Multiple nonnegative solutions of systems with coupled nonlinear boundary conditions

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Using the theory of fixed point index, we discuss the existence and multiplicity of nonnegative solutions of a wide class of boundary value problems with coupled nonlinear boundary conditions. Our approach is fairly general and covers a variety of situations. We illustrate in an example that all the constants that occur in our theory can be computed. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction

The aim of this paper is to present a theory for the existence of positive solution for a fairly general class of systems of ODEs subject to nonlinear, nonlocal boundary conditions (BCs). In particular, we are interested in systems that present a coupling in the BCs; this type of problems has been studied in [1–8] and often occurs in applications, for example, when modeling the displacement of a suspension bridge subject to nonlinear controllers.

In [9], Lu and coauthors, by means of the Krasnosel’skiĭ–Guo theorem on cone compressions and cone expansions, studied the existence of positive solutions of the system of ODEs

\[ u''(t) + f_1(t, u(t)) = 0, \quad t \in (0, 1), \]
\[ v^{(4)}(t) = f_2(t, u(t)), \quad t \in (0, 1), \]

subject to the BCs

\[ u(0) = u(1) = v(0) = v(1) = v''(0) = v''(1) = 0. \]

The motivation, given in [9], for studying the BVP (1)–(2) is that it can be seen as the stationary case of a model for the oscillations of the center span of a suspension bridge, where the fourth order equation represents the road bed (seen as an elastic beam) and the second order equation models the main cable (seen as a vibrating string). The BCs in this case illustrate the fact that the beam is simply supported and that the two ends of the cable are supposed to be immovable (see also, e.g., [10, 11]).

The existence of positive solutions of a coupled system with an elastic beam equation of the type

\[ u''(t) + f_1(t, v(t)) = 0, \quad t \in (0, 1), \]
\[ v^{(4)}(t) = f_2(t, u(t), v(t)), \quad t \in (0, 1), \]

has been studied by Sun in [12], by monotone iterative techniques, under the BCs

\[ u(0) = u(1) = v(0) = v(1) = v'(0) = v'(1) = 0. \]

A common feature of the systems (3)–(4) and (1)–(2) is that the BCs under consideration are local and homogeneous.

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In [13], Infante and coauthors, by means of classical fixed point index theory, provided a fairly general theory suitable to study the existence of nonnegative solutions of a variety of systems of ODEs subject to linear, nonlocal conditions, with one example being the system

\begin{align*}
  u''(t) + g_1(t)f_1(t, u(t), v(t)) &= 0, \quad t \in (0, 1), \\
  v^{(4)}(t) &= g_2(t)f_2(t, u(t), v(t)), \quad t \in (0, 1),
\end{align*}

with the BCs

\begin{align*}
  u(0) &= \beta_{11}[u], \quad u(1) = \delta_{12}[v], \quad v(0) = \beta_{21}[v], \quad v''(0) = 0, \quad v(1) = 0, \quad v''(0) + \delta_{22}[u] = 0,
\end{align*}

where \( \beta_{ij}[\cdot], \delta_{ij}[\cdot] \) are bounded linear functionals given by Riemann–Stieltjes integrals, namely

\begin{align*}
  \beta_{ij}[w] &= \int_0^1 w(s) \, d\beta_{ij}(s), \quad \delta_{ij}[w] = \int_0^1 w(s) \, d\delta_{ij}(s).
\end{align*}

This type of formulation includes, as special cases, multipoint or integral conditions, when

\begin{align*}
  \alpha_{ij}[w] &= \sum_{i=1}^m \alpha_{ij}w(\eta_{ij}) \quad \text{and} \quad \alpha_{ij}[w] = \int_0^1 \alpha_{ij}(s)w(s) \, ds
\end{align*}

(see, e.g., [14–23]).

In the case of the systems (5)–(6), the BCs

\begin{align*}
  u(0) = \beta u(\xi), \quad u(1) = v(1) = v''(0) = v(0) = 0, \quad v''(1) + \delta u(\eta) = 0
\end{align*}

can be interpreted as a cable-beam model with two devices of feedback control, where the displacement of the left end of the cable is related to displacement of another point \( \xi \) of the cable and the bending moment in the right end of the beam depends upon the displacement registered in a point \( \eta \) of the string. We point out that not necessarily the response of the controllers needs to be of linear type; for example, this happens with conditions of the type

\begin{align*}
  u(0) = H(u(\xi)), \quad u(1) = v(1) = v''(0) = v(0) = 0, \quad v''(1) + L(u(\eta)) = 0;
\end{align*}

we refer to [24] for more details regarding the illustration of nonlinear controllers on a beam.

