An alternative way to solve the Cohen-Macaulay-ness conjecture in the quiver variety

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Abstract

This short paper is another way to say that one can attack the Cohen-Macaulay-ness conjecture in the geometry of quiver variety using homological algebra.

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1 Introduction

We fix an algebraically closed field $k$. A quiver $Q$ is a quadruple $Q = (Q_0, Q_1, s, t)$, where $Q_0$ is a finite set of vertices, $Q_1$ is a finite set of arrows, and $s, t : Q_1 \to Q_0$ are two applications assigning to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and target $t(\alpha) \in Q_0$. $Q$ is a Dynkin quiver if its underlying graph $\overline{Q}$ is one of the following:

- $A_n$
- $D_n$
- $E_6$
- $E_7$
- $E_8$

when $\overline{Q}$ is connected without cycles, we say that $Q$ is a tree quiver. Clearly, a Dynkin quiver is a tree quiver.
A representation of $Q$ is a couple $M = ((V_i)_{i \in Q_0}, (f_a)_{a \in Q_1})$ of $k$-vector spaces and linear applications such that for any couple of point $(i, \alpha) \in Q_0 \times Q_1$ we associates a $k$-vector space $V_i$ and a linear application $f_a$.

Fix a dimension vector $d = (d_i)_{i \in Q_0}$, and denote by $rep(Q, d)$ the $k$-vector space $\prod_{a \in Q_1} \text{Hom}_k(k^{d(\alpha)}, k^{t(\alpha)})$, this is a topological space under Zariski topology, we will always denoted by $\mathbb{A}^l$, where $l = \sum_{a \in Q_1} d_a t(a)$, thus any representation of $Q$ is a point of $\mathbb{A}^l$.

The algebraic group $\prod_{i \in Q_0} \text{GL}_d(k)$ acts on $\mathbb{A}^l$ by conjugation, this mean that for any $(g_i)_{i \in Q_0} \in \prod_{i \in Q_0} \text{GL}_d(k)$ and $(f_a)_{a \in Q_1} \in \mathbb{A}^l$, $(g_i g)_{i \in Q_0} = (g_i(g_a f_a g_i^{-1}))_{a \in Q_1} \in \mathbb{A}^l$. Let $O_M$ be the orbit of the representation $M$ and $\overline{O}_M$ its closure, it’s very interesting to study the elements which lie on the boundary of $\overline{O}_M$ (degeneration), and the geometric properties such as regularity, normality, Cohen-Macaulayness, Gorensteinness... somehow this help to understand the category of finite-dimensional modules over the path algebra $kQ$, especially the classification problem of representations. Let $k[\overline{O}_M]$ be the coordinate ring of the variety $\overline{O}_M$, this is isomorphic to the affine $k$-algebra $k[rep(Q, d)]/I(\overline{O}_M)$, where $k[rep(Q, d)]$ is the polynomial algebra $k[x_{i,j}, \alpha \in Q_1, 1 \leq i \leq t(\alpha), 1 \leq j \leq s(\alpha)]$ and $I(\overline{O}_M)$ is the ideal of vanishing polynomials on $\overline{O}_M$. A fundamental question arises: what is the projective dimension of the $k[rep(Q, d)]$-module $k[\overline{O}_M]$? Hilbert’s syzygy theorem [5,6] ensures that the projective dimension of the $k[rep(Q, d)]$-module $k[\overline{O}_M]$ is finite, this mean that there exist a $k[rep(Q, d)]$-free resolution

$$0 \to F_p \to \cdots \to F_1 \to F_0 \to k[\overline{O}_M] \to 0$$

where $F_0, F_1, ..., F_p$ are finitely generated free $k[rep(Q, d)]$-modules. Sometimes the projective dimension of the coordinate ring $k[\overline{O}_M]$ (as a $k[rep(Q, d)]$-module) can tell us something on the ideal of the variety $\overline{O}_M$, see for example Hilbert-Burch theorem, more precisely if $\text{pd} k[\overline{O}_M] = 2$, then $I(\overline{O}_M) = a I_4(A)$ where $a \in k[rep(Q, d)]$ and $I_4(A)$ is the ideal generated by the $t \times t$ subdeterminants of the matrix $A$ with entries in $k[rep(Q, d)]$, therefore we know exactly what are the equations that define the variety $\overline{O}_M$ (see [5,6] for more details). it has been proven that when $Q$ is of type $A_n$ or $D_n$ the orbit closures are normal and Cohen-Macaulay with rational singularities (see [1,2,3]), unfortunately for the quivers $E_6$, $E_7$ and $E_8$ the answer is unknown. In this paper, we will restrict our consideration to the Cohen-Macaulayness case. According to the theorem below (theorem 3), it seems that there is a strong connection of this last property and the projective dimension of the $k[rep(Q, d)]$-module $k[\overline{O}_M]$. We believe that this connection can help to solve the Cohen-Macaulay problem for the quivers $E_6$, $E_7$ and $E_8$, however finding this invariant is not an easy task at all, see [6].

