Yang–Mills–Scalar–Matter Fields in the Two-Point Space

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Abstract
The purpose of this work is to present a Non–Commutative Geometrical version of the Yang–Mills Theory and the Yang–Mills Scalar Matter Theory by constructing a concrete example using M. Durdevich’s theory of quantum principal bundles.

Keywords: Quantum Principal Bundles (qpb), Yang–Mills Theory, Yang–Mills Scalar Matter Theory.

1 Introduction
The study of principal bundles and principal connections have applications in Topology and Differential Geometry among other areas. In Physics, it provides the natural framework to talk about Classical and Quantum Field Theory; in particular, it is possible to talk about the Yang–Mills theory and the theory of matter fields. For the purpose of this work, we will consider that for a Yang–Mills Scalar Matter model it is necessary:

1. A Riemannian or a Pseudo–Riemannian closed and oriented manifold \((M, g)\).
2. A principal \(G\)–bundle over \(M\), \(\pi : GM \to M\) with \(G\) a compact group.
3. An ad–invariant inner product in the Lie algebra \(g\) of \(G\).
4. A unitary representation \(\alpha\) of \(G\) acting on \(V^\alpha\).
5. A smooth function \(V : \mathbb{R} \to \mathbb{R}\) called the potential.
Using these objects we define the Lagrangian

$$L_{YMSM}(\omega, T) := -\frac{1}{2} \langle R^\omega, R^\omega \rangle + \frac{1}{2} \langle \nabla^\omega T, \nabla^\omega T \rangle - V(T)$$

(1)

where $\omega$ is a principal connection, $R^\omega$ is its curvature viewed as $g_M$–valued 2–form, $T \in \Gamma(M, V^\alpha M)$, $\nabla^\omega$ is the induced linear connection and $V(T) = V((T, T))$.

This Lagrangian is gauge–invariant and the critical points of its associated action

$$\mathcal{A}_{YMSM}(\omega, T) = \int_M L_{YMSM}(\omega, T) \, dvol_g$$

(2)

satisfy

$$\langle \nabla^\omega T \mid \lambda \rangle = \langle d^{\nabla^\omega \ast} R^\omega \mid \lambda \cdot T \rangle$$

(3)

for all $g_M$–valued 1–form $\lambda$, where $d^{\nabla^\omega \ast}$ is the adjoint operator of the exterior covariant derivative of the induced linear connection $\nabla^\omega$; and

$$\nabla^\omega \ast \nabla^\omega T - V'(T) = 0.$$  

(4)

These equations describe the dynamics of scalar matter particles coupled to gauge bosons with symmetry $G$.

In this way, it should be natural to think that a non–commutative geometrical version of the Yang–Mills theory and scalar matter fields has to arise from the non–commutative geometrical version of the theory of principal bundles and principal connections.

M. Durdevich developed in [D1], [D2] (among other works) a complete theory of quantum principal bundles (qpbs) and quantum principal connections (qpcs) using the concept of quantum group studied by S. Woronovicz in [W1], [W2] as the structure group of the bundle. Durdevich’s theory uses the universal differential envelope $\ast$–calculus as quantum differential forms on the group; however, every differential $\ast$–calculus that extends the comultiplication can be used, for example, the braided exterior $\ast$–calculus. The universal differential envelope $\ast$–calculus is maximal with this property. If the read wants to know in detail the theory, we recommend the book written by S. Sontz [So], where the reader can find explicit calculations and exercises.

