Topological phase transition in the quench dynamics of a one-dimensional Fermi gas with spin–orbit coupling

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Abstract

We study the quench dynamics of a one-dimensional ultracold Fermi gas with synthetic spin–orbit coupling. At equilibrium, the ground state of the system can undergo a topological phase transition and become a topological superfluid with Majorana edge states. As the interaction is quenched near the topological phase boundary, we identify an interesting dynamical phase transition of the quenched state in the long-time limit, characterized by an abrupt change of the pairing gap at a critical quenched interaction strength. We further demonstrate the topological nature of this dynamical phase transition from edge-state analysis of the quenched states. Our findings provide interesting clues for the understanding of topological phase transitions in dynamical processes, and can be useful for the dynamical detection of Majorana edge states in corresponding systems.

1. Introduction

Topological phases and phase transitions [1] in quantum many-body systems have recently attracted much attention in various physical contexts. Characterized by topologically nontrivial properties in the Hilbert space rather than by the local order parameters associated with symmetry breaking, a topological state typically features gapless edge states and fractionalized quasi-particle excitations [2, 3]. A particularly interesting topological state is the topological superfluid (TSF) phase. A gapped superfluid state in the bulk, the TSF state is best known for the existence of gapless Majorana edge states, or the Majorana zero modes, at its boundary [2, 3]. Being their own anti-particles, Majorana zero modes are exotic quasi-particles that have drawn a lot of interest due to their potential applications in fault-tolerant quantum computations [4]. Besides systems with intrinsic chiral $p$-wave pairing superfluidity [5–9], TSF phases can also be induced from an $s$-wave pairing superfluid in the presence of an effective Zeeman field and spin-orbit coupling (SOC), i.e., the interaction between the spin and momentum degrees of freedom [10, 11]. While the search for Majorana zero modes is being actively pursued in condensed matter systems such as the semiconductor/superconductor heterostructures [12–19], the possibility of realizing topological superfluidity and hence Majorana zero modes in ultracold atomic gases has also been extensively studied due to the recently implemented synthetic SOC in these systems [20–25].

An advantage of ultracold atomic gases is the controllability, which not only allows for experimental characterization of the system over a wide range of parameters, but also provides a convenient way to study the dynamical processes. In recent years, dynamical processes in ultracold atomic gases, quench dynamics in particular, play an increasingly important role in revealing key properties of the system [26–31]. In a typical quench process, the system evolves from a pure state following a sudden change of parameters [26]. In the long-time limit, the quenched state is typically quite different from an equilibrium state described by the ensemble
theory [32, 33]. More importantly, it has been shown that depending on the quench parameters, the quenched state in this limit can exhibit different dynamical phases [34–36].

With the prospect of realizing topological orders in ultracold atomic gases, it is timely to study dynamical processes in systems with topological order and ask questions like whether topological properties are robust against dynamical perturbations and how topology should be defined and probed in dynamical processes. On the one hand, in an equilibrium state with topological order, features like edge states are protected by a bulk gap [4], while such protection may not be effective in dynamical processes, especially in quench dynamics, where the sudden change inevitably induces high-energy excitations [37]. On the other hand, despite many recent studies on topological phases and edge states in quantum quenches, topology in a dynamical process is still an on-going area of research that needs further characterization [34–36, 38–42].

In this work, we theoretically study the quench dynamics of a one-dimensional ultracold Fermi gas under the synthetic SOC that has recently been realized experimentally [21, 22]. Starting from a ground state of the system, we implement a sudden change of the interaction strength and focus on the dynamics of the pairing gap as the system evolves under the quenched Hamiltonian. At sufficiently long evolution times, the pairing gap either approaches a finite value or oscillates in time, depending on the quench parameters. Interestingly, we identify a critical interaction strength in the quenched Hamiltonian, where the pairing gap in the long-time limit undergoes an abrupt jump. We associate this critical interaction strength with a dynamical phase transition. We further demonstrate that edge states exist in the long-time limit only for processes with quenched interaction strength below the critical value of the dynamical phase transition, which suggests that the dynamical phase transition is also of a topological nature. The topological property of the quenched state is thus determined by the quenched Hamiltonian, regardless of the initial state. Our findings provide clues for the understanding of topological phase transitions in dynamical processes, and may have interesting implications for future experiments.

