HÖLDER INDEX AT A GIVEN POINT FOR DENSITY STATES
OF SUPER-$\alpha$-STABLE MOTION OF INDEX $1 + \beta$

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Running head: Hölder index for superprocesses at a given point.
Abstract. A Hölder regularity index at given points for density states of $(\alpha, 1, \beta)$-superprocesses with $\alpha > 1 + \beta$ is determined. It is shown that this index is strictly greater than the optimal index of local Hölder continuity for those density states.

1. Introduction and statement of results

For $0 < \alpha \leq 2$ and $1 + \beta \in (1, 2)$, the so-called $(\alpha, d, \beta)$-superprocess $X = \{X_t : t \geq 0\}$ in $\mathbb{R}^d$ is a finite measure-valued process related to the log-Laplace equation

\[
\frac{d}{dt} u = \Delta_\alpha u + au - bu^{1+\beta},
\]

where $a \in \mathbb{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\Delta_\alpha := -\langle -\Delta \rangle^{\alpha/2}$ determining a symmetric $\alpha$-stable motion in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$ (Brownian motion if $\alpha = 2$), whereas its continuous-state branching mechanism

\[
v \mapsto -av + bv^{1+\beta}, \quad v \geq 0,
\]

belongs to the domain of attraction of a stable law of index $1 + \beta \in (1, 2)$ (the branching is critical if $a = 0$).

From now on we assume that $d < \frac{2\alpha}{\alpha - 2}$. Then $X$ has a.s. absolutely continuous states $X_t(dx)$ at fixed times $t > 0$ (cf. Fleischmann [Fle88] with the obvious changes for $a \neq 0$). Moreover, as is shown in Fleischmann, Mytnik, and Wachtel [FMW10 Theorem 1.2(a),(c)], there is a dichotomy for their density function (also denoted by $X_t$): There is a continuous version $\tilde{X}_t$ of the density function if $d = 1$ and $\alpha > 1 + \beta$, but otherwise the density function $X_t$ is locally unbounded on open sets of positive $X_t(dx)$-measure. (The case $\alpha = 2$ had been derived earlier in Mytnik and Perkins [MP03].) In the case of continuity, Hölder regularity properties of $\tilde{X}_t$ had been studied in [FMW10], too.

Let us first recall the notion of an optimal Hölder index at a point (see e.g. Jaffard [Jaf99]). We say a function $f$ is Hölder continuous with index $\eta \in (0, 1]$ at the point $x$ if there is an open neighborhood $U(x)$ of $x$ and a constant $C$ such that

\[
|f(y) - f(x)| \leq C|y - x|^{\eta} \quad \text{for all } y \in U(x).
\]

The optimal Hölder index $H(x)$ of $f$ at the point $x$ is defined as

\[
H(x) := \sup\{\eta \in (0, 1) : f \text{ is Hölder continuous at } x \text{ with index } \eta\},
\]

and set to 0 if $f$ is not Hölder continuous at $x$.

Going back to the continuous (random) density function $\tilde{X}_t$, in what follows, $H(x)$ will denote the (random) optimal Hölder index of $\tilde{X}_t$ at $x \in \mathbb{R}$. In [FMW10 Theorem 1.2(a),(b)], the so-called optimal index for local Hölder continuity of $\tilde{X}_t$ had been determined by

\[
\eta_c := \frac{\alpha}{1+\beta} - 1 \in (0, 1).
\]

This means that in any non-empty open set $U \subset \mathbb{R}$ with $X_t(U) > 0$ one can find (random) points $x$ such that $H(x) = \eta_c$. This however left unsolved the question whether there are points $x \in U$ such that $H(x) > \eta_c$.

The purpose of this note is to verify the following theorem conjectured in [FMW10 Section 1.3]. To formulate it, let $\mathcal{M}_t$ denote the set of finite measures on $\mathbb{R}^d$, and $B_\epsilon(x)$ the open ball of radius $\epsilon > 0$ around $x \in \mathbb{R}^d$. 
Theorem 1.1 (Hölder continuity at a given point). Fix $t > 0$, $z \in \mathbb{R}$, and $X_0 = \mu \in \mathcal{M}_1$. Let $d = 1$ and $\alpha > 1 + \beta$.

(a) (Hölder continuity at a given point): For each $\eta > 0$ satisfying
\[
\eta < \tilde{\eta}_c := \min \left\{ \frac{1 + \alpha}{1 + \beta} - 1, 1 \right\},
\]
with probability one, the continuous version $\tilde{X}_t$ of the density is Hölder continuous of order $\eta$ at the point $z$:
\[
\sup_{x \in B(z), x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x - z|^\eta} < \infty, \quad \epsilon > 0.
\]

(b) (Optimality of $\tilde{\eta}_c$): If additionally $\beta > (\alpha - 1)/2$, then with probability one for any $\epsilon > 0$,
\[
\sup_{x \in B(z), x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x - z|^\eta} = \infty \quad \text{whenever } X_t(z) > 0.
\]

Theorem 1.1(b) states the optimality of $\tilde{\eta}_c$ in the case $\beta > (\alpha - 1)/2$. But it is easy to see that the opposite case $\beta \leq (\alpha - 1)/2$ implies that $\tilde{\eta}_c = 1$. Therefore the optimality of $\tilde{\eta}_c$ follows here automatically from the definition of $H(z)$. But opposed to the local unboundedness of the ratio $|\tilde{X}_t(x) - \tilde{X}_t(z)|/|x - z|^\eta$ in the case $\beta > (\alpha - 1)/2$, we conjecture that $\tilde{X}_t$ is even Lipschitz continuous at the given $z$ for $\beta < (\alpha - 1)/2$.

Since $\eta_c < \tilde{\eta}_c$, at each given point $z \in \mathbb{R}$ the density $\tilde{X}_t$ allows some Hölder exponents $\eta$ larger than $\eta_c$, the optimal Hölder index for local domains. Thus, Theorem 1.1 nicely complements the main result of [FMW10].

The full program however would include proving that for any $\eta \in (\eta_c, \tilde{\eta}_c)$ with probability one there are (random) points $x \in \mathbb{R}$ such that the optimal Hölder index $H(x)$ of $\tilde{X}_t$ at $x$ is exactly $\eta$. Moreover, one would like to establish the Hausdorff dimension, say $D(\eta)$, of the (random) set $\{x : H(x) = \eta\}$. The function $\eta \mapsto D(\eta)$ then reveals the so-called multifractal spectrum related to the optimal Hölder index at points. As we already mentioned in [FMW10], we conjecture that
\[
\lim_{\eta \downarrow \eta_c} D(\eta) = 0 \quad \text{and} \quad \lim_{\eta \uparrow \tilde{\eta}_c} D(\eta) = 1 \text{ a.s.}
\]

The investigation of such multifractal spectrum is left for future work.

The multifractal spectrum of random functions and measures has attracted attention for many years and has been studied for example in Dembo et al. [DPRZ01], Durand [Dur99], Hu and Taylor [HT00], Klenke and Mörters [KM05], Le Gall and Perkins [LGP95], Mörters and Shieh [MS04], and Perkins and Taylor [PT98]. The multifractal spectrum of singularities that describe the Hausdorff dimension of sets of different Hölder exponents of functions was investigated for deterministic and random functions in Jaffard [Ja99, Ja00, Ja04] and Jaffard and Meyer [JM96].

Note also that in the case $\alpha = 2$ for the optimal exponents $\eta_c$ and $\tilde{\eta}_c$ we have
\[
\eta_c \downarrow 0 \quad \text{and} \quad \tilde{\eta}_c \downarrow \frac{1}{2} \quad \text{as } \beta \uparrow 1,
\]
whereas for continuous super-Brownian motion ($\beta = 1$) one would have $\eta_c = \frac{1}{2} = \tilde{\eta}_c$. This discontinuity reflects the essential differences between continuous and discontinuous super-Brownian motion concerning Hölder continuity properties of density states, as discussed already in [FMW10] Section 1.3].
After some preparation in the next section, the proof of Theorem 1.1(a), (b) will be given in Sections 3 and 4, respectively.

2. Some proof preparation

Let \( p^\alpha \) denote the continuous \( \alpha \)-stable transition kernel related to the fractional Laplacian \( \Delta_\alpha = -(-\Delta)^{\alpha/2} \) in \( \mathbb{R}^d \), and \( S^\alpha \) the related semigroup. Fix \( X_0 = \mu \in \mathcal{M}_f \setminus \{0\} \).

First we want to recall the martingale decomposition of the \((\alpha, d, \beta)\)-superprocess \( X \) (valid for any \( \alpha, d, \beta \); see, e.g., [FMW10, Lemma 1.6]): For all sufficiently smooth bounded non-negative functions \( \varphi \) on \( \mathbb{R}^d \) and \( t \geq 0 \),

\[
\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t ds \langle X_s, \Delta_\alpha \varphi \rangle + M_t(\varphi) + a I_t(\varphi)
\]

with discontinuous martingale

\[
t \mapsto M_t(\varphi) := \int_{(0,t] \times \mathbb{R}^d \times \mathbb{R}_+} \tilde{N}(d(s, x, r)) \, r \varphi(x)
\]

and increasing process

\[
t \mapsto I_t(\varphi) := \int_0^t ds \langle X_s, \varphi \rangle.
\]

Here \( \tilde{N} := N - \hat{N} \), where \( N(d(s, x, r)) \) is a random measure on \((0, \infty) \times \mathbb{R}^d \times (0, \infty)\) describing all the jumps \( r\delta_x \) of \( X \) at times \( s \) at sites \( x \) of size \( r \) (which are the only discontinuities of the process \( X \)). Moreover,

\[
\hat{N}(d(s, x, r)) = \varrho \, ds \, X_s(dx \, r^{-2-\beta} \, dr
\]

is the compensator of \( N \), where \( \varrho := b(1 + \beta)\beta / \Gamma(1 - \beta) \) with \( \Gamma \) denoting the Gamma function.

