Absolutely Minimizing Lipschitz Extensions and Infinity Harmonic Functions on the Sierpinski gasket

Fabio Camilli*  Raffaela Capitanelli*  Maria Agostina Vivaldi*

June 20, 2018

Abstract

Aim of this note is to study the infinity Laplace operator and the corresponding Absolutely Minimizing Lipschitz Extension problem on the Sierpinski gasket in the spirit of the classical construction of Kigami for the Laplacian. We introduce a notion of infinity harmonic functions on pre-fractal sets and we show that these functions solve a Lipschitz extension problem in the discrete setting. Then we prove that the limit of the infinity harmonic functions on the pre-fractal sets solves the Absolutely Minimizing Lipschitz Extension problem on the Sierpinski gasket.

MSC2010: 31C20, 28A80.

Keywords: Sierpinski gasket, McShane-Whitney extensions, Absolute Minimizing Lipschitz Extension, infinity Laplacian.

1 Introduction

The theory of Absolutely Minimizing Lipschitz Extension ([1, 4]) concerns the classical problem of extending a Lipschitz continuous function $f$ defined on the boundary of an open set $U \subset \mathbb{R}^d$ to the interior of $U$ without increasing the Lipschitz constant. In other words, to find a Lipschitz continuous function $u : \overline{U} \to \mathbb{R}$ such that $u = f$ on $\partial U$ and $\text{Lip}(u, U) = \text{Lip}(f, \partial U)$ (Lip denotes the Lipschitz constant).

The previous problem has several solutions, all in between a maximal and a minimal one called respectively the McShane’s extension and the Whitney’s extension. But, among all these possible solutions, the “canonical” one is the so called Absolutely Minimizing Lipschitz Extension (AMLE in short). This function is characterized by satisfying the extension problem not only in $U$, but also in any open subset $V$ of $U$, that is $\text{Lip}(u, V) = \text{Lip}(u, \partial V)$ for any open $V \subset U$. The relevance of the

---

*1 Dip. di Scienze di Base e Applicate per l’Ingegneria, “Sapienza” Università di Roma, via Scarpa 16, 00161 Roma, Italy, (e-mail: camilli, capitanelli, vivaldi@sbai.uniroma1.it)
notion of AMLE relies in the several additional properties that this function satisfies, for example it is the unique viscosity solution of the Dirichlet problem for the infinity Laplace equation
\[ \Delta_\infty u(x) = 0, \quad x \in U. \] (1.1)
Here \( \Delta_\infty u = \sum_{i,j=1}^d \partial_{x_i} u \partial_{x_j} u \partial_{x_i}^2 u \), the infinity Laplacian, is a nonlinear degenerate second order operator and a function satisfying (1.1) is said infinity harmonic.

The theory of AMLE has been extended to length spaces (see [1, 3, 9, 10, 15]), hence it applies in particular to the Sierpinski gasket \( S \) endowed with its geodesic distance. Prescribed a boundary data \( f \) on the vertices of the initial simplex from which \( S \) is obtained via iteration, there exists a unique AMLE of \( f \) to \( S \). Moreover this function can be characterized as in the Euclidean case by an intuitive geometric property, called Comparison with Cones.

After the seminal work of Kigami [11], a standard way to define a harmonic function on the Sierpinski gasket, and more in general for the class of post-critically finite fractals, is to consider the uniform limit of solutions of suitable scaled finite difference differences on the pre-fractals.

For the infinity Laplace operator this approach has been pursued in [5]. In this thesis, a graph infinity Laplacian on pre-fractal sets is defined and an algorithm to compute explicitly infinity harmonic functions is studied. By means of a constructive approach based on the previous algorithm, it is proved that the sequence of the infinity harmonic functions on the pre-fractal sets converges to a function defined on the limit fractal set. It is worth noting that the same graph infinity Laplacian is used in the Euclidean case to approximate the viscosity solution of (1.1) (see [12, 13, 14]).

Following an approach similar to [5], we aim to define an infinity harmonic function on \( S \) as the limit of solutions of finite difference equations on pre-fractals. We study the Lipschitz extension problem on the pre-fractal sets and we show that an appropriate notion of AMLE can be introduced in this framework. We also prove that an AMLE is a solution of the graph infinity Laplacian and it satisfies a Comparison with Cone property with respect to the path distance. The Comparison with Cone property on the pre-fractals is crucial since it allows us to show that the limit of the AMLEs on the pre-fractals is an AMLE on the Sierpinski. The convergence result allows us to define an infinity harmonic function on the Sierpinski as the limit of infinity harmonic functions on the pre-fractals and to conclude the equivalence, as in Euclidean case, between AMLE and infinity harmonic functions.

Hence, besides giving a simpler proof of the convergence result in [5], we also obtain as in the Euclidean case the equivalence among the various properties which characterize the AMLE theory. We remark that the previous construction on the Sierpinski gasket can be readily extended to the class of post-critically finite fractals since it is only based on the convergence of the path distance defined on the pre-fractal sets to the path distance (intrinsic length) on the limit fractal set ([2], [7]).

The paper is organized as follows. In Section 2 we introduce notations and definitions and we prove some preliminary properties for the graph infinity Laplacian. Section 3 is devoted to the AMLE problem on pre-fractals. In Section 4 we recall the definition of AMLE in metric spaces and we prove the convergence result for the pre-fractals invading the Sierpinski gasket. Finally, in Section 5 we
describe the algorithm in [5] and we study the relation between infinity and $p$-harmonic function in the pre-fractals.

2 Notations and preliminary results

In this section we introduce notations and definitions and we collect some preliminary results on infinity harmonic functions on graphs.

Consider an unitary equilateral triangle $V^0$ of vertices $\{q_1, q_2, q_3\}$ in $\mathbb{R}^2$ and the maps $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$\psi_i(x) := q_i + \frac{1}{2}(x - q_i).$$

Iterating the $\psi_i$'s, we get the set

$$V^\infty = \bigcup_{n=0}^\infty V^n$$

where each $V^n$ is given by the union of the images of $V^0$ under the action of the maps $\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$ with $w = (w_1, \ldots, w_n)$, $w_i \in \{1, 2, 3\}$, a word of length $|w| = n$. Then the Sierpinski gasket $S$ is the closure of $V^\infty$ (with respect to the Euclidean topology) and it is the unique non empty compact set $F$ which satisfies

$$F = \bigcup_{i=1}^3 \psi_i(F).$$

For any $n$, we can identify $V^n$ with the graph $(V^n, \sim_n)$, where $\sim_n$ is the following relation on $V^n$: for $x, y \in V^n$, $x \sim_n y$ if and only if the segment connecting $x$ and $y$ is the image of a side of the starting simplex under the action of some $\psi_w$ with $|w| = n$. If $x, y \in V^n$ and $x \sim_n y$, then we will say that $x, y$ are adjacent in $V^n$.

Given a set $K \subset V^n \setminus V^0$, we define the boundary and the closure of $K$ by

$$\partial K = \{y \in V^n \setminus K : \exists x \in K \text{ s.t. } y \sim_n x\},$$

$$\overline{K} = K \cup \partial K.$$

The distance between two adjacent vertices $x, y \in V^n$ is

$$d_n(x, y) = \frac{1}{2^n} =: \delta_n,$$

while the distance between $x, y \in V^n$ is the vertex distance

$$d_n(x, y) := \min\{d_n(x_0, x_1) + d_n(x_1, x_2) + \cdots + d_n(x_{N-1}, x_N)\}$$  \hspace{1cm} (2.1)

where the minimum is over all the finite path $\{x_0 = x, x_1, \ldots, x_N = y\}$ with $x_i \sim_n x_{i+1}$, $i = 0, \ldots, N - 1$, connecting $x$ to $y$.

