REGULAR LOCAL RINGS OF DIMENSION FOUR AND GORENSTEIN SYZYGETIC PRIME IDEALS

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Dedicated to the memory of Wolmer V. Vasconcelos.

ABSTRACT. Let $R$ be a Noetherian local ring. We prove that $R$ is regular of dimension at most 4 if, and only if, every prime ideal, defining a Gorenstein quotient ring, is syzygetic. We deduce a characterization of these rings in terms of the André-Quillen homology.

1. Introduction

Let $R$ be a Noetherian commutative ring and $I$ an ideal of $R$. Let $\alpha : S(I) \to R(I)$ be the graded surjective morphism from the symmetric algebra of $I$ to the Rees algebra of $I$. The ideal $I$ is said to be syzygetic if the second component $\alpha_2 : S_2(I) \to I^2$ is an isomorphism; it is said to be of linear type if $\alpha$ is an isomorphism. Ideals generated by a regular sequence are of linear type, hence syzygetic. Noetherian rings of global dimension at most 1, 2 and 3, were recently characterized in [13] in terms of the syzygetic and linear type conditions. Recall that the global dimension of $R$ is defined as the supremum of the projective dimensions of all $R$-modules. For a Noetherian ring $R$, having global dimension at most $N$ is equivalent to $R_m$ being regular local with dim($R_m$) $\leq N$, for every maximal ideal $m$ of $R$. The purpose of this note is to extend [13] to dimension 4, but restricted to the local case.

Theorem. Let $(R, m)$ be a Noetherian local ring. The following conditions are equivalent:

(i) $R$ is regular and dim($R$) $\leq 4$;
(ii) Every prime ideal $p$ of $R$, with $R/p$ Gorenstein, is syzygetic;

The main ingredient in the implication (i) $\Rightarrow$ (ii) is the following result of Herzog, Simis and Vasconcelos: if $R$ is regular local, with $1/2 \in R$, then a Gorenstein ideal of height 3 is syzygetic (see [5, Proposition 2.8]). In order to avoid the condition $1/2 \in R$, we give a different approach borrowing ideas from Ulrich in [16, (2.2)]. This is done in Lemma 2.1

As for the reverse implication, suppose that (ii) holds. Using another nice result of Herzog, Simis and Vasconcelos one deduces that $R$ is regular. Indeed, a Noetherian local ring whose maximal ideal is syzygetic is regular (see the proof of Corollary 3.8 and, particularly, Proposition 2.5, in [5]).

Thus, one must show that any regular local ring $R$ of dimension at least 5 admits a prime ideal $p$, whose factor ring $R/p$ is Gorenstein, and such that $p$ is not syzygetic. We first address the five dimensional case. Our candidate, call it $I$, is inspired by the following affine example: consider the most simple five dimensional Gorenstein curve which is not a complete intersection ([7, Theorem 4.4]). Then take a minimal system of generators of its equations, namely, the kernel of the ring homomorphism $\mathbb{C}[[X,Y,Z,T,U]] \to \mathbb{C}[[V]]$, which sends $X,Y,Z,T,U$ to $V^6, V^7, V^8, V^9, V^{10}$. On substituting the variables by the regular parameters $x, y, z, t, u$ of the regular local ring $R$, we obtain a minimal system of generators of the candidate ideal $I$.

Lemma 2.2 proves that $I$ is a perfect ideal of height 4, such that $R/I$ is Gorenstein, and such that $I$ is not syzygetic. To see that $I$ is perfect we use the well-known acyclicity criterion of Buchsbaum

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and Eisenbud. We would like to stress here that our proof holds in any regular local ring, non
necessarily an algebra essentially of finite type over a field. In that sense, Singular \([3]\) is of great
help in finding and checking products of matrices, as well as guessing which minors will conform
the regular sequences of the required length. However, our specific full proof of the perfectness of
I can not be deduced, at least just by transport, from Singular.

The key point of the paper is to show that \(I\) is in fact a prime ideal. First, we reduce to the
complete case. Then we take any associated prime \(p\) to \(I\), necessarily different from the maximal
ideal. Since \(R\) is complete, the integral closure of \(R/p\) is a DVR. Using the valuation corresponding
to \(R/p\), we are able to deduce the equality \(I = p\). This is done in Proposition\(2.3\).

