Semiclassical analysis of level widths for one-dimensional potentials

Gert-Ludwig Ingold
Institut für Physik, Universität Augsburg, Universitätsstraße 1, D-86135 Augsburg, Germany

Rodolfo A. Jalabert
Université Louis Pasteur, IPCMS-GEMME, 23 rue du Loess, F-67037 Strasbourg Cedex, France

Klaus Richter
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, D-01187 Dresden, Germany

(September 10, 1999)

We present a semiclassical study of level widths for a class of one-dimensional potentials in the presence of an ohmic environment. Employing an expression for the dipole matrix element in terms of the Fourier transform of the classical path we obtain the level widths within the Golden rule approximation. It is found that for potentials with an asymptotic power-law behavior, which may in addition be limited by an infinite wall, the width that an eigenstate of the isolated system acquires due to the coupling to the environment is proportional to its quantum number.

I. INTRODUCTION

Semiclassical approaches were essential at the advent of quantum mechanics and have since remained a privileged tool for learning this subject, for developing our physical intuition on new problems and for performing analytical calculations [1]. In particular, in the one-dimensional (1d) case the semiclassical (WKB) approximation treats a wide class of confining potentials $V(q)$ in the limit of large quantum numbers. We can write down closed expressions for eigenenergies and eigenfunctions, and for power-law potentials the scaling of the eigenenergies with the quantum numbers and the classical actions can be established [2,3]. Once we have an almost complete solution of a 1d problem we can use it as the basis for understanding more complicated physical situations. This is the approach of this work, where we analyze the problem of a particle in $d = 1$ coupled to a dissipative environment.

For an isolated quantum system, the ground state and the excited states are stable and the corresponding density of states is a series of delta functions. Any coupling to external degrees of freedom usually renders the excited states unstable, thus the levels acquire a finite lifetime [4] which is related to the level width through the time-energy uncertainty relation. The density of states of the system including the additional degrees of freedom still consists of (closely spaced) delta functions, however the reduced density of states corresponding to the system alone is smeared out compared to that of the uncoupled case.

An example of such a situation is an atom coupled to the electromagnetic modes of the radiation field. If the atom were isolated from the field there would be no transitions from excited states to states lower in energy. This changes if we take the coupling to the radiation field into account. Then transitions between states may occur, and the finite lifetime of the excited states broadens the spectral lines associated with the transition.

Two types of transitions are caused by the coupling to the radiation field: stimulated processes and spontaneous emission. In the first case photons have to be present, while in the second case the mere presence of field modes is sufficient to allow for transitions to energetically lower states accompanied by the emission of a photon. In the limit of zero temperature, the mean number of photons in the radiation field vanishes and the broadening of atomic levels is entirely due to spontaneous emission.

In the following, we replace the atom with a particle moving in a one-dimensional potential. The environment is modeled by a set of harmonic oscillators coupled bilinearly to the particle [5]. Using semiclassical methods, we extend the scaling of Refs. [2,3] to level widths and demonstrate that, for a wide class of confining potentials, level widths are simply proportional to quantum number. To establish this general result we first present the theory for the level widths (Sec. II) and then apply it to a few specific examples: harmonic oscillator (Sec. III), 1d box (Sec. IV), half-oscillator (Sec. V) and Coulomb potential (Sec. VI). The semiclassical approximation for level widths (Sec. V) simplifies the calculations in the examples considered and can be applied to general power-law potentials (Sec. VIII). In the concluding chapter (Sec. IX) we analyze the experimental implications of our findings and their possible extensions to higher dimensions.

II. GENERAL EXPRESSION FOR THE LEVEL WIDTHS

As our model we consider a particle of mass $M$ moving in a one-dimensional potential $V(q)$. The spectrum
of the corresponding Hamiltonian

\[ H_S = \frac{p^2}{2M} + V(q) \]  

(1)
is assumed to consist of a discrete part at low energies which may be followed by a continuous part at higher energies. It is on the discrete part of the spectrum (consisting of eigenenergies \( E_n, n = 0, 1, 2, \ldots \)) where we focus our analysis. To be consistent, we always denote the ground state by \( n = 0 \) although this may lead to a slightly unusual notation.

Since we consider the limit of large quantum numbers we require that the number of discrete eigenstates is infinite or at least can be made sufficiently large. This includes for example the radial part of the Coulomb problem but excludes the Morse potential.