We also consider the distance $d_{n,K}$ between $x, y \in \overline{K}$ defined as in (2.1) where the minimum is taken over all the finite paths restricted to stay inside $K$, i.e. $\{x = x_0, \ldots, x_N = y\}$ with $x_i \sim_n x_{i+1}$ for $i = 0, \ldots, N - 1$ and $x_i \in K$ for $i = 1, \ldots, N - 1$ (if there is no path in $K$ connecting $x$ to $y$ we set $d_{n,K}(x, y) = +\infty$). Observe that in general $d_n(x, y) \leq d_{n,K}(x, y)$ for $x, y \in \overline{K}$. 

3
Definition 2.1. A set $K \subset V^n \setminus V^0$ is said to be connected if $d_{n,K}(x,y) < +\infty$ for any $x,y \in K$.

For $u : V^n \to \mathbb{R}$ and a non empty subset $K \subset V^n \setminus V^0$, we define

$$\text{Lip}^n(u,K) = \max_{x,y \in K, x \neq y} \left| \frac{u(x) - u(y)}{d_{n,K}(x,y)} \right|,$$  \hspace{0.5cm} (2.2)

and

$$\text{Lip}^n(u,\partial K) = \max_{x,y \in \partial K, x \neq y} \left| \frac{u(x) - u(y)}{d_{n,K}(x,y)} \right|,$$  \hspace{0.5cm} (2.3)

(note that in (2.2) the maximum is taken over the set $\bar{K} = K \cup \partial K$).

For $x \in V^n \setminus V^0$

$$\Delta_{n,\infty} u(x) = \max_{y \sim_n x} \left\{ u(y) - u(x) \right\} + \min_{y \sim_n x} \left\{ u(y) - u(x) \right\},$$  \hspace{0.5cm} (2.4)

and

$$F^n(u,x) = \max_{y \sim_n x} \left| \frac{u(x) - u(y)}{\delta_n} \right|.$$  \hspace{0.5cm} (2.5)

We give some basic properties of the previous operators.

Definition 2.2. A function $u : V^n \to \mathbb{R}$ is said infinity harmonic in $K \subset V^n \setminus V^0$ if

$$\Delta_{n,\infty} u(x) = 0 \hspace{0.5cm} \text{for all} \hspace{0.5cm} x \in K.$$  

Proposition 2.3. Let $K \subset V^n \setminus V^0$ be a connected set.

(i) If $u,v : V^n \to \mathbb{R}$ satisfy

$$\Delta_{n,\infty} u \geq 0, \hspace{0.5cm} \Delta_{n,\infty} v \leq 0 \hspace{0.5cm} \text{in} \hspace{0.5cm} K$$

and $u \leq v$ on $\partial K$, then $u \leq v$ in $K$.

(ii) For any $g : \partial K \to \mathbb{R}$, there exists a unique infinity harmonic function $u$ in $K$ and such that $u = g$ on $\partial K$. Moreover $\min_{\partial K} g \leq u \leq \max_{\partial K} g$ and

$$\text{either} \hspace{0.5cm} \min_{y \sim_n x} \left\{ u(y) - u(x) \right\} < 0 < \max_{y \sim_n x} \left\{ u(y) - u(x) \right\} \hspace{0.5cm} \text{or} \hspace{0.5cm} u(y) = u(x) \hspace{0.5cm} \forall y \sim_n x.$$  \hspace{0.5cm} (2.6)

Proof. If $K$ is connected, then it is connected with the boundary, i.e. for any $x \in K$ there is $y \in \partial K$ such that $d_{n,K}(x,y) < \infty$. Under this assumption, the comparison principle in (i) is proved in [13, Theorem 4]. Existence can be proved either as in [13, Theorem 5] by means of a fixed point argument or via the constructive approach given by the Lazarus algorithm, see Section 5.1 for details. The estimate at the boundary is again consequence of the comparison principle and since the constants are infinity harmonic. For property (2.6), see [13, Lemma 3].

In the next proposition we consider properties of the distance function with respect to the graph infinity Laplacian (2.4).

Proposition 2.4. Let $K \subset V^n \setminus V^0$ be connected and $x_0 \in \partial K$. Then the function $u(x) = d_{n,K}(x_0,x)$ (respectively, $v(x) = -d_{n,K}(x_0,x)$) satisfies $\Delta_{n,\infty} u \leq 0$ (respectively, $\Delta_{n,\infty} v \geq 0$) in $K$. 

4
Proof. Consider the function \( u(x) = d_{n,K}(x_0, x) \). Given \( x \in K \), then the minimum of \( u(y) - u(x) \), \( y \sim_n x \), will be a negative one, along an adjacent point \( y \) which is contained in the shortest path from \( x \) to \( x_0 \). Since the path from \( y \) to \( x_0 \) will be one vertex shorter, then \( \min_{y \sim_n x} \{u(y) - u(x)\} = u(y) - u(x) = -\delta_n \).

If \( u \) does reach the maximum in \( K \) at \( x \), then \( \max_{y \sim_n x} \{u(y) - u(x)\} \leq 0 \); otherwise the maximum of \( u(y) - u(x) \), \( y \sim_n x \), will be positive attained at a point \( y \) such that \( x \) is contained in the shortest path from \( y \) to \( x_0 \) and \( \max_{y \sim_n x} \{u(y) - u(x)\} = u(y) - u(x) = \delta_n \). Hence

\[
\Delta_n u(x) = \max_{y \sim_n x} \{u(y) - u(x)\} + \min_{y \sim_n x} \{u(y) - u(x)\} \leq \delta_n - \delta_n = 0.
\]

In a similar way it is possible to prove that \( v(x) = -d_{n,K}(x_0, x) \) satisfies \( \Delta_n v \geq 0 \) in \( K \).

In the next proposition we prove that a function \( u \) is linear along any minimal path joining two points which realizes the maximum of the slope at the boundary, see (2.3). This is a crucial property that will be exploited in Section 5.1 to explicitly compute an infinity harmonic function on \( V^n \).

**Proposition 2.5.** Let \( K \subset V^n \setminus V^0 \) be connected, \( u : K \to \mathbb{R} \) be such that \( \text{Lip}^n(u, K) = \text{Lip}^n(u, \partial K) \) and \( x, y \in \partial K \) such that

\[
\text{Lip}^n(u, \partial K) = \frac{|u(x) - u(y)|}{d_{n,K}(x, y)}.
\]

If \( \gamma = \{x = x_0, x_1, \ldots, x_N = y\} \) is a minimal path for \( d_{n,K} \) joining \( x \) to \( y \), then \( u \) is linear along \( \gamma \), i.e.

\[
u(x_i) = \frac{d_{n,K}(x_i, x)u(y) + d_{n,K}(x_i, y)u(x)}{d_{n,K}(x, y)}, \quad i = 0, \ldots, N
\]

**Proof.** Set \( L = \text{Lip}^n(u, K) \). If \( u \) is not linear along \( \gamma \) and since \( \text{Lip}^n(u, \partial K) = L \) there exists an index \( i \in \{1, \ldots, N\} \) such that

\[
|u(x_i) - u(x_{i-1})| < Ld_{n,K}(x_i, x_{i-1}). \tag{2.7}
\]

Then, by \( \text{Lip}^n(u, K) = L \) and (2.7)

\[
Ld_{n,K}(y, x) = |u(y) - u(x_{N-1}) + \cdots + u(x_i+1) - u(x_i) + u(x_i) - u(x_{i-1}) + \cdots + u(x_1) - u(x)|
< L[d_{n,K}(y, x_N) + \cdots + Ld_{n,K}(x_i, x_{i-1}) + \cdots + d_{n,K}(x_1, x)] = Ld_{n,K}(y, x)
\]

and therefore a contradiction. \( \square \)

The next proposition connects the functional \( F^n(u, x) \), which can be interpreted as the Lipschitz constant of \( u \) at \( x \), with the Lipschitz constant \( \text{Lip}^n(u, K) \) in \( K \).

