Differentially Private Stochastic Coordinate Descent

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Abstract

In this paper we tackle the challenge of making the stochastic coordinate descent algorithm differentially private. Compared to the classical gradient descent algorithm where updates operate on a single model vector and controlled noise addition to this vector suffices to hide critical information about individuals, stochastic coordinate descent crucially relies on keeping auxiliary information in memory during training. This auxiliary information provides an additional privacy leak and poses the major challenge addressed in this work. Driven by the insight that under independent noise addition, the consistency of the auxiliary information holds in expectation, we present DP-SCD, the first differentially private stochastic coordinate descent algorithm. We give a convergence analysis of our new method, analyze its privacy-utility trade-off and demonstrate competitive performance against the popular stochastic gradient descent alternative while requiring significantly less tuning.

1 Introduction

The availability of a huge amount of data has given rise to a plethora of machine learning (ML) applications. The main algorithmic challenges for these applications are two-fold: (i) minimize the learning time (including training and hyperparameter tuning) and (ii) maximize the output quality (e.g., the prediction accuracy made by the learnt model). Stochastic coordinate descent (SCD) [27] is a very popular optimization algorithm in both academia and industry due to its favorable convergence behavior and the absence of hyperparameters that need to be tuned [11,13,14,15]. In particular, for training generalized linear models it is the algorithm of choice for many applications and has been implemented as a default solver in several popular packages such as Scikit-learn, TensorFlow and Liblinear [13].

However, SCD is not designed with privacy concerns in mind. Many applications employ sensitive data for training a model, often in an iterative manner. Clearly, providing rigorous privacy guarantees is very appealing. For example, a cloud infrastructure that trains ML models on sensitive data (e.g., hospital records) can benefit from sending these models (e.g., by selling them) without the risk of leaking sensitive information.

Differential privacy (DP) [12] is particularly well-suited for iterative ML algorithms in the presence of a strong adversary. The composability property of DP enables a fine-grained way of measuring privacy along with modularity for the DP mechanisms. Nevertheless, increasing the level of privacy comes at a cost: a reduction in the predictive power of the resulting model (known as the privacy-utility trade-off). ML algorithms with DP guarantees typically offer hyperparameters to tune this trade-off, such as the lot size [1]. While such a knob is desirable, the tuning cost (in terms of time) grows exponentially with the number of

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additional hyperparameters (e.g., when tuning via grid search). This tuning also costs in terms of privacy as it may employ sensitive data. Thus, ML algorithms with no additional hyperparameters (i.e., no learning rate), such as SCD, make good candidates for DP learning. To the best of our knowledge, SCD has never been studied through the lens of differential privacy.

**Challenge.** We study how existing applications, built on SCD, can be extended to guarantee differentially private model training. Making SCD private is however not trivial due to the noise addition that is vital for providing DP guarantees. An efficient implementation of SCD stores and updates not only the model vector $a$ but also an auxiliary vector $v := Xa$ to avoid recurring computations. These vectors need to be consistent for standard convergence results to hold. However, independent noise addition to both vectors, necessary for DP, prohibits this consistency.

**Contribution.** We present DP-SCD, a differentially private version of the standard SCD algorithm and formally prove the DP guarantees. We theoretically analyze SCD under noise addition and provide a bound on the maximum level of noise that can be tolerated to achieve a given level of utility. We empirically show (based on real datasets) that for applications for which DP-SCD performs exact minimization per update (e.g., ridge regression and support vector machines), DP-SCD achieves a better privacy-utility trade-off compared to the popular differentially private stochastic gradient descent algorithm (DP-SGD) while, at the same time, being free of a learning rate parameter that needs tuning. We also compare and discuss the primal and dual formulation of SCD in the differentially private setting.

## 2 Preliminaries

Before we dive into the details of making SCD differentially private, we first formally define the problem class and provide the necessary background on SCD and differential privacy.

### 2.1 Setup

We target the training of Generalized Linear Models (GLMs), the class of models that SCD is most popular for. This class includes convex optimization problems with the following form.

$$\min_{\theta} \mathcal{F}(X, \theta) \text{ with } \mathcal{F}(X, \theta) := \frac{1}{N} \sum_{i=1}^{N} \ell_i(\theta^\top x_i) + \frac{\lambda}{2} \|\theta\|^2 \quad (1)$$

where the model vector $\theta$ is learnt from the training data $X \in \mathbb{R}^{M \times N}$ with $N$ training examples $x_i \in \mathbb{R}^{M}$ as columns, $\lambda$ denotes the regularization parameter, and $\ell_i$ the convex loss functions. The norm $\|\cdot\|$ refers to the $L_2$-norm. For the rest of the paper we use the common assumption that the data examples $x_i$ are normalized, i.e., $\|x_i\| = 1$ (as in [7]), and that the loss functions $\ell_i$ are $1/\mu$-smooth. A wide range of ML models fall into this setup including ridge regression, $L_2$-regularized support vector machines (SVMs) and logistic regression [22].

**Threat model.** We assume the threat model of [1]. We consider an adversary that has white-box access to the training procedure (algorithm, hyperparameters, and intermediate output) and can have access even to $[\cdots x_{i-1}, x_i, x_{i+1} \cdots]$, where $x_i$ is the data instance the adversary is targeting. However, the adversary cannot have access to the intermediate results of any update computation. We make this assumption more explicit in §3.1.

### 2.2 Primal-Dual Stochastic Coordinate Descent

The primal SCD algorithm proceeds iteratively and repeatedly selects a coordinate $j \in [M]$ at random, solves a one dimensional auxiliary problem and updates the parameters $\theta$:

$$\theta^+ \leftarrow \theta + e_j \delta^* \text{ where } \delta^* = \arg\min_{\delta} \mathcal{F}(X, \theta + e_j \delta) \quad (2)$$
where $e_j$ denotes the unit vector with value 1 at position $j$. This problem often has a closed form solution; otherwise $F$ is generally replaced by its second order Taylor approximation.

A crucial approach for improving the time complexity of each SCD update is to keep an auxiliary vector $v := X^\top X$ (also called the \textit{shared vector}) in memory in order to avoid recurring computations, as the loss function applies on data through the linear map $\theta^\top X$ (Problem 1). This auxiliary vector is updated in each iteration as $v^+ \leftarrow v + \delta^* x_i$.

### Dual SCD.

SCD can be equivalently applied to the dual formulation of Problem 1, often referred to as SDCA [22]. The dual optimization problem has the following objective.

$$\min_\alpha F^*(X, \alpha) \text{ where } F^*(X, \alpha) := \frac{1}{N} \sum_{i=1}^{N} \ell_i^*(-\alpha_i) + \frac{1}{2\lambda N^2} \|X\alpha\|^2$$

where $\alpha \in \mathbb{R}^N$ denotes the dual model vector and $\ell_i^*$ the convex conjugate of the loss function $\ell_i$. For the dual problem, the auxiliary vector is $v := X\alpha$.

