On the consistency of cell division processes

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Abstract

For a class of cell division processes, generating tessellations of the Euclidean space $\mathbb{R}^d$, spatial consistency is investigated. This addresses the problem whether the distribution of these tessellations, restricted to a bounded set $V$, depends on the choice of a larger region $W \supset V$ where the construction of the cell division process is performed. This can also be understood as the problem of boundary effects in the cell division procedure. In \cite{5} it was shown that the STIT tessellations are spatially consistent There were hints that the STIT tessellation process might be the only translation-invariant cell division process that has such a consistency property. In the present paper it is shown that, within a reasonable wide class of cell division processes, the STIT tessellations are the only ones that are consistent.

Keywords: stochastic geometry; random tessellation; iteration/nesting of tessellations; stability of distribution; STIT tessellation

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1 Introduction

In Stochastic Geometry the well established models for random tessellations of the Euclidean space $\mathbb{R}^d$, $d \geq 2$, are the Poisson-Voronoi tessellations and the Poisson hyperplane tessellations, see \cite{1, 6}. Moreover, there are many suggestions in the literature to construct tessellations by sequential division of the cells, i.e. of the polytopes which constitute a tessellation. A systematization including many of such constructions was recently given in \cite{2}. Usually, these tessellations are constructed in a bounded window $W \subset \mathbb{R}^d$. This yields a key problem for this kind of constructions: Are the tessellations consistent in space, i.e. does the distribution of the resulting tessellation depend on the window where the construction is performed? More precisely, if $Y(W, t)$ and $Y(V, t)$ are the random tessellations generated by cell division until time $t > 0$ in bounded windows $V \subset W$, are then $Y(W, t) \cap V$ and $Y(V, t)$ identically distributed? The consistency of a model implies the existence of a tessellation $Y(t)$ of the whole space $\mathbb{R}^d$ such that the restrictions $Y(t) \cap W$ have the same law as $Y(W, t)$. In the present paper we consider a certain class of cell division processes, and the main result is that in this

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class the STIT tessellations (introduced in [5]) are the only ones which are consistent.

2 Random tessellations and consistency

For the $d$-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, denote by $\partial B$, $\text{int}(B)$ and $\text{cl}(B)$ the boundary, the interior and the topological closure, respectively, of a set $B \subset \mathbb{R}^d$. Let $[\mathcal{H}, \mathcal{F}]$ denote the measurable space of hyperplanes in $\mathbb{R}^d$ with its Borel $\sigma$-algebra $\mathcal{F}$ w.r.t. the topology of closed convergence for closed subsets of $\mathbb{R}^d$, see [6]. For a set $B \subset \mathbb{R}^d$ we write $[B] = \{ h \in \mathcal{H} : B \cap h \neq \emptyset \}$. Further, let $\mathcal{P}$ denote the set of all polytopes (i.e. convex hulls of finite point sets) with interior points in $\mathbb{R}^d$. A set $\{ C_1, C_2, \ldots \}$ with $C_i \in \mathcal{P}$ is a tessellation if $\bigcup_{i=1}^{\infty} C_i = \mathbb{R}^d$ and $\text{int}(C_i) \cap \text{int}(C_j) = \emptyset$ for $i \neq j$. Moreover, a locally finiteness condition must be satisfied, $\#\{ i : C_i \cap B \neq \emptyset \} < \infty$ for all bounded $B \subset \mathbb{R}^d$, i.e. the number of polytopes intersecting a bounded set is finite.

A tessellation can be considered as a set $\{ C_1, C_2, \ldots \}$ of polytopes – referred to as the cells – as well as a closed set $\bigcup_{i=1}^{\infty} \partial C_i \subset \mathbb{R}^d$, the union of the cell boundaries. There is an obvious one-to-one relation between both descriptions of a tessellation, and also the $\sigma$-algebras which are used for them can be related appropriately. Let $\mathbb{T}$ denote the set of all tessellations of $\mathbb{R}^d$. By $\mathbb{T} = \{ y \} \in \mathbb{T}$ we mean the closed set of cell boundaries of the tessellation $y$. Then $\mathbb{T}$ can be endowed with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{T})$ of the topology of closed convergence. A random tessellation $Y$ is a random variable with values in $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. For $W \in \mathcal{P}$, the set of tessellations restricted to $W$ is denoted $\mathbb{T} \wedge W$. In particular, for $y \in \mathbb{T}$ we have $y \cap W \in \mathbb{T} \wedge W$, and the boundary of $W$ does not belong to this restricted tessellation. Here $W$ is referred to as a window.

