Mathematical Models of Station Keeping for Low Orbit Spacecraft with Electric Propulsion and Limited Power Supply

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Abstract. The paper describes mathematical models used to select design parameters of an electric propulsion unit for station keeping on low orbit of an Earth observation satellite. Mathematical models are developed with consideration for limited power supply on board of spacecraft.

1. Introduction

High resolution geoimaging is essential for many Earth observation problems. From the point of view of obtaining high quality imaging it is most convenient to place the observation satellite on low near-circular orbits, where atmospheric drag has a significant impact on orbit evolution. This leads to gradual lowering of the orbit, and consequently the spacecraft enters dense atmospheric layers.

To keep the orbit parameters in allowable range, it is necessary to correct the trajectory of motion at certain intervals. A necessary condition for correction is the ability to apply to the spacecraft a moment of thrust with higher value than the total force of perturbations.

The propellant expense can be reduced by using electric propulsion for station keeping. This is possible due to small size of both the electric propulsion unit, and the propellant for it. Electric propulsion allows high precision in orbit placement and station keeping for practically indefinite time. High jet stream velocity characteristic for electric propulsion allows to decrease the propellant supply necessary for station keeping, and increase the service life of a low orbit spacecraft. Service life depends on number of corrections, average length of powered flight intervals, propellant supply, which parameters are determined by the type of the thruster. Higher specific impulse ensures longer service life of a spacecraft with the same amount of propellant due to significantly lower fuel consumption. However, the higher the specific impulse, the more powerful the thruster, and the more powerful electric power station is required. Consequently, the area of solar panels also increases. With atmospheric drag, greater area of solar panels translates into more frequent and lengthy orbit correction thrusts.

These contradictions require an analysis of feasibility of using electric propulsion for low orbit station keeping during a satellite's service life.

The problem of optimal design of a technical system (and in particular a space system with electric propulsion) is interpreted as a kind of optimization problem and is reduced to finding such values of the system's design parameters that provide an extremum for a certain quality functional (efficiency criterion). The language of optimization here is natural and convenient, with the issues having a lot in common, and optimization methods are well developed, but far from the only possible ones.
2. Mathematical models for analysis of the processes of low orbit station keeping for an electric propulsion powered spacecraft

2.1 Spacecraft design model. General optimization problem. Main criterion. Criteria translatable into limitations

Design layout of a spacecraft is a number of functional modules that form the design and layout scheme. For the selected scheme the vector of design parameters $p$, that can include boundary values of control functions.

$$p = (p_1, p_2, ..., p_i)^T \in P$$  \hspace{2cm} (1)

Dynamics of the spacecraft as an object of control is described by common differential equations in general view

$$\frac{dx}{dt} = f(t, x, u, p, v)$$

$$x(t_0) \in X_0(z), x(t_f) \in X_f(z)$$

$$u = u(t, x) \in U(p)(x, u) = y \in Y(p) \subset D$$  \hspace{2cm} (2)

Here $x$ – is the state vector, $x = (x_1, x_2, ..., x_n)^T, x \in X$; $u$ is the vector of controlling functions $u = (u_1, u_2, ..., u_n)^T, u = u(t, x)$ out of allowable range $U(p)$, $Z$ signifies a specific dynamic maneuver (space transfer) out of a certain multitude $Z$. Various objects correspond to different $p$ vectors, varying in at least one component; however, for the same object various controls $u(t, x)$ can be applied within the set boundaries.

Therefore, as we shall define a problem of finding design parameters $p \in P$ and a set of variables $\bar{u}(t, x, z), \bar{z} = (t, z)$ that belong to an allowable range $D$ and ensuring successful dynamic maneuvers from the range $Z$ at the minimal (or maximal) value of the set efficiency criterion as the common problem of compatible optimization.

