A note on Kähler-Ricci soliton

Xiuxiong Chen, Song Sun, Gang Tian

Abstract

In this note we provide a proof of the following: Any compact KRS with positive bisectional curvature is biholomorphic to the complex projective space. As a corollary, we obtain an alternative proof of the Frankel conjecture by using the Kähler-Ricci flow.

The purpose of this note is to give a proof of the following theorem, which does not rely on the previous solutions of Frankel conjecture:

**Theorem 1.** An $n$ dimensional compact complex manifold admitting a Kähler-Ricci soliton with positive bisectional curvature is biholomorphic to the complex projective space $\mathbb{C}P^n$.

**Remark 2.** Since the Futaki invariant of $\mathbb{C}P^n$ vanishes, the Kähler-Ricci soliton must be Kähler-Einstein. In addition, by a theorem of Berger (cf. [2], [11]), it is actually a constant multiple of the standard Fubini-Study metric. It also follows from the uniqueness theorem in [27].

**Remark 3.** Using a method which originated in [22], the above theorem was proved in [12] without using the uniformization theorem. The proof given here is different and we use some Morse theory.

As a by-product, one can use the method of Kähler-Ricci flow to prove the following Frankel conjecture.

**Corollary 4.** Every compact Kähler manifold with positive bisectional curvature is biholomorphic to the complex projective space.

**Remark 5.** The Frankel conjecture was proved by Siu-Yau ([26]) using harmonic maps and by Mori ([21]) via algebraic methods. In Siu-Yau’s proof, by using the theorem of Kobayashi-Ochiai ([19]), the key thing is to show the existence of a rational curve representing the generator of $H_2(M; \mathbb{Z})/\text{Tor}$. They first proved using...
the second variation formula that in the case of positive bisectional curvature, a stable harmonic map from the sphere is either holomorphic or anti-holomorphic. Such a harmonic map can be constructed by an energy-minimizing process and applying Sacks-Uhlenbeck’s blowing-up analysis. Consequently, the generator of the second homotopy class can be represented by a single holomorphic sphere. In Mori’s proof, such a rational curve was constructed using deformation theory of curves and some algebraic geometry of positive characteristic.

There has been much work on attempting to prove the Frankel conjecture using the method of Kähler-Ricci flow. By a theorem of Berger (c.f. [2], [11]), it suffices to show that the flow converges to a Kähler-Einstein metric with positive bisectional curvature. The one dimensional case was completely settled by R. Hamilton ([17]), B. Chow ([7]) and Chen-Lu-Tian ([9]). In higher dimensions, assuming \( c_1 > 0 \) and the existence of a Kähler-Einstein metric, Chen-Tian (c.f. [10], [11]) proved that if the initial metric has positive bisectional curvature, then the flow converges exponentially fast to a Kähler-Einstein metric with positive bisectional curvature. It follows that the space of Kähler metrics with positive bisectional curvature is path-connected. In a talk at MIT, G. Perelman showed uniform estimates on the scalar curvature and diameter along the Kähler-Ricci flow and announced that the flow with arbitrary initial metric converges to a Kähler-Einstein metric if there exists one. In [28], using an estimate of Perelman, Tian-Zhu gave a proof of the convergence of the Kähler-Ricci flow on any Kähler-Einstein manifold via a different method. In [8], X. Chen proved that an irreducible Fano Kähler manifold with positive orthogonal bisectional curvature is biholomorphic to \( \mathbb{C}P^n \) using the Frankel conjecture. Recently, Phong-Song-Sturm-Weinkove ([23]) proved that the Kähler-Ricci flow starting from a metric of positive bisectional curvature will converge to a Kähler-Einstein metric under various extra conditions.

Our proof of theorem 1 uses induction on the dimension \( n \), starting from the trivial case \( n = 0 \). Denote by \( A_n \) the statement of theorem 1 for dimension \( n \), and by \( B_n \) the statement of corollary 4 for dimension \( n \). We will prove first that \( A_n \) implies \( B_n \), and then that \( B_k (k < n) \) implies \( A_n \). So theorem 1 and corollary 4 are proved simultaneously.

