Corrigendum

Naked singularity formation in Brans–Dicke theory
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Abstract
In this corrigendum, we have corrected the equations governing the behavior of the BD scalar field by redefining the effective energy density. We have shown the effects of the BD scalar field on the formation or otherwise of the trapped surfaces. Also, the time behavior of the Kretschmann scalar has been studied and it has been shown that the event horizon can fail to form during the dynamical evolution of the collapse scenario, which results in a naked singularity as the collapse endstate.

The basic equations of motion for Brans–Dicke theory are corrected by redefining the effective energy density and pressure. Then equations (10) and (11) are corrected as follows:

\[ \rho_{\text{eff}} = \frac{F'}{R^2 R} = \frac{1}{\phi} (\rho_\phi + \rho_m) = \frac{1}{\phi} \left( \rho_m + \frac{\omega \dot{\phi}^2}{2} - \frac{3}{a} \ddot{a} + \frac{V(\phi)}{2} \right), \]  

\[ p_{\text{eff}} = -\frac{F}{R^2 R} = \frac{1}{\phi} (p_\phi + p_m) = \frac{1}{\phi} \left( p_m + \frac{\omega \dot{\phi}^2}{2} + \ddot{\phi} + 2 \frac{\dot{a}}{a} \dot{\phi} - \frac{V(\phi)}{2} \right). \]  

From the LHS of equation (1) we have

\[ F = \frac{R^3}{3\phi} (\rho_\phi + \rho_m), \quad \dot{a}^2 = \frac{a^2}{3\phi} (\rho_\phi + \rho_m). \]  

Adding equations (1) and (2) we arrive at the following differential equation:

\[ \frac{\dot{\phi}_a}{\dot{\phi}^2} \left[ \left( n + 2 \right) a^{1-n} + \rho_{\text{tot}} a^{-(2+3w)} \left( \frac{5 + 3w}{6} \right) \right] - \frac{\dot{\phi}^3}{\dot{\phi}^3} \left[ \frac{2\omega + 1}{6} (a^{2-n} + \rho_{\text{tot}} a^{-(1+3w)}) \right] + \frac{n a^{-n}}{3} \frac{\phi_{,\phi}}{\dot{\phi}} - \frac{\phi_{,\phi}}{3 \dot{\phi}^2} [a^{2-n} + \rho_{\text{tot}} a^{-(1+3w)}] = 0. \]  

The above equation can be solved by taking the ansatz \( \phi = a^\alpha \) for the BD scalar field and \( n = 3(1+w) \), where \( \alpha \) satisfies the following equation:

\[ -\alpha^2 (1 + \rho_{\text{tot}}) (3 + 2\omega) + \alpha (n + 4 + \rho_{\text{tot}} (7 + 3w)) + 2n = 0. \]  

We then find the following expressions for the mass function, the time behavior of the scale factor and the singular epoch:

\[ F = \frac{r^3}{3} (1 + \rho_{\text{tot}}) a^{-(\alpha+3w)}. \]

\[ a(\tau) = \left( a_{*}^{3(1+3\alpha+3w)} - \frac{1}{2} \sqrt{1 + \frac{\rho_{\text{tot}}}{3}} (\alpha + 3(1 + w)) (\tau - \tau_* + \frac{\alpha}{3(1+3w)}) \right)^{\frac{1}{3(1+w)}}, \]

\[ \tau_* = \frac{2\sqrt{3}}{\sqrt{1 + \rho_{\text{tot}} (\alpha + 3(1 + w))}}. \]
Substituting the expression obtained for \( n \) into equation (5), we find \( \alpha \) and the ratio \( F/R \) for each case as

\[
\alpha = -\frac{12}{7(1 + \rho_{\text{tot}}) + \sqrt{(1 + \rho_{\text{tot}})(121 + 49\rho_{\text{tot}} + 48\omega)}},
\]

\[
\frac{F}{R} = \frac{r^2}{3(1 + \rho_{\text{tot}})}a^{-(1+\alpha)}
\]

for dust;

\[
\alpha = -\frac{4}{3(1 + \rho_{\text{tot}}) + \sqrt{(1 + \rho_{\text{tot}})(21 + 9\rho_{\text{tot}} + 8\omega)}}
\]

\[
\frac{F}{R} = \frac{r^2}{3(1 + \rho_{\text{tot}})}a^{-\alpha}
\]

for cosmic strings;

\[
\alpha = -\frac{4}{5(1 + \rho_{\text{tot}}) + \sqrt{(1 + \rho_{\text{tot}})(49 + 25\rho_{\text{tot}} + 16\omega)}}
\]

\[
\frac{F}{R} = \frac{r^2}{3(1 + \rho_{\text{tot}})}a^{-(1-\alpha)}
\]

for domain walls, and

\[
\alpha = -\frac{4}{2(1 + \rho_{\text{tot}}) + \sqrt{2(1 + \rho_{\text{tot}})(5 + 2\rho_{\text{tot}} + 2\omega)}}
\]

\[
\frac{F}{R} = \frac{r^2}{3(1 + \rho_{\text{tot}})}a^{-(\alpha+4)}
\]

for radiation fluid. Note that since the physical quantities such as the BD scalar field diverge at the singularity, the negative values of \( \alpha \) have been chosen here.

