1. Introduction

The aim of this paper is to construct an immersion of the Drinfeld moduli schemes in a finite product of infinite Grassmannians, such that they will be locally closed subschemes of these Grassmannians which represent a kind of flag varieties. This construction is derived from two results: the first is that the moduli functor of vector bundles with an \( \infty' \)-formal level structure (defined below) over a curve \( X \) is representable, in a natural way, by a closed subscheme of the infinite Grassmannian. The second is an equivalence (see \([8],[19],[7],[6],[6]\)) between Drinfeld \( A \)-modules of rank \( n \) and elliptic sheaves, extended for level structures in \([?]\).

Let us detail these results. Let \( X \) be a smooth, proper and geometrically irreducible curve over a finite field \( \mathbb{F}_q \), \( \infty \) a rational point of \( X \), \( A = H^0(X - \infty, \mathcal{O}_X) \), \( D \) an effective divisor over \( \text{Spec}(A) \), and \( S \) an arbitrary scheme over \( \mathbb{F}_q \). \( \hat{A}^\infty \) denotes the ring of adeles outside \( \infty \) and

\[
\hat{O}^\infty = \lim_{\longleftarrow I} A/I
\]

\( I \) ideal of \( A \)

The moduli functor of vector bundles of rank \( n \), with a \( \infty' \)-formal level structure is:

\[
\mathcal{M}_{\infty}^n(S) = \lim_{\longleftarrow D > 0} \mathcal{M}_D^n(S)
\]

\( \infty \notin \text{supp}(D) \)

where

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\[ M^n_D(S) = \begin{cases} \text{D-level structures, } (M, f_D), \text{ over a locally free sheaf } M \text{ of rank } n \text{ over } X \times S, \text{ up to isomorphism} \end{cases} \]

In this paper, we consider the moduli functor \[ \widehat{M}^n = \lim_{\longrightarrow} D^n = \lim_{\longrightarrow} D^n_{D} \]

where \( D^n_D \) is the moduli functor of Drinfeld \( \mathcal{A} \) modules endowed with a \( D \)-level structure with characteristic away from \( \text{supp}(D) \). Or equivalently elliptic sheaves with a twisted \( \infty' \)-level structure (see Remark 3).

The results will be obtained bearing in mind that the infinite Grassmannian scheme \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \) \((\mathcal{O}^1_{\mathcal{A}/k} \otimes_{A} (\hat{\mathcal{A}}^\infty)^n, \Omega^1_{\mathcal{A}/k} \otimes_{A} (\hat{\mathcal{O}}^\infty)^n)\) represents the functor \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \) and that the Krichever map \((17), (23), (21)\) given in (3.3)

\[ \varphi: \widehat{M}^n \to \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \]

makes \( \widehat{M}^n \) a closed subfunctor of \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \). Therefore, it is representable by a closed subscheme of \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \). In the literature this morphism is considered when \( S \) is a field, recently Ines Quandt extended it for noetherian schemes. In this paper it is considered for an arbitrary scheme, although fixing the curve \( X \), (when \( S \) is the base field, those results are the Weil’s uniformation of vector bundles, in [3] are settled similar results but in another setting of infinite Grassmannian).

Putting together the Krichever morphism with the equivalence between Drinfeld \( \mathcal{A} \) modules and elliptic sheaves, we obtain in a natural way, a locally closed immersion of functors

\[ \psi: \widehat{D}^n \hookrightarrow \prod_{i=0}^{n-1} \text{Gr}(\Omega^1_{\mathcal{A}/k} \otimes_{A} (\hat{\mathcal{A}}^\infty)^n, \Omega^1_{\mathcal{A}/k} \otimes_{A} (\hat{\mathcal{O}}^\infty)^n) \]

(\( \Omega^1_{\mathcal{A}/k} \) is the sheaf of differentials over \( A \))

This infinite Grassmannian is a slight modification of \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \). Therefore, \( \widehat{D}^n \) is representable by a locally closed subscheme of a finite product of infinite Grassmannians, moreover it can be seen as a point of an infinite flag variety, in this setting, one can see an analogy with Deligne-Lusztig variety for the Coxeter element [?].

Similar results could be obtained when \( \lim_{\longrightarrow} D^n_{mD} \) and \( \lim_{\longrightarrow} M^n_{mD}(S) \) are considered instead of the projective limits:

\[ \lim_{\longrightarrow} D^n_{D} \quad \lim_{\longrightarrow} M^n_{D}(S) \]

Let us now briefly state the contents of the different sections of this paper. In the second and third sections we study the infinite Grassmannian and the Krichever morphism over an arbitrary scheme and conclude that \( \widehat{M}^n \) is representable by a closed subscheme of \( \text{Gr}((\hat{\mathcal{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \). Using this, in the fourth section, we show that the moduli functors of Drinfeld \( \mathcal{A} \)-modules with level structures are representable by locally closed schemes of a finite product of infinite Grassmannians.
2. Preliminaries on the Sato’s infinite Grassmannian.

