Fixed-time Synchronization of Networked Uncertain Euler-Lagrange Systems

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Abstract—This paper considers the fixed-time control problem of a multi-agent system composed of a class of Euler-Lagrange dynamics with parametric uncertainty and a dynamic leader under a directed communication network. A distributed fixed-time observer is first proposed to estimate the desired trajectory and then a fixed-time controller is constructed by transforming uncertain Euler-Lagrange systems into second-order systems and utilizing the backstepping design procedure. The overall design guarantees that the synchronization errors converge to zero in a prescribed time independent of initial conditions. The control design conditions can also be relaxed for a weaker finite-time control requirement.

Index Terms—Finite-time control, fixed-time control, multi-agent systems, Euler-Lagrange systems, directed graph

I. INTRODUCTION

Fixed-time control for multi-agent systems, requiring exact achievement of a collective behavior in a prescribed time independent of initial conditions, or finite-time control of a weaker requirement allowing the prescribed time dependent on initial conditions, has attracted researchers’ extensive attention over the past years due to its potential advantages in transient performance and robustness property [1]. The early work on finite-time formation control of single-integrator multi-agent systems can be found in [2]. For the leader-following consensus problem of general linear multi-agent systems, [3] proposed two classes of finite-time observers to estimate the second-order leader dynamics, which can work in undirected and directed communication networks, respectively. More efforts have also been devoted to nonlinear systems. For example, [4] considered the finite-time control of first-order multi-agent systems with unknown nonlinear dynamics, while both first-order and second-order nonlinear systems were considered in [5]. In particular, observer-based control was proposed to solve the leader-following fixed-time consensus problem under the strongly connected communication network. The fixed-time consensus problem was also investigated for double-integrator systems under directed communication network and more general multi-agent systems with high-order integrator dynamics in [6], [7], respectively.

Euler-Lagrange systems capture a large class of contemporary engineering problems and finite-time control of this class of systems has been intensively investigated, especially in the individual setting. For example, [8] considered finite-time control for an Euler-Lagrange system based on the method for a double-integrator system, while [9], [10] further dealt with nonlinear systems in the presence of uncertainties. The work in [11] studied a non-singular sliding surface and constructed a continuous finite-time control strategy for uncertain Euler-Lagrange system. Furthermore, [12] designed an adaptive controller to track a desired trajectory in finite time and [13] proposed a method for handling both uncertain dynamics and globally unbounded disturbances.

The research on fixed-time or finite-time control of uncertain Euler-Lagrange systems in a network setting is relatively rare. Some related results can be found in [14] where, by adaptive control technique, a finite-time synchronization controller was constructed for a multi-agent system modeled by some mechanical nonlinear systems with a connected communication network. The recent work reported in [15] studied finite-time coordination behavior of a multiple Euler-Lagrange system with an undirected network in the absence of uncertainties. In particular, with the introduction of auxiliary variables, the system can be converted into a simpler form such that the adding a power integrator method can be applied to ensure the convergence.

This paper provides a solution to the leader-following fixed-time synchronization problem for multiple Euler-Lagrange systems with parametric uncertainty. The strategy is based on a class of observers that can accurately estimate a dynamic trajectory in a fixed time. The design relaxes the undirected and connected assumption for the communication network in [3], [9], [13], [15] and considers a directed network graph. Then an observer-based controller is proposed for the multi-agent system composed of a dynamic leader and multiple heterogeneous Euler-Lagrange dynamics, as opposed to the finite-time control method for multiple special mechanical systems in [14]. In particular, the distributed control law is able to guarantee each Euler-Lagrange system can track a desired trajectory in a prescribed time, independent of initial conditions. It is worth mentioning that the control design conditions can be relaxed for a weaker finite-time control requirement. Also, a reduced continuous controller can be directly applied to the fixed-time synchronization problem for second-order nonlinear systems with a directed graph.

II. PROBLEM FORMULATION

Consider a group of $m$-link robotic manipulators of the following Euler-Lagrange dynamics

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i, \quad i = 1, \cdots, N,$$  \hspace{1cm} (1)

where $q_i \in \mathbb{R}^m$, $\dot{q}_i \in \mathbb{R}^m$ are the vectors of generalized position and velocity of the $i$-th robotic manipulator, also called agent $i$, $M_i(q_i) \in \mathbb{R}^{m \times m}$ is a symmetric and positive definite inertia matrix,
the existence of a settling-time function $T_0$ as the root.

The time synchronization problem is called a finite-time synchronization problem, which is based on the definition of finite-time stability \[1\]. Assumption 2.1: The graph $G$ contains a spanning tree with node 0 as the root.

Remark 2.2: Under Assumption 2.1 all the eigenvalues of $H$ have positive real parts; see, e.g., \[17\]. By Theorem 2.5.3 of \[18\], there exists a positive definite diagonal matrix $D \in \mathbb{R}^{N \times N}$ such that $H^T D + D H$ is positive definite. Let $\lambda_0 > 0$ be the smallest eigenvalue of $H^T D + D H$ and $D = \text{diag}(d_1, \cdots, d_N) = 2D/\lambda_0$. One has $H^T D + D H \geq 2N$.

We end this section with some technical lemmas from, e.g., \[6\], \[16\], \[19\] and \[20\], which will be used in the proofs of the main results in this paper.

Lemma 2.1: For any $\xi_i \in \mathbb{R}$, $i = 1, \cdots, n$, and any $p \in (0, 1]$, \( \sum_{i=1}^{n} |\xi_i|^p \leq \sum_{i=1}^{n} |\xi_i|^p \leq n^{-p}\sum_{i=1}^{n} |\xi_i|^p \) \[19\]. For any $p > 1$, \( \sum_{i=1}^{n} |\xi_i|^p \leq \sum_{i=1}^{n} |\xi_i|^p \leq n^{-p}\sum_{i=1}^{n} |\xi_i|^p \) \[2\].

