A Wong-Zakai theorem for $\Phi^4_3$ model *

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Abstract

We prove a version of the Wong-Zakai theorem for the dynamical $\Phi^4_3$ model driven by space-time white noise on $T^3$. For the $\Phi^4_3$ model it is proved in [Hai14] that a renormalisation has to be performed in order to define the nonlinear term. Compared to the results in [Hai14] we consider piecewise linear approximation to space-time white noise and the renormalisation corresponds to adding the solution multiplied by a function depending on $t$ to the equation.

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1 Introduction

In this paper we consider a Wong-Zakai approximation (c.f. [WZ65a, WZ65b]) in the case of the dynamical $\Phi^4_3$ model driven by space-time white noise on $T^3$:

$$d\Phi = \Delta \Phi dt - \Phi^3 dt + dW(t). \quad (1.1)$$

Here $W$ is a cylindrical Wiener process. This model is an important model in the stochastic quantisation of Euclidean quantum field theory (see [GJ87] and the reference therein). It is considered as a universal model for phase coexistence near the critical point [GLP99]. In two spatial dimensions, this problem was previously treated in [AR91] and [DD03]. In three spatial dimensions this equation (1.1) is ill-posed and the main difficulty in this case is that $W$ and hence $\Phi$ are so singular that the non-linear term is not well-defined in the classical sense. It was a long-standing open problem to give a meaning to this equation in three dimensional case.

A breakthrough result was achieved recently by Martin Hairer in [Hai14], where he introduced a theory of regularity structures and gave a meaning to this equation (1.1) successfully. Also by using the paracontrolled distribution proposed by Gubinelli, Imkeller and Perkowski in [GIP13] existence and uniqueness of local solutions to (1.1) has been obtained in [CC13]. Recently, these two approaches have been successful in giving a meaning to a lot of ill-posed stochastic PDEs like the Kardar-Parisi-Zhang (KPZ) equation ([KPZ86], [BG97], [Hai13]), the

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dynamical $\Phi^4_3$ model ([Hai14], [CC13]), the Navier-Stokes equation driven by space-time white noise ([ZZ14a], [ZZ14b]) and so on (see [HP14] for more interesting examples). From a philosophical perspective, the theory of regularity structures and the paracontrolled distribution method is a global approach using Fourier analysis.

An interesting question to the SDE especially the dynamical $\Phi^4_3$ model is as follows: Given a sequence $W_\varepsilon$ of regularization of the noise $W$ (for example convolution with a mollifier), can we obtain a non-trivial solution associated with $W$ by taking the limit of $\Phi_\varepsilon$ as $\varepsilon$ goes to 0, where $\Phi_\varepsilon$ is the solution associated to $W_\varepsilon$. In finite dimensional case a series of classical results has been obtained by Wong and Zakai [WZ65a, WZ65b]. However, the answer to this question for the dynamical $\Phi^4_3$ model is no (see [HRW12]). Indeed, we have to consider the following modified equation

$$\partial_t \Phi_\varepsilon = \Delta \Phi_\varepsilon + C_\varepsilon \Phi_\varepsilon - \Phi^3_\varepsilon + \varepsilon \xi,$$  \hspace{1cm} (1.2)

In [Hai14] Martin Hairer considered an $\varepsilon$-approximation $\xi_\varepsilon$ to space-time white noise. Here $\xi_\varepsilon$ is given by convolution with a mollifier: $\int_0^\infty \langle \rho_\varepsilon(t-s, x-\cdot), dW(s) \rangle$, where $x \in \mathbb{T}^3$, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{T}^3)$ and $W$ is the cylindrical Wiener process and $\rho_\varepsilon(t, x)$ is a compactly supported smooth mollifier that is scaled by $\varepsilon$ in the spatial directions and by $\varepsilon^2$ in the time direction, i.e. $\rho_\varepsilon(t, x) = \varepsilon^{-5} \rho(\varepsilon^{-2} t, \varepsilon^{-1} x)$ for some smooth, compactly supported function $\rho$. Denote by $\Phi_\varepsilon$ the solution to (1.2). It is proved in [Hai14] that there exist choices of constants $C_\varepsilon$ diverging as $\varepsilon \to 0$, as well as a process $\Phi$ such that $\Phi_\varepsilon \to \Phi$ in probability. Furthermore, while the constants $C_\varepsilon$ do depend crucially on the choice of mollifiers $\rho_\varepsilon$, the limits $\Phi$ do not depend on them. Also in [CC13] purely spatial regularization has been considered and a similar result has been obtained.

In this paper we consider another approximation given by piecewise linear approximation combined with convolution with a mollifier. First we also do convolution with a mollifier: $W_\varepsilon(t) = \int_0^t \xi_\varepsilon(s) ds$, $t \in \mathbb{R}$, for $\xi_\varepsilon$ given as above and consider piecewise linear approximation: for $t \in [k\vartheta, (k+1)\vartheta)$, $k \in \mathbb{Z}$, $\vartheta > 0$

$$W_{\varepsilon, \vartheta}(t) = W_\varepsilon(k\vartheta) + \frac{t - k\vartheta}{\vartheta} (W_\varepsilon((k+1)\vartheta) - W_\varepsilon(k\vartheta)),$$

and $\xi_{\varepsilon, \vartheta}(t) = \partial_t W_{\varepsilon, \vartheta} = \frac{1}{\vartheta} \int_{k\vartheta}^{(k+1)\vartheta} \xi_\varepsilon(u) du$ for $t \in [k\vartheta, (k+1)\vartheta)$, $k \in \mathbb{Z}$, which is our regularised noise.

This approximation is the celebrated Wong-Zakai approximation of the solution and is related to a classical problem: approximating solutions in terms of a simpler model, where the stochastic integral is changed into a deterministic one, replacing the noise by its piecewise linear interpolation on a time grid. For finite-dimensional diffusion processes, this kind of approximation is well-known (see, e.g. [T96], [LQZ02] and the references therein). There is a substantial number of publications devoted to Wong-Zakai approximations of infinite dimensional stochastic equations (see [N04] and [CM11] and the references therein).

In this paper we use the theory of regularity structures to study this approximation to the dynamical $\Phi^4_3$ model. The key idea of the theory of regularity structures is as follows: we perform an abstract Taylor expansion on both sides of the equation. Originally Taylor expansions are only for functions. Here the right objects, e.g. regularity structure that could possibly take
the place of Taylor polynomials, can be constructed. Given a noise $\xi$, the regularity structure can be endowed with a model $i\xi$, which is a concrete way of associating every element in the abstract regularity structure to the actual Taylor polynomial at every point. Multiplication, differentiation, the state space of solutions, and the convolution with singular kernels can be defined on this regularity structure, which is the major difficulty when trying to give a meaning to such singular stochastic partial differential equations as above. On the regularity structure, a fixed point argument can be applied to obtain local existence and uniqueness of the solutions $\bar{\Phi}$ to the equation lifted onto the regularity structure. Furthermore, we can go back to the real world with the help of another central tool of the theory, namely the reconstruction operator $R$. If $\xi$ is a smooth process, $\Phi = R\bar{\Phi}$ coincides with the classic solution of the equation. Now we have the following maps

$$
\xi \mapsto i\xi \mapsto \bar{\Phi} \mapsto R\bar{\Phi}.
$$

The last two maps are continuous with respect to suitable topologies, while the above sequence $i\xi_\varepsilon$ of canonical models fails to converge with $\xi_\varepsilon$ a smooth approximation to the noise $\xi$. It may, however, still be possible to renormalize the model $i\xi_\varepsilon$ into some converging model $\hat{i}\xi_\varepsilon$, which in turn can be related to a specific renormalised equation (1.2).

In this paper for the approximating sequence $\xi_\varepsilon,\vartheta$ we build the associated model $i\xi_\varepsilon,\vartheta$, which fails to converge. We also renormalize the model $i\xi_\varepsilon,\vartheta$ into some converging model $\hat{i}\xi_\varepsilon,\vartheta$, which in turn can be related to the following renormalised equation (1.3):

$$
\partial_t\Phi_{\varepsilon,\vartheta}(t) = \Delta\Phi_{\varepsilon,\vartheta}(t) + C^{(\varepsilon,\vartheta)}(t)\Phi_{\varepsilon,\vartheta}(t) - \Phi_{\varepsilon,\vartheta}^3(t) + \xi_{\varepsilon,\vartheta}(t). \tag{1.3}
$$

Here $C^{(\varepsilon,\vartheta)}$ are functions depending only on time $t$.

With these notations at hand, the main result of this article is as follows:

**Theorem 1.1** Let $\xi_\varepsilon,\vartheta$ defined as above. Denote by $\Phi_{\varepsilon,\vartheta}$ the solution to (1.3). Suppose that $\rho(t,x) = \rho_1(t)\rho_2(x)$ for smooth functions $\rho_1,\rho_2$ and that $\varepsilon^2 \leq C_0\vartheta$. Then there exist choices of functions $C^{(\varepsilon,\vartheta)}$ diverging as $\varepsilon,\vartheta \to 0$ such that $\Phi_{\varepsilon,\vartheta} \to \Phi$ in probability locally in time. Here $\Phi$ is the solution to the dynamical $\Phi^4_3$ model obtained in [Hai14].

**Remark 1.2**

(i) The aim of this paper is to study the effect of piecewise linear approximation to the noise. If $\varepsilon = 0$ it corresponds to piecewise linear approximation. However, in this case, we cannot obtain the classical solution directly because of the spatial singularity of the noise. Hence we do convolution with a mollifier first. Here the condition $\varepsilon^2 \leq C_0\vartheta$ is not strict, and it means that $\varepsilon$ could go to zero first. If we consider the case that $\vartheta \to 0$ for fixed $\varepsilon$, it corresponds to the smooth noise case and we could conclude the results easily.

(ii) We could also first do purely spatial regularization corresponding to $\rho(t,x) = \delta(t)\rho_2(x)$ and then do piecewise linear approximation. In this case the results in Theorem 1.1 still holds (see Remark 3.8). In fact, the only difference is the proof of Theorem 3.7.

In our case it is required in (1.3) to minus $\Phi_{\varepsilon,\vartheta}$ multiplied by a function $C^{(\varepsilon,\vartheta)}$ depending on $t$ such that the associated solutions $\Phi_{\varepsilon,\vartheta}$ converge to the solution to the $\Phi^4_3$ model as $\varepsilon,\vartheta \to 0$. We introduce the symbol $C$ to represent $C^{(\varepsilon,\vartheta)}(t)$ in the regularity structure and define a bigger regularity structure $\mathfrak{T}^1$ to include $C$ and the original regularity structure $\mathfrak{T}_F$ constructed in [Hai14] associated with the $\Phi^4_3$ model, which helps us to construct a suitable renormalised model corresponding to (1.3) for $\mathfrak{T}_F$ (see Remark 3.4).
We would also like to emphasize that the proof in this paper are not restricted to the specific equation (1.1). A similar argument would yield similar results for all the models that can be treated with the methods developed in [Hai14].

In Section 2 we present a summary of some notions of the theory of regularity structures. In Section 3 we construct the renormalised model and prove the main results. The convergence of the renormalised model is proved in Section 4.

2 Regularity structures

In this section we recall some preliminaries for the theory of regularity structures from [Hai14].

**Definition 2.1** A regularity structure \( T = (A, T, G) \) consists of the following elements:

(i) An index set \( A \subset \mathbb{R} \) such that \( 0 \in A \), \( A \) is bounded from below and locally finite.

(ii) A model space \( T \), which is a graded vector space \( T = \oplus_{\alpha \in A} T_\alpha \), with each \( T_\alpha \) a Banach space. Furthermore, \( T_0 \) is one-dimensional and has a basis vector \( 1 \). Given \( \tau \in T \) we write \( \|\tau\|_\alpha \) for the norm of its component in \( T_\alpha \).

(iii) A structure group \( G \) of (continuous) linear operators acting on \( T \) such that for every \( \Gamma \in G \), every \( \alpha \in A \) and every \( \tau_\alpha \in T_\alpha \) one has

\[
\Gamma \tau_\alpha - \tau_\alpha \in T_{< \alpha} := \bigoplus_{\beta < \alpha} T_\beta.
\]

Furthermore, \( \Gamma 1 = 1 \) for every \( \Gamma \in G \).

The canonical example is the space \( \bar{T} = \bigoplus_{n \in \mathbb{N}} \bar{T}_n \) of abstract polynomials in finitely many indeterminates, with \( A = \mathbb{N} \) and \( \bar{T}_n \) denoting the space of monomials that are homogeneous of degree \( n \). In this case, a natural group of transformations \( G \) acting on \( \bar{T} \) is given by the group of translations.

Given a scaling \( s = (s_0, s_1, \ldots, s_d) \) of \( \mathbb{R}^{d+1} \). We call \( |s| = s_0 + s_1 + \ldots + s_d \) scaling dimension. We define the associate metric on \( \mathbb{R}^{d+1} \) by

\[
\|z - z'|_s := d_s(z, z') := \sum_{i=0}^d |z_i - z'_i|^{1/s_i}.
\]

For \( k = (k_0, \ldots, k_d) \) we define \( |k|_s = \sum_{i=0}^d s_i k_i \).

2.1 Specific regularity structures

We consider the regularity structure \( \bar{T} \) given by all polynomials in \( d + 1 \) indeterminates, let us call them \( X_0, \ldots, X_d \), which denote the time and space directions respectively. Denote \( X^k = X_0^{k_0} \cdots X_d^{k_d} \) with \( k \) a multi-index. For the case of the dynamical \( \Phi^4_3 \) model, \( d = 3 \), \( s = (2, 1, 1, 1) \). In the regularity structure we use the symbol \( \Xi \) to replace the driving noise \( \xi \). We introduce the integration map \( I_k \) associated with the operation of convolution with the heat kernel \( G \) and the integration map \( I_k \) for a multi-index \( k \), which represents integration against \( D^k G \).

We recall the following notations from [Hai14]: defining a set \( \mathcal{F} \) by postulating that \( \{1, \Xi, X_j \} \subset \mathcal{F} \) and whenever \( \tau, \tilde{\tau} \in \mathcal{F} \), we have \( \tau \tilde{\tau} \in \mathcal{F} \) and \( I_k(\tau) \in \mathcal{F} \); defining \( \mathcal{F}_+ \) as the set of all elements
\( \tau \in \mathcal{F} \) such that either \( \tau = 1 \) or \( |\tau|_s > 0 \) and such that, whenever \( \tau \) can be written as \( \tau = \tau_1 \tau_2 \) we have either \( \tau_i = 1 \) or \( |\tau_i|_s > 0 \); \( \mathcal{H}, \mathcal{H}_+ \) denote the sets of finite linear combinations of all elements in \( \mathcal{F}, \mathcal{F}_+ \), respectively. Here for each \( \tau \in \mathcal{F} \) a weight \( |\tau|_s \) which is obtained by setting \( |1|_s = 0 \),

\[
|\tau \bar{\tau}|_s = |\tau|_s + |\bar{\tau}|_s,
\]

for any two formal expressions \( \tau \) and \( \bar{\tau} \) in \( \mathcal{F} \) such that

\[
|\Xi|_s = \alpha, \quad |X_i|_s = s_i, \quad |\mathcal{I}(\tau)|_s = |\tau|_s + 2 - |k|_s,
\]

with \(-\frac{18}{7} < \alpha < -\frac{5}{2} \).

As in [Hai14] we construct the regularity structure, which contains those that are actually useful for the abstract reformulation of the equation (1.1). Define

\[
\mathfrak{M}_F = \{ \Xi, U^n : n \leq 3 \},
\]

and the sets \( \mathcal{W}_0 = \mathcal{U}_0 = \emptyset \) and \( \mathcal{W}_n, \mathcal{U}_n \) for \( n > 0 \) recursively by

\[
\mathcal{W}_n = \mathcal{W}_{n-1} \cup \bigcup_{Q \in \mathfrak{M}_F} Q(\mathcal{U}_{n-1}, \Xi),
\]

\[
\mathcal{U}_n = \{ X^k \} \cup \{ \mathcal{I}(\tau) : \tau \in \mathcal{W}_n \},
\]

and

\[
\mathcal{F}_F := \bigcup_{n \geq 0} (\mathcal{W}_n \cup \mathcal{U}_n).
\]

Then \( \mathcal{F}_F \) contains the elements required to describe both the solutions and the terms in the equation. We denote by \( \mathcal{H}_F \) the set of finite linear combinations of elements in \( \mathcal{F}_F \).

Now we follow [Hai14] to construct the structure group \( G_F \). Define a linear projection operator \( P_+ : \mathcal{H} \to \mathcal{H}_+ \) by imposing that

\[
P_+ \tau = \tau, \quad \tau \in \mathcal{F}_+, \quad P_+ \tau = 0, \quad \tau \in \mathcal{F} \setminus \mathcal{F}_+,
\]

and two linear maps \( \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+ \) and \( \Delta^+ : \mathcal{H}_+ \to \mathcal{H}_+ \otimes \mathcal{H}_+ \) by

\[
\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}, \quad \Delta^+ \mathbf{1} = \mathbf{1} \otimes \mathbf{1},
\]

\[
\Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i, \quad \Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i,
\]

\[
\Delta \Xi = \Xi \otimes \mathbf{1},
\]

and recursively by

\[
\Delta(\tau \bar{\tau}) = (\Delta \tau)(\Delta \bar{\tau})
\]

\[
\Delta(\mathcal{I}_k \tau) = (\mathcal{I}_k \otimes \mathbf{1}) \Delta \tau + \sum_{l,m} \frac{X^l}{l!} \otimes \frac{X^m}{m!} (P_+ \mathcal{I}_{k+l+m} \tau),
\]

\[
\Delta^+(\tau \bar{\tau}) = (\Delta^+ \tau)(\Delta^+ \bar{\tau})
\]

\[
\Delta^+(\mathcal{I}_k \tau) = (\mathbf{1} \otimes \mathcal{I}_k \tau) + \sum_{l} (P_+ \mathcal{I}_{k+l} \otimes \frac{(-X)^l}{l!}) \Delta \tau.
\]
By using the theory of regularity structures (see [Hai14, Section 8]) a structure group $G_F$ of linear operators acting on $\mathcal{H}_F$ satisfying Definition 2.1 can be defined as follows: For group-like elements $g \in \mathcal{H}_F^*$, the dual of $\mathcal{H}_+$, we construct the following regularity structure.

Now that we have fixed our algebraic regularity structure $\mathcal{F}_F = (A, \mathcal{H}_F, G_F)$, we introduce a family of objects which is a concrete way of associating every $\tau \in \mathcal{H}_F$ and $x_0 \in \mathbb{R}^{d+1}$ with the actual "Taylor polynomial based at $x_0$" represented by $\tau$ in order to allow us to describe solutions to (1.1) locally.

First we introduce some notations: Given a smooth compactly supported test function $\varphi$ and a space-time coordinate $z = (t, x_1, ..., x_d) \in \mathbb{R}^{d+1}$, we denote by $\varphi^\lambda_z$ the test function

$$\varphi^\lambda_z(s, y_1, ..., y_d) = \lambda^{-|\lambda|} \varphi\left(\frac{s - t}{\lambda^{\delta_0}}, \frac{y_1 - x_1}{\lambda^{\delta_1}}, ..., \frac{y_d - x_d}{\lambda^{\delta_d}}\right).$$

Denote by $B_\alpha$ the set of smooth test functions $\varphi : \mathbb{R}^{d+1} \mapsto \mathbb{R}$ that are supported in the centred ball of radius 1 and such that their derivatives of order up to $1 + |\alpha|$ are uniformly bounded by 1. We denote by $\mathcal{S}'$ the space of all distributions on $\mathbb{R}^{d+1}$ and denote by $L(E, F)$ the set of all continuous linear maps between the topological vector spaces $E$ and $F$. With these notations at hand we give the definition of a model:

**Definition 2.2** Given a regularity structure $\mathfrak{T} = (A, T, G)$, a model for $\mathfrak{T}$ consists of maps

$$\mathbb{R}^{d+1} \ni z \mapsto \Pi_z \in L(T, \mathcal{S}'), \quad \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \ni (z, z') \mapsto \Gamma_{zz'} \in G,$

satisfying the algebraic compatibility conditions

$$\Pi_z \Gamma_{zz'} = \Pi_{z'}, \quad \Gamma_{zz'} \circ \Gamma_{z'z''} = \Gamma_{zz''},$$

as well as the analytical bounds

$$|\Pi_z \tau(\varphi^\lambda_z)| \lesssim \lambda^\alpha \|\tau\|_\alpha, \quad \|\Gamma_{zz'} \tau\|_\beta \lesssim \|z - z'|^{\alpha - \beta} \|\tau\|_\alpha.$$

Here, the bounds are imposed uniformly over all $\tau \in T_\alpha$, all $\beta < \alpha < \gamma$, $\gamma > 0$, and all test functions $\varphi \in B_\gamma$ with $r = \inf A$. They are imposed locally uniformly in $z$ and $z'$.