Efficiency of a single maneuver $Z$ shall be characterized by a certain optimality criterion $\mu$, that depends on trajectory $x(t)$, control $u(t, x)$, the parameters of the object $p$ and the maneuver $z$. For the sake of certainty we shall set it that the goal of optimal synthesis is the maximum $\mu$:

$$\bar{\mu} = \max_{u(t, x) \in U(p)} \mu(z, p, x(t), u(t, x))$$ \hspace{2cm} (3)

Obviously, in this case every maneuver $z$ out of the range $Z$ corresponds to its own singular optimal design solution, characterized by the parameters vector $\bar{p}(z)$ and the control vector $\bar{u}(t, x, z)$.

The difficulty of solving the general problem of finding optimal trajectories and parameters is that the trajectories are significantly influenced by design parameters, and design parameters of a spacecraft are in many respects determined by the chosen trajectories. Here we face the problem of connection of these problems, and conditions for their correct separation.

Let us introduce a functional I, that depends on both the parameters of the maneuver $Z$, of the spacecraft $p$, and the trajectory $x(t)$, control $u(t, x)$ and maneuver success conditions, including undefined factors. Let us call the problem of finding the extremum of the functional I at set maneuver parameters $z$ and spacecraft parameters a dynamic optimization model.

In problems of dynamics and motion control the functional I is usually set as an integral terminal criterion where the integral member characterizes an expense of a resource for completion of a dynamic maneuver, and the terminal member $F$ - the input of total discrepancy on boundary conditions.

Let us define the problem of maximum optimality criterion $\mu(z, p, x, u)$ as divisible into dynamic (finding optimal trajectories and control laws) and parametric (finding optimal parameters), if it is possible to single out in the criterion $\mu$ a criterion of a lower level - functional I, that depends only on trajectories and controls, but independent of design parameters, the minimum for which for every
maneuver \( f \in Z \) is achieved on the pair \((\bar{r}(t), \bar{u}(t))\) \(\in D\) and ensures a local minimum of the criterion \( \mu \) at any choice of the parameter vector \( p \in P \).

\[
\mu = \mu(z, p, I(z, x, u))
\]

Let, in accordance with the above:

\[
\min_{D} I[z, x(t), u(t, x)] = S(z) \forall p \in P,
\]

then \( \mu(z, p, \bar{r}, \bar{u}) = \mu(z, p, S(z)) \).

Obviously, if \( \frac{\partial \mu}{\partial S} < 0 \), then the solution of optimality criterion problem is found in two independent operations:

\[
(\bar{r}, \bar{u}) = \arg \min_{(r, u) \in D} I[z, x, u] I(z, r, u) = S(z),
\]

\[
\bar{p} = \arg \max_{p \in P} \mu(z, p, S(z)).
\]

Strict division of the optimization problems is possible only on perfectly regulated low thrust engine, when its dynamics is described by a simplest model of a variable mass point with ideal and "free" control. In a general case the dynamic characteristic \( S \) is not invariant in respect to parameters of the spacecraft, for which reason the division of optimization problem is often only relative.

The problem of the maximum of the criterion \( \mu(z, p, x, u) \) we shall define as conditionally divisible into dynamic and parametric parts, if the minimum of the functional \( I \), dependent on the trajectory, control and parameters, for each fixed maneuver \( z \in Z \) and fixed parameter vector \( p \in P \) is achieved on the pair \((\bar{r}(t), \bar{u}(t))\) \(\in D\) and provides local maximum for the criterion \( \mu \). The global maximum \( \mu \) is found in the form of two consequent operations:

\[
(\bar{r}, \bar{u}) = \arg \min_{(r, u) \in D} I[z, p, x, u] I(z, r, u, p) = S(z, p)
\]

\[
\bar{p} = \arg \max_{p \in P} \mu(z, p, S(z, p)).
\]

Solution of this problem is complicated by the need of having the dependency \( S(z, p) \), defined on the manifold of \( P \) over the complete maneuver range \( Z \).

