Infinite Conformal Algebras in Supersymmetric Theories on Four Manifolds

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Abstract

We study a supersymmetric theory twisted on a Kähler four manifold $M = \Sigma_1 \times \Sigma_2$, where $\Sigma_{1,2}$ are 2D Riemann surfaces. We demonstrate that it possesses a "left-moving" conformal stress tensor on $\Sigma_1$ ($\Sigma_2$) in a BRST cohomology, which generates the Virasoro algebra with the conventional commutation relations. The central charge of the Virasoro algebra has a purely geometric origin and is proportional to the Euler characteristic $\chi$ of the $\Sigma_2$ ($\Sigma_1$) surface. It is shown that this construction can be extended to include a realization of a Kac-Moody algebra in BRST cohomology with a level proportional to the Euler characteristic $\chi$. This structure is shown to be invariant under renormalization group. A representation of the algebra $W_{1+\infty}$ in terms of a free chiral supermultiplet is also given. We discuss the role of instantons and a possible relation between the dynamics of 4D Yang-Mills theories and those of 2D sigma models.

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1 Introduction

Two-dimensional $N = 2$ superconformal theories have been intensively studied in recent years because of their relevance to superstrings, topological theories, and integrable systems. The large class of $N = 2$ superconformal theories can be realized as the infrared fixed points of the Landau-Ginzburg models $[1, 2, 3, 4, 5, 6]$. The difficulty of such a description is that the corresponding Landau-Ginzburg models do not possess a conformal symmetry. The proposal of an effective field theoretical description of $N = 2$ superconformal theories in terms of Landau-Ginzburg models has recently received much further support. Witten $[7]$ has shown that under the operation of a half-twist the Landau-Ginzburg model turns out to be left-moving and conformally invariant. This superconformal algebra is realized on the classes of cohomology for one of the supergenerators of the $N = 2$ SUSY algebra and turns out to be invariant under the renormalization group. This enables us to extract the essential information about the $N = 2$ superconformal algebra realized at the infrared fixed point of the Landau-Ginzburg model. In particular Witten computed the corresponding elliptic genus. Witten also extended his analysis to the case of $(0, 2)$ SUSY models coupled to abelian gauge multiplets $[8]$. In Ref.$[9]$ this construction has been extended to more complicated 2D $N = 2$ models where $W_N$ algebras are realized in the cohomology for one of the supergenerators.

The idea of identifying different theories in terms of appropriate cohomologies was also used in $[10]$ where an equivalence of a special superconformal coset (with $\hat{c} = 3$) and $c = 1$ matter coupled to two dimensional gravity was demonstrated. Moreover this coset was shown to be connected with a twisted version of an euclidean two dimensional black hole in which the ghost and matter systems are mixed. This approach has also been used to identify bosonic strings and $N = 1$ superstrings with particular classes of vacua for $N = 2$ superstrings $[11]$ (for a further development see also $[12]$ and the references therein).

A natural question is if it is possible to extend similar constructions to the case of $D = 4$ theories. First, the existence of cohomological structures which are invariant under the renormalization group can give some important information about the dynamics of the theory. Second, it has been observed $[13]$ that there is a relation between two-dimensional integrable systems and four-dimensional self-dual Yang-Mills equations. The mathematical conjecture $[14]$ (see also $[15, 16, 17, 18, 19, 20, 21, 22$ and the references therein) that all possible bosonic integrable systems in lower dimensions originate in the Yang-Mills equations for self-dual connections (in $D = (2, 2)$) provides an additional motivation to look for such a correspondence. It has been also demonstrated that the supersymmetric integrable systems can be extracted from supersymmetric self-dual Yang-Mills theory (see $[23]$ and references therein). An important element of such an extraction is the dimensional reduction of four dimensional self-dual Yang-Mills equations to two dimensions.

The important role of self-dual Yang-Mills connections has been supported by recent results in $N = 2$ superstring theory. It is found $[24, 23, 20]$ that the consis-
tent space-time background is $N = 4$ self-dual supersymmetric Yang-Mills theory for open $N = 2$ superstrings, or $N = 8$ self-dual supergravity for $N = 2$ closed (heterotic) superstrings [27]. This correspondence could reveal some hidden symmetries of string theory and help to describe the structure of the string vacua from a unique symmetry principle.

The importance of a relation between self-dual Yang-Mills connections and two dimensional integrable systems for superstring theory provides a motivation to look for such a correspondence at the quantum level.

A possible tool for such a study is given by the topological theories. An important example of such a theory is given by the topological Yang-Mills theory [28] which is a twisted version of the $N = 2$ supersymmetric Yang-Mills theory. The space of physical operators in the topological Yang-Mills theory is defined as cohomology classes of an appropriate BRST operator. The action of the theory turns out to be BRST exact and, hence, the physical correlators are independent of the external metric and the gauge coupling constant. Therefore they can be calculated semiclassically around instanton configurations which turn out to be an essential ingredient of the theory because the physical correlators can be represented as integrals over the instanton moduli space. These correlators can depend on moduli of a differential structure on the curved manifold and give a realization of the Donaldson map [29] that relates a smooth structure of a 4-dimensional manifold to the topology of the instanton moduli space in terms of physical correlators. The physical correlators turn out to be Donaldson’s invariants which characterize smooth structures on 4D-manifolds.

Notice however that the topological Yang-Mills theory [28] has a finite number of physical degrees of freedom as a result of a reduction of the space of physical states of the $N = 2$ supersymmetric theory by using the BRST operator. In turn a four dimensional analogue of 2D half-twisted theories can be an intermediate situation between the usual dynamical 4D theories and 4D topological theories and provide us with a sort of a quantum dimensional reduction.

Recently [31] it has been shown that $N = 1 D = 4$ SUSY gauge theories with an appropriate representation of matter can be twisted on a Kähler manifold. These twisted models have an appropriate BRST charge, which is one of the N=1 supergenerators. The BRST charge does not depend on the external metric. If the theory does not contain any superpotential for the matter supermultiplet then the action is BRST exact. Otherwise it is BRST closed. The physical operators are defined as the classes of cohomology of the BRST operator.

Such a twisting is a four dimensional analogue of a half-twisting of 2D theories [0]. The topological Yang-Mills theory [28] is actually a particular case of such a twisted model with the fields of matter in the adjoint representation of the gauge

\[2\text{The topological Yang-Mills theory is also intriguing from the physical point of view. In particular in the heterotic string theory the scattering amplitudes of spacetime axions at zero momenta are found to be proportional to the Donaldson invariants [29] in the form as they are presented in Witten’s theory [28].} \]
group. However in the topological Yang-Mills theory \[28\] and in its twisted $N = 1$ supersymmetric version one considers the cohomologies of the different BRST operators \[3\]. An extension of the construction of ref. \[31\] to supersymmetric theories without gauge interactions is straightforward (we illustrate it in the present paper). We will henceforth generically name the twisted $N = 1$ supersymmetric theories heterotic topological theories.

The space of physical operators in the heterotic theory is in general more complicated than in the topological Yang-Mills theory. This space contains a ground ring of the local operators which are BRST invariant off-shell. This is a purely topological part of the model and this ring coincides with the ring of local physical operators in the Witten’s topological Yang-Mills theory provided that the matter is in the adjoint representation of the gauge group. However there is also an infinite set of operators in the cohomology of the BRST operator which can depend on the external metric and commute with the BRST operator only up to the equations of motion. This structure of the space of physical operators is similar to that of 2D half-twisted models \[7, 8\]. In general the correlators with insertions of such operators depend on the external metric due to a presence of the external metric in the inserted operators. They can also depend on holomorphic coordinates on the four dimensional Kähler manifold \[31\] but do not depend on the anti-holomorphic ones. The latter fact implies that the twisting procedure provides us with a dimensional reduction from four dimensional space into two dimensional one.

In turn in the heterotic topological gauge theories (without a superpotential) the physical correlators allow for a localization near the solutions of equations of motion in the path integral due to the BRST exactness of the action and, hence, are determined semiclassically by the fluctuations near instantons similar to the topological Yang-Mills theory. Therefore we may expect that the heterotic topological gauge theory can give a relation between self-dual connections and 2D integrable systems at the quantum level.

In the present paper we focus to the theories on a four manifold $M = \Sigma_1 \times \Sigma_2$, where $\Sigma_{1,2}$ are 2D Riemann surfaces. We show that the twisted supersymmetric theory on a four manifold $M = \Sigma_1 \times \Sigma_2$ possesses two left-moving conformal stress tensors which generate two chiral Virasoro algebras (on $\Sigma_1(\Sigma_2)$) in the cohomology of a BRST operator \[33\]. The central charges of these algebras are shown to be invariant under the renormalization group and can be calculated in the weak coupling limit. These central charges turn out to be proportional to the Euler characteristic of a Riemann surface $\Sigma_1(\Sigma_2)$. In addition a twisted theory can contain BRST closed holomorphic (in a BRST cohomology) currents which generate a Kac-Moody algebra on $\Sigma_1(\Sigma_2)$ with a level proportional to the Euler characteristic of $\Sigma_2(\Sigma_1)$. We also show that the $W_{1+\infty} \[34, 35, 36\]$ algebra can be realized in terms of a free chiral supermultiplet.

It is interesting that the central charge manifests itself as a gravitational anomaly

\[3\] Recently \[32\] Witten calculated the Donaldson invariants by using the formulation of the topological Yang-Mills theory \[28\] in terms of the twisted $N=1$ supersymmetry.
of the four dimensional twisted theory. Actually we demonstrate that due to this anomaly two Liouville theories (for the external metric) are generated on the surfaces $\Sigma_{1,2}$. It turns out that the multiloop corrections to the gravitational anomaly vanish due to the BRST invariance of the theory. Therefore we are tempted to look at the effective two dimensional conformal theories from the point of view of two-dimensional gravity.

It turns out that in the case of matter in the adjoint representation of the gauge group the central charge of the Virasoro algebra $c = 0$. A tempting possibility is that one could identify this formulation (twisted $\mathcal{N}=1$ supersymmetry) of the topological Yang-Mills theory with a topological 2D gravity coupled to topological matter in terms of a BRST cohomology.

One can also hope that such a construction can give an important information on the renormalization group (RG) in $D = 4 \, \mathcal{N} = 1$ super QCD. We discuss below the instanton effects in the heterotic topological Yang-Mills theory and demonstrate an appearance of a possible correspondence between RG flows and instantons in 4D super QCD and in 2D sigma models.

The paper is organized as follows. In section 2 we review the twisting procedure of supersymmetric theories on Kähler manifolds and formulate the heterotic topological theory. This construction is also extended to self-interacting chiral supermultiplet without gauge interactions. We define the physical operators as classes of the BRST cohomology. We discuss the anomalies in a decoupling of the external metric from the physical correlators and the conditions of the anomaly cancellation and the renormalization properties of the gravitational anomaly. In section 3 we demonstrate a realization of the $W_{1+\infty}$ algebra in terms of a free chiral supermultiplet in the ghost number $G = 0$ BRST cohomology. We also briefly discuss the BRST cohomology at $G \neq 0$. In section 4 we give a realization of the Virasoro algebra in terms of a self-interacting chiral supermultiplet. In section 5 we focus to twisted supersymmetric gauge theories. We discuss the procedure of gauge fixing and the properties of physical correlators. We then study the BRST cohomology at the ghost number $G = 0$ from the point of view of $U(1)$ and Konishi anomalies. We give a realization of Kac-Moody and Virasoro algebras. We finally discuss the instanton effects and a possible relation of the dynamics of the four dimensional Yang-Mills theory to that of two dimensional sigma models. We conclude by summarizing the results of the paper. Some possible generalizations of these results are also discussed.

### 2 Supersymmetric theory on a four dimensional Kähler manifold

#### 2.1 Coupling to a reduced multiplet of supergravity

Let us consider $\mathcal{N} = 1$ supermultiplet in a euclidean compact four manifold $M$. On a curved manifold all the supersymmetries are broken by an external metric.
However as it is shown in \[31\], on a Kähler manifold, if we introduce an external vector field $V_\mu$ coupled to an appropriate axial fermionic current constructed out of quantum fields one of the four supergenerators is associated with an unbroken fermionic symmetry of the theory even for non-trivial external Kähler metric. This vector field should depend on the external metric and is fixed by the condition that one of the supergenerators survives in the curved metric (for a similar approach to the twisting of the $N = 2$ supersymmetric theories see \[37, 38, 39\]).

In order to implement twisting it is convenient to consider a supermultiplet coupled to an external $N = 1$ supergravity multiplet that contains the vierbein field $e^a_\mu$, the gravitino (Weyl spinor) field $\chi_\mu$, $\bar{\chi}_\mu$, $U(1)$ vector field $V_\mu$ and auxiliary fields. We consider these supergravity fields as external ones. The total system is invariant under simultaneous local supertransformations both of matter and supergravity fields. In a second step we must reduce the external supergravity multiplet in order to get a supergravity background which is invariant under a global supertransformation. It turns out that such a reduction is possible only on Kähler manifolds \[31\]. We carry out the reduction to the vierbein field $e^a_\mu$ and the vector field $V_\mu$. By using the formulas of supertransformations of the ("new minimal") supergravity \[40\] we observe that this reduced supermultiplet is invariant under a SUSY transformation if the variation of the gravitino field vanishes

$$\delta \chi_\mu = (\partial_\mu - \frac{i}{4} \omega_\mu^{ab} \sigma_{ab} + V_\mu) \epsilon,$$

$$\delta \bar{\chi}_\mu = (\partial_\mu - \frac{i}{4} \omega_\mu^{ab} \sigma_{ab} - V_\mu) \bar{\epsilon}.$$  

Here the letters $\mu, \nu...$ stand for world indices while $a, b...$ correspond to the Lorentz indices in the tangent frame; $\sigma^{ab} = [\sigma^a, \sigma^b]/2i$, where $\sigma^a$, $a = 1, 2, 3, 4$ are the Pauli matrices. The left- (right-)handed spinors $\epsilon(\bar{\epsilon})$ are parameters of the supertransformation. The spin-connection $\omega_\lambda^{ab}$ reads

$$\omega_\lambda^{ab} = \frac{1}{2} [e^{\alpha \nu} e^{\beta \mu} e_{\lambda \nu} (\partial_\nu e_\mu^\alpha - \partial_\mu e_\nu^\alpha) - e^{\alpha \nu} (\partial_\lambda e_\nu^b - \partial_\nu e_\lambda^b) - e^{b \nu} (\partial_\nu e_\alpha^a - \partial_\alpha e_\nu^a)].$$

On the Kähler manifold the holonomy group is reduced to $U(2)$ (see, e.g. \[41\]) and in the sector of right handed spinors we can choose for definiteness the matrix $i \omega_\mu^{ab} \sigma_{ab}/4$ to be $\omega_\mu \sigma_3$ (with some $\omega_\mu$) for all $\mu$ in an open region of the 4-manifold. We can then take $V_\mu = \pm \omega_\mu$. The global supersymmetry that survives in this background is generated by one of the right-handed $N = 1$ supercharges and corresponds to a constant parameter $\bar{\epsilon}$.

From now on we shall consider the reduced supergravity multiplet with the vierbein field and the vector $V_\mu$ obeying

$$\left(\frac{i}{4} \omega_\mu^{ab} \sigma_{ab} + V_\mu\right) \bar{\epsilon} = 0,$$

for a non-vanishing constant spinor $\bar{\epsilon} \neq 0$. 

5
Let us first consider the gauge supermultiplet coupled to the external metric and the vector field \( V_\mu \). The \( N = 1 \) gauge supermultiplet contains a gauge field \( A_\mu \), a fermionic gaugino field \( \lambda_\alpha \) and \( \bar{\lambda}^\dot{\alpha} \), and an auxiliary scalar field \( D \) which is necessary for SUSY algebra to be closed. The lagrangian of the SUSY Yang-Mills theory reads as follows \cite{12}:

\[
L = \sqrt{g} \frac{1}{e^2} \text{Tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\lambda} i \slashed{D} \lambda + \frac{1}{2} D^2 \right] \tag{2.4}
\]

where \( \bar{\lambda} \) and \( \lambda \) stand for the right and left-handed Weyl spinor correspondingly, \( F_{\mu\nu} \) is the strength tensor for the gauge field; \( e^2 \) stands for the gauge coupling constant, \( g \) is a determinant of the external metric tensor, \( \slashed{D} = \sigma_\mu \slashed{D}_\mu \), \( \slashed{D}_\mu = \nabla_\mu (\omega, V) - i A_\mu \), and \( \nabla_\mu = \partial_\mu - i \omega_\mu \bar{a} \sigma_{ab} / 4 + V_\mu \). This Lagrangian is invariant under a global supertransformation generated by the right-handed supergenerator \( Q \) which corresponds to the constant spinor \( \bar{\epsilon} \) obeying eq.(2.3). This supertransformation reads as follows

\[
\delta A_\mu = (\bar{\epsilon} \sigma_\mu a) e_\mu^a, \quad \delta \lambda = 0, \tag{2.5}
\]

\[
\delta \bar{\lambda} = D \bar{\epsilon} - \frac{1}{2} F_{\mu\nu} \bar{\epsilon} \sigma^{ab} e^\mu_a e^\nu_b, \quad \delta D = -\bar{\epsilon} i \slashed{D} \lambda.
\]

By a direct calculation one can check that \( Q \) is nilpotent, i.e. \( Q^2 = 0 \) and the Lagrangian is \( Q \)-exact

\[
L = \sqrt{g} \frac{1}{2e^2} \left\{ Q, \bar{\lambda} \left( D + \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \bar{\eta} \right\}, \tag{2.6}
\]

where a constant spinor \( \bar{\eta} \) is linearly independent of \( \bar{\epsilon} \) and is normalized by \( \bar{\epsilon} \bar{\eta} = 1 \). Eq. (2.6) for the Lagrangian is equivalent to Eq. (2.4) up to a term which is proportional to a topological charge of the gauge field.

Let us now consider a chiral supermultiplet in a representation \( R \) of the gauge group. The components of a chiral supermultiplet are a complex, scalar field \( \phi \), a Weyl fermion \( \psi \) and a complex scalar auxiliary field \( F \). It is important however that the axial current of fermions which belong to the supermultiplet of matter be coupled to the twisting vector field \( V_\mu \) with an opposite charge to the one of the gaugino current. This fact will allow us to cancel an anomaly of the total axial current which is used for the twisting. The absence of the anomaly for this current is a necessary condition for an interpretation of the twisted theory as a topological one (see below).

The Lagrangian for such a supermultiplet coupled to the gauge supermultiplet in the presence of the external Kähler metric and a vector (twisting) field \( V_\mu \) reads

\[
L = \sqrt{g} (D_\mu \bar{\phi} D^\mu \phi + \bar{\psi} i \slashed{D} \psi + \bar{F} F + \bar{\phi} D \phi - 2 \bar{\phi} \lambda \psi - \bar{\psi} \lambda \phi). \tag{2.7}
\]

This Lagrangian is invariant under the following SUSY transformation

\[
\delta \phi = 0, \quad \delta \psi = -i \sigma_\mu \bar{\epsilon} D^\mu \phi, \quad \delta F = -i \bar{\epsilon} \slashed{D} \psi + \bar{\lambda} \bar{\epsilon} \phi, \tag{2.8}
\]

\[
\delta \bar{\psi} = \bar{\epsilon} \bar{\psi}, \quad \delta \bar{\psi} = \bar{F} \bar{\epsilon}, \quad \delta \bar{F} = 0.
\]
Here the constant spinor $\bar{\epsilon}$ obeys Eq. (2.3). The generator $Q$ of this transformation is nilpotent ($Q^2 = 0$) and the Lagrangian (2.7) is $Q$-exact

$$L = \{Q, \bar{\eta}(-\bar{\psi} F + i\bar{\phi} \bar{D} \psi - \bar{\phi} \lambda \phi)\}. \quad (2.9)$$

Notice that the generator $Q$ of the supertransformations (2.5), (2.8) depends on the external metric and the covariant derivative acting to the spinors includes an additional vector field $V_\mu$, i.e. $\bar{D} = \sigma^\mu(\partial_\mu - i\sigma_{ab}\omega_{ab}/4 + V_\mu - iA_\mu)$. Now we want to interpret the generator $Q$ as a BRST charge in order to define the theory as a topological one. However the dependence of the BRST charge on external metric does not allows us to do that since the variation of the action with respect to the external metric is not $Q$-exact. To avoid this difficulty we shall change the spins of the quantum fields in order to remove the metric from the definition of the generator $Q$ and to formulate the Lagrangian in terms of usual covariant derivatives. In this way we shall introduce new fundamental fields which absorb some components of the metric tensor.

### 2.2 Twisting of a supersymmetric theory

It is convenient at this point to introduce a complex structure $J^\mu_\nu$ on $M$ associated to the spinors $\bar{\epsilon}$ and $\bar{\eta}$

$$J^\mu_\nu = (\bar{\eta}\sigma^{ab}\bar{\epsilon})e^\mu_a e^\nu_b. \quad (2.10)$$

We have

$$D_\lambda J^\mu_\nu = 0, \quad (J^2)^\mu_\nu = -\delta^\mu_\nu, \quad (2.11)$$

where $D_\lambda$ is the untwisted covariant derivative on $M$. With this complex structure one can define the holomorphic ($z^m$, $m = \bar{1}, \bar{2}$) and antiholomorphic ($\bar{z}^\bar{m}$, $\bar{m} = 1, 2$) coordinates on $M$ so that

$$J^m_n z^n = iz^m, \quad J^{\bar{m}}_{\bar{n}} \bar{z}^{\bar{n}} = -iz^{\bar{m}}. \quad (2.12)$$

In this well adapted frame (see, e.g. [11]) the complex structure has the simplest form

$$J^m_n = i\delta^m_n, \quad J^{\bar{m}}_{\bar{n}} = -i\delta^{\bar{m}}_{\bar{n}}, \quad J^m_n = 0, \quad J^{\bar{m}}_{\bar{n}} = 0 \quad (2.13)$$

and, hence,

$$J^{mn} = ig^{mn}, \quad (2.14)$$

while the other components of the metric tensor $g^{\mu\nu}$ vanish.

