Predicting the $D_s^{(*)}D_s^{(*)}$ bound states as the partners of $X(3872)$

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In this work, we investigate the SU(3) flavor symmetry, heavy quark spin symmetry and their breaking effects in the di-meson systems. We prove the existence of the $[\bar{D}_s^*D_s^{(*)}]^{0+}$, $[D_s^*D_s/D_s^*D_s^{(*)}]^{1+}$, and $[D_s^*D_s^{(*)}]^{1+}$ bound states as the consequence of two prerequisites in the SU(3) flavor symmetry and heavy quark spin symmetry. The first prerequisite, the $X(3872)$ as the weakly $D^*D/\bar{D}D^*$ bound state is supported by its mass and decay branching ratios. The second prerequisite, the existence of the $[\bar{D}_s^*D_s^{(*)}]^{0+}$ bound state is supported by the lattice QCD calculation [1] and the observation of $X_{c0}(3930)$ by the LHCb Collaboration [2, 3]. We hope the future experimental analyses can search for these bound states in the $B \to D_{s0}(s)D_s^{(*)}h$ processes ($h$ denotes the light hadrons). The $[\bar{D}_s^*D_s^{(*)}]^{0+}$ bound state is also expected to be reconstructed in the $B \to J/\psi\phi K$ decay.

Keywords: SU(3) flavor symmetry; hadronic molecule; $X(3872)$

I. INTRODUCTION

The SU(3) flavor symmetry [SU(3)$_F$] is an approximate symmetry of QCD Lagrangian, which manifests itself in the hadron spectra. From the observation of $\Omega^-$ in the 1960s [4], the SU(3)$_F$ symmetry was well used to classify the mesons and baryons into multiplets. However, for a long time, the SU(3) flavor symmetry was seldom investigated in the superstructures of QCD, such as the di-meson systems.

Since the observation of $X(3872)$ in 2003 [5], more and more hidden charm/bottom exotic hadrons have been reported in experiments [6–11]. Many of these states are in the proximity of di-hadron thresholds. For example, the $X(3872)$ [5], $Z_c(3900)$ [12], $Z_c(4020)$ [13, 14] and $P_c$ [15, 16] states are near the $D^0\bar{D}^0$, $D^+D^-$, $D^*\bar{D}^*$, $D^\ast_0D^*$, and $\Sigma_cD^*$ thresholds, respectively, which indicates these states might be the di-hadron bound states or resonances. Recently, the strange hidden charm pentaquark candidate $P_{cs}(4459)$ [17] was reported by the LHCb Collaboration, which is in good agreement with our previous prediction of the $\Xi_c\bar{D}^*$ bound state with a mass 4456.9 MeV [18]. Very recently, the $Z_{cs}(3985)^-$ state was observed by the BESIII Collaboration [19], which could be interpreted as the $\bar{D}_s^*D_s^{(*)}/\bar{D}_s^*D_s^{(*)}$ di-meson states as the $U$-spin partner of $Z_c(3900)^-$ [20, 21]. The $P_{cs}(4459)^0$ and $Z_{cs}(3985)^-$ states are good candidates of the strange partners of $P_c$ and $Z_c(3900)$ states in the SU(3)$_F$ symmetry, which inspired many works about the SU(3)$_F$ symmetry for the di-hadron systems [22–39].

The heated discussion on the strange partners of $P_c$ and $Z_c$ states reminds us of the SU(3)$_F$ partner states of $X(3872)$, the super star in the exotic hadron family. In Refs. [40–42], the heavy quark spin symmetry (HQSS) partners of $X(3872)$ with the $J^{PC}$ quantum numbers $2^{++}$ was proposed. In this work, we will consider the $\bar{D}_s^*D_s^{(*)}$ states as the counterparts of $X(3872)$ in the HQSS and SU(3)$_F$ symmetry. Recently, the lattice QCD calculation with $m_{\pi} \simeq 280$ MeV indicated the existence of the scalar $D_sD_s$ bound state [1], which might correspond to the $X_{c0}(3930)$ observed by the LHCb Collaboration [2, 3]. In this work, we will show that the existence of the $\bar{D}_s^*D_s^{(*)}$ bound states is the natural consequence of two prerequisites in the SU(3)$_F$ symmetry and HQSS,

- The $X(3872)$ is the molecular state with its mass coinciding exactly with the $D_0^0D_0^0$ threshold;
- There exist the $\bar{D}_sD_s$ bound states with $J^{PC} = 0^{++}$.

This work is organized as follows. In Sec. II, we discuss the SU(3)$_F$ symmetry, HQSS and their breaking effects for the $\bar{D}_s^*D_s^{(*)}$ di-meson systems. In Sec. III, we prove the existence of $\bar{D}_s^*D_s^{(*)}$ bound states from the perspective of SU(3)$_F$ symmetry and HQSS as the partners of $X(3872)$. In Sec. IV, we show that the predictions of the $\bar{D}_s^*D_s^{(*)}$ bound states are valid when the coupled-channel formalism is adopted for $X(3872)$. We conclude with a brief discussion and a short summary in Sec. V.

II. SU(3)$_F$ SYMMETRY AND HQSS FOR DI-MESON SYSTEMS

We will take the $\bar{D}_s^*D_s^{(*)}$ as an example to discuss the SU(3)$_F$ symmetry and HQSS for the di-meson systems. In Fig. 1, we present the $\bar{D}_s^*D_s^{(*)}$ multiplets and their flavor wave functions in the SU(3)$_F$ symmetry. Their other HQSS partners have the similar structures. We can see that the hidden strange $D_sD_s$ system will mix with the $\bar{D}D$ system in the SU(3)$_F$ limit. For the spin wave function, we list the inner products of HQSS basis and di-meson basis in Table I. In the heavy quark limit, the mixture also occurs in the two $0^{++}$ di-meson states and two $1^{++}$ di-meson states, respectively. The $C$-parity is only for the neutral states here and below.
For the charmed mesons, the breaking effects will result in two kinds of mass splittings,
\[ D_s^{(*)} - D_s^{(*)} \approx 100 \text{ MeV}, \]
\[ D_s^{(*)} - D_s^{(*)} \approx 140 \text{ MeV}, \]
which arises from the SU(3)$_F$ symmetry and HQSS breaking effects, respectively. However, the observed di-meson candidates are in the proximity of thresholds about several MeVs, which are much smaller than the mass splittings in Eqs. (1) and (2). In other words, the interactions accounting for these molecular di-meson systems are too weak to lead to significant mixture between states with mass difference over 100 MeV. Thus, when the large mass splittings in Eqs. (1) and (2) are involved, the states in the real world will be distinguishable according to the di-meson thresholds rather than the SU(3)$_F$ symmetry and HQSS.