Our approach allows us to deal with a larger class of nonlinear, nonlocal BCs, with one example given by the BCs

\begin{align*}
  u(0) = H_1(\beta_{11}[u]) + L_{11}(\delta_{11}[v]), \quad u(1) = H_2(\beta_{12}[u]) + L_{12}(\delta_{12}[v]), \\
  v(0) = H_2(\beta_{21}[v]) + L_{21}(\delta_{21}[u]), \quad v'(0) = 0, \quad v(1) = 0, \\
  v''(1) + H_2(\beta_{22}[v]) + L_{22}(\delta_{22}[u]) = 0,
\end{align*}

where \( H_{ij}, L_i \) are continuous functions. For earlier contributions on problems with nonlinear BCs, we refer the reader to [24–33] and references therein.

Here, we develop an existence theory for multiple positive solutions of the perturbed Hammerstein integral equations of the type

\begin{align*}
  u(t) &= \sum_{i=1,2} \gamma_{1i}(t) \left( H_{1i}(\beta_{1i}[u]) + L_{1i}(\delta_{1i}[v]) \right) + \int_0^1 k_{1i}(t, s)g_{1i}(s)f_1(s, u(s), v(s)) \, ds, \\
  v(t) &= \sum_{i=1,2} \gamma_{2i}(t) \left( L_{2i}(\delta_{2i}[u]) + H_{2i}(\beta_{2i}[v]) \right) + \int_0^1 k_{2i}(t, s)g_{2i}(s)f_2(s, u(s), v(s)) \, ds.
\end{align*}

Similar systems of perturbed Hammerstein integral equations were studied in [4, 5, 13, 34–38]. Our theory covers, as a special case, the systems (5)–(7), and we show in an example that all the constants that occur in our theory can be computed.

We make use of the classical fixed point index theory (see, e.g., [39,40]) and also benefit of ideas from the papers [13,32,36,41,42].

2. Positive solutions for systems of perturbed integral equations

We begin with stating some assumptions on the terms that occur in the system of perturbed Hammerstein integral equations

\begin{align*}
  u(t) &= \sum_{i=1,2} \gamma_{1i}(t) \left( H_{1i}(\beta_{1i}[u]) + L_{1i}(\delta_{1i}[v]) \right) + F_1(u, v)(t), \\
  v(t) &= \sum_{i=1,2} \gamma_{2i}(t) \left( L_{2i}(\delta_{2i}[u]) + H_{2i}(\beta_{2i}[v]) \right) + F_2(u, v)(t),
\end{align*}

\( i, j = 1, 2, \) \( i, j = 1, 2, \)
where

\[ F_i(u, v)(t) := \int_0^1 k_i(t, s)g_i(s)f_i(s, u(s), v(s)) \, ds, \]

namely

- For every \( i = 1, 2, f_i : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty) \) satisfies Carathéodory conditions, that is, \( f_i(\cdot, u, v) \) is measurable for each fixed \((u, v)\) and \( f_i(t, \cdot, \cdot) \) is continuous for almost every \((a.e.) \ t \in [0, 1] \), and for each \( r > 0 \), there exists \( \phi_{ij} \in L^\infty[0, 1] \) such that
  \[ f_i(t, u, v) \leq \phi_{ij}(t) \quad \text{for } u, v \in [0, r] \text{ and a.e. } t \in [0, 1], \]

- For every \( i = 1, 2, k_i : [0, 1] \times [0, 1] \to [0, \infty) \) is measurable, and for every \( \tau \in [0, 1] \), we have
  \[ \lim_{t \to \tau} |k_i(t, s) - k_i(\tau, s)| = 0 \quad \text{for a.e. } s \in [0, 1]. \]

