Algebraic $K$-theory of planar cuspal curves

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Introduction

The purpose of this paper is to evaluate the algebraic $K$-groups of a planar cuspal curve over a perfect $\mathbb{F}_p$-algebra relative to the cusp point. A conditional calculation of these groups was given in [4, Theorem A], assuming a conjecture on the structure of certain polytopes. Our calculation here, however, is unconditional and illustrates the advantage of the new setup for topological cyclic homology by Nikolaus–Scholze [10], which we will be using throughout. The only input necessary for our calculation is the evaluation by the Buenos Aires Cyclic Homology group [3] and Larsen [8] of the structure of Hochschild complex of the coordinate ring as a mixed complex, that is, as an object of the $\infty$-category of chain complexes with circle action.

We consider the planar cuspal curve “$y^a = x^b$,” where $a, b \geq 2$ are relatively prime integers. For $m \geq 0$, we define $\ell(a, b, m)$ to be the number of pairs $(i, j)$ of positive integers such that $ai + bj = m$, and for $r \geq 0$, define $S(a, b, r)$ to be the set of positive integers $m$ such that $\ell(a, b, m) \leq r$. The subset $S = S(a, b, r) \subset \mathbb{Z}_{>0}$ is a truncation set in the sense that if $m \in S$ and $d$ divides $m$, then $d \in S$, and hence, the ring of (big) Witt vectors $\mathbb{W}(k)$ with underlying set $k^S$ is defined. We refer to [5, Section 1] for a detailed introduction to Witt vectors.

**Theorem A.** Let $k$ be a perfect $\mathbb{F}_p$-algebra, and let $a, b \geq 2$ be relatively prime integers. There is a canonical isomorphism

$$K_j(k[x, y]/(y^a - x^b), (x, y)) \simeq \mathbb{W}(k)/(V_a \mathbb{W}_{S/a}(k) + V_b \mathbb{W}_{S/b}(k)),$$

if $j = 2r \geq 0$ with $S = S(a, b, r)$, and the remaining $K$-groups are zero.

We remark that recently Angeltveit [1] has given a different proof of this result that, unlike ours, employs equivariant homotopy theory. We also remark that the strategy employed in this paper was used by Speirs [11] to significantly simplify the calculation in [6] of the relative algebraic $K$-groups of a truncated polynomial algebra over a perfect $\mathbb{F}_p$-algebra.

We recall from [4, Section 1] that the group in the statement is a module over the ring $\mathbb{W}(k)$ of big Witt vectors in $k$ of finite length $\frac{1}{2}(2r + 1)(a - 1)(b - 1)$. The calculation of the length goes back to Sylvester [12]. Moreover, it admits a $p$-typical product decomposition indexed by positive integers $m'$ not divisible by $p$. To state this, we let $s = s(a, b, r, p, m')$ be the unique integer such that

$$\ell(a, b, p^{s-1}m') \leq r < \ell(a, b, p^sm'),$$
if such an integer exists, and 0, otherwise. We further assume that $p$ does not divide $b$ and write $a = p^n a'$ with $a'$ not divisible by $p$. Then

$$\mathbb{W}_S(k)/(V_a \mathbb{W}_{S/\alpha}(k) + V_b \mathbb{W}_{S/\beta}(k)) \simeq \prod_{m' \in \mathbb{N}} W_h(k)$$

by a canonical isomorphism, where

$$h = h(a, b, r, p, m') = \begin{cases} 
  s, & \text{if neither } a' \text{ nor } b \text{ divides } m', \\
  \min\{s, u\}, & \text{if } a' \text{ but not } b \text{ divides } m', \\
  0, & \text{if } b \text{ divides } m'.
\end{cases}$$

It is a pleasure to acknowledge the generous support that we have received while working on this paper. Hesseholt was funded in part by the Isaac Newton Institute as a Rothschild Distinguished Visiting Fellow and by the Mathematical Sciences Research Institute as a Simons Professor. Nikolaus was funded in part by the Deutsche Forschungsgemeinschaft under Germany’s Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics–Geometry–Structure. Finally, we are grateful to Tyler Lawson for pointing out that our arguments in an earlier version of this paper could be simplified significantly.

