On the quantization of the nonintegrable phase in electrodynamics

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Using the fact that the nonintegrable phase factor can reformulate the gauge theory in terms of path dependent vector potentials, the quantization condition for the nonintegrable phase is investigated. It is shown that the path-dependent formalism can provide compact description of the flux quantization and the charge quantization at the existence of a magnetic monopole. Moreover, the path-dependent formalism gives suggestions for searching of the quantized flux in different configurations and for other possible reasons of the charge quantization. As an example, the developed formalism is employed for a (1+1) dimensional world, showing the relationship between the fundamental unit of the charge and the fine structure constant for this world.

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I. INTRODUCTION

The homogeneous Maxwell equations, Gauss’ law for magnetism and Faraday’s law of induction, identifies the electromagnetic field strength tensor \(F_{\mu\nu}\), whose components are the electric and magnetic field, in terms of gauge dependent vector potentials. Furthermore, any other vector potential, related by a so-called gauge transformation, describes the same electric and magnetic field. Although, all the physical observables are independent from the vector potentials, hence they are gauge invariant, it is essential to introduce them for the Hamiltonian formulation of the dynamics. Moreover, as elegantly described by Aharanov and Bohm, the vector potential has a significant role in quantum mechanics [1]. The vector potential does not only provide a compact mathematical formulation of the associated field strength tensor but also it leads to new predictions such as Aharanov-Bohm effect [1–4], flux quantization [4–7] and the Dirac’s charge quantization condition in the existence of a magnetic monopole [8–9]. Later, in their celebrated paper [10], Wu and Yang gave a complete description of electromagnetic phenomena may have the same field strength tensor \(F_{\mu\nu}\). Historically, such kind of line integrals of the potentials have previously been suggested in [11–14]. Moreover, it was shown by DeWitt [13] and Mandelstam [14] that the nonintegrable phase factor Eq. (1) can eliminate the gauge freedom from the formalism. However, the expense is that the vector potentials, which depend on the field strength tensor, become path dependent. Every gauge functions in the conventional gauge theory have a counterpart path in this equivalent formulation [15–16].

In the present manuscript, we discuss the quantization condition of the nonintegrable phase in the light of the path-dependent formalism of the gauge theory (shortly, we will call it the path-dependent formalism), and explore topological electromagnetic effects in quantum mechanics. It is demonstrated that the path-dependent formalism provides a clear description of the electromagnetic flux quantization which could point out possible reasons for the charge quantization. We apply the developed formalism for a (1+1) dimensional world and find a relationship between the fundamental unit of the charge and the fine structure constant for this world.

After briefly summarizing the gauge theory in Sec. II, we show explicit derivation of the path-dependent formalism in Sec. III and the correspondence between the conventional gauge theory and the path-dependent formalism in a pedestrian level. The condition on the quantization of the nonintegrable phase is discussed in Sec. IV where the flux and the Dirac’s charge quantization conditions are given. In Sec. V the quantization of the charge and the estimation of its fundamental units are illustrated in a (1+1) dimensional world. The conclusion and further remarks are given in Sec. VI.

The CGS units and the metric convention \(g = (+, -, -, -)\) are used throughout the paper.

II. CONVENTIONAL GAUGE THEORY

The abelian gauge theory can be summarized in the light of [17]. In classical electrodynamics the Maxwell equations in (3+1) spacetime dimensions read

\[
\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu ,
\]

\[
\epsilon^{\alpha\beta\mu\nu} \partial_\mu F_{\alpha\beta} = 0 ,
\]

where the components of the electromagnetic field strength tensor \(F^{\mu\nu}\) are the electric field \(F^{0\nu} = E^\nu\) and the magnetic field \(F^{\mu\kappa} = \epsilon^{\mu\kappa\lambda} B_\lambda\), and the four-vector current is defined \(j^\nu = (\epsilon^\nu, J)\). The homogeneous Maxwell equation (5) allow to express the electric and magnetic fields in terms of a four-vector potential \(A^\mu = (\phi, A)\) as

\[
F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .
\]
Furthermore, any other four-vector potential, related by a so-called gauge transformation, describes the same electric and magnetic field. In other words, the transformation

\[ A'^\mu \to A'^\mu + \partial'^\mu \chi \]  

(5)

leaves the electromagnetic field strength tensor invariant and, consequently, all the physically measurable quantities related to the electrodynamics such as the Maxwell equations, the Lorentz force law become gauge invariant.

