On a conjecture by Eckhoff and Dolnikov concerning line transversals to Euclidean disks

Alexander Magazinov

October 31, 2017

Abstract

Let $K$ be a convex body in the Euclidean plane $\mathbb{R}^2$. We say that a point set $X \subseteq \mathbb{R}^2$ satisfies the property $T(K)$ if the family of translates $\{K + x : x \in X\}$ has a line transversal. A weaker property, $T(K, s)$, of the set $X$ is that every subset $Y \subseteq X$ consisting of at most $s$ elements satisfies the property $T(K)$.

The following question goes back to Grünbaum: given $K$ and $s$, what is the minimal positive number $\lambda = \lambda(K, s)$ such that every finite point set in $\mathbb{R}^2$ with the property $T(K, s)$ also satisfies the property $T(\lambda K)$? The constant $\lambda_{\text{disj}}(K, s)$ is defined similarly, with the only additional assumption that the translates $x + K$ and $y + K$ are disjoint for every $x, y \in X, x \neq y$.

One case of particular interest is $s = 3$ and $K = B$, where $B$ is a unit Euclidean ball. Namely, it was conjectured by Eckhoff and, independently, Dolnikov that $\lambda(B, 3) = 1 + \sqrt{5}/2$.

In this paper we propose a stronger conjecture, which, on the other hand, admits an algebraic formulation in a finite alphabet. We verify our conjecture numerically on a sufficiently dense grid in the space of parameters and thereby obtain an estimate $\lambda_{\text{disj}}(B, 3) \leq 1.645$. This is an improvement on the previously known upper bounds $\lambda(B, 3) \leq 1 + \sqrt{1 + 4 \sqrt{2}}/2 \approx 1.79$ (Jerónimo Castro and Roldán-Pensado, 2011) and $\lambda_{\text{disj}}(B, 3) \leq 1.65$ (Heppes, 2005).

1 Introduction

Let $K$ be a convex body in the Euclidean plane $\mathbb{R}^2$. We will consider families

$$\mathcal{F} = \{K + x : x \in X\} \quad (X \subseteq \mathbb{R}^2)$$

(1)

of translates of $K$ with the following property: every $s$-tuple of translates from $\mathcal{F}$ has a line transversal. (Here $s > 2$ is a fixed integer number.) The following question has been proposed by Grünbaum [5]: given $K$ and $s$, what is the minimum value of a constant $\lambda = \lambda(K, s)$ such that for every finite family $\mathcal{F}$ satisfying the above property one can guarantee that the family of blow-ups $\lambda F = \{\lambda K + x : x \in X\}$ can be stabbed by a single line?

Let us introduce a convenient notation. We will rather consider point sets $X \subseteq \mathbb{R}^2$ instead of the associated families of translates (1). If a family (1) can be stabbed by a single line, we say that the associated point set $X$ satisfies the property $T(K)$. If every subset of $X$, consisting of at most $s$ elements, satisfies the property $T(K)$, then we say that $X$ satisfies the property $T(K, s)$. Then Grünbaum’s question can be formulated as follows.

*Supported in part by ERC Starting Grant 678520.
**Question.** Given a convex body $K \subset \mathbb{R}^2$ and an integer $s > 2$, what is the minimum possible value $\lambda(K, s)$ such that the following holds: every finite point set $X \subset \mathbb{R}^2$ with the property $T(K, s)$ necessarily satisfies the property $T(\lambda K)$?

Applying additional restriction to the point set $X$ may also make sense. For instance, let us call $X$ $K$-separated if the family $X$ consists of pairwise disjoint translates. Then one can ask an analogous question.

**Question.** Given a convex body $K \subset \mathbb{R}^2$ and an integer $s > 2$, what is the minimum possible value $\lambda_{\text{disj}}(K, s)$ such that the following holds: every $K$-separated finite point set $X \subset \mathbb{R}^2$ with the property $T(K, s)$ necessarily satisfies the property $T(\lambda K)$?

The case $K = B$, $s = 3$, where $B = \{(x, y) : x^2 + y^2 \leq 1\}$, i.e., $B$ is the unit ball centered at the origin, has attracted some particular attention. The following conjecture was posed in 1969 by Eckhoff [2] and, independently, in 1972 by Dolnikov (see [8]).

**Conjecture 1** (Dolnikov, Eckhoff).

\[ \lambda(B, 3) = \frac{1 + \sqrt{5}}{2}. \]

Conjecture 1 remains unresolved so far (see [3]). Recently, it was highlighted in the Handbook of Discrete and Convex Geometry (see [6, Conjecture 4.2.25]) as an important problem in the theory of geometric transversals.

Some partial results towards Conjecture 1 have been achieved. It is known that $\lambda(B, 3) \geq \lambda_{\text{disj}}(B, 3) \geq \frac{1 + \sqrt{5}}{2}$, as implied by considering the vertex set of a regular pentagon with each side equal to $\frac{2}{\sqrt{1 + 4\sqrt{2}}} \approx 1.79$ due to Jerónimo Castro and Roldán-Pensado [9], and $\lambda_{\text{disj}}(B, 3) \leq 1.65$ due to Heppes [7].

In this paper we improve the known upper bounds for both $\lambda(B, 3)$ and $\lambda_{\text{disj}}(B, 3)$. Moreover, we provide some significant evidence indicating that our approach can resolve Conjecture 1 completely.

2 The “finite” conjecture, the parametrization and the restriction to a grid

Let us pose a conjecture, which is, apparently, stronger than Conjecture 1.

**Conjecture 2.** Let $E \subset \mathbb{R}^2$ be an elliptical disk, $Z \subset \partial E$ be a point set such that $\#Z \leq 5$ and $E$ has the minimum area of all elliptic disks containing $E$. Consider the set

\[ R = R(Z) = \{x \in \mathbb{R}^2 : \{x\} \cup Z \text{ satisfies the property } T(B, 3)\}. \]

Then the set $R \cap E$ satisfies the property $T\left(\frac{1 + \sqrt{5}}{2} B\right)$.

We will use the following parametrization of Conjecture 2. Consider the cartesian coordinate system $(x, y)$, where

\[ E = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} \leq 1\right\}, \quad r_1 \geq r_2. \]

If $Z = \{z_1, z_2, \ldots, z_k\}$, $k \leq 5$, then

\[ z_i = (r_1 \cos \alpha_i, r_2 \sin \alpha_i), \quad \text{for} \quad i = 1, 2, \ldots, k. \]

Without loss of generality one can assume

\[ 0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < 2\pi. \]

Thus Conjecture 2 gets parameterized by $r_1, r_2, \alpha_1, \alpha_2, \ldots, \alpha_k$. 

