Research Article

Improved Multiple Lyapunov Functions of input-to-output Stability and input-to-state Stability for Switched Systems

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Received 16 January 2022; Accepted 7 February 2022; Published 26 March 2022

Academic Editor: Muhammad Arif

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This paper investigates the Lyapunov characterizations of input-to-output stability (IOS) and input-to-state stability (ISS) for nonlinear switched systems. The restriction on the negativity of derivatives of Lyapunov functions is relaxed; additionally, subsystems are not restricted to be ISS/IOS throughout the state space. These problems are overcome by a newly proposed method, which extends multiple Lyapunov functions (MLFs) to an indefinite form. Indefinite multiple Lyapunov functions (iMLFs) are proposed on the basis of an inequality to estimate the upper boundary of the system state. Moreover, we apply the iMLFs method to verify the extended notions of IOS in a switched system. More relaxed sufficient conditions for ISS and IOS are proposed, a simulation experiment is carried out on a numerical example of the system, which demonstrates their effectiveness and superiority.

1. Introduction

As a specific kind of dynamical systems, the switched system consists of a series of subsystems together with a designed law-derived switching signal [1], which is very compatible with the modeling and analysis of practical systems, such as dynamic systems, chemical industrial systems, and automotive control [2–4]. Nonlinear switched systems received considerable attention over the last decade [5–7]. However, the study of switched systems remains a challenging subject, primarily because of the continuous and discrete dynamics caused by the subsystems and switching signal. Additionally, in general, the characterizations of subsystems have no direct influence on the whole switched system [8–11].

As the basic concept of system robustness, ISS has been recognized as a useful method to verify the stability of nonlinear systems [9, 11–14]. Furthermore, some output variables are more important than full states in some practical applications [15]; therefore, IOS is presented as a framework in relation to the output robustness with respect to system inputs [16–18]. Both of these properties have been extensively studied and exploited in nonswitched systems over the past few years [19, 20]. Therefore, this work aims to extend ISS and IOS to switched systems and investigating sufficient conditions for verifying them.

Nevertheless, the stability of the nonlinear switched system does not appear easy to ensure, as it concerns both the characterization of the subsystems and the influence caused by the switching signal [21, 22]. For example, [8] indicated that a switched system might also be unstable under some switching signals even though it consists of asymptotically stable subsystems. Additionally, nonlinear conditions are more complex than the linear conditions. Several methods have been presented over the past few years to perform ISS/IOS verification for switched systems [2, 10, 23, 24]. Morse [23] proposed a common Lyapunov function (CLF) method. Then, [24] showed that a constrained switching law, combined with slow switches in a switched system, could also guarantee asymptotic stability. This inspired researchers to propose several more general methods; one of the commonly applied methods which exerts restrictions on the time period that subsystems being activated named average dwell-time (ADT) [25–28], and another effective method is the multiple Lyapunov functions
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(MLFs) [4, 24, 29, 30]. For example, in [10, 31–34], ISS for
switched systems exploited by ADT, and MLF-based results
are presented in [24, 29, 35–37].

However, it should be noted that there are some com-
mon restrictions in the aforementioned works that concern
switched systems. First, some or all subsystems are assumed
to be ISS [38–43]; for example, [2, 40, 42] required all
subsystems to be ISS and [7, 8, 43] argued that the stability of
switched systems could be guaranteed with some subsystems
that are non- ISS, but the switched system should not switch
too frequently or stay within the subregions of non-ISS
subsystems for long periods of time. Second, the derivatives
of multiple Lyapunov functions with respect to time must be
strictly negative-definite, such as the results in [6, 29, 36, 39].
Moreover, in some works, the considered systems are time-
invariant [9, 35], and time-varying situations are more
complicated because the switches as well as Lyapunov
functions are both time-dependent.

To address the aforementioned problems, in our pre-
vious work [44], we presented a new method which extends
the Lyapunov function to an indefinite format, and by which
we gave more relaxed verifying conditions for integral ISS
(ISS) and ISS characterizations of typical nonlinear systems.
Then, [35] exploited and applied the method to switched
systems, and [45] extended our method to the verification of
systems with time-delay. Therefore, this work aims to extend
the multiple Lyapunov functions to an indefinite form and
address ISS/IOS verification for switched systems.

Under the motivation of the considerations mentioned
above, this paper aims to present more relaxed ISS/IOS
conditions for nonlinear switched systems. Compared with
existing works, this paper possesses the following distinct
features: First, none of the subsystems are restricted to ISS/
IOS, and the individual ISS/IOS property for subsystems is
only required in the subregions when they are activated;
second, the system state-based switching law is more flexible
in system modeling than time-dependent law. Moreover, the
derivatives of Lyapunov functions are no longer restricted to
be strictly negative-definite; instead, they are permitted to be
indefinite over the state space, and some sufficient condi-
tions for extended notions of IOS are also provided. Finally,
we provide an individual upper boundary $\lambda_k$ for every
Lyapunov function.