Lemma 2.2: The inequality $|\xi_i^p - \xi_j^p| \leq 2^{-p}(\xi_i - \xi_j)^p$ holds for $\forall \xi_i, \xi_j \in \mathbb{R}$ and $0 < p \leq 1$ and $p$ is a ratio of two odd integers.

Lemma 2.3: \( \lim_{t \to T^*} (\cdot) = 0 \) in fixed-time stable and there is a constant settling-time $T^* \leq 1/p_0(1-p_k) + 1/q_0(q_k-1)$.

### III. Distributed Observer Design

As the agents not connected to agent 0 do not have access to the information of the dynamic leader \[2\], its state needs to be estimated by a properly designed fixed-time observer as follows:

\[
\hat{y}_k = S\hat{y}_k - c_1 y_k - c_2 \text{sign}(y_k) - c_3 \text{sign}(y),
\]

\[
y_k = \sum_{j=0}^{N} a_{ij}(\eta_i - \eta_j), \quad i = 1, \cdots, N. \tag{5}
\]

In this section, we construct a lemma based on the fixed-time observer \[5\]. Let $\eta^T = [\eta_0^T, \eta_1^T, \cdots, \eta_N^T]^T$ for the convenience of presentation.

Lemma 3.1: Consider the system composed of \[2\] and \[5\] under Assumption 2.1 with $0 < a < 1$, $b > \frac{1}{2} > 1$, $c_1 > \|D \circ S\|$ and $c_2, c_3 > 0$. There exists a constant settling-time $T_1^* \geq 0$ such that, \(\forall \eta(0) \in \mathbb{R}^{(N+1)n}\),

\[
\lim_{t \to T_1^*} (\eta_k(t) - \eta_k(0)) = 0,
\]

\[
\eta_k(t) - \eta_k(0) = 0, \quad t \geq T_1^*, \quad \forall k \in \mathbb{R}^{n_x}, \eta_0 \in \mathbb{R}^n. \tag{6}
\]

Proof: Let $\bar{\eta}_k = \eta_k - \eta_k, \quad 0, i = 0, 1, \cdots, N$. The observer \[5\] can be rewritten as

\[
\hat{\bar{y}}_k = S\bar{y}_k - c_1 y_k - c_2 \text{sign}(y_k) - c_3 \text{sign}(y),
\]

\[
y_k = \sum_{j=0}^{N} a_{ij}(\bar{y}_j - \bar{y}_i), \quad i = 1, \cdots, N. \tag{7}
\]

Let $Y_k = c_1 y_k + c_2 \text{sign}(y_k) + c_3 \text{sign}(y)$ and $\bar{y}, \bar{y}_k, Y_k$ be the column stacks of $\bar{y}_1, \cdots, N$. Note $y = (H \circ I_n)\bar{y}$ and $\bar{y} = (I_N \circ S)y - (H \circ I_n)Y$.

And, let

\[
V(y) = \sum_{i=1}^{N} \left( \frac{c_2 d_i}{1 + a} \|y_{i+1}^{l_i}a\|_1 + \frac{c_3 d_i}{1 + b} \|y_{i+1}^{l_i+b}\|_1 \right)
+ \frac{c_4}{2} y^T (D \circ I_n)y. \tag{9}
\]
Along the trajectory of $\mathbf{3}$, the time derivative of $V(y)$ satisfies
\begin{align*}
\dot{V}(y) &= \sum_{i=1}^{N} d_i (c_2 \sin^a(y_i) + c_3 \sin^b(y_i))^T \dot{y}_i + c_1 y^T (D \otimes I_n) \dot{y} \\
&= Y^T (D \otimes I_n) \dot{y} \\
&= Y^T (D \otimes S) y - \frac{1}{2} Y^T (H T D + DH) \otimes I_n) Y \\
&\leq \frac{1}{2} Y^{T} Y + \frac{1}{2} \|D \otimes S\| \|y\|^2 - Y^T Y \\
&= -\frac{1}{2} \|y\|^2 - \|D \otimes S\|^2 \|y\|^2.
\end{align*}
Further calculation shows that
\begin{align*}
\|y\|^2 &= \|y_1\|^2 + \|y_2\|^2 + \|y_3\|^2 + \|y_4\|^2 \\
&\leq c_1 \|y_1\|^2 + c_2 (\|y_2\|^2) + c_3 \|y_3\|^2 + c_4 \|y_4\|^2
\end{align*}
for $y_i(0) = 0$. By Lemma 2.1, for $0 < \alpha < 1$ and $b > 1$,
\begin{align*}
\sum_{i=1}^{N} \|y_i\|^2 &\leq \sum_{i=1}^{N} (\|y_i\|_1)^{\alpha} = \|y\|_1^\alpha, \\
\sum_{i=1}^{N} \|y_i\|^2 &\leq \frac{1}{(nN)^{\alpha-1}} \|y\|_1^\alpha.
\end{align*}
As a result,
\begin{align*}
\|y\|^2 &= \sum_{i=1}^{N} \|y_i\|^2 \\
&\geq c_1 \|y\|^2 + c_2 (\|y\|^2) + c_3 \|y\|^2 + c_4 \|y\|^2
\end{align*}
and hence
\begin{align*}
\dot{V}(y) &\leq -\frac{1}{2} (c_1 - \|D \otimes S\|^2) \|y\|^2 - c_2 (\|y\|^2) \\
&= -\frac{1}{2} \left( \|y\|^2 - \frac{c_2}{2} (\|y\|^2) \right) \\
&\leq -\hat{c}_1 \left( \|y\|^2 + (\|y\|^2)^{\alpha} + (\|y\|^2)^b \right)
\end{align*}
for some constant $\hat{c}_1 = \min \left\{ \frac{1}{2} (c_1 - \|D \otimes S\|^2), \frac{c_2}{2} \right\} > 0$.

