Representations of reduced enveloping algebras and cells in the affine Weyl group

J.E. Humphreys

Abstract. Let $G$ be a semisimple algebraic group over an algebraically closed field of characteristic $p > 0$, and let $\mathfrak{g}$ be its Lie algebra. The crucial Lie algebra representations to understand are those associated with the reduced enveloping algebra $U^\chi(\mathfrak{g})$ for a “nilpotent” $\chi \in \mathfrak{g}^\ast$. We conjecture that there is a natural assignment of simple modules in a regular block to left cells in the affine Weyl group $W_a$ (for the dual root system) lying in the two-sided cell which corresponds to the orbit of $\chi$ in Lusztig’s bijection. This should respect the action of the component group of $C_G(\chi)$ and fit naturally into Lusztig’s enriched bijection involving the characters of $C_G(\chi)$. Some evidence will be described in special cases.

In order to explain the conjecture, we have to review some facts about three logically independent topics:

(A) cells in affine Weyl groups,
(B) nilpotent orbits,
(C) Lie algebra representations in characteristic $p > 0$.

Subtle connections between (A) and (B) have been discovered by Lusztig, while connections between (B) and (C) have emerged over several decades (notably in the work of Kac–Weisfeiler, Friedlander–Parshall, Premet, and others cited below). We hope to build further links between (A) and (C), with the goal of finding a representation-theoretic model for Lusztig’s formal conjecture in [22] §10.

Notation varies considerably in the literature (and sometimes clashes). Our conventions here start with a simple, simply connected algebraic group $G$ over an algebraically closed field $K$ of characteristic $p > 0$. Let $T$ be a maximal torus and $W$ the Weyl group. Denote by $\Phi$ the root system, with positive system $\Phi^+$ relative to a simple system $\Delta$. The character group $X = X(T)$ is the full weight lattice for $\Phi$. Let $Q = \mathbb{Z}\Phi$ be the root lattice.

1. Cells in affine Weyl groups

1.1. First we recall some basic results about cells. These arise in the work of Kazhdan and Lusztig on arbitrary Coxeter groups and their Hecke algebras, but
here we focus just on the case of affine Weyl groups. (See Lusztig’s papers \[18 - 22\] as well as Shi \[31\], Xi \[35, 36\].)

Define $W_a := W \ltimes Q$ (the affine Weyl group) and $\tilde{W}_a := W \ltimes X$ (the extended affine Weyl group). The latter is not usually a Coxeter group, but is important for Lusztig’s $p$-adic group program; here we focus just on $W_a$. We say $W_a$ is of type $\tilde{X}_n$ if $\Phi$ is of type $X_n$. It is important to note that $W_a$ is a dual version of the usual affine Weyl group constructed by Bourbaki via the coroot lattice; this reflects the influence of Langlands duality in Lusztig’s program.

As in the case of an arbitrary Coxeter group, the group $W_a$ is partitioned into two-sided cells (here denoted $\Omega$). Each of these is in turn partitioned into left cells or equally well into right cells, each of which is the set of inverses of elements in some left cell. These partitions arise (together with Kazhdan–Lusztig polynomials) from comparison of the Kazhdan–Lusztig basis for the Hecke algebra with the standard basis.

The definition of cells yields a natural partial ordering on the collection of two-sided cells. The highest cell in this ordering contains just the identity element 1 of $W_a$. Since $W_a$ acts simply transitively on the alcoves in the affine space $E := \mathbb{R} \otimes \mathbb{Z} X$, the various cells can be identified with sets of alcoves. In this picture $W$ labels the family of alcoves around the special point 0. (Conventions differ in the literature; for example, some authors work with right actions rather than left actions in this context.)

1.2. Beyond these generalities, Lusztig develops more special features of cells for $W_a$. Generalizing the case of a Weyl group, he defines in \[19\] an $a$-invariant $a(w)$ for each $w \in W_a$, constant on each two-sided cell and denoted $a(\Omega)$. This is an integer between 0 and $N := |\Phi^+|$, defined combinatorially in terms of the Hecke algebra. The $a$-invariant respects (inversely) the partial ordering of two-sided cells. For example, $a(\Omega) = 0$ precisely when $\Omega = \{1\}$. At the other extreme, it turns out that there is a unique cell $\Omega$ with $a(\Omega) = N$; this is the lowest two-sided cell \[32\].

