Operator amenability of Fourier–Stieltjes algebras

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Abstract

In this paper, we investigate, for a locally compact group $G$, the operator amenability of the Fourier-Stieltjes algebra $B(G)$ and of the reduced Fourier-Stieltjes algebra $B_r(G)$. The natural conjecture is that any of these algebras is operator amenable if and only if $G$ is compact. We partially prove this conjecture with mere operator amenability replaced by operator $C$-amenability for some constant $C < 5$. In the process, we obtain a new decomposition of $B(G)$, which can be interpreted as the non-commutative counterpart of the decomposition of $M(G)$ into the discrete and the continuous measures. We further introduce a variant of operator amenability — called operator Connes-amenability — which also takes the dual space structure on $B(G)$ and $B_r(G)$ into account. We show that $B_r(G)$ is operator Connes-amenable if and only if $G$ is amenable. Surprisingly, $B(\mathbb{F}_2)$ is operator Connes-amenable although $\mathbb{F}_2$, the free group in two generators, fails to be amenable.

Keywords: locally compact groups, amenability, Fourier–Stieltjes algebra, reduced Fourier–Stieltjes algebra, operator amenability, almost periodic functions.

2000 Mathematics Subject Classification: 22D25, 43A30, 46H25, 46L07, 46L89, 46M18, 47B47, 47L25, 47L50 (primary).

Introduction

In his now classic memoir \cite{johnson1972}, B. E. Johnson initiated the theory of amenable Banach algebras. The choice of terminology is motivated by \cite{johnson1972} Theorem 2.5: a locally compact group $G$ is amenable if and only if its group algebra $L^1(G)$ is an amenable Banach algebra. There are other Banach algebras associated with a locally compact $G$ which are as natural objects of study as $L^1(G)$, e.g. the measure algebra $M(G)$. If $G$ is discrete and amenable, then $M(G) = \ell^1(G) = L^1(G)$ is amenable by Johnson’s theorem. It was conjectured by A. T.-M. Lau and R. J. Loy that $M(G)$ is amenable only if $G$ is discrete and amenable (\cite{laualoy2003}), a conjecture that was ultimately confirmed by H. G. Dales, F. Ghahramani, and A. Ya. Helemski˘ı (\cite{dales2003}).

*Research supported by NSERC under grant no. 227043-00.
In [7], P. Eymard introduced, for an arbitrary locally compact $G$, its Fourier algebra $A(G)$ and its Fourier–Stieltjes algebra $B(G)$. If $G$ is abelian with dual group $\hat{G}$, then the Fourier and Fourier–Stieltjes transform, respectively, yield $A(G) \cong L^1(\hat{G})$ and $B(G) \cong M(\hat{G})$. Disappointingly, the amenability of $A(G)$ reflects the amenability of $G$ rather inadequately: there are compact groups $G$, e.g. $G = \text{SO}(3)$, for which $A(G)$ fails to be amenable ([12]). It would seem that the only locally compact groups $G$ for which $A(G)$ is known to be amenable are those which have a closed, abelian subgroup with finite index ([15] or [8]).

Being the predual of the group von Neumann algebra $\text{VN}(G)$, the Fourier algebra $A(G)$ has a canonical operator space structure. In [20], Z.-J. Ruan introduced a variant of amenability — called operator amenability — which takes the operator space structure of $A(G)$ into account. As it turns out, operator amenability is the “right” notion of amenability for $A(G)$ in the sense that it characterizes the amenable, locally compact groups: $A(G)$ is operator amenable if and only if $G$ is amenable ([20, Theorem 3.6]).

Let $C^*(G)$ and $C^*_r(G)$ denote the full and the reduced $C^*$-algebra of $G$, respectively. Then $B(G) = C^*(G)^*$ and the reduced Fourier–Stieltjes algebra $B_r(G) = C^*_r(G)^*$ also have canonical operator space structures turning them into completely contractive Banach algebras. It is thus natural to ask for which $G$, the algebras $B(G)$ and $B_r(G)$, respectively, are operator amenable ([21, Problem 32]). Since $B_r(G) = B(G) = M(\hat{G})$ for abelian $G$, [21, Theorem 1.1] suggests that this is the case if and only if $G$ is compact. We have not been able to prove this conjecture in full. However, if we replace mere operator amenability by what we shall call operator $C$-amenability: the Fourier–Stieltjes algebra $B(G)$ — and, equivalently, $B_r(G)$ — is operator $C$-amenable for some $C < 5$ if and only if $G$ is compact.

