Nucleon axial-vector and pseudoscalar form factors, and PCAC relations

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We use a continuum quark+diquark approach to the nucleon bound-state problem in relativistic quantum field theory to deliver parameter-free predictions for the nucleon axial and induced pseudoscalar form factors, \(G_A\) and \(G_P\), and unify them with the pseudoscalar form factor \(G_5\) or, equivalently, the pion-nucleon form factor \(G_{πNN}\). We explain how partial conservation of the axial-vector current and the associated Goldberger-Treiman relation are satisfied once all necessary couplings of the external current to the building blocks of the nucleon are constructed consistently; in particular, we fully resolve the seagull couplings to the diquark-quark vertices associated with the axial-vector and pseudoscalar currents. Among the results we describe, the following are worth highlighting. A dipole form factor defined by an axial charge \(g_A = G_A(0) = 1.25(3)\) and a mass-scale \(M_A = 1.23(3)m_N\), where \(m_N\) is the nucleon mass, can accurately describe the pointwise behaviour of \(G_A\). Concerning \(G_P\), we obtain the pseudoscalar charge \(g_P^0 = 8.80(23)\), and find that the pion pole dominance approach delivers a reliable estimate of the directly computed result. Our computed value of the pion-nucleon coupling constant, \(g_{πNN}/m_N = 14.02(33)/\text{GeV}\) is consistent with recent precision determinations.

I. INTRODUCTION

The properties of the proton and neutron (nucleons) are determined by the strong interaction; and a central aim of on-going experimental and theoretical efforts is to understand their structure as composite objects made of three valence light quarks [1]. Electron+nucleon scattering is a well developed experimental technique in such studies and it has delivered, for instance, precise measurements of nucleon electromagnetic form factors [2–8]. While they are relatively well determined, less constrained are the axial and induced-pseudoscalar form factors derived from the isovector axial-vector current that describes the neutrino-nucleon scattering process. These nucleon axial form factors are important quantities for the understanding of weak interactions, neutrino-nucleus scattering and parity violation experiments.

The axial form factor, \(G_A(Q^2)\), is experimentally determined from neutrino-proton scattering, \(νμ + p → μ^+ + n\) [9–12], while the induced pseudoscalar form factor, \(G_P(Q^2)\), can be extracted from the longitudinal cross-section in pion electroproduction [13–15] or, potentially, from single-spin asymmetries in ν(ν) charged-current quasielastic scattering on free nucleons [16]. At zero momentum transfer the axial form factor gives the axial charge \(g_A := G_A(0)\), measured with high precision from \(β\)-decay experiments [17–20]. The induced pseudoscalar coupling \(g_P^0\) can be determined via the muon capture process \(μ^- + p → n + ν_μ\) from the singlet state of the muonic hydrogen atom at the muon capture point, \(Q^2 = 0.88m_π^2\) [21–25], where \(m_π\) is the muon mass.

On the theory side, reliable extractions of the axial form factors can only be made within nonperturbative quantum chromodynamics (QCD). Prime contenders in this respect are lattice QCD (lQCD) and nonperturbative functional methods, like the Dyson-Schwinger equations (DSEs).

lQCD has made important progress in the past years. For example, several groups have computed the axial form factors [26–34], some of them using simulations generated directly at the physical pion mass and with large volumes. The problem of excited state contamination [35] has also been analysed in detail with chiral perturbation theory methods [36–38] and a variety of lattice improvements have been implemented [32, 33, 39–41]. The continuum limit and the physical pion mass are mandatory in lQCD if one wants to check important low-energy QCD relations, such as the partially conserved axial-vector current (PCAC) which, at the form factor level, connects \(G_A(Q^2)\) and \(G_P(Q^2)\) with the pseudoscalar form factor, \(G_5(Q^2)\). Another important feature is that, at low \(Q^2\), assuming pion pole dominance (PPD), \(G_A(Q^2)\) and \(G_P(Q^2)\) are directly connected, leading to the Goldberger-Treiman relation.

These observables can also be studied using the DSEs, see, e.g. Refs. [42–49]. This theoretical framework ranges...
among different levels of complexity, from symmetry-preserving QCD-kindred model studies [50–54] to systematic analyses using a truncation derived from n-particle-irreducible effective interactions [55–60].

The three-valence-body problem in relativistic quantum field theory is a challenge, only solved in Refs. [61–65]. Fortunately, the quark+diquark approximation [66, 67] has delivered comparable results for baryon properties [47, 68, 69]. This is because any interaction capable of creating pseudo-Nambu-Goldstone modes as bound-states of a light dressed-quark and -antiquark, and reproducing the measured value of their leptonic decay constants, must also generate strong colour-antitriplet correlations between any two dressed quarks contained within a hadron. This assertion is supported by evidence accumulated in two decades of studying two- and three-body bound-state problems in hadron physics, e.g., Refs. [47, 66, 67, 70–75]. No realistic counter examples are known; and the existence of such diquark correlations is also supported by simulations of IQCD [69].

In connection with the form factors of the nucleon axial current, Ref. [76] reported their consistent calculation using the Poincaré-covariant three-valence-body rainbow-ladder truncation [77, 78]. However, the study underestimates the experimental value of the axial charge by 20–25% and algorithmic inadequacies prevented computation of the axial form factors beyond $Q^2 \sim 2 \text{GeV}^2$. The quark+diquark approximation can be used, inter alia, to reach higher transferred momenta.

In the first computation within the quark+diquark picture, the interaction between the axial current and the bystander quark [79] was the only one considered. Subsequently, Refs. [80–82] included the contributions from the interaction with diquarks and also from the so-called exchange-quark diagram. However, they still neglected the “seagull terms”, which are essential in order to maintain, for instance, PCAC relations.

Herein, we construct all elements necessary for calculation of the nucleon’s axial-vector current; in particular, we determine, for the first time, the seagull contributions. We choose to approach the problem using the Poincaré-covariant quark+diquark QCD-kindred framework, which has successfully been employed in the description and unification of an array of properties of the nucleon, ∆-baryon, and their low-lying excitations [50–54, 83–85]. With its inputs tuned elsewhere via comparisons with different observables, many aspects of emergent hadron mass are implicitly expressed in the formulation. First results for the axial and induced pseudoscalar form factors are reported in Ref. [86]. This study expands significantly upon that analysis, including all mathematical details involved in proving PCAC on an analytic level, quantification of the range over which PCAC holds numerically, and a concise discussion of numerical results for the pion-nucleon form factor.

Section II is devoted to the description of the matrix elements of the nucleon axial-vector and pseudoscalar currents, and their associated form factors. In Sec. III, we briefly survey our theoretical framework, introducing the elements necessary for a practical calculation of the axial and induced pseudoscalar form factors. Section IV discusses our numerical results and associated comparisons with available experimental data and results from lQCD. Section V supplies a summary and perspective.

II. NUCLEON AXIAL AND PSEUDOSCALAR CURRENTS

The axial form factor, $G_A(Q^2)$, and the induced-pseudoscalar form factor, $G_P(Q^2)$, are defined via the nucleon matrix element

$$J_{5i,\mu}^I(K, Q) := \langle N(P_f)|\mathbf{A}_{5\mu}^I(0)|N(P_i)\rangle \quad (1a)$$

$$\bar{u}(P_f)\tau^I\gamma_\mu G_A(Q^2) + i \frac{Q_\mu}{2m_N} G_P(Q^2) u(P_i) , \quad (1b)$$

where $P_i$ and $P_f$ are, respectively, the initial and final momenta of the nucleon, defined such that the on-shell condition is fulfilled, $P_{2i}^2 = -m_N^2$, with $m_N$ the nucleon mass and $u(P)$ the associated Euclidean spinor (here we have suppressed the spin label, see Ref. [83, Appendix B] for details). Furthermore, we denote $K = (P_i + P_f)/2$ as the average momentum of the system and $Q = P_f - P_i$ the transferred momentum between initial and final states. Throughout this paper we work in the Euclidean space and use the $SU(2)_F$ isospin limit $m_u = m_d = 4m_q$. The flavour structure is given by the Pauli matrices $\{\tau^I|j = 1, 2, 3\}$, where $\tau^3$ represents the neutral current and $\tau^{1\pm i2} := (\tau^1 \pm i\tau^2)/2$ the charged currents. Moreover,

$$\mathbf{A}_{5\mu}^I(x) = \bar{\psi}(x) \frac{\tau^I}{2} \gamma_\mu \psi(x) , \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} , \quad (2)$$

is the isovector axial current operator.

Similarly, the pseudoscalar form factor $G_5(Q^2)$ is defined by the kindred pseudoscalar current

$$J_{5i}^J(K, Q) := \langle N(P_f)|\mathbf{P}_5^J(0)|N(P_i)\rangle \quad (3a)$$

$$\bar{u}(P_f)\frac{\tau^J}{2} \gamma_5 G_5(Q^2) u(P_i) , \quad (3b)$$

where

$$\mathbf{P}_5(x) = \bar{\psi}(x) \frac{\tau^J}{2} \gamma_5 \psi(x) , \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} , \quad (4)$$

is the isovector pseudoscalar current operator.

The form factors $G_A$, $G_P$ and $G_5$ can be obtained directly from these currents using any set of sensible projection operators. For example, with

$$J_{5i}^I(K, Q) := \sum_{\text{spins} (i,f)} u(P_f) J_{5i,\mu}^I(K, Q) \bar{u}(P_i) \quad (5)$$
and obvious analogues, then
\begin{align}
G_A &= -\frac{1}{4(1+\tau)}\text{tr}_D[J_{5\mu}\gamma_5\gamma_\tau^\mu], \\
G_P &= \frac{1}{\tau}(G_A - \frac{Q_\mu}{4m_N\tau}\text{tr}_D[J_{5\mu}\gamma_5]), \\
G_5 &= -\frac{1}{2\tau}\text{tr}_D[J_5\gamma_5],
\end{align}
where the trace is over spinor indices; \(\tau = Q^2/4m_N^2\); \(\gamma_\mu = \gamma_\mu - (\gamma \cdot Q)Q_\mu/Q^2\); \(J_{5\mu}\) and \(J_5\) represent the Dirac parts of the corresponding currents, viz. \(J_{5\mu}^I =: (\tau^I/2)J_{5\mu}\) and \(J_5^I =: (\tau^I/2)J_5\); and the flavour matrix \(\tau^I/2\) must be projected onto the isospinors of the proton, \(p \equiv (1,0)^T\), or neutron, \(n \equiv (0,1)^T\).

A detailed analysis of the consequences of PCAC within functional methods can be found in Ref. [76]. We briefly recapitulate here. By using standard Ward-Green-Takahashi identities one can immediately infer the PCAC relation between the current operators:
\[ \partial_\mu J_{\mu}^5(x) + 2m_\pi P_\pi^5(x) = 0. \]

Using the nucleon matrix element expression of the identity above, one then obtains the PCAC relation at the nucleon level
\[ Q_\mu J_5^\mu(K, Q) + 2im_\pi J_5^\mu(K, Q) = 0, \]
and this entails
\[ G_A(Q^2) - \frac{Q^2}{4m_N^2}G_P(Q^2) = \frac{m_\pi}{m_N}G_5(Q^2). \]

It should be pointed out that PCAC is an operator relation and thus any realistic results for \(G_A\), \(G_P\) and \(G_5\) should precisely satisfy Eq. (9).

At the pion mass pole: \(Q^2 = -m_\pi^2\), the residue of \(G_5\) is the pion-nucleon coupling constant \(g_{\pi NN}\). Thus, one defines the pion-nucleon form factor \(G_{\pi NN}(Q^2)\) via
\[ G_5(Q^2) =: \frac{m_\pi^2}{Q^2 + m_\pi^2}G_{\pi NN}(Q^2), \]
with \(G_{\pi NN}(-m_\pi^2) = g_{\pi NN}\) and \(f_\pi\) the pion's leptonic decay constant. Since \(G_P(Q^2)\) is analytic at \(Q^2 = 0\), contraction of Eq. (1b) with \(Q_\mu\) and using Eq. (10) together with the PCAC relation (9) leads to the original Goldberger-Treiman relation,
\[ G_A(0) = \frac{f_\pi}{m_N}G_{\pi NN}(0). \]
This relation is valid for all current quark masses.

### III. THEORETICAL FRAMEWORK

#### A. Diquark correlations

The bulk of QCD’s particular features and nonperturbative phenomena can be traced to the evolution of the strong running coupling. Its unique characteristics are primarily determined by the three-gluon vertex: the four-gluon vertex does not contribute dynamically at leading order in perturbative analyses of matrix elements; and nonperturbative continuum analyses of QCD’s gauge sector indicate that satisfactory agreement with gluon propagator results from QCD simulations is typically obtained without reference to dynamical contributions from the four-gluon vertex, e.g. Refs. [60, 87–96].

Analyses of the three valence-quark scattering problem reveals that the binding energy contribution of the diagram in which each leg of the three-gluon vertex is attached to one quark vanishes when projected onto the baryon’s colour-singlet wave function, i.e.
\[ \varepsilon_{f_1,f_2,f_3} \otimes \{f^{abc}\} \varepsilon_{\ell_1,\ell_2} \delta_{\ell'_1,\ell'_2} \equiv 0. \]

Here \(\varepsilon_{\ell_1,\ell_2}\) and \(\varepsilon_{f_1,f_2,f_3}\) are the Levi-Civita tensors representing the colour structure of initial and final baryon states, \(\{\lambda^\mu\}\) are \(SU(3)_c\) Gell-Mann matrices and \(f^{abc}\) is the structure tensor of \(SU(3)_c\).

Consequently, the three-gluon vertex is the primary factor in generating the class of renormalisation-group-invariant running interactions [97] that produce strong attraction between two quarks, creating tight diquark correlations in analyses of the three valence-quark scattering problem.

Further details on diquark physics can be found elsewhere [69] however, some technical details are needed here. The quark+quark correlation can be described by the 4-point Green function
\[ G(x_1, x_2; y_1, y_2) = \langle 0 | T\{ \psi(x_1)\bar{\psi}(x_2)\bar{\psi}(y_1)\bar{\psi}(y_2) \} | 0 \rangle = \langle T\{ \psi(x_1)\bar{\psi}(x_2)\bar{\psi}(y_1)\bar{\psi}(y_2) \} \rangle, \]
and its Fourier transformation, \(G(k_1, k_2, q_1, q_2)\), defined in momentum space is
\[ (2\pi)^4\delta^{(4)}(k_1 + k_2 + q_1 + q_2)G(k_1, k_2, q_1, q_2) = \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{-ik_1x_1}e^{-ik_2x_2}e^{-iq_1y_1}e^{-iq_2y_2} \times \langle T\{ \psi(x_1)\bar{\psi}(x_2)\bar{\psi}(y_1)\bar{\psi}(y_2) \} \rangle. \]

One can study the phenomenological consequences of having diquark correlations inside baryons using the following Ansatz (depicted in Fig. 1):
\[ G(k_1, k_2, q_1, q_2) = \sum_{JP=0^+, 1^+, 1^+,...} \Lambda^{JP}(k; K)\Delta^{JP}(K)\tilde{\Lambda}^{JP}(q; -K). \]
where \{\Lambda^J^P\} (\{\bar{\Lambda}^J^P\}) is the (charge conjugated) Bethe-Salpeter (BS) wave function describing the diquark correlation characterised by the quantum numbers \(J^P\). The diquark propagator is \(\Delta^J^P(K)\) and \(K\) is the total momentum of the quark+quark pair. Furthermore, \(q = (q_1 - q_2)/2\) and \(k = (k_1 - k_2)/2\) are the relative momenta between quarks in the initial and final diquark states.

