Graphs and Generalized Witt identities

G.A.T.F.da Costa
Departamento de Matemática
Universidade Federal de Santa Catarina
88040-900-Florianópolis-SC-Brasil

Abstract

This paper is about two generalizations of the Witt identity. Both are associated with graphs. One of the identities is well known in association with the Ihara zeta function of a graph. The other identity is associated with the Bowen-Lanford zeta function of a directed graph. We show that the Witt identity is a special case of the second identity. Several combinatorial and algebraic aspects of both identities are investigated in connection with graphs, Strehl identity, Carlitz and Metropolis-Rota relations, free Lie algebras, and a coloring problem. New interpretations of the Ihara and Bowen-Lanford zeta functions are presented.

This paper is an enlarged version of the preprint [5].

1 Introduction

First, I will recall Witt identity and some facts about it because they are the basic motivation for this paper. Let $N$ be a positive integer, $R$ a real number, $\mu$ the classical Möbius function defined by the rules: a) $\mu(+1) = +1$, b) $\mu(g) = 0$, if $g = p_1^{a_1} \cdots p_q^{a_q}$, $p_1, \ldots, p_q$ primes, and any $a_i > 1$, c) $\mu(p_1 \cdots p_q) = (-1)^q$. The polynomial of degree $N$ in $R$ with rational coefficients given in terms of Möbius function,

$$\mathcal{M}(N; R) = \frac{1}{N} \sum_{g|N} \mu(g) R^{\frac{N}{g}},$$

is called the Witt polynomial or the Witt dimension formula according to the context where it appears, combinatorics or algebra. See [18, 23, 21]. The Witt dimension formula is so called because $\mathcal{M}(N; R)$ is the dimension of the homogeneous subspace $L_N$ of the free Lie algebra $L = \bigoplus_{N=1}^{\infty} L_N$ generated by a vector space $V$ with $\dim V = R$. Furthermore, it satisfies the formal relation

$$\prod_{N=1}^{\infty} (1 - z^N)^{\mathcal{M}(N; R)} = 1 - Rz$$

called the Witt identity.
A symmetrical form of this identity is given by the Strehl identity [34],

$$\prod_{k \geq 1} \left( \frac{1}{1 - \beta z^k} \right)^{\mathcal{M}(k, \alpha)} = \prod_{j \geq 1} \left( \frac{1}{1 - \alpha z^j} \right)^{\mathcal{M}(j, \beta)}.$$

The Witt polynomial is also called the \textit{necklace polynomial} because $\mathcal{M}(N; R)$ gives the number of inequivalent non-periodic colorings of a circular string of $N$ beads - a necklace - with at most $R$ colors. In reference [4] L. Carlitz proved that

$$\mathcal{M}(N, \alpha \beta) = \sum_{[s, t] = N} \mathcal{M}(s, \alpha) \mathcal{M}(t, \beta) \quad (1.4)$$

as a special case of a more general result. The summation is over the set of positive integers $\{s, t \mid [s, t] = N\}$, $[s, t]$ being the least common multiple of $s$ and $t$. In [18] Metropolis and Rota gave a new proof of this result and of other identities that are satisfied by the Witt polynomials. For instance,

$$\mathcal{M}(N, \alpha^l) = \sum_{[t, l] = Nl} \mathcal{M}(t, \alpha) \quad (1.5)$$

where $l$ is a positive integer. They gave a ring theoretical interpretation to these relations in order to obtain several results about the \textit{necklace ring} and \textit{Witt vectors}. In [21], P. Moree proved that analogous relations are satisfied by the \textit{formal series Witt transform}.

Another interesting result about $\mathcal{M}$ is that $\mathcal{M}(N; R)$ is the number of equivalence classes of closed non-periodic paths of length $N$ which traverse counterclockwise without backtracking the edges of a graph - $G_1$, Figure 1 - which has $R$ loops counterclockwise directed and hooked to a single vertex [6, 7, 26].

This paper considers two generalizations of the Witt identity and investigates several combinatorial and algebraic aspects of both identities in connection with graphs, Strehl identity, Carlitz and Metropolis-Rota relations, free Lie algebras, and a coloring problem. One of the identities is well known in association with the \textit{Ihara zeta function} of a non directed graph. A great deal of work has been done on the Ihara zeta function. I refer the reader to [27, 28, 29] and the book [30] for a comprehensive overview and references therein. The second identity resembles very much the first but it is defined on directed graphs. See [15, 19, 20, 22]. Also, associated to it there is a zeta function, the \textit{Bowen-Lanford zeta function} [22]. Both identities have paths counting formulas associated with them. We will show that the Witt identity is the second identity when the graph is $G_1$ in Figure 1 or an equivalent with respect to the zeta function. New interpretations of the Ihara and Bowen-Lanford zeta functions will be presented.

The paper is organized as follows. In section 2, the zeta functions and respective identities and paths counting formulas are defined. In section 3, it is proved
that the paths counting formulas satisfy several relations of the sort that Strehl and Carlitz, Metropolis and Rota proved for the Witt polynomials. In section 4, we interpret the zeta functions in terms of data associated to a free Lie superalgebra. In section 5, the paths counting formulas are interpreted in terms of a restricted necklace coloring problem. In this context, another interpretation of the zeta functions arises.

2 Graph zeta functions and associated identities

2.1 The Ihara zeta function.

Let $G = (V, E)$ be a finite connected non directed graph with no 1-degree vertices, $V$ is the set of $|V|$ vertices, $E$ is the set of $|E|$ edges with elements labeled $e_1, ..., e_{|E|}$. The graph may have multiple edges and loops. In order to define a closed path in a non directed graph we shall orient the edges. The orientation is arbitrary but fixed. An oriented edge has origin and end given by its orientation. Let $G'$ be the graph with $2|E|$ oriented edges built from the previous oriented graph $G$ by adding in the opposing oriented edges $e_{|E|+1} = (e_1)^{-1}, ..., e_{2|E|} = (e_{|E|})^{-1}, (e_i)^{-1}$ being the oriented edge opposite to $e_i$ and with origin (end) the end (origin) of $e_i$. In the case that $e_i$ is an oriented loop, $e_{i+|E|} = (e_i)^{-1}$ is just an additional oriented loop hooked to the same vertex.

A path of length $N$ in the non oriented graph $G$ is the ordered sequence $(e_{i_1}, ..., e_{i_N})$ of oriented edges in $G'$ such that the end of $e_{i_k}$ is the origin of $e_{i_{k+1}}$. Sometimes, it will be useful to represent a path by a word in the alphabet of the symbols in the set $\{e_1, ..., e_{2E}\}$, a word being a concatenated product of symbols which respect the order of the symbols in the sequence. A cycle is a non-backtracking tail-less closed path, that is, the end of $e_{i_N}$ coincides with the origin of $e_{i_1}$, subjected to the non-backtracking condition that $e_{i_{k+1}} \neq e_{i_k+|E|}$. In another words, a cycle never goes immediately backwards over a previous edge. Tail-less means that $e_{i_j} \neq e_{i_N}^{-1}$. The length of a cycle is the number of edges in its sequence. A cycle $p$ is called periodic if $p = q^r$ for some $r > 1$, the integer $r$ is the period of $p$, $q$ is a non periodic cycle. The cycle $(e_{i_N}, e_{i_1}, ..., e_{i_{N-1}})$ is called a circular permutation of $(e_{i_1}, ..., e_{i_N})$ and $(e_{i_N}^{-1}, ..., e_{i_1}^{-1})$ is an inversion of the latter. A cycle and its inverse are taken as distinct paths whereas circular permutations are taken as equivalent.