Then for every compact set $\mathfrak{R} \subset \mathbb{R}^{d+1}$ and any two models $Z = (\Pi, \Gamma)$ and $\bar{Z} = (\bar{\Pi}, \bar{\Gamma})$ we define

$$|||Z; \bar{Z}|||_{\mathfrak{R}} := \sup_{z \in \mathfrak{R}} \left[ \sup_{\varphi, \lambda, \alpha, \tau} \lambda^{-\alpha} |(\Pi_z \tau - \Pi_{\bar{z}} \tau)(\varphi^\lambda_z)| + \sup_{\|z - z'| \leq 1} \sup_{\alpha, \beta, \tau} \|z - z'|^{\beta - \alpha} \|\Gamma_{zz'} \tau - \bar{\Gamma}_{zz'} \tau\|_\beta \right],$$

where the suprema are taken over the same sets as in Definition 2.2, but with $\|\tau\|_\alpha = 1$. This gives a natural topology for the space of all models for a given regularity structure.
To describe the models for the regularity $\mathfrak{T}_F$ we are interested in, we fix a kernel $K : \mathbb{R}^{d+1} \to \mathbb{R}$ with the following properties:

(i) $K = \sum_{n \geq 0} K_n$ where each $K_n : \mathbb{R}^{d+1} \to \mathbb{R}$ is smooth and compactly supported in a ball of radius $2^{-n}$ around the origin. Furthermore, we assume that for every multi-index $k$, one has a constant $C$ such that

$$\sup_x |D^k K_n(x)| \leq C 2^{n(d-1+|k|)},$$

holds uniformly in $n$. Finally, we assume that $\int K_n(x)P(x)dx = 0$ for every polynomial $P$ of degree at most $r$ for some sufficiently large value of $r$.

(ii) $K(t, x) = 0$ for $t \leq 0$ and $K(t, -x) = K(t, x)$.

(iii) For $(t, x)$ with $|x|^2 + t < 1/2$ and $t > 0$ $K(t, x) = \frac{1}{|\pi t/|x|^2|} e^{-|x|^2/4t}$, and $K$ is smooth on $\{|x|^2 + t \geq 1/4\}$.

The kernel $K$ satisfying these properties can be obtained from the heat kernel $G$ as in [Hai14, Lemma 5.5].

**Definition 2.3** A model $(\Pi, \Gamma)$ for $\mathfrak{T}_F$ is admissible if it satisfies $(\Pi_x X^k)(y) = (y - x)^k$ as well as

$$(\Pi_x \mathcal{I} \tau)(y) = \int K(y - z)(\Pi_x \tau)(z)dz + \sum_l \frac{(y - x)^l}{l!} f_x(P_+ \mathcal{I} \tau), \quad (2.1)$$

for $\tau \in \mathcal{H}_F$ with $\mathcal{I}(\tau) \in \mathcal{H}_F$. Here $f_x(\mathcal{I} \tau)$ are defined by

$$f_x(\mathcal{I} \tau) = -\int D^l_x K(x - z)(\Pi_x \tau)(z)dz. \quad (2.2)$$

Furthermore, we impose $f_x(X_i) = -x_i$, $f_x(\tau \bar{\tau}) = f_x(\tau)f_x(\bar{\tau})$ and extend this to all of $\mathcal{H}_+$ by linearity. $\Gamma$ is given by

$$\Gamma_{xy} = (\Gamma_{f_x})^{-1} \circ \Gamma_{f_y}, \quad (2.3)$$

where $\Gamma_{f_x} \tau := (I \otimes f_x) \Delta \tau$ for $\tau \in \mathcal{H}_F$.

Let $\xi$ be periodic space-time white noise and $\rho : \mathbb{R}^4 \to \mathbb{R}$, $\rho(t, x) = \rho_1(t)\rho_2(x)$ be a smooth compactly supported function integrating to $1$, set $\rho_x(t, x) = \varepsilon^{-5} \rho(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$. Given the following approximation $\xi_{\varepsilon, \vartheta}$ to $\xi$, there is a canonical way of lifting it to an admissible model $(\Pi^{(\varepsilon, \vartheta)}, \Gamma^{(\varepsilon, \vartheta)})$ as follows. We set for $k \in \mathbb{Z}$,

$$\xi_{\varepsilon} = \rho_\varepsilon \ast \xi, \quad \xi_{\varepsilon, \vartheta}(t, x) = \frac{1}{\vartheta} \int_{k\vartheta}^{(k+1)\vartheta} \xi_{\varepsilon}(u, x)du, \quad t \in (k\vartheta, (k + 1)\vartheta).$$

$$(\Pi^{(\varepsilon, \vartheta)} X^k)(z) = (\Pi^{(\varepsilon, \vartheta)} X^k)(z) = (z - x)^k,$$

and recursively define

$$(\Pi^{(\varepsilon, \vartheta)} \tau \bar{\tau})(z) = (\Pi^{(\varepsilon, \vartheta)} \tau)(z)(\Pi^{(\varepsilon, \vartheta)} \bar{\tau})(z),$$

and

$$(\Pi^{(\varepsilon, \vartheta)} \mathcal{I} \tau)(z) = \int K(z - z_1)(\Pi^{(\varepsilon, \vartheta)} \tau)(z_1)dz_1 + \sum_l \frac{(z - x)^l}{l!} f^{(\varepsilon, \vartheta)}(P_+ \mathcal{I} \tau). \quad (2.4)$$
Here \( f^{(x,\theta)}_x(I_\tau) \) are defined by
\[
f^{(x,\theta)}_x(I_\tau) = -\int D^1_r K(x - z_1)(\Pi^{(x,\theta)}_{1/})^2(z_1)d\gamma_1.
\] (2.5)

Furthermore we impose \( f^{(x,\theta)}_x(X_i) = -x_i \), \( f^{(x,\theta)}_x(\tau) = f^{(x,\theta)}_x(\tau)f^{(x,\theta)}_x(\bar{\tau}) \) and extend this to all of \( \mathcal{H}_+ \) by linearity. Then define
\[
\Gamma^{(x,\theta)}_{xy} = \Gamma^{(x,\theta)}_x \circ (\Gamma^{(x,\theta)}_y)^{-1},
\] (2.6)
where \( \Gamma^{(x,\theta)}_x \tau := (I \otimes f^{(x,\theta)}_x)\Delta \tau \) for \( \tau \in \mathcal{H}_F \).

Then by [Hai14, Proposition 8.27] it is easy to check that \((\Pi^{(x,\theta)}, \Gamma^{(x,\theta)})\) is an admissible model for the regularity structure \( \mathfrak{T}_F \) constructed in Section 2.1.

Now we give the following definition for the spaces of distributions \( \mathcal{C}_\alpha^\eta \), \( \alpha < 0 \), which is an extension of the definition of Hölder space to include \( \alpha < 0 \).

**Definition 2.4** Let \( \eta \in \mathcal{S}' \) and \( \alpha < 0 \). We say that \( \eta \in \mathcal{C}_\alpha^\eta \) if the bound
\[
|\eta(\varphi^\lambda_\delta)| \lesssim \lambda^\alpha,
\]
holds uniformly over all \( \lambda \in (0, 1] \), all \( \varphi \in \mathcal{B}_\alpha \) and locally uniformly over \( z \in \mathbb{R}^{d+1} \).

For every compact set \( \mathcal{R} \subset \mathbb{R}^{d+1} \), we will denote by \( \|\eta\|_{\alpha;\mathcal{R}} \) the seminorm given by
\[
\|\eta\|_{\alpha;\mathcal{R}} := \sup_{z \in \mathcal{R}} \sup_{\varphi \in \mathcal{B}_\alpha} \sup_{\lambda \leq 1} \lambda^{-\alpha} |\eta(\varphi^\lambda_\delta)|.
\]

We also write \( \|\cdot\|_\alpha \) for the same expression with \( \mathcal{R} = \mathbb{R}^{d+1} \). In the following we also use \( \mathcal{C}^\alpha \) to denote \( \mathcal{C}_\alpha^\eta \) on \( \mathbb{R}^d \) for the scaling \( \mathfrak{g} := (s_1, \ldots, s_d) \).

We also have the following definition of spaces of modelled distributions, which are the Hölder spaces on the regularity structure. Set \( \mathfrak{T} = \{(t, x) : t = 0\} \). Given a subset \( \mathcal{R} \subset \mathbb{R}^{d+1} \) we denote by \( \mathfrak{R}_{\mathfrak{T}} \) the set
\[
\mathfrak{R}_{\mathfrak{T}} = \{(z, \bar{z}) \in (\mathcal{R} \setminus \mathfrak{T})^2 : z \neq \bar{z} \text{ and } \|z - \bar{z}\|_\mathfrak{g} \leq |t|^{1/\mathfrak{s}_0} \land \bar{t}^{1/\mathfrak{s}_0} \land 1\},
\]
where \( z = (t, x), \bar{z} = (\bar{t}, \bar{x}) \).

**Definition 2.5** Given a model \((\Pi, \Gamma)\) for the regularity structure \( \mathfrak{T}_F \) and \( \mathfrak{T} \) as above. Then for any \( \gamma > 0 \) and \( \eta \in \mathbb{R} \), the space \( \mathcal{D}^{\gamma, \eta} \) consists of all functions \( f : \mathbb{R}^{d+1} \setminus \mathfrak{T} \to \bigoplus_{\alpha < \gamma} T_\alpha \) such that for every compact set \( \mathcal{R} \subset \mathbb{R}^{d+1} \) one has
\[
\|f\|_{\gamma, \eta; \mathcal{R}} := \sup_{z \in \mathfrak{R}_{\mathfrak{T}}} \sup_{\lambda \leq \gamma} \|f(z)\|_\mathfrak{g}^{\lambda} + \sup_{(z, \bar{z}) \in \mathfrak{R}_{\mathfrak{T}}} \sup_{\lambda \leq \gamma} \frac{\|f(z) - \Gamma_{z\bar{z}} f(\bar{z})\|_\mathfrak{g}^{\lambda}}{\|z - \bar{z}\|_\mathfrak{g}^{\lambda - 1}} < \infty.
\]
Here we wrote \( \|\tau\|_\mathfrak{T} \) for the norm of the component of \( \tau \) in \( T_\mathfrak{T} \) and also used \( t \) and \( \bar{t} \) as shorthands for the time components of the space-time points \( z \) and \( \bar{z} \).
For $f \in \mathcal{D}^{\gamma,\eta}$ and $\bar{f} \in \mathcal{D}^{\gamma,\eta}$ (denoting by $\mathcal{D}^{\gamma,\eta}$ the space built over another model $(\tilde{\Pi}, \tilde{\Gamma})$), we also set

$$\|f; \bar{f}\|_{\gamma,\eta,\mathbb{R}} := \sup_{z \in \mathbb{R}^d} \sup_{t \leq \gamma} \|f(z) - \bar{f}(z)\|_t + \sup_{(z, \bar{z}) \in \mathbb{R}^{2d}} \sup_{t \leq \gamma} \|f(z) - \bar{f}(\bar{z}) - \Gamma_{z\bar{z}}f(\bar{z}) + \Gamma_{z\bar{z}}\bar{f}(\bar{z})\|_t,$$

which gives a natural distance between elements $f \in \mathcal{D}^{\gamma,\eta}$ and $\bar{f} \in \mathcal{D}^{\gamma,\eta}$.

Given a regularity structure, we say that a subspace $V \subset T$ is a sector of regularity $\alpha$ if it is invariant under the action of the structure group $G$ and it can be written as $V = \oplus_{\beta \in A} V_\beta$ with $V_\beta \subset T_\beta$, and $V_\beta = \{0\}$ for $\beta < \alpha$. We will use $\mathcal{D}^{\gamma,\eta}(V)$ to denote all functions in $\mathcal{D}^{\gamma,\eta}$ taking values in $V$.

Under suitable regularity assumptions, we can reconstruct from a given modelled distribution $f$, a distribution $\mathcal{R}f$ in the real world which "looks like $\Pi_x f(x)$ near $x$". This result, which defines the so-called reconstruction operator, is one of the most fundamental result in the theory of regularity structures.

**Theorem 2.6** (cf. [Hai14, Proposition 6.9]) Given a regularity structure and a model $(\Pi, \Gamma)$. Let $f \in \mathcal{D}^{\gamma,\eta}(V)$ for some sector $V$ of regularity $\alpha \leq 0$, some $\gamma > 0$, and some $\eta \leq \gamma$. Then provided that $\alpha \wedge \eta > -\delta_0$, there exists a unique distribution $\mathcal{R}f \in \mathcal{C}_{\mathbb{R}^\alpha}$ such that

$$\|\mathcal{R}f - \Pi_x f(z)\|_{\gamma,\eta} \lesssim \lambda^\gamma,$$

holds uniformly over $\lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_r$ with $\varphi_r^\lambda$ compactly supported away from $\mathfrak{P}$ and locally uniformly over $z \in \mathbb{R}^{d+1}$.

Moreover, $(\Pi, \Gamma, f) \rightarrow \mathcal{R}f$ is jointly (locally) Lipschitz continuous with respect to the metric for $(\Pi, \Gamma)$ and $f$ defined in Definitions 2.2 and 2.5.

### 2.3 Abstract fixed point problem

We reformulate (1.1) as a fixed point problem in $\mathcal{D}^{\gamma,\eta}$ for suitable $\gamma$ and $\eta$. By Duhamel’s formula, (1.1) is equivalent for smooth $\xi$ to the integral equation

$$u = G * ((\xi - w^3)_{t > 0}) + Gu_0.$$

Here, $G$ denotes the heat kernel, $*$ denotes space-time convolution, and $Gu_0$ denotes the solution to the heat equation with initial condition $u_0$. In order to interpret this equation as an identity in $\mathcal{D}^{\gamma,\eta}$, we need the following results from [Hai14, Proposition 6.16].

**Theorem 2.7** Let $\mathcal{S}_F = (A, \mathcal{H}_F, G_F)$ be the regularity structure constructed as above and $(\Pi, \Gamma)$ be an admissible model for $\mathcal{S}_F$. Let $\gamma > 0$, $\eta \leq \gamma$. Let $\mathcal{I}$ act on some sector $V$ of regularity $\alpha \leq 0$. Then provided that $\alpha \wedge \eta > -2$, $\gamma + 2, \eta + 2$ not in $\mathbb{N}$, there exists a continuous linear operator $\mathcal{K}_\gamma : \mathcal{D}^{\gamma,\eta}(V) \rightarrow \mathcal{D}^{\gamma,\eta'}$ with $\gamma' = \gamma + 2$ and $\eta' = (\eta \wedge \alpha) + 2$, such that

$$\mathcal{R}\mathcal{K}_\gamma f = K \ast \mathcal{R}f,$$

holds for $f \in \mathcal{D}^{\gamma,\eta}(V)$.  

\[ 9 \]
In the following we will only consider (1.1) with periodic boundary conditions. By the theory of regularity structures proposed in [Hai14] we can define translation maps and use it to define the periodic modelled distribution. Here the fundamental domain of the translation theory of regularity structures proposed in [Hai14] we can define translation maps and use it for
\[ ||| \cdot ||| \]
\[ K_\xi \] an admissible model for \( T(2.7) \).

Now we reformulate the fixed point map as
\[ v = (K_\gamma + R_\gamma \mathcal{R})(1_{t>0}^3), \]
\[ u = - (K_\gamma + R_\gamma \mathcal{R})(1_{t>0}^3) + v + \mathcal{G}u_0. \]

Here for smooth function \( R = G - K \),
\[ R_\gamma : \mathcal{C}_\alpha^\gamma \rightarrow \mathcal{D}^{\gamma, \eta}, (R_\gamma f)(z) = \sum_{|k|_\alpha < \gamma} \frac{X^k}{k!} \int D_1^k R(z - \bar{z}) f(\bar{z}) d\bar{z}, \]
\[ \mathcal{G}u_0 = \sum_{|k|_\alpha < \gamma} \frac{X^k}{k!} D^k (G u_0)(z), \]
where \( \gamma, \bar{\gamma} \) will be chosen below and we define \( \mathcal{R}(1_{t>0}^3) \) as the distribution \( \xi 1_{t>0}^3 \).

We consider the second equation in (2.7): Define
\[ V := \mathcal{I}(\mathcal{F}) \oplus \bar{T}. \]

Now for \( \gamma > |2\alpha + 4|, \eta \leq \alpha + 2 \) and \( u_0 \in \mathcal{C}^\eta(\mathbb{R}^3) \), periodic, [Hai14, Lemma 7.5] implies that \( \mathcal{G}u_0 \in \mathcal{D}^{\gamma, \eta} \).

We define for any \( \beta < 0 \) and compact set \( \mathcal{R} \) the norm
\[ ||| \xi |||_{\beta, \mathcal{R}} = \sup_{s \in \mathcal{R}} ||| \xi 1_{t \geq s} \|||_{\beta, \mathcal{R}}, \]
and we denote by \( \mathcal{C}_\beta^\alpha \) the intersections of the completions of smooth functions under \( | \cdot |_{\beta, \mathcal{R}} \) for all compact sets \( \mathcal{R} \). By [Hai14, Proposition 9.5] we know that for every \( \alpha \in (-3, -\frac{3}{2}) \), space-time white noise \( \xi \in \mathcal{C}_\alpha^\alpha \) almost surely and \( K * \xi \in \mathcal{C}(\mathbb{R}, C^{\alpha+2}(\mathbb{R}^3)) \) almost surely. With these notations at hand, we recall the following results from [Hai14].

**Proposition 2.8** ([Hai14, Proposition 9.8]) Let \( \Xi_\mathcal{F} \) be the regularity structure associated to \( (\Phi^4) \) with \( \alpha \in (-\frac{15}{2}, -\frac{3}{2}) \). Let \( \eta \in (-\frac{2}{3}, \alpha + 2) \), \( \gamma > |2\alpha+4| \), \( \bar{\gamma} = \gamma + 2\alpha + 4 \) and let \( Z = (\Pi, \Gamma) \) be an admissible model for \( \Xi_\mathcal{F} \) with the additional properties that \( \xi := \mathcal{R} \Xi \) belongs to \( \mathcal{C}_\alpha^\alpha \) and that \( K * \xi \in \mathcal{C}(\mathbb{R}, C^{\alpha+2}(\mathbb{R}^3)) \). Then there exists a maximal solution \( S^L(u_0, Z) \in \mathcal{D}^{\gamma, \eta}(V) \) to the equation (2.7).

Furthermore, let \( T^L(u_0, Z) \in \mathbb{R}^+ \cup \{ +\infty \} \) be the first time such that \( |||(\mathcal{R} S^L(u_0, Z))(t.)\|||_\eta \geq L \) and set \( O = [-1, 2] \times \mathbb{R}^d \). Then, for every \( \varepsilon > 0 \) and \( C > 0 \) there exists \( \delta > 0 \) such that, setting \( T = 1 \wedge T^L(u_0, Z) \wedge T^L(\tilde{u}_0, \tilde{Z}), \) one has the bound \( |||S^L(u_0, Z) - S^L(\tilde{u}_0, \tilde{Z})|||_{\gamma, \eta, T} \leq \varepsilon, \) for all \( u_0, \tilde{u}_0, Z, \tilde{Z} \) such that \( |||Z|||_{\gamma, O} \leq C, |||\tilde{Z}|||_{\gamma, O} \leq C, |||u_0|||_\eta \leq L/2, |||\tilde{u}_0|||_\eta \leq L/2, |||u_0 - \tilde{u}_0|||_\eta \leq \delta, \)
and \( ||Z; \tilde{Z}||_{\gamma;O} \leq \delta \), and satisfy the bounds \( |\xi|_{\alpha;O} + |\tilde{\xi}|_{\alpha;O} \leq C \), \( \sup_{t \in [0,1]} \|(K \ast \xi)(t, \cdot)\|_{\eta} \leq C \), as well as
\[
|\xi - \tilde{\xi}|_{\alpha;O} \leq \delta, \sup_{t \in [0,1]} \|(K \ast \xi)(t, \cdot) - (K \ast \tilde{\xi})(t, \cdot)\|_{\eta} \leq \delta.
\]
Here we have set \( \tilde{\xi} = \mathcal{R} \Xi \), where \( \mathcal{R} \) is the reconstruction operator associated to \( \tilde{Z} \).

### 3 Renormalisation procedure and main result

In this section we have constructed a model associated with \( \xi_{\varepsilon, \vartheta} \) and in this section we will prove the convergence result required in Proposition 2.8, which at last implies Theorem 1.1. As we mentioned in the introduction, there is no hope to prove the sequence of models converges to a limit. We have to renormalize the model into some converging renormalised model.