2
Remark. One can eliminate the trigonometric expressions in the parametrization by the standard substitution $t_i = \tan \alpha_i^2$. The condition that $E$ is the ellipsoid of minimal area containing $Z$ is algebraic in $t_i$ (see, for example, [4]). The set $R$ is defined algebraically in $r_1$, $r_2$ and $t_i$. Therefore Conjecture 2 has algebraic parametrization with at most 7 variables.

We are ready to state the main results of this paper.

**Theorem 3.** If Conjecture 2 holds, then Conjecture 1 holds, too.

**Theorem 4.** Let $(k; r_1, r_2; \alpha_1, \alpha_2, \ldots, \alpha_k)$ be the parameters as in (3), (4) and (5). Then Conjecture 2 is true in the following cases:

(a) $k = 3$.

(b) $k = 4$, $r_1, r_2 \in 0.015\mathbb{Z}$.

(c) $k = 5$, $r_1, r_2 \in 0.015\mathbb{Z}$, and $\alpha_i \in \frac{\pi}{960}(\mathbb{Z} + 1/2)$ for each $i = 1, 2, \ldots, 5$.

**Theorem 5.** $\lambda_{\text{disj}}(B, 3) \leq \lambda(B, 3) \leq 1.645$.

We conclude this section with a brief guide over the contents of the rest of the paper.

- Section 3 contains the proof of Theorem 3.
- Section 4 accommodates a number of auxiliary statements necessary for the further argument. We formulate some useful corollaries of the so-called John representation associated with the minimum area elliptic disk. Then we provide several simple tools for extrapolating estimates on a finite subset of the parameter space to the subspace covered by Theorem 4 and then to the entire parameter space. Finally, we state two lemmas that allow us eliminate all “too long” elliptical disks from consideration. The proofs are given in the subsequent sections.
- Sections 5–7 are devoted to the proofs of statements formulated in Section 4. The only proof that remains postponed is the one of Lemma 15.
- Section 8 addresses the computer-assisted proofs, namely the ones of Lemma 15 and parts (b)–(c) of Theorem 4.
- Section 9 contains a short (non-computer-assisted) proof of part (a) of Theorem 4.
- Section 10 reduces Theorem 5 to Theorem 4, which is proved earlier.

3 Reduction of the Dolnikov–Eckhoff conjecture to Conjecture 2

Let us show that Conjecture 2 indeed implies Conjecture 1.

**Proof of Theorem 3** Let $X$ be a counterexample to Conjecture 1. One can choose a sufficiently small constant $\varepsilon > 0$ so that every sufficiently small perturbation $X'$ of the set $X$ is still a counterexample to Conjecture 1. Let us choose $X'$ to be sufficiently generic so there is no ellipse passing through 6 or more points of $X'$.

Let $E$ be an elliptical disk of minimal area containing the set $X'$. Denote $Z = X' \cap \partial E$. Since $X'$ is generic, we have $\# Z \leq 5$. In addition, by [ref:Ball], $E$ is the (only) elliptical disk of minimal area containing $Z$.

Let $R$ be defined according to (2). Then, since $X'$ satisfies the property $T(B, 3)$, we have

$$X' \subset E \cap R.$$ 

But if Conjecture 2 holds, then $X'$ satisfies the property $T\left(\frac{1 + \sqrt{5}}{2}B\right)$. This contradicts our previous assumption that $X'$ is a counterexample to Conjecture 1. \qed
4 Some auxiliary results

4.1 John–Ball criterion of the minimum area elliptic disk

This subsection is based on the following Proposition 6. The proof, in any dimension, not only in the plane, can be found in [10] (the necessary property of the minimum area ellipsoid) and in [1] (the sufficiency). See also [4] for, perhaps, a more accessible exposition.

**Proposition 6.** Let \( X \subset \mathbb{R}^2 \) be a finite set. Then

1. There exists an elliptical disk \( E \supset X \) such that every elliptical disk \( E' \supset X \) that is distinct from \( E \) satisfies \( |E'| > |E| \). (I.e., the minimum area elliptical disk containing \( X \) is unique.)

2. The following assertions are equivalent.
   
   (i) \( B \) is the minimum area elliptical disk containing \( X \).
   
   (ii) \( X \subset B \) and there exists a subset \( \{x_1, x_2, \ldots, x_k\} \subset X \cap \partial B \) and \( k \) positive numbers \( c_1, c_2, \ldots, c_k \) such that
   
   \[
   c_1 x_1 + c_2 x_2 + \ldots + c_k x_k = 0, \\
   c_1 x_1 \otimes x_1 + c_2 x_2 \otimes x_2 + \ldots + c_k x_k \otimes x_k = \text{Id}
   \]  
   (6)

Proposition 6 will be used through the three corollaries below. Before we turn to the corollaries, let us introduce a functional playing a crucial role in the subsequent argument. Namely, define

\[
F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \cos \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) + \cos \frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) + \cos \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4).
\]  

(7)

**Corollary 7.** Let

\[
X = \{ (\cos \alpha_i, \sin \alpha_i) : i = 1, 2, \ldots, k \}.
\]

Then the following assertions are equivalent.

(i) \( B \) is the minimum area elliptical disk containing \( X \).

(ii) \( (0, 0, 0, 0) \in \text{conv} \{ (\cos \alpha_i, \sin \alpha_i, \cos 2\alpha_i, \sin 2\alpha_i) : i = 1, 2, \ldots, k \} \).

**Corollary 8.** Let

\[
X = \{ (\cos \alpha_i, \sin \alpha_i) : i = 1, 2, \ldots, k \}
\]

be a finite set such that \( B \) is the minimum area elliptical disk containing \( X \). Assume, additionally, that

\[
0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < 2\pi.
\]

If \( \alpha_{k+1} = \alpha_1 + 2\pi \), then the inequality \( \alpha_{i+1} - \alpha_i \leq \frac{2\pi}{k} \) holds for every \( i = 1, 2, \ldots, k \).

**Corollary 9.** Let \( x_1, x_2, x_3, x_4 \in \partial B \) be four points such that the identities (7) hold with some positive coefficients \( c_1, c_2, c_3, c_4 \). Let \( x_i = (\cos \alpha_i, \sin \alpha_i) \), where Then \( F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0 \).

