Modular symmetry in the SMEFT

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Abstract

We study the modular symmetric standard-model effective field theory. We employ the stringy Ansatz on coupling structure that 4-point couplings $y^{(4)}$ of matter fields are written by a product of 3-point couplings $y^{(3)}$ of matter fields, i.e., $y^{(4)} = y^{(3)}y^{(3)}$. In this framework, we discuss the flavor structure of bilinear fermion operators and 4-fermion operators, where the holomorphic and anti-holomorphic modular forms appear. From the Ansatz, the $A_4$ modular-invariant semileptonic four-fermion operator $[E_R \Gamma E_R][D_R \Gamma D_R]$ does not lead to the flavor changing (FC) processes since this operator would be constructed in terms of gauge couplings $g$ as $y^{(3)} \sim g$. The chirality flipped bilinear operator $[D_R \Gamma D_L]$ also does not lead FC if the mediated mode corresponds to the Higgs boson $H_d$. In this case, the flavor structure of this operator is the exactly same as the mass matrix. On the other hand, if the flavor structure of the operator is not the exactly same as the mass matrix, the situation would change drastically. Then, we obtain the non-trivial relations of FC transitions at nearby fixed points $\tau = i, \omega, i\infty$, which are testable in the future. As an application, we discuss the relations of the lepton flavor violation processes $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$ at nearby $\tau_e = i$, where the successful lepton mass matrix was obtained. We also study the flavor changing 4-quark operators in the $A_4$ modular symmetry of quarks. As a result, the minimal flavor violation could be realized by taking relevant specific parameter sets of order one.
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1 Introduction

The Standard Model (SM) has been regarded as the low-energy limit of an full theory at the high energy scale. After the first years of running of the LHC, no new physics (NP) has been discovered. That is, there is a mass gap between the SM spectrum and these hypothetical additional degrees of freedom such as new particles.

Describing possible physics beyond the SM in general terms gets increasingly important, and the systematic study goes under the name of the SMEFT: the SM Effective Field Theory (EFT) based on the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry, and the SM field content as dynamical degrees of freedom below a cut-off scale $\mu > G_F^{-1/2}$. Several years after the pioneering analysis in Ref. [1], the first complete list of non-redundant SMEFT Lagrangian terms up to dimension-six has been presented in Ref. [2].

When all the possible flavor structures are taken into account in absence of any flavor symmetry, a large proliferation in the number of independent coefficients in the SMEFT occurs: there are 1350 CP-even and 1149 CP-odd independent coefficients for the dimension-six operators [3]. The flavor symmetry is a challenging hypothesis to reduce the number of independent parameters of the flavor sector. Above all, the flavor symmetries $U(3)^5$ and $U(2)^5$ have been successfully applied to the SMEFT [4]. The $U(3)^5$ flavor symmetry is the maximal flavor symmetry allowed by the SM gauge sector, while $U(2)^5$ is the corresponding subgroup acting only on the first two (light) families. The $U(3)^5$ allows us to apply the Minimal Flavor Violation (MFV) hypothesis [5, 6], which is the most restrictive hypothesis consistent with the SMEFT, and suppress non-standard contributions to flavor-violating observables [6]. In the $U(2)^5$ symmetry [7–9], it retains most of the MFV virtues and allows us to have a much richer structure as far as the dynamics of third family is concerned.

In the $U(3)^5$ and $U(2)^5$ flavor symmetric scenario, the Yukawa couplings are understood as spurions (symmetry-breaking terms) which have non-trivial representations under the symmetry. Assuming that flavor- and CP-violating effects in NP is also controlled by the Yukawa, the flavor structure of higher-dimensional operators are expressed by the spurions. Then, by counting different order in the breaking terms of these symmetries, we can classify the number of independent dimension-six operators, as studied in Ref. [4]. These flavor symmetries are not the only options to efficiently suppress flavor-violating observables in the SMEFT, and the non-Abelian discrete symmetry would be one of the alternative choice.

The non-Abelian discrete groups have been discussed to challenge the flavor problem of quarks and leptons [10–19]. Indeed, supersymmetric (SUSY) modular invariant theories give us an attractive framework to address the flavor symmetry of quarks and leptons with non-Abelian discrete groups [20]. In this approach, the quark and lepton mass matrices are written in terms of modular forms which are holomorphic functions of the modulus $\tau$. The arbitrary symmetry breaking sector of the conventional models based on flavor symmetries is replaced by the moduli space, and then Yukawa couplings are given by modular forms.

The well-known finite groups $S_3$, $A_4$, $S_4$ and $A_5$ are isomorphic to the finite modular groups $\Gamma_N$ for $N = 2, 3, 4, 5$, respectively [21]. The lepton mass matrices have been given successfully in terms of $A_4$ modular forms [20]. Modular invariant flavor models have been also proposed on the $\Gamma_2 \simeq S_3$ [22], $\Gamma_4 \simeq S_4$ [23] and $\Gamma_5 \simeq A_5$ [24]. Based on these modular forms, the flavor mixing of quarks and leptons has been discussed intensively in these years. Phenomenological studies of the lepton flavors have been done based on $A_4$ [25–27], $S_4$ [28–30] and $A_5$ [31]. A clear prediction of the
neutrino mixing angles and the Dirac CP phase was given in the simple lepton mass matrices with the $A_4$ modular symmetry \[\text{[26]}\]. The Double Covering groups $T'$ [32,33] and $S'_4$ [34,35] were also realized in the modular symmetry. Furthermore, phenomenological studies have been developed in many works [36–101].

Superstring theory is a promising candidate for the unified theory including gravity. Various string compactifications lead to four-dimensional low energy field theories with the specific structure, where 4-point couplings $y^{(4)}_{ijkl}$ of matter fields can be written by a product of 3-point couplings $y^{(3)}_{ijm}$ of matter fields,

$$y^{(4)}_{ijkl} = \sum_m y^{(3)}_{ijm} y^{(3)}_{mkl} \tag{1.1}$$

up to an overall factor, where the modes corresponding to $m$ may be light or heavy modes. Furthermore, $n$-point couplings $y^{(n)}$ can also be written by products of 3-point couplings $y^{(3)}$, i.e., $y^{(n)} = (y^{(3)})^{n-2}$. Thus, the symmetries in 3-point couplings are still symmetries even for higher-dimensional operators, and the flavor structures of higher-dimensional operators are controlled by 3-point couplings. This structure in the string-derived low-energy effective field theory meets the MFV hypothesis \[\text{[102]}\]. Note that the string EFTs satisfy the relation (1.1) at the compactification scale or the string scale, but it holds at the low-energy scale. This is because new operators appearing through the vacuum expectation value (VEV) of scalar fields and integrating out heavy states keep the relation (1.1).

In addition, these couplings in the string-derived effective field theory depend on moduli, which represent geometrical characters of string compact spaces such as size and shape. When we ignore the dynamic of moduli fields, these moduli-dependent couplings behave as spurions. Then, the geometrical symmetry, under which moduli transform non-trivially, would be important from the viewpoint of the MFV, although Yukawa spurions transform $(\bar{3}, \bar{3}, 1, 1, 1)$, $(3, 1, \bar{3}, 1, 1)$, and $(1, 1, 1, 3, \bar{3})$ in the $U(3)^5 = U(3)_Q \otimes U(3)_U \otimes U(3)_D \otimes U(3)_L \otimes U(3)_E$ flavor symmetric MFV scenario, where $\{Q, U, D, L, E\}$ denote the five independent types of the SM fermions. The $U(2)^5$ flavor symmetric scenario can also be realized in string models due to the fact that matter Yukawa couplings have rank 1 at the leading order \[\text{[103]}\]. The modular symmetry is the geometrical symmetry, which changes the basis of cycles of the torus $T^2$ as well as the orbifold $T^2/\mathbb{Z}_2$.\footnote{The possible discrete modular symmetries on higher-dimensional toroidal orbifolds were classified in the context of Type IIB string theory \[\text{[104]}\].} Moreover, zero-modes transform non-trivially under the modular symmetry (see, e.g., for heterotic string theory on orbifolds Ref. [105] and for magnetized brane models Ref. [106]). That is, the modular symmetry corresponds to the flavor symmetry in the low-energy effective field theory. Yukawa couplings as well as other couplings transform non-trivially under the modular symmetry, because these couplings depend on the modulus. Calabi-Yau threefolds have many moduli, and their geometrical symmetries are symplectic modular symmetries \[\text{[107,108]}\]. Their phenomenological implications were studied in Refs. \[\text{[109,110]}\]. Hence, these observations strongly support that flavor structures of higher-dimensional operators as well as Yukawa couplings in the string EFTs are determined by the modular flavor symmetry.

A drawback of the MFV hypothesis is that it does not allow us to define a clear power-counting in the SMEFT. This is because one of the breaking term, namely Yukawa coupling $Y_t$, is large. It is therefore not obvious why one should not consider more powers of $Y_t$ in the counting of
independent operators. On the other hand, it defines a clear power counting in the modular symmetry due to the modular weights.

In this paper, we study the SMEFT with the $\Gamma_N$ modular flavor symmetry. The modular flavor symmetry is regarded as a remnant of the geometrical symmetry of the extra-dimensional space. Constraints on higher-dimensional operators only by $\Gamma_N$ would be weak, in particular for $\Gamma_N$ singlets and many parameters would be allowed. Hence, we employ the relation (1.1) as Ansatz in the SMEFT. Through this Ansatz, higher-dimensional operators are related with 3-point couplings, although the $m$ mode in Eq. (1.1) may be known or unknown.

We take the level 3 finite modular groups, $\Gamma_3$ for the flavor symmetry since the property of $A_4$ flavor symmetry has been well known [111–117]. Based on the tensor decomposition of $A_4$ modular group, we discuss the bilinear and 4-fermion operators with flavor changing (FC) process at nearby fixed points. As an application, we discuss the lepton flavor violation (LFV).

The paper is organized as follows. In section 2, we review on the $A_4$ modular flavor symmetry for flavors. In section 3, we discuss the 4-fermion operators of the SMEFT in $A_4$ modular symmetry. In sections 4 and 5, we study the flavor structure of the bilinear fermion operators with chirality flip and chirality conserve, respectively. In section 6, we discuss the 4-quark operator with $\Delta F = 2$. Section 7 is devoted to the summary. In Appendix A, we summarizes the SMEFT operators. In Appendix B, we give a brief review on an explicit $A_4$ modular flavor model. The $S$ and $ST$ transformations of its mass matrix are given in Appendix C. In addition, the mass matrix at nearby $\tau = i$ is given in Appendix D. In Appendix E, we present tensor products of $A_4$ including modular forms. Appendix F summarizes briefly the $U(2)$ flavor symmetry of the quark sector.

## 2 $A_4$ modular symmetry and flavor of quarks and leptons

In this section, we briefly review the models with $A_4$ modular symmetry.

### 2.1 Modular flavor symmetry

The modular group $\Gamma$ is the group of linear fractional transformations $\gamma$ acting on the modulus $\tau$, belonging to the upper-half complex plane as:

$$\tau \longrightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, $\text{Im}[\tau] > 0$, (2.1)

which is isomorphic to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{1, -1\}$ transformation. This modular transformation is generated by $S$ and $T$,

$$S: \tau \longrightarrow -\frac{1}{\tau}, \quad T: \tau \longrightarrow \tau + 1,$$

(2.2)

which satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I.$$  

(2.3)

We introduce the series of groups $\Gamma(N)$, called principal congruence subgroups, where $N$ is the level 1, 2, 3, .... These groups are defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

(2.4)
For $N = 2$, we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\}$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\Gamma(N)$ are finite modular groups. In these finite groups $\Gamma_N$, $T^N = I$ is imposed. The groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3, A_4, S_4$ and $A_5$, respectively [21].

Modular forms $f_i(\tau)$ of weight $k$ are the holomorphic functions of $\tau$ and transform as

$$f_i(\tau) \longrightarrow (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau), \quad \gamma \in \bar{\Gamma}, \quad (2.5)$$

under the modular symmetry, where $\rho(\gamma)_{ij}$ is a unitary matrix under $\Gamma_N$.

Under the modular transformation of Eq. (2.1), chiral superfields $\psi_i$ ($i$ denotes flavors) with weight $-k$ transform as [118]

$$\psi_i \longrightarrow (c\tau + d)^{-k} \rho(\gamma)_{ij} \psi_j. \quad (2.6)$$

We study global SUSY models. The superpotential which is built from matter fields and modular forms is assumed to be modular invariant, i.e., to have a vanishing modular weight. For given modular forms, this can be achieved by assigning appropriate weights to the matter superfields.

The kinetic terms are derived from a Kähler potential. The Kähler potential of chiral matter fields $\psi_i$ with the modular weight $-k$ is given simply by

$$\frac{1}{|i(\bar{\tau} - \tau)|^k} \sum_i |\psi_i|^2, \quad (2.7)$$

where the superfield and its scalar component are denoted by the same letter, and $\bar{\tau} = \tau^*$ after taking VEV of $\tau$. The canonical form of the kinetic terms is obtained by changing the normalization of parameters [26]. The general Kähler potential consistent with the modular symmetry possibly contains additional terms [119]. However, we consider only the simplest form of the Kähler potential.

For $\Gamma_3 \simeq A_4$, the dimension of the linear space $M_k(\Gamma(3))$ of modular forms of weight $k$ is $k + 1$ [120–122], i.e., there are three linearly independent modular forms of the lowest non-trivial weight 2, which form a triplet of the $A_4$ group. These modular forms have been explicitly given [20] in the symmetric base of the $A_4$ generators $S$ and $T$ for the triplet representation as shown in the next subsection.

### 2.2 Modular forms

The holomorphic and anti-holomorphic modular forms with weight 2 compose the $A_4$ triplet as:

$$Y_3^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \quad Y_3^{(2)}(\tau) \equiv Y_3^{(2)*}(\tau) = \begin{pmatrix} Y_1^*(\tau) \\ Y_2^*(\tau) \\ Y_3^*(\tau) \end{pmatrix}. \quad (2.8)$$

In the representation of the generators $S$ and $T$ for $A_4$ triplet:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (2.9)$$
where $\omega = e^{i\frac{2\pi}{3}}$, modular forms are given explicitly in terms of the Dedekind eta function $\eta(\tau)$ and its derivative [20]:

$$
Y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)/\eta(3\tau)}{\eta(3\tau)} \right),
$$

$$
Y_2(\tau) = -i \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),
$$

$$
Y_3(\tau) = -i \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right).
$$

Those are also expressed in the $q$ expansions, $q = \exp(2i\pi\tau)$:

$$
\begin{pmatrix}
Y_1(\tau) \\
Y_2(\tau) \\
Y_3(\tau)
\end{pmatrix} =
\begin{pmatrix}
1 + 12q + 36q^2 + 12q^3 + \ldots \\
-6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\
-18q^{2/3}(1 + 2q + 5q^2 + \ldots)
\end{pmatrix}. \tag{2.11}
$$

### 2.3 Representation of down-type quarks and charged leptons

Assign the left-handed down-type quarks to $A_4$ triplets $\mathbf{3}$ and the three right-handed ones to $A_4$ three different singlets. Then, those are expressed as follows:

$$
D_L = \begin{pmatrix}
d_L \\
s_L \\
b_L
\end{pmatrix}, \quad \bar{D}_L = \begin{pmatrix}
\bar{d}_L \\
\bar{s}_L \\
\bar{b}_L
\end{pmatrix}, \quad (d_R, s_R, b_R) = (1, 1'', 1'), \quad (d_R, s_R, b_R) = (1, 1', 1''). \tag{2.12}
$$

It is noticed that quarks of second and third families are exchanged each other in $\bar{D}_L$. The weight of $D_L$ and $\bar{D}_L$, $k$ are assigned to 2 and $-2$, respectively. On the other hand, $k = 0$ for singlets $d_R^c, d_R$, etc.

The charged leptons are like down-type quarks as:

$$
E_L = \begin{pmatrix}
e_L \\
\mu_L \\
\tau_L
\end{pmatrix}, \quad \bar{E}_L = \begin{pmatrix}
\bar{e}_L \\
\bar{\mu}_L \\
\bar{\tau}_L
\end{pmatrix}, \quad (e_R^c, \mu_R^c, \tau_R^c) = (1, 1'', 1'), \quad (e_R, \mu_R, \tau_R) = (1, 1', 1''). \tag{2.13}
$$

The weight of $E_L$ and $\bar{E}_L$, $k$, are also 2 and and $-2$, respectively. On the other hand, $k = 0$ for $e_R^c, e_R$, etc.

Most of modular flavor models, which have been studied, are supersymmetric models. In the following sections, we study models below the supersymmetry breaking scale. We assume that the light modes are exactly the same as the SM with two doublet Higgs models.

### 3 SMEFT 4-fermion operators in $A_4$ modular symmetry

We write down 4-fermion operators as well as dipole operators in terms of modular forms $Y(\tau)$. We also follow the Ansatz (1.1) that those higher-dimensional operators are related with 3-point couplings, e.g., Yukawa couplings with Higgs fields. Here, the higher-dimensional operators are supposed to be generated by integrating out heavy superpartners, massive gauge bosons and
stringy modes. We have already many modular flavor symmetric models, which lead to realistic quark and lepton mass matrices separately. However, when we use the common value of the modulus $\tau$ for both quark and lepton sectors, the models are severely constrained and very difficult to realize all the experimental values of quark and lepton masses and their mixing angles at the same time. In order to cover many modular flavor models, we assume that the $A_4$ modular flavor symmetry in the lepton sector is independent of the $A_4$ symmetry in the quark sector, i.e., $A_4^E \otimes A_4^Q$ symmetry. They have two independent moduli, $\tau_q$ and $\tau_e$ for the quark sector and the lepton sector, respectively. Such a setup can be realized through the compactification, that the compact space includes $T^2 \times T^2$, and the flavor structure in the quark sector originates from one $T^2$, while the lepton flavor structure originates from the other $T^2$. Indeed, a similar setup was studied e.g., in Ref. [45]. Using this setup and Ansatz, we study their implications on flavor changing processes.

As examples, consider the semileptonic flavor changing neutral processes,

$$b \to s \bar{\mu} \mu \,(s \bar{e} e)\,, \quad b \to d \bar{\mu} \mu \,(d \bar{e} e)\,, \quad s \to d \bar{\mu} \mu \,(d \bar{e} e)\,, \quad (3.1)$$

which are caused by the flavor changing $\Delta F = 1$ operator. Impose the modular $A_4$ symmetry on quarks and leptons, respectively, that is $A_4^E \otimes A_4^Q$. The triplet modular forms with weight 2 are denoted as $Y(\tau_q)$ and $Y(\tau_e)$, which are different for quarks and charged leptons because $\tau_q$ and $\tau_e$ are different. In order to discuss relevant operators, we take a $A_4$ modular model, which leads to the successful fermion mass matrices. Suppose that three left-handed quark and lepton doublets are of a triplet of the $A_4$ group. The three right-handed quarks and charged leptons are different singlets of $A_4$. On the other hand, the Higgs doublets are supposed to be singlets of $A_4$. The generic assignments of representations and modular weights to the fields are presented in Table 1, where right-handed up-type quarks are omitted since those are not necessary in following discussions.

|       | $Q_L$ | $(d_R^c, s_R^c, b_R^c)$ | $L_L$ | $(e_R^c, \mu_R^c, \tau_R^c)$ | $H_d$ | $Y(\tau_q)$, $Y(\tau_e)$ |
|-------|-------|-------------------------|-------|-----------------------------|-------|--------------------------|
| $SU(2)$ | 2     | 1                       | 2     | 1                           | 2     | 1                        |
| $A_4$  | 3     | $(1, 1'', 1')$          | 3     | $(1, 1'', 1')$              | 1     | 3                        |
| $k$    | 2     | $(0, 0, 0)$             | 2     | $(0, 0, 0)$                 | 0     | 2                        |

Table 1: The assignment of $A_4$ representations and weights $k$ for down-type quarks, charged leptons, down-type Higgs doublet and the modular forms.

We discuss the semileptonic 4-fermion operators, which are categorized as

$$[\bar{E}_L\Gamma E_L] [\bar{D}_L\Gamma D_L] : Q_{\ell q}^{(1)} , Q_{\ell q}^{(3)} ,$$
$$[\bar{E}_R\Gamma E_R] [\bar{D}_R\Gamma D_R] : Q_{\ell d} ,$$
$$[\bar{E}_L\Gamma E_L] [\bar{D}_R\Gamma D_R] : Q_{\ell d} ,$$
$$[\bar{E}_R\Gamma E_R] [\bar{D}_L\Gamma D_L] : Q_{\ell q} ,$$
$$[\bar{E}_L\Gamma E_R] [\bar{D}_R\Gamma D_L] : Q_{\ell d\ell q} , \quad (3.2)$$

where $L$ and $R$ denote the left-handed and the right-handed fields, and $\Gamma$ denotes a generic combination of Dirac matrices, color and $SU(2)_L$ generators, which play no role as far as the flavor structure is concerned. Corresponding SMEFT operators $Q$ of which explicit expression are shown in Appendix A, are also listed.
First of all, let us construct the modular $A_4$-invariant 4-fermion operators $[\bar{E}_L \Gamma E_R][\bar{D}_R \Gamma D_L]$ by using the holomorphic and anti-holomorphic modular forms:

$$[\bar{E}_L \Gamma E_R][\bar{D}_R \Gamma D_L] \Rightarrow [\bar{E}_L Y^*(\tau_e) \Gamma E_R]_1 [\bar{D}_R Y(\tau_q) D_L]_1, \quad (3.3)$$

where the subscript 1 denotes the $A_4$ trivial singlet. After the decomposition of $A_4$ tensor product, we obtain coefficients for each operator, which are written explicitly in the following sections. From the viewpoint of the Ansatz, $y^{(4)} = y^{(3)} y^{(3)}$, these $[\bar{E}_L \Gamma E_R][\bar{D}_R \Gamma D_L]$ operators would be constructed in terms of Yukawa couplings with the Higgs fields. Thus, we assume that these operators include the same coefficients as the Yukawa couplings, i.e. mass matrices, as written explicitly later.

