SAMPLING AND INTERPOLATION IN BARGMANN-FOCK SPACES OF POLYANALYTIC FUNCTIONS

LUIŠ DANIEL ABREU

Abstract. Using Gabor analysis, we give a complete characterization of all lattice sampling and interpolating sequences in the Fock space of polyanalytic functions, displaying a "Nyquist rate" which increases with $n$, the degree of polyanalyticity of the space. Such conditions are equivalent to sharp lattice density conditions for certain vector-valued Gabor systems, namely superframes and Gabor super-Riesz sequences with Hermite windows, and in the case of superframes they were studied recently by Gröchenig and Lyubarskii. The proofs of our main results use variations of the Janssen-Ron-Shen duality principle and reveal a duality between sampling and interpolation in polyanalytic spaces, and multiple interpolation and sampling in analytic spaces. To connect these topics we introduce the polyanalytic Bargmann transform, a unitary mapping between vector valued Hilbert spaces and polyanalytic Fock spaces, which extends the Bargmann transform to polyanalytic spaces. Motivated by this connection, we discuss a vector-valued version of the Gabor transform. These ideas have natural applications in the context of multiplexing of signals. We also point out that a recent result of Balan, Casazza and Landau, concerning density of Gabor frames, has important consequences for the Gröchenig-Lyubarskii conjecture on the density of Gabor frames with Hermite windows.

1. Introduction

In this paper we find a new, and perhaps unexpected, connection between polyanalytic functions and time-frequency analysis, and use it to obtain a complete characterization of all lattice sampling and interpolating sequences in the Bargmann-Fock space of polyanalytic functions or, equivalently, of all lattice vector-valued Gabor frames and vector-valued Gabor Riesz sequences for $L^2(\mathbb{R}, \mathbb{C}^n)$.

1.1. Overview. The Bargmann-Fock space of polyanalytic functions, $F^n(\mathbb{C}^d)$, consists of all functions satisfying the equation

$$\left(\frac{d}{dz}\right)^n F(z) = 0,$$

and such that

$$\int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} \, dz < \infty.$$
Functions satisfying (1.1) are known as polyanalytic functions of order $n$. Since (1.1) generalizes the Cauchy-Riemann equation

$$\frac{d}{dz}F(z) = 0,$$

then the space $F^n(\mathbb{C}^d)$ is a generalization of the Bargmann-Fock space of analytic functions, $\mathcal{F}(\mathbb{C}^d) = F^1(\mathbb{C}^d)$. In the case of $\mathcal{F}(\mathbb{C})$, a complete description of the sets of sampling and interpolation is known [38], [46], [47].

Polyanalytic functions inherit some of the properties of analytic functions, often in a nontrivial form. However, as in the theory of several complex variables, many of the properties break down once we leave the analytic setting. An obvious difference lies in the structure of the zeros. For instance, while nonzero entire functions do not have sets of zeros with an accumulation point, polyanalytic functions can vanish along closed curves: just take $F(z) = \frac{z}{|z|^2 - 1}$, a polyanalytic function of order 2. Polyanalytic functions have been investigated thoroughly, notably by Balk and his students [8]. They are naturally related to polyharmonic functions, which have an intriguing structure of zero sets [27], [9], [41].

We will study the spaces $F^n(\mathbb{C}^d)$ using time-frequency analysis, and think about the polyanalytic functions in a different way from the classical approach.

The link to time-frequency analysis has impressive consequences: by endowing the spaces $F^n(\mathbb{C}^d)$ with the structure inherent to translations and modulations, one can use tools that were unavailable with complex variables. This will provide $F^n(\mathbb{C}^d)$ with properties reminiscent of the classical analytic Fock space.

In order to state our main results, we briefly recall some definitions. See sections 4 and 5 for more details. We associate the sequence $\Lambda = \{(x, w)\}$ with the sequence of complex numbers $\Gamma = \{(x+ iw)\}$. Then $\Gamma$ is an interpolating sequence for $F^n(\mathbb{C}^d)$ if, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists $F \in F^n(\mathbb{C}^d)$ such that

$$e^{ixz - \frac{2}{\pi}|z|^2}F(z) = \alpha_{i,j},$$

for every $z \in \Gamma$. We say that $\Gamma$ is a sampling sequence for $F^n(\mathbb{C}^d)$ if there exist $A, B > 0$ such that, for every $F \in F^n(\mathbb{C}^d)$,

$$A \|F\|_{F^n(\mathbb{C}^d)}^2 \leq \sum_{z \in \Gamma} |F(z)|^2 e^{-\pi|z|^2} \leq B \|F\|_{F^n(\mathbb{C}^d)}^2.$$

The concept of interpolating sequences has its roots in deep problems in complex analysis, and sampling sequences are a major issue in signal processing, since they correspond to the case where stable numerical reconstructions of a function from its samples is possible. Monograph [49] is a good introduction to sampling and interpolation and its interconnections with other branches of pure and applied mathematics.

Our main results, Theorem 4 and Theorem 6 below, use the concept of Beurling density, which, in the lattice case, is given by $D(\Gamma) = D(\Lambda) = |\det A|^{-1}$, where $\Lambda = AZ^2$.

**Theorem 4.** The lattice $\Gamma$ is a sampling sequence for $F^n(\mathbb{C})$ if and only if:

$$D(\Gamma) > n.$$

**Theorem 6.** The lattice $\Gamma$ is an interpolating sequence for $F^n(\mathbb{C})$ if and only if:

$$D(\Gamma) < n.$$
These results follow by establishing one duality between sampling in $F^n(C^d)$ and multiple interpolation in $F(C^d)$ and a second duality between interpolation in $F^n(C^d)$ and multiple sampling in $F(C^d)$. When $d = 1$, these properties allow us to directly apply the results in [12]. Taking $n = 1$ we recover the well known duality between sampling and interpolation in $F(C^d)$.

Theorems exhibiting a ”Nyquist rate” phenomenon tend to be hard to prove. They have been studied, for general sequences, in spaces of analytic functions, first in the Paley-Wiener space [11], [36], [37] and then in Bargmann-Fock [38], [46], [47] and Bergman [48] spaces of analytic functions. There are two reasons for us to believe that Beurling type methods do not work here. First, they use complex variables tools that are not available in the polyanalytic situation. Second, in all of these spaces, there were known ”doubly orthogonal systems” [45], which provided the eigenfunctions for the fundamental equation involving the ”concentration operator”. We are not aware of a system with this double orthogonality property in the polyanalytic situation.

Therefore, we introduce new tools.

With a view to relating our problem to one concerning the density of vector valued Gabor systems, we extend Bargmann’s work [10] to the setting of polyanalytic functions. Once we do this, our argument, which is conceptual in nature, follows smoothly.

It is also worth noting that the density Theorem in Gabor analysis has itself a very rich story, beginning with fundamental but imprecise statements by John Von Neumann and Dennis Gabor, which caught the attention of mathematicians after conjectures by Daubechies and Grossman [15]. See the survey article [28], [13] and the important special windows studied in [35].

To give a context to our approach, recall the connection between the classical (analytic) Bargmann-Fock space and time-frequency analysis.

It is well known that, up to a certain weight, the Gabor transform with a gaussian window belongs to the Fock space of analytic functions. Moreover, it has been shown that this is the only choice leading to spaces of analytic functions [3].

However, a nice picture emerges when we take Hermite functions as windows. The analytic situation generated by the gaussian window then becomes the tip of the iceberg of a larger structure involving spaces of polyanalytic functions. Indeed, the Gabor transform with the $n$th Hermite function is, up to a certain weight (the same as in the analytic case), a polyanalytic function of order $n + 1$.

To fully understand the situation, we will need the spaces constituted by the functions satisfying (1.2), which are polyanalytic of order $n$, but are not polyanalytic of any lower order (in particular they have no analytic functions). These are the true polyanalytic Fock spaces $F^n(C^d)$. The polyanalytic Fock and true polyanalytic Fock spaces are related by the following orthogonal decomposition (see Corollary 1 in section 3):

$$F^n(C^d) = F^0(C^d) \oplus \ldots \oplus F^{n-1}(C^d).$$

Then, each space $F^n(C^d)$ is associated with Gabor transforms with the $n$th Hermite window. Such occurrence, which seems to have been hitherto unnoticed, will be fundamental in our discussion. This observation is related to some recent developments in Gabor analysis with Hermite functions [23], [24], [20], to Janssen’s approach to the density Theorem [31], [33] and also to the techniques used in [29], [30], [53].
which suggest that wavelet spaces and polyanalytic functions share intriguing patterns.

Fock spaces of polyanalytic functions are briefly mentioned in Balk’s monograph \[8\] and they are implicit in quantum mechanics, in connection with the Landau levels of the Schrödinger operator with magnetic field \[14,21\] and displaced Fock states \[51\]. However, we were not able to find any reference to polyanalytic functions in the mathematical physics literature, apart from \[52\], where creation and annihilation operators are used.