The eigenstates acquire a finite width if we weakly couple the particle to environmental degrees of freedom. We assume that the environment consists of a set of harmonic oscillators coupled bilinearly to the particle. This leads to the full Hamiltonian

\[ H = H_S + \sum_{J=0}^{\infty} \left[ \frac{p^2}{2m_J} + \frac{m_J \omega_j^2}{2} \left( x_J - \frac{c_J}{m_J \omega_j^2} q \right)^2 \right] \]  

(2)

implying a coupling between system and environment through the Hamiltonian

\[ H_1 = - \sum_{J=0}^{\infty} c_J x_J q. \]  

(3)

By eliminating the environmental degrees of freedom, we obtain an effective operator equation of motion \([4, 5]\),

\[ \ddot{q} + \int_0^t ds \gamma(t-s) \dot{q}(s) + \frac{1}{M} \frac{dV}{dq} = \frac{1}{M} \xi(t), \]  

(4)

with damping kernel

\[ \gamma(t) = \frac{2}{M} \int_0^\infty \frac{d\omega}{\omega} J(\omega) \cos(\omega t), \]  

(5)
spectral density of bath oscillators

\[ J(\omega) = \pi \sum_{J=0}^{\infty} \frac{c_J^2}{2m_J \omega_j} \delta(\omega - \omega_j), \]  

(6)

and an operator-valued fluctuating force \( \xi(t) \) which we do not need to specify further.

Of great importance is the special case of \( J(\omega) = M \gamma/\omega \). Then the damping kernel becomes \( \gamma(t) = 2\gamma \delta(t) \). Noting that in Eq. (6) only half of the delta function contributes, the second term becomes \( \gamma \dot{q}(t) \) describing the well-known classical damping proportional to the particle velocity. This type of damping is often referred to as ohmic because such a damping term appears also in equations describing electrical circuits containing an ohmic resistor.

The Hamiltonian (3) provides a microscopic model for dissipation in quantum systems in the sense that dissipation is due to coupling to additional degrees of freedom. It does not, however, pretend that in a real resistor we can identify environmental oscillators microscopically.

Without going into details, we mention that the Hamiltonian (3), besides the fact that the environment can be eliminated analytically, provides a good description of many realistic systems. It has been well studied over the years \([7, 8]\) and more recently became known as Caldeira-Leggett model \([3]\) in the context of macroscopic quantum phenomena.

Using the Hamiltonian (2) and assuming weak coupling between the particle and its environment, we calculate the zero temperature width of the \( n \)-th level by means of the Fermi golden rule

\[ \Gamma_n = \frac{2\pi}{\hbar} \sum_{m,j=0}^{\infty} |\langle m, 1 | H_1 | n, 0 \rangle|^2 \delta(E_n - E_m - \hbar \omega_j). \]  

(7)

This expression describes the decay of the state \( n \) to an energetically lower state \( m \) by one excitation of the \( j \)-th environmental mode that changes its occupation number from 0 to 1. Inserting the dipole matrix element

\[ \langle 1 | x_J | 0 \rangle = \left( \frac{\hbar}{2m_J \omega_j} \right)^{1/2} \]  

(8)
of the \( j \)-th environmental oscillator we get

\[ \Gamma_n = \pi \sum_{m,j=0}^{\infty} \frac{c_J^2}{m_J \omega_j} |d_{nm}|^2 \delta(E_n - E_m - \hbar \omega_j) \]  

(9)

where

\[ d_{nm} = \langle m | q | n \rangle \]  

(10)
is the dipole matrix element of the system. The properties of the environmental modes appearing in Eq. (8) can be expressed in terms of their spectral density (6), so we write the level width as

\[ \Gamma_n = \frac{2}{\hbar} \sum_{m=0}^{n-1} |d_{nm}|^2 J \left( \frac{E_n - E_m}{\hbar} \right). \]  

(11)

This result is valid for arbitrary bath density. Employing a cubic frequency dependence for \( J(\omega) \) would lead to the natural decay width of an excited atomic state due to spontaneous emission, apart from prefactors arising from a proper treatment of the polarization of the emitted photons.

While the following calculations could in principle be performed for arbitrary bath densities, we will consider only the important case of ohmic damping where the level widths are

\[ \Gamma_n = \frac{2M \gamma}{\hbar^2} \sum_{m=0}^{n-1} |d_{nm}|^2 (E_n - E_m). \]  

(12)
The sum over the system eigenstates is restricted since an environment at zero temperature cannot excite the system into states of higher energy.

Eq. (12) constitutes the starting point for a calculation of level widths that will be performed in the following sections. It represents, up to the factor $\gamma$, a sum over oscillator strengths

$$f_{nm} = \frac{2M}{\hbar^2} |d_{nm}|^2 (E_n - E_m).$$

The finiteness of the upper limit prevents us from evaluating the level widths by the standard Thomas-Reiche-Kuhn sum rule [9] for oscillator strengths,

$$\sum_{m=0}^{\infty} f_{nm} = 1.$$  \hspace{1cm} (14)

### III. HARMONIC OSCILLATOR

The maybe simplest example is that of a harmonic potential with frequency $\omega_0$. The dipole matrix element in this case couples only nearest neighbors,

$$d_{nm} = \sqrt{\frac{\hbar}{2M\omega_0}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}).$$  \hspace{1cm} (15)

This leads to a simple and well-known result for the level widths of a damped harmonic oscillator [10]

$$\Gamma_n = n\gamma.$$  \hspace{1cm} (16)

