Neutrino and CP-even Higgs boson masses in a nonuniversal \( U(1)' \) extension

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Abstract

We propose a new anomaly-free and family nonuniversal \( U(1)' \) extension of the standard model with the addition of two scalar singlets and a new scalar doublet. The quark sector is extended by adding three exotic quark singlets, while the lepton sector includes two exotic charged lepton singlets, three right-handed neutrinos and three sterile Majorana leptons to obtain the fermionic mass spectrum of the standard model. The lepton sector also reproduces the elements of the PMNS matrix and the squared-mass differences data from neutrino oscillation experiments. Also, analytical relations of the PMNS matrix are derived via the inverse see-saw mechanism, and numerical predictions of the parameters in both normal and inverse order scheme for the mass of the phenomenological neutrinos are obtained. We employed a simple seesaw-like method to obtain analytical mass eigenstates of the CP-even \( 3 \times 3 \) mass matrix of the scalar sector.

1 Introduction

Despite all its success, the Standard Model (SM) of Glashow, Weinberg and Salam [1] has some unexplained features, which has motivated many models and extensions. In particular, the observed fermion mass hierarchies, their mixing and the three family structure are not explained in the SM. From the phenomenological point of view, it is possible to describe some features of the mass hierarchy by assuming zero-texture Yukawa matrices [2]. Models with spontaneously broken flavor symmetries may also produce hierarchical mass structures. For example, in models with gauge symmetry \( SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \), the electroweak doublets exhibit a discrete symmetry after the spontaneous symmetry breaking, obtaining Fritzsch zero-texture mass matrices [3] in the basis \( U = (u_0, c_0, t_0) \) of the form:

\[
- \langle \mathcal{L}_{Y,U} \rangle_0 = U_L \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix} U_R + \text{h.c.} \tag{1}
\]

The zero-texture of the above matrix can describe the mass spectrum in the quark sector and the CP violation phase observed in the experiments. This mass structure can also be obtained in the lepton sector, as shown by Fukugita, Tanimoto y Yanagida [4], where very small mass values are predicted

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through a seesaw mechanism. In addition, these type of models contain Majorana neutrinos which induce matter-antimatter asymmetry through leptogenesis.

Another issue that the SM can not explain is the observation of neutrino oscillations. These observations have been confirmed by many experiments from four different sources: solar neutrinos as in Homestake, SAGE, GALLEX & GNO, SNO, Borexino and Super-Kamiokande experiments, atmospheric neutrinos as in IceCube, neutrinos from reactors as KamLAND, and from accelerators as in MINOS, T2K and NOνA. The experimental data are compatible with the hypothesis that at least two species of neutrinos have mass, where the left-handed flavor neutrino fields are linear combinations of mass eigenstates

$$|\nu^\alpha_L\rangle = \sum_{i=1,2,3} U_{ai} |\nu^i_L\rangle, \quad a = e, \mu, \tau$$

where $$U$$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, which can be parameterized as function of three mixing angles and one CP violating phase. However, the experiments can not determine the true nature of the active neutrinos (Majorana or Dirac) nor the absolute values of their mass. Table 1 shows the parameters from references and available at NuFIT 3.0, where two hierarchies are assumed: normal ordering (NO), where the squared mass difference between the third and first species accomplish $$\Delta m^2_{31} > 0$$, and inverted ordering (IO), where $$\Delta m^2_{32} < 0$$ between the second and third species.

On the other hand, in order to obtain tiny neutrino masses, two method can be used: radiative corrections and see-saw mechanism. The latter scheme has been studied in the literature and is considered as one of the most traditional schemes for the explanation of smallness of neutrino masses. The see-saw mechanism implies the addition of a lepton number-violating high energy scale ($$M$$), which gives masses to light neutrinos as $$m_\nu = v^2/\sqrt{M}$$. There are some basic ways to implement this mechanism: a heavy right-handed Majorana neutrino $$\nu_R$$ mixed to the corresponding left-handed neutrino $$\nu_L$$ via the SM scalar doublet (type I seesaw), a heavy scalar triplet bosons (type II), or a heavy fermionic triplet (type III). Since the new scale $$M$$ associated with the new fields is high ($$\sim 10^{12}$$ GeV), this mechanism has the problem that is not accessible to be test in experiments. However, there is another possibility: the inverse see-saw mechanism (ISS), where a very light Majorana neutrino $$N_R$$ is incorporated, such that in the basis ($$\nu_L, \nu_R^c, N_R^c$$) the mass matrix has the form of Fristzsch zero-texture:

$$M_\nu = \begin{pmatrix}
0 & m_\nu^T & 0 \\
m_\nu & 0 & m_N^T \\
0 & m_N & M_N
\end{pmatrix}$$

where the submatrix $$m_N$$ has components of the order of the TeV scale, while $$M_N$$ is of the order of the KeV scale, in order to obtain active neutrinos at the sub-eV scale. The inverse see-saw mechanism was proposed in [26]. This mechanism has also been implemented in SU(3)$_L \otimes$ U(1)$_X$ models in order to study the $$\mu \to e\gamma$$ decay [27].

On the other hand, the discovery of the Higgs boson at ATLAS [28] and CMS [29] whose mass is 125 GeV opens the window to propose other scalar fields. A new scalar sector is considered as extension to the SM in order to explain some phenomenological aspects. One of the most studied SM extension is the two-Higgs-doublet-model (2HDM) which proposes the existence of two scalar doublets whose
scalar potential mixes them together obtaining two charged scalar bosons $H^\pm$, a CP-odd pseudoscalar $A^0$ and two CP-even scalar bosons $h$ and $H$ [30]. This model was motivated in order to give masses to up-like and down-like quarks [31] where vacuum expectation values (VEV) $v_2$ and $v_1$ are related to the electroweak VEV by $v^2 = v_2^2 + v_1^2$.

There are also extensions to the 2HDM adding a new scalar singlet $\chi$, as in the Next-to-Minimal 2HDM (N2HDM) [33]. In some cases, this additional singlet implement the spontaneous symmetry breaking (SSB) of an additional $U(1)'$ gauge symmetry through the acquisition of non-vanishing VEV $v_\chi$, and consequently its imaginary part become in the would-be Goldstone boson eaten by the corresponding gauge boson of $U(1)'$ [34]. Furthermore, if this SSB happens at a higher scale than the electroweak ($v \ll v_\chi$), the CP-even mass matrix exhibits an internal hierarchy which allows us to employ a perturbative see-saw-like method in order to obtain analytical expressions for the mass eigenvalues and angles of the corresponding mixing matrix.

Models with extra $U(1)'$ symmetry are one of the most studied extensions of the SM, which implies many phenomenological and theoretical advantages including flavor physics [35], neutrino physics [36], dark matter [37], among other effects [38]. A complete review of the above possibilities can be found in reference [39]. In particular, family non-universal $U(1)'$ symmetry models have many well-established motivations. For example, they provide hints for solving the SM flavor puzzle, where even though all the fermions acquire masses at the same scale, $v = 246$ GeV, experimentally they exhibit very different mass values. These models also imply a new $Z'$ neutral boson, which contains a large number of phenomenological consequences at low and high energies [40]. In addition to the new neutral gauge boson $Z'$, an extended fermion spectrum is necessary in order to obtain an anomaly-free theory. Also, the new symmetry requires an extended scalar sector in order to (i) generate the breaking of the new Abelian symmetry and (ii) obtain heavy masses for the new $Z'$ gauge boson and the extra fermion content. A non-universal $U(1)'$ model in the quark sector was proposed in [34], obtaining zero-texture quark mass matrices with hierarchical structures, where three quarks [up, down and strange] acquire masses at the MeV scale, and three quarks [charm, bottom and top] exhibit masses at the GeV scale. Additional phenomenological consequences of this model were studied in [41, 42, 43] including effects on scalar DM.

The main purpose of this paper is to construct an anomaly-free and family non-universal $U(1)'$ symmetry model in both the quark and leptonic sector, with extra lepton and quark singlets, two scalar doublets, and two scalar singlets. The leptonic sector includes new charged and right-handed neutral leptons, and sterile Majorana neutrinos in order to reproduce the PMNS matrix and the observed mass structure of the leptons. In Sec. 2 we describe the spectrum and most important properties of the model. We also show the scalar and gauge Lagrangians, including rotations into mass eigenvectors. In Sec. 3 we show how mass structures in the fermion sector are predicted in the model, first for the quark sector in subsection 3.1 and later for the leptonic sector in subsection 3.2. Sec. 4 is devoted to obtain some phenomenological parameters from neutrino oscillation data at 1 $\sigma$. Finally, the Sec. 5 outlines the main results of the article.

2 Non universal model with extra $U(1)_X$ symmetry

The model proposes the existence of a new non-universal gauge group $U(1)'$ whose gauge boson and coupling constant are $Z'_\mu$ and $g_X$, respectively. It brings the following triangle anomaly equations:
Leptons \(\ell\)

Non-SM Leptons

where the sums in \(Q\) outlined below:

corresponding \(Z\)

where column is composed by ordinary SM particles but also new exotic non-SM particles, as shown in table 2,

is the corresponding weak hypercharge. The fermion content compatible with the above conditions

2. In order to ensure cancellation of the gauge chiral anomalies, the model includes extra isospin

both left- and right-handed sectors:

1 has \(X\) both left- and right-handed neutrinos \(\nu\)

Table 2: Non-universal \(X\) quantum number and \(Z_2\) parity for SM and non-SM fermions.

