SMALL FIBERWISE OSCILLATION OF THE EIGENFUNCTIONS OF COLLAPSING EINSTEIN MANIFOLDS

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ABSTRACT. By Cheeger-Colding’s almost splitting theorem, if a domain in a Ricci flat manifold is pointed-Gromov-Hausdorff close to a lower dimensional Euclidean domain, then there is a harmonic almost splitting map. We show that any eigenfunction of the Laplace operator is almost constant along the fibers of the almost splitting map, in the $L^2$-average sense. This generalizes an estimate of Fukaya in the case of collapsing with bounded diameter and sectional curvature.

1. Introduction and background

A natural and important theme in geometric analysis is to understand the uniform behavior of the eigenvalues and eigenfunctions of the Laplace operators associated to a given family of Riemannian manifolds. In the seminal work [18], by introducing the concept of measured Gromov-Hausdorff convergence, Fukaya proved that the eigenvalues of the Laplace operators of Riemannian manifolds are continuous with respect to such topology, provided that the manifolds in consideration have uniformly bounded diameter and sectional curvature. In proving such continuity, the main difficulty is that a sequence of Riemannian manifolds may collapse in volume, and a key tool to overcome this difficulty is the following functional inequality, which is referred to as the Key Lemma in Fukaya’s work [18, §3]:

**Theorem 1.1 (Fukaya’s Key Lemma).** Suppose a sequence of $m$-dimensional Riemannian manifolds $\{(M_i, g_i)\}$ satisfy the regularity assumptions

$$\forall l \in \mathbb{N}, \quad \exists C_l > 0, \quad \sup_{M_i} \|\nabla^l R_{g_i}\| \leq C_l,$$

and suppose that there is a $k$-dimensional ($k < m$) closed Riemannian manifold $(N, h)$, such that

$$\lim_{i \to \infty} d_{GH}(M_i, N) = 0.$$

Then for any $i$ sufficiently large, and any $u \in C^\infty(M_i)$, we have the estimate

$$\|\nabla^T u\|_{L^2(M_i)} \leq C_F(m, k) \left(\|u\|_{L^2(M_i)} + \|\Delta_g^q u\|_{L^2(M_i)}\right) d_{GH}(M_i, N),$$

for some dimensional constants $C_F(m, k) > 0$ and some large $q \in \mathbb{N}$, with $\| \cdot \|_{L^2(M_i)}$ denoting the $L^2$-average of a given function on $M_i$.

Here we recall that when the Riemannian manifold $(M_i, g)$ is sufficiently close to, in the Gromov-Hausdorff sense, another lower dimensional Riemannian manifold $(N, h)$, then the regularity assumptions $\square$ guarantee that there is a fibration $\Phi_i : M_i \to N$, which is also an almost Riemannian submersion. Moreover, the $\Phi_i$ fibers, as embedded submanifolds in $M_i$, are all homeomorphic.
to some infranil manifold. Here for any $x \in \mathcal{M}$, $\nabla^T u(x) \in T_x \Phi^{-1}(\Phi(x))$ denotes the restriction of $\nabla u(x)$ to the directions tangential to the fiber of $\Phi$ through $x$.

Roughly speaking, Theorem 1.1 tells that when a sequence of Riemannian manifolds collapses to a lower dimensional one with (1.1) satisfied, then on those sufficiently collapsed ones in the sequence, any "reasonable" function behaves almost like constants along the fiber directions. In fact, besides playing a crucial rule in proving the continuity of the eigenvalues, the Key Lemma provides more information than needed: when the collapsing limit is a manifold, the eigenfunctions of the collapsing sequence converge in the $C^1$-sense (see [19] §3), and this in turn helped prove that the collapsing limit can be embedded into a finite dimensional Euclidean space by heat kernel methods (see [19] §4).

The curvature assumption (1.1) is crucial here — without this condition, we cannot expect any topological structure of $\mathcal{M}$ relating to $N$, and Fukaya’s proof heavily relies on the fact that the fibers are infranil manifolds: he worked locally on a tangent space of a point, on which the pull-back metrics do not collapse, and the pull-back functions under consideration are locally periodic with shorter and shorter periods (see [18] §3).

However, with many natural examples of Einstein manifolds collapsing to lower dimensional metric spaces without a priori curvature bounds — for instance, the collapsing of Ricci flat $K3$ surfaces constructed by Gross and Wilson [20] and recently by Hein, Sun, Viaclovsky and Zhang [21] — one wonders if there should be any sort of extension of Fukaya’s Key Lemma to Einstein manifolds that are pointed Gromov-Hausdorff close to lower dimensional metric spaces at a given scale, without assuming (1.1). To explain the basic setup in this situation, let us focus on a small piece of a very collapsed Riemannian manifold with almost non-negative Ricci curvature, and recall the following fundamental theorem due to Cheeger and Colding (see [7] Theorem 1.2 and [9] Lemma 1.21):

**Theorem 1.2** (Cheeger-Colding’s Almost Splitting Theorem). Let $(M^m, g)$ be a Riemannian manifold. There there exists $\varepsilon(m) > 0$ and $l(m) > 0$ to the following effect: suppose a geodesic $l$-ball $B(p, l) \subset M$ satisfies $d_{GH}(B(p, l), B^k(l)) \leq \varepsilon r$ for some integer $l > l(m)$ and some $\varepsilon \in (0, \varepsilon(m))$, where $B^k(l)$ denotes the $k$-Euclidean $l$-ball centered at the origin, then there is a harmonic map $\Phi : B(p, 4r) \to \mathbb{R}^k$ such that

1. $\Phi(B(p, 2r)) \subset B^k(2r)$;
2. $\sup_{B(p, 2r)} |\nabla \Phi^a| \leq C(m)$;
3. $\int_{B(p, 2r)} |\nabla \Phi^a, \nabla \Phi^b| - \delta^{ab} \leq \Psi(\varepsilon, l^{-1}|m|)$; and
4. $\int_{B(p, 2r)} |\text{Hess} \Phi|^2 \leq \Psi(\varepsilon, l^{-1}|m|)$.

for some $C(m) > 0$ and $\Psi(\varepsilon, l^{-1}|m|) \to 0$ as $\varepsilon \to 0$ and $l \to \infty$. Here $\delta^{ab}$ denotes the Kronecker delta and $\Phi^a$ $(a = 1, \ldots, k)$ denotes a component function of the vector valued harmonic map $\Phi$.

