Acyclic resolutions for arbitrary groups

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Abstract

We prove that for every abelian group $G$ and every compactum $X$ with $\dim_G X \leq n \geq 2$ there is a $G$-acyclic resolution $r: Z \rightarrow X$ from a compactum $Z$ with $\dim_G Z \leq n$ and $\dim Z \leq n + 1$ onto $X$.

Keywords: cohomological dimension, acyclic resolution

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1 Introduction

Spaces denoted by $X$ are assumed to be separable metrizable. A compactum is a metrizable compact space.

Let $G$ be an abelian group. A space $X$ has the cohomological dimension $\dim_G X \leq n$ if $\check{H}^{n+1}(X, A; G) = 0$ for every closed subset $A$ of $X$. The case $G = \mathbb{Z}$ is an important special case of cohomological dimension. It was known long ago that $\dim X = \dim_{\mathbb{Z}} X$ if $X$ is finite dimensional. Solving an outstanding problem in cohomological dimension theory Dranishnikov constructed in 1987 an infinite dimensional compactum of $\dim_{\mathbb{Z}} = 3$. A few years earlier a deep relation between $\dim_{\mathbb{Z}}$ and $\dim$ was established by the Edwards cell-like resolution theorem [4, 8] saying that a compactum of $\dim_{\mathbb{Z}} \leq n$ can be obtained as the image of a cell-like map defined on a compactum of $\dim \leq n$. A compactum $X$ is cell-like if any map $f: X \rightarrow K$ from $X$ to a CW-complex $K$ is null homotopic. A map is cell-like if its fibers are cell-like. The reduced Čech cohomology groups of a cell-like compactum are trivial with respect to any group $G$.

Acyclic resolutions originated in the Edwards cell-like resolution. A compactum $X$ is $G$-acyclic if $\check{H}^*(X; G) = 0$ and a map is $G$-acyclic if its fibers are $G$-acyclic. Thus a cell-like map is $G$-acyclic with respect to any abelian group $G$. By the Vietoris-Begle theorem a $G$-acyclic map cannot raise the cohomological dimension $\dim_G$. Dranishnikov proved the following important

Theorem 1.1 ([2, 3]) Let $X$ be a compactum with $\dim_{\mathbb{Q}} X \leq n$, $n \geq 2$. Then there are a compactum $Z$ with $\dim_{\mathbb{Q}} Z \leq n$ and $\dim Z \leq n + 1$ and a $\mathbb{Q}$-acyclic map $r: Z \rightarrow X$ from $Z$ onto $X$. 
It has been widely conjectured that Theorem 1.1 holds for any abelian group \( G \). A substantial progress in solving this conjecture was made by Koyama and Yokoi [5] who proved it for a large class of groups including \( \mathbb{Q} \) and very recently by Rubin and Schapiro [7] who settled the case \( G = \mathbb{Z}_{p^\infty} \).

The purpose of this note is to finally answer this conjecture affirmatively by proving

**Theorem 1.2** Let \( G \) be an abelian group and let \( X \) be a compactum with \( \dim_G X \leq n, n \geq 2 \). Then there are a compactum \( Z \) with \( \dim_G Z \leq n \) and \( \dim Z \leq n + 1 \) and a \( G \)-acyclic map \( r : Z \to X \) from \( Z \) onto \( X \).

In general the dimension \( n + 1 \) of \( Z \) in Theorem 1.2 is best possible [5]. However, it is unknown if the dimension of \( Z \) in Theorem 1.1 can be reduced to \( n \). In this connection let us also mention the following interesting result of Dranishnikov.

**Theorem 1.3** ([1]) Let \( X \) be a compactum with \( \dim_{\mathbb{Z}_p} X \leq n \). Then there are a compactum \( Z \) with \( \dim Z \leq n \) and a \( \mathbb{Z}_p \)-acyclic map \( r : Z \to X \) from \( Z \) onto \( X \).

Our proof of Theorem 1.2 essentially uses Dranishnikov’s idea of constructing a \( \mathbb{Q} \)-acyclic resolution presented in [3] and involves some methods of [6]. The proof is self-contained and does not rely on previous results concerning acyclic resolutions. The paper [3] is an excellent source of basic information on cohomological dimension theory.

