Intermittency and infinite variance: the case of integrated supOU processes

Danijel Grahovac\(^1\)*, Nikolai N. Leonenko\(^2\)†, Murad S. Taqqu\(^3\)‡

1 Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, 31000 Osijek, Croatia
2 School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, Wales, UK, CF24 4AG
3 Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA

Abstract: SupOU processes are superpositions of Ornstein-Uhlenbeck type processes with a random intensity parameter. They are stationary processes whose marginal distribution and dependence structure can be specified independently. Integrated supOU processes have then stationary increments and satisfy central and non-central limit theorems. Their moments, however, can display an unusual behavior known as “intermittency”. We show here that intermittency can also appear when the processes have a heavy tailed marginal distribution and, in particular, an infinite variance.

Keywords: supOU processes, Ornstein-Uhlenbeck process, absolute moments, limit theorems, infinite variance

MSC2010: 60F05, 60G52, 60G10

1 Introduction

Superpositions of Ornstein-Uhlenbeck type (supOU) processes form a rich class of stationary processes with a flexible dependence structure. Their distribution is determined by the characteristic quadruple

\[(a, b, \mu, \pi),\] (1)

where \((a, b, \mu)\) is some Lévy-Khintchine triplet (see e.g. Sato (1999)) and \(\pi\) is a probability measure on \(\mathbb{R}_+\). In the construction of the supOU process \(\{X(t), t \in \mathbb{R}\}\), the choice of \((a, b, \mu)\) uniquely characterizes the one-dimensional marginals which are independent on the choice of \(\pi\). On the other hand, the probability distribution \(\pi\) affects the dependence structure. See Barndorff-Nielsen (2001), Barndorff-Nielsen & Stelzer (2011), Barndorff-Nielsen & Stelzer (2013), Barndorff-Nielsen & Veraart (2013), Barndorff-Nielsen et al. (2018), Grahovac, Leonenko, Sikorskii & Taqqu (2019) for details.

SupOU processes provide models with analytically and stochastically tractable dependence structure displaying either weak or strong dependence and also having marginal distributions that are infinitely divisible. They have applications in environmental studies, ecology, meteorology, geophysics, biology, see e.g. Barndorff-Nielsen et al. (2015), Podolskij (2015), Barndorff-Nielsen et al. (2018) and the references therein. The supOU processes are particularly relevant in finance and the statistical theory of turbulence since they can model key stylized features of observational series from finance and turbulence. Recently in Kelly et al. (2013), the supOU process have been used to assess the mass of black hole.

\*dgrahova@mathos.hr
\†LeonenkoN@cardiff.ac.uk
\‡murad@bu.edu
By aggregating the supOU process \( \{X(t), \ t \in \mathbb{R}\} \) one obtains the integrated supOU process

\[
X^*(t) = \int_0^t X(s)ds.
\]

A suitably normalized integrated process exhibits complex limiting behavior. Indeed, if the underlying supOU process has finite variance, then four classes of processes may arise in a classical limiting scheme (Grahovac, Leonenko & Taqqu (2019)). Namely, the limit process may be Brownian motion, fractional Brownian motion, a stable Lévy process or a stable process with dependent increments. The type of limit depends on whether Gaussian component is present in (1), on the behavior of \( \pi \) in (1) near origin and on the growth of the Lévy measure \( \mu \) in (1) near origin (see Grahovac, Leonenko & Taqqu (2019) for details). In the infinite variance case, the limiting behavior is even more complex as the limit process may additionally depend on the regular variation index of the marginal distribution (see Grahovac et al. (2018) for details).

The limiting behavior of the integrated process has practical significance since supOU processes may be used as stochastic volatility models, see Barndorff-Nielsen (1997), Barndorff-Nielsen & Shephard (2001) and the references therein. In this setting the integrated process \( X^* \) represents the integrated volatility (see e.g. Barndorff-Nielsen & Stelzer (2013)). Moreover, the limiting behavior is important for statistical estimation (see Stelzer et al. (2015), Nguyen & Veraart (2018)).

The integrated supOU process may exhibit another interesting limiting property related to behavior of their absolute moments in time. Although a suitably normalized integrated process satisfies a limit theorem, it may happen than its moments do not converge beyond some critical order. One way to investigate this behavior is to measure the rate of growth of moments by the scaling function, defined for the process \( Y = \{Y(t), \ t \geq 0\} \) as

\[
\tau_Y(q) = \lim_{t \to \infty} \frac{\log \mathbb{E}|Y(t)|^q}{\log t},
\]

assuming the limit in (3) exists and is finite. We will often focus on

\[
\frac{\tau(q)}{q} = \lim_{t \to \infty} \frac{\log (\mathbb{E}|Y(t)|^q)^{1/q}}{\log t}
\]

which has the advantage of involving \((\mathbb{E}|Y(t)|^q)^{1/q}\) which has the same units as \(Y(t)\). The values \(q\) are assumed to be in the range of finite moments \(q \in (0, \overline{q}(Y))\), where

\[
\overline{q}(Y) = \sup\{q > 0 : \mathbb{E}|Y(t)|^q < \infty \ \forall t\}.
\]

Different scaling procedure play a key role in physics de Gennes (1979), risky asset modeling Heyde (2009), statistics Grahovac et al. (2015) and ambit stochastics Barndorff-Nielsen et al. (2018).

To see how this is related to limit theorems, suppose that \( Y \) satisfies a limit theorem in the form

\[
\left\{ \frac{Y(Tt)}{A_T} \right\} \overset{d}{\to} \{Z(t)\},
\]

with \(A_T\) a sequence of constants and convergence in the sense of convergence of all finite-dimensional distributions as \(T \to \infty\). By Lamperti’s theorem (see, for example, (Pipiras & Taqqu 2017, Theorem 2.8.5)), the limit \( Z \) is \(H\)-self-similar for some \(H > 0\), that is, for any constant \(c > 0\), the finite-dimensional distributions of \(Z(ct)\) are the same as those of \(c^H Z(t)\). Moreover, the normalizing sequence is of the form \(A_T = \ell(T)T^H\) for some \(\ell\) slowly varying at infinity. For self-similar process, the moments evolve as a power function of time since \(\mathbb{E}|Z(t)|^q = \mathbb{E}|Z(1)|^{qT^Hq}\).
and therefore the scaling function of $Z$ is $\tau_Z(q) = Hq$. If the convergence of moments would hold
\[
\frac{\mathbb{E}|Y(Tt)|^q}{A_T^q} \to \mathbb{E}|Z(t)|^q, \quad \forall t \geq 0,
\]
then the scaling function of $Y$ would also be $\tau_Y(q) = Hq$ (see Grahovac, Leonenko, Sikorskii & Taqqu 2019, Theorem 1). In particular, $q \mapsto \tau_Y(q)/q$ would be constant over $q$ values for which (4) holds.

It has been showed in Grahovac, Leonenko, Sikorskii & Taqqu (2019) that the integrated supOU process $X^*$ may have a scaling function which does not correspond to some self-similar process, namely
\[
\tau_X^*(q) = q - \alpha
\]
for a certain range of $q$. This happens, in particular, for a non-Gaussian integrated supOU process with marginal distribution having exponentially decaying tails and probability measure $\pi$ in (1) regularly varying at zero. This implies that the function
\[
q \mapsto \frac{\tau_X^*(q)}{q} = \frac{q - \alpha}{q} = 1 - \frac{\alpha}{q}
\]
is not constant. It is strictly increasing, a property referred to as intermittency. Hence, intermittency implies the convergence of moments (4) must fail to hold beyond some critical value of $q$. See Grahovac et al. (2016), Grahovac, Leonenko, Sikorskii & Taqqu (2019), Grahovac, Leonenko & Taqqu (2019) which provide a complete picture on the behavior of moments in the case where $X^*(t)$ has finite variance.

Intermittency refers in general to an unusual moment behavior. It is of major importance in many fields of science, such as the study of rain and cloud studies, magnetohydrodynamics, liquid mixtures of chemicals and physics of fusion plasmas, see e.g. Zel’dovich et al. (1987). Another area of possible application is turbulence. In turbulence, the velocities or velocity derivatives (or differences) under a large Reynolds number could be modeled with infinitely divisible distributions, they allow long range dependence and there seems to exist a kind of switching regime between periods of relatively small random fluctuation and period of “higher” activity. This phenomenon is also referred to as intermittency, see e.g. (Frisch 1995, Chapter 8) or Zel’dovich et al. (1987).

In this paper we focus on the limiting behavior of moments and on intermittency in the case where $X^*(t)$ has infinite variance.

To establish the rate of growth of moments we make use of the limit theorems established in Grahovac et al. (2018). The type of the limiting process depends heavily on the structure of the underlying supOU process. Hence, the form of the scaling function of the integrated process will depend on the several parameters related to the quadruple (1). Special care is needed since the range of finite moments is limited. We show that the scaling function may look like a broken line indicating that there is a change-point in the rate of growth of moments. Hence, infinite variance integrated supOU processes may also exhibit the phenomenon of intermittency. Our results also indicate that in some cases, if we decompose the process into several components, the intermittency of the finite variance component may remain hidden by the infinite moments of the infinite variance component. We conclude that moments may have limited capability in identifying unusual limiting behavior.

The paper is organized as follows. In Section 2 we introduce notation and assumptions. Section 3 introduces the decomposition of the process which serves as a basis for the further analysis. The scaling functions of the components in this decomposition are obtained in Section 4. These results are then combined in Section 5 giving the proofs of the main results. Sections 6 and 7 contain the proofs of two lemmas used to derive the main results.
2 Preliminaries

We shall use the notation
\[ \kappa_Y(\zeta) = C \{ \zeta \overset{Y}{\mapsto} \} = \log \mathbb{E} e^{i\zeta Y} \]

to denote the cumulant (generating) function of a random variable \( Y \). For a stochastic process \( Y = \{ Y(t) \} \) we write \( \kappa_Y(t, t) = \kappa_Y(\zeta, \zeta) \), and by suppressing \( t \) we mean \( \kappa_Y(\zeta) = \kappa_Y(\zeta, 1) \), that is the cumulant function of the random variable \( Y(1) \).

The class of supOU processes has been introduced by Barndorff-Nielsen in Barndorff-Nielsen (2001) as follows. Let \( \Lambda \) denote a homogeneous infinitely divisible random measure (Lévy basis) on \( \mathbb{R}_+ \times \mathbb{R} \) and suppose that the cumulant function of the random variable \( \Lambda(A) \), where \( A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \), equals
\[ C \{ \zeta \overset{\Lambda}{\mapsto} \} = m(A)\kappa_L(\zeta) = (\pi \times \text{Leb}) (A)\kappa_L(\zeta). \tag{5} \]
The measure \( m \) is called the control measure and it is the product \( m = \pi \times \text{Leb} \) of a probability measure \( \pi \) on \( \mathbb{R}_+ \) and the Lebesgue measure on \( \mathbb{R} \). Finally, \( \kappa_L(\zeta) \) in (5) is the cumulant function \( \kappa_L(\zeta) = \log \mathbb{E} e^{i\zeta L(1)} \) of some infinitely divisible random variable \( L(1) \) with Lévy-Khintchine triplet \((a, b, \mu)\) i.e.
\[ \kappa_L(\zeta) = i\zeta a - \frac{\zeta^2}{2} b + \int_{\mathbb{R}} \left( e^{i\zeta x} - 1 - i\zeta 1_{[-1,1]}(x) \right) \mu(dx). \tag{6} \]
The Lévy process \( L = \{ L(t), t \geq 0 \} \) associated with the triplet \((a, b, \mu)\) is called the background driving Lévy process. It has independent stationary increments and thus, its finite-dimensional distributions depend only on the distribution of \( L(1) \).

The supOU process is a strictly stationary process \( X = \{ X(t), t \in \mathbb{R} \} \) given by the stochastic integral (Barndorff-Nielsen (2001))
\[ X(t) = \int_{\xi=0}^{\infty} \int_{s=-\infty}^{\infty} e^{-\xi t + s} 1_{[0,\infty)}(\xi t - s) \Lambda(d\xi, ds). \tag{7} \]

By appropriately choosing the background driving Lévy process \( L \), one can obtain any self-decomposable distribution as a marginal distribution of \( X \). Recall that an infinitely divisible random variable \( X \) is selfdecomposable if its characteristic function \( \phi(\zeta) = \mathbb{E} e^{i\zeta X}, \zeta \in \mathbb{R} \), has the property that for every \( c \in (0,1) \) there exists a characteristic function \( \phi_c \) such that \( \phi(\zeta) = \phi(c\zeta)\phi_c(\zeta) \) for all \( \zeta \in \mathbb{R} \) (see e.g. Sato (1999)). Equivalently, for every \( c \in (0,1) \) there is a random variable \( Y_c \) such that the random variable \( X \) has the same distribution as \( cX + Y_c \). Note that the one-dimensional marginals of the supOU process are independent on the choice of \( \pi \). The probability measure \( \pi \) “randomizes” the rate parameter \( \xi \) in (7) and the Lebesgue measure \( ds \) is associated with the moving average variable \( s \). The quadruple \((a, b, \mu, \pi)\) given in (1) determines the law of the supOU process \( \{ X(t), t \in \mathbb{R} \} \). More details about supOU processes can be found in Barndorff-Nielsen (2001), Barndorff-Nielsen & Leonenko (2005), Barndorff-Nielsen et al. (2013), Barndorff-Nielsen & Stelzer (2011), Barndorff-Nielsen et al. (2018) and Grahovac, Leonenko, Sikorskii & Taqqu (2019).