When $k < m$, the map $\Phi$ resembles, in a certain sense, the fibration in the case of collapsing with bounded sectional curvature. However, $\Phi$ is far from being even a topological fibration, since it may well have singular values. Considering each point $x \in \Phi^{-1}(R)$, with $R \subset \Phi(B(p, 2r))$ denoting the regular values of $\Phi$, we define for any function $u \in C^\omega(B(p, 2r))$, the vector $\nabla^T u(x) \in T_x M$ as the part of $\nabla u(x)$ tangential to the fiber $\Phi^{-1}(\Phi(x))$. In contrast to the case of collapsing with bounded curvature, even over the regular values of $\Phi$, we have no information about the specific
structure of the fibers of $\Phi$, except their being closed and embedded submanifolds in $B(p, 3r)$. The purpose of this note is to show that even in this very rough case, there is still an analogue of Fukaya’s Key Lemma:

**Theorem 1.3** (Smallness of the Fiberwise Gradient $L^2$-Average). Let $B(p, l r)$ be a geodesic ball in an $m$-dimensional Ricci flat manifold, and suppose that $d_{GH}(B(p, l r), B^l(p)) \leq \varepsilon r$ for some $l > l(m)$ and $r \in (0, \varepsilon(m))$, where $l(m), \varepsilon(m)$ are positive dimensional constants determined by Theorem [1.2]. Then for any $u \in C^\infty(B(p, 2r))$ satisfying $\Delta u = \theta u$ for some $\theta \in \mathbb{R}$, we have

$$r\|\nabla^T u\|_{L^2(B(p,r))} \leq C(m, k, \theta)\|u\|_{L^\infty(B(p,2r))}\left(e^{\frac{1}{\varepsilon^2}} + \Psi(\varepsilon, l^{-1}|m|)\right).$$

Here the constant $C(m, k, \theta) > 0$ is determined only by $m, k$ and $\theta$, and $\|\cdot\|_{L^2(B(p,2r))}$ denotes the $L^2$-average of a function on $B(p, 2r)$.

**Remark 1.** Our estimate here only relies on the lower bound of the Ricci curvature. The Ricci flatness guarantees that the metric is real analytic, and the theorem still holds for any manifold with an analytic metric, whose Ricci curvature being bounded below; see Theorem [3.3]. Especially, this theorem works for Kähler manifolds with holomorphic bisectional curvature bounded below.

Notice that the definition of $\nabla^T u$ is not canonical — it depends on the harmonic almost splitting map $\Phi$. The same phenomenon even occurs in the case of collapsing with bounded curvature, where the fibration map also faces many choices, see [17].

We would now like to mention our novelty in proving Theorem [1.3] and briefly forecast the contents of the note. The conventional strategy in dealing with collapsing when [1.1] is satisfied, is to pull the metric back to a local universal covering space, which approximately looks like a product, and Fukaya’s proof of Theorem [1.1] hinges upon this property. When [1.1] is weakened to a mere Ricci curvature lower bound, while this strategy works in certain situations like controlling the fundamental group or understanding the infinitesimal behavior of the metric (see e.g. [26] and [23]), it cannot lead us to the more delicate gradient estimate as [1.3], without any extra assumption (like Ricci bounded covering geometry). This is mainly due to the lack of a fibration (or submersion) structure, as well as the absence of the structural information of the $\Phi$ fibers. We realize, however, that the specific structure of a $\Phi$ fiber do not affect our estimate on the change of a function along the fiber — only size matters. Therefore, our main efforts are plunged into the immersion side of the picture: the regular fibers of the almost submersion $\Phi$ are compactly embedded submanifolds. In §2, we will investigate the variation of $|\nabla^T u|^2$ along the dynamics driven by $\nabla^T u$, and obtain the estimate

$$|\nabla^T u|^2(\gamma_s(t)) \geq e^{-Ct} |\nabla^T u|^2(x),$$

where $\gamma_s$ is the flow line of the vector field $\nabla^T u$ along $\Phi^{-1}(\Phi(x))$, starting from $x$ with $\Phi(x) \in \mathcal{R}$. On the other hand, for any $t > 0$, $\gamma_s(t)$ stays within $\Phi^{-1}(\Phi(x))$, so the a priori gradient estimate of $u$ and the diameter bound $\text{diam}(\Phi^{-1}(\Phi(x)), g) < 2\varepsilon r$ ensure that $|u(x) - u(\gamma_s(t))| < C\varepsilon r$. Therefore, $|\nabla^T u|(x)$ cannot be too large compared to $\varepsilon$, because integrating the above inequality with respect to $t$ controls $|\nabla^T u|(x)$ from above. In §3, this argument will be extended, in the $L^2$-average sense, across all fibers of $\Phi$, with the integral Jacobian and Hessian estimates provided by Theorem [1.2].

Here in the process of extension, we need the analyticity of the metric to ensure that the singular fibers have zero total measure.

The estimate in Theorem [1.3] is an effective $L^2$-average gradient estimate rather than the original $L^\infty$-gradient estimate — replacing the pointwise estimates by the local $L^2$-average estimate as we
change from sectional curvature bounds to the corresponding Ricci curvature bounds is a natural and necessary phenomenon ever since the early works of Colding on the local $L^2$-average estimates of the regularized angle and distance functions, see [11, 12, 13]. Such $L^2$-average gradient estimates usually suffice to relate the metric measure properties of the collapsing manifolds with those of the collapsing limit spaces.

In [8], Cheeger and Colding generalized Fukaya’s eigenvalue continuity theorem to manifolds with only Ricci curvature lower bound, without referring to an analogue of Fukaya’s Key Lemma, but we believe the $L^2$-average tangential gradient estimate in Theorem 1.3 may provide some new tool to sharpen our understanding of the collapsing limits. Let us mention one potential application of Theorem 1.3 as an example: suppose a sequence of Einstein manifolds $(M_i, g_i)$ of uniformly bounded diameters and Einstein constants Gromov-Hausdorff converges to a limit metric space $(X, d)$ of lower dimension (in the sense of [14, Theorem 1.12]), then by using a covering argument, as well as [8, Theorem 3.23], we can apply Theorem 1.3 to show that the eigenfunctions converge to those on the limit in the $H^1$-sense (see [5] and [8]), whence the $H^1$-convergence of the related heat kernels (see [8, §6] and [15]), and consequently, this will lead to an embedding theorem of $(X, d)$ into the infinite dimensional Hilbert space $L^2(X)$, in view of the related arguments in [19]. Yet we will not put more details here, since the above mentioned $H^1$-convergence and embedding results have recently been obtained by Ambrosio, Honda, Portegies and Tewodrose [3] in more general settings. Their approach relies on the more abstract theory of $RCD^*$-spaces developed by Ambrosio, Gigli and Savaré, see [12]; and this is in turn based on the Lott-Sturm-Villani characterization of the Ricci curvature lower bound via optimal transport, see [25, 28, 29].

2. A priori estimate along the regular fibers

In this section we control the size of the tangential derivatives of a function on a regular fiber of $\Phi$. Here we will overcome the main technical difficulty — the lack of a geometric structure of any regular fiber of $\Phi$ — by investigated the dynamics driven by the tangential gradient vector fields along the regular fibers, only relying on the fact that they are small compactly embedded sub-manifolds.

Let $\Phi : B(p, 4r) \to \mathbb{R}^k$ be given as in Theorem 1.2. Since we may assume that $\Phi$ provides an $\varepsilon r$-Gromov-Hausdorff approximation of $B(p, 2r)$ to $B^{q}(2r)$ (see [6, Lemma 9.16]), we have

\begin{equation}
\forall \vec{v} \in \Phi(B(p, 4r)), \quad \text{diam}(\Phi^{-1}(\vec{v}), g) \leq 2\varepsilon r.
\end{equation}

Moreover, let us denote the Jacobian matrix of $\Phi$ as

\begin{equation}
\forall x \in B(p, r), \quad J_k(\nabla \Phi)(x) := \left[\langle \nabla \Phi^a(x), \nabla \Phi^b(x) \rangle \right]_{a \leq k},
\end{equation}

with $\lambda(x)$ and $\Lambda(x)$ denoting its least and largest eigenvalues, respectively.