## 2 Preliminaries

All groups below are abelian and functions between groups are homomorphisms. \( \mathcal{P} \) stands for the set of primes. For a non-empty subset \( \mathcal{A} \) of \( \mathcal{P} \) let \( S(\mathcal{A}) = \{p_1^{n_1}p_2^{n_2}...p_k^{n_k} : p_i \in \mathcal{A}, n_i \geq 0\} \) be the set of positive integers with prime factors from \( \mathcal{A} \) and for the empty set define \( S(\emptyset) = \{1\} \). Let \( G \) be a group and \( g \in G \). We say that \( g \) is \( \mathcal{A} \)-torsion if there is \( n \in S(\mathcal{A}) \) such that \( ng = 0 \) and \( g \) is \( \mathcal{A} \)-divisible if for every \( n \in S(\mathcal{A}) \) there is \( h \in G \) such that \( nh = g \). \( \text{Tor}_A G \) is the subgroup of the \( \mathcal{A} \)-torsion elements of \( G \). \( G \) is \( \mathcal{A} \)-torsion if \( G = \text{Tor}_A G \), \( G \) is \( \mathcal{A} \)-torsion free if \( \text{Tor}_A G = 0 \) and \( G \) is \( \mathcal{A} \)-divisible if every element of \( G \) is \( \mathcal{A} \)-divisible.

**Proposition 2.1**

(i) If \( G \) is \( \mathcal{A} \)-torsion then \( G \) is \( (\mathcal{P} \setminus \mathcal{A}) \)-divisible and \( (\mathcal{P} \setminus \mathcal{A}) \)-torsion free.

(ii) A factor group of an \( \mathcal{A} \)-divisible group is \( \mathcal{A} \)-divisible and a factor group of an \( \mathcal{A} \)-torsion group is \( \mathcal{A} \)-torsion.

(iii) The direct sum of \( \mathcal{A} \)-divisible groups is \( \mathcal{A} \)-divisible and the direct sum of \( \mathcal{A} \)-torsion groups is \( \mathcal{A} \)-torsion.

Let \( f : G \to H \) be a homomorphism of groups \( G \) and \( H \) and let \( H \) be \( \mathcal{B} \)-torsion. Then \( G/\text{Tor}_B G \) is
Proof. The proof of (i), (ii), (iii) is obvious.

Let \( \phi : G \to G/\text{Tor}_B G \) be the projection and \( \phi(x) = y \). Then there is \( n \in S(B) \) such that \( n f(x) = f(nx) = 0 \) and hence \( nx \in \ker f \).

(iv) Let \( m \in S(A) \). Since \( B \subset A \), \( nm \in S(A) \). Then there is \( z \in \ker f \) such that \( nmz = nx \). Hence \( n(mz - x) = 0 \) and therefore \( \phi(mz - x) = 0 \). Thus \( m \phi(z) = \phi(x) = y \) and \( G/\text{Tor}_B G \) is \( A \)-divisible.

(v) By (i) \( \ker f \) is \((P \setminus A)\)-divisible and therefore there is \( z \in \ker f \) such that \( nz = nx \). Then \( n(z - x) = 0 \) and there is \( m \in S(A) \) such that \( mz = 0 \). Hence \( \phi(z) = \phi(x) = y \) and \( my = \phi(mz) = 0 \) and (v) follows.

(vi) By (v) \( G/\text{Tor}_B G \) is \( A \)-torsion. By (i) \( \ker f \) is \((P \setminus A)\)-divisible and since \( \ker f \) is \( A \)-divisible, \( \ker f \) is \( P \)-divisible. Then by (iv) \( G/\text{Tor}_B G \) is \( A \)-divisible. \( \square \)

The notation \( e - \dim X \leq Y \) is used to indicate the property that every map \( f : A \to Y \) of a closed subset \( A \) of \( X \) into \( Y \) extends over \( X \). It is known that \( \dim G X \leq n \) if and only if \( e - \dim X \leq K(G, n) \) where \( K(G, n) \) is the Eilenberg-Mac Lane complex of type \((G, n)\). A map between CW-complexes is combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain.

Let \( M \) be a simplicial complex and let \( M[n] \) be the \( n \)-skeleton of \( M \) (= the union of all simplexes of \( M \) of \( \dim \leq n \)). By a resolution \( EW(M, n) \) of \( M \) we mean a CW-complex \( EW(M, n) \) and a combinatorial map \( \omega : EW(M, n) \to M \) such that \( \omega \) is 1-to-1 over \( M[n] \). The resolution is said to be suitable for a map \( f : M[n] \to Y \) if the map \( f \circ \omega_{|W_{n-1}(M[n])} \) extends to a map from \( EW(M, n) \) to \( Y \). The resolution is said to be suitable for a compactum \( X \) if for every simplex \( \Delta \) of \( M \), \( e - \dim X \leq \omega^{-1}(\Delta) \).

Note that if \( \omega : EW(M, n) \to M \) is a resolution suitable for \( X \) then for every map \( \phi : X \to M \) there is a map \( \psi : X \to EW(M, n) \) such that for every simplex \( \Delta \) of \( M \), \( (\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta \). We will call \( \psi \) a combinatorial lifting of \( \phi \).