Using Durdevich’s theory one can get a very natural non–commutative geometrical version of the Yang–Mills theory and the Yang–Mills Scalar Matter theory, i.e., one can get the non–commutative geometrical version of Equations (2), (3) and (4) [Sa11], [Sa12], [Sa13]. This work aims is to show a concrete example of this theory. Specifically, we will work on the quantum trivial principal bundle given by the two–point space as the base space and the symmetric group of order 2, $S_2$, as the structure group; and on this quantum principal bundle we will show the respective Lagrangians and the non–commutative geometrical associated field equations, as well as their solutions. It is worth mentioning again that this is just one example of the general theory.
To accomplish our purpose this paper breaks down into 9 parts. In the second one we present the two-point space in the Non-Commutative Geometry’s framework, the graded differential $\ast$–algebra that we will use as well as some other geometrical concepts, for example, the codifferential. In the third one we will talk about the structure quantum group $S_2$. In the fourth section we are going to present the quantum principal bundle on which we will work as well a characterization of its quantum principal connections, curvatures and covariant derivatives. In the fifth section we will present the theory of associated quantum vector bundles which includes induced quantum linear connections, exterior covariant derivatives and their adjoint operators. The sixth section is to present our definition of the quantum gauge group which is based on the one presented by T. Brzeziński in [Br]. The seventh section is for the non–commutative geometrical version of the Yang–Mills theory and the eighth one is for the non–commutative geometrical version of the Yang–Mills Scalar Matter theory. In the last section we will talk about some final and important comments.

Unfortunately, I did not meet Zbigniew Oziewicz, but I have heard to many stories about him, particularly, because he was a regular loved member of the Seminar of Quantum Geometry directed by M. Durdevich at UNAM. He used to work in Category Theory and I think that the categorical formulation of this theory presented in [Sa12] would have liked him.

2 The Two–Point Space

Let us start taking the two-point space $\{x_0, x_1\}$ and the quantum space ($C^*$–algebra)

\[ M := C_C(\{x_0, x_1\}), \cdot, \beta(x), || ||_\infty, \ast ) \]

A canonical basis of $M$ as $C$–vector space is given by the functions

\[ \beta_M := \{ \{x_0, x_1\} : M \rightarrow C, \, p_1 : \{x_0, x_1\} \rightarrow C \}, \]

which are defined by

\[ p_0(x) = \begin{cases} 0 & \text{if } x = x_1 \\ 1 & \text{if } x = x_0 \end{cases}, \quad p_1(x) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1. \end{cases} \]

Our next step is considering the universal graded differential $\ast$–algebra of $M$ without $n$–forms for $n \geq 3$

\[ (\Omega^\ast(M), d, \ast) \]

We can represent this algebra as follows:

\[ M \leftrightarrow \left\{ p = \begin{pmatrix} z_0 & 0 \\ 0 & z_1 \end{pmatrix} \middle| z_0, z_1 \in C \right\}, \]

\[ \Omega^1(M) \leftrightarrow \left\{ \mu = \begin{pmatrix} 0 & z_0 \\ z_1 & 0 \end{pmatrix} \middle| z_0, z_1 \in C \right\}, \]
\(\Omega^2(M) \leftrightarrow \left\{ \mu = \begin{pmatrix} z_0 & 0 \\ 0 & z_1 \end{pmatrix} \ | \ z_0, z_1 \in \mathbb{C} \right\}.\)

This space will play the role of quantum differential forms on \(M\).

Consider
\[dvol := p_0 \, dp_1 \, dp_0 + p_1 \, dp_1 \, dp_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \Omega^2(M).\]

We can define left and right hermitian structures on \(\Omega^\bullet(M)\):

\[\langle - , - \rangle : M \times M \to M \]
\[(\hat{\mu}, p) \mapsto \hat{p} \hat{p}^*;\]

\[\langle - , - \rangle : \Omega^1(M) \times \Omega^1(M) \to M \]
\[(\hat{\mu}, \mu) \mapsto \hat{\lambda}_0 \lambda_0^* p_0 + \hat{\lambda}_1 \lambda_1^* p_1\]

if \(\hat{\mu} = \begin{pmatrix} 0 & \hat{\lambda}_0 \\ \hat{\lambda}_1 & 0 \end{pmatrix}, \mu = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix}\):

\[\langle - , - \rangle : \Omega^2(M) \times \Omega^2(M) \to M \]
\[(\hat{p} \, dvol, p \, dvol) \mapsto \hat{p} \hat{p}^*.\]

and finally
\[\langle \hat{\mu}, \mu \rangle_r = \langle \hat{\mu}^*, \mu^* \rangle_l,\]

which will play the role of left and right quantum riemannian structures on \(M\).

