RESOLVENT SET OF SCHRÖDINGER OPERATORS AND UNIFORM HYPERBOLICITY

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Abstract. All the main results in this notes are well known. Yet it’s still interesting to give a concise, detailed and self-contained descriptions of some of the basic relations between the one dimensional discrete Schrödinger operators and the corresponding Schrödinger cocycles. In particular, this notes gives a detailed proof of the equivalence between two different descriptions of uniformly hyperbolic \( \text{SL}(2, \mathbb{R}) \)–matrix sequences in Section 1. A self-contained and detailed proof of the equivalence between resolvent set of the Schrödinger operators and the uniform hyperbolicity of the Schrödinger cocycles is established in Section 2 and 3.

1. Uniformly Hyperbolic \( \text{SL}(2, \mathbb{R}) \)-sequence

Fix \( M > 0 \). Consider a map \( A : \mathbb{Z} \to \text{SL}(2, \mathbb{R}) \) with \( \|A(k)\| \leq M, \forall k \in \mathbb{Z} \). Let

\[
A_n(k) = \begin{cases} 
A(k+n-1) \cdots A(k), & n \geq 1, \\
I, & n = 0, \\
A(k+n)^{-1} \cdots A(k-1)^{-1}, & n \leq -1.
\end{cases}
\]

Let \( B \cdot (x) \) denotes induced transformation of \( B \in \text{SL}(2, \mathbb{R}) \) acting on projective space \( \mathbb{RP}^1 \ni x \). Throughout this notes, \( C, c \) will be some universal constants, where \( C \) is large and \( c \) small. We first give the following definition.

Definition 1. We say that \( A \) is uniformly hyperbolic (\( \mathcal{UH} \)) if there are two maps \( u, s : \mathbb{Z} \to \mathbb{RP}^1 = \mathbb{R}/(\pi \mathbb{Z}) \) such that

- \( u, s \) are \( A \)-invariant, which means that \( A(k) \cdot u(k) = u(k+1) \) and \( A(k) \cdot s(k) = s(k+1) \) \( \forall k \in \mathbb{Z} \);
- there exists \( C > 0, \lambda > 1 \) such that \( \|A_{-n}(k)v\|, \|A_n(k)w\| \leq C\lambda^{-n} \) for every \( n \geq 1, k \in \mathbb{Z} \) and all unit vectors \( v \in u(k), w \in s(k) \).

Here \( u \) is called the unstable direction and \( s \) is the stable direction of \( A \).

We have the following equivalent condition for \( \mathcal{UH} \) sequence.

Theorem 1. \( A \in \mathcal{UH} \) if and only if there exists \( c > 0, \lambda > 1 \) such that \( A \) satisfies the following uniform exponential growth condition

\[
\|A_n(k)\| \geq c\lambda^n, \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}.
\]

Proof. Clearly, Definition 1 implies the uniform exponential growth of \( A \). So we need to show the converse is also true. Without loss of generality, we may assume for some fixed \( c > 0, \lambda > 1 \) is the maximum of all \( \lambda \)'s such that \( A \) has the corresponding uniform exponential growth. For \( B \in \text{SL}(2, \mathbb{R}) \), let \( s(B) \in \mathbb{RP}^1 \) denotes the most
Lemma 1. There exist \( u \) and \( s : \mathbb{Z} \to \mathbb{R}^1 \) such that
\[
\lim_{n \to \infty} u_n(k) = u(k), \quad \lim_{n \to \infty} s_n(k) = s(k),
\]
where the convergence is uniform in \( k \in \mathbb{Z} \). Furthermore, \( u(k) \neq s(k), \forall k \in \mathbb{Z} \), are \( A \)-invariant.

Proof. By definition we have
\[
\|A_n(k)\hat{s}_{n+1}(k)\| = \|A(k+n)^{-1}A_{n+1}(k)\hat{s}_{n+1}(k)\| \leq M\|A_{n+1}(k)\|^{-1} \leq C\lambda^{-n}. 
\]
Since \( \|A_n(k)\hat{s}_n(k)\| = \|A_n(k)\|^{-1} \leq C\lambda^{-n} \), a direct computation shows that we must have
\[
|s_n(k) - s_{n+1}(k)| \leq C\lambda^{-n}\|A_n(k)\|^{-1} \leq C\lambda^{-2n}.
\]
Thus \( \{s_n(k)\}_{n \in \mathbb{Z}} \) is a Cauchy sequence for each \( k \in \mathbb{Z} \) and convergence is independent of \( k \in \mathbb{Z} \). Thus \( \lim_{k \to \infty} s_n(k) = s(k) \) for some \( s : \mathbb{Z} \to \mathbb{R}^1 \). Note we also have the estimate
\[
\|s(k) - s_n(k)\| \leq C\lambda^{-2n}, \quad \forall n \geq N,
\]
where \( N \) is some large positive integer that is independent of \( k \). Similarly, we get all the estimate for \( u \).