- For every \( i = 1, 2 \), there exist a subinterval \([a_i, b_i] \subseteq [0, 1], \) a function \( \Phi_i \in L^\infty[0, 1] \), and a constant \( c_i \in (0, 1] \), such that
  \[ k_i(t, s) \leq \Phi_i(s) \quad \text{for } t \in [0, 1] \text{ and a.e. } s \in [0, 1], \]

\[ k_i(t, s) \geq c_i \Phi_i(s) \quad \text{for } t \in [a_i, b_i] \text{ and a.e. } s \in [0, 1], \]

- For every \( i = 1, 2, g_i, \Phi_i \in L^1[0, 1], g_i \geq 0 \text{ a.e.}, \) and \( \int_{a_i}^{b_i} \Phi_i(s)g_i(s) \, ds > 0. \)

- For every \( i, j = 1, 2, \beta_{ij}[\cdot] \) and \( \delta_{ij}[\cdot] \) are linear functionals given by
  \[ \beta_{ij}[w] = \int_0^1 w(s) \, dB_{ij}(s), \quad \delta_{ij}[w] = \int_0^1 w(s) \, dC_{ij}(s), \]

involving Riemann–Stieltjes integrals; \( B_{ij} \) and \( C_{ij} \) are of bounded variation, and \( dB_{ij}, dC_{ij} \) are positive measures.

- \( H_{ij}, L_{ij} : [0, \infty) \to [0, \infty) \) are continuous functions such that there exist \( h_{ij1}, h_{ij2}, l_{ij2} \in [0, \infty), i, j = 1, 2, \) with
  \[ h_{ij1}w \leq H_{ij}(w) \leq h_{ij2}w, \quad L_{ij}(w) \leq l_{ij2}w, \]

for every \( w \geq 0. \)

- \( \gamma_{ij} \in C[0, 1], \gamma_{ij}(t) \geq 0 \) for every \( t \in [0, 1], \) \( h_{ij2}\beta_{ij}[\gamma_{ij}] < 1, \) and there exists \( c_{ij} \in (0, 1] \) such that
  \[ \gamma_{ij}(t) \geq c_{ij}\|\gamma_{ij}\|_{\infty} \text{ for every } t \in [a_i, b_i], \]

where \( \|w\|_{\infty} := \max\{|w(t)|, t \in [0, 1]\}. \)

- \( D_i := (1 - h_{ij2}\beta_{ij}[\gamma_{ij}])/(1 - h_{ij2}\beta_{ij}[\gamma_{ij}]) - h_{ij2}h_{ij1}\beta_{ij}[\gamma_{ij}]. \) \( D_i > 0, i = 1, 2. \)

It follows from \( D_i > 0 \) that

\[ D_i := (1 - h_{ij1}\beta_{ij}[\gamma_{ij}])/(1 - h_{ij1}\beta_{ij}[\gamma_{ij}]) - h_{ij2}h_{ij1}\beta_{ij}[\gamma_{ij}] > 0. \]

We work in the space \( C[0, 1] \times C[0, 1] \) endowed with the norm

\[ \|(u, v)\| := \max\{|u|_{\infty}, |v|_{\infty}\}. \]

Let

\[ K := \left\{ w \in C[0, 1] : w(t) \geq 0 \text{ for } t \in [0, 1] \text{ and } \min_{t \in [a_i, b_i]} w(t) \geq c_i\|w\|_{\infty} \right\}, \]

where \( c_i = \min\{c_i, c_1, c_2\}, \) and consider the cone \( K \in C[0, 1] \times C[0, 1] \) defined by

\[ K := \{(u, v) \in K_1 \times K_2\}. \]

For a positive solution of the system (8), we mean a solution \((u, v) \in K \) of (8) such that \( \|(u, v)\| > 0 \).