Let $V$ be a vector space over a field $k$ and $A, B$ vector subspaces of $V$.

**Definition 2.1.** Two vector subspaces $A, B$ of $V$ are said to be commensurable if $A + B/A \cap B$ is a vector space over $k$ of finite dimension. 

Let us consider a collection, $F = \{V_i\}_{i \in I}$, of subspaces of $V$, such that $\cap_i V_i = 0$ and $V_i$ is commensurable to $V_j$, for all $i, j$, with $F$ it is possible to define a Hausdorff topology on $V$ (Tate’s topology): $F$ is a basis of neighborhoods of 0.

$A \sim F$ will denote an open subspace of $V$ commensurable with any $\{V_i\}_{i \in I}$ (or equivalently with some $V_i$).

**Definition 2.2.** The completion of $V$ with respect to the $F$-topology is defined by:

$$\hat{V} = \lim_{A \sim F} V/A$$

Analogously, given a vector subspace $B \subseteq V$ we can define the completions of $B$ and $V/B$ with respect to $B \cap F$ and $B + F/B = \{B + V_i/B\}_{i \in I}$, respectively.

**Example 1.**
- $(V, F = \{0\}); V$ is complete.
- $V = k((t)), F = \{t^n k[[t]]\}_{n \in \mathbb{Z}}; V$ is complete.
- Let $(X, \mathcal{O}_X)$ be a smooth, proper and irreducible curve over the field $k$, and let $V$ be the ring of adeles of the curve and $F = \{\prod_p \widehat{\mathcal{O}_p}\}_{1 \subseteq \mathcal{O}_X}$ ($\widehat{\mathcal{O}_p}$ being the $m_p$-adic completion of the local ring of $X$ in the point $p$) and $I$ ideals of $\mathcal{O}_X$; $V$ is complete with respect the $F$-topology.

**Definition 2.3.** Given a $k$-scheme $S$ and a vector subspace $B \subseteq V$, we define the sheaves over $S$:

- $\hat{V}_S = \lim_{A \sim F} (V/A \otimes \mathcal{O}_S)$.
- $\hat{B}_S = \lim_{A \sim F} ((B/A \cap B) \otimes \mathcal{O}_S)$.
- $(V/B)_S = \lim_{A \sim F} ((V/A + B) \otimes \mathcal{O}_S)$.

We have the exact sequence (II.9.1 and II.9.2.1): 

$$0 \rightarrow \hat{B}_S \rightarrow \hat{V}_S \rightarrow (V/B)_S$$

Remark that $\hat{V}_S$ and $\hat{B}_S$ are not quasi-coherent $\mathcal{O}_S$-modules but $(V/B)_S$ it is.

A discrete vector subspace of $V$ is a vector subspace, $L \subseteq V$, such that $L \cap V_i$ and $V/L + V_i$ are $k$-vector spaces of finite dimension (for some $i \in I$).

**Definition 2.4.** Given a $k$-scheme $S$, a discrete submodule of $\hat{V}_S$ is a subfunctor $L \subseteq \hat{V}_S$ in the category of $S$-schemes, given by a quasi-coherent sheaf of $\mathcal{O}_S$-modules, verifying: for each $s \in S$ there exists an open neighborhood $U_s$ of $s$ and an open commensurable $k$-vector subspace $B \sim F$ such that: $L_{U_s} \cap \hat{B}_{U_s}$ is free of type finite and $\hat{V}_{U_s}/L_{U_s} + \hat{B}_{U_s} = 0$. 
We define the Grassmannians functor over the category of \( k \)-schemes by:

\[
\text{Gr}(V,F)(S) = \left\{ \text{discrete sub-O}_S\text{-modules of } \hat{V}_S \text{ with respect to the Tate's topology for } F \right\}
\]

Remark 1. Note that if \( V \) is a finite dimensional \( k \)-vector space and \( F = \{0\} \), then \( \text{Gr}(V,\{0\}) \) is the usual Grassmannian functor defined by Grothendieck [12]. ( \( \mathcal{L} \) is a subfunctor of \( V \otimes O_S \) in the category of \( S \)-schemes if and only if \( V \otimes O_S / \mathcal{L} \) is flat.)