In order to count cycles of a given length a crucial tool is the edge adjacency matrix of $G$ [27]. This is the $2|E| \times 2|E|$ matrix $T$ defined as follows: $T_{ij} = 1$, if end vertex of edge $i$ is the start vertex of edge $j$ and edge $j$ is not the inverse edge of $i$; $T_{ij} = 0$, otherwise. Notice that $T$ is the vertex adjacency matrix of the oriented line graph $L(G')$ of $G'$, namely, the graph whose vertices are the edges of $G'$ and given $v_i, v_j \in L(G')$ there is an oriented edge from vertex $v_i$ to vertex $v_j$ if $T_{ij} = 1$.
The Ihara zeta function $\zeta_G(z)$ of an undirected graph $G$ is here formally defined by

$$\zeta_G(z) = \prod_{[p]} (1 - z^{N(p)})^{-1}$$

where the product is over the equivalence classes of non periodic cycles in $G$, $N(p)$ the length of a cycle in $[p]$. The basic fact about $\zeta$ is that $\zeta^{-1}$ is a polynomial. This is expressed in the following identity:

$$\zeta_G(z)^{-1} = \prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N,T)} = \det (I - zT),$$

$$\Omega(N, T) = \frac{1}{N} \sum_{g|N} \mu(g) \Tr T^N.$$  \hspace{1cm} (2.2)

$\Omega(N, T)$ is the number of equivalence classes of nonperiodic cycles of length $N$ in $G$, $g$ ranges over the positive divisors of $N$. The function $\zeta_G$ have another important representation as

$$\zeta_G(z)^{-1} = (1 - z^2)^r \det(I - zA + z^2Q),$$

where $A$ is the vertex adjacency matrix of the undirected graph $G$ and $Q$ is a diagonal matrix with $Q_{ii} = d(v_i)$, the degree of vertex $v_i$, and $r = |V| - |E|$ is the Euler number of $G$.

The Ihara zeta function was first defined by Y. Ihara in his paper on discrete subgroups [10]. However, it was J.-P. Serre in his book on trees [24] who sujested that Ihara’s definition could be interpreted in graph theoretical terms. Ihara’s formulation worked only for regular graphs. Afterwards, H. Bass [1] obtained an expression for the zeta function in terms of a determinant that could be applied to all graphs. The literature on the zeta function of a graph is quite vast. We refer the reader to the papers [27, 28, 29], the book [30] and references therein for a very nice overview about this function and its properties.

### 2.2 The Bowen-Lanford zeta function

Let $G = (V, E)$ be a finite connected and directed graph with no 1-degree vertices, $V$ is the set of $|V|$ vertices, $E$ is the set of $|E|$ edges with elements labeled $e_1, ..., e_{|E|}$. The graph may have multiple edges and loops.

A path in a finite connected and directed graph $G$ is given by an ordered sequence $(e_{i_1}, ..., e_{i_N})$, $i_k \in \{1, ..., |E|\}$, of oriented edges such that the end of $e_{i_k}$ is the origin of $e_{i_{k+1}}$. Notice that in this case and contrary to the previous one there are no inverse edges, hence, paths are backtrack-less, tail-less and have no inverse. A path has a natural orientation which is induced by the orientations of the edges in the sequence.
We shall consider directed graphs which are strongly connected. A directed graph is called strongly connected if it contains a directed path from \( a \) to \( b \) and a directed path from \( b \) to \( a \) for every pair of vertices \( a, b \). In general, an oriented graph may not be strongly connected but it may have strongly connected components which are the maximal strongly connected subgraphs. The graph become acyclic if each component is contracted to a single vertex.

The Bowen-Lanford zeta function of a finite connected and directed graph \( G \), \( \zeta_{dG}(z) \), is defined as

\[
\zeta_{dG}(z) = \prod_{[p]} (1 - z^{N(p)})^{-1},
\]

where the product is over the equivalence classes of non periodic cycles in \( G \), \( N(p) \) is the length of a cycle in \([p]\). See [15, 19, 20,22].

The Bowen-Lanford zeta function was first defined by R. Bowen and O. E. Lanford in their paper on shift transformations [3]. However, it was Kotani and Sunada in their paper [15] who interpreted Bowen and Lanford results in graph theoretical terms. Kotani and Sunada presentation of \( \zeta_{dG} \) make use of the Perron-Frobenius operator formalism. Instead, in the sequel we shall associate \( \zeta_{dG} \) to adjacency matrices of vertices and edges.

In order to count cycles of a given length in a directed graph \( G \) the directed vertex adjacency matrix \( A_d(G) \) can be used. See [2, 33]. This is the matrix of order \(|V| \times |V|\) with entries defined as follows. Label the vertices of \( G \), 1, 2, ..., \(|V|\). Then, \((A_d)_{ij}\) is the number of edges directed from vertex \( i \) to vertex \( j \). The number of cycles of length \( N \) in \( G \) is given by \( \text{Tr} A_d^N \). An important property of the directed vertex adjacency matrix of a graph which is strongly connected is that it is an irreducible matrix.

Also, one can use the directed edge adjacency matrix of \( G \) to count cycles of a given length in a directed graph. This is the \(|E| \times |E|\) matrix \( S \) defined as follows: \( S_{ij} = 1 \), if end vertex of edge \( i \) is the start vertex of edge \( j \); \( S_{ij} = 0 \), otherwise. If \( G \) is strongly connected, then the matrix \( S(G) \) is also irreducible in view of the fact that \( S \) is the vertex adjacency matrix of the line graph of \( G \) and the line graph of a strongly connected graph has the same property. Let’s prove that the number of cycles of length \( N \) in \( G \) is given by \( \text{Tr} S^N \), hence, \( \text{Tr} S^N = \text{Tr} A_d^N \).

**Theorem 2.1** The number \( \text{Tr} S^N \) (or \( \text{Tr} A_d^N \)) (over)counts cycles of length \( N \) in a direct graph \( G \).

**Proof.** Suppose \( G \) is strongly connected. Let \( a \) and \( b \) be two edges of \( G \). The \((a,b)^{th}\) entry of matrix \( S^N \) is

\[
(S^N)_{(a,b)} = \sum_{e_{i_1}, \ldots, e_{i_{N-1}}} S_{(a,e_{i_1})}S_{(e_{i_2},e_{i_3})} \ldots S_{(e_{i_{N-1}},b)}.
\]
From the definition of the entries of $S$ it follows that $(S^N)_{(a,b)}$ counts the number of paths of length $N$ from edge $a$ to edge $b$. For $b = a$, only cycles are counted. Taking the trace gives the number of cycles with every edge taken into account as starting edge, hence, the trace overcounts cycles because every edge in a cycle is taken into account as starting edge.

**Theorem 2.2** Denote by $\theta(N, S)$ the number of equivalence classes of non periodic cycles of length $N$ which traverse a graph $G$. This number is given by the following formulas:

$$\theta(N, S) = \frac{1}{N} \sum_{g | N} \mu(g) \text{Tr} \, S^N_\frac{N}{g},$$

$$\theta(N, S) = \frac{\text{Tr} \, S^N}{N} - \sum_{N \neq g | N} \frac{1}{g} \theta \left( \frac{N}{g}, S \right),$$

where $S$ can be replaced by $A_d$.

**Proof.** In the set of $\text{Tr} \, S^N$ cycles there is the subset with $N\theta(N, S)$ elements formed by the non periodic cycles of length $N$ plus their circular permutations and the subset with

$$\sum_{g \neq 1|N} \frac{N}{g} \theta \left( \frac{N}{g}, S \right)$$

elements formed by the periodic cycles of length $N$ (whose periods are the common divisors of $N$) plus their circular permutations. (A cycle of period $g$ and length $N$ is of the form $(e_{k_1}e_{k_2}...e_{k_t})^g$ where $t = N/g$, and $(e_{k_1}e_{k_2}...e_{k_t})$ is a non periodic cycle so that the number of periodic cycles with period $g$ plus their circular permutations is given by

$$\frac{N}{g} \theta \left( \frac{N}{g}, S \right).$$

Hence,

$$\text{Tr} \, S^N = \sum_{g | N} \frac{N}{g} \theta \left( \frac{N}{g}, S \right).$$

Möbius inversion formula gives the result. Now, isolate the term with $g = 1$ in the previous result to get the recursive relation for $\theta$.

