Sasaki-Einstein Twist of Kerr-AdS Black Holes

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Abstract

We consider Kerr-AdS black holes with equal angular momenta in arbitrary odd spacetime dimensions $\geq 5$. Twisting the Killing vector fields of the black holes, we reproduce the compact Sasaki-Einstein manifolds constructed by Gauntlett, Martelli, Sparks and Waldram. We also discuss an implication of the twist in string theory and M-theory.
Kerr-AdS black holes are characterized by mass, angular momenta and cosmological constant. In spacetime dimension $d$, the number of angular momenta is equal to the rank of the rotation group $\text{SO}(d - 1)$. The five-dimensional Kerr-AdS black holes with two angular momenta were constructed in [1], and recently the general form in arbitrary dimension was found by using the Kerr-Schild ansatz [2].

On the other hand, the Wick rotation of the black holes leads to Riemannian metrics. However, the metrics in general do not extend smoothly to compact manifolds. In [3][4][2], it was shown that this can be achieved by taking a certain limit (Page limit) which enhances the isometry of the metric. Indeed, the infinite series of Einstein metrics on compact manifolds were explicitly constructed [4][2], and analyzed in detail in [5].

Recently, infinite series of Sasaki-Einstein metrics on compact manifolds were presented in [6][7]. It is expected that these metrics can be related to some Kerr-AdS black holes by a certain limit. Our aim in this letter is to clarify the relation between them.

We begin with the $(2n + 3)$-dimensional Kerr-AdS black hole with a negative cosmological constant $(2n + 2)\lambda < 0 \ (n \geq 1)$ as follows [1][2]:

$$
\hat{g} = -\frac{\hat{W}(r)}{\hat{b}(r)}dt^2 + \frac{dr^2}{\hat{W}(r)} + r^2 \left(g_{\mathbb{C}P^n} + \hat{b}(r) \left(d\psi + A + \hat{f}(r)dt\right)^2\right),
$$

(1)

where

$$
\hat{W}(r) = 1 - \lambda r^2 - \frac{2M(\delta^2 + \lambda J^2)}{r^{2n}} + \frac{2MJ^2}{r^{2n+2}} = (1 - \lambda r^2)\hat{b}(r) - \frac{2M\delta^2}{r^{2n}},
$$

$$
\hat{b}(r) = 1 + \frac{2MJ^2}{r^{2n+2}},
$$

$$
\hat{f}(r) = \frac{1}{J} \left(1 - \frac{\delta}{\hat{b}(r)}\right).
$$

(2)

The metric $g_{\mathbb{C}P^n}$ is the Fubini-Study metric on $\mathbb{C}P^n$ with a normalization $\text{Ric}_{\mathbb{C}P^n} = (2n + 2)g_{\mathbb{C}P^n}$, and the 1-form $A$ is the U(1) connection associated with the Kähler form $dA/2$ on $g_{\mathbb{C}P^n}$. The black hole is parameterized by the mass $M$, the angular momentum $J$ and a trivial parameter $\delta$. The parameter $\delta$ is related to the parameter $\beta$ introduced in [8] as $\delta = -\lambda J^2 \beta + 1$. This metric is a special case that all angular momenta are set to be equal.

The metric (1) reduces to the AdS metric at $r \to \infty$ because the metric of the circle
bundle over \( \mathbb{C}P^n \) tends to the standard metric of \( S^{2n+1} \). A horizon appears for sufficiently small \( J \). If we set \( \delta^2 = -\lambda J^2 \), the \( \bar{W}(r) \) does not have positive roots so that the curvature singularity at \( r = 0 \) is not screened by the horizon, and so is naked. As will be seen below, in the Euclidean picture this solution is shown to be related to the Sasaki-Einstein metrics.

The Euclidean Einstein metric with a positive cosmological constant \( (2n+2)\lambda > 0 \) is extracted from the Kerr-AdS black hole \( (1) \) by the substitution \( t \to i\tau \) and \( J \to iJ \):

\[
g = \frac{W(r)}{b(r)} d\tau^2 + \frac{dr^2}{W(r)} + r^2 \left( g_{\mathbb{C}P^n} + b(r) \left( d\psi + A + f(r) d\tau \right)^2 \right),
\]

where

\[
W(r) = 1 - \lambda r^2 - \frac{2M(\delta^2 - \lambda J^2)}{r^{2n}} - \frac{2MJ^2}{r^{2n+2}} = (1 - \lambda r^2)b(r) - \frac{2M\delta^2}{r^{2n}},
\]

\[
b(r) = 1 - \frac{2MJ^2}{r^{2n+2}},
\]

\[
f(r) = \frac{1}{J} \left( 1 - \frac{\delta}{b(r)} \right).
\]