For the invariance property, we only need to note that \( \|A_n(k+1)A(k)\hat{s}_{n+1}(k)\| = \|A_{n+1}(k)\hat{s}_{n+1}(k)\| = \|A_{n+1}(k)\|^{-1} \leq C\lambda^{-(n+1)} \).

Thus, we must have that
\[
|A(k) \cdot s_{n+1}(k) - s_n(k+1)| \leq C\lambda^{-2n}, \quad \forall n \geq 1 \text{ and } \forall k \in \mathbb{Z}.
\]
So we have
\[
A(k) \cdot s(k) = A(k) \cdot \left[ \lim_{n \to \infty} s_{n+1}(k) \right] = \lim_{n \to \infty} A(k) \cdot s_{n+1}(k) = \lim_{n \to \infty} s_n(k+1) = s(k+1).
\]
Similarly, we get that \( u \) is also \( A \)-invariant.

To show \( u(k) \neq s(k) \), by invariance property, we only need to show that \( u(k_0) \neq s(k_0) \) for some \( k_0 \). By maximality of \( \lambda, \forall \varepsilon > 0 \), there exists \( \{k_l\}_{l \in \mathbb{Z}^+} \subset \mathbb{Z} \) and \( \{n_l\}_{l \in \mathbb{Z}^+} \subset \mathbb{Z}^+ \), \( n_l \to \infty \) as \( l \to \infty \), such that
\[
\|A_{n_l}(k_l)\| \leq c\lambda^{n_l(1+\varepsilon)}, \quad \forall l \in \mathbb{Z}^+.
\]
So we may fix a \( \varepsilon < 1 \) small and \( l \) large. Since \( |s(k_l) - s_{n_l}(k_l)| \leq C\lambda^{-2n_l} \), we must have
\[
\|A_{n_l}(k_l)\hat{s}(k_l)\| \leq C\lambda^{n_l(\varepsilon-1)} < 1.
\]
Similarly, we have \( \|A_{-n_l}(k_l+n_l)\hat{u}(k_l+n_l)\| < 1 \). Since \( A_{n_l}(k_l)\cdot u(k_l) = u(k_l+n_l) \) and \( A_{n_l}(k_l)^{-1} = A_{-n_l}(k_l+n_l) \), we have
\[
\|A_{n_l}(k_l)\hat{u}(k_l)\| > 1,
\]
which implies that \( u(k_l) \neq s(k_l) \).

\[ \square \]

Next we show the following lemma.

Lemma 2. There exists a \( \gamma > 0 \) in \( \mathbb{R}^1 = \mathbb{R}/(\pi\mathbb{Z}) \) such that
\[
|s(k) - u(k)| \geq \gamma, \quad \forall k \in \mathbb{Z}.
\]
Proof. Let’s consider the sequence space $B^2_M[SL(2, \mathbb{R})]$ which is equipped with product topology, where $B_M$ denotes the ball in $SL(2, \mathbb{R})$ with radius $M$. Thus $A \in B^2_M[SL(2, \mathbb{R})]$. Consider the shift map

$$T : B^2_M[SL(2, \mathbb{R})] \rightarrow B^2_M[SL(2, \mathbb{R})], \ T(\omega)(k) = \omega(k+1), \ \omega \in B^2_M[SL(2, \mathbb{R})].$$

Let $\Omega = \{T^n(A)\}_{n \in \mathbb{Z}}$, which is a compact topological space. This is called the hull of $A$ inside $B^2_M[SL(2, \mathbb{R})]$ and denoted by $Hull(A)$. Clearly, $T : \Omega \rightarrow \Omega$ is a homeomorphism. Let $F : \Omega \rightarrow SL(2, \mathbb{R})$, $F(\omega) = \omega(0)$, be the evaluation map at the 0-position. Let $F_n(\omega) = F(T^{n-1}(\omega)) \cdot F(\omega)$. Then it’s not difficult to see that

$$\|F_n(\omega)\| \geq c\lambda^n, \ \forall \omega \in \Omega.$$

Indeed, $\forall \omega \in \Omega$, there exists a sequence $n_l \rightarrow \infty$ such that $\lim_{l \rightarrow \infty} T^{n_l}(A) = \omega$ in product topology. Then the estimate follows from the fact that

$$\|F_n(T^k(A))\| = \|A_n(k)\| \geq c\lambda^n, \ \forall n, \forall k \in \mathbb{Z}.$$

Now if we define $s_n(\omega) = s(F_n(\omega))$, then by the argument that $s_n(\omega)$ converges to $s(k)$ uniformly in $k$, we conclude

$$\lim_{n \rightarrow \infty} u_n(\omega) = u(\omega), \ \lim_{n \rightarrow \infty} s_n(\omega) = s(\omega)$$

for some $u, s : \Omega \rightarrow \mathbb{RP}^1$ and the convergence is uniform in $\omega \in \Omega$. It’s not difficult to see that $u_n(\omega)$ and $s_n(\omega)$ are continuous in $\omega$ for all large $n$, see [Z] Lemma 10. Thus $u$ and $s$ are also continuous. By the same proof of Lemma 1 we may show that $u$ and $s$ are $(T, F)$-invariant. In other words, $F(\omega) \cdot u(\omega) = u(T(\omega))$. And $u(\omega) \neq s(\omega), \ \forall \omega \in \Omega$. Thus by compactness of $\Omega$ and continuity of $u, s$, we have for some $\gamma > 0$

$$|u(\omega) - s(\omega)| \geq \gamma, \ \forall \omega \in \Omega.$$

Since $A \in \Omega$ and $u(k) = u[T^k(A)]$, we get $|u(k) - s(k)| \geq \gamma, \ \forall k \in \mathbb{Z}$. □