It is worth remarking here that in a dynamical theory based on SU(2)-colour, diquarks are colour-singlets. They would thus exist as asymptotic states and form mass-degenerate multiplets with mesons composed from like-flavoured quarks. (These properties are a manifestation of Pauli-Gürsey symmetry [98, 99].) Consequently, the isoscalar-scalar, \([ud]\) \(J^P = 0^+\), diquark would be massless in the presence of dynamical chiral symmetry breaking (DCSB), matching the pion, and the isovector-pseudovector, \([ud]\) \(J^P = 1^+\), diquark would be degenerate with the theory’s \(\rho\)-meson. Such identities are lost in changing the gauge group to SU(3)-colour; but clear and instructive similarities between mesons and diquarks nevertheless remain, such as: (i) isoscalar-scalar and isovector-pseudovector diquark correlations are the strongest, but others could appear inside a hadron so long as their quantum numbers are allowed by Fermi-Dirac statistics; (ii) the associated diquark mass-scales express the strength and range of the correlation and are each bounded below by the partner meson’s mass; and (iii) realistic diquark correlations are soft, i.e. they possess an electromagnetic size that is bounded below by that of the analogous mesonic system.

**B. Faddeev quark+diquark amplitude**

The existence of tight diquark correlations simplifies analyses of the three valence-quark scattering problem, hence, baryon bound states, because it reduces that task to solving a Poincaré covariant matrix-valued function \(\Psi\), the Faddeev amplitude for a nucleon with total momentum \(P = p_q + p_d = k_q + k_d\). \(\Psi\) describes the relative momentum correlation between the dressed-quarks and -diquarks. Legend. *Shaded rectangle* – Faddeev kernel; *single line* – dressed-quark propagator; \(\Gamma\) – diquark correlation amplitude; and *double line* – diquark propagator. Ground-state nucleons (\(n\) - neutron, \(p\) - proton) contain both isoscalar-scalar diquarks, \([ud]\) \(\in (n,p)\), and isovector-axial-vector diquarks \([dd]\) \(\in n,\ [ud]\) \(\in (n,p)\), \((su)\) \(\in p\). Other possible correlations play no measurable role in nucleons [69].

![FIG. 2. Quark+diquark Faddeev equation, a linear integral equation for the Poincaré-covariant matrix-valued function \(\Psi\), the Faddeev amplitude for a nucleon with total momentum \(P = p_q + p_d = k_q + k_d\). \(\Psi\) describes the relative momentum correlation between the dressed-quarks and -diquarks. Legend. *Shaded rectangle* – Faddeev kernel; *single line* – dressed-quark propagator; \(\Gamma\) – diquark correlation amplitude; and *double line* – diquark propagator. Ground-state nucleons (\(n\) - neutron, \(p\) - proton) contain both isoscalar-scalar diquarks, \([ud]\) \(\in (n,p)\), and isovector-axial-vector diquarks \([dd]\) \(\in n,\ [ud]\) \(\in (n,p)\), \((su)\) \(\in p\). Other possible correlations play no measurable role in nucleons [69].*](image)

legs from their BS wave functions:

\[
\Gamma^J^P(p;P) := S^{-1}(p_+)\Lambda^J^P(p;P)(S^{-1})^T(-p_-),
\]

where \(p_\pm = p \pm P/2\) and “\(T\)” denotes matrix transpose.

In working towards realistic QCD-connected predictions, one can adapt the pattern used for mesons, viz. solve the DSE for the dressed-quark propagator and BS equations for the diquark correlation amplitudes; thereby, build the Faddeev kernel and determine baryon masses and corresponding Faddeev amplitudes. This \textit{ab initio} approach has delivered many successful results, but it is computationally intensive and limited in application by existing algorithms. An alternative [80, 102, 103] is to construct a QCD-kindred framework, in which all elements of the Faddeev kernel and interaction currents are momentum dependent, and consistent with QCD scaling laws.

A QCD-kindred framework has recently provided a large range of predictions for various baryons, including their spectrum [50, 52] and form factors [51, 53, 54, 67, 83, 104, 105]. This approach uses an efficacious algebraic parametrisation for the dressed light-quark propagator which is consistent with contemporary numerical results [50], expresses confinement and dynamical chiral symmetry breaking, retains the leading diquark amplitudes and describes diquark propagation in a manner consistent with colour confinement and asymptotic freedom.

Our QCD-kindred framework is sketched in Appendix A. Here, we just mention some key points: accounting for Fermi-Dirac statistics, five types of diquark correlations are possible in a \(J = 1/2\) bound-state: isoscalar-scalar, isovector-pseudovector, isoscalar-pseudoscalar, isoscalar-vector, and isovector-vector. However, only the first two are quantitatively important in positive-parity systems [69]; thus, the only ones considered herein. For the scalar and axial-vector
with the quarks and diquarks in various ways. In order to perform practical computations, we need to specify three more building blocks, viz. the coupling of the external current to: the quark lines; the diquark lines; and the diquark-quark wave functions, i.e. the seagull terms.

2. Axial-vector and pseudoscalar vertices

We begin with the dressed-quarks’ axial-vector vertex, \( \Gamma_{5\mu}^{j}(k_{+}, k_{-}) \), and the pseudoscalar vertex, \( \Gamma_{5}^{j}(k_{+}, k_{-}) \). They are related by the axial-vector Ward-Takahashi identity (AXWTI)

\[
Q_{\mu} \Gamma_{5\mu}^{j}(k_{+}, k_{-}) + 2i m_{q} \Gamma_{5}^{j}(k_{+}, k_{-}) = S^{-1}(k_{+}) i \gamma_{5} \frac{\tau^{j}}{2} + \frac{\tau^{j}}{2} i \gamma_{5} S^{-1}(k_{-}),
\]

(18)

where \( Q \) is the incoming momentum of the weak boson, \( k_{-} \) and \( k_{+} \) are the quark’s incoming and outgoing momenta, \( k_{\pm} = k \pm Q/2 \).

The general expressions for \( \Gamma_{5\mu}^{j}(k_{+}, k_{-}) \) and \( \Gamma_{5}^{j}(k_{+}, k_{-}) \) can be found in Ref. [106] and are given by

\[
\Gamma_{5\mu}^{j}(k_{+}, k_{-}) = \frac{\tau^{j}}{2} \gamma_{5} \left[ \gamma_{\mu} F_{R}(k, Q) + \gamma \cdot k k_{\mu} G_{R}(k, Q) - \sigma_{\mu\nu} k_{\nu} H_{R}(k, Q) + \Gamma_{5\mu}^{j}(k, Q) \right]
\]

(19)

and

\[
i \Gamma_{5}^{j}(k_{+}, k_{-}) = \frac{\tau^{j}}{2} \gamma_{5} \left[ i E_{R}^{\mu}(k, Q) + \gamma \cdot F_{P}(k, Q) + \gamma \cdot k k_{\cdot} Q G_{P}(k, Q) + \sigma_{\mu\nu} k_{\nu} Q_{\mu} H_{P}^{\nu}(k, Q) \right]
\]

(20)

\[
+ \frac{r_{P}}{Q^{2} + m_{\pi}^{2}} \Gamma_{5}^{j}(k, Q),
\]

respectively, where \( F_{R}, G_{R}, H_{R}, E_{R}^{\mu}, F_{P}, G_{P}^{\mu}, H_{P}^{\mu} \) and \( \Gamma_{5\mu}^{j} \) are regular functions when \( Q^{2} \rightarrow -m_{\pi}^{2} \), with \( Q_{\mu} \Gamma_{5\mu}^{j}(k, Q) \sim O(Q^{2}) \), and the residue \( r_{P} \) at the pion pole is given by \( r_{P} = f_{\pi} m_{\pi}^{2} / 2m_{\pi} \). The pion mass and decay constant are given by \( m_{\pi} = 0.14 \) GeV and \( f_{\pi} = 0.092 \) GeV [80].

The Bethe-Salpeter amplitude of the pion, \( \Gamma_{5}^{j} \), has the general form [106]

\[
\Gamma_{5}^{j}(k, Q) = \tau^{j} \gamma_{5} \left[ i E_{\pi}(k, Q) + \gamma \cdot Q F_{\pi}(k, Q) + \gamma \cdot k k_{\cdot} Q G_{\pi}(k, Q) + \sigma_{\mu\nu} k_{\nu} Q_{\mu} H_{\pi}(k, Q) \right].
\]

(21)

Substituting Eqs. (19) – (21) into Eq. (18), and equating both sides, one obtains

\[
\frac{\tau^{j}}{2} \gamma_{5} \left[ \gamma \cdot Q (F_{R}(k, Q) + 2m_{q} F_{P}(k, Q)) \right]
\]
The leading tensor component of the pion.

\[ F \in \{ A, B \} \]

where \( F \) are the dressing functions of the quark propagator (see Eq. (A.1b) in Appendix A).

For the BS amplitude of the pion, \( \Gamma_\pi \), we use the following Ansatz \([79, 80, 82]\)

\[ E_\pi(k, Q) = \frac{1}{2f_\pi} (B(k_+^2) + B(k_-^2)) \]

\[ \Delta_\pi(k, Q) = \frac{1}{f_\pi} \Sigma_B(k_+^2, k_-^2) \]

and

\[ F_\pi(k, Q) = G_\pi(k, Q) = H_\pi(k, Q) = 0. \]

This retains the leading tensor component of the pion. The subleading components typically provide corrections at the level of 10\% in pion-related observables. We then obtain

\[ E_R^P(k, Q) = 0, \]

\[ F_R(k, Q) + 2m_q F_R^P(k, Q) = \Sigma_A(k_+^2, k_-^2), \]

\[ G_R(k, Q) + 2m_q G_R^P(k, Q) = 2\Delta_A(k_+^2, k_-^2), \]

\[ H_R(k, Q) + 2m_q H_R^P(k, Q) = 0. \]

Considering now that \( F_R, G_R, H_R, F_R^P, G_R^P \) and \( H_R^P \) are functions which do not depend explicitly on \( m_q \), the set of equations is solved by

\[ F_R(k, Q) = \Sigma_A(k_+^2, k_-^2), \]

\[ G_R(k, Q) = 2\Delta_A(k_+^2, k_-^2), \]

\[ H_R(k, Q) = 0. \]

and

\[ E_R^P = F_R^P = G_R^P = H_R^P = 0. \]

Therefore, we finally arrive at

\[ \Gamma_{A\pi}^j(k_+, k_-) = \frac{\tau^j}{2} \gamma_5 \left[ \gamma_\mu \Sigma_A(k_+^2, k_-^2) + 2\gamma \cdot k k_\mu \Delta_A(k_+^2, k_-^2) + 2i \frac{Q_\mu}{Q^2 + m_q^2} \Sigma_B(k_+^2, k_-^2) \right], \]

as the expressions for the axial-vector and pseudoscalar vertices, which are completely determined by the dressed-quark propagator.

Note that, in the chiral limit, Eq. (29) becomes

\[ \Gamma_{A\pi}^{j, m_q=0}(k_+, k_-) = \frac{\tau^j}{2} \gamma_5 \left[ \gamma_\mu \Sigma_A(k_+^2, k_-^2) + 2\gamma \cdot k k_\mu \Delta_A(k_+^2, k_-^2) + 2i \frac{Q_\mu}{Q^2 + m_q^2} \Sigma_B(k_+^2, k_-^2) \right], \]

which is precisely the solution of the chiral-limit AXWTTI, Eq. (31) has been employed previously \([79, 80, 82]\).

It is worth emphasising here that Eqs. (29), (30) are minimal Ansätze, deliberately constructed to be consistent with all other elements in our QCD-kindred framework. Since there are no associated parameters, these Ansätze could lead to poor results for the form factors considered herein. As will become evident, they do not. On the contrary, they yield sound predictions for the entire array of studied form factors. If one were to add additional terms, then other elements of the framework would require compensating modifications. That would serve no useful purpose herein. We therefore leave such considerations to the future instant when \textit{ab initio} treatments are available for every Schwinger function needed to deliver predictions for all nucleon form factors on the entire domain of empirically accessible \( Q^2 \).

3. Seagull terms

Since diquark correlations are nonpointlike, any considered current must couple to the diquark BS amplitudes, leading to the seagull terms. Within the nucleon’s quark+diquark picture, the seagull diagrams for the electromagnetic current were first derived in Ref. \([102]\): they are essential to guarantee current conservation in any realistic study. However, the contribution coming from the seagull diagrams when considering the axial and pseudoscalar currents has eluded understanding for more than twenty years, partly because there is no trivial relation between the vertices and propagators of the diquarks for these currents.

We derive the axial-vector and pseudoscalar seagull terms here by using a strategy similar to the one followed in Ref. \([102]\). First, the equal-time commutators of
the axial current operator can be calculated straightforwardly from Eq. (2):

\[
\mathcal{A}_{\mu=4}^j(x, \psi(y))_{x=x_1} = \frac{\tau^j}{2} \gamma_5 \psi(x) \delta^{(4)}(x-y), \quad (33a)
\]

\[
\mathcal{A}_{\mu=4}^j(x, \bar{\psi}(y))_{x=x_1} = \bar{\psi}(x) \gamma_5 \frac{\tau^j}{2} \delta^{(4)}(x-y). \quad (33b)
\]

Second, we define the five-point Schwinger function \( G_{5\mu}^j \) as the coupling of the quark-+quark scattering kernel with the axial current,

\[
(2\pi)^4 \delta^{(4)}(k_1 + k_2 + q_1 + q_2 - Q) G_{5\mu}^j(k, K'; q, K)
:= \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 d^4 z e^{-ik_1 x_1} e^{-ik_2 x_2} e^{-i q_1 y_1} e^{-i q_2 y_2}
\times e^{i Q z} \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \mathcal{A}_{\mu}^j(z) \} \}, \quad (34)
\]

and, correspondingly, \( G_{5}^j \) is the coupling of the same quark+quark scattering kernel with the pseudoscalar current,

\[
(2\pi)^4 \delta^{(4)}(k_1 + k_2 + q_1 + q_2 - Q) G_{5}^j(k, K'; q, K)
:= \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 d^4 z e^{-ik_1 x_1} e^{-ik_2 x_2} e^{-i q_1 y_1} e^{-i q_2 y_2}
\times e^{i Q z} \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \mathcal{P}_\mu^j(z) \} \}, \quad (35)
\]

where \( k = (k_1 - k_2)/2, q = (q_1 - q_2)/2, k_1 + k_2 = K', -q_1 - q_2 = K' = K + Q \).

