Codes for the Z-Channel
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Abstract—This paper is a collection of results on combinatorial properties of codes for the Z-channel. A Z-channel with error fraction \( \tau \) takes as input a length-\( n \) binary codeword and injects an adversarial manner up to \( n \tau \) asymmetric errors, i.e., errors that only zero out bits but do not flip 0's to 1's. It is known that the largest \(( L - 1)\)-list-decodable code for the Z-channel with error fraction \( \tau \) has exponential size (in \( n \)) if \( \tau \) is less than a critical value that we call the \(( L - 1)\)-list-decoding Plotkin point and has constant size if \( \tau \) is larger than the threshold. The \(( L - 1)\)-list-decoding Plotkin point is known to be \( L^{-\frac{1}{2} - \frac{1}{L - 1}} \), which equals 1/4 for unique-decoding with \( L = 1 \). In this paper, we derive various results for the size of the largest codes above and below the list-decoding Plotkin point. In particular, we show that the largest \(( L - 1)\)-list-decodable code \( \varepsilon \)-above the Plotkin point, for any given sufficiently small positive constant \( \varepsilon > 0 \), has size \( \Theta_L(\varepsilon^{-2}/2) \) for any \( L - 1 \geq 1 \). We also devise upper and lower bounds on the exponential size of codes below the list-decoding Plotkin point.

Index Terms—Channel coding, communication channels, channel capacity, combinatorial mathematics, error correction codes.

I. INTRODUCTION

In coding theory, the Z-channel is used to model some asymmetric data storage and transmission systems. In this binary-input binary-output channel, the symbol 0 is always transmitted correctly, whereas the transmitted symbol 1 can be received as 0.

In this paper, we consider the combinatorial setting where the encoder transmits \( n \) symbols, and the maximum number of errors inflicted by an adversary is proportional to \( n \). The error model under consideration is also known as the adversarial/Hamming/zero error model [1], [2] which is standard in coding theory. It stands in contrast to the stochastic setting in Shannon theory [3] where the error model is assumed to be average-case and a vanishing probability of decoding error is desired. For a given word \( x \in \{0,1\}^n \), we define the Z-ball centered at \( x \) with radius \( \tau n \) (where \( \tau \in [0,1] \) and \( n \in \mathbb{Z}_{\geq1} \)) as a set of all possible words that can be transmitted over the Z-channel with at most \( \tau n \) errors such that \( x \) is received. Given \( \tau \) and \( n \), the main goal for \(( L - 1)\)-list-decoding is to construct a code \( C \subseteq \{0,1\}^n \) such that for any \( x \in \{0,1\}^n \), the Z-ball centered at \( x \) contains at most \( L - 1 \) codewords from \( C \). For \( L = 2 \), we say that \( C \) is a uniquely-decodable code tolerating a fraction \( \tau \) of (asymmetric) errors. For \( L \geq 3 \), \( C \) is referred to as an \(( L - 1)\)-list-decodable code for the Z-channel with list-decoding radius \( \tau \). Finding fundamental limits of error-correcting codes is one of the major problems in coding theory. Uniquely-decodable codes for asymmetric errors have been a subject for extensive studies in the numerous papers [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]. Codes for Z-channels with feedback are studied in [16], [17], [18], and [19]. Bounds and constructions of codes correcting a single asymmetric error are obtained in [20], and [21]. Up to our best knowledge, there are only two papers [19], [22] in the literature that discuss properties of list-decodable codes for the Z-channel.

First, we recall some results concerning the unique-decoding case. It is known [4], [5] that the rate of optimal codes tolerating a fraction \( \tau \) of asymmetric errors is asymptotically equal to the rate of optimal codes correcting a fraction \( \tau \) of symmetric errors.\(^3\) Hence there exist exponential-sized uniquely-decodable codes for any fraction of errors \( \tau < 1/4 \). The Plotkin bound [23] implies that the size of a code capable of correcting a fraction \( \tau = 1/4 + \varepsilon \) of symmetric errors is bounded above by \( 1 + 1/(4\varepsilon) \). One might ask a similar question for asymmetric errors. Specifically, can we bound the size of a code \( C \subseteq \{0,1\}^n \) tolerating a fraction \( \tau = 1/4 + \varepsilon \) of asymmetric errors using a function that depends only on \( \varepsilon \)? The paper [5] claims that such a bound exists only for \( \varepsilon > 1/12 \). We disprove this statement and provide an order-optimal uniquely-decodable code of size \( \Theta(\varepsilon^{-3/2}) \) as \( \varepsilon \to 0 \). From the results of [23], [24], and [25], it follows that the maximal size of a code tolerating a fraction \((1/4 + \varepsilon ) \) of symmetric errors is \((4\varepsilon)^{-1}(1+o(1)) \) as \( \varepsilon \to 0 \).

1We use \( \mathbb{Z}_{\geq a} \) to denote the set of integers at least \( a \).

2Formally, \( n \tau \) needs to be an integer. Otherwise, in the proofs of upper (resp. lower) bounds on the maximal code size, one can take \( \varepsilon \) such that \( n \tau \varepsilon \) is the largest (resp. smallest) integer less (resp. bigger) than \( n \tau \). Asymptotically in \( n \to \infty \), the effect of this quantization on our bounds is negligible. Therefore, we at times drop ceiling/floor for notational convenience. Similarly, other quantities such as the Hamming weight \( n \omega \) that will show up later in the paper can be suitably rounded up or down and the same results continue to hold.

3Hereafter, errors are called symmetric if any transmitted symbol from the alphabet \{0,1\} can be bit-flipped.
Much less is known about list-decodable codes for the Z-channel. By [19], and [22], exponential-sized (or positive-rate) \((L - 1)\) list-decodable codes with list-decoding radius \(\tau\) exist only for \(\tau < \tau_L\), where \(\tau_L = w_{\text{max}} - w_{\text{max}}\) and \(w_{\text{max}}\) equals \(L^{-\frac{1}{\tau+1}}\). We call \(\tau_L\) the \((L - 1)\)-list-decoding Plotkin point. We extend the above results from unique-decoding to list-decoding and obtain the same characterization \(\Theta_L(\varepsilon^{-3/2})\) for list-decodable codes with arbitrary list size \(L - 1\) correcting \(\tau_L + \varepsilon\) fraction of asymmetric errors. The same question for symmetric errors was also studied before. In a recent work [25], the results in [23], and [24] for unique-decoding were generalized to list-decoding with any odd list size that is at least one and the optimal code size was shown to be \(\Theta_L(\varepsilon^{-1})\). For even list size, the problem seems significantly more difficult and [25] showed that the optimal code size is \(\Theta_L(\varepsilon^{-3/2})\) for list size two.

II. OVERVIEW OF OUR RESULTS AND TECHNIQUES

This paper is a collection of results on combinatorial properties of codes for the Z-channel with adversarial errors. The most technically challenging part of our results has to do with obtaining the order-optimal size of codes that correct a fraction of asymmetric errors \(\varepsilon\)-above the Plotkin point. We start with the unique-decoding case and show in Sec. V that the optimal size of codes which correct \(1/4 + \varepsilon\) fraction of asymmetric errors is exactly \(\Theta(\varepsilon^{-3/2})\). This follows from an upper bound (Theorem 4 in Sec. V-A) and a matching construction (Theorem 5 in Sec. V-B). We then generalize these results to list-decoding for any list size at least one. We show in Sec. VI-C-VI-G that the same bound \(\Theta_L(\varepsilon^{-3/2})\) is also optimal for list-decodable codes which correct \(\tau_L + \varepsilon\) fraction of asymmetric errors, where \(\tau_L\) is the list-decoding Plotkin point. See Sec. VI-A and VI-B for definitions and properties of the list-decoding radius and list-decoding Plotkin point.

We briefly explain below the ideas behind our results on codes that correct a fraction of errors beyond the Plotkin bound.

1) (Upper bound for (approximate) constant-weight codes.) For constant-weight codes (i.e., a code in which all codewords have the same Hamming weight; see [26]), it follows from a standard double-counting argument that the size of any code \(\varepsilon\)-above the Plotkin point (i.e., \(1/4\)) is at most \(O_L(\varepsilon^{-1})\). For unique-decoding, this is known by [23]; for list-decoding, this follows from [19] (see Theorem 6 in Sec. VI-C). Furthermore, these results can be extended to approximate constant-weight codes (i.e., a code in which all codewords have approximately the same Hamming weight). This can be done by either repeating the double-counting argument with additional care of the deviation in weights or carefully augmenting the codewords in such a way that the augmented codewords all have the same weight. See Theorem 8 and 9 in Sec. VI-E for details.

2) (Upper bound for general codes.) For general codes in which codewords can have any weight between 0 and \(n\), given the above results, it is tempting to partition \(\{0, 1\}^n\) into \(n - 1\) slices each of weight between \(nw\) and \((n - 1)/n\) for some \(w < 1\). The subcode in each slice is therefore approximate constant-weight and has size at most \(O_L(\varepsilon^{-1})\). In total, we get an upper bound \(O_L(\varepsilon^{-2})\) on the size of the whole code. However, this bound turns out to be suboptimal! We improve it by partitioning the space in a more delicate way. The width of the slice is wider for weights far from the critical (relative) weight \(w_{\text{max}} = L^{-\frac{1}{\tau+1}}\), and thinner for smaller weights close to \(w_{\text{max}}\). In particular, we choose the width to be \(\varepsilon(1/2)\) on average for \(w_{\text{max}} - \Omega(L^{-1})\) and \(\varepsilon(1/2)\) on average for \(\Omega(L^{-1})\). We choose it to be \(\varepsilon\) for \(w_{\text{max}} - \Omega(L^{-1})\), and keep the subcode in each slice having size \(O_L(\varepsilon^{-1})\). In total, there are at most \(O_L(1/\varepsilon^{1/2}) = O_L(\varepsilon^{-1/2})\) slices for small and large weights (i.e., weights close to \(w_{\text{max}}\)) and at most \(O_L(\varepsilon^{-1/2})\) slices for moderate weights (i.e., weights close to \(w_{\text{max}}\)). This gives an improved upper bound \(O_L(\varepsilon^{-1/2})\) for the size of the whole code. The rigorous analyses for \(L = 2\) and \(L > 2\) are presented in the proofs of Theorem 4 in Sec. V-A and Theorem 10 in Sec. VI-F, respectively.

3) (Construction of constant-weight codes.) In Sec. VI-D, we analyze a code formed by rows of a matrix whose columns are all possible constant-weight words. In Theorem 7, such a code is shown to be \(\varepsilon\)-above the list-decoding Plotkin point and have order-optimal size \(\Omega_L(\varepsilon^{-1})\). This code generalizes a construction proposed in [25] in the context of list-decodable codes for symmetric errors and has a similar flavour as weak-flip codes discussed in [27].

4) (Construction of general codes.) We note that purely random codes that correct a fraction of errors larger than the Plotkin bound do not have large size. However, we show how to use randomness in order to build a code \(\varepsilon\)-above the Plotkin point of size \(\Omega_L(\varepsilon^{-3/2})\). The non-uniform partition used in the proof of the converse bound suggests a matching construction. We reuse the constant-weight construction in Theorem 7 in a consistent way with the non-uniform partition. Specifically, we first build \(\Theta_L(\varepsilon^{-1/2})\) constant-weight codes such that the \(i\)th code has size \(\Theta_L(\varepsilon^{-1})\) and contains codewords with relative weight \(w_{\text{max}} - i\varepsilon\). The asymmetry property of the Z-channel comes to play when we apply \(\Theta_L(\varepsilon^{-1/2})\) independent random permutations on the set of coordinates within each code and consider the union of all these codes. For \(L = 2\), we carefully analyze the unique-decoding radius of the resulting construction in the proof of Theorem 5 in Sec. V-B. For \(L > 2\), we investigate the list-decoding radius of our construction in the proof of Theorem 12.
In Sec. VII-A and VII-B, we study codes for the Z-channel below the list-decoding Plotkin point and obtain upper and lower bounds on the list-decoding capacity. The upper bound in Theorem 15 follows the classical idea of Elias and Bassigny [4]. Specifically, the space is multicovered by special balls and the size of all balls is carefully adjusted such that each of them contains a constant number of codewords only. The lower bound in Theorem 17 uses the standard random coding with expurgation technique. The question of obtaining the exact list-decoding capacity for arbitrary list sizes is difficult and our upper and lower bounds do not match. However, we manage to derive the list-decoding capacity for asymptotically large list sizes in Theorem 20 in Sec. VII-C.

Sec. VIII contains discussion on the capacity of Z-channels with stochastic errors. This is a direct consequence of the seminal channel coding theorem by Shannon [3].

We end the paper with open questions in Sec. IX.

III. RELATION TO PRIOR WORKS

The \((L - 1)\)-list-decoding Plotkin point was studied in [22] for a large family of adversarial channels. Specialized to Z-channels considered in the current paper, [22] provided a characterization of \(\tau_L\) which reads as follows. For a finite set \(\Sigma\), let \(\mathcal{P}(\Sigma)\) denote the set of distributions on \(\Sigma\). For any given \(L \in \mathbb{Z}_{\geq 2}\) and \(\tau \in [0, 1]\), define the confusability set \(\mathcal{K}(\tau) \subset \mathcal{P}(\{0, 1\}^L)\) as

\[
\mathcal{K}(\tau) := \left\{ \sum_{y \in \{0, 1\}} P_{X_1, \ldots, X_L, Y} \in \mathcal{P}(\{0, 1\}^{L+1}) : \right. \\
\left. \forall i \in [L], P_{X_1, \ldots, X_L, Y}(0, 1) = 0, P_{X_i, Y}(1, 0) \leq \tau \right\}.
\]

Here, for a distribution \(P_{A,B} = \sum b P_{A,B=b}\) denotes the marginal on \(A\); for \(P_{X_1, \ldots, X_L, Y}, P_{X,Y}\) denotes the marginal on \((X_i, Y)\). The confusability set should be interpreted as the set of types (i.e., empirical distributions; see Definition 3) of \((X_i, Y)\) such that there exists a \(y \in \{0, 1\}\) that can be obtained by changing at most \(\tau\) fraction of ones to zeros in each \(x_i\) \((i \in [L])\). It can be shown [22] that \(\mathcal{K}(\tau)\) is convex and satisfies \(\mathcal{K}(\tau) \subset \mathcal{K}(\tau')\) for any \(0 \leq \tau \leq \tau' \leq 1\). Define also the set of completely positive tensors \(\text{CP} \subset (\mathbb{R}^2_{\geq 0})^{\otimes L}\) as

\[
\text{CP} := \left\{ \sum_{i=1}^k P_i^{\otimes L} \in (\mathbb{R}^2_{\geq 0})^{\otimes L} : \right. \\
\left. k \in \mathbb{Z}_{\geq 1}, (p_1, \ldots, p_k) \in (\mathbb{R}^2_{\geq 0})^k \right\}.
\]

It is known that \(\text{CP}\) is a convex cone (see [28, Theorem 6.9]). With the above definitions, [22, Theorem 47, 54] imply that the \((L - 1)\)-list-decoding Plotkin point \(\tau_L\) for Z-channels is equal to

\[
\tau_L = \inf \{ \tau \in [0, 1] : \text{CP} \cap \mathcal{P}(\{0, 1\}^L) \subset \mathcal{K}(\tau) \}.
\]

Though the characterization (1) is dimension-free (i.e., independent of \(n\)), it is formulated as an optimization problem involving checking inclusion relation between \(2^L\)-dimensional sets. Since evaluation of the RHS of (1) seems challenging, (1) does not directly provide an explicit expression of \(\tau_L\). [19] later derived the Plotkin point of constant-weight codes from the first principle following the methodology in [29] (see also Sec. VI-B). The current work differs from [19], and [22] in the following aspects.