A quantum integral can be defined as the linear map
\[\int_M : \Omega^2(M) \to \mathbb{C} \text{ s.t. } \int_M p_0 \, dp_1 \, dp_0 = \frac{1}{2} = \int_M p_1 \, dp_1 \, dp_1.\]

All of these allow us to introduce the inner products
\[\langle - | - \rangle_l = \int_M \langle - , - \rangle_l \, dvol, \quad \langle - | - \rangle_r = \int_M \langle - , - \rangle_r \, dvol.\]

The left and right quantum Hodge operators \(\star_l, \star_r\) can be defined and with that, the left and right codifferential:
\[d^{\star_l} \mu = \begin{pmatrix} i(z_0 - z_1) & 0 \\ 0 & -i(z_0 - z_1) \end{pmatrix}\]
for \(\mu \in \Omega^1(M)\),
\[d^{\star_r} \mu = \begin{pmatrix} 0 & -(z_0 + z_1) \\ -(z_0 + z_1) & 0 \end{pmatrix}\]
for \(\mu \in \Omega^2(M)\) and
\[d^{\star} := \star \circ d^{\star_l} \circ \star.\]

Of course
\[\langle \eta | d\eta \rangle_l = \langle d^{\star_l} \eta | \eta \rangle_l \text{ and } \langle \eta | d\eta \rangle_r = \langle d^{\star_r} \eta | \eta \rangle_r.\]
3 The Group $S_2$

The quantum space $(G := C_\mathbb{C}(S_2), \cdot, \|x\|_\infty, \star)$ associated to the group $S_2$ is basically the same as $M$ but we will use the notation

$$\{\Delta_0, \Delta_1\}$$

for its basis. The structure of Hopf algebra will be denoted by $(\phi, \epsilon, \kappa)$, where $\phi$ is the coproduct, $\epsilon$ is the counity and $\kappa$ is the coinverse. Also we are going to use the universal graded differential $*$-algebra

$$\left(\Gamma^\wedge, d, *\right)$$

as quantum differential forms on $G$.

In this way the space $\text{inv} \Gamma := \text{Ker}(\epsilon)$ will play the role of the quantum Lie algebra and

$$\{\varsigma := (\Delta_0 - \Delta_1)d\Delta_1\}$$

is a basis of $\text{inv} \Gamma$.

4 The Quantum Principal Bundle

We will consider the trivial quantum principal bundle (qp) $\zeta := (M \otimes G, M, \Phi)$, where $\Phi(x \otimes g) = x \otimes \phi(g)$.

The differential calculus $(\Omega^*(M \otimes G), d, *)$ on the bundle will be the tensor product of $(\Omega^*(M), d, *)$ and $(\Gamma^\wedge, d, *)$. The extension of $\Phi$ will be given by $\Psi(\mu \otimes \vartheta) = x \otimes \phi(\vartheta)$, where here, $\phi$ is extension of the coproduct on $(\Gamma^\wedge, d, *)$.

In this case, quantum principal connections (qpcs)

$$\omega : \text{inv} \Gamma \rightarrow \Omega^1(M \otimes G)$$

are completely defined by the equation

$$\omega(\varsigma) = \mu \otimes 1 + 1 \otimes \varsigma,$$

for some $\mu \in \Omega^1(M)$.

The curvature of any qpc

$$R^\omega : \text{inv} \Gamma \rightarrow \Omega^2(M \otimes G)$$

is defined by

$$R^\omega(\varsigma) = d\mu - 2\mu \mu \in \Omega^2(M);$$

It is worth mentioning that in this case, the universal differential envelope $*$-calculus agrees with the universal graded differential $*$-algebra.
while the covariant derivative
\[ D^\omega : \text{Hor}^* (M \otimes G) = \Omega^* (M) \otimes G \longrightarrow \text{Hor}^* (M \otimes G) = \Omega^* (M) \otimes G \]
is given by
\[ D^\omega (\eta \otimes g) = d(\eta \otimes g) - (-1)^{\eta \cdot \omega} \eta \cdot \omega (\kappa (g^{(1)}) dg^{(2)}) \]
if \( \phi (g) = g^{(1)} \otimes g^{(2)} \) (in Sweedler’s notation) [So].

