Research Article

On the Approximate Solutions of the Constant Forced (Un) Damping Helmholtz Equation for Arbitrary Initial Conditions

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This paper presents some novel solutions to the family of the Helmholtz equations (including the constant forced undamping Helmholtz equation (equation (1)) and the constant forced damping Helmholtz equation (equation (2))) which have been reported. In the beginning, equation (1) is solved analytically using two different techniques (direct and indirect solutions): in the first technique (direct solution), a new assumption is introduced to find the analytical solution of equation (1) in the form of the Weierstrass elliptic function with arbitrary initial conditions. In the second case (indirect solution), the solution of the undamping (standard) Duffing equation is devoted to determining the analytical solution to equation (1) in the form of Jacobian elliptic function with arbitrary initial conditions. Moreover, equation (2) is solved using a new ansatz and with the help of equation (1) solutions. Also, the evolution equations (equations (1) and (2)) are solved numerically via the Adomian decomposition method (ADM). Furthermore, a comparison between the approximate analytical solution and approximate numerical solutions using the fourth-order Runge–Kutta method (RK4) and ADM is reported. Furthermore, the maximum distance error for the obtained solutions is estimated. As a practical application, the Helmholtz-type equation will be derived from the fluid governing equations of quantum plasma particles with(out) taking the ionic kinematic viscosity into account for investigating the characteristics of (un)damping oscillations in a degenerate quantum plasma model.

1. Introduction

The ordinary and partial differential equations have played an important role in explaining many natural phenomena, in addition to their applications in many engineering and physical problems. Due to the great role played by these equations, many authors focused their efforts on finding some solutions to these equations [1–10]. The undamping Helmholtz equation \( \ddot{q}(t) + \sum_{i=1}^{n} k_i \dot{q}(t) = 0 \), where \( i \) is the odd number, i.e., \( i = 1, 3, 5, \ldots \) and undamping Duffing equation \( \ddot{q}(t) + \sum_{i=1}^{n} k_i \dot{q}(t) = 0 \), where \( i = 1, 2, 3, \ldots \) in addition to their family (including friction/damping force in addition to excitation/perturbation force) are among the most famous differential equations in dynamic, electrical, and engineering systems [11–16]. A lot of physical and engineering problems such as the human eardrum oscillations, the dynamics of the ships, the electrical circuits signal oscillations, heavy symmetric gyroscope, and microperforated panel absorber [17–21] have been investigated using different solutions of the Helmholtz-type oscillator. The Helmholtz-type oscillator is a second-order differential equation with a quadratic nonlinear term in addition to some other terms. For realistic physical situations, we cannot ignore both the friction/dissipation force
and excitation/perturbation forcing. Accordingly, the general form of the Helmholtz equation reads \[22–24\]
\[\begin{align*}
\ddot{q} + 2\gamma \dot{q} + \alpha q + \beta q^2 &= F, & q(0) &= q_0 & & \ddot{q}(0) = \dot{q}_0,
\end{align*}\]
where \(q\) denotes the displacement of the system, \(\alpha\) is the natural frequency, \(\beta\) is a nonlinear system parameter, \(\gamma\) represents the damping factor, and \(F\) is a constant/excitation force. The first equation in system (1) is called the constant forced damping Helmholtz equation (equation (2)). If the coefficient \(\gamma\) of the damping term (2\(\gamma\)) is neglected, equation (2) reduces to the constant forced Helmholtz equation \((\ddot{q} + \alpha q + \beta q^2 = F)\) (equation (1)). Also, if both the coefficient \(\gamma\) of the damping term and the constant force \(F\) are neglected, the traditional form of Helmholtz equation is covered \((\ddot{q} + \alpha q + \beta q^2 = 0)\). The initial value problem (IVP) (1) and its family have many applications in several fields, starting from analyzing the signals that propagate in electrical circuits, plasma physics, general relativity, betatron oscillations, vibrations of shells, vibrations of the acoustically driven human eardrum, solid-state physics, etc. [26–33].