1) Both [19], [22] focused on constant-weight (or more generally constant-type; see Definition 3) codes. This is without loss of generality for studying achievable rates of codes below the Plotkin point. However, above the Plotkin point, constant-weight codes exhibit different behaviours from weight-unconstrained codes. We characterize the scaling of the optimal sizes of both code ensembles.

2) The converse bound in [22, Theorem 54] implies that constant-weight codes correcting \(\tau_L + \varepsilon\) fraction of errors have size at most \(K(\varepsilon^{-1})\) for some function \(K(\cdot)\) that grows as fast as the hypergraph Ramsey number [30], [31] which is far from being optimal. Then the relation (8) was proved in [19, Lemma 4] for constant-weight codes which, as we show in Sec. VI-C, implies the optimal upper bound \(O(1/\varepsilon)\).

3) No construction of codes above the Plotkin point was given in [19], and [22]. Our work provides sharp lower bounds for such codes with constant or arbitrary weight by extending the construction in [25, Theorem 1].

For codes below the Plotkin point, [19, Theorem 3] also derived lower bounds on the achievable rates. Here we follow the approach in [22, Theorem 47] and derive an alternative expression of achievable rates which is formulated more compactly using the confusability set \(\mathcal{K}(w, \tau)\) (see [19] for the definition). No upper bound on the achievable rates was given in [19]. Here we derive an upper bound (see Theorem 15) using ideas from [4]. We also characterize the list-decoding capacity when the list size is sufficiently large.

IV. PRELIMINARIES

We use the standard Bachmann–Landau notation. For a set \(S\) and an integer \(0 \leq t \leq |S|\), \(\binom{S}{t} := \{S' \subset S : |S'| = t\}\). The set \(\{i, i + 1, \ldots, j\}\), for some integers \(i\) and \(j\) with \(i \leq j\), will be denoted as \([i, j]\). The set \([1, j]\) is shortly denoted as \([j]\). By slight abuse of notation, we write \([w_1, w_2]\) to also denote the closed interval of real numbers between \(w_1\) and \(w_2\). A vector of length \(n\) is denoted by bold lowercase letters, such as \(x\), and the \(i\)th entry of the vector \(x\) is referred to as \(x_i\). The all-zero vector, whose length will be clear from the context, is written as \(0\). Define the asymmetric function \(\Delta(x, y)\) to be the number of positions \(i \in [n]\) such that \(x_i = 1\) and \(y_i = 0\). The Hamming distance between \(x, y \in \{0, 1\}^n\) is then \(d_H(x, y) := \Delta(x, y) + \Delta(y, x)\). The (Hamming) weight of \(x \in \{0, 1\}^n\) is \(wt(x) = d_H(x, 0)\); the relative weight is \(wt(x)/n\). The Z-distance between \(x, y \in \{0, 1\}^n\) is defined as \(d_Z(x, y) := \max(\Delta(x, y), \Delta(y, x))\). Note that the Hamming distance and the Z-distance are related by the following identity: \(d_Z(x, y) = \frac{1}{2}(d_H(x, y) + |wt(x) - wt(y)|)\).
By this relation, we see that \( d_Z(\cdot, \cdot) \) is indeed a distance, and it equals \( d_H(\cdot, \cdot)/2 \) if the Hamming weights of \( x \) and \( y \) are the same. Define the Z-ball and the H-ball centered at \( x \in \{0,1\}^n \) with radius \( t \) as follows

\[
B_t^Z(x) := \{ y \in \{0,1\}^n : \Delta(y, x) = 0, \Delta(x, y) \leq t \},
\]

\[
B_t^H(x) := \{ y \in \{0,1\}^n : d_H(x, y) \leq t \}.
\]

Similarly, define the Z-sphere and the H-sphere centered at \( x \in \{0,1\}^n \) with radius \( t \) as follows

\[
S_t^Z(x) := \{ y \in \{0,1\}^n : \Delta(y, x) = 0, \Delta(x, y) = t \},
\]

\[
S_t^H(x) := \{ y \in \{0,1\}^n : d_H(x, y) = t \}.
\]

A (binary block) code \( C \subseteq \{0,1\}^n \) is an arbitrary subset of vectors of the same length \( n \). The size of a code \( C \) is denoted as \(|C|\). The rate of \( C \) is defined as \( R(C) := \frac{\log |C|}{n} \). A code \( C \subseteq \{0,1\}^n \) is called \( w \)-constant-weight if the weight of all codewords \( x \in C \) is \( wt(x) = nw \).

**Remark 1:** With a slight abuse of terminology, we will interchangeably use weight/distance/radius to refer to relative weight/distance/radius, that is, the former quantities normalized by \( 1/n \). Without further specification, the exact meaning should be clear from the context.

**Definition 1 (Uniquely-Decodable Code):** We say that a code \( C \subseteq \{0,1\}^n \) corrects \( t \) asymmetric (symmetric) errors if for any \( x \in \{0,1\}^n \) the respective Z-ball (H-ball) centered at \( x \) with radius \( t \) contains at most one codeword from \( C \), i.e., it holds that \( |B_t^Z(x) \cap C| \leq 1 \) (\( |B_t^H(x) \cap C| \leq 1 \)).

Note that a code \( C \) corrects \( t \) asymmetric errors if and only if for any two distinct codewords \( x, y \in C \), it holds that \( d_Z(x, y) > t \); see, e.g., [32], [33].

A code \( C \subseteq \{0,1\}^n \) is said to correct a fraction \( \tau \) of asymmetric (symmetric) errors if it corrects \( t \) asymmetric (symmetric) errors for \( t = \lceil \tau n \rceil \).

In the following statement, we first recall an observation from [7].

**Lemma 1:** Let a code \( C \subseteq \{0,1\}^n \) be \( w \)-constant-weight. Then \( C \) corrects \( t \) asymmetric errors if and only if it corrects \( t \) symmetric errors which is in turn satisfied if and only if the minimum distance of \( C \) is larger than \( 2t \).

We state two classical coding theory results, which were proved in [4] and [23] respectively.

**Lemma 2:** Let \( C \subseteq \{0,1\}^n \) be a code that corrects \( t \) symmetric errors, where \( 4t + 3 > n \). Then its size is bounded above as follows

\[
|C| \leq 2 \left( \frac{2t + 2}{4t + 3 - n} \right).
\]

**Lemma 3:** Let a code \( C \subseteq \{0,1\}^n \) be \( w \)-constant-weight and corrects \( t \) symmetric errors. If the inequality \( w^2 - (w - t)n > 0 \) is fulfilled, then the size of \( C \) is bounded above as follows

\[
|C| \leq \frac{tn}{w^2 - (w - t)n}.
\]

We introduce below the concept of list-decodable codes.

**Definition 2 (List-decodable code):** Let \( L \) be a positive integer at least 2. A code \( C \subseteq \{0,1\}^n \) is said to be \((L - 1)\)-list-decodable with list-decoding radius \( \tau \) if for any \( x \in \{0,1\}^n \) the respective Z-ball (H-ball) centered at \( x \) with radius \( t := \lceil \tau n \rceil \) contains at most \( L - 1 \) codewords from \( C \), i.e., it holds that \( |B_t^Z(x) \cap C| \leq L - 1 \) (\( |B_t^H(x) \cap C| \leq L - 1 \)).

At last, we need the binary entropy function defined as \( H(x) := -x \log x - (1 - x) \log(1 - x) \) for any \( x \in [0,1] \).

**V. Uniquely-Decodable Codes Above the Plotkin Point**

In this section, we obtain upper and lower bounds on the size of an optimal code capable of correcting a fraction \( \tau = \frac{1}{4} + \varepsilon \) of asymmetric errors. For simplicity of arguments, we naturally assume in this section that if an \( n \)-length code corrects a fraction \( \tau = \frac{1}{4} + \varepsilon \) of errors, then \( \tau n = n/4 + \varepsilon n \) is an integer and, consequently, \( \varepsilon n \geq 1/4 \).

**A. Upper Bound**

In the following statement, we derive an upper bound on the size of a code capable of correcting a large fraction of asymmetric errors. The idea of the proof is to partition a code into \( O(\varepsilon^{-1/2}) \) subcodes with approximate constant-weight. By lengthening codewords within each subcode, we obtain constant-weight codes correcting a large fraction of asymmetric errors and show that their size can be bounded by \( O(\varepsilon^{-1}) \).

**Theorem 4:** Let \( C \subseteq \{0,1\}^n \) be a code correcting a fraction \( \tau = \frac{1}{4} + \varepsilon \) of asymmetric errors for some real number \( \varepsilon \leq 1/12 \), where \( \tau n \) is an integer. Then the size of the code can be bounded as follows

\[
|C| \leq 1 + \frac{3\sqrt{\varepsilon} + 4\varepsilon}{\varepsilon^{1/2}} + 14.
\]

**Remark 2:** A suboptimal upper bound \( O(\varepsilon^{-2}) \) was presented in the first version of [34] for uniquely-decodable codes. The authors of the current manuscript then managed to improve this bound to the optimal order \( O(\varepsilon^{-3/2}) \). (In fact, we proved the upper bound \( O_L(\varepsilon^{-3/2}) \) for \((L - 1)\)-list-decodable codes with any \( L \geq 2 \); see Theorem 10.) During the finalization of the current manuscript, the first author updated the other manuscript [19] and presented the improved bound \( O(\varepsilon^{-3/2}) \) for \( L = 2 \) there. However, the latter bound was first obtained by the authors of the current manuscript.

**Proof of Theorem 4:** Without loss of generality, we assume that the number of codewords of weight at most \( n/2 \) is at least as the number of codewords of weight at least \( n/2 \). Otherwise, we can consider the code that is obtained by replacing 0 by 1 and 1 by 0 in all codewords of the original code because the Z-distance \( d_Z(x, y) = \max(\Delta(x, y), \Delta(y, x)) \) is not changed after such swapping. For a non-negative integer \( i \), define \( \rho_i := \frac{i}{\tau n} \). Define the subcode

\[
A_i := \{ x \in C : \rho_i n < wt(x) \leq \rho_{i+1} n \}.
\]

Extend all codewords of \( A_i \) by appending \( \rho_{i+1} n - \rho_i n - 1 \) extra positions such that all codewords have weight \( \rho_{i+1} n \). Note that this can be done in different ways. From Lemma 1 it follows that the obtained code \( A_i' \subseteq \{0,1\}^n \) with \( n' := n + \rho_{i+1} n - \rho_i n - 1 \) contains codewords of weight \( w_i' := \rho_{i+1} n \) and corrects \( t := (1/4 + \varepsilon)n \) symmetric errors.
To apply Lemma 3 for code $A'_i$, one needs to check that the condition of this lemma is fulfilled, specifically $w_i^2 - (w'_i - t) n'_i > 0$. Note that the function $f_i(z) := z^2 - (z - t)n'_i$ of argument $z$ is monotonically decreasing for $z \leq n'_i/2$. Clearly, $\rho_{i+1} n \leq n'_i/2$. Thus,

\[
f_i(w'_i) = f_i((\rho_{i+1} n)) \\
\geq f_i(\rho_{i+1} n) \\
= \rho^2_{i+1} n^2 - (\rho_{i+1} n - (1/4 + \varepsilon) n) \\
\times (n + [n_{i+1} - 1] - (1 + \rho_{i+1} - \rho_i)) \\
\geq n^2 \left( \rho^2_{i+1} - (\rho_{i+1} - (1/4 + \varepsilon))(1 + \rho_{i+1} - \rho_i) \right),
\]

where the last inequality holds for $\rho_{i+1} - (1/4 + \varepsilon) \geq 0$ or $\varepsilon \leq 1/12$ as $\rho_{i+1} \geq 1/3$. Observe that $\rho^2_{i+1} - (\rho_{i+1} - 1/4) \geq 0$ since $\rho_i = 1 + 1/(4 + \varepsilon)$. Therefore, we can proceed as follows

\[
f_i(w'_i) \geq n^2 \varepsilon (1 + \rho_{i+1} - \rho_i) > 0.
\]

Thus, we can apply Lemma 3 for code $A'_i$ and estimate its size

\[
|A_i| = |A'_i| \\
\leq \frac{tn'_i}{f_i(w'_i)} \\
\leq \frac{(1/4 + \varepsilon)n(n + [\rho_{i+1} n] - [\rho_i n] - 1)}{n^2 \varepsilon(1 + \rho_{i+1} - \rho_i)} \\
\leq 1 + 1/(4\varepsilon).
\]

Let $i_0 := \lceil 1/(2\sqrt{\varepsilon}) \rceil$ and, thus, $\rho_{i_0} = \rho_{2i_0+1} \geq 1 - 2\sqrt{\varepsilon} \geq 2 + 2\sqrt{\varepsilon}$. For a non-negative integer $j$, consider the subcode

$B_j := \{ x \in C : |\rho_{i_0} n| + j[2\varepsilon n] < \text{wt}(x) \leq |\rho_{i_0} n| + (j + 1)[2\varepsilon n] \}$.

We extend all codewords of $B_j$ by appending $[2\varepsilon n]$ extra positions such that all obtained codewords have the same weight. We get the code $B''_j \subseteq \{0, 1\}^n$ with $n'' := n + [2\varepsilon n]$ which contains codewords of weight $w''_j := |\rho_{i_0} n| + (j + 1)[2\varepsilon n]$ and corrects $t = (1/4 + \varepsilon) n$ symmetric errors. We shall apply Lemma 2 for code $B''_j$. The condition of that lemma is satisfied as long as $4t + 3 > 2n''$ or

\[
(1 + 4\varepsilon)n + 3 > n + 2\varepsilon n + 1 \geq n + [2\varepsilon n],
\]

which is correct. By Lemma 2, we estimate

\[
|B''_j| = |B'_j| \\
\leq \frac{2t + 2}{4t + 3 - n''} \\
\leq \frac{n + 4\varepsilon n + 3 - n - 2\varepsilon n - 1}{2\varepsilon n + 2} \\
= 1 + 1/(2\varepsilon).
\]

Each codeword of $C$, having weight within the interval $[1, n/2]$, is included in $A_i$ with some $i \in [0, i_0 - 1]$ or $B_j$ with some $j \in [0, j_0 - 1]$ if $j_0 := \lceil 3/(4\sqrt{\varepsilon}) + 2 \rceil$. Indeed, all codewords with weight in interval $[1, |\rho_{i_0} n| + j_0[2\varepsilon n]]$ are considered by this partition and the largest weight is at least $n/2$:

\[
|\rho_{i_0} n| + j_0[2\varepsilon n] \geq \frac{1 - 2\sqrt{\varepsilon}}{2 + 2\sqrt{\varepsilon}} n - 1 + \left( \frac{3}{4\sqrt{\varepsilon}} + 2 \right) 2\varepsilon n \\
= \left[ \frac{(4\sqrt{\varepsilon} - 8\varepsilon)n - 8\sqrt{\varepsilon} - 8\varepsilon}{8\varepsilon^2 + 8\varepsilon} + 12\varepsilon n + 12\varepsilon^3/2 n + 32\varepsilon^3/2 n + 32\varepsilon^2 n \right] \\
= \left( \frac{2 + 2\sqrt{\varepsilon}}{2\sqrt{\varepsilon}} \right) 4\sqrt{\varepsilon} \\
\geq \frac{n(4\sqrt{\varepsilon} + 4\varepsilon) - 8\varepsilon^2 + 8\varepsilon}{8\varepsilon^2 + 8\varepsilon} > \frac{n}{2},
\]

where in the second-last inequality we applied the fact that $\varepsilon \geq 1/4$.