2.2 Compatible optimization of ballistic and design parameters

Because design parameters of the spacecraft influence dynamic characteristics of the maneuver, and, vice versa, the ballistic scheme and the trajectories of the transfer in no small degree determine the choice of design parameters, the spacecraft parameters and the family of optimal trajectories must be optimized compatibly [1]. Any change in design parameters makes it necessary to repeat the calculations of the optimal trajectories family and control modes, as well as ballistic parameters. The procedure for compatible optimization of trajectories and design is as follows:

A common problem of compatible optimization of a spacecraft intended to perform a range of dynamic maneuvers is set. A design model of the spacecraft is introduced, as well as the model of mass distribution across the spacecraft components.

The dynamic part is singled out from the common optimization problem. The dynamic problem is solved step by step, with models of varying degree of completeness, detail and precision, for selected subareas of the phase space.

A problem of compatible optimization of trajectories, ballistic and design parameters, is solved. As the result, the main parameters for the spacecraft design layout, energy expense for the maneuver, trajectories and control modes are chosen in first approximation.

The impact of undefined factors on both the results of the dynamic problem and the values of design parameters is looked into. If necessary, parameters ensuring minimum of loss in the global optimality criterion for the whole maneuver range are defined.
2.3 Mathematical model of limitations on available electric power on board of spacecraft

Parameters of the main elements of the electric power supply system (solar panels, batteries) are defined on the stage of shaping the principle design layout of the observation spacecraft, so as to meet the demand of all energy consumers. The primary electric power source are solar panels, that can be attached devices, and an increase in their area may also increase aerodynamic drag. On the other hand, solar panels position relative to the Sun should be optimized for maximum performance. The total area of solar panels is chosen from the limit:

\[ N_{\text{v},\text{sol}} \cdot S_{\text{C} \Sigma} \cdot \cos \alpha_{\text{sol}}(t) \cdot t_{\text{sol}} \geq C_{\text{AB}} \cdot I_{\text{wp}} \cdot R_{\text{wp}} + \sum_{j=1}^{k} N_j(t) \cdot \Delta t_j, \]

where \( N_{\text{v},\text{sol}} \) is specific power of solar panels; \( S_{\text{C} \Sigma} \) is total area of solar panels; \( \cos \alpha_{\text{sol}}(t) \) is cyclogram of change in the angle between the normal to the solar panel and the direction to the Sun; \( C_{\text{AB}} \) is the volume of the battery; \( I_{\text{wp}} \) current of the battery; \( N_j(t) \) cyclogram of operation of j-th unit on lighted interval of the flight; \( \Delta t_j \) is operation time of the j-th unit on lighted interval.

Different various of positioning the solar panel with area \( S_{\text{C} \Sigma} \) and the angle to the aerodynamic drag vector are considered (See Fig. 1).

![Figure 1. Method of choosing characteristics of solar panels of a low orbit spacecraft with limits on energy and geometry.](image1)

The starting requirements to structure and position of the solar panels in relation to the spacecraft’s body are formed with consideration to the limits on external geometry of the low orbit space platform as well as limits on design characteristics of the power supply system (See Fig. 2.2).

\[ \sum_{i=1}^{k} (S_{\text{C} \Sigma} \cdot \cos \alpha_{\text{sol}}(t)) \geq S_{\text{C} \Sigma} \cdot \cos \alpha_{\text{sol}}(t), \]

where \( k \) is the number of solar panels; \( S_{\text{C} \Sigma} \) is the area of the i-th solar panel (see Fig 2); \( \cos \alpha_{\text{sol}}(t) \) is cyclogram of change in cosines of the i-th angle between the normal to the solar panel and the direction to the Sun.

![Figure 2. Structure and position of solar panel with consideration of the limits on geometry and power plant.](image2)

The limits on geometry and energy are used on consequent stages of development of the low orbit space platform.
With given characteristics of operational orbit, the first parameter to be chosen is the volume of the batteries necessary for operation of the spacecraft on the shaded intervals of flight.

\[ C_{AB} \cdot I_{pep} \cdot R_{cSYM} \geq \sum_{i=1}^{n} N_i(t) \cdot \Delta t_i, \]

where \( C_{AB} \) is the battery volume; \( R_{cSYM} \) is average resistance in on-board electric circuits; \( I_{pep} \) battery discharge current; \( N_i(t) \) is cyclogram of electric power consumption on the shaded part of the flight; \( \Delta t_i \) is operation time of the i-th unit on the shaded part of the flight.