5 Associated Quantum Vector Bundles

For any unitary finite–dimensional corepresentation \( \alpha \) on \( V^\alpha \) of \( G \), the space \( \text{Mor}(\alpha, \Phi) \) (morphisms of corepresentations) is a \( M \)–bimodule. Even more, it is a finitely generated projective left and right \( M \)–module. Taking the left (right) structure, we define the left associated quantum vector bundle (left associated quantum vector bundle) as \( \Gamma^l (M, V^\alpha M) := \text{Mor}(\alpha, \Phi) \) (left qvb–valued differential forms) and \( \text{Mor}(\alpha, \Phi) \otimes_M \Omega^1 (M) \) (right qvb–valued differential forms). Taking a qpc \( \omega \) and denoting the last isomorphisms by \( \Upsilon_\alpha \) and \( \hat{\Upsilon}_\alpha \), respectively, the linear maps
\[ \nabla^\omega_\alpha := \Upsilon_\alpha \circ D^\omega : \Gamma^l (M, V^\alpha M) \longrightarrow \Omega^1 (M) \otimes_M \Gamma^l (M, V^\alpha M) \]
\[ \hat{\nabla}^\omega_\alpha := \hat{\Upsilon}_\alpha \circ \ast \circ D^\omega \circ \ast : \Gamma^r (M, V^\alpha M) \longrightarrow \Gamma^r (M, V^\alpha M) \otimes_M \Omega^1 (M) \]
satisfy the left and right Leibniz rule, respectively, and they are known as the induced quantum linear connections (induced qlcs). The exterior covariant derivatives also fulfill
\[ d\nabla^\omega_\alpha = \Upsilon_\alpha \circ D^\omega \circ \Upsilon^{-1}_\alpha, \quad d\hat{\nabla}^\omega_\alpha = \hat{\Upsilon}_\alpha \circ \ast \circ D^\omega \circ \ast \circ \hat{\Upsilon}^{-1}_\alpha. \]

In Differential Geometry, for a linear representation \( \alpha \) of the Lie group of a principal bundle, the Gauge Principle ([KMS], [SW]) gives us a characterization of vector–bundle–valued differential forms in terms of basic differential forms of type \( \alpha \), which we can observe here with the fact that \( \Upsilon_\alpha, \hat{\Upsilon}_\alpha \) are isomorphisms. The Gauge Principle also tells us that induced linear connections and their exterior covariant derivatives on associated vector bundles are given by covariant derivatives of principal connections. The theory presented shows the non–commutative geometrical counterpart of the Gauge Principle [Sa11], [Sa12].
Following the classical case, the inner products \{\langle -| - \rangle_l\} and \{\langle -| - \rangle_r\} can be extended to qvb–valued differential forms and the maps

\[
d^\nabla^\star_l := (-1)^k (\ast^{-1}_l \circ \ast) \otimes \text{id} \circ d^\nabla \circ ((\ast \circ \ast) \otimes \text{id}),
\]

\[
d^\nabla^\star_r := (-1)^k (\text{id} \otimes \ast^{-1}_r \circ \ast) \circ d^\nabla \circ (\text{id} \otimes \ast \circ \ast),
\]

are the adjoint operators of \(d^\nabla\) and \(d^\nabla\), respectively \[Sa13\], i.e.

\[
\langle \psi' | d^\nabla^\star_l \psi \rangle_l = \langle d^\nabla^\star_l \psi' | \psi \rangle_l
\]

\[
\langle \varphi' | d^\nabla^\star_r \varphi \rangle_r = \langle d^\nabla^\star_r \varphi' | \varphi \rangle_r.
\]