Since the number of codewords with weight from the interval $[1, n/2]$ is not less than the number of codewords of weight from the interval $[n/2, n - 1]$, it holds

\[
|C| \leq 2 \sum_{i=0}^{i_0-1} |A_i| + \sum_{j=0}^{j_0-1} |B'_j| + 2 \\
\leq 2i_0 \left( \frac{1}{4}\right) + 2j_0 \left( \frac{1}{4} \varepsilon + \frac{3}{4\sqrt{\varepsilon}} \right) \left( 4 + \frac{1}{\varepsilon} \right) + 2 \\
= \frac{1}{4} + \varepsilon + 3/4 + 3\sqrt{\varepsilon} + 3\varepsilon \geq 14 \\
= \frac{1 + 3\sqrt{\varepsilon} + 4\varepsilon}{\varepsilon^3/2} + 14.
\]

$\Box$

B. Construction

In the following statement, we prove that there exists a code of size $\Omega(\varepsilon^{-3/2})$ and length $\exp(\Theta(\varepsilon^{-3/2}))$ capable of correcting a fraction $1/4 + \varepsilon$ of asymmetric errors. We use the intuition from the proof of Theorem 4. First, we build $O(\varepsilon^{-1/2})$ codes such that the $j$th code is a uniquely-decodable code containing codewords with the relative weight $1/2 - j\varepsilon$. We borrow an idea for such code with $j = 0$ from [25], where the authors constructed list-decodable codes for symmetric errors. By performing simple repetition, we construct longer codes of the same size while the error-correction capability of all those codes remains unchanged. Applying a random permutation on the set of coordinates within each code, we guarantee that two codewords from different codes have a large asymmetric distance with overwhelming probability.

Theorem 5: There exists a code of length $\exp(O(\varepsilon^{-3/2}))$, capable of correcting a fraction $\tau = \frac{1}{4} + \varepsilon$ of asymmetric errors. Furthermore, its size is at least $\frac{3\sqrt{\varepsilon}}{128} \varepsilon^{-3/2}(1 + o(1))$ as $\varepsilon \to 0$.

Proof: Consider the positive integer $m := \lceil 3/(32\varepsilon) \rceil$ and define the real number $c := 2^{-3/2}$. For every $j \in \{-\lfloor c\sqrt{m} \rfloor, \ldots, \lfloor c\sqrt{m} \rfloor \}$, denote $n_j := \lfloor c^m n \rfloor$. Consider a binary matrix $A_j$ of size $2m \times n_j$, whose columns are all possible binary vectors of length $2m$ and weight $m - j$. For two distinct rows $x$ and $y$ of matrix $A_j$, we compute
the number of positions in which the rows are different. We have
\[ \Delta(x, y) = \Delta(y, x) = \binom{2m - 2}{m - j - 1}. \]

This means that a code, whose codewords are rows of \( A_j \), corrects a fraction \( \tau_j \) of asymmetric errors, where
\[ \tau_j := \frac{(2m - 2j - 1)}{(2m - j)} = \frac{(m - j)(m + j)}{2m(2m - 1)} - \frac{1}{2m(2m - 1)}. \]

Let us show that for small enough \( \varepsilon \), it holds that \( \tau_j \geq \frac{1}{4} + \varepsilon \).
We have
\[ \frac{m/2 - j^2}{4m^2 - 2m} - \frac{1}{(2m - j)} \geq \frac{m^2 - c^2 m}{4m^2 - 2m} - \frac{1}{2m - 1} \]
\[ = \frac{32m - 16 - 3}{(2m - 1)} \]
\[ = \frac{3}{32m} + \frac{3}{2m(32m - 16)} - \frac{1}{(2m - 1)} \]
\[ \geq \varepsilon. \]

To show the last inequality, we used that \( m = \lfloor 3/(32\varepsilon) \rfloor \), \( c = 2^{-3/2} \) and the fact that for sufficiently large \( m \) (small \( \varepsilon \)), it holds
\[ \frac{3}{64m^2} > \frac{1}{(2m - 1)}. \]

For a positive integer \( z \), consider a matrix \( A_j(z) \) of size \( 2m \times zn_j \) which is composed of \( z \) copies of the matrix \( A_j \). We write copies of \( A_j \) to the right, i.e., \( A_j(z) = (A_j, A_j, \ldots, A_j) \).

By \( \hat{A}_j(z) \) denote a random matrix obtained from \( A_j(z) \) by applying a random permutation of its columns. This permutation is taken uniformly at random from the set of all permutations. Note that a code whose codewords are rows of \( \hat{A}_j(z) \) corrects the same fraction of errors as the original code obtained from \( A_j \).

Define integers
\[ z_j := \prod_{i \neq j} \left( \frac{2m}{m - i} \right), \quad N := \prod_{i = -\lfloor \sqrt{m} \rfloor}^{\lfloor \sqrt{m} \rfloor} \left( \frac{2m}{m - i} \right), \]
\[ M := 2m(2\lfloor \sqrt{m} \rfloor + 1). \]

Consider a random matrix \( A \) of size \( M \times N \) containing \( \hat{A}_j(z) \) as a submatrix for all \( j \in \{-\lfloor \sqrt{m} \rfloor, \ldots, \lfloor \sqrt{m} \rfloor \} \). We assume that matrices \( \hat{A}_j(z) \) are written one below the other using the ascending order of parameter \( j \). For \( \varepsilon \to 0 \), we estimate the number of rows \( M \) and the number of columns \( N \) in \( A \) as follows
\[ M = \frac{3\sqrt{3}}{128\varepsilon \sqrt{\varepsilon}} (1 + o(1)), \quad N = \exp(\Theta(\varepsilon^{-3/2})). \]

Define the random code \( C \) of length \( N \) and size \( M \) to be the set of rows in \( A \). Let us show that for two distinct rows \( \hat{x} \) and \( \hat{y} \) in \( A \), the value \( d_2(\hat{x}, \hat{y}) \) is large enough with overwhelming probability. The latter ensures that the code \( C \) can correct the required fraction of asymmetric errors. More formally, let \( \hat{x} \) and \( \hat{y} \) be rows from \( A_j(z) \) and \( A_{j'}(z) \), where \( j < j' \). Clearly, \( wt(\hat{x}) = \frac{m - j}{2m} > \frac{m - j}{2m} \) and \( d_2(\hat{x}, \hat{y}) = \Delta(\hat{x}, \hat{y}) \).

The probability distribution for the random variable \( \Delta(\hat{x}, \hat{y}) \) is given by:
\[ \Pr\{\Delta(\hat{x}, \hat{y}) = t\} = \left( \frac{\rho(\hat{x})}{\rho(\hat{y})} \right)^t \]
\[ = \left( \frac{\rho(\hat{x})}{\rho(\hat{y})} \right)^t \]
for \( t \in \{\rho(\hat{x}) - \rho(\hat{y}), \ldots, \min(\rho(\hat{x}), N - \rho(\hat{y}))\} \) and 0 otherwise. We estimate the probability of the event that \( \Delta(\hat{x}, \hat{y}) \) is small as follows
\[ \Pr\{\Delta(\hat{x}, \hat{y}) \leq N \left( \frac{1}{4} + \varepsilon \right)\} \]
\[ \leq N \max_{t \in [0, [N(1/4 + \varepsilon)]]} \Pr\{\Delta(\hat{x}, \hat{y}) = t\}. \]

Let an integer \( t \) be equal to \( \alpha N \) for some real number \( \alpha \in [1/2m, \min(m/2m, m/2m)] \). Note that \( M \) and \( N \) are functions of \( \varepsilon \). Define the function
\[ g_{i,j}(\alpha, \varepsilon) := \frac{1}{N} \log \Pr\{\Delta(\hat{x}, \hat{y}) = t\}. \]

For arbitrary integers \( u \) and \( v \) so that \( u > v \geq 1 \), the binomial coefficient \( \binom{u}{v} \) satisfies
\[ \sqrt{\frac{u}{8v(u - v)}} \cdot 2^{|H(v/u)} \leq \left( \frac{u}{v} \right) \leq 2^{|H(v/u)} \cdot \left( \frac{u}{v} \right). \]

Thus, for \( \alpha \in (1/2m, \min(m/2m, m/2m)] \), it holds that
\[ g_{i,j}(\alpha, \varepsilon) \leq \frac{m - j}{2m} H \left( \frac{2am}{m - j} \right) + \frac{m + j}{2m} H \left( \frac{j - i + 2\alpha m}{m + j} \right) - H \left( \frac{m - i}{2m} \right) \]
\[ \leq r_{i,j}(\alpha, \varepsilon) + \delta(\varepsilon), \]
where functions \( r_{i,j}(\alpha, \varepsilon) \) and \( \delta(\varepsilon) \) are defined as follows
\[ r_{i,j}(\alpha, \varepsilon) := \frac{m - j}{2m} H \left( \frac{2am}{m - j} \right) + \frac{m + j}{2m} H \left( \frac{j - i + 2\alpha m}{m + j} \right) - H \left( \frac{m - i}{2m} \right), \]
\[ \delta(\varepsilon) := \frac{\log(2N)}{2N}. \]

Using the relation \( \frac{\partial H(x)}{\partial x} = \log(1 - x) \), we compute the derivative of \( r_{i,j}(\alpha, \varepsilon) \): 
\[ g_{i,j}(\alpha, \varepsilon) := \frac{\partial r_{i,j}(\alpha, \varepsilon)}{\partial \alpha} \]
\[ = \frac{\log \left( \frac{m - j - 2\alpha m}{j - i - 2\alpha m} \right)}{2am} + \frac{\log \left( \frac{m + i - 2\alpha m}{j - i + 2\alpha m} \right)}{2am}. \]

Let \( \alpha_{i,j} = \alpha_{i,j}(\varepsilon) := (m - j)N + (m - j)(2m - i) \). Clearly, the function \( g_{i,j}(\alpha, \varepsilon) \) is positive for \( \alpha < \alpha_{i,j} \), and the function \( r_{i,j}(\alpha, \varepsilon) \leq 0 \) for all required \( \alpha \). Furthermore, \( r_{i,j}(\alpha_{i,j}, \varepsilon) = 0 \).
Since $m = \lfloor 3/(2\varepsilon) \rfloor$ and $-|c\sqrt{m}| \leq j < i \leq |c\sqrt{m}|$, we obtain that $\alpha_{i,j}(\varepsilon) - \left(\frac{1}{4} + \varepsilon\right) > 0$ and
\[
\alpha_{i,j}(\varepsilon) - \left(\frac{1}{4} + \varepsilon\right) = \frac{(m-j)(m+i)}{4m^2} - \left(\frac{1}{4} + \varepsilon\right)
\]
\[
= \frac{(i-j)m-ij-4\varepsilon m^2}{4m^2}.
\]
Since the derivative of $r_{i,j}(\alpha, \varepsilon)$ is positive for $\alpha \leq 1/4 + \varepsilon$, we get
\[
\sup_{\frac{1}{4} + \varepsilon < \alpha < \frac{1}{4} + \varepsilon + \varepsilon} g_{i,j}(\alpha, \varepsilon) \leq r_{i,j}(1/4 + \varepsilon, \varepsilon) + \delta(\varepsilon). \tag{3}
\]
Observe that we have the partial sum of the Taylor series with the remainder in Lagrange’s form
\[
\begin{align*}
  r_{i,j}(\alpha_{i,j}, \varepsilon) &= r_{i,j}(1/4 + \varepsilon, \varepsilon) + (\alpha_{i,j} - 1/4 - \varepsilon)q_{i,j}(1/4 + \varepsilon, \varepsilon) \\
  &\quad + (\alpha_{i,j} - 1/4 - \varepsilon)^2 \sigma_{i,j}(\theta, \varepsilon), \tag{4}
\end{align*}
\]
where $\sigma_{i,j}(\alpha, \varepsilon) := \frac{\partial q_{i,j}(\alpha, \varepsilon)}{\partial \alpha}$ and $\theta$ is some point within the interval $[1/4 + \varepsilon, \alpha_{i,j}]$. Let us show that the function $\sigma_{i,j}(\alpha, \varepsilon)$ in the interval $(1/4 + \varepsilon, \alpha_{i,j})$ can be bounded below
\[
\frac{\sigma_{i,j}(\alpha, \varepsilon)}{\log \varepsilon} = \frac{\log \varepsilon}{\alpha(m-j-2\alpha m)} - \frac{2\alpha m + j m - j}{(j-i-2\alpha m)(m + i - 2\alpha m)}.
\]
For $\varepsilon \to 0$, $m = \Theta(\varepsilon^{-1})$. Thus, $\sigma_{i,j}(\theta, \varepsilon) = -16 \log \varepsilon(1 + o(1))$ as $\varepsilon \to 0$. Indeed, it holds that
\[
\lim_{\varepsilon \to 0} \sup_{1/4 + \varepsilon < \alpha < \alpha_{i,j}} \sigma_{i,j}(\alpha, \varepsilon) = -16 \log \varepsilon,
\]
\[
\lim_{\varepsilon \to 0} \inf_{1/4 + \varepsilon < \alpha < \alpha_{i,j}} \sigma_{i,j}(\alpha, \varepsilon) = -16 \log \varepsilon.
\]
Now we estimate $q_{i,j}(1/4 + \varepsilon, \varepsilon)$ as $\varepsilon \to 0$
\[
q_{i,j}(1/4 + \varepsilon, \varepsilon) = \log \left(1 + \frac{mi - 4\varepsilon m^2 - jm - ji}{(m/2 + 2\varepsilon m)(m/2 + j - i + 2\varepsilon m)}\right)
\]
\[
= 4 \log \varepsilon \frac{m(m-j) - 4\varepsilon m^2 - ji}{m^2} (1 + o(1)).
\]
Recall that $r_{i,j}(\alpha_{i,j}, \varepsilon) = 0$. Combining the above bounds with (4), we get
\[
r_{i,j}(1/4 + \varepsilon, \varepsilon) = -\left(\alpha_{i,j} - \frac{1}{4} - \varepsilon\right)
\]
\[
\times \left(m(i-j) - 4\varepsilon m^2 - ji \right) \frac{4 \log \varepsilon - 2 \log \varepsilon}{m^2} \left(1 + o(1)\right)
\]
\[
\leq -\lambda \varepsilon^2 + o(\varepsilon^2)
\]
for some constant $\lambda > 0$ and $\varepsilon \to 0$. Using this inequality, the bound $\delta(\varepsilon) = o(\varepsilon^2)$ and the inequality (3), we estimate the LHS of (2) as follows
\[
\Pr\left\{\Delta(\hat{x}, \hat{y}) \leq N\left(\frac{1}{4} + \varepsilon\right)\right\} \leq N^{-2} = N^2 = o(1),
\]
since $\varepsilon^2 N = \exp(\Omega(\varepsilon^{-3/2}))$. The probability of the event that $\max(\Delta(\hat{x}, \hat{y}), \Delta(\hat{y}, \hat{x})) \leq N\left(\frac{1}{4} + \varepsilon\right)$ for some distinct rows $\hat{x}, \hat{y}$ in matrix $A$ can be bounded above as follows
\[
\left\{\frac{M}{2}\right\}_{\hat{x}, \hat{y} \in A} \Pr\left\{\max(\Delta(\hat{x}, \hat{y}), \Delta(\hat{y}, \hat{x})) \leq N\left(\frac{1}{4} + \varepsilon\right)\right\} = o(1).
\]
It follows that for $\varepsilon \to 0$, with overwhelming probability, the code composed of rows of matrix $A$ can correct a fraction $\tau = \frac{1}{4} + \varepsilon$ of asymmetric errors. \hfill \Box