Once the result is proved in dimension 5, we extend the example in any arbitrary higher dimension
in Corollary \(2.4\).

The paper finishes with a characterization of Noetherian local rings which are regular and of
dimension at most 4 in terms of the Andrè-Quillen homology (see Corollary \(3.1\)).

For any unexplained notation we refer to \([10], [2], [5], [1]\) and \([14]\).

2. PROOF OF THE MAIN RESULT

Lemma 2.1. Let \((R, m)\) be a Gorenstein local ring. Let \(I\) be an ideal of height 3, generically a
complete intersection, having finite projective dimension, and such that \(R/I\) is Gorenstein. Then
\(I\) is syzygetic.

Proof. Let \(S = R/I\) and let \(H_1(I)\) denote the first Koszul homology group associated to a minimal
system of generators of \(I\). By \([2]\) Proposition 1.2.13], for every associated prime \(q\) to \(H_1(I)\),

\[
\text{depth}(H_1(I)) \leq \dim(S/q) \leq \dim(S/Ann_S(H_1(I))) = \dim(H_1(I)) \leq \dim S.
\]

By hypothesis, \(I\) is in the linkage class of a complete intersection, so \(H_1(I)\) is a Cohen-Macaulay
module of maximum dimension (see, e.g., \([6]\) Remark 1.3, Theorem 1.14 and Example 2.2]). Therefore
\(\dim(S/q) = \dim(S)\) and every associated prime \(q\) of \(H_1(I)\) is a minimal prime of \(S\). Hence
the set of zero divisors of \(H_1(I)\) is included in the set of zero divisors of \(S\), which implies \(H_1(I)\) is
torsion-free. On the other hand, by \([11, 15.12]\) or \([5]\) Discussion before Proposition 2.5], one has the exact sequence:

\[
0 \to H_2(R, S, S) \to H_1(I) \to F \otimes S \to I/I^2 \to 0,
\]

where \(F \to I \to 0\) is a minimal free presentation of \(I\) and \(H_2(R, S, M)\) stands for the second Andrè-
Quillen homology group of the \(R\)-algebra \(S\) with coefficients in the \(S\)-module \(M\) (see \([1], [14], [9], [8]\)). Since \(I\) is generically a complete intersection at its associated primes, for every associated
prime \(p\) to \(I\),

\[
H_2(R, S, S)_p = H_2(R_p, R_p/I_p, R_p/I_p) = 0.
\]

Hence \(H_2(R, S, S)\) is a \(S\)-torsion module, thus included in the torsion of \(H_1(I)\), which is zero.
Therefore \(H_2(R, S, S) = 0\). Since \(H_2(R, R/I, R/I) \cong \ker(\alpha_2 : S_2(I) \to I^2)\), it follows that \(I\) is
syzygetic (see, e.g., \([1]\) Corollaire 15.10] or \([11] Corollaire 3.2\]).

Proof of Theorem \((i) \Rightarrow (ii)\). Suppose that \((R, m)\) is regular local with \(\dim(R) \leq 4\). Then \(m\) is
generated by a regular sequence, so \(m\) is syzygetic (see, e.g., \([5]\) Corollary 3.8]); since \(R\) is a UFD,
then every height 1 prime ideal is principal (generated by a nonzero divisor), and so again syzygetic;
furthermore, every height 2 Gorenstein prime ideal in a regular local ring is generated by a regular
sequence, thus syzygetic (see, e.g. \([7]\) §4]). Let \(p\) be a prime ideal of height 3, such that \(R/p\) is
Gorenstein. Since \(R\) is regular, \(p\) has finite projective dimension and \(p\) is generically a complete
intersection. By Lemma\(2.1\) \(p\) is syzygetic.
Lemma 2.2. Let \((R, m, k)\) be a regular local ring of dimension 5. Let \(x, y, z, t, u\) be a regular system of parameters. Let \(I\) be the ideal of \(R\) generated by
\[
\begin{align*}
&f_1 = y^2 - xz, 
&f_2 = yz - xt, 
&f_3 = z^2 - yt, 
&f_4 = yt - xu, 
&f_5 = zt - yu, 
&f_6 = t^2 - zu, 
&f_7 = zu - x^3, 
&f_8 = tu - x^2y, 
&f_9 = u^2 - x^2z.
\end{align*}
\]
Then \(I\) is a perfect ideal of height 4, such that \(R/I\) is Gorenstein, and such that \(I\) is not syzygetic.