The paper is organized as follows in section 2, we present the model Hamiltonian and discuss the typical ground-state phase diagram in an equilibrium state. In section 3, we study the quench dynamics of the system and identify a topological phase transition in the non-equilibrium quench dynamics. The topological nature of this phase transition, as well as the associated steady states, is then confirmed by the edge-state analysis in section 4. Finally, we summarize in section 5.

2. Model Hamiltonian and phases at equilibrium

We study a quasi-one-dimensional, optically trapped Fermi gas, where the transverse confinement is much larger than the longitudinal one. In the presence of an additional optical lattice potential in the longitudinal direction, and considering only the lowest band, we may write the Hamiltonian in a tight-binding model [11, 43, 44]:

\[
\hat{H} = -g_s \sum_{j, \sigma} (\hat{c}_j^{\dagger} \hat{c}_{j+1, \sigma} + h.c.) + h \sum_j (\hat{c}_j^{\dagger} \hat{c}_{j+1} - \hat{c}_{j+1}^{\dagger} \hat{c}_j) + \alpha \sum_j (\hat{c}_j^{\dagger} \hat{c}_{j+1} - \hat{c}_{j+1}^{\dagger} \hat{c}_j + h.c.) - U \sum_j \hat{c}_j^{\dagger} \hat{c}_j \hat{c}_{j+1}^{\dagger} \hat{c}_{j+1},
\]

where \(\hat{c}_j^{\dagger}\) is the creation operator of fermions on site \(j\) with spin \(\sigma = \uparrow, \downarrow\), \(g_s\) is the hopping rate between two neighboring sites, and the on-site interaction is attractive with \(U > 0\). In current cold atom experiments, the effective Zeeman field \(h\) is proportional to the effective Rabi frequency of the Raman process generating the synthetic SOC, and the SOC strength \(\alpha\) is proportional to the momentum transfer of the Raman process [20–22].

Following the standard BCS-type mean-field theory, we can Fourier-transform Hamiltonian (1) into momentum space and write the effective mean-field Hamiltonian in the grand canonical ensemble:

\[
\hat{H}_{\text{eff}} = \sum_{k, \sigma} \epsilon_k \hat{c}_k^{\dagger} \hat{c}_k + \sum_k (\Delta \hat{c}_k^{\dagger} \hat{c}_{-k} + h.c.) + \sum_k \left( \alpha \hat{c}_k^{\dagger} \hat{c}_{k+1}^{\dagger} \hat{c}_{k+1}^\dagger \hat{c}_k + h.c. \right) - \frac{\Delta^2}{U},
\]

where \(\hat{c}_k^{\dagger}\) is the creation operator for fermions of momentum \(k\) and spin \(\sigma\), with \(k\) defined on the first Brillouin zone. The dispersion of the lowest band \(\epsilon_{k,1/2} = -2 \cos k = \mu \pm h\), the SOC \(\alpha = 2 \pi \sigma \sin k\), and the pairing gap \(\Delta = U \sum_k \langle \hat{c}_{-k} \hat{c}_k \rangle\). The chemical potential \(\mu\) is related to the on-site particle density \(n\), which is kept fixed throughout the lattice. Here, we have \(0 < n < 1\) for \(\mu < 0\), and \(1 < n < 2\) for \(\mu > 0\). Due to the particle-hole symmetry of the effective Hamiltonian, we will focus on the case of \(\mu > 0\) henceforth. Note that in a strictly one-
dimensional system, the BCS mean-field approach outlined earlier becomes invalid, due to the lack of long-range order. However, realistic one-dimensional cold atomic gases are typically quasi-one-dimensional, with tight confinement in the transverse direction. As the residue degrees of freedom in this tightly confined transverse direction effectively suppress quantum fluctuation, the mean-field description in this case should give a qualitatively valid picture at zero temperature [45].