VI. LIST-DECODABLE CODES ABOVE THE PLOTKIN POINT

A. List-Decoding Radius

Fix $L \in \mathbb{Z}_{>2}$. For an $L$-list $\mathcal{L} := \{x_1, \ldots, x_L\} \in \{0, 1\}^n$ of distinct vectors, define its Chebyshev radius with respect to $Z$-distance as the smallest $t$ such that all sequences in $\mathcal{L}$ are contained in a $Z$-ball of radius $t$:
\[
\text{rad}(\mathcal{L}) := \min\{t \in \{0, 1, \ldots, n\} : \exists y \in \{0, 1\}^n, \mathcal{L} \subset B_Z^t(y)\}. \tag{5}
\]
In other words, $\text{rad}(\mathcal{L})$ is the smallest number of asymmetric errors that are sufficient to change all sequences in $\mathcal{L}$ to a same sequence $y$. Let $\mathcal{C} \subseteq \{0, 1\}^n$ be an arbitrary code for the $Z$-channel. The $(L - 1)$-list-decoding radius $\tau_L(\mathcal{C})$ of $\mathcal{C}$ is defined as
\[
\tau_L(\mathcal{C}) := \min_{\mathcal{L} \in \mathcal{C}} \text{rad}(\mathcal{L}).
\]
It is easy to see that the minimum in the definition of $\text{rad}(\mathcal{L})$ can be achieved by the following vector $y_{L} = (y_1, \ldots, y_n) \in \{0, 1\}^n$:
\[
y_j = \begin{cases} 
1, & x_{L,j} = \cdots = x_{L,j-1} = 1 \\
0, & \text{otherwise}
\end{cases}
\tag{6}
\]
for each $j \in [n]$. Here $x_{i,j}$ denotes the $j$th entry of $x_i$. In words, the smallest $Z$-ball containing $\mathcal{L}$ is centered at the vector whose support is the intersection of the supports of all vectors in $\mathcal{L}$. Observe that since $\text{supp}(y_{L}) \subseteq \text{supp}(x_i)$ for all $i \in [L]$, we have $d_2(x_i, y_{L}) = |\text{supp}(x_i)| - |\text{supp}(y_{L})| = \text{wt}(x_i) - \text{wt}(y_{L})$.\footnote{For a vector $v$, we use $\text{supp}(v)$ to denote its support, i.e., $\text{supp}(v) = \{i : v_i \neq 0\}$.} Therefore
\[
\text{rad}(\mathcal{L}) := \max_{i \in [L]} \text{wt}(x_i) - \text{wt}(y_{L})
\]
\[
= \max_{i \in [L]} \text{wt}(x_i) - \left| \bigcup_{i \in [L]} \text{supp}(x_i) \right|.
\]
Furthermore, if all $x_i$'s have the same weight $nw$, then
\[
\text{rad}(\mathcal{L}) = nw - \left| \bigcup_{i \in [L]} \text{supp}(x_i) \right|. \tag{7}
\]
For a constant-weight code $\mathcal{C}$ of weight $w$, we have
\[
\tau_L(\mathcal{C}) = nw - \max_{\mathcal{L} \in \mathcal{C}} \left| \bigcup_{x \in \mathcal{L}} \text{supp}(x) \right|.
\]
It is not hard to see that $C \subseteq \{0, 1\}^n$ is $(L - 1)$-list-decodable with list-decoding radius $\tau$ if and only if $\tau_L(C) > n\tau$. The $(L - 1)$-list-decoding capacity of a Z-channel with input weight $w$ and fraction of asymmetric errors $\tau$ is defined as

$$C_{L-1}(w, \tau) := \limsup_{n \to \infty} \max_{\tau_L(C_n) > n\tau} \frac{R(C_n)}{n\tau}.$$ 

The $(L - 1)$-list-decoding radius of a Z-channel with input weight $w$ and rate $R$ is defined as

$$\tau_L(w, R) := \limsup_{n \to \infty} \max_{\tau_L(C_n) \geq R} \frac{\tau_L(C_n)}{n\tau}.$$ 

### B. List-Decoding Plotkin Point

Let $\tau_L(w) := w - w^L$ and $\tau_L := \max \{ \tau_L(w) \}$. With slight abuse of notation, we write $w_{\max}$ to denote the argument $w$ that attains the maximum of $\tau_L(w)$. It is not hard to check that $\tau_L(w)$ is concave in $w$ and the maximizer $w_{\max}$ equals $L - \frac{1}{w}$. Therefore $\tau_L = L - \frac{1}{w} - L - \frac{w}{w} = (1 - L^{-1}) L - \frac{1}{w} = (L - 1) \frac{1}{w}$. Note that $w_{\max}$ is increasing in $L$. The minimum value of $w_{\max}$ over all $L \in \mathbb{Z}_{\geq 2}$ is $1/2$ when $L = 2$ and $w_{\max} = 1$.

Note that $\tau_L$ is increasing in $L$ and attains its minimum value $1/4$ at $L = 2$. Furthermore, $\tau_L(\cdot) \sim 1$.

### C. Upper Bound for Constant-Weight Codes

Fix $w \in (0, 1)$. Consider an arbitrary code $C$ of constant-weight $w$ for the Z-channel with noise level $\tau = \tau_L(w) + \epsilon$. Let $M := |C|$. It was shown in [19] via a double-counting argument that

$$\frac{M}{M(M - 1) \cdots (M - L + 1)} \geq \frac{\tau}{\tau_L(w)}.$$ 

Rearranging terms, we get

$$(M(M - 1) \cdots (M - L + 1)) \tau_L(w) + \epsilon \leq M^L \tau_L(w) \leq (M(M - 1) \cdots (M - L + 1)) \tau_L(w) \leq \frac{(M - M(M - 1) \cdots (M - L + 1)) \tau_L(w)}{M^L - M(M - 1) \cdots (M - L + 1)} \leq \frac{\tau_L(w)}{\epsilon}.$$ 

Applying Taylor expansion to the LHS of the above inequality (as a function of $M$) at $M \to \infty$, we get

$$\frac{M}{(\frac{M}{2})} - O(L(1)) \leq \frac{\tau_L(w)}{\epsilon} \leq \frac{C_{L,w}}{\epsilon} + O(L(1),$$

where $C_{L,w} := \tau_L(w)(\frac{L}{2}) = (w - w^L)(\frac{L}{2}).$

We have therefore proved the following theorem.

**Theorem 6:** Any $w$-constant-weight $(L - 1)$-list-decodable code $C$ for the Z-channel with list-decoding radius $w - w^L + \epsilon$ has size at most $\frac{w^L}{w} + O(L(1))$ where $C_{L,w} := (w - w^L)(\frac{L}{2}).$

**Remark 3:** From Theorem 6 it follows that the rate $R(C)$ of a $w$-constant-weight code $C \subseteq \{0, 1\}^n$ for the Z-channel with $(L - 1)$-list-decoding radius $w - w^L + \epsilon$ is asymptotically zero as $n \to \infty$. This observation motivates us to call the discussed codes list-decodable zero-rate codes.

### D. Construction of Constant-Weight Codes

In this section, we construct codes of constant-weight $w$ whose list-decoding radius is $\epsilon$-above the list-decoding Plotkin point $\tau_L(w)$ for Z-channels. The code we construct has size $\Theta(1/\epsilon)$, which is optimal by the upper bound in the preceding section. Our construction is inspired by [25].

We prove the following theorem.

**Theorem 7:** There exists a $w$-constant-weight $(L - 1)$-list-decodable code $C$ for the Z-channel with list-decoding radius $w - w^L + \epsilon$ of size at least $w^L w^{L-1}(\frac{L}{2})$.

**Proof:** Fix the relative weight $0 < w < 1$ to be rational without loss of generality. Let $m$ be a sufficiently large integer.

Since $m$ is sufficiently large, we may assume that $m/w$ is an integer. Let $M := m/w$ and $n := (m/w)$. Our construction of $C$ can be viewed as an $M \times n$ matrix, each row of which is a codeword denoted by $x_i$ ($1 \leq i \leq M$). The code $C$ consists of all possible length-$\frac{m}{w}$ vectors of Hamming weight $m$ as its columns.

To examine $(L - 1)$-list-decodability of $C$, we compute the $(L - 1)$-list-decoding radius $\tau_L(C)$ of $C$. By symmetry, the Chebyshev radius rad($L$) of any $L$-list $L \in \binom{M}{L}$ of codewords does not depend on the choice of $L$. Therefore, $\tau_L(C) = E_L \{ \text{rad} (L) \}$ where the expectation is over $L$ uniformly chosen from $\binom{M}{L}$. We now compute the latter quantity.

$$E_L \{ \text{rad} (L) \} = nw - E_L \left\{ \left| \bigcap_{x \in L} \text{supp}(x) \right| \right\}$$

$$= nw - E_L \left\{ \sum_{i \in [n]} 1 \left\{ \forall x \in L, \ x_i = 1 \right\} \right\}$$

$$= nw - \sum_{i \in [n]} \text{Pr} \left\{ \forall x \in L, \ x_i = 1 \right\}$$

$$= nw - \sum_{i \in [n]} \frac{\binom{m}{w}}{\binom{m}{w}}.$$ 

The last equality follows since the probability in the summation can be viewed as the probability that one gets $L$ ones when sampling without replacement $L$ bits from a length-$\frac{m}{w}$ vector of weight $m$.

Taking the Taylor expansion of the above expression at $m \to \infty$, we get

$$\tau_L(C) = E_L \{ \text{rad} (L) \} = n \left( w - w^L + \epsilon \right) = \left( \frac{L}{2} \right)^{-1} + O_L(w^{-1})$$

Recall that we want $C$ to be $\epsilon$-above the Plotkin point, i.e.,

$$\tau_L(C) \geq n(\tau_L(w) + \epsilon) = n(w - w^L + \epsilon).$$

This is satisfied as

Note that there are $\binom{m}{w} = n$ such columns in total.
long as
\[ \varepsilon \leq (1 - w)w^L\left(\frac{L}{2}\right)^{m-1} + o_{L,w}(m^{-2}) \]
\[ \Longleftrightarrow \quad \varepsilon \leq (1 - w)w^{L-1}\left(\frac{L}{2}\right)^{M-1} + o_{L,w}(M^{-2}) \]
\[ \Longleftrightarrow \quad M \leq \frac{c_{L,w}}{\varepsilon} + o_{L,w}(1), \]
where \( c_{L,w} := (1 - w)w^{L-1}\left(\frac{L}{2}\right) \). This finishes the proof. \( \square \)

E. Upper Bound for Approximate Constant-Weight Codes

The proof in Sec. VI-C can be modified to work for approximate constant-weight codes.

**Theorem 8:** Let \( C \) be an arbitrary \((L - 1)\)-list-decodable code for the Z-channel with list-decoding radius \( \tau = w - w^L + \varepsilon \). Suppose that every codeword in \( C \) has weight in \([nw(1 - \delta), nw(1 + \delta)]\) for some \( \delta \in [0, \tau/(2w)] \). Then \( M := |C| \) satisfies the following relation:

\[ \frac{M^L}{M(M - 1) \cdots (M - L + 1)} \geq \frac{\tau - 2w\delta}{w(1 + \delta) - (w(1 - \delta))^L}. \]

**Remark 4:** If \( \delta = 0 \), the above theorem recovers Theorem 6.

**Proof:** To derive an upper bound on \( M \), we follow a double-counting argument commonly used in coding theory. We bound from both sides the following quantity:

\[ \sum_{L \in [M]^{M - j} \in L} \sum_{j \in L} d_L(x_j, y_L) \]

where \( y_L \) is defined in Eqn. (6).

We first give a lower bound on Eqn. (9). We drop all terms in Eqn. (9) with an \( L \) whose elements are not distinct, for any \( L \) whose elements are all distinct, by list-decodability,

\[ \max_{j \in L} d_L(x_j, y_L) = \max_{j \in L} wt(x_j) - wt(y_L) > n\tau. \]

Since \( C \) is approximate constant-weight, we have \([wt(x_i) - wt(x_j)] \leq 2nw\delta\) for any \( i \neq j \). Therefore, Eqn. (9) is at least \( M(M - 1) \cdots (M - L + 1)(n\tau + (L - 1)(n\tau - 2nw\delta)) \).

We then give an upper bound on Eqn. (9).

\[ \sum_{(i_1, \ldots, i_L) \in [M]^L} \sum_{j \in L} \left(\sum_{i \in [M]} (wt(x_{i,j}) - wt(y_{i,j})) \right) \]

\[ = \sum_{(i_1, \ldots, i_L) \in [M]^L \ni j} \left(\sum_{i \in [M]} (wt(x_{i,j})) \right) \]

\[ - \sum_{k \in [n]} \left(\prod_{i \in [M]} \left(\sum_{i \in [M]} 1 \{ x_{i,k} = 1 \}\right) \right) \]

\[ \leq M^L n w(1 + \delta) - \sum_{j \in [L]} \sum_{k \in [n]} \left(\prod_{i \in [M]} 1 \{ x_{i,k} = 1 \}\right) \]

\[ = M^L n w(1 + \delta) - L \sum_{k \in [n]} S_k \]

where \( S_k := \sum_{i \in [M]} 1 \{ x_{i,k} = 1 \} \) denotes the weight of the \( k \)th column of \( C \in \{0, 1\}^{M \times n} \). By the norm comparison inequality

\[ \|x\|_p \leq n^{1/p - 1/q}\|x\|_q \]

for any \( x \in \mathbb{R}^n \) and \( 0 < p < q \), we have

\[ \sum_{k \in [n]} S_k \geq n^{1 - L} \left(\sum_{k \in [n]} S_k \right)^L \geq n^{1 - L}(nw(1 - \delta))^L = n(Mw(1 - \delta))^L. \]

Therefore, Eqn. (9) is upper bounded by \( M^L n w(1 + \delta) - nL(Mw(1 - \delta))^L \).

Finally, putting the lower and upper bounds together, we get

\[ M(M - 1) \cdots (M - L + 1)(n\tau + (L - 1)(n\tau - 2nw\delta)) \]

\[ \leq M^L n w(1 + \delta) - nL(Mw(1 - \delta))^L \]

\[ \implies M(M - 1) \cdots (M - L + 1)(L - 1) \cdot 2w\delta \]

\[ \leq M^L n w(1 + \delta) - Ln^2(Mw(1 - \delta))^L \]

\[ \implies M(M - 1) \cdots (M - L + 1)(\tau - 2w\delta) \]

\[ \leq M^L (w(1 + \delta) - (w(1 - \delta))^L) \]

\[ \implies \frac{M^L}{M(M - 1) \cdots (M - L + 1)} \geq \frac{\tau - 2w\delta}{w(1 + \delta) - (w(1 - \delta))^L}. \]

\[ \square \]

In fact, one can apply a similar trick as that used in the proof of Theorem 4 and obtain a cheaper version of Theorem 8.

**Theorem 9:** Let \( C \) be an \((L - 1)\)-list-decodable code for the Z-channel with list-decoding radius \( \tau > \tau_L \). Suppose that every codeword in \( C \) has weight between \( nw_1 \) and \( nw_2 \) for some \( 0 \leq w_1 \leq w_2 \leq 1 \). Then

\[ |C| \leq \frac{L - 1}{\left(\frac{w_2/(1 + w_2 - w_1) - w_2/(1 + w_2 - w_1)}{\tau/(1 + w_2 - w_1)}\right)^{\tau/w}}. \]

**Proof:** The proof follows by augmenting the code \( C \) to reduce it from approximate constant-weight to exact constant-weight.

Let \( C \) be as described in the theorem statement. One can append \((w_2 - w_1)n\) coordinates to each codeword in \( C \) in such a way that all codewords of length \((1 + w_2 - w_1)n\) have weight exactly \( w_2n \). Note that the relative weight of each codeword is now \( w_2/(1 + w_2 - w_1) \). Moreover, the augmented code is \((L - 1)\)-list-decodable with relative list-decoding radius \( \tau/(1 + w_2 - w_1) \).