Proof. Let \(\varphi_1 = (f_1, \ldots, f_9)\) be the \(1 \times 9\) matrix given by the nine aforementioned binomials:
\[
\varphi_1 = \left( y^2 - xz, yz - xt, z^2 - yt, yt - xu, zt - yu, t^2 - zu, zu - x^3, tu - x^2y, u^2 - x^2z \right).
\]
Let \(\varphi_2, \varphi_3\) and \(\varphi_4\) be the \(9 \times 16, 16 \times 9\) and \(9 \times 1\), matrices defined as:
\[
\varphi_2 = \begin{pmatrix}
0 & 0 & -u & t & z & 0 & x^2 & 0 & u & t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u & t & 0 & -y & x^2 & 0 & u & 0 & -z & 0 & x^2 & 0 & u & 0 \\
u & 0 & 0 & x & 0 & u & 0 & 0 & y & x^2 & 0 & u & -t & 0 & 0 \\
u & 0 & 0 & -y & 0 & 0 & 0 & -t & -z & 0 & x^2 & 0 & 0 & -t & 0 & x^2 \\
0 & 0 & -y & x & 0 & u & 0 & 0 & y & 0 & 0 & 0 & -t & z & x^2 & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & z & 0 & 0 & -u & 0 \\
-z & -y & 0 & 0 & 0 & -t & -z & 0 & 0 & 0 & -u & -t & 0 & 0 & 0 & -u \\
x & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & z & 0 & 0 & -u & t & 0
\end{pmatrix},
\]
\[
\varphi_3 = \begin{pmatrix}
z & -t & 0 & 0 & y & 0 & 0 & 0 & 0 \\
-t & u & 0 & 0 & -z & 0 & 0 & 0 & 0 \\
u & 0 & y & -z & 0 & 0 & 0 & 0 & xy \\
0 & -x^2 & -z & t & u & 0 & 0 & 0 & -xz \\
x^2 & 0 & t & -u & 0 & 0 & 0 & 0 & xt \\
0 & 0 & 0 & 0 & -x & z & -t & 0 & 0 \\
-z & t & 0 & 0 & 0 & -t & u & 0 & 0 \\
0 & -u & -x & 0 & 0 & u & 0 & -z & -x^2 \\
u & 0 & y & 0 & 0 & 0 & -x^2 & t & xy \\
0 & 0 & 0 & -t & -u & x^2 & 0 & -u & 0 \\
-x & 0 & 0 & 0 & 0 & -y & 0 & 0 & -t \\
y & 0 & 0 & 0 & x & 0 & t & 0 & u \\
0 & 0 & 0 & x & 0 & 0 & -u & y & 0 \\
0 & 0 & x & -y & 0 & -u & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 & 0 & y & 0 & z \\
x & -y & 0 & 0 & 0 & y & -z & 0 & 0
\end{pmatrix},
\]
\[
\varphi_4^\top = (zt - yu, z^2 - yt, u^2 - x^2y, tu - x^2y, -t^2 + zu, -yt + xu, -yz + xt, -zu + x^3, y^2 - xz).
\]
Since \(\varphi_1 \cdot \varphi_2 = 0, \varphi_2 \cdot \varphi_3 = 0\) and \(\varphi_3 \cdot \varphi_4 = 0\), then
\[
0 \leftarrow R/I \leftarrow R = F_0 \varphi_1^\top R^9 = F_1 \varphi_2^\top R^{16} = F_2 \varphi_3^\top R^9 = F_3 \varphi_4 F_4 \leftarrow 0
\]
is a complex of \(R\)-modules. To see that this complex is exact, we use the acyclicity criterion of Buchsbaum and Eisenbud (see, e.g., [2, Theorem 1.4.12]). Set \(r_i = \sum_{j=i}^{4} (-1)^{j-i} \text{rank } F_j\), so that \(r_1 = 1, r_2 = 8, r_3 = 8\) and \(r_4 = 1\). Thus we have to prove that \(\text{grade}(I_1(\varphi_1)) \geq 1\), \(\text{grade}(I_8(\varphi_2)) \geq 2\), \(\text{grade}(I_8(\varphi_3)) \geq 3\) and \(\text{grade}(I_1(\varphi_4)) \geq 4\).