At equilibrium, the ground state of the system can be determined by minimizing the thermodynamic potential, which at zero temperature is reduced to

\[ \Omega = \langle \hat{H}_{\text{eff}} \rangle. \]

Here \( \langle \cdots \rangle \) is taken with respect to the ground state. Previous studies show that the ground state of \( \hat{H}_{\text{eff}} \) is a TSF with Majorana edge states for

\[ \mu \Delta \leq \Delta \leq \mu + \Delta, \]

and is a conventional superfluid otherwise [46]. In figure 1, we map out the ground state phase diagram at zero temperature, which serves as a basis for further studies of quench dynamics. Consistent with previous works, the system is in the TSF phase in the weakly interacting regime and undergoes a topological phase transition to become a conventional superfluid (SF) phase in the strongly interacting regime with the phase boundary labeled \( U_c \). With our choice of parameters (\( h = 2g_s \) and \( \alpha = g_s \)), the transition boundary in figure 1 is of the first order. This is manifested not only in the discontinuity of the pairing gap \( \Delta \) at the transition boundary, but also in the shape of the thermodynamic potential \( \Omega (\Delta) \). As illustrated in the right panel of figure 1, a double-well structure emerges in the thermodynamic potential close to the phase boundary. The global minimum of the thermodynamic potential jumps from the left well to the right well across the phase boundary, indicating its first-order nature [9, 24, 47]. Note that similar to the TSF state in a two-dimensional Fermi gas [24], the TSF state here persists in the weak-coupling limit. For the convenience of later discussions, we label the critical pairing gap of the topological phase transition \( \Delta_c \), where

\[ \Delta_c = \sqrt{h^2 - (\mu - 2g_s)^2}. \]

3. Quench dynamics

We now study the quench dynamics of the system where the interaction \( U \) undergoes a sudden change. We assume that the system is initially in the zero-temperature ground state with \( U = U_i \). At the start of the quench \((t = 0)\), the interaction is suddenly switched to \( U = U_f \), driving the system out of equilibrium. The system then evolves under the Hamiltonian with the quenched interaction \( U_f \).

To characterize the dynamics of the system, we introduce operators:

\[ \hat{c}_{\mathbf{k} \tau} = \hat{c}^\dagger_{\mathbf{k} \tau} \hat{c}_{\mathbf{k} \tau}, \quad \hat{c}_{\mathbf{k} \tau} = \hat{c}^\dagger_{\mathbf{k} \tau} \hat{c}_{\mathbf{k} \tau}, \quad \hat{\sigma}_{\mathbf{k}} = \hat{c}_{\mathbf{k} \uparrow} \hat{c}_{\mathbf{k} \downarrow}. \]

Together with the momentum-space density operator \( \hat{n}_{\mathbf{k} \tau} = \hat{c}^\dagger_{\mathbf{k} \tau} \hat{c}_{\mathbf{k} \tau} \), we may express the time-dependent mean-field effective Hamiltonian as:

\[
\text{Figure 1.} \text{ The left panel shows the equilibrium phase diagram with the x-axis denoting the on-site interaction and y-axis the on-site occupation. The right four panels (from up to down) show, respectively, the thermodynamic potential as a function of } \Delta \text{ at four different points (from left to right) in the phase diagram.}
\]
where \( \tau \) and \( \sigma \) are defined as above, and \( \epsilon \) is given by \( \epsilon = -2 \cos k - \mu. \) From the Schrödinger’s equations, one may derive a set of closed equations for the expectation values of these operators. The dynamical parameters of the system can be calculated numerically from these equations (see appendix A for the detailed derivation of dynamical equations). Note that a similar approach has been applied to study the quench dynamics in a Fermi gas throughout the BCS-BEC crossover without SOC [48–51], and more recently, the quench dynamics of a chiral \( p \)-wave SF near the topological phase transition [34]. We note that the mean-field approximation adopted in these studies should provide a qualitatively correct understanding of the quench dynamics, and will serve as the basis for future studies. Here, we focus on the dynamics of the pairing gap \( \Delta \) during the evolution. Interestingly, the dynamics and the quenched states are quite different depending on the topological nature of the initial state.

The right panel of figure 2 shows typical quench dynamics of the pairing gap when the initial ground state is a conventional SF, while the left panel illustrates the ground state pairing gap as a function of the on-site interaction. When the evolution time following the quench is sufficiently long, the pairing gap typically approaches a finite value, suggesting that the system relaxes to a steady state. For large \( \tilde{U}_f \) the pairing gap of the steady state is close to that in the ground state of the quenched Hamiltonian. For small \( \tilde{U}_f \) the steady state pairing gap and that of the ground state at \( \tilde{U}_f \) are noticeably different. In comparison, we demonstrate in figure 3 the quench dynamics where the system is initially in a TSF state. Different from the previous case, strong oscillations of \( \Delta(t) \) are observed in the quenched state when \( \tilde{U}_f \) is deep in the conventional SF regime. We observe no significant damping in the oscillations within the longest evolution time scale that we can numerically implement. This is reminiscent of the quench processes in a two-dimensional \( p \)-wave fermionic superfluid, where the weak-to-strong pairing quenches have also been shown to give rise to strong oscillations, in contrast to strong-to-weak quenches [34].