Analysis on $\mathbf{9}$ using Lemma 2.1 and noting $0 < \frac{2a}{1+a} < 1$ and $a(1+b) > 1$ gives
\begin{align*}
\dot{V}(\mathbf{9}) &\leq \sum_{i=1}^{N} \left( \frac{c_1 d_i}{2} \right) \frac{2a}{1+a} \|y_i\|^{\frac{2a}{1+a}} \|y_i\|_1 + \frac{c_2 d_i}{1+a} \frac{2a}{1+a} \|y_i\|_1 \\
&= \frac{c_1 d_i}{1+a} \frac{2a}{1+a} \|y_i\|_1 \left( \|y_i\|^{\frac{2a}{1+a}} + \frac{c_2}{1+a} \|y_i\|_1 \right)
\end{align*}
for $d_{\max} = \max \{d_1, \ldots, d_N\}$ and
\begin{align*}
\hat{c}_2 &= \max \left\{ \frac{c_1 d_{\max}}{2} \frac{2a}{1+a} \left( nN \right)^{1-\frac{2a}{1+a}}, \frac{c_2 d_{\max}}{1+a} \frac{2a}{1+a} \left( nN \right)^{1-\frac{2a}{1+a}} \right\}.
\end{align*}
By Lemma 2.2, the system $\mathbf{3}$ is fixed-time stable. In particular, there exists a constant
\begin{align*}
\hat{T}_1(y(0)) &= \frac{1}{\hat{c}_1 (1-a)} + \frac{1}{\hat{c}_2 (1-b)} + \frac{1}{\hat{c}_3 (1-c)}.
\end{align*}
Consider the system composed of $\mathbf{3}$ and $\mathbf{13}$ under Assumption 2.1 with $0 < \alpha < 1$, $c_1 > \|D \otimes S\|$ and $c_2 > 0$. There exists a setting-time function $T_1(\eta(0)) \geq 0$ such that, $\forall \eta(0) \in \mathbb{R}^{(N+1)n}$,
\begin{align*}
\lim_{t \to T_1} (\eta_1(t) - \eta_0(t)) &= 0, \quad i = 1, \ldots, N, \\
\eta_1(t) - \eta_0(t) &= 0, \quad t \geq T_1(\eta(0)).
\end{align*}
In other words, $\dot{V} + \rho V^{\frac{2a}{1+a}}$ is negative definite for any $\rho < \frac{\hat{c}_1}{c_1}$. By Theorem 1 in $\mathbf{11}$, the system $\mathbf{3}$ is finite-time stable. In particular, there exists a finite settling-time function
\begin{align*}
\tilde{T}_1(y(0)) &= \frac{3c_2 (a+1) \|y(0)\|^{1-\frac{2a}{1+a}}}{\hat{c}_1 (1-a)},
\end{align*}
such that $\lim_{t \to \tilde{T}_1(\eta(0))} (y(t) - \eta(t)) = 0$ and $y(t) = 0$, $t > \tilde{T}_1(\eta(0))$, $\forall \eta(0) \in \mathbb{R}^{(N+1)n}$. Under Assumption 2.1, we have $\tilde{\eta}_1 = (H^{-1} \otimes I_n) y$ and hence
\begin{align*}
\tilde{T}_1(\eta(0)) &= \tilde{T}_1((H \otimes I_n) \eta(0)).
\end{align*}
IV. ROBUST CONTROLLER DESIGN

Based on the fixed-time observer (5), we further propose a distributed robust control law to solve the leader-following fixed-time synchronization problem for the multiple Euler-Lagrange systems. It is assumed the model (1) contains uncertainties and the terms $M_i(q_i)$, $C_i(q_i, v_i)$, and $G_i(q_i)$ are not completely known, but they satisfy the following bounded conditions

\[
k_{\text{max}} I_m \leq M_i(q_i) \leq k_{\text{max}} I_m,
\]
\[
\|C_i(q_i, \dot{q}_i)\| \leq k_C \|\dot{q}_i\|, \|G_i(q_i)\| \leq k_G, \forall q_i, \dot{q}_i \in \mathbb{R}^m,
\]
for some positive constants $k_{\text{max}}$, $k_C$, $k_M$, and $k_G$. Throughout the section, we consider every individual agent $i = 1, \cdots, N$.

First, the equations in (1) can be rewritten as, with $v_i = \dot{q}_i,
\[
\dot{q}_i = v_i - E_i \hat{q}_i, \dot{v}_i = v_i - E_i \hat{v}_i, \tau_i = \dot{M}_i u_i, \dot{M}_i = \frac{2I_m}{k_{\text{max}}^2 + k_{\text{max}}^2}.
\]

Also, let $u_i = u_{i1} + u_{i2}$, with $u_{i1}$ and $u_{i2}$ to be designed. As a result, the above equations become

\[
\dot{q}_i = \dot{v}_i, \dot{v}_i = M_i^{-1}(q_i)\tau_i + F_i(q_i, v_i) - E_i \dot{v}_i = u_{i1} + u_{i2} + (M_i^{-1}(q_i)\hat{M} - I_m)(u_{i1} + u_{i2}) + F_i(q_i, v_i) - E_i \dot{v}_i,
\]
where $F_i(q_i, v_i) = -M_i^{-1}(q_i)C_i(q_i, v_i) + G_i(q_i))$. Moreover, it can be put in a compact form

\[
\dot{q}_i = \dot{v}_i, \dot{v}_i = u_{i1} + E_i \hat{q}_i + Z_i
\]
with

\[
Z_i = u_{i1} + (M_i^{-1}(q_i)\hat{M} - I_m)(u_{i1} + u_{i2}) + F_i(q_i, v_i).
\]

Inspired by [13], we construct a lemma that motivates the design of $u_{i1}$.