We now recall the upper bound for \( w \)-constant-weight code given by Eqn. (8):

\[ \frac{M^L}{M(M - 1) \cdots (M - L + 1)} \geq \frac{\tau}{w - w^L} \]

\[ \implies \frac{M(L - 1)^{L - 1}\tau}{w - w^L} \leq M \leq \frac{L - 1}{1 - \left(\frac{w - w^L}{\tau}\right)^{\tau/w}}. \]

Normalizing the parameters by \( \frac{1}{1 + w_2 - w_1} \), we get the desired bound. \( \square \)

In the following statement, we estimate the size of a code \( \varepsilon \)-above the list-decoding Plotkin point if all codewords are of weight \( \varepsilon \)-close to each other.
Corollary 1: Let $C$ be an $(L-1)$-list-decodable code for the Z-channel with list-decoding radius $\tau = \tau_L + \varepsilon$. Suppose that the relative weight of all codewords is in the range $[w_1, w_2]$, where $w_2 - w_1 \leq \phi_L \varepsilon$ with $\phi_L$ being a positive real number so that $\phi_L \leq \frac{1 - \tau_L}{2\tau_L}$. Then $|C| \leq \frac{(L-1)^2}{\varepsilon^2}$ for small enough $\varepsilon$.

Proof: By Theorem 9 we get that

$$|C| \leq \frac{L - 1}{1 - \left(\frac{w_2/(1+w_2-w_1) - w_2/(1+w_2-w_1)L}{\tau/(1+w_2-w_1)}\right)^{\frac{L-1}{\varepsilon}}}. $$

After simple manipulation, it remains to show that

$$w_2 = \frac{w_2^L}{(1+\varepsilon\phi_L)L-1} \leq \left(1 - \frac{\varepsilon}{L-1}\right)^{L-1}(\tau_L + \varepsilon).$$

The LHS of the above inequality achieves its maximum at $w_2 = \frac{1+\varepsilon\phi_L}{L-1}$. Thus, after simplification, we get a stronger sufficient condition

$$(1+\varepsilon\phi_L)\tau_L \leq (1-\varepsilon)(\tau_L + \varepsilon)$$

or

$$\phi_L \leq \frac{(1-\tau_L)\varepsilon - \varepsilon^2}{\varepsilon\tau_L}.$$ 

The latter holds for sufficiently small $\varepsilon$, and the corollary follows. □

Remark 5: Readers who are familiar with the literature may have noticed that unlike results for symmetric errors [25], [29], to prove Theorem 8, we did not use Ramsey-theoretic machinery to first extract a subcode which is (approximately) "equidistant". The benefit of the Ramsey reduction is that the Chebyshev radius of the subcode turns out to be (approximately) the same as another stronger notion of radius called "average radius". The average radius is analytically much easier to deal with since unlike the Chebyshev radius, it does not involve a minimax expression. However, the drawback is that the size of the subcode is much smaller than the original one. The optimal size of codes correcting a fraction of symmetric errors above the Plotkin bound remains open.

The reason why in Theorem 8 we did not need Ramsey reduction while still managed to obtain the optimal bound on the code sizes is as follows. For asymmetric errors, the Chebyshev radius of (approximate) constant-weight codes admits an explicit expression since the Chebyshev center (i.e., the minimizer $y$ in Eqn. (5)) can be easily identified with Eqn. (6). Note, however, that in the symmetric case there is no simple formula for the Chebyshev center.

F: Upper Bound for General Codes

For $L \geq 2$ and $\varepsilon > 0$, denote the maximal size of a list-decodable code for the Z-channel with $(L-1)$-list-decoding radius $\tau_L + \varepsilon$ by $M_L(\varepsilon)$. From Corollary 1 one can immediately see that $M_L(\varepsilon) = O_L(\varepsilon^{-2})$. However, it is possible to improve this bound. In principle, we will follow the same approach as in the proof of Theorem 4. For simplicity of arguments, we assume that some numbers that are introduced in the discussion in this and next subsections are integers since this will not affect the main conclusion.

Theorem 10: For $L \geq 2$, it holds that $M_L(\varepsilon) = O_L(\varepsilon^{-3/2})$ as $\varepsilon \rightarrow 0$.

Remark 6: In the above Theorem 10 and Theorem 12 in Sec. VI-G below, we make no effort to optimize the constants implicit in $O_L(\varepsilon^{-3/2})$ and $\Omega_L(\varepsilon^{-3/2})$. Even if we did, they are likely to be suboptimal. An interesting open question is to characterize (or provide bounds on) the leading constant which depends on $L$.

Proof: Let $C$ be an $(L-1)$-list-decodable code of length $n$ with $(L-1)$-list-decoding radius $\tau(C) = \tau_L + \varepsilon$ for some $\varepsilon > 0$. Recall that $w_{\text{max}}$ is the argument achieving the maximum of the function $w - w^L$ and $\tau_L = w_{\text{max}} - w_{\text{max}}^L$. Consider all codewords whose Hamming weight is between $w_1n$ and $w_2n$ with $w_1 < w_2$.

Small Weight $w_1 = w_{\text{max}} - \Omega_L(\sqrt{\varepsilon})$: In the following, we analyze the case when $w_1 = w_{\text{max}} - \Omega_L(\sqrt{\varepsilon})$, where $w_{\text{max}}$ is the maximizer of $w - w^L$, i.e., $w_{\text{max}}$ satisfies $1 - Lw_{\text{max}}^{-1} = 0$. In our analysis, we assume that $w_1$ is fixed and $\Delta := w_2 - w_1 > 0$ is to be specified. By Theorem 9, the number of codewords of these weights is at most

$$\frac{L - 1}{1 - \left(\frac{w_2/(1+w_2-w_1) - w_2/(1+w_2-w_1)L}{\tau/(1+w_2-w_1)}\right)^{\frac{L-1}{\varepsilon}}}.$$

This quantity is at most $\frac{L-1}{\varepsilon}$ if $\Delta = \frac{w_1}{\Delta}$ satisfies

$$w_2 - w_2(w_2/(1 + \Delta))^{L-1} \leq \tau(C)(1 - \varepsilon)^{L-1}. $$

Recall that $\tau(C) = \tau_L + \varepsilon = w_{\text{max}} - w_{\text{max}}^L + \varepsilon$. For any $\gamma > 0$, there exists a sufficiently small $\varepsilon > 0$ such that the RHS of inequality (10) is lower bounded as follows

$$\tau(C)(1 - \varepsilon)^{L-1} \geq (w_{\text{max}} - w_{\text{max}}^L + \varepsilon)(1 - (L-1)\varepsilon)$$

$$\geq w_{\text{max}} - w_{\text{max}}^L + \varepsilon(1 - \gamma - (L-1)(w_{\text{max}} - w_{\text{max}}^L)).$$

Define $C_L$ to be $1 - \gamma - (L-1)(w_{\text{max}} - w_{\text{max}}^L)$. Clearly, for $2 \leq L \leq 3$ and sufficiently small $\gamma$, $C_L > 0$, whereas for $L > 3$, $C_L < 0$. Note that $(w_1 + \Delta)(1 + \Delta)^{-1} \geq w_1$. Now we elaborate the LHS of (10):

$$w_2 - w_2(w_2/(1 + \Delta))^{L-1} \leq w_2 - w_2((w_1 + \Delta)(1 + \Delta)^{-1})^{L-1} \leq w_2 - w_1^L.$$ 

Given (10)-(12), we conclude that it is sufficient to take $w_2$ such that it satisfies

$$w_2 \leq w_{\text{max}} - w_{\text{max}}^L + w_1^L + C_L\varepsilon.$$ (13)

Consider the iterative process

$$x_i := w_{\text{max}} - w_{\text{max}}^L + x_{i-1}^L + C_L\varepsilon$$

with starting point $x_1 = 0$. We shall prove by induction that $x_i \geq w_{\text{max}} - \frac{D_i}{i}$ for all $i \in [1, e^{-1/2}]$ and some
absolute constant $D_L \geq 0$. The base case $i = 1$ holds true if $D_L \geq w_{\text{max}}$. Now assume that the inductive hypothesis is true for some $i > 1$. Recall the Bernoulli inequality $(1-y)^s \geq 1 - sy$ for any real $y \leq 1$ and $s \geq 1$. Then we obtain

$$x_{i+1} \geq w_{\text{max}} - w_{\text{max}}^L + w_{\text{max}}^L \left(1 - \frac{D_L}{i w_{\text{max}}} \right)^L + C_L \varepsilon$$

$$\geq w_{\text{max}} - w_{\text{max}}^L + w_{\text{max}}^L \left(1 - \frac{2D_L}{i w_{\text{max}}} + \frac{D_L^2}{2i^2 w_{\text{max}}^2} \right)^{L/2} + C_L \varepsilon$$

$$\geq w_{\text{max}} - w_{\text{max}}^L + w_{\text{max}}^L \left(1 - \frac{L D_L}{i w_{\text{max}}} + \frac{L D_L^2}{2w_{\text{max}}^2} \right) + C_L \varepsilon$$

Note that for $D_L \geq \max(4w_{\text{max}}, |C_L|)$ and $i \leq \varepsilon^{-1/2}$, the sum of the last three terms is non-negative and the inductive hypothesis follows.

Define the subcode $C' \subseteq C$ that includes all codewords of $C$ with the Hamming weight within the range $[0,n(w_{\text{max}} - D_L \varepsilon^{L/2})]$. The above arguments imply that the size of $C'$ is $O_L(\varepsilon^{-3/2})$.

**Large Weight $w_1 = w_{\text{max}} + \Omega(\varepsilon)$:** Consider the iterative process

$$x_i := w_{\text{max}} - w_{\text{max}}^L + x_{i+1}^L + C_L \varepsilon$$

with starting point $x_1 = w_{\text{max}} + E_L \varepsilon^{1/2}$, where an absolute constant $E_L$ is to be specified. We shall prove by induction that $x_i \geq w_{\text{max}} + \varepsilon^{-1/2} E_L$ for all $i \in [1, \varepsilon^{-1/2}]$. The base case $i = 1$ holds true. Assume the hypothesis is true for some $i > 1$. Then we obtain

$$x_{i+1} \geq w_{\text{max}} - w_{\text{max}}^L + \left(\frac{E_L}{\varepsilon^{-1/2} + 1 - i} \right)^L + C_L \varepsilon$$

$$= w_{\text{max}} - w_{\text{max}}^L + w_{\text{max}}^L \left(1 + \frac{2E_L}{w_{\text{max}}(\varepsilon^{-1/2} + 1 - i)} \right)^{L/2} + C_L \varepsilon$$

$$\geq w_{\text{max}} + \varepsilon^{-1/2} + 1 - i + \frac{E_L}{2w_{\text{max}}(\varepsilon^{-1/2} + 1 - i)^2} + C_L \varepsilon$$

Observe that the sum of the last three terms is non-negative for $E_L \geq \max(8w_{\text{max}}, |C_L|)/2$.

Define the subcode $C'' \subseteq C$ that includes all codewords of $C$ with the Hamming weight within the range $[n(w_{\text{max}} + D_L \varepsilon^{1/2}), n]$. The above arguments imply that the size of $C''$ is $O_L(\varepsilon^{-3/2})$.

**Moderate Weight** $|w_1 - w_{\text{max}}| = O_L(\sqrt{\varepsilon})$: By Corollary 1 we can partition the set of codewords with weight $[n(w_{\text{max}} - D_L \varepsilon^{1/2}), n(w_{\text{max}} + E_L \varepsilon^{1/2})]$ into $O_L(\varepsilon^{-1/2})$ subcodes such that each of them has size $O_L(\varepsilon^{-1})$.

Summing up the above discussion, we conclude that the size of $C$ can be bounded as $O_L(\varepsilon^{-3/2})$ as $\varepsilon \to 0$. \(\square\)

**G. Construction of General Codes**

In this section we construct $(L - 1)$-list-decodable codes of size $O_L(\varepsilon^{-3/2})$ and length $\exp(\Theta(\varepsilon^{-3/2}))$ whose list-decoding radius is $\tau_L + \varepsilon$ as $\varepsilon \to 0$. First we prove an auxiliary technical lemma.

**Lemma 11:** Let $x_1, \ldots, x_L$ be binary vectors of length $N$ such that the Hamming weight of these vectors is $w_1 N, \ldots, w_L N$ with $0 \leq w_1 \leq \ldots \leq w_L \leq 1$. Define $\bar{x}_1, \ldots, \bar{x}_L$ to be random vectors obtained from $x_1, \ldots, x_L$ by applying independent random permutations over the set of coordinates (each of the $N!$ permutations is equally likely to appear). Let $\tilde{W}/N$ denote the number of coordinates $i \in [N]$ such that $\tilde{x}_{1,i} = \tilde{x}_{2,i} = \ldots = \tilde{x}_{L,i} = 1$, i.e., random variable $\tilde{W}$ is defined as the fraction of coordinates where all vectors $\bar{x}_1, \ldots, \bar{x}_L$ are ones. Then for any $\gamma > 0$ it holds that

$$\Pr\left\{\tilde{W} \geq \gamma + \frac{1}{L} \sum_{i=1}^{L} w_i \right\} \leq (L + 1) \exp(-N\gamma^2 2^{-2L+1}).$$

**Proof:** Fix a real number $\delta$ such that $0 < \delta < \gamma/(2^L - 1)$. Let $\bar{x}_i$ be a random binary vector such that each coordinate of $\bar{x}_i$ is an independent random variable which has Bernoulli distribution with parameter $\eta_i := \min(1, w_i + \delta)$. We note that the random vectors $\bar{x}_i$ and $\bar{x}_i$ can be equivalently defined (in terms of distributions) using the following three steps:

1. sample an independent binomial random variable $\xi_i$ with parameters $\text{Bin}(N, \eta_i)$ and set $\eta_i$ to be constant $w_i N$;
2. define $y_i$ and $z_i$ to be the binary vectors whose first $\xi_i$ and, respectively, $\eta_i$ coordinates are ones and the remaining coordinates are zeros;
3. apply an independent random permutation $\pi_i$, defined over the set of coordinates $[N]$, to $y_i$ and $z_i$ to obtain $\bar{x}_i$ and $\tilde{x}_i$.

