The profinite topology of free groups and weakly generic tuples of automorphisms

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Let $\mathcal{A}$ be a countable first order structure and endow the universe of $\mathcal{A}$ with the discrete topology. Then the automorphism group $\text{Aut}(\mathcal{A})$ of $\mathcal{A}$ becomes a topological group. A tuple of automorphisms $(g_0, \ldots, g_{n-1})$ is defined to be weakly generic iff its diagonal conjugacy class (in the algebraic sense) is dense (in the topological sense) and the $\langle g_0, \ldots, g_{n-1} \rangle$-orbit of each $a \in \mathcal{A}$ is finite. Existence of tuples of weakly generic automorphisms are interesting from the point of view of model theory as well as from the point of view of finite combinatorics. The main results of the present work are as follows. In Theorem 2.6 we characterize the existence of tuples of weakly generic automorphisms with the aid of the profinite topology of free groups. In Corollary 2.12 we will show that if $\text{Aut}(\mathcal{A})$ has finite topological rank $r$ (and satisfies a further, mild technical condition) then the existence of a weakly generic tuple in $\text{Aut}(\mathcal{A})^r$ implies the existence of weakly generic tuples in $\text{Aut}(\mathcal{A})^n$ for all natural number $n \geq 1$. Finally, in Theorem 3.2 we show that if $\mathcal{A}$ is a countable model of an $\aleph_0$-categorical, simple theory in which all types over the empty set are stationary and $\mathcal{A}$ has a pair of weakly generic automorphisms then it has tuples of weakly generic automorphisms of arbitrary finite length. At the technical level we will combine elementary investigations about the profinite topology of free groups with the results of [11] about topological ranks of the automorphism groups of some structures.

1 Introduction

Let $\mathcal{A}$ be a countable first order structure and endow the universe of $\mathcal{A}$ with the discrete topology. Then the automorphism group $\text{Aut}(\mathcal{A})$ of $\mathcal{A}$ becomes a topological group (with the subspace topology inherited from the suitable topological power of the discrete topology on $\mathcal{A}$). We start by recalling the following definition.

**Definition 1.1** A finite tuple $(g_0, \ldots, g_{n-1}) \in \text{Aut}(\mathcal{A})^n$ is defined to be weakly generic iff it satisfies the following conditions:

1. The diagonal conjugacy class of $(g_0, \ldots, g_{n-1})$ (in the group theoretic sense) is dense in the topological sense, i.e., $\{(f^{-1}g_0f, \ldots, f^{-1}g_{n-1}f) : f \in \text{Aut}(\mathcal{A})\}$ is dense in $\text{Aut}(\mathcal{A})^n$.
2. If $G$ is the subgroup of $\text{Aut}(\mathcal{A})$ generated by $(g_0, \ldots, g_{n-1})$, then the $G$-orbit of each $a \in \mathcal{A}$ is finite.

The problem of existence of tuples of automorphisms satisfying (1) of Definition 1.1 was originally motivated by the model theoretic Small Index Property; in this respect we refer to [3, 5, 12] and the references therein. Because of stipulation (2) of Definition 1.1, existence of weakly generic automorphisms is closely related to interesting results in finite combinatorics. More concretely, a partial function $f$ of a structure $B$ is defined to be a partial isomorphism of $B$ iff $f$ is an isomorphism between the substructures of $B$ generated by the domain and range of it. A class $K$ of finite structures (of the same relational language) is defined to have Hrushovski’s Extension Property ($\text{EP}$ for short) iff for all $B \in K$ and all finite sequences $(f_0, \ldots, f_{n-1})$ of partial isomorphisms of $B$ there exists $C \in K$ such that $B$ can be embedded into $C$ and each $f_i$ can be extended to an automorphism of $C$. It is well known (cf., e.g., [14, Lemma 2.4]) that for a Fraïssé class $K$, the extension property of $K$ is closely

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related to the existence of weakly generic automorphisms of the Fraïssé limit of $K$. For further details we refer, e.g., to [1, 2, 4, 6, 7, 14, 17–19]; for a survey on weakly generic tuples of automorphisms we also refer to [15].

Now we turn to sum up the main results of the present paper. At the technical level, often it is much more easy to establish the existence of a single weakly generic automorphism than the existence of tuples of weakly generic automorphisms of arbitrary finite length. In fact, one has a feeling that if there exists a pair $(g_0, g_1)$ of weakly generic automorphisms, then there exist tuples of weakly generic automorphisms of arbitrary finite length. The aim of the present paper is to make this impression more precise. We are trying to be as general as we can. We note that (for a finite, relational language) in the following list each item implies the next:

1. $\mathcal{A}$ is a Fraïssé limit;
2. $\mathcal{A}$ is $\aleph_0$-categorical;
3. $\mathcal{A}$ is saturated;
4. each finite elementary mapping of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$.

Unfortunately, there is an ambiguity in the terminology of related literature. Structures having the last property of the above list are sometimes called “homogeneous”, but sometimes “homogeneous” means the stronger requirement that all partial isomorphisms can be extended to automorphisms. Throughout this paper we are assuming only that each finite elementary mapping of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$. In order to avoid ambiguity, we will always explicitly indicate this.

The main results of the present paper are as follows. In Theorem 2.6 we characterize the existence of tuples of weakly generic automorphisms with the aid of the profinite topology of free groups. In Corollary 2.12 we will show that if $Aut(\mathcal{A})$ has finite topological rank $r$ (and satisfies a further, mild technical condition) then the existence of a weakly generic tuple in $\text{Aut}(\mathcal{A})^r$ implies the existence of weakly generic tuples in $\text{Aut}(\mathcal{A})^n$ for all natural number $n \geq 1$. In order to illustrate applicability of Corollary 2.12, we combine it with the results of [11] and obtain Theorem 3.2: If $\mathcal{A}$ is a countable model of an $\aleph_0$-categorical, simple theory in which all types over the empty set are stationary and $\mathcal{A}$ has a pair of weakly generic automorphisms, then it has tuples of weakly generic automorphisms of arbitrary finite length.

The structure of the paper is as follows. At the end of this section we sum up our system of notation. § 2.1 deals with the profinite topology of free groups. § 2.2 contains some results about automorphism groups of finite topological rank. Finally, § 3 contains investigations about models of $\aleph_0$-categorical simple theories.

1.1 Notation

Our notation is mostly standard, but the following list may be helpful.

Throughout $\omega$ denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, \ldots, n - 1\}$. Let $A$ and $B$ be sets. Then $A^B$ denotes the set of functions whose domain is $A$ and whose range is a subset of $B$. For a topological space $X$ and $n \in \omega$, the $n$th power of $X$ (in the topological sense) will be denoted by $X^n$. In addition, $|A|$ denotes the cardinality of $A$ and $\mathcal{P}(A)$ denotes the power set of $A$, i.e., $\mathcal{P}(A)$ consists of all subsets of $A$. If $\kappa$ is a cardinal then $[A]^\kappa = \{x \in \mathcal{P}(A) : |x| = \kappa\}$ and $[A]^<\kappa = \{x \in \mathcal{P}(A) : |x| < \kappa\}$.

Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions $f, g$ we define $f \circ g(x) = f(g(x))$.

If $G$ is a group and $f_0, \ldots, f_{n-1} \in G$, then $(f_0, \ldots, f_{n-1})$ denotes the subgroup of $G$ generated by $\{f_0, \ldots, f_{n-1}\}$. If $G$ is a group acting on a set $X$ and $a \in X$, then $O_G(a)$ denotes the orbit of $a$ with respect to the action of $G$. Further, $\mathcal{F}(x_0, \ldots, x_{n-1})$ denotes the free group with free generators $\{x_0, \ldots, x_{n-1}\}$. For a group $G$ and a subgroup $H$ of $G$, the index of $H$ in $G$ will be denoted by $(G : H)$. If $A$ is any set, then $\text{Sym}(A)$ denotes the symmetric group of $A$.