1.2. The results of Gröchenig-Lyubarskii and of Balan-Casazza-Landau.

Our results are connected to a very recent result of Gröchenig and Lyubarskii, which deserves more specific comment in this introduction. Denote by \(G(h_n, \Lambda)\) the set of the translations and modulations indexed by the lattice \(\Lambda\) and acting coordinate-wise on the vector \(h_n\). Then, the following result holds.

**Theorem [24]:** Let \(h_n = (h_0, ..., h_{n-1})\) be the vector of the first \(n\) Hermite functions. Then \(G(h_n, \Lambda)\) is a frame for \(L^2(\mathbb{R}, \mathbb{C}^n)\) if and only if

\[ D(\Lambda) > n. \]

One may wonder if the equivalence of this condition to the one in Theorem 4 reflects causality or casuality. As we will see, causality is the answer. Actually, one of the key steps in our approach consists of showing that Theorem 4 is equivalent to the above Theorem.

The original proof of this Theorem in [24] combines the use of the so-called Wexler-Raz biorthogonality relations with complex analysis techniques based on properties of the Weierstrass sigma function. We will give an alternative proof, which is considerably shorter (at the cost of using deeper results from the literature) and has the advantage of also characterizing the vector valued Gabor Riesz sequences with Hermite windows in the following statement, which is equivalent to Theorem 6:

**Theorem 7:** \(G(h_n, \Lambda)\) is a Riesz sequence for \(L^2(\mathbb{R}, \mathbb{C}^n)\) if and only if

\[ D(\Gamma) < n. \]

One should also remark that the necessity of the condition \(D(\Lambda) \geq n\) for vector valued frames, follows from a result of Balan \[5\]. Moreover, this condition holds for general sets, since a Ramanathan-Steger \[40\] type argument is used.

Now let us look closer at the problem of deciding when the translations and modulations of a single Hermite function constitute a frame.

It can be easily seen as a corollary of the above Theorem that \(D(\Lambda) > n\) is sufficient for the system \(G(h_{n-1}, \Lambda)\) to be a frame. It has been observed by Gröchenig and Lyubarskii that there are some examples which support the intriguing conjecture that such a result might be sharp.

A recent result of Balan, Casazza and Landau shows that this conjecture cannot hold for general sets. Let \(S_0\) stand for the Feichtinger algebra \[16\].

**Theorem 7** Assume \(G(g, \Lambda)\) is a Gabor frame for \(L^2(\mathbb{R}^d)\) with \(g \in S_0\). Then, for every \(\epsilon > 0\), there exists a subset \(J_\epsilon \subset \Lambda\) so that \(G(g, J_\epsilon)\) is a Gabor frame for \(L^2(\mathbb{R}^d)\) and its upper Beurling density satisfies \(D^+(J_\epsilon) \leq 1 + \epsilon\).
This theorem implies that Gröchenig-Lyubarskii conjecture can only be true for lattices. Indeed, for every \( n \), the \( S_0 \) norm of the Hermite functions (see [34]) is
\[
\|h_n\|_{S_0} = 2^{n+1} \frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\sqrt{n!}},
\]
and therefore,
\[h_n \in S_0.\]
Thus, the Balan-Casazza-Landau Theorem tells us that we can always find, for every \( n \), a set \( J \) with Beurling density arbitrarily close to one and such that \( G(h_n, \Lambda) \) is a frame. However, it is still possible that the conjecture is true for lattices, since a subset of a lattice may not be a lattice.

**Conjecture 1.** Let \( \Lambda \) be a lattice in \( \mathbb{R}^2 \). If \( G(h_n, \Lambda) \) is a Gabor frame for \( L^2(\mathbb{R}^d) \), then
\[D(\Lambda) > n.\]

### 1.3. New Concepts

- We first introduce what we call the *true-polyanalytic Bargmann transform*:

\[
(B^n f)(z) = (z^{n!})^{-\frac{1}{2}} e^{\pi |z|^2} \frac{d^n}{dz^n} \left[ e^{-\pi |z|^2} F(z) \right].
\]

Here \( F \) stands for the Bargmann transform of \( f \). As we will see, this is a unitary mapping from \( L^2(\mathbb{R}^d) \) to \( \mathcal{F}^n(\mathbb{C}^d) \). This mapping relates to Gabor transforms with Hermite windows \( \Phi_n \) in the following way:

\[
V_{\Phi_n} f(x, \omega) = e^{i\pi x \omega - \frac{1}{\pi} |z|^2} (B^n f)(z),
\]
and we will provide its basic theory starting from this relation, following as much as possible the presentation of the Bargmann transform given in Gröchenig’s book [22, section 3.3].

- For vector-valued functions \( f = (f_0, ..., f_{n-1}) \), we define the *polyanalytic Bargmann transform*,

\[
(B^n f) = \sum_{0 \leq k \leq n-1} (B^k f_k),
\]

which will be unitary between \( L^2(\mathbb{R}^d, \mathbb{C}^n) \) and \( \mathcal{F}^n(\mathbb{C}^d) \).

- In the last section we will see that the polyanalytic Bargmann transform is a special case of a vector-valued version of the Gabor transform. Although this transform plays no direct role in the proofs of the main Theorems, it must be in the picture for completeness, since it is the natural time-frequency transformation of which the polyanalytic Bargmann transform is a special case:

\[
V_{g} f(x, \omega) = \sum_{k=0}^{n-1} V_{g_k} f_k(x, \omega).
\]

In the case where \( \{g_k\}_{k=0}^{n-1} \) constitutes an orthonormal sequence, it provides an isometry between \( L^2(\mathbb{R}^d, \mathbb{C}^n) \) and between \( L^2(\mathbb{R}^{2d}) \). This transform is the continuous counterpart of the *superframes* discussed in [26, 4, 24, Theorem 2.7].
1.4. **Technical summary of proofs.** With the tools described above at hand, our main argument will depend on two profound results. More specifically, we will combine variations on the Janssen-Ron-Shen duality principle [43] with the characterization of multiple sampling and interpolation sequences in the Fock space [12]. The duality principles reflect all the rich inner structure of Gabor frames. The second result uses a deep elaboration on Beurling’s balayage technique [11] developed by Seip in [48].

We will proceed as follows. First, using an orthogonal basis for the polyanalytic Fock spaces, we prove the unitarity of $B^n$ and $B^n$. Then we study sampling in $F^n(C)$. Using the unitary mapping $B^n$, we show that the problem is equivalent to the study of vector valued frames with Hermite windows, also known as superframes [4], [24]. This problem has been recently studied in [24], but we provide an alternative proof, which is more natural in the context of sampling and interpolation: applying a vector valued version of the Janssen-Ron-Shen duality we translate the statement into a problem concerning unions of Riesz sequences. After noticing that the latter is equivalent to a multiple interpolation problem in Fock spaces of analytic functions, we apply the interpolation result in [12]. We then study interpolation in $F^n(C)$. In order to do this, we ”dualize” the arguments that we have used in the sampling part, once again using the Janssen-Ron-Shen duality, this time between vector-valued Riesz sequences and multi-frames with Hermite functions. This translates our interpolation problem into one of multiple sampling. Noticing that this problem is equivalent to multiple sampling in Fock spaces, we apply the sampling result from [12].

1.5. **Organization of the paper.** The next section contains the classical tools that we are going to use. We list the basic properties of the Gabor transform, the Bargmann transform and the Hermite functions.

In the third section, we introduce the true polyanalytic Bargmann and the polyanalytic Bargmann transforms. By making a connection with the Gabor transform, we study their basic properties, find an orthogonal basis for the polyanalytic Fock spaces and prove the unitarity properties.

Our main results are in the fourth and fifth sections, where we derive the duality principles and study sampling and interpolation for $F^n(C)$.

In our last section we introduce the super Gabor transform and make some remarks of a more informal character concerning applications and open problems.