The ideal \((f_1, f_3 + f_4, f_6 + f_7, f_9, x)\) is equal to \((y^2, z^2, t^2, u^2, x)\), so has grade 5. By [2, Corollary 1.6.19], \(f_1, f_3 + f_4, f_6 + f_7, f_9\) is an \(R\)-regular sequence in \(I = I_1(\varphi_1)\) of length four. In particular, \(\text{grade}(I) \geq 4 \geq 1\). Similarly, one has the equality:
\[
((\varphi_4)_{9,1}, (\varphi_4)_{2,1} - (\varphi_4)_{6,1}, (\varphi_4)_{5,1} + (\varphi_4)_{8,1}, (\varphi_4)_{3,1}, x) = (y^2, z^2, t^2, u^2, x).
\]
Therefore $(\varphi_4)_{9,1}, (\varphi_4)_{2,1} - (\varphi_4)_{6,1}, (\varphi_4)_{5,1} + (\varphi_4)_{8,1}, (\varphi_4)_{3,1}$ is an $R$-regular sequence in $I_1(\varphi_4)$ of length four and \(\text{grade}(I_1(\varphi_4)) \geq 4\).

In order to prove \(\text{grade}(I_2(\varphi_2)) \geq 2\), we look for minors of \(\varphi_2\) with monic pure terms in one of the parameters. For instance, up to sign, the minor \(g_1 := y^8 - 3xy^6 z + 3x^2y^4z^2 - x^3y^2z^3 \in I_8(\varphi_2)\), with monic pure term in \(y\), is obtained from the \(8 \times 8\) sub-matrix given by the rows 2, 3, 4, 5, 6, 7, 8, 9 and the columns 2, 4, 5, 6, 7, 8, 9, 10. Similarly, we get \(g_2 := t^8 - 3zt^6 u + 3z^2t^4u^2 - z^3t^2u^3 \in I_8(\varphi_2)\) from the \(8 \times 8\) sub-matrix given by the rows 1, 2, 3, 4, 5, 7, 8, 9 and the columns 3, 8, 10, 12, 13, 14, 15, 16. Since \((g_1, g_2, x, u) = (x, y^8, t^4u, t)\), then \(\text{grade}(g_1, g_2, x, u) = 4\), \(g_1, g_2\) is an $R$-regular sequence in \(I_8(\varphi_2)\), and \(\text{grade}(I_8(\varphi_2)) \geq 2\). Observe that this argument does not depend on the characteristic of the ring \(R\).

As before, let us seek for minors of \(\varphi_3\) with monic pure terms in one of the parameters. For instance, \(h_1 := y^8 - 3xy^6z + 3x^2y^4z^2 - x^3y^2z^3 \in I_8(\varphi_3)\) is obtained from the \(8 \times 8\) sub-matrix given by the rows 1, 3, 11, 12, 13, 14, 15, 16 and the columns 1, 2, 3, 4, 5, 6, 7, 8; \(h_2 := z^8 - 3yz^6t + 3y^2z^4t^2 - y^3z^2t^3 \in I_8(\varphi_3)\) is obtained from the rows 1, 2, 3, 4, 6, 8, 15, 16 and the columns 1, 3, 4, 5, 6, 7, 8, 9. Finally, \(h_3 := t^8 - 3zt^6u + 3z^2t^4u^2 - z^3t^2u^3 \in I_8(\varphi_3)\) is obtained from the rows 1, 2, 4, 5, 6, 7, 9, 11 and the columns 1, 2, 3, 4, 6, 7, 8, 9. Note that \(\text{rad}(h_1, h_2, h_3, x, u) = m\). Hence \(\text{grade}(h_1, h_2, h_3, x, u) = 5\). It follows that \(h_1, h_2, h_3\) is an $R$-regular sequence in \(I_8(\varphi_3)\) and that \(\text{grade}(I_8(\varphi_3)) \geq 3\).

We conclude that the complex above is a free resolution of \(R/I\). It is minimal since \(\varphi_3(F_i) \subseteq m F_{i-1}\), for every \(i = 1, \ldots, 4\). Therefore

\[
4 \leq \text{grade}(I) = \min\{i \geq 0 \mid \text{Ext}^i_R(R/I, R) \neq 0\} \leq \text{proj dim}_R(R/I) \leq 4.
\]

Thus \(I\) is a perfect ideal of grade 4 and \(R/I\) is Gorenstein (see, e.g., [2] Theorem 1.2.5, page 25 and [7] Proposition 3.2).