Now, taking the partial derivative of the five-point correlation function \( \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \mathcal{A}_{\mu}^j(z) \} \rangle \), and using the PCAC relation, Eq. (7), we get

\[
\frac{\partial}{\partial z_\mu} \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \mathcal{A}_{\mu}^j(z) \} \rangle
+ 2m_q \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \mathcal{P}_\mu^j(z) \} \rangle
= \delta^{(4)}(z - x_1) \langle T \{ \frac{\tau^j}{2} \gamma_5 \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \} \rangle
+ \delta^{(4)}(z - x_2) \langle T \{ \psi(x_1) \left( \frac{\tau^j}{2} \gamma_5 \psi(x_2) \right) \bar{\psi}(y_1) \bar{\psi}(y_2) \} \rangle
+ \delta^{(4)}(z - y_1) \langle T \{ \psi(x_1) \bar{\psi}(x_2) \left( \bar{\psi}(y_1) \gamma_5 \frac{\tau^j}{2} \right) \bar{\psi}(y_2) \} \rangle
+ \delta^{(4)}(z - y_2) \langle T \{ \psi(x_1) \bar{\psi}(x_2) \bar{\psi}(y_1) \left( \bar{\psi}(y_2) \gamma_5 \frac{\tau^j}{2} \right) \} \rangle .
\quad (36)
\]

Performing a Fourier transform of both sides and introducing a compact notation for the sum over diquark contributions in the resulting Fourier amplitudes, where \( j \) expresses the evident isospin structure:

\[
G_{5\mu}^j(k, K'; q, K) = \sum_{J^P} G_{5\mu}^{j, J^P}(k, K'; q, K), \quad (37a)
\]

\[
G_{5}^j(k, K'; q, K) = \sum_{J^P} G_{5}^{j, J^P}(k, K'; q, K), \quad (37b)
\]

one can use Eqs. (15), (34), (35), to arrive at the momentum space form of Eq. (36), viz.,

\[
Q_\mu G_{5\mu}^{J^P}(k, K'; q, K) + 2im_q G_{5}^{J^P}(k, K'; q, K)
= \sum_{J^P} \left( Q_\mu G_{5\mu}^{j, J^P} + 2im_q G_{5}^{j, J^P} \right)(k, K'; q, K), \quad (38a)
\]

\[
= \sum_{J^P} \sum_{i=1}^{4} G_{5\mu}^{j, J^P}(k, K'; q, K) + 2im_q G_{5}^{j, J^P}(k, K'; q, K).
\]

Here, the identification of terms is straightforward: the first sum is over the separate diquark contributions; the second sum expresses that over the Fourier amplitudes of the four terms on the right-hand-side in Eq. (36); and Eq. (38c) serves as a formal expedient for subsequent use. In Eq. (38b), writing diquark wave functions as \( \Lambda^{J^P} \):

\[
\tilde{G}_{5\mu}^{j, J^P} = \frac{\tau^j}{2} i \gamma_5 \Lambda^{J^P}(k - Q/2; K') \times \Delta^{J^P}(K) \Lambda^{J^P}(q; K), \quad (39a)
\]

\[
\tilde{G}_{5}^{j, J^P} = \Lambda^{J^P}(k + Q/2; K) \left( i \gamma_5 \frac{\tau^j}{2} \right)^T \times \Delta^{J^P}(K) \Lambda^{J^P}(q; K), \quad (39b)
\]

\[
\tilde{G}_{5}^{j, J^P} = \Lambda^{J^P}(k, K') \Delta^{J^P}(K') \times (i \gamma_5 \frac{\tau^j}{2})^T \Lambda^{J^P}(q - Q/2; -K'), \quad (39c)
\]

\[
\tilde{G}_{5}^{j, J^P} = \Lambda^{J^P}(k, K') \Delta^{J^P}(K') \times \Lambda^{J^P}(q + Q/2; -K') \frac{\tau^j}{2} \gamma_5 . \quad (39d)
\]

Continuing toward expressions for the one-particle-irreducible 4-point seagull couplings of the axial-vector and pseudoscalar currents to the quarks and the diquarks, we exploit Eq. (38c) and define \( \chi_{5\mu + (i)}^{j, J^P} \) and \( \chi_{5 + (i)}^{j, J^P} \) via

\[
(S(k_1) \chi_{5\mu + (i)}^{j, J^P}(k_1, k_2, K) S^T(k_2)) \Delta^{J^P}(K)
:= (Z^{J^P})^{-1} \int \frac{d^4 q}{(2\pi)^4} G_{5\mu + (i)}^{j, J^P}(k, K'; q, K) \Gamma^{J^P}(q; K), \quad (40)
\]

and

\[
(S(k_1) \chi_{5 + (i)}^{j, J^P}(k_1, k_2, K) S^T(k_2)) \Delta^{J^P}(K)
:= (Z^{J^P})^{-1} \int \frac{d^4 q}{(2\pi)^4} G_{5 + (i)}^{j, J^P}(k, K'; q, K) \Gamma^{J^P}(q; K). \quad (41)
\]

Here

\[
Z^{J^P} := \int \frac{d^4 q}{(2\pi)^4} \text{trCDF} \left[ \tilde{\Lambda}^{J^P}(q; -K) \Gamma^{J^P}(q; K) \right], \quad (42)
\]
and the trace is over colour, flavour and Dirac matrices. Then, Eq. (39a) entails

\[
(S(k_1)\{Q_{\mu} \chi_{5,1}^{j,P}(k_1, k_2, K)
+ 2im_q \chi_{5,1}^{j,P}(k_1, k_2, K)\} S^T(k_2)) \Delta_{j,P}(K)
= (Z_{j,P}^{-1}) \int \frac{d^4_q}{(2\pi)^4} \tilde{G}_{j,P}^{(1)}(k'; q, K) \Gamma_{j,P}(q; K)
= (Z_{j,P}^{-1}) \int \frac{d^4_q}{(2\pi)^4} \left(\frac{\tau_j}{2} \gamma^5 \Lambda_{j,P}(q - K/2; K)\right)
\times \Delta_{j,P}(K) \text{tr}_{\text{CDF}} \left[\tilde{\Lambda}_{j,P}(q; -K) \Gamma_{j,P}(q; K)\right]
\]

\[
= \frac{\tau_j}{2} \gamma^5 \Lambda_{j,P}(k - Q/2; K) \Delta_{j,P}(K),
\]

(43)

and using the AXWTI, Eq. (18), we arrive at

\[
Q_{\mu} \chi_{5,1}^{j,P}(k_1, k_2, K) + 2im_q \chi_{5,1}^{j,P}(k_1, k_2, K) = (Q_{\mu} \Gamma_{5}^{(1)}(k_1, k_1 - Q) + 2im_q \Gamma_{5}^{(1)}(k_1, k_1 - Q))
\times S(k_1 - Q) \Gamma_{j,P}(k - Q/2; K)
- \frac{\tau_j}{2} \gamma^5 \Gamma_{j,P}(k - Q/2; K).
\]

(44)

Proceeding similarly, we arrive at

\[
Q_{\mu} \chi_{5,2}^{j,P}(k_1, k_2, K) + 2im_q \chi_{5,2}^{j,P}(k_1, k_2, K)
= \Gamma_{j,P}(k + Q/2; K) S^T(k_2 - Q)(Q_{\mu} \Gamma_{5}^{(1)}(k_2, k_2 - Q)
+ 2im_q \Gamma_{5}^{(1)}(k_2, k_2 - Q))^T
- \Gamma_{j,P}(k + Q/2; K)(i\gamma^5 \frac{\tau_j}{2})^T,
\]

(45)

for the second contribution, Eq. (39b). Besides, the third and fourth contributions are

\[
(S(k_1)\{Q_{\mu} \chi_{5,2}^{j,P}(k_1, k_2, K)
+ 2im_q \chi_{5,2}^{j,P}(k_1, k_2, K)\} S^T(k_2)) \Delta_{j,P}(K)
= (Z_{j,P}^{-1}) \int \frac{d^4_q}{(2\pi)^4} \tilde{G}_{j,P}^{(1)}(k'; k') \Delta_{j,P}(K') \times
\text{tr}_{\text{CDF}} \left[\left(i\gamma^5 \frac{\tau_j}{2}\right)^T \tilde{A}_{j,P}(q - Q/2; -K') \Gamma_{j,P}(q; K)\right],
\]

(46)

for Eq. (39c), and

\[
(S(k_1)\{Q_{\mu} \chi_{5,4}^{j,P}(k_1, k_2, K)
+ 2im_q \chi_{5,4}^{j,P}(k_1, k_2, K)\} S^T(k_2)) \Delta_{j,P}(K)
= (Z_{j,P}^{-1}) \int \frac{d^4_q}{(2\pi)^4} \tilde{G}_{j,P}^{(1)}(k'; k) \Delta_{j,P}(K') \times
\text{tr}_{\text{CDF}} \left[\tilde{A}_{j,P}(q + Q/2; -K') \frac{\tau_j}{2} i\gamma^5 \Gamma_{j,P}(q; K)\right],
\]

(47)

for Eq. (39d), respectively.

Let us first study the third and fourth contributions in more detail. Their flavour structure involves four different diquark configurations: scalar diquark, \([ud]_{10}^+,\) and axial-vector diquarks, \([ud]_{1+}, [uu]_{1+}, [dd]_{1+}\). When the correlations \([ud]_{10}, [ud]_{1+}\) are involved, we obtain trivial zeros because the traces of the flavour parts vanish (the diquarks’ flavour matrices are listed in Appendix A 2):

\[
\text{tr}_{F} \left[\left(t_{ij}^0\right)^{1/2} (a_{ij}^0)\right] = \text{tr}_{F} \left[\left(t_{ij}^0\right)\right] = 0,
\]

(48)

\[
\text{tr}_{F} \left[\left(t_{ij}^0\right)^{1/2} (a_{ij}^0)\right] = \text{tr}_{F} \left[\left(t_{ij}^0\right)\right] = 0.
\]

(49)

However, when the correlations \([uu]_{1+}\) and \([dd]_{1+}\) are involved, the flavour traces are not zero and thus we need to investigate them more closely. For \([uu]_{1+}\) we have

\[
(S(k_1)\{Q_{\mu} \chi_{5,4}^{j,P}(k_1, k_2, K)
+ 2im_q \chi_{5,4}^{j,P}(k_1, k_2, K)\} S^T(k_2)) \Delta_{1+}(K)
= (Z_{j,P}^{-1}) \frac{1}{\Lambda_{1+}^{(1)}(k; K')} \Delta_{1+}(K') \times
\int \frac{d^4_q}{(2\pi)^4} \text{tr}_{\text{CDF}} \left[\left(i\gamma^5 \frac{\tau_j}{2}\right)^T \tilde{A}_{1+}(q - Q/2; -K') \Gamma_{1+}(q; K')\right]
\]

\[
= (Z_{j,P}^{-1}) \frac{1}{\Lambda_{1+}^{(1)}(k; K')} \Delta_{1+}(K') \times \sigma_{V}((q - K/2)^2/\omega_{1+}^2) \sigma_{V}((q + K/2)^2)
\times \left[\epsilon_{\rho\sigma\beta\gamma} q_{\rho}(K_{\sigma} + Q_{\sigma}) - \epsilon_{\rho\sigma\beta\gamma} q_{\rho} K_{\sigma}/2\right],
\]

(50)

where \(\sigma_{V}\) is a quark-propagator dressing function, Eq. (A.1a), and

\[
Z_{1+}^{(1)} := \int \frac{d^4_q}{(2\pi)^4} \text{tr}_{\text{CDF}} \left[\tilde{A}_{1+}^{(1)}(q; -K) \Gamma_{1+}(q; K)\right].
\]

(51)

In this derivation, we used the explicit form of the axial-vector BS amplitude, Eq. (A.6b), which only depends on the relative momentum. Similarly, we obtain

\[
(S(k_1)\{Q_{\mu} \chi_{5,4,1+}^{j,P}(k_1, k_2, K) + 2im_q \chi_{5,4,1+}^{j,P}(k_1, k_2, K)\} S^T(k_2)) \Delta_{1+}(K)
= (Z_{j,P}^{-1}) \frac{1}{\Lambda_{1+}^{(1)}(k; K')} \Delta_{1+}(K') \times \sigma_{V}((q + K/2)^2/\omega_{1+}^2) \sigma_{V}((q - K/2)^2)
\times \left[\epsilon_{\rho\sigma\beta\gamma} q_{\rho}(K_{\sigma} + Q_{\sigma}) + \epsilon_{\rho\sigma\beta\gamma} Q_{\rho} K_{\sigma}/2\right].
\]

(52)

Obviously, neither Eq. (50) nor Eq. (52) vanishes if \(j = 3\). Nevertheless, if we add them together and subsequently take the limit \(Q \to 0\), we find

\[
0 = Q_{\mu} \chi_{5,4,1+}^{j,P}(k_1, k_2, K) + \chi_{5,4,1+}^{j,P}(k_1, k_2, K)
+ 2im_q \chi_{5,4,1+}^{j,P}(k_1, k_2, K) + \chi_{5,4,1+}^{j,P}(k_1, k_2, K).
\]

(53)
The same result can be shown for \( \{ dd \}_1^+ \).

Therefore, for all kinds of diquark correlations we can write a unified result\(^1\)

\[
Q_\mu \chi_{5\mu}^{J^p}(k_1, k_2, K) + 2im_q \chi_5^{J^p}(k_1, k_2, K) \\
:= \sum_{i=1}^{4} Q_\mu \chi_{5,\mu,\{i\}}^{J^p}(k_1, k_2, K) + 2im_q \chi_{5,\{i\}}^{J^p}(k_1, k_2, K) \\
\approx (Q_\mu \Gamma_{5\mu}^{J^p} + 2im_q \Gamma_{5}^{J^p})S(k_1 - Q)\Gamma^{J^p}(k - Q/2) \\
+ \Gamma^{J^p}(k + Q/2)S^{T}(k_2 - Q)(Q_\mu \Gamma_{5\mu}^{J^p} + 2im_q \Gamma_{5}^{J^p})^{T} \\
- \frac{\tau^j}{2}i\gamma_5 \Gamma^{J^p}(k - Q/2) - \Gamma^{J^p}(k + Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T}. 
\]

In arriving at Eq. (54), we used Eq. (53) and supposed it to be valid outside the neighbourhood \( Q^2 \approx 0 \).

The approximation expressed in Eq. (54) is kindred to that employed in deriving the seagull terms for the electromagnetic current \cite{102}; nevertheless, one might ask whether Eq. (54) is saturated. It is worth highlighting that Eqs. (56), (57) are complete because axial-vector/pseudoscalar probes do not couple to scalar targets and the axial-vector diquark propagator is symmetric. Hence, in contrast to the electromagnetic case, the seagull identity, Eq. (57), does not have a diquark part. As a consequence, AXWThs of the relevant current-diquark vertices are both equal to zero, see Section III C 4.

As in the electromagnetic case \cite{102}, \( \chi_{5,\mu}^{J^p} \) and \( \chi_5^{J^p} \) can be separated into two parts

\[
\chi_{5,\mu}^{J^p} = \chi_{5,\mu,\{\text{legs}\}}^{J^p} + \chi_{5,\mu,\{\text{sg}\}}^{J^p}, \\
\chi_5^{J^p} = \chi_5^{J^p,\{\text{legs}\}} + \chi_5^{J^p,\{\text{sg}\}}, 
\]

where: \( \chi_{5,\mu,\{\text{legs}\}}^{J^p} \) and \( \chi_{5,\mu,\{\text{sg}\}}^{J^p} \) describe the coupling of the axial and pseudoscalar currents to the amputated quark legs separately; and \( \chi_5^{J^p,\{\text{legs}\}} \) and \( \chi_5^{J^p,\{\text{sg}\}} \) describe the axial-vector and pseudoscalar one-particle irreducible seagull couplings.