#### 3.1 Renormalised model

In this subsection we construct the renormalised model and prove that it is also an admissible model for the regularity structure \( \mathfrak{S}_F \) associated with the dynamical \( \Phi_3^4 \) model. In our case we will prove that \( K \ast \xi_{\varepsilon, \vartheta} \cdot K \ast \xi_{\varepsilon, \vartheta} - C_1^{(\varepsilon, \vartheta)} \) converges to a non-trivial limit as \( \varepsilon, \vartheta \) goes to 0. Here \( C_1^{(\varepsilon, \vartheta)} \) is a function depending on \( t \), which is a main difference from the case in [Hai14], where all the terms be subtracted in the renormalisations are constants. To construct an admissible renormalised model in our case, we introduce the symbols \( C_1, C_2 \) to replace \( C_1^{(\varepsilon, \vartheta)}(t), C_2^{(\varepsilon, \vartheta)}(t) \) and define a bigger regularity structure \( \mathfrak{S}_1 \) to include the original regularity structure \( \mathfrak{S}_F \) and \( C_1, C_2 \), where \( C_1, C_2 \) will be specified below. We build a model for \( \mathfrak{S}_1 \) and use it to construct the required renormalised model for \( \mathfrak{S}_F \).

First we construct the regularity structure \( \mathfrak{S}_1 \). Define a set \( \mathcal{F}_1 \) by postulating that \( \{1, \Xi, X_j, C_1, C_2\} \subset \mathcal{F}_1 \) and whenever \( \tau, \bar{\tau} \in \mathcal{F}_1 \), we have \( \tau \bar{\tau} \in \mathcal{F}_1 \) and \( \mathcal{I}_k(\tau) \in \mathcal{F}_1 \); defining \( \mathcal{F}_+^1 \) as the set of all elements \( \tau \in \mathcal{F}_1 \) such that either \( \tau = 1 \) or \( |\tau|_s > 0 \) and such that, whenever \( \tau \) can be written as \( \tau = \tau_1 \tau_2 \), \( \tau_1, \tau_2 \in \mathcal{F}_1 \), we have then \( \tau_i = 1 \) or \( |\tau_i|_s > 0 \); \( \mathcal{H}_1, \mathcal{H}_+^1 \) denote the sets of finite linear combinations of all elements in \( \mathcal{F}_1, \mathcal{F}_+^1 \), respectively. Here for each \( \tau \in \mathcal{F}_1 \) a weight \( |\tau|_s \) is obtained as above and by setting \( |C_1|_s = |C_2|_s = -\delta_0 \) with \( 4\alpha + 10 < -\delta_0 < 0 \). Recall \( \mathfrak{M}_F = \{ \Xi, U^n : n \leq 3 \} \). We define the sets \( \mathcal{W}_n^1, \mathcal{U}_n^1 \) for \( n \geq 0 \) recursively by
\[
\mathcal{W}_0^1 = \mathcal{U}_0^1 = \emptyset,
\]
\[
\mathcal{W}_n^1 = \mathcal{W}_{n-1}^1 \cup \bigcup_{Q \in \mathfrak{M}_F} Q(\mathcal{U}_{n-1}^1, \Xi),
\]
\[
\mathcal{U}_n^1 = \{X^k, C_1, C_2\} \cup \{\mathcal{I}(\tau) : \tau \in \mathcal{W}_n\},
\]
and
\[
\mathcal{F}_1^F := \bigcup_{n \geq 0} (\mathcal{W}_n^1 \cup \mathcal{U}_n^1).
\]

We denote by \( \mathcal{H}_F^1 \) the set of finite linear combinations of elements in \( \mathcal{F}_F^1 \) and denote by \( \mathcal{F}_F^1, \mathcal{H}_F^1 \) the set of those basis vectors \( \bar{\tau} \in \mathcal{F}_F^1 \) that can be written as \( \bar{\tau} = X^l_{i_0} \Pi_i \mathcal{I}_{i_0} \tau_i \) for some
multiindices $l_i$ and some elements $\tau_i \in F^1_F$. Denote $H^{1+}_F$ the set of finite linear combinations of elements in $F^{1+}_F$.

Now we construct the structure group $G^1_F$. We also define the operators as $\Delta, \Delta^+$ in Section 2. We still use $\Delta, \Delta^+$ to denote them for the notation’s simplicity. $\Delta$ on $1, X_1, \Xi$ and $\Delta^+$ on $1, X_1$ can be defined as in Section 2. Define

$$\Delta C_1 = C_1 \otimes 1, \quad \Delta C_2 = C_2 \otimes 1.$$ 

Then $\Delta, \Delta^+$ on other terms can also be defined recursively as in Section 2.

By using the theory of regularity structures (see [Hai14, Section 8]) we can define a structure group $G^1_F$ of linear operators acting on $H^1_F$ satisfying Definition 2.1 as follows: For group-like elements $g \in H^1_+, \Gamma : H^1 \to H^1, \Gamma_g \tau = (I \otimes g)\Delta \tau$. By [Hai14, Theorem 8.24] we construct the following regularity structure.

**Theorem 3.1** Let $T = H^1_F$ with $T_\gamma = \langle \{\tau \in F^1_F : |\tau|_s = \gamma\} \rangle$, $A^1 = \{|\tau|_s : \tau \in F^1_F\}$. Then $\Xi^1 = (A^1, H^1_F, G^1_F)$ defines a regularity structure $\Xi^1$.

We emphasize that we do not replace the regularity structure associated with $\Phi^4$ in our case. Here the introduction of $\Xi^1$ is to prove that the renormalised model is an admissible model for $\Xi_F$. In the following we extend the model $(\Pi^{(\varepsilon, \varrho)}, \Gamma^{(\varepsilon, \varrho)})$ constructed in Section 2 to a model for $\Xi^1$, which is used to construct the renormalised model. We still denote it by $(\Pi^{(\varepsilon, \varrho)}, \Gamma^{(\varepsilon, \varrho)})$ for simplicity.

Given continuous functions $C^{(\varepsilon, \varrho)}_1(t), C^{(\varepsilon, \varrho)}_2(t)$, for $z = (t, y)$ define $\Pi^{(\varepsilon, \varrho)}_x \Xi$ and $\Pi^{(\varepsilon, \varrho)}_x X^k$ as in Section 2 and define

$$(\Pi^{(\varepsilon, \varrho)}_x C_1)(z) = C^{(\varepsilon, \varrho)}_1(t), \quad (\Pi^{(\varepsilon, \varrho)}_x C_2)(z) = C^{(\varepsilon, \varrho)}_2(t),$$

and recursively define $(\Pi^{(\varepsilon, \varrho)}_x \tau \varrho)(z) = (\Pi^{(\varepsilon, \varrho)}_x \tau)(z)(\Pi^{(\varepsilon, \varrho)}_x \varrho)(z)$, and use (2.4) to define $\Pi^{(\varepsilon, \varrho)}_x (\mathcal{I}_x \tau)$ with $f^{(\varepsilon, \varrho)}_x (\mathcal{I}_x \tau)$ defined by (2.5) and $f^{(\varepsilon, \varrho)}_x (X_i) = -x_i, f^{(\varepsilon, \varrho)}_x (\tau \varrho) = f^{(\varepsilon, \varrho)}_x (\tau)f^{(\varepsilon, \varrho)}_x (\varrho)$ and extend this to all of $H^{1+}_F$ by linearity. Then define $\Gamma^{(\varepsilon, \varrho)}_{xy}$ still by (2.6).

**Proposition 3.2** $(\Pi^{(\varepsilon, \varrho)}, \Gamma^{(\varepsilon, \varrho)})$ is a model for the regularity structure $\Xi^1$ constructed in Theorem 3.1.

**Proof** Since $C^{(\varepsilon, \varrho)}_1, C^{(\varepsilon, \varrho)}_2$ are continuous functions, by a similar argument as in the proof of [Hai14, Proposition 8.27] the result follows. \qed

Define

$$\mathcal{F}_0 := \{1, \Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_i, \mathcal{I}(\Psi^3)\Psi, \mathcal{I}(\Psi^3)\Psi^2, \mathcal{I}(\Psi^2)\Psi, \mathcal{I}(\Psi)\Psi, \mathcal{I}(\Psi)\Psi^2, X_i\},$$

$$\mathcal{F}_* := \{\Psi, \Psi^2, \Psi^3\},$$

where the index $i$ corresponds to any of the three spatial directions.

Then $\mathcal{F}_0 \subset \mathcal{F}_F$ contains every $\tau \in \mathcal{F}_F$ with $|\tau|_s \leq 0$ and for every $\tau \in \mathcal{F}_0$, $\Delta \tau \in H_0 \otimes H^{1+}_0$. Here $H_0$ denotes the linear span of $\mathcal{F}_0$ and $H^{1+}_0$ denotes the linear span of the elements in $\mathcal{F}^+_0$ of the form $X^k \prod_i \mathcal{I}_{l_i} \tau_i$ for some multiindices $k$ and $l_i$ such that $|\mathcal{I}_{l_i} \tau_i|_s > 0$ and $\tau_i \in \mathcal{F}_*$. 

\[12\]
We define a linear map \( M \) from \( \mathcal{H}_0 \) to \( \mathcal{H}_F^1 \) by

\[
M \Psi^2 = \Psi^2 - C_1, \\
M(\Psi^2 X_i) = \Psi^2 X_i - C_1 X_i, \\
M \Psi^3 = \Psi^3 - 3C_1 \Psi, \\
M(I(\Psi^2)) = I(\Psi^2) - I(C_1), \\
M(I(\Psi^2)\Psi^2) = I((\Psi^2) - I(C_1))((\Psi^2) - C_1) - C_2, \\
M(I(\Psi^3)\Psi) = I((\Psi^3) - 3I(C_1\Psi))\Psi, \\
M(I(\Psi^3)\Psi^2) = I((\Psi^3) - 3I(C_1\Psi))((\Psi^2) - C_1) - 3C_2 \Psi, \\
M(I(\Psi)\Psi^2) = I(\Psi)((\Psi^2) - C_1),
\]

as well as \( M \tau = \tau \) for the remaining basis elements \( \tau \in \mathcal{F}_0 \). Define linear map \( \Delta^M : \mathcal{H}_0 \to \mathcal{H}_F^1 \times \mathcal{H}_F^{1+} \) by

\[
\Delta^M \tau = (M \tau) \otimes 1,
\]

for those elements \( \tau \in \mathcal{F}_0 \) which do not contain a factor \( I(\Psi^2), I(\Psi^3) \). For the remaining elements, we define

\[
\Delta^M I(\Psi^2) = (M(I(\Psi^2))) \otimes 1 + X_i \otimes I_i(C_1). \\
\Delta^M I(\Psi^2)\Psi^2 = (M(I(\Psi^2)\Psi^2)) \otimes 1 + (\Psi^2 - C_1)X_i \otimes I_i(C_1). \\
\Delta^M I(\Psi^3)\Psi = (M(I(\Psi^3)\Psi)) \otimes 1 + 3\Psi X_i \otimes I_i(C_1\Psi). \\
\Delta^M I(\Psi^3)\Psi^2 = (M(I(\Psi^3)\Psi^2)) \otimes 1 + 3(\Psi^2 - C_1)X_i \otimes I_i(C_1\Psi).
\]

Moreover, we define a linear map \( \hat{M} : \mathcal{H}_0^+ \to \mathcal{H}_F^{1+} \), which is a multiplicative morphism and leaves \( X^k \) invariant, and

\[
\hat{M}I(\Psi^n) = I(M\Psi^n), \quad \hat{M}I_i(\Psi) = I_i(\Psi).
\]

By this we have

\[
\hat{M}I_k = M(I_k \otimes I)\Delta^M, \\
(I \otimes \mathcal{M})(\Delta \otimes I)\Delta^M = (M \otimes \hat{M})\Delta.
\]

Here \( \mathcal{M} : \mathcal{H}_F^{1+} \times \mathcal{H}_F^{1+} \to \mathcal{H}_F^{1+} \) denotes the multiplication map.

Define a linear multiplicative morphism: \( \hat{\Delta}^M : \mathcal{H}_0^+ \to \mathcal{H}_F^{1+} \times \mathcal{H}_F^{1+} \) by

\[
\hat{\Delta}^M X^k = X^k \otimes 1, \\
\hat{\Delta}^M I(\Psi^n) = I(M\Psi^n) \otimes 1 + 3\delta_{n3}(X_i \otimes I_i(C_1\Psi) - X_i I_i(C_1\Psi) \otimes 1) \\
+ \delta_{n2}(X_i \otimes I_i(C_1) - X_i I_i(C_1) \otimes 1).
\]

We can easily check that

\[
(A \hat{M} A \otimes \hat{M}) \Delta^+ = (I \otimes \mathcal{M})(\Delta^+ \otimes I)\hat{D}^M.
\]

Here \( A \) is given in [Hai14, Section 8] for the regularity structure \( \Xi^1 \).
Define for \( \tau \in \mathcal{H}_0, \tau_1 \in \mathcal{H}_0^+ \),
\[
\Pi^{M,\varepsilon,\vartheta}_x \tau = (\Pi^{(\varepsilon,\vartheta)}_x \otimes f^{(\varepsilon,\vartheta)}_x) \Delta M \tau, \quad f^{M,\varepsilon,\vartheta}_x \tau_1 = f^{(\varepsilon,\vartheta)}_x \hat{M} \tau_1,
\]
and define \( \Gamma^M = (F^M_x)^{-1} \circ F^M_y \) with \( F^M_x := (I \otimes f^M_x) \Delta \). Then by a similar argument as [Hai14, Theorem 8.44] we have the following result.

**Proposition 3.3** \((\Pi^M, \Gamma^M)\) is an admissible model for \( \mathfrak{T}_F \) on \( \mathcal{H}_0 \). Furthermore, it extends uniquely to an admissible model for all of \( \mathfrak{T}_F \).

**Proof** By the definition for \( \Pi^M \) and the expression for \( \Delta^M \) we know that \( (\Pi^M_x \tau)(\varphi^\lambda_x) \) can be written as a finite linear combination of terms of the type \( (\Pi^M_x \bar{\tau})(\varphi^\lambda_x) \) with \( |\bar{\tau}|_s \geq |\tau|_s \) and \( \bar{\tau} \in \mathcal{H}_F^1 \). Then by Proposition 3.2 the required scaling as a function of \( \lambda \) follows.

Define \( \gamma_{xy} := (f_x \mathcal{A} \otimes f_y) \Delta^+ \) and we have \( \Gamma_{xy} = (I \otimes \gamma_{xy}) \Delta \). Since \( (\Pi^{(\varepsilon,\vartheta)}, \Gamma^{(\varepsilon,\vartheta)}) \) is a model for \( \mathfrak{T}^1 \), this implies that for \( \tau \in \mathcal{H}_F^{1,+} \)
\[
|\gamma_{xy} \tau| \lesssim \|x - y\|_{\tau} \langle \tau \rangle_s.
\]
Since \( \Gamma_{xy}^M = (I \otimes \gamma_{xy}^M) \Delta \) with \( \gamma_{xy}^M = (\gamma_{xy} \otimes f_y) \hat{\Delta}^M \) and \( \hat{\Delta}^M \tau = \tau \otimes 1 + \sum \tau^1 \otimes \tau^2 \) with \( |\tau|^1_s > |\tau|_s \), it follows from the expression of \( \hat{\Delta}^M \) that for \( \tau \in \mathcal{H}_0^+ \)
\[
|\gamma_{xy}^M \tau| \lesssim \|x - y\|_{\tau} \langle \tau \rangle_s.
\]
Thus \( (\Pi^M, \Gamma^M) \) is a model on \( \mathcal{H}_0 \). By (3.2), (3.3) and similar arguments as in [Hai14, Section 8] we know that it is also an admissible model on \( \mathcal{H}_0 \). Finally applying [Hai14, Theorem 5.14, Proposition 3.31] \((\Pi^M, \Gamma^M)\) can be extended uniquely to all of \( \mathfrak{T}_F \).

**Remark 3.4**
(i) It is a little different from the case in [Hai14] to construct the renormalised model. In [Hai14] the renormalised map \( M \) is a linear map from \( \mathcal{H}_0 \) to \( \mathcal{H}_0 \), which is enough for the construction of the renormalised model. In our case we have to subtract some functions \( C_1, C_2 \) to make the diverging terms converge in some sense. As we explained at the beginning of the section, we construct a new regularity structure \( \mathfrak{T}^1 \) including \( C_1, C_2 \) which represent functions \( C_1 \) and \( C_2 \), respectively. The renormalised map \( M \) is a linear map from \( \mathcal{H}_0 \) to \( \mathcal{H}_F^1 \), which does not belong to the renormalisation group defined in [Hai14, Definition 8.41]. However, we could still use it to define \( \hat{M}, \hat{\Delta}^M, \hat{\Delta}^M \) and construct the renormalised model on \( \mathcal{H}_F \). We emphasize that the renormalised model is associated with the old regularity structure \( \mathfrak{T}_F \). Below we still use the old regularity structure \( \mathfrak{T}_F \). \( \mathfrak{T}^1 \) is a tool to prove that the renormalised model is an admissible model for \( \mathfrak{T}_F \).

(ii) In fact, we can also define the renormalised model for the bigger regularity structure \( \mathfrak{T}^1 \) and apply directly the results in [Hai14, Section 4] to conclude that the renormalised model is an admissible model for \( \mathfrak{T}^1 \), which is also the required model when restricted on \( \mathfrak{T}_F \). For this argument we need to define the corresponding \( \mathcal{F}_0 \) for \( \mathfrak{T}^1 \), which is a little bit complicated. As a result, we use the above proof for simplicity.

### 3.2 Renormalised solutions

Denote by \( u_{\varepsilon,\vartheta} = \mathcal{S}^L(u_0, \xi_{\varepsilon,\vartheta}) \) the classical solution map to the equation
\[
\partial_t u_{\varepsilon,\vartheta} = \Delta u_{\varepsilon,\vartheta} - u_{\varepsilon,\vartheta}^3 + \xi_{\varepsilon,\vartheta}.
\]
Here $u_0 \in \mathcal{C}^0(\mathbb{T}^3)$. The renormalised map $\bar{S}_M^L(u_0, \xi_{\varepsilon, \theta})$ is given by the classical solution map to the equation

$$\partial_t u_{\varepsilon, \theta} = \Delta u_{\varepsilon, \theta} + (3C_1^{(\varepsilon, \theta)} - 9C_2^{(\varepsilon, \theta)})u_{\varepsilon, \theta} - u_{3\varepsilon, \theta} + \xi_{\varepsilon, \theta}.$$ 

By the same argument as the proof of [Hai14, Proposition 9.10] we obtain the following result:

**Proposition 3.5** Denote by $Z_{\varepsilon, \theta} = (\Pi^{(\varepsilon, \theta)}, \Gamma^{(\varepsilon, \theta)})$ the model given above, and by $Z_{\varepsilon, \theta}^M = (\Pi_{M, \varepsilon, \theta}, \Gamma_{M, \varepsilon, \theta})$ the renormalised model. Then for every symmetric $u_0 \in \mathcal{C}^0(\mathbb{R}^3)$ one has the identities

$$RS^L(u_0, Z_{\varepsilon, \theta}) = \bar{S}_L^L(u_0, \xi_{\varepsilon, \theta}), \quad RS^L(u_0, Z_{\varepsilon, \theta}^M) = \bar{S}_M^L(u_0, \xi_{\varepsilon, \theta}).$$

**3.3 Main results**

In this subsection we prove Theorem 1.1. We first prove the required convergence in Proposition 2.8 for $\xi_{\varepsilon, \theta}$ and $K^* \xi_{\varepsilon, \theta}$. Our argument essentially follows [Hai14, Proposition 9.5].

**Proposition 3.6** Let $\xi$ be white noise on $\mathbb{R} \times \mathbb{T}^3$, which we extend periodically to $\mathbb{R}^4$, and define $\xi_{\varepsilon, \theta}$ as in Subsection 2.2. Then for every compact set $\mathcal{R} \subset \mathbb{R}^4$ and every $0 < \theta < -\alpha - \frac{5}{2}$ we have

$$E|\xi_{\varepsilon, \theta} - \xi|_{\alpha, \mathcal{R}} \lesssim \varepsilon^\theta + \vartheta^\theta. \quad (3.4)$$

Finally for every $0 < \kappa < -\frac{2\alpha + 5}{4}$, the bound

$$E \sup_{t \in [0, 1]} \|K \ast \xi_{\varepsilon, \theta}(t, \cdot) - K \ast \xi(t, \cdot)\|_{\alpha + 2} \lesssim \varepsilon^{2\kappa} + \vartheta^\kappa,$$

holds uniformly over $\varepsilon, \vartheta \in (0, 1]$.

