Gaussian bounds and Collisions of variable speed random walks on lattices with power law conductances

Xinxing Chen

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Abstract

We consider a weighted lattice $\mathbb{Z}^d$ with conductance $\mu_e = |e|^{-\alpha}$. We show that the heat kernel of a variable speed random walk on it satisfies a two-sided Gaussian bound by using an intrinsic metric. We also show that when $d = 2$ and $\alpha \in (-1, 0)$, two independent random walks on such weighted lattice will collide infinite many times while they are transient.

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1 Introduction

In [16], Hebisch and Saloff-Coste proved that when a group has polynomial volume growth of order $D$, the heat kernel of a constant speed random walk on the group satisfies a two-sided Gaussian estimate, i.e.,

$$c_1t^{-D/2} \exp\left(-c_2\frac{\rho(x,y)^2}{t}\right) \leq p_t(x,y) \leq c_3t^{-D/2} \exp\left(-c_4\frac{\rho(x,y)^2}{t}\right).$$

where $\rho(x,y)$ is a metric on the group. Delmotte [12] gave equivalence of Gaussian bounds, parabolic Harnack inequalities, and the combination of volume regularity and Poincaré inequality. Later, there are many papers, such as [1, 2, 3, 4, 5, 25], showing that Gaussian bounds hold for lattice $\mathbb{Z}^d$ with different random conductances. In this paper, we consider a deterministic weighted lattice which does not satisfy Poincaré inequalities for all (sufficiently large) balls or volume doubling property, show that a variable random walk on it also satisfies the two-side Gaussian bound, but with a metric which is not comparable to the Euclidean metric.

Let $\alpha \in \mathbb{R}$. For $x, y \in \mathbb{Z}^d$ with $|x - y|_1 = 1$, we set $\mu_{xy} = (|x|_\infty \vee |y|_\infty)^{-\alpha}$ for the conductance of $(x,y)$. For convenience, we set $\mu_{xy} = 0$ if $x$ and $y$ are not nearest neighbor.
Write $\mu_x = \sum_y \mu_{xy}$ and $\nu_x = (|x|_\infty \vee 1)^\alpha$ for each $x \in \mathbb{Z}^d$. Let $X = \{X_t : t \geq 0\}$ be a continuous time random walk on the lattice $\mathbb{Z}^d$ with generator

$$\mathcal{L} f(x) = \frac{1}{\nu_x} \sum_{y \in \mathbb{Z}^d} (f(y) - f(x)) \mu_{xy}.$$ 

Then $X$ is a variable speed random walk waiting for an exponentially distributed time with mean $\nu_x \approx |x|_\infty^{2\alpha}$ before jumping. The transition density of $X$ with respect to $\nu$ is denoted by

$$p_t(x,y) = \frac{\mathbb{P}_x(X_t = y)}{\nu_y}.$$ 

To show the Gaussian bounds hold, we introduce a metric $\rho$ of $\mathbb{Z}^d$. We call $x_0 \cdots x_m$ a path if $|x_{i+1} - x_i| = 1$ for each $i < m$. Let $\rho(x,x) = 0$ for $x \in \mathbb{Z}^d$, and for $x,y \in \mathbb{Z}^d$ with $y \neq x$ set

$$\rho(x,y) = \min \left\{ \sum_{i=0}^{m} \nu_{z_i} : z_0 z_1 \cdots z_m \text{ is a path with } z_0 = x \text{ and } z_m = y \right\}.$$ 

Then there exists a constant $C = C(\alpha, d)$, such that

$$\frac{1}{\nu_x} \sum_{y \sim x} \rho(x,y)^2 \mu_{xy} \leq C \quad \text{for all } x. \tag{1.1}$$

Metrics satisfying (1.1) are called intrinsic metrics, see [14, 27]. One may expect that analogues of diffusion processes on manifolds hold using the intrinsic metrics for random walks on graphs. For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$, write $B_\rho(x,r) = \{y \in \mathbb{Z}^d : \rho(x,y) \leq r\}$ for a $\rho$-ball. We extend $\nu$ to a measure on $\mathbb{Z}^d$ and set

$$V_\rho(x,r) = \nu(B_\rho(x,r)).$$

**Theorem 1.1** Let $\alpha > -1$. Let $x,y \in \mathbb{Z}^d$ and $t > 0$. If $t \leq (\nu_x \vee \nu_y) \rho(x,y)$, then

$$p_t(x,y) \leq c_1(\nu_x \nu_y)^{-1/2} \exp \left( -c_2 \frac{\rho(x,y)}{\nu_x \vee \nu_y} \left( 1 + \log \left( \frac{\nu_x \vee \nu_y}{t} \right) \right) \right). \tag{1.2}$$

If $t \geq (\nu_x \vee \nu_y) \rho(x,y)$, then

$$p_t(x,y) \leq \frac{c_3}{\sqrt{V_\rho(x,t^{1/2}) V_\rho(y,t^{1/2})}} \exp \left( -c_4 \frac{\rho(x,y)^2}{t} \right) \tag{1.3}$$

and

$$p_t(x,y) \geq \frac{c_5}{\sqrt{V_\rho(x,t^{1/2}) V_\rho(y,t^{1/2})}} \exp \left( -c_6 \frac{\rho(x,y)^2}{t} \right). \tag{1.4}$$

**Remark 1.2** (1) In Lemmas 2.2 and 2.4, we give the bounds of $\rho(x,y)$ and $V_\rho(x,t^{1/2})$, respectively.

(2) Note that if $\alpha < -1$ then $\sup_{x,y} \rho(x,y) < \infty$ and $X$ will explode in a finite time. However, we still do not know whether the heat kernel of $X$ has Gaussian bounds at the critical point $\alpha = -1$. 

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Next, we consider the collision problem of random walks on these weighted lattices. As usual, we say that two walks $X$ and $X'$ collide infinitely often if almost surely there exists a sequence of (random) times $\{t_i : i \geq 1\}$ with $\lim t_i = \infty$ such that $X_{t_i} = X'_{t_i}$ for all $i$. In [24], Pólya first studied whether two independent simple random walks on $\mathbb{Z}^d$ collide infinitely often. He reduced it to the problem of a single walker returning to his starting point. Later Jain and Pruitt in [23] showed the Hausdorff dimension of the intersection of two independent stable processes, and Shieh in [26] gave a sufficient condition for infinitely collisions of Lévy processes in $\mathbb{R}$. However, if the walks are not on a homogeneous space, the problem will be complicated. Recently in [17], Hutchcroft and Peres use the Mass-Transport Principle to prove that a recurrent reversible random rooted graph has the infinite collision property. Examples that two recurrent random walks will never meet, were shown in [6, 7, 18]. Here, we give another example that two transient random walks will collision in finite often.

**Theorem 1.3** Let $\alpha > -1$. Let $X'$ be an independent copy of $X$.

1. Process $X$ is recurrent if and only if $\alpha \geq d - 2$.
2. If $d \leq 2$, then $X$ and $X'$ collide infinitely often.
3. If $d \geq 3$, then $X$ and $X'$ collide finitely often.

**Remark 1.4** It is much interesting that $X$ is not recurrent while $X$ and $X'$ collide infinitely often when $d = 2$ and $\alpha \in (-1, 0)$. Similarly, when $d \geq 3$ and $\alpha \geq d - 2$, $X$ is recurrent while $X$ and $X'$ collide finitely often.

In Section 2, we obtain some geometric properties of the weighted lattice $\mathbb{Z}^d$. In Section 3, we obtain an upper bound on $p_T(w, w)$ by using the approach of Barlow and Chen [4], which in turn is based on [19, 2]. In Section 4, we obtain the lower bounds of near diagonal transition probability by using the result of Delmette [12] directly and a chain argument. In Section 5, we give the proof of Theorem 1.1. Section 6 deals with the proof of Theorem 1.3 by the two-sided Gaussian bounds.

Throughout this paper, we use the notation $c, c'$ etc to denote fixed positive constants which may vary on each appearance, and $c_i$ to denote positive constants which are fixed in each argument. If we need to refer to constant $c_1$ of Lemma 2.1 elsewhere we will use the notation $c_{2.1.1}$. For any two functions $f$ and $g$, we say $f \asymp g$ if there exists $c_1(\alpha, d) > 0$ such that $c_1 f \leq g \leq c_2 f$. For brevity, we write $| \cdot |_p$ for the $L^p$-norm of the Euclidean space $\mathbb{R}^d$, while $| \cdot |$ instead of $| \cdot |_\infty$ for the $L^\infty$-norm. Write $B(x, r) = \{y \in \mathbb{Z}^d : |y - x| \leq r\}$ for an $L^\infty$-ball.

## 2 Some geometric properties

Fix $\alpha > -1$ henceforth. In this section, we shall estimate the metric $\rho(x, y)$ and the volume $V_\rho(x, r)$, and give Poincaré inequalities. Let us begin with the volume of a path.

**Lemma 2.1** Let $z_0 \cdots z_n$ be a path with $\max(|z_0|, |z_n|, |z_0 - z_n|) \geq n \geq 1$. Then

$$c_1 n(|z_0| \lor |z_n|)^\alpha \leq \sum_{i=0}^{n} \nu_{z_i} \leq c_2 n(|z_0| \lor |z_n|)^\alpha.$$  (2.1)
Proof. Without loss generality, we may assume that $|z_0| \geq |z_n|$ in the following. (Otherwise, relabel $z_{n-k}$ with $z_k$ for all $k$.) Then

$$|z_0 - z_n|_1 \leq d|z_0 - z_n| \leq d|z_0| + d|z_n| \leq 2d|z_0|.$$  

Using the condition $\max\{|z_0|, |z_n|, |z_0 - z_n|_1\} \geq n \geq 1$, we get

$$|z_0| \geq \frac{n}{2d} \lor 1. \quad (2.2)$$  

Since $z_0 \cdots z_n$ is a path, we have $|z_i - z_0| \leq i$ for each $i$. So, $\nu_{z_i} = (|z_i| \lor 1)^{\alpha}$ takes value between $(|z_0| + i)^{\alpha}$ and $(|z_0| - i) \lor 1)^{\alpha}$. Hence $\nu_{z_i} \geq c|z_0|^\alpha$ for $i \leq \frac{n}{2d}$, which implies

$$\sum_{i=0}^{n} \nu_{z_i} \geq \sum_{i \leq n/(4d)} \nu_{z_i} \geq c[n/4d] \nu_{z_0} \geq c'n|z_0|^\alpha = c'n(|z_0| \lor |z_n|)^\alpha. \quad (2.3)$$  

We have proved the lower bound of (2.1). For the upper bound, we consider two cases.