## 2 Basic Concepts

In this section, we recall some background of representation theory and algebraic geometry. To gain a deeper understanding we refer the reader to [4,5,6,7].

Let $Q = (Q_0, Q_1, s,t)$ be a finite quiver.

$Q$ is said to be finite if $Q_0$ and $Q_1$ are finite sets, and connected if the underlying graph $\overline{Q}$ of $Q$ is a connected graph. Consider two representations $M = ((M_i)_{i \in Q_0}, (f_a)_{a \in Q_1})$ and $N = ((N_i)_{i \in Q_0}, (g_a)_{a \in Q_1})$ of $Q$, a morphism $(h_i)_{i \in Q_0} : M \to N$ of representations is given by a family of linear applications
Theorem 1. [Artin-Voigt, 7]

If \( M \) is a representation of dimension \( d \), then \( \dim(\text{rep}(Q,d)) - \dim(O_M) = \dim_k \text{Ext}^1_Q(M, M) \).

We finish this section recalling several well known and useful tools about commutative algebra. Let \( L \) be a finitely generated module over the polynomial algebra \( R = k[x_1, \ldots, x_l] \) and \( p \) a prime ideal of \( R \). We define these five homological invariants: depth, Krull dimension, height, grade and the projective dimension by:

* \( \text{depth}(m, L) = \min \{ n : \text{Ext}_R^n(k, L) \neq 0 \} \), with \( m = (x_1, x_2, \ldots, x_l) \),
* \( \dim(L) = \dim(R/\text{ann}(L)) \),
* \( \text{ht}(p) = \sup \{ p_0 \subset p_1 \subset \cdots \subset p_s = p \}, \) where \( p_0, \ldots, p_s \in \text{spec}(R) \),
* \( \text{grade}(p) = \text{grade}(R/p, R) = \min \{ n : \text{Ext}_R^n(R/p, R) \neq 0 \} \),
* \( \text{pd}(L) = \sup \{ n : \text{Ext}_R^n(L, -) \neq 0 \} \) We say that \( L \) is Cohen-Macaulay \( R \)–module if \( \text{depth}(m, L) = \dim(L) \), when \( L = R \), this is equivalent to saying that \( R_p \) is Cohen-Macaulay for every prime ideal \( p \) of \( R \). An affine variety \( X \) is said to be Cohen-Macaulay if its coordinate ring \( k[X] \) is Cohen-Macaulay.
Theorem 2. [Auslander-buchsbaum,[4,5]]

Let $L$ be a finitely generated graded module over the polynomial algebra $k[x_1, ..., x_i]$. Then, $pd(L) + \text{depth}(m, L) = \dim(R)$.

3 Main results

Definition 3.1. Let $M$ be a representation of some bound quiver $(Q, I)$, where $I$ is a homogeneous graded ideal. $M$ will be called homogeneous if $\mathcal{O}_M$ is an affine cone, that is $\mathcal{O}_M$ contains all lines spanned by its elements.

We denote by $\lambda M$ the representation with the point $\lambda m$ in the variety $\text{rep}(Q,d)$.

Lemma 3.2. $M$ is homogeneous $\iff \lambda M \cong M$ for all $\lambda \neq 0$.

Proof. If $M$ is homogeneous, then $\lambda m \in \mathcal{O}_M$, thus there exist a morphism $\phi : k \rightarrow \mathcal{O}_M$ such that $\phi(t) = m$, $\forall t \neq 0$ and $\phi(0) = \lambda m$. Using this fact, we can easily get a decreasing sequence of orbit closures $\mathcal{O}_M \supseteq \mathcal{O}_{\lambda M} \supseteq \mathcal{O}_{\lambda^2 M} \supseteq ... \supseteq \mathcal{O}_{\lambda^k M} \supseteq ...$. Now by the noetherianity of $k[\text{rep}(Q,d)]$ there exists $N^*$ such that $\mathcal{O}_{\lambda^N M} = \mathcal{O}_{\lambda^{N+1} M}$, but we know that orbits are constructible, therefore $\lambda^p M \cong \lambda^{p+1} M$ and in particular $\lambda M \cong M$. For the converse, let $\psi : k \rightarrow \text{rep}(Q,d)$ defined by $\psi(t) = tm$. The inverse image of the orbit closure of $M$ is closed set of $k$ and by the hypothesis, it is infinite. Thus it must be $k$ and $\psi(0) = 0 \in \mathcal{O}_M$.