Now we are ready to show that for some $\lambda_0 > 1$,

$$\|A_{-n}(k)u(k)\| \leq C\lambda_0^{-n}, \ \forall n \in \mathbb{Z}^+, \ \forall k \in \mathbb{Z}^+.$$

By $A$-invariance of $u$, we may equivalently show

$$\|A_n(k)u(k)\| \geq c\lambda_0^n, \ \forall n \in \mathbb{Z}^+, \ \forall k \in \mathbb{Z}.$$

By Lemma 1 and Lemma 2 $\forall \epsilon > 0$ small and $\forall \beta > 0$ large, there exists a $N \in \mathbb{Z}^+$ such that

$$\|u_N(k) - s_N(k)\| \geq \frac{\gamma}{2}, \ \|u(k) - s_N(k)\| \geq \frac{\gamma}{2}$$

and $\|A_N(k)\| \geq c\lambda N > \beta$. Let $R_\theta$ be the rotation matrix with rotation angle $\theta$. By definition, we have

$$A_N(k) = R_{\frac{\theta}{2}} R_{\frac{\pi}{2} - s_N(k)} R_{\frac{\theta}{2}}^{-1} R_{\frac{\pi}{2} - s_N(k)}^{-1}.$$

Then we may consider a new map $B : \mathbb{Z} \rightarrow SL(2, \mathbb{R})$ such that

$$B(k) = \left(\begin{array}{cc} \|A_N(k)N\| & 0 \\ 0 & \|A_N(k)N\|^{-1} \end{array}\right) R_{\frac{\pi}{2} - s_N(kN)}.$$

By [Z] Lemma 11, we have that if $\beta$ satisfies that $|\tan \frac{\gamma}{2}| > \frac{2}{C\lambda^{N-\epsilon}}$, then there is a constant invariant cone field of $B(k)$ around $0 \in \mathbb{RP}^1$, of which the size only depends on $\gamma$, such that each vector in this cone field is expanded under iteration by $B(k)$. In
particular, by choosing $N$ large, so that $\beta$ is sufficiently large and $\varepsilon > 0$ sufficiently small, we get for some $\alpha > 1$ and for each unit vector $v \in [u_N(kN) − u(kN)]$

$$\|B_n(k)v\| \geq c\alpha^n, \forall k \in \mathbb{Z}, \forall n \in \mathbb{Z}^+.$$ 

Clearly, this is equivalent to

$$\|A_n(k)\hat{u}(k)\| \geq c\alpha^n, \forall k \in \mathbb{Z}, \forall n \in \mathbb{Z}^+,$$

which is exactly what we want with $\lambda_0 = \alpha^{\frac{1}{n}}$. Similar, we may get the estimate for $s$. Thus, as the notation suggested, we show that $u$ is the unstable direction of $A$ as in Definition 1 and $s$ is the stable direction, which completes the proof of the Theorem 1.

If we look at everything carefully, we may see that there is no reason we should restrict the Definition 1 and Theorem 1 to $A$ defined on $\mathbb{Z}$. In fact, the proof of Lemma 2 has already given some hints since we’ve embedded the sequence $A$ to a map defined on a compact topological space. Fix $M > 0$. Given a set $\Omega$, a bijection $T : \Omega \rightarrow \Omega$ and a map $A : \Omega \rightarrow B_M[SL(2, \mathbb{R})]$, we may define a dynamical system on $\Omega \times \mathbb{R}^2$ as follows.

$$(T, A) : \Omega \times \mathbb{R}^2 \rightarrow \Omega \times \mathbb{R}^2, (T, A)(\omega, v) = (T(\omega), A(\omega)v).$$

Let $(T^n, A_n) = (T, A)^n$ denotes the iteration of the dynamics. Here $A$ is called a cocycle map. For simplicity of notations, $(T, A)$ may also denote the induced projective dynamics of $(T, A)$ on $\Omega \times \mathbb{RP}^1$.

**Definition 2.** $(T, A) \in \mathcal{UH}$ if there are two maps $u, s : \Omega \rightarrow \mathbb{RP}^1 = \mathbb{R}/(\pi\mathbb{Z})$ such that

- $u, s$ are $(T, A)$-invariant, which means that

  $$A(\omega) \cdot u(\omega) = u[T(\omega)], A(\omega) \cdot s(\omega) = s[T(\omega)], \forall \omega \in \Omega$$

  and

- there exists $C > 0, \lambda > 1$ such that $\|A_{-n}(\omega)v\|, \|A_n(\omega)w\| \leq C\lambda^{-n}$ for every $n \geq 1, \omega \in \Omega$ and all unit vectors $v \in u(\omega), w \in s(\omega)$

Here $u$ is called the unstable section and $s$ the stable section.