In [21], it was conjectured that, if we want to capture the amenability of a locally compact group $G$ in terms of an amenability condition for $B(G)$ or $B_r(G)$, this notion of amenability needs to take both the operator space and the dual space structure of $B(G)$ and $B_r(G)$ into account. We introduce such a notion — called operator Connes-amenability — and show that, indeed, $B_r(G)$ is operator Connes-amenable if and only if $G$ is amenable. Surprisingly, there are non-amenable, locally compact groups $G$ — including $F_2$ — for which $B(G)$ is operator Connes-amenable.

1 Completely contractive Banach algebras and operator amenability

Since there are now several expository sources on the theory of operator spaces available ([6], [19], and [28]), we refrain from introducing the basics of operator space theory. We will adopt the notation from [6]: in particular, $\hat{\otimes}$ stands for the projective tensor product.
of operator spaces and not of Banach spaces.

We briefly recall a few definitions and results from [20].

**Definition 1.1** A Banach algebra $\mathfrak{A}$ which is also an operator space is called *completely contractive* if the multiplication of $\mathfrak{A}$ is a completely contractive bilinear map.

Clearly, $\mathfrak{A}$ is completely contractive if and only if the multiplication of $\mathfrak{A}$ induces a complete contraction $\Delta : \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$.

**Examples**
1. For any Banach algebra $\mathfrak{A}$, the maximal operator space $\text{max} \mathfrak{A}$ is completely contractive.

2. If $\mathfrak{H}$ is a Hilbert space, then any closed subalgebra of $B(\mathfrak{H})$ is completely contractive.

3. We denote the $W^*$-tensor product by $\bar{\otimes}$. A *Hopf–von Neumann algebra* is a pair $(\mathfrak{M}, \nabla)$, where $\mathfrak{M}$ is a von Neumann algebra, and $\nabla$ is a co-multiplication: a unital, $w^*$-continuous, and injective $^*$-homomorphism $\nabla : \mathfrak{M} \to \mathfrak{M} \bar{\otimes} \mathfrak{M}$ which is co-associative, i.e. the diagram

$$
\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\nabla} & \mathfrak{M} \bar{\otimes} \mathfrak{M} \\
\downarrow \nabla & & \downarrow \nabla \otimes \text{id}_\mathfrak{M} \\
\mathfrak{M} \bar{\otimes} \mathfrak{M} & \xrightarrow{\text{id}_\mathfrak{M} \otimes \nabla} & \mathfrak{M} \bar{\otimes} \mathfrak{M} \bar{\otimes} \mathfrak{M}
\end{array}
$$

commutes. Let $\mathfrak{M}_*$ denote the unique predual of $\mathfrak{M}$. By [6] Theorem 7.2.4, we have $\mathfrak{M} \bar{\otimes} \mathfrak{M} \cong (\mathfrak{M}_* \bar{\otimes} \mathfrak{M}_*)^*$. Thus $\nabla$ induces a complete contraction $\nabla_* : \mathfrak{M}_* \bar{\otimes} \mathfrak{M}_* \to \mathfrak{M}_*$ turning $\mathfrak{M}_*$ into a completely contractive Banach algebra.

4. Let $G$ be a locally compact group, and let $W^*(G) := C^*(G)^{**}$. There is a canonical $w^*$-continuous unitary representation $\omega : G \to W^*(G)$, the *universal representation* of $G$, with the following universal property: for any representation (always WOT-continuous and unitary) $\pi$ of $G$ on a Hilbert space, there is unique $w^*$-continuous $^*$-homomorphism $\theta : W^*(G) \to \pi(G)^{**}$ such that $\pi = \theta \circ \omega$. Applying this universal property to the representation

$$
G \to W^*(G) \bar{\otimes} W^*(G), \quad x \mapsto \omega(x) \otimes \omega(x)
$$

yields a co-multiplication $\nabla : W^*(G) \to W^*(G) \bar{\otimes} W^*(G)$. Hence, $B(G) := C^*(G)^*$, the *Fourier–Stieltjes* algebra of $G$, is a completely contractive Banach algebra. Since $B_r(G)$ and $A(G)$ are closed ideals of $B(G)$ (see [7]), they are also completely contractive Banach algebras. (It is not hard to see that the operator space structures on $B_r(G)$ and $A(G)$ inherited from $B(G)$ coincide with those they have as the preduals of $C_r^*(G)^{**}$ and $\text{VN}(G)$, respectively.)
Definition 1.2 Let $\mathfrak{A}$ be a completely contractive Banach algebra. An operator $\mathfrak{A}$-bimodule $E$ is an $\mathfrak{A}$-bimodule $E$ which is also an operator space such that the module actions

$$\mathfrak{A} \times E \to E, \quad (a, x) \mapsto a \cdot x$$

and

$$E \times \mathfrak{A} \to E, \quad (x, a) \mapsto x \cdot a$$

are completely bounded.