Following [6] we will construct a resolution of an \((n + 1)\)-dimensional simplicial complex \( M \) which is suitable for \( X \) with \( \dim_G X \leq n \) and a map \( f : M[n] \to K(G, n) \). In the sequel we will refer to this resolution as the standard resolution for \( f \). Fix a CW-structure on \( K(G, n) \) and assume that \( f \) is cellular. We will obtain a CW-complex \( EW(M, n) \) from \( M[n] \) by attaching the mapping cylinder of \( f|_{\partial \Delta} \) to \( \partial \Delta \) for every \((n + 1)\)-simplex \( \Delta \) of \( M \). Let \( \omega : EW(M, n) \to M \) be the projection sending each mapping cylinder to the corresponding \((n + 1)\)-simplex \( \Delta \) such that \( \omega \) is the identity map on \( \partial \Delta \), the \( K(G, n) \)-part of the cylinder is sent to the barycenter of \( \Delta \) and \( \omega \) is 1-to-1 on the rest of the cylinder. Clearly \( f|_{\partial \Delta} \) extends over its mapping cylinder and therefore \( f \circ \omega_{|W_{n-1}(M[n])} \) extends over \( EW(M, n) \). For each simplex \( \Delta \) of \( M \), \( \omega^{-1}(\Delta) \) is either contractible or homotopy equivalent to \( K(G, n) \). Define a
CW-structure on $EW(M, n)$ turning $\omega$ into a combinatorial map. Thus we get that the standard resolution is indeed a resolution suitable for both $X$ and $f$. Note that from the construction of the standard resolution $\omega : EW(M, n) \to M$ it follows that for every subcomplex $T$ of $M$, $\omega^{-1}(T)$ is the standard resolution of $T$ for $f|_{T^{[n]}}$ and $\omega^{-1}(T)$ is $(n - 1)$-connected if $T$ is $(n - 1)$-connected.

Proposition 2.2 Let $M$ be an $(n + 1)$-dimensional finite simplicial complex and let $\omega : EW(M, n) \to M$ be the standard resolution for $f : M^{[n]} \to K(G, n)$, $n \geq 2$. Then for $\omega_* : H_n(EW(M, n)) \to H_n(M)$, $\ker \omega_*$ is a factor group of the direct sum $\oplus G$ of finitely many $G$.

Proof. Inside each $(n+1)$-simplex of $M$ cut a small closed ball around the barycenter and not touching the boundary and split $M$ into two subspaces $M = M_1 \cup M_2$ where $M_1 =$ the closure of the complement to the union of the balls and $M_2 =$ the union of the balls. Then $\omega$ is 1-to-1 over $M_1$, $H_{n-1}(M_1 \cap M_2) = 0$, $H_n(M_2) = 0$ and the preimage under $\omega$ of each ball is homotopy equivalent to $K(G, n)$ and hence $H_n(\omega^{-1}(M_2))$ is the direct sum $\oplus G$ of finitely many $G$. Consider the Mayer-Vietoris sequences for the pairs $(M_1, M_2)$ and $(\omega^{-1}(M_1), \omega^{-1}(M_2))$, in which we identify $M_1$ and $M_1 \cap M_2$ with $\omega^{-1} M_1$ and $\omega^{-1}(M_1 \cap M_2)$ respectively.

From the Mayer-Vietoris sequences it follows that $j_* (H_n(\omega^{-1}(M_1)) \oplus H_n(\omega^{-1}(M_2))) = H_n(\omega^{-1}(M_1 \cup M_2))$ and $j_* (0 \oplus H_n(\omega^{-1}(M_2))) \subset \ker \omega_*$. Let us show that $j_* (0 \oplus H_n(\omega^{-1}(M_2))) \subset \ker \omega_*$. Let $j_* (a \oplus b) \in \ker \omega_*$. Then in the Mayer-Vietoris sequence for the pair $(M_1, M_2)$, $i_* (a \oplus 0) = 0$ and therefore there is $c \in H_n(M_1 \cap M_2)$ such that $i_*(c) = a \oplus 0$. Then in the Mayer-Vietoris sequence for the pair $(\omega^{-1}(M_1), \omega^{-1}(M_2))$, $i_* (c) = a \oplus d$ and $j_*(a \oplus d) = 0$. Thus $j_* (a \oplus b) = j_* (0 \oplus (b - d))$ and therefore $j_* (0 \oplus H_n(\omega^{-1}(M_2))) = \ker \omega_*$. Recall that $H_n(\omega^{-1}(M_2)) = \oplus G$ and the proposition follows. □

Proposition 2.3 Let $M = M_1 \cup M_2$ be a $CW$-complex with subcomplexes $M_1$ and $M_2$ such that $M_1, M_2$ and $M_1 \cap M_2$ are $(n - 1)$-connected, $n \geq 2$. Then $M$ is $(n - 1)$-connected and

(i) $H_n(M)$ is $A$-divisible if $H_n(M_1)$ and $H_n(M_2)$ are $A$-divisible;

(ii) $H_n(M)$ is $A$-torsion if $H_n(M_1)$ and $H_n(M_2)$ are $A$-torsion.