Unlike the di-meson thresholds, it is reasonable to presume the interactions between two mesons satisfy the SU(3)$_F$ and HQSS. We define the relative energy $\Delta E$ and binding energy $E_b$ of di-meson system as
\[ \Delta E = -E_b = M_{\text{dimeson}} - M_{\text{threshold}}. \]
The $\Delta E$ of the di-meson systems are at the order of 10 MeV. The relative momentum is at the order of $\sqrt{\Delta E m_{D^0}} \approx 130$ MeV, which is a small scale compared to the charm quark mass. Thus, the interactions between two mesons are very soft, which manifests the heavy quark symmetry. The interactions between mesons originating from either the flavor-blind gluonic interaction or exchanging SU(3)$_F$ multiplet mesons will result in the approximate SU(3)$_F$ symmetry. Thus, in this work, we will consider the SU(3)$_F$ symmetry and HQSS breaking effects in constructing the spin and flavor functions but take the symmetry limits in modeling the hadronic interaction.

We could embed the HQSS and SU(3)$_F$ symmetry with the interactions at the quark level,
\[ V_{q\bar{q}} = c_1 + c_2 s_1 \cdot s_2 + c_3 C_2 + c_4 (s_1 \cdot s_2) C_2, \]
where $V_{q\bar{q}}$ denotes the interactions between the light quark and antiquark [The interactions involving the heavy (anti)quark is neglected]. The $s_1$ is the spin operator of the light (anti)quark. $C_2 = -\sum_{i=1}^{8} \lambda_i^2 \lambda_i^2$ is the Casimir operator in the flavor space. Eq. (3) represents the general interaction of the $S$-wave $D_s^{(*)} D_s^{(*)}$ channel in the SU(3)$_F$ symmetry and HQSS, which could be realized in other equivalent approaches [43, 44].

III. $X(3872)$ AS THE $D_s^{(*)} D_s^{(*)}$ MOLECULAR STATE

The $X(3872)$ mass coincides exactly with the $D_s^{(*)} D_s^{(*)}$ threshold [45],
\[ m_{D^0} + m_{D^{*0}} - m_{X(3872)} = (0.00 \pm 0.18) \text{ MeV}. \]
Meanwhile, it has the large branching fraction of $X(3872) \rightarrow D_s^{(*)} D_s^{(*)}$ [45]. Its mass and branch fraction indicate that the main component of $X(3872)$ is $D_s^{(*)} D_s^{(*)}$. The scarcity of $D^{*+} D^+ / D^- D^{*+}$ component induce to large isospin violation effect. The large ratio of the branching fraction $B(X \rightarrow J/\psi \omega) / B(X \rightarrow J/\psi \pi^+ \pi^-)$ also support that the $X(3872)$ is not an eigenstate of isospin [46]. The $D^{*+} D^+ / D^- D^{*+}$ threshold is above the $X(3872)$ by about 8 MeV, which could be regarded as a large scale compared with the width and binding energy of $X(3872)$ as the $D_s^{(*)} D_s^{(*)}$ di-meson system. Similar perspectives were addressed in the refined one-boson-exchanged calculation [47] and XEFT [48]. In this section, we will treat the $X(3872)$ as the $D_s^{(*)} D_s^{(*)}$ di-meson system.

As a $D_s^{(*)} D_s^{(*)}$ di-meson system, the flavor wave function of $X(3872)$ in the light part will be $|u\bar{u}\rangle$. As a consequence, it has the same flavor matrix element with the $D_s^{(*)} D_s^{(*)}$ systems,
\[ \langle C_2 \rangle_{u\bar{u}} = \langle C_2 \rangle_{s\bar{s}}. \]

In other words, the $D_s^{(*)} D_s^{(*)}$ systems are the $V$-spin partners of the $X(3872)$ [20]. Thus, we could reparameterize the interactions of $X(3872)$ and $D_s^{(*)} D_s^{(*)}$ systems as,
\[ V_{q\bar{q}} = \tilde{c}_1 + \tilde{c}_2 s_1 \cdot s_2, \]
where their common flavor information has been absorbed into the new coupling constants $\tilde{c}_1$ and $\tilde{c}_2$.

In the spin space, the matrix elements read [20],
\[ \langle s_1 \cdot s_2 \rangle_{\{pp, vv\}}^{0^+} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1 \end{pmatrix}, \]
\[ \langle s_1 \cdot s_2 \rangle_{\{pv, vv\}}^{1^+} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} \end{pmatrix}, \]
\[ \langle s_1 \cdot s_2 \rangle_{\{pv\}}^{2^+} = \frac{1}{4}, \quad \langle s_1 \cdot s_2 \rangle_{\{vv\}}^{2^+} = \frac{1}{4}. \]
TABLE I. The HQSS basis, di-meson basis and their inner product in the spin space. In the heavy quark symmetry basis, the spin wave functions are denoted as $|S^\text{H}/L; S^\text{C}/L; J^{PC}\rangle$, where the subscript $H/L$ represents the heavy/light degree of freedom ($C$-parity only for the neutral states).