Despite the difference in the detailed dynamics, for both cases, we may numerically identify a critical final interaction \( \tilde{U}_d \). As the quench parameter \( \tilde{U}_f \) crosses \( \tilde{U}_d \), the pairing gap \( |\Delta| \) in the long-time limit features a jump. This is demonstrated in figure 4, where we plot the typical pairing gap \( |\Delta| \) in the long-time limit as a function of \( \tilde{U}_f. \) Here, \( |\Delta|_{\inf} \) is the asymptotic value of the pairing gap if the system approaches a steady state in the long-time limit, and is the time average of the pairing gap in many periods in the case of strong oscillations. We associate this critical parameter \( \tilde{U}_d \) with a dynamical phase transition. For a non-topological initial state, \( \tilde{U}_d \) is numerically close but not equal to the first-order phase boundary in the ground state; while for a topological initial state, \( \tilde{U}_d \)
lies in the conventional SF regime, whose deviation from the topological phase boundary is much clearer. We
have further checked numerically that $U_d$ is not sensitive to the initial interaction $U_i$.

A particularly interesting observation here is that in either case, for $U < U_{fd}$, $\Delta_{\text{inf}}$ is smaller than the critical value $\Delta_c$ that lies on the boundary of the topological phase transition of the initial state. In contrast, when $U > U_{fd}$, $\Delta_{\text{inf}}$ is larger than $\Delta_c$. Naturally, one is tempted to conclude that the dynamical phase transition should be a topological one as well. We will show that this is indeed the case. To provide conclusive evidence for the topological nature of the dynamical phase transition, we will characterize the edge states of the dynamical processes.

4. Edge states in dynamical processes

Because the quenched state in the long-time limit is quite different from the ground state of the quenched Hamiltonian, the characterization of the topology of the quenched state is, in general, a highly non-trivial problem. Here, we propose to invoke the bulk-edge correspondence and examine the existence of edge states in the long-time limit [52]. In heterostructure nanowires, a widely studied detection method for the Majorana edge state is by coupling the nanowire to a normal metal lead and measuring the tunneling conductance at zero bias [16]. In the presence of a Majorana edge state, the resulting conductance is quantized to $2e^2/h$ [53]. Here, we
borrow this idea to theoretically study the existence of edge states in the quench dynamics. In particular, we examined the topology of the quenched state in the long-time limit. In practice, the transport measurement on cold atoms has been realized in recent experiments [54], suggesting the possibility of carrying out similar measurements in the future.

We consider a spin-orbit coupled Fermi gas, which is conveniently simulated by a lattice of $L$ sites, labeled 0 to $L-1$, and with open boundaries. A non-interacting Fermi gas in a half-infinite chain serves as a reservoir (lead), and is coupled to the edge site of the lattice gas labeled 0. The Hamiltonian of the reservoir is then:

$$H_{\text{res}} = -g_l \sum_{j=-\infty}^{-1} (\hat{c}_j^\dagger \hat{c}_{j+1} + \text{h.c.})$$

with the coupling Hamiltonian:

$$H_V = g_m \sum_{\sigma} (\hat{c}_{1,\sigma}^\dagger \hat{c}_{0,\sigma} + \text{h.c.})$$

where $\hat{c}_{1,\sigma}$ are field operators at the end of the interacting and noninteracting lattice, respectively. While $g_l$ denotes the hopping rate between two neighboring sites in the reservoir, $g_m$ is the coupling rate between the reservoir and the SOC Fermi gas. The full Hamiltonian is then:

$$H_{\text{tot eff}}(t) = H_{\text{res}}(t) + H_V + H_t$$

where $H_{\text{eff}}(t)$ is the mean-field Hamiltonian of the SOC Fermi gas expressed in real space.