Recall that we assumed \( d < \frac{\alpha}{\beta} \), and fix \( t > 0 \). Then the random measure \( X_t(dx) \) is a.s. absolutely continuous. From the Green’s function representation related to (2.1) (see, e.g., [FMW10] (1.9)) we obtain the following representation of a version of the density function of \( X_t(dx) \) (see, e.g., [FMW10] (1.12)):

\[
X_t(x) = \mu * p^\alpha_t(x) + \int_{(0,t] \times \mathbb{R}^d} M(d(s, y)) \, p^\alpha_{t-s}(y-x)
\]

\[
+ a \int_{(0,t] \times \mathbb{R}^d} I(d(s, y)) \, p^\alpha_{t-s}(y-x) =: Z^1_t(x) + Z^2_t(x) + Z^3_t(x), \quad x \in \mathbb{R}^d,
\]

(with notation in the obvious correspondence). Here \( M(d(s, y)) \) is the martingale measure related to (2.2) and \( I(d(s, y)) \) the random measure related to (2.3).

Let \( \Delta X_s := X_s - X_{s-} \), \( s \in (0, t) \), denote the jumps of the measure-valued process \( X \) by time \( t \). Recall that they are of the form \( r\delta_x \). By an abuse of notation, we also write \( r := \Delta X_s(x) \). Put

\[
f_{s,x} := \log((t-s)^{-1}) \, 1_{\{x \neq 0\}} \log(|x|^{-1})
\]

As a further preparation we turn to the following lemma. Recall that \( t > 0 \) is fixed.
Lemma 2.1 (A jump mass estimate). Fix $X_0 = \mu \in \mathcal{M}_1 \setminus \{0\}$. Suppose $d = 1$ and $\alpha > 1 + \beta$. Let $\varepsilon > 0$ and $q > 0$. There exists a constant $c_{2.10} \in \mathbb{R}$ such that

\[(2.7) \quad \mathbb{P} \left( \Delta X_s(x) > c_{2.10} \left( (t-s)|x| \right) \left( f_{s,x} \right)^{\ell} \right) \quad \text{for some } s < t, \quad x \in B_{1/r}(0) \leq \varepsilon, \]

where

\[(2.8) \quad \ell := \frac{1}{1 + \beta} + q. \]

Proof. For any $c > 0$ (later to be specialized to some $c_{2.12}$) set

\[
Y := N \left( (s,x,r) : (s,x) \in (0,t) \times B_{1/r}(0), \ r \geq c \left( (t-s)|x| \right)^{1/(1+\beta)} \left( f_{s,x} \right)^{\ell} \right),
\]

Clearly,

\[(2.9) \quad \mathbb{P} \left( \Delta X_s(x) > c \left( (t-s)|x| \right)^{1/(1+\beta)} \left( f_{s,x} \right)^{\ell} \right) \quad \text{for some } s < t \text{ and } x \in B_{1/r}(0) = \mathbb{P}(Y \geq 1) \leq \mathbb{E}Y,
\]

where in the last step we have used the classical Markov inequality. From (2.4),

\[(2.10) \quad \mathbb{E}Y = \theta \mathbb{E} \int_0^t \int_{\mathbb{R}} X_s(dx) 1_{B_{1/r}(0)}(x) \int_c \left( (t-s)|x| \right)^{1/(1+\beta)} \left( f_{s,x} \right)^{\ell} dr \ r^{-2-\beta}
\]

Now, writing $C$ for a generic constant (which may change from place to place),

\[
\int_{\mathbb{R}} \mathbb{E}X_s(dx) 1_{B_{1/r}(0)}(x) \left| x \right|^{-1} \log^{-1-q(1+\beta)} \left( \left| x \right|^{-1} \right)
\]

\[
\leq \ e^{\alpha t} \mu(dy) \int_{\mathbb{R}} dx \ p_{s}^{\alpha}(x-y) 1_{B_{1/r}(0)}(x) \left| x \right|^{-1} \log^{-1-q(1+\beta)} \left( \left| x \right|^{-1} \right)
\]

\[
\leq \ C \mu(R) s^{-1/\alpha} \int_{\mathbb{R}} dx \ 1_{B_{1/r}(0)}(x) \left| x \right|^{-1} \log^{-1-q(1+\beta)} \left( \left| x \right|^{-1} \right)
\]

\[
= \ c_{2.11} s^{-1/\alpha},
\]

where $c_{2.11} = c_{2.11}(q)$ (recall that $t$ is fixed). Consequently,

\[
\mathbb{E}Y \leq \theta c_{2.11} c^{-1-\beta} \int_0^t ds s^{-1/\alpha} (t-s)^{-1} \log^{-1-q(1+\beta)} \left( (t-s)^{-1} \right)
\]

\[
= \ c_{2.12} c^{-1-\beta}
\]

with $c_{2.12} = c_{2.12}(q)$. Choose now $c$ such that the latter expression equals $\varepsilon$ and write $c_{2.12}$ instead of $c$. Recalling (2.9), the proof is complete. 

Since $\sup_{0<\gamma<1} y^\gamma \log^{1/y} < \infty$ for every $\gamma > 0$, we get from Lemma 2.1 the following statement.
Corollary 2.2 (A jump mass estimate). Fix $X_0 = \mu \in \mathcal{M}_1\{0\}$. Suppose $d = 1$ and $\alpha > 1 + \beta$. Let $\varepsilon > 0$ and $\gamma \in (0,(1+\beta)^{-1})$. There exists a constant $c_{2.13} = c_{2.13}(\varepsilon, \gamma)$ such that
\begin{equation}
(2.13) \quad P\left( \Delta X_s(x) > c_{2.13} \left( (t-s)|x| \right)^{\lambda} \text{ for some } s < t \text{ and } x \in B_2(0) \right) \leq \varepsilon,
\end{equation}
where
\begin{equation}
(2.14) \quad \lambda := \frac{1}{1 + \beta} - \gamma.
\end{equation}

Several times we will use the following estimate concerning the $\alpha$-stable transition kernel $p^\alpha$ taken from \textbf{[FMW10]} Lemma 2.1.

Lemma 2.3 ($\alpha$-stable density increment). For every $\delta \in [0,1]$,
\begin{equation}
(2.15) \quad |p^\alpha_t(x) - p^\alpha_t(y)| \leq C \frac{|x-y|^\delta}{t^{\beta/\alpha}} \left( p^\alpha_t(x/2) + p^\alpha_t(y/2) \right), \quad t > 0, \ x, y \in \mathbb{R}.
\end{equation}

In the proof of our main result we need also a further technical result we quote from \textbf{[FMW10]} Lemma 2.3. Let $L = \{L_t : t \geq 0\}$ denote a spectrally positive stable process of index $\kappa \in (1,2)$. Per definition, $L$ is an $\mathbb{R}$-valued time-homogeneous process with independent increments and with Laplace transform given by
\begin{equation}
(2.16) \quad E e^{-\lambda L_t} = e^{t\lambda^\kappa}, \quad \lambda, t \geq 0.
\end{equation}

Note that $L$ is the unique (in law) solution to the following martingale problem:
\begin{equation}
(2.17) \quad t \mapsto e^{-\lambda L_t} - \int_0^t ds \ e^{-\lambda L_s} \lambda^\kappa \text{ is a martingale for any } \lambda > 0.
\end{equation}

Let $\Delta L_s := L_s - L_{s-} > 0$ denote the (positive) jumps of $L$.

Lemma 2.4 (Big values of the process in the case of bounded jumps). We have
\begin{equation}
(2.18) \quad P\left( \sup_{0 \leq u \leq t} L_u \{ \sup_{0 \leq v \leq u} \Delta L_v \leq y \} \geq x \right) \leq \left( \frac{C t}{xy^{\kappa-1}} \right)^{x/y}, \quad t > 0, \ x, y > 0.
\end{equation}

3. Hölder continuity at a given point: proof of Theorem 1.1(a)

We will use some ideas from the proofs in Section 3 of \textbf{[FMW10]}. However, to be adopted to our case, those proofs require significant changes. Let $d = 1$ and fix $t, z, \mu, \alpha, \beta, \eta$ as in the theorem. Consider an $x \in B_1(z)$. Without loss of generality we will assume that $t \leq 1$ and, changing $\mu$ appropriately, that $z = 0$ and $0 < x < 1$.