**Proposition 2.6.** For \( u : V^n \to \mathbb{R} \) and for any connected set \( K \subset V^n \setminus V^0 \),

\[
\text{Lip}^n(u, K) = \max_{x \in K} F^n(u, x) \tag{2.8}
\]
Proof. It is clear that $F^n(u, x) \leq \text{Lip}^n(u, K)$ for any $x \in K$, hence $\max_{x \in K} F^n(u, x) \leq \text{Lip}^n(u, K)$. Let $x, y \in K$ be such that $\text{Lip}^n(u, K) = |u(x) - u(y)|/d_{n,K}(x, y)$. Consider a minimal path $\gamma = \{x = x_0, x_1, \ldots, x_N = y\}$ from $x$ to $y$. Hence

$$|u(x) - u(y)| \leq \sum_{i=0}^{N-1} |u(x_{i+1}) - u(x_i)| \leq \frac{\delta_n}{d_{n,K}(x, y)} \left( \sum_{i=1}^{N-1} F^n(u, x_i) + F^n(u, x_1) \right)$$

Since $\gamma$ is composed by $N$ arcs, then $d_{n,K}(x, y) = N\frac{1}{2\pi} = N\delta_n$, hence

$$|u(x) - u(y)| \leq \frac{1}{N} \sum_{i=0}^{N-1} \max_{x \in K} F^n(u, x) = \max_{x \in K} F^n(u, x)$$

and therefore (2.8).

3 The Absolutely Minimizing Lipschitz Extension problem in $V^n$

In this section we fix $n \in \mathbb{N}$ and we consider the Lipschitz extension of a function $g : V^0 \to \mathbb{R}$ to $V^n$. The following result is the analogous of the classical Whitney-McShane solution to the Lipschitz extension problem.

**Proposition 3.1.** Given $K \subset V^n \setminus V^0$ connected and a function $g : \partial K \to \mathbb{R}$, set $L_0 = \text{Lip}^n(g, \partial K)$ and define

$$M_*^n(x) = \max_{y \in \partial K} \{g(y) - L_0d_{n,K}(x, y)\}, \quad M^*^n(x) = \min_{y \in \partial K} \{g(y) + L_0d_{n,K}(x, y)\}.$$

Then $\text{Lip}^n(M^*, K) = \text{Lip}^n(M_*, K) = L_0$ and for any $u : V^n \to \mathbb{R}$ such that $u = g$ on $\partial K$ and $\text{Lip}(u, K) = L_0$, we have

$$M_*(x) \leq u(x) \leq M^*(x), \quad x \in K. \quad (3.1)$$

**Proof.** By the very definition of $\text{Lip}(u, K)$, for any $x \in K, y \in \partial K$

$$g(y) - L_0d_{n,K}(x, y) = u(y) - L_0d_{n,K}(x, y) \leq u(x)$$

and therefore (3.1). Let us prove that $\text{Lip}(M_*(x), V^n) = L_0$. Given $x_1, x_2 \in K$, let $y_1 \in \partial K$ be such that $M_*(x_1) = g(y_1) - L_0d_n(x_1, y_1)$. Hence,

$$M_*(x_1) - M_*(x_2) \leq g(y_1) - L_0d_{n,K}(x_1, y_1) - g(y_1) + L_0d_{n,K}(x_2, y_1) \leq L_0d_{n,K}(x_1, x_2).$$

The proof that $M_*(x_1) - M_*(x_2) \geq -L_0d_{n,K}(x_1, x_2)$ is similar. 

Our approach to the Absolutely Minimizing Lipschitz Extension problem on $V^n$ is based on the following properties/observations:
1. If $\Delta^\infty_n u(x) = 0$, then $t = u(x)$ minimizes the functional $I(t) = \max_{y \sim_n x} \frac{|t - u(y)|}{\delta_n}$ giving the Lipschitz constant of $u$ at $x$ (see [14, Theorem 5] and [12]).

2. $\text{Lip}^n(u, K) = \max_{x \in K} F_n(u, x)$ (see Prop. 2.6).

3. If $u$ solves $\Delta^\infty_n u(x) = 0$ in $K \subset V^n \setminus V^0$ connected, then $u$ is linear along a minimal path for $d_{n,K}$ joining two points which realize the maximal slope $\text{Lip}^n(u, \partial K)$ at the boundary ([5, 17]). The same property holds also if $\text{Lip}^n(u, K) = \text{Lip}^n(u, \partial K)$ (see Prop. 2.5).

4. The functions $M^*(x)$, $M_*(x)$, defined by means of the distance function $d_{n,K}$, give the minimal and maximal solution to the Lipschitz extension problem on $K$. Moreover, for $x_0 \in \partial K$, $d_{n,K}(x_0, \cdot)$ and $-d_{n,K}(x_0, \cdot)$ are a supersolution and a subsolution of $\Delta^\infty_n u = 0$ in $K$ (Prop. 2.4)

Indeed it is clear that all the previous concepts are strictly related and, in analogy with the Euclidean case, we introduce the following definitions

**Definition 3.2.**

(i) A function $u : V^n \to \mathbb{R}$ is said an absolute minimizer for the functional $\text{Lip}^n$ on $V^n$ if for any connected set $K \subset V^n \setminus V^0$ and for any $v : V^n \to \mathbb{R}$ such that $u = v$ on $\partial K$, then $\text{Lip}^n(u, K) \leq \text{Lip}^n(v, K)$. We denote by $\text{AMLE}(V^n)$ the set of the absolute minimizers for $\text{Lip}^n$ in $V^n$.

(ii) A function $u : V^n \to \mathbb{R}$ satisfies the Comparison with Cones property (noted CC property) in $V^n$ if for any connected set $K \subset V^n \setminus V^0$, for any $x_0 \in \partial K$, $\lambda \geq 0$ and $\alpha \in \mathbb{R}$

$$u \leq \lambda d_{n,K}(x_0, \cdot) + \alpha \quad \text{on } \partial K \implies u \leq \lambda d_{n,K}(x_0, \cdot) + \alpha \quad \text{on } K,$$

$$u \geq -\lambda d_{n,K}(x_0, \cdot) + \alpha \quad \text{on } \partial K \implies u \geq -\lambda d_{n,K}(x_0, \cdot) + \alpha \quad \text{on } K.$$

(iii) A function $u$ is said an absolute minimizer for the functional $F_n$ on $V^n$ if for any $x \in V^n \setminus V^0$ and for any $v : V^n \to \mathbb{R}$ such that $v(y) = u(y)$ for all $y \sim_n x$, then $F_n(u, x) \leq F_n(v, x)$. We denote by $\text{AM}(V^n)$ the set of the absolute minimizer for $F_n$ in $V^n$.

We have the following result.