Proof. The connectedness of $M$ follows from van Kampen and Hurewicz’s theorems and the Mayer-Vietoris sequence. (i) and (ii) follow from the Mayer-Vietoris sequence and (ii) and (iii) of Proposition 2.1. □

Let $X$ be a compactum and let $\sigma(G)$ be the Bockstein basis of a group $G$. By Bockstein’s theory $\dim_G X \leq n$ if and only if $\dim_E X \leq n$ for every $E \in \sigma(G)$. Denote:

$\mathcal{T}(G) = \{ p \in \mathcal{P} : \mathbb{Z}_p \in \sigma(G) \}$;

$\mathcal{T}_\infty(G) = \{ p \in \mathcal{P} : \mathbb{Z}_{p^{\infty}} \in \sigma(G) \}$. 


\[D(G) = P \text{ if } Q \in \sigma(G) \text{ and } D(G) = P \setminus \{p \in P : Z(p) \in \sigma(G)\} \text{ otherwise} ;
\]
\[F(G) = D(G) \setminus (T(G) \cup T_{\infty}(G)).\]

Note that \(T(G), T_{\infty}(G)\) and \(F(G)\) are disjoint and \(G\) is \(F(G)\)-torsion free.

**Proposition 2.4** Let \(X\) be a compactum and let \(G\) be a group such that \(G/\text{Tor}G \neq 0\) and \(\dim_G X \leq n\). Then \(\dim_E X \leq n\) for every group \(E\) such that \(E\) is \(D(G)\)-divisible and \(F(G)\)-torsion free.

**Proof.** The proof is based on Bockstein’s theorem and inequalities.

If \(p \in P \setminus D(G)\) and therefore \(Z(p) \in \sigma(G)\) and \(\dim_{Z(p)} X \leq \dim_{Z(p)} X \leq n\).

If \(p \in P \setminus D(G)\) and \(\dim_{Z(p)} X \leq n\) or \(p \in D(G) \setminus F(G)\) and then either \(p \in T(G)\) and \(\dim_{Z(p)} X \leq \dim_{Z(p)} X \leq n\).

If \(p \in T(G)\) and \(\dim_{Z(p)} X \leq n\) or \(p \in T_{\infty}(G)\) and \(\dim_{Z(p)} X \leq n\).

If \(p \in P \setminus D(G)\) and therefore \(\dim_{Z(p)} X \leq n\).

If \(Q \in \sigma(E)\) then consider the following cases:

(i) \(D(G) = P\). Then since \(G/\text{Tor}G \neq 0\), \(Q \in \sigma(G)\) and therefore \(\dim_Q X \leq n\) (this is the only place where we use that \(G/\text{Tor}G \neq 0\));

(ii) there is \(p \in P \setminus D(G)\). Then \(\dim_Q X \leq \dim_{Z(p)} X \leq n\).

\[\square\]

### 3 Proof of Theorem 1.2

Represent \(X\) as the inverse limit \(X = \lim \limits_{\leftarrow} \{K_i, h_i\}\) of finite simplicial complexes \(K_i\) with combinatorial bonding maps \(h_{i+1} : K_{i+1} \rightarrow K_i\) onto and the projections \(p_i : X \rightarrow K_i\) such that for every simplex \(\Delta\) of \(K_i\), \(\text{diam}(p_i^{-1}(\Delta)) \leq 1/i\). Following A. Dranishnikov [3] we construct by induction finite CW-complexes \(L_i\) and maps \(g_{i+1} : L_{i+1} \rightarrow L_i, \alpha_i : L_i \rightarrow K_i\) such that

(a) \(L_i\) is \((n+1)\)-dimensional and obtained from \(K_i^{[n+1]}\) by replacing some \((n+1)\)-simplexes by \((n+1)\)-cells attached to the boundary of the replaced simplexes by a map of degree \(\in S(F(G))\). Then \(\alpha_i\) is a projection of \(L_i\) taking the new cells to the original ones such that \(\alpha_i \cup 1\) over \(K_i^{[n]}\). We define a simplicial structure on \(L_i\) for which \(\alpha_i\) is a combinatorial map and refer to this simplicial structure while constructing resolutions of \(L_i\). Note that for \(F(G) = \emptyset\) we don’t replace simplexes of \(K_i^{[n+1]}\) at all;

(b) the maps \(h_i, g_i\) and \(\alpha_i\) combinatorially commute. By this we mean that for every simplex \(\Delta\) of \(K_i\), \((\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta\).

