Quantum supervaluationist account of the EPR paradox

Arkady Bolotin

Ben-Gurion University of the Negev, Beersheba (Israel)

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Abstract

In the paper, the EPR paradox is explored by the approach of quantum supervaluationism that leads to a “gappy” semantics with the propositions giving rise to truth-value gaps. Within this approach, the statement, which asserts that in the singlet state the system of two (i.e., A and B) spin-$\frac{1}{2}$ particles possesses the a priori property “spin A is up and spin B is down along the same axis” or “spin A is down and spin B is up along the same axis”, does not have the truth-value at all. Consequently, after the verification of, say, the proposition “spin $A$ is up along the $z$-axis”, the statistical population describing the valuation of the logical connective “spin $B$ is down along the $z$-axis and spin $B$ is up (down) along the $x$-axis” would have no elements.

Keywords: Quantum mechanics; EPR paradox; Truth values; Bivalence; Supervaluationism.

1 Introduction

Let $|s, m_j\rangle$ denote the vector of the Hilbert space describing the state of particle’s spin, where $s$ stands for the spin quantum number and $m_j$ specifies the spin projection quantum number along the $j \in \{x, y, z\}$ axis. Consider a system with two (A and B) spin-$\frac{1}{2}$ particles (that is, $s^{(A)} = s^{(B)} = \frac{1}{2}$) which is prepared in a singlet state (i.e., a state with total spin angular momentum 0) described by the vector $|0, 0\rangle_j$, namely,

$$|0, 0\rangle_j = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, m_j^{(A)} = +\frac{1}{2}\rangle \otimes |\frac{1}{2}, m_j^{(B)} = -\frac{1}{2}\rangle - |\frac{1}{2}, m_j^{(A)} = -\frac{1}{2}\rangle \otimes |\frac{1}{2}, m_j^{(B)} = +\frac{1}{2}\rangle \right).$$

Suppose that after being prepared in the singlet, these particles travel away from each other in a region of zero magnetic field where by means of a Stern-Gerlach magnet an observer “Alice” measures the spin of the particle A and by means of another Stern-Gerlach magnet an observer “Bob” measures the spin of the particle B. Assume that the measurements are space-like separated, such that neither of the observers can act upon or exercise influence on the result of the other.

Let $\uparrow_j$ denote the proposition asserting that the spin-$\frac{1}{2}$ particle exists in the spin state “$j$-up” (i.e., the statement “$m_j = +\frac{1}{2}$”) whereas $\downarrow_j$ signify the alternative proposition that this particle is in
the spin state “$j$-down” (i.e., the statement “$m_j = -\frac{1}{2}$”).

Let the double-bracket notation $[\bullet]_v$, where the symbol $\bullet$ stands for any proposition (compound or simple) denote a *valuation in a circumstance* $v$, that is, a mapping from a set of propositions \{\bullet\} to a set of truth-values $V_N = \{0, 1\}$ having the cardinality $N$ and the range with the upper bound 1 (which represents the *truth*) and the lower bound 0 (representing the *falsehood*), relative to a particular circumstance of evaluation indicated by $v$.

Let us consider the propositions Same$_j$ and Diff$_j$:

\[
[\text{Same}_j]_v = [\uparrow_j(A) \land \uparrow_j(B) \lor \downarrow_j(A) \land \downarrow_j(B)]_v,
\]

(2)

\[
[\text{Diff}_j]_v = [\uparrow_j(A) \land \downarrow_j(B) \lor \downarrow_j(A) \land \uparrow_j(B)]_v,
\]

(3)

where $\lor$ stands for “exclusive or” logical connective. The proposition Same$_j$ asserts that the two particles have the same directions of their spins along the $j$-axis, while the proposition Diff$_j$ declares that their spin directions are different along $j$.

Because the system is prepared in the singlet state, Alice and Bob can affirm that prior to the verification the proposition Same$_j$ has the value of falsehood at the same time as the proposition Diff$_j$ has the value of truth, i.e., $[\text{Same}_j]_v = 0$ and $[\text{Diff}_j]_v = 1$.