2. **Background**

2.1. **The Gabor transform.** Fix a function $g \neq 0$. Then the Gabor (short-time) Fourier transform of a function $f$ with respect to the ”window” $g$ is defined, for every $x, \omega \in \mathbb{R}^d$, as

\[
V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \omega} dt.
\]

The following relations are usually called the orthogonal relations for the short-time Fourier transform. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then $V_{g_1} f_1, V_{g_2} f_2 \in L^2(\mathbb{R}^{2d})$ and

\[
\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}.
\]
The Gabor transform provides an isometry
\[ V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}), \]
that is, if \( f, g \in L^2(\mathbb{R}^d) \), then
\[ \|V_gf\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \]
For every \( x, \omega \in \mathbb{R}^d \) define the operators translation by \( x \) and modulation by \( \omega \) as
\[ T_x f(t) = f(t - x), \]
\[ M_\omega f(t) = e^{2\pi i \omega t} f(t). \]
Using these operators we can write (2.1) as
\[ V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle_{L^2(\mathbb{R}^d)}. \]

2.2. The Bargmann transform. Here we will use multi-index notation: \( z = (z_1, ..., z_d) \), \( n = (n_1, ..., n_d) \) and \( |n| = n_1 + ... + n_d \). The Bargmann transform, defined by
\[ (Bf)(z) = \int_{\mathbb{R}^d} f(t) e^{2\pi t z - \pi |z|^2} dt, \]
is an isomorphism
\[ B : L^2(\mathbb{R}^d) \to \mathcal{F}(\mathbb{C}^d), \]
where \( \mathcal{F}(\mathbb{C}^d) \) stands for the Bargmann-Fock space of analytic functions in \( \mathbb{C}^d \) with the norm
\[ \|F\|^2_{\mathcal{F}(\mathbb{C}^d)} = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi |z|^2} dz. \]
The collection of the monomials of the form
\[ e_n(z) = \left( \frac{2^{\frac{|n|}{2}}}{n!} \right)^\frac{1}{2} z^n = \prod_{j=1}^d \frac{\pi^{\frac{n_j}{2}}}{\sqrt{n_j!}} z^{n_j}, \]
where \( n = (n_1, ..., n_d) \), with \( n_i \geq 0 \), constitutes an orthonormal basis of \( \mathcal{F}(\mathbb{C}^d) \). The reproducing kernel of \( \mathcal{F}(\mathbb{C}^d) \) is the function \( e^{\pi w} \). This means that, for every \( F \in \mathcal{F}(\mathbb{C}^d) \),
\[ \langle F(w), e^{\pi w} \rangle_{\mathcal{F}(\mathbb{C}^d)} = F(z). \]
Differentiating \( n - k \) times the corresponding reproducing equation, we obtain
\[ \langle F(w), w^{n-k} e^{\pi w} \rangle_{\mathcal{F}(\mathbb{C}^d)} = \pi^{k-n} F^{(n-k)}(z), \]
where for \( d > 1 \) we are using the multi-index derivative
\[ \frac{d}{dz} f = \frac{df}{dz_1...dz_d}. \]

A simple calculation shows that the Bargmann transform is related to the Gabor transform with the Gaussian window \( \varphi(t) = 2^d e^{-\pi t^2} \) by the formula
\[ V_\varphi f(x, -\omega) = e^{i\pi x \omega - \frac{i\pi |\omega|^2}{4}} (Bf)(z), \]
where \( z = x + i\omega \).

We will need one more operator. Define a ”translation” \( \beta_z \) on \( \mathcal{F}(\mathbb{C}^d) \) by
\[ \beta_z F(\zeta) = e^{i\pi x \omega - \frac{i\pi |\omega|^2}{4}} e^{i\pi \zeta} F(\zeta - z). \]
The operator $\beta_z$ satisfies the intertwining property
\begin{equation}
\beta_z \mathcal{B} = \mathcal{B} M_\omega T_x, \quad z = x + i\omega.
\end{equation}

2.3. The Hermite functions. The Hermite functions can be defined via the so-called Rodrigues Formula
\begin{equation}
h_n(t) = c_n e^{\pi t^2} \left( \frac{d}{dt} \right)^n \left( e^{-2\pi t^2} \right).
\end{equation}
where $c_n$ is chosen in such a way that they can provide an orthonormal basis of $L^2(\mathbb{R})$. Now let $n = (n_1, \ldots, n_d)$ and $x \in \mathbb{R}^d$. The $d$-dimensional Hermite functions are
\begin{equation}
\Phi_n(x) = \prod_{j=1}^d h_{n_j}(x_j).
\end{equation}
They form a complete orthonormal system of $L^2(\mathbb{R}^d)$.

A very important property of the Hermite functions (see for instance [34]) is that they are mapped onto a basis of the Bargmann-Fock space via the Bargmann transform.
\begin{equation}
(\mathcal{B}\Phi_n)(z) = e_n(z).
\end{equation}

3. Polyanalytic Fock spaces and polyanalytic Bargmann transforms

3.1. Definitions. In this section we use multi-index notation in such a way that there will be no difference between the one and the $d$-dimensional case. Only at the end of the last two sections is it necessary to specialize $d = 1$.

It is well known [8] that every polyanalytic function of order $n$ can be uniquely expressed in the form
\begin{equation}
F(z) = \sum_{0 \leq k \leq n-1} \overline{z}^k \varphi_k(z),
\end{equation}
where $\{\varphi_p(z)\}_{p=0}^{n-1}$ are analytic functions, each of them with a power series expansion,
\begin{equation}
\varphi_p(z) = \sum_{j \geq 0} c_{j,p} z^j,
\end{equation}
As a result, there is also a power series expansion for the polyanalytic function $F$:
\begin{equation}
F(z) = \sum_{0 \leq p \leq n-1} \sum_{j \geq 0} c_{j,p} z^j.
\end{equation}
We will often use the inner product in the polyanalytic Fock space, given by
\begin{equation}
\langle F, G \rangle_{\mathcal{F}^n(\mathbb{C}^d)} = \int_{\mathbb{C}^d} F(z) \overline{G(z)} e^{-\pi |z|^2} \, dz.
\end{equation}
Observe also that this implies
\begin{equation}
\langle F, G \rangle_{\mathcal{F}^n(\mathbb{C}^d)} = \left\langle e^{-\pi |\cdot|^2} F, e^{-\pi |\cdot|^2} G \right\rangle_{L^2(\mathbb{R}^{2d})}.
\end{equation}
3.2. The true polyanalytic Bargmann transform.

**Definition 1.** The true polyanalytic Bargmann transform of order $n$, of a function on $\mathbb{R}^d$, is defined by the formula

\begin{equation}
(\mathcal{B}^n f)(z) = (\pi |n| n!)^{-\frac{d}{2}} e^{\pi |z|^2} \left( e^{-\pi |z|^2} F(z) \right),
\end{equation}

where $F(z) = (\mathcal{B} f)(z)$.

Clearly $\mathcal{B}^0 f = \mathcal{B} f$ and $\mathcal{B}^n$ is a generalization of the Bargmann transform. We now provide the fundamental properties of $\mathcal{B}^n$. We try to stay as close as possible to the presentation of section 3.4 in [22]. The next proposition is the departing point of our study.

**Proposition 1.** If $f$ is a function on $\mathbb{R}^d$ with polynomial growth, then its true polyanalytic Bargmann transform $\mathcal{B}^n f$ is a polyanalytic function of order $n + 1$ on $\mathbb{C}^d$. If we write $z = x + i\omega$, then this transform is related to the Gabor transform with Hermite windows in the following way:

\begin{equation}
V_{\Phi_n} f(x, \omega) = e^{i\pi x \omega - \frac{1}{4} |z|^2} (\mathcal{B}^n f)(z).
\end{equation}

Moreover, if $f \in L^2(\mathbb{R})$, then

\begin{equation}
\|\mathcal{B}^n f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.
\end{equation}

**Proof.** Let $F = \mathcal{B} f$. The following calculation is from Proposition 3.2 in [23], where (2.10) is used:

\begin{align*}
V_{\Phi_n} f(x, \omega) &= \langle f, M_{x T_{\omega}} \Phi_n \rangle_{L^2(\mathbb{R}^d)} \\
&= \langle F, \beta_z \mathcal{B} \Phi_n \rangle_{F(\mathbb{C}^d)} \\
&= (\pi |n| n!)^{-\frac{d}{2}} e^{i\pi x \omega - \frac{1}{4} |z|^2} \langle F(w), e^{i\pi \omega (w - z)} \rangle_{F(\mathbb{C}^d)} \\
&= (\pi |n| n!)^{-\frac{d}{2}} e^{i\pi x \omega - \frac{1}{4} |z|^2} \sum_{0 \leq k \leq n} \binom{n}{k} (-\pi z)^k F(n-k)(z).
\end{align*}

Now, since the Bargmann transform of a function in $\mathbb{R}^d$ is an entire function [22, Proposition 3.4.1], the functions $F[n-k](z)$ are also entire, and from (3.1) we recognize the sum as a polyanalytic function of order $n + 1$. To prove (3.4) observe that the last expression can be written as

\begin{equation}
e^{i\pi x \omega - \frac{1}{4} |z|^2} (\pi |n| n!)^{-\frac{d}{2}} e^{\pi |z|^2} \left( e^{-\pi |z|^2} F(z) \right) = e^{i\pi x \omega - \frac{1}{4} |z|^2} (\mathcal{B}^n f)(z).
\end{equation}

The isometric property (3.5) is an immediate consequence of (3.4) and (2.3). \hfill \Box

3.3. Orthogonal decomposition.

**Definition 2.** For $k, m \in \mathbb{N}_d^+$, consider the functions $e_{k,m}$ defined as

\begin{equation}
e_{k,m}(z) = (\pi |k| k!)^{-\frac{d}{2}} e^{\pi |z|^2} \left( \frac{d}{dz} \right)^k \left[ e^{-\pi |z|^2} c_m(z) \right].
\end{equation}

From (2.10) one can easily see that

\begin{equation}
e_{k,m}(z) = (\mathcal{B}^k \Phi_m)(z).
\end{equation}

**Proposition 2.** The set \{e_{k,m}\}_{0 \leq k \leq n-1; m \geq 0} is an orthogonal basis of $\mathcal{F}^n(\mathbb{C}^d)$. 