Theorem 2.5. ([3], [6], [4], [14]) The functor \( \text{Gr}(V,F) \) is representable by a \( k \)-scheme \( \text{Gr}(V,F) \).

Proof. If for each \( B \sim F \) and \( \mathcal{L}_o \) transversal to \( B \), we consider the representable functors \( F_B(S) = \text{Hom}_{O_S}(L_oS, \hat{B}_S) \), then we obtain an open covering of schemes for \( \text{Gr}(V,F) \).

Fixing \( V^+ \sim F \), the index of an element \( \mathcal{L} \in \text{Gr}(V,F)(S) \), is defined as the locally constant function \( i_{\mathcal{L}} : S \rightarrow \mathbb{Z} \) defined by:

\[
i_{\mathcal{L}}(s) = \dim_{k(s)}(\mathcal{L}_s \cap \hat{V}^+_s) - \dim_{k(s)}\hat{V}_s / \mathcal{L}_s + \hat{V}^+_s
\]

Remark 2. Let \( \text{Gr}^d(V,F) \) be the moduli scheme of discrete submodules of index \( d \). One then has:

\[
\text{Gr}(V,F) = \coprod_{d \in \mathbb{Z}} \text{Gr}^d(V,F)
\]

For further details about this construction see [4].

For our purpose \( F' = \{I(\hat{O}^\infty)^n\}_{I \subseteq A} \) and \( V = (\hat{A}^\infty)^n \) (I ideal of \( A \)). We denote now \( \text{Gr}((\hat{A}^\infty)^n, F') \) by \( \text{Gr}((\hat{A}^\infty)^n, (\hat{O}^\infty)^n) \)

3. The Krichever morphism as a closed immersion in the Sato’s infinite Grassmannian.

Let \( S \) be scheme over \( k \) (\( k \) not necessarily finite), \( X \) a proper, smooth and geometrically irreducible over \( k \), \( \pi : X \times S \rightarrow S \) the natural projection, and \( M \) a locally free sheaf on \( X \times S \) of rank \( n \). We shall denote by \( O^\infty_M \) the sheaf on \( S \), \( \pi_*(i_*M|_{\text{Spec}(O_\infty \times S)}) \), where \( O_\infty \) is the stalk of \( O_X \) in \( \in \in X \) and \( i : \text{Spec}(O_\infty) \times S \rightarrow X \times S \) is the natural morphism. For an effective divisor \( D \) on \( X \), such that \( \in \not\in \text{supp}(D) \), \( O_{X \times S}(D) \) will be denote by \( O_X(D) \otimes_k O_S \).

Definition 3.1. A \( D \)-level structure on \( S \), \( (M, f_D) \), is a surjective morphism of sheaves of \( O_{X \times S} \)-modules

\[
f_D : M \rightarrow (O_X/O_X(-D))^n \otimes_k O_S
\]

Two \( D \)-level structures, \( (M, f_D) \) and \( (M', f_{D'}) \), are said to be equivalent if there exists an isomorphism of sheaves of \( O_{X \times S} \)-modules, \( \phi : M \rightarrow M' \), such that the following diagram is commutative.
Lemma 3.3. There exists a natural morphism of functors in the category of \( k \)-schemes:

\[
\varphi : \hat{\mathcal{M}}_{\infty}^n \to \hat{O}_S(\hat{\mathcal{A}}_{\infty}^n) 
\hat{\mathcal{O}}_{\infty}^n(\hat{\mathcal{A}}_{\infty}^n)
\]

being \( \varphi(M, f^\infty) = \pi_*(f^\infty \otimes 1)(\hat{\mathcal{M}}_{\infty}^n) \).

Definition 3.2. A \( \infty^\prime \)-formal level structure on \( S \), \( (M, f^\infty) \), is an element of

\[
\hat{\mathcal{M}}_{\infty}^n(S) = \lim_{\substack{D > 0 \\ \infty \notin \text{supp}(D)}} \hat{\mathcal{M}}_{\infty}^n(S)
\]

We denote by \( \hat{\mathcal{O}}_{\infty}^\wedge \) and \( \hat{\mathcal{A}}_{\infty}^\wedge \) the sheaves of \( \mathcal{O}_{X \times S} \)-modules:

\[
\lim_{\substack{D > 0 \\ \infty \notin \text{supp}(D)}} \mod \mathcal{O}_X / \mathcal{O}_X (-D) \otimes \mathcal{O}_S \text{ and } \hat{\mathcal{O}}_{\infty}^\wedge \otimes_{\mathcal{O}_{X \times S}} i_* (\mathcal{O}_\infty \otimes \mathcal{O}_S)
\]

respectively.