**Proposition 3.3.** The following properties are equivalent

(i) $u \in \text{AMLE}(V^n)$;

(ii) $u$ satisfies the CC property in $V^n$;

(iii) $u$ is infinity harmonic in $V^n \setminus V^0$;

(iv) $u \in \text{AM}(V^n)$.
Proof. To show that (i) implies (ii), assume by contradiction that there exists a connected set $K \subset V^n \setminus V^0$, $x_0 \in \partial K$, $\alpha \in \mathbb{R}$ and $\lambda \geq 0$ such that

$$u(x) \leq \alpha + \lambda d_{n,K}(x_0, x) \quad \forall x \in \partial K$$

and that the set $W = \{y \in K : u(y) > \alpha + \lambda d_{n,K}(x_0, y)\}$ is not empty (we assume that $W$ is connected otherwise we consider a connected component of $W$). Observe that $\partial W \subset K$ and

$$u(x) \leq \alpha + \lambda d_{n,K}(x_0, x) \quad \forall x \in \partial W. \tag{3.2}$$

Let $y_1, y_2 \in \partial W$ such that, defined $L := \text{Lip}^n(u, \partial W)$, then

$$L \leq \frac{u(y_2) - u(y_1)}{d_{n,W}(y_1, y_2)}.$$ 

Let $\gamma$ be a minimal path for $d_{n,W}(y_1, y_2)$, $y_0 \in \gamma \cap W$ (this point exists by the definition of $\text{Lip}^n(u, \partial W)$) and $z_0 \in \partial W$ such that $d_{n,K}(y_0, x_0) = d_{n,K}(y_0, z_0) + d_{n,K}(z_0, x_0) = d_{n,W}(y_0, z_0) + d_{n,K}(z_0, x_0)$. Hence by (3.2)

$$u(x) \leq \beta + \lambda d_{n,W}(z_0, x) \quad \forall x \in \partial W,$$

$$u(y_0) > \beta + \lambda d_{n,W}(z_0, y_0), \tag{3.3}$$

where $\beta = \alpha + \lambda d_{n,K}(z_0, x_0)$. Since $u \in \text{AMLE}(V^n)$, by Prop. 2.5 $u$ is linear along $\gamma$ and therefore $u(y_2) - u(y_0) = Ld_{n,W}(y_0, y_2)$. Hence by (3.3)

$$\beta + \lambda d_{n,W}(z_0, y_2) \geq u(y_2) = u(y_0) + Ld_{n,W}(y_0, y_2)$$

$$> \beta + \lambda d_{n,W}(z_0, y_0) + Ld_{n,W}(y_0, y_2)$$

$$\geq \beta + \lambda d_{n,W}(z_0, y_2) + (L - \lambda)d_{n,W}(y_0, y_2).$$

Since $d_{n,W}(y_0, y_2) > 0$, then $L < \lambda$. On the other hand, by (3.3) and $\text{Lip}^n(u, \partial W) = \text{Lip}^n(u, W)$, we have $u(z_0) \leq \beta$ and

$$u(z_0) + Ld_{n,W}(y_0, z_0) \geq u(y_0) \geq \beta + \lambda d_{n,W}(y_0, z_0)$$

and therefore $\lambda < L$, hence a contradiction to $W$ not empty.

To prove that (ii) implies (i), assume now that $u$ satisfies (ii) and set $L = \text{Lip}^n(u, \partial K)$. Fix $x \in K$, then by the CC property

$$u(z) - Ld_{n,K}(x, z) \leq u(x) \leq u(z) + Ld_{n,K}(x, z) \tag{3.4}$$

for all $z \in \partial K$. Set $J_i$ a connected component of $K \setminus \{x\}$, then $\partial J_i \subset \partial K \cup \{x\}$ and by (3.4) $\text{Lip}^n(u, \partial J_i) \leq L$. Again by the CC property we have

$$u(x) - Ld_{n,J_i}(x, y) \leq u(y) \leq u(x) + Ld_{n,J_i}(x, y), \quad \text{for all } y \in J_i.$$

Since $d_{n,J_i}(x, y) = d_{n,K}(x, y)$ we obtain $|u(x) - u(y)| \leq Ld_{n,K}(x, y)$. 

\hfill 8
As $K$ is connected, then for any $z_1 \in J_1$ and for any $z_2 \in J_2$ (where $J_1$ and $J_2$ denote two different connected components of $K \setminus \{x\}$) we have
\[
d_{n,K}(z_1, z_2) = d_{n,J_1}(z_1, x) + d_{n,J_2}(x, z_2)
\]
and therefore, since $x$ is arbitrary in $K$, $\text{Lip}^n(u, K) = L$. We prove that (iii) implies (ii). If $u$ is infinity harmonic in $K$, $x_0 \in \partial K$, $\alpha \in \mathbb{R}$ and $\lambda \geq 0$ are such that
\[u(y) \leq \alpha + \lambda d_{n,K}(x_0, y), \quad \forall y \in \partial K,
\]
then by Prop. 2.3 and Prop.2.4, it immediately follows that
\[u(x) \leq \alpha + \lambda d_{n,K}(x_0, x), \quad \forall x \in K.
\]
Similarly for the other relation in the definition of the CC property.

To show that (ii) implies (iii), assume by contradiction that there exists $x \in V^n \setminus V^0$ such that
\[\Delta^n_{\infty} u(x) = 2\eta > 0. \tag{3.5}\]
Consider the set $K = \{x\}$. Hence $\partial K = \{y_i\}_{i=1}^4$ where $y_i$ are the four points adjacent to $x$ in $V^n$. Let $y_1, y_2 \in \partial K$ be the points where $u$ attains its maximum, respectively minimum, on $\partial K$ and $\lambda \geq 0$ be such that $u(y_1) - u(y_2) = \lambda d_{n,K}(y_1, y_2) = 2\lambda \delta_n$. Hence
\[u(y_j) \geq u(y_2) = u(y_1) - \lambda d_{n,K}(y_1, y_2) = u(y_1) - \lambda d_{n,K}(y_1, y_j), \quad j = 1, 2, 3, 4,
\]
and therefore by the CC property in $K$
\[u(x) \geq u(y_1) - \lambda d_{n,K}(y_1, x) = u(y_1) - \lambda \delta_n. \tag{3.6}\]
By (3.5), we have
\[u(x) + \eta = \frac{1}{2} \left( \min_{i=1, \ldots, 4} \{u(y_i)\} + \max_{i=1, \ldots, 4} \{u(y_i)\} \right) = \frac{u(y_1) + u(y_2)}{2}
\]
\[= \frac{u(y_1)}{2} + \frac{u(y_1) - \lambda d_{n,K}(y_1, y_2)}{2} = u(y_1) - \lambda \delta_n = u(y_1) - \lambda d_{n,K}(y_1, x)
\]
It follows that $u(x) + \eta = u(y_1) - \lambda d_{n,K}(y_1, x)$ and therefore a contradiction to (3.6).

The equivalence between (iii) and (iv) is proved in [14, Theorem 5] observing that $\Delta^n_{\infty} u(x) = 0$ if and only if $u(x)$ is such that
\[\mathcal{J}^n(u, x) = \min_{t \in \mathbb{R}} \left\{ \max_{y \sim_n x} \frac{|t - u(y)|}{\delta_n} \right\},
\]
i.e. $u(x)$ minimizes the functional $I(t) = \max_{y \sim_n x} |t - u(y)|/\delta_n$.

\[\square\]

**Corollary 3.4.** *Given $g : V^0 \to \mathbb{R}$ and defined $L_0 = \max_{i,j=1,2,3} \{|g(q_i) - g(q_j)|\}$, where $\{q_1, q_2, q_3\}$ are the vertices of $V^0$, then for any $n \in \mathbb{N}$ there exists a unique $u^n \in \text{AMLE}(V^n)$ such that $u = g$ on $V^0$.***
Proof. The previous result is an immediate consequence of Prop. 2.3 and Prop. 3.3.

