A new criterion for rational equivalence of cycles on a projective variety over an algebraically closed field is given, and some consequences considered.

Introduction
The motivation for this paper was Bloch’s conjecture [4] on injectivity of the Abel Jacobi mapping for complex projective surfaces with $p_g = 0$, a surprising conjecture in that for $p_g \neq 0$ this map has infinite-dimensional kernel [15]. This and related conjectures (for example in [5,14]) imply the existence of many more rational equivalences under certain conditions (for example if $p_g = 0$ [4], or if the variety is defined over a number field [5]) than otherwise. In cases where this can be proved, the methods rely on special properties such as the variety having enough automorphisms. So a step towards finding a general method might be to reconsider the definitions of rational equivalence of cycles on a general variety.

The main result of this paper (theorem 2.1) is a criterion for rational equivalence, derived from standard definitions (1.1). For nonsingular $V$ in $\mathbb{P}^n$, 2.1 states that $X$ is rationally equivalent to $Y$ if and only if $X - Y =$
$(A - B) \cdot D \cdot V)$, where $A, B$ are hypersurfaces of like degree and $D$ is a complete-intersection cycle in a strict sense. This generalizes the well-known fact that divisors $X$ and $Y$ are rationally equivalent if and only if $X - Y = A - B$, where $A$ and $B$ are hypersurface sections of like degree. It extends results of Severi and Samuel (see section 1). The intersection product used here is Samuel’s (see section 0).

In section 3, the description of the families of pairs of rationally equivalent cycles implied by 2.1 is given. In terms of Mumford’s description of these sets, this amounts to replacing the Chow varieties of rational curves in $S^n V$ (in the 0-cycle case) in his correspondences by rational subvarieties which are easy to describe. One consequence (corollary 3.7) is that complete intersection expressions for multiples of points on $V$, like those found by Roitman in [17] for complete intersections $V$ with $p_g = 0$ (but not necessarily effective), and similar to those sometimes implied when there are enough automorphisms (e.g. [12]) (see 3.8.2), must always exist if all points on $V$ are rationally equivalent to each other. One still needs a key for constructing these expressions for noncomplete intersections $V$.

In section 4, there are more details for the case of surfaces over $C$ or $\bar{Q}$, with some examples of conjectures equivalent to those of [4,5,14]. Since the criterion of (2.1) can be stated as existence of solutions to a polynomial equation of sufficiently high degree (see 2.4), there is some scope for algebraic methods or computer experiments. If one wanted to devise an algorithm to test for rational equivalence between two points, one would need to know if the above degree could be bounded, which suggests connections with the problem of whether there are infinitely many rational points on the surface.

A reason why these equations should be expected to have solutions under the condition $p_g = 0$ or when the underlying field is $Q$ is still lacking. This is discussed in 4.3. As for the role of $p_g$, the more geometric versions of the conjectures coming from (3) are considered in 4.5. These require finding bounds for the dimensions of certain “special position” loci, and suggest links with vector bundles, K-theory [4,14,16], or instantons [8]. The calculations seem difficult.

In 4.6 we give an example of describing rational equivalences explicitly. The families of pairs of points $X, Y$ on a generic surface in $P^3$ which are made rationally equivalent by $X - Y$ being a difference of two intersections with lines are described. It might be interesting to find the corresponding families obtained by intersecting with pairs of complete intersections of higher degree.
Not pursued here is the possibility of using the construction of section 3 backwards, i.e. to obtain information about Chow varieties of varieties such as $\mathbb{P}^n$ known to have trivial Chow groups.

0. Preliminaries on cycles and intersections.

For a projective variety $V \subset \mathbb{P}^n$ over an algebraically closed field $k$, the $r$-cycles (formal sums of irreducible subvarieties of dimension $r$) form a group $Z_r(V)$. The $r$-cycles rationally equivalent (written $\sim$) to zero form a subgroup $B_r(V)$; and the quotient $A_r(V) = Z_r(V)/B_r(V)$, also called $CH_r(V)$, is the Chow group.

The degree of a cycle $\sum n_i X_i$ on $V$ is $\sum n_i \deg X_i$. The classes of degree zero cycles form a subgroup $A_0^r(V)$. The distinct cycles $X_i = \sum n_i X_i$ of degree $l$, with $n_i \geq 0$ for all $i$, are in one-to-one correspondence with the points of a projective algebraic set $\text{Chow}_l(V)$ (see [18]). In particular for $r = 0$, $\text{Chow}_l^0(V)$ is isomorphic to the symmetric product $S^l V$.

If $W$ is a subscheme of $V$, then $[W]$ denotes the associated cycle $\sum l(\mathcal{O}_{W_i,W})_W$ (see [9]), where the $W_i$ are the irreducible components of $W$ and $l(\mathcal{O}_{W_i,W})$ is the length of the Artinian ring $\mathcal{O}_{W_i,W}$. When $W$ is a subvariety, we write $[W] = W$.

If $X$ and $Y$ are cycles meeting properly on $V$, i.e. such that $\text{codim } X \cap Y = \text{codim } X + \text{codim } Y$, Samuel’s intersection cycle $(X \cdot Y)_V$ is a well-defined member of $Z(V)$ when $V$ is nonsingular at generic points of $X \cap Y$. It agrees as a formal sum with Fulton’s more generally defined product $X \cdot Y$ in $A_*(X \cap Y)$ ([9] chapter 7), and suffices for this paper. In fact, we only use it for the case when at least one of $X$ or $Y$ is the cycle associated to a local complete intersection scheme in $V$, when the multiplicities of components are again lengths of local rings. So if $X$ and $Y$ are cycles associated to schemes $X_0$ and $Y_0$ which meet properly, and $Y_0$ is a local complete intersection in $V$ with $V$ nonsingular along $X_0 \cap Y_0$, we have $(X \cdot Y)_V = [X_0 \cap Y_0]$. When $V = \mathbb{P}^n$ we usually drop the suffix $V$.

For example, if $f$ is a rational function on $\mathbb{P}^n$ represented by $a/b$, then the divisor $(f)$ of $f$ is defined by $(f) = \{a = 0\} - \{b = 0\}$, and the divisor $(f_W)$ of $f$ restricted to a variety $W \subset \mathbb{P}^n$ (on which it is defined and nonzero) could be defined by $(f_W) = ((f) \cdot W)_{\mathbb{P}^n}$. This was Samuel’s definition of $(f_W)$, and it agrees with the $[\text{div } f_W]$ in chapter 1 of [9]. If $W \subset V \subset \mathbb{P}^n$, we also have $(f_W) = ((f_V) \cdot W)_V$ if $V$ is nonsingular at generic points of $(f_W)$, but not always (see [18], page 103).
1. Some characterizations of rational equivalence.

1.1 Definitions

Let $X$ and $Y$ be $r$-cycles on $V$. Then $X$ is rationally equivalent to $Y$ on $V$, written $X \sim Y$, if and only if one of the following three conditions holds. The equivalence of these three, at least for nonsingular $V$, is proved in [19]. Samuel used (2) as a definition; (1) and (3) are used in [9] and [15] respectively, for example.

1. $X - Y = \sum_{i=1}^{t} (f_{D_i})$, where $D_1, \ldots, D_t$ are subvarieties of dimension $r + 1$ and $f_{D_i}$ is a nonzero rational function on $D_i$. Note that the $D_i$ can be assumed distinct.

2. $X - Y = \pi_* (V \times \{0\} \cdot D) - \pi_* (V \times \{\infty\} \cdot D)$, where $D$ is an $(r+1)$-cycle on $V \times \mathbb{P}^1$ and $\pi$ is projection onto $V$.

3. (Assuming that $X$ and $Y$ are effective of degree $l$ and identifying them with their Chow forms) there exists an integer $e$ and a morphism $f: \mathbb{P}^1 \to \text{Chow}^{l+e}_r(V)$ such that $f(0) = X + Z$ and $f(\infty) = Y + Z$ for some $Z$ in $\text{Chow}^{e}_r(V)$.