6 The Quantum Gauge Group

Let \(f_1, f_2 : \Gamma^\wedge \rightarrow \Omega^\bullet(M \otimes G)\) be two graded–preserving linear maps. The convolution product of \(f_1\) with \(f_2\) is defined by

\[
f_1 \ast f_2 = m \circ (f_1 \otimes f_2) \circ \phi : \Gamma^\wedge \rightarrow \Omega^\bullet(M \otimes G),
\]

where \(m\) is the product of \(\Omega^\bullet(M \otimes G)\). \(f\) is a convolution invertible map if there exists \(f^{-1} : \Gamma^\wedge \rightarrow \Omega^\bullet(M \otimes G)\) such that

\[
f \ast f^{-1} = f^{-1} \ast f = \mathbb{1}; \quad f(\mathbb{1}) = \mathbb{1};
\]

and

\[
\begin{array}{c}
\Gamma^\wedge \\
\downarrow f \\
\Omega^\bullet(M \otimes G)
\end{array}
\xrightarrow{Ad}
\begin{array}{c}
\Gamma^\wedge \otimes \Gamma^\wedge \\
\downarrow \otimes \\
\Omega^\bullet(M \otimes G) \otimes \Gamma^\wedge
\end{array}
\]

where \(Ad : \Gamma^\wedge \rightarrow \Gamma^\wedge \otimes \Gamma^\wedge\) is the extension of the right adjoint coaction \(Ad : G \rightarrow G \otimes G\) \([W2]\). The quantum gauge group \(GG\) is defined as the group of all convolution invertible maps \[Sa12\]. \(GG\) acts on the space of qpcs and on \(\Gamma^\wedge(M, V^\alpha M)\), \(\Gamma^\wedge(M, V^\alpha M)\) by means of

\[
F_l \circ \omega, \quad F_l \circ T, \quad * \circ F_l \circ * \circ T,
\]

respectively, where

\[
F_l := m \circ (\text{id}_{\Omega^\bullet(M \otimes G)} \otimes f) \circ \Psi.
\]

7 Yang–Mills Theory

We can define the adjoint copresentation on \(\text{inv}, \Gamma\) as

\[
ad(\varsigma) = \varsigma \otimes \mathbb{1},
\]
so we can consider the qvbs associated to ad. Since \( R^\omega, \tilde{R}^\omega := \ast \circ R^\alpha \circ \ast \in \text{Mor}(\text{ad},\Psi|_{\text{Hor}^\bullet(M\otimes G)}) \), the expression

\[
L_{\text{YM}}(\omega) := -\frac{1}{4} \left( \langle R^\omega, R^\omega \rangle_l + \langle \tilde{R}^\omega, \tilde{R}^\omega \rangle_r \right)
\]

is well-defined and it is called the non-commutative geometrical Yang–Mills Lagrangian [Sa13]. It is gauge-invariant just for a subgroup

\[
GG_{\text{YM}} := \{ f \in GG | f \circ \omega^{\text{triv}} = \omega^{\text{triv}} \text{ for all } \omega \} \supset S_2,
\]

where \( \omega^{\text{triv}}(\varsigma) = 1 \otimes \varsigma \).

Critical points of its associated action

\[
S_{\text{YM}}(\omega) := -\frac{1}{4} \left( \langle R^\omega, R^\omega \rangle_l + \langle \tilde{R}^\omega, \tilde{R}^\omega \rangle_r \right)
\]

satisfy

\[
\langle \lambda | (d\nabla^* - dS^* - d\tilde{S}^*) R^\omega \rangle_l + \langle \ast \circ \lambda \circ \ast | (d\nabla^* - dS^* - d\tilde{S}^*) R^\omega \rangle_r = 0,
\]

for all \( \lambda \in \text{Mor}^k(\text{ad},\Psi|_{\text{Hor}^\bullet(M\otimes G)}) \) (morphisms whose image lies on the space of quantum 1-forms), where \( dS^* \), \( d\tilde{S}^* \) are the adjoint of the operators of \( \Upsilon_{\text{ad}} \circ S^\omega, \tilde{\Upsilon}_{\text{ad}} \circ \ast \circ S^\omega \circ \ast \), respectively. The operator

\[
S^\omega : \text{Mor}(\text{ad},\Psi|_{\text{Hor}^\bullet(M\otimes G)}) \longrightarrow \text{Mor}(\text{ad},\Psi|_{\text{Hor}^\bullet(M\otimes G)})
\]

in this case is given by

\[
S^\omega(\tau)(\varsigma) = 2 \left( \omega(\varsigma) \tau(\varsigma) - (-1)^k \tau(\varsigma) \omega(\varsigma) \right),
\]

if \( \tau \in \text{Mor}^k(\text{ad},\Psi|_{\text{Hor}^\bullet(M\otimes G)}) \).