We use the first order optimality conditions to relate the primal and the dual model vectors as $\theta(\alpha) = X^\top \alpha$, which leads to the important definition of the duality gap [10]:

$$\text{Gap}(\alpha) := F^*(X, \alpha) + F(X, \theta(\alpha)) = (X\alpha, \theta(\alpha)) + \frac{\lambda}{2} \|\theta(\alpha)\|^2 + \frac{\lambda}{2} \|\theta\|^2$$

By the construction of the two problems, the optimal values for the objectives match in the convex setting and the duality gap attains zero [22]. Therefore, the model $\theta$ can be learnt from solving either Problem 1 or Problem 2, i.e., the primal and the dual problems are identical from an algorithmic perspective.

However, the primal and dual problems are quite different from a privacy perspective. When applied to the dual, SCD computes each update by processing one example, whereas the primal SCD processes one coordinate across all the examples. Several implications arise as differential privacy is defined on a per-example basis.

### 2.3 Differentially Private Machine Learning

Differential privacy (DP) is a guarantee for a function $f$ applied on a database of sensitive data [12]. In the context of supervised ML, the objective function is the update function of the algorithm and the data is typically a set of input-label pairs $(x_i, y_i)$ that is used during the model training. Two input datasets are \textit{adjacent} if they differ only in a single input-label pair. Querying the model translates into making predictions for the label of some new input.

**Definition 1** (Differential privacy). A randomized mechanism $\mathcal{M} : D \to \mathbb{R}$ satisfies $(\epsilon, \delta)$-DP if for any two adjacent inputs $d, d' \in D$ and for any subset of outputs $S \subseteq \mathbb{R}$ it holds that:

$$\Pr[\mathcal{M}(d) \in S] \leq e^\epsilon \Pr[\mathcal{M}(d') \in S] + \delta$$

(4)

The \textit{Gaussian mechanism} is a popular method for making a deterministic function $f : D \to \mathbb{R}$ differentially private (according to Definition 1). By adding Gaussian noise to the output of the function we can hide particularities of individual input values. The resulting mechanism is defined as: $\mathcal{M}(d) := f(d) + \mathcal{N}(0, S_f^2 \sigma^2)$ where $S_f$ denotes the sensitivity of the function $f$. This definition can be readily extended to the multi-dimensional case in order to fit the ML setting: An iterative machine learning algorithm can be viewed as a function $f : \mathbb{R}^{M \times N} \to \mathbb{R}^M$ that repeatedly computes model updates from the data and thus requires a multi-dimensional noise addition at each iteration $t$:

$$\mathcal{M}_t(d) = f(d) + \mathcal{N}(0, S_f^2 \sigma^2 I), \ I \in \mathbb{R}^{M \times M}$$

(5)

The sensitivity is defined as:

$$S_f := \max_{\text{adjacent } d,d'} \|f(d) - f(d')\|$$

(6)
by the sensitivity of the update function, i.e., $S$ by using the Gaussian mechanism. In particular, DP-SGD adds noise to each update with the variance given $\theta$ applied to make ML differentially private. SGD iteratively updates the model vector as $\theta$ used for solving Problem 1, is an example of how the Gaussian mechanism can be applied to make ML differentially private. SGD is a stochastic coordinate descent algorithm (DP-SCD) in Algorithm 1. The crucial extension in comparison with the standard dual SCD (SDCA [22]) is that we consider mini-batch based updates, that independently process a random sample of $L$ coordinates ($B$) in each inner iteration (Steps 6-11). This is not only beneficial from a performance perspective, as the updates can be executed in parallel, but it also serves as a hyperparameter that steers the privacy-utility trade-off of our algorithm (similar to the lot size in [1]). We formalize our parallel updates based on [15]. In particular, we reuse the local subproblem formulation for the special case where each parallel process updates only a single example $j \in B$.

$$F_j^*(\alpha_j, \Delta, v, x_j) := \frac{1}{N} \ell_j^*(\alpha_j - \Delta) + \frac{1}{2\lambda N^2} \left( \frac{1}{L} \|v\|^2 + 2v^\top x_j \Delta + L \|x_j\|^2 \Delta^2 \right)$$ (7)

**Algorithm 1:** DP-SCD (for Problem 3)

**Input:** $N$ examples $x_i \in \mathbb{R}^M$, $y_i$: labels, $\lambda$: regularization, $T$: iterations, $L$: mini-batch size, $(\epsilon, \delta)$: DP parameters, $C$: scaling factor

1. **Init:** $\alpha = 0$; $v = 0$; shuffle examples $x_i$
2. $\sigma \leftarrow$ smallest noise magnitude, s.t., $MA(\delta, \sigma, L, T) = \epsilon$

3. for $t = 1, 2, \ldots T$

4. $\Delta v = 0$

5. for $j \in B$

6. $\Delta_j = \arg\min_{\Delta} F_j^*(\alpha_j, \Delta, v, x_j)$

7. $\Delta_j = \max \left(1, \frac{\|\Delta_j\|}{\sigma} \right)$ // scale

8. $\Delta v += \Delta_j x_j$

9. $\Delta \alpha += e_j \Delta_j$

10. end // update the model

11. $\alpha += e_B(\Delta \alpha + N(0, \sigma^2 2C^2 I_1))$ // $I_1 \in \mathbb{R}^{L \times L}$

12. $v += \Delta v + N(0, \sigma^2 2C^2 I_2)$ // $I_2 \in \mathbb{R}^{M \times M}$

13. end

14. return $\theta = \frac{1}{\lambda N} v$ // retrieve primal model
Note that the minimizer $\Delta_j$ in Step 7 can be often computed in closed form, e.g., for ridge regression, or SVMs. Exact minimization is however not necessary for our algorithm to converge. Approximate solutions are sufficient for convergence (e.g., [23, Assumption 1]). Hence, for logistic regression we use a single Newton step to efficiently update the coordinates.

Finally, to guarantee differential privacy, we bound the sensitivity of each coordinate update to be $C$ by scaling $\Delta_j$ (Step 8). We then use the Gaussian mechanism to make $\alpha$ and $v$ differentially private. We address two main questions regarding DP-SCD:

1. How much noise do we need to add to guarantee $(\epsilon, \delta)$-differential privacy? (§3.1)
2. Can we still give convergence guarantees for this new algorithm under noise addition? (4)

We answer the first question in §3.1 by analyzing the sensitivity of our update function. For the second question (that we answer in §4), the main challenge is that independent noise addition destroys the consistency between $\alpha$ and $v$, i.e., $v \neq X\alpha$. We show how to address this challenge and prove convergence for our method.