Similarly, one defines left and right operator $\mathfrak{A}$-modules. If $E$ is a left and $F$ is a right operator $\mathfrak{A}$-module, then $E \otimes F$ becomes an operator $\mathfrak{A}$-bimodule in a canonical fashion via

$$a \cdot (x \otimes y) := a \cdot x \otimes y$$

and

$$(x \otimes y) \cdot a := x \otimes y \cdot a \quad (a \in \mathfrak{A}, x \in E, y \in F).$$

In particular, $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ is an operator $\mathfrak{A}$-bimodule in a canonical way.

For any operator $\mathfrak{A}$-bimodule $E$, its dual module $E^*$ is also an operator $\mathfrak{A}$-bimodule. We shall call an operator $\mathfrak{A}$-bimodule $E$ dual if it is of the form $E = (E_*)^*$ for some operator $\mathfrak{A}$-bimodule $E_*$.

Definition 1.3 A completely contractive Banach algebra $\mathfrak{A}$ is called operator amenable if every completely bounded derivation from $\mathfrak{A}$ into a dual operator $\mathfrak{A}$-bimodule is inner.

There is an intrinsic characterization of amenable Banach algebras in terms of approximate diagonals ([11]). This characterization has an analogue for operator amenable, completely contractive Banach algebras ([20, Proposition 2.4]):

Theorem 1.4 The following are equivalent for a completely contractive Banach algebra $\mathfrak{A}$:

(i) $\mathfrak{A}$ is operator amenable.

(ii) There is an approximate operator diagonal for $\mathfrak{A}$, i.e. a bounded net $(d_\alpha)_\alpha$ in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad \text{and} \quad a \Delta d_\alpha \to a \quad (a \in \mathfrak{A}).$$

(iii) There is a virtual operator diagonal for $\mathfrak{A}$, i.e. an element $D \in (\mathfrak{A} \hat{\otimes} \mathfrak{A})^{**}$ such that

$$a \cdot D = D \cdot a \quad \text{and} \quad a \Delta^{**} D = a \quad (a \in \mathfrak{A}).$$

In analogy with the classical situation, Theorem 1.4 allows for a refinement of the notion of operator amenability:

Definition 1.5 Let $C \geq 1$. A completely contractive Banach algebra $\mathfrak{A}$ is called operator $C$-amenable if there is an approximate operator diagonal for $\mathfrak{A}$ bounded by $C$. 
Example  For any amenable, locally compact group $G$, the Fourier algebra $A(G)$ is operator 1-amenable (this is implicitly shown in [20]).

We conclude this preliminary section with a lemma, which is the operator analogue of a classical result ([21, Theorem 2.3.7]); given Theorem 1.4, the proof from [21] carries over with the obvious modifications:

**Lemma 1.6** Let $\mathfrak{A}$ be an operator amenable, completely contractive Banach algebra. Then the following are equivalent for a closed ideal $I$ of $\mathfrak{A}$:

(i) $I$ is operator amenable.

(ii) $I$ has a bounded approximate identity.

(iii) $I$ is completely weakly complemented, i.e. there is a completely bounded projection from $\mathfrak{A}^*$ onto $I^\perp$.

Remark  Of course, (iii) is satisfied whenever $I$ is completely complemented, i.e. if there is a completely bounded projection from $\mathfrak{A}$ onto $I$.

2  **A decomposition for $B(G)$**

Let $G$ be a locally compact group. Then we have a direct sum decomposition $M(G) = \ell^1(G) \oplus M_c(G)$, where $M_c(G)$ denotes the ideal of continuous measures in $M(G)$. This decomposition was crucial in the proof of [4, Theorem 1.1]. In this section, we establish an analogous decomposition for $B(G)$.

Let $G$ be a abelian with dual group $\hat{G}$ whose Bohr compactification we denote by $b\hat{G}$; we write $G_d$ for the group $G$ equipped with the discrete topology. For $\mu \in M(G)$, we denote its Fourier–Stieltjes transform in $B(\hat{G})$ by $\hat{\mu}$. Then we have for $\mu \in M(G)$:

$$\mu \in \ell^1(G) \iff \mu \in M(G_d) \iff \hat{\mu} \in B(\hat{G}_d) \iff \hat{\mu} \in B(b\hat{G}) \iff \hat{\mu} \in B(\hat{G})$$

where the last equivalence holds by [7, (2.27) Corollaire 4].