Proof. The orthogonality follows from (3.7), (3.4) and (2.2), since
\[
\langle e_{k,m}, e_{l,j} \rangle_{L^2(\mathbb{R}^d)} = \langle B^k \Phi_m, B^l \Phi_j \rangle_{\mathcal{F}(\mathbb{C}^d)}
\]
\[
= \left\langle e^{\pi |z|^2 - i\pi x\omega} \Phi_k \Phi_m, e^{\pi |z|^2 - i\pi x\omega} V \Phi_j \right\rangle_{\mathcal{F}(\mathbb{C}^d)}
\]
\[
= \langle V_k \Phi_m, V_l \Phi_j \rangle_{L^2(\mathbb{R}^d)} = \langle \Phi_m, \Phi_j \rangle_{L^2(\mathbb{R}^d)} = \delta_{m,j} \delta_{k,l}.
\]
To prove completeness of \( \{e_{k,m}\} \) in \( \mathcal{F}^n(\mathbb{C}^d) \), suppose that \( F \in \mathcal{F}^n(\mathbb{C}^d) \) is such that
\[
\langle F, e_{k,m} \rangle_{\mathcal{F}(\mathbb{C}^d)} = 0, \quad 0 \leq k \leq n - 1; \quad m \geq 0.
\]
For \( k = 0 \), we can use the representation of \( F \) in power series (3.2). Interchanging the sums with the integrals and using the orthogonality of the functions (2.5), the result is
\[
\langle F, e_{0,m} \rangle_{\mathcal{F}(\mathbb{C}^d)} = \sum_{0 \leq p \leq n-1} c_{p+m,p} \frac{(p+m)!}{\sqrt{m!}|2p+m|} = 0, \quad m \geq 0.
\]
For \( k \geq 1 \), a calculation using integration by parts gives:
\[
\langle F, e_{k,m} \rangle_{\mathcal{F}(\mathbb{C}^d)} = \int_{\mathbb{C}^d} F(z) e_{k,m}(z) e^{-\pi |z|^2} \, dz
\]
\[
= \int_{\mathbb{C}^d} e^{-\pi |z|^2} \sum_{0 \leq p \leq n-1} c_{p+m,p} \frac{(p+m)!}{\sqrt{m!}|2p+m|} \delta_{m,j} \delta_{k,l} \, dz
\]
\[
= \sum_{0 \leq p \leq n-1} \sum_{j \geq 0} c_{j,p} \frac{p...(p-k+1)\pi^{|m|}}{\sqrt{m!}} \int_{\mathbb{C}^d} z^{j+p-k} e^{-\pi |z|^2} \, dz.
\]
As a result,
\[
\sum_{0 \leq p \leq n-1} \frac{p...(p-k+1)(p+m-k)!}{\pi^{m+2p-k}2^{p-k} \sqrt{m!}} c_{m+p-k,p} = 0, \quad m \geq 0, \quad 0 \leq k \leq n - 1,
\]
resulting in a triangular system for each \( m \). Solving this system we obtain \( c_{j,p} = 0 \) for \( k \leq p \leq n - 1 \) and \( j \geq 0 \). Therefore, \( F = 0 \).

Remark 1. Clearly the orthogonality in Proposition 2 can be obtained directly by integration by parts and moving to polar coordinates. For \( k = 0 \) this has the advantage of showing that the functions are also orthogonal in the polydisk, providing the useful "double orthogonality" property as in the proof of [22, Theorem 3.4.2]. However, for \( k \geq 0 \), the boundary behavior required in the integration by parts eliminates this advantage, making the functions \( e_{k,m} \) less likely to possess such a property.

Remark 2. It is clear that these functions are reminiscent of the so-called special Hermite functions, which are the Wigner transforms of two Hermite functions [50]. They also appear in the study of Landau levels in [21].

Definition 3. The true polynomial Fock space of order \( n \) is defined as
\[
\mathcal{F}^n(\mathbb{C}^d) = \text{Span} \left\{ e_{m,n}(z) \right\}_{m \geq 0}.
\]
Remark 3. Observe that
\[
\left( \frac{d}{dz} \right)^n \left[ e^{-\pi|z|^2} z^m \right] = \frac{d^{m+n}}{dz^n dz^m} \left[ e^{-\pi|z|^2} \right].
\]
Therefore, our functions \(e_{n,m}\) are essentially the complex Hermitian functions introduced in [44, pag. 126] and, as a result, according to Theorem 7.1 in [44], the true polyanalytic Fock spaces are the eigenspaces of the Schrödinger operator with magnetic field in \(\mathbb{R}^2\), associated with the eigenvalue \(n + \frac{1}{2}\). Also, observe that the basis used in [39] approaches this one by a formal limit procedure.

The orthogonal basis property has the following consequence:

Corollary 1. The polyanalytic Fock space, \(F^n(\mathbb{C}^d)\), admits the following decomposition in terms of true polyanalytic Fock spaces \(F^k(\mathbb{C}^d)\).

\[
F^n(\mathbb{C}^d) = F^0(\mathbb{C}^d) \oplus \cdots \oplus F^{n-1}(\mathbb{C}^d).
\]

This results in a definition equivalent to the one in [53], where the spaces were defined using the decomposition.Observe that \(F^1(\mathbb{C}^d) = F^0(\mathbb{C}^d) = F(\mathbb{C}^d)\) and that functions in \(F^n(\mathbb{C}^d)\) are polyanalytic of order \(n + 1\).

3.4. Unitarity of \(B^n\). The true polyanalytic Bargmann transform keeps the unitarity property

Theorem 1. The true polyanalytic Bargmann transform is an isometric isomorphism

\[
B^n : L^2(\mathbb{R}^d) \rightarrow F^n(\mathbb{C}^d).
\]

Proof. Since we know from (3.5) that \(B^n\) is isometric, we only need to show that \(B^n[L^2(\mathbb{R}^d)]\) is dense in \(F^n(\mathbb{C}^d)\). This is now easy, since the Hermite functions constitute a basis of \(L^2(\mathbb{R}^d)\) and, by (3.7), they are mapped onto the basis \(\{e_{n,m}(z)\}\) of \(F^n(\mathbb{C})\). Since \(B^n[L^2(\mathbb{R}^d)]\) contains a basis of \(F^n(\mathbb{C}^d)\), then it must be dense. \(\square\)

3.5. The polyanalytic Bargmann transform. Now consider the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^n)\) consisting of vector-valued functions \(f = (f_0, \ldots, f_{n-1})\) with the inner product

\[
\langle f, g \rangle_\mathcal{H} = \sum_{0 \leq k \leq n-1} \langle f_k, g_k \rangle_{L^2(\mathbb{R}^d)}.
\]

The polyanalytic Bargmann transform of a function \(f = (f_0, \ldots, f_{n-1})\) is defined as

\[
(B^n f)(z) = \sum_{0 \leq k \leq n-1} (B^k f_k)(z).
\]

The next Theorem, which may have independent interest as a generalization of Bargmann’s unitary transform, will be the cornerstone in the proof of our main results regarding sampling and interpolation.