Setting the chemical potential of the reservoir to $\mu_R$ while noticing that the gap of the SF is always symmetric $c$.

Figure 5 shows the zero bias conductance as a function of quenched interaction $U_f$ (left panel) and $U_d$ (right panel). The initial state is a topological SF $U_f < U_c$, where $U_c$ marks the onset of a dynamical topological phase transition. Typically, the dynamical critical point $U_c$ is close to the equilibrium phase boundary $U_c$: in the left panel of figure 5, $U_f \approx U_d = 7.25g_c$; while in the right panel, they are $U_f = 8.15g_c$ and $U_d \approx 7.6g_c$.

Although the existence of edge states in the quench is not sensitive to the initial condition, the conductance in figure 5 behaves differently for $U_f < U_d$ and $U_d > U_f$. For $U_f < U_d$, $G$ reaches the quantized value at $U_f = U_d$ (no quench), and is close to $2/h$ in a wide range of $U_f$ near $U_d$. This suggests that the edge modes in the quenched states are well protected before $U_f$ crosses the critical point, despite the inevitable excitations, which suppress the conductance for $U_f \neq U_d$. In contrast, for $U_f > U_d$, the emergence of edge states is accompanied by the crossing.
of the critical $U_c$ of $U_f$ in the quenched Hamiltonian. As the bulk gap in the ground state closes at $U_c$, the excitations in the quenched state make the signature of the edge states less obvious. This is more apparent when $U_f$ is deep in the TSF regime ($U_f < U_c$), where the suppression of conductance leads to a peak structure therein.

5. Conclusions

In summary, we find a dynamical phase transition in cold fermions with synthetic SOC under an interaction quench. This transition is characterized by an abrupt change of asymptotic pairing gap, which also crosses the boundary distinguishing the topological and the trivial ground states. The edge state analysis strongly supports that this dynamical transition is a topological one, with the topological properties of the quenched state sensitively relying on the quench parameters. With the recent achievement of synthetic SOC and the high controllability of cold atomic gases, the dynamical topological phase transitions discussed here may be probed in future experiments.

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Appendix A. Differential equations

In this section, we derive the equations of motion for the expectation values of the operators. As an example, we consider the expectation value of $\tau_k$ in the Schrödinger’s picture:

$$\tau_k(t) = \langle \psi(t) | \hat{\tau}_k | \psi(t) \rangle,$$

where $| \psi(t) \rangle$ denotes the wave function at time $t$. The Schrödinger’s equation gives:

$$i \frac{d}{dt} | \psi(t) \rangle = \hat{H}_{\text{eff}}(t) | \psi(t) \rangle.$$

The equation of motion for $\tau_k(t)$ is then:

$$-i \frac{d\tau_k(t)}{dt} = \langle \psi(t) | \left[ \hat{H}_{\text{eff}}(t), \hat{\tau}_k \right] | \psi(t) \rangle.$$

The commutator on the right-hand side consists of six possible operators: $[\hat{\nu}_{k\uparrow}, \hat{\nu}_{k\downarrow}, \hat{n}_{k\uparrow}, \hat{n}_{k\downarrow}, \hat{\tau}_k, \hat{\sigma}_k]$. By calculating the derivative of their expectation values to $| \psi(t) \rangle$, we get a set of closed equations for the expectation values of the operators:

$$-i \frac{d\nu_{k\uparrow}}{dt} = 2\nu_{k\uparrow}v_{k\uparrow} + \Delta^* \sigma_{-k} - \Delta^* \sigma_k + a_k \tau_{-k} + a_k \tau_k,$n_{k\downarrow} + a_k \tau_k,$$

$$\frac{dn_{k\downarrow}}{dt} = -\Delta^* \tau_{k} + a_k \sigma_k^* - a_k \sigma_k,$$

$$\frac{dn_{k\uparrow}}{dt} = -\Delta^* \tau_{k} + a_k \sigma_k - a_k \sigma_k^*,$$

$$\frac{d\tau_{k}}{dt} = \langle \psi(t) | \left[ \hat{H}_{\text{eff}}(t), \hat{\tau}_k \right] | \psi(t) \rangle.$$ (A.4)