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
\text{Quarks} & X & \text{Z}_2 & \text{Leptons} & X & \text{Z}_2 \\
\hline
\text{SM Fermionic Isospin Doublets} & & & & & \text{SM Fermionic Isospin Singlets} \\
\hline
q_L^1 = \left( \begin{array}{c} U^1_L \\ D^1_L \end{array} \right) & +1/3 & + & \ell_L^e = \left( \begin{array}{c} \nu^e \\ e^e \end{array} \right)_L & 0 & + \\
q_L^2 = \left( \begin{array}{c} U^2_L \\ D^2_L \end{array} \right) & 0 & - & \ell_L^\mu = \left( \begin{array}{c} \nu^\mu \\ e^\mu \end{array} \right)_L & 0 & + \\
q_L^3 = \left( \begin{array}{c} U^3_L \\ D^3_L \end{array} \right) & 0 & + & \ell_L^\tau = \left( \begin{array}{c} \nu^\tau \\ e^\tau \end{array} \right)_L & -1 & + \\
\hline
\text{Non-SM Quarks} & & & & & \text{Non-SM Leptons} \\
\hline
U_R^{1,3} & +2/3 & + & e_R^\tau & -4/3 & - \\
U_R^{2} & +2/3 & - & e_R^\mu & -1/3 & - \\
D_R^{1,2,3} & -1/3 & - & N_R^{\mu,\tau} & 0 & - \\
\hline
\end{array}
\]

Table 2: Non-universal \(X\) quantum number and \(Z_2\) parity for SM and non-SM fermions.

where column \(X\) contains the quantum numbers of the extra U(1) \(X\) and the \(Z_2\) column presents their corresponding \(Z_2\)-parity under a new \(Z_2\) discrete symmetry. Some properties of this spectrum are outlined below:

1. The U(1) \(X\) symmetry is only non-universal in the left-handed SM quark sector: the first family

1 has \(X = 1/3\) while the last two 2, 3 have \(X = 0\). Leptons exhibit non-universal charges in both left- and right-handed sectors: \(X = 0\) for the left-handed components \(e, \mu\) and \(X = -1\) for \(\tau\), while for the right-handed components \(X = -4/3\) for \(e, \tau\) and \(X = -1/3\) for \(\mu\). We use the following assignment for the phenomenological families:

\[
U^{1,2,3} = (u, c, t), \quad D^{1,2,3} = (d, s, b), \quad e^{e,\mu,\tau} = (e, \mu, \tau), \quad \nu^{e,\mu,\tau} = (\nu^e, \nu^\mu, \nu^\tau).
\]

2. In order to ensure cancellation of the gauge chiral anomalies, the model includes extra isospin

singlets. The quark sector has an up \(T\) and two down \(J^{1,2}\) quarks. For the lepton sector, three

right-handed neutrinos \(\nu_R^{e,\mu,\tau}\) and two charged leptons \(E\) and \(\mathcal{E}\) are added with non-trivial U(1) \(X\) charges, as shown in Tab. 2.
Scalar bosons

Higgs Doublets

\[ \phi_1 = \left( \begin{array}{c} \frac{h_1 + v_1 + i\eta_1}{\sqrt{2}} \\ \frac{\phi_2^+}{\sqrt{2}} \end{array} \right) \]

\[ \phi_2 = \left( \begin{array}{c} \frac{h_2 + v_2 + i\eta_2}{\sqrt{2}} \end{array} \right) \]

Higgs Singlets

\[ \chi = \left( \begin{array}{c} \frac{\xi_\chi + v_\chi + i\eta_\chi}{\sqrt{2}} \\ \frac{\sigma}{\sqrt{2}} \end{array} \right) \]

| Scalar | X | Z_2 |
|--------|---|------|
| \( \phi_1 \) | 2/3 | + |
| \( \phi_2 \) | 1/3 | - |
| \( \chi \) | -1/3 | + |
| \( \sigma \) | -1/3 | - |

Table 3: Non-universal X quantum number for Higgs fields.

3. The most natural way to obtain massive neutrinos, according to neutrino oscillations, is through a see-saw mechanism, which requires the introduction of extra Majorana neutrinos. Thus, for obtaining a realistic model compatible with massive neutrinos, three sterile Majorana neutrinos \( N_{R^{\mu \tau}} \) are included.

The scalar sector of the model is shown in table 3, which exhibits the following properties:

1. Two scalar doublets \( \phi_{1,2} \) are included with \( U(1)_X \) charges +2/3 and +1/3 respectively, whose vacuum expectation values (VEVs) are related to the electroweak VEV by \( v = \sqrt{v_1^2 + v_2^2} \). The internal \( Z_2 \) symmetry is introduced in order to obtain adequate zero texture matrices.

2. An extra scalar singlet \( \chi \) with VEV \( v_\chi \) is required for the SSB of \( U(1)_X \) and also to generate masses to exotic isospin singlets. We assume that it happens at a larger scale \( v_\chi \gg v \) than electroweak.

3. Another scalar singlet \( \sigma \) is introduced. Since it is not essential for the symmetry breaking mechanisms, we may choose \( v_\sigma = 0 \) for its VEV.

Finally, in the vector sector, an extra gauge boson \( Z_{\mu}^{\prime} \) is required to obtain a local \( U(1)_X \) symmetry. The covariant derivative of the model is

\[ D_{\mu} = \partial_{\mu} - igW_{\mu}^a T^a - igY_2 B_{\mu} - igXZ_{\mu}, \]

where \( 2T^a \) corresponds to the Pauli matrices for isospin doublets and \( T^a = 0 \) for isospin singlets. The electric charge is defined by the Gell-Mann-Nishijima relation:

\[ Q = T_3 + \frac{Y_2}{2}. \]

2.1 Scalar masses

The scalar potential of the model is

\[ V = \mu_1^2 \phi_1^+ \phi_1 + \mu_2^2 \phi_2^+ \phi_2 + \mu_\chi^2 \chi^* \chi + \mu_\sigma^2 \sigma^* \sigma \\
+ \frac{f}{\sqrt{2}} \left( \phi_1^+ \phi_2 \chi^* + \text{h.c.} \right) + \frac{f'}{\sqrt{2}} \left( \phi_1^+ \phi_2 \sigma^* + \text{h.c.} \right) \\
+ \lambda_1 \left( \phi_1^+ \phi_1 \right)^2 + \lambda_2 \left( \phi_2^+ \phi_2 \right)^2 + \lambda_3 (\chi^* \chi)^2 + \lambda_4 (\sigma^* \sigma)^2 \\
+ \lambda_5 \left( \phi_1^+ \phi_1 \right) \left( \phi_2^+ \phi_2 \right) + \lambda_6 \left( \phi_1^+ \phi_2 \right) \\
+ \left[ \phi_1^+ \phi_1 \right] \left[ \lambda_6 (\chi^* \chi) + \lambda_6' (\sigma^* \sigma) \right] \\
+ \left[ \phi_2^+ \phi_2 \right] \left[ \lambda_7 (\chi^* \chi) + \lambda_7' (\sigma^* \sigma) \right] \\
+ \lambda_8 (\chi^* \chi) (\sigma^* \sigma) + \lambda_9 \left[ (\chi^* \chi) (\sigma^* \sigma) + \text{h.c.} \right]. \]
After symmetry breaking the mass matrices for the scalar sector are found. For the charged scalar bosons the mass matrix is obtained in the basis \((\phi_1^\pm, \phi_2^\pm)\)

\[
M_C^2 = \frac{1}{4} \begin{pmatrix}
-f \frac{v_1 v_2}{v_1} - \lambda_5' v_2^2 & f v_1 + \lambda_5' v_1 v_2 \\
-f \frac{v_1 v_1}{v_2} + \lambda_5' v_1 v_2 & -f \frac{v_1 v_1}{v_2} - \lambda_5' v_1^2
\end{pmatrix}
\]

which is diagonalized by

\[
R_C = \begin{pmatrix}
c_\beta & s_\beta \\
-s_\beta & c_\beta
\end{pmatrix},
\]

where \(\tan \beta = s_\beta / c_\beta = v_1 / v_2\). The mass matrix has the eigenvalues

\[
m_{G^\pm}^2 = 0,
\]
\[
m_{H^\pm}^2 = -\frac{1}{4} f v_1 \frac{v_1 v_1}{v_2} - \frac{1}{4} \lambda_5' v_2^2,
\]

yielding the Goldstone bosons \(G_{W}^\pm\), which provide the mass to the physical \(W_\mu^\pm\) gauge bosons, and two physical charged Higgs bosons \(H^\pm\).