Equivalently, if \( \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_1 + 2\pi \) and \( \phi_i = \frac{\alpha_{i+1} - \alpha_i}{2} \) \((i = 1, 2, 3, 4, \alpha_5 = \alpha_1 + 2\pi)\), then

\[
\cos(\phi_2 - \phi_4) + \cos(\phi_2 + \phi_4) + \cos(2\phi_1 + \phi_2 + \phi_4) = 0.
\]  

(8)

Remark. A cyclic shift of the 4-tuple \( (\phi_1, \phi_2, \phi_3, \phi_4) \) turns (7) into itself.
Corollary 10. Let $x_1, x_2, \ldots, x_5 \in \partial B$ be five points such that the identities (6) hold with some positive coefficients $c_1, c_2, \ldots, c_5$. Let $x_1 = (\cos \alpha_1, \sin \alpha_1)$, where
\[ \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_1 + 2\pi. \]
Then the following five real numbers:
\[ F(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad F(\alpha_2, \alpha_3, \alpha_4, \alpha_5), \quad -F(\alpha_3, \alpha_4, \alpha_5, \alpha_1), \quad F(\alpha_4, \alpha_5, \alpha_1, \alpha_2), \quad -F(\alpha_5, \alpha_1, \alpha_2, \alpha_3) \]
are either all negative or all positive. Conversely, if $x_1$ and $\alpha_1$ are as above, and the values (9) are either all negative or all positive, then $B$ is the minimum area elliptical disk containing the set $\{x_1, x_2, x_3, x_4, x_5\}$.

Corollaries 7–10 are proved in Section 5.

4.2 Approximation lemmas

Since the computer verification is possible only for a finite subset (though a dense one) in the space of parameters, we will need to perform an extrapolation to the entire parameter space. This subsection provides some simple tools for such an extrapolation.

Lemma 11. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-degenerate affine map such that for every $x, y \in \mathbb{R}^2$ the inequality
\[ \|Mx - My\| \leq \|x - y\| \]
holds. Assume that a finite set $X$ satisfies the property $T(rB)$ for some $r > 0$. Then the set $MX$ satisfies the property $T(rB)$, too.

Lemma 12. Assume that a finite set $X \subseteq \mathbb{R}^2$ satisfies the property $T(rB)$ for some $r > 0$. Let $\varepsilon > 0$ and let a finite set $Y \subseteq \mathbb{R}^2$ satisfy $Y \subseteq X + \varepsilon B$. Then $Y$ satisfies the property $T((r + \varepsilon)B)$.

Lemma 13. Let $\alpha_i, \alpha'_i \in \mathbb{R}$ ($i = 1, 2, 3, 4$) and $\varepsilon > 0$ be given such that $\max |\alpha_i - \alpha'_i| \leq \varepsilon$. Then
\[ |F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - F(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4)| < 3\varepsilon. \]
The proofs are provided in Section 6.

4.3 Some a priori bounds

The results of this subsection will allow us eliminate all “too long” elliptical disks from consideration. In other words, by using the lemmas below we will restrict ourselves to a compact subset of the parameter space.

Lemma 14. Let $r > 2$. Let $X$ be a finite set such that $rB$ is the minimum area elliptical disk that contains $X$. Then $X$ violates the property $T(B, 3)$.

Remark. By Lemma 14 and Lemma 11 if $X$ satisfies the property $T(B, 3)$ and $E$ is the elliptical disk of minimum volume containing $X$, then the smaller radius of $E$ does not exceed 2. Therefore $X$ satisfies the property $T(2B)$. This gives an alternative proof for the Eckhoff’s bound $\lambda(B, 3) \leq 2$.

Lemma 14 is proved in Section 7.

Lemma 15. Consider the elliptical disk
\[ E = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{3^2} + \frac{y^2}{1.62^2} \leq 1 \right\}. \]
Let $X$ be a finite set such that $E$ is the minimum area elliptical disk that contains $X$. Then $X$ violates the property $T(B, 3)$.

The proof of Lemma 15 is computer-assisted. As all the other computer-assisted proofs, it is addressed in Section 8.
5 Minimum area elliptic disk: proofs of the key properties

The aim of this section is to prove Corollaries 7–10.

Proof of Corollary 7. One can rewrite the identities (6) as follows:

\[ c_1 \cos \alpha_1 + c_2 \cos \alpha_2 + \ldots + c_k \cos \alpha_k = 0, \]
\[ c_1 \sin \alpha_1 + c_2 \sin \alpha_2 + \ldots + c_k \sin \alpha_k = 0, \]
\[ c_1 \cos^2 \alpha_1 + c_2 \cos^2 \alpha_2 + \ldots + c_k \cos^2 \alpha_k = 1, \]
\[ c_1 \sin^2 \alpha_1 + c_2 \sin^2 \alpha_2 + \ldots + c_k \sin^2 \alpha_k = 1, \]
\[ c_1 \sin \alpha_1 \cos \alpha_1 + c_2 \sin \alpha_2 \cos \alpha_2 + \ldots + c_k \sin \alpha_k \cos \alpha_k = 0. \]

Taking the difference of the third and the fourth lines of (11) yields

\[ c_1 \cos 2\alpha_1 + c_2 \cos 2\alpha_2 + \ldots + c_k \cos 2\alpha_k = 0. \]

At the same time, multiplying the fifth line of (11) by 2 yields

\[ c_1 \sin 2\alpha_1 + c_2 \sin 2\alpha_2 + \ldots + c_k \sin 2\alpha_k = 0. \]

Aggregately, one concludes that

\[ \sum_{i=1}^{k} c_i (\cos \alpha_i, \sin \alpha_i, \cos 2\alpha_i, \sin 2\alpha_i) = 0. \]

This proves the implication \((i) \Rightarrow (ii)\).

Now assume that \((ii)\) holds. Then there are non-negative coefficients \(c'_1, c'_2, \ldots, c'_k\), not all of which are zero, such that

\[ \sum_{i=1}^{k} c'_i (\cos \alpha_i, \sin \alpha_i, \cos 2\alpha_i, \sin 2\alpha_i) = 0. \]

Then

\[ c'_1 x_1 + c'_2 x_2 + \ldots + c'_k x_k = 0 \quad \text{and} \]
\[ c'_1 x_1 \otimes x_1 + c'_2 x_2 \otimes x_2 + \ldots + c'_k x_k \otimes x_k = \lambda \text{Id}, \]

where \(\lambda > 0\). Taking \(c_i = \sqrt{\lambda} c'_i\) proves \((ii)\) and therefore the implication \((ii) \Rightarrow (i)\). \(\square\)

Proof of Corollary 8. We argue by contradiction. Let the conclusion of Corollary 8 be false. Then, with no loss of generality, we can assume that \(\alpha_1 = 0, \alpha_2 > \frac{2\pi}{3}\).

