GEODESIC ORBIT METRICS IN A CLASS OF HOMOGENEOUS BUNDLES OVER QUATERNIONIC STIEFEL MANIFOLDS

ANDREAS ARVANITOEYORGOS, NIKOLAOS PANAGIOTIS SOURIS AND MARINA STATHA

Abstract. Geodesic orbit spaces (or g.o. spaces) are defined as those homogeneous Riemannian spaces \( M = G/H, g \) whose geodesics are orbits of one-parameter subgroups of \( G \). The corresponding metric \( g \) is called a geodesic orbit metric. We study the geodesic orbit spaces of the form \( (\text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s), g) \), with \( 0 < n_1 + \cdots + n_s \leq n \). Such spaces include spheres, quaternionic Stiefel manifolds, Grassmann manifolds and quaternionic flag manifolds. The present work is a contribution to the study of g.o. spaces \( (G/H, g) \) with \( H \) semisimple.

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1. Introduction

Geodesic orbit spaces \( (M = G/H, g) \) are defined by the simple property that any geodesic \( \gamma \) has the form

\[ \gamma(t) = \exp(tX) \cdot o, \]

where \( \exp \) is the exponential map on \( G, o = \gamma(0) \) is a point in \( M \) and \( \cdot \) denotes the action of \( G \) on \( M \). These spaces were initially considered in [14] and up to today they have been extensively studied within various geometric contexts, including the Riemannian ([13]), pseudo-Riemannian ([10]), Finsler ([24]) and affine ([12]) context. The classification of g.o. spaces remains an open problem.

There are diverse examples of g.o. spaces, including the classes of symmetric spaces, weakly symmetric spaces ([9], [23]), isotropy irreducible spaces ([22]), \( \delta \)-homogeneous spaces ([6]), Clifford-Wolf homogeneous spaces ([7]) and naturally reductive spaces. Reviews about g.o. spaces up to 2017 can be found in [5] and in the introduction of [15]. Various results up to 2020 are included in the recently published book [8].

Determining the g.o. metrics among the \( G \)-invariant metrics on a space \( G/H \) presents some challenges. The main challenge lies in the fact that the space of \( G \)-invariant metrics may have complicated structure, depending on whether the isotropy representation of \( H \) on the tangent space \( T_o(G/H) \) contains pairwise equivalent submodules. To remedy this obstruction, various simplification results for g.o. metrics have been established (e.g. [15], [19]). A general observation is that the existence and the form of the g.o. metrics on \( G/H \) depends to a large extent on the structure of the tangent space \( T_o(G/H) \) induced from the isotropy representation and on the Lie algebraic relations between the corresponding submodules (e.g. [11]).
When $G$ is compact semisimple, the classification of the g.o. spaces $(G/H, g)$ with $H$ abelian and $H$ simple has been obtained in the works [20] and [11] respectively. On the other hand, the classification of compact g.o. spaces $(G/H, g)$ with $H$ semisimple remains open, while no general results are known for this case. The present paper is a continuation of our work [4] towards a study of g.o. metrics on a general family of spaces $G/H$, such that the isotropy representation of all of its members has a similar description. In that work we studied the geodesic orbit metrics on the spaces $\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s)$ and $\text{U}(n)/\text{U}(n_1) \times \cdots \times \text{U}(n_s)$ with $0 < n_1 + \cdots + n_s \leq n$. These are spaces $G/H$ where $G$ is a compact classical Lie group and $H$ is a diagonally embedded product of Lie groups of the same type as $G$.

In the present paper we study the geodesic orbit metrics on the spaces $\text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s)$ with $0 < n_1 + \cdots + n_s \leq n$. These spaces properly include the spheres $S^{4n-1} = \text{Sp}(n)/\text{Sp}(n-1)$, the quaternionic Stiefel manifolds $\text{Sp}(n)/\text{Sp}(n-k)$, the Grassmann manifolds $\text{Sp}(n)/\text{Sp}(k) \times \text{Sp}(n-k)$, and the quaternionic flag manifolds $\text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s)$, $n_1 + \cdots + n_s = n$ ([18]). If $n_1 + \cdots + n_s < n$, each of these spaces can be viewed as a total space over a quaternionic Stiefel manifold, with fiber a quaternionic flag manifold, i.e.

$$\text{Sp}(m)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s) \to \text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s) \to \text{Sp}(n)/\text{Sp}(m),$$

with $m = n_1 + \cdots + n_s$.

Our main result is the following.

**Theorem 1.1.** Let $G/H$ be the space $\text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s)$, where $0 < n_1 + \cdots + n_s \leq n$. If $G/H \neq \text{Sp}(n)/\text{Sp}(n-1)$ (i.e. if $n - (n_1 + \cdots + n_s) \neq 1$ or $s > 1$) then a $G$-invariant Riemannian metric on $G/H$ is geodesic orbit if and only if it is the standard metric induced from the Killing form on the Lie algebra $\mathfrak{sp}(n)$ of $\text{Sp}(n)$.

If $G/H = \text{Sp}(n)/\text{Sp}(n-1)$ (i.e. $s = 1$ and $n - n_1 = 1$) then a $G$-invariant metric $g$ on $G/H$ is geodesic orbit if and only if $g = g_\mu$, $\mu > 0$, where $g_\mu$ denotes a one-parameter family of deformations of the standard metric $g_1$ of the fibration $\text{Sp}(1)$ of the fibration $\text{Sp}(n)/\text{Sp}(n-1) \to \text{Sp}(n)/\text{Sp}(1) \times \text{Sp}(n-1)$.

We remark that the non standard geodesic orbit metric $g_\mu$ appears in [16] and [21].

The paper is structured as follows: In Section 2 we present preliminary facts about homogeneous spaces and some special properties of the isotropy representation, to be used in our study. In Section 3 we state and prove some useful properties of g.o. spaces. In Section 4 we describe the isotropy representation of the spaces $G/H = \text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s)$ and the structure of the tangent space $T_o(G/H)$ induced from the isotropy representation. In Section 5 we state and prove some preliminary propositions required for the proof of Theorem 1.1. Finally, in Section 6 we prove Theorem 1.1.