This suggests that the appropriate replacement for $\ell^1(G)$ in the Fourier–Stieltjes algebra context is $B(G) \cap \text{AP}(G)$, where $\text{AP}(G)$ denotes the algebra of all almost periodic functions on $G$. It is well known (see [17, 3.2.16], for example) that $\text{AP}(G)$ is a commutative $C^*$-algebra whose character space is a compact group denoted by $aG$ (for abelian $G$, we have $aG = bG$). We will first give an alternative description of $B(G) \cap \text{AP}(G)$ which will turn out to be useful later on.
Let $G$ be a locally compact group, and let $\mathcal{R}$ be any family of representations of $G$. We denote by $A_{\mathcal{R}}(G)$ the closed linear span in $B(G)$ of the coefficient functions of all representations in $\mathcal{R}$, i.e., of all functions of the form
\[ G \to \mathbb{C}, \; x \mapsto \langle \rho(x)\xi, \eta \rangle, \]
where $\rho \in \mathcal{R}$, and $\xi$ and $\eta$ are vectors in the corresponding Hilbert space. If $\mathcal{R}$ is the family of all representations of $G$, then $A_{\mathcal{R}}(G) = B(G)$, and if $\mathcal{R}$ just consists of the left regular representation, then $A_{\mathcal{R}}(G) = A(G)$. Let $\mathcal{F}$ denote the family of all finite-dimensional representations of $G$. Since $\mathcal{F}$ is closed under taking tensor products, it is immediate that $A_{\mathcal{F}}(G)$ is a (completely contractive) Banach algebra.

**Proposition 2.1** Let $G$ be a locally compact group. Then $A_{\mathcal{F}}(G) = B(G) \cap AP(G)$, and we have a canonical completely isometric isomorphism between $A_{\mathcal{F}}(G)$ and $B(aG)$.

**Proof** In view of [7, (2.27) Corollaire 4], it is sufficient to prove the second assertion only.

Let $\iota : G \to aG$ denote the (not necessarily injective) canonical map. It is easy to see that $A_{\mathcal{F}}(G) \cong B(aG)$ via
\[ B(aG) \to B(G), \quad f \mapsto f \circ \iota. \] (1)

We claim that (1) is a complete isometry. To see this, let $\omega_G : G \to W^*(G)$ and $\omega_{aG} : aG \to W^*(aG)$ denote the universal representations of $G$ and $aG$, respectively. Applying the universal property of $\omega_G : G \to W^*(G)$ to $\omega_{aG} \circ \iota : G \to W^*(aG)$ yields a (necessarily surjective) $w^*$-continuous *-homomorphism $\pi : W^*(G) \to W^*(aG)$. It is immediate that (1) is the adjoint of $\pi$. Hence, (1) is a complete isometry by [6, Theorem 4.1.8].

**Remarks**

1. Note that $A_{\mathcal{F}}(G)$ can be very small relative to $B(G)$: for example, if $G = \text{SL}(2, \mathbb{R})$, we have, $A_{\mathcal{F}}(G) = \mathbb{C}$.

2. Suppose that $G$ is non-compact. Since $B(G)$ is a complete invariant for $G$ ([26, Corollary]), it follows that $B(G) \not\cong B(aG)$ and thus $A_{\mathcal{F}}(G) \subsetneq B(G)$ by Proposition 2.1.

Let $G$ be a locally compact group. For any function $f$ on $G$ and $x \in G$, we define the left and the right translate of $f$ by $x$ by letting
\[ (L_x f)(y) := f(xy) \quad \text{and} \quad (R_x f)(y) := f(yx) \quad (y \in G). \]

A linear space $E$ of functions on $G$ is said to be translation invariant if $L_x f, R_x f \in E$ for all $f \in E$ and $x \in G$.

We record the following well known lemma for convenience:
Lemma 2.2 Let $G$ be a locally compact group. Then the following are equivalent for a closed subspace $E$ of $B(G)$:

(i) $E$ is translation invariant;

(ii) $E$ is a $W^*(G)$-submodule of $B(G)$;

(iii) $E = p \cdot B(G)$ for a unique central projection $p \in W^*(G)$.

Proof (i) $\iff$ (ii) is [26 Proposition 1.(i)], and (ii) $\iff$ (iii) is a well known general fact about von Neumann algebras (which can be found in [25], for instance). $\square$

Let $\mathcal{R}$ be a family of representations of $G$. Then it is clear that $A_{\mathcal{R}}(G)$ is translation invariant. Hence, there is a unique central projection $p_{\mathcal{R}} \in W^*(G)$ such that $A_{\mathcal{R}}(G) = p_{\mathcal{R}} \cdot B(G)$.