Let us now redefine the quantum fields. After an appropriate redefinition [31] the gauge multiplet contains a gauge field $A_\mu = (A_n, A_{\bar{n}})$ (with the holomorphic and anti-holomorphic components), a scalar fermion $\lambda = i\bar{\eta}\lambda$ and a fermion $(0, 2)$ form $\lambda_{\bar{m}a} = (\bar{\eta}\sigma_{\bar{m}a}\bar{\eta})(\bar{\epsilon}\lambda)$ (they both originate from the right handed gaugino), a fermion $(1, 0)$-form $\chi_n = \bar{\epsilon}\sigma_a \lambda$ (it corresponds to the left handed gaugino), and an auxiliary field $D' = iD + ig^{mn}F_{nm}$, where $F_{nm}$ is the strength tensor components of the gauge field. Similarly for the chiral multiplet we have a scalar complex bosonic field
\( \phi (\bar{\phi}) \), a scalar fermion \( \bar{\psi} \), the fermionic fields \( \psi_{\bar{m}} \) and \( \bar{\psi}_{mn} \) which are \((0, 1)\) and \((2, 0)\) forms respectively, and the auxiliary bosonic \((0, 2)\) and \((2, 0)\) forms \( N_{\bar{m}n} \) and \( \bar{N}_{mn} \). The total twisted Lagrangian reads as follows \[31\]

\[
L = \sqrt{g} \frac{1}{\epsilon^2} \text{Tr}[F^{\bar{m}n} F_{\bar{m}n} + i \bar{\lambda}^{mn} D_m \chi_n - \frac{1}{2} D^2 + ig^{\bar{m}n} D' F_{\bar{m}n} + i \bar{\lambda} D^m \chi_m] + \quad (2.15)
\]

\[
+ \sqrt{g} (\bar{\phi} D^{\bar{m}} D_{\bar{m}} \phi + \bar{\psi}^{\bar{m}n} D_{\bar{m}} \psi_n + \bar{\psi} D^{\bar{m}} \psi_{\bar{m}} + N_{\bar{m}n} \bar{N}^{\bar{m}n} - i \bar{\phi} \chi_n \psi^n - i \bar{\psi} \bar{\lambda} \phi + \frac{1}{4} \bar{\psi} \bar{\lambda}^{mn} \phi + i \bar{\phi} D' \phi) =
\]

\[
= \sqrt{g} \{ Q, \frac{1}{\epsilon^2} \text{Tr}[-i \bar{\lambda} g^{\bar{m}n} F_{\bar{m}n} + \frac{1}{2} D' \bar{\lambda} - \frac{i}{2} \bar{\lambda} F^{\bar{m}n}]\} + \quad (2.19)
\]

\[+ \sqrt{g} \{ Q, -\frac{1}{2} \bar{\psi}^{\bar{m}n} N_{\bar{m}n} + \bar{\phi} D^{\bar{m}} \psi_{\bar{m}} - i \bar{\phi} \bar{\lambda} \phi \}.
\]

Here \( Q \) is the scalar nilpotent generator of BRST transformations which for the gauge multiplet read

\[
\delta A_n = \chi_n, \quad \delta A_{\bar{n}} = 0, \quad \delta \chi_n = 0, \quad \delta \bar{\lambda} = -D', \quad \delta \bar{\lambda}_{\bar{m}n} = 2i F_{\bar{m}n}, \quad \delta D' = 0,
\]

while for the chiral multiplet we have

\[
\delta \phi = 0, \quad \delta \psi_{\bar{m}} = D_{\bar{m}} \phi, \quad \delta N_{\bar{m}n} = D_{\bar{m}} \psi_n - D_n \psi_{\bar{m}} + \frac{1}{2} \bar{\lambda} \lambda_{\bar{m}n} \phi, \quad (2.17)
\]

\[
\delta \bar{\phi} = \bar{\psi}, \quad \delta \bar{\psi} = 0, \quad \delta \bar{\psi}^{\bar{m}n} = -2 \bar{N}^{\bar{m}n}, \quad \delta \bar{N}^{\bar{m}n} = 0.
\]

It is easy to see that the generator of these transformations does not depend on the external metric.

One can see that all the fields are combined into different supermultiplets. Indeed in the gauge sector we have the following supermultiplets: \((A_n, \chi_n), (\bar{\lambda}, D')\) and \((\bar{\lambda}_{\bar{m}n}, F_{\bar{m}n})\). In the matter sector the multiplets are \((\phi, \psi_{\bar{m}}, N_{\bar{m}n}), (\bar{\phi}, \bar{\psi}), (\psi_{mn}, \bar{N}_{mn})\).

By fixing the ghost number of the BRST charge to be 1 we have the following dimensions \((d)\) and ghost numbers \((G)\) for the fields:

\[
(d, G)(A_n) = (d, G)(A_{\bar{n}}) = (1, 0), \quad (d, G)(\chi_n) = (1, 1) \quad (2.18)
\]

\[
(d, G)(\bar{\lambda}) = (d, G)(\bar{\lambda}_{\bar{m}n}) = (2, -1), \quad (d, G)(D') = (2, 0).
\]

and

\[
(d, G)(\phi) = (0, 2), \quad (d, G)(\bar{\phi}) = (2, -2), \quad (d, G)(\psi_{\bar{m}}) = (1, 1), \quad (2.19)
\]

\[
(d, G)(\bar{\psi}) = (d, G)(\bar{\psi}_{mn}) = (2, -1), \quad (d, G)(N_{\bar{m}n}) = (2, 0), \quad (d, G)(\bar{N}_{mn}) = (2, 0).
\]

Notice also that for the case of hyperkähler manifold \( M \) the holonomy group is reduced to the \( SU(2) \) one and twisted and untwisted theories coincide \[31, 32\].
### 2.3 Twisted self-interacting chiral multiplet

Let us now consider the theory without gauge interactions. The Lagrangian for a twisted free chiral multiplet reads as follows

\[
L_0 = \sqrt{g}(\bar{\phi}D^m \phi + \bar{\psi}^m \gamma_m \psi + N_{\bar{m}\bar{n}} \bar{N}^{{m\bar{n}}}) = (2.20)
\]

\[
= \sqrt{g}\{Q, -\frac{1}{2}\bar{\psi}^{m\bar{n}} N_{m\bar{n}} + \bar{\phi} D^m \psi_m\}. \tag{2.21}
\]

Here \(Q\) is the scalar nilpotent generator of BRST transformations

\[
\delta \phi = 0, \quad \delta \bar{\psi}_m = \partial_m \phi, \quad \delta N_{\bar{m}\bar{n}} = D_{\bar{m}} \psi_{\bar{n}} - D_{\bar{n}} \psi_{\bar{m}},
\]

\[
\delta \bar{\phi} = \bar{\psi}, \quad \delta \bar{\psi} = 0, \quad \delta \bar{\psi}_{mn} = -2 N_{mn}, \quad \delta \bar{N}_{mn} = 0. \tag{2.22}
\]

If there is a nontrivial holomorphic \((2,0)\) form \(E_{mn}\) on \(M\) (i.e. \(H^{2,0}(M) \neq 0\)) then it is possible to introduce a superpotential \(W(x)\) which induces masses and self-interactions for the quantum fields. The corresponding interacting Lagrangian reads

\[
L_{\text{int}} = \sqrt{g}[E^{\bar{m}\bar{n}}(\bar{\psi}_m \psi_n W''(\phi) + N_{\bar{m}\bar{n}} W'(\phi)) - S_{mn}(\bar{\psi}_m \psi_n W''(\phi) + 2 \bar{N}_{mn} W'(\phi))]
\]

\[
= \sqrt{g}[E^{\bar{m}\bar{n}}(\bar{\psi}_m \psi_n W''(\phi) + N_{mn} W'(\phi)) + \{Q, S_{mn} \bar{\psi}_m \psi_n W'(\phi)\}], \tag{2.23}
\]

where \(S_{mn}\) is an arbitrary non-singular \((0,2)\) form on \(M\). It is easy to see that

\[
\{Q, \int_M L_{\text{int}}\} = 0,
\]

and hence the total action of the theory with a superpotential is \(Q\)-closed (but not \(Q\)-exact due to the interaction terms).

Notice that a superpotential can also be introduced into the heterotic topological gauge theory. In this case the action is \(Q\)-closed but not \(Q\)-exact. Moreover as we shall see below the heterotic topological gauge theory has a moduli space corresponding to deformations of the Lagrangian by a superpotential.

The condition of renormalizability of four dimensional theory implies that \(W(x)\) should be a polynomial of a degree not higher than 3. In the case of the superpotential of a degree 2 a Majorana mass is induced for the chiral supermultiplet, while for the cubic superpotential the \(|\phi|^4 + \text{Yukawa interactions} \) are induced. As it is well known \(\|\) the superpotential in SUSY theories is non-renormalizable \(\|\). The only renormalization of the action comes from \(D\)-terms \(\|\). This fact was very important in the analysis of 2D \(\mathcal{N}=2\) SUSY theories \(\|\), where the \(\mathcal{N} = 2\) superconformal theories were associated to quasihomogeneous superpotentials. As we demonstrate below the non-renormalizability of a superpotential is also important for a conformal structure that appears in the heterotic topological theory. For simplicity we shall also consider the case of \(W(x) = \lambda x^{n+1}/(n + 1)\), where \(\lambda\) is a coupling constant, and formally we shall consider all positive integer values of \(n\).

\(\|\)Notice however that some subtleties can appear due to infrared effects \(\|\).
2.4 Physical operators

We can now define the physical operators as classes of cohomology of the BRST operator $Q$. The local observable for the sector of the gauge multiplet becomes a $(2,0)$ form (of dimension 2)

$$O_{mn}^{(0)} = \text{Tr} \chi_m \chi_n. \quad (2.24)$$

It is to be noticed that the situation here is different from the ordinary topological theories where the local observables are zero-forms; non-zero forms should usually be integrated over closed cycles to get non-local observables (in the case of highest forms one gets moduli of the topological theory). The difference here is due to the splitting of four coordinates into holomorphic and anti-holomorphic ones, so that the $(2,0)$ form is effectively a scalar with respect to anti-holomorphic derivatives. We have

$$\partial \bar{k} \text{Tr} \chi_m \chi_n = \{Q, \ldots\}. \quad (2.25)$$

It follows from this equation that the physical correlators under an insertion of this operator are holomorphic with respect to its coordinate. This is of course quite similar to the left-moving nature of the cohomology in 2D half-twisted theories [7].

One can also construct the non-local observables using the descent procedure [28]. We have

$$\partial \bar{k} O_{mn}^{(0)} = \{Q, H_{mn,k}^{(1)}\},$$

$$\partial [\bar{p} H_{mn,k}] = \{Q, H_{mn,\bar{k}\bar{p}}^{(2)}\}$$

where

$$H_{mn,k}^{(1)} = \text{Tr} (F_{km} \chi_n - F_{kn} \chi_m), \quad (2.26)$$

$$H_{mn,\bar{k}\bar{p}}^{(2)} = \frac{1}{2} \text{Tr} (F_{km} F_{\bar{p}n} - F_{kn} F_{\bar{p}m} + F_{\bar{p}k} F_{mn}) + i \frac{1}{4} \text{Tr} (\partial m \bar{\lambda}_{\bar{p}k} \chi_n - \partial n \bar{\lambda}_{\bar{p}k} \chi_m). \quad (2.27)$$

The operator $H_{mn,\bar{k}\bar{p}}^{(2)}$ is obviously the density of the topological charge of the gauge field up to an exact form. These relations allow us to construct the following $Q$-closed non-local observables

$$O^{(1)} = \int \bar{\omega} \wedge H^{(1)}, \quad O^{(2)} = \int H^{(2)}, \quad (2.28)$$

where $\omega$ is a closed $(0,1)$ form. Here we used that the forms $H^{(1)}$ and $H^{(2)}$ are $Q$-closed up to exact differential forms according to eqs. (2.26).

The local operators in the matter sector are given by the same gauge invariant functions of the (dimensionless) scalar field $\phi$ as in the supersymmetric version of the theory (because this scalar field has zero axial charge and therefore does not change its spin under the twisting) which correspond to flat directions of the classical moduli space of vacua [13]. If the cohomology space $H^{2,0}(M) \neq 0$ (i.e. there are holomorphic $(2,0)$ forms on $M$) then it is also possible to construct non-local operators. Let the matter supermultiplets transform as a reducible representation $R$ of the gauge group, where each irreducible representation $R_i$ enters with a repetition
n_i. In particular let us consider two such irreducible representations \( R_i \) and \( R_j \) for which the tensor product \( R_i \otimes R_j \) contains a singlet. One can construct the following bilinear BRST-invariant operator

\[
O_{\text{matter}} = \sum_{IJ} a_{IJ} \int_M E \wedge <\psi^I \wedge \psi^J + \frac{1}{2} N^I \phi^J + \frac{1}{2} \phi^I N^J >. \tag{2.29}
\]

Here \(<...>\) stands for a gauge invariant pairing of the fields (its definition depends on a representation of the gauge group for the fields of matter), the indices \( I \) and \( J \) stand for copies of the irreducible representations \( R_i \) and \( R_j \) of the matter fields, and \( E \) is a holomorphic \((2,0)\) form on \( M \). \( a_{IJ} \) is a constant matrix (a choice of it depends on a representation of the fields of matter). For example for the case when the multiplet of matter is in an adjoint representation of the gauge group (this example corresponds to the twisted \( N = 2 \) supersymmetric Yang-Mills theory) the matrix \( a_{IJ} \) is just 1. For the case of the \( SU(2) \) gauge group where the fields of matter fall in four copies of a spinor representation, \( a_{IJ} \) is an antisymmetric \( 4 \times 4 \) matrix. The operator Eq. (2.29) is actually a twisted version of an \( F \) term for a bilinear combination of the chiral superfields \([12]\). This operator corresponds to mass deformations of the theory (see Eq. (2.22)).

Similarly one can write down a BRST closed operator which is an integral of a \((2,2)\) form and contains both the fields of the gauge and matter sectors. We have

\[
O_{\text{mix}} = \sum_{IJ} a_{IJ} \int_M [H^{(1)} \wedge <\phi^I \psi^J + \psi^I \phi^J > + H^{(2)} <\phi^I \phi^J > + \]

\[+ O^{(0)} \wedge <\psi^I \wedge \psi^J + \frac{1}{2} \phi^I N^J + \frac{1}{2} N^I \phi^J >]. \tag{2.30}
\]

This operator exists even for the manifolds with \( H^{2,0}(M) = 0 \). One can of course construct more complicated physical operators.

It is worth noticing that we could use the vector field \(-V_\mu\) for a twisting of the theory on a Kähler manifold. Such a modification of the model corresponds to a change \( \epsilon, \eta \to \eta, \epsilon \). The local operators and their correlators in this mirror model are antiholomorphic up to BRST exact operators (for example, \( \partial_\mu \phi = \{ Q, \ldots \} \)).

Notice also that for the case of a hyperkähler manifold there is also another BRST charge since in this case the holonomy group is \( SU(2) \). This BRST charge corresponds to a different right-handed supergenerator with a parameter which is proportional to the spinor \( \bar{\eta} \) defined in section 2.1. The holomorphic derivatives of the local physical operators of the matter sector turn out to be exact with respect to this second BRST charge. Therefore the correlators of the local physical operators do not depend on all coordinates.

### 2.5 Anomalies

In this section we consider the problem of the quantum anomaly for the BRST symmetry in the heterotic topological gauge theories.
Let us consider such a theory without a superpotential for the matter fields. The Lagrangian of its twisted version is BRST exact at the classical level. Due to the $Q$-exactness of the Lagrangian the metric is expected to decouple from the physical correlators. However as it is shown in ref. [31] on a curved Kähler manifold an anomaly at the quantum level can prevent the decoupling of the metric. This anomaly appears because in the case of an arbitrary representation of the matter multiplet the fermionic current which is coupled to the twisting vector field $V_{\mu}$ is anomalous.

This anomaly can be easily computed at the one-loop level [31]. Let us consider a variation of the effective action for external gravitational and gauge fields (we define the effective action by $S_{\text{eff}} = -\log Z$, where $Z$ is the partition function). This one-loop effective action is given by a ratio of the different Laplace operators on $M$ in the external gravitational and gauge fields [31]. The infrared contributions to the effective action appear due to the zero modes of the Laplace operators and are not related to the BRST anomaly that we consider (this anomaly is related to the necessary ultraviolet regularization of the theory). The ultraviolet contribution to the variation of the effective action can be represented as follows

$$\delta S_{\text{gr}}(g) + V_{\text{mix}}[A_m, A_{\bar{m}}, g_{m\bar{m}}]. \quad (2.31)$$

Here $\delta S_{\text{gr}}(g)$ does not depend on the gauge field. The term $V_{\text{mix}}[A_m, A_{\bar{m}}, g_{m\bar{m}}]$ is a local functional of the external gauge field because it is determined by ultraviolet contributions.

Let us consider the purely gravitational part of the variation in Eq. (2.31). It is obvious that $S_{\text{gr}}$ is proportional to the dimension of the representation of the fields over which we integrate in the path integral. The contribution from the gauge sector is proportional to the dimension of the gauge group $\dim G$, while that from the matter sector is proportional to $-\dim R$ [31] (dim $R$ stands for the dimension of the representation of the matter multiplet). The difference in sign comes from the different statistics of vector fields in the gauge and in the matter multiplets. If we integrate the gravitational part of the variation of the effective action and normalize the path integral that divides it by the same path integral without any external gauge field then we get the following purely gravitational factor for the partition function $Z$

$$\left( \frac{T_2(\zeta, M)}{T_2(0, M)} \right)^{\dim R - \dim G}, \quad (2.32)$$

where $T_2(\zeta, M)$ is the Ray-Singer torsion [46] ($\zeta$ stands for a flat connection on $M$ associated with a particular representation of the fundamental group on $M$) and $T_2(0, M)$ is an ultraviolet contribution to the Ray-Singer torsion. It can be shown that this factor depends only on the Kähler class of the metric [46].

It is worth emphasizing that the gravitational anomaly is $Q$-closed since the external metric is BRST invariant.

Let us now consider the second part of the variation in Eq. (2.31). Since it is a local functional of the gauge field one can determine it by a calculation in the case
of a gauge field for which the Laplace operators have no zero modes, in particular for small values of the gauge field. It is useful at this point to compare the present situation with the supersymmetric version of the theory (with the twisted spin connection). Actually there are two sorts of anomalies. The first one is a conformal anomaly which is proportional to \( \int \text{Tr}^* F \wedge F \log g \) while the second anomaly appears due to the coupling of the anomalous axial current to the external vector field \( V_\mu \).

The axial anomaly for the gaugino current gives the following term in the effective action \( \log Z \)

\[
\frac{1}{16\pi^2} \int_M \text{Tr}_A dF \wedge F \frac{1}{\Delta} D_\mu V_\mu = \frac{1}{32\pi^2} \int_M \text{Tr}_A dF \wedge F \log e/\bar{e}. \tag{2.33}
\]

where we used the fact that locally the vector field \( V_\mu \) is a total derivative \[31\]. Here \( \text{Tr}_A \) stands for the trace taken in an adjoint representation. The contribution from the matter sector to the axial anomaly is given by the same expression (2.33) with an opposite sign and with the trace \( \text{Tr}_R \) taken in the representation \( R \) of the gauge group.

For the twisted theory formulated in terms of sections of holomorphic and anti-holomorphic vector bundles over \( M \) the anomaly can be determined by a direct calculation \[31\]. We get the following expression for \( V_{mix} \)

\[
- \frac{1}{16\pi^2} \int_M \text{Tr}_A [g^{nm} \delta g_{km} (-F^{kl} F_{ln} + F^p_{\mu} F^k_{mn}) - g^{nm} \delta g_{mn} (-F^{kl} F_{lk} + (F^p_{\mu})^2)] + \frac{1}{32\pi^2} \int_M \text{Tr}_R [g^{nm} \delta g_{km} (-F^{kl} F_{ln} + F^p_\mu F^k_n) - g^{mn} \delta g_{mn} (-F^{kl} F_{lk} + (F^p_{\mu})^2)] \tag{2.34}
\]

Let the variation of the metric be purely Kählerian, i.e. \( \delta g_{\bar{m}m} = \partial_n \omega_{\bar{m}n} + \partial_{\bar{m}} \omega_m \) where \( \omega_m \) and \( \omega_{\bar{n}} \) are (1,0) and (0,1) forms so that the Kähler forms \( J \) and \( J + d\omega \) belong to the same cohomology class in \( H^2(M, R) \). Then this variation can be integrated and we get the following expression for an anomalous contribution to the effective action \( \log Z \)

\[
- \frac{1}{32\pi^2} \int_M \text{Tr}_A dF \wedge F \log g + \frac{1}{32\pi^2} \int_M \text{Tr}_R F \wedge F \log g. \tag{2.35}
\]

The anomaly in this form is similar to that in Eq. (2.33) but it takes into account the change of the path integral measure when we translate the supersymmetric theory into the twisted one.