As an illustration we discuss the density of states for a damped harmonic oscillator. In view of the relation between the partition function of a canonical ensemble,

$$Z(\beta) = \int_0^{\infty} \mathrm{d}E \rho(E) \exp(-\beta E),$$  \hspace{1cm} (17)

and the density of states $\rho(E)$, one may obtain the latter by inverse Laplace transformation [11] from the partition function. For a damped harmonic oscillator, the partition function is [12]

$$Z(\beta) = \frac{1}{\hbar\beta\omega_0} \prod_{\nu_n} \frac{\nu_n^2}{\nu_n^2 + \nu_n \gamma (\nu_n) + \omega_0^2}.$$  \hspace{1cm} (18)

Here, $\tilde{\gamma}$ denotes the Laplace transform of the damping kernel (3) and $\nu_n = 2\pi n / \hbar \beta$ are the Matsubara frequencies. The product in Eq. (18) does not converge in the case of an ohmic environment where $\tilde{\gamma}(\nu_n) = \gamma$ and we therefore introduce a cutoff in the spectral density of states. After inverse Laplace transform, according to (17), we arrive at the density of states $\rho(E)$, for which the ohmic limit can be computed if the (infinite) ground state energy of system plus environment is subtracted. The result is shown in Fig. 1 for ohmic damping with $\gamma / 2\omega_0 = 0.1$, omitting the delta function corresponding to the stable ground state. The levels broaden with increasing energy and for large energies only the average density of states $\rho = 1 / \hbar \omega_0$ is seen.

![Graph of density of states](image)

**FIG. 1.** Density of states for a harmonic oscillator of frequency $\omega_0$ coupled to an ohmic environment with $\gamma / 2\omega_0 = 0.1$. The delta function corresponding to the stable ground state is not shown.

According to Eq. (14) the level widths for the harmonic oscillator are proportional to $n$ or, equivalently, to $E_n$. In general, however, $E_n$ is not proportional to $n$. Then the question arises whether the widths are simply determined by $n$, or by $E_n$ and whether or not there is a universal behavior.

### IV. ONE-DIMENSIONAL BOX

A more non-trivial example is that of a particle in a one-dimensional box of length $L$. The eigenenergies are

$$E_n = \frac{\hbar^2 \pi^2}{2ML^2} (n+1)^2,$$  \hspace{1cm} (19)

and dipole matrix elements can be easily calculated. For reasons of symmetry a dipole coupling can induce transitions only between states of different parity. For $n - m$ odd one obtains

$$d_{nm} = -\frac{8L}{\pi^2} \left( \frac{1}{4} \right)^{\frac{n}{2}} \frac{(n+1)(m+1)}{[(n+1)^2 - (m+1)^2]^2}. \hspace{1cm} (20)$$

We are interested in the universal behavior for large quantum numbers, where $d_{nm}$ decreases as $(n-m)^2$. Therefore, we define $l = n - m$ and write approximately for $l \ll m, n$

$$d_{nm} = \frac{2L}{\pi^2} \frac{1}{l^2}. \hspace{1cm} (21)$$

We can now evaluate the level widths for large quantum number $n$ by assuming that the sum over all final states $m$ converges fast enough that only differences
at large quantum numbers are therefore given by

$$\Gamma_{n,sc} = \frac{8\gamma}{\pi^2} n \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^3}. \quad (22)$$

The sum can be expressed in terms of Riemann’s zeta function as $(7/8)\zeta(3) = 1.051 \ldots$ \textsuperscript{[13]} The level widths at large quantum numbers are therefore given by

$$\Gamma_{n,sc} = 0.852 \ldots \gamma n. \quad (23)$$

The subindex “sc” stands for “semiclassical” since, as seen in the next section, Eq. (23) results directly from a WKB approximation. The quality of our approximate result is seen in Fig. 2 where we show the ratio between the WKB approximation. The ratio between the exact width $\Gamma_n$ and the semiclassical width $\Gamma_{n,sc}$. The relative error drops below 1% at about $n \approx 38$.

The result (23) indicates that the proportionality in the case of the harmonic oscillator should be interpreted to be to the quantum number and not to the eigenenergy. The prefactor resulting for the 1d box is different from that of the harmonic oscillator. Therefore a potentially universal behavior can only concern the scaling with the quantum number, while the prefactor will be specific to the given potential.

![FIG. 2. Ratio between the result (23) and the exact width $\Gamma_n$ for a particle in a box as a function of the state $n$.](image)

V. SEMICLASSICAL EXPRESSION FOR THE DIPOLE MATRIX ELEMENT

In the two examples treated above, the harmonic oscillator and the 1d box, the wave functions are known and the dipole matrix elements are readily calculated. To treat generic potentials in the limit of large quantum numbers we use a semiclassical approximation for the dipole matrix element. Such an expression (see Eq. (34) below) was first derived by Debye \textsuperscript{[14]} and we present it in this section to emphasize the physical assumptions on which it is based.

