Embeddings of integrable models in supergravity and their perturbative stability

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Motivation

Gravity and Scalars

The supergravity solution

Stability analysis

Conclusions
Motivation

- Our story starts with the $\lambda$-model which is an integrable deformation of a CFT interpolating between a WZW model and the non-Abelian T-dual of the PCM [K. Sfetsos, '13]

- Various $\lambda$-models on group and coset spaces have been embedded to the type-II supergravity [K. Sfetsos, D.C. Thompson, '14; S. Demulder, K. Sfetsos & D.C. Thompson, '15; B. Hoare & A.A. Tseytlin, '15; R. Borsato, A.A. Tseytlin & L. Wulff, '16; Y. Chervonyi & O. Lunin, '16]

- Embeddings including undeformed AdS spaces have been constructed only recently [G. I. & K. Sfetsos, '19]

- Our desire is to study these new solutions in the context of the AdS/CFT correspondence

- Lack of supersymmetry suggests that the stability of these solutions must be analysed

- This is also related to the Ooguri-Vafa conjecture [H. Ooguri & C. Vafa, '16]
Gravity and Scalars

Consider a theory of gravity in $D$-dimensions with $n$ scalars:

$$S(g, X) = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{|g|} \left( \mathcal{R} - \gamma_{ij} \partial X^i \cdot \partial X^j - V(X) \right)$$

$i, j = 1, \ldots, n$

The equations of motion for the metric and the scalars are:

$$\nabla^2_{g} X^i - \frac{1}{2} \gamma^{ij} \partial_j V(X) = 0$$

$$\mathcal{R}_{\mu\nu} - \gamma_{ij} \partial_\mu X^i \partial_\nu X^j - \frac{g_{\mu\nu}}{D-2} V(X) = 0$$

We focus on AdS solutions with constant scalars:

$$ds_D^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = L^2 \left( r^2 \eta_{\alpha\beta} dx^\alpha dx^\beta + \frac{dr^2}{r^2} \right), \quad \bar{X}^i = \text{const}$$
The potential in this case must satisfy:

\[ V(\bar{X}) = -\frac{(D-1)(D-2)}{L^2} \quad \& \quad \partial_i V(\bar{X}) = 0, \quad \forall \ i = 1, \ldots, n \]

We are interested in studying the fluctuations around the background:

\[ g_{\mu \nu} = \bar{g}_{\mu \nu} + \delta g_{\mu \nu}, \quad X^i = \bar{X}^i + \delta X^i, \quad i = 1, \ldots, n \]

The linearised equations of motion read:

\[ \nabla^2_{\bar{g}} \delta X^i - (M^2)^{i j} \delta X^j = 0 \quad \& \quad \delta R_{\mu \nu} + \frac{D-1}{L^2} \delta g_{\mu \nu} = 0 \]

where:

\[ (M^2)^{i j} = \frac{1}{2} \gamma^{i k} \partial_j \partial_k V(X) \bigg|_{X=\bar{X}} \]
The scaling dimensions for the scalar fluctuations are extracted from their asymptotic behaviour at the boundary of AdS

Requiring reality of the scaling dimensions implies that the eigenvalues of $M^2$ satisfy the Breitenlohner-Freedman (BF) bound [P. Breitenlohner & D. Z. Freedman, '82]:

$$d_i \geq -\left(\frac{D-1}{2L}\right)^2, \quad \forall \ i = 1, \ldots, n$$

The equation for the metric fluctuations greatly simplifies at the transverse-traceless gauge:

$$\nabla^\mu \hat{g} \delta g_{\mu\nu} = 0, \quad g^{\mu\nu} \delta g_{\mu\nu} = 0$$

where it reduces to:

$$\nabla^2 \hat{g} \delta g_{\mu\nu} + \frac{2}{L^2} \delta g_{\mu\nu} = 0$$
The supergravity solution

The solution of our interest [G. I. & K. Sfetsos, ’19] has metric:

\[
\begin{align*}
    ds^2 &= \frac{2}{\ell} \left( - r^2 dt^2 + r^2 dx^2 + \frac{dr^2}{r^2} + d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 \right) \\
    &\quad + e^{2\phi_y} \left( \lambda_+^2 dy_1^2 + \lambda_-^2 dy_2^2 \right) + e^{2\phi_z} \left( \lambda_+^2 dz_1^2 + \lambda_-^2 dz_2^2 \right)
\end{align*}
\]

where

\[
\phi_y(y) = -\frac{1}{2} \ln \left(1 - y_1^2 - y_2^2\right) \quad \& \quad \phi_z(z) = -\frac{1}{2} \ln \left(z_1^2 + z_2^2 - 1\right)
\]