More fundamentally, the gauge invariance can appear as a consequence of the conservation of the electric charge under local symmetry transformation via Noether’s theorem. In quantum theory, the conservation of the electric charge follows from the local phase invariance of the wave function. Further, the local phase invariance of the wave function imposes an interaction between the associated conserved quantity and the gauge field such that the Schrödinger equation which governs the time evolution of the wave function becomes invariant under the gauge-transformation. For instance, the Dirac equation for a relativistic spin-1/2 particle

\[ i\hbar \gamma^\mu \left( \partial_\mu - \frac{i e}{\hbar c} A_\mu \right) \psi(x) = 0 \]  

(6)

is invariant under the transformations (5) as long as the wave function transforms as

\[ \psi(x) \to \exp \left( \frac{i e \chi}{\hbar c} \right) \psi(x) \]  

(7)

where in last line we have used \( A_{\nu}(y) = A_{\nu}(y) - F_{\nu\lambda}(y) \). Further, the first two integrand terms in Eq. (11) can be written as \( \frac{\partial}{\partial s} \left( A_{\nu}(y) \frac{\partial y^\nu}{\partial x^\mu} \right) \) and using the boundary conditions (10),

\[ \mathcal{A}_\mu(x) = \int_0^1 F_{\nu\lambda}(y) \frac{\partial y^\nu}{\partial s} \frac{\partial y^\lambda}{\partial s} ds \]  

(12a)

is obtained. Furthermore, since the filed strength tensor \( F_{\mu\nu} \) is antisymmetric, Eq. (12a) can be written as

\[ \mathcal{A}_\mu(x) = \frac{1}{2} \int_0^1 F_{\nu\lambda}(y) \left( \frac{\partial y^\nu}{\partial s} \frac{\partial y^\lambda}{\partial s} - \frac{\partial y^\lambda}{\partial s} \frac{\partial y^\nu}{\partial s} \right) ds \]  

(12b)

The expression (12) is gauge independent because it is written solely in terms of the gauge invariant field strength tensor

In general, the gauge theory can be patterned as follows: First, for every conservation law there is an associated symmetry via Noether’s theorem; second, the local ones among them lead to the existence of gauge fields; and third, the gauge field theory imposes interactions between the gauge field and the conserved quantity. Such an generalization of the local gauge invariance leads to, for instance, the existence of the non-abelian gauge field [19].

### III. THE PATH-DEPENDENT FORMALISM

As it was discussed in [10], the fundamental concept which describes complete electromagnetism is the nonintegrable phase factor [1]. The nonintegrable phase factor can eliminate the vector potential from the formalism [13, 14]. In fact, let us define the gauge function \( \chi \) via the path integral

\[ \chi = -\int_0^x A_\mu dy^\nu, \]  

(8)

then the associated Schrödinger equation becomes invariant under the following gauge transformation

\[ A_\mu(x) \to A_\mu(x) - \frac{\partial}{\partial x^\nu} \int_0^x A_\nu dy^\nu. \]  

(9)

The latter yields to the gauge invariant vector potential \( \mathcal{A}_\mu(x) \). Explicitly, if one parametrizes the path \( y = y(s, x) \) as

\[ y(1, x) = x, \quad y(0, x) = x_0, \]  

(10)

where the electromagnetic field vanishes at \( x_0 \), at which \( A_\mu \) may, without loss of generality, be set equal to zero. Then, Eq. (9) becomes

\[ F_{\mu\nu}. \]  

However, the expense is that the vector potential is path dependent and every gauge functions in the conventional gauge theory have a counterpart in the path-dependent formalism [15, 16]. As a consequence, we will label both the vector potential \( \mathcal{A}_\mu \) and the wave function \( \Psi \) with the path index \( \mathcal{P} \) which refers to a certain path as

\[ i\hbar \gamma^\mu \left( \partial_\mu - \frac{i e}{\hbar c} \mathcal{A}_\mu(\mathcal{P}, x) \right) - mc \right) \Psi(\mathcal{P}, x) = 0. \]  

(13)