If $\mathcal{A}$ and $\mathcal{B}$ are structures, then $\mathcal{A} \leq \mathcal{B}$ denotes the fact that $\mathcal{A}$ is a substructure of $\mathcal{B}$. In addition, $\text{Aut}(\mathcal{A})$ denotes the automorphism group of $\mathcal{A}$ and $\text{FEM}(\mathcal{A})$ denotes the set of all finite elementary mappings of $\mathcal{A}$. If $p_0, \ldots, p_{n-1} \in \text{FEM}(\mathcal{A})$, then the basic open set $N_{p_0, \ldots, p_{n-1}}$ of $\text{Aut}(\mathcal{A})^n$ is defined to be $N_{p_0, \ldots, p_{n-1}} = \{(f_0, \ldots, f_{n-1}) \in \text{Aut}(\mathcal{A})^n : p_0 \subseteq f_0, \ldots, p_{n-1} \subseteq f_{n-1}\}$. 

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As it is well known, \( \{ N_{p_0, \ldots, p_{n-1}} : p_0, \ldots, p_{n-1} \in FEM(\mathcal{A}) \} \) is a base of the usual topology of \( Aut(\mathcal{A})' \). Further, if each finite elementary mapping of \( \mathcal{A} \) can be extended to an automorphism of \( \mathcal{A} \), then \( N_{p_0, \ldots, p_{n-1}} \neq \emptyset \) holds for all \( p_0, \ldots, p_{n-1} \in FEM(\mathcal{A}) \).

### 2 Weakly generic automorphisms and the profinite topology on groups

In this section we study the connections between the profinite topology on free groups and the existence of weakly generic tuples of automorphisms. Related investigations are initiated in [4], here we present much more elementary observations.

We start by recalling the definition of profinite topologies on groups: the profinite topology \( \tau \) on a group \( G \) is defined as follows. A base of \( \tau \) is the family of all (left) cosets of all subgroups of \( G \) of finite index; more formally, \( \tau \) is the topology on \( G \) generated by

\[
\{aN : a \in G, N \leq G \text{ and the index of } N \text{ in } G \text{ is finite} \}.
\]

Throughout this work, we always endow free groups with their profinite topology and treat them as topological groups.

**Lemma 2.1** Suppose \( \mathcal{A} \) is a countable structure such that each finite elementary mapping of \( \mathcal{A} \) can be extended to an automorphism of \( \mathcal{A} \) and suppose

\[
h : \mathcal{F}(x_0, \ldots, x_{n-1}) \rightarrow Aut(\mathcal{A})
\]

is a continuous homomorphism. Let \( G = \text{ran}(h) \). Then the \( G \)-orbit of each \( a \in \mathcal{A} \) is finite.

The proof is based on some obvious topological arguments, we present the proof only for completeness.

**Proof.** Assume \( a \in \mathcal{A} \) is arbitrary. Let \( q = \{ \langle a, a \rangle \} \). As \( \text{Id}_\mathcal{A} \in Aut(\mathcal{A}) \) extends \( q \), it follows that \( q \) is a finite elementary mapping (so \( N_q \) is a nonempty basic open subset of \( Aut(\mathcal{A}) \)). Further, \( h \) is continuous at the point \( \mathcal{F}(x_0, \ldots, x_{n-1}) \), hence there exists a subgroup \( \mathcal{H} \leq \mathcal{F}(x_0, \ldots, x_{n-1}) \) of finite index such that for all elements \( b \) of \( \mathcal{H} \) we have \( h(b) \in N_q \).

Let \( G_a = \{ g \in G : g(a) = a \} \). Denoting the universe of \( \mathcal{H} \) by \( H \), by the last sentence of the previous paragraph we have \( h[H] \subseteq G_a \) and hence the index of \( G_a \) in \( G \) is also finite. It is well known that the size of the \( G \)-orbit of \( a \) is equal with the index of \( G_a \) in \( G \). As \( a \in \mathcal{A} \) was arbitrary, the proof is complete. \( \square \)

#### 2.1 Characterization theorems

In this subsection we provide necessary and sufficient conditions for the existence of weakly generic tuples of automorphisms with the aid of the profinite topology of free groups. The main result of this subsection is Theorem 2.6.

**Lemma 2.2** Let \( \mathcal{A} \) be a countable structure such that each finite elementary mapping of \( \mathcal{A} \) can be extended to an automorphism of \( \mathcal{A} \), let \( n \in \omega \) and let \( p_0, \ldots, p_{n-1} \in FEM(\mathcal{A}) \). Then the following statements are equivalent.

1. For each finite subset \( A_0 \) of \( \mathcal{A} \) there exists a finite substructure \( A_1 \) of \( \mathcal{A} \) such that \( A_0 \subseteq A_1 \), for all \( i < n, \text{dom}(p_i), \text{ran}(p_i) \subseteq A_1 \) and each \( p_i \) can be extended to an automorphism \( q_i \) of \( A_1 \) such that \( q_i \) is an elementary mapping of \( A_1 \).
2. there exist \( f_0, \ldots, f_{n-1} \in Aut(\mathcal{A}) \) such that for all \( i < n \) we have \( p_i \subseteq f_i \) and the function that maps each \( x_i \) onto \( f_i \) extends to a continuous group homomorphism \( h : \mathcal{F}(x_0, \ldots, x_{n-1}) \rightarrow Aut(\mathcal{A}) \).

**Proof.** To show that (1) implies (2), fix an enumeration \( \{ a_k : k \in \omega \} \) of \( \mathcal{A} \). We define \( A_k \) and \( \{ f_{i,k} : i < n \} \) for all \( k \in \omega \) by recursion such that the following stipulations are satisfied for all \( i < n \) and \( k \in \omega \):

(a) \( a_k \in A_k \subseteq \mathcal{A} \) is finite;
(b) \( A_k \subseteq A_{k+1} \);
(c) \( f_{i,k} \) is a finite elementary mapping of \( \mathcal{A} \) and \( f_{i,k+1} \in Aut(\mathcal{A}_{k+1}) \).

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(d) \( p_i \subseteq f_{i,k} \subseteq f_{i,k+1} \).

We define \( A_0 = \{ a_0 \} \) and \( f_{i,0} = p_i \) for all \( i < n \). Next, assume that \( A_m \) and \( f_{i,m} \) have already been defined for all \( m \leq k \) and \( i < n \). Applying (1) to \( A_k \cup \{ a_{k+1} \} \) and \( \langle f_{i,k} : i < n \rangle \) we get a finite substructure \( A_{k+1} \) and \( \langle f_{i,k+1} : i < n \rangle \) such that \( A_k \cup \langle a_{k+1} \rangle \subseteq A_{k+1} \), for all \( i < n \) we have \( f_{i,k} \subseteq f_{i,k+1} \in \text{Aut}(A_{k+1}) \), moreover \( f_{i,k+1} \) is a finite elementary mapping of \( A \). So (a)-(d) hold for \( k + 1 \), as well. This completes our recursive construction: \( A_k \) and \( \langle f_{i,k} : i < n \rangle \) have been defined for all \( k \in \omega \).

For each \( i < n \) let \( f_i = \cup_{k \in \omega} f_{i,k} \). Combining (a) and (c) we conclude that each \( f_i \) is an automorphism of \( A \) extending \( p_i \) (in particular, \( f_i \) is surjective because for all \( k \in \omega \) we have \( a_k \in \text{dom}(f_{i,k}) \) by (a); further, \( f_{i,k} \subseteq f_{i,k+1} \) and \( f_{i,k+1} \in \text{Aut}(A_{k+1}) \) by (c); hence \( a_k \in \text{ran}(f_{i,k+1}) \subseteq \text{ran}(f_i) \), as well).