**Lemma 4.1:** Consider the quantity $Z_i$ defined in (17) with the control law

\[
u_{i1} = \begin{cases} -\frac{k_{\text{max}}}{k_{\text{max}}+k_{\text{max}}} \|\nu_{i1}\| + f_i(v_i), & \|\nu_i\| \neq 0 \\ 0, & \|\nu_i\| = 0 \end{cases}
\]

\[
k \geq 1, \epsilon = \frac{k_{\text{max}}^{-1} - k_{\text{max}}^{-1}}{k_{\text{max}} + k_{\text{max}}}, f_i(v_i) = k_{\text{max}}^{-1}(k_{\text{max}} \|v_i\|^2 + k_g),
\]
for an arbitrary $C_i \in \mathbb{R}^m$. Then, $\dot{C}_i^T Z_i \leq 0$ holds for any $u_{i2} \in \mathbb{R}^m$.

**Proof:** From the properties of Euler-Lagrange system, we have the following facts:

\[
\|M_i^{-1}(q_i)\hat{M} - I_m\| = \|\frac{2M_i}{k_{\text{max}}^2 + k_{\text{max}}^2} - I_m\| \leq \epsilon
\]
\[
\|F_i(q_i, v_i)\| \leq k_M^{-1}(k_{\text{max}} \|v_i\|^2 + k_g) = f_i(v_i),
\]
which will be used in the calculation below.

When $\|\nu_i\| = 0$, $\dot{C}_i^T Z_i(t) \leq 0$ holds trivially. Otherwise, one has the following direct calculation

\[
\dot{C}_i^T Z_i \leq \dot{C}_i^T u_{i1} + |C_i|\|\|M_i^{-1}(q_i)\hat{M} - I_m\|\|u_{i1}\| + \|u_{i2}\|) + \|F_i(q_i, v_i)\|
\]
\[
\leq \epsilon |C_i|\|\|u_{i1}\| + \|u_{i2}\| + f_i(v_i))
\]
\[
\leq (\frac{k}{1 - \epsilon} + 1 + \frac{\epsilon k}{1 - \epsilon}) |C_i|\|\|u_{i1}\| + \|u_{i2}\| + f_i(v_i))
\]
\[
= - (\frac{k}{1 - \epsilon} - 1)|C_i|\|\|u_{i1}\| + \|u_{i2}\| + f_i(v_i)) \leq 0,
\]
which completes the proof.

**Remark 4.1:** As the system dynamics considered in this paper contain uncertainties, a robust control approach is used in the design of $u_{i1}$. In particular, to guarantee $\zeta_i^T Z_i(t) \leq 0$, which will be used later for proof of convergence, $u_{i1}$ is designed based on the boundaries of the uncertainties characterized by (15) via high gain domination. It is worth noting that the control gains in $u_{i1}$ become higher if $k_{\text{max}}$ and $k_{\text{max}}$ are larger and/or $\epsilon$ is closer to 1 (corresponding to a bigger difference between $k_{\text{max}}$ and $k_{\text{max}}$), i.e., the size of uncertainties is larger. It is a general principle in robust control that control gains depend on the size of uncertainties. In practice, when system parameters cannot be precisely measured, a smaller range of uncertainties would be beneficial for controller design.

With Lemma 4.1 ready for $u_{i1}$, the remaining task is to select a specific $\xi_i$ and design $u_{i2}$ such that the second-order system (16) is fixed-time stable, which is more complicated than finite-time control; see some existing methods in [5], [6], [21]. For solving such problem, we first introduce an explicit procedure of designing a set of parameters to be used for the controller design. It is worth noting that these parameters are independent of system dynamics. Let $\frac{1}{2} \leq \alpha < 1$ and $\beta > 1$ be two specified rational numbers of ratio of two odd integers. Define four constants

\[
p_1 = 0, p_2 = \beta - \alpha, p_3 = \frac{\beta}{\alpha} - \beta + \alpha - 1, p_4 = \frac{\beta}{\alpha} - 1
\]

and four functions, for $p \geq 0$ and $\lambda > 0$,

\[
l_1(p) = 2^{1-\alpha} \frac{p + \beta}{p + \alpha} + \frac{p + \alpha}{p + \alpha + 1}, l_2(p) = \frac{p + \beta}{p + \alpha} + 1,
\]
\[
l_3(p, \lambda) = \frac{2^{1-\alpha} (1 + \alpha)}{p + \alpha + 1} + (\lambda \gamma_1)^{\frac{p + \beta + 1}{\lambda}} = \frac{2^{1-\alpha} (1 + \alpha)}{p + \alpha + 1}, \lambda \gamma_1 \leq \frac{p + \beta + 1}{\lambda}
\]
\[
l_4(p, \lambda) = \frac{2^{1-\alpha} \gamma_2 (p + \beta + 1)}{\lambda}
\]