Actually it is easy to check that if we take into account the contributions to the effective action which contain the gaugino fields then the total anomaly in the effective action is proportional to

\[
\int_M H^{(2)} \log g, \tag{2.36}
\]

where the operator \( H^{(2)} \) is defined as in the previous section. This anomaly is obviously not BRST invariant and breaks the BRST invariance of the theory at the quantum level.
For an external anti-instanton field this term in the effective action obviously mixes the dependence of the metric and of the moduli of instanton. Hence in order to get a topological theory we have to cancel this anomaly by contributions of matter in an appropriate representation of the gauge group.

Finally for the total contribution of the non-zero modes to the partition function we get

$$Z = \frac{\hat{T}_2(M, \zeta, 0, R)}{\hat{T}_2(M, \zeta, A, R)} \frac{\hat{T}_2(M, \zeta, A, Ad)}{\hat{T}_2(M, \zeta, 0, Ad)},$$

(2.37)

where $\hat{T}_2(M, \zeta, A, R)$ stands for a generalized Ray-Singer torsion in an external non-abelian gauge field $A$ and in the presence of a flat connection $\zeta$ associated with a particular representation of the fundamental group of $M$.

From Eq. (2.35) we can see that the mixed anomaly $V_{mix}[A_m, A_{\bar{m}}, g_{m\bar{m}}]$ is cancelled if

$$C_2(G) - T(R) = C_2(G) - \sum_i T(R_i) = 0.$$

(2.38)

Here $R = \sum_i R_i$, $R_i$ are irreducible representations of the gauge group; $C_2(G)$ stands for the Casimir operator ($\text{Tr}_A t^a t^b = C_2(G) \delta^{ab}$) and $T(R_i)$ is the Dynkin index of an irreducible representation $R_i$ of the group $G$ ($\text{Tr}_R t^a t^b = T(R) \delta^{ab}$, $t^a$ and $t^b$ stand for the generators of the gauge group). The condition for the cancellation of the gravitational anomaly (the Ray-Singer torsion) reads

$$\dim G - \sum_i \dim R_i = 0,$$

(2.39)

Where $\dim G$ and $\dim R_i$ stand for the dimensions of the adjoint and $R_i$ representations of the gauge group respectively. A condition (2.38) has been analyzed (for a different problem) in ref.[47]. From their analysis one can easily extract that if the condition (2.38) is fulfilled then

$$\sum_i \dim R_i - \dim G \geq 0.$$

(2.40)

The equality in the above equation is reached only for matter in the adjoint representation of the gauge group, i.e. for the twisted $N = 2$ Yang-Mills theory. In the Appendix we list the representations of all classical groups $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ which obey Eq. (2.38) and give the corresponding values of $\sum_i \dim R_i - \dim G$.

From now on we shall consider only theories where the mixed anomaly is cancelled. Notice that the condition for a cancellation of the mixed anomaly does not imply that the variation of the external metric is purely Kählerian, i.e. does not change the Kähler class of the metric. It is easy to see that the mixed anomaly is cancelled provided that the condition (2.38) is satisfied for an arbitrary variation of the Kähler external metric.

5The mixed anomaly is clearly absent in the case of a twisted model of a chiral supermultiplet without gauge interactions.
As to the purely gravitational part of the (normalized) partition function it can depend only on the Kähler class of the metric. Thus the whole physical correlator remains unmodified under smooth variations of the external Kähler metric provided that they do not change its Kähler class. Therefore we do not impose the condition \((2.39)\). Moreover as we shall see below the gravitational part of the non-normalized partition function has an interesting interpretation as a conformal anomaly of an embedded conformal theory \(6\).

Our analysis was restricted above to the one-loop level. It is important to understand if the above arguments can be extended to the multiloop level. The absence of a mixed anomaly at the one-loop level is sufficient for the vanishing of multiloop contributions to the mixed anomaly. This is because the anomaly originates essentially from the axial anomaly of the fermionic current coupled to the external vector field \(V_\mu\). The multiloop corrections to the axial anomaly at the level of matrix elements can appear only due to a rescattering of gluons \(13\) provided the one-loop anomaly is not cancelled. Thus the mixed anomaly is absent provided that the condition \((2.38)\) is satisfied.

If the mixed anomaly is cancelled the BRST invariance is not broken by the quantum corrections since the external metric and, hence, the effective action is \(Q\)-invariant (the multiloop corrections to the gravitational anomaly are discussed below). It is also easy to see that the variation of this Lagrangian both in the gauge coupling constant is \(Q\)-exact. Therefore the physical correlators do not depend on the gauge coupling constant and the theory formally allows for a localization near the solutions of the classical equations of motion similar to usual topological theories \(28\). In such theory one can try to calculate the physical correlators in the limit of weak coupling \(e^2 \rightarrow 0\). However there is a subtlety in such an approach. The point is that the theory under consideration is, in general, strongly interacting and the coupling constant \(e^2\) may become of order 1 due to infrared contributions since it depends on a renormalization scale. In order to keep the coupling constant small one has to “freeze” it by a Higgs mechanism or by suppressing the infrared effects via the introduction of a mass gap into the theory. Indeed Witten \(32\) recently calculated Donaldson invariants on four-dimensional Kähler manifolds in the infrared limit by introducing such a mass gap into the topological Yang-Mills theory. The option to introduce a mass gap is technically very important but not, strictly speaking, necessary for the physical correlators to be topologically invariant. In fact this has been demonstrated by formulas \(32\) for the Donaldson invariants which allow for their expansion into series in powers of the mass parameter.

Actually the infrared problem which appears in the topological theories is a counterpart of those in the supersymmetric theories on flat space-time \(19, 13\). The so-called non-renormalizability theorems are based essentially on the existence of an infrared cut-off in the theory. Such an infrared cut-off can be provided by a mass gap (as in the case of the non-renormalizability theorem for a superpotential) or by

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\(^6\)There exists one more restriction to the matter sector: the theory should not have (both local and global) gauge anomalies.
an appropriate external field (as in the case of the non-renormalizability theorem for an effective action in an external instanton field) [50, 51].

In next subsection we consider the problem of renormalization of the gravitational anomaly.

2.6 Induced gravity and renormalizations

The variation of the Lagrangian in the external metric is not strictly speaking $Q$-exact at the quantum level since the effective action is $Q$-closed but not $Q$-exact. However such a dependence of the effective action (and of the physical correlators) on the external metric is factorized out at the one-loop level. Therefore it does not spoil the BRST invariance of the theory provided the multiloop corrections to the gravitational anomaly vanish.

Actually in four dimensional quantum field theories coupled to the external gravitational the multiloop contributions to the induced gravitational effective action are not usually vanishing [52]. We shall argue however that in the heterotic topological theories the gravitational anomaly does not acquire any multiloop corrections. The absence of multiloop corrections maintains the BRST invariance of the theory at the quantum level because otherwise the ultraviolet logarithms that appear due to interactions of quantum fields could induce mixed gravity-gauge field terms in the effective action (such terms would spoil the BRST invariance of the theory). In what follows we give the arguments which are valid for any heterotic topological theory.

From the point of view of supersymmetry the gravitational anomaly originates in the anomaly of the axial fermionic current which is coupled to the external vector field $V_\mu$ (see section 2). This axial current has an anomaly that depends only on the classical external gravitational fields. The anomalous dimension of the current (which could be responsible for multiloop corrections to the axial anomaly at the level of matrix elements) is due to a rescattering of the fields which enter the anomaly. In our case the gravitational field is the external one and does not induce any diagrams with virtual gravitons. Therefore the current does not have any anomalous dimension and hence the anomaly is not renormalizable.

A different argument is based on the background superfield formalism of supergravity [53]. As it has been demonstrated in ref. [53] a superfield Feynmann diagram for multiloop corrections to the effective action can be represented as an integral over four Grassmann variables with an integrand which is a local expression with respect to the Grassmann coordinates. When we formulate our theory in terms of the usual fields (with unmodified spins) the quantum supermultiplets are coupled to a reduced external gravitational superfield that is invariant under one of four supercharges. This means that the integrands in the superfield diagrams in such a background do not depend on one of the Grassmann variables. Therefore such multiloop corrections to the effective action in this special supergravity background vanish due to an integration over Grassmann coordinates. It is straightforward to reformulate this conclusion in terms of twisted fields. We thus see that there is no
multiloop corrections to the gravitational anomaly \(2.49, 2.51\). This argument is an extension of the usual theorems of non-renormalizability for a superpotential and of the effective action in the instanton background \([50, 51]\) to the case of a special supergravitational background.

We shall demonstrate by an explicit calculation the vanishing of the 2-loop correction to the effective gravitational action in the model of a single chiral supermultiplet with a superpotential \(W(x)\) (we assume that \(H^{2,0}(M) \neq 0\)). For simplicity we take \(W(x) = x^3/12\). In this model the interaction is described by the following terms in the Lagrangian (see Eq. \(2.22\))

\[
L_{\text{int}} = \sqrt{g}\left[ E_{12}^4(\psi_1\psi_2\phi + \frac{1}{2}N_{12}\phi^2) - S_{12}^4(\bar{\psi}_1\bar{\psi}_2\bar{\phi} + \bar{N}_{12}\bar{\phi}^2) \right],
\]

where \(E_{12}\) is a holomorphic \((2,0)\) form and \(S_{12}\) stands for a non-singular \((0,2)\) form. Actually the vanishing of multiloop corrections to the gravitational effective action formally follows from the \(Q\)-exactness of the terms in the Lagrangian \(2.41\) which are proportional to \(S_{12}\). For the correction to the effective action we have

\[
\int_M d^4x \sqrt{g(x)} \int_M d^4y \sqrt{g(y)} S_{12}(y) E_{12}(x)
\]

\[
\left\{ - < N_{12}(x)N_{12}(y) > < \phi(x)\bar{\phi}(y) > < \bar{\phi}(y)\phi(x) > +
\right.
\]

\[
\left. + < \phi(x)\bar{\phi}(y) > [ < \psi_1(x)\bar{\psi}_1(y) > \cdot < \psi_2(x)\bar{\psi}_2(y) > -
\right.
\]

\[
\left. - < \psi_2(x)\bar{\psi}_2(y) > \cdot < \psi_1(x)\bar{\psi}_1(y) > \right\}.
\]

By substituting in the free propagators given in Appendix (Eq. \(A.8\)) we have

\[
\int_M d^4x \sqrt{g(x)} \int_M d^4y g^{-1}(y) S_{12}(y) E_{12}(x) \left\{ - \frac{1}{2} \delta^4(x-y) \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right)^2 - \right.
\]

\[
\left. - \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \left[ \left( D_2^2 \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \cdot \left( \frac{1}{\Delta_{01}} D^4 \delta^4(x-y) \right) +
\right. \right.
\]

\[
\left. + \left( D_1^4 \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \cdot \left( \frac{1}{\Delta_{01}} D^4 \delta^4(x-y) \right) \right\}.
\]

By using

\[
\left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right) D_n \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right) = \frac{1}{2} D_n \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right)^2
\]

and taking into account a holomorphicity of the form \(E_{12}\) one can integrate by parts in the second line of Eq. \(2.43\). Thus we get

\[
\int_M d^4x \sqrt{g(x)} \int_M d^4y g^{-1}(y) S_{12}(y) E_{12}(x) \left\{ - \frac{1}{2} \delta^4(x-y) \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right)^2 + \right.
\]

\[
\left. - \left( \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \left[ \left( D_2^2 \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \cdot \left( \frac{1}{\Delta_{01}} D^4 \delta^4(x-y) \right) +
\right. \right.
\]

\[
\left. + \left( D_1^4 \frac{1}{\Delta_{00}} \delta^4(x-y) \right) \cdot \left( \frac{1}{\Delta_{01}} D^4 \delta^4(x-y) \right) \right\}.
\]
\[ + \frac{1}{2} \left( \frac{1}{\Delta_{0,0}} \right)^2 \left( D_2 \left( \frac{1}{\Delta_{0,1}} \right)_{11} D^2 \delta^4(x - y) + D_1 \left( \frac{1}{\Delta_{0,1}} \right)_{22} D^1 \delta^4(x - y) \right) = 0. \]

The latter equality follows from the following identity
\[ D_2 \left( \frac{1}{\Delta_{0,1}} \right)_{11} D^2 \delta^4(x - y) + D_1 \left( \frac{1}{\Delta_{0,1}} \right)_{22} D^1 \delta^4(x - y) = \delta^4(x - y), \] (2.45)

which is a manifestation of the supersymmetric Ward identity (for the free fields)
\[ 2 < N_{12}(x) \bar{N}_{12}(y) > - (D_1 \bar{\psi}_2 - D_2 \psi_1)(x) \bar{\psi}_{12}(y) >= \{ Q, N_{12}(x) \bar{\psi}_{12}(y) \} >= 0. \] (2.46)

This calculation is of course formal since we did not introduce any ultraviolet regularization. But it demonstrates the origin of the cancellation of multiloop corrections.

Notice that one could expect the BRST cohomology not to be well defined in the presence of the gravitational anomaly since it is BRST closed but not exact. However because the gravitational anomaly is not renormalizable one can cancel it, for example, by adding to the model \((\text{dim } R - \text{dim } G)\) free \(U(1)\) gauge supermultiplets. This means that the BRST cohomology is defined and not spoiled by this gravitational anomaly.

The gravitational anomaly for heterotic topological theories has a natural interpretation as a conformal anomaly of the Virasoro algebra which appears in the \(Q\) cohomology (see below). Indeed the form of the corresponding term in the effective action can be extracted from the 4D conformal anomaly in the trace of the energy-momentum tensor (see, e.g. [54]) which in our case reads
\[ < \theta^\mu_\mu > = (\text{dim } R - \text{dim } G) \frac{1}{3 \cdot 128 \pi^2} * R^*_{\mu \nu \lambda \rho} R^\mu \nu \lambda \rho, \] (2.47)

where \(R_{\mu \nu \lambda \rho}\) is the Riemann tensor. The variation of the effective action \(S_{\text{eff}}\) with respect to the metric \(g_{\mu \nu}\) reads
\[ \delta S_{\text{eff}} = (\text{dim } R - \text{dim } G) \frac{1}{6 \cdot 128 \pi^2} \int_M d^4x \sqrt{g} * R^*_{\mu \nu \lambda \rho} R^\mu \nu \lambda \rho \delta \log g, \] (2.48)

and hence for the induced gravitational action we get
\[ (\text{dim } G - \text{dim } R) \frac{1}{3 \cdot 128 \pi^2} \int_M d^4x \sqrt{g} * R^*_{\mu \nu \lambda \rho} R^\mu \nu \lambda \rho \frac{1}{\Delta} R, \] (2.49)

where \(\Delta\) is the Laplace operator, and \(R = g^{m \bar{m}} \partial_m \partial_{\bar{m}} \log \det g_{m \bar{m}}\) (see, e.g. [41]).

For the case \(M = \Sigma_1 \times \Sigma_2\) with a block diagonal metric
\[ g_{\mu \nu} = \begin{pmatrix} g^{(1)} & 0 \\ 0 & g^{(2)} \end{pmatrix}, \] (2.50)

we get for the effective action for the external gravitational field
\[ S_{\text{eff}} = (\text{dim } R - \text{dim } G) \left( \frac{\chi_1}{48 \pi^2} \int_{\Sigma_2} R^{(2)} \frac{1}{\Delta^{(2)}} R^{(2)} + \frac{\chi_2}{48 \pi^2} \int_{\Sigma_1} R^{(1)} \frac{1}{\Delta^{(1)}} R^{(1)} \right), \] (2.51)
where $\chi_{1,2}$ are the Euler characteristics of $\Sigma_{1,2}$, $R^{(1)}$, $R^{(2)}$ are the Riemann tensors of $\Sigma_{1,2}$, and $\Delta^{(1)}$, $\Delta^{(2)}$ are the corresponding Laplace operators. We observe that this is a sum of two induced Liouville actions for the conformal theories on $\Sigma_2$ and $\Sigma_1$ with the central charge $\chi_1$ and $\chi_2$ respectively. If the gauge interactions are absent the factor $(\dim R - \dim G)$ changes into $N_f$ (the number of chiral supermultiplets).

Thus the anomaly in the decoupling of the external metric in the heterotic topological theory can be interpreted as a conformal anomaly in the 2D conformal theories embedded into the manifold $M = \Sigma_1 \times \Sigma_2$. In the following sections we will demonstrate that such conformal theories indeed appear in the cohomology of the BRST operator.

### 3 Realization of the $W_{1+\infty}$ algebra in terms of free chiral supermultiplet

#### 3.1 Ghost number $G = 0$ cohomology

In this section we consider a single chiral supermultiplet and focus to the case of four dimensional manifold $M = \Sigma_1 \times \Sigma_2$, where $\Sigma_{1,2}$ are 2D Riemann surfaces. For simplicity we take $\Sigma_1 = T^2$ to be a torus with a flat two-dimensional metric. The metric of $M = \Sigma_1 \times \Sigma_2$ can be chosen block diagonal $g = (g^{(1)}, g^{(2)})$, where $g^{(1,2)}$ are two dimensional metrics for $\Sigma_{1,2}$ respectively.

We introduce below some bosonic $Q$-exact operators with the ghost number $G = 0$, which are relevant for our construction of the $W_{1+\infty}$ algebra.

The equations of motion for a free chiral supermultiplet read (see Eq. (2.20))

$$
D_{\bar{m}}\psi_{\bar{n}} - D_{\bar{n}}\psi_{\bar{m}} = 0, \quad D^m\psi_m = 0, \quad D_{\bar{m}}\bar{\psi}^{\bar{m}\bar{n}} + D_{\bar{n}}\bar{\psi} = 0, \quad (3.1)
$$

$$
D^mD_{\bar{m}}\phi = 0, \quad D^\bar{m}D_m\bar{\phi} = 0.
$$

Let us define the following operators

$$
W_{n+1}(z, \bar{z}) = 2\pi \int_{\Sigma_2} \sqrt{|g^{(2)}|} d^2u(-\partial_1\bar{\phi}\partial^n\phi + g^{2\bar{2}}\bar{\psi}_{12}\partial^n\psi_2), \quad (3.2)
$$

where $z$ and $u$ are the complex coordinates on $\Sigma_1$ and $\Sigma_2$ respectively; $\partial_1 = \partial/\partial z$, $\partial_1 = \partial/\partial z$, $\partial_2 = \partial/\partial u$ and $\partial_2 = \partial/\partial u$. By using the definition of $Q$ in Eq. (2.21) it is easy to see that these operators are $Q$-closed on mass-shell

$$
\{Q, W_{n+1}\} = 2\pi \int_{\Sigma_2} \sqrt{|g^{(2)}|} d^2u(\partial_1\bar{\psi}\partial^n\phi - g^{2\bar{2}}\bar{\psi}_{12}\partial^n\bar{\psi}_2) = \quad (3.3)
$$

$$
= 2\pi \int_{\Sigma_2} \sqrt{|g^{(2)}|} d^4u(-\partial^n\phi)(\partial_1\bar{\psi} + g^{2\bar{2}}\partial_2\bar{\psi}_{21})) = 0.
$$
Moreover these operators are holomorphic on $\Sigma_1$ in the $Q$-cohomology

$$
\partial_1 W_{n+1} = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (-\partial_1 \partial_1 \phi \partial^n_1 \phi - \partial_1 \bar{\phi} \partial^n_1 \bar{\partial}_1 \bar{\phi} + g^{22} (\partial_1 \bar{\psi}_{12}) \partial^n_1 \psi_2 + g^{22} \bar{\psi}_{12} \partial^n_1 \bar{\partial}_1 \bar{\psi}_{12}) = 2\pi \{ Q, \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (-\partial_1 \bar{\phi} \partial^n_1 \bar{\psi}_1 - g^{22} \partial_2 \bar{\phi} \partial^n_1 \bar{\psi}_2) \}.
$$

Thus we see that $\partial_1 W_{n+1}$ is $Q$-exact and hence trivial in cohomology.

It is worth emphasizing that the operators $T = W_2$ and $J = W_1$ are 11- and 1-component of 2D tensors on $\Sigma_1$. These are actually the components of the four dimensional energy momentum tensor and $U(1)$ current integrated over $\Sigma_2$ respectively. This fact is responsible for the holomorphicity of $T$ and $J$. Indeed the four dimensional current $J_\mu$ corresponds to an unbroken $U(1)$ $R$-symmetry of the action and obeys

$$
\partial_n J^n + \partial^n J_n = 0, \tag{3.5}
$$

while for the energy-momentum tensor $\theta_{\mu\nu}$ we have

$$
\partial^n \theta_{n1} + \partial^n \theta_{n1} = 0. \tag{3.6}
$$

When integrated over $\Sigma_2$ the terms which are the total derivatives in $z^2$ and $\bar{z}^2$ in these equations vanish. In turn it is easy to check that the components $J_1$ and $\theta_{11}$ are $Q$-exact because of the $Q$-exactness of the action (for example, $\theta_{11}$ is a variation of the action with respect to the component $g_{11}$ of the metric).

Similarly we can also define the $Q$-closed operators which are local with respect to the coordinates of the Riemann surface $\Sigma_2$ and are represented by integrals over $\Sigma_1$.

These operators do not of course exhaust the full ghost number $G = 0$ cohomology of the BRST operator $Q$. In particular there exist also fermionic $Q$-closed operators which are represented by integrals of composite operators over $\Sigma_2$ ($\Sigma_2$). We postpone a detailed description of the $Q$-cohomology for a next publication.