Let \( h : \mathcal{F}(x_0, \ldots, x_{n-1}) \to \text{Aut}(A) \) be the group homomorphism that extends the function mapping each \( x_i \) onto \( f_i \); we shall show that \( h \) is continuous. To do so, assume \( c \in \mathcal{F}(x_0, \ldots, x_{n-1}) \) and assume \( N \) is an open set of \( \text{Aut}(A) \) with \( h(c) \in N \). Then there exists a finite elementary mapping \( q \) of \( A \) such that the basic open set \( N_q \) satisfies \( h(c) \in N_q \subseteq N \). As \( q \) is finite, (a) implies that there exists \( k \in \omega \) such that \( k \geq 1 \), \( \text{dom}(q) \cap \text{ran}(q) \cap \text{dom}(p_0) \cap \text{ran}(p_0) \cap \ldots \cap \text{dom}(p_{n-1}) \cap \text{ran}(p_{n-1}) \subseteq A_k \). Let \( h' : \mathcal{F}(x_0, \ldots, x_{n-1}) \to \text{Aut}(A_k) \) be the homomorphism extending the function that maps each \( x_i \) onto \( f_{i,k} \). As \( A_k \) is finite, it follows that \( k(\text{ker}(h')) \) has finite index and hence \( c \cdot \text{ker}(h') \) is an open subset of \( \mathcal{F}(x_0, \ldots, x_{n-1}) \) (which of course contains \( c \)). Hence, it is enough to show that

\[(\forall b \in c \cdot \text{ker}(h')) (h(b) \in N_q).\]  

So assume \( b \in c \cdot \text{ker}(h') \). Then \( c^{-1} b \in \text{ker}(h') \), i.e.,

\[Id_{A_k} = h'(c^{-1} b) = h'(c^{-1})h'(b) = (h'(c))^{-1}h'(b)\]

whence \( h'(c) = h'(b) \). It follows that \( h'(b) \in N_q \). Further, by construction, \( h(b)|_{A_k} = h'(b) \), hence \( h(b) \in N_q \), i.e., \((*)\) holds, as desired.

Now we turn to show that (2) implies (1). As in (1), fix a finite \( A_0 \subseteq A \). By (2), there exist \( f_0, \ldots, f_{n-1} \in \text{Aut}(A) \) such that for all \( i < n \) we have \( p_i \subseteq f_i \) and the function that maps each \( x_i \) onto \( f_i \) extends to a continuous group homomorphism \( h : \mathcal{F}(x_0, \ldots, x_{n-1}) \to \text{Aut}(A) \). Let \( G = \text{ran}(h) \); we treat it as a subgroup of \( \text{Aut}(A) \). By Lemma 2.1 we have

\[(\forall a \in A)(\text{the } G\text{-orbit of } a \text{ is finite}).\]  

For any \( a \in A \) denote the \( G\)-orbit of \( a \) by \( O_G(a) \). Now let \( B_0 = A_0 \cup \text{dom}(p_0) \cup \text{ran}(p_0) \cup \ldots \cup \text{dom}(p_{n-1}) \cup \text{ran}(p_{n-1}) \), let

\[A_1 = \bigcup_{a \in B_0} O_G(a)\]

and for each \( i < n \) let \( q_i = h(x_i)|_{A_i} \). Then \( A_1 \) is finite because of \((**)\), and by construction, \( p_i \subseteq q_i \subseteq \text{Aut}(A_1) \) for all \( i < n \). Further, each \( q_i \) is an elementary mapping of \( A \), because \( q_i \) is the restriction of \( h(x_i) \in \text{Aut}(A) \). This witnesses that (1) holds, as desired.

**Definition 2.3** Let \( G \) be a topological group. Then \( \langle g_0, \ldots, g_{n-1} \rangle \in G^n \) is defined to be a tuple of co-continuity iff the function that maps each \( x_i \) onto \( g_i \) can be extended to a continuous group homomorphism \( h : F(x_0, \ldots, x_{n-1}) \to G \).

Keeping the notation introduced in Lemma 2.2, one can rephrase (2) of Lemma 2.2 as claiming, that there exists a tuple of co-continuity \( \langle f_0, \ldots, f_{n-1} \rangle \in \text{Aut}(A)^n \) such that \( p_i \subseteq f_i \) for all \( i < n \).

For completeness we insert here a corollary of Lemma 2.2 which we do not use later in this paper.

**Corollary 2.4** Let \( K \) be a Fraïssé class and let \( A \) be the Fraïssé limit of \( K \). Then the following statements are equivalent.

1. \( K \) has Hrushovski’s extension property;
2. for each \( n \in \omega \), the set of the tuples of co-continuity in \( \text{Aut}(A)^n \) is dense in \( \text{Aut}(A)^n \).

**Proof.** First suppose (1) and let \( n \in \omega, p_0, \ldots, p_{n-1} \in \text{FEM}(A) \). Since \( K \) has Hrushovski’s extension property, (1) of Lemma 2.2 holds. Therefore, by Lemma 2.2(2) there exists a tuple \( \langle f_0, \ldots, f_{n-1} \rangle \in \text{Aut}(A)^n \) of co-
Specifically, conjugating with \( f \) is a finite elementary mapping of \( \langle a, f, g(a) : a \in A \rangle \) is a tuple of co-continuity. By Lemma 2.2, \( \exists A_0 \subseteq A \) is finite, for all \( i < n \) we have \( h^{-1}g_i \in N_{p_i} \), i.e., \( p_i \subseteq h^{-1}g_i \). In addition, again because \( \langle g_0, \ldots, g_{n-1} \rangle \) is a weakly generic tuple, for all \( a \in A \), the \( \langle h^{-1}g_0, h^{-1}g_1, \ldots, h^{-1}g_{n-1} \rangle \)-orbit of \( a \) is finite.

Let \( B = A_0 \cup \text{dom}(p_0) \cup \text{ran}(p_0) \cup \ldots \cup \text{dom}(p_{n-1}) \cup \text{ran}(p_{n-1}) \), let

\[
A_1 = \bigcup_{a \in B} O_{(h^{-1}g_0, \ldots, h^{-1}g_{n-1})} b(a)
\]

and for each \( i < n \) let \( q_i = h^{-1}g_i \). Then, by the end of the previous paragraph, \( A_0 \subseteq A_1 \) is finite, for all \( i < n \) we have \( \text{dom}(p_i), \text{ran}(p_i) \subseteq A_1, p_i \subseteq q_i \in A_{n+1} \), and (as \( q_i \) is a restriction of an automorphism of \( A \)), \( q_i \) is a finite elementary mapping of \( A \). Thus, (1) of Lemma 2.2 holds, as desired.

Next, we show that (1) implies (ii) of (2). Let \( r_0 = \langle r_{00}, \ldots, r_{0n-1} \rangle \) and \( r_1 = \langle r_{10}, \ldots, r_{1n-1} \rangle \) be \( n \)-tuples of finite elementary mappings of \( A \). As \( \langle g_0, \ldots, g_{n-1} \rangle \) is weakly generic, there exist \( k_0, k_1 \in Aut(A) \) such that for all \( i < n \) and \( j < 2 \) we have \( k_j^{-1}g_k \in N_{p_k} \), i.e., \( r_i \subseteq k_j^{-1}g_k \). Let \( k = k_0^{-1} \circ k_1 \). As we observed in Remark 2.5,

\[
k_j^{-1}g_k = k_j^{-1}[g_k] \quad \text{and} \quad k_jr_i^{-1}k_j^{-1} = k_j[r_i]
\]
hence, for all $i < n$
\[ k[r_i^n] = (k_0^n \circ k_1)[r_i^n] \subseteq k_0^{-1}[g_i]. \]

Consequently,

\[ r_i^0 \cup k[r_i^n] \subseteq k_0^{-1}[g_i] = k_0^{-1}g_0k_0 \in \text{Aut}(A), \]

so, for all $i < n$, $r_i^0 \cup k[r_i^n]$ is a finite elementary mapping of $A$, i.e., $k$ satisfies (ii) of (2).

Now we turn to show that (2) implies (1). Let $\{t_i : i < n\} \subseteq \omega$ be an enumeration of $\#\text{FEM}(A)$ and $\{a_m : m \in \omega\}$ an enumeration of $A$. For each $m \in \omega$ and $i < n$ we define $A_m$ and $g_{m,i}$ by recursion such that the following stipulations are satisfied for all $m \in \omega$ and $i < n$.

