Entanglement is a key ingredient for quantum technologies and a fundamental signature of quantumness in a broad range of phenomena encompassing many-body physics, thermodynamics, cosmology and life sciences. For arbitrary multiparticle systems, entanglement quantification typically involves nontrivial optimisation problems, and it may require demanding tomographical techniques. Here, we develop an experimentally feasible approach to the evaluation of geometric measures of multiparticle entanglement. Our framework provides analytical results for particular classes of mixed states of $N$ qubits, and computable lower bounds to global, partial, or genuine multiparticle entanglement of any general state. For global and partial entanglement, useful bounds are obtained with minimum effort, requiring local measurements in just three settings for any $N$. For genuine entanglement, a number of measurements scaling linearly with $N$ are required. We demonstrate the power of our approach to estimate and quantify different types of multiparticle entanglement in a variety of $N$-qubit states useful for quantum information processing and recently engineered in laboratories with quantum optics and trapped ion setups.

**INTRODUCTION**

The fascination with quantum entanglement has evolved over the last eight decades, from the realm of philosophical debate to a very concrete recognition of its resource role in a range of applied sciences. Although considerable progress has been achieved in the detection of entanglement, its experimentally accessible quantification remains an open problem for any real implementation of an entangled system. Quantifying entanglement is yet necessary to gauge precisely the quantum enhancement in information processing and computation, and to pin down exactly how much a physical or biological system under observation departs from an essentially classical behaviour. This is especially relevant in the case of complex, multiparticle systems, for which only quite recently have notable advances been reported on the control of entanglement.

An intuitive framework for quantifying the degree of multiparticle entanglement relies on a geometric perspective. Within this approach, one first identifies a hierarchy of non-entangled multiparticle states, also referred to as $M$-separable states for $2 \leq M \leq N$, where $N$ is the number of particles composing the quantum system of interest; see Figure 1. Introducing then a distance functional $D$ with respect to the natural properties of contractivity under quantum operations and joint convexity (see Materials and Methods), the quantity $E_M^D$ defined as

$$E_M^D(\rho) = \inf_{\rho, \zeta \text{ M- separable}} D(\rho, \zeta),$$

is a valid geometric measure of ($M$-inseparable) multiparticle entanglement in the state $\rho$.

Some special cases are prominent in this hierarchy. For $M=N$, the distance from $N$-separable (also known as fully separable) states defines the global multiparticle entanglement $E_N^D$, which accounts for any form of entangled distribution among two or more of the $N$ particles. Geometric measures of global entanglement have been successfully employed to characterise quantum phase transitions in many-body systems and directly assess the usefulness of initial states for Grover’s search algorithms. On the other extreme of the hierarchy, for $M=2$, the distance from two-separable (also known as biseparable) states defines instead the genuine multiparticle entanglement $E_2^D$, which quantifies the entanglement shared by all the $N$ particles, that is the highest degree of inseparability. Genuine multiparticle entanglement is an essential ingredient for quantum technologies including multiuser quantum cryptography, quantum metrology and measurement-based quantum computation. Finally, for any intermediate $M$, we can refer to $E_M^D$ as partial multiparticle entanglement. The presence of partial entanglement is relevant in quantum informational tasks such as quantum secret sharing and may have a relevant role in biological phenomena.

Probing and quantifying different types of entanglement can shed light on which nonclassical features of a mixed multiparticle state are necessary for quantum-enhanced performance in specific tasks and can guide the understanding of the emergence of classicality in multiparticle quantum systems of increasing complexity.

The quantitative amount of multiparticle entanglement, be it global or genuine (or any intermediate type), has an intuitive operational meaning when adopting the geometric approach. Namely, $E_M^D$ measures how distinguishable a given state $\rho$ is from the closest $M$-separable state. Given some widely adopted metrics, such a distinguishability is directly connected to the usefulness of $\rho$ for quantum information protocols relying on multiparticle entanglement. For instance, the trace distance of entanglement is a valid geometric measure of ($M$-inseparable) multiparticle entanglement in the state $\rho$. A number of measurements scaling linearly with $N$ are required. We demonstrate the power of our approach to estimate and quantify different types of multiparticle entanglement in a variety of $N$-qubit states useful for quantum information processing and recently engineered in laboratories with quantum optics and trapped ion setups.

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**Author contributions**

M Cianciaruso and T R Bromley contributed equally to this work.

**Competing interests**

The authors declare no competing interests.

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bounds on the number of orthogonal states that can be discriminated by local operations and classical communication (LOCC). The geometric entanglement based on infidelity (monotonically related to Bures distance) has also a dual interpretation based on the convex roof construction; that is, it quantifies the minimum price (in units of pure-state entanglement) that has to be spent on average to create a given density matrix $\rho$ as a statistical mixture of pure states.

It is therefore clear that finding the minimum in Equation (1), and hence evaluating geometric measures of multiparticle entanglement defined by meaningful distances, is a central challenge to benchmark quantum technologies. However, obtaining such a solution for general multiparticle states is in principle a formidable problem. Even if possible, there would remain major challenges for experimental evaluation, which would in general require a complete reconstruction of the state through full tomography. For multiparticle states of any reasonable number of qubits, full state tomography places significant demands on experimental resources, and it is thus highly desirable to provide quantitative guarantees on the geometric multiparticle entanglement present in a state, via nontrivial lower bounds, in an experimentally accessible way.3–13

Here we provide substantial advances towards addressing this problem in a general fashion. We identify a general framework for the provision of experimentally friendly quantitative guarantees on the geometric multiparticle entanglement present in a state. This approach consists of:

1. Choosing a set of reference states: find a restricted family of $N$-qubit states with the property that any state may be mapped into this family through a fixed procedure of single-qubit LOCC. This reference family should be simple to characterise, and can be chosen from experimental or theoretical considerations.

2. Identifying $M$-separable reference states: Apply the fixed LOCC procedure to the general set of $M$-separable states, hence identifying the subset of $M$-separable states within the reference family.

3. Calculating $E^D_M$ for the reference states: solve the optimisation problem for the geometric entanglement of reference states. This is markedly simplified by using the properties of contractivity and joint convexity, that hold for any distance functional $D$ defining a valid entanglement measure, and imply in particular that one of the closest $M$-separable states to any reference state is to be found itself within the reference family.

4. Deriving optimised lower bounds for any state: exploit the freedom to apply single-qubit unitaries to any $N$-qubit state $\rho$ in order to find the corresponding reference state with the highest geometric entanglement, providing an optimised lower bound to $E^D_M(\rho)$.