If we take the direct image by \( \pi : X \times S \to S \) in \( \hat{\mathcal{O}}_{\infty}^\wedge \) and \( \hat{\mathcal{A}}_{\infty}^\wedge \), we obtain two sheaves of \( \mathcal{O}_S \)-modules, which will be called \( \hat{\mathcal{O}}_{\infty}^\wedge \) and \( \hat{\mathcal{A}}_{\infty}^\wedge \). In the case \( S = \text{Spec}(k) \), \( \hat{\mathcal{O}}_{\infty}^\wedge \) is precisely \( \hat{\mathcal{O}}_{\infty} \) and \( \hat{\mathcal{A}}_{\infty} \) is \( \hat{\mathcal{A}}_{\infty} \).

If \( (M, f^\infty) \) is a \( \infty^\prime \)-formal level structure on \( S \) we can obtain an exact sequence of sheaves of \( \mathcal{O}_{X \times S} \)-modules:

\[
0 \to M \to i_*(M_{\text{Spec}(\mathcal{O}_\infty) \times S}) \to f^\infty (\hat{\mathcal{A}}_{\infty}^\wedge / \hat{\mathcal{O}}_{\infty}^\wedge)^n \to 0
\]

being \( f^\infty \) the morphism obtained from the level structure and \( f^\infty \) the composition of the morphisms:

\[
f^\infty \otimes 1 : M \otimes_{\mathcal{O}_{X \times S}} i_* (\mathcal{O}_\infty \otimes \mathcal{O}_S) = i_*(M_{\text{Spec}(\mathcal{O}_\infty) \times S}) \to (\hat{\mathcal{O}}_{\infty}^\wedge)^n \otimes_{\mathcal{O}_{X \times S}} i_* (\mathcal{O}_\infty \otimes \mathcal{O}_S) = (\hat{\mathcal{A}}_{\infty}^\wedge)^n
\]

and the natural projection \( (\hat{\mathcal{A}}_{\infty}^\wedge)^n \to (\hat{\mathcal{A}}_{\infty}^\wedge)^n / (\hat{\mathcal{O}}_{\infty}^\wedge)^n \).

It is easily obtained tensoring by \( M \) the sequence of \( \mathcal{O}_{X \times S} \)-modules:

\[
0 \to \mathcal{O}_{X \times S} \xrightarrow{h} i_* (\mathcal{O}_\infty \otimes \mathcal{O}_S) \to j_* \hat{\mathcal{A}}_{\infty}^\wedge / \hat{\mathcal{O}}_{\infty}^\wedge \to 0
\]

where \( h \) and \( j \) are the natural morphisms.

So by tensoring by \( \mathcal{O}_X (D) \) this exact sequence and taking \( \pi_* \), we obtain:

\[
\pi_* M(D) = O^M_{\infty} \cap (\hat{\mathcal{O}}_{\infty}^\wedge(D))^n \quad R^1 \pi_* M(D) = \mod (\hat{\mathcal{A}}_{\infty}^\wedge)^n / O^M_{\infty} + (\hat{\mathcal{O}}_{\infty}^\wedge(D))^n
\]

\( \hat{\mathcal{O}}_{\infty}^\wedge(D) \) being \( \pi_*(\hat{\mathcal{O}}_{\infty}^\wedge \otimes_{\mathcal{O}_{X \times S}} \mathcal{O}_{X \times S}(D)) \).

\[
\varphi : \hat{\mathcal{M}}_{\infty}^n \to \hat{O}_S(\hat{\mathcal{A}}_{\infty}^n) \quad (\hat{\mathcal{O}}_{\infty}^n)
\]

being \( \varphi(M, f^\infty) = \pi_*(f^\infty \otimes 1)(\hat{\mathcal{M}}_{\infty}^n) \).
Proof. This morphism can be obtained adapting the proof of \[17\], \[23\] using the following vanishing theorem.

Theorem 3.4. If \( M \) is a locally free sheaf of rank \( n \) on \( X \times S \), there exists an affine open covering \( \{U_i\}_{i \in I} \) of \( S \) and effective divisors \( D_i \) over \( X - \{\infty\} \), with \( \text{deg}(D_i) \gg 0 \), such that: \( R^1 \pi_* M(D_i)|_{U_i} = 0 \) and \( \pi_* M(D_i)|_{U_i} \) is a free sheaf of \( \mathcal{O}_{U_i} \)-modules of finite type. Moreover, there exists \( -D'_i \subset X - \infty \) with \( \text{deg}(D'_i) \ll 0 \) such that \( \pi_* M(D'_i)|_{U_i} = 0 \)

Proof. Classically this result is proved by assuming that \( M \) is noetherian. Since an exact sequence of modules over a ring

\[
0 \to M' \to M \to M'' \to 0
\]

is split when \( M'' \) is free and if \( M \) is free \( M' \) is locally free, it is possible to avoid the noetherian hypothesis.