Case I: $|z_0| \geq |z_n| \lor n$. Directly calculate

$$\sum_{i=0}^{n} \nu_{z_i} \leq \sum_{i=0}^{n} (|z_0| + i)^{\alpha} + (|z_0| - i) \lor 1)^{\alpha} \leq 2 \sum_{i=|z_0| - n}^{i=|z_0| + n} (i \lor 1)^{\alpha}$$

$$\leq c_1 \int_{|z_0| - n}^{n|z_0| + n} x^{\alpha} \, dx = \frac{c_1}{1 + \alpha} (|z_0| + n)^{\alpha + 1} - (|z_0| - n)^{\alpha + 1}).$$  

Since $\lim_{t \to 0^+} ((1 + t)^{\alpha + 1} - (1 - t)^{\alpha + 1})t^{-1} = 2(\alpha + 1)$, we obtain

$$\sup_{t \in (0, 1]} |((1 + t)^{\alpha + 1} - (1 - t)^{\alpha + 1})t^{-1}| \leq c_2.$$  

Substituting $t = \frac{n}{|z_0|} \leq 1$ into the above inequality gives

$$\sum_{i=0}^{n} \nu_{z_i} \leq \frac{c_1}{1 + \alpha} (|z_0| + n)^{\alpha + 1} - (|z_0| - n)^{\alpha + 1}) \leq \frac{c_1c_2}{1 + \alpha} n|z_0|^\alpha = cn(|z_0| \lor |z_n|)^\alpha.$$  

Case II: $|z_n| \leq |z_0| < n$ and $|z_0 - z_n|_1 = n$. Then $z_0 \cdots z_n$ is an $L^1$-geodesic, which implies $\{z_0, \cdots, z_n\} \subset B(0, n)$ and $\{|i : z_i \in B(0, r)|\} \leq 2dr$ for each $r$. Write

$$k = \lceil \log_2 n \rceil, \; \; T_0 = B(0, 1) \; \; \text{and} \; \; T_l = B(0, 2^l) - B(0, 2^{l-1}) \; \; \text{for} \; \; l \geq 1.$$  

Then

$$\sum_{i=0}^{n} \nu_{z_i} = \sum_{l=0}^{k} \sum_{i : z_i \in T_l} \nu_{z_i} \leq c \sum_{l=0}^{k} 2^{l\alpha} |\{i : z_i \in T_l\}| \leq c \sum_{l=0}^{k} 2^{l\alpha} |\{i : z_i \in B(0, 2^l)\}|$$

$$\leq c \sum_{l=0}^{k} 2^{l\alpha} (2d \cdot 2^l) = 2dc \sum_{l=0}^{k} 2^{(\alpha + 1)l} \leq c'2^{(1 + \alpha)(k-1)} \leq c'n^{1 + \alpha}.$$  

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Since \( \frac{n}{2^n} \leq |z_0| < n \), we still have \( \sum_{i=0}^{n} \nu_i \leq c_2 n (|z_0| \vee |z_n|)^{\alpha} \) and prove the lemma. \( \square \)

For \( x \in \mathbb{Z}^d \) and \( r \in \mathbb{R}^+ \), we set
\[
\rho_x(r) = (|x| \vee r)^\alpha r.
\]
Then \( \rho_x(\cdot) \) is strictly increasing and
\[
\left( \frac{r}{s} \right)^{c_1} \leq \frac{\rho_x(r)}{\rho_x(s)} \leq \left( \frac{r}{s} \right)^{c_2}, \quad \forall \ r \geq s > 0.
\]
A simple calculation gives, if \( x, y \in \mathbb{Z}^d \) and \( r \geq \kappa |x-y| \), then there exists \( C = C(\alpha, \kappa) > 0 \) such that
\[
C^{-1} \rho_y(r) \leq \rho_x(r) \leq C \rho_y(r).
\]
Set \( \rho_x^{-1}(r) = (|x| \vee r^{1/(1+\alpha)})^{-\alpha} r \), which is the inverse function of \( \rho_x \). Then \( \rho_x^{-1}(\cdot) \) also satisfies (2.5) and (2.6).

**Lemma 2.2** Let \( x, y \in \mathbb{Z}^d \). Let \( \gamma \) be an \( L^1 \)-geodesic path from \( x \) to \( y \). Then
\[
\left\{ \rho(x, y), \sum_{u \in V(\gamma)} \nu_u, \sum_{(u,v) \in E(\gamma)} \mu_u^{-1} \right\} \subset [c_1 \rho_x(|x-y|), c_1^{-1} \rho_x(|x-y|)].
\]

**Proof.** By (2.6), we have \( \rho_x(|x-y|) \asymp \rho_y(|x-y|) \). So, we may assume \( |x| \geq |y| \) without loss generality. (Otherwise, exchange \( y \) with \( x \).) Hence \( |x| \geq \frac{1}{2} (|x| + |y|) \geq \frac{1}{2} |x-y| \), which implies
\[
\rho_x(|x-y|) \asymp |x-y| \cdot |x|^\alpha.
\]
Let \( z_0 z_1 \cdots z_m \) be a \( \rho \)-geodesic path with \( z_0 = x \) and \( z_m = y \), then by Lemma 2.1
\[
\rho(x, y) \geq \frac{1}{2} \sum_{k=0}^{[|x-y|/2]} \nu_{2^k} \geq c \frac{|x-y|/2}{|x|^\alpha}.
\]
By the definition of \( \rho(x, y) \), it is clear that \( \sum_{u \in V(\gamma)} \nu_u \geq \rho(x, y) \). Moreover, by Lemma 2.1
\[
\sum_{u \in V(\gamma)} \nu_u \leq c|x-y| (|x| \vee |y|)^\alpha \leq 2dc|x-y|^\alpha.
\]
Since \( \mu_u^{-1} \asymp (|u| \vee 1)^\alpha = \nu_u \) whenever \( u \sim v \), we also have
\[
\sum_{(u,v) \in E(\gamma)} \nu_u^{-1} \asymp \sum_{u \in V(\gamma)} \nu_u
\]
Combining these inequalities together, we complete the proof. \( \square \)

Since \( \rho_x(r) \) is increasing in \( r \), Lemma 2.2 immediately implies Corollary 2.3 as follows. Recall that \( B_\rho(x, r) \) is a \( \rho \)-ball. One can compare it with an \( L^1 \)-ball.
Corollary 2.3 For any \( x \in \mathbb{Z}^d \) and \( r > 0 \),
\[
B(x, \rho_x^{-1}(c_1 r)) \subset B_{\rho}(x, r) \subset B(x, \rho_x^{-1}(c_2 r)).
\]
Recall that \( V_{\rho}(x, r) \) is the volume of \( B_{\rho}(x, r) \). Set \( V(x, r) = \nu(B(x, r)) \), similarly.

Lemma 2.4 Let \( x \in \mathbb{Z}^d \) and \( r > 0 \).

1. \( V(x, r) \asymp r^d(|x| \vee r)^\alpha \) if \( r \geq 1 \).
2. \( V_{\rho}(x, r) \asymp V(x, \rho_x^{-1}(r)) \asymp \begin{cases} 
\nu_x & \text{if } r < \nu_x; \\
r^d|x|^{-(d-1)\alpha} & \text{if } \nu_x \leq r \leq |x|^{1+\alpha}; \\
r^{(d+\alpha)/(1+\alpha)} & \text{if } r > |x|^{1+\alpha}.
\end{cases} \)

Proof. (1) Let \( x_1 \) be the first coordinate of \( x \) and set
\[
\Lambda = \{ s = (s_1, \ldots, s_d) \in B(x, r) : s_1 = x_1 \}.
\]
Write \( e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^d \). By Lemma 2.1 for each \( s \in \Lambda \) we have
\[
\sum_{t = -r}^{r} \nu_{s+te_1} \asymp r(|s-re_1| \vee |s+re_1|)^\alpha \asymp r(|s| \vee r)^\alpha \asymp r(|x| \vee r)^\alpha.
\]
Hence,
\[
V(x, r) = \sum_{s \in \Lambda} \sum_{t = -r}^{r} \nu_{s+te_1} \asymp |\Lambda| \cdot r(|x| \vee r)^\alpha \asymp r^d(|x| \vee r)^\alpha. \tag{2.8}
\]
(2) Using (2.8) and Corollary 2.3 we get the desired result.

Lemma 2.5 Let \( w \in \mathbb{Z}^d \) and \( R \geq 1 \). Then for any \( x \in B(w, R) \) and \( r \in [1, R] \),
\[
V(w, R) \leq c_1 \left( \frac{R}{r} \right)^{c_1} V(x, r). \tag{2.9}
\]
Especially, \( V(w, R) \leq c_1 R^{c_1} \nu_x \).

Proof. It follows directly from Lemma 2.1 (1).

So, \( \nu(B(w, R)) \) satisfy the volume doubling property in any case. However, \( \mu(B(w, R)) = \sum_{x \in B(w, R)} \mu_x \) do not satisfy the volume doubling property since \( \mu(\mathbb{Z}^d) < \infty \) when \( \alpha > d \).

In [28] Virág, extending the early result of [22], showed that Poincaré inequalities hold in any convex lattices. We shall apply their technique to our weighted lattices.
Lemma 2.6  Let $x \in \mathbb{Z}^d$, $r > 0$. Then for any function $f$ on $B(x, r)$,

$$\min_{a} \sum_{u \in B(x, r)} (f(u) - a)^2 \nu_u \leq c_1[\rho_x(r)]^2 \sum_{u, v \in B(x, r)} (f(u) - f(v))^2 \mu_{uv}. \quad (2.10)$$

Proof. If $r \in (0, 1)$ then $B(x, r) = \{x\}$ and (2.10) holds since both side of the inequality are zero. So, we may assume that $r \geq 1$ in the following.