For any representation $\pi$ of $G$, we denote its canonical $w^*$-continuous extension to $W^*(G)$ by $\pi$ as well. We call a representation $\pi$ of $G$ purely infinite-dimensional if $\pi(p_F) = 0$. We denote the family of all purely infinite-dimensional representations of $G$ by $\mathcal{PIF}$; note that $\mathcal{PIF} \neq \emptyset$ if $G$ is not compact.

Theorem 2.3 Let $G$ be a locally compact group. Then the following are equivalent and true:

(i) $A_{\mathcal{PIF}}(G)$ is an ideal of $B(G)$.

(ii) The map

$$B(G) \to A_F(G), \quad f \mapsto p_F \cdot f$$

is an algebra homomorphism.

(iii) $\nabla p_F = p_F \otimes p_F$.

Proof It is immediately checked that (i) and (ii) are equivalent.

Let $x \in W^*(G)$ and $f, g \in B(G)$, and note that

$$\langle x, p_F \cdot (fg) - (p_F \cdot f)(p_F \cdot g) \rangle = \langle xp_F, fg \rangle - \langle x, (p_F \cdot f)(p_F \cdot g) \rangle$$

$$= \langle \nabla(xp_F), f \otimes g \rangle - \langle \nabla x, (p_F \cdot f) \otimes (p_F \cdot g) \rangle$$

$$= \langle \nabla(x)(p_F) - (\nabla x)(p_F \otimes p_F), f \otimes g \rangle$$

This proves the equivalence of (ii) and (iii).

We shall now verify that (iii) is indeed true.

Let $WAP(G)$ denote the weakly almost periodic functions on $G$ (see [3] for the definition of $WAP(G)$ and further information). By [3 Theorem 3.1], we have $B(G) \subset WAP(G)$. 7
Taking the adjoint of this inclusion map, we obtain a canonical map $\pi : \text{WAP}(G)^* \to W^*(G)$. Since WAP$(G)$ is an introverted subspace of $\ell^\infty(G)$, its dual $\text{WAP}(G)^*$ is a Banach algebra in a canonical manner. It is routinely verified — e.g. by checking multiplicativity on $M(G)$ — that $\pi$ is a *-homomorphism. The character space $wG$ of $\text{WAP}(G)$ is a compact, semitopological semigroup containing a topologically isomorphic copy of $G$. The kernel $K(wG)$ of $wG$ is intersection of all ideals of $wS$; it is non-empty by [3, Theorems 2.1 and 2.2], and by [3, Theorems 2.7], it is a compact group. Let $e_{K(wG)}$ denote its identity element. Then by (the proof of) [3, Theorem 2.22], we have $\text{AP}(G) = e_{K(wG)} \cdot \text{WAP}(G)$. It follows that $p_F = \pi(e_{K(wG)})$. In particular, $p_F$ is a character on $B(G)$. By [26, Theorem 1.(ii)], this implies (iii). $\square$

Remarks

1. Let $G$ be a non-discrete locally compact group. Then we have a further decomposition of $M_c(G)$, namely $M_c(G) = M_s(G) \oplus L^1(G)$, where $M_s(G)$ denotes the measures in $M_c(G)$ which are singular with respect to left Haar measure. The decomposition of $M(G)$ into $\ell^1(G) \oplus M_s(G)$ and $L^1(G)$ has long been known to have a $B(G)$-analogue (see [1] and [16]). In view of Theorem 2.3, we now have a complete analogue for $B(G)$ of the decomposition of the measure algebra into its discrete part, its singular, continuous part, and its absolutely continuous part.

2. Let $G$ be a non-compact, locally compact group. Then $A(G)$ is a translation invariant subspace of $B(G)$ having zero intersection with $A_{\mathcal{F}}(G)$. It follows that $A(G) \subset A_{\mathcal{PTF}}(G)$. Since for a non-discrete, locally compact group, the absolutely continuous measures are properly contained in the continuous measures, the natural conjecture is that $A(G) \subsetneq A_{\mathcal{PTF}}(G)$. This conjecture seems to be open for general locally compact groups, even in the amenable case.

3 Operator non-amenability for $B(G)$ and $B_r(G)$ if $G$ is not compact

We will now use Theorem 1.3 to show that $B(G)$ — and, equivalently, $B_r(G)$ — cannot be operator $C$-amenable with $C < 5$ unless $G$ is compact.

We first need a purely operator space theoretic lemma.