**Proof of Corollary 4** \((A_n \rightarrow B_n)\). Suppose that \((M, J, g)\) is a \( n \) dimensional Kähler manifold with positive bisectional curvature. Then the tangent bundle \( TM \) is ample, so in particular, \( c_1(TM) > 0 \). By applying a Bochner type formula (c.f. [3]), we know \( b_2(M) = 1 \). In fact, \( b_{2k} = 1 \) and \( b_{2k+1} = 0 \) for all \( k = 0, \ldots, n \). Therefore we can assume the Kähler form \( \omega \) lies in the canonical class \( c_1(TM) \).
Now we run the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g_{ij}(t) = g_{ij}(t) - R_{ij}(t),$$

with $g(0) = g$. In [6], H. D. Cao proved that the flow exists globally. By S. Bando ([1]) in dimension 3 and N. Mok ([21]) in all dimensions, the positivity of the bisectional curvature is preserved under the Kähler-Ricci flow. To study convergence, we make use of a theorem of Perelman that the scalar curvature and the diameter are uniformly bounded along the flow (cf. [25]). Thus the bisectional curvature is also uniformly bounded along the flow, then all higher order derivative estimates of the curvature follow easily (cf. [10]). Hence there exist a subsequence $g(t_i)$ and diffeomorphisms $f_i$, such that $f_i^* g(t_i)$ converges to $g_\infty$ smoothly. Moreover, we can assume $f_i^* J$ converges smoothly to $J_\infty$ (possibly different from $J$) and $g_\infty$ is Kähler with respect to $J_\infty$. Using Perelman’s $W$ functional, we can prove that $(g_\infty, J_\infty)$ is a Kähler-Ricci soliton (cf. [24]). Clearly it has non-negative bisectional curvature. We claim it actually has positive bisectional curvature. Indeed, by Lemma 6 proved near the end of this note, it suffices to show that the Ricci curvature of $g_\infty$ is positive. It follows from the strong maximum principle for tensors along the Ricci flow (cf. [16]) that if $Ric(g_\infty)$ has a null direction at some point, then the manifold $(M, g_\infty, J_\infty)$ splits holomorphically isometrically into a product of a flat factor $P$ and another factor $Q$ with strictly positive Ricci curvature. Both $P$ and $Q$ are Kähler manifolds, so $b_2(P), b_2(Q) \geq 1$. Hence $b_2(M) \geq 2$, which is a contradiction. Therefore we have proved the claim. Now by Theorem 1, we know $(M, J_\infty)$ is biholomorphic to $\mathbb{C}P^n$. Since $\mathbb{C}P^n$ has trivial local deformation, i.e. $H^1(\mathbb{C}P^n, \Theta) = 0$ (c.f. [4], [20]), $(M, J)$ is also biholomorphic to $\mathbb{C}P^n$, and this proves Corollary 4. Note that by [11], the Kähler Ricci flow here indeed converges exponentially fast to the Fubini-Study metric. □

Now we come to prove that $B_k(k < n)$ implies $A_n$. It uses the theorem of Kobayashi-Ochiai as in the previous solutions for the Frankel conjecture. By Bishop-Goldberg (c.f. [3]), a compact Kähler manifold with positive bisectional curvature has the second Betti number equal to 1. Suppose an $n$ dimensional compact complex manifold $(M, J)$ admits a Kähler-Ricci soliton $g$, with the Kähler form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, i.e.

$$Ric(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} f,$$

where $f$ is a real function whose gradient is a holomorphic vector field, i.e. the $(2,0)$ part of $\nabla \nabla f$ vanishes. In particular, we know $f$ is a Morse-Bott function (see [14]), i.e. the set of critical points of $f$ consists of smooth compact submanifolds of $M$ on which $Hess(f)$ is non-degenerate along the normal directions. We may assume $f$
is not a constant, since otherwise $(M, J, g)$ is Kähler-Einstein and $(M, J)$ is biholomorphic to the complex projective space by [2]. A critical point of $f$ is the same as a zero of the vector field $Im \nabla f$. Since $Im \nabla f$ is Killing, the critical submanifolds are totally geodesic in $M$; since $Im \nabla f$ is also holomorphic, the critical submanifolds are all Kähler submanifolds, and the Morse indices (the number of negative eigenvalues of $Hess(f)$) are all even. By Kobayashi-Ochiai [19] (c.f. [26]), to prove theorem 1, it suffices to show the generator of $\pi_2(M) \simeq H_2(M; \mathbb{Z})$ modulo torsion could be represented by a rational curve, i.e. a holomorphic map from $\mathbb{C}P^1$ to $M$. Obviously the critical submanifolds are all Kähler manifolds with positive bisectional curvature with dimension less than $n$, thus by induction hypothesis they are all biholomorphic to the complex projective spaces of various dimension. By a general theorem of Frankel (see [14]), the following holds:

$$b_i(M) = \sum_{\alpha} b_{i-\lambda_\alpha}(F_\alpha),$$

where $F_\alpha$'s are all critical submanifolds of $f$, and $\lambda_\alpha$ is the Morse index of $F_\alpha$. Let $i = 0, 2$, we have

$$1 = b_0(M) = \sum_{\alpha: \lambda_\alpha = 0} b_0(F_\alpha),$$

and

$$1 = b_2(M) = \sum_{\alpha: \lambda_\alpha = 0} b_2(F_\alpha) + \sum_{\alpha: \lambda_\alpha = 2} b_0(F_\alpha).$$

Therefore we have two alternatives: either there are no critical submanifolds of index 2, or there is exactly one critical submanifold of index 2. In both cases the minimal submanifold of $f$ (i.e. the set of critical points achieving the minimum of $f$) is connected. In the first case, the minimal submanifold is a $\mathbb{C}P^k$ for some $k \geq 1$, and the complement of the stable manifold$^1$ of the minimal submanifold has codimension at least 4. Hence a generic representative of the generator of $\pi_2(M)$ (modulo torsion) will be deformed to a surface on the minimal manifold under the negative gradient flow of $f$. It is easy to see the latter must represent the generator of $\pi_2(\mathbb{C}P^k)$ and is thus homotopic to a rational curve. In the second case, the minimal submanifold must be a point. Denote the unique critical submanifold of index 2 by $M_2$, and the minimal submanifold by $M_0$. Then all the critical submanifolds other than $M_2$ and $M_0$ have Morse indices at least 4. Denote $Stab(N)$ the stable manifold of the critical submanifold $N$. Then we have the following stratification:

$$M = M_0 \cup M_2 \cup_{\alpha} Stab(N_\alpha),$$

$^1$The stable manifold of a critical submanifold $N$ is defined to be the set of points which will converge to $N$ under the negative gradient flow of $f$. It is an open manifold of dimension complementary to the Morse index of $N$. 

4
where \( N_\alpha \) runs over all critical submanifolds of \( M \) with Morse index greater or equal to 4. We claim that the closure \( \overline{\text{Stab}(M_2)} \) is an analytic subvariety of \( M \). Indeed, \( \text{Stab}(M_2) \) is a complex submanifold of \( M \setminus \bigcup_\alpha \text{Stab}(N_\alpha) \) with complex dimension \( n-1 \). Using the holomorphic gradient flow of \( f \) it suffices to prove this near \( M_2 \). The latter follows from the fact that \( X = \nabla^{1,0} f \) is linearizable at its critical points (c.f. [15]). More precisely, near a point \( p \) in \( M_2 \), by [5], we can find a coordinate chart \( (U, \{z^i\}) \) such that \( X = a_i z^i \frac{\partial}{\partial z^i} \), where \( a_i \geq 0, i = 1, \ldots, n-1 \) and \( a_n < 0 \). Thus \( \text{Stab}(M_2) \cap U \) is given by the hyperplane \( z_n = 0 \). Since \( \bigcup_\alpha \text{Stab}(N_\alpha) \) is the union of finitely many complex subvarieties of complex codimension at least 2 in \( M \), the analyticity of \( \overline{\text{Stab}(M_2)} \) follows from the Levi extension theorem (see [15]). Therefore, the cycle \( \overline{\text{Stab}(M_2)} \) represents an element in \( H_{2n-2}(M; \mathbb{Z}) \), and it is smooth near \( M_2 \). On the other hand, near a point \( p \) in \( M_2 \), we have the aforementioned linearized coordinate \( (U, \{z^i\}) \), then locally the unstable manifold of \( p \) is given by the \( z_n \) axis. Clearly the orbits of \( Im \nabla f \) on the \( z_n \) axis are all periodic. Pick such an orbit \( \theta : S^1 \to M \), then we can construct a map \( R : S^1(\simeq \mathbb{R}/T\mathbb{Z}) \times \mathbb{R} \to M \) by defining \( R(s, t) = \phi_t(\theta(s)) \), where \( \phi_t \) is the integral curve of the negative gradient flow of \( f \), and \( T \) is the period of the orbit of \( \theta \). Since \( [\nabla f, J \nabla f] = 0 \), \( R \) is holomorphic. By the Riemann removable singularity theorem, \( R \) extends to a rational curve, still denoted by \( R : \mathbb{C}P^1 \to M \). It coincides with the unstable manifold of \( p \) in \( U \), thus \( R \) is smoothly embedded near \( p \), intersecting \( \overline{\text{Stab}(M_2)} \) transversally at exactly one point \( p \). So the intersection number of integral homology classes represented by \( \overline{\text{Stab}(M_2)} \) and \( R \) is 1. It follows that \( R \) represents a generator of \( \pi_2(M) \) (modulo torsion). Now the remaining follows from standard arguments. By a well-known result of Grothendieck, the quotient bundle \( R^*(TM/T \mathbb{C}P^1) \) splits into a direct sum of line bundles \( Q_2, \ldots, Q_n \), each \( Q_i \) (\( 2 \leq i \leq n \)) is positive. It follows that