By [28, Proposition 2], for each $u, v \in \mathbb{Z}^d$ we can choose a path $\gamma_{uv}$ such that, (1) $\gamma_{uv}$ is an $L^1$-geodesic path from $u$ to $v$; (2) each site in $\gamma_{uv}$ has $L^\infty$-distance less than 1 from the Euclidean line $\overline{uv}$. For $u, y \in \mathbb{Z}^d$, write

$$\Lambda_{uy} = \{s + z : s \in \gamma_{y, 2y - u}, |z| \leq 4, z \in \mathbb{Z}^d\}.$$  

By the construction, we have

$$1_{\{y \in \gamma_{uv}\}} \leq 1_{\{v \in \Lambda_{uy}\}} + 1_{\{u \in \Lambda_{vy}\}} \text{ for all } u, v, y.$$  

By Lemma 2.2

$$\sum_{v \in \Lambda_{uy}} \nu_v \leq \sum_{s \in \gamma_{y, 2y - u}, z \in \mathbb{Z}^d, |z| \leq 4} \sum_{s \in \gamma_{y, 2y - u}} \nu_{s + z} \leq c \sum_{s \in \gamma_{y, 2y - u}} \nu_s \leq c' \rho_y(|y - u|).$$

So, if $u, y \in B(x, r)$, we can use (2.10) and get

$$\sum_{v \in \Lambda_{uy}} \nu_v \leq c' \rho_y(2r) \leq c' \rho_x(r).$$

By Lemma 2.2 if $u, v \in B(x, r)$ then

$$\sum_{(y, z) \in E(\gamma_{uv})} \mu_{yz}^{-1} \leq c \rho_u(|u - v|) \leq c \rho_x(r). \quad (2.11)$$

Therefore, writing $B = B(x, r)$,

$$\sum_{u \in B} (f(u) - \overline{f})^2 \nu_u \leq \frac{1}{\nu(B)} \sum_{u, v \in B} (f(u) - f(v))^2 \nu_u \nu_v = \frac{1}{\nu(B)} \sum_{u, v \in B} \left( \sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \right) \nu_u \nu_v$$

$$\leq \frac{1}{\nu(B)} \sum_{u, v \in B} \left( \sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \right) \left( \sum_{(y, z) \in E(\gamma_{uv})} \mu_{yz}^{-1} \right) \nu_u \nu_v$$

$$\leq \frac{c \rho_x(r)}{\nu(B)} \sum_{u, v \in B} \sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \nu_u \nu_v$$

$$\leq \frac{c \rho_x(r)}{\nu(B)} \sum_{y, z \in B} \sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \sum_{u, v \in B} 1_{\{y \in \gamma_{uv}\}} \nu_u \nu_v$$

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where the second inequality is by the Cauchy-Schwarz inequality. \hfill \Box

**Lemma 2.7** Let \( w \in \mathbb{Z}^d \), \( R \geq 1 \) and \( r \in (0, \rho_w(R)] \). Let \( g : B(w, R) \to \mathbb{R}^+ \) with \( \sum_{x \in B(w,R)} g(x) \nu_x \leq 1 \). Then

\[
\sum_{x,y \in B(w,R)} (g(x) - g(y))^2 \mu_{xy} \geq c_1 r^{-2} \left( \sum_{x \in B(w,R)} g(x)^2 \nu_x - \frac{c_2}{V(w,R)} \left( \frac{\rho_w(R)}{r} \right)^{c_3} \right). \tag{2.12}
\]

**Proof.** Let \( \tilde{r} = \min \{ \rho_x^{-1}(r) : x \in B(w,R) \} \cap R \). Since \( r \leq \rho_w(R) \) and \( \rho_x(R) \asymp \rho_w(R) \) for each \( x \in B(w,R) \), we have

\[
\rho_x(\tilde{r}) \leq c_1 r. \tag{2.13}
\]

Note that for any \( x \in B(w,R) \),

\[
\frac{\rho_x^{-1}(r)}{R} = \frac{\rho_x^{-1}(r)}{\rho_x^{-1}(\rho_x(R))} \geq c_1 \left( \frac{r}{\rho_x(R)} \right)^{c_1} \geq c_2 \left( \frac{r}{\rho_w(R)} \right)^{c_2}.
\]

So, \( \frac{\tilde{r}}{R} \geq c_2 \left( \frac{r}{\rho_w(R)} \right)^{c_2} \). Using Lemma 2.5 we then have

\[
\frac{V(w,R)}{V(x,\tilde{r})} \leq c \left( \frac{R}{\tilde{r}} \right)^{c_1} \left( \frac{\rho_w(R)}{r} \right)^{c_3}. \tag{2.14}
\]

Choose \( B_i = B(x_i, r_i), i = 1, \ldots, N \) such that \( B(w,R) = \bigcup_{i=1}^N B(x_i, r_i) \) and \( \tilde{r} \leq r_i \leq 2\tilde{r} \) for each \( i \), and

\[
|\{i : x \in B(x_i, r_i)\}| \leq c_4 \quad \text{for all } x \in B(w,R). \tag{2.15}
\]

Use Lemmas 2.6

\[
\sum_{x,y \in B} (g(x) - g(y))^2 \mu_{xy} \geq c_4^{-1} \sum_{i=1}^N \sum_{x,y \in B_i} (g(x) - g(y))^2 \mu_{xy}
\]

\[
\geq c_4^{-1} \sum_{i=1}^N [\rho_x(\tilde{r})]^{-2} \sum_{x \in B_i} (g(x) - \tilde{g}_i)^2 \nu_x
\]

\[
\geq c_4^{-1} \sum_{i=1}^N (c_1 r)^{-2} \left( \sum_{x \in B_i} g(x)^2 \nu_x - \frac{(\sum_{x \in B_i} g(x) \nu_x)^2}{V(x_i, \tilde{r})} \right)
\]

\[
\geq (c_4 c_1^2 r)^{-1} r^{-2} \left( \sum_{x \in B} g(x)^2 \nu_x - \frac{c_3}{V(w,R)} \left( \frac{\rho_w(R)}{r} \right)^{c_3} \sum_{i=1}^N (\sum_{x \in B_i} g(x) \nu_x)^2 \right),
\]

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where $\mathcal{B} = B(w, R)$ and $\overline{g}_i$ is the mean of $g$ on $B_i$. Using (2.15), we get
\[
\sum_{i=1}^{N} \sum_{x \in B_i} g(x) \nu_x \leq c \sum_{x \in \mathcal{B}} g(x) \nu_x \leq c.
\]
Combining these inequalities with $\sum_i a_i^2 \leq \left( \sum_i a_i \right)^2$ for all $a_i \geq 0$, we complete the proof. \(\square\)

**Remark 2.8** One cannot expect to improve Lemma 2.7 to the whole space such as
\[
\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \geq c_1 r^{-2} \left( \sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x \right) - \frac{c_2}{V(w, R)} \left( \frac{\rho_w(R)}{r} \right)^{c_3}
\]
for all $r \in (0, \rho_w(R)]$, and $g : \mathbb{Z}^d \to \mathbb{R}^+$ with $\sum_{x \in \mathbb{Z}^d} g(x) \nu_x \leq 1$.

To see this, we fix $\alpha \in (-1, 0)$ and $d \geq 2$. On the one hand, choose $R \geq 1$ and $w \in \mathbb{Z}^d$ with $|w| = R^{-\alpha - 1}$. Then $\rho_w(R) = 1$, and hence one can take $r = 1$ further. Such,
\[
V(w, R) \asymp R^{d-1} \rho_w(R) = R^{d-1} \to \infty.
\]
On the other hand, let $s \geq 1$, and take
\[
g(x) = A(s - |x|)1_{B(0,s)}(x), \quad x \in \mathbb{Z}^d,
\]
where $A$ is the constant which such that $\sum_x g(x) \nu_x = 1$. Then
\[
\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \leq A^2 \sum_{x, y \in B(0,s)} \mu_{xy} \leq cA^2 s^{d-\alpha},
\]
and
\[
\sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x \geq \frac{A^2 s^2}{4} \sum_{x \in B(0,s/2)} \nu_x \geq cA^2 s^{d+2+\alpha}.
\]
So, as $s$ goes to infinity,
\[
\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \ll \sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x.
\]
By (2.18) and (2.17), the inequality (2.16) fails.

### 3 On-diagonal upper bound estimates

Fix $w \in \mathbb{Z}^d$, $R \geq 1$ and $T = \rho_w(R)^2$. In this section, our aim is to give an upper bound of $p_T(w, w)$. As Lemma 2.7 and Remark 2.8 say, we have a good ball $B(w, R)$ only. So, we turn to the random walk $X$ with reflection at $\partial_i B(w, R)$. By the approach of Barlow and
Chen [4], we obtain upper bounds of the heat kernel of the reflection process, and then bring these bounds back to the original process.

Write $\mathcal{B} = B(w, R)$ for short. Let $Y$ be the continuous time random walk on $\mathcal{B}$ with generator

$$\mathcal{L}_\mathcal{B} f(x) = \frac{1}{\nu_x} \sum_{y \in \mathcal{B}} (f(y) - f(x))\mu_{xy}. $$

For $x \in \mathbb{Z}^d$ and $r > 0$, set

$$\tau_{x,r} = \inf \lbrace t \geq 0 : X_t \not\in B(x, r) \rbrace. $$

If $Y$ and $X$ start at the same vertex in $B(w, R - 1)$, then we can couple $Y$ and $X$ on the same probability space such that

$$Y_s = X_s \quad \text{for} \quad 0 \leq s \leq \tau_{w,R-1}. $$

We use $\mathbb{P}_x$ for both $X$ and $Y$. Denote the heat kernel of $Y$ by

$$q_t(x, y) = \frac{\mathbb{P}_x(Y_t = y)}{\nu_y}. $$

**Proposition 3.1** For $u \in \mathcal{B}$ and $t \in (0, T]$,

$$q_t(u, u) \leq \frac{c_1}{V(w, R)} \left( \frac{T}{t} \right)^{c_2}. $$

Especially, $q_T(w, w) \leq \frac{c_1}{V(w, R)}$.