| $|\bar{D}D_1; 0^{++}\rangle$ | $|\frac{1}{\sqrt{2}}(\bar{D}^*D + \bar{D}D^*); 1^{+-}\rangle$ | $|\frac{1}{\sqrt{2}}(\bar{D}^*D - \bar{D}D^*); 1^{+-}\rangle$ | $|\bar{D}^*D^*; 0^{++}\rangle$ | $|\bar{D}^*D^*; 1^{+-}\rangle$ | $|\bar{D}^*D^*; 2^{++}\rangle$ |
|---|---|---|---|---|---|
| $|0^{++}_H; 0^{++}_L; 0^{++}\rangle$ | $-\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $|0^{+-}_H; 1^{+-}_L; 1^{+-}\rangle$ | $1$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $|1^{+-}_H; 0^{+-}_L; 1^{+-}\rangle$ | $-\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $|1^{+-}_H; 1^{+-}_L; 0^{+-}\rangle$ | $-\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $|1^{+-}_H; 1^{+-}_L; 1^{+-}\rangle$ | $1$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |
| $|1^{+-}_H; 2^{++}\rangle$ | $1$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | $1$ |

![Diagram](image)

FIG. 2. The interactions of $\bar{D}^*_sD^*_s$ systems in the order of becoming more attractive.

where $P$ and $V$ represent the pseudoscalar and vector heavy mesons, receptively. The superscript denotes the $J^{PC}$ of the di-meson channel. Though we present the off-diagonal matrix elements, their mixing effect is negligible as discussed in Sec. II. We can easily obtain the following relation,

$$
(\frac{V^1_{pp'}}{V^0_{VV'}} - \frac{V^0_{pp'}}{V^1_{VV'}}): (\frac{V^1_{VV'}}{V^0_{pp'}} - \frac{V^0_{pp'}}{V^1_{VV'}}) = 1 : -2 : -1.
$$

The recent lattice QCD calculation yielded a shallow $[\bar{D}_sD_s]^{0^{++}}$ bound state with $\Delta E = -6.2^{+3.8}_{-2.0}$ MeV [1]. The $X(3872)$ is a marginal bound state and its binding energy is less than 1 MeV. Since the $D_s D_s$ is a deeper bound state than $X(3872)$, the interaction in the $[D_s D_s]^{0^{++}}$ channel is more attractive than that in the $[\bar{D}^0 D^0]^{0^{++}}$ channel. According to Eqs. (7)-(9), the $[\bar{D}_s D_s]^{0^{++}}$ channel is more attractive than $[\bar{D}_s D_s/ DJ^* D_s]^{1^{++}}$ one, which indicates the coupling constant in Eq. (6) $\tilde{c}_2 > 0$. Thus, we obtain the order of interactions for $D^*_s(D^*_s)$ in Fig. 2, which become more attractive along the arrow. As a consequence, the more attractive interactions in the $[\bar{D}_s D_s/ DJ^* D_s]^{1^{++}}$, $[D_s^* D_s^*]^{1^{++}}$ and $[\bar{D}_s D_s]^{0^{++}}$ will give rise to deeper binding solutions. Among them, the $[\bar{D}_s D_s]^{0^{++}}$ bound state is the deepest one.

Apart from the evidence of the $D_s D_s$ bound states in lattice QCD, the $\chi_{c0}(3930)$ observed by LHCb collaboration was a good candidate in experiment [2, 3]. The $\chi_{c0}(3930)$ is below the $D_s D_s$ threshold about 12.9 MeV, which would lead to the deeper $[D_s^* D_s/ DJ^* D_s]^{1^{++}}$, $[D_s^* D_s^*]^{1^{++}}$ and $[D_s^* D_s]^{0^{++}}$ binding solutions. The existence of the $[D_s^* D_s/ DJ^* D_s]^{1^{++}}$, $[D_s^* D_s^*]^{1^{++}}$ and $[D_s^* D_s]^{0^{++}}$ bound states arises from the SU(3)$_F$ symmetry and HQSS, which is independent of the specific interaction mechanism.

We adopt a simple model to demonstrate the argument proposed in Sec. III numerically. We assume the hadronic interaction of the $X(3872)$ and $\bar{D}^*_sD^*_s$ systems are contact interaction, as shown in Eq. (6). We introduce the $g(p) = \exp(-p^2/\Lambda^2)$ to regulate divergence in Lippmann-Schwinger equations (LSEs),

$$
T(p', p; E) = V(p', p) + \int d^3p'' V(p'', p') T(p'', p; E),
$$

(11)

where $\mu$ is the reduced mass. The potential reads

$$
V(p', p'') = v g(p') g(p''),
$$

(12)

where $v$ is the coupling constant. In the calculation, we take the physical masses of the corresponding channels [45]. With substitution $T(p', p; E) = t(E) g(p') g(p'')$ in Eq. (11), we obtain the algebraic equation,

$$
t(E) = v + v F(E) t(E),
$$

(13)

where

$$
F(E) = \int_0^\infty dp'' \frac{4\pi}{(2\pi)^3} \frac{2p''^2}{2\mu E - p''^2 + i\epsilon},
$$

(14)

The solution is

$$
t^{-1}(E) = a(\Lambda) - F(E; \Lambda), \text{ with } a(\Lambda) \equiv \frac{1}{v(\Lambda)},
$$

(15)

where the cutoff-dependence of $a$ and $F$ cancels out. We can obtain the pole of $t(E)$ below the threshold which corresponds to the bound state.