Earlier, “rational equivalence” between $X$ and $Y$ was defined by Severi [21, 2] as “belonging to the same intersection series” (i.e. $X = \sum m_i X_i$ and $Y = \sum m_i X_i'$, where $X_i = H_{i,1} \cdot \ldots \cdot H_{i,m-r}$ and $X_i' = H'_{i,1} \cdot \ldots \cdot H'_{i,m-r}$ with divisors $H_{i,j} \sim H'_{i,j}$ for all $i,j$). This was shown to be equivalent to insisting that $X - Y = \sum m_i (X_i - X_i')$ with $X_i$ and $X_i'$ as above, by [23]. (See [9] for an historical outline.)

These definitions were shown to be equivalent to a version of (2) above by Baldassari [2], using his generalization of 1.2 (1) below to cycles on $V \times \mathbb{P}^n$, and 1.2 (2).

1.2 Severi’s theorems

1. (See [18].) Every $r$-cycle $X$ on $\mathbb{P}^n$ can be written as $X = H_1 \cdot \ldots \cdot H_{n-r}$, where the $H_i$ are divisors (in general not effective).

2. (See [21].) On $V \subset \mathbb{P}^n$ nonsingular, $X \sim Y$ if and only if $X$ and $Y$ “belong to the same series of intersection away from some fixed and semi-fixed components”.

Some related work of Severi is discussed in [27]. It was assumed that $A^0_0$ is always finite-dimensional; this was disproved by Mumford [15].

1.3 Samuel’s theorem

In theorem 10 of [19], Samuel proved the following natural generalization of linear equivalence of divisors.
(1) Theorem. On a nonsingular variety \( V \subset \mathbb{P}^n \), \( r \)-cycles \( X \) and \( Y \) are rationally equivalent if and only if \( X - Y = (f) \cdot D \) for some rational function \( f \) on \( \mathbb{P}^n \) and \( r \)-cycle \( D \) on \( V \).

To prove this, after showing that his definition ((2) above) implied (1), Samuel used the following trick to replace the sum of terms \( \sum_{i=1}^t (f_{D_i}) \) or \( \sum_{i=1}^t ((f_i) \cdot D_i) \) (see the previous section) by a single expression, with \( D = \sum D_i \).

(2) Trick. Let \( f_i \) be a rational function on \( \mathbb{P}^n \) restricting to \( f_{D_i} \) on \( D_i \) for \( i = 1, \ldots, t \). Let \( g_i \) be a form vanishing on every \( D_j \) \( (j \neq i) \) but not on \( D_i \). Choose \( g_1, \ldots, g_t \) all of the same (high enough) degree. Let \( f = \sum g_i f_i / \sum g_i \). Then \( f = f_i \) on \( D_i \).

This requires \( D_i \neq D_j \) for \( i \neq j \), which can be assumed in definition (1) by combining: \((f_{D_i} + (f'_{D_j})) = (f_{D_i} f'_{D_j})\).

2. Complete intersection expression for rational equivalence.

We prove 2.1, which is like a version of Severi’s theorem 1.2 (2) with hypersurface sections (in particular effective divisors) replacing arbitrary virtual divisors. Severi also stated in [20] that effective divisors would suffice for 0-cycles on surfaces (but deferred publishing the complete proof). The proof of 2.1 given here amounts to showing that in Samuel’s theorem 1.3 we can choose \( D \) to be a complete intersection cycle on \( V \). The condition of 2.1 can be expressed compactly as an equation on Chow forms (Corollary 2.4). There is some choice in the expression (see 2.5), and we may try to simplify it by re-embedding the variety (see 2.6).

2.1 Theorem

Let \( V \) be a variety of dimension \( m \) in \( \mathbb{P}^n \), nonsingular away from a set of dimension \( r \), where \( r \leq m - 2 \). Then \( r \)-cycles \( X \) and \( Y \) on \( V \) are rationally equivalent on \( V \) if and only if \( X - Y = V_X - V_Y \), where \( V_X = \{(a = h_1 = \ldots = h_{m-r-1} = 0) \cap V\} \) and \( V_Y = \{(b = h_1 = \ldots = h_{m-r-1} = 0) \cap V\} \) for some forms \( a \) and \( b \) of like degree and forms \( h_1, \ldots, h_{m-r-1} \) such that \( V_X \) and \( V_Y \) are in \( Z_r(V) \).

Remarks. (1) We could also characterize rational equivalence of \( X \) and \( Y \) modulo some fixed set \( Z \subset V \) in this way, i.e. for \( X \) and \( Y \) in \( Z_r(V \backslash Z) \), \( X - Y = V_X - V_Y + \varepsilon \), with \( \varepsilon \) supported on \( Z \), and for this could allow \( V \) to be singular on \( Z \) of higher dimension.

(2) If \( V \) is nonsingular in dimension \( r \), we can write \( V_X = (A \cdot D)_V \) and \( V_Y = (B \cdot D)_V \), where \( A = \{(a = 0) \cap V\} \), \( B = \{(b = 0) \cap V\} \) and \( D = \{(h_1 = \ldots = h_{m-r-1} = 0) \cap V\} \), as in 0.
Proof. The sufficiency of the conditions is clear. So assume $X \sim Y$. By
definition (1), $X - Y = \sum_{i=1}^{t_0} (f_{D_i})$ where $D_1, \ldots, D_{t_0}$ are distinct
$(r - 1)$-dimensional subvarieties. Assuming Proposition 2.2 below, we can find a
complete intersection cycle $D = [\{h_1 = \ldots = h_{m-r-1} = 0\} \cap V]$ such
that $D = \sum_{i=1}^{t} D_i$ for $t \geq t_0$, and $D_i \neq D_j$ for $i \neq j$. Let $\lambda_{t_0+1}, \ldots, \lambda_t$
be arbitrary nonzero constants and set $f_{D_i} = \lambda_i$ for $i \geq t_0 + 1$. Then
since $(f_{D_i}) = 0$, we still have $X - Y = \sum_{i=1}^{t} (f_{D_i})$. Now apply Samuel’s
trick 1.3. Let $f = \sum g_i f_i / \sum g_i$, where $f_i$ is an extension of $f_{D_i}$ to
$\mathbb{P}^n$ and $g_i$ is a form vanishing on every $D_j (j \neq i)$ but not on $D_i$.
Then $(f_{D_i}) = ((f) \cdot D_i)_{|D_i}$, so $X - Y = ((f) \cdot D)_{|D_i}$. Let $a$ and $b$ be forms such that
$a/b$ represents $f$ in lowest terms. Then $V_X = \{(a = h_1 = \ldots = h_{m-r-1} = 0) \cap V\}$
and $V_Y = \{(b = h_1 = \ldots = h_{m-r-1} = 0) \cap V\}$ gives 2.1.

2.2 Proposition
Let $V \subset \mathbb{P}^n$ be an algebraic set of dimension $m$; and let $D_0$ be an $l$-cycle
on $V$ with $l \leq m - 1$, such that $D_0 = \sum_{i=1}^{t_0} D_i$ with $D_i$ irreducible and
$D_i \neq D_j$ for $i \neq j$. Suppose that $V$ is nonsingular away from at most a
set $Z$ of dimension $l - 1$. Then there exist forms $h_1, \ldots, h_{m-l}$ on $\mathbb{P}^n$ such
that $\{(h_1 = \ldots = h_{m-l} = 0) \cap V\} = D$ where $D = \sum_{i=1}^{t} D_i$ for $t \geq t_0$ and
$D_i \neq D_j (i \neq j)$.
To prove this, we use the following Lemma.

2.3 Lemma
Let $V$, $D_0$ and $Z$ be as in 2.2. Then there exists a form $h$ vanishing on
$D_0$ such that $\{h = 0\} \cap V = V'$, where $V'$ has dimension $m - 1$ and is
nonsingular away from a set $Z'$, still of dimension at most $l - 1$.