Under our assumptions, it is routine to show that the integral operator

\[ T(u, v)(t) := \left( \sum_{i=1}^{1,2} \gamma_{ij} f_i(t, u(s), v(s)) \right) \left( \sum_{i=1}^{1,2} \gamma_{ij} h_i(t, u(s), v(s)) \right) = \left( \frac{F_1(u, v)(t)}{F_2(u, v)(t)} \right) \]

leaves the cone \( K \) invariant and is compact (see, e.g., Lemma 1 of [13]).
We use the following (relative) open bounded sets in $K$:

$$K_\rho = \{ (u, v) \in K : \| (u, v) \| < \rho \},$$

and

$$V_\rho = \left\{ (u, v) \in K : \min_{t \in [a, b_1]} u(t) < \rho \text{ and } \min_{t \in [a_0, b_2]} v(t) < \rho \right\}.$$ 

The set $V_\rho$ (in the context of systems) was introduced by the authors in [35] and is equal to the set called $\Omega^{\rho/c}$ in [34]. $\Omega^{\rho/c}$ is an extension to the case of systems of a set given by Lan [43]. For our index calculations, we make use of the fact that

$$K_\rho \subset V_\rho \subset K_{\rho/c},$$

where $c = \min\{c_1, c_2\}$. We denote by $\partial K_\rho$ and $\partial V_\rho$ the boundary of $K_\rho$ and $V_\rho$ relative to $K$.

We utilize the following results of [42] regarding order preserving matrices:

**Definition 2.1**

A $2 \times 2$ matrix $Q$ is said to be order preserving (or positive) if $p_1 \geq p_0, q_1 \geq q_0$ imply that

$$Q \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \geq Q \begin{pmatrix} p_0 \\ q_0 \end{pmatrix},$$

in the sense of components.

**Lemma 2.2** ([42])

Let

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

with $a, b, c, d \geq 0$ and $\det Q > 0$. Then, $Q^{-1}$ is order preserving.

**Remark 2.3**

It is a consequence of Lemma 2.2 that if

$$\mathcal{N} = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$$

satisfies the hypotheses of Lemma 2.2, $p \geq 0, q \geq 0$, and $\mu > 1$, then

$$\mathcal{N}^{-1} = \begin{pmatrix} p \\ q \end{pmatrix} \leq \mathcal{N}^{-1} \begin{pmatrix} p \\ q \end{pmatrix},$$

where

$$\mathcal{N}^{-1} = \begin{pmatrix} \mu-a & -b \\ -c & \mu-d \end{pmatrix}.$$ 

In the sequel of the paper, we use the following notation.

$$K_{ij}(s) := \int_0^1 k_i(t, s) d\beta_j(t), \quad Q_1 = \sum_{i=1,2} \beta_1[\gamma_2][\delta_1][1], \quad S_1 = \sum_{i=1,2} \beta_2[\gamma_2][\delta_1][1],$$

$$\theta_1 = \frac{1 - h_{12}^2\beta_2[\gamma_2]}{D_i}, \quad \theta_2 = \frac{h_{12}^2\beta_1[\gamma_2]}{D_i}, \quad \theta_3 = \frac{h_{12}^2\beta_1[\gamma_1]}{D_i}, \quad \theta_4 = \frac{1 - h_{12}^2\beta_1[\gamma_1]}{D_i}.$$ 

We are now able to prove a result concerning the fixed point index on the set $K_\rho$.

**Lemma 2.4**

Assume that

$$f_{0,\rho}(\| \gamma_1 \| \infty h_{12}^2 \theta_1 + \| \gamma_2 \| \infty h_{12}^2 \theta_3) \int_0^1 K_{11}(s) g_1(s) ds + (\| \gamma_1 \| \infty h_{12}^2 \theta_2 + \| \gamma_2 \| \infty h_{12}^2 \theta_4) \int_0^1 K_{12}(s) g_2(s) ds + \frac{1}{m_i})$$

$$+ \| \gamma_1 \| \infty h_{12}^2 (\theta_1 O_i + \theta_2 S_i) + \| \gamma_2 \| \infty h_{12}^2 (\theta_3 Q_i + \theta_4 S_i) + \sum_{j=1,2} \| \gamma_j \| \infty h_{12}^2 [1] < 1$$

(9)
Thus, we have

\[
\begin{align*}
\rho f_0^\rho &= \sup \left\{ \frac{f(t,u,v)}{\rho} : (t,u,v) \in [0,1] \times [0,\rho] \times [0,\rho] \right\}
\quad \text{and} \quad \frac{1}{m_1} = \sup_{t \in [0,1]} \int_0^1 \kappa(t,s) g(s) \, ds.
\end{align*}
\]

Then, the fixed point index, \( i_K(T,K_\rho) \), is equal to 1.