2.1 Basic assumptions

We now state a set of assumptions for the class of supOU processes we consider. The marginal distribution is assumed to be in the domain of attraction of some infinite variance stable law. The next assumption concerns the dependence structure controlled by the probability distribution \( \pi \). Finally, Lévy measure \( \mu \) is assumed to have a power law behavior near origin which will give rise to another parameter affecting the limiting behavior.
2.1.1 Marginal distribution

We consider infinite variance supOU processes. Recall first that a random variable \(Z\) has an infinite variance stable distribution \(S_{\gamma}(\sigma, \rho, c)\) with parameters \(0 < \gamma < 2, \sigma > 0, -1 \leq \rho \leq 1\) and \(c \in \mathbb{R}\), if it has a cumulant function of the form:

\[
\kappa_{S_{\gamma}(\sigma, \rho, c)}(\zeta) := C\{\zeta \dagger Z\} = i\zeta - \sigma^{\gamma}(1 - i\rho \text{sign}(\zeta)\chi(\zeta, \gamma)), \quad \zeta \in \mathbb{R},
\]

where

\[
\chi(\zeta, \gamma) = \begin{cases} 
\tan\left(\frac{\pi \zeta}{2}\right), & \gamma \neq 1, \\
\frac{\pi}{2} \log|\zeta|, & \gamma = 1.
\end{cases}
\]

For simplicity of exposition, wherever it applies we will assume \(Z\) is symmetric (\(\rho = 0\)) when \(\gamma = 1\), hence we can write

\[
\chi(\zeta, \gamma) = \chi(\gamma) = \begin{cases} 
\tan\left(\frac{\pi \zeta}{2}\right), & \gamma \neq 1, \\
0, & \gamma = 1.
\end{cases}
\]

We suppose that the marginal distribution of the supOU process \(\{X(t), t \in \mathbb{R}\}\) in (7) belongs to the domain of attraction of stable law, that is, \(X(1)\) has balanced regularly varying tails:

\[
P(X(1) > x) \sim pk(x)x^{-\gamma_1} \quad \text{and} \quad P(X(1) \leq -x) \sim qk(x)x^{-\gamma}, \quad \text{as} \quad x \to \infty,
\]

for some \(p, q \geq 0, p + q > 0, 0 < \gamma < 2\) and some slowly varying function \(k\). If \(\gamma = 1\), we assume \(p = q\). In particular, the variance is infinite. Moreover, when the mean is finite, that is when \(\gamma > 1\), we assume \(\mathbb{E}X(1) = 0\). These assumptions imply that \(X(1)\) is in the domain of attraction of stable distribution \(S_{\gamma}(\sigma, \rho, 0)\) with (Ibragimov & Linnik 1971, Theorem 2.6.1)

\[
\sigma = \left(\frac{\Gamma(2 - \gamma)}{1 - \gamma}(p + q)\cos\left(\frac{\pi \gamma}{2}\right)\right)^{1/\gamma}, \quad \rho = \frac{p - q}{p + q}.
\]

By (Fasen & Kluppelberg 2007, Proposition 3.1), the tail of the distribution function of \(X(1)\) is asymptotically equivalent to the tail of the background driving Lévy process \(L(t)\) at \(t = 1\). More precisely, as \(x \to \infty\)

\[
P(L(1) > x) \sim \gamma P(X(1) > x) \quad \text{and} \quad P(L(1) \leq -x) \sim \gamma P(X(1) \leq -x).
\]

Hence, (9) implies

\[
P(L(1) > x) \sim p\gamma k(x)x^{-\gamma} \quad \text{and} \quad P(L(1) \leq -x) \sim q\gamma k(x)x^{-\gamma}, \quad \text{as} \quad x \to \infty,
\]

and \(L(1)\) is in the domain of attraction of stable distribution \(S_{\gamma}(\gamma^{1/\gamma}\sigma, \rho, 0)\).

2.1.2 Dependence structure

The second set of assumptions deals with the dependence structure dictated by the behavior near the origin of the probability measure \(\pi\) in the characteristic quadruple (1). If variance is finite \(\mathbb{E}X(t)^2 < \infty\), then the correlation function of the supOU process \(X\) is the Laplace transform of \(\pi\):

\[
r(t) = \int_{\mathbb{R}_+} e^{-t\xi}\pi(d\xi), \quad t \geq 0.
\]

Hence, by a Tauberian argument, the decay of the correlation function at infinity is related to the decay of the distribution function of \(\pi\) at zero (see (Fasen & Kluppelberg 2007, Proposition
We will assume that the probability measure $\pi$ is regularly varying at zero, that is for some $\alpha > 0$ and some slowly varying function $\ell$

$$\pi((0,x]) \sim \ell(x^{-1})x^\alpha, \quad \text{as } x \to 0.$$  

(14)

Note that $\alpha$ can take any positive value.

To simplify the proofs of some of the results below, we will assume that $\pi$ has a density $p$ which is monotone on $(0,x')$ for some $x' > 0$, so that (14) implies

$$p(x) \sim \alpha \ell(x^{-1})x^{\alpha-1}, \quad \text{as } x \to 0.$$  

(15)

Note that if variance of the supOU process is finite and $\alpha \in (0,1)$, then the correlation function is not integrable, and the finite variance supOU process may be said to exhibit long-range dependence. On the other hand, note that the tail distribution of $\pi$ does not affect the tail behavior of $\tau(t)$, and in particular the decay of correlations. Hence it is not restrictive to assume in order to simplify the presentation of the results that

$$\int_0^\infty \xi \pi(d\xi) < \infty.$$  

(16)

2.1.3 Behavior of the Lévy measure at the origin

Consider the Lévy measure $\mu$ in (6). Somewhat surprisingly, the limiting behavior of the integrated supOU process $X^*(t)$ is affected by the growth of the Lévy measure $\mu$ near the origin. We will quantify this growth by assuming a power law behavior of the Lévy measure near the origin. Let

$$M^+(x) = \mu([x,\infty)), \quad x > 0$$

$$M^-(x) = \mu((\infty,-x]), \quad x > 0,$$

denote the tails of $\mu$. We will assume that there exist $\beta \geq 0$, $c^+, c^- \geq 0$, $c^+ + c^- > 0$ such that

$$M^+(x) \sim c^+ x^{-\beta} \quad \text{and} \quad M^-(x) \sim c^- x^{-\beta} \quad \text{as } x \to 0.$$  

(17)

Since $\mu$ is the Lévy measure, we must have $\beta < 2$. If (17) holds, then $\beta$ is the Blumenthal-Getoor index of the Lévy measure $\mu$ defined by (see Grahovac, Leonenko & Taqqu (2019))

$$\beta_{BG} = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \mu(dx) < \infty \right\}.$$  

(18)

Note that by (Kyprianou 2014, Lemma 7.15) $M^+(x) \sim P(L(1) > x)$ and $M^-(x) \sim P(L(1) \leq -x)$ as $x \to \infty$, hence we can express (12) equivalently as

$$M^+(x) \sim p \gamma k(x)x^{-\gamma} \quad \text{and} \quad M^-(x) \sim q \gamma k(x)x^{-\gamma}, \quad \text{as } x \to \infty.$$  

The condition (17) may be equivalently stated in terms of the Lévy measure of $X(1)$. Indeed, if $\nu$ is the Lévy measure of $X(1)$, then (17) is equivalent to

$$\nu((x,\infty)) \sim \beta^-1 c^+ x^{-\beta} \quad \text{and} \quad \nu((\infty,-x]) \sim \beta^-1 c^- x^{-\beta} \quad \text{as } x \to 0.$$  

(19)

See Grahovac, Leonenko & Taqqu (2019) for details.
3 The basic decomposition

As stated in the introduction, we are interested in establishing the rate of growth of moments of the integrated process \( X^* \), measured by the scaling function \( \tau_X^* \) defined by (3). The situation is more delicate than in the finite variance case since the range of finite moments is limited and the scaling function of the integrated process \( X^* \) is well-defined only over the interval \((0, \varphi(X^*)) = (0, \gamma)\).

To investigate the behavior of moments, we make a decomposition of the integrated process \( X^* \) into components that have different limiting behavior. The decomposition is based on the Lévy-Itô decomposition of the background driving Lévy process \( L \). Let

\[
\mu_1(dx) = \mu(dx) \mathbf{1}_{\{|x| > 1\}}(dx),
\]

\[
\mu_2(dx) = \mu(dx) \mathbf{1}_{\{|x| \leq 1\}}(dx),
\]

where \( \mu \) is the Lévy measure of the Lévy process \( L \). Then we can make a decomposition of the Lévy basis into:

- \( \Lambda_1 \) with characteristic quadruple \((a, 0, \mu_1, \pi)\),
- \( \Lambda_2 \) with characteristic quadruple \((0, 0, \mu_2, \pi)\),
- \( \Lambda_3 \) with characteristic quadruple \((0, b, 0, \pi)\).

Let \( L_1(t) \), \( L_2(t) \) and \( L_3(t) \), \( t \in \mathbb{R} \) denote the corresponding background driving Lévy processes so that we have the following cumulant functions:

\[
C\{\zeta \hat{\Delta} L_1(1)\} = i\zeta a + \int_{\mathbb{R}} \left( e^{i\zeta x} - 1 \right) \mu_1(dx) = i\zeta a + \int_{|x| > 1} \left( e^{i\zeta x} - 1 \right) \mu(dx), \tag{20}
\]

\[
C\{\zeta \hat{\Delta} L_2(1)\} = \int_{\mathbb{R}} \left( e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x) \right) \mu_2(dx)
= \int_{|x| \leq 1} \left( e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x) \right) \mu(dx),
\]

\[
C\{\zeta \hat{\Delta} L_3(1)\} = -\frac{\zeta^2}{2} b.
\]

Note that \( L_1 \) is a compound Poisson process and \( L_3 \) is Brownian motion. Consequently, we can represent \( X(t) \) as

\[
X(t) = \int_0^\infty \int_{-\infty}^{\xi t} e^{-\xi t + s} \Lambda_1(d\xi, ds) + \int_0^\infty \int_0^{\xi t} e^{-\xi t + s} \Lambda_2(d\xi, ds)
+ \int_0^\infty \int_{-\infty}^{\xi t} e^{-\xi t + s} \Lambda_3(d\xi, ds)
\tag{21}
= X_1(t) + X_2(t) + X_3(t),
\]

with \( X_1 \), \( X_2 \) and \( X_3 \) independent. Let \( X_1^* \), \( X_2^* \) and \( X_3^* \) denote the corresponding integrated processes which are independent. We next investigate the scaling functions of each process \( X_1^* \), \( X_2^* \) and \( X_3^* \) separately. These results will then be combined to give the scaling function of the integrated process.

4 Evaluation of the three scaling functions

For reference, we summarize the assumptions on the class of supOU processes considered. We exclude some boundary cases to simplify the presentation of the results.
Assumption 1. The supOU process \( \{ X(t), t \in \mathbb{R} \} \) is such that

- \( \pi \) has a density \( p \) satisfying (15) with \( \alpha > 0 \) and some slowly varying function \( \ell \) and (16) holds,
- the marginal distribution satisfies (9) with \( 0 < \gamma < 2 \),
- the behavior at the origin of the Lévy measure \( \mu \) is given by (17) with \( 0 \leq \beta < 2 \), \( \beta \neq 1+\alpha \).

4.1 The scaling function of \( X_1^\ast \)

The process \( X_1^\ast \) has infinite moments of order greater than \( \gamma \) and its scaling function \( \tau_{X_1^\ast} \) is well-defined for \( q \in (0, \gamma) \). Following (Grahovac et al. 2018, Lemma 5.1 and 5.2), two processes may arise as a limit of \( X_1^\ast \) after normalization.

