THE CURVATURE TENSOR OF $(\kappa, \mu, \nu)$-CONTACT METRIC MANIFOLDS

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Abstract. We study the Riemann curvature tensor of $(\kappa, \mu, \nu)$-contact metric manifolds, which we prove to be completely determined in dimension 3, and we observe how it is affected by $D_{\alpha}$-homothetic deformations. This prompts the definition and study of generalized $(\kappa, \mu, \nu)$-space forms and of the necessary and sufficient conditions for them to be conformally flat.

1. Introduction

All researchers who are currently working on contact metric geometry and related topics agree on the great importance of $(\kappa, \mu)$-spaces, since they were introduced by D. E. Blair, T. Koufogiorgos and V. J. Papantoniou in [4] as those contact metric manifolds satisfying the equation

$$R(X, Y)\xi = \kappa \{\eta(Y)X - \eta(X)Y\} + \mu \{\eta(Y)hX - \eta(X)hY\},$$

for every $X, Y$ on $M$, where $\kappa$ and $\mu$ are constants, $h = 1/2L_\xi \phi$ and $L$ is the usual Lie derivative. These spaces include the Sasakian manifolds ($\kappa = 1$ and $h = 0$), but the non-Sasakian examples have proven to be even more interesting. Actually, their name was really given by E. Boeckx in [5], who also provided a classification the next year in [6]. R. Sharma extended the notion in [25], by considering $\kappa$ and $\mu$ to be differentiable functions on the manifold and called those new spaces generalized $(\kappa, \mu)$-spaces. Later, T. Koufogiorgos and C. Tsichlias proved in [22] that in dimensions greater than or equal to 5, the functions $\kappa, \mu$ must be constant and presented examples in dimension 3 with non-constant functions. There have been more papers dealing with these spaces, some of them replacing the contact metric structure by a different one, but let us emphasize the recent work published by B. Cappelletti Montano and L. Di Terlizzi in [7] as a proof of their relevance and possibilities.

Starting from the paper [20], in which T. Koufogiougos gave an expression for the curvature tensor of a $(\kappa, \mu)$-space with pointwise constant $\phi$-sectional curvature and dimension greater than or equal to 5, the second and third authors (jointly with M. M. Tripathi) recently defined in [9] a generalized $(\kappa, \mu)$-space form as an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6,$$

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where \( f_1, \ldots, f_6 \) are differentiable functions on \( M \) and \( R_1, \ldots, R_6 \) are the tensors given by
\[
R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \\
R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\
R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \\
R_4(X, Y)Z = g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX, \\
R_5(X, Y)Z = g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y, \\
R_6(X, Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi,
\]
for any vector fields \( X, Y, Z \). Such a manifold was denoted by \( M(f_1, \ldots, f_6) \) and several examples of it were presented in [3]. This notion also includes that of generalized Sasakian-space-forms, which can be obtained by putting \( f_4 = f_5 = f_6 = 0 \) in [2]. For more details about these spaces, see [1] and [2].

Also in [3], after the formal definition of a generalized \((\kappa, \mu, \nu)\)-space form was given, those with contact metric structure were deeply studied. It was proved that they are generalized \((\kappa, \mu, \nu)\)-spaces with \( \kappa = f_1 - f_3 \) and \( \mu = f_4 - f_6 \). Furthermore, if their dimension is greater than or equal to 5, then they are \((-6, -1, 1)-spaces\) with constant \( \phi \)-sectional curvature \( 2f_6 - 1 \), where \( f_4 = 1, f_5 = 1/2 \) and \( f_1, f_2, f_3 \) depend linearly on the constant \( f_6 \). A method for constructing infinitely many examples of this type was also presented.

Moreover, it was proved that the curvature tensor of a generalized \((\kappa, \mu, \nu)\)-space form is not unique in the 3-dimensional case and that several properties and results are also satisfied. Examples of generalized \((\kappa, \mu, \nu)\)-space forms with non-constant functions \( f_1, f_3 \) and \( f_4 \) were also given.

Later, in [5] the study of generalized \((\kappa, \mu)\)-spaces was continued by analysing the behaviour of such spaces under \( D_\alpha \)-homothetic deformations. An alternative definition of this type of manifold was introduced and it was proved that these spaces remain so after a \( D_\alpha \)-homothetic deformation, albeit with different functions. Infinitely many examples of this type of manifold were also showed in dimension 3 with some non-constant functions.