For any smooth function $u$ on $B(p, 2r)$, we now present the following a priori estimate of the tangential derivative in terms of $\lambda$, $\Lambda$, $|\text{Hess}_u|$ and $|\text{Hess}_u|$:  

**Lemma 2.1 (A priori estimate).** Fix $r \in (0, 1)$. Let $\vec{v} \in \Phi(B(p, 2r)) \subset \mathbb{R}^k$ be a regular value of $\Phi$, and we employ the following notations:

\begin{equation}
\lambda := \inf_{\Phi^{-1}(\vec{v})} \lambda(x), \quad \Lambda := \sup_{\Phi^{-1}(\vec{v})} \Lambda(x), \quad \text{and} \quad C_0 := \max_{1 \leq b \leq k} \sup_{\Phi^{-1}(\vec{v})} |\text{Hess}_u|^2,
\end{equation}

\begin{equation}
|J_k(\nabla \Phi)|^2 \leq C_0^2 |\text{Hess}_u|.
\end{equation}
and

\[(2.4) \quad K := \sup_{B(\Phi^{-1}(\bar{r}), 2r)} \left( r|\nabla u| + r^2 |\text{Hess}_u| \right).\]

Let us also recall that \(\nabla^T u\) denotes the part of \(\nabla u\) tangential to \(\Phi^{-1}(\bar{r})\), then

\[(2.5) \quad \sup_{\Phi^{-1}(\bar{r})} r \left| \nabla^T u \right| \leq 2 \left( 1 + \lambda^{-1} \sqrt{kC_0} \right)^{1/2} K \sqrt{e}.\]

The basic idea in proving this lemma resembles that of the Key Lemma in [18 (4.3)]: with the Hessian bound of the given function, its gradient cannot alter too much within a small fiber; therefore the large gradient at one point will create a marked difference from nearby points, in terms of the value of the function; but this difference is in turn controlled by the uniform gradient bound and the size of the fiber. However, in [18 (4.3)], such difference is captured along a geodesic which is almost tangential to the fiber (by [17 Lemma 4-8]), whereas in our case, there is no such geometric structure available, and instead, we follow the flow lines of the tangential gradient fields to compute the difference.

**Proof.** For the given \(x \in \Phi^{-1}(\bar{r})\), let us denote \(\sup_{\Phi^{-1}(\bar{r})} \left| \nabla^T u \right| = \delta_0\). We may assume \(\delta_0 > 0\) since otherwise \((2.5)\) is trivial.

Since \(\bar{r}\) is a regular value of \(\Phi\) and \(\Phi\) is continuous, we know that \(\Phi^{-1}(\bar{r})\) is a closed embedded submanifold of \(B(p, 3r)\) (see [30 Theorem 1.38]), and by \((2.1)\) we know that \(\Phi^{-1}(\bar{r})\) is compact, implying that \(\lambda > 0\). Moreover, there exists some \(x \in \Phi^{-1}(\bar{r})\) such that \(\delta_0 = |\nabla^T u| (x)\), and we could consider the integral curve \(\gamma_x\) of \(\nabla^T u\) with initial value \(x \in \Phi^{-1}(\bar{r})\). The curve \(\gamma_x(t)\) is defined at least up to some small \(t > 0\), and for any such \(t\) we have

\[\frac{d}{dt} \left| \nabla^T u \right|^2 (\gamma_x(t)) = \nabla_\gamma u \left| \nabla^T u \right|^2 = 2 \langle \nabla_\gamma u \nabla^T u, \nabla^T u \rangle.\]

We also notice that for any smooth vector fields \(X\) and \(Y\) tangential to \(\Phi^{-1}(\bar{r})\),

\[\nabla_X \langle \nabla^T u, Y \rangle = \nabla_X \langle \nabla u, Y \rangle = \text{Hess}_u(X, Y) + \langle \nabla u, (\nabla_X Y)^T \rangle + \langle \nabla u, (\nabla_X Y)^\perp \rangle = \text{Hess}_u(X, Y) + \langle \nabla^T u, \nabla_X Y \rangle - \sum_{a,b=1}^k J_k(\nabla \Phi)^{-1}_{ab} \langle \nabla u, \nabla \Phi^a \rangle \langle \nabla \Phi^b, \nabla_X Y \rangle,
\]

where \((\nabla_X Y)^T\) and \((\nabla_X Y)^\perp\) are respectively the parts of \(\nabla_X Y\) tangential and perpendicular to \(\Phi^{-1}(\bar{r})\), and \(J_k(\nabla \Phi)^{-1}_{ab}\) is \((a, b)\)-entry of the inverse of the Jacobian matrix of \(\Phi\).

Notice that \(\langle \nabla \Phi^b, \nabla_X Y \rangle\) is the \(|\nabla \Phi^b|\) multiple of the second fundamental form of \(\Phi^{-1}(\bar{r})\) in the direction of \(\nabla \Phi^b\), and we have

\[\langle \nabla \Phi^b, \nabla_X Y \rangle = -\text{Hess}_\Phi(X, Y)\]

Therefore we have

\[\langle \nabla_X (\nabla^T u), Y \rangle = \nabla_X \langle \nabla^T u, Y \rangle - \langle \nabla^T u, \nabla_X Y \rangle = \text{Hess}_u(X, Y) + \sum_{a,b=1}^k J_k(\nabla \Phi)^{-1}_{ab} \langle \nabla u, \nabla \Phi^a \rangle \text{Hess}_\Phi(X, Y),\]

\[(2.6)\]
and by (2.3) we could estimate
\[ \left| \sum_{a,b=1}^{k} J_k(\nabla \Phi)^{-1}_{ab}(\nabla u, \nabla \Phi^0) Hess_{g^0}(X, Y) \right| \leq \lambda^{-1} \sqrt{\lambda k C_0 r^{-1}} |\nabla u||X||Y|. \]

Further considering (2.4), we obtain from (2.6) and the last inequality that
\[ (2.7) \quad \sup_{\Phi^{-1}(\tilde{y})} |\nabla^T (\nabla^T u)| \leq (1 + \lambda^{-1} \sqrt{\lambda k C_0}) K r^{-2}. \]

Especially, for \( X = Y = \nabla^T u \), since \( \gamma_x \) is the integral curve of \( \nabla^T u \), we have
\[ (2.8) \quad \left| \frac{d}{dt} |\nabla^T u|^2 (\gamma_x(t)) \right| \leq 2(1 + \lambda^{-1} \sqrt{\lambda k C_0}) K r^{-2} |\nabla^T u|^2 (\gamma_x(t)) \]
whenever the curve \( \gamma_x(t) \) is defined up to \( t > 0 \).

Now integrating along \( \gamma_x \) up to the time \( t > 0 \) when \( \gamma_x(t) \) remains being defined, we have
\[ (2.9) \quad \langle \nabla^T u, \dot{\gamma}_x(t) \rangle = |\nabla^T u|^2 (\gamma_x(t)) \]
\[ \geq e^{-2(1 + \lambda^{-1} \sqrt{\lambda k C_0}) K r^{-2} t} \delta_0^2. \]

This inequality tells that \( |\nabla^T u|^2 (\gamma_x(t)) \) is always comparable to its initial value \( \delta_0^2 \), and it helps us extend the flow line \( \gamma_x(t) \), i.e., we have the following

Claim 2.2. \( \gamma_x(t) \) is defined for any \( t \geq 0 \).