1. Some recollections necessary for the proof

We first recall the Nikolaus–Scholze formula for topological cyclic homology from [10]; see also [7]. We write $\mathbb{T}$ for the circle group and $C_p \subset \mathbb{T}$ for the subgroup of order $p$. If $R$ is a ring, then we write

$$\text{TC}^-(R) \xrightarrow{\text{can}} \text{TP}(R)$$

for the canonical map from the homotopy fixed points to the Tate construction of the spectrum with $\mathbb{T}$-action $\text{THH}(R)$. The Frobenius map

$$\text{THH}(R) \xrightarrow{\varphi} \text{THH}(R)^{tC_p}$$

is $\mathbb{T}$-equivariant, provided that we let $\mathbb{T}$ act on the target through the isomorphism $\rho: \mathbb{T} \to \mathbb{T}/C_p$ given by the $p$th root, and therefore, it induces a map

$$\text{TC}^-(R) = \text{THH}(R)^{h\mathbb{T}} \xrightarrow{\varphi^{h\mathbb{T}}} (\text{THH}(R)^{tC_p})^{h(\mathbb{T}/C_p)}.$$ 

The Tate-orbit lemma [10, I.2.1] identifies the $p$-completion of the target of this map with that of $\text{TP}(R)$, and Nikolaus–Scholze show that, after $p$-completion, the topological cyclic homology of $R$ is the equalizer

$$\text{TC}(R) \xrightarrow{\varphi} \text{TC}^-(R) \xrightarrow{\text{can}} \text{TP}(R)$$

of these two parallel maps. If $p$ is nilpotent in $R$, as is the case in the situation that we consider, then the spectra in question are already $p$-complete.

The normalization of $A = k[x, y]/(y^a - x^b)$ is the $k$-algebra homomorphism to $B = k[t]$ that to $x$ and $y$ assigns $t^a$ and $t^b$, respectively, and this homomorphism identifies $A$ with sub-$k$-algebra $k[t^a, t^b] \subset k[t] = B$. The comparison theorems of
McCarthy [9] and Geisser–Hesselholt [2] show that the square

\[
\begin{array}{ccc}
K(A) & \longrightarrow & TC(A) \\
\downarrow & & \downarrow \\
K(B) & \longrightarrow & TC(B)
\end{array}
\]
is cartesian. Moreover, the fundamental theorem of algebraic K-theory shows that the map \( K(B) \to K(k) \) induced by the \( k \)-algebra homomorphism that to \( t \) assigns 0 is an equivalence. Hence the relative \( K \)-groups that we wish to determine are canonically identified with the homotopy groups of the common fibers of the vertical maps in the diagram above.

The \( k \)-algebras \( A \) and \( B \) are both monoid algebras. In general, if \( k[\Pi] \) is the monoid algebra of an \( E_1 \)-monoid \( \Pi \) in spaces, then, as cyclotomic spectra,

\[
\text{THH}(k[\pi]) \simeq \text{THH}(k \otimes S[\Pi]) \simeq \text{THH}(k) \otimes B^{\text{cy}}(\Pi)",
\]

where \( B^{\text{cy}}(\Pi) \) denotes the unstable cyclic bar-construction of \( \Pi \). In addition, on the right-hand side, the Frobenius map factors as a composition

\[
\text{THH}(k) \otimes B^{\text{cy}}(\Pi)_+ \xrightarrow{\phi \otimes \tilde{\phi}} \text{THH}(k)^{tC_p} \otimes B^{\text{cy}}(\Pi)^{hC_p}_+ \xrightarrow{\text{can}} (\text{THH}(k) \otimes B^{\text{cy}}(\Pi)_+)^{tC_p}
\]
of the map induced by the Frobenius \( \phi: \text{THH}(k) \to \text{THH}(k)^{tC_p} \) and the unstable Frobenius \( \tilde{\phi}: B^{\text{cy}}(\Pi) \to B^{\text{cy}}(\Pi)^{hC_p} \) and a canonical map. Moreover, we have a map of spectra with \( \mathbb{T} \)-action