1.3. With the help of the $a$-invariant, Lusztig shows that $W_a$ has only finitely many two-sided cells, each partitioned into finitely many left (or right) cells. It is then natural to ask how many two-sided and one-sided cells there are. These questions are extremely difficult to approach in a purely combinatorial way, though they have been answered for type $\tilde{A}_n$ by Shi \[31\] and in some isolated low rank cases. To formulate and prove general conjectures, some connection with the geometry of the nilpotent variety and flag variety seems to be essential.

1.4. The lowest two-sided cell $\Omega$ has been explored thoroughly by Shi \[32\]. It contains $|W|$ left cells, each obtained by intersecting $\Omega$ with a Weyl chamber. The entire antidominant chamber is one left cell. On the other hand, the intersection $\Gamma$ of $\Omega$ with the dominant chamber is a shifted version of this chamber: Taking 0 as the origin in $E$, consider the special point $p$ (the sum of fundamental weights) at which translates of all root hyperplanes meet. Then $\Gamma$ consists of the alcoves lying on the positive sides of all these hyperplanes.

1.5. In \[20\], Lusztig defines a set $\mathcal{D}$ of distinguished involutions in $W_a$, as follows. For $w \in W_a$, let $\ell(w)$ be its length and let $\delta(w)$ be the degree of the Kazhdan–Lusztig polynomial $P_{1,w}(q)$. Then $w \in \mathcal{D}$ if $a(w) = \ell(w) - 2\delta(w)$, in
which case \( w \) is shown to be an involution. Each left cell contains a unique distinguished involution. For example, in the dominant left cell of the lowest two-sided cell \( \Omega \) described above, the distinguished involution belongs to the lowest of the \(|W|\) alcoves around the special point \( 2\rho \). The distinguished involution in the antidominant left cell of \( \Omega \) is the longest element of \( W \) (if 1 corresponds to the lowest dominant alcove).

1.6. Lusztig and Xi [28] show that each two-sided cell of \( W_a \) contains a canonical left cell, whose corresponding alcoves all lie in the dominant Weyl chamber \( \mathcal{C} \subset E \). In this way, \( \mathcal{C} \) is partitioned into canonical left cells belonging to the two-sided cells.

Chmutova–Ostrik [9] develop an algorithm to compute the distinguished involutions in all canonical left cells, with explicit tables given in low ranks. But it seems to be more difficult to locate these involutions in arbitrary left cells.

1.7. Pictures of the cells for the affine Weyl groups of types \( \tilde{A}_2, \tilde{B}_2, \tilde{G}_2 \) are given by Lusztig [19, \S 11]. Paul Gunnells has used computer graphics to investigate all three-dimensional cells as well.

Jian-yi Shi [31] has worked out the combinatorics in considerable detail for type \( \tilde{A}_n \), while developing general tools such as “sign types” for the study of cells. Other affine Weyl groups of low rank have been studied in a similar spirit by him and a number of other people, including Robert Bédard, Cheng Dong Chen, Jie Du, Gregory Lawton, Feng Li, Jia Chun Liu, He Bing Rui, Nanhua Xi, Xin Fa Zhang.

2. Nilpotent orbits and cells

2.1. Denote by \( \mathcal{N} \) the set of nilpotent elements in \( g := \text{Lie} G \). This is the nilpotent variety (or nullcone). It consists of finitely many orbits under the adjoint action of \( G \), partially ordered by inclusion of one orbit in the closure of another. The orbits range from \( \{0\} \) to the regular orbit, which is dense in \( \mathcal{N} \) and therefore has dimension \( 2N = |\Phi| \). Whenever \( p \) is a “good” prime (as in 3.1 below), there is a \( G \)-equivariant isomorphism between \( \mathcal{N} \) and the unipotent variety of \( G \). Moreover, the partially ordered set of \( G \)-orbits in \( \mathcal{N} \) is isomorphic to the corresponding set for the Lie algebra over \( \mathbb{C} \) of the same type. Although Lusztig’s use of unipotent classes is based in characteristic 0, the ideas therefore transfer readily to our situation. (Jantzen [17] gives a helpful account with emphasis on characteristic \( p \).)