Going a step further from \((\kappa, \mu)\)-spaces, T. Koufogiorgos, M. Markellos and V. J. Papantoniou introduced in [21] the notion of \((\kappa, \mu, \nu)\)-contact metric manifold, where now the equation to be satisfied is
\[
R(X, Y)\xi = \kappa \{ \eta(Y)X - \eta(X)Y \} + \mu \{ \eta(Y)hX - \eta(X)hY \} \\
+ \nu \{ \eta(Y)\phi hX - \eta(X)\phi hY \}, \tag{3}
\]
for some smooth functions \( \kappa, \mu, \) and \( \nu \) on \( M \). They proved that, in dimension greater than or equal to 5, \( \kappa \) and \( \mu \) are necessarily constant and that \( \nu \) is zero, hence the \((\kappa, \mu, \nu)\)-contact metric manifolds are in particular \((\kappa, \mu)\)-spaces. They also proved that if a \( D_\alpha \)-homothetic deformation is applied to them, they keep being \((\kappa, \mu, \nu)\)-contact metric manifolds, although with different functions, result that they used to provide examples in dimension 3 with \( \nu \) a non-zero function. Some other authors also studied manifolds satisfying condition [3], but with a non-contact metric structure, as we will point out later.

In the present paper, after reviewing some concepts and results on almost contact metric manifolds in section [2] we prove in section [3] that the curvature tensor of a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold is completely determined and can
be written in terms of $\kappa, \mu, \nu$ and its $\phi$-sectional curvature $F$. We apply this result in order to give the curvature tensor in a particular example and we study how $D_\nu$-homothetic deformations affect it. In section 4 we define generalized $(\kappa, \mu, \nu)$-space forms as a generalization of generalized $(\kappa, \mu)$-space forms that englobes the $(\kappa, \mu, \nu)$-contact metric manifolds and we provide some properties and examples. Finally, in section 5 we study some necessary and sufficient conditions for generalized $(\kappa, \mu, \nu)$-space forms of dimension greater or equal to 5 to be conformally flat.

In conclusion, by introducing generalized $(\kappa, \mu, \nu)$-space forms we offer a very general frame in which many previous theories can be included and unified, opening new possibilities for further studies.

2. Preliminaries

In this section, we recall some general definitions and basic formulas which will be used later. For more background on almost contact metric manifolds, we recommend the reference [3].

An odd-dimensional Riemann manifold $(M, g)$ is said to be an almost contact metric manifold if there exist on $M$ a $(1, 1)$-tensor field $\phi$, a vector field $\xi$ (called the structure vector field) and a 1-form $\eta$ such that $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X) \xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields $X, Y$ on $M$. In particular, in an almost contact metric manifold we also have $\phi^2 = 0$ and $\eta \circ \phi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(\phi X, \phi Y)$ is the fundamental 2-form of $M$. If, in addition, $\xi$ is a Killing vector field, then $M$ is said to be a $K$-contact manifold. It is well-known that a contact metric manifold is a $K$-contact manifold if and only if

$$\nabla_X \xi = -\phi X$$

for all vector fields $X$ on $M$. Even an almost contact metric manifold satisfying the equation (4) becomes a $K$-contact manifold.

On the other hand, the almost contact metric structure of $M$ is said to be normal if the Nijenhuis torsion $[\phi, \phi]$ of $\phi$ equals $-2d\eta \otimes \xi$. A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$\nabla_X \phi Y = g(X, Y)\xi - \eta(Y)X$$

for any vector fields $X, Y$ on $M$. Moreover, for a Sasakian manifold the following equation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Given an almost contact metric manifold $(M, \phi, \xi, \eta, g)$, a $\phi$-section of $M$ at $p \in M$ is a section $\Pi \subseteq T_p M$ spanned by a unit vector $X_p$ orthogonal to $\xi_p$, and $\phi X_p$. The $\phi$-sectional curvature of $\Pi$ is defined by $K(X, \phi X) = R(X, \phi X, \phi X, X)$. A Sasakian manifold with constant $\phi$-sectional curvature $c$ is called a Sasakian space form. In such a case, its Riemann curvature tensor is given by equation $R = f_1 R_1 + f_2 R_2 + f_3 R_3$ with functions $f_1 = (c + 3)/4$, $f_2 = f_3 = (c - 1)/4$.