**Proof.** Given Lemma 2.7, the proof is similar to [2, Proposition 3.1] and [4, Proposition 3.2], so we omit it. $\square$

**Lemma 3.2** Let $x_1, x_2 \in \mathcal{B}$ with $|x_1 - x_2| \geq \frac{1}{16} R$. If $t \leq c_1 T$ and $R \geq c_2$, then

$$q_t(x_1, x_2) \leq \frac{1}{4V(w, R)}. $$

**Proof.** Write $\eta = \max_{x \in \mathcal{B}} \nu_x$. By (2.5) and (2.6), we have

$$\frac{T^{1/2}}{\eta} = \frac{\rho_w(R)}{\max_{x \in \mathcal{B}} \rho_x(1)} = \inf_{x \in \mathcal{B}} \left\{ \frac{\rho_w(R)}{\rho_x(R)} \cdot \frac{\rho_x(R)}{\rho_x(1)} \right\} \geq c_1 R^{c_1}. $$

Set $c_2 = 2^{\lceil \alpha \rceil + 2}d$. Let $\tilde{\nu}_x = \eta^{-1} \nu_x$, $\tilde{\mu}_{xy} = \eta \mu_{xy}$ and $\tilde{\rho}(x, y) = c_2^{-1} \eta^{-1} \rho(x, y)$ for $x, y \in \mathcal{B}$. Then

$$\left\{ \begin{array}{l}
\frac{1}{\nu_x} \sum_{y \in \mathcal{B}} \tilde{\rho}(x, y)^2 \tilde{\mu}_{xy} \leq 1; \\
\tilde{\rho}(x, y) \leq 1 \quad \text{whenever} \quad x \sim y.
\end{array} \right. $$

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Hence \( \tilde{\rho}(\cdot, \cdot) \) is an adapted metric, which was introduced by Davies \[20\] and \[21\]. Let \( Z_s = Y_{\eta^2 s} \), for \( s \geq 0 \). Then \( Z \) has the generator

\[
\tilde{\mathcal{L}}_B f(x) = \frac{1}{\nu_x} \sum_{y \in B} (f(y) - f(x)) \tilde{\mu}_{xy}.
\]

We state that there exists constant \( c, c' > 0 \) such that if \( s \leq c \eta^{-2} T \) and \( R \geq c' \) then

\[
\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{\nu_{x_2}}{4V(w, R)}.
\]  

(3.7)

If this is true, then we have (3.4) and prove the lemma.

We now prove (3.7). Set \( c_3 = (3.1)_{c_1} + c_1^{-1} (2.3). \) For each \( i \in \{1, 2\} \), define

\[
f_{x_i}(s) = \frac{V(w, R)}{\nu_{x_i}} \left( \frac{\eta^2 s}{T} \right)^{c_3}, \quad s \geq 0.
\]

Then by Proposition (3.1) for \( s \leq \eta^{-2} T \),

\[
\mathbb{P}_{x_i}(Z_s = x_i) = \mathbb{P}_{x_i}(Y_{\eta^2 s} = x_i) = q_{\eta^2 s}(x_i, x_i) \nu_{x_i} \leq \frac{1}{f_{x_i}(s)}.
\]  

(3.8)

Next we shall estimate the off-diagonal transition probability \( \mathbb{P}_{x_1}(Z_s = x_2) \) by using the 'two-point' method of Grigor’yan-see \[15, 11, 13, 8\]. The metric \( d_{\nu}(x, y) \) in \[8\] is just \( \tilde{\rho}(x, y) \) and one can easily check that \( f_{x_i}(s) \) is \((1, 2)\)-regular on \((0, T]\): see \[15, 8\] for the definition. By (3.5) and Lemma (2.5) for \( s \leq \eta^{-2} T \),

\[
\frac{f_{x_i}(s)}{s^{c_3}} = \frac{V(w, R)}{\nu_{x_i}} \left( \frac{\eta^2 s}{T} \right)^{c_3} \leq \frac{(2.5)_1 R (2.5)_1}{(3.1)_{c_1}} \cdot (c_1 R^{c_1})^{-2c_3} c' R^{-2c_1 c_3} \leq c' R^{-2.5_{1}} \leq c'.
\]

Therefore, by \[8\] Theorem 1.1] for \( s \in (\tilde{\rho}(x_1, x_2), \eta^{-2} T] \),

\[
\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{c_4 \tilde{\nu}_{x_2} \tilde{\nu}_{x_1}^{1/2}}{\sqrt{f_{x_1}(c_5 s) f_{x_2}(c_5 s)}} \exp \left(-c_6 \frac{\tilde{\rho}(x_1, x_2)^2}{s} \right)
\]

\[
= c_7 \nu_{x_2} \left( \frac{T}{\eta^2 s} \right)^{c_3} \exp \left(-c_6 c_2^{-2} \rho(x_1, x_2)^2 \frac{T}{\eta^2 s} \right).
\]  

(3.9)

(3.10)

By Lemma (2.2) and the condition \( |x_1 - x_2| \geq \frac{1}{10} R \), we have

\[
\rho(x_1, x_2) \geq (2.2)_2 \rho_{x_1}(\frac{1}{10} R) \geq c_8 \rho_w(R) = c_8 T^{1/2}.
\]  

(3.11)

Substituting (3.11) into (3.10) gives

\[
\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{c_7 \nu_{x_2} (T)}{\eta^2 s} \exp \left(-c_6 c_2^{-2} c_8^2 \frac{T}{\eta^2 s} \right),
\]

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which implies (3.7) holds for each \( s \in (\tilde{\rho}(x_1, x_2), c_9 \eta^{-2} T] \), provided \( c_9 > 0 \) is small enough.

On the other hand, by [8, Corollary 2.8] we have the ‘long range’ bounds, that is, if \( s \leq \tilde{\rho}(x_1, x_2) \) then

\[
P_{x_1}(Z_s = x_2) \leq c'(\tilde{\nu}_x / \tilde{\nu}_y)^{1/2} e^{-c\tilde{\rho}(x_1, x_2)}. \tag{3.12}
\]

Using (3.5) and (3.11), we have

\[
\tilde{\rho}(x_1, x_2) = \frac{c_9^{-1} \eta^{-1} \rho(x_1, x_2)}{1} \geq c_9^{-1} T^{1/2} \geq c' R^{\epsilon''}. \tag{3.13}
\]

Combining these inequalities with Lemma 2.5, we get

\[
P_{x_1}(Z_s = x_2) \leq c(\nu_x / \nu_y)^{1/2} e^{-c' R^{\epsilon''}} = \frac{cV(w, R)}{V(w, R)}(\nu_x / \nu_y)^{1/2} e^{-c' R^{\epsilon''}} \leq \frac{cV(w, R)}{V(w, R)}(\nu_x / \nu_y)^{1/2} e^{-c' R^{\epsilon''}}.
\]

So, (3.7) holds again if \( s \leq \tilde{\rho}(x_1, x_2) \) and \( R \geq c \).

### Lemma 3.3

Let \( t \leq c_1 T \) and \( x \in B(w, \frac{7}{8} R) \). If \( R \geq c_2 \) then

\[
P_x(Y_t \not\in B(x, \frac{1}{16} R)) \leq \frac{1}{4}.
\]

**Proof.** By Lemma 3.2 we get

\[
P_x(Y_t \not\in B(x, \frac{1}{16} R)) = \sum_{y \in B-B(x, \frac{1}{16} R)} q_t(x, y) \nu_y \leq \sum_{y \in B-B(x, \frac{1}{16} R)} \nu_y / 4V(w, R) \leq \frac{1}{4}.
\]

Now we bring these bounds of the reflection process back to the original process. Note that \( X \) and \( Y \) agree until time \( \tau_{w,R} \).

### Lemma 3.4

If \( R \geq c_1 \) then for \( x \in B(w, \frac{5}{8} R) \),

\[
P_x(\tau_{x,R/8} < c_2 T) \leq \frac{1}{2}.
\]

**Proof.** Given Lemma 3.3 the proof is similar to [4, Lemma 4.1], so we omit it.

### Proposition 3.5

Let \( w \in \mathbb{Z}^d, R > 0 \) and \( T = \rho_w(R)^2 \). Then

\[
P_w(X_T = w) \leq \frac{c_1 \nu_w}{V(w, R)}.
\]
Proof. If $R < (3.4_1 \vee 3.2_2)$ then by Lemma 2.5
\[
\frac{\nu_w}{V(w, R)} \geq c'R^{-c} \geq c'(3.4_1 \vee 3.2_2)^{-c} \geq c_1^{-1}p_w(X_T = w).
\]
So, let $R \geq (3.4_1 \vee 3.2_2)$. Given Lemma 3.4 similar to the inequality (4.6) of Barlow and Chen \[4\] we obtain
\[
p_{czT}(w, w) \leq q_{czT}(w, w) + \sup_{0 < s \leq c_T} \max_{y \in A} q_s(y, w),
\]
where $c_2 = (3.4_1 \wedge 3.2_1 \wedge 1$ and $A = B(w, 5R/8) - B(w, 5R/8 - 1)$. By Proposition 3.1 and Lemma 3.2
\[
p_T(w, w) \leq p_{czT}(w, w) \leq \frac{c_3}{V(w, R)}.
\]
\[
\Box
\]

4 Near diagonal lower bound estimates

In this section, we shall prove the following lower bounds for the near diagonal transition probabilities. Recall $\tau_{x, r}$ from section 3. Fix $\delta \in (0, 1/2)$. We will use the notation $K_i$ to denote constants which depend only $\delta, \alpha$ and $d$, while $c_i = c_i(\alpha, d)$ as before.