The symbol notations utilized and system descriptions
are illustrated thoroughly in Section 2. Then the state-based
switching law and more relaxed ISS sufficient conditions are
proposed as the main results, and then the verification of IOS
is investigated. In addition, new conditions for verifying the
extended notions of IOS properties are discussed in Section
4. Finally, a simulation experiment is carried out based on a
numerical example of a switched system, which demonstrates
the proposed methods is effective, and we give di-
rections for further work in the conclusion.

2. Preliminaries

In this work, we employ the following notations: The Eu-
iclidan supreme norms of the real vector are repre-
sented by $|\cdot|$ and $\|\cdot\|$, respectively. Let $\mathbb{R}$ consist of all the real
numbers, whose nonnegative subset is denoted by $\mathbb{R}^+$. All
integers and the integers in $[0, +\infty)$ are represented by $\mathbb{Z}$
and $\mathbb{Z}^+$, respectively. The maximum value between $x$ and $y$ is
denoted as $x\vee y$. The truncation of $u(t)$, more specifically,
$u(a)_{[t, t+1]} = 0$ when $s \geq t$, and $u(a)_{[t, t+1]} = u(a)$
when $a \in [t, t+1]$ is represented as $u(a)_{[t, t+1]}$. The state trajectory with
initial state $x_0 \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ is represented as
$x(t, x_0, u)$. As before, we denote a continuous function
$\chi(s) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \mathcal{H}$ if $\chi$ strictly increases for $s \geq 0$
and satisfies $\chi(0) = 0$. Function $\chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \mathcal{H}^c$ if $\chi$
satisfies $\lim_{s \rightarrow +\infty} \chi(s) = +\infty$ and $\chi \in \mathcal{H}$. Considering a function
$\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, if $\varphi(t, s) = 0$ when $t \rightarrow +\infty$ for
a fixed $s$; and $\varphi \in \mathcal{H}$ for a fixed $t$, then we call function $\varphi$ a
$\mathcal{H}$-$\varphi$-function.

The system considered in this work is provided as:

$$\dot{x} = f_\sigma(t, u, \xi); \quad y = h_\sigma(t, x),$$

(1)

where $\xi$ denotes the initial state of switched System (1), based
on a designed switching law, right-continuous function
$\sigma : \mathbb{R}^+ \rightarrow \mathbb{Z}$ controls the subsystem sequence. $u \in \mathbb{R}^m$
denotes the measurable and bounded input, vectors $f_\sigma$ and
$h_\sigma$ satisfying $f_\sigma(t, 0, 0) = 0$, $h_\sigma(t, 0) = 0$ are both locally
Lipschitz continuous. $y = h_\sigma(t, x)$ denotes the output.

Function $\sigma$ is illustrated as:

$$\sigma = |x(t_0); (k_0, t_0), (k_1, t_1), \ldots, (k_n, t_n), \ldots | \in \mathbb{M}, i \in \mathbb{Z}^+,$$

(2)

where $t_0$ is the initial time of System (1), $k_i \neq k_{i+1}$,
$\mathbb{M} : \{0, \ldots, m\}$ is a right-continuous set consists of the
numbers of subsystems. To be more specific, when
$t \in [t_i, t_{i+1})$, then $\sigma(t) = k_i$, and the $k_i \in M$-th system is
activated.

Note that this work is constructed under the following
assumptions: First, the system state should not change
abruptly between switching intervals; second, in a limited
time interval, switches between subsystems only occur a
finite number of times.

The definitions of ISS and IOS are given as:

Definition 1. [14] Considering System (1), suppose that $x(t)$
always exists as $t \geq t_0$ for any measurable $u(t)$ and $x(t_0)$, and
there exist some continuous functions $\varphi \in \mathcal{H}$ and $\chi \in \mathcal{H}$
satisfying:

$$|x(t)| \leq \varphi(t_0) + \chi(\|u(t)\|),$$

(3)

then the considered system is ISS.

Definition 2. [14] Considering System (1), suppose that $x(t)$
always exists as $t \geq t_0$ for any measurable $u(t)$, output $y(t)$
and initial state $x(t_0)$; some functions $\varphi \in \mathcal{H}$ and $\chi \in \mathcal{H}$
satisfying:

$$|y(t, x)| \leq \varphi(t_0) + \chi(\|u(t)\|),$$

(4)

then the considered system is IOS.

Our main intention is to derive more relaxed sufficient
conditions for verifying ISS/IOS in switched systems by
extending MLF to an indefinite notion.
3. Main Results

We present a new method called indefinite multiple Lyapunov functions (iMLFs) in the following parts. The new iMLFs are then applied to the analysis of System (1) and some new ISS/IOS sufficient conditions are derived. The details can be divided into several steps: This section first introduce a designed switching law based on the system state; then, a lemma for estimating the upper boundary on the solution of a set of inequalities is presented, which derives the iMLF method. Finally, new sufficient conditions for verifying ISS/IOS properties are presented on the basis of the designed switching law.

3.1. A designed switching law based on the system state. As discussed above, the stability of a switched system concerns the characterization of subsystems as well as the construction of switching signals. Our objective in this work is to introduce a designed switching law based on the system state. For any system of switching signals. Our objective in this work is to introduce a designed switching law based on the system state.