\[
c_1(R^*TM) = c_1(\mathbb{C}P^1) + \sum_{i=2}^{n} c_1(Q_i).
\]

Hence \( c_1(TM) \) evaluated at \([R]\) is bigger or equal to \( n+1 \). That is, \( c_1(TM) \geq (n+1)c_1(F) \), where \( F \) is the line bundle on \( M \) such that \( \langle c_1(F), [R] \rangle = 1 \). By the result of Kobayashi-Ochiai ([19]), \( M \) is biholomorphic to \( \mathbb{C}P^n \). This finishes the proof of Theorem 1. □

We need the following lemma in the proof of Corollary 4.

**Lemma 6.** Along the Kähler-Ricci flow, suppose there are positive constants \( T_0 \) and \( C \), such that for \( t \in [T_0, +\infty) \), \( g(t) \) has positive bisectional curvature and the Ricci curvature of \( g(t) \) satisfies \( \text{Ric}(g(t)) \geq C \cdot g(t) \). Then the bisectional curvature of \( g(t) \) is uniformly bounded below from 0.
Proof. We follow an idea in [8] which is essentially an application of the maximum principle. Along the Kähler-Ricci flow, we have the following evolution equations:

\[
\frac{\partial g_{ij}}{\partial t} = g_{ij} - R_{ij} ;
\]

\[
\frac{\partial R}{\partial t} = \Delta R + |Ric|^2 - R ;
\]

\[
\frac{\partial Ric}{\partial t} = \Delta Ric + Ric \cdot Rm - Ric^2 ;
\]

\[
\frac{\partial R_{ijkl}}{\partial t} = \Delta R_{ijkl} + R_{ijpq}R_{qpk\ell} - R_{ijkq}R_{pjq\ell} + R_{ipqk}R_{qj\ell} + R_{ij\ell k} - \frac{1}{2} (R_{ip}R_{pj\ell k} + R_{pj}R_{iq\ell k} + R_{kp}R_{ij\ell} + R_{pm}R_{ijkl}) .
\]

(1)

Here we define

\[
(Ric \cdot Rm)_{ij} = R_{ik}R_{ij\ell} ,
\]

and

\[
(Ric^2)_{ij} = R_{ik}R_{kj} ,
\]

Now we put \( S = Rm - \mu (g \ast Ric) \), where \( \mu \) is a function of \( t \), and

\[
(g \ast Ric)_{ijkl} = g_{ij}R_{k\ell} + g_{k\ell}R_{ij} + g_{i\ell}R_{kj} + g_{kj}R_{i\ell} .
\]