By definition (2.5) of $Z_t^\alpha$,
\begin{equation}
(3.1) \quad Z_t^\alpha(x) - Z_t^\alpha(0) = \int_{[0,t] \times \mathbb{R}} M\left( d(s,y) \right) \varphi_+(s,y) - \int_{[0,t] \times \mathbb{R}} M\left( d(s,y) \right) \varphi_-(s,y),
\end{equation}
where $\varphi_+(s,y)$ and $\varphi_-(s,y)$ are the positive and negative parts of $p^\alpha_t(y-x) - p^\alpha_t(y-\eta)$ (for the fixed $x$). It is easy to check that $\varphi_+$ and $\varphi_-$ satisfy the assumptions in \textbf{[FMW10]} Lemma 2.15. Thus, there exist spectrally positive stable processes $L^+_t$ and $L^-_t$ such that
\begin{equation}
(3.2) \quad Z_t^\alpha(x) - Z_t^\alpha(0) = L^+_t - L^-_t,
\end{equation}
where
where \( T_{\pm} := \int_0^t ds \int \mathbb{R} X_s(dy) \left( \varphi_{\pm}(s, y) \right)^{1+\beta}. \) Fix any
\[
(3.3) \quad \varepsilon \in \left(0, \frac{1}{3}\right) \quad \text{and} \quad \gamma \in \left(0, \min \left\{ \frac{\eta_c}{2\alpha}, \frac{\bar{\eta}_c}{2(2\alpha + 1)} \right\} \right).
\]
Also fix some \( J = J(\gamma) \) and
\[
(3.4) \quad 0 =: \rho_0 < \rho_1 < \cdots < \rho_J := 1/\alpha
\]
such that
\[
(3.5) \quad \rho_{\ell}(\alpha + 1) - \frac{\rho_{\ell+1}}{1+\beta} \geq \frac{\gamma}{2}, \quad 0 \leq \ell \leq J - 1.
\]
According to \cite{FMW10} Lemma 2.11, there exists a constant \( c_\varepsilon \) such that
\[
(3.6) \quad \mathbf{P}(V \leq c_\varepsilon) \geq 1 - \varepsilon,
\]
where
\[
(3.7) \quad V := \sup_{0 \leq s \leq t, \ y \in B_2(0)} S_{2^{\alpha}(t-s)} X_s(y)
\]
(note that there is no difference in using \( B_2(0) \) or its closure for taking the supremum). By Lemma 2.2 we can fix \( c_{\ell} \) sufficiently large such that the probability of the event
\[
(3.8) \quad A_{\varepsilon, 1} := \left\{ \Delta X_s(y) \leq c_{\ell} \left( (t-s) |y| \right)^{\lambda} \text{ for all } s < t \text{ and } y \in B_2(0) \right\}
\]
is larger than \( 1 - \varepsilon. \) Moreover, according to \cite{FMW10} Lemma 2.14, there exists a constant \( c^* = c^*(\varepsilon, \gamma) \) such that the probability of the event
\[
(3.9) \quad A_{\varepsilon, 2} := \left\{ \Delta X_s(y) \leq c^*(t-s)^{\lambda} \text{ for all } s < t \text{ and } y \in \mathbb{R} \right\}
\]
is larger than \( 1 - \varepsilon. \) Set
\[
(3.10) \quad A_\varepsilon := A_{\varepsilon, 1} \cap A_{\varepsilon, 2} \cap \{ V \leq c_\varepsilon \}.
\]
Evidently,
\[
(3.11) \quad \mathbf{P}(A_\varepsilon) \geq 1 - 3\varepsilon.
\]
Define \( Z_{\varepsilon}^{2,\lambda} := Z_{\varepsilon}^{2,1}(A_\varepsilon). \) We first show that \( Z_{\varepsilon}^{2,\lambda} \) has a version which is locally H"{o}lder continuous of all orders \( \eta \) less than \( \bar{\eta}_c. \) It follows from (3.2) that, for any \( k > 0,$
\[
(3.12) \quad \mathbf{P}\left( \left| Z_{\varepsilon}^{2,\lambda}(x) - Z_{\varepsilon}^{2,\lambda}(0) \right| \geq 2k x^\eta \right) \\
\leq \mathbf{P}(L_{T_+}^{\lambda} \geq kx^\eta, A_\varepsilon) + \mathbf{P}(L_{T_-}^{\lambda} \geq kx^\eta, A_\varepsilon).
\]
Define:
\[
(3.13) \quad \tilde{D}_0 := \left\{ (s, y) \in [0, t) \times B_2(0) : y \in (-2(t-s)^{1/\alpha - \rho_1}, x+2(t-s)^{1/\alpha - \rho_1}) \right\}
\]
and, for \( 1 \leq \ell \leq J - 1,$
\[
(3.14) \quad D_0 := \tilde{D}_0 \quad \text{and} \quad D_\ell := \tilde{D}_\ell \setminus \tilde{D}_{\ell-1}, \quad 1 \leq \ell \leq J - 1.
\]
Recall the following standard estimate on
\[ (3.22) \]
\[ (3.23) \]
for some constant \( c \).

Using Lemma 2.3 with \( \alpha \), we have
\[ \varphi_{\pm}(s, y) \]
are bounded by
\[ (3.16) \]
\[ c \sup_{(s, y) \in D_{\ell}} \left| (t - s)|y| \right|^\lambda \varphi_{\pm}(s, y). \]

For \( 0 \leq \ell < J \), put
\[ D_{\ell,1} := \{ (s, y) \in D_{\ell} : (t - s)^{1/\alpha - \rho_{\ell} + 1} \leq x \}, \]
\[ D_{\ell,2} := \{ (s, y) \in D_{\ell} : (t - s)^{1/\alpha - \rho_{\ell} + 1} > x \}, \]
\[ D_{\ell,1}(s) := \{ y \in B_{2}(0) : (s, y) \in D_{\ell,1} \}, \]
\[ s \in [0, t), \]
\[ D_{\ell,2}(s) := \{ y \in B_{2}(0) : (s, y) \in D_{\ell,2} \}, \]
\[ s \in [0, t). \]

Since obviously \( D_{\ell} = D_{\ell,1} \cup D_{\ell,2} \) we get that \( (3.16) \) is bounded by
\[ c \sup_{0 < s < t} (t - s)^{\lambda} \sup_{y \in D_{\ell,1}(s)} |y|^{\lambda} \varphi_{\pm}(s, y) \]
\[ + c \sup_{0 < s < t} (t - s)^{\lambda} \sup_{y \in D_{\ell,2}(s)} |y|^{\lambda} \varphi_{\pm}(s, y) =: c \lambda(I_{1} + I_{2}). \]

Clearly,
\[ \varphi_{\pm}(s, y) \leq \left| p_{t-s}^{\alpha}(y - x) - p_{t-s}^{\alpha}(y) \right|, \]
for all \( s, y \).

First let us bound \( I_{1} \). Note that for any \( (s, y) \in D_{\ell,1} \),
\[ |y| \leq x + 2(t - s)^{1/\alpha - \rho_{\ell} + 1} \leq 3x. \]

Therefore we have
\[ I_{1} \leq 3^{\lambda} x^{\lambda} \sup_{0 < s < t} (t - s)^{\lambda} \sup_{y \in D_{\ell,1}(s)} |p_{t-s}^{\alpha}(y - x) - p_{t-s}^{\alpha}(y)|. \]

Using Lemma 2.3 with \( \delta = \eta - 2\alpha \gamma \) gives
\[ \sup_{y \in D_{\ell,1}(s)} \left| p_{t-s}^{\alpha}(y - x) - p_{t-s}^{\alpha}(y) \right| \]
\[ \leq C x^{\eta - 2\alpha \gamma} (t - s)^{-\eta_{\ell,\alpha} + 2\gamma} \]
\[ \times \sup_{y \in D_{\ell,1}(s)} \left( p_{t-s}^{\alpha}((y - x)/2) + p_{t-s}^{\alpha}(y/2) \right) \]
\[ = C x^{\eta - 2\alpha \gamma} (t - s)^{-\eta_{\ell,\alpha} + 2\gamma - 1/\alpha} \]
\[ \times \sup_{y \in D_{\ell,1}(s)} \left( p_{t-s}^{\alpha}((t - s)^{-1/\alpha}(y - x)/2) + p_{t-s}^{\alpha}(y/2) \right). \]

Recall the following standard estimate on \( p_{t-s}^{\alpha} \):
\[ p_{t-s}^{\alpha}(y) \leq c \left| y \right|^{(\alpha + 1)}, \quad y \in \mathbb{R}, \]
for some constant \( c \). Thus on \( D_{\ell,1}(s) \), we have \( |y| \geq 2(t - s)^{1/\alpha - \rho_{\ell}} \), implying
\[ p_{t-s}^{\alpha}((t - s)^{-1/\alpha} y/2) \leq p_{t-s}^{\alpha}((t - s)^{-\rho_{\ell}}) \leq c \left| (t - s)^{\rho_{\ell}(\alpha + 1)} \right|. \]
where the last inequality follows by (3.23). A similar estimate holds for the second $p_t^\alpha$-expression in (3.22). Thus, (3.22) yields

\[
(3.25) \quad \sup_{y \in D_{t,1}(s)} \left| p_{t-s}^\alpha (y - x) - p_{t-s}^\alpha (y) \right| \leq C x^{\eta_c - 2\alpha \gamma} (t - s)^{-\eta_c / \alpha + 2\gamma - 1 / \alpha + \rho_t (\alpha + 1)}.
\]

Now let us check that

\[
(3.26) \quad \sup_{0 < s < t} (t - s)^{\lambda} (t - s)^{-\eta_c / \alpha + 2\gamma - 1 / \alpha + \rho_t (\alpha + 1)} \leq 1.
\]

Recall that $\eta_c = \frac{\alpha}{1 + \beta} - 1$. Then one can easily get that

\[
(3.27) \quad \lambda - \eta_c / \alpha + 2\gamma - 1 / \alpha + \rho_t (\alpha + 1) = \gamma + \rho_t (\alpha + 1) \geq \gamma,
\]

where the last inequality follows by (3.5). Therefore (3.26) follows immediately. Combining (3.21), (3.25), and (3.26), we see that

\[
(3.28) \quad I_1 \leq C x^{\lambda + \eta_c - 2\alpha \gamma} \leq C x^{\tilde{\eta}_c - (2\alpha + 1)\gamma},
\]

where we used definitions of $\eta_c$ and $\tilde{\eta}_c$, given in (1.5) and Theorem 1.1(a), respectively.