Suppose that using the outcome of the measurement, Alice proves (disproves) the statement “$m_j(A) = +\frac{1}{2}$” as well as the statement “$m_j(A) = -\frac{1}{2}$”. This act destroys information about the projection of the spin of the particle $A$ along any other axis $k \in \{x, y, z\}$ not equal to $j$ (that might previously have been obtained) and, for that reason, one can write

\[
\{[\uparrow_j(A)]_v\} = \{0, 1\} \implies \{[\uparrow_k(A)]_v\} \neq \{0, 1\},
\]

(4)

where the symbol $\uparrow$ must be replaced by either $\uparrow$ or $\downarrow$.

However, if both of the following premises are assumed, namely,

1. the truth-values of the propositions $\downarrow_j$ exist before the act of verification,

2. these truth-values are elements of the two-valued set $V_2 = \{0, 1\},$

then the valuation $[\text{Diff}_j]_v = 1$ will imply

\[
[\uparrow_j(A)]_v \cdot [\downarrow_j(B)]_v + [\downarrow_j(A)]_v \cdot [\uparrow_j(B)]_v = 1
\]

(5)

or, explicitly,

2
\[
\begin{align*}
\lceil \downarrow_j(B) \rceil_v &= \lceil \uparrow_j(A) \rceil_v , \\
\lceil \uparrow_j(B) \rceil_v &= \lceil \downarrow_j(A) \rceil_v .
\end{align*}
\]

Hence, the verification of \( \uparrow_j(A) \) will produce the bivaluation \( \{ \lceil \downarrow_j(B) \rceil_v \} = \{0, 1\} \) without destroying information about the spin projection of the particle \( B \) along any axis \( k \neq j \).

Let us consider, for example, the product \( \{ \lceil \downarrow_x(B) \rceil_v \} \times \{ \lceil \uparrow_x(B) \rceil_v \} \). Its statistical population is the “cross product” of the sets \( \{ \lceil \downarrow_x(B) \rceil_v \} \times \{ \lceil \uparrow_x(B) \rceil_v \} \) and defined such that

\[
\{ \lceil \downarrow_x(B) \rceil_v \} \times \{ \lceil \uparrow_x(B) \rceil_v \} = \{( \lceil \downarrow_x(B) \rceil_v , \lceil \uparrow_x(B) \rceil_v ) \} .
\]

Following the verification of the proposition \( \uparrow_x(A) \), the said statistical population will contain two pairs – each for every possible preexisting truth-value of \( \uparrow_x(B) \), namely,

\[
\{ \lceil \downarrow_x(B) \rceil_v \} \times \{ \lceil \uparrow_x(B) \rceil_v \} = \{1\} \times \{ \lceil \uparrow_x(B) \rceil_v \} = \{(1, 1), (1, 0)\} .
\]

In consequence, the verification (refutation) of \( \uparrow_x(A) \) carried out in another experimental run will bring the definiteness to this product. This way, the statement “\( m_x(B) = -\frac{1}{2} \) and \( m_x(B) = +\frac{1}{2} \)” will have the truth value contrary to the basic principles of quantum theory.

As it can be readily seen, to resolve this paradox (known as the EPR paradox [1]) upon the supposition that the particle \( B \) cannot be affected by measurements carried out on the particle \( A \), one can deny either the premise (1) or the premise (2).

This paper presents a logic approach to the EPR paradox where the premise (1) is denied.

## 2 Quantum Supervaluationism

Consider the lattice \( L(\mathcal{H}) \) formed by the column spaces (ranges) of the projection operators \( \hat{P}_\alpha \), \( \hat{P}_\beta \), . . . on the Hilbert space \( \mathcal{H} \).

In the lattice \( L(\mathcal{H}) \) the ordering relation \( \leq \) corresponds to the subset relation \( \text{ran}(\hat{P}_\alpha) \subseteq \text{ran}(\hat{P}_\beta) \); the operation \textit{meet} \( \cap \) corresponds to the interception \( \text{ran}(\hat{P}_\alpha) \cap \text{ran}(\hat{P}_\beta) \); the operation \textit{join} \( \cup \) corresponds to the smallest closed subspace of \( \mathcal{H} \) containing the union \( \text{ran}(\hat{P}_\alpha) \cup \text{ran}(\hat{P}_\beta) \). The lattice \( L(\mathcal{H}) \) is bounded, i.e., it has the greatest element \( \text{ran}(\hat{1}) = \mathcal{H} \) and the least element \( \text{ran}(\hat{0}) = \{0\} \) that satisfy the following subset relation for every \( \text{ran}(\hat{P}) \) in \( L(\mathcal{H}) \):

