On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game

Masayuki Kumon
Risk Analysis Research Center
Institute of Statistical Mathematics

and

Akimichi Takemura
Graduate School of Information Science and Technology
University of Tokyo

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Abstract

In the framework of the game-theoretic probability of Shafer and Vovk (2001) it is of basic importance to construct an explicit strategy weakly forcing the strong law of large numbers (SLLN) in the bounded forecasting game. We present a simple finite-memory strategy based on the past average of Reality’s moves, which weakly forces the strong law of large numbers with the convergence rate of $O(\sqrt{\log n/n})$. Our proof is very simple compared to a corresponding measure-theoretic result of Azuma (1967) on bounded martingale differences and this illustrates effectiveness of game-theoretic approach. We also discuss one-sided protocols and extension of results to linear protocols in general dimension.

Keywords and phrases: Azuma-Hoeffding-Bennett inequality, capital process, game-theoretic probability, large deviation.

1 Introduction

The book by Shafer and Vovk (2001) established the whole new field of game-theoretic probability and finance. Their framework provides an attractive alternative foundation of probability theory. Compared to the conventional measure theoretic probability, the game theoretic probability treats the sets of measure zero in a very explicit way when proving various probabilistic laws, such as the strong law of large numbers. In a game-theoretic proof, we can explicitly describe the behavior of the paths on a set of measure zero, whereas in measure-theoretic proofs the sets of measure zero are often simply ignored.
This feature of game-theoretic probability is well illustrated in the explicit construction of Skeptic’s strategy forcing SLLN in Chapter 3 of Shafer and Vovk (2001).

However the strategy given in Chapter 3 of Shafer and Vovk (2001), which we call a mixture $\epsilon$-strategy in this paper, is not yet satisfactory, in the sense that it requires combination of infinite number of “accounts” and it needs to keep all the past moves of Reality in memory. We summarize their construction in Appendix in a somewhat more general form than in Chapter 3 of Shafer and Vovk (2001). In fact in Section 3.5 of their book, Shafer and Vovk pose the question of required memory for strategies forcing SLLN.

In this paper we prove that a very simple single strategy, based only on the past average of Reality’s moves is weakly forcing SLLN. Furthermore it weakly forces SLLN with the convergence rate of $O(\sqrt{\log n/n})$. In this sense, our result is a substantial improvement over the mixture $\epsilon$-strategy of Shafer and Vovk. Since $\epsilon$-strategies are used as essential building blocks for the “defensive forecasting” (Π), the performance of defensive forecasting might be improved by incorporating our simple strategy.

Our thinking was very much influenced by the detailed analysis by Takeuchi (9 and Chapter 5 of 8) of the optimum strategy of Skeptic in the games, which are favorable for Skeptic. We should also mention that the intuition behind our strategy is already discussed several times throughout the book by Shafer and Vovk (see e.g. Section 5.2). Our contribution is in proving that the strategy based on the past average of Reality’s moves is actually weakly forcing SLLN.

In this paper we only consider weakly forcing by a strategy. A strategy weakly forcing an event $E$ can be transformed to a strategy forcing $E$ as in Lemma 3.1 of Shafer and Vovk (2001). We do not present anything new for this step of the argument.

The organization of this paper is as follows. In Section 2 we formulate the bounded forecasting game and motivate the strategy based on the past average of Reality’s moves as an approximately optimum $\epsilon$-strategy. In Section 3 we prove that our strategy is weakly forcing SLLN with the convergence rate of $O(\sqrt{\log n/n})$. In Section 4 we consider one-sided protocol and prove that one-sided version of our strategy weakly forces one-sided SLLN with the same order. In Section 5 we treat a multivariate extension to linear protocols. In Appendix we give a summary of the mixture $\epsilon$-strategy in Chapter 3 of Shafer and Vovk (2001).

2 Approximately optimum single $\epsilon$-strategy for the bounded forecasting game

Consider the bounded forecasting game in Section 3.2 of Shafer and Vovk (2001).

**Bounding Forecasting Game**

**Protocol:**

$K_0 := 1.$

FOR $n = 1, 2, \ldots$:

Skeptic announces $M_n \in \mathbb{R}$.  