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2. Preliminaries

2.1. Invariant metrics on homogeneous spaces. Let $G/H$ be a homogeneous space with origin $o = eH$ and assume that $G$ is compact. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of $G, H$ respectively. Moreover, let $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ and $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ be the adjoint representations of $G$ and $\mathfrak{g}$ respectively, where $\text{ad}(X)Y = [X, Y]$. Since $G$ is compact, there exists an $\text{Ad}$-invariant (and hence $\text{ad}$ skew-symmetric) inner product $B$ on $\mathfrak{g}$, which we henceforth fix. In turn, we have a $B$-orthogonal reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where the subspace $\mathfrak{m}$ is $\text{Ad}(H)$-invariant (and $\text{ad}(\mathfrak{h})$-invariant) and is naturally identified with the tangent space of $G/H$ at the origin.

A Riemannian metric $g$ on $G/H$ is called $G$-invariant if for any $x \in G$ the left translations $\tau_x : G/H \rightarrow G/H, pH \mapsto (xp)H$, are isometries of $(G/H, g)$. The $G$-invariant metrics are in one to one correspondence with $\text{Ad}(H)$-invariant inner products $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$. Moreover, any such product corresponds to a unique endomorphism $A : \mathfrak{m} \rightarrow \mathfrak{m}$, called the corresponding metric endomorphism, that satisfies

$$\langle X, Y \rangle = B(AX, Y) \quad \text{for all } X, Y \in \mathfrak{m}. \quad (2)$$

It follows from Equation $(2)$ that the metric endomorphism $A$ is symmetric with respect $B$, positive definite and $\text{Ad}(H)$-equivariant, that is $(\text{Ad}(h) \circ A)(X) = (A \circ \text{Ad}(h))(X)$ for all $h \in H$ and $X \in \mathfrak{m}$. Conversely, any endomorphism on $\mathfrak{m}$ with the above properties determines a unique $G$-invariant metric on $G/H$.

Since $A$ is diagonalizable, there exists a decomposition $\mathfrak{m} = \bigoplus_{j=1}^{l} \mathfrak{m}_{\lambda_j}$ into eigenspaces $\mathfrak{m}_{\lambda_j}$ of $A$, corresponding to distinct eigenvalues $\lambda_j$. Each eigenspace $\mathfrak{m}_{\lambda_j}$ is $\text{Ad}(H)$-invariant. When an $\text{Ad}$-invariant inner product $B$ and a $B$-orthogonal reductive decomposition $(1)$ have been fixed, we will make no distinction between a $G$-invariant metric $g$ and its corresponding metric endomorphism $A$.

The form of the $G$-invariant metrics on $G/H$ depends on the isotropy representation $\text{Ad}^{G/H} : H \rightarrow \text{Gl}(\mathfrak{m})$, defined by $\text{Ad}^{G/H}(h)X := (d\tau_h)_o(X)$, $h \in H$, $X \in \mathfrak{m}$. We consider a $B$-orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s,$$

into $\text{Ad}^{G/H}$-invariant and irreducible submodules. We recall that two submodules $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are equivalent if there exists an $\text{Ad}^{G/H}$-equivariant isomorphism $\phi : \mathfrak{m}_i \rightarrow \mathfrak{m}_j$. The simplest case occurs when all the submodules $\mathfrak{m}_i$ are pairwise inequivalent. Then any $G$-invariant metric $A$ on $G/H$ has a diagonal expression with respect to decomposition $(3)$. In particular, $A|_{\mathfrak{m}_j} = \lambda_j \text{Id}, \ j = 1, \ldots, s$.

The next proposition is useful to compute the isotropy representation of a reductive homogeneous space.

**Proposition 2.1.** Let $G/H$ be a homogeneous space and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$. Let $h \in H$, $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$. Then

$$\text{Ad}^G(h)(X + Y) = \text{Ad}^H(h)X + \text{Ad}^{G/H}(h)Y$$
that is, the restriction $\text{Ad}^G|_H$ splits into the sum $\text{Ad}^H \oplus \text{Ad}^{G/H}$. We denote by $\chi$ the $\text{Ad}^{G/H}$.

The following lemma provides a simple condition for proving that two $\text{Ad}^{G/H}$-submodules are inequivalent.

**Lemma 2.2.** Let $G/H$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\mathfrak{m}_i, \mathfrak{m}_j \subseteq \mathfrak{m}$ be submodules of the isotropy representation $\text{Ad}^{G/H}$. Assume that for any pair of non-zero vectors $X \in \mathfrak{m}_i$, $Y \in \mathfrak{m}_j$ there exists a vector $\mathfrak{a} \in \mathfrak{h}$ such that $[\mathfrak{a}, X] = 0$ and $[\mathfrak{a}, Y] \neq 0$. Then the submodules $\mathfrak{m}_i, \mathfrak{m}_j$ are $\text{Ad}^{G/H}$-inequivalent.

**Proof.** If $\mathfrak{m}_i, \mathfrak{m}_j$ are equivalent, then there exists an $\text{Ad}^{G/H}$-equivariant isomorphism $\phi : \mathfrak{m}_i \to \mathfrak{m}_j$. The $\text{Ad}^{G/H}$-equivariance of $\phi$ implies that $\phi$ is $\text{ad}_{\mathfrak{h}}$-equivariant. In other words,

$$\phi([\mathfrak{a}, X]) = [\mathfrak{a}, Y],$$

for any $\mathfrak{a} \in \mathfrak{h}$. However, $\phi$ is an isomorphism, therefore, $[\mathfrak{a}, X]$ is non-zero if and only if $[\mathfrak{a}, Y]$ is non-zero, which contradicts the hypothesis of the lemma. Hence, the submodules $\mathfrak{m}_i, \mathfrak{m}_j$ are inequivalent. \hfill $\square$

**Remark 2.3.** For any two $\text{Ad}^{G/H}$-submodules $\mathfrak{m}_1, \mathfrak{m}_2$, we denote by $[\mathfrak{m}_1, \mathfrak{m}_2]$ the space generated by the vectors $[X_1, X_2]$ where $X_1 \in \mathfrak{m}_1$ and $X_2 \in \mathfrak{m}_2$. Similarly, denote by $[\mathfrak{h}, \mathfrak{m}_1]$ the space generated by the vectors $[\mathfrak{a}, X_1]$ where $\mathfrak{a} \in \mathfrak{h}$ and $X_1 \in \mathfrak{m}_1$. If $\mathfrak{m}_1, \mathfrak{m}_2$ are $B$-orthogonal then $[\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathfrak{m}$. Indeed, $[\mathfrak{m}_1, \mathfrak{m}_2]$ is $B$-orthogonal to $\mathfrak{h}$ because $B([\mathfrak{m}_1, \mathfrak{m}_2], \mathfrak{h}) \subseteq B(\mathfrak{m}_1, \mathfrak{m}_2) = \{0\}$. Moreover, by the Jacobi identity, $[\mathfrak{m}_1, \mathfrak{m}_2]$ is also an $\text{Ad}^{G/H}$-submodule and $[\mathfrak{h}, \mathfrak{m}_1]$ is an $\text{Ad}^{G/H}$-submodule of $\mathfrak{m}_1$.