The “legs” and “sg” terms separately satisfy the following Ward-Green-Takahashi identities:

\[
Q_\mu \chi_{5,\mu,\{\text{legs}\}}^{J^p}(k_1, k_2, K) + 2im_q \chi_{5,\mu,\{\text{legs}\}}^{J^p}(k_1, k_2, K) \\
= (Q_\mu \Gamma_{5\mu}^{J^p} + 2im_q \Gamma_{5}^{J^p})S(k_1 - Q)\Gamma^{J^p}(k - Q/2) \\
+ \Gamma^{J^p}(k + Q/2)S^{T}(k_2 - Q)(Q_\mu \Gamma_{5\mu}^{J^p} + 2im_q \Gamma_{5}^{J^p})^{T}, \\
\]

\[
Q_\mu \chi_{5,\mu,\{\text{sg}\}}^{J^p}(k, Q) + 2im_q \chi_{5,\mu,\{\text{sg}\}}^{J^p}(k, Q) \\
= -\frac{\tau^j}{2}i\gamma_5 \Gamma^{J^p}(k - Q/2) - \Gamma^{J^p}(k + Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T}, 
\]

whereby Eq. (54) is saturated. It is worth highlighting that Eqs. (56), (57) are complete because axial-vector/pseudoscalar probes do not couple to scalar targets and the axial-vector diquark propagator is symmetric. Hence, in contrast to the electromagnetic case, the seagull identity, Eq. (57), does not have a diquark part. As a consequence, AXWThs of the relevant current-diquark vertices are both equal to zero, see Section III C 4.

In this work, we therefore employ the following \textit{Ansätze}

\[
\chi_{5,\mu,\{\text{legs}\}}^{J^p}(k, Q) = -\frac{Q_\mu}{Q^2 + m_\pi^2} \left[ \frac{\tau^j}{2}i\gamma_5 \Gamma^{J^p}(k - Q/2) + \Gamma^{J^p}(k + Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right], \\
\chi_5^{J^p}(k, Q) = \frac{1}{2m_q Q^2 + m_\pi^2} \left[ \frac{\tau^j}{2}i\gamma_5 \Gamma^{J^p}(k - Q/2) + \Gamma^{J^p}(k + Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right]. 
\]

and

\[
i\chi_{5,\mu,\{\text{legs}\}}^{J^p}(k, Q) = -\frac{Q_\mu}{Q^2 + m_\pi^2} \left[ \Gamma^{J^p}(k + Q/2)\frac{\tau^j}{2}i\gamma_5 + \Gamma^{J^p}(k - Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right]. \\
i\chi_5^{J^p}(k, Q) = \frac{1}{2m_q Q^2 + m_\pi^2} \left[ \Gamma^{J^p}(k + Q/2)\frac{\tau^j}{2}i\gamma_5 + \Gamma^{J^p}(k - Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right]. 
\]

In a similar way, we can obtain the seagulls’ charge conjugations:

\[
\bar{\chi}_{5,\mu,\{\text{legs}\}}^{J^p}(k, Q) = \frac{Q_\mu}{Q^2 + m_\pi^2} \left[ \Gamma^{J^p}(k + Q/2)\frac{\tau^j}{2}i\gamma_5 + \Gamma^{J^p}(k - Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right], \\
\bar{\chi}_5^{J^p}(k, Q) = \frac{1}{2m_q Q^2 + m_\pi^2} \left[ \Gamma^{J^p}(k + Q/2)\frac{\tau^j}{2}i\gamma_5 + \Gamma^{J^p}(k - Q/2)(i\gamma_5 \frac{\tau^j}{2})^{T} \right]. 
\]
and

\[ i\chi^{j^F}_{5,[sg]}(k; Q) = -\frac{1}{2m_Q Q^2 + m_\pi^2} \left[ \Gamma^F(k + Q/2) \frac{\tau^j}{2} + (i\gamma_5 \frac{\tau^j}{2})^T \Gamma^F(k - Q/2) \right]. \]

Equations (58)-(61) are novel and the most important results in this section.

The axial-vector Ward-Takahashi identity in Eq (57), cannot completely determine the seagull terms, e.g. their transverse parts are neglected herein. It is possible to use more complicated Ansätze; however, Eqs. (58) – (61) are simple expressions which ensure that the computed Goldberger-Treiman relations.

It should also be stressed that the starting point for the construction of the electromagnetic seagull terms was to keep them regular [102], which is not a suitable strategy here. The axial-vector and pseudoscalar seagulls could be singular, just like the analogous quark vertices. Consequently, Eqs. (58) – (61) have a pole at the pion mass.

With the dressed quark vertices and the seagull terms in hand, we can construct the remaining elements: the axial-vector and pseudoscalar current-diquark vertices. Here, we use the same strategy as in Ref. [81]. In short, we focus on the dominant tensor structure and compute its strength in the on-shell nucleon by: inserting Fig. 4 into diagrams (2) and (3) of Fig. 3; evaluating the result in each case; then equating the values with those obtained by computing the original one-loop diagrams from Fig. 3 with the probe-diquark vertex written as the appropriate coupling times the dominant associated tensor structure. Plainly, each current-diquark vertex receives four contributions, viz. those depicted in Fig. 4. Two of them are generated by the coupling of the current to the upper and lower quark lines of the resolved diquark. The other two are the current couplings to the diquark amplitudes, which are again seagull terms. In Ref. [81], the authors only took into account the first two contributions; all four are included in this work.

4. Current-diquark vertices

Furthermore, we need to study the flavour structures of the currents and diquark correlations. Given that the scalar diquark is also isoscalar, associated with \( \tau^5 \), then there is no \([ud]_0^+ \to [ud]_0^+\) contribution. On the other hand, transitions between scalar and axial-vector diquarks are possible in the form of: charged currents, \( \{dd\}_1^+ \to [ud]_0^+ \) and \( \{uu\}_1^+ \to [ud]_0^+ \); and neutral currents, \( \{ud\}_1^+ \to [ud]_0^+ \). Note too that, in the isospin limit, the coupling strengths are identical in magnitude. Moreover, the constructed vertices must properly express the momentum-dependence of the diquark amplitudes, which is a natural consequence of the nonpointlike characteristic of diquark correlations.

Taking the above arguments into account, we obtain the following expressions:

i) \( \{qq\}_1^+ \) – pseudoscalar-current vertex

\[ \Gamma^{aa}_{5,\alpha\beta}(p_d, k_d) = -\frac{1}{2m_Q Q^2 + m_\pi^2} M^E \epsilon_{\alpha\beta\delta}(p_d + k_d)_{\gamma\delta} d(\tau^{aa}) , \]

ii) \( \{qq\}_1^+ \) – axial-current vertex

\[ i\Gamma^{aa}_{5,\alpha\beta}(p_d, k_d) = \frac{2}{2m_Q Q^2 + m_\pi^2} \left[ 2i\kappa_{ps}^a M^E Q_{\gamma} \right] d(\tau^{aa}) , \]

iii) Pseudoscalar-current induced \( 0^+ \leftrightarrow 1^+ \) transition vertex

\[ \Gamma^{sa}_{5,\alpha}(p_d, k_d) = -\frac{1}{2m_Q Q^2 + m_\pi^2} \left( 2i\kappa_{ps}^a M^E Q_{\beta} \right) d(\tau^{sa}) , \]

iv) Axial-current induced \( 0^+ \leftrightarrow 1^+ \) transition vertex

\[ \Gamma^{sa}_{5,\alpha}(p_d, k_d) = \frac{M^E}{Q^2 + m_\pi^2} \left( 2i\kappa_{ps}^s M^E Q_{\beta} \right) d(\tau^{sa}) . \]
compositeness via form factor suppression on $Q^2 > 0$. It is noteworthy that using dipole suppression instead, viz. $d(x)^2$, no prediction in any image drawn herein changes by more than the linewidth because, in all cases, diagram (1) in Fig. 3 is both dominant and hard, whereas all weak-boson–diquark interactions are soft and subdominant.

On the other hand, using Eqs. (18) and (57), we can get the AXWTIs of the current-diquark vertices:

$$0 = Q_{\mu} \Gamma_{\mu\alpha\beta}^{a}(p_d, k_d) + 2im_{q} \Gamma_{\mu\alpha\beta}^{a}(p_d, k_d), \quad (66a)$$

$$0 = Q_{\mu} \Gamma_{\mu\alpha\beta}^{a}(p_d, k_d) + 2im_{q} \Gamma_{\mu\alpha\beta}^{a}(p_d, k_d). \quad (66b)$$

It should be emphasised that Eqs. (66) are independent of the explicit forms of the current-diquark vertices involved; hence, can be used as constraints for the vertices’ algebraic forms. It is easy to see that Eqs. (62) – (65) satisfy the above AXWTIs precisely, from which we can get

$$\kappa_{ps}^{aa} = \frac{m_{N}}{2M_{E}} \kappa_{ax}^{aa}, \quad (67a)$$

$$\kappa_{ps}^{sa} = \frac{m_{N}}{2M_{E}} \kappa_{ax}^{sa}. \quad (67b)$$

We verified these constraints numerically

$$\kappa_{ax}^{sa} = 0.73, \quad (68a)$$

$$\kappa_{ps}^{sa} = 1.31, \quad (68b)$$

$$\kappa_{ax}^{sa} = 0.75, \quad (68c)$$

$$\kappa_{ps}^{sa} = 1.34, \quad (68d)$$

from the direct computation of the four diagrams in Fig. 4.

As anticipated following Eq. (57), Eqs. (66) both equate to zero because, unlike the electromagnetic case, the corresponding seagull terms have no contribution from the diquarks’ leg.

We have now specified all the necessary building blocks to construct the diagrams for $J_{5\mu}^{J}(K, Q)$ and $J_{5\mu}^{J}(K, Q)$ drawn in Fig. 3, with the corresponding expressions given in Appendix B. We are therefore in a position to analytically verify the PCAC relation, Eq. (8). This is completed in Appendix D.

**IV. NUMERICAL RESULTS**

Our predictions for the nucleon’s axial and induced pseudoscalar form factors were sketched elsewhere [86]. Herein, so as to provide a complete, self-contained report, we recapitulate those results. We subsequently describe predictions for the pseudoscalar (pion-nucleon) form factor and quantify the domain on which PCAC is satisfied.

The axial form factor, $G_{A}(Q^2)$, can be expanded at small momenta as

$$G_{A}(Q^2) = g_{A} \left(1 - \frac{\langle r_{A}^2 \rangle}{6} Q^2 + \ldots \right), \quad (69)$$

FIG. 5. Upper panel – A. $G_{A}(x)/g_{A}$ calculated herein – blue curve within lighter blue uncertainty band, compared with lQCD results from Ref. [32] – green diamonds. With respect to our central results, this comparison may be quantified by reporting the mean-\(\chi^2\) value, which is 0.27. Lower panel – B. $G_{A}(x)/g_{A}$ calculated herein – blue curve within lighter blue uncertainty band, compared with lQCD results from Ref. [33] – purple boxes [dipole] and golden triangles [z expansion]. The mean-\(\chi^2\) values are 10.86 [dipole] and 1.88 [z expansion].

where $g_{A} := G_{A}(0)$ is the nucleon axial charge, and $\langle r_{A}^2 \rangle^{1/2}$ is the axial mean square radius with

$$\langle r_{A}^2 \rangle = -\frac{d}{dQ^2} \left(\frac{G_{A}(Q^2)}{G_{A}(0)}\right)_{Q^2=0}. \quad (70)$$

Sometimes, when comparing with other theoretical estimations, it is convenient to use a dipole Ansatz for the axial form factor:

$$G_{A}(Q^2) = \frac{g_{A}}{(1 + Q^2/m_{A}^2)^2}, \quad (71)$$

where $m_{A}$ is the so-called axial mass. It should be stressed that we use Eq. (71) to interpolate the global $Q^2$-behaviour of $G_{A}$, instead of relating it to $\langle r_{A}^2 \rangle$ via $m_{A} = \sqrt{12/\langle r_{A}^2 \rangle}$.

**A. The axial and induced pseudoscalar form factors**

Our Faddeev equation result for $G_{A}(x)/g_{A}$ is depicted in Fig. 5, together with results from lQCD [32, 33]. The
lighter blue band expresses the impact of ±5% variations in our diquark masses. Notably, scalar and axial-vector diquark variations interfere destructively, e.g. reducing $m_{ud}$ increases $g_A$, whereas $g_A$ decreases with the same change in the axial-vector mass. It is clear from Fig. 5 that our prediction agrees with the QCD results in Refs. [32] and is similar to that in Ref. [33] obtained from $z$ expansion. We will return to this point when we discuss PCAC below. The extracted values for $g_A$, the charge radius and the axial mass are

$$g_A = 1.25(3),$$

$$\langle r^2_A \rangle^{1/2} m_N = 3.25(4),$$

$$m_A/m_N = 1.23(3).$$

As signalled above, the listed uncertainties in our values express the impact of varying the diquark masses.

In Table I, referring to Fig. 3, we list the relative strengths of each diagram’s contribution to the nucleon’s axial charge. Diagram (1), with the weak-boson striking the dressed-quark in association with a spectator scalar diquark, is dominant. On the other hand, the contribution from diagrams (5) and (6) are zero in this case because the seagull terms, Eqs. (58) and (60), are proportional to $Q_n$ and thus they do not contribute to $G_A$, which is determined by the $Q$-transverse piece of the nucleon axial current, see $G_A$’s projection, Eq. (6a). The impacts on all form factors of the pion pole terms in our current construction are detailed in Appendix E.

In Fig. 6 we depict our prediction for the induced pseudoscalar form factor $G_P(x)$, as well as corresponding QCD results. The model-uncertainty band surrounding this curve cannot be distinguished when using the scale necessary to draw the figure.

| $J^S_q$ | $J^A_q$ | $J^{AA}_q$ | $J^{SA+AS}_q$ | $J_{ex}$ | $J_{sg}$ |
|--------|--------|--------|-------------|--------|--------|
| $G_A(0)$ | 0.71±22 ±0.064±0.025±0.130±0.022±0.19±0.072±0.0 | 0 |
| $G_P(0)$ | 0.74±22 ±0.070±0.025±0.130±0.022±0.19±0.0 | 0 |
| $G_5(0)$ | 0.74±22 ±0.069±0.025±0.130±0.022±0.19±0.0 | 0 |

FIG. 6. Upper panel – A. Result for $G_P(x)$ calculated herein – blue curve within lighter blue model-uncertainty band, compared with QCD results from Ref. [32] – green diamonds. With respect to our central results, this comparison may be quantified by reporting the mean-$\chi^2$ value, which is 1.04. Lower panel – B. Similar comparison with QCD results from Ref. [33] – purple boxes [dipole] and golden triangles [$z$ expansion]. The mean-$\chi^2$ values are 15.14 [dipole] and 3.84 [$z$ expansion].

