Research Article

Dynamic Cournot Duopoly Game with Delay

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The delay Cournot duopoly game is studied. Dynamical behaviors of the game are studied. Equilibrium points and their stability are studied. The results show that the delayed system has the same Nash equilibrium point and the delay can increase the local stability region.

1. Introduction

It is well known that the duopoly game is one of the fundamental oligopoly games [1, 2]. Even the duopoly situation is an oligopoly of two producers that can be more complex than one might imagine since the duopolists have to take into account their actions and reactions when decisions are made [3]. Oligopoly theory is one of the oldest branches of mathematical economics dating back to 1838 when its basic model was proposed by Bisch et al. [4]. In repeated duopoly game all players maximize their profits. Recently, the dynamics of duopoly game has been studied [5–11]. Bischi and Naimzada [8] gave the general formula of duopoly game with a form of bounded rationality. Agiza et al. [10] examined the dynamical behavior of Bowley’s model with bounded rationality. They also have studied the complex dynamics of bounded rationality duopoly game with nonlinear demand function [11]. In general, a player, in order to adjust his output, can choose his strategy rule among many available techniques. Naïve, adaptive, and boundedly rational strategies are only a few examples. When literature deals with duopoly games, most papers focus on games with homogeneous players. Another branch of literature is interested in games with heterogeneous players. In this type of literatures, the assumption of players adopting heterogeneous rules to decide their production is, in our opinion, more realistic than the opposite case; see [12]. Agiza and Elsadany [13, 14] were of the first authors who studied games with heterogeneous players, and in particular they analyze the dynamic behaviors emerging in this kind of games. Recently, Zhang et al. [15] and Dubiel-Teleszynski [16] used the same technique to analyze a duopoly game with heterogeneous players and nonlinear cost function. In addition, Angelini et al. [17] and Tramontana [18] studied a duopoly game with heterogeneous players with isoelastic demand function. Other studies on the dynamics of oligopoly models with more firms and other modifications have been studied [19–21]. Also in the past decade, there has been a great deal of interest in chaos control of duopoly games because of its complexity [22–25]. Recently Askar [26] has shown complex dynamics such as bifurcation and chaos, in a Cournot duopoly game with log-concave demand function. Zhao and Zhang [27] studied a duopoly game with heterogeneous players participating in carbon emission trading and analyzed the asymptotic stability of the equilibrium points of the game. In these literatures, adjustment speed or other parameters are taken as bifurcation parameters, and complex results such as period doubling bifurcation, unstable period orbits, and chaos are found.
The delayed discrete monopoly game is considered by Ahmed et al. [28]. Yassen and Agiza [29] and Hassan [30] showed that the stability region of the Nash equilibrium can become larger in a delayed duopoly game. Even so, a more realistic case of bounded rationality game with delay was introduced and discussed in [31]. Yali [32] reconsidered the duopoly game in Elsadany [31] for the case of increasing marginal costs instead of constant marginal costs. Recently, continuous dynamic monopoly with a single or two fixed delays is investigated in [33]. The cases of continuously distributed delays are studied in [34]. Time delay on discrete monopoly models is also considered in [35, 36].

In this work, the study is based on the duopoly market with delay, to investigate situations that one player considers delayed decisions and another player makes without delayed decisions. According to the analysis of numerical simulations, the parameters effects on the stability of this system are obtained. We formulate a duopoly game with heterogeneous bounded rationality: one is boundedly rational player without delay and the other is delayed boundedly rational player and studying its dynamical behaviors. This idea combines two realistic ideas; in our opinion, the first is heterogeneous players and the second is bounded rationality method with delay.

This paper is organized as follows. In Section 2, heterogeneous bounded rationality is briefly described. In Section 3, we analyze the dynamics for simple case of duopoly game with heterogeneous bounded rationality. Explicit parametric conditions of the existence and local stability of equilibrium points will be given. In Section 4, we present the numerical simulations, to verify our results which take place by the theoretical analysis. Finally, some remarks are confined in Section 5.