Various other varieties and groups are associated with \( \mathcal{N} \). The flag variety \( B \) of \( G \) may be identified with the collection of Borel subalgebras of \( g \). If \( e \in \mathcal{N} \), the set of Borel subalgebras containing \( e \) is denoted by \( \mathcal{B}_e \). It plays an essential role in the Springer resolution of singularities of \( \mathcal{N} \), where it is referred to as a Springer fiber.

Let \( C_G(e) \) be the centralizer of \( e \) in \( G \), and denote by \( A(e) \) the finite component group \( C_G(e)/C_G(e) \). The cohomology of \( \mathcal{B}_e \) with suitable coefficients (complex or \( l \)-adic) vanishes in odd degrees and has commuting actions by the finite groups \( W \) and \( A(e) \). We write simply \( H^i(\mathcal{B}_e) \). This is the framework for the Springer Correspondence (see for example [17, \S 13]).

2.2. Soon after the Kazhdan–Lusztig theory was developed, Lusztig [18, 3.6] conjectured the existence of a bijection between the collection of two-sided cells of \( W_a \) (based as above on the root lattice rather than coroot lattice) and the collection
of unipotent classes in $G$ (or equivalently, the collection of nilpotent orbits in $\mathfrak{g}$). This bijection should respect the natural partial orderings, with the cell $\{1\}$ corresponding to the regular nilpotent orbit and the lowest two-sided cell corresponding to the zero orbit. (His ideas were formulated in characteristic 0 but adapt to our setting when $p$ is good.)

By combining a number of deep techniques, Lusztig was able to construct a suitable bijection in \cite{lusztig22}. Under his bijection, if the two-sided cell $\Omega$ corresponds to the orbit of some $e \in \mathcal{N}$, then $a(\Omega) = \dim \mathcal{B}_e$. But the order-preserving property remained elusive except in low ranks. This was later proved combinatorially for type $A_n$ by Shi, while the general case follows from recent work of Bezrukavnikov \cite[Thm. 4]{bezrukavnikov}.

2.3. In \cite[3.6]{lusztig}, Lusztig formulated a further conjecture on left cells in terms of the fixed points of $A(e)$ on the cohomology of $\mathcal{B}_e$:

(LC) The number of left cells in the two-sided cell corresponding to a nilpotent $e$ should be equal to $\sum_i (-1)^i \dim H^i(\mathcal{B}_e)^{A(e)}$.

Due to the vanishing of cohomology in odd degrees, the contributions here are all nonnegative.

While (LC) has not yet been proved in general, it agrees with direct calculations in low ranks and with the results of Shi for type $\tilde{A}_n$ \cite[14.4.5,15.1,17.4]{shi}. Here all component groups are trivial, while on the other hand the representation of $W$ on the cohomology is known to be induced from the trivial character of a parabolic subgroup $W_I$ generated by reflections relative to a set $I$ of simple roots. (See the discussion in \cite[p. 203]{lusztig}.) When translated into the language of partitions, the number $|W|/|W_I|$ agrees with the number of left cells found by Shi for a corresponding two-sided cell.

2.4. As part of his more refined study of the “asymptotic Hecke algebra” in connection with $p$-adic representations, Lusztig \cite[§10]{lusztig} formulated more detailed conjectures relating the cells with geometry. Fix a two-sided cell $\Omega$ corresponding in his bijection to the orbit of $e \in \mathcal{N}$, and let $\Gamma_\Omega$ be its canonical left cell. Denote by $F$ a maximal reductive subgroup of $C_G(e)$, so $F/F^0 \cong A(e)$. Write $\tilde{F}$ for the set of isomorphism classes of irreducible representations of $F$.

Lusztig’s conjectural set-up involves a finite set $Y$, acted on by $A(e)$, with cardinality equal to the Euler characteristic of $B_e$. The orbits of $A(e)$ in $Y$ should be in bijection with the left cells in $\Omega$, with a singleton orbit expected to correspond to the canonical left cell. In general, the isotropy group in $A(e)$ of an element $y \in Y$ corresponds to an intermediate subgroup $F \supset F_y \supset F^0$. The representations of $F$ or $F_y$ enter via a notion of “$F$-vector bundle” on $Y$ or $Y \times Y$.