If \( \gamma < 1 + \alpha \), then as \( T \to \infty \)

\[
\frac{1}{T^{1/\gamma} k^\#(T)^{1/\gamma}} X_1^\ast(T) \xrightarrow{d} \{ L_\gamma(t) \},
\]

where \( k \) is the slowly varying function in (9), \( k^\# \) is the de Bruijn conjugate of \( 1/k \) \( (x^{1/\gamma}) \) and the limit \( \{ L_\gamma \} \) is a \( \gamma \)-stable Lévy process such that \( L_\gamma(1) \xrightarrow{d} \mathcal{S}_\gamma(\bar{\sigma}_{1,\gamma}, \rho, 0) \) with

\[
\bar{\sigma}_{1,\gamma} = \sigma \left( \gamma \int_0^{\infty} \xi^{1-\gamma} \pi(d\xi) \right)^{1/\gamma},
\]

and \( \sigma \) and \( \rho \) given by (10). Recall that the de Bruijn conjugate (Bingham et al. 1989, Subsection 1.5.7) of some slowly varying function \( h \) is a slowly varying function \( h^\# \) such that

\[
h(x)h^\#(xh(x)) \to 1, \quad h^\#(x)h(xh^\#(x)) \to 1,
\]

as \( x \to \infty \). By (Bingham et al. 1989, Theorem 1.5.13) such function always exists and is unique up to asymptotic equivalence.

If, on the other hand \( \gamma > 1 + \alpha \), then as \( T \to \infty \)

\[
\frac{1}{T^{1/(1+\alpha)} \ell^\#((T))^{1/(1+\alpha)}} X_1^\ast(T) \xrightarrow{d} \{ L_{1+\alpha}(t) \},
\]

where \( \ell^\# \) is de Bruijn conjugate of \( 1/\ell \) \( (x^{1/(1+\alpha)}) \) and the limit \( \{ L_{1+\alpha} \} \) is \( (1+\alpha) \)-stable Lévy process such that \( L_{1+\alpha}(1) \xrightarrow{d} \mathcal{S}_\gamma(\bar{\sigma}_{1,\alpha}, \tilde{\rho}_1, 0) \) with

\[
\bar{\sigma}_{1,\alpha} = \left( \frac{\Gamma(1-\alpha)}{\alpha}(c_1^- + c_1^+) \cos \left( \frac{\pi(1+\alpha)}{2} \right) \right)^{1/(1+\alpha)}, \quad \tilde{\rho}_1 = \frac{c_1^--c_1^+}{c_1^-+c_1^+},
\]

and \( c_1^-, c_1^+ \) given by

\[
c_1^- = \frac{\alpha}{1+\alpha} \int_{-\infty}^{-1} y^{1-\alpha} \mu(dy), \quad c_1^+ = \frac{\alpha}{1+\alpha} \int_1^{\infty} y^{1+\alpha} \mu(dy).
\]

We now consider convergence of moments in these limit theorems. First, if \( \gamma < 1 + \alpha \), then we get the following scaling function for the process \( X_1^\ast \).

Lemma 4.1. If Assumption 1 holds and \( \gamma < 1 + \alpha \), then

\[
\tau_{X_1^\ast}(q) = \frac{1}{\gamma} q, \quad 0 < q < \gamma.
\]
For moments of order \( q \) in the range \((1 + \alpha, \gamma)\) we are not able to obtain the exact form of the scaling function \( \tau_{X^*_1}(q) \). However, we provide a bound which will be enough for the proof of the main results later on. We conjecture that equality holds in (26).

**Lemma 4.2.** If Assumption 1 holds and \( \gamma > 1 + \alpha \), then

\[
\tau_{X^*_1}(q) = \begin{cases} 
\frac{1}{\gamma} q, & 0 < q \leq 1 + \alpha, \\
q - \alpha, & 1 + \alpha < q < \gamma.
\end{cases}
\]  

(26)

The proofs of Lemma 4.1 and Lemma 4.2 are particularly delicate because of the presence of infinite second moments. They are given in Sections 6 and 7, respectively. Figure 1 shows the two forms of the scaling function of \( X^*_1 \).

### 4.2 The scaling function of \( X^*_2 \)

By the decomposition (21), \( X^*_2 \) is the integrated supOU process corresponding to a characteristic quadruple \((0, 0, \mu_2, \pi)\) where \( \mu_2(dx) = \mu(dx)1_{|x| \leq 1}(dx) \) and we assume \( \mu_2 \neq 0 \). In particular, \( X^*_2 \) has finite variance since \( \int_{|x| > 1} x^2 \mu_2(dx) < \infty \). Moreover, \( \int_{|x| > 1} e^{a|x|} \mu_2(dx) < \infty \) and exponential moment of \( X_2(1) \) is finite which by (Lukacs 1970, Theorem 7.2.1) implies that the cumulant function of \( X_2(1) \) is analytic in the neighborhood of the origin and all moments are finite. Hence, we may use the results of Graovac, Leonenko & Taqqu (2019), namely Eq. (4.9), Theorem 4.2 and Theorem 4.3 from Graovac, Leonenko & Taqqu (2019). These results are stated here in the following lemma.

**Lemma 4.3.** Suppose that Assumption 1 holds. Then the scaling function \( \tau_{X^*_2}(q) \) of the process \( X^*_2 \) is as follows:

(a) If \( \alpha > 1 \), then

\[
\tau_{X^*_2}(q) = \begin{cases} 
\frac{1}{2} q, & 0 < q \leq q^*, \\
q - \alpha, & q \geq q^*.
\end{cases}
\]

where \( q^* \) is the largest even integer less than or equal to \( 2\alpha \) and \( q^* \) is the smallest even integer greater than \( 2\alpha \).
(b) If $\alpha \in (0, 1)$ and $\beta < 1 + \alpha$, then

$$\tau_{X^*_2}(q) = \begin{cases} \frac{q}{1 + \alpha}, & 0 < q \leq 1 + \alpha, \\ q - \alpha, & q \geq 1 + \alpha. \end{cases}$$

(c) If $\alpha \in (0, 1)$ and $1 + \alpha < \beta < 2$, then

$$\tau_{X^*_2}(q) = \begin{cases} \left(1 - \frac{q}{\beta}\right) q, & 0 < q \leq \beta, \\ q - \alpha, & q \geq \beta. \end{cases}$$

Lemma 4.3(a) and convexity of the scaling function imply that for $q_* \leq q \leq q^*$

$$\tau_{X^*_2}(q) \leq \frac{q^* - \alpha - q_*/2}{q^* - q_*}(x - q_*) + \frac{q_*}{2}.$$ 

Note also that Lemma 4.3(a) implies that $\tau_{X^*_2}(q) = q/2$ for $q \leq 2$ which will be enough for the proofs of Theorems 5.1 and 5.2 below.

In contrast with the component $X^*_1$, the scaling function of $X^*_2$ displays intermittency in any case covered by Assumption 1. Even in the short-range dependent scenario $\alpha > 1$, intermittency appears for higher order moments. Scaling functions of $X^*_2$ are shown in Figure 2.

4.3 The scaling function of $X^*_3$

The process $X^*_3$ defined in (21) is a Gaussian process. Its scaling function is given in (Grahovac, Leonenko & Taqqu 2019, Theorem 4.1 and 4.4). Gaussian supOU processes do not display intermittency and their scaling function is linear over positive reals (Figure 3). This result is stated here in Lemma 4.4.

Lemma 4.4. Suppose that Assumption 1 holds. Then the scaling function $\tau_{X^*_3}(q)$ of the process $X^*_3$ is as follows:

(a) If $\alpha > 1$, then

$$\tau_{X^*_3}(q) = \frac{1}{2} q, \quad \forall q > 0.$$ 

(b) If $\alpha \in (0, 1)$, then

$$\tau_{X^*_3}(q) = \left(1 - \frac{\alpha}{2}\right) q, \quad \forall q > 0.$$ 

5 The scaling function of the integrated process $X^*$

To derive the scaling function of the integrated process $X^* = X^*_1 + X^*_2 + X^*_3$ we will use the expressions for the scaling functions of components in the decomposition (21) and the following proposition which shows how to compute the scaling function of a sum of independent processes.

Proposition 5.1. Let $Y_1 = \{Y_1(t), t \geq 0\}$ and $Y_2 = \{Y_2(t), t \geq 0\}$ be two independent processes with scaling functions $\tau_{Y_1}$ and $\tau_{Y_2}$, respectively, and suppose that $EY_1(t) = EY_2(t) = 0$ for every $t \geq 0$ if the mean is finite. If $q \in (0, \overline{Y}(Y_1)) \cup (0, \overline{Y}(Y_2))$ and $\tau_{Y_1}(q)$ and $\tau_{Y_2}(q)$ are well-defined and positive, then the scaling function of the sum $Y_1 + Y_2 = \{Y_1(t) + Y_2(t), t \geq 0\}$, evaluated at point $q$, equals

$$\tau_{Y_1+Y_2}(q) = \max \{\tau_{Y_1}(q), \tau_{Y_2}(q)\}.$$
Figure 2: Scaling function of $X^*_2$ (see Lemma 4.3)

Proof. Suppose that $\max \{ \tau_{Y_1}(q), \tau_{Y_2}(q) \} = \tau_{Y_1}(q)$. For $\varepsilon > 0$ we can take $t$ large enough so that

$$\frac{\log E |Y_1(t)|^q}{\log t} \geq \frac{\log E |Y_2(t)|^q}{\log t} - \varepsilon$$

and hence

$$E |Y_1(t)|^q \geq E |Y_2(t)|^q t^{-\varepsilon}. \quad (27)$$

From the inequality

$$E |Y_1(t) + Y_2(t)|^q \leq c_q E |Y_1(t)|^q + c_q E |Y_2(t)|^q, \quad c_q = \max \{1, 2^{q-1} \}, \quad (28)$$
we have that

\[
\tau_{Y_1+Y_2}(q) = \lim_{t \to \infty} \frac{\log \mathbb{E}|Y_1(t) + Y_2(t)|^q}{\log t}
\]

\[
\leq \lim_{t \to \infty} \left( \frac{\log c_q}{\log t} + \frac{\log (\mathbb{E}|Y_1(t)|^q + \mathbb{E}|Y_2(t)|^q)}{\log t} \right)
\]

\[
= \lim_{t \to \infty} \frac{\log \mathbb{E}|Y_1(t)|^q + \log \left(1 + \frac{\mathbb{E}|Y_2(t)|^q}{\mathbb{E}|Y_1(t)|^q}\right)}{\log t}
\]

\[
\leq \lim_{t \to \infty} \frac{\log \mathbb{E}|Y_1(t)|^q + \log (1 + t^\varepsilon)}{\log t}
\]

\[
= \tau_{Y_1}(q) + \varepsilon,
\]

where we used (27). Since \(\varepsilon\) was arbitrary, we conclude that \(\tau_{Y_1+Y_2}(q) \leq \max \{\tau_{Y_1}(q), \tau_{Y_2}(q)\}\).

We prove the reverse inequality for the \(q \geq 1\) case first. Note that in this case \(\mathbb{E}|Y_1(t)| = \mathbb{E}|Y_2(t)| = 0\) for every \(t \geq 0\). For \(x \in \mathbb{R}\) we have by using Jensen’s inequality that

\[
|x|^q = |x + \mathbb{E}Y_2(t)|^q \leq \mathbb{E}|x + Y_2(t)|^q.
\]

Letting \(F_{Y_1(t)}\) and \(F_{Y_2(t)}\) denote the distribution functions of \(Y_1(t)\) and \(Y_2(t)\), respectively, we get by independence

\[
\mathbb{E}|Y_1(t)|^q = \int_{-\infty}^{\infty} |x|^q dF_{Y_1(t)}(x) \leq \int_{-\infty}^{\infty} \mathbb{E}|x + Y_2(t)|^q dF_{Y_1(t)}(x)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x + y|^q dF_{Y_2(t)}(y) dF_{Y_1(t)}(x) = \mathbb{E}|Y_1(t) + Y_2(t)|^q.
\]

From here it follows that

\[
\tau_{Y_1+Y_2}(q) \geq \tau_{Y_1}(q).
\]

Suppose now that \(q < 1\) and let \(Y_2' = \{Y_2'(t), t \geq 0\}\) be an independent copy of the process \(Y_2 = \{Y_2(t), t \geq 0\}\), independent of \(Y_1\). From (28) we have that

\[
\mathbb{E}|Y_1(t) + Y_2(t)|^q \geq 2^{1-q} \mathbb{E}|Y_1(t) + Y_2(t) - Y_2'(t)|^q - \mathbb{E}|Y_2(t)|^q.
\]
Since $Y_2(t) - Y'_2(t)$ is symmetric it follows that $Y_1(t) + Y_2(t) - Y'_2(t) \overset{d}{=} Y_1(t) - Y_2(t) + Y'_2(t)$.
From the identity
\[
Y_1(t) = \frac{1}{2} (Y_1(t) + Y_2(t) - Y'_2(t) + Y_1(t) - Y_2(t) + Y'_2(t))
\]
we get by using (28) that
\[
E |Y_1(t)|^q \leq 2^{-q} 2^q - 1 \left( E |Y_1(t) + Y_2(t) - Y'_2(t)|^q + E |Y_1(t) - Y_2(t) + Y'_2(t)|^q \right)
\]
\[
= E |Y_1(t) + Y_2(t) - Y'_2(t)|^q.
\]
Returning back to (29) we have
\[
E |Y_1(t) + Y_2(t)|^q \geq 2^{1-q} E |Y_1(t)|^q - E |Y_2(t)|^q = E |Y_1(t)|^q \left( 2^{1-q} - \frac{E |Y_2(t)|^q}{E |Y_1(t)|^q} \right). \tag{30}
\]
Without loss of generality we may assume $\tau_{Y_1}(q) \geq \tau_{Y_2}(q)$.
Assume first that this inequality is strict, namely that $\tau_{Y_1}(q) > \tau_{Y_2}(q)$. For $\varepsilon > 0$ small enough we can take $t$ large enough so that
\[
\frac{\log E |Y_1(t)|^q}{\log t} \geq \frac{\log E |Y_2(t)|^q}{\log t} + \varepsilon
\]
and hence
\[
E |Y_1(t)|^q \geq E |Y_2(t)|^q t^\varepsilon.
\]
We conclude that
\[
\frac{E |Y_2(t)|^q}{E |Y_1(t)|^q} \to 0, \text{ as } t \to \infty.
\]
By taking logarithms in (30), dividing by $\log t$ and letting $t \to \infty$, we get
\[
\tau_{Y_1 + Y_2}(q) \geq \tau_{Y_1}(q).
\]
If $\tau_{Y_1}(q) = \tau_{Y_2}(q)$, we may have three cases as a limit of $E |Y_2(t)|^q / E |Y_1(t)|^q$ as $t \to \infty$: either the limit is 0, $\infty$ or some constant $C > 0$.