\[
\text{ran}(\hat{0}) \subseteq \text{ran}(\hat{P}) \subseteq \text{ran}(\hat{1}) .
\]
Let \( \hat{P}_\alpha \cap \hat{P}_\beta, \hat{P}_\alpha \cup \hat{P}_\beta \) and \( \hat{P}_\alpha + \hat{P}_\beta \) denote projections on the interception \( \text{ran}(\hat{P}_\alpha) \cap \text{ran}(\hat{P}_\beta) \), the union \( \text{ran}(\hat{P}_\alpha) \cup \text{ran}(\hat{P}_\beta) \) and the sum \( \text{ran}(\hat{P}_\alpha) + \text{ran}(\hat{P}_\beta) \), respectively. One can write then

\[
\text{ran}(\hat{P}_\alpha \cap \hat{P}_\beta) = \text{ran}(\hat{P}_\alpha) \cap \text{ran}(\hat{P}_\beta),
\]

(11)

\[
\text{ran}(\hat{P}_\alpha \cup \hat{P}_\beta) = \text{ran}(\hat{P}_\alpha) \cup \text{ran}(\hat{P}_\beta),
\]

(12)

\[
\text{ran}(\hat{P}_\alpha + \hat{P}_\beta) = \text{ran}(\hat{P}_\alpha) + \text{ran}(\hat{P}_\beta).
\]

(13)

Clearly, if the column spaces \( \text{ran}(\hat{P}_\alpha) \) and \( \text{ran}(\hat{P}_\beta) \) are orthogonal, their union coincides with their sum, i.e.,

\[
\text{ran}(\hat{P}_\alpha) \cap \text{ran}(\hat{P}_\beta) = \text{ran}(\hat{P}_\alpha) = \text{ran}(\hat{P}_\beta) = \text{ran}(\hat{P}_\alpha) + \text{ran}(\hat{P}_\beta).
\]

(14)

On the other hand, if the projection operators \( \hat{P}_\alpha \) and \( \hat{P}_\beta \) on \( \mathcal{H} \) are orthogonal, then \( \hat{P}_\alpha \hat{P}_\beta = \hat{P}_\beta \hat{P}_\alpha = 0 \). Hence, in this case one must get

\[
\hat{P}_\alpha \cap \hat{P}_\beta = \hat{P}_\alpha \hat{P}_\beta = \hat{0},
\]

(15)

\[
\hat{P}_\alpha \cup \hat{P}_\beta = \hat{P}_\alpha + \hat{P}_\beta.
\]

(16)

Now, consider the truth-value assignments of the projection operators in the lattice \( L(\mathcal{H}) \).

Let \( v \) be the truth-value assignment function and \( \hat{P}_\diamond \) denote the projection operator associated with the proposition \( \diamond \). Assume that the following valuational axiom holds:

\[
v(\hat{P}_\diamond) = [\left[ \diamond \right]]_v.
\]

(17)

Suppose that a system is in a pure state \( |\Psi_\alpha\rangle \) lying in the column space of the projection operator \( \hat{P}_\alpha \). Since being in \( \text{ran}(\hat{P}_\alpha) \) means \( \hat{P}_\alpha |\Psi_\alpha\rangle = 1 \cdot |\Psi_\alpha\rangle \), one can assume that in the state \( |\Psi_\alpha\rangle \in \text{ran}(\hat{P}_\alpha) \), the truth-value assignment function \( v \) assigns the truth value \( 1 \) to the operator \( \hat{P}_\alpha \) and, in this way, the proposition \( \alpha \), specifically, \( v(\hat{P}_\alpha) = [\left[ \alpha \right]]_v = 1 \). Contrariwise, if \( v(\hat{P}_\alpha) = [\left[ \alpha \right]]_v = 1 \), then one can assume that the system is in the state \( |\Psi_\alpha\rangle \in \text{ran}(\hat{P}_\alpha) \). These two assumptions can be written down together as the logical biconditional, namely,

\[
|\Psi_\alpha\rangle \in \text{ran}(\hat{P}_\alpha) \iff v(\hat{P}_\alpha) = [\left[ \alpha \right]]_v = 1.
\]