### 3. Properties of geodesic orbit spaces

**Definition 3.1.** A $G$-invariant metric $g$ on $G/H$ is called a geodesic orbit metric (g.o. metric) if any geodesic of $(G/H, g)$ through $o$ is an orbit of a one parameter subgroup of $G$. Equivalently, $g$ is a geodesic orbit metric if for any geodesic $\gamma$ of $(G/H, g)$ though $o$ there exists a non-zero vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX) \cdot o$, $t \in \mathbb{R}$. The space $(G/H, g)$ is called a geodesic orbit space (g.o. space).

Let $G/H$ be a homogeneous space with $G$ compact, fix an $\text{Ad}$-invariant inner product $B$ on $\mathfrak{g}$ and consider the $B$-orthogonal reductive decomposition \cite{11}. Moreover, identify each $G$-invariant metric on $G/H$ with the corresponding metric endomorphism $A : \mathfrak{m} \to \mathfrak{m}$.

We have the following condition.

**Proposition 3.2.** \cite{11,19} The metric $A$ on $G/H$ is geodesic orbit if and only if for any vector $X \in \mathfrak{m} \setminus \{0\}$ there exists a vector $\mathfrak{a} \in \mathfrak{h}$ such that

$$[\mathfrak{a} + X, AX] = 0. \quad (4)$$

The following result, which we will call the normalizer lemma, can be used to simplify the necessary form of the g.o. metrics on $G/H$ by using the normalizer $N_G(H^0)$.

**Lemma 3.3.** \cite{14} The inner product $(\cdot, \cdot)$, generating the metric of a geodesic orbit Riemannian space $(G/H, g)$, is not only $\text{Ad}(H)$-invariant but also $\text{Ad}(N_G(H^0))$-invariant, where $N_G(H^0)$ is the normalizer of the unit component $H^0$ of the group $H$ in $G$. 

As a result of the normalizer lemma, the metric endomorphism $A$ of a g.o. metric on $G/H$ is $\text{Ad}(N_G(H^0))$-equivariant. We will now state a complementary result to the normalizer lemma for compact spaces, that characterizes the restriction of a g.o. metric to the compact Lie group $N_G(H^0)/H^0$.

**Lemma 3.4.** Let $(G/H, g)$ be a compact geodesic orbit space with the $B$-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $B$ is an $\text{Ad}$-invariant inner product on $\mathfrak{g}$. Let $A : \mathfrak{m} \to \mathfrak{m}$ be the corresponding metric endomorphism of $g$, and let $\mathfrak{n} \subseteq \mathfrak{m}$ be the Lie algebra of the compact Lie group $N_G(H^0)/H^0$. Then the restriction of $A$ to $\mathfrak{n}$ defines a bi-invariant metric on $N_G(H^0)/H^0$.

**Proof.** We denote by $\mathfrak{n}_0(\mathfrak{h}) \subset \mathfrak{g}$ the Lie algebra of $N_G(H^0)$. We have a $B$-orthogonal decomposition $\mathfrak{n}_0(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ coincides with the Lie algebra of $N_G(H^0)/H^0$. Moreover, we have a $B$-orthogonal decomposition $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$, where $\mathfrak{p}$ coincides with the tangent space of $G/N_G(H^0)$ at the origin. By the normalizer lemma, the restriction of $A$ on $\mathfrak{p}$ defines an invariant metric on $G/N_G(H^0)$, and thus $A \mathfrak{p} \subseteq \mathfrak{p}$. By taking into account the symmetry of $A$ with respect to the product $B$, we deduce that $B(A\mathfrak{n}, \mathfrak{p}) = B(\mathfrak{n}, A\mathfrak{p}) \subseteq B(\mathfrak{n}, \mathfrak{p}) = \{0\}$. Hence, the image $A\mathfrak{n}$ is $B$-orthogonal to $\mathfrak{p}$ which, along with decomposition $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$, yields $A\mathfrak{n} \subseteq \mathfrak{n}$. Therefore, the restriction $A|_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$ defines a left-invariant metric on $N_G(H^0)/H^0$. Since $A$ is a g.o. metric on $G/H$, Proposition 3.2 implies that for any $X \in \mathfrak{n}$ there exists a vector $a \in \mathfrak{h}$ such that $0 = [a + X, AX] = [a + X, A|_{\mathfrak{n}} X]$. Therefore, by the same proposition, $A|_{\mathfrak{n}}$ defines a g.o. metric on $N_G(H^0)/H^0$. On the other hand, any left-invariant g.o. metric on a Lie group is necessarily bi-invariant ([2]), and hence $A|_{\mathfrak{n}}$ is a bi-invariant metric on $N_G(H^0)/H^0$. \hfill \Box

**Remark 3.5.** Let $\mathfrak{p} \subset \mathfrak{m}$ be the tangent space of $G/N_G(H^0)$, let $\mathfrak{n} \subset \mathfrak{m}$ be the Lie algebra of $N_G(H^0)/H^0$, and consider the decomposition $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$. By combining lemmas 3.3 and 3.4, we conclude that the metric endomorphism $A$ corresponding to a g.o. metric has the block-diagonal form

$$A = \begin{pmatrix} A|_{\mathfrak{n}} & 0 \\ 0 & A|_{\mathfrak{p}} \end{pmatrix},$$

where $A|_{\mathfrak{n}}$ defines a bi-invariant metric on $N_G(H^0)/H^0$ and $A|_{\mathfrak{p}}$ defines a g.o. metric on $G/N_G(H^0)$.

