Superlinear amplitude amplification

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Abstract
Quantum search/amplitude amplification algorithms are designed to be able to amplify the amplitude in the target state linearly with the number of operations. Since the probability is the square of the amplitude, this results in the success probability rising quadratically with the number of operations. This paper presents a new kind of quantum search algorithm in which the amplitude of the target state, itself increases quadratically with the number of operations. However, the domain of applications of this is much more limited than standard amplitude amplification.

1 Background
Quantum searching was invented to speed up the searching process in databases. It was realized by Hoyer et al[2] and independently by me [3] that this searching was a special case of amplitude amplification whereby the amplitude in a target state could be amplified linearly with the number of operations. This realization considerably increased the power of the algorithm, no longer was it limited to database searching but was applicable to a host of physics and computer science problems. In fact it gave a square-root speedup for almost any classical probabilistic algorithm.

The idea behind this speedup was realized later on to be a two dimensional rotation through which the state-vector got driven from the source to a target state through a sequence of small rotations in the two dimensional space defined by the source and the target state.

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This is easily seen by considering the basic transformation: $-U I_s U^{-1} I_t U$ say $V$. Then if we calculate $V_{ts}$, by definition of the $I_t$ & $I_s$ operations, it easily follows that

$$V_{ts} = (-U I_s U^{-1} I_t U)_{ts} = 3U_{ts} - 4|U_{ts}|^2 \approx 3U_{ts} \quad (1a)$$

Note that this is true for any unitary $U$. It stays true if we replace $U$ by $V$ which yields:

$$(-VI_s V^{-1} I_t V)_{ts} = 3V_{ts}$$

Substituting for $V$ as $-UI_s U^{-1} I_t U$ in $V_{ts}$ from $(1a)$, it follows that:

$$(-UI_s U^{-1} I_t UI_s U^{-1} I_t UI_s U^{-1} I_t U)_{ts} \approx 9U_{ts}$$

Similarly by recursing multiple times, we can prove the transformation:

$$U(-I_s U^{-1} I_t U)^p_{ts} \approx (2p + 1)U_{ts}$$

to be true for large $p$.

### 2 Quadratic Amplitude Amplification

This paper gives a new kind of amplitude amplification in which the amplitude in the target grows quadratically with the number of iterations. Instead of choosing the basic transformation to be $V = -UI_s U^{-1} I_t U$, we choose $V$ to be $-U^{-1} I_t I_s U$. It follows by using the definitions of $I_s$ & $I_t$, that

$$V_{ts} = (-U^{-1} I_t I_s U)_{ts} = 2U_{ss} U_{ts}^* + 2U_{tt}^* U_{st} \quad (2)$$

In case $U_{ss} \approx U_{tt}^*$ and $U_{st} \approx U_{ts}^*$, then $V_{ts} \approx 4U_{ss} U_{ts}$. Unlike the recursion equation of the previous transformation which only depended on $U_{ts}$, this equation depends on both $U_{tt}$ and $U_{ss}$ and even $U_{st}$. So we need to investigate how $U_{tt}$ and $U_{ss}$ vary in successive recursions.

Consider $V_{ss}$. Again assuming $U_{st} \approx U_{ts}^*$ and $U_{ss} \approx U_{tt}^*$

$$V_{ss} = (-U^{-1} I_t I_s U)_{ss} = -1 + 2|U_{ss}|^2 - 2|U_{ts}|^2$$

Note that if we denote $U_{ss} = (1 - \delta)$, and $V_{ss}$ by $1 - \gamma$ assuming all terms to be real and neglecting $2|U_{ts}|^2$ on the RHS, the above equation may be written as:

$$\gamma \approx 4\delta$$
Therefore $V_{ss}$ stays close to 1 for approximately $\ln \frac{\delta}{1}$ recursions. In $i$ recursions, provided $U_{tt} \approx 1$, $U_{ts}$ rises by a factor of approximately $4^i$; therefore in $\frac{\ln \frac{\delta}{4}}{\ln 4}$ recursions $U_{ts}$ rises by approximately a factor of $\frac{1}{\delta}$. The number of queries is approximately $2^{\ln \frac{\delta}{4}}$ which is $\frac{1}{\sqrt{\delta}}$, as expected the amplification of $U_{ts}$ is quadratic in the region when $U_{ss}$ is approximately 1.

