Abstract

It is shown that any linear estimator that satisfies the moment conditions up to order $p$ is equivalent to a local polynomial regression of order $p$ with some non-negative weight function if and only if the kernel has at most $p$ sign changes. If the data points are placed symmetrically about the estimation point, a linear weighting function is equivalent to the standard quadratic weighting function.
1 Local Polynomial Regression

We consider a linear estimate of a function or its derivatives given a sequence of measurements, \( \{y_i, i = 1 \ldots N\} \) at the locations, \( \{x_i\} \). In nonparametric estimation, typical assumptions are: \( f(t) \) has \( p \) continuous derivatives (\( q < p \)) and \( y_i = f(x_i) + \varepsilon_i \), where the errors, \( \varepsilon_i \), are independent random variables with zero mean and variance equal to \( \sigma_i^2 \). These assumptions motivate our work, but are not necessary for our results.

One method to select the coefficients of a linear estimator is local polynomial regression (LPR) as described in works by Cleveland (1979), Fan and Gijbels (1992), Fan (1993), Hastie and Loader (1993)). Not every weighted linear estimate arises from LPR. We show that any linear estimator that satisfies the moment conditions up to order \( p - 1 \) is equivalent to a local polynomial regression of order \( p - 1 \) if and only if the kernel has at most \( p - 1 \) sign changes.

Let \( \{(x_i, y_i), i = 1, \ldots N\} \) be given, where \( N \) is the number of measurements, \( x_i \) is the \( i \)th measurement location and \( y_i \) is the corresponding measured value. We consider linear estimators of the \( q \)th derivative of an unknown function, \( f^{(q)}(t) \), of the form:

\[
\hat{f}^{(q)}(t) = \sum_{i=1}^{N} K_i(t)y_i , \tag{1.1}
\]

where the \( K_i(t) \) depend on the design, \( \{x_i\} \), but are independent of \( \{y_i\} \). For a given value of \( t \), we say the weight coefficients, \( \{K_i(t), i = 1 \ldots N\} \), are of type \( (q, p) \) if it satisfies the moment conditions:

\[
\frac{1}{m!} \sum_{i=1}^{N} (x_i - t)^m K_i(t) = \delta_{m,q} , \quad m = 0, \ldots, p - 1 . \tag{1.2}
\]

In local polynomial regression, at each point, \( t \), a set of nonnegative weights is specified, \( w_i(t) \), and a low order polynomial is fitted to the weighted sum of squares. (The weights are usually scaled as \( w_i(t) = W \left( \frac{x_i - t}{h} \right) \), where \( W \) is a non-negative function on \([-1,1]\) and \( h \) is the bandwidth parameter.) At point \( t \), the local estimate of \( f(x) \) is \( \sum_{j=0}^{p-1} a_j(t)x^j \), where \( p - 1 \) the order of the polynomial approximation. The
coefficients, \( a_j(t) \), are determined by minimizing

\[
F(a_0, a_1, \ldots, a_{p-1}) = \sum_{i=1}^{N} w_i(t) \left( \sum_{j=0}^{p-1} a_j(x_i - t)^j - y_i \right)^2.
\]

The resulting estimate of \( f^{(q)}(t) \) is \( q!a_q \). Since the functional is quadratic and non-negative, the minimum exists and satisfies

\[
0 = \frac{\partial F}{\partial a_k} = \sum_{j=0}^{p-1} \left[ \sum_{i=1}^{N} (x_i - t)^{k+j} w_i(t) \right] a_j - \sum_{i=1}^{N} (x_i - t)^k w_i(t) y_i
\]

for \( k = 0, 1, \ldots, p - 1 \). This system of linear equations can be rewritten as

\[
\sum_{j=0}^{p-1} d_{kj}(t) \left( a_j h^j \right) = m_k(t), \quad k = 0, 1, \ldots, p - 1, \tag{1.3}
\]

where

\[
d_{kj}(t) = \frac{1}{Nh} \sum_{i=1}^{N} \left( \frac{x_i - t}{h} \right)^{k+j} w_i(t), \quad m_k(t) = \frac{1}{Nh} \sum_{i=1}^{N} \left( \frac{x_i - t}{h} \right)^k w_i(t) y_i.
\]

In (1.3), \( h \) is used solely to scale the equations for numerical stability.

