Identifiability in Blind Deconvolution under Minimal Assumptions

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Abstract

Blind deconvolution (BD) arises in many applications. Without assumptions on the signal and the filter, BD is ill-posed. In practice, subspace or sparsity assumptions have shown the ability to reduce the search space and yield the unique solution. However, existing theoretical analysis on uniqueness in BD is rather limited. In an earlier paper of ours [1], we provided the first algebraic sample complexities for BD with almost all bases or frames. We showed that for BD of vectors in $\mathbb{C}^n$, with two subspace constraints of dimensions $m_1$ and $m_2$, the required sample complexity is $n \geq m_1 m_2$. This result is suboptimal, since the number of degrees of freedom is merely $m_1 + m_2 - 1$. We provided analogous results, with similar suboptimality, for BD with sparsity or mixed subspace and sparsity constraints. In this paper, taking advantage of the recent progress on the information-theoretic limits of low-rank matrix recovery [2], we are finally able to bridge this gap, and derive an optimal sample complexity result for BD with generic bases or frames. We show that for BD of vectors in $\mathbb{C}^n$, with two subspace constraints of dimensions $m_1$ and $m_2$, the required sample complexity is $n > m_1 + m_2$. We present the extensions of this result to BD with sparsity constraints or mixed constraints, with the sparsity level replacing the subspace dimension.

Index terms— uniqueness, sample complexity, bilinear inverse problem, low-rank matrix recovery

1 Introduction

Blind deconvolution (BD) is the bilinear inverse problem of recovering the signal and the filter simultaneously given the their convolution or circular convolution. It arises in many applications,
including blind image deblurring \cite{3}, blind channel equalization \cite{4}, speech dereverberation \cite{5}, and seismic data analysis \cite{6}. Without further assumptions, BD is an ill-posed problem, and does not yield a unique solution. In this paper, we focus on subspace or sparsity assumptions on the signal and the filter. These priors serve as constraints or regularizers \cite{7-11}. With a reduced search space, BD can be better-posed. However, despite the success in practice, the theoretical results on the uniqueness in BD with a subspace or sparsity constraint are limited.

Recently, recasting bilinear or quadratic inverse problems, such as blind deconvolution \cite{11} and phase retrieval \cite{12}, as rank-1 matrix recovery problems by “lifting” has attracted a lot of attention. Choudhary and Mitra \cite{13} showed that the identifiability in BD (or any bilinear inverse problem) hinges on the set of rank-2 matrices in a certain nullspace. In particular, they showed a negative result that the solution to blind deconvolution with a canonical sparsity prior, that is, sparsity over the natural basis, is not identifiable \cite{14}. However, the authors did not analyze the identifiability of signals that are sparse over other dictionaries.

Using the lifting framework, Ahmed et al. \cite{11}, Ling and Strohmer \cite{15}, and Lee et al. \cite{16} proposed algorithms to solve BD with with subspace constraints, mixed constraints, and sparsity constraints, respectively. Chi \cite{17} solved BD with mixed constraints, where the sparse spikes do not necessarily lie on a grid. They all showed successful recovery with high probability using convex programming or alternating minimization. A common drawback of these works is that the probabilistic assumptions on the bases or frames are very limiting in practice. On the positive side, these identifiability results are constructive, being demonstrated by establishing performance guarantees of algorithms. However, these guarantees too are shown only in some probabilistic sense.

In an earlier paper \cite{1}, we addressed the identifiability in single channel blind deconvolution up to scaling under subspace or sparsity constraints. We presented the first algebraic sample complexities for BD with fully deterministic signal models. In particular, we showed that for BD of vectors in $\mathbb{C}^n$, with two generic subspace constraints of dimensions $m_1$ and $m_2$, the bilinear mapping is injective if $n \geq m_1 m_2$. However, the number of degrees of freedom is $m_1 + m_2 - 1$, hence the above result is suboptimal. Similarly, the sample complexities for BD with sparsity and mixed constraints are $n \geq 2s_1 s_2$ and $n \geq 2s_1 m_2$, respectively, where $s_1$ and $s_2$ denote the sparsity levels of the signal and the filter. The cost for the unknown support is an extra factor of 2. These results suffer from the same suboptimality as the results for the subspace constraints, in comparison to the number of degrees of freedom of the continuous-valued unknowns.