The pairing gap $\Delta(t)$ in equation (A.4) can be expressed as

$$\Delta(t) = U_f \sum_k \tau_k(t),$$

which can be solved self-consistently from equations (A.4) and (A.5).
Appendix B. Calculation of tunneling conductances

To calculate the tunneling conductance, we employ the numerical operator method first introduced in [55] for the real-time Kondo problem, which was then extended to time-dependent systems [56] and topological superconductors [57]. The full Hamiltonian describing the reservoir and the superfluid is expressed as

$$H_{\text{tot}}(t) = H_{\text{eff}}(t) + H_{\text{rs}} + H_{\text{V}}.$$  \hspace{1cm} (B.1)

We define the operators \(\hat{d}_{\alpha}(t)\) and \(\hat{d}_{\alpha}^\dagger(t)\) in the Heisenberg picture defined as

$$\hat{d}_{\alpha}(t) = \lim_{\tau \to 0} e^{iH_{\text{rs}}(t_\tau)} \cdots e^{iH_{\text{rs}}(t_1)} \hat{a}_{\alpha} e^{-iH_{\text{rs}}(t_1)} \cdots e^{-iH_{\text{rs}}(t_\tau)},$$  \hspace{1cm} (B.3)

where \(\alpha\) is the abbreviation of \((j, \sigma, s) (s = 0, 1)\), \(H_{\text{rs}}\) is the time-dependent Hamiltonian, \(N\) is the total number of steps, \(t_0 = 0\) is the initial time, and the intermediate time is \(t_n = n\tau\), where the time step \(\tau = t/N\).

The calculation of equation (B.3) is divided into \(N\) steps with the time evolving from \(t_{N-1}\) to \(t_0\). In the \(n\)th step, we undress the pair of operators \(e^{iH_{\text{rs}}(t_{n-1})}\) and \(e^{-iH_{\text{rs}}(t_{n-1})}\) from \(\hat{d}_{\alpha}\). Since \(H_{\text{rs}}\) is quadratic, in principle, the intermediate results at each step can be written as

$$\hat{d}_{\alpha}(t_n) = \sum_{\alpha'} W_{\alpha,\alpha'}(t_n) \hat{d}_{\alpha'},$$  \hspace{1cm} (B.4)

with \(\hat{d}_{\alpha}(t_n)\) defined as

$$\hat{d}_{\alpha}(t_n) = e^{iH_{\text{rs}}(t_{n-1})} \cdots e^{iH_{\text{rs}}(t_1)} \hat{a}_{\alpha} e^{-iH_{\text{rs}}(t_1)} \cdots e^{-iH_{\text{rs}}(t_{n-1})},$$  \hspace{1cm} (B.5)

and \(W_{\alpha,\alpha'}(t_n)\) denoting the propagator. At \(n = N\), we get

$$\hat{d}_{\alpha}(t) = \sum_{\alpha'} W_{\alpha,\alpha'}(t_N) \hat{d}_{\alpha'}.$$  \hspace{1cm} (B.6)

To calculate the propagator \(W_{\alpha,\alpha'}(t_N)\), we derive an iterative relation. According to equation (B.5), we have

$$\hat{d}_{\alpha}(t_n) = e^{iH_{\text{rs}}(t_{n-1})} \hat{d}_{\alpha}(t_{n-1}) e^{-iH_{\text{rs}}(t_{n-1})}.$$  \hspace{1cm} (B.7)

Substituting equation (B.4) into the previous equation, we find

$$\sum_{\alpha'} W_{\alpha,\alpha'}(t_n) \hat{d}_{\alpha} = \sum_{\alpha'} W_{\alpha,\alpha'}(t_{n-1}) \left( \hat{d}_{\alpha'} + i\tau \left[ H_{\text{eff}}(t_{N-n}), \hat{d}_{\alpha'} \right] \right) + \left( i\tau \right)^2 \left[ H_{\text{tot}}(t_{N-n}), \left[ H_{\text{tot}}(t_{N-n}), \hat{d}_{\alpha'} \right] \right] + O(\tau^3).$$  \hspace{1cm} (B.8)

The calculation of these commutators is straightforward with the results

$$\left[ H_{\text{tot}}(t_j), \hat{d}_{\beta} \right] = \sum_\rho G_{\alpha,\beta}(t_j) \hat{d}_\rho.$$  \hspace{1cm} (B.9)