(a) $a_m \in A_{m+1} \subseteq A$ is finite;
(b) $A_m \subseteq A_{m+1}$;
(c) $g_{m,i} : \text{Aut}(A_m) \to \text{Aut}(A)$ is a finite elementary mapping of $A$;
(d) $g_{m,i} \subseteq g_{m+1,i}$;
(e) if $l < n$, then there exists $k \in \text{Aut}(A)$ such that $k^{-1}t_{m,k} \subseteq g_{m,i}$.

Let $A_0 = t_0, 0 = \cdots = t_{0,n-1} = \emptyset$, then (a)-(e) hold. Suppose that $A_l$ and $g_l$, $i$ have already been defined for all $l \leq m$ and $i < n$. Then, by (ii) of (2), there exist $f_0, \ldots, f_{n-1} \in \text{Aut}(A)$ such that $f_i = f_0 \in N_{n_0, \ldots, f_{n-1} \in N_{n,e-1}$ and the function that maps each $x_i$ onto $f_i$ can be extended to a continuous homomorphism from $\mathcal{F}(x_0, \ldots, x_{n-1})$ into $\text{Aut}(A)$, i.e., (2) of Lemma 2.2 holds. Moreover, by Lemma 2.2, there exists a finite substructure $B_{m+1}$ of $A$ such that for all $i < n$, dom$(t_{m,i})$, ran$(t_{m,i}) \subseteq B_{m+1}$ and each $t_{m,i}$ can be extended to an automorphism $g_{m,i}$ of $B_{m+1}$ such that $g_{m,i}$ is an elementary mapping of $A$. By (ii) of (2), there exists $k \in \text{Aut}(A)$ such that for all $i < n$, $g_{m,i} \cup \{q_{m,i} \in \text{FEM}(A)\}$.

Let $A_{m+1} = A_m \cup k[B_{m+1}]$ and for all $i < n$ let $g_{m+1,i} = g_{m,i} \cup k[q_{m,i}]$. It follows from (i) of (2) that (2) of Lemma 2.2 holds for $g_{m+1,0}, \ldots, g_{m+1,n-1}$. Applying Lemma 2.2 to $A_{m+1} \cup \{a_m\}$ and $g_{m+1,0}, \ldots, g_{m+1,n-1}$, we obtain a finite $A_{m+1} \subseteq A$ such that $A_{m+1} \cup \{a_m\} \subseteq A_{m+1}$, for all $i < n$, dom$(g_{m+1,i})$, ran$(g_{m+1,i}) \subseteq A_{m+1}$ and each $g_{m+1,i}$ can be extended to an automorphism $g_{m+1,i}$ of $A_{m+1}$ such that $g_{m+1,i}$ is an elementary mapping of $A$. It is easy to see that with this choice (a)-(e) remain true for $m + 1$.

Thus $A_m$ and $g_{m,i}$ have been defined for all $m \in \omega$ and $i < n$. For each $i < n$ let

$$g_i = \bigcup_{m \in \omega} g_{m,i}.$$ 

Combining (a), (c) and (d), we conclude that each $g_i$ is an automorphism of $A$. Further, by (e) the conjugacy class of $(g_0, \ldots, g_{n-1})$ is dense in $\text{Aut}(A)^\omega$. Moreover, if $a \in A$, then there exists $m \in \omega$ with $a = a_m$, so by (a), $a \in A_{m+1}$, hence by construction and (c),

$$O_{(g_0, \ldots, g_{n-1})}(a) = O_{(g_m \circ g_0, \ldots, g_m \circ g_{n-1})}(a) \subseteq A_{m+1}$$

which is finite by (a). This means that $(g_0, \ldots, g_{n-1})$ is a weakly generic tuple, so (1) holds, and we are done. \hfill \qed

### 2.2 Joint embeddings of finite elementary mappings

Let $A$ be a countable structure such that each finite elementary mapping of $A$ can be extended to an automorphism of $A$. For $n \in \omega$ define $\text{Age}(A)^n$ as follows:

\[ \text{Age}(A)^n = \{(B, p_0, \ldots, p_{n-1}) : B \subseteq A \text{ is finite, } p_0, \ldots, p_{n-1} \in \text{FEM}(A) \text{ and } \text{dom}(p_0), \ldots, \text{dom}(p_{n-1}), \text{ran}(p_0), \ldots, \text{ran}(p_{n-1}) \subseteq B\}. \]

Then (2)(ii) of Theorem 2.6 can be rephrased that $\text{Age}(A)^n$ has the Joint Embedding Property (JEP, for short). If $A$ is a Fraïssé limit (of its age), then according to [12, Theorem 1.1] $\text{Aut}(A)$ has a dense conjugacy class if $\text{Age}(A)^n$ has the JEP. Further, in some intuitive sense, the role of (2)(i) of Theorem 2.6 is to strengthen (2)(ii) of Theorem 2.6 to guarantee the existence of $(g_0, \ldots, g_{n-1}) \in \text{Aut}(A^n)$ with a dense conjugacy class such that the $(g_0, \ldots, g_{n-1})$-orbit of each $a \in A$ is finite.
In this subsection we are investigating (2)(ii) of Theorem 2.6 and provide a sufficient condition implying JEP for $\text{Age}(A)^n_r$.

The main result of this subsection is Corollary 2.12 where we show that if $A$ satisfies the sufficient condition mentioned above, the topological rank of $\text{Aut}(A)$ is at most $r$ and there is a weakly generic tuple in $\text{Aut}(A)^n_r$, then there are weakly generic tuples in $\text{Aut}(A)^n_r$, for all $n \in \omega$, $n \geq 1$.

We start with the definition needed for the sufficient condition we mentioned above. We will compare our definition with other notions studied earlier in [8–10].

**Definition 2.7** Let $A$ be any structure. Then $A$ is defined to have square Stone spaces iff for any $p \in S(\emptyset)$ and for any $X \subseteq A$ with $|X| < |A|$ there exists a realization $\alpha \subseteq A$ of $p$ such that

$$(\forall \bar{b}, \bar{c} \subseteq X, \forall \bar{c}', \bar{c}' \subseteq \alpha)(tp(\bar{b}/\emptyset) = tp(\bar{c}/\emptyset), tp(\bar{c}/\emptyset) = tp(\bar{c}'/\emptyset) \implies tp(\bar{b}, \bar{c}/\emptyset) = tp(\bar{b}, \bar{c}'/\emptyset)).$$

**Remark 2.8** We note that Definition 2.7 is closely related to the definition of UJEP introduced in [14, Definition 2.3]. Here we do not recall the definition of UJEP because we do not need it for the present investigations.

Let $A$ be an infinite structure and let $B \subseteq A$. In order to compare the “square Stone space property” with other thoroughly investigated notions, first we recall from [9] that a type $p \in S(B)$ is defined to be strongly determined over $B$ iff for any elementary extension $M$ of $A$, $p$ can be extended to a complete type $q \in S(M)$ such that

(*) $\text{if } \bar{c} \text{ realizes } q \text{ (in an elementary extension of } M \text{ or in the monster model of the theory of } A) \text{ then any elementary map of } M \text{ which fixes acl}^q(B) \text{ pointwise is elementary over } \bar{c}.$

Further, a theory $T$ is defined to be strongly determined (over $\emptyset$) iff each type (over $\emptyset$) of it is strongly determined (over $\emptyset$).

Moreover, $T$ being strongly determined is closely related to amenability properties of $T$. More concretely, according to a comment after [8, Corollary 4.20], if $T$ is extremely amenable then $T$ is strongly determined over $\emptyset$. In fact, modulo certain technical conditions, $T$ is amenable if and only if $T$ is strongly determined over $\emptyset$.

We claim that the theory $TH(A)$ of $A$ is “strongly determined over $\emptyset$” is ‘almost the same as’ “$A$ has square Stone spaces”.

More concretely, supposing that $A$ is saturated, we claim the following:

1. If $\text{acl}^{eq}(\emptyset) = \text{acl}^{eq}(\emptyset)$ holds for $A$, then $A$ has square Stone spaces iff its theory is strongly determined over $\emptyset$ and

2. $TH(A)$ is strongly determined over $\emptyset$ iff $A^{eq}$ has square Stone spaces.