Proof. Suppose $V = \cup_{i=1}^{s} V_i$ where $V_i$ is irreducible. We show below that
there exist forms $g_i (for i = 1 \ldots s)$, all of some high enough degree $d$,
vvanishing on $\{D_0 \cup (\cup_{j \neq i} V_j)\} = X_i$, such that $\{g_i = 0\} \cap V_i$ is nonsingular
on $V_i$ not only away from $Z$ and $D_0$, but also at generic points of $D_0$. Then
we can take $h = \sum g_i$ to give 2.3.

By Hilbert’s basis theorem, the base locus of the linear system $I_d(X_i)$,
(forms of degree $d$ vanishing on $X_i$), is just $X_i$ for large enough $d$. By
Bertini’s theorem II ([1]), generic $g_i \in I_d(X_i)$ has $\{g_i = 0\} \cap V_i$ nonsingular
away from $X_i \cup \text{sing} V_i$, i.e. away from $D_0 \cup \text{sing} V_i$.
At a generic point $p \in D_0 \cap V_i$, both $D_0$ and $V_i$ are nonsingular. Since
dim $D_0 < \dim V_i$, we have a strict inclusion $\{\partial f_i |_p : f \in I_d(V_i)\} \subset \{\partial f |_p : f \in I_d(X_i)\}$ (when $d$ is large enough). So for generic $g_i \in I_d(X_i)$, $\{g_i = 0\} \cap V_i$ is nonsingular at $p$.
To prove 2.2, we apply 2.3 $m - l$ times in succession.
2.4 Corollary of 2.1

Let $\phi_X$ denote the Chow form for a cycle $X$ (see [18]). Then in the notation of 2.1, $X \sim Y$ on $V$ if and only if $\phi_X \phi_{G_1} = \phi_Y \phi_{G_2}$ for some $r$-cycles $G_1$ and $G_2$ both cut out on $V$ by complete intersections of the same multidegrees.

The nontrivial solutions for the resulting system of equations in $(\text{Chow}_r(V) \times \Pi_{i=0}^{n-1} P^n)^2$ would give information about $A_r^0(V)$ (or about Chow$_r(V)$, if $A_r^0(V) = 0$ as for $V = P^n$). For $r \neq 0$, this may be more accessible than it is in 2.1. The $\phi_{G_i}$ are simpler if $V$ is a complete intersection.

That is one reason for allowing $V$ to be singular above and considering new embeddings (2.5–2.6). We do not address the question of how many expressions $X - Y = \sum (f_{D_i})$ with a given value of $\sum \deg f_{D_i}$ there may be, but rather ask how to build new complete intersection expressions for $X - Y$ out of given ones.

2.5 Varying expressions

If $V$ is nonsingular in dimension $r$ and $X - Y = (A \cdot D)_V - (B \cdot D)_V$ with $A = [(a = 0) \cap V], B = [(b = 0) \cap V], D = [(h_1 = \ldots = h_{m-r-1} = 0) \cap V]$ as in 2.1, we can choose new $A, B, D$ as follows. The corresponding changes in $a, b, h_i$ give similar new expressions $V_X - V_Y$ in the general case.

(1) We can choose $A$ and $B$ with $A \cap B$ proper. This is proved below.

(2) $X - Y = ((A + E) \cdot D)_V - ((B + E) \cdot D)_V$, for any divisor $E$ meeting $D$ properly.

(3) If $A \cap B$ is proper and $[(h_i = 0) \cap V] = H_i$, so that $D = (H_1 \cdot \ldots \cdot H_{m-r-1})_V$, then $X - Y = ((A - B) \cdot (H_1 \cdot \ldots \cdot H_j \cdot (H_j + rC) \cdot H_{j+1} \cdot \ldots \cdot H_{m-r-1})_V$ for generic $C$ in the pencil $\langle A, B \rangle$, for any $r \in \mathbb{N}$.

(4) Applying (3) to each $H_i$ and then applying (2), we can construct new $A, B, H_i$ for $i = 1, \ldots, m - r - 1$ all of the same sufficiently high degree.

To prove (1), suppose that $X - Y = ((f_0) \cdot D)_V$, where $f_0$ is given in lowest terms by $a_0/b_0$ with $\{a_0 = b_0 = 0\} \cap D$ improper. Let $a = \gamma a_0, b = \gamma b_0 + \delta, \gamma$ is any form of degree $d - a_0$ not vanishing on any component of $\{a_0 = 0\} \cap V$ or $D$. Let $\delta$ be in $I_d(D)$. Then for sufficiently large $d$, $\{a = b = 0\} \cap V$ is proper for generic $\delta$, and $f = a/b$ restricted to $D$ is the same as $f_0$.

2.6 Multiple embeddings and projections

(1) Linear rational equivalences. By (4) above, whenever $X \sim Y$ on $V$, there is an embedding $V \subset \mathbb{P}^n$ (some $s$-tuple of our original embedding) such that $X - Y = (L_1 \cdot V)_{\mathbb{P}^n} - (L_2 \cdot V)_{\mathbb{P}^n}$ with $L_1, L_2$ linear subspaces of $\mathbb{P}^n$ of codimension $m-r$, and $\dim L_1 \cap L_2 = \dim L_1 - 1$. The existence
of such an expression for $X - Y$ on general $V \subset P^n$ (so with $s = 1$ above) would put some conditions on $V$ (cf. 4.6), but it is not obvious how these conditions meet on the set of $s$-tuple embedded $V$.

(2) Projections. Another way to try to simplify a search for rational equivalences would be to project $V \subset P^n$ into a smaller space $P^{n'}$ ($n > n'$).

Not all rational equivalences on the image $V'$ lift to $V$ (if $V \to V'$ is not an isomorphism), as the example below illustrates. On the other hand, if $V \to V'$ is a birational morphism of surfaces (for example) then $A^0_0(V)$ is finite-dimensional (see [4]) if and only if $A^0_0(V')$ is.

2.7 Example
Let $X$ and $Y$ be any points on a surface $V$ in $P^n$. Then there is a birational map $\phi : V \to V' \subset P^3$ such that $\phi(X) - \phi(Y) = ((L_1 - L_2) \cdot V')_{P^3}$, where $L_1$ and $L_2$ are lines.

Proof. Choose hypersurface sections $A$, $B$, $D$ with $(A \cdot D) = X + X_1$, $(B \cdot D) = Y + Y_1$, for some $X_1, Y_1$ disjoint from $X, Y$. Then choose $E$ with $(E \cdot D) = X_1 + Y_1 + W$ disjoint from $X + Y$. Adjusting the $(((A + E) - (B + E)) \cdot D)$ as in 2.5 we obtain $A', B', D'$ all of the same degree $s$ with $(A' \cdot D') = X + 2X_1 + Y_1 + W'$ and $(B' \cdot D') = Y + X_1 + 2Y_1 + W'$. Then let $\phi$ be $s$-tuple embedding $\psi$ followed by projection to $P^3$ from a centre in $\psi(\{a' = b' = d' = 0\})$, where $a'$ defines $A'$ etc.

3. Families of pairs of rationally equivalent cycles.

Let $V$ be a projective variety of dimension $m$, with a chosen embedding $V \subset P^n$ of degree $d_0$. Let $U^i$ be the subset of $(\text{Chow}^i(V))^2$ consisting of pairs of effective cycles which are rationally equivalent.

If $V$ has singular locus of dimension at most $r$, theorem 2.1 and the following constructions hold. If $V$ has a singular locus $S$ of dimension greater than $r$, we could consider pairs $X, Y$ of cycles on $V \setminus S$ — still allowing rational equivalences with auxiliary cycles $Z \subset S$ — and obtain similar results.

3.1 Definitions
Let $P^n$ be the projective space of forms $h_i$ on $P^n$ of degree $e_i$, for $i = 0, \ldots, m - r - 1$. Call $e_0 = s$ and $h_0 = a$. Let $e = (e_1, \ldots, e_{m-r-1})$ be the multidegree of the complete intersection $\{h_1 = \ldots h_{m-r-1} = 0\}$, and let $d = d_0 s \Pi_{i=1}^{m-r-1} e_i$ be the degree of $\{(a = h_1 = \ldots h_{m-r-1} = 0) \cap V\}$.