Next, the $A_4$ modular-invariant semileptonic 4-fermion operators $[\bar{E}_R \Gamma E_R][\bar{D}_R \Gamma D_R]$ can be written by

$$[\bar{E}_R \Gamma E_R][\bar{D}_R \Gamma D_R] \Rightarrow \left[ \sum_{i=1}^{3} r_{ei} \bar{e}_{iR} \Gamma e_{iR} \right] \left[ \sum_{i=1}^{3} r_{qi} \bar{d}_{iR} \Gamma d_{iR} \right], \quad (3.4)$$

where $r_{ei}$ and $r_{qi}$ are arbitrary real constants. From the viewpoint of the Ansatz (1.1), $y^{(4)} = y^{(3)} y^{(3)}$, these operators would be constructed in terms of gauge couplings $g$ as $y^{(3)} \sim g$. Therefore, one expects $r_{e(q)1} = r_{e(q)2} = r_{e(q)3}$, which do not lead to the FC processes. Similarly, for the other operators, one of possible modular $A_4$-invariant operators in our Ansatz (1.1) would be the type like 4-fermion operators mediated by gauge bosons. However, they do not lead to the FC processes similar to the $[\bar{E}_R \Gamma E_R][\bar{D}_R \Gamma D_R]$ operator. Operators including holomorphic and anti-holomorphic modular forms would lead to the FC processes. Hence, another possibility for $A_4$ modular-invariant semileptonic 4-fermion operators are constructed by using the holomorphic and anti-holomorphic modular forms:

$$[\bar{E}_L \Gamma E_L][\bar{D}_L \Gamma D_L] \Rightarrow [\bar{E}_L Y^*(\bar{\tau}_e) \Gamma Y(\tau_e) E_L]_1 [\bar{D}_L Y^*(\bar{\tau}_q) \Gamma Y(\tau_q) D_L]_1, \quad (3.5)$$

$$[\bar{E}_L \Gamma E_L][\bar{D}_R \Gamma D_R] \Rightarrow [\bar{E}_L Y^*(\bar{\tau}_e) \Gamma Y(\tau_e) E_L]_1 \left[ \sum_{i=1}^{3} r_{qi} \bar{d}_{iR} \Gamma d_{iR} \right], \quad (3.6)$$

$$[\bar{E}_R \Gamma E_R][\bar{D}_L \Gamma D_L] \Rightarrow \left[ \sum_{\epsilon_i = \bar{e}, \mu, \tau} r_{e_i} \bar{e}_{iR} \Gamma e_{iR} \right] [\bar{D}_L Y^*(\bar{\tau}_q) \Gamma Y(\tau_q) D_L]_1. \quad (3.7)$$

After decomposition of $A_4$ tensor products, we will give coefficients explicitly in the following sections. These operators could be consistent with the Ansatz, but the mode $m$ in Eq. (1.1) is unknown. Then, many free parameters, which are not related with couplings in the renormalizable SM Lagrangian, appear in these operators. At any rate, these operators are $A_4$ modular invariant. The above $[\bar{E}_L \Gamma E_L][\bar{D}_R \Gamma D_R]$ and $[\bar{E}_R \Gamma E_R][\bar{D}_L \Gamma D_L]$ operators do not lead to the FC processes in the quark sector and lepton sector, respectively. In the following sections, we concentrate on these $[\bar{E}_L \Gamma E_R][\bar{D}_R \Gamma D_L], [\bar{E}_R \Gamma E_L][\bar{D}_L \Gamma D_R]$ and $[\bar{E}_L \Gamma E_L][\bar{D}_L \Gamma D_L]$ operators leading to the FC processes as well as dipole operators. Moreover, 4-quark operators are also constructed in a similar way.

---

2Note that our kinetic terms (2,7) for left-handed fermions are not canonical. They couple with gauge bosons as $\frac{1}{(\tau_e - \tau_q)^2} A_{\mu} [\bar{E}_L \Gamma E_L]_1$ and $\frac{1}{(\tau_q - \tau_q)^2} A_{\mu} [\bar{D}_L \Gamma D_L]_1$. 


4 Bilinear fermion operators $[\tilde{D}_R \Gamma D_L]$ and $[\tilde{D}_L \Gamma D_R]$ 

The 4-fermion scalar operator $[\bar{E}_L E_R][\tilde{D}_R D_L]$ does not appear at the tree level in the standard model (SM). Indeed, it is not allowed by the exact $U(3)$ flavor symmetry. On the other hand, it appears in three-point couplings of two fermions and modular forms in the modular flavor symmetry. Moreover, due to the modular weights, they define a clear power counting of modular forms.

Since the 4-fermion operators are given by the products of bilinear fermion operators, interesting features from Ansatz (1.1) appear in the bilinear fermion operators. For example, the 3-point couplings are realized in terms of the modular forms typically in $[\tilde{D}_R \Gamma D_L]$ operators.

In this section, we discuss the bilinear operators of quarks, $[\tilde{D}_R \Gamma D_L]$ and $[\tilde{D}_L \Gamma D_R]$, and corresponding ones of charged leptons. Those correspond the SMEFT operators $Q$ of which explicit expression are shown in Appendix A as follows:

$$
\begin{align*}
[\tilde{D}_R \Gamma D_L] & : Q_{dH}, Q_G, Q_{dW}, Q_{dB}, \\
[\tilde{D}_R \Gamma D_L] & : Q_{dH}', Q_G', Q_{dW}', Q_{dB}', \\
[\tilde{E}_L \Gamma E_R] & : Q_{eH}, Q_{eW}, Q_{eB}, \\
[\tilde{E}_L \Gamma E_L] & : Q_{eH}', Q_{eW}', Q_{eB}'.
\end{align*}
$$

4.1 $[\tilde{D}_R \Gamma D_L]$ and $[\tilde{D}_L \Gamma D_R]$ bilinears in the flavor space

At first, let us begin with discussing the holomorphic operator $[\tilde{D}_R \Gamma D_L]$ and anti-holomorphic operator $[\tilde{D}_L \Gamma D_R]$ in the flavor space. The magnitudes of $LR$ couplings are proportional to modular forms. Taking account of $\tilde{D}_R = (d^c, s^c, b^c)$, we can decompose the operator in the base of Eq. (2.9) for $S$ and $T$ as:

$$
[\tilde{D}_R \Gamma D_L] \Rightarrow [\tilde{D}_R \Gamma Y(\tau_q)D_L]_1 = \\
[\alpha_d \tilde{d}_R \Gamma (Y_1(\tau_q)d_L + Y_2(\tau_q)b_L + Y_3(\tau_q)s_L) + \beta_d \tilde{s}_R \Gamma (Y_3(\tau_q)b_L + Y_1(\tau_q)s_L + Y_2(\tau_q)d_L)],
$$

$$
[\tilde{D}_L \Gamma D_R] \Rightarrow [\tilde{D}_L \Gamma Y^*(\tau_q)\Gamma D_R]_1 = \\
[\beta^*_d \tilde{d}_R \Gamma (Y_1^*(\tau_q)d_L + Y_2^*(\tau_q)b_L + Y_3^*(\tau_q)s_L) + \gamma_d \tilde{s}_R \Gamma (Y_3^*(\tau_q)b_L + Y_1^*(\tau_q)s_L + Y_2^*(\tau_q)d_L)],
$$

where the subscript 1, 1’, 1” denote the $A_4$ singlets, respectively. The parameters $\alpha_d$, $\beta_d$ and $\gamma_d$ are constants. These expressions are written in the matrix representation as:

$$
[\tilde{D}_R \Gamma Y(\tau_q)D_L]_1 = (\tilde{d}_R, \tilde{s}_R, \tilde{b}_R) \Gamma \begin{pmatrix}
\alpha_d & 0 & 0 \\
0 & \beta_d & 0 \\
0 & 0 & \gamma_d
\end{pmatrix}
\begin{pmatrix}
Y_1(\tau_q) & Y_3(\tau_q) & Y_2(\tau_q) \\
Y_2(\tau_q) & Y_1(\tau_q) & Y_3(\tau_q) \\
Y_3(\tau_q) & Y_2(\tau_q) & Y_1(\tau_q)
\end{pmatrix}
\begin{pmatrix}
d_L \\
s_L \\
b_L
\end{pmatrix},
$$

$$
[\tilde{D}_L \Gamma Y^*(\tau_q)\Gamma D_R]_1 = (\tilde{d}_L, \tilde{s}_L, \tilde{b}_L) \Gamma \begin{pmatrix}
\alpha^*_d & 0 & 0 \\
0 & \beta^*_d & 0 \\
0 & 0 & \gamma^*_d
\end{pmatrix}
\begin{pmatrix}
Y_1^*(\tau_q) & Y_3^*(\tau_q) & Y_2^*(\tau_q) \\
Y_2^*(\tau_q) & Y_1^*(\tau_q) & Y_3^*(\tau_q) \\
Y_3^*(\tau_q) & Y_2^*(\tau_q) & Y_1^*(\tau_q)
\end{pmatrix}
\begin{pmatrix}
d_R \\
s_R \\
b_R
\end{pmatrix}.
$$
It is useful to compare them with the down-type quark mass matrix $M_d$ in the assignment of Table 1. The mass matrix is given in terms of weight 2 modular forms as:

$$
M_d = v_d \begin{pmatrix}
\alpha_{d(m)} & 0 & 0 \\
0 & \beta_{d(m)} & 0 \\
0 & 0 & \gamma_{d(m)}
\end{pmatrix}
\begin{pmatrix}
Y_1(\tau_q) & Y_3(\tau_q) & Y_2(\tau_q) \\
Y_2(\tau_q) & Y_1(\tau_q) & Y_3(\tau_q) \\
Y_3(\tau_q) & Y_2(\tau_q) & Y_1(\tau_q)
\end{pmatrix}_{RL},
$$

(4.4)

where the VEV of the Higgs field $H_q$ is denoted by $v_d$. Parameters $\alpha_{d(m)}, \beta_{d(m)}, \gamma_{d(m)}$ can be taken to be real constants. Since the bilinear operators appear in four-field operators, it is reasonable to assume

$$
\alpha_d = c\alpha_{d(m)}, \beta_d = c\beta_{d(m)}, \gamma_d = c\gamma_{d(m)},
$$

(4.5)

from the viewpoint of the Ansatz Eq. (1.1), where the mode $m$ may correspond to $H_d$. Here, $c$ is a common constant. Hereafter, we set $c = 1$ for simplicity. In this case, the matrix structure of bilinear operators $[D_R D_L]$ appearing four-field operators is exactly the same as the mass matrix. Obviously, the bilinear operator matrix is diagonal in the basis for mass eigenstates. The FC processes such as $b \to s, b \to d, s \to d$ never happen. Hence, we obtain the very clear results in the modular symmetric SMEFT with the Ansatz Eq. (1.1).

In what follows, we study larger violations such that $\alpha_d, \beta_d, \gamma_d$ are of $O(\alpha_{d(m)}), O(\beta_{d(m)}), O(\gamma_{d(m)})$, respectively, but they are different by factors from $\alpha_{d(m)}, \beta_{d(m)}, \gamma_{d(m)}$, i.e.

$$
\alpha_d - \alpha_{d(m)} \sim \alpha_d, \quad \beta_d - \beta_{d(m)} \sim \beta_d, \quad \gamma_d - \gamma_{d(m)} \sim \gamma_d.
$$

(4.6)

Unknown modes $m$ in Eq. (1.1) may contribute to such violations.

For the charged lepton operators $[E_R \Gamma E_L]$ and $[E_L \Gamma E_R]$, we obtain the decompositions by replacing $\tau_q, \alpha_d, \beta_d, \gamma_d$ with $\tau_e, \alpha_e, \beta_e, \gamma_e$ in Eq. (4.1). As in the down-sector quarks, the FC processes such as $\mu \to e, \tau \to e, \tau \to \mu$ never occur when we assume $\alpha_e = \alpha_{e(m)}, \beta_e = \beta_{e(m)}, \gamma_e = \gamma_{e(m)}$, where $\alpha_{e(m)}, \beta_{e(m)}, \gamma_{e(m)}$ are parameters in the charged lepton mass matrix as shown in Appendix B. This is the clear result in the modular symmetric SMEFT with the Ansatz Eq. (1.1).

On the other hand, violations of the above parameter relation may lead to the FC processes. In what follows, we study such violations such as

$$
\alpha_e - \alpha_{e(m)} \sim \alpha_e, \quad \beta_e - \beta_{e(m)} \sim \beta_e, \quad \gamma_e - \gamma_{e(m)} \sim \gamma_e.
$$

(4.7)

The $A_4$ flavor coefficients are given in Table 2 for relevant bilinear operators of down-type quarks and charged leptons, where the overall strength of the NP effect is not included. Hereafter, without specifying them, we denote $\alpha_{d,e(m)}, \beta_{d,e(m)}, \gamma_{d,e(m)}$ by $\alpha_{d,e}, \beta_{d,e}, \gamma_{d,e}$, too, because they are the same orders.

---

3The overall strength of the NP effect is omitted in coefficients of other Tables.
Therefore, the flavor structure of the these operators is predicted if the modulus $\tau_{q,e}$ is fixed. It is noticed that above operators are given in the flavor base. In order to move the mass eigenstate on models, for example, Eqs. (4.10) and (4.11). The interesting value of $\tau_{q,e}$ is fixed points of the modulus in the fundamental domain of $SL(2,\mathbb{Z})$ since the moduli stabilization is realized in a controlled way at nearby fixed points [123, 124]. Furthermore, the fixed points are statistically favored in the string landscape [125]. We discuss the phenomenology at nearby fixed points in the next subsection.

### 4.2 Diagonal matrix $M_E^\dagger M_E$ and $M_{q}^\dagger M_q$ at fixed points

Residual symmetries arise whenever the VEV of the modulus $\tau$ breaks the modular group $\Gamma$ only partially. Here and in what follows, we denote $\tau = \tau_{q,e}$ unless we specify it. Fixed points of modulus are the case. There are only 2 inequivalent finite points in the fundamental domain of $\Gamma$, namely, $\tau = i$ and $\tau = \omega = -1/2 + i\sqrt{3}/2$. The first point is invariant under the $S$ transformation $\tau' = -1/\tau$. In the case of $A_4$ symmetry, the subgroup $\mathbb{Z}_2^S = \{I, S\}$ is preserved at $\tau = i$. The second point is the left cusp in the fundamental domain of the modular group, which is invariant under the $ST$ transformation $\tau' = -1/(\tau + 1)$. Indeed, $\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$ is one of subgroups of $A_4$ group. The right cusp at $\tau = -\omega^2 = 1/2 + i\sqrt{3}/2$ is related to $\tau = \omega$ by the $T$ transformation. There is also infinite point $\tau = i\infty$, in which the subgroup $\mathbb{Z}_3^T = \{I, T, T^2\}$ of $A_4$ is preserved. We summarize at three cases of the transformation:

$$S \text{ invariant : } \tau = i, \quad ST \text{ invariant : } \tau = \omega, \quad T \text{ invariant : } \tau = i\infty. \quad (4.9)$$

If a residual symmetry of $S$ and $T$ in $A_4$ is preserved in mass matrices of $\tau_{q,e}$, we have commutation relations between the mass matrices and the generator $G \equiv S, T, ST$ as:

$$[M_{RL}^\dagger M_{RL}, G] = 0, \quad (4.10)$$

where $M_{RL}$ denotes the mass matrix of charged leptons and quarks, $M_E$ and $M_{q}$ ($q = u, d$).

Then the mass matrices $M_E^\dagger M_E$ and $M_q^\dagger M_q$ could be diagonal in the diagonal basis of $G$ at the fixed points. Therefore, the hierarchical structures of flavor mixing are easily realized near those fixed points.
4.2.1 Mass matrix and operators $\bar{D}_L \Gamma D_R$, $\bar{D}_R \Gamma D_L$ at the fixed point $\tau = i$

At $\tau = i$, holomorphic and anti-holomorphic modular forms of weight 2 are given as:

$$Y(\tau_q = i) = Y_1(i) \begin{pmatrix} 1 \\ 1 - \sqrt{3} \\ -2 + \sqrt{3} \end{pmatrix}, \quad Y^*(\tau_q = i) = Y_1(i) \begin{pmatrix} 1 \\ -2 + \sqrt{3} \\ 1 - \sqrt{3} \end{pmatrix},$$

$$Y(\tau_e = i) = Y_1(i) \begin{pmatrix} 1 \\ 1 - \sqrt{3} \\ -2 + \sqrt{3} \end{pmatrix}, \quad Y(\tau_e = i) = Y_1(i) \begin{pmatrix} 1 \\ -2 + \sqrt{3} \\ 1 - \sqrt{3} \end{pmatrix}, \quad (4.11)$$

in the base of Eq. (2.9). At this fixed point, we transform the left-handed quarks and charged lepton fields as:

$$D_L \rightarrow D^S_L \equiv U_S D_L, \quad \bar{D}_L \rightarrow \bar{D}^S_L \equiv \bar{D}_L U_S^\dagger,$$

$$E_L \rightarrow E^S_L \equiv U_S E_L, \quad \bar{E}_L \rightarrow \bar{E}^S_L \equiv \bar{E}_L U_S^\dagger, \quad (4.12)$$

where the unitary matrix $U_S$ is

$$U_S = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & 2 & 2 \\ \sqrt{3} + 1 & -2 & \sqrt{3} - 1 \\ \sqrt{3} - 1 & -2 & \sqrt{3} + 1 \end{pmatrix}, \quad (4.13)$$

(see Eq. (D.1) of Appendix D). On the other hand, the right-handed quarks and charged lepton fields are unchanged since they are $A_4$ singlets.

At $\tau_q = i$, under the transformation of Eq. (4.12), the down-type quark mass matrix is given as (see in Appendix D)

$$M_d = \frac{1}{2} v_d \begin{pmatrix} 0 & 3(\sqrt{3} - 1)\tilde{\alpha}_{d(m)} & -(3 - \sqrt{3})\tilde{\alpha}_{d(m)} \\ 0 & -3(\sqrt{3} - 1)\tilde{\beta}_{d(m)} & -(3 - \sqrt{3})\tilde{\beta}_{d(m)} \\ 0 & 2(3 - \sqrt{3})\tilde{\gamma}_{d(m)} \end{pmatrix}_{RL},$$

$$M_d^\dagger M_d = \frac{1}{2} v_d^2 \begin{pmatrix} 0 & 9(2 - \sqrt{3})(\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) & 3(3 - 2\sqrt{3})(\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) \\ 0 & 3(3 - 2\sqrt{3})(\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) & 3(2 - \sqrt{3})(\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2 + 4\tilde{\gamma}_{d(m)}^2) \end{pmatrix}_{LL}, \quad (4.14)$$

where $\tilde{\alpha}_{d(m)} = (6 - 3\sqrt{3})Y_1(i)^2\alpha_{d(m)}$, $\tilde{\beta}_{d(m)} = (6 - 3\sqrt{3})Y_1(i)^2\beta_{d(m)}$ and $\tilde{\gamma}_{d(m)} = (6 - 3\sqrt{3})Y_1(i)^2\gamma_{d(m)}$ and $\tilde{\gamma}_{d(m)}$ is supposed to be much larger than $\tilde{\alpha}_{d(m)}$ and $\tilde{\beta}_{d(m)}$.

Since two eigenvalues of $S$ are degenerate such as $(1, -1, -1)$, there is still a freedom of the 2–3 family rotation. Therefore, $M_d^\dagger M_d$ could be diagonal after the small 2–3 family rotation of $\mathcal{O}(\tilde{\alpha}_{d(m)}^2/\gamma_{d(m)}^2, \tilde{\beta}_{d(m)}^2/\gamma_{d(m)}^2)$. The charged lepton mass matrix is the same one in Eq. (4.14) by replacing the subscript $d$ with $e$.