Theorem 2. The polyanalytic Bargmann transform is an isometric isomorphism

\[
B^n : \mathcal{H} \rightarrow F^n(\mathbb{C}^d).
\]
Proof. For the isometry, first observe that, from (2.2) and (3.4),
\[
\langle B^k f_k, B^j f_j \rangle_{F^n(\mathbb{C}^d)} = \delta_{k,j}.
\]
Then, using the isometric property of \(B^n\),
\[
\|B^n f\|_{F^n(\mathbb{C}^d)}^2 = \sum_{0 \leq k \leq n-1} \|B^k f_k\|_{F^n(\mathbb{C}^d)}^2 = \sum_{0 \leq k \leq n-1} \|f_k\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{H^2}^2.
\]
Moreover, \(B^n[L^2(\mathbb{R}^d)]\) is dense in \(F^n(\mathbb{C}^d)\), since, by the decomposition (3.10), every element \(F \in F^n(\mathbb{C}^d)\) can be written as
\[
F = \sum_{0 \leq k \leq n-1} F_k,
\]
with \(F_k \in F^k(\mathbb{C}^d)\), \(0 \leq k \leq n-1\). Since \(B^k\) is unitary, there exists \(f_k \in L^2(\mathbb{R}^d)\) such that \(F_k = B^k f_k\), for every \(0 \leq k \leq n-1\). It follows that \(F = B^n f\), with \(f = (f_0, \ldots, f_{n-1})\). \(\square\)

4. Sampling in \(F^n(\mathbb{C})\)

4.1. Definitions. We will work with lattices. A lattice is a discrete subgroup in \(\mathbb{R}^{2d}\) of the form \(\Lambda = AZ^{2d}\), where \(A\) is an invertible \(2d \times 2d\) matrix. We will define the density of the lattice by
\[
D(\Lambda) = \frac{1}{|\det A|}.
\]
If we write \(z = x + i\omega\) and \(\pi_z g = M_\omega T_x g\), the adjoint lattice \(\Lambda^0\) is defined by the commuting property as
\[
\Lambda^0 = \{ \mu \in \mathbb{R}^{2d} : \pi_z \pi_\mu = \pi_\mu \pi_z, \text{ for all } z \in \Lambda \}.
\]
If \(\Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z}\), then \(\Lambda^0 = \beta^{-1} \mathbb{Z} \times \alpha^{-1} \mathbb{Z}\). In general,
\[
\Lambda^0 = J(A^T)^{-1} \mathbb{Z}^{2d},
\]
where \(A^T\) is the transpose of \(A\) and \(J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\) (consisting of \(d \times d\) blocks) is the matrix defining the standard sympletic form (see [24] and [17]). Therefore,
\[
D(\Lambda^0) = \frac{1}{D(\Lambda)}.
\]
We will use the notation \(\Gamma = \{ z = x + i\omega \}\) to indicate the complex sequence associated with the sequence \(\Lambda = (x, \omega)\). The density of \(\Gamma\) will be the density of the associated lattice, that is \(D(\Gamma) = D(\Lambda)\).

Definition 4. \(\Gamma\) is a sampling sequence for the space \(F^n(\mathbb{C}^d)\) if there exist positive constants \(A\) and \(B\) such that, for every \(F \in F^n(\mathbb{C}^d)\),
\[
A \|F\|^2_{F^n(\mathbb{C}^d)} \leq \sum_{z \in \Gamma} |F(z)|^2 e^{-\pi |z|^2} \leq B \|F\|^2_{F^n(\mathbb{C}^d)}.
\]
The definition of sampling in the spaces $\mathcal{F}^k(\mathbb{C}^d)$ is exactly the same.

Now we take the following definition, obtained from [12, page 114], by making
a small simplification (in the notation of [12, page 114] we set $\nu(z) = n$) and
rewriting it in our context (observe that the weight $e^{i\pi x \omega}$ makes no difference).

**Definition 5.** A sequence $\Gamma_n$, consisting of $n$ copies of $\Gamma$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C}^d)$ if, for every sequence $\{\alpha_{i,j}^{(k)}\}_{k=0,\ldots,n-1}$ such that $\{\alpha_{i,j}^{(k)}\}_{k=0,\ldots,n-1} \in l^2$, there exists $F \in \mathcal{F}(\mathbb{C}^d)$ such that $\langle F, \beta_{\varepsilon_k} \rangle = \alpha_{i,j}^{(k)}$, for all $0 \leq k \leq n - 1$ and every $\varepsilon \in \Gamma$.

Consider again the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^n)$ consisting of vector-valued functions $\mathbf{f} = (f_0, \ldots, f_{n-1})$ with the inner product $(\mathbf{f}, \mathbf{g})$. The time-frequency shifts act coordinate-wise in an obvious way.

**Definition 6.** The vector valued system $\mathcal{G}^{(g, \Lambda)} = \{M_{\omega}T_x g\}_{(x,\omega) \in \Lambda}$ is a Gabor superframe for $\mathcal{H}$ if there exist constants $A$ and $B$ such that, for every $\mathbf{f} \in \mathcal{H}$,

$$A \|\mathbf{f}\|_{\mathcal{H}}^2 \leq \sum_{(x,\omega) \in \Lambda} |\langle \mathbf{f}, M_{\omega}T_x g \rangle|_\mathcal{H}^2 \leq B \|\mathbf{f}\|_{\mathcal{H}}^2.$$  

Superframes were introduced in a more abstract form in [20] and in the context of "multiplexing" in [4]. We will need a fundamental structure Theorem from time-frequency analysis, namely the following version of the Janssen-Ron-Shen duality principle [24, Theorem 2.7].

**Theorem A.** Let $\mathbf{g} = (g_0, \ldots, g_{n-1})$. The vector valued system $\mathcal{G}^{(g, \Lambda)}$ is a Gabor superframe for $\mathcal{H}$ if and only if the union of Gabor systems $\cup_{k=0}^{n-1} \mathcal{G}^{(g_k, \Lambda^0)}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$.

4.2. **Duality principle.** In this section we will obtain the following duality principle.

**Theorem 3.** $\Gamma$ is a sampling sequence for $\mathcal{F}^n(\mathbb{C}^d)$ if and only if the adjoint sequence $\Gamma_n^0$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C}^d)$.

We first prove two Lemmas. Combining them with Theorem A gives Theorem 3.

**Lemma 1.** The union of Gabor systems $\cup_{k=0}^{n-1} \mathcal{G}^{(h_k, \Lambda)}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ if and only if $\Gamma_n$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C}^d)$.

**Proof.** The union of Gabor systems $\cup_{k=0}^{n-1} \mathcal{G}^{(h_k, \Lambda)}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ if, for every sequence $\{\alpha_{i,j}^{(k)}\}_{k=0,\ldots,n-1} \in l^2$, there exists a $f \in L^2(\mathbb{R}^d)$ such that $\langle f, M_{\omega}T_x h_k \rangle = \alpha_{i,j}^{(k)}$, for all $k = 0, \ldots, n-1$ and every $(x,\omega) \in \Lambda$. Using the unitarity of $\mathcal{B}$ and the intertwining property (2.10) gives

$$\langle f, M_{\omega}T_x h_k \rangle = \langle \mathcal{B} f, \beta_{\varepsilon_k} \rangle,$$

and setting $F = \mathcal{B} f$ shows that $\Gamma_n$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C}^d)$.

The next Lemma is a key step in our argument and it is at this point that the unitarity of the polyanalytic Bargmann transform is essential.
Lemma 2. Let $h_n = (h_0, \ldots, h_{n-1})$. Then the set $G(h_n, \Lambda)$ is a Gabor superframe for $H = L^2(\mathbb{R}^d, \mathbb{C}^n)$ if and only if the associated complex sequence $\Gamma$ is a sampling sequence for $F^n(\mathbb{C}^d)$.

Proof. Using the definition of the inner product (3.11), identity (3.4) and the definition of the polyanalytic Bargmann transform, it is clear that

$$
\langle f, M_{\omega} T_x h_n \rangle_H = \sum_{0 \leq k \leq n-1} \langle f_k, M_{\omega} T_x h_k \rangle_{L^2(\mathbb{R}^d)}
$$

$$
= \sum_{0 \leq k \leq n-1} e^{i\pi x \cdot -\frac{1}{2} |z|^2} (B^k f_k)(z)
$$

(4.5)

$$
= e^{i\pi x \cdot -\frac{1}{2} |z|^2} (B^n f)(z).
$$

(4.6)

Therefore, setting $F = B^n f$, the unitarity of $B^n$ shows that the inequalities (4.4) are equivalent to (4.3). □

4.3. Main result. We will need the concept of Beurling density of a sequence.

Let $I$ be a compact set of measure 1 in the complex plane and let $n^-(r)$ denote the smallest (and $n^+(r)$ the biggest) number of points from $\Gamma$ to be found in a translate of $rI$. We define the lower and the upper Beurling density of $\Gamma$ to be

$$
D^-(\Gamma) = \lim_{r \to \infty} \sup \frac{n^-(r)}{r^2} \quad \text{and} \quad D^+(\Gamma) = \lim_{r \to \infty} \sup \frac{n^+(r)}{r^2},
$$

respectively. When $\Gamma$ is associated with the lattice $\Lambda$, $D^-(\Gamma) = D^+(\Gamma) = D(\Gamma) = D(\Lambda)$.

We will now use the following result, which is Theorem 2.2 in [12]. Observe that we can remove the uniformly discrete condition from the statement in [12] since we are dealing only with lattices.

Theorem B. The sequence $\Gamma_n$ is a multiple interpolating lattice sequence in the Fock space $F(\mathbb{C})$ if and only if $D(\Gamma_n) < 1$.

From this we obtain the characterization of sampling lattices in $F^n(\mathbb{C})$.