Set \(H := (f_1, \ldots, f_7, f_9) \subset I\). Since the aforementioned resolution of \(R/I\) is minimal, \(f_8 \notin H\) and \(H : f_8 \subset R\). However, one can check that \(f_8^2 = -xf_1f_7 - xf_2f_5 + x^2f_3f_4 + x^2f_4 + f_6f_9 + f_7f_9\). Thus \(f_8^2 \in HI\) and \(HI : f_8^2 = R\). Therefore, \(H : f_8 \subset HI : f_8^2\) and \(I\) is not syzygetic (see [12, Lemma 4.2]).

**Proposition 2.3.** Let \((R, m, k)\) be a regular local ring of dimension 5. Let \(x, y, z, t, u\) be a regular system of parameters. Let \(I = (f_1, \ldots, f_9)\) be the ideal of \(R\) defined as in the preceding lemma. Then \(I\) is a prime ideal of height 4, such that \(R/I\) is Gorenstein, and such that \(I\) is not syzygetic.

**Proof.** By Lemma 2.2 we only have to prove that \(I\) is prime. Let \((\hat{R}, \hat{m})\) be the completion of \((R, m)\), which is a five dimensional regular local ring with maximal ideal \(\hat{m} = \hat{m} \hat{R} = (x, y, z, t, u) \hat{R}\) generated by the regular system of parameters \(x, y, z, t, u\). By Lemma 2.2 again, \(I \hat{R} = (f_1, \ldots, f_9) \hat{R}\) is a perfect ideal of height 4. If we prove that \(I \hat{R}\) is prime, since \(\hat{R}\) is faithfully flat, then \(I = I \hat{R} \cap R\) and \(I\) is prime as well. Therefore we can suppose that \(R\) is complete.

Since \(I\) is perfect of height 4, then \(I\) is height unmixed and so \(m\) is not an associated prime to \(I\). Let \(p\) be any associated prime to \(I\) and set \(D = R/p\). Thus \(D\) is a one dimensional complete Noetherian local domain. Let \(V\) be the integral closure of \(D\) in its quotient field \(K\). Then \(V\) is a finitely generated \(D\)-module and a DVR (see [15, Theorem 4.3.4]). Let \(\nu\) be the valuation on \(K\) corresponding to \(V\). Set \((\nu_x, \nu_y, \nu_z, \nu_t, \nu_u) = (\nu(x), \nu(y), \nu(z), \nu(t), \nu(u))\). In \(V\), \(f_i = 0\), for \(i = 1, \ldots, 9\). Applying \(\nu\) to these equalities, one gets \((\nu_x, \nu_y, \nu_z, \nu_t, \nu_u) = (6n, 7n, 8n, 9n, 10n)\), for some integer \(n \geq 1\) (see, e.g., [4] Proof of Proposition 2.6). In particular, \(\nu_z \geq 6\).

Let \((S, n, k)\) be the regular local ring with \(S = R[x R]\) and \(n = m x R = (y, z, t, u)\), by abuse of notation. One has \(x R + I = (x, y^2, z t, y u - z t^2, z u, t^2, t u, u^2)\) and \(R/(x R + I) \cong S/J\), where \(J\) is the ideal of \(S\) defined as \(J = (y^2, y z, y t, y u - z t^2, z u, t^2, t u, u^2)\). Since \(x R = \text{Ann}_R(S)\), then \(\text{length}_R(R/(x R + I)) = \text{length}_S(S/J)\). Note that \(n^3 \subset J \subset n^2\). Consider the following two exact sequences of \(S\)-modules:

\[
0 \to n/J \to S/J \to S/n \to 0 \quad \text{and} \quad 0 \to n^2/J \to n/J \to n/n^2 \to 0.
\]
On taking lengths,
\[ \text{length}_R(R/(xR + I)) = \text{length}_S(S/J) = \text{length}_S(S/n) + \text{length}_S(n/n^2) + \text{length}_S(n^2/J) = 6. \]
Since \( xR + I \subseteq xR + p \) and \( R/(xR + p) \cong (R/p)/(x \cdot R/p) = D/Dx \), then
\[ 6 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + p)) = \text{length}_D(D/Dx). \]
Since \( f_1 = 0, f_3 + f_4 = 0, f_6 + f_7 = 0 \) and \( f_9 = 0 \) in \( D \), then \( y^2, z^2, t^2, u^2 \in xD \), and so \( D \) is a parameter ideal of the one dimensional Cohen-Macaulay local domain \((D, m/p, k)\). Since \( V \) is a finitely generated Cohen-Macaulay \( D \)-module of rank \( 1 \), then \( \text{length}_D(D/xD) = \text{length}(V/xV) \) (see [2 Corollary 4.6.11, (c)]). Moreover \( \text{length}_D(V/xV) = [k_V : k] \cdot \text{length}_V(V/xV) = [k_V : k] \cdot \nu(x) \)
Therefore, \( \text{length}_D(D/xD) = [k_V : k] \cdot \nu_x \). Recapitulating,
\[ 6 = \text{length}_R(R/(xR + I)) \geq \text{length}_R(R/(xR + p)) = [k_V : k] \cdot \nu_x \geq 6. \]
Hence \( \text{length}_R(R/(xR + I)) = \text{length}_R(R/(xR + p)) \). By the additivity of the length with respect to short exact sequences, \( xR + I = xR + p \).

Note that \( x \notin p \), otherwise \( p \supset xR + I \supset (x, y^2, z^2, t^2, u^2) \) and \( p = m \), a contradiction. Then \( p \cap xR = xp \). In particular, on tensoring \( 0 \rightarrow p/I \rightarrow R/I \rightarrow R/p \rightarrow 0 \) by \( R/xR \), one obtains the exact sequence \( 0 \rightarrow L/xL \rightarrow R/(xR+I) \rightarrow R/(xR+p) \rightarrow 0 \), where \( L = p/I \). Since \( xR+I = xR+p \), then \( L = xL \). By Nakayama’s Lemma, \( L = 0 \) and \( I = p \). Therefore, \( I \) is a prime ideal. \( \square \)

We extend Proposition 2.3 to higher dimension, just by adding to the ideal \((f_1, \ldots, f_9)\) the parameters of \( R \) not involved in the definition of the \( f_i \).

**Corollary 2.4.** Let \((R, m, k)\) be a regular local ring of dimension \( n \geq 6 \). Let \( x_1, \ldots, x_n \) be a regular system of parameters. Let \( I \) be the ideal of \( R \) generated by \( f_1, f_2, f_3, \ldots, f_9 \), where
\[
\begin{align*}
 f_1 &= x_2^2 - x_1 x_3 , & f_2 &= x_2 x_3 - x_1 x_4 , & f_3 &= x_3^2 - x_2 x_4 , \\
 f_4 &= x_2 x_4 - x_1 x_5 , & f_5 &= x_3 x_4 - x_2 x_5 , & f_6 &= x_4^2 - x_3 x_5 , \\
 f_7 &= x_3 x_5 - x_1^2 , & f_8 &= x_4 x_5 - x_3^2 , & f_9 &= x_5^2 - x_1^2 .
\end{align*}
\]
Then \( I \) is a prime ideal of height \( n - 1 \), such that \( R/I \) is Gorenstein, and such that \( I \) is not syzygetic.

**Proof.** Set \( I = (f_1, \ldots, f_9, x_6, \ldots, x_n) \) and let \( J = (x_6, \ldots, x_n) \) be the ideal generated by \( x_6, \ldots, x_n \). Set \( \bar{R} = R/J \) and let \( \bar{g} \) stand for the class modulo \( J \) of an element \( g \) of \( R \). Set \( \bar{I} = I/J = (f_1, \ldots, f_9) \).

Note that \( \bar{x}_1, \ldots, \bar{x}_5 \) is a regular system of parameters of the regular local ring \( \bar{R} \). By Proposition 2.3, \( \bar{I} \) is a prime ideal of height 4, such that \( R/I \) is Gorenstein, and such that \( I \) is not syzygetic.