Moreover, the Dirac equation (13) is invariant under the following path transformation

\[ \mathcal{A}_\mu(\mathcal{P}', x) = \mathcal{A}_\mu(\mathcal{P}, x) + \partial_\mu \int_{x_0}^{x} A_\nu dy^\nu, \]  

(14)
as long as the wave function satisfies

$$\Psi[\mathcal{P}', x] = \exp \left( \frac{ie}{\hbar c} \int_0^x A_\mu dy^\mu \right) \Psi[\mathcal{P}, x],$$  \hspace{1cm} (15)$$
with the closed loop $\partial \Sigma = \mathcal{P} - \mathcal{P}'$. Furthermore, using the four-dimensional Stokes’ law, the loop integral can be converted to surface integral

$$\int_{\partial \Sigma} A_\mu dy_\mu = \frac{1}{2} \int_\Sigma F_{\mu\nu} d\sigma_{\mu\nu} = \Phi_{\text{EM}}(x),$$  \hspace{1cm} (16)$$
with the electromagnetic flux $\Phi_{\text{EM}}$ [19]. In addition to that, using the definition of the path dependent vector potential [9], the electromagnetic flux for a nonconfined field can also be identified as

$$\int_{\mathcal{P}} A_\mu(dy^\mu) = \Phi_{\text{EM}}(x),$$  \hspace{1cm} (17)$$
which implies that for any path $\mathcal{P}$

$$\int_{\mathcal{P}} A_\mu dy^\mu = 0$$  \hspace{1cm} (18)$$
always holds [20]. In conclusion, true electromagnetism can be described by path invariance of the so-called nonintegrable phase factor which is known as Wilson loop when the gauge group is nonabelian [21].

In order to illustrate the equivalence between the conventional gauge theory and the path-dependent formalism, let us specify some certain paths which have a well-known counterpart in the conventional gauge theory. For the sake of simplicity, assume that there is only a constant and uniform electric field $E_0$ and further, without loss of generality, let the initial point be $x_0 = (0, 0)$. If one chooses the path $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ with the each segments

$$\mathcal{P}_1 : \ y^\mu(s, x) = (0, s x), \quad 0 \leq s \leq 1,$$
$$\mathcal{P}_2 : \ y^\mu(s, x) = (s ct, x), \quad 0 \leq s \leq 1,$$
then path dependent vector potential becomes

$$A^\mu(x) = F_{0i} \int_0^1 \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} ds,$$
$$= (-c t E_0).$$  \hspace{1cm} (21)$$
This gauge is called velocity gauge. On the other hand, if we choose the segments of the path as

$$\mathcal{P}_1' : \ y^\mu(s, x) = (s ct, 0), \quad 0 \leq s \leq 1,$$
$$\mathcal{P}_2' : \ y^\mu(s, x) = (ct, s x), \quad 0 \leq s \leq 1,$$
then the vector potential yields

$$A^\mu(x) = F_{0i} \int_0^1 \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} ds,$$
$$= (-x \cdot E_0, 0).$$  \hspace{1cm} (22)$$
which is know as length gauge. Furthermore, if we trace a straight line as

$$\mathcal{P}' : \ y^\mu(s, x) = (s ct, s x), \quad 0 \leq s \leq 1,$$
then, the vector potential is given by

$$A^\mu(x) = F_{0i} \int_0^1 \left( \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} - \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} \right) ds,$$
$$= \left( -\frac{1}{2} x \cdot E_0, -\frac{1}{2} c t E_0 \right).$$  \hspace{1cm} (23)$$
which is known as Fock-Schwinger gauge $x_\mu A^\mu = 0$. Following the path transformation [14], relation between different gauges is given by the electromagnetic flux. For instance, gauge transformation between velocity and length gauge is given by

$$\Phi_{\text{EM}}(x) = -c t x \cdot E_0$$  \hspace{1cm} (24)$$
with $\partial \Sigma = \mathcal{P} - \mathcal{P}'$. Similarly, transformation between the length gauge and the Fock-Schwinger gauge can be accomplished by

$$\Phi_{\text{EM}}(x) = -\frac{1}{2} c t x \cdot E_0$$  \hspace{1cm} (25)$$
and further, without loss of generality, let the initial point be $x_0 = (0, 0)$. If one chooses the path $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ with the each segments