**Theorem 4.1** Let $w \in \mathbb{R}^d$ and $R \geq 1$. For $x_1, x_2 \in B(w, R)$ and $t \in [\delta R^2, 2\rho_w(R)^2]$,
\[
\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, c_1R} > t) \geq K_2\frac{\nu_{x_2}}{V(w, R)}.
\]
(4.1)

Since $\mu(B(w, R))$ do not satisfy the volume doubling property, we cannot obtain the lower bound by a general approach. Let us begin with a ball far from the origin.

**Lemma 4.2** Let $w \in \mathbb{Z}^d$ and $R \geq 1$ with $|w| \geq 32R$. Then for any $x_1, x_2 \in B(w, R)$ and $t \in [\delta R^2, 2\rho_w(R)^2]$,
\[
\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, 8R} > t) \geq K_1 R^{-d}.
\]
(4.2)

**Proof.** Since $|w| \geq 32R$, $\rho_w(R) = R|w|^\alpha$, moreover, for any $x, y \in B(w, 16R)$ with $x \sim y$,
\[
\nu_x \in [4d c_1^{-1}|w|^\alpha, c_1|w|^\alpha] \quad \text{and} \quad \mu_{xy} \in [c_1^{-1}|w|^{-\alpha}, c_1|w|^{-\alpha}].
\]

By the application of Lemma 3.4 on $B(w, 8R)$, there exists $c_2 \in (0, 1/2)$ such that
\[
\mathbb{P}_x(\tau_{x, R} > c_2 \rho_w(R)^2) \geq \frac{1}{2}, \quad \text{for all } x \in B(w, R).
\]
(4.3)

For each $x, y \in B(w, 16R)$, we set
\[
\nu_x = c_1|w|^{-\alpha}\nu_x \quad \text{and} \quad \mu_{xy} = \begin{cases} c_1^{-1}|w|^{\alpha}\mu_{xy}, & \text{if } x \neq y; \\ \nu_x - c_1^{-1}|w|\sum_{z \in B(w, 16R) \setminus \{x\}} \mu_{xz}, & \text{if } x = y. \end{cases}
\]

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So, $\tilde{\nu}_x, \tilde{\mu}_{xy} \in [c_3, c_3^{-1}]$ for all $x \in B(w, 16R)$ and $y \in B(w, 16R) \cap B(x, 1)$. Let $Z$ be the continuous time (constant speed) random walk on $B(w, 16R)$ with generator

$$\tilde{\mathcal{L}} f(u) = \frac{1}{\nu_u} \sum_{v \in B(w, 16R)} (f(u) - f(v)) \tilde{\mu}_{uv}. $$

Then $Z$ and $X$ can be coupled in the same probability such that

$$Z_s = X_{c_1^{-2}w^{2a}, s}, \quad \text{for all } s < \sigma = c_1^2|w|^{-2a}r_{w, 8R},$$

where $\sigma := \inf \{ s \geq 0 : Z_s \notin B(w, 8R) \}$. Fix $x_1, x_2 \in B(w, R)$, and let $u(s, y) = \mathbb{P}_{x_1}(Z_s = y, \sigma > s)/\tilde{\nu}_y$ for each $y \in B(w, 16R)$ and $s \geq 0$. Then $u$ is a positive solution of the heat equation $\partial u/\partial s = \tilde{\mathcal{L}} u$ on $(0, \infty) \times B(w, 4R)$. One can easily check that $DV(C_1), P(C_2)$ and $\Delta(\alpha)$ hold for the weighted graph with vertex set $B(w, 16R)$ and edge weight $\tilde{\mu}_{xy}$, and so $u(s, y)$ satisfies the Harnack inequality, see [12, Theorem 1.7]. Therefore,

$$\max_{[\frac{1}{2}s_0, s_0]} u \leq K_1^{-1} \min_{[\delta R^2, 2\delta R^2]} u,$$

where $s_0 = \delta c_2 c_1^2 R^2$. Furthermore, for any $s \in [\delta c_1^2 R^2, 2\delta c_1^2 R^2],$

$$\mathbb{P}_{x_1}(Z_s = x_2, \sigma > s) \geq K_1 \left( \sum_{z \in B(w, 2R)} \tilde{\nu}_z \right)^{-1} \sum_{z \in B(w, 2R)} \mathbb{P}_{x_1}(Z_{s_0} = x_2, \sigma > s_0) \geq K_1 c_3 |B(w, 2R)|^{-1} \mathbb{P}_{x_1}(Z_{s_0} \in B(w, 2R), \sigma > s_0) \geq K_1 c_3 (5R)^{-d} \mathbb{P}_{x_1}(\inf \{ h : Z_h \notin B(x_1, R) \} > s_0). \quad (4.4)$$

Since $X_t = Z_{c_1^{-2}w^{-2a}t}$ for all $t < \tau$, inequality (4.4) can be rewrote as

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, 8R} > t) \geq K_2 R^{-d} \mathbb{P}_{x_1}(\tau_{x, R} > \delta c_2 R^2 |w|^{2a}), \quad t \in [\delta R^2 |w|^{2a}, 2R^2 |w|^{2a}].$$

Using (4.3), we finish the proof. \hfill \Box

**Lemma 4.3** For any $t \in [\delta R^{2+2a}, R^{2+2a}]$ and $x \in B(0, R),$

$$\mathbb{P}_x(|X_t| > K_1 R, \tau_{x, c_3 R} > t) \geq \frac{1}{4}.$$

**Proof.** By Proposition 3.5, for any $x, y \in \mathbb{Z}$ and $t > 0,$

$$p_t(x, y) \leq (p_t(x, x)p_t(y, y))^{1/2} \leq c_1 (V(x, \rho_x^{-1}(t^{1/2}))V(y, \rho_y^{-1}(t^{1/2})))^{-1/2}.$$

So, from Lemma 2.4 we can get, if $x, y \in B(0, R)$ and $t \in [\delta R^{2+2a}, R^{2+2a}]$ then

$$p_t(x, y) \leq K_1 t^{-(d+\alpha)/(2+2\alpha)} \leq K_2 R^{-d-\alpha}. $$

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Fix \( x \in B(0, R) \) and \( t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}] \). By Lemma 2.4 again, for each \( \varepsilon \in (0, 1) \),

\[
\mathbb{P}_x(|X_t| \leq \varepsilon R) = \sum_{y \in B(0, \varepsilon R)} p_t(x, y) \nu_y \leq V(0, \varepsilon R) \cdot K_2 R^{-(d+\alpha)} \\
\leq c_2 (\varepsilon R)^{d+\alpha} \cdot K_2 R^{-(d+\alpha)} = K_2 c_2 \varepsilon^{d+\alpha}.
\]

Hence there exists \( \varepsilon_0 = \varepsilon_0(\delta, \alpha, d) > 0 \) such that

\[
\mathbb{P}_x(|X_t| \leq \varepsilon_0 R) \leq \frac{1}{4}.
\] (4.5)

On the other hand, applying Lemma 3.4 gives

\[
\mathbb{P}_x(\tau_{x, \varepsilon R} < t) \leq \mathbb{P}_x(\tau_{x, \varepsilon R} < R^{2+2\alpha}) \leq \frac{1}{2}.
\] (4.6)

Combing (4.6) with (4.5), we finish the proof. \( \square \)

**Lemma 4.4** Let \( R \geq 1 \). Let \( x_1, x_2 \in B(0, R) \setminus B(0, \delta R) \) and \( t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}] \). Then

\[
\mathbb{P}_x(\tau_{x_1} < \tau_{x_2}, \tau_{0,10R} > t) \geq K_1 R^{-d}.
\] (4.7)

**Proof.** Write \( T = B(0, R) \setminus B(0, \delta R) \) for short. If \( d \geq 2 \), then \( T \) is connected. Note that \( \rho_w(R) \subset [K_1^{-1} R^{1+\alpha}, K_1 R^{1+\alpha}] \) and \( B(w, 8R) \subset B(0, 10R) \) for all \( w \in T \), and there exist vertices \( w_i \in T, i \leq K_2 \) such that \( T = \bigcup_{i=1}^{K_2} B(w_i, \frac{\delta}{64} R) \). A standard chaining argument using Lemma 1.2 on \( B(w_i, \frac{\delta}{64} R) \), proves (4.7) for \( d \geq 2 \). Next, we consider \( d = 1 \). Since \( T = ([0, -\delta R] \cup [\delta R, R]) \cap \mathbb{Z} \) is not connected, we have to discuss the problem on several cases.

Case I: \( x_1, x_2 > 0 \). Then \( x_1 \) and \( x_2 \) can be joint with a sequence of balls \( B(w_i, \frac{\delta}{32} R) \) within \( [\delta R, R] \cap \mathbb{Z} \) as before. Hence (4.7) holds for this case, too.