We sketch the proof of (1) only; one can prove (2) similarly. To see (1), first assume $A$ has square Stone spaces. Let $p \in S(\emptyset)$ be arbitrary, we shall show that $p$ is strongly determined. To do so, let $M$ be any elementary extension of $A$ and let $M^*$ be any strongly homogeneous, saturated enough elementary extension of $M$. Combining the square Stone space property of $A$ with a compactness argument, we obtain a realization $\bar{a} \in M^*$ of $p$ such that (for $b = \bar{b} = \bar{a}$ and) for any $\bar{c}, \bar{c}' \in M$ if $tp(\bar{a}/\emptyset) = tp(\bar{c}/\emptyset)$, then $tp(\bar{a}/\emptyset) = tp(\bar{c}/\emptyset)$. Let $q = tp(\bar{a}/M)$. Then it is straightforward to check, that each elementary map of $M$ remains elementary over any realization of $q$, i.e., $p$ is strongly determined, indeed.

Conversely, assume $TH(A)$ is strongly determined over $\emptyset$. To show that $A$ has square Stone spaces assume $p \in S(\emptyset)$ and $X \subseteq A$, $|X| < |A|$. Since $p$ is strongly determined, it has an extension $q \in S(A)$ satisfying (*). Since $A$ is saturated, $q_{|X}$ has a realization $\bar{a}$ in $A$. Now assume $\bar{b}, \bar{c} \subseteq X$ and $\bar{c}' \subseteq \bar{a}$ are such that $tp(\bar{b}/\emptyset) = tp(\bar{c}/\emptyset)$ and $tp(\bar{c}/\emptyset) = tp(\bar{c}'/\emptyset)$. The function $g_0$ mapping $\bar{c}$ onto $\bar{c}'$ is elementary. Since $A$ is saturated, there exists $g \in \text{Aut}(A)$ extending $g_0$. Then $tp(\bar{b}c/\emptyset) = tp(g(\bar{b})\bar{c}'/\emptyset)$. Further, the function $f$ mapping $g(\bar{b})$ onto $\bar{b}$ is elementary. We assumed $\text{acl}^{eq}(\emptyset) = \text{acl}^{eq}(\emptyset)$, hence $f$ fixes $\text{acl}^{eq}(\emptyset)$ pointwise. It follows that $f$ is elementary over $\bar{a}$, particularly, $tp(g(\bar{b})\bar{c}'/\emptyset) = tp(\bar{b}\bar{c}'/\emptyset)$. Therefore $tp(\bar{b}c/\emptyset) = tp(\bar{b}\bar{c}'/\emptyset)$, i.e., $A$ has square Stone spaces, indeed.

[9, § 2] contains several examples for theories admitting strongly determined types. We close this remark with some examples for theories with the square Stone space property. With these examples we also would like to demonstrate that in many cases, the square Stone space property can be verified easily.
The Rado graph (countable random graph) $\mathcal{G}$ has square Stone spaces because of the following. Suppose we are given some $n \in \omega$, an $n$-type $p$ of $\mathcal{G}$ over $\emptyset$ (i.e., $p \in S^n_0(\emptyset)$) and $X \subseteq G$. Let 
\[ q = p \cup \{a \neq v, \neg E(a, v) : a \in X, v \text{ is a variable with a free occurrence in } p\} \]
where $E$ is the relation symbol denoting the edge relation of $\mathcal{G}$. Thus, $q$ is a (partial) type over $X$, which was obtained from $p$ by requiring that the values of the variables $v$ of $p$ should lie outside $X$ and there are no edges between elements of $X$ and (the values of) the variables of $p$. It is easy to see that any realization $\bar{a}$ of $q$ satisfies the implication in Definition 2.7. One can show similarly that the following structures have square Stone spaces:

1. the Fraïssé limit of the class of all finite structures of a given relational language;
2. the Fraïssé limit of all finite tournaments;
3. the Fraïssé limit of all finite metric spaces (with rational distances);
4. the Henson graphs $\mathcal{H}_n$ (recall that the $n$th Henson graph $\mathcal{H}_n$ is the Fraïssé limit of the class of all finite graphs not containing an isomorphic copy of the complete graph with $n$ vertices).

In Theorem 3.1 of § 3 we will see that if $\mathcal{A}$ is a model of a simple theory in which all types over the empty set are stationary, then $\mathcal{A}$ has square Stone spaces. We note also that variants of this requirement play an important role in [11] as well (we refer to [11, Definition 3.10] in particular and [11, § 3] in general).

Roughly, in the next lemma we show that having square Stone spaces implies JEP for $\text{Age}(\mathcal{A})^0_n$.

**Lemma 2.9** Let $\mathcal{A}$ be a countable structure such that each finite elementary mapping of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$. Suppose $\mathcal{A}$ has square Stone spaces and let $n \in \omega$, $n \geq 1$. Assume $r^0_n, \ldots, r^0_0, r^1_n, \ldots, r^1_0$ are finite elementary mappings of $\mathcal{A}$. Then there exists $k \in \text{Aut}(\mathcal{A})$ such that for all $i < n$ we have $r^i_0 \cup k[r^i_1] \in \text{FEM}(\mathcal{A})$.

**Proof.** For $i < 2$ let $X_i = \bigcup_{j<i}(\text{dom}(r^i_j) \cup \text{ran}(r^i_j))$ and let $p = \text{tp}(X_i/\emptyset)$. Since $\mathcal{A}$ has square Stone spaces, there exists a realization $\bar{a}$ of $p$ such that for all $\bar{b}, \bar{b}' \subseteq X_0$ and $\bar{c}, \bar{c}' \subseteq \bar{a}$,

\[ \text{tp}(\bar{b}/\emptyset) = \text{tp}(\bar{b}'/\emptyset), \text{tp}(\bar{c}/\emptyset) = \text{tp}(\bar{c}'/\emptyset) \implies \text{tp}(\bar{b}^{-} \bar{c}/\emptyset) = \text{tp}(\bar{b}'^{-} \bar{c}'/\emptyset). \]  

(\ast)

As $\bar{a}$ realizes $p$, there exists an elementary mapping $X_i$ onto $\bar{a}$; let $k$ be an automorphism of $\mathcal{A}$ extending $k'$. It remains to show that for all $i < n$ we have $r^i_0 \cup k[r^i_1] \in \text{FEM}(\mathcal{A})$. To do so, fix $i < n$. As $r^i_1$ is an elementary mapping,

\[ \text{tp}(\text{dom}(r^i_1)/\emptyset) = \text{tp}(\text{ran}(r^i_1)/\emptyset). \]  

(a)

Further, $k$ is an automorphism, hence

\[ \text{tp}(\text{dom}(r^i_0)/\emptyset) = \text{tp}(k[\text{dom}(r^i_1)]/\emptyset) \quad \text{and} \quad \text{tp}(\text{ran}(r^i_0)/\emptyset) = \text{tp}(k[\text{ran}(r^i_1)]/\emptyset). \]  

(b)

Combining (a) and (b) we get $\text{tp}(k[\text{dom}(r^i_1)]/\emptyset) = \text{tp}(k[\text{ran}(r^i_1)]/\emptyset)$. Further $\text{dom}(r^i_0), \text{ran}(r^i_0) \subseteq X_0$ and as $r^i_0 \in \text{FEM}(\mathcal{A})$, we have $\text{tp}(\text{dom}(r^i_0)/\emptyset) = \text{tp}(\text{ran}(r^i_0)/\emptyset)$. It follows from (\ast) that

\[ \text{tp}(\text{dom}(r^i_0) \cap \text{dom}(r^i_1)/\emptyset) = \text{tp}(\text{ran}(r^i_0) \cap \text{ran}(r^i_1)/\emptyset). \]

i.e., $r^i_0 \cup k[r^i_1] \in \text{FEM}(\mathcal{A})$, as desired. \qed

**Lemma 2.10** Suppose $\mathcal{F}$ and $\mathcal{G}$ are topological groups and $h : \mathcal{F} \to \mathcal{G}$ is a homomorphism. Let $a, b \in \mathcal{F}$.