$$\mathcal{P}_1 : \ y^\mu(s, x) = (0, s x), \quad 0 \leq s \leq 1,$$
$$\mathcal{P}_2 : \ y^\mu(s, x) = (s ct, x), \quad 0 \leq s \leq 1,$$
then path dependent vector potential becomes

$$A^\mu(x) = F_{0i} \int_0^1 \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} ds,$$
$$= (-c t E_0).$$  \hspace{1cm} (26)$$
This gauge is called velocity gauge. On the other hand, if we choose the segments of the path as

$$\mathcal{P}_1' : \ y^\mu(s, x) = (s ct, 0), \quad 0 \leq s \leq 1,$$
$$\mathcal{P}_2' : \ y^\mu(s, x) = (ct, s x), \quad 0 \leq s \leq 1,$$
then the vector potential yields

$$A^\mu(x) = F_{0i} \int_0^1 \frac{\partial y^0}{\partial s} \frac{\partial y^i}{\partial x_\mu} ds,$$
$$= (-x \cdot E_0, 0).$$  \hspace{1cm} (27)$$
which is known as length gauge. Furthermore, if we trace a straight line as

$$\mathcal{P}' : \ y^\mu(s, x) = (s ct, s x), \quad 0 \leq s \leq 1,$$
how it leads to the topological electromagnetic effects in quantum mechanics. Consider two arbitrary paths $P_1$ and $P_2$ as shown in Fig. 1 which have the same starting and terminating points. Using Eq. (14) and Eq. (16), the path dependent vector potential defined on $P_2$ can be written

$$A_\mu(P_2, x) = A_\mu(P_1, x) + \partial_\mu \Phi_{EM}(x).$$

(33)

If the electromagnetic flux $\Phi_{EM}$ which is defined on the surface bounded by the loop $\partial \Sigma = P_1 - P_2$ is constant in both space and time, then the path dependent vector potentials $A_\mu(P_1, x)$ and $A_\mu(P_2, x)$ coincide. Since the vector potential already overdescribes the true electromagnetism [10], i.e., different vector potentials can describe the same physics, the wave function for a given vector potential has to be unique. As a consequence the wave function defined on the path $P_2$

$$\Psi[P_2] = \exp \left( \frac{ie}{\hbar c} \Phi_{EM} \right) \Psi[P_1],$$

(34)

should match with $\Psi[P_1]$ [22]. Then it follows that the nonintegrable phase has to be quantized as

$$\frac{e \Phi_{EM}}{\hbar c} = 2\pi n, \quad n = \pm 1, \pm 2, \ldots,$$

(35)

which can be interpreted as the least condition of the flux quantization [23].

The validity of such a requirement can be further confirmed in the following way. If there exists a constant and uniform flux $\Phi_{EM}$, then it is also possible to find another path $P_3$ whose winding number $N$ is greater than 1, i.e., a path which can wrap the flux more than one time such that each turn can be defined on different hypersurfaces. In this case the wave function defined for the path $P_3$

$$\Psi[P_3] = \exp \left( \frac{ie}{\hbar c} N \Phi_{EM} \right) \Psi[P_1]$$

(36)

would depend on the winding number $N$, which is inconsistent with physical realization unless there exists an experiment which can differentiate the winding number $N$.

B. Examples

Let us illustrate the quantization condition for the nonintegrable phase in the following examples. In a certain field configuration, we choose two paths which provide the same vector potential and show that the electromagnetic flux confined by these paths is constant and uniform and, therefore, should be quantized.

As a first application of the electromagnetic flux quantization law for the constant and uniform flux Eq. (35), we derive the Dirac’s charge quantization condition from the latter. Although quantum mechanics does not require the existence of magnetic monopoles, it does not also prohibit its presence even in the current formulation of the electromagnetism. The fundamental relation, as it stands, $B = \nabla \times A$ with a non singular free vector potential $A$ allows to modify the associated Maxwell equation to $\nabla \cdot B = 4\pi \rho_m$ with the magnetic monopole charge density $\rho_m$. Further, it was shown by Dirac that if there exists a magnetic monopole, it would explain why the electric charges in the nature are quantized [8, 9]. Dirac’s original derivation was based on the singular vector potential whose singularity corresponds to the so-called Dirac’s string. Later, the same result was obtained in [10] using a nonsingular vector potential defined on a domain which is divided into two overlapping regions. In the path-dependent formalism such a derivation was done in [19] where the surface invariance of the closed path is used.