The following lemma describes the invariant g.o. metrics on compact Lie groups in terms of their metric endomorphism with respect to an $\text{Ad}$-invariant inner product $B$. Recall that the Lie algebra $\mathfrak{g}$ of a compact Lie group $G$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{j}$, where $\mathfrak{g}_j$ are simple ideals of $\mathfrak{g}$ and $\mathfrak{j}$ is its center.

**Lemma 3.6.** ([2], [19]) Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{j}$. A left invariant metric $A$ on $G$ is a g.o. metric if and only if it is bi-invariant. In particular, $A$ is a g.o. metric if and only if
The following terminology will be useful. Let $W$ be a subspace of a vector space $V$ and we write $V = W ⊕ W^\perp$ with respect to some inner product $B$ on $V$. Then, for $v ∈ V$ it is $v = w + w'$, where $w ∈ W$ and $w' ∈ W^\perp$.

**Definition 3.7.** The vector $v$ has non zero projection on $W$ if $w' ≠ 0$. A subset $S$ of $V$ has non zero projection on $W$ if there exists a vector $v ∈ S$ that has non zero projection on $W$.

Finally, the following lemma is useful, since it will enable us to equate some of the eigenvalues of a g.o. metric.

**Lemma 3.8.** ([19]) Let $(G/H, g)$ be a g.o. space with $G$ compact and with corresponding metric endomorphism $A$ with respect to an $\text{Ad}$-invariant inner product $B$. Let $m$ be the $B$-orthogonal complement of $h$ in $g$.

1. Assume that $m_1, m_2$ are $\text{ad}(h)$-invariant, pairwise $B$-orthogonal subspaces of $m$ such that $[m_1, m_2]$ has non zero projection on $(m_1 ⊕ m_2)^\perp$. Let $λ_1, λ_2$ be eigenvalues of $A$ such that $A|m_i = λ_i \text{Id}, i = 1, 2$. Then $λ_1 = λ_2$.

2. Assume that $m_1, m_2, m_3$ are $\text{ad}(h)$-invariant, pairwise $B$-orthogonal subspaces of $m$ such that $[m_1, m_2]$ has non-zero projection on $m_3$. Let $λ_1, λ_2, λ_3$ be eigenvalues of $A$ such that $A|m_i = λ_i \text{Id}, i = 1, 2, 3$. Then $λ_1 = λ_2 = λ_3$.

4. The space $\text{Sp}(n)/\text{Sp}(n_1) × \cdots × \text{Sp}(n_s), \sum n_i ≤ n$

We compute the isotropy representation of $M = G/H = \text{Sp}(n)/\text{Sp}(n_1) × \cdots × \text{Sp}(n_s), n_1 + \cdots + n_s ≤ n$. Denote by $ν_{2n}$ the standard representation of $\text{Sp}(n)$, that is $ν_{2n} : \text{Sp}(n) → \text{Aut}(\mathbb{C}^{2n})$. Then the complexified adjoint representation of $\text{Sp}(n)$ is given by $\text{Ad}^{\text{Sp}(n)} ⊗ \mathbb{C} = S^2ν_{2n}$, where $S^2$ is the second symmetric power of $ν_{2n}$. Let $σ_{n_i} : \text{Sp}(n_1) × \cdots × \text{Sp}(n_s) → \text{Sp}(n_i)$ be the projection onto the $i$-factor and $φ_i = ν_{2n_i} × σ_{n_i}$ be the projection of the standard representation of $H$, i.e.

$$\text{Sp}(n_1) × \cdots × \text{Sp}(n_s) → \text{Sp}(n_i) → \text{Aut}(\mathbb{C}^{2n_i}).$$

We set $n_0 := n - (n_1 + n_2 + \cdots + n_s)$.

Then we have:

$$\text{Ad}^G ⊗ \mathbb{C}|_H = S^2ν_{2n} |_H = S^2(φ_1 ⊕ \cdots ⊕ φ_s ⊕ 1_{2n_0}) = S^2φ_1 ⊕ S^2φ_2 ⊕ \cdots \oplus S^2φ_s ⊕ S^21_{2n_0}$$

$$⊕((φ_1 ⊗ φ_2) ⊕ \cdots ⊕ (φ_1 ⊗ φ_s)) ⊕ ((φ_2 ⊗ φ_3) ⊕ \cdots (φ_2 ⊗ φ_s)) ⊕ \cdots (φ_{s-1} ⊗ φ_s)$$

$$⊕(φ_1 ⊗ 1_{2n_0}) ⊕ (φ_2 ⊗ 1_{2n_0}) ⊕ \cdots ⊕ (φ_s ⊗ 1_{2n_0}),$$

where $S^21_{2n_0} = 1 ⊕ \cdots ⊕ 1$ is the sum of $\binom{2n_0+1}{2} = n_0(2n_0 + 1)$ trivial representations.

The representation $S^2φ_1 ⊕ S^2φ_2 ⊕ \cdots ⊕ S^2φ_s$ is equal to $\text{Ad}^H ⊗ \mathbb{C}$, the complexified adjoint representation of $H = \text{Sp}(n_1) × \cdots × \text{Sp}(n_s)$. 
Then, in view of Equation (5), Proposition 2.1 implies that the complexified isotropy representation of $G/H$ is given by

$$
\chi \otimes \mathbb{C} = S^2 1_{2n_0} \oplus \{(\varphi_1 \otimes \varphi_2) \oplus \cdots \oplus (\varphi_1 \otimes \varphi_s)\} \oplus \{(\varphi_2 \otimes \varphi_3) \oplus \cdots \oplus (\varphi_2 \otimes \varphi_s)\} \oplus \cdots \oplus (\varphi_{s-1} \otimes \varphi_s) \\
\oplus (\varphi_1 \otimes 1_{2n_0}) \oplus (\varphi_2 \otimes 1_{2n_0}) \oplus \cdots \oplus (\varphi_s \otimes 1_{2n_0}) \\
= S^2 1_{2n_0} \bigoplus_{1 \leq i < j \leq s} (\varphi_i \otimes \varphi_j) \bigoplus_{j=1}^s (\varphi_j \otimes 1_{2n_0}).
$$

Expression (6) induces a decomposition of the complexified tangent space $m$ of $G/H$ as

$$
m \otimes \mathbb{C} = n'_1 \oplus \cdots \oplus n'_{n_0(2n_0+1)} \bigoplus_{1 \leq i < j \leq s} n_{ij} \bigoplus_{j=1}^s n_{0j};
$$

where $\dim_\mathbb{C}(n'_j) = 1$, $\dim_\mathbb{C} n_{ij} = 4n_in_j$ and $n_{0j} = n'_1 \oplus n'_2 \oplus \cdots \oplus n'_{2n_0}$, with $n'_\alpha \cong n'_\beta$, $\alpha \neq \beta$ and $\dim_\mathbb{C}(n'_0) = 2n_j$, $\ell = 1, 2, \ldots, 2n_0$.