In this work, we choose the cutoff $\Lambda = 1.0$ GeV. One can choose different regulators and cutoffs, or alternatively solve the Schrödinger equation [43, 44] in coordinate space, which gives the same physical implications. We choose the 0 and 1 MeV as the lower and upper limits of the binding energy of $X(3872)$, and adopt the binding energy of the $[\bar{D}_s D_s]^{0^{++}}$ in the range of lattice QCD [1] and experimental measurement [2, 3] as inputs. We present the predictions of the $[\bar{D}_s D_s]^{0^{++}}$, $[\bar{D}_s D_s/ DJ^* D_s]^{1^{++}}$, and $[D_s^* D_s^*]^{1^{++}}$ systems in Table II. There do exist the bound states for these systems with the binding energy from several MeVs to several.
tens of MeVs. One can see that they are bound more deeply than the $[\bar{D}_s D_s]^{0+}$ state. Among them, the $[\bar{D}_s D_s]^{0+}$ is the deepest one. The mass ranges of the $[\bar{D}_s D_s]^{0+}$, $[\bar{D}_s D_s/\bar{D}_s D_s]^{1+}$ and $[\bar{D}_s D_s]^{1+}$ systems are predicted to be $[4140.1, 4216.1]$, $[4036.8, 4075.6]$ and $[4177.2, 4218.1]$ MeV, respectively. Though the binding energies are sensitive to the input, the existence of these bound states is persistent.

We take the $[\bar{D}_s D_s]^{0+}$ state as an example to show the related parameter regions in Fig. 3. In the overlap parameter region of the $X(3872)$ and $[\bar{D}_s D_s]^{0+}$ as the bound states, the $[\bar{D}_s D_s]^{0+}$ state is also bound. Taking either the lattice QCD $[\bar{D}_s D_s]^{0+}$ result [1] or the $\chi_{c0}(3930)$ as the $[\bar{D}_s D_s]^{0+}$ bound state [2, 3] gives the same implication.

We adopt the renormalizable effective field theory that describes two scattering channels [52–54]. We introduce the two channel potential for $X(3872)$,

$$
\begin{bmatrix}
v_{11} & v_{12} \\
v_{12} & v_{22}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
V^{I=1} + V^{I=0} & V^{I=0} - V^{I=1} \\
V^{I=0} - V^{I=1} & V^{I=1} + V^{I=0}
\end{bmatrix},
$$

(16)

where the channel 1 is the $\bar{D}D^0/\bar{D}D^0$ channel and channel 2 is the $D^+D^-/D^-D^+$ channel. $V^{I=0}$ and $V^{I=1}$ are the interactions for $I = 0$ and $I = 1$ channels, respectively. The potential in Eq. (16) satisfies the isospin symmetry. We introduce the regulator $g(p^2)$ and adopt the separable potentials like Eq. (12). We can reduce the coupled-channel LSEs into algebraic equations,

$$
v_{ij}(E) = v_{ij} + \sum_{a=1,2} v_{ia} F_a t_{aj}(E),
$$

(17)

where

$$
F_1(E) = \int_0^\infty dp d^2p' \frac{4\pi}{(2\pi)^3} \frac{2\mu p'' g(p'')^2}{2\mu E - p'' + i\epsilon},
$$

$$
F_2(E) = \int_0^\infty dp d^2p' \frac{4\pi}{(2\pi)^3} \frac{2\mu p'' g(p'')^2}{2\mu E - p'' + i\epsilon},
$$

(18)

$\delta = m_{D^+} + m_{D^-} - m_{D^0} - m_{\bar{D}0} \approx 8$ MeV. For convenience, we introduce the hard regulator $g(p) = \theta(\Lambda - p)$. For $E < 0$, we obtain

$$
F_1 = \frac{8\mu\pi}{(2\pi)^3} \left[ -\Lambda + \gamma \arctan \left( \frac{\Lambda}{\gamma} \right) \right],
$$

(19)

where $\gamma = \sqrt{-2\mu E}$ is the binding momentum. If we set the $\Lambda \gg \gamma$,

$$
F_1 \approx \frac{8\mu\pi}{(2\pi)^3} \left[ -\Lambda + \frac{\gamma}{2} \right].
$$

(20)

We define $\omega = \sqrt{2\mu\delta}$. For $\Lambda \gg \omega$, $F_2$ reads

$$
F_2 \approx \frac{8\mu\pi}{(2\pi)^3} \left[ -\Lambda + \frac{\omega^2}{2} \right].
$$

(21)

Solving Eq. (17), we obtain

$$
t^{-1} = \begin{bmatrix}
\frac{v_{12}}{v_{12} - v_{11}} & -F_1 & -F_2 \\
\frac{v_{12}}{v_{12} - v_{11}} & \frac{v_{12}}{v_{12} - v_{11}} & -F_2
\end{bmatrix},
$$

(22)

In order to discuss the cutoff-dependence, we introduce $a_{11}, a_{12}$ and $a_{22}$ as the combination of $v_{ij}$ to make

$$
t^{-1} = \begin{bmatrix}
a_{11} - F_1 & a_{12} \\
a_{12} & a_{22} - F_2
\end{bmatrix}.
$$

(23)

The relations of $a_{ij}$ and $v_{ij}$ read

$$
v_{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2}, \quad v_{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2},$$

$$
v_{12} = -\frac{a_{12}}{a_{11}a_{22} - a_{12}^2}.
$$

(24)

IV. $X(3872)$ IN THE COUPLING-CHANNEL FORMALISM

There are different interpretations for the large isospin breaking decays of $X(3872)$. In Refs. [50, 51], the authors interpreted the isospin breaking effect as the consequence of the mass splitting in the propagators of the $D^0 D^0 / D^0 D^0$ and $D^+D^- / D^-D^+$ components, which is amplified by the different effective phase spaces of $J/\psi p$ and $J/\psi \omega$. In this section, we include $D^+D^- / D^-D^+$ components for $X(3872)$ with the coupled-channel formalism. We will demonstrate that a less attractive (more repulsive) $[\bar{D}_s D_s/\bar{D}_s D_s]^{1+}$ interaction is obtained than in the single-channel case. Thus, the conclusions in Sec. III are still valid.
TABLE II. The $\Delta E$ and masses for the [\(\tilde{D}^* D_s^*\)]^{0++, 1++}, [\(\tilde{D}^* D_s^*/\tilde{D} D_s^*\)]^{1++, 1++} and [\(\tilde{D}^* D_s^*\)]^{1++, 1++} systems (the results are given in units of MeV). The binding energies of $X(3872)$ and [\(\tilde{D} D_s\)]^{0++, 1++} systems are adopted as the inputs [1–3]. In the last two rows, we list the numerical results from Ref. [49].