Here, we will derive the same condition using the flux quantization condition Eq. (35). Consider two paths $P_1$ and $P_2$ whose initial and final points coincide, as shown in Fig. 2. Each path generates the associated vector potential of the magnetic field due to a magnetic monopole with charge $g$. Then, there exists a surface $\Sigma_2$ which encloses the magnetic monopole $g$. Further, if one of the paths, say $P_2$, is deformed in a way that the two paths overlap each other, then the surface $\Sigma_1$ vanishes, and $\Sigma_2$ turns into a closed surface whose associated electromagnetic flux becomes $4\pi g$ [24]. Consequently, stemming from Eq. (35), the Dirac charge quantization condition is obtained:

$$\frac{2e g}{\hbar c} = n.$$

(37)

Explicitly, let us take the monopole located at the origin $x = 0$, then the magnetic field becomes

$$B(r) = \frac{g \hat{r}}{r^2}.$$

(38)

If one chooses the path $P_1$, see Fig. 3, which connects the points

$$x_0 \rightarrow (0, 0, -z) \rightarrow (x, 0, -z) \rightarrow (x, 0, z) \rightarrow (0, 0, z) \rightarrow x$$

(39)

with the starting point $x_0$, say $x_0 = (\infty, 0, -\infty)$, where the vector potential is zero, and terminating point $x = (x, y, z)$,
whose counterpart in spherical coordinate corresponds to the

which is valid on the north hemisphere $0 \leq \phi \leq \pi$. Nevertheless, the path $P_2$ which connects the following points

$x_0 \rightarrow (0, 0, -\varepsilon) \rightarrow (x, y, -\varepsilon) \rightarrow x$

yields the vector potential

$A(P_2, x) = \frac{g}{\sqrt{x^2 + y^2}} (y(1 + z/r), -x(1 + z/r), 0)$,

whose counterpart in spherical coordinate corresponds to the vector potential defined on the south hemisphere,

$A(P_2, x) = A^S(x) = -\frac{g(1 + \cos(\theta))}{r \sin(\theta)} \hat{\phi}$,

where the surface $\Sigma$ is the surface of a $\phi$ sphere slice. Furthermore, as illustrated in Fig. 3 if we increase the angle $\phi$ to $2\pi$ which corresponds to choosing the path $P_1$ as

$x_0 \rightarrow (0, 0, -\varepsilon) \rightarrow (x, y, -\varepsilon) \rightarrow (x, y, z) \rightarrow (0, 0, z) \rightarrow x$, (46)

two paths coincide and the surface of a $\phi$ sphere slice $\Sigma$ becomes the surface of a full sphere which leads to a constant flux $\Phi_{EM} = 4\pi g$. Then one obtains the Dirac’s charge quantization condition Eq. (35). We should stress that in this method, we have used neither periodicity, nor single-valuedness of the wave function. In addition, we did not need further quantization conditions like quantization of the angular momentum or the energy.

As a second example, let us consider a confined static magnetic field

$B(x) = B_0 (1 - \theta(r - r_0)) \hat{z}$,

with $r = \sqrt{x^2 + y^2}$, $r_0 = \sqrt{x_0^2 + y_0^2}$, and the Heaviside step function $\theta(x)$. It is clear that there exist two paths whose enclosed loop give the following electromagnetic flux $\Phi_{EM} = B_0 \pi r_0^2$. Then, the quantization condition Eq. (35) becomes

$\frac{e B_0 \pi r_0^2}{\hbar c} = 2\pi n$

which corresponds to the flux quantization phenomenon in the superconducting rings. Specifically, one of the mentioned paths $P_1$ can be chosen to connect the points

$x_0 \rightarrow (x, 0, 0) \rightarrow (0, 0, 0) \rightarrow x$, (49)

with a starting point $x_0 = (\infty, 0, 0)$ at which the vector potential is zero, then the path dependent vector potential yields

$A(P_1, x) = \begin{cases} B_0 (y, x, 0)/2, & r \leq r_0, \\ B_0 r_0^2 (y, x, 0)/2r^2, & r > r_0. \end{cases}$
will also give the same vector potential Eq. (50). Although the same potential is realized \[1\].