- If the limit is 0, then we can apply the same argument as in the case $\tau_{Y_1}(q) > \tau_{Y_2}(q)$.
- If the limit is $\infty$, by interchanging the roles of $Y_1$ and $Y_2$ in the previous part of the proof we obtain the following analog of (30), namely
\[
E |Y_1(t) + Y_2(t)|^q \geq 2^{1-q} E |Y_2(t)|^q - E |Y_1(t)|^q = E |Y_2(t)|^q \left( 2^{1-q} - \frac{E |Y_1(t)|^q}{E |Y_2(t)|^q} \right). \tag{31}
\]
Since $E |Y_1(t)|^q / E |Y_2(t)|^q \to 0$ as $t \to \infty$, we get $\tau_{Y_1 + Y_2}(q) \geq \tau_{Y_2}(q) = \tau_{Y_1}(q)$.
- If the limit is $C < 2^{1-q}$, then $2^{1-q} - \frac{E |Y_2(t)|^q}{E |Y_1(t)|^q}$ is eventually positive and logarithm can be applied in (30) to obtain the claim.
- If the limit is $C \geq 2^{1-q}$, then the limit of $\frac{E |Y_1(t)|^q}{E |Y_2(t)|^q}$ is $1/C$. Since $2^{1-q} - C^{-1} \geq 2^{1-q} - 2^{q-1} > 0$, then $2^{1-q} - \frac{E |Y_1(t)|^q}{E |Y_2(t)|^q}$ is eventually positive and the logarithm can be applied in (31) to obtain the claim.
We are now ready to state the main results. We will show that infinite variance supOU processes may exhibit the phenomenon of intermittency. We first consider the case when the underlying supOU process has no Gaussian component ($b = 0$). The obtained scaling functions for this case are shown in Figures 4a-4d.

**Theorem 5.1.** Suppose that Assumption 1 holds. Then the scaling function $\tau_{X^*}(q)$ of the process $X^*$ is as follows:

(a) If $\alpha > 1$ or if $\alpha \in (0, 1)$ and $\gamma < 1 + \alpha$, then

$$\tau_{X^*}(q) = \frac{1}{\gamma} q, \quad 0 < q < \gamma.$$  

(b) If $\beta < 1 + \alpha < \gamma$, then

$$\tau_{X^*}(q) = \begin{cases} \frac{1}{1+\alpha} q, & 0 < q \leq 1 + \alpha, \\ q - \alpha, & 1 + \alpha \leq q < \gamma. \end{cases}$$

(c) If $1 + \alpha < \beta \leq \gamma$, then

$$\tau_{X^*}(q) = \begin{cases} \frac{1 - \frac{\alpha}{\beta}}{1 + \alpha} q, & 0 < q \leq \beta, \\ q - \alpha, & \beta \leq q < \gamma. \end{cases}$$

(d) If $1 + \alpha < \gamma < \beta$, then

$$\tau_{X^*}(q) = \frac{1 - \frac{\alpha}{\beta}}{1 + \alpha} q, \quad 0 < q < \gamma.$$  

**Proof.** We shall combine the results of Lemmas 4.1, 4.2 and 4.3 by using Proposition 5.1.

(a) Suppose that $\gamma < 1 + \alpha$ and split cases depending on the scale function of $X_2^*$.

- If $\alpha > 1$, then from Lemma 4.3 $\tau_{X_2^*}(q) = q/2$ for $q \in (0, 2)$. Since $1/\gamma > 1/2$, we have for $q \in (0, \gamma)$

  $$\tau_{X^*}(q) = \max \{ \tau_{X_1^*}(q), \tau_{X_2^*}(q) \} = \max \left\{ \frac{1}{\gamma} q, \frac{1}{2} q \right\} = \frac{1}{\gamma} q.$$  

- If $\alpha \in (0, 1)$ and $\beta < 1 + \alpha$, then we have for $q \in (0, \gamma)$

  $$\tau_{X^*}(q) = \max \{ \tau_{X_1^*}(q), \tau_{X_2^*}(q) \} = \max \left\{ \frac{1}{\gamma} q, \frac{1}{1 + \alpha} q \right\} = \frac{1}{\gamma} q,$$

  since $\frac{1}{\gamma} > \frac{1}{1 + \alpha}$.

- If $\alpha \in (0, 1)$ and $\beta > 1 + \alpha$, then for $q \in (0, \gamma)$

  $$\tau_{X^*}(q) = \max \{ \tau_{X_1^*}(q), \tau_{X_2^*}(q) \} = \max \left\{ \frac{1}{\gamma} q, \left( 1 - \frac{\alpha}{\beta} \right) q \right\} = \frac{1}{\gamma} q,$$

  since $1 - \frac{\alpha}{\beta} < 1 + \frac{1 - \gamma}{\beta} < 1 + \frac{1 - \gamma}{\gamma} = \frac{1}{\gamma}$.

(b) If $\gamma > 1 + \alpha$ and $\beta < 1 + \alpha$, then necessarily $\alpha \in (0, 1)$ and by Lemmas 4.2 and 4.3 we have

$$\tau_{X^*}(q) = \begin{cases} \max \left\{ \frac{1}{1 + \alpha} q, \frac{1}{1 + \alpha} q \right\}, & 0 < q \leq 1 + \alpha, \\ \max \left\{ \frac{1}{1 + \alpha} q, q - \alpha \right\}, & 1 + \alpha \leq q < \gamma, \end{cases} = \begin{cases} \frac{1}{1 + \alpha} q, & 0 < q \leq 1 + \alpha, \\ q - \alpha, & 1 + \alpha \leq q < \gamma. \end{cases}$$
Figure 4: Scaling functions obtained in Theorems 5.1 and 5.2. There is intermittency in the cases (b) and (c).
(c) If \( \gamma > 1 + \alpha, \beta > 1 + \alpha \) and \( \beta \leq \gamma \), we have

\[
\tau_{X^\ast}(q) = \begin{cases} 
\max \left\{ \frac{1}{1+\alpha} q, \left(1 - \frac{\alpha}{\beta}\right) q \right\}, & 0 < q \leq 1 + \alpha, \\
\max \left\{ \tau_{X_1^\ast}(q), \left(1 - \frac{\alpha}{\beta}\right) q \right\}, & 1 + \alpha < q \leq \beta, \\
\max \left\{ \tau_{X_1^\ast}(q) - \alpha, q - \alpha \right\}, & \beta < q < \gamma.
\end{cases}
\]

For the case \( q \leq 1 + \alpha \), note that because \( \beta > 1 + \alpha \) we have \( 1 - \frac{\alpha}{\beta} > 1 - \frac{\alpha}{1+\alpha} = \frac{1}{1+\alpha} \). In Lemma 4.2 we showed that \( \tau_{X_1^\ast}(q) \leq q - \alpha \) for \( 1 + \alpha < q < \gamma \) and for \( q \leq \beta \) we have \( q - \frac{\alpha}{\beta} q \geq q - \alpha \). Hence we obtain

\[
\tau_{X^\ast}(q) = \begin{cases} 
\left(1 - \frac{\alpha}{\beta}\right) q, & 0 < q \leq 1 + \alpha, \\
\left(1 - \frac{\alpha}{\beta}\right) q, & 1 + \alpha < q \leq \beta, \\
q - \alpha, & \beta < q < \gamma.
\end{cases}
\]

(d) If \( \gamma > 1 + \alpha, \beta > 1 + \alpha \) and \( \beta > \gamma \), then by using the same arguments as in the previous case we get

\[
\tau_{X^\ast}(q) = \begin{cases} 
\max \left\{ \frac{1}{1+\alpha} q, \left(1 - \frac{\alpha}{\beta}\right) q \right\}, & 0 < q \leq 1 + \alpha, \\
\tau_{X_1^\ast}(q) - \alpha, & \beta < q < \gamma.
\end{cases}
\]

One may follow the proof of Theorem 5.1 from Figure 5. Each subfigure shows the scaling function of \( X_1^\ast \) and the scaling function of \( X_2^\ast \). In Figure 5, following Proposition 5.1, the scaling function of the integrated process \( X^\ast \) is obtained by taking the maximum of these two functions (thick). The vertical dotted line indicates the range of finite moments. The scaling function of \( X^\ast \) is well-defined only in this range.

Note that the scaling function has a change-point in only two of the cases of Theorem 5.1. Hence intermittency appears only in cases (b) and (c) of Theorem 5.1 shown in Figures 5d and 5e, respectively.

One may notice that infinite order moments hide the intermittency property as they limit the domain of the scaling function. This can be seen in Figure 5. The finite variance component \( X_2^\ast \) exhibits intermittency in all cases, however, this is not always apparent from the scaling function of the process \( X^\ast \). This is due to the fact that infinite order moments may hide the behavior of the intermittent component. In these cases, the change point in the scaling function of \( X_2^\ast \) is to the right of the moment index \( \gamma \), hence the scaling function of \( X^\ast \) remains linear on \((0, \gamma)\) (see Figures 5a, 5b, 5c and 5f).

We next state the result for the supOU process with Gaussian component \((b \neq 0)\). The scaling functions for this case are shown in Figures 4e-4f.

**Theorem 5.2.** Suppose that Assumption 1 holds. Then the scaling function \( \tau_{X^\ast}(q) \) of the process \( X^\ast \) is as follows:

(a) If \( \alpha > 1 \) or if \( \alpha \in (0,1) \) and \( \gamma < \frac{2}{2-\alpha}, \) then

\[
\tau_{X^\ast}(q) = \frac{1}{\gamma} q, \quad 0 < q < \gamma.
\]

(b) If \( \alpha \in (0,1) \) and \( \gamma > \frac{2}{2-\alpha}, \) then

\[
\tau_{X^\ast}(q) = \left(1 - \frac{\alpha}{2}\right) q, \quad 0 < q < \gamma.
\]
Figure 5: Scaling functions of $X^*$ when $b = 0$ (no Gaussian component). Each plot shows the scaling functions $\tau_{X^*_1}$ (blue), $\tau_{X^*_2}$ (red) and $\tau_{X^*}$ (thick green). Dashed parts of the plots denote the upper bounds. The vertical thick dotted line denotes the position of $\gamma$, beyond which the moments are infinite.
Proof of Theorem 5.2. We will use the results of Theorem 5.1 and Lemma 4.4 and combine them using Proposition 5.1 so that
\[ \tau_{X^*}(q) = \max \left\{ \tau_{X_1^*}(q), \tau_{X_2^*}(q), \tau_{X_3^*}(q) \right\}. \]

(a) If \( \alpha > 1 \), then for \( q < \gamma \)
\[ \tau_{X^*}(q) = \max \left\{ \frac{1}{\gamma} q, \frac{1}{2} q \right\} = \frac{1}{\gamma} q. \]

If \( \alpha \in (0, 1) \) and \( \gamma < \frac{2}{2-\alpha} \), then also \( \gamma < 1 + \frac{\alpha}{2-\alpha} < 1 + \alpha \) and hence
\[ \tau_{X^*}(q) = \max \left\{ \frac{1}{\gamma} q, \left(1 - \frac{\alpha}{2}\right) q \right\} = \frac{1}{\gamma} q, \]
since \( \frac{1}{\gamma} > 1 - \alpha/2 \).

(b) Suppose now that \( \alpha \in (0, 1) \) and \( \gamma > \frac{2}{2-\alpha} \).

- If \( \frac{2}{2-\alpha} < \gamma < 1 + \alpha \), then
\[ \tau_{X^*}(q) = \max \left\{ \frac{1}{\gamma} q, \left(1 - \frac{\alpha}{2}\right) q \right\} = \left(1 - \frac{\alpha}{2}\right) q, \]
since \( \frac{1}{\gamma} < 1 - \alpha/2 \).