Reality announces \( x_n \in [-1, 1] \).

\[
K_n := K_{n-1} + M_n x_n.
\]

END FOR

For a fixed \( \epsilon, |\epsilon| < 1 \), the \( \epsilon \)-strategy sets \( M_n = \epsilon K_{n-1} \). Under this strategy Skeptic’s capital process \( K_n \) is written as \( K_n = \prod_{i=1}^n (1 + \epsilon x_i) \) or

\[
\log K_n = \sum_{i=1}^n \log(1 + \epsilon x_i).
\]

For sufficiently small \( |\epsilon| \), \( \log K_n \) is approximated as

\[
\log K_n \simeq \epsilon \sum_{i=1}^n x_i - \frac{1}{2} \epsilon^2 \sum_{i=1}^n x_i^2.
\]

The right-hand side is maximized by taking

\[
\epsilon = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2}.
\]

In particular in the fair-coin game, where \( x_n \) is restricted as \( x_n = \pm 1 \), approximately optimum \( \epsilon \) is given as

\[
\epsilon = \bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i.
\]

Actually, as shown by Takeuchi (9), it is easy to check that \( \epsilon = \bar{x}_n \) exactly maximizes \( \prod_{i=1}^n (1 + \epsilon x_i) \) for the case of the fair-coin game. Recently Kumon, Takemura and Takeuchi (2005) 5 give a detailed analysis of Bayesian strategies for the biased-coin games, which include the strategy \( \epsilon = \bar{x}_n \) as a special case.

Of course, the above approximately optimum \( \epsilon \) is chosen in hindsight, i.e., we can choose optimum \( \epsilon \) after seeing the moves \( x_1, \ldots, x_n \). However it suggests to choose \( M_n \) based on the past average \( \bar{x}_{n-1} \) of Reality’s moves. Therefore consider a strategy \( \mathcal{P} = \mathcal{P}^c \)

\[
M_n = c \bar{x}_{n-1} K_{n-1}.
\]

In the next section we prove that for \( 0 < c \leq 1/2 \) this strategy is weakly forcing SLLN. The restriction \( 0 < c \leq 1/2 \) is just for convenience for the proof and in Kumon, Takemura and Takeuchi (2005) we consider \( c = 1 \) for biased-coin games.

Compared to a single fixed \( \epsilon \)-strategy \( M_n = \epsilon K_{n-1} \) or the mixture \( \epsilon \)-strategy in Chapter 3 of Shafer and Vovk (2001), letting \( \epsilon = c \bar{x}_{n-1} \) depend on \( \bar{x}_{n-1} \) seems to be reasonable from the viewpoint of effectiveness of Skeptic’s strategy. The basic reason is that as \( \bar{x}_{n-1} \) deviates more from the origin, Skeptic should try to exploit this bias in Reality’s moves by betting a larger amount. Clearly this reasoning is shaky because for each round Skeptic has to move first and Reality can decide her move after seeing Skeptic’s move. However in the next section we show that the strategy in (1) is indeed weakly forcing SLLN with the convergence rate of \( O(\sqrt{\log n/n}) \).
3 Weakly forcing SLLN by past averages

In this section we prove the following result.

**Theorem 3.1** In the bounded forecasting game, if Skeptic uses the strategy (1) with 
\( 0 < c \leq \frac{1}{2} \), then 
\[ \limsup_n \sqrt{n} |\bar{x}_n| \leq \frac{1}{\sqrt{\log n} > 1.} \]

This theorem states that the strategy (1) weakly forces that \( \bar{x}_n \) converges to 0 with the convergence rate of \( O(\sqrt{\log n}/n) \). Therefore it is much stronger than the mixture \( \epsilon \)-strategy in Chapter 3 of Shafer and Vovk (2001), which only forces convergence to 0. A corresponding measure theoretic result was stated in Theorem 1 of Azuma (1967) as discussed in Remark 3.1 at the end of this section. The rest of this section is devoted to a proof of Theorem 3.1.

By comparing \( 1, 1/2, 1/3, \ldots \), and the integral of \( 1/x \) we have
\[
\log(n + 1) = \int_1^{n+1} \frac{1}{x} \, dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \int_1^n \frac{1}{x} \, dx = 1 + \log n.
\]

Next, by summing up the term
\[
\frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}
\]
from \( k = i \) to \( n \), we have
\[
\sum_{k=i}^{n} \frac{1}{(k-1)k} = \frac{1}{i-1} - \frac{1}{n}
\]
or
\[
\frac{1}{i-1} = \sum_{k=i}^{n} \frac{1}{(k-1)k} + \frac{1}{n}.
\]

Now consider the sum
\[
\sum_{i=2}^{n} \frac{i}{i-1} \bar{x}_i^2 = \sum_{i=2}^{n} \frac{1}{(i-1)i} (x_1 + \cdots + x_i)^2.
\]