### 3.2 Central charge

We will now show that the operators $W_{n+1}$ generate the $W_{1+\infty}$ algebra in the cohomology of $Q$. To this end let us consider the operator product expansion (OPE) of the operators $W_{n+1}$. We have

$$
W_{n+1}(z, \bar{z}) W_{p+1}(w, \bar{w}) = (2\pi)^2 \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 v \left( \partial^n \phi(z, \bar{z}, \bar{u}) \partial_1 \phi(w, \bar{w}, v, \bar{v}) \right) + \left( \partial^n \bar{\psi}_{12}(z, \bar{z}, \bar{u}) \bar{\psi}_{12}(w, \bar{w}, v, \bar{v}) \right) + \ldots
$$

$$
(3.7)
$$

$$
= \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 v \left( \partial^n \phi(z, \bar{z}, \bar{u}) \partial_1 \phi(w, \bar{w}, v, \bar{v}) \right) + \left( \partial^n \bar{\psi}_{12}(z, \bar{z}, \bar{u}) \bar{\psi}_{12}(w, \bar{w}, v, \bar{v}) \right) + \ldots
$$

(3.7)
\[
< \partial^p_1 \phi(w, \bar{w}, v, \bar{v}) \partial_1 \bar{\phi}(z, \bar{z}, u, \bar{u}) > + \\
+ \partial^p_0 \phi(w, \bar{w}, v, \bar{v}) \partial_1 \bar{\phi}(z, \bar{z}, u, \bar{u}) < \partial^p_1 \phi(z, \bar{z}, u, \bar{u}) \partial_1 \bar{\phi}(w, \bar{w}, v, \bar{v}) > - \\
- \partial^p_1 \psi_2(z, \bar{z}, u, \bar{u}) \bar{\psi}_{12}(w, \bar{w}, v, \bar{v}) < \partial^p_1 \psi_2(w, \bar{w}, v, \bar{v}) \bar{\psi}_{12}(z, \bar{z}, u, \bar{u}) > - \\
- \partial^p_1 \psi_2(w, \bar{w}, v, \bar{v}) \bar{\psi}_{12}(z, \bar{z}, u, \bar{u}) < \partial^p_1 \psi_2(z, \bar{z}, u, \bar{u}) \bar{\psi}_{12}(w, \bar{w}, v, \bar{v}) > + \\
+ \text{terms non-singular at } (z \to w).
\]

Here \( z (\bar{z}) \) and \( w (\bar{w}) \) are the complex coordinates on \( \Sigma_1 \) while \( u (\bar{u}) \) and \( v (\bar{v}) \) are the complex coordinates on \( \Sigma_2 \) (\( \partial_1 = \partial/\partial z \) or \( \partial_1 = \partial/\partial w \) and \( \partial_2 = \partial/\partial u \) or \( \partial_2 = \partial/\partial v \)).

The propagators of free matter fields are given in the Appendix. Let us calculate first the most singular (at \( z \to w \)) terms proportional to the unity operator in Eq. (3.9). These contributions can be represented as follows

\[
< W_{n+1}(z, \bar{z}) W_{p+1}(w, \bar{w}) > = -4\pi^2 \int_{\Sigma_2} d^2 u \int_{\Sigma_2} d^2 v \\
\left[ \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(z, \bar{z}, u, \bar{u}) \delta^2(z - w) \delta^2(u - v) \right) \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(w, \bar{w}, v, \bar{v}) \delta^2(w - z) \delta^2(v - u) \right) - \\
- \left( \frac{\partial^{p+1}_1}{\Delta_{01}}(z, \bar{z}, u, \bar{u}) \delta^2(z - w) \delta^2(u - v) \right) \left( \frac{\partial^{p+1}_1}{\Delta_{01}}(w, \bar{w}, v, \bar{v}) \delta^2(w - z) \delta^2(v - u) \right) \right],
\]

where

\[
\Delta_{00} = \partial_1 \partial_1 + e^{-\rho} \partial_2 \partial_2, \quad \Delta_{01} = \partial_1 \partial_1 + \partial_2 e^{-\rho} \partial_2
\]

are the Laplace operators on \( (0, 0) \) and \( (0, 1) \) forms on \( M \), \( \exp \rho = g_{22} \) (the metric on \( \Sigma_1 \) is assumed to be flat, and \( g_{22} \) does not depend on coordinates on \( \Sigma_1 \)).

In order to compute this contribution we consider an expansion in derivatives of the external metric in Eq. (3.9) (we can do it in the limit \( |z - w| \to 0 \)). It is easy to see that for a constant metric (\( \rho = \text{const} \)) this expression vanishes due to the cancellation between bosonic and fermionic contributions. In the linear approximation in \( \partial_\mu \rho \) we have

\[
< W_{n+1}(z, \bar{z}) W_{p+1}(w, \bar{w}) > = -4\pi^2 \int_{\Sigma_2} d^2 u \int_{\Sigma_2} d^2 v \\
\left[ \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(\partial_\rho \partial_\mu) \partial^2 \frac{1}{\Delta_{00}}(z, \bar{z}, u, \bar{u}) \delta^2(z - w) \delta^2(u - v) \right) \times \\
\times \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(w, \bar{w}, v, \bar{v}) \delta^2(w - z) \delta^2(v - u) \right) + \\
+ \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(\partial_\rho \partial_\mu) \partial^2 \frac{1}{\Delta_{00}}(z, \bar{z}, u, \bar{u}) \delta^2(z - w) \delta^2(u - v) \right) \times \\
\times \left( \frac{\partial^{p+1}_1}{\Delta_{00}}(w, \bar{w}, v, \bar{v}) \delta^2(w - z) \delta^2(v - u) \right) \right].
\]
We use the fact that
\[
\frac{1}{\partial_1 \partial_1 + \partial_2 \partial_2} \delta^2(z) \delta^2(u) = \frac{1}{4\pi^2} \frac{1}{|z|^2 + |u|^2};
\] (3.11)
and integrate over \(u - v\) in Eq. (3.10). We get
\[
< W_{n+1}(z, \bar{z}) W_{p+1}(w, \bar{w}) >= \frac{n!p!(-1)^n}{(z-w)^{n+p+2}} \cdot \frac{1}{4\pi} \int_{\Sigma_2} d^2 u (\partial_2 \partial_2 \rho) = \]
\[
= \frac{n!p!(-1)^{n+1} \chi}{2(z-w)^{n+p+2}}
\]
where \(\chi = (-1/2\pi) \int_{\Sigma_2} \sqrt{g} \partial_2 \partial_2 \rho = 2(1-g)\) is the Euler characteristic of the Riemann surface \(\Sigma_2\).

Actually we can do even better and show that this is an exact result which does not depend on our assumption that the metric changes slowly as compared to the distance \(|z - w|\). To this end we split the contributions of zero and non-zero modes of the 2D Laplace operators \(\Delta_{00}^{(2)} = \exp(-\rho) \partial_2 \partial_2\) and \(\Delta_{01}^{(2)} = \partial_2 \exp(-\rho) \partial_2\) which act on the \((0,0)\) and \((0,1)\) forms on \(\Sigma_2\) respectively. The only zero mode of \(\Delta_{00}^{(2)}\) on a compact Riemann surface \(\Sigma_2\) corresponds to a constant wave function \(|0\rangle\), while there are \(g\) zero modes of \(\Delta_{01}^{(2)}\) with anti-holomorphic wave functions \(|1_i\rangle\), \(i = 1, \ldots, g\), where \(g = \text{dim } H^{1,0}(\Sigma_2)\). Thus we have
\[
< W_{n+1}(z, \bar{z}) W_{p+1}(w, \bar{w}) >= 4\pi^2 \int_{\Sigma_2} d^2 u
\]
\[
\left[ < 0| \left( \frac{\partial_1^{n+1}}{\partial_1 \partial_1 + e^{-\rho} \partial_2 \partial_2} \right)_{z,w} \left( \frac{\partial_1^{p+1}}{\partial_1 \partial_1 + e^{-\rho} \partial_2 \partial_2} \right)_{w,z} |0 > - \right.
\]
\[
- \sum_{i=1}^{g} < 1_i| \left( \frac{\partial_1^{n+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right)_{z,w} \left( \frac{\partial_1^{p+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right)_{w,z} |1_i > + \]
\[
+ 4\pi^2 \lim_{\epsilon \to 0} \int_{\Sigma_2} d^2 u
\]
\[
\left[ \left( \frac{e^{-\rho} \partial_2 \partial_2}{e^{-\rho} \partial_2 \partial_2 - \epsilon^2} \right)_{z,w} \left( \frac{\partial_1^{n+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right)_{w,z} \delta^2(u-v)|_{u=v}- \right.
\]
\[
- \left( \frac{\partial_1^{n+1} e^{-\rho_2} \partial_2}{\partial_2 e^{-\rho_2} \partial_2 - \epsilon^2} \right)_{z,w} \left( \frac{\partial_1^{p+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right)_{w,z} \delta^2(u-v)|_{u=v} \right],
\]
where \((...)_{z,w}\) stands for an operator on \(\Sigma_2\) which is the \((z,w)\) matrix element with respect to the coordinates on \(\Sigma_1\), i.e., for example,
\[
\left( \frac{\partial_1^{p+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right)_{z,w} = \left( \frac{\partial_1^{p+1}}{\partial_1 \partial_1 + e^{-\rho_2} \partial_2 \partial_2} \right) (z, \bar{z}, u, \bar{u}) \delta^2(z-w) \] (3.14)
and
\[
P_{00} = \lim_{\epsilon \to 0} \frac{e^{-\rho} \partial_2 \partial_2}{e^{-\rho} \partial_2 \partial_2 - \epsilon^2}, \quad P_{01} = \lim_{\epsilon \to 0} \frac{\partial_2 e^{-\rho} \partial_2}{\partial_2 e^{-\rho} \partial_2 - \epsilon^2}
\]
are the projectors to non-zero modes of the operators \(e^{-\rho} \partial_2 \partial_2\) and \(\partial_2 e^{-\rho} \partial_2\) respectively; \(\epsilon\) is a real parameter.

We integrate by parts (on \(\Sigma_2\)) the second term and observe that the contribution from the non-zero modes vanishes. In turn for the contribution from the zero modes we easily get
\[
4\pi^2(1 - g) \left( \frac{\partial^{n+1}}{\partial_1 \partial_1} \right) (z, \bar{z}) \delta^2(z - w) \left( \frac{\partial^{n+1}}{\partial_1 \partial_1} \right) (w, \bar{w}) \delta^2(w - z) = (3.16)
\]
\[
= (1 - g)(-1)^{p+1}(\partial^{n+1}_1 \log(z - w))(\partial^{n+1}_1 \log(z - w)) = \frac{\chi n! p! (-1)^{n+1}}{2(z - w)^{n+p+2}}.
\]
We see that this result is exact and agrees with Eq. (3.12). It is worth emphasizing that this contribution is remarkably holomorphic on \(\Sigma_1\).

### 3.3 OPE

Other terms in the OPE for \(W_{n+1} W_{p+1}\) can be represented after some calculations in the following form (at this point for simplicity we do not take into account the presence of the external metric)
\[
\int_{\Sigma_2} d^2u \int_{\Sigma_2} d^2v \left[ \left( \frac{\partial^n}{\partial z} \phi(z, \bar{z}, u, \bar{u}) \right) \partial_w \bar{\phi}(w, \bar{w}, v, \bar{v}) + \right.
\]
\[
+ \partial^n_2 \psi_2(z, \bar{z}, u, \bar{u}) \bar{\psi}_{12}(w, \bar{w}, v, \bar{v}) \left( \frac{p + 1}{(z - w)^{n+1}} \right) + (3.17)
\]
\[
- \left( \partial^p_w \phi(w, \bar{w}, v, \bar{v}) \partial_z \bar{\phi}(z, \bar{z}, u, \bar{u}) + \partial^p_w \psi_2(w, \bar{w}, v, \bar{v}) \bar{\psi}_{12}(z, \bar{z}, u, \bar{u}) \right) \frac{(n + 1)!(-1)^{n+1}(\bar{z} - \bar{w})^{n+1}}{|z - w|^2 + |u - v|^2}.
\]
where \(\partial_z = \partial / \partial z\) and \(\partial_w = \partial / \partial w\). By expanding the integrand in powers of \(z - w\) \((\bar{z} - \bar{w})\) and \(u - v\) \((\bar{u} - \bar{v})\) and integrating over \(u - v\) and \(\bar{u} - \bar{v}\) we get

\[
2\pi \sum_{k=0}^{\infty} \frac{(z - w)^k}{k!} \int_{\Sigma_2} \sqrt{g(2)} d^2v \left[ \left( \partial_w \bar{\phi} \partial_w^{n+k} \phi - g^2 \bar{\psi}_{12} \partial_w^{n+k} \psi_2 \right) \frac{p!(-1)}{(z - w)^{p+1}} \right. + (3.18)
\]
\[
\left. \left( \partial^k_w \bar{\phi} \partial_w^p \phi - g^2 \bar{\psi}_{12} \partial_w^p \psi_2 \right) \frac{n!(-1)^n}{(z - w)^{n+1}} \right]
\]
\[
+ Q\text{-exact terms} = \frac{p + n}{(z - w)^2} W_{p+n}(w, \bar{w}) + \frac{n}{z - w} \partial_w W_{n+p} + f(W_{n+p-1}, ..., W_1) +
\]
Here we used that \( \partial_n \phi = \{ Q, \psi_n \} \), \( \bar{\partial} \psi = \{ Q, \bar{\psi} \} \) and the equations of motion. Thus the OPE for \( W_{n+1}W_{p+1} \) is holomorphic on \( \Sigma_1 \) in \( Q \)-cohomology.

It is easy to see that the operator \( T = W_2 \) generates the holomorphic Virasoro algebra with the central charge \( \chi \) which is the Euler characteristic of \( \Sigma_2 \). Moreover one can see that the operators \( W_{n+1} \) generates the holomorphic \( W_{1+\infty} \) algebra.

Indeed by introducing the Fourier modes

\[
W_n = \oint \frac{dz}{2\pi i} z^{n+\frac{g}{2}} W_{n+\frac{g}{2}}(z, \bar{z})
\]

we get (in the \( Q \)-cohomology)

\[
[W_n, W_p] = \frac{\chi(-1)^{n+1} n!}{2} \frac{(s + n)\ldots(s - p)}{(n + p + 1)!} \delta_{s+\nu,0} + (sp - ns') W_{n+p-1}^{n+p-1} + R.
\]

Here \( R \) stands for terms which depend only on \( W^k_i \) with \( k < n + p - 1 \). The standard \( W_{1+\infty} \) commutation relations correspond to \( R = 0 \) \([34, 35]\), see also \([36]\) while in our algebra \( R \neq 0 \). However by adding to the operators \( W_{n+1} \) appropriate linear combinations \( \sum_{k=0}^{n-1} a_k \bar{\partial}_z^{-k} W_{k+1} \), where \( a_k \) are constant coefficients, one can recover the standard commutation relations with \( R = 0 \).

Similarly we can construct the \( W_{1+\infty} \) algebra on the Riemann surface \( \Sigma_2 \). Its central charge is given by the Euler characteristics of \( \Sigma_1 \). Thus we have a direct sum of two \( W_{1+\infty} \) algebras embedded into the four dimensional theory of free chiral supermultiplet.

### 3.4 Ghost number \( G \neq 0 \)

We considered only the cohomology with the ghost number \( G = 0 \). For \( G \neq 0 \) in the free theory one can also define the following operators

\[
W^{(n)}_{k+1} = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} \left[ -\phi^n \partial_1^{k+1} \bar{\phi} + g^{22} n \phi^{n-1} (\partial_1^{k} \bar{\psi} \psi_2) \right].
\]

These operators are \( Q \)-closed and holomorphic in the \( Q \)-cohomology:

\[
\{ Q, W^{(n)}_{k+1} \} = 0, \quad \partial_1 W^{(n)}_{k+1} = \{ Q, \ldots \}.
\]

One can calculate the OPE for these operators. In particular the most singular ("central") term in the OPE has the following form

\[
W^{(n)}_{k+1}(z, \bar{z}) W^{(m)}_{p+1}(w, \bar{w}) = \frac{m nk! p! (-1)^p+1}{(z - w)^{p+k}}
\]

\[
\int_{\Sigma_2} \sqrt{g^{(2)} d^2 u} \left( |\omega_0|^2 - \sum_{i=1}^{g} |\omega_i|^2 \right) (u, \bar{u}) \cdot \phi^{n-1}(z, \bar{z}, u, \bar{u}) \phi^{m-1}(w, \bar{w}, u, \bar{u}) + \ldots,
\]

where \( \omega_0 \) and \( \omega_i, i = 1, \ldots, g \), are the normalized to one wave functions of the zero modes of the operators \( e^{-\tau} \partial_2 \bar{\partial}_2 \) and \( \partial_2 e^{-\tau} \partial_2 \) respectively. Unfortunately I do not know of any natural interpretation for this extended algebra.
4 Virasoro algebra for a self-interacting chiral supermultiplet

Let us now consider a manifold \( M = \Sigma_1 \times \Sigma_2 \) where the genera of both Riemann surfaces \( \Sigma_{1,2} \) are non-zero. In this case the manifold \( M \) has \( H^{2,0}(M) \neq 0 \). Let \( E_{mn} \) be a non-trivial holomorphic \((2,0)\) form. For simplicity we shall also assume that \( \Sigma_1 = T^2 \).

In this case it is possible to introduce a superpotential into our model. Let us consider the model with a quasihomogeneous superpotential \( W(x) = \lambda x^{N+1}/(N+1) \), where \( \lambda \) is a coupling constant (from now on we shall suppress \( \lambda \) for simplicity). We shall show that in this case the \( W_{1+\infty} \) algebra is broken down to a Virasoro one.

The equations of motion in this case read

\[
\begin{align*}
\bar{N}_{mn} &= -E_{mn}\phi^N, \quad N_{\bar{m}n} = 2S_{m\bar{n}}\bar{\phi}^N, \\
D_{\bar{m}}\bar{\psi}^{\bar{m}n} + D^n\bar{\psi} &= 2NE^{\bar{m}n}\bar{\psi}_{\bar{m}}\phi^{N-1}, \quad D^m\psi_m = -2S^{12}\bar{\psi}_{12}N\bar{\phi}^{N-1}, \\
D_{\bar{m}}\bar{\psi}_{\bar{n}} - D_n\psi_{\bar{m}} &= 2S_{\bar{m}n}\bar{\psi}N\bar{\phi}^{N-1}, \\
D^mD_{\bar{m}}\phi &= -4NS^{12}E_{12}\phi^N\bar{\phi}^{N-1} + 2N(N-1)S^{12}\bar{\psi}_{12}\bar{\phi}^{N-2}, \\
D^mD_{\bar{m}}\bar{\phi} &= -4NS^{12}E_{12}\phi^N\bar{\phi}^{N-1} - N(N-1)E^{\bar{m}n}\bar{\psi}_{\bar{m}}\bar{\psi}_{\bar{n}}\phi^{N-2}.
\end{align*}
\]

Here \( S_{m\bar{n}} \) is a non-singular \((0,2)\) form.

As a consequence of these equations of motion one can see that the local operator \( E_{mn}\phi^N = 0 \) in \( Q \)-cohomology since it is \( Q \)-exact

\[
\{Q, \frac{1}{2}\bar{\psi}_{mn}\} = -\bar{N}_{mn} = E_{mn}\phi^N.
\]

This is a four-dimensional analog of the relations that define the ground ring in 2D \( N = 2 \) supersymmetric theories. If the \((0,2)\)-form \( E_{mn} \) has no zeros (i.e. \( \Sigma_1 = \Sigma_2 = T^2 \)) we get

\[
\phi^N = 0.
\]

Hence the algebra of local scalar operators is generated by monomials \( \phi^k \), \( k = 0, ..., N-1 \) modulo \( \phi^N \). In the particular case of \( N = 1 \) which corresponds to the free massive theory the operator \( \phi = 0 \) in the \( Q \)-cohomology. In turn as we shall see below in this case the central charge of a Virasoro algebra which appears in the \( Q \)-cohomology turns out formally to be zero. Actually in such a special case the corresponding stress turns out to be \( Q \)-exact due to the equations of motion \([1,1]\).

In the case when \( E_{mn} \) has zeros the analysis of the algebra of local scalar operators is not so straightforward because the operator \( \bar{\psi}_{mn}/E_{12} \) is not well defined due to a singularity in \( 1/E_{12} \). Correspondingly a Virasoro algebra which appears in the \( Q \)-cohomology has a non-zero central charge even in the case \( N = 1 \).
We now construct a Virasoro algebra on $\Sigma_2$. It is convenient to introduce the following operators

$$W_{p,q} = \int_{\Sigma_2} \sqrt{g^{(2)}} d^2u \left( -\partial_1^{q+1} \bar{\phi} \partial_1^p \phi + g^{22} \partial_1^{p} \bar{\psi} \partial_1^q \psi_2 \right). \quad (4.4)$$

By using the equations of motion (4.1) and taking $E_{12}$ to be independent of the coordinates on $\Sigma_1$ one can check that

$$\{Q, W_{p,q}\} = 2 \int_{\Sigma_2} \sqrt{g^{(2)}} d^2u E_{12} g^{22} \left[ N(\partial_1^p \phi) \partial_1^q (\psi_2 \phi^{N-1}) + (\partial_1^p \phi^N) (\partial_1^q \psi_2) \right]. \quad (4.5)$$

This expression is $Q$-closed but non-vanishing, and hence the operators $W_{p,q}$ do not belong to the $Q$-cohomology.