2.4 Mathematical model of optimal structure of control over low orbit station keeping of an electric propulsion powered spacecraft

When the mass and dimensional model, dependent on orbit parameters and spacecraft controlled law, described above, is united with the energy model which sets the limits on electric power available for electric propulsion engine, the result is a mathematical model of station keeping on low and evolving due to atmospheric drag orbit with the help of electric propulsion. Limiting ourselves with correction of only the flat elements of the orbit \( A, e \), as well as the state angle \( \Delta u \), we arrive at the system of equations (14) that describe the process of station keeping of a model spacecraft with electric propulsion on low orbit.

\[
\begin{align*}
&M_{KA} = M_{rK} + M_{3D} + \frac{1}{1-\alpha_K} \left( \int_{t_0}^{\tau_{SYM}} \frac{2 \cdot N_{PMAPP} \cdot M_{PT}}{T_M} \cdot \eta_{\Sigma} \cdot \eta_{3D} \cdot \left( 1 + k_{CT} \right) \cdot M_{PT} \right), \\
&M_{3D} = S_{CK} \cdot M_{3D_{CK}} + \frac{N_{PMAPP} \cdot \chi_{AC}}{E_{\rho r, AK}} + \delta M_{3K}, \quad \sigma_{3D} = \frac{1}{T_0} \frac{\int \sigma(t) dt}{}, \\
&N_{PMAPP} = \int_{t_0}^{\tau_{SYM}} \frac{N_i(t) \cdot T_{CYT}}{T_{CYT}}, \quad \sum_{i=1}^{n} \frac{N_i(t) \cdot T_{CYT}}{T_{CYT}}, \\
&F_K = 2 \cdot N_{PMAPP} \cdot \eta_{3D} \cdot \eta_{3D}, \\
&\frac{\Delta A}{\Delta t} = \frac{4 \Delta A}{\pi} \sqrt{\frac{A}{\mu}} \left( \frac{A}{\mu} \left( \frac{a + a^2}{2} \right) - 2 \sigma_K \rho(t) \sqrt{\mu A} \right), \\
&\frac{\Delta u}{\Delta t} = \frac{a_k}{\pi} \sqrt{\frac{A}{\mu}} \left( 3 \xi + a^2 \right) + 4 \sin \left( \frac{\xi}{2} \right) \cos \left( \frac{\eta}{2} \right) \cos \eta - \frac{\varepsilon}{2} \sin \left( \frac{\xi}{2} \right) \cos \alpha \cos 2\eta - 2 e \sigma_K \rho(t) \sqrt{\mu A}, \\
&\rho(t) = K_i(t) \cdot K_j(t) \cdot K_k(t) \cdot \rho_0 \cdot \exp \left[ a_i - \frac{\left( r - 6371 \right) - a_j}{a_j} \right], \quad \rho(t) \in [\rho_{min}(t), \rho_{max}(t)].
\end{align*}
\]

In this model the structure of the station keeping on the powered loop, considering limits of power supply, is as follows: on active loops the electric propulsion unit is fired on the parts of the orbit where control over orbital parameters is optimal (see Fig. 3), conditional to availability of power supply for the electric propulsion unit at a given moment. As the result, the general switch-on scheme for electric propulsion unit on active loops can be expressed as on Figure 3.

3. Conclusion

The paper considered applicability of electric propulsion unit for low orbit station keeping of Earth probing satellites. It is suggested to solve the problem of station keeping with an electric propulsion unit based on domestic stationary plasma thrusters. Models for choosing powered flight intervals on the loop optimized for propellant consumption were developed.
Figure 3. The scheme of forming cyclograms of correction on active loops considering energy limits and indefinite state of the atmospheric density.

4. References
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