Regarding to the neutral scalar sector, the mass matrix of the CP-odd sector in the basis \((\eta_1, \eta_2, \zeta_\chi)\) is:

\[
M_I^2 = -\frac{f}{4} \begin{pmatrix}
v_2 v_1 & -v_1 & v_2 \\
v_1 & v_1 v_1 & -v_1 \\
v_2 & -v_1 & v_1 v_1
\end{pmatrix},
\]

which can be diagonalized by the following transformation

\[
R_I = \begin{pmatrix}
c_\beta & s_\beta & 0 \\
-s_\beta & c_\beta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \(\gamma\) describes the doublet-singlet mixing \(\tan \gamma = s_\gamma / c_\gamma = v_\chi / v_\beta c_\beta\). When \(R_I\) acts on \(M_I^2\) the following eigenvalues are obtained

\[
m_{G_2}^2 = 0,
\]
\[
m_{G_{2'}}^2 = 0,
\]
\[
m_{A_0}^2 = -\frac{1}{4} f v_1 \frac{v_1 v_1}{v_2} s_\beta c_\gamma^2,
\]

where the first two are the would-be Goldstone bosons of the neutral vector bosons \(Z_\mu\) and \(Z'_{\mu}\), respectively, while the latter is a physical CP-odd pseudoscalar boson \(A_0\).

On the other hand, the CP-even scalar mass matrix is

\[
M_R^2 = \begin{pmatrix}
\lambda_1 v_1^2 & \frac{1}{4} f v_1 v_2 & \lambda_5 v_1 v_2 + \frac{1}{4} f v_1 \\
\lambda_5 v_1 v_2 & \lambda_2 v_2^2 & \frac{1}{4} \lambda_6 v_1 v_1 + \frac{1}{4} f v_1 \\
\frac{1}{4} \lambda_6 v_1 v_1 + \frac{1}{4} f v_2 & \lambda_7 v_1 v_1 + \frac{1}{4} f v_1 & \lambda_3 v_2^2 + \frac{1}{4} f v_1 v_2
\end{pmatrix},
\]

where \(\lambda_5 = (\lambda_5 + \lambda_5') / 2\). Since this matrix exhibits a characteristic third order polynomial with non-trivial eigenvalues, it is convenient to use another approximation in order to obtain the eigenvalues and mixing angles. We propose a seesaw-like mechanism by assuming a hierarchy of VEVs through
the condition \(|fv_x, v_x^2 \gg v^2|\) in the matrix elements. Thus, the matrix (20) can be written in blocks as

\[
M_R^2 = \begin{pmatrix}
M_1 & M_{12}^T \\
M_{12} & M_2
\end{pmatrix},
\]

where

\[
M_1 = \begin{pmatrix}
\lambda_1 v_1^2 - \frac{1}{4} f v_x v_2 \\
\lambda_5 v_1 v_2 + \frac{1}{4} f v_x
\end{pmatrix},
\]

\[
M_{12}^T = \begin{pmatrix}
\frac{\lambda_5 v_1 v_2}{4} + \frac{f v_2}{v_1} \\
\frac{\lambda_7 v_2 v_1}{4} + \frac{f v_1}{v_2}
\end{pmatrix},
\]

\[
M_2 = \lambda_3 v_x^2 - \frac{1}{4} f v_1 v_2 v_x \approx \lambda_3 v_x^2.
\]

According to the block diagonalization procedure shown in Appendix A, the mass matrix (21) can be decoupled into two independent blocks through a unitary transformation as:

\[
R_T M_R^2 R_S = \begin{pmatrix}
M_{2H} & 0 \\
0 & m_{R_x}^2
\end{pmatrix},
\]

where the transformation matrix can be approximately written as

\[
R_S = \begin{pmatrix}
1 & F_R^T \\
- F_R & 1
\end{pmatrix},
\]

with

\[
F_R \approx -M_2^{-1} M_{12}, \\
m_{R_x}^2 \approx M_2 = \lambda_3 v_x^2, \\
M_{2H} \approx M_1 - M_{12}^T M_2^{-1} M_{12}.
\]

and

\[
M_{2H}^2 = \begin{pmatrix}
\tilde{\lambda}_1 v_x^2 s_\beta^2 - \frac{1}{4} f v_x \\
\tilde{\lambda}_5 v_x^2 c_\beta^2 + \frac{1}{4} f v_x
\end{pmatrix},
\]

where the new tilde constants are

\[
\tilde{\lambda}_1 = \lambda_1 - \frac{\lambda_6^2}{4 \lambda_3} - \frac{\lambda_7^2}{4 \lambda_3 s_\beta^2},
\]

\[
\tilde{\lambda}_2 = \lambda_2 - \frac{\lambda_6^2 t_\beta^2}{4 \lambda_3} - \frac{\lambda_7^2}{4 \lambda_3},
\]

\[
\tilde{\lambda}_5 = \tilde{\lambda}_5 - \frac{\lambda_6^2 t_\beta^2}{2 \lambda_3} - \frac{\lambda_7^2}{2 \lambda_3 s_\beta^2}.
\]

In order to obtain the largest eigenvalue of \(M_{2H}^2\), we neglect non-dominant terms from the condition that \(fv_x \gg v_2^2, v_1^2, v_2 v_1\), which leads us to

\[
M_{2H}^2 \approx - \frac{1}{4} f v_x \begin{pmatrix}
\cot \beta & -1 \\
-1 & \tan \beta
\end{pmatrix}.
\]
Due to this approximation, the new matrix has null determinant and its trace is of the order of the largest eigenvalue

$$m^2_H \approx \text{Tr} \left[ M^2_{HH} \right] \approx -\frac{1}{4} \frac{f v_x}{s_\beta c_\beta},$$

(29)

The lightest mass eigenvalue can be calculated through the ratio of the determinant and the trace of (26), i.e.

$$\frac{\text{Det} \left[ M^2_{HH} \right]}{\text{Tr} \left[ M^2_{HH} \right]} = \frac{m^2_h m^2_H}{m^2_h + m^2_H} \approx m^2_h,$$

(30)

obtaining

$$m^2_h \approx \left( \tilde{\lambda}_1 s_\beta^2 + 2 \tilde{\lambda}_5 c_\beta s_\beta + \tilde{\lambda}_2 s_\beta^2 \right) v^2,$$

(31)

which we associate to the observed 125 GeV Higgs boson. The mixing angle associated to (26) is defined as

$$t_{2\alpha} = \frac{f v_x + 2 \tilde{\lambda}_5 c_\beta s_\beta v^2}{f v_x + 2 t_{2\beta} (s_\beta^2 \tilde{\lambda}_1 - c_\beta^2 \tilde{\lambda}_2) v^2} t_{2\beta},$$

(32)

Finally, the diagonalization of the CP-even matrix (20) is achieved by $R_R$ parametrized by a CKM-like matrix

$$R_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_\alpha & s_\alpha & 0 \\ -s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(33)

where $t_\alpha = s_\alpha/c_\alpha$ and

$$s_{13} = \frac{1}{2} \frac{\lambda_6 v s_\beta}{\lambda_3 v_x}, \quad s_{23} = \frac{1}{2} \frac{\lambda_7 v c_\beta}{\lambda_3 v_x},$$

(34)

whose corresponding cosines are approximated as $c_{13} \approx 1 - s_{13}^2/2$ and $c_{23} \approx 1 - s_{23}^2/2$. In fact, the $R_S$ matrix which block-diagonalizes $M^2_R$ is the product of the former two rotation matrices with mixing angles $\theta_{23}$ and $\theta_{13}$. In conclusion, the scalar spectrum of the model is:

- Four would-be Goldstone bosons: $G^\pm_W, G^0_Z, G^0_{Z'}$.
- Three scalar CP-even $h, H, H_\chi$ fields with mass

$$m^2_h \approx \left( \tilde{\lambda}_1 c_\beta^4 + 2 \tilde{\lambda}_5 c_\beta^2 s_\beta + \tilde{\lambda}_2 s_\beta^2 \right) v^2,$$

$$m^2_H \approx -\frac{f v_x}{4 s_\beta c_\beta},$$

$$m^2_{H_\chi} \approx \lambda_3 v_\chi^2,$$

(35)

- A pseudoscalar CP-odd $A^0$ whose mass is

$$m^2_{A^0} = -\frac{1}{4} \frac{f v_x}{s_\beta c_\beta s_\gamma^2},$$

(36)

- Two charged scalar bosons $H^\pm$ with mass

$$m^2_{H^\pm} = -\frac{1}{4} \frac{f v_x}{s_\beta c_\beta s_\gamma^2} - \frac{1}{4} \tilde{\lambda}_5^\prime v^2.$$  

(37)
2.2 Gauge boson masses

The kinetic terms of the scalar fields are

\[ \mathcal{L}_{\text{kin}} = \sum_i (D_{\mu} S)^\dagger (D^{\mu} S). \]  

(38)

After the symmetry breaking, the charged bosons \( W_{\mu}^\pm = (W_{\mu}^1 \mp W_{\mu}^2)/\sqrt{2} \) acquire masses \( M_W = g v/2 \), while the masses for neutral gauge bosons are obtained from the following squared mass matrix in the basis \((W_{\mu}^3, B_{\mu}, Z_{\mu}')\):

\[
M_0^2 = \frac{1}{4} \begin{pmatrix}
  g^2 v^2 & -g' v^2 & -\frac{2}{3} g g_X v^2 (1 + 2 c_\beta^2) \\
  * & g'^2 v^2 & \frac{2}{3} g' g_X v^2 (1 + 2 c_\beta^2) \\
  * & * & \frac{4}{9} g_X^2 v^2 \left[ 1 + (1 + 3 c_\beta^2) c^2 \right] 
\end{pmatrix},
\]