By $\overset{D}{=} \text{the identity of the distributions of random variables. Our investigation of consistency of random tessellations will be based on the following proposition. In [6], Theorem 2.3.1, a more general form is given; we specify it here for } \mathbb{R}^d$. 

**Theorem 2.1.** (Schneider and Weil) Let $(Z_i : i \in \mathbb{N})$ be a sequence of random closed sets in $\mathbb{R}^d$, and $(G_i : i \in \mathbb{N})$ a sequence of open, bounded sets with $\text{cl}(G_i) \subset G_{i+1}$ for $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} G_i = \mathbb{R}^d$. If $Z_m \cap \text{cl}(G_i) \overset{D}{=} Z_i$ for all $m > i$, then there exists a random closed set $Z$ in $\mathbb{R}^d$ with

$$Z \cap \text{cl}(G_i) \overset{D}{=} Z_i$$

for all $i \in \mathbb{N}$. 

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Definition 2.2. A family \((Y(W) : W \in \mathcal{P})\) of random tessellations with \(Y(W) \in \mathcal{T} \land W\) is called consistent if and only if for any two windows \(V, W \in \mathcal{P}\) with \(V \subset W\) holds
\[
Y(V) \overset{D}= Y(W) \cap V.
\]

Obviously, if \(Y\) is a random tessellation of \(\mathbb{R}^d\), then \((Y(W) := Y \cap W, W \in \mathcal{P})\) is a consistent family of tessellations. On the other hand, Theorem 2.1 yields that for any consistent family \((Y(W), W \in \mathcal{P})\) exists a random tessellation \(Y\) such that \(Y \cap W \overset{D}= Y(W)\).

We will go a step further and study continuous-time processes of tessellations. Therefore, also the consistency of finite-dimensional distributions of these processes have to be considered.

Definition 2.3. For any \(W \in \mathcal{P}\) let \((Y(t, W), t > 0)\) be a random process of tessellations with values in \(\mathcal{T} \land W\). The family of tessellation processes \(((Y(t, W), t > 0) : W \in \mathcal{P})\) is called consistent in space if and only if for any two windows \(V, W \in \mathcal{P}\) with \(V \subset W\) and for all \(0 < t_1 < \ldots < t_n, n \in \mathbb{N}\) holds
\[
(Y(t_1, V), \ldots, Y(t_n, V)) \overset{D}= (Y(t_1, W \cap V, \ldots, Y(t_n, W \cap V)).
\]

Again, if \((Y(t), t > 0)\) is a random process of tessellations of \(\mathbb{R}^d\), then \(((Y(t, W), t > 0) : W \in \mathcal{P})\) is a consistent family of tessellation processes. Vice versa, if \(((Y(t, W), t > 0) : W \in \mathcal{P})\) is a consistent family of tessellation processes, then for all \(0 < t_1 < \ldots < t_n, n \in \mathbb{N}\) exist tessellations \(Y(t_1), \ldots, Y(t_n)\) with
\[
(Y(t_1, W), \ldots, Y(t_n, W)) \overset{D}= (Y(t_1) \cap W, \ldots, Y(t_n) \cap W)
\]
for all \(W \in \mathcal{P}\). Because the laws of \((Y(t_1), \ldots, Y(t_n))\) with \(0 < t_1 < \ldots < t_n, n \in \mathbb{N}\), form a projective family of distributions, and the measurable space \((\mathcal{T}, \mathcal{B}(\mathcal{T}))\) of tessellations is a Polish space (see [4]), Kolmogorov’s extension Theorem (see e.g. [3]) yields that there is a process \((Y(t), t > 0)\) with the respective finite-dimensional distributions.

3 A class of cell division processes

Inspired by Cowan’s paper [2] we study a certain class of random tessellation processes which are generated by sequential cell division. But we will consider continuous-time processes only. A cell division process is defined by the
distributions of life times (that corresponds to Cowan’s selection rule) and by a division rule for the extant cells.