It is well known that on a contact metric manifold $(M, \phi, \xi, \eta, g)$, the tensor $h$, defined by $2h = L_\xi \phi$, satisfies the following relations [4]

$$h \xi = 0, \quad \nabla_X \xi = -\phi X - \phi h X, \quad h \phi = -\phi h, \quad tr h = 0, \quad \eta \circ h = 0.$$

Therefore, it follows from equations (4) and (5) that a contact metric manifold is $K$-contact if and only if $h = 0$. 
On the other hand, a contact metric manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) is said to be a generalized \((\kappa, \mu)\)-space if its curvature tensor satisfies the condition (1) for some smooth functions \(\kappa\) and \(\mu\) on \(M\) independent of the choice of vectors fields \(X\) and \(Y\). If \(\kappa\) and \(\mu\) are constant, the manifold is called a \((\kappa, \mu)\)-space. T. Koufogiorgos proved in [20] that if a \((\kappa, \mu)\)-space \(M\) has pointwise constant \(\phi\)-sectional curvature \(F\) and dimension greater than or equal to 5, the curvature tensor of this \((\kappa, \mu)\)-space form is given by equation (2), where

\[
\begin{align*}
  f_1 &= \frac{F + 3}{4}, \\
  f_2 &= \frac{F - 1}{4}, \\
  f_3 &= \frac{F + 3}{4} - \kappa, \\
  f_4 &= \frac{1}{2}, \\
  f_5 &= 1, \\
  f_6 &= 1 - \mu.
\end{align*}
\]

Recently, a \((\kappa, \mu, \nu)\)-contact metric manifold was defined in [21] as a contact metric manifold \((M, \phi, \xi, \eta, g)\) whose curvature tensor satisfies (3) for some smooth functions \(\kappa, \mu, \text{and } \nu\) on \(M\) independent of the choice of vectors fields \(X\) and \(Y\). According to the above notations, we could also refer to them as contact metric generalized \((\kappa, \mu, \nu)\)-spaces.

It was showed in [21] that every \((\kappa, \mu, \nu)\)-contact metric manifold of dimension greater than or equal to 5 is a \((\kappa, \mu)\)-space, but that there exist examples in dimension 3 with \(\nu \neq 0\).

Given an almost contact metric manifold \((M, \phi, \xi, \eta, g)\), we recall that a \(D\)-homothetic deformation is defined by

\[
\begin{align*}
  \phi &= \phi, \\
  \xi &= \frac{1}{\alpha} \xi, \\
  \eta &= a \eta, \\
  g &= ag + a(a - 1) \eta \otimes \eta,
\end{align*}
\]

where \(\alpha\) is a positive constant (see [24]). It is clear that \((M, \phi, \xi, \eta, g)\) is also an almost contact metric manifold and that

\[
\tau = \frac{1}{a} h.
\]

Finally, we will denote by \(Q\) the Ricci operator on \(M\) and define the scalar curvature as \(\tau = trQ\). We will also assume that all the functions considered in this paper will be differentiable functions on the corresponding manifolds.

### 3. The curvature tensor of \((\kappa, \mu, \nu)\)-contact metric manifolds

In this section we will study the curvature tensor of \((\kappa, \mu, \nu)\)-contact metric manifolds, which is completely determined in dimension 3. We will also see how a \(D\)-homothetic deformation affects it.

We know from [21] that a \((\kappa, \mu, \nu)\)-contact metric manifold satisfies \(\kappa \leq 1\) and that the condition \(\kappa = 1\) is equivalent to being Sasakian. Therefore, we will concentrate on the case \(\kappa < 1\).

**Theorem 3.1.** Let \(M\) be a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold with \(\kappa < 1\). Then its curvature tensor can be written as

\[
R = \left(\frac{\tau}{2} - 2 \kappa\right) R_1 + \left(\frac{\tau}{2} - 3 \kappa\right) R_3 + \mu R_4 + \nu R_7,
\]

where \(R_1, R_3, R_4\) are the same tensors appearing in \((\kappa)\) and \(R_7\) is the following one:

\[
R_7(X, Y)Z = g(Y, Z)\phi h X - g(X, Z)\phi h Y + g(\phi h Y, Z) X - g(\phi h X, Z) Y.
\]
Proof. It is well known that a 3-dimensional contact metric manifold satisfies:

\[ R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \]

(11)

\[ - \frac{\tau}{2} (g(Y,Z)X - g(X,Z)Y). \]

Thanks to Proposition 3.1 from [21] we also know that:

\[ Q = \left( \frac{\tau}{2} - \kappa \right) I + \left( -\frac{\tau}{2} + 3\kappa \right) \eta \otimes \xi + \mu h + \nu \phi h. \]

(12)

Substituting equation (12) in (11) we obtain:

\[ R(X,Y)Z = \left( \frac{\tau}{2} - 2\kappa \right) R_1(X,Y)Z + \left( \frac{\tau}{2} - 3\kappa \right) R_3(X,Y)Z + \mu R_4(X,Y)Z + \nu \{ g(Y,Z)\phi hX - g(X,Z)\phi hY + g(\phi hY,Z)X - g(\phi hX,Z)Y \}. \]

We only need to define the tensor \( R_7 \) as written in (10) in order to get the desired result.