**Proof**

We show that \( \mu(u,v) \neq T(u,v) \) for every \((u,v) \in \partial K_\rho\) and for every \( \mu \geq 1 \); this ensures that the index is 1 on \( K_\rho \). In fact, if this does not happen, there exist \( \mu \geq 1 \) and \((u,v) \in \partial K_\rho\) such that \( \mu(u,v) = T(u,v) \). Assume, without loss of generality, that \( \|u\|_\infty = \rho \) and \( \|v\|_\infty \leq \rho \). Then,

\[
\mu u(t) = \sum_{i=1,2} \gamma_{11}(t) (H_{1}(\beta_{11}[u]) + L_{11}(\delta_{11}[v])) + F_{1}(u,v)(t),
\]

and therefore, because \( v(t) \leq \rho \), for all \( t \in [0,1] \),

\[
\mu u(t) \leq \sum_{i=1,2} \gamma_{11}(t) h_{12} \beta_{11}[u] + \sum_{i=1,2} \gamma_{11}(t) h_{12} \delta_{11}[\rho] + F_{1}(u,v)(t) = \sum_{i=1,2} \gamma_{11}(t) h_{12} \beta_{11}[u] + \rho \sum_{i=1,2} \gamma_{11}(t) h_{12} \delta_{11}[1] + F_{1}(u,v)(t).
\]  

Applying \( \beta_{11} \) and \( \beta_{12} \) to both sides of (10) gives

\[
\mu \beta_{11}[u] \leq \sum_{i=1,2} \beta_{11}[\gamma_{11}] h_{12} \beta_{11}[u] + \rho \sum_{i=1,2} \beta_{11}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{11}[F_{1}(u,v)],
\]

\[
\mu \beta_{12}[u] \leq \sum_{i=1,2} \beta_{12}[\gamma_{11}] h_{12} \beta_{12}[u] + \rho \sum_{i=1,2} \beta_{12}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{12}[F_{1}(u,v)].
\]

Thus, we have

\[
(\mu - h_{12} \beta_{11}[\gamma_{11}]) \beta_{11}[u] - h_{12} \beta_{11}[\gamma_{11}] \beta_{12}[u] \leq \rho \sum_{i=1,2} \beta_{11}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{11}[F_{1}(u,v)],
\]

\[
-h_{12} \beta_{12}[\gamma_{11}] \beta_{11}[u] + (\mu - h_{12} \beta_{12}[\gamma_{11}]) \beta_{12}[u] \leq \rho \sum_{i=1,2} \beta_{12}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{12}[F_{1}(u,v)],
\]

that is,

\[
\begin{pmatrix}
\mu - h_{12} \beta_{11}[\gamma_{11}] & -h_{12} \beta_{11}[\gamma_{11}] \\
-h_{12} \beta_{12}[\gamma_{11}] & \mu - h_{12} \beta_{12}[\gamma_{11}]
\end{pmatrix}
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\leq
\begin{pmatrix}
\rho \sum_{i=1,2} \beta_{11}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{11}[F_{1}(u,v)] \\
\rho \sum_{i=1,2} \beta_{12}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{12}[F_{1}(u,v)]
\end{pmatrix}.
\]

The matrix

\[
M_{\mu} = \begin{pmatrix}
\mu - h_{12} \beta_{11}[\gamma_{11}] & -h_{12} \beta_{11}[\gamma_{11}] \\
-h_{12} \beta_{12}[\gamma_{11}] & \mu - h_{12} \beta_{12}[\gamma_{11}]
\end{pmatrix}
\]

satisfies the hypotheses of Lemma 2.2; thus, \( (M_{\mu})^{-1} \) is order preserving. If we apply \( (M_{\mu})^{-1} \) to both sides of the inequality (11), then we obtain

\[
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\leq
\begin{pmatrix}
\frac{1}{\det(M_{\mu})}
\end{pmatrix}
\begin{pmatrix}
\mu - h_{12} \beta_{12}[\gamma_{11}] & h_{12} \beta_{12}[\gamma_{11}] \\
h_{12} \beta_{12}[\gamma_{11}] & \mu - h_{12} \beta_{12}[\gamma_{11}]
\end{pmatrix}
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\]