(39)

where \( \epsilon = v/\nu_X \). Taking into account \( c^2 \ll 1 \), the matrix can be diagonalized with only two angles, obtaining the following mass eigenstates:

\[
\begin{pmatrix}
  A_\mu \\
  Z_{1\mu} \\
  Z_{2\mu}
\end{pmatrix} \approx R_0 \begin{pmatrix}
  W_{\mu}^3 \\
  B_{\mu} \\
  Z_{\mu}'
\end{pmatrix},
\]

(40)

with:

\[
R_0 = \begin{pmatrix}
  s_W & c_W & 0 \\
  c_W c_Z & -s_W c_Z & s_Z \\
  -c_W s_Z & s_W s_Z & c_Z
\end{pmatrix},
\]

(41)

where \( \tan \theta_W = s_W/c_W = g'/g \) defines the Weinberg angle, and \( s_Z = \sin \theta_Z \) is a small mixing angle between the SM neutral gauge boson \( Z \) and the \( U(1)_X \) gauge boson \( Z' \) such that in the limit \( s_Z \to 0 \), \( Z_1 = Z \) and \( Z_2 = Z' \). This mixing angle is approximately

\[
s_Z \approx (1 + c_\beta^2) \frac{2 g_X c_W}{3 g} \left( \frac{M_Z}{M_{Z'}} \right)^2,
\]

(42)

where the neutral masses are:

\[
M_Z \approx \frac{g v}{2 c_W}, \quad M_{Z'} \approx \frac{g_X v}{3}.
\]

(43)

3 Fermion masses

3.1 Quark sector

We find the Yukawa Lagrangian compatible with the \( SU(2)_L \otimes U(1)_Y \otimes U(1)_X \) gauge symmetry. For the quark sector we obtain

\[
-\mathcal{L}_Q = \overline{q}_L \left( \tilde{\phi}_2 h_2^L \right)_{12} U_R^{ij} + \overline{q}_L \left( \tilde{\phi}_1 h_1^L \right)_{aj} U_R^{ij} + \overline{q}_L \left( \phi_1 h_1^P \right)_{1j} D_R^{ij} + \overline{q}_L \left( \phi_2 h_2^P \right)_{aj} D_R^{ij}
\]

\[
+ \overline{q}_L \left( \phi_1 h_1^L \right)_{1m} J_{1R}^m + \overline{q}_L \left( \phi_2 h_2^L \right)_{am} J_{1R}^m + \overline{q}_L \left( \phi_2 h_2^L \right)_{1j} T_R^j + \overline{q}_L \left( \phi_1 h_1^L \right)_{aj} T_R^j
\]

\[
+ \overline{q}_L \left( \sigma h_2^L \right)_{aj} U_R^{ij} + \overline{q}_L \left( \sigma h_1^L \right)_{am} J_{1R}^m + \overline{q}_L \left( \sigma h_1^L \right)_{aj} J_{1R}^m + \overline{q}_L \left( \sigma h_2^L \right)_{aj} J_{1R}^m + h.c.,
\]

(44)

where \( \tilde{\phi}_{1,2} = i \sigma_2 \phi_{1,2}^* \) are conjugate fields, \( a = 2, 3 \) label the second and third quark doublets and \( n(m) = 1, 2 \) is the index of the exotic \( J^{n(m)} \) quarks. A sum over the indices \( i, a \) and \( n \) is understood. We can see in the quark Lagrangian that due to the non-universality of the \( U(1)_X \) symmetry, not all
couplings between quarks and scalars are allowed by the gauge symmetry, which leads us to specific zero-texture Yukawa matrices. However, these structures are not inherited by the mass matrices of the quarks, due to the interactions of the four scalar fields $\phi_1$, $\phi_2$, $\sigma_0$ and $\chi_0$ that couple simultaneously to all quark flavors. In order to reproduce the observed mass spectrum, we must restrict further the number of couplings in the Lagrangian, which can be done by assuming the $Z_2$ discrete symmetries shown in tables 2 and 3. Assuming these discrete symmetries, the Lagrangian (44) after the symmetry breaking leads us to the following mass terms at tree level:

$$-(\mathcal{L}_Q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & v_2 a_{12} & 0 \\ v_1 a_{31} & 0 & v_1 a_{33} \end{pmatrix}, \quad M_D = \frac{v_2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the mass matrices generate the following zero structures:

$$M'_{U} = \left( \begin{array}{c|c} M_U & M_{UT} \\ \hline M_{TU} & M_T \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & v_2 a_{12} & 0 & v_2 y_1 \\ v_1 a_{31} & 0 & v_1 a_{33} & 0 \\ v_1 a_{22} & 0 & 0 & v_1 y_2 \\ 0 & 0 & v_1 y_2 & 0 \end{pmatrix},$$

$$M'_{D} = \left( \begin{array}{c|c} M_D & M_{DJ} \\ \hline M_{JD} & M_J \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & v_1 j_{11} & v_1 j_{12} \\ 0 & 0 & 0 & v_2 j_{21} & v_2 j_{22} \\ v_2 B_{31} & v_2 B_{32} & v_2 B_{33} & 0 & 0 \\ 0 & 0 & 0 & v_3 k_{11} & v_3 k_{12} \\ 0 & 0 & 0 & v_3 k_{21} & v_3 k_{22} \end{pmatrix}.$$

After diagonalization, the above structures lead us to hierarchies of the phenomenological quarks, as detailed below.

**Up sector**

First, we consider the up-type matrix $M'_{U}$ in equation (47). We obtain its symmetrical quadratic form as:

$$M_{U}^{2} = (M'_{U})^T = \frac{1}{2} \begin{pmatrix} v_2^2 \left( a_{12}^2 + y_1^2 \right) & v_1 v_2 \left( a_{12} a_{22} + y_1 y_2 \right) & 0 & v_2 v_\chi \left( a_{12} c_2 + y_1 h_\chi^T \right) \\ v_1 v_2 \left( a_{12} a_{22} + y_1 y_2 \right) & v_1^2 \left( a_{22}^2 + y_2^2 \right) & 0 & v_1 v_\chi \left( a_{22} c_2 + y_2 h_\chi^T \right) \\ 0 & 0 & v_1^2 \left( a_{31}^2 + a_{33}^2 \right) & 0 \\ v_2 v_\chi \left( a_{12} c_2 + y_1 h_\chi^T \right) & v_1 v_\chi \left( a_{22} c_2 + y_2 h_\chi^T \right) & 0 & v_2^2 \left( c_2^2 + h_\chi^T h_\chi^T \right) \end{pmatrix}. \tag{48}$$
The above mass matrix can be written as
\[ M_U^2 = \begin{pmatrix} A & C \\ C^T & D \end{pmatrix}, \quad (49) \]
which has the same structure as the general form of equation \([129]\) in the Appendix \(A\), where each block is:

\[
A = \frac{1}{2} \begin{pmatrix} v_2^2 (a_{11}^2 + y_1^2) & v_1 v_2 (a_{12} a_{22}^* + y_1 y_2) & 0 \\ v_1 v_2 (a_{12} a_{22}^* + y_1 y_2) & v_1^2 (a_{22}^2 + y_2^2) & 0 \\ 0 & 0 & v_1^2 (a_{31}^2 + a_{33}^2) \end{pmatrix},
\]

\[
C = \frac{1}{2} \begin{pmatrix} v_2 v_\chi (a_{12} c_2 + y_1 h_\chi^T) \\ v_1 v_\chi (a_{22} c_2 + y_2 h_\chi^T) \\ 0 \end{pmatrix},
\]

\[
D = \frac{1}{2} v_\chi^2 \left( c_\chi^2 + h_\chi^T h_\chi \right). \quad (50)
\]

We can see that each block are of the order \(A \sim v_2^2, C \sim v_{1,2} v_\chi\), and \(D \sim v_\chi^2\), respectively, obeying the hierarchy from equation \([130]\). Thus, according to Appendix \(A\), the mass matrix \((49)\) can be block diagonalized as

\[ m_{U}^2 = \left(V_L^{(U)} \right)^T M_U^2 V_L^{(U)} = \begin{pmatrix} m_U^2 & 0 \\ 0 & m_T^2 \end{pmatrix}, \quad (51) \]

where:

\[ m_U^2 \approx A - CD^{-1} C^T, \]

\[ m_T^2 \approx D, \quad (52) \]

and the rotation matrix has the approximated form:

\[ V_L^{(U)} \approx \begin{pmatrix} I & F_U \\ -F_U^T & I \end{pmatrix}, \quad F_U \approx CD^{-1}. \quad (53) \]

Since the block \(D\) is just a number (see equation \([50]\)), from \([52]\) we obtain directly the mass of the heavy \(T\) quark:

\[ m_T^2 \approx \frac{1}{2} v_\chi^2 \left( c_\chi^2 + h_\chi^T h_\chi \right). \quad (54) \]