It is well known that, in the absence of both friction and the excitation forces from the IVP (1), the unforced and undamping Helmholtz equations are covered. The exact analytic solutions to the unforced and undamping Helmholtz equation have been derived in detail in the literature in terms of the Weierstrass elliptic function [34–38] and Jacobi elliptic functions [38–41]. Generally, to solve any quadratic or cubic nonlinear second-order differential equation, firstly, we should transform it to an elliptic integral and then we solve it [24]. It is known that the unforced and undamping Helmholtz equations are completely integrable, so they have exact solutions, but if the friction force (damping term) is included, then the unforced damping Helmholtz equation becomes nonintegrable and cannot support an exact solution for arbitrary values of its coefficients \((\gamma, \alpha, \beta)\). Thus, under certain condition, the unforced damping Helmholtz equation has been solved analytically in terms of the Jacobi elliptic functions by Johannessen [24]. Also, Almendral and Sanjuán [27] derived an exact solution to the undamped Helmholtz equation using the Lie theory under certain conditions for the coefficients \((\gamma, \alpha, \beta)\).

In many realistic physical models, both the damping and the excitation/external terms are very important to be included, and thus the problem becomes more complicated to find its analytical solutions. In this paper, we will derive some analytical solutions to equation (1) in the terms of the Weierstrass and Jacobian elliptic functions. Also, an approximate analytical solution to equation (2) for arbitrary values to the coefficients and the initial conditions will be derived in detail. Moreover, the problem under consideration will be solved numerically via using the RK4 and ADM to make a comparison between the obtained solutions and the approximate numerical solutions. Furthermore, the maximum distance error between the approximate analytical solution and the approximate numerical solutions will be estimated. Also, the dynamics of nonlinear oscillations that can be generated in the RLC electronic circuits and quantum plasma will be investigated using the solution of equation (2).

The rest of this work is organized in the following manner: in Section 2, we will introduce in detail our methodology for solving the family of the Helmholtz-type equations. Also, we will introduce some new approaches for solving equation (1) as well as the exact solution of the undamped and unforced Helmholtz equation, which will be devoted to finding an approximate analytical solution to equation (2). In Section 3, a comparison between the obtained solutions and the approximate numerical solution using the ADM will be investigated. In Section 4, some realistic applications will be introduced. Finally, our results will be summarized in Section 5.

2. Our Methodology for Solving the Family of the Helmholtz Equations

Before proceeding in solving the IVP (1), it is necessary to refer to two fundamental equations and their solutions: the first one is called the Duffing equation and the other is called the constant forced Helmholtz equation.

2.1. Duffing Equation and Its Solution. The analytical solution to the following IVP, which is called Duffing equation [30],
\[\begin{align*}
\ddot{\eta} + R\dot{\eta} + S\eta + \eta^3 &= 0, \\
\eta(0) &= \eta_0 & & \dot{\eta}(0) = \dot{\eta}_0,
\end{align*}\]
is given by the following formula:
\[\eta(t) = c_1 \text{cn} \left( \sqrt{R + Sc_1^2} t + c_2, m \right). \tag{3}\]

By inserting this relation into the IVP (2) and after several tedious but simple math operations, we finally get the values of \(c_1\) and \(c_2\) as follows:
\[\begin{align*}
c_1 &= \pm \sqrt{\frac{-R + \sqrt{\Delta}}{S}}, \\
c_2 &= c_1^{-1} \left( \frac{\eta_0}{\eta_1} \frac{Sc_1^2}{2(R + Sc_1^2)} \right).
\end{align*}\]

where \(R, S, \eta_0, \text{ and } \eta_0\) are real numbers and \(\Delta\) is called the discriminant to Duffing (2):
\[\Delta = (R + \eta_0^2 S)^2 + 2\eta_0^2 S > 0. \tag{5}\]

Solution (3) could be expressed as
\[\eta(t) = \eta_0 \text{cn} (\omega t|\mu) + (\eta_0 / \omega) \text{dn} (\omega t|\mu) \text{sn} (\omega t|\mu) \left(1 - (1/2)(1 - \left( R + S\eta_0 / \sqrt{\Delta} \right)) \text{sn} (\omega t|\mu)^2 \right). \tag{6}\]
\[
\begin{align*}
\omega &= \sqrt{\Delta}, \\
m &= \frac{1}{2} - \frac{R}{2\sqrt{\Delta}}.
\end{align*}
\]