| $X(3872)$ input | $[\tilde{D}^* D_s^*]$^{0++, 1++} | $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]$^{1++, 1++} | $[\tilde{D}^* D_s^*]$^{1++, 1++} |
|-----------------|---------------------------------|---------------------------------|---------------------------------|
| $\Delta E$ | $\Delta E$ | $M$ | $\Delta E$ | $M$ | $\Delta E$ | $M$ |
| 0.0 | -2.4 | 3934.3 | -20.3 | 4204.1 | -9.5 | 4071.0 | -11.4 | 4213.0 |
| 0.0 | -6.2 | 3930.5 | -45.5 | 4178.9 | -22.5 | 4058.0 | -25.2 | 4199.2 |
| 0.0 | -8.2 | 3928.5 | -57.6 | 4166.8 | -29.0 | 4051.5 | -32.0 | 4192.4 |
| 0.0 | -12.9 | 3923.8 | -84.3 | 4140.1 | -43.7 | 4036.8 | -47.2 | 4177.2 |
| -1.0 | -2.4 | 3934.3 | -8.3 | 4216.1 | -4.9 | 4075.6 | -6.3 | 4218.1 |
| -1.0 | -6.2 | 3930.5 | -28.9 | 4195.5 | -15.9 | 4064.6 | -18.2 | 4206.2 |
| -1.0 | -8.2 | 3928.5 | -39.6 | 4184.8 | -21.7 | 4058.8 | -24.4 | 4200.0 |
| -1.0 | -12.9 | 3923.8 | -64.1 | 4160.3 | -35.2 | 4045.3 | -38.5 | 4185.9 |
| cutoff-I [49] | -13 | 3924 | -84 | 4140 | -46 | 4035 | -47 | 4177 |
| cutoff-II [49] | -9 | 3928 | -84 | 4140 | -41 | 4040 | -44 | 4180 |

In order to make the $t$-matrix cutoff-independent, the $a_{11}$ and $a_{22}$ depend on $\Lambda$ as

$$a_{11}(\Lambda) = a_{11}(\Lambda_0) - \frac{\mu}{\pi^2}(\Lambda - \Lambda_0),$$

$$a_{22}(\Lambda) = a_{22}(\Lambda_0) - \frac{\mu}{\pi^2}(\Lambda - \Lambda_0).$$

(25)

The above relations make $(a_{11} - F_1)$ and $(a_{22} - F_2)$ cutoff independent. $a_{12}$ itself is cutoff-independent.

From Eq. (23), the $t$-matrix reads

$$t = \begin{bmatrix}
\frac{1}{a_{11} - F_1 - a_{12}} & \frac{-a_{12}}{(a_{11} - F_1)(a_{22} - F_2) - a_{12}} \\
\frac{-a_{12}}{(a_{11} - F_1)(a_{22} - F_2) - a_{12}} & \frac{1}{(a_{22} - F_2) - a_{12}}
\end{bmatrix}$$

(26)

The pole of the $t$-matrix is obtained by

$$[a_{11} - F_1(\gamma)][a_{22} - F_2(\gamma)] - a_{12}^2 = 0,$$

(27)

where

$$a_{11} - F_1(\gamma) = C - \frac{\mu}{2\pi}\gamma,$$

$$a_{22} - F_2(\gamma) = C - \frac{\mu}{2\pi}\sqrt{\gamma^2 + \omega^2}.$$

(28)

$C$ is a constant. One of the solutions is $\gamma_1 = \gamma_{X(3872)}$. If $a_{11} - F_1(\gamma_1) > 0$ and $a_{22} - F_2(\gamma_1) > 0$, there should exist the second binding solution with $\gamma_2 > \gamma_1$. As shown in the left plot in Fig. 4, $a_{11} - F_1(\gamma)$ and $a_{22} - F_2(\gamma)$ will decrease with $\gamma$ and the second binding solution $\gamma_2$ appear with $a_{11} - F_1(\gamma_2) < 0$ and $a_{22} - F_2(\gamma_2) < 0$. However, there is no such a state observed in experiments. Thus, $a_{11} - F_1(\gamma_1) < 0$.

When we try to relate the $v_{11}$ in the coupled-channel formalism (for [\(\tilde{D}^* D_s^*/\tilde{D} D_s^*\)]^{1++, 1++}) to the $v$ in the single-channel calculation (for the [\(\tilde{D}^* D_s^*/\tilde{D} D_s^*\)]^{1++, 1++} system), we cannot expect $v_{11}(\Lambda) = v(\Lambda)$ is always valid for all the cutoff scale because the cutoff-dependence of $v_{11}$ is affected by the second channel. However, the relation of $a_{11}$ and $a$ reads

$$a_{11}(\Lambda) = a(\Lambda).$$

(29)

The validity of above relation is cutoff-independent. In the single-channel calculation, $a$ depends on $\Lambda$ as $a(\Lambda) = a(\Lambda_0) - \frac{\mu}{\pi^2}(\Lambda - \Lambda_0)$, which is similar to Eq. (25). Thus, the difference of $a(\Lambda)$ and $a_{11}(\Lambda)$ is a cutoff-independent constant. We will choose a $\Lambda_0$ to fix their difference. The cutoff-off in Eqs. (17) and (18) is a scale to regulate the divergence in LSEs. The freedom and dynamics at the larger scale than $\Lambda$ are absorbed into the cutoff-dependent coupling constants $v_{11}$. At a very large scale, the tiny coupled-channel effect is absorbed into the coupling constants. Thus, we can expect the $v = v_{11}$ at the large $\Lambda$ limit. When we take $\Lambda_0 \geq \sqrt{2\mu\delta}$ and $\Lambda_0 \gg \frac{\pi^2}{\mu}a_{12}$, we have,

$$\frac{1}{v_{11}(\Lambda_0)} = a_{11}(\Lambda_0) - \frac{a_{12}^2}{a_{22}(\Lambda_0)} = a_{11}(\Lambda_0),$$

(30)

$$\frac{1}{v(\Lambda_0)} = \frac{1}{v(\Lambda)} = a(\Lambda_0).$$

(31)

We can obtain Eq. (29) from $a_{11}(\Lambda_0) = a(\Lambda_0)$. This relation can also be rewritten as

$$\frac{1}{v(\Lambda)} + \frac{a_{12}^2}{a_{22}(\Lambda)} = \frac{1}{v(\Lambda)}.$$  

(32)

Thus, we relate the interaction of $X(3872)$ in the coupled-channel formalism to the $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ interaction.