Remark 3.3. In general, one can find representations which are not homogeneous. Fix $k = \mathbb{C}$ and consider the following quiver:

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\beta
\end{array}
\]

$\alpha^2 = 0$ and $\beta^3 = 0$. Then take

\[
M_\alpha = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad M_\beta = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

one can easily prove that for $\lambda \neq \pm 1$, $\lambda M$ is not isomorphic to $M$. More generally we have this direct implication: If any representation $M$ of a quiver $Q$ is homogeneous, then $Q$ has no oriented cycle. This later come from the fact that homogeneous representations are nilpotent.

Theorem 3. Let $Q$ be a tree quiver and $M$ a representation of $Q$.

\[\mathcal{O}_M \text{ is Cohen-Macaulay} \iff pd(k[\mathcal{O}_M]) = \dim(k) + \dim_k \text{End}_Q(M) - \dim_k \prod_{i \in Q_0} \text{End}(M_i)\]

Proof. Assume that $\mathcal{O}_M$ is CM, and let $0 = M_0 \subsetneq M_1 \subsetneq ... \subsetneq M_r = M$ be a composition series of $M$.

Since $Q$ is an acyclic quiver, every composition factor $M_i/M_{i-1}$ is isomorphic to some simple representation of the form $S_j$ for $j \in Q_0$. This means that $M$ degenerates to $\bigoplus_{i=1}^r S_1$, hence $0 \in \mathcal{O}_M$.

Let $\alpha : 1 \rightarrow 2$ be an arrow in $Q_1$. Put $\phi_1 = \text{Id}_{M_1}$ and $\phi_2 = \lambda M_{M_2}$, now if $\beta : d \rightarrow c$, we take
\( \phi_d = \lambda' \text{Id}_{M_d} \) if \( \phi_c = \lambda'^{-1} \text{Id}_{M_c} \) or \( \phi_c = \lambda' \text{Id}_{M_c} \) if \( \phi_d = \lambda'^{-1} \text{Id}_{M_d} \). Thus, the diagram below

\[
\begin{array}{ccc}
M_a & \xrightarrow{f_a} & M_b \\
\phi_c \downarrow & & \downarrow \phi_b \\
M_a & \xrightarrow{\lambda M_a} & M_b
\end{array}
\]

commutes for every arrow : \( a \to b \). Therefore, the representations \( M \) and \( \lambda M \) lie in the same orbit, i.e., \( O_M = O_{\lambda M} \). Note that we have proved that \( \overline{O}_M \) is an affine cone, thus the ideal \( I(O_M) \) is graded.

By Auslander-Buchsbaum and Artin-Voit formulas, we have the equality \( pd(k[\overline{O}_M]) = dim(k[\text{rep}(Q,d)]) - \text{depth}(m,k[\overline{O}_M]) = \text{dim}_k \text{Ext}^1_Q(M,M) + \text{dim}(k[\overline{O}_M]) - \text{depth}(m,k[\overline{O}_M]) \), this can be written \( pd(k[\overline{O}_M]) = \text{dim}_k \text{Ext}^1_Q(M,M) = \text{dim}(k[\overline{O}_M]) - \text{depth}(m,k[\overline{O}_M]) \).

Now by the Cohen-Macaulayness of the orbits closure, the projective dimension of \( k[\overline{O}_M] \) is exactly the dimension of the space \( \text{Ext}^1_Q(M,M) \). Finally using Ringel’s canonical exact sequence [8] :

\[
\begin{array}{c}
0 \longrightarrow \text{End}_Q(M) \longrightarrow \prod_{i \in \mathbb{Q}_0} \text{End}_k(M_i) \longrightarrow \text{rep}(Q,d) \longrightarrow \text{Ext}^1_Q(M,M) \longrightarrow 0
\end{array}
\]

We obtain the desired result, \( pd(k[\overline{O}_M]) = \text{dim}(\Lambda^1) + \text{dim}_k \text{End}_Q(M) - \text{dim}_k \prod_{i \in \mathbb{Q}_0} \text{End}_k(M_i) \).