Let us consider a linear combination of $W_{p,q}$ with a definite spin

$$\tilde{W}^{(n)} = \sum_{p+q=n} a^{(n)}_{p,q} W_{p,q}, \quad (4.6)$$

where $a^{(n)}_{p,q}$ are some constants. We want to find out all the sets of $a^{(n)}_{p,q}$ for which $\{Q, \tilde{W}^{(n)}\} = 0$. We will actually show that the only non-trivial solution appears for $n = 1$. To show this let us formally consider the integrals of $\tilde{W}^{(n)}$ over the holomorphic coordinate $z_1$ (with vanishing boundary conditions)

$$A^{(n)} = \int dz_1 \tilde{W}^{(n)} = \sum_{p+q=n} a^{(n)}_{p,q} (-1)^q \int dz_1 W_{n,0} = b \int dz_1 W_{n+1}. \quad (4.7)$$

We then have

$$\{Q, A^{(n)}\} = 2b \int dz_1 \int_{\Sigma_2} \sqrt{g^{(2)}} E_{12} g^{22} [N(\partial_1^p \phi) \psi_2 \phi^{N-1} + \phi^N \partial_1^p \psi_2]. \quad (4.8)$$

This expression vanishes only at $b = 0$ or at $n = 1$ (in the latter case the integrand is a total derivative in $z_1$). At $b = 0$ however the operator $\tilde{W}^{(n)}$ is a total derivative in $z_1$ since its formal integral over $z_1$ vanishes, and therefore it reduces to the operators $\tilde{W}^{(k)}$ with $k < n$. We thus have inductively shown that the only non-trivial $Q$-closed operator $\tilde{W}$ can appear at $n = 1$. It is easy to check that for any $N$ such an operator does exist and it has the following form

$$T_N = W_2 - \frac{1}{N+1} \partial_1 W_1, \quad (4.9)$$

so that

$$\{Q, T_N\} = 0, \quad \partial_1 T_N = \{Q, ...\}. \quad (4.10)$$

Notice that the 1-component of the $U(1)$ current does not belong to the $Q$-cohomology since the phase symmetry is broken by the superpotential.

The operator $T_N$ has a spin 2 and generates the holomorphic Virasoro algebra on $\Sigma_1$. In order to check it it is necessary to calculate the OPE for $T_N(z, \bar{z})T_N(w, \bar{w})$. It can be done in the limit of weak coupling similarly with the calculation of Ref. [7, 8].
The point is that the superpotential terms in the Lagrangian are dimensionful (for any superpotential in contrast to the untwisted supersymmetric theory) and make less singular contributions to the OPE as compared to the free ones. Formally the same follows from the fact that the OPE for this operator can not depend on a choice of the form $S_{\bar{m}\bar{n}}$ in the $Q$ cohomology since the terms in the Lagrangian which depend on $S_{\bar{m}\bar{n}}$ are $Q$-exact (see Eq. (2.22)). In particular it is easy to calculate the central charge which is given by

$$c_N = \chi \left(1 - \frac{6}{N+1} + \frac{6}{(N+1)^2}\right).$$

(4.11)

The model is renormalizable only if $N = 1, 2$. For the case $N = 1$ which corresponds to the free massive model we have $c_2 = -\chi/2 = g - 1$, while for $N = 2$ $c_3 = -\chi/3 = 2(g - 1)/3$. In both cases the central charge is non-negative since we assumed that $g > 0$.

Let us now discuss the behaviour of this algebra with respect to the renormalization group. The superpotential is not $Q$-exact. Hence the physical correlators can depend on the coupling constant in the superpotential. This is a difference of the present case from the twisted $N = 1$ gauge theory without a superpotential for the matter fields [31] where all the action is $Q$-exact (see below for a discussion). However the superpotential is not renormalizable [42] while the $D$-terms which are $Q$-exact are only renormalizable. We thus conclude that our construction is invariant under the renormalization group. This fact can also be understood as follows. The quantum effects result in only a renormalization of the wave functions by a factor $Z$. In turn the same factor $Z$ appears in the operator $T_N$ due to quantum effects. These effects are non-trivial for the operator $T_N$ because it is $Q$-closed only on mass-shell in contrast to the case of operators $\phi^k$ which are $Q$-closed off mass-shell and do not acquire any multiplicative factor under renormalization [55]. Notice that since the operator $T_N$ is an integral of a component of the conserved energy-momentum tensor it does not acquire any its own renormalization factor. After a redefinition of the quantum fields the factor $Z$ disappears in the OPE for $T_N$ which is therefore invariant under the renormalization group.

Notice that similarly we can construct the Virasoro algebra on the Riemann surface $\Sigma_2$. The central charge of this algebra will be given by the same expression Eq. (4.11) with a change of the Euler characteristics of the surface $\Sigma_2$ into that of $\Sigma_1$. Thus we have a direct sum of two Virasoro algebras embedded into the four dimensional theory.

5 A twisted supersymmetric gauge theory

In this section we construct the Virasoro algebra in the $Q$-cohomology of a heterotic topological gauge theory without a superpotential for the matter fields and discuss the instanton effects and a possible relation to two dimensional supersymmetric sigma models.
5.1 Gauge fixing and equations of motion

In order to formulate the theory at the quantum level one has to fix a gauge. To this end we introduce the ghost field $c$, the antighost field $c^{+}$ and an auxiliary field $B$. The BRST transformations responsible for the gauge fixing read

$$Q_{BRST} A_n = D_n c, \quad Q_{BRST} A_0 = D_0 c, \quad Q_{BRST} \chi_n = i \{ c, \chi_n \}, \quad (5.1)$$
$$Q_{BRST} \lambda = i \{ c, \lambda \}, \quad Q_{BRST} \lambda^{mn} = i \{ c, \lambda^{mn} \}, \quad Q_{BRST} D' = i \{ c, D' \},$$
$$Q_{BRST} c = \frac{i}{2} \{ c, c \}, \quad Q_{BRST} c^{+} = B, \quad Q_{BRST} B = 0.$$  

The generator of this symmetry anticommutes with the generator $Q$ of the twisted supersymmetry (the generator $Q$ is assumed to annihilate the fields $c$, $c^{+}$ and $B$). Therefore the total symmetry of the theory at the quantum level should be generated by $Q' = Q + Q_{BRST}$. Let us fix the gauge by adding the following $Q + Q_{BRST}$-exact operator to the Lagrangian

$$L_{\text{fix}} = \text{Tr} \{ Q', c^{+} \partial^\mu A_\mu + \frac{1}{2\alpha} c^{+} B \} = (5.2)$$
$$= \text{Tr} [ B \partial^\mu A_\mu - c^{+} \partial^\mu D_\mu c - c^{+} \partial^m \chi_m + \frac{1}{2\alpha} B^2 ],$$

where $\alpha$ is a gauge fixing parameter. The term $c^{+} \partial^m \chi_m$ is unusual for gauge fixing. To recover the background field gauge one has to change the usual derivatives $\partial_\mu$ in the above equations into $D^{(0)}_\mu = \partial_\mu - i A^{(0)}_\mu$, where $A^{(0)}_\mu$ is an external gauge field. For definiteness we shall fix the gauge parameter $\alpha = -1$.

For the total Lagrangian $L_{\text{tot}} = L + L_{\text{fix}}$, where $L$ is given by Eq. (2.15) the equations of motion for the gauge supermultiplet read

$$(2D_m F^{mn} + i D_n D' - \{ \lambda^{mn}, \chi_m \} + D^{(0)n} B - i \{ D^{(0)n} c^{+}, c \})^{a} + (5.3)$$
$$+ i \bar{\psi}_t^{a} \psi^{n} + i \bar{\phi} t^{a} D^n \phi = 0,$$
$$+ i \bar{\psi}^{mn} t^{a} \psi^{m} - i D_{n} \bar{\phi} t^{a} \phi = 0,$$
$$D^{a} = ig^{mn} F_{mn}^{a} + i \bar{\phi} t^{a} \phi, \quad i D_{m} \chi_{n} - i D_{n} \chi_{m} - \frac{1}{2} \bar{\psi}_{mn} t^{a} \phi = 0,$$
$$i D^{m} \chi_{m} + i \bar{\psi} \phi = 0, \quad i D^{m} \bar{\lambda}^{mn} + i D^{a} \bar{\lambda} - i \bar{\phi} \bar{\psi}^{n} = 0,$$

where $t^{a}$ stands for a generator of the gauge group, while for the multiplet of matter we have

$$D^{m} D_{m} \phi - i \chi_{n} \psi^{n} + i D' \phi = 0, \quad D_{m} D^{m} \phi + \frac{1}{4} \bar{\psi}_{mn} \lambda^{mn} + i \bar{\phi} D' - i \bar{\psi} \lambda = 0, \quad (5.4)$$
$$D_{m} \psi_{n} - D_{n} \psi_{m} + \frac{1}{2} \bar{\lambda}_{mn} \phi = 0, \quad D^{m} \psi_{m} - i \bar{\lambda} \phi = 0, \quad D_{m} \bar{\psi}^{mn} + D^{n} \bar{\psi} + i \bar{\phi} \bar{\lambda} = 0.$$  

Thus the quantized theory is invariant under a combined BRST transformations.
5.2 Physical correlators

It follows from section 2. that the physical correlators are sections of a holomorphic bundle on $M$.

Let us consider the structure of a physical correlator. We observe that the term $(1/2) Tr D^2$ in the Lagrangian (2.13) is BRST exact and can be taken out from the Lagrangian without any change of the correlator. After this modification one can integrate out over the field $D'$ that leads to the following constraint

$$g^{m\bar{n}} F_{m\bar{n}} = 0. \quad (5.5)$$

It is easy to see that this condition is necessary for the anti-self-duality of the Yang-Mills field. In order to show this it is convenient to use the identity $\epsilon_{mnkl} = g_{m\bar{l}} g_{kn} - g_{m\bar{n}} g_{kl}$. One can then check that

$$\epsilon_{nkml} F_{m\bar{l}} = -F_{nk} - g_{nk} (g_{\bar{p}q} F_{\bar{p}q}), \quad (5.6)$$

$$\epsilon_{nkml} F_{km} = 2F_{k\bar{l}}, \quad \epsilon_{nkml} F_{\bar{n}\bar{l}} = 2F_{k\bar{m}}.$$

In turn the anti-self-duality condition means that

$$\epsilon_{nkml} F_{m\bar{l}} = -F_{nk}, \quad \epsilon_{nkml} F_{km} = -2F_{k\bar{l}}, \quad \epsilon_{nkml} F_{\bar{n}\bar{l}} = -2F_{k\bar{m}}. \quad (5.7)$$

From these equations it is easy to see that the anti-self-duality condition is equivalent to the following equations:

$$g_{\bar{p}q} F_{\bar{p}q} = 0, \quad F_{km} = F_{\bar{n}\bar{l}} = 0. \quad (5.8)$$

Eq. (5.5) coincides with the first of the anti-self-duality conditions. The other conditions in eqs. (5.7) appear in the limit of a weak gauge coupling constant because in this case the functional integral is dominated by the fields with $F_{km} = F_{\bar{n}\bar{l}} = 0$ which correspond to the minimum of the action. In turn we can consider this limit since the action of the theory is BRST exact and, hence, the correlators are independent of the value of the coupling constant. The theory is therefore similar to the topological Yang-Mills theory [28] and the physical correlators can be computed semiclassically in the presence of an anti-instanton field.

In such a case some of the fields of the model have zero modes. These zero modes are absorbed by the operators inserted into the correlator. Technically the wave functions of the zero modes should be substituted into the preexponential factor in the path integral for the correlator (directly or by using Yukawa couplings). Moreover one should integrate over quadratic fluctuations near the anti-instanton field. Notice that the classical action (2.13) for the anti-instanton equals zero.

We assume that the anti-instanton configuration is non-trivial only for the gauge field while the other fields vanish classically. Let us analyse the zero modes (for more details see ref.[31]). Actually it is easy to see from eqs. (5.7) that the variation of
the self-duality equations in the vector field $A_n$ (fixing the gauge of the variation of the gauge field by a condition $D^\mu \delta A_\mu = 0$) gives the following equations

$$D_{[m} \delta A_{n]} = 0, \quad g^{mn} D_m \delta A_n = 0,$$  \hspace{1cm} (5.9)

$$D_{[\bar{m}} \delta A_{\bar{n}]} = 0, \quad g^{\bar{m} \bar{n}} D_{\bar{m}} \delta A_{\bar{n}}.$$  \hspace{1cm} (5.10)

These equations determine the zero modes of the gauge field near the anti-instanton field up to gauge transformations.

The equations of motion for the field $\chi_n$ coupled to the anti-instanton field read as follows

$$D_{[m} \chi_{n]} = 0, \quad g^{mn} D_m \chi_n = 0.$$  \hspace{1cm} (5.11)

Due to the similarity of these equations to eqs. (5.9) and (5.10) we see that the zero modes of the fermionic field $\chi_n$ and $A_n$ coincide. Therefore the zero modes of $\chi_n$ correspond to a half of tangent vectors to the moduli space $M$ of the anti-instanton because there are no superpartners to $A_n$.

The zero modes of the gauge field near the anti-instanton configuration determine the collective coordinates which correspond to coordinates in the moduli anti-instanton space. For a compact manifold $M$ the moduli space $M$ of the anti-instanton is a manifold of dimension $[56, 57]$

$$d = p_1(G) - \frac{1}{2} \dim G(\chi + \tau),$$  \hspace{1cm} (5.12)

where $p_1(G)$ is the first Pontryagin class of the adjoint bundle over $M$, $G$ stands for the gauge group, $\chi$ is the Euler characteristic of $M$, and $\tau$ is the signature of $M$.

For the case of a Kähler manifold $M$ the moduli space $M$ is a complex manifold of a complex dimension $d/2$ [58, 60]. The Kähler structure on $M$ is induced by a Kähler structure on $M$. The corresponding Kähler form on $M$ turns depend on only the Kähler class of the Kähler metric on $M$ (see also [31]).

In this paper we focus to the case of a generically irreducible anti-instanton field and assume that only vector fields can have zero modes.

Notice that the fermionic field $\chi_m$ has $d/2$ zero modes but not $d$ despite the similarity of eqs. (5.9), (5.10) and (5.11). The point is that for the gluonic field we should consider the pairs $(\delta A_m, \delta \bar{A}_m)$ while the fermionic field has only components with a holomorphic index. It is easy to show that if $(\delta A_m, \delta \bar{A}_m)$ is a wave function for a gluonic zero mode then so is $(i\delta A_m, -i\delta \bar{A}_m)$ as it can be rewritten as $J^\mu \delta A_\mu$ ($J^\mu$ is the complex structure) which satisfies the same equation with $\delta A_\mu$ as the tensor $J^\mu$ is covariantly constant.

Let us now consider the matter sector of the fields. The equations of motion for the fermionic vector field $\psi_m$ which is coupled to the anti-instanton field read

$$D_m^\dagger \psi_m - D_m \psi_m = 0, \quad D^m \psi_m = 0.$$  \hspace{1cm} (5.14)
These equations are similar to those for the zero modes of the gauge field but the \( \psi_{\bar{n}} \) matter field can belong to an arbitrary representation \( R \) of the gauge group (not necessarily to an adjoint one). The number of zero modes of the field \( \psi_{\bar{n}} \) depends on the representation \( R \) and is determined by the index theorem for a corresponding Dirac operator \[53\].

We also have to consider the integration over quadratic fluctuations around the anti-instanton field. The result of such an integration provides us with a combination of determinants of the Laplace-type operators. As a result we get for the physical correlator a representation as an integral over moduli space \( \mathcal{M} \) of anti-instanton of a \((d/2, d/2)\) form on \( \mathcal{M} \) \[31\].

5.3 \( U(1) \) and Konishi anomalies

We now want to study the cohomology of the BRST operator. We will not give here an exhaustive analysis (it will be given elsewhere). Instead we will consider a part of this cohomology for the ghost number \( G = 0 \) which is relevant for the construction of the conformal algebra and for an analysis of some dynamical properties of the theory which is given below.

To this end we consider a twisted formulation of the Konishi anomaly \[60\]. Let the matter multiplet transform as a representation \( R \) of the gauge group which consists of irreducible representations \( R_i \), \( i = 1, \ldots, f \). Let a representation \( R_i \) enter \( R \) with a repetition \( n_i \).

Let us first consider the case of massless matter fields. For a particular irreducible representation \( R_i \) at the classical level we have

\[
\{ Q, \bar{\psi}^{I}_{\bar{n}} \phi^{I} \} = 0, \tag{5.15}
\]

where the index \( I \) stands for the \( I \)'s copy of the representation \( R_i \) in \( R \). At the quantum level we have to introduce an ultra-violet regularization. To this end it is convenient to use the Pauli-Villars ultraviolet regularization for the matter multiplet in external gauge \( (A^{cl}_{\mu}) \) and gaugino \( (\chi^{cl}_{\mu}) \) fields. The Lagrangian for the regulator fields of matter reads (a subscript \( \text{reg} \) stands in order to label the regulator fields) read

\[
L^{m}_{\text{reg}} = \sqrt{g} (\bar{\phi}_{\text{reg}} D^{\bar{m}} \bar{D}_{\bar{n}} \phi_{\text{reg}} + \bar{\psi}^{\bar{m}_{\text{reg}}}_{\bar{n}} D_{\bar{n}} \psi_{\text{reg}, \bar{n}} + \bar{\psi}^{\bar{m}_{\text{reg}}}_{\bar{n}} D_{\bar{n}} \psi_{\text{reg}, \bar{n}} + N_{\text{reg}, \bar{m}_{\text{reg}}} N^{\bar{m}_{\text{reg}}} - \tag{5.16}
\]

\[-i \bar{\phi}_{\text{reg}} \chi_{\mu}^{\text{cl}} \psi_{\text{reg}} - e^{2} \bar{\phi}_{\text{reg}} F_{\mu}^{\text{cl}} \phi_{\text{reg}} +
\]

\[+
\frac{1}{2} ME_{m,n} \sum_{i} \sum_{I,J} a^{(i)}_{I,J} (\bar{\psi}^{I}_{\text{reg}} \psi_{\text{reg}} + N^{I}_{\text{reg}} \phi_{\text{reg}} - \frac{1}{2} MS_{\bar{m}_{\text{reg}}} \sum_{i} (\sum_{I,J} a^{(i)}_{I,J} \bar{\psi}^{I}_{\text{reg}} \psi_{\text{reg}} + 2 N^{I}_{\text{reg}} \phi_{\text{reg}}).
\]

Here \( S_{\bar{m}_{\text{reg}}} \) stands for a non-singular \((0, 2)\) form on the manifold \( M \); \( F_{\mu
u} \) is the strength of external gauge fields and the covariant derivatives are assumed to be taken in external gauge fields; the indices \( I \) and \( J \) stand for copies of \( R_i \) if the repetition of \( R_i \) in \( R \) is more than 1. We assume that the fields which do not
have indices $I$ and $J$ (the first two lines in Eq. (5.16)) correspond to the total representation $R$. We also assume a gauge invariant scalar product of the fields in the mass terms. $a_{IJ}^{(i)}$ is an appropriate non-degenerate constant matrix defined above (its choice depends on the representation $R_i$). For example for the adjoint representation of the matter field it can be taken to be unity; if we consider a model with the $SU(2)$ gauge group and the matter multiplet given by four copies of the spinor representation of $SU(2)$ then $a_{IJ}^{(i)}$ is to be a non-degenerate antisymmetric $4 \times 4$ matrix. We normalize it by the condition

$$\sum_J a_{IJ}^{(i)} a_{JK}^{(i)} = \delta_{IK}$$

For simplicity we shall restrict ourselves to the case of the forms $E_{mn}$ and $S_{\bar{m}\bar{n}}$ being non-zero everywhere (this corresponds to the case $\Sigma_1 = \Sigma_2 = T^2$) and $E_{12}S^{12} = -1$. The mass terms of the regulator fields are gauge invariant under simultaneous gauge transformations of the background and regulator fields. Moreover they are invariant under the BRST transformations defined in the same manner as for the physical fields.

We define the regularized operator

$$\langle \bar{\psi}_{12}^I \phi^J \rangle_{reg} = \bar{\psi}_{12}^I \phi^J - \bar{\psi}_{12}^{reg,I} \phi_{reg,I}.$$  \hspace{1cm} (5.17)

Due to the presence of regulator fields, we now have

$$\{Q, \langle \bar{\psi}_{12}^I \phi^J \rangle_{reg} \} = 2N_{12}^{reg,I} \phi_{reg,I} = ME_{12} \sum_J a_{IJ}^{(i)} \phi^I \phi^J.$$  \hspace{1cm} (5.18)

By integrating over the regulator fields we get

$$\{Q, \langle \bar{\psi}_{12}^I \phi^J \rangle_{reg} \} = -\frac{1}{4\pi^2} \text{Tr}_{R_i} \chi_1 \chi_2,$$  \hspace{1cm} (5.19)

where $\text{Tr}_{R_i}$ stands for the trace in the representation $R_i$. The operator $\text{Tr}_{\chi_1 \chi_2}$ coincides with the operator $O^{(0)}_{12}$ defined in eq.(2.24) and is $Q$-closed. However Eq. (5.19) shows that it is $Q$-exact. Hence the physical correlators in the massless theory with insertions of the operator $O^{(0)}_{12}$ vanish.