When we expand the right-hand side, the coefficient of the term \( x_j x_k, j < k \), is given by
\[
2 \times \left( \frac{1}{(k-1)k} + \frac{1}{k(k+1)} + \cdots + \frac{1}{(n-1)n} \right) = 2 \times \left( \frac{1}{k-1} - \frac{1}{n} \right).
\]

We now consider the coefficient of \( x_j^2 \). In (1) we have the sum from \( i = 2 \) and we need to treat \( x_1 \) separately. The coefficient of \( x_1^2 \) is \( 1 - 1/n \) from (3). For \( j \geq 2 \), the coefficient of
\( x_j^2 \) is given by \( 1/(j - 1) - 1/n \), as in the case of the cross terms. Therefore

\[
\sum_{i=2}^{n} \frac{i}{i-1} \bar{x}_i^2 = 2 \sum_{1 \leq j < i \leq n} \left( \frac{1}{i-1} - \frac{1}{n} \right) x_j x_i + \left( 1 - \frac{1}{n} \right) x_1^2 + \sum_{i=2}^{n} \left( \frac{1}{i-1} - \frac{1}{n} \right) x_i^2 \\
= 2 \sum_{1 \leq j < i \leq n} \frac{1}{i-1} x_j x_i - \frac{2}{n} \sum_{1 \leq j < i \leq n} x_j x_i + x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2 - \frac{1}{n} \sum_{i=1}^{n} x_i^2 \\
= 2 \sum_{i=2}^{n} \bar{x}_{i-1} x_i - n \bar{x}_n^2 + x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2. \tag{5}
\]

Write \( \bar{x}_0 = 0 \). Then the first sum on the right-hand side can be written from \( i = 1 \). Under this notational convention \( \bar{x}_0 \) is rewritten as

\[
\sum_{i=1}^{n} \bar{x}_{i-1} x_i = \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \bar{x}_i^2 + \frac{n}{2} \bar{x}_n^2 - \frac{1}{2} \left( x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2 \right). \tag{6}
\]

Now the capital process \( K_n = K_n^P \) of \( \Pi \) is written as

\[
K_n = \prod_{i=1}^{n} (1 + c \bar{x}_{i-1} x_i).
\]

As in Chapter 3 of Shafer and Vovk (2001) we use

\[
\log(1 + t) \geq t - t^2, \quad |t| \leq 1/2.
\]

Then for \( 0 < c \leq 1/2 \) we have

\[
\log K_n = \sum_{i=1}^{n} \log (1 + c \bar{x}_{i-1} x_i) \geq c \sum_{i=1}^{n} \bar{x}_{i-1} x_i - c^2 \sum_{i=1}^{n} \bar{x}_{i-1}^2 x_i^2,
\]

and under the restriction \( |x_n| \leq 1, \forall n \), we can further bound \( \log K_n \) from below as

\[
\log K_n \geq c \sum_{i=1}^{n} \bar{x}_{i-1} x_i - c^2 \sum_{i=1}^{n} \bar{x}_{i-1}^2. \tag{7}
\]
By considering the restriction $|x_n| \leq 1$, (3) is bounded from below as

\[
\sum_{i=1}^{n} \bar{x}_{i-1} x_i \geq \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \bar{x}_i^2 + \frac{n}{2} \bar{x}_n^2 - \frac{1}{2} \left( 1 + \sum_{i=2}^{n} \frac{1}{i-1} \right)
\]

\[
\geq \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \bar{x}_i^2 + \frac{n}{2} \bar{x}_n^2 - \frac{1}{2} (2 + \log(n-1))
\]

\[
\geq \frac{1}{2} \sum_{i=2}^{n} \bar{x}_i^2 + \frac{n}{2} \bar{x}_n^2 - \frac{1}{2} (2 + \log(n-1))
\]

\[
\geq \frac{1}{2} \sum_{i=1}^{n} \bar{x}_i^2 + \frac{n}{2} \bar{x}_n^2 - \frac{1}{2} (3 + \log(n-1)).
\]

Since $c \leq 1/2$, substituting this into (7) yields

\[
\log K_n \geq c \left( \frac{1}{2} - c \right) \sum_{i=1}^{n} \bar{x}_{i-1} + c \frac{n}{2} \bar{x}_n^2 - \frac{c}{2} (3 + \log(n-1))
\]

\[
\geq c \frac{n}{2} \bar{x}_n^2 - \frac{c}{2} (3 + \log(n-1))
\]

\[
\geq \frac{c}{2} (n \bar{x}_n^2 - \log n) - \frac{3}{2} c
\]

\[
= \frac{c}{2} \log \left( \frac{n \bar{x}_n^2}{\log n} - 1 \right) - \frac{3}{2} c.
\]

Now if $\limsup_n \sqrt{n} |\bar{x}_n|/\sqrt{\log n} > 1$, then $\limsup_n \log K_n = +\infty$ because $\log n \uparrow \infty$. This proves the theorem.