### 3.1 Privacy Analysis

We view the training procedure as a sequence of mechanisms $\mathcal{M}_t$ where each mechanism corresponds to one outer iteration and computes an update on $L$ examples. We assume these mechanisms to be atomic from an adversary point of view, i.e., we assume no access to the individual coordinate updates. For determining the sensitivity of this mechanism it is important to note that all updates within mechanism $\mathcal{M}_t$ touch different data points and are computed independently. The output of each mechanism $\mathcal{M}_t$ is the concatenation $[\alpha^T, v^T]$ of the dual and the auxiliary vectors. The sensitivity of the output is given as:

**Lemma 1** (Sensitivity of DP-SCD). The sensitivity of each mechanism $\mathcal{M}_t$ in Algorithm 1 is bounded: $S_f \leq \sqrt{2C}$.

**Proof:** Each mini-batch processing accesses sensitive information and thus needs to have DP guarantees. The input of each mini-batch processing consists of $\alpha$ and $v$ and the output consists of the updates, i.e., $f = [\Delta_1, \ldots, \Delta_L, \sum_{i=1}^n \Delta v_i] \in \mathbb{R}^{L+M}$.

The sensitivity with respect to a single example is bounded by Step 8 of Algorithm 1. Given that $|\Delta_i| \leq C$ (per-example scaling) and $\|x_i\| \leq 1$ (normalized data), it holds that $\|\Delta v\| \leq \|x_i\| * |\Delta_i| \leq C$. Hence, the sensitivity of $f$ is as follows. Assume $X \setminus X' = x_k$:

$$f(X) = [\Delta_1, \ldots, \Delta_k, \ldots, \Delta_L, \Delta v_1 + \cdots + \Delta v_k + \cdots + \Delta v_L] \in \mathbb{R}^{L+M}$$

$$f(X') = [\Delta_1, \ldots, 0, \ldots, \Delta_L, \Delta v_1 + \cdots + 0 + \cdots + \Delta v_L] \in \mathbb{R}^{L+M}$$

$$S_f^2 := \max_{X' \setminus X = x_k} \|f(X) - f(X')\|^2 = \|0, \ldots, \Delta_k, 0, \Delta v_k\|^2 = \Delta_k^2 + \|\Delta v_k\|^2 \leq C^2 + C^2 = 2C^2$$

\[\square\]

**Theorem 1** (Privacy bound for DP-SCD). Algorithm 1 is $(\epsilon, \delta)$ differentially private for any $\epsilon = O(q^2T)$ and $\delta > 0$ if we choose $\sigma = \Omega\left(\frac{q\sqrt{T \ln(1/\delta)}}{\epsilon}\right)$.

**Proof:** Each mini-batch processing (Steps 6-11) accesses sensitive information and thus needs to have DP guarantees. We make the output of each mini-batch processing differentially private by using the Gaussian mechanism (Equation 5) with $M$ replaced by $M + L$ in our case. The moments of each mechanism $\mathcal{M}_i$ are bounded (given Lemma 1 and Lemma 3). Hence, based on Theorem 1, we can derive the lower bound for $\sigma$ that guarantees $(\epsilon, \delta)$-DP for the output model.

In practice, we choose the smallest $\sigma$ that provides the given privacy guarantee, i.e.,

$$\text{MA}\left(\delta, \sigma, \frac{L}{N}, T\right) \leq \epsilon$$

(8)

where $T$ denotes the iterations of Algorithm 1. Given that $\epsilon$ decreases monotonically with increasing $\sigma$, we perform binary search until the variance of the output of the MA gets smaller than 1% of the given $\epsilon$. 


**Data-dependent constraints.** The update computation involves certain dataset-dependent constraints for applications such as logistic regression or SVMs. For example, logistic regression employs the labels to ensure that the logarithms in the delta computation are properly defined [22]. The noise addition breaks these constraints. An approach that enforces these constraints after the noise addition would leak privacy and break the DP guarantee. We thus enforce these constraints in the beginning of the delta computation. As a result, the output model does not respect these constraints (as opposed to SCD). Nevertheless, there are no negative implications as the purpose of these constraints is to enable valid delta computations.

### 3.2 Cost Analysis

The performance overhead of DP-SCD with respect to SCD boils down to the cost of sampling the Gaussian distribution. This cost is proportional to the mini-batch size, i.e., larger size means less frequent noise additions and less frequent sampling. The noise addition also prohibits any performance optimizations that accelerate training for sparse datasets.

The time complexity for Algorithm 1 is \(O(T \cdot M)\). The updates for the coordinates with a given mini-batch can be parallelized. We discuss parallelizable variants of SCD in §6.

### 3.3 Primal Version

The primal formulation of DP-SCD (shown in Algorithm 2) computes the updates in a coordinate-wise manner, thus making differentially private learning more challenging than in the dual formulation. At each iteration of the inner loop (similar to Steps 6-11 of Algorithm 1) the primal version updates a given coordinate \(j\) for all the examples. Therefore, the sampling ratio \((q)\) is 1 as each \(\Delta\)-computation touches one coordinate of the entire dataset. This invalidates the important property of the mini-batch size \((L)\) to regulate the privacy-utility trade-off. Additionally, the noise addition to make the primal version DP is significantly larger than the dual version.

#### Algorithm 2: PRIMALDP-SCD (for Problem 1)

| Line | Description |
|------|-------------|
| 1    | \(\theta = 0\); \(v = 0\); shuffle examples \(x_i\) |
| 2    | \(\sigma\) ← smallest noise magnitude, \(s.t., MA(\delta, \sigma, 1, T) = \epsilon\) |
| 3 for | \(t = 1, 2, \cdots T\) do |
| 4    | \(\Delta v = 0\) |
| 5    | Randomly a block of \(L\) coordinates \(B \subset [M]\) |
| 6 for | \(j \in B\) do |
| 7    | \(\Delta_j = \arg\min_{\Delta_j} F_j(\theta, \Delta, v, X[j,:])\) // update |
| 8    | \(\Delta_j = \max\left(1, \frac{\Delta_j}{C}\right)\) // scale |
| 9    | \(\Delta v = \Delta_j X[j,:]\) |
| 10   | \(\Delta \theta = e_j \Delta_j\) |
| 11 end |
| 12   | \(\theta = e_\theta(\Delta \theta + N(0, \sigma^2(4L^2 - 2)C^2 I_1))\) // \(I_1 \in \mathbb{R}^{L \times L}\) |
| 13   | \(v = \Delta v + N(0, \sigma^2(4L^2 - 2)C^2 I_2)\) // \(I_2 \in \mathbb{R}^{N \times N}\) |
| 14 end |
| 15 return | \(\theta\) // DP model |

Each mini-batch processing accesses sensitive information and thus needs to have DP guarantees. We device the PRIMALDP-SCD by assuming that the norm of each row of \(X\) (i.e., a given coordinate for all the examples) instead of each column, is bounded \((\|X[j,:]\| = 1)\).

#### Lemma 2 (Sensitivity of PRIMALDP-SCD)

The sensitivity of each mechanism \(M_t\) in Algorithm 2 is bounded: \(S_f \leq \sqrt{(4L^2 - 2)C}\).