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\tau_0} & \text{TC}(k) \\
\downarrow & & \downarrow \\
\text{THH}(k)
\end{array}
\]

from \( \mathbb{Z} \) with trivial \( \mathbb{T} \)-action, and therefore, we can rewrite

\[
\text{THH}(k) \otimes B^{\text{cy}}(\Pi)_+ \simeq \text{THH}(k) \otimes \mathbb{Z} \otimes B^{\text{cy}}(\Pi)_+.
\]

Accordingly, we do not need to understand the homotopy type of the space with \( \mathbb{T} \)-action \( B^{\text{cy}}(\Pi) \). It suffices to understand the homotopy type of the chain complex with \( \mathbb{Z} \)-action \( \mathbb{Z} \otimes B^{\text{cy}}(\Pi)_+ \), which, in the case at hand, is exactly what the Buenos Aires Cyclic Homology group [3] and Larsen [8] have done for us.

To state their result, we let \( \langle t^a, t^b \rangle \subset \langle t \rangle \) be the free monoid on a generator \( t \) and the submonoid generated by \( t^a \) and \( t^b \), respectively, and set

\[
B^{\text{cy}}(\langle t \rangle) \simeq B^{\text{cy}}(\langle t^a, t^b \rangle) / B^{\text{cy}}(\langle t^a \rangle).
\]

Counting powers of \( t \) gives a \( \mathbb{T} \)-equivariant decomposition of pointed spaces

\[
B^{\text{cy}}(\langle t \rangle, \langle t^a, t^b \rangle) \simeq \bigvee_{m \in \mathbb{Z}_{>0}} B^{\text{cy}}(\langle t \rangle, \langle t^a, t^b \rangle; m).
\]

We can now state the result of the calculation by the Buenos Aires Cyclic Homology group [3] and by Larsen [8] as follows.
Theorem 1. Let $a, b \geq 2$ be relatively prime integers, and let $m \geq 1$ be an integer. In the ∞-category $D(\mathbb{Z})^{BT}$ of chain complexes with $\mathbb{T}$-action, there is a canonical equivalence between

$$\mathbb{Z} \otimes B^\text{cy}((t), (t^a, t^b); m)$$

and the total cofiber of the square

$$\begin{array}{c}
\mathbb{Z} \otimes (\mathbb{T}/C_m) + [2\ell(a, b, m)] \\
\downarrow \\
\mathbb{Z} \otimes (\mathbb{T}/C_m) + [2\ell(a, b, m)] \\
\end{array}$$

Here, all maps in the square are induced by the respective canonical projections, and if $c$ does not divide $m$, then $\mathbb{T}/C_m/c$ is understood to be the empty set.

The result is not stated in this form in op. cit., and therefore, some explanation is in order. The ∞-category $D(\mathbb{Z})^{BT}$ is equivalent to the ∞-category of modules over the $\mathbb{E}_1$-algebra $C_*(\mathbb{T}, \mathbb{Z})$ given by the singular chains on the circle. This $\mathbb{E}_1$-algebra, in turn, is a Postnikov section of a free $\mathbb{E}_1$-algebra over $\mathbb{Z}$, and therefore, it is formal in the sense that, as an $\mathbb{E}_1$-algebra over $\mathbb{Z}$, it is equivalent to the Pontryagin ring $H_*(\mathbb{T}, \mathbb{Z})$ given by its homology. As a model for the ∞-category of modules over the latter, we may use the dg-category of dg-modules over $H_*(\mathbb{T}, \mathbb{Z})$. But a dg-module over $H_*(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}[d]/(d^2)$ is precisely what is called a mixed complex in op. cit. This shows that $D(\mathbb{Z})^{BT}$ is equivalent to the ∞-category of mixed complexes. Some additional translation is necessary to bring the results in the above form for which we refer to [4 Section 5].