Conversely, \( pd(k[\overline{O}_M]) = \text{dim}(\Lambda^1) + \text{dim}_k \text{End}_Q(M) - \text{dim}_k \prod_{i \in \mathbb{Q}_0} \text{End}_k(M_i) = \text{dim}_k \text{Ext}^1_Q(M,M) = \text{dim}(k[\text{rep}(Q,d)]) - \text{dim} \overline{O}_M = h(I(\overline{O}_M)) = \text{grade}(I(\overline{O}_M)). \) Hence, \( I(\overline{O}_M) \) is a perfect ideal, and by exercise 19.9 in [5], the variety \( \overline{O}_M \) is CM. \( \square \)

**Corollary 3.4.** Let \( Q \) be a tree quiver and \( M \) a representation of \( Q \). Assume that \( \overline{O}_M \) is CM. Then,

1- \( pd(k[\overline{O}_M]) = \min\{pd(k[\overline{O}_N]), M \leq_{d_{Qg}} N\} \)

2- \( O_M \) is closed \( \Leftrightarrow \) \( pd(k[\overline{O}_M]) = pd(k[\overline{O}_N]) \) for every \( n \in \overline{O}_M \)

**Proof.** If \( \overline{O}_M \) is CM then by the previous theorem \( pd(k[\overline{O}_M]) = \text{dim}_k \text{Ext}^1_Q(M,M) \leq \text{dim}_k \text{Ext}^1_Q(N,N) = \text{grade}(I(\overline{O}_M)) = \min\{n : \text{Ext}^n_k(k[\overline{O}_M],k[\text{rep}(Q,d)]) \neq 0 \} \leq \text{pd}(k[\overline{O}_N]) \).

If \( O_M \) is closed, then its boundary is empty, thus every degeneration of \( M \) has a closed orbit with equal to \( O_M \).

If \( pd(k[\overline{O}_M]) = pd(k[\overline{O}_N]) \) for every \( n \in \overline{O}_M \), then \( \overline{O}_M = \overline{O}_N \). We know that orbits are locally closed, thus \( M \) is isomorphic to \( N \). \( \square \)

**Corollary 3.5.** Let \( Q \) be a quiver of type \( \mathbb{A}_n \) or \( \mathbb{D}_n \).

Then, \( pd(k[\overline{O}_M]) = \text{dim}(\Lambda^1) + \text{dim}_k \text{End}_Q(M) - \text{dim}_k \prod_{i \in \mathbb{Q}_0} \text{End}_k(M_i). \)

**Proof.** by [1,2,3] orbits closure are CM in \( \mathbb{A}_n \) and \( \mathbb{D}_n \). \( \square \)

**Remark 3.6.** There exist a quiver \( Q \) and a representation \( M \) such that the projective dimension formula does not hold. In fact, we can take

\[
Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1
\]

and \( m \) is the point \( (M_\alpha, M_\beta) \) in \( \text{rep}(Q,d = (3,3)) \) with:
\[
M_\alpha = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad M_\beta = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

See [9].

4 References

[1] Abeasis.S, Fra.A.del, Kraft.H, The geometry of representations of \(A_m\), AMS,256(1981),401-418.

[2] Bobinski.G,Zwara.G, Normality of orbit closures for Dynkin quivers of type \(A_n\), Manuscripta Math.105 (2001), 103-109.

[3] Bobinsky.G,Zwara.G, Schubet varieties and representations of Dynkin quivers, Colloq.Math-94(2002),285-309.

[4] Bruns.W, Herzog.J, Cohen-Macaulay rings, Cambridge stud.Adv Math,39,Cambridge Univ Press (1993).

[5] Eisenbud.D, Commutative algebra with a view toward algebraic geometry, volume 150, Springer-Verlag,New York,1995.

[6] Eisenbud.D, The geometry of syzygies, volume 229, Springer-Verlag, New York,2005.

[7] Hazewinkel.M, Gubareni.N Kirichenko.V.V, Algebras,rings and Modules, volume 2, Springer netherlands, volume586,2007.

[8] Ringel.C.M, Representations of K-species and modules, J.Algebra,41(2),269-302,1976.

[9] Zwara.G An orbit closure for a representation of the Kronecker quiver with bad singularities, Colloq.Math.97,(2003),81-86.

[10] Skowronski.A, Yamagata.Kunio Representations of algebras and related topics, EMP, Switzerland,2011.