Given two operator spaces $E_1$ and $E_2$, their operator space $\ell^\infty$-direct sum $E_1 \oplus_\infty E_2$ is defined by taking the Banach space $\ell^\infty$-direct sum on each matrix level. It is then immediate that $E_1 \oplus_\infty E_2$ is again an operator space. If $F$ is another operator space, then it is immediately checked that

\[ CB(F, E_1 \oplus_\infty E_2) \cong CB(F, E_1) \oplus_\infty CB(F, E_2) \] (2)
canonically as Banach spaces. From the definition of the operator space structures on $\mathcal{CB}(E_1 \oplus_\infty E_2, F)$, $\mathcal{CB}(E_1, F)$, and $\mathcal{CB}(E_2, F)$ (see [5, p. 45]), it follows that the identification $\mathcal{CB}(E_1 \oplus E_2, F)$ is even a complete isometry.

The canonical embedding of $E_1 \oplus E_2$ into $(E_1^* \oplus_\infty E_2^*)^*$ equips $E_1 \oplus E_2$ with another operator space structure, denoted by $E_1 \oplus_1 E_2$. On the Banach space level, this is just the ordinary $\ell^1$-direct sum of Banach spaces. Replacing $E_1^*$ and $E_2^*$ with $E_1$ and $E_2$, respectively, in (2) and combining the duality result [6, Corollary 7.1.5] with the commutativity of $\hat{\otimes}$, we obtain:

**Lemma 3.1** Let $E_1$, $E_2$, and $F$ be operator spaces. Then we have a canonical completely isometric isomorphism

$$(E_1 \oplus_1 E_2) \hat{\otimes} F \cong (E_1 \hat{\otimes} F) \oplus_1 (E_2 \hat{\otimes} F).$$

We can now prove the main result of this section:

**Theorem 3.2** For a locally compact group, the following are equivalent:

(i) $G$ is compact.

(ii) $B_r(G)$ is operator $C$-amenable for some $C < 5$.

(iii) $B(G)$ is operator $C$-amenable for some $C < 5$.

**Proof** (i) $\implies$ (ii): If $G$ is compact, then $B_r(G) = B(G) = A(G)$. Since $A(G)$ is operator 1-amenable, this proves (ii).

(ii) $\implies$ (iii): Since $A(G)$ is a closed $C_r^*(G)$-submodule of $B_r(G)$, there is a projection $p \in C_r^*(G)^{**}$ such that $A(G) = p \cdot B_r(G)$. In particular, $A(G)$ is a completely complemented ideal of $B_r(G)$ and thus operator amenable by Lemma [16]. By [20, Theorem 3.6], this implies the amenability of $G$ and thus $B_r(G) = B(G)$ by [18, (4.21) Theorem].

(iii) $\implies$ (i): Assume towards a contradiction that $G$ is not compact. Let $(d_\alpha)_{\alpha \in A}$ be an approximate operator diagonal for $B(G)$ bounded by $C < 5$. Without loss of generality, suppose that $\Delta d_\alpha = 1$ for all $\alpha \in A$. We then have

$$d_\alpha = p_F \cdot d_\alpha \cdot p_F + p_F \cdot d_\alpha \cdot p_{pTF} \cdot d_\alpha \cdot p_F + p_{pTF} \cdot d_\alpha \cdot p_{pTF} \quad (\alpha \in A).$$

Since $B(G) = p_F \cdot B(G) \oplus_1 p_{pTF} \cdot B(G)$ in the operator space sense, Lemma 3.1 and [3] yield

$$\|p_F \cdot d_\alpha \cdot p_F\| + \|p_F \cdot d_\alpha \cdot p_{pTF}\| + \|p_{pTF} \cdot d_\alpha \cdot p_F\| + \|p_{pTF} \cdot d_\alpha \cdot p_{pTF}\| = \|d_\alpha\| \leq C < 5 \quad (\alpha \in A).$$

First note that, by Theorem 2.3, we have

$$\Delta(p_F \cdot d_\alpha \cdot p_F) = p_F \cdot \Delta d_\alpha = p_F \cdot 1 = 1.$$
Since $\Delta$ is a (complete) contraction, this yields in turn that
\[ \|p_F \cdot d_\alpha \cdot p_F\| \geq 1 \quad (\alpha \in A). \quad (6) \]

Let $\mathcal{U}$ be an ultrafilter on $A$ that dominates the order filter. Then we have for $f \in A(G) \subset A_{\mathcal{P}I}(G)$:
\[ f(w^* \lim_{\mathcal{U}} \Delta(d_\alpha \cdot p_F)) = w^* \lim_{\mathcal{U}} \Delta((f \cdot d_\alpha) \cdot p_F) = w^* \lim_{\mathcal{U}} \Delta(d_\alpha \cdot f) \cdot p_F = 0. \]

It follows that $w^* \lim_{\mathcal{U}} \Delta(d_\alpha \cdot p_F) = 0$. Combining this with (5), we obtain
\[ w^* \lim_{\mathcal{U}} \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_F) = -1. \]

and therefore
\[ \lim_{\mathcal{U}} \|p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_F\| \geq \lim_{\mathcal{U}} \|\Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_F)\| \geq 1. \quad (7) \]