In the following, we will use the abbreviation

$$B_m = B^\text{cy}((t), (t^a, t^b); m).$$

Theorem 1 shows, in particular, that the connectivity of $B_m$ tends to infinity with $m$. Hence, tensoring with $\text{THH}(k)$, we obtain an equivalence

$$\text{THH}(k) \otimes B^\text{cy}((t), (t^a, t^b)) \simeq \bigoplus_{m \in \mathbb{Z}_{>0}} \text{THH}(k) \otimes B_m \simeq \prod_{m \in \mathbb{Z}_{>0}} \text{THH}(k) \otimes B_m,$$

which, in turn, implies a product decomposition

$$(\text{THH}(k) \otimes B^\text{cy}((t), (t^a, t^b)))^{tC_p} \simeq \prod_{m \in \mathbb{Z}_{>0}} (\text{THH}(k) \otimes B_m)^{tC_p}. $$

We will use the following result repeatedly below.

Lemma 2. The unstable Frobenius induces an equivalence

$$\text{THH}(k)^{tC_p} \otimes B_{m/p} \xrightarrow{\text{id} \otimes \overline{\varphi}} (\text{THH}(k) \otimes B_m)^{tC_p},$$

where the left-hand side is understood to be zero, if $p$ does not divide $m$.

Proof. We recall two facts from [4 Section 3]. The first is that the pointed space $B_{m/p}$ is finite, and the second is that the cofiber of the composition

$$B_{m/p} \xrightarrow{\overline{\varphi}} (B_m)^{hC_p} \xrightarrow{\text{can}} B_m$$

of the unstable Frobenius map and the canonical map is a finite colimit of free pointed $C_p$-cells. Here $C_p$ acts trivially on the left-hand and middle terms.
the map in the statement factors as the composition
\[
\text{THH}(k)^{C_p} \otimes B_{m/p} \longrightarrow (\text{THH}(k) \otimes B_{m/p})^{C_p} \longrightarrow (\text{THH}(k) \otimes B_m)^{C_p}
\]
of the canonical colimit interchange map, where \(B_{m/p}\) is equipped with the trivial \(C_p\)-action, and the map of Tate spectra induced from the composite map above. The first fact implies that the left-hand map is an equivalence, and the second fact implies that the right-hand map is an equivalence.

**Proposition 3.** Let \(G\) be a compact Lie group, let \(H \subset G\) be a closed subgroup, let \(\lambda = T_H(G/H)\) be the tangent space at \(H = eH\) with the adjoint left \(H\)-action, and let \(S^\lambda\) the one-point compactification of \(\lambda\). For every spectrum with \(G\)-action \(X\), there are canonical natural equivalences
\[
\begin{align*}
(X \otimes (G/H)_+)^{hG} &\simeq (X \otimes S^\lambda)^{hH}, \\
(X \otimes (G/H)_+)^{tG} &\simeq (X \otimes S^\lambda)^{tH}.
\end{align*}
\]

**Proof.** We recall that for every map of spaces \(f: X \to Y\), the restriction functor \(f^* : \text{Sp}^Y \to \text{Sp}^X\) has both a left adjoint \(f_!\) and a right adjoint \(f^*\). For the unique map \(p: BG \to pt\), we have \(p_! (X) \simeq X^{hG}\) and \(p^* (X) \simeq X^{hG}\). We consider the following diagram of spaces.

\[
\begin{array}{ccc}
BH & \xrightarrow{f} & BG \\
\downarrow{q} & & \downarrow{p} \\
pt & & pt
\end{array}
\]