Note in Definition 2 no topological structure or $\sigma$-algebra structure is assumed for $\Omega$. Then we have the following corollary of the proof of Theorem 1.

**Corollary 1.** Let $(\Omega, T, A)$ be as in Definition 2. Assume in addition $\Omega$ is a compact topological space, $T$ a homeomorphism and $A$ continuous. Then $(T, A) \in \mathcal{UH}$ if and only if there exists $c > 0, \lambda > 1$ such that $(T, A)$ satisfies the following uniform exponential growth condition

$$\|A_n(\omega)\| \geq c\lambda^n, \forall n \in \mathbb{Z}, \forall \omega \in \Omega.$$

Furthermore, the corresponding unstable and stable sections are continuous on $\Omega$.

See [Y] for some similar discussions of this section. As suggested by the proof of Theorem 1, $\mathcal{UH}$ is also equivalent to the existence of an invariant cone field. See, for example, [A, Section 2.1], for some detailed description.
2. Schrödinger Operators Defined on a Sequence

Fix $M > 0$. Let $v \in [-M, M]^\mathbb{Z}$ be a sequence of bounded real numbers. Then we may define an operator $H_v$ on $\ell^2(\mathbb{Z})$ as follows:

$$(H_v u)_n = u_{n+1} + u_{n-1} + v(n)u_n, \quad u = (u_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Clearly, $H_v$ is bounded self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z})$. Here $H_v$ is called a onedimensional discrete Schrödinger operator, $v$ is the corresponding potential. We are interested in the spectrum, $\sigma(H_v)$, of the operator $H_v$, which is the following:

$$\sigma(H_v) = \{ E \in \mathbb{C} : H_v - E \text{ is not invertible} \}.$$  

It’s standard result that $\sigma(H_v)$ is a closed set. Let $g : \sigma(H_v) \to \mathbb{R}$, $g(x) = x$ be the identity function. It’s an easy estimate that the operator norm $\|H_v\| \leq 2 + M$. By continuous functional calculus, $\|g\|_\infty = \|H_v\|$ (see, for example, [RS]). Thus, by self-adjointness, we must have that $\sigma(H_v) \subset [-M - 2, M + 2]$ is compact. Let $\rho(H_v) = \mathbb{R} \setminus \sigma(H_v)$ denotes the resolvent set of $H_v$ inside real numbers. We first have the following lemma.

**Lemma 3.** $E \in \sigma(H_v)$ if and only if $\forall \varepsilon > 0$, there exists a finitely supported unit vector $u \in \ell^2(\mathbb{Z})$ such that

$$\| (H_v - E)u \| < \varepsilon.$$  

**Proof.** Recall the Weyl’s Criterion (see, for example, [RS]) says that $E \in \sigma(H_v)$ if and only if $\forall \varepsilon > 0$, there exists a unit $u \in \ell^2(\mathbb{Z})$ such that $\| (H_v - E)u \| < \varepsilon$. Thus the if part follows immediately.

For the proof of the only if part, again by Weyl’s criterion, $\forall \varepsilon > 0$, there exists a unit vector $\hat{u} \in \ell^2(\mathbb{Z})$ such that $\| (H_v - E)\hat{u} \| < \frac{\varepsilon}{2}$. For $l \in \mathbb{Z}^+$, we define

$$u^l(n) = \begin{cases} \hat{u}(n), & n \in [-l,l] \\ 0, & n \notin [-l,l]. \end{cases}$$

Then we may choose $L$ large enough such that

$$\|u^L\| \geq \frac{1}{2}, \quad \| (H_v - E)u^L \| < \frac{\varepsilon}{2}.$$  

Clearly, $u = \frac{u^L}{\|u^L\|}$ can be our choice. \hfill \Box

Given $x \in \mathbb{R}$, we define $A(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. For the potential $v$, we define the map

$$A^{(E-v)} : \mathbb{Z} \to \text{SL}(2, \mathbb{R}), \quad A^{(E-v)}(n) = A^{(E-v(n))},$$

which is called a Schrödinger cocycle map associated with Schrödinger operator $H_v$. The relation between the operator and the cocycle map is the following. $u \in \mathbb{C}^\mathbb{Z}$ is a solution to the eigenfunction equation $H_v u = Eu$ if and only if

$$A^{(E-v)}(n) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}.$$  

Then we have the following basic relation between the spectral behavior of Schrödinger operator and dynamics of the Schrödinger cocycle.

**Theorem 2.** $\sigma(H_v) = \{ E \in \mathbb{R} : A^{(E-v)} \notin \mathcal{UH} \}$. 


Proof. Let’s first show \( \sigma(H_v) \subset \{ E : A(E-v) \notin \mathcal{U}\mathcal{H} \} \). Equivalently, we show
\( \{ E : A(E-v) \in \mathcal{U}\mathcal{H} \} \subset \rho(H_v) \).

