt{\gamma_1} x) + \tanh(\sqrt{\gamma_2} y) \tanh(\sqrt{\xi} z) \right],
$$

(17.a)

$$
\varphi_2(x, y, z) = \frac{1}{D} \left[ \tanh(\sqrt{\gamma_2} y) + \tanh(\sqrt{\gamma_1} x) \tanh(\sqrt{\xi} z) \right],
$$

(17.b)

$$
\varphi_3(x, y, z) = \frac{1}{D} \left[ \tanh(\sqrt{\xi} z) + \tanh(\sqrt{\gamma_1} x) \tanh(\sqrt{\gamma_2} y) \right],
$$

(17.c)

and $D = 1 + \tanh(\sqrt{\gamma_1} x) \tanh(\sqrt{\gamma_2} y) \tanh(\sqrt{\xi} z)$.

It would be interesting to investigate the full phase diagram obtained from the above equations. However, similar to what has been done in previous studies of $p = 2$ Ashkin-Teller models, the most interesting case is the isotropic one, for which $J_2 = J_1$. In such a case, besides the usual spin reversal symmetries, there is a new symmetry in which the $\sigma$ and $\tau$ spins variables may be interchanged.

For $p > 1$ there is always a disordered, paramagnetic, phase where $q_1 = q_2 = r = 0$ and $\gamma_1 = \gamma_2 = \xi = 0$, with the free energy given by

$$
f_P = -\frac{1}{4T} (2J_2^2 + J_4^2) - 2T \ln 2,
$$

(21)

where $T$ is the temperature (herein we work in units such that $k_B = 1$). From the above expression we obtain the entropy density

$$
s_P = -\frac{1}{4T^2} (2J_2^2 + J_4^2) + 2 \ln 2.
$$

(22)

For low temperatures this entropy becomes negative and the system gets frozen. The paramagnetic phase can exist only above the curve shown in Fig. 1, which is given by
Fig. 1. Phase diagram of the $p$-spin interaction Ashkin-Teller SG, within the RS solution, for arbitrary values of $p$. At low temperatures, the paramagnetic solution (P) becomes unstable, and there are no stable non-trivial solutions.

\[
\frac{T}{J_2} = \frac{1}{2\sqrt{\ln 2}} \left( 1 + \frac{J_2^2}{2J_2^2} \right)^{1/2}.
\]  

(23)

There are many other non-trivial RS solutions. The stability analysis of those solutions against replica fluctuations can be performed along the lines pioneered by de Almeida and Thouless [5] for the $p = 2$ infinite-range Ising SG model. In general, the eigenvalues are roots of cubic equations, and become too lengthy to quote their expressions, except in a few particular cases. For instance, to the $q_1 = q_2 = 0$, $\gamma_1 = \gamma_2 = 0$, $r \neq 0$ solution there corresponds, in the limit $n \to 0$, the following longitudinal

\[
\lambda_L = \frac{1}{2} (\beta J_4)^2 p(p-1)r^{p-2} \left[ 1 - \frac{1}{2} (\beta J_4)^2 p(p-1)r^{p-2} (1 - 4r + 3t) \right],
\]  

(24)

and transversal

\[
\lambda_T = \frac{1}{2} (\beta J_4)^2 p(p-1)r^{p-2} \left[ 1 - \frac{1}{2} (\beta J_4)^2 p(p-1)r^{p-2} (1 - 2r + t) \right]
\]  

(25)

eigenvalues, where

\[
t = \langle \varphi_1^4(x, y, z) \rangle_{x, y, z}.
\]  

(26)

When $r = 0$, one gets the paramagnetic solution and, from the vanishing of the above eigenvalues, such a solution is marginally stable for $p > 2$. On the other hand, in the $p \to \infty$ the $r = 1$ solution presents both eigenvalues negative, being completely unstable. The same happens for any other non-trivial RS solution. Thus, we must look for RSB solutions; this will be done in the next section.
4 Replica Symmetry Breaking Solution