Proof of claim. Clearly, \( \gamma_x(t) \) exists at least up to some small positive time. Now let \( T \) be the supremum of the existence time for \( \gamma_x(t) \) and assume, for the purpose of a contradiction argument, that \( T < \infty \). For any sequence \( t_i \nearrow T \), since \( \{\gamma_x(t_i)\} \subset \Phi^{-1}(\tilde{y}) \equiv B(x, 3\varepsilon r) \) and \( \Phi \) is continuous, there exists some \( y \in \Phi^{-1}(\tilde{y}) \) and a subsequence, still denoted by \( \{\gamma_x(t_i)\} \), such that
\[ \lim_{i \to \infty} d_M(\gamma_x(t_i), y) = 0. \]

We need to show that the above convergence is also with respect to the intrinsic metric \( d_{\Phi^{-1}(\tilde{y})} \) — this is because (2.7) only provides a derivative control of \( \nabla^T u \) in the directions tangential to \( \Phi^{-1}(\tilde{y}) \), and in order to apply this to guarantee that (2.9) persists in taking limit, we need to show that the convergence \( \gamma_x(t_i) \rightarrow y \) actually takes place within the Riemannian manifold \( (\Phi^{-1}(\tilde{y}), g|_{\Phi}) \), where \( g|_{\Phi} \) denotes the metric tensor \( g \) restricted to \( T \Phi^{-1}(\tilde{y}) \).

To this end, we notice that since \( \tilde{y} \) is a regular value of \( \Phi \), there is a positive radius \( r_0 \leq \varepsilon r \), such that the following conditions are satisfied for the geodesic ball \( B(y, r_0) \):

(a) there is a co-ordinate chart of \( B(y, r_0) \subset B(p, r) \) in which \( B(y, r_0) \cap \Phi^{-1}(\tilde{y}) \) is a single slice, see [10] Theorem 1.38;
(b) \( B(y, r_0) \) is contained in a normal neighborhood of \( y \) — especially, within \( B(y, r_0) \) the distance \( d_M(y, \cdot) \) is realized by the length of \( g \)-geodesic segments emanating from \( y \) and entirely contained in \( B(y, r_0) \), see [16] Proposition 3.6].

Now regarding \( y \) as a point on the Riemannian manifold \( (\Phi^{-1}(\tilde{y}), g|_{\Phi}) \), and let \( r_1 \leq r_0 \) be a radius such that \( B_{d_{\Phi^{-1}(\tilde{y})}}(y, 2r_1) \) is contained in a normal neighborhood of \( y \) for \( g|_{\Phi} \). Now there is a radius \( r_2 \in (0, r_1) \) so that within \( B(y, r_2) \cap \Phi^{-1}(\tilde{y}) \), the intrinsic distance \( d_{\Phi^{-1}(\tilde{y})}(y, \cdot) \) is realized by the length of \( g|_{\Phi} \)-geodesic segments emanating from \( y \) and staying within \( B_{d_{\Phi^{-1}(\tilde{y})}}(y, r_1) \subset \Phi^{-1}(\tilde{y}) \).
entirely. Since under the local co-ordinate chart in (a) any such $g_{\Phi^{-1}(\gamma)}$-geodesic segment stays within the slice, we can obviously see that the intrinsic distance $d_{\Phi^{-1}(\gamma)}(y, \cdot)$ and the extrinsic distance $d_M(y, \cdot)$ are comparable — up to a factor controlled by $\|\Phi\|_{C^1(B(y, 4\epsilon r))}$. Therefore, since each $\gamma(t_i) \in \Phi^{-1}(\bar{v})$, the convergence $\lim_{i \to \infty} d_M(y, \gamma_s(t_i)) = 0$ is equivalent to the convergence $\lim_{i \to \infty} d_{\Phi^{-1}(\gamma)}(\gamma_s(t_i), y) = 0$.

Now we see that (2.7) and (2.9) guarantee that $\nabla T u(\gamma(t_i)) \to \nabla T u(y)$ as $i \to \infty$ in $T\Phi^{-1}(\bar{v})$, and consequently $|\nabla^T u|^2(y) \geq e^{-2(1+\lambda^{-1}\sqrt{\lambda k C_0}) Kr^{-2} t} \delta_0^2 > 0$; on the other hand, since $B(y, r_2) \cap \Phi^{-1}(\bar{v})$ is a slice in $B(y, r_1)$, it is easy to see that we can smoothly extend $\gamma_s$ beyond $T$ with the initial value $y$, in the direction of $\nabla T u(y)$ — breaking the supposed supremum of the existence time. \hfill \Box

Continuing our discussion, we could now integrate the inequality above for all $t > 0$:

$$u(\gamma_s(t)) - u(x) = \int_0^t \langle \nabla u, \dot{\gamma}_s(s) \rangle \, ds$$  \hfill (2.10)

$$= \int_0^t \langle \nabla^T u, \dot{\gamma}_s(s) \rangle \, ds$$

$$\geq \frac{1 - e^{-2(1+\lambda^{-1}\sqrt{\lambda k C_0}) Kr^{-2} t}}{2(1 + \lambda^{-1}\sqrt{\lambda k C_0}) Kr^{-2}} \delta_0^2.$$  

On the other hand, since $\Phi^{-1}(\bar{v}) \subset B(x, 4\epsilon r)$, by (2.1) and (2.4) we see that

$$\forall y, y' \in \Phi^{-1}(\bar{v}), \quad |u(y) - u(y')| \leq K r^{-1} d_g(y, y') \leq 2K \epsilon.$$  \hfill (2.11)

Since $\gamma_s$ is the integral curve of a vector field tangent to $\Phi^{-1}(\bar{v})$, $\gamma_s(t) \in \Phi^{-1}(\bar{v})$ for all $t \geq 0$, combining (2.10) and (2.11) we get the following upper bound for $t$:

$$t \leq \frac{\ln \delta_0^2 - \ln \left(\delta_0^2 - 4(1 + \lambda^{-1}\sqrt{\lambda k C_0}) K^2 \epsilon r^{-2}\right)}{2(1 + \lambda^{-1}\sqrt{\lambda k C_0}) Kr^{-2}}.$$  \hfill (2.12)

However, since we know (2.10) is valid for all $t \geq 0$, the right-hand side of (2.12) must be $\infty$, that is to say, we have

$$\delta_0 \leq 2 \left(1 + \lambda^{-1}\sqrt{\lambda k C_0}\right)^{\frac{1}{2}} K \sqrt{\epsilon} r^{-1}.$$  

This is the desired derivative bound. \hfill \Box

**Remark 2.** In the proof of the *a priori* estimate above, we notice that as long as $B(p, 2r) \subset M$ as smooth manifolds, and $\Phi : B(p, 2r) \to \mathbb{R}^k$ is a smooth map whose regular fibers are bounded in $B(p, 2r)$, then the tangential flow lines of a $C^2$ function $u$ is always defined for any $t > 0$ within a regular $\Phi$-fiber — this is guaranteed, via Claim 2.2 by the local bounds of $J_{\Phi}(\nabla \Phi)$, $Hess_u$ and $Hess_{\Phi}$ around the fiber, and such bounds are in turn guaranteed by the smoothness of $M$ and $\Phi$, and the compactness and regularity of the fiber in consideration. The key point is that we do not need to assume any uniform bound on $J_{\Phi}(\nabla \Phi)$, $Hess_u$ or $Hess_{\Phi}$ to conclude the long-time existence of the dynamics driven by $\nabla T u$ on any regular $\Phi$-fiber, and therefore we are free to differentiate and integrate along the flows lines of $\nabla T u$. 