That the reciprocal of $\zeta_{dG}(z)$ is a polynomial of degree at most equal to the number of vertices $|V|$ of $G$ is expressed in the identity:

$$\zeta_{dG}(z)^{-1} = \prod_{N=1}^{\infty} (1 - z^N)^{\theta(N)} = \det(I - zS) = \det(I - zA_d),$$

(2.8)
where $\theta(N) := \theta(N, S) = \theta(N, A_d)$. A proof is postponed to a later section.

**Remark 2.1.** If graph $G$ is not strongly connected but it has strongly connected components, then the graph becomes acyclic if each component is contracted to a cycle. In this case, $\zeta_d G(z)$ decomposes into a product of the zeta functions of the components.

**Remark 2.2.** The identity (2.8) becomes the Witt identity (1.2) when the graph has only one vertex and $R$ directed loops hooked to it as in Figure 1. In this case, $A_d = (R)$, $\text{Tr} A_d^N = R^N$, and $\det(1-zA_d) = 1-Rz$, $S$ is the $R \times R$ matrix with all entries equal to one, $\text{Tr} S^N = R^N$, $\det(1-zA) = 1-Rz$, and $\theta(N, A_s) = \theta(N, S) = \mathcal{M}(N; R)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Graph $G_1$}
\end{figure}

**Remark 2.3.** Let $T$ stand either for the matrix $T$, $S$ or $A_d$ and $C(N, T)$ stand for the respective cycle counting number formula so that

$$C(N, T) = \frac{1}{N} \sum_{g|N} \mu(g) \text{Tr} T_g^N,$$  \hspace{1cm} (2.9)\]

and

$$\prod_{N=1}^{+\infty} (1-z^N)C(N,T) = \det(I-zA(T)).$$  \hspace{1cm} (2.10)\]

Then, $C(N, T)$ can be expressed as a summation over Witt polynomials defined on the eigenvalues $\lambda_i$ of $T$. By the Schur decomposition method there is an orthogonal matrix $P$ and an upper triangular matrix $J$ with eigenvalues of $T$ along the diagonal such that $T = PJP^{-1}$, hence,

$$\text{Tr} T^N = \text{Tr}(PJP^{-1})^N = \text{Tr} J^N = \sum_{i=1}^{P} \lambda_i^N,$$
where \(D = 2|E|\), if \(T = T\), \(D = |V|\), if \(T = A_d\), and \(D = |E|\) if \(T = S\). Therefore,

\[
C(N, T) = \frac{1}{N} \sum_{g \in G} \mu(g) \text{Tr} T_N^g = \sum_{i=1}^D \frac{1}{N} \sum_{g \in G} \mu(g) \lambda_i^g = \sum_{i=1}^D M(N; \lambda_i)
\]

where \(M\) is the Witt polynomial on \(\lambda_i\). Using Witt identity, it follows that

\[
\zeta^{-1}(z) = \prod_{N=1}^{+\infty} (1 - z^N)^{C(N, T)} = \prod_{i=1}^D (1 - \lambda_i z).
\]

Let's consider the directed case. Expanding the product,

\[
\zeta^{-1}_{dG}(z) = \prod_{i=1}^{|V|} (1 - \lambda_i z) = 1 - \text{Tr}(A_d) z + \cdots + (-1)^{|V|} \det(A_d) z^{|V|}.
\]

From this we conclude that the coefficients of the terms of the polynomial \(\text{det}(1 - zS)\) with degrees \(|V| < n \leq |E|\) are equal to zero, hence, \(\text{det} S = 0\).

**Remark 2.4.** Let \(G\) be a strongly connected graph with \(|V|\) vertices and no loops. Suppose that \(G\) is regular, that is, the vertices have same out and in degree \(d\). The number of out-trees and in-trees with respect to any vertex are the same [8]. Let \(t(G)\) be the number of spanning out-trees (or in-trees) in \(G\). Then, the number of directed spanning trees in a regular directed graph without loops can be computed from its Bowen-Lanford zeta function. From (2.8) we get

\[
- \frac{1}{d^2} \frac{d}{dz} \left[ z^{-|V|} \zeta^{-1}_{dG}(z) \right]_{z=d^{-1}} = \frac{d}{du} [\det(u - A_d)]_{u=d} = |V| t(G).
\]

In the last step we used a result from [8].

### 3 Strehl identity, Carlitz and Metropolis-Rota-type relations

#### 3.1 Strehl-type identity

The Strehl identity [34] is a symmetric generalization of the Witt identity. In this subsection we prove a generalization of this result which follows directly from relation (2.10).

**Theorem 3.1** Given two graphs \(G_1\) and \(G_2\) with matrices \(T_1\) and \(T_2\), respectively, the following identity holds:

\[
\prod_{k \geq 1} \left[ \frac{1}{\text{det}(1 - z^k T_1)} \right]^{C(k, T_2)} = \prod_{j \geq 1} \left[ \frac{1}{\text{det}(1 - z^j T_2)} \right]^{C(j, T_1)}.
\]
**Proof.** Call $I$ and $II$ the left and right hand sides of (3.1). Starting from $I$, using (2.2), we get

$$I = \prod_{k,j \geq 1} \left( \frac{1}{1 - z^{kj}} \right)^{C(j; T_1)C(k; T_2)} = \prod_{j \geq 1} \prod_{k \geq 1} \left( \frac{1}{1 - z^{jk}} \right)^{C(k; T_2)C(j; T_1)} = II.$$ 

Let $Z_G$ stand either for the Ihara zeta function, $\zeta_G(z)$, or the Bowen-Lanford zeta function, $\zeta_{dG}(z)$. Then, in terms of the zeta functions of two graphs $G_1$ and $G_2$ the previous result can be stated as

$$\prod_{k \geq 1} [Z_{G_1}(z^k)]^{C(k; T_2)} = \prod_{j \geq 1} [Z_{G_2}(z^j)]^{C(j; T_1)}$$

which follows from (3.1). This statement is trivial if the graphs have the same zeta function.

**Remark 3.1.** See remark 2.2. Let $G_1$ and $G_2$ be two graphs of the type of $G_1$ in Figure 1 with $R_1$ and $R_2$ oriented loops and edge adjacency matrices $S_1$ and $S_2$, respectively. In this case, the identity (3.1) becomes the classical Strehl identity.

### 3.2 Carlitz and Metropolis-Rota-type relations.

In [4] Carlitz investigated the properties of the function $\psi(k) = \sum_{st=k} \mu(s)g(t)$ for an arbitrary integral-valued function $g(t)$. Among his results in [4] is the relation (1.4) which follows from a more general result when $g(t) = R^t/k$. In this subsection we take $g(t) = \text{Tr} T^t/k$ and use some ideas from [4] and [21] rather than the combinatorial arguments of Metropolis and Rota in [18] to prove several relations which are satisfied by $C(N, T)$, given by (2.9).

The relations we are about to prove involve the **Kronecker product** of adjacency matrices, hence, they can be understood as relations for computing the number of cycles of a given length in the graph product in terms of the numbers of cycles in each graph.

Given two graphs $G_1$ and $G_2$ with adjacency matrices $T_1$ and $T_2$, respectively, the **Kronecker product** $G_1 \otimes G_2$ is a graph with adjacency matrix $T_1 \otimes T_2$. This is the matrix having the element $(T_1)_{ij}$ replaced by the matrix $(T_1)_{ij}T_2$. Many properties of the Kronecker product of graphs have been proved since Weichsel introduced it in [35] for non directed graphs. The product of directed graphs was investigated in [17]. A basic result is that the product graph need not be (strongly) connected even if graphs $G_1$ and $G_2$ are. An explicit formula for the number of components is given in [35] and [17] together with conditions for the product to be connected. The Ihara zeta function on Kronecker products have been investigated in [25].