In this paper, we are finally ready to bridge this gap. Inspired by the brilliant work of Riegler
et al. [2] on information-theoretic limits of low-rank matrix recovery, we show nearly optimal sufficient conditions for blind deconvolution with generic bases or frames that match the number of degrees of freedom.

2 Problem Statement

2.1 Notations

We state the notations that will be used throughout the paper. We use lower-case letters $x$, $y$, $z$ to denote vectors, and upper-case letters $D$ and $E$ to denote matrices. We use $F$ to denote the normalized discrete Fourier transform (DFT) matrix. Unless otherwise stated, all vectors are column vectors. The dimensions of all vectors and matrices are made clear in the context. We use superscript letters to denote subvectors or submatrices. For example, the scalar $x^{(j)}$ represents the $j$th entry of $x$. The vector $D^{(j,:)}$ represents the $j$th row of the matrix $D$. The colon notation is borrowed from MATLAB.

The transpose and conjugate transpose to a matrix $A$ are denoted by $A^T$ and $A^*$, respectively. The inner product of two matrices $A$ and $M$ are denoted by $\langle A, M \rangle = \text{trace}(A^* M)$. We use $\| \cdot \|_0$ to denote the $\ell_0$ “norm”, or number of nonzero entries. We use $\| \cdot \|_2$ and $\| \cdot \|_F$ to denote the $\ell_2$ norm of a vector and the Frobenious norm of a matrix, respectively. The open ball in $\mathbb{R}^{m_1}$ of radius $\rho$ centered at $x$ is denoted by $B_{\mathbb{R}^{m_1}}(x, \rho)$. Similarly, the open ball in $\mathbb{C}^{m_1 \times m_2}$ of radius $\rho$ centered at $M$ is denoted by $B_{\mathbb{C}^{m_1 \times m_2}}(M, \rho)$. We use $\odot$ to denote entrywise product. Circular convolution is denoted by $\ast$. The nullspace of a linear operator are denoted by $\mathcal{N}(\cdot)$.

We say a property holds for almost all vectors/matrices (generic vectors/matrices) if the property holds for all vectors/matrices but a set of Lebesgue measure zero.

2.2 Blind Deconvolution

In this paper, we study the blind deconvolution problem with the circular convolution model. It is the joint recovery of two vectors $u_0 \in \mathbb{C}^n$ and $v_0 \in \mathbb{C}^n$, namely the signal and the filter\footnote{Due to symmetry, the name “signal” and “filter” can be used interchangeably.} given their circular convolution $z = u_0 \ast v_0$, subject to subspace or sparsity constraints. The
constraint sets $\Omega_\mathcal{U}$ and $\Omega_\mathcal{V}$ are subsets of $\mathbb{C}^n$.

$$\begin{align*}
\text{find } & (u, v), \\
\text{s.t. } & u \odot v = z, \\
& u \in \Omega_\mathcal{U}, \ v \in \Omega_\mathcal{V}.
\end{align*}$$

We consider the following scenarios:

1. **(Subspace Constraints)** The signal $u$ and the filter $v$ reside in lower-dimensional subspaces spanned by the columns of $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$, respectively. The matrices $D$ and $E$ have full column ranks. The signal $u = Dx$ for some $x \in \mathbb{C}^{m_1}$. The filter $v = Ey$ for some $y \in \mathbb{C}^{m_2}$.

2. **(Sparsity Constraints)** The signal $u$ and the filter $v$ are sparse over given dictionaries formed by the columns of $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$, with sparsity level $s_1$ and $s_2$, respectively. The matrices $D$ and $E$ are bases or frames that satisfy the spark condition [18]: the spark, namely the smallest number of columns that are linearly dependent, of $D$ (resp. $E$) is greater than $2s_1$ (resp. $2s_2$). The signal $u = Dx$ for some $x \in \mathbb{C}^{m_1}$ such that $\|x\|_0 \leq s_1$. The filter $v = Ey$ for some $y \in \mathbb{C}^{m_2}$ such that $\|y\|_0 \leq s_2$.