Fix \( E \) such that \( A(E-v) \in \mathcal{U}\mathcal{H} \). Then by Definition [1] there are unstable direction \( u \) and stable directions \( s \). Let \( u^u, u^s \in \mathbb{R}^2 \) be solution to the eigenfunction equation \( H_v u = Eu \) and be such that
\[
\begin{bmatrix}
u^u(0) \\
u^u(-1)
\end{bmatrix} \in u(0), \begin{bmatrix}
u^s(0) \\
u^s(-1)
\end{bmatrix} \in s(0).
\]

Let’s normalize them so that
\[
\det \begin{bmatrix} u^s(0), & u^u(0) \\ u^s(-1), & u^u(-1) \end{bmatrix} = 1.
\]

Thus we have \( \forall n \in \mathbb{Z}, \)
\[
\det \begin{bmatrix} u^s(n), & u^u(n) \\ u^s(n-1), & u^u(n-1) \end{bmatrix} = \det \begin{bmatrix} A_n(E-v)(0) \begin{bmatrix} u^s(0), & u^u(0) \\ u^s(-1), & u^u(-1) \end{bmatrix} = 1.
\]

Then we construct the function \( G : \mathbb{Z}^2 \to \mathbb{R} \)
\[
G(p, q) = \begin{cases} u^u(p)u^s(q) & \text{if } p < q, \\ u^u(q)u^s(p) & \text{if } q < p. \end{cases}
\]

Since \( |u(n) - s(n)| \geq \gamma > 0 \text{ in } \mathbb{R}^1 \), one readily see that there exist \( C > 0, \lambda > 1 \), independent of \( (p, q) \), such that
\[
|G(p, q)| \leq Ce^{-\lambda|p-q|}, \forall (p, q) \in \mathbb{Z}^2.
\]

Thus if we define the operator \( S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) such that
\[
(Su)_n = \sum_{p \in \mathbb{Z}} G(p, n)u_p, \ u \in \ell^2(\mathbb{Z}),
\]
then \( S \) is a bounded operator. Indeed, \( \forall u \in \ell^2(\mathbb{Z}) \)
\[
\|S(u)\|^2 = \sum_{n \in \mathbb{Z}} |(Su)_n|^2 = \sum_{n \in \mathbb{Z}} | \sum_{p \in \mathbb{Z}} G(p, n)u_p|^2
\]
\[
\leq C \sum_{n \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} |G(p, n)u_p|^2 = C \sum_{p \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |G(p, n)u_p|^2
\]
\[
= C \sum_{p \in \mathbb{Z}} |u_p|^2 \sum_{n \in \mathbb{Z}} |G(p, n)|^2 \leq C \sum_{p \in \mathbb{Z}} |u_p|^2
\]
\[
= C\|u\|^2,
\]
here we used Jensen’s Inequality and Fubini’s Theorem. Also, by definition of \( G \),
one readily check
\[
[(H_v - E) \circ S(u)]_n = (Su)_{n+1} + (Su)_{n-1} + [v(n) - E](Su)_n
\]
\[
= \sum_{p \in \mathbb{Z}} [G(p, n + 1) + G(p, n - 1) + (v(n) - E)G(p, n)] u_p
\]
\[
= [G(n, n + 1) + G(n, n - 1) + (v(n) - E)G(n, n)]u_n
\]
\[
= \{u^u_{n+1}u^s_n + u^s_n[u^u_{n-1} + (v(n) - E)u^s_n]\}u_n
\]
\[
= (u^u_{n+1}u^s_n - u^u_nu^s_{n+1})u_n
\]
\[
= u_n.
\]
Hence, $H_v - E$ is invertible with inverse $G(p, q)$, which implies that $E \in \rho(H_v)$. Here $G$ is the so called Green’s function for $H_v - E$.

Now we show $\{ E : A(E - v) \notin \mathcal{UH} \} \subset \sigma(H_v)$. We need the following lemma.

**Lemma 4.** Let $(\Omega, T, A)$ be such that $\Omega$ is a compact topological space, $T$ homeomorphism on $\Omega$ and $A : \Omega \to \text{SL}(2, \mathbb{R})$ is continuous. Then $(T, A) \notin \mathcal{UH}$ if and only if there exists a $\omega \in \Omega$, $v \in \mathbb{S}^1 \subset \mathbb{R}^2$ such that

$$\| A_n(\omega)v \| \leq 1, \forall n \in \mathbb{Z}.$$  

**Proof.** For the proof of if part, $\mathcal{UH}$ implies that for some bounded constants $c_1$, $c_2$, we have

$$v = c_1 \hat{u}(\omega) + c_2 \hat{s}(\omega), \forall v \in \mathbb{R}^2, \forall \omega \in \Omega.$$  

Thus for $v \neq 0$, we must have that $\| A_n(\omega)v \|$ grows exponentially fast either as $n \to \infty$ or as $n \to -\infty$.