For negative discriminant (\(\Delta < 0\)), the solution may be written in the following form:

\[
\eta(t) = \rho - \frac{2\rho}{1 + \kappa \sec \left(\sqrt{\omega} t + \frac{1}{2\omega} \left(\rho + \eta_0/\kappa (\rho - \eta_0)\right) m\right)},
\]

where

\[
m = \frac{4\rho \sqrt{2S} \sqrt{\rho^2 S - R}}{2\rho \sqrt{2S} \sqrt{\rho^2 S - R + R - 3\rho^2 S}},
\]

\[
\omega = \frac{1}{4} \left(2\rho \sqrt{2S} \sqrt{\rho^2 S - R + R - 3\rho^2 S}\right),
\]

\[
\kappa = \sqrt{2\rho \sqrt{2S} \sqrt{\rho^2 S - R + R + 3\rho^2 S}} / \sqrt{R + \rho^2 S},
\]

\[
\rho = \pm \sqrt{\frac{2R\rho_0^2 + S\eta_0^4 + 2\eta_0^2}{-S}}.
\]

For a zero discriminant (\(\Delta = 0\)), the solution of Duffing equation (2) will be

\[
\eta(t) = c_1 \tan h \left(c_1 \sqrt{-\frac{S}{2}} t + c_2\right),
\]

with

\[
c_1 = \pm \sqrt{\frac{\sqrt{-3\eta_0^4 - \sqrt{-2\eta_0^2\eta_0}}}{\sqrt{-3\eta_0^4 + \sqrt{\eta_0^4}}}},
\]

\[
c_2 = \frac{1}{2} \log \left(\frac{\eta_0 - \sqrt{\frac{3\eta_0^4 - \sqrt{-2\eta_0^2\eta_0}}{\eta_0}}}{\sqrt{-3\eta_0^4 + \sqrt{\eta_0^4}}\eta_0}\right),
\]

where \(\eta_0\eta_0 \neq 0\).

When \((R + \eta_0^2 S)^2 + 2\eta_0^2 S = \eta_0 = 0\), the solution reads

\[
\eta(t) = \frac{\sqrt{-S}}{2\eta_0} \eta_0 \tan h \left(\frac{-S}{2\eta_0} t\right).
\]

In case \((R + \eta_0^2 S)^2 + 2\eta_0^2 S = \eta_0 = 0\), the solution becomes the constant function \(\eta(t) = \eta_0\).

2.2. The Analytical Solution to the Constant Forced Helmholtz Equation. The solution of the constant forced Helmholtz equation

\[
\begin{align*}
\ddot{\xi}(t) + a + b \xi(t) + c \xi^2(t) &= 0, \\
\dot{\xi}(0) &= \xi_0 & \xi(0) &= \xi_0,
\end{align*}
\]

may be expressed in either one of the following forms. Note that \(a = -F\).

2.2.1. First Formula. Suppose that the solution of system (13) has the following form:

\[
B = \frac{6(a + A(Ac + b))}{2Ac + b}
\]

\[
C = \frac{12}{2Ac + b}
\]

\[
g_2 = \frac{1}{12} (b^2 - 4ac),
\]

\[
g_3 = \frac{1}{216} (2Ac + b) (b^2 - 2c(3a + A^2c) - 2Abc),
\]

\[
d_1 = \frac{(A - \xi_0)(2Ac + b)}{6(a + A(Ac + b))},
\]

\[
d_2 = \frac{\dot{\xi}_0 (2Ac + b)}{6(a + A(Ac + b))}.
\]

The value of parameter \(A\) represents a root to the following quartic equation:
Forced and Damped Helmholtz Equation. Let us rewrite system (1) in the following traditional form:

\[ \begin{cases} x' + 2\varepsilon x + ax + \beta x^2 = F, & x(0) = x_0, \quad x'(0) = \dot{x}_0, \end{cases} \]  

(24)

with \( \beta \neq 0 \).