3. **(Mixed Constraints)** The signal $u$ is sparse over a given dictionary $D \in \mathbb{C}^{n \times m_1}$, and the filter $v$ resides in a lower-dimensional subspace spanned by the columns of $E \in \mathbb{C}^{n \times m_2}$. The matrix $D$ satisfies the spark condition, and $E$ has full column rank. The signal $u = Dx$ for some $x \in \mathbb{C}^{m_1}$ such that $\|x\|_0 \leq s_1$. The filter $v = Ey$ for some $y \in \mathbb{C}^{m_2}$.

In all three scenarios, the vectors $x$, $y$, and $z$ reside in Euclidean spaces $\mathbb{C}^{m_1}$, $\mathbb{C}^{m_2}$ and $\mathbb{C}^n$. With the representations $u = Dx$ and $v = Ey$, it is easy to verify that $z = u \odot v = (Dx) \odot (Ey)$ is a bilinear function of $x$ and $y$. Given the measurement $z = (Dx_0) \odot (Ey_0)$, the blind deconvolution problem can be rewritten in the following form:

$$\text{(BD)} \quad \text{find } (x, y), \\
\text{s.t. } (Dx) \circ (Ey) = z, \\
x \in \Omega_\mathcal{X}, \ y \in \Omega_\mathcal{Y}.$$ 

If $D$ and $E$ satisfy the full column rank condition or spark condition, then the uniqueness of $(u, v)$ is equivalent to the uniqueness of $(x, y)$. For simplicity, we will discuss problem (BD) from now on. The constraint sets $\Omega_\mathcal{X}$ and $\Omega_\mathcal{Y}$ depend on the constraints on the signal and
the filter. Unlike our prequel paper \cite{1}, to take advantage of the analysis by Riegler et al. \cite{2}, we assume that $\Omega_X$ and $\Omega_Y$ are bounded. For subspace constraints, $\Omega_X$ and $\Omega_Y$ are bounded subsets of $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$, respectively. For sparsity constraints, $\Omega_X$ and $\Omega_Y$ are bounded subsets of $\{x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1\}$ and $\{y \in \mathbb{C}^{m_2} : \|y\|_0 \leq s_2\}$, respectively.

### 2.3 Identifiability up to Scaling

An important question concerning the blind deconvolution problem is to determine when it admits a unique solution. The BD problem suffers from scaling ambiguity. For any nonzero scalar $\sigma \in \mathbb{C}$ such that $\sigma x_0 \in \Omega_X$ and $\frac{1}{\sigma} y_0 \in \Omega_Y$, $(D(\sigma x_0)) \otimes (E(\frac{1}{\sigma} y_0)) = (Dx_0) \otimes (Ey_0) = z$. Therefore, BD does not yield a unique solution if $\Omega_X, \Omega_Y$ contain such scaled versions of $x_0, y_0$.

Any valid definition of unique recovery in BD must address this issue. If every solution $(x, y)$ is a scaled version of $(x_0, y_0)$, then we must say $(x_0, y_0)$ can be uniquely identified up to scaling.

We define identifiability as follows.

**Definition 2.1.** For the constrained BD problem, the solution $(x_0, y_0)$, in which $x_0 \neq 0$ and $y_0 \neq 0$, is said to be identifiable up to scaling, if every solution $(x, y) \in \Omega_X \times \Omega_Y$ satisfies $x = \sigma x_0$ and $y = \frac{1}{\sigma} y_0$.

For blind deconvolution, there exists a linear operator $G_{DE} : \mathbb{C}^{m_1 \times m_2} \rightarrow \mathbb{C}^n$ such that $G_{DE}(xy^T) = (Dx) \otimes (Ey)$. Given the measurement $z = G_{DE}(x_0y_0^T) = (Dx_0) \otimes (Ey_0)$, one can recast the BD problem as the recovery of the rank-1 matrix $M_0 = x_0y_0^T \in \Omega_M = \{xy^T : x \in \Omega_X, y \in \Omega_Y\}$. This procedure is called “lifting”.

\[
\text{(Lifted BD)} \quad \text{find } M, \\
\text{s.t. } G_{DE}(M) = z, \\
M \in \Omega_M.
\]

The uniqueness of $M_0$ is equivalent to the identifiability of $(x_0, y_0)$ up to scaling.