This formalism is then subject to several requirements in \cite[10.5]{lusztig}. For example, the representation of $A(e)$ on $H^*(\mathcal{B}_e)$ should be equivalent to the permutation representation of $A(e)$ on $Y$. (This recovers the statement (LC) above.) As a consequence, one should have a natural bijection between $\Gamma_\Omega \cap \Gamma_\Omega^{-1}$ and $\tilde{F}$. For an arbitrary left cell $\Gamma$ corresponding to the orbit of $y \in Y$, the group $F$ should be replaced by the group $F_y$.

Out of this abstract framework emerges a conjectural bijection between pairs $(\mathcal{O}_e, \varphi)$ and $X^+$, where $\mathcal{O}_e$ is a nilpotent orbit and $\varphi$ an irreducible representation of $C_G(e)$. (Such a bijection was conjectured independently by Vogan.) Note that when we work with $W_a$ rather than $\tilde{W}_a$, the root lattice $Q$ replaces $X$. 

Bezrukavnikov has found suitable bijections in [4] and [3]: these are shown in [5] Remark 6] to coincide. For other work related to Lusztig’s conjectures (especially in this last formulation), see the individual and joint papers by Achar and Sommers [33, 2, 1], Bezrukavnikov and Ostrik [30, 7], Lusztig [23], Xi [34, 35, 36].

3. Lie algebra representations in characteristic $p > 0$

3.1. The representation theory of $\mathfrak{g}$ has been studied over a long period of time: for surveys of earlier work, see [10] and [11]. In a series of papers, Jantzen [13–16] has extended the theory considerably. Here we focus on just the simple modules for the universal enveloping algebra $U(\mathfrak{g})$. These all occur as modules for reduced enveloping algebras $U_\chi(\mathfrak{g})$, which are finite-dimensional quotients of $U(\mathfrak{g})$ parametrized by $\chi \in \mathfrak{g}^*$. Those $U_\chi(\mathfrak{g})$ for $\chi$ in a single orbit under the coadjoint action of $G$ are isomorphic, so one looks for a well-chosen orbit representative $\chi$.

In order to obtain uniform results, Jantzen imposes several relatively weak hypotheses (H1)–(H3) on $\mathfrak{g}$ and $p$, which we also assume. For a simply connected group, he requires the prime $p$ to be good for $\Phi$, which eliminates some root systems when $p = 2, 3, 5$. Moreover, the algebras $\mathfrak{sl}(n, K)$ with $p | n$ should be omitted (or replaced by the Lie algebras of corresponding general linear groups). Then there is always a $G$-equivariant isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, which transports the Jordan decomposition in $\mathfrak{g}$ to $\mathfrak{g}^*$.

Earlier work of Kac–Weisfeiler shows that the crucial case to study is that of a nilpotent $\chi \in \mathfrak{g}^*$ (corresponding to some nilpotent $e \in \mathfrak{g}$). Here one begins to make connections with the results on nilpotent orbits summarized above and with related conjectures arising in Lusztig’s work [24, 25, 26, 27]. From now on we consider only the nilpotent case, subject to the above restrictions on $p$ and $\Phi$.

3.2. The blocks of $U_\chi(\mathfrak{g})$ have been determined by Brown and Gordon [8]. As summarized by Jantzen [16 C.5], there is a natural bijection between the blocks and the “central characters”, which in turn are parametrized by the $W$-orbits in $X/pX$ under the dot action $w \cdot \lambda := w(\lambda + \rho) - \rho$. This is a Lie algebra version of the Linkage Principle.

If $e$ is the nilpotent element corresponding to $\chi$, the component group $A(e)$ permutes the simple modules in a block. This action is understood only in some special cases.

In general the simple modules in a given block are not easy to parametrize by weights, though each can be obtained as a quotient of one or more “baby Verma modules”: these are induced from one-dimensional modules for a Borel subalgebra $\mathfrak{b}$ satisfying $\chi(\mathfrak{b}) = 0$. The choice of $\mathfrak{b}$ affects this construction when $\chi \neq 0$ if $B_e$ has more than one irreducible component.

3.3. To make contact with the geometry of $\mathcal{N}$, we look only at regular blocks: those for which the weight parameters attached to simple modules lie inside alcoves. This requires $p > h$ (where $h$ is the Coxeter number). Jantzen’s translation functors then furnish information about other blocks.