We will construct \(L_i\) in such a way that \(Z = \lim \limits_{\leftarrow} \{L_i, g_i\}\) will be of \(\dim_G \leq n\) and \(Z\) will admit a \(G\)-acyclic map onto \(X\).
Set \( L_1 = K^{[n+1]}_i \) with \( \alpha_i : L_1 \to K_1 \) the embedding and assume that the construction is completed for \( i \). Let \( E \in \sigma(G) \) and let \( f : L_i^{[n]} \to K(E, n) \) be a cellular map. Let \( \omega_L : EW(L_i, n) \to L_i \) be the standard resolution of \( L_i \) for \( f \). We are going to construct from \( EW(L_i, n) \) a resolution of \( K_i \) suitable for \( X \). On the first step of the construction we will obtain from \( EW(L_i, n) \) a resolution \( \omega_{n+1} : EW(K^{[n+1]}_i, n) \to K^{[n+1]}_i \) such that \( EW(L_i, n) \) is a subcomplex of \( EW(K^{[n+1]}_i, n) \) and \( \omega_{n+1} \) extends \( \alpha_i \circ \omega_L \). On the second step we will construct resolutions \( \omega_j : EW(K^{[j]}_i, n) \to K^{[j]}_i \), \( n+2 \leq j \leq \dim K_i \) such that \( EW(K^{[j]}_i, n) \) is a subcomplex of \( EW(K^{[j+1]}_i, n) \) and \( \omega_{j+1} \) extends \( \omega_j \). The construction is carried out as follows.

Step 1. For every simplex \( \Delta \) of \( K_i \) of \( \dim = n+1 \) consider separately the subcomplex \( (\alpha_i \circ \omega_L)^{-1}(\Delta) \) of \( EW(L_i, n) \). Note that from (a) and the properties of the standard resolution it follows that the preimage under \( \alpha_i \circ \omega_L \) of an \( (n-1) \)-connected subcomplex of \( K_i \) is \((n-1)\)-connected. Then \( (\alpha_i \circ \omega_L)^{-1}(\Delta) \) is \((n-1)\)-connected. Enlarge \( (\alpha_i \circ \omega_L)^{-1}(\Delta) \) by attaching cells of \( \dim = n+1 \) in order to kill \( \operatorname{Tor}_{\mathcal{F}(G)} H_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) \) and attaching cells of \( \dim > n+1 \) in order to kill all homotopy groups of the enlarged subcomplex in \( \dim > n \). Define \( EW(K^{[n+1]}_i, n) \) as \( EW(L_i, n) \) with all the cells attached for all \((n+1)\)-dimensional simplexes \( \Delta \) of \( K_i \) and let a map \( \omega_{n+1} : EW(K^{[n+1]}_i, n) \to K^{[n+1]}_i \) extend \( \alpha_i \circ \omega_L \) by sending the interior points of the attached cells to the interior of the corresponding \( \Delta \).

Step 2. Assume that a resolution \( \omega_j : EW(K^{[j]}_i, n) \to K^{[j]}_i \), \( n+1 \leq j < \dim K_i \) is constructed such that the \( n \)-skeleton of \( EW(K^{[j]}_i, n) \) coincides with the \( n \)-skeleton of \( EW(L_i, n) \). For every simplex \( \Delta \) of \( K_i \) of \( \dim = j+1 \) consider separately the subcomplex \( \omega_j^{-1}(\partial \Delta) \) of \( EW(K^{[j]}_i, n) \). Then the \( n \)-skeleton of \( \omega_j^{-1}(\partial \Delta) \) coincides with the \( n \)-skeleton of \( (\alpha_i \circ \omega_L)^{-1}(\partial \Delta) \) and therefore \( \omega_j^{-1}(\partial \Delta) \) is \((n-1)\)-connected. Enlarge \( \omega_j^{-1}(\partial \Delta) \) by attaching cells of \( \dim = n+1 \) in order to kill \( \operatorname{Tor}_{\mathcal{F}(G)} H_n(\omega_j^{-1}(\partial \Delta)) \) and attaching cells of \( \dim > n+1 \) in order to kill all homotopy groups of the enlarged subcomplex in \( \dim > n \). Define \( EW(K^{[j+1]}_i, n) \) as \( EW(K^{[j]}_i, n) \) with all the cells attached for all \((j+1)\)-simplexes of \( K_i \) and let a map \( \omega_{j+1} : EW(K^{[j+1]}_i, n) \to K^{[j+1]}_i \) extend \( \omega_j \) by sending the interior points of the attached cells to the interior of the corresponding \( \Delta \).

Finally denote \( EW(K_i, n) = EW(K^{[m]}_i, n) \) and \( \omega = \omega_m : EW(K_i, n) \to K_i \) where \( m = \dim K_i \). From the construction it follows that the \( n \)-skeleton of \( EW(K_i, n) \) is contained in \( EW(L_i, n) \) and for every simplex \( \Delta \) of \( K_i \), \( \omega^{-1}(\Delta) \) is contractible if \( \dim \Delta \leq n \) and \( \omega^{-1}(\Delta) \) is homotopy equivalent to \( K/H_n(\omega^{-1}(\Delta)), n \) if \( \dim \Delta > n+1 \).