For a 1d confining potential $V(q)$ the WKB approximation \textsuperscript{[15]} for the wave function with eigenenergy $E_n$ ($n \gg 1$) is

$$\psi_{n,sc}(q) = \left( \frac{4M}{T p(q, E_n)} \right)^{1/2} \cos \left( \frac{1}{\hbar} S(q, E_n) - \frac{\pi}{4} \right). \quad (24)$$

Here, $q$ lies between the two turning points $q_1$ and $q_2$ ($V(q_1) = V(q_2) = E_n$, $q_1 < q_2$) which determine the limits of the classical motion at the energy $E_n$. The classical action is computed from $q_1$ to $q$ according to

$$S(q, E_n) = \int_{q_1}^{q} dq \frac{p(q, E_n)}{p(q, E_n)} \quad (25)$$

with

$$p(q, E_n) = [2M(E_n - V(q))]^{1/2} \quad (26)$$

The quantization condition determining $E_n$ depends on the action along a period of the classical motion

$$S(E_n) = 2S(q_2, E_n) = 2\pi\hbar \left( n + \frac{1}{2} \right), \quad (27)$$

and the period $T$ is given by

$$T(E) = \frac{dS}{dE} = \int_{q_1}^{q_2} dq \frac{2M}{p(q, E_n)}. \quad (28)$$

The commutation relation $(i/\hbar)[H, q] = p/M$ allows us to express the dipole matrix element in terms of the momentum matrix element. We can then write

$$d_{nm} = \frac{\hbar^2}{M(E_n - E_m)} \int_{-\infty}^{+\infty} dq \psi_{n,sc}^*(q) \frac{\partial}{\partial q} \psi_{n,sc}(q). \quad (29)$$

Under the assumption $l \ll n, m$, both eigenenergies have almost the same classically allowed region. Neglecting the contribution of the classically forbidden region, we restrict the integration to the interval $(q_1, q_2)$. Moreover, we assume that $V(q)$ is a smoothly varying function, and thus the derivative with respect to $q$ is carried out only in the cosine, which has a small factor $\hbar$ in its denominator, but not in the prefactor. With these assumptions we have

$$d_{nm} = -\frac{2\hbar}{T(E_n - E_m)} \int_{q_1}^{q_2} dq \left\{ \sin \left[ \frac{1}{\hbar} (S(q, E_n) - S(q, E_m)) \right] - \cos \left[ \frac{1}{\hbar} (S(q, E_n) + S(q, E_m)) \right] \right\}. \quad (30)$$

The second term in the integral is rapidly oscillating and is negligible upon integration. For the first term we use the relation

$$d_{nm} = -\frac{\hbar^2}{M(E_n - E_m)} \int_{q_1}^{q_2} dq \left\{ \sin \left[ \frac{1}{\hbar} (S(q, E_n) - S(q, E_m)) \right] - \cos \left[ \frac{1}{\hbar} (S(q, E_n) + S(q, E_m)) \right] \right\}. \quad (31)$$
where \( t(q, E_n) \) is the time for a classical particle to move from \( q_1 \) to \( q \) at energy \( E_n \). With Eqs. (27) and (28) we can write
\[
E_n - E_m \approx \frac{\partial E}{\partial n} = \frac{2\pi \hbar}{T} t(q),
\]
and then find from Eq. (30)
\[
d_{n,m} = -\frac{1}{\pi l} \int_{q_1}^{q_2} dq \sin (2\pi l t(q))\]
\[
= \frac{1}{T(E)} \int_0^{T(E)} dt q(t, E) \cos (2\pi l t/T(E))
\]
where the second line has been obtained by means of the substitution \( dq = \dot{q} dt \) and partial integration.

Eq. (33) constitutes the interesting result found by Debye which allows to relate the dipole matrix elements in the semiclassical limit to the Fourier components of the classical motion of the particle. It extends Bohr’s correspondence principle which states that in the limit \( \hbar \to 0 \) the frequencies of an atomic transition should agree with electrodynamics. By virtue of Eq. (33), a correspondence between quantum theory and electrodynamics can not only be established for frequencies but also for intensities which according to Eq. (14) are related to the square of dipole matrix elements.

Eq. (33) is the leading order approximation in \( \hbar \). Higher order corrections have been derived in Ref. [11]; however we will make no use of them in this work.

In the presence of hard walls the WKB approximation is still applicable and only a change in the phase of the semiclassical wavefunction (24) is needed to recover the result for the 1D box. Then the semiclassical approximation to the dipole matrix element is the same as for smooth potentials (Eq. (33)). The period of a classical trajectory with energy \( E \) in the box is
\[
T(E) = L \left( \frac{2M}{E} \right)^{1/2},
\]
leading in the limit of large \( n \) to the forms (21) and (23) for the matrix elements and the level widths, respectively.