In the next section, we present the main results on the identifiability in blind deconvolution up to scaling, i.e., the optimal sample complexity that guarantees the uniqueness of the solution to (Lifted BD). There are two caveats. First, unlike our results on BD with generic bases or frames in \cite{1}, this paper is not about the injectivity of the bilinear mapping of circular convolution, which only depends on $D$ and $E$, and applies uniformly to all pairs $(x_0, y_0)$. Instead, if the generic matrices $D$ and $E$ are fixed, then under the conditions presented in this paper, almost all pairs $(x_0, y_0)$ are identifiable up to scaling. Secondly, unlike \cite{1}, besides the subspace or
sparsity constraints, we also need the constraint sets $\Omega_X$ and $\Omega_Y$ to be bounded.

3 Blind Deconvolution with Minimal Assumptions

3.1 Main Results

Subspace membership and sparsity have been used as priors in blind deconvolution for a long time. Previous works either use these priors without theoretical justification [7–10,19], or impose probabilistic models and show successful recovery with high probability [11,15–17]. The sufficient conditions for the identifiability in BD in our prequel paper [1] are fully deterministic, but (except for a special class of so-called sub-band structured signals or filters) suboptimal. In this section, we present sufficient conditions for identifiability in BD with minimal assumptions, which, to within one sample, are sharp.

Theorem 3.1 (Subspace Constraints). In (BD) with subspace constraints where $\Omega_X$ and $\Omega_Y$ are bounded subsets of $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$, respectively, $(x_0, y_0) \in \Omega_X \times \Omega_Y (x_0 \neq 0, y_0 \neq 0)$ is identifiable up to scaling, for almost all $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$, if $n > m_1 + m_2$.

Theorem 3.2 (Mixed Constraints). In (BD) with mixed constraints where $\Omega_X$ is a bounded subset of $\{x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1\}$ and $\Omega_Y$ is a bounded subset of $\mathbb{C}^{m_2}$, $(x_0, y_0) \in \Omega_X \times \Omega_Y (x_0 \neq 0, y_0 \neq 0)$ is identifiable up to scaling, for almost all $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$, if $n > s_1 + m_2$.

Theorem 3.3 (Sparsity Constraints). In (BD) with sparsity constraints where $\Omega_X$ and $\Omega_Y$ are bounded subsets of $\{x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1\}$ and $\{y \in \mathbb{C}^{m_2} : \|y\|_0 \leq s_2\}$, respectively, $(x_0, y_0) \in \Omega_X \times \Omega_Y (x_0 \neq 0, y_0 \neq 0)$ is identifiable up to scaling, for almost all $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$, if $n > s_1 + s_2$.

These results will be proved in Section 3.3. In fact, for almost all $D$, $E$, and almost all pairs $(x_0, y_0)$, the pair $(x_0, y_0)$ is identifiable up to scaling. By symmetry, we can derive another sufficient condition for the scenario where $u = Dx$ resides in a $m_1$-dimensional subspace spanned by the columns of $D$, and $v = Ey$ is $s_2$-sparse over $E$. The above sufficient conditions are appealing since they approach the information-theoretic limits of blind deconvolution. In BD with subspace, mixed, and sparsity constraints, the number of degrees of freedom is $m_1 + m_2 - 1$, $s_1 + m_2 - 1$, and $s_1 + s_2 - 1$, respectively. Therefore, to within one sample difference, the sample complexities presented above are optimal.
3.2 Information-Theoretic Limits of Low-Rank Matrix Recovery