Let us show that \( EW(K_i, n) \) is suitable for \( X \). In order to verify that \( \dim H_n(\omega^{-1}(\Delta)) X \leq \).
n for every simplex $\Delta$ of $K_i$ of $\dim \geq n+1$ we first consider Step 1 of the construction. Let $\Delta$ be an $(n+1)$-dimensional simplex of $K_i$. By $\omega_L|_\alpha$ we will denote the map $\omega_L|_{(\alpha_1, \omega_L)^{-1}(\Delta)} : (\alpha_1 \circ \omega_L)^{-1}(\Delta) \longrightarrow \alpha_1^{-1}(\Delta)$ with the range restricted to $\alpha_1^{-1}(\Delta)$. Note that by (a), $H_\alpha(\alpha_1^{-1}(\Delta))$ is $\mathcal{F}(G)$-torsion and $H_\alpha((\alpha_1 \circ \omega_L)^{-1}(\Delta)) = H_n((\alpha_1 \circ \omega_L)^{-1}(\Delta))/\text{Tor}_{\mathcal{F}(G)}H_n((\alpha_1 \circ \omega_L)^{-1}(\Delta))$. Let $(\omega_L|_{_\alpha})_\ast : H_n((\alpha_1 \circ \omega_L)^{-1}(\Delta)) \longrightarrow H_n(\alpha_1^{-1}(\Delta))$. Consider the following cases.

**Case 1-1.** $E = \mathbb{Z}_p$. By Proposition 2.2 $\ker(\omega_L|_{_\alpha})_\ast$ is $p$-torsion. Then since $p$ is not in $\mathcal{F}(G)$, by Proposition 2.1, (v), $H_n(\omega_1^{-1}(\Delta))$ is $p$-torsion and by Bockstein’s theorem $\dim_{H_n(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p} X \leq n$.

**Case 1-2.** $E = \mathbb{Z}_p^\infty$. By Proposition 2.2 $\ker(\omega_L|_{_\alpha})_\ast$ is $p$-torsion and $p$-divisible. Then since $p$ is not in $\mathcal{F}(G)$, by Proposition 2.1, (vi), $H_n(\omega_1^{-1}(\Delta))$ is $p$-torsion and $p$-divisible and by Bockstein’s theorem $\dim_{H_n(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p^\infty} X \leq n$.

**Case 1-3.** $E = \mathbb{Z}_{(p)}$ or $E = \mathbb{Q}$. By Proposition 2.2 $\ker(\omega_L|_{_\alpha})_\ast$ is $\mathcal{D}(G)$-divisible. Then since $\mathcal{F}(G) \subset \mathcal{D}(G)$, by Proposition 2.1, (iv), $H_n(\omega_1^{-1}(\Delta))$ is $\mathcal{D}(G)$-divisible and since $H_n(\omega^{-1}(\Delta))$ is $\mathcal{F}(G)$-torsion free, by Proposition 2.4, $\dim_{H_n(\omega^{-1}(\Delta))} X \leq n$.

Now let us pass to Step 2 of the construction. We will show that the properties of the homology groups established above will be preserved for simplexes of higher dimensions. Let $\Delta$ be a $(j+1)$-dimensional simplex of $K_i$, $j \geq n+1$ and recall that $H_n(\omega^{-1}(\Delta)) = H_n(\omega^{-1}(\Delta))/\text{Tor}_{\mathcal{F}(G)}H_n(\omega^{-1}(\partial \Delta))$. Note that from the construction it follows that the preimage under $\omega$ of an $(n-1)$-connected subcomplex of $K_i$ is $(n-1)$-connected. Also note that the intersection of a $j$-dimensional simplex of $\Delta$ with the union of any collection of $j$-dimensional simplexes of $\Delta$ is $(n-1)$-connected. These facts allow us to apply below Proposition 2.3 for assembling $\omega_1^{-1}(\partial \Delta)$ from $\omega_1^{-1}(\Delta')$ for $j$-dimensional simplexes $\Delta'$ of $\Delta$ to show that $\omega_1^{-1}(\partial \Delta)$ has properties corresponding to properties of $\omega_1^{-1}(\Delta')$. Once again we consider separately the following cases.

**Case 2-1.** $E = \mathbb{Z}_p$. If for every $j$-dimensional simplex $\Delta'$ of $\Delta$, $H_n(\omega_1^{-1}(\Delta'))$ is $p$-torsion then by Proposition 2.3, $H_n(\omega_1^{-1}(\partial \Delta))$ is $p$-torsion and hence $H_n(\omega_1^{-1}(\Delta))$ is $p$-torsion. Therefore $\dim_{H_n(\omega_1^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p} X \leq n$.