Remark 3.5. The definition of Lipschitz constant at the boundary, which is computed considering paths staying inside the domain, see (2.3), rules out the pathological example considered in [9, Example 2.2] where a function defined on the boundary does not have an absolutely minimizing extension. Note that the theory developed in this section applies to a generic graph. Let \((X, d)\) be the metric space with \(X = \{x, y, z\}\) and \(d(x, y) = 3/2, d(x, z) = d(y, z) = 1\). Consider \(K = \{z\}\), hence \(\partial K = \{x, y\}\), and \(f : \partial K \to \mathbb{R}\) defined by \(f(x) = 0, f(y) = 1\). Hence \(\text{Lip}(f, \partial K) = 1/2\) (consider the path inside \(K\) given by \(x_0 = y, x_1 = z, x_2 = x\)). If \(u\) is infinity harmonic in \(K\), then 
\[
\max\{u(x), u(y)\}/2 + \min\{u(x), u(y)\}/2 = 1/2
\]
and \(\text{Lip}(u, K) = \text{Lip}(u, \partial K) = 1/2\). If the Lipschitz constant of \(f\) at the boundary is computed as \(|f(y) - f(x)|/d(x, y) = 2/3\), then it is shown in [9] that an absolutely minimizing Lipschitz extension of \(f\) to \(X\) does not exist.

4 The Absolutely Minimizing Lipschitz Extension problem in \(S\)

Following [1, 9, 10], we introduce absolute minimizing Lipschitz extensions on \(S\). We consider the length space \((S, d)\) where \(d\) is the path distance defined by
\[
d(x, y) := \inf \{\ell(\gamma) : \gamma \text{ is a path joining } x \text{ to } y\}
\]
with \(\ell(\gamma)\) the length of \(\gamma\) (see [7]). We consider as boundary of \(S\), as for all the sets \(V^n\), the initial set \(V^0 = \{q_1, q_2, q_3\}\) and we assume that a function \(g : V^0 \to \mathbb{R}\) is given.

Given \(A \subset S\) and \(f : A \to \mathbb{R}\), we define the Lipschitz constant of \(f\) on \(A\) to be
\[
\text{Lip}(f, A) := \sup_{x, y \in A, x \neq y} \frac{|f(y) - f(x)|}{d(x, y)}
\]

Definition 4.1.

- A continuous function \(u : S \to \mathbb{R}\) is said an absolute minimizer for the functional \(\text{Lip}\) on \(S\) if for any proper, open, connected set \(A \subset S \setminus V^0\) and for any \(v : S \to \mathbb{R}\) such that \(u = v\) on \(\partial A\), then \(\text{Lip}(u, A) \leq \text{Lip}(v, A)\). We denote by \(\text{AMLE}(S)\) the set of the absolute minimizer for \(\text{Lip}\) in \(S\).

- A function \(u : S \to \mathbb{R}\) satisfies the Comparison with Cones property (noted CC property) if for any proper, open, connected set \(A \subset S \setminus V^0\), for any \(x_0 \in S \setminus A\), \(\lambda \geq 0\) and \(\alpha \in \mathbb{R}\)

\[
\begin{align*}
    u &\leq \lambda d(x_0, \cdot) + \alpha \quad \text{on } \partial A \text{ implies } u \leq \lambda d(x_0, \cdot) + \alpha \quad \text{on } A \\
    u &\geq -\lambda d(x_0, \cdot) + \alpha \quad \text{on } \partial A \text{ implies } u \geq -\lambda d(x_0, \cdot) + \alpha \quad \text{on } A
\end{align*}
\]

In the following proposition we summarize the results in [9, 10] concerning the existence of AMLE in metric spaces which in particular applies to \((S, d)\).
Proposition 4.2.

(i) A function $u$ is of class AMLE($\mathcal{S}$) if and only if it satisfies the Comparison with Cones property.

(ii) For any given $g : V^0 \to \mathbb{R}$, there exists a unique function $u \in \text{AMLE}(\mathcal{S})$ such that $u = g$ on $V^0$.

Proof. For the proof of (i), see [10, Proposition 4.1], for the one of (ii), we refer to [9, Theorem 4.3] for existence and to [15, Theorem 1.4] for the uniqueness. \hfill \Box

Theorem 4.3. Given $g : V^0 \to \mathbb{R}$, let $u^n$ be the AMLE($V^n$) of $g$ to $V^n$. Then

$$\lim_{n \to \infty} u^n(x) = u(x), \quad \text{uniformly in } \mathcal{S},$$

i.e. $\lim_{n \to \infty} \sup_{x \in V^n} |u^n(x) - u(x)| = 0$, where $u$ is the AMLE($\mathcal{S}$) of $g$ to $\mathcal{S}$.

Proof. Since the sequence $\{u^n\}_{n \in \mathbb{N}}$ is uniformly bounded and equi-Lipschitz continuous, there exists a function $u : \mathcal{S} \to \mathbb{R}$ such that, up to a subsequence, $\lim_{n \to \infty} u^n(x) = u(x)$ and the convergence is also uniform. Moreover $u$ is Lipschitz continuous with Lipschitz constant $L$. We prove that $u$ satisfies the CC property on $\mathcal{S}$. Given a proper, open set any proper, open, connected set $A \subset \mathcal{S} \setminus V^0$, let $x_0 \in \mathcal{S} \setminus A$, $\alpha \in \mathbb{R}$, $\lambda > 0$ (if $\lambda = 0$ the argument is similar) be such that

$$u(y) \leq \alpha + \lambda d(x_0, y), \quad \forall y \in \partial A,$$

and assume by contradiction that the set $W = \{y \in A : u(y) > \alpha + \lambda d(x_0, y)\}$ is not empty (we assume that $W$ is connected otherwise we consider a connected component of $W$). Observe that

$$u(y) = \alpha + \lambda d(x_0, y) \quad \forall y \in \partial W.$$

Let $y_0 \in W$ and $\eta > 0$ be such that

$$u(y_0) \geq \alpha + \lambda d(x_0, y_0) + 4\eta. \quad (4.1)$$

Defined $W_n = W \cap V^n$, let $\varepsilon = \frac{\eta}{L + \lambda}$. As $W$ is open, there exists an integer $n_0$ and $y_0^* \in W_{n_0}$ such that $d(y_0, y_0^*) < \varepsilon$. Since $W_n \subset W_{n+1}$, then $y_0^* \in W_n$ for any $n \geq n_0$.

Therefore by (4.1) and the continuity of $u$, we have

$$u(y_0) \geq u(y_0) - L\varepsilon \geq \alpha + \lambda d(x_0, y_0^*) + 3\eta.$$

Similarly, there exists a point $x_0^* \in V^{n_1}$ such that $d(x_0, x_0^*) < \frac{\eta}{L + \lambda}$: then

$$u(y_0^*) \geq \alpha + \lambda d(x_0, y_0^*) + 3\eta \geq \alpha + \lambda (d(y_0^*, x_0^*) + d(x_0, x_0^*) - 2d(x_0, x_0^*)) + 3\eta \geq \alpha + \lambda (d(y_0^*, x_0^*) + d(x_0, x_0^*)) + 2\eta.$$

Defined $\beta = \alpha + \lambda d(x_0, x_0^*)$, we have

$$u(y_0^*) \geq \beta + \lambda d(y_0^*, x_0^*) + 2\eta.$$
Moreover
\[ u(y) \leq \beta + \lambda d(x_0^*, y) \quad \forall y \in A \setminus W. \]