Now let us bound $I_2$. Note that for any $(s, y) \in D_{t,2}$,

\[
(3.29) \quad |y| \leq x + 2 (t - s)^{1 / \alpha - \rho_t + 1} \leq 3 (t - s)^{1 / \alpha - \rho_t + 1}.
\]

Therefore we have

\[
(3.30) \quad I_2 \leq 3^\lambda \sup_{0 < s < t} \left( (t - s)^{\lambda + (1 / \alpha - \rho_t + 1)\lambda} \sup_{y \in D_{t,2}(s)} \left| p_{t-s}^\alpha (y - x) - p_{t-s}^\alpha (y) \right| \right).
\]

Using again Lemma 2.3 but this time with $\delta = \tilde{\eta}_c - (2\alpha + 1)\gamma$ gives

\[
(3.31) \quad \sup_{y \in D_{t,2}(s)} \left| p_{t-s}^\alpha (y - x) - p_{t-s}^\alpha (y) \right| \leq C x^{\tilde{\eta}_c - (2\alpha + 1)\gamma} (t - s)^{-\tilde{\eta}_c / \alpha + 2\gamma + \gamma / \alpha - \frac{\rho_t + 1}{1 + \beta} + \rho_t (\alpha + 1)}.
\]

By definition (2.13) of $\lambda$,

\[
\lambda + \left( \frac{1}{\alpha} - \rho_t + 1 \right) \lambda - \tilde{\eta}_c / \alpha + 2\gamma + \frac{\gamma}{\alpha} - \frac{1}{\alpha} + \rho_t (\alpha + 1) = \frac{1}{\alpha} \left( \frac{1 + \alpha}{1 + \beta} - 1 - \tilde{\eta}_c \right) + \gamma + \gamma \rho_t + 1 - \frac{\rho_t + 1}{1 + \beta} + \rho_t (\alpha + 1)
\]

\[
(3.32) \quad \geq \gamma / 2
\]

where in the last step we used definition of $\tilde{\eta}_c$ given in Theorem 1.1(a), and (3.5). Thus

\[
(3.33) \quad \sup_{0 < s < t} (t - s)^{\lambda + (1 / \alpha - \rho_t + 1)\lambda - \tilde{\eta}_c / \alpha + 2\gamma + \gamma / \alpha - 1 / \alpha + \rho_t (\alpha + 1)} \leq 1.
\]

Combining estimates (3.30), (3.31), and (3.33), we obtain

\[
(3.34) \quad I_2 \leq C x^{\tilde{\eta}_c - (2\alpha + 1)\gamma}.
\]
If the jumps of $M(d(s,y))$ are smaller than $c^*(t-s)^\lambda$ on $R \setminus B_2(0)$ (where $c^*$ is from \(3.39\)), then the jumps of the process $u \mapsto \int_{(0,u] \times (R \setminus B_2(0))} M(d(s,y)) \varphi_\pm (s,y)$ are bounded by
\[
(3.35) \quad c^*(t-s)^\lambda \sup_{y \in R \setminus B_2(0)} \varphi_\pm(s,y).
\]

Using Lemma 2.3 once again but this time with $\delta = \bar{\eta}_e - 2\alpha \gamma$, we have
\[
|p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)| \leq C x^{\bar{\eta}_e - 2\alpha \gamma} (t-s)^{-\bar{\eta}_e/\alpha + 2\gamma}
\times (p^\alpha_t ((y-x)/2) + p^\alpha_{t-s}(y/2)).
\]

Since $0 < x < 1$,
\[
\sup_{y \in R \setminus B_2(0)} \left( p^\alpha_{t-s}((y-x)/2) + p^\alpha_{t-s}(y/2) \right)
\leq C (t-s)^{-1/\alpha} p^\alpha_t ((t-s)^{-1/\alpha}/2) \leq C (t-s).
\]

Therefore, (3.19), (3.36), and (3.37) imply
\[
(3.38) \quad c^*(t-s)^\lambda \sup_{y \in R \setminus B_2(0)} \varphi_\pm(s,y) \leq C x^{\bar{\eta}_e - 2\alpha \gamma} (t-s)^{\lambda - \bar{\eta}_e/\alpha + 2\gamma + 1}
\]
for some constant $c \in \mathbb{R}^{\{3.39\}}$. Here we have used that $\bar{\eta}_e \leq (1 + \alpha)/(1 + \beta) - 1$ implies $\lambda - \bar{\eta}_e/\alpha + 2\gamma + 1 \geq 1$.

Combining (3.16), (3.18), (3.28), (3.34), and (3.38), we see that all jumps of the process $u \mapsto \int_{(0,u] \times R} M(d(s,y)) \varphi_\pm(s,y)$ on the set $A^\delta$ are bounded by
\[
(3.39) \quad c \in \mathbb{R}^{\{3.39\}} x^{\bar{\eta}_e - (2\alpha + 1)\gamma}
\]
for some constant $c \in \mathbb{R}^{\{3.39\}}$. Therefore, by an abuse of notation writing $L$ for $L^+$ and $L^-$,
\[
(3.40) \quad P(L_{T_\pm} \geq k_\epsilon, A^\delta)
= P(L_{T_\pm} \geq k_\epsilon, \sup_{0 < u < T_\pm} \Delta L_u \leq c \in \mathbb{R}^{\{3.39\}} x^{\bar{\eta}_e - (2\alpha + 1)\gamma}, A^\delta)
\leq P \left( \sup_{0 < u < T_\pm} L_u 1\left( \sup_{0 < u < v} \Delta L_u \leq c \in \mathbb{R}^{\{3.39\}} x^{\bar{\eta}_e - (2\alpha + 1)\gamma} \right) \geq k_\epsilon, A^\delta \right).
\]

Since
\[
(3.41) \quad T_\pm \leq \int_0^t ds \int_R X_s(dy) \left| p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y) \right|^{1+\beta},
\]
applying [FMW10] Lemma 2.12 with
\[
(3.42) \quad \theta = 1 + \beta \quad \text{and} \quad \delta = 1_{\beta<\alpha/2} + \frac{\alpha - \beta - \epsilon}{1 + \beta} 1_{\beta \geq \alpha/2},
\]
we may fix $\epsilon \in (0, \alpha \gamma \beta)$ to get the bound
\[
(3.43) \quad T_\pm \leq c \in \mathbb{R}^{\{3.43\}} \left( x^{1+\beta} 1_{\beta<\alpha/2} + x^{\alpha - \beta - \epsilon \in (0, \alpha \gamma \beta)} 1_{\beta \geq \alpha/2} \right)
\]
on \(\{V \leq c]\) for some constant \(c_{3.43} = c_{3.43}(\varepsilon)\). Consequently,

\[
P(L_{T_\lambda} \geq kx^n, A^c) 
\leq \mathbb{P}\left(\sup_{0 < u < c} (x^{1+\beta} 1_{\beta < (\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1} 1_{\beta \geq (\alpha-1)/2}) \times 1 \left\{ \sup_{0 < u < v} \Delta L_u \leq c \right\} \geq kx^n \right).
\]

Now use Lemma 2.4 with \(\kappa = 1 + \beta\), \(t = \eta(3.45)\), \(kx^n\) instead of \(x\), and \(y = c_{3.49} x^{\eta} (2\alpha+1)\). This gives

\[
P(L_{T_\lambda} \geq kx^n, A^c) 
\leq \left( \frac{C \mathbb{E} [3.45] (x^{1+\beta} 1_{\beta < (\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1} 1_{\beta \geq (\alpha-1)/2}) x^{\eta} (2\alpha+1)^2} {kx^{\eta}(c_{3.39} x^{\eta} (2\alpha+1) \beta)} \right)^{\frac{2-k\alpha}{2(2\alpha+1)}}.
\]

Now we need additionally the following simple inequalities, which are easy to derive:

\[
-\eta - \beta(\eta - (2\alpha + 1)\gamma) + 1 + \beta \geq (2\alpha + 1)\gamma \beta \quad \text{on} \quad \beta \leq \frac{\alpha - 1}{2},
\]

and

\[
-\eta - \beta(\eta - (2\alpha + 1)\gamma) + \alpha - \varepsilon_1 \geq (2\alpha + 1)\gamma \beta - \varepsilon_1 \geq \alpha \gamma \beta \quad \text{on} \quad \beta \geq \frac{\alpha - 1}{2}.
\]

In fact, \(\tilde{\eta} = 1\) under \(\beta < (\alpha - 1)/2\), whereas the other case in the definition of \(\tilde{\eta}\) applies under \(\beta \geq (\alpha - 1)/2\). Then, using the above inequalities and (3.45), we obtain

\[
P(L_{T_\lambda} \geq kx^n, A^c) \leq \left( c_{3.48} k^{-1} \eta^{\alpha \gamma \beta} \right)^{\frac{2-k\alpha}{2(2\alpha+1)}}
\]

for some constant \(c_{3.48} = c_{3.48}(\varepsilon)\). Applying this bound with \(\gamma = \frac{\bar{\eta} - \eta}{2(2\alpha+1)}\), to the summands at the right hand side in inequality (3.13), and noting that \(\alpha \gamma \beta\) is also a positive constant here, we have

\[
P\left( |Z_t^{2,\varepsilon}(x) - Z_t^{2,\varepsilon}(0)| \geq 2kx^n \right) \leq 2(c_{3.49} k^{-1} x) \left( c_{3.49} kx^{\eta - \eta} (2\alpha + 1)^2 \right)
\]

for some constant \(c_{3.49}\). This inequality yields

\[
\lim_{k \to \infty} \sum_{n=1}^{\infty} P\left( |Z_t^{2,\varepsilon}(n^{-q}) - Z_t^{2,\varepsilon}(0)| \geq k n^{-q} \right) = 0
\]

for every positive \(q\).

Recall that our purpose is to show that

\[
\sup_{0 < x < 1} \frac{|Z_t^2(x) - Z_t^2(0)|}{x^n} < \infty \quad \text{almost surely},
\]

or, in other words,

\[
\lim_{k \to \infty} \mathbb{P} \left( \sup_{0 < x < 1} \frac{|Z_t^2(x) - Z_t^2(0)|}{x^n} > k \right) = 0.
\]
It is easy to see that

\[
(3.53) \quad \left\{ \sup_{0<x<1} \frac{|Z^2_i(x) - Z^2_i(0)|}{x^q} > k \right\} \subseteq \bigcup_{n=1}^{\infty} \left\{ \sup_{x \in I_n} |Z^2_i(x) - Z^2_i(0)| > \frac{k}{2^q} n^{-q} \right\},
\]

where \( I_n := \{ x : (n+1)^{-q} \leq x < n^{-q} \} \). Moreover, by the triangle inequality,

\[
(3.54) \quad |Z^2_i(x) - Z^2_i(0)| \leq |Z^2_i(x) - Z^2_i(n^{-q})| + |Z^2_i(n^{-q}) - Z^2_i(0)|, \quad x \in I_n.
\]

Furthermore, for all \( R > 0 \),

\[
\left\{ \sup_{0<x<y<1} \frac{|Z^2_i(x) - Z^2_i(y)|}{|x-y|^{q/(q+1)}} \leq R \right\}
\]

\[
\subseteq \left\{ |Z^2_i(x) - Z^2_i(n^{-q})| \leq R q^{q/(q+1)} n^{-q}, \quad x \in I_n \right\}.
\]