Theorem 3.5. Let \( \mathcal{F} \) be the subfunctor of the infinite Grassmannian defined by:

\[
\mathcal{F}(S) = \{ \mathcal{L} \in \text{Gr}((\hat{\mathbb{A}}_S^n)\!^n, (\hat{\mathcal{O}}_S^n)) (S) : \text{are } \mathcal{O}_\infty\text{-modules} \}
\]

There then exists a natural morphism of functors: \( T : \mathcal{F} \to \hat{\mathcal{M}}_\infty^n \), such that \( T \circ \varphi = \text{Id}_{\hat{\mathcal{M}}_\infty^n} \) and \( \varphi \circ T = \text{Id}_\mathcal{F} \). The definition of \( T \) will be clear from the proof.

Proof. If \( \mathcal{L} \) is a \( \mathcal{O}_\infty \otimes \mathcal{O}_S \)-module, \( \mathcal{L} \) has an associated quasicoherent \( \mathcal{O}_{X \times S} \)-module \( i_* \hat{\mathcal{L}} \), where \( \hat{\mathcal{L}} \) is the quasicoherent module over \( \text{Spec} \mathcal{O}_\infty \times S \) given by \( \mathcal{L} \). From the definition of \( \hat{\mathcal{L}} \) it is possible to deduce a morphism of sheaves of \( \mathcal{O}_{X \times S} \)-modules:

\[
g : i_* \hat{\mathcal{L}} \to (\hat{\mathbb{A}}_S^n)^n
\]

On the other hand, since \( \mathcal{L} \) is a discrete submodule, there exists an affine open covering \( \{U_i\}_{i \in I} \) of \( S \), and \( D_i \) effective divisors over \( X - \{\infty\} \) such that

\[
(\hat{\mathbb{A}}_U^n)/\hat{\mathcal{O}}_{U_i}^n(D_i)^n + i_* \mathcal{L}_{X \times U_i} = 0
\]

Note that \( \mathcal{O}_{X \times S}(D_i) \) are faithfully flat \( \mathcal{O}_{X \times S} \)-modules, and hence

\[
g : i_* \hat{\mathcal{L}} \to (\hat{\mathbb{A}}_S^n)^n / (\hat{\mathcal{O}}_S^n)^n
\]

is a surjective morphism of quasicoherent sheaves of \( \mathcal{O}_{X \times S} \)-modules.

Let \( M \) be the quasicoherent sheaf defined by \( \text{Ker}(g) \). \( M \) has associated in a natural way a \( \infty \)-formal level structure, \( f^\infty \). This is deduced for each \( D \subset X - \{\infty\} \) from the commutative diagrams:

\[
\begin{array}{cccc}
0 & \longrightarrow & M & \longrightarrow & i_* \hat{\mathcal{L}} & \overline{g} & (\hat{\mathbb{A}}_S^n)^n & \longrightarrow & 0 \\
\uparrow & & \uparrow= & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & M(-D) & \longrightarrow & i_* \hat{\mathcal{L}}(\overline{g}(-D)) & (\hat{\mathcal{O}}_S^n(-D))^n & \longrightarrow & 0
\end{array}
\]

and the snake Lemma.

To finish the proof, the two following Lemmas will show that \( M \) is locally free of rank \( n \) over \( X \times S \). Then, \( T(\mathcal{L}) \) will be defined by \( (M, f^\infty) \).

From the very definition of \( T \) and \( \varphi \), it is not hard to prove that \( T \circ \varphi = \text{Id}_{\hat{\mathcal{M}}_\infty^n} \) and \( \varphi \circ T = \text{Id}_\mathcal{F} \). 

\[ \square \]
(3.5.2) From the exact sequence
\[ 0 \rightarrow M \rightarrow i_\ast \hat{L} \rightarrow (\hat{A}_S^\infty)^n \rightarrow 0 \]
for deg(D') >> 0, πₘ M(D') is locally free on S, M(D') is locally generated by global sections on X × S and R¹πₘ M(D') = 0. Moreover, \( M_s = M \otimes_{\mathcal{O}_{X \times S}} (\mathcal{O}_X \otimes k(s)) \), for each \( s \in S \), is a locally free sheaf of rank \( n \) on \( X \times k(s) \) and \( M \) is flat over \( S \), from this last statement we deduce that if \( S \) is locally noetherian, \( M \) is locally of finite presentation and thus \( M \) is locally free of rank \( n \). And if \( S \) is not locally noetherian, we have to prove this by other methods:

**Lemma 3.6.** \( M \) is a locally free sheaf of rank \( n \) on \( X \times S \).

**Proof.** We can assume that \( k \) is algebraically closed, \( S = \text{Spec}(B), p \in \text{Spec}(A) \) verifying \( H^1(X \times \text{Spec}(B), M(-p)) = 0, H^0(X \times \text{Spec}(B), M(-p)) \) is a \( B \)-module locally free of finite rank and that there exists a surjective morphism of sheaves
\[ (3.6.1) \mathcal{O}_X \rightarrow \otimes_k H^0(X \times \text{Spec}(B), M) \rightarrow M \rightarrow 0 \]

Taking global sections in the exact sequence of the \( p \)-level structure on \( M \) deduced from the snake lemma for (3.5.1) we obtain an exact sequence of \( B \)-modules
\[ (3.6.2) 0 \rightarrow H^0(X \times \text{Spec}(B), M(-p)) \rightarrow H^0(X \times \text{Spec}(B), M) \xrightarrow{f_p} B^n \rightarrow 0 \]

Let \( t_1, ..., t_n \) be elements of \( H^0(X \times \text{Spec}(B), M) \) such that their images by the morphism \( f_p \) are a basis of \( B^n \). Let us consider the natural morphism
\[ t_1(\mathcal{O}_X \otimes B) \oplus ... \oplus t_n(\mathcal{O}_X \otimes B) \rightarrow M \]
This morphism is injective since that \( M/m_p^k M = (O_p/m_p^k O_p)^n \otimes B \), for all \( k \in \mathbb{N} \).

In conclusion, if for a given \( y \in X \times \text{Spec}(B) \) (\( y \) is not necessarily of the form \( p \times s \)) it suffices to find elements \( t_1', ..., t_n' \in H^0(X \times \text{Spec}(B), M) \) such that
\[ \left( t_1'(\mathcal{O}_X \otimes B) \oplus ... \oplus t_n'(\mathcal{O}_X \otimes B) \right)_{|k(y)} = 0 \]
(recall that \( M \) is of finite type).

By (3.5.2), we obtain for each \( y \in X \times \text{Spec}(B) \) the isomorphism \( M|_{k(y)} \cong k(y)^n \). Moreover, we have the surjective morphism (3.6.1) and by the exact sequence (3.6.2)
\[ H^0(X \times \text{Spec}(B), M) = (t_1 B \oplus ... \oplus t_n B) \oplus H^0(X \times \text{Spec}(B), M(-p)) \]
Thus, in the case of \( \{ t_1, ..., t_n \} \) not being a basis in \( M|_{k(y)} \), as \( k \) has enough elements, one could find \( \{ z_1, ..., z_n \} \in H^0(X \times \text{Spec}(B), M(-p)) \) such that \( \{ t_1 + z_1, ..., t_n + z_n \} \), would be a basis in \( M|_{k(y)} \). Taking \( t'_i = t_i + z_i \), we conclude. \( \square \)

We are now able to prove the main statement of this section, namely, that \( \mathcal{F} \) (and therefore \( \hat{M}_\infty^n \)) is a closed subfunctor of \( \text{Gr}((\hat{A}_\infty)^n, (\hat{O}_\infty)^n) \).

**Theorem 3.7.** The functor \( \hat{M}_\infty^n \) is representable by a closed subscheme \( \hat{M}_\infty^n \) of \( \text{Gr}((\hat{A}_\infty)^n, (\hat{O}_\infty)^n) \).
Proof. The condition \( L' \subset L \) defines a closed subscheme in \( \text{Gr}(\hat{\mathbb{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n) \), taking \( a.\mathcal{L} = L' \) for all \( a \in \mathcal{O}_\infty \) we conclude.

As consequence of Remark 2, it is easy to check

\[
\hat{\mathcal{M}}^n_\infty = \bigsqcup_{d \in \mathbb{Z}} \hat{\mathcal{M}}^{n,d}_\infty
\]

where \( \hat{\mathcal{M}}^{n,d}_\infty \) is the moduli scheme of vector bundles of rank \( n \) and degree \( d+n(g-1) \), with \( \infty' \)-formal level structures.