The dilaton is:

\[
\Phi(y, z) = \phi_y(y) + \phi_z(z)
\]

The RR sector contains only a self-dual five-form:

\[
\begin{align*}
    F_5 &= 2k \; dz_1 \wedge dy_2 \wedge \left( \sqrt{\frac{\ell - \mu}{2}} \Vol(AdS_3) + \sqrt{\frac{\ell + \mu}{2}} \Vol(S^3) \right) \\
    &\quad - 2k \; dz_2 \wedge dy_1 \wedge \left( \sqrt{\frac{\ell + \mu}{2}} \Vol(AdS_3) + \sqrt{\frac{\ell - \mu}{2}} \Vol(S^3) \right)
\end{align*}
\]
The rest of the fields are trivial

\[ H_3 = F_1 = F_3 = 0 \]

We also define:

\[ \lambda_{\pm} = \sqrt{k \frac{1 \pm \lambda}{1 \mp \lambda}} \], \quad \mu = \frac{4 \lambda}{k(1 - \lambda^2)} \], \quad \lambda \in [0, 1), \quad \ell \geq \mu

Dilaton and Einstein equations:

\[
R + 4 \nabla^2 \Phi - 4 (\partial \Phi)^2 - \frac{1}{12} H_3^2 = 0, \\
R_{MN} + 2 \nabla_M \nabla_N \Phi - \frac{1}{4} (H_3^2)_{MN} = \frac{e^{2\Phi}}{2} \left[ (F_1^2)_{MN} + \frac{1}{2} (F_3^2)_{MN} + \frac{1}{48} (F_5^2)_{MN} \\
- G_{MN} \left( \frac{1}{2} F_1^2 + \frac{1}{12} F_3^2 \right) \right]
\]

Bianchi and flux equations:

\[
dH_3 = 0, \quad dF_1 = 0, \quad dF_3 = H_3 \wedge F_1, \quad dF_5 = H_3 \wedge F_3 \\
d \star F_1 + H_3 \wedge \star F_3 = 0 = 0, \quad d \star F_3 + H_3 \wedge F_5 \\
d(e^{-2\Phi} \star H_3) - F_1 \wedge \star F_3 - F_3 \wedge F_5 = 0
\]
Stability analysis

- We will study the perturbative stability of our solution from a lower dimensional point of view
- We adopt the following reduction ansatz for the metric:

$$d\hat{s}^2 = e^{2A} \left[ ds^2_{\mathcal{M}_3} + \frac{2e^{2\psi}}{\ell} \left( d\theta_1^2 + \sin^2 \theta_1 \, d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_2^2 \right) \ight. \\
\left. + e^{2\phi_y} \left( \lambda_+ e^{2\chi_1} \, dy_1^2 + \lambda_- e^{2\chi_2} \, dy_2^2 \right) + e^{2\phi_z} \left( \lambda_+ e^{2\chi_3} \, dz_1^2 + \lambda_- e^{2\chi_4} \, dz_2^2 \right) \right]$$

- For the dilaton and the five-form we take:

$$\hat{\Phi}(x, y, z) = 4A(x) + \phi_y(y) + \phi_z(z),$$

$$\hat{F}_5 = dz_1 \wedge dy_2 \wedge \left( c_1 e^{x_2-x_1+x_3-x_4-3\psi} \Vol(\mathcal{M}_3) + c_2 \Vol(S^3) \right)$$

$$\quad - dz_2 \wedge dy_1 \wedge \left( c_2 e^{x_1-x_2-x_3+x_4-3\psi} \Vol(\mathcal{M}_3) + c_1 \Vol(S^3) \right)$$

- The rest of the fields are taken to be zero
- The various functions that enter in the ansatz are taken to depend only on the coordinates of $\mathcal{M}_3$
The constants $c_1$ and $c_2$ are:

$$c_1 = 2k \sqrt{\frac{\ell - \mu}{2}} , \quad c_2 = 2k \sqrt{\frac{\ell + \mu}{2}}$$

To recover our solution we set:

\[
\begin{align*}
    ds^2_{M_3} &= \bar{g}_{\mu\nu} \, dx^\mu \, dx^\nu = \frac{2}{\ell} \left( r^2 \eta_{\alpha\beta} \, dx^\alpha \, dx^\beta + \frac{dr^2}{r^2} \right), \\
    \bar{A} &= \bar{\psi} = \bar{\chi}_1 = \bar{\chi}_2 = \bar{\chi}_3 = \bar{\chi}_4 = 0
\end{align*}
\]

The reduction ansatz satisfies the Bianchi and flux equations.