2. Heterogeneous Bounded Rationality

We consider a Cournot duopoly game where \( q_i \) denotes the quantity supplied by firm \( i, i = 1, 2 \). In addition let \( P(q_i + q_j), i \neq j \); denote by a twice differentiable and nonincreasing inverse demand function and let \( C_i(q_i) \) denote by the twice differentiable increasing cost function. For firm \( i \) the profit resulting from the above Cournot game is given by

\[
\Pi_i(t) = P\left(q_i(t) + q_j(t + 1)\right) q_i(t) - C_i\left(q_i(t)\right). \tag{1}
\]

Maximizing profit function of player \( i \), taking quantity supplied the opponent \( j, i \neq j \), as given, results in the well-known reaction function for firm \( i \),

\[
q_i(t + 1) = r_i\left(q_j(t + 1)\right) = \arg \max_{q_i} \left[P\left(q_i(t) + q_j(t + 1)\right) q_i(t) - C_i\left(q_i(t)\right)\right]. \tag{2}
\]

Cournot assumed that the expected quantity in the next time step \( q_j(t + 1) \) is given by

\[
q_j(t + 1) = q_i(t). \tag{3}
\]

Then the Cournot duopoly game defined as a discrete dynamical system has the form

\[
q_i(t + 1) = R_i\left(q_j(t)\right), \quad i, j = 1, 2, \quad i \neq j. \tag{4}
\]

However, as pointed out by Bischi and Naimzada [8] there is an unrealistic assumption in this approach. It is implicitly assumed that the duopolist knows the market demand function. A more realistic approach is to assume bounded rationality, that is, each firm (say \( i \)th one) modifies its production according to its marginal profit \( \partial \Pi_i(q_i(t), q_j(t + 1))/\partial q_i, i = 1, 2, i \neq j \). This will help each player to increase or decrease the quantity produced at time \( t + 1 \) depending on whether its own marginal profit at time \( t \) is positive or negative. Hence the dynamical system of duopoly game with bounded rationality is

\[
q_i(t + 1) = q_i(t) + \alpha_i(q_i) \frac{\partial \Pi_i\left(q_i(t), q_j(t + 1)\right)}{\partial q_i}, \quad i = 1, 2, \quad i \neq j. \tag{5}
\]

where \( \alpha_i(q_i) \) is the adjustment of the \( i \)th firm \( i = 1, 2 \). They assumed that \( q_j(t + 1) = q_j(t) \) in the term of bounded rationality and also assumed \( \alpha_i(q_i) = \alpha q_i \). Then Bischi-Naimzada bounded rationality duopoly game has the form

\[
q_i(t + 1) = q_i(t) + \alpha_i(q_i) \frac{\partial \Pi_i\left(q_i(t), q_j(t)\right)}{\partial q_i}, \quad i = 1, 2, \quad i \neq j. \tag{6}
\]

This means that if the marginal profit is positive/negative the \( i \)th player increases/decreases its production \( q_i \) in the next period output.

Also they assumed that the expected product of a firm \( q_j(t + 1) \) is equal to its previous quantity \( q_j(t) \) in bounded rationality term. However it may make more sense to use previous productions, that is, \( q_j(t - 1), q_j(t - 2), \ldots, q_j(t - T) \) with different weights. Ahmed et al. [28] and Agiza et al [10] have examined this point in monopoly case only. In [29, 30], they assumed that delay was put in the full term of bounded rationality for all players in the game. In the duopoly game discussed in [31], each producer considers time delay only for opponents output and averages every opponent’s previous outputs. We suggested putting delay in the term of bounded rationality for all players except the \( i \)th player. Here both realistic ideas of bounded rationality and delay are combined. The dynamical system will be

\[
q_i(t + 1) = q_i(t) + \alpha_i(q_i) \frac{\partial \Pi_i\left(q_i(t), q_P\right)}{\partial q_i}, \quad i = 1, 2, \tag{7}
\]

where \( q_P = q_j(t + 1) = \sum_{l=0}^{T} q_j(t - l) \omega_l \) and \( \omega_l \geq 0, \sum_{l=0}^{T} \omega_l = 1 \). The factors \( \omega_l, l = 0, 1, 2, \ldots, T \) are the weights given to previous productions.