For \( h \in \mathcal{H} \) we denote by \( h^+ \) and \( h^- \) the closed half-spaces of \( \mathbb{R}^d \) generated by \( h \), with the following definition: If \( u \in \mathcal{S}^{d-1}_+ \) is a vector in the upper half-sphere of \( \mathbb{R}^d \) and \( a \in \mathbb{R} \) such that \( h = \{ x \in \mathbb{R}^d : \langle x, u \rangle = a \} \), then \( h^+ := \{ x \in \mathbb{R}^d : \langle x, u \rangle \geq a \} \) and \( h^- := \{ x \in \mathbb{R}^d : \langle x, u \rangle \leq a \} \).

Assumptions:

(i) Let \( \lambda : \mathcal{P} \to (0, \infty) \) be a function with

\[
\forall C \in \mathcal{P}, \exists k_C \geq 1, \forall h \in [C] : \lambda(C \cap h^\pm) \leq k_C \lambda(C). \tag{1}
\]

(ii) Let \( \{ \Lambda_{[C]} : C \in \mathcal{P} \} \) be a family of probability measures on \((\mathcal{H}, \mathcal{B})\) where \( \Lambda_{[C]} \) is concentrated on \([C] \), but \( \Lambda_{[C]} \) is not concentrated on a set of hyperplanes which are all parallel to one line (i.e. the directional distribution is not concentrated on a great subsphere, cf. [6], Subsection 10.3)

The construction:

Let \( W \in \mathcal{P} \), referred to as a window. The cell division process \((Y(t, W) : t \geq 0)\) with states in \( \mathcal{T} \land W \) is defined by the following construction.

1. Let \( \tau_0, \tau_1, \ldots \sim \mathcal{E}(1) \) be a sequence of i.i.d. random variables which are exponentially distributed with parameter 1.

2. \( Y(t, W) = \emptyset \) for all \( 0 \leq t < \frac{1}{\lambda(W)} \tau_0 \)

3. At time \( t = \frac{1}{\lambda(W)} \tau_0 \) the window \( W \) is divided by a random hyperplane \( h_0 \) with law \( \Lambda_{[W]} \). Two new cells \( C_1, C_2 \) are born with \( C_1 = W \cap h_0^+ \) and \( C_2 = W \cap h_0^- \). The state of the process is \( Y(t, W) = W \cap h_0^+ \) for \( \frac{1}{\lambda(W)} \tau_0 \leq t < \min\{ \frac{1}{\lambda(C_1)} \tau_1, \frac{1}{\lambda(C_2)} \tau_2 \} \).

4. Any cell \( C_i \) which appears during the construction has a life time \( \frac{1}{\lambda(C_i)} \tau_i \). At the end of its life time it is divided by a random hyperplane \( h_i \) with the law \( \Lambda_{[C_i]} \), into the new cells \( C_i \cap h_i^+ \) and \( C_i \cap h_i^- \), respectively. Always, \( h_i \) is assumed to be conditionally independent of all the other dividing hyperplanes, given the cell \( C_i \). Thus the process \( Y(t, W) \) jumps into another state exactly at those times when the life time of one of the extant cells elapses and the respective cell is divided.

Example 3.1. Some examples for the functional \( \lambda \) (corresponding to Cowan’s selection rule) are:
\[ \lambda(C) = W_i(C), \text{ where } W_i \text{ denotes the } i\text{-th intrinsic volume (see e.g. } \mathbb{E} \text{), } i = 0, \ldots, d. \] These functionals are monotonically increasing, i.e.

\[ \forall C, C' \in \mathcal{P} : C' \subset C \Rightarrow \lambda(C') \leq \lambda(C). \]

In particular \( W_0(C) = 1 \) for all \( C \in \mathcal{P} \) and \( W_d \) is the volume.

\[ \lambda(C) \ldots \text{ number of vertices of } C \in \mathcal{P}. \] This functional is not monotone in \( C \), but condition (1) is satisfied.

\[ \lambda(C) \] for a given (non-zero) translation invariant and locally finite measure \( \Lambda^* \) on \((\mathcal{H}, \mathcal{F})\), chose \( \lambda(C) = \Lambda^*([C]), C \in \mathcal{P} \). If \( \Lambda^* \) is also rotation invariant, then this functional coincides, up to a constant factor, with the intrinsic volume \( W_1 \).