Let \( \gamma_n \subset V^n \) be a path \( \{x_0^n = x_0^n, x_1^n, \ldots, x_{N}^n = y_0^n\} \) with \( x_i^n \sim_n x_{i+1}^n, i = 0, \ldots, N-1 \), connecting \( x_0^n \) to \( y_0^n \) such that \( d_n(x_0^n, y_0^n) = \ell(\gamma_n) \). Since \( d_n(x_0^n, y_0^n) \rightarrow d(x_0^*, y_0^*) \) for \( n \rightarrow \infty \) (see for example [2, Corollary 5.1], [7]), then
\[ d(x_0^*, y_0^*) \geq \ell(\gamma_n) - \frac{\eta}{\lambda} \]
for \( n \) sufficiently large, where \( \ell(\gamma_n) \) is the length of \( \gamma_n \). Denoted by \( z_n \) the first point \( x_i^n \in \gamma_n, i = 1, \ldots, N-1 \), such that \( x_i^n \not\in W \),
\[ d_n(x_0^n, y_0^n) = d_n(x_0^n, z_n) + d_n(z_n, y_0^n). \]
and by (4.4)
\[ d(x_0^*, y_0^*) \geq d_n(x_0^n, z_n) + d_n(z_n, y_0^n) - \frac{\eta}{\lambda}. \]
Set \( \beta_n = \beta + \lambda d_n(x_0^n, z_n) \), then
\begin{align*}
    u(y) &\leq \beta + \lambda d(x_0^*, y) \leq \beta + \lambda d_n(x_0^n, y) \leq \beta + \lambda d_n(x_0^n, z_n) + \lambda d_n(z_n, y) = \beta_n + \lambda d_n(z_n, y) \quad \forall y \in A \setminus W, \\
    u(y_0^n) &\geq \beta + \lambda d(x_0^*, y_0^n) + 2\eta \geq \beta + \lambda d_n(x_0^n, z_n) + \lambda d_n(z_n, y_0^n) - \eta + 2\eta = \beta_n + \lambda d_n(z_n, y_0^n) + \eta. \quad (4.3)
\end{align*}

For any \( y \in \partial W_n \) we have in particular that \( y \in A \setminus W \) and by the uniform convergence of \( u^n \) to \( u \) and (4.2), there exists \( \varepsilon_n \rightarrow 0 \) for \( n \rightarrow \infty \) such that
\[ u^n(y) \leq \beta_n + \varepsilon_n + \lambda d_n(W^n, z_n, y) \quad \forall y \in \partial W_n. \]
Then by the CC property for \( u^n \), we get (in particular)
\[ u^n(y_0^n) \leq \beta_n + \varepsilon_n + \lambda d_n(W^n, z_n, y_0^n). \quad (4.4) \]
Passing to the limit for \( n \rightarrow \infty \) in (4.4) we get a contradiction to (4.3). In fact
\[ u(y_0^n) \leq \liminf(\beta_n + 2\varepsilon_n + \lambda d_n(W^n, z_n, y_0^n)) \]
and by definition of \( z_n \), we have \( d_n(W^n, z_n, y_0^n) \leq d_n(z_n, y_0^n) \) so
\[ u(y_0^n) \leq \liminf(\beta_n + 2\varepsilon_n + \lambda d_n(z_n, y_0^n)) = \liminf(\beta_n + \lambda d_n(z_n, y_0^n)) \]
while from (4.3) we can deduce
\[ u(y_0^n) \geq \limsup(\beta_n + \lambda d_n(z_n, y_0^n)) + \eta \]
Arguing in a similar way for the other relation, we conclude that \( u \) satisfies the CC property on \( S \) and therefore \( u \in \text{AMLE}(S) \). By the uniqueness of \( u \), see Prop. 4.2, we get that all the sequence \( u^n \) converges uniformly to \( u \). \( \square \)
**Definition 4.4.** We say that a function \( u : S \to \mathbb{R} \) is infinity harmonic if it is the limit of infinity harmonic functions \( u^n : V^n \to \mathbb{R} \) such that \( u^n = u \) on \( V^0 \).

By Theorem 4.3, we have

**Corollary 4.5.** The following properties are equivalent

(i) \( u \in \text{AMLE}(S) \);

(ii) \( u \) satisfies the Comparison with Cones property;

(iii) \( u \) is infinity harmonic.

**Remark 4.6.** As for the Laplacian on the Sierpinski gasket (see [16]), one would be tempted to say that, given \( f : V^{n-1} \to \mathbb{R} \), \( \tilde{f} \) is an infinity harmonic extension of \( f \) to \( V^n \) if \( \tilde{f} \in \text{AMLE}(V^n) \) and \( \tilde{f} = f \) on \( V^{n-1} \). But a function \( \tilde{f} \) with such property could not exist. This can be seen with the following example. Let \( g : V^0 \to \mathbb{R} \) such that \( g(q_1) = 0 \), \( g(q_2) = e \in [0,1/7] \) and \( g(q_3) = 1 \) and denote by \( q_{ij} = q_{ji} \), \( i,j = 1,2,3 \) and \( i \neq j \), the point in \( V^1 \setminus V^0 \) on the segment of vertices \( q_i \) and \( q_j \). By the Lazarus algorithm (see Section 5.1) the unique solution of \( \Delta^1 u = 0 \) in \( V^1 \setminus V^0 \) and \( u = g \) on \( V^0 \) is given by \( u^1(q_{12}) = (1+e)/4 \) and \( u^1(q_{23}) = (1+e)/2 \), \( u^1(q_{13}) = 1/2 \).

If we consider the problem \( \Delta^2 u = 0 \) in \( V^2 \setminus V^0 \) and \( u = g \) on \( V^0 \), by applying again the Lazarus algorithm we find \( u^2(q_{12}) = (3+4e)/12 \) and therefore \( u^2(q_{12}) \neq u^1(q_{12}) \) if \( e \neq 0 \). Hence the function \( u^2 \) does not satisfy \( \Delta^1 u^2 = 0 \) in \( V^1 \), since otherwise \( u^2 \equiv u^1 \) on \( V^1 \). Note that the set \( K = V^1 \setminus V^0 \) is not connected as a subset of \( V^2 \) and \( \partial K = V^2 \setminus V^0 \), hence the values of \( u^2 \) on \( V^2 \) are used to compute \( \Delta^2 u^2 = 0 \) at \( x \in V^1 \).

This is a main difference with the case of the Laplacian where the harmonic extension \( \tilde{f} \) of a function \( f \) to \( V^n \) still satisfies the Laplace equation on \( V^{n-1} \). The main consequence of this observation is that we cannot define an infinity harmonic function on \( S \) as a continuous function whose restriction to \( V^n \) is infinity harmonic for all \( n \in \mathbb{N} \).

The following property can be useful in order to characterize the limit of the sequence \( u^n \).

**Proposition 4.7.** For \( u : S \to \mathbb{R} \), the functional

\[
F^n(u, V^n) =: \max_{x \in V^n \setminus V^0} F^n(u, x)
\]

is increasing in \( n \in \mathbb{N} \).

**Proof.** Let \( x \in V^n \setminus V^0 \) be such that \( F^n(u, V^n) = F^n(u, x), y \in V^n \) such that \( F^n(u, x) = |u(x) - u(y)|/\delta_n \) and \( z \) the point in \( V^{n+1} \) in between \( x \) and \( y \). Then

\[
\frac{|u(x) - u(y)|}{\delta_n} \leq \frac{1}{2} \frac{|u(x) - u(z)|}{\delta_{n+1}} + \frac{1}{2} \frac{|u(z) - u(y)|}{\delta_{n+1}}
\]

\[
\leq \frac{1}{2} F^{n+1}(u, z) + \frac{1}{2} F^{n+1}(u, z) \leq F^{n+1}(u, V^{n+1})
\]

hence \( F^n(u, V^n) \leq F^{n+1}(u, V^{n+1}) \).
5 Some complements to the AMLE problem in $V^n$

We discuss in this section two further properties of the AMLE problem in $V^n$:

- an algorithm to compute explicitly an infinity harmonic function on $V^n$;
- the relation between $p$-harmonic and infinity harmonic functions on $V^n$.

5.1 A constructive approach: The Lazurus algorithm

We describe an algorithm introduced in [17] and extensively studied in [5] which allows to compute an infinity harmonic function on $V^n$.