For the proof of only if part, assume that $\exists \varepsilon > 0, \exists L \in \mathbb{Z}^+$ such that $\forall (\omega, v) \in \Omega \times \mathbb{S}^1$, $\exists |l| \leq L$ such that

$$\| A_l(\omega)v \| \geq 1 + \varepsilon.$$  

Then we claim $(T, A)$ satisfies uniform exponential grow condition. For $(\omega, v) \in \Omega \times \mathbb{S}^1$, let $l(\omega, v)$ be smallest $|l| \leq L$ such that $\| A_l(\omega)v \| \geq 1 + \varepsilon$. Let

$$l_0 = 0, \ v_0 = v, \text{ and } \omega_0 = \omega.$$  

Inductively, we define

$$l_k = l(\omega_{k-1}, v_{k-1}), \ v_k = \frac{A_{l_k}(\omega_{k-1})v_{k-1}}{\| A_{l_k}(\omega_{k-1})v_{k-1} \|}, \text{ and } \omega_k = T^{l_k}(\omega_{k-1}).$$  

By definition, we have for each $k$

$$l_k \notin I_{k-1} := \bigcup_{j=1}^{k-1} I_{l_j} (l_{j-1}),$$  

where $I_p(q) \subset \mathbb{Z}$ is the interval around $q$ with radius $p$. Clearly $I_k$ is a connected interval in $\mathbb{Z}$ and

$$|I_{k+1}| \geq |I_k| + 1, \forall k \geq 0.$$  

Thus there exists a $K \leq L$ such that $|I_K| \geq L$. Now since $|l_k| \leq L, \forall k \geq 0$, we must have that $l_k$ have same sign for all $k > K$. Similarly, if for all $1 \leq l \leq L$, $\| A_l(\omega)v \| < 1 + \varepsilon$, then $l_k < 0$ for all $k > 0$. For simplicity, let’s fix some $n \in \mathbb{Z}^+$.

Then for the given $(\omega, v)$, we have either $l_k > 0$ for all $k > 0$ or there is some $N > n + L$ such that

$$l_k \left( T^N(\omega), \frac{A_N(v)}{\| A_N(v) \|} \right) < 0, \forall k > 0.$$  

For simplicity, let’s assume the first case. Let $p$ be the first integer such that $\sum_{k=0}^{p} l_k > n$, then $p > \frac{n}{\varepsilon}$ and we have

$$\| A_n(\omega) \| \geq \| A_n(\omega)v \| \geq e \prod_{k=1}^{p-1} \| A_{l_k}(\omega_{k-1})v_{k-1} \| \geq e(1 + \varepsilon)^{\frac{n}{\varepsilon}}.$$  

Thus by Theorem [1], we have $(T, A) \in \mathcal{UH}$. Hence if we assume $(T, A) \notin \mathcal{UH}$, then $\forall \varepsilon > 0, \forall L > 0$, $\exists (\omega, v) \in \Omega \times \mathbb{S}^1$ such that $\forall |l| \leq L$ we have

$$\| A_l(\omega)v \| < 1 + \varepsilon.$$
Thus for each \( l \in \mathbb{Z}^+ \), we get for \( \varepsilon = \frac{1}{L} \), \( L = \frac{1}{\varepsilon} \), a \((\omega^l, v^l)\) \( \in \Omega \times S^1 \) satisfies the above condition. By compactness of the space \( \Omega \times S^1 \), we may assume

\[
\lim_{l \to \infty} (\omega^l, v^l) = (\omega, v), \text{ for some } (\omega, v) \in \Omega \times S^1.
\]

Thus we have

\[
\|A_n(\omega)v\| \leq 1, \forall n \in \mathbb{Z},
\]

which completes the proof. \( \square \)

As in the proof of Lemma 2, let \( \Omega = \text{Hull}(v) \) inside \([-M, M]^{\mathbb{Z}}\), which is equipped with product topology. Let \( T : \Omega \to \Omega \) be the shift operator \( T(\omega)_n = \omega_{n+1} \). Then for each \( \omega \in \Omega \), we may define the corresponding operator and denotes it by \( H_\omega \).

Let \( f : \Omega \to \mathbb{R}, f(\omega) = \omega_0 \), be the evaluation map at 0–position. Then for the family of operators \( \{H_\omega\}_{\omega \in \Omega} \), we define the corresponding Schrödinger cocycle map

\[
A^{(E-f)} : \Omega \to \text{SL}(2, \mathbb{R}), A^{(E-f)}(\omega) = A^{(E-f)(\omega)}.
\]

Clearly, \( A^{(E-v)} \notin \mathcal{U} \mathcal{H} \) implies that \( (T, A^{(E-f)}) \notin \mathcal{U} \mathcal{H} \). By Lemma 4 there is a \((\omega, v) \in \Omega \times S^1\) such that

\[
\|A_n^{(E-f)}(\omega)v\| \leq 1, \forall n \in \mathbb{Z}.
\]