Case II: \( x_1 > 0 > x_2 \). For conciseness, we write \( \hat{P} \) for the measure of the process \( X \) killed on exiting \( B(0, 10R) \). Let \( \varepsilon_0 = \varepsilon_0(\delta, \alpha, d) \in (0, 1) \) be a small constant, whose value will be taken later. Set \( x_* = [\varepsilon_0 R] \). By the result of Case I, we have

\[
\inf_{s \in [0, \hat{R}^{2+2\alpha}, R^{2+2\alpha}]} \inf_{x, y \in B(0, R) \setminus B(0, \hat{R})} \hat{P}_x(X_s = y) \geq K_3 R^{-1},
\]

where \( \hat{R} = \min\{\frac{1}{2} \varepsilon_0, \frac{1}{3} \delta \} \). So,

\[
\hat{P}_{x_1}(X_{t/3} \in (\frac{1}{2} \varepsilon_0 R, \varepsilon_0 R)) \geq \frac{1}{4} K_3 \varepsilon_0 \quad \text{and} \quad \inf_{s \in [t/3, t]} \hat{P}_{-x_1}(X_s = x_2) \geq K_3 R^{-1}.
\]

For \( x \in \mathbb{Z} \), we define \( \sigma_x = \inf\{t \geq 0 : X_t = x\} \), the first time of visiting vertex \( x \). By the strong Markov property,

\[
\mathbb{P}_x(\tau_{x_1} < \tau_{x_2}, \tau_{0,10R} > t) \geq \hat{P}_{x_1}(\sigma_{x_*} < \frac{4}{3}, \sigma_{-x_*} < \frac{2}{3}, X_t = x_2) \\
\geq \hat{P}_{x_1}(\sigma_{x_*} < \frac{4}{3}) \hat{P}_{x_2}(\sigma_{-x_*} < \frac{2}{3}) \inf_{s \in [t/3, t]} \hat{P}_{-x_1}(X_s = x_2) \\
\geq \hat{P}_{x_1}(X_{t/3} \in (\frac{1}{2} \varepsilon_0 R, \varepsilon_0 R)) \hat{P}_{x_2}(\sigma_{-x_*} < \frac{2}{3}) \inf_{s \in [t/3, t]} \hat{P}_{-x_1}(X_s = x_2)
\]
So, we need a lower bound of $\mathbb{P}_x^*(\sigma_{-x} < \frac{t}{3})$. By Lemma 4.2, for any $x \in \mathbb{N}$ and $r, s \geq 2|x|$, 
\begin{align*}
\mathbb{P}_x(\sigma_{x-r} > \sigma_{x+s}) = \frac{\sum_{i=x-r}^{x+r-1} \mu_i^{\frac{1}{2}}}{\sum_{i=x-s}^{x+s-1} \mu_i^{\frac{1}{2}}} \leq \frac{1}{(\mu_{x-r})^{\frac{1}{2}}(\mu_{x+s})^{\frac{1}{2}}} \leq c_1 \left( \frac{r}{r+s} \right)^{\alpha}. \tag{4.9}
\end{align*}

So, there exists $c_2 \in \mathbb{N}$ such that
\begin{align*}
\mathbb{P}_x(\sigma_{x} > \sigma_{c_2x}) \leq \frac{1}{8}, \text{ for all } x \in \mathbb{N}. \tag{4.10}
\end{align*}

By Lemma 4.3, there exist $c_3 = c_3(\alpha, d) \in (0, 1)$ and $K_4 = K_4(\delta, \alpha, d) \in (0, 1)$ such that
\begin{align*}
\mathbb{P}_x(|X_{t/3}| > K_4 R, \tau_{x,R} > t/3) \geq \frac{1}{4}, \text{ } x \in B(0, c_3 R). \tag{4.11}
\end{align*}

Now we choose $\varepsilon_0 = c_2^{-1} K_4 c_3$. Then $x_0 = \lfloor c_2^{-1} K_4 c_3 R \rfloor \in B(0, c_3 R)$ and so,
\begin{align*}
\mathbb{P}_{x_0}(\sigma_{-x_0} \wedge \sigma_{c_2x_0} < \frac{t}{3}, \tau_{0,10R} > \frac{t}{3}) \geq \mathbb{P}_{x_0}(|X_{t/3}| > K_4 R, \tau_{c_2x_0,R} > t/3) \geq \frac{1}{4}. \tag{4.12}
\end{align*}

Combining (4.11) with (4.10), we get
\begin{align*}
\mathbb{P}_{x_0}(\sigma_{-x_0} < \frac{t}{3}) \geq \frac{1}{8}.
\end{align*}

Substituting the above inequality into (4.8), we prove (4.7) for the second case. By symmetry, we have (4.7) as $x_1 < 0$. Therefore, (4.7) holds in any case. \hfill \Box

Proof of Theorem 4.1 If $|w| \geq 32R$, then one can take $c_1 = 8$ in (4.1) and the problem is reduced to Lemma 4.2. So, let $R > |w|/32$ in the following. Then $\rho_w(R) \in [c_1 R^{1+\alpha}, c_2 R^{1+\alpha}]$ and $B(w, R) \subset B(0, 40R)$. Fix $t \in [\delta \rho_w(R)^2, 2\rho_w(R)^2]$. Then $t \in [c_1 \delta R^{2+2\alpha}, c_2 R^{2+2\alpha}]$. By Lemma 4.3, for any $x \in B(0, 40R)$,
\begin{align*}
\mathbb{P}_x(|X_{t/3}| > K_4 R, \tau_{0,c_3R} > t/3) \geq \frac{1}{4}. \tag{4.13}
\end{align*}

Write $T = B(0, c_3 R) \setminus B(0, K_4 R)$. By Lemma 4.4 for all $x, y \in T$,
\begin{align*}
\mathbb{P}_x(X_{t/3} = y, \tau_{0,10c_3R} > t/3) \geq K_2 R^{-d}.
\end{align*}

Therefore, for any $x_1, x_2 \in B(w, R) \subset B(0, 40R),$
\begin{align*}
\mathbb{P}_{x_1}(X_t = x_2, \tau_{0,10c_3R} > t) &\geq \sum_{x,y \in T} \mathbb{P}_{x_1}(X_{t/3} = x, X_{2t/3} = y, X_t = x_2) \\
&= \sum_{x,y \in T} \mathbb{P}_{x_1}(X_{t/3} = x) \mathbb{P}_x(X_{t/3} = y) \mathbb{P}_y(X_{t/3} = x_2) \\
&\geq K_2 R^{-d} \sum_{x,y \in T} \mathbb{P}_{x_1}(X_{t/3} = x) \mathbb{P}_y(X_{t/3} = x_2)
\end{align*}
where we use \( \hat{\mathbb{P}} \) to denote the measure of the process \( X \) killed on exiting \( B(0, 10c_3R) \). Substituting \( V(w, R) \leq cR^{d+\alpha} \) and \( \tau_{0,10c_3R} \leq \tau_{w,cR} \) into (4.13), we complete the proof. \( \square \)

## 5 Proof of Theorem 1.1

**Lemma 5.1** There exists constant \( c_1 > 0 \) such that for any \( x, y \in \mathbb{Z}^d \),

\[
(v_x \vee v_y) | \log \nu_x - \log \nu_y|^3 \leq c_1 \rho(x, y).
\]

**Proof.** Let \( |x| > |y| \geq 1 \). Directly calculate

\[
\frac{(v_x \vee v_y)| \log \nu_x - \log \nu_y|^3}{(|x| \vee |y|)^\alpha \cdot |x - y|} = \frac{|x|^{\alpha} \vee |y|^{\alpha}}{|x|^{\alpha}} \cdot \frac{|x - y|}{|x|^{\alpha}} \cdot \frac{|x - y|}{|y|^{\alpha}} \cdot \frac{\log(|x|)}{(|x| - |y|)}
\]

\[
= \left( \frac{|y|}{|x|} \right)^{\alpha} \cdot |x|^{\alpha} \cdot \left( \frac{\log(|x|)}{|x| - |y|} \right) \cdot \frac{\log(|x|)}{|x| - |y|}
\]

\[
\leq |x|^{\alpha} \sup_{t > 1} \left\{ t^{(-\alpha)\wedge 0} \cdot \frac{\log^3 t}{t - 1} \right\}.
\]

Since \( \alpha > -1 \), the supremum of the right side is finite and hence if \( |x| > |y| \geq 1 \) then

\[
(v_x \vee v_y) | \log \nu_x - \log \nu_y|^3 \leq c(|x| \vee |y|)^\alpha |x - y| \leq c' \rho(x, y).
\]

The proof of the rest case is the same and so we omit the details. \( \square \)

**Proof of Theorem 1.1** We obtain the Gaussian upper bounds by the same way as Lemma [3.2] Write \( \eta = \nu_x \vee v_y \) for short. Set \( \tilde{\nu}_u = \eta^{-1} \nu_v \) and \( \tilde{\mu}_{uv} = \eta \mu_{uv} \) for each \( u, v \in \mathbb{Z}^d \). Denote \( \tilde{\rho} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+ \) by

\[
\tilde{\rho}(u, v) = \left( (2|\alpha|+2)^{-1} \cdot \eta^{-1} \rho(u, v) \right) \wedge |u - v|.
\]

Then \( \tilde{\rho}(\cdot, \cdot) \) is an adapted metric of \( \mathbb{Z}^d \), that is, for all \( u \in \mathbb{Z}^d \),

\[
\left\{ \frac{1}{\nu_u} \sum_{v \in \mathbb{Z}^d} \tilde{\rho}(u, v)^2 \tilde{\mu}_{uv} \leq 1 \right\} \quad \text{and} \quad \tilde{\rho}(u, v) \leq 1 \quad \text{whenever} \quad v \sim u.
\]