Let $\tilde{W}/N$ denote the number of coordinates $i \in [N]$ such that $\pi_{1,i} = \ldots = \pi_{L,i} = 1$, i.e., $\tilde{W}$ is the fraction of coordinates where all vectors $\bar{x}_1, \ldots, \bar{x}_L$ are ones. Clearly, $\tilde{W}/N$ has a Binomial distribution with parameters $N$ and $\prod_{i=1}^{L} \eta_i$. Let $A_i$ denote the event that $\xi_i \geq w_i N$. Clearly, $\tilde{W}$ is stochastically dominated by the random variable $\tilde{W}$ conditioned on events $A_1, \ldots, A_L$. If $\pi_{1,i} = 1$, then $A_i$ happens with probability 1. Thus, by Hoeffding’s inequality, we obtain

$$\Pr\{A_i\} \geq 1 - \exp(-2\delta^2 N).$$

Hence,

$$\Pr\{A_1 \cap \ldots \cap A_L\} \geq 1 - L \exp(-2\delta^2 N). \quad (14)$$
Then
\[
\Pr \left\{ \tilde{W} \geq \gamma + \sum_{i=1}^{L} w_i \right\} \\
\leq \Pr \left\{ \tilde{W} \geq \gamma + \sum_{i=1}^{L} w_i \left| A_1, \ldots, A_L \right. \right\} \\
\leq \Pr \left\{ \tilde{W} \geq \gamma + \sum_{i=1}^{L} w_i \right\} \\
= 1 - \Pr \left\{ \{A_1 \cap \ldots \cap A_L\} \right\} \\
+ \Pr \left\{ \text{Bin}(N, w') \geq N \left( \gamma + \sum_{i=1}^{L} w_i \right) \right\} \tag{15},
\]
where \(w' := \sum_{i=1}^{L} w_i\). One can easily prove by induction on \(L\) that \(w' = \sum_{i=1}^{L} \min(1, w_i + \delta) \leq \sum_{i=1}^{L} w_i + (2^{L-1} - 1)\delta\). Indeed, to verify
\[
\sum_{i=1}^{L} (w_i + \delta) \leq \sum_{i=1}^{L} w_i + (2^{L-1} - 1)\delta
\]
for \(0 \leq w_1 \leq \cdots \leq w_L \leq 1\) and \(\delta \in [0,1]\), first observe that equality holds when \(L = 1\). Then assuming the inequality for \(L-1\), we have
\[
\sum_{i=1}^{L} (w_i + \delta) \leq \left( \sum_{i=1}^{L-1} w_i + (2^{L-1} - 1)\delta \right) \left( w_L + \delta \right) \\
= \sum_{i=1}^{L} w_i + \left[ (2^{L-1} - 1)w_L \right. \\
+ \left. \sum_{i=1}^{L-1} w_i + (2^{L-1} - 1)\delta \right] \delta \\
\leq \sum_{i=1}^{L} w_i + \left[ (2^{L-1} - 1) + 1 + (2^{L-1} - 1) \right] \delta,
\]
as desired. By Hoeffding’s inequality, we get
\[
\Pr \left\{ \text{Bin}(N, w') \geq N \left( \gamma + \sum_{i=1}^{L} w_i \right) \right\} \\
\leq \exp \left( -2(\gamma - (2^{L-1} - 1)\delta)^2 N \right).
\]
By combining the latter inequality with inequalities (14)-(15), we obtain
\[
\Pr \left\{ \tilde{W} \geq \gamma + \sum_{i=1}^{L} w_i \right\} \\
\leq L \exp(-2\delta^2 N) + \exp \left( -2(\gamma - (2^{L-1} - 1)\delta)^2 N \right).
\]
After choosing \(\delta\) that satisfies the equality \(\delta = \gamma - (2^{L-1} - 1)\delta\), i.e., \(\delta = \gamma/2^L\), we derive the required statement. \(\square\)

Now we are ready to present the main statement concerning the existence of list-decodable codes. In principle, we follow the same arguments as used in the case of uniquely-decodable codes. The suggested construction has order-optimal size as \(\varepsilon \to 0\) by the upper bound for general codes.

**Theorem 12:** There exists an \((L - 1)\)-list-decodable code code of length \(\exp(O(L(\varepsilon^{-3/2}))\) whose list-decoding radius is \(\tau_L + \varepsilon\). Furthermore, its size is \(\Omega(L(\varepsilon^{-3/2}))\) as \(\varepsilon \to 0\).

**Proof:** Recall that \(w_{\text{max}}\) is the argument attaining the maximum of the function \(\tau_L(w) = w - w^L\), i.e., \(w_{\text{max}}\) satisfies \(w_{\text{max}}^{k-1} = \frac{1}{j}\). Consider the positive integer \(m := \frac{1}{\varepsilon}\). Define \(J\) to be the set of consecutive integers between \(-\sqrt{(w_{\text{max}} - w_{\text{max}}^L)m/2}\) and \(\sqrt{(w_{\text{max}} - w_{\text{max}}^L)m/2}\). For every \(j \in J\), denote \(n_j := \left(\frac{m}{w_{\text{max}} - w_{\text{max}}^L} - j\right)\). Consider a binary matrix \(A_j\) of size \(m \times n_j\), whose columns are all possible binary vectors of length \(m\) and weight \(w_{\text{max}} - j\). By the proof of Theorem 7, we get that the code formed by rows of matrix \(A_j\) is a \(w_j\)-constant-weight \((L - 1)\)-list-decodable code with the list-decoding radius \(\tau_{j,L}\), where
\[
\tau_{j,L} = w_j - w_j^L + (1 - w_j)w_j^{L-1}\left(\frac{L}{2}\right)m^{-1} + O_L(m^{-2}),
\]
where
\[
w_j = \frac{m(w_{\text{max}} - j)}{m}.
\]
Clearly, we have
\[
\min\{\tau_{j,L} : j \in J\}
\]
\[
= w_{\text{max}} - w_{\text{max}}^L + \min_{j \in J} \left( \frac{jLw_j^{L-1}m - j - j^2(L/w_{\text{max}}^L)^{L-2}}{m^2} \right) + (1 - w_{\text{max}})w_j^{L-1}\left(\frac{L}{2}\right)m^{-1} + o_L(m^{-1})
\]
\[
= w_{\text{max}} - w_{\text{max}}^L + \left( 1 - w_{\text{max}} \right)\left( \frac{L - 1}{4m} \right) + o_L(m^{-1})
\]
\[
= \tau_L + O_L(\varepsilon).
\]
For a positive integer \(z\), consider a matrix \(A_j^{(z)}\) of size \(2z \times zn_j\) that is composed of \(z\) copies of the matrix \(A_j\). We write copies of \(A_j\) to the right, i.e., \(A_j^{(z)} = (A_j, A_j, \ldots, A_j)\). By \(\tilde{A}_j^{(z)}\) denote a matrix obtained from \(A_j^{(z)}\) by a random permutation of its columns. This permutation is taken uniformly at random from the set of all permutations. Note that the list-decoding radius of the code formed by \(\tilde{A}_j^{(z)}\) is the same as for the code given by \(A_j\). Define integers
\[
z_j := \prod_{i \in J \setminus \{j\}} \left( \frac{m}{w_{\text{max}} - j} \right), \quad N := \prod_{i \in J} \left( \frac{m}{w_{\text{max}} - j} \right), \quad M := m|J|.
\]
Consider a random matrix \(\tilde{A}\) of size \(M \times N\) containing \(\tilde{A}_j^{(z)}\) as a submatrix for all \(j \in J\). We assume that the matrices \(\tilde{A}_j^{(z)}\) are written one below the other using the ascending order of parameter \(j\). Let \(C\) denote a code formed by rows of \(\tilde{A}\). For \(\varepsilon \to 0\), we estimate the number of rows \(M\) and the number of columns \(N\) in \(\tilde{A}\) as follows
\[
M = \Theta_L(\varepsilon^{-3/2}), \quad N = \exp(\Theta_L(\varepsilon^{-3/2})).
\]
We claim that with high probability, \(C\) is an \((L - 1)\)-list-decodable code with the list-decoding radius \(\tau_L + O_L(\varepsilon)\). In the remainder of the proof, we prove this claim.

Let us take \(L\) distinct codewords \(\mathcal{L} = \{\tilde{x}_1, \ldots, \tilde{x}_L\}\) from \(C\) such that \(L_1\) codewords \(L_1 = \{\tilde{x}_1, \ldots, \tilde{x}_{L_1}\}\) are rows from \(\tilde{A}_j^{(z_1)}\), \(L_2\) codewords \(L_2 = \{\tilde{x}_{L_1+1}, \ldots, \tilde{x}_{L_1+L_2}\}\) are rows from \(\tilde{A}_j^{(z_2)}\), \ldots, \(L_k\) codewords \(L_k = \ldots\).
Recall that $\tau_L = w_{\text{max}} - w_L^{L_{\text{max}}}$ and $L_1 + \ldots + L_k = L$. Then we obtain

$$w_{j_1} - j_1 \varepsilon - \frac{k}{L} \leq \tau_L + \varepsilon \left( -j_1 + \sum_{i=1}^{k} \frac{w_{\text{max}}^{L_{\text{max}}} - j_i \varepsilon}{L_i} \right) + o_L(\varepsilon)$$

(a)

$$\geq \tau_L + \varepsilon \left( -j_1 + \frac{1}{L} \sum_{i=1}^{k} j_i L_i \right) + o_L(\varepsilon)$$

(b)

$$= \tau_L + \Omega_L(\varepsilon),$$

where (a) follows from the fact $L_i \geq 1$ and $w_{\text{max}}^{L_{\text{max}} - 1} = \frac{1}{L}$, (b) is implied by the fact that $L_i \geq 1$. For $j_1 < j_2 < \ldots < j_k$ and $L_1, \ldots, L_k$, we shall prove that the Chebyshev radius for $\mathcal{L}$ is at least $\tau_L + \Omega_L(\varepsilon)$.

**VII. LIST-DECODEABLE CODES BELOW THE PLOTKIN POINT**

In the following two subsections, we will bound $(L - 1)$-list-decoding capacity $C_{L-1}(w, \tau)$ for $\tau < w - w_L^L$. We note that for such a purpose, it suffices to consider constant-weight codes. This is because for any general code $\mathcal{C}$ of size $M$, one can find a constant-weight subcode $\mathcal{C}' \subseteq \mathcal{C}$ of size at least $M/(n + 1)$. The rate of $\mathcal{C}$ and $\mathcal{C}'$ is asymptotically (in $n$) the same.

Since the analyses involve application of the method of types, we need to first introduce the notion of \textit{types} of a tuple of vectors. Let $\mathcal{P}(\Sigma)$ denote the set of distributions supported on a finite set $\Sigma$.

**Definition 3 (Type):** Let $\Sigma_1, \ldots, \Sigma_k$ be finite alphabets. Let $(x_1, \ldots, x_k) \in \Sigma_1 \times \ldots \times \Sigma_k$. The joint type $T_{x_1, \ldots, x_k} \in [\Sigma_1 \times \ldots \times \Sigma_k]$ is defined as their empirical distribution (also known as histogram), i.e., for any $(x_1, \ldots, x_k) \in \Sigma_1 \times \ldots \times \Sigma_k$,

$$T_{x_1, \ldots, x_k}(x_1, \ldots, x_k) := \frac{1}{n} \left| \left\{ i \in [n] : x_{1,i} = x_1, \ldots, x_{k,i} = x_k \right\} \right|.$$

Sanov’s theorem determines the large deviation exponent of the type of an i.i.d. vector.

**Theorem 13 (Sanov [35]):** Let $Q \subseteq \mathcal{P}(\Sigma)$ be a subset of distributions on a finite set $\Sigma$ such that it is equal to the closure of its interior. Let $x \in \Sigma^n$ be distributed according to $P^\otimes n$ for some $P \in \mathcal{P}(\Sigma)$. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \Pr\{T_x \in Q\} = -\inf_{Q \subseteq \mathcal{P}(\Sigma)} D(Q || P),$$

where $D(Q || P)$ denotes the Kullback–Leibler divergence between $Q$ and $P$ defined as

$$D(Q || P) := \sum_{x \in \Sigma} Q(x) \log \frac{Q(x)}{P(x)}.$$

**A. Upper Bound on $(L - 1)$-List-Decoding Capacity**

In this section, we derive an upper bound on the size of any list-decodable code for the Z-channel. At a high level, the proof follows the idea of Elias and Bassalygo [4].
The idea is to cover the space where the code is living using Hamming balls. The radius of the balls is carefully chosen so that there can only be a constant number of codewords in the each ball satisfying the list-decodability condition. Then the total number of codewords is bounded by the covering number times a constant. We flesh out the detail below.

We first present a covering lemma that will be useful in the proof of the upper bound.

**Lemma 14:** Let \( w, v \in (0, 1) \) and \( \max(0, w + v - 1) \leq a \leq \min(w, v) \). Define
\[
I(w, v, a) := (1 - w - v + a) \log \frac{1 - w - v + a}{1 - w + v - a} + (v - a) \log \frac{v - a}{1 - w - v + a} + (w - a) \log \frac{w - a}{w - v} + a \log \frac{a}{wv},
\]
to be the mutual information of the joint distribution
\[
P_{U,X} := \left[ 1 - w - v + a \quad v - a \quad w - a \quad \frac{a}{wv} \right].
\]
Then for any \( \varepsilon > 0 \), there exists a covering \( D \subseteq S^H_n(0) \) of \( S^H_n(0) \) satisfying: for any \( x \in S^H_n(0) \), there is a \( u \in D \) such that \( T_{u,x} = P_{U,X} \). Furthermore, the size of \( D \) is at most \( 2^{nI(w, v, a) + \varepsilon} \).

**Proof:** We will show that with high probability a random subset \( D \) of \( S^H_n(0) \) is a covering of \( S^H_n(0) \). Indeed, we sample \( M := 2^{nI(w, v, a) + \varepsilon} \) vectors uniformly at random from all weight-\( n_v \) vectors and call such a set \( D \). The probability that some sequence \( x \) in \( S^H_n(0) \) is not covered by any vector in \( D \) can be bounded as follows:

\[
\Pr\{\exists x \in S^H_n(0), \forall u \in D, T_{u,x} \neq P_{U,X}\} \\
\leq \sum_{x \in S^H_n(0)} \prod_{u \in D} \left(1 - \Pr\{T_{u,x} = P_{U,X}\}\right) \\
= \sum_{x \in S^H_n(0)} \prod_{u \in D} \left(1 - \frac{\binom{n}{n_v} \binom{n}{n_w}}{\binom{n}{n_{w+v}}}ight) \\
= \binom{n}{n_{w+v}} \cdot \left(1 - 2^n(1 - w)H\left(\frac{n_v}{n_{w+v}}\right) + n_vH\left(\frac{n_w}{n_{w+v}}\right) + n_{w+v}H(v) + o(n)\right)^M \\
\leq 2^nH(w) \cdot \left(1 - 2^nH(w, v, a) + o(n)\right)^{2^nI(w, v, a) + 2^n\varepsilon} \\
\leq 2^nH(w) \cdot e^{-2^n\varepsilon} + O_L(n) \leq e^{-2^n\varepsilon} + O_L(n).
\]

In Eqn. (17), we used the fact \( \lim_{n \to \infty} (1 - 1/n)^n = 1/e. \)
Therefore, with probability at least \( 1 - e^{-2^n\varepsilon} \), \( D \) is a covering of \( S^H_n(0) \) with respect to \( P_{U,X} \). This finishes the proof of the lemma. \( \square \)

We are now ready to prove the upper bound which reads as follows.

**Theorem 15:** Fix \( 0 < w < 1 \). Let \( C \) be a \( w \)-constant-weight list-decodable code for the Z-channel of length \( n \) with list-decoding radius \( \tau \). Then for \( n \to \infty \)
\[
R(C) \leq \min_{0 \leq v \leq 1, \max(0, w + v - 1) \leq a \leq \min(w, v)} I(w, v, a) + o(1),
\]
where \( I(w, v, a) \) was defined in Lemma 14.

**Proof:** By Lemma 14, for any list-decodable code \( C \) with constant-weight \( nw \), one can find a weight-\( nw \) covering \( D \) with respect to a joint distribution \( P_{U,X} \) as specified in Lemma 14, where the parameters \( v \) and \( a \) are to be optimized later and \( D \) satisfies the properties given by the lemma.
For each \( u \in S^H_n(0) \), define the following jointly typical set \( A(u, P_{U,X}) \) with respect to the joint distribution \( P_{U,X} \):
\[
A(u, P_{U,X}) := \{ x \in S^H_n(0) : T_{u,x} = P_{U,X} \}. \quad (18)
\]
By the covering property of \( D \), we have
\[
S^H_n(0) \subseteq \bigcup_{u \in D} A(u, P_{U,X}) \\
\implies S^H_n(0) \cap C = \left( \bigcup_{u \in D} A(u, P_{U,X}) \right) \cap C \\
\implies C = \bigcup_{u \in D} (A(u, P_{U,X}) \cap C) \\
\implies |C| \leq \sum_{u \in D} |A(u, P_{U,X}) \cap C|.
\]
For each \( u \in D \), define \( C_u := A(u, P_{U,X}) \cap C \). By Markov’s inequality, there must be a \( u^* \in D \) such that \( |C_{u^*}| \geq |C|/|D| \). We further define the punctured subcodes of \( C_{u^*} \) with respect to \( u^* \). For \( u \in \{0, 1\}^n \), let \( C_{u^*}.u := \{(x_i)_{i \in [n], u^*_i = u} : x \in C \} \) be the subcode obtained by restricting codewords in \( C \) to the coordinates \( i \)’s where \( u^*_i = u \). Note that \( C_{u^*}.1 \in \{0, 1\}^{nw} \) and \( C_{u^*}.0 \in \{0, 1\}^{n(1-v)} \).