**Proof.** The input of each mini-batch processing consists of \(\theta\) and \(v\) and the output consists of the updates, i.e., \(f = [\Delta_1, \cdots, \Delta_L, \sum_B \Delta v] \in \mathbb{R}^{L+N}\). The sensitivity with respect to a single example is bounded by
Step 8 of Algorithm 2 which ensures $|\Delta_j| \leq C$ (per-example scaling). Moreover, given the normalized data, it holds that $|\Delta v| \leq \|X[j, \xi]\| * \Delta_j \leq C$. Hence the sensitivity of $f$ can be computed by choosing $X, X'$ such that $\|f(X) - f(X')\|^2$ is maximized. The sensitivity is maximized when $X \setminus X' = x_0$:

$$f(X) = [\Delta_1, \Delta_2, \ldots, \Delta_L, \Delta v_1 + \Delta v_2 + \cdots + \Delta v_L] \in \mathbb{R}^{L+N}$$

$$f(X') = [0, \Delta'_2, \ldots, \Delta'_L, 0 + \Delta v'_2 + \cdots + \Delta v'_L] \in \mathbb{R}^{L+N}$$

The difference among $f(X), f(X')$ is the result of the difference due to the missing example for $X'$ that affects all the updates as each update employs all the examples for coordinate $j$. Moreover, the subsequent values-vectors can, in the worst case, be opposite. Therefore, the sensitivity is as follows by using the triangle inequality.

$$S_j^2 := \max_{x \in \mathbb{R}^{N}} \|f(X) - f(X')\|^2$$

$$= \|(|\Delta_1|, |\Delta_2|, \ldots, |\Delta_L|, |\Delta v_1 + \Delta v_2 + \cdots + |\Delta v_L|)|^2$$

$$\leq (L - 3)C^2 + (2L - 1)C^2 = (4L^2 - 2)C^2$$

We therefore conclude that the dual version (Algorithm 1) is preferable over the primal in the DP setting.

### 3.4 Sequential version

We present a baseline algorithm namely seqDP-SCD to depict the importance of independent updates (inside a given mini-batch) for DP-SCD, and focus on the dual problem. As shown in Algorithm 3, seqDP-SCD adopts the natural (i.e., as in vanilla SCD) method of performing sequential and thus correlated updates. In particular, the updates for both $\alpha$ and $v$ at sample $j$ (Steps 10 and 11 of Algorithm 3) depend on all the previous samples of the same mini-batch ($B$). In contrast, in our main DP-SCD algorithm (Algorithm 1), there is no such correlation and the updates can be also executed in parallel. On the one hand this correlation is better for convergence in terms of iterations (not time) [14]. On the other hand, due to this correlation, Algorithm 3 requires significantly more noise than Algorithm 1 that makes the overall performance of our proposed algorithm (Algorithm 1) significantly better as we show in §5.

**Algorithm 3: seqDP-SCD** (for Problem 3)

```
// same as Steps 1-3 of Algorithm 1
3 for $t = 1, 2, \ldots, T$ do
4 $\Delta v = 0$
5 Randomly sample $L$ examples $B \subseteq [N]$
6 for $j \in B$ do
7 $\Delta_j = \arg min_{\Delta} \mathcal{F}_j(\alpha_j, \Delta, v, x_j)$
8 $\Delta_j / = \max \left(1, \frac{|\Delta_j|}{C} \right)$ // scale
9 $\Delta v += \Delta_j x_j$
10 $\alpha += e_j \Delta_j$ // update the model
11 $v += \Delta v$
12 end
13 $\alpha += e_\xi N(0, \sigma^2(4L^2 - 2)C^2 I_1)$ // $I_1 \in \mathbb{R}^{L \times L}$
14 $v += N(0, \sigma^2(4L^2 - 2)C^2 I_2)$ // $I_2 \in \mathbb{R}^{M \times M}$
15 end
16 return $\theta = \frac{1}{N} v$ // retrieve primal model
```
Lemma 3 (Sensitivity of seqDP-SCD). The sensitivity of each mechanism $\mathcal{M}_t$ in Algorithm 3 is bounded: $S_f \leq \sqrt{(4L^2 - 2)C}$.

Proof. The proof is similar to Lemma 2. The difference among $f(X), f(X')$ consists of (a) the difference due to the missing example for $X'$ and (b) the difference due to all the subsequent values (correlated updates). Moreover, the subsequent values-vectors can, in the worst case, be opposite. Therefore, the sensitivity follows by using the triangle inequality.

4 Convergence Analysis

We recall the main challenge for generalizing the convergence guarantees of SCD to DP-SCD, namely the need to handle potential inconsistencies between the auxiliary vector $v$ and the model vector $\alpha$, i.e., $v \neq X\alpha$. Note that a variant of Algorithm 1 that only updates $\alpha$ and recomputes $v$ in every iteration would overcome this issue. However, such a variant involves two disadvantages that make it impractical: (i) significant computational overhead and (ii) on the final step this variant would need to employ the entire dataset to map the dual model to the primal model ($\theta := \frac{1}{\lambda N} X\alpha$), which creates a massive privacy leakage and alleviates the effect of the mini-batch (i.e., $q=1$ for the moments accountant).

To analyze the convergence of Algorithm 1 we split each mini-batch iteration of Algorithm 1 in two steps: (i) the update step includes the computation of $L$ coordinate updates (Steps 6-11) and (ii) the perturbation step adds Gaussian noise to the two vectors $\alpha$ and $v$ independently (Steps 12 and 13).

We base our analysis on [15] and thus include the parameters $\gamma$ and $\sigma'$ (not related to the standard deviation of the noise $\sigma$) for ease of analysis. We specify our general result to the case of Algorithm 1 by setting $\gamma = 1$ and $\sigma' = L$. We therefore consider the following privacy preserving model sequence $\{\alpha\}_i$ with intermediate, non-public models $\{\hat{\alpha}\}_i$. The corresponding sequence for $\{v\}_i$ can be derived in a similar way.

\[
\alpha_0 \rightarrow \hat{\alpha}_1 := \alpha_0 + \gamma \sum_{j=1}^{L} \Delta \alpha_{0,j} \rightarrow \alpha_1 := \hat{\alpha}_1 + \eta_{\alpha,1} \\
\rightarrow \hat{\alpha}_2 := \alpha_1 + \gamma \sum_{j=1}^{L} \Delta \alpha_{1,j} \rightarrow ... \\
\rightarrow \alpha_n := \alpha_0 + \gamma \sum_{i=1}^{n-1} \Delta \alpha_i + \sum_{i=1}^{n-1} \eta_{\alpha,i}
\]

Here, $\eta \sim \mathcal{N}(0, \sigma^2)$ is Gaussian noise added to preserve privacy and $\Delta \alpha_i := \sum_{j=1}^{L} \Delta \alpha_{i,j}$ with $\Delta \alpha_{i,j} = \arg \min_{\Delta} F^*(\alpha_i + e \Delta)$.