Recently, Riegler et al. [2] derived sample complexity results of low-rank matrix recovery, and the recovery of matrices of low description complexity, that match the number of degrees of freedom. They considered the case where the matrices are real. Define the measurement operator \( A : \mathbb{R}^{m_1 \times m_2} \rightarrow \mathbb{R}^n \) as

\[
z = A(M_0) = [\langle A_1, M_0 \rangle, \langle A_2, M_0 \rangle, \cdots, \langle A_n, M_0 \rangle]^T \in \mathbb{R}^n,
\]

where the measurement matrices \( A_j \in \mathbb{R}^{m_1 \times m_2} \) \((j = 1, 2, \cdots, n)\). The matrix recovery problem is:

\[
(MR) \quad \text{find } M, \\
\text{s.t. } A(M) = z, \\
M \in \Omega_M.
\]

The Minkowski dimension of the nonempty bounded constraint set \( \Omega_M \subset \mathbb{R}^{m_1 \times m_2} \) is defined as follows.

**Definition 3.4.** The lower and upper Minkowski dimensions of nonempty bounded set \( \Omega_M \subset \mathbb{R}^{m_1 \times m_2} \) are

\[
\dim_B(\Omega_M) := \liminf_{\rho \to 0} \frac{\log N_{\Omega_M}(\rho)}{\log \frac{1}{\rho}}, \quad \overline{\dim_B}(\Omega_M) := \limsup_{\rho \to 0} \frac{\log N_{\Omega_M}(\rho)}{\log \frac{1}{\rho}},
\]

where \( N_{\Omega_M}(\rho) \) denotes the covering number of \( \Omega_M \) given by

\[
N_{\Omega_M}(\rho) = \min \left\{ k \in \mathbb{N} : \Omega_M \subset \bigcup_{i=1,2,\cdots,k} B_{\mathbb{R}^{m_1 \times m_2}}(M_i, \rho), \ M_i \in \mathbb{R}^{m_1 \times m_2} \right\}.
\]

The Minkowski dimension of the constraint set \( \Omega_M \) can be used to represent its description complexity. Riegler et al. showed that the solution to (MR) is unique if the sample complexity is greater than the description complexity. For almost all measurement matrices \( A_1, A_2, \cdots, A_n \in \mathbb{R}^{m_1 \times m_2} \), the recovery of \( M_0 \in \Omega_M \) is unique if \( n > \operatorname{dim_B}(\Omega_M) \) (See [2] Theorem 1). An even more amazing result is that the same sample complexity can be achieved by rank-1 measurement matrices. For almost all \( a_j \in \mathbb{R}^{m_1} \) and \( b_j \in \mathbb{R}^{m_2} \) \((j = 1, 2, \cdots, n)\), the recovery of \( M_0 \in \Omega_M \) from measurements \( \langle a_j b_j^T, M_0 \rangle = a_j^T M_0 b_j \) \((j = 1, 2, \cdots, n)\) is unique if \( n > \overline{\dim_B}(\Omega_M) \) (See [2] Theorem 2). We state the extension of this result to the case where the constraint set and the
measurement matrices are complex.

**Lemma 3.5.** In the recovery of a complex matrix \( M_0 \in \Omega_M \subset \mathbb{C}^{m_1 \times m_2} \), for almost all \( a_j \in \mathbb{C}^{m_1} \) and \( b_j \in \mathbb{C}^{m_2} \) \((j = 1, 2, \ldots, n)\), the recovery of \( M_0 \) from measurements \( \langle a_j b_j^T, M_0 \rangle = a_j^* M_0 b_j \) \((j = 1, 2, \ldots, n)\) is unique if \( 2n > \dim_B(\Omega_M) \).

The Minkowski dimension of the constraint set of complex matrices \( \Omega_M \subset \mathbb{C}^{m_1 \times m_2} \) can be defined as in Definition 3.4, with the real number field \( \mathbb{R} \) replaced by the complex number field \( \mathbb{C} \). As will be shown in the next section, by simply changing the number field to complex, the Minkowski dimension of a set doubles. Meanwhile, by taking \( n \) complex measurements, the sample complexity (in terms of real-valued measurements) also doubles, i.e., the sample complexity becomes \( 2n \). Except for the extra factor of 2, the proof in [2] remains intact.