VI. HALF-OSCILLATOR

Our next example is the harmonic oscillator with an infinite wall at \( q = 0 \). The particle is therefore confined to \( q \geq 0 \). In contrast to the usual harmonic oscillator, the dipole matrix element no longer couples only nearest neighbor states. The eigenstates of the half-oscillator are given by the odd eigenstates of the harmonic oscillator with the prefactor adjusted to account for the restricted interval of normalization. The dipole matrix element \( d_{nm} \) may then be evaluated by expressing the Hermite polynomial with the higher quantum number \( n > m \) by means of the Rodrigues formula,
\[
d_{nm} = -\left( \frac{\hbar}{M\omega \pi} \right)^{1/2} \frac{1}{2^{n+m}[(2n+1)!(2m+1)!]^{1/2}}
\times \int_0^{\infty} d\xi \xi H_{2m+1}(\xi) \frac{d^{2n+1}}{d\xi^{2n+1}} \exp(-\xi^2).
\]
After repeated partial integration and use of
\[
\frac{d^k}{d\xi^k} (\xi H_{2m+1}(\xi)) \bigg|_{\xi=0} = (-1)^{n+1-k/2} \frac{2^{k-1} k(2m+1)!}{(m+1-k/2)!},
\]
which holds for \( k \) even and yields zero for \( k \) odd, one arrives at
\[
d_{nm} = 2 \left( \frac{\hbar}{M\pi \omega} \right)^{1/2} \frac{(2m+1)!}{2^{2(m+n+1)}(2n+1)!} \frac{1}{2^{n-m+1}} \sum_{l=1}^{m+1} 2^n l \frac{[2(n-l)]!}{(m-l+1)!(n-l)!}.
\]
The sum may be evaluated to obtain
\[
d_{nm} = \left( \frac{\hbar}{M\pi \omega} \right)^{1/2} \frac{1}{2^{n-m+1}} \frac{[(2n+1)!(2m+1)!)^{1/2}}{n!m!}
\times \frac{1}{4(n-m)^2-1}.
\]
A corresponding semiclassical evaluation of the dipole matrix element using Eq. (33) leads to
\[
d_{n,n-1} = -\frac{4}{\pi} \left( \frac{\hbar}{M\omega} \right) \left( n - \frac{1}{4} \right) \frac{1}{4l^2-1}.
\]
For \( n \gg l \) this agrees with the exact result up to an irrelevant sign. For the level widths we then obtain
\[
\Gamma_{n,sc} = \frac{8 \pi^2}{2} \gamma n = 0.810 \ldots 3n
\]
and thus again a proportionality to the quantum number. Even though the eigenenergies of the one-dimensional box and the half-oscillator depend differently on the quantum number and in spite of the fact that the dipole matrix elements of oscillator and half-oscillator are quite different, the final result for the level widths is proportional to the quantum number \( n \) in both cases.

VII. RADIAL PART OF COULOMB POTENTIAL

A less trivial dependence of the dipole matrix elements on quantum number is found for our last example, the radial part of the three-dimensional Coulomb problem for vanishing angular momentum. In this case, an infinite potential wall again restricts the particle to positive \( q \) where it moves in an attractive Coulomb potential
$-A/q$. Unlike the previous examples, the spectrum of the Coulomb problem contains a continuous part at positive energies. The discrete part of the spectrum consists of the eigenenergies

$$E_n = -\frac{MA^2}{2n^2} \frac{1}{(n+1)^2}. \quad (41)$$

Dipole matrix elements can be obtained by solving the Schrödinger equation in momentum representation and read

$$d_{nm} = \frac{8\hbar^2}{M\pi A} \frac{[(n+1)(m+1)]^{5/2}}{(n+1)^2 - (m+1)^2} I_{nm} \quad (42)$$

with

$$I_{nm} = \int_0^\infty du \frac{u \sin[f_n(u) - f_m(u)]}{1 + (n+1)^2 u^2} [1 + (m+1)^2 u^2]$$

and

$$f_n(u) = 2(n+1)\arctan[(n+1)u]. \quad (44)$$

For $n, m \gg 1$ and $n > m$ this becomes

$$d_{nm} = (-1)^{n-m+1} \frac{2^{4/3} 3^{1/6} \hbar^2}{M\pi A} \Gamma \left( \frac{2}{3} \right) \left( \frac{(nm)^{11/6}}{(n^2 - m^2)^{5/3}}. \quad (45)$$

It is instructive to derive this result from the semiclassical expression (43). Even though the trajectory $q(t)$ cannot be obtained explicitly, we can obtain the matrix elements by observing that the behavior of the Fourier coefficients of $q(t)$ is dominated by the most singular part of the original function which in our case is the motion around the reflection point at $q = 0$.