The (real) decomposition of the tangent space of $G/H$ is given by

$$
m = n_1 \oplus \cdots \oplus n_{n_0(2n_0+1)} \bigoplus_{1 \leq i < j \leq s} m_{ij} \bigoplus_{j=1}^s m_{0j},
$$

where $m_{ij} \otimes \mathbb{C} = n_{ij}$, $m_{0j} \otimes \mathbb{C} = n_{0j}$, $\dim_\mathbb{R}(n_1) = 1$, $\dim_\mathbb{R}(m_{ij}) = 4n_in_j$, $\dim_\mathbb{R}(m_{0j}) = 4n_0n_j$.

Also, $m_{0j} = m_1^j \oplus \cdots \oplus m_{2n_0}^j$ with $m_\ell^j \otimes \mathbb{C} = n_\ell^j$, $m_\alpha^j \cong m_\beta^j$, $\alpha \neq \beta$ and $\dim_\mathbb{R}(m_\ell^j) = 2n_j$, $\ell = 1, \ldots, 2n_0$. Note that $n_1 \oplus \cdots \oplus n_{n_0(2n_0+1)} \cong \mathfrak{sp}(n_0)$.

**Remark 4.1.** If $n_0 = 0$ the isotropy representation of $G/H$ has no equivalent representations and all above expressions simplify.

We now give explicit matrix representations of $m_{ij}, m_{0j}$. Recall the Lie algebra of $\text{Sp}(n)$,

$$
\mathfrak{sp}(n) = \left\{ \begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{pmatrix} \mid X \in \mathfrak{u}(n), \ Y \text{ is a } n \times n \text{ complex symmetric matrix} \right\} \subset \mathfrak{u}(2n).
$$

For $i = 1, \ldots, s$, we embed $\mathfrak{sp}(n_i) = \left\{ \begin{pmatrix} X_i & -\bar{Y}_i \\ Y_i & \bar{X}_i \end{pmatrix} \right\}$ in $\mathfrak{sp}(n)$ as
We consider the Ad(Sp(n))-invariant inner product $B : \mathfrak{sp}(n) \times \mathfrak{sp}(n) \rightarrow \mathbb{R}$, given by

$$B(X, Y) = -\text{Trace}(XY), \quad X, Y \in \mathfrak{sp}(n),$$

and we obtain a $B$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathfrak{sp}(n_1) \oplus \cdots \oplus \mathfrak{sp}(n_s)$ and $\mathfrak{m} \cong T_o(G/H)$. We note that $B$ is a multiple of the Killing form of $\mathfrak{sp}(n)$.

Next, we consider a basis for $\mathfrak{g} = \mathfrak{sp}(n)$ as follows. Let $M_{2n, \mathbb{C}}$ be the set of $2n \times 2n$ complex matrices and we consider the following matrices in $M_{2n, \mathbb{C}}$ with zeros in all entries except the ones indicated:

- $E_{ab}$ with 1 in $(a, b)$-entry and 1 in $(n + a, n + b)$-entry.
- $F_{ab}$ with $i$ in $(a, b)$-entry and $-i$ in $(n + a, n + b)$-entry.
- $G_{ab}$ with $-1$ in $(a, n + b)$-entry and 1 in $(n + b, a)$-entry.
- $H_{ab}$ with $i$ in $(a, n + b)$-entry and $i$ in $(n + b, a)$-entry.

For $1 \leq a < b \leq 2n$ we set

$$e_{ab} = E_{ab} - E_{ba}, \quad f_{ab} = F_{ab} + F_{ba}, \quad g_{ab} = G_{ab} + G_{ba}, \quad h_{ab} = H_{ab} + H_{ba}. $$

Then the set $\mathcal{B} = \{e_{ab}, f_{ab}, g_{ab}, h_{ab} : 1 \leq a < b \leq n; f_{aa}, g_{aa}, h_{aa} : 1 \leq a \leq n\}$ constitutes a basis of $\mathfrak{sp}(n)$, which is orthogonal with respect to $B$.

The above matrices have the form

$$e_{ab} = \begin{pmatrix}
\begin{array}{c|c}
\begin{array}{cccc}
\vdots & \vdots & \ddots & \\
\end{array} & \begin{array}{cccc}
\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
\end{array}
\end{array}
\end{array}
\end{pmatrix}$$

$$f_{ab} = \begin{pmatrix}
\begin{array}{c|c}
\begin{array}{cccc}
\vdots & \vdots & \ddots & \\
\end{array} & \begin{array}{cccc}
\begin{array}{cccc}
i & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -i \\
\end{array}
\end{array}
\end{array}
\end{pmatrix}$$

$$g_{ab} = \begin{pmatrix}
\begin{array}{c|c}
\begin{array}{cccc}
\vdots & \vdots & \ddots & \\
\end{array} & \begin{array}{cccc}
\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{array}
\end{array}
\end{array}
\end{pmatrix}$$

$$h_{ab} = \begin{pmatrix}
\begin{array}{c|c}
\begin{array}{cccc}
\vdots & \vdots & \ddots & \\
\end{array} & \begin{array}{cccc}
\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{array}
\end{array}
\end{array}
\end{pmatrix}$$
Also, for isomorphic to \( m = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab} = g_{ab} \text{ and } h_{ba} = h_{ab}\}. The next lemma follows from straightforward calculations.