Then every \(\alpha_i\) satisfies the inequality

\[ \cos(\alpha_i + \pi/3) - \cos(2\alpha_i - \pi/3) \leq 0, \]

and the equality can be achieved only if \(\alpha_i = 0\) or \(\alpha_i = \frac{4\pi}{3}\). Then all 4-tuples \((\cos \alpha_i, \sin \alpha_i, \cos 2\alpha_i, \sin 2\alpha_i)\) belong to the half-space \(\ell(t_1, t_2, t_3, t_4) < 0\), where

\[ \ell(t_1, t_2, t_3, t_4) = \frac{1}{2} t_1 - \frac{\sqrt{3}}{2} t_2 - \frac{\sqrt{3}}{2} t_3 - \frac{\sqrt{3}}{2} t_4. \]

Moreover, if the 4-tuple belongs to the boundary of that half-space, then either \(\alpha_i = 0\) or \(\alpha_i = \frac{4\pi}{3}\). This contradicts the conclusion of Corollary 7. \(\square\)
Proof of Corollary 9. With no loss of generality, assume from the very beginning that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_1 + 2\pi$.

Corollary 7 implies that the affine dimension of the 5-tuple of points
\[
\{0\} \cup \{(\cos \alpha_j, \sin \alpha_j, \cos 2\alpha_j, \sin 2\alpha_j) : j = 1, 2, 3, 4\}
\]
cannot be equal to 4. Therefore
\[
\Delta(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{vmatrix}
\cos \alpha_1 & \cos \alpha_2 & \cos \alpha_3 & \cos \alpha_4 \\
\sin \alpha_1 & \sin \alpha_2 & \sin \alpha_3 & \sin \alpha_4 \\
\cos 2\alpha_1 & \cos 2\alpha_2 & \cos 2\alpha_3 & \cos 2\alpha_4 \\
\sin 2\alpha_1 & \sin 2\alpha_2 & \sin 2\alpha_3 & \sin 2\alpha_4 \\
\end{vmatrix} = 0.
\]

One can check that
\[
\Delta = C \cdot e_1^2 e_2^2 + e_1^2 e_3^2 + e_1^2 e_4^2 + e_2^2 e_4^2 + e_3^2 e_4^2 \times \frac{(e_1^2 - e_2^2)(e_1^2 - e_3^2)(e_1^2 - e_4^2)(e_2^2 - e_3^2)(e_2^2 - e_4^2)(e_3^2 - e_4^2)}{(e_1 e_2 e_3 e_4)^4},
\]
where $C \in \mathbb{R}$ is a fixed constant and $e_j = \exp(i\alpha_j/2)$. But if $j < j'$, then $\alpha_{j'} - \alpha_j \in (0, 2\pi)$. Therefore
\[
e_{j'}^2 - e_j^2 \in i\mathbb{R} - \cdot e_{j'},
\]
Consequently, the fraction \[
\frac{(e_1^2 - e_2^2)(e_1^2 - e_3^2)(e_1^2 - e_4^2)(e_2^2 - e_3^2)(e_2^2 - e_4^2)(e_3^2 - e_4^2)}{(e_1 e_2 e_3 e_4)^4}
\]
attains only real negative values, hence the sign of $\Delta(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is completely determined by the sign of
\[
\frac{e_1^2 e_2^2 + e_1^2 e_3^2 + e_1^2 e_4^2 + e_2^2 e_4^2 + e_3^2 e_4^2}{e_1 e_2 e_3 e_4} = 2F(\alpha_1, \alpha_2, \alpha_3, \alpha_4).
\]
(Of course, the last expression is a real number.) In particular, if $\Delta(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$, then $F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$, as required.

Proof of Corollary 10. We have to check whether the origin $0 \in \mathbb{R}^4$ belongs to the interior of the simplex
\[
\text{conv}\{(\cos \alpha_j, \sin \alpha_j, \cos 2\alpha_j, \sin 2\alpha_j) : j = 1, 2, \ldots, 5\}.
\]
The necessary and sufficient condition is that the determinants
\[
\Delta(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \Delta(\alpha_5, \alpha_1, \alpha_2, \alpha_3), \ldots, \Delta(\alpha_2, \alpha_3, \alpha_4, \alpha_5)
\]
have the same sign. Equivalently, the 5 values $\Delta$ have the same sign. (The equivalence is established similarly to the proof of Corollary 9.)

6 Perturbation lemmas: the proofs

The goal of this section is to prove Lemmas 11–13.

Proof of Lemma 11. Let $l$ be a line such that $\text{dist}(x, l) \leq r$ for every $x \in X$. Such a line exists because $X$ satisfies the property $T(rB)$. Then $\text{dist}(MX, MI) \leq \text{dist}(x, l) \leq r$, therefore the line $MI$ intersects every translate $rB + y$, where $y$ runs through the set $MX$. Hence the set $MX$ indeed satisfies the property $T(rB)$. \qed
Proof of Lemma 12. Let $l$ be a line such that $\text{dist}(x,l) \leq r$ for every $x \in X$. Such a line exists because $X$ satisfies the property $T(rB)$.

Let $y \in Y$. Then there exists a point $x \in X$ such that $\|x - y\| \leq \varepsilon$. Consequently, $\text{dist}(y,l) \leq \text{dist}(x,l) + \varepsilon \leq r + \varepsilon$. Thus the line $l$ intersects every translate $(r + \varepsilon)B + y$, where $y$ runs through the set $Y$. Therefore the set $Y$ indeed satisfies the property $T((r + \varepsilon)B)$. \hfill \Box

Proof of Lemma 13. Let $\beta = \alpha'_i - \alpha$. Then

\[
|F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - F(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4)|
\leq \frac{1}{2}(|\beta_1 + \beta_2 - \beta_3 - \beta_4| + |\beta_1 - \beta_2 + \beta_3 - \beta_4| + |\beta_1 - \beta_2 - \beta_3 + \beta_4|)
\leq \frac{1}{2} \cdot 6\varepsilon = 3\varepsilon,
\]

as required.

Indeed, the expression in the second line is convex in $(\beta_1, \beta_2, \beta_3, \beta_4)$, therefore it is sufficient to check the inequality only at the vertices of the cube $[-\varepsilon, \varepsilon]^4$. It is also clear that the inequalities cannot turn into equalities simultaneously. \hfill \Box

7 The minimum area elliptical disk has width $\leq 2$

In this section we prove Lemma 14. Once the proof is complete, we immediately conclude that Conjecture 2 holds whenever $r_1 > 2$. Indeed, in this case we necessarily have $R(X) = \emptyset$ (see the remark after Lemma 14).