Remark 1

1. The applied switching law controls the switches between subsystems based on specific changes in system states. For any fixed time $t$, the function $\sigma(t)$ selects a fixed $k$ for System (1), which denotes the current subsystem. When the state satisfies $\tilde{\Omega}_{kj}(t)$, the current subsystem switches to the next subsystem in the sequence.

2. Also note that $\cup_{k=1}^m \Omega_k(t) = \mathbb{R}^n$. Now, we present a brief proof. From $\mu_{kj}(t, x) + \mu_{ij}(t, x) \leq \min(0, \mu_{ij}(t, x))$, we can derive that $\sum_{i=1}^m \mu_{kij} \leq 0$ holds for any $i, j \in M$, where $k_{j+1}$ is considered as $k_1$. Assume that there exist some $x \in \mathbb{R}^n$ but not in $\Omega_k(t)$ for any $k \in M$. Then, there exist some $V_k(t, x) + \mu_{kij} \geq V_{kj}(t, x)$ that holds for some subsystem $i$. However, considering (3), $\sum_{i=1}^m V_k(t, x) + \mu_{kij} \geq V_{kj}(t, x) = \sum_{i=1}^m \mu_{kij}(t, x) \leq 0$, from that we can see the consequence opposite the assumption before. Hence, we can affirm that $\cup_{k=1}^m \Omega_k(t) = \mathbb{R}^n$. In this regard, we do not need to consider that some states might not be included under this switching signal.

3.2. ISS conditions of switched nonlinear systems. First, we introduce a lemma presented in our previous work [44], where the derivative of the solution is indefinite rather than strictly negative-definite. Then, this characterization is applied to System (1) under the switching law so that some new sufficient conditions can be derived for ISS/IOS.

Lemma 1. [44] Considering a positive-definite $C^1$ function $\eta(t): \mathbb{R}^n \rightarrow \mathbb{R}^n$, a function $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n \in \mathbb{K}$, $\lambda(t): \mathbb{R}^n \rightarrow \mathbb{R}^n$, $u(t): \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any $t \geq t_0$ and $\eta(t) \geq \chi(|u(t)|)$:

$$\dot{\eta}(t) \leq \lambda(t)r(t),$$

then

$$\eta(t) \leq \eta(t_0)e^{\int_{t_0}^t \lambda(s)ds} + \text{esssup}_{t_0 \leq s \leq t} \chi(|u(s)|) \int_{t_0}^t \lambda^*(\delta ds),$$

where $\lambda(\delta)/0$ is denoted by $\lambda^*(\delta)$.

Then, this lemma is applied to the verification of ISS for System (1) under our switching law.

Theorem 2. Considering a switched system (1) with a continuous, essentially bounded input $u(t)$, suppose that there exist positive-definite and differentiable functions $V_k(t, x)$, some $\mathbb{K}_\infty$-functions $\alpha_i$ and $\alpha_i(s) \geq s$, some continuous functions $\beta_{kj}(t, x) \leq 0$, $\mu_{ij}(t, x)$, $\lambda_k(t)$ and $\mathbb{K}$-function $\chi$ satisfying:

$$\alpha_i(|u|) \leq V_k(t, x) \leq \alpha_2(|u|).$$

2.

$$\frac{\partial V_k(t, x)}{\partial t} + \frac{\partial V_k(t, x)}{\partial x} f_k(t, x, u) + \sum_{j=1}^m \beta_{kj}(t, x)[V_k(t, x) + \mu_{kj}(t, x) - V_j(t, x)] \leq \lambda_k(t)V_k(t, x),$$

where $k, j \in M; k \neq j$ and $V_k(t, x) \geq \chi(|u(t)|); \chi,$

$$\int_{t_0}^{t_\infty} \lambda^*_m(\delta)ds = C \leq \infty$$
where $\lambda_m = \max\{\lambda_{k_1}, \ldots, \lambda_{k_i}\}$, $\lambda_m^+ (\delta) = \lambda_m (\delta) > 0$, $C$ is defined as a nonnegative constant and

$$
\int_{t_0}^{t} \lambda_m^- (\delta) \mathrm{d}\delta \geq \varepsilon (t - t_0),
$$

(11)

where $\lambda_m^- (\delta) = -\lambda_m (\delta) > 0$, $t \geq t_0$, and $\varepsilon$ is defined as a sufficiently small constant.4.

Proof. We divide the proof into several parts as follows:

Step 1. Prove that Lyapunov functions $V_{\sigma(t)} (t, x)$ are bounded and find the upper boundary in time interval $[t_i, t]$.

\[
V_{\sigma(t)} (t, x) \\
\leq V_{k_i} (t, x(t)) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta} + \text{esssup}_{t_i \leq s \leq t} \chi (\|u(s)\|) e^{\int_{t_i}^{s} \lambda_{k_i}^- (\delta) \mathrm{d}\delta} \\
= V_{k_i} (t, x(t)) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta} + \chi (\|u(t)\|) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta}.
\]

(15)

Then, in Step (ii), considering $V_{k_i} (t_i, x(t))$, $t_i$ is the switching instant; from the switching law, we know that for any $i \in \mathbb{Z}^+$,

$$
V_{k_i} (t_i, x(t_i)) = V_{k_i} (t_i, x(t_i)) + \mu_{k_i} (t_i, x(t_i)),
$$

(16)

holds in any switching time-interval, from (10), we have

\[
V_{k_i} (t, x(t)) \\
\leq \left[ V_{k_i} (t_i, x(t_i)) + \mu_{k_i} (t_i, x(t_i)) \right] e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta} \\
+ \chi \left( \|u(t)\|_{[t_i, t]} \right) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta}.
\]

(17)

Step 2. Apply the switching law to find the relationship between the Lyapunov functions of two adjacent intervals.