The punctured subcodes \( C_{u^*}.0 \) and \( C_{u^*}.1 \) enjoy the following property. For \( u \in \{0, 1\} \), all codewords in \( C_{u^*}.u \) have the same type \( P_{X|U=u} \). That is, every codeword in \( C_{u^*}.u \) has weight \( w - a \) and every codeword in \( C_{u^*}.1 \) has weight \( a \). Clearly, \( |C_{u^*}.0| \leq |C_{u^*}.0| \cdot |C_{u^*}.1| \).

Suppose that there are \( n\tau_0 \) errors in the locations \( i \)’s such that \( u^*_i = 0 \) and there are \( n\tau_1 \) errors in the locations \( j \)’s such that \( u^*_j = 1 \). The parameters \( \tau_0 \) and \( \tau_1 \) satisfy \( \tau_0 + \tau_1 \leq \tau \).
As long as the following two conditions are satisfied,
\[
\frac{\tau_0}{1 - v} > \frac{w - a}{1 - v} - \left(\frac{w - a}{1 - v}\right)^L, \quad \frac{\tau_1}{v} > \frac{a}{v} - \left(\frac{a}{v}\right)^L,
\]
both \( |C_{u^*}.0| \) and \( |C_{u^*}.1| \) are at most a constant (independent of \( n \)) by Theorem 6. Specifically, if
\[
\frac{\tau_0}{1 - v} > \frac{w - a}{1 - v} - \left(\frac{w - a}{1 - v}\right)^L + \varepsilon, \quad \frac{\tau_1}{v} > \frac{a}{v} - \left(\frac{a}{v}\right)^L + \varepsilon,
\]
then
\[
|C_{u^*}.0| \leq \frac{C_0}{\varepsilon} + O_L(1), \quad |C_{u^*}.1| \leq \frac{C_1}{\varepsilon} + O_L(1),
\]
where
\[ C_0 := \left[ \frac{w - a}{1 - v} - \left( \frac{w - a}{1 - v} \right)^L \right] \left( \frac{L}{2} \right), \]
\[ C_1 := \left[ \frac{a - \frac{a}{v}}{v} \right] \left( \frac{L}{2} \right). \]
Therefore
\[ |C| \leq |C_{w,v} \cdot |D| \leq |C_{w,v,0} \cdot |C_{w,v,1} \cdot |D| \leq \left( \frac{C_0}{\varepsilon} + O_L(1) \right) \left( \frac{C_1}{\varepsilon} + O_L(1) \right) \cdot 2^{n(I(w,v,a) + \varepsilon)}, \]
and \( R(C) \leq I(w,v,a) + \varepsilon + o(1) \). Taking \( \varepsilon \to 0 \) finishes the proof.

Though not used in the proof of Theorem 15, the upper bound on the size of a covering given in Lemma 14 is actually tight (on the exponential scale). Such a converse follows from a simple sphere-covering-type argument as below.

**Lemma 16:** Let \( I(w,v,a) \) and \( P_{U,X} \) be as defined in Lemma 14. Then any covering \( D \subset S_{nw}^H(0) \) of \( S_{nw}^H(0) \) satisfying the property in Lemma 14 has size at least \( 2^{nH(w,v,a) - o(n)} \).

**Proof:** For any \( u \in S_{nw}^H(0) \), we first compute the size of \( A(u, P_{U,X}) \) defined in Eqn. (18).

\[ |A(u, P_{U,X})| = \left( \frac{n(1-v)}{(n(w-a)} \right) \left( \frac{nw}{na} \right) \leq 2^{n[(1-v)H(\frac{w-a}{v}) + vH(\frac{a}{v})]} \]

Now, by the covering property of \( D \), we have
\[ S_{nw}^H(0) = \bigcup_{u \in D} A(u, P_{U,X}) \]
\[ \implies |S_{nw}^H(0)| \leq \sum_{u \in D} |A(u, P_{U,X})| \]
\[ \implies 2^{nH(w) - o(n)} \leq |D| 2^{n[(1-v)H(\frac{w-a}{v}) + vH(\frac{a}{v})]} \]
\[ \implies |D| \geq 2^{n[H(w) - (1-v)H(\frac{w-a}{v}) - vH(\frac{a}{v}) - o(n)]} \]
\[ = 2^{nI(w,v,a) - o(n)}. \]

\[ \square \]

**B. Lower Bound on \((L-1)\)-List-Decoding Capacity**

In this section, our goal is to construct an \((L-1)\)-list-decodable code \( \mathcal{C} \subset \{0,1\}^n \) for a Z-channel with noise level \( \tau \). We would like to obtain a lower bound on the rate \( R(C) \) that can be achieved by such a \( C \).

Before deriving the bound, we first introduce a set of distributions that will play an important role in the proceeding analysis. For any \( w \in [0,1] \) and \( \tau \in [0, w] \), define the following set of distributions:
\[ \mathcal{K}(w, \tau) := \left\{ P_{X_1, \ldots, X_L} \in \mathcal{P}([0,1]^L) : \right. \]
\[ \left. \forall i \in [L], P_{X_i}(1) = w \right. \]
\[ \left. w - P_{X_1, \ldots, X_L}(1, \ldots, 1) \leq \tau \right\}. \]

In words, \( \mathcal{K}(w, \tau) \) is the collection of distributions \( P_{X_1, \ldots, X_L} \) on length-\( L \) binary strings satisfying: (i) each of its marginal \( P_{X_i} \) (for \( 1 \leq i \leq L \)) is \( \text{Ber}(w) \); (ii) the probability mass \( P_{X_1, \ldots, X_L}(1, \ldots, 1) \) on the all-one string is at least \( w - \tau \).

We now describe our construction and analyze it. Our approach follows the standard random coding with expurgation technique. We sample a codebook \( \mathcal{C} \in \{0,1\}^{M \times n} \) each entry of which is i.i.d. according to \( \text{Ber}(w) \).

First, we claim that in expectation a \( 1/poly(n) \) fraction of the codewords have weight exactly \( nw \). To see this, note that for any \( x \in \mathcal{C} \),
\[ \Pr\{w(x) = nw\} = \left( \frac{n}{nw} \right)^w (1 - w)^{n(1-w)} \]
\[ = \frac{\sqrt{2\pi nw(n/w)e^n}}{\sqrt{2\pi n(w/e)(nw/e)^n}} \times (1 - O_L(n^{-1}))^{nw(1-w)^{n(1-w)}} \]
\[ = \frac{1}{\sqrt{2\pi nw(1-w)^n}} (1 - O_L(n^{-1})), \]
where the second step is by Stirling’s approximation \( n! = \sqrt{2\pi n(n/e)^n} \). Therefore,
\[ E|\{x \in \mathcal{C} : w(x) = nw\}| = M \frac{n}{\sqrt{2\pi nw(1-w)^n}} (1 - O_L(n^{-1})). \]

Second, we compute the expected number of \( bad \) \( L \)-lists, i.e., those \( L \)-lists whose list-decoding radius is at most \( \tau \). For any list \( \mathcal{L} \in \left( \mathcal{C}^L \right) \cup \{0,1\}^n \) of codewords all of weight \( nw \), it is clear that \( \text{rad} \mathcal{L} \leq \tau \) (where \( \text{rad} \mathcal{L} \) was defined in Eqn. (7)) if and only if the joint type \( T_{\mathcal{L}} \) of \( \mathcal{L} \) is in \( \mathcal{K}(w, \tau) \). By Sanov’s theorem (Theorem 13),
\[ -\frac{1}{n} \log \Pr\{\text{rad} \mathcal{L} \leq \tau\} = -\frac{1}{n} \log \Pr\{T_{\mathcal{L}} \in \mathcal{K}(w, \tau)\} \]
\[ = n \to \infty \min_{P_{X_1, \ldots, X_L} \in \mathcal{K}(w, \tau)} \mathcal{D}(P_{X_1, \ldots, X_L} \| \text{Ber}^L(w)) \]
\[ = E(w, \tau). \]

Therefore, for sufficiently large \( n \), we have
\[ E \left\{ \mathcal{L} \in \left( \mathcal{C}^L \cup \mathcal{K}(w, \tau) \right) : \text{rad} \mathcal{L} \leq \tau \right\} = \left( \frac{M}{L} \right)^{2 - nE(w, \tau) - o(1)}. \]

Finally, we expurgate all codewords whose weights are not exactly \( nw \); we also expurgate one codeword from each of the \( bad \) \( L \)-lists \( \mathcal{L} \), i.e., those \( \mathcal{L} \) such that \( \text{rad} \mathcal{L} \leq \tau \). If
\[ \left( \frac{M}{L} \right)^{2 - nE(w, \tau) - o(1)} \leq \frac{M}{2\sqrt{2\pi nw(1-w)}} \times (1 - O_L(n^{-1})) \]
\[ \leq M \frac{2\sqrt{2\pi nw(1-w)}}{2\sqrt{2\pi nw(1-w)}} \times (1 - O_L(n^{-1})) \]
\[ \leq RL - E(w, \tau) - (1 - o(1)) \leq R(1 - o(1)) \]
\[ \leq R \leq E(w, \tau) - (1 - o(1)), \]
then after expurgation, we are left with at least \( M \frac{M}{2\sqrt{2\pi nw(1-w)}} (1 - O(n^{-1})) \) codewords which form an
(L - 1)-list-decodable code \( C' \subset C \). Note that the rate \( R(C') \) is asymptotically equal to \( R(C) \).

Therefore, we have proved the following theorem.

**Theorem 17:** There exist \( w \)-constant-weight \((L - 1)\)-list-decodable codes for the Z-channel with list-decoding radius \( \tau < \tau_L(w) \) of rate at least

\[
1 - \frac{1}{L - 1} \min_{P_{X_1, \ldots, X_L} \in \mathcal{K}(w, \tau)} D\left(P_{X_1, \ldots, X_L} \middle| \text{Ber}^{\otimes L}(w)\right),
\]

where \( \mathcal{K}(w, \tau) \) was defined in Eqn. 19.

**C. List-Decoding Capacity**

Obtaining the exact \((L - 1)\)-list-decoding capacity for the Z-channel is a difficult question and we are only able to derive nonmaching upper and lower bounds in Sec. VII-A and VII-B respectively. Nevertheless, we can compute the list-decoding capacity \( C_{L-1}(w, \tau) \) when \( L \) is sufficiently large. Specifically, we determine the limit \( \lim_{L \to \infty} C_{L-1}(w, \tau) \) using the following two lemmas. The proofs below follow the outline that one uses to prove the standard list-decoding capacity theorem for symmetric errors.

Define

\[
C_{LD}(w, \tau) := -(1 - w + \tau) \log(1 - w + \tau) + \tau \log \tau - w \log w.
\]

Note that for any fixed \( w \), \( C_{LD}(w, \tau) \) is convex and decreasing in \( \tau \). It attains its maximum value \( H(w) \) at \( \tau = 0 \) and attains its minimum value 0 at \( \tau = w \). Furthermore, \( C_{LD}(w, \tau) \) is concave in \( w \) and attains its maximum value at \( w = \frac{1 + \tau}{2} \) for any fixed \( \tau \). The corresponding maximum value is \( C_{LD}(\tau) := -(1 + \tau) \log\frac{1 + \tau}{2} + \tau \log \tau \) which is in turn convex and decreasing with maximum value 1 at \( \tau = 0 \) and minimum value 0 at \( \tau = 1 \).

**Lemma 18 (Upper Bound):** For any \( \delta \in (0, 1) \), any \( w \)-constant-weight code of rate \( C_{LD}(w, \tau) + \delta \) for the Z-channel with error fraction \( \tau \) is not \((L - 1)\)-list-decodable for any \( L < 2n\delta + o(n) \).

**Proof:** Let \( C \) be any \( w \)-constant-weight code of rate \( C_{LD}(w, \tau) + \delta \). To show non-list-decodability, we need to exhibit a bad center \( y \) such that the Z-ball around \( y \) of radius \( \tau \) contains at least \( 2^{n\delta + o(n)} \), codewords from \( C \). Indeed, we will show that at random there is such a property. Specifically, let \( y \) be uniformly distributed among all vectors of weight \( n(w - p) \). We compute the expected number of codewords in \( B_{n\tau}^Z(y) \).

\[
\mathbb{E}|B_{n\tau}^Z(y) \cap C| = \sum_{x \in C} \Pr\{B_{n\tau}^Z(y) \ni x\} = \sum_{x \in C} \Pr\{\text{supp}(y) \subseteq \text{supp}(x)\}
\]

\[
= \sum_{x \in C} \left(\frac{nw}{n(w-p)}\right)^{n-w} = \sum_{x \in C} \left(\frac{n}{n(w-p)}\right)^{n-w} = 2^{n\tau} C_{LD}(w, \tau) + n\delta 2^{n\tau} H\left(\frac{w-\tau}{1-w}\right) - H(w-p) + o(n)
\]

\[
= 2^{n\delta + o(n)}.
\]

This finishes the proof.

**Lemma 19 (Construction):** For any \( \delta \in (0, 1) \) and any \( L > 1/\delta \), there exist \((L - 1)\)-list-decodable codes for the Z-channel with error fraction \( \tau \) of rate \( C_{LD}(w, \tau) - \delta \).

**Proof:** First note that if \( \tau \geq w \), the capacity is trivially zero under any constant list size since the channel can zero out all bits in any codeword and the list size has to be as large as the code size. In the proof we therefore assume \( \tau < w \).

Let \( C \subset \{0, 1\}^n \) be a collection of \( M := 2^{nw} \) (where \( R = C_{LD}(w, \tau) - \delta \) random vectors \( x_1, \ldots, x_M \) each of which is i.i.d. chosen from the set of all weight-\( nw \) binary vectors. If a codeword \( x \in C \) is input to the channel, the weight of the output vector \( y \in \{0, 1\}^n \) is in the range \([n(w - \tau), nw]\). A vector \( y \in \{0, 1\}^n \) of weight in \([n(w - \tau), nw]\) can result from \( x \in C \) through the channel if and only if \( \text{supp}(y) \subseteq \text{supp}(x) \). We compute the probability that this happens to a random codeword \( x \in C \). For any fixed \( y \in \{0, 1\}^n \) of weight \( nw \) where \( u \in [w - \tau, w] \), we have

\[
\Pr\{\text{supp}(y) \subseteq \text{supp}(x)\} = \left(\frac{n(1-u)}{nw}\right) \leq 2^{n(1-u)H\left(\frac{w-\tau}{1-w}\right) - H(w) + o(n)}.
\]

Therefore, the probability that such a random code \( C \) is not \((L - 1)\)-list-decodable is:

\[
\Pr\{C \text{ is not } (L-1)\text{-list-decodable}\} \leq \Pr\left\{ \exists i_1, \ldots, i_L \in \left\{\frac{[M]}{L}\right\} \exists y \in \{0, 1\}^n, \text{ s.t. wt}(y) \in [n(w - \tau), nw]; \forall j \in [L], \text{supp}(y) \subseteq \text{supp}(x_{i_j})\right\} 
\]

\[
\leq \frac{M}{L} 2^n \max_{w - \tau \leq u \leq w} \left(\frac{n(1-u)H\left(\frac{w-\tau}{1-w}\right) - H(w) + o(n)}{n}\right)^L 
\]

\[
\leq \frac{M}{L} 2^n \left(\frac{2^n(1-u)H\left(\frac{w-\tau}{1-w}\right) - H(w) + o(1)}{n}\right)^L 
\]

\[
\leq 2^n \left(R_{L+1} + n(L(1-w+\tau)H\left(\frac{w-\tau}{1-w+\tau}\right) - H(w) + o(1))\right).
\]

The last inequality follows since the function \( f_w(u) := (1-u) H\left(\frac{w-u}{1-w}\right) \) has the following property. For any fixed \( w \), \( f_w(u) \) is concave and decreasing in \( u \in [0, w] \), and therefore in the domain \( u \in [w - \tau, w] \) it attains its maximum at \( u = w - \tau \). By elementary algebraic manipulation, we have

\[
(1 - w + \tau) H\left(\frac{\tau}{1 - w + \tau}\right) - H(w) = -C_{LD}(w, \tau).
\]

Recall \( R = C_{LD}(w, \tau) - \delta \). Then the exponent (normalized by \( n^{-1} \)) of the RHS of the inequality (21) equals

\[
(C_{LD}(w, \tau) - \delta)L + 1 - C_{LD}(w, \tau)L + o(1)
\]

\[
= -L + 1 + o(1),
\]

which is negative if \( L > 1/\delta + o(1) \). That is, the probability that \( C \) is \((L - 1)\)-list-decodable is at least \( 1 - 2^{-\Omega(n)} \) for any \( L > 1/\delta + o(1) \).