To analyze the convergence behavior of Algorithm 1, we consider the two intermediate steps $\alpha_i \rightarrow \hat{\alpha}_{i+1}$ (update step) and $\hat{\alpha}_{i+1} \rightarrow \alpha_{i+1}$ (perturbation step) separately. The main idea is to show that:

(i) $\alpha_i \rightarrow \hat{\alpha}_{i+1}$ decreases the objective even if the update is computed based on a noisy version of $\alpha, v$.

(ii) the damage on the objective when adding noise and going from $\hat{\alpha}_{i+1} \rightarrow \alpha_{i+1}$ is bounded.

The intuition is that as long as the damage is smaller than the gain achieved with the update, the objective decreases and thus the algorithm converges.

The key observation that allows us to derive convergence guarantees in this setting is the following.

Remark 1 (Consistency in expectation). Algorithm 1 preserves the consistency between $\alpha$ and $v$ in expectation, i.e., $E[v] = XE[\alpha]$.

Proof. This follows from the construction of the model updates and the independent noise with zero mean that is added to both sequences $\{\alpha\}_i, \{v\}_i$. 


4.1 Update Step

Each iteration of Algorithm [1] computes a mini-batch update $\Delta \alpha$ that is applied to the model $\alpha$ and indirectly to the auxiliary vector $v$ in Steps 12 and 13 respectively. We denote by $\Delta \hat{\alpha}$ the unscaled version of this update, i.e., the update computed excluding Step 8. We add this step back later in our analysis. Lemma 4 gives a lower bound for the decrease in the objective achieved by performing this update even if $\Delta \alpha$ is computed based on noisy versions of $\alpha$, $v$ where $E[v] = E[X\alpha]$ but $v \neq X\alpha$.

Lemma 4 (Update step - objective decrease lower bound). Assuming $\ell_i$ are $1/\mu$-smooth, then the update step of Algorithm [1] decreases the objective, even if computed based on a noisy version of $\alpha, v$. The decrease is as follows:

$$E[S(\alpha) - S(\alpha + \Delta \alpha)] \geq \frac{\mu \lambda L}{\mu \lambda N + L} E[S(\alpha)]$$

where $S$ denotes the dual suboptimality defined as: $S(\alpha) := F^*(\alpha) - \min_\alpha F^*(\alpha)$.

Proof: Consider the optimization step $\alpha_i \rightarrow \alpha_{i+1}$ (Sequence 9) that employs a set of unscaled updates ($\Delta \hat{\alpha}_j$). Given the $\mu$-strong convexity of $\ell_i^*$ (follows from $1/\mu$-smoothness of $\ell_i$), the decrease in the dual objective is:

$$\Delta F := F^*(\alpha) - \min_{\Delta \hat{\alpha}_j} F^*(\alpha + \gamma \sum_{j=1}^L e_j \Delta \hat{\alpha}_j)$$

$$\geq F^*(\alpha) - F^*(\alpha + \gamma \sum_{j=1}^L e_j \Delta \hat{\alpha}_j) \forall \Delta \hat{\alpha}_j \in \mathbb{R}$$

Using Lemma 3, we have:

$$= F^*(\alpha) - (1 - \gamma) F^*(\alpha) - \gamma \sum_{i=1}^L \ell_i^{-\gamma}(\Delta \hat{\alpha}_j; v, \alpha_j)$$

$$= \gamma \left( F^*(\alpha) - \sum_{i=1}^L \ell_i^{-\gamma}(\Delta \hat{\alpha}_j; v, \alpha_j) \right)$$

where $\ell_i^{-\gamma}$ is the subproblem that each parallel loop is solving:

$$\ell_i^{-\gamma} := \frac{1}{N} \ell_i^{-\gamma}(\alpha_j - \Delta \hat{\alpha}_j) + \frac{1}{L \lambda N^2} \|v\|^2 + \frac{\sigma^{-\gamma}}{2 \lambda N^2} \|x_j \Delta \hat{\alpha}_j\|^2 + \frac{1}{\lambda N^2} x_j^\top v \Delta \hat{\alpha}_j$$

Therefore:

$$\frac{1}{\gamma} \Delta F = \frac{1}{N} \sum_{j=1}^L \ell_j^{-\gamma}(\alpha_j) + \frac{1}{L \lambda N^2} \|v\|^2 - \frac{1}{N} \sum_{j=1}^L \ell_j^{-\gamma}(\alpha_j + \Delta \hat{\alpha}_j)) - \frac{1}{2 \lambda N^2} \|v\|^2 - \frac{\sigma^{-\gamma}}{2 \lambda N^2} \sum_{j=1}^L \|x_j \Delta \hat{\alpha}_j\|^2 - \frac{1}{\lambda N^2} \sum_{j=1}^L x_j^\top v \Delta \hat{\alpha}_j$$

Consider $\Delta \hat{\alpha}_j := s(u_j - \alpha_j)$ for any $s \in (0, 1]$ and for $u_j = -\nabla_j \ell_j(\frac{1}{\lambda N} x_j^\top E[v])$. In some sense, $s$ denotes the deviation from the “optimal” $\Delta$-value $(u_j - \alpha_j)$. Hence we have:

$$\frac{N}{\gamma} \Delta F \geq \sum_{j=1}^L (\ell_j^{-\gamma}(\alpha_j) - \ell_j^{-\gamma}(\alpha_j + s(u_j - \alpha_j))) - \frac{\sigma^{-\gamma}}{2 \lambda N} \sum_{j=1}^L \|x_j s(u_j - \alpha_j)\|^2 - \frac{1}{\lambda N} \sum_{j=1}^L x_j^\top v s(u_j - \alpha_j)$$

By the $\mu$-strong convexity of $\ell_i^*$ (follows from $1/\mu$-smoothness of $\ell_i$) we have:

$$\ell_j^*(\alpha_j + s(u_j - \alpha_j)) = \ell_j^*(s(u_j) + (1 - s)(u_j - \alpha_j)) \leq \lambda \ell_j^*(-u_j) + (1 - s)\ell_j^*(-u_j) - \frac{\mu}{2} s(1 - s)(u_j - \alpha_j)^2$$

Hence we have:

$$\frac{N}{\gamma} \Delta F \geq \sum_{j=1}^L (\ell_j^{-\gamma}(\alpha_j) - \lambda \ell_j^{-\gamma}(-u_j) - (1 - s)\ell_j^{-\gamma}(-\alpha_j) + \frac{\mu}{2} s(1 - s)(u_j - \alpha_j)^2)$$
\[-\sum_{j=1}^{L} \left( \frac{\sigma's^2(u_j - \alpha_j)^2}{2\lambda N} \|x_j\|^2 + \frac{1}{\lambda N} x_j^\top \nu_s(u_j - \alpha_j) \right) \]
\[= \sum_{j=1}^{L} \left( -s\ell_j^*(-u_j) + s\ell_j^*(-\alpha_j) + \frac{\mu}{2}s(1-s)(u_j - \alpha_j)^2 - \frac{\sigma's^2(u_j - \alpha_j)^2}{2\lambda N} \|x_j\|^2 - \frac{1}{\lambda N} x_j^\top \nu_s(u_j - \alpha_j) \right) \]