**Lemma 4.2.** The Lie-bracket relations among the vectors (9) are given as follows:

\[
\begin{align*}
[e_{ij}, e_{lm}] &= \delta_{jl}e_{im} + \delta_{im}e_{jl} - \delta_{it}e_{jm} - \delta_{jm}e_{it} \\
[e_{ij}, g_{lm}] &= -\delta_{jl}g_{im} + \delta_{im}g_{jl} - \delta_{it}g_{jm} + \delta_{jm}g_{it} \\
f_{ij}, f_{lm}] &= -\delta_{jl}f_{im} - \delta_{im}f_{jl} - \delta_{it}f_{jm} - \delta_{jm}f_{it} \\
g_{ij}, h_{lm}] &= -\delta_{jl}h_{im} + \delta_{im}h_{jl} - \delta_{it}h_{jm} + \delta_{jm}h_{it} \\
g_{ij}, h_{lm}] &= -\delta_{jl}g_{im} + \delta_{im}g_{jl} + \delta_{it}g_{jm} + \delta_{jm}g_{it} \\
g_{ij}, h_{lm}] &= -\delta_{jl}h_{im} + \delta_{im}h_{jl} + \delta_{it}h_{jm} + \delta_{jm}h_{it} \\
g_{ij}, h_{lm}] &= -\delta_{jl}g_{im} + \delta_{im}g_{jl} + \delta_{it}g_{jm} + \delta_{jm}g_{it}
\end{align*}
\]

A choice for the modules in the decomposition (7) is the following:

\[
m_{ij} = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab}, h_{ab} : n_0 + n_1 + \cdots + n_i - 1 + 1 \leq a \leq n_0 + n_1 + \cdots + n_i, \\
n_0 + n_1 + \cdots + n_i + 1 \leq b \leq n_0 + n_1 + \cdots + n_i, \ 1 \leq i < j \leq s\}
\]

\[
m_{0j} = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab}, h_{ab} : 1 \leq a \leq n_0, n_0 + n_1 + \cdots + n_j - 1 + 1 \leq b \leq n_0 + n_1 + \cdots + n_j, \ j = 1, \ldots, s\}
\]

The \( n_0(2n_0 + 1) \) trivial representations in (7) generate the Lie algebra \( \mathfrak{n} = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab}, h_{ab} : 1 \leq a < b \leq n_0; f_{aa}, g_{aa}, h_{aa} : 1 \leq a \leq n_0\}, \) isomorphic to \( \mathfrak{sp}(n_0)\).

The equivalent modules in the decomposition of \( \mathfrak{m}_{0j} \) are given by

\[
m_{\ell} = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab}, h_{ab} : n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b \leq n_0 + n_1 + \cdots + n_j, \ \ell = 1, \ldots, 2k\}
\]

Also, for \( j = 1, \ldots, s \), we have

\[
\mathfrak{sp}(n_j) = \text{span}_\mathbb{R}\{e_{ab}, f_{ab}, g_{ab}, h_{ab} : n_0 + n_1 + \cdots + n_{j-1} + 1 \leq a < b \leq n_0 + n_1 + \cdots + n_j; \\
f_{aa}, g_{aa}, h_{aa} : n_0 + n_1 + \cdots + n_{j-1} + 1 \leq a \leq n_0 + n_1 + \cdots + n_j\}
\]

In summary, we obtain the \( B \)-orthogonal decomposition

\[
m = \mathfrak{n} \oplus \mathfrak{p},
\]

where

\[
\begin{align*}
\mathfrak{n} &\cong \mathfrak{sp}(n_0), \\
\mathfrak{p} &= \bigoplus_{1 \leq i < j \leq s} \mathfrak{m}_{ij} \bigoplus \mathfrak{m}_{0j} = \bigoplus_{0 \leq i < j \leq s} \mathfrak{m}_{ij}.
\end{align*}
\]
Decomposition (10) as a subspace of $\mathfrak{sp}(n)$ can be depicted in the following $2n \times 2n$ complex matrix, where the positions represented by $\Box$ are zero matrices (corresponding to positions of $\mathfrak{sp}(n_i), i = 1, \ldots, s$):

\[
\begin{pmatrix}
\begin{array}{cccc}
  n & m_{01} & \cdots & m_{0s} \\
  m_{01} & 0 & \cdots & m_{1s} \\
  m_{02} & m_{12} & \ddots & \vdots \\
  \vdots & \vdots & \ddots & m_{s-1,s} \\
  m_{0s} & m_{1s} & \cdots & 0
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{cccc}
  n & m_{01} & \cdots & m_{0s} \\
  m_{01} & 0 & \cdots & m_{1s} \\
  m_{02} & m_{12} & \ddots & \vdots \\
  \vdots & \vdots & \ddots & m_{s-1,s} \\
  m_{0s} & m_{1s} & \cdots & 0
\end{array}
\end{pmatrix}
\]

**Proposition 4.3.** The following relations are satisfied by the modules in the decomposition (7):

\[
[\mathfrak{sp}(n_0), \mathfrak{sp}(n_0)] \subseteq \mathfrak{sp}(n_0),
\]

\[
[m_{ij}, m_{jk}] \subseteq m_{ik}, \quad 0 \leq i < j < k \leq s.
\]

\[
[\mathfrak{sp}(n_i), m_{lm}] = \begin{cases} 
  m_{lm}, & \text{if } i = l \text{ or } i = m, \\
  \{0\}, & \text{otherwise}
\end{cases}, \quad l < m, \quad i, l, m = 0, \ldots, s.
\]

**Proof.** The first relation is true since $\mathfrak{sp}(n_0)$ is a subalgebra of $\mathfrak{g} = \mathfrak{sp}(n)$. The second and the third relations follow from Lemma 4.2 and the expression of the submodules $m_{ij}$ in terms of the basis $B$. \qed

**Remark 4.4.** The above relations are not the only relations among the modules in decomposition (7) which are valid, however these are the ones which we use in our study.

5. Some preliminary results

In order to prove our main theorem 1.1 we will need some propositions and lemmas, which we collect in the present section.

The following can be easily proved by induction on $s$.

**Lemma 5.1.** Let $R_s = \{\lambda_{ij} : 0 \leq i < j \leq s\}$ be a set of real numbers such that $\lambda_{ij} = \lambda_{jk} = \lambda_{ik}$ for all $0 \leq i < j < k \leq s$. Then $R_s$ is a singleton.

Consider the space $G/H = Sp(n)/Sp(n_1) \times \cdots \times Sp(n_s)$ and recall the description of its tangent space given in Section 4. Let $g$ be a $G$-invariant g.o. metric on $G/H$ with corresponding metric endomorphism $A : m \rightarrow m$ (cf. Equation (2)), where we take $B$ to be the multiple of the Killing form (3). Recall the spaces $\mathfrak{n} = \mathfrak{sp}(n_0)$ ($n_0 = n - (n_1 + \cdots + n_s)$) and $\mathfrak{p}$, which were defined in decomposition (10).