Proof of Lemma 14. With no loss of generality we can assume that $X \subseteq \partial(rB)$ and $\#X \leq 5$. Indeed, if this is not the case, apply a sufficiently small perturbation to $X$, yielding a set $X'$ with no 6 point lying on the same ellipse. If $E$ is the minimum area elliptical disk containing $X'$, consider the affine map $F$ such that $F(E) = \frac{2\pi}{3} \cdot B$. Clearly, $F$ is a contraction (if $X$ and $X'$ are sufficiently close to each other). Therefore, by Lemma 14 it will be sufficient to argue for $F(X') \cap F(E)$ instead of $X$ and for $\frac{2\pi}{3}$ instead of $r$.

We proceed by case analysis.

Case 1. $\#X = 3$. By condition of the lemma, the ball $rB$ is the minimum area ellipsoid containing the triangle $\text{conv} X$. By a well-known fact [ref], this is only possible if $\text{conv} X$ is a regular triangle. But then each height of $\text{conv} X$ equals $\frac{2\pi}{3} > 3 > 2$, which contradicts the $T(B,3)$ property.

Case 2. $\#X = 4$. In the notation of Corollary 9 we can, with no loss of generality, assume that $\phi_1 = \max(\phi_1, \phi_2, \phi_3, \phi_4)$. In particular, since $\phi_1 + \phi_2 + \phi_3 + \phi_4 = \pi$, we have $\phi_1 \geq \frac{\pi}{4}$.

Now we claim that

\[
\max(\phi_2, \phi_4) \geq \frac{\pi}{4}, \tag{12}
\]

Indeed, otherwise

\[
\cos(\phi_2 - \phi_4) + \cos(\phi_2 + \phi_4) = 2 \cos \phi_2 \cos \phi_4 > 2 \cos^2 \frac{\pi}{4} = 1 \geq -\cos(2\phi_1 + \phi_2 + \phi_4),
\]

which contradicts (9). The claim (12) is proved. With no loss of generality, let $\phi_2 \geq \frac{\pi}{4}$.

Finally, using Corollary 8 we get

\[
\frac{2\pi}{3} \geq \phi_1 + \phi_2 = \pi - (\phi_3 + \phi_4) \geq \pi - \frac{2\pi}{3} = \frac{\pi}{3}.
\]

From the above we conclude that each angle of the triangle $T = \text{conv}\{x_1, x_2, x_3\}$ belongs to the range $\left[\frac{\pi}{4}, \frac{2\pi}{3}\right]$. A standard formula from elementary geometry $h_a = 2R \sin \beta \sin \gamma$ [ref], where $R = r$ is the radius.
of the circumcircle of $T$ yields that each height of $T$ is at least $2r \sin^2 \frac{\pi}{4} = r > 2$. This contradicts the $T(B, 3)$ property of $X$.

**Case 3.** $X = 5$. Let the points of $X$ be enumerated as $x_1, x_2, \ldots, x_5$ so that the polygonal line $x_1x_2 \ldots x_5x_1$ is the boundary of the convex pentagon $\text{conv} X$. Let, finally, $\phi_i$ ($i = 1, 2, \ldots, 5$) be half the central angular measure of the arc $x_ix_{i+1}$ ($i_6 = i_1$) of $\partial(rB)$ that does not contain other points $x_j$. We consider two subcases.

**Subcase 3.1.** $\text{max}(\phi_1, \phi_2, \ldots, \phi_5) \leq \frac{\pi}{2}$. With no loss of generality assume that $\phi_1 \geq \frac{\pi}{2}$. Let $y_1, y_2 \in \partial(rB)$ be two points such that the polygonal line $x_1x_2y_2y_1x_1$ bounds a convex quadrangle, and the arcs $x_1y_1$ and $x_2y_2$ have central angular measure $\frac{\pi}{2}$ each. There are two smaller subcases.

**Subcase 3.1.1.** The arc $y_1y_2$ (the one that does not contain $x_1$ and $x_2$) contains no points of $X$. Then $X$ is contained in the union of arcs $x_1y_1$ and $x_2y_2$. Let us start moving all points of $X$ simultaneously over $\partial(rB)$ towards either $x_1$ or $x_2$, depending on which of the two arcs the particular point belongs to, until John’s condition degenerates. (Clearly, John’s condition will degenerate before all points of $X$ arrive at either $x_1$ or $x_2$.)

At the moment John’s condition degenerates, the modified set $X$ contains a 4-tuple $\{z_1, z_2, z_3, z_4\}$ satisfying the condition of Corollary 6. By Corollary 8 one concludes that exactly two of the points $z_i$ (say, $z_1$ and $z_2$) belong to the arc $x_1y_1$, while the other two belong to the arc $x_2y_2$. But this produces a contradiction, similarly to the proof of Lemma 12.

**Subcase 3.1.2.** Some point $x_j$ belongs to the arc $y_1y_2$. Then each angle of the triangle $\text{conv}\{x_1, x_2, x_j\}$ is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Similarly to Case 2, the property $T(B, 3)$ does not hold for $X$.

**Subcase 3.2.** $\text{max}(\phi_1, \phi_2, \ldots, \phi_5) < \frac{\pi}{2}$. With no loss of generality, let $\phi_1 = \text{max}(\phi_1, \phi_2, \ldots, \phi_5)$. By the Pigeonhole Principle, $\phi_1 \geq \frac{\pi}{6}$.

Consider the triangle $\text{conv}\{x_1, x_2, x_4\}$. Let $\alpha$ be the angle at $x_2$. Then

$$\frac{\pi}{2} \geq \phi_4 + \phi_5 = \alpha = \pi - \phi_1 - \phi_2 - \phi_3 \geq \pi - 3\phi_1.$$ 

Consequently, the height $h_{1,24}$ of the triangle from $x_1$ satisfies

$$h_{1,24} = 2r \sin \phi_1 \sin \alpha \geq 2r \sin \phi_1 \sin 3\phi_1 > 2.$$ 

Similarly, $h_{2,14} > 2$. Finally, $h_{4,12} \geq 2r \sin^2 3\phi_1 > 2$. Again, $X$ violates the property $T(B, 3)$.

\[ \square \]

8 Statements with computer-assisted proofs: the algorithms

The proofs of Lemma 15 and parts (b)–(c) of Theorem 4 are computer-assisted. In this section we describe our approach towards the proof.