From (7), it is clear that the delay was put in the bounded rationality term for all players except the \( i \)th player. This
argument is the basic difference between our paper and the other papers [29, 30].

In this work, we consider that each player has different bounded rationality method in duopoly game. We assume player 1 is a boundedly rational player without delay and player 2 is a boundedly rational player with delay. With above assumptions, we can express the process of duopoly game with heterogeneous bounded rationality defined as

\[ q_1 (t + 1) = q_1 (t) + \alpha_1 q_1 \frac{\partial \Pi_1}{\partial q_1} (q_1 (t), q_2 (t)), \]

\[ q_2 (t + 1) = q_2 (t) + \alpha_2 q_2 \frac{\partial \Pi_2}{\partial q_2} (q_1 (t), q_2 (t)). \]

3. Duopoly Game with Heterogeneous Bounded Rationality

For simplicity we take \( T = 1 \), and we consider the duopoly game with heterogeneous bounded rationality. The profit of \( i \)th firm is given by

\[ \Pi_i = q_i (a - b (q_1(t) + q_2(t))) - c_i q_i, \quad i = 1, 2. \]  

Under above assumption, the delayed duopoly game with heterogeneous bounded rationality (7) is given by

\[ q_1 (t + 1) = q_1 (t) + \alpha_1 q_1 [a - c_1 - 2b q_1 (t) - b q_2 (t)], \]

\[ q_2 (t + 1) = q_2 (t) + \alpha_2 q_2 [a - c_2 - 2b q_2 (t) - b \times \omega q_1 (t) + (1 - \omega) q_1 (t - 1)]. \]

To study the stability of dynamical system (10), rewrite it as a three-dimensional system in the form

\[ p (t + 1) = q_1 (t), \]

\[ q_1 (t + 1) = q_1 (t) + \alpha_1 q_1 (a - c_1 - 2b q_1 (t) - b q_2 (t)), \]

\[ q_2 (t + 1) = q_2 (t) + \alpha_2 q_2 (a - c_2 - 2b q_2 (t) - b \times \omega q_1 (t) + (1 - \omega) p (t)). \]

Game (11) is a three-dimensional map which depends on seven parameters. We are concerned with the qualitative changes of the asymptotic and/or long-run dynamics of the iterated map (11). We will study the asymptotic behavior of the dynamical model using the localization of fixed points of the dynamical system and the determination of the parameters sets for given local stable fixed points.

3.1. Equilibrium Points and Local Stability. It is clear that the system (11) has four fixed points in the following form:

\[ E_0 = (0, 0, 0), \]

\[ E_1 = \left( \frac{a - c_1}{2b}, \frac{a - c_2}{2b}, 0 \right), \quad \text{if } c_1 < a, \quad (12) \]

\[ E_2 = (0, 0, a - c_2), \quad \text{if } c_2 < a, \]

\[ E_\ast = (q_1^\ast, q_1^\ast, q_2^\ast) \]

such that

\[ q_1^\ast = \frac{a + c_2 - 2c_1}{3b}, \quad q_2^\ast = \frac{a + c_1 - 2c_2}{3b}. \]

Obviously, \( E_0, E_1, E_2 \) are boundary equilibrium points. The fixed point \( E_\ast \) is a Nash equilibrium point and has economic meaning when

\[ 2c_1 - c_2 < a, \quad (14) \]

\[ 2c_2 - c_1 < a. \]

To investigate the local stability of the equilibrium points \( E_0, E_1, E_2 \) and \( E_\ast \), we have found the Jacobian matrix of the system equations (11). The Jacobian matrix for the model system (11) at any point \( J(p,q_1,q_2) \) takes the form

\[ J(p,q_1,q_2) = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha_2 b (1 - \omega) q_2 & -\alpha_2 b a q_2 & B \end{bmatrix}, \]  

where

\[ A = 1 + \alpha_1 (a - c_1 - 4b q_1 - b q_2), \]

\[ B = 1 + \alpha_2 (a - c_2 - 4b q_2 - b (\omega q_1 + (1 - \omega) p)). \]

The stability of equilibrium points will be determined by the nature of the eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points.