Let us consider the bilinear operators of the subsection 4.1 in the diagonal base of the generator $S$. The $A_4$ triplet left-handed fields are transformed as in Eq. (4.12). Putting the modular forms of Eq. (4.11) into the coefficients of Table 2, we can predict the flavor structure of the FC bilinear operators in the new base of $S$. Those coefficients are listed in Table 3 at $\tau = i$. The left-handed fields are not yet the mass eigenstate, but close to it. We should move the left-handed fields to the mass eigenstate by the small rotation in the flavor space.
4.2.2 Mass eigenstate at nearby $\tau = i$

In order to get the observed fermion masses and CKM elements, the modulus $\tau$ is deviated from the fixed points $\tau = i$. Indeed, the successful quark mass matrices have been obtained at nearby $\tau = i$ [47]. By using a small dimensionless parameter $\epsilon$, we put the modulus value as $\tau = i + \epsilon$. Then, approximate behaviors of the ratios of modular forms are [71]:

$$
\frac{Y_2(\tau)}{Y_1(\tau)} \approx (1 + \epsilon_1)(1 - \sqrt{3}), \quad \frac{Y_3(\tau)}{Y_1(\tau)} \approx (1 + \epsilon_2)(-2 + \sqrt{3}), \quad \epsilon_1 = \frac{1}{2}\epsilon_2 \approx 2.05i\epsilon.
$$

(4.15)

These approximate forms are agreement with exact numerical values within 0.1% for $|\epsilon| \leq 0.05$. Since the modulus $\tau$ is different ones for the quark and lepton sectors each other, we use the notation $\epsilon^q_i$ for the quark sector and $\epsilon^\ell_i$ for the lepton sector hereafter.

The quark mass matrix is diagonalized by the transformation which is shown in Appendix D:

$$
D_L \rightarrow D_L^m \equiv U_{md}^T U_{12}^T(90^\circ) U_S D_L, \quad D_L \rightarrow D_L^a \equiv D_L U_S^T U_{12}(90^\circ) U_{md},
$$

(4.16)

where

$$
U_{md} \simeq \begin{pmatrix}
1 & s_{12}^d e^{i\eta_d} & 0 \\
-s_{12}^d e^{-i\eta_d} & 1 & s_{23}^d \\
s_{12}^d s_{23}^d & -s_{23}^d & 1
\end{pmatrix} \simeq \begin{pmatrix}
1 & O(\epsilon^q_i) & 0 \\
O(\epsilon^q_i) & 1 & O(\epsilon^q_i) \\
O(\epsilon^q_i) & O(\epsilon^q_i) & 1
\end{pmatrix}.
$$

(4.17)

In these transformations, $U_{12}^T(90^\circ)$ denotes an extra rotation of $90^\circ$ between the first and second families. It is required to realize the hierarchy of three CKM mixing angles simply. Owing to $U_{12}^T(90^\circ)$, the mixing angle $s_{13}^d$ between the first and third families is negligibly small. Details are presented in Eq.(D.4) of Appendix D. In the quark sector, $s_{12}^d$ may be expressed in terms of CKM elements. Since $V_{CKM} = U_{mu}^T U_{md}$, where $U_{mu}$ is the mixing matrix of the up-type quark mass matrix, $s_{12}^d$ is approximately the ratio of CKM elements $|V_{ud}/V_{ts}|$ if the mixing angle $s_{13}^u$ is also negligibly small as well as $s_{13}^d$. Then, we have

$$
s_{12}^d e^{i\eta_d} \simeq -\frac{V_{ud}^*}{V_{ts}^*}.
$$

(4.18)

In the mass eigenstate, $A_4$ flavor coefficients of quark bilinear operators in Eq. (4.1) and Table 2 are given in terms of mixing angles $s_{12}^d$, $s_{23}^d$ and $\epsilon_i^q$ at $\tau_q = i + \epsilon$ as well as the case at $\tau_q = i$ as seen in Table 3. Due to the extra rotation of the left-handed quarks $U_{12}(90^\circ)$ of Eq. (4.16), the magnitudes of coefficients between $\tau_q = i$ and $\tau_q = i + \epsilon$ are exchanged with respect to $d_L$ and $s_L$ ($d_L$ and $s_L$) in Table 3.

The mixing angles are $s_{12}^d = O(0.1)$, $s_{23}^d = O(\epsilon^q_1)$ and $s_{23}^d = O(\epsilon^q_1)$ as seen in Appendix D. It is remarked that ratios among the $b \rightarrow s$ and $b \rightarrow d$ transitions are

$$
-\frac{1}{2}\bar{\alpha}_d [\bar{d}_R \Gamma b_L^m] - \frac{1}{2}\bar{\beta}_d [\bar{s}_R \Gamma b_L^m], \quad -\frac{s_{23}^d \epsilon^*_1}{\sqrt{3}s_{23}^d + \epsilon^*_1} [\bar{d}_L \Gamma b_R^m] - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}s_{23}^d + \epsilon_1^*} \bar{\gamma}_d [\bar{s}_L \Gamma b_R^m],
$$

(4.19)

where $\bar{\alpha}_d = (6 - 3\sqrt{3})Y_1(i)\alpha_d$, $\bar{\beta}_d = (6 - 3\sqrt{3})Y_1(i)\beta_d$ and $\bar{\gamma}_d = (6 - 3\sqrt{3})Y_1(i)\gamma_d$. The superscript $m$ of the left-handed quarks denotes the mass eigenstate of the transformation in Eq.(4.16). The
Table 3: $A_4$ flavor coefficients of the FC quark bilinear operators in $Y_1(i)$ unit at $\tau_q = i$ and $\tau_q = i + \epsilon$.

The magnitude of this ratio depends on the detail of the model, especially, the up-type quark sector. For example, we obtained the best fit parameters:

$$\frac{\tilde{\beta}_d(m)}{\tilde{\gamma}_d(m)} = 4.26 \times 10^{-3}, \quad \frac{\tilde{\alpha}_d(m)}{\tilde{\beta}_d(m)} = 3.40,$$

in the model of Appendix B [71]. Since $\tilde{\alpha}_d \sim \tilde{\alpha}_d(m)$, $\tilde{\beta}_d \sim \tilde{\beta}_d(m)$ and $\tilde{\gamma}_d \sim \tilde{\gamma}_d(m)$, it is found that the $\bar{s}_L b^m_R$ transition is the dominant among others as seen in Table 3. The $\bar{d}_L b^m_R$ transition is smaller of one order than the $\bar{s}_L b^m_R$ transition. Both $\bar{d}_R s^m_L$ and $\bar{d}_L s^m_R$ transitions are significantly suppressed compared with $\bar{s}_L b^m_R$.

For charged leptons $e$, $\mu$ and $\tau$, the transformation is somewhat different from the down-type quark sector in Eq. (4.16). The charged lepton mass matrix is diagonalized by the transformation which is also shown in Appendix D:

$$E_L \rightarrow E_L^m \equiv U^T_{me} U_S E_L, \quad \bar{E}_L \rightarrow \bar{E}_L^m \equiv \bar{E}_L U^T_S U_{me},$$

(4.21)

where

$$U_{me} \simeq \begin{pmatrix} 1 & s^e_{12} & s^e_{13} \\ -s^e_{12} & 1 & 0 \\ -s^d_{13} & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & O(0.1) & O(|\epsilon|) \\ O(0.1) & 1 & 0 \\ O(|\epsilon|) & 0 & 1 \end{pmatrix}. \quad (4.22)$$

Indeed, the numerical fit was succeeded as shown in Appendix D [77]. In these transformations, an extra rotation of $U^T_{12}(90^\circ)$ is not required in the transformation of Eq. (4.21) because the observed lepton mixing angles are quite large. The large mixing angles are realized in the neutrino mass matrix of Appendix B.

In the mass eigenstate, the $A_4$ coefficients of charged lepton bilinear operators are given in terms of mixing angles $s^e_{12}$, $s^e_{13}$ and $\epsilon^e_1$ at $\tau_e = i + \epsilon$ in Table 4. These coefficients are different from the quark ones in Table 3.
| $\tau_e$ | $\bar{\mu}_R \Gamma_{\tau_L}$ | $\bar{\mu}_L \Gamma_{\tau_R}$ | $\bar{e}_R \Gamma_{\tau_L}$ | $\bar{e}_L \Gamma_{\tau_R}$ | $\bar{e}_R \Gamma_{\mu_L}$ | $\bar{e}_L \Gamma_{\mu_R}$ |
|---|---|---|---|---|---|---|
| $i$ | $-\frac{1}{2} (3 - \sqrt{3}) \tilde{\beta}_e$ | 0 | $-\frac{1}{2} (3 - \sqrt{3}) \tilde{\alpha}_e$ | 0 | $\frac{3}{2} (\sqrt{3} - 1) \tilde{\alpha}_e$ | 0 |
| $i + \epsilon$ | $-\frac{1}{2} (3 - \sqrt{3}) \tilde{\beta}_e (1 - \sqrt{3}) s_{12}^2 \epsilon_1 \gamma_e$ | $(1 - \sqrt{3})(\sqrt{3} s_{13}^2 + \epsilon_1) \gamma_e$ | $\frac{3}{2} (\sqrt{3} - 1) \tilde{\alpha}_e$ | 0 | $\frac{1}{2} (\sqrt{3} - 1)(3 s_{12}^2 + \sqrt{3} s_{13}^2 - 2 \epsilon_1) \tilde{\beta}_e$ | 0 |

Table 4: $A_4$ flavor coefficients of the FC lepton bilinear operators in $Y_1(i)$ unit at $\tau_e = i$ and $\tau_e = i + \epsilon$.

### 4.3 Diagonal bases of $ST$ for quarks

#### 4.3.1 At $\tau = \omega$

At $\tau = \omega$, as presented in Eq. (4.9), holomorphic and anti-holomorphic modular forms are given as:

$$Y^{(2)}(\tau_q = \omega) = Y_1(\omega) \begin{pmatrix} 1 \\ \omega \\ -\frac{1}{2} \omega^2 \end{pmatrix}, \quad Y^{(2)*}(\tau_q = \omega) = Y_1(\omega) \begin{pmatrix} 1 \\ -\frac{1}{2} \omega \\ \omega^2 \end{pmatrix}. \quad (4.23)$$

The left-handed quark fields are transformed as:

$$D_L \to D_{L}^{ST} \equiv U_{ST_i} D_L, \quad \bar{D}_L \to \bar{D}_{L}^{ST} \equiv \bar{D}_L U_{ST_i}^{\dagger}, \quad (4.24)$$

where $U_{ST_i}$ is presented in Eq. (C.7) in Appendix C.2. It is noticed there are independent 6 unitary transformation $U_{ST_i}$ to diagonalize $M_d^d M_d$. At this stage, we cannot fix $U_{ST_i}$ among six ones. Once the up-type quark mass matrix is given to reproduce the CKM matrix, $U_{ST_i}$ is fixed.

Putting modular forms of Eq. (4.23) into coefficients of bilinear operators in Table 2, we obtain coefficients of FC operators in the diagonal base of $ST$. We summarize them for each $U_{ST_i}$. They correspond to the mass matrix as discussed below Eq. (4.4).

#### 4.3.2 Mass eigenstate at nearby $\tau = \omega$

In order to get the observed fermion masses and CKM elements, the modulus $\tau$ is deviated from the fixed points $\tau = \omega$. By using a small dimensionless parameter $\epsilon$, we put the modulus value as $\tau = i + \epsilon$. Then, approximate behaviors of the ratios of modular forms are given in Ref. [71]. Small deviation of $\tau$ like $\tau = \omega + \epsilon$ leads to the approximate behavior of the ratios of modular forms:

$$\frac{Y_2(\tau)}{Y_1(\tau)} \simeq \omega (1 + \epsilon_1), \quad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq -\frac{1}{2} \omega^2 (1 + \epsilon_2), \quad \epsilon_1 \simeq \frac{1}{2} \epsilon_2 \simeq 2.1 i \epsilon. \quad (4.25)$$

These approximate forms are agreement with exact numerical values within a few % for $|\epsilon| \leq 0.05$. 

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Table 4: $A_4$ flavor coefficients of the FC lepton bilinear operators in $Y_1(i)$ unit at $\tau_e = i$ and $\tau_e = i + \epsilon$. 

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| \(U_{ST1}\) | \(M_d\) | \(M_d^\dagger M_d\) | \(\tilde{s}_R \Gamma_{L}^{ST}\) | \(\bar{s}_L \Gamma_{L}^{ST}\) | \(\bar{d}_R \Gamma_{L}^{ST}\) | \(\tilde{d}_R \Gamma_{L}^{ST}\) |
|---|---|---|---|---|---|---|
| \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(0\) | \(\frac{3}{2} \omega \tilde{d}_{\tilde{d}}\) | \(0\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |
| \(U_{ST2}\) | \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(0\) | \(\frac{3}{2} \omega \tilde{d}_{\tilde{d}}\) | \(0\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |
| \(U_{ST3}\) | \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(0\) | \(0\) | \(\frac{3}{2} \omega \tilde{d}_{\tilde{d}}\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |
| \(U_{ST4}\) | \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(\omega \tilde{d}_{\tilde{d}}\) | \(0\) | \(0\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |
| \(U_{ST5}\) | \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(0\) | \(0\) | \(0\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |
| \(U_{ST6}\) | \(\frac{9}{4} v_d\) | \(\begin{pmatrix} 0 & 0 & \omega \tilde{d}_{d(m)} \\ -\omega^2 \beta_{d(m)} & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)} & 0 \end{pmatrix}\) | \(\begin{pmatrix} \beta_{d(m)}^2 & 0 & 0 \\ 0 & \tilde{\gamma}_{d(m)}^2 & 0 \\ 0 & 0 & \tilde{\beta}_{d(m)}^2 \end{pmatrix}\) | \(\omega \tilde{d}_{\tilde{d}}\) | \(0\) | \(0\) | \(-\frac{3}{2} \omega \beta_{\tilde{d}}\) |

Table 5: Down-type quark mass matrices and \(A_4\) flavor coefficients of the FC quark bilinear operators in \(Y_1(\omega)\) unit for each \(U_{ST}\), at \(\tau_q = \omega\). Here, \(\tilde{\alpha}_d = Y_1(\omega)\alpha_d\), \(\tilde{\beta}_d = Y_1(\omega)\beta_d\) and \(\tilde{\gamma}_d = Y_1(\omega)\gamma_d\).

As a representative, we show the \(M_d^\dagger M_d\) including corrections up to \(O(\epsilon)\) for the case of \(U_{ST4}\) in Table 5:

\[
M_d^\dagger M_d \approx \frac{9}{4} v_d^2 \begin{pmatrix} \tilde{\alpha}_{d(m)}^2 & -\frac{2}{3} \tilde{\gamma}_{d(m)} | \epsilon_1^q | \\ \tilde{\beta}_{d(m)}^2 & 0 \\ \tilde{\gamma}_{d(m)} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_{d(m)} & \tilde{\beta}_{d(m)} \end{pmatrix} P^* ,
\]

where \(\tilde{\alpha}_{d(m)} = Y_1(\omega)\alpha_{d(m)}\), \(\tilde{\beta}_{d(m)} = Y_1(\omega)\beta_{d(m)}\) and \(\tilde{\gamma}_{d(m)} = Y_1(\omega)\gamma_{d(m)}\), and \(O(\tilde{\beta}_{d(m)}^2 \epsilon_1^q, \tilde{\alpha}_{d(m)}^2 \epsilon_1^q)\), terms are neglected. The phase matrix \(P\) is

\[
P = \begin{pmatrix} e^{2i\eta} & 0 \\ 0 & e^{i\eta} \end{pmatrix}, \quad \eta = \arg \epsilon_1^q .
\]

The mass matrix \(M_d^\dagger M_d\) is diagonalized as:

\[
U_{md}^T M_d^\dagger M_d P U_{md} = \text{diag} \left( m_{d_q}^2, m_{s_q}^2, m_{b_q}^2 \right),
\]

where we have up to \(O(|\epsilon_1^q|)\):

\[
U_{md} \approx \begin{pmatrix} 1 & \begin{pmatrix} s_{d2}^d & s_{d3}^d \end{pmatrix} \begin{pmatrix} s_{d2}^d & s_{d3}^d \end{pmatrix} \begin{pmatrix} s_{d2}^d & s_{d3}^d \end{pmatrix} \ \begin{pmatrix} 1 & \begin{pmatrix} 2/3 |\epsilon_1^q| \ O(|\epsilon_1^q|) \\ 2/3 |\epsilon_1^q| \ O(|\epsilon_1^q|) \end{pmatrix} \begin{pmatrix} 1 & 2/3 |\epsilon_1^q| \ O(|\epsilon_1^q|) \end{pmatrix} \end{pmatrix} \end{pmatrix} \approx \begin{pmatrix} 1 & -2/3 |\epsilon_1^q| \\ \begin{pmatrix} 2/3 |\epsilon_1^q| \\ -2/3 |\epsilon_1^q| \end{pmatrix} \begin{pmatrix} 2/3 |\epsilon_1^q| \\ -2/3 |\epsilon_1^q| \end{pmatrix} \end{pmatrix} ) .
\]
Here, the 1-3 mixing angle $s_{13}^d$ negligibly small. The magnitude of $s_{12}^d$ is also approximately the ratio of CKM elements $|V_{td}/V_{ts}|$ if the mixing angle $s_{13}^u$ is also negligibly small as well as $s_{13}^d$. For other cases of $U_{STi}$ in Table 5, the mixing matrix $U_{md}$ is the same one as in Eq. (4.29).

The mass eigenstate is realized by the transformation:

$$d_L \rightarrow d_L^m \equiv U^T_{md} U_{STi} d_L, \quad \bar{d}_L \rightarrow \bar{d}_L^m \equiv \bar{d}_L U^\dagger_{STi} U_{md}. \quad (4.30)$$

In Table 6, $A_4$ flavor coefficients of the FC operators are summarized.

| $U_{ST1}$ ($\bar{c}_d \gg \bar{\gamma}_d \gg \bar{\beta}_d$) | $\frac{s_R \Gamma b^m_L}{s_L \Gamma b^m_R}$ | $\frac{d_R \Gamma b^m_L}{d_L \Gamma b^m_R}$ | $\frac{d_R \Gamma s^m_L}{d_L \Gamma s^m_R}$ |
|---|---|---|---|
| $U_{ST2}$ ($\bar{\gamma}_d \gg \bar{\beta}_d \gg \gamma_d$) | $\frac{\omega^2 s^d_{23} e^d_1 \gamma_d}{\frac{3}{2} \tilde{r}_d}$ | $\frac{5}{3} \omega \bar{\alpha}_d$ | $\frac{3}{3} \gamma_d$ | $\frac{3}{2} \omega \bar{\beta}_d$ |
| $U_{ST3}$ ($\gamma_d \gg \alpha_d \gg \bar{\beta}_d$) | $-\omega^2 e^d_1 \beta_d$ | $\frac{1}{2} s^d_{12} (3 s^d_{23} + 2 e^d_1) \bar{\alpha}_d$ | $\frac{1}{2} s^d_{12} (3 s^d_{23} + 2 e^d_1) \bar{\gamma}_d$ | $\frac{1}{2} \omega (3 s^d_{23} + 2 e^d_1) \bar{\beta}_d$ |
| $U_{ST4}$ ($\bar{\beta}_d \gg \gamma_d \gg \alpha_d$) | $-\frac{3}{2} \omega^2 \beta_d$ | $\frac{1}{2} (3 s^d_{12} + 2 e^d_1) \bar{\alpha}_d$ | $-\frac{1}{2} (3 s^d_{12} + 2 e^d_1) \bar{\gamma}_d$ | $\frac{3}{2} \omega s^d_{12} \bar{\beta}_d$ |
| $U_{ST5}$ ($\bar{\beta}_d \gg \bar{\alpha}_d \gg \gamma_d$) | $-\frac{3}{2} \omega^2 \beta_d$ | $\frac{1}{2} (3 s^d_{12} + 2 e^d_1) \bar{\alpha}_d$ | $-\frac{1}{2} (3 s^d_{12} + 2 e^d_1) \bar{\gamma}_d$ | $\frac{3}{2} \omega s^d_{12} \bar{\beta}_d$ |
| $U_{ST6}$ ($\bar{\beta}_d \gg \bar{\alpha}_d \gg \bar{\gamma}_d$) | $\frac{1}{2} (3 s^d_{12} - 2 e^d_1) \bar{\gamma}_d$ | $\frac{1}{2} \omega (3 s^d_{23} + 2 e^d_1) \bar{\alpha}_d$ | $\frac{1}{2} \omega (3 s^d_{23} + 2 e^d_1) \bar{\gamma}_d$ | $\frac{1}{2} \omega |\bar{\beta}_d$ |

Table 6: $A_4$ flavor coefficients of the FC quark bilinear operators in $Y_1(\omega)$ unit for each $U_{STi}$ at nearby $\tau_q = \omega$.