Let $W_{e}$ be the incidence correspondence in $\text{Chow}^1_{e}(V) \times \text{Chow}^{d-1}_{e}(V) \times \Pi_{i=0}^{m-r-1} P_{a_i}$, consisting of the closure of the set of all $(X, Z, a, h_1, \ldots, h_{m-r-1})$ such that $\{(a = h_1 = \ldots h_{m-r-1} = 0) \cap V\} = X + Z$. Let
\[ \Delta^{d-l}_{se} \subset W^l_{se} \times W^l_{se} \] be the set of pairs with matching Chow\(^{d-l}_r(V)\) and \(h_1, \ldots, h_{m-r-1}\) terms. This is a closed set.

### 3.2 Theorem

With the above notation, let \(\pi : (W^l_{se})^2 \rightarrow (\text{Chow}^l_r(V))^2\) be projection. Let \(U^l_{se} = \pi(\Delta^{d-l}_{se})\) (a closed set). Then \(U^l = \cup_{s,e} U^l_{se}\).

### Remarks
1. \(U^l\) is also expressed as a countable union of closed sets, in Lemma 3 of [15]: we give an alternative description of these sets. (2) The sets \(U^l_{se}\) belong to \(r\)-cycles on \(V\) with its chosen embedding \(\phi : V \rightarrow \mathbb{P}^n\), so if there is any ambiguity, one could label them \(U^l_{se}(r, V, \phi)\). If \(\{U^l_i\}_{i \in I}\) is the collection of all irreducible components of all of the \(U^l_{se}\), then \(U^l = \cup_i U^l_i\) is independent of \(\phi\).
2. The set \(U^l_{se}\) includes the diagonal \(\Delta\) of \((\text{Chow}^l_r(V))^2\) (for large enough \(s, e\)). Also there may be “improper” components on which the rational equivalence is not given as a difference of proper complete intersections as in 2.1, but only as a limit of such. If we let \(V^l_{se}\) be the set of components of \(U^l_{se}\) for which some (and hence generic, by remark below) \((X, Y)\) in \(V^l_{se}\) has a proper expression as in 2.1, we have \(U^l = \cup_{s,e} V^l_{se} = \cup_{s,e} \left( V^l_{se}(\Delta) \cup \Delta \right) \).

### Proof of 3.2

By 2.1, we know that \(U^l \subseteq \cup_{s,e} U^l_{se}\). To show that \(U^l_{se} \subset U^l\), we consider the correspondence for all complete intersections of type \(s, e\), i.e. \(W^d_{se}\). Since \(W^d_{se}\) maps birationally onto \(\Pi_{i=0}^{m-r-1} \mathbb{P}^n\), its image \(J\) in \(\text{Chow}^d(V)\) is at least unirational. So any two points on \(J\) are joined by a rational curve (Lemma 4 of [19]). If \((X, Y) \in U^l_{se}\), there is some \(Z\) such that \(X + Z\) and \(Y + Z\) are in \(J\). By definition 1.2 (3), \(X \sim Y\).

### Remark

If we label the improper part of \(W^d_{se}\) as \(T\), in other words \(T\) is the set of all \((X, Z, a, h_1, \ldots, h_{m-r-1})\) in \(W^d_{se}\) such that \(\{a = h_1 = \ldots = h_{m-r-1} = 0\} \cap V\) has a component of dimension \(r + 1\), then \(T\) is closed.

We could let \(\Delta_0\) be the union of the components of \(\Delta^{d-l}_{se}\) not contained in \((T \times W) \cup (W \times T)\) and let \(V^l_{se} = \pi(\Delta_0)\) is as in 3.1 (3).

Often it is convenient to know that we only need to look at one of the sets \(U^l_{se}\). This follows from the next proposition.

### 3.3 Proposition

Let \(P = \cup_{\alpha=1}^a U^l_{s(\alpha), e(\alpha)}\), where \((s(\alpha), e(\alpha)) = (s(\alpha), e_1(\alpha), \ldots, e_{m-r-1}(\alpha))\) \(\in \mathbb{N}^{m-r}\). Then there exists \((s, e) \in \mathbb{N}^{m-r}\) such that \(P \subset U^l_{se}\).

### Proof

It is clear from 2.4 that the \(U^l_{se}\) will form a system of nests. We may assume that \(q = 2\). If \(U^l_{s(\alpha), e(\alpha)} \subset U^l_{s'(\alpha), e'(\alpha)}\) for \(\alpha = 1, 2\) with \(s(\alpha) = s'(\alpha)\) or \(e_i(\alpha) = e'_i(\alpha)\), we say the \(s\) (respectively \(e_i\)) term can be matched. Rules
(2) and (3) from 2.5 become

\[
\begin{align*}
(2) \quad U^t_{s,e} &\subset U^t_{s+t,e} \quad \text{(any } t \in \mathbb{N}) \\
(3) \quad U^t_{s,e} &\subset U^t_{s,e'}
\end{align*}
\]

where \( e'_i = e_i \) for \( i \neq j \) and \( e'_j = e_j + rs \) for some \( r \in \mathbb{N} \). (In different notation, 2.5 still holds for \( V \) singular away from \( \text{dim } r \).) It is easy to match the \( s \) terms using (2), so we do this last. To match the \( e_j \) term, one finds (3) alone may not suffice, but (2) followed by (3) does. This requires two pairs of positive integers \( t(\alpha) \) and \( r(\alpha) \), \( \alpha = 1, 2 \), satisfying

\[
e_j(1) + r(1)(s(1) + t(1)) = e_j(2) + r(2)(s(2) + t(2)) = e'_j.
\]

There are many solutions. For example, assuming \( e_j(1) > e_j(2) \), choose \( t(\alpha) \) (\( \alpha = 1, 2 \)) such that \( s(\alpha) + t(\alpha) = \rho(\alpha) \), where \( \rho(\alpha) \) (\( \alpha = 1, 2 \)) are distinct primes for which the expression \( \lambda \rho(1) + \mu \rho(2) = 1 \) has integer solutions \( \lambda > 0 \) and \( \mu < 0 \). Then let \( r(1) = \lambda(e_j(2) - e_j(1)) \) and \( r(2) = -\mu(e_j(2) - e_j(1)) \).

3.4 Corollary

If \( P \) is a closed subset of \( U^l \) and \( k \) is uncountable, then \( P \subset U^l_{se} \) for some \( (s, e) \in \mathbb{N}^{n-r} \).

3.5 Corollary

A variety \( V \subset \mathbb{P}^n \) has has \( A^0_0(V) = 0 \) if and only if \( (\text{Chow}^l_1(V))^2 = \cup_{s,e} U^l_{se} \).

Over uncountable fields \( k \), this holds if and only if \( (\text{Chow}^l_1(V))^2 = U^l_{se} \) for some \( s, e \).

This gives, for example, the following.

3.6 Corollary

A nonsingular variety \( V \subset \mathbb{P}^n \) has \( A^0_0(V) = 0 \) if and only if given any zero-cycle \( Z \) of degree \( N \) on \( V \), and any point \( X \) on \( V \), there exist complete intersection cycles \( D_i \) and hypersurface sections \( A_i, B_i \) (of equal degree) giving

\[
NX = \sum_{i=1}^t ((A_i - B_i) \cdot D_i) + Z.
\]

Here \( t \geq N \), with equality if \( Z \) is effective. Furthermore over uncountable \( k \), we may assume that \( A_i \) and \( D_i \) have the same degrees for all \( i \).

Proof. By 3.5, \( A^0_0(V) = 0 \) if and only if \( V \times V = U^l_{se} \) for some \( s, e \). In fact, using 3.4 we can also assume \( s, e \) large enough that proper expressions as in 2.1 for \( X \sim Y \) are given by \( U^l_{se} \) for all \( (X, Y) \). For the converse, we use the fact that the kernel of the Albanese mapping is divisible [14].
3.7 Corollary

A variety $V \subset \mathbb{P}^n$ over $\mathbb{C}$ has $A^0_0(V) = 0$ if and only if there exists $N \in \mathbb{N}$ such that for each point $X$ on $V$ there is a cycle $E_X$ in $\mathbb{P}^n$ with $(E_X \cdot V) = NX$.