Step 3. Find the relationship between the initial $V_{\sigma_i} (t_0, x(t_0))$ and $V_{k_i} (t, x)$ for any $t \in \mathbb{Z}^+$. Prove that $V_{k_i} (t, x(t)) \leq V_{k_i} (t_0, x(t_0)) e^{\int_{t_0}^{t} \lambda_{m}^- (\delta) \mathrm{d}\delta} + i \chi (\|u(t)\|) e^{\int_{t_0}^{t} \lambda_{m}^- (\delta) \mathrm{d}\delta}$.

Step 4. Prove that function $V_{\sigma(t)} (t, x)$ satisfies the requirement of ISS.

Step (i). Assuming that $t_i \leq t$, then $t_i$ is the switching-time of the last subsystem, and we have $\sigma(t) = k_i$, which indicates that the current subsystem is $k_i$. From (7) we can derive that if $V_{k_i} (t, x(t)) \geq \chi (\|u(t)\|)$,

$$
\frac{\partial V_{k_i} (t, x(t))}{\partial t} + \frac{\partial V_{k_i} (t, x(t))}{\partial x} f_k (t, x, u) \leq \lambda_k (t) V_{k_i} (t, x(t)),
$$

(14)

as function $\beta_{k_i}$ is strict-negative. We can derive from (14) and Lemma 3.1 that the following equation holds:

$$
V_{k_i} (t, x(t)) \\
\leq V_{k_i} (t_i, x(t_i)) + \mu_{k_i} (t_i, x(t_i)) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta} \\
+ \chi \left( \|u(t)\|_{[t_i, t]} \right) e^{\int_{t_i}^{t} \lambda_{k_i}^- (\delta) \mathrm{d}\delta}.
$$

(18)
From (13), we can see that \( \mu_{kj} \) also fits in Lemma 1. Therefore, we can deduce that

\[
\mu_{kj}(t, x) \leq \mu_{kj}(t, x(t)) \int_{t_i}^{t} \lambda_k(\theta) \, d\theta.
\]  

(19)

Then, we see that

\[
\mu_{kj}(t, x(t)) \leq \mu_{kj}(t, x(t)) \int_{t_i}^{t} \lambda_k(\theta) \, d\theta
\]

(20)

where \( \lambda_{k_{i,j}} = \max\{\lambda_{k_{i,j}}, \lambda_k\} \). Note that (16) holds for any switching interval. Therefore, combining with (18) yields:

\[
V_{\sigma(t)}(t, x(t))
\]

\[
\leq V_{k_{i-1}}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-1}}(\theta) \, d\theta
\]

\[
+ \mu_{k_{i-1},k_i}(t, x(t)) \int_{t_i}^{t} \lambda_{k_i}(\theta) \, d\theta
\]

\[
+ \chi(\|u(\theta\|_{[t_i,t]} \int_{t_i}^{t} \lambda_{k_i}(\theta) \, d\theta
\]

\[
\leq [V_{k_{i-1}}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-1}}(\theta) \, d\theta
\]

\[
+ \mu_{k_{i-1},k_i}(t, x(t)) \int_{t_i}^{t} \lambda_{k_i}(\theta) \, d\theta
\]

\[
+ \chi(\|u(\theta\|_{[t_i,t]} \int_{t_i}^{t} \lambda_{k_i}(\theta) \, d\theta
\]

\[
\leq V_{k_{i-2}}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-2}}(\theta) \, d\theta
\]

\[
+ [\mu_{k_{i-2},k_{i-1}}(t, x(t))
\]

\[
+ \mu_{k_{i-1},k_i}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-1},k_i}(\theta) \, d\theta
\]

\[
+ \mu_{k_{i-1},k_i}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-1},k_i}(\theta) \, d\theta
\]

\[
+ 2\chi(\|u(\theta\|_{[t_i,t]} \int_{t_i}^{t} \lambda_{k_{i-1},k_i}(\theta) \, d\theta
\]

\[
\leq V_{k_{i-2}}(t, x(t)) \int_{t_i}^{t} \lambda_{k_{i-2}}(\theta) \, d\theta
\]

\[
+ 2\chi(\|u(\theta\|_{[t_i,t]} \int_{t_i}^{t} \lambda_{k_{i-1},k_i}(\theta) \, d\theta
\]
For simplicity, redefine $\chi(\|u(s)\|) = \chi(\|u(s\|_{[t_0,t]})).$ Recalling that $\lambda_m = \max\{\lambda_k\},$ from (21), we can summarize that, for $l \in \mathbb{Z}^+$ and $l \leq i/2,$ after $l$ times the same procedure of (12-19),

$$V_{\sigma(t)}(t, x(t)) \leq V_{k_0}(t_{i-1}, x(t_{i-1}))e^{\int_{t_i}^{t_{i+1}} \lambda_m(\delta) d\delta} + 2\chi(\|u(s\|_{[t_{i-1},t_i]})e^{\int_{t_i}^{t_{i+1}} \lambda_m(\delta) d\delta}.$$  \hfill (22)

Then we have completed step (ii), we have found a rule between the switches.