This process presents a versatile and comprehensive approach to obtain lower bounds on geometric multiparticle entanglement measures according to any valid distance. While building on some previously utilised methods for steps (1)3–13,16 and (4)13,15,16, it introduces novel techniques in steps (2) and most importantly (3), which are crucial for completing the framework and making it effective in practice (see e.g., Appendix C, D, E and F of the Supplementary Material).

To illustrate the power of our approach, we focus initially on a reference family of mixed states $\sigma$ of $N$ qubits, that we label $\mathcal{M}^\sigma_N$ states, which form a subset of the class of states having all maximally mixed marginals. This family includes maximally entangled Bell states of two qubits and their mixtures, as well as multiparticle bound entangled states.35–46 For any $N$ these states are completely specified by three easily measurable quantities, given by the correlation functions $c_j = \langle a_j^{(N)} \rangle$, where $\{a_j\}_{j=1,2,3}$ are the Pauli matrices. In the following we show how every entanglement monotone $E^D_M$ can be evaluated exactly for any even $N$ on these states, by revealing an intuitive geometric picture common to all valid distances $D$. For odd $N$, the results are distance-dependent; we show nonetheless that $E^D_M$ can still be evaluated exactly if $D$ denotes the trace distance. The results are nontrivial for all $M > [N/2]$ in the hierarchy of Figure 1. A central observation, in line with the general framework, is that an arbitrary state of $N$ qubits can be transformed into an $\mathcal{M}^\sigma_N$ state by an LOCC procedure, which cannot increase entanglement by definition. This implies that our exact formulae readily provide practical lower bounds to the degree of global and partial multiparticle entanglement in completely general states. Importantly, the bounds are obtained by measuring only the three correlation functions $c_j$ for any number of qubits, and can be further improved by adjusting the local measurement basis (see Figure 2 for an illustration).

Furthermore, we discuss how our results can be extended to allow for the quantitative estimation of genuine multiparticle entanglement as well, at the cost of performing extra measurements. As $\mathcal{M}^\sigma_N$ states are always biseparable, we must consider a different reference family. We focus on the class of $N$-qubit states obtained as mixtures of Greenberger–Horne–Zeilinger (GHZ) states,5,47 the latter being central resources for quantum communication and estimation; this class of states depends on $2^N - 1$ real parameters. We calculate exactly distance-based measures of genuine multiparticle entanglement $E^D_M$ for these states, for every valid $D$. Once more, these analytical results provide lower bounds to geometric measures of genuine entanglement for any general state of $N$ qubits, obtainable
Experimentally friendly protocol to quantify global and partial $N$-particle entanglement. Top row: (a) A state $\rho$ of $N$ qubits is shared by $N$ parties, named Alice, Bob, Charlie, ..., Natalie. Each party, labelled by $a = A, ..., N$, locally measures her or his qubit in three orthogonal directions $\{\sigma^{(a)}_j\}$, with $j = 1, 2, 3$, indicated by the solid arrows. If the shared state $\rho$ is completely unknown, a standard choice can be to measure the three canonical Pauli operators for all the qubits (corresponding to the directions of the dashed axes); if instead some partial information on $\rho$ is available, the measurement directions can be optimised a priori. Once all the data are collected, the $N$ parties communicate classically to construct the three correlation functions $\{c_j\}$, with $c_j = \langle \otimes a \sigma^{(a)}_j^{(a)} \rangle$. Middle row: for any $N$, one can define a reference subset of $N$-qubit states with all maximally mixed marginals ($\mathcal{M}_N^0$ states), which are completely specified by a triple of orthogonal correlation functions $\{c\}$. These states enjoy a convenient representation in the space of $\{c_1, c_2, c_3\}$. (b) For even $N$, $\mathcal{M}_N^0$ states fill the tetrahedron with vertices $\{1, (-1)^{N/2}, 1\}, \{-1, (-1)^{N/2}, 1\}, \{1, (-1)^{N/2}, -1\}$ and $\{-1, (-1)^{N/2}, -1\}$. For odd $N$, they are instead contained in the unit Bloch ball. For any $M > \lceil N/2 \rceil$, $M$-separable $\mathcal{M}_N^0$ states are confined to the octahedron with vertices $\{\pm 1, 0, 0\}$, $\{0, \pm 1, 0\}$ and $\{0, 0, \pm 1\}$, illustrated in red in both panels; conversely, for $M \leq \lceil N/2 \rceil$, all $\mathcal{M}_N^0$ states are $M$-separable. Bottom row: geometric analysis of multiparticle entanglement. The bottom panels depict zooms of (d) a corner of the tetrahedron for even $N$ and (e) a sector of the unit sphere for odd $N$, opposite a face of the octahedron of $M$-separable $\mathcal{M}_N^0$ states (for $M > \lceil N/2 \rceil$). Instances of inseparable $\mathcal{M}_N^0$ states are indicated by blue circles, and their closest $M$-separable states by red crosses. The cyan surfaces in each of the two bottom panels contain states with equal global and partial multiparticle entanglement $E^0_M$, which we compute exactly. The results are valid for any contractive and jointly convex distance $D$ in the even $N$ case, and for the trace distance in the odd $N$ case. The entanglement of an $\mathcal{M}_N^0$ state with correlation functions $\{c_1, c_2, c_3\}$ provides an analytical lower bound for the entanglement of any $N$-qubit state with the same correlation functions, such as the state $\rho$ initially shared by the $N$ parties in (a). The bound is effective for the most relevant families of $N$-qubit states in theoretical and experimental investigations of quantum information processing, as we show in this article.

experimentally in this case by performing at least $N+1$ local measurements.\textsuperscript{48}

We demonstrate that our results provide overall accessible quantitative assessments of global, partial and genuine multiparticle entanglement in a variety of noisy states produced in recent experiments,\textsuperscript{40,44,49–52} going beyond mere detection,\textsuperscript{4–12} yet with a significantly reduced experimental overhead. Compared with some recent complementary approaches to the quantification of multiparticle entanglement,\textsuperscript{13–23} we find that our results, obtained via the general quantitative framework discussed above, fare surprisingly well in their efficiency and versatility despite the minimal experimental requirements (see Table 1 for an in-depth comparison).