Let us now consider the theory with massive matter fields. The mass terms for the physical fields have the following form (we assume that $H^{2,0}(M) \neq 0$; see Eq. (2.22) with a quadratic superpotential)

$$\frac{1}{2} m E_{mn} \sum_i \sum_J a_{IJ}^{(i)} (\psi^I, m_i \psi^J, n_i + 2 N_{mn}^{I, mn} \bar{\psi}^J + 2 N_{mn}^{I, \bar{m} \bar{n}} \bar{\phi}^J).$$  \hspace{1cm} (5.20)

Here $E_{mn} \in H^{2,0}$, $S_{\bar{m}\bar{n}}$ stands for a non-singular $(0,2)$ form on $M$; $m$ is a mass parameter, $a_{IJ}^{(i)}$ is an above defined constant matrix; the indices $I$ and $J$ stand for copies of $R_i$ as long as the repetition of $R_i$ in $R$ is more than 1.
In the massive theory when the matter fields have non-trivial mass terms instead of Eq. (5.19) we get the twisted formulation of the Konishi anomaly

\[ \frac{m}{2} E_{12} \sum J^{(i)} a_J^{(i)} \phi^J = \frac{1}{4\pi^2} \text{Tr} R_i \chi_1 \chi_2. \]  

(5.21)

In a particular case of matter in the adjoint representation we have

\[ \frac{m}{2} E_{12} \text{Tr} \text{Ad} \phi = \frac{1}{4\pi^2} \text{Tr} \text{Ad} \chi_1 \chi_2. \]  

(5.22)

Hence in the massive theory the operator \( O_{12}^{(0)} \) is not \( Q \)-exact.

The Konishi anomaly has the following consequence. Let us consider a correlator of \( N_\phi \) local operators \(< \phi \phi >\) (\(< \ldots >\) stands for a gauge invariant scalar product in the representation \( R \)) and \( N_\chi \) local operators \( \text{Tr} \chi_1 \chi_2 \). Such a correlator is a holomorphic form on \( M \). Due to Eq. (5.22) the tensor structure of the correlator is factorized out. As a consequence it reduces to a correlator of \( N_\phi + N_\chi \) operators \(< \phi \phi >\). In the case of a massless theory the correlator with insertions of the operator \( O_{12}^{(0)} \) vanishes due to a triviality of \( O_{12}^{(0)} \) in \( Q \)-cohomology. On the other hand it vanishes due to the Lorentz invariance of the theory. To be more precise one can consider an infinitesimal holomorphic transformation of complex coordinates \( z_1 \) and \( z_2 \) on the manifold \( M \). It is easy to check that the variation of the action is \( Q \)-exact under such a transformation since it reduces to a variation of the external metric. Hence the correlator does not change. However the operators that enter the correlator modify their coordinate dependence. Therefore the correlator should be a form on \( M \) which is invariant under such a holomorphic transformation. Obviously the only non-trivial invariant form is a constant scalar. We thus maintain the above conclusion of triviality of a correlator with insertions of \( O_{12}^{(0)} \). In the case of massive matter fields the variation of the action under holomorphic transformations of coordinates is \( Q \)-closed but not \( Q \)-exact. Therefore the correlator can be non-vanishing. The same holds true for the correlators with insertions of the operator (2.29) constructed out of the fields of matter.

Let us now consider once again a correlator of \( N_\phi \) local operators \(< \phi \phi >\) and \( N_\chi \) local operators \( \chi_1 \chi_2 \) in the massless theory. In the weak coupling limit it can be calculated by a substitution of the zero modes of the fields \( \psi_n \) and \( \chi_n \) in an instanton field. It is tempting to interpret the vanishing of this correlator as a manifestation of a lack of fields in the preexponential factor which could absorb these zero modes. Since the number of zero modes of the fields \( \psi_n \) and \( \chi_n \) is fixed by the index theorem one can extract a relation between the Euler characteristic and the signature of the manifold \( M \). Unfortunately this argument is rather speculative because the vanishing of the correlator in the ultraviolet limit can be due to an integration over the moduli space of the anti-instanton.

Let us now restrict ourselves to the case of the four-dimensional manifold \( M = \Sigma_1 \times \Sigma_2 \), where \( \Sigma_{1,2} \) are two-dimensional Riemann surfaces. We shall assume the external metric to be block-diagonal (see Eq. (2.50)). One can construct \( U(1) \) currents which are in the BRST cohomology at the classical level. Let us consider the
massless theory. If we take one \((R_i)\) of irreducible representations of matter fields which enter the model one can define for the fields which belong to the \(I\)'s copy of the representation \(R_i\) representation the singlet following current

\[
J^{(I)} = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (D_1 \bar{\phi}^I \cdot \phi^I - \bar{\psi}_{12}^I \psi^{I,2}).
\]  

(5.23)

At the classical level by using the equations of motion one can check that \(\{Q, J^{(I)}\} = 0\) and \(\partial_I J^{(I)} = \{Q, ...\}\). However this current contains a chiral fermionic current in terms of untwisted fields and hence may have an anomaly at the quantum level. We shall show below that this current is indeed anomalous.

To this end it is convenient to use the Pauli-Villars regularization introduced above. We define the regularized current as

\[
J^{(I)} = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (D_1 \bar{\phi}^I \cdot \phi^I - \bar{\psi}_{12}^I \psi^{I,2}) - 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (D_1 \bar{\phi}_{\text{reg}}^I \phi_{\text{reg}}^I - \bar{\psi}_{12}^{I,\text{reg}} \psi^{I,2,\text{reg}}).
\]  

(5.24)

It is easy to check by using the equations of motion for the Lagrangians (2.15) and (5.16) that

\[
\{Q, J^{(I)}\} = 2\pi M \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u E_{12} \sum_J a_{IJ} (\phi_{\text{reg}}^I \psi_{\text{reg}}^{J,2} + \psi_{\text{reg}}^{I,2} \phi_{\text{reg}}^J).
\]  

(5.25)

By using the vertex \(- \sum_I i \phi_{\text{reg}}^I \chi_n \psi_{\text{reg}}^{I,n}\) in the Lagrangian (5.16) one can integrate over the regulator fields and get the anomaly for the current \(J^{(I)}\). The anomaly is obviously proportional to a Dynkin index of the representation \(R_i\). Therefore in order to keep our computation as simple as possible we consider a matter multiplet in the adjoint representation of the gauge group (this matter would correspond to the twisted \(N = 2\) supersymmetric Yang-Mills theory) while the correct answer for the anomaly of the current constructed out of the matter fields in a representation \(R_i\) is given by the anomaly for the matter in the adjoint representation times a factor \(T(R_i)/C_2(G)\).

In the case of matter in an adjoint representation the matrix \(a_{IJ}\) is reduced to 1. In this case \((J^{(I)} \rightarrow J^{Ad})\)

\[
\{Q, J^{Ad}\} = 4\pi M \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u E_{12} \text{Tr}_{Ad} \phi_{\text{reg}}^2 \psi_{\text{reg}}^2.
\]  

(5.26)

By using the propagators of the regulator matter fields in the external gauge field (the propagators for massive fields in the external gauge field are given in Appendix) we get for a current of matter in an irreducible representation \(R_i\)

\[
\{Q, J^{(I)}\} = \frac{T(R_i)}{C_2(G)} \cdot \frac{1}{2\pi} \int_{\Sigma_2} d^2 u \text{Tr}_{Ad} (\chi_2 F_{21} - \chi_1 F_{22}) = \frac{1}{2\pi} \text{Tr}_{R_i} \int_{\Sigma_2} d^2 u (\chi_2 F_{21}^{\text{reg}} - \chi_1 F_{22}^{\text{reg}}).
\]  

(5.27)
This anomaly is proportional to the operator $H_{12,2}^{(1)}$ \textnormal{(see eq.(2.27)) integrated over $\Sigma_2$. It is easy to see that this operator is $Q$-closed. Actually Eq. (5.27) shows that this operator is $Q$-exact. And hence (in the massless theory) the physical correlators with insertions of this operator should vanish.}

Let us now consider an anomaly in $\partial_1 J^{(I)}$. By similar calculations one can show that

$$\partial_1 J^{(I)} = \frac{1}{2\pi} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u H_{12,12}^{(2)}.$$ (5.28)

This result agrees with eq.(2.26) \textnormal{(the operator $H^{(2)}$ is defined in eq.(2.27)). Indeed}

$$\{Q, \partial_1 J^{(I)}\} = \frac{1}{2\pi} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \{Q, H_{12,12}^{(2)}\} = \frac{1}{2\pi} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \partial_1 H_{12,12}^{(1)} = \partial_1 \{Q, J^{(I)}\}. \quad (5.29)$$

\textbf{5.4 Holomorphic currents and stress tensor on $\Sigma_2$}

In this section and in the next one we will consider the massless theory and we focus to the case of four-dimensional manifold $M = \Sigma_1 \times \Sigma_2$.

In the case when the matter multiplet does not belong to the adjoint representation of the gauge group one can still take non-anomalous linear combinations of the currents $J^{R_i}$. We can also define non-singlet \textnormal{(with respect to the flavour group symmetry)} currents. Let the representation of the matter fields be a sum of irreducible representations $R_i$ with a repetition $n_i$. It is easy to see that the non-anomalous flavour group symmetry is

$$G_F = SU(n_1) \times \ldots \times SU(n_f) \times U(1)^{n-1}, \quad (5.30)$$

where $f$ is a number of different irreducible representations that enter the model and $n = \sum_i n_i$. We introduce the multiplet of currents constructed out of the fields which belong to the representation $R_i$ as follows

$$J^{a}_{(i)} = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u (T^{(i)})^{a}_{IJ}(D_1 \bar{\phi}^I \cdot \phi^J - g^{22} \bar{\psi}_{12}^I \psi_{12}^J), \quad (5.31)$$

where $(T^{(i)})^{a}_{IJ}$ stands for a generator of the flavour group $SU(n_i)$ for the representation $R_i$ and index $a$ corresponds to an adjoint representation of the $SU(n_i)$ group. It is easy to check that these currents are $Q$ closed at the classical level and non-anomalous at the quantum level. It is easy also to check that they are also holomorphic in the $Q$-cohomology. In the massless theory they together with the non-anomalous singlet ones generate a Kac-Moody algebra that corresponds to a flavour group symmetry of the theory. The defining relations for such a Kac-Moody algebra can be calculated in the weak coupling limit since the action is $Q$-exact. Hence the calculation can be done in a model of free chiral supermultiplet, while the
space of the physical operators is restricted by the $Q$-cohomology in the interacting theory.

By using the techniques of section 3 it is easy to check that the linear combinations of the currents $J^a_{(i)}(z,\bar{z})$ which correspond to the group given by Eq. (5.30) indeed generate the Kac-Moody algebra with the $G_F$ structure constants

$$J^a_{(i)}(z,\bar{z})J^b_{(j)}(0,0) = \frac{k_i\delta^{ab}}{z^2} + \frac{i\epsilon^{abc}}{z}J^c_{(i)}(0,0) + \text{regular terms} + Q - \text{exact terms},$$

where $z$ and $\bar{z}$ are the complex coordinates on $\Sigma_1$ and the level $k$ is given by

$$k_i = -\chi T(R_i)$$

($\chi$ is the Euler characteristic of $\Sigma_2$ and $T(R_i)$ stands for Dynkin index of representation $R_i$). Notice that the level of this Kac-Moody algebra is negative for $\Sigma_2 = S^2$, zero for $\Sigma_2 = T^2$ and positive for higher genera.

We now construct a stress tensor on $\Sigma_2$ which is $Q$-closed and holomorphic in the $Q$-cohomology. Let us introduce the following gauge invariant operator

$$T(z,\bar{z}) = T_g(z,\bar{z}) + T_m(z,\bar{z}),$$

where

$$T_g(z,\bar{z}) = \frac{2\pi}{e^2} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2u [2g^{\bar{2}2}F_{12}F_{12} - i(D_1\bar{\lambda})\chi_1],$$

$$T_m(z,\bar{z}) = 2\pi \int_{\Sigma_2} \sqrt{g^{(2)}} d^2u [-D_1\bar{\phi})(D_1\phi) + g^{\bar{2}2}\bar{\psi}_{12}(D_1\psi_2)].$$

Here $z = z^1$ and $\bar{z} = z^\dagger$ are coordinates on the surface $\Sigma_1$. We imply that the matter fields in Eq. (5.36) belong to the (reducible) representation $R$, i.e. all the present matter fields contribute to Eq. (5.36). At the classical level this operator $T(z,\bar{z})$ obeys

$$\{Q, T\} = 0, \quad \partial_i T = \{Q, ...\}.$$ (5.37)

When the gauge is fixed by adding $L_{\text{fix}}$ (see Eq. (5.2)) we have to modify the stress tensor. Thus instead of Eq. (5.35) we introduce $T^{\text{qu}}_g$

$$T^{\text{qu}}_g(z,\bar{z}) = \frac{2\pi}{e^2} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2u [2g^{\bar{2}2}F_{12}F_{12} - i(D_1\bar{\lambda})\chi_1 - (D_1^{(0)}B)A_1 + (D_1^{(0)}c^+)(D_1c)],$$

and

$$T(z,\bar{z}) = T^{\text{qu}}_g(z,\bar{z}) + T_m(z,\bar{z}).$$ (5.39)

By using the equations of motion (5.3) and (5.4) one can check that eqs. (5.37) are still valid if we use the total BRST charge $Q' = Q + Q_{\text{BRST}}$ (see Eq.(5.1)) instead of $Q$.

One may therefore expect that the operator algebra generated by $T$ is holomorphic in the $Q$-cohomology. Actually we should still check that there is no anomaly
at the quantum level which could spoil the Q-closeness and holomorphicity of $T$. Such an anomaly could appear due to the presence of the gauge interactions. By an explicit calculation we shall demonstrate (at the one loop level) that the “anomalous” contributions coming from $T_g$ and $T_m$ cancel each other. To this end we have to introduce a gauge invariant ultraviolet regularization. For simplicity we shall consider the case of $H^{2,0} \neq 0$. In this case there exists a non-trivial $(2,0)$ holomorphic form on $M$. As a consequence it allows us to introduce the Pauli-Villars ultra-violet regularization. At the one loop level in the background gauge such a regularization is sufficient to maintain the gauge invariance of the theory (for similar anomaly calculations see, e.g. [18]).

In our one-loop approximation we shall calculate the matrix elements of the operators $\{Q, T\}$ and $\partial_i T$ in external gauge ($A_{\mu}^{cl}$) and gluino ($\chi_{\lambda}^{cl}$) fields.

In the quadratic approximation the Lagrangian for physical quantum fields read

$$L = \sqrt{-g} \frac{1}{\epsilon^2} \text{Tr}[-2A^n(D^m D_m \delta^k_k + i F^k_n - i F_p \delta^k_n)A_k + i A_m F^{mn} A_n + i A^m F_{mn} A^n + (5.40)]$$

$$+i \bar{\lambda}_m D_m \chi_n + \bar{\lambda}_m A_m \chi_n + i \bar{\lambda} D^m \chi_n + \bar{\lambda} A^m \chi_n - c^+ (D^m D_m + D_m D^m) c - c^+ D^{(0)}_m \chi^m] +$$

$$+\sqrt{-g} \left(\bar{\phi} D^m D_m \phi + \bar{\psi} m \bar{m} D_m \bar{\psi} n + \psi D^m \psi m + N_{mn} \bar{N}^m_{\bar{m}} - i \bar{\phi} \chi_n \psi^m - e^2 \phi F^m_{mn} \phi \right).$$

Here $F_{mn}$, $F^{mn}$ and $F_{mn}$ are the components of the strength of an external gauge field $A_{\mu}^{cl}$, and the covariant derivatives are taken in the external gauge field.

The Lagrangian for the regulator fields read (a subscript $\text{reg}$ stands in order to label the regulator fields)

$$L_{\text{reg}} = L_{\text{reg}}^g + L_{\text{reg}}^m,$$

where

$$L_{\text{reg}}^g = \sqrt{g} \frac{1}{\epsilon^2} \text{Tr}[-2A^n_{\text{reg}}(D^m D_m \delta^k_k + i F^k_n - i F_p \delta^k_n)A_k + i A_m^{\text{reg}} F^{mn} A_n + i A^m_{\text{reg}} F_{mn} A^n_{\text{reg}} + (5.42)]$$

$$+i \bar{\lambda}_{\text{reg}}^m D_m \chi_n + \bar{\lambda}_{\text{reg}}^m A_m \chi_n + i \bar{\lambda}_{\text{reg}} D^m \chi_n + \bar{\lambda}_{\text{reg}} A^m \chi_n - c^+_{\text{reg}} (D^m D_m + D_m D^m) c_{\text{reg}} - c^+ D^{(0)}_m \chi^m] +$$

$$+\sqrt{g} \frac{1}{\epsilon^2} \text{Tr} \left(\frac{M}{2} E_{mn} \bar{\lambda} \lambda_{\text{reg}} + \frac{M}{2} S_{mn} \chi \chi_{\text{reg}} + 2M^2 \bar{A}^n A_n + 2M^2 c^+_{\text{reg}} c_{\text{reg}} \right)$$

and $L_{\text{reg}}^m$ is given by Eq. (5.16). For simplicity we shall restrict ourselves to the case of the forms $E_{mn}$ and $S_{mn}$ being non-zero everywhere (this corresponds to the case $\Sigma_1 = \Sigma_2 = T^2$) and $E_{12} S^{12} = -1$. These mass terms are gauge invariant under gauge transformations of the background and regulator fields. However the mass terms for the regulator fields of the gauge multiplet are not BRST invariant (the regulator fields are assumed to transform in the same manner as the physical ones under BRST transformations). We shall see that despite of this fact no anomaly appears in the relations (5.33) for the stress tensor $T$.

The regularized stress tensor $T_{\text{reg}}$ is defined as a difference of the stress tensor of the physical fields and that of the regulator ones as given in eqs. (5.34), (5.35),
contributions to exactly that which is given by Eq. (5.44) with an opposite sign. Thus the anomalous quantum level.

We have cohomology at the quantum level due to a cancellation of anomalous contributions of matter fields. Therefore the operator $T$ the contribution of the gauge sector comes with an opposite sign and cancels the cohomology at the quantum level. By using the propagators given in Appendix and by expanding the expression \( \{Q', T_{reg}\} \) can be calculated similarly to the case of an anomaly in the current

\[
\{Q', T_{reg}\} = -2\pi M \sum_{IJ} a_{I,J} \int_{\Sigma_2} \sqrt{g(2)} d^2 u \ E_{12} \partial_1 (\psi_{reg}^I \phi_{reg}^J) + \tag{5.43}
\]

\[
+ \frac{2\pi}{e^2} \int_{\Sigma_2} \sqrt{g(2)} d^2 u \text{Tr}[-2i ME_{12}(D_1 A_{reg}^2 - D^2 A_{reg}, 1) \bar{\lambda}_{reg} + 2M^2 A_{reg}, 1 \chi_{reg}, 1].
\]

The contribution from the matter sector (the first term on the right hand side of Eq. (5.43)) can be calculated similarly to the case of an anomaly in the current $J^{(I)}$. Integrating over the regulator fields we get for this contribution the following expression

\[
- \frac{1}{4\pi} \partial_1 \int_{\Sigma_2} d^2 u \text{Tr}_R (\chi_2 F_{21} - \chi_1 F_{22}),
\]

where we have omitted the subscript $cl$ for external (low energy) fields.

Let us consider now the contribution from the gauge sector, i.e. the terms in the second line of Eq. (5.43). By using appropriate vertices in Eq. (5.42) we have for that contribution

\[
\frac{1}{e^2} < 4\pi i M \int_{\Sigma_2} \sqrt{g(2)} d^2 u \ E_{12}(D_1 A_{reg}^2 - D^2 A_{reg}, 1) \bar{\lambda}_{reg}(z, \bar{z}, u, \bar{u}) \tag{5.45}
\]

\[
\int d^4 y \text{Tr}_{\chi_{reg}^{12}} [A_{reg}, 1 \chi_{-2}^c - A_{reg}, 2 \chi_{1}^c](y) - 4\pi M^2 \int_{\Sigma_2} \sqrt{g(2)} d^2 u \ A_{reg}, 1 \chi_{1}^c \int d^4 y \bar{\lambda}_{reg} [A_{reg}, 1 \chi_{1}^c + A_{reg}, 2 \chi_{2}^c](y) > .
\]

By using the propagators given in Appendix and by expanding the expression Eq. (5.45) in powers of the external gauge field and gaugino $\chi_\alpha$, it is straightforward to calculate the contribution of the gauge sector into $\{Q', T_{reg}\}$. It turns out to be exactly that which is given by Eq. (5.44) with an opposite sign. Thus the anomalous contributions to $\{Q', T_{reg}\}$ cancel and the operator $T$ is BRST closed at the quantum level.

In a similar way we can check that this operator is holomorphic in the $Q'$-cohomology at the quantum level due to a cancellation of anomalous contributions from the matter and gauge sectors. We have

\[
\partial_1 T_{reg} = \tag{5.46}
\]

\[
= -\frac{2\pi}{e^2} \int_{\Sigma_2} \sqrt{g(2)} d^2 u \text{Tr}[-ME^{12} \bar{\lambda}_{reg} D_1 \bar{\lambda}_{reg}^{12} - iM^2 \bar{\lambda}_{reg}^{12} \chi_1^{reg} - 2M^2 A_{reg}^{12} (D^n A_{reg}^{12} - D_1 A_{reg}^{12})] + \]

\[
+ 2\pi \partial_1 \int_{\Sigma_2} \sqrt{g(2)} d^2 u \left[ M^2 \sum_I \phi_{reg}^I \partial_1 \phi_{reg}^I - ME^{12} \sum_{I,J} a_{I,J} \psi_{reg}^I \psi_{reg}^J \right] + \{Q, \ldots\}.
\]

We actually can calculate the individual contributions from the sectors of matter fields and of the gauge multiplet in an easier way by observing that the operator in Eq. (5.44) is just the operator $H_{12,2}^{(1)}$ integrated over $\Sigma_2$. Hence by using eq.(2.26) we get, e.g. that the individual contribution of the matter sector is given by $H_{12,12}^{(2)}/(2\pi)$. The contribution of the gauge sector comes with an opposite sign and cancels the contribution of matter fields. Therefore the operator $T$ is holomorphic in the $Q'$-cohomology at the quantum level.
5.5 Virasoro algebra on $\Sigma_2$

We can now consider an operator algebra generated by the operator $T$. Since the action of the theory is BRST exact (we consider the massless theory) the operator algebra generated by $T$ can be determined in the weak coupling limit. In this limit the operator $T$ is a sum of non-interacting stress tensors for the matter $T_m$ and for the gauge multiplet $T^q_g$ (we consider the usual $\alpha$ gauge fixing without background gauge field). The gauge multiplet is effectively reduced to a superposition of $\dim G$ abelian gauge supermultiplets and we take into account only a quadratic part of $T^q_g$

\[ T^q_g = \frac{2\pi}{e^2} \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \text{Tr} [2(D_1A_2 - D_2A_1)(D_1A_2 - D_2A_1) - \] (5.47)

\[ -i(D_1\bar{\lambda})\chi_1 - A_1D_1(D^nA_n + D_nA^n) + D_1c^+D_1c, \]

where $D_n$ and $D^n$ stand for covariant derivatives in the external gravitational field; we also used in Eq. (5.47) an expression for the field $B$ in terms of the gauge potential that results from the equations of motion.