Solutions to the non-commutative geometrical Yang–Mills equation (eq.7) are flat qpcs in the sense that \( R^\omega = 0 \), and \( \omega_{\text{YM}} \) with

\[
\omega_{\text{YM}}(\varsigma) = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix} \otimes 1 + 1 \otimes \varsigma.
\]

The curvature of \( \omega_{\text{YM}} \) is given by

\[
R_{\text{YM}}(\varsigma) = \begin{pmatrix} -i/2 & 0 \\ 0 & -i/2 \end{pmatrix} \in \Omega^2(M).
\]

It is worth mentioning that the action of \( GG_{\text{YM}} \) on the set of critical points is trivial.

In terms of a physical interpretation, the Yang–Mills part of this example models a space-time of just two points with just one gauge boson field with symmetry \( GG_{\text{YM}} \supset S_2 \), which is given by \( \omega_{\text{YM}} \). However, it seems like the vacuum is full of non-gauge-equivalent boson fields (flat qpcs).
Yang–Mills Scalar Matter Theory

The non–commutative geometrical Yang–Mills–Scalar–Matter Lagrangian is given by \[ \text{Sa13} \]

\[
L_{\text{YMMSM}}(\omega, T_1, T_2) := -\frac{1}{4} \left( \langle R^\omega, R^\omega \rangle_l + \langle \hat{R}^\omega, \hat{R}^\omega \rangle_r \right) + \\
\frac{1}{4} \left( \langle \nabla_\alpha^\omega T_1, \nabla_\alpha^\omega T_1 \rangle_l - V_1(T_1) - \langle \hat{\nabla}_\alpha^\omega T_2, \hat{\nabla}_\alpha^\omega T_2 \rangle_r + V_r(T_2) \right),
\]
where \( \hat{\alpha} \) is the conjugate corepresentation of \( \alpha \) (\text{W1, Sa11}) and \( V : M \rightarrow M \) is a Fréchet differentiable map called the potential; and

\[
V_l(T_1) = V((T_1, T_1)_l), \quad V_r(T_2) = V((T_2, T_2)_r).
\]

Critical points of its associated action satisfy

\[
\langle Y_\alpha \circ K^\lambda(T_1) | \nabla_\alpha^\omega T_1 \rangle_l - \langle \hat{\nabla}_\alpha \circ (\ast \circ K^\lambda \circ \ast)(T_2) | \hat{\nabla}_\alpha^\omega T_2 \rangle_r = \\
\langle \lambda | (d\nabla^\omega \ast \ast - dS^\omega \ast \ast)R^\omega \rangle_l + (\ast \circ \lambda \circ \ast | (d\hat{\nabla}^\omega \ast \ast - d\hat{\nabla}^\omega \ast \ast)\hat{R}^\omega \rangle_r
\]

for all \( \lambda \in \text{Mor}^1(\text{ad}, \Psi|_{\text{Hor}_{\text{MSG}}}) \), and

\[
\nabla_\alpha^\omega \ast \ast (\nabla_\alpha^\omega T_1) - V'_l(T_1)^* T_1 = 0,
\]

\[
\hat{\nabla}_\alpha^\omega \ast \ast (\hat{\nabla}_\alpha^\omega T_2) - T_2 V'_r(T_2)^* = 0,
\]

with \( K^\lambda(T) = -T(0)\lambda(\pi(T(0))) \) for all \( T \in \text{Mor}(\alpha, \Phi) \) if \( \Phi(T(v)) = T(0)(v) \otimes T(1)(v) \) (in Sweedler’s notation), where \( \pi \) is the quantum germ map \( \text{Sa3} \).

In Non–Commutative Geometry, it is necessary to take the left structure of \( \text{Mor}(\alpha, \Phi) \) and the right structure of \( \text{Mor}(\hat{\alpha}, \Phi) \). In the classical case, the left and right structure are the same and \( D^\omega = \ast \circ D^\omega \circ \ast \), so our theory recreates the classical case (Equations (3) and (4) for a real potential) but considering antiparticles too. Therefore, Equations (8), (9) and (10) are the non–commutative geometrical version or a non–commutative geometrical generalization of Equations (3), (4).