We claim that each of the statements has to be verified for a finite number of pairs $(r_1, r_2)$. Indeed, Lemma 15 concerns a particular pair (3, 1.62). For Theorem 4 the case $r_2 < 1.62$ is immediate, while the case $r_2 \geq 2$ is impossible due to Lemma 14. Hence we are interested only in the values $1.62 \leq r_2 \leq 2$. But after proving Lemma 15 we conclude that the case $r_1 > 3$ is impossible. Thus

$$(r_1, r_2) \in (0.15Z^2) \cap ([1.62, 3] \times [1.62, 2]),$$

which leaves us with a finite set of pairs to consider. Each pair $(r_1, r_2)$ is considered separately, therefore in the follow-up we assume that $r_1$ and $r_2$ are fixed.

Denote

$$P_n^p = \left[\frac{2p\pi}{n}, \frac{2(p + 1)\pi}{n}\right], \quad (p = 0, 1, \ldots, n - 1).$$

In order to prove each of the results, we use the so-called divide-and-conquer technique. The particular details are provided below.
8.1 Lemma [15]

It is sufficient to consider the case \( \#X = 5 \) with the additional assumption that the set \( NX \) satisfies [4], where \( N \) is an affine map such that \( NE = B \). Indeed, the case \( \#X > 5 \) is ruled out by a small generic perturbation. In turn, for the case \( \#X \leq 5 \) there is a set \( X' \subset \partial E \) such that \( \#X' = 5, \) \( NX' \) satisfies [4] and each point of \( X' \) is arbitrarily close to some point of \( X \).

Denote

\[
Q(p_1, p_2, \ldots, p_5; n) = I_{n_1}^{p_1} \times I_{n_1}^{p_2} \times \ldots \times I_{n_1}^{p_5}.
\]

We start with \( n = n_0 = 60 \). Consider the set \( Q_0 \) of all cubes \( Q(p_1, p_2, \ldots, p_5; n_0) \) such that \( 0 \leq p_1 \leq p_2 \leq \ldots \leq p_5 < n_0 \).

For each fixed \( Q = Q(p_1, p_2, \ldots, p_5; n_0) \in Q_0 \) denote \( \beta_i = \frac{2(p_i + 1/2)\pi}{n_0} \). Then one of the following holds:

1. The set \( \{(r_1 \cos \beta_i, r_2 \sin \beta_i) : i = 1, 2, \ldots, 5\} \) violates the property \( T \left( \left( 1 + r_1 \frac{2}{n} \right) B, 3 \right) \). Then, by Lemma [12] every 5-tuple \( (\alpha_1, \alpha_2, \ldots, \alpha_5) \in Q \) violates the property \( T(B, 3) \).

2. Consider the five values as in [4] with the arguments \( \beta_1, \beta_2, \ldots, \beta_5 \). If the largest of those values, \( F_{\max} \) and the smallest, \( F_{\min} \), satisfy

\[
F_{\max} > \frac{3\pi}{n}, \quad F_{\min} < -\frac{3\pi}{n},
\]

then no 5-tuple \( \{(\cos \alpha_i, \sin \alpha_i) : i = 1, 2, \ldots, 5\} \), where \( (\alpha_1, \alpha_2, \ldots, \alpha_5) \in Q \) and \( \alpha_1 < \alpha_2 < \ldots < \alpha_5 \), satisfies [6].

3. None of the above holds. Then we include all the cubes

\[
\{Q(p'_1, p'_2, \ldots, p'_5; 2n_0) : p'_i \in \{2p_i, 2p_i + 1\}, p'_1 \leq p'_2 \leq \ldots \leq p'_5\}
\]

in the new set \( Q_1 \).

We apply the same procedure to the set \( Q_1 \) and \( n = n_1 = 2n_0 \). Then we continue in the same fashion with \( (Q_2, n_2) \), etc. One obtains that \( Q_5 = \emptyset \), which immediately implies Lemma [15].

8.2 Theorem [4], part (b)

The setting of Theorem [4] part (b) refers to 4-tuples of points. Therefore we consider the 4-dimensional cubes

\[
Q(p_1, p_2, p_3, p_4; n) = I_{n_1}^{p_1} \times I_{n_1}^{p_2} \times \ldots \times I_{n_1}^{p_4}.
\]

We start with \( n = n_0 = 120 \). Consider the set \( Q_0 \) of all cubes \( Q(p_1, p_2, p_3, p_4; n_0) \) such that \( 0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 < n_0 \).

For each fixed \( Q = Q(p_1, p_2, p_3, p_4; n_0) \in Q_0 \) denote \( \beta_i = \frac{2(p_i + 1/2)\pi}{n_0} \). Then one of the following holds:

1. The set \( Z = \{(r_1 \cos \beta_i, r_2 \sin \beta_i) : i = 1, 2, \ldots, 4\} \) violates the property \( T \left( \left( 1 + r_1 \frac{2}{n} \right) B, 3 \right) \). Then, by Lemma [12] every 5-tuple \( (\alpha_1, \alpha_2, \ldots, \alpha_5) \in Q \) violates the property \( T(B, 3) \).

2. \( |F(\beta_1, \beta_2, \beta_3, \beta_4)| > \frac{3\pi}{n} \). Then every 4-tuple \( Z' = \{(r_1 \cos \alpha_i, r_2 \sin \alpha_i) : i = 1, 2, 3, 4\} \), where \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in Q \) and \( \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \), admits an elliptical disk \( E' \supset Z' \) such that \( |E'| < |E| \).

3. \( \beta_{i+1} - \beta_i > \frac{2\pi}{3} \) for some \( i \in \{1, 2, 3, 4\} \), where \( \beta_5 = \beta_1 + 2\pi \). Then, with \( \alpha_i \) and \( Z' \) as above, the condition of Corollary [8] is violated, thus \( Z' \) admits an elliptical disk \( E' \supset Z' \) such that \( |E'| < |E| \).

(Here we use that \( n \) is a multiple of 3.)
4. With Z as above, the set \( E \cap R(Z, r_1, \frac{\varepsilon}{n}) \) satisfies the property \( T\left(\sqrt[n]{\frac{r_1}{2}} B\right) \), where
\[
R(Z, \varepsilon) = \{x \in \mathbb{R}^2 : Z \cup \{x\} \text{ satisfies the property } T((1 + \varepsilon)B, 3)\}.
\]
Then \( R(Z') \subseteq R(Z, \varepsilon) \), hence the conclusion of Theorem 4, part (b) holds whenever \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in Q \) and \( \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \).

5. None of the above holds. Then we include all the cubes
\[
\{Q(p'_1, p'_2, p'_3, p'_4) : 2n_0) : p'_i \in \{2p_i, 2p_i + 1\}, p'_1 \leq p'_2 \leq p'_3 \leq p'_4\}
\]
in the new set \( Q_1 \).