**Proof** For any scaling $s$ of $\mathbb{R}^4$ and any $n \in \mathbb{Z}$, define

$$\Lambda_s^n = \left\{ \sum_{j=0}^3 2^{-nj}k_j e_j : k_j \in \mathbb{Z} \right\},$$

where we denote by $e_j$ the $j$th element of the canonical basis of $\mathbb{R}^4$. We choose a wavelet basis $\{\psi_x^{n,s} = 2^{-\frac{3}{2}n} \psi_x^{2-n}, \varphi(\cdot - y), n \geq 0, x \in \Lambda_s^n \cap \mathcal{R}, y \in \Lambda_s^n \cap \mathcal{R}, \psi \in \Psi \}$ as in [Hai14, Section 3.2] on $\mathbb{R}^4$. Writing $\Psi_* = \Psi \cap \{\varphi\}$, we note that for every $p > 1$, we have the bound

$$E\|\xi_{\varepsilon, \theta} - \xi\|_{1_t \in [0, s], \alpha, \mathcal{R}}^{2p} \leq \sum_{\psi \in \Psi_*} \sum_{n \geq 0} \sum_{x \in \Lambda_s^n \cap \mathcal{R}} E2^{2\alpha np - |s||p|} \langle (\xi_{\varepsilon, \theta} - \xi) 1_{t \in [0, s]}, \psi_x^{n,s} \rangle^{2p} \lesssim \sum_{\psi \in \Psi_*} \sum_{n \geq 0} \sum_{x \in \Lambda_s^n \cap \mathcal{R}} 2^{2\alpha np - |s||p|} (E\|\xi_{\varepsilon, \theta} - \xi\|_{1_t \in [0, s], \psi_x^{n,s}}^2)^p.$$
Here we wrote $\mathcal{R}$ for the 1-fattening of $\mathcal{R}$. By a straightforward calculation we have

$$E|\langle (\xi_\varepsilon - \xi)1_{t \in [0,s)}, \psi^n_x \rangle|^2$$

$$= E\left\{ \sum_{k=0}^{\infty} 1_{t \in [k, (k+1), \frac{1}{\vartheta}]} \int_{[k, \vartheta]}^{(k+1) \vartheta} \xi_\varepsilon(u)du - \xi_\varepsilon(t), 1_{t \in [0,s]} |\psi^n_x|^2 \right\}$$

$$\lesssim E\left\{ \sum_{k=0}^{[\frac{t}{\vartheta}]-1} 1_{t \in [k, (k+1), \frac{1}{\vartheta}]} \int_{[k, \vartheta]}^{(k+1) \vartheta} (\xi_\varepsilon(u) - \xi_\varepsilon(t))du, 1_{t \in [0,s]} |\psi^n_x|^2 \right\}$$

$$+ E\left\{ 1_{t \in \left( [\frac{t}{\vartheta}], [\vartheta], s \right]} \int_{[\vartheta]}^{(\vartheta+1) \vartheta} \xi_\varepsilon(u)du, 1_{t \in [0,s]} |\psi^n_x|^2 \right\}$$

$$+ E|\langle \xi_\varepsilon - \xi, 1_{t \in [0,s]} |\psi^n_x|^2 \rangle|^2 + E|\langle \xi_\varepsilon_\varepsilon - \xi_\varepsilon, 1_{t \in [0,s]} |\psi^n_x|^2 \rangle|^2$$

$$\lesssim I + II + III + IV,$$

where

$$I := E\left\{ \int \sum_{k=0}^{[\frac{t}{\vartheta}]-1} 1_{t \in [k, (k+1), \frac{1}{\vartheta}]} \int_{[k, \vartheta]}^{(k+1) \vartheta} (\psi^n_x(u, y) - \psi^n_x(t, y))\xi_\varepsilon(t)du dtdy \right\},$$

$$II := \frac{1}{\vartheta} \int \left( \int_{[\vartheta]}^{(\vartheta+1) \vartheta} |\psi^n_x(u, y)|du \right)^2 dy,$$

$$III = \int_{[\vartheta]}^{(\vartheta+1) \vartheta} |\psi^n_x(t, y)|^2 dtdy,$$

$$IV := E|\langle \xi_\varepsilon - \xi, 1_{t \in [0,s]} |\psi^n_x|^2 \rangle|^2.$$

Here we used $\|\rho_\varepsilon f\|_{L^2} \leq \|f\|_{L^2}$. Now we estimate each term separately. For $I$ we have

$$I \lesssim \int \int \left( \sum_{k=0}^{[\frac{t}{\vartheta}]-1} 1_{t \in [k, (k+1), \frac{1}{\vartheta}]} \int_{[k, \vartheta]}^{(k+1) \vartheta} (\psi^n_x(u, y) - \psi^n_x(t, y))du \right)^2 dtdy$$

$$\lesssim \int \int \left( \sum_{k=0}^{[\frac{t}{\vartheta}]-1} 1_{t \in [k, (k+1), \frac{1}{\vartheta}]} \int_{[k, \vartheta]}^{(k+1) \vartheta} |D_\tilde{u}\psi^n_x(\tilde{u}, y)|d\tilde{u}du \right)^2 dtdy$$

$$\lesssim \frac{1}{\vartheta^2} \int \sum_{k=0}^{[\frac{t}{\vartheta}]-1} \int_{[k, \vartheta]}^{(k+1) \vartheta} \left( \int_{[k, \vartheta]}^{(k+1) \vartheta} |D_\tilde{u}\psi^n_x(\tilde{u}, y)|d\tilde{u} \right)^2 dtdy$$

$$\lesssim \vartheta^2 \int_{[\vartheta]}^{(\vartheta+1) \vartheta} \int |D_\tilde{u}\psi^n_x(\tilde{u}, y)|^2 d\tilde{u}dy$$

$$\lesssim 1 \wedge (2^n s) \wedge (2^n \vartheta^2).$$

Similarly,

$$II + III \lesssim 1 \wedge (2^n s) \wedge (2^n \vartheta^2).$$

By the proof in [Hai14, Proposition 9.5] we also have

$$IV \lesssim 1 \wedge (2^n s) \wedge (2^n \varepsilon^2),$$

16
which implies that

\[
E[(\xi_{\epsilon, \theta} - \xi)1_{t \in [0,s]}, \psi^{n, \delta}_x)^2] \leq 1 \wedge (2^{2n} s) \wedge (2^{2n} \varepsilon^2) + 1 \wedge (2^{2n} s) \wedge (2^{2n} \vartheta).
\]

Thus it follows that for \(0 < \kappa < -\frac{5}{2} - \alpha\),

\[
E\|(\xi_{\epsilon, \theta} - \xi)1_{t \in [0,s]}\|^{2p}_{\alpha + 2} \lesssim \left[ \vartheta^{\frac{5}{2} - \alpha - \kappa} + \varepsilon^{\frac{5}{2} - \alpha - \kappa/2} s^{5/2 - \kappa/2} \right].
\]

Then the required bound (3.4) follows from Kolomogorov criterion by choosing \(p\) large enough.

Now we prove the second result: we choose scaling \(s = (1,1,1)\) and choose a wavelet basis on \(\mathbb{R}^3\): \(\psi^{n, \delta}_x = 2^{2n} \psi(2^n (\cdot - x)), \varphi(\cdot - y), n \geq 0, x \in \Lambda^n_s, y \in \Lambda^n_s, \psi \in \Psi\) as in [Hai14, Section 3.2] on \(\mathbb{R}^3\) with \(\Lambda^n_s = \{ \sum_{j=1}^3 2^{-nk} \varepsilon_j : k_j \in \mathbb{Z}\}\). Writing \(\Psi_* = \Psi \cap \{ \varphi \}\). We would like to estimate \(E\| (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(s, \cdot) \|^{2p}_{a+2}\) for \(t > s \geq 0\) and use Kolmogorov continuity test. Here we only consider the case that \(s = 0\) for simplicity. For general \(s\), we could obtain the desired estimates similarly. We note that for every \(p > 1\), we have the bound

\[
E\| (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(0, \cdot) \|^{2p}_{a+2} \leq \sum_{\psi \in \Psi_*} \sum_{n \geq 0} \sum_{x \in \Lambda^n_s} E 2^{(a+1)p(n+|s|p)} \| ((K \ast \xi_{\epsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(0, \cdot), \psi^{n, \delta}_x) \|_{a+2}^{2p}.
\]

We have the following identity

\[
K \ast \xi_{\epsilon, \theta}(t, y) = \sum_{k = -\infty}^{[\frac{t}{T}] - 1} \int_{k\theta}^{(k+1)\theta} \int_{k\theta}^{(k+1)\theta} K(t - u, y - y_1) \frac{1}{\vartheta} \int_{k\theta}^{(k+1)\theta} \xi_{\epsilon}(u_1, y_1) du_1 dy_1 du
\]

\[
+ \int_{[\frac{t}{T}]\theta}^{t} \int_{[\frac{t}{T}]\theta}^{(\frac{t}{T}+1)\theta} K(t - u, y - y_1) \frac{1}{\vartheta} \int_{[\frac{t}{T}]\theta}^{(\frac{t}{T}+1)\theta} \xi_{\epsilon}(u_1, y_1) du_1 dy_1 du,
\]

which implies that

\[
E\| (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\epsilon, \theta} - K \ast \xi)(0, \cdot, \psi^{n, \delta}_x(\cdot)) \|^2 \lesssim J_1 + J_2 + J_3 + J_4,
\]

where

\[
J_1 := E\| \sum_{k = 0}^{[\frac{t}{T}] - 1} \int_{k\theta}^{(k+1)\theta} \frac{1}{\vartheta} \int_{k\theta}^{(k+1)\theta} [K(t - u, \cdot - y_1) - K(t - u_1, \cdot - y_1)] du
\]

\[
\xi_{\epsilon}(u_1, y_1) du_1 dy_1, \psi^{n, \delta}_x(\cdot)) |^2
\]

\[
J_2 := E\| \int_{[\frac{t}{T}]\theta}^{t} K(t - u, \cdot - y_1) du_1 dy_1, \psi^{n, \delta}_x(\cdot)) |^2,
\]

\[
J_3 := E\| \int_{[\frac{t}{T}]\theta}^{t} K(t - u_1, \cdot - y_1) \xi_{\epsilon}(u_1, y_1) du_1 dy_1, \psi^{n, \delta}_x(\cdot)) |^2.
\]
\[ J^4 := E\left| \int \sum_{k=0}^{\frac{1}{2}-1} \int_{k^2}^{(k+1)^2} \left[ K(t-u, \cdot - y_1) - K(t-u_1, \cdot - y_1) - K(u, \cdot - y_1) + K(-u, \cdot - y_1) \right] du \right|^2. \]

Now we bound each term separately: For \( J^1 \) we have

\[ J^1 \lesssim \int \sum_{k=0}^{\frac{1}{2}-1} \int_{k^2}^{(k+1)^2} \left| K(t-u, \cdot - y_1) - K(t-u_1, \cdot - y_1) \right| du \left( \frac{1}{y_1} \right)^{1-\alpha} |\psi_x(n,\tilde{\alpha}(\cdot))| \, du_1 \, dy_1. \]

We introduce the notation: for \((t, y) \in \mathbb{R}^4, \alpha \in \mathbb{R}^+\)

\[ G^0_\alpha(t, y) := \frac{1}{|t|^{\frac{3}{2}} + |y|^\alpha} 1_{\{|t| + |y|^2 \leq C\}}. \]

Here \( C \) is a constant. Now we use [Hai14, Theorem 10.18] to control \(|K(t-u, y-y_1) - K(t-u_1, y-y_1)|\) by \( \vartheta^\delta(G^{(3+\delta)}_0(t-u, y-y_1) + G^{(3+\delta)}_0(t-u_1, y-y_1))\), which implies that

\[ J^1 \lesssim \int \sum_{k=0}^{\frac{1}{2}-1} \int_{k^2}^{(k+1)^2} \left( G^{(3+\delta)}_0(t-u, y-y_1) + G^{(3+\delta)}_0(t-u_1, y-y_1) \right) \vartheta^\delta \left( \frac{1}{y_1} \right)^{1-\alpha} |\psi_x(n,\tilde{\alpha}(\cdot))| \, du_1 \, dy_1. \]

Here and in the following we use the first \( \int \) to denote the integration with respect to \( dy_1 dy dy \)

if there’s no confusion. Since \(|t-u|^{1-\beta} |y-y_1|^{2+\delta+\beta} \lesssim |t-u|^{\frac{3}{2}+\beta} + |y-y_1|^{3+\delta}\) for \(0 < \beta < 1\),

the above term can be bounded by

\[ \vartheta^\delta \int \sum_{k=0}^{\frac{1}{2}-1} \int_{k^2}^{(k+1)^2} \left( G^{(3+\delta)}_0(t-u, y-y_1) + G^{(3+\delta)}_0(t-u_1, y-y_1) \right) \vartheta^\delta \left( \frac{1}{y_1} \right)^{1-\alpha} |\psi_x(n,\tilde{\alpha}(\cdot))| \, du_1 \, dy_1 \]

for \(\beta, \delta > 0, 2\beta+4\delta < -(2\alpha+5)\), which is the required bound for \( J^1 \). Here in the first inequality we used Young’s inequality and in the last inequality we used [Hai14, Lemma 10.14]. For \( J^2, J^3 \)
by similar calculations and the fact that $|K(z)| \lesssim \|z\|_{3}^{-3}$ we have

$$J^2 + J^3 \lesssim \int \int_{[\frac{1}{2}]^{\theta}} \int_{[\frac{1}{2}]^{\theta}} |K(t, \cdot, y_1)|^2 du_1 dy_1$$

$$+ \int \int_{[\frac{1}{2}]^{\theta}} |\langle K(t, u_1, \cdot, y_1), \psi_{x}^{n_{y}}(\cdot) \rangle|^2 du_1 dy_1$$

$$\lesssim \int \int_{[\frac{1}{2}]^{\theta}} \int_{[\frac{1}{2}]^{\theta}} G_0^{(3)}(t - u, y - y_1)G_0^{(3)}(t - \bar{u}, \bar{y} - y_1)du_1 dy_1$$

$$+ \int \int_{[\frac{1}{2}]^{\theta}} |G_0^{(3)}(t - u_1, y - y_1)G_0^{(3)}(t - u_1, \bar{y} - y_1)|^2 du_1 dy_1 dy_1.$$

Since $|t - u|^{1-\beta - \delta} |y - y_1|^{2+\delta + \beta} \lesssim |t - u|^\frac{3}{2} + |y - y_1|^3$ for $0 < \beta < 1 - \delta$, the above term can be bounded by

$$\int \int \int_{[\frac{1}{2}]^{\theta}} \int \int \int_{[\frac{1}{2}]^{\theta}} \frac{1}{|t - u|^{1-\beta - \delta}} |y - y_1|^{2+\delta + \beta} |\bar{y} - y_1|^{2+\delta + \beta} du_1 dy_1 dy_1$$

$$\lesssim g_\delta |t|^\beta \int \int \int_{[\frac{1}{2}]^{\theta}} \|\psi_{x}^{n_{y}}(y)\psi_{x}^{n_{y}}(\bar{y})\||y - \bar{y}|_{3}^{-1-2\delta - 2\beta} dyd\bar{y},$$

which is also the desired bound for $J^2 + J^3$. Here we also used [Hai14, Lemma 10.14] in the last inequality. Now we consider $J^4$:

$$J^4 \lesssim \int \int \sum_{k=-[\frac{1}{2}]}^{-1} \int_{[k, (k+1)^{\theta}]^{\theta}} \int_{[k, (k+1)^{\theta}]^{\theta}} |K(t - u, \cdot, y_1) - K(t - u_1, \cdot, y_1) - K(-u, \cdot, y_1)|$$

$$+ K(-u_1, \cdot, y_1)|du_1 \frac{1}{\|u_1\|_{k\theta, (k+1)^{\theta}, \psi_{x}^{n_{y}}(\cdot)}|^2 du_1 dy_1$$

$$\lesssim \int \int \sum_{k=-[\frac{1}{2}]}^{-1} \frac{1}{\|u_1\|_{k\theta}^2} \int_{[k, (k+1)^{\theta}]^{\theta}} \int_{[k, (k+1)^{\theta}]^{\theta}} \int_{[k, (k+1)^{\theta}]^{\theta}} |K(t - u, y - y_1) - K(t - u_1, y - y_1)|$$

$$+ K(-u, y - y_1) + K(-u_1, y - y_1) |du_1 \int_{[k, (k+1)^{\theta}]^{\theta}} |K(t - \bar{u}, \bar{y} - y_1) - K(t - u_1, \bar{y} - y_1)|$$

$$+ K(-\bar{u}, \bar{y} - y_1) + K(-u_1, \bar{y} - y_1) |du_1 \psi_{x}^{n_{y}}(y) \psi_{x}^{n_{y}}(\bar{y})| du_1 dy_1 dy_1 dy_1 dy_1.$$
imply that

$$J^4 \lesssim \int \sum_{k=-[\frac{1}{2}]}^{-1} \frac{1}{|y|^2} \int_{k\theta}^{(k+1)\theta} \int_{k\theta}^{(k+1)\theta} \vartheta^{|t|^\beta}(G_0^{(3+\delta+\beta)}(t-u, y-y_1) + G_0^{(3+\delta+\beta)}(t-u_1, y-y_1))$$

$$+ G_0^{(3+\delta+\beta)}(-u, y-y_1) + G_0^{(3+\delta+\beta)}(-u_1, y-y_1))du$$

$$\int_{k\theta}^{(k+1)\theta} \left( G_0^{(3+\delta+\beta)}(t-\bar{u}, \bar{y}-y_1) + G_0^{(3+\delta+\beta)}(t-u_1, \bar{y}-y_1) \right. $$

$$+ G_0^{(3+\delta+\beta)}(-\bar{u}, \bar{y}-y_1) + G_0^{(3+\delta+\beta)}(-u_1, \bar{y}-y_1))d\bar{u}|\psi_x^n(y)|\psi_x^n(\bar{y})|dy_1dy_1(dy_1dy_1).$$

Since $$|t-u|^{\frac{1}{2+2\delta+\beta}} \lesssim |t-u|^{\frac{1}{2+2\delta+\beta}} + |y-y_1|^{3+\delta+\beta}$$ for $$\beta > 0, 0 < \delta < 1$$, we have

$$G_0^{(3+\delta+\beta)}(t-u, y-y_1) \lesssim \frac{1}{|t-u|^{\frac{1}{2+2\delta+\beta}} |y-y_1|^{2+2\delta+\beta}}.$$

which combining with

$$\frac{1}{|t-u|^{\frac{1}{2+2\delta+\beta}} |y-y_1|^{2+2\delta+\beta}} \leq \frac{1}{|t-u|^{1-\delta}} + \frac{1}{|t-u|^{1-\delta}}$$

implies that the above term can be bounded by

$$\int \int_{C} \vartheta^{|t|^\beta} \left( \frac{1}{|t-u|^{1-\delta}} |y-y_1|^{2+2\delta+\beta} + \frac{1}{|t-u|^{1-\delta}} |y-y_1|^{2+2\delta+\beta} \right) du$$

$$\lesssim \int \int |\vartheta^{|t|^\beta} \left( |\psi_x^n(y)|\psi_x^n(\bar{y})||y-y_1|^{\frac{1}{2+2\delta+\beta}} dy_1dy_1.$$

Combining the above estimates we obtain that

$$E \langle \langle (K \ast \xi_{\varepsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\varepsilon, \theta} - K \ast \xi)(0, \cdot), \psi_x^n \rangle \rangle^2$$

$$\lesssim E \langle \langle (K \ast \xi_{\varepsilon, \theta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\varepsilon, \theta} - K \ast \xi)(0, \cdot), \psi_x^n \rangle \rangle^2 $$

$$+ E \langle \langle (K \ast \xi - K \ast \xi)(t, \cdot) - (K \ast \xi - K \ast \xi)(0, \cdot), \psi_x^n \rangle \rangle^2 $$

$$\lesssim (\vartheta^{|t|^\beta} \int \int |\psi_x^n(y)|\psi_x^n(\bar{y})||y-y_1|^{\frac{1}{2+2\delta+\beta}} dy_1dy_1$$

$$+ \int \int |\psi_x^n(y)|\psi_x^n(\bar{y})|\hat{\mathcal{W}}^{(2)}(t, y) - \hat{\mathcal{W}}^{(2)}(0, y), \hat{\mathcal{W}}^{(2)}(t, \bar{y}) - \hat{\mathcal{W}}^{(2)}(0, \bar{y})| dy_1dy_1$$

$$\lesssim (\varepsilon^{2\delta} + \vartheta^{|t|^\beta} \int \int |\psi_x^n(y)|\psi_x^n(\bar{y})||y-y_1|^{\frac{1}{2+\delta-2\beta}} dy_1dy_1$$

$$\lesssim (\varepsilon^{2\delta} + \vartheta^{|t|^\beta} \int \int |\psi_x^n(y)|\psi_x^n(\bar{y})||y-y_1|^{\frac{1}{2+\delta-2\beta}} dy_1dy_1,$$

where $$\beta, \delta > 0, 2\beta + 4\delta < -(2\alpha + 5),$$

$$\hat{\mathcal{W}}^{(2)}(t, y; u, y_1) = K \ast \rho_{\varepsilon}(t-u, y-y_1) - K(t-u, y-y_1),$$

and we used [Hai14, Lemmas 10.17, 10.18] and the interpolation to deduce that

$$|\langle \hat{\mathcal{W}}^{(2)}(t, y) - \hat{\mathcal{W}}^{(2)}(0, y), \hat{\mathcal{W}}^{(2)}(t, \bar{y}) - \hat{\mathcal{W}}^{(2)}(0, \bar{y}) \rangle |$$

$$\lesssim \int \int |t|^\beta \varepsilon^{2\delta} \left( G_0^{(3+\delta+\beta)}(-u_1, y-y_1) + G_0^{(3+\delta+\beta)}(-u_1, y-y_1) \right)$$

$$\left. (G_0^{(3+\delta+\beta)}(t-u_1, y-y_1) + G_0^{(3+\delta+\beta)}(-u_1, \bar{y}-y_1))du_1dy_1. \right)$$
Thus the above estimates yield that
\[
E \| (K \ast \xi_{\varepsilon, \vartheta} - K \ast \xi)(t, \cdot) - (K \ast \xi_{\varepsilon, \vartheta} - K \ast \xi)(0, \cdot) \|_{\alpha+2}^{2p} \leq \sum_{\psi \in \Psi^*} \sum_{n \geq 0} 2^{2(\alpha+\frac{3}{2})np+\frac{3}{2}n(\varepsilon^{2\delta} + \vartheta^\delta)p} |t|^{\beta p} 2^{-3np+np(4\delta+2\beta)}.
\]
Then the results follow from Kolmogorov continuity test (in time) if we choose \( p \) sufficiently large.