For a regular block of $U_\chi(\mathfrak{g})$, the work of Bezrukavnikov, Mirković, and Rumynin provides a geometric interpretation. Under the assumption that $p > h$, they prove that the number of nonisomorphic simple modules in the block is equal to the Euler characteristic of the Springer fiber $B_e$: see [6, 5.4.3, 7.1.1].
3.4. The best understood case involves a nilpotent orbit in \( g^* \) containing some \( \chi \) in standard Levi form, which means that the corresponding nilpotent element \( e \) is regular in some Levi subalgebra of a parabolic subalgebra \( \mathfrak{p}_I \) of \( g \) (determined by a set \( I \) of simple roots). All nilpotent orbits satisfy this condition for \( g = \mathfrak{sl}(n,K) \), but in general things get more complicated. (See [13], §10, [15], §2, [16], D.1.)

Jantzen has studied simple \( U_\chi(\mathfrak{g}) \)-modules (and their projective covers) in considerable detail when \( \chi \) has standard Levi form. In particular, each simple module can be labelled as \( L_\chi(\lambda) \) for one or more \( \lambda \in X \). Here \( L_\chi(\lambda) \cong L_\chi(\mu) \) if and only if \( \mu \in W_I \cdot \lambda + pX \), where \( W_I \) is the subgroup of \( W \) generated by simple reflections for \( \alpha \in I \) and \( w \cdot \lambda := w(\lambda + \rho) - \rho \).

This can be pictured in terms of the alcove geometry of \( W_a \), with the origin of the affine space \( E \) taken to be \( -\rho \) and the translations all multiplied by \( p \). Jantzen calls the group \( W_p \) in this setting. Fixing a weight \( \lambda \) inside the lowest dominant alcove, the orbit \( W_p \cdot \lambda \) under the natural dot action contains (with periodic repetitions) all weights needed to parametrize the simple modules in a single regular block. In fact, it suffices to work with the \( |W| \) alcoves surrounding a single special point such as \( -\rho \). Then the induced action of \( W_I \) on these alcoves identifies those which correspond to the same simple module.

3.5. In [14, 16], Jantzen has also studied in depth the case of a subregular \( \chi \): its \( G \)-orbit has dimension \( 2N - 2 \), where \( N = |\Phi^+| \). Only in types \( A_n \) and \( B_n \) does such an orbit have a representative in standard Levi form. But the simple modules in a regular block of \( U_\chi(\mathfrak{g}) \) can be correlated closely with the irreducible components of \( B_e \) (here a Dynkin curve), which helps to bypass the problem of labelling by weights.

4. Simple modules and left cells

4.1. Here we suggest closer connections between the representation theory discussed in §3 and the cells in \( W_p \). While our ideas are speculative, they have some support from computations in special cases (including unpublished work of Jantzen as well as [12]).

Fix a regular block of \( U_\chi(\mathfrak{g}) \), with \( \chi \) nilpotent, and denote by \( S \) a complete set of nonisomorphic simple modules in this block. As suggested by Bezrukavnikov, this is a candidate for the finite set \( Y \) in Lusztig’s formulation discussed in §2. If \( \chi \) corresponds to \( e \in \mathfrak{g} \), denote by \( \mathcal{L} \) the collection of left cells of the two-sided cell \( \Omega \) corresponding in Lusztig’s bijection to the orbit of \( e \). In case the component group \( A(e) \) is trivial, the cardinalities of \( S \) and \( \mathcal{L} \) are expected to be the same: compare the theorem of [6] cited in §3 with the conjecture (LC) in §2.

**Conjecture.** Fix notation as above.

(a) There is a natural map \( \varphi \) from \( S \) onto \( \mathcal{L} \), whose fibers are the orbits of \( A(e) \) in \( S \).

(b) A simple module fixed by \( A(e) \) maps under \( \varphi \) to the canonical left cell \( \Gamma \) in \( \Omega \). (We call this module “canonical”.)

4.2. The meaning of “natural” in part (a) of the conjecture has to be clarified. What we have in mind is a simple recipe for assigning modules to left cells, but it has only been made rigorous in special cases. Consider for example the case when \( \chi \) has standard Levi form, so the modules in \( S \) can be parametrized by weights in a \( W_p \)-orbit which lie in alcoves surrounding any given special point \( v \in E \). Suppose
v can be chosen inside the dominant Weyl chamber in such a way that weights in those surrounding alcoves which lie in the canonical left cell $\Gamma$ suffice to parametrize $S$. If $w \in W_p \cong W_a$ labels one of these alcoves, assign the corresponding simple module to the left cell in $\Omega$ containing the alcove labelled by $w^{-1}$. (It would still have to be shown that this assignment is independent of the choice of the special point.)