The top horizontal map is the map induced by the inclusion of \(H\) in \(G\). It is a fiber bundle, whose fibers are compact manifolds. Therefore, by parametrized Atiyah duality, its relative dualizing spectrum \(Df \in \text{Sp}^{BH}\) is given by the sphere bundle associated with the fiberwise normal bundle, which is \(Df \simeq S^{-\lambda}\). By definition of the dualizing spectrum, we have for all \(Y \in \text{Sp}^{BH}\), a natural equivalence
\[
f_!(Y \otimes S^{-\lambda}) \simeq f_*(Y)
\]
in \(\text{Sp}^{BG}\). It follows that for all \(Y \in \text{Sp}^H\), we have a natural equivalence
\[
p_* f_!(Y \otimes S^{-\lambda}) \simeq p_* f_*(Y) \simeq q_*(Y)
\]
in \(\text{Sp}\), which we also write as
\[
((Y \otimes S^{-\lambda}) \otimes_H G_+)^{hG} \simeq Y^{hH}.
\]
By \cite{10} Theorem I.4.1 (3)], we further deduce a natural equivalence
\[
((Y \otimes S^{-\lambda}) \otimes_H G_+)^{tG} \simeq Y^{tH}.
\]
Indeed, the left-hand side vanishes for \(Y = 
\Sigma^\infty H_+\), and the fiber of the map
\[
((Y \otimes S^{-\lambda}) \otimes_H G_+)^{hG} \xlongrightarrow{\text{can}} ((Y \otimes S^{-\lambda}) \otimes_H G_+)^{tG}
\]
preserves colimits in \(Y\).

Finally, given \(X \in \text{Sp}^{BG}\), we set \(Y = f^*(X) \otimes S^\lambda\) to obtain the equivalences in the statement. \(\Box\)

\footnote{Strictly speaking this statement does not make sense, since \(BH\) and \(BG\) are only defined as homotopy types. What we mean is that the map is classified by a map \(BG \to B\text{Diff}(G/H)\), where the latter is the diffeomorphism group of the compact manifold \(G/H\).}
2. Proof of Theorem A

We consider the diagram with horizontal equalizers

\[
\begin{array}{c}
\text{TC}(A) \xrightarrow{\varphi} \text{TP}(A) \\
\downarrow \quad \downarrow \\
\text{TC}(B) \xrightarrow{\varphi} \text{TP}(B)
\end{array}
\]

and wish to evaluate the cofiber of the left-hand vertical map. We have

\[
\text{cofiber} (\text{THH}(A) \to \text{THH}(B)) \simeq \bigoplus_{m \geq 1} \text{THH}(k) \otimes B_m \simeq \prod_{m \geq 1} \text{THH}(k) \otimes B_m.
\]

The right-hand equivalence follows from the fact that the connectivity of $B_m$ tends to infinity with $m$. Therefore, the cofiber of $\text{TC}(A) \to \text{TC}(B)$ is identified with the equalizer of the induced maps

\[
\prod_{m \geq 1} (\text{THH}(k) \otimes B_m)^{ht} \xrightarrow[\text{can}]{} \prod_{m \geq 1} (\text{THH}(k) \otimes B_m)^{tT}.
\]

While the canonical map “can” preserves this product decomposition, the Frobenius map “$\varphi$” takes the factor indexed by $m$ to the factor indexed $pm$. Therefore, we write $m = pm'$ with $m'$ not divisible by $p$ and rewrite the diagram as

\[
\prod_{m'} \prod_{v \geq 0} (\text{THH}(k) \otimes B_{pm'})^{ht} \xrightarrow[\text{can}]{} \prod_{m'} \prod_{v \geq 0} (\text{THH}(k) \otimes B_m)^{tT},
\]

where both maps now preserve the outer product decomposition indexed by positive integers $m'$ not divisible by $p$. We abbreviate and write

\[
\text{TC}(m') \xrightarrow{\varphi} \text{TP}(m')
\]

for the equalizer diagram given by the factors indexed by $m'$. To complete the proof, we evaluate the induced diagram on homology groups.

We fix $m = pm'$. It follows from Theorem 1 that

\[
\text{THH}(k) \otimes B_m \simeq \text{THH}(k) \otimes \mathbb{Z} (\mathbb{Z} \otimes B_m)
\]

agrees, up to canonical equivalence, with the total cofiber of the square

\[
\begin{array}{c}
\text{THH}(k) \otimes (\mathbb{T}/C_{m/a}) + [2\ell(a, b, m)] \xrightarrow{} \text{THH}(k) \otimes (\mathbb{T}/C_{m/a}) + [2\ell(a, b, m)] \\
\downarrow \\
\text{THH}(k) \otimes (\mathbb{T}/C_{m/b}) + [2\ell(a, b, m)] \xrightarrow{} \text{THH}(k) \otimes (\mathbb{T}/C_{m/b}) + [2\ell(a, b, m)].
\end{array}
\]