By taking the expectation w.r.t. the randomization in the noise, using Jensen’s inequality for convex functions \(E[\ell_j^*(-x)] \geq \ell_j^*(E[x])\), the fact that \(\sigma^2 = E[\alpha_j^2] - E[\alpha_j]^2\) and the fact that the noise on \(\alpha\) and \(\nu\) is independent we have:

\[
\frac{N}{\gamma} E[\Delta_F] \geq \sum_{j=1}^{L} \left( -sE[\ell_j^*(-u_j)] + sE[\ell_j^*(-\alpha_j)] + \frac{\mu}{2}s(1-s)E[(u_j - \alpha_j)^2] - \frac{\sigma's^2}{2\lambda N} \|x_j\|^2 \right)
\]
\[
= \sum_{j=1}^{L} \left( -s\ell_j^*(-u_j) + s\ell_j^*(-\alpha_j) - \frac{\sigma's^2}{2\lambda N} \|x_j\|^2 \right) + \left( \frac{\mu}{2}s(1-s) - \frac{\sigma's^2}{2\lambda N} \|x_j\|^2 \right) \left( u_j^2 - 2u_jE[\alpha_j] + E[\alpha_j]^2 \right)
\]

The following is the Fenchel-Young inequality that holds as equality given \(u_j = -\nabla_j \ell_j \left( \frac{1}{\lambda N} x_j^\top E[\nu] \right)\):

\[
\ell_j \left( \frac{1}{\lambda N} x_j^\top E[\nu] \right) + \ell_j^*(-u_j) = -\frac{1}{\lambda N} x_j^\top E[\nu]u_j
\]

Hence we have:

\[
\frac{N}{\gamma} E[\Delta_F] \geq \sum_{j=1}^{L} \left( s\ell_j \left( \frac{1}{\lambda N} x_j^\top E[\nu] \right) + sE[\ell_j^*(-\alpha_j)] + \frac{\sigma's^2}{2\lambda N} \|x_j\|^2 \right) + \left( \frac{\mu}{2}s(1-s) - \frac{\sigma's^2}{2\lambda N} \|x_j\|^2 \right) \left( u_j^2 - 2u_jE[\alpha_j] + E[\alpha_j]^2 + \sigma^2 \right)
\]

We then employ the definition of the duality gap \(Gap(\alpha) := F(\theta(\alpha)) - (-F^*(\alpha))\) and take the expectation w.r.t. the randomization in the noise along with the Jensen inequality for convex functions.

\[
E[Gap(\alpha)] = F(\theta(E[\alpha])) - (-E[F^*(\alpha)]) = \frac{1}{N} \sum_{j=1}^{N} \left( \ell_j(x_j^\top \theta) + E[\ell_j^*(-\alpha_j)] \right) + \frac{\lambda}{2} \|\theta(E[\alpha])\|^2 + \frac{1}{2\lambda N^2} \|XE[\alpha]\|^2
\]

We then apply the Fenchel-Young inequality:

\[
g(\theta(E[\alpha])) + g^*(XE[\alpha]) \geq \theta(E[\alpha])^\top XE[\alpha]\]
\[
\Leftrightarrow \frac{\lambda}{2} \|\theta(E[\alpha])\|^2 + \frac{1}{2\lambda N^2} \|XE[\alpha]\|^2 \geq \frac{1}{\lambda N} E[\nu]^\top XE[\alpha]
\]

The above inequality holds as equality in light of Remark [1] and the primal-dual map. Therefore by using uniform mini-batch sampling the duality gap becomes:

\[
E[Gap(\alpha)] = \frac{1}{N} \sum_{j=1}^{N} \left( \ell_j(x_j^\top \theta) + E[\ell_j^*(-\alpha_j)] + \frac{1}{\lambda N} E[\alpha_j | x_j^\top \nu] \right)
\]
perturbation step of Algorithm 1 increases the objective by at most:

\[ \Delta \]

Lemma 5 (Perturbation step - objective increase upper bound). Assume \( \ell_i^* \) are \( \nu \)-smooth. Then, the perturbation step of Algorithm [2] increases the objective by at most:

\[
E[|S(\alpha + \Delta \alpha + \eta) - S(\alpha + \Delta \alpha)|] \leq \frac{1}{2} \left( \nu + \frac{1}{\lambda N^2} \right) L \sigma^2
\]  

4.2 Perturbation Step

To prove the convergence of DP-SCD (Algorithm [1]), it remains to show that adding noise at the end of each mini-batch update is not increasing the objective more than the decrease achieved by the rescaled update \( \Delta \alpha \).
Proof. Given that \( \ell_{i}\) is \( \nu\)-smooth and the regularization term of \( \mathcal{F}\) (Problem 3) is \( \frac{1}{\kappa N^2}\)-smooth, \( \mathcal{F}\) is \( \nu' = \nu + \frac{1}{\kappa N^2}\) smooth. We thus have \( \mathcal{F} ((\alpha + \Delta \alpha + \eta) \leq \mathcal{F} (\alpha + \Delta \alpha) + \eta^\top \nabla \mathcal{F} + \nu' \| \eta \|^2 \). Then we subtract \( \min_{\theta} \mathcal{F} (\alpha) \) on both sides and take expectations w.r.t the randomness in the perturbation noise. The claim follows from \( \mathbb{E}[\eta] = 0 \) and \( \mathbb{E}[\| \eta \|^2] = \sigma^2 \).

Combining Inequality 12 Lemmas 4 and 5 yields our main result stated in the following theorem.

**Theorem 2** (Utility guarantee for Algorithm 1). Suppose that \( \ell_i \) is \( 1/\mu \)-smooth and \( \nu\)-strongly convex. If we choose \( L, C \) such that \( L(2(1-\kappa))\mu \lambda - 1) > \mu \lambda N \) for \( \kappa \in (0, 1) \), and \( T \) such that \( T = \mathcal{O} \left( \log \left( \frac{N^2 \epsilon^2}{\kappa N^2} \right) \right) \), then the utility of Algorithm 1 is bounded:

\[
\mathbb{E}[\mathcal{S}(\alpha^{(T)})] \leq \mathcal{O} \left( \frac{1}{\frac{1}{\lambda N^2}} L^3 \log \left( \frac{N\epsilon}{\nu + \frac{1}{\lambda N^2}} \right) \right) \ln (1/\delta) / (N^2 \epsilon^2)
\]

Proof. We reorder terms in Inequality 13 and subtract \( \mathcal{S}(\alpha) \) on both sides. We then combine Inequalities 10 and 12 and get that the suboptimality decreases per round by:

\[
\mathbb{E}[\mathcal{S}(\alpha) - \mathcal{S}(\alpha + \Delta \alpha + \eta)] \geq \frac{(1-\kappa)\mu \lambda L}{\mu \lambda N} \mathbb{E}[\mathcal{S}(\alpha)] - \frac{1}{2} \left( \frac{1}{\lambda N^2} \right) \nu' L \sigma^2
\]