Moreover, bearing in mind Lemma 3.4 of [5] and the above results, we obtain that \( SL^n(k((t)))/SL^n(A) \) is a scheme in groups which is a closed subscheme of the infinite Grassmannian \( \text{Gr}((k((t)))^n, (k[[t]])^n) \), being \( t \) a local parameter of \( \infty \in X \).

In some sense the ind-group scheme \( SL^n(A) \), is a parabolic subgroup of \( SL^n(k((t))) \).

4. Immersion of Drinfeld moduli schemes in a finite product of infinite Grassmannians.

Now \( k \) is a finite field \( \mathbb{F}_q \). Let \( S \) be a \( \mathbb{F}_q \) scheme.

**Definition 4.1.** [8], [19] An elliptic sheaf is a diagram of vector bundles of rank \( n \) over \( X \times S \):

\[
\ldots \longrightarrow M_{-1} \overset{i_0}{\longrightarrow} M_0 \overset{i_1}{\longrightarrow} \ldots \overset{i_n}{\longrightarrow} M_n \overset{i_{n+1}}{\longrightarrow} \ldots
\]

\[
\ldots \longrightarrow F^*M_{-2} \overset{F^{*i_{-1}}}{\longrightarrow} F^*M_{-1} \overset{F^{*i_0}}{\longrightarrow} \ldots \overset{F^{*i_{n-1}}}{\longrightarrow} F^*M_{n-1} \overset{F^{*i_n}}{\longrightarrow} \ldots
\]

satisfying:

a) For any \( s \in S \), \( \text{deg}((M_{-1})_s) = n(1 - g) \). \( \text{deg} \) denotes the degree.

b) For all \( i \in \mathbb{N} \), \( M_{i+n} = M_i(\infty)(= M \otimes_{\mathcal{O}_X(\infty) \otimes \mathcal{O}_S}) \).

c) \( M_i + t(F^*M_i) = M_{i+1} \) (\( t \) is a morphism of \( \mathcal{O}_{X \times S} \)-modules).

\( F \) denotes the morphism of schemes:

\[
\text{Id} \times \sigma : \text{Spec}(A) \times S \longrightarrow \text{Spec}(A) \times S
\]

\( \sigma \) being the morphism of Frobenius on \( S \).

**Definition 4.2.** A \( D \)-level structure in a Drinfeld diagram is a \( D \)-level structure in each vector bundle \( M_i \) compatible with the diagram.

It is not hard to see that if a Drinfeld diagram has a \( D \)-level structure, then for all \( i \in \mathbb{Z} \) is \( (M_i/t(F^*M_{i-1}))|_{D \times S} = 0 \).
Remark 3. It is known that there exists an equivalence of categories, which commutes with base change:

\[
\begin{align*}
\{ \text{Drinfeld modules of rank } n \text{ over } S \\
\text{with characteristic away from } \text{supp}(D) \text{ with an } I_D\text{-level structure} \}
\leftrightarrow
\{ \text{Elliptic sheaves of rank } n \text{ over } S \\
\text{s.t. } (M_i/t(F^*M_i-1))_{|D \times S} = 0 \\
\text{for all } i, \text{ with a twisted } D\text{-level structure} \}
\end{align*}
\]

A twisted $D$-level structure on a locally free $\mathcal{O}_{X \times S}$-module $M$, is a surjective morphism of $\mathcal{O}_{X \times S}$-modules

\[
f_D : M \to (\Omega^1_X/\Omega^1_X(-D))^n \otimes \mathcal{O}_S
\]

($\Omega^1_X$ is the sheaf of differentials of $X$).

For comodity in the notation, we are going to work with the definition of level structure given in [1.2], with a easy modification can be changed this results for twisted $D$-level structures.

Regarding the Remark 3 and the results of the above sections, we shall show that the moduli functor of Drinfeld $A$-modules with $\infty'$-formal level structures is representable by a subscheme of a finite product of infinite Grassmannians.

Let us consider the moduli functor:

\[
\hat{\mathcal{D}}_n^\infty = \lim_{\rightarrow \infty} \tilde{\mathcal{D}}_D^n
\]

where $\tilde{\mathcal{D}}_D^n$ is the moduli functor of elliptic sheaves with level structures on the effective divisor $D$.