3. The $L^2$-Average Gradient Control of the Tangential Derivatives

In this section we prove our main estimate, Theorem 1.3 We will begin with extending the fiber-wise estimate in Lemma 2.1 across all regular fibers of $\Phi$. In fact, once we know the long-time existence of the flow lines of $\nabla^T u$ on the regular fibers, an integral version of the argument employed in Lemma 2.1 enables us to weaken the assumptions on $|\text{Hess}_u|$ and $|\text{Hess}_\varphi|$ to more natural integral bounds:

**Proposition 3.1** (Interior $L^2$ estimate for the tangential gradients). Fix $r \in (0, 1)$ and $C_0 \in (0, 1)$. Let $B(p, 4r) \subset M$ be a geodesic ball in a smooth $m$ dimensional Riemannian manifold $(M, g)$. With the given $\Phi$ defined on $B(p, 4r)$ as before, assuming besides (2.1), that $u \in C^\infty(B(p, 2r))$ satisfies

\[
\sup_{B(p, 2r)} (|u|^2 + r^2|\nabla u|^2) + r^4 \int_{B(p, 2r)} |\text{Hess}_u|^2 \leq K^2,
\]

and that $\Phi$ satisfies the estimates

\[
\sup_{B(p, 4r)} \max_{1 \leq a \leq k} |\nabla \Phi^a| \leq 1 + C_0 \quad \text{and} \quad r^2 \sum_{b=1}^k \int_{B(p, 4r)} |\text{Hess}_\varphi^b|^2 \leq C_0^2,
\]

then for $C_1 = C_1(C_0, k) = 8k^2(1 + C_0)^{k-1}$, we have

\[
r^2 \int_{\Phi^{-1}(R)} |\nabla^T u|^2 |J_k(\nabla \Phi)| \, dV_g \leq C_1 |B(p, r)| K^2 \left( \sqrt{K} + C_0 \right),
\]

where $|J_k(\nabla \Phi)| = \sqrt{\det J_k(\nabla \Phi)}$ is the Jacobian of the map $\Phi$ (see (2.2), and $R$ denotes the regular values of $\Phi$ in $\Phi(B(p, r - 4\varepsilon r))$).

**Proof.** First notice that since $R$ consists of regular values of $\Phi$ in $\Phi(B(p, r - 4\varepsilon r))$, we must have $\Phi^{-1}(R) \subset B(p, r)$ by (2.1). Moreover, each $\Phi$-fiber over $R$ is a compact, embedded sub-manifold of $B(p, r)$. By the observation just discussed in Remark 2.4 we know that the flow lines $\gamma_a(t)$ of $\nabla^T u$, with initial values $x \in \Phi^{-1}(R)$ exists for all $t \geq 0$. It is also clear that $\Phi^{-1}(R)$, as a union of regular $\Phi$-fibers, each of which being invariant under the diffeomorphism (of the fiber) generated by $\nabla^T u$, is itself invariant under the evolution driven by $\nabla^T u$.

By (2.6) we could compute for any $t > 0$ to see

\[
\frac{d}{dt} \int_{\Phi^{-1}(R)} \left( |\nabla^T u|^2 |J_k(\nabla \Phi)| \right)(\gamma_a(t)) \, dV_g(x)
\]

\[
= \int_{\Phi^{-1}(R)} \left( \frac{d}{dt} |\nabla^T u|^2 |J_k(\nabla \Phi)| + |\nabla^T u|^2 \frac{d}{dt} |J_k(\nabla \Phi)| \right)(\gamma_a(t)) \, dV_g(x)
\]

\[
= 2 \int_{\Phi^{-1}(R)} \text{Hess}_u(\gamma_a(t), \gamma_a(t)) |J_k(\nabla \Phi)|(\gamma_a(t)) \, dV_g(x)
\]

\[
- 2 \sum_{a,b=1}^k \int_{\Phi^{-1}(R)} \left( J_{k(\nabla \Phi)^{-1}_{ab}}(\nabla \Phi^a, \nabla u) \text{Hess}_{\varphi^b}(\nabla^T u, \nabla^T u) |J_k(\nabla \Phi)| \right)(\gamma_a(t)) \, dV_g(x)
\]

\[
+ \sum_{a,b=1}^k \int_{\Phi^{-1}(R)} \left( |\nabla^T u|^2 J_{k(\nabla \Phi)^{-1}_{ab}} \text{Hess}_{\varphi^b}(\nabla^T u, \nabla \Phi^b) |J_k(\nabla \Phi)| \right)(\gamma_a(t)) \, dV_g(x).
\]
In order to obtain a uniform estimate only depending on (3.1) and (3.2), we need to analyze the following (multi-)linear quantities

\[
\begin{align*}
F(\nabla u, \nabla^T u, \nabla^T u) &:= \sum_{a,b=1}^k J_k(\nabla \Phi)^{-1}_{ab}(\nabla \Phi^a, \nabla u) \text{Hess}_{\Phi^b}(\nabla^T u, \nabla^T u) \left| J_k(\nabla \Phi) \right| \\
G(\nabla^T u) &:= \sum_{a,b=1}^k J_k(\nabla \Phi)^{-1}_{ab} \text{Hess}_{\Phi^b}(\nabla^T u, \nabla \Phi^a) \left| J_k(\nabla \Phi) \right|
\end{align*}
\]

(3.5)

at each \( y = \gamma_x(t) \in \Phi^{-1}(\mathcal{R}) \). We will rely on the invariance of these sum under the orthogonal transformations.

To this end, fix some \( y = \gamma_x(t) \in \Phi^{-1}(\mathcal{R}) \), and let \( Q = [q_{ab}] \in O(k) \) be an orthogonal matrix that diagonalizes \( J_k(\nabla \Phi)(y) \); the specific value of \( Q \) depends on \( y \in \Phi^{-1}(\mathcal{R}) \) but \( Q \) is a constant matrix. We denote \( \Phi^a = \sum_{b=1}^k q_{ab} \Phi^b \), and \( \Phi = [\Phi^1, \ldots, \Phi^k] \) (the transpose of the row vector), then

\[
J_k(\nabla \Phi) = \begin{bmatrix} \Phi^1 & \cdots & \Phi^k \end{bmatrix} \cdot [\nabla \Phi^1, \ldots, \nabla \Phi^k] = \begin{bmatrix} \Phi^1 \\ \vdots \\ \Phi^k \end{bmatrix} \cdot [\nabla \Phi^1, \ldots, \nabla \Phi^k] Q' = Q J_k(\nabla \Phi) Q',
\]

where the dot “\( \cdot \)” denotes the inner product of vector fields with respect to the metric tensor \( g \). Notice that since \( Q \) is a constant matrix and each \( \Phi^b \) is a constant linear combination of \( \Phi^1, \ldots, \Phi^k \), the covariant derivatives land directly on \( \Phi^1, \ldots, \Phi^k \) — besides \( \nabla \Phi^a = \sum_{a=1}^k q_{ab} \nabla \Phi^b \), we also have \( \text{Hess}_{\Phi^a} = \sum_{a=1}^k q_{ab} \text{Hess}_{\Phi^b} \).