**Case 2-2.** $E = \mathbb{Z}_p^\infty$. If for every $j$-dimensional simplex $\Delta'$ of $\Delta$, $H_n(\omega_1^{-1}(\Delta'))$ is $p$-torsion and $p$-divisible then by Proposition 2.3, $H_n(\omega_1^{-1}(\partial \Delta))$ is $p$-torsion and $p$-divisible and hence $H_n(\omega_1^{-1}(\Delta))$ is $p$-torsion and $p$-divisible. Therefore $\dim_{H_n(\omega_1^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p^\infty} X \leq n$.

**Case 2-3.** $E = \mathbb{Z}_{(p)}$ or $E = \mathbb{Q}$. If for every $j$-dimensional simplex $\Delta'$ of $\Delta$, $H_n(\omega_1^{-1}(\Delta'))$ is $\mathcal{D}(G)$-divisible then by Proposition 2.3, $H_n(\omega_1^{-1}(\partial \Delta))$ is $\mathcal{D}(G)$-divisible. Then $H_n(\omega_1^{-1}(\Delta))$ is $\mathcal{D}(G)$-divisible and $\mathcal{F}(G)$-torsion free and by Proposition 2.4, $\dim_{H_n(\omega_1^{-1}(\Delta))} X \leq n$.

Thus we have shown that $EW(K_i, n)$ is suitable for $X$. Now replacing $K_{i+1}$ by
a $K_i$ with a sufficiently large $l$ we may assume that there is a combinatorial lifting of $h_{i+1}$ to $h'_{i+1} : K_{i+1} \rightarrow EW(K_i, n)$. Replace $h'_{i+1}$ by its cellular approximation preserving the property of $h'_{i+1}$ of being a combinatorial lifting of $h_{i+1}$.

Consider the $(n + 1)$-skeleton of $K_{i+1}$ and let $\Delta_{i+1}$ be an $(n + 1)$-dimensional simplex in $K_{i+1}$. Let $\Delta_i$ be the smallest simplex in $K_i$ containing $h_{i+1}(\Delta_{i+1})$. Then $h'_{i+1}(\Delta_{i+1}) \subset \omega^{-1}(\Delta_i)$. Let $\tau : (\alpha_i \circ \omega_L)^{-1}(\Delta_i) \rightarrow \omega^{-1}(\Delta_i)$ be the inclusion. Note that from the construction it follows that for $h_{i+1}$ we have that there is $k < \omega^{-1}(\Delta_i))$, ker $\tau$ is $\mathcal{F}(G)$-torsion. Recall that the $n$-skeleton of $\omega^{-1}(\Delta_i)$ is contained in $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$ and consider $h'_{i+1}|_{\partial \Delta_{i+1}}$ as a map to $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$. Let $a$ be the generator of $H_n(\partial \Delta_{i+1})$. Since $h'_{i+1}|_{\partial \Delta_{i+1}}$ extends over $\Delta_{i+1}$ as a map to $\omega^{-1}(\Delta_i)$ we have that $\tau_a((h'_{i+1}|_{\partial \Delta_{i+1}})_*(a)) = 0$. Hence $(h'_{i+1}|_{\partial \Delta_{i+1}})_*(a) \in \ker \tau_a$ and therefore there is $k \in S(\mathcal{F}(G))$ such that $k((h'_{i+1}|_{\partial \Delta_{i+1}})_*(a)) = 0$. Replace $\Delta_{i+1}$ by a cell $C$ attached to the boundary of $\Delta_{i+1}$ by a map of degree $k$ if $(h'_{i+1}|_{\partial \Delta_{i+1}})_*(a) \neq 0$ and set $C = \Delta_{i+1}$ if $(h'_{i+1}|_{\partial \Delta_{i+1}})_*(a) = 0$. Then $h'_{i+1}|_{\partial \Delta_{i+1}}$ can be extended over $C$ as a map to $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$ and we will denote this extension by $g'_{i+1}|_{C : C} : (\alpha_i \circ \omega_L)^{-1}(\Delta_i)$. Thus replacing if needed $(n + 1)$-simplexes of $K_{i+1}^{[n+1]}$ we construct from $K_{i+1}^{[n+1]}$ a CW-complex $L_{i+1}$ and a map $g'_{i+1} : L_{i+1} \rightarrow EW(L_i, n)$ which extends $h'_{i+1}$ restricted to the $n$-skeleton of $K_{i+1}$. Now define $g_{i+1} = \omega_L \circ g'_{i+1} : L_{i+1} \rightarrow L_i$ and finally define a simplicial structure on $L_{i+1}$ for which $\alpha_{i+1}$ is a combinatorial map. It is easy to check that the properties (a) and (b) are satisfied. Since the triangulation of $L_{i+1}$ can be replaced by any of its barycentric subdivisions we may also assume that

\[(c) \text{ diam}^j g_{i+1}^j(\Delta) \leq 1/i \text{ for every simplex } \Delta \text{ in } L_{i+1} \text{ and } j \leq i\]

where $g^j_i = g_{j+1} \circ g_{j+2} \circ \ldots \circ g_i : L_i \rightarrow L_j$.