Also, let us assume that

\[ a^2 + 4F\beta \geq 0. \]  

(25)

Now, suppose that the solution of system (24) is given by

\[ x(t) = d + \exp(-et)y(t), \]  

(26)

where \( y = y(t) \) is a solution to the following Helmholtz equation:

\[ \begin{cases} y'' + py + \beta y^2 = 0, & y(0) = y_0 = x_0 - d, \quad y'(0) = \dot{y}_0 = \varepsilon x_0 + \dot{x}_0 - dc. \end{cases} \]  

(27)

Inserting (26) into the first equation in system (24), \( \mathbb{R}(t) \equiv \ddot{x} + 2\varepsilon \dot{x} + ax + \beta x^2 - F = 0 \), we obtain

\[ \mathbb{R}(t) = \beta d^2 + \alpha d - F + e^{-2\varepsilon t}y(t)[\beta y(t) - e^\varepsilon(-\alpha - 2\beta d + \varepsilon^2 + p + \beta y(t))]. \]  

(28)

Expression (28) suggests the following choices:

\[ -\alpha - 2\beta d + \varepsilon^2 + p = 0, \]  

(29)

\[ \beta d^2 + \alpha d - F = 0, \]  

(30)

giving us the values of \( p \) and \( d \) as follows:

\[ p = \varepsilon^2, \quad d = \frac{-\alpha + \sqrt{\alpha^2 + 4\varepsilon^2}}{2\beta}. \]  

(30)

The solution to the following IVP,
\begin{align*}
    &y'' + (\alpha + 2\beta d - \epsilon^2) y + \beta y^2 = 0, \\
    &y(0) = y_0 = x_0 - d, \\
    &y'(0) = \dot{y}_0 = \varepsilon x_0 + \dot{x}_0 - de,
\end{align*}

(31)

may be expressed in different forms as we explained in the previous section.

3. A Comparison between Our Solutions and ADM Solution

There are many numerical methods that could be used to find approximate solutions to IVP (24). Here, we make use of the ADM [42–44] to solve IVP (24) with(out) forcing term \((F)\). According to this method, the first iteration/approximation for the unforced case \((F)\) reads

\[ x_{\text{ADM0}}(t) = \left( e^{-\frac{t}{\sqrt{\alpha}}} \right) \left[ (\epsilon x_0 + \dot{x}_0)\Theta_1 + x_0\Theta_2 \right], \]

(32)

where \(\Theta_1 = \sin(t\,F)\), \(\Theta_2 = \cos(t\,F)\), and \(F = \sqrt{\alpha - \epsilon^2}\).

For the second iteration/approximation, we have

\[ x_{\text{ADM1}}(t) = \beta \left( e^{-t\frac{\alpha}{\sqrt{\alpha}}\sqrt{F}} \right) \left[ -e^{-t\epsilon\Theta_1 + \epsilon\Theta_2} + \epsilon\Theta_1 \right] \left[ \alpha x_0\Theta_2 + (\epsilon x_0^2 + \dot{x}_0\,F^2 + (1 + \epsilon^2) x_0 + \epsilon\dot{x}_0)\Theta_1 \right]^2. \]

(33)

It is clear that the semianalytical solution (26) \((F = 0)\) gives good results as compared to both the RK4 and ADM approximate numerical solutions.

Now, let us find an approximate solution for IVP (24) in the presence of the forcing term, using the ADM. Accordingly, the first approximation is given by

\[ x_{\text{ADM0}}(t) = \left( e^{-d\frac{\alpha}{\sqrt{\alpha}}\sqrt{F}} \right) \left[ (\epsilon(x_0 - d) + \dot{x}_0)\Theta_1 + (x_0 - d)\Theta_2 \right], \]

(36)

where \(\Theta_1 = \sin(t\,F)\), \(\Theta_2 = \cos(t\,F)\), \(F = \sqrt{\alpha - \epsilon^2}\), \(\Lambda = (\alpha + 2\beta d)\), and \(d = (-\alpha \pm \sqrt{\alpha^2 + 4\beta^2})/(2\beta)\).