For step (iii), from the above equation, we can draw the conclusion that when $i$ is even,

$$V_{\sigma(t)}(t, x(t)) \leq V_{k_0}(t_{0}, x(t_{0}))e^{\int_{t_0}^{t_{i}} \lambda_m(\delta) d\delta} + (i - 1)\chi(\|u(s\|_{[t_0,t_1]})e^{\int_{t_0}^{t_{i+1}} \lambda_m(\delta) d\delta} \leq [V_{k_0}(t_{0}, x(t_{0}))e^{\int_{t_0}^{t_{i}} \lambda_m(\delta) d\delta} + \chi(\|u(s\|_{[t_0,t_1]})e^{\int_{t_0}^{t_{i}} \lambda_m(\delta) d\delta} + (i - 1)\chi(\|u(s\|_{[t_0,t_1]})e^{\int_{t_0}^{t_{i+1}} \lambda_m(\delta) d\delta}.$  \hfill (23)

When $i$ is odd,
\[
V_{\sigma(t)}(t, x(t)) \\
\leq V_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
+ (i-1)\chi(\|u(s)\|)e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
\leq [V_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta}] \\
+ \chi(\|u(s)\|)[t_{i-1}, t_i]e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
+ \mu_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
+ (i-1)\chi(\|u(s)\|)e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
\leq \mu_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
+ \mu_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} + i\chi(\|u(s)\|)e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
\leq [V_{k_1}(t_1, x(t_1)) + \mu_{k_1}(t_1, x(t_1))e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta} \\
+ i\chi(\|u(s)\|)e^{\int_{t_1}^{t} \lambda_m(\theta)d\theta}, 
\] (24)

Step (iv): Considering the definition of \(\lambda_m^+(t)\) and \(\lambda_m^-(t)\), we have

\[
e^{\int_{t_0}^{t} \lambda_m^-(\theta)d\theta} = e^{\int_{t_0}^{t} \lambda_m^+(\theta)d\theta} = e^{\int_{t_0}^{t} \lambda_m^+(\theta)d\theta} - e^{\int_{t_0}^{t} \lambda_m^-(\theta)d\theta} = e^{C_1}e^{\int_{t_0}^{t} \lambda_m^+(\theta)d\theta} = C_1e^{\int_{t_0}^{t} \lambda_m^-(\theta)d\theta},
\] (25)

where \(C_1 = e^C\). Putting together (6), (8), (9), (20), (22) and (21) yields

\[
\alpha_1(\|x(t)\|) \\
\leq V_{k_1}(t_0, x(t_0)) + \mu_{k_1}(t_0, x(t_0))e^{\int_{t_0}^{t} \lambda_m(\theta)d\theta} \\
+ i\chi(\|u(s)\|)e^{\int_{t_0}^{t} \lambda_m(\theta)d\theta} \\
\leq (\alpha_2(\|x(t_0)\|)) + \mu_{k_1}(t_0, x(t_0))e^{\int_{t_0}^{t} \lambda_m(\theta)d\theta} \\
+ i\chi(\|u(s)\|)e^{\int_{t_0}^{t} \lambda_m(\theta)d\theta} \\
\leq 2C_1\alpha_2(\|x(t_0)\|) + \mu_{k_1}(t_0, x(t_0))e^{-\int_{t_0}^{t} \lambda_m(\theta)d\theta} \\
+ iC_1\chi(\|u(s)\|),
\] (26)

where \(\alpha_2(s) \geq s\). We know from the Caratheodory conditions [46, 47] that \(x(t)\) always exists, then constructing the \(Z\mathcal{L}\) function \(\varphi(s, t)\) with respect to a minimum real number \(T \in [t_0, t]\) satisfying (9) as follows

\[
\varphi(s, t) = \begin{cases} 
\alpha_1^{-1}(4C_1\alpha_2(s)), & t \in [t_0, T) \\
\alpha_1^{-1}(4C_1[e^{-c(T-1)}(t-T) - t + 1 + T]a_2(s)), & t \in [T, T + 1) \\
\alpha_1^{-1}(4C_1e^{-ct}a_2(s)), & t \geq T + 1 
\end{cases}
\] (27)

Recalling the definition of \(\chi(\|u(s)\|)\), not hard to point out that \(\chi(\|u(s)\|) \in \mathcal{H}\) function; also, we have proven that \(\varphi(s, t)\) is a \(Z\mathcal{L}\)-function in our previous work [44]. As a conclusion, by means of (27), the condition in Definition 2.1 has been satisfied. Now, we have completed the proof, i.e., the considered system with the designed switching law is ISS.

