Gravitomagnetic effect in gravitational waves

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Abstract

After an introduction emphasizing the importance of the gravitomagnetic effect in general relativity, with a resume of some space-based applications, we discuss the so-called magnetic components of gravitational waves (GWs), which have to be taken into account in the context of the total response functions of interferometers for GWs propagating from arbitrary directions.

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In the weak-field and slow motion approximation, the Einstein field equations of general relativity, which establish how the mass-energy distribution determines the spacetime metric, get linearized resembling to the Maxwellian equations of electromagnetism. As a consequence, a “gravitomagnetic” field $B_g$, induced by the off-diagonal components $g_{0i}$, $i = 1, 2, 3$ of the spacetime metric tensor related to the mass-energy currents of the source of the gravitational field, arises [1]. The gravitomagnetic field affects orbiting test particles, precessing gyroscopes, moving clocks and atoms and propagating electromagnetic waves [2][3]. Perhaps, the most famous gravitomagnetic effects are the precession of the axis of a gyroscope [4][5] and the Lense-Thirring precessions of the orbit of a test particle [6], both occurring in the field of a central slowly rotating mass like, e.g., our planet. Direct, undisputable measurements of such fundamental predictions of general relativity are not yet available.

Some attempts to detect the Lense-Thirring effect have been more or less recently performed in the gravitational fields of the Sun [7], Earth [8][9][10][11].
and Mars \[11\] with natural (the inner planets of the Solar System) and artificial bodies (the terrestrial LAGEOS satellites and the martian Mars Global Surveyor probe); some of them have raised debates concerning their reliability and/or realistic level of accuracy reached \[12\] \[13\] \[14\]. The LARES satellite, recently approved by ASI with the claimed goal of measuring the Lense-Thirring effect together with the exiting LAGEOS and LAGEOS II at a ≈ 1% level, should be launched with a VEGA rocket in 2010-2011, but doubts exist that it will effectively be able to reach the expected level of accuracy \[15\].

The dedicated GP-B mission \[16\] \[17\], aimed to measure the Pugh-Schiff effect with four superconducting gyroscopes carried on board a spacecraft in a polar orbit around the Earth has not (yet?) obtained the expected accuracy (1% or better) because of the occurrence of some unexpected competing systematic effects \[18\] \[19\] \[20\].

The gravitomagnetic field plays also a fundamental role in some astrophysical scenarios like rotating black holes and neutron stars \[21\] \[22\] \[23\]. Recently, starting by the analysis in \[24\], some papers in the literature have shown the importance of the gravitomagnetic effects in the framework of the GWs detection too \[25\] \[26\] \[27\]. In fact, the so-called “magnetic” components of GWs have to be taken into account in the context of the total response functions of interferometers for GWs propagating from arbitrary directions, see \[27\] for a review. In this proceeding paper, the interferometric response functions for the magnetic components are re-analysed following the lines of \[25\]. As interferometric GWs detection is performed in a laboratory environment on Earth, the coordinate system in which the space-time is locally flat is typically used \[28\] and the distance between any two points is given simply by the difference in their coordinates in the sense of Newtonian physics. In this frame, called the frame of the local observer, GWs manifest themselves by exerting tidal forces on the masses (the mirror and the beam-splitter in the case of an interferometer). We work with \( G = 1, \ c = 1 \) and \( h = 1 \) and we call \( h_+(t_{tt} + z_{tt}) \) and \( h_\times(t_{tt} + z_{tt}) \) the weak perturbations due to the + and the \( \times \) polarizations which are expressed in terms of synchronous coordinates \( t_{tt}, x_{tt}, y_{tt}, z_{tt} \) in the transverse-traceless (TT) gauge. In this way, the most general GW propagating in the \( z_{tt} \) direction can be written in terms of plane monochromatic waves \[25\]

\[
h_{\mu\nu}(t_{tt} + z_{tt}) = h_+(t_{tt} + z_{tt})e_{\mu\nu}^{(+)} + h_\times(t_{tt} + z_{tt})e_{\mu\nu}^{(\times)} = \\
= h_+0 \exp i\omega(t_{tt} + z_{tt})e_{\mu\nu}^{(+)} + h_\times0 \exp i\omega(t_{tt} + z_{tt})e_{\mu\nu}^{(\times)},
\]