Since there is no stable SG solution in the $p \to \infty$ limit, within the RS assumption, we need to look for RSB solutions. Fortunately, it is enough to consider just the first step in Parisi's RSB procedure in order to get the correct solution in such a limit, as verified in other $p$-spin-interaction models studied before [11–13,33]. In the present case, since we deal with three distinct pairs of conjugated matrices, namely $(q_1, \gamma_1)$, $(q_2, \gamma_2)$, and $(r, \xi)$, one should apply the RSB scheme for all six matrices. In principle, one could apply the RSB scheme for each matrix in an independent way, i.e., each matrix should have its own block sizes. However, it is easy to convince oneself that this is not a physically acceptable procedure, due to the symmetries involving the interchanging between the spins in the Hamiltonian. Therefore, herein we will apply the same RSB for each matrix, i.e., we will divide all six $n \times n$ matrices into $n/m$ groups of size $m$. Following Parisi [6], we denote the elements of the off-diagonal blocks by $q_{1,0}, q_{2,0}, r_0, \gamma_{1,0}, \gamma_{2,0}$ and $\xi_0$, and those of the diagonal blocks by $q_{1,1}, q_{2,1}, r_1, \gamma_{1,1}, \gamma_{2,1}$ and $\xi_1$. Thus, we obtain the free-energy density functional

$$f = -\frac{\beta}{4}(J_1^2 q_{1,1} + J_2^2 q_{2,1} + J_4^2 r_0)(m - 1) + \frac{\beta m}{4}(J_1^2 q_{1,0} + J_2^2 q_{2,0} + J_4^2 r_0)$$

$$+ \frac{1}{2\beta}(\gamma_{1,1} q_{1,1} + \gamma_{2,1} q_{2,1} + \xi_1 r_1)(m - 1) - \frac{m}{2\beta}(\gamma_{1,0} q_{1,0} + \gamma_{2,0} q_{2,0} + \xi_0 r_0)$$

$$- \frac{1}{4}(J_1^2 + J_2^2 + J_4^2) + \frac{1}{2\beta}(\gamma_{1,1} + \gamma_{2,1} + \xi_1)$$

$$- \frac{1}{\beta m} \langle \ln Z(x_0, x_1, x_2) \rangle_{x_0, x_1, x_2} , \tag{27}$$

where

$$Z(x_0, x_1, x_2) = \langle \langle A^m(x_0, x_1, x_2, y_0, y_1, y_2) \rangle \rangle_{y_0, y_1, y_2} , \tag{28}$$

$$A = 4 \cosh(u_0) \cosh(u_1) \cosh(u_2) + 4 \sinh(u_0) \sinh(u_1) \sinh(u_2) , \tag{29}$$

with $u_0$, $u_1$ and $u_2$ defined by

$$u_0 = \sqrt{\gamma_{1,0}} x_0 + \sqrt{\gamma_{1,1} - \gamma_{1,0}} y_0 , \quad u_1 = \sqrt{\gamma_{2,0}} x_1 + \sqrt{\gamma_{2,1} - \gamma_{2,0}} y_1 ,$$

$$u_2 = \sqrt{\xi_0} x_2 + \sqrt{\xi_1 - \xi_0} y_2 . \tag{30}$$
By extremizing the above free-energy density one gets the equations of state within the one-step RSB procedure (see Appendix A); it is easy to see that such equations present several non-trivial solutions in the limit $p \to \infty$. A careful analysis of the free energy shows that only four of those solutions are physically acceptable. For low temperatures there is always a SGI phase (see Fig. 2), in which $q_{1,1} = q_{2,1} = r_1 = 1$, $q_{1,0} = q_{2,0} = r_0 = 0$, and

$$m = \frac{2\sqrt{2 \ln 2}}{\sqrt{2 + (J_4/J_2)^2} J_2} \frac{T}{T}. \quad (31)$$

The corresponding free-energy density is given by

$$f_I = -\sqrt{2 J_2^2 + 2 J_4^2} \ln 2, \quad (32)$$

which yields a vanishing entropy. The transition between the SGI phase and the paramagnetic one occurs along a line given by

$$\frac{T}{J_2} = \frac{1}{2\sqrt{2 \ln 2}} \sqrt{2 + (J_4/J_2)^2}, \quad (33)$$

where both phases present zero entropy, i.e., there is no latent heat. This line holds as long as $J_4/J_2 \leq \sqrt{2}$. Beyond that point, we find two other first-order transition lines confining a SGII phase, as shown in Fig. 2, throughout which one has $r_1 = 1$, $m = 2\sqrt{\ln 2} T/J_4$, and all other order parameters zero. This phase presents a free-energy density