Now suppose \( J_k(\nabla \Phi)(y) \) has eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \) on the diagonal, then for each \( a = 1, \ldots, k \), we have \( \lambda_a = |\nabla \Phi^a|^2(y) \). With the notation \( c_a := \langle \nabla u, \nabla \Phi^a \rangle \), we see \( |c_a| \leq k r^{-1} \). Also notice that by the invariance of the determinant, we have \( \det J_k(\nabla \Phi) = \det J_k(\nabla \Phi) = \lambda_1 \lambda_2 \cdots \lambda_k \). Now by orthogonality of \( Q \) we can compute

\[
\begin{align*}
\sum_{a,b=1}^k J_k(\nabla \Phi)^{-1}_{ab}(\nabla \Phi^a, \nabla u) \text{Hess}_{\Phi^b}(\nabla^T u, \nabla^T u) \left| J_k(\nabla \Phi) \right| \\
&= \langle \nabla \Phi^1, \nabla u \rangle, \ldots, \langle \nabla \Phi^k, \nabla u \rangle \rangle Q' \begin{bmatrix} \lambda_1^{-1} & & \cdots \\ & \ddots & \\ & & \lambda_k^{-1} \end{bmatrix} Q \begin{bmatrix} \text{Hess}_{\Phi^1}(\nabla^T u, \nabla^T u) \\ \vdots \\ \text{Hess}_{\Phi^k}(\nabla^T u, \nabla^T u) \end{bmatrix} \sqrt{\lambda_1 \cdots \lambda_k} \\
&= \sum_{a=1}^k c_a \lambda_a^{-\frac{1}{2}} \text{Hess}_{\Phi^a}(\nabla^T u, \nabla^T u) \sqrt{\lambda_1 \cdots \lambda_k} \\
&= \sum_{a=1}^k c_a \text{Hess}_{\Phi^a}(\nabla^T u, \nabla^T u) \prod_{b \neq a} \lambda_b^{\frac{1}{2}}.
\end{align*}
\]

(3.6)

To estimate the last line above, we notice that all terms come with a positive power of \( \lambda_1, \ldots, \lambda_k \) — this is crucial for us, as we do not have any uniformly positive lower bound of \( \lambda_1 \) when \( y \in \Phi^{-1}(\mathcal{R}) \) varies. We will also rely on the assumption (3.2), the linearity of taking covariant derivatives, as
well as the fact that $Q \in O(k)$ to see that for each $a = 1, \ldots, k$,

$$\lambda_a^{\frac{1}{2}} = |\nabla \phi^a(y)| \leq \max_{1 \leq b \leq k} |\nabla \phi^b(y)| \leq 1 + C_0$$

and therefore by (3.8)

$$\Phi \leq (3.8)$$

across uniform estimate across $\Phi^{-1}(\mathcal{R})$:

(3.7) $$\left| F(\nabla u, \nabla^T u, \nabla^T u) \right| \leq k(1 + C_0)^{k-1} K^3 r^{-3} \sum_{b=1}^{k} |\text{Hess}_{\phi^b}|.$$

In the same setting as above, we have at $y \in \Phi^{-1}(\mathcal{R})$ the matrix $Q = [q_{ab}] \in O(k)$ to help compute

(3.8) $$\sum_{a,b=1}^{k} J(\nabla \Phi)^{-1}_{ab} \text{Hess}_{\phi^b}(\nabla^T u, \nabla \phi^b) |J_k(\nabla \Phi)|$$

$$= \sum_{a,b=1}^{k} \sum_{c=1}^{k} q_{ca} \lambda_c \lambda_{cb} \text{Hess}_{\phi^b}(\nabla^T u, \nabla \phi^b) \sqrt{\lambda_1 \cdots \lambda_k}$$

$$= \sum_{a=1}^{k} \lambda_a^{\frac{1}{2}} \text{Hess}_{\phi^a}(\nabla^T u, \nabla \phi) \prod_{b \neq a} \lambda_b^{\frac{1}{2}},$$

and therefore by $\lambda_a^{\frac{1}{2}} |\nabla \phi^a| = 1$ we could estimate as before to see

(3.9) $$\left| G(\nabla^T u) \right| \leq k(1 + C_0)^{k-1} K r^{-1} \sum_{b=1}^{k} |\text{Hess}_{\phi^b}|,$$

as an estimate holds uniformly across $\Phi^{-1}(\mathcal{R})$.

Recalling (3.4), we could now control the integral by integrating the above estimates (3.7) and (3.9) across $\Phi^{-1}(\mathcal{R})$ as following:

(3.10) $$\frac{d}{dt} \int_{\Phi^{-1}(\mathcal{R})} \left( |\nabla^T u|^2 |J_k(\nabla \Phi)| \right) (y_\mathcal{X}(t)) \ dV_g(x)$$

$$\geq -2(1 + C_0)^k K^2 r^{-2} \int_{\Phi^{-1}(\mathcal{R})} |\text{Hess}_{\phi^a}| (y_\mathcal{X}(t)) \ dV_g(x)$$

$$- 3k(1 + C_0)^{k-1} K^3 r^{-3} \int_{\Phi^{-1}(\mathcal{R})} \sum_{b=1}^{k} |\text{Hess}_{\phi^b}| (y_\mathcal{X}(t)) \ dV_g(x).$$

To bound the last two integrals, we notice that $\Phi^{-1}(\mathcal{R})$ is invariant under the flow of $\nabla^T u$, meaning that these integrals are the same as the ones in (3.11) and (3.2), restricted on the subset $\Phi^{-1}(\mathcal{R})$. Therefore we arrive at the following lower bound:

(3.11) $$\frac{d}{dt} \int_{\Phi^{-1}(\mathcal{R})} \left( |\nabla^T u|^2 |J_k(\nabla \Phi)| \right) (y_\mathcal{X}(t)) \ dV_g(x) \geq -4k^2(1 + C_0)^k |B(p, r)| K^3 r^{-4}. $$
Integrating the above inequality in \( t \), we see for any \( t > 0 \) fixed,

\[
\int_{\Phi^{-1}(\mathcal{R})} \left( \left| \nabla^T u \right|^2 |J_k(\nabla \Phi)| \right) (\gamma_x(t)) \, dV_g(x)
\]

\[
\geq \int_{\Phi^{-1}(\mathcal{R})} \left| \nabla^T u \right|^2 (x)|J_k(\nabla \Phi)|(x) \, dV_g(x) - 4k^2(1 + C_0)^k|B(p, r)|K^3 r^{-4} t.
\]

Now we compute the variation of \( \int u|J_k(\nabla \Phi)| \) driven by \( \nabla^T u \):

\[
\frac{d}{dt} \int_{\Phi^{-1}(\mathcal{R})} (u|J_k(\nabla \Phi)|) (\gamma_x(t)) \, dV_g(x)
\]

\[
= \int_{\Phi^{-1}(\mathcal{R})} \left( \left| \nabla^T u \right|^2 |J_k(\nabla \Phi)| + uG(\nabla^T u) \right) (\gamma_x(t)) \, dV_g(x)
\]

and by (3.1), (3.2), (3.9) and (3.12) we can estimate

\[
\frac{d}{dt} \int_{\Phi^{-1}(\mathcal{R})} (u|J_k(\nabla \Phi)|) (\gamma_x(t)) \, dV_g(x)
\]

\[
\geq \int_{\Phi^{-1}(\mathcal{R})} \left| \nabla^T u \right|^2 (x)|J_k(\nabla \Phi)|(x) \, dV_g(x) - 4k^2(1 + C_0)^k|B(p, r)|K^3 r^{-4} t
\]

\[
- k(1 + C_0)^{k-1} K r^{-1} \sum_{b=1}^k \left| \text{Hess}_{\Phi^0} \right| (\gamma_x(t)) \, dV_g(x)
\]

\[
\geq \int_{\Phi^{-1}(\mathcal{R})} \left| \nabla^T u \right|^2 (x)|J_k(\nabla \Phi)|(x) \, dV_g(x) - 4k^2(1 + C_0)^k|B(p, r)|K^3 r^{-4} t
\]

\[
- k^2(1 + C_0)^{k-1} |B(p, r)|K^2 C_0 r^{-2}.
\]