**Proposition 5.2.** $A|_{\mathfrak{p}} = \lambda \text{Id}$, where $\lambda > 0$. 
Proof. By the normalizer Lemma 3.3 the metric endomorphism $A$ is $\text{Ad}(N_G(H^0))$-equivariant. As a result, $A$ is $\text{ad}(n^\circ(h))$-equivariant, where $n^\circ(h) = \{Y \in g : [Y, h] \subseteq h\}$ is the Lie algebra of $N_G(H^0)$. In our case $h = sp(n_1) \oplus \cdots \oplus sp(n_s)$ and

$$n^\circ(h) = sp(n_0) \oplus sp(n_1) \oplus \cdots \oplus sp(n_s).$$

By taking into account Lemma 4.2 and the expressions of the subspaces $m_{0j}, m_{ij}, sp(n_j), j = 0, 1, \ldots, s$ in terms of the basis $\mathcal{B}$, we deduce that the submodules $m_{0j}, m_{ij}$ are $\text{ad}(n^\circ(h))$-invariant and $\text{ad}(n^\circ(h))$-irreducible. Apart from being $\text{ad}(n^\circ(h))$-irreducible, the $\text{Ad}(N_G(H^0))$-submodules $m_{ij}$, $0 \leq i < j \leq s$, are pairwise inequivalent. To see this, firstly observe that if $m_{ij}$ and $m_{lm}$, with $0 \leq i < j \leq s$ and $0 \leq l < m \leq s$, are two distinct $\text{Ad}(N_G(H^0))$-submodules then there exists at least one index $i_0$ such that either

1) $i_0 = i$ or $i_0 = j$ and $i_0 \neq l, m$, or
2) $i_0 = l$ or $i_0 = m$ and $i_0 \neq i, j$.

For case 1), taking into account Equation (13) we obtain $[sp(n_{i_0}), m_{ij}] = m_{ij}$ and $[sp(n_{i_0}), m_{lm}] = \{0\}$. For case 2), Equation (13) yields $[sp(n_{i_0}), m_{ij}] = \{0\}$ and $[sp(n_{i_0}), m_{lm}] = m_{lm}$. By virtue of Lemma 2.2 we deduce that the submodules $m_{ij}$ and $m_{lm}$ are inequivalent in both cases.

On the other hand, and in view of Remark 3.3 the restriction $A|_p = A|_{\mathcal{B}_{i_0 \leq i < j \leq s}}$ is given by Proposition 3.1. The algebra $n = sp(n_0)$ coincides with the Lie algebra of $N_G(H^0)/H^0$. By Lemma 3.4, $A|_n$ defines a bi-invariant metric on $N_G(H^0)/H^0$, which in turn corresponds to an $\text{Ad}$-equivariant inner product on $sp(n_0)$. Note that $sp(n_0)$ is simple and the only $\text{Ad}$-invariant inner product is a scalar multiple of the Killing form. Therefore,

$$A|_n = \mu \text{Id}, \quad \mu > 0.$$  

Next, we have the following:

**Proposition 5.3.** We assume that $n_0 > 1$. Then $\lambda = \mu$, where $\lambda$ is given by Proposition 5.2 and $\mu$ is given by Equation (15).
Proof. Recall the Ad(G/H)-submodules $m_i \subseteq m_0 \subseteq p$ defined in Section \textsection 4. By Proposition \textsection 5.2, we have

$$A|_{m_j} = \lambda \text{Id}.$$ 

We choose the vectors $e_{12} \in n$ and $e_{1k+1} \in m_1$. Since $n_0 > 1$, the above vectors do not coincide, hence by Lemma 4.2 it follows that

$$[e_{12}, e_{1k+1}] = -e_{2k+1} \in m_2.$$ 

Therefore, $[n, m_1]$ has non zero projection on $(n \oplus m_1)$. Along with the ad(h)-invariance of $n$ and $m_1$ and the facts that $A|_{m_j} = \lambda \text{Id}$ and $A|_{n} = \mu \text{Id}$, part 1. of Lemma 5.5 yields

$$\lambda = \mu.$$ 

Finally, we consider the case $n_0 = 1$.

Proposition 5.4. Let $n_0 = 1$ and $s > 1$. Then $\lambda = \mu$, where $\lambda$ is given by Proposition \textsection 5.2 and $\mu$ is given by Equation (15).

The proof requires the following.

Lemma 5.5. Let $m_{ij}$, $i > 0$, be one of the submodules defined in Section \textsection 4. Let $a \in h = \text{sp}(n_1) \oplus \cdots \oplus \text{sp}(n_s)$ such that $[a, m_{ij}] = \{0\}$. Then the projection of $a$ on $\text{sp}(n_i) \oplus \text{sp}(n_j)$ is zero, i.e. $a \in \text{sp}(n_1) \oplus \cdots \oplus \text{sp}(n_{i-1}) \oplus \text{sp}(n_{i+1}) \oplus \cdots \oplus \text{sp}(n_{j-1}) \oplus \text{sp}(n_{j+1}) \oplus \cdots \oplus \text{sp}(n_s)$.

Proof. Let $\pi_j(a)$ be the projection of $a$ on $\text{sp}(n_j)$ and set $h_{ij} := \text{sp}(n_i) \oplus \text{sp}(n_j)$. By relation (13), we deduce that $[h_{ij}, m_{ij}] = [h_{ij}, m_{ij}] = m_{ij}$. Then we have that $\{0\} = [a, m_{ij}] = [(\pi_i + \pi_j)(a), m_{ij}]$, therefore $c_{ij} := \{X \in h_{ij} : [X, Y] = 0 \text{ for all } Y \in m_{ij}\}$ lies in the space $c_{ij} := \{X \in h_{ij} : [X, Y] = 0 \text{ for all } Y \in m_{ij}\}$. Using the Jacobi identity and the ad(h)-invariance of $m_{ij}$, it is not hard to verify that the space $c_{ij}$ is an ideal of the Lie algebra $h_{ij}$. Since $h_{ij}$ is semisimple, $c_{ij}$ is necessarily one of the ideals $\{0\}, \text{sp}(n_i), \text{sp}(n_j)$ or $h_{ij}$. On the other hand, relation (13) implies that $[\text{sp}(n_i), m_{ij}] \neq \{0\}, [\text{sp}(n_j), m_{ij}] \neq \{0\}$ and $[\text{sp}(n_i) \oplus \text{sp}(n_j), m_{ij}] \neq \{0\}$. We conclude that

$$c_{ij} = \{0\}$$ 

and thus $(\pi_i + \pi_j)(a) = 0$.