**Theorem 1.** The equilibrium point \( E_0 \) of system (11) is unstable equilibrium point.

**Proof.** At the equilibrium point \( E_0 (0, 0, 0) \), the Jacobian matrix given by (15) takes the form

\[ J (E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 + \alpha_1 (a - c_1) & 0 \\ 0 & 0 & 1 + \alpha_2 (a - c_2) \end{bmatrix}. \]

The eigenvalues of \( J(E_0) \) are given by \( \lambda_1 = 0, \lambda_2 = 1 + \alpha_1 (a - c_1) \) and \( \lambda_3 = 1 + \alpha_2 (a - c_2) \). From the conditions that \( a, a_i, c_i \) (\( i = 1, 2 \)) are positive parameters and \( c_i < a, \quad i = 1, 2 \). We have that \( |\lambda_{2,3}| > 1 \). Hence the equilibrium point \( E_0 \) is unstable.

**Theorem 2.** If the Nash equilibrium point \( E_\ast \) is strictly positive, then the equilibrium points \( E_1, E_2 \) of system (11) are saddle points.
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Proof. At the boundary equilibrium point \( E_1((a-c_1)/2b, (a-c_1)/2b, 0) \), the Jacobian matrix (15) takes the form of

\[
J(E_1) = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 - \alpha_1(a-c_1) & -\alpha_1(a-c_1)/2 \\
0 & 0 & 1 + \alpha(a-2c_2 + c_1)/2 \\
\end{bmatrix}
\] (18)

with eigenvalues \( \lambda_1 = 0, \ \lambda_2 = 1 + (\alpha_1/2)(a-2c_2 + c_1) \) and \( \lambda_3 = 1 - \alpha_1(a-c_1) \). If the Nash equilibrium point has positive coordinates, then \( 2c_2 - c_1 < a; \) hence, \( |\lambda_2| > 1 \). Then \( E_1 \) is a saddle point. Similarly we can prove that \( E_2 \) is also saddle point.

Now we investigate the local stability of Nash equilibrium point \( E_\ast \). The Jacobian matrix (15) at \( E_\ast \) is

\[
J(E_\ast) = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 - 2\alpha_1bq_1^* & -\alpha_1bq_1^* \\
-\alpha_2b(1-\omega)q_2^* & -\alpha_2bq_2^* & 1 - 2\alpha_2bq_2^* \\
\end{bmatrix}
\] (19)

The Nash equilibrium point is given in (12) and stability conditions are that all roots of the equation \( P(\lambda) = 0 \) satisfy \( |\lambda| < 1 \), where

\[
P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3
\] (20)

such that

\[
a_1 = 2b\alpha_1q_1^* + 2b\alpha_2q_2^* - 2, \\
a_2 = 1 - 2\alpha_1bq_1^* - 2\alpha_1bq_2^* \\
+ 4\alpha_1\alpha_2b^2q_1^*q_2^* - \alpha_1\alpha_2b^2\omega q_1^*q_2^*, \\
a_3 = \alpha_1\alpha_2b^2q_1^*q_2^* (\omega - 1).
\] (21)

A necessary and sufficient condition for (19) to have only roots of absolute value less than one is the following (see [37]):

\[
1 + a_1 + a_2 + a_3 > 0, \\
1 - a_1 + a_2 - a_3 > 0, \\
|a_2 - a_1a_3| < 1 - a_3^2, \\
|a_3| < 1.
\] (22)

According to Jury criteria, the Nash equilibrium point is locally asymptotically stable if conditions (22) are satisfied. When \( \omega \) is sufficiently small and for \( 0 < b < 1 \) the Nash equilibrium point is stable; see Figures 8 and 9. Hence, we deduced from above analysis that delay has stabilization effect for Nash equilibrium point.