For the operator $T_m$ it is easy to extract from section 3 that its contribution to the OPE is given by

\[ T_m(z, \bar{z})T_m(w, \bar{w}) = \frac{\chi \dim R}{2(z - w)^4} + \frac{2T_m}{(z - w)^2} + \frac{\partial_z T_m}{z - w} + \text{regular terms} + \{Q, \ldots\}, \] (5.48)

where $z$ and $w$ are complex coordinates on $\Sigma_1$ and $\chi$ is the Euler characteristic of $\Sigma_2$.

Similarly with the calculation of OPE given in section 3 one can find the OPE for the operator $T^q_g$. It is easy to check that the contributions to the central term $T^q_g(z, \bar{z})T^q_g(w, \bar{w})$ of the two last terms in Eq. (5.47) cancel each other. Thus for the central term in the OPE we essentially have

\[ < T^q_g(z, \bar{z})T^q_g(w, \bar{w}) > = 4\pi^2 \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 u \int_{\Sigma_2} \sqrt{g^{(2)}} d^2 v \] (5.49)

\[ [4 < (D_1A_2(z, u))(D_1A_2(w, v)) > \cdot < (D_1A_2(w, v))(D_1A_2(z, u)) > + \]

\[ + \cdot < D_1\bar{\lambda}(z, u)\chi_1(w, v) > \cdot < D_1\bar{\lambda}(w, v)\chi_1(z, u) > ], \]

where $z, w$ and $u, v$ are the complex coordinates on the $\Sigma_1$ and $\Sigma_2$ surfaces respectively. By using the propagators given in Appendix (with $M = 0$ and without an external gauge field) and the techniques of section 3 we get for the central term in OPE

\[ < T^q_g(z, \bar{z})T^q_g(w, \bar{w}) > = -\frac{\chi \dim G}{2(z - w)^4}. \] (5.50)

The sign “-” at the central term appears here in contrast to the contribution of the matter sector because the vector field ($A_2$) on $\Sigma_2$ has an opposite statistics as
compared to the field $\psi_2$. Similarly one can calculate the less singular terms in the OPE. Thus we get
\[ T(z, \bar{z})T(w, \bar{w}) = \frac{c}{2(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial\bar{z}T}{z-w} + \text{regular terms} + \{Q,...\}, \quad (5.51) \]

where
\[ c = \chi(\text{dim } R - \text{dim } G). \quad (5.52) \]

Thus we see that the operator $T$ generates a Virasoro algebra in the $Q$-cohomology with the central charge $c$ which coincides with that which we can expect from the expression (2.39) for the gravitational anomaly.

The currents $J_{(i)}^a$ which generate the Kac-Moody algebra are the primary fields with respect to this stress tensor and have a conformal dimension 1
\[ T(z, \bar{z})J_{(i)}^a(w, \bar{w}) = \frac{1}{(z-w)^2} J_{(i)}^a(w, \bar{w}) + \frac{1}{z-w} \partial_1 J_{(i)}^a(w, \bar{w}) + \text{regular terms} + \{Q,...\}. \quad (5.53) \]

We again observe that a different Virasoro algebra on $\Sigma_2$ can be also constructed. Its central charge is equal to the Euler characteristic of the surface $\Sigma_1$. Hence we have a direct sum of two Virasoro algebras.

### 5.6 Instanton effects

In four dimensional supersymmetric gauge theories without a superpotential for the matter supermultiplet the vacuum state is degenerate due to a valley in the potential. This degeneracy is not lifted in perturbation theory due to the non-renormalizability theorem [50]. However instantons can induce an effective superpotential and lift a degeneracy of the vacuum [52, 53, 56].

In the heterotic topological gauge theory instantons do not induce any superpotential just because it is forbidden by the Lorentz invariance of the theory. Therefore this theory has a moduli space of ground states. Actually the absence of an effective superpotential in this case is a counterpart to the analysis of an impossible structure of an effective superpotential in untwisted supersymmetric theories [52, 53, 56, 55, 53] where a superpotential does not appear for such a particular representation of the matter fields which enter the heterotic topological Yang-Mills theory due to the global phase symmetries of the theory. On a curved manifold the dynamics could differ from that of the theory in a flat space-time. However for the Wilsonian effective action that we shall consider in this section it is natural to assume that the main conclusions drawn for the untwisted supersymmetric theory in flat space-time remains true because the Wilsonian action is determined by ultraviolet contributions [53].

For simplicity let us consider the supersymmetric massless gauge theory with the $SU(2)$ gauge group and four copies of the fundamental representation for the matter fields (the mixed anomaly in the twisted version of this theory is cancelled,
see Appendix). This is just the case when no effective superpotential is induced by instantons and hence the classical vacuum degeneracy is not lifted. Therefore the quantum theory has a moduli space of ground states.

An effective low-energy theory can be described in terms of “mesons” constructed as gauge invariant composite chiral fields $V^{ij} = Q^i Q^j$, where $Q^i$ stands for a “quark” matter chiral superfield; indices $i,j = 1,2,3,4$ correspond to flavours of “quarks”. In the massless theory the flavour group of symmetry is $SU(4)$. At the classical level the Pfaffian of the matrix $V^{ij}$ trivially vanishes by Bose symmetry. However due to instantons it acquires a non-zero vacuum expectation value

$$\text{Pf}V = \Lambda^4,$$

where $\Lambda$ is a scale of the gauge theory. This vacuum expectation value corresponds (in an appropriate basis) to

$$V = \Lambda \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

and breaks the group symmetry $SU(4) \to Sp(4)$. The low energy fields are the fluctuations of $V$ around the expectation value Eq. (5.54) subject to constraint Eq. (5.54). Thus the low energy fields belong to $SU(4)/Sp(4) = SU(3)/SU(2) = S^5$.

We assume that the glueball field $S = \text{Tr} W^2$, where $W$ is the superstrength of the gauge supermultiplet corresponds to heavy excitations. In order to incorporate Eq. (5.54) into a low energy effective theory one may use a Lagrange multiplier field $X$ with the superpotential

$$W_{\text{eff}} = X(\text{Pf} V - \Lambda^4).$$

In the twisted theory a superfield $X$ becomes a $(2,0)$-form while the $F$ term $[W_{\text{eff}}]_F$ becomes a $(2,2)$-form. This $F$ term is $Q$-closed in the twisted theory. For example the glueball superfield $S$ can play the role of the Lagrange multiplier field $X$. In such a case the twisted version of the $F$ term for the effective potential is given by the $Q$-closed operator Eq. (2.30). Let us denote

$$V^{12} = x_1, \quad V^{13} = x_2, \quad V^{14} = x_3, \quad V^{34} = y_1, \quad V^{24} = y_2, \quad V^{23} = y_3,$$

and put $\lambda = 1$. Then the superpotential Eq. (5.56) can be rewritten as follows

$$W_{\text{eff}} = X(\sum_{i=1}^3 x_i y_i - 1).$$

We now want to determine the ground ring of local scalar operators constructed out of the matter chiral fields $x_i$ and $y_i$. By solving the equations of extremum for the superpotential Eq. (5.58) we get $X = 0$ and Eq. (5.54). The ground ring is generated by $x_i$ and $y_i$, $i = 1,2,3$, modulo Eq. (5.54) (this construction is similar to the chiral rings in two-dimensional $N = 2$ supersymmetric theories). Since the Witten index is 2 for the $SU(2)$ supersymmetric gauge theory such a ground
ring should be generated by two elements. Actually the massless limit of the theory that we considered up to now is not well defined in the following sense. In the massless theory we can take any constant matrix conjugate to that of Eq. (5.55) for a vacuum expectation value of the field $V^{ij}$. In turn if we consider a massive theory with small mass parameters $m_{ij}$ \((m_{ij} < \Lambda)\)

$$\sum_{i,j=1}^{4} m_{ij} V^{ij}$$

(5.59)

then the above degeneracy for the vacuum expectation value of $V^{ij}$ is lifted \[64\]. Hence the massless limit depends on the way the masses vanish.

Therefore in order to control the behaviour of the theory we add masses to the fields $Q^i$ and hence for the effective potential we get

$$W_{\text{eff}} = X(\sum_{i=1}^{3} x_i y_i - 1) + \sum_{i=1}^{3} (a_i x_i + b_i y_i),$$

(5.60)

where $a_i$ and $b_i$ are mass parameters. At the extremum of the potential (5.60) we get

$$\sum_i x_i y_i - 1 = 0, \quad X x_i + b_i = 0, \quad X y_i + a_i = 0.$$  

(5.61)

In the twisted version of the theory the expressions given in Eq. (5.61) are $Q$-exact by the equations of motion (we imply that $X, x_i$ and $y_i$ are the lowest components of chiral supermultiplets).

It is easy to see that from Eq. (5.61) follows

$$X^2 = \sum_i a_i b_i.$$  

(5.62)

Let us take for definiteness $a_1 = b_1 = \beta, a_2 = a_3 = b_2 = b_3 = 0$. Then we have

$$x_2 = x_3 = y_2 = y_3 = 0, \quad X = -\beta x_1 = -\beta y_1, \quad X^2 = \beta^2.$$  

(5.63)

Thus the ground ring is generated by the operators $1, X$ modulo $X^2 = \beta^2$. It is easy to see that if we do not take into account the instanton effects then we should put $\Lambda = 0$ in the superpotential Eq. (5.56) and the ground ring is generated by the operator $1, X$ modulo $X^2 = 0$.

The ground ring in this model is quite similar to the one in $N = 2$ supersymmetric $CP^1$ models \[67\]. Moreover \[68, 69, 70, 67\] the effect of instantons in the $CP^1$ supersymmetric sigma model is exactly to change the condition $X^2 = 0$ into $X^2 = \beta^2$. Because in our heterotic topological theory we have embedded conformal theories in $Q$-cohomology we may conjecture that there is a correspondence between instantons and renormalization group in four dimensional supersymmetric theory with this particular choice of representation for the matter fields and those in an appropriate two dimensional $N = 2$ supersymmetric sigma model \[71\]. A half-twisted version of such a sigma model is to have a chiral conformal stress tensor
with the same central charge as the chiral conformal tensor in the twisted 4D theory. In the present particular example the central charge \( c = 5\chi \).

We may also conjecture that there exists a relation between the moduli space of heterotic topological theories and that of two dimensional \( N = 2 \) supersymmetric sigma models.

6 Conclusions

We have shown that in a twisted \( N = 1 \) SUSY model with a single free chiral supermultiplet on the four manifold \( M = \Sigma_1 \times \Sigma_2 \) there exist two chiral infinite dimensional symmetries \( W_{1+\infty} \) in the cohomology of the BRST operator. The generators of such an algebra are integrals over \( \Sigma_2(\Sigma_1) \) of the bilinear composite operators. The central charge of the \( W_{1+\infty} \) algebra is the Euler characteristic \( \chi_2(\chi_1) \) of \( \Sigma_2(\Sigma_1) \).

It is worth noticing that the representation of \( W_{1+\infty} \) given here is very close to a representation of this algebra in terms of 2D free (fermionic or bosonic) fields \([72, 73, 36]\). It is amusing however that in our representation the central charge has a purely geometric origin.

The theory becomes a dynamical one if we introduce a superpotential. We have shown in a particular example of a non-trivial quasihomogeneous superpotential for a single chiral supermultiplet that the two algebras \( W_{1+\infty} \) are reduced to two chiral Virasoro algebras with central charges proportional to \( \chi_2(\chi_1) \). We point out at this point that one can try to extend our construction to a model with any number of chiral supermultiplets \([71]\). In such a model it will be interesting to see if a realization of \( W_N \) and \( W_{p\infty} \) \([72]\) algebras can similarly be obtained directly from a four dimensional quantum field theory.

In the heterotic topological gauge theory (without a superpotential for the matter multiplet) there exist two chiral Virasoro algebras with the central charges proportional to \( \chi_2(\chi_1) \) and a Kac-Moody algebra in the BRST cohomology corresponding to the group of the flavour symmetry of the theory with the level proportional to \( \chi_2(\chi_1) \).

Notice that the cohomology of the BRST operator is larger than the one we discussed in this paper. We postpone a detailed analysis of it for a next publication.

We also point out that the supersymmetric theory can be twisted differently as we discussed in section 2. For such a mirror model we could get an anti-chiral conformal algebras in the corresponding BRST cohomology with the same central charges.

It is important that this (extended) conformal structure is invariant under the renormalization group due to the BRST symmetry. Therefore it contains an important information on the dynamics of the untwisted supersymmetric theory. We demonstrated a similarity of the ground ring structure of the heterotic topological theory to that of twisted \( N = 2 \) supersymmetric sigma models. This fact hints
the existence of a relation between the renormalization group flow and instantons in four dimensional supersymmetric gauge theories and those in $N = 2$ supersymmetric sigma models. An interesting question is also if this conformal structure corresponds to a sort of an infrared fixed point in the untwisted four dimensional supersymmetric gauge theory.

It would also be of some interest to extend such a construction to the heterotic topological gauge theories with a superpotential for the matter multiplet [71].

We thus arrive at an interpretation for the chiral Virasoro algebras in $Q$-cohomology as algebras on a surface $\Sigma_1(\Sigma_2)$ with a classical metric. It is tempting to try to extend such an interpretation allowing the metric on $\Sigma_1(\Sigma_2)$ to be a quantum one. In such a case we could get a two dimensional quantum gravity extracted from the four dimensional induced quantum gravity theory. An important ingredient here would be that the induced gravity action (gravitational anomaly) is non-renormalizable at the multiloop level. We therefore may think that such a realization of the two dimensional quantum gravity in terms of the four dimensional theory on $M = \Sigma_1 \times \Sigma_2$ is self-consistent.

It would also be interesting to extend the above construction of the conformal algebras to the case of a four dimensional Kähler manifold $M$ which is more complicated than a direct product of two Riemann surfaces. One may hope that in such a case it will be also possible to recover two dimensional integrable structures embedded into four dimensional quantum field theory. A different way to reveal integrable structures could be a deformation of a heterotic topological theory which possesses a conformal structure through the introduction of a superpotential for the matter fields [71].

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A Appendix

A.1 Solution to the condition of cancellation of the mixed anomaly

The condition (2.38) of cancellation of the mixed anomaly has the following solutions.

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(A). $SU(m) = A_{m-1}$. Table 1 gives all the dimensions and Dynkin indices of the representations of $SU(m)$ that can occur in heterotic topological theories.

Table 1  

| Representation $R_i$ | Dimension | $T(R_i)$ | Range of $m$ |
|----------------------|-----------|----------|--------------|
| $R_1 = \square$     | $m$       | $1/2$    | $2 \leq m$   |
| $R_2 = \square$     | $m(m-1)/2$| $(m-1)/2$| $4 \leq m$   |
| $R_3 = \square$     | $m(m+1)/2$| $(m+2)/2$| $3 \leq m$   |
| $R_4 = \square$     | $m(m-1)(m-2)/6$| $(m-2)(m-3)/4$| $6 \leq m \leq 8$ |
| $R_5 = (m-1)\square$| $m^2 - 1$ | $m$      | $2 \leq m$   |

The representation $R_5$ stands for the adjoint representation of $SU(m)$. Let the repetitions of representations $R_i$ be $n_i$. The following cases are the solutions of Eq. (2.38).

(A1). $n_5 = 1$, $N - 1 = n_2 = n_3 = n_4 = 0$. In this case the theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(A2). $n_1 = 2m$, $n_2 = n_3 = n_4 = n_5 = 0$. In this case $\dim R - \dim G = 2m^2 - m^2 + 1 = m^2 + 1$.

(A3). $n_1 = m - 2$, $n_2 = 0$, $n_3 = 1$, $n_4 = n_5 = 0$, $m \geq 3$. In this case $\dim R - \dim G = m(m - 3)/2 + 1$.

(A4). $n_2 = p$, $n_1 = 2m - p(m - 2)$, $n_3 = n_4 = n_5 = 0$, $m \geq 4$. In this case $\dim R - \dim G = m^2 + 1 - pm(m - 3)/2 > 0$.

(A5). $n_2 = n_3 = 1$, $n_1 = n_4 = n_5 = 0$, $m \geq 4$. In this case $\dim R - \dim G = 1$.

(A6). $n_4 = 2$, $n_1 = n_2 = n_3 = n_5 = 0$, $m = 6$. In this case $\dim R - \dim G = 5$.

(A7). $n_1 = (9m - m^2 - 6)/2$, $n_4 = 1$, $n_2 = n_3 = n_5 = 0$, $6 \leq m \leq 8$. In this case $\dim R - \dim G = 21$ at $m = 6$, $33$ at $m = 7$ and $50$ at $m = 8$.

(A8). $n_1 = 2$, $n_2 = n_4 = 1$, $n_3 = n_5 = 0$, $m = 6$. In this case $\dim R - \dim G = 12$.

(B) $SO(2m+1) = B_m$. The representations of this gauge group with the Dynkin index less than $C_2(G)$ are shown in table 2.
Table 2
The dimensions and Dynkin index of $SO(2m + 1)$ representations.

| Representation $R_i$ | Dimension $T(R_i)$ | Range of $m$ |
|----------------------|---------------------|--------------|
| $R_1$ fundamental    | $2m + 1$            | $2 \leq m$   |
| $R_2$ spinor         | $2^m$               | $2 \leq m \leq 6$ |
| $R_3$ adjoint        | $m(2m + 1)$         | $2 \leq m$   |

The cases for the solutions of Eq. (2.38) are the following.

(B1). $n_3 = 1, n_1 = n_2 = 0, m \geq 2$. This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(B2). $n_1 = 2m - 1, n_2 = n_3 = 0, m \geq 2$. In this case $\dim R - \dim G = 2m^2 - m - 1 > 0$.

(B3). $n_2 = 6, n_1 = n_3 = 0$ for $SO(5)$. In this case $\dim R - \dim G = 14$.

(B4). $n_2 = 5, n_1 = n_3 = 0$ for $SO(7)$. In this case $\dim R - \dim G = 19$.

(B5). $n_2 = p, n_1 = (2m - 1) - 2^{m-3}p$. There are the following cases.

$SO(5), n_1 = n_2 = 2, \dim R - \dim G = 8$.
$SO(5), n_1 = 1, n_2 = 4, \dim R - \dim G = 11$.
$SO(7), n_1 = 4, n_2 = 1, \dim R - \dim G = 15$.
$SO(7), n_1 = 3, n_2 = 2, \dim R - \dim G = 16$.
$SO(7), n_1 = 2, n_2 = 3, \dim R - \dim G = 17$.
$SO(7), n_1 = 1, n_2 = 4, \dim R - \dim G = 18$.
$SO(9), n_1 = 5, n_2 = 1, \dim R - \dim G = 25$.
$SO(9), n_1 = 3, n_2 = 2, \dim R - \dim G = 23$.
$SO(9), n_1 = 1, n_2 = 3, \dim R - \dim G = 21$.
$SO(11), n_1 = 5, n_2 = 1, \dim R - \dim G = 32$.
$SO(11), n_1 = 1, n_2 = 2, \dim R - \dim G = 20$.
$SO(13), n_1 = 3, n_2 = 1, \dim R - \dim G = 25$.

(C). $Sp(2m) = C_m$. The representations allowed by Eq. (2.38) are given in table 3.
Table 3
The dimensions and Dynkin index of $Sp(2m)$ representations.

| Representation $R_i$ | Dimension | $T(R_i)$ | Range of $m$ |
|----------------------|-----------|----------|--------------|
| $R_1$                | $2m$      | 1        | $3 \leq m$   |
| $R_2$                | $m(2m-1)-1$ | $2m-2$  | $3 \leq m$   |
| $R_3$ adjoint        | $m(2m+1)$ | $2m+2$   | $3 \leq m$   |

The cases for the solutions of Eq. (2.38) are the following.

(C1). $n_3 = 1, n_1 = n_2 = 0$. This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(C2). $n_1 = 2m + 2, n_2 = n_3 = 0$. In this case $\dim R - \dim G = 2m^2 + 3m$.

(C3). $n_1 = 4, n_2 = 1, n_3 = 0$. In this case $\dim R - \dim G = 6m - 1$.

(D). $SO(2m) = D_m$. The representations that have the Dynkin index $T(R_i)$ less than or equal to $(2m - 2)$ are given in table 4. As in the case of $SO(2m + 1)$ group only the fundamental, spinor and the adjoint representations are allowed. Since $SO(4)$ and $SO(6)$ are isomorphic to $SU(2) \times SU(2)$ and $SU(4)$, only $m \geq 4$ are relevant.