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By Lemma 2.2
\[ \eta^{-1} \rho(x, y) \leq (\nu_x \vee \nu_y)^{-1} \cdot c(|x| \vee |y|)\eta^2 |x - y| \leq c|x - y|_1. \]
So,
\[ c_1^{-1} \eta^{-1} \rho(x, y) \leq \tilde{\rho}(x, y) \leq c_1 \eta^{-1} \rho(x, y). \] (5.1)
Set \( Z_s = X_{\eta^2 s} \) for \( s \geq 0 \). Then \( Z \) has the generator
\[ \mathcal{L} f(u) = \frac{1}{\nu_u} \sum_{v \in \mathcal{V}} (f(v) - f(u))\tilde{\mu}_{uv}. \]
By Proposition 3.5 for each \( s \in \mathbb{R} \)
\[ \mathbb{P}_z(Z_s = z) = \mathbb{P}_z(X_{\eta^2 s} = z) \leq \frac{c_2 \nu_z}{V(z, \rho_z^{-1}(\eta s^{1/2}))} := \frac{1}{f_z(s)}. \]
By Lemma 2.5 and the inequality (2.5), for each \( s \geq (\log \nu_x - \log \nu_y)^2 \) we have
\[ f_z(s) \leq \frac{V(z, \rho_z^{-1}(\eta s^{1/2}))}{c_2 \nu_z} \leq c(\rho_z^{-1}(\eta s^{1/2}))^c \leq c' \left( \frac{\eta \nu_z}{\rho_z(1)} \right)^c \leq c' \left( \frac{\nu_x \vee \nu_y}{\nu_x \wedge \nu_y} \right)^{c'} \leq c' e^{c' s^{1/2}}. \]
Therefore, similar to (3.9) we can apply [8, Theorem 5.1] and get
\[ \mathbb{P}_x(Z_s = y) \leq \frac{c_3 (\nu_y / \nu_x)^{1/2}}{\sqrt{f_z(s/c_3) f_y(s/c_3)}} \exp \left( -\frac{\tilde{\rho}(x, y)^2}{c_3 s} \right) \quad \text{for all } s \geq |c_3 \log(\nu_x / \nu_y)|^3 \vee \tilde{\rho}(x, y). \]
By the inequality (5.1) and Lemma 5.1
\[ |c_3 \log(\nu_x / \nu_y)|^3 \leq c_4 \tilde{\rho}(x, y). \]
So, for each \( t \geq c_4 c_1 \eta \rho(x, y) \), we have \( \eta^{-2} t \geq |c_3 \log(\nu_x / \nu_y)|^3 \vee \tilde{\rho}(x, y) \) and
\[ \mathbb{P}_x(X_t = y) = \mathbb{P}_x(Z_{\eta^2 t} = y) \leq \frac{c_5 (\nu_y / \nu_x)^{1/2}}{\sqrt{f_z(\eta^{-2} t/c_3) f_y(\eta^{-2} t/c_3)}} \exp \left( -\frac{\tilde{\rho}(x, y)^2}{c_3 \eta^{-2} t} \right) \]
\[ \leq \frac{c \nu_y}{\sqrt{V(x, \rho_x^{-1}(t^{1/2})) V(y, \rho_y^{-1}(t^{1/2}))}} \exp \left( -c' \frac{\rho(x, y)^2}{t} \right). \] (5.2)
Further, by Lemma 2.4 we conclude that
\[ p_t(x, y) \leq \frac{c}{\sqrt{V_\rho(x, t^{1/2}) V_\rho(y, t^{1/2})}} \exp \left( -c' \frac{\rho(x, y)^2}{t} \right), \quad t \geq c_4 c_1 \eta \rho(x, y). \] (5.3)
On the other hand, by [8, Corollary 2.8], if \( s \leq c_4 c_1 \tilde{\rho}(x, y) \) then
\[ \mathbb{P}_x(Z_s = y) \leq c(\tilde{\nu}_y / \tilde{\nu}_x)^{1/2} \exp \left( -c' \tilde{\rho}(x, y) \left( 1 \vee \log \left( \tilde{\rho}(x, y) / s \right) \right) \right). \]
Hence, for each $t \leq c_4 c_1 \eta \rho(x, y)$,
\[
\mathbb{P}_x(X_t = y) = \mathbb{P}_x(Z_{\eta^{-1} t} = y) \leq c (\tilde{\nu}_y / \tilde{\nu}_x)^{1/2} \exp \left( -c' \rho(x, y) \left( 1 + \log \left( \eta^2 \rho(x, y) / t \right) \right) \right) \\
\leq c (\nu_y / \nu_x)^{1/2} \exp \left( -c' \eta^{-1} \rho(x, y) \left( 1 + \log \left( \eta \rho(x, y) / t \right) \right) \right) .
\]  
(5.4)
Combining (5.4) with (5.3), we conclude that both (1.2) and (1.3) are true.

The Gaussian lower bound is proved by a standard chaining argument. If $t \geq \rho(x, y)^2$, then there exists $c_1 > 1$ such that $t \geq c_1^2 \rho_x(|x-y|^2)$. Applying Theorem 4.1 on $B(x, \rho^{-1}_x(c_1 t^{1/2}))$, we get
\[
p_t(x, y) \geq \frac{c}{V(x, \rho^{-1}_x(c_1 t^{1/2}))} \geq \frac{c'}{V^\rho(x, t^{1/2})} .
\]  
(5.5)
So, let $(\nu_x \lor \nu_y) \rho(x, y) \leq t \leq \rho(x, y)^2$. Fix an $L_1$-geodesic path $\gamma$ from $x$ to $y$. By Lemma 2.2 there exists $c_2 > 1$ such that
\[
\nu(\gamma) \leq c_2 \rho(x, y) .
\]
Set $r = t / \rho(x, y)$, then
\[
\rho(x, y) \geq r \geq \nu_x \lor \nu_y = \max_{z \in \gamma} \nu_z .
\]
Hence there exists a sequence of vertices $y = z_0, z_1, \ldots, z_m = x$ on the path $\gamma$, such that
\[
m \leq 2 c_2 \rho(x, y) / r = 2 c_2 \frac{\rho(x, y)^2}{t} \quad \text{and} \quad r \leq \rho(z_{i-1}, z_i) \leq 2 r \quad \text{for} \quad i \leq m .
\]
As a result,
\[
|z_{i-1} - z_{i-2}| \leq c_3 \rho^{-1}_{z_{i-1}}(\rho(z_{i-1}, z_{i-2})) \leq c_3 \rho^{-1}_{z_{i-1}}(2 \rho(z_{i-1}, z_i)) \leq (c_3 - 1) |z_i - z_{i-1}| .
\]
Write $r_i = |z_i - z_{i-1}|$, $F_i = B(z_i, r_i)$ and $F_i^* = B(z_i, c_3 r_i)$ for $i \leq m$. Then
\[
F_{i-1} \cup F_i \subset F_i^* .
\]
Set $s = (4 c_2)^{-1} r^2$. Then $s \asymp \rho(z_i, z_{i-1})^2 \asymp \rho_z(r_i)^2$. As (5.5), we have
\[
p_s(y', x') \geq \frac{c_4}{\nu(F_i^*)} \quad \text{for} \quad y' \in F_{i-1}, x' \in F_i .
\]  
(5.6)
By Lemma 2.5 for $y' \in F_{i-1}$,
\[
\mathbb{P}_{y'}(X_s \in F_i) \geq c_4 \frac{\nu(F_i)}{\nu(F_i^*)} \geq c_5 .
\]
Note that
\[
t - ms = t - m \cdot (4 c_2)^{-1} r^2 \geq t - (2 c_2 \rho(x, y) / r) \cdot (4 c_2)^{-1} r \cdot (t / \rho(x, y)) = \frac{t}{2} .
\]
So, as (5.5) we can get
\[
p_{t - ms}(x, y') \geq \frac{c_6}{V^\rho(x, t^{1/2})} \quad \text{for} \quad y' \in F_m .
\]  
(5.7)
Therefore,
\[
p_t(x, y) = p_t(y, x) \geq \mu_x^{-1} \mathbb{P}_y(X_{is} \in F_i, 1 \leq i \leq m, X_t = x)
\]
\[
\geq c_5^m \min_{y' \in F_m} \mathbb{P}_y(X_{t} - ms = x) \mu_x^{-1}
\]
\[
= c_5^m \min_{y' \in F_m} p_{t - ms}(x, y') \geq c_5^m \frac{c_6}{V_\rho(x, t^{1/2})}
\]
\[
\geq \frac{c_6}{V_\rho(x, t^{1/2})} \exp\{-c'_5 m\} \geq \frac{c_6}{V_\rho(x, t^{1/2})} \exp\{-2c_5 c'_5 \rho(x, y)^2\},
\]
which implies (1.4). We have completed the proof of Theorem 1.1.

\[\square\]

6 Proof of Theorem 1.3

Proof of Theorem 1.3. (1) By Theorem 1.1 and Lemma 2.4, if \(\alpha < d - 2\) then
\[
\int_1^\infty p_t(0, 0) dt \leq c \int_1^\infty t^{-(d+\alpha)/(2+2\alpha)} dt = \frac{2 + 2\alpha}{d - 2 - \alpha} c < \infty.
\]
Hence if \(\alpha < d - 2\) then \(X\) is transient. Similarly, if \(\alpha \geq d - 2\) then \(\int_1^\infty p_t(0, 0) dt = \infty\) and so \(X\) is recurrent.

(2) Let \(X'\) be an independent copy of \(X\). We use \(\mathbb{P}_{x,x'}\) for the probability measure of the processes \(X\) and \(X'\) which start from \(x\) and \(x'\) respectively.

If \(d = 1\) then
\[
\int_1^\infty \mathbb{P}_0(X_t = X'_t = 0) dt = \int_1^\infty \mathbb{P}_0(X_t = 0) \mathbb{P}_0(X'_t = 0) dt
\]
\[
= \int_1^\infty p_t(0, 0)^2 dt \geq c \int_1^\infty t^{-2(1+\alpha)/(2+2\alpha)} dt = \infty.
\]
So, \((X, X')\) is recurrent, which implies \(X\) and \(X'\) collide at the origin infinitely often.

Let \(d = 2\). Fix \(\lambda = \lceil 100 \text{[1.4]} \rceil > 100\). For \(k \geq 1\), we set
\[
t_k = \lambda^{2k(1+\alpha)},
\]
\[
T_k = B(0, 2\lambda^k) - B(0, \lambda^k),
\]
\[
\theta_k = \inf\{t \geq 0 : |X_t| \geq \lambda^{k+1}\}, \quad \theta'_k = \inf\{t \geq 0 : |X'_t| \geq \lambda^{k+1}\}
\]
and
\[
H_k = \int_0^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_t = X'_t \in T_k\}} dt.
\]
So, if \(H_k > 0\) then there exists at least one collision of \(X\) and \(X'\) before their breaking out of \(B(0, \lambda^{k+1})\). We shall use the second moment method to estimate the probability of the event \(\{H_k > 0\}\) as the approach of [9, 10]. Fix \(x, y \in B(0, \lambda^k)\). Then
\[
\mathbb{E}_{x,y}(H_k) = \int_0^{2t_k} \mathbb{P}_{x,y}(X_t = X'_t \in T_k, \theta_k > t, \theta'_k > t) dt
\]
inequality (6.1) becomes

By the strong Markov property, we get for each \(u, v \in B(0, 2\lambda^k)\) and \(t \in [t_k, 2t_k]\),

\[
P_u(X_t = v, \theta_k > t) \geq \frac{c \nu_v}{V(0, 2\lambda^k)}.
\]