In particular, when $w$ labels the distinguished involution in $\Gamma$, then $w = w^{-1}$; so the simple module in $S$ corresponding to this alcove is assigned to $\Gamma$. That this “canonical” simple module should be fixed by $A(e)$ is suggested by the parallel discussion in [22, 10.7].

In rank 2 cases all of this can be observed directly. But in general there are serious combinatorial difficulties in working with the geometry of the cells even in the good case when $\chi$ has standard Levi form. The first problem is to locate a suitable special point $v$. One might look at the alcove containing the distinguished involution in the canonical left cell $\Gamma$: this will be the lowest alcove in $\Gamma$ attached to some special point $v$. Do the surrounding alcoves which lie in $\Gamma$ suffice to account for all simple modules in $S$? In rank 3, where Gunnells has constructed pictures of the cells, the evidence about the number of available alcoves is encouraging.

(But there is one nilpotent orbit of type $C_3$ which seems to require an alternate choice of special point. This orbit has an element in standard Levi form, while the component group $A(e)$ has order 2.)

4.3. The highest two-sided cell corresponds to the regular nilpotent orbit. Here the related representation theory is quite transparent, since a regular block has only one simple module (of dimension $p^N$).

At the other extreme, one can say quite a bit about the lowest two-sided cell $\Omega$, which corresponds to the zero orbit. Here the canonical left cell is just a shifted version of the dominant chamber, whose geometry is transparent. The associated representation theory comes from the group $G$, with simple modules parametrized in the usual way by highest weights.

Using suggestions of Shi, we can argue as follows. Start with a special point for $W_p$ lying in $Q$ such as $v = 2(p-1)\rho$; the surrounding $|W|$ alcoves lie inside the canonical left cell $\Gamma$. If we write $v = x \cdot (-\rho)$ (with $x$ a translation from $pQ$), these alcoves are obtained by applying $x$ to the alcoves around $-\rho$ labelled by the elements $w \in W$, and thus are labelled by elements $xw$. Now $x^{-1} \cdot \rho$ lies inside the antidominant chamber, which is a single left cell of $\Omega$. Since $W$ acts simply transitively on the Weyl chambers, we see that the alcoves labelled by the various $(xw)^{-1} = w^{-1}x^{-1}$ all lie in distinct Weyl chambers and thus in distinct left cells of $\Omega$.

It is easy to see that the resulting bijection between $S$ and $L$ is independent of the choice of $v$, since the role of $W$ is independent of translations by elements of $pQ$.

4.4. Jantzen’s study of the subregular case makes it possible to say something, even though $\chi$ can be chosen to have standard Levi form only for root systems of type $A_n$ and $B_n$. In a regular block there is always an isolated simple module, denoted $L_0$ in [14, D.6] and associated with the longest element $w_0$ of $W$. This module is characterized in terms of its “$\kappa$-invariant” and has a projective cover of smallest possible dimension.
The dominant alcove $A$ obtained by reflecting the lowest alcove across its upper wall $H$ contains the distinguished involution in the canonical left cell $\Gamma$; it is the lowest alcove in $\Gamma$ among those sharing the vertex obtained by reflecting $-\rho$ in the hyperplane $H$. In our framework it is natural to assign the simple module $L_0$ to $A$ and thus to the left cell $\Gamma$. (This is motivated in part by the approach to computing dimensions in [12], where $H$ plays a key role.) Low rank evidence indicates that the translate of $A$ attached to the special point $-\rho$ is in the same $W_I$-orbit as the alcove labelled by $w_0$ in types $A_n$ and $B_n$. Here $I$ is the set of simple roots involved in Jantzen's choice of subregular nilpotent element.

For type $G_2$, there are five simple modules in a regular block, three of equal dimension being permuted by $A(e) \cong S_3$. Here the two-sided cell is finite, with three left cells: the canonical left cell (to which $L_0$ should be assigned) has 8 elements, while the others have respectively 8 and 7. Comparison with Lusztig's model, as developed by Xi [35], shows that the triple of simple modules should be assigned to the cell with 7 elements: here the isotropy group in $S_3$ has order 2. However, it is unclear for root systems other than $A_n, B_n, G_2$ how to assign the simple modules other than $L_0$ to left cells.