At iteration \( t \) we thus have:

\[
\mathbb{E}[\mathcal{S}(\alpha^{(t-1)})] - \mathbb{E}[\mathcal{S}(\alpha^{(t)})] \geq \frac{(1-\kappa)\mu \lambda L}{\mu \lambda N + L} \mathbb{E}[\mathcal{S}(\alpha^{(t-1)})] - \frac{1}{2} \left( \frac{1}{\lambda N^2} \right) \nu' L \sigma^2
\]

We apply the previous inequality recursively and get:

\[
\mathbb{E}[\mathcal{S}(\alpha^{(T)})] \leq A^T \mathbb{E}[\mathcal{S}(\alpha^{(0)})] + \mathcal{O} \left( \nu' L \sigma^2 \right)
\]

If we choose \( L \) and \( C \) such that \( A < \frac{1}{2} \iff L(2(1-\kappa))\mu \lambda - 1) > \mu \lambda N \) and \( T \) such that \( T = \mathcal{O} \left( \log \left( \frac{N^2 \epsilon^2}{\kappa N^2} \right) \right) \), we get the bound on the utility:

\[
\mathbb{E}[\mathcal{S}(\alpha^{(T)})] \leq \mathcal{O} \left( \frac{\nu' L^3 \ln (1/\delta)}{N^2 \epsilon^2} \right) + \mathcal{O} \left( \frac{\nu' L^3 T \ln (1/\delta)}{N^2 \epsilon^2} \right)
\]

By omitting the \( \ln \) term the bound on \( T \) simplifies as: \( T = \mathcal{O} \left( \log \left( \frac{N^2 \epsilon^2}{\kappa N^2} \right) \right) \). Hence the utility bound becomes:

\[
\mathbb{E}[\mathcal{S}(\alpha^{(T)})] \leq \mathcal{O} \left( \frac{\nu' L^3 \ln (1/\delta)}{N^2 \epsilon^2} \right) + \mathcal{O} \left( \frac{\nu' L^3 T \ln (1/\delta)}{N^2 \epsilon^2} \right)
\]

The suboptimality is proportional to the magnitude of the noise and hence, finding the exact minimizer requires \( \sigma \to 0 \) (i.e., \( \epsilon \to \infty \)). The smaller the \( \sigma \) the larger the \( \epsilon \) and thus the less private the learning is. We empirically confirm that DP-SCD converges smoother with a smaller \( \sigma \) in Section 5.

Theorem 2 constitutes the first analysis of coordinate descent in the differentially private setting and it can be a stepping stone for future theoretical results in this setting as we discuss in Section 7.
5 Experiments

Our empirical results compare our new DP-SCD algorithm (Algorithm 1) against SCD, SGD, DP-SGD. We include seqDP-SCD (Algorithm 3) as a baseline, to depict the importance of independent updates (inside a given mini-batch) for DP-SCD. We test the performance on three popular applications that belong to GLMs (§2.1), namely ridge regression, logistic regression and SVMs.

5.1 Setup

Datasets. We employ public real datasets. In particular, we report on YearPredictionMSD\footnote{https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/regression.html} for ridge regression, Phishing\footnote{https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html} for logistic regression, and Adult\footnote{https://archive.ics.uci.edu/ml/datasets/Adult} for SVMs. We preprocess each dataset by scaling each coordinate by its maximum absolute value, followed by scaling each example to unit norm (normalized data). For YearPredictionMSD we center the labels at the origin. Based on \cite{4} and regarding Adult, we convert the categorical variables to dummy/indicator ones and replace the missing values with the most frequently occurring value of the corresponding feature. We employ a training/test split for our data, to train and test the performance of our algorithms. YearPredictionMSD and Adult include a separate training and test set file. Phishing consists of a single file that we split with 75\%:25\% ratio into a training and a test set. Finally, we hold-out a random 25\% of the training set for tuning the hyperparameters (validation set). The resulting training/validation/test size is \{347786/115929/51630, 24420/8141/16281, 6218/2073/2764\} and the number of coordinates are \{90, 81, 68\} for \{YearPredictionMSD, Adult, Phishing\} respectively.

Performance metrics. Accuracy measures the classification performance as the fraction of correct predictions among all the predictions. The larger the accuracy, the better the utility. Mean squared error (MSE) measures the prediction error as: \(MSE = \frac{1}{N} \sum_{i=1}^{N} (\hat{Y}_i - Y_i)^2\) where \(\hat{Y}_i\) is the predicted value and \(Y_i\) is the actual value. The lower the MSE, the better the utility. We quantify convergence by showing the decrease in the primal objective \(\mathcal{F}(X^{\text{(training)}}, \theta)\) from Problem \cite{1} on the training set.

Hyperparameters. We fix \(\lambda\) to \(10^{-4}\) for YearPredictionMSD and Phishing and to \(10^{-5}\) for the Adult dataset based on the best performance of SCD and SGD for \(\lambda\) in \{10\(^{-8}\), 10\(^{-7}\), \ldots, 1, 10, \ldots, 10^8\}. For a fair comparison of the DP algorithms, the iterations need to be fixed. Based on \cite{28}, we test the DP algorithms for \{5, 10, 50\} epochs and fix the number of iterations to \(T = 50N\) (i.e., 50 epochs) for YearPredictionMSD and \(T = 10N\) for the other datasets. Based on \cite{20,29}, we vary \(\epsilon\) in \{0.1, 0.5, 1, 2\} and fix \(\delta = 0.001\). We choose the other hyperparameters by selecting the combination with the best performance (lowest MSE for ridge regression and largest accuracy for logistic regression and SVMs) on the validation set. The range of tested values is as follows.

- \(C, \eta \in \{10^{-8}, 10^{-7}, \ldots, 1, \ldots, 10^4\}\)
- \(|\xi|, L \in \{0, 5, 10, 50, 100, 200, 500, 1000, 1250, 1500, 1750, 2000\}\)

Deployment. We run our experiments on commodity Linux machines. There are no special hardware requirements for our code other than enough RAM to load the datasets. We report the median result across 10 different runs by changing the seeding, i.e., the randomization due to initialization, sampling and Gaussian noise.

5.2 Results

Tuning cost. The hyperparameters of seqDP-SCD, DP-SCD, SGD, DP-SGD are \((L, C), (L, C), (\eta, |\xi|), (\eta, L, C)\) respectively; SCD requires no tuning. We tested a total of 156 configurations for DP-SCD as opposed to a total of 2028 for DP-SGD. For large-scale datasets that require significant amount of training time and resources, the difference in the number of hyperparameters constitutes an appealing property of
Figure 1: Privacy-utility trade-off. Better utility means lower MSE or larger accuracy. DP-SCD outperforms DP-SGD for the applications that enable exact update steps (namely ridge regression and SVMs) despite DP-SCD having less hyperparameters.

DP-SCD. Noteworthy, the SGD tuning is not useful for the DP-SGD tuning as the best choice for $\eta$ depends on the choice of $C$ [25].