Motivated by the structure of their relations Metropolis and Rota gave them a ring theoretical interpretation which they investigated in connection with the
necklace ring, unital series and Witt vectors. It is possible that the relations below will have a similar interpretation.

**Theorem 3.2** Given matrices $T_1$ and $T_2$ and a positive integer $N$ define

$$S(N, T_i) = \sum_{g \mid N} \mu(g) \text{Tr} T_i^g, \quad i = 1, 2,$$

(3.3)

and denote by $T_1 \otimes T_2$ the Kronecker product of $T_1$ and $T_2$. Then,

$$\sum_{[s,t] = N} S(s, T_1)S(t, T_2) = S(N, T_1 \otimes T_2).$$

(3.4)

The summation is over the set of positive integers $\{s,t \mid [s,t] = N\}$, $[s,t]$ being the least common multiple of $s$ and $t$. Furthermore, we have

$$S(N, T^l) = \sum_{[l,t] = NL} S(t, T).$$

(3.5)

**Proof.** In order to prove (3.4) it suffices to consider the equivalent formula (see [4])

$$\sum_{k \mid N} \sum_{[s,t] = k} S(s, T_1)S(t, T_2) = \sum_{k \mid N} S(k, T_1 \otimes T_2).$$

Using the Möbius inversion formula, the left hand side is equal to

$$\sum_{s \mid N} S(s, T_1) \sum_{t \mid N} S(t, T_2) = (\text{Tr} T_1^N)(\text{Tr} T_2^N).$$

But $(\text{Tr} T_1^N)(\text{Tr} T_2^N) = \text{Tr}(T_1 \otimes T_2)^N$. By the Möbius inversion formula this gives the right hand side of the equivalent formula. Using ideas from [21], the next identity can be proved using the following equivalent formula:

$$\sum_{g \mid N} S(N, T^l) = \sum_{g \mid N} \sum_{[l,t] = NL} S(t, T).$$

The right hand side is equal to $\sum_{d \mid N} S(t, T) = \text{Tr} T^U = \text{Tr}(T^l)^N$. Apply the Möbius inversion formula to get the result.

**Remark 3.2.** Formula (3.4) may be generalized to the case $\mathcal{T} = T_1 \otimes T_2 \otimes ... \otimes T_l$ to give

$$\sum_{[s_1, ..., s_l] = N} S(s_1, T_1) \ldots S(s_l, T_l) = S(N, \mathcal{T}).$$

(3.6)

Also, it can be proved that

$$S(N, T_1^s \otimes T_2^r) = \sum_{[rp, sq] = Nrs} S(p, T_1)S(q, T_2),$$

(3.7)
where \( r \) and \( s \) are relatively prime and the summation is over all positive integers \( p \) and \( q \) such that \([rp, sq] = Nrs\). The proof is an application of previous identities as in Theorem 5 of [18].

**Theorem 3.3** Let \((s, t)\) denote the maximum common divisor of the positive integers \( s \) and \( t \). Then,

\[
\sum_{[s, t] = N} (s, t)C(s, T_1)C(t, T_2) = C(N, T_1 \otimes T_2).
\] (3.8)

The summation is over the set of positive integers \( \{ s, t \mid [s, t] = N \} \), \([s, t]\) being the least common multiple of \( s \) and \( t \). Also,

\[
C(N, T^t) = \sum_{[t, t] = Nt} \frac{t}{N} C(t, T).
\] (3.9)

and

\[
(r, s)C\left( N, T_1^{s/(r, s)} \otimes T_2^{t/(r, s)} \right) = \sum_{[rp, sq] = Nrs} (rp, sq)C(p, T_1)C(q, T_2).
\] (3.10)

The sum is over \( p \) and \( q \) such that \( pq/(pr, qs) = N/(r, s) \).

**Proof.** Use that \( S(s, T) = s\Omega(s, T) \) and \([s, t](s, t) = st\) to get (3.6), and (3.7) also follows. In terms of \( C \), (3.5) becomes

\[
NC(N, T_1^s \otimes T_2^r) = \sum_{[rp, sq] = Nrs} pqC(p, T_1)C(q, T_2).
\]

Using \((rp, sq)[rp, sq] = rpsq\) with \([rp, sq] = Nrs\) implies \((rp, sq)N = pq\) and

\[
C(N, T_1^s \otimes T_2^r) = \sum_{[rp, sq] = Nrs} (rp, sq)C(p, T_1)C(q, T_2).
\]

Replace \( s \) and \( r \) by \( s/(r, s) \) and \( r/(r, s) \) to get (3.8).

**Remark 3.3.** Identity (3.8) can be extended to the general case \( T = T_1 \otimes T_2 \otimes \ldots \otimes T_l \) to give

\[
\sum_{[s_1, \ldots, s_l] = N} (s_1, \ldots, s_l)C(s_1, T_1) \ldots C(s_l, T_l) = C(N, T),
\] (3.11)

where \((s_1, \ldots, s_l)\) is the greatest common divisor of \( s_1, \ldots, s_l \) and the sum runs over all positive integers \( s_1, \ldots, s_l \) with least common multiple \([s_1, \ldots, s_l]\) equal to \( N \).

**Remark 3.4.** Let \( G_1 \) and \( G_2 \) be two graphs of the type of \( G_1 \) in Figure 1 with \( R_1 \) and \( R_2 \) oriented loops and directed adjacency matrices \( T_1 \) and \( T_2 \), respectively. In view of remark 2.3, \( C(N, T_i) = M(N; R_i) \), \( i = 1, 2 \) and \( C(N, T_1 \otimes T_2) = M(N; R_1 R_2) \).

The above relations become the Metropolis-Rota identities. The graph product in this case has \( R_1 R_2 \) edges hooked to a single vertex.
4  Graph zeta functions and free Lie superalgebras

Let $\mathcal{T}, \mathcal{C}(N, \mathcal{T})$ and $D$ be as in remark 2.3.

**Theorem 4.1** Define

$$g(z) := \sum_{N=1}^{\infty} \frac{\text{Tr} \mathcal{T}^N}{N} z^N. \quad (4.1)$$

Then,

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\pm \mathcal{C}(N, \mathcal{T})} = e^{\mp g(z)} = [\det (1 - z \mathcal{T})]^{\pm} = 1 \mp \sum_{i=1}^{+\infty} d_{\pm}(i) z^i, \quad (4.2)$$

and

$$d_{\pm}(i) = \sum_{m=1}^{i} \lambda_{\pm}(m) \sum \prod_{k=1}^{i} \frac{(\text{Tr} \mathcal{T}^k)^{a_k}}{a_k! k^{a_k}}, \quad (4.3)$$

with $\lambda_{\pm}(m) = (-1)^{m+1}$, $\lambda_{-}(m) = +1$, $d_{+}(i) = 0$ for $i > D$, and $d_{-}(i) \geq 0$, for all $i$’s. Furthermore,

$$\text{Tr} \mathcal{T}^N = N \sum_{s \in S(N)} (\pm 1)^{|s|+1} \frac{(|s| - 1)!}{s!} \prod d_{\pm}(i)^{s_i}, \quad (4.4)$$

where

$$S(N) = \left\{ s = (s_i)_{i \geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum is_i = N \right\}, \quad (4.5)$$

and $| s | = \sum s_i, s! = \prod s_i!$.