**Remark 3.1** In the framework of the conventional measure theoretic probability, a strong law of large numbers analogous to Theorem 3.1 can be proved using Azuma-Hoeffding-Bennett inequality (Appendix A.7 of Vovk, Gammerman and Shafer (2005), Section 2.4 of Dembo and Zeitouni (1998), Azuma (1967), Hoeffding (1963), Bennett (1962)). Let $X_1, X_2, \ldots$ be a sequence of martingale differences such that $|X_n| \leq 1, \forall n$. Then for any $\epsilon > 0$

\[
P(|\bar{X}_n| \geq \epsilon) \leq 2 \exp(-n\epsilon^2/2).
\]

Fix an arbitrary $\alpha > 1/2$. Then for any $\epsilon > 0$

\[
\sum_n P(|\bar{X}_n| \geq \epsilon (\log n)^{\alpha}/\sqrt{n}) \leq \sum_n \exp\left(-\frac{\epsilon^2}{2} (\log n)^{2\alpha}\right) < \infty.
\]

Therefore by Borel-Cantelli $\sqrt{n} |\bar{X}_n|/(\log n)^{\alpha} \to 0$ almost surely. Actually Theorem 1 of Azuma (1967) states the following stronger result

\[
\limsup_{n \to \infty} \frac{\sqrt{n} \bar{x}_n}{\sqrt{\log n}} \leq \sqrt{2} \quad \text{a.s.}
\]

(8)
Although our Theorem 3.1 is better in the constant factor of $\sqrt{2}$, Azuma’s result (8) and our result (9) are virtually the same. However we want to emphasize that our game theoretic proof requires much less mathematical background than the measure theoretic proof. Also see the factor of $\sqrt{3/2}$ in the one-sided version of our result in Theorem 4.1 below.

4 One-sided protocol

In this section we consider one-sided bounded forecasting game where $M_n$ is restricted to be nonnegative ($M_n \geq 0$), i.e. Skeptic is only allowed to buy tickets. We also consider the restriction $M_n \leq 0$. In Chapter 3 of Shaver and Vovk, weak forcing of SLLN is proved by combining positive and negative one-sided strategies, whereas in the previous section we proved that a single strategy $\mathcal{P} = \mathcal{P}^c$ weakly forces SLLN. Therefore it is of interest to investigate whether one-sided version of our strategy weakly forces one-sided SLLN. We adopt the same notations as Section 5 of Kumon, Takemura and Takeuchi (2005), where one-sided protocols for biased-coin games are studied.

For the positive one-sided case consider the strategy $\mathcal{P}^+$ with

$$M_n = c\bar{x}_{n-1}^+ K_{n-1}, \quad \bar{x}_{n-1}^+ = \max(\bar{x}_{n-1}, 0).$$

Similarly we consider negative one-sided strategy $\mathcal{P}^-$ with $M_n = -c\bar{x}_{n-1}^- K_{n-1}, \quad \bar{x}_{n-1}^- = \max(-\bar{x}_{n-1}, 0)$.

For these protocols we have the following theorem.

**Theorem 4.1** If Skeptic uses the strategy $\mathcal{P}^+$ with $0 < c \leq 1/2$, then $\limsup_n K_n = \infty$ for each path $\xi = x_1 x_2 \ldots$ of Reality’s moves such that

$$\limsup_n \frac{\sqrt{n} x_n}{\sqrt{\log n}} > \sqrt{\frac{3}{2}}.$$ 

Similarly if Skeptic uses the strategy $\mathcal{P}^-$ with $0 < c \leq 1/2$, then $\limsup_n K_n = \infty$ for each path $\xi = x_1 x_2 \ldots$ of Reality’s moves such that $\liminf_n \sqrt{n} x_n / \sqrt{\log n} < -\sqrt{3/2}$.