(1)

and the correspondent line element will be

\[
ds^2 = dt_{tt}^2 - dz_{tt}^2 - (1 + h_+)dx_{tt}^2 - (1 - h_+)dy_{tt}^2 - 2h_\times dx_{tt}dy_{tt}.
\]

(2)

The wordlines \( x_{tt}, y_{tt}, z_{tt} = \text{const.} \) are timelike geodesics representing the histories of free test masses \[24\] \[25\]. The coordinate transformation \( x^\alpha = x^\alpha(x^\mu_{tt}) \) from the TT coordinates to the frame of the local observer is \[25\].
\[ t = t_{tt} + \frac{1}{4}(x_{tt}^2 - y_{tt}^2)\dot{h}_+ - \frac{1}{2}x_{tt}y_{tt}\dot{h}_x \]

\[ x = x_{tt} + \frac{1}{2}y_{tt}\dot{h}_+ - \frac{1}{2}y_{tt}\dot{h}_x + \frac{1}{2}x_{tt}z_{tt}\dot{h}_+ - \frac{1}{2}x_{tt}x_{tt}\dot{h}_x \]

\[ y = y_{tt} + \frac{1}{2}y_{tt}\dot{h}_+ - \frac{1}{2}x_{tt}\dot{h}_x + \frac{1}{2}x_{tt}y_{tt}\dot{h}_+ - \frac{1}{2}x_{tt}z_{tt}\dot{h}_x \]

\[ z = z_{tt} - \frac{1}{4}(x_{tt}^2 - y_{tt}^2)\dot{h}_+ + \frac{1}{2}x_{tt}y_{tt}\dot{h}_x, \]

where it is \( \dot{h}_+ \equiv \frac{\partial h_+}{\partial t} \) and \( \dot{h}_x \equiv \frac{\partial h_x}{\partial t} \). The coefficients of this transformation (components of the metric and its first time derivative) are taken along the central wordline of the local observer \[24, 25\]. It is well known from \[24\] that the linear and quadratic terms, as powers of \( x_{tt}^n \), are unambiguously determined by the conditions of the frame of the local observer, while the cubic and higher-order corrections are not determined by these conditions. Thus, at high-frequencies, the expansion in terms of higher-order corrections breaks down \[24, 25\]. Considering a free mass riding on a timelike geodesic \((x = l_1, y = l_2, z = l_3) \) \[24, 25\], eqs. \[3\] define the motion of this mass with respect to the introduced frame of the local observer. In concrete terms one gets

\[ x(t) = l_1 + \frac{1}{2}[l_1 h_+(t) - l_2 h_x(t)] + \frac{1}{2}l_1 l_2 \dot{h}_+(t) + \frac{1}{2}l_2 l_3 \dot{h}_x(t) \]

\[ y(t) = l_2 - \frac{1}{2}[l_2 h_+(t) + l_1 h_x(t)] - \frac{1}{2}l_2 l_3 \dot{h}_+(t) + \frac{1}{2}l_1 l_3 \dot{h}_x(t) \]

\[ z(t) = l_3 - \frac{1}{4}(l_1^2 - l_2^2)\dot{h}_+(t) + 2l_1 l_2 \dot{h}_x(t). \]

In absence of GWs the position of the mass is \((l_1, l_2, l_3)\). The effect of the GW is to drive the mass to have oscillations. Thus, in general, from eqs. \[4\] all three components of motion are present \[24, 25\]. Neglecting the terms with \( \dot{h}_+ \) and \( \dot{h}_x \) in eqs. \[4\], the “traditional” equations for the mass motion are obtained \[24, 25, 26, 27, 28\]

\[ x(t) = l_1 + \frac{1}{2}[l_1 h_+(t) - l_2 h_x(t)] \]

\[ y(t) = l_2 - \frac{1}{2}[l_2 h_+(t) + l_1 h_x(t)] \]

\[ z(t) = l_3. \]