Remark 2. Theorem 3.2 extends MLF to an indefinite form, which relaxes the restriction in existing results on the Lyapunov functions for each subsystem is restricted to be decreasing in the domain [2, 3, 10, 29]; the sufficient conditions raise the upper bound of the derivatives of Lyapunov functions such that they could be positive-definite in some time.
intervals. Furthermore, the switching law is presented in our best way to utilize the ISS/IOS property of subsystems; the summation of $\beta_{k_j} V_k(t,x) + \mu_{k_j}(t,x) - V_j(t,x)$ ensures that all subsystems are only supposed to be IOS during activated time intervals in the switching sequence. Compared with [29, 36], the verifying work of the ISS property for switched systems is much simplified due to the construction of the switching law and the new sufficient conditions.

Moreover, when $m = 1$, Theorem 1 is the indefinite Lyapunov function for typical nonlinear nonswitched systems, so this work is more general than [44]. The extension of this work will focus on the introduction of impulsive signals.

Remark 3. Note that we also provide new sufficient conditions based on the CLF method. Consider switched nonlinear System (1) with an essentially bounded input $u$ and a founded common Lyapunov function $V(t,x)$, if there exist two $\mathcal{H}_\infty$ functions $\alpha_1, \alpha_2(s) \geq s$, continuous functions $\lambda$ and a $\mathcal{K}$-function $\chi$ satisfying

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$

$$V(t,x) \leq \lambda(u)V(t,x) \Rightarrow \chi(u(t)) \leq V(t,x).$$

and conditions (8), (9) are satisfied. Then, the considered system with a CLF is ISS, regardless of the design of switching signals. This proposition is another special case of Theorem 1. When a CLF could be found for System (1), the verifying work could obviously be further simplified.

3.3. IOS for nonlinear switched systems. Now, we extend our main result to IOS verification for time-varying nonlinear switched systems, and some new sufficient conditions are proposed. The system output $h_{k_j}(t,x)$ is now considered.

**Corollary 3.** Considering System (1) with essentially bounded $u(t)$, if there exist two $\mathcal{H}_\infty$ functions $\alpha_1, \alpha_2(s) \geq s$, continuous functions $\lambda_k$, $\beta_{k_j}(t,x)$, $\beta_{k_j}(t,x) \geq 0$ and a $\mathcal{K}$-function $\chi$ satisfying:

$$\alpha_1(|h_k(t,x)|) \leq V_k(t,x) \leq \alpha_2(|x|),$$

and conditions (7)-(9) hold. Then, the considered system with the designed switching law is IOS.

The proof part is easy to derive from Theorem 1, so it is omitted here. The details can be found in the Appendix.

Remark 4. The sufficient conditions for IOS in this corollary are extensions of our main result. We also employ Lemma 1 and the switching law in this corollary. Compared with existing results concerning IOS for switched systems, the proposed sufficient conditions have distinct advantages. On the one hand, the subsystem is not restricted to be IOS, and the IOS property is only demanded in some subregions in the state space when the subsystem is activated. On the other hand, the iMLF method relaxes the restrictions on derivatives of Lyapunov functions and permits them to be indefinite for any $t \in \mathbb{Z}^+$ [16]. With the help of the designed switching law and iMLF, candidates of Lyapunov functions are more convenient to select, and IOS verification is also much simplified.

4. Extended notions of IOS

In this section, we also investigate some extensions of IOS, and the individual new sufficient conditions are also obtained. Note that these new sufficient conditions also allow the derivative of Lyapunov functions to be indefinite, and the subsystems are also permitted to be unstable.

**Definition 1.** [14] (State-independent input-to-output stability, or SIIOS) Considering System (1), suppose that $x(t)$ always exists as $t \geq t_0$ for any $x(t_0)$ and bounded $u(t)$; $y(t)$ is detectable, then the system is SIIOS for some $\phi \in \mathbb{ZL}$ and $y \in \mathcal{K}$ satisfying:

$$\|y(t,x)\| \leq \phi(|h(t_0,x(t_0))|, t) + y(\|u\|),$$  

**Definition 2.** [14] (Output-Lagrange input-to-output stability) Considering an IOS system conforms to (1), suppose that $x(t)$ always exists when $t \geq t_0$ for any $x(t_0)$ and bounded $u(t)$, then we say the system is OLIIOS if:

$$\|y(t,x)\| \leq \max\{\gamma_1(|h(t_0,x(t_0))|), \gamma_2(\|u\|)\},$$

where functions $\gamma_1, \gamma_2$ are $\mathcal{H}$-functions.

**Definition 3.** [14] (Robustly output stability, or ROS) Considering a constructed system such that:

$$x(t) = g_o(t,x,d) = f_o(t,x,d(t)\phi(|h_o(x(t))|));$$

$$y(t) = h_o(t,x),$$

where $d(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is the system input of (30), and $\phi \in \mathcal{K}_\infty$. Suppose that state trajectory $x(t)$ always exists when $t \geq t_0$, for any $x(t_0)$ and bounded $u(t)$ such that

$$|y\phi(t,x)| \leq \phi(|x(t_0)|, t),$$

where function $\phi \in \mathbb{ZL}$, then, we say that the system (1) is ROS.