1. If $h$ is continuous at $a$, then it is also continuous at $a^{-1}$.
2. If $h$ is continuous at $a$ and $b$, then it is also continuous at $a \cdot b$.

This lemma is a part of folklore, its proof is based on some obvious topological arguments. We present the proof only for completeness.

**Proof.** To show (1), assume $h$ is continuous at $a$. Let $i^\mathcal{F} : \mathcal{F} \to \mathcal{F}$, $i^\mathcal{F}(x) = x^{-1}$ and let $i^\mathcal{G} : \mathcal{G} \to \mathcal{G}$, $i^\mathcal{G}(x) = x^{-1}$. Observe that $h = i^\mathcal{G} \circ h \circ i^\mathcal{F}$, because $h$ is a homomorphism. But $i^\mathcal{F}$ and $i^\mathcal{G}$ are continuous, hence $i^\mathcal{G} \circ h \circ i^\mathcal{F}$ is continuous at $a^{-1}$, which means that $h$ is continuous at $a^{-1}$, as desired.
To show (2), assume \( h \) is continuous at \( a \) and \( b \) and assume \( N \subseteq \mathcal{G} \) is open such that \( N \ni h(a \cdot b) = h(a) \cdot h(b) \).

Since the multiplication operation of \( \mathcal{G} \) is continuous, there exist open sets \( N_a, N_b \subseteq \mathcal{G} \) such that \( h(a) \in N_a, h(b) \in N_b \) and for all \( (c, d) \in N_a \times N_b \) we have \( c \cdot d \in N \). Further, since \( h \) is continuous, \( h^{-1}(N_a) \) and \( h^{-1}(N_b) \) are nonempty, open subsets of \( \mathcal{F} \). Let \( g : \mathcal{F} \to \mathcal{F} \) be the function \( g(x) = a^{-1} \cdot x \). Since \( g \) is continuous, \( g^{-1}(h^{-1}(N_b)) \) is open in \( \mathcal{F} \) and clearly, \( ab \in g^{-1}(h^{-1}(N_b)) \).

To complete the proof, it suffices to show that if \( d \in g^{-1}(h^{-1}(N_b)) \), then \( h(d) \in N \). So assume \( d \in g^{-1}(h^{-1}(N_b)) \). Then

\[
h(d) = h(aa^{-1}d) = h(ag(d)) = h(a)h(g(d)).
\]

In addition, \( h(a) \in N_a \) and \( h(g(d)) \in N_b \). Consequently \( h(d) = h(a)h(g(d)) \in N \), as desired. \( \square \)

Let \( \mathcal{G} \) be a topological group. Recall that \( \mathcal{G} \) is defined to be topologically finitely generated iff it has a finitely generated dense subgroup \( \mathcal{G}_0 \). In addition, if for some \( r \in \omega \), \( \mathcal{G}_0 \) can be generated by \( r \) elements, then \( \mathcal{G} \) is defined to have topological rank (at most) \( r \).

**Theorem 2.11** Let \( A \) be a countable structure such that each finite elementary mapping of \( A \) can be extended to an automorphism of \( A \). Suppose the topological rank of \( \text{Aut}(A) \) is at most \( r \in \omega \) and there is a weakly generic tuple in \( \text{Aut}(A)^r \). Then for all \( n \in \omega \) the set of all \( n \)-tuples of co-continuity is dense in \( \text{Aut}(A)^n \).

**Proof.** Let \( r_0, \ldots, r_{n-1} \in \text{FEM}(A) \); we shall find a tuple of co-continuity in \( N_{r_0,\ldots,r_{n-1}} \).

We need the following notation. If \( w = w_0 \ldots w_k \) is a word over the alphabet \( \{x_0, \ldots, x_{r-1}\} \) and \( f_0, \ldots, f_{r-1} \in \text{Aut}(A) \), then \( \sigma_w(f_0, \ldots, f_{r-1}) \) denotes the function "naturally described by \( w \) if one substitutes \( f_i \) in place of \( x_i \)" for all \( i < r \); more formally, \( \sigma_w(f_0, \ldots, f_{r-1}) = \text{Id}_{A}, \sigma_w(f_0, \ldots, f_{r-1}) = f_i \) and \( \sigma_{x_i}^{-1}(f_0, \ldots, f_{r-1}) = f_i \circ \sigma(w)(f_0, \ldots, f_{r-1}) \).

By assumption, the topological rank of \( \text{Aut}(A) \) is at most \( r \), hence there exist \( d_0, \ldots, d_{r-1} \in \text{Aut}(A) \) such that \( \mathcal{D} := (d_0, \ldots, d_{r-1}) \) is a dense subgroup of \( \text{Aut}(A) \). In particular, there exist words \( w_0, \ldots, w_{n-1} \) such that for all \( i < n \) we have

\[
\sigma_{w_i}(d_0, \ldots, d_{r-1}) \in N_{r_i}.
\]

Further, as the group operations of \( \text{Aut}(A) \) are continuous, the functions mapping the tuple \( (f_0, \ldots, f_{r-1}) \) onto \( \sigma_{w_i}(f_0, \ldots, f_{r-1}) \) are continuous for all \( i < n \). Hence there exist \( s_0, \ldots, s_{r-1} \in \text{FEM}(A) \) such that \( f_0 \in N_{s_0}, \ldots, f_{r-1} \in N_{s_{r-1}} \) imply

\[
\sigma_{w_i}(f_0, \ldots, f_{r-1}) \in N_{s_0}, \ldots, \sigma_{w_{r-1}}(f_0, \ldots, f_{r-1}) \in N_{s_{r-1}}.
\]

By assumption \( \text{Aut}(A)^r \) has a weakly generic tuple of automorphisms, hence by Theorem 2.6, for each \( i < r \) there exists \( f_i \in N_i \) such that the homomorphism \( k : \mathcal{F}(x_0, \ldots, x_{r-1}) \to \text{Aut}(A) \) that maps each \( x_i \) onto \( f_i \) is continuous. Let \( h : \mathcal{F}(x_0, \ldots, x_{r-1}) \to \text{Aut}(A) \) be the homomorphism that maps \( x_i \) onto \( \sigma_{w_i}(f_0, \ldots, f_{r-1}) \) for all \( i < n \).

Our aim is to show that \( h \) is continuous (this is enough, because this would witness that

\[
\{\sigma_{w_0}(f_0, \ldots, f_{r-1}), \ldots, \sigma_{w_{r-1}}(f_0, \ldots, f_{r-1})\}
\]

is a tuple of co-continuity in \( N_{r_0,\ldots,r_{n-1}} \), as desired). For continuity of \( h \) it is enough to show that

(a) \( (\forall i < n)(h \text{ is continuous at } x_i) \)

because according to (a), \( h \) is continuous at each point of a set of generators of \( \mathcal{F}(x_0, \ldots, x_{r-1}) \); combining this with Lemma 2.10 one obtains that \( h \) is continuous at each point of \( \mathcal{F}(x_0, \ldots, x_{r-1}) \).

Turning to prove (a), let \( i < n \) be fixed and let \( N_i \) be any basic open subset of \( \text{Aut}(A) \) containing \( \sigma_{w_i}(f_0, \ldots, f_{r-1}) \) (i.e., \( z \subseteq \sigma_{w_i}(f_0, \ldots, f_{r-1}) \) is a finite elementary mapping). For (a), it suffices to show

(b) there exists a subgroup \( N \subseteq \mathcal{F}(x_0, \ldots, x_{r-1}) \) such that \( (\mathcal{F}(x_0, \ldots, x_{r-1}) : N) \) is finite and for all \( b \in x_i N \) we have \( h(b) \in N_i \).