\[
\begin{pmatrix}
\rho \sum_{i=1,2} \beta_{11}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{11}[F_{1}(u,v)] \\
\rho \sum_{i=1,2} \beta_{12}[\gamma_{11}] h_{12} \delta_{11}[1] + \beta_{12}[F_{1}(u,v)]
\end{pmatrix},
\]

and by Remark 2.3, we have

\[
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\leq
\begin{pmatrix}
\frac{1}{D_1}
\end{pmatrix}
\begin{pmatrix}
1 - h_{12} \beta_{12}[\gamma_{11}] & h_{12} \beta_{12}[\gamma_{11}] \\
h_{12} \beta_{12}[\gamma_{11}] & 1 - h_{12} \beta_{12}[\gamma_{11}]
\end{pmatrix}
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\]

\[
\begin{pmatrix}
\rho Q_1 + \beta_{11}[F_{1}(u,v)] \\
\rho S_1 + \beta_{12}[F_{1}(u,v)]
\end{pmatrix}.
\]

Thus, we have

\[
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\leq
\begin{pmatrix}
\theta_{11} & \theta_{12} \\
\theta_{13} & \theta_{14}
\end{pmatrix}
\begin{pmatrix}
\rho Q_1 + \beta_{11}[F_{1}(u,v)] \\
\rho S_1 + \beta_{12}[F_{1}(u,v)]
\end{pmatrix}.
\]


Thus,
\[
\left( \frac{\beta_{11}[u]}{\beta_{12}[u]} \right) \leq \left( \frac{\rho(\theta_1 Q_1 + \theta_2 S_1) + \theta_{11} \beta_{11} [F_1(u, v)] + \theta_{12} \beta_{12} [F_1(u, v)]}{\rho(\theta_3 Q_1 + \theta_4 S_1) + \theta_{13} \beta_{11} [F_1(u, v)] + \theta_{14} \beta_{12} [F_1(u, v)]} \right).
\]
Substituting into (10) gives
\[
\mu u(t) \leq \rho \left( \gamma_{11}(t) h_{12}(\theta_1 Q_1 + \theta_2 S_1) + \gamma_{12}(t) h_{122}(\theta_3 Q_1 + \theta_4 S_1) + \sum_{i=1,2} \gamma_{11}(t) h_{12} \delta_{i1}[1] \right)
+ (\gamma_{11}(t) h_{12} \theta_{11} + \gamma_{12}(t) h_{122} \theta_{13}) \beta_{11} [F_1(u, v)]
+ (\gamma_{11}(t) h_{12} \theta_{12} + \gamma_{12}(t) h_{122} \theta_{14}) \beta_{12} [F_1(u, v)]
+ F_1(u, v)(t)
= \rho \left( \gamma_{11}(t) h_{12}(\theta_1 Q_1 + \theta_2 S_1) + \gamma_{12}(t) h_{122}(\theta_3 Q_1 + \theta_4 S_1) + \sum_{i=1,2} \gamma_{11}(t) h_{12} \delta_{i1}[1] \right)
+ (\gamma_{11}(t) h_{12} \theta_{11} + \gamma_{12}(t) h_{122} \theta_{13}) \int_0^1 K_{11}(s) g_1(s, u(s), v(s)) \, ds
+ (\gamma_{11}(t) h_{12} \theta_{12} + \gamma_{12}(t) h_{122} \theta_{14}) \int_0^1 K_{12}(s) g_1(s, u(s), v(s)) \, ds + F_1(u, v)(t).
\]
Taking the supremum over [0, 1] gives
\[
\mu \rho \leq \rho \left( \|\gamma_{11}\|_{\infty} h_{12}(\theta_1 Q_1 + \theta_2 S_1) + \|\gamma_{12}\|_{\infty} h_{122}(\theta_3 Q_1 + \theta_4 S_1) + \sum_{i=1,2} \|\gamma_{11}\|_{\infty} h_{12} \delta_{i1}[1] \right)
+ \rho \theta_{11} (\|\gamma_{11}\|_{\infty} h_{12} \theta_{11} + \|\gamma_{12}\|_{\infty} h_{122} \theta_{13}) \int_0^1 K_{11}(s) g_1(s) \, ds
+ \rho \theta_{12} (\|\gamma_{11}\|_{\infty} h_{12} \theta_{12} + \|\gamma_{12}\|_{\infty} h_{122} \theta_{14}) \int_0^1 K_{12}(s) g_1(s) \, ds + 1 C_1 M_1 = 1.
\]
Using the hypothesis (9), we obtain \( \mu \rho < \rho \). This contradicts the fact that \( \mu \geq 1 \) and proves the result.