- If \( \gamma > 1 + \alpha \) and \( \beta < 1 + \alpha \), then we have
\[ \tau_{X^*}(q) = \begin{cases} \max \left\{ \frac{1}{1+\alpha} q, \left(1 - \frac{\alpha}{2}\right) q \right\}, & 0 < q \leq 1 + \alpha, \\ \max \left\{ q - \alpha, \left(1 - \frac{\alpha}{2}\right) q \right\}, & 1 + \alpha < q < \gamma. \end{cases} \]

Now \( \alpha < 1 \) implies \( \frac{1}{1+\alpha} = 1 - \frac{\alpha}{1+\alpha} < 1 - \frac{\alpha}{2} \) and for \( q < 2 \) we have \( q - \frac{\alpha}{2} q > q - \alpha \). Hence,
\[ \tau_{X^*}(q) = \left(1 - \frac{\alpha}{2}\right) q, \quad 0 < q < \gamma. \]

- If \( \gamma > 1 + \alpha \), \( 1 + \alpha < \beta \) and \( \beta < \gamma \), then
\[ \tau_{X^*}(q) = \begin{cases} \max \left\{ \frac{1}{\beta} q, \left(1 - \frac{\alpha}{2}\right) q \right\}, & 0 < q \leq \beta, \\ \max \left\{ q - \alpha, \left(1 - \frac{\alpha}{2}\right) q \right\}, & \beta < q < \gamma. \end{cases} \]

since \( 1 - \frac{\alpha}{2} < 1 - \frac{\alpha}{2} \) and by the same argument as in the previous case.

- The same argument applies to case \( \gamma > 1 + \alpha \), \( 1 + \alpha < \beta \) and \( \beta > \gamma \).

Figures 6 and 7 illustrate the proof of Theorem 5.2. The scaling functions \( \tau_{X_1^*}, \tau_{X_2^*} \) and \( \tau_{X_3^*} \) of each component are shown on each plot and their maximum is denoted by the thick line. Figure 6 is related to the case (a) of Theorem 5.2 and Figure 7 to the case (b) of Theorem 5.2. The figures are split based on different forms of the scaling functions of the three components \( X_1^*, X_2^* \) and \( X_3^* \). Note that if the Gaussian component is present, then the scaling function displays no intermittency. For example, even if the scaling functions of two components \( X_1^* \) and \( X_2^* \) have a change-point, this cannot be seen from the scaling function of \( X^* \) due to infinite moments (see Figures 7c, 7d, 7e).
(a) $\alpha > 1$

(b) $\alpha \in (0, 1)$, $\gamma < \frac{2}{\alpha} < 1 + \alpha$ and $\beta < 1 + \alpha$

(c) $\alpha \in (0, 1)$, $\gamma < \frac{\beta}{2 - \alpha} < 1 + \alpha$ and $\beta > 1 + \alpha$

Figure 6: Scaling functions of $X^*$ when $b \neq 0$: case (a) of Theorem 5.2. Each plot shows the scaling functions $\tau_{X_1}$ (blue), $\tau_{X_2}$ (red), $\tau_{X_3}$ (purple) and $\tau_{X^*}$ (thick green). Dashed part of the plot denotes the upper bound. The vertical thick dotted line denotes the position of $\gamma$, beyond which the moments are infinite.
Figure 7: Scaling functions of $X^*$ when $b \neq 0$: case (b) of Theorem 5.2. Each plot shows the scaling functions $\tau_{X^*}(q)$ (blue), $\tau_{X^*}^1$ (red), $\tau_{X^*}^2$ (purple) and $\tau_{X^*}$ (thick green). Dashed part of the plot denotes the upper bound. The vertical thick dotted line denotes the position of $\gamma$, beyond which the moments are infinite.
6 Proof of Lemma 4.1

Let \( q < \gamma \) and \( A_T = T^{1/\gamma} \kappa^{\#}(T)^{1/\gamma} \). We will show that \( \{ |A_T^{-1}X_t^1(Tt)|^q \} \) is uniformly integrable so that \( \mathbb{E}[A_T^{-1}X_t^1(Tt)]^q \to \mathbb{E}[L_\gamma(t)]^q \) as \( T \to \infty \), where \( \{ L_\gamma \} \) is a Lévy process from (22).

First we recall some known results. If \( Y \) is some random variable, let \( \tilde{Y} \) denote its symmetrization, i.e. \( \tilde{Y} = Y - Y' \) with \( Y' = \text{d} Y \) and independent of \( Y \). By (von Bahr & Esseen 1965, Lemma 4), if \( r \in [1,2] \), \( \mathbb{E}|Y|^r < \infty \) and \( \mathbb{E}Y = 0 \), then

\[
\mathbb{E}|Y|^r \leq \mathbb{E}|\tilde{Y}|^r.
\]

On the other hand, if \( r < 1 \) and \( \mathbb{E}|Y|^r < \infty \), then we obtain from (Gut 2013, Proposition 3.6.4) that

\[
\mathbb{E}|Y|^r \leq 2\mathbb{E}|\tilde{Y}|^r + 2|\text{med}(Y)|^r,
\]

where med\((Y)\) denotes the median of \( Y \). Furthermore, one may express \( r \)-th absolute moment, \( 0 < r < 2 \) as (von Bahr & Esseen 1965, Lemma 2)

\[
\mathbb{E}|Y|^r = k_r \int_{-\infty}^{\infty} (1 - \text{Re} \exp \kappa Y(\zeta)) |\zeta|^{-r-1} d\zeta
\]

where \( k_r > 0 \) is a constant.

We consider now the symmetrized random variable \( \tilde{X}_1^*(Tt) \). The characteristic function of \( \tilde{X}_1^*(Tt) \) is \( |\exp \kappa X_1(\zeta, Tt)|^2 \), hence from (34) we get

\[
\mathbb{E} \left| A_T^{-1}\tilde{X}_1^*(Tt) \right| = k_q \int_{-\infty}^{\infty} (1 - |\exp \kappa X_1(A_T^{-1}\zeta, Tt)|^2) |\zeta|^{-q-1} d\zeta.
\]

From (7) we get the decomposition

\[
X_1^*(Tt) = \int_{u=0}^{Tt} \int_{\xi=0}^{\infty} \int_{s=-\infty}^{\xi u} e^{-\xi u + s} \Lambda_1(d\xi, ds) du
\]

\[
= \int_{u=0}^{Tt} \int_{\xi=0}^{\infty} \int_{s=-\infty}^{0} e^{-\xi u + s} d\Lambda_1(d\xi, ds) + \int_{u=0}^{Tt} \int_{\xi=0}^{\infty} \int_{s=0}^{\xi u} e^{-\xi u + s} d\Lambda_1(d\xi, ds)
\]

\[
= \int_{\xi=0}^{\infty} \int_{s=-\infty}^{\xi u} \int_{u=0}^{Tt} e^{-\xi u + s} d\Lambda_1(d\xi, ds) + \int_{\xi=0}^{\infty} \int_{s=0}^{\xi u} \int_{u=0}^{Tt} e^{-\xi u + s} d\Lambda_1(d\xi, ds)
\]

\[
= \int_{\xi=0}^{\infty} \int_{s=0}^{\xi u} \int_{u=0}^{Tt} e^{-\xi u + s} d\Lambda_1(d\xi, ds) + \int_{\xi=0}^{\infty} \int_{s=0}^{\xi u} \int_{u=0}^{Tt} e^{-\xi u + s} d\Lambda_1(d\xi, ds)
\]

\[
=: \Delta X_{1,1}^*(Tt) + \Delta X_{1,2}^*(Tt).
\]

The equality of the integrals on the right-hand side follows from

\[
1_{\{0 \leq u \leq Tt\}} 1_{\{0 \leq s / \xi \leq u \leq Tt\}} = 1_{\{0 \leq s / \xi \leq u \leq Tt\}}.
\]

Since \( \Delta X_{1,1}^*(Tt) \) and \( \Delta X_{1,2}^*(Tt) \) are independent, we get

\[
|\kappa X_1(A_T^{-1}\zeta, Tt)| \leq |\kappa \Delta X_{1,1}^*(A_T^{-1}\zeta, Tt)| + |\kappa \Delta X_{1,2}^*(A_T^{-1}\zeta, Tt)|.
\]

Now we consider bounds for each term separately.

- For the first term on the right hand side we use some parts of the proof of (Grahovac et al. 2018, Lemma 5.1). From the integration formula for the stochastic integral, for any \( \Lambda \)-integrable function \( f \) on \( \mathbb{R}_+ \times \mathbb{R} \), one has (see Rajput & Rosinski (1989))

\[
C \left\{ \zeta \int_{\mathbb{R}_+ \times \mathbb{R}} f \, d\Lambda \right\} = \int_{\mathbb{R}_+ \times \mathbb{R}} \kappa_L(\zeta f(\xi, s)) ds \, d\pi(d\xi)
\]

21
and we get that

\[
\kappa_{\Delta X_{1,1}}(A_T^{-1}\zeta, Tt) = \int_{-\infty}^{\infty} \int_{0}^{t} \kappa_{L_1}(\zeta A_T^{-1} \int_{0}^{Tt} e^{-\xi u + s} du) ds \pi(d\xi) \\
= \int_{-\infty}^{\infty} \int_{0}^{t} \kappa_{L_1}(\zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T})) ds \pi(d\xi). 
\]

(38)

The assumption (12) implies that (Ibragimov & Linnik 1971, Theorem 2.6.4)

\[
\kappa_{L_1}(\zeta) \sim k (1/|\zeta|) \kappa_{S_1(\gamma/\gamma, \sigma, \rho, 0)}(\zeta), \quad \text{as } \zeta \to 0.
\]

(39)

Since \(|\kappa_{S_1(\gamma/\gamma, \sigma, \rho, 0)}(\zeta)| = C|\zeta|^\gamma\) and \(k\) is slowly varying at infinity, then for arbitrary \(\delta > 0\), in some neighborhood of the origin one has

\[
|\kappa_{L_1}(\zeta)| \leq C_1 |\zeta|^{\gamma - \delta}, \quad |\zeta| \leq \varepsilon.
\]

On the other hand, since \(|e^{i\xi x} - 1| \leq 2\), we have from (20) that

\[
|\kappa_{L_1}(\zeta)| \leq |a||\zeta| + 2 \int_{|x| > 1} 1_{\{x > 1\}}(x) \mu(dx) \leq |a||\zeta| + C_2,
\]

since the Lévy measure is integrable on \(|x| > 1\). By taking \(C_3\) large enough we arrive at the bound

\[
|\kappa_{L_1}(\zeta)| \leq C_1 |\zeta|^{\gamma - \delta} 1_{\{|\zeta| \leq \varepsilon\}}(\zeta) + C_3 |\zeta| 1_{\{|\zeta| > \varepsilon\}}(\zeta).
\]

(40)

Now we have from (38)

\[
\left| \kappa_{\Delta X_{1,1}}(A_T^{-1}\zeta, Tt) \right| \\
\leq C_1 \int_{0}^{t} \int_{-\infty}^{\infty} \left| \zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T}) \right|^{\gamma - \delta} 1_{\{|\zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T})| \leq \varepsilon\}}(\zeta) ds \pi(d\xi) \\
+ C_3 \int_{-\infty}^{\infty} \left| \zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T}) \right| 1_{\{|\zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T})| > \varepsilon\}}(\zeta) ds \pi(d\xi) \\
\leq C_1 |\zeta|^{\gamma - \delta} A_T^{\gamma + \delta} \int_{0}^{t} \int_{-\infty}^{\infty} e^{(\gamma - \delta)s} (\xi^{-1} (1 - e^{-\xi T}))^{\gamma - \delta} ds \pi(d\xi) \\
+ C_3 |\zeta| t A_T^{-1} \int_{-\infty}^{\infty} e^{s(\xi T)} - 1 (1 - e^{-\xi T}) 1_{\{|\zeta A_T^{-1} e^\xi - 1 (1 - e^{-\xi T})| > \varepsilon\}}(\zeta) ds \pi(d\xi) \\
\leq C_1 \frac{1}{\gamma - \delta} |\zeta|^{\gamma - \delta} t A_T^{-1} \int_{0}^{t} (\xi T)^{-1} (1 - e^{-\xi T}) \int_{0}^{\infty} \left( (\xi T)^{-1} (1 - e^{-\xi T}) \right)^{\gamma - \delta} \pi(d\xi) \\
+ C_3 |\zeta| t A_T^{-1} \int_{0}^{t} (\xi T)^{-1} (1 - e^{-\xi T}) \int_{0}^{\infty} \left( (\xi T)^{-1} (1 - e^{-\xi T}) \right)^{\gamma - \delta} \pi(d\xi).
\]

(41)