### 3.3 Proof of the Main Results

The identifiability of \((x_0, y_0)\) up to scaling in (BD) is equivalent to the uniqueness of \( M_0 = x_0 y_0^T \) in (Lifted BD). Note that

\[
z = G_{DE}(M_0) = (Dx_0) \oplus (Ey_0) = \sqrt{n}F^*[(FDx_0) \oplus (FEy_0)],
\]

\[
\frac{1}{\sqrt{n}}(Fz)^{(j)} = (FD)^{(j,:)}x_0(FE)^{(j,:)}y_0 = (FD)^{(j,:)}x_0y_0^T(FE)^{(j,:)^T} = a_j^* M_0 b_j,
\]

where \( a_j = (FD)^{(j,:)*} \) is the conjugate transpose of the \( j \)th row of \( FD \), and \( b_j = (FE)^{(j,:)*} \) is the conjugate transpose of the \( j \)th row of \( FE \). Rewriting (Lifted BD) in the frequency domain:

\[
\text{(Lifted BD) } \quad \text{find } M, \\
\text{s.t. } \quad a_j^* M b_j = \frac{1}{\sqrt{n}}(Fz)^{(j)}, \quad j = 1, 2, \ldots, n \\
\text{ } \quad M \in \Omega_M = \{xy^T : x \in \Omega_X, y \in \Omega_Y\}.
\]

By Lemma 3.5, the recovery of \( M_0 \) is unique for almost all \( D \in \mathbb{C}^{n \times m_1} \) and \( E \in \mathbb{C}^{n \times m_2} \) if \( 2n > \dim_B(\Omega_M) \). Hence, Theorems 3.1, 3.2, and 3.3 follow from the upper bounds on Minkowski dimensions in Lemma 3.6.

**Lemma 3.6.** The lower Minkowski dimensions of the bounded constraint sets in (Lifted BD) with subspace, mixed, and sparsity constraints are bounded above by \( 2(m_1 + m_2) \), \( 2(s_1 + m_2) \), and \( 2(s_1 + s_2) \), respectively.

**Proof of Lemma 3.6** Recall that, when the signal \( x \in \Omega_X \) satisfies a subspace constraint, \( \Omega_X \) is
a bounded subset of $\mathbb{C}^{m_1}$. Consider the real and imaginary part of the set $\Omega_X$, which we denote by $\text{Re}(\Omega_X)$ and $\text{Im}(\Omega_X)$. The set $\text{Re}(\Omega_X)$ is a bounded subset of $\mathbb{R}^{m_1}$. Hence there exists an large enough constant $L$ such that

$$\text{Re}(\Omega_X) \subset B_{\mathbb{R}^{m_1}}(0, L),$$

i.e., $\text{Re}(\Omega_X)$ is contained in a ball of radius $L$. By a standard volume argument,

$$N_{\text{Re}(\Omega_X)}(\rho) \leq \left( \frac{3L\sqrt{m_1}}{\rho} \right)^{m_1},$$

$$\limsup_{\rho \to 0} \frac{\log N_{\text{Re}(\Omega_X)}(\rho)}{\log \frac{1}{\rho}} \leq \limsup_{\rho \to 0} m_1 \frac{\log \frac{1}{\rho} + \log 3L\sqrt{m_1}}{\log \frac{1}{\rho}} = m_1.$$

The same derivation can be repeated for $\text{Im}(\Omega_X)$. It follows that

(subspace constraint) $\overline{\dim}_B(\text{Re}(\Omega_X)) \leq m_1, \quad \overline{\dim}_B(\text{Im}(\Omega_X)) \leq m_1.$ \hfill (1)