3 Example - U is the Inversion about Average Operation

Consider the situation when $s$, the starting state is an arbitrary basis state and $U$ is the inversion about average transformation. Then, assuming there to be $N$ states to be searched, $U_{ss}$ is $-1 + \frac{2}{N}$ and $U_{ts}$ is $\frac{2}{N}$. Then analyzing the sequence of operations for a few steps-

- $U = WI_0W$
- $-U^{-1}I_tI_sU = -WI_0W(I_tI_sW)I_0W = WI_0W (I_tI_s) W I_0 W$
- $U^{-1}I_tI_sU I_tI_s U^{-1}I_tI_sU = WI_0W(I_tI_sW)I_0W(I_tI_sW)I_0W (I_tI_sW)I_0W (I_tI_sW)I_0W$
  
  $= \cdots WI_0W (I_tI_s) W I_0 W (I_tI_s) W I_0 W$

(3)

Looks something like the search algorithm, which is:

$= \cdots WI_t W I_t W I_s W I_t W I_s W$

However, any similarity is superficial, as we discuss in the following section, this algorithm is not a rotation of the state vector in two-dimensional Hilbert space.

Nevertheless, the dynamics of the algorithm are fairly simple to understand and analyze iteratively: The state just before an inversion about average is described by 3 parameters, the amplitudes in the target state, source state and that in the other states.

The evolution is obtained by the following equations ($A(t)$ denotes the average amplitude over all states):
3.1 Observations

The number of iterations required for searching with certainty is $\sqrt{2}$ times more than required by the search algorithm.

The variation of the amplitude in the target state is $\frac{1}{2} - \frac{1}{2} \cos \left( \frac{2\sqrt{2}t}{\sqrt{N}} \right)$. In the initial stages (when $t$ is close to 0), the amplitude varies as $\frac{2t^2}{N}$. As
expected, the rate of increase is quadratic. However, once the probability in the target state become significant (also affecting $U_{ss}$), the quadratic nature of the increase is destroyed.

The algorithm of this paper may be useful in applications where the basic $U_{ts}$ that needs to be amplified is small (in the above example where the U transform was the inversion about average, $U_{ts}$ was only $\frac{2}{N}$ whereas in the search algorithm it is about $\frac{1}{\sqrt{N}}$).

It is possible that there would exist applications where a few applications of this algorithm provided the driving transform for amplitude amplification algorithms. That way, we would get the quadratic speedup plus the flexibility of the amplitude amplification algorithms.

3.2 This is not the search algorithm

One might be tempted to conclude that the above algorithm was a variant of the search algorithm because, overall, it gave a square-root speedup; also it consists of similar sequences of unitary transformations. However, that is not the case.

The chief characteristic of the search algorithm and all its variants (amplitude amplification algorithms) was a rotation of the state vector in appropriately defined two dimensional space. The algorithm of this paper needs more than two dimensions to operate in. To see this consider the basic recursion equation used to develop the algorithm: $V_{ts} = (-U^{-1}I_s U)_{ts} = 2U_{ss}U_{ts}^* + 2U_{tt}^* U_{st}$. Given large $U_{ss}$ & $U_{tt}$ and $U_{st} \approx U_{ts}^*$, we had argued that $V_{ts}$ was amplified significantly in each recursion. In order to satisfy this condition needs more than two dimensions. This is because if there were only two dimensions it would follow from unitarity of $U$ that $2U_{ss}U_{ts}^* + 2U_{tt}^* U_{st} = 0$ (any two columns of a unitary matrix are orthogonal) - therefore, additional dimensions are necessary.

4 Conclusion

The above algorithm gives a quadratic amplification under certain conditions. The quadratic amplification offers something new beyond the search algorithm, even though it is not as universally applicable. To borrow a term
Figure 2: Hierarchy of search algorithms - The quantum search algorithm attained a square-root speedup over classical but the price paid was more sensitivity. The present algorithm provides a square-root speedup over quantum searching, but it still more sensitive.

from analog amplifiers: this only has a limited dynamic range - outside of this range it has to be supplemented by other more robust algorithms.

Just as [1] ([2],[3]), this algorithm provides yet another tool in the quantum algorithm designer’s toolkit. Whereas, ([1]) is independently useful to design quantum algorithms, the fixed point algorithms & the algorithm of this paper may be useful in combination with other algorithms - ([2],[3]) to improve the robustness and the algorithm of this paper to increase the amplification in selected ranges.

As described in the Observations section, the algorithm of this paper may be useful in conjunction with the standard quantum search algorithm. This is somewhat similar in spirit to applications where the search algorithm is combined with a classical algorithm. One such example is the counting algorithm of [2] where one gets round the cyclical nature of the search algorithm by making appropriately timed observations (which is the classical algorithm). The counting algorithm is not usually looked at this way, but in the context of robustness versus speed, it is insightful to look upon it as a combination of classical and quantum search algorithms,
References

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