In [Hai14, Theorem 10.22] a random model \( \hat{Z} \) has been obtained by taking the limit of the models associated with the convolution approximation. Define \( \Phi = RS^L(u_0, \hat{Z}) \). Then \( \Phi \) is a local solution to the dynamical \( \Phi^3 \) model. In our case we also have the following main convergence result at the level of models:

**Theorem 3.7** Let \( \mathcal{F} \) be the regularity structure associated to the dynamical \( \Phi^3 \) model, and \( \rho(t, x) = \rho_1(t) \rho_2(x) \), let \( \xi_{\varepsilon, \vartheta} \) as above and let \( Z_{\varepsilon, \vartheta} \) be the associated model. If \( \varepsilon^2 \leq C_0 \varepsilon \) for \( C_0 \) independent of \( \varepsilon, \vartheta \), then there exist choices of \( C_1(\varepsilon, \vartheta), C_2(\varepsilon, \vartheta) \) such that \( \hat{Z}_{\varepsilon, \vartheta} = Z_{\varepsilon, \vartheta} \rightarrow \hat{Z} \) in probability.

More precisely, for any \( \theta < -\frac{5}{2} - \alpha \), any compact set \( \mathfrak{R} \), and any \( \gamma < r \) one has the bound
\[
E \| Z_{\varepsilon, \vartheta}; \hat{Z} \|_{\gamma; \mathfrak{R}} \lesssim \varepsilon^{\theta} + \vartheta^{\theta/2},
\]
uniformly over \( \varepsilon, \vartheta \in (0, 1] \) satisfying \( \varepsilon^2 \leq C_0 \varepsilon \).

The proof of Theorem 3.7 is the content of Section 4 below.

**Remark 3.8** (i) If \( \rho(t, x) = \delta(t) \rho_2(x) \) the convergence result in Proposition 3.6 and Theorem 3.7 still holds in this case. In fact, if \( |K(z)| \lesssim \|z\|_\zeta^\delta \) for \( -4 < \zeta < 0 \) it is sufficient to control \( |K \ast \rho_2(x) - K| \) by \( (t^{-\frac{5}{2}} \varepsilon^{\zeta-\zeta} |x|^{\zeta+\delta}) \wedge |t|^{\frac{5}{2}} \) for \( \zeta + \delta > -3 \) and obtain similar calculations as in Section 4. Here \( \rho_2(\varepsilon y) = \varepsilon^{-3} \rho_2(\varepsilon y) \).

(ii) It is possible to show that \( C_1(\varepsilon, \vartheta) \sim \varepsilon^{-1} \) and \( C_2(\varepsilon, \vartheta) \sim |\log \varepsilon| \).

**Proof of Theorem 1.1** The proof of the theorem is essentially a collection of the results of this paper. As obtained in Proposition 3.5 \( RS^L(u_0, Z_{\varepsilon, \vartheta}^L) = \Phi_{\varepsilon, \vartheta} \). Define \( \Phi = RS^L(u_0, \hat{Z}) \). By the continuity of the map \( R \) and Proposition 2.8 we obtain that there exists a sequence of random time \( \tau_L \) converging to the explosion time \( \tau \) of \( \Phi \) such that
\[
\sup_{t \in [0, \tau_L]} \| \Phi_{\varepsilon, \vartheta} - \Phi \|_\eta \rightarrow^P 0, \quad \text{as } \varepsilon, \vartheta \rightarrow 0.
\]

## 4 Convergence of the renormalised model

In this section we will prove Theorem 3.7. To prove the main convergence result Theorem 3.7 we first prove some useful lemmas. We will use the following notations:
Define for \((t, y), (t_2, y_2) \in \mathbb{R}^4\)

\[
K_{\varepsilon, \vartheta}(t, y, t_2, y_2) := \sum_{k=-\infty}^{[\frac{k}{2}]-1} \int_{k\vartheta}^{(k+1)\vartheta} \int K(t - u, y - y_1) \frac{1}{\vartheta^2} \int_{k\vartheta}^{(k+1)\vartheta} \rho_\varepsilon(u_1 - t_2, y_1 - y_2) du_1 dy_1 du
\]

\[
+ \int_{(\frac{1}{2})\vartheta}^{t} \int \int K(t - u, y - y_1) \frac{1}{\vartheta^2} \int_{(\frac{1}{2})\vartheta}^{(\frac{1}{2})\vartheta} \rho_\varepsilon(u_1 - t_2, y_1 - y_2) du_1 dy_1 du
\]

:= K_{\varepsilon, \vartheta}^{(1)}(t, y, t_2, y_2) + K_{\varepsilon, \vartheta}^{(2)}(t, y, t_2, y_2).

Then for \(\tau = \mathcal{I}(\Xi) = \Psi\) we have

\[
(\hat{\Pi}^{(\varepsilon, \vartheta)}(\Psi))(z) = K \ast \xi_{\varepsilon, \vartheta}(z) = \int K_{\varepsilon, \vartheta}(z, z_1) \xi(z_1) dz_1.
\]

For \(\varphi\) smooth and \(x \in \mathbb{R}^4\) we have that

\[
E|\langle K \ast \xi_{\varepsilon, \vartheta}, \varphi_x^\lambda \rangle|^2 = \int \int f^{(\varepsilon, \vartheta)}(z, \bar{z}) \varphi_\lambda^x(z) \varphi_\lambda^x(\bar{z}) dz d\bar{z}.
\]

Here for \(z = (t, y), \bar{z} = (\bar{t}, \bar{y})\)

\[
f^{(\varepsilon, \vartheta)}(z, \bar{z}) := J^1 + J^2 + J^3 + J^4,
\]

with

\[
J^1 = \int K_{\varepsilon, \vartheta}^{(1)}(z, z_1) K_{\varepsilon, \vartheta}^{(1)}(z, z_1) dz_1, \quad J^2 = \int K_{\varepsilon, \vartheta}^{(2)}(z, z_1) K_{\varepsilon, \vartheta}^{(2)}(z, z_1) dz_1,
\]

\[
J^3 = \int K_{\varepsilon, \vartheta}^{(1)}(z, z_1) K_{\varepsilon, \vartheta}^{(2)}(z, z_1) dz_1, \quad J^4 = \int K_{\varepsilon, \vartheta}^{(2)}(z, z_1) K_{\varepsilon, \vartheta}^{(1)}(z, z_1) dz_1.
\]

(4.1)

Recall that \(\rho(t, x) = \rho_1(t)\rho_2(x)\). By a similar argument as the proof in [Hai14, Lemma 10.17] we first give an estimate for \(K \ast \rho_{2, \varepsilon}\), which is required for the estimate of \(f^{(\varepsilon, \vartheta)}\). Here \(\rho_{2, \varepsilon}(y) = \varepsilon^{-3}\rho_2(\frac{y}{\varepsilon})\).

**Lemma 4.1** If \(|K| \lesssim \|z\|_s\) for \(\zeta \in (-4, 0)\), then \(K \ast \rho_{2, \varepsilon}\) has bounded spatial derivatives of all orders. Furthermore, one has the bound

\[
|K \ast \rho_{2, \varepsilon}(z)| \leq Ct^{-\frac{\zeta}{2}} \|z\|_s^{\zeta + \delta},
\]

for \(0 < \delta < 1, \zeta + \delta > -3\).

**Proof** We can write

\[
K \ast \rho_{2, \varepsilon}(t, x) = \int K(t, x - y) \rho_{2, \varepsilon}(y) dy.
\]

We use the notation \(z = (t, x)\) and \(|K(t, x - y)|\) can be bounded by \(C|t|^{\frac{\zeta}{2}}\), and

\[
|K \ast \rho_{2, \varepsilon}(t, x)| \leq C|t|^{\frac{\zeta}{2}}.
\]

follows from the fact that \(\rho_{2, \varepsilon}\) integrates to 1. Without loss of generality we assume that \(\rho_2\) is supported in the set \(\{x : |x| \leq 1\}\). For \(|x| \geq 2\varepsilon\), we have \(|x - y| \geq \frac{|x|}{2}\), which implies that

\[
|K \ast \rho_{2, \varepsilon}(t, x)| \leq C t^{-\frac{\zeta}{4}} |x|^{\zeta + \delta}.
\]
For $|x| \leq 2\varepsilon$ we use the fact that $|\rho_{2,\varepsilon}|$ is bounded by a constant multiple of $\varepsilon^{-3}$

$$|K \ast \rho_{2,\varepsilon}(t, x)| \lesssim \varepsilon^{-3} \int_{|y| \leq 3\varepsilon} t^{-\frac{3}{2}} |y|^\xi dy \lesssim C t^{-\frac{3}{2}} |x|^\xi.$$

Combining all the estimates the result follows. \hfill \Box

In the following we give useful estimates for $f^{(\varepsilon, \vartheta)}$, which is used for the proof of Theorem 3.7.

**Lemma 4.2** If $\varepsilon^2 \leq C_0 \vartheta$ for some $C_0$ independent of $\varepsilon, \vartheta$, we obtain that for every $\delta > 0$

$$|f^{(\varepsilon, \vartheta)}(z, \bar{z})| \lesssim \|z - \bar{z}\|_{\mathcal{S}}^{-1-\delta}. \quad (4.2)$$

**Proof** In the following we use the notations $z = (t, y), \bar{z} = (\bar{t}, \bar{y})$. Since $\varepsilon^2 \leq C_0 \vartheta$, we have for $\rho_{1,\varepsilon}(t) := \varepsilon^{-2} \rho_1(\frac{t}{\varepsilon})$ that when $|k - k_1| \leq C(C_0)$

$$\int_{k_0}^{(k+1)\vartheta} \int_{k_1\vartheta}^{(k+1)\vartheta} \rho_{1,\varepsilon} \ast \rho_{1,\varepsilon}(u_1 - u_2) du_1 du_2 \neq 0,$$

which implies that

$$J^1 \lesssim \sum_{k = -\infty}^{[\frac{t}{\varepsilon}] - 1} \sum_{k_1 = -\infty}^{[\frac{\bar{t}}{\varepsilon}] - 1} \int_{k \vartheta}^{(k+1)\vartheta} \int_{k_1 \vartheta}^{(k+1)\vartheta} |K(t - u, y - y_1)| du \int_{k_1 \vartheta}^{(k+1)\vartheta} |K(\bar{t} - \bar{u}, \bar{y} - y_2)| d\bar{u}$$

$$\lesssim \sum_{k = -[\frac{\bar{t}}{\varepsilon}]}^{[\frac{t}{\varepsilon}] - 1} \sum_{k_1 = -[\frac{\bar{y}}{\varepsilon}]}^{[\frac{y}{\varepsilon}] - 1} \int_{k \vartheta}^{(k+1)\vartheta} \int_{k_1 \vartheta}^{(k+1)\vartheta} \frac{1}{\vartheta} \int_{k \vartheta}^{(k+1)\vartheta} |t - u|^{-\frac{1}{2}} G_0^{(3-\delta)}(t - u, y - y_1) du$$

$$\int_{k_1 \vartheta}^{(k+1)\vartheta} |\bar{t} - \bar{u}|^{-\frac{1}{2}} G_0^{(3-\delta)}(\bar{t} - \bar{u}, \bar{y} - y_1) d\bar{u} dy_1, \quad (4.3)$$

for $\delta > 0$, where we used Lemma 4.1 in the last inequality. Now we estimate this term in the following three cases:

If $\bar{t} - t \geq (2C(C_0) + 2)\vartheta$, we have that $|u - \bar{u}| \leq (C(C_0) + 1)\vartheta$, which implies that

$$\bar{t} - \bar{u} > t - u. \quad (4.4)$$

Furthermore, we also have that

$$\bar{t} - \bar{u} = \bar{t} - t + t - u + u - \bar{u} \geq (C(C_0) + 1)\vartheta \geq \bar{u} - u + u - t \geq \bar{u} - t,$$

which also implies that

$$|\bar{t} - \bar{u}| \geq \frac{|\bar{t} - t|}{2}. \quad (4.5)$$
By (4.4), (4.5) and \((t-u)^{1-3\delta/4}|y-y_1|^{1+\delta/2} \lesssim |t-u|^3 - \delta + |y-y_1|^{3-\delta}\) we obtain that

\[
J^1 \lesssim \sum_{k=-(\frac{3}{2})}^{(\frac{3}{2})-1} \int_{\frac{1}{k}}^{(k+1)\theta} \left( \frac{1}{(t-u)^{1-\delta/4}|y-y_1|^{1+\delta/2}} \frac{1}{(t-t)^{1/2} + \frac{\delta}{2} |y-y_1|^{2+\delta}} \right) du dy_{1|y_1| \leq C} \tag{4.6}
\]

valids for every \(\delta > 0\), where we used [Hai14, Lemma 10.14] in the last inequality.

If \(t-\bar{t} \geq (2C(C_0) + 2)\theta\), we also obtain the same bounds.

If \(|\bar{t} - t| \leq (2C(C_0) + 2)\theta\), we have for every \(\delta > 0\)

\[
\int_{\frac{1}{k}}^{(k+1)\theta} \frac{1}{(t-u)^{1-\delta/4}|y-y_1|^{1+\delta/2}} \frac{1}{(t-t)^{1/2} + \frac{\delta}{2} |y-y_1|^{2+\delta}} du dy_{1|y_1| \leq C} \lesssim |t-\bar{t}|^{-\frac{1}{2} - \frac{\delta}{2}},
\]

which implies that

\[
J^1 \lesssim \left( \sum_{k=-(\frac{3}{2})}^{(\frac{3}{2})-1} \int_{\frac{1}{k}}^{(k+1)\theta} \left( \frac{1}{(t-u)^{1-\delta/4}|y-y_1|^{1+\delta/2}} \frac{1}{(t-t)^{1/2} + \frac{\delta}{2} |y-y_1|^{2+\delta}} \right) du dy_{1|y_1| \leq C} \right)
\]

\[
\lesssim |t-\bar{t}|^{-\frac{1}{2} - \frac{\delta}{2}} \wedge |y-y_1|^{-1-\delta},
\]

valids for every \(\delta > 0\), where we used [Hai14, Lemma 10.14] and Young’s inequality to deduce that \(\frac{1}{(t-u)^{1/2} + \frac{\delta}{2}} \lesssim \frac{1}{(t-u)^{1/2} + \frac{\delta}{2}} + \frac{1}{(t-u)^{1/2} + \frac{\delta}{2}}\) in the last inequality. Combining (4.6) and (4.7) we obtain that

\[
J^1 \lesssim \|z - \tilde{z}\|_{s}^{-1-\delta},
\]

valid for every \(\delta > 0\).

We now turn to \(J^2\) and \(J^2 \neq 0\) if and only if \(|t-\bar{t}| \leq (2C(C_0) + 2)\theta\). Similar arguments as the estimate (4.7) imply that for every \(\delta > 0\)

\[
J^2 \lesssim \left( \int_{\frac{1}{k}}^{(k+1)\theta} \frac{1}{(t-u)^{1/2} + \frac{\delta}{2}} |y-y_1|^{2+\delta} \frac{1}{(t-t)^{1/2} + \frac{\delta}{2} |y-y_1|^{2+\delta}} du dy_{1|y_1| \leq C} \right)
\]

\[
\lesssim |t-\bar{t}|^{-\frac{1}{2} - \frac{\delta}{2}} \wedge |y-y_1|^{-1-\delta}.
\]

\(J^3, J^4\) can be estimated similarly. Thus (4.2) follows.
Moreover we have the following estimate:

**Lemma 4.3**  
If \( \varepsilon^2 \leq C_0 \theta \) for some \( C_0 \) independent of \( \varepsilon, \theta \), we obtain that

\[
|f^{(\varepsilon, \theta)}(z, \bar{z}) - f(z, \bar{z})| \lesssim (\theta + \varepsilon^{2\theta}) \|z - \bar{z}\|^{-1-2\theta - \delta},
\]

holds uniformly over \( \varepsilon, \theta \in (0, 1) \) satisfying \( \varepsilon^2 \leq C_0 \theta \), provided that \( \theta < 1 \) and that \( \delta > 0 \), where \( f(z, \bar{z}) = K \ast K(z - \bar{z}) \).

**Proof**  
We have

\[
|f^{(\varepsilon, \theta)}(z, \bar{z}) - f(z, \bar{z})| 
\lesssim |f^{(\varepsilon, \theta)}(z, \bar{z}) - f^{(\varepsilon)}(z, \bar{z})| + |f^{(\varepsilon)}(z, \bar{z}) - f(z, \bar{z})|
\]

where \( f^{(\varepsilon)}(z, \bar{z}) = K_\varepsilon \ast K_\varepsilon(z - \bar{z}) \), \( K_\varepsilon = K \ast \rho_\varepsilon \). For \( (t, y), (t_2, y_2) \in \mathbb{R}^4 \) denote

\[
K_\varepsilon(t - t_2, y - y_2) = \int \int_{-\infty}^{\frac{1}{2}\theta} K(t - u, y - y_1)\rho_\varepsilon(u - t_2, y_1 - y_2)du dy_1 \\
+ \int \int_{\frac{1}{2}\theta}^{t} K(t - u, y - y_1)\rho_\varepsilon(u - t_2, y_1 - y_2)du dy_1 \\
= K^{(1)}_\varepsilon(t, y, t_2, y_2) + K^{(2)}_\varepsilon(t, y, t_2, y_2).
\]

Here \( |f^{(\varepsilon, \theta)}(z, \bar{z}) - f^{(\varepsilon)}(z, \bar{z})| \) can be separated as \( J^1, J^2, J^3, J^4 \) with

\[
J^1 = \left| \int (K^{(1)}_\varepsilon(z, z_1)K^{(1)}_{\varepsilon, \theta}(\bar{z}, z_1) - K^{(1)}_\varepsilon(z, z_1)K^{(1)}_{\varepsilon, \theta}(\bar{z}, z_1))dz_1 \right|,
\]

\[
J^2 = \int \left| K^{(2)}_\varepsilon(z, z_1)K^{(2)}_{\varepsilon, \theta}(\bar{z}, z_1) - K^{(2)}_\varepsilon(z, z_1)K^{(2)}_{\varepsilon, \theta}(\bar{z}, z_1) \right|dz_1,
\]

\[
J^3 = \int \left| K^{(1)}_\varepsilon(z, z_1)K^{(2)}_{\varepsilon, \theta}(\bar{z}, z_1) - K^{(1)}_\varepsilon(z, z_1)K^{(2)}_{\varepsilon, \theta}(\bar{z}, z_1) \right|dz_1,
\]

\[
J^4 = \int \left| K^{(2)}_\varepsilon(z, z_1)K^{(1)}_{\varepsilon, \theta}(\bar{z}, z_1) - K^{(2)}_\varepsilon(z, z_1)K^{(1)}_{\varepsilon, \theta}(\bar{z}, z_1) \right|dz_1.
\]

Each term can be estimated as in the proof of Lemma 4.2. We take \( J^1 \) as an example:

\[
J^1 = \left| \sum_{k = -\infty}^{[\frac{1}{2}\theta]-1} \sum_{k_1 = -\infty}^{[\frac{1}{2}\theta]-1} \int \int_{k\theta}^{(k+1)\theta} \int \int_{k_1\theta}^{(k_1+1)\theta} \frac{1}{\theta^2} \int_{k\theta}^{(k+1)\theta} \int_{k_1\theta}^{(k_1+1)\theta} [K(t - u, y - y_1) \\
- K(t - u_1, y - y_1)K(\bar{t} - \bar{u}, \bar{y} - y_2) + K(t - u_1, y - y_1)(K(\bar{t} - \bar{u}, \bar{y} - y_2) \\
- K(\bar{t} - u_2, \bar{y} - y_2))] dud\bar{u}\rho_\varepsilon \ast \rho_\varepsilon(u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right|.
\]