We apply the same procedure to the set \( Q_1 \) and \( n = n_1 = 2n_0 \). Then we continue in the same fashion with \((Q_2, n_2)\), etc. One obtains that \( Q_5 = \emptyset \), which immediately implies Theorem 4, part (b).

8.3 Theorem 4, part (c)

The algorithm repeats the one from the previous subsection with the following changes.

1. Five-dimensional cubes are used instead of four-dimensional, since this part of Theorem 4 refers to 5-tuples of points.
2. The condition for the minimality of \( E \) is treated similarly to Lemma 15.
3. Having obtained the pair \((Q_4, n_4)\), we observe that \( n_4 = 1920 \). Therefore for every each cube \( Q \in \mathcal{Q}_4 \) it is sufficient to check its center. This is accomplished straightforwardly.

9 Proof of Theorem 4, part (a)

If \( r_2 \leq \sqrt[5]{\frac{r_1}{2}} \), then \( E \) satisfies the property \( T\left(\sqrt[5]{\frac{r_1}{2}} B\right) \). We will show that the case \( r_2 > \sqrt[5]{\frac{r_1}{2}} \) is impossible. Namely, since \#\( Z = 3 \), it will be sufficient to show that \( Z \) violates the property \( T(B) \).

Let \( N \) be an affine map such that \( NE = \sqrt[5]{\frac{r_1}{2}} B \). \( N \) satisfies (11), hence it is sufficient to prove that \( NZ \) violates the property \( T(B) \). But this is clear, because \( \text{conv } NZ \) is a regular triangle of height \( \frac{1}{2} \cdot \sqrt[5]{\frac{r_1}{2}} > 2 \).

10 Reduction of Theorem 5 to Theorem 4

Now, assuming that Theorem 4 is verified, we turn to the proof of Theorem 5.

Proof of Theorem 5: We argue by contradiction. Assume that Theorem 5 is false. Then there exists a set \( X_0 \) and a constant \( \varepsilon > 0 \) such that \( X \) satisfies the property \( T(B, 3) \), but does not satisfy the property \( T((c + \varepsilon)B) \). Then the set \( X_1 = \frac{\frac{c}{c + \varepsilon}}{2} X_0 \) satisfies the property \( T\left(\frac{\frac{c}{c + \varepsilon}}{2} B, 3\right) \), but not the property \( T((c + \varepsilon/2)B) \). Finally, every sufficiently small perturbation \( X \) of the set \( X_1 \) satisfies the property \( T(B, 3) \), but does not satisfy the property \( T(cB) \). As in the previous section, one can choose \( X \) to be sufficiently generic to guarantee that no ellipse passes through six different points of \( X \).

Let \( E \) be the elliptical disk of minimal area containing \( X \). Consider an arbitrary affine map \( N \) such that \( NE = B \). Then there exists a finite subset \( Z = \{z_1, z_2, \ldots, z_k\} \subseteq X \cap \partial E \) and positive coefficients \( c_1, c_2, \ldots, c_k \) such that the identities (10) hold for \( x_i = N z_i \). Of course, \( k \leq 5 \), because we assume \( X \) to be generic. On the other hand, \( k \geq 3 \), since (10) cannot hold for \( k = 1, 2 \). As in Conjecture 2 we use the notation \( R(Z) \) defined by (2). Now we proceed by case analysis.
Case 1. $k = 3$. Let $Z = X \cap \partial E$. Using case (a) of Theorem 11 we conclude that the set $E \cap R(Z)$ satisfies the property $T \left( \frac{r_1 + 1}{2} \cdot B \right)$. But $X \subseteq E \cap R(Z)$. Therefore the set $X$ satisfies the property $T \left( \frac{r_1 + 1}{2} \cdot B \right)$, too.

Case 2. $k = 4$. Consider two subcases.

Subcase 2.1. $r_2 \leq 1.62$. In this subcase it is clear that $E$ satisfies the property $T(1.62B)$. Since $X \subseteq E$, the set $X$ satisfies the property $T(1.62B)$, too.

Subcase 2.2. $r_2 > 1.62$. Let $r_1', r_2' \in 0.015\mathbb{Z}$ satisfy

$$r_1' \leq r_1 < r_1' + 0.015, \quad r_2' \leq r_2 < r_2' + 0.015.$$ 

Consider the following affine maps $M_1$ and $M_2$:

$$(x, y) \mapsto M_1 \left( \frac{r_1'}{r_1} x, \frac{r_2'}{r_2} y \right), \quad (x, y) \mapsto M_2 \left( \frac{r_2' + 0.015}{r_2} x, \frac{r_2' + 0.015}{r_2} y \right).$$

Consider an arbitrary triple $\{x_1, x_2, x_3\} \subseteq X$. By condition of the lemma, it satisfies the property $T(B)$. Since $M_1$ satisfies (10), the triple $\{M_1x_1, M_1x_2, M_1x_3\}$ satisfies the property $T(B)$, too. Therefore the set $M_1X$ satisfies the property $T(B, 3)$. Hence $M_1X \subseteq M_1E \cap R(M_1Z)$.

By Theorem 11 case (b), the set $M_1E \cap R(M_1Z)$ satisfies the property $T \left( \frac{r_1 + 1}{2} \cdot B \right)$. Therefore the set $M_1X$ satisfies the property $T \left( \frac{r_1 + 1}{2} \cdot B \right)$.

Consequently, the set $(M_2M_1)X$ satisfies the property $T \left( \frac{r_1 + 0.015}{r_2} + 0.015, \frac{r_1 + 1}{2} \cdot B \right)$. Since

$$\frac{r_1' + 0.015}{r_2'} \cdot \frac{\sqrt{5} + 1}{2} \leq \frac{1.635}{1.62} \cdot \frac{\sqrt{5} + 1}{2} < 1.635,$$

we conclude that $(M_2M_1)X$ satisfies the property $T(1.635B)$.

Finally, the map $(M_2M_1)^{-1}$ satisfies (10). Hence the set $X$ satisfies the property $T(1.635B)$.

Case 3. $k = 5$. Consider three subcases.

Subcase 3.1. $r_2 \leq 1.645$. By definition of the subcase, $E$ satisfies the property $T(1.645B)$. But $X \subseteq E$, hence the set $X$ satisfies the property $T(1.645B)$, too.

Subcase 3.2. $r_2 > 1.645$, $r_1 > 3$. Consider the map $M$ defined by

$$(x, y) \mapsto M \left( \frac{3}{r_1} x, \frac{1.62}{r_2} y \right).$$

The map $M$ satisfies (10), and the set $X$ satisfies the property $T(3, B)$. Therefore, by Lemma 11 $M_X$ satisfies the property $T(3, B)$ as well. But this is a contradiction to Lemma 15, hence the subcase is impossible.