### 4.4 Diagonal bases of $T$ for quarks

#### 4.4.1 At $\tau = i\infty$

At $\tau = i\infty$ as presented in Eq. (4.9), holomorphic and anti-holomorphic modular forms are given as:

$$Y^{(2)}(\tau_q = i\infty) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y^{(2)*}(\tau_q = i\infty) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (4.31)$$

Since the generator $T$ is already diagonal in Eq. (2.9), the unitary matrix to transform $D_L$ is the unit matrix. Then,

$$D_L \rightarrow D_L, \quad \bar{D}_L \rightarrow \bar{D}_L. \quad (4.32)$$

The down-type quark mass matrix is simply given as:

$$M_d = v_d \begin{pmatrix} \alpha_{d(m)} & 0 & 0 \\ 0 & \beta_{d(m)} & 0 \\ 0 & 0 & \gamma_{d(m)} \end{pmatrix}_{RL}, \quad M^\dagger_d M_d = v^2_d \begin{pmatrix} \alpha_{d(m)}^2 & 0 & 0 \\ 0 & \beta_{d(m)}^2 & 0 \\ 0 & 0 & \gamma_{d(m)}^2 \end{pmatrix}, \quad (4.33)$$
where \( Y_1(i\infty) = 1 \) is taken.

At \( \tau_q = i\infty \), \( A_4 \) flavor coefficients of the relevant operators are summarized in Table 7.

| \( \tau_q = i\infty \) | \( s_R \Gamma b_L \) | \( s_L \Gamma b_R \) | \( d_R \Gamma b_L \) | \( d_L \Gamma b_R \) |
| \( \tau_q \to i\infty \) | \( s^d_{23} \beta_d \) | \( s^d_{23} \delta_{\tau_d} \) | \( -s^d_{12} \delta_{\tau_d} \) | \( -s^d_{23} \delta_{\tau_d} \) |


\[
\text{Table 7: } A_4 \text{ flavor coefficients of the FC quark bilinear operators at } \tau_q = i\infty \text{ and toward } \tau_q = i\infty.
\]

### 4.4.2 Mass eigenstate towards \( \tau_q = i\infty \)

Taking leading terms of Eq. (2.11), we can express modular forms approximately as:

\[
Y_1(\tau) \simeq 1 + 12p \epsilon, \quad Y_2(\tau) \simeq -6p^\frac{1}{2} \epsilon^\frac{1}{2}, \quad Y_3(\tau) \simeq -18p^\frac{1}{2} \epsilon^\frac{1}{2}, \quad p = e^{2\pi i \text{Re } \tau}, \quad \epsilon = e^{-2\pi i \text{Im } \tau}, \tag{4.34}
\]

where \( \text{Im } \tau \gg 1 \). Then, the down-type quark mass matrix

\[
M_d^T M_d \simeq v_d^2 \begin{pmatrix}
\alpha^2_{d(m)} & \beta^2_{d(m)} & \delta^\alpha_d \\
\beta^2_{d(m)} & \gamma^2_{d(m)} & \delta^\beta_d \\
\delta^\alpha_d & \delta^\beta_d & \delta^\gamma_d
\end{pmatrix}, \tag{4.35}
\]

where \( \delta = -6p^\frac{1}{2} \epsilon^\frac{1}{2} \).

Taking account of the quark mass hierarchy, that is \( \gamma^2_q \gg \beta^2_q \gg \alpha^2_q \), the mass matrix \( M_d^T M_d \) is rewritten as:

\[
M_d^T M_d \simeq v_d^2 P \begin{pmatrix}
\alpha^2_{d(m)} & \beta^2_{d(m)} |\delta| & 0 \\
\beta^2_{d(m)} |\delta| & \gamma^2_{d(m)} & 0 \\
0 & 0 & |\delta|
\end{pmatrix} P^*, \tag{4.36}
\]

where \( \mathcal{O}(\alpha^2_d \delta) \) is neglected and

\[
P = \begin{pmatrix}
e^{2i\eta} & 0 & 0 \\
0 & e^{i\eta} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \eta = \text{arg } \delta^*. \tag{4.37}
\]

Then, the mass matrix is diagonalized as:

\[
U^T_{md} P^* M_d^T M_d P U_{md} = \text{diag } (m_d^2, m_s^2, m_b^2), \tag{4.38}
\]

where we have up to \( \mathcal{O}(|\delta|) \):

\[
U_{md} \simeq \begin{pmatrix}
1 & s^d_{12} & s^d_{13} \\
-s^d_{12} & 1 & s^d_{23} \\
s^d_{12} & -s^d_{23} & 1
\end{pmatrix} \simeq \begin{pmatrix}
1 & |\delta| & \mathcal{O}(\delta^2) \\
-|\delta| & 1 & |\delta| \\
\mathcal{O}(\delta^2) & -|\delta| & 1
\end{pmatrix}. \tag{4.39}
\]
Here, the 1-3 mixing angle $s_{13}^d$ is negligibly small. The mixing angle $s_{12}^d$ is also approximately the ratio of CKM elements $|V_{td}/V_{ts}|$ if the mixing angle $s_{13}^u$ is also negligibly small as well as $s_{13}^d$.

The mass eigenstate of the down-type quarks is obtained by the transformation:

$$d_L \rightarrow d_L^m \equiv U_{md}^T d_L, \quad \bar{d}_L \rightarrow \bar{d}_L^m \equiv \bar{d}_L U_{md}.$$  \hspace{1cm} (4.40)

Then, towards $\tau_q = i\infty$, $A_4$ flavor coefficients of the FC operators are obtained. We summarize them in Table 7. Since $\gamma_{d(m)} \gg \beta_{d(m)} \gg \alpha_{d(m)}$, the $[\bar{s}_L \Gamma b_R]$ transition is much larger than others.

### 4.5 Bilinear FC operators in $U(2)$ flavor symmetry

As well known, the $U(2)$ flavor symmetry is a powerful hypothesis to reduce the number of independent parameters of the quark sector. Indeed, the flavor symmetries $U(2)^5$ have been successfully applied to the SMEFT [4]. Let us compare our predictions with ones of $U(2)^5$ flavor symmetry. The down-type quark transitions with chirality flip allowed by different $U(2)$ breaking terms are summarized in Table 8, where we follow the result of Table 4 in Ref. [4]. Details of parameters are given as presented in Appendix F. The magnitudes of parameters are

$$s_d = \mathcal{O}(10^{-1}), \quad \epsilon_q = \mathcal{O}(10^{-1}), \quad \delta_d = \mathcal{O}(10^{-2}), \quad \delta_d' = \mathcal{O}(10^{-3}),$$ \hspace{1cm} (4.41)

and others are of $\mathcal{O}(1)$. \hspace{1cm} (4.41)

|     | $s \rightarrow d$         | $b \rightarrow s$         | $b \rightarrow d$         |
|-----|---------------------------|---------------------------|---------------------------|
| $RL$| $(\rho_1 s_d \delta_d')^*[\bar{d}_R s_L]$ | $(\sigma_1 \epsilon_q \delta_d)[\bar{s}_R b_L]$ | $(\sigma_1 \epsilon_q s_d \delta_d')^*[\bar{d}_R b_L]$ |
| $LR$| $-\rho_1 s_d \delta_d[\bar{d}_L s_R]$ | $\beta_1 \epsilon_q [\bar{s}_L b_R]$ | - |

Table 8: Left-right fermion bilinears allowed by different $U(2)$ breaking terms. The $\epsilon_q, \delta_d, \delta_d'$ stand for an order of spurion, $s_d$ is a mixing angle and $\rho_1, \sigma_1, \beta_1$ are independent coefficients. The detail definition of the parameters are presented in Appendix F.

Those results are compared with our ones at $\tau = i + \epsilon$ of Table 3, at $\tau = \omega + \epsilon$ of Table 6 and towards $\tau = i\infty$ of Table 7. In $U(2)$ flavor symmetry, the $[\bar{s}_L b_R]$ transition is much larger than other ones as seen in Table 8. The $[\bar{d}_R s_L]$ transition is suppressed of $10^{-5}$. These results are contrast to our ones. For example, the $\bar{d}_L b_R$ transition is the dominant one in our case of Table 3. The $\bar{s}_L b_R$ transition is one order suppressed. Taking numerical values in Eq. (4.20) and $\epsilon_1^q = \mathcal{O}(0.1)$, it is found that $\bar{d}_R s_L$ and $\bar{d}_L s_R$ transitions are suppressed of $\mathcal{O}(10^{-3})$ and $\mathcal{O}(10^{-5})$, respectively as seen in Table 3.

Thus, the predictions of $A_4$ modular symmetry are considerably different from ones of $U(2)$. This difference comes from the flavor structure of the right-handed fermions. In our framework, the right-handed fermions are assigned to be $A_4$ singlets contrast to the left-handed ones of the $A_4$ triplet typically, that is, the flavor structure is drastically different from the left-handed one. On the other hand, both left-handed and right-handed ones are singlets (3rd family) and doublets (1st and 2nd families) in the framework of $U(2)^5$ symmetry. Then, operators $RL$ and $LR$ have the typical flavor structures like CKM, respectively.
4.6 Application to the lepton flavor violation

As well known, the LFV such as the $\mu \rightarrow e\gamma$ decay is suppressed enough in the SM, and is a good probe to discover the NP. If the flavor structure of NP is ruled with the $A_4$ modular symmetry, we can expect the correlation among decays of $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$.

For $\mu \rightarrow e\gamma$, the relevant effective Lagrangian which is valid below some scale $\Lambda$ with $m_W \geq \Lambda \gg m_b$, is given as [126, 127]

$$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda^2} (C_{RL}^{di} O_{RL}^{di} + C_{LR}^{di} O_{LR}^{di}) + h.c.,$$

(4.42)

with the dipole operators $O_{RL}^{di} = cm_\mu (\bar{e}_R \sigma^{\mu\nu} \mu_L) F_{\mu\nu}$ and $O_{LR}^{di} = cm_\mu (\bar{e}_L \sigma^{\mu\nu} \mu_R) F_{\mu\nu}$, while $C_{RL}^{di}$ and $C_{LR}^{di}$ are the dimensionless Wilson coefficients. Here, we consider tree-level matching between the SMEFT and this effective Lagrangian. Then, $O_{LR}^{di}$ matches to the SMEFT operators $Q_{eW}$ and $Q_{eB}$ (see Table 12 in Appendix A). The matching relation for $C_{LR}^{di}$ is given as the linear combination of $C_{eW}$ and $C_{eB}$, which are Wilson coefficients of the SMEFT operators. Since both $C_{eW}$ and $C_{eB}$ are proportional to $A_4$ flavor coefficients of the bilinear operators $\bar{E}_R \Gamma \Gamma L$ and $\bar{E}_L \Gamma \Gamma R$, respectively, we can simply take $C_{LR}^{di}$ to be equal to the coefficients in Table 9 apart from the absolute values. In Table 9, $A_4$ flavor coefficients are listed for the decay processes $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$ at nearby $\tau_e = i$.

| $\mu \rightarrow e\gamma$ | $\tau \rightarrow \mu\gamma$ | $\tau \rightarrow e\gamma$ |
|--------------------------|--------------------------|--------------------------|
| $\frac{3}{2} (\sqrt{3} - 1) \tilde{\alpha}_e [\bar{e}_R \sigma^{\mu\nu} \mu_L]$ | $-\frac{1}{2} (3 - \sqrt{3}) \tilde{\beta}_e [e_R \sigma^{\mu\nu} \tau_L]$ | $\frac{3}{2} (\sqrt{3} - 1)(3 s_{i2}^2 + \sqrt{3} s_{i1} s_{i3} - 2 e_1) \tilde{\beta}_e [\bar{e}_R \sigma^{\mu\nu} \tau_L]$ |
| $(1 - \sqrt{3}) (\sqrt{3} s_{i1}^2 + e_1^2) \tilde{\beta}_e [\bar{e}_L \sigma^{\mu\nu} \mu_R]$ | $(1 - \sqrt{3}) s_{i1} s_{i3} e_1 \tilde{\gamma}_e [\bar{e}_L \sigma^{\mu\nu} \tau_R]$ | $(1 - \sqrt{3})(\sqrt{3} s_{i1}^2 + e_1) \tilde{\gamma}_e [\bar{e}_L \sigma^{\mu\nu} \tau_R]$ |

Table 9: $A_4$ flavor coefficients of the charged lepton FC bilinear operators at nearby $\tau_e = i$, where $\tilde{\gamma}_e \gg \tilde{\alpha}_e \gg \tilde{\beta}_e$.

At the tree level, the Lagrangian in Eq. (4.42) results in the branching ratio

$$\text{BR}(\mu \rightarrow e\gamma) = \frac{\alpha m_{\mu}^5}{\Lambda^4 \Gamma_{\mu}} \left| C_{RL}^{di} \right|^2 + \left| C_{LR}^{di} \right|^2,$$

(4.43)

where $\Gamma_{\mu}$ is the width of the muon and $\alpha$ is the electromagnetic fine-structure constant. The flavor structure of the dipole operators $[\bar{E}_R \sigma^{\mu\nu} E_L F_{\mu\nu}]$ and $[\bar{E}_L \sigma^{\mu\nu} E_R F_{\mu\nu}]$ in the flavor space are presented in Table 9. Therefore, we can calculate Wilson coefficients at nearby fixed points of the modulus $\tau_e = i$, where the successful lepton mass matrix has been reproduced [77].

At nearby $\tau_e = i$, the best fit point is given by the parameters [77]:

$$\tau_e = -0.080 + 1.007 i, \quad \frac{\tilde{\alpha}_e(m)}{\tilde{\gamma}_e(m)} = 6.82 \times 10^{-2}, \quad \frac{\tilde{\beta}_e(m)}{\tilde{\gamma}_e(m)} = 1.02 \times 10^{-3}.$$

(4.44)

That is $\tilde{\gamma}_e(m) \gg \tilde{\alpha}_e(m) \gg \tilde{\beta}_e(m)$. Since the Wilson coefficients are proportional to the coefficients in Table 9, the $O_{RL}^{di}$ operator dominates the decay amplitude for $\mu \rightarrow e\gamma$. On the other hand, the $O_{LR}^{di}$ operator dominates the decay amplitude for $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$. Then, we predict

$$\text{BR}(\mu \rightarrow e\gamma) \geq \text{BR}(\tau \rightarrow e\gamma) \gg \text{BR}(\tau \rightarrow \mu\gamma).$$

(4.45)
Those ratios are:

\[
\begin{align*}
\frac{\text{BR}(\tau \to \mu \gamma)}{\text{BR}(\tau \to e \gamma)} & \approx \frac{s_{12}^e s_{13}^e}{\sqrt{3} s_{13}^e} \simeq 1.3 \times 10^{-3}, \\
\frac{\text{BR}(\mu \to e \gamma)}{\text{BR}(\tau \to e \gamma)} & \approx \left( \frac{m_\mu}{m_\tau} \right)^5 \left( \frac{\Gamma_\tau}{\Gamma_\mu} \right) \frac{1}{\sqrt{3 s_{13}^e + \epsilon_1}} \left( \frac{\bar{\alpha}_e}{\bar{\gamma}_e} \right)^2 \simeq 4,
\end{align*}
\]

where \( s_{12}^e = s_{13}^e = \epsilon_1 = 0.1 \) are put for a benchmark as seen in Appendix D, and \( \bar{\alpha}_e/\bar{\gamma}_e \simeq \bar{\alpha}_{e(m)}/\bar{\gamma}_{e(m)} \) is put. Inputting the current upper bound of the branching ratio \( \mu \to e \gamma \), \( \text{BR}(\mu \to e \gamma) < 4.2 \times 10^{-13} \) [128], we obtain \( \text{BR}(\tau \to \mu \gamma) < 1.4 \times 10^{-16} \) and \( \text{BR}(\tau \to e \gamma) < 1.1 \times 10^{-13} \). These current experimental upper bounds are \( 3.3 \times 10^{-8} \) and \( 4.4 \times 10^{-8} \), respectively [129], which are far away from our predictions.

Let us compare above predictions with ones of \( U(2)^5 \) flavor symmetry. The chirality-flipped \( \ell_i \to \ell_j \gamma \) transitions are summarized in Table 10, where we follow the result of Table 4 in Ref. [4]. Magnitudes of parameters are given as presented in Appendix F,

\[
s_e = \mathcal{O}(10^{-1}), \quad \epsilon_\ell = \mathcal{O}(10^{-1}), \quad \delta_e = \mathcal{O}(10^{-2}), \quad \delta'_e = \mathcal{O}(10^{-3}),
\]

and others are of \( \mathcal{O}(1) \). As seen in Table 10, \( O^R_{LR} \) operator dominates the decay amplitude for \( \mu \to e \gamma \) and \( \tau \to \mu \gamma \) while \( O^R_{RL} \) operator dominates the decay amplitude for \( \tau \to e \gamma \). Then, we predict

\[
\text{BR}(\tau \to \mu \gamma) \gg \text{BR}(\mu \to e \gamma) \gg \text{BR}(\tau \to e \gamma).
\]

This relation is completely different from the prediction at nearby \( \tau = i \).

| \( \bar{\mu} \to e \gamma \) | \( \bar{\tau} \to \mu \gamma \) | \( \bar{\tau} \to e \gamma \) |
|---|---|---|
| \( \bar{RL} \) \( (\rho_1 s_e \delta'_e)^* [\bar{e}_R \sigma^\mu \sigma^\nu \mu L] \) | \( \sigma_1 \epsilon_\ell \delta_e [\bar{\mu}_R \sigma^\mu \sigma^\nu \tau L] \) | \( \sigma_1 \epsilon_\ell s_e \delta'_e [\bar{e}_R \sigma^\mu \mu \tau L] \) |
| \( \bar{LR} \) \( -\rho_1 s_e \delta_e [\bar{e}_L \sigma^\mu \mu R] \) | \( \beta_1 \epsilon_\ell [\bar{\mu}_L \sigma^\mu \tau R] \) | - |

Table 10: Left-right fermion bilinears allowed by different \( U(2) \) breaking terms. The \( \epsilon_\ell, \delta_e, \delta'_e \) stand for an order of spurion, \( s_e \) is a mixing angle and \( \rho_1, \sigma_1, \beta_1 \) are independent coefficients. The detail definition of the parameters are presented in Appendix F.

5 **Bilinear fermion operators** \( [\bar{D}_L \Gamma D_L] \) and \( [\bar{E}_L \Gamma E_L] \)

In order to study the semileptonic operators \( [\bar{E}_L \Gamma E_L][\bar{D}_L \Gamma D_L] \), which dominates \( b \to s \bar{\mu} \mu \), etc. transitions, we discuss the bilinear operators \( [\bar{D}_L \Gamma D_L] \) and \( [\bar{E}_L \Gamma E_L] \). Corresponding SMEFT operators \( Q \) including fermion bilinear are

\[
[\bar{D}_L \Gamma D_L] : Q^{(1)}_{Hq}, Q^{(3)}_{Hq},
\]

\[
[\bar{E}_L \Gamma E_L] : Q^{(1)}_{H\ell}, Q^{(3)}_{H\ell}.
\]
5.1 \([\bar{D}_L \Gamma D_L]\) and \([\bar{E}_L \Gamma E_L]\) in flavor space

At first, we construct the \(A_4\) invariant bilinear quark operator \(\bar{D}_L \Gamma D_L\). As presented in Appendix E, the \(A_4\) invariant bilinear quark operator is obtained by the two type of combinations.

\[
(Y^* \bar{D}_L)_{1,1',1''} \otimes (YD_L)_{1,1',1''}, \quad [(Y^* \bar{D}_L)_{3s,3a} \otimes (YD_L)_{3s,3a}]_{1,1',1''}. \tag{5.2}
\]

The singlets combination gives three \(A_4\) invariant bilinear quark operators as

\[
(Y^* \bar{D}_L)_{1} \gamma^\mu(YD_L)_1, \quad (Y^* \bar{D}_L)_{1'} \gamma^\mu(YD_L)_{1'}, \quad (Y^* \bar{D}_L)_{1''} \gamma^\mu(YD_L)_{1''}. \tag{5.3}
\]

On the other hand, the triplets combination gives four \(A_4\) invariant bilinear quark operators as

\[
(Y^* \bar{D}_L \Gamma YD_L)_{1} \Gamma(YD_L)_1 + s_{q1} (Y^* \bar{D}_L)_{3s} \Gamma(YD_L)_{3s}, \quad (Y^* \bar{D}_L \Gamma YD_L)_{1} \Gamma(YD_L)_{1'}, \quad (Y^* \bar{D}_L \Gamma YD_L)_{1} \Gamma(YD_L)_{1''}, \quad (Y^* \bar{D}_L \Gamma YD_L)_{1} \Gamma(YD_L)_{1''). \tag{5.4}
\]

Finally, we have the \(A_4\) invariant bilinear quark operator, which is expressed as:

\[
\bar{D}_L \Gamma D_L \rightarrow s_{q1} (Y^* \bar{D}_L)_{1} \Gamma(YD_L)_1 + s_{q2} (Y^* \bar{D}_L)_{1'} \Gamma(YD_L)_{1'}, \quad s_{q3} (Y^* \bar{D}_L)_{1''} \Gamma(YD_L)_{1''} + t_{qas} (Y^* \bar{D}_L \Gamma YD_L)_{1(a,s)} + t_{qas} (Y^* \bar{D}_L \Gamma YD_L)_{1(a,s)} + t_{qas} (Y^* \bar{D}_L \Gamma YD_L)_{1(a,s)}. \tag{5.5}
\]

Here, \(s_{qi}(i = 1, 2, 3)\) and \(t_{qas}\), etc., are arbitrary constants. For the lepton sector, we have similar forms as:

\[
\bar{E}_L \Gamma E_L = s_{e1} (Y^* \bar{E}_L)_{1} \Gamma(YE_L)_1 + s_{e2} (Y^* \bar{E}_L)_{1'} \Gamma(YE_L)_{1'}, \quad s_{e3} (Y^* \bar{E}_L)_{1''} \Gamma(YE_L)_{1''} + t_{eas} (Y^* \bar{E}_L \Gamma YE_L)_{1(a,s)} + t_{eas} (Y^* \bar{E}_L \Gamma YE_L)_{1(a,s)} + t_{eas} (Y^* \bar{E}_L \Gamma YE_L)_{1(a,s)}. \tag{5.6}
\]

Since both \(\bar{D}_L \gamma_\mu D_L\) and \(\bar{E}_L \gamma_\mu E_L\) are Hermitian, we have parameter relations as:

\[
\text{Im } s_{qi} = 0, \quad s_{q3} = s_{q2}^*, \quad t_{qas} = t_{qas}^*, \quad \text{Im } s_{e1} = 0, \quad s_{e3} = s_{e2}^*, \quad t_{eas} = t_{eas}^*. \tag{5.7}
\]

By using Eqs. (5.5) and (5.6), we can calculate the bilinear operators \([\bar{D}_L \Gamma D_L]\) and \([\bar{E}_L \Gamma E_L]\).