When the signal $x \in \Omega_X$ satisfies sparsity constraint, $\Omega_X$ is a bounded subset of $\{x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1\}$. The real part $\text{Re}(\Omega_X)$ is a bounded subset of $\{x \in \mathbb{R}^{m_1} : \|x\|_0 \leq s_1\}$, the union of $\binom{m_1}{s_1}$ $s_1$-dimensional subspaces. Suppose $\text{Re}(\Omega_X)$ is contained in a ball of radius $L$. By a standard volume argument,

$$N_{\text{Re}(\Omega_X)}(\rho) \leq \binom{m_1}{s_1} \left( \frac{3L\sqrt{s_1}}{\rho} \right)^{s_1} \leq \left( \frac{e m_1}{s_1} \right)^{s_1} \left( \frac{3L\sqrt{s_1}}{\rho} \right)^{s_1},$$

where the second inequality follows from Stirling’s approximation. Hence,

$$\limsup_{\rho \to 0} \frac{\log N_{\text{Re}(\Omega_X)}(\rho)}{\log \frac{1}{\rho}} \leq \limsup_{\rho \to 0} s_1 \frac{\log \frac{1}{\rho} + \log 3L\sqrt{s_1}}{\log \frac{1}{\rho}} = s_1.$$

The same derivation can be repeated for $\text{Im}(\Omega_X)$. It follows that

(sparsity constraint) $\overline{\dim}_B(\text{Re}(\Omega_X)) \leq s_1, \quad \overline{\dim}_B(\text{Im}(\Omega_X)) \leq s_1.$ \hfill (2)

Similarly, for the filter $y \in \Omega_Y$, the upper Minkowski dimensions of the real and imaginary parts
of the subspace and sparsity constraint sets satisfy:

(subspace constraint) \[ \dim_B(\Re(\Omega_Y)) \leq m_2, \quad \dim_B(\Im(\Omega_Y)) \leq m_2. \] (3)

(sparsity constraint) \[ \dim_B(\Re(\Omega_Y)) \leq s_2, \quad \dim_B(\Im(\Omega_Y)) \leq s_2. \] (4)

Next, we apply the above results and derive a bound for the Minkowski dimension of \( \Omega_M = \{ xy^T : x \in \Omega_X, y \in \Omega_Y \} \), which is the constraint set of (Lifted BD). Since the sets \( \Re(\Omega_X), \Im(\Omega_X), \Re(\Omega_Y), \) and \( \Im(\Omega_Y) \) are all bounded, there exists an large enough constant \( L \) such that

\[ \Re(\Omega_X), \Im(\Omega_X) \subset \mathcal{B}_{\mathbb{R}^{m_1}}(0, L), \quad \Re(\Omega_Y), \Im(\Omega_Y) \subset \mathcal{B}_{\mathbb{R}^{m_2}}(0, L). \]

We first cover \( \Re(\Omega_X), \Im(\Omega_X), \Re(\Omega_Y), \) and \( \Im(\Omega_Y) \) with open balls of radius \( \rho \) centered at the following four sets of points, respectively:

\[ \{ x_i^{\Re} \}_{i=1}^{N_{\Re}(\Omega_X)(\rho)}, \quad \{ x_i^{\Im} \}_{i=1}^{N_{\Im}(\Omega_X)(\rho)} \subset \mathcal{B}_{\mathbb{R}^{m_1}}(0, L), \]

\[ \{ y_i^{\Re} \}_{i=1}^{N_{\Re}(\Omega_Y)(\rho)}, \quad \{ y_i^{\Im} \}_{i=1}^{N_{\Im}(\Omega_Y)(\rho)} \subset \mathcal{B}_{\mathbb{R}^{m_2}}(0, L). \]

Given any point \( M = xy^T \in \Omega_M \), we can find centers of the above coverings, \( x_i^{\Re}, x_i^{\Im}, y_i^{\Re}, \) and \( y_i^{\Im} \), such that

\[ \| \Re(x) - x_i^{\Re} \|_2 \leq \rho, \quad \| \Im(x) - x_i^{\Im} \|_2 \leq \rho, \quad \| \Re(y) - y_i^{\Re} \|_2 \leq \rho, \quad \| \Im(y) - y_i^{\Im} \|_2 \leq \rho. \]