The second approximation to IVP (24) according to the ADM reads

\[ x_{\text{ADM1}}(t) = \left( e^{-\frac{2\epsilon\alpha}{\sqrt{\alpha}}\sqrt{F}} \right) \left[ (d - x_0)^2(-2\,d\beta - \alpha) + 2\dot{x}_0\epsilon(d - x_0) - x_0^2 \right] \]

\[ + \left( \frac{\beta e^{-\frac{2\epsilon}{\sqrt{2}}}}{\sqrt{\alpha}} \right) \left[ \epsilon(d - x_0)^2 \left( 13\Lambda - 8\epsilon^2 \right) - 6\dot{x}_0(d - x_0) + 2\dot{x}_0^2 \right] \Theta_1 \]

\[ + \left( \frac{\beta e^{-\frac{2\epsilon}{\sqrt{2}}}}{\sqrt{\alpha}} \right) \left[ \epsilon(d - x_0)^2 \left( 8\epsilon^2 - 5\Lambda \right) + \dot{x}_0(d - x_0)(3\Lambda - 8\epsilon^2) + 2\dot{x}_0^2 \right] \Theta_1 \]

\[ - \left( \frac{\beta e^{-\frac{2\epsilon}{\sqrt{2}}}}{\sqrt{\alpha}} \right) \left[ (d - x_0)^2 \left( 3\Lambda + 8\epsilon^2 \right) + 16\dot{x}_0\epsilon(x_0 - d) + 6\dot{x}_0^2 \right] \Theta_2 \]

\[ - \left( \frac{\beta e^{-\frac{2\epsilon}{\sqrt{2}}}}{2\sqrt{\alpha} \sqrt{2}} \right) \left[ 2\dot{x}_0\epsilon(d - x_0)(7\Lambda - 8\epsilon^2) + \dot{x}_0^2(-3\Lambda + 4\epsilon^2) + \right. \]

\[ \left. (d - x_0)^2(-18\epsilon^2\Lambda + 3\Lambda^2 + 16\epsilon^3) \right] \Theta_2, \]

(37)
where \( \eta = (8 \epsilon^2 - 9 (2 \delta + \alpha)) \).

The approximate Adomian approximate solution is given by

\[
x(t) = x_{\text{ADM0}}(t) + x_{\text{ADM1}}(t) + \cdots.
\]

(38)

For \((\epsilon, \alpha, \beta, F) = (0.1, 2, 1, 1)\) and \(x(0) = 0 \& x'(0) = 0.2\), a comparison between the semianalytical solution (26) (for \(F \neq 0\)) and the ADM approximate numerical solution (38) and the RK4 approximate numerical solution has been investigated as shown in Figure 2. Furthermore, the maximum distance error has been calculated as follows:

\[
L_D = \max_{0 \leq t \leq 40} |x_{\text{RK4}}(t) - x_{\text{semi-analy}}(t)| = 0.00332902,
\]

\[
L_D = \max_{0 \leq t \leq 40} |x_{\text{RK4}}(t) - x_{\text{ADM}}(t)| = 0.00132421.
\]

(39)

Also, the semianalytical solution (26) for \(F = 0\) and \(F \neq 0\) gives excellent results as compared to the ADM approximate numerical solution (38).

### 4. Quantum Plasma Oscillations

In this section, we will reduce the fluid governing equations of a quantum plasma model to an evolution equation using the RPT [45–49]. After a suitable transformation, we will be able to convert the obtained evolution equation to a Helmholtz-type equation in order to investigate the characteristics behavior of the damping oscillations in the model under consideration. Now, let us assume that we have a collisionless and unmagnetized electron-ion quantum plasma consisting of inertialless degenerate trapped electrons which obey the Fermi–Dirac distribution and classical fluid cold positive nondegenerate ion. Thus, the basic normalized fluid equations that govern the nonlinear dynamics of various structures could be presented as [51, 52]

\[
\frac{\partial n + \partial_x (\mu n)}{\partial t} = 0,
\]