Since \( k \) is a continuous homomorphism, by Lemma 2.1, the \( (f_0, \ldots, f_{r-1}) \)-orbit of each \( a \in A \) is finite. Let \( Z = \text{dom}(z) \cup \text{ran}(z) \) and let \( B = \bigcup_{a \in Z} O_{(f_0,\ldots,f_{r-1})}(a) \). Let \( B \) be the substructure of \( A \) generated (in fact, spanned) by \( B \). It follows that for all \( j < n \), \( B \) is closed under \( \sigma_{w_j}(f_0, \ldots, f_{r-1}) \), i.e., \( \sigma_{w_j}(f_0, \ldots, f_{r-1}) |_B \) is an automorphism of \( B \).
Let \( h^* : F(x_0, \ldots, x_{n-1}) \rightarrow Aut(B) \) be the homomorphism that maps each \( x_j \) onto \( \sigma_{w_j}(f_0, \ldots, f_{r-1})|_B = h(x_j)|_B \) and let
\[
N = \ker(h^*) = \{ g \in F(x_0, \ldots, x_{n-1}) : h^*(g) = Id_B \}.
\]
As \( B \) is finite, so is \( Sym(B) \), therefore \( |F(x_0, \ldots, x_{n-1})|/|N| \leq |Sym(B)| \) and hence \( N \) has finite index in \( F(x_0, \ldots, x_{n-1}) \).

Let \( c \in N \). Since \( c \) is a word over the alphabet \( \{x_0, \ldots, x_{n-1}\} \cup \{x_0^{-1}, \ldots, x_{n-1}^{-1}\} \), there exists \( m \in \omega \) such that \( c = c_0 \ldots c_{m-1} \) where each \( c_l \) is either \( x_{j(l)} \) or \( x_{j(l)}^{-1} \) for some \( j(l) < n \). Now observe that
\[
h(c)|_B = (h(c_0) \circ \ldots \circ h(c_{m-1}))|_B
\]
\[
= h(c_0)|_B \circ \ldots \circ h(c_{m-1})|_B
\]
\[
= h^*(c_0) \circ \ldots \circ h^*(c_{m-1})
\]
\[
= h^*(c_0 \ldots c_{m-1})
\]
\[
= h^*(c) \in N.
\]
It follows that for any \( c \in N \) we have
\[
h(x)c \supseteq h(x)c|_B = h(x)c|_B \circ Id_B = h(x)c|_B = \sigma_{w_j}(f_0, \ldots, f_{r-1})|_B \supseteq c.
\]
specifically, \( h(x)c \in N \), so (b) holds, as desired.

Now we can state and prove one of the main results of the paper.

**Corollary 2.12** Let \( A \) be a countable structure such that each finite elementary mapping of \( A \) can be extended to an automorphism of \( A \). Suppose \( A \) has square Stone spaces and the topological rank of \( Aut(A) \) is at most \( r \in \omega \). Then the following statements are equivalent.

1. There exists a weakly generic tuple \( \langle g_0, \ldots, g_{r-1} \rangle \in Aut(A)^r \);
2. for all \( n \in \omega \), \( n \geq 1 \) there exists a weakly generic tuple \( \langle g_0, \ldots, g_{n-1} \rangle \in Aut(A)^n \).

**Proof.** Clearly, (2) implies (1), hence it is enough to show that (1) implies (2). So assume (1) and let \( n \in \omega \) be fixed. It is enough to show that (2) of Theorem 2.6 holds. By assumption, \( A \) has square Stone spaces, hence Lemma 2.9 implies that (ii) of Theorem 2.6(2) is satisfied. Finally, (i) of Theorem 2.6(2) follows from Theorem 2.11. \( \Box \)

### 3 Simplicity and weakly generic automorphisms

In this section we apply the results obtained in previous sections to countable models of simple theories. In more detail, in Lemma 3.1 we prove that if \( A \) is a model of a simple theory and all types of \( A \) over the empty set are stationary, then \( A \) has square Stone spaces. We conclude the paper with Theorem 3.2 which is an easy observation based on Corollary 2.12 and the results of [11]: If \( A \) is a countable model of an \( \aleph_0 \)-categorical, simple theory, all types of \( A \) over the empty set are stationary and there exists a weakly generic pair \( \langle g_0, g_1 \rangle \in Aut(A)^2 \), then for all \( n \in \omega \), \( n \geq 1 \) there exists a weakly generic tuple \( \langle g_0, \ldots, g_{n-1} \rangle \in Aut(A)^n \).

**Lemma 3.1** Suppose \( A \) is a model of a simple theory \( T \) (i.e., forking is symmetric in models of \( T \)). Suppose further that all types of \( A \) over the empty set are stationary. Then

1. \( A \) has square Stone spaces;
2. if \( \mathbf{b}, \mathbf{b}', \mathbf{c}, \mathbf{c}' \) are finite tuples from \( A \) such that \( \mathbf{b} \downarrow_\varnothing \mathbf{c}, \mathbf{b}' \downarrow_\varnothing \mathbf{c}' \), \( tp(\mathbf{b}/\varnothing) = tp(\mathbf{b}'/\varnothing) \) and \( tp(\mathbf{c}/\varnothing) = tp(\mathbf{c}'/\varnothing) \), then \( tp(\mathbf{b}^-\mathbf{c}/\varnothing) = tp(\mathbf{b}'^-\mathbf{c}'/\varnothing) \).

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Proof. To show (1), suppose we are given \( p \in S^d(\varnothing) \) and \( X \subseteq A \). Let \( p' \) be the non-forking extension of \( p \) to \( X \) and let \( \tau \subseteq A \) be any realization of \( p' \). Let \( \overline{b}, \overline{b} \subseteq X \) and \( \overline{\tau}, \overline{\tau}' \subseteq \tau \) be arbitrary such that \( tp(\overline{b}/\varnothing) = tp(\overline{b}'/\varnothing) \) and \( tp(\overline{\tau}/\varnothing) = tp(\overline{\tau}'/\varnothing) \). Now observe that

\[
\overline{b} \downarrow_{\varnothing} \overline{\tau}, \quad \overline{b} \downarrow_{\varnothing} \overline{\tau}', \quad tp(\overline{b}/\varnothing) = tp(\overline{b}'/\varnothing) \quad \text{and} \quad tp(\overline{\tau}/\varnothing) = tp(\overline{\tau}'/\varnothing),
\]

hence it is enough to prove (2). To do so, let \( \gamma = tp(\overline{\tau}/\varnothing) \), \( \gamma' = tp(\overline{\tau}'/\varnothing) \) and let \( \gamma' = tp(\overline{\tau}'/\varnothing) \). Since the function \( k \) mapping \( \overline{b} \) onto \( \overline{b} \) is elementary, it follows, that \( k[\gamma] = \gamma' \): otherwise, by construction, \( k[\gamma] \) and \( \gamma' \) would be different extensions of \( \gamma \) to \( \overline{b} \) both of which do not fork over \( \varnothing \), and this is impossible because by assumption each type of \( A \) over \( \varnothing \) is stationary. Therefore we have

\[
\text{as desired both in (1) and (2).}
\]

Theorem 3.2 Suppose \( A \) is a countable model of an \( \aleph_0 \)-categorical, simple theory. Suppose further that all types of \( A \) over the empty set are stationary. Then the following statements are equivalent.

1. There exists a weakly generic pair \( \langle g_0, g_1 \rangle \in Aut(A)^2 \);
2. for all \( n \in \omega \), \( n \geq 1 \) there exists a weakly generic tuple \( \langle g_0, \ldots, g_{n-1} \rangle \in Aut(A)^n \).