Some examples for the distributions \( \Lambda_{[C]}, C \in \mathcal{P} \) (correspond to Cowan’s division rule) are:

\[ \Lambda \text{ be a (non-zero) translation invariant and locally finite measure on } (\mathcal{H}, \mathcal{F}) \text{ that is not concentrated on a set of hyperplanes which are all parallel to one line. For this measure define } \Lambda_{[C]}(\cdot) = \Lambda(\cdot \cap [C])/(\Lambda([C])). \]

\[ \text{Define } \Lambda_{[C]} \text{ by the following procedure: throw a random point uniformly into } C, \text{ and then choose a random hyperplane through this point with a certain directional distribution.} \]

Remark 3.2. The homogeneous STIT tessellations as they were first introduced in [2] fit into this scheme, choosing a translation invariant measure \( \Lambda \text{ on } (\mathcal{H}, \mathcal{F}) \text{ and using } \lambda(C) = \Lambda([C]), \Lambda_{[C]}(\cdot) = \Lambda(\cdot \cap [C])/(\Lambda([C])) \text{ for all } C \in \mathcal{P}. \]

4 Necessary and sufficient conditions for consistency

**Theorem 4.1.** Let the family of tessellation processes \((Y(t, W), t > 0) : W \in \mathcal{P}\) be a family of cell division processes determined by \( \lambda \) and \( \{\Lambda_{[C]} : C \in \mathcal{P}\} \) which satisfy assumptions (i) and (ii) above. If this family of processes is consistent in space then there exists a measure \( \nu \) on \([\mathcal{H}, \mathcal{F}]\) such that for all \( C \in \mathcal{P} \)

\[ \lambda(C) = \nu([C]) \] (2)

and

\[ \Lambda_{[C]}(\cdot) = \frac{1}{\nu([C])} \nu(\cdot \cap [C]). \] (3)
Now we consider consistent families of cell division processes which yield homogeneous (i.e. spatially stationary) tessellations in \( \mathbb{R}^d \). It was already known that homogeneous STIT tessellation processes are consistent. The following theorem states that STIT are the only consistent cell division processes.

**Theorem 4.2.** Let the family of tessellation processes \( ((Y(t, W), t > 0) : W \in \mathcal{P}) \) be a family of cell division processes determined by \( \lambda \) and \( \{ \Lambda_C : C \in \mathcal{P} \} \) which satisfy assumptions (i) and (ii) above. This family of processes is consistent in space and all \( Y_t, t > 0 \), are homogeneous (spatially stationary), if and only if this process has the same distribution as the homogeneous STIT process driven by the hyperplane measure \( \nu \) given in Theorem 4.1.

**Remark 4.3.** It is easily seen by examples that the choices of \( \lambda \) and \( \{ \Lambda_C : C \in \mathcal{P} \} \) mentioned in the Example 3.1 and different from S3, D1 do not fulfill the necessary conditions for consistency.

## 5 Proofs

**Lemma 5.1.** Fix \( \lambda \) and \( \{ \Lambda_C : C \in \mathcal{P} \} \) which satisfy (i) and (ii) above. Let be \( V, W \in \mathcal{P} \) and \( V \subset W \). If for all \( t > 0 \)

\[
Y(V, t) \overset{D}{=} Y(W, t) \cap V
\]

then for all \( H \in \Pi, H \subset [V] \)

\[
\lambda(V)\Lambda_{[V]}(H) = \lambda(W)\Lambda_{[W]}(H)
\]

**Proof.** Consider polytopes \( V, W \in \mathcal{P} \) with \( V \subset W \) and a Borel set \( B \subset V \). For the window \( V \) the life time until the first division is \( \frac{1}{\lambda(V)} \tau_0 \). This division generates two new cells, \( C_1, C_2 \), say. Condition (i) ensures that the waiting time until the next division in \( V \) (i.e. a division of \( C_1 \) or \( C_2 \) respectively) is

\[
\min\left\{ \frac{1}{\lambda(C_1)} \tau_1, \frac{1}{\lambda(C_2)} \tau_2 \right\} \geq \min\left\{ \frac{1}{k_V \lambda(V)} \tau_1, \frac{1}{k_V \lambda(V)} \tau_2 \right\} = \frac{1}{k_V \lambda(V)} \min\{\tau_1, \tau_2\},
\]

i.e. it is greater or equal than an exponentially distributed random variable with parameter \( 2k_V \lambda(V) \).