Subcase 3.3. $1.645 < r_2 \leq r_1 \leq 3$. We need some further notation. Define the constants $r_1', r_2' \in 0.015\mathbb{Z}$ by the inequalities

$$r_1' \leq 0.995r_1 < r_1' + 0.015, \quad r_2' \leq 0.995r_2 < r_2' + 0.015.$$ 

We introduce the maps $M_1$ and $M_2$ defined by

$$(x, y) \mapsto M_1 \left( \frac{r_1'}{r_1} x, \frac{r_2'}{r_2} y \right), \quad (x, y) \mapsto M_2 \left( \frac{r_2' + 0.015}{0.995r_2} x, \frac{r_2' + 0.015}{0.995r_2} y \right).$$

If $N$ is the map defined by

$$(x, y) \mapsto N \left( \frac{1}{r_1} x, \frac{1}{r_2} y \right),$$

12
then the identity $NE = B$ holds. Then, by definition of the subcase, there is a subset $Z = \{x_1, x_2, x_3, x_4, x_5\} \subseteq X \cap \partial E$ such that the points $x_i = N z_i$ satisfy (6). The points $x_i$ can be parameterized by the parameters $\alpha_i$ so that $x_i = (\cos \alpha_i, \sin \alpha_i)$. With no loss of generality, assume that

$$0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_5 < 2\pi.$$ 

Define $\alpha'_i \in [\pi/(5+1/2), \pi/1920] (i = 1, 2, \ldots, 5)$ by the inequalities

$$\alpha_i - \pi/1920 \leq \alpha'_i < \alpha_i + \pi/1920.$$

Finally, for every $t \in [0,1]$ let

$$\alpha_i(t) = (1 - t)\alpha_i + t\alpha'_i, \quad x_i(t) = (\cos \alpha_i(t), \sin \alpha_i(t)).$$

Now consider two subcases.

**Subcase 3.3.1.** For every $t \in [0,1]$ there exist positive coefficients $c_i(t) (i = 1, 2, \ldots, 5)$ such that substitution of $x_i(t)$ instead of $x_i$ and $c_i(t)$ instead of $c_i$ turns the identities (6) into correct ones. By definition of $r'_1$ and $r'_2$, one has $\max \left(\frac{r'_1}{r_1}, \frac{r'_2}{r_2}\right) \leq 0.995$. Therefore the set $M_1 X$ satisfies the property $T(0.995,B,3)$. Next,

$$\|M_1 z_1 - M_1(N^{-1}x_1(1))\| \leq r'_1(\alpha_i - \alpha'_i) < \frac{3\pi}{1920} < 0.005.$$ 

Thus, by Lemma 12 the set $Y = M_1(X \cup \{N^{-1}x_1(1), N^{-1}x_2(1), \ldots, N^{-1}x_5(1)\})$ satisfies the property $T(B,3)$. But $M_1 N^{-1}x_1(1) \in Y$, therefore the case (c) of Theorem 4 is applicable. Hence the set $M_1 X \supseteq Y$ satisfies the property $T \left(\frac{\sqrt{\pi} + 1}{2}, B\right)$. Consequently, the set $(M_2 M_1) X$ satisfies the property $T \left(\frac{\sqrt{\pi} + 0.015}{0.995 r'_2}, \frac{\sqrt{\pi} + 1}{2}, B\right)$. Since

$$\frac{r'_2 + 0.015}{0.995 r'_2}, \frac{\sqrt{\pi} + 1}{2} \leq \frac{1.635}{0.995 \cdot 1.62}, \frac{\sqrt{\pi} + 1}{2} < 1.645,$$

we conclude that $(M_2 M_1) X$ satisfies the property $T(1.645 B)$. Finally, the map $(M_2 M_1)^{-1}$ satisfies (10). Hence the set $X$ satisfies the property $T(1.645 B)$.

**Subcase 3.3.2.** The condition of Subcase 3.3.1 does not hold for $t \in T$, where $T \in [0,1]$ is a non-empty set. (By definition, $x_i(0) = x_i$, hence the condition necessarily holds for $t = 0$.) Corollary 7 immediately implies that for $t_0 = \inf T$ there is a proper subset $X_0 \subseteq \{x_1(t_0), x_2(t_0), \ldots, x_5(t_0)\}$ such that $B$ is the minimum area elliptical disk containing $X_0$. Similarly to Subcase 3.3.1, the set $Y = M_1(X \cup N^{-1}X_0)$ satisfies the property $T(B,3)$. But we have either $\#X_0 = 3$ or $\#X_0 = 4$, therefore either case (a) or case (b) of Theorem 4 is applicable. Hence the set $M_1 X \supseteq Y$ satisfies the property $T \left(\frac{\sqrt{\pi} + 1}{2}, B\right)$. The rest of the argument proceeds exactly as in Subcase 3.3.1. 

\[\square\]

**References**

[1] Ball, K. (1992). Ellipsoids of maximal volume in convex bodies. Geometriae Dedicata, 41(2), 241–250.

[2] Eckhoff, J. (1969). Transversalenprobleme vom Gallai’schen Typ. Dissertation, Georg-August-Universität Göttingen. (In German.)

[3] Eckhoff, J. (2016). Problems in Discrete Geometry. In: Convexity and Discrete Geometry Including Graph Theory (pp. 269–273). Springer.

13
[4] Gruber, P. M. (2011). John and Loewner ellipsoids. Discrete & Computational Geometry, 46(4), 776–788.

[5] Grünbaum, B. (1958). On common transversals. Archiv der Mathematik, 9(6), 465–469.

[6] Handbook of Discrete and Computational Geometry (2017). 3rd edition, eds.: Cs. D. Tóth, J. O’Rourke, and J. E. Goodman. CRC press, to appear.

[7] Heppes, A. (2005). New upper bound on the transversal width of T (3)-families of discs. Discrete & Computational Geometry, 34(3), 463–474.

[8] Jerónimo Castro, J. (2007). Line transversals to translates of unit discs. Discrete & Computational Geometry, 37(3), 409–417.

[9] Jerónimo Castro, J., and Roldán-Pensado, E. (2011). Line transversals to translates of a convex body. Discrete & Computational Geometry, 45(2), 329–339.

[10] John, F. (1948) Extremum problems with inequalities as subsidiary conditions. In Courant Anniversary Volume (pp. 187–204). Interscience, New York, 1948.