RESULTS

Global and partial multiparticle entanglement

We begin by choosing as our reference family the set of $N$-qubit $\mathcal{M}_N^0$ states. An $\mathcal{M}_N^0$ state is defined as

$$\sigma = \frac{1}{I} \left( I^0 \otimes I^0 + \sum_{j=1}^3 c_j \sigma^{(a)}_j \right),$$

where $I$ is the $2 \times 2$ identity matrix. These states are invariant under permutations of any pair of qubits and enjoy a nice
geometrical representation in the space of the three correlation functions \( c_\varphi \) corresponding to a tetrahedron for even \( N \) and to the unit ball for odd \( N \), as depicted in Figure 2. We can then characterise the subset of \( M \)-separable \( \mathcal{M}_M^N \) states for any \( N \). We find that, if \( M > [N/2] \), then the \( M \)-separable \( \mathcal{M}_M^N \) states fill a subset corresponding to an octahedron in the space of the correlation functions (Figure 2). When \( M < [N/2] \), all \( \mathcal{M}_M^N \) states are instead \( M \)-separable. The proofs are deferred to the Supplementary Material.

We can now tackle the quantification of global and partial multiparticle entanglement in these states; for the latter, we will always focus on the nontrivial case \( M > [N/2] \) throughout this section. First, we observe that the closest \( M \)-separable state \( \varphi_m \) to an \( \mathcal{M}_M^N \) state \( \varphi \), which solves the optimisation in Equation (1), can always be found within the subset of \( M \)-separable \( \mathcal{M}_M^N \) states, yielding a considerable simplification of the general problem. To find the exact form of \( \varphi_m \) and consequently of \( E_M^D(\varphi) \), we approach the cases of even and odd \( N \) separately.

For even \( N \), we prove that there exists a unique solution to the minimisation in Equation (1), independent of the specific choice of contractive and jointly convex distance \( D \). Namely, the closest \( M \)-separable state \( \varphi_m \) in the octahedron bounding the corner of the tetrahedron in which \( \varphi \) is located, and is identified by the intersection of such octahedron face with the line connecting \( \varphi \) to the vertex of the tetrahedron corner, as depicted in Figure 2d. It follows that, for any nontrivial \( M \), valid \( D \) and even \( N \), the multiparticle entanglement \( E_M^D(\varphi_m) \) of an \( \mathcal{M}_M^N \) state \( \varphi_m \) with correlation functions \( c_m \) is only a monotonically increasing function of the Euclidean distance between the point of coordinates \( c_m \) and the closest octahedron face, which is in turn proportional to \( h_m = \frac{1}{2} \sum_{j=1}^{N} |c_j| - 1 \) (notice that \( h_m \) equals the bipartite measure known as concurrence for \( N=2 \)). We have then a closed formula for any valid geometric measure of global and partial multiparticle entanglement on an arbitrary \( \mathcal{M}_M^N \) state \( \varphi_m(\cdot) \) with even \( N \), given by

\[
E_M^D(\varphi_m(\cdot)) = \begin{cases} 
0, & h_m \leq 0 \text{ (or } M \leq [N/2]); \\
\min \left\{ \frac{1}{2} \sqrt{|c_j|^2 + \frac{1}{2}(2h_m - |c_j|)^2} \right\}, & \text{otherwise}.
\end{cases}
\]

The usefulness of the just-derived analytical results for multiparticle entanglement is not limited to the \( \mathcal{M}_M^N \) states. In accordance with our general framework, a crucial observation is that the \( \mathcal{M}_M^N \) states are extremal among all quantum states with given correlation functions \( c_m \). Specifically, any general state \( \rho \) of \( N \) qubits can be transformed into an \( \mathcal{M}_M^N \) state with the same \( c_m \).
Table 3. Applications of our framework to construct accessible lower bounds on global and partial (M-inseparable) multiparticle entanglement (which are nonzero for any $M > |N/2|$ when $\sum_j |c_j| > 1$), for the families of $N$-qubit states listed as follows

| $N$ | State | $\{c_1, c_2, c_3\}$ | $\sum_{j=1}^3 |c_j|$ | $\{\theta, \psi, \phi\}$ |
|-----|-------|-----------------|----------------|------------------|
| $N=3$ | GHZ$^{(3)}$ | $\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ | $2\sqrt{\frac{2}{3}}$ | $\{1, 1, 1\}$ |
| | $W^{(3)}$ | $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ | $\sqrt{\frac{2}{3}}$ | $\{1, 1, 1\}$ |
| $N=4$ | GHZ$^{(4)}$ | (1,1,1) | 3 | (0,0,0) |
| | $W^{(4)}$ | $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$ | $\frac{5}{\sqrt{3}}$ | $\{1, 1, 1\}$ |
| | $[\rho_{\\text{Wei}}^{(4)}(x)]$ | $\{x, x, 2x - 1\}$ | $2x + |2x - 1|$ | (0,0,0) |
| | $[c_1]$ | (1,1,1) | 3 | * |
| | $[c_2]$ | (1,1,1) | 3 | * |
| | $[\rho_1]$ | (1,1,1) | 3 | * |
| $N=5$ | GHZ$^{(5)}$ | $\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\}$ | $\sqrt{2}$ | $\{0, 0, 0, 0\}$ |
| | $W^{(5)}$ | $\{\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{3}}\}$ | $\sqrt{\frac{2}{3}}$ | $\{1, 1, 1, 1\}$ |
| | $[\rho_{\\text{Wei}}^{(5)}(x)]$ | $\{x, x, 2x - 1\}$ | $2x + |2x - 1|$ | (0,0,0) |
| | $[c_1]$ | (1,1,1) | 3 | * |
| $N=6$ | GHZ$^{(6)}$ | (1, -1, 1) | 3 | (0,0,0) |
| | $[\rho_{\\text{Wei}}^{(6)}(x)]$ | $\{x, x, 2x - 1\}$ | $2x + |2x - 1|$ | (0,0,0) |
| | $[c_1]$ | (1, -1, 1) | 3 | * |
| | $[c_2]$ | (1,1,1) | 3 | * |
| | $[\rho_1]$ | (1,1,1) | 3 | * |
| $N=7$ | GHZ$^{(7)}$ | $\{\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, 0\}$ | $\sqrt{2}$ | $\{0, 0, 0, 0, 0\}$ |
| | $[\rho_{\\text{Wei}}^{(7)}(x)]$ | $\{x, x, 2x - 1\}$ | $2x + |2x - 1|$ | (0,0,0) |
| | $[c_1]$ | (1, -1, 1) | 3 | * |
| $N=8$ | GHZ$^{(8)}$ | (1,1,1) | 3 | (0,0,0) |
| | $[\rho_{\\text{Wei}}^{(8)}(x)]$ | $\{x, x, 2x - 1\}$ | $2x + |2x - 1|$ | (0,0,0) |