Hence, the time of the second division is greater or equal to the sum of two independent exponentially distributed random variables, the first one with parameter \( \lambda(V) \) and the second one with parameter \( 2k_V \lambda(V) \).

Then the properties of the exponential distribution yield for the construction within the window \( V \) that for small \( \Delta t > 0 \) up to a probability \( o(\Delta t) \) not
more than one division of $V$ takes place in the time interval $(0, \Delta t)$. Note
the the event that $V$ is divided exactly once until $\Delta t$ can also be written as
$Y(V, \Delta t) = h_0 \cap V$. Hence

$$\mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset)$$

$$= \mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset, Y(V, \Delta t) = h_0 \cap V)$$

$$+ \mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset, Y(V, \Delta t) \text{ contains more than two cells })$$

$$= \mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset, Y(V, \Delta t) = h_0 \cap V) + o(\Delta t)$$

$$= \mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset | Y(V, \Delta t) = h_0 \cap V) \cdot P(Y(V, \Delta t) = h_0 \cap V) + o(\Delta t)$$

$$= \Lambda_{[V]}([B]) \cdot (\lambda(V) \Delta t + o(\Delta t)) + o(\Delta t)$$

Analogously, for the same set $B$ we obtain for the construction in $W$

$$\mathbb{P}(Y(W, \Delta t) \cap B \neq \emptyset) = \Lambda_{[W]}([B]) \lambda(W) \Delta t + o(\Delta t).$$

The identity (4) implies that for $B \subset V$ and $\Delta t > 0$ holds

$$\mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset) = \mathbb{P}(Y(V, \Delta t) \cap V \cap B \neq \emptyset) = \mathbb{P}(Y(W, \Delta t) \cap B \neq \emptyset).$$

Consequently, for the limits

$$\Lambda_{[V]}([B]) \lambda(V) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(Y(V, \Delta t) \cap B \neq \emptyset)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{P}(Y(W, \Delta t) \cap B \neq \emptyset)}{\Delta t}$$

$$= \Lambda_{[W]}([B]) \lambda(W).$$

for all Borel sets $B \subset V \subset W$.

As the $\Lambda_{[V]}$ and $\Lambda_{[W]}$ are probability measures and $\Lambda_{[V]}([V]) = 1,$

$$\lambda(V) \Lambda_{[V]}([V] \setminus [B])$$

$$= \lambda(V) (1 - \Lambda_{[V]}([B]))$$

$$= \lambda(V) - \lambda(V) \Lambda_{[V]}([B])$$

$$= \lambda(W) \Lambda_{[W]}([V]) - \lambda(W) \Lambda_{[W]}([B])$$

$$= \lambda(W) \Lambda_{[W]}([V] \setminus [B]).$$
Note that for $B_1, B_2 \subset V$ holds $([V] \setminus [B_1]) \cap ([V] \setminus [B_2]) = [V] \setminus ([B_1] \cup [B_2])$, and therefore the set $D = \{ [V] \setminus [B] : B \subset V, B \text{ open} \}$ is a $\cap$-stable generating system for $\mathcal{F} \cap [V]$, see [6, Lemma 2.1.1] Thus, because both the measures $\lambda([V]) \Lambda_{[W]}([V])$ and $\lambda([W]) \Lambda_{[V]}([V])$ are finite and coincide on $D$, they are equal on $\mathcal{F} \cap [V]$.

For a sequence $W_1 \subset W_2 \subset \ldots$ with all $W_n \in \mathcal{F}$ and $\bigcup_{n=1}^{\infty} W_n = \mathbb{R}^d$ we define a set function $\nu$ on $[\mathcal{H}, \mathcal{F}]$ by

$$
\nu(H) = \lim_{n \to \infty} \lambda(W_n) \Lambda_{[W]}(H \cap [W_n]), \quad \forall H \in \mathcal{F}.
$$

(6)

Lemma 5.2. The function $\nu$ defined by (6) does not depend on the particular choice of the sequence $(W_n, n \in \mathbb{N})$, and $\nu$ is a measure on $[\mathcal{H}, \mathcal{F}]$.