To calculate solutions for these equations, we have to choose the corepresentation and the potential \( V \). For example, for the trivial corepresentation on \( \mathbb{C} \) and a potential \( V \) such that for all \( p \in M \)

\[
V'(p) = \begin{pmatrix}
2 - 2\frac{y}{x} & 0 \\
0 & 2 - 2\frac{x}{y}
\end{pmatrix}
\]

for some fixed \( x, y \in \mathbb{R} \), critical points are triplets \( (\omega, T_1, T_2) \), where \( \omega \) is a flat qpc or \( \omega_{\text{YM}} \) and \( T_1 \in \Gamma^4(M, \mathcal{C}M), T_2 \in \Gamma^r(M, \mathcal{C}M) \) are defined by

\[
T_1(1) = T_2(1) = \begin{pmatrix}
x & 0 \\
0 & y
\end{pmatrix}.
\]
In this case, we can ensure that at least, the Lagrangian is gauge–invariant for $GG_{YM}$.

In terms of a physical interpretation, this example models left scalar matter and right scalar antimatter fields in the trivial corepresentation coupled to gauge boson fields in a space–time of just two points.

Now let us consider the alternating quantum representation on $C$, $\alpha^{\text{alt}}$. In this case a left–right $M$ basis of $\text{Mor}(\alpha^{\text{alt}}, \Phi)$ is given by the element

\[ T^{\text{alt}} : C \longrightarrow M \otimes G \]

\[ \lambda \longmapsto \lambda \mathbb{I} \otimes \mathbb{I}^{\text{alt}} \]

where $\mathbb{I}^{\text{alt}} = \Delta_0 - \Delta_1$ and hence, every $T \in \text{Mor}(\alpha^{\text{alt}}, \Phi)$ is of the form $T = p^T T^{\text{alt}} = T^{\text{alt}} p^T$ where $p^T = T(1)(\mathbb{I} \otimes \mathbb{I}^{\text{alt}})$.

In general, for a qpc $\omega$ with $q\omega(\varsigma) = \left( \begin{array}{cc} 0 & \lambda_0 \\ \lambda_1 & 0 \end{array} \right) \otimes \mathbb{I} + \mathbb{I} \otimes \varsigma$ and $T_1 = \left( \begin{array}{cc} \tilde{p}_0 & 0 \\ 0 & \tilde{p}_1 \end{array} \right) T^{\text{alt}}$, $T_2 = T^{\text{alt}} \left( \begin{array}{cc} \tilde{p}_0 & 0 \\ 0 & \tilde{p}_1 \end{array} \right)$, Equation 8 turns into

\[ i(\tilde{p}_1^2 - \tilde{p}_0 \tilde{p}_1 + \tilde{p}_0 \tilde{p}_1 - |\tilde{p}_1|^2) + 2(|\tilde{p}_0|^2 - |\tilde{p}_1|^2) \lambda_0^* = u^*(1 + 2i \lambda_1) \]

\[ i(\tilde{p}_1^2 - \tilde{p}_0 \tilde{p}_1 + \tilde{p}_0 \tilde{p}_1 - |\tilde{p}_1|^2) + 2(|\tilde{p}_0|^2 - |\tilde{p}_1|^2) \lambda_1^* = u^*(1 + 2i \lambda_0), \]

where $u = -(\lambda_0 + \lambda_1) - 2i \lambda_0 \lambda_1$; while Equation 9 turns into

\[ \nabla_{\alpha^{\text{alt}}_{\omega}}(\nabla_{\alpha^{\text{alt}}_{\omega}} T_1) = \left( \begin{array}{cc} \tilde{u}_0 & 0 \\ 0 & \tilde{u}_1 \end{array} \right) T^{\text{alt}} , \]

\[ \nabla_{\alpha^{\text{alt}}_{\omega}^*}(\nabla_{\alpha^{\text{alt}}_{\omega}^*} T_2) = T^{\text{alt}} \left( \begin{array}{cc} \tilde{u}_0 & 0 \\ 0 & \tilde{u}_1 \end{array} \right) , \]