(i) $N$-qubit GHZ states$^{49}$ $[\text{GHZ}^{(N)}] = \frac{1}{\sqrt{2}}(|00\ldots 00\rangle + |11\ldots 11\rangle)$ with $N \geq 3$. (ii) $N$-qubit W states$^{54} [\text{W}^{(N)}] = \frac{1}{\sqrt{N}}(|00\ldots 01\rangle + |00\ldots 10\rangle + \ldots + |10\ldots 0\rangle)$, with $N \geq 3$. (iii) $N$-qubit Wei states$^{55,56} [\rho_{\text{Wei}}^{(N)}(x)] = x \langle \text{GHZ}^{(N)}|\langle \text{GHZ}^{(N)}| + |\text{H}\rangle \langle \text{H}| + |\text{V}\rangle \langle \text{V}| + |\text{E}\rangle \langle \text{E}| + |\text{W}\rangle \langle \text{W}| + |\text{C}\rangle \langle \text{C}| + |\text{P}\rangle \langle \text{P}|$. (iv) $N$-qubit linear cluster states $[C_i^{(N)}]$ corresponding to the $N$-vertex linear graph $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$. (v) $N$-qubit rectangular cluster states $[C_{\text{r}}]^{(N)}$ corresponding to the $N$-vertex ladder-type graph $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$. (vi) $N$-qubit (symmetric) Dicke states $[D_k^{(N)}] = \frac{1}{\sqrt{2^n}} \sum_{\Pi \in \{0, 1\}^n} \left(|0\rangle^{N-k} \otimes |1\rangle^k\right)$, which are superpositions of all states with $k$ qubits in the excited state $|1\rangle$ and $N-k$ qubits in the ground state $|0\rangle$, with the symbol $\{(\Pi_i)_{i=1}^n\}$ denoting all the $Z \equiv \frac{N!}{k!(N-k)!}$ distinct permutations of $0's$ and $1's$ we focus on half-excited Dicke states, given by $k=N/2$ for any even $N$, and $k=N-1$ for any odd $N$. (vii) $4$-qubit singlet state$^{52} [W^{(4)}] = \frac{1}{\sqrt{2}}(|0100\rangle + |1010\rangle + |0110\rangle + |1001\rangle + |0011\rangle + |1100\rangle + |0101\rangle + |1010\rangle + |1101\rangle + |0010\rangle)$. (viii) $N$-qubit generalised Smolin states$^{43,45,57} [\rho_{\text{Smolin}}^{(N)}]$ for even $N > 4$, which are instances of $M_4^{(N)}$ states with correlation triple $(-1)^{N/2}, (-1)^{N/2}, (-1)^{N/2}$, hence their entanglement quantification is exact. The asterisk indicates non-permutationally invariant states for which the optimisation of the bounds requires different angles for each qubit (not reported here). Notice that in the table we listed mostly pure states. In general, if the triple $\{c_j\}$ is optimal for a pure $N$-qubit state $|\Phi^{(N)}\rangle$, then for the mixed state $\rho^{(N)}(q) = q|\Phi^{(N)}\rangle \langle \Phi^{(N)}| + (1-q)|\Omega^{(N)}\rangle \langle \Omega^{(N)}|$, obtained by mixing $|\Phi^{(N)}\rangle$ with white noise, one still gets nonzero lower bounds to global and partial entanglement for all $q > 1/\sum_{j=1}^{3} |c_j|$, as shown in Figure 3 for some representative examples.

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by means of a procedure that we name \( \mathcal{M}_N^{p_{\text{f}}-\text{fication}} \), involving only LOCC (see Materials and Methods). This immediately implies that, for any \( M > \lfloor N/2 \rfloor \), the multipartite entanglement \( E_M^{\rho} \) of \( \rho \) can have a nontrivial exact lower bound given by the corresponding multipartite entanglement of the \( \mathcal{M}_N^{p_{\text{f}}-\text{fication}} \) state \( \bar{\sigma} \) with the same \( \{c_j\} \):

\[
E_M^{\rho}(\bar{\sigma}(c)) \leq E_M^{\rho}(\bar{\rho}) \quad \forall \rho : \text{Tr}(\rho \bar{\sigma}_j^N) = c_j \quad (j = 1, 2, 3). \tag{4}
\]

From a practical point of view, one needs only to measure the three correlation functions \( \{c_j\} \), as routinely done in optical, atomic, and spin systems, to obtain an estimate of the global and partial multipartite entanglement content of an unknown state \( \rho \), with no need for a full state reconstruction.

Furthermore, the lower bound can be improved if a partial knowledge of the state \( \rho \) is assumed, as is usually the case for experiments aiming to produce specific families of states for applications in quantum information processing. In those realisations, one typically aims to detect entanglement by constructing optimised entanglement witnesses tailored on the target states. By exploiting similar ideas, we can optimise the quantitative lower bound in Equation (4) over all possible single-qubit local unitaries applied to the state \( \rho \) before the \( \mathcal{M}_N^{p_{\text{f}}-\text{fication}} \),

\[
\sup_{U_{\rho}} E_M^{\rho}(U_{\rho} \bar{\sigma}_j^N) \leq E_M^{\rho}(U_{\rho} \rho U_{\rho}^\dagger) = E_M^{\rho}(\bar{\rho}). \tag{5}
\]

where \( \text{Tr}(U_{\rho} \rho U_{\rho}^\dagger \sigma_j^N) = \bar{c}_j \) and \( U_{\rho} = \otimes_{q=1}^{N-1} U_q^{(\theta_q)} \) denote a single-qubit local unitary operation. Experimentally, the optimised lower bound can then be still accessed by measuring a triple of correlation functions \( \{\bar{c}_j\} \) given by the expectation values of correspondingly rotated Pauli operators on each qubit, \( \bar{c}_j = \langle U_j \sigma_j^{(\theta_j)} U_j^\dagger \rangle \), as illustrated in Figure 2a. Optimality in Equation (5) can be achieved by the choice of \( U_{\rho} \) such that \( h_{\rho} = \frac{1}{2} \sum_{j=1}^3 |\bar{c}_j| - 1 \) is maximum. The optimisation procedure can be significantly simplified when considering a state \( \rho \), which is invariant under permutations of any pair of qubits. In such a case, one may need to optimise only over three angles \( \{\theta_q, \psi_q, \phi_q\} \) parameterising a generic unitary applied to each single qubit; the optimisation can be equivalently performed over an orthogonal matrix acting on the Bloch vector of each qubit (see Materials and Methods).