The \(A_4\) flavor coefficients of the relevant operators are summarized in the base of Eq. (2.9) for \(S\) and \(T\) in Table 11, where coefficients of \(\bar{\mu}_L \mu_L\) and \(\bar{e}_L e_L\) are presented for the lepton sector since we focus on \(b \rightarrow s \mu\mu\), etc. transitions.

5.2 Diagonal base of \(S\) for both quarks and leptons

5.2.1 Semileptonic operator at \(\tau = i\)

In order to move the diagonal base of \(S\) for both quark \(D_L\) and lepton \(E_L\), we transform the quark and lepton fields like Eqs. (4.12) and (4.13). Furthermore, we put \(\tau = i\) for modular forms as in Eq. (4.11). Then, \(A_4\) triplet left-handed fields are transformed as in Eq. (4.12):

\[
D_L \rightarrow D^S_L \equiv U_S D_L, \quad \bar{D}_L \rightarrow \bar{D}^S_L \equiv \bar{D}_L U_S^\dagger, \quad E_L \rightarrow E^S_L \equiv U_S E_L, \quad \bar{E}_L \rightarrow \bar{E}^S_L \equiv \bar{E}_L U_S^\dagger, \tag{5.8}
\]
where \( U_S \) is presented in Eq. (4.13) and

\[
D_L^S = \begin{pmatrix}
d_L^S \\ s_L^S \\ \mu_L^S \\ \tau_L^S
\end{pmatrix}, \quad E_L^S = \begin{pmatrix}
e_L^S \\ \mu_L^S \\ \tau_L^S
\end{pmatrix}.
\] (5.9)

Then, we have relevant semileptonic operators

\[
C_{\alpha \beta \nu L}^S [\bar{\alpha} \Gamma \beta_L^S],
\] (5.10)

where \( \alpha, \beta \) denote \( d, s, b, e, \mu, \tau \). Coefficients of Table 11 are transformed to simple ones as:

\[
\begin{align*}
C_{bLbL}^S &= \frac{1}{2} Y_1(i)^2 \left[ (2 - \sqrt{3})(s_{q1} + s_{q2} + 4s_{q3} - t_{qaa}) \right], \\
C_{sLsL}^S &= \frac{1}{2} Y_1(i)^2 \left[ (2 - \sqrt{3})(9s_{q1} + 9s_{q2} + 4t_{qss}) \right], \\
C_{dLdL}^S &= \frac{1}{2} (2 - \sqrt{3})Y_1(i)^2(4t_{qss} - 3t_{qaa}), \\
C_{sLbL}^S &= \frac{1}{2} Y_1(i)^2 \left[ (3 - 2\sqrt{3})(3s_{q1} - 3s_{q2} + 2t_{qsa}) \right], \\
C_{bLsL}^S &= \frac{1}{2} Y_1(i)^2 \left[ (3 - 2\sqrt{3})(3s_{q1} - 3s_{q2} - 2t_{qas}) \right], \\
C_{dLbL}^S &= C_{dLsL}^S = 0, \\
C_{\mu L\mu L}^S &= \frac{1}{2} (2 - \sqrt{3})Y_1(i)^2(9s_{e1} + 9s_{e2} + 4t_{eess}), \\
C_{eLeL}^S &= \frac{1}{2} (2 - \sqrt{3})Y_1(i)^2(4t_{eess} - 3t_{eaa}).
\end{align*}
\] (5.11)

In this base of quarks and leptons, the off diagonal elements of the mass matrix \( (M_d^\dagger M_d)_{LL} \) vanish or are tiny under the condition \( \tilde{\alpha}_d \gg \tilde{\alpha}_d, \tilde{\beta}_d \) as seen in Eq. (4.14). We also expect that \( C_{sLbL}^S \) and \( C_{bLsL}^S \) are suppressed. Therefore, we impose the constraints of parameters in Eq. (5.11) as:

\[
3s_{q1} - 3s_{q2} + 2t_{qaa} = 0, \quad \text{Re} t_{qaa} = 0,
\] (5.12)
which lead to $C_{sLbL}^S = C_{bLsL}^S = 0$.

Next, we shift $\tau_q = i$ to $\tau_q = i + \epsilon$ and transform $D_L \rightarrow D_L^\prime \equiv U_{md}^T U_{12}^T (90^\circ) U_S D_L$ as well as in Eq. (4.16), we obtain

$$
C_{sLbL}^S = Y_1(i)^2 \left[ \frac{1}{2}(\sqrt{3} - 2)[3(s_q + s_{q2} + 4s_{q3}) - 4t_{qss}]s_{23}^d - 3(\sqrt{3} - 2)(s_{q1} - s_{q2})\epsilon_1 
+ 3(2\sqrt{3} - 3)(s_{q1} + s_{q2} - 2s_{q3})\epsilon_1^* \right],
$$

$$
C_{dLbL}^S = Y_1(i)^2 \left[ \frac{1}{2}(\sqrt{3} - 2)[3(s_q + s_{q2} + 4s_{q3}) - 4t_{qss}]s_{23}^d + 3(\sqrt{3} - 2)((s_{q1} - s_{q2})(\epsilon_1 + \epsilon_1^*)) (-s_{12}^d) 
+ \left\{ 3(\sqrt{3} - 2)(s_{q1} - s_{q2})\epsilon_1 + \frac{1}{6}(3 - 2\sqrt{3})(3t_{qaa} + 4t_{qss})\epsilon_1^* \right\} s_{23}^d \right],
$$

$$
C_{dLsL}^S = Y_1(i)^2 \left[ -\frac{3}{2}(\sqrt{3} - 2)[3(s_q + s_{q2}) + 4t_{qaa}]s_{12}^d + 3(\sqrt{3} - 2)(s_{q1} - s_{q2})\epsilon_1 
+ \frac{1}{6}(3 - 2\sqrt{3})(3t_{qaa} + 4t_{qss})\epsilon_1^* \right],
$$

where $s_{12}^d$ is redefined to be complex one including a phase as seen in Eq. (4.18), and $s_{23}^d$ is real.

For the coefficients of charged leptons, we transform $E_L \rightarrow E_L^\prime \equiv U_{me}^T U_S E_L$ as well as in Eq. (4.16), we obtain

$$
C_{\mu L\mu L}^S \simeq \frac{1}{2}(2 - \sqrt{3})Y_1(i)^2(9s_{e1} + 9s_{e2} + 4t_{eess}), \quad C_{eLeL}^S \simeq \frac{1}{2}(2 - \sqrt{3})Y_1(i)^2(4t_{eess} - 3t_{eaa}).
$$

Thus, $C_{eLeL}^S$ is comparable to $C_{\mu L\mu L}^S$ unless specific relations are set.

It is found that there is no relation among three coefficients of quarks because of many parameters. As well known, the $U(2)$ flavor symmetry predicts [4]

$$
\frac{C_{dLbL}^S}{C_{sLbL}^S} \simeq -s_{12}^d = \frac{V_{td}^*}{V_{ts}},
$$

where Eq. (4.18) is taken account. This relation could be reproduced if we impose following relations in addition to ones in Eq. (5.12)

$$
s_{q1} = s_{q2}, \quad 3t_{qaa} + 4t_{qss} = 0,
$$

which lead to $t_{as} = t_{sa} = 0$ and $s_{q1} = s_{q2} = s_{q3}$ finally. We arrive at

$$
C_{sLbL}^S = \frac{1}{2}(\sqrt{3} - 2)(9s_{q1} - t_{qss})s_{23}^d Y_1(i)^2,
$$

$$
C_{dLbL}^S = -s_{12}^d C_{sLbL}^T Y_1(i)^2,
$$

$$
C_{dLsL}^S = \frac{3}{2}(2 - \sqrt{3})(6s_{q1} + t_{qao})s_{12}^d Y_1(i)^2.
$$

In addition, imposing $s_{q1} = 0$, we obtain

$$
\frac{C_{dLbL}^S}{C_{sLbL}^S} = -s_{12}^d = \frac{V_{td}^*}{V_{ts}},
$$

25
where \( C_{sLsL}^{ST} \simeq C_{sLsL}^{S} \) is given in Eq. (5.11). This relation is also predicted in the \( U(2) \) flavor symmetry. Thus, a simple setup of parameters gives well known relations of the coefficients in the \( U(2) \) symmetry. However, we have no principle to choose such a parameter set at this stage unless an additional symmetry is put.

### 5.3 Diagonal bases of \( ST \) for quarks

We consider the case of the diagonal \( ST \) only for quarks because the successful lepton mass matrices are realized at nearby \( \tau_c = i \). The \( A_4 \) triplet left-handed fields are transformed as in Eq. (4.24):

\[
D_L \to D_L^{ST} \equiv U_{STi} D_L, \quad \bar{D}_L \to \bar{D}_L^{ST} \equiv \bar{D}_L U_{STi}^{\dagger},
\]

where \( U_{STi} \) is presented in Eq. (C.7) and

\[
D_L^{ST} = \begin{pmatrix} d_L^{ST} \\ s_L^{ST} \\ b_L^{ST} \end{pmatrix}.
\]

As a representative, we discuss the case of \( U_{ST4} \), which is shown explicitly in Eq. (C.7) of Appendix C.2. The down-type quark mass matrix is given in Table 5. Then, we have relevant semileptonic operators

\[
C_{\alpha L, \beta L}^{ST}[\alpha^{ST} \Gamma^{\beta L}] ,
\]

where \( \alpha, \beta \) denote \( d, s, b \). Putting \( \tau_q = \omega \), those coefficients are given as:

\[
C_{bLbL}^{ST} = Y_1(\omega)^2 \frac{1}{4} (9s_{q2} + 4t_{qss}) ,
\]

\[
C_{sLsL}^{ST} = Y_1(\omega)^2 \frac{1}{16} (36s_{q3} + 4t_{qss} - 9t_{qaa} + 6t_{qas} - 6t_{qsa}) ,
\]

\[
C_{dLdL}^{ST} = Y_1(\omega)^2 \frac{1}{16} (36s_{q1} + 4t_{qss} - 9t_{qaa} - 6t_{qas} + 6t_{qsa}) ,
\]

\[
C_{sLsL}^{ST} = C_{dLbL}^{ST} = C_{dLsL}^{ST} = 0 .
\]

Thus, the flavor changing operators vanish.

Next step, we shift \( \tau_q = \omega \) to \( \tau_q = \omega + \epsilon \) and transform \( d_L \to d_L^m \equiv U_{md}^{T} U_{ST4} d_L \) and obtain

\[
C_{sLsL}^{ST} = -Y_1(\omega)^2 \left[ \frac{3}{16} s_{23}^d (12s_{q2} + 4t_{qss} + 3t_{qaa} - 12s_{q3} - 2t_{qas} + 2t_{qsa}) - \frac{1}{6} \epsilon_1^* (9s_{q2} - 2t_{qss} + 3t_{qas}) \right] ,
\]

\[
C_{dLsb}^{ST} = Y_1(\omega)^2 \frac{1}{24} \left[ \frac{3}{16} s_{23}^d (12s_{q2} + 4t_{qss} + 3t_{qaa} - 12s_{q3} - 2t_{qas} + 2t_{qsa}) - \frac{1}{6} \epsilon_1^* (9s_{q2} - 2t_{qss} + 3t_{qas}) \right]
\]

\plus \frac{1}{24} \left[ Y_1(\omega)^2 \epsilon_1^* s_{23}^d (36s_{q1} + 4t_{qss} + 9t_{qaa} - 6t_{qas} - 6t_{qsa}) - \frac{1}{6} \epsilon_1^* (9s_{q2} - 2t_{qss} + 3t_{qas}) \right] ,
\]

\[
C_{dLsL}^{ST} = Y_1(\omega)^2 \left[ \frac{1}{4} s_{12}^d (9s_{q1} - 9s_{q3} + 3t_{qaa} - 3t_{qsa}) - \frac{1}{6} \epsilon_1^* (9s_{q3} - 2t_{qss} + 3t_{qsa}) \right] .
\]
In order to obtain the $U(2)$ symmetry like relation in Eq. (5.15),
\begin{equation}
\frac{C_{dtSS}^{STm}}{C_{sLbL}^{STm}} = -s^{d}_{12} = \frac{V^*_{td}}{V^*_{ts}} ,
\end{equation}
we impose relations of parameters:
\begin{equation}
9s_{q3} - 2t_{qss} + 3t_{qsa} = 0 , \quad 36s_{q1} + 4t_{qss} + 9t_{qaa} - 6t_{qas} - 6t_{qsa} = 0 .
\end{equation}
Putting them into $C_{dLSL}^{STm}$ in Eq. (5.23), we obtain
\begin{equation}
C_{dLSL}^{STm} = \frac{1}{4} Y_1(\omega)^2 s^{d}_{12} (9s_{q1} - 9s_{q3} + 3t_{qsa} - 3t_{qas}) .
\end{equation}
In addition, imposing $t_{sa} = t_{aa} = 0$, we obtain
\begin{equation}
\frac{C_{dLSL}^{STm}}{C_{sLSL}^{STm}} = -s^{d}_{12} = \frac{V^*_{td}}{V^*_{ts}} ,
\end{equation}
where $C_{sLSL}^{STm} \simeq C_{sLSL}^{ST}$ is given in Eq. (5.22). This relation is also predicted in the $U(2)$ flavor symmetry.

### 5.4 Diagonal bases of $T$ for quarks

Let us consider the case of the diagonal $T$ only for quarks. In the diagonal base of $T$ of Eq. (2.9), the mass matrix $M_q^\dag M_q$ is given at $\tau_q = i\infty$ as:
\begin{equation}
M_q^{2(0)} \equiv M_q^\dag M_q = v^2_d \begin{pmatrix}
\alpha^2_q & 0 & 0 \\
0 & \beta^2_q & 0 \\
0 & 0 & \gamma^2_q
\end{pmatrix} ,
\end{equation}
where $Y_i(i\infty) = 1$ is taken. Mixing angles appear through the finite effect of $\text{Im} [\tau]$.

Since the base of Eq. (2.9) is already the diagonal base of $T$, the transformation is trivial as seen in Eq. (4.32). Then, we have
\begin{equation}
D_T^T = D_L = \begin{pmatrix}
\alpha^T_d \\
\beta^T_d \\
\gamma^T_d
\end{pmatrix} .
\end{equation}
Therefore, the relevant bilinear operators are given in subsection 5.1 for the diagonal case of $T$.

Then, we have relevant semileptonic operators
\begin{equation}
C_{aL\beta L}^T \left[ \tilde{\alpha}^T \Gamma \tilde{\beta}^T_L \right] ,
\end{equation}
where $\alpha, \beta$ denote $d, s, b$. Those coefficients are given at $\tau = i\infty$ as:
\begin{align}
C_{bLbL}^T &= \frac{1}{36} \left( 36s_{q3} + 4t_{qss} - 9t_{qaa} - 6t_{qas} + 6t_{qsa} \right) , \\
C_{sLSL}^T &= \frac{1}{36} \left( 36s_{q2} + 4t_{qss} - 9t_{qaa} + 6t_{qas} - 6t_{qsa} \right) , \\
C_{dLdL}^T &= \frac{1}{9} \left( 9s_{q1} + 4t_{qss} \right) , \\
C_{sLbL}^T = C_{dLbL}^T = C_{dLSL}^T &= 0 .
\end{align}
It is noticed that the flavor changing operators vanish at $\tau = i\infty$.

Next step, we include the finite effect of the modular forms and transform $d_L \to d_L^m \equiv U^T_{md}d_L$ as given in Eq. (4.40) and obtain
\[
C_{sLbL}^{Tm} = \frac{1}{3} s_{23}^d (3s_{q2} - 3s_{q3} + t_{qsa} - t_{qsa}) + \frac{1}{9} \delta^s(9s_{q3} - 3t_{qsa} - 2t_{qss}),
\]
\[
C_{dLSb}^{Tm} = -\frac{1}{3} s_{12}^d s_{23}^d (3s_{q2} - 3s_{q3} + t_{qsa} - t_{qsa}) - \frac{1}{9} s_{12}^d \delta^s(9s_{q3} - 3t_{qsa} - 2t_{qss})
\]
\[
+ \frac{1}{9} \delta(9s_{q1} + 3t_{qsa} - t_{qsa})) + \frac{1}{36} s_{12}^d (36s_{q2} + 4t_{qss} + 9t_{qaa} - 6t_{qsa} - 6t_{qss}),
\]
\[
C_{dLS}^{Tm} = s_{12}^d (s_{q1} - s_{q2} + \frac{1}{3} t_{qss} + \frac{1}{4} t_{qaa} + \frac{1}{6} t_{qsa} - \frac{1}{6} t_{qas})
\]
\[
+ \delta^s(s_{q2} + \frac{1}{3} t_{qss} + \frac{1}{4} t_{qaa} - \frac{1}{6} t_{qsa} - \frac{1}{6} t_{qas}).
\]

In order to obtain the $U(2)$ symmetry like relation
\[
\frac{C_{dLS}^{Tm}}{C_{sLbL}^{Tm}} = -s_{12}^d = \frac{V_{td}^*}{V_{ts}},
\]
we impose relations of parameters
\[
t_{qss} = 9s_{q1} + 3t_{qsa}, \quad t_{qaa} = -4(s_{q1} + 3s_{q2}),
\]
with all real parameters. Then, we have a reasonable result for $C_{dLS}^{Tm}$
\[
C_{dLS}^{Tm} = s_{12}^d (3s_{q1} - 2s_{q2} + t_{qsa}),
\]
which is proportional to $s_{12}^d$.

In addition, imposing $t_{sa} = t_{ss} = 0$, which give $s_{q1} = 0$, we obtain
\[
\frac{C_{dLS}^{Tm}}{C_{sLbL}^{Tm}} = -s_{12}^d = \frac{V_{td}^*}{V_{ts}},
\]
which is also predicted in the $U(2)$ flavor symmetry.

### 5.5 $\Delta F = 1$ semileptonic operators in $A_4^E \otimes A_4^Q$ symmetry

Let us discuss the semileptonic flavor changing neutral processes,
\[
b \to s \bar{\mu} \mu (s \bar{e}e), \quad b \to d \bar{\mu} \mu (d \bar{e}e), \quad s \to d \bar{\mu} \mu (d \bar{e}e),
\]
which are caused by the flavor changing $\Delta F = 1$ operators of $[E_L \gamma_{\mu} E_L][\bar{D}_L \gamma_{\mu} D_L]$.

Suppose the $A_4$ modular symmetry on quarks and leptons, respectively, that is $A_4^E \otimes A_4^Q$. Then, those operators are given simply by the products of bilinear fermion operators; We can predict the correlations among processes in Eq. (5.37) by using the results of the previous subsections.

The $b_L \to s_L$, $b_L \to d_L$ and $s_L \to d_L$ transitions have been discussed in subsections 5.2, 5.3 and 5.4. The transition ratio of $b \to s \bar{e}e$ to $b \to s \bar{\mu} \mu$ is given by the ratio of $C_{sLbL}^{Sm}$ to $C_{sLbL}^{Sm}$, which are presented in Eq. (5.14). Since $C_{sLbL}^{Sm}$ is comparable to $C_{sLbL}^{Sm}$, the $b \to s \bar{e}e$ process is not suppressed compared with $b \to s \bar{\mu} \mu$. Since the $A_4$ modular symmetry controls the flavor structure of NP in our framework, the investigation of the ratio of $B \to K^{(*)} \bar{\mu} \mu$ to $B \to K^{(*)} \bar{e}e$ provides an important test. Also $B \to \bar{\mu} \mu$ and $B \to \bar{e}e$ are interesting processes because those include $[E_L E_R][D_R D_L]$ operator.