Proof. First we shall show that \( Aut(A) \) has topological rank at most 2. Recall from [11] the following: a topological group \( G \) is defined to have a cyclically dense conjugacy class iff there exist \( f, g \in G \) such that \( \{ f^{-n}gf^n : n \text{ is an integer} \} \) is dense in \( G \). Clearly, if \( G \) has a cyclically dense conjugacy class, then the topological rank of \( G \) is at most 2. Further, according to [11, Definition 3.10], a ternary relation \( \downarrow \) on finite subsets of a countable structure is defined to be a Canonical Independence Relation (CIR for short) iff it satisfies stipulations \((\text{CIR}_0)-(\text{CIR}_1)\) below; moreover by [11, Corollary 3.14], if a countable, \( \aleph_0 \)-categorical structure admits a CIR, then its automorphism group has a cyclically dense conjugacy class. Hence, it is enough to show that there exists a ternary relation \( \downarrow \) on finite subsets of \( A \) which is a canonical independence relation, i.e., satisfies the following stipulations for all \( X, Y, Z, U \in [A]^{<\omega} \):

\((\text{CIR}_0)\) (stationarity over \( \varnothing \)) If \( \overline{b}, \overline{b}, \overline{\tau}, \overline{\tau}' \) are finite tuples from \( A \) such that

\[
\begin{align*}
&\text{ran}(\overline{b}) \downarrow_{\varnothing} \text{ran}(\overline{\tau}), \quad \text{ran}(\overline{b}') \downarrow_{\varnothing} \text{ran}(\overline{\tau}'), \quad tp(\overline{b}/\varnothing) = tp(\overline{b}'/\varnothing), \quad tp(\overline{\tau}/\varnothing) = tp(\overline{\tau}'/\varnothing),
\end{align*}
\]

then \( tp(\overline{b} \uparrow \overline{\tau}/\varnothing) = tp(\overline{b}' \uparrow \overline{\tau}'/\varnothing) \).

\((\text{CIR}_1)\) (extension on the right) If \( X \downarrow_Z Y \), then for each finite tuple \( \overline{d} \) of \( A \) there exists \( \overline{d}' \) such that \( tp(\overline{d}/YZ) = tp(\overline{d}'/YZ) \) and \( X \downarrow_Z Y \cup \text{ran}(\overline{d}) \).

\((\text{CIR}_2)\) (transitivity on both sides) If \( X \downarrow_{UZ} Y \) and \( U \downarrow_Z Y \), then \( XU \downarrow_Z Y \) and similarly, if \( X \downarrow_{UZ} Y \) and \( X \downarrow_Z U \), then \( X \downarrow_Z YU \).

\((\text{CIR}_3)\) (monotonicity) If \( X \downarrow_Z Y \) and \( X' \subseteq X, Y' \subseteq Y \), then \( X' \downarrow_Z Y' \).

\((\text{CIR}_4)\) (existence) \( X \downarrow_Z Z \) and \( Z \downarrow_Z X \).

Note that a canonical independence relation may not satisfy the usual symmetry property of forking independence in simple theories.

Turning back to the proof of our theorem, let \( \downarrow^f \) be the forking-independence relation of \( A \); we claim that this is a canonical independence relation.

First observe that \((\text{CIR}_0)\) holds, because it is the same as Lemma 3.1(2). Moreover, \((\text{CIR}_1)-(\text{CIR}_4)\) can be established straightforwardly from the usual forking calculus in simple theories (for further details we refer to [11, Example 4.2] where the stable case had been mentioned). For completeness, we prove \((\text{CIR}_2)\) for \( \downarrow^f \). So assume \( X \downarrow^f_{UZ} Y \) and \( U \downarrow^f_Z Y \). Then, using the usual calculus of forking in simple theories (presented, e.g., in [20]), we have

\[
X \downarrow^f_{UZ} Y \implies XUZ \downarrow^f_{UZ} Y
\]

and

\[
U \downarrow^f_Z Y \implies UZ \downarrow^f_Z Y.
\]
Combining the last two lines with the usual transitivity of $\downarrow^f$ one obtains

$$XUZ \downarrow^f Y,$$

whence $XU \downarrow^f Y$, as desired. The other part of $(CIR_2)$ can be established similarly.

Summing up, $\mathpzc{A}$ has a $CIR$ and hence (as we already mentioned) it follows from [11, Corollary 3.14] that $\text{Aut}(\mathpzc{A})$ has a cyclically dense conjugacy class, in particular, the topological rank of $\text{Aut}(\mathpzc{A})$ is at most 2. Moreover (as $\mathpzc{A}$ is assumed to be a model of a simple theory such that all types of $\mathpzc{A}$ over the empty set are stationary) by Lemma 3.1, $\mathpzc{A}$ has square Stone spaces. Then the statement of the present theorem follows from Corollary 2.12. \hfill $\square$

Let $\mathpzc{A}$ be a countable structure. In Theorem 3.2 we assume that all types of $\mathpzc{A}$ over the empty set are stationary. (⋆)

We briefly analyze this condition.

If $\mathpzc{A}$ is stable then all of its strong types are stationary, in particular, all types of $\mathpzc{A}^\text{eq}$ over the empty set are stationary. Hence, on the one hand, for stable structures (⋆) seems to be a rather mild requirement. On the other hand, stability theory seems to be the natural context of (⋆). Therefore it is adequate to ask if our methods can be applied for some simple, unstable structures. We close this work with Proposition 3.4 below which is analogous to Theorem 3.2 and applies for many examples listed at the end of Remark 2.8; these examples include simple unstable structures (e.g., the Rado graph) as well as non-simple structures (e.g., the Henson graphs).

Definition 3.3 Let $\mathpzc{A}$ be a structure and let $X, Y, Z \subseteq A$. Then $X$ and $Y$ are defined to be free over $Z$ iff $X \cap Y \subseteq Z$ and “there are no basic relations between $X - Z$ and $Y - Z$”, i.e., if $\vec{u} \in X - Z$, $\vec{v} \in Y - Z$ are nonempty sequences and $\psi$ is an atomic formula then $\mathpzc{A} \not\models \psi(\vec{u}, \vec{v})$.

Further, we define a ternary relation $\downarrow^\text{free}$ on finite subsets of $\mathpzc{A}$ as follows: for finite $X, Y, Z \subseteq A$

$$X \downarrow^\text{free}_Z Y \text{ holds iff } X \text{ and } Y \text{ are free over } Z.$$ 

Suppose $\mathcal{L}$ is a finite relational language containing at most binary relation symbols. Suppose $K$ is a Fraïssé class of finite $\mathcal{L}$-structures which is closed under free amalgams and let $\mathpzc{A}$ be the Fraïssé limit of $K$. It is easy to check that $\downarrow^\text{free}$ is a canonical independence relation on $\mathpzc{A}$ (i.e., $\downarrow^\text{free}$ satisfies $(CIR_0)$-$(CIR_4)$ recalled in the proof of Theorem 3.2; for $(CIR_2)$ we use that each relation symbol in $\mathcal{L}$ is at most binary). In particular, $\downarrow^\text{free}$ is a canonical independence relation on the Rado graph and on the Henson graphs.

Proposition 3.4 Suppose $\mathpzc{A}$ is a countable model of an $\aleph_0$-categorical theory which eliminates quantifiers. $\downarrow^\text{free}$ is a canonical independence relation on $\mathpzc{A}$. Then the following statements are equivalent.

1. There exists a weakly generic pair $(g_0, g_1) \in \text{Aut}(\mathpzc{A})^2$;
2. for all $n \in \omega$, $n \geq 1$ there exists a weakly generic tuple $(g_0, \ldots, g_{n-1}) \in \text{Aut}(\mathpzc{A})^n$.

Proof. We check that the conditions of Corollary 2.12 are satisfied. Clearly, each finite elementary mapping of $\mathpzc{A}$ can be extended to an automorphism of $\mathpzc{A}$.

Further, $\mathpzc{A}$ has square Stone spaces, because of the following. Let $X \subseteq A$ be finite and $p \in S^4(\emptyset)$ be arbitrary. Let $\overline{\mathpzc{T}}$ be any realization of $p$. By $(CIR_4)$ we have

$$X \downarrow^\text{free}_\emptyset \emptyset,$$

hence, by $(CIR_1)$, there exists a realization $\overline{\mathpzc{T}} \subseteq A$ of $p$ such that $X \downarrow^\text{free}_\emptyset \text{ ran}(\overline{\mathpzc{T}})$. This $\overline{\mathpzc{T}}$ satisfies the implication in Definition 2.7 because of $(CIR_3)$, $(CIR_0)$ and elimination of quantifiers.

Since by assumption, $\downarrow^\text{free}$ is a canonical independence relation on $\mathpzc{A}$, it follows from [11, Corollary 3.14] that the topological rank of $\text{Aut}(\mathpzc{A})$ is at most 2.

Thus, all the conditions of Corollary 2.12 are satisfied; one can apply this Corollary to complete the proof. \hfill $\square$

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