Consequently, for all \( n \geq 1 \),

\[
\left\{ \sup_{x \in I_n} |Z^2_i(x) - Z^2_i(0)| > \frac{k}{2^q} n^{-q} \right\}
\]

\[
\subseteq \left\{ \sup_{0<x<y<1} \frac{|Z^2_i(x) - Z^2_i(y)|}{|x-y|^{q/(q+1)}} > c(q)k \right\} \cup \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^q} n^{-q} \right\},
\]

where \( c(q) \) is some positive constant. If we choose \( q \) so small that \( \eta q/(q+1) < \eta_c \), then

\[
(3.56) \quad \lim_{k \to \infty} \mathbf{P} \left( \sup_{0<x<y<1} \frac{|Z^2_i(x) - Z^2_i(y)|}{|x-y|^{q/(q+1)}} > c(q)k \right) = 0,
\]

since, by Theorem 1.2(a) of \( \text{FMW10} \), \( Z^2_i \) is locally Hölder continuous of every index smaller than \( \eta_c \). Therefore, it suffices to show that

\[
(3.57) \quad \lim_{k \to \infty} \mathbf{P} \left( \bigcup_{n=1}^{\infty} \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^{q+1}} n^{-q} \right\} \right) = 0.
\]

But

\[
(3.58) \quad \mathbf{P} \left( \bigcup_{n=1}^{\infty} \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^{q+1}} n^{-q} \right\} \right) \leq \mathbf{P} \left( \bigcup_{n=1}^{\infty} \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^{q+1}} n^{-q} \right\} \right) + \mathbf{P}(A^{c,c}),
\]

where \( A^{c,c} \) denotes the complement of \( A^c \). It follows from \( (3.50) \) that

\[
(3.59) \quad \lim_{k \to \infty} \mathbf{P} \left( \bigcup_{n=1}^{\infty} \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^{q+1}} n^{-q} \right\} \right) = 0.
\]

Moreover, \( \mathbf{P}(A^{c,c}) \leq 3\varepsilon \), see \( (3.11) \). As a result we have

\[
(3.60) \quad \lim_{k \uparrow \infty} \sup \mathbf{P} \left( \bigcup_{n=1}^{\infty} \left\{ |Z^2_i(n^{-q}) - Z^2_i(0)| > \frac{k}{2^{q+1}} n^{-q} \right\} \right) \leq 3\varepsilon.
\]

Since \( \varepsilon \) may be arbitrarily small, this implies \( (3.32) \). This yields the desired Hölder continuity of \( Z^2_i \) at 0, for all \( \eta < \tilde{\eta}_c \). Since \( Z^1_i \) and \( Z^2_i \) are a.s. Lipschitz continuous at 0 (cf. \( \text{FMW10} \) Remark 2.13), recalling \( (2.5) \), the proof of Theorem 1.1(a) is complete. \( \square \)
4. Optimality of $\bar{\eta}_k$: proof of Theorem 1.1(b)

We continue to consider $d = 1$, to fix $t, z, \mu, \alpha, \beta, \eta$ as in the theorem, and to assume $0 < t < 1$ and $z = 0$.

In analogy to the proof of optimality of $\eta_\kappa$ in [FMW10, Section 5], our strategy is to find a sequence of “big” jumps that occur close to time $t$. But in contrast to the case of the local Hölder continuity, we need to find these “big” jumps in the vicinity of 0, where these jumps should destroy the Hölder continuity of any index greater or equal than $\bar{\eta}_k$. This needs to overcome some new technical difficulties.

Recall that we need to prove the optimality in the case $\beta > (\alpha - 1)/2$ only. This implies that $\bar{\eta}_k = \frac{\alpha + 1}{\beta + 1} - 1 < 1$.

First let us give two technical lemmas that we need for the proof.

**Lemma 4.1 (Some left-hand continuity).** For all $c, \theta > 0$,

$$\mathbb{P}\left( X_t(0) > \theta, \liminf_{s \to t} S_{t-s}^c X_s(c (t-s)^{1/\alpha}) \leq \theta \right) = 0. \tag{4.1}$$

*Proof.* For brevity, set

$$A := \left\{ \liminf_{s \to t} S_{t-s}^c X_s(c (t-s)^{1/\alpha}) \leq \theta \right\} \tag{4.2}$$

and for $n > 1/t$ define the stopping times

$$\tau_n := \left\{ \inf \left\{ s \in (t-1/n, t) : S_{t-s}^c X_s(c (t-s)^{1/\alpha}) \leq \theta + 1/n \right\}, \quad \omega \in A, \right\} \tag{4.3}$$

Define also

$$x_n := c (t - \tau_n)^{1/\alpha}. \tag{4.4}$$

Then, using the strong Markov property, we get

$$\mathbb{E}[X_t(x_n) \mid \mathcal{F}_{\tau_n}] = S_{t-\tau_n}^c X_{\tau_n}(x_n) = X_t(0) 1_A + S_{t-\tau_n}^c X_{\tau_n}(x_n) 1_A. \tag{4.5}$$

We next note that $x_n \to 0$ almost surely as $n \uparrow \infty$. This implies, in view of the continuity of $X_t$ at zero, that $X_t(x_n) \to X_t(0)$ almost surely. Recalling that $\mathbb{E}\sup_{|x| \leq 1} X_t(x) < \infty$ in view of Corollary 2.8 of [FMW10], we conclude that

$$X_t(x_n) \to X_t(0) \quad \text{in } \mathcal{L}_1. \tag{4.6}$$

This, in its turn, implies that

$$\mathbb{E}[X_t(x_n) \mid \mathcal{F}_{\tau_n}] - \mathbb{E}[X_t(0) \mid \mathcal{F}_{\tau_n}] \to 0 \quad \text{in } \mathcal{L}_1. \tag{4.7}$$

Furthermore, it follows from the well known Levy theorem on convergence of conditional expectations that

$$\mathbb{E}[X_t(0) \mid \mathcal{F}_{\tau_n}] \to_{n \uparrow \infty} \mathbb{E}[X_t(0) \mid \mathcal{F}_\infty] \quad \text{in } \mathcal{L}_1, \tag{4.8}$$

where $\mathcal{F}_\infty := \sigma(\bigcup_{n > 1/t} \mathcal{F}_{\tau_n})$.

Noting that $\tau_n \uparrow t$, we conclude that

$$\mathcal{F}_{\tau_n} \subseteq \mathcal{F}_\infty \subseteq \mathcal{F}_t. \tag{4.9}$$

Since $X_t(0)$ is continuous for every fixed $t$, we have $X_t(0) = \mathbb{E}[X_t(0) \mid \mathcal{F}_{\tau_n}]$ almost surely. Consequently, $\mathbb{E}[X_t(0) \mid \mathcal{F}_\infty] = X_t(0)$ almost surely, and we get, as a result,

$$\mathbb{E}[X_t(0) \mid \mathcal{F}_{\tau_n}] \to_{n \uparrow \infty} X_t(0) \quad \text{in } \mathcal{L}_1. \tag{4.10}$$
Combining (4.7) and (4.10), we have
\[ \mathbb{E}[X_t(x_n) \mid \mathcal{F}_{t_n}] \to_{n \to \infty} X_t(0) \] in $\mathcal{L}_1$.

From this convergence and (4.11) we get finally
\[ \mathbb{E} 1_A \left| X_t(0) - S_{t-t_n}^\alpha x_n(x_n) \right| \to_{n \to \infty} 0. \]

Since $S_{t-t_n}^\alpha x_n(x_n) \leq \theta + 1/n$ on $A$, for all $n > 1/t$, the latter convergence implies that $X_t(0) \leq \theta$ almost surely on the event $A$. Thus, the proof is finished. \[\square\]

**Lemma 4.2 (Some local boundedness).** Fix any non-empty compact $K \subset \mathbb{R}$. Then
\[ W_K := \sup_{c \geq t^{-\alpha}, s \geq t^{-1/\alpha}, x \in K} \frac{X_s(B_{c(t-s)^{1/\alpha}}(x))}{c(t-s)^{1/\alpha}} < \infty \text{ almost surely.} \]

**Proof.** Every ball of radius $c(t-s)^{1/\alpha}$ can be covered with at most $[c] + 1$ balls of radius $(t-s)^{1/\alpha}$. Therefore,
\[ \sup_{c \geq t^{-\alpha}, s \geq t^{-1/\alpha}, x \in K} \frac{X_s(B_{c(t-s)^{1/\alpha}}(x))}{c(t-s)^{1/\alpha}} \leq 2 \sup_{0 < s \leq t, x \in K_1} \frac{X_s(B_{(t-s)^{1/\alpha}}(x))}{(t-s)^{1/\alpha}}, \]

where $K_1 := \{ x : \text{dist}(x, K) \leq 1 \}$. We further note that
\[ S_{t-t_n}^\alpha x_n(x) = \int_\mathbb{R} dy \ p_{t-t_n}^\alpha(x-y)X_s(y) \geq \int_{B_{(t-s)^{1/\alpha}}(x)} dy \ p_{t-t_n}^\alpha(x-y)X_s(y). \]

Using the monotonicity and the scaling property of $p^\alpha$, we get the bound
\[ S_{t-t_n}^\alpha x_n(x) \geq (t-s)^{-1/\alpha} p_{t-t_n}^\alpha(1)X_s(B_{(t-s)^{1/\alpha}}(x)). \]

Consequently,
\[ \sup_{0 < s \leq t, x \in K_1} \frac{X_s(B_{(t-s)^{1/\alpha}}(x))}{(t-s)^{1/\alpha}} \leq \frac{1}{p_{t}^\alpha(1)} \sup_{0 < s \leq t, x \in K_1} S_{t-t_n}^\alpha x_n(x). \]

It was proved in Lemma 2.11 of [FMW10], that the random variable at the right hand side is finite. Thus, the lemma is proved. \[\square\]

Introduce the event
\[ D_\theta := \left\{ X_t(0) > \theta, \sup_{0 < s \leq t} X_s(R) \leq \theta^{-1}, \ W_{B_2(0)} \leq \theta^{-1} \right\}. \]