The more relaxed SIIOS sufficient conditions are given as:

**Corollary 1.** Considering System (1) with a measurable, bounded input $u(t)$, if there exist two $\mathcal{H}_\infty$ functions $\alpha_1, \alpha_2(s) \geq s$, continuous functions $\lambda_k$, $\beta_{k_j}(t,x) \leq 0$ and a $\mathcal{K}$-function $\chi$ satisfying

$$\alpha_1(|h_k(t,x(t))|) \leq V_k(t,x) \leq \alpha_2(|x(t)|),$$

and conditions (7)-(9) are satisfied, then System (1) is SIIOS.
Remark 5. To be more specific, the differences between IOS and SIIOS is that the decay rate of \( y(t) \) of a SIIOS system is supposed not to be dependent on system states, which is manifested in this system:

\[
\begin{align*}
    x_1(t) &= \frac{-2x_1(t) + u(t)}{1 + t^2 + x_2^2} \\
    x_2(t) &= 0 \\
    y &= x_1(t)
\end{align*}
\]

(35)

considering \( V(t, x) = x^2/2 \), Then one can deduce that

\[
\dot{V}(t, x) = \frac{-V(t, x)}{1 + t^2 + x_2^2(t_0)}
\]

(36)

holds for all \( \|u\|^2 \leq V(t, x) \). Then it is IOS. However, due to the decreasing rate of \( y(t) \) is dependent on \( x_2(t_0) \), the system is not state-independent IOS. Therefore, this notion is much stricter than IOS.

Remark 6. Observing that output-layer input-to-output stability could be regarded as IOS under output redefinition, the definition of OLIOS is equal to

\[
|h(t, x, u)| \leq \varphi(h(t, x, u)),
\]

(37)

where \( \varphi \in \mathcal{L}, \varphi \in \mathcal{K}, t \geq 0 \). The specific relationship between the output-stability notions is SIIOS \( \longrightarrow \) OLIOS \( \longrightarrow \) IOS \( \longrightarrow \) ROS, where SIIOS is the strongest notion, but the reverse does not hold.

As a weaker notion of IOS, ROS presents a function for estimating the margin of the output feedback without breaking the robustness. We also provide sufficient conditions for ROS as follows.

Corollary 2. System (1) with a measurable, bounded input \( u(t) \) is ROS if there exist two \( \mathcal{K}_\infty \) functions \( \alpha_1, \alpha_2(s) \geq s \), continuous functions \( \lambda_k, \beta_{kj}(x, t) \leq 0 \) and a \( \mathcal{K} \)-function \( \chi \) satisfying

\[
\alpha_1(h(t, x(t))) \leq V_k(t, x(t)) \leq \alpha_2(x(t)),
\]

(38)

and conditions (7)-(9) hold.

The proofs of Corollary 2 and Corollary 3 are also easy to derive from Corollary 1, so we omit them here.

5. Demonstrative example

A switched system is constructed to show the effectiveness of our main result by depicting the simulation results of state trajectories as well as switching signals.

Considering a system conforms to (1) which is composed of two subsystems, and the state is defined as \( x = [x_1, x_2]^T \). We define \( M = \{1, 2\} \) such that
It is easy to see that the $K$– functions satisfying condition (6) are easy to find; moreover, we have proved in [44] that the Lyapunov function for a system admits

$$\lambda_m(t) = \frac{2}{1 + t^2 - t |\cos t|},$$

then conditions (8) and (9) are satisfied. (Then, we can learn from Theorem 3.2 that System (35) is an ISS system. The superiority of proposed iMLFs is obvious. In traditional Lyapunov regard, a function $V(t)$ for some systems with an indefinite derivative could not be regarded as a right choice, then sometimes the system could not be classified as an ISS system. But this example shows that the system with iMLFs could also be ISS. This paper not only broaden the field that the functions being choosen, but also simplified the verification of some switched systems. The simulation results of System (35) are depicted as follows. Considering the state trajectories are too flat and the fluctuation is not obvious when initial states are not close enough to the equilibrium, the system state trajectory with or without input is provided in Fig 1 and Fig 2, and the switching signal which jumps between two subsystems is provided in Fig 3 and Fig 4. To show our result is also effective when initial states are not close to the equilibrium, we also depict the state response of System (35) with smaller initial state in Fig 5. This example reveals the effectiveness of our method.

To verify the IOS of this example, we select $y = h_x(t,x) = x_1x_2$. Additionally, it is easy to see that we can find the proper functions $\alpha_1$ and $\alpha_2(x)$ which meet the condition (27); moreover, as we have proved above, this system admits a $\lambda_m(t) = \frac{2}{1 + t^2 - t |\cos t|}$, and then we can learn from Corollary 3.3 that this system is an IOS system.