Theorem 4. The lattice $\Gamma$ is a sampling sequence for $F^n(\mathbb{C})$ if and only if $D(\Gamma) > n$.

Proof. We know by the duality principle that $\Gamma$ is a sampling sequence for $F^n(\mathbb{C})$ if and only if the adjoint sequence $\Gamma_n^0$ is a multiple interpolating sequence in the Fock space $F(\mathbb{C})$. By definition of Beurling density, it is obvious that

$$
D(\Gamma_n^0) = n D(\Gamma^0).
$$

Therefore, Theorem B states that $\Gamma^0$ is a multiple interpolating sequence in the Fock space $F(\mathbb{C})$ if and only if $D(\Gamma^0) < \frac{1}{n}$. Using (4.2), we conclude that $\Gamma$ is a sampling sequence for $F^n(\mathbb{C})$ if and only if $D(\Gamma) > n$. □

Using Lemma 1, we recover Theorem 1.1 of [24].

Corollary 2. Let $h_n = (h_0, \ldots, h_{n-1})$. Then the set $G(h_n, \Lambda)$ is a Gabor superframe for $H = L^2(\mathbb{R}, \mathbb{C}^n)$ if and only if $D(\Gamma) > n$. 

5. Interpolation in $\textbf{F}^n(\mathbb{C})$

5.1. Definitions.

Definition 7. The sequence $\Gamma$ is an interpolating sequence for $\textbf{F}^n(\mathbb{C}^d)$ if, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists $F \in \textbf{F}^n(\mathbb{C}^d)$ such that $e^{i\pi x \omega - \frac{1}{2} |z|^2} F(z) = \alpha_{i,j}$, for every $z \in \Gamma$.

Definition 8. The sequence $\Gamma_n$, consisting of $n$ copies of $\Gamma$ is said to be a multiple sampling sequence for $\mathcal{F}(\mathbb{C}^d)$ if there exist numbers $A$ and $B$ such that

\[
\text{(5.1)} \quad A \|F\|^2_{\mathcal{F}(\mathbb{C}^d)} \leq \sum_{z \in \Gamma} \sum_{0 \leq k \leq n-1} |\langle F, \beta_{z^k} \rangle|^2 \leq B \|F\|^2_{\mathcal{F}(\mathbb{C}^d)}.
\]

Definition 9. The set $\cup_{k=0}^{n-1} \mathcal{G}(g_k, \Lambda)$ is said to generate a Gabor multi-frame in $L^2(\mathbb{R}^d)$ if there exist constants $A$ and $B$ such that, for every $f \in L^2(\mathbb{R}^d)$,

\[
\text{(5.2)} \quad A \|f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{(x, \omega) \in \Lambda} \sum_{0 \leq k \leq n-1} \left| \langle f, M_{x}T_{z^k} \rangle_{L^2(\mathbb{R}^d)} \right|^2 \leq B \|f\|^2_{L^2(\mathbb{R}^d)}.
\]

The dual of the duality principle contained in Theorem A is now required. As stated at the end of [25], it reads as follows:

Theorem C. The set $\mathcal{G}(g, \Lambda)$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ if and only if $\cup_{k=0}^{n-1} \mathcal{G}(g_k, \Lambda^0)$ is a Gabor multi-frame in $L^2(\mathbb{R}^d)$.

5.2. Duality principle. Now we prove the following duality.

Theorem 5. The sequence $\Gamma$ is an interpolating sequence for $\textbf{F}^n(\mathbb{C}^d)$ if and only if $\Gamma_n$ is a multiple sampling sequence for $\mathcal{F}(\mathbb{C}^d)$.

As in the sampling section, we first prove two Lemmas which, combined with Theorem C, give the result. The next Lemma requires only the unitarity of the Bargmann transform.

Lemma 3. The set $\cup_{k=0}^{n-1} \mathcal{G}(h_k, \Lambda)$ is a Gabor multi-frame in $L^2(\mathbb{R}^d)$ if and only if $\Gamma_n$ is a multiple sampling sequence for $\mathcal{F}(\mathbb{C}^d)$.

Proof. Similar to Lemma 1: using the unitarity of $\mathcal{B}$ and the intertwining property (2.9) gives $\langle f, M_{x}T_{z^k} h_k \rangle = \langle \mathcal{B} f, \beta_{z^k} \rangle$; setting $F = \mathcal{B} f$ it follows from the unitarity of the Bargmann transform that (5.1) and (5.2) are equivalent. \vspace{10pt}

Again, we make the key connection in the next step, where the unitarity of the polyanalytic Bargmann transform is required.

Lemma 4. The sequence $\Gamma$ is an interpolating sequence for $\textbf{F}^n(\mathbb{C}^d)$ if and only if $\mathcal{G}(h_n, \Lambda)$ is a Riesz sequence for $\mathcal{H}$.

Proof. The sequence $\Gamma$ is an interpolating sequence for $\textbf{F}^n(\mathbb{C}^d)$ if, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists $F \in \textbf{F}^n(\mathbb{C}^d)$ such that $e^{i\pi x \omega - \frac{1}{2} |z|^2} F(z) = \alpha_{i,j}$, for every $z \in \Gamma$. Using the unitarity of $\mathcal{B}^n$, we find, for every $F \in \textbf{F}^n(\mathbb{C}^d)$, a vector valued function $f \in \mathcal{H}$ such that $\mathcal{B}^n f = F$ or, by (1.5) - (1.6), $\langle f, M_{x}T_{z^k} h_n \rangle_{\mathcal{H}} = e^{i\pi x \omega - \frac{1}{2} |z|^2} F$. Therefore, the first assertion is equivalent to say that, for every sequence $\{\alpha_{i,j}\} \in l^2$, there exists a $f \in \mathcal{H}$ such that $\langle f, M_{x}T_{z^k} h_n \rangle_{\mathcal{H}} = \alpha_{i,j}$, for every $z \in \Gamma$. This says that $\mathcal{G}(h_n, \Lambda)$ is a Riesz sequence for $\mathcal{H}$. \vspace{10pt}
5.3. Main result. We will need the following result, which is contained in Theorem 2.1 in [12]:

**Theorem D.** The sequence $\Gamma_n$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if $D(\Gamma_n) < 1$.

As before, we can obtain our main result from this one.

**Theorem 6.** The lattice $\Gamma$ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if $D(\Gamma) < n$.

**Proof.** We know from the duality principle that $\Gamma$ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if $\Gamma_n$ is a multiple sampling sequence for $\mathcal{F}(\mathbb{C})$. Once again we have $D(\Gamma_0^n) = nD(\Gamma^0)$. Therefore, Theorem D states that $\Gamma_0^n$ is a multiple interpolating sequence in the Fock space $\mathcal{F}(\mathbb{C})$ if and only if $D(\Gamma^0) > \frac{1}{n}$. As in Theorem 5 it follows that $\Gamma$ is an interpolating sequence for $\mathbf{F}^n(\mathbb{C})$ if and only if $D(\Gamma) < n$.

□

From this and Lemma 4 we obtain a new result characterizing all the lattices which generate vector valued Gabor Riesz sequences with Hermite functions. This reveals, at least for lattices, the existence of a critical density for vector-valued Gabor systems with Hermite functions.

**Theorem 7.** $\mathcal{G}(h_n, \Lambda)$ is a Riesz sequence for $\mathcal{H}$ if and only if $D(\Gamma) < n$.

**Remark 4.** We should remark that the reason we did not care about the Bessel condition in the equivalence of the Riesz sequence and interpolating property, used several times in the previous section, is that the Hermite functions belong to Feichtinger’s algebra $S_0$ (see [18, 16]):

$$\|h_n\|_{S_0} = \int_{\mathbb{R}} |\langle h_n, M_{\omega} T_x \phi \rangle| \, dz < \infty,$$

where $\phi$ is the $L^2$-normalized Gaussian. As a result they satisfy the Bessel condition [28, theorem 12].

### 6. Generalizations, applications and open problems

6.1. **The super Gabor transform.** The polyanalytic Bargmann transform is an instance of a more general transform. Although it plays no role in the proofs of our main results, we briefly describe it here for completeness of the picture.

The super Gabor transform of a function $f$ with respect to the "window" $g = (g_0, \ldots, g_{n-1})$ is defined, for every $x, \omega \in \mathbb{R}^d$, as

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle_{\mathcal{H}} = \sum_{0 \leq k \leq n-1} \langle f_k, M_\omega T_x g_k \rangle_{L^2(\mathbb{R}^d)}.$$  \hspace{1cm} (6.1)

That is to say,

$$V_g f(x, \omega) = \sum_{0 \leq k \leq n-1} V_{g_k} f_k(x, \omega).$$

This defines a map

$$V_g f : \mathcal{H} \to L^2(\mathbb{R}^{2d}).$$
In the case when the vector $g$ is extracted from an orthogonal sequence $\{g_k\}_{k \geq 0}$, the essential properties of the Gabor transform are kept. As an example of how the results concerning Gabor analysis can be generalized to this setting, we obtain the isometric properties and the orthogonality relations (the latter valid under the slightly weaker condition of biorthogonality).