We can now investigate how useful our results are on concrete examples. Table 3 presents a compendium of optimised analytical lower bounds on the global and partial multipartite entanglement of several relevant families of \( N \)-qubit states up to \( N = 8 \). All the bounds are experimentally accessible by measuring the three correlation functions \( \{\bar{c}_j\} \), corresponding to optimally rotated Pauli operators (Figure 2).

Let us comment on some cases where our analysis is particularly effective. For GHZ states, cluster states, and half-excited Dicke states, which constitute primary resources for quantum computation and metrology, we get the maximum in our bounds for any even \( N \). This means that our bounds remain robust to estimate global and partial entanglement in noisy versions of these states \( \{\} \), when one considers mixtures of any of these states with probability \( q \) and the maximally mixed state with probability \( 1 - q \) for all \( q > 1/3 \). Notably, for values of \( q \) sufficiently close to 1, our bounds to global entanglement can be tighter than the (more experimentally demanding) ones derived very recently in ref. 13, as shown in Figure 3a. Focusing on noisy GHZ states, we observe, however, that our scale-invariant threshold \( q > 1/3 \), obtained by measuring the three canonical Pauli operators for each qubit, is weaker than the well-established inseparability threshold \( q > 1/(1 + 2^{N/2}) \). Nevertheless, we note that our simple quantitative bound given by Equation (4) becomes tight in the paradigmatic limit of pure GHZ states \( q = 1 \) of any even number \( N \) of qubits, thus returning the exact value of their global multipartite entanglement via Equation (2), despite the fact that such states are not (and are very different from) \( \mathcal{M}_N^{p_{\text{f}}-\text{fication}} \). Equation (4) also provides a useful nonvanishing lower bound to the global (and partial) \( N \)-particle entanglement of GHZ states in the interval \( x \in [\frac{1}{2}, 1] \), for any even \( N \). A comparison between such a bound (with \( D \) denoting the relative entropy), which requires only three local measurements, and the true value of the relative

\[
E_M^{\rho}(\bar{\rho}) = E_M^{\rho}(\rho^{(N)}) - \sum_{j=1}^N E_M^{\rho}(\rho_j^{(N)}) - \sum_{j=1}^{N-1} E_M^{\rho}(\rho_{j,j+1}^{(N)}) - \sum_{j=1}^{N-2} E_M^{\rho}(\rho_{j,j+2}^{(N)}) - \cdots.
\]

and given explicitly by

\[
E_M^{\rho}(\bar{\rho}) = -\frac{1}{2} \left( 1 - \sum_{j=1}^3 |\bar{c}_j| - 1 \right)
\]

in the next section of this paper. The results are scale-invariant and hold for any even \( N \).
entropy of global $N$-particle entanglement for these states, whose experimental evaluation would conventionally require a complete state tomography, is presented in Figure 3b.

### Genuine multiparticle entanglement

We now show how general analytical results for geometric measures of genuine multiparticle entanglement can be obtained as well within our approach. The results from the previous section, while quite versatile, cannot provide useful bounds for the complete hierarchy of multiparticle entanglement, because while quite versatile, cannot provide useful bounds for the genuine multiparticle concurrence and negativity have been calculated exactly via Equation (6); interestingly, for even $N$ it is found to coincide with the lower bound to their global entanglement that we had obtained by $M_1$-ification, plotted as a solid line in Figure 3b. This means that for these states also the genuine multiparticle entanglement can be quantified entirely by measuring the three canonical correlation functions ($c_j$), for any $N$. More generally, for arbitrary $G_N$ states, all the genuine entanglement measures given by equation (6) can be obtained by measuring the maximum GHZ overlap $\rho_{\text{max}}$, which requires $N+1$ local measurement settings given explicitly in ref. 48. This is remarkable, as with the same experimental effort needed to detect full inseparability we have now a complete quantitative picture of genuine entanglement in these states based on any geometric measure, agreeing with and extending the findings of refs 14,18. Furthermore, as evident from Equation (6), all the geometric measures are monotonic functions of each other: our analysis thus reveals that there is a unique ordering of genuinely entangled $G_N$ states within the distance-based approach of Figure 1.

In the same spirit as the previous section, and in compliance with our general framework, we note that the exact results obtained for the particular reference family of $G_N$ states provide quantitative lower bounds to the genuine entanglement of general $N$-qubit states. This follows from the observation that any $N$-qubit state $\rho$ can be transformed into a $G_N$ state with eigenvalues $\rho_i^\pm = (\beta_i^\pm | \rho | \beta_i^\pm )$ by an LOCC procedure that we may call GHZ-diagonalisation. Therefore, given a completely general state $\rho$, one only needs to measure its overlap with a suitable reference GHZ state; if this overlap is found larger than 1/2, then by using Equation (6) with $\rho_{\text{max}}$ equal to the measured overlap one obtains analytical lower bounds to the genuine multiparticle entanglement $E_2^G$ of $\rho$ with respect to any desired distance $D$. As before, the bounds can be optimised in situations of partial prior knowledge—e.g., by applying local unitaries on each qubit before the GHZ-diagonalisation, which has the effect of maximising the overlap with a chosen particular GHZ vector in the basis $\{ \beta_i^\pm \}$. The bounds then remain accessible for any state $\rho$ by $N+1$ local measurements, with exactly the same demand as for just witnessing entanglement.

For instance, for the singlet state $|\psi^-(N)\rangle$, which is a relevant resource in a number of quantum protocols including multiserver secret sharing, one has $\rho_{\text{max}} = (\beta_1^- | \psi^-(N) \rangle \langle \psi^-(N) | \beta_1^- ) = 2/3 > 1/2$, obtainable by measuring the overlap with the GHZ basis state $|\beta_1^- \rangle = (|001\rangle + |110\rangle) / \sqrt{2}$. Optimised bounds to the genuine multiparticle entanglement of half-excited Dicke states $D_{N/2}^N$ for even $N \geq 4$, defined in Table 3, can be found as well based on GHZ-diagonalisation, and are expressed by $\rho_{\text{max}} = (N/2)^{1-N}$, meaning that they become looser with increasing $N$ and stay nonzero only up to $N=8$. In this respect, we note that alternative methods to detect full inseparability of Dicke states for any $N$ are