Integrating the last inequality in \( t \) once again we see for any \( t > 0 \),

\[
\int_{\Phi^{-1}(\mathcal{R})} (u|J_k(\nabla \Phi)|) (\gamma_x(t)) \, dV_g(x) - \int_{\Phi^{-1}(\mathcal{R})} u(x)|J_k(\nabla \Phi)|(x) \, dV_g(x)
\]

\[
\geq t \int_{\Phi^{-1}(\mathcal{R})} \left| \nabla^T u \right|^2 (x)|J_k(\nabla \Phi)|(x) \, dV_g(x) - 2k^2(1 + C_0)^k|B(p, r)|K^3 r^{-4} t^2
\]

\[
- k^2(1 + C_0)^{k-1} |B(p, r)|K^2 C_0 r^{-2}.
\]

On the other hand, by the uniform gradient controls (5.1) and (5.2), we have \( \left| \nabla^T u \right| \leq Kr^{-1} \) and \( |J_k(\nabla \Phi)| \leq (1 + C_0)^k \) on \( \Phi^{-1}(\mathcal{R}) \), and by the small fiber assumption (2.1), we know that \( d_g(\gamma_x(t), \gamma_x(0)) \leq 2er \) for any \( t > 0 \) and initial value \( x \in \Phi^{-1}(\mathcal{R}) \). Therefore

\[
\forall x \in \Phi^{-1}(\mathcal{R}), \forall t > 0, \quad |u(\gamma_x(t)) - u(x)||J(\nabla \Phi)|(\gamma_x(t)) \leq 2(1 + C_0)^k K e.
\]

Moreover, since for any smooth curve \( \gamma \), we have

\[
\left| \nabla \gamma J_k(\nabla \Phi) \right| = |G(\gamma)| \leq k(1 + C_0)^{k-1} |\gamma| \sum_{b=1}^k \left| \text{Hess}_{\Phi^0} \right|,
\]

which is obtained following the same path leading to (3.9). Now fixing some \( t > 0 \), we see that almost every pair of points \( (x, \gamma_x(t)) \) with \( x \in \Phi^{-1}(\mathcal{R}) \) could be connected by a unique minimal
geodesic $\sigma_{x,\gamma_s(t)}$ of speed $d(x, \gamma_s(t))$, whose image is entirely contained in $B(p, r)$. Therefore, integrating the inequality

$$\|J_k(\nabla \Phi)(\gamma_s(t)) - J_k(\nabla \Phi)(x)\| \leq k(1 + C_0)^{k-1} d(x, \gamma_s(t)) \sum_{b=1}^{k} \int_{0}^{1} \|Hess\Phi\|_s(\sigma_{x,\gamma_s(t)}(s)) \, ds$$

in $x \in \Phi^{-1}(R)$, we see that

$$\int_{\Phi^{-1}(R)} |J_k(\nabla \Phi)(\gamma_s(t)) - J_k(\nabla \Phi)(x)| \, dV_g(x) \leq 2e k(1 + C_0)^{k-1} \sum_{b=1}^{k} \int_{B(p, r)} |Hess\Phi|$$

$$\leq 2e k(1 + C_0)^{k} |B(p, r)| K e.$$ 

Combining the above estimates we see for and fixed $t > 0$ that

$$(3.16) \quad \left| \int_{\Phi^{-1}(R)} (u|J_k(\nabla \Phi)|) (\gamma_s(t)) \, dV_g(x) - \int_{\Phi^{-1}(R)} u(x)|J_k(\nabla \Phi)|(x) \, dV_g(x) \right| \leq 4k(1 + C_0)^{k} |B(p, r)| K e.$$

Now recalling the previous lower bound \(3.15\), we obtain the following estimate for any $t \geq 0$:

$$\int_{\Phi^{-1}(R)} |\nabla u|^2 \|J_k(\nabla \Phi)\| \, dV_g \leq \left(4r^{-1} e + 2tK^2 r^{-4}\right) k^2 (1 + C_0)^{k} |B(p, r)| K + k^2 (1 + C_0)^{k-1} |B(p, r)| K^2 C_0 r^{-2}.$$

Now choosing $t = \sqrt{e} K^{-1} r^2$, we immediately have

$$r^2 \int_{\Phi^{-1}(R)} |\nabla u|^2 \|J_k(\nabla \Phi)\| \, dV_g \leq 16k^2 (1 + C_0)^{k-1} |B(p, r)| K^2 \left( \sqrt{e} + C_0 \right),$$

whence the desired estimate \(3.3\). \(\square\)

**Remark 3.** It is is necessary to integrate against $|J_k(\nabla \Phi)|$ in \(3.3\), and this is in correspondence with the measured Gromov-Hausdorff convergence.

**Remark 4.** The lack of a uniform positive lower bound of $\lambda$ is also a crucial problem even in the non-collapsing setting, see especially the Transformation Theorem of Cheeger and Naber \[9, Theorem 1.32\], which is the major technical input of \[9\] towards their solution of the Codimension Four Conjecture. The invariance of the canonical quantities related to $J_k(\nabla \Phi)$, under the action of the orthogonal group, has also been explored in \[24\] to understand the asymptotic behavior of $|J_k(\nabla \Phi)|$ on complete non-compact manifolds with non-negative Ricci curvature.

In order to apply our previous estimate in Proposition \(3.1\) we will extend the domain of integral across the singular fibers of $\Phi$, while still end up with the same estimates. As discussed in the introduction, the situation we consider is a (volume) collapsing sequence of $m$ dimensional Riemannian manifolds with a uniform Ricci curvature lower bound.

Besides the desired estimates in Theorem \(1.2\) that the $\Phi(\epsilon)$-splitting map $\Phi$ satisfies, one key property is that $\Phi$ is actually a *harmonic map*. If the domain of $\Phi$ is contained in an Einstein manifold, then around any point we could write down the equations $\Delta_g \Phi^a = 0$ ($a = 1, \ldots, k$)
under the harmonic coordinates. Under such coordinates, the components of $g$ are real analytic and determine the coefficients of $\Delta_g$, implying that each $\Phi^a$ ($a = 1, \ldots, k$) is an analytic function. Therefore $\Phi$ is an analytic map once we assume its domain is in an Einstein manifold. To extend the estimate in Proposition 3.1 across the singular fibers of $\Phi$, let us recall the following fact for analytic maps (see [22], §3.1, Exercise 4(a)):

**Lemma 3.2 (Nullity of singular fibers).** Suppose $\Phi : M \to N$ is an analytic map, whose domain is connected and has dimension no less than that of the codomain. Let $\Sigma \subset M$ denote the set of singular points of $\Phi$, then $\Phi^{-1}(\Phi(\Sigma))$ has measure zero in $M$.