We want to prove:

**Theorem 4.3.** There exists an injective morphism of functors in the category of $\mathbb{F}_q$-schemes:

\[
\psi : \hat{\mathcal{D}}_n^\infty \to \prod_{i=0}^{n-1} \text{Gr}^{i+1}(\hat{\mathbb{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n
\]

such that $\hat{\mathcal{D}}_\infty^n$ is a locally closed subfunctor of $\prod_{i=0}^{n-1} \text{Gr}^{i+1}(\hat{\mathbb{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n$. So $\hat{\mathcal{D}}_\infty^n$ is representable by a scheme $\tilde{\mathcal{D}}_\infty^n$ of the infinite Grassmannian $\prod_{i=0}^{n-1} \text{Gr}^{i+1}(\hat{\mathbb{A}}^\infty)^n, (\hat{\mathcal{O}}^\infty)^n$.

**Proof.** To prove this Theorem we need a previous result about $\infty'$-formal level structures:

If $(M, f^\infty)$ and $(M', f'^\infty)$ are two $\infty$-formal level structures on $X \times S$, $\mathcal{L}$ and $\mathcal{L}'$ the discrete submodules associated with this $\infty$-formal level structures and $g : (M, f^\infty) \to (M', f'^\infty)$ a morphism between this level structures

i.e.: a commutative diagram of sheaves of $\mathcal{O}_{X \times S}$-modules

\[
\begin{array}{ccc}
M & \xrightarrow{g} & M' \\
\downarrow f^\infty & & \downarrow f'^\infty \\
(\mathcal{O}_S^\infty)^n & \xrightarrow{id} & (\mathcal{O}_S^\infty)^n
\end{array}
\]
Then $g$ is unique. Moreover, the morphism $g$ exists if and only if $L \subseteq L'$.

Now we are going to prove the theorem. If $H$ denotes the Drinfeld diagram together with a $\infty$-level structure:

\[
\cdots \rightarrow (M_0, f_0^\infty) \rightarrow \cdots \rightarrow (M_n, f_n^\infty) \rightarrow \cdots
\]

\[
\cdots \rightarrow (F^* M_{-1}, F^* f_{-1}^\infty) \rightarrow \cdots \rightarrow (M_{-1}, f_{-1}^\infty) \rightarrow \cdots
\]

the morphism $\psi$ is defined as follows

$\psi(H) = ((\varphi(M_0, f_0^\infty), \cdots, (\varphi(M_{-1}, f_{-1}^\infty))$)

($\varphi$ being the Krichever morphism (3.3)).

The injectivity of $\psi$ follows easily from b) of (4.1) and noting that $g$ is fixed for the $\infty'$-formal level structure.

On the other hand, the image functor of $\psi$ lies in $\prod_{i=0}^{n-1} \hat{M}_{\infty,i}^{n,i+1}$, and by the above statement and (4.1), the necessary and sufficient condition for an element $(L_1, \cdots, L_n) \in \prod_{i=0}^{n-1} \hat{M}_{\infty,i}^{n,i+1}(S) \subseteq \prod_{i=0}^{n-1} \hat{M}_{i+1}^{i+1}((\hat{A}^\infty)^n, (\hat{O}^\infty)^n)(S)$

to be in $\text{Img}(\varphi)(S)$ is that:

- For all $1 \leq i \leq n-2$, $L_i \subset L_{i+1}$ and $L_{n-1} \subset L_0(\infty), L_0(\infty)$ being the discrete submodule obtained from the $\infty$-formal level structure:

  $(M_0 \otimes \mathcal{O}_{X \times S}(\infty), f_0^\infty \otimes 1)$.

- For all $1 \leq i \leq n-2$, $(\sigma)^* L_i \subset L_{i+1}$ and $(\sigma)^* L_{n-1} \subset L_0(\infty)$.

- For all $1 \leq i \leq n-2$, $(\sigma)^* L_i + L_i = L_{i+1}$ and $(\sigma)^* L_{n-1} + L_{n-1} = L_0(\infty)$

Recalling that given two discrete submodules $L$ and $L'$ in $(\hat{A}^\infty)^n$ the subset of $S$ where $L \subseteq L'$ is a closed subscheme of $S$, then an element of $\prod_{i=0}^{n-1} \hat{M}_{\infty,i}^{n,i+1}(S)$ fulfills conditions 1) and 2) is a closed subscheme and by Nakayama's Lemma, applied to $L_{i+1}/L_i + (\sigma)^* L_i$, condition 3) is fulfilled in an open subscheme of $S$.

$(L_{i+1}/L_i$ is a finite $B$-module since that there exists a surjective morphism of $B$-modules $L_{i+1}/L_i : L_{i+1}/L_i(-\infty) \rightarrow L_{i+1}/L_i$. Recall that $L_{i+1}/L_i(-\infty) \simeq \pi_*(M_{i+1}/M_{i+1}(-\infty))$ is a $B$-module locally free of finite rank). We thus conclude. $\square$
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