### 4. Numerical Simulations

To provide some numerical evidence for the dynamical behavior of model (11), we present various numerical results here to show that the delay has effect to increase stability domain. In order to study local stability properties of the equilibrium points it is convenient to take the parameters values as follows: \( a = 11, \ b = 0.5, \ c_1 = 3, \ c_2 = 5. \)

To show the influence of delay weight \( \omega \) on the dynamical behaviors of the delayed Cournot duopoly game, four cases of \( \omega \) are considered. Figure 1 shows that the bifurcation diagram for the nondelay case \( \omega = 1: q_i \ (i = 1, 2) \) converges to the Nash equilibrium point as \( \alpha_1 < 0.265 \) approximately; when \( \alpha_1 \) increases, the equilibrium point becomes unstable, period doubling bifurcations appear, and finally chaotic behaviors occur. Figure 2 (for \( \omega = 0.8 \)) shows that the game keeps stable for \( \alpha_1 \) taking its value up to nearly 0.29.

Also, Figure 3 (for \( \omega = 0.6 \)) shows that the game keeps stable for \( \alpha_1 \) taking its value up to nearly 0.32. Comparing these three diagrams, we can see that instability for a case of proper delay (Figures 2 and 3) occurs later than that for the nondelay case (Figure 1). The bifurcation diagram is illustrated in Figure 4, which shows that the stability loss is evidently due to a Neimark-Sacker bifurcation. The corresponding two-dimensional phase portraits for various values of \( \omega \) are shown in Figures 5, 6, and 7.
Figure 3: Bifurcation diagram of game (II) with respect to $\alpha_1$ when $\omega = 0.6$.

Figure 4: Bifurcation diagram of game (II) with respect to $\alpha_1$ when $\omega = 0.5$.

Figure 5: Phase portrait for game (II) when $\omega = 1$.

Figure 6: Phase portrait for game (II) when $\omega = 0.8$.

Figure 7: Phase portrait for game (II) when $\omega = 0.5$.

Figure 8 shows the bifurcation diagram of $q_1$ with respect to the delay parameter $\omega$, while other parameters being fixed ($a = 11$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$, $\alpha_1 = 0.35$ and $\alpha_2 = 0.3$). It also shows that the bifurcation diagram of $q_1$ convergent to Nash equilibrium point as $\omega$ is small. It is clear that the Nash equilibrium is convergence as $\omega$ is small. This means that delay parameter $\omega$ has a stabilization effect for duopoly game.

A bifurcation diagram of $q_2$ with respect to $\omega$, while other parameters are fixed ($a = 11$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$, $\alpha_1 = 0.35$ and $\alpha_2 = 0.3$) is shown in Figure 9. Also, from the bifurcation diagram of $q_2$ the Nash equilibrium point is convergent as $\omega$ is small.

From Figures 8 and 9, every one can deduce that the delay has effect to convergent of the Nash equilibrium. So, the delay has a stabilization effect for the Nash equilibrium point.

Using the stability conditions (22) of the Nash equilibrium point we can draw the region of stability in the parameter plane ($\alpha_1$, $\alpha_2$). Figure 10 shows the stability region of the Nash equilibrium point in nondelay case (i.e. $\omega = 1$). Also Figure 11 shows the stability region of the Nash
equilibrium point in delay case when $\omega = 0.1$. Comparison between Figures 10 and 11, one can see that the stability region in delay case is larger than those in the nondelay case. So, we can see that an intermediate delay weight can expand the stability region. Hence, we say that a proper delay has a stabilization effect on the delayed Cournot duopoly game.

5. Conclusion

In this paper, we established delayed Cournot duopoly game with heterogeneous bounded rationality. The stability conditions of the equilibrium points of this game are investigated. Basic properties of the game have been analyzed by means of bifurcation diagrams and stability regions. This paper shows that increasingly large delay speed act as an instability factor, which causes period doubling bifurcation and chaotic behavior. This means that the delay has a stabilization effect on the game, when it is sufficiently small and for $0 < b < 1$. We concluded that the delay weight can delay the occurrence of complex behaviors such as bifurcation and chaos and can expand the stability region for the system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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