(By Severi’s theorem 1.2 (1), $E_X$ is an intersection of virtual divisors, so a difference of two sums of intersections of hypersurfaces.)

Proof. Let $Z$ be a complete intersection cycle in $3.6$, to get necessity. If for any point $X$, $NX = (E_X \cdot V)$ for some $E_X$, then $N\delta \sim 0$ for all $\delta$ in $Z^0_0(V)$, so $A^0_0(V) = 0$.

Question. How is 3.7 related to the result from [16]: for an affine surface $V$, $A^0_0(V) = 0$ if and only if every point is a complete intersection?

3.8 Further questions

(1) Roitman [17] proved that a complete intersection $V$ with $p_g = 0$ has $A^0_0(V) = 0$, by constructing a cone $E_X$ with $(E_X \cdot V) = NX$ for each $X$. This $E_X$ is defined by the parts $f_{ij}$ of the Taylor expansions at $X$, $f_i = \sum f_{ij}$, of all the equations $f_i$ for $V$ (see [14]) with some extra equations to correct the dimension if necessary. Are complete intersections $V$ are the only varieties for which $E_X$ can be assumed to be effective in 3.7? How can the $E_X$ in 3.7 be found in general?

(2) Expressions $NX = (E_X \cdot V) + Z$, with $Z$ in some fixed locus, can be found for a surface $V$ with $p_g = q = 0$ and “enough automorphisms” (see [3]), if in addition the relevant quotients $V/H_i$ by subgroups $H_i$ of Aut$V$ are rational. The original example with enough automorphisms was the Godeaux surface $V = Q/\mathbb{Z}_5$, where $Q$ is the Fermat quintic (see [12,4]). Here $NX$ is expressed as a combination of fibres of maps $V \to V_i = V/H_i$ with $H_i < \text{Aut}V$. Such fibres can be written as complete intersections minus points on a fixed divisor, when the surfaces $V_i$ are rational, as in this case. Can these expressions for this special $V$ be used to give any information on the solutions for generic Godeaux surface $V$, now known to have $A^0_0 = 0$ by other methods [25]?

3.9 Families of varieties

Let $V \to T$ be a morphism of projective varieties such that for generic $t \in T$, $V_t$ is a nonsingular variety of dimension $m$. Suppose that $V \subset \mathbb{P}^n$, so that $V_t$ is a subvariety of $\mathbb{P}^n$ of degree $d$, for generic $t$. Let $W_{se}(T)$ be the collection of the $W_{se}$ for each $V_t$, i.e. $W_{se}(T)$ is the closure of the set of all $(X, Z, a, h_1, \ldots, h_{m-r-1}, t)$ such that $\{a = h_1 = \ldots = h_{m-r-1} = 0\} \cap V_t = X + Z$, and is a subset of $\text{Chow}_t^r(V) \times \text{Chow}_{d-r}^r(V) \times \Pi_{i=0}^{m-r-1} \mathbb{P}^n_i \times T$. 
Then let \( \Delta^{d-l}_{sc}(T) \) be the set of pairs in \((W_{sc}(T))^2 \) with \( Z, h_i \) (all \( i \)) and \( t \) terms matched. Let \( \pi : \Delta^{d-l}_{sc}(T) \to \text{Chow}_l(V)^2 \times T \) be projection, and let \( U^j_{sc}(T) = \pi(\Delta^{d-l}_{sc}) \). Then if \( U^j(T) \) is the set of all \((X, Y, t)\) such that \((X, Y) \in \text{Chow}_l(V)^2 \) and \( X \sim Y \) on \( V_t \), we have \( U^j(T) = \cup_{s, e} U^j_{se}(T) \).

### 3.10 Examples of families

1. \( r \)-cycles as families of 0-cycles:

   If \( X \) and \( Y \) are \( r \)-cycles on \( P^n \) of degree \( l \) (or more generally on \( V \subset P^n \)), we could view \( X \) as a family of 0-cycles \((X \cdot \Lambda)\) where \( \Lambda \in \text{Gr}(n-r+1, n+1) \), on the sections \( \Lambda \) of \( P^n \), and describe the triples \((A, B, D)\) \((A, B \text{ hypersurfaces and } D \text{ a complete intersection in } P^n)\) giving rise to rational equivalences \( X - Y = (A \cdot D) - (B \cdot D) \) in terms of correspondences \( \Delta^{d-l}_{sc}(T) \) for 0-cycles on \( P^n \) (which are easier to describe explicitly) using the following.

   **Lemma.** \( X - Y = (A \cdot D) - (B \cdot D) \) if and only if \((X \cdot \Lambda) - (Y \cdot \Lambda) = \big( ((A - B) \cdot D) \cdot \Lambda \big) \) for generic \( \Lambda \) meeting \( X \) and \( Y \) properly.

2. Families of surfaces:

   If \( V \to T \) is a family of surfaces \( V_t \subset P^n \) with \( A^0_0(V_t) = 0 \) for generic \( t \), then there exists \((s, e) \in \mathbb{N}^2 \) such that given points \( X, Y \) on \( V_t \) there exist hypersurface sections \( A, B, D \) of \( V_t \) defined by equations of degree \( s, e \) respectively, such that \( X - Y = (A \cdot D)_{V_t} - (B \cdot D)_{V_t} \). In other words, \( V_t \times V_t = V_{se}^1 \) (for generic \( V_t \subset P^n \)) in the notation of 3.2 (3).

### 4. 0-cycles on surfaces defined over \( C \) or a number field.

#### 4.1 Background: conjectures A and B

For more details see [4, 14]. Let \( F \) be a complex projective surface. Mumford showed that if \( p_g \) is nonzero, then \( A^0_0(F) \) is “infinite-dimensional” [15], by showing that every component of \( U^l \cap \pi^{-1}(X) \) has dimension at most \( l \), for any \( X \) in \( S^l F \). From Roitman’s theorems [17], one can deduce that \( A^0_0(F) \) is finite-dimensional if and only if \( \dim U^l_{sc} \geq 3l \) for some \( s, e, l \). So such a surface must have \( p_g = 0 \), and the Bloch-Mumford conjecture [4], states that all surfaces with \( p_g = 0 \) have finite-dimensional \( A^0_0(F) \). We refer to this as conjecture A.

Rokitman showed that the Albanese mapping \( \theta : A^0_0(F) \to \text{Alb}(F) \) is an isomorphism on torsion. Since also the kernel of this map is divisible [6], \( A^0_0(F) \) is finite-dimensional, if and only if \( \theta \) is an isomorphism.

Conjecture A was shown to hold for surfaces with \( p_g = 0 \) and not of general type in [6], and for some examples of general type in (for example) [12, 3, 13, 25].
On the other hand, even surfaces with \( p_g \neq 0 \) have “many” rational equivalences, for example coming from linear equivalences on curves on the surface, and moreover Roitman’s theorem “closure of rational equivalence is albanese equivalence” [17] implies that the Zariski closure of \( U^l \) generates the Albanese kernel. When \( p_g \neq 0 \) this implies that \( U^l \) has infinitely many components for large \( l \). Furthermore, a conjecture of Bloch and Beilinson [5] (which we refer to as conjecture B) implies that if \( F \) is defined over \( \overline{\mathbb{Q}} \), then \( \theta \) is an isomorphism from \( A^0(F_{\overline{\mathbb{Q}}}) \) onto the \( \overline{\mathbb{Q}} \) part of \( \text{Alb}(F) \). In this form, conjecture B is given as a question in [17]. So far no surfaces with \( p_g \neq 0 \) are known to satisfy conjecture B.