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6 4-quark operator with $\Delta F = 2$ in $A_4$ symmetry

Now we discuss the left-handed 4-quark operator of which corresponding SMEFT operators $Q$ are given as

$$[\bar{D}_L \Gamma D_L] [\bar{D}_L \Gamma D_L] : Q_{qq}^{(1)}, Q_{qq}^{(3)}.$$  

In the modular symmetry, the $A_4$ flavor coefficient of the 4-quark operator $\bar{L} L L L L$ is not given by the products of coefficients of the bilinear quark operators. Therefore, we discuss the 4-quark operators directly, which are obtained by extracting the $A_4$ singlet component from

$$[\bar{D}_L \gamma_\mu (Y^*(\tau_q)Y(\tau_q)) D_L] [\bar{D}_L \gamma_\mu (Y^*(\tau_q)Y(\tau_q)) D_L].$$  

This is expressed by three parts:

$$\left\{ [\bar{D}_L \gamma_\mu (Y^*(\tau_q)Y(\tau_q)) D_L] [\bar{D}_L \gamma_\mu (Y^*(\tau_q)Y(\tau_q)) D_L] \right\}_1 = \Omega_1 + \Omega_2 + \Omega_3,$$

where $\Omega_1$, $\Omega_2$ and $\Omega_3$ are decomposed as:

$$\begin{align*}
\Omega_1 &= \left[ (\bar{D}_L Y^*)_1 \gamma_\mu (Y D_L)_1 \oplus (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1'' \oplus (\bar{D}_L Y^*)_1'' \gamma_\mu (Y D_L)_1' \right]_1 \\
&\times \left[ (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1 \oplus (\bar{D}_L Y^*)_1'' \gamma_\mu (Y D_L)_1' \oplus (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1'' \right]_1 \\
&+ \left[ (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1' \oplus (\bar{D}_L Y^*)_1'' \gamma_\mu (Y D_L)_1' \oplus (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1'' \right]_1 \\
&\times \left[ (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1' \oplus (\bar{D}_L Y^*)_1'' \gamma_\mu (Y D_L)_1' \oplus (\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1'' \right]_1' \\
\Omega_2 &= \left\{ \sum_{i,j,s,a} (\bar{D}_L Y^*)_3 i \gamma_\mu (Y D_L)_3 j \right\}_1' \left\{ \sum_{i'j'=s,a} (\bar{D}_L Y^*)_3 i' \gamma_\mu (Y D_L)_3 j' \right\}_1' \\
&+ \left\{ \sum_{i,j,s,a} (\bar{D}_L Y^*)_3 i \gamma_\mu (Y D_L)_3 j \right\}_1' \left\{ \sum_{i'j'=s,a} (\bar{D}_L Y^*)_3 i' \gamma_\mu (Y D_L)_3 j' \right\}_1', \\
\Omega_3 &= \left\{ \sum_{I,J,s,a} \left[ \sum_{i,j,s,a} (\bar{D}_L Y^*)_3 i \gamma_\mu (Y D_L)_3 j \right]_3 I \left[ \sum_{i'j'=s,a} (\bar{D}_L Y^*)_3 i' \gamma_\mu (Y D_L)_3 j' \right]_3 J \right\}_1.
\end{align*}$$

As seen in Eq. (6.3), $\Omega_1$, $\Omega_2$ and $\Omega_3$ have 18 terms, 32 terms and 64 terms, respectively. Finally, we have 114 terms in this decompositions (see also Appendix E).

We find that the MFV cannot be reproduced if all terms contribute to the FC processes. Therefore, we present three cases, in which a specific parameter set dominates the 4-quark operator with $\Delta F = 2$.

6.1 4-quark operator with $\Delta F = 2$ at nearby $\tau_q = i$

As far as $\Omega_2$ and $\Omega_3$ dominate the FC processes, the MFV cannot be reproduced without fine tuning of parameters. As a simple example, if the following terms dominate the FC processes;

$$s_{Q_{1x}} [(\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1] [(\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1]$$

$$+ s_{Q_{1y}} [(\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1''] [(\bar{D}_L Y^*)_1' \gamma_\mu (Y D_L)_1''],$$

(6.4)
where \( s_{Q1z} \) and \( s_{Q1y} \) are complex parameters, we obtain simple results:

\[
C_{s_{LbLsLbL}}^{sm} = (4\sqrt{3} - 7)Y_1(i)^2(9s_{23}^d - 6\sqrt{3}e_1^s s_{23}^d + \epsilon_1^2)(s_{Q1x} + 4s_{Q1y}),
\]
\[
C_{d_{LbLdLbL}}^{sm} = (4\sqrt{3} - 7)Y_1(i)^2(s_{12}^d)^2(9s_{23}^d - 6\sqrt{3}e_1^s s_{23}^d + \epsilon_1^2)(s_{Q1x} + 4s_{Q1y}),
\]
\[
C_{d_{lsLdLsL}}^{sm} = -4(4\sqrt{3} - 7)Y_1(i)^2(s_{12}^d)^2|\epsilon_1|^4(s_{Q1x} + 4s_{Q1y}),
\]

(6.5)

where \( C_{s_{LbLsLbL}}^{sm} \) is the coefficient of the operator \([\bar{s}_L\gamma_\mu b_L)(\bar{s}_L\gamma_\mu b_L)]\), and so on. Then, we have a typical relation of the MFV:

\[
\left| \frac{C_{d_{LbLdLbL}}^{sm}}{C_{s_{LbLsLbL}}^{sm}} \right|^2 = (s_{12}^d)^2 = \left| \frac{V_{td}^*}{V_{ts}} \right|^2,
\]

(6.6)

where Eq. (4.18) is put.

### 6.2 4-quark operator with \( \Delta F = 2 \) at nearby \( \tau_q = \omega \)

As a simple example, if the following terms dominate the FC processes:

\[
s_{Q1z}[(\bar{D}_L Y_q^*(\tau))_{1\nu} \gamma_\mu (Y_\nu(\tau) D_L)_{1\nu}][[(\bar{D}_L Y_q^*(\tau))_{1\nu} \gamma_\mu (Y_\nu(\tau) D_L)_{1\nu}],
\]

(6.7)

where \( s_{Q1z} \) is a complex parameter, we obtain simple results:

\[
C_{s_{LbLsLbL}}^{STm} = \frac{9}{16} Y_1(\omega)^2(3s_{23}^d - 2\epsilon_1^s)^2 s_{Q1z},
\]
\[
C_{d_{LbLdLbL}}^{STm} = (s_{12}^d)^2 \frac{9}{16} Y_1(\omega)^2(3s_{23}^d - 2\epsilon_1^s)^2 s_{Q1z},
\]
\[
C_{d_{LSdLsL}}^{STm} = (s_{12}^d)^2 \frac{1}{16} Y_1(\omega)^2|3s_{23}^d - 2\epsilon_1^s|^4 s_{Q1z}.
\]

(6.8)

These give the MFV like relation in Eq. (6.6).

### 6.3 4-quark operator with \( \Delta F = 2 \) towards \( \tau_q = i\infty \)

As a simple example, if the following terms dominate the FC processes:

\[
s_{Q1z}^*[[(\bar{D}_L Y_q^*(\tau))_{1\nu} \gamma_\mu (Y_\nu(\tau) D_L)_{1\nu}][[(\bar{D}_L Y_q^*(\tau))_{1\nu} \gamma_\mu (Y_\nu(\tau) D_L)_{1\nu}],
\]

(6.9)

where \( s_{Q1z} \) is the same one in Eq. (6.7), we obtain simple results:

\[
C_{s_{LbLsLbL}}^{TM} = (s_{23}^d + \delta^s)^2 s_{Q1z},
\]
\[
C_{d_{LbLdLbL}}^{TM} = (s_{12}^d)^2 (s_{23}^d + \delta^s)^2 s_{Q1z},
\]
\[
C_{d_{LSdLsL}}^{TM} = (s_{12}^d)^2 |s_{23}^d + \delta^s|^4 s_{Q1z}.
\]

(6.10)

These also give MFV like relation in Eq. (6.6).
7 Summary

We have studied the modular symmetric standard-model effective field theory. We have employed the stringy Ansatz on the coupling structure that 4-point couplings \( y^{(4)} \) of matter fields are written by a product of 3-point couplings \( y^{(3)} \) of matter fields, i.e., \( y^{(4)} = y^{(3)} y^{(3)} \). In this framework, we have discussed the flavor structure of bilinear fermion operators and 4-fermion operators.

In order to cover many modular flavor models, we take a setup that the \( A_4 \) modular flavor symmetry in the lepton sector is independent of the \( A_4 \) symmetry in the quark sector, i.e., \( A_4^L \otimes A_4^Q \) symmetry. They have two independent moduli, \( \tau_q \) and \( \tau_e \) for the quark sector and the lepton sector, respectively. Moreover, we take the holomorphic and anti-holomorphic modular forms with weight 2, which couples to quarks and leptons.

From the viewpoint of the Ansatz \( y^{(4)} = y^{(3)} y^{(3)} \), the \( A_4 \) modular-invariant semileptonic 4-fermion operator \( [ \tilde{E}_R \Gamma E_R ] [ \tilde{D}_R \Gamma D_R ] \) does not lead to the FC processes since this operator would be constructed in terms of gauge couplings \( g \) as \( y^{(3)} \sim g \).

The bilinear operator \( [ \tilde{D}_R \Gamma D_L ] \) also does not lead FC if the mediated mode corresponds to the Higgs boson \( H_d \) in the viewpoint of the Ansatz. In this case, the flavor structure of this operator is the exactly same as the mass matrix. Then, the bilinear operator matrix is diagonal in the basis for mass eigenstates. The FC processes such \( b \to s, b \to d, s \to d \) never happen. On the other hand, if the flavor structure of the operator is not the exactly same as the mass matrix, the situation would change drastically. Then, we have obtained the non-trivial relations of the FC transitions at nearby fixed points \( \tau = i, \omega, i \infty \), which are testable in the future.

As an application, we have discussed the LFV processes, \( \mu \to e\gamma, \tau \to e\gamma \) and \( \tau \to \mu\gamma \). We have estimated the branching ratios by using \( A_4 \) flavor coefficients at nearby \( \tau_e = i \) since the successful lepton mass matrix has been obtained there. We have predicted \( \text{BR}(\mu \to e\gamma) \gtrsim \text{BR}(\tau \to e\gamma) \gg \text{BR}(\tau \to \mu\gamma) \) at nearby \( \tau_e = i \). This prediction is different from the one of the \( U(2) \) flavor symmetry.

We also have studied the flavor changing 4-quark operators in the \( A_4 \) modular symmetry of quarks. Then, the MFV could be realized by taking relevant specific parameter sets of order one.

The flavor symmetry is a useful way for a systematic NP analysis of the SMEFT. The finite modular group is one of the attractive choice for the flavor symmetry, and the characteristic predictions are derived. This application on the SMEFT would be a first step of a systematic analysis for the modular symmetric model, connecting quark and lepton flavor observables.

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Appendix

A SMEFT operators

| Class 5–7: Fermion Bilinears | (LR) | (RR) | (RR') |
|-------------------------------|------|------|-------|
| 5: $\psi^2 H^3 + \text{h.c.}$ | $(e\ell)$ | $Q_{\ell H} (H^H H) (\ell_{\mu} e_{\nu} H)$ | $Q_{\ell W} (\ell_{\mu} \sigma^{\mu\nu} e_{\nu}) T^I H W_{\mu\nu}^I$ | $Q_{\ell B} (\ell_{\mu} \sigma^{\mu\nu} e_{\nu}) H B_{\mu\nu}$ |
| 6: $\psi^2 X H + \text{h.c.}$ | $(q\ell)$ | $Q_{q H} (H^H H) (q_{\mu u_{\nu}} H)$ | $Q_{q G} (q_{\mu u_{\nu}} T^A u_{\nu}) H G_{\mu\nu}^A$ | $Q_{q W} (q_{\mu u_{\nu}} T^A u_{\nu}) T^I H W_{\mu\nu}^I$ | $Q_{q B} (q_{\mu u_{\nu}} T^A u_{\nu}) H B_{\mu\nu}$ |
| 7: $\psi^2 H^2 D$ | $(\ell\ell)$ | $Q_{\ell H} (H^H D_{\mu} H) (\ell_{\mu} e_{\nu} H)$ | $Q_{\ell c} (H^H D_{\mu} H) (\ell_{\mu} e_{\nu} H)$ | $Q_{\ell t} (H^H D_{\mu} H) (\ell_{\mu} e_{\nu} H)$ | $Q_{\ell B} (H^H D_{\mu} H) (\ell_{\mu} e_{\nu} H)$ |
| | $(\ell q)$ | $Q_{\ell q} (H^H D_{\mu} H) (\ell_{\mu} q_{\nu})$ | $Q_{\ell u} (H^H D_{\mu} H) (\ell_{\mu} q_{\nu})$ | $Q_{\ell d} (H^H D_{\mu} H) (\ell_{\mu} q_{\nu})$ | $Q_{\ell B} (H^H D_{\mu} H) (\ell_{\mu} q_{\nu})$ |

| Class 8: Fermion Quadrilinears | (LR)(LR) | (RR)(RR) | (LL)(RR) |
|-------------------------------|----------|----------|----------|
| semileptonic | $Q_{\ell q}^{(1)} (\ell_{\mu} e_{\nu} q_{\lambda}) (\ell_{\mu} q_{\nu} e_{\lambda})$ | $Q_{\ell u} (\ell_{\mu} e_{\nu} q_{\lambda}) (\ell_{\mu} q_{\nu} e_{\lambda})$ | $Q_{\ell t} (\ell_{\mu} e_{\nu} q_{\lambda}) (\ell_{s\mu} q_{\nu} e_{\lambda})$ | $Q_{\ell t} (\ell_{\mu} e_{\nu} q_{\lambda}) (\ell_{s\mu} q_{\nu} e_{\lambda})$ |
| 4-quark | $Q_{\ell q}^{(2)} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\nu} e_{\lambda})$ | $Q_{\ell u} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\mu} e_{\lambda})$ | $Q_{\ell d} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\mu} e_{\lambda})$ | $Q_{\ell t} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{s\mu} q_{\mu} e_{\lambda})$ |
| 4-lepton | $Q_{\ell t} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\nu} e_{\lambda})$ | $Q_{\ell u} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\mu} e_{\lambda})$ | $Q_{\ell d} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{\mu} q_{\mu} e_{\lambda})$ | $Q_{\ell t} (\ell_{\mu} q_{\nu} e_{\lambda}) (\ell_{s\mu} q_{\mu} e_{\lambda})$ |

Table 12: List of all fermionic SMEFT operators in the Warsaw basis [2]. The division in classes is adopted from Ref. [3]. The $p, r, s, t$ are flavor index, and $j, k$ stand for SU(2) index. The operator classes 1–4 without fermion fields are irrelevant in this paper, and not listed here.

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B A model of mass matrices in the $A_4$ modular symmetry

We present a viable model for quarks [71]. Suppose that three left-handed quark doublets are of a triplet of the $A_4$ group. The three right-handed quarks are three different singlets of $A_4$. On the other hand, the Higgs doublets are supposed to be singlets of $A_4$. The generic assignments of representations and modular weights to the minimal supersymmetric standard model (MSSM) fields are presented in Table 13, where weight 2 and 6 modular forms are presented.

| $SU(2)$ | $A_4$ | $k$ | $Q_L$ | $(d_R^c, s_R^c, b_R^c)$ | $(u_R^c, c_R^c, t_R^c)$ | $H_q$ | $Y_{q}^{(2)}$ | $Y_{q}^{(6)}$ | $Y_{q}^{(6)}$ |
|---------|-------|-----|-------|-------------------------|-------------------------|-------|---------------|---------------|---------------|
| 2       | 3     | 2   | (2)   | 1                       | (1, 1″, 1′)             | 1     | 2             | 1             | 6             |

Table 13: An example of assignments of weights $k$ for quarks and modular forms.

Then, up-type $M_u$ and down-type $M_d$ quark mass matrices are given, respectively as:

$M_u = v_u \begin{pmatrix} \alpha_{u(m)} & 0 & 0 \\ 0 & \beta_{u(m)} & 0 \\ 0 & 0 & \gamma_{u(m)} \end{pmatrix} \begin{pmatrix} Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \\ Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \\ Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \end{pmatrix} + \begin{pmatrix} \alpha'_{u(m)} & 0 & 0 \\ 0 & \beta'_{u(m)} & 0 \\ 0 & 0 & \gamma'_{u(m)} \end{pmatrix} \begin{pmatrix} Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \\ Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \\ Y_1^{(6)} & Y_2^{(6)} & Y_3^{(6)} \end{pmatrix} \right]_{RL}$

$M_d = v_d \begin{pmatrix} \alpha_{d(m)} & 0 & 0 \\ 0 & \beta_{d(m)} & 0 \\ 0 & 0 & \gamma_{d(m)} \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix} \right]_{RL}$

(B.1)

where $\tilde{Y}_i^{(6)} = Y_i^{(6)} + g_i Y_i^{(6)}$ with $g_{u1} = \alpha'_{u(m)}/\alpha_{u(m)}$, $g_{u2} = \beta'_{u(m)}/\beta_{u(m)}$, $g_{u3} = \gamma'_{u(m)}/\gamma_{u(m)}$ and $g_q = \alpha'_{q(m)}/\alpha_{q(m)}$. The VEV of the Higgs field $H_q$ is denoted by $v_q$. Parameters $\alpha_{q',} \beta_q, \gamma_q$ can be taken to be real, on the other hand, $g_{u1}, g_{u2}, g_{u3}$ and $g_u$ are complex parameters.

For the lepton sector, we also present mass matrices of a successful model [77], where the neutrino mass matrix is given in terms of weight 4 modular forms by using Weinberg operator. The assignments of representations and modular weights to the lepton fields are presented in Table 14.

| $SU(2)$ | $A_4$ | $k$ | $L_L$ | $(\epsilon_R^c, \mu_R^c, \tau_R^c)$ | $H_q$ | $Y_{\ell}^{(2)}$ | $Y_{\ell}^{(4)}$ |
|---------|-------|-----|-------|-------------------------|-------|---------------|---------------|
| 2       | 3     | 2   | (1, 1″, 1′) | 1                       | 3     | 3             | \{3, 1, 1′\} |

Table 14: Representations and weights $k$ for MSSM fields and modular forms of weight 2 and 4.
The charged lepton mass matrix and neutrino ones are given as:

\[
M_E = v_d \begin{pmatrix} \alpha_{e(m)} & 0 & 0 \\ 0 & \beta_{e(m)} & 0 \\ 0 & 0 & \gamma_{e(m)} \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL},
\]

\[
M_\nu = \frac{v^2}{\Lambda} \left[ \begin{array}{ccc}
2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\
-Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\
-Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} 
\end{array} \right] + g'_1 Y_1^{(4)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g'_2 Y_1^{(4)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

respectively, where \(\alpha_{e(m)}, \beta_{e(m)}\) and \(\gamma_{e(m)}\) are real parameters and \(g'_1, g'_2\) are supposed to be real.

C Unitary transformation of \(S\) and \(ST\) and mass matrix

The mass matrix is transformed by the unitary transformation, which transforms the generator \(S\) and \(ST\). We discuss details at the fixed points \(\tau = i\) and \(\tau = \omega\).

C.1 Diagonal base of \(S\) for \(A_4\) triplet

The generators of \(A_4\) group for the triplet are:

\[
S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

where \(\omega = e^{i\frac{2\pi}{3}}\) for a triplet. The eigenvalues of \(S\) is \((1, -1, -1)\). This is the diagonal base of \(T\).

In order to present the mass matrices in the diagonal base of \(S\), we move to the diagonal base of \(S\) as follows:

\[
U_{S1} S U_{S1}^\dagger = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_{S2} S U_{S2}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_{S3} S U_{S3}^\dagger = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where

\[
U_{Si} \equiv P_i \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Then, the generator \(T\) is not anymore diagonal. However, the eigenvalue \(-1\) of \(S\) is degenerated, there is a freedom of the rotation between corresponding rows and between columns.

As seen in subsection C.3, the Dirac mass matrix \(M_{RL}\) is transformed as:

\[
\hat{M}_{RL} = M_{RL} U_{S1}^\dagger,
\]

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If there is a residual symmetry of $A_4$ in the Dirac mass matrix, $\mathbb{Z}_2^S = \{I, S\}$, which is realized at the fixed point $\tau = i$ as presented in Eq. 4.9, the generator $S$ commutes with $\hat{M}_{RL}^\dagger \hat{M}_{RL}$,

$$\left[\hat{M}_{RL}^\dagger \hat{M}_{RL}, S\right] = 0. \quad (C.5)$$

Therefore, the mass matrix could be diagonal in the diagonal base of $S$.