Clearly, this is the analogous of the electric component of motion in electrodynamics \[24, 25\], while equations

\[ x(t) = l_1 + \frac{1}{2}l_1 l_3 \dot{h}_+(t) + \frac{1}{2}l_2 l_3 \dot{h}_x(t) \]

\[ y(t) = l_2 - \frac{1}{2}l_2 l_3 \dot{h}_+(t) + \frac{1}{2}l_1 l_3 \dot{h}_x(t) \]

\[ z(t) = l_3 - \frac{1}{4}(l_1^2 - l_2^2)\dot{h}_+(t) + 2l_1 l_2 \dot{h}_x(t), \]
are the analogous of the magnetic component of motion. One could think
that the presence of these “magnetic” components is a “frame artefact” due
to the transformation (3), but in Section 4 of [24] eqs. (4) have been directly
obtained from the geodesic deviation equation too, thus the magnetic compo-
nents have a real physical significance. The fundamental point of [24, 25]
is that the “magnetic” components become important when the freque
ncy of the wave increases but only in the low-frequency regime. This can be understood directly
from eqs. (4). In fact, using eqs. (1) and (3), eqs. (4) become

\[
x(t) = l_1 + \frac{1}{2} l_1 l_3 \dot{\omega} h_+(t) - l_2 h_{\times}(t) + \frac{1}{2} l_1 l_3 \omega \dot{h}_+(t) - \frac{\pi}{2} + \frac{1}{2} l_1 l_3 \omega h_{\times}(t - \frac{\pi}{2})
\]

\[
y(t) = l_2 - \frac{1}{2} l_2 h_+(t) + l_1 h_{\times}(t) - \frac{1}{2} l_2 l_3 \omega h_+(t) - \frac{\pi}{2} + \frac{1}{2} l_1 l_3 \omega h_{\times}(t - \frac{\pi}{2})
\]

\[
z(t) = l_3 - \frac{1}{4} (l_1^2 - l_2^2) \omega h_+(t) - \frac{\pi}{2} + 2 l_1 l_2 \omega h_{\times}(t - \frac{\pi}{2}).
\]

Thus, the terms with \(\dot{h}_+\) and \(\dot{h}_{\times}\) in eqs. (4) can be neglected only when the
wavelength goes to infinity, while, at high-frequencies, the expansion in terms of \(\omega l_i l_j\) corrections, with \(i, j = 1, 2, 3\), breaks down [24, 25]. Now, let us compute
the total response functions of interferometers for the “magnetic” components.
Equations (4), that represent the coordinates of the mirror of the interferometer
in presence of a GW in the frame of the local observer, can be rewritten for the
pure magnetic component of the + polarization as

\[
x(t) = l_1 + \frac{1}{2} l_1 l_3 \dot{h}_+(t)
\]

\[
y(t) = l_2 - \frac{1}{2} l_2 h_+(t) + l_1 h_{\times}(t) - \frac{1}{2} l_2 l_3 \omega h_+(t) - \frac{\pi}{2} + \frac{1}{2} l_1 l_3 \omega h_{\times}(t - \frac{\pi}{2})
\]

\[
z(t) = l_3 - \frac{1}{4} (l_1^2 - l_2^2) \omega h_+(t) - \frac{\pi}{2} + 2 l_1 l_2 \omega h_{\times}(t - \frac{\pi}{2}).
\]

where \(l_1, l_2\) and \(l_3\) are the unperturbed coordinates of the mirror. To compute the response functions for an arbitrary propagating direction of the GW,
we recall that the arms of the interferometer are in general in the \(\mathbf{u}\) and \(\mathbf{v}\)
directions, while the \(x, y, z\) frame is adapted to the propagating GW (i.e. the
observer is assumed located in the position of the beam splitter) [24, 25]. Then,
a spatial rotation of the coordinate system has to be performed