From the dilaton and Einstein equations we obtain differential equations for the functions $A, \psi, \chi_1, \ldots, \chi_4$ and the metric $g_{\mu\nu}$.
The dilaton equation:

\[ R_g + \#\nabla^2_g \text{scalars} + \#\partial \text{scalars} \cdot \partial \text{scalars} + \text{exps of scalars} = 0 \]

Einstein equations on $\mathcal{M}_3$:

\[ R^g_{\mu \nu} + \#\nabla_\mu \nabla_\nu \text{scalars} + \#\partial_\mu \text{scalars} \partial_\nu \text{scalars} \]
\[ + g_{\mu \nu} \left[ \#\nabla^2_g \text{scalars} + \#\partial \text{scalars} \cdot \partial \text{scalars} + \text{exps of scalars} \right] = 0 \]

- The two can be combined to eliminate $R_g$ from the dilaton equation:

\[ \nabla^2_g \text{scalars} + \#\partial \text{scalars} \cdot \partial \text{scalars} + \text{exps of scalars} = 0 \]

- From the transverse directions we obtain 5 more equations of the same form and 2 first order equations:

\[ \partial_\mu (2A + \chi_1 + \chi_2) = 0, \quad \partial_\mu (2A + \chi_3 + \chi_4) = 0 \]
We eliminate $\chi_2$ & $\chi_4$ and we are left with $A, \psi, \chi_1, \chi_3, g_{\mu\nu}$ and an equal number of equations for each object.

We move to a more convenient frame where:

$$g_{\mu\nu} = e^{8A-6\psi}g_{\mu\nu}$$

In this frame the equations for $A, \psi, \chi_1, \chi_3, g_{\mu\nu}$ can be derived from a 3D action of gravity with scalars with:

$$\gamma_{ij} = \begin{pmatrix} 32 & -12 & 2 & 2 \\ -12 & 12 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

and

$$V(X) = -3\ell e^{8A-8\psi} - 2e^{8A-6\psi} \left( \frac{e^{-2\chi_1}}{\lambda_+^2} + \frac{e^{4A+2\chi_1}}{\lambda_-^2} - \frac{e^{-2\chi_3}}{\lambda_+^2} - \frac{e^{4A+2\chi_3}}{\lambda_-^2} \right) + \frac{e^{12A-12\psi}}{2k^2} \left( c_1^2 e^{2\chi_3-2\chi_1} + c_2^2 e^{2\chi_1-2\chi_3} \right)$$
We have all the ingredients to compute the matrix $M^2$ and its eigenvalues.

The change of frame didn’t affect the radius of $AdS_3$:

$$ L = \sqrt{\frac{2}{\ell}} $$

The BF bound now reads:

$$ d_i \geq -\frac{\ell}{2}, \quad \forall \ i = 1, \ldots, 4 $$

The undeformed case:

$$ M^2 = 4\ell \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{\ell} & 0 & \frac{1}{2} - \frac{1}{\ell} & -\frac{1}{2} \\ \frac{1}{\ell} & 0 & -\frac{1}{2} & \frac{1}{2} + \frac{1}{\ell} \end{pmatrix} $$

$$ d_1 = 0, \quad d_2 = 2\ell \left( 1 + \sqrt{1 + \frac{4}{\ell^2}} \right) $$

$$ d_3 = 4\ell, \quad d_4 = 2\ell \left( 1 - \sqrt{1 + \frac{4}{\ell^2}} \right) $$

The BF bound is not violated for:

$$ \hat{\ell} \geq \frac{8}{3}, \quad \hat{\ell} := k\ell $$
The deformed case:

- One of the eigenvalues of $M^2$ is zero ($d_1 = 0$) and the other three depend non-trivially on $\lambda$.
- Violation of the BF bound means negative values for:

  \[ b_i := d_i + \frac{\ell}{2} \]

- The allowed region is:

  \[ \ell \geq \mu \quad \Rightarrow \quad \hat{\ell} \geq \frac{4\lambda}{1-\lambda^2} \]
The fourth eigenvalue of $M^2$ leads to a more interesting structure:

The region that does not violate the BF bound is defined by:

$$\hat{\ell} \geq \frac{8}{3 \sqrt{3}} \frac{\sqrt{3 + 22 \lambda^2 + 3 \lambda^4}}{1 - \lambda^2}$$
Conclusions

- We studied the perturbative stability of a non-supersymmetric solution of the type-IIB supergravity whose geometry contains an $AdS_3$, a round $S^3$ and two $\lambda$-deformed spaces.

- Our approach is based on the study of scalar fluctuations in a three-dimensional effective theory of gravity with scalars.

- The analysis we performed revealed a sub-region in the allowed parametric space where the BF bound is not violated.

- A more complete treatment requires the study of the full spectrum.

- Non-perturbative instabilities should be also tested.
Thank you!

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