Applying a standard technique due to Cheeger and Colding we could further relax the assumption of $\|\text{Hess}_u\|_{L^2}$ to a local integral bound of $\Delta u$ in the following

**Theorem 3.3.** With the same setting as in Theorem 1.2 we further assume that $g$ is real analytic. Let $u$ be a smooth function on $B(p, 2r)$ with

\begin{equation}
(3.17) \quad \sup_{B(s, 2r)} |u| + r|\nabla u| \leq K,
\end{equation}

then there is some positive constant $C_2(m, k)$, independent of $\varepsilon \in (0, \varepsilon(m))$ and $r \in (0, 1)$, such that

\begin{equation}
(3.18) \quad r^2 \int_{B(p, r)} |\nabla^T u|^2 dV_g \leq C_2(m, k) \left(\varepsilon^2 + \Psi(\varepsilon, l^{-1}|m|) \left(K^2 + \|\Delta u\|_{L^2(B(p, 2r))}^2\right)^2\right),
\end{equation}

where the $L^2$ norm denotes the $L^2$-average, and $\Psi(\varepsilon, l^{-1}|m|)$ is the same as the one obtained in Theorem 1.2

**Proof.** Let $\mathcal{R} \subset \Phi(B(p, r))$ be the regular values of $\Phi$, then by Lemma 3.2 we clearly have

\begin{equation}
(3.19) \quad |\Phi^{-1}(\mathcal{R})| = |B(p, r)|.
\end{equation}

Moreover, the assumption (3.2) of $\Phi$ in Proposition 3.1 is satisfied with $C_0 = \Psi = \Psi(\varepsilon, l^{-1}|m|)$.

We only need to further control $\|\text{Hess}_u\|_{L^2(B(p, r))}$ following Cheeger and Colding’s well-known argument based on their construction of a controlled cut-off function $\varphi$ supported on $B(p, 2r)$, such that $\varphi \equiv 1$ within $B(p, r + 4\varepsilon r)$ and $r|\nabla \varphi| + r^2|\Delta \varphi| \leq 2C_{\text{ctf}}(m)$. Testing the Weitzenböck formula applied to $u$ against $\varphi$ on $B(p, 2r)$, and applying integration by parts and Hölder’s inequality we see

\begin{align*}
\int_{B(p, 2r)} \varphi |\text{Hess}_u|^2 &\leq \frac{1}{2} \int_{B(p, 2r)} (|\Delta \varphi| + 2\varphi (m - 1))|\nabla u|^2 + \int_{B(p, 2r)} (\Delta u)^2 \varphi + |\Delta u||\langle \nabla \varphi, \nabla u \rangle| \\
&\leq \left(8C_{\text{ctf}}(m)r^{-2} + (m - 1)(\varepsilon r)^2\right)K^2r^{-2} + \frac{3}{2}||\Delta u||_{L^2(B(p, 2r))}^2.
\end{align*}

Therefore, by assuming $C_{\text{ctf}}(m) > 1$ without loss of any generality, we get

\begin{equation}
(3.20) \quad \int_{B(p, r)} |\text{Hess}_u| \leq 4mC_{\text{ctf}}(m)Kr^{-2} + 2||\Delta u||_{L^2(B(p, 2r))}.
\end{equation}

Applying this estimate to (3.10) and integrating in $t$ we immediately see

\begin{align*}
\int_{\Phi^{-1}(\mathcal{R})} \left(|\nabla^T u|^2 |\gamma_k(t)| \right) dV_g(x) - \int_{\Phi^{-1}(\mathcal{R})} |\nabla^T u|^2 (x) \left|J_k(\nabla \Phi)(x) \right| dV_g(x) \\
\geq -16tm^2C_{\text{ctf}}(m)(1 + \Psi)^4|B(p, 2r)|K^3r^{-4} - 4(t + 1 + \Psi)^4\|\Delta u\|_{L^2(B(p, 2r))}|B(p, 2r)|K^2r^{-2},
\end{align*}
and applying this inequality to (3.13) and integrating in $t$ we have
\[
\int_{\Phi^{-1}(\mathbb{R})} |\nabla^2 u|^2 |J_k(\nabla \Phi)| \ dV_g \leq 16m^2C_{ef}(m)(1 + \Psi)^k|B(p, 2r)|K|e + \Psi K^2 r^{-2}
\]
Now setting $t = \sqrt{\varepsilon K^{-1}r^2}$ we have
\[
\int_{\Phi^{-1}(\mathbb{R})} |\nabla^2 u|^2 |J_k(\nabla \Phi)| \ dV_g \leq 16m^2C_{ef}(m)\sqrt{\varepsilon}(1 + \Psi)^k|B(p, 2r)|K^{2}r^{-2}
\]
and dividing by $|B(p, r)|$ on both sides we get, by (3.19) and the uniform boundedness of $|\nabla u|$ and $|J_k(\nabla \Phi)|$, that
\[
r^2 \int_{B(p,r)} |\nabla^2 u|^2 |J_k(\nabla \Phi)| \ dV_g \leq C_2(m,k)\left(e^{\frac{1}{2}} + \Psi(\varepsilon, r^{-1}|m|)\right) K^2 + ||\Delta u||_{L^2(B(p,2r))}K^2r^2,
\]
where
\[
C_2(m,k) := \left(48m^2C_{ef}(m)\right)^k \sup_{\varepsilon \in (0,\varepsilon(m))} \frac{\Lambda_{m,-\varepsilon^2}(2)}{\Lambda_{m,-\varepsilon^2}(1)}.
\]
is clearly independent of $\varepsilon \in (0, \varepsilon(m))$ and $r \in (0,1)$. \hfill \square

With Theorem 3.3 we could now prove our main theorem by invoking the Cheng-Yau gradient estimate in [10].

Proof of Theorem 3.3 Since $(M,g)$ is Ricci flat, the metric $g$ is analytic. Then the conditions of Theorem 3.3 are fulfilled. Now if $u \in C^\infty(B(p, 2r))$ further satisfies the eigenfunction equation $\Delta u = \theta u$, then by [10] Theorem 6, we could estimate, for some positive constant $C_{CY}(m, \theta)$, that
\[
\sup_{B(p,r)} r|\nabla u| \leq C_{CY}(m, \theta) \sup_{B(p,2r)} |u|.
\]
Plugging this estimate in (3.17) we get $K = (1 + C_{CY}(m, \theta)||u||_{L^\infty(B(p,2r))}$, whence the desired estimate (1.3) with $C(m,k,\theta) = C_2(m,k)(1 + C_{CY}(m, \theta))$. \hfill \square
Remark 5. In fact, for a general function \( u \in C^\infty(B(p, 2r)) \), the quantity \( \|u\|_{L^\infty(B(p,2r))} \) on the right-hand side of the estimate (1.3) could be replaced by certain \( L^q \)-average like
\[
\|u\|_{\bar{L}^q(B(p,2r))} + \|\Delta u\|_{\bar{L}^q(B(p,2r))}
\]
for some \( q > 2m \). Once we recall, through the work of Anderson [4] and more generally the work by Saloff-Coste [27], that the Sobolev inequality on a Riemannian manifold with Ricci curvature lower bound involves the correct power of the volume \( |B(p,2r)| \), this bound could be obtained by following the routine of De Giorgi iteration, starting from the Weitzenböck formula applied to \( u \).

Here we will save the extra lines of details, as our main concern is about the “nice” functions like the eigenfunctions of the Laplace operator.

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