(18)

On the other hand, the vector \( |\Psi_\alpha\rangle \) must lie in the null space of any projection operator \( \hat{P}_\beta \) orthogonal to \( \hat{P}_\alpha \). Since being in \( \ker(\hat{P}_\beta) \) means \( \hat{P}_\beta |\Psi_\alpha\rangle = 0 \cdot |\Psi_\alpha\rangle \), one can assume then that
\[ |\Psi_\alpha\rangle \in \ker(\hat{P}_\beta) \iff v(\hat{P}_\beta) = [\beta]_v = 0. \] (19)

Suppose by contrast that the system is in the pure state \(|\Psi\rangle\) that does not lie in the column or null space of the projection operator \(\hat{P}_\beta\), i.e., \(|\Psi\rangle \notin \text{ran}(\hat{P}_\beta)\) and \(|\Psi\rangle \notin \ker(\hat{P}_\beta)\). Under the valuation assumptions (18) and (19), the truth-value function \(v\) must assign neither 1 nor 0 to \(\hat{P}_\beta\), namely, \(v(\hat{P}_\beta) \neq 1\) and \(v(\hat{P}_\beta) \neq 0\). Hence, in this case the proposition \(\Diamond\) associated with \(\hat{P}_\beta\) cannot be bivalent, namely, \([\Diamond]_v \notin \mathcal{V}_2\).

Using a supervaluationary semantics (see, for example [2] or [3]), this failure of bivalence can be described as a truth-value gap for the proposition \(\Diamond\), explicitly,

\[ |\Psi\rangle \notin \left\{ \begin{array}{c} \text{ran}(\hat{P}_\beta) \\ \ker(\hat{P}_\beta) \end{array} \right\} \iff \{v(\hat{P}_\beta)\} = \{[\Diamond]_v\} = \emptyset. \] (20)

Within the said semantics, the operators \(\hat{1}\) and \(\hat{0}\) can be equated with “the super-truth” and “the super-falsity” since under the valuations (18) and (19) these operators are true and false, respectively, in any arbitrary state \(|\Phi\rangle\) in the Hilbert space \(\mathcal{H}\), that is,

\[ |\Phi\rangle \in \left\{ \begin{array}{c} \text{ran}(\hat{1}) = \mathcal{H} \\ \ker(\hat{0}) = \mathcal{H} \end{array} \right\} \iff \begin{cases} v(\hat{1}) = 1 \\ v(\hat{0}) = 0 \end{cases}. \] (21)

In this way, the approach based on the assumption (20) results in a “gappy” logic with the propositions giving rise to truth-value gaps. Accordingly, one can call this approach quantum supervaluationism (for other details of the approach see [4]).

### 3 Logic EPR account

The entangled particles \(A\) and \(B\) can be represented by observables \(\sigma_j^{(A)} \otimes \sigma_j^{(B)}\) created by the Pauli matrices, namely,

\[
\begin{align*}
\sigma_z^{(A)} \otimes \sigma_z^{(B)} &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, & \sigma_x^{(A)} \otimes \sigma_x^{(B)} &= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}, & \sigma_y^{(A)} \otimes \sigma_y^{(B)} &= \begin{bmatrix}
1 & 0 & 0 & \bar{1} \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\bar{1} & 0 & 0 & 0
\end{bmatrix},
\end{align*}
\] (22)

where \(\bar{1} \equiv -1\).

Take the observable \(\sigma_z^{(A)} \otimes \sigma_z^{(B)}\): Its eigenvectors \(|\Psi_{zz}(\lambda_{1,2})\rangle\) corresponding to the negative eigenvalues \(\lambda_{1,2} = -1\) are
\[ |\Psi_{zz}(\lambda_1 = -1)\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \text{ran}(\hat{P}^\dagger_z) = \left\{ \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\} , \tag{23} \]

\[ |\Psi_{zz}(\lambda_2 = -1)\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \text{ran}(\hat{P}^\uparrow z) = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\} , \tag{24} \]

where \( \hat{P}^\dagger_z \) and \( \hat{P}^\uparrow_z \) denote the orthogonal projection operators defined as

\[ \hat{P}^\dagger_z = |\uparrow_z^{(A)}\rangle \langle \uparrow_z^{(B)}| \otimes |\downarrow_z^{(A)}\rangle \langle \downarrow_z^{(B)}| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{25} \]

\[ \hat{P}^\uparrow_z = |\downarrow_z^{(A)}\rangle \langle \downarrow_z^{(B)}| \otimes |\uparrow_z^{(A)}\rangle \langle \uparrow_z^{(B)}| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{26} \]