The metric has the isometry \( SU(n+1) \times U(1) \times \mathbb{R} \). The generator of \( U(1) \times \mathbb{R} \) is given by \((\partial_{\psi}, \partial_{\tau})\). It is easy to see that under the Page limit and a special choice of the parameters \[3\, [4] \, [2]\] this metric reduces to a homogeneous Einstein metric with the isometry \( SU(n+1) \times SU(2) \times U(1) \) on a circle bundle over \( \mathbb{C}P^n \times S^2 \). Indeed, the metric can be written as

\[
g_0 = \frac{1}{W_0} \left( d\chi^2 + \sin^2 \chi d\eta^2 \right) + \frac{r_0^2}{(2(n+1) - (n+2)b_0)} \left( g_{\mathbb{C}P^n} + b_0 (d\psi + A + \frac{k}{2} \cos \chi d\eta)^2 \right),
\]

where

\[
W_0 = \frac{2\lambda(n+1) (2(n+1) - (n+2)b_0)}{n+1 - b_0},
\]

\[
r_0^2 = \frac{n+1 - b_0}{\lambda(n+1)},
\]

\[
k = \pm \frac{2 \sqrt{(n+1)b_0(1-b_0)}}{b_0(2(n+1) - (n+2)b_0)} \in \mathbb{Z},
\]

and \( b_0 \) is a constant with \( 0 < b_0 < 1 \). In the case of \( n = 1 \), this reproduces the metric given in Theorem 2 of \[3\]. Further, for \( k = 1 \), it gives the homogeneous Sasaki-Einstein manifold \( T^{1,1} \).

\[5\]If we replace the Fubini-Study \( \mathbb{C}P^n \) by an arbitrary Einstein-Kähler manifold with the same scalar curvature, we obtain another Kerr black hole with different asymptotic behavior.
We now transform the metric to inhomogeneous Sasaki-Einstein metrics on circle bundles over \( \mathbb{C}P^n \times S^2 \) (\( S^2 \) bundle over \( \mathbb{C}P^n \)) presented in [6] [7].

First, we set \( \delta^2 = \lambda J^2 \), then the coefficient of \( 1/r^{2n} \) in \( W \) vanishes.

Twisting the \( U(1) \times \mathbb{R} \) coordinates as
\[
\tilde{\tau} = \tau + J\psi, \tag{7}
\]
we obtain
\[
g = g_K + (Jd\psi - \sigma)^2, \tag{8}
\]
where the metric \( g_K \) is a local positive Kähler-Einstein metric in dimension \( 2n + 2 \),
\[
g_K = \frac{dr^2}{W(r)} + r^2 g_{\mathbb{C}P^n} + r^2 W(r) \left( \frac{d\tilde{\tau}}{J} + A \right)^2, \tag{9}
\]
and the Kähler form of \( g_K \) is given by \( d\sigma/2\sqrt{\lambda} \),
\[
\sigma = \left( 1 - \frac{\sqrt{\lambda}r^2}{J} \right) d\tilde{\tau} - \sqrt{\lambda}r^2 A. \tag{10}
\]
Thus, as is well known, the metric \( g \) in (8) turns out to be locally Sasaki-Einstein. If we write the metric \( g \) by the coordinates \((\tau, \tilde{\tau})\), instead of \((\tau, \psi)\) or \((\tilde{\tau}, \psi)\), we can eliminate the parameter \( \delta \) after rescaling \( M\delta^2 \rightarrow M \) and \( J\delta^{-1} \rightarrow J \) \(^4\).

On the other hand, twisting the coordinates as
\[
\tilde{\psi} = \psi - \frac{c}{J} \tilde{\tau}, \tag{11}
\]
we have
\[
g = g_C + \omega(r) \left( d\tilde{\tau} + f(r)(d\tilde{\psi} + A) \right)^2, \tag{12}
\]
where
\[
g_C = \frac{dr^2}{W(r)} + r^2 g_{\mathbb{C}P^n} + q(r)(d\tilde{\psi} + A)^2, \tag{13}
\]
and the components are given by
\[
\omega(r) = k^2 r^2 W(r) + (k\sqrt{\lambda}r^2 - 1)^2,
\]
\[
f(r) = \frac{r^2}{\omega(r)} \left( kW(r) + \sqrt{\lambda}(k\sqrt{\lambda}r^2 - 1) \right),
\]
\[
q(r) = \frac{r^2 W(r)}{\omega(r)} \tag{14}
\]
\(^4\)The authors are grateful to Gary Gibbons, Malcolm Perry and Chris Pope for this remark.
with \( k = (c + 1)/J \). The metric \( g_C \) is conformally Kähler \([7]\).