**Proposition 3.** If

\[
\langle g_i, g_j \rangle_{L^2(\mathbb{R}^d)} = \delta_{i,j},
\]

then $V_g f$ is an isometry between Hilbert spaces, that is

\[
\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{\mathcal{H}}.
\]

**Proof.** Using (6.2) and (2.3) gives

\[
\|V_g f\|_{L^2(\mathbb{R}^{2d})}^2 = \sum_{0 \leq k \leq n-1} \langle V_{g_k} f_k, V_{g_k} f_k \rangle_{L^2(\mathbb{R}^{2d})}
\]

\[
= \sum_{0 \leq k \leq n-1} \|V_{g_k} f_k\|_{L^2(\mathbb{R}^{2d})}^2
\]

\[
= \sum_{0 \leq k \leq n-1} \|f_k\|_{L^2(\mathbb{R}^d)}^2
\]

\[
= \|f\|_{\mathcal{H}}^2.
\]

The orthogonality relations are valid under the slightly weaker condition of biorthogonality.

**Proposition 4.** If

\[
\langle g_{1,i}, g_{2,j} \rangle_{L^2(\mathbb{R}^d)} = \delta_{i,j},
\]

then $V_g f$ satisfies

\[
\langle V_{g_{1,i}} f_1, V_{g_{2,j}} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{\mathcal{H}}.
\]

**Proof.** Using (6.3) and (2.3) gives

\[
\langle V_{g_{1,i}} f_1, V_{g_{2,j}} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \sum_{0 \leq k \leq n-1} \langle f_{1,k}, f_{2,k} \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{\mathcal{H}}.
\]

Clearly, when we take $g = (\Phi_0, \ldots, \Phi_{m-1})$, we have the following relation with the polyanalytic Bargmann transform:

\[
(B^n \Gamma)(z) = e^{i\pi\omega - \pi \frac{|z|^2}{2}} V_g f(x, \omega).
\]

This observation removes some of the mystery from the previous sections. Now we are in a position to say that the polyanalytic Bargmann-Fock spaces play exactly the same role as the Bargmann-Fock space in the scalar case. We have thus all the basic ingredients to build a theory of vector valued (or super) Gabor analysis:

- A vector valued Gabor transform.
- A discrete theory for $L^2(\mathbb{R}^d)$ frames and Riesz basis.
- A special vector of windows providing the connection with complex analysis, where, in the case $d = 1$, a Nyquist rate phenomenon can be observed.
The analyzing vector can be extracted from orthogonal systems other than the Hermite functions, though they probably do not lead to very structured situations. As a first example we may think of the Haar basis. Other wavelets may also be used, but we will not pursue this question further in this paper.

6.2. Applications. Although we are here dealing mostly with questions of a conceptual nature, there are potential applications of these results in multiplexing, an important method in telecommunications, computer networks and digital video, as indicated in [4] and [24]. The idea of multiplexing is to encode $n$ independent signals $f_k \in L^2(\mathbb{R}^d)$ as a single sequence $f$ that captures the time-frequency information of each one. With suitable windows $g = (g_0, \ldots, g_{n-1})$, the time-frequency content of each signal can be measured by the associated super-Gabor systems. The super Gabor transform $V_g f(x, \omega)$, gives the precise value one wants to approximate via the discrete systems.

6.3. Further questions. As in the scalar-valued case, when the connection to complex analysis is missing the lattices generating super-frames should be hard to describe as, for instance, the case of a "rectangular" window. What we mean by rectangular window is one obtained from the Haar basis, which is the natural generalization of the characteristic function of an interval. We wonder how Janssen’s tie [32] would generalize to this situation.

It is still unclear if there is a relation between the Bargmann-Fock space of polyanalytic functions and vector valued coherent states, as considered in [1] and [2].

A rather mysterious topic is sampling and interpolation in the true polyanalytic space $F_{n-1}(\mathbb{C})$. Partial results follow from ours. Using decomposition (3.10), it is easy to see that we obtain two propositions concerning $F_{n-1}(\mathbb{C})$: one, a sufficient condition for sampling, the other a necessary condition for interpolation.

**Proposition 5.** If $D(\Lambda) > n$ then $\Gamma$ is a sampling sequence for $F_{n-1}(\mathbb{C})$.

**Proof.** This is obvious from decomposition (3.10) because, if $D(\Lambda) > n$, then inequality (6.3) holds for every $F \in F_n(\mathbb{C})$. In particular it also holds for every $F \in F^n(\mathbb{C})$. □

**Proposition 6.** If $\Gamma$ is an interpolating sequence for $F_{n-1}(\mathbb{C})$, then

$$D(\Lambda) < n.$$ 

**Proof.** If $D(\Lambda) > n$, then $\Gamma$ is not an interpolating sequence for $F^n(\mathbb{C})$. As a result, there exists $\{\alpha_{i,j}\} \in l^2$, such that it is impossible to find $F \in F^n(\mathbb{C})$, verifying $e^{i\pi x\omega - \pi |z|^2} F(z) = \alpha_{i,j}$. Again, from the decomposition (3.10) one sees that in particular it is impossible to find $F \in F^n(\mathbb{C})$ verifying $e^{i\pi x\omega - \pi |z|^2} F(z) = \alpha_{i,j}$. □

We may wonder whether these conditions are sharp. In the case of Proposition 6 the answer is a definite "no". This is because, if $\Gamma$ is an interpolating sequence for $F_{n-1}(\mathbb{C})$, then $\{(f, M_{x} f h_{n-1})_{L^2(\mathbb{R})}\}_{(x, w) \in \Lambda}$ is a Riesz basis for $L^2(\mathbb{R})$. As a result, $\{(f, M_{x} f h_{n-1})_{L^2(\mathbb{R})}\}_{(x, w) \in \Lambda}$ is a frame for $L^2(\mathbb{R})$ and consequently, using Rieffel-Ramanathan-Steger Theorem [42], [40], we must have $D(\Lambda^o) > 1$ and, by scalar Ron-Shen duality, this implies $D(\Lambda) < 1$. Therefore, estimate (6.5) is far from being sharp.

Let us look closer at Proposition 5. Here things become rather intriguing.
It is easy to see that the set $G(h_n, \Lambda)$ is a Gabor frame if and only if the associated sequence $\Gamma$ is a sampling sequence for $F^n(C)$: since $\langle f, M_\omega T_x h_n^{-1} \rangle_{L^2(\mathbb{R})} = V_{h_n} f(x, w)$, then (3.4) can be written as

$$\langle f, M_\omega T_x h_n^{-1} \rangle_{L^2(\mathbb{R})} = e^{i\pi x \omega - \frac{\pi}{4} |\omega|^2} B^n f(z).$$

By setting $F = B^n f$, the isometry of $B^n : L^2(\mathbb{R}) \to F^n(\mathbb{C})$ and the relation (6.6) show that definition (4.3) is equivalent to the fact that $\{\langle f, M_\omega T_x h_n^{-1} \rangle_{L^2(\mathbb{R})} \}_{(x,w)\in \Lambda}$ is a frame.

Therefore, Proposition 5 is equivalent to the sufficient condition obtained in [23], where the authors prove that, if $D(\Lambda) > n$, then $\{\langle f, M_\omega T_x h_n^{-1} \rangle_{L^2(\mathbb{R})} \}_{(x,w)\in \Lambda}$ is a frame and give some evidence to support their conjecture that the result is sharp.

If true, this would be quite a surprising statement, in face of decomposition (3.10).

Moreover, by duality, it would bring estimate (6.5) down to $D(\Lambda) < 1/n$.

Acknowledgement. I would like to thank Hans Feichtinger for his postdoctoral guidance and the NuHAG group at the University of Vienna, where I first had contact with some of the ideas developed in this paper. Moreover, specific acknowledgement is due to Karlheinz Gröchenig, who gave generous local seminars with fresh ideas on Gabor frames with Hermite windows and on the vectorial duality of Gabor frames, and to Yuri Lyubarskii, who read the first version of the manuscript, providing comments and corrections which are incorporated in this version. The Referees and the Editor helped to improve the readability of the paper and they have drawn my attention to the recent preprint [7]. Finally, Radu Balan explained me the results in [8] and gave insightful remarks on the whole topic while I was preparing the revised version of the manuscript.