We are now in a position to study the effect of the BD scalar field on the formation or otherwise of the apparent horizon as the dynamical procedure of the collapse scenario evolves (we set \( \omega = -1 \) in the rest of this paper). The establishment of a weak energy condition during the collapse scenario implies that the initial energy density of matter must be positive; then equation (7) implies that \( |\alpha| < 1 \), which causes the ratio \( F/R \) to grow and the expansion parameter to tend to negative infinity. Thus there exist no radial null geodesics emerging from the singularity. For the case of cosmic strings, since \( \alpha \) is always less than zero, the ratio \( F/R \) stays finite till the singular epoch and causes the expansion parameter to be positive up to the singularity; then if no trapped surfaces exist initially none would form until the epoch \( a(\tau) = 0 \). For the case of domain walls, from equation (9) it is seen that \( \alpha < 0 \), implying that \( 1 - \alpha > 0 \). Thus the ratio \( F/R \) tends to zero as the scale factor vanishes, causing the expansion parameter to be positive up to the singularity. Thus the formation of trapped surfaces fails to form. For the last case, from equation (10) it is seen that \( |\alpha| < 4 \), so the ratio \( F/R \) increases and the expansion parameter tends to negative infinity as the scale factor vanishes. Thus, trapped surfaces do form as the collapse reaches the singularity. Since we are concerned with a collapse procedure \( \dot{a} < 0 \), the rate of change of mass function with respect to time is negative for both cases with \( w = -1/3 \) and \( w = -2/3 \), which means that the mass function contained in the collapsing shell with a constant coordinate radius keeps decreasing. Then there exists an observable outward energy flux during the dynamical evolution of the collapse scenario. Also, the weak energy condition \((\rho_{\text{eff}} > 0 \text{ and } \rho_{\text{eff}} + p_{\text{eff}} > 0)\) is satisfied by the collapsing configuration as follows:

\[
(1 + \rho_{\text{tot}})a^{-(\alpha+3(1+w))} > 0,
\]
Figure 1. The behavior of the potential as a function of the BD scalar field for $\omega = -1$ and different values of $\rho_{\text{om}}$ and $w$: $\rho_{\text{om}} = 1$ and $w = 0$ (dotted curve), $\rho_{\text{om}} = 1$ and $w = -\frac{1}{3}$ (dashed curve), $\rho_{\text{om}} = 2$ and $w = -\frac{2}{3}$ (solid curve), $\rho_{\text{om}} = 2$ and $w = \frac{1}{3}$ (thick-dashed curve).

and

$$w = 0, \quad \alpha < 0 \Rightarrow \left[\left(1 - \frac{\alpha}{3}\right)(1 + \rho_{\text{om}})\right]^{-\omega \left(\frac{\alpha+1}{3}\right)} > 0 \Rightarrow (\rho_{\text{eff}} + p_{\text{eff}})_{\text{dust}} > 0,$$

$$w = -\frac{1}{3}, \quad \alpha < 0 \Rightarrow \left[\left(\frac{4 - \alpha}{3}\right)(1 + \rho_{\text{om}})\right]^{-\omega \left(\frac{\alpha+2}{3}\right)} > 0 \Rightarrow (\rho_{\text{eff}} + p_{\text{eff}})_{\text{cosmic strings}} > 0,$$

$$w = -\frac{2}{3}, \quad \alpha < 0 \Rightarrow \left[\left(\frac{5 - \alpha}{3}\right)(1 + \rho_{\text{om}})\right]^{-\omega \left(\frac{\alpha+1}{3}\right)} > 0 \Rightarrow (\rho_{\text{eff}} + p_{\text{eff}})_{\text{domain walls}} > 0,$$

$$w = \frac{1}{3}, \quad \alpha < 0 \Rightarrow \left[\left(\frac{2 - \alpha}{3}\right)(1 + \rho_{\text{om}})\right]^{-\omega \left(\frac{\alpha+4}{3}\right)} > 0 \Rightarrow (\rho_{\text{eff}} + p_{\text{eff}})_{\text{radiation}} > 0.$$