For the rest of the paper take an arbitrary $\varepsilon \in (0, t \wedge 1/8)$. For constants $c, Q > 0$, define the stopping time
\[ \tau_{\varepsilon, c, Q} := \inf \left\{ s \in (t-\varepsilon, t) : \Delta X_s(y) > Q (y(t-s))^{1/(1+\beta)} \log^{1/(1+\beta)}((t-s)^{-1}), \right. \]
\[ \left. \text{for some } \frac{c}{2} (t-s)^{1/\alpha} \leq y \leq \frac{3c}{2} (t-s)^{1/\alpha} \right\}. \]

In the next lemma we are going to show the finiteness of $\tau_{\varepsilon, c, Q}$, which means that there is a “big” jump close to time $t$ and to the spatial point $z = 0$.\[\square\]
Lemma 4.3 (Finiteness of \( \tau_{\varepsilon,c}^{(4.19)} Q \)). For each \( \theta > 0 \) there exists a constant \( c_{4.19} \) such that

\[
P \left( \tau_{\varepsilon,c}^{(4.19)} Q = \infty \big| D_\theta \right) = 0, \quad Q > 0.
\]

Proof. Analogous to the proof of Lemma 4.3 in [FMW10], it suffices to show that

\[
I_{\varepsilon,c} := \int_{t-\varepsilon}^{t} \frac{ds}{(t-s) \log((t-s)^{-1})} \int_{\frac{2}{\varepsilon}(t-s)^{1/\alpha}}^{\frac{3}{\varepsilon}(t-s)^{1/\alpha}} dy y^{-1} X_s(y) = \infty
\]

almost surely on the event \( \{ X_t(0) > \theta, \sup_{0 \leq s \leq t} X_s(R) \leq \theta^{-1} \} \).

Since \( y \leq \frac{2}{\varepsilon} (t-s)^{1/\alpha} \) and \( p^{\alpha}(x) \leq p^{\alpha}(0) \) for all \( x \in R \), we have

\[
I_{\varepsilon,c} \geq \frac{2}{3c} \int_{t-\varepsilon}^{t} \frac{ds}{(t-s)^{1+1/\alpha} \log((t-s)^{-1})} \times \int_{\frac{2}{\varepsilon}(t-s)^{1/\alpha}}^{\frac{3}{\varepsilon}(t-s)^{1/\alpha}} dy \frac{p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y)}{p^{\alpha}_{t-s}(0)} X_s(y).
\]

Then, using the scaling property of \( p^{\alpha} \), we obtain

\[
I_{\varepsilon,c} \geq \frac{2}{3c p^{\alpha}_{1}(0)} \int_{t-\varepsilon}^{t} \frac{ds}{(t-s)^{1+1/\alpha} \log((t-s)^{-1})} \left( S_{t-s}^{\alpha} X_s \left( c(t-s)^{1/\alpha} \right) \right.
\]

\[
- \left. \int_{|y-c(t-s)^{1/\alpha}| > \frac{3}{\varepsilon}(t-s)^{1/\alpha}} dy p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y) X_s(y) \right).
\]

If

\[
c \left( \frac{1}{2} + j \right) (t-s)^{1/\alpha} < |y - c(t-s)^{1/\alpha}| < c \left( 2 + \frac{1}{2} + j \right) (t-s)^{1/\alpha},
\]

then

\[
p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y) \leq p^{\alpha}_{t-s}(c(j + 1/2)(t-s)^{1/\alpha})
\]

\[
= (t-s)^{-1/\alpha} p^{\alpha}_{1}(c(j + 1/2)) \leq c_{3.23} \epsilon^{-1}(t-s)^{-1/\alpha}(1/2 + j)^{-\alpha - 1}.
\]

From this bound and Lemma 4.2, we conclude that

\[
\int_{|y-c(t-s)^{1/\alpha}| > \frac{3}{\varepsilon}(t-s)^{1/\alpha}} dy p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y) 1_{B_{1/2}(0)}(y) X_s(y)
\]

\[
\leq 2W_{B_{1/2}(0)} c_{3.23} \epsilon^{-1} \sum_{j=0}^{\infty} (1/2 + j)^{-\alpha - 1} \leq CW_{B_{1/2}(0)} c^{-\alpha - 1}.
\]

Furthermore, if \(|y| \geq 2 \) and \((t-s) \leq c^{-\alpha} \), then

\[
p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y) \leq p^{\alpha}_{t-s}(1) = (t-s)^{-1/\alpha} p^{\alpha}_{1}((t-s)^{-1/\alpha}) \leq c_{3.23} (t-s).
\]

This implies that

\[
\int_{R \setminus B_{1/2}(0)} dy p^{\alpha}_{t-s}(c(t-s)^{1/\alpha} - y) X_s(y) \leq c_{3.23} (t-s) X_s(R)
\]

\[
\leq c_{3.23} c^{-\alpha} X_s(R).
\]
Combining this bound with (4.25), we obtain
\( \int_{|y - c(t-s)^{1/\alpha}| > \frac{\theta}{2}(t-s)^{1/\alpha}} dy \, p_{t-s}^{\alpha}(c(t-s)^{1/\alpha} - y)X_s(y) \leq Cc^{-\alpha}\left(W_{B_2(0)} + \sup_{0 < s \leq t} X_s(R)\right). \)

Thus, we can choose \( c \) so large that the right hand side in the previous inequality does not exceed \( \theta/2 \). Since, in view of Lemma 4.1, \( \lim \inf_{s \to t} S^\alpha_{t-s} X_s(c(t-s)^{1/\alpha}) > \theta \), we finally get
\( \lim \inf_{s \to t} \left( S^\alpha_{t-s} X_s(c(t-s)^{1/\alpha}) - \int_{|y - c(t-s)^{1/\alpha}| > \frac{\theta}{2}(t-s)^{1/\alpha}} dy \, p_{t-s}^{\alpha}(c(t-s)^{1/\alpha} - y)X_s(y) \right) \geq \theta/2. \)

From this bound and (4.22) follows the desired property of \( I_{\varepsilon,c} \). \( \square \)

Fix any \( \theta > 0 \), and to simplify notation write \( c := c_{(4.19)} \). For all \( n \) sufficiently large, define
\( A_n := \left\{ \Delta X_s \left( \left( \frac{c}{2}\, 2^{-n}, \frac{3c}{2}\, 2^{-n} \right) \right) \geq 2^{-(\bar{\eta}c+1)n} n^{1/(1+\beta)} \right\} \)
for some \( s \in (t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}) \).

Based on Lemma 4.3 we will show in the following lemma that, conditionally on \( D_\theta \), one of the \( A_n \)'s occurs for \( n \) sufficiently large. This then gives us a bit more precise information on the “big” jumps we are looking for.

**Lemma 4.4 (Existence of big jumps).** For all \( N \) sufficiently large,
\( \mathbb{P}\left( \bigcup_{n=N}^{\infty} A_n \Big| D_\theta \right) = 1. \)

**Proof.** If \( y \in \left( \frac{c}{2}(t-s)^{1/\alpha}, \frac{3c}{2}(t-s)^{1/\alpha} \right) \) and \( s \in (t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}) \), then
\( \left( (t-s) y \log((t-s)^{-1}) \right)^{1/(1+\beta)} \geq \left( 2^{-\alpha(n+1)} \frac{c}{2} 2^{-n-1} \alpha \log 2n \right)^{1/(1+\beta)} = c_{(4.32)} 2^{-(\bar{\eta}c+1)n} n^{1/(1+\beta)}. \)

This implies that
\( A_n \supseteq \left\{ \Delta X_s \left( \left( \frac{c}{2}, \frac{3c}{2}(t-s)^{1/\alpha} \right) \right) \geq c_{(4.32)} \left( (t-s) y \log((t-s)^{-1}) \right)^{1/(1+\beta)} \right\} \)
for some \( s \in (t - 2^{-\alpha n}, t - 2^{-\alpha(n+1)}) \).
In what follows we denote with some abuse of notation \( \tau_{\varepsilon,c} := \tau_{\varepsilon,c,c} \). Consequently, from (4.32) we get
\[
\bigcup_{n=N}^{\infty} A_n \supseteq \{ \tau_{2^{-n},c} < \infty \}.
\]
Applying Lemma 4.3 completes the proof. \( \Box \)

Now it is time to explain our

**Detailed strategy of proof of Theorem 1.1(b).** Define
\[
A^\varepsilon := \{ \Delta X_s(y) \leq c 22.7 ((t-s)|y|^{1/(1+\beta)} f_{s,x}^\ell \text{ for all } s < t \text{ and } y \in B_{1/c}(0) \} \cap \{ \Delta X_s(y) \leq c^* (t-s)^{1/(1+\beta)-\gamma} \text{ for all } s < t \text{ and } y \in \mathbb{R} \} \cap \{ V \leq c_\varepsilon \},
\]
where \( f_{s,x}, \ell \) and \( c^* \) are defined in (2.6), (2.8) and (3.9), respectively. Note that \( D_\theta \uparrow \{ X_t(0) > 0 \} \) as \( \theta \downarrow 0 \) and by (3.6), (3.9) and Lemma 2.1 we have \( A^\varepsilon \uparrow \Omega \) as \( \varepsilon \downarrow 0 \). Hence, for the proof of Theorem 1.1(b) it is sufficient to show that
\[
P \left( \sup_{x \in B_{1}(0), x \neq 0} \frac{|\tilde{X}_t(x) - \tilde{X}_t(0)|}{|x|^{\gamma_c}} = \infty \ \big| D_\theta \cap A^\varepsilon \right) = 1.
\]
Moreover, since \( Z_t^1 \) and \( Z_t^3 \) are a.s. Lipschitz continuous at 0, the latter will follow from the equality
\[
P (Z_t^2(c 2^{-n-2}) - Z_t^2(0) \geq 2^{-\gamma_c n} n^{1/(1+\beta)-\varepsilon} \text{ infinitely often} \ | D_\theta \cap A^\varepsilon ) = 1.
\]
To verify (4.37), we will again use our method of representing \( Z_t^2 \) as a time-changed stable process. To be more precise, applying (4.37) with \( x = c 2^{-n-2} \) (for \( n \) sufficiently large) and using \( n \)-dependent notation as \( L_n^\pm, T_{n,\pm} \) (and \( \varphi_{n,\pm} \)), we have
\[
(4.38) \quad Z_t^2(c 2^{-n-2}) - Z_t^2(0) = L_n^+(T_{n,+}) - L_n^-(T_{n,-}).
\]
Let us define the following events
\[
B_n^+ := \{ L_n^+(T_{n,+}) \geq 2^{1-\gamma_c n} n^{1/(1+\beta)-\varepsilon} \}, \quad B_n^- := \{ L_n^-(T_{n,-}) \leq 2^{-\gamma_c n} n^{1/(1+\beta)-\varepsilon} \}
\]
and
\[
(4.39) \quad B_n := B_n^+ \cap B_n^-.
\]
Then, obviously,
\[
(4.40) \quad \{ Z_t^2(c 2^{-n-2}) - Z_t^2(0) \geq 2^{-\gamma_c n} n^{1/(1+\beta)-\varepsilon} \} \supseteq B_n \supseteq B_n \cap A_n.
\]
Thus, (4.37) will follow once we verify
\[
(4.41) \quad \lim_{N \uparrow \infty} P \left( \bigcup_{n=N}^{\infty} (B_n \cap A_n) \big| D_\theta \cap A^\varepsilon \right) = 1.
\]
Taking into account Lemma 4.3 we conclude that to get (4.41) we have to show
\[
(4.42) \quad \lim_{N \uparrow \infty} P \left( \bigcup_{n=N}^{\infty} (B_n^- \cap A_n) \big| D_\theta \cap A^\varepsilon \right) = 0.
\]
Hence, the proof of Theorem 1.1(b) will be complete once we demonstrated statement (4.42). □