On the other hand, from the matrices in \([50]\), and after some algebra, the matrix \(m_U^2\) in \([52]\), which contains the SM sector, can be put into the form:

\[ m_U^2 \approx \frac{1}{2} \begin{pmatrix} v_2^2 r_1^2 & v_1 v_2 r_1 r_2 & 0 \\ v_1 v_2 r_1 r_2 & v_1^2 r_2^2 & 0 \\ 0 & 0 & v_1^2 (a_{31}^2 + a_{33}^2) \end{pmatrix}, \quad (55) \]

where:

\[ r_1 = \frac{(a_{12} h_\chi^T - y_1 c_2)}{\sqrt{c_\chi^2 + h_\chi^T h_\chi}}, \]

\[ r_2 = \frac{(a_{22} h_\chi^T - y_2 c_2)}{\sqrt{c_\chi^2 + h_\chi^T h_\chi}}. \quad (56) \]
We see that the 33 component of (55) appears decoupled, which corresponds to one of the eigenvalues. We associate this component to the top quark:

\[ m_t^2 = \frac{1}{2} v_1^2 (a_{33}^2 + a_{33}^2), \]  

(57)

which leaves us with the 2 \times 2 submatrix

\[ m_{uc}^2 \approx \frac{1}{2} \begin{pmatrix} v_1^2 r_1^2 & v_1 v_2 r_1 r_2 \\ v_1 v_2 r_1 r_2 & v_1^2 r_2^2 \end{pmatrix}. \]

(58)

It is evident that the above matrix has null determinant, which leads us to at least one null eigenvalue. In fact, this structure produces one massless quark, which we associate to the lightest quark: the up quark \((u)\), while the other eigenvalue, associated to the charm quark, corresponds to the trace of the matrix:

\[ m_c = \text{Tr}[m_{uc}] = \frac{1}{2} (v_1^2 r_1^2 + v_2^2 r_2^2) \approx \frac{1}{2} v_1^2 r_2^2, \]

(59)

Since the mass of the top quark in (57) depends only on \(v_1\), we take \(v_2 \ll v_1\), which leads us to the approximation in equation (59).

In order to generate mass to the \(u\) quark, we consider the one-loop radiative correction shown in figure 1-(a). This contribution add an input into the 11 component in the original 4 \times 4 matrix \(M_U^2\) in (47), which produces the one-loop quadratic mass matrix

\[ M_U^2_{(1)} = M_U^2 + \Delta M_U^2, \]

(60)

where the small one-loop contribution is:

\[ \Delta M_U^2 = \frac{1}{2} \begin{pmatrix} v_1^2 \Sigma_{11} & 0 & v_1^2 a_{31} \Sigma_{11} & 0 \\ 0 & 0 & 0 & 0 \\ v_1^2 a_{31} \Sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

(61)

and \(\Sigma_{11}\) the value of the diagram in figure 1-(a) which obey the following analytical expression:

\[ \Sigma_{11} = -\frac{1}{16\pi^2} \frac{f'}{(h_\sigma)^2 (h_\sigma)^2} C_0 \left( \begin{array}{c} M_2 \\ M_T \\ M_T \end{array} \right), \]

(62)

where:

\[ C_0 (x_1, x_2) = \frac{1}{(1 - x_1^2) (1 - x_2^2) (x_1^2 - x_2^2)} \left[ x_1^2 x_2^2 \ln \left( \frac{x_1^2}{x_2^2} \right) - x_1^2 \ln x_1^2 + x_2^2 \ln x_2^2 \right], \]

(63)

and \(M_2\) is a characteristic mass arised from the internal \(\phi_2\) line as linear combinations of mass eigenvalues. The new one-loop contribution only has effect on the 3 \times 3 block matrix \(m_U^2\) in (55), which change into the one-loop mass matrix

\[ m_{U(1-loop)}^2 \approx \frac{1}{2} \begin{pmatrix} v_1^2 r_1^2 + v_1^2 \Sigma_{11} & v_1 v_2 r_1 r_2 & v_1^2 a_{31} \Sigma_{11} \\ v_1 v_2 r_1 r_2 & v_1^2 r_2^2 & 0 \\ v_1^2 a_{31} \Sigma_{11} & 0 & 2 m_t^2 \end{pmatrix}, \]

(64)

where \(m_t\) is the top mass at tree level obtained in (57). The new 13 component emerged from the 1 loop diagram will correct the top mass. However, we will neglect this correction, which leads us again to a 2 \times 2 matrix

\[ m_{uc(1-loop)}^2 \approx \frac{1}{2} \begin{pmatrix} v_1^2 r_1^2 + v_1^2 \Sigma_{11} & v_1 v_2 r_1 r_2 \\ v_1 v_2 r_1 r_2 & v_1^2 r_2^2 \end{pmatrix}, \]

(65)
which exhibits determinant different from zero. The trace of the matrix corresponds to the sum of the eigenvalues, i.e.:

$$
\text{Tr}[m_{u,c(1\text{-loop})}^2] = m_u^2 + m_c^2 = \frac{1}{2} \left( v_1^2 r_1^2 + v_2^2 r_1^2 \right) + \frac{1}{2} v_1^2 v_2^2 
$$  \hspace{1cm} (66)

If we approximate the mass of the charm quark according to (59), we obtain for the quark $u$ that:

$$
m_u^2 = \frac{1}{2} v_1^2 \Sigma_{11}^2 
$$  \hspace{1cm} (67)

**Down sector**

For the down-type matrix $M'_D$ in (47), for simplicity we take in the heavy sector, proportional to $v_\chi$, a diagonal form, i.e., $k_{ij} = 0$ for $i \neq j$. In this scenery, its quadratic form can also be put in the block form

$$
M_D^2 = \begin{pmatrix} A & C \\ C^T & D \end{pmatrix},
$$  \hspace{1cm} (68)

where:

$$
A = \frac{1}{2} \begin{pmatrix}
  v_1^2 (j_1^1 + j_2^1) & v_1 v_2 (j_{11} j_{21} + j_{12} j_{22}) & 0 \\
  v_1 v_2 (j_{11} j_{21} + j_{12} j_{22}) & v_2^2 (j_1^2 + j_2^2) & 0 \\
  0 & 0 & v_2^2 (B_{31} + B_{32} + B_{33})
\end{pmatrix},
$$

$$
C = \frac{1}{2} \begin{pmatrix}
  v_1 v_\chi j_{11} k_{11} & v_1 v_\chi j_{12} k_{22} \\
  v_2 v_\chi j_{21} k_{11} & v_2 v_\chi j_{22} k_{22}
\end{pmatrix},
$$

$$
D = \frac{v_\chi^2}{2} \begin{pmatrix} k_{11}^2 & 0 \\ 0 & k_{22}^2 
\end{pmatrix}.
$$  \hspace{1cm} (69)

After block diagonalization, the matrix become:

$$
m_D^2 = \begin{pmatrix} V_L^{(D)} \end{pmatrix}^T M_D^2 V_L^{(D)} = \begin{pmatrix} m_D^2 & 0 \\ 0 & m_J^2 \end{pmatrix},
$$  \hspace{1cm} (70)

where:

$$
m_D^2 \approx A - C D^{-1} C^T,
$$

$$
m_J^2 \approx D,
$$  \hspace{1cm} (71)
where the one-loop correction is:

\[ V_{L(D)}^{(1)} \approx \begin{pmatrix} I & F_D \\ -F_D^T & I \end{pmatrix}, \quad F_D \approx CD^{-1}. \] (72)

First, since the matrix \( D \) appears diagonal, we obtain directly the mass of the heavy down-type quarks:

\[ m^2_{j_1} = \frac{1}{2} v_\chi^2 k_{11}, \quad m^2_{j_2} = \frac{1}{2} v_\chi^2 k_{22}. \] (73)

Second, for the SM down sector, the matrix \( m^2_D \) in (71) gives:

\[ m^2_D = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & v_2^2 (B_{31}^2 + B_{32}^2 + B_{33}^2) \end{pmatrix}, \] (74)

which exhibits two massless quarks: the down \((d)\) and strange \((s)\) quarks, and one massive quark associated to the bottom \((b)\):

\[ m_b^2 = \frac{1}{2} v_2^2 (B_{31}^2 + B_{32}^2 + B_{33}^2). \] (75)

In order to obtain mass for \( d \) and \( s \), we again consider the one-loop contribution shown in figure 1(b), which produces new entrances different from zero in (74) as follows:

\[ m^2_{D(1-loop)} = \frac{1}{2} \begin{pmatrix} v_2^2 (\Sigma_{11}^2 + \Sigma_{12}^2 + \Sigma_{13}^2) & v_1 v_2 (\Sigma_{11} \Sigma_{21} + \Sigma_{12} \Sigma_{22} + \Sigma_{13} \Sigma_{23}) & v_2^2 (\Sigma_{11} B_{31} + \Sigma_{12} B_{32} + \Sigma_{13} B_{33}) \\ v_1 v_2 (\Sigma_{21}^2 + \Sigma_{22}^2 + \Sigma_{23}^2) & v_1^2 (\Sigma_{21} \Sigma_{21} + \Sigma_{22} \Sigma_{22} + \Sigma_{23} \Sigma_{23}) & 2 m_b^2 \\ 0 & 0 & 0 \end{pmatrix}, \] (76)

where the one-loop correction is:

\[ \Sigma_{ij} = \frac{-1}{16 \pi^2} \frac{f'}{\sqrt{2} M_j} \left( \frac{h^j_l}{h^j_u} \right)_m C_0 \begin{pmatrix} M_l & M_s \\ M_j & M_j \end{pmatrix}. \] (77)