$$f_{II} = -\sqrt{\ln 2} J_4 - \frac{J_2^2}{2T} - T \ln 2, \quad (34)$$

which leads to the entropy density

$$s_{II}(T) = -\frac{J_2^2}{2T^2} + \ln 2. \quad (35)$$

Due to discontinuities in the order parameters, the transition separating the paramagnetic and SGII phases is first-order; this critical frontier may be determined by demanding the free-energy densities given by Eqs. (21) and (34) to be equal, leading to the straight line,

$$T = \frac{J_4}{2\sqrt{\ln 2}}. \quad (36)$$

Along this critical frontier there is also no latent heat. For lower temperatures another transition occurs, this time from the SGII to the SGI phase. From
The borders of the paramagnetic phase (P) present discontinuities in the order parameters, but no latent heat, whereas the critical frontier SGI-SGII is a genuine first-order phase transition, with discontinuities in the order parameters and a finite latent heat. The phases P, SGI, and SGII coexist at a triple point (TP), following the standard Gibbs phase rule. A distinct SG solution (SGIII) becomes possible at the multiphase point (MP), where the phases P, SGI, and SGIII coexist.

Their free-energy densities, given respectively by Eqs. (34) and (32), we find the corresponding critical frontier,

$$\frac{T}{J_2} = \frac{1}{2\sqrt{\ln 2}} \left[ (2 + \sqrt{2}) \sqrt{1 + \frac{J_4^2}{2J_2^2}} - (1 + \sqrt{2}) \frac{J_4}{J_2} \right].$$

It is easy to verify that along the SGI-SGII critical frontier there is a finite latent heat. Therefore, in this case one has a genuine first-order transition line. The three transition lines given by eqns. (33), (36) and (37) merge together at a triple point (TP), located at $J_4/J_2 = \sqrt{2}$ and $J_2/T = \sqrt{2\ln 2}$, as shown in Fig. 2. Although two of these transition lines do not represent conventional first-order transitions in the thermodynamic sense, since we do not observe any discontinuity in the first derivatives of their corresponding free-energy densities, i.e., no latent heat, the picture around the triple point follows the standard Gibbs phase rule [34].

Besides the solutions discussed above, there are also two equivalent solutions (which we call SGIII): $q_{1,1} = 1$, $m = 2\sqrt{\ln 2} T/J_2$, with all remaining order parameters zero, and its symmetric one, i.e., $q_{2,1} = 1$, $m = 2\sqrt{\ln 2} T/J_2$, with all remaining order parameters equal to zero. However, the SGIII phase is realized only at the multiphase point (MP), $[J_4 = 0, T/J_2 = 1/(2\sqrt{\ln 2})]$, where it coexists with the paramagnetic and SGI phases.
5 Conclusion

We have studied a version of the Ashkin-Teller SG model, with \( p \)-spin interactions, by means of the replica approach. The RS solution leads to a simple phase diagram, for arbitrary values of \( p \), with a paramagnetic phase that is stable at high temperatures, becoming unstable at low temperatures; within such a solution, there are no stable non-trivial solutions at low temperatures. By applying a one-step RSB procedure, we have found a rich phase diagram in the \( p \to \infty \) limit, with four distinct phases, namely, a paramagnetic one at high temperatures, and three SG phases (SGI, SGII, and SGIII) at lower temperatures. The borders of the paramagnetic phase present discontinuities in the order parameters, but no latent heat, whereas the critical frontier separating phases SGI and SGII is a genuine first-order phase transition, exhibiting both discontinuities in the order parameters, as well as a finite latent heat. These critical frontiers all meet at a triple point according to the standard Gibbs phase rule [34], similarly to what happens in a previously investigated model [13]. The SGIII solution is stable only at a multiphase point, where it coexists with the paramagnetic and SGI phases. Also, in the \( p \to \infty \) limit, it is possible to show the equivalence of the model considered herein with a random energy model, as defined by Derrida [7,8], which can be solved by other methods; this equivalence is shown in Appendix B. A detailed analysis of the corresponding random-energy model will be published elsewhere.