If the number of data points with non-zero weights is at least \( p \), the matrix \([d_{kj}(t)]\) is non-singular. Let \([\tilde{d}_{jk}(t)]\) be the inverse matrix. Solving for \( a_q h^q \) shows that a local polynomial regression corresponds to a linear estimate (1.1) with weighting coefficients, \( K_i(t) \):

\[
K_i(t) = w_i(t) \left[ \frac{q!}{Nh^{q+1}} \sum_{k=0}^{p-1} \tilde{d}_{qk}(t) \left( \frac{x_i - t}{h} \right)^k \right]. \tag{1.4}
\]

Let \( \tilde{x} \) be a dummy variable and define

\[
\tilde{P}(\tilde{x}; t, \{x_i\}) = w_i(t) \left[ \frac{q!}{Nh^{q+1}} \sum_{k=0}^{p-1} \tilde{d}_{qk}(t; \{x_i\}) \tilde{x}^k \right]. \tag{1.5}
\]

Here \( \tilde{P}(\tilde{x}; t, \{x_i\}) \) is a polynomial of order \( p - 1 \) in \( \tilde{x} \) given \( t \) and \( \{x_i\} \). We name \( \tilde{P}(\tilde{x}; t, \{x_i\}) \) the factor polynomial. The \( p \) coefficients, \( \tilde{d}_{qk}(t; \{x_i\}) \) determine the \( N \) values of the linear weights: \( K_i(t) = w_i(t) \tilde{P}(\tilde{x}; t, \{x_i\}) \) (see Müller (1987)).

Thus for a given estimation point \( t \) and weights \( w_i \), the local polynomial regression estimator is equivalent to a kernel estimator whose kernel is the product of the weights
with a polynomial in $\frac{x-t}{h}$ of order $p-1$. The equivalent kernel automatically satisfies the moment conditions and thus is a kernel of type $(q, p)$.

We say that a discrete function $Q(x_i)$ has a sign change between $x_j$ and $x_{j+k}$ if $Q(x_j)Q(x_{j+k}) < 0$ and $Q(x_{j+1}) = \ldots = Q(x_{j+k-1}) = 0$. The weights, $w_i(t)$, are non-negative, and the factor polynomial has at most $p-1$ roots. Therefore, for the given $t$, the equivalent kernel $K(t, x_i)$ has at most $p-1$ sign changes. Answering the question: “which kernel estimators can be represented as a local polynomial regression?” we show that the necessary condition is also sufficient.

**Theorem 1.** A linear estimator of type $(q, p)$ is generated by local polynomial regression of degree $p-1$ with non-negative weights if and only if the kernel has no more than $p-1$ sign changes.

It is known (see Müller (1985)) that any kernel of type $(q, p)$ has at least $p-2$ sign changes. This implies

**Corollary.** The order of the factor polynomial for a degree $(p-1)$ LPR is at either $p-1$ or $p-2$.

## 2 Equivalence of Linear and Quadratic Weightings

It is known (Müller (1987), Fan(1993)) that the optimal interior kernel of type $(q, p)$, $p-q \equiv 0 \mod 2$, is produced by the scaling weight function, $W(y) = 1 - y^2$, in the limit of nearly equi-spaced measurement points as $N \to \infty$. We show that this choice is not unique.

**Theorem 2.** Let $p-q$ be even. If data points, $x_i$, are symmetric around the estimation point, $t$, and their weights are chosen as $w_i = W\left(\frac{x_i-t}{h}\right)$, then each of the functions $W_1(y) = 1 - y$, $W_2(y) = 1 + y$, $W_3(y) = 1 - y^2$ produces the same estimator.

The weights, $W_1(y)$ and $W_2(y)$, assign less weight to the estimation point, $t$, than to one side of the data. This surprising result is useful in constructing optimal boundary kernels which depend continuously on the estimation point.

Because of the optimality in the interior, the Bartlett-Priestley weighting, $W(y) =
1 − y^2, is used often in the boundary region as well (Hastie and Loader (1993)). In a future work, we show that a linear weighting has a lower asymptotic MSE than the Bartlett-Priestley weighting in the boundary region. Theorem 2 shows that one can switch the weighting function from \( W(y) = 1 − y^2 \) to \( W(y) = 1 − y \) without generating a discontinuity in the estimate.