Analogously, we see that $w^* \lim_{\mathcal{U}} \Delta(p_F \cdot d_\alpha \cdot p_{\mathcal{P}IF}) = -1$ and consequently
\[ \lim_{\mathcal{U}} \|p_F \cdot d_\alpha \cdot p_{\mathcal{P}IF}\| \geq 1. \quad (8) \]

Since
\[ 1 = w^* \lim_{\mathcal{U}} \Delta(p_F \cdot d_\alpha \cdot p_F) = w^* \lim_{\mathcal{U}} \Delta(d_\alpha) - \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_F) - \Delta(p_F \cdot d_\alpha \cdot p_{\mathcal{P}IF}) - \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_{\mathcal{P}IF}) \]
\[ = 1 - w^* \lim_{\mathcal{U}} \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_F) - w^* \lim_{\mathcal{U}} \Delta(p_F \cdot d_\alpha \cdot p_{\mathcal{P}IF}) \]
\[ - w^* \lim_{\mathcal{U}} \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_{\mathcal{P}IF}) \]
\[ = 3 - w^* \lim_{\mathcal{U}} \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_{\mathcal{P}IF}), \]

it follows that $w^* \lim_{\mathcal{U}} \Delta(p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_{\mathcal{P}IF}) = 2$. We thus obtain
\[ \lim_{\mathcal{U}} \|p_{\mathcal{P}IF} \cdot d_\alpha \cdot p_{\mathcal{P}IF}\| \geq 2. \quad (9) \]

Altogether, (6), (7), (8), and (9) contradict (4). \qed

Remarks

1. The proof of (ii) $\implies$ (i) shows that, whenever $B_r(G)$ — or, equivalently, $B(G)$ — is operator amenable, then $G$ is amenable.

2. We strongly suspect that $B(G)$ and $B_r(G)$ are operator amenable only if $G$ is compact. One possible way of proving this would be to follow the route outlined (for measure algebras) in [4]: assume that $B(G)$ is operator amenable, but that $G$ is not compact. Then Lemma 1.6 implies that $A_{\mathcal{P}IF}(G)$ is operator amenable and thus
has a bounded approximate identity. This, in turn, would imply that every element of $A_{PIF}(G)$ is a product of two elements in $A_{PIF}(G)$ by Cohen’s factorization theorem ([17, 5.2.4 Corollary]). We believe that this is not true, but have been unable to confirm this belief with a proof.

3. Another open question related to Theorem 3.2 is for which locally compact groups $G$, the Fourier–Stieltjes algebra $B(G)$ is amenable in the classical sense. The corresponding question for the Fourier algebra is also still open: as mentioned in the introduction, the only locally compact groups $G$ for which $A(G)$ is known to be amenable are those with an abelian subgroup of finite index, and it is plausible to conjecture that these are indeed the only ones. The plausible conjecture for $B(G)$ is that it is amenable if and only if $G$ is compact and has an abelian subgroup of finite index.

4 Operator Connes-amenability

Amenability in the sense of [10] is not the “right” notion of amenability for von Neumann algebras because it is too restrictive to allow for the development of a reasonably rich theory ([27]). In [13], a variant of amenability — christened Connes-amenability in [9] — was introduced for von Neumann algebras, which takes the normal structure in von Neumann algebras into account. This notion of amenability has turned out to be equivalent to a number of important $W^*$-algebraic properties, such as injectivity and semidiscreteness; see [21, Chapter 6] for a self-contained exposition.

Similarly, [4, Theorem 1.1] suggests that Johnson’s original definition of amenability is too strong to deal with measure algebras. In [22], the first-named author extended the notion of Connes-amenability to the class of dual Banach algebras. This class includes — besides $W^*$-algebras — all measure algebras and all algebras $B(E)$ for a reflexive Banach space $E$. In [23], we proved that a locally compact group $G$ is amenable if and only if $M(G)$ is Connes-amenable.

We shall now introduce a hybrid of operator amenability and Connes-amenability, which will turn out to be the “right” notion of amenability for the reduced Fourier–Stieltjes algebra in the sense that it singles out precisely the amenable, locally compact groups.

**Definition 4.1** A completely contractive Banach algebra is called dual if $\mathfrak{A}^*$ has a closed $\mathfrak{A}$-submodule $\mathfrak{A}_*$ such that $\mathfrak{A} = (\mathfrak{A}_*)^*$.