Energy conservation leads to the trajectory at energy $E$ in implicit form as

$$\left( \frac{2}{M} \right)^{1/2} \left| \frac{E}{A} \right|^{3/2} - t + \frac{\pi}{2} = \arcsin(q') - q'^{1/2}(1 - q')^{1/2} \quad (46)$$

with $q' = (|E|/A)q$. From this result we compute the period

$$T = \pi \left( \frac{M}{2} \right)^{1/2} \frac{A}{|E|^{3/2}} \quad (47)$$

and by expanding the arcsin we find, for the trajectory around the reflection point encountered at $t = 0$,

$$q = \left( \frac{9A}{2M} \right)^{1/3} t^{2/3}. \quad (48)$$

We then write the trajectory for $0 \leq t < T$ as

$$q = \left( \frac{9A}{2M} \right)^{1/3} t^{2/3} \left( 1 - \frac{t}{T} \right)^{2/3} + q_{reg} \quad (49)$$

where the first term reproduces the most singular behavior of the trajectory around $t = 0$ and $t = T$ while the second term contains the more regular part and is of no further interest. From Eq. (49) we find with Eq. (43) for $n \gg l \gg 1$ and up to an irrelevant sign

$$d_{n,n-l} = \frac{\hbar^2}{M\pi A} \frac{3^{1/6}}{2^{1/3}} \Gamma \left( \frac{2}{3} \right) \frac{n^2}{t^{5/3}}. \quad (50)$$

Written in symmetric form this agrees with Eq. (43) in the limit $n \gg l$.

The level widths can now be obtained from Eq. (12) with (41) and (50) as

$$\Gamma_{n,sc} = \frac{6^{1/3} \Gamma(2/3)^2}{\pi^2} \zeta \left( \frac{7}{3} \right) \gamma n = 0.477 \ldots \gamma n. \quad (51)$$

Again, we find the proportionality to the quantum number $n$ which, in view of the complicated structure of the dipole matrix element, may come as a surprise. This indicates that this proportionality might be more universal. We will indeed prove it in the following section for power-law potentials with and without walls.

**VIII. SEMICLASSICAL TREATMENT FOR POWER-LAW POTENTIALS**

In this section we generalize the previous results to power-law potentials of the form $V(q) = A|q|^{\alpha}$. The amplitude $A$ and the exponent $\alpha$ should have opposite signs. The possible coordinates might be limited by an infinite potential wall, and we assume that such a wall is present at $q = 0$ whenever $\alpha < 0$. The case $\alpha = 0$ will be excluded because it requires two walls and thus becomes the box potential already treated in Sec. [V]. Furthermore, we have to restrict the exponent to $\alpha > -2$. At $\alpha = -2$ the action becomes independent of energy as will become clear from Eq. (56) below so that we have to exclude this pathological case. We emphasize, that an attractive $1/q^2$ potential can never appear in an radial equation of motion in $d > 1$ where the angular degrees of freedom have been eliminated.

Since our semiclassical approach requires sufficiently large energies, the discussion applies also to potentials which effectively behave like a power-law potential at higher energies, e.g. the quartic double-well potential.

According to Eq. (12) the $n$ dependence of the level width is determined from the dipole matrix element $d_{nm}$ and the energy difference $(E_n - E_m)$. Within the hypothesis established in Sec. [V] (which was more general than for power-law potentials) we saw that $(E_n - E_m)$ is simply inverse proportional to the period of the classical motion between the turning points (Eq. (23)).

To evaluate the dependence on $n$ of the level widths [12] we employ scaling properties of the energy conservation condition.
\[ E = \frac{M}{2} q'^2 + A q'^\alpha. \] (52)

Introducing a dimensionless coordinate
\[ q' = \left( \frac{|A|}{E} \right)^{1/\alpha} q \] (53)
as in (50) and dimensionless time
\[ t' = \frac{A^{1/\alpha} E^{(\alpha-2)/2\alpha}}{M^{1/2}} t, \] (54)
we then find for the energy eigenvalues
\[ S(E) = \frac{M^{1/2} E^{(2+\alpha)/2\alpha}}{|A|^{1/\alpha}} \int dq' q'^2 = \frac{2\pi\hbar n}{\alpha}, \] (56)
where the integral runs over one period. On the right-hand-side we have omitted an \( n \)-independent term which depends on the presence of walls in the potential. With \( S' = \frac{d}{dq'} S \) we then find for the energy eigenvalues
\[ E = \left[ \frac{2\pi\hbar}{S'} \right]^{2\alpha} \left( \frac{|A|^2}{M\alpha} \right)^{1/(2+\alpha)} n^{2\alpha/(3+\alpha)}. \] (57)

This is in agreement with previous results for power-law potentials \([21]\) where \( S' \) has been evaluated explicitly. On the other hand, Eq. (57) is still correct for sufficiently large \( n \) if the potential behaves only asymptotically like a power-law. Then, in general, \( S' \) can no longer be evaluated analytically.