Proof of Proposition 5.4. In view of Remark 3.5, Proposition 5.2 and Equation (15), the g.o. metric on $G/H$ has the form

$$A = \begin{pmatrix} \mu \text{Id}|_{n} & 0 \\ 0 & \lambda \text{Id}|_{p} \end{pmatrix}.$$ 

(16)

Let $X \in m = n \oplus p$ and write $X = X_n + X_p$, where $X_n$ is the projection of $X$ on $n$ and $X_p$ is the projection of $X$ on $p$. Further, we write

$$X_p = \sum_{0 \leq i < j \leq s} X_{ij},$$ 

(17)

where $X_{ij}$ denotes the projection of $X$ on the space $m_{ij}$. Taking into account the definition of the spaces $n$, $m_{ij}$ in terms of the basis $B$, along with Lemma 4.2 we deduce that

$$[n, m_{ij}] \subseteq \begin{cases} m_{ij}, \text{ if } i = 0 \\ \{0\}, \text{ otherwise.} \end{cases}$$ 

(18)
Since $A$ is a g.o. metric, Proposition 3.2 along with expression (16) of $A$, implies that there exists a vector $a \in \mathfrak{h} = \mathfrak{sp}(n_1) \oplus \cdots \oplus \mathfrak{sp}(n_s)$ such that

$$0 = [a + X_n + X_p, \mu X_n + \lambda X_p] = \mu[a, X_n] + \lambda[a, X_p] + (\lambda - \mu)[X_n, X_p].$$

Taking into account Equation (17) along with relation (18) and the fact that $[\mathfrak{n}, \mathfrak{h}] = \{0\}$ (recall that $\mathfrak{n} = \mathfrak{sp}(n_0)$), the above equation is equivalent to

$$0 = \lambda \sum_{0 \leq i<j \leq s} [a, X_{ij}] + (\lambda - \mu) \sum_{j=1}^{s} [X_n, X_{0j}]. \quad (19)$$

We choose $X_n := f_{11}$, $X_{01} := e_{12}$ and $X_p := X_{01} + X_{12} = e_{12} + X_{12} \in \mathfrak{m}_{01} + \mathfrak{m}_{12}$ (since $s > 1$, $\mathfrak{m}_{12} \neq \{0\}$), where we consider $X_{12}$ as arbitrary. Then Equation (19) along with Lemma 4.2 yield

$$0 = \lambda[a, e_{12}] + \lambda[a, X_{12}] + 2(\lambda - \mu)f_{21}. \quad (20)$$

By the $\text{ad}(\mathfrak{h})$-invariance of $\mathfrak{m}_{ij}$ and the fact that $a \in \mathfrak{h}$, the first two terms of Equation (20) lie in $\mathfrak{m}_{01}$ and $\mathfrak{m}_{12}$ respectively, while the last term lies in $\mathfrak{m}_{01}$. Therefore, Equation (20) yields the system

$$\lambda[a, e_{12}] + 2(\lambda - \mu)f_{21} = 0 \quad (21)$$

$$\lambda[a, X_{12}] = 0. \quad (22)$$

Since $X_{12} \in \mathfrak{m}_{12}$ is arbitrary, Equation (22) implies that $[a, \mathfrak{m}_{12}] = \{0\}$. Lemma 5.5 then yields $(\pi_1 + \pi_2)(a) = 0$, where $\pi_j$ denotes the projection of $a$ on $\mathfrak{sp}(n_j)$. Hence $\pi_1(a) = \pi_2(a) = 0$. On the other hand, the fact that $e_{12} \in \mathfrak{m}_{01}$ along with relation (13), imply that $[a, e_{12}] = [\pi_1(a), e_{12}] = 0$. Substituting into Equation (21), we obtain $2(\lambda - \mu)f_{21} = 0$ and thus $\lambda = \mu$. \hfill \Box

6. PROOF OF THEOREM 1.1

We can now combine the results of the previous section to give a proof of Theorem 1.1.

The standard metric on $G/H = Sp(n)/Sp(n_1) \times \cdots \times Sp(n_s)$ is a g.o. metric, hence the sufficiency part of the theorem holds trivially. For the necessity part, let $g$ be a $G$-invariant g.o. metric on $G/H$. If $n_0 = 0$, i.e. $\mathfrak{n} = \{0\}$, then Proposition 5.2 implies that $g$ is the standard metric.

Now assume that $n_0 > 0$. If $n_0 \neq 1$, then the theorem follows from Remark 3.5 and Propositions 5.2 and 5.3. If $n_0 = 1$ then $\mathfrak{n} = \mathfrak{sp}(1)$ and we consider two cases, $s = 1$ and $s > 1$. If $s = 1$, then $G/H = Sp(n)/Sp(n-1)$ and in this case any metric endomorphism $A$ such that $A|_p = \lambda \text{Id}$ and $A|_n = \mu \text{Id}$ is a g.o. metric and vice versa (16). Therefore, the $G$-invariant g.o. metrics on $G/H$ are precisely the metrics of the form $A = \left( \begin{array}{cc} \mu \text{Id}|_n & 0 \\ 0 & \lambda \text{Id}|_p \end{array} \right)$. It follows that the $G$-invariant g.o. metrics on $G/H$ are homothetic to the metrics $g_\mu = \left( \begin{array}{cc} \mu \text{Id}|_n & 0 \\ 0 & \text{Id}|_p \end{array} \right)$. This settles Theorem 1.1 for the case $n_0 = 1$ and $s = 1$. Finally, if $n_0 = 1$ and $s > 1$, Theorem 1.1 follows from Proposition 5.4. \hfill \Box
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University of Patras, Department of Mathematics, GR-26500 Rion, Greece
Email address: arvanito@math.upatras.gr

University of Patras, Department of Mathematics, GR-26500 Rion, Greece
Email address: nsouris@upatras.gr

University of Patras, Department of Mathematics, GR-26500 Rion, Greece
Email address: statha@math.upatras.gr