On the other hand, the singlet state \( |0, 0_z\rangle \) is the vector

\[ |0, 0_z\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_z^{(A)}\rangle \otimes |\downarrow_z^{(B)}\rangle - |\downarrow_z^{(A)}\rangle \otimes |\uparrow_z^{(B)}\rangle \right) , \tag{27} \]

where

\[ |\uparrow_z^{(A)}\rangle \otimes |\downarrow_z^{(B)}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \tag{28} \]

\[ |\downarrow_z^{(A)}\rangle \otimes |\uparrow_z^{(B)}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \tag{29} \]

and so

\[ |0, 0_z\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} . \tag{30} \]
As follows, the vector \(|0, 0_z\rangle\) lies in the column space of the sum of two projection operators \(\hat{P}^{\downarrow \uparrow}_z\) and \(\hat{P}^{\uparrow \downarrow}_z\):

\[
|0, 0_z\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \text{ran}(\hat{P}^{\downarrow \uparrow}_z + \hat{P}^{\uparrow \downarrow}_z) = \text{ran}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.
\]

One can conclude from here that the projection operator \(\hat{P}^\text{Diff}_z\) associated with the proposition \(\text{Diff}_z\) can be presented as the sum:

\[
\hat{P}^\text{Diff}_z = \hat{P}^{\downarrow \uparrow}_z + \hat{P}^{\uparrow \downarrow}_z.
\]

Likewise, one can get

\[
|0, 0_x\rangle = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \text{ran}(\hat{P}^{\downarrow \uparrow}_x + \hat{P}^{\uparrow \downarrow}_x) = \left\{ \begin{bmatrix} a - c \\ b \\ -b \\ a + c \end{bmatrix} : a, b, c \in \mathbb{R} \right\},
\]

where

\[
\hat{P}^\text{Diff}_x = \hat{P}^{\downarrow \uparrow}_x + \hat{P}^{\uparrow \downarrow}_x = \frac{1}{4} \begin{bmatrix} 1 & \bar{1} & 1 & 1 \\ \bar{1} & 1 & \bar{1} & 1 \\ 1 & \bar{1} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & \bar{1} & 1 & \bar{1} \\ \bar{1} & 1 & 1 & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \\ \bar{1} & \bar{1} & 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & \bar{1} \\ 0 & 1 & \bar{1} & 0 \\ \bar{1} & 0 & 0 & 1 \\ 0 & \bar{1} & 1 & 0 \end{bmatrix},
\]

\[
\text{ran}(\hat{P}^{\downarrow \uparrow}_x) = \left\{ \begin{bmatrix} a \\ -a \\ a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}, \quad \text{ran}(\hat{P}^{\uparrow \downarrow}_x) = \left\{ \begin{bmatrix} a \\ -a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\},
\]

and

\[
|0, 0_y\rangle = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \text{ran}(\hat{P}^\text{Diff}_y) = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} : a, b \in \mathbb{C} \right\},
\]

in which

\[
\hat{P}^\text{Diff}_y = \hat{P}^{\downarrow \uparrow}_y + \hat{P}^{\uparrow \downarrow}_y = \frac{1}{4} \begin{bmatrix} 1 & i & -i & 1 \\ -i & 1 & \bar{1} & -i \\ i & \bar{1} & 1 & i \\ 1 & -i & i & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & -i & i & 1 \\ i & 1 & \bar{1} & i \\ -i & \bar{1} & 1 & -i \\ 1 & -i & i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & \bar{1} \\ 0 & 1 & \bar{1} & 0 \\ \bar{1} & 0 & 0 & 1 \\ 0 & \bar{1} & 1 & 0 \end{bmatrix}.
\]
\[
\text{ran}(\hat{P}_{y}^{\uparrow \downarrow}) = \left\{ \begin{bmatrix} a \\ -ia \\ ia \\ a \end{bmatrix} : a \in \mathbb{C} \right\}, \quad \text{ran}(\hat{P}_{y}^{\uparrow}) = \left\{ \begin{bmatrix} a \\ ia \\ -ia \\ a \end{bmatrix} : a \in \mathbb{C} \right\}.
\]