4.2 Examples of restatements of conjectures

(1) General type surfaces over \( \mathbb{C} \). Let \( F \) be a surface of general type with \( p_g = 0 \), with canonical divisor \( K_F \). Then conjecture A holds, i.e. \( A^0(F) = 0 \), if and only if there exist integers \( s, e \) such that for any pair of points \( X \) and \( Y \) on \( F \) there exist effective divisors \( A, B \) in \( |sK_F| \) and \( D \) in \( |eK_F| \) with \( X - Y = (A \cdot D)_F - (B \cdot D)_F + W \), where \( W \) is a zero-cycle supported on the \( -2 \) curves of \( F \) (and \( -1 \) curves, if \( F \) is non-minimal). Furthermore, if \( F \) belongs to a family, then \( s \) and \( e \) can be chosen to work for all \( F \) in the family.

Proof. This follows from 3.10 (2), applied to the 5-canonical model of \( F \), say (in which case \( s \) and \( e \) are divisible by 5). The map from \( F \) to this model contracts any \( -1 \) and \( -2 \) curves, but is one-to-one elsewhere, when \( F \) is of general type (see [7]).

(2) Surfaces over a number field. Let \( F \) be a nonsingular surface in \( \mathbb{P}^3 \) defined over \( \overline{\mathbb{Q}} \). Then conjecture B implies that \( A^0(F_{\overline{\mathbb{Q}}}) = 0 \) (since \( q = \dim \text{Alb}(F) = 0 \)), and by 2.1 this holds if and only if given points \( X \) and \( Y \) on \( F_{\overline{\mathbb{Q}}} \) there are surface sections \( A, B, D \) of \( F \) such that \( (A \cdot D)_F - (B \cdot D)_F = X - Y \).

If \( F \) is any projective surface over \( \overline{\mathbb{Q}} \), conjecture B holds if and only if there is a curve \( E \) on \( F \) such that given \( X \) and \( Y \) on \( F \), there are hypersurface sections \( A, B, D \) and a zero cycle \( Z \) supported on \( E \) with \( (A \cdot D)_F - (B \cdot D)_F + Z = X - Y \).

4.3 Algebraic formulation.

When the ideal defining \( F \) is known, one can use (2.4) to obtain other restatements. For example, if \( F \) is a surface given in \( \mathbb{P}^3 \) by the homogeneous equation \( f = 0 \), then two cycles \( X, Y \) in \( S^l F \) are rationally equivalent if and only if for some \( s, e \) the equation

\[
(*) \quad \Phi_X(g) R(b, h, f, g) = \Phi_Y(g) R(a, h, f, g)
\]
for all $g$ in the dual $\mathbb{P}^3$, has a solution $a,b,h$ (forms of degree $s,s,e$ on $\mathbb{P}^3$) which is nontrivial (i.e. neither side of this equation vanishes). Here $R$ denotes a resultant of 4 polynomials in $\mathbb{P}^3$ (see [24] or [9]). Since for fixed $b,h,f$ the resultant $R(b,h,f,g)$ vanishes if and only if $g = 0$ meets the intersection $\{b = h = f = 0\}$, it is the Chow polynomial (see [22]) for this intersection. So conjecture B holds for nonsingular $F$ in $\mathbb{P}^3$ defined over $\bar{\mathbb{Q}}$ if and only if equation (*) has nontrivial solutions for every pair of points on $F$.

If $F$ is singular, one does not expect every pair of nonsingular points to become rationally equivalent on the resolution. For example take $F$ to be a quartic nonsingular away from two disjoint double lines, whose resolution gives an elliptic ruled surface. But existence of solutions to (*) seems unlikely to depend on $F$ being nonsingular. If solutions exist, they must either be forced to give cycles $A.D,B.D$ meeting the singular locus and with high enough multiplicities there that the pull-backs can differ, or else degenerate to trivial solutions. For the above example this would give the correct result $A^0_0(F'\setminus E) = 0$, where $F'$ is the normalization of $F$ and $E$ the elliptic curve over which $F'$ is ruled (the double cover of one of the double lines on $F$). This suggests the following version of conjecture B.

Conjecture B. Let $F$ be any surface in $\mathbb{P}^3$ defined over $\bar{\mathbb{Q}}$, and $X,Y$ any two points in its nonsingular locus. Then equation (*) has nontrivial solutions.

This with 3.8 and also 4.4, 4.6 below suggest the following questions.

(1) Let $F$ be any surface in $\mathbb{P}^3$ and $X,Y$ any two points on it. Is there a sequence of triples of forms $a_n,b_n,h_n$ of increasing degrees, constructed from the defining equations for $X,Y$, the equation for $F$, and their Taylor series, such that whenever either $X,Y$ and $F$ are defined over $\mathbb{Q}$ or $p_g(F) = 0$, the triple gives a solution to (*) for large $n$ (the value of $n$ depending on the coordinates in the former case, see 4.4)?

(2) Must there be such a sequence if the conjectures of [4] and [5] are true?

4.4 Problem of finding rational equivalences

We would like to see how to write down some explicit rational equivalences on a surface $F$, and find $s,e$ as in 3.10 (2) (or 4.2 (1) for general type) if $F$ is known to have $A^0_0(F) = 0$. For a trivial example, if $F$ is a plane and $X,Y$ two points on it, then we can write $X−Y = ((L_1−L_2)\cdot L)$, where $L$ is the line joining $X$ to $Y$ and $L_1, L_2$ are other lines through $X$ and $Y$ respectively. So here $s = e = 1$ will do. In 4.6 we consider rational equivalences of points on surfaces in $\mathbb{P}^3$, obtained by intersecting with lines. Next one could look at the surfaces treated in [6], elliptic surfaces with $p_g = 0$. Here
the Abel-Jacobi theorem for curves (see [10]) is invoked, and the problem reduces to writing down explicit rational equivalences on a curve. For an example of general type, the generic Godeaux surface $F = Q/Z_5$, where $Q$ is a nonsingular quintic in $P^3$ on which the cyclic group $Z_5$ acts freely, was shown to have trivial Chow group by Voisin [25]. Rational equivalences of points on $F$ correspond to $Z_5$-equivariant rational equivalences of orbits of points on the quintic, so this would also serve as an example of the problem of writing down some rational equivalences of cycles of low degree on a general surface in $P^3$. This suggests the following.

**Conjecture:** For a surface $F$ in $P^n$, given $X,Y$ in $S^l F \times S^l F$, there exist integers $s,e$ (given in terms of $l$, invariants of $F$ and its embedding, and of the coordinate field of $X$ and $Y$), such that $X \sim Y$ if and only if $(X,Y) \in U_{s,e}$. The existence of such $s,e$ is equivalent to the existence of an upper bound for the $s,e$ needed to express a rational equivalence, by 3.4. An analogous bound for divisors on curves, which is independent of coordinate fields, does exist. For a nonsingular curve $C$ in $P^n$ of genus $g$ and degree $d$, divisors $X$ and $Y$ (effective of degree $l$) are rationally equivalent if and only if $X - Y = A - B$ for some sections $A,B$ of $C$ by hypersurfaces of degree $s$, where Riemann-Roch gives that any $s$ more than both $(2g - 2 + l)/d$ and $r$ (where $r$ is the smallest for which the $r$-tuple embedding of $C$ is projectively normal), will work.

One consequence of the above conjecture is that solving an equation similar to that in 4.3 (given by 2.4) gives at least in principle a procedure for deciding whether or not $X \sim Y$ over a chosen algebraically closed field (or whether $X - Y$ is torsion over a number field; see [4]), and for writing down a rational equivalence. Unfortunately it seems likely that the $s,e$ (if they exist) will be too large for this to be practical.

Also, if conjecture B is assumed, then the existence of bounds dependent on coordinate field is equivalent to there being only finitely many points on a surface over a number field (away from a finite collection of rational and elliptic curves), when the surface has $p_g \neq 0$. This follows from Mumford’s theorem together with Faltings’ Mordell theorem (curves of genus at least 2 have finitely many rational points).