By Proposition 3, the induced square of $\mathbb{T}$-homotopy fixed points takes the form

\[
\begin{array}{c}
\text{THH}(k)^{hC_{m/a}} [2\ell(a, b, m) + 1] \xrightarrow{} \text{THH}(k)^{hC_{m/a}} [2\ell(a, b, m) + 1] \\
\downarrow \\
\text{THH}(k)^{hC_{m/b}} [2\ell(a, b, m) + 1] \xrightarrow{} \text{THH}(k)^{hC_{m}} [2\ell(a, b, m) + 1]
\end{array}
\]

\[\text{Here we use in an essential way that, as a spectrum with } \mathbb{T} \text{-action, } \text{THH}(k) \text{ is a } \mathbb{Z} \text{-module.}\]
with the maps in the diagram given by the corestriction maps on homotopy fixed points. We now write \( a = p^ua' \) with \( a' \) not divisible by \( p \) and assume (without loss of generality) that \( p \) does not divide \( b \). If \( a \) and \( b \) both do not divide \( m \), then
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq \text{THH}(k)^{hC_m}[2\ell(a, b, m) + 1]
\]
\[
\simeq \text{THH}(k)^{hC_{pv}}[2\ell(a, b, m) + 1],
\]
and if \( a = p^ua' \) but not \( b \) divides \( m \), then
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq \text{cofiber}(\text{THH}(k)^{hC_m/a} \to \text{THH}(k)^{hC_m})[2\ell(a, b, m) + 1]
\]
\[
\simeq \text{cofiber}(\text{THH}(k)^{hC_{pv-u}} \to \text{THH}(k)^{hC_{pv}})[2\ell(a, b, m) + 1]
\]
\[
\simeq 0,
\]
and if \( a \) and \( b \) both divide \( m \), then
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq 0,
\]
since \( B_m \simeq 0 \). By the same reasoning, we find that
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq \text{THH}(k)^{tC_{pv}}[2\ell(a, b, m) + 1],
\]
if \( a \) and \( b \) both do not divide \( m \), that
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq \text{cofiber}(\text{THH}(k)^{tC_{pv-u}} \to \text{THH}(k)^{tC_{pv}})[2\ell(a, b, m) + 1],
\]
if \( a = p^ua' \) divides \( m \) but \( b \) does not divide \( m \), and that
\[
(\text{THH}(k) \otimes B_m)^{ht} \simeq 0,
\]
otherside.

At the level of homotopy groups, the diagram
\[
\begin{array}{ccc}
\text{THH}(k)^{hC_{pv}} & \text{can} & \text{THH}(k)^{tC_{pv}} \\
\downarrow \text{cor} & & \downarrow \text{cor} \\
\text{THH}(k)^{hC_{pv}} & \text{can} & \text{THH}(k)^{tC_{pv}}
\end{array}
\]
becomes
\[
\begin{array}{ccc}
W(k)[t, x]/(tx - p, p^at) & \text{can} & W(k)[t^{\pm 1}, x]/(tx - p, p^at) \\
\downarrow & & \downarrow \\
W(k)[t, x]/(tx - p, p^st) & \text{can} & W(k)[t^{\pm 1}, x]/(tx - p, p^st),
\end{array}
\]
where the horizontal maps are the unique graded \( W(k) \)-algebra homomorphisms that take \( t \) to \( t \) and \( x \) to \( x = pt^{-1} \), and where the vertical maps are the unique maps of graded \( W(k)[t, x] \)-modules that take \( 1 \) to \( p^{v-u} \).