If the confined magnetic field, on the other hand, depends on the coordinate \(z\) or the time (for instance its amplitude may vary in time) then the flux, in general, does not have to be quantized, hence there may exist an experiment which detects the phase.

FIG. 5. Geometric configuration for the electric flux quantization.

Similarly, as it is shown in Fig. [4] the path \(P_2\) connecting the points
\[
\begin{align*}
x(0,0,0) & \rightarrow (x,0,0) \rightarrow (x,y,0) \rightarrow (-x,-y,0) \rightarrow (-x,y,0) \rightarrow (x,0,0) \rightarrow (0,0,0) \rightarrow x
\end{align*}
\]
will also give the same vector potential Eq. (50). Although these two paths give the same vector potential, there is a non zero magnetic flux in the surface bounded by these paths, which appears as a constant phase in the wave function defined on the path \(P_2\). In order to satisfy the uniqueness of the wave function, the phase and therefore the flux of the confined static magnetic field has to be quantized, which gives the condition Eq. (55).

If the confined magnetic field, on the other hand, depends on the coordinate \(z\) or the time (for instance its amplitude may vary in time) then the flux, in general, does not have to be quantized due to the fact that Eq. (50) implies different vector potentials for different paths. Note that in the experiment of the Aharonov-Bohm effect the existence of the confined magnetic field is realized [1].

In the third example, we consider a constant and uniform electric field along \(x\)-direction, which is confined on a specific region of the spacetime as
\[
E(t,x) = E_0 (\theta(ct) - \theta(ct - c\Delta t)) (\theta(x) - \theta(x - \Delta x)) \hat{x}.
\]
Then, there exist two paths whose enclosed area includes the confined electric field, which yields the electromagnetic flux
\[
\Phi_{EM} = c \int E(t',x') dx' dt' = c E_0 \Delta x \Delta t.
\]

Following Fig. [5] both the path \(P_1\) and the path \(P_2\) give the same potential
\[
\mathcal{P} = \begin{cases}
-E_0(-x,ct,0,0)/2, & \Delta x \geq x \geq 0 \land \Delta t \geq t \geq 0, \\
-\frac{E_0 \Delta x^2}{2} \left(1 - \frac{ct}{x^2},0,0\right), & x > \Delta x > 0 \land \Delta t > t > 0, \\
-\frac{E_0 \Delta t^2}{2} \left(\frac{x}{t^2}, \frac{c}{t},0,0\right), & \Delta x > x > 0 \land t > \Delta t > 0.
\end{cases}
\]

Since there is a non-zero electric flux in the loop enclosed by the paths, the phase has to be quantized such that
\[
\frac{e c E_0 \Delta x \Delta t}{\hbar c} = 2 \pi n
\]
holds. Further, if the given electric field \(E_0\) depends on other coordinates \(y\) and/or \(z\) then the flux does not have to be quantized, hence there may exist an experiment which detects the phase.

V. ON THE ELECTRIC CHARGE QUANTIZATION

The quantization of the nonintegrable phase in the presence of the magnetic monopole, as we have shown in the previous section, explains why the electric charges are quantized. However, quantum mechanics cannot require magnetic monopoles to exist. Furthermore, since the Dirac’s 1931 paper which predicts magnetic monopoles to be able to quantize the electric charges, there has never been reproducible evidence for the existence of magnetic monopoles [24]. If magnetic monopoles do not exist, then how could one explain the charge quantization? In general, not only a magnetic monopole, but also the existence of the fundamental value for the constant and uniform confined electromagnetic flux, independent of the field configuration, would explain the charge quantization. In terms of mathematical jargon, if there exists a non simply connected region in spacetime, then the phase can be quantized.

Let us discuss the charge quantization in a \((1+1)\) dimensional spacetime world. The Maxwell equations in an arbitrary \((d+1)\) dimensional spacetime is given [26]
\[
\begin{align*}
\partial_{\mu} F_{\mu\nu} &= \frac{2\pi^{d/2}}{\Gamma(d/2)} J^\nu, \\
e^{\alpha \beta} \partial_{\mu} F_{\alpha \beta} &= 0,
\end{align*}
\]
with the Gamma function \(\Gamma(x)\). The causal electric field of a point charge \(q\) moving on an arbitrary world line \(r^\alpha(\tau) = (r^0(\tau), r^i(\tau))\) can be found via solving the Maxwell equation [50] with the retarded propagator [27].