For the first term we proceed as in the proof of (Grahovac et al. 2018, Lemma 5.1). If \(\gamma \in (0, 1)\), then from the inequality \(x^{-1}(1 - e^{-x}) \leq 1, x > 0\), we get

\[
C_1 \frac{1}{\gamma - \delta} |\zeta|^{\gamma - \delta} t A_T^{-1} \int_{0}^{t} (\xi T)^{-1} (1 - e^{-\xi T}) \int_{0}^{\infty} \left( (\xi T)^{-1} (1 - e^{-\xi T}) \right)^{\gamma - \delta} \pi(d\xi) \\
\leq C_1 \frac{1}{\gamma - \delta} |\zeta|^{\gamma - \delta} t A_T^{-1} \int_{0}^{t} (\xi T)^{-1} (1 - e^{-\xi T}) \int_{0}^{\infty} \left( (\xi T)^{-1} (1 - e^{-\xi T}) \right)^{\gamma - \delta} \pi(d\xi) \\
\leq C_4 |\zeta|^{\gamma - \delta},
\]

22
since \( T^{\gamma-\delta-1+\delta/\gamma} k#(T)^{(-\gamma+\delta)/\gamma} \to 0 \) as \( T \to \infty \), due to \( \gamma - \delta - 1 + \delta/\gamma < 0 \). If \( \gamma \in (1, 2) \), then from the inequality \( x^{-1}(1 - e^{-x}) \leq x^{(1-\gamma)/(\gamma-\delta)} \) it follows

\[
C_1 \frac{1}{\gamma - \delta} \left| \zeta \right| \gamma - \delta t^{\gamma - \delta} A_T^{-\gamma + \delta} T^{\gamma - \delta} \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{-\xi T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_1 \frac{1}{\gamma - \delta} \left| \zeta \right| \gamma - \delta t^{\gamma - \delta} T^{\gamma - \delta - 1} A_T^{-\gamma + \delta} T^{\gamma - \delta} \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{-\xi T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_1 \frac{1}{\gamma - \delta} \left| \zeta \right| \gamma - \delta t^{\gamma - \delta} T^{\gamma - \delta} A_T^{-\gamma + \delta} k#(T)^{(-\gamma+\delta)/\gamma} \int_0^\infty \xi^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_5 |\zeta| |\gamma - \delta|.
\]

since \( T^{\delta/\gamma} k#(T)^{(-\gamma+\delta)/\gamma} \to 0 \) as \( T \to \infty \) and \( \int_0^\infty \xi^{\gamma - \delta} \pi(d\xi) < \infty \) due to (16). For \( \gamma = 1 \) case we may use the fact that \( x^{-1}(1 - e^{-x}) \leq x^{-\eta/(\gamma-\delta)}, \eta > 0 \), to obtain

\[
C_1 \frac{1}{\gamma - \delta} \left| \zeta \right| \gamma - \delta t^{\gamma - \delta} A_T^{-\gamma + \delta} T^{\gamma - \delta} \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{-\xi T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_1 \frac{1}{\gamma - \delta} \left| \zeta \right| \gamma - \delta t^{\gamma - \delta - \varepsilon} T^{-\varepsilon} k#(T)^{(-\gamma+\delta)/\gamma} \int_0^\infty \xi^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_6 |\zeta| |\gamma - \delta|.
\]

\( \circ \) Returning now to the second term (41), from the inequality \( x^{-1}(1 - e^{-x}) \leq 1, x > 0 \), we get

\[
C_3 |\zeta| t A_T^{-1} T \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{-\xi T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_3 |\zeta| t A_T^{-1} T \int_0^\infty \left( (\xi A_T^{-1} T^{-1} - 1 - e^{t T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_3 |\zeta| t A_T^{-1} T \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{t T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_3 |\zeta| t A_T^{-1} T \pi \left( (0, e^{t T T}) \right).
\]

By (15), for arbitrary \( 0 < \eta < 1 + \alpha - \gamma \), in some neighborhood of the origin it holds that \( \pi ((0, x)) \leq C_7 x^{\alpha - \eta} \). Hence we have

\[
C_3 |\zeta| t A_T^{-1} T \int_0^\infty \left( (\xi T T^{-1} - 1 - e^{t T T}) \right)^{\gamma - \delta} \pi(d\xi)
\]

\[
\leq C_8 |\zeta|^{1 + \alpha - \eta} A_T^{-1 - \alpha + \eta} T
\]

\[
= C_8 |\zeta|^{1 + \alpha - \eta} T^{1 - (1 + \alpha)/\gamma + \eta/\gamma}
\]

\[
\leq C_9 |\zeta|^{1 + \alpha - \eta}
\]

since \( 1 + \alpha > \gamma \). We conclude finally

\[
|\kappa_{\Delta x_{1,1}}(A_T^{-1} T, T)| \leq C_5 |\zeta|^{\gamma - \delta} + C_9 |\zeta|^{1 + \alpha - \eta} \leq \begin{cases} C_{10} |\zeta|^{\gamma - \delta}, & |\zeta| \leq 1, \\ C_{11} |\zeta|^{1 + \alpha - \eta}, & |\zeta| > 1. \end{cases}
\]

(42)

\( \bullet \) We now consider \( |\kappa_{\Delta x_{1,2}}(A_T^{-1} T, T)| \) in (37). Because of (39) we can write

\[
\kappa_{L_1}(\zeta) = k(\zeta)\kappa_{S_1}(\gamma_{\sigma, \rho, 0})(\zeta),
\]
with $\overline{k}$ slowly varying at zero such that $\overline{k}(\zeta) \sim k(1/\zeta)$ as $\zeta \to 0$. By (Grahovac et al. 2018, Eq. (34)) we have that

$$\kappa_{\Delta X_{1,2}^{-1} A_T^{-1} \zeta, T t} = \kappa_{\mathcal{S}_1(\gamma^1/\gamma, \sigma, \rho, 0)}(\zeta) \times \int_0^\infty \int_0^t \xi^{1-\gamma} \left(1 - e^{-\xi T(t-s)}\right) \overline{k} \frac{\left((T k^\#(T))^{-1/\gamma} \xi^{-1} (1 - e^{-\xi T(t-s)})\right)}{k^\#(T)} ds \pi(d\xi). \quad (43)$$

where $\kappa_{\mathcal{S}_1(\gamma^1/\gamma, \sigma, \rho, 0)}$ is a cumulant function of stable distribution as in (8). The definition of $k^\#$ implies that (Bingham et al. 1989, Theorem 1.5.13)

$$\frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma} \xi^{-1} (1 - e^{-\xi T(t-s)})\right)} \sim \frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma}\right)} \rightarrow 1, \quad as \ T \rightarrow \infty,$$

and due to slow variation of $\overline{k}$, for any $\zeta \in \mathbb{R}$, $\xi > 0$ and $s \in (0, t)$, as $T \rightarrow \infty$

$$\frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma} \xi^{-1} (1 - e^{-\xi T(t-s)})\right)} \equiv \frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma}\right)} \rightarrow 1. \quad (44)$$

By using Potter’s bounds (Bingham et al. 1989, Theorem 1.5.6) we have from (44) that for any $\varepsilon > 0$

$$\frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma} \xi^{-1} (1 - e^{-\xi T(t-s)})\right)} \leq C_{12} \max \left\{ \xi^{-\varepsilon}, \xi^\varepsilon \right\} \max \left\{ \zeta^{-\varepsilon}, \zeta^\varepsilon \right\},$$

for $T$ large enough. By taking $\varepsilon < \gamma$ we get

$$\xi^{1-\gamma} \left(1 - e^{-\xi T(t-s)}\right) \frac{k^\#(T)}{\overline{k} \left((T k^\#(T))^{-1/\gamma} \xi^{-1} (1 - e^{-\xi T(t-s)})\right)} \leq C_{12} \xi^{1-\gamma} \left(1 - e^{-\xi T(t-s)}\right) \gamma^{-\varepsilon} \max \left\{ \xi^{-\varepsilon}, \xi^\varepsilon \right\} \max \left\{ \zeta^{-\varepsilon}, \zeta^\varepsilon \right\} \leq C_{12} \xi^{1-\gamma} \max \left\{ \xi^{-\varepsilon}, \xi^\varepsilon \right\} \max \left\{ \zeta^{-\varepsilon}, \zeta^\varepsilon \right\}. $$

Since $\gamma < 1 + \alpha$ and (16) holds, we have

$$\int_0^\infty \int_0^t \xi^{1-\gamma} \max \left\{ \xi^{-\varepsilon}, \xi^\varepsilon \right\} ds \pi(d\xi) = t \int_0^1 \xi^{1-\gamma-\varepsilon} \pi(d\xi) + t \int_1^\infty \xi^{1-\gamma+\varepsilon} \pi(d\xi) < \infty.$$

We finally conclude from (43) that

$$\left|\kappa_{\Delta X_{1,2}^{-1} A_T^{-1} \zeta, T t}\right| \leq C_{13} \left|\kappa_{\mathcal{S}_1(\gamma^1/\gamma, \sigma, \rho, 0)}(\zeta)\right| \max \left\{ \zeta^{-\varepsilon}, \zeta^\varepsilon \right\} \leq C_{14} |\zeta|^\gamma \max \left\{ \zeta^{-\varepsilon}, \zeta^\varepsilon \right\}. \quad (45)$$

- We shall now put the terms together. By using (42) and (45) one has from (37) that

$$|\kappa_{X_1^{-1}}(A_T^{-1} \zeta, T t)| \leq \begin{cases} C_{10} |\zeta|^{\gamma-\delta} + C_{14} |\zeta|^\gamma, & |\zeta| \leq 1, \\ C_{11} |\zeta|^{1+\alpha-\eta} + C_{14} |\zeta|^{\gamma+\varepsilon}, & |\zeta| > 1. \end{cases}$$
Since $\gamma < 1 + \alpha$ and $\varepsilon$, $\delta$ and $\eta$ are arbitrary, we may choose them so that $\varepsilon < \delta < \gamma - q$ and $1 + \alpha - \eta > \gamma + \varepsilon$, hence

$$|\kappa_{X_1^*}(A_T^{-1} \zeta, Tt)| \leq \begin{cases} C_{15} |\zeta|^{\gamma - \delta}, & |\zeta| \leq 1, \\ C_{16} |\zeta|^{1 + \alpha - \eta}, & |\zeta| > 1. \end{cases} \tag{46}$$

\begin{itemize}
  \item To get the bound for the moment $\mathbb{E} |A_T^{-1} \tilde{X}_1^*(Tt)|^q$, we use \eqref{35}, \eqref{46} and

$$|\exp \kappa_{X_1^*}(A_T^{-1} \zeta, Tt)|^2 = \exp \{2 \Re \kappa_{X_1^*}(A_T^{-1} \zeta, Tt)\} \geq \exp \{-2|\kappa_{X_1^*}(A_T^{-1} \zeta, Tt)|\}, \tag{47}$$

and get

$$\mathbb{E} |A_T^{-1} \tilde{X}_1^*(Tt)|^q \leq k_q \int_{-\infty}^{\infty} \left(1 - \exp\{-2|\kappa_{X_1^*}(A_T^{-1} \zeta, Tt)|\}\right) |\zeta|^{-q-1} d\zeta$$

$$\leq k_q \int_{|\zeta| \leq 1} \left(1 - \exp\{-2C_{15} |\zeta|^{\gamma - \delta}\}\right) |\zeta|^{-q-1} d\zeta$$

$$+ k_q \int_{|\zeta| > 1} \left(1 - \exp\{-2C_{16} |\zeta|^{1 + \alpha - \eta}\}\right) |\zeta|^{-q-1} d\zeta$$

$$\leq k_q \int_{-\infty}^{\infty} \left(1 - \exp\{-2C_{15} |\zeta|^{\gamma - \delta}\}\right) |\zeta|^{-q-1} d\zeta$$

$$+ k_q \int_{-\infty}^{\infty} \left(1 - \exp\{-2C_{16} |\zeta|^{1 + \alpha - \eta}\}\right) |\zeta|^{-q-1} d\zeta.$$ \end{itemize}

By \eqref{34}, the terms on the right-hand side are $q$-th absolute moments of $(\gamma - \delta)$-stable and $(1 + \alpha - \eta)$-stable random variables with characteristic functions $\exp\{-2C_{15} |\zeta|^{\gamma - \delta}\}$ and $\exp\{-2C_{16} |\zeta|^{1 + \alpha - \eta}\}$, respectively. Since $q < \gamma - \delta$ and $q < 1 + \alpha - \eta$, both integrals are finite.