\[
\frac{\partial u + u \partial_x u + \partial_x \phi}{\partial t} = \eta \partial_x^2 u,
\]

\[
\partial_x^2 \phi = (n_e - n),
\]

where \(n_e\) and \(n\) represent the normalized electron and ion number densities, respectively, \(u\) gives the normalized ion speed, \(\eta\) is the normalized kinematic viscosity of the ions, and \(\phi\) indicates the normalized electrostatic wave potential. Therefore, we shall adopt the adiabatic trapped degeneracy for electrons, by relying on notations similar to those in [52], wherein the fundamental algebra is expressed in detail. The electron normalized number density according to Fermi–Dirac distribution reads

\[
n_e = \sqrt{(1 + \phi)^3 + \frac{\tau^2}{\sqrt{(1 + \phi)}}}
\]

(41)

\[
\approx s_0 + s_1 \phi + s_2 \phi^2 + s_3 \phi^3,
\]

where \(T\) expresses the normalized temperature of the degenerate electron, \(s_0 = 1 + T^2\), \(s_1 = (3 - T^2)/2\), \(s_2 = 3(1 + T^2)/8\), and \(s_3 = -(1 + 5T^2)/6\). Note that expression (41) is obtained under the approximation \(\phi \ll 1\) for small wave amplitude.

For investigating the nonlinear structures and oscillations in the present model, the RPM will be employed for this purpose. Accordingly, the stretching and expansions for the independent and dependent variables are, respectively, introduced as follows:

\[
\xi = e^{(1/2)}(x - V_{ph} t),
\]

\[
\tau = e^{(3/2)} t,
\]

\[
n(x, t) = 1 + en^{(1)} + \epsilon^2 n^{(2)} + \epsilon^3 n^{(3)} + \cdots,
\]

\[
u(x, t) = u + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} + \cdots,
\]

\[
\phi(x, t) = \epsilon^{(1)} + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} + \cdots.
\]

(43)
where \( \varepsilon \) is a small \((\varepsilon \ll 1)\) and real parameter and \( V_{\text{ph}} \) represents the normalized phase velocity of the unmodulated structures. It is assumed that the impact of ionic kinematic viscosity is small according to many lab experiments [53, 54]. Thus, we could set \( \eta = \varepsilon^{1/2} \tilde{\eta} \), where \( \tilde{\eta} \) is \( O(1) \). By substituting stretching (42) and expansions (43) into system (40) and by collecting the terms of different powers of \( \varepsilon \), we could get a system of reduced equations. Solving the system of reduced equations for the first-two orders of \( \varepsilon \) by following the same procedures in [46–49], we finally obtain the Korteweg–de Vries Burgers (KdVB) equation [50].

\[
\partial_t \phi + A_p \partial_x \phi + B_p \partial_x^3 \phi - C_p \partial_x \phi = 0, \tag{44}
\]

with

\[
\begin{align*}
A_p &= B_p \left( \frac{3}{V_{\text{ph}}^2} - 2s_2 \right), \\
B_p &= \frac{V_{\text{ph}}^3}{2}, \\
C_p &= \frac{\tilde{\eta}}{2}, \\
V_{\text{ph}} &= \frac{1}{\sqrt{s_1}}
\end{align*}
\]

where \( A_p, B_p, \) and \( C_p \) represent the coefficients of the nonlinear, dispersion, and dissipative terms, respectively, and \( \phi \equiv \psi^{(1)} \).

Using the traveling wave transformation \( \phi(\xi + V_f t) = q(\zeta), \) where \( \zeta = \xi + V_f t \) and \( V_f \) gives the frame velocity, into the KdVB equation (44) and integrating once over \( \zeta \), we get the constant forced and damped Helmholtz equation as follows:

\[
q''(\zeta) + 2yq'(\zeta) + aq(\zeta) + \beta q(\zeta)^2 = C, \tag{46}
\]

where \( C \) is the integration constant, \( y = -C_p/(2B_p), \) \( \alpha = V_f/B_p, \) and \( \beta = A_p/(2B_p) \). Now, we can apply the above solution of the constant forced and damped Helmholtz equation that is given in equation (26) to equation (46) for investigating the characteristics of the damped oscillations in a quantum plasma.