Denote $Z = \varprojlim_i(L_i, g_i)$ and let $r_i : Z \rightarrow L_i$ be the projections. For constructing $L_{i+1}$ we used an arbitrary map $f : L_i^{[n]} \rightarrow K(E, n), E \in \sigma(G)$. Let us show that choosing $E \in \sigma(G)$ and $f$ in an appropriate way for each $i$ we can achieve that $\text{dim}_E Z \leq n$ for every $E \in \sigma(G)$ and hence $\text{dim}_G Z \leq n$.

Let $\psi : F \rightarrow K(E, n)$ be a map of a closed subset $F$ of $L_j$. Then by (c) for a sufficiently large $i > j$ the map $\psi \circ g^j_i|_{(g^j_i)^{-1}(F)}$ extends over a subcomplex $N$ of $L_i$ to a map $\phi : N \rightarrow K(E, n)$. Extending $\phi$ over $L_i^{[n]}$ we may assume that $L_i^{[n]} \subset N$ and replacing $\phi$ by its cellular approximation we assume that $\phi$ is cellular. Now define the map $f : L_i^{[n]} \rightarrow K(E, n)$ that we use for constructing $L_{i+1}$ as $f = \phi|_{L_i^{[n]}}$. Since $g_{i+1}$ factors through $EW(L_i, n)$, the map $f \circ g_{i+1}|_{(g_{i+1})^{-1}(L_i^{[n]})} : g_{i+1}^{-1}(L_i^{[n]}) \rightarrow K(E, n)$ extends to a map $f' : L_{i+1} \rightarrow K(E, n)$. Define $\psi' : L_{i+1} \rightarrow K(E, n)$ by $\psi'(x) = (\phi \circ g_{i+1})(x)$ if $x \in g_{i+1}^{-1}(N)$ and $\psi'(x) = f'(x)$ otherwise. Then $\psi'|_{(g_{i+1})^{-1}(F)} : (g_{i+1})^{-1}(F) \rightarrow K(E, n)$ is homotopic to $\psi \circ g_{i+1}|_{(g_{i+1})^{-1}(F)} : (g_{i+1})^{-1}(F) \rightarrow K(E, n)$ and hence $\psi \circ g_{i+1}|_{(g_{i+1})^{-1}(F)}$ extends
over $L_{i+1}$. Now since we need to solve only countably many extension problems for every $L_j$ with respect to $K(E,n)$ for every $E \in \sigma(G)$ we can choose for each $i$ a map $f : L_i^{[n]} \rightarrow K(E,n)$ in the way described above to achieve that $\dim_E Z \leq n$ for every $E \in \sigma(G)$ and hence $\dim_G Z \leq n$.

The property (b) implies that for every $x \in X$ and $z \in Z$,

(d1) $g_{i+1}(\alpha_i^{-1}(st(p_i(x)))) \subset \alpha_i^{-1}(st(p_i(x)))$ and

(d2) $h_{i+1}(st((\alpha_i+1 \circ r_i+1)(z))) \subset st((\alpha_i \circ r_i)(z))$

where $st(a) =$the union of all the simplexes containing $a$.

Define a map $r : Z \rightarrow X$ by $r(z) = \cap \{p_i^{-1}(st((\alpha_i \circ r_i)(z)))) : i = 1, 2, \ldots\}$. Then (d1) and (d2) imply that $r$ is indeed well-defined and continuous.

The properties (d1) and (d2) also imply that for every $x \in X$

$r^{-1}(x) = \lim_{\leftarrow} \{p_i^{-1}(st(p_i(x))) : i = 1, 2, \ldots\}$,

where the map $g_i|_{\alpha_i^{-1}(st(p_i(x)))}$ is considered as a map to $\alpha_i^{-1}(st(p_i(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, $r$ is a map onto and let us show that $r^{-1}(x)$ is $G$-acyclic.

Since $st(p_i(x))$ is contractible, $T = \alpha_i^{-1}(st(p_i(x)))$ is $(n-1)$-connected. From (a) and Proposition 2.3 it follows that $H_n(T)$ is $\mathcal{F}(G)$-torsion. Then, since $G$ is $\mathcal{F}(G)$-torsion free, by the universal-coefficient theorem $H^n(T;G) = \text{Hom}(H_n(T),G) = 0$. Thus $\tilde{H}^k(r^{-1}(x);G) = 0$ for $k \leq n$ and since $\dim_G Z \leq n$, $\tilde{H}^k(r^{-1}(x);G) = 0$ for $k \geq n+1$. Hence $r$ is $G$-acyclic and this completes the proof.

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