Proof. Equation (5) and the non-negativity of $\lambda(W_n)$ and $\Lambda_{[W_n]}$ yield

$$
\lambda(W_{n+1}) \Lambda_{[W_{n+1}]}(H \cap [W_{n+1}]) \\
= \lambda(W_{n+1}) \Lambda_{[W_{n+1}]}(H \cap ([W_n] \cup ([W_{n+1}] \setminus [W_n]))) \\
= \lambda(W_{n+1}) \Lambda_{[W_{n+1}]}(H \cap [W_n]) + \lambda(W_{n+1}) \Lambda_{[W_{n+1}]}(H \cap ([W_{n+1}] \setminus [W_n])) \\
= \lambda(W_n) \Lambda_{[W_n]}(H \cap [W_n]) + \lambda(W_{n+1}) \Lambda_{[W_{n+1}]}(H \cap ([W_{n+1}] \setminus [W_n])) \\
\geq \lambda(W_n) \Lambda_{[W_n]}(H \cap [W_n]),
$$

i.e., the sequence $\lambda(W_n) \Lambda_{[W_n]}(H \cap [W_n])$ is monotonically increasing. Thus, the limit exists with the possibility of the limit being $\infty$.

In order to prove that this limit does not depend on the particular choice of the sequence of windows, consider two such monotone sequences $(W_n, n \in \mathbb{N})$ and $(W'_n, n \in \mathbb{N})$. Then for any $n \in \mathbb{N}$ there is a $m \in \mathbb{N}$ such that $W_n \subset W_m'$ and hence

$$
\lim_{n \to \infty} \lambda(W_n) \Lambda_{[W_n]}(H \cap [W_n]) \leq \lim_{n \to \infty} \lambda(W'_n) \Lambda_{[W'_n]}(H \cap [W'_n]).
$$

Exchanging the roles of $W_n$ and $W'_n$, it is seen that the opposite inequality holds as well, and hence both limits are equal.
Let $H_1, H_2, \ldots \in \mathcal{H}$ be pairwise disjoint sets from $\mathcal{H}$. Then
\[
\nu(\bigcup_{i=1}^{\infty} H_i) = \lim_{n \to \infty} \lambda(W_n)\Lambda_{[W_n]}\left((\bigcup_{i=1}^{\infty} H_i) \cap [W_n]\right)
\]
\[
= \lim_{n \to \infty} \lambda(W_n)\Lambda_{[W_n]}\left(\bigcup_{i=1}^{\infty} (H_i \cap [W_n])\right)
\]
\[
\equiv (a) \lim_{n \to \infty} \lambda(W_n)\sum_{i=1}^{\infty} \Lambda_{[W_n]}(H_i \cap [W_n])
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{\infty} \lambda(W_n)\Lambda_{[W_n]}(H_i \cap [W_n])
\]
\[
\equiv (b) \sum_{i=1}^{\infty} \lim_{n \to \infty} \lambda(W_n)\Lambda_{[W_n]}(H_i \cap [W_n])
\]
\[
= \sum_{i=1}^{\infty} \nu(H_i).
\]

Here, (a) is correct because the $\Lambda_{[W_n]}$ are measures themselves. Equation (b) is due to the monotone convergence theorem. Thus, the $\sigma$-additivity is proved and hence $\nu$ is a measure on $[\mathcal{H}, \mathcal{S}]$. 

**Corollary 5.3.** For all $W \in \mathcal{P}$ and $H \in \mathcal{S}$ with $H \subset [W]$ 
\[
\nu(H) = \lambda(W)\Lambda_{[W]}(H).
\]

**Proof.** If $W \in \mathcal{P}$ there is an $n_0$ such that $W \subset W_{n_0}$. Thus $H \subset [W] \subset [W_{n_0}]$ and hence for all $n \geq n_0$ equation (5) yields
\[
\lambda(W)\Lambda_{[W]}(H) = \lambda(W_n)\Lambda_{[W_n]}(H) = \nu([W]).
\]

**Proof.** (of Theorem 4.1) Putting $H = [W]$, this corollary and (5) immediately yield equations (2) and (3).

**Proof.** (of Theorem 4.2) According to Theorem 4.1 the spatial consistency yields the existence of a measure $\nu$ which controls $\lambda$ and $\{\Lambda_C : C \in \mathcal{P}\}$. Now it is sufficient to show that the homogeneity (in space) of the $Y(t)$ implies that $\nu$ is translation invariant. This can be done analogously to the proof of Lemma 5.1, choosing a Borel set $B$, a translation vector $x \in \mathbb{R}^d$ and then a window $W$ such that $B, B + x \subset W$. 

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