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A Equations of State for the Replica-Symmetry-Breaking Solution

In this appendix we derive the equations of state of the one-step RSB solution discussed in section 4. We shall consider the same matrix block sizes, i.e., the same value of \( m \) for all matrices; denoting the elements of the off-diagonal blocks by \( q_{1,0}, q_{2,0}, r_0, \gamma_1, \gamma_2, \) and \( \xi_0 \), whereas those of the diagonal blocks by \( q_{1,1}, q_{2,1}, r_1, \gamma_{1,1}, \gamma_{2,1} \) and \( \xi_1 \), one gets the free-energy density defined in Eqs. (27) – (30). The extremization of such a free-energy density with respect to this set of parameters leads to the following equations of state,
\[ \gamma_{1,0} = \frac{1}{2} p(\beta J_1) q_{1,0}^{p-1} \]  
(A.1)

\[ \gamma_{1,1} = \frac{1}{2} p(\beta J_1) q_{1,1}^{p-1} \]  
(A.2)

\[ \gamma_{2,0} = \frac{1}{2} p(\beta J_2) q_{2,0}^{p-1} \]  
(A.3)

\[ \gamma_{2,1} = \frac{1}{2} p(\beta J_2) q_{2,1}^{p-1} \]  
(A.4)

\[ \xi_0 = \frac{1}{2} p(\beta J_4) r_0^{p-1} \]  
(A.5)

\[ \xi_1 = \frac{1}{2} p(\beta J_4) r_1^{p-1} \]  
(A.6)

\[ q_{1,0} = \left< \left< \left[ \frac{\langle \langle A^{m-1} B \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.7)

\[ q_{1,1} = \left< \left< \left[ \frac{\langle \langle A^{m-2} B^2 \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.8)

\[ q_{2,0} = \left< \left< \left[ \frac{\langle \langle A^{m-1} C \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.9)

\[ q_{2,1} = \left< \left< \left[ \frac{\langle \langle A^{m-2} C^2 \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.10)

\[ r_0 = \left< \left< \left[ \frac{\langle \langle A^{m-1} D \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.11)

\[ r_1 = \left< \left< \left[ \frac{\langle \langle A^{m-2} D^2 \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right> \right>_{x_0,x_1,x_2} \]  
(A.12)

where we have used the notation introduced in section 3, i.e., the double brackets \( \langle \rangle \) stand for Gaussian averages with respect to the set of variables \( (x, y, z) \). The extremization with respect to the parameter associated with the block sizes, \( m \), leads to

\[
0 = -\frac{\beta}{4} \left[ J_1^2 (q_{1,1}^p - q_{1,0}^p) + J_2^2 (q_{2,1}^p - q_{2,0}^p) + J_4^2 (r_1^0 - r_0^0) \right] \\
+ \frac{1}{2\beta} (\gamma_{1,1} q_{1,1} - \gamma_{1,0} q_{1,0} + \gamma_{2,1} q_{2,1} - \gamma_{2,0} q_{2,0} + \xi_1 r_1 - \xi_0 r_0) \\
+ \frac{1}{\beta m^2} \left< \ln \langle \langle A^m \rangle \rangle_{y_0,y_1,y_2} \right>_{x_0,x_1,x_2} \\
- \frac{1}{\beta m} \left< \left[ \frac{\langle \langle A^m \ln A \rangle \rangle_{y_0,y_1,y_2}}{\langle \langle A^m \rangle \rangle_{y_0,y_1,y_2}} \right]^2 \right>_{x_0,x_1,x_2} 
\]  
(A.13)
In the equations above one has,

\[
A = 4 \cosh(u_0) \cosh(u_1) \cosh(u_2) + 4 \sinh(u_0) \sinh(u_1) \sinh(u_2) \quad (A.14)
\]

\[
B = 4 \sinh(u_0) \cosh(u_1) \cosh(u_2) + 4 \cosh(u_0) \sinh(u_1) \sinh(u_2) \quad (A.15)
\]

\[
C = 4 \cosh(u_0) \sinh(u_1) \cosh(u_2) + 4 \sinh(u_0) \cosh(u_1) \sinh(u_2) \quad (A.16)
\]

\[
D = 4 \cosh(u_0) \cosh(u_1) \sinh(u_2) + 4 \sinh(u_0) \sinh(u_1) \cosh(u_2) \quad (A.17)
\]

with

\[
u_0 = \sqrt{\gamma_{1,0}} x_0 + \sqrt{\gamma_{1,1} - \gamma_{1,0}} y_0 \quad (A.18)
\]

\[
u_1 = \sqrt{\gamma_{2,0}} x_1 + \sqrt{\gamma_{2,1} - \gamma_{2,0}} y_1 \quad (A.19)
\]

\[
u_2 = \sqrt{\xi_0} x_2 + \sqrt{\xi_1 - \xi_0} y_2 \quad . \quad (A.20)
\]