**Proof.** Define $P_{\pm}$ by

$$P_{\pm}(z) = \prod_{N'=1}^{+\infty} (1 - z^{N'})^{\pm \mathcal{C}(N', \mathcal{T})}. \quad (4.6)$$

Take the logarithm of both sides and use (2.9) to get

$$lnP_{\pm} = \mp \sum_{N'} \sum_{k} \frac{1}{k} \mathcal{C}(N', \mathcal{T}) z^{N'k} = \mp \sum_{N=1}^{+\infty} \sum_{k \mid N} \frac{1}{k} \mathcal{C} \left( \frac{N}{k}, \mathcal{T} \right) z^N \quad (4.7)$$

$$= \mp \sum_{N=1}^{+\infty} \frac{\text{Tr} \mathcal{T}^N}{N} z^N = \mp g(z) \quad (4.8)$$

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from which the first equality in (4.2) follows. From the definition of $g(z)$, it follows that

$$
\mp g(z) := \mp \sum_{N=1}^{\infty} \frac{\Tr T^N}{N} z^N = \mp \Tr \sum_{N=1}^{\infty} \frac{1}{N} T^N z^N = \pm \Tr \ln(1 - zT)
$$

$$
= \pm \ln \det(1 - zT)
$$

proving the second equality in (4.2).

The third equality is obtained by formally expanding the exponential. As the formal Taylor expansion of $1 - e^{\mp g}$, the coefficients $c_{\pm}$ are given by

$$
d_{\pm}(i) = \frac{1}{i!} \frac{d^i}{dz^i} \left[ \pm (1 - e^{\mp g}) \right] |_{z=0}.
$$

Using Faa di Bruno’s formula as in [6, 7], the derivatives can be computed explicitly and (4.3) follows. The determinant is a polynomial of maximum degree $D$, hence, $d_{+}(i) = 0$ for $i > D$. Clearly, $d_{-}(i) \geq 0$.

To prove (4.4) write

$$
F := \mp \ln \left( 1 \mp \sum_{i} d_{\pm}(i) z^i \right) = \mp \sum_{l=1}^{\infty} \frac{1}{l} \left( \pm \sum_{i} d_{\pm}(i) z^i \right)^l.
$$

Expand the right hand side in powers of $z$ to get:

$$
F = \pm \sum_{l=1}^{\infty} \sum_{s \in T(l)} \left( \frac{1}{l} \prod_{i=1}^{s} (s_i)! \right) \left( \prod_{i} d_{\pm}(i)^{s_i} \right) z^{\sum s_i l}
$$

$$
= \sum_{k=1}^{\infty} z^k \sum_{s \in T(k)} (\pm)^{|s|+1} \left( \frac{|s| - 1}{s!} \right) \prod_{i} d_{\pm}(i)^{s_i}.
$$

The second equality in (4.2) applied to the left hand side yields

$$
F = \sum_{k=1}^{\infty} \frac{\Tr T^k}{k} z^k.
$$

Comparing the coefficients give the result.

The general expressions (4.3) for the coefficients $d_{\pm}(i)$ given in theorem 4.1 are complicated. They can be computed recursively as the next theorem shows.
Theorem 4.2 Set $\omega(n) := \text{Tr} T^n$. Then,

\[ d_\pm(1) = \omega(1), \quad (4.6) \]

\[ nd_\pm(n) = \omega(n) \mp \sum_{k=1}^{n-1} \omega(n-k)d_\pm(k), \quad n \geq 1, \quad (4.7) \]

\[ d_-(n) = d_+(n) + \sum_{i=1}^{n-1} d_+(i)d_-(n-i), \quad n \geq 1, \quad (4.8) \]

\[ |d_+(n)| \leq d_-(n), \quad (4.9) \]

\[ C(n, T) = d_+(n) + \frac{1}{n} \sum_{k=1}^{n-1} \left( \sum_{g|k} gC(g, T) \right) d_+(n-k) - \sum_{n\neq g|n} \frac{g}{n} C(g, T). \quad (4.10) \]

Proof. Define $f(z) := e^{\mp h(z)}$, $g_n := \mp \omega_n$, and denote by $f'$ the formal derivative of $f$. Also, define $c(n) := \mp d_\pm(n)$. Then,

\[ f'(z) = f(z) \sum_{n=1}^{\infty} g(n)z^{n-1} = \sum_{n=1}^{\infty} c(n)nz^{n-1}. \]

Thus,

\[ \left( 1 + \sum_{n=1}^{\infty} c(n)z^n \right) \left( \sum_{n=1}^{\infty} g(n)z^{n-1} \right) = \sum_{n=1}^{\infty} c(n)nz^{n-1}. \]

Equating the coefficients of both sides, we get

\[ nc(n) = g(n) + \sum_{k=1}^{n-1} g(n-k)c(k). \]

From the definitions the result follows. Now, using that $e^{-h(z)}e^{h(z)} = 1$, (4.8) follows.

To prove (4.9) add $d_+(n)$ to $d_-(n)$ to get that

\[ d_-(n) + d_+(n) = \frac{w(n)}{n} \geq 0. \]

Then, using this result, subtract $d_+$ from $d_-$ to get

\[ n(d_-(n) - d_+(n)) = \sum_{k=1}^{n-1} w(n-k)(d_-(k) + d_+(k)) \geq 0. \]

Relation (4.10) follows from (4.7) using that $\omega(n) = \sum_{g|n} gC(g, T)$. 

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Remark 4.1. Using the recurrence relations one can ferret out the following determinant

\[
d_+(n) = \frac{(-1)^{n-1}}{n!} \begin{vmatrix}
\omega(1) & 1 & 0 & 0 & \cdots & 0 \\
\omega(2) & \omega(1) & 2 & 0 & \cdots & 0 \\
\omega(3) & \omega(2) & \omega(1) & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\omega(n-1) & \omega(n-2) & \omega(n-3) & \cdots & \omega(1) & n-1 \\
\omega(n) & \omega(n-1) & \omega(n-2) & \cdots & \omega(2) & \omega(1) \\
\end{vmatrix}
\]

Multiply $-1$ to every entry $\omega(l)$ with $l$ even in the above determinant to get $d_-(n)$.

The Witt dimension formula and identity are associated with the following result:

**Proposition 4.3 (23)** If $V$ is an $R$-dimensional vector space and $L$ is the free Lie algebra generated by $V$ then $L = \bigoplus_{N=1}^{\infty} L_N$, and $L_N$ has dimension given by Witt polynomial $\mathcal{M}(N;R)$. The generating function for the dimensions of the homogeneous subspaces of the enveloping algebra of $L$ is given by the reciprocal of the Witt identity, relation (1.2).

In two papers S. -J. Kang and M. -H Kim [11, 12] generalized the above proposition to the case that the free Lie algebra $L$ is generated by an infinite graded vector space. They obtained a generalized Witt formula for the dimensions of the homogeneous subspaces of $L$ which satisfies a generalized Witt identity. In [13] S. -J. Kang extended the results to superspaces and Lie superalgebras. The following proposition summarizes Kang’s results from [13] which are relevant for our objectives.

**Proposition 4.4** Let $V = \bigoplus_{N=1}^{\infty} V_N$ be a $\mathbb{Z}_{>0}$-graded superspace with finite dimensions $\dim V_N = |t_N|$ and superdimensions $\dim V_N = t_N \in \mathbb{Z}$, \forall $N \geq 1$. Let $L = \bigoplus_{N=1}^{\infty} L_N$ be the free Lie superalgebra generated by $V$ with a $\mathbb{Z}_{>0}$-gradation induced by that of $V$. Then, the $L_N$ superdimension is

\[
\text{Dim } L_N = \sum_{g|N} \frac{\mu(g)}{g} W \left( \frac{N}{g} \right). \tag{4.11}
\]

The summation ranges over all positive divisors $g$ of $N$ and $W$ is given by

\[
W(N) = \sum_{s \in T(N)} \frac{(|s|-1)!}{s!} \prod t(i)^{s_i}, \tag{4.12}
\]

where $T(N) = \{s = (s_i)_{i \geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum i s_i = N \}$ and $|s| = \sum s_i$, $s! = \prod s_i!$. Furthermore,

\[
\prod_{N=1}^{\infty} (1 - z^N)^{\pm \text{Dim } L_N} = 1 \mp \sum_{N=1}^{\infty} f_\pm(N) z^N, \tag{4.13}
\]
with \( f_+(N) = t_N \) and \( f_-(N) = \text{Dim} U(\mathcal{L})_N \), where \( \text{Dim} U(\mathcal{L})_N \) is the dimension of the \( N \)-th homogeneous subspace of the universal enveloping algebra \( U(\mathcal{L}) \) and the generating function for the \( W \)'s,

\[
g(z) := \sum_{N=1}^{\infty} W(N) z^N, \tag{4.14}
\]

satisfies

\[
e^{-g(z)} = 1 - \sum_{N=1}^{\infty} t_N z^N. \tag{4.15}
\]

**Remark 4.2.** In [13], (4.11) is called the generalized Witt formula; \( W \) is called the Witt partition function; and the + case of (4.13) is called the generalized Witt identity.