For the $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ system, $a - F(\gamma_1) = a_{11} - F_1(\gamma_1) < 0$. Thus, there are two possibilities for its binding solution as shown in the middle plot and right plot in Fig. 4, respectively. In the first case, the possible binding solution of $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ appears with binding momentum $\gamma_{[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}} < \gamma_1$. In other words, $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ is more shallowly bound state than $X(3872)$, which indicates there is tiny parameter region for $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ to form bound state. It is more likely that there is no binding solution for $[\tilde{D}^* D_s^*/\tilde{D} D_s^*]^{1++, 1++}$ system as shown in the right plot in Fig. 4. In both cases, the
FIG. 4. The possible solutions of Eq. (27). We use the blue dashed line and orange dot-dashed line to denote the $a_{11} - F_1$ and $a_{22} - F_2$, respectively. Their product is presented in blue solid line. The black dashed line represents the $a_1^2$. The cross points of black dashed line and blue solid line represent the binding solutions of Eq. (27), which are labeled by red points. We use the green points to denote the solutions of $a - F(\gamma) = 0$ (the same as $a_{11} - F_1(\gamma) = 0$), which represent the binding solutions of $[D_s D_s^*/D_s D_s^*]^{1+ +}$. The left subfigure represents the case with $a_{11} - F_1(\gamma_1) > 0$ and $a_{22} - F_2(\gamma_1) > 0$. The middle one and the right one represent cases with $a_{11} - F_1(\gamma_1) < 0$ and $a_{22} - F_2(\gamma_1) < 0$. In the middle subfigure, there is a binding solution for $[D_s D_s^*/D_s D_s^*]^{1+ +}$. In the right subfigure, there does not exist $[D_s D_s^*/D_s D_s^*]^{1+ +}$ bound state.

$[D_s^* D_s/D_s D_s^*]^{1+ +}$ interaction extracted from the coupled-channel scheme is less attractive (more repulsive) than that from single-channel calculation in Sec. III (In Appendix A, we will rule out the possibility corresponding to middle subfigure in Fig. 4.). According to Fig. 2, the less attractive (more repulsive) $[D_s D_s/D_s D_s^*]^{1+ +}$ interaction is, the more attractive $[D_s^* D_s/D_s D_s^*]^{1+ +}$, $[D_s^* D_s^*]^{1+ +}$, and $[D_s^* D_s^*]^{0+ +}$ channels are. Thus, the existences of $[D_s^* D_s/D_s D_s^*]^{1+ +}$, $[D_s^* D_s^*]^{1+ +}$, and $[D_s^* D_s^*]^{0+ +}$ bound states will not change. One can notice that we do not specify the mechanism of large isospin violation decays of $X(3872)$. Our predictions are not affected by the underlying mechanism qualitatively.

V. SUMMARY AND DISCUSSION

In this work, we prove the existence of the $[D_s^* D_s]^{0+ +}$, $[D_s^* D_s/D_s D_s^*]^{1+ +}$, and $[D_s^* D_s^*]^{1+ +}$ bound states, which is the natural consequence of SU(3)$_F$ symmetry and HQSS. These three states are all deeper bound states than the $[D_s D_s]^{0+ +}$ system, while the $[D_s^* D_s^*]^{1+ +}$ is the deepest one.

As shown in Sec. II, the strange quark destroys the SU(3)$_F$ symmetry for the di-meson systems, which suppresses the mixture of the $D^*(s) D^*(s)$ and $D(s) D^{(*)}$ systems. However, the isospin symmetry is still a very good approximation for the di-meson systems, which is supported by the observation of the almost degenerate $Z_c(3900)/Z_c(4020)$ isospin triplet [45, 55–57]. The neutral states are the half-and-half mixture of the $D^{(*)} D^{(*)}$ and $D^{(*)} D^{(*)}$. Compared with $Z_c(3900)/Z_c(4020)$ states, the $X(3872)$ with a large isospin breaking effect is very unusual even in the exotic hadron family. The peculiarities of the $X(3872)$ might stem from either the accidental fine-tuning of the $D^* D/\bar{D} D^*$ interaction or the accidental fine-tuning of $P$-wave charmonium state to the $D^{(*)} D^{(*)}$ threshold [58]. Our proof is established upon the first fine-tuning. If the $X(3872)$ is a consequence of the second fine-tuning, the $P$-wave charmonium state coinciding with the $D^{(*)} D^{(*)}$ threshold, the absence of the $[D^{(*)} D^{(*)}/D^{(*)} D^{(*)}]^{1+ +}$ di-meson state would imply its weakly attractive or repulsive hadronic interaction. The unbound $[D^{(*)} D^{(*)}/D^{(*)} D^{(*)}]^{1+ +}$ system together with the existence of the $[D_s D_s]^{0+ +}$ bound state would induce to the same
consequence in the SU(3)$_F$ symmetry and HQSS limit.