Privacy-utility trade-off. Figure 1 quantifies the trade-off between privacy and utility for different privacy levels (i.e., $\epsilon$ values). We observe that seqDP-SCD has the worst performance due to the significantly larger noise compared to the other algorithms. DP-SCD performs better than DP-SGD for ridge regression and SVMs, and worse for logistic regression, that can be attributed to the following. On the one hand, DP-SCD requires $\sqrt{2}$ more noise than DP-SGD (for the same privacy guarantee) due to the need of a shared vector ($\S3$). On the other hand, each update of DP-SCD finds an exact solution to the minimization problem for ridge regression and SVMs and an approximate one for logistic regression, whereas DP-SGD takes a direction opposite to the gradient. Apparently, the steps of DP-SCD in the case of ridge regression and SVMs are more precise despite suffering more noise than DP-SGD.

Convergence. Figure 2 shows the impact of noise on the convergence behavior for the DP algorithms on the YearPredictionMSD dataset. In particular, for a given $\epsilon$, we select the best (in terms of validation MSE) configuration (also used in Figure 1(a)), and measure the decrease in the objective with respect to epochs (not time as that would be implementation-dependent). We empirically verify the results of Theorem 2 by observing that the distance between the convergence point and the optimum depends on the level of privacy. Moreover, DP-SCD and DP-SGD converge with similar speed for $\epsilon = 0.1$. Decreasing the amount of noise ($\epsilon = 1$), makes DP-SCD converge almost as fast as SGD and with more stability comparing to $\epsilon = 0.1$. This is aligned with the results of $\S4$, i.e., the fact that the larger amount of noise (decrease in $\epsilon$) makes the decrease in the suboptimality more noisy.

Mini-batch size. The mini-batch size is an important parameter for DP-SCD that affects the privacy-utility trade-off while also controls the level of parallelism ($\S3$). Figure 3 shows that the utility improves (i.e., MSE drops) as the mini-batch size increases till a saturation point. The larger the noise, the smaller the value of this saturation point ($L = 1000$ for $\epsilon = 0.1$ and $L = 2000$ for $\epsilon = 1$). We observe that increasing
Figure 2: Impact of noise on convergence. Differential privacy does not prevent convergence but increases the noise in reducing the objective and the distance to the optimum (aligned with the result of Theorem 2).

the level of parallelism can only improve the utility of DP-SCD thus making our algorithm suitable for large-scale parallel processing.

Figure 3: Impact of mini-batch size on utility for DP-SCD. Utility increases with increasing mini-batch size, till a saturation point that depends on the level of privacy.

6 Related Work

Perturbation methods for DP. Existing works achieve differentially private ML by perturbing the query output (i.e., model prediction). These works target both convex and non-convex optimization algorithms and focus on a specific application (e.g., a subclass of optimization functions (properties of the loss function) or a particular optimization algorithm. These approaches can be divided into three main classes. The first class involves input perturbation approaches that add noise to the input data. These approaches are easy to implement but often prohibit the ML model from providing accurate predictions. The second class involves output perturbation approaches that add noise to the model after the training procedure finishes, i.e., without modifying the vanilla training algorithm. This noise addition can be model-specific or model-agnostic. The third class involves inner perturbation approaches that modify the learning algorithm such that the noise is injected during learning. One method for inner perturbation is to modify the objective of the training procedure. Another approach involves adding noise to the output of each update step of the training without modifying the objective. Our new DP-SCD algorithm belongs to the third class.

DP - Empirical Risk Optimization (ERM). Various works address the problem of ERM (similar to our setup), through the lens of differential privacy. Table 1 compares the utility bounds between DP-SCD and representative works for each perturbation method for DP-ERM. We simplify the bounds following for easier comparison. The assumptions of these methods, described in Table 1 and S1.
Table 1: Comparison of utility bounds of ($\varepsilon, \delta$)-DP algorithms for empirical risk minimization.

| Method   | Perturbation Method | Utility Bound                                      |
|----------|---------------------|---------------------------------------------------|
| [29]     | Output              | $O\left(\frac{m}{N^2\varepsilon^2}\right)$      |
| [6,7]    | Inner (objective)   | $O\left(\frac{m}{N^2\varepsilon^2}\right)$      |
| [26]     | Inner (update)      | $O\left(\frac{m \cdot \log(N)}{N^2\varepsilon^2}\right)$ |
| DP-SCD   | Inner (update)      | $O\left(\frac{L \cdot \log(N/L)}{N^2\varepsilon^2}\right)$ |

are similar. We highlight that the bound for DP-SCD is independent of the dimensionality of the problem ($m$) due to the dual updates, while also incorporates the mini-batch size ($L$) for quantifying the impact of the varying degree of parallelism.

Existing DP-ERM methods based on SGD typically require the tuning of an additional hyperparameter (learning rate or step size) similar to DP-SGD [1]. The value of this hyperparameter for certain loss functions can be set based on properties of these functions [28]. Furthermore regarding [28], the authors build upon permutation-based SGD and employ output perturbation, but tolerate only a constant number of iterations.

Coordinate descent. SCD algorithms do not require parameter tuning if they update one coordinate (or block) at a time, by exact minimization (or Taylor approximation). One such algorithm is SDCA [22] that is similar to DP-SCD when setting $\varepsilon \to \infty$, $L = 1$ and $C \to \infty$. Alternative SCD algorithms take a gradient step in the coordinate direction that requires a step size [3116].

Parallelizable variants of SCD (such as [520]) have shown remarkable speedup when deployed on multiple CPUs/GPUs [3111192130], or multiple machines [1115]. These works employ sampling to select the data to be updated in parallel. DP-SCD also employs sampling via the mini-batch size ($L$), similar to the lot size of DP-SGD [1], to (a) enable parallel updates and (b) steer the privacy-utility trade-off. DP-SCD is also similar to CoCoA [15] with $K = L$ and $T_{local} = 1$. A first differentiation is that each $\Delta_j$ is computed on a single example and not on a partition of the dataset ($\mathcal{P}_j$) when split among $K$ workers. A second one is the noise addition after each CoCoA update (Steps 12 and 13 of Algorithm 1) that is necessary for differential privacy.

7 Conclusion

This paper introduces the first differentially private stochastic coordinate descent algorithm (DP-SCD). We formally derive its privacy bounds and study the convergence. We show that the dual formulation of DP-SCD is preferable over the primal due to the difference in the sensitive data access patterns. We empirically compare DP-SCD against a popular SGD alternative (DP-SGD) under the same privacy guarantees. DP-SCD has a better privacy-utility trade-off than DP-SGD for ridge regression and SVMs, while DP-SCD also requires less hyperparameter tuning than DP-SGD.

Our work is only a first step towards differentially private machine learning based on coordinate descent. We empirically confirm our theoretical result by showing that convergence is achieved despite the presence of noise. The convergence rate of DP-SCD is comparable to the one of DP-SGD and we plan to also study it theoretically.

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