In the sequel we apply ideas from section 2.3 of [13] to interpret algebraically the zeta functions of section 2.1 and 2.2.

Given a formal power series \( \sum_{N=1}^{\infty} t_N z^N \) with \( t_N \in \mathbb{Z} \), for all \( i \geq 1 \), the coefficients in the series can be interpreted as the superdimensions of a \( \mathbb{Z}_{>0} \)-graded superspace \( V = \bigoplus_{i=N}^{\infty} V_N \) with dimensions \( \dim V_N = |t_N| \) and superdimensions \( \text{Dim} V_N = t_N \in \mathbb{Z} \). Let \( \mathcal{L} \) be the free Lie superalgebra generated by \( V \). Then, it has a gradation induced by \( V \) and its homogeneous subspaces have dimension given by (4.11) and (4.12). Apply this interpretation to the determinant \( \det(1 - z\mathcal{T}) \) which is a polynomial of degree \( D \) in the formal variable \( z \). It can be taken as a power series with coefficients \( t_N = 0 \), for \( N > D \). Comparison of the formulas in Theorem 4.1 with the formulas in the above Proposition yields that given a graph \( G \), \( \mathcal{T} \) its associated matrix, let \( V = \bigoplus_{N=1}^{\infty} V_N \) be a \( \mathbb{Z}_{>0} \)-graded superspace with finite dimensions \( \dim V_N = |d_+(N)| \) and the superdimensions \( \text{Dim} V_N = d_+(N) \) given by (4.3), the coefficients of \( \det(1 - z\mathcal{T}) \). Let \( \mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) be the free Lie superalgebra generated by \( V \). Then, the \( \mathcal{L}_N \) superdimension is \( \text{Dim} \mathcal{L}_N = \mathcal{C}(N,T) \). This means that the equivalence classes of nonperiodic cycles of length \( N \) in \( G \) form a basis of \( \mathcal{L}_N \). The algebra has generalized Witt identity given by (2.10). Moreover, the following new interpretation of the zeta function of a graph follows:

The zeta function \( \zeta_G(z) \) of a graph \( G \) is the generating function for the dimensions \( \text{Dim} U(\mathcal{L})_N = d_-(N) \) given by (4.3) of the subspaces of the enveloping algebra \( U(\mathcal{L}) \).

**Remark 4.3** In [14], S.-J. Kang, J.-H. Kwon, and Y.-T. Oh derived Peterson-type dimension formulas for graded Lie superalgebras. In particular, see the example 3.6, p. 118 of [14]. Then, the Peterson-type formulas are recursive relations between the dimensions \( \text{Dim} \mathcal{L}_N \) and the coefficients \( f_+(N) \) in (4.13). See the formula after relation (3.15) in [14]. Using our notation, this formula is exactly the relation (4.10) in theorem 4.2.
Remark 4.4 Given two graphs and the free Lie algebras generated by them the algebra generated by the Kronecker product graph will have dimensions that can be expressed in terms of the dimensions of the algebras generated by the individual graphs. They are given by the Carlitz-Metropolis-Rota-type identities derived in subsection 3. It would be interesting to investigate how the vector spaces that generate the algebras of the individual graphs are related to the vector space that generate the algebra of the graph product.

Remark 4.5 When the coefficients $d_+(N)$ are all positive the free Lie superalgebra is just a free Lie algebra, that is, no superspaces involved. In the case of the Ihara zeta function the positivity of all the coefficients $d_+(N)$ seems to be a rare event. In the case of the Bowen-Lanford zeta function, it is not difficult to find examples.

Remark 4.6 It is known from the works of several authors that two graphs can have the same Ihara zeta function. See [31, 32] and references therein. This is also true for the Bowen-Lanford zeta function. Graphs with the same zeta function will generate the same algebra. Since the functions have the same coefficients $d_+$ it follows from (4.10) that they have the same numbers of cycles of same length.

Example 4.1. $G_2$, the graph shown in Figure 2. The edge and vertex adjacency matrices of $G_2$ are

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The loops 1 and 3 are hooked to vertices 3 and 2, respectively. The matrices have the traces $\text{Tr} S^N = \text{Tr} A_d^N = 2^N$, and the determinants

$$\det(1 - zS) = \det(1 - zA_d) = 1 - 2z.$$

![Figure 2: Graph $G_2$](image-url)
The number of classes of nonperiodic cycles of length $N$ is

$$\theta(N) = \frac{1}{N} \sum_{g|N} \mu(g)2^{\frac{N}{g}}.$$

For instance, $\theta(1) = 2$, $\theta(2) = 1$, $\theta(3) = 2$, $\theta(4) = 3$, $\theta(5) = 6$, $\theta(6) = 9$. For $N = 1$, the cycles are $e_1$, $e_3$. For $N = 2$, $[e_4e_5]$. For $N = 3$, $[e_3e_4e_5]$ and $[e_2e_4e_6]$. The Bowen-Lanford zeta function of $G_2$ is

$$\zeta_{dG_2}^{-1}(z) = \prod_{N=1}^{+\infty} (1 - z^N)^{\theta(N)} = 1 - 2z.$$

**Example 4.2.** $G_1$, the graph with $R \geq 2$ edges counterclockwise oriented and hooked to a single vertex. See Figure 1. The edge adjacency matrix for $G_1$ is the $2R \times 2R$ symmetric matrix

$$T_{G_1} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where $A$ is the $R \times R$ matrix with all entries equal to 1 and $B$ is the $R \times R$ matrix with the main diagonal entries equal to 0 and all the other entries equal to 1. This matrix has trace given by

$$\text{Tr} T_{G_1}^N = 1 + (R - 1)(1 + (-1)^N) + (2R - 1)^N, \quad N = 1, 2, \ldots,$$

and determinant

$$\det(1 - z T_{G_1}) = (1 - z) [1 - (2R - 1)z] (1 - z^2)^{R-1} \quad \text{and} \quad 1 - \sum_{i=1}^{2R} c(i) z^i,$$

where $c(2R) = (-1)^R(2R - 1)$,

$$c(2i) = (-1)^i(2i - 1) \begin{pmatrix} R \\ i \end{pmatrix}, \quad i = 1, \ldots, R - 1,$$

and

$$c(2i + 1) = 2R(-1)^i \begin{pmatrix} R - 1 \\ i \end{pmatrix}, \quad i = 0, 1, \ldots R - 1.$$

Furthermore,

$$[\det(1 - z T_{G_1})]^{-1} = \sum_{q=0}^{+\infty} z^q \sum_{i=0}^q a_i (2R - 1)^{q-i},$$

where

$$a_i = \sum_{k=0}^{i} (-1)^i-k \begin{pmatrix} k + R - 1 \\ R - 1 \end{pmatrix} \begin{pmatrix} i - k + R - 2 \\ R - 2 \end{pmatrix}.$$
Let’s consider the case $R = 2$. In this case,
\[
\text{Tr} T_{G_1}^N = 2 + (-1)^N + 3^N, \quad \det(1 - zT_{G_1}) = 1 - 4z + 2z^2 + 4z^3 - 3z^4,
\]
so that the number of classes of nonperiodic cycles of length $N$ is given by the formula
\[
\Omega(N, T_{G_1}) = \frac{1}{N} \sum_{g|N} \mu(g) \left( 2 + (-1) \frac{N}{g} + 3 \frac{N}{g} \right).
\]