If the matrix in (76) is grouped as

\[ m^2_{D(1-loop)} = \begin{pmatrix} m_{2s}^2 & 0 \\ 0 & 2 m_b^2 \end{pmatrix}, \] (78)

where the bottom mass is dominant, we can block diagonalize it as:

\[ R^T_L m^2_{D(1-loop)} R_L \approx \begin{pmatrix} m_{2s}^2 & 0 \\ 0 & 2 m_b^2 \end{pmatrix}, \] (79)

with:

\[ m_{2s}^2 = m_1^2 - \frac{m m^T}{2 m_b^2} = \frac{1}{2 m_b^2} \begin{pmatrix} s_{11} v_2^2 & s_{12} v_1 v_2 \\ s_{12} v_1 v_2 & s_{22} v_1^2 \end{pmatrix}, \] (80)

and

\[ s_{11} = (\Sigma_{11} B_{32} - \Sigma_{12} B_{31})^2 + (\Sigma_{11} B_{33} - \Sigma_{13} B_{31})^2 + (\Sigma_{12} B_{33} - \Sigma_{13} B_{32})^2, \]

\[ s_{22} = (\Sigma_{21} B_{32} - \Sigma_{22} B_{31})^2 + (\Sigma_{21} B_{33} - \Sigma_{23} B_{31})^2 + (\Sigma_{22} B_{33} - \Sigma_{23} B_{32})^2, \]

\[ s_{12} = B_{31}^2 (\Sigma_{11} \Sigma_{23} + \Sigma_{12} \Sigma_{21}) + B_{32}^2 (\Sigma_{11} \Sigma_{12} + \Sigma_{13} \Sigma_{23}) + B_{33}^2 (\Sigma_{11} \Sigma_{21} + \Sigma_{12} \Sigma_{22}) - B_{31} B_{32} (\Sigma_{12} \Sigma_{21} + \Sigma_{11} \Sigma_{22}) - B_{31} B_{33} (\Sigma_{11} \Sigma_{23} + \Sigma_{13} \Sigma_{21}) - B_{32} B_{33} (\Sigma_{13} \Sigma_{22} + \Sigma_{12} \Sigma_{23}). \] (81)
The eigenvalues of $m_{d_s}^2$ will lead us to the down and strange masses. For example, if the mixing component $s_{12}$ is null, we obtain:

\[
\begin{align*}
    m_d^2 &\approx \frac{s_{11}v^2}{2m_\nu^2}, \\
    m_s^2 &\approx \frac{s_{22}v^2}{2m_\nu^2}.
\end{align*}
\]

\[\text{(82)}\]

### 3.2 Lepton sector

The non-universal $U(1)_X$ also forbids some Yukawa couplings between leptons and scalar bosons. The allowed couplings are shown below for neutral and charged leptons, respectively:

\[
-\mathcal{L}_{Y,N} = h_{\nu e} \nu_L^e \phi_2 \nu_R^e + h_{\nu \mu} \nu_L^\mu \phi_2 \nu_R^\mu + h_{\nu \tau} \nu_L^\tau \phi_2 \nu_R^\tau + h_{\nu e} \nu_L^e \phi_2 \nu_R^e + h_{\nu \mu} \nu_L^\mu \phi_2 \nu_R^\mu + h_{\nu \tau} \nu_L^\tau \phi_2 \nu_R^\tau
\]

\[
+ h_{\chi \nu e}^j \chi^* N_R + \frac{1}{2} N^T R M^j N_R + \text{h.c.}
\]

\[\text{(83)}\]

\[
-\mathcal{L}_{Y,E} = h_{\bar{e}_L e_R} \phi_2 e_R^e + h_{\bar{\nu}_L \phi_2 \nu_R^e} + \epsilon_{\bar{\nu}_L} \phi_2 \nu_R^e + H_{\bar{\nu}_L} \phi_2 \nu_R^e + q_{11} \nu_L^1 \phi_1 E_R + q_{21} \nu_L^2 \phi_1 E_R
\]

\[
+ h_{\sigma e} E_L^e \sigma e_R^e + h_{\sigma e} \sigma E_L^e e_R^e + h_{E} E_L^e \sigma e_R^e + H_1 E_L^e \nu R + H_2 E_L^e \nu R + \text{h.c.}
\]

\[\text{(84)}\]

Since the Higgs doublet $\phi_2$ has the discrete symmetry $\phi_2 \rightarrow -\phi_2$, all the right-handed leptons except $E_R$ and $E_R$ also have $Z_2$ negative parities in order to obtain the adequate zero textures, i.e.:

\[
\nu_R^{e,\mu,\tau} \rightarrow -\nu_R^{e,\mu,\tau}, \quad \nu_R^{\mu,\tau} \rightarrow -\nu_R^{\mu,\tau}, \quad N_R^{e,\mu,\tau} \rightarrow -N_R^{e,\mu,\tau}.
\]

\[\text{(85)}\]

### Neutral leptons

Evaluating in the VEVs, the terms obtained from \[\text{(83)}\] can be written in the following mass term using the basis $N_L = \left( \nu_L^{e,\mu,\tau}, (\nu_R^{e,\mu,\tau})^C, (N_R^{e,\mu,\tau})^C \right)^T$ for the neutral sector

\[
-\mathcal{L}_{Y,N} = \frac{1}{2} \overline{N_L} M_N N_L,
\]

\[\text{(86)}\]

where the mass matrix is

\[
M_N = \left( \begin{array}{ccc} 0 & 0 & 0 \\ m_D & 0 & M_D^T \\ 0 & M_D & M_M \end{array} \right),
\]

\[\text{(87)}\]

with $M_D = h_{\chi}^e \nu_\chi / \sqrt{2}$ being a Dirac mass between $\nu_R^e$ and $N_R$, where $h_{\chi}$ is a $3 \times 3$ matrix, and

\[
m_D = \frac{\nu_2}{\sqrt{2}} \begin{pmatrix}
    h_{\nu e}^e & h_{\nu e}^\mu & h_{\nu e}^\tau \\
    h_{\nu \mu}^e & h_{\nu \mu}^\mu & h_{\nu \mu}^\tau \\
    h_{\nu \tau}^e & h_{\nu \tau}^\mu & h_{\nu \tau}^\tau
\end{pmatrix},
\]

\[\text{(88)}\]

is a Dirac mass matrix between $\nu_L$ and $\nu_R$. $M_M$ is the mass of the Majorana neutrino $N_R$.

Considering that $M_M \ll m_D$ and $M_D$, the matrix $M_N$ can be diagonalized through the inverse seesaw mechanism \[\text{[26]}\text{[27]}. \] If the following blocks are defined

\[
M_N = \begin{pmatrix} M_D \\ 0 \end{pmatrix},
\]

\[\text{(89)}\]

the mass matrix becomes

\[
M_\nu = \begin{pmatrix} M_D & M_N^T \\ M_N & M_N \end{pmatrix},
\]

\[\text{(90)}\]
which has the same form as the block matrix (129) from Appendix A in the limit with $A = 0$. Thus, we define the rotations

$$\mathcal{W}_{SS}^T M_\nu \mathcal{W}_{SS} = \begin{pmatrix} m_{\text{light}} & 0 \\ 0 & m_{\text{heavy}} \end{pmatrix}, \quad (91)$$

with

$$\mathcal{W}_{SS} \approx \begin{pmatrix} I \\ - (F^N)^T \end{pmatrix}, \quad F^N \approx (\mathcal{M}_N)^{-1} \mathcal{M}_\nu, \quad (92)$$

and

$$m_{\text{light}} \approx -\mathcal{M}_\nu^T \mathcal{M}_N^{-1} \mathcal{M}_\nu, \quad (93)$$

$$m_{\text{heavy}} \approx \mathcal{M}_N. \quad (94)$$

Since

$$\mathcal{M}_N^{-1} = \begin{pmatrix} - (M_D)^{-1} M_M (M_D)^{-1} & M_D^{-1} \\ (M_D)^{-1} & 0 \end{pmatrix}, \quad (95)$$

the light mass term is

$$m_{\text{light}} = n_D^T (M_D)^{-1} M_M (M_D)^{-1} m_D. \quad (96)$$

Now, a unitary matrix $\mathcal{V}$ is considered which diagonalizes the $3 \times 3$ block matrix $\mathcal{M}_N$ [27]:

$$\mathcal{V}^T \mathcal{M}_N \mathcal{V} = \mathcal{V}^T \begin{pmatrix} 0 & M_D^T \\ M_D & M_N \end{pmatrix} \mathcal{V} = \begin{pmatrix} V_1^* M_1^{\text{diag}} V_1^T & 0 \\ 0 & V_2^* M_2^{\text{diag}} V_2^T \end{pmatrix}, \quad (97)$$

with $V_1$ and $V_2$ sub-rotation matrices. $\mathcal{V}$ may be formally expressed as [27]

$$\mathcal{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 - S/2 & S \\ -S/2 & 1 - S/2 \end{pmatrix}. \quad (98)$$