\[
u = -x \cos \theta \cos \phi + y \sin \phi + z \sin \theta \cos \phi
\]

\[
v = -x \cos \theta \sin \phi - y \cos \phi + z \sin \theta \sin \phi
\]

\[
w = x \sin \theta + z \cos \theta,
\]

or, in terms of the \(x, y, z\) frame:
\[ x = -u \cos \theta \cos \phi - v \cos \theta \sin \phi + w \sin \theta \]
\[ y = u \sin \phi - v \cos \phi \]
\[ z = u \sin \theta \cos \phi + v \sin \theta \sin \phi + w \cos \theta. \]

In this way the GW is propagating from an arbitrary direction \( \mathbf{r} \) to the interferometer (see figure 2 in [25]). As the mirror of eqs. (8) is situated in the \( \mathbf{u} \) direction, using eqs. (8), (9) and (10) the \( u \) coordinate of the mirror is given by
\[ u = L + \frac{1}{4} L^2 A \dot{h}_+(t), \]
where
\[ A \equiv \sin \theta \cos \phi (\cos^2 \theta \cos^2 \phi - \sin^2 \phi) \]
and \( L = \sqrt{l_1^2 + l_2^2 + l_3^2} \) is the length of the interferometer arms.
The computation for the \( v \) arm is similar to the one above. Using eqs. (8), (9) and (10), the coordinate of the mirror in the \( v \) arm is
\[ v = L + \frac{1}{4} L^2 B \dot{h}_+(t), \]
where
\[ B \equiv \sin \theta \sin \phi (\cos^2 \theta \cos^2 \phi - \sin^2 \phi). \]

Equations (11) and (13) represent the distance of the two mirrors of the interferometer from the beam-splitter in presence of the GW (note that only the contribution of the magnetic component of the + polarization of the GW is taken into account). A “signal” can also be defined in the time domain \( T = L \) in our notation
\[ \frac{\delta T(t)}{T} \equiv \frac{u - v}{L} = \frac{1}{4} L (A - B) \dot{h}_+(t). \]

The quantity (15) can be computed in the frequency domain using the Fourier transform of \( \dot{h}_+ \), defined by
\[ \tilde{h}_+(\omega) = \int_{-\infty}^{\infty} dh_+(t) \exp(i\omega t), \]

obtaining
\[ \frac{\tilde{\delta T}(\omega)}{T} = H^+_{\text{magn}}(\omega) \tilde{h}_+(\omega), \]

where the function
\[ H_{\text{magn}}^{+}(\omega) = -\frac{i}{2} \omega L (A - B) = -\frac{i}{2} \omega L \sin \theta [(\cos^2 \theta + \sin 2\phi \frac{1+\cos^2 \theta}{2})(\cos \phi - \sin \phi)] \]

is the total response function of the interferometer for the magnetic component of the + polarization.

The analysis for the magnetic component of the \(\times\) polarization is similar. At the end one gets (see [25] for details)

\[ H_{\text{magn}}^{\times}(\omega) = -i \omega T (C - D) = -i \omega L \sin 2\phi (\cos \phi + \sin \phi) \cos \theta. \]

These response functions increase with increasing frequency. Thus, one understands why the “magnetic” components of GWs are important: because of the increasing with increasing frequency, if one neglects the “magnetic” contributions, a portion of about the 15% of the signal could be, in principle, lost in the high frequency portion of the ground based GWs interferometers.

**Conclusions**

After an introduction in which the importance of the gravitomagnetic field in general relativity has been emphasized, with a resume of some astronomical and astrophysics applications, the so-called “magnetic” components of GWs, which have to be taken into account in the context of the total response functions of interferometers for GWs propagating from arbitrary directions, have been discussed. We have shown that the magnetic contributions turn out to be important for the total interferometric response functions of both of the polarizations. In fact, if one neglects such contributions, a portion of about the 15% of the signal could be lost in the high frequency portion of interferometers.

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