The graph generates the following algebra. Let $V = \bigoplus_{i=1}^4 V_i$ be a $\mathbb{Z}_{>0}$-graded super space with dimensions $\dim V_1 = 4$, $\dim V_2 = 2$, $\dim V_3 = 4$, $\dim V_4 = 3$ and super dimensions $\text{Dim} V_1 = -4$, $\text{Dim} V_2 = 2$, $\text{Dim} V_3 = 4$, $\text{Dim} V_4 = -3$. Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free graded Lie super algebra generated by $V$. The dimensions of the subspaces $\mathcal{L}_N$ are given by the generalized Witt formula
\[
\text{Dim} \mathcal{L}_N = \frac{1}{N} \sum_{g|N} \mu(g) \left( 2 + (-1) \frac{N}{g} + 3 \frac{N}{g} \right).
\]

For instance, $\text{Dim} \mathcal{L}_1 = 4$, $\text{Dim} \mathcal{L}_2 = 4$, $\text{Dim} \mathcal{L}_3 = 8$, with basis:

$\mathcal{L}_1 : \{ [e_1], [e_1^{-1}], [e_2], [e_2^{-1}] \}$,
$\mathcal{L}_2 : \{ [e_1 e_2], [e_1 e_2^{-1}], [e_1^{-1} e_2], [e_1^{-1} e_2^{-1}] \}$,
$\mathcal{L}_3 : \{ [e_1^2 e_2^2], [e_1^{-1} e_2^2], [e_1^2 e_2^{-1}], [e_1^{-1} e_2^{-1}], [e_1^{-2} e_2^2], [e_1 e_2^{-2}], [e_1 e_2^2], [e_1^{-2} e_2^{-1}] \}$.

The generalized Witt identity is
\[
\prod_{N=1}^{+\infty} \left( 1 - z^N \right)^{\Omega(N, T_{G_1})} = 1 - 4z + 2z^2 + 4z^3 - 3z^4.
\]

The dimensions of the subspaces $U_n(\mathcal{L})$ of the enveloping algebra $U(\mathcal{L})$ have dimensions generated by the Ihara zeta function of the graph,
\[
\zeta_{G_1}(z) = \prod_{N=1}^{+\infty} \left( 1 - z^N \right)^{-\Omega(N, T_{G_1})} = 1 + \frac{1}{16} \sum_{n=1}^{\infty} \left( (-1)^n + 3^{n+3} - 12 - 4n \right) z^n.
\]

The first few terms give $\text{Dim} U_1 = 4$, $\text{Dim} U_2 = 14$, $\text{Dim} U_3 = 44$, $\text{Dim} U_4 = 135$.

**Example 4.3.** $G_3$, the bipartite graph shown in Figure 3. The edge adjacency matrix of $G_3$ is
\[
T_{G_3} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
The matrix has the trace $\text{Tr} T_{G_3}^N = 0$ if $N$ is odd and $\text{Tr} T_{G_3}^N = 4 + 2 \cdot 2^N$ if $N$ is even, and the determinant

$$\det(1 - zT_{G_3}) = 1 - (6z^2 - 9z^4 + 4z^6).$$

The number of classes of nonperiodic cycles of length $N$ is $\Omega(N, T_{G_3}) = 0$, if $N$ is odd, and

$$\Omega(N, T_{G_3}) = \frac{1}{N} \sum_{g|N \text{ even}} \mu(g) \left( 4 + 2\frac{N}{g} + 1 \right),$$

if $N$ is even. The graph generates the following algebra. Let $V = \bigoplus_{i=1}^3 V_{2i}$ be a $\mathbb{Z}_{>0}$-graded superspace with dimensions $\dim V_2 = 6$, $\dim V_4 = 9$, $\dim V_6 = 4$ and superdimensions $\dim V_2 = 6$, $\dim V_4 = -9$, $\dim V_6 = 4$. Let $\mathcal{L} = \bigoplus_{N=1}^{\infty} \mathcal{L}_N$ be the free graded Lie superalgebra generated by $V$. The dimension of $\mathcal{L}_N$ subspace is $\dim \mathcal{L}_N = \Omega(N, T_{G_3})$. For instance, $\dim \mathcal{L}_2 = 6$ and $\dim \mathcal{L}_4 = 6$. The basis are:

$$\mathcal{L}_2 : \{ [e_1 e_2^{-1}], [e_1^{-1} e_2], [e_1 e_3^{-1}], [e_1^{-1} e_3], [e_2 e_3^{-1}], [e_2^{-1} e_3] \},$$

$$\mathcal{L}_4 : \{ [e_1 e_2^{-1} e_3 e_2^{-1}], [e_1^{-1} e_2 e_1 e_3^{-1}], [e_1 e_3^{-1} e_2 e_3^{-1}], [e_1^{-1} e_2 e_1^{-1} e_3], [e_1^{-1} e_2 e_3^{-1} e_2],$$

$$[e_1^{-1} e_3 e_2^{-1} e_3] \}.$$

The dimensions satisfy the generalized Witt identity

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\Omega(N, T_{G_3})} = 1 - 6z^2 + 9z^4 - 4z^6.$$

The generating function for the dimensions of the subspaces $U_n(\mathcal{L})$ of the enveloping algebra $U(\mathcal{L})$ is given by the Ihara zeta function of $G_3$:

$$\zeta_{G_3}(z) = \prod_{N=1}^{+\infty} (1 - z^N)^{-\Omega(N, T_{G_3})} = 1 + \frac{1}{18} \sum_{n=1}^{\infty} (2^{2n+5} - 6n - 14) z^{2n}.$$
Example 4.4. $G_3$, the graph shown in Figure 3, with vertex $v_1$ on the left, is strongly connected. The directed edge and vertex adjacency matrices of $G_3$ are

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$  

The matrices have the traces $\text{Tr} S^N = \text{Tr} A^N_d = 0$ if $N$ is odd and $\text{Tr} S^N = \text{Tr} A^N_d = 2^{\frac{N}{2}} + 1$ if $N$ is even, and the determinants

$$\det(1 - zS) = \det(1 - zA_d) = 1 - 2z^2.$$  

The number of classes of nonperiodic cycles of length $N$ is $\theta(N) = 0$, if $N$ is odd, and

$$\theta(N) = \frac{1}{N} \sum_{\substack{gN \text{ even} \\ g | N}} \mu(g) 2^{\frac{N}{2} + 1},$$  

if $N$ is even. The first few values are $\theta(2) = 2$, $\theta(4) = 1$, $\theta(6) = 2$, $\theta(8) = 3$, $\theta(10) = 6$. For $N = 2$, the classes are $[e_1e_2]$ and $[e_2e_3]$. For $N = 4$, only $[e_1e_2e_3e_2]$.