The rest of this section is devoted to a proof of this theorem for $\mathcal{P}^+$. If $\bar{x}_n$ is eventually all nonnegative, then the behavior of the capital process $K_n^P$ and $K_n^{P^+}$ are asymptotically equivalent except for a constant factor reflecting some initial segment of Reality’s path $\xi$. Then the theorem follows from Theorem 3.1. On the other hand if $\bar{x}_n$ is eventually negative, then $K_n^{P^+}$ stays constant and Theorem 4.1 holds trivially. Therefore we only need to consider the case that $\bar{x}_n$ changes sign infinitely often. Note that at time $n$ when $\bar{x}_n$ changes the sign, the overshoot is bounded as

$$|\bar{x}_n| \leq 1/n.$$
We consider capital process after a sufficiently large time $n_0$ such that $\bar{x}_{n_0} \simeq 0$, and proceed to divide the sequence $\{\bar{x}_n\}$ into the following two types of blocks. For $n_0 \leq k \leq l - 1$, consider a block $\{k, \ldots, l - 1\}$. We call it a nonnegative block if

$$\bar{x}_{k-1} < 0, \bar{x}_k \geq 0, \bar{x}_{k+1} \geq 0, \ldots, \bar{x}_{l-1} \geq 0, \bar{x}_l < 0.$$ 

Similarly we call it a negative block if

$$\bar{x}_{k-1} \geq 0, \bar{x}_k < 0, \bar{x}_{k+1} < 0, \ldots, \bar{x}_{l-1} < 0, \bar{x}_l \geq 0.$$ 

By definition, negative and nonnegative blocks are alternating.

For a nonnegative block $K_l^P = K_k^P \prod_{i=k+1}^{l-1} (1 + c\bar{x}_{i-1}x_i)$ (9)

whereas for a negative block $K_l^P = K_k^P$. Taking the logarithm of (9) we have

$$\log K_l^P - \log K_k^P = \sum_{i=k+1}^{l} \log(1 + c\bar{x}_{i-1}x_i)$$

$$\geq c \sum_{i=k+1}^{l} \bar{x}_{i-1}x_i - c^2 \sum_{i=k+1}^{l} \bar{x}_{i-1}^2x_i^2$$

$$\geq c \sum_{i=k+1}^{l} \bar{x}_{i-1}x_i - c^2 \sum_{i=k+1}^{l} \bar{x}_{i-1}^2.$$ 

From (6) it follows

$$\sum_{i=k+1}^{l} \bar{x}_{i-1}x_i = \frac{1}{2} \sum_{i=k+1}^{l} \frac{i}{i-1} \bar{x}_{i-1}^2 + \frac{l}{2} \bar{x}_1^2 - \frac{k}{2} \bar{x}_k^2 - \frac{1}{2} \sum_{i=k+1}^{l} \frac{1}{i-1} x_i^2$$

$$\geq \frac{1}{2} \sum_{i=k+1}^{l} \frac{i}{i-1} \bar{x}_{i-1}^2 - \frac{1}{2} \frac{l}{k} - \frac{1}{2} \left( \frac{1}{k} \log \frac{l}{k} \right).$$

In the above, we used the approximation formula

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} \leq \int_m^n \frac{dx}{x} = \log \frac{n}{m} + \frac{1}{m}.$$ 

Thus we obtain

$$\log K_l^P - \log K_k^P \geq -c \frac{1}{k} - c \log \frac{l}{k}. \quad (10)$$

Now starting at $n_0$, we consider adding the right-hand side of (10) for nonnegative blocks and $0 = \log K_l^P - \log K_k^P$ for negative blocks. Then after passing sufficiently
many blocks, $\log K^P_\epsilon - \log K^P_k$ behave as (11) during half number of the entire blocks. Therefore at the beginning $n_k$ of the last nonnegative block, we have

$$\log K^P_{n_k} - \log K^P_{n_0} \geq -\frac{c}{2} \log \frac{n_k}{n_0} - \frac{c}{4} \log \frac{n_k}{n_0} + o(1) = -\frac{3c}{4} \log \frac{n_k}{n_0} + o(1). \quad (11)$$

To finish the proof of Theorem 4.1 let $n$ be in a middle of the last nonnegative block $\{n_k, \ldots, n_l\}$. Then as above, we have

$$\log K^P_n - \log K^P_{n_k} \geq \frac{cn}{2} \bar{x}^2 - \frac{c}{2} \log n - \frac{c}{4} \log n_k + \frac{3c}{4} \log n_0 + o(1). \quad (12)$$

Adding (11) and (12) we have

$$\log K^P_n - \log K^P_{n_0} \geq \frac{cn}{2} \bar{x}^2 - \frac{c}{2} \log n - \frac{c}{4} \log n_k + \frac{3c}{4} \log n_0 + o(1).$$

Noting that $n_k = O(n)$, we derive

$$\log K^P_n \geq \frac{cn}{2} \bar{x}^2 - \frac{3c}{4} \log n + O(1) = \frac{c}{2} \log n \left( \frac{n \bar{x}^2}{\log n} - \frac{3}{2} \right) + O(1).$$

This completes the proof of Theorem 4.1.