We can also prove the following:

**Proposition 3.2.** Let \( M \) be a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold with \( \kappa < 1 \). Then its \( \phi \)-sectional curvature is \( F = \frac{\tau}{2} - 2\kappa \).

Proof. There exists a \( \phi \)-basis \( \{ e, \phi e, \xi \} \) with \( hX = \lambda X \) (where \( \lambda = \sqrt{1 - \kappa} \)) because \( \kappa < 1 \) (equation (4-8) from [21]). Due to the fact that the \( \phi \)-sectional curvature \( F = R(X, \phi X, \phi X, X) \) on a point \( P \in M \) does not depend on the choice of \( X \), then \( F = R(e, \phi e, \phi e, e) \).

If we use now Proposition 3.1 we get:

\[ F = R(e, \phi e, \phi e, e) = \left( \frac{\tau}{2} - 2\kappa \right) R_1(e, \phi e, \phi e, e) + \left( \frac{\tau}{2} - 3\kappa \right) R_3(e, \phi e, \phi e, e) + \mu R_4(e, \phi e, \phi e, e) + \nu R_7(e, \phi e, \phi e, e). \]

An straightforward computation gives us that

\[ R_1(e, \phi e, \phi e, e) = 1, \]

\[ R_3(e, \phi e, \phi e, e) = R_4(e, \phi e, \phi e, e) = R_7(e, \phi e, \phi e, e) = 0. \]

We conclude that

\[ F = R(e, \phi e, \phi e, e) = \frac{\tau}{2} - 2\kappa, \]

as stated above.

Therefore, Proposition 3.1 can be rewritten as:

**Corollary 3.3.** Let \( M \) be a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold with \( \kappa < 1 \). Then its curvature tensor can be written as

\[ R = FR_1 + (F - \kappa)R_3 + \mu R_4 + \nu R_7, \]

where \( F \) is the \( \phi \)-sectional curvature and \( R_1, R_3, R_4, R_7 \) are the previously defined tensors.
Remark 3.4. Corollary [3, 3] could also be obtained analogously to how it was proved in [20] that a \((\kappa, \mu)\)-space form of dimension greater than or equal to 5 has curvature tensor \(R = f_1R_1 + \cdots + f_6R_6\), with \(f_1, \ldots, f_6\) functions as in [7].

The hypothesis on the dimension was only used to prove that the \(\phi\)-sectional curvature is constant, not the form of the tensor \(R\), so the reasoning is also valid in dimension 3, where \(K(X, \phi X)\) is always independent of the choice of \(X\). Adapting that proof to the case of the \((\kappa, \mu, \nu)\)-contact metric manifolds, we would obtain that the formula of \(R(\tilde{X}, \tilde{Y})Z\) does not vary if \(\tilde{X}, \tilde{Y}\) are vector fields orthogonal to \(\xi\):

\[
R(\tilde{X}, \tilde{Y})Z = \frac{F + 3}{4} R_1(\tilde{X}, \tilde{Y})Z + \frac{F - 1}{4} R_2(\tilde{X}, \tilde{Y})Z + R_4(\tilde{X}, \tilde{Y})Z + \frac{1}{2} R_5(\tilde{X}, \tilde{Y})Z. \tag{14}
\]

If we use the fact that every vector field can be written as \(X = \tilde{X} + \eta(X)\xi\), where \(\tilde{X}\) is orthogonal to \(\xi\), and the formula of \(R(\xi, X)Y\) for a \((\kappa, \mu, \nu)\)-contact metric manifold (see equation (4-10) of [21]), we get

\[
R(X, Y)Z = R(\tilde{X}, \tilde{Y})Z - \eta(Y)R(\xi, \tilde{X})Z + \eta(X)R(\xi, \tilde{Y})Z
\]

\[
= \frac{F + 3}{4} R_1(\tilde{X}, \tilde{Y})Z + \frac{F - 1}{4} R_2(\tilde{X}, \tilde{Y})Z + R_4(\tilde{X}, \tilde{Y})Z + \frac{1}{2} R_5(\tilde{X}, \tilde{Y})Z
\]