By [Hai14, Lemma 10.18] we could control \( |K(t - u, y - y_1) - K(t - u_1, y - y_1)| \) by \( \phi^\theta(G_0^{(3+2\theta)}(t - u, y - y_1) + G_0^{(3+2\theta)}(t - u_1, y - y_1)) \) provided that \( \theta > 0 \). Furthermore by Lemma 4.1 and [Hai14,}
Lemma 10.14] we obtain
\[
J^1 \lesssim \sum_{k=-\infty}^{[\frac{1}{\delta}]^{-1}} \frac{}{[\frac{1}{\delta}]^{-1}} \sum_{k=1}^{[\frac{1}{\delta}]^{-1}} \int \frac{\partial^\theta}{\partial \vartheta} \int_{k \delta}^{(k+1)\delta} |t - u|^{-\frac{\delta}{2}} G_0^{(3-\delta+2\vartheta)}(t - u, y_1) du \\
\int_{k_1 \delta}^{(k_1+1)\delta} |\bar{t} - \bar{u}|^{-\frac{\delta}{2}} G_0^{(3-\delta)}(\bar{t} - \bar{u}, \bar{y} - y_1) d\bar{u} d\bar{y}_1 \\
+ \vartheta^\theta \int \int G_0^{(3)}(t - u_1, y - y_1) G_0^{(3+2\vartheta)}(\bar{t} - u_1, \bar{y} - y_1) du_1 dy_1.
\]

Then by similar arguments as the proof of Lemma 4.2 we obtain that \(J^1 \lesssim \vartheta^\theta \|z - \bar{z}\|_{\delta}^{-1-2\theta-\delta}.\)

Other terms can be estimated similarly, which implies that
\[
|f^{(\varepsilon, \vartheta)}(z, \bar{z}) - f^{(\varepsilon)}(z - \bar{z})| \lesssim \vartheta^\theta \|z - \bar{z}\|_{\delta}^{-1-2\theta-\delta},
\]
holds uniformly over \(\varepsilon, \vartheta \in (0, 1] \) satisfying \(\varepsilon^2 \leq C_0 \vartheta,\) provided that \(\theta < 1\) and that \(\delta > 0.\)

Since \(K\) and \(K_\varepsilon\) are of order \(-3,\) by [Hai14, Lemma 10.17] we obtain that
\[
|f^{(\varepsilon)}(z - \bar{z}) - f^{(\varepsilon)}(z - \bar{z})| \lesssim \varepsilon^{2\theta} \|z - \bar{z}\|_{\delta}^{-1-2\theta-\delta},
\]

Combining the above estimates we obtain
\[
|f^{(\varepsilon, \vartheta)}(z, \bar{z}) - f^{(\varepsilon, \vartheta)}(z - \bar{z})| \\
\lesssim |f^{(\varepsilon, \vartheta)}(z, \bar{z}) - f^{(\varepsilon)}(z - \bar{z})| + |f^{(\varepsilon)}(z - \bar{z}) - f^{(\varepsilon)}(z - \bar{z})| \\
\lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \|z - \bar{z}\|_{\delta}^{-1-2\theta-\delta},
\]
holds uniformly over \(\varepsilon, \vartheta \in (0, 1] \) satisfying \(\varepsilon^2 \leq C_0 \vartheta,\) provided that \(\theta < 1\) and that \(\delta > 0.\)

In the following we use the same notations as in Lemma 4.3 and have
\[
K_{\varepsilon, \vartheta}(z, z_1) - K_{\varepsilon}(z, z_1) = (K^{(1)}_{\varepsilon, \vartheta} - K^{(1)}_{\varepsilon})(z, z_1) + (K^{(2)}_{\varepsilon, \vartheta} - K^{(2)}_{\varepsilon})(z, z_1).
\]
In the proof of Theorem 3.7 we have to estimate \(\langle (K_{\varepsilon, \vartheta} - K)(z, \cdot), (K_{\varepsilon, \vartheta} - K)(\bar{z}, \cdot) \rangle.\) By the proof in [Hai14, Theorem 10.11] it is sufficient to estimate \(\langle (K^{(i)}_{\varepsilon, \vartheta} - K^{(i)}_{\varepsilon})(z, \cdot), (K^{(i)}_{\varepsilon, \vartheta} - K^{(i)}_{\varepsilon})(\bar{z}, \cdot) \rangle\) and \(\langle (K_{\varepsilon} - K)(z, \cdot), (K_{\varepsilon} - K)(\bar{z}, \cdot) \rangle\) separately.

**Lemma 4.4** If \(\varepsilon^2 \leq C_0 \vartheta\) for some \(C_0\) independent of \(\varepsilon, \vartheta,\) we obtain that for \(i = 1, 2\)
\[
|\langle (K^{(i)}_{\varepsilon, \vartheta} - K^{(i)}_{\varepsilon})(z, \cdot), (K^{(i)}_{\varepsilon, \vartheta} - K^{(i)}_{\varepsilon})(\bar{z}, \cdot) \rangle| \lesssim \vartheta^\theta \|z - \bar{z}\|_{\delta}^{-1-2\theta-\delta},
\]
holds uniformly over \(\varepsilon, \vartheta \in (0, 1] \) satisfying \(\varepsilon^2 \leq C_0 \vartheta,\) provided that \(\theta < 1\) and that \(\delta > 0.\)

**Proof** We have for \(z = (t, y), \bar{z} = (\bar{t}, \bar{y})\)
\[
\langle (K^{(1)}_{\varepsilon, \vartheta} - K^{(1)}_{\varepsilon})(z, \cdot), (K^{(1)}_{\varepsilon, \vartheta} - K^{(1)}_{\varepsilon})(\bar{z}, \cdot) \rangle \\
= \sum_{k=-\infty}^{[\frac{1}{\delta}]^{-1}} \sum_{k_1=-\infty}^{[\frac{1}{\delta}]^{-1}} \int \int_{k \delta}^{(k+1)\delta} \int_{k_1 \delta}^{(k_1+1)\delta} \int_{k \delta}^{(k+1)\delta} \int_{k_1 \delta}^{(k_1+1)\delta} (K(t - u, y - y_1) - K(t - u_1, y - y_1)) \\
(K(\bar{t} - \bar{u}, \bar{y} - y_2) - K(\bar{t} - u_2, \bar{y} - y_2)) d\bar{u} \frac{1}{\rho_\varepsilon} \rho_\varepsilon(u - u_1, y_1 - y_2) du_1 du_2 dy_1 dy_2.
\]
By [Hai14, Lemma 10.18] we could control $|K(t-u,y-y_1) - K(t-u_1,y-y_1)|$ by $\vartheta^\theta(G_0^{(3+2\theta)}(t-u,y-y_1) + C_0^{(3+2\theta)}(t-u_1,y-y_1))$. Then by similar arguments as the proof of Lemmas 4.2, 4.3 we obtain that

$$|\langle (K^{(1)}_{\varepsilon,\theta} - K^{(1)}_{\varepsilon})_\varepsilon, z \rangle, (K^{(1)}_{\varepsilon,\theta} - K^{(1)}_{\varepsilon})_\varepsilon, \zeta \rangle| \lesssim \vartheta^\theta \|z - \zeta\|_{s}^{-1-2\theta - \delta}.$$

For $i = 2$ we could also estimate similarly as the calculation in (4.8) and the result follows. □

**Lemma 4.5** If $\varepsilon^2 \leq C_0 \vartheta$ for some $C_0$ independent of $\varepsilon, \vartheta$, we have that for every $\delta > 0$

$$|f^{(\varepsilon,\theta)}(z, \bar{z}_1) - f^{(\varepsilon,\theta)}(z, \bar{z}_2)| \lesssim \|\bar{z}_1 - \bar{z}_2\|^2_{s} \|z - \bar{z}_1\|_{s}^{-1-2\delta} + \|z - \bar{z}_2\|_{s}^{-1-2\delta}). \quad (4.11)$$

**Proof** Without loss of generality for $\bar{z}_1 = (\bar{t}_1, \bar{y}_1), \bar{z}_2 = (\bar{t}_2, \bar{y}_2)$ we assume that $\bar{t}_1 \leq \bar{t}_2$. By a similar argument as the proof of Lemma 4.2 we have

$$|f^{(\varepsilon,\theta)}(z, \bar{z}_1) - f^{(\varepsilon,\theta)}(z, \bar{z}_2)| \lesssim \sum_{i=1}^{6} J_i,$$

where

$$J^1 = \left[ \sum_{k=-\infty}^{[\frac{\bar{t}_1}{\vartheta}]} \sum_{k_1=-\infty}^{[\frac{\bar{t}_2}{\vartheta}]} \int \int K(t-u,y-y_1)du \int K(\bar{t}_1 - \bar{u}, \bar{y}_1 - y_2)K(t-u,y-y_1)du \int K(\bar{t}_2 - \bar{u}, \bar{y}_2 - y_2)du \right] - \int \int K(t-u,y-y_1)du \int K(\bar{t}_1 - \bar{u}, \bar{y}_1 - y_2)K(t-u,y-y_1)du \int K(\bar{t}_2 - \bar{u}, \bar{y}_2 - y_2)du \left| \right|,$$

if $[\frac{\bar{t}_1}{\vartheta}] < [\frac{\bar{t}_2}{\vartheta}]$, $J^2 = \sum_{i=1}^{3} J^{2i}$ with

$$J^{21} = \left[ \sum_{k=-\infty}^{[\frac{\bar{t}_1}{\vartheta}]} \sum_{k_1=-\infty}^{[\frac{\bar{t}_2}{\vartheta}]} \int \int K(t-u,y-y_1)du \int \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1du_2dy_1dy_2 \right]$$

$$J^{22} = \left[ \sum_{k=-\infty}^{[\frac{\bar{t}_1}{\vartheta}]} \sum_{k_1=-\infty}^{[\frac{\bar{t}_2}{\vartheta}]} \int \int K(t-u,y-y_1)du \int \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1du_2dy_1dy_2 \right]$$

$$J^{23} = \left[ \sum_{k=-\infty}^{[\frac{\bar{t}_1}{\vartheta}]} \sum_{k_1=-\infty}^{[\frac{\bar{t}_2}{\vartheta}]} \int \int K(t-u,y-y_1)du \int \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1du_2dy_1dy_2 \right].$$
if \( \frac{[\frac{a}{b}]}{[\frac{c}{d}]} = \frac{[\frac{a}{b}]}{[\frac{c}{d}]} \),

\[
J_2^2 = \left| \sum_{k=\infty}^{[\frac{a}{b}]-1} \int \int \int_{t_1}^{(k+1)\vartheta} K(t-u, y-y_1)du \int_{t_1}^{\tilde{t}_2} K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2)d\tilde{u} \right|
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(k+1)\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

\[
J_3^3 = \left| \sum_{k=\infty}^{[\frac{a}{b}]-1} \int \int \int_{t_1}^{(k+1)\vartheta} K(t-u, y-y_1)du \int_{t_1}^{\tilde{t}_1} (K(\tilde{t}_1 - \tilde{u}, \tilde{y}_1 - y_2) - K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2))d\tilde{u} \right|
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(k+1)\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

if \( \frac{[\frac{a}{b}]}{[\frac{c}{d}]} < \frac{[\frac{a}{b}]}{[\frac{c}{d}]} \), \( J^4 = \sum_{i=1}^{3} J_i^{4i} \) with

\[
J_{41}^1 = \left| \int \int \int_{t_1}^{(\frac{a}{b})\vartheta} K(t-u, y-y_1)du \int_{t_1}^{(\frac{a}{b})\vartheta} K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2)d\tilde{u} \right|
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(\frac{a}{b})\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

\[
J_{42}^2 = \left| \int \int \int_{t_1}^{\tilde{t}_2} K(t-u, y-y_1)du \int_{t_1}^{\tilde{t}_2} K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2)d\tilde{u} \right|
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(\frac{a}{b})\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

\[
J_{43}^3 = \sum_{k_1 = [\frac{a}{b}] + 1}^{[\frac{a}{b}]} \int \int \int_{t_1}^{(k_1+1)\vartheta} K(t-u, y-y_1)du \int_{t_1}^{(k_1+1)\vartheta} K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2)d\tilde{u}
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(k_1+1)\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

if \( \frac{[\frac{a}{b}]}{[\frac{c}{d}]} = \frac{[\frac{a}{b}]}{[\frac{c}{d}]} \),

\[
J_4^4 = \left| \int \int \int_{t_1}^{\tilde{t}_2} K(t-u, y-y_1)du \int_{t_1}^{\tilde{t}_2} K(\tilde{t}_2 - \tilde{u}, \tilde{y}_2 - y_2)d\tilde{u} \right|
\]

\[
\frac{1}{\vartheta^2} \left| \int \int \int_{t_1}^{(\frac{a}{b})\vartheta} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ;
\]

28
\[ J^5 = \left| \int \int \int_{[\frac{t}{\delta^2}]} t \ K(t-u, y-y_1)du \int_{[\frac{t}{\delta^2}]} (K(t_1 - \bar{u}, y_1 - y_2) - K(t_2 - \bar{u}, y_2 - y_2))d\bar{u} \right| \]
\[ \frac{1}{\delta^2} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| ; \]
\[ J^6 = \left| \sum_{k_{1} = -\infty}^{[\frac{t}{\delta^2}]-1} \int \int \int_{[\frac{t}{\delta^2}]} t \ K(t-u, y-y_1)du \int_{[\frac{t}{\delta^2}]} (K(t_1 - \bar{u}, y_1 - y_2) - K(t_2 - \bar{u}, y_2 - y_2))d\bar{u} \right| \]
\[ \frac{1}{\delta^2} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \rho_\varepsilon * \rho_\varepsilon (u_1 - u_2, y_1 - y_2)du_1 du_2 dy_1 dy_2 \right| . \]

By the computation as in the proof of Lemma 4.2 we have
\[ J^1 \lesssim \sum_{k = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \sum_{k_1 = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \int \int \int_{[\frac{t}{\delta^2}]} K(t-u, y-y_1)du \int_{[\frac{t}{\delta^2}]} (K(t_1 - \bar{u}, y_1 - y_2) - K(t_2 - \bar{u}, y_2 - y_2))d\bar{u} \]
\[ - K(t_2 - \bar{u}, y_2 - y_2))d\bar{u} \frac{1}{\delta^2} \rho_2, \varepsilon (y_1 - y_2)dy_1 dy_2 . \]

We now use Lemma 4.1 and [Hai14, Lemma 10.18] and obtain that for \( \delta > 0 \)
\[ J^1 \lesssim \sum_{k = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \sum_{k_1 = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \frac{1}{\delta^2} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} |t-u|^{-\delta/2} G_0^{(3-\delta)} (t-u, y-y_1)du \]
\[ \| z_1 - z_2 \|_s \delta \int_{[\frac{t}{\delta^2}]} \left( |t_1 - \bar{u}|^{-\delta} G_0^{(3-\delta)} (t_1 - \bar{u}, y_1 - y_2) + |t_2 - \bar{u}|^{-\delta} G_0^{(3-\delta)} (t_2 - \bar{u}, y_2 - y_1) \right) d\bar{u} dy_1 \]
\[ \lesssim \| z_1 - z_2 \|_s \left( \| z - z_1 \|_s^{-1-2\delta} + \| z - z_2 \|_s^{-1-2\delta} \right) , \]
where in the last inequality we used similar arguments as in the proof of Lemma 4.2 for \( J^1 \) there.

We now turn to \( J^2 \) and we only consider \( J^{23} \) and other terms in \( J^2 \) can be estimated similarly. Applying Lemma 4.1 we have that for \( \delta > 0 \)
\[ J^{23} \lesssim \sum_{k = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \sum_{k_1 = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} |t-u|^{-\delta/2} G_0^{(3-\delta)} (t-u, y-y_1)du \]
\[ \int_{[\frac{t}{\delta^2}]} \left| t_2 - \bar{u} \right|^{-\delta/2} G_0^{(3-\delta)} (t_2 - \bar{u}, y_2 - y_1) d\bar{u} \frac{1}{\delta^2} dy_1 . \]

First by \( \left( t_2 - \bar{u} \right)^{3/2} \delta \| y_2 - y_1 \|^{2+\delta} \leq \left( t_2 - \bar{u} \right)^{3/2-\delta/2} + \| y_2 - y_1 \|^{3-\delta} \) and \( |t_2 - t_1| \geq |t_2 - \bar{u}| \) we obtain
\[ J^{23} \lesssim \sum_{k = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \sum_{k_1 = -[\frac{t}{\delta^2}]}^{[\frac{t}{\delta^2}]-1} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \int_{[\frac{t}{\delta^2}]} \frac{1}{|u|^{\frac{1}{2}} - \frac{\delta}{2} |y-y_1|^{2+\delta}} du \]
\[ \frac{|t_2 - t_1|^{\delta/4}}{(t_2 - \bar{u})^{1/2} - \frac{\delta}{2} \| y_2 - y_1 \|^{2+\delta}} \frac{1}{\delta^2} dy_1 . \]
Moreover, by Young’s inequality we have \( \frac{|t_2-t_1|^{3/4}}{(t_2-u)^{1/4} - (t-u)^{1/4}} \leq \frac{|t_2-t_1|^{3/2}}{|t-u|^{1/2}} + \frac{1}{(t_2-u)^{1/2}} \), which implies that
\[
J^{23} \lesssim |t_2 - t_1|^{3/4} |y - \bar{y}_2|^{-1-2\delta}.
\]

If \( t - \bar{t}_2 \geq (2C(C_0)+2)^\delta \), by similar arguments as in the proof of Lemma 4.2 we have that \( t - u \geq \frac{t - \bar{t}_2}{2} \), which implies that for \( \delta > 0 \)
\[
J^{23} \lesssim \sum_{k_1=\left\lfloor \frac{r_2}{2} \right\rfloor+1}^{\left\lfloor \frac{r_2}{2} \right\rfloor} \int_{k_1}^{(k_1+1)\theta} \frac{1}{(t - \bar{t}_2)^{\frac{1}{2} + \delta} |y - y_1|^{2-2\delta} (t_2 - u)^{1-\delta/2} |\bar{y}_2 - y_1|^{1+\delta}} d\mu \lesssim |t - \bar{t}_2|^{-\frac{1}{2} - \delta} |t_2 - \bar{t}_1|^{\delta/4}.
\]

If \( \bar{t}_2 - t \geq (2C(C_0)+2)^\delta \), we also obtain \( \bar{t}_2 - \bar{u} \geq \frac{\bar{t}_2 - t}{2} \), which combining with \( |\bar{t}_2 - \bar{t}_1| \geq |\bar{t}_2 - \bar{u}| \) implies that
\[
J^{23} \lesssim \sum_{k=-\infty}^{\left\lfloor \frac{r_2}{2} \right\rfloor - 1} \int_{k\theta}^{(k+1)\theta} \frac{1}{(t - u)^{1-\delta} |y - y_1|^{1+\delta} (t_2 - t)^{1-\delta/2} |\bar{y}_2 - y_1|^{2-\delta}} d\mu \lesssim |t - \bar{t}_2|^{-\frac{1}{2} - \delta},
\]
which implies that
\[
J^{23} \lesssim \sum_{k_1=\left\lfloor \frac{r_2}{2} \right\rfloor+1}^{\left\lfloor \frac{r_2}{2} \right\rfloor} \int_{k_1}^{(k_1+1)\theta} \frac{1}{|t - \bar{t}_2|^{\frac{1}{2} + \delta} |y - y_1|^{2-2\delta} (t_2 - u)^{1-\delta/2} |\bar{y}_2 - y_1|^{1+\delta}} d\mu \lesssim |t - \bar{t}_2|^{-\frac{1}{2} - \delta} |t_2 - \bar{t}_1|^{\delta/4}.
\]

Hence we obtain that
\[
J^{23} \lesssim \| \tilde{z}_1 - \tilde{z}_2 \|_s \| z - \tilde{z}_2 \|_s^{-1-2\delta}.
\]

\( J^4 \) can be estimated similarly as \( J^2 \) and \( J^3, J^5, J^6 \) can be estimated similarly as \( J^1 \). Thus the result follows.

Now we are ready to prove Theorem 3.7. By Lemmas 4.2-4.5 we could obtain the results by a similar argument as the proof of [Hai14, Theorem 10.22]. Here for the completeness of the paper we give all details of the proof.