| Distance $D$ | $g_D(\rho_{\text{max}})$ |
|------------|-------------------|
| Relative entropy $D_{\text{RE}}$ | $1 + \rho_{\text{max}} \log_2(1 - \rho_{\text{max}}) = \rho_{\text{max}}(1 - \rho_{\text{max}})$ |
| Trace $D_T$ | $\rho_{\text{max}}^{\frac{-1}{2}}$ |
| Infidelity $D_I$ | $\frac{1}{2} - \sqrt{\rho_{\text{max}}(1 - \rho_{\text{max}})}$ |
| Squared Bures $D_B$ | $2 - \frac{1}{2} \sqrt{1 - \rho_{\text{max}} + \rho_{\text{max}}}$ |
| Squared Hellinger $D_H$ | $2 - \frac{1}{2} \sqrt{1 - \rho_{\text{max}} + \rho_{\text{max}}}$ |
In this section, we benchmark the applicability of our results to real data from recent experiments.\textsuperscript{44,49} In the speciﬁc experimental implementation of ref. 44 for $N = 4$, the global entanglement was detected (but not quantified) via a witness constructed by measuring precisely the three correlation functions $\{c_i\}$. On the basis of the existing data alone (and without assuming that the produced state is within the $\mathcal{M}_D$ family), we can then provide a quantitative estimate to the multipartite entanglement of this experimental bound entangled state in terms of any geometric measure $E^D_{\text{ed}}$, by using Table 2. The results are reported in Table 5a for the illustrative case of the trace distance.

Remaining within the domain of quantum optics, recently two laboratories reported the creation of six-photon Dicke states $|D^{(6)} \rangle$. Dicke states\textsuperscript{58} are valuable resources for quantum metrology, computation and networked communication and emerge naturally in many-body systems as ground states of the isotropic Lipkin–Meschkov–Glick model.\textsuperscript{66} On the basis of the values of the three correlation functions $\{c_i\}$, which were measured in refs 51,52 to construct some entanglement witnesses, we can provide quantitative bounds to their global and partial geometric entanglement $E^D_{\text{ed}}$ (for $4 \leq M \leq 6$) from Equation (4); see Table 5a. A series of experiments at Innsbruck\textsuperscript{60,49,50,67} resulted in the generation of a variety of relevant multi-qubit states with trapped ion setups, for explorations of fundamental science and for the

### Table 5. Accessible lower bounds to global, partial and genuine multiparticle entanglement for a variety of experimental quantum states

| a | Global and partial multiparticle entanglement |
|---|---|
| State | Reference | Fidelity (%) | $\{\xi_1, \xi_2, \xi_3\}$ | $\sum_{j=1}^{3} |\xi_j|$ | $E^D_{\text{ed}}$ |
| $\rho_5^{(4)}$ | 44 | 96.83 ± 0.05 | $(0.401 \pm 0.004,0.362 \pm 0.004,0.397 \pm 0.008)$ | 1.16 ± 0.01 | 0.040 ± 0.002 |
| $\rho_6^{(4)}$ | 51 | 56 ± 2 | $(0.63 \pm 0.02,0.63 \pm 0.02, -0.3 \pm 0.1)$ | 1.69 ± 0.04 | 0.17 ± 0.01 |
| $\rho_8^{(4)}$ | 52 | 65 ± 2 | $(0.63 \pm 0.02,0.63 \pm 0.02, -0.42 \pm 0.02)$ | 1.35 | 0.102 |
| $\rho_{\text{GHZ}}^{(3)}$ | 67 | 87.9 | $(-0.497,0.515, -0.341)$ | 2.25 | 0.312 |
| $\rho_{\text{GHZ}}^{(4)}$ | 67 | 80.3 | $(0.663,0.683,0.901)$ | 1.24 | 0.0589 |
| $\rho_{\text{W}}^{(4)}$ | 67 | 19.4 | $(-0.404,0.454, -0.378)$ | 1.39 | 0.0963 |
| $\rho_{\text{W}}^{(4)}$ | 67 | 31.4 | $(0.472, -0.468, -0.446)$ |

| b | Genuine multiparticle entanglement |
|---|---|
| State | Reference | Fidelity (%) | $E^D_{\text{ed}}$ | $E^D_{\text{ed}}$ | $E^D_{\text{ed}}$ |
| $\rho_{\text{GHZ}}^{(3)}$ | 50 | 97.0 ± 0.3 | 0.81 ± 0.02 | 0.470 ± 0.003 | 0.329 ± 0.008 | 0.36 ± 0.01 |
| $\rho_{\text{GHZ}}^{(4)}$ | 50 | 95.7 ± 0.3 | 0.74 ± 0.01 | 0.457 ± 0.003 | 0.297 ± 0.007 | 0.323 ± 0.008 |
| $\rho_{\text{GHZ}}^{(5)}$ | 50 | 94.4 ± 0.5 | 0.69 ± 0.02 | 0.444 ± 0.005 | 0.27 ± 0.01 | 0.29 ± 0.01 |
| $\rho_{\text{GHZ}}^{(6)}$ | 50 | 89.2 ± 0.4 | 0.51 ± 0.01 | 0.392 ± 0.004 | 0.190 ± 0.005 | 0.200 ± 0.006 |
| $\rho_{\text{GHZ}}^{(10)}$ | 50 | 81.7 ± 0.4 | 0.313 ± 0.009 | 0.317 ± 0.004 | 0.113 ± 0.003 | 0.117 ± 0.003 |
| $\rho_{\text{GHZ}}^{(14)}$ | 50 | 62.6 ± 0.6 | 0.046 ± 0.004 | 0.126 ± 0.006 | 0.016 ± 0.002 | 0.016 ± 0.002 |
| $\rho_{\text{GHZ}}^{(18)}$ | 50 | 50.8 ± 0.9 | 0.0002 ± 0.0004 | 0.008 ± 0.009 | 0.0001 ± 0.0001 | 0.0001 ± 0.0001 |

(a) Accessible lower bounds to global and partial multiparticle entanglement of some experimentally prepared states, given by Equation (5) and evaluated in particular for the trace distance of entanglement $E^D_{\text{ed}}$ by using Equation (2) for even $N$ and Equation (3) for odd $N$. Following the theoretical analysis of Table 3, data obtained by direct measurements of the canonical correlation functions were used to construct bounds for a noisy Smolin state of 4 photons,\textsuperscript{44} noisy Dicke states of 6 photons\textsuperscript{51,52} and noisy GHZ states of four ions.\textsuperscript{67} For noisy GHZ states of three ions and noisy W states of four ions (two implementations labelled as A and B),\textsuperscript{57} full data sets were used to extract the optimised correlation functions $\{c_j\}$ required for the bounds. For all the presented experimental states (whose fidelities with the ideal target states are reported for reference), we are able to provide a reliable estimate of the multipartite entanglement $E^D_{\text{ed}}$ for any $M > |N/2|$.