4.5. For a fixed nilpotent orbit, our broader hope is to model Lusztig's conjectural set-up in full detail. Besides taking for the finite set $Y$ the set $S$ above, one needs to bring in the action of $C_G(e)$. Still missing is a construction (presumably based on $B_e$) of suitable modules which carry compatible actions of $g$ and $F$.

But there is a reasonable prototype in the case $\chi = 0$. Here one starts with Weyl modules $V(\lambda)$ with $\lambda \in X^+$. Their duals are realized as spaces of global sections of line bundles on $B$ (the Springer fiber in this case). With these modules one has a Kazhdan–Lusztig theory, conjectured by Lusztig (for $p$ not too small) to determine simple modules $L(\lambda)$ via an alternating sum formalism with coefficients depending on Kazhdan–Lusztig polynomials for $W_p$. In turn $L(\lambda)$ factors (by Steinberg's theorem) into a tensor product of a simple $U_\chi(g)$-module and the Frobenius twist of a simple module for $G$ (which looks like the characteristic 0 version if $\lambda$ is suitably bounded relative to $p$).

One would like to find a similar construction for all $\chi$. A geometric construction of $g$-modules using the Springer fiber has been proposed by Mirković–Rumynin [29], but without the additional features indicated above.

4.6. When $\chi$ is fixed, motivation for correlating simple modules with left cells comes indirectly from the experimental calculations reported in [12]. These are reinforced by Jantzen's unpublished calculations in higher rank cases. The idea here is that the geometry of lower boundaries of canonical left cells, together with the placement of weights in alcoves, should play a key role in predicting the dimensions (and formal characters) of simple modules. The experimental evidence also reinforces the suggestion above about the existence of a tensor product decomposition of Steinberg type.

4.7. Lusztig's conjectural framework works with a fixed nilpotent orbit or two-sided cell. But there is additional motivation for assigning simple modules to left cells when we compare one orbit with an orbit in its closure. When $\psi$ is in the closure of the $G$-orbit of $\chi$, one expects that a simple $U_\chi(g)$-module will “deform” into a not necessarily simple $U_\psi(g)$-module.
On the level of Grothendieck groups, this would imply a recipe for writing the dimension of the given simple $U_q(g)$-module as a sum of dimensions of simple $U_q(g)$-modules. In all known cases these dimension formulas are given uniformly by polynomials in $p$ and the weight coordinates (compare [6 §6]). Experimentation in low ranks by Jantzen and the author suggests that such decompositions may be possible in a unique way.

Ostrik proposes that deformation should be studied in the context of projective covers of simple modules. He suggests an interpretation of the process in terms of comparison of Lusztig’s equivariant $K$-theory bases for the two Springer fibers: these bases may be comparable even when the Springer fibers themselves are not. Using this viewpoint he recovers for example our dimension comparisons in the case of type $G_2$.

In low ranks, the cell pictures related to our hypothetical assignment of simple modules to left cells show an intriguing correlation with the computed degeneration in dimension formulas. But all of this remains to be placed in a rigorous theoretical setting, beginning with the process of deformation.