The semiclassical dipole matrix element \([23]\) reads after scaling
\[ d_{n,n-1} = \left( \frac{E}{|A|} \right)^{1/\alpha} d'_1 \] (58)
with
\[ d'_1 = -\frac{1}{\pi t} \int_{q'_1}^{q'_n} dq' \sin \left( 2\pi t \frac{q'}{T'} \right) \] (59)
where \( T' \) denotes the period \( T \) scaled according to \([24]\).

In view of Eqs. (28), (31) and (58), the level width \([23]\) can be expressed as
\[ \Gamma_{n,sc} = \gamma \frac{8\pi^2}{S'^2} \frac{2 + \alpha}{2\alpha} \left[ \sum_{l=1}^{n} l d'^2_l \right] n. \] (60)

Apart from the factor \( n \) this result still depends on the state number \( n \) via the upper limit of the sum. As a last step, we therefore have to consider the convergence properties of this sum.

For \( \alpha > 0 \) and in the absence of an infinite potential wall, the particle velocity is always continuous and, as a consequence, the scaled dipole moment \([23]\) will decay at least as \( l^{-2} \). This still holds for a wall with finite potential to its one side because in the worst case the reflection will lead to a triangular cusp singularity in the trajectory. In this case, the dipole moment will decay as \( l^{-3} \).

The case of negative exponent \( \alpha \) with an infinite potential wall at the divergence at \( q = 0 \) is more interesting. Close to \( q = 0 \) we may neglect the constant on the left-hand-side of (53). Assuming that the reflection happens at \( t' = 0 \) we find for the trajectory close to the reflection point \( q' \sim |t'|^{2/(2-\alpha)} \). Proceeding as in Sec. VII one finds for the scaled dipole matrix element \( d'_1 \sim |t'|^{(2-\alpha)/(2-\alpha)} \).

For \( \alpha = -1 \) this is in agreement with (60). For \( \alpha > -2 \) it follows that \( d'_1 \) decays always faster than \( l^{3/2} \).

As a consequence, the argument of the sum in Eq. (60) decays faster than \( 1/l^2 \) for all potentials under consideration. Neglecting terms of order \( 1/n \), as is consistent with our previous approximations, we may extend the upper summation limit to infinity and arrive at our final result
\[ \Gamma_{n,sc} = \gamma \frac{8\pi^2}{S'^2} \frac{2 + \alpha}{2\alpha} \left[ \sum_{l=1}^{\infty} l d'^2_l \right] n. \] (61)

For sufficiently large energies, the level width is therefore proportional to the state number \( n \). We point out that the proportionality constant depends on \( \alpha \) and \( \gamma \) but not on \( M \) and \( A \). While these properties are special to the case of ohmic damping, an extension to other bath densities would be straightforward along the lines presented here.

**IX. CONCLUSIONS**

Semiclassical methods for radiative lifetimes, ubiquitous in spectroscopic measurements, have mainly been derived for excited states of hydrogenlike atoms \([21]\). Here, we applied such methods to a whole class of one-dimensional potentials which behave like a power-law for large energies and may contain one or two walls. Using a semiclassical expression for the dipole matrix element in terms of the Fourier transform of the classical paths we have shown that for ohmic damping the level width is proportional to the number \( n \) of the state. This result is valid for sufficiently large \( n \), therefore requiring weak coupling to the environment in order to be observable. The proportionality of level widths with the state number of the harmonic oscillator could thereby be generalized to a large class of one-dimensional potentials. The prefactor of the linear law depends on the specific potential.

Extensions of our results to higher dimensions are interesting since we will be approaching experimental reality and will be able to investigate the relevance of the integrability of the classical system on level widths. The
chaotic or integrable character of the classical dynamics has been shown to be of crucial importance for other observables like the orbital magnetic response [21].

The problem of an integrable 2d system, where we can separate the dynamics, can be reduced to two 1d problems. For the 2d box it is readily demonstrated that the mean behavior of $\Gamma_n$ is again a linear increase with $n$.

The problem becomes more involved in chaotic systems where the energy is the only conserved quantity and the explicit forms of individual wavefunctions are not known. When increasing the number of degrees of freedom of the system from 1 to 2 we start seeing fluctuations of level widths with respect to a secular behavior. Since level widths are given by matrix elements of the dipole operator we expect that the fluctuations will be more important in the integrable case, where selection rules impose wide variations according to the quantum numbers of the initial and final states.

ACKNOWLEDGMENTS

We have benefited from discussions with J.-Y. Bigot, H. Grabert, S. Kohler, and S. Otto. Part of this work has been carried out while one of us (GLI) was at the Centre d’Etudes de Saclay with financial support from the Volkswagen-Stiftung.