For the convenience of presentation, we also define

\[
\ell_1(p) = (2 - \alpha)2^{1-\alpha} l_1(p), \ell_2(p) = (2 - \alpha)2^{1-\alpha} l_2(p), \ell_3(p, \lambda) = (2 - \alpha)2^{1-\alpha} l_3(p, \lambda), \ell_4(p, \lambda) = (2 - \alpha)2^{1-\alpha} l_4(p, \lambda).
\]

Next, pick $L_1 = \max(l_2(p_2), l_1(p_3))$ and two positive parameters

\[
\gamma_1 > \max \left\{ \frac{2^{1-\alpha} p + \ell_1(p_1) + 2L_1 \frac{2^{1-\alpha} p + \ell_2(p_3) + \ell_1(p_4)}{\alpha + \beta} \right\}
\]
\[
\gamma_2 > \max \left\{ \ell_2(p_4) + 2L_1 \frac{2^{1-\alpha} p + \ell_1(p_1) + \ell_1(p_2)}{\alpha + \beta} \right\}.
\]

Now, it is ready to select

\[
\lambda_1 = \gamma_1 \ell_1, \lambda_2 = \gamma_1 \frac{2^{1-\alpha} \gamma_2 (p + \beta + 1)}{\alpha}, \lambda_3 = \gamma_1 \frac{2^{1-\alpha} \gamma_2 (p + \beta + 1)}{\alpha}, \lambda_4 = \gamma_1 \frac{2^{1-\alpha} \gamma_2 (p + \beta + 1)}{\alpha}.
\]

Then, pick

\[
L_2 = \max \left\{ \frac{2^{1-\alpha} p + \ell_4(p_3, \lambda_4) + \ell_3(p_4, \lambda_4)}{\beta} \right\}
\]
\[
\ell_4(p_1, \lambda_1) + \ell_3(p_2, \lambda_2), \ell_4(p_2, \lambda_2), \ell_4(p_3, \lambda_3) \right\}.
\]

Finally, we select the following two parameters

\[
k_1 > \frac{2^{1-\alpha} p + \ell_4(p_3, \lambda_4) + 4L_2, k_2 > \ell_4(p_4, \lambda_4) + 4L_2.
\]

With these parameters obtained, it is ready to have the following lemma.

**Lemma 4.2:** Consider the system (16) where $u_{i1}$ is given in (15) with

\[
\zeta_i = \varepsilon_i^{2-\alpha}
\]
\[ u_{21} = -k_1 e^{2\alpha - 1} - k_2 e^{\beta + \alpha - 2} + ESS\eta, \]
\[ \epsilon_i = \bar{v}_i^\alpha + (\gamma_1 q_i^\alpha + \gamma_2 q_i^\beta)^\alpha. \]

(22)

Suppose the observer governing \( \eta \) satisfies Lemma 5.1. If the control parameters \( \gamma_1, \gamma_2, k_1, k_2 \) satisfy (19) and (20), then the equilibrium of (16) at the origin is fixed-time stable. In particular, there exists a constant settling-time \( T_{2*}^i \geq 0 \) such that
\[ \lim_{t \to T_{2*}^i + T_{2*}^i} [\bar{q}_i(t), \bar{v}_i(t)] = 0, \quad t \geq T_{2*}^i + T_{2*}^i, \quad \forall \bar{q}_i(T_{2*}^i), \bar{v}_i(T_{2*}^i) \in \mathbb{R}^m. \]

(23)

**Proof:** For the convenience of proof, we define the following variables
\[ \bar{e}_i = \epsilon_i^{(2\alpha - 1)}, \quad \tilde{e}_i = \epsilon_i^{(\beta + \alpha - 2)} \]
\[ \bar{v}_i^\alpha = -\gamma_1 q_i^\alpha - \gamma_2 q_i^\beta, \quad \epsilon_i = \bar{v}_i^\alpha - \bar{v}_i^\beta \]

Let
\[ W_1(\bar{q}_i) = \frac{1}{2} \| \bar{q}_i \|^2 + \frac{1}{2} \| \bar{q}_i^\alpha \|_1. \]

Along the trajectory of \( \bar{q}_i \)-th subsystem in (16), the derivative of \( W_1(\bar{q}_i) \) satisfies
\[ \dot{W}_1(\bar{q}_i) = (\bar{q}_i + \bar{q}_i^\alpha)^T \bar{v}_i = (\bar{q}_i + \bar{q}_i^\alpha)^T (\bar{v}_i - \bar{v}_i^\alpha + \bar{v}_i^\beta) \]
\[ = (\bar{q}_i + \bar{q}_i^\alpha)^T (\bar{v}_i - \bar{v}_i^\alpha) - (\bar{q}_i + \bar{q}_i^\alpha)^T (\gamma_1 q_i^\alpha + \gamma_2 q_i^\beta). \]

By Lemma 2.3 for \( 0 < \alpha < 1 \), one has
\[ |\epsilon_i - \bar{v}_i^\beta| = |(\bar{v}_i^\alpha)^\beta| - (\bar{v}_i^\alpha)^\beta| \leq 2^{1-\alpha}|\epsilon_i^\alpha| - |\epsilon_i^\alpha| \leq 2^{1-\alpha} - |\epsilon_i^\alpha| \leq 2^{1-\alpha} - |\epsilon_i^\alpha|. \]

(24)

And, by Lemma 2.3
\[ |\bar{v}_i^\alpha - \bar{v}_i^\beta| = |(\bar{q}_i^\alpha + 1)| + \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| \leq 2^{1-\alpha} \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| \leq 2^{1-\alpha} \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| \leq 2^{1-\alpha} \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha|. \]

(25)