If $\gamma > 1$, we may assume that $q > 1$ and from \eqref{32} we have

$$\mathbb{E} |A_T^{-1} X_1^*(Tt)|^q \leq \mathbb{E} |A_T^{-1} \tilde{X}_1^*(Tt)|^q.$$ \text{if $\gamma \leq 1$, then from \eqref{32}}

$$\mathbb{E} |A_T^{-1} X_1^*(Tt)|^q \leq \mathbb{E} |A_T^{-1} \tilde{X}_1^*(Tt)|^q + 2 |\text{med}(A_T^{-1} X_1^*(Tt))|^q.$$ 

Since $\{A_T^{-1} X_1^*(Tt)\}$ converges in distribution, the median $\text{med}(A_T^{-1} X_1^*(Tt))$ also converges (see e.g. \cite[Lemma 21.2]{VdV2000}), hence we can bound the second term on the right. This completes the proof of uniform integrability of $\{|A_T^{-1} X_1^*(Tt)|^q\}$, hence the convergence of moments. Since the limiting process is $1/\gamma$-self-similar, from \cite[Theorem 1]{Graham2019} we conclude that

$$\tau_{X_1^*}(q) = \frac{1}{\gamma} q, \quad \text{for } q < \gamma.$$ \text{7 Proof of Lemma 4.2}

We first consider the case $q < 1 + \alpha$. The proof is similar to the proof of Lemma 4.1. We will prove that $\{|A_T^{-1} X_1^*(Tt)|^q\}$ is uniformly integrable where now $A_T = T^{1/(1+\alpha)} \mathcal{I}^\#(T) T^{1/(1+\alpha)}$. We can assume $q > 1$. From \eqref{32}, \eqref{35} and \eqref{47} it follows that

$$\mathbb{E} |A_T^{-1} X_1^*(Tt)|^q \leq \mathbb{E} |A_T^{-1} \tilde{X}_1^*(Tt)|^q \leq k_q \int_{-\infty}^{\infty} \left(1 - \exp\{-2|\kappa_{X_1^*}(A_T^{-1} \zeta, Tt)|\}\right) |\zeta|^{-q-1} d\zeta. \tag{48}$$

We now derive bound for $|\kappa_{X_1^*}(A_T^{-1} \zeta, Tt)|$. Again we use the decomposition \eqref{36} and bound $|\kappa_{X_1,1}(A_T^{-1} \zeta, Tt)|$ and $|\kappa_{X_1,2}(A_T^{-1} \zeta, Tt)|$ separately.
We consider first $\kappa_{\Delta X_{i,1}^1}(A_T^{-1}\zeta, Tt)$. From (40) we also have the following bound for $\varepsilon < 1 + \alpha - q$

$$|\kappa_{L_i}(\zeta)| \leq C_1|\zeta|^{1+\alpha-\varepsilon}.$$ 

and by using Potter’s bounds (Bingham et al. 1989, Theorem 1.5.6) we have for $0 < \delta < \varepsilon\alpha/(1 + \alpha)$

$$\tilde{\ell}(T\xi^{-1}) = \frac{\tilde{\ell}(T\xi^{-1})}{\ell(\xi^{-1})} \leq C_2 \max \left\{ T^{-\delta}, T^\delta \right\} \ell(\xi^{-1}).$$

By (15), we can write the density $p$ of $\pi$ in the form $p(x) = \alpha(\tilde{\ell}(x^{-1})x^{\alpha-1}$ with $\tilde{\ell}$ slowly varying at infinity such that $\tilde{\ell}(t) \sim \ell(t)$ as $t \to \infty$. Hence from (38) we have

$$\kappa_{\Delta X_{i,1}^1}(A_T^{-1}\zeta, Tt) = \int_0^\infty \int_{-\infty}^0 \kappa_{L_1} \left( A_T^{-1}Te^s\xi^{-1} \left(1 - e^{-\xi t}\right) \right) ds\pi(T^{-1}d\xi)$$

$$= \int_0^\infty \int_{-\infty}^0 \kappa_{L_1} \left( A_T^{-1}Te^{s}\xi^{-1} \left(1 - e^{-\xi t}\right) \right) \alpha(\tilde{\ell}(T\xi^{-1})\xi^{\alpha-1}T^{-\alpha}d\xi ds\xi.$$ 

and

$$|\kappa_{\Delta X_{i,1}^1}(A_T^{-1}\zeta, Tt)| \leq C_3|\zeta|^{1+\alpha-\varepsilon}T^{-(1+\alpha-\varepsilon)/(1+\alpha)+1+\alpha-\varepsilon-\alpha+\delta} \ell^\#(T)^{-1/(1+\alpha)}$$

$$\times \int_0^\infty \int_{-\infty}^0 e^s \left(\xi^{-1} \left(1 - e^{-\xi t}\right)\right)^{1+\alpha-\varepsilon} \ell(\xi^{-1})\xi^{\alpha-1}d\xi ds\xi.$$ 

$$\leq C_3|\zeta|^{1+\alpha-\varepsilon}T^{-\varepsilon/(1+\alpha)+\delta} \ell^\#(T)^{-1/(1+\alpha)} \int_0^\infty \ell(\xi^{-1})\xi^{\alpha-1}d\xi$$

$$\leq C_4|\zeta|^{1+\alpha-\varepsilon}.$$ 

(49)

We consider now $|\kappa_{\Delta X_{i,2}^1}(A_T^{-1}\zeta, Tt)|$. Analogous to (38) we obtain

$$\kappa_{\Delta X_{i,2}^1}(A_T^{-1}\zeta, Tt) = \int_0^\infty \int_{0}^t \kappa_{L_1} \left( A_T^{-1}\xi^{-1} \left(1 - e^{-\xi(t-s)}\right) \right) \xi Tds\pi(d\xi)$$

$$= \int_0^\infty \int_{0}^t \kappa_{L_1} \left( A_T^{-1}\xi^{-1} \left(1 - e^{-\xi(t-s)}\right) \right) \alpha(\tilde{\ell}(\xi^{-1})\xi^{\alpha}Tds\xi.$$ 

We shall assume that $\zeta > 0$, the other case is similar. Change of variables $x = \xi A_T^{-1}\xi^{-1}$ yields

$$\kappa_{\Delta X_{i,2}^1}(A_T^{-1}\zeta, Tt) = \zeta^{1+\alpha} \int_0^\infty \int_{0}^t \kappa_{L_1} \left( x \left(1 - gt(\xi, x, s)\right) \right) A_T^{-1}(x)T\tilde{\ell}(A_T\xi^{-1}) \alpha x^{-\alpha-2}dsdx$$

$$= \zeta^{1+\alpha} \int_0^\infty \int_{0}^t \kappa_{L_1} \left( x \left(1 - gt(\xi, x, s)\right) \right) \frac{\tilde{\ell}\left( T^{1/(1+\alpha)}\ell^\#(T)^{1/(1+\alpha)} x\xi^{-1}\right)}{\ell^\#(T)} \alpha x^{-\alpha-2}dsdx, (50)$$

where $gt(\xi, x, s) = e^{-x^{-1}\xi^{-1}(s-t)}$. From Potter’s bounds (Bingham et al. 1989, Theorem 1.5.6), for $0 < \eta < \min\{\gamma - 1 - \alpha, \alpha\}$ there is $C_1$ such that

$$\frac{\tilde{\ell}\left( T^{1/(1+\alpha)}\ell^\#(T)^{1/(1+\alpha)} x\xi^{-1}\right)}{\ell\left( T^{1/(1+\alpha)}\ell^\#(T)^{1/(1+\alpha)}\right)} \leq C_1 \max\{x^{-\eta}\xi^{-\eta}, x^{-\eta}\xi^{-\eta}\}.$$ 

and by the definition of de Bruijn conjugate (Bingham et al. 1989, Theorem 1.5.13)

$$\frac{\ell^\#(T)}{\ell\left( T\ell^\#(T)^{1/(1+\alpha)}\right)} \sim 1, \text{ as } T \to \infty.$$
Hence, for $T$ large enough
\[
\frac{\ell \left( T^{1/(1+\alpha)} \ell^# (T)^{1/(1+\alpha)} x \zeta^{-1} \right)}{\ell^# (T)} \leq C_2 \max \{ x^{-\eta} \zeta^{\eta}, x^{\eta} \zeta^{-\eta} \},
\]
and by inserting in (50) we get
\[
\left| \kappa \Delta X_{1,2}^* (A_T^{-1}, T t) \right| \\
\leq \alpha C_2 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t |\kappa L_1 (x (1 - g_T (\zeta, x, s)))| \max \{ x^{-\eta}, x^{\eta} \} x^{-\alpha-2} ds dx.
\]
Now we use the bound (40) valid for arbitrary $\delta > 0$ to obtain
\[
\left| \kappa \Delta X_{1,2}^* (A_T^{-1}, T t) \right| \leq C_3 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t (x (1 - g_T (\zeta, x, s)))^{\gamma - \delta} 1_{\{x (1 - g_T (\zeta, x, s)) \leq \varepsilon\}} \\
\times \max \{ x^{-\eta}, x^{\eta} \} x^{-\alpha-2} ds dx \\
+ C_4 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t (x (1 - g_T (\zeta, x, s))) 1_{\{x (1 - g_T (\zeta, x, s)) > \varepsilon\}} \\
\times \max \{ x^{-\eta}, x^{\eta} \} x^{-\alpha-2} ds dx \\
=: I_1 + I_2.
\]
We consider each term separately.

- For $I_1$ we make change of variables $y = x (1 - g_T (\zeta, x, s))$ and get
  \[
  I_1 = C_3 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t y^{\gamma - \delta} 1_{\{y \leq \varepsilon\}} \max \{ y^{-\eta}, (1 - g_T (\zeta, x, s))^{\eta}, y^{\eta} (1 - g_T (\zeta, x, s))^{-\eta} \} \\
  \times y^{-\alpha-2} (1 - g_T (\zeta, x, s))^{\alpha+1} dy \\
  \leq C_5 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t y^{\gamma - \alpha - 2 - \delta - \eta} 1_{\{y \leq \varepsilon\}} (1 - g_T (\zeta, x, s))^{\alpha+1-\eta} dy \\
  \leq C_6 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\varepsilon y^{\gamma - \alpha - 2 - \delta - \eta} dy \\
  = C_7 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \},
  \]
  where we used the fact that the integral in the last line is finite due to $\gamma > 1 + \alpha$ and the choice of $\eta$ and $\delta$.

- We now consider $I_2$. Since $x (1 - g_T (\zeta, x, s)) > \varepsilon$ implies $x > \varepsilon$, we have for $I_2$,
  \[
  I_2 \leq C_7 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_0^\infty \int_0^t x^{-\alpha-1} 1_{\{x > \varepsilon\}} \max \{ x^{-\eta}, x^{\eta} \} dx ds \\
  \leq C_8 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \} \int_\varepsilon^\infty x^{-\alpha-1+\eta} dx \\
  = C_9 \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \}.
  \]

Returning back to (51) we conclude that
\[
\left| \kappa \Delta X_{1,2}^* (A_T^{-1}, T t) \right| \leq C_{10} \zeta^{1+\alpha} \max \{ \zeta^\eta, \zeta^{-\eta} \}.
\]
From (37), (49) and (52) it follows that for $\varepsilon > 0$ and $\eta > 0$ arbitrary small there are constants $C_1, C_2 > 0$ such that

$$|\kappa_{X_1}(A_T^{-1}\zeta, Tt)| \leq \begin{cases} C_1|\zeta|^{1+\alpha-\varepsilon} + C_2|\zeta|^{1+\alpha-\eta}, & |\zeta| \leq 1, \\ C_1|\zeta|^{1+\alpha-\varepsilon} + C_2|\zeta|^{1+\alpha+\eta}, & |\zeta| > 1. \end{cases}$$

Assuming e.g. that $\varepsilon < \eta$ we have

$$|\kappa_{X_1^{-1}\zeta}(A_T^{-1}\zeta, Tt)| \leq \begin{cases} C_3|\zeta|^{1+\alpha-\eta}, & |\zeta| \leq 1, \\ C_4|\zeta|^{1+\alpha+\eta}, & |\zeta| > 1. \end{cases}$$

As in the proof of Lemma 4.1, it follows from (48) that

$$\mathbb{E} \left| A_T^{-1}X_1(Tt) \right|^q \leq k_q \int_{|\zeta|\leq 1} \left( 1 - \exp\{-2C_3|\zeta|^{1+\alpha-\eta} \} \right) |\zeta|^{q-1} d\zeta$$

$$+ k_q \int_{|\zeta|> 1} \left( 1 - \exp\{-2C_4|\zeta|^{1+\alpha+\eta} \} \right) |\zeta|^{q-1} d\zeta$$

$$\leq k_q \int_{-\infty}^{\infty} \left( 1 - \exp\{-2C_3|\zeta|^{1+\alpha-\eta} \} \right) |\zeta|^{q-1} d\zeta$$

$$+ k_q \int_{-\infty}^{\infty} \left( 1 - \exp\{-2C_4|\zeta|^{1+\alpha+\eta} \} \right) |\zeta|^{q-1} d\zeta.$$ 

The terms on the right-hand side are $q$-th absolute moments of $(1 + \alpha - \eta)$-stable and $(1 + \alpha + \eta)$-stable random variables with characteristic functions $\exp\{-2C_3|\zeta|^{1+\alpha-\eta}\}$ and $\exp\{-2C_4|\zeta|^{1+\alpha+\eta}\}$, respectively. We are considering the case $q < 1 + \alpha$, hence these moments are finite if we choose $\eta$ small enough. Hence, $\{ |A_T^{-1}X_1(Tt)|^q \}$ is uniformly integrable, the moments converge and from (Grahovac, Leonenko, Sikorskii & Taqqu 2019, Theorem 1) we have that $\tau_{X_1}(q) = q/(1 + \alpha)$ for $q < 1 + \alpha$. Since the scaling function is convex Grahovac et al. (2016), hence continuous, we obtain

$$\tau_{X_1}(q) = \frac{1}{1 + \alpha}q,$$

for $q \leq 1 + \alpha$.