**Proof of Theorem 3.7.** By [Hai14, Theorem 10.7] we only need to show that the renormalised model converges for those elements \( \tau \in \mathcal{F}_k \) with non-positive homogeneity. In the case of the dynamical \( \Phi^4_3 \) model, these elements are given by
\[
\mathcal{F}_- = \{ \Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_1, \mathcal{I}(\Psi^3)\Psi, \mathcal{I}(\Psi^2)\Psi^2, \mathcal{I}(\Psi^3)\Psi^2 \}.
\]
By [Hai14, Theorem 10.7] it is sufficient to prove that for $\tau \in \mathcal{F}_F$ with $|\tau|_s < 0$, any test function $\varphi \in \mathcal{B}$ and every $x \in \mathbb{R}^4$, and for some $0 < \theta < \frac{5}{2} - \alpha$,

$$\mathbf{E}[(\hat{\Pi}_x \tau - \hat{\Pi}_x^{(\varepsilon, \vartheta)}(\tau))(\varphi)]^2 \lesssim (e^{2\theta} + \vartheta^2)^{2|\tau|_s + \kappa}, \quad (4.12)$$

where $\hat{\Pi}_x \tau$ is obtained from the proof of [Hai14, Theorem 10.22]. Since the map $\varphi \mapsto (\hat{\Pi}_x^{(\varepsilon, \vartheta)}(\tau))(\varphi)$ is linear, we can find some functions $\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}$ such that

$$(\hat{\Pi}_x^{(\varepsilon, \vartheta)}(\tau))(\varphi) = \sum_{k \leq ||\tau||} I_k \left( \int \varphi(y)(\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(y)dy \right),$$

where $||\tau||$ denotes the number of occurrences of $\Xi$ in the expression $\tau$ and $I_k$ is defined as in [Hai14, Section 10.1]. We also have the following notation as in [Hai14, Section 10]:

$$(\hat{\Pi}_x \tau)(\varphi) = \sum_{k \leq ||\tau||} I_k \left( \int \varphi(y)(\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(y)dy \right),$$

where $\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)} \in L^2(\mathbb{R} \times \mathbb{T}^3)^{\otimes k}$. Now we want to estimate the terms $|\langle (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(z), (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(\bar{z}) \rangle|$ and $|\langle (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(z), (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau))(\bar{z}) \rangle|$, where $\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau) = \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau) - \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, k)}(\tau)$.

We have

$$(\hat{\Pi}_x^{(\varepsilon, \vartheta)}(\Psi))(z) = K * \xi, \vartheta(z) := \int K_{\varepsilon, \vartheta}(z, z_1) \xi(z_1)dz_1,$$

which implies that

$$\langle (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 1)}(\Psi))(z), (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 1)}(\Psi))(\bar{z}) \rangle = K_{\varepsilon, \vartheta}(z, z_1).$$

In the following we use $\xrightarrow{z_1 \rightarrow z}$ to represent a factor $K(z-z_1)$ and $\xrightarrow{z \rightarrow z}$ to represent $K_{\varepsilon, \vartheta}(z, z_1)$. We also use the convention that if a vertex is drawn in grey, then the corresponding variable is integrated out. Now we have

$$f^{(\varepsilon, \vartheta)}(z, \bar{z}) = \int_1^z.$$  

By Lemma 4.2 we obtain that

$$|\langle (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 1)}(\Psi))(z), (\hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 1)}(\Psi))(\bar{z}) \rangle| = \left| \int_{z_1}^z \right| \lesssim \|z - \bar{z}\|_{s}^{-1 - \delta}, \quad (4.13)$$

holds uniformly over $\varepsilon, \vartheta$. Now for $\hat{\Pi}_x \Psi = K * \xi$ as in [Hai14, Theorem 10.22] we also have

$$(\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 1)}(\Psi))(z, z_1) = (K_{\varepsilon, \vartheta}(z, z_1) - K(z - z_1))$$

$$= (K_{\varepsilon, \vartheta}^{(1)}(z, z_1) - K_{\varepsilon}^{(1)}(z, z_1)) + (K_{\varepsilon, \vartheta}^{(2)}(z, z_1) - K_{\varepsilon}^{(2)}(z, z_1))$$

$$+ (K_{\varepsilon}(z - z_1) - K(z - z_1))$$

$$:= (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 11)}(\Psi))(z, z_1) + (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 12)}(\Psi))(z, z_1) + (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 13)}(\Psi))(z, z_1).$$

By Lemma 4.4 and [Hai14, Lemmas 10.14, 10.17] we have for $i = 1, 2$

$$|\langle (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 11)}(\Psi))(z), (\delta \hat{\mathcal{W}}_x^{(\varepsilon, \vartheta, 11)}(\Psi))(\bar{z}) \rangle| \lesssim \theta^\delta \|z - \bar{z}\|_{s}^{-1 - \delta - 2\delta}, \quad (4.14)$$
and
\[ |\langle (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;13)} \Psi)(z), (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;13)} \Psi)(\bar{z}) \rangle \rangle | \lesssim \varepsilon^{2\theta} \|z - \bar{z}\|_{s}^{-1-\delta-2\theta}, \tag{4.15} \]
holds uniformly over \( \varepsilon, \vartheta \in (0,1] \) satisfying \( \varepsilon^{2} \leq C_{0}\vartheta \), provided that \( 0 < \theta < 1 \) and that \( 0 < \delta < 1 \), which deduces (4.12) for \( \tau = \Psi \) easily. In the following we use \(|\zeta - \hat{\mathcal{L}}_{-}\zeta|\) to represent \( \sum_{i=1}^{3} |\langle (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;11)} \Psi)(z), (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;11)} \Psi)(\bar{z}) \rangle \rangle | \). By (4.14) and (4.15) we have
\[ |\zeta| \lesssim (\vartheta^{2} + \varepsilon^{2\theta}) \|z - \bar{z}\|_{s}^{-1-\delta-2\theta}, \]
for simplicity.

For \( \tau = \Psi^{2} \) we could choose for \( z = (t, y) \)
\[ C_{1}^{(\varepsilon,\vartheta)}(t) = \int K_{\varepsilon,\vartheta}(z, z_{1})^{2} dz_{1}. \]
Here since \( \rho = \rho_{1}\rho_{2} \) we can easily deduce that \( C_{1} \) only depends on \( t \). We obtain that
\[ (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;2)} \Psi^{2})(z) = \nabla_{z}. \]
By Lemma 4.2 we have that for every \( \delta > 0 \)
\[ |\langle (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;2)} \Psi^{2})(z), (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;2)} \Psi^{2})(\bar{z}) \rangle \rangle | = f^{(\varepsilon,\vartheta)}(z, \bar{z})^{2} \lesssim \|z - \bar{z}\|_{s}^{-2-\delta}, \]
holds uniformly over \( \varepsilon, \vartheta \) satisfying \( \varepsilon^{2} \leq C_{0}\vartheta \). As in the proof of [Hai14, Theorem 10.22] \( \hat{\mathcal{W}}_{x}^{(2)} \Psi^{2} = K^{2} \). By (4.14), (4.15) and Lemma 4.2 we have
\[ |\langle (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;2)} \Psi^{2})(z), (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;2)} \Psi^{2})(\bar{z}) \rangle \rangle | \lesssim (\vartheta^{2} + \varepsilon^{2\theta}) \|z - \bar{z}\|_{s}^{-2-2\theta-\delta}, \]
holds uniformly over \( \varepsilon, \vartheta \in (0,1] \) satisfying \( \varepsilon^{2} \leq C_{0}\vartheta \), provided that \( \theta < 1 \) and that \( \delta > 0 \), which implies that (4.12) holds for \( \tau = \Psi^{2} \).

For \( \tau = \Psi^{3} \) we have
\[ (\hat{\Pi}_{x}^{(\varepsilon,\vartheta;3)} \Psi^{3})(z) = (\hat{\Pi}_{x}^{(\varepsilon,\vartheta)} \Psi)(z)^{\circ 3}, \]
and
\[ (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;3)} \Psi^{3})(z) = \nabla_{z}. \]
By Lemma 4.2 we have for every \( \delta > 0 \)
\[ |\langle (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;3)} \Psi^{3})(z), (\hat{\mathcal{W}}_{x}^{(\varepsilon,\vartheta;3)} \Psi^{3})(\bar{z}) \rangle \rangle | = |f^{(\varepsilon,\vartheta)}(z, \bar{z})^{3}| \lesssim \|z - \bar{z}\|_{s}^{-3-\delta}, \]
holds uniformly over $\varepsilon, \vartheta$ satisfying $\varepsilon^2 \leq C_0 \vartheta$. By (4.14), (4.15) and Lemma 4.2 we have

$$
|\langle (\delta \hat{W}_x^{(e,\vartheta,3)} \Psi^3)(z), (\delta \hat{W}_x^{(e,\vartheta,3)} \Psi^3)(\bar{z}) \rangle| 
\lesssim \left\| z - \bar{z} \right\|_s^{-2-\delta-\varepsilon} \left\| z - \bar{z} \right\|_s^{-3-2\vartheta-\delta},
$$

holds uniformly over $\varepsilon, \vartheta \in (0, 1]$ satisfying $\varepsilon^2 \leq C_0 \vartheta$, provided that $0 < \theta < 1$ and that $0 < \delta < 1$, which implies that (4.12) holds for $\tau = \Psi^3$.

Regarding $\tau = \Psi^2 X_i$ the corresponding bound follows from those for $\tau = \Psi^2$.

As in [Hai14, Theorem 10.22] we also use the notation $\dashrightarrow$ to represent $\left\| z - \bar{z} \right\|_s \leq C$ for a constant $C$.

Now for $\tau = \mathcal{I}(\Psi^3) \Psi$ we have

$$(\hat{\Pi}_x^{(e,\vartheta)}\tau)(z) = (\hat{\Pi}_x^{(e,\vartheta)}\Psi)(z) [K \ast (\hat{\Pi}_x^{(e,\vartheta)}\Psi^3)(z) - K \ast (\hat{\Pi}_x^{(e,\vartheta)}\Psi^3)(x)].$$

For the term in the fourth Wiener chaos we have

$$(\hat{W}_x^{(e,\vartheta,4)}\tau)(z) = \Psi - \Psi_{4,z}.$$}

We have the following estimates:

$$
|\langle (\hat{W}_x^{(e,\vartheta,4)}\tau)(z), (\hat{W}_x^{(e,\vartheta,4)}\tau)(\bar{z}) \rangle| 
\lesssim \int \int \left\| z - \bar{z} \right\|_s^{-1-\delta} |K(z - z_1) - K(x - z_1)| \left\| z_1 - \bar{z}_1 \right\|_s^{-3-\delta}
|K(z - z_1) - K(x - z_1)| dz_1 d\bar{z}_1,
$$

where we used Lemma 4.2 to obtain the estimate. We now use [Hai14, Lemma 10.18] to control $|K(z - z_1) - K(x - z_1)|$ by $\left\| z - x \right\|_s^{1-\delta} \left( \left\| z - z_1 \right\|_s^{3.5+\delta} + \left\| x - z_1 \right\|_s^{-3.5+\delta} \right)$ with $0 < \delta < \frac{1}{2}$ and obtain that

$$
|\langle (\hat{W}_x^{(e,\vartheta,4)}\tau)(z), (\hat{W}_x^{(e,\vartheta,4)}\tau)(\bar{z}) \rangle| 
\lesssim \left\| z - \bar{z} \right\|_s^{-1-\delta} \left\| z - x \right\|_s^{1-\delta} \left\| z - \bar{z} \right\|_s^{3.5+\delta} + \left\| x - z_1 \right\|_s^{-3.5+\delta} \left( G(z - x) + G(z - \bar{z}) + G(z - z) \right),
$$

holds uniformly over $\varepsilon, \vartheta$ satisfying $\varepsilon^2 \leq C_0 \vartheta$. Here the function $G$ is a bounded function and is given by

$$G(z - \bar{z}) = \Psi^{3.5+\delta} - \Psi^{3.5+\delta} - \Psi^{3.5+\delta} - \Psi^{3.5+\delta}.$$
the proof of [Hai14, Theorem 10.22]. By (4.14), (4.15) we obtain

$$|\langle (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(z), (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(\bar{z}) \rangle|$$

$$\lesssim \int \int \left[ |z - z_1|^{-\delta} |K(z-z_1) - K(x-z_1)| |z_1 - \bar{z}_1|^2 \right] dz_1 d\bar{z}_1$$

$$|\langle (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(z), (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(\bar{z}) \rangle|$$

$$\lesssim (\vartheta^\varepsilon + \varepsilon^2) \int \int \left[ |z - z_1|^{-1-\delta} |K(z-z_1) - K(x-z_1)| |z_1 - \bar{z}_1|^{-3-\vartheta-\delta} 
+ |z - \bar{z}|^{-1-\vartheta-\delta} |z - x|^{-3-\vartheta-\delta} \right] dz_1 d\bar{z}_1,$$

where we used Lemmas 4.2, 4.3 to obtain the estimate. We now apply [Hai14, Lemma 10.18] to control $|K(z-z_1) - K(x-z_1)|$ by $\|z - x\|^{1-\delta} \left( \|z - z_1\|_{s}^{3.5+\delta} + \|z - z_1\|_{s}^{3.5+\delta} \right)$ for $0 < \delta < \frac{1}{2}$ and use similar arguments as (4.17) to deduce that

$$|\langle (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(z), (\delta \hat{W}_x^{(\varepsilon, \vartheta, 4)} \tau)(\bar{z}) \rangle| \lesssim (\vartheta^\varepsilon + \varepsilon^2) \left( \|z - \bar{z}\|_{s}^{-1-\theta-\delta} \|z - x\|^{1-\theta-\delta} \right)$$

for $0 < \theta < 1$ and that $1 > \theta > 0$. For the term in the second Wiener chaos, we also have the following identity:

$$(\hat{W}_x^{(\varepsilon, \vartheta, 2)} \tau)(z) = 3\left( \begin{array}{c} \cdots \bar{z} \cdots \end{array} \right) := 3(\langle \hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau)(z) \rangle - \langle \hat{W}_x^{(\varepsilon, \vartheta, 22)} \tau)(z) \rangle).$$

For $\hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau$ we have that for every $\delta > 0$

$$|\langle (\hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau)(z), (\hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau)(\bar{z}) \rangle|$$

$$\lesssim \int \int \left[ \|z - z_1\|_{s}^{-1-\delta} |K(z-z_1)| |z_1 - \bar{z}_1|^{-2-\delta} |K(z - \bar{z}_1)| |\bar{z} - \bar{z}_1|_{s}^{-1-\delta} \right] dz_1 d\bar{z}_1$$

$$\lesssim \|z - \bar{z}\|_{s}^{-3\delta},$$

where we used Lemma 4.2 in the first inequality and [Hai14, Lemma 10.14] in the last inequality. Choose $\hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau$ as $\hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau$ with each instance of $K_{\varepsilon, \vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. By Lemmas 4.2, 4.3 and (4.14), (4.15) we have

$$|\langle (\delta \hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau)(z), (\delta \hat{W}_x^{(\varepsilon, \vartheta, 21)} \tau)(\bar{z}) \rangle|$$

$$\lesssim (\vartheta^\varepsilon + \varepsilon^2) \int \int \left[ \|z - z_1\|_{s}^{-1-\delta} |K(z-z_1)| |z_1 - \bar{z}_1|^{-2-\vartheta-\delta} |K(z - \bar{z}_1)| |\bar{z} - \bar{z}_1|_{s}^{-1-\delta} 
+ \|z - \bar{z}_1\|_{s}^{-1-\theta-\delta} |K(z-z_1)| |z_1 - \bar{z}_1|^{-2-\delta} |K(z - \bar{z}_1)| |\bar{z} - \bar{z}_1|_{s}^{-1-\theta-\delta} \right] dz_1 d\bar{z}_1$$

$$\lesssim (\vartheta^\varepsilon + \varepsilon^2) \|z - \bar{z}\|_{s}^{-2\vartheta-3\delta},$$
holds uniformly over \( \varepsilon, \vartheta \in (0, 1] \) satisfying \( \varepsilon^2 \leq C_0 \vartheta \), provided that \( 0 < \vartheta < 1 \) and that \( 0 < \delta < 1 \), and we used \([Hai14, \text{Lemma 10.14}]\) in the last inequality. For \( \hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau \) by Lemma 4.2 we have

\[
|\langle (\hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau)(z), (\hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau)(\bar{z}) \rangle| \\
\lesssim \int \int f^{(e, \vartheta)}(z, z_1)|K(x - z_1)|f^{(e, \vartheta)}(z_1, \bar{z}_1)^2|K(x - \bar{z}_1)|f^{(e, \vartheta)}(\bar{z}, \bar{z}_1)dz_1d\bar{z}_1 \\
\lesssim \int \int \|z - x\|^{-\bar{4}\delta}(G^1(z - x) + G^1(\bar{z} - x) + G^1(z - \bar{z}) + G^1(0)),
\]

holds uniformly over \( \varepsilon, \vartheta \in (0, 1] \) satisfying \( \varepsilon^2 \leq C_0 \vartheta \), provided that \( 0 < \vartheta < 1 \) and that \( 0 < \delta < 1 \), where we used \([Hai14, (10.37)]\) in the last inequality and the function \( G^1 \) is a bounded function and given by

\[
G^1(z - \bar{z}) = \frac{z - \bar{z}}{3\delta - 4} - \frac{z - \bar{z}}{-2 - \delta} - \frac{z - \bar{z}}{-1 - \delta}.
\]

Define \( \hat{\mathcal{W}}_x^{(22)} \tau \) as \( \hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau \) with each instance of \( K_{e, \vartheta} \) replaced by \( K \). For the difference by Lemmas 4.2, 4.3 and (4.14), (4.15) we have

\[
|\langle (\hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau)(z), (\hat{\mathcal{W}}_x^{(e, \vartheta, 22)} \tau)(\bar{z}) \rangle| \\
\lesssim (\vartheta^\theta + \varepsilon^{2\vartheta}) \int \int \left[ \|z - z_1\|_s^{-1-\delta}|K(x - z_1)||z_1 - \bar{z}_1\|_s^{-2-2\vartheta-\delta}|K(x - \bar{z}_1)||\bar{z} - \bar{z}_1\|_s^{-1-\delta} \\
+ \|z - z_1\|_s^{-1-\theta-\delta}|K(x - z_1)||z_1 - \bar{z}_1\|_s^{-2-\delta}|K(x - \bar{z}_1)||\bar{z} - \bar{z}_1\|_s^{-1-\theta-\delta} \right] dz_1d\bar{z}_1 \\
\lesssim (\vartheta^\theta + \varepsilon^{2\vartheta}) \|z - x\|^{-2\vartheta-\delta},
\]

holds uniformly over \( \varepsilon, \vartheta \in (0, 1] \) satisfying \( \varepsilon^2 \leq C_0 \vartheta \), provided that \( 0 < \vartheta < 1 \) and that \( 0 < \delta < 1 \), where we used similar arguments as above in the last inequality. Combining all the estimates above we obtain that (4.12) holds for \( \tau = \mathcal{I}(\Psi^2)\Psi \).