Note that if the ionic kinematic viscosity is neglected, i.e., \( C_p = 0 \), then the KdV and undamped Helmholtz equations could be covered. The KdV equation, i.e., equation (44) for \( C_p = 0 \), is one of the most popular soliton and cnoidal equations and has been extensively investigated. For the soliton solution, the following conditions must be fulfilled: \( (q(\zeta), q'(\zeta), q''(\zeta)) \rightarrow 0 \) at \( \zeta \rightarrow \pm \infty \), so the integration constant \( C \) in equation (46) must vanish. Accordingly, equation (46) could be reduced to the undamped Helmholtz equation:

\[
q''(\zeta) + a(q(\zeta) + \beta q(\zeta)^2) = 0. \tag{47}
\]

It is well known that equation (47) supports many solutions such as periodic solution (see the solutions to the constant forced Helmholtz equation above) and solitons. The soliton solution to equation (47) in the form of the Weierstrass elliptic function \( \wp \) could be written in the following manner:

\[
\phi(x, t) = \frac{2a\beta + \sqrt{\alpha^2(\beta^2 + 2aC)}}{\alpha^2} - \frac{(9\beta/2\alpha)}{1 + 6\wp\left(x - \frac{t\sqrt{\alpha^2(\beta^2 + 2aC)/\alpha}}{(1/12)} \right)}, \tag{48}
\]

where \((x, t) \equiv (\zeta, \tau) \).

Here, the obtained approximate analytical solution (26) to the constant forced and damped Helmholtz equation (46) will be analyzed numerically according to the quantum plasma parameters \((T, \tilde{\eta}) = (0.1, 0.055)\), i.e., \((\alpha, \beta, \gamma, F) = (1.828, 2.98, -0.05, F)\), and \((T, \tilde{\eta}) = (0.9, 0.18)\), i.e., \((\alpha, \beta, \gamma, F) = (1.146, 1.7273, -0.1, F)\). The behavior of the quantum plasma oscillations according to the approximate analytical solution (26) and the approximate numerical solution according to the RK4 method is presented in Figure 3 for different values of quantum plasma parameters. It is clear from Figure 3 that our approximate analytical solution (26) is more accurate than the RK4 numerical solution. On the contrary, the RK4 numerical solution gives poor results and with increasing time this solution becomes unstable.
5. Conclusions

The Helmholtz-type equations including the constant forced undamping Helmholtz equation (equation (1)) and the constant forced damping Helmholtz equation (equation (2)) have been solved analytically and numerically. Two techniques were used to get the analytical solutions to equation (1). In the first technique, we used a new assumption to find an analytical solution to equation (1) in the form of Weierstrass elliptic function. In the second case, the solution of the standard Duffing equation has been utilized to find an analytical solution to equation (1) in the form of Jacobian elliptic function. However, the main goal of this paper is to solve equation (2), using the obtained solutions of equation (1). Moreover, both equation (1) and equation (2) have been solved numerically via the ADM. The analytical and approximate analytical solutions of equations (1) and (2) have been compared to the RK4 and ADM approximate numerical solutions. Furthermore, the maximum distance error between the RK4 approximate numerical solution and the approximate analytical solutions in addition to the approximate numerical solution using the ADM has been estimated. It was found that the obtained solutions are generally consistent with both RK4 and ADM solutions. Moreover, the obtained solutions have been applied for analyzing the oscillations that may arise in the quantum plasma. During the analysis, it was found sometimes that the approximate analytical solution is better than the RK4 numerical solution as shown in the quantum plasma model. Finally, these solutions may help us understand the oscillations that may arise in the different physical and engineering systems.

In future work, the similar approaches could be used for analyzing and solving higher-order nonlinear oscillator equations. Also, a damping Helmholtz–Duffing equation with time-dependent forced term is considered one of the most important and vital problems due to its great role in explaining many natural phenomena in different branches of science. Thus, in the next work, some new approaches will be devoted to find some solutions for these problems.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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