Then

\[
S_{kl} = (1 - (n + 2)\mu)R_{kl} - \mu R \cdot g_{kl} ,
\]

i.e

\[
Sic = (1 - (n + 2)\mu)Ric - \mu R \cdot g ,
\]

where \( Sic_{ij} = g^{kl}S_{ijkl} \). Therefore, by a straightforward calculation, we obtain

\[
\frac{\partial R_{ijkl}}{\partial t} = \Box S_{ijkl} + \mu g \ast (\Delta Ric)_{ijkl} + \mu (g \ast Ric)_{ijkl} + \mu [(g \ast (Ric \cdot S))_{ijkl} + (Ric \ast Sic)_{ijkl}] + I
\]

\[-\mu [(Ric \ast Ric)_{ijkl} + (Ric^2 \ast g)_{ijkl}] ,
\]

where \( \Box S_{ijkl} \) denotes the right side of (1) with \( R_{ijkl} \) replaced by \( S_{ijkl} \), and

\[
I = \mu^2 [(g \ast Ric)_{ijpq}(g \ast Ric)_{qpk\ell} - (g \ast Ric)_{ipqk}(g \ast Ric)_{pjq\ell} + (g \ast Ric)_{iqpj}(g \ast Ric)_{qpkj}] .
\]
Finally we calculate:

\[-\frac{\partial}{\partial t}(\mu(g \ast \text{Ric})_{\bar{ij}k\ell}) = -\mu'(g \ast \text{Ric})_{\bar{ij}k\ell} - \mu\left(\frac{\partial g}{\partial t} \ast \text{Ric}\right)_{\bar{ij}k\ell} - \mu(g \ast \frac{\partial \text{Ric}}{\partial t})_{\bar{ij}k\ell}\]

\[= -\mu'(g \ast \text{Ric})_{\bar{ij}k\ell} - \mu(g \ast \text{Ric})_{\bar{ij}k\ell} + \mu(\text{Ric} \ast \text{Ric})_{\bar{ij}k\ell}\]

\[-\mu(g \ast (\Delta \text{Ric}))_{\bar{ij}k\ell} - \mu(g \ast (\text{Ric} \cdot \text{Rm}))_{\bar{ij}k\ell} + \mu(g \ast (\text{Ric}^2))_{\bar{ij}k\ell}.\]

It follows that

\[
\frac{\partial S_{\bar{ij}k\ell}}{\partial t} = \Box S_{\bar{ij}k\ell} + \mu g \ast (\Delta \text{Ric})_{\bar{ij}k\ell} + \mu(g \ast \text{Ric})_{\bar{ij}k\ell} + \mu[(g \ast (\text{Ric} \cdot S))_{\bar{ij}k\ell} + (\text{Ric} \ast S_i G)_{\bar{ij}k\ell}] + I
\]

\[-\mu[(\text{Ric} \ast \text{Ric})_{\bar{ij}k\ell} + (\text{Ric}^2 \ast g)_{\bar{ij}k\ell}] - \mu'(g \ast \text{Ric})_{\bar{ij}k\ell} - \mu(g \ast \text{Ric})_{\bar{ij}k\ell}\]

\[+ \mu(\text{Ric} \ast \text{Ric})_{\bar{ij}k\ell} - \mu(g \ast (\Delta \text{Ric}))_{\bar{ij}k\ell} - \mu(g \ast (\text{Ric} \cdot \text{Rm}))_{\bar{ij}k\ell} + \mu(g \ast (\text{Ric}^2))_{\bar{ij}k\ell}\]

\[= \Box S_{\bar{ij}k\ell} + \mu^2(g \ast (\text{Ric} \cdot (g \ast \text{Ric}))_{\bar{ij}k\ell} + \mu(\text{Ric} \ast S_i G)_{\bar{ij}k\ell} - \mu'(g \ast \text{Ric})_{\bar{ij}k\ell} + I\]

\[= \Box S_{\bar{ij}k\ell} + \mu(\text{Ric} \ast \text{Ric})_{\bar{ij}k\ell} + \mu^2(g \ast (\text{Ric} \cdot (g \ast \text{Ric}))_{\bar{ij}k\ell}\]

\[-\mu^2(\text{Ric} \ast ((n + 2) \text{Ric} + R \ast g)_{\bar{ij}k\ell} - \mu'(g \ast \text{Ric})_{\bar{ij}k\ell} + I.\]