Muon capture experiments ($\mu + p \rightarrow \nu + n$) determine the induced pseudoscalar charge

$$g_P^* = \frac{m_\mu}{2m_N} G_P(Q^2 = 0.88 m_\mu^2).$$

We obtain $g_P^* = 8.80(23)$, which is slightly larger than the recent MuCap Collaboration value, $g_P^* = 8.06(55)$ [24, 25], but agrees with the world average value $g_P^* = 8.79(1.92)$ [36].

In Table I, referring to Fig. 3, we list again the relative strengths of each diagram’s contribution to $G_P(0)$. Once again, diagram (1), with the weak-boson striking the dressed-quark in association with a spectator scalar diquark, is overwhelmingly dominant. In this case, however, there is an active cancellation between the contributions from diagrams (4), (5) and (6).

We find $g_A^p/g_A^p = -0.16(2)$ at the hadronic scale. This is a significant suppression of the magnitude of the $d$-quark component relative to that found in quark models, $g_A^d/g_A^u = -0.25$ [108], and QCD analyses, $g_A^d/g_A^u = -0.40(2)$ [109] and $g_A^d/g_A^u = -0.58(3)$ [110]. Notably,
whilst $g_A$ is a conserved charge, invariant under QCD evolution, the separation into component contributions from different quark flavours is not. This effect can potentially reconcile our flavour separation results with the IQCD predictions: our Faddeev wave function is defined at the hadronic scale, $\zeta_H \simeq 0.33 \text{ GeV}$ [111, 112], whereas the IQCD values are renormalised at $\zeta \simeq 2 \text{ GeV}$. The size reduction shown in our framework owes to the presence of strong diquark correlations in our nucleon wave function, with the calculated value reflecting the relative strength of scalar and axial-vector diquarks: the isoscalar–scalar correlations are dominant, but the isovector–axial-vector diquarks have a measurable influence.

It is also interesting to find the pole position of $G_P$. Although we are not readily able to compute form factors in the time-like region, we can perform an extrapolation since the pion mass pole is very close to the origin. To do this, we use $1/G_P$, instead of $G_P$, and extrapolate to the region of small negative momentum. The result is depicted in Fig. 7: $1/G_P$ has a zero at $Q^2 = -m_\pi^2$, verifying that $G_P$ has a pole at this position.

The low momentum behaviour of the induced pseudoscalar form factor can be well approximated by using the pion pole dominance (PPD) assumption, which relates $G_P$, at low momenta, to the axial form factor $G_A$:

$$G_P \approx \frac{4m_\pi^2 G_A}{Q^2 + m_\pi^2} \, .$$  \hspace{1cm} (74)

PPD is not an exact relation but there is every reason to expect that it is satisfied to a high level of precision – see Eq. (E.3) and the associated discussion. Moreover, recent IQCD analyses [32, 33] find that PPD is satisfied with a discrepancy $\lesssim 5\%$.

We have computed the PPD ratio

$$R_{\text{PPD}} := \frac{4m_\pi^2 G_A}{(Q^2 + m_\pi^2)G_P} \, ,$$  \hspace{1cm} (75)

and depict it in Fig. 8: $R_{\text{PPD}}$ is very close to one; however, a deviation of about 1\% is found on $x \simeq 0$, which becomes smaller as $x$ gets larger. This situation is genuine and can be explained within the quark+diquark picture. First, we define a PCAC ratio according to Eq. (9)

$$R_{\text{PCAC}} := \frac{4m_\pi^2 G_A}{Q^2G_P + 4m_qm_NG_5} \, ,$$  \hspace{1cm} (76)

in the chiral limit $m_q = m_\pi = 0 \text{ GeV}$, $R_{\text{PCAC}}$ and $R_{\text{PPD}}$ are equivalent, and they should precisely be equal to 1:

$$R_{\text{PCAC}}^{m_q=0} = R_{\text{PPD}}^{m_q=0} = \frac{4m_\pi^2 G_A^{m_q=0}}{Q^2 G_P^{m_q=0}} = 1 \, .$$  \hspace{1cm} (77)

Second, focusing on the axial current, the singular parts of the dressed-quark vertex, Eq. (29), seagull term, Eq. (58), and diquark vertices, Eqs. (63), (65), do not contribute to $G_A$; and their contributing regular parts do not explicitly depend on the current quark mass $m_q$. Thus, the chiral limit and the $m_q$-dependent expressions will generate the same axial form factor, i.e. $G_A^{m_q=0} \equiv G_A$, provided that their parameters are the same.\footnote{In reality, the parameters should vary with the current quark mass, therefore, produce different $G_A$.}

Third, regular and singular parts contribute to the induced pseudoscalar form factor, so one may write

$$G_P := G_{P,\text{regular}} + G_{P,\text{singular}} \, .$$  \hspace{1cm} (78)

It is obvious that

$$G_{P,\text{regular}}^{m_q=0} = G_{P,\text{regular}}$$ \hspace{1cm} (79a)

$$Q^2G_{P,\text{singular}}^{m_q=0} = (Q^2 + m_\pi^2)G_{P,\text{singular}} \, ;$$  \hspace{1cm} (79b)

hence, Eq. (75) becomes

$$R_{\text{PPD}} = \frac{4m_\pi^2 G_A^{m_q=0}}{Q^2 G_P^{m_q=0} + m_\pi^2 G_{P,\text{regular}}} \, .$$  \hspace{1cm} (80a)
Our predicted value of $(m_q/m_N)G_5(0)$ is 1.25(3) = $G_A(0)$. (81)

(Recall, the theoretical uncertainty expresses the impact of ±5% variations in the diquark masses. Clearly, our results satisfy the Goldberger-Treiman relation precisely.

In Table I, referring to Fig. 3, we list the relative strengths of each diagram’s contribution to $G_5(0)$, i.e. $G_{\pi NN}(0)$. The results for $G_5(0)$ are almost identical to those for $G_P(0)$. This can be understood in the following way: according to our argument on PPD, we know that the singular part of the axial current provides almost the total contribution (≥99%) to the induced pseudoscalar form factor $G_P$ at $Q^2 \simeq 0$ GeV$^2$, and thus we can safely focus on the singular part of $J_5\mu$. Next, considering $G_P$’s projection, Eq. (6b), we immediately find that $G_A$ does not contribute because of its transverse nature. Meanwhile, if one takes into account the detailed structure of the building blocks and currents, and compares Eq. (6b) with $G_5$’s projection, Eq. (6c), one arrives at

$$G_P \simeq \frac{Q_\mu}{\tau^2} \text{tr}[J_{5\mu}\gamma_5] \times c_1,$$

(82a)

$$G_5 = \frac{1}{\tau} \text{tr}[J_5\gamma_5] \times c_2$$

(82b)

$$\simeq \frac{Q_\mu}{\tau^2} \text{tr}[J_{5\mu}\gamma_5] \times c'_2$$

(82c)

$$\simeq G_P \times (c'_2/c_1).$$

(82d)

when $Q^2 \simeq 0$ GeV$^2$, where $c_1$, $c_2$ and $c'_2$ are constants.

The pseudoscalar form factor $G_5$, or equivalently the pion-nucleon form factor, $G_{\pi NN}$, cannot be directly measured except at the pion mass point $Q^2 = -m_\pi^2$, where we can obtain the pion-nucleon coupling constant $g_{\pi NN}$. Our predicted value of $(m_q/m_N)G_5(Q^2)$ is depicted in Fig. 9, compared with recent IQCD results[32, 33] also presented in the same figure. We have also extrapolated the function $1/G_5$ from the space-like region onto a small domain of timelike momenta in order to determine its

\[ \frac{f_\pi}{m_N}G_{\pi NN}(0) = 1.25(3) = G_A(0). \]
This value overlaps with that extracted from pion-nucleon scattering \[113\] \((g_{\pi NN}/m_N = 13.97(10)/\text{GeV})\); and compares well with a determination based on the Granada 2013 \(np\) and \(pp\) scattering database \[114\] \((g_{\pi NN}/m_N = 14.11(3)/\text{GeV})\) and a recent analysis of nucleon-nucleon scattering using effective field theory and related tools \[115\] \((g_{\pi NN}/m_N = 14.09(4)/\text{GeV})\). The IQCD study in Ref.\[33\] produces \(g_{\pi NN}/m_N = 13.77(85)/\text{GeV}\) (dipole) and \(g_{\pi NN}/m_N = 15.74(1.93)/\text{GeV}\) \((z\text{-expansion})\). All these comparisons are drawn in Fig. 11 A.

It is also worth noting that on \(-m_N^2 < Q^2 < 2m_N^2\), a good interpolation of our central result is provided by \((x = Q^2/m_N^2)\):

\[
G_{\pi NN}(x) = \frac{(13.52 - 2.291x)m_N}{1 + 2.383x + 0.5563x^2 + 1.434x^3}.
\]  

Eq. (83) is obtained with the following dipole form:

\[
G_{\pi NN}^d(x) = \frac{13.47m_N}{(1 + x/0.845^2)^2},
\]  

which corresponds to a \(\pi NN\) dipole scale \(\Lambda_{\pi NN} = 0.845m_N = 0.79\text{GeV}\), viz. a soft form factor. (A similar value was obtained previously in a rudimentary quark+scalar-diquark model \[80\].) Our prediction is \(\sim 20\%\) larger than, hence qualitatively equivalent to, the \(\pi NN\) dipole mass inferred from a dynamical coupled-channels analysis of \(\pi N, \gamma N\) interactions \[117\]. Future such studies may profit by implementing couplings and range parameters determined in analyses like ours.

We now can compute the so-called Goldberger-Treiman discrepancy:

\[
\Delta_{GT} = 1 - \frac{G_A(0)}{G_{\pi NN}(m_N^2)}\]  

\[
= 1 - \frac{G_{\pi NN}(0)}{G_{\pi NN}(m_N^2)},
\]  

which measures the difference of \(G_{\pi NN}\) values at \(Q^2 = 0\) and the pion's on-shell mass, for a given current-quark mass; and thereby the distance from the chiral limit. We predict \(\Delta_{GT} = 0.030(1)\), which is loosely consistent with the estimate in Ref.\[116\], viz. 0.023(5), and matches, within errors, the IQCD results in Ref.\[33\]. These comparisons are drawn in Fig. 11 B.

Finally, we check whether our computed \(G_A(Q^2)\), \(G_P(Q^2)\) and \(G_5(Q^2)\) form factors satisfy the PCAC relation, Eq. (9). Figure 12 displays the PCAC ratio, defined in Eq.\[76\]; one observes that the computed \(R_{PCAC}\) is practically indistinguishable from unity for the whole range of momentum-transfer depicted. Therefore, we have shown analytically but also numerically that our theoretical approach is consistent with fundamental symmetry requirements. It is worth emphasizing here that this result does not rely on any fine-tuned set of para-
meters; instead, it is automatically satisfied owing to our careful construction of the currents, discussed above.

V. SUMMARY

Using a Poincaré-covariant quark+diquark Faddeev equation treatment of the nucleon and weak interaction currents that ensure consistency with relevant Ward-Green-Takahashi identities, we delivered a unified set of predictions for the nucleon’s axial, induced pseudoscalar, and explicit pseudoscalar form factors. The presentation included a detailed discussion of partial conservation of the nucleon axial current and associated Goldberger-Treiman relations, and all technical details relating to current constructions in the quark+diquark approach, including the seagull terms.

Concerning the axial form factor, \( G_A \), we found that it can reliably be represented by a dipole form factor, normalised by the axial charge \( g_A = 1.25(3) \) and characterised by a mass-scale \( M_A = 1.23(3)m_N \), where \( m_N \) is the nucleon mass. Moreover, the \( Q^2 \)-behaviour of \( G_A \) is in good agreement with recent lattice-QCD (lQCD) results [32].

Regarding the induced pseudoscalar form factor, we found that \( G_P \) depends on the transferred momentum in such a way that it agrees favourably with data obtained from low-energy pion electroproduction [13] and also the lQCD results in Ref. [32]. Moreover, the pion pole dominance Ansatz provides a sound estimate of the directly computed result. Additionally, the induced pseudoscalar charge \( g^p_\pi \) is consistent with the value determined from a recent \( \mu \)-capture experiment [24].

With respect to \( G_S \), recent lQCD computations [32] agree with our theoretical result in the whole range of \( Q^2 \) studied. Furthermore, our prediction for the pion-nucleon coupling constant \( g_{\pi NN}/m_N = G_{\pi NN}(Q^2 = -m_X^2)/m_N = 14.02(33)/\text{GeV} \) agrees with a recent effective field theory analysis [115].

We proved that our nucleon’s axial, induced-pseudoscalar and pseudoscalar form factors analytically satisfy the PCAC relation and verified this numerically; establishing thereby that our theoretical approach is consistent with key symmetries in Nature.

It is here worth highlighting that we find \( g_A^d/g_A^s = -0.16(2) \) at the hadronic scale. This is a significant suppression of the magnitude of the \( d \)-quark component relative to that found in nonrelativistic quark models. The size reduction owes to the presence of strong diquark correlations in our nucleon wave function, with the calculated value reflecting the relative strength of scalar and axial-vector diquarks: the isoscalar–scalar correlations are dominant, but the isovector–axial-vector diquarks have a measurable influence.

Finally, as noted elsewhere [86], a natural next step is to improve upon the quark+diquark approximation and use the more fundamental three-quark Faddeev equation approach to the nucleon bound-state problem, extending the analysis in Ref. [76]. In finding and implementing an approach to improving the expressions of emergent hadronic mass in both the Faddeev kernel and interaction current, one could deliver continuum predictions for all nucleon form factors that posses tighter links to QCD’s Schwinger functions.

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Appendix A: The QCD-kindred framework

The QCD-kindred model for mesons and baryons that we exploit herein was introduced in Refs. [118–120] and refined in a series of subsequent analyses that may be traced from Refs. [83, 107]. Consistency between the various Schwinger function elements is guaranteed through their mutual interplay in the description and prediction of hadron observables. Combined with the material above, the information in this appendix is sufficient for an interested practitioner to implement the framework for themself.

1. Dressed quark propagator

The dressed-quark propagator is:

\[
S(p) = -i\gamma \cdot p \sigma V(p^2) + \sigma S(p^2) \quad (A.1a)
\]

\[
= 1/[i\gamma \cdot p A(p^2) + B(p^2)] . \quad (A.1b)
\]

Regarding light-quarks in QCD, the wave function renormalisation and dressed-quark mass:

\[
Z(p^2) = 1/A(p^2) , \quad M(p^2) = B(p^2)/A(p^2) , \quad (A.2)
\]

respectively, receive significant momentum-dependent corrections at infrared momenta [121–124]: \( Z(p^2) \) is suppressed and \( M(p^2) \) enhanced. These features are an expression of emergent hadronic mass [125].