## 5 Multivariate linear protocol

In this section we generalize Theorem 5.1 to multivariate linear protocols. See Section 3 of Vovk, Nouretdinov, Takeamura and Shafer (2005) [12] and Section 6 of Takeamura and Suzuki (2005) [7] for discussions of linear protocols. Since the following generalization works for any dimension, including the case of infinite dimension, we assume that Skeptic and Reality choose elements from a Hilbert space $H$. The inner product of $x, y \in H$ is denoted by $x \cdot y$ and the norm of $x \in H$ denoted by $\|x\| = (x \cdot x)^{1/2}$. Actually we do not specifically use properties of infinite dimensional space and readers may just think of $H$ as a finite dimensional Euclidean space $\mathbb{R}^m$. For example spectral resolution below just corresponds to the spectral decomposition of a nonnegative definite matrix.

Let $\mathcal{X} \subset H$ denote the move space of Reality, and assume that $\mathcal{X}$ is bounded. Then by rescaling we can say without loss of generality that $\mathcal{X}$ is contained in the unit ball

$$\mathcal{X} \subset \{ x \in H \mid \|x\| \leq 1 \}.$$

In this case the closed convex hull $\overline{\text{co}}(\mathcal{X})$ of $\mathcal{X}$ is contained in the unit ball. As in [7] we also assume that the origin 0 belongs to $\overline{\text{co}}(\mathcal{X})$. Note also that the average $\bar{x}_n$ of Reality’s moves always belongs to $\overline{\text{co}}(\mathcal{X})$ and hence $\|\bar{x}_n\| \leq 1$. In order to be clear, we write out our game for multivariate linear protocol.
Bounded Linear Protocol Game in General Dimension

Protocol:
\( K_0 := 1. \)
FOR \( n = 1, 2, \ldots \):
Skeptic announces \( M_n \in H. \)
Reality announces \( x_n \in X. \)
\( K_n := K_{n-1} + M_n \cdot x_n. \)
END FOR

As a natural multivariate generalization of the strategy \( P^c \) given by (1), we consider the strategy \( P = P^A \)

\[ M_n = A\bar{x}_{n-1}K_{n-1}, \quad (13) \]

where \( A \) is a self-adjoint operator in \( H. \) Then \( A \) has the spectral resolution

\[ A = \int_{-\infty}^{\infty} \lambda E(d\lambda), \quad (14) \]

where \( E \) denotes the real spectral measure of \( A, \) or the resolution of the identity corresponding to \( A. \) Let \( \sigma(A) \) denote the spectrum of \( A \) (i.e. the support of \( E \)) and let

\[ c_0 = \inf\{ \lambda \mid \lambda \in \sigma(A) \}, \quad c_1 = \sup\{ \lambda \mid \lambda \in \sigma(A) \}. \]

In the finite dimensional case, \( c_0 \) and \( c_1 \) correspond to the smallest and the largest eigenvalue of the matrix \( A, \) respectively.

Now we have the following generalization of Theorem 3.1

**Theorem 5.1** In the bounded linear protocol game in general dimension, if Skeptic uses the strategy (13) with \( 0 < c_0 \leq c_1 \leq 1/2. \) Then \( \limsup_n K_n = \infty \) for each path \( \xi = x_1x_2 \ldots \) of Reality’s moves such that

\[ \limsup_n \sqrt{n} \frac{\|\bar{x}_n\|}{\sqrt{\log n}} > \sqrt{\frac{c_1}{c_0}}. \]

**Proof:** In the expression

\[ K_n = \prod_{i=1}^{n} (1 + A\bar{x}_{i-1} \cdot x_i), \quad (15) \]

we have

\[ A\bar{x}_{i-1} \cdot x_i = \int_{c_0}^{c_1} \lambda(E(d\lambda)\bar{x}_{i-1} \cdot x_i) = \bar{y}_{i-1} \cdot y_i, \quad (16) \]