Notice that $I = O(\mu^2)$. Since $\text{Ric}(g(t)) \geq C \cdot g(t)$ for $t \geq T_0$, if $\mu(t) \equiv \mu(0) \equiv \mu > 0$ is sufficiently small, we have

\[
(\frac{\partial}{\partial t} - \Box)S_{\bar{ij}k\ell} \geq 0.\]

Now by the maximum principle as in [21], we see that $S_{\bar{ij}j} \geq 0$ for all $t \geq T_0$. Thus $R_{\bar{ij}j} \geq \mu(g \ast \text{Ric})_{\bar{ij}j} \geq \mu C(g \ast g)_{\bar{ij}j}$. □

**Remark 7.** By a more careful study of critical submanifolds of $f$ and using the localization formula, it may be possible to give a direct proof of vanishing of the Futaki invariant associated to the holomorphic vector field induced by $f$. If so, one would obtain a proof of Theorem 1 which does not rely on the theorem of
Acknowledgments: Both the first and third authors are partially supported by NSF grants. We would like to thank the anonymous referee for pointing out an error in the earlier version of this paper.

References

[1] S. Bando. *On the three dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Diff. Geom. 19(1984), 283C297.

[2] M. Berger, *Sur les variétés d’Einstein compactes*, Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d’Expression Latine (Namur, 1965), 35-55.

[3] R. L. Bishop, S. I. Goldberg, *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. 16 (1965), 119-122.

[4] R. Bott, *Homogeneous Vector Bundles*, Annals of Math, 66 (1957), 203-248.

[5] R. Bryant, *Gradient Kähler Ricci Solitons*, arxiv: math/0407453v2.

[6] H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359–372.

[7] B. Chow, *The Ricci flow on the 2-sphere*, J. Diff. Geom. 33 (1991), 325-334.

[8] X. X. Chen, *On Kähler Manifolds with Positive Orthogonal Bisectional Curvature*, Adv. Math. 215 (2007), no. 2, 427–445.

[9] X. X. Chen, P. Lu, G. Tian, *A Note on Uniformization of Riemann Surfaces by Ricci Flow*, Proc. Amer. Math. Soc. 134 (2006), no. 11, 3391C3393.

[10] X. X. Chen, G. Tian, *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487–544.

[11] X. X. Chen, G. Tian, *Ricci flow on Kähler-Einstein manifolds*, Duke Math. J. 131 (2006), no. 1, 17–73.

[12] X. X. Chen, G. Tian, *unpublished notes*.

[13] X.X. Chen, M. J. Zhu, *Liouville energy on a topological two sphere*, Arxiv preprint arXiv:0710.4320, 2007
[14] T. Frankel, *Fixed Points and Torsion on Kähler Manifolds*, Ann. of Math. (2) 70(1959), 1-8.

[15] P. Griffiths, J. Harris. *Principles of Algebraic Geometry*, Wiley Classics Library Edition, 1994.

[16] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differ. Geom. 24 (1986), 153-179.

[17] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math. 71 AMS, Providence, RI (1988), 237-262.

[18] S. Kobayashi, *Transformation groups in Differential Geometry*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[19] S. Kobayashi, T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. 13 (1973), 31-47.

[20] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Grundlehren Math. Wiss. 283, Springer-Verlag, 1985.

[21] N. Mok, *The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bisectional curvature*, J. Differential Geom. 27 (1988) 179-14.

[22] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. 76 (2) (1979), 213-234.

[23] D. H. Phong, J. Song, J. Sturm, B. Weinkove, *The Kähler-Ricci Flow with Positive Bisectional Curvature*, arxiv: math/0706.2852.

[24] N. Sesum, *Convergence of a Kähler-Ricci flow*, Math. Res. Lett, 12 (2005), 623-632.

[25] N. Sesum, G. Tian, *Bounding the scalar curvature and diameter along the Kähler Ricci flow (after Perelman)*, J. Inst. Math. Jussieu 7 (2008), no. 3, 575–587.

[26] Y. T. Siu, S. T. Yau, *Compact Kähler Manifolds of Positive Bisectional Curvature*, Invent. Math. 59 (1980), no. 2, 189-204.

[27] G. Tian, X. H. Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. 184 (2000), no. 2, 271–305.

[28] G. Tian, X. H. Zhu, *Convergence of Kähler-Ricci Flow*, J. Amer. Math. Soc, 20(2007), no. 3, 675-699.