Observing that the set of infinity harmonic functions is invariant by addition and multiplication for constants, if $u$ is infinity harmonic, then

$$v(x) = \frac{u(x) - \min_{V^0} \{u\}}{\max_{V^0} \{u\} - \min_{V^0} \{u\}}$$

(5.1)

is also infinity harmonic. Moreover $v$ assumes the boundary values 0, $e$, 1 for some $e \in [0, 1]$. Since the values at the boundary determine univocally an infinity harmonic function, to compute $u$ is sufficient to consider the case of the boundary values 0, $e$, 1 and then to invert the affine transformation (5.1).

It is possible to further reduce the computation by considering $e \in [0, 1/2]$. In fact if $e \in [1/2, 1]$, then

$$w(x) = 1 - v(x)$$

is infinity harmonic and assume the boundary values 0, 1, $1 - e$ for some $e \in [0, 1/2]$. Since the order of the vertices of $V^0$ where these values are assumed is not relevant. Computed $w$, we have $v(x) = 1 - w(x)$.

We start with the boundary values $v(q_1) = 0$, $v(q_2) = e$, $v(q_3) = 1$ (see figure 5.1.(a)) and we compute the corresponding infinity harmonic function on $V^1$. We denote by $q_{ij} = q_{ji}$, $i, j = 1, 2, 3$ and $i \neq j$, the point in $V^1 \setminus V^0$ on the segment of vertices $q_i$ and $q_j$. Exploiting Prop.2.5, since

$$\text{Lip}^1(v, \partial(V^1 \setminus V^0)) = \frac{v(q_3) - v(q_1)}{d_1(q_1, q_3)} = 1,$$

the function $v$ is linear along the minimal path \{q_1, q_{13}, q_3\} for $d_1(q_1, q_3)$, hence $v(q_{13}) = 1/2$.

To compute the other values we consider the connected set $K = \{q_{12}, q_{23}\}$ with $\partial K = V^0 \cup \{q_{13}\}$. We distinguish two cases:

(i): If $e \in [1/3, 1/2]$, then

$$\text{Lip}^1(v, \partial K) = \frac{v(q_3) - v(q_1)}{d_{1,K}(q_1, q_3)} = \frac{1}{3/2}$$

and a minimal path for $d_{1,K}(q_1, q_3)$ is given by $\{q_1, q_{12}, q_{23}, q_3\}$. Applying again Prop. 2.5, we get $v(q_{12}) = 1/3$, $v(q_{23}) = 2/3$ (see figure 5.1.(b)).

(ii): If $e \in [0, 1/3]$, then

$$\text{Lip}^1(v, \partial K) = \frac{v(q_3) - v(q_2)}{d_{1,K}(q_2, q_3)} = \frac{1 - e}{1}$$
and a minimal path for $d_{1,K}(q_2,q_3)$ is given by $\{q_2,q_{23},q_3\}$. Hence $v(q_{23}) = (1 + e)/2$. To determine $v(q_{12})$, we consider the connected set $J = \{q_{12}\}$ with boundary $\partial J = \{q_1,q_{13},q_{23},q_2\}$. Since

$$\text{Lip}^1(v,\partial J) = \frac{v(q_{23}) - v(q_1)}{d_{1,K}(q_1,q_{23})} = \frac{(1 + e)/2}{1}$$

and a minimal path for $d_{1,K}(q_1,q_{23})$ is given by $\{q_1,q_{12},q_{23}\}$, we get $v(q_{12}) = (1 + e)/4$ (see figure 5.1.(c)).

Arguing as above it is possible (in principle) to compute infinity harmonic function on $V^n$ for any $n \in \mathbb{N}$ (the case $n = 2$ is detailed in [5]). Alternatively it is possible to use the iterative scheme developed in [14] in the framework of numerical approximation of infinity harmonic functions in Euclidean space. Since this scheme works for general graph, it can be applied to $V^n$.

It is worth noticing that a somewhat stronger result than Theorem 4.3 is obtained in [5]. In fact it is proved that the restriction of an infinity harmonic function $u^n$ on $V^n$ is eventually unchanging on $V^k$ for $n$ sufficiently larger than $k$. This result is based on the explicit construction of the optimal paths for the Lazarus algorithm and the argument is rather involved.

![Figure 1: infinity harmonic function on $V^1$: (a) boundary condition, (b) $e \in [1/3, 1/2]$, (c) $e \in [0, 1/3]$](image)

### 5.2 Infinity and $p$-harmonic function in $V^n$

Another important point of view of the AMLE theory is the relation between $p$-harmonic and infinity harmonic functions. In the Euclidean case it can be proved that the limit of $p$-harmonic functions for $p \to \infty$ is an infinity harmonic function [1, 4], while this property may fail in general metric-measure spaces [10]. We start to prove a similar result for the pre-fractal $V^n$. Fixed $n \in \mathbb{N}$ and given $g : V^0 \to \mathbb{R}$, we introduce for $p \in [1, \infty)$ the $p$-energy functional

$$I^n_p(u) = \left( \sum_{x \in V^n} \sum_{y \sim_n x} \left| \frac{u(y) - u(x)}{\delta_n} \right|^p \right)^{1/p}.$$  \hspace{1cm} (5.2)

A function which achieves the minimum in (5.2) is said a $p$-harmonic function on $V^n$. The existence of a $p$-harmonic function on graphs is studied in [8].
**Proposition 5.1.** Let \( \{u^n_p\}_{p \geq 1} \) be the family of the \( p \)-harmonic functions on \( V^n \) such that \( u^n_p = g \) on \( V^0 \). Then \( u^n_p \to u^n \) for \( p \to \infty \) where \( u^n \in \text{AMLE}(V^n) \) and \( u^n = g \) on \( V^0 \).

We need a preliminary lemma, expressing the local character of \( p \)-harmonic functions. For \( x \in V^n \setminus V^0 \) and \( v : V^n \to \mathbb{R} \), set

\[
I^n_p(v, x) = \left( \sum_{y \sim_n x} \frac{|v(y) - v(x)|^p}{\delta_n} \right)^{\frac{1}{p}}.
\]

**Lemma 5.2.** If \( u^n_p \) is \( p \)-harmonic, then for any \( v : V^n \to \mathbb{R} \) such that \( v(y) = u^n_p(y) \) for \( y \sim_n x \), we have \( I^n_p(u^n_p, x) \leq I^n_p(v, x) \).

**Proof.** If by contradiction there exists \( v : V^n \to \mathbb{R} \) such that \( v(y) = u^n_p(y) \) for \( y \sim_n x \) and such that \( I^n_p(v, x) > I^n_p(u^n_p, x) - \varepsilon \) for some \( \varepsilon > 0 \), then the function \( \tilde{u} : V^n \to \mathbb{R} \) defined by

\[
\tilde{u}(y) = \begin{cases} 
  u^n_p(y), & y \neq x; \\
  v(x), & y = x.
\end{cases}
\]

is such that \( I^n_p(\tilde{u}) < I^n_p(u^n_p) \). Hence a contradiction to the definition of \( p \)-harmonic function. \( \square \)