Let $V$ be the vector space with $\dim V = 2$. It generates the graded free Lie algebra $\bigoplus_{N=1}^{\infty} \mathcal{L}_N$ with $\dim \mathcal{L}_N = 0$, if $N$ is odd, and $\dim \mathcal{L}_N = \theta(N)$, if $N$ is even. For instance, $\dim \mathcal{L}_2 = 2$, $\dim \mathcal{L}_4 = 1$. $\mathcal{L}_2$ and $\mathcal{L}_4$ have basis

$$\mathcal{L}_2 : \{[e_1e_2], [e_2e_3]\}, \quad \mathcal{L}_4 : \{[e_1e_2e_3e_2]\}.$$  

The algebra has generalized Witt identity

$$\prod_{N=1}^{+\infty} (1 - z^N)^{\theta(N)} = 1 - 2z^2.$$  

and the dimensions of the spaces $U_n(L)$ of the enveloping algebra are generated by the Bowen-Lanford zeta function

$$\zeta_{dG_3}(z) = \prod_{N=1}^{+\infty} (1 - z^N)^{-\theta(N)} = (1 - 2z^2)^{-1} = \sum_{n=0}^{\infty} 2^n z^{2n}.$$  

Example 4.5. $G_4$, the graph shown in Figure 4. Call $v_4$ the upper vertex and the others, $v_1, v_2, v_3$, from left to right. The directed edge adjacency and the vertex adjacency matrices of $G_4$ are

$$S = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$  

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The matrices have determinants
\[
\det (1 - z S) = \det (1 - z A_d) = 1 - (2z^2 + z^4).
\]
If \( N \) is odd, \( \mathrm{Tr} S^N = 0 \). If \( N \) is even, it follows from (4.4) that
\[
\mathrm{Tr} S^N = \mathrm{Tr} A_d^N = N \sum_{2a+4b=N} \frac{(a+b-1)!}{a!b!} 2^a.
\]

\[
\begin{array}{c}
\text{Figure 4: Graph } G_4
\end{array}
\]

The number of classes of nonperiodic cycles for \( N \) odd is zero. For \( N \) even,
\[
\theta(N) = \sum_{N/g \text{ even}} \frac{\mu(g)}{g} \sum_{2a+4b=N/g} \frac{(a+b-1)!}{a!b!} 2^a.
\]
For instance, \( \theta(2) = 2, \theta(4) = 2, \theta(6) = 2 \). The cycles are \([e_1 e_2]\) and \([e_3 e_4]\), for \( N = 2 \), \([e_1 e_3 e_4 e_2]\) and \([e_2 e_5 e_6 e_4]\), for \( N = 4 \), and \([e_2 e_5 e_6 e_4 e_2 e_5] \) \([e_2 e_5 e_6 e_4 e_3 e_4] \) \([e_1 e_3 e_4 e_2 e_1 e_2] \) \([e_1 e_3 e_4 e_3 e_4 e_2]\), for \( N = 6 \). Also,
\[
\prod_{N=1}^{+\infty} (1 - z^N)^{\theta(N)} = 1 - 2z^2 - z^4
\]

Let \( V = \bigoplus_{i=1}^{4} \), \( \dim V_1 = \dim V_3 = 0 \) \( \dim V_2 = 2 \) \( \dim V_4 = 1 \). It generates the graded free Lie algebra \( \bigoplus_{N=1}^{\infty} \mathcal{L}_N \) with \( \dim \mathcal{L}_N = 0 \), if \( N \) is odd, and \( \dim \mathcal{L}_N = \theta(N) \), if \( N \) is even. For instance, \( \dim \mathcal{L}_2 = 2 \) \( \dim \mathcal{L}_4 = 2 \). \( \mathcal{L}_2 \) and \( \mathcal{L}_4 \) have basis
\[
\mathcal{L}_2 : \quad \{[e_1 e_2], [e_3 e_4]\},
\]
\[
\mathcal{L}_4 : \quad \{[e_1 e_3 e_4 e_2], [e_2 e_5 e_6 e_4]\}.
\]
The algebra has generalized Witt identity
\[
\prod_{N=1}^{+\infty} (1 - z^N)^{\theta(N)} = 1 - 2z^2 - z^4,
\]
and the dimensions of the spaces $U_n(L)$ of the enveloping algebra are generated by the Bowen-Lanford zeta function

$$\zeta_{dG}(z) = \prod_{N=1}^{+\infty} (1 - z^N)^{-\theta(N)} = (1 - 2z^2 - z^4)^{-1} = \sum_{n=0}^{\infty} d_-(n)z^n.$$ 

The coefficients can be computed recursively in the following manner. We have that $d_+(1) = 0$, $d_+(2) = 2$, $d_+(3) = 0$, $d_+(4) = 1$, and $d_+(n) = 0$, if $n \geq 5$. Using the relation (4.8), theorem 4.2, we find that

$$d_-(n) = d_+(n) + d_+(2)d_-(n-2) + d_+(4)d_-(n-4)$$

from which we get that $d_-(n) = 0$, if $n$ is odd and $d_-(2) = 2$, $d_-(4) = 5$, $d_-(6) = 12$, etc.

## 5 Restricted necklace colorings

In this section we interpret the cycle counting formulas (2.3) and (2.6) as counting formulas for the number of classes of non periodic colorings of a necklace with $N$ beads.

First, let’s consider (2.3). Given a graph $G$ with $|E|$ edges and the colors $c_1, \ldots, c_{2|E|}$, assign $c_i$, $c_{i|E|+i}$ to the edges $e_i$, $e_{|E|+i}^{-1} \in G'$, respectively, so that to a cycle of length $N$ in $G$ corresponds with an ordered sequence of $N$ colors. Assign each color in this sequence to a bead in a circular string with $N$ beads - a necklace - in such a manner that two adjacent colors in the sequence are assigned to adjacent beads. The non backtracking condition for cycles implies that no two adjacent beads are painted with colors, say, $c_i$ and $c_{i|E|+i}$. There is a correspondence between the classes of nonperiodic cycles of length $N$ in $G$ and classes of nonperiodic colorings of a necklace with $N$ beads with at most $2|E|$ distinct colors induced by the cycles so that the number of inequivalent colorings is $\Omega(N,T)$, given by (2.3). Of course, the structure of the graph reflects itself in the coloring so that the coloring is restricted by that structure. For instance, the presence of loops in the graph means that their assigned colors may appear repeated in a string of adjacent beads. This can not happen to a color assigned to an edge which is not a loop. The edge adjacency matrix $T$ may be called the color matrix. It basically tells what colors are allowed to be adjacent to a given color in the necklace. Element $T_{ij} = 1$, if a color $c_j$ can be adjacent to color $c_i$ and $c_j \neq c_{|E|+i}$; $T_{ij} = 0$, otherwise.

Inversely, any matrix $T$ with zeros and ones as entries and even order and that has the correct structure (see [27], Lemma 4 on p. 151, and [9], Remark 1.5 on p. 7) is a necklace color matrix that can be interpreted as the edge adjacency matrix of some graph $G$, hence, $\Omega(N,T)$ is the number of classes of nonperiodic colorings in the necklace and of cycles in the graph.
The same ideas apply to directed graphs. Given a strongly connected graph $G$ with $|E|$ edges and edge adjacency matrix $S$, assign to each edge $e_i$ a color $c_i \in \{c_1, ..., c_{|E|}\}$ so that to a cycle of length $N$ in $G$ corresponds with an ordered sequence of $N$ colors. Assign each color in this sequence to a bead in a necklace with $N$ beads in such a manner that two adjacent colors in the sequence are assigned to adjacent beads. Then, the number of equivalence classes of nonperiodic colorings of the necklace with $N$ beads and color matrix $S$ is $\theta(N, S)$, given by (2.6).

Inversely, any irreducible $n \times n$ matrix $S$ with zeros and ones as entries is a color matrix that can be interpreted as the edge adjacency matrix of a strongly connected graph $G$. Recall that a matrix with zeros and ones is the adjacency matrix of a strongly connected graph if and only if it is irreducible [16], hence, $\theta(N, S)$ is the number of equivalence classes of nonperiodic cycles of length $N$ in $G$ and of nonperiodic colorings of a necklace with $N$ beads with at most $|E| = n$ colors.

The above ideas yield the following interpretation of the zeta functions considered in this paper:

Given a color matrix and an infinite sequence of necklaces indexed by the number of beads $N$ the Ihara (or the Bowen-Lanford) zeta function can be interpreted as the function that generates the sequence $(\Omega(1,T), \Omega(2,T), \ldots) ((\theta(1,T), \theta(2,T), \ldots))$ of the numbers of classes of nonperiodic restricted colorings of the necklaces.

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