Now we will present two lemmas, from which (4.42) will follow immediately. To this end, split
\begin{equation}
B_n^c \cap A_n = (B_n^+, c_n \cap A_n) \cup \left( B_n^-, c_n \cap A_n \right).
\end{equation}

**Lemma 4.5 (First term in (4.43)).** We have
\begin{equation}
\lim_{N \to \infty} \sum_{n=N}^{\infty} P(B_n^{+, c} \cap A_n \cap A^c) = 0.
\end{equation}

The proof of this lemma is a word-for-word repetition of the proof of Lemma 5.3 in [FMW10] (it is even simpler as we do not need additional indexing in \( k \) here), and we omit it. The idea behind the proof is simple: Whenever \( X \) has a “big” jump guaranteed by \( A_n \), this jump corresponds to the jump of \( L_n^+ \) and then it is very difficult for a spectrally positive process \( L_n^+ \) to come down, which is required by \( B_n^{+, c} \).

**Lemma 4.6 (Second term in (4.43)).** We have
\begin{equation}
\lim_{N \to \infty} \sum_{n=N}^{\infty} P(B_n^{-, c} \cap A_n \cap A^c \cap D_\theta) = 0.
\end{equation}

The remaining part of the paper will be devoted to the proof of Lemma 4.6 and we prepare now for it.

One can easily see that \( B_n^{-, c} \) is a subset of a union of two events (with the obvious correspondence):
\begin{equation}
B_n^{-, c} \subseteq U_n^1 \cup U_n^2 := \{ \Delta L_n^- > 2^{-\tilde{\eta}_n} n^{1/(1+\beta) - 2\varepsilon} \} \\
\cup \{ \Delta L_n^- \leq 2^{-\tilde{\eta}_n} n^{1/(1+\beta) - 2\varepsilon}, \; L_n^-(T_{n-}) > 2^{-\tilde{\eta}_n} n^{1/(1+\beta) - \varepsilon} \},
\end{equation}
where
\begin{equation}
\Delta L_n^- := \sup_{0<s \leq T_{n-}} \Delta L_n^-(s).
\end{equation}

The occurrence of the event \( U_n^1 \) means that \( L_n^- \) has big jumps. If \( U_n^2 \) occurs, it means that \( L_n^- \) gets large without big jumps. It is well-known that stable processes without big jumps can not achieve large values. Thus, the statement of the next lemma is not surprising.

**Lemma 4.7 (No big values of \( L_n^- \) in case of absence of “big” jumps).** We have
\begin{equation}
\lim_{N \to \infty} \sum_{n=N}^{\infty} P(U_n^2 \cap A^c) = 0.
\end{equation}

The proof of this lemma is a word-for-word repetition of the proof of Lemma 5.6 in [FMW10] (again with obvious simplifications).

**Lemma 4.8 (Big jumps of \( L_n^- \) caused by several big jumps of \( M \)).** There exist constants \( \rho \) and \( \xi \) such that, for all sufficiently large values of \( n \),
\begin{equation}
A^c \cap A_n \cap U_n^1 \subseteq A^c \cap E_n(\rho, \xi),
\end{equation}
where \( E_n(\rho, \xi) \) is a set defined in the next section.
where

\[ E_n(\rho, \xi) := \left\{ \begin{array}{ll}
\text{There exist at least two jumps of } M \text{ of the form } r\delta_{(s,y)} \text{ such that} \\
r \geq \left( (t-s) \max \{(t-s)^{1/\alpha}, |y|\} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}((t-s)^{-1}) , \\
|y| \leq (t-s)^{1/\alpha} \log^\varepsilon((t-s)^{-1}) , \\
s \in \left[ t - 2^{-\alpha n^\rho}, t - 2^{-\alpha n^{-\rho}} \right] \right\}.
\]

\[ (4.50) \]

**Proof.** By the definition of \( A_n \), there exists a jump of \( M \) of the form \( r\delta_{(s,y)} \) with \( r, s \) as in \( E_n(\rho, \xi) \), and \( y > c2^{-n-1} \). Furthermore, noting that \( \varphi_{n,-}(y) = 0 \) for \( y \geq c2^{-n-3} \), we see that the jumps \( r\delta_{(s,y)} \) of \( M \) contribute to \( L_n^-(T_{n,-}) \) if and only if \( y < c2^{-n-3} \). Thus, to prove the lemma it is sufficient to show that \( U_n^1 \) yields the existence of at least one further jump of \( M \) on the half-line \( \{ y < c2^{-n-3} \} \) with properties mentioned in the statement. Denote

\[ D := \left\{ (r, s, y) : r \geq \left( (t-s) \max \{(t-s)^{1/\alpha}, |y|\} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}((t-s)^{-1}) , \right. \]

\[ y \in \left( -(t-s)^{1/\alpha} \log^\varepsilon((t-s)^{-1}) , c2^{-n-3} \right) , \]

\[ s \in \left[ t - 2^{-\alpha n^\rho}, t - 2^{-\alpha n^{-\rho}} \right] \}

\[ (4.51) \]

Then we need to show that \( U_n^1 \) implies the existence of a jump \( r\delta_{(s,y)} \) of \( M \) with \( (r, s, y) \in D \).

Note that

\[ D = D_1 \cap D_2 \cap D_3 \]

\[ := \left\{ (r, s, y) : r \geq 0 , s \in [0, t] , y \in \left( -(t-s)^{1/\alpha} \log^\varepsilon((t-s)^{-1}) , c2^{-n-3} \right) \right\} \]

\[ \cap \left\{ (r, s, y) : r \geq 0 , y \in (-\infty, c2^{-n-3}) , s \in \left[ t - 2^{-\alpha n^\rho}, t - 2^{-\alpha n^{-\rho}} \right] \right\} \]

\[ \cap \left\{ (r, s, y) : y \in (-\infty, c2^{-n-3}) , s \in [0, t] , \right. \]

\[ r \geq \left( (t-s) \max \{(t-s)^{1/\alpha}, |y|\} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}((t-s)^{-1}) \right\} . \]

Therefore,

\[ (4.52) \]

\[ D^c \cap \{ y < c2^{-n-3} \} = \left( D_1^c \cap \{ y < c2^{-n-3} \} \right) \cup \left( D_2 \cap D_3 \right) \cup (D_1 \cap D_2 \cap D_3) , \]

where the complements are defined with respect to the set

\[ (4.53) \]

\[ \{ (r, s, y) : r \geq 0 , s \in [0, t] , y \in \mathbb{R} \} . \]

We first show that any jumps of \( M \) in \( D_1^c \cap \{ y < c2^{-n-3} \} \) cannot be the course of a jump of \( L_n^- \) such that \( U_n^1 \) holds. Indeed, using Lemma 2.23 with \( \delta = \bar{\eta}_c \), we get
for $y < 0$ the inequality

$$
\varphi_{n,-}(y) = p^{\alpha}_{t-s}(y) - p^{\alpha}_{t-s}(y - c 2^{-n-1}) \leq 2^{1-\tilde{\eta}_n} (t-s)^{1-\tilde{\eta}_n/\alpha} p^{\alpha}_{t-s}(y)
$$

\leq C 2^{-\tilde{\eta}_n} (t-s)^{-(1+\tilde{\eta}_n/\alpha)} \left( \frac{y}{(t-s)^{1/\alpha}} \right)^{-\alpha-1}

\leq C 2^{-\tilde{\eta}_n} (t-s)^{1-\tilde{\eta}_n/\alpha} |y|^{-\alpha-1},

(4.54)
in the second step we used the scaling property and (3.23).