\[
- \eta(Y)\left\{\kappa(g(\tilde{X}, Z)\xi + \eta(Z)\tilde{X}) + \mu(g(\tilde{h}X, Z)\xi - \eta(Z)\tilde{h}X)
\right.
\]

\[
\left. + \nu(g(\phi hZ, \tilde{X})\xi - \eta(Z)\phi h\tilde{X})\right\}
\]

\[
+ \eta(X)\left\{\kappa(g(\tilde{Y}, Z)\xi - \eta(Z)\tilde{Y}) + \mu(g(\tilde{h}Y, Z)\xi - \eta(Z)\tilde{h}Y)
\right.
\]

\[
\left. + \nu(g(\phi hZ, \tilde{Y})\xi - \eta(Z)\phi h\tilde{Y})\right\}. \tag{14}
\]

After some calculations where we use the definition of the tensors \(R_1, \ldots, R_6\) and that \(\tilde{X} = X - \eta(X)\xi\), it follows that the formula of the curvature tensor \(R(X, Y)Z\) for any vector fields \(X, Y, Z\) is:

\[
R(X, Y)Z = \frac{F + 3}{4} R_1(X, Y)Z + \frac{F - 1}{4} R_2(X, Y)Z + \left(\frac{F + 3}{4} - \kappa\right) R_3(X, Y)Z
\]

\[
+ R_4(X, Y)Z + \frac{1}{2} R_5(X, Y)Z + (1 - \mu) R_6(X, Y)Z
\]

\[
- \nu\left\{\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX + g(\phi hX, Z)\eta(Y)\xi - g(\phi hY, Z)\eta(X)\xi\right\}. \tag{14}
\]

If we denote by \(R_8\) the factor that multiplies \(\nu\), i.e., if we define the tensor

\[
R_8(X, Y)Z = \eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX
\]

\[
+ g(\phi hX, Z)\eta(Y)\xi - g(\phi hY, Z)\eta(X)\xi,
\]

we can write the Riemann curvature tensor as

\[
R = \frac{F + 3}{4} R_1 + \frac{F - 1}{4} R_2 + \left(\frac{F + 3}{4} - \kappa\right) R_3 + R_4 + \frac{1}{2} R_5 + (1 - \mu) R_6 - \nu R_8.
\]

We know from Lemma 3.8 of [8] that \(R_2 = 3(R_1 + R_3)\), \(R_5 = 0\) and \(R_6 = -R_4\) on every contact metric manifold, so we obtain:

\[
R = FR_1 + (F - \kappa) R_3 + \mu R_4 - \nu R_8.
\]

This new way of writing the curvature tensor coincides with [13] thanks to the fact that \(R_8 = -R_7\) in every 3-dimensional contact metric manifold \((M, \phi, \xi, \eta, g)\), which can be easily proved by checking that it is true for a \(\phi\)-basis \(\{E, \phi E, \xi\}\).
Corollary 3.3 also implies that the examples of 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifolds with non-constant functions given in [21] have curvature tensors written like (13), so we only need to calculate the \(\phi\)-sectional curvature \(F\) in order to know the tensor explicitly.

For instance, Example 4.2 of [21], which is a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold with \(\kappa = 1 - \frac{e^{2c}x^2}{4}, \mu = 2 + e^{cz}z^2\) and \(\nu = c \neq 0\) constant, has Riemann curvature tensor \(R = FR_1 + (F - \kappa)R_3 + \mu R_4 + \nu R_7\), where \(F\) is the \(\phi\)-sectional curvature

\[
F = -\left(3 + \frac{3}{2}c^2y^2 + 3c^2yz + \frac{3c}{4z^2} + c^2z^2 - \frac{1}{4}e^{2cz}z^2\right).
\]

We will now study how a \(Da\)-homothetic deformation affects the curvature tensor of a \((\kappa, \mu, \nu)\)-contact metric manifold.

We already know from [21] that applying a \(Da\)-homothetic deformation \((a > 0)\) to a \((\kappa, \mu, \nu)\)-contact metric manifold yields a new \((\kappa, \mu, \nu)\)-contact metric manifold with

\[
\kappa = \kappa + \frac{a^2 - 1}{a^2}, \quad \mu = \mu + 2a - 2, \quad \nu = \nu/a.
\]

Applying Corollary 13 we get that the deformed manifold has a Riemann curvature tensor that can be written as:

\[
\mathcal{R} = F\mathcal{R}_1 + (\mathcal{F} - \phi)\mathcal{R}_3 + \mu\mathcal{R}_4 + \nu\mathcal{R}_7,
\]

where \(\mathcal{F}\) is the \(\phi\)-sectional curvature and \(\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_7\) are the already defined tensors on the deformed manifold.