Using (24) and (25), one has
\[ W_{1,1}(\bar{q}_i) \leq -\left( \gamma_1 - \frac{2^{1-\alpha}}{1 + \alpha} \right)|\bar{q}_i^\alpha + 1| + \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| + \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| \leq -\left( \gamma_1 - \frac{2^{1-\alpha}}{1 + \alpha} \right)|\bar{q}_i^\alpha + 1| + \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha| + \left( \frac{\alpha}{1 + \alpha} \right)|\epsilon_i^\alpha|. \]

(26)

Next, we define a vector function
\[ d(\bar{q}_i, \bar{v}_i) = \int_{\epsilon_i^\alpha}^{\bar{v}_i^\alpha} (s - \bar{v}_i^\beta)^{1-\alpha} ds, \]

and hence
\[ W_{2,1}(\bar{q}_i, \bar{v}_i) = ||d(\bar{q}_i, \bar{v}_i)||_1. \]

Before the analysis on its derivative, we give the following calculation in order:
\[ \int_{\epsilon_i^\alpha}^{\bar{v}_i^\alpha} (s - \bar{v}_i^\beta)^{1-\alpha} ds \leq \text{diag}(|\bar{v}_i - \bar{v}_i^\beta|)|\epsilon_i|^{1-\alpha}, \]
\[ |\bar{v}_i|^{1-\alpha} \text{diag}(|\bar{v}_i - \bar{v}_i^\beta|)|\epsilon_i|^{1-\alpha} \leq 2^{1-\alpha}|\epsilon_i|^1 |\epsilon_i|. \]

(27)

Then, the derivative of \( W_{2,1}(\bar{q}_i, \bar{v}_i) \) along the trajectory of (16) satisfies, using (24),
\[ W_{2,1}(\bar{q}_i, \bar{v}_i) \\
= (2 - \alpha)q_i^\alpha \frac{\partial \bar{v}_i^\alpha}{\partial \bar{q}_i^\alpha} \int_{\epsilon_i^\alpha}^{\bar{v}_i^\alpha} (s - \bar{v}_i^\beta)^{1-\alpha} ds + (\epsilon_i^{2-\alpha}) \bar{v}_i^\beta \leq A_1 + A_2. \]

(28)

for
\[ A_1 = (2 - \alpha) \| \bar{v}_i^\alpha \|_1 \left( \frac{\partial \bar{v}_i^\alpha}{\partial \bar{q}_i^\alpha} \right) \left( \frac{\partial \bar{v}_i^\beta}{\partial \bar{q}_i^\alpha} \right) |\epsilon_i|. \]
\[ A_2 = (\epsilon_i^{2-\alpha}) \left( T_{2*}^i - E \bar{q}_i + Z_i \right). \]

By Lemma 2.1 we can obtain
\[ \frac{\partial (\bar{v}_i^\alpha)}{\partial \bar{q}_i} = \text{diag}((\gamma_1 q_i^\alpha + \gamma_2 q_i^\beta)^\alpha - 1) \text{diag}((\gamma_1 q_i^\alpha + \gamma_2 q_i^\beta)^\alpha) \]
\[ \leq \frac{\gamma_1}{\alpha} \int_{\epsilon_i^\alpha}^{\bar{v}_i^\alpha} \left( \frac{\partial \bar{v}_i^\alpha}{\partial \bar{q}_i^\alpha} \right) \left( \frac{\partial \bar{v}_i^\beta}{\partial \bar{q}_i^\alpha} \right) |\epsilon_i|. \]

(29)

that implies \( A_1 \leq (2 - \alpha) \sum_{j=1}^4 \lambda_j |\bar{v}_i|^1 \text{diag}((\bar{q}_i^\alpha)|\epsilon_i|. \]

To simplify the presentation, we introduce the following operator
\[ \langle x, y \rangle_\alpha = x |\bar{q}_i^\alpha|_1, \quad \langle x, y \rangle_\epsilon = x |\epsilon_i^\alpha|_1. \]

Then, using Lemma 2.3a and a similar argument as (24) gives
\[ \lambda |\bar{v}_i|^1 \text{diag}((\bar{q}_i^\alpha)|\epsilon_i|. \]

(30)
Under the conditions for $\gamma_1$, $\gamma_2$, $k_1$, and $k_2$, there exist $\hat{\gamma} > 0$ and $\hat{k} > 0$ satisfying

$$\gamma_1 - \frac{2^{1-\alpha}}{1 + \alpha} - \ell_1(p_1) \geq \hat{\gamma} + 2L_1, \quad \gamma_2 - \ell_2(p_2) \geq \hat{\gamma} + 2L_1$$

and

$$k_1 - \frac{2^{1-\alpha}}{\alpha + 1} - \ell_3(p_1, \lambda_1) \geq \hat{k} + 4L_2, \quad k_2 - \ell_4(p_4, \lambda_4) \geq \hat{k} + 4L_2.$$ 

Then, combining (26), (28), (29), and (30) gives, for $t \geq T_1$,

$$W_i(\tilde{q}_i, \tilde{v}_i) \leq -B_1 - B_2,$$

where

$$B_1 = (\gamma + 2L_1, \alpha + 1)q + (\hat{\gamma} + 2L_1, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)q$$

$$- (L_1, 2\beta - \alpha + 1)\eta + (L_1, \frac{2^{1-\alpha}}{1 + \alpha} - \beta + 2\alpha)\eta_1$$

$$B_2 = (\hat{k} + 4L_2, \alpha + 1)\epsilon + (\hat{k} + 4L_2, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)\epsilon$$

$$- (L_2, \frac{2^{1-\alpha}}{1 + \alpha} + \beta + \alpha)\epsilon - (L_2, \beta + 1)\epsilon$$

$$- (L_2, 2\beta - \alpha + 1)\epsilon - (L_2, \frac{2^{1-\alpha}}{1 + \alpha} - \beta + 2\alpha)\epsilon.$$ 