We have now determined the diagram
\[
\begin{array}{ccc}
\text{TC}(m') & \text{can} & \text{TP}(m') \\
\varphi
\end{array}
\]
at the level of homotopy groups, and it is only a matter of bookkeeping to see that
the statement of the theorem ensues. We recall the functions \( s = s(a, b, r, p, m') \)
and \( h = h(a, b, r, p, m') \) from the \( p \)-typical decomposition

\[
\mathbb{W}_S(k)/(V_a \mathbb{W}_{S/a}(k) + V_b \mathbb{W}_{S/b}(k)) \simeq \prod_{m' \in \mathbb{N}_0} W_h(k).
\]

Suppose first that neither \( a' \) nor \( b \) divides \( m' \). Then

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{hT}) \simeq \begin{cases} W_{v+1}(k), & \text{if } 0 \leq v < s, \\ W_v(k), & \text{if } s \leq v, \end{cases}
\]

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{tT}) \simeq W_v(k),
\]

with \( s = s(a, b, r, p, m') \). The Frobenius map

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{hT}) \xrightarrow{\varphi} \pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{tT})
\]

is an isomorphism for \( 0 \leq v < s \), and the canonical map

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{hT}) \xrightarrow{\text{can}} \pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{tT})
\]

is an isomorphism for \( s \leq v \). Hence, we have a map of exact sequences

\[
0 \xrightarrow{} \prod_{s \leq v} W_v(k) \xrightarrow{} \text{TC}_{2r+1}(m') \xrightarrow{} \prod_{0 \leq v < s} W_{v+1}(k) \xrightarrow{} 0
\]

where the left-hand vertical map is an isomorphism, and where the right-hand vertical map is an epimorphism with kernel \( W_s(k) \). Since \( h = s \), we conclude that \( \text{TC}_{2r+1}(m') \simeq W_s(k) \) and \( \text{TC}_{2r}(m') \simeq 0 \) as desired.

Suppose next that \( a' \) but not \( b \) divides \( m' \). If \( u \leq s \), then

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{hT}) \simeq \begin{cases} W_{v+1}(k), & \text{if } 0 \leq v < u, \\ W_u(k), & \text{if } u \leq v, \end{cases}
\]

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^m m'})^{tT}) \simeq \begin{cases} W_v(k), & \text{if } 0 \leq v < u, \\ W_u(k), & \text{if } u \leq v. \end{cases}
\]

Moreover, we see as before that the Frobenius and canonical maps are isomorphisms for \( 0 \leq v < s \) and \( s \leq v \), respectively, so we have a map of exact sequences

\[
0 \xrightarrow{} \prod_{s \leq v} W_u(k) \xrightarrow{} \text{TC}_{2r+1}(m') \xrightarrow{} \prod_{0 \leq v < s} W_c(k) \xrightarrow{} 0
\]

\[
0 \xrightarrow{} \prod_{s \leq v} W_u(k) \xrightarrow{} \text{TP}_{2r+1}(m') \xrightarrow{} \prod_{0 \leq v < s} W_d(k) \xrightarrow{} 0,
\]

where \( c = \min\{u, v + 1\} \) and \( d = \min\{u, v\} \). The left-hand vertical map is an isomorphism, and the right-hand vertical map is an epimorphism with kernel \( W_u(k) \), so again \( \text{TC}_{2r+1}(m') \simeq W_h(k) \), since \( h = u \), and \( \text{TC}_{2r}(m') \simeq 0 \).
If \( s < u \), then

\[
\pi_{2r+1}((\text{THH}(k) \otimes B_{p^r m'})^{hT}) \simeq \begin{cases} 
W_{v+1}(k), & \text{if } 0 \leq v < s, \\
W_v(k), & \text{if } s \leq v < u, \\
W_u(k), & \text{if } u \leq v,
\end{cases}
\]

and a similar argument shows that \( \text{TC}_{2r+1}(m') \simeq W_h(k) \), since \( h = s \), and that \( \text{TC}_{2r}(m') \simeq 0 \).

Finally, if \( b \) divides \( m' \), then \( \text{TC}_{2r+1}^{-}(m') \) and \( \text{TP}_{2r+1}(m') \) are both zero, and therefore, so is \( \text{TC}_{2r+1}(m') \) and \( \text{TC}_{2r}(m') \). This completes the proof.

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