If we use (15) and the fact that \(\mathcal{F} = \frac{1}{a}F - \frac{a - 1}{a^2}(3a + 1 - \kappa)\) under the previous hypothesis, we conclude that the curvature tensor \(\mathcal{R}\) of the deformed \((\kappa, \mu, \nu)\)-contact metric manifold is

\[
\mathcal{R} = \mathcal{F}_1\mathcal{R}_1 + \mathcal{F}_3\mathcal{R}_3 + \mathcal{F}_4\mathcal{R}_4 + \mathcal{F}_7\mathcal{R}_7,
\]

where

\[
\begin{align*}
\mathcal{F}_1 &= \frac{1}{a}F - \frac{a - 1}{a^2}(3a + 1 - \kappa), \\
\mathcal{F}_3 &= \frac{1}{a}F + \frac{1}{a^2}((a - 2)\kappa - 4a^2 + 2a + 2), \\
\mathcal{F}_4 &= \frac{1}{a}(\mu + 2a - 2), \\
\mathcal{F}_7 &= \frac{\nu}{a},
\end{align*}
\]

and \(F\) is the \(\phi\)-sectional curvature of the original manifold. Therefore, we can completely determine the curvature tensor of the deformed manifold, just by knowing the original \(\kappa, \mu, \nu\) and \(F\).

4. Generalized \((\kappa, \mu, \nu)\)-space forms

We will extend the notion of generalized \((\kappa, \mu)\)-space form to englobe the \((\kappa, \mu, \nu)\)-contact metric manifolds.
Definition 4.1. A generalized $(\kappa, \mu, \nu)$-space form is an almost contact metric manifold whose curvature tensor can be written as
\begin{equation}
R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6 + f_7 R_7 + f_8 R_8,
\end{equation}
where $f_1, \ldots, f_8$ are arbitrary functions on $M$, $R_1, \ldots, R_8$ are the tensors in \cite{2}, $R_7$ the one that appears in \cite{10} and $R_8$ the one in \cite{11}. We will denote it by $M(f_1, \ldots, f_8)$.

Firstly, we study the contact metric case. We can easily obtain the following result:

Proposition 4.1. If $M(f_1, \ldots, f_8)$ is a contact metric generalized $(\kappa, \mu, \nu)$-space form, then it is a $(\kappa, \mu, \nu)$-contact metric manifold with $\kappa = f_1 - f_3$, $\mu = f_4 - f_6$ and $\nu = f_7 - f_8$.

Proof. We already knew from \cite{9} that
\begin{equation}
(f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6)(X,Y)\xi =
= (f_1 - f_3)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).
\end{equation}
It is easy to check that $R_7(X,Y)\xi = -R_8(X,Y)\xi = \eta(Y)\phi hX - \eta(X)\phi hY$.
Hence, we can conclude that the manifold is a $(f_1 - f_3, f_4 - f_6, f_7 - f_8)$-contact metric manifold. \hfill $\square$

Using Theorem 4.8 from \cite{9} and Theorem 4.1 from \cite{21}, it is obvious that the following theorem holds true:

Theorem 4.2. Let $M(f_1, \ldots, f_8)$ be a contact metric generalized $(\kappa, \mu, \nu)$-space form of dimension greater than or equal to 5. Then
\begin{equation}
\begin{align*}
f_1 &= \frac{f_6 + 1}{2}, & f_2 &= \frac{f_6 - 1}{2}, & f_3 &= \frac{3f_6 + 1}{2}, \\
f_4 &= 1, & f_5 &= \frac{1}{2}, & f_6 &= \text{constant} > -1, \\
\kappa &= -f_6 &= \text{constant} < 1, \\
\mu &= 1 - f_6 &= \text{constant} < 2, \\
\nu &= f_7 - f_8 &= 0, \\
F &= 2f_6 - 1 &= \text{constant} > -3.
\end{align*}
\end{equation}
Hence $M$ is a $(-f_6, 1-f_6)$-space with constant $\phi$-sectional curvature $F = 2f_6 - 1 > -1$.