with

\[ \bar{y}_{i-1} = \int_{c_0}^{c_1} \sqrt{\lambda}E(d\lambda)\bar{x}_{i-1}, \quad y_i = \int_{c_0}^{c_1} \sqrt{\lambda}E(d\lambda)x_i. \]
By the Schwarz’s inequality,
\[ |A\bar{x}_{n-1} \cdot x_i| \leq \|\bar{y}_{i-1}\|\|y_i\| \leq c_1 \|\bar{x}_{i-1}\|\|x_i\| \leq \frac{1}{2}. \]
Hence as in (17), we can bound \(\log K_n\) from below as
\[ \log K_n \geq \sum_{i=1}^{n} \bar{y}_{i-1} \cdot y_i - c_1 \sum_{i=1}^{n} \|\bar{y}_{i-1}\|^2. \tag{17} \]
The first term also can be expressed as in (6), and it is bounded from below as follows.
\[ \sum_{i=1}^{n} \bar{y}_{i-1} \cdot y_i = \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \|\bar{y}_i\|^2 + \frac{n}{2} \|\bar{y}_n\|^2 - \frac{1}{2} \left( \|y_1\|^2 + \sum_{i=2}^{n} \frac{1}{i-1} \|y_i\|^2 \right) \geq \frac{1}{2} \sum_{i=1}^{n} \|\bar{y}_i\|^2 + \frac{n}{2} \|\bar{y}_n\|^2 - \frac{c_1}{2} (3 + \log(n-1)). \tag{18} \]
Combining (17) and (18), we have
\[ \log K_n \geq \left( \frac{1}{2} - c_1 \right) \sum_{i=1}^{n} \|\bar{y}_{i-1}\|^2 + \frac{n}{2} \|\bar{y}_n\|^2 - \frac{c_1}{2} (3 + \log(n-1)) \geq \frac{n}{2} \|\bar{y}_n\|^2 - \frac{c_1}{2} (3 + \log(n-1)) \geq \frac{n}{2} \|\bar{y}_n\|^2 - \frac{c_1}{2} \log n - \frac{3}{2} c_1 = \frac{c_1}{2} \log n \left( \frac{n \|\bar{y}_n\|^2}{c_1 \log n} - 1 \right) - \frac{3}{2} c_1. \tag{19} \]
It follows that if \(\limsup_n \sqrt{n} \|\bar{y}_n\|/\sqrt{c_1 \log n} > 1\), then \(\limsup_n \log K_n = +\infty\). Now the theorem follows from \(c_0 \|\bar{x}_n\|^2 \leq \|\bar{y}_n\|^2\).

Remark 5.1 Suppose that \(\{A_m\}\) is a sequence of positive definite degenerate self-adjoint operators with finite dimensional ranges \(R_{A_m}(H) \subsetneq R_{A_{m+1}}(H) \cdots\), and with the supports (ranges of eigenvalues) \(\sigma(A_m) \subsetneq \sigma(A_{m+1}) \cdots \subset (0, 1/2]\). Also suppose that \(A_\infty\) is a compact operator with infinite dimensional range \(R_{A_\infty}(H) \subset H\), and \(A_\infty\) is obtained in the limit \(A_m \to A_\infty\), \(m \to \infty\). Then \(c_0(A_m) \to c_0(A_\infty) = 0\), \(m \to \infty\), so that in Theorem 2.1 \(\sqrt{c_1(A_m)/c_0(A_m)} \to \sqrt{c_1(A_\infty)/c_0(A_\infty)} = \infty\), implying that the strategy \(\mathcal{P}_{A_\infty}\) cannot weakly force SLLN with any rate. This phenomenon reflects one feature that the dimension of Skeptic’s move space is related to the effective weakly force of his strategy. It is a subject we will treat in the forthcoming paper.
A Summary of mixture $\epsilon$-strategy in Chapter 3 of Shafer and Vovk (2001)

Here we summarize the mixture $\epsilon$-strategy in Chapter 3 of Shafer and Vovk (2001) for the bounded forecasting game in such a way that its resource requirement (computational time and memory) becomes explicit.

For a single fixed $\epsilon$-strategy $P^\epsilon$, which sets $M_n = \epsilon K_{n-1}$, the capital process is given as $K_n^\epsilon(x_1 \ldots x_n) = \prod_{i=1}^{n}(1 + \epsilon x_i)$. Shafer and Vovk combine $P^\epsilon$ for many values of $\epsilon$. Let $1/2 \geq \epsilon_1 > \epsilon_2 > \cdots > 0$ be a sequence of positive numbers converging to 0. Write $\epsilon_{-k} = -\epsilon_k$, $k = 1, 2, \ldots$. Furthermore let $\{p_k\}_{i=0, \pm 1, \pm 2, \ldots}$ be a probability distribution on the set of integers $\mathbb{Z}$, such that $p_0 = 0$ and $p_{-k} = p_k$ (symmetric). Actually $p_k$ is the initial amount (out of 1 dollar) put into the account with the strategy $P^\epsilon_k$, $k = \pm 1, \pm 2, \ldots$. For example we could take