**Proof of Prop. 5.1.** Let \( u^n_p \) be a \( p \)-harmonic function and \( u^n \in \text{AMLE}(V^n) \) such that \( u^n = g \) on \( V^0 \). Then

\[
I_p(u^n_p) \leq I_p(u^n) = \left( \sum_{x \in V^n \setminus V^0} \sum_{y \sim_n x} |\mathcal{F}^n(u^n, x)|^p + \sum_{i=1}^{3} \sum_{x \sim_n q_i} \left| \frac{u^n(x) - u^n(q_i)}{\delta_n} \right|^p \right)^{\frac{1}{p}}
\]

where \( \mathcal{F}^n \) is defined as in (2.5) and \( q_i, i = 1, 2, 3 \) are the vertices of \( V^0 \). Hence, by (2.8)

\[
I_p(u^n_p) \leq (4N + 6)^{\frac{1}{p}} \max_{V^n \setminus V^0} \{ \mathcal{F}^n(u^n, x) \} = (4N + 6)^{\frac{1}{p}} \text{Lip}^n(u^n, V^n) = (4N + 6)^{\frac{1}{p}} L_0
\]

where \( L_0 = \max_{j, i=1,2,3} \{ g(q_i) - g(q_j) \} \) and \( N = \#(V^n \setminus V^0) \) (hence \( N = (3^{n+1} - 3)/2 \)). It follows that for any \( x \in V^n \setminus V^0 \) and \( y \sim_n x \), \( |u^n_p(y) - u^n_p(x)| \) is bounded, uniformly in \( p \in [1, \infty) \). Since \( u^n_p = g \) on \( V^0 \), then the functions \( u^n_p \) are uniformly bounded in \( p \) on \( V^n \). Passing to a subsequence, we get that there exists a sequence \( \{u^n_{p_j}\}_{j \in \mathbb{N}} \) which converges to a function \( \tilde{u} \) on \( V^n \) such that \( \tilde{u} = g \) on \( V^0 \).

We claim that \( \tilde{u} \in \text{AM}^n(V^n) \) (see Def. 3.2.(iii)). Let \( x \in V^n \) and \( v : V^n \to \mathbb{R} \) such that \( v(y) = u^n_{p_j}(y) \) for \( y \sim_n x \). Then by Lemma 5.2 we have

\[
\left( \sum_{y \sim_n x} \left| \frac{u^n_{p_j}(y) - u^n_{p_j}(x)}{\delta_n} \right|^p \right)^{\frac{1}{p}} \leq \left( \sum_{y \sim_n x} \left| \frac{u^n_{p_j}(y) - v(x)}{\delta_n} \right|^p \right)^{\frac{1}{p}}, \quad \forall j \in \mathbb{N}.
\]

Passing to the limit for \( j \to \infty \) in the previous inequality, since the \( p \)-norm in \( \mathbb{R}^N \) converges to the \( \infty \)-norm we get

\[
\max_{y \sim_n x} \left| \frac{\tilde{u}(y) - \tilde{u}(x)}{\delta_n} \right| \leq \max_{y \sim_n x} \left| \frac{\tilde{u}(y) - v(x)}{\delta_n} \right|.
\]
Hence $F^n(\bar{u}, x) \leq F^n(v, x)$ and therefore the claim. Since there exists a unique $u^n \in AM^n(V^n)$ such that $u^n = g$ on $V^0$, see Prop. 3.3 and Cor. 3.4, it follows that $\bar{u} = u^n$. Moreover by the uniqueness of $u^n$, any convergent sequence of $\{u^n_p\}_{p \geq 1}$ tends to $u^n$. Therefore we conclude that $\lim_{p \to \infty} u^n_p = u^n$.

We proved the convergence $u^n_p \to u^n$ for $p \to \infty$ in Proposition 5.1 and the convergence $u^n \to u$ for $n \to \infty$ in Theorem 4.3. A natural question is if it possible to invert the order of the limits. Actually we have only partial results. A notion of $p$-harmonic functions on $V^n$ is studied in [6] with the aim of defining a $p$-Laplace operator on the Sierpinski gasket in the spirit of Kigami’s approach. The discrete $p$-energy $E_p^{(n)}$ considered in [6] is different from (5.2), even if it is dominated from below and from above by $(I^n_p)^p$. The sequence of the discrete energies $E_p^{(n)}$ is increasing, hence it is possible to define the $p$-energy $E_p$ on the Sierpinski gasket $S$ as the limit

$$E_p(u) = \lim_{n \to \infty} E_p^{(n)}(u).$$

(5.3)

Note that a minimizer $v^n_p$ of the energy $E_p^{(n)}$ defined in [6] such that $v^n_p = g$ on $V^0$ could be different from a minimizer $u^n_p$ of the $p$-energy in (5.2) with the same boundary datum.

In [6, Cor.2.4], it is proved that, for $p$ fixed, any sequence $\{v^n_p\}$ of the $p$-harmonic-extensions with respect to the energy $E_p^{(n)}$ on $V^n$ (such that $v^n_p = g$ on $V^0$) converges uniformly as $n \to \infty$ to a function $v_p$ that is a $p$-harmonic extension of $g$ on $S$ with respect the limit $p$-energy defined in (5.3).

Passing to the limit for $p \to \infty$ we can prove that any sequence $v_p$ of $p$-harmonic extensions with respect the $p$-energy $E_p$ on $S$ such that $v_p = g$ on $V^0$ converges (up to a subsequence) to a function $v : S \to \mathbb{R}$ uniformly in $S$ but up to now we are not able to prove that $v$ is a AMLE of $g$ to $S$.

References

[1] Aronsson, G.; Crandall, M.G.; Juutinen, P. A tour of the theory of absolutely minimizing functions. Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505.

[2] Camilli, F.; Capitanelli, R.; Marchi, C. Eikonal equations on the Sierpinski gasket, Math. Ann. 364 (2016), no.3-4, 1167–1188.

[3] Champion, T.; De Pascale, L. Principles of comparison with distance functions for absolute minimizers. Journal of Convex Analysis 14 (2007), no. 3, 515-541.

[4] Crandall, M.G. A visit with the $\infty$-laplace equation. Calculus of variations and nonlinear partial differential equations, 75-122, Lecture Notes in Math., 1927, Springer, Berlin, 2008.

[5] Guay, M. Infinity-harmonic functions on SG, Master thesis, 2011, http://www.math.cornell.edu/files/Research/SeniorTheses/guayThesis.pdf.

[6] Herman, P.E.; Peirone, R.; Strichartz, R. $p$-Energy and $p$-Harmonic Functions on Sierpinski Gasket Type Fractals. Potential Analysis 20 (2004), no.2, 125–148.
[7] Hinz, A.M.; Schief, A. The average distance on the Sierpinski gasket. Probab. Theory Related Fields 87 (1990), no.1, 129–138.

[8] Holopainen, I.; Soardi, P.M. p-harmonic functions on graphs and manifolds. Manuscripta Math. 94 (1997), no. 1, 95–110.

[9] Juutinen, P. Absolutely minimizing Lipschitz extensions on a metric space. Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 1, 57–67.

[10] Juutinen, P.; Shanmugalingam, N. Equivalence of AMLE, strong AMLE, and comparison with cones in metric measure spaces. Math. Nachr. 279 (2006), no. 9-10, 1083–1098.

[11] Kigami, J. Analysis on fractals, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2001.

[12] Le Gruyer, E. On absolutely minimizing Lipschitz extensions and PDE $\Delta_\infty(u) = 0$. NoDEA Nonlinear Differential Equations Appl. 14 (2007), no. 1-2, 29-55.

[13] Manfredi, J.; Oberman, A.; Sviridov, A. Nonlinear elliptic Partial Differential Equations and p-harmonic functions on graphs. Differential Integral Equations 28 (2015), no.1-2, 79–102.

[14] Oberman, A.M. A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions. Math. Comp. 74 (2005), no. 251, 1217-1230.

[15] Peres, Y.; Schramm, O.; Sheffield, S.; Wilson, D.B. Tug-of-war and the infinity Laplacian. J. Am. Math. Soc. 22, 167-210 (2009).

[16] Strichartz, R. Differential equations on fractals, a tutorial. Princeton University Press, Princeton, 2006.

[17] Stromquist, J.; Lazarus, W.; Propp, A.; Ullman, D. Combinatorial games under auction play. Games and Economic Behavior, 2 (1999), 229–264.