**4.5 Geometry of the sets $U^l_{s,e}$.**

Two overlapping approaches to these sets are to look directly at the construction in 3.2, and to look at the determinantal equation given by 2.4 (as in 4.2). This section consists of preliminary remarks on the former.
(1) Expected role of $p_g$.
To prove conjecture A in the form in 4.1, it would suffice to prove that for a surface with $p_g = 0$, there exist $s, e, l$ with $\dim U_{s,e}^l \geq 3l$. This leads to the problem of finding $\dim \Delta'$ (where $\Delta'$ is the complement of the cover of the diagonal in $(S^lF)^2$), and its fibre dimension over $U_{s,e}^l$. Equivalently, we can consider the image $T$ of $\Delta'$ in $(S^lF)^2 \times S^{d-l}F$, and its projections onto $U_{s,e}^l$ and onto its image $S$ in $S^{d-l}F$. Let $\lambda$ be the fibre dimension of $T$ over $U_{s,e}^l$. Combining Bloch's conjecture and Mumford's theorem gives:

Conjecture A: $\dim(T) - \lambda \geq 3l$ for some $s, e, l$ if and only if $p_g = 0$.

(2) Approximate calculations.
The dimension of $T$ is at most $m + 2k - x$, where $m = h^0(\mathcal{O}_F(e)) - 1$, $k \leq d - g$ and $g$ is the genus of generic section of $F$ by an equation of degree $e$, and $x$ is the number of independent conditions imposed on a pair of complete intersections of type $s, e$ on $F$ with their degree $e$ equation in common, by making them share $d - l$ points (counted with multiplicities). For small $l$, $T$ will for a general surface be empty unless $S^{d-l}F$ contains cycles imposing fewer than $d - l$ conditions, and it is hard to find $x$ when $T$ is nonempty without more knowledge of $S$. On the other hand for $l$ large (i.e. greater than $g$), $U_{s,e}^l$ would be expected to have dimension less than $3l$ (with exceptions for special surfaces such as $\mathbb{P}^2$). This suggests that we should first try to analyse the set $S$ for small $l$ (see (3)). For a very rough check, if we just assume $T$ is nonempty, then its dimension is at most $m + 2k$, which by Riemann-Roch is linear in $\chi(\mathcal{O}_F)$. On the other hand, using the nesting rules of 2.5 and 3.3, we can construct whole families of rational equivalences $X \sim Y$ from a given one, by adding “redundant” expressions (so increasing $s$ and $e$). For this new $s, e$, the fibre of $T$ over $X, Y$ has dimension depending on at least $\chi(\mathcal{O}_F)^2$. This allows the possibility of a role for $p_g(F)$ compatible with the above conjecture — although these contributions might turn out to be insignificant. For a proof we would need to find more precise upper and lower bounds for the fibre dimension, as well as bounds for $\dim T$.

(3) The set $S$
For a given $l$, it is probably necessary to consider the sets $S$ for the case $l \leq g$ where $g$ is as in (2), and because of “nesting” (3.3) it is also sufficient. The set $S$ is then automatically the “special position locus” for the image $J$ of $W_{s,e}$ in $S^{d-l}F$, which we define to be the set of cycles $Z$ in $J$ imposing fewer conditions on complete intersections of type $s, e$ than does the generic cycle in $J$. In this range ($d - l$ large), it seems hard to apply intersection theory (as in [26] or [10]) to find $S$ because it is an excess intersection and
in general will lie in the singular locus of $S^{d-l}F$.

In [26], the collection of sets of $m$ points in special position (in the ordinary sense of not spanning a $\mathbf{P}^{m-1}$) on a general surface in $\mathbf{P}^{3m-2}$ was shown to be finite, as conjectured by Donaldson [8] in connection with Yang-Mills. It would be interesting to know if any similar conjecture could be made for the geometry of $S$. A possible means of finding $S$ would be to try to generalize Serre’s construction of rank 2 vector bundles associated to special position sets (where the scheme structure is also important; see [11]), to obtain some moduli space of vector bundles associated to $S$ from which to estimate its dimension.

Often $\dim S$ can be used to find $\dim T$ (so this would be an alternative to finding the $x$ in (2) above). For example, by increasing $s, e$ if necessary according to the rules of 2.5, we may assume in addition to (i) $l \leq g$ as above, the condition (ii) generic $Z$ in $S$ lies on a unique section $D$ of $F$ by an equation of degree $e$. Then $T \rightarrow S$ has infinite fibre over generic $Z$ if and only if some component of the corresponding $D$ has a linear system of degree at most $l$. Choosing $s, e$ so that both $\dim S$ and $\dim T$ can be calculated might present a problem, since the most natural candidate for a vector bundle construction would be $s = e$. Also it may be that increasing $s$ and $e$ artificially conceals information (2.6 (1)). This applies particularly when it comes to estimating $\lambda$ (see (1)), unless the following holds.

**Conjecture:** The component of $T$ with maximum fibre dimension over $X, Y$ is that for which the corresponding expression $((A-B) \cdot D)_V = X - Y$ has $D = D_0 + D_1$ with $(A \cdot D_0)_V = (B \cdot D_0)_V$ with $D_0$ of maximum degree.

(4) The case $l = 1$, $F$ of general type.

If $F$ is a surface of general type with $A_0^0(F) = 0$, i.e. satisfying conjecture A, then for sufficiently large $s, e$ we have $U_{s,e}^1 = F^2$. Here the cover $T$ of $S$ must be generically finite on any component of $T$ covering $F^2$ (since otherwise $F$ would be covered by rational curves). So with the above notation, conjecture A can be stated as follows.

**Conjecture A:** $\dim(S) - \lambda = 4$, where $\lambda$ is the fibre dimension of $T$ over $F^2$.

The problem breaks into finding the two dimensions (or suitable bounds). It might be hoped that the geometry of $S$ would give information about $\lambda$. Here typical $Z$ in $S$ must determine a singular $D$, and have at least a triple point at a double point of $D$ (i.e. multiplicity higher than the multiplicity of the singularity on $D$).
4.6 Generic surfaces in \( \mathbb{P}^3 \)

(1) Proposition. The generic surface \( F \) of degree \( d \geq 6 \) in \( \mathbb{P}^3 \) has \( U_{1,1}^1 \setminus \Delta = \emptyset \).

Proof. The set \( U_{1,1}^1 \) constructed in 3.2 for \( F \subseteq \mathbb{P}^3 \) is the set of pairs \((q_1, q_2) \in F \times F\) with \( q_1 - q_2 = (L_1 \cdot F) - (L_2 \cdot F)\) for some pair of lines \( L_1, L_2 \) in \( \mathbb{P}^3 \). If \( F_d \) is the set of surfaces of degree \( d \) in \( \mathbb{P}^3 \), so \( F_d \) is isomorphic to the projective space of dimension \((d+3)/3\) - 1, it is easy to show that the correspondence \( X_r \) below has dimension \( \dim F_d - 2r + 8 \), which gives the result. The set \( X_r \), a subset of \( \mathbb{P}^3 \times \text{Gr}(2, 4)^2 \times F_d \), is defined to be the closure of the set of all \((p, L_1, L_2, F)\) such that \( L_i \not\subseteq F \), \( L_1 \not= L_2 \), \((L_i \cdot F) \not\supseteq rp\) for some pair \( r \).

Using the Taylor series for \( F \) at \( p \) (see (3) below), we can prove the following.

(2) Proposition. Let \( U \) be the closure of \( U_{1,1}^1 \setminus \Delta \). For generic quintic \( F \), \( U \) is finite (and nonempty). For generic quartic \( F \), \( U \) is a surface with projection onto \( F \) of degree 24. For \( d \leq 3 \), \( U = F \times F \) (hence \( A_0^0(F) = 0 \), as is well-known).

Corollary. For quartic \( F \), where \( U^1 = \{(q_1, q_2) \in F \times F : q_1 \sim q_2\}\) as in 3.1, we have \( U^1 = F \times F \).