Table 4
The dimensions and Dynkin index of $SO(2m)$ representations.

| Representation $R_i$ | Dimension | $T(R_i)$ | Range of $m$ |
|----------------------|-----------|----------|--------------|
| $R_1$ fundamental    | $2m$      | 1        | $4 \leq m$   |
| $R_2$ spinor         | $2^{m-1}$ | $2^{m-4}$| $5 \leq m \leq 7$ |
| $R_3$ adjoint        | $m(2m-1)$ | $2m-2$   | $4 \leq m$   |

The cases for the solutions of Eq. (2.38) are the following.

(D1). $n_3 = 1, n_1 = n_2 = 0$. This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(D2). $n_1 = 2m - 2, n_2 = n_3 = 0$. In this case $\dim R - \dim G = 2m^2 - 3m > 0$.

(D3). $n_2 = 4, n_1 = n_3 = 0$ for $SO(10)$. In this case $\dim R - \dim G = 19$.

(D4). $n_1 = (2m - 2) - p2^{m-4}, n_2 = p, n_3 = 0$. In this case there are the following solutions to Eq. (2.38).

$SO(10), n_1 = 6, n_2 = 1, \dim R - \dim G = 31$.
$SO(10), n_1 = 4, n_2 = 2, \dim R - \dim G = 27$.
$SO(10), n_1 = 2, n_2 = 3, \dim R - \dim G = 23$. 47
$SO(12)$, $n_1 = 6$, $n_2 = 1$, $\dim R - \dim G = 38$.
$SO(12)$, $n_1 = 2$, $n_2 = 2$, $\dim R - \dim G = 22$.
$SO(14)$, $n_1 = 4$, $n_2 = 1$, $\dim R - \dim G = 29$.

$E_6$. The solutions of Eq. (2.38) are

(1). $R$ is the adjoint representation ($78$, $C_2(E_6) = 12$). This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(2). $R$ is four copies of the fundamental representation ($27$) with Dynkin index 3. In this case $\dim R - \dim G = 30$.

$E_7$. The solutions of Eq. (2.38) are

(1). $R$ is the adjoint representation ($133$, $C_2(E_7) = 18$). This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(2). $R$ consists of three copies of the fundamental representation ($56$) with Dynkin index 6. In this case $\dim R - \dim G = 35$.

$E_8$. The only solution is the adjoint representation $R = Ad$. This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

$F_4$. The solutions to Eq. (2.38) are

(1). $R$ is the adjoint representation ($52$, $C_2(F_4) = 9$). This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(2). $R$ consists of three copies of the fundamental representation ($26$) with Dynkin index 3. In this case $\dim R - \dim G = 26$.

$G_2$. The solutions to Eq. (2.38) are

(1). $R$ is the adjoint representation ($14$, $C_2(G_2) = 112$). This theory coincides with Witten’s topological Yang-Mills theory and $\dim R - \dim G = 0$.

(2). $R$ consists of 4 copies of fundamental representation ($7$). In this case $\dim R - \dim G = 14$.

A.2 Propagators

The propagators of the massive fields in the external gauge fields read

\begin{equation}
< \phi(x) \bar{\phi}(y) > = - \frac{1}{\sqrt{g(y)}} \frac{1}{M^2 - \Delta_{00}} \delta^4(x - y), \tag{A.1}
\end{equation}

\begin{equation}
< \psi_n(x) \bar{\psi}_{12}(y) > = \frac{1}{\sqrt{g(y)}} \left[ \left( \frac{1}{M^2 - \Delta_{01}} \right)^{n_1} D_2 - \left( \frac{1}{M^2 - \Delta_{01}} \right)^{n_2} D_1 \right] \delta^4(x - y), \tag{A.2}
\end{equation}

\begin{equation}
< \bar{\psi}(x) \bar{\psi}_{12}(y) > = \frac{1}{\sqrt{g(y)}} ME_{12} \frac{1}{M^2 - \Delta_{00}} \delta^4(x - y), \tag{A.3}
\end{equation}
\[
< \psi_n(x) \psi_m(y) > = \frac{1}{\sqrt{g(y)}} \left[ \frac{M}{2} \left( \frac{1}{M^2 - \Delta_{0,1}} \right) \right] S^p_m \left[ \frac{M}{2} \frac{1}{M^2 - \Delta_{1,0}} \right]_{kn} \delta^4(x - y),
\]
(A.4)

\[
< \bar{\psi}(x) \psi_n(y) > = -\frac{1}{\sqrt{g(y)}} \frac{1}{M^2 - \Delta_{00}} D^a \delta^4(x - y),
\]
(A.5)

\[
< N_{12}(x) \bar{N}^{12}(y) > = \frac{1}{2} \delta^4(x - y),
\]
(A.6)

where
\[
\Delta_{00} = D^m D_m, \quad (\Delta_{01})_{m}^n = D^m D^n \delta^0_k - [D^n, D_k],
\]
(A.7)

\[
D_n, D^n \text{ are the covariant derivatives in the external gauge and gravitational fields, and } (1/\Delta_{01})_{nn} \text{ stand for the } (\bar{nn}) \text{ components of an operator } 1/\Delta_{01} \text{ respectively.}
\]

The propagators of free massless fields read as follows

\[
< \phi(x) \bar{\phi}(y) > = \frac{1}{\sqrt{g(y)}} \frac{1}{\Delta_{00}} \delta^4(x - y),
\]
(A.8)

\[
< \psi_1(x) \bar{\psi}_{12}(y) > = -\frac{1}{\sqrt{g(y)}} \left( \frac{1}{\Delta_{01}} \right)_{11} D_2 \delta^4(x - y),
\]

\[
< \psi_2(x) \bar{\psi}_{12}(y) > = \frac{1}{\sqrt{g(y)}} \left( \frac{1}{\Delta_{01}} \right)_{22} D_1 \delta^4(x - y),
\]
where \(\Delta_{00}\) and \(\Delta_{01}\) stand for the Laplace operators acting on \((0,0)\) and \((0,1)\) forms on \(M\) respectively.

Let us consider the gauge supermultiplet. For the gauge field we have

\[
< A_\mu(x) A_\nu(y) > = \frac{e^2}{2 \sqrt{g(y)}} \left( \frac{1}{M^2 - \Delta_{0,1}} \right)_{\mu \nu} \delta^4(x - y),
\]
(A.9)

where \(\tilde{\Delta}_{0,1}\) stands for the operator in the quadratic form in Eq. (5.40) for the gauge field. In a particular case of the external anti-instanton field for which \(F_{mn} = F^{mn} = F_p = 0\) we have

\[
< A_{\bar{n}}(x) A_n(y) > = \frac{e^2}{2 \sqrt{g(y)}} \left( \frac{1}{M^2 - \Delta_{0,1}} \right)_{\bar{n} n} \delta^4(x - y)
\]
(A.10)

while \(< A_{\bar{n}}(x) A_n(y) >= < A_m(x) A_n(y) >= 0\).

For the fermionic fields we have

\[
< \chi_n(x) \bar{\chi}(y) >= \frac{e^2}{\sqrt{g(y)}} i D_n \frac{1}{M^2 - \Delta_{00}} \delta^4(x - y),
\]
(A.11)
\[<\chi_n(x)\bar{\lambda}_{12}(y)> = \frac{e^2}{\sqrt{g(y)}} i \left[ \left( \frac{1}{M^2 - \Delta_{1,0}} \right)_{n_2} D_1 - \left( \frac{1}{M^2 - \Delta_{1,0}} \right)_{n_1} D_2 \right] \delta^4(x - y), \]  
(A.12)

\[<\chi_n(x)\chi_m(y)> = \frac{e^2}{\sqrt{g(y)}} \left[ \frac{M}{2} \left( \frac{1}{M^2 - \Delta_{1,0}} \right)_{\tilde{n}_\rho} E_{\tilde{n}}^\rho - \frac{M}{2} E_{\tilde{n}}^\rho \left( \frac{1}{M^2 - \Delta_{0,1}} \right)_{\tilde{m}_\rho} \right] \delta^4(x - y), \]  
(A.13)

\[<\bar{\lambda}(x)\bar{\lambda}_{12}(y)> = \frac{1}{\sqrt{g(y)}} MS_{12}(x) \frac{1}{M^2 - \Delta_{00}} \delta^4(x - y), \]  
(A.14)

\[<c(x)c^+(y)> = \frac{e^2}{\sqrt{g(y)}} \frac{1}{2M^2 - D^m D_m - D_m D^m} \delta^4(x - y). \]  
(A.15)

References

[1] E. Martinec, ”Criticality, Catastrophes, and Compactifications”, in Physics And Mathematics Of Strings, ed. L. Brink, D. Friedan, and A.M. Polyakov (World Scientific, 1990).

[2] C. Vafa and N. Warner, ‘Catastrophes And The Classification Of Conformal Field Theories, Phys. Lett. 218 B (1989) 51.

[3] W. Lerche, C. Vafa, and N. Warner, Chiral Rings In N = 2 Superconformal Theory, Nucl. Phys. B 324 (1989) 427.

[4] P. S. Howe and P. C. West, Chiral Correlators In Landau-Ginsburg Models And N = 2 Superconformal Models, Phys. Lett. 227 B (1989) 397; Fixed Points In Multi-Field Landau-Ginsburg Models, Phys. Lett. 244 B (1990) 270.

[5] S. Cecotti, L. Girardello, and A. Pasquinucci, Non-perturbative Aspects And Exact Results For The N = 2 Landau-Ginsburg Models, Nucl. Phys. B 328 (1989) 701; Singularity Theory And N = 2 Supersymmetry, Int. J. Mod. Phys. A 6 (1991) 2427.

[6] S. Cecotti, N = 2 Landau-Ginsburg vs. Calabi-Yau Sigma Models, Int. J. Mod. Phys. A6 (1991) 1749; Nucl. Phys. B 355 (1991) 755.

[7] E. Witten, On the Landau-Ginzburg description of N = 2 minimal models. Preprint IASSNS-HEP-93/10.

[8] E. Silverstein and E. Witten. “Global U(1) R-symmetry And Conformal Invariance of (0, 2) Models”, Phys. Lett. B 328 (1994) 307.

[9] K. Mohri, N = 2 super W algebra in half-twisted Landau-Ginzburg model. UTHEP-260, hep-th/9307029.
[10] S.Mukhi and C.Vafa, “Two Dimensional Black-hole as a Topological Coset Model of $c = 1$ String Theory”, Nucl. Phys. B 407 (1993) 667.

[11] N.Berkovits and C.Vafa, On the Uniqueness of String Theory. Mod. Phys. Lett. A9 (1994) 653; hep-th/9310170.
[12] D.Ghoshal and S.Mukhi, Preprint MRI-PHY/13/93, TIFR/TH/93-62; hep-th/9312189.

N.Ohta and J.L.Petersen, N = 1 from N = 2 Superstrings, Phys. Lett. 325 B (1994) 67.
F.Bastianelli, N.Ohta, and J.L.Petersen, Toward The Universal Theory Of Strings, Phys. Lett. 327 B (1994) 35; A Hierarchy of Superstrings. NBI-HE-94-20; hep-th/9403150.

[13] E.Witten, Some Exact Multi-instanton Solutions Of The Classical Yang-Mills Theory, Phys. Rev. Lett. 38 (1977) 121; A.N.Leznov and M.V.Saveliev, Representation Theory And Integration Of Non-linear Spherically Symmetric Equations To Gauge Theories, Comm. Math. Phys. 74 (1980) 111.

[14] R.S.Ward. Phil. Trans. R. Soc. Lond. A 315 (1985) 451.

[15] R.S.Ward, Phys. Lett. 61 A (1977) 81; in Field Theory, Quantum Gravity and Strings, eds. H.J.de Vega and N.Sanchez, Springer Lecture Notes in Physics Vol. 246 (1986).

[16] L.Mason and G.Sparling. Phys. Lett. A 137 (1989) 29; J. Geom. Phys. 8 (1992) 243.

[17] I.Bakas and D.A.Depireux, Mod. Phys. Lett. A6 (1991) 399; Int. J. Mod. Phys. A7 (1992) 1767; Mod. Phys. Lett. A6 (1991) 1561.

[18] M.J.Ablowitz, S.Chakravarty, and L.A.Takhtajian, A Self-Dual Yang-Mills Hierarchy and Its Reductions to Integrable Systems in 1+1 and 2+1 Dimensions. Colorado preprint, August 1992.

[19] Q-Han Park. Phys. Lett. 236 B (1990) 429.

[20] J.Schiff, Self-dual Yang-Mills and the hamiltonian structures of integrable systems. IASSNS-HEP-92/34, hep-th/9211070.

[21] F.Guil and M.Manas, Two-Dimensional Integrable Systems and Self-Dual Yang-Mills Equations. PRINT-93-0539 (Madrid).

[22] C.Castro, W Gravity, N = 2 Strings and 2 + 2 SU*(∞) Yang-Mills Instantons. J. Math. Phys. 35 (1994) 3013, hep-th/9308102; The KP Equations From Plebanski and SU(∞) Self-Dual Yang-Mills. IAEC-7-93-REV.

[23] H.Nishino, Selfdual Supersymmetric Yang-Mills Theory Generates Witten’s Topological Field Theory, Phys. Lett. 309 B (1993) 68.
[24] H.Ooguri and C.Vafa, *Selfduality And N = 2 String Magic*, Mod. Phys. Lett. A5 (1990) 1389; *Geometry Of N = 2 Strings*, Nucl. Phys. B361 (1991) 469; *N = 2 Heterotic Strings*, ibid. 367 (1991) 83.

[25] H.Nishino and S.J.Gates, Jr., *N = (2, 0) Superstrings As The Underlying Theory Of Selfdual Yang-Mills Theory*, Mod. Phys. Lett. A7 (1992) 2543; *Soluble Supersymmetric Systems Embedded In Supersymmetric Selfdual Yang-Mills Theory*, Phys. Lett. 299 B (1993) 255.

[26] W.Siegel, *N = 2, N = 4 String Theory Is Selfdual N = 4 Yang-Mills Theory*, Phys. Rev. D 46 (1992) 3235.

[27] W.Siegel, *Selfdual N = 8 Supergravity As Closed N = 2 (N = 4) Strings*, Phys. Rev. D 47 (1993) 2504.

[28] E.Witten, *Topological Quantum Field Theory*, Comm. Math. Phys. 117 (1988) 353.

[29] S.Donaldson, *Topology* 29 (1990) 257. 139 (1991) 377.

[30] J.A.Harvey and A.Strominger, *String Theory and the Donaldson Polynomial*, Comm. Math. Phys. 151 (1993) 221.

[31] A.Johansen, *Twisting of N=1 SUSY gauge theories and heterotic topological theories*. FERMILAB-PUB-93/062-T; hep-th/9403017.

[32] E.Witten, *Supersymmetric Yang-Mills on a four-manifold*. Preprint IASSNS-HEP-94/5, hep-th/9403195.

[33] A.Johansen, *Realization of W_{1+\infty} and Virasoro algebras in supersymmetric theories on four manifolds*; hep-th/9406156.

[34] I.Bakas, Phys. Lett. B 228 (1989) 406; Comm. Math. Phys. 134 (1990) 487.

[35] C.N.Pope, L.J.Romans and X.Shen, Nucl. Phys. B 339 (1990) 191; Phys. Lett. B 242 (1990) 401.

[36] I.Bakas and E.B.Kirstis. Progress of Theor. Physics, Supplement, 102 (1990) 15.

[37] J.P.Yamron, *Topological Actions From Twisted Supersymmetric Theories*, Phys. Lett. 213B (1988) 325.

[38] A.Karlhede and M.Rocek, *Topological Quantum Field Theory And N = 2 Supergravity*, Phys. Lett. 212B (1988) 51.

[39] A.Galperin and O.Ogievetsky, *Holonomy Group, Complex Structures And D = 4 Topological Yang-Mills Theory*, Comm. Math. Phys. 139 (1991) 377.
[40] M.F. Sohnius and P.C. West, Nucl. Phys. B 198 (1981) 493; Phys. Lett. 105 B (1981) 353.
S. Ferrara, S. Sabharwal and M. Villasante, Phys. Lett. 205 B (1988) 302.

[41] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, vol. 2 (Cambridge Univ. Press, Cambridge, 1987).

[42] S.J. Gates, Jr., M.T. Grisaru, M. Rocek, and W. Siegel, Superspace, or One Thousand And One Lessons In Supersymmetry (Benjamin-Cummings, 1983).
J. Wess and J. Bagger, Supersymmetry And Supergravity, Princeton University Press (second edition, 1992).

[43] M.A. Shifman and A.I. Vainshtein, On Holomorphic Dependence And Infrared Effects In Supersymmetric Gauge Theories, Nucl. Phys. B 359 (1991) 571.

[44] C. Vafa, Topological Landau-Ginsburg Models, Mod. Phys. Lett. A6 (1991) 337.

[45] N. Seiberg, Exact Results on the Space of Vacua of Four Dimensional SUSY Gauge Theories. Preprint RU-94-18, hep-th/9402044.

[46] D.B. Ray and I.M. Singer, R-Torsion and the Laplacian on Riemann Manifold, Advan. Math. 7 (1971) 145; Analytic Torsion For Complex Manifolds, Ann. Math. 98 (1974) 154.

[47] I.G. Koh and S. Rajpoot, Finite N = 2 Extended Supersymmetric Field Theories, Phys. Lett. 135B (1984) 397.

[48] A.A. Anselm and A.A. Johansen, Radiative Corrections To The Axial Anomaly, Sov. Phys. JETP 69 (1989) 670.

[49] M.A. Shifman and A.I. Vainshtein, Solution Of The Anomaly Puzzle in SUSY Gauge Theories And The Wilson Operator Expansion, Nucl. Phys. B 277 (1986) 456.

[50] M.T. Grisaru, W. Siegel and M. Rocek, Improved Methods For Supergraphs, Nucl. Phys. B 159 (1979) 429.

[51] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B 229 (1983) 381; Phys. Lett. B 166 (1986) 329.

[52] I.T. Drummond and S.J. Hathrell, Phys. Rev. D 21 (1980) 958; I.T. Drummond and G.M. Shore, Phys. Rev. D 19 (1979) 1134; S.J. Hathrell, Ann. Phys. 139 (1982) 136.

[53] M.T. Grisaru and D. Zanon, Covariant Supergraphs. 1. Yang-Mills Theory, Nucl. Phys. B 252 (1985) 578; Covariant Supergraphs. 2. Supergravity, Nucl. Phys. B 252 (1985) 591.

[54] S.M. Christensen and M.J. Duff, Axial And Conformal Anomalies For Arbitrary Spin In Gravity And Supergravity, Phys. Lett. 76B (1978) 571.
[55] A. Johansen, *Generating Functional For Donaldson Invariants And Operator Algebra In Topological D = 4 Yang-Mills Theory*, Sov. J. Nucl. Phys. **55** (1992) 1434.

[56] M. F. Atiyah, F. R. S., N. J. Hitchin and I. M. Singer, *Self-Duality In Four-Dimensional Riemannian Geometry*, Proc. R. Soc. Lond. **A 362** (1978) 425.

[57] D. S. Freed and K. K. Uhlenbeck, Instantons and four-manifolds, Springer, 1984.

[58] M. Itoh, *Geometry Of Anti-Self-Dual Connections And Kuranishi Map*, J. Math. Soc. Jpn. **40** (1988) 9.

[59] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Gravitation, Gauge Theories And Differential Geometry*, Phys. Rep. **66** (1980) 213.

[60] K. Konishi, *Anomalous Supersymmetry Transformation Of Some Composite Operators In SQCD*, Phys. Lett. **135 B** (1984) 439.

[61] A. C. Davis, M. Dine and N. Seiberg, Phys. Lett. **125 B** (1983) 487.

[62] I. Affleck, M. Dine, and N. Seiberg, Phys. Rev. Lett. **51** (1983) 1026; Nucl. Phys. **B241** (1984) 493; Nucl. Phys. **B256** (1985) 557.

[63] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. **B260** (1985) 157.

[64] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, *Non-perturbative Aspects In Supersymmetric Gauge Theories*, Phys. Rep. **162** (1988) 169, and references therein.

[65] N. Seiberg, *Naturalness Versus Supersymmetric Non-renormalization Theorems*. Preprint RU-93-45, hep-ph/9309333; K. Intriligator, R. G. Leigh and N. Seiberg, *Exact Superpotentials in Four Dimensions*. Preprint RU-93-26, hep-th/9403198.

[66] E. Witten, *Constraints on Supersymmetry Breaking*, Nucl. Phys. **B 202** (1982) 253.

[67] S. Cecotti and C. Vafa, *Exact Results For Supersymmetric Sigma Models*, Phys. Rev. Lett. **68** (1992) 903.

[68] F. Wilczek and A. Zee, *Appearance of Gauge Structures in Simple Dinamical Systems*, Phys. Rev. Lett. **52** (1984) 2111.

[69] E. Witten, *Topological Sigma Models*, Comm. Math. Phys. **118** (1988) 411.

[70] E. Witten, *On the Structure of the Topological Phase of Two-Dimensional Gravity*, Nucl. Phys. **B340** (1990) 281.

[71] A. Johansen, work in progress.
[72] I.Bakas and E.B.Kiristis, Nucl. Phys. B343 (1990) 185; Mod. Phys. Lett. A5 (1990) 2039.

[73] E.Bergshoeff, C.Pope, L.Romans, E.Sezgin and X.Shen, Phys. Lett. 240 B (1990) 105.
D.A.Depireux, Phys. Lett. 252B (1990) 586.