### B Equivalence with Random Energy Model

In this appendix we consider two particular cases of the model defined in Eq. (1), in the limit \( p \to \infty \), and find the corresponding equivalences with random-energy models. Let us first consider spin configurations for which \( \{\sigma_i\} \) and \( \{\tau_i\} \) are completely independent of each other; this is expected to be valid at high temperatures, and the corresponding random-energy model should yield the same results as the model of Eq. (1) in the paramagnetic phase. The probability that such a configuration presents a given energy \( E \) is

\[
P_1(E) = \langle \delta(E - \mathcal{H}(\{\sigma_i, \tau_i\})) \rangle, \quad (B.1)
\]

where the average is taken over all possible realizations of \( J_{i_1 \ldots i_p} \). Using Eqs. (1) and (2) we find

\[
P_1(E) = \frac{1}{\sqrt{N\pi J_{eff}^2}} \exp \left( -\frac{E^2}{NJ_{eff}^2} \right), \quad (B.2)
\]

where we have introduced the effective variance \( J_{eff}^2 = J_1^2 + J_2^2 + J_4^2 \). The total number of such configurations is \( 4^N \); therefore, the average number of macroscopic states with energies between \( E \) and \( E + dE \) is given by

\[
n_1(E) = 4^N P_1(E) \approx \exp \left[ N \left( 2 \ln 2 - \frac{E^2}{NJ_{eff}^2} \right) \right]. \quad (B.3)
\]
In the thermodynamic limit, $N \to \infty$, it is convenient to introduce the energy $u = E/N$, and entropy $s(u) = S(E)/N$ densities. From Eq. (B.3) we obtain

$$s(u) = -\frac{u^2}{2 J_{\text{eff}}^2} + 2 \ln 2. \quad (B.4)$$

Using the thermodynamic definition of temperature $1/T = \partial s/\partial u$ we may write

$$s(T) = -\frac{J_{\text{eff}}^2}{4T^2} + 2 \ln 2. \quad (B.5)$$

When $J_1 = J_2$ we recover the entropy density of the paramagnetic phase, given by Eq. (22).

Let us now consider another particular case, namely, situations in which the $\{\sigma_i\}$ and $\{\tau_i\}$ configurations are the same; this is expect to hold throughout the SGII phase. In this case, the last term in (1) does not contribute, and we get the simplified Hamiltonian

$$\mathcal{H} = -\sum_{1 \leq i_1 < \ldots < i_p \leq N} \left[ J^{(1)}_{i_1 \ldots i_p} + J^{(2)}_{i_1 \ldots i_p} \right] \sigma_{i_1} \ldots \sigma_{i_p}. \quad (B.6)$$

The number of such configurations is $2^N$. The probability that one of these configurations presents an energy $E$ is given by

$$P_2(E) = \langle \delta(E - \mathcal{H}(\{\sigma_i\})) \rangle, \quad (B.7)$$

with $\mathcal{H}$ given by Eq. (B.6). From the above expression and the probability distribution for the couplings, we find

$$P_2(E) = \frac{1}{\sqrt{N\pi(J_1^2 + J_2^2)}} \exp \left[-\frac{E^2}{N(J_1^2 + J_2^2)}\right]. \quad (B.8)$$

Therefore, the mean number of such configurations is

$$n_2(E) = 2^N P_2(E) \approx \exp \left\{ N \left[ \ln 2 - \frac{E^2}{N(J_1^2 + J_2^2)} \right] \right\}. \quad (B.9)$$

For $J_1 = J_2$, the entropy density as a function of the energy density is given by

$$s(u) = -\frac{u^2}{2 J_2^2} + \ln 2. \quad (B.10)$$
Expressing this entropy as a function of the temperature, one gets the same entropy density of the phase SGII [cf. Eq. (35)],

\[ s_{II}(T) = -\frac{J^2}{2T^2} + \ln 2. \] (B.11)

In summary, we have found the equivalence of our model with random-energy models in two particular cases: (i) spin configurations \( \{\sigma_i\} \) and \( \{\tau_i\} \) independent from each other. This leads to a random-energy model, appropriated for the paramagnetic phase, with \( 4^N \) independent random-energy levels \( E_i \) following the distribution given by (B.2); (ii) the same spin configurations \( \{\sigma_i\} \) and \( \{\tau_i\} \). In this case one gets a random-energy model, appropriated for the SGII phase, with \( 2^N \) independent random-energy levels \( E_i \) distributed according to Eq. (B.8). To obtain the free-energy density for the phase SGII we can use the method introduced by Derrida [8], which avoids the use of replicas.

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