Among the six S-wave $\bar{D}_s^0 X_s^+$ systems, we have discussed four of them. The remaining $[\bar{D}_s^* D_s^s/\bar{D}_s^* D_s^s]^{1++}$ and $[\bar{D}_s^* D_s^s]^{2++}$ have the same interaction as the $X(3872)$ in the single-channel analysis and have the less attractive (more repulsive) interaction in the coupled-channel interaction. However, we do not expect these two systems have the similar fine-tuning mechanism as the $X(3872)$. Therefore, there may not exist the di-meson states in these two channels. In Ref. [59], the $X(4140)$ was dynamically generated as $[\bar{D}_s^* D_s^s]$ molecule with $J^{P_C} = 0^{++}$.

In Ref. [49], the authors treated $X(3872)$, $X(3915)$, and $Y(4140)$ as $[D^* D/\bar{D}^* D^* D]^{1++}$, $[D^* D^*]^{0++}$, $[D_s^* D_s^s]^{0++}$ molecules, respectively and predicted the spectrum of heavy meson molecules. We compare their predictions with our predictions in Table II. Their results agree with our predictions, though we choose the different states as the inputs. Meanwhile, they neither obtained the $[\bar{D}_s^* D_s^s/\bar{D}_s^* D_s^s]^{1++}$ and $[\bar{D}_s^* D_s^s]^{2++}$ molecular states. In our analysis, we do not choose $X(3915)$ as the input due to its controversial quantum number and nature. The experimental analysis prefers $J^{P_C} = 0^{++}$ [60]. However, a reanalysis presented in Ref. [61] showed that a $J^{P_C} = 2^{++}$ is also possible. Apart from the molecular candidate of $[D^* D^*]^{0++}$ [49], the $X(3915)$ was also interpreted as the $[D_s^* D_s^s]^{0++}$ bound states [62], c$\bar{c}$s$\bar{s}$s$\bar{s}$ tetrakistate [63] and $P$-wave charmonium [64].

If $X(3872)$ is not a bound state but a threshold effect with a virtual pole near the threshold, we can infer that the $D^{(*)}D^{(*)}D^{(*)}D^{(*)}$ interaction is either repulsive or not as attractive as we expected in Sections III and IV. In this case, our predictions will not change qualitatively. According to Eq. (10), the less attractive $[\bar{D}_s^* D_s^s/\bar{D}_s^* D_s^s]^{1++}$ will lead to more attractive $[\bar{D}_s^* D_s^s/\bar{D}_s^* D_s^s]^{1++}$, $[\bar{D}_s^* D_s^s]^{1++}$ and $[\bar{D}_s^* D_s^s]^{0++}$ interactions. Thus, our prediction will not change qualitatively by the isospin breaking effect.

In the lattice QCD simulation [1], there exists the $0^{++} D_s D_s$ bound state in the one-channel approximation. After considering the coupled-channel effect of $D D$, there is an indication for a narrow $0^{++}$ resonance just below the $D_s D_s$ threshold with a large coupling to $D_s D_s$ and a very small coupling to $D D$. The suppression of coupled-channel effect between $D_s D_s$ and $D D$ is in agreement with our analysis in Sec. II. Apart from the $D D$ channel, the $D_s D_s$ states might be affected by the $D^* D^*$ channel [59, 65].

We hope the future analyses in experiments can search for the $[\bar{D}_s^* D_s^s]^{0++}$, $[\bar{D}_s^* D_s^s/\bar{D}_s^* D_s^s]^{1++}$, and $[\bar{D}_s^* D_s^s]^{1++}$ bound states in the $B \rightarrow D^{(*)}_s D^{(*)} h$ processes ($h$ denotes the light hadrons). The $[\bar{D}_s^* D_s^s]^{0++}$ bound state is also expected to be observed in the $J/\psi \phi$ final state in the $B \rightarrow J/\psi \phi K$ decay [54]. In fact, we notice there seems to exist some excess near 4200 MeV in the $J/\psi \phi$ invariant mass spectrum in the previous analysis of LHCb Collaboration [66, 67].

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**Appendix A: Pole trajectories in the coupled-channel formalism**

In the coupled-channel formalism, the existence of the pole corresponding to $X(3872)$ will reduce the three parameters, $a_{11}$, $a_{22}$ and $a_{12}$, to two independent ones in Eq. (27). If we presume the coupling constants satisfy the SU(3)$_F$ symmetry ($a_{11} = a_{22} \equiv a_{ii}$), only one free parameter $a_{ii}$ is left. The $a_{12}$ reads,

$$a_{12}^2 = |a_{ii} - F_1(\gamma_{X(3872)})||a_{ii} - F_2(\gamma_{X(3872)})|.$$  
(A1)

Meanwhile, the non-existence of the other $1^{++}$ states around the $D D^*$ threshold except $X(3872)$, will also constrain the range of the only parameter $a_{ii}$. By ruling out the existence of the second bound state (see the left subfigure in Fig. 4), we obtain

$$[a_{ii} - F_1(\gamma_{X(3872)})] < [a_{ii} - F_2(\gamma_{X(3872)})] < 0.$$  
(A2)

In this section, we will explore the constraint from the non-existence of the near-threshold poles in the complex energy plane.
Under the constraints of Eqs. (A1) and (A2), we decrease the parameter $a_{ii}$ (equivalently increase $a_{12}^2$) and obtain the trajectories of the poles in the four Riemann sheets. We use the RS$_{a_{12}}$ to denote the Riemann sheets, where two subscripts stand for the sign of Im$k_1$ and Im$k_2$, respectively. $k_1$ is the momentum in the center of mass frame for the $i$th channel. In order to demonstrate the trajectories of the poles conveniently, we fix the binding energy of the $X(3872)$ as 1 MeV. The slight variation of the bind energy will not change the results qualitatively. We start from $a_{12}^2 = 0$, which corresponds to the vanishing coupled-channel effect. There exist poles in the real axes of RS$_{+-}$ and RS$_{-+}$. The pole in the RS$_{-+}$ represents the bound solution corresponding to the charged $D^*_s/D/D^*$ systems. In the range (a), the pole in the lower plane of RS$_{-+}$ corresponds to a resonance state. Its width increases with the $a_{12}^2$ and finally achieves its maximum at the threshold of $D^+D^{*-}$. At the end of the range (a), the pole in RS$_{+-}$ disappears at the $D^0D^{*-0}$ threshold. The pole in the RS$_{-+}$ appears in the range (b) and (c) and moves along the real axis to be away from the $D^0D^*$ threshold. The poles in the RS$_{-+}$ will finally disappear in the real axis at the end of range (b), where the poles in RS$_{+-}$ will appear at the beginning of range (c). Finally, with the increasing of $a_{12}^2$, the poles except the one corresponding to $X(3872)$ are away from the $D^0D^*$ threshold. In the other words, decreasing $a_{ii}$ (equivalently increasing $a_{12}^2$) would achieve the range of parameters corresponding to the experimental observations around the $D^0D^*$ threshold.