Figure 2: The switching signal of System (35) without inputs

Figure 3: The state response of Example (35) with a bounded input $u(t) = \cos(96t)$

Figure 4: The switching signal of System (35) with a bounded input $u(t) = \cos(96t)$

Figure 5: The switching signal of System (35) with a bounded input $u(t) = \cos(96t)$
6. Conclusion

This paper investigates the ISS and IOS characterizations for nonlinear switched systems by extending the MLFs method to an indefinite form. Some more relaxed ISS/IOS sufficient conditions are proposed, and the main result is applied to the extensions of IOS. Based on the designed switching law and the proposed iMLFs methods, the provided conditions permit the subsystems to admit the Lyapunov functions with indefinite time derivatives. Furthermore, the subsystems are allowed to be non-ISS/IOS. These approaches greatly extend the range of candidate Lyapunov functions and subsystems being selected. Moreover, the work of designing and verifying switched systems is much simplified. Finally, we construct numerical example of the switched system and a simulation for nonlinear switched systems by extending the MLFs applied to the extensions of IOS. Based on the designed switching law and the proposed iMLFs methods, the switching law and the proposed iMLFs methods, the

6.1. Proof of Corollary 3.3. Now we provide the proof of Corollary 3.3. Considering (7), not hard to see that (12) holds. According to Lemma 3.1, we have

\[ V_k(t, x) \leq V_k(t, x(t)) e^{\int_t^T \lambda_k(t) ds} \]

\[ + \chi(||u(s)||_{[t,T]} e^{\int_t^T \lambda_k(t) ds}], \]

where \( t \leq t \leq t+1 \). As dictated by the designed switching law, when \( t \leq t < t+1 \) and \( i \in \mathbb{Z}^+ \), then in the current subsystem we also have (14) holds. Taking (37) into consideration:

\[ V_{a(t)}(t, x(t)) \]

\[ \leq (V_{k_{i,1}}(t, x(t)) + \mu_{k_{i,1},k_{i,2}}(t, x(t))) e^{\int_{t_1}^{t_1} \lambda_{k_{i,1}}(s) ds} \]

\[ + \chi(||u(s)||_{[t_1,T]} e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds}], \]

\[ = V_{k_{i,1}}(t, x(t)) e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds} \]

\[ + \mu_{k_{i,1},k_{i,2}}(t, x(t)) e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds} \]

\[ + \chi(||u(s)||_{[t_1,T]} e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds}], \]

Also, from (12) we can derive that

\[ \mu_{k_{i,1},k_{i,2}}(t, x(t)) e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds} \]

\[ \leq \mu_{k_{i,1},k_{i,2}}(t_1) e^{\int_{t_1}^{T} \lambda_{k_{i,1}}(s) ds}, \]

combining (38) with (39) yields

\[ V_k(t, x(t)) \leq V_{k_{i,2}}(t, x(t)) e^{\int_{t_{i-1}}^{T} \lambda_{k_{i,2}}(s) ds} \]

\[ + 2 \chi(||u(s)||_{[t_{i-1},T]} e^{\int_{t_{i-1}}^{T} \lambda_{k_{i,2}}(s) ds}]. \]

Redefine \( \chi(||u(s)||) = \chi(||u(s)||_{[t_{i-1},T]} \), when \( i \) is even, with a similar method we proved in (21) indicates

\[ V_{a(t)}(t, x(t)) \]

\[ \leq [V_{k_{a}}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{a}(s) ds} \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{a}(s) ds}], \]

\[ \leq V_{k_{a}}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{a}(s) ds} \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{a}(s) ds}], \]

When \( i \) is odd, then

\[ V_{a(t)}(t, x(t)) \]

\[ \leq [V_{k_{a}}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{a}(s) ds} \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{a}(s) ds}], \]

\[ \leq V_{k_{a}}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{a}(s) ds} + \mu_{k_{a},k_{a}}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{a}(s) ds}] \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{a}(s) ds}], \]

Putting together (41), (27), and (42), we can reach the following conclusion:

\[ \alpha_1(||h_k(t, x)||) \]

\[ \leq [V_{k_1}(t_0, x(t_0)) + \mu_{k_1,k_2}(t_0, x(t_0)) \]

\[ e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}], \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}], \]

\[ \leq [\alpha_2(|x(t_0)|) + \mu_{k_1,k_2}(t_0, x(t_0)) \]

\[ e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}], \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}], \]

\[ \leq 2 \alpha_1 \alpha_2(|x(t_0)| + \mu_{k_1,k_2}(t_0, x(t_0)) e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}] \]

\[ + \chi(||u(s)||_{[t_0,T]} e^{\int_{t_0}^{T} \lambda_{k_1}(s) ds}], \]

since \( \alpha_2(s) \geq s \) holds. Constructing function
where $T$ satisfies (9) for any $t \geq T$. Note that $\chi([u(s)])$ is the $\mathcal{X}$-function and $\varphi(s,t)$ is a $\mathcal{Z}$-function. By means of (43), we have the conditions in Definition 2.2 have been satisfied. Now we have completed the proof, i.e. the considered system with the designed switching law is IOS.

Data Availability

No data were used during this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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