An efficacious parametrisation of \( S(p) \), which exhibits the features described above, has been used extensively in hadron studies [50–54]. It is defined via

\[
\bar{\sigma}_S(x) = 2 \bar{m} F(2(x + \bar{m}^2))
\]
2. Correlation amplitudes

As mentioned in Sec. III B, for the nucleon we only need to include the two positive-parity diquark correlations: the isoscalar-scalar \((0^+\)) and isovector-pseudovector \((1^+\)) diquarks. Their dominant structures are:

\[
\Gamma^{0+}(k; K) = g_0 + \gamma_3 C \bar{t}^0 J_c F(k^2/\omega_{0^+}^2), \quad (A.6a)
\]

\[
\Gamma_\mu^{1+}(k; K) = ig_1 + \gamma_\mu C \bar{t}_f J_c F(k^2/\omega_{1^+}^2), \quad (A.6b)
\]

where: \(K\) is the total momentum of the correlation, \(k\) is a two-body relative momentum, \(\mathcal{F}\) is the function in Eq. (A.4), \(\omega_{J^P}\) is a size parameter, and \(g_{J^P}\) is a coupling into the channel, which is fixed by normalisation; \(C = \gamma_2\gamma_4\) is the charge-conjugation matrix; \(t^0\) and \(\bar{t} = (t^1, t^2, t^3)\) are the flavour matrices:

\[
t_0 = \frac{i}{\sqrt{2}} \tau^2, \quad (A.7a)
\]

\[
t_1 = \frac{1}{\sqrt{2}} (\tau^0 + \tau^3), \quad (A.7b)
\]

\[
t_2 = \frac{1}{\sqrt{2}} \tau^1, \quad (A.7c)
\]

\[
t_3 = \frac{1}{\sqrt{2}} (\tau^0 - \tau^3), \quad (A.7d)
\]

\(\tau^0 = \text{diag}[1, 1, 1; \{\tau^i, j = 1, 2, 3\}\) are the Pauli matrices; and \(\bar{t}_f = \pm\bar{t}_f\) are the Pauli matrices; \(\gamma_\mu\) are the Lorentz indices \(\mu, \nu\), the left-hand-side of Eq. (A.8b) also includes a factor \(\delta_{\mu\nu}\). Plainly, the coupling strength in each channel, \(g_{J^P}\) in Eq. (A.6), is fixed by the associated value of \(\omega_{J^P}\).

3. Diquark propagators and couplings

A propagator is associated with each quark+quark correlation in Fig. 2. We use [83]:

\[
\Delta_{0^+}(K) = \frac{1}{m_{0^+}^2} F(k^2/\omega_{0^+}^2), \quad (A.9a)
\]

\[
\Delta_{\mu\nu}^{1+}(K) = \left[ \delta_{\mu\nu} + \frac{K_\mu K_\nu}{m_{1^+}^2} \right] \frac{1}{m_{1^+}^2} F(k^2/\omega_{1^+}^2). \quad (A.9b)
\]

These forms ensure that the diquarks are confined within the baryons: whilst the propagators are free-particle-like at spacelike momenta, they are pole-free on the timelike
axis. This is enough to ensure confinement via the violation of reflection positivity (see, e.g., Ref. [129], Sec. 3). The diquark masses and widths are related via [83]

\[ m_{f \mu}^2 = 2 \omega_{f \mu}^2. \]  

This correspondence accentuates the diquarks’ free-particle-like propagation characteristics within the baryon.

Using Eq. (17) and Eqs. (A.6), (A.8), one finds

\[ g_{0+} = 14.8, \quad g_{1+} = 12.7, \]  

Since it is the coupling-squared which appears in the Faddeev kernels, 0+ diquarks will dominate the Faddeev amplitudes of J = 1/2 baryons; but 1+ diquarks must also play a measurable role because \( g_{1+}^2 / g_{0+}^2 \approx 0.7. \)

4. Faddeev amplitudes

Solving the Faddeev equation, Fig. 2, yields both the mass-squared and bound-state amplitude of all baryons with a given value of \( J^P. \) The form of the Faddeev amplitude fixes the channel. A baryon is described by

\[ \Psi^B = \psi_1^B + \psi_2^B + \psi_3^B, \]  

where the subscript identifies the bystander quark, viz. the quark that is not participating in a diquark correlation. \( \psi_{1,2,3}^B \) are obtained from \( \psi_3^B = \psi^B \) by cyclic permutations of all quark labels.

For the nucleon \((B = N),\)

\[ \psi^N(p_1, \sigma_1, \alpha_1) = \left[ \Gamma_{0+}(k; K) \right]^{\alpha_1 \alpha_2} \Psi_{\sigma_1 \sigma_2}^{0+}(K) \left[ \Psi_{\sigma_1}(\ell; P) \right]^{\alpha_2 \sigma_3} + \left[ \Gamma_{1+}(k; K) \right]^{\alpha_1 \alpha_2} \Delta^{1+}_{\sigma_1 \sigma_2}(K) \left[ \Psi_1^{1+}(\ell; P) \right]^{\alpha_2 \sigma_3} u(P), \]  

where \((p_1, \sigma_1, \alpha_1)\) are the momentum, spin and isospin labels of the quarks comprising the bound state; \( P = p_1 + p_2 + m_3 = p_4 + p_5 \) is the baryon’s total momentum; \( k = (p_1 - p_2) / 2, \) \( K = p_1 + p_2 = p_3 + \ell = (-K + 2p_3) / 3; \) and \( u(P) \) is a Euclidean spinor (see Ref. [83], Appendix B for details). The remaining terms in Eq. (A.13) are the following matrix-valued functions:

\[ \Phi^{0+}(\ell; P) = S(\ell + \eta P) \Delta^{0+}(\eta P - \ell) \Psi^{0+}(\ell; P) \]  

\[ = \sum_{k=1}^2 \tilde{s}_k(\ell^2, \ell \cdot P) S^k(\ell; P) \frac{\lambda_0}{\sqrt{3}} s^0_f, \]  

\[ \tilde{S}_{\nu}^{1+}(\ell; P) = S(\ell + \eta P) \Delta_{\ell \nu}^{1+}(\eta P - \ell) \Psi_{\nu}^{1+}(\ell; P) \]  

\[ = \sum_{k=1}^6 \tilde{a}_k(\ell^2, \ell \cdot P) \gamma_5 A_{\nu k}(\ell; P) \frac{\lambda_0}{\sqrt{3}} \tilde{s}^f, \]  

where \( \eta, \tilde{\eta} \in [0, 1] \) and \( \eta + \tilde{\eta} = 1. \)

Both the Faddeev amplitude and wave function are Poincaré covariant: each of the scalar functions that appears is frame independent, but the chosen frame determines just how the elements should be combined. Hence, the manner by which the dressed quarks’ orbital angular momentum \( L \) and spin \( S \) add to form a particular \( J^P \) combination is frame dependent: \( L \) and \( S \) are not independently Poincaré invariant.

The Dirac tensors of different \( J^P \) are:

\[ 2S : \right \} S^1, A_{\nu}^5, (A_{\nu}^3 + A_{\nu}^6); \]  

\[ 2P : \right \} S^2, A_{\nu}^5, (A_{\nu}^3 + A_{\nu}^6); \]  

\[ 4P : \right \} (2A_{\nu}^4 - A_{\nu}^5)/3; \]  

\[ 4D : \right \} (2A_{\nu}^4 - A_{\nu}^5)/3; \]  

\[ \nu \]  

viz. the scalar functions associated with these Dirac matrix combinations in a Faddeev wave function possess the indicated angular momentum correlation between the quark and diquark. Those functions are:

\[ 2S : \right \} \tilde{s}_1, \tilde{s}_2, (\tilde{s}_3 + 2\tilde{s}_5)/3; \]  

\[ 2P : \right \} \tilde{s}_2, \tilde{s}_1, (\tilde{s}_3 + 2\tilde{s}_5)/3; \]  

\[ \nu \]
4P : \( \hat{a}_4 - \hat{a}_5 \); \hspace{1cm} (A.19c)
4D : \( \hat{a}_3 - \hat{a}_5 \). \hspace{1cm} (A.19d)

Appendix B: Current diagrams

To compute the form factors, one must specify how the probe couples to the constituents of the composite hadrons. Herein, this amounts to specifying the couplings of the axial (5\(\mu\)) or pseudoscalar (5) current to the dressed quarks and diquarks.

In Fig. 3, we have separated the different contributions to the currents into six terms. \(N.B.\) Diagrams 1, 2 and 3 represent the impulse-approximation: they are one-loop integrals, which we evaluate by Gaussian quadrature. The remainder, Diagram 4, 5 and 6 describe the probe’s coupling to the Faddeev kernel: they are two-loop diagrams, whose evaluation requires the use of Monte-Carlo methods. For explicit calculations, we work in a Breit frame: \( P_i = K - Q/2, \) \( P_f = P + Q/2 \) and \( K = (0, 0, 0, i\sqrt{m_N^2 + Q^2}/4) \).

1. Diagram 1

Probe coupling directly to the bystander quark:

\[ J_{5(\mu)}^{q1}(K, Q) = J_{5(\mu)}^{q,0^+(K, Q)} + J_{5(\mu)}^{q,1^+(K, Q)} , \] \hspace{1cm} (B.1)

with

\[ J_{5(\mu)}^{q,0^+}(K, Q) = \int_p \tilde{\Psi}^0(p_f'; -P_f)S(p_{q+}, p_{q-}) \Gamma_{5(\mu)}^q(p_{q+}, p_{q-}) \times S(p_{q-}) \Delta^{0^+}(p_d) \Psi^{0^+}(p_l'; P_1), \] \hspace{1cm} (B.2)

and

\[ J_{5(\mu)}^{q,1^+}(K, Q) = \int_p \tilde{\Psi}^1(p_f'; -P_f)S(p_{q+}, p_{q-}) \Gamma_{5(\mu)}^q(p_{q+}, p_{q-}) \times S(p_{q-}) \Delta^{1^+}(p_d) \Psi^{1^+}(p_l'; P_1), \] \hspace{1cm} (B.3)

where \( \Gamma_{5(\mu)}^q \) is the dressed-quark’s pseudoscalar (axial-vector) vertex, Eq. (30) or (29); and, with \( p \) the loop momentum:

\[ p_1' = p - \eta Q/2, \] \hspace{1cm} (B.4a)
\[ p_f' = p + \eta Q/2, \] \hspace{1cm} (B.4b)
\[ p_{q-} = p_1' + \eta P_i, \] \hspace{1cm} (B.4c)
\[ p_{q+} = p_f' + \eta P_f, \] \hspace{1cm} (B.4d)
\[ p_d = \hat{\eta} P_i - p_f'. \] \hspace{1cm} (B.4e)

2. Diagram 2

Probe coupling to an axial-vector diquark correlation:

\[ J_{5(\mu),aa}^{dq}(K, Q) = \int_p \tilde{\Psi}^{1^+}(p_f'; -P_f) \Delta^{0^+}(p_d) \times \Gamma_{5(\mu),aa}^{so}(p_{d+}, p_{d-}) \Delta^{1^+}(p_d) S(p_q) \Psi^{1^+}(p_l'; P_1), \] \hspace{1cm} (B.5)

where \( \Gamma_{5(\mu),aa}^{so} \) is the axial-vector diquark correlation’s pseudoscalar (axial-vector) vertex, Eq. (62) or (63); and the momenta are

\[ p_1'' = p + \eta Q/2, \] \hspace{1cm} (B.6a)
\[ p_f'' = p - \eta Q/2, \] \hspace{1cm} (B.6b)
\[ p_{d-} = \hat{\eta} P_i - p_f'', \] \hspace{1cm} (B.6c)
\[ p_{d+} = \hat{\eta} P_f - p_f'', \] \hspace{1cm} (B.6d)
\[ p_q = p_1'' + \eta P_i \] \hspace{1cm} (B.6e)
\[ = p_f'' + \eta P_f. \] \hspace{1cm} (B.6f)

3. Diagram 3

Probe-induced transition between scalar and axial-vector diquarks. It is the sum of the following transitions: charged currents, \{\(dd\)\}_{1+} \leftrightarrow \{\(uu\)\}_{1+} \leftrightarrow \{\(ud\)\}_{0+}; and neutral currents, \{\(ud\)\}_{1+} \leftrightarrow \{\(ud\)\}_{0+}, where

\[ J_{5(\mu),as}^{dq}(K, Q) = \int_p \tilde{\Psi}^{1^+}(p_f'; -P_f) \Delta^{0^+}(p_d) \times \Gamma_{5(\mu),as}^{so}(p_{d+}, p_{d-}) \Delta^{1^+}(p_d) S(p_q) \Psi^{1^+}(p_l'; P_1), \] \hspace{1cm} (B.7)

and

\[ J_{5(\mu),as}^{dq}(K, Q) = \int_p \tilde{\Psi}^{1^+}(p_f'; -P_f) \Delta^{0^+}(p_d) \times \Gamma_{5(\mu),as}^{so}(p_{d+}, p_{d-}) \Delta^{1^+}(p_d) S(p_q) \Psi^{1^+}(p_l'; P_1), \] \hspace{1cm} (B.8)

where \( \Gamma_{5(\mu),as}^{so} \) and \( \Gamma_{5(\mu),as}^{so} \) are the probe-induced transition vertices in Eqs. (64), (65).

4. Diagram 4

Probe coupling to the quark that is exchanged as one diquark breaks up and another is formed:

\[ J_{5(\mu)}^k(K, Q) = \sum \int_{p_{k+}, p_{k-}} \int_p \tilde{\Psi}^{j_1p_1}(\tilde{k}_f) \Gamma_{j_1p_1}(\tilde{k}_f) \] \hspace{1cm} (B.9)

\footnote{Owing to their flavour structures, neither the axial-vector nor the pseudoscalar probe couples to the scalar diquark, see Sec. III C 4.}
write the nucleon’s Faddeev equation, pictured in Fig. 2:

\[ \times \{ S(q') \Gamma_{J^P}^{2(J^P)}(q', \bar{q}) S(q) \}^T \Gamma_{J^P}^{2(J^P)} (p', \Phi_{J^P}) \, . \quad (B.9) \]

The process of quark exchange provides the attraction required in the Faddeev equation to bind the nucleon. It also ensures the Faddeev amplitude has the correct antisymmetry under the exchange of any two dressed-quarks. These essential features are absent in models with elementary (noncomposite) diquarks. The complete contribution is obtained by summing over \( J^P_{1} \) and \( J^P_{2} \), which can each take the values 0\(^+\), 1\(+\).

5. Diagrams 5 and 6

Two-loop seagull diagrams appearing as partners to Diagram 4:

\[ J_{5(\mu)}^{sg}(K, Q) = \sum_{J^P_{1}, J^P_{2}=0,+,1+} \int \int \Phi_{J^P_{1}, J^P_{2}} (p, q, \Phi_{J^P_{1}}) \times S^T(q') \Gamma_{J^P_{1} J^P_{2}} (p', q', \Phi_{J^P_{1}}) \, . \quad (B.10) \]

for Diagram 5; and

\[ J_{5(\mu)}^{sg}(K, Q) = \sum_{J^P_{1}, J^P_{2}=0,+,1+} \int \int \Phi_{J^P_{1}, J^P_{2}} (p, q, \Phi_{J^P_{1}}) \times S^T(q') \Gamma_{J^P_{1} J^P_{2}} (p', q', \Phi_{J^P_{1}}) \, . \quad (B.11) \]

for Diagram 6; the momenta are:

\[ \begin{align*}
\bar{p}_q &= k + \eta P_f , \quad (B.12a) \\
\bar{p}_{q'} &= p + \eta P_f , \quad (B.12b) \\
k_1 &= \frac{\bar{p}_q - \bar{q}'}{2} , \quad (B.12c) \\
k_2 &= \frac{\bar{p}_{q'} - \bar{q}'}{2} . \quad (B.12d)
\end{align*} \]

and, again, \( J^P_{1} \) and \( J^P_{2} \) are summed.