References

[1] S. T. Ali, A general theorem on square-integrability: vector coherent states, J. Math. Phys. 39 (1998), no. 8, 3954–3964.
[2] S. T. Ali, M. Engliš, J. P. Gazeau, Vector coherent states from Plancherel’s theorem, Clifford algebras and matrix domains, J. Phys. A 37 (2004), no. 23, 6067–6089.
[3] G. Ascensi, J. Bruna, Model space results for the Gabor and Wavelet transforms, IEEE Trans. Inform. Theory 55 (2009), no. 5, 2250-2259.
[4] R. Balan, Multiplexing of signals using superframes, In SPIE Wavelets applications, volume 4119 of Signal and Image processing XIII, pag. 118-129 (2000).
[5] R. Balan, Density and Redundancy of the Noncoherent Weyl-Heisenberg Superframes. The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), 29-41 Contemp. Math., 247, Amer. Math. Soc., Providence R.I. (1999).
[6] R. Balan, Extensions of No-Go Theorems to Many Signal Systems, AMS Special Session on Wavelets, San Antonio 1997, Contemp. Math 216, 3-15 (1997).
[7] R. Balan, P. Casazza, Z. Landau, Redundancy for localized and Gabor frames, http://arxiv.org/abs/0904.4471
[8] M. B. Balk, Polyanalytic Functions, Akad. Verlag, Berlin (1991).
[9] M. B. Balk, M. Y. Mazalov, On the Hayman uniqueness problem for polyharmonic functions, in Clifford Algebras and Their Application in Mathematical Physics (Aachen, Germany, 1996), Fund. Theories Phys. 94, Kluwer, Dordrecht, Germany, 1998, 11 – 16.
[10] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform. Comm. Pure Appl. Math. 14 187–214 (1961).
[11] A. Beurling, The Collected Works Of Arne Beurling, Vol. 2 Harmonic Analysis, Boston 1989.
[12] S. Brekke, K. Seip, Density Theorems for sampling and interpolation in the Bargmann-Fock space, III. Math. Scand. 73 (1993), no. 1, 112–126.
[13] O. Christensen, B. Deng, C. Heil, Density of Gabor frames. Appl. Comput. Harmon. Anal. 7 (1999), no. 3, 292-304.
[14] I. Daubechies, "Ten Lectures On Wavelets", CBMS-NSF Regional conference series in applied mathematics (1992).
[15] I. Daubechies, A. Grossmann, Frames in the Bargmann space of entire functions. Comm. Pure Appl. Math. 41 (1988), no. 2, 151–164.
[16] H. G. Feichtinger, On a new Segal algebra. Monatsh. Math. 92 (1981), no. 4, 269–289.
[17] H. G. Feichtinger, W. Kozek, Quantization of TF lattice-invariant operators on elementary LCA groups. Gabor analysis and algorithms, 233–266, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.
[18] H. G. Feichtinger, G. A. Zimmermann, A Banach space of test functions for Gabor analysis. Gabor analysis and algorithms, 123–170, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.
[19] G. B. Folland, Harmonic analysis in phase space. Annals of Mathematics Studies, 122. Princeton University Press, Princeton, NJ, 1989.
[20] H. Führ, Simultaneous estimates for vector-valued Gabor frames of Hermite functions. Adv. Comput. Math. 29, no. 4, 357–373, (2008).
[21] M. Gosson, Luef, Spectral and regularity properties of a Weyl calculus related to Landau quantization, preprint arXiv:0810.3874v1 [math-ph].
[22] K. Gröchenig, "Foundations Of Time-Frequency Analysis", Birkhäuser, Boston, (2001).
[23] K. Gröchenig, Y. Lyubarskii, Gabor frames with Hermite functions, C. R. Acad. Sci. Paris, Ser. I 344 157-162 (2007).
[24] K. Gröchenig, Y. Lyubarski, Gabor (Super)Frames with Hermite Functions, Math. Ann., 345, no. 2, 267-286 (2009).
[25] K. Gröchenig, Gabor frames without inequalities. Int. Math. Res. Not. IMRN 2007, no. 23.
[26] D. Han, D. R. Larson, Frames, bases and group representations. Mem. Amer. Math. Soc. 147 (2000), no. 697.
[27] W. K. Hayman and B. Korenblum, Representation and uniqueness theorems for polyharmonic functions, J. Anal. Math. 60 (1993), 113 – 133.
[28] C. Heil, History and evolution of the density Theorem for Gabor frames. J. Fourier Anal. Appl. 13 (2007), no. 2, 113–166.
[29] O. Hutnik, On the structure of the space of wavelet transforms, C. R. Math. Acad. Sci. Paris, Ser. I 346 649–652, (2008).
[30] O. Hutnik, A note on wavelet subspaces, Monatsh. Math. online published.
[31] A. J. E. M. Janssen, Signal analytic proofs of two basic results on lattice expansions. Appl. Comput. Harmon. Anal. 1 (1994), pp. 350–354.
[32] A. J. E. M. Janssen, Zak transforms with few zeros and the tie, in "Advances in Gabor Analysis" (H.G. Feichtinger, T.Strohmer, eds.), Boston, 2003, pp. 31-70.
[33] A. J. E. M. Janssen, Some Weyl-Heisenberg frame bound calculations. Indag. Math. (N.S.) 7 (1996), no. 2, 165–183.
[34] A. J. E. M. Janssen, Hermite function description of Feichtinger’s space $S_0$. J. Fourier Anal. Appl. 11 (2005), no. 5, 577-588.
[35] A. J. E. M. Janssen, T. Strohmer, Hyperbolic secants yield Gabor frames. Appl. Comput. Harmon. Anal. 12 (2002), no. 2, 259-267.
[36] Landau, H. J., Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967), 37–52.
[37] Landau, H. J., Sampling, data transmission, and the Nyquist rate, Proc. IEEE, 55, 1701-1706, (1967).
[38] Y. Lyubarskii, Frames in the Bargmann space of entire functions, Entire and subharmonic functions, 167-180, Adv. Soviet Math., 11, Amer. Math. Soc., Providence, RI (1992).
[39] A. K. Ramazanov, Representation of the space of polyanalytic functions as the direct sum of orthogonal subspaces. Application to rational approximations. (Russian) Mat. Zametki 66 (1999), no. 5, 741–759; translation in Math. Notes 66 (1999), no. 5-6, 613–627 (2000).
[40] J. Ramanathan, T. Steger, Incompleteness of sparse coherent states. Appl. Comput. Harmon. Anal. 2, no. 2, 148–153 (1995).
[41] H. Render, Real Bargmann spaces, Fischer decompositions, and sets of uniqueness for polyharmonic functions. Duke Math. J. 142 (2008), no. 2, 313–352.
[42] M. A. Rieffel, Von Neumann algebras associated with pairs of lattices in Lie groups, Math. Ann. 257, 403-418 (1981).
[43] A. Ron, Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, Duke Math. J. 89 (1997), 237–282.
[44] I. Shigekawa, Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemannian manifold. J. Funct. Anal. 75 (1987), no. 1, 92–127.
[45] K. Seip, Reproducing formulas and double orthogonality in Bargmann and Bergman spaces, SIAM J. Math. Anal. 22, No. 3 pp. 856-876 (1991).
[46] K. Seip, Density Theorems for sampling and interpolation in the Bargmann-Fock space I, J. Reine Angew. Math. 429, 91-106 (1992).
[47] K. Seip, R. Wallstén, Density Theorems for sampling and interpolation in the Bargmann-Fock space II, J. Reine Angew. Math. 429 (1992), 107-113.
[48] K. Seip, Beurling type density Theorems in the unit disc, Invent. Math., 113, 21-39 (1993).
[49] K. Seip, Interpolation and sampling in spaces of analytic functions. University Lecture Series, 33. American Mathematical Society, Providence, RI, 2004.
[50] S. Thangavelu, Lectures on Hermite And Laguerre Expansions. With a preface by Robert S. Strichartz. Mathematical Notes, 42. Princeton University Press, Princeton, NJ, 1993.
[51] A. Wünsche, Displaced Fock states and their connection to quasiprobabilities, Quantum Opt. 3 (1991) 359-383.
[52] N. L Vasilevski, Poly-Fock spaces, Differential operators and related topics, Vol. I (Odessa, 1997), 371–386, Oper. Theory Adv. Appl., 117, Birkhäuser, Basel, (2000).
[53] N. L. Vasilevski, On the structure of Bergman and poly-Bergman spaces. Integral Equations Operator Theory 33, no. 4, 471–488, (1999).

CMUC, DEPARTMENT OF MATHEMATICS OF UNIVERSITY OF COIMBRA, SCHOOL OF SCIENCE AND TECHNOLOGY (FCTUC) 3001-454 COIMBRA, PORTUGAL, AND NuHAG, FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSSE 15, A-1090 WIEN, AUSTRIA  
E-mail address: daniel@mat.uc.pt  
URL: http://www.mat.uc.pt/~daniel/