Now we come to the case \( \tau = \mathcal{I}(\Psi^2)\Psi^2 \). We have for \( z = (t, y) \)

\[
(\hat{\Pi}_x^{(e, \vartheta)})(z) = (\hat{\Pi}_x^{(e, \vartheta)})(z)[K \ast (\hat{\Pi}_x^{(e, \vartheta)}\Psi^2)(z)] - K \ast (\hat{\Pi}_x^{(e, \vartheta)}\Psi^2)(z) - C_2(e, \vartheta)(t).
\]

For the term in the fourth Wiener chaos, we have

\[
(\hat{\mathcal{W}}_x^{(e, \vartheta, 4)})(z) = \frac{\psi}{z} - \frac{\psi}{z}.
\]

By Lemma 4.2 we have

\[
|\langle (\hat{\mathcal{W}}_x^{(e, \vartheta, 4)} \tau)(z), (\hat{\mathcal{W}}_x^{(e, \vartheta, 4)} \tau)(\bar{z}) \rangle| \\
\lesssim \int \int \|z - \bar{z}\|_s^{-2-\delta}|K(z - z_1) - K(x - z_1)||z_1 - \bar{z}_1\|_s^{-2-\delta} \\
|K(\bar{z} - z_1) - K(x - z_1)|dz_1d\bar{z}_1.
\]

35
We now apply [Hai14, Lemma 10.18] to control $|K(z-z_1) - K(x-z_1)|$ by $\|z-x\|^{1-\delta}_s \left( \|z - z_1\|^{-4+\delta}_s + \|x - z_1\|^{-4+\delta}_s \right)$ for $0 < \delta < 1$ and obtain that the above term can be bounded by

$$\|z - \bar{z}\|^{-2-\delta}_s \|z - x\|^{-1-\delta}_s \|\bar{z} - x\|^{-1-\delta}_s (G^2(z - x) + G^2(\bar{z} - x) + G^2(z - \bar{z}) + G^2(0)),$$

holds uniformly over $\varepsilon, \vartheta$ satisfying $\varepsilon^2 \leq C_0 \vartheta$. Here the function $G^2$ is a bounded function and is given by

$$G^2(z - \bar{z}) = z \quad \delta - 4 \rightarrow -2 - \delta \rightarrow -4 + \delta \rightarrow \bar{z}.$$

Choose $\hat{\mathcal{W}}^{(4)}_x (z, \bar{z})$ as $\hat{\mathcal{W}}^{(4, \vartheta, 4)}_x (z, \bar{z})$ with each instance of $K_{\varepsilon, \vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. Similarly by Lemmas 4.2, 4.3 and (4.14), (4.15) we have that

$$|\langle (\delta \hat{\mathcal{W}}^{(\varepsilon, \vartheta, 4)}_x (z, \bar{z}), (\delta \hat{\mathcal{W}}^{(\varepsilon, \vartheta, 4)}_x (z, \bar{z})) \rangle| \lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \int \int |z - \bar{z}|^{-2-\delta}_s |K(z - z_1) - K(x - z_1)| \|z_1 - \bar{z}_1\|^{-2-2\theta-\delta}_s |K(z - z_1) - K(x - z_1)| \|z_1 - \bar{z}_1\|^{-2-\delta}_s \|K(z - z_1) - K(x - z_1)\| \ dz_1 d\bar{z}_1 \lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \|z - \bar{z}\|^{-2-\delta}_s \|z - x\|^{1-\theta-\delta}_s \|\bar{z} - x\|^{1-\delta}_s + \|z - \bar{z}\|^{-2-2\theta-\delta}_s \|z - x\|^{1-\delta}_s \|\bar{z} - x\|^{1-\delta}_s] \|z - x\|^{1-\delta}_s \|\bar{z} - x\|^{1-\delta}_s,$$

holds uniformly over $\varepsilon, \vartheta \in (0, 1]$ satisfying $\varepsilon^2 \leq C_0 \vartheta$, provided that $\theta < 1$ and that $\delta > 0$. For the term in the second Wiener chaos, we have the following identity

$$(\hat{\mathcal{W}}^{(\varepsilon, \vartheta, 2)}_x (z, \bar{z}) = 4(\hat{\mathcal{W}}^{(\varepsilon, \vartheta, 2)}_x (z, \bar{z})).$$

Then by Lemma 4.2 we obtain that

$$|\langle (\hat{\mathcal{W}}^{(\varepsilon, \vartheta, 2)}_x (z, \bar{z}), (\hat{\mathcal{W}}^{(\varepsilon, \vartheta, 2)}_x (z, \bar{z})) \rangle| \lesssim \int \int |z - \bar{z}|^{-1-\delta}_s |z - z_1|^{-1-\delta}_s |K(z - z_1) - K(x - z_1)| \|z_1 - \bar{z}_1\|^{-1-\delta}_s \|K(z - z_1) - K(x - z_1)\| \ dz_1 d\bar{z}_1 \lesssim \|z - \bar{z}\|^{-1-\delta}_s \|z - x\|^{\frac{1}{2}-\delta}_s \|\bar{z} - x\|^{\frac{1}{2}-\delta}_s (G^3(z, \bar{z}) + G^3(z, x) + G^3(x, \bar{z}) + G^3(x, x)),$$

holds uniformly over $\varepsilon, \vartheta$ satisfying $\varepsilon^2 \leq C_0 \vartheta$. Here the function $G^3$ is a bounded function and is given by

$$G^3(a, b) = \begin{cases} a \rightarrow -a \rightarrow b, \\ z \rightarrow -z \rightarrow \bar{z} \rightarrow \bar{z} \rightarrow z. \end{cases}$$
and we used Young's inequality to obtain $G^2$ is bounded. Choose $\hat{\mathcal{W}}_{x}^{2} \tau$ as $\hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 2)} \tau$ with each instance of $K_{\varepsilon, \vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. Similarly by Lemmas 4.2, 4.3 and (4.14), (4.15) we have

\[ |(\delta \hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 2)} \tau)(z), (\delta \hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 2)} \tau)(\bar{z})| \lesssim \varepsilon^{2\vartheta} + 2\vartheta \|z - \bar{z}\|_{s}^{-2 - \delta} \|z - x\|_{s}^{2 - 2\delta - \theta} + \|z - \bar{z}\|_{s}^{1 - 2\delta - \theta} \|z - x\|_{s}^{1 - 2\delta} \|z - x\|_{s}^{1 - 2\delta}, \]

which is valid uniformly over $\varepsilon, \vartheta \in (0, 1]$ satisfying $\varepsilon^2 \leq C_0 \vartheta$, provided that $\theta < 1$ and that $\delta > 0$.

Still considering $\tau = \mathcal{I}(\Psi^2)\Psi^2$. We now turn to the component in the 0th Wiener chaos. For $z = (t, y)$, choose $C_{2}^{(\varepsilon, \vartheta)}(t) = 2 \int f^{(\varepsilon, \vartheta)}(z, z_{1})^{2} K(z - z_{1}) dz_{1} = \mathcal{W}$.

We have

\[ (\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \tau)^{(0)}(z) = -2 \int f^{(\varepsilon, \vartheta)}(z, z_{1})^{2} K(x - z_{1}) dz_{1}, \]

which combining with Lemma 4.2 implies that

\[ |(\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \tau)^{(0)}(z)| \lesssim \int \|z - z_{1}\|_{s}^{-2 - \delta} |K(x - z_{1})| dz_{1} \lesssim \|z - x\|_{s}^{-\delta}, \]

for every $\delta > 0$. Choose $\hat{\mathcal{W}}_{x}^{(0)} \tau$ as above with each instance of $K_{\varepsilon, \vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. Moreover, by Lemma 4.2 and Lemma 4.3 we have

\[ |(\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \tau)^{(0)}(z) - (\hat{\Pi}_{x} \tau)^{(0)}(z)| \lesssim (\varepsilon^{2\vartheta} + \vartheta) \int \|z - z_{1}\|_{s}^{-2 - \delta - 2\vartheta} |K(x - z_{1})| dz_{1} \lesssim (\varepsilon^{2\vartheta} + \vartheta) \|z - x\|_{s}^{-\delta - 2\vartheta}, \]

which is valid uniformly over $\varepsilon, \vartheta \in (0, 1]$ satisfying $\varepsilon^2 \leq C_0 \vartheta$, provided that $\theta < 1$ and that $\delta > 0$.

For $\tau = \mathcal{I}(\Psi^3)\Psi^2$, we have the following identity

\[ (\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \tau)(z) = (\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \Psi^2)(z) [K * (\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \Psi^3)(z) - K * (\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \Psi^3)(x) - 3C_{2}^{(\varepsilon, \vartheta)}(\hat{\Pi}_{x}^{(\varepsilon, \vartheta)} \Psi)(z). \]

For the term in the fifth Wiener chaos, we have

\[ (\hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 5)} \tau)(z) = \mathcal{W} \quad - \quad \mathcal{W}_{x}. \]

\[ |(\hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 5)} \tau)(z), (\hat{\mathcal{W}}_{x}^{(\varepsilon, \vartheta, 5)} \tau)(\bar{z})| \lesssim \int \int \|z - \bar{z}\|_{s}^{-2 - \delta} |K(z - z_{1}) - K(x - z_{1})| \|z_{1} - \bar{z}_{1}\|_{s}^{-3 - \delta} |K(\bar{z} - \bar{z}_{1}) - K(x - \bar{z}_{1})| dz_{1} d\bar{z}_{1} \lesssim \|z - \bar{z}\|_{s}^{-2 - \delta} \|z - x\|_{s}^{\frac{1}{2} - \delta} \|\bar{z} - x\|_{s}^{\frac{1}{2} - \delta} (G^4(z - x) + G^4(\bar{z} - x) + G^4(0) + G^4(z - \bar{z})), \]

37
where we used Lemma 4.2 in the first inequality and [Hai14, Lemma 10.18] in the second inequality. Here the function $G^4$ is a bounded function and is given by

$$G^4(z - \hat{z}) = z - 3.5\delta - 3.5\delta - 3.5\delta - 3.5\delta - \hat{z}.$$ 

Choose $\hat{\mathcal{W}}^{(5)}_x$ as $\hat{\mathcal{W}}^{(e,\vartheta,5)}_x$ with each instance of $K_{\epsilon,\vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. For the difference by Lemmas 4.2, 4.3 (4.14), (4.15) we have similar estimates:

$$\left|\langle (\delta \hat{\mathcal{W}}^{(e,\vartheta,5)}_x)(z), (\delta \hat{\mathcal{W}}^{(e,\vartheta,5)}_x)(\hat{z}) \rangle\right| \lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \int \int \left| z - \hat{z}\right|_{s}^{-2-\delta} |K(z - z_1)| \left| z_1 - \hat{z}_1\right|_{s}^{-3-2\theta-\delta}$$

$$|K(\hat{z} - \hat{z}_1) - K(x - \hat{z})| + \left| z - \hat{z}\right|_{s}^{-2-2\theta-\delta} |K(z - z_1)| \left| z_1 - \hat{z}_1\right|_{s}^{-3-\delta}$$

$$|K(\hat{z} - \hat{z}_1) - K(x - \hat{z}_1)| dz_1 d\hat{z}_1,$$

which is valid uniformly over $\varepsilon, \vartheta \in (0, 1]$ satisfying $\varepsilon^2 \leq C_0 \vartheta$, provided that $\theta < 1$ and that $\delta > 0$.

The component in the third Wiener chaos is very similar to what was obtained previously. Indeed, we have

$$(\hat{\mathcal{W}}^{(e,\vartheta,3)}_x)(z) = 6(\hat{\mathcal{W}}^{(e,\vartheta,3)}_x - \hat{\mathcal{W}}^{(e,\vartheta,3)}_x) := 6((\hat{\mathcal{W}}^{(e,\vartheta,31)}_x)(z) - (\hat{\mathcal{W}}^{(e,\vartheta,32)}_x)(z)).$$

Then we obtain that for every $\delta > 0$

$$\left|\langle (\delta \hat{\mathcal{W}}^{(e,\vartheta,31)}_x)(z), (\delta \hat{\mathcal{W}}^{(e,\vartheta,31)}_x)(\hat{z}) \rangle\right| \lesssim \int \int \left| z - \hat{z}\right|_{s}^{-1-\delta} |z - z_1|_{s}^{-1-\delta} |K(z - z_1)| \left| z_1 - \hat{z}_1\right|_{s}^{-2-\delta} |K(\hat{z} - \hat{z}_1)| \left| \hat{z} - \hat{z}_1\right|_{s}^{-1-\delta} dz_1 d\hat{z}_1$$

$$\lesssim \left| z - \hat{z}\right|_{s}^{-1-4\delta},$$

where we used Lemma 4.2 in the first inequality and [Hai14, Lemma 10.14] in the last inequality. Choose $\hat{\mathcal{W}}^{(31)}_x$ as $\hat{\mathcal{W}}^{(e,\vartheta,31)}_x$ with each instance of $K_{\epsilon,\vartheta}$ replaced by $K$, which is the same as in the proof of [Hai14, Theorem 10.22]. Similarly by Lemmas 4.2, 4.3 and (4.14), (4.15) we have

$$\left|\langle (\delta \hat{\mathcal{W}}^{(e,\vartheta,31)}_x)(z), (\delta \hat{\mathcal{W}}^{(e,\vartheta,31)}_x)(\hat{z}) \rangle\right| \lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \int \int \left| z - \hat{z}\right|_{s}^{-1-\delta} \left| z - z_1\right|_{s}^{-1-\delta} |K(z - z_1)| \left| z_1 - \hat{z}_1\right|_{s}^{-2-2\theta-\delta} |K(\hat{z} - \hat{z}_1)| \left| \hat{z} - \hat{z}_1\right|_{s}^{-1-\delta}$$

$$+ \left| z - z_1\right|_{s}^{-1-\delta} |K(z - z_1)| \left| z_1 - \hat{z}_1\right|_{s}^{-2-\delta} |K(\hat{z} - \hat{z}_1)| \left| \hat{z} - \hat{z}_1\right|_{s}^{-1-\delta} dz_1 d\hat{z}_1$$

$$\lesssim (\vartheta^\theta + \varepsilon^{2\theta}) \left| z - \hat{z}\right|_{s}^{-1-2\theta-4\delta},$$

38
which is valid uniformly over \(\varepsilon, \vartheta \in (0, 1]\) satisfying \(\varepsilon^2 \leq C_0 \vartheta\), provided that \(\vartheta < 1\) and that \(\delta > 0\). Similarly we have

\[
| \langle (\tilde{W}^{(x, \varepsilon, 3\delta, 2)}_x) (z), (\tilde{W}^{(x, \varepsilon, 3\delta, 2)}_x (\tilde{z}) \rangle |
\]

\[
\lesssim \int \int \| z - \tilde{z} \|_{s}^{-1-\delta} \| z - z_1 \|_{s}^{-1-\delta} |K(x - z_1)| \| z_1 - \tilde{z}_1 \|_{s}^{-2-\delta} |K(x - \tilde{z}_1)| \| \tilde{z} - \tilde{z}_1 \|_{s}^{-1-\delta} d\tilde{z}_1 d\tilde{z}_1
\]

\[
\lesssim | \| z - \tilde{z} \|_{s}^{-1-\delta} (\| z - x \|_{s}^{-3\delta} + \| z - \tilde{z} \|_{s}^{-3\delta})
\]

where we used Young’s inequality in the third inequality and [Hai14, Lemma 10.14] in the last inequality. By Lemmas 4.2, 4.3 (4.14, 4.15) and similar arguments as above we have

\[
| \langle (\delta \tilde{W}^{(x, \varepsilon, 3\delta, 2)}_x (z), (\delta \tilde{W}^{(x, \varepsilon, 3\delta, 2)}_x (\tilde{z}) \rangle |
\]

\[
\lesssim (\vartheta^\theta + \varepsilon^{-2\theta}) (\| z - \tilde{z} \|_{s}^{-1-\delta} (\| z - x \|_{s}^{-2\vartheta - 3\delta} + \| z - \tilde{z} \|_{s}^{-2\vartheta - 3\delta}) + \| z - \tilde{z} \|_{s}^{-1-\delta} (\| z - x \|_{s}^{-3\delta} + \| z - \tilde{z} \|_{s}^{-3\delta}))
\]

which is valid uniformly over \(\varepsilon, \vartheta \in (0, 1]\) satisfying \(\varepsilon^2 \leq C_0 \vartheta\), provided that \(0 < \vartheta < 1\) and that \(\delta > 0\).

We turn to the first Wiener chaos:

\[
(\tilde{W}^{(x, \varepsilon, 1)}_x (z) = \frac{1}{6} \left( \int_{s} \frac{1}{s} - 3C_2^{(x, \varepsilon)} z \right) - 6 \int_{s} \frac{1}{(s)^2} \int_{s} \frac{1}{(s)^3} dz_1 d\tilde{z}_1
\]

\[
:= 6 \left[ (\tilde{W}^{(x, \varepsilon, 11)}_x (z) - (\tilde{W}^{(x, \varepsilon, 12)}_x (z) \right] .
\]

By Lemma 4.2 and Lemma 4.5 we have that for every \(\delta > 0\)

\[
| \langle (\tilde{W}^{(x, \varepsilon, 11)}_x (z), (\tilde{W}^{(x, \varepsilon, 11)}_x (\tilde{z}) \rangle |
\]

\[
\lesssim \int \int f(\varepsilon, \vartheta, 1) K(z - z_1) \left( f(\varepsilon, \vartheta, 1) (z_1, \tilde{z}_1) - f(\varepsilon, \vartheta, 1) (z, \tilde{z}_1) - f(\varepsilon, \vartheta, 1) (z_1, \tilde{z}) + f(\varepsilon, \vartheta, 1) (z, \tilde{z}) \right)
\]

\[
\lesssim \int \int \| z - z_1 \|_{s}^{-5+\delta} \| \tilde{z} - \tilde{z}_1 \|_{s}^{-5+\delta} + \| z - \tilde{z}_1 \|_{s}^{-1-8\delta} + \| z - \tilde{z}_1 \|_{s}^{-1-8\delta} + \| z_1 - \tilde{z}_1 \|_{s}^{-1-8\delta} + \| z - \tilde{z}_1 \|_{s}^{-1-8\delta}
\]

\[
\lesssim \| z - \tilde{z} \|_{s}^{-1-8\delta},
\]

where we used the interpolation in the second inequality and [Hai14, Lemma 10.14] in the last inequality. Choose \(\tilde{W}^{(11)}_x (z, z_1) = \int (L(z - z_2) (K(z_2 - z_1) - K(z_2 - z)) dz_2\) as in the proof of [Hai14, Theorem 10.22], where \(L = (K * K)^2 K\).

Moreover, by similar arguments as the proof of Lemma 4.5 we have

\[
| \langle (K_{\varepsilon, \vartheta} - K_{\varepsilon})(z, \cdot), (K_{\varepsilon, \vartheta} - K_{\varepsilon})(\tilde{z}_1, \cdot) - (K_{\varepsilon, \vartheta} - K_{\varepsilon})(\tilde{z}_2, \cdot) \rangle |
\]

\[
\lesssim \vartheta^{\delta} \| \tilde{z}_1 - \tilde{z}_2 \|_{s}^{\delta} (\| \tilde{z}_1 - z \|_{s}^{-1-2\delta - 2\vartheta} + \| \tilde{z}_2 - z \|_{s}^{-1-2\delta - 2\vartheta}),
\]
and
\[
\left| (K_\varepsilon - K)(z, \cdot), (K_\varepsilon - K)(\bar{z}_1, \cdot) - (K_\varepsilon - K)(\bar{z}_2, \cdot) \right|
\lesssim \varepsilon^{2\theta} \|z_1 - \bar{z}_1\|_s^\delta (\|\bar{z}_1 - z\|_s^{1-2\delta-2\theta} + \|\bar{z}_2 - z\|_s^{1-2\delta-2\theta}),
\]
which combining with Lemma 4.3 imply that
\[
\left| \langle (\delta\hat{V}_x^{(\varepsilon, \vartheta, 11)}\tau)(z), (\delta\hat{V}_x^{(\varepsilon, \vartheta, 11)}\tau)(\bar{z}) \rangle \right|
\lesssim (\varepsilon^{2\theta} + \vartheta^{\theta}) \int \int \left[ |z - z_1|_s^{-5+\delta} |\bar{z} - \bar{z}_1|_s^{-5+\delta} \left| |z_1 - \bar{z}_1|_s^{-1-8\delta-2\theta} + |z - \bar{z}_1|_s^{-1-8\delta-2\theta} \right| + |z - z_1|_s^{-5+\delta-\theta} |\bar{z} - \bar{z}_1|_s^{-5+\delta-\theta} \right]
\left[ |z_1 - \bar{z}_1|_s^{-1-8\delta} + |z - \bar{z}_1|_s^{-1-8\delta} + |z_1 - \bar{z}_1|_s^{-1-8\delta} + |z - \bar{z}_1|_s^{-1-8\delta} \right] dz_1 d\bar{z}_1
\lesssim (\varepsilon^{2\theta} + \vartheta^{\theta}) \|z - \bar{z}\|_s^{-1-8\delta-2\theta},
\]
which is valid uniformly over \(\varepsilon, \vartheta \in (0, 1]\) satisfying \(\varepsilon^2 \leq C_0 \vartheta\), provided that \(0 < \theta < \delta < 1\). Here we used [Hai14, Lemma 10.14] in the last inequality. We also use Lemma 4.2 to obtain that
\[
\left| \langle (\hat{V}_x^{(\varepsilon, \vartheta, 12)}\tau)(z), (\hat{V}_x^{(\varepsilon, \vartheta, 12)}\tau)(\bar{z}) \rangle \right|
\lesssim \int \int |z - z_1|_s^{-2-\delta} |K(x - z_1)||z_1 - \bar{z}_1|_s^{-1-\delta} |\bar{z} - \bar{z}_1|_s^{-2-\delta} |K(x - \bar{z}_1)| dz_1 d\bar{z}_1
\lesssim \|z - x\|_s^{-\frac{1}{2}-2\delta} (G^5(z - \bar{z}) + G^5(z - x) + G^5(\bar{z} - x) + G^5(0)),
\]
holds uniformly over \(\varepsilon, \vartheta \in (0, 1]\) satisfying \(\varepsilon^2 \leq C_0 \vartheta\), provided that \(\theta < 1\) and that \(\delta > 0\). Here the function \(G^5\) is a bounded function and is given by
\[
G^5(z - \bar{z}) = \begin{array}{ccccc}
& & & & \\
& & -1-\delta & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\delta - 4.5 & & & & -4.5 + \delta
\end{array}
.
\]
Similarly by Lemmas 4.2, 4.3 and (4.14), (4.15), we have
\[
\left| \langle (\delta\hat{V}_x^{(\varepsilon, \vartheta, 12)}\tau)(z), (\delta\hat{V}_x^{(\varepsilon, \vartheta, 12)}\tau)(\bar{z}) \rangle \right|
\lesssim (\varepsilon^{2\theta} + \vartheta^{\theta}) \|z - x\|_s^{-\frac{1}{2}-2\theta} \|\bar{z} - x\|_s^{-\frac{1}{2}-2\theta},
\]
holds uniformly over \(\varepsilon, \vartheta\) satisfying \(\varepsilon^2 \leq C_0 \vartheta\), provided that \(\theta < 1\) and that \(\delta > 0\). Hence we conclude that (4.12) holds for all \(\tau \in \mathcal{F}^-\), which implies the results.

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