Using (97), and assuming that $M_D = M_D^T$, $M_M S^T = S^T M_M$, $M_M S = S^* M_M$, $M_D S^T = S^T M_D$ and $M_D S = S^* M_D$, from the off-diagonal elements we find

$$S = S^* = - \frac{1}{4} M_D^{-1} M_M, \quad (99)$$

and substituting for the diagonal elements, we get the mass matrices

$$V_1^* M_1^{\text{diag}} V_1^T = \frac{M_M}{2} - M_D - \frac{1}{8} M_M M_D^{-1} M_M \approx -M_D, \quad (100)$$

$$V_2^* M_2^{\text{diag}} V_2^T = \frac{M_M}{2} + M_D + \frac{1}{8} M_M M_D^{-1} M_M \approx M_D. \quad (101)$$

The mass eigenstates $n_L$ are constructed as:

$$N_L = U_N n_L, \quad (102)$$

with $n_L = (\nu_L^{1,2,3}, N_{1L}^{1,2,3}, N_{2L}^{1,2,3})$, and the rotation matrix as

$$U_N = \mathcal{W}_{SS} \mathcal{W}_H \mathcal{W}_B, \quad (103)$$

with $\mathcal{W}_{SS}$ the seesaw matrix rotation from [92],

$$\mathcal{W}_H = \begin{pmatrix} 1 \\ 0 \\ \mathcal{V} \end{pmatrix} \quad (104)$$

the matrix rotation of the heavy neutrinos, and

$$\mathcal{W}_B = \text{block diag} (U_\nu, V_1, V_2) \quad (105)$$

the matrices that diagonalize each $3 \times 3$ block.
For the charged sector in the flavor basis \( E = (e^e, e^\mu, e^\tau, E') \), the mass terms obtained from (84) after the symmetry breaking are

\[
-\mathcal{L}_{Y,E} = \mathbf{E}^T \mathbf{M}_E \mathbf{E}_R + \frac{H_2 v}{\sqrt{2}} \mathbf{E}_L \mathbf{E}_R + h.c.,
\]

where the lepton mass matrix \( \mathbf{M}_E \) has the following form:

\[
\mathbf{M}_E = \frac{v^2}{\sqrt{2}} \begin{pmatrix}
0 & \eta & 0 & q_{11} \ell_\beta \\
0 & h & 0 & q_{21} \ell_\beta \\
\zeta & 0 & H & 0 \\
0 & 0 & 0 & H_1 v_\chi/v_2
\end{pmatrix},
\]

which exhibits one massless lepton (the electron). To obtain a massive electron, we include the one-loop correction shown in figure 2 which add a new term

\[
\mathbf{M}_{E(1)} = \mathbf{M}_E + \Delta \mathbf{M}_E,
\]

with:

\[
\Delta \mathbf{M}_E = \frac{v^2}{2} \begin{pmatrix}
\Sigma_{11} & 0 & \Sigma_{13} & 0 \\
\Sigma_{21} & 0 & \Sigma_{23} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since \( \mathbf{M}_{E(1)} \) is not hermitian, there are two rotation matrices \( \mathbf{V}_L^E \) and \( \mathbf{V}_R^E \) for left- and right-handed electrons. Hence, the left-handed rotation is obtained by diagonalizing \( \mathbf{M}_E \mathbf{M}_L^E \) obtaining the corresponding eigenvalues

\[
\begin{align*}
m_e^2 &= \frac{h^2 \Sigma_{11} v_\chi^2}{2 (\eta^2 + h^2)} \approx \frac{v_2^2}{2} \Sigma_{11}, \\
m_\mu^2 &= \frac{v_2^2}{2} (\eta^2 + h^2) \approx \frac{v_2^2}{2} h^2, \\
m_\tau^2 &= \frac{v_2^2}{2} (\zeta^2 + H^2) \approx \frac{v_2^2}{2} H^2, \\
m_E^2 &= \frac{H_1^2 v_\chi^2}{2}.
\end{align*}
\]

In addition, the flavor eigenstates are related to mass eigenstates \( \mathbf{e} = (e, \mu, \tau, E')^T \) by:
\[ E_L = V_E^T e_L \]

where the corresponding left-handed rotation matrix can be expressed as:

\[ V_L^E = V_{SS,L}^E V_{SM,L}^E, \]

which diagonalizes as:

\[ M_{E \Sigma E} = \frac{1}{2} \begin{pmatrix} M_{ee}^2 & M_{eE}^2 \\ M_{eE}^T & M_{EE}^2 \end{pmatrix}, \]

whose blocks are

\[ M_{ee}^2 = \frac{v_e^2}{2} \begin{pmatrix} q_{11}^2 t_\beta^2 + \eta^2 + \Sigma_{11}^2 + \Sigma_{13}^2 & q_{11} q_{21} t_\beta^2 + h \eta + \Sigma_{11} \Sigma_{21} + \Sigma_{13} \Sigma_{23} \\ q_{21}^2 t_\beta^2 + h^2 + \Sigma_{21}^2 + \Sigma_{23}^2 & \zeta \Sigma_{11} + H \Sigma_{13} \end{pmatrix}, \]

\[ M_{eE}^2 = \frac{v_1 v_\mu}{2} H_1 \begin{pmatrix} q_{11} \\ 0 \end{pmatrix}, \]

\[ M_{EE}^2 = \frac{v_2 H_1}{2}. \]

The former matrix \( V_{SS,L}^E \) is

\[ V_{SS,L}^E = \begin{pmatrix} I & F^E \\ -F^E \end{pmatrix}, \]

with \( F^E = M_{eE}^2 (M_{EE}^2)^{-1} \). The latter rotation is:

\[ V_{SM,L}^E = \begin{pmatrix} V_{SM,L}^E \\ 0 \end{pmatrix}, \]

where the top-left block diagonalizes the SM charged lepton masses

\[ V_{SM,L}^E = \begin{pmatrix} c_{\alpha_{e\mu}} & s_{\alpha_{e\mu}} & \Sigma_{11} \\ -s_{\alpha_{e\mu}} & c_{\alpha_{e\mu}} & \Sigma_{21} \\ \frac{\eta}{H} & \frac{\eta}{H} & 1 \end{pmatrix}. \]

The angle \( \alpha_{e\mu} \) is defined by \( t_{\alpha_{e\mu}} = \tan \alpha_{e\mu} \approx \eta/h \), which is a free parameter of the model as shown below.

### 4 PMNS matrix

To explore some phenomenological consequences of the above structures, we assume for simplicity that \( M_D \) is diagonal and \( M_M \) is proportional to the identity

\[ M_D = \begin{pmatrix} h_{N_X} & 0 & 0 \\ 0 & h_{N_Y} & 0 \\ 0 & 0 & h_{N_Z} \end{pmatrix}, \]

\[ M_M = \mu_N \mathbb{1}_{3 \times 3}. \]

Thus, \( V_1 = V_2 = \mathbb{1}_{3 \times 3} \) in (97). On the other hand, replacing the Dirac matrix from (88) into the light mass eigenvalues in (96), we obtain

\[ m_{\text{light}} = \mu_N \nu_2^2 h_{N_X}^2 v_\chi^2 \begin{pmatrix} (h_{2e}^0)^2 + (h_{2\mu}^0)^2 \rho^2 & \rho h_{2e} v_{2e} & h_{2e} h_{2\mu} + h_{2\mu} h_{2e} \rho^2 \\ \rho h_{2e} v_{2e} & (h_{2\mu} v_{2e})^2 + (h_{2e} v_{2\mu})^2 & h_{2\mu} h_{2\mu} + h_{2\mu} h_{2\mu} \rho^2 \\ h_{2e} h_{2\mu} + h_{2\mu} h_{2e} \rho^2 & (h_{2\mu} v_{2e})^2 + (h_{2e} v_{2\mu})^2 & (h_{2\mu} v_{2\mu})^2 + (h_{2\mu} v_{2\mu})^2 \rho^2 \end{pmatrix}, \]

(120)
where $\rho = h_{N\chi_1}/h_{N\chi_2}$. The matrix $m_{\text{light}}$ has zero determinant, obtaining at least, one massless neutrino. The above matrix is diagonalized through

$$U^T \nu_m \nu = m_{\text{diag}}^{\text{light}},$$

where $U$ contains the mixing angles that transform the weak eigenstates $\nu^L_{\mu,\tau}$ into mass eigenstates $\nu^L_{1,2,3}$. The PMNS matrix is defined as the product of the above rotation matrix and the rotation matrix of the charged sector $V_{\text{SM}}^{E,L}$

$$U_{\text{PMNS}} = (V_{\text{SM}}^{E,L})^\dagger U. \quad (122)$$

We use the following parametrization for the PMNS matrix [11]:

$$U_{\text{PMNS}} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & -s_{23}s_{13}e^{i\delta} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}s_{13}
\end{pmatrix}. \quad (123)$$

The mixing angles can be obtained from some matrix components as

$$s_{13} = |U_{e3}|^2,$$
$$s_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2},$$
$$s_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}. \quad (124)$$

**Parameter values**

In order to have a model consistent with neutrino oscillation data [22], the values of the Yukawa parameters $h_{\nu e}^{\mu}$, $h_{\nu \mu}^{\mu}$, $h_{\nu \tau}^{\mu}$, $h_{\nu e}^{\nu}$, $h_{\nu \mu}^{\nu}$, $h_{\nu \tau}^{\nu}$ and $\alpha_{\nu\mu}$ must be properly adjusted. To achieve this, we implement a MonteCarlo method to generate random numbers in the parameter space, where only the numbers which match up the mass matrix to experimental data are accepted, while the others are rejected. It is worth mentioning that the other two rotation parameters described by $\Sigma_{13}/H$ and $\Sigma_{23}/H$ were approximated to $m_e/m_\tau$, while $h_{2\nu}^{\mu}$ was chosen null to simplify the search.