**Remark** In general, there is no need for $\mathfrak{A}_*$ to be unique.

**Examples**

1. If $\mathfrak{A}$ is a dual Banach algebra in the sense of [22, Definition 1.1], then $\max \mathfrak{A}$ is a dual, completely contractive Banach algebra.
2. Every $W^*$-algebra is a dual, completely contractive Banach algebra.

3. For any locally compact group $G$, the Fourier–Stieltjes algebras $B(G)$ and $B_r(G)$ are dual, completely contractive Banach algebras.

**Definition 4.2** Let $\mathfrak{A}$ be a dual, completely contractive Banach algebra. A dual operator $\mathfrak{A}$-bimodule $E$ is called *normal* if, for each $x \in E$, the maps

$$\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are $w^*$-continuous.

**Definition 4.3** A dual, completely contractive Banach algebra $\mathfrak{A}$ is called *operator Connes-amenable* if every $w^*$-continuous, completely bounded derivation from $\mathfrak{A}$ into a normal, dual operator $\mathfrak{A}$-bimodule is inner.

For the reduced Fourier–Stieltjes algebra, we obtain:

**Theorem 4.4** The following are equivalent for a locally compact group $G$:

(i) $G$ is amenable.

(ii) $B_r(G)$ is operator Connes-amenable.

*Proof* (i) $\implies$ (ii): By [20, Theorem 3.6], $A(G)$ is operator amenable. The $w^*$-density of $A(G)$ in $B_r(G)$ then yields the operator Connes-amenability of $B_r(G)$ (compare [22, Proposition 4.2(i)]).

(ii) $\implies$ (i): The same argument as in the proof of [22, Proposition 4.1] yields that $B_r(G)$ has an identity. Since $B_r(G)$ is a closed ideal of $B(G)$ by [7, (2.16) Proposition], it follows that $B_r(G) = B(G)$ and thus $C^*_r(G) = C^*(G)$. By [18, (4.21) Theorem], this is equivalent to $G$ being amenable. ⊓⊔

It is, of course, tempting to conjecture that $B_r(G)$ in Theorem 4.4(ii) can be replaced by $B(G)$. The implication (i) $\implies$ (ii) then still holds because $B(G) = B_r(G)$ for amenable $G$. The argument used to establish the converse, however, does no longer work for $B(G)$ instead of $B_r(G)$. As well shall now see, not only the proof no longer works, but the statement becomes false: there are non-amenable, locally compact groups for which $B(G)$ is operator Connes-amenable.

**Lemma 4.5** Let $G$ be a locally compact group. Then $A_F(G)$ is operator amenable.
Proof Since $aG$ is compact, we have $B(aG) = A(aG)$. Since $aG$ is amenable, $A(aG) = B(aG)$ is operator amenable by [20 Theorem 3.6]. By Proposition 2.1, the completely contractive Banach algebras $B(aG)$ and $A_F(G)$ are completely isometrically isomorphic. Hence, $A_F(G)$ is operator amenable. □

Remark Should our conjecture that $B(G)$ is operator amenable only for compact $G$ be correct, then Lemma 4.5 would yield immediately that $A(G) \subseteq A_{PIF}(G)$ for non-compact, amenable $G$: otherwise, we would have a short exact sequence

$$\{0\} \rightarrow A(G) \rightarrow B(G) \rightarrow A_F(G) \rightarrow \{0\}$$

of completely contractive Banach algebras whose endpoints are operator amenable. The straightforward analogue of a hereditary property of amenability in the classical sense ([21 Theorem 2.3.10]) would then yield the operator amenability of $B(G)$, which is impossible.

Recall that a $C^*$-algebra $A$ is called residually finite-dimensional if the family of finite-dimensional *-representations of $A$ separates the points of $A$. For locally compact groups $G$, the property of $C^*(G)$ being residually finite-dimensional implies that $G$ is maximally almost periodic ([21 Theorem 1.1]), though the converse need not be true ([2]).

**Theorem 4.6** Let $G$ be a locally compact group such that $C^*(G)$ is residually finite-dimensional. Then $B(G)$ is operator Connes-amenable.

Proof By Lemma 4.5, $A_F(G)$ is operator amenable. Since $C^*(G)$ is residually finite-dimensional, a simple Hahn–Banach argument shows that $A_F(G)$ is $w^*$-dense in $B(G)$. Then (the operator analogue of) [22 Proposition 4.2(i)] yields the operator Connes-amenability of $B(G)$. □

**Example** Let $\mathbb{F}_2$ denote the free group in two generators. Then $C^*(\mathbb{F}_2)$ is residually finite-dimensional by [5 Proposition VII.6.1], so that $B(\mathbb{F}_2)$ is operator Connes-amenable by Theorem 4.6. However, $\mathbb{F}_2$ is not amenable.

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