- We now turn to the case $1 + \alpha < q < \gamma$ in Lemma 4.2. We will show that for arbitrary $\varepsilon > 0$

$$\mathbb{E} \left| T^{-\alpha+q-\frac{\varepsilon}{q}} X_1(T) \right|^q \leq C,$$  

(53)

for some constant $C > 0$ and $T$ large enough. This implies that $\tau_{X_1}(q) \leq q - \alpha + \varepsilon$ and completes the proof since $\varepsilon$ is arbitrary. To show (53), we will use (48) with $A_T = T^{1-\alpha/q+\varepsilon/q}$. First, by (Grahovac, Leonenko & Taqqu 2019, Lemma 5.1), we may express cumulant function of $X_1(T)$ as

$$\kappa_{A_T^{-1}X_1(T)}(\zeta) = \int_0^\infty \int_{-\infty}^0 \kappa_L \left( A_T^{-1}\zeta \xi - 1 e^{\xi s} \left( 1 - e^{-\xi T} \right) \right) ds \xi \pi(d\xi)$$

$$+ \int_0^\infty \int_0^T \kappa_L \left( A_T^{-1}\zeta \xi - 1 e^{-\xi (T-s)} \right) ds \xi \pi(d\xi).$$

28
Making change of variables and writing \( p(x) = \alpha \tilde{\ell}(x^{-1})x^{\alpha - 1} \), with \( \tilde{\ell}(t) \sim \ell(t) \) as \( t \to \infty \), yields

\[
\kappa_{\alpha^{-1}X_1(T)}(\zeta) = \int_0^\infty \int_{-\infty}^0 \kappa_{L_1}(T_{\alpha - \frac{\alpha}{q}}(1 - e^{-x}))dsx^{-1}T^{-1}\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 \kappa_{L_1}(T_{\alpha - \frac{\alpha}{q}}(1 - e^{-x}))dsx\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 \kappa_{L_1}(T_{\alpha - \frac{\alpha}{q}}(1 - e^{-x}))dsx\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 \kappa_{L_1}(T_{\alpha - \frac{\alpha}{q}}(1 - e^{-x}))dsx\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 \kappa_{L_1}(T_{\alpha - \frac{\alpha}{q}}(1 - e^{-x}))dsx\pi(T^{-1}dx)
\]

Take \( \delta > 0 \) such that \( q + \delta < \gamma \) and \( \delta < \frac{\alpha q}{\alpha - 1} \) and note that from (40) we have the bound

\[
|\kappa_{L_1}(\zeta)| \leq C|\zeta|^{q+\delta}, \quad \zeta \in \mathbb{R}.
\]

Hence,

\[
|\kappa_{\alpha^{-1}X_1(T)}(\zeta)| \leq C|\zeta|^{q+\delta} \int_0^\infty \int_{-\infty}^0 x^{\alpha - q - \delta}e^{(q+\delta)x}\left(1 - e^{-x}\right)^{q+\delta}d\alpha \tilde{\ell}(T^{-1})T\left(\alpha - \frac{\alpha}{q}\right)\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 x^{\alpha - q - \delta}e^{(q+\delta)x}\left(1 - e^{-x}\right)^{q+\delta}d\alpha \tilde{\ell}(T^{-1})T\left(\alpha - \frac{\alpha}{q}\right)\pi(T^{-1}dx)
\]

\[
= \int_0^\infty \int_{-\infty}^0 x^{\alpha - q - \delta}e^{(q+\delta)x}\left(1 - e^{-x}\right)^{q+\delta}d\alpha \tilde{\ell}(T^{-1})T\left(\alpha - \frac{\alpha}{q}\right)\pi(T^{-1}dx)
\]

Note that by the choice of \( \delta \), we have \( \left(\frac{\alpha}{q} - \frac{\alpha}{q}\right)(q + \delta) - \alpha < 0 \). By Potter’s bounds (Bingham et al. 1989, Theorem 1.5.6), for any \( \eta > 0 \) we have that \( \tilde{\ell}(T^{-1}) \leq C_1 \tilde{\ell}(x^{-1})T^\eta \). Taking \( \eta < \alpha - \left(\frac{\alpha}{q} - \frac{\alpha}{q}\right)(q + \delta) \) yields

\[
|\kappa_{\alpha^{-1}X_1(T)}(\zeta)| \leq C_2 \int_0^\infty \int_{-\infty}^0 x^{\alpha - q - \delta}e^{(q+\delta)x}\left(1 - e^{-x}\right)^{q+\delta}d\alpha \tilde{\ell}(x^{-1})dx
\]

\[
= \int_0^\infty \int_{-\infty}^0 x^{\alpha - q - \delta}e^{(q+\delta)x}\left(1 - e^{-x}\right)^{q+\delta}d\alpha \tilde{\ell}(x^{-1})dx
\]

\[
\leq C_2|\zeta|^{q+\delta} \int_0^\infty x^{\alpha - q - \delta}d\alpha \tilde{\ell}(x^{-1})dx
\]

\[
\leq C_2|\zeta|^{q+\delta} \int_0^\infty x^{\alpha - q - \delta}d\alpha \tilde{\ell}(x^{-1})dx
\]

\[
\leq C_2|\zeta|^{q+\delta} + C_4|\zeta|^{q+\delta} \int_0^\infty x\pi(dx)
\]

where we have used the inequality \( x^{-1}(1 - e^{-x}) \leq 1, x > 0 \), (16) and the fact that \( \pi \) is probability measure. Now we use (48) to get that

\[
\mathbb{E}\left|T^{-1+\frac{\alpha}{q}-\frac{\alpha}{q}}X_1(T)\right|^q \leq k_q \int_{-\infty}^\infty \left(1 - \exp\{-2C_5|\zeta|^{q+\delta}\}\right)|\zeta|^{-q-1}d\zeta.
\]

The right hand side corresponds to \( q \)-th absolute moment of \( (q + \delta) \)-stable random variable which is finite. Hence, (53) holds and this completes the proof.
Acknowledgements

Nikolai N. Leonenko was supported in particular by Cardiff Incoming Visiting Fellowship Scheme, International Collaboration Seedcorn Fund, Australian Research Council’s Discovery Projects funding scheme (project DP160101366) and the project MTM2015-71839-P of MINECO, Spain (co-funded with FEDER funds). Murad S. Taqqu was supported in part by the Simons foundation grant 569118 at Boston University.

References

Barndorff-Nielsen, O. E. (1997), ‘Processes of normal inverse Gaussian type’, Finance and Stochastics 2(1), 41–68.

Barndorff-Nielsen, O. E. (2001), ‘Superposition of Ornstein–Uhlenbeck type processes’, Theory of Probability & Its Applications 45(2), 175–194.

Barndorff-Nielsen, O. E., Benth, F. E. & Veraart, A. E. D. (2015), ‘Recent advances in ambit stochastics with a view towards tempo-spatial stochastic volatility/intermittency’, Banach Center Publications 104(1), 25–60.

Barndorff-Nielsen, O. E., Benth, F. E. & Veraart, A. E. D. (2018), Ambit Stochastics, Springer International Publishing.

Barndorff-Nielsen, O. E. & Leonenko, N. N. (2005), ‘Spectral properties of superpositions of Ornstein-Uhlenbeck type processes’, Methodology and Computing in Applied Probability 7(3), 335–352.

Barndorff-Nielsen, O. E., Pérez-Abreu, V. & Thorbjørnsen, S. (2013), ‘Lévy mixing’, ALEA 10(2), 1013–1062.

Barndorff-Nielsen, O. E. & Shephard, N. (2001), ‘Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics’, Journal of the Royal Statistical Society: Series B (Statistical Methodology) 63(2), 167–241.

Barndorff-Nielsen, O. E. & Stelzer, R. (2011), ‘Multivariate supOU processes’, The Annals of Applied Probability 21(1), 140–182.

Barndorff-Nielsen, O. E. & Stelzer, R. (2013), ‘The multivariate supOU stochastic volatility model’, Mathematical Finance 23(2), 275–296.

Barndorff-Nielsen, O. E. & Veraart, A. E. (2013), ‘Stochastic volatility of volatility and variance risk premia’, Journal of Financial Econometrics 11(1), 1–46.

Bingham, N. H., Goldie, C. M. & Teugels, J. L. (1989), Regular Variation, Cambridge University Press.

de Gennes, P.-G. (1979), Scaling Concepts in Polymer Physics, Cornell University Press.

Fasen, V. & Kluppelberg, C. (2007), Extremes of supOU processes, in ‘Stochastic Analysis and Applications: The Abel Symposium 2005’, Vol. 2, Springer Science & Business Media, pp. 339–359.

Frisch, U. (1995), Turbulence: The Legacy of A.N. Kolmogorov, Cambridge University Press, Cambridge.
Grahovac, D., Jia, M., Leonenko, N. & Taufer, E. (2015), ‘Asymptotic properties of the partition function and applications in tail index inference of heavy-tailed data’, *Statistics* **49**(6), 1221–1242.

Grahovac, D., Leonenko, N. N., Sikorskii, A. & Taqqu, M. S. (2019), ‘The unusual properties of aggregated superpositions of Ornstein-Uhlenbeck type processes’, *Bernoulli*. In press.

Grahovac, D., Leonenko, N. N., Sikorskii, A. & Tešnjak, I. (2016), ‘Intermittency of superpositions of Ornstein–Uhlenbeck type processes’, *Journal of Statistical Physics* **165**(2), 390–408.

Grahovac, D., Leonenko, N. N. & Taqqu, M. S. (2018), ‘The multifaceted behavior of integrated supOU processes: The infinite variance case’, *Preprint*. arXiv:1806.09811.

Grahovac, D., Leonenko, N. N. & Taqqu, M. S. (2019), ‘Limit theorems, scaling of moments and intermittency for integrated finite variance supOU processes’, *Stochastic Processes and their Applications*. In press.

Gut, A. (2013), *Probability: A Graduate Course*, Springer Science & Business Media.

Heyde, C. C. (2009), ‘Scaling issues for risky asset modelling’, *Mathematical Methods of Operations Research* **69**(3), 593–603.

Ibragimov, I. & Linnik, Y. V. (1971), *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff.

Kelly, B. C., Treu, T., Malkan, M., Pancoast, A. & Woo, J.-H. (2013), ‘Active galactic nucleus black hole mass estimates in the era of time domain astronomy’, *The Astrophysical Journal* **779**(2), 187.

Kyprianou, A. E. (2014), *Fluctuations of Lévy Processes with Applications: Introductory Lectures*, Springer Science & Business Media.

Lukacs, E. (1970), *Characteristic Functions*, Hafner Publishing Company.

Nguyen, M. & Veraart, A. E. (2018), ‘Bridging between short-range and long-range dependence with mixed spatio-temporal Ornstein–Uhlenbeck processes’, *Stochastics* pp. 1–30.

Pipiras, V. & Taqqu, M. S. (2017), *Long-Range Dependence and Self-Similarity*, Cambridge University Press, Cambridge, UK.

Podolskij, M. (2015), Ambit fields: Survey and new challenges, in R. H. Mena, J. C. Pardo, V. Rivero & G. Uribe Bravo, eds, ‘XI Symposium on Probability and Stochastic Processes’, Springer International Publishing, Cham, pp. 241–279.

Rajput, B. S. & Rosinski, J. (1989), ‘Spectral representations of infinitely divisible processes’, *Probability Theory and Related Fields* **82**(3), 451–487.

Sato, K.-i. (1999), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, UK.

Stelzer, R., Tosstorff, T. & Wittlinger, M. (2015), ‘Moment based estimation of supOU processes and a related stochastic volatility model’, *Statistics & Risk Modeling* **32**(1), 1–24.

Van der Vaart, A. W. (2000), *Asymptotic Statistics*, Vol. 3, Cambridge University Press.

von Bahr, B. & Esseen, C.-G. (1965), ‘Inequalities for the $r$-th absolute moment of a sum of random variables, $1 \leq r \leq 2$', *The Annals of Mathematical Statistics* **36**(1), 299–303.

Zel’dovich, Y. B., Molchanov, S., Ruzmaǐkin, A. & Sokolov, D. D. (1987), ‘Intermittency in random media’, *Soviet Physics Uspekhi* **30**(5), 353.