Since the non-existence of the other resonance states except $X(3872)$ constrains a relative large $a_{12}^2$ (the large coupled-channel effect), the case in the middle subfigure of Fig. 4 could be ruled out. Thus, there does not exist the $[D_s^+D_s^*/D_s^*/D_s^*]^1^{++}$ bound state.

![FIG. 5. The trajectories of poles in the four Riemann sheets with the decreasing of the parameter $a_{ii}$ (equivalently increasing $a_{12}^2$). We use arrow to denote the direction of trajectories of poles. The (a), (b), (c) are three ranges of $a_{12}^2$ in ascending order. We fix the binding energy of $X(3872)$ as 1 MeV.](image-url)

[1] S. Prelovsek, S. Collins, D. Mohler, M. Padmanath, and S. Piemonte, (2020), arXiv:2011.02542 [hep-lat].
[2] R. Aaij et al. (LHCb), Phys. Rev. Lett. **125**, 242001 (2020), arXiv:2009.00025 [hep-ex].
[3] R. Aaij et al. (LHCb), Phys. Rev. D **102**, 112003 (2020), arXiv:2009.00026 [hep-ex].
[4] V. E. Barnes et al., Phys. Rev. Lett. **12**, 204 (1964).
[5] S. K. Choi et al. (Belle), Phys. Rev. Lett. **91**, 262001 (2003), arXiv:hep-ex/0309032.
[6] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. E. Thomas, A. Vairo, and C.-Z. Yuan, Phys. Rept. **873**, 1 (2020), arXiv:1907.07583 [hep-ex].
[7] Y.-R. Liu, H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Prog. Part. Nucl. Phys. **107**, 237 (2019), arXiv:1903.11976 [hep-ph].
[8] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Rev. Mod. Phys. **90**, 015004 (2018), arXiv:1705.00141 [hep-ph].
[9] S. L. Olsen, T. Skwarnicki, and D. Zieminska, Rev. Mod. Phys. **90**, 015003 (2018), arXiv:1708.04012 [hep-ex].
[10] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Phys. Rept. **639**, 1 (2016), arXiv:1601.02092 [hep-ph].
[11] A. Esposito, A. Pilloni, and A. D. Polosa, Phys. Rept. **668**, 1 (2017), arXiv:1611.07920 [hep-ph].
[12] A. Bondar et al. (Belle), Phys. Rev. Lett. **108**, 122001 (2012), arXiv:1110.2251 [hep-ex].
[13] M. Ablikim et al. (BESIII), Phys. Rev. Lett. **111**, 242001 (2013), arXiv:1309.1896 [hep-ex].
[14] M. Ablikim et al. (BESIII), Phys. Rev. Lett. **112**, 132001 (2014), arXiv:1308.2760 [hep-ex].
[15] R. Aaij et al. (LHCb), Phys. Rev. Lett. **115**, 072001 (2015), arXiv:1507.03414 [hep-ex].
[16] R. Aaij et al. (LHCb), Phys. Rev. Lett. **122**, 222001 (2019), arXiv:1904.03947 [hep-ex].
[17] R. Aaij et al. (LHCb), (2020), arXiv:2012.10380 [hep-ex].
[18] B. Wang, L. Meng, and S.-L. Zhu, Phys. Rev. D **101**, 034018 (2020), arXiv:1912.12592 [hep-ph].
[19] M. Ablikim et al. (BESIII), (2020), arXiv:2011.07855 [hep-ex].
[20] L. Meng, B. Wang, and S.-L. Zhu, Phys. Rev. D **102**, 111502 (2020), arXiv:2011.08656 [hep-ph].
[21] B. Wang, L. Meng, and S.-L. Zhu, Phys. Rev. D **103**, L021501 (2021), arXiv:2011.10922 [hep-ph].
[22] Q.-N. Wang, W. Chen, and H.-X. Chen, (2020), arXiv:2011.10495 [hep-ph].
[23] Z.-F. Sun and C.-W. Xiao, (2020), arXiv:2011.09404 [hep-ph].
[24] M.-C. Du, Q. Wang, and Q. Zhao, (2020), arXiv:2011.09225 [hep-ph].
[25] X. Cao, J.-P. Dai, and Z. Yang, Eur. Phys. J. C **81**, 184 (2021), arXiv:2011.09244 [hep-ph].
[26] R. Chen and Q. Huang, Phys. Rev. D **103**, 034008 (2021), arXiv:2011.09156 [hep-ph].
[27] Z. Yang, X. Cao, F.-K. Guo, J. Nieves, and M. P. Valderrama, (2020), arXiv:2011.08725 [hep-ph].
[28] J.-Z. Wang, Q.-S. Zhou, X. Liu, and T. Matsuki, Eur. Phys. J. C **81**, 51 (2021), arXiv:2011.08628 [hep-ph].
[29] K. Azizi and N. Er, Eur. Phys. J. C **81**, 61 (2021), arXiv:2011.11488 [hep-ph].
[30] X. Jin, X. Liu, Y. Xue, H. Huang, and J. Ping, (2020), arXiv:2011.07920 [hep-ph].