Lemma 18 and 19 imply the following characterization.

**Theorem 20 (List-decoding capacity):** The \((L - 1)\)-list-decodability of the Z-channel with error fraction \( \tau \) under
input constraint \( w \) is given by \( \lim_{L \to \infty} \ C_{L-1}(w, \tau) = C_{\text{LD}}(w, \tau) \) as defined in Eqn. (20).

VIII. CAPACITY OF STOCHASTIC Z-CHANNELS

In all other sections of this paper, we considered Z-channels with adversarial errors. In this section, we derive the capacity of stochastic Z-channels from Shannon’s seminal channel coding theorem.

To this end, we first state Shannon’s theorem for general discrete memoryless channels (DMCs). Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two finite sets denoting the input and output alphabets of the channel. A DMC is a stochastic matrix \( W_{\mathcal{Y}|\mathcal{X}} \) that maps a distribution over \( \mathcal{X} \) to a distribution over \( \mathcal{Y} \). When the channel is used \( n \) times with an input sequence \( \mathbf{x} \in \mathcal{X}^n \), the output sequence \( \mathbf{y} \in \mathcal{Y}^n \) follows a product law:

\[
\Pr\{\mathbf{y} = (b_1, \ldots, b_n) | \mathbf{x} = (a_1, \ldots, a_n)\} = \prod_{i=1}^{n} W_{\mathcal{Y}|\mathcal{X}}(b_i|a_i),
\]

for any \((a_1, \ldots, a_n) \in \mathcal{X}^n\) and \((b_1, \ldots, b_n) \in \mathcal{Y}^n\).

For a code \( \mathcal{C} = \{ \mathbf{x}_1, \ldots, \mathbf{x}_M \} \subset \mathcal{X}^n \), its rate is defined as \( R(\mathcal{C}) := \frac{1}{n} \log M \). The average (over messages) probability of error of \( \mathcal{C} \) when used over a DMC \( W_{\mathcal{Y}|\mathcal{X}} \) with a decoder \( \text{Dec} : \mathcal{Y}^n \to [M] \) is defined as

\[
P_{e, \text{avg}}(\mathcal{C}) := \frac{1}{M} \sum_{i=1}^{M} \sum_{y \in \mathcal{Y}^n} \prod_{j=1}^{n} W_{\mathcal{Y}|\mathcal{X}}(y_j | x_{i,j}) \mathbb{1}\{\text{Dec}(y) \neq i\}.
\]

The capacity of the channel is then defined as

\[
C(W_{\mathcal{Y}|\mathcal{X}}) := \lim_{n \to \infty} \sup_{\mathcal{C} \in \mathcal{P}(\mathcal{X}^n) : P_{e, \text{avg}}(\mathcal{C}) \leq \tau} R(\mathcal{C}_n),
\]

i.e., the largest rate for which there exists a sequence of codes with vanishing error probability. The DMC is said to be equipped with input constraints \( \mathcal{Q} \subset \mathcal{P}^\infty(\mathcal{X}) \) if the type \( T_x \) of any input sequence \( x \in \mathcal{C} \) is required to belong to \( \mathcal{Q} \). The capacity of \( W_{\mathcal{Y}|\mathcal{X}} \) with input constraints \( \mathcal{Q} \) can be similarly defined.

**Theorem 21 (Channel Coding Theorem [3]):** The capacity \( C(W_{\mathcal{Y}|\mathcal{X}}) \) of a DMC \( W_{\mathcal{Y}|\mathcal{X}} \) in \( \mathcal{P}(\mathcal{Y}|\mathcal{X}) \) with input constraints \( \mathcal{Q} \) is given by

\[
C(W_{\mathcal{Y}|\mathcal{X}}) = \max_{P_X \in \mathcal{Q}} I(X;Y),
\]

where the mutual information is evaluated with respect to the joint law \( P_{X,Y} \).

A discrete memoryless Z-channel is defined as follows. Both the input and output alphabets are binary: \( \mathcal{X} = \mathcal{Y} = \{0, 1\} \).

The channel transition law is parameterized by the zeroing-out probability \( \tau \), i.e., \( W_{\mathcal{Y}|\mathcal{X}}(0|0) = 1, W_{\mathcal{Y}|\mathcal{X}}(0|1) = \tau \). The input constraint is such that all input sequences should have Hamming weight at most \( nw \) for some \( w \in [0,1] \). It is easy to evaluate Eqn. (22) which yields \( C(W_{\mathcal{Y}|\mathcal{X}}) = C(w, \tau) = H\left(w(1-\tau)\right) - wH(\tau) \).

We note that for any fixed \( w \in [0, 1] \), \( C(w, \tau) \) is convex and decreasing in \( \tau \). It attains its maximum value \( H(w) \) at \( \tau = 0 \) and attains its minimum value 0 at \( \tau = 1 \).

We also note that \( C(w, \tau) \) is concave in \( w \) for any fixed \( \tau \). The maximizing \( w \) is given by

\[
w_{\text{max}} := (1 - \tau + \tau^{-\frac{1}{\tau}})^{-1},
\]

and the corresponding capacity is given by \( C(\tau) := H(w_{\text{max}}(1-\tau)) - w_{\text{max}}H(\tau) \). Equivalent expressions have also been presented in [16]:

\[
w_{\text{max}} = \left(\left(1 + \frac{H(\tau)}{1-\tau}\right)(1-\tau)\right)^{-1},
\]

\[
C(\tau) = \log \left(1 + \tau^{-\frac{1}{\tau}} - \tau^\frac{1}{\tau}\right).
\]

Note that \( C(\tau) \) is convex and decreasing with maximum value 1 at \( \tau = 0 \) and minimum value 0 at \( \tau = 1 \).

For symmetric errors and erasures, it happens that the capacities of binary symmetric channels and binary erasure channels coincide with the respective list-decoding capacities (under adversarial errors). Comparing the expressions of \( C(w, \tau) \) and \( C_{\text{LD}}(w, \tau) \), we see that this is no longer true for Z-channels.

IX. OPEN PROBLEMS

1) For codes correcting symmetric errors, the Elias–Bassalygo bound can be improved using Delsarte’s linear program [36]. We do not know how to derive a linear programming-type bound for Z-channels. As far as we know, the linear programming framework, in its most general form, assumes that the ambient space that the code lives in (which is \([0, 1]^n \) for general codes and \( S_m^n(0) \) for \( w \)-constant-weight codes) can be defined as an association scheme. We do not see how to do so under the Z-metric \( d_Z(\cdot, \cdot) \) since the optimization of the intersection of two Z-spheres is not invariant under shifts.

2) The largest code size is exponential in \( n \) if the fraction of asymmetric errors the list-decodable code can correct is less than the Plotkin bound \( \tau_L \), and we gave bounds on the exponent in Sec. VII-A and VII-B; whereas it is \( \Theta_L(\varepsilon^{-3/2}) \) if the fraction of errors is \( \varepsilon \)-above \( \tau_L \). There is one missing case which we did not solve, that is, what is the largest code size with error fraction being exactly \( \tau_L \)? We conjecture that in this case the answer is \( \Theta_L(n^{3/2}) \). Note that the answer to the same question for symmetric errors is \( 2n \) proved by a geometric argument.

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REFERENCES

[1] R. W. Hamming, “Error detecting and error correcting codes,” Bell Syst. Tech. J., vol. 29, no. 2, pp. 147–160, Apr. 1950.

[2] C. Shannon, “The zero error capacity of a noisy channel,” IEEE Trans. Inf. Theory, vol. IT-2, no. 3, pp. 8–19, Sep. 1956.

[3] C. E. Shannon, “A mathematical theory of communication,” Bell Syst. Tech. J., vol. 27, no. 3, pp. 379–423, Jul. 1948.

[4] L. A. Bassalygo, “New upper bounds for error correcting codes,” Problemy Peredachi Informatsii, vol. 1, no. 4, pp. 41–44, 1965.
[5] J. Borden, “A low-rate bound for asymmetric error-correcting codes (corresp.),” *IEEE Trans. Inf. Theory*, vol. IT-29, no. 4, pp. 600–602, Jul. 1983.

[6] W. Kim and C. Freiman, “Single error-correcting codes for asymmetric binary channels,” *IEEE Trans. Inf. Theory*, vol. IT-5, no. 2, pp. 62–66, Jun. 1959.

[7] S. R. Varshamov, “On the theory of asymmetric codes,” in *Doklady Akademii Nauk*, vol. 164, no. 4, Moscow, Russia: Russian Academy of Sciences, 1965, pp. 757–760.

[8] T. Klove, “Upper bounds on codes correcting asymmetric errors (corresp.),” *IEEE Trans. Inf. Theory*, vol. IT-27, no. 1, pp. 128–131, Jan. 1981.

[9] T. Klove, “Error correcting codes for the asymmetric channel,” Dept. Inform., Univ. Bergen, Bergen, Norway, 1981.

[10] F.-W. Fu, S. Ling, and C. Xing, “New lower bounds and constructions for binary codes correcting asymmetric errors,” *IEEE Trans. Inf. Theory*, vol. 49, no. 12, pp. 3292–3299, Dec. 2003.

[11] B. Bose and S. Cunningham, “Asymmetric error correcting codes,” in *Sequences II*. Cham, Switzerland: Springer, 1993, pp. 24–35.

[12] J. Zhang and F.-W. Fu, “A construction of Vn, sequences and its application to binary asymmetric error-correcting codes,” *Finite Fields Appl.*, vol. 55, pp. 216–230, Jan. 2019.

[13] M. Blaum, *Codes for Detecting and Correcting Unidirectional Errors* (IEEE Computer Society Press Reprint Collections). Los Alamitos, CA, USA: IEEE Computer Society Press, 1993.

[14] L. G. Tallini and B. Bose, “On some new Zm linear codes based on elementary symmetric functions,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2018, pp. 1665–1669.

[15] S. Al-Bassam and B. Bose, “Asymmetric/unidirectional error correcting and detecting codes,” *IEEE Trans. Comput.*, vol. 43, no. 5, pp. 590–597, May 1994.

[16] L. G. Tallini, S. Al-Bassam, and B. Bose, “Feedback codes achieving the capacity of the Z-channel,” *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1357–1362, Mar. 2008.

[17] L. G. Tallini, S. Al-Bassam, and B. Bose, “Correction to ‘feedback codes achieving the capacity of the Z-channel’ [Mar 08, 1357–1362],” *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 4348–4350, Sep. 2009.

[18] C. Deppe, V. Lebedev, G. Maringer, and N. Polyanskii, “Coding with noiseless feedback over the Z-channel,” in *Proc. Int. Comput. Combinatorics Conf. (ITCS)*, Jun. 2020, pp. 98–109.

[19] A. Lebedev, V. Lebedev, and N. Polyanskii, “Two-stage coding over the Z-channel,” *IEEE Trans. Inf. Theory*, vol. 68, no. 4, pp. 2290–2299, Apr. 2022.

[20] S. D. Constantine and T. R. N. Rao, “On the theory of binary asymmetric error correcting codes,” *Inf. Control*, vol. 40, no. 1, pp. 20–36, Jan. 1979.

[21] B. Bose and S. A. Al-Bassam, “On systematic single asymmetric error-correcting codes,” *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 669–672, Mar. 2000.

[22] Y. Zhang, A. J. Budkuley, and S. Jaggi, “Generalized list decoding,” in *Proc. 11th Innov. Theor. Comput. Sci. Conf. (ITCS)*, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020, pp. 1–83.

[23] M. Plotkin, “Binary codes with specified minimum distance,” *IEEE Trans. Inf. Theory*, vol. IT-6, no. 4, pp. 445–450, Sep. 1960.

[24] V. I. Levenshtein, “Application of Hadamard matrices on coding problem,” *Problems Cybern.*, vol. 5, pp. 123–136, 1961.

[25] N. Alon, B. Bukh, and Y. Polyanskii, “List-decodable zero-rate codes,” *IEEE Trans. Inf. Theory*, vol. 65, no. 3, pp. 1657–1667, Mar. 2019.

[26] R. Graham and N. Sloane, “Lower bounds for constant weight codes,” *IEEE Trans. Inf. Theory*, vol. IT-26, no. 1, pp. 37–43, Jan. 1980.

[27] H. Lin, S. M. Moser, and P. Chen, “Weak flip codes and their optimality on the binary erasure channel,” *IEEE Trans. Inf. Theory*, vol. 64, no. 7, pp. 5191–5218, Jul. 2018.

[28] L. Qi and Z. Luo, *Tensor Spectral Theory and Special Tensors*. Philadelphia, PA, USA: SIAM, 2017.

[29] M. V. Blinovsky, “Bounds for codes in the case of list decoding of finite volume,” *Problems Inf. Transmiss.*, vol. 22, no. 1, pp. 7–19, 1986.

[30] T. P. Ramsey, “On a problem of formal logic,” *Proc. London Math. Soc.*, vols. s2–30, no. 1, pp. 264–286, 1930, doi: 10.1112/plms/s2-30.1.264.

[31] P. Erdős and R. Rado, “Combinatorial theorems on classifications of subsets of a given set,” *Proc. London Math. Soc.*, vols. s3–2, no. 1, pp. 417–439, 1952, doi: 10.1112/plms/s3-2.1.417.

[32] J. H. Weber, C. de Vrootd, and D. E. Boekee, “Necessary and sufficient conditions on block codes correcting errors of various types,” *IEEE Trans. Comput.*, vol. 41, no. 9, pp. 1189–1193, 1992, doi: 10.1109/12.165401.

[33] L. G. Tallini and B. Bose, “On a new class of error control codes and symmetric functions,” in *Proc. IEEE Int. Symp. Inf. Theory*. Toronto, ON, Canada, F. R. Kschischang and E. Yang, Eds. Jul. 2008, pp. 980–984, doi: 10.1109/ISIT.2008.4595133.

[34] A. Lebedev, V. Lebedev, and N. Polyanskii, “Two-stage coding over the Z-channel,” 2020, arXiv:2010.16362v1.

[35] I. N. Sanov, *On the Probability of Large Deviations of Random Variables*. Washington, DC, USA: United States Air Force, Office of Scientific Research, 1958.

[36] R. McEliece, E. Rodemich, H. Rumsey, and L. Welch, “New upper bounds on the rate of a code via the Delsarte–MacWilliams inequalities,” *IEEE Trans. Inf. Theory*, vol. IT-23, no. 2, pp. 157–166, Mar. 1977.

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