Define \( u \in \mathbb{R}^\mathbb{Z} \) such that

\[
\left(\begin{array}{c}
u_n \\
u_{n-1}
\end{array}\right) = A_n^{(E-f)}(\omega)v, \forall n \in \mathbb{Z}.
\]

Then \( \|u\|_\infty \leq C \) and by the relation between the operators and cocycles, we have that

\[
H_\omega u = Eu.
\]

We claim that \( E \in \sigma(H_\omega) \). Indeed, if \( \|u\| \leq C \), then \( E \) is an eigenvalue of \( H_\omega \), hence, \( E \in \sigma(H_\omega) \). Otherwise, if we define \( \hat{u}^L \) such that

\[
\hat{u}^L_n = \begin{cases} u_n, & \text{if } |n| \leq L, \\ 0, & \text{otherwise}, \end{cases}
\]

then \( \|\hat{u}^L\| \to \infty \) as \( L \to \infty \) and

\[
[(H_\omega - E)\hat{u}^L]_n = \begin{cases} \pm u_n, & \text{if } n = \pm L, \pm(L + 1) \\ 0, & \text{otherwise}. \end{cases}
\]

Thus if we define \( u^L = \frac{\hat{u}^L}{\|u^L\|} \), then we have

\[
\|(H_\omega - E)u^L\| \leq \frac{C}{\|u^L\|},
\]

which can be arbitrary small as \( L \to \infty \). Thus by Lemma 3 \( E \in \sigma(H_\omega) \). In any case, \( \forall \varepsilon > 0 \), there exists a finitely supported unit vector \( u \in \ell^2(\mathbb{Z}) \) such that \( \|(H_\omega - E)u\| < \varepsilon \). By definition, there exists a \( N \in \mathbb{Z} \) such that \( T^N(v) \) is sufficiently close to \( \omega \) in product topology. Since \( u \) is finitely supported, we must have

\[
\|(H_{T^Nv} - E)u\| < \varepsilon.
\]

Equivalently, we have

\[
\|(H_v - E)[T^{-N}(u)]\| < \varepsilon,
\]

where
where \((Tu)_n = u_{n+1}\) is an unitary operator on \(\ell^2(\mathbb{Z})\). Since \(T^{-N}(u)\) is again finitely supported with norm 1, we have \(E \in \sigma(H_u)\) by Lemma 3 which completes the proof. \(\square\)

One can basically find Lemma 4 in \[SS\], see for example \[SS\] Theorem 1.7. The proof of Lemma 4 presented here is from a course of Artur Avila in Fields Institute in Jan-Mar, 2011.

3. DYNAMICALLY DEFINED SCHröDINGER OPERATORS

In Section 1 and 2, we mainly considered the sequence of \(SL(2,\mathbb{R})\)–matrix and the Schrödinger operators defined on a sequence of bounded real numbers. Note that in the proof of Theorem 1 and 2, we embedded the sequences into some dynamical systems. Thus, it’s important to consider the Schrödinger operators and cocycles defined on dynamical systems. Let’s start with the following stronger version of Lemma 3.

Lemma 5. Fix \(M > 0\). Then \(\forall v \in [-M, M]^\mathbb{Z}, E \in \sigma(H_v)\) if and only if \(\forall \varepsilon > 0, \exists L = L(M, \varepsilon)\) such that there exists a unit vector \(u \in \ell^2(\mathbb{Z})\) supported on an interval, \(I \subset \mathbb{Z}\), with \(|I| \leq L\) and \(\|(H_v - E)u\| < \varepsilon\).

Proof. By Lemma 3, we only need to show the only if part. In fact, we only need to show that \(L = L(M, \varepsilon)\) is independent of \((v, E) \in [-M, M]^\mathbb{Z} \times [-M - 2, M + 2]\).

Assume this is false, then there exists a \(\varepsilon > 0\), for each \(l \in \mathbb{Z}^+\), there exists a \((v^l, E^l), E^l \in \sigma(H_v)\) such that any unit vector \(u \in \ell^2(\mathbb{Z})\) satisfies \(\|(H_{v^l} - E^l)u\| < \varepsilon\) must be that \(u\) is not supported on any interval \(I \subset \mathbb{Z}\) of length less than or equal to \(l\).

Let \(\Omega_l = \text{Hull}(v_l)\) as usual. Then it’s not difficult to see that \(\forall \omega \in \Omega_l\), any unit vector \(u\) satisfying \(\|(H_{v^l} - E^l)u\| < \varepsilon\) is not supported on any interval \(I \subset \mathbb{Z}\) of length less than or equal to \(l\). By the proof of Theorem 2, \(\forall l \in \mathbb{Z}^+, \exists \omega^l \in \Omega_l\) and \(u^l \in \ell^\infty(\mathbb{Z})\) with \(\|u^l\|_\infty \leq 1\) such that \((H_{v^l} - E^l)u^l = 0\). By the way we construct \(u^l\) in the proof of Theorem 2 by shifting both \(\omega^l\) and \(v^l\) and normalizing \(u^l\), we may assume

\[u^l(0) = 1, \|u^l\|_\infty < C, \forall l \in \mathbb{Z}^+\]

By compactness, we may assume

\[\lim_{l \to \infty} (\omega^l, u^l) = (\omega, u) \in [-M, M]^\mathbb{Z} \times [-1, 1]^\mathbb{Z}, \lim_{l \to \infty} E^l = E \in [-M - 2, M + 2]\]

where the convergence of \((\omega^l, u^l)\) to \((\omega, u)\) is with respect to product topology. Thus, we must have

\((H_E - E)u = 0, u(0) = 1, \|u\|_\infty < C\).