Consequently, one gets

\[
|0, 0_{j}\rangle = C \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{C} \right\} \subseteq \left\{ \begin{bmatrix} a-c \\ b \\ -b \\ a+c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} \subseteq \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} : a, b \in \mathbb{C} \right\},
\]

where \( C \in \mathbb{C} \). Under the supervaluationist postulation (20), this brings the following valuations in the singlet state \(|0, 0_{j}\rangle\):

\[
|0, 0_{j}\rangle \in \text{ran}(\hat{P}_{j}^{\text{Diff}}) \iff v(\hat{P}_{j}^{\uparrow \downarrow} + \hat{P}_{j}^{\uparrow}) = \llbracket \text{Diff}_{j} \rrbracket_{v} = 1,
\]

\[
|0, 0_{j}\rangle \notin \text{ran}(\hat{P}_{j}^{\uparrow \downarrow}) \iff \begin{cases} \text{ran}(\hat{P}_{j}^{\uparrow \downarrow}) & v(\hat{P}_{j}^{\uparrow}) = \left\{ \llbracket 1_{j}^{(A)} \wedge \downarrow_{j}^{(B)} \rrbracket_{v} \right\} = \emptyset, \\ \text{ker}(\hat{P}_{j}^{\uparrow \downarrow}) & v(\hat{P}_{j}^{\uparrow}) = \left\{ \llbracket 1_{j}^{(A)} \wedge \downarrow_{j}^{(B)} \rrbracket_{v} \right\} = \emptyset. \end{cases}
\]

As follows, even though the proposition \( \text{Diff}_{j} \) has the preexisting value of truth in the state \(|0, 0_{j}\rangle\), the statement \( m_{j}^{(A)} = \pm \frac{1}{2} \) and \( m_{j}^{(B)} = \mp \frac{1}{2} \) does not have the truth-value at all prior to its verification. Otherwise stated, in the singlet state the valuation \( \llbracket \text{Diff}_{j} \rrbracket_{v} = \llbracket 1_{j}^{(A)} \wedge \downarrow_{j}^{(B)} \rrbracket_{v} \) does not accept the principle of truth-functionality, namely, \( \llbracket \text{Diff}_{j} \rrbracket_{v} \) cannot be presented as a function of \( \llbracket 1_{j}^{(A)} \rrbracket_{v} \) and \( \llbracket \downarrow_{j}^{(B)} \rrbracket_{v} \).
Suppose Alice verifies experimentally that the proposition $\uparrow_z^{(A)}$ is true but the proposition $\downarrow_z^{(A)}$ is false. After Alice’s verification, the spin state of the two-particle system $|\Psi^{(AB)}\rangle$ turns into separable $|\Psi_z^{(A)}\rangle \otimes |\Psi_z^{(B)}\rangle$ where

$$
|\Psi_z^{(A)}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \begin{cases} \text{ran}(|\uparrow_z^{(A)}\rangle \langle \uparrow_z^{(A)}|) & \iff v(|\uparrow_z^{(A)}\rangle \langle \uparrow_z^{(A)}|) = [\uparrow_z^{(A)}]_v = 1 \\ \text{ker}(|\downarrow_z^{(A)}\rangle \langle \downarrow_z^{(A)}|) & \iff v(|\downarrow_z^{(A)}\rangle \langle \downarrow_z^{(A)}|) = [\downarrow_z^{(A)}]_v = 0 \end{cases}
$$

and $|\Psi_z^{(B)}\rangle = \begin{bmatrix} b \\ a \end{bmatrix}$ where $a$ and $b$ are unknown real numbers. This gives

$$
|\Psi_z^{(A)}\rangle \otimes |\Psi_z^{(B)}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \\ 0 \\ 0 \end{bmatrix} \in \begin{cases} \text{ran}(\hat{P}_z^{\uparrow\uparrow}) , \text{ if } a = 0 \\ \text{ran}(\hat{P}_z^{\downarrow\downarrow}) , \text{ if } b = 0 \end{cases}
$$