(b) Lower bounds to genuine multiparticle entanglement of noisy GHZ states of up to 14 ions,\textsuperscript{38} as quantified in terms of all the distance-based entanglement measures $E^D_{\text{ed}}$ reported in Table 4, obtained by Equation (6) with $p_{\text{max}}$ given in each case by the measured fidelity with the pure reference GHZ state. All the reported entanglement estimates are obtained from the same data needed to witness full inseparability, which for general $N$-qubit states can be accessed by $N+1$ local measurements without the need for a full tomography.

Available,\textsuperscript{45,51,52} but quantitative results are lacking in general. Nevertheless, applying our general approach to an alternative reference family more tailored to the Dicke states could yield tighter lower bounds that do not vanish beyond $N = 8$.

Finally, notice that a lower bound to a distance-based measure of genuine multipartite entanglement, as derived in this section, is automatically also a lower bound to corresponding measures of global and any form of partial entanglement, as evident by looking at the geometric picture in Figure 1. However, for states that are entangled yet not genuinely entangled, the simple bound from the previous section remains instrumental to assess their inseparability with minimum effort. $\mathcal{M}_N$ states are themselves instances of such states (in fact, for even $N$, $\mathcal{M}_N$ states are also $G_N$ states, but with $p_{\text{max}} \leq 1/2$ for $N > 2$).

### Applications to experimental states

In this section, we discuss the applicability of our results to real data from recent experiments.\textsuperscript{44,49,52,67}

In refs 44,45 the authors used quantum optical setups to prepare an instance of a bound entangled four-qubit state, known as Smolin state.\textsuperscript{68} Such a state cannot be written as a convex mixture of product states of the four qubits, yet no entanglement can be distilled out of it, thus incarnating the irreversibility in entanglement manipulation, while still representing a useful resource for information locking and quantum secret sharing.\textsuperscript{3,46} It turns out that noisy Smolin states are particular types of $\mathcal{M}_N$ states (for any even $N$),\textsuperscript{43,46} that in the representation of Figure 2b are located along the segment connecting the tetrahedron vertex $((-1)^{N/2}, -1)^{N/2}, -1)^{N/2}$ with the origin. Therefore, this work provides exact analytical formulae for all the nontrivial hierarchy of their global and partial entanglement, as mentioned in Table 3.

In the speciﬁc experimental implementation of ref. 44 for $N = 4$, the global entanglement was detected (but not quantified) via a witness constructed by measuring precisely the three correlation functions $\{c_j\}$. On the basis of the existing data alone (and without assuming that the produced state is within the $\mathcal{M}_D$ family), we can then provide a quantitative estimate to the multipartite entanglement of this experimental bound entangled state in terms of any geometric measure $E^D_{\text{ed}}$, by using Table 2. The results are reported in Table 5a for the illustrative case of the trace distance.
implementation of quantum protocols. In those realisations, data acquisition and processing for the purpose of entanglement verification was often a more demanding task than running the experiment itself.\textsuperscript{69} Focusing first on global and partial entanglement, we obtained full datasets for experimental density matrices corresponding to particularly noisy GHZ and W states of up to four qubits, produced during laboratory test runs.\textsuperscript{67} Despite the relatively low fidelity with their ideal target states, we still obtain meaningful quantitative bounds from Equation (5). The results are compactly presented in Table 5a.

Regarding now genuine multiparticle entanglement, the authors of ref. 50 reported the creation of (noisy) GHZ states of up to \( N = 14 \) trapped ions. In each of these states, full inseparability was witnessed by measuring precisely the maximum overlap \( P_{\text{max}} \) with a reference pure GHZ state, without the need for complete state tomography. Thanks to Equation (6), we can now use the same data to obtain a full quantification of the genuine \( N \)-particle entanglement of these realistic states, according to any measure \( E^B_N \) at no extra cost in terms of experimental or computational resources. The results are in Table 5b, for all the representative choices of distances enumerated in Table 4. Notice that we do not need to assume that the experimentally produced states are in the \( G_N \) set: the obtained results can be still safely regarded as lower bounds.

## DISCUSSION

We have introduced a general framework for estimating and quantifying geometric entanglement monotonies. This enabled us to achieve a compendium of exact results on the quantification of general distance-based measures of (global, partial and genuine) multiparticle entanglement in some pivotal reference families of \( N \)-qubit mixed states. In turn, these results allowed us to establish faithful lower bounds to various forms of multiparticle entanglement for arbitrary states, accessible by few local measurements and effective on prominent resource states for quantum information processing.

Our results can be regarded as realising simple yet particularly convenient instances of quantitative entanglement witnesses,\textsuperscript{21,23} with the crucial advance that our lower bounds are analytical (in contrast to conventional numerical approaches requiring semidefinite programming) and hold for all valid geometric measures of entanglement, which are endowed with meaningful operational interpretations yet have been traditionally hard to evaluate.\textsuperscript{13,69}

A key aspect of our analysis lies in fact in the generality of the adopted techniques, which rely on natural information-theoretic requirements of contractivity and joint convexity of any valid distance \( D \) entering Equation (1). We can expect our general framework to be applicable to other reference families of states (for example, states diagonal in a basis of cluster states,\textsuperscript{14,69} or more general states with X-shaped density matrices\textsuperscript{18}), thereby leading to alternative entanglement bounds for arbitrary states, which might be more tailored to different classes, or to specific measurement settings in laboratory.

Furthermore, our framework lends itself to numerous other applications. These include the obtainment of accessible analytical results for the geometric quantification of other useful forms of multiparticle quantum correlations, such as Einstein–Podolsky–Rosen steering,\textsuperscript{70,71} and Bell nonlocality in many-body systems.\textsuperscript{66} This can eventually lead to a unifying characterisation, resting on the structure of information geometry, of the whole spectrum of quantum signatures of coherence in experimental phenomena. We attempt to extend our approach in this sense in subsequent works.