where $\tilde{u}_0 := 2(\tilde{p}_0 - \tilde{p}_1) + 2i \tilde{p}_0(\lambda_0 + \lambda_1) - 4i \tilde{p}_1 \lambda_1 - 4 \tilde{p}_0 \lambda_0 \lambda_1$, $\tilde{u}_1 := -2(\tilde{p}_0 - \tilde{p}_1) + 2i \tilde{p}_1(\lambda_0 + \lambda_1) - 4i \tilde{p}_0 \lambda_0 \lambda_1$, $\tilde{u}_0 := 2(\tilde{p}_0 - \tilde{p}_1) + 2i \tilde{p}_0(\lambda_0 + \lambda_1) - 4i \tilde{p}_1 \lambda_1 - 4 \tilde{p}_0 \lambda_0 \lambda_1$, $\tilde{u}_1 := -2(\tilde{p}_0 - \tilde{p}_1) + 2i \tilde{p}_1(\lambda_0 + \lambda_1) - 4i \tilde{p}_0 \lambda_0 \lambda_1$. This allows us to find solutions, for example, taking $\omega_{YM}$ and any $T_1$, $T_2$, Equation 10 turns into

\[ \tilde{p}_0 \tilde{p}_1 = \tilde{p}_0^* \tilde{p}_1 ; \]

while Equation 11 turns into

\[ \nabla_{\alpha^{\text{alt}}_{\omega}^*}(\nabla_{\alpha^{\text{alt}}_{\omega}} T_1) = T_1 , \quad \nabla_{\alpha^{\text{alt}}_{\omega}^*}(\nabla_{\alpha^{\text{alt}}_{\omega}^*} T_2) = T_2 . \]

Of course, there are more solutions.

Finally it is easy to show that

\[ (d^{\nabla_{\alpha}^*} - d^{S^{\text{alt}}_{\omega}^*})^2 = (d^{\nabla_{\alpha}^*} - d^{S^{\text{alt}}_{\omega}^*})^2 = 0 ; \]

so in this case a kind of left–right noncommutative continuity equation holds.
At least we can ensure that
\[ \{ f \in GG \mid f(\mathbb{I}^{alt}) = e^{it} \mathbb{I} , F_f \circ \omega^{triv} = \omega^{triv} \text{ with } t \in \mathbb{R} \} \]
is a subgroup of \( GG \) for which the Lagrangian is always gauge–invariant, no matter the form of the potential \( V \).

In terms of a physical interpretation, \textit{this example models left scalar matter and right scalar antimatter fields in the alternating representation coupled to gauge boson fields in a space–time of just two points.}

9 Final Comments

To conclude this work we have to talk about some interesting points

1. There is a general theory that supports all the concepts presented. This was just an example.

2. The operator \( S^\omega \) does not have a \textit{classical} counterpart in the sense that in Differential Geometry, \( S^\omega = 0 \). Its participation in Equations (7) and (8) is completely a \textit{quantum phenomena}.

3. Our definition of the quantum gauge group is based on the one presented by T. Brzezinski in [Br]; nevertheless, it does not recreate the \textit{classical} case. One option is considering the group generated by all convolution invertible maps \( \hat{f} \) that also are graded differential \( * \)–algebra morphisms. Another option is considering the group of all maps \( \hat{\mathcal{M}} \) such that they are graded differential \( * \)–algebra difeomorphisms. However, applying these ideas in other examples, there are not enough gauge transformations.

4. Although all these constructions are developed in the framework of Non–Commutative Geometry, the quantum gauge group is a \textit{classical} group. Therefore, an exciting way of research would be trying to define the quantum gauge group as a quantum group.

5. In the Lagrangian \( L_{YMSM} \) we work with left and right structures and with a corepresentation and its conjugate. In the \textit{quantum case} both parts are necessary and this can be appreciated when we work with the \textit{quantum Hopf fibration} [Sa14]. In this case, considering both parts one can easily find solutions. It looks like if in the \textit{quantum case left particles and right antiparticles cannot be separated}; they appear naturally interconnected.

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