Using equation (1) one can easily find the behavior of the potential with respect to the BD scalar field as

$$V(\phi) = \beta \phi^{-\frac{\omega \left(\frac{\alpha+1}{3}\right)}{3}},$$

where $\beta$ is given by

$$\beta = \left(2 + \alpha(1 + \rho_{\text{om}})\left(\frac{6 - \omega}{3}\right)\right).$$

Figure 1 shows the behavior of $V(\phi)$. Up until now we have shown that the formation of trapped surfaces for $w = -\frac{1}{3}$ and $w = -\frac{2}{3}$ can be avoided. In what follows we investigate the behavior of the Kretschmann invariant as a function of time for these two cases and it is shown that the formation of the event horizon can fail. In order to calculate this quantity, one has to utilize equation (6), its first and second time derivative and substitute the results into the following equation:

$$K = 12 \left[\left(\frac{\dot{a}}{a}\right)^4 + \left(\frac{\ddot{a}}{a}\right)^2\right].$$

Figure 2 shows the behavior of the Kretschmann scalar as a function of time. It is seen that this quantity diverges for both cases $w = -\frac{1}{3}$ and $w = -\frac{2}{3}$ at $\tau_s = 1.455$ and $\tau_s = 2.29$, respectively. It then converges at late times which implies the failure of formation of the event horizon.
Figure 2. The behavior of the Kretschmann scalar (in units of $s^{-4}$) as a function of the proper time (right, domain walls and left, cosmic strings).

Figure 3. The behavior of the BD scalar field with respect to $\tau$ and $r$ for $\omega = -1$ and $w = -\frac{1}{3}$. For the initial energy density, scale factor and proper time we have adopted the values $\rho_{0m} = 1$, $a^* = 1$ and $\tau^* = 0$, respectively.

Figure 4. The behavior of the BD scalar field with respect to $\tau$ and $r$ for $\omega = -1$ and $w = 0$. For the initial energy density, scale factor and proper time, we have adopted the values $\rho_{0m} = 1$, $a^* = 1$ and $\tau^* = 0$, respectively.
Figure 5. The behavior of the BD scalar field with respect to $\tau$ and $r$ for $\omega = -1$ and $w = -\frac{1}{3}$.
For the initial energy density, scale factor and proper time, we have adopted the values $\rho_{0m} = 2$, $a^* = 1$ and $\tau^* = 0$, respectively.

Figure 6. The behavior of the BD scalar field with respect to $\tau$ and $r$ for $\omega = -1$ and $w = 1$.
For the initial energy density, scale factor and proper time, we have adopted the values $\rho_{0m} = 2$, $a^* = 1$ and $\tau^* = 0$, respectively.

Finally, using the equation
\[
\Box \phi = \frac{1}{2\omega + 3} \left( T^m + \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right),
\]
(16)
together with equation (13), we arrive at a differential equation for $\phi(a(\tau), r)$ as
\[
\left[ \frac{n - 8}{6} a^{1-n} + \frac{\rho_{0m}}{6} (3w - 5)a^{-(3w+2)} \right] \frac{\phi_{,a}}{\phi} + \frac{1}{6} \left[ a^{2-n} + \rho_{0m}a^{-(1+3w)} \right] \left( \frac{\phi_{,a}}{\phi} \right)^2
- \left[ a^{2-n} + \rho_{0m}a^{-(1+3w)} \right] \frac{\phi_{,aa}}{3\phi} + \frac{\phi''}{r a^2} + 2 \frac{\phi'}{r a^2} + C \frac{a^{3(1+w)}}{2\omega + 3} = 0,
\]
(17)
where $C$ is given by

$$C = \frac{3\beta(1 + w)}{\alpha} + 2\beta - \rho_{\text{lim}}(3w - 1). \quad (18)$$

Equation (17) can be more simplified and the result is as follows:

$$\phi'' + \frac{2\phi'}{r} - Da^{1-\alpha-3w} = 0, \quad (19)$$

where $D$ is a constant and is given by

$$D = \frac{\alpha(\alpha - 1)}{3} + \left(\frac{5\alpha - \alpha^2}{6} - \frac{\alpha w}{2}\right)(1 + \rho_{\text{lim}}) - \frac{C}{2\omega + 3}. \quad (20)$$

Figures 3–6 show the behavior of BD scalar field as a function of proper time and coordinate radius.

**Conclusion**

We have corrected the equations governing the behavior of the BD scalar field by redefining the effective energy density. We have shown the effects of the BD scalar field on the formation or otherwise of the trapped surfaces. Also, the time behavior of the Kretschmann scalar has been studied and it has been shown that the event horizon can fail to form during the dynamical evolution of the collapse scenario, which results in a naked singularity as the collapse endstate.