The singularities coming from the roots \( r = r_i \) of \( W = 0 \) can be resolved by the restriction of the range of the angle \( \tilde{\psi} \); putting \( R^2 = 4(r - r_i)/W'(r_i) \) one has in the limit \( r \to r_i \),

\[
\frac{dr^2}{W(r)^2} + q(r)d\tilde{\psi}^2 \to dR^2 + K_i^2 R^2 d\tilde{\psi}^2,
\]

where

\[
K_i = \frac{(n + 2)\lambda r_i^2 - (n + 1)}{k\sqrt{\lambda r_i^2 - 1}}.
\]

If we set \( \lambda(n + 2)/(n + 1) = k\sqrt{\lambda} \), that is,

\[
c = \frac{n + 2}{n + 1}\sqrt{\lambda}J - 1,
\]

then \( K_i \) is independent of \( r_i \). Under a suitable condition on the parameter \( MJ^2 \), the corresponding metric \( g \) has an \( SU(n + 1) \times U(1) \times U(1) \) symmetry, and it reproduces a Sasaki-Einstein metric on a compact manifold given by Gauntlett et al. in \([6][7]\).

We shall comment on the implication of our method in the higher dimensional context. As explained above, the higher dimensional backgrounds are related each other as follows:

\[
\begin{align*}
\text{AdS}_p \times S^q & \quad \text{AdS}_p \times M_{SE}^q \\
\uparrow \text{Wick rot.} & \quad \uparrow \text{Wick rot. and } \delta^2 = \lambda J^2 \\
H^p \times dS_q & \quad H^p \times M_{dS}^q \\
\downarrow \text{cosmo.} & \quad \uparrow \text{cosmo.} \\
S^p \times \text{AdS}_q & \quad S^p \times M_{AdS}^q
\end{align*}
\]

where \((p, q) = (5, 5), (4, 7)\), and a \( p \)-form flux is associated with them. The left hand side shows that the maximally supersymmetric backgrounds are related to each other. Under the Wick rotation and a sign change of the cosmological constant, the \( \text{AdS}_5 \times S^5 \) solution in the type-IIB string theory is mapped to itself, while \( \text{AdS}_4 \times S^7 \) becomes \( S^4 \times \text{AdS}_7 \). In the right hand side, we have generalized \( S^q \) to \( M_{SE}^q \), where \( M_{SE}^q \) stands for the \( q \)-dimensional Sasaki-Einstein manifold specified by \([12]\). This shows the relation between \( \text{AdS}_p \times M_{SE}^q \) and \( S^p \times M_{AdS}^q \), where \( M_{AdS}^q \) means the \( q \)-dimensional Kerr-AdS black hole.

It is known that the former solution admits supersymmetry due to the Sasaki-Einstein
structure of $M^q_{SE}$, and that string/M-theory on $\text{AdS}_p \times M^q_{SE}$ is dual to supersymmetric Yang-Mills theory in $(p-1)$-dimensions. Though the condition $\delta^2 = \lambda J^2$ implies a naked singularity for $M^q_{AdS}$ and cannot be imposed consistently on $M^q_{dS}$, where $M^q_{dS}$ means the Kerr-dS black hole, it may be interesting to examine string/M-theory on $S^p \times M^q_{AdS}$ and the dual Yang-Mills theory.

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References

[1] S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, “Rotation and the AdS/CFT correspondence,” Phys. Rev. D 59 (1999) 064005 [arXiv:hep-th/9811056].

[2] G.W. Gibbons, H. Lü, D.N. Page and C.N. Pope, “The General Kerr-de Sitter Metrics in All Dimension,” [arXiv:hep-th/0404008].

[3] D. N. Page, “A Compact Rotating Gravitational Instanton,” Phys. Lett. B 79 (1978) 235.

[4] Y. Hashimoto, M. Sakaguchi and Y. Yasui, “New infinite series of Einstein metrics on sphere bundles from AdS black holes,” [arXiv:hep-th/0402199] Commun. Math. Phys. in press.

[5] G. W. Gibbons, S. A. Hartnoll and Y. Yasui, “Properties of some five dimensional Einstein metrics,” [arXiv:hep-th/0407030]
[6] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki-Einstein metrics on $S^2 \times S^3$,” arXiv:hep-th/0403002.

[7] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, “A new infinite class of Sasaki-Einstein manifolds,” arXiv:hep-th/0403038.

[8] M. Cvetič, H. Lü and C. N. Pope, “Charged Kerr-de Sitter black holes in five dimensions,” arXiv:hep-th/0406196.