Using Lemma 3.8 from \cite{8}, we can easily see that the curvature tensor of a 3-dimensional contact metric generalized $(\kappa, \mu, \nu)$-space form $M(f_1, \ldots, f_8)$ can be written as
\begin{equation}
R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6 + f_7 R_7 + f_8 R_8 =
= f_1 R_1 + 3f_2 R_2 + f_3 R_3 + f_4 R_4 - f_6 R_4 + f_7 R_7 - f_8 R_7 =
= (f_1 + 3f_2)R_1 + (f_3 + 3f_2)R_3 + (f_4 - f_6)R_6 + (f_7 - f_8)R_7.
\end{equation}
By an straightforward computation, we also have that the $\phi$-sectional curvature of that manifold would be $F = f_1 + 3f_2$, so its curvature tensor could be written as $R = FR_1 + (F-(f_1 - f_3))R_3 + (f_4 - f_6)R_6 + (f_7 - f_8)R_7 = FR_1 + (F-\kappa)R_3 + \mu R_4 + \nu R_7$, where $f_1, \ldots, f_8$ are arbitrary functions on $M$, $R_1, \ldots, R_8$ are the tensors in \cite{2}, $R_7$ the one that appears in \cite{10} and $R_8$ the one in \cite{11}. We will denote it by $M(f_1, \ldots, f_8)$.
which coincides with equation (13) from Corollary 3.3.

In conclusion, contact metric generalized $(\kappa, \mu, \nu)$-space forms are either $(\kappa, \mu)$-spaces (in dimension greater than or equal to 5) or $(\kappa, \mu, \nu)$-contact metric manifolds (in dimension 3). This fact does not detract from the interest of defining such manifolds because there are generalized $(\kappa, \mu, \nu)$-spaces that are not contact metric ones. For instance, G. Dileo and A. M. Pastore study in [17] and [18] and A.M. Pastore and V. Salterelli in [24] almost Kenmotsu manifolds which are also generalized $(\kappa, \mu)$-spaces or generalized $(\kappa, 0, \nu)$-spaces, though they use a different notation. They give examples in dimension 3.

Almost cosymplectic $(\kappa, \mu, \nu)$-spaces have also been widely studied. For $\mu = \nu = 0$, P. Dacko published [10], where he proved that $\kappa$ must be constant and H. Endo presented multiple results in [13] and [14]. This last author also examined in [15] and [16] these spaces for $\nu = 0$ and $\kappa, \mu$ constants. Afterwards, P. Dacko and Z. Olszak studied in [11] and [12] almost cosymplectic $(\kappa, \mu, \nu)$-spaces with $\kappa, \mu$ and $\nu$ functions that only vary in the direction of the vector field $\xi$, presenting multiple examples.

Moreover, H. Özturk, N. Aktan and C. Murathan examine in [23] the almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-spaces. They also provide an example of almost $\alpha$-cosymplectic $(\kappa, \mu)$-space of dimension 3 with non-constant functions $\kappa$ and $\mu$.

Are these generalized $(\kappa, \mu, \nu)$-spaces also generalized $(\kappa, 0, \nu)$-space forms? It can be proved that Theorem 3.1, Proposition 3.2 and Corollary 3.3 are also true for 3-dimensional $(\kappa, \mu, \nu)$-spaces with almost cosymplectic or almost Kenmotsu structures. Therefore, the previously mentioned examples are generalized $(\kappa, \mu, \nu)$-space forms with functions $f_1 = F$, $f_3 = F - \kappa$, $f_4 = \mu$, $f_7 = \nu$ and the rest zero.

5. CONFORMALLY FLAT GENERALIZED $(\kappa, \mu, \nu)$-SPACE FORMS

In this section, we will give necessary and sufficient conditions for a generalized $(\kappa, \mu, \nu)$-space form to be conformally flat if its dimension is greater than or equal to 5 and the tensor $h$ satisfies some properties.

It is easy to see that $R_1, \ldots, R_s$ must be zero if $h = 0$. Therefore, a generalized $(\kappa, \mu, \nu)$-space form with $h = 0$ is a generalized Sasakian space form, which was already studied under the hypothesis of conformal flatness by U. K. Kim in [19]. One of the results he proved was that a generalized Sasakian space form $M(f_1, f_2, f_3)$ of dimension greater than or equal to 5 is conformally flat if and only if $f_2 = 0$. We can give a similar result in our case if $h \neq 0$.

We recall that a Riemann manifold is said to be conformally flat if it is locally conformal to a flat manifold. The Schouten tensor of a manifold $M^{2n+1}$ is defined as

$$L = -\frac{1}{2n-1}Q + \frac{\tau}{4n(2n-1)}I,$$

and the Weyl tensor as

$$W(X, Y)Z = R(X, Y)Z - (g(LX, Z)Y - g(LY, Z)X + g(X, Z)LX - g(Y, Z)LX),$$

for all $X, Y, Z$ vector fields on $M$.