[1] Matthias Brack and Rajat K. Bhaduri, *Semiclassical Physics*, Frontiers in Physics Vol. 96 (Addison-Wesley, Reading, 1997).
[2] Uday P. Sukhatme, “WKB Energy Levels for a Class of One-Dimensional Potentials,” Am. J. Phys. 41, 1015–1016 (1973).
[3] J. F. Cariñena, C. Farina, and Cássio Sigaud, “Scale invariance and the Bohr-Wilson-Sommerfeld (BWS) quantization for power law one-dimensional potential wells,” Am. J. Phys. 61, 712–717 (1993).
[4] V. Weisskopf and E. Wigner, “Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie,” Z. Phys. 63, 54–73 (1930).
[5] see e.g. Thomas Dittrich, Peter Hänggi, Gert-Ludwig Ingold, Bernhard Kramer, Gerd Schön, and Wilhelm Zwerger, *Quantum transport and dissipation* (Wiley-VCH, Weinheim, 1998), Chap. 4.
[6] V. B. Magalinskii, “Dynamical Model in the Theory of the Brownian Motion,” Zh. Eksp. Teor. Fiz. 36, 1942–1944 (1959) [Sov. Phys. JETP 9, 1381–1382 (1959)].
[7] see e.g. Robert J. Rubin, “Statistical Dynamics of Simple Cubic Lattices. Model for the Study of Brownian Motion,” J. Math. Phys. 1, 309–318 (1960) and 2, 373–386 (1961); I. R. Senitzky, “Dissipation in Quantum Mechanics. The Harmonic Oscillator,” Phys. Rev. 119, 670–679 (1960) and 124, 642–648 (1961); G. W. Ford, M. Kac, and P. Mazur, “Statistical Mechanics of Assemblies of Coupled Oscillators,” J. Math. Phys. 6, 504–515 (1965); P. Ullersma, “An exactly solvable model for Brownian motion,” Physica 32, 27–55, 56–73, 74–89, 90–96 (1966); Robert Zwanzig, “Nonlinear Generalized Langevin Equations,” J. Stat. Phys. 9, 215–220 (1973).
[8] A. O. Caldeira and A. J. Leggett, “Quantum Tunneling in a Dissipative System,” Ann. Phys. (N.Y.) 149, 374–456 (1983).
[9] W. Thomas, “Über die Zahl der Dispersionselektroden, die einem stationären Zustand zugeordnet sind,” Naturwissenschaften 13, 627 (1925); W. Kuhn, “Über die Gesamtstärke der von einem Zustande ausgehenden Absorptionslinien,” Z. Phys. 33, 408–412 (1925); F. Reiche and W. Thomas, “Über die Zahl der Dispersionselektroden, die einem stationären Zustand zugeordnet sind,” Z. Phys. 34, 510–525 (1925); for a presentation in English language see e.g. C. Cohen-Tannoudji, B. Din, and F. Laboë, *Quantum Mechanics* (Wiley, New York, 1978), Vol. 2, p. 1318.
[10] see e.g. Claude Cohen-Tannoudji, Jacques Dupont-Roc, and Gilbert Grynberg, *Atom-Photon Interactions* (Wiley, New York, 1992), p. 326.
[11] A. Hanke and W. Zwerger, “Density of states of a damped quantum oscillator,” Phys. Rev. E 52, 6875–6878 (1995).
[12] see e.g. Ulrich Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993), p. 51.
[13] *Handbook of Mathematical Functions*, edited by Milton Abramowitz and Irene A. Stegun (Dover, New York, 1972), p. 807.
[14] P. Debye, “Wellenmechanik und Korrespondenzprinzip,” Physik. Zeitschr. 28, 170–174 (1927).
[15] Martin C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990).
[16] Robert Karrlein and Hermann Grabert, “Semiclassical theory of vibrational energy relaxation,” J. Chem. Phys. 108, 4972–4983 (1998).
[17] Eugene V. Ivash, “A Momentum Representation Treatment of the Hydrogen Atom Problem,” Am. J. Phys. 40, 1095–1097 (1972).
[18] S. M. Susskind and R. V. Jensen, “Numerical calculations of the ionization of one-dimensional hydrogen atoms using hydrogenic and Sturmian basis functions,” Phys. Rev. A 38, 711–728 (1988).
[19] S. P. Gorelovski˘ı, N. B. Delone, and V. P. Krainov, “Procedures for estimating hydrogenic and Sturmian basis functions,” Zh. Eksp. Teor. Fiz. 82, 1789–1797 (1982) [Sov. Phys. JETP 55, 1032–1036 (1982)].
[20] Klaus Richter, Denis Ulm, and Rodolfo A. Jalabert, “Orbital magnetism in the ballistic regime: geometrical effects,” Phys. Rep. 276, 1–84 (1996).
[21] H. A. Kramers, “On the Theory of X-Ray Absorption and of the Continuous X-Ray Spectrum,” Philos. Mag. 46, 836–871 (1923); K. Omidvar, “Semiclassical formula for the radiative mean lifetime of the excited state of the hydrogenlike atom,” Phys. Rev. A 26, 3053–3061 (1982); Hermann Markov and Larry Spruch, “Semiclassical estimation of the radiative mean lifetimes of hydrogenlike states,” Phys. Rev. A 43, 1268–1274 (1991).