In a \((1+1)\) dimensional spacetime, then, the causal electric field can be written as
\[
E(t,x) = q (\theta(x - r(t_+)) - \theta(r(t_-) - x)),
\]
where the retarded time is given by \[ct - r(t_+) = \pm (x - r_0(t_+))\].

In addition to the \((1+1)\) dimensional causal spacetime, let us assume that there exists a quantum vacuum with a possibility of virtual creation and annihilation of electron-position pairs. Then, due to an electron-positron pair creation in this universe, a confined electric field arises on the spacetime region bounded by the world lines of the each pair as shown in Fig. [6]. Consequently, the flux defined on the spacetime area \(A\) bounded by the world line of the electron and the world line of the positron reads
\[
\Phi_{EM} = 2 e A
\]
FIG. 6. The spacetime are bounded by the world line of each pair implies a constant uniform confined electric field in a (1+1) dimensional spacetime, which leads to the phase quantization.

with the charge of the electron $e$. Then, the flux quantization condition implies

$$\alpha^{(1)} A = \pi n,$$  \hspace{1cm} (60)

where $\alpha^{(1)}$ is the fine structure constant for the (1+1) dimensional spacetime. There are many ways to read the condition of Eq. (60). First of all, if the area $A$ has a fundamental value which is determined by the universe, then the condition of Eq. (60) not only could explain why the charge is quantized in the universe, but also would predict the fundamental units of the charge. Secondly, in the other way around, the area $A$ bounded by the world lines of the pair has to take discrete values, which leads to the existence of only discrete set of allowed pair wordlines.

Moreover, the condition (60) may also be regarded as an expression which estimates the fine structure constant for the (1+1) dimensional spacetime. Namely, the area $A$ is bounded as $A < c^2 \tau^2 / 2$ with the lifetime of the lightest charged particle $\tau$. Further, using the Heisenberg uncertainty relation

$$\Delta t \Delta E \geq \frac{\hbar}{2},$$  \hspace{1cm} (61)

the flux quantization condition (60) estimates

$$\alpha^{(1)} \sim \lambda_C^{-2}$$  \hspace{1cm} (62)

with the Compton wavelength $\lambda_C = h/(me c)$, as long as the Planck constant $\hbar$, the speed of light $c$, the mass of the electron $m_e$, and the fundamental charge $e$ remain same in the (1+1) dimensional world.

It would be very remarkable to test the derived scaling law for the fine structure constant of the one dimensional world, Eq. (62), in an effectively one-dimensional solid layers like quantum wires.

Since the world is at least (3+1) dimensional in spacetime, the explanation of the charge quantization is missing in the absence of a magnetic monopole. Nonetheless, we give us a liberty to briefly speculate the possible reasons of the electric charge quantization in a (3+1) dimensional world. Note that the existence of a confined constant and uniform field anywhere in the universe would explain the charge quantization everywhere. If we imagine that the spatial dimensions are emergent in a row from the Big Bang, instead of at the same time, one could think that the charge is quantized due to the constant uniform confined electromagnetic field in the (1+1) spacetime structure of the early universe.

VI. CONCLUSION

The nonintegrable phase factor which describes the complete theory of electromagnetism can replace the gauge freedom on the vector potential with the path freedom. In this equivalent formulation of the gauge theory, we have shown the quantization of the electromagnetic flux generated from the constant and uniform confined field. This would imply and explain the charge quantization, if the existence of the fundamental confined field with a uniform and constant flux is proven. Finally, in the absence of the evidence for magnetic monopoles, we are naively questioning the possible reasons of the charge quantization via investigating the quantization condition of the nonintegrable phase. It was shown that the (1+1) dimensional world could allow to explain the quantization of the charge as well as the estimation of its fundamental unit.

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[22] Here it should be stressed that the constant phase appears in Eq. (34) is a nontrivial phase based on the requirement of the local gauge invariance. On the contrary, the trivial phase of the global gauge invariance \( \chi \) can be set equal to zero without loss of generality.
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