In particular, \( I \) is a prime ideal of such that \( R/I \cong \bar{R}/\bar{I} \) is Gorenstein. Moreover,
\[ \text{ht}(I) = \dim(R) - \dim(R/I) = n - \dim(\bar{R}/\bar{I}) = n - (\dim(\bar{R}) - \text{ht}(\bar{I})) = n - 1. \]
Suppose that \( f_8 = \sum_{i=1, i \neq 8}^{g} a_i f_i + \sum_{i=0}^{9} b_i x_i \), for some \( a_i, b_j \in R \). Then, on taking classes modulo \( J \), \( f_8 \in (f_1, \ldots, f_7, f_9) \), which is a contradiction with the proof of Lemma 2.2 where one shows that \( \bar{I} \) is minimally generated by \( f_1, \ldots, f_9 \). Hence \( f_8 \notin (f_1, \ldots, f_7, f_9, x_6, \ldots, x_9) =: \bar{H} \) and \( H : f_8 \subseteq R \). The same equality at the end of the proof of Lemma 2.2 shows that \( f_8^2 \in HI \), thus \( H : f_8 \subseteq HI : f_8^2 = R \). It follows that \( I \) is not syzygetic. \( \square \)

**Proof of Theorem (ii) \( \Rightarrow \) (i).** By hypothesis (ii), the maximal ideal \( m \) of \( R \) is syzygetic. Hence \( R \) is regular local (see the proof of [5 Corollary 3.8]). By hypothesis (ii) and using Proposition 2.3 and Corollary 2.4 one deduces that \( \dim(R) \leq 4 \). \( \square \)

3. Final remarks

From the Theorem and the isomorphism \( H_2(R, R/I, R/I) \cong \ker(\alpha_2) \) ([I Corollaire 15.10] or [11 Corollaire 3.2]), we deduce a characterization of Noetherian local rings which are regular of dimension at most 4 in terms of the André-Quillen homology.

**Corollary 3.1.** Let \((R, m)\) be a Noetherian local ring. The following conditions are equivalent:
(a) $R$ is regular and $\dim(R) \leq 4$;
(b) $H_2(R, S, S) = 0$ for every Gorenstein quotient domain $S$ of $R$.

Remark 3.2. The “global” argument used in [13] does not seem to work here. Indeed, let $R$ be a Noetherian ring, not necessarily local. For the sake of easiness, let $\text{Spec}(R)$ denote the set of prime ideals of $R$, $\text{Gor}(R)$ be the set of ideals $I$ of $R$ such that $R/I$ is Gorenstein and, finally, $\text{Syz}(R)$ be the set of syzygetic ideals of $R$. Part of [13, Theorem (C)] states: “If $\text{Spec}(R) \subset \text{Syz}(R)$, then $R$ has global dimension at most 3”. The proof has two steps. First, it exhibits a non syzygetic height 3 prime ideal in any regular local ring of dimension 4. Subsequently, it supposes that $\dim(R) \geq 4$, then it localizes at a prime ideal $q$ of height 4, obtaining the four dimensional regular local ring $R_q$. Using the first step, it deduces the existence of a prime ideal $pR_q \in \text{Spec}(R_q)$, which is not syzygetic. Therefore $p$ is prime, but not syzygetic. Thus $\text{Spec}(R) \not\subset \text{Syz}(R)$, which finishes the proof. The analogous result in dimension four would be: “If $\text{Spec}(R) \cap \text{Gor}(R) \subset \text{Syz}(R)$, then $R$ has global dimension at most 4”. As a first step, we have shown Proposition 2.3 (and even Corollary 2.4). Trying to proceed as before, suppose that $\dim(R) \geq 5$ and localize at a prime ideal $q$ of height 5, obtaining the five dimensional regular local ring $R_q$. Using Proposition 2.3 one deduces the existence of a prime ideal $pR_q \in \text{Spec}(R_q)$, such that $R_q/pR_q \cong (R/p)_q$ is Gorenstein, and such that $pR_q$ is not syzygetic. It follows that $p$ is prime, but not syzygetic. However, we can not assure that $R/p$ is Gorenstein, in other words, $p$ is not necessarily in $\text{Spec}(R) \cap \text{Gor}(R)$, and so we are not able to deduce $\text{Spec}(R) \cap \text{Gor}(R) \not\subset \text{Syz}(R)$.

With the preceding notations and using [13, Theorem (C)], one concludes:

Corollary 3.3. Let $(R, m)$ be a Noetherian local ring. Then $R$ is regular of dimension 4 if, and only if, $\text{Spec}(R) \cap \text{Gor}(R) \subset \text{Syz}(R)$ and $\text{Spec}(R) \not\subset \text{Syz}(R)$.

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