Further, by (4.35), on the set $A^c$ we have

$$
\Delta X_s(y) \leq C \left( |y|(t-s) \right)^{1/(1+\beta)} (f_{s,y})^\ell, \quad |y| \leq 1/e,
$$

and

$$
\Delta X_s(y) \leq C (t-s)^{1/(1+\beta)-\gamma}, \quad |y| > 1/e,
$$

(4.55)

and recall that $f_{s,x} = \log((t-s)^{-1}) 1_{\{|x| \neq 0\}} \log(|x|^{-1})$. Combining (4.54) and (4.55), we conclude that the corresponding jump of $L_n$, henceforth denoted by $\Delta L_n[r_{\delta(s,y)}]$, is bounded by

$$
\Delta L_n[r_{\delta(s,y)}] \leq C 2^{-\tilde{\eta}_n} \log(t-s)^{1/(1+\beta)} (|y|)^{-1/\gamma}.
$$

(4.56)

Since $|y|^{-\alpha+1/(1+\beta)} \log(t-s)^{1/(1+\beta)} (|y|)^{-1}$ is monotone decreasing, we get, maximizing over $y$, for $y < -(t-s)^{1/\alpha} \log(t-s)^{1/(1+\beta)}$ the bound

$$
\Delta L_n[r_{\delta(s,y)}] \leq C 2^{-\tilde{\eta}_n} \log(t-s)^{1/(1+\beta)} (|y|)^{-1/\gamma}.
$$

(4.57)

Choosing $\xi \geq \frac{1+2q(1+\beta)}{1+1/(1+\alpha)}$, we see that

$$
\Delta L_n[r_{\delta(s,y)}] \leq C 2^{-\tilde{\eta}_n}, \quad |y| < 1/e.
$$

(4.58)

Moreover, if $y < -1/e$, then it follows from (4.54) and (4.56) that the jump $\Delta L_n[r_{\delta(s,y)}]$ is bounded by

$$
C 2^{-\tilde{\eta}_n} (t-s)^{1-\tilde{\eta}_n/\alpha} \log(t-s)^{1/(1+\beta)} |y|^{-\alpha-1} \leq C 2^{-\tilde{\eta}_n}.
$$

(4.59)

Combining (4.55) and (4.59), we see that all the jumps of $M$ in $D_1 \cap \{y < c 2^{-n-3}\}$ do not produce jumps of $L_n$ such that $U_n^1$ holds.

We next assume that $M$ has a jump $r_{\delta(s,y)}$ in $D_1 \cap D_2^c$. If, additionally, $s \leq t-2^{-\alpha n} n^\rho$, then, using Lemma 2.3 with $\delta = 1$, we get

$$
\varphi_{n,-}(y) = p^{\alpha}_{t-s}(y) - p^{\alpha}_{t-s}(y - c 2^{-n-1}) \leq 2^{1-n} (t-s)^{-2/\alpha}.
$$

(4.60)

From this bound and (4.55) we obtain

$$
\Delta L_n[r_{\delta(s,y)}] \leq C 2^{-n} (t-s)^{-2/\alpha} \log(t-s)^{1/(1+\beta)} (|y|)^{-1/\gamma}
$$

\leq C 2^{-n} (t-s)^{-2/(1+\beta)}/\alpha \log(t-s)^{2 \alpha q/\beta} + 2q (t-s)^{-1}.
$$

(4.61)

Using the assumption $t-s \geq 2^{\alpha n} n^\rho$, we arrive at the inequality

$$
\Delta L_n[r_{\delta(s,y)}] \leq C 2^{-\tilde{\eta}_n} n^{-\rho(1-\tilde{\eta}_n)/\alpha + 2q/(1+\beta)} + 2q.
$$

(4.62)

From this we see that if we choose $\rho \geq \frac{\alpha(1+2q(1+\beta))}{(1+\beta)(1-\tilde{\eta}_n)}$, then the jumps of $L_n$ are bounded by $C 2^{-\tilde{\eta}_n}$, and hence $U_n^1$ does not occur.

If $M$ has a jump in $D_1 \cap D_2$ at time $s \geq t-2^{\alpha n} n^{-\rho}$, then, using (4.55) and the bound

$$
\varphi_{n,-}(y) = p^{\alpha}_{t-s}(y) - p^{\alpha}_{t-s}(y - c 2^{-n-1}) \leq p^{\alpha}_{t-s}(0) \leq C (t-s)^{-1/\alpha},
$$

(4.63)
we get for \( y \in \left( -(t - s)^{1/\alpha} \log \xi ((t - s)^{-1}), c2^{-n-3} \right) \) and \( t - s \leq 2^{-\alpha n} n^{-\rho} \) the inequality
\[
\Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C \left( t - s \right)^{\frac{1}{\alpha} + \frac{1}{\alpha + 1}} \log \frac{2^{C_2} + 2q}{(t - s)^{-1}} \\
\leq C 2^{-\bar{n} n} \left( n^{-\rho(\bar{n}/\alpha)} + \frac{1}{\alpha} \right) 2^{\bar{n} n} + 2q.
\] (4.65)

Choosing \( \rho \geq \frac{\alpha(\xi + 2 + 2q(1 + \beta))}{(1 + \beta)\bar{n}} \), we conclude that \( \Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C 2^{-\bar{n} n} \), and again \( U_n^1 \) does not occur.

Finally, it remains to consider the jumps of \( M \) in \( D_1 \cap D_2 \cap D_3^c \). If the value of the jump does not exceed \( (t - s)^{\frac{\alpha + 1}{\alpha + 1 + \beta}} \log \frac{1}{(t - s)^{-1}} \), then it follows from Lemma 4.8 with \( \delta = \bar{n} \) that
\[
\Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C 2^{-\bar{n} n} \log \frac{1}{(t - s)^{-1}}.
\] (4.66)

Then, on \( D_2 \), that is, for \( t - s > 2^{-\alpha n} n^{-\rho} \),
\[
\Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C 2^{-\bar{n} n} n^{-\frac{1}{\alpha+1} - 2\epsilon}.
\] (4.67)

Furthermore, if \( y < -(t - s)^{1/\alpha} \) and the value of the jump is less than \( (|y| \cdot (t - s)|^{\frac{\alpha + 1}{\alpha + 1 + \beta}} \log \frac{1}{(t - s)^{-1}} \), then, using (4.54), we get
\[
\Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C 2^{-\bar{n} n} \left( t - s \right)^{1 - \bar{n}/\alpha} \log \frac{1}{(t - s)^{-1}} |y|^{-\alpha - 1 + \frac{1}{\alpha + 1}}
\]
\[
\leq C 2^{-\bar{n} n} \log \frac{1}{(t - s)^{-1}}.
\] (4.68)

Then, on \( D_2 \), that is, for \( t - s > 2^{-\alpha n} n^{-\rho} \),
\[
\Delta L_n^* \left[ r \delta_{(s,y)} \right] \leq C 2^{-\bar{n} n} n^{-\frac{1}{\alpha+1} - 2\epsilon}.
\] (4.69)

By (4.67) and (4.69), we see that the jumps of \( M \) in \( D_1 \cap D_2 \cap D_3^c \) do not produce jumps such that \( U_n^1 \) holds. Combining all the above we conclude that to have \( \Delta L_n^* \left[ r \delta_{(s,y)} \right] > C 2^{-\bar{n} n} n^{-\frac{1}{\alpha+1} - 2\epsilon} \) it is necessary to have a jump in \( D_1 \cap D_2 \cap D_3 \). Thus, the proof is finished. \( \square \)

**Proof of Lemma 4.6** In view of the Lemmas 4.7 and 4.8, it suffices to show that
\[
\lim_{N \uparrow \infty} \sum_{n=N}^{\infty} P \left[ E_n(\rho, \xi) \cap A^c \cap D_\theta \right] = 0.
\] (4.70)

The intensity of the jumps in \( D \) [the set defined in (4.51) and satisfying conditions in \( E_n(\rho, \xi) \)] is given by
\[
\int_{t - 2^{-\alpha n} n^{-\rho}}^{t - 2^{-\alpha n} n^{-\rho}} ds \int_{|y| \leq (t - s)^{1/\alpha} \log \xi ((t - s)^{-1})} \left[ \frac{\log 2^{(1 + \beta)(t - s)^{-1}}}{(t - s) \max \{ (t - s)^{1/\alpha}, |y| \} } \right] X_s(dy).
\] (4.71)

We first note that
\[
\int_{|y| \leq (t - s)^{1/\alpha}} \frac{X_s(dy)}{\max \{ (t - s)^{1/\alpha}, |y| \} } \leq \frac{1}{\theta^{-1}} X_s \left( \frac{1}{\theta^{-1}} \right).
\] (4.72)
on $D_\theta$. Further, for every $j \geq 1$,\(^{(4.73)}\)
\[
\int_{(t-s)^{1/\alpha} |y| \leq (j+1)(t-s)^{1/\alpha}} \frac{X_s(dy)}{\max\{ (t-s)^{1/\alpha} |y| \}} 
\leq j^{-1} (t-s)^{-1/\alpha} \max\{ (t-s)^{1/\alpha} |y| \} 
\leq 2 \theta^{-1} j^{-1}
\]
on $D_\theta$. As a result, on the event $D_\theta$ we get the inequality\(^{(4.74)}\)
\[
\int_{|y| \leq (t-s)^{1/\alpha} \log^5((t-s)^{-1})} \frac{1}{X_s(dy)} \frac{1}{\max\{ (t-s)^{1/\alpha} |y| \}} 
\leq C \theta^{-1} \log \left( \log \left( \log \left( \log \left( \log \left( \log \left( (t-s)^{-1} \right) \right) \right) \right) \right) \right).
\]
Substituting this into \(^{(4.71)}\), we conclude that the intensity of the jumps is bounded by\(^{(4.75)}\)
\[
C \theta^{-1} \int_{t^{-2-\alpha n^{-\rho}}} ds \log^{2\varepsilon(1+\beta)-1} \left( \frac{(t-s)^{-1} \log \log \left( (t-s)^{-1} \right)}{(t-s)} \right).
\]
Simple calculations show that the latter expression is less than\(^{(4.76)}\)
\[
C \theta^{-1} n^{2\varepsilon(1+\beta)-1} \log^{1+2\varepsilon(1+\beta)} n.
\]
Consequently, since $E_n(\rho, \xi)$ holds when there are two jumps in $D$, we have\(^{(4.77)}\)
\[
P\left( E_n(\rho, \xi) \cap A^\epsilon \cap D_\theta \right) \leq C \theta^{-2} n^{2\varepsilon(1+\beta)-2} \log^{2+4\varepsilon(1+\beta)} n.
\]
Because $\varepsilon < 1/8 \leq 1/4(1+\beta)$, the sequence $P\left( E_n(\rho, \xi) \cap A^\epsilon \cap D_\theta \right)$ is summable, and the proof of the lemma is complete. \(\square\)

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