On the other hand, the appropriate mass scale and mass ordering can be obtained by adjusting the outer factor of the mass matrix and the ratio $\rho$. For NO the Yukawa coupling can be set by

$$h_{N\chi_1}^2 = 0.02,$$
$$\rho^2 = 0.5, \quad (125)$$

while for IO

$$h_{N\chi_1}^2 = 0.025,$$
$$\rho^2 = 0.625. \quad (126)$$

In the same way, the mass scale is set by

$$v_2 = 7 \text{ GeV},$$
$$v_\chi = 7 \text{ TeV},$$
$$\mu_N = 1 \text{ keV}. \quad (127)$$

The above values fix the outer factor of the mass matrix [120] at 50 meV, which yields to the correct squared-mass differences. Nevertheless, there exist other possible values for the parameters $\mu_N$, $h_{N1\chi}$, $v_\chi$ and $\tan \beta$ that lead us the factor at 50 meV.

If the following constraint is assumed

$$\frac{\mu_N v_2^2}{h_{N\chi_1}^2 v_\chi^2} = 50 \text{ meV}, \quad (128)$$

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Figure 3: Contour plots of $v_\chi$ vs. $v_2$ from eq. (128) for different values of $h_{N\chi 1}^2$ and $\mu_N$. From below to above there are the corresponding contour plots for the following values of $\mu_N$: 500 eV (gray, line), 1 keV (black, line), 5 keV (gray, dashed), 10 keV (black, dashed), 50 keV (gray, dot-dashed) and 100 keV (black, dot-dashed).

Table 4: Yukawa coupling domain which fulfill at 1σ neutrino oscillation data for NO reported by [22]. $h_{2\mu}^{\nu e} = 0$ for simplifying the MonteCarlo search.

| $\alpha_{e\mu}$ | $\alpha_{e\mu} = 0^\circ$ | $\alpha_{e\mu} = 15^\circ$ | $\alpha_{e\mu} = 30^\circ$ |
|------------------|--------------------------|--------------------------|--------------------------|
| $h_{ee}^{\nu\nu}$ | $0.264 \rightarrow 0.278$ | $0.285 \rightarrow 0.299$ | $0.237 \rightarrow 0.270$ |
| $h_{e\mu}^{\nu\nu}$ | $-0.707 \rightarrow -0.244$ | $-0.726 \rightarrow -0.335$ | $-0.796 \rightarrow -0.547$ |
| $h_{\mu\mu}^{\nu\nu}$ | $-0.491 \rightarrow -0.190$ | $-0.464 \rightarrow -0.173$ | $-0.342 \rightarrow -0.039$ |

Table 5: Yukawa coupling domain which fulfill at 1σ neutrino oscillation data for IO reported by [22]. $h_{2\mu}^{\nu e} = 0$ for simplifying the MonteCarlo search.

| $\alpha_{e\mu}$ | $\alpha_{e\mu} = 0^\circ$ | $\alpha_{e\mu} = 1^\circ$ | $\alpha_{e\mu} = 2^\circ$ |
|------------------|--------------------------|--------------------------|--------------------------|
| $h_{ee}^{\nu\nu}$ | $1.094 \rightarrow 1.107$ | $1.091 \rightarrow 1.105$ | $1.090 \rightarrow 1.103$ |
| $h_{e\mu}^{\nu\nu}$ | $-0.122 \rightarrow -0.106$ | $-0.127 \rightarrow -0.113$ | $-0.128 \rightarrow -0.118$ |
| $h_{\mu\mu}^{\nu\nu}$ | $0.970 \rightarrow 1.060$ | $0.980 \rightarrow 1.070$ | $1.010 \rightarrow 1.080$ |
| $h_{e\mu}^{\nu\nu}$ | $0.110 \rightarrow 0.127$ | $0.122 \rightarrow 0.138$ | $0.135 \rightarrow 0.149$ |
| $h_{\mu\mu}^{\nu\nu}$ | $0.930 \rightarrow 1.030$ | $0.920 \rightarrow 1.010$ | $0.910 \rightarrow 0.980$ |
contour plots can be done for different values of $\mu_N$ in the $v_\chi$ vs. $v_2$ plane, as shown in figure 3.

The tables 4 and 5 show regions where the neutrino Yukawa couplings and the angle $\alpha_{e\mu}$ make consistent this model with neutrino oscillation data reported by [22] at 3$\sigma$.

The Yukawa coupling $h_{\chi N_3}$ is not fixed by oscillations of the light neutrinos; however, they may contribute into the total rotation matrix $U_N$ in (103). Thus, the neutral spectrum of the model is composed by three active light neutrinos $\nu_L^{1,2,3}$ and six quasidegenerated sterile neutrinos $N_1^{1,2,3}$ and $N_2^{1,2,3}$ at the TeV scale.

5 Conclusions

Abelian nonuniversal gauge extensions of the SM are very well-motivated models which involve a wide number of theoretical aspects. In this work, by requiring nonuniversality in the left-handed quark sector and in lepton sector, we propose a new $G_{SM} \times U(1)'$ gauge model. We obtained a free-anomaly theory with invariant Yukawa interactions, predicting hierarchical mass structures in the quark and charged lepton sector with few free parameters.

For the quark sector, we identify three energy scales. First, at the breaking scale of the $U(1)_X$ symmetry, we obtain heavy masses to the extra heavy quarks $J$ and $T$, with $M_{J,T} \approx \nu_X$. Second, at tree level, we obtain masses at the electroweak scale for the $c$, $t$ and $b$ quarks, with $M_{c,t,b} \approx \nu_1,2$. Finally, at one-loop level, we obtain light masses for the $u$, $d$ and $s$ quarks, with $M_{u,d,s} \approx \nu_1,2/\nu_X$. For the leptonic sector, we also obtain the same hierarchical structure, where the extra leptons $E$ and $\ell$ acquire masses at the $\nu_X$ scale, the $\mu$ and $\tau$ have masses at the electroweak scale, and the electron obtains masses at one-loop, which is suppressed as $\nu_1,2/\nu_X$.

On the other hand, with the addition of extra Majorana neutrinos, we found that neutrinos may acquire tiny masses via the inverse seesaw mechanism. The selection of a small Majorana mass term (from eV to KeV scale) and the experimental limits on observables from neutrino oscillations allows us to perform numerical adjustment for the values of the Yukawa couplings of neutrinos in NO and IO scenarios. In addition, because the non-universal $U(1)_X$ charges, the electron remains massless at tree level but a non-vanishing mass term emerges at one-loop corrections which gives a viable explanation for its small mass compared to the electroweak scale.

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A Block Diagonalization

Let us take a generic matrix with arbitrary dimension of the form:

$$M^2 = \begin{pmatrix} A & C \\ C^T & D \end{pmatrix},$$

(129)

with $A, D$ and $C$ sub-matrices whose elements obey the hierarchy

$$A \ll C \ll D.$$

(130)

The matrix (129), as shown in reference [46], can be block diagonalized approximately by a unitary rotation of the form:

$$V = \begin{pmatrix} I & F \\ -F^T & I \end{pmatrix},$$

(131)

where $I$ is an identity matrix, and $F$ a small sub-rotation with $F \ll 1$. Keeping only up to linear terms on $F$, the rotation gives:

$$V^T M^2 V = \begin{pmatrix} A - CF^T - FCT & C + AF - FD \\ CT + FTA - DFT & D + CF + F^TC \end{pmatrix},$$

(132)
which, by definition, must lead us to a diagonal block form

\[ \text{m}^2 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \tag{133} \]

with \( a \) and \( d \) non-diagonal matrices, and 0 the null matrix. By matching the upper right non-diagonal block in \( \text{m}^2 \) and \( \text{m}^2 \), we obtain that \( C + AF - FD = 0 \). Taking into account the hierarchy in \( \text{m}^2 \), we may neglect the term with \( A \), finding the following approximate solution:

\[ F \approx CD^{-1}. \tag{134} \]

On the other hand, if we match the diagonal blocks in \( \text{m}^2 \) and \( \text{m}^2 \), and using the solution \( \text{m}^2 \), we can obtain the form of the submatrices \( a \) and \( b \) in terms of the original blocks \( A \), \( C \) and \( D \). We obtain at dominant order that:

\[ a \approx A - CD^{-1}C^T \]
\[ b \approx D. \tag{135} \]

The above matrices can be diagonalized independently.

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