To prove this, we consider the surface \( U \) above and its “iterates”, i.e. sets of pairs \((q_1, q_2) \in F \times F\) such that \( q_1 - q_2 = \sum_{i=1}^{k} ((L_{i1} - L_{i2}) \cdot F)\) for \( k \) pairs of lines. This also verifies Roitman’s theorem on closure of rational equivalence in this case. Also, \( U^1 \) contains self-products of infinite sequences of curves, derived from e.g. (a) rational curves on \( F \), and (b) the “touch” curve \( R \) on \( F \) (i.e. the curve of points \( p \) where some line \( L \) gives \( (L \cdot F) = 4p \)). Starting with a curve \( C \subset F \) of points rationally equivalent to each other, and adding \( \pi_1(\pi_2^{-1}(C) \cap U_{1,1}^1) = C' \), gives \( (C' \cup C) \times (C' \cup C) \subset U^1 \). This construction is then repeated. This supports conjecture B. It would be interesting to know if the union of \( U \) and its iterates should contain all of \( F_\mathbb{Q} \times F_\mathbb{Q} \), when \( F \) is a quartic defined over a number field satisfying conjecture B.

(3) Taylor series. Let \( p \) be a point on the surface \( F \) in \( \mathbb{P}^3 \). Let \( f = \sum (f_i)_p \) be the Taylor expansion for a polynomial \( f \) defining \( F \) at \( p \). If coordinates \((X, Y, Z, T)\) are chosen for \( \mathbb{P}^3 \), and \( p \in \{T \neq 0\} \), then with respect to \( x = X/T, y = Y/T, \) and \( z = Z/T, \) we have

\[
(f_i)_p(a, b, c) = \sum_{l+m+n=1}^{l, m, n \geq 0} \frac{\partial^i f}{\partial x^l \partial y^m \partial z^n} \bigg|_{(l, m, n)} a^l b^m c^n \frac{f}{l! m! n!}.
\]

Lemma. For \( p \in F \), we have \((f_0)_p = 0\). The map \( L \mapsto L \cap \{T = 0\} \) is a one-to-one correspondence between the set of lines \( L \) in \( \mathbb{P}^3 \) such that \((L \cdot F) \supseteq rp\), and the set of points \((a, b, c, 0)\) in \( \mathbb{P}^3 \) with \((a, b, c)\) a solution of \((f_1)_p = \ldots = (f_{r-1})_p = 0\).
Definition. The “polar locus” \( F^r_q \) for a point \( q \) (which may be on \( F \)) is the set of points \( p \) in \( F \) for which there is some line \( L \) giving \( (L \cdot F) \supseteq rp + q \).

This generalizes the classical polar locus (which is \( F^2_q \)).

Corollary of Lemma. If \( q = (a, b, c, 0) \), then \( F^r_q \) is the set of points \( p \in F \) such that \((f_1)p(a, b, c) = \ldots = (f_{r-1})p(a, b, c) = 0\).

(4) Examples and proof of (2). If \( q \) lies on a quartic \( F \), then \( F^3_q \) consists of 24 points in general. At a generic point \( p_i \in F \), there are two lines \( L_1 \) and \( L_2 \) for which \((L_i \cdot F) \supset 3p_i \) (by the lemma); so if \( p_i \) is one of the 24 points in \( F^3_q \), one of these lines (say \( L_1 \)) has \((L_1 \cdot F) = 3p_i + q \). Then \((L_2 \cdot F) = 3p_i + q_i \), some \( q_i \). This gives rise to 24 points \( q_i \) rationally equivalent to \( q \) (as in 4.6 (2)).

To prove that \( U \) in 4.6 (2) is finite for a generic quintic, a dimension count shows that it is enough to find some nonsingular quintic for which \( U \) is nonempty. The generic quintic \( F \) defined on \( \{T \neq 0\} \) by \( f \in \mathcal{L} \) below works:

\[ \mathcal{L} = \langle x + y + z, xy, xyz, f_4, f_5 \rangle, \]

with \( f_4 \) and \( f_5 \) generic forms of degree 4 and 5. For let \( p = (0, 0, 0, 1) \), \( r = (0, -1, 1, 0) \), \( s = (1, 0, -1, 0) \), \( L_1 = \langle p, r \rangle \) and \( L_2 = \langle s, r \rangle \). Then \((L_i \cdot F) \supset 4p_i \), but not \( 5p_i \) in general (for \( i = 1, 2 \)), so \( U \neq \emptyset \). By Bertini’s theorem, \( F \) is nonsingular, since the base locus of \( \mathcal{L} \) on \( \mathbb{P}^3 \) consists of \( p \), and generic \( F \) is nonsingular at \( p \).

Acknowledgements. The author is grateful to the Max Planck Institut in Bonn, where this work was begun; and thanks the referee for helpful comments. In particular, the referee pointed out that in the proof of 2.3 the underlying field need only be infinite and perfect, and suggested a different method for finite fields.

References.

1. Akizuki, Y.: Theorems of Bertini on linear systems. J. Math. Soc. Jpn. \textbf{3}, 170–180 (1951)
2. Baldassari, M.: Algebraic Varieties. (Ergebnisse der Mathematik) Berlin Heidelberg New York: Springer 1956
3. Barlow, R.: Rational equivalence of zero-cycles for some more surfaces with \( p_g = 0 \). Invent. Math. \textbf{79}, 303–308 (1985)
4. Bloch, S.: Lectures on Algebraic Cycles. (Duke University Math. Series IV) Durham, NC: Duke University 1980
5. Bloch, S.: Algebraic cycles and values of L-functions. J. Reine Angew. Math. \textbf{350}, 94–108 (1984)
6. Bloch, S., Kas, A. and Lieberman, D.: Zero cycles on surfaces with \( p_g = 0 \). Compositio Math. \textbf{33}, 135–145 (1976)
7. Bombieri, E.: Canonical models of surfaces of general type. Publ. Math IHES 42, 171–220 (1973)
8. Donaldson, S. K.: Instantons in Yang-Mills theory. In: Interactions between particle physics and mathematics, pp. 59–75. Oxford: Oxford University Press 1989
9. Fulton, W.: Intersection Theory. (Ergebnisse der Mathematik) Berlin Heidelberg New York: Springer 1984
10. Griffiths, P.A.: An introduction to the theory of special divisors on algebraic curves. (CBMS regional conference series 44) Providence: American Mathematical Society 1980
11. Griffiths, P. and Harris, J.: Residues and zero-cycles on algebraic varieties. Ann. Math. 108, 461–505 (1978)
12. Inose, H. and Mizukami, M.: Rational equivalence of zero-cycles on some surfaces with $p_g = 0$. Math. Ann. 244, 205–217 (1979)
13. Keum, J.H.: Some new surfaces of general type with $p_g = 0$. Preprint, University of Utah, Salt Lake City.
14. Lewis, J.D.: A Survey of the Hodge Conjecture. Montreal: CRM publications 1991
15. Mumford, D.: Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9, 195–204 (1969)
16. Murthy, M. and Swan, R.: Vector bundles over affine surfaces. Invent. Math. 36, 125–165 (1976)
17. Roitman, A.A.: The torsion of the group of zero-cycles modulo rational equivalence. Ann. Math. 111, 553–569 (1980)
18. Samuel, P.: Méthodes d’Algèbre Abstraite en Géométrie Algébrique. Berlin Heidelberg New York: Springer 1955
19. Samuel, P.: Rational equivalence of arbitrary cycles. Am. J. Math. 78, 383–400 (1956)
20. Severi, F.: Un’altra proprietà fondamentale delle serie di equivalenza sopra una superficie. Rend. Accad. Linc. 21, 3–7 (1935)
21. Severi, F.: Serie, sistemi d’equivalenza, e corrispondenze algebriche sulle varietà algebriche (vol 1). Rome: Cremonese 1942
22. Shafarevich, I. R.: Basic Algebraic Geometry. Berlin Heidelberg New York: Springer 1974
23. Todd, J. A.: Some group-theoretic considerations in algebraic geometry. Ann. Math. 35, 702–704 (1934)
24. Van der Waerden, B.L.: Modern Algebra, volume II. New York: Unger 1950
25. Voisin, C.: Sur les zéro-cycles de certaines hypersurfaces munies d’un automorphisme. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19, 473–492 (1992)
26. Xu, M.W.: The configuration of a finite set on surface. Preprint, MPI Bonn (1990)
27. Zariski, O.: Algebraic Surfaces. (Ergebnisse der Mathematik) Berlin Heidelberg New York: Springer 1971