By Theorem 2, \(E \in \sigma(H_v)\). Furthermore, for all sufficiently large \(l \in \mathbb{Z}^+\), any unit vector \(u\) satisfying \(\|(H_{v^l} - E)u\| < \varepsilon\) is not supported on any interval \(I \subset \mathbb{Z}\) of length less than or equal to \(l\). This is clearly a contradiction with Lemma 4 concluding the proof. \(\square\)

Lemma 5 is first mentioned and used in \[ADZ\] Lemma 12.

From now on, let \((\Omega, d)\) be a compact metric space with distance \(d\), \(T : \Omega \to \Omega\) a homeomorphism and \(f : \Omega \to \mathbb{R}\) a continuous function. Abusing the notation lightly, \(T\) also denotes the left shift operator on the sequence space. \((\Omega, T)\) is said
to be topological transitive if it has a dense $T$–orbit. $(\Omega, T)$ is said to be minimal if each $T$–orbit is dense.

For $\omega \in \Omega$, we consider the Schrödinger operator $H_\omega$,

$$(H_\omega u)_n = u_{n+1} + u_{n-1} + f(T^n \omega) u_n, \; u \in \ell^2(\mathbb{Z}).$$

This is called dynamically defined family of Schrödinger operators with underlying dynamics $(\Omega, T)$. As in the proof of Theorem 2, we define the family of Schrödinger cocycle maps

$$A^{(E-f)} : \Omega \to \text{SL}(2, \mathbb{R}), \; A^{(E-f)}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}, \; E \in \mathbb{R}.$$ 

As in the discussion following Theorem 1, we have a family of dynamical systems $(T, A, f)$ such that $H_\omega = -\omega^T A \omega$. Then by uniform continuity of $f$, $\exists \delta > 0$, independent of $\omega \in \Omega$, if $d(x, T^n \omega) < \delta$, then

$$\|((H_\omega - E)(T^{-n} u))\| < \varepsilon.$$ 

Then by uniform continuity of $f$, $\exists \delta > 0$, independent of $\omega \in \Omega$, if $d(x, T^n \omega) < \delta$, then

$$\|((H_x - E)(T^{-n} u))\| < \varepsilon.$$ 

Thus, we must have $E \in B_\varepsilon(\sigma(H_x))$. Indeed, if $E \in \sigma(H_x)$, we have nothing to say. Otherwise, it’s straightforward that the above inequality implies that

$$\|((H_x - E)^{-1})\| > \frac{1}{\varepsilon}.$$ 

Let $g : \sigma(H_x) \to \mathbb{R}$ be the identity function on $\sigma(H_x)$. Then the continuous functional calculus implies

$$\|(g - E)^{-1}\|_{\infty} > \frac{1}{\varepsilon},$$

which implies that $E \in B_\varepsilon(\sigma(H_x))$. Clearly, in $\|(H_x - E)(T^{-n} u)\| < \varepsilon$, by shifting $u$ more, we may replace $x$ by any $T^n(x)$, $n \in \mathbb{Z}$ while we still have $E \in B_\varepsilon(\sigma(H_x))$, which completes the proof. 

Now we have the following two corollaries, which are widely used in dynamically defined Schrödinger operators.
Corollary 2. Let $(\Omega, T, f)$ be as in Theorem 3. Assume in addition that $(\Omega, T)$ is topological transitive. Let $x$ be that $\text{Orb}(x) = \Omega$ and let $\Sigma = \sigma(H_x)$. Then $\forall \omega \in \Omega$, we have $\sigma(H_\omega) \subset \Sigma$. Furthermore, we have

$$\Sigma = \{E : (T, A^{(E-f)}) \notin \mathcal{UH}\}.$$ 

Proof. By Theorem 3, we clearly have $\sigma(H_\omega) \subset \Sigma$, $\forall \omega \in \Omega$. Let

$$A : \mathbb{Z} \to \text{SL}(2, \mathbb{R}), \quad A^E(n) = A^{(E-f)}(T^n x).$$

Then, $\text{Orb}(x) = \Omega$ and Theorem 1 clearly imply that

$$A^E \in \mathcal{UH} \iff (T, A^{(E-f)}) \in \mathcal{UH}.$$ 

Hence, $\Sigma = \{E : (T, A^{(E-f)}) \notin \mathcal{UH}\}$ follows from Theorem 2.

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