It is easy to verify the following inequalities

$$\frac{\beta}{\alpha} + \beta > 2\beta - \alpha + 1 > \alpha + 1$$

$$\frac{\beta}{\alpha} + \beta > \frac{\beta}{1 + \alpha} - \beta + 2\alpha > \alpha + 1$$

$$\frac{\beta}{\alpha} + \beta > \frac{\beta}{1 + \alpha} + \alpha > \alpha + 1$$

$$\frac{\beta}{\alpha} + \beta > 1 + \alpha > 1.$$ 

Therefore,

$$(2L_1, \alpha + 1)q + (2L_1, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)q$$

$$\geq (L_1, 2\beta - \alpha + 1)\eta + (L_1, \frac{2^{1-\alpha}}{1 + \alpha} - \beta + 2\alpha)\eta_1$$

that implies $B_1 \geq (\gamma, \alpha + 1)q + (\hat{\gamma}, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)q$. Similarly, one has $B_2 \geq (\hat{k}, \alpha + 1)\epsilon + (\hat{k}, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)\epsilon$. The above two inequalities conclude

$$W_i(\tilde{q}_i, \tilde{v}_i) \leq - (\gamma, \alpha + 1)q - (\gamma, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)q$$

$$- (\hat{k}, \alpha + 1)\epsilon - (\hat{k}, \frac{2^{1-\alpha}}{1 + \alpha} + \beta)\epsilon.$$ \hfill (31)

Next, since

$$\|d(\tilde{q}_i, \tilde{v}_i)\|_1 \leq \text{diag}((\|\tilde{v}_i - \tilde{v}_i\|_1^{2-\alpha})^2 \leq 2^{1-\alpha}\|\tilde{v}_i\|^2,$$

one has

$$W_i(\tilde{q}_i, \tilde{v}_i) \leq \frac{1}{2}\|\tilde{q}_i\|_1^2 + \frac{1}{\beta + 1}\|\tilde{q}_i\|^2 + 2^{1-\alpha}\|\tilde{v}_i\|^2.$$ 

Direct calculation, using Lemma 27.2 gives

$$W_i \leq (\nu_1, \alpha + 1)q + (\nu_1, \frac{\beta}{\alpha} + 1)\eta + (\nu_1, \alpha + 1)\epsilon$$

and

$$W_i \leq (\nu_2, \frac{2\beta + 1}{\alpha + 1})q + (\nu_2, \frac{2\beta + 1}{\alpha + 1})\eta + (\nu_2, \frac{2\beta + 1}{\alpha + 1})\epsilon.$$

for some constants $\nu_1, \nu_2 > 0$. Again, it is easy to verify the following inequalities

$$\frac{\beta}{\alpha} + \beta > \frac{\beta}{\alpha} + 1 + \alpha > 1.$$

$$\frac{\beta}{\alpha} + \beta > \frac{2^{1-\alpha} + \beta}{\alpha + 1} > \alpha + 1.$$ 

Therefore,

$$W_i \leq (2\nu_1, \alpha + 1)q + (\nu_1, \frac{2^{1-\alpha} + \beta}{\alpha + 1})\eta + (\frac{\beta}{\alpha} + \beta)\epsilon.$$

$$W_i \leq (2\nu_2, \alpha + 1)q + (\nu_2, \frac{2^{1-\alpha} + \beta}{\alpha + 1})\eta + (\frac{2^{1-\alpha} + \beta}{\alpha + 1})\epsilon + (\nu_2, \alpha + 1)\epsilon + (\nu_2, \frac{\beta}{\alpha} + \beta)\epsilon.$$ \hfill (32)

Comparing (31) with (32), one can conclude

$$W_i \leq -k_1W_i \frac{2^{1-\alpha}}{\alpha + 1} - k_2W_i \frac{2^{1-\alpha}}{\alpha + 1}$$

for $\rho_1 = \min \left\{ \frac{\frac{\beta}{\alpha} + 1}{2^{1-\alpha}}, \frac{\beta}{\alpha} + 1 \right\}$ and $\rho_2 = \min \left\{ \frac{\frac{\beta}{\alpha} + 1}{2^{1-\alpha}}, \frac{\beta}{\alpha} + 1 \right\}$. By Lemma 4.2, the equilibrium of (16) is fixed-time stable. In particular, there exists

$$T^*_2 \leq \frac{2}{\rho_1(1 - \alpha) + \beta + \alpha (\beta + \alpha)}$$

such that (33) holds.

Finally, based on Lemma 3.1 and Lemma 4.2, we can obtain the following theorem for the solvability of the fixed-time synchronization problem with $T^* = T^*_1 + T^*_2$.

**Theorem 4.1:** The fixed-time synchronization problem for the multi-agent system composed of (1) and (2) under Assumption 3.1 is solvable by the observer (13) and the controller $\tau_i = M(u_{1i} + u_{2i})$ of the form (18) and (22) with all the parameters given in Lemma 3.1 and Lemma 4.2.