Appendix C: Colour and flavour coefficients

The nucleon’s Faddeev equation and the current diagrams of Appendix B need to be augmented with appropriate colour and flavour coefficients. Using the colour and flavour matrices of the diquark amplitudes, Eqs. (A.6), and the quark–diquark amplitudes, Eqs. (A.14), projecting them onto the isospinors of the proton \( p = (1, 0)^T \) or the neutron \( n = (0, 1)^T \), we can write the nucleon’s Faddeev equation, pictured in Fig. 2:

\[ \begin{pmatrix}
\Psi^0_p (p; P) \\
\Psi^1_p (p; P)
\end{pmatrix}
= \int \begin{pmatrix}
\left(-\frac{1}{2}\right) \mathcal{K}^{0+0+}_v (\mathcal{X}^{0+}_v) \mathcal{K}^{1+0+}_v \\
\left(-\frac{1}{2}\right) \mathcal{K}^{1+0+}_v (\mathcal{X}^{1+}_v)
\end{pmatrix}
\begin{pmatrix}
\Phi^0_k (k; P) \\
\Phi^1_k (k; P)
\end{pmatrix} , \quad (C.1)
\]

where \( -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2} \) are the associated colour-flavour coefficients:

\[ \mathcal{K}^{J^P_{1}} J^P_{2} \equiv \mathcal{K}^{J^P_{1}} J^P_{2} (p, k, P) := \Gamma_{J^P_{1} J^P_{2}} (k_r) S^T(q) \Gamma_{J^P_{1}} (p_r) , \quad (C.2) \]

with the momenta

\[ \begin{align*}
\tilde{p}_q &= p + \eta P , \quad (C.3a) \\
\tilde{k}_q &= k + \eta P , \quad (C.3b) \\
\tilde{p}_d &= -p + \eta P , \quad (C.3c) \\
\tilde{k}_d &= -k + \eta P , \quad (C.3d) \\
q &= \tilde{p}_d - \tilde{k}_q , \quad (C.3e) \\
p_r &= \frac{\tilde{k}_q - q}{2} , \quad (C.3f) \\
k_r &= \frac{\tilde{p}_d - q}{2} . \quad (C.3g)
\end{align*} \]

For the form-factor diagrams of Fig. 3, their flavour factors are obtained by using Eqs. (A.7) and (A.16). We have

\[ \sum_{m,n=0}^3 \left[ \delta_{m,n} (s^m_f) \left( \frac{\tau^j_f}{2} \right) (s^m_f) \right] , \quad (C.4) \]

for the probe-quark diagram (Diagram 1):

\[ \sum_{m,n=0}^3 \left[ (s^m_f)^+ (s^m_f) 2tr_F \left[ (t^m_f)^+ (t^m_f) \right] \right] , \quad (C.5) \]

for the probe-diquark diagram (Diagram 2 and 3); and

\[ \sum_{m,n=0}^3 \left[ (s^m_f)^+ (t^m_f) \left( \frac{\tau^j_f}{2} \right) (s^m_f) \right] , \quad (C.6) \]

for the exchange diagram (Diagram 4). The seagull case is a little more complicated: we need to treat the bystander and exchange quarks’ legs separately. From the seagull diagram (Diagram 5) we have

\[ \sum_{m,n=0}^3 \left[ (s^m_f)^+ (t^m_f) \left( \frac{\tau^j_f}{2} \right) (s^m_f) \right] , \quad (C.7) \]

for the exchange leg; and

\[ \sum_{m,n=0}^3 \left[ (s^m_f)^+ \left( \frac{\tau^j_f}{2} \right) (t^m_f) (t^m_f)^+ (s^m_f) \right] , \quad (C.8) \]

for the bystander leg. From the diagram of the seagull’s conjugation (Diagram 6), we have

\[ \sum_{m,n=0}^3 \left[ (s^m_f)^+ (t^m_f) \left( \frac{\tau^j_f}{2} \right) (s^m_f) \right] , \quad (C.9) \]
for the exchange leg; and
\[
\sum_{m,n=0}^{3} \left[ (s_{p}^{m}) (t_{p}^{n}) (t_{p}^{n}) \left( \frac{r_{p}}{2} \right) (s_{p}^{m}) \right], \quad (C.10)
\]
for the bystander leg.

At last, again, we need to project these matrices, Eqs. (C.4)–(C.10), onto the isospinors of the proton \( p = (1,0)^{T} \) or the neutron \( n = (0,1)^{T} \) to obtain the flavour coefficients.

The colour factors of these diagrams are the same as for the electromagnetic case: for the impulse-approximation diagrams (Diagrams 1–3), their colour coefficients are \( \pm 1 \); and for the exchange or seagull diagrams (Diagrams 4–6), their colour coefficients are \( \pm 1 \).

**Appendix D: Proof of PCAC**

We verify here that the interaction current we have constructed is sufficient to ensure that the identity in Eq. (8) is preserved. Observe first that \( J_{5}^{\mu}(K,Q) \) and \( J_{5}^{\tilde{\mu}}(K,Q) \) are both a sum of six terms (drawn in Fig. 3):

\[
J_{5}^{\mu} = J_{5}^{\mu}(\mu) + J_{5}^{\mu}(\mu) + J_{5}^{\mu}(\mu) + J_{5}^{\mu}(\mu) + J_{5}^{\mu}(\mu) + J_{5}^{\mu}(\mu), \quad (D.1)
\]
each one of which must be considered. Note, too, that we will consider either the neutral (\( \tau^{+} \)) or the charged (\( \tau^{1\pm2} \) ) currents. In the isospin-symmetry limit, their flavour coefficients are the same.

**Diagram 1: flavour coupling to quark line**

For Diagram 1 in Fig. 3, contracting Eq. (B.2) with \( Q_{\mu} \) and using Eq. (18), we obtain\(^4\)

\[
Q_{\mu}J_{5}^{\rho_{0}+}(K,Q) + 2im_{q}J_{5}^{\rho_{0}+}(K,Q) = \frac{1}{2} \int p \tilde{\Psi}^{\rho_{0}+}(p_{f}^{i} - P_{f}) S(p_{q}^{+}) Q_{\mu} \Gamma_{\rho_{0}+} (p_{q}^{+}, p_{q}^{-}) + 2im_{q} \Gamma_{\rho_{0}+} (p_{q}^{+}, p_{q}^{-}) \Psi^{\rho_{0}+} (p_{f}^{i}; P_{f}) = \frac{1}{2} \int p \tilde{\Psi}^{\rho_{0}+}(p_{f}^{i} - P_{f}) i \Gamma_{\rho_{0}+}(p_{f}^{i}; P_{f}) + \frac{1}{2} \int p \tilde{\Psi}^{\rho_{0}+}(p_{f}^{i} - P_{f}) i \Gamma_{\rho_{0}+}(p_{f}^{i}; P_{f}), \quad (D.2)
\]
where \( \int p := \int (4\pi)^{4} \) and \( 1/2 \) is the colour-flavour coefficient computed via Eq. (C.4). The Dirac parts of the dressed-quark’s axial-vector and pseudoscalar vertices (19) and (20) are denoted by \( \Gamma_{5\mu} \) and \( \Gamma_{5} \): \( \Gamma_{5\mu}^{\rho_{0}} \), and \( \Gamma_{5}^{\rho_{0}} := (\tau^{\rho_{0}}/2)\Gamma_{5\mu} \) and \( \Gamma_{5}^{\rho_{0}} := (\tau^{\rho_{0}}/2)\Gamma_{5} \).

Next, using the nucleon’s Faddeev equation, Eq. (C.1), to substitute \( \Psi^{0+}(p_{f}^{i} - P_{f}) \) and \( \Psi^{0+}(p_{f}^{i}; P_{f}) \) in Eq. (D.2), some algebra leads to

\[
Q_{\mu}J_{5}^{\rho_{0}+}(K,Q) + 2im_{q}J_{5}^{\rho_{0}+}(K,Q) = (-\frac{1}{2}) \int p \int \Phi_{f}^{\rho_{0}+}(\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) i \gamma_{5} \Phi_{f}^{\rho_{0}+} + (-\frac{1}{2}) \int p \int \Phi_{f}^{\rho_{0}+} i \gamma_{5} \Phi_{f}^{\rho_{0}+} \Gamma_{\rho_{0}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{0}+} + \sqrt{\frac{3}{4}} \int p \int \Phi_{f}^{\rho_{0}+} i \gamma_{5} \Phi_{f}^{\rho_{0}+} \Gamma_{\rho_{0}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{0}+} + \sqrt{\frac{3}{4}} \int p \int \Phi_{f}^{\rho_{0}+} i \gamma_{5} \Phi_{f}^{\rho_{0}+} \Gamma_{\rho_{0}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{0}+}, \quad (D.3)
\]
where we use the abbreviations \( \Phi_{f}^{\rho_{0}+} \equiv \Phi_{f}^{\rho_{0}+} (p_{f}^{i} - P_{f}), \Phi_{f}^{\rho_{0}+} \equiv \Phi_{f}^{\rho_{0}+} (k_{f}^{i}; P_{f}), \Psi_{f}^{0+} \equiv \Psi_{f}^{0+} (p_{f}^{i} - P_{f}) \) and \( \Psi_{f}^{0+} \equiv \Psi_{f}^{0+} (k_{f}^{i}; P_{f}); \) and the momenta are

\[
\bar{k}_{r} = \frac{1}{2} ((k + \eta Q) + p + (3\eta - 1) P_{f}), \quad (D.4a) \]

\[
\bar{p}_{r} = \frac{1}{2} (p + 2(k + \eta Q) + (3\eta - 1) P_{f}), \quad (D.4b)
\]

\[
\bar{q} = -p - (k + \eta Q) + (1 - 2\eta) P_{f}, \quad (D.4c)
\]

\[
\bar{k}_{r}^{'} = \bar{k}_{r} - Q, \quad (D.4d)
\]

\[
\bar{q}^{'} = \bar{q} + Q. \quad (D.4f)
\]

Similarly, for the axial-vector part, i.e. \( J_{5}^{\rho_{1}+} \) and \( J_{5}^{\rho_{1}+} \) in Eq. (B.3), we obtain

\[
Q_{\mu}J_{5}^{\rho_{1}+}(K,Q) + 2im_{q}J_{5}^{\rho_{1}+}(K,Q) = (-\frac{3}{12}) \int p \int \Phi_{f}^{\rho_{1}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) i \gamma_{5} \Phi_{f}^{\rho_{1}+} + (-\frac{3}{12}) \int p \int \Phi_{f}^{\rho_{1}+} \Gamma_{\rho_{1}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{1}+} + (-\frac{1}{12}) \int p \int \Phi_{f}^{\rho_{1}+} \Gamma_{\rho_{1}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{1}+} + (-\frac{1}{12}) \int p \int \Phi_{f}^{\rho_{1}+} \Gamma_{\rho_{1}+} (\bar{k}_{r}) S^{T}(q)(\bar{p}_{r}) \Phi_{f}^{\rho_{1}+}, \quad (D.5)
\]

Adding together Eqs. (D.3) and (D.5), and taking into account Eq. (B.1), we find

\[
Q_{\mu}J_{5}^{\rho_{0}+}(K,Q) + 2im_{q}J_{5}^{\rho_{0}+}(K,Q) = \sum_{J_{1}^{p},J_{2}^{p}=0,1+} (Q_{\mu}J_{5}^{\rho_{1}+} J_{1}^{p} J_{2}^{p} + 2im_{q}J_{5}^{\rho_{1}+} J_{1}^{p} J_{2}^{p}), \quad (D.6)
\]

\(^{4}\) For the sake of completeness, we will write out all the colour-flavour factors explicitly.
where
\[ Q_\mu J^{\alpha_0+1+}_5 + 2i m_q J^{\alpha_{q,a}}_5 = \int_p \int_{k_f} \Phi^{\alpha}_{f} \left[ \left( -\frac{1}{4} \right) \Gamma^\alpha_{\beta}(\hat{k}_r) S^T(\hat{q}_T) \Gamma^\beta_{\alpha}(\hat{p}_r) i\gamma_5 + \left( 0 \right) \right] \Phi^{\beta}_{i} \] \tag{D.7}
\[ Q_\mu J^{\alpha_{q,a}}_5 + 2i m_q J^{\alpha_{q,a}}_5 = \int_p \int_{k_f} \Phi^{\alpha}_{f} \left[ \left( -\frac{3}{4} \right) \Gamma^\alpha_{\beta}(\hat{k}_r) S^T(\hat{q}_T) \Gamma^\beta_{\alpha}(\hat{p}_r) i\gamma_5 + \left( \frac{1}{12} \right) \right] \Phi^{\beta}_{i} \] \tag{D.8}
and
\[ Q_\mu J^{\alpha_{q,a}}_5 + 2i m_q J^{\alpha_{q,a}}_5 = \int_p \int_{k_f} \Phi^{\alpha}_{f} \left[ \left( -\frac{1}{12} \right) \Gamma^\alpha_{\beta}(\hat{k}_r) S^T(\hat{q}_T) \Gamma^\beta_{\alpha}(\hat{p}_r) i\gamma_5 + \left( \frac{1}{12} \right) \right] \Phi^{\beta}_{i} \] \tag{D.9}

Diagrams 2 and 3: current coupling to diquark line

For diagram 2, using Eq. (B.5) and the corresponding AXWTI (66a), we have
\[ Q_\mu J^{\alpha_{q,a}}_5 + 2i m_q J^{\alpha_{q,a}}_5 = \left( \frac{2}{3} \right) \int_p \Psi^{\alpha}_{\alpha_f} (\hat{q}_f') - P_f \Delta^{\alpha_{q,a}}_{\alpha_f} (p_d) \right] Q_\mu \Gamma^{\alpha_{q,a}}_{\mu,\alpha_f} (p_d) + 2i m_q \Gamma^{\alpha_{q,a}}_{\mu,\alpha_f} (p_d +, p_d -) \Delta^{\alpha_{q,a}}_{\beta} (p_d) S (p_q) \Psi^{\beta}_{\beta_f} (p_q') P_i = 0 \] \tag{D.10}

Similarly, using Eqs. (B.7), (B.8) and the corresponding AXWTI, Eq. (66b), the 0–1 transition of diagram 3 is expressed by
\[ Q_\mu J^{\alpha_{q,a}}_5 + 2i m_q J^{\alpha_{q,a}}_5 = \left( \frac{\sqrt{3}}{3} \right) \int_p \Psi^{\alpha}_{\alpha_f} (\hat{q}_f') - P_f \Delta^{\alpha_{q,a}}_{\alpha_f} (p_d +) \right] Q_\mu \Gamma^{\alpha_{q,a}}_{\mu,\alpha_f} (p_d +, p_d -) \Delta^{\alpha_{q,a}}_{\beta} (p_d) S (p_q) \Psi^{\beta}_{\beta_f} (p_q') P_i = 0 \] \tag{D.11}

The flavour coefficients are calculated via Eq. (C.6).
Diagram 5: first seagull contribution

For the seagull diagram 5, using Eq. (B.10) and the seagull’s AXWTHI, Eq. (57), we get

\[ Q_{\mu}J_{5\mu}^{\text{sg}}(K, Q) + 2im_qJ_{5\mu}^{\text{sg}}(K, Q) \]

= \sum_{J_1^\gamma, J_2^\gamma = 0, +1} \left( Q_{\mu}J_{5\mu}^{\text{sg},i J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\text{sg}, i J_1^\gamma J_2^\gamma} \right),

(D.19)

where

\[ Q_{\mu}J_{5\mu}^{\text{sg},0^+0^+} + 2im_qJ_{5}^{\text{sg},0^+0^+} \]

= \int_p \int_k \Phi_1^{0^+} \left( \frac{1}{4} i \gamma_5 \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) + \right.