Substituting equation (B.9) into equation (B.8) and comparing the left and the right sides, we obtain the iterative relation

$$W_{\alpha,\alpha'}(t_n) = W_{\alpha,\alpha'}(t_{n-1}) + i\tau \sum_\rho W_{\alpha,\rho}(t_{n-1}) G_{\rho,\alpha'}(t_{N-n}) - \frac{\tau^2}{2} \sum_{\rho,\rho'} W_{\alpha,\rho}(t_{n-1}) G_{\rho,\rho'}(t_{N-n}) G_{\rho',\alpha'}(t_{N-n}).$$  \hspace{1cm} (B.10)

Here we drop terms of the order \(O(\tau^3)\). In principle, the error caused by the discretization of time can be made arbitrarily small by letting \(\tau \to 0\). In practice, we adaptively choose \(\tau\) to make the discretization error negligible.

According to equation (B.10), we start from \(n = 0\) when \(W_{\alpha,\alpha'}(t_0) = \delta_{\alpha,\alpha'}\), and iteratively work out \(W_{\alpha,\alpha'}(t_n)\) until \(t_N = t\). This approach typically works well, due to the critical fact that the matrix \(G_{\alpha,\beta}\) has only a few non-zero elements, such that the number of non-zero \(W_{\alpha,\beta}\), which needs to be kept at each step, increases slowly.
according to equation (B.10). However, when $N$ is very large, which is inevitable since we must choose a small $\tau$ and at the same time a large $t$ to obtain the asymptotic limit of $G(t)$, there will be too many non-zero $W_{\alpha \beta}$, which cannot be simultaneously kept. To work around this, we apply a truncation scheme and keep only a fixed number of non-zero propagators (the number is denoted by $M$) with the largest magnitudes at each step. This truncation scheme is necessary to obtain the steady limit of $G(t)$. We decide the value of $M$ adaptively, i.e., set an original $M$ and increase it until the desired precision is obtained. The logic in doing so is that the truncation error goes to zero in the limit $M \to \infty$.

Substituting the expression of $\hat{d}_{\alpha \beta}(t)$ into equation (B.2), we have

$$G(t) = -2g_{\alpha} \text{Im} \sum_{\alpha \beta} W_{\alpha \beta}(t) W_{\alpha \beta}(0) \left( \frac{d}{d \mu_R} \left( \hat{d}_{\beta \alpha}(t) \right) \right)_{\mu_R=0}.$$  \hspace{1cm} (B.11)

Next we calculate the derivative of correlation functions. The correlation functions of the initial state are non-zero only if the two field operators are both in the superfluid or both in the reservoir, since the superfluid and the reservoir are decoupled at the initial time. Considering that the initial state of the superfluid located on sites from 0 to $L - 1$ is independent of $\mu_R$, we immediately get

$$\frac{d}{d \mu_R} \left( \hat{d}_{\alpha \beta} \hat{d}_{\beta \alpha}^{*} \right) = 0,$$  \hspace{1cm} (B.12)

as $j, j' \geq 0$. As $j, j' < 0$, we notice that the reservoir is in the ground state of a free Fermi gas at the chemical potential $\mu_R$, and get

$$\left\{ \begin{array}{l}
\{ \hat{d}_{\alpha \beta} \hat{d}_{\beta \alpha}^{*} \} = \delta_{\alpha \beta} \frac{\sin ((j-j') \theta)}{\pi (j-j')}, \\
\{ \hat{d}_{\alpha \beta} \hat{d}_{\beta \alpha} \} = \delta_{\alpha \beta} \left( \frac{\sin ((j-j') \theta)}{\pi (j-j')} \right)
\end{array} \right.$$  \hspace{1cm} (B.13)

with $\theta = \arccos(\mu_R / 2g_{\beta})$ and $g_{\beta}$ the hopping between two neighbor sites of the reservoir. Finally we find

$$\left( \frac{d}{d \mu_R} \left( \hat{d}_{\alpha \beta} \hat{d}_{\beta \alpha}^{*} \right) \right)_{\mu_R=0} = \left[ \left( 1 - \delta_{\alpha \beta} \right) \left( 2\delta_{\alpha \beta} - 1 \right) \right] \frac{\cos ((j-j') \xi)}{2g_{\beta} \pi}.$$  \hspace{1cm} (B.14)

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