References

1. P.N. Achar, On the equivariant $K$-theory of the nilpotent cone in the general linear group, Represent. Theory 8 (2004), 180–211.
2. P.N. Achar and E.N. Sommers, Local systems of nilpotent orbits and weighted Dynkin diagrams, Represent. Theory 6 (2002), 190–201.
3. R. Bezrukavnikov, On tensor categories attached to cells in affine Weyl groups, Representation Theory of Algebraic Groups and Quantum Groups, 69–90, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.
4. R. Bezrukavnikov, Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone, Represent. Theory 7 (2003), 1–18.
5. R. Bezrukavnikov, Perverse sheaves on affine flags and nilpotent cone of the Langlands dual group, arXiv:math.RT/0201256.
6. R. Bezrukavnikov, I. Mirković, D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic, arXiv:math.RT/0205144, to appear in Ann. of Math.
7. R. Bezrukavnikov and V. Ostrik, On tensor categories attached to cells in affine Weyl groups II, Representation Theory of Algebraic Groups and Quantum Groups, 101–119, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.
8. K.A. Brown and I. Gordon, The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras, Math. Z. 238 (2001), 733–779.
9. T. Chmutova and V. Ostrik, Calculating canonical distinguished involutions in the affine Weyl groups, Experiment. Math. 11 (2002), 99–117.
10. I. Gordon, Representations of semisimple Lie algebras in positive characteristic and quantum groups at roots of unity, pp. 149–167, Quantum Groups and Lie Theory, ed. A. Pressley, Proc. Durham 1999, London Math. Soc. Lecture Note Ser., 290, Cambridge Univ. Press, Cambridge, 2001.
11. J.E. Humphreys, Modular representations of simple Lie algebras, Bull. Amer. Math. Soc. (N.S.) 35 (1998), 105–122.
12. J.C. Jantzen, Analogues of Weyl’s formula for reduced enveloping algebras, Experiment. Math. 11 (2002), 567–573.
13. J.C. Jantzen, Representations of Lie algebras in prime characteristic, Notes by Iain Gordon, pp. 185–235, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, 1997), Kluwer Acad. Publ., Dordrecht, 1998.
14. J.C. Jantzen, Subregular nilpotent representations of Lie algebras in prime characteristic, Represent. Theory 3 (1999), 153–222.
15. J.C. Jantzen, Modular representations of reductive Lie algebras, J. Pure Appl. Algebra 152 (2000), 133–185.
16. ______, *Representations of Lie algebras in positive characteristic*, Representation Theory of Algebraic Groups and Quantum Groups, 175–218, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.

17. ______, *Nilpotent orbits in representation theory*, pp. 1–211, *Lie Theory*, ed. J.-P. Anker and B. Orsted, Progr. Math., vol. 228, Birkhäuser, Boston, 2004.

18. G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), 623–653.

19. ______, *Cells in affine Weyl groups*, Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), 255–287, Adv. Stud. Pure Math. **6**, Math. Soc. Japan, Tokyo, 1985.

20. ______, *Cells in affine Weyl groups II*, J. Algebra **109** (1987), 536–548.

21. ______, *Cells in affine Weyl groups III*, J. Fac. Sci. Univ. Tokyo Sect IA Math. **34** (1987), 223–243.

22. ______, *Cells in affine Weyl groups IV*, J. Fac. Sci. Univ. Tokyo Sect IA Math. **36** (1989), 297–328.

23. ______, *Cells in affine Weyl groups and tensor categories*, Adv. Math. **129** (1997), 85–98.

24. ______, *Periodic W-graphs*, Represent. Theory **1** (1997), 207–279.

25. ______, *Bases in equivariant K-theory*, Represent. Theory **2** (1998), 298–369.

26. ______, *Subregular nilpotent elements and bases in K-theory*, Canad. J. Math. **51** (1999), 1194–1225.

27. ______, *Bases in equivariant K-theory*, II, Represent. Theory **3** (1999), 281–353.

28. G. Lusztig and N. Xi, *Canonical left cells in affine Weyl groups*, Adv. Math. **72** (1988), 284–288.

29. I. Mirković and D. Rumynin, *Geometric representation theory of restricted Lie algebras*, Transform. Groups **6** (2001), 175–191.

30. V. Ostrik, *On the equivariant K-theory of the nilpotent cone*, Represent. Theory **4** (2000), 296–305.

31. Jian Yi Shi, *The Kazhdan–Lusztig cells in certain affine Weyl groups*, Lect. Notes in Math. 1179, Springer-Verlag, Berlin, 1986.

32. ______, *A two-sided cell in an affine Weyl group*, II, J. London Math. Soc. (2) **37** (1988), 253–264.

33. E. Sommers, *Lusztig’s canonical quotient and generalized duality*, J. Algebra **243** (2001), 790–812.

34. N. Xi, *The based ring of the lowest two-sided cell of an affine Weyl group*, II, Ann. Sci. École Norm. Sup. (4) **27** (1994), 47–61.

35. ______, *Representations of affine Hecke algebras*, Lect. Notes in Math. 1587, Springer-Verlag, Berlin, 1994.

36. ______, *The based ring of two-sided cells of affine Weyl groups of type \( \tilde{A}_{n-1} \)*, Mem. Amer. Math. Soc. **157** (2002), no. 749.

Dept. of Mathematics & Statistics, U. Massachusetts, Amherst, MA 01003

E-mail address: jeh@math.umass.edu