### C.2 Diagonal bases of $ST$ and $T$ for $A_4$ triplet

We can move $ST$ to the diagonal base by the six-type unitary transformations $V_{ST}$ as follows:

$$U_{STi} ST U_{STi}^\dagger = P_i \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} P_i^T, \quad (C.6)$$

where

$$U_{STi} = \frac{1}{3} P_i \begin{pmatrix} -2\omega^2 & -2\omega & 1 \\ -\omega^2 & 2\omega & 2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (C.7)$$

and $P_i (i = 1, 2, 3)$ are given in Eq. (C.3). The order of eigenvalues of $ST$ depends on $P_i$. We have eigenvalues $\{\omega^2, \omega, 1\}$ for $P_1$, $\{\omega, \omega^2, 1\}$ for $P_2$, $\{\omega^2, 1, \omega\}$ for $P_3$, $\{1, \omega, \omega^2\}$ for $P_4$, $\{1, \omega^2, \omega\}$ for $P_5$ and $\{\omega, 1, \omega^2\}$ for $P_6$, respectively. Thus, there are independent six unitary transformations to move $ST$ to the diagonal base.

As seen in subsection C.3, the Dirac mass matrix $M_{RL}$ is transformed as:

$$\hat{M}_{RL} = M_{RL} U_{STi}^\dagger. \quad (C.8)$$

If there exists the residual symmetries of the $A_4$ group $\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$ which is realized at the fixed point $\tau = \omega$ as presented in Eq. (4.9), we have

$$\left[\hat{M}_{RL}^\dagger \hat{M}_{RL}, ST\right] = 0, \quad (C.9)$$

which leads to the diagonal $\hat{M}_{RL}^\dagger \hat{M}_{RL}$ because $ST$ has three different eigenvalues.

On the other hand, the generator $T$ is already diagonal in the original base of Eq. (C.1). If there exists the residual symmetries of the $A_4$ group $\mathbb{Z}_3^T = \{I, T, T^2\}$, which is realized at $\tau = i\infty$,

$$\left[M_{RL}^\dagger M_{RL}, T\right] = 0, \quad (C.10)$$

which also gives the diagonal $M_{RL}^\dagger M_{RL}$. 

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C.3 Mass matrix in the new bases of generators $S$ and $ST$

Define the new basis of generators, $\tilde{G}$ ($G = S, ST$) by a unitary transformation $U$ as:

$$\tilde{G} = UGU^\dagger,$$  \hspace{1cm} (C.11)

where $\tilde{G}$, $G$ and $U$ are $3 \times 3$ matrices. Since the $A_4$ triplet transforms under the unitary transformation as:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_3 \rightarrow G \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_3 = U^\dagger \tilde{G} U \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_3.$$  \hspace{1cm} (C.12)

Thus, in the new basis, the $A_4$ triplet transforms as:

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}_3 \rightarrow \tilde{G} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}_3,$$  \hspace{1cm} (C.13)

where

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}_3 = U \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_3.$$  \hspace{1cm} (C.14)

Let us rewrite the Dirac mass matrix $M_{RL}$ in the new base ($\tilde{G}$) of the triplet left-handed fields. Denoting $L$ and $\hat{L}$ to be triplets of the left-handed fields in the base of $G$ and $\tilde{G}$, respectively, and $R$ to be right-handed singlets, the Dirac mass matrix is written as:

$$\bar{R}M_{RL}L = \bar{R}M_{RL}U^\dagger \hat{L},$$  \hspace{1cm} (C.15)

where

$$L = U^\dagger \hat{L}.$$  \hspace{1cm} (C.16)

Then, the Dirac mass matrix $\hat{M}_{RL}$ in the new base is given as:

$$\hat{M}_{RL} = M_{RL}U^\dagger.$$  \hspace{1cm} (C.17)

D Mass matrix at nearby $\tau = i$

Both down-type quark mass matrix of Eq. (B.1) and the charged lepton mass matrix of Eq. (B.2) are given in terms of weight 2 modular forms.

Perform the unitary transformation of $D_L \rightarrow U_SD_L$, where $U_S = U_{23}(15^\circ)U_{S2}$ (see Appendix C.3)

$$U_S = U_{23}(-15^\circ)U_{S2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 15^\circ & \sin 15^\circ \\ 0 & -\sin 15^\circ & \cos 15^\circ \end{pmatrix} U_{S2} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & 2 & 2 \\ \sqrt{3} + 1 & -2 & \sqrt{3} - 1 \\ \sqrt{3} - 1 & -2 & \sqrt{3} + 1 \end{pmatrix}.$$  \hspace{1cm} (D.1)
Then, the down-type quark mass matrix at $\tau = i$ in Eq. (B.1) is simply given as:

$$M_d = \frac{1}{2} v_d \begin{pmatrix} 0 & 3(\sqrt{3} - 1) \tilde{\alpha}_{d(m)} & -3(\sqrt{3} - 1) \tilde{\beta}_{d(m)} \\ 0 & -3(\sqrt{3} - 1) \tilde{\beta}_{d(m)} & -3(\sqrt{3} - 1) \tilde{\alpha}_{d(m)} \\ 2(3 - \sqrt{3}) \tilde{\gamma}_{d(m)} & 0 & 0 \end{pmatrix},$$

where $\tilde{\alpha}_{d(m)} = (6 - 3\sqrt{3}) Y_1(i) \alpha_{d(m)}$, $\tilde{\beta}_{d(m)} = (6 - 3\sqrt{3}) Y_1(i) \beta_{d(m)}$, and $\tilde{\gamma}_{d(m)} = (6 - 3\sqrt{3}) Y_1(i) \gamma_{d(m)}$.

The rotation $U_{23}(-15^\circ)$ realizes that the $(3,3)$ entry is much larger than other entries. Since two eigenvalues of $S$ are degenerate such as $(1, -1, -1)$, there is still a freedom of the 2–3 family rotation. Therefore, $M_d^U M_d$ could be diagonal after the small 2–3 family rotation of $O(\tilde{\alpha}_{d(m)}^2/\tilde{\gamma}_{d(m)}^2, \tilde{\beta}_{d(m)}^2/\tilde{\gamma}_{d(m)}^2)$. The charged lepton mass matrix is the same one in Eq. (D.2) by replacing the subscript $d$ with $e$.

Since the quark and lepton mass matrices cannot reproduce the observed CKM and PMNS matrices at fixed points as discussed in Ref. [71]. Therefore, the deviations from the fixed points are required to realize observed masses and mixing angles.

Let us consider the small deviation from $\tau = i$. By using the approximate modular forms of weight 2 at $\tau = i + \epsilon$, we present $M_d^U M_d$ including of order $\epsilon$. By the unitary transformation $D_L \rightarrow U_{12} U_S D_L$, where $U_{12}$ is the first-second family exchange,

$$U_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the mass matrix of Eq. (D.2) is modified as:

$$M_d^U M_d = v_d^2 \begin{pmatrix} \frac{\alpha_{d(m)}^2}{2} (2 - \sqrt{3}) (\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) & \frac{3(\sqrt{3} - 2)}{2} (\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) \epsilon_1 & \frac{3(\sqrt{3} - 2)}{4} (\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) \epsilon_1 \\ \frac{3(\sqrt{3} - 2)}{2} (\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) \epsilon_1 & \frac{2(3 - \sqrt{3})}{2} (\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) \epsilon_1 & \frac{2(3 - \sqrt{3})}{4} (\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) \epsilon_1 \\ \frac{3(\sqrt{3} - 2)}{4} (\tilde{\alpha}_{d(m)}^2 - \tilde{\beta}_{d(m)}^2) \epsilon_1 & \frac{2(3 - \sqrt{3})}{4} (\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) \epsilon_1 & \frac{2(3 - \sqrt{3})}{4} (\tilde{\alpha}_{d(m)}^2 + \tilde{\beta}_{d(m)}^2) \epsilon_1 \end{pmatrix},$$

where $\gamma_{d(m)}^2 \gg \alpha_{d(m)}^2, \beta_{d(m)}^2$ is taken and $\epsilon_2 = 2 \epsilon_1$ in Eq. (4.15) is put. The extra transformation $U_{12}$ is performed to keep the hierarchy among matrix elements $(3,3) \gg (2,3) \gg (1,3)$ because the CKM mixing angles are hierarchical.

Then, we can estimate the mixing matrix $U_{dm}$ of order $O(\epsilon_1)$ to diagonalize $M_d^U M_d$:

$$U_{12}^\dagger M_d^U M_d U_{12} = \text{diag}(m_d^2, m_s^2, m_b^2), \quad U_{md} \simeq \begin{pmatrix} 1 & s_{12} d e^{-i \eta} & 0 \\ -s_{12} d e^{-i \eta} & 1 & s_{23} \\ s_{12} s_{23} d e^{-i \eta} & -s_{23} & 1 \end{pmatrix},$$

where $\eta = 2 \text{arg} \{\epsilon_1\}$. The mixing angle $s_{23}^d$ is of order $O(\epsilon_1)$. On the other hand, $s_{12}^d$ depends on $\alpha_{d(m)}^2/\gamma_{d(m)}^2, \beta_{d(m)}^2/\gamma_{d(m)}^2$ and $\epsilon_1$. In the numerical fit, $s_{12}^d \sim 0.1$ has been obtained [47].

On the other hand, the transformation of the charged leptons is still $E_L \rightarrow U_S E_L$, where $U_{12}$ of Eq. (D.3) does not appear because the large mixing angle between the second and third families
is reproduced in the neutrino mass matrix under this transformation, which is preferable for the lepton mixing matrix. Then, at \( \tau = i + \epsilon \), we have

\[
M_{E\text{--}E}^R \simeq \nu _d \begin{pmatrix}
2(2 - \sqrt{3})(\alpha _{e(m)}^2 + \tilde{\alpha }_{e(m)}^2 + \tilde{\delta }_{e(m)}^2)\epsilon _1 | & 3(\sqrt{3} - 2)(\alpha _{e(m)}^2 - \tilde{\beta }_{e(m)}^2)\epsilon _1 | & (3 - 2\sqrt{3})(\alpha _{e(m)}^2 - \tilde{\beta }_{e(m)}^2)\epsilon _1 | \\
3(\sqrt{3} - 2)(\tilde{\alpha }_{e(m)}^2 - \tilde{\beta }_{e(m)}^2)\epsilon _1 | & \frac{3}{2}(2 - \sqrt{3})(\alpha _{e(m)}^2 + \tilde{\beta }_{e(m)}^2)\epsilon _1 | & \frac{3}{2}(3 - 2\sqrt{3})(\alpha _{e(m)}^2 + \tilde{\beta }_{e(m)}^2)\epsilon _1 |\\
(3 - 2\sqrt{3})(\tilde{\alpha }_{e(m)}^2 - \tilde{\beta }_{e(m)}^2)\epsilon _1 | & \frac{3}{2}(2 - \sqrt{3})(\tilde{\alpha }_{e(m)}^2 + \tilde{\beta }_{e(m)}^2)\epsilon _1 | & \frac{3}{2}(3 - 2\sqrt{3})(\tilde{\alpha }_{e(m)}^2 + \tilde{\beta }_{e(m)}^2)\epsilon _1 |
\end{pmatrix},
\]

where \( \gamma _{e(m)}^2 \gg \alpha _{e(m)}^2, \beta _{e(m)}^2 \) is taken and \( \epsilon _2 = 2\epsilon _1 \) in Eq. (4.15) is put. It is remarked that the phase of \( \epsilon _1 \) can be removed. That is a real matrix. Since \( \alpha _{e(m)}^2/\gamma _{e(m)}^2 \) and \( \beta _{e(m)}^2/\gamma _{e(m)}^2 \) are smaller than \( 10^{-3} \), it is noticed that this matrix is a rank one matrix in the limit of \( \alpha _{e(m)}^2 = \beta _{e(m)}^2 = 0 \). Taking relevant value of \( \alpha _{e(m)}^2/\gamma _{e(m)}^2, \beta _{e(m)}^2/\gamma _{e(m)}^2 \) and \( |\epsilon _1| \), desired lepton masses could be obtained.

The mixing matrix \( U_{me} \) to diagonalize \( M_{E\text{--}E}^R \):

\[
U_{me}^\dagger M_{E\text{--}E}^R U_{me} = \text{diag}(m_{\nu e}^2, m_{\mu}^2, m_{\tau}^2),
\]

\[
U_{me} \simeq \begin{pmatrix}
1 & s_{12}^c & s_{13}^c \\
-s_{12}^c & 1 & 0 \\
-s_{13}^c & 0 & 1
\end{pmatrix},
\]

where \( s_{13}^c \) is of order \( \mathcal{O}(|\epsilon _1|) \). On the other hand, \( s_{12}^c \) depends on \( \alpha _{e(m)}^2/\gamma _{e(m)}^2, \beta _{e(m)}^2/\gamma _{e(m)}^2 \) and \( |\epsilon _1| \). In the numerical fit, \( s_{12}^c \sim 0.1 \) has been obtained [77].

### E Decomposition of products of triplets

We take the generators of \( A_4 \) group for the triplet as follows:

\[
S = \frac{1}{3} \begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix},
\]

where \( \omega = e^{i \frac{2\pi}{3}} \) for a triplet. In this base, the multiplication rule is

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}_3 \otimes \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}_3 = (a_1 b_1 + a_2 b_3 + a_3 b_2)_1 + (a_3 b_3 + a_1 b_2 + a_2 b_1)_1',
\]

\[
\begin{pmatrix}
1 \\
1' \\
1''
\end{pmatrix}_3 \otimes \begin{pmatrix}
1 \\
1' \\
1''
\end{pmatrix}_3 = \begin{pmatrix}
1 \\
1' \\
1''
\end{pmatrix}_3, \quad 1' \otimes 1'' = 1', \quad 1'' \otimes 1' = 1', \quad 1' \otimes 1'' = 1',
\]

where

\[
T(1') = \omega, \quad T(1'') = \omega^2.
\]

More details are shown in the review [11, 12].
By using these multiplication rules, we have

\[
(YD_L)_1 = Y_1d_{L1} + Y_2d_{3L} + Y_3d_{2L},
\]
\[
(YD_L)_1' = Y_3d_{3L} + Y_1d_{2L} + Y_2d_{1L},
\]
\[
(YD_L)_1'' = Y_2d_{2L} + Y_1d_{3L} + Y_3d_{1L},
\]

(E.4)

\[
(YD_L)_{3s} = \begin{pmatrix}
(YD_L)_{3s1} \\
(YD_L)_{3s2} \\
(YD_L)_{3s3}
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
2Y_1d_{1L} - Y_2d_{3L} - Y_3d_{2L} \\
2Y_3d_{3L} - Y_1d_{2L} - Y_2d_{1L} \\
2Y_2d_{2L} - Y_1d_{3L} - Y_3d_{1L}
\end{pmatrix}_{3s},
\]

(E.5)

\[
(YD_L)_{3a} = \begin{pmatrix}
(YD_L)_{3a1} \\
(YD_L)_{3a2} \\
(YD_L)_{3a3}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
Y_2d_{3L} - Y_3d_{2L} \\
Y_1d_{2L} - Y_2d_{1L} \\
Y_3d_{1L} - Y_1d_{3L}
\end{pmatrix}_{3a},
\]

(E.6)

where the $A_4$ triplets $Y_3$ and $D_{L3}$ are abbreviated as $Y$ and $D_L$, respectively.

The similar formulae are obtained for $Y^*_i d^L_j$, but it is noticed that the exchange between $1'$ and $1''$, and the second entry and third one in the triplet representation, respectively, through their complex conjugates. That is:

\[
(Y^* D_L)_1 = Y_1^* d^L_1 + Y_2^* d^L_3 + Y_3^* d^L_2,
\]
\[
(Y^* D_L)_1' = Y_3^* d^L_3 + Y_1^* d^L_2 + Y_2^* d^L_1,
\]
\[
(Y^* D_L)_1'' = Y_2^* d^L_2 + Y_1^* d^L_3 + Y_3^* d^L_1,
\]

(E.7)

\[
(Y^* D_L)_{3s} = \begin{pmatrix}
(Y^* D_L)_{3s1} \\
(Y^* D_L)_{3s2} \\
(Y^* D_L)_{3s3}
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
2Y_1^* d^L_{1L} - Y_2^* d^L_{3L} - Y_3^* d^L_{2L} \\
2Y_3^* d^L_{3L} - Y_1^* d^L_{2L} - Y_2^* d^L_{1L} \\
2Y_2^* d^L_{2L} - Y_1^* d^L_{3L} - Y_3^* d^L_{1L}
\end{pmatrix}_{3s},
\]

(E.8)

\[
(Y^* D_L)_{3a} = \begin{pmatrix}
(Y^* D_L)_{3a1} \\
(Y^* D_L)_{3a2} \\
(Y^* D_L)_{3a3}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
Y_2^* d^L_{3L} - Y_3^* d^L_{2L} \\
Y_1^* d^L_{2L} - Y_2^* d^L_{1L} \\
Y_3^* d^L_{1L} - Y_1^* d^L_{3L}
\end{pmatrix}_{3a},
\]

(E.9)

By using these decompositions, we can calculate the product of following triplets:

\[
[(Y^* D_L)_{3s} \otimes (YD_L)_{3s}]_{3s} = \frac{1}{3} \begin{pmatrix}
2(Y^* D_L)_{3s1}(YD_L)_{3s1} - (Y^* D_L)_{3s2}(YD_L)_{3s3} - (Y^* D_L)_{3s3}(YD_L)_{3s2} \\
2(Y^* D_L)_{3s3}(YD_L)_{3s1} - (Y^* D_L)_{3s1}(YD_L)_{3s3} - (Y^* D_L)_{3s2}(YD_L)_{3s1} \\
2(Y^* D_L)_{3s2}(YD_L)_{3s1} - (Y^* D_L)_{3s1}(YD_L)_{3s2} - (Y^* D_L)_{3s3}(YD_L)_{3s1}
\end{pmatrix},
\]

(E.10)

\[
[(Y^* D_L)_{3a} \otimes (YD_L)_{3a}]_{3a} = \frac{1}{2} \begin{pmatrix}
(Y^* D_L)_{3a1}(YD_L)_{3a1} - (Y^* D_L)_{3a2}(YD_L)_{3a3} - (Y^* D_L)_{3a3}(YD_L)_{3a2} \\
(Y^* D_L)_{3a3}(YD_L)_{3a1} - (Y^* D_L)_{3a1}(YD_L)_{3a3} - (Y^* D_L)_{3a2}(YD_L)_{3a1}
\end{pmatrix},
\]

Other six decompositions $[(Y^* D_L)_{3s} \otimes (YD_L)_{3s}]_{3s}$, $[(Y^* D_L)_{3a} \otimes (YD_L)_{3a}]_{3a}$, $[(Y^* D_L)_{3a} \otimes (YD_L)_{3a}]_{3a}$, $[(Y^* D_L)_{3a} \otimes (YD_L)_{3a}]_{3a}$ and $[(Y^* D_L)_{3a} \otimes (YD_L)_{3a}]_{3a}$ are written similarly.
\section{U(2) flavor symmetry}

The U(2)\textsuperscript{5} flavor symmetry is the subgroup of U(3)\textsuperscript{5} and distinguishes the first two families of fermions from the third one \cite{7-9}. The flavor symmetry is decomposed as

\[ U(2)^5 = U(2)_L \otimes U(2)_Q \otimes U(2)_E \otimes U(2)_U \otimes U(2)_D. \]  

\text{(F.1)}

Under this symmetry, the first two families transform as doublets of the U(2) subgroups, whereas third family ones as a singlet. Then, the third-family Yukawa couplings are allowed by the symmetry, and it provides a natural explanation of why third-family Yukawa couplings are large. A minimal set of U(2)\textsuperscript{5} breaking terms (spurions) which reproduce the observed SM flavor parameters, without tuning and with minimal size for the breaking terms, is given by \cite{7}

\[ V_\ell \sim (2,1,1,1,1), \quad V_q \sim (1,2,1,1,1), \]

\[ \Delta_e \sim (2,1,\bar{2},1,1), \quad \Delta_u \sim (1,2,1,\bar{2},1), \quad \Delta_d \sim (1,2,1,1,\bar{2}), \]  

\text{(F.2)}

where V_{q,\ell} are complex two-vectors and \Delta_{e,u,d} are complex 2 \times 2 matrices. In terms of these spurions, the 3 \times 3 Yukawa matrices can be decomposed as

\[ Y_e = y_\ell \left( \begin{array}{cc} \Delta_e & x_\ell V_\ell \\ 0 & 1 \end{array} \right), \quad Y_u = y_t \left( \begin{array}{cc} \Delta_u & x_t V_q \\ 0 & 1 \end{array} \right), \quad Y_d = y_b \left( \begin{array}{cc} \Delta_d & x_b V_q \\ 0 & 1 \end{array} \right), \]  

\text{(F.3)}

where y_{\ell,t,b} and x_{\ell,t,b} are free complex parameters, expected to be of order \mathcal{O}(1). Using the residual U(2)\textsuperscript{5} invariance, we can transform the spurions to the following explicit form (see Ref. \cite{4})

\[ V_{q(\ell)} = e^{i\varphi_{q(\ell)}} \left( \begin{array}{c} 0 \\ \epsilon_{q(\ell)} \end{array} \right), \quad \Delta_e = O_e^{\dagger} \left( \begin{array}{cc} \delta_e' & 0 \\ 0 & \delta_e \end{array} \right), \quad \Delta_u = U_u^{\dagger} \left( \begin{array}{cc} \delta_u' & 0 \\ 0 & \delta_u \end{array} \right), \quad \Delta_d = U_d^{\dagger} \left( \begin{array}{cc} \delta_d' & 0 \\ 0 & \delta_d \end{array} \right), \]  

\text{(F.4)}

where O and U represent 2 \times 2 orthogonal and complex unitary matrices, respectively

\[ O_e = \left( \begin{array}{cc} c_e & s_e \\ -s_e & c_e \end{array} \right), \quad U_q = \left( \begin{array}{cc} c_q e^{i\alpha_q} & s_q \\ -s_q e^{-i\alpha_q} & c_q \end{array} \right), \]  

\text{(F.5)}

with \( s_i \equiv \sin \theta_i \) and \( c_i \equiv \cos \theta_i \). The \( \epsilon_i \) and \( \delta_i^{(t)} \) are small positive real parameters controlling the overall size of the spurions. From the observed hierarchies of the Yukawa couplings, it is expected that they have following order relation

\[ 1 \gg \epsilon_i \gg \delta_i \gg \delta_i' > 0 \]  

\text{(F.6)}

where

\[ \epsilon_i = \mathcal{O}(y_\ell |V_{ts}|) = \mathcal{O}(10^{-1}), \quad \delta_i = \mathcal{O} \left( \frac{y_c}{y_t}, \frac{y_s}{y_b}, \frac{y_\mu}{y_\tau} \right) = \mathcal{O}(10^{-2}), \quad \delta_i' = \mathcal{O} \left( \frac{y_u}{y_t}, \frac{y_d}{y_b}, \frac{y_s}{y_\tau} \right) = \mathcal{O}(10^{-3}). \]  

\text{(F.7)}

In terms of the spurion expressions discussed above, we can classify the number of independent operators in the SMEFT with a U(2)\textsuperscript{5} flavor symmetry. Following the complete classification in Ref. \cite{4}, we focus on the chirality-flipped \( d_i \rightarrow d_j \) and \( \ell_i \rightarrow \ell_j \) transitions in this paper. Here, we classify the operators up to \( \mathcal{O}(V^3, \Delta V) \sim \mathcal{O}(10^{-3}) \), given the size of the spurions in Eq. (F.7). The \( b_i \rightarrow b_j \) transition allowed by different U(2) breaking terms are summarized in Table 8, where we follow the result of Table 4 in Ref. \cite{4}. The case of \( \ell_i \rightarrow \ell_j \) transition is given in the same way, and is summarised in Table 10. We denote with latin (greek) letters the real (complex) couplings appearing in hermitian (non-hermitian) structures. Terms with the same number of spurions are denoted by the same latin or greek letter with different subscript.
References

[1] W. Buchmuller and D. Wyler, Nucl. Phys. B 268 (1986), 621-653.