$$\epsilon_k = \frac{1}{2|k|}, \quad p_k = \frac{1}{2^{|k|+1}}, \quad k = \pm 1, \pm 2, \ldots,$$

as is done in Chapter 3 of Shafer and Vovk (2001). However it is more convenient here to leave $\epsilon_k$ and $p_k$ to be general. Then the mixture strategy weakly forcing SLLN is given by the weighted average of the strategies $P^\epsilon_k$, $k = \pm 1, \pm 2, \ldots$, with respect to the weights $\{p_k\}_{k=0, \pm 1, \pm 2, \ldots}$, namely

$$P^* = \sum_{k=-\infty}^{\infty} p_k P^\epsilon_k.$$

Consider the value $M^*_n$ of $P^*$. It is written as

$$M^*_n = \sum_{k=-\infty}^{\infty} p_k e_k \prod_{i=1}^{n-1}(1 + \epsilon_k x_i).$$

Now introduce the elementary symmetric functions $e_{n,0}, e_{n,1}, \ldots, e_{n,n}$ of the numbers $x_1, \ldots, x_n$ as

$$e_{n,0} = 1, \quad e_{n,1} = \sum_{i=1}^{n} x_i, \quad e_{n,2} = \sum_{1 \leq i < j \leq n} x_i x_j, \quad \ldots, \quad e_{n,n} = \prod_{i=1}^{n} x_i.$$

Recall that the set of values $\{x_1, \ldots, x_n\}$ and the set of values $\{e_{n,1}, \ldots, e_{n,n}\}$ are in one-to-one relationship. Therefore computing all the values of the elementary symmetric functions of $\{x_1, \ldots, x_n\}$ is computationally equivalent to keep all the values of $\{x_1, \ldots, x_n\}$. Using elementary symmetric functions, $M^*_n$ is written as

$$M^*_n = \sum_{k=-\infty}^{\infty} p_k \sum_{i=0}^{n-1} \epsilon_k^{i+1} e_{n-1,i} = \sum_{i=0}^{n-1} e_{n-1,i} \sum_{k=-\infty}^{\infty} p_k \epsilon_k^{i+1}.$$

Let $F$ denote the discrete probability distribution on $[-1/2, 1/2]$ with

$$F(\{\epsilon_k\}) = p_k, \quad k = \pm 1, \pm 2, \ldots,$$
and let
\[ \mu_m = E_F(X^m) = \sum_{k=\infty}^{\infty} p_k \epsilon_k^m = \int_{-1/2}^{1/2} x^m F(dx) \]
denote the \( m \)-th moment of \( F \). Then (21) is written as
\[ M_n^* = \sum_{i=0}^{n-1} \mu_{i+1} e_{n-1,i}. \] (22)

At this point it becomes clear that we can take an arbitrary probability distribution \( F \) on \([-1/2, 1/2]\) provided that \( F \) is not degenerate at 0 and 0 is the point of support, i.e., for all \( \epsilon > 0 \)
\[ F(\epsilon) - F(-\epsilon) > 0. \]
\( M_n^* \) in (22) for such an \( F \) is a strategy weakly forcing SLLN. Now the simplest \( F \) seems to be the uniform distribution on \([-1/2, 1/2]\). Actually (20) is in a sense close to the uniform distribution. Then
\[ \mu_m = \int_{-1/2}^{1/2} x^m dx = \begin{cases} \frac{1}{m+1} \frac{1}{2^m} & m \text{ : even} \\ 0 & m \text{ : odd.} \end{cases} \]

In this case there is virtually no computational resource is needed to compute \( \mu_m \), since it is explicitly given. Furthermore the strategy weakly forcing SLLN based on the uniform distribution is given by
\[ M_n^* = \sum_{i=0 \text{ : odd}}^{n-1} \frac{1}{i+1} \frac{1}{2^{i+1}} e_{n-1,i}. \]

In order to compute the \( M_n^* \) we need all the values of the elementary symmetric functions \( e_{n-1,i}, i = 1, \ldots, n-1 \), which is equivalent to keeping all the values of \( \{x_1, \ldots, x_{n-1}\} \) as discussed above. Therefore the mixture \( \epsilon \)-strategy needs a memory of size proportional to \( n \) at time \( n \), whereas our strategy in (11) only needs to keep track of the values of \( \bar{x}_{n-1} \) and \( K_{n-1} \).

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