If the dimension of the manifold is greater than or equal to 5, it is well known that $M$ is conformally flat if and only if the Weyl tensor $W$ is identically zero. In
dimension 3, this tensor is always zero and the manifold is conformally flat if and only if the Schouten tensor is a Codazzi tensor, i.e., if it satisfies that \((\nabla_X L)Y - (\nabla_Y L)X = 0\), for all \(X,Y\) vector fields on \(M\).

Before presenting the main theorem of this section, let us see a result which will be used in its proof:

**Lemma 5.1.** Let \(M^{2n+1}(f_1, \ldots, f_8)\) be a generalized \((\kappa, \mu, \nu)\)-space form. If \(h \neq 0\), \(h\) is symmetric and \(h \phi + \phi h = 0\), then its Ricci operator is written as

\[
Q = (2n f_1 + 3 f_2 - f_3) I - (3 f_2 + f_3 (2n - 1)) \eta \otimes \xi + (f_4 (2n - 1) - f_6) + ((2n - 1) f_7 - f_8) \phi h.
\]

Therefore, its scalar curvature is

\[
\tau = 2n ((2n + 1) f_1 + 3 f_2 - 2 f_3).
\]

**Theorem 5.2.** Let \(M^{2n+1}(f_1, \ldots, f_8)\) a generalized \((\kappa, \mu, \nu)\)-space form of dimension greater than or equal to 5. If \(h \neq 0\), \(h\) is symmetric and \(h \phi + \phi h = 0\), then \(M\) is conformally flat if and only if \(f_2 = f_5 R_5 = f_6 = f_8 = 0\).

**Proof.** Substituting the formulas of the Ricci operator \((20)\) and the scalar curvature \((21)\) on a generalized \((\kappa, \mu, \nu)\)-space form in the definition of the Schouten tensor \((18)\), we get that

\[
L = - \frac{1}{2} \left( f_1 + \frac{3}{2n - 1} f_2 \right) I + \left( \frac{3}{2n - 1} f_2 + f_3 \right) \eta \otimes \xi - \left( f_4 - \frac{1}{2n - 1} f_6 \right) h - \left( f_7 - \frac{1}{2n - 1} f_8 \right) \phi h.
\]

Using now equations \((16)\) and \((22)\) in the definition of the Weyl tensor \((19)\), we obtain that it can be written as

\[
W = - \frac{3 f_2}{2n - 1} R_1 + f_2 R_2 - \frac{3 f_2}{2n - 1} R_3 + \frac{f_6}{2n - 1} R_4 + f_5 R_5 + f_6 R_6 + \frac{1}{2n - 1} f_8 R_7 + f_8 R_8.
\]

If \(f_2 = f_5 R_5 = f_6 = f_8 = 0\), it is obvious that \(W = 0\).

If \(W = 0\), then we have in particular that \(W(X, \xi) \xi = 0\) for every vector field \(X\) orthogonal to \(\xi\). Thanks to equation \((23)\), this means that

\[
\frac{2(1 - n)}{2n - 1} (f_6 h X + f_8 \phi h X) = 0.
\]

Now, \(2n + 1 > 3\) so \(f_6 h X + f_8 \phi h X = 0\). Moreover, \(h \neq 0\), so the vector fields \(h X\) and \(\phi h X\) are mutually orthogonal and not zero. Hence \(f_6 = f_8 = 0\) and \((23)\) could be written as

\[
W = - \frac{3 f_2}{2n - 1} R_1 + f_2 R_2 - \frac{3 f_2}{2n - 1} R_3 + f_5 R_5.
\]

Taking now \(X = \phi Y\) and \(Z = Y\), with \(Y\) an unit vector field orthogonal to \(\xi\), we obtain \(f_2 = 0\).

Therefore, the Weyl tensor would be \(W = f_5 R_5 = 0\) and we conclude the proof. \(\square\)

**Remark 5.3.** The hypothesis \(f_5 R_5 = 0\) of the previous theorem is not always equivalent to \(f_5 = 0\) because the tensor \(R_5\) could be identically zero, like it occurs in dimension 3. However, if \(f_5 = 0\) it is obvious that \(f_5 R_5 = 0\) and if \(R_5 = 0\), then \(f_5\) is an arbitrary function, so we can choose \(f_5 = 0\) in particular.
The properties “$h$ is symmetric” and “$\omega \phi + \phi h = 0$” are satisfied in some well-known cases. For example, if the manifold has a contact metric, an almost Kenmotsu or an almost cosymplectic structure.

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