[2] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, JHEP 10 (2010), 085 [arXiv:1008.4884 [hep-ph]].

[3] R. Alonso, E. E. Jenkins, A. V. Manohar and M. Trott, JHEP 04 (2014), 159 [arXiv:1312.2014 [hep-ph]].

[4] D. A. Faroughy, G. Isidori, F. Wilsch and K. Yamamoto, JHEP 08 (2020), 166 [arXiv:2005.05366 [hep-ph]].

[5] R. S. Chivukula and H. Georgi, Phys. Lett. B 188 (1987), 99-104.

[6] G. D’Ambrosio, G. F. Giudice, G. Isidori and A. Strumia, Nucl. Phys. B 645 (2002), 155-187 [arXiv:hep-ph/0207036 [hep-ph]].

[7] R. Barbieri, G. Isidori, J. Jones-Perez, P. Lodone and D. M. Straub, Eur. Phys. J. C 71 (2011), 1725 [arXiv:1105.2296 [hep-ph]].

[8] R. Barbieri, D. Buttazzo, F. Sala and D. M. Straub, JHEP 07 (2012), 181 [arXiv:1203.4218 [hep-ph]].

[9] G. Blankenburg, G. Isidori and J. Jones-Perez, Eur. Phys. J. C 72 (2012), 2126 [arXiv:1204.0688 [hep-ph]].

[10] G. Altarelli and F. Feruglio, Rev. Mod. Phys. 82 (2010) 2701 [arXiv:1002.0211 [hep-ph]].

[11] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, Prog. Theor. Phys. Suppl. 183 (2010) 1 [arXiv:1003.3552 [hep-th]].

[12] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu and M. Tanimoto, Lect. Notes Phys. 858 (2012) 1, Springer.

[13] D. Hernandez and A. Y. Smirnov, Phys. Rev. D 86 (2012) 053014 [arXiv:1204.0445 [hep-ph]].

[14] S. F. King and C. Luhn, Rept. Prog. Phys. 76 (2013) 056201 [arXiv:1301.1340 [hep-ph]].

[15] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, New J. Phys. 16, 045018 (2014) [arXiv:1402.4271 [hep-ph]].

[16] M. Tanimoto, AIP Conf. Proc. 1666 (2015) 120002.

[17] S. F. King, Prog. Part. Nucl. Phys. 94 (2017) 217 [arXiv:1701.04413 [hep-ph]].

[18] S. T. Petcov, Eur. Phys. J. C 78 (2018) no.9, 709 [arXiv:1711.10806 [hep-ph]].

[19] F. Feruglio and A. Romanino, arXiv:1912.06028 [hep-ph].
[20] F. Feruglio, in From My Vast Repertoire ...: Guido Altarelli’s Legacy, A. Levy, S. Forte, Stefano, and G. Ridolfi, eds., pp.227–266, 2019, arXiv:1706.08749 [hep-ph].

[21] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, Nucl. Phys. B 858, 437 (2012) [arXiv:1112.1340 [hep-ph]].

[22] T. Kobayashi, K. Tanaka and T. H. Tatsuishi, Phys. Rev. D 98, no. 1, 016004 (2018) [arXiv:1803.10391 [hep-ph]].

[23] J. T. Penedo and S. T. Petcov, Nucl. Phys. B 939, 292 (2019) [arXiv:1806.11040 [hep-ph]].

[24] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 1904, 174 (2019) [arXiv:1812.02158 [hep-ph]].

[25] J. C. Criado and F. Feruglio, SciPost Phys. 5 (2018) no.5, 042 [arXiv:1807.01125 [hep-ph]].

[26] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, JHEP 11 (2018), 196 [arXiv:1808.03012 [hep-ph]].

[27] G. J. Ding, S. F. King and X. G. Liu, JHEP 1909 (2019) 074 [arXiv:1907.11714 [hep-ph]].

[28] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, JHEP 1904 (2019) 005 [arXiv:1811.04933 [hep-ph]].

[29] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, JHEP 02 (2020), 097 [arXiv:1907.09141 [hep-ph]].

[30] X. Wang and S. Zhou, JHEP 05 (2020), 017 [arXiv:1910.09473 [hep-ph]].

[31] G. J. Ding, S. F. King and X. G. Liu, Phys. Rev. D 100 (2019) no.11, 115005 [arXiv:1903.12588 [hep-ph]].

[32] X. G. Liu and G. J. Ding, JHEP 1908 (2019) 134 [arXiv:1907.01488 [hep-ph]].

[33] P. Chen, G. J. Ding, J. N. Lu and J. W. F. Valle, Phys. Rev. D 102 (2020) no.9, 095014 [arXiv:2003.02734 [hep-ph]].

[34] P. P. Novichkov, J. T. Penedo and S. T. Petcov, Nucl. Phys. B 963 (2021), 115301 [arXiv:2006.03058 [hep-ph]].

[35] X. G. Liu, C. Y. Yao and G. J. Ding, Phys. Rev. D 103 (2021) no.5, 056013 [arXiv:2006.10722 [hep-ph]].

[36] I. de Medeiros Varzielas, S. F. King and Y. L. Zhou, Phys. Rev. D 101 (2020) no.5, 055033 [arXiv:1906.02208 [hep-ph]].

[37] T. Asaka, Y. Heo, T. H. Tatsuishi and T. Yoshida, JHEP 2001 (2020) 144 [arXiv:1909.06520 [hep-ph]].

[38] G. J. Ding, S. F. King, C. C. Li and Y. L. Zhou, JHEP 08 (2020), 164 [arXiv:2004.12662 [hep-ph]].
[39] T. Asaka, Y. Heo and T. Yoshida, Phys. Lett. B 811 (2020), 135956 [arXiv:2009.12120 [hep-ph]].

[40] M. K. Behera, S. Mishra, S. Singirala and R. Mohanta, [arXiv:2007.00545 [hep-ph]].

[41] S. Mishra, [arXiv:2008.02095 [hep-ph]].

[42] F. J. de Anda, S. F. King and E. Perdomo, Phys. Rev. D 101 (2020) no.1, 015028 [arXiv:1812.05620 [hep-ph]].

[43] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, arXiv:1906.10341 [hep-ph].

[44] P. P. Novichkov, S. T. Petcov and M. Tanimoto, Phys. Lett. B 793 (2019) 247 [arXiv:1812.11289 [hep-ph]].

[45] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi and H. Uchida, Phys. Lett. B 794 (2019) 114 [arXiv:1812.11072 [hep-ph]].

[46] H. Okada and M. Tanimoto, Phys. Lett. B 791 (2019) 54 [arXiv:1812.09877 [hep-ph]].

[47] H. Okada and M. Tanimoto, Eur. Phys. J. C 81 (2021) no.1, 52 [arXiv:1905.13421 [hep-ph]].

[48] T. Nomura and H. Okada, Phys. Lett. B 797, 134799 (2019) [arXiv:1904.03937 [hep-ph]].

[49] H. Okada and Y. Orikasa, Phys. Rev. D 100, no.11, 115037 (2019) [arXiv:1907.04716 [hep-ph]].

[50] Y. Kariyazono, T. Kobayashi, S. Takada, S. Tamba and H. Uchida, Phys. Rev. D 100, no.4, 045014 (2019) [arXiv:1904.07546 [hep-th]].

[51] T. Nomura and H. Okada, Nucl. Phys. B 966 (2021), 115372 [arXiv:1906.03927 [hep-ph]].

[52] H. Okada and Y. Orikasa, arXiv:1908.08409 [hep-ph].

[53] T. Nomura, H. Okada and O. Popov, Phys. Lett. B 803 (2020) 135294 [arXiv:1908.07457 [hep-ph]].

[54] J. C. Criado, F. Feruglio and S. J. D. King, JHEP 2002 (2020) 001 [arXiv:1908.11867 [hep-ph]].

[55] S. F. King and Y. L. Zhou, Phys. Rev. D 101 (2020) no.1, 015001 [arXiv:1908.02770 [hep-ph]].

[56] G. J. Ding, S. F. King, X. G. Liu and J. N. Lu, JHEP 1912 (2019) 030 [arXiv:1910.03460 [hep-ph]].

[57] I. de Medeiros Varzielas, M. Levy and Y. L. Zhou, JHEP 11 (2020), 085 [arXiv:2008.05329 [hep-ph]].

[58] D. Zhang, Nucl. Phys. B 952 (2020) 114935 [arXiv:1910.07869 [hep-ph]].
[59] T. Nomura, H. Okada and S. Patra, Nucl. Phys. B 967 (2021), 115395 [arXiv:1912.00379 [hep-ph]].

[60] T. Kobayashi, T. Nomura and T. Shimomura, Phys. Rev. D 102 (2020) no.3, 035019 [arXiv:1912.00637 [hep-ph]].

[61] J. N. Lu, X. G. Liu and G. J. Ding, Phys. Rev. D 101 (2020) no.11, 115020 [arXiv:1912.07573 [hep-ph]].

[62] X. Wang, Nucl. Phys. B 957 (2020), 115105 [arXiv:1912.13284 [hep-ph]].

[63] S. J. D. King and S. F. King, JHEP 09 (2020), 043 [arXiv:2002.00969 [hep-ph]].

[64] M. Abbas, Phys. Rev. D 103 (2021) no.5, 056016 [arXiv:2002.01929 [hep-ph]].

[65] H. Okada and Y. Shoji, Phys. Dark Univ. 31 (2021), 100742 [arXiv:2003.11396 [hep-ph]].

[66] H. Okada and Y. Shoji, Nucl. Phys. B 961 (2020), 115216 [arXiv:2003.13219 [hep-ph]].

[67] G. J. Ding and F. Feruglio, JHEP 06 (2020), 134 [arXiv:2003.13448 [hep-ph]].

[68] T. Nomura and H. Okada, [arXiv:2007.04801 [hep-ph]].

[69] T. Nomura and H. Okada, arXiv:2007.15459 [hep-ph].

[70] H. Okada and M. Tanimoto, [arXiv:2005.00775 [hep-ph]].

[71] H. Okada and M. Tanimoto, Phys. Rev. D 103 (2021) no.1, 015005 [arXiv:2009.14242 [hep-ph]].

[72] K. I. Nagao and H. Okada, JCAP 05 (2021), 063 [arXiv:2008.13686 [hep-ph]].

[73] K. I. Nagao and H. Okada, [arXiv:2010.03348 [hep-ph]].

[74] C. Y. Yao, X. G. Liu and G. J. Ding, [arXiv:2011.03501 [hep-ph]].

[75] X. Wang, B. Yu and S. Zhou, Phys. Rev. D 103 (2021) no.7, 076005 [arXiv:2010.10159 [hep-ph]].

[76] M. Abbas, Phys. Atom. Nucl. 83 (2020) no.5, 764-769.

[77] H. Okada and M. Tanimoto, JHEP 03 (2021), 010 doi:10.1007/JHEP03(2021)010 [arXiv:2012.01688 [hep-ph]].

[78] C. Y. Yao, J. N. Lu and G. J. Ding, JHEP 05 (2021), 102 [arXiv:2012.13390 [hep-ph]].

[79] F. Feruglio, V. Gherardi, A. Romanino and A. Titov, [arXiv:2101.08718 [hep-ph]].

[80] S. F. King and Y. L. Zhou, JHEP 04 (2021), 291 [arXiv:2103.02633 [hep-ph]].

[81] P. Chen, G. J. Ding and S. F. King, JHEP 04 (2021), 239 [arXiv:2101.12724 [hep-ph]].
[82] P. P. Novichkov, J. T. Penedo and S. T. Petcov, JHEP 04, 206 (2021) [arXiv:2102.07488 [hep-ph]].

[83] X. Du and F. Wang, JHEP 02, 221 (2021) [arXiv:2012.01397 [hep-ph]].

[84] T. Kobayashi, T. Shimomura and M. Tanimoto, Phys. Lett. B 819, 136452 (2021) [arXiv:2102.10425 [hep-ph]].

[85] G. J. Ding, S. F. King and C. Y. Yao, [arXiv:2103.16311 [hep-ph]].

[86] H. Kuranaga, H. Ohki and S. Uemura, [arXiv:2105.06237 [hep-ph]].

[87] C. C. Li, X. G. Liu and G. J. Ding, [arXiv:2108.02181 [hep-ph]].

[88] M. Tanimoto and K. Yamamoto, JHEP 10 (2021), 183 [arXiv:2106.10919 [hep-ph]].

[89] H. Okada and Y. h. Qi, [arXiv:2109.13779 [hep-ph]].

[90] T. Kobayashi, H. Okada and Y. Orikasa, [arXiv:2111.05674 [hep-ph]].

[91] A. Dasgupta, T. Nomura, H. Okada, O. Popov and M. Tanimoto, [arXiv:2111.06898 [hep-ph]].

[92] T. Nomura and H. Okada, [arXiv:2109.04157 [hep-ph]].

[93] K. I. Nagao and H. Okada, [arXiv:2108.09984 [hep-ph]].

[94] T. Nomura, H. Okada and Y. Orikasa, Eur. Phys. J. C 81 (2021) no.10, 947 [arXiv:2106.12375 [hep-ph]].

[95] T. Nomura and H. Okada, [arXiv:2106.10451 [hep-ph]].

[96] H. Okada, Y. Shimizu, M. Tanimoto and T. Yoshida, JHEP 07 (2021), 184 [arXiv:2105.14292 [hep-ph]].

[97] G. J. Ding, S. F. King and J. N. Lu, JHEP 11 (2021), 007 [arXiv:2108.09655 [hep-ph]].

[98] B. Y. Qu, X. G. Liu, P. T. Chen and G. J. Ding, Phys. Rev. D 104 (2021) no.7, 076001 [arXiv:2106.11659 [hep-ph]].

[99] X. Zhang and S. Zhou, JCAP 09 (2021), 043 [arXiv:2106.03433 [hep-ph]].

[100] X. Wang and S. Zhou, JHEP 07 (2021), 093 [arXiv:2102.04358 [hep-ph]].

[101] X. Wang, Nucl. Phys. B 962 (2021), 115247 doi:10.1016/j.nuclphysb.2020.115247 [arXiv:2007.05913 [hep-ph]].

[102] T. Kobayashi and H. Otsuka, [arXiv:2108.02700 [hep-ph]].

[103] L. E. Ibanez and A. M. Uranga, “String theory and particle physics: An introduction to string phenomenology”.

45
[104] T. Kobayashi and H. Otsuka, Phys. Rev. D 101 (2020) no.10, 106017 [arXiv:2001.07972 [hep-th]].

[105] S. Ferrara, D. Lust and S. Theisen, Phys. Lett. B 233 (1989), 147-152; W. Lerche, D. Lust and N. P. Warner, Phys. Lett. B 231 (1989), 417-424; J. Lauer, J. Mas and H. P. Nilles, Nucl. Phys. B 351, 353 (1991).

[106] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba and T. H. Tatsuishi, Phys. Rev. D 97, no. 11, 116002 (2018); T. Kobayashi and S. Tamba, Phys. Rev. D 99 (2019) no.4, 046001; H. Ohki, S. Uemura and R. Watanabe, Phys. Rev. D 102, no.8, 085008 (2020); S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi and H. Uchida, Phys. Rev. D 102, no.10, 105010 (2020); S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada and H. Uchida, JHEP 11, 101 (2020) [arXiv:2007.06188 [hep-th]]; S. Kikuchi, T. Kobayashi and H. Uchida, Phys. Rev. D 104, no.6, 065008 (2021) [arXiv:2101.00826 [hep-th]]; Y. Almumin, M. C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz and S. Shukla, JHEP 05, 078 (2021) [arXiv:2102.11286 [hep-th]].

[107] A. Strominger, Commun. Math. Phys. 133 (1990), 163-180.

[108] P. Candelas and X. de la Ossa, Nucl. Phys. B 355 (1991), 455-481.

[109] K. Ishiguro, T. Kobayashi and H. Otsuka, [arXiv:2010.10782 [hep-th]].

[110] K. Ishiguro, T. Kobayashi and H. Otsuka, [arXiv:2107.00487 [hep-th]].

[111] E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001) [arXiv:hep-ph/0106291].

[112] K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552, 207 (2003) [arXiv:hep-ph/0206292].

[113] G. Altarelli and F. Feruglio, Nucl. Phys. B 720 (2005) 64 [hep-ph/0504165].

[114] G. Altarelli and F. Feruglio, Nucl. Phys. B 741 (2006) 215 [hep-ph/0512103].

[115] Y. Shimizu, M. Tanimoto and A. Watanabe, Prog. Theor. Phys. 126 (2011) 81 [arXiv:1105.2929 [hep-ph]].

[116] S. T. Petcov and A. V. Titov, Phys. Rev. D 97 (2018) no.11, 115045 [arXiv:1804.00182 [hep-ph]].

[117] S. K. Kang, Y. Shimizu, K. Takagi, S. Takahashi and M. Tanimoto, PTEP 2018, no. 8, 083B01 (2018) [arXiv:1804.10468 [hep-ph]].

[118] S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B 225, 363 (1989).

[119] M. C. Chen, S. Ramos-Sánchez and M. Ratz, Phys. Lett. B 801, 135153 (2020) [arXiv:1909.06910 [hep-ph]].

[120] R. C. Gunning, Lectures on Modular Forms (Princeton University Press, Princeton, NJ, 1962).
[121] B. Schoeneberg, *Elliptic Modular Functions* (Springer-Verlag, 1974)

[122] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms* (Springer-Verlag, 1984)

[123] H. Abe, T. Kobayashi, S. Uemura and J. Yamamoto, Phys. Rev. D 102 (2020) no.4, 045005 [arXiv:2003.03512 [hep-th]].

[124] T. Kobayashi and H. Otsuka, Phys. Rev. D 102 (2020) no.2, 026004 [arXiv:2004.04518 [hep-th]].

[125] K. Ishiguro, T. Kobayashi and H. Otsuka, JHEP 03 (2021), 161 [arXiv:2011.09154 [hep-ph]].

[126] A. Crivellin, S. Davidson, G. M. Pruna and A. Signer, JHEP 05 (2017), 117 [arXiv:1702.03020 [hep-ph]].

[127] S. Davidson, JHEP 02 (2021), 172 [arXiv:2010.00317 [hep-ph]].

[128] A. M. Baldini et al. [MEG], Eur. Phys. J. C 76 (2016) no.8, 434 [arXiv:1605.05081 [hep-ex]].

[129] P. A. Zyla et al. [Particle Data Group], PTEP 2020 (2020) no.8, 083C01.