\left. \left( -\frac{1}{4} \Gamma^{0^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) \right) \Phi_1^{0^+}_{\gamma,i} \right);

(D.20)

\[ Q_{\mu}J_{5\mu}^{\text{sg},0^+1^+} + 2im_qJ_{5}^{\text{sg},0^+1^+} \]

= \int_p \int_k \Phi_0^{1^+} \left( \frac{\sqrt{3}}{4} i \gamma_5 \Gamma^{1^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) + \right.

\left. \left( -\frac{\sqrt{3}}{12} \Gamma^{1^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) \right) \Phi_{\gamma,i}^{1^+} \right);

(D.21)

\[ Q_{\mu}J_{5\mu}^{\text{sg},1^+0^+} + 2im_qJ_{5}^{\text{sg},1^+0^+} \]

= \int_p \int_k \Phi_0^{1^+} \left( \frac{\sqrt{3}}{12} i \gamma_5 \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) + \right.

\left. \left( -\frac{\sqrt{3}}{12} \Gamma^{0^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) \right) \Phi_{\gamma,i}^{0^+} \right);  \quad (D.22)

and

\[ Q_{\mu}J_{5\mu}^{\text{sg},1^+1^+} + 2im_qJ_{5}^{\text{sg},1^+1^+} \]

= \int_p \int_k \Phi_1^{1^+} \left( \frac{1}{12} i \gamma_5 \Gamma^{1^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) + \right.

\left. \left( -\frac{5}{12} \Gamma^{1^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) \right) \Phi_{\gamma,i}^{1^+} \right).  \quad (D.23)

The colour/flavour coefficients in the first lines of Eqs. (D.20)–(D.23) are calculated via Eq. (C.8), i.e. the bystander legs of the seagulls; and the coefficients in the second lines are calculated via Eq. (C.7), the exchange legs of the seagulls.

Diagram 6: second seagull contribution

For the conjugated seagull contribution, diagram 6, using Eq. (B.11) and the AXWTHI of the seagulls’ conjugations, one finds

\[ Q_{\mu}J_{5\mu}^{\text{sg}}(K, Q) + 2im_qJ_{5}^{\text{sg}}(K, Q) \]

= \sum_{J_1^\gamma, J_2^\gamma = 0, +1} \left( Q_{\mu}J_{5\mu}^{\text{sg},i J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\text{sg}, i J_1^\gamma J_2^\gamma} \right),

(D.24)

where

\[ Q_{\mu}J_{5\mu}^{\text{sg},0^+0^+} + 2im_qJ_{5}^{\text{sg},0^+0^+} \]

= \int_p \int_k \Phi_1^{0^+} \left( \frac{1}{4} i \gamma_5 \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) + \right.

\left. \left( -\frac{1}{4} \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) \right) \Phi_{\gamma,i}^{0^+} \right);  \quad (D.25)

\[ Q_{\mu}J_{5\mu}^{\text{sg},0^+1^+} + 2im_qJ_{5}^{\text{sg},0^+1^+} \]

= \int_p \int_k \Phi_{\alpha,f}^{1^+} \left( \frac{\sqrt{3}}{4} \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) + \right.

\left. \left( -\frac{\sqrt{3}}{12} \Gamma^{0^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) \right) \Phi_{\alpha,f}^{1^+} \right);  \quad (D.26)

\[ Q_{\mu}J_{5\mu}^{\text{sg},1^+0^+} + 2im_qJ_{5}^{\text{sg},1^+0^+} \]

= \int_p \int_k \Phi_{\alpha,f}^{1^+} \left( \frac{\sqrt{3}}{12} \Gamma^{0^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) + \right.

\left. \left( -\frac{5}{12} \Gamma^{0^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{0^+}(\bar{p}_r) \right) \Phi_{\alpha,f}^{0^+} \right);  \quad (D.27)

and

\[ Q_{\mu}J_{5\mu}^{\text{sg},1^+1^+} + 2im_qJ_{5}^{\text{sg},1^+1^+} \]

= \int_p \int_k \Phi_{\alpha,f}^{1^+} \left( \frac{1}{12} i \gamma_5 \Gamma^{1^+}(\bar{k}_S) S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) + \right.

\left. \left( -\frac{5}{12} \Gamma^{1^+}(\bar{k}_S) i \gamma_5 S^T(\bar{q}) \Gamma^{1^+}(\bar{p}_r) \right) \Phi_{\alpha,f}^{1^+} \right).  \quad (D.28)

The colour/flavour coefficients in the first lines of Eqs. (D.25)–(D.28) are calculated via Eq. (C.10), i.e. the bystander legs of the seagulls’ conjugations; and the coefficients in the second lines are calculated via Eq. (C.9), the exchange legs.

Sum of all contributions

Using Eqs. (D.1), (D.6), (D.11)–(D.14), (D.19) and (D.24), it is straightforward to obtain their sum:

\[ Q_{\mu}J_{5\mu}^{\prime}(K, Q) + 2im_qJ_{5}^{\prime}(K, Q) = \sum_{J_1^\gamma, J_2^\gamma = 0, +1} \left[ Q_{\mu}J_{5\mu}^{\prime, i J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\prime, i J_1^\gamma J_2^\gamma} \right] \]

+ \left[ Q_{\mu}J_{5\mu}^{\prime, e J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\prime, e J_1^\gamma J_2^\gamma} \right] \]

+ \left[ Q_{\mu}J_{5\mu}^{\prime, e J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\prime, e J_1^\gamma J_2^\gamma} \right] (K, Q) \]

+ \left[ Q_{\mu}J_{5\mu}^{\prime, e J_1^\gamma J_2^\gamma} + 2im_qJ_{5}^{\prime, e J_1^\gamma J_2^\gamma} \right] (K, Q) \]
where \( j = 3 \) for the neutral current, or \( j = 1 \pm i2 \) for the charged currents.

There are interesting features behind the details of this proof. First, Eqs. (D.11) – (D.13) explicitly show that the three processes of Diagrams 2 and 3 satisfy PCAC separately. This is natural, and expected, since the axial-vector and pseudoscalar seagull terms do not involve a diquark part, in contrast to their electromagnetic counterparts. Second, the contributions from the coupling to the quark leg are cancelled by the contributions from the bystander legs of the seagull terms, and their conjugations. Furthermore, the contributions of the quark-exchange diagrams are cancelled by the those from the exchange legs of the seagulls, and their conjugations. This is analogous to the electromagnetic case [102]. Third, in an analysis of Nambu–Jona-Lasinio like models, it was found that PCAC can be ensured by including an isoscalar-vector diquark correlation in the nucleon in addition to scalar and axial-vector diquarks [130]. Our analysis has shown that whilst this may be useful in some cases, it is not necessary.

Appendix E: Impacts of pion pole terms

Regarding \( G_A(Q^2) \), consider Eq. (6a). Evidently, \( G_A(Q^2) \) only receives contributions from axial-vector \( Q \)-transverse pieces of the nucleon current. Reviewing our construction, the axial-vector terms are Eqs. (29), (58), (60), (63), (65). Each of the associated pion-pole parts is \( Q \)-longitudinal. Thus, \( G_A(Q^2) \) is unaffected by the pion-pole terms in our current construction. It is completely determined by the regular parts alone. This is apparent in Fig. 5.

Turn now to Eq. (9), an identity valid \( \forall Q^2 \). In the chiral limit, it entails:

\[
G_P(Q^2) = \frac{4m_N^2}{Q^2} G_A(Q^2) .
\]  

(E.1)

This is an exact statement of chiral symmetry and the pattern by which it is broken at the level of the nucleon. It is true when using any and all symmetry-consistent current constructions; consequently, true in our formulation.

Eq. (E.1) states that \( G_A \) and \( G_P \) are not independent functions in the chiral limit. Instead, the behaviour of \( G_A \) is completely determined by that of \( Q^2 G_P(Q^2) \) and vice versa. This can be understood at a deeper level by recognising that Eq. (E.1) is equivalent to the symmetry constraint

\[
\text{tr}_Q Q_\mu \gamma_5 J_{5\mu}(Q) = 0.
\]  

(E.2)

In words, this means that the current, \( J_{5\mu}(Q) \), may contain both regular and \( Q_\mu / Q^2 \) pole terms; but when contracted with \( Q_\mu \), the resulting regular and pole pieces combine to cancel exactly. An analogous process takes place at the dressed-quark level in QCD, as shown for the dressed-quark+axial-vector vertex in Ref. [106, Eqs. (10)- (13)].

Consequently, in the chiral limit, \( J_{5\mu}(Q) \) is purely \( Q \)-transverse. Further, the \( 1/Q^2 \) in Eq. (E.1) is completely absorbed into the transverse projection operator in Eq (6a). One may interpret this as follows: in the chiral limit, the pion pole contributions to the nucleon axial-vector current serve to cancel any and all longitudinal axial-vector couplings of the nucleon, ensuring that only \( Q \)-transverse interactions survive. In this sense, the essentially dynamical chiral-limit \( Q^2 \)-dependence of \( G_P(Q^2) \) is also completely determined by the regular parts of our current construction, with the \( 1/Q^2 \) in Eq. (E.1) expressing what is effectively a kinematic factor.

At physical values of the light-quark current masses, Eq. (E.1) returns to the identity in Eq. (9). In consequence, the nucleon axial-vector current is no longer purely \( Q \)-transverse. It possesses a \( Q \)-divergence that is measured by \( 2m_q G_5(Q^2) \). This means that, at physical current-quark masses, the pion pole contribution to the nucleon axial-vector current is insufficient to completely cancel all longitudinal axial-vector couplings of the nucleon; and the surviving longitudinal components are measured by the pseudoscalar form factor, \( G_5(Q^2) \).

Focusing now on \( m_q G_5(Q^2) \), which is obtained using Eq. (6c), one notes that every term which can contribute to the \( J_5(Q) \) current includes an overall multiplicative factor of \( m_q^2/[Q^2 + m_q^2] \). The associated (pole) pieces are represented in Eqs. (30), (59), (61), (62), (64). The form of Eq. (30) is fixed by DCSB at the dressed-quark level. The forms of Eqs. (59), (61), (62), (64) are determined by the axial-vector Ward-Green-Takahashi identity at the nucleon level, compliance with which is itself driven by the dressed-quark-level expression of DCSB. Of these five terms, Eq. (30), which generates the diagram 1 part in Fig. 3, produces the dominant contribution at all values of \( Q^2 \). All other contributions are soft, providing symmetry-constrained contributions on \( Q^2 \approx 0 \), but dropping away more rapidly than the Eq. (30) contributions as \( Q^2 \) increases. Thus, ratios of the type \( |n/column 1|, n = 2, \ldots, 6 \), formed from the entries in row 3 of Table I, provide upper bounds on the magnitudes of every such \( G_5(Q^2) \)-related diagram contribution ratio \( \forall Q^2 \geq 0 \).

Eq. (9) also explains the success of the pion pole dominance (PPD) Ansatz – Eq. (74) – for \( G_P(Q^2) \) on \( Q^2 \geq 0 \). In fact, using Eq. (10) and the Goldberger-Treiman relation – Eq. (11), whose accuracy is displayed in Fig. 11B, Eq. (9) entails

\[
G_P(Q^2) = \frac{4m_\pi^2}{Q^2} G_A(Q^2) \times \left[ 1 - \frac{m_\pi^2}{Q^2 + m_\pi^2} \frac{G_{\pi NN}(Q^2)/G_{\pi NN}(0)}{G_A(Q^2)/G_A(0)} \right].
\]  

(E.3)

These identities are to be compared with the PPD as-
sumption, Eq. (74):
\[
G_P(Q^2) = \frac{4m^2}{Q^2 + m^2} G_A(Q^2) \quad \text{(E.4a)}
\]
\[
= \frac{4m^2}{Q^2} G_A(Q^2) \left[ 1 - \frac{m^2}{Q^2 + m^2} \right]. \quad \text{(E.4b)}
\]

Forming the ratio of these two expressions, one finds
\[
\delta_{G_P}(Q^2) := \frac{G_P(Q^2)}{G_P(Q^2)} - 1 = \frac{m^2}{Q^2} \left[ 1 - \frac{G_{\pi NN}(Q^2)/G_{\pi NN}(0)}{G_A(Q^2)/G_A(0)} \right]. \quad \text{(E.5a)}
\]
\[
= \frac{m^2}{Q^2} \left[ 1 - \frac{G_{\pi NN}(Q^2)/G_{\pi NN}(0)}{G_A(Q^2)/G_A(0)} \right]. \quad \text{(E.5b)}
\]

Now,
\[
\delta_{G_P}(Q^2 \approx 0) = \frac{1}{6} m^2 \left[ r_{\pi NN}^2 - r_A^2 \right], \quad \text{(E.6)}
\]
where \( r_{\pi NN}, r_A \) are the associated form factor radii.

Inserting typical values for these quantities, then
\[
|\delta_{G_P}(Q^2 \approx 0)| < 0.01. \quad \text{(E.7)}
\]

Furthermore, using the fact that both \( G_A(Q^2) \) and \( G_{\pi NN}(Q^2) \) possess the same power-law behaviour at large \( Q^2 \), and approximating each by a dipole function, Eqs. (70), (84), then
\[
\delta_{G_P}(Q^2) \approx m^2 \left[ 1 - \frac{\Lambda_{\pi NN}^2}{Q^2} \right]. \quad \text{(E.8)}
\]

Hence, the ratio very rapidly approaches unity as \( Q^2 \) increases. Additional details are presented in connection with Eq. (75).

Collecting these remarks, one arrives at the following observations. (a) \( G_A(Q^2) \) is regular on \( Q^2 \geq m^2 \) and receives no contributions from any term of the form \( Q^2/[Q^2 + m^2] \) in general and using our current construction. (b) The product \( Q^2 G_P(Q^2) \) is directly proportional to \( G_A(Q^2) \) in the chiral limit; hence, likewise, receives no pion pole contributions. At the physical current-quark mass, \( Q^2 G_P(Q^2) \) retains its regular part but receives, in addition, the same pion pole contributions as \( G_A(Q^2) \); these corrections vanish as \( m^2_n/[Q^2 + m^2_n] \). (c) The pion pole pieces of our current construction are largely lodged in \( G_{\pi NN}(Q^2) \). The feed-in to \( G_P(Q^2) \), required by Eq. (9), is indicated by Table I. Given that diagram 1 in Fig. 3 is always dominant, then the Table I ratios [column n/column 1, \( n = 2, \ldots, 6 \], are upper bounds on the magnitude of the given ratio \( \forall Q^2 \).
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