Let \( M_c = x_c y_c^T \), where \( x_c = x_{i_1}^{\Re} + \sqrt{-1} x_{i_2}^{\Im} \) and \( y_c = y_{i_3}^{\Re} + \sqrt{-1} y_{i_4}^{\Im} \). Then

\[ \| M - M_c \|_F = \| xy^T - x_c y_c^T + x_c y_c^T - x_c y_c^T \|_F \]

\[ \leq \| x - x_c \|_2 \| y \|_2 + \| y - y_c \|_2 \| x_c \|_2 \]

\[ = \sqrt{\| \Re(x) - x_{i_1}^{\Re} \|_2^2 + \| \Im(x) - x_{i_2}^{\Im} \|_2^2} \sqrt{\| \Re(y) \|_2^2 + \| \Im(y) \|_2^2} \]

\[ + \sqrt{\| \Re(y) - y_{i_3}^{\Re} \|_2^2 + \| \Im(y) - y_{i_4}^{\Im} \|_2^2} \sqrt{\| x_{i_1}^{\Re} \|_2^2 + \| x_{i_2}^{\Im} \|_2^2} \]

\[ \leq 4L \rho. \]

Therefore, the set \( \Omega_M \) can be covered by \( N_{\Re}(\Omega_X)(\rho) N_{\Im}(\Omega_X)(\rho) N_{\Re}(\Omega_Y)(\rho) N_{\Im}(\Omega_Y)(\rho) \) open balls in \( \mathbb{C}^{m_1 \times m_2} \), centered at the rank-1 matrices (like \( M_c \)) generated by the centers of the
coverings of $\text{Re}(\Omega_X)$, $\text{Im}(\Omega_X)$, $\text{Re}(\Omega_Y)$, and $\text{Im}(\Omega_Y)$. It follows that

$$N_{\Omega_M}(4L\rho) \leq N_{\text{Re}(\Omega_X)}(\rho)N_{\text{Im}(\Omega_X)}(\rho)N_{\text{Re}(\Omega_Y)}(\rho)N_{\text{Im}(\Omega_Y)}(\rho).$$

Therefore,

$$\dim_B(\Omega_M) \leq \limsup_{\rho \to 0} \log N_{\Omega_M}(4L\rho) \leq \limsup_{\rho \to 0} \log N_{\text{Re}(\Omega_X)}(\rho)N_{\text{Im}(\Omega_X)}(\rho)N_{\text{Re}(\Omega_Y)}(\rho)N_{\text{Im}(\Omega_Y)}(\rho)$$

$$= \limsup_{\rho \to 0} \frac{\log N_{\text{Re}(\Omega_X)}(\rho)}{\log \frac{1}{4L\rho}} + \limsup_{\rho \to 0} \frac{\log N_{\text{Im}(\Omega_X)}(\rho)}{\log \frac{1}{4L\rho}} + \limsup_{\rho \to 0} \frac{\log N_{\text{Re}(\Omega_Y)}(\rho)}{\log \frac{1}{4L\rho}} + \limsup_{\rho \to 0} \frac{\log N_{\text{Im}(\Omega_Y)}(\rho)}{\log \frac{1}{4L\rho}}$$

$$= \dim_B(\text{Re}(\Omega_X)) + \dim_B(\text{Im}(\Omega_X)) + \dim_B(\text{Re}(\Omega_Y)) + \dim_B(\text{Im}(\Omega_Y)).$$

The lemma follows from the above relation and the inequalities in (1), (2), (3), and (4). □

4 Conclusions

We studied the identifiability of blind deconvolution problems with subspace or sparsity constraints. The sample complexity results in Section 3.1 are, to within one sample, optimal. Our results are derived with generic bases or frames, which means they are violated on a set of Lebesgue measure zero.

There are two caveats for these results. First, this paper is not about the injectivity of the bilinear mapping of circular convolution. If the generic matrices $D$ and $E$ are fixed, then under the conditions presented in this paper, there could exist an unidentifiable pair $(x_0, y_0)$. Secondly, besides the subspace or sparsity constraints, we also need the constraint sets $\Omega_X$ and $\Omega_Y$ to be bounded. These caveats could be artifacts of our approach. Whether one can derive sufficient conditions without these limitations is an interesting open question.

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