**Remark 4.2:** Suppose the sub-controller $u_{1i}$ follows (18) but the sub-controller $u_{2i}$ reduces to the following finite-time controller, by setting $k_2 = 0$ and $\gamma_2 = 0$,

$$u_{2i} = -k_1\epsilon_i^{2-\alpha} + \text{ESS}\eta_i, \quad \epsilon_i = \frac{1}{\gamma_1}\hat{q}_i - \frac{1}{\gamma_1}\tilde{q}_i,$$ \hfill (33)

where $\eta_i$ is governed by the finite-time observer (13). If $\frac{1}{\gamma_1} < \alpha < 1$ is a ratio of two odd integers and $\gamma_1, k_1$ satisfy

$$\gamma_1 > \frac{2^{1-\alpha}}{1 + \alpha} + \frac{\alpha(2 - \alpha)2^{1-\alpha}}{1 + \alpha} + \frac{2^{1-\alpha}}{1 + \alpha} \left( \frac{2^{1-\alpha} + \gamma_1}{1 + \alpha} \right),$$

then the equilibrium of the closed-loop system composed of (16) at the origin is finite-time stable. In particular, there exists a finite settling-time function $T_2 \leq \tilde{q}_i(T_1(\eta(0)))$, $\tilde{v}_i(T_1(\eta(0))) \leq 0$ such that

$$\lim_{t \to T_1 + T_2} \tilde{q}_i(\tilde{q}_i, \tilde{v}_i(t)) = 0,$$

$$\tilde{v}_i(t), \tilde{v}_i(t) = 0, t \geq T_1 + T_2, \forall \tilde{q}_i(\tilde{q}_i, \tilde{v}_i(t)) \in \mathbb{R}^n.$$ 

As a result, the finite-time synchronization problem for the multi-agent system composed of (1) and (2) under Assumption 2.1 is solvable by the observer (18) and the controller $\tau_i = M(u_{1i} + u_{2i})$ of the form (18) and (33). The proof can similarly follow that of Lemma 4.2 and is thus omitted.
Fig. 1. Illustration of the communication network topology.

V. AN EXAMPLE

Consider a group of six robotic manipulators given by (1) where $q_i = [q_{i1}, q_{i2}]^T \in \mathbb{R}^2$ and

$$M_i(q_i) = \begin{bmatrix} \theta_{i1} + \theta_{i2} + 2\theta_{i3} \cos(q_{i2}) & \theta_{i2} + \theta_{i3} \cos(q_{i2}) \\ \theta_{i2} + \theta_{i3} \cos(q_{i2}) & \theta_{i3} \end{bmatrix}$$

$$C_i(q_i, \dot{q}_i) = \begin{bmatrix} -\theta_{i3} \sin(q_{i2}) \dot{q}_{i1}^2 - 2\theta_{i3} \sin(q_{i2}) \dot{q}_{i1} \dot{q}_{i2} \\ \theta_{i3} \sin(q_{i2}) \dot{q}_{i1}^2 \end{bmatrix}$$

$$G_i(q_i) = \begin{bmatrix} \theta_{i3} g \cos(q_{i1}) + \theta_{i4} g \cos(q_{i1} + q_{i2}) \\ \theta_{i4} g \cos(q_{i1} + q_{i2}) \end{bmatrix}$$

for $i = 1, \ldots, 6$. In the equations, $\theta_{ij}, j = 1, \ldots, 6, i = 1, \ldots, 6$, represent unknown parameters. The leader system is given by (2) with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $E = I_2$. The information flow among all the subsystems and the leader is described by the digraph in Fig. 1 which contains a spanning tree with node 0 as the root, satisfying Assumption 2.1. Let $D = 8I_6$. Then $DH + H^T D \geq 2I_N$.

By Lemma 5.1 let $c_1 = 5.4, c_2 = 1, c_3 = 1, a = 3/5, b = 3$. Now we can construct the fixed-time observer (5) whose performance is shown in Fig. 2. It is observed that the estimation errors $\eta_i - \eta_0, i = 1, \ldots, 6$, approach zero at the time instants marked by the vertical lines. In the simulation, the error tolerance of numerical calculation is set as $10^{-5}$ which is used as the criterion of approaching zero.

Next, we apply the observer (5) to solve the fixed-time control problem of Euler-Lagrange systems and design the fixed-time control law $u = M(u_{i1} + u_{i2})$ where $u_{i1}$ is given by (19) and $u_{i2}$ is given by (22). Although we do not know the exact value of $M_i(q_i), C_i(q_i, \dot{q}_i)$ and $G_i(q_i)$, it is assumed that the unknown parameters in the following ranges $\theta_{i1} \in [6, 8], \theta_{i2} \in [0.8, 1], \theta_{i3} \in [1, 1.4], \theta_{i4} \in [1.5, 2]$, and $\theta_{i5} \in [1, 1.3]$. Simple calculation verifies that the properties in (19) are satisfied for $k_{i1} = 0.3, k_{i2} = 0.08, k_{i3} = 3$, and $k_{i4} = 50$. We select the parameters in (19) and (22) as $\kappa = 3, \epsilon = 11/19, \gamma_1 = 10, \gamma_2 = 10, k_1 = 20, k_2 = 15, \alpha = 7/9, \beta = 9/7$.

For the purpose of simulation, we provide the values for uncertain parameters $\theta_{i1} = 7, \theta_{i2} = 0.96, \theta_{i3} = 1.2, \theta_{i4} = 5.96, \theta_{i5} = 2$, and $\theta_{i6} = 1.2$. The simulation is conducted with arbitrarily selected initial conditions. Fig. 3 shows $q_i, \dot{q}_i$ respectively converge to $q_0, \dot{q}_0$ in fixed time instants.

VI. CONCLUSION

This paper has proposed the fixed-time robust control design for the consensus problem of networked Euler-Lagrange systems based on a distributed observer, which is capable of estimating the desired trajectory of the leader in a fixed time under a directed graph. The heterogeneous uncertain Euler-Lagrange systems are converted into second-order systems by a partial design of the control law, and then the backstepping procedure for second-order systems are utilized to accomplish the fixed-time control design.

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