But since system’s state is prepared in the singlet state lying in the sum $\text{ran}(\hat{P}_z^{\uparrow\downarrow}) + \text{ran}(\hat{P}_z^{\downarrow\uparrow})$, one can infer that after Alice’s verification the value of the projection operator $\hat{P}_z^{\uparrow\downarrow}$ becomes true, namely, $v(\hat{P}_z^{\uparrow\downarrow}) = 1$, which implies $b = 0$ and so

$$
|\Psi_z^{(B)}\rangle = \begin{bmatrix} 0 \\ a \end{bmatrix} \iff \begin{cases} v(|\uparrow_z^{(B)}\rangle \langle \uparrow_z^{(B)}|) = [\uparrow_z^{(B)}]_v = 0 \\ v(|\downarrow_z^{(B)}\rangle \langle \downarrow_z^{(B)}|) = [\downarrow_z^{(B)}]_v = 1 \end{cases}
$$

even without Bob’s verification.

It is straightforward that

$$
\begin{bmatrix} 0 \\ a \end{bmatrix} : a \in \mathbb{R} \notin \begin{cases} \text{ran}(|\uparrow_z^{(B)}\rangle \langle \uparrow_z^{(B)}|) = \begin{bmatrix} a \\ a \end{bmatrix} : a \in \mathbb{R} \\ \text{ran}(|\downarrow_z^{(B)}\rangle \langle \downarrow_z^{(B)}|) = \begin{bmatrix} a \\ -a \end{bmatrix} : a \in \mathbb{R} \end{cases}
$$

so, under the supervaluationist postulation, in the state $|\Psi_z^{(B)}\rangle$ neither $\uparrow_z^{(B)}$ nor $\downarrow_z^{(B)}$ can carry the truth value, explicitly,

$$
|\Psi_z^{(B)}\rangle \notin \begin{cases} \text{ran}(|\uparrow_z^{(B)}\rangle \langle \uparrow_z^{(B)}|) \iff \begin{cases} v(|\uparrow_z^{(B)}\rangle \langle \uparrow_z^{(B)}|) = [\uparrow_z^{(B)}]_v = \emptyset \\ v(|\downarrow_z^{(B)}\rangle \langle \downarrow_z^{(B)}|) = [\downarrow_z^{(B)}]_v = \emptyset \end{cases} \end{cases}
$$

Consequently, after the verification of $\uparrow_z^{(A)}$ the statistical population describing the product $[\downarrow_z^{(B)}]_v \cdot [\uparrow_z^{(B)}]_v$ would have no elements at all:

$$
\{[\downarrow_z^{(B)}]_v\} \times \{[\uparrow_z^{(B)}]_v\} = \{1\} \times \emptyset = \emptyset
$$

This implies that the statement “$m_z^{(B)} = -\frac{1}{2}$ and $m_z^{(B)} = \frac{1}{2}$” would have no truth value, and so the verification (refutation) of $\uparrow_z^{(A)}$ in the next experimental run would not bring the definiteness
to the product $\left[ \downarrow_z^{(B)} \right]_v \cdot \left[ \uparrow_z^{(B)} \right]_v$.

### 4 Conclusion remarks

As the expression (9) demonstrates, it is counter-factual definiteness that, together with the principle of truth-functionality, led to the bivaluation of the statement $m_z^{(B)} = \mp \frac{1}{2}$ and $m_x^{(B)} = \pm \frac{1}{2}$.

Then again, from the point of view of the logical matrix which fixes a model of logic \[5\], counter-factual definiteness can be interpreted as the assertion that the experimentally testable propositions – like $\downarrow_z^{(B)}$ and $\downarrow_x^{(B)}$ – possess intrinsic truth values that exist even when these propositions have not been verified (see to that end the definition of counter-factual definiteness in \[6, 7, 8\]).

This suggests that to permit realist interpretations of quantum mechanics (whose characteristic details can be found, e.g., in papers \[9, 10\]), models of logic underpinning such interpretations must reject preexisting truth-values in general – as the described in this paper quantum supervaluationist EPR account does.

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