By Lemma 2.4 for \(v \in \mathbb{T}_k\),

\[
\frac{\nu_v}{V(0, 2\lambda^k)} \geq c \frac{|v|^\alpha}{(2\lambda^k)^{2+\alpha}} \geq c' \lambda^{-2k}.
\]

Hence \(P_u(X_t = v, \theta_k > t) \geq c' \lambda^{-2k}\) for each \(u \in \{x, y\}, v \in \mathbb{T}_k\) and \(t \in [t_k, 2t_k]\). Therefore, inequality (6.1) becomes

\[
E_{x,y}(H_k) \geq (c\lambda^{-2k})^2 \cdot |T_k| \cdot t_k \geq c^2 \lambda^{-4k} \cdot c'(\lambda^k)^2 \cdot \lambda^{2k(1+\alpha)} = c'' \lambda^{2k\alpha}.
\]

On the other hand, for any \(u \in \mathbb{T}_k\),

\[
E_{u,u}(H_k) \leq \int_0^{2t_k} \sum_{w \in \mathbb{T}_k} \left[\mathbb{P}_u(X_t = w)\right]^2 dt
\]

\[
\leq \frac{\max_{w \in \mathbb{T}_k} \nu_w}{\nu_u} \int_0^{2t_k} \sum_{w \in \mathbb{T}_k} \mathbb{P}_u(X_t = w) \mathbb{P}_w(X_t = u) dt
\]

\[
\leq c \int_0^{2t_k} \mathbb{P}_u(X_{2t} = u) dt \leq c^2 \nu_u^2 + c \int_{\nu_u^2}^{2t_k} \mathbb{P}_u(X_{2t} = u) dt
\]

\[
\leq c \nu_u^2 + \int_{\nu_u^2}^{2t_k} \frac{c' \nu_u}{V_p(u, t^{1/2})} dt
\]

\[
\leq c \nu_u^2 + c'' \nu_u^2 \int_{\nu_u^2}^{2t_k} t^{-1} dt,
\]

where the last second inequality is by (1.3), while the last by Lemma 2.4. Hence

\[
E_{u,u}(H_k) \leq c \nu_u^2 (1 + \log(2t_k) - \log(\nu_u^2)) \leq c' \lambda^{2k\alpha} \cdot (\log(\lambda^{2k(1+\alpha)}) - \log(\lambda^{2k\alpha})) = c'' k \lambda^{2k\alpha}.
\]

By the strong Markov property,

\[
E_{x,y}(H_k^2) = 2E_{x,y} \left( \int_0^{\theta_k \wedge \theta_k' \wedge 2t_k} 1_{\{X_s = X_s' \in \mathbb{T}_k\}} ds \int_t^{\theta_k \wedge \theta_k' \wedge 2t_k} 1_{\{X_s = X_s' \in \mathbb{T}_k\}} ds \right)
\]

\[
\leq 2E_{x,y} \left( \int_0^{\theta_k \wedge \theta_k' \wedge 2t_k} 1_{\{X_s = X_s' \in \mathbb{T}_k\}} E_{X_{s},X_{s}'}(H_k) ds \right)
\]

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\[ \leq 2 \sup_{u \in \mathbb{T}_k} E_{u,u}(H_k) E_{x,y}(H_k). \]

So, by (6.3), (6.2) and the Cauchy Schwarz inequality,

\[ \mathbb{P}_{x,y}(H_k > 0) \geq \frac{[E_{x,y}(H_k)]^2}{E_{x,y}(H_k)} \geq \frac{E_{x,y}(H_k)}{2 \sup_{u \in \mathbb{T}_k} E_{u,u}(H_k)} \geq \frac{c}{k}. \]

Therefore, when \( X \) and \( X' \) start from \( x, y \in B(0, \lambda^k) \) respectively, the probability that they will collide before their breaking out \( B(0, \lambda^{k+1}) \), is not less than \( \frac{c}{k} \). Note that \( \sum_k \frac{1}{k} = \infty. \) Using the second Borel-Cantelli Lemma as [10, Theorem 1.1], we prove that \( X \) and \( X' \) collide infinitely often when \( d = 2 \).

(3) Let \( d \geq 3 \). For \( k \geq 0 \), set

\[ T_k = B(0, 2^{k+1}) - B(0, 2^k) \quad \text{and} \quad Z_k = \int_0^\infty 1_{\{X_t = x' \in \mathbb{T}_k\}} dt. \]

Then

\[ E_{0,0}(Z_k) = \sum_{u \in \mathbb{T}_k} \int_0^\infty [\mathbb{P}_0(X_t = u)]^2 dt \]

\[ = \sum_{u \in \mathbb{T}_k} \int_0^{t_k} [\mathbb{P}_0(X_t = u)]^2 dt + \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} [\mathbb{P}_0(X_t = u)]^2 dt + \sum_{u \in \mathbb{T}_k} \int_0^{s_k} [\mathbb{P}_0(X_t = u)]^2 dt \]

\[ = I_1 + I_2 + I_3, \]

where \( s_k = (1 \lor 2^k \alpha) 2^{k(1+\alpha)} \) and \( t_k = 2^{k(2+2\alpha)} \). We shall deal with the three sums separately. Since \( t_k \geq c\rho(0, u)^2 \) for \( u \in \mathbb{T}_k \), we can use Theorem 1.1 and Lemma 2.4, and get

\[ I_1 \leq \sum_{u \in \mathbb{T}_k} \int_0^{t_k} \frac{c v_u^2}{V_p(0, t^{1/2}) V_p(u, t^{1/2})} dt \]

\[ \leq |\mathbb{T}_k| \cdot \max_{u \in \mathbb{T}_k} v_u^2 \cdot \int_{t_k}^{\infty} c t^{-(d+\alpha)/(1+\alpha)} dt \]

\[ \leq 2^{dk} \cdot c'' 2^{2k \alpha} \cdot c''' (2^{2k(1+\alpha)})^{1-(d+\alpha)/(1+\alpha)} \]

\[ = c 2^{k(2+2\alpha-d)}. \]

Next, since \( (1 \lor \nu_u) \rho(0, u) \geq c s_k \) and \( t_k^{1/2} \leq c' |u|^{1+\alpha} \) for \( u \in \mathbb{T}_k \), using Theorem 1.1 and Lemma 2.4 again gives

\[ I_2 \leq \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} \frac{c v_u^2}{V_p(0, t^{1/2}) V_p(u, t^{1/2})} \exp \left( -\frac{\rho(0, u)^2}{c t} \right) dt \]

\[ \leq \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} \frac{c v_u^2}{t^{(d+\alpha)/(2+2\alpha)} \cdot t^{d/2} |u|^{(d-1)\alpha}} \exp \left( -\frac{2^{2k(1+\alpha)}}{c t} \right) dt \]
\[
\leq |T_k| \cdot \max_{u \in T_k} \{ \nu_u^2 |u|^{(d-1)\alpha} \} \cdot \int_0^\infty c_t^{-(d+\alpha)/(2+2\alpha)-d/2} \exp \left( -\frac{2^{2k(1+\alpha)}}{c_t} \right) dt
\]
\[
\leq 2^d k \cdot c'' \cdot (d+1)\alpha \cdot (2^{2k(1+\alpha)})^{1-(d+\alpha)/(2+2\alpha)-d/2} \cdot c'' \int_0^\infty x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx
\]
\[
= 2^d k \cdot c'' \cdot (1 \lor 2^{k(1+\alpha)}) \cdot 2^{k(1+\alpha)} \exp \left( -c'(1 \lor 2^{k(1+\alpha)})^{-1} \cdot 2^{2k(1+\alpha)} \right)
\]
\[
= 2^d k c'' \cdot 2^{2k(1+\alpha)-d} \cdot e^{-c'' 2^{2k(1+\alpha)-d}}.
\]

Therefore,
\[
E_{0,0}(Z_k) \leq 2^d k \cdot c'' \cdot 2^{2k(1+\alpha)-d}.
\]

For the remaining term, applying Theorem 1.1 we still have
\[
I_3 \leq \sum_{u \in T_k} \int_0^{s_k} (\nu_u/\nu_0) \exp \left( -c(\nu_0 \lor \nu_u)^{-1} \rho(0, u) \left( 1 \lor \log \left( (\nu_0 \lor \nu_u)\rho(0, u)/t \right) \right) \right) dt
\]
\[
\leq |T_k| \cdot s_k \cdot \max_{u \in T_k} \nu_u \exp \left( -c(\nu_0 \lor \nu_u)^{-1} \rho(0, u) \right)
\]
\[
\leq 2^d k \cdot (1 \lor 2^{k\alpha})^2 \cdot 2^{k(1+\alpha)} \cdot 2^{k\alpha} \exp \left( -c'(1 \lor 2^{k\alpha})^{-1} \cdot 2^{2k(1+\alpha)} \right)
\]
\[
= 2^d k \cdot c'' \cdot 2^{2k(1+\alpha)-d} \cdot e^{-c'' 2^{2k(1+\alpha)-d}}.
\]

Therefore,
\[
E_{0,0}(Z_k) \leq 2^d k \cdot c'' \cdot 2^{2k(1+\alpha)-d}.
\]

On the other hand, once \(X\) and \(X'\) collide at some vertex \(u\) and some time \(t\), then with at least \(e^{-2}\) probability they will stick together during time \([t, t + \nu_u/\mu_u]\), which implies
\[
E_{0,0}(Z_k | Z_k > 0) \geq c \min_{u \in T_k} \nu_u/\mu_u \geq c' 2^{2k\alpha}.
\]

So, for each \(k \geq 0\),
\[
\mathbb{P}_{0,0}(Z_k > 0) = \frac{E_{0,0}(Z_k)}{E_{0,0}(Z_k | Z_k > 0)} \leq e^{-(d-2)k}.
\]

Therefore,
\[
\sum_k \mathbb{P}_{0,0}(Z_k > 0) \leq c \sum_k e^{-(d-2)k} < \infty.
\]

By the Borel-Cantelli Lemma, we completed the proof of (3). \(\square\)

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Xinxing Chen
Department of Mathematics, Shanghai Jiaotong University, Shanghai, China, 200240.
E-mail: chenxinx@sjtu.edu.cn