**Appendix. Proofs**

**Lemma.** Let \( K_{1,i} \) and \( K_{2,i} \) be kernels of type \((q,p)\) with the same estimation point and the same support such that \( K_{r,i} = W_i Q_r(x_i) \), \( r = 1, 2 \), where \( W_i \geq 0 \) for all data points \( x_i \) in the support. If \( Q_1(x) \) and \( Q_2(x) \) are polynomials of order \( p − 1 \) then \( K_{1,i} = K_{2,i} \) for every data point \( x_i \).

**Proof.** Since \( K_1 \) and \( K_2 \) satisfy the same moment conditions, their difference is orthogonal to any polynomial \( P(x_i) \) of order \( p − 1 \): \( \sum_i (K_{1,i} − K_{2,i}) P(x_i) = 0 \). When we choose \( P(x_i) = Q_1(x_i) − Q_2(x_i) \), we have \( \sum_i W_i (Q_1(x_i) − Q_2(x_i))^2 = 0 \). Since \( W_i \geq 0 \), it implies \( W_i (Q_1(x_i) − Q_2(x_i)) = 0 \) for every \( x_i \).

**Proof of Theorem 1.** Let a kernel \( K(x_i) \) have \( m \leq p − 1 \) sign changes. We enumerate the sign changes: \( z_1, z_2, \ldots, z_m \). Namely, if the \( l \)th sign change occurs at \( x_j \) or between \( x_j \) and \( x_{j+k} \), we set \( z_l = x_j + \varepsilon \) where \( \varepsilon < \min\{x_2 − x_1, x_3 − x_2, \ldots, x_N − x_{N−1}\} \). Now we define \( H(x) = (-1)^s \prod_{l=1}^m (x − z_l) \), \( W(x_i) = K(x_i)/H(x_i) \). The function \( W(x_i) \) has no sign changes. We choose \( s \) to make all of the values \( W(x_i) \) non-negative. Let \( Q \) be the factor polynomial for the local polynomial regression with the weights \( w_i = W(x_i) \). Since \( K = WH \) and \( WQ \) are kernels of type \((q,p)\), and \( H, Q \) are polynomials of order \( p − 1 \), The lemma implies that \( K(x_i) = W(x_i) H(x_i) = W(x_i) Q(x_i) \) for every data point \( x_i \). Thus \( K \) is the equivalent kernel for the local polynomial regression with the weights \( w_i \).

**Proof of Theorem 2.** It is sufficient to check that weightings \( W_1(y) = 1 − y \) and \( W_3(y) = 1 − y^2 \) have the same equivalent kernel. Let \( Q_1(y) \) and \( Q_3(y) \) be their respective factor polynomials. Since \( Q_3 \) is a polynomial of order \( p − 1 \), then \( W_3 Q_3 \)
is a polynomial of order $p + 1$. Since $W_3$ is even and the placement of data points is symmetric, the equivalent kernel, $W_3Q_3$, is an even function (if $q$ is even) or an odd function (if $q$ is odd). The difference $p - q$ is even, thus $W_3Q_3$ can not have term $y^{p+1}$. Therefore, $W_3Q_3$ is a polynomial of order $p$, and the true order of $Q_3$ is at most $p - 2$. Now we notice that $W_3(y)Q_3(y) = W_1(y)\left[(1+y)Q_3(y)\right]$. Both $(1+y)Q_3(y)$ and $Q_1(y)$ are polynomials of order $p - 1$. Thus the lemma implies that $W_3(y)Q_3(y) = W_1(y)Q_1(y)$ when $y = \frac{x_i - t}{h}$.

References

Cleveland, W. S. (1979). Robust locally weighted regression and smoothing scatterplots. *J. Amer. Statist. Assoc.* 74 829-836.

Fan, J. and Gijbels, I. (1992). Variable bandwidth and local linear regression smoothers. *Ann. Stat.* 20 2008-2036.

Fan, J. (1993). Local linear regression smoothers and their minimax efficiencies. *Ann. Stat.* 21 196-216.

Hastie, T. and Loader, C. (1993). Local regression: automatic kernel carpentry. *Statistical Science* 8 120-143.

Müller, H. G. (1985). On the number of sign changes of a real function. *Periodica Mathematica Hungarica* 16 209-213.

Müller, H. G. (1987). Weighted local regression and kernel methods for nonparametric curve fitting. *J. Amer. Statist. Assoc.* 82 231-238.