Another key feature of our results is the experimental accessibility. Having tested our entanglement bounds on a selection of very different families of theoretical and experimentally produced states with high levels of noise, we can certify their usefulness in realistic scenarios. We recall that, for instance, three canonical local measurements suffice to quantify exactly the global entanglement of GHZ states of any even number \( N \) of qubits, whereas \( N+1 \) local measurements provide their exact genuine entanglement, according to every geometric measure for any \( N \), when such states are realistically mixed with white noise. Compared with other complementary studies of accessible quantification of multiparticle entanglement,\textsuperscript{15–23} our study retains not only a comparably low resource demand but also crucial aspects such as efficiency and versatility, as shown in Table 1. This can lead to a considerable simplification of quantitative resource assessment in future experiments based on large-scale entangled registers, involving e.g., two quantum bytes (16 qubits) and beyond.\textsuperscript{50,67}

## MATERIALS AND METHODS

### Distance-based measures of multiparticle entanglement

A general distance-based measure of multiparticle entanglement \( E^D \) is defined in Equation (1). In this work, the distance \( D \) is required to satisfy the following two physical constraints:

- (D.i) Contractivity under quantum channels, i.e., \( D(\Omega(\rho), \Omega(\rho')) \leq D(\rho, \rho') \), for any states \( \rho, \rho' \), and any completely positive trace preserving map \( \Omega \);
- (D.ii) Joint convexity, i.e., \( D(\rho_1 + \rho_2, \rho_1' + \rho_2') \leq D(\rho_1, \rho_1') + D(\rho_2, \rho_2') \), for any states \( \rho_1, \rho_1', \rho_2, \rho_2' \), and any \( q \in [0, 1] \).

Constraint (D.ii) implies that \( E^D \) is invariant under local unitaries and monotonically nonincreasing under LOCC (i.e., it is an entanglement monotone\textsuperscript{23}). Constraint (D.iii) implies that \( E^D \) is also convex. A selection of distance functionals with respect to these properties is given in Table 2.

### Multiparticle \( M^q \)-fication Theorem

Any \( N \)-qubit state \( \rho \) can be transformed into a corresponding \( M^q \)-state through a fixed transformation, \( \Theta \), consisting of single-qubit LOCC, such that \( \Theta(\rho) = \sigma = \frac{1}{2}(\mathbb{I}^N + \sum_{i=1}^N c_i \sigma_i(\Omega)) \), where \( c_i = \text{Tr}(\rho \sigma_i(\Omega)) \).

**Proof.** Here we sketch the form of the \( M^q \)-fication channel \( \Theta \). We begin by setting 2\((N-1)\) single-qubit local unitaries \( \{ U_j \}_{j=1}^{2(N-1)} = \{ (\sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^N - 1), (\sigma_1 \otimes \sigma_1 \otimes \mathbb{I}^N - 1), \ldots, (\mathbb{I}^N - 1 \otimes \sigma_1 \otimes \sigma_1), (\mathbb{I}^N - 1 \otimes \sigma_1 \otimes \sigma_1), \ldots, (\mathbb{I}^N - 1 \otimes \sigma_1 \otimes \sigma_1) \} \) from equation (3). We then fix a sequence of states \( \{ \rho_0, \rho_1, \ldots, \rho_{2(N-1)} \} \) defined by \( \rho_j = \frac{1}{2}(\rho_j + \rho_j' + U_j \rho_j U_j^\dagger) \), where \( j \in \{ 1, 2, \ldots, 2(N-1) \} \). By setting \( \rho_0 = \rho_0(N-1) = \Theta(\rho) \), we define the required channel \( \Theta(\rho) = \frac{1}{2N-1} \sum_{j=1}^{2(N-1)} \rho_j U_j^\dagger U_j \), where \( \{ U_j \}_{j=1}^{2(N-1)} = \{ \mathbb{I}^N, U_1, \ldots, U_j, \ldots, U_{2(N-1)} \} \). Notice that \( \{ U_j \}_{j=1}^{2(N-1)} \) is still a sequence of single-qubit local unitaries. Since \( \Theta \) is a convex mixture of such local unitaries, it belongs to the class of single-qubit LOCC, mapping any \( M \)-separable set into itself. In the Supplementary Material, we show that \( \Theta(\rho) = \sigma \), concluding the proof.

### Lower-bound optimisation

For any valid distance-based measure of global and partial multiparticle entanglement \( E^D \), the maximisation in Equation (5) is equivalent (for even \( N \)) to maximising \( |\xi_1| + |\xi_2| + |\xi_3| \), where \( \xi_j = \text{Tr}(U_j \rho \sigma \sigma^\dagger) \), over local single-qubit unitaries \( \rho_0 = \sigma = \rho_0^N \), \( \{ \sigma_i \} \). By using the well-known correspondence between the special unitary group \( SU(2) \) and special orthogonal group \( SO(3) \), we have that to any one-qubit unitary \( U_0 \) corresponds the orthogonal 3 \times 3 matrix \( O_0(U) \) such that \( U_0 \vec{n} \cdot \vec{\sigma} U_0^\dagger \) = \( O_0(U) \vec{n} \cdot \vec{\sigma} \), where \( \vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3 \) and \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) is the vector of Pauli matrices. We have then that \( \text{sup}_{U_0 \in O_0(U)} |\xi_1| + |\xi_2| + |\xi_3| = \text{sup}_{O_0(U) \in SO(3)} \{ |\xi_1| + |\xi_2| + |\xi_3| \} \), where \( \vec{\xi}_{\alpha} = \sum_{\beta} \text{Tr}(\sigma_\alpha \otimes \sigma_\beta \otimes \sigma_\gamma O_0(U_0)_{\alpha \beta \gamma} \otimes \ldots \otimes O_0(U_0)_{\alpha \beta \gamma} \ldots) \), and \( \vec{\xi}_{\alpha} = \text{Tr}(\sigma_\alpha \otimes \sigma_\beta \otimes \sigma_\gamma \otimes \ldots \otimes \sigma_\gamma) \). In the case of permutationally invariant states \( \rho \), the 3 \times 3 \times \ldots \times 3 tensor \( \vec{T}_{\alpha \beta \gamma} \) is fully symmetric, i.e., \( \vec{T}_{\alpha \beta \gamma} = \vec{T}_{\beta \gamma \alpha} = \vec{T}_{\gamma \alpha \beta} \) for any permutation \( \Omega \) of the indices, so that the optimisation can be achieved.
when $O^{(i)} = O^{(j)} = \cdots = O^{(N)}$ (ref. 72). As indicated in the main text, we then need to perform the maximisation over just the three angles $(\theta, \phi, \phi)$, which determine the orthogonal matrix $O^{(i)}$ corresponding to an arbitrary single-qubit unitary

$$U^{(i)} = \begin{pmatrix}
\cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} & e^{i \phi} \\
-i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\
e^{i \phi} & 0 & \cos \frac{\theta}{2}
\end{pmatrix}.$$ 

As a special case, for a two-qubit state $(N = 2)$ the optimal local operation is the one which diagonalises the correlation matrix $(T_{ij})$.

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CONTRIBUTIONS

All the authors conceived the idea, derived the technical results, discussed all stages of the project, and prepared the manuscript and figures.

COMPETING INTERESTS

The authors declare no conflict of interest.

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