We give the first lemma that shows that the index is 0 on a set \( V_\rho \).

**Lemma 2.5**

Assume that
\[
\begin{align*}
f_{1}(\rho, \rho/c) &= \inf \left\{ \frac{f_1(t, u, v)}{\rho} : (t, u, v) \in [a_1, b_1] \times [\rho, \rho/c] \times [0, \rho/c] \right\}, \\
f_{2}(\rho, \rho/c) &= \inf \left\{ \frac{f_2(t, u, v)}{\rho} : (t, u, v) \in [a_2, b_2] \times [0, \rho/c] \times [\rho, \rho/c] \right\} \text{ and } \frac{1}{\bar{M}_j} = \inf_{t \in [a_1, b_1]} \int_{a_1}^{b_1} k_1(t, s) g_1(s) \, ds.
\end{align*}
\]

Then, \( i_K(T, V_\rho) = 0 \).

**Proof**

Let \( e(t) = 1 \) for \( t \in [0, 1] \). Then, \( (e, e) \in K \). We prove that
\[
(u, v) \neq T(u, v) + \mu (e, e) \quad \text{for} \ (u, v) \in \partial V_\rho \quad \text{and} \quad \mu \geq 0.
\]
In fact, if this does not happen, there exist \( (u, v) \in \partial V_\rho \) and \( \mu \geq 0 \) such that \( (u, v) = T(u, v) + \mu (e, e) \). Without loss of generality, we can assume that for all \( t \in [a_1, b_1] \), we have
\[
\rho \leq u(t) \leq \rho/c, \quad \min u(t) = \rho \quad \text{and} \quad 0 \leq v(t) \leq \rho/c.
\]
Then, for $t \in [a_1, b_1]$, we obtain

$$u(t) = \sum_{i=1,2} \gamma_{1i}(t) (H_{1i}(\beta_{1i}[u]) + L_{1i}(\delta_{1i}[v])) + F_1(u, v)(t) + \mu e,$$

and therefore,

$$u(t) \geq \sum_{i=1,2} \gamma_{1i}(t)H_{1i}(\beta_{1i}[u]) + F_1(u, v)(t) + \mu e \geq \sum_{i=1,2} \gamma_{1i}(t)h_{1i}(\beta_{1i}[u]) + F_1(u, v)(t) + \mu e. \quad (13)$$

Applying $\beta_{11}$ and $\beta_{12}$ to both sides of (13) gives

$$\beta_{11}[u] \geq h_{111}\beta_{11}[\gamma_{11}][\beta_{11}[u] + h_{122}\beta_{12}[\gamma_{12}][\beta_{12}[u] + \beta_{11}[F_1(u, v)] + \mu \beta_{11}[e],$$

$$\beta_{12}[u] \geq h_{112}\beta_{12}[\gamma_{11}][\beta_{11}[u] + h_{122}\beta_{12}[\gamma_{12}][\beta_{12}[u] + \beta_{12}[F_1(u, v)] + \mu \beta_{12}[e].$$

Thus, we have

$$\begin{align*}
(1 - h_{111}\beta_{11}[\gamma_{11}])\beta_{11}[u] - h_{122}\beta_{12}[\gamma_{12}] \beta_{12}[u] & \geq \beta_{11}[F_1(u, v)] + \mu \beta_{11}[e], \\
-h_{111}\beta_{12}[\gamma_{11}])\beta_{11}[u] + (1 - h_{122}\beta_{12}[\gamma_{12}])\beta_{12}[u] & \geq \beta_{12}[F_1(u, v)] + \mu \beta_{12}[e],
\end{align*}$$

that is,

$$\begin{pmatrix}
(1 - h_{111}\beta_{11}[\gamma_{11}]) & -h_{122}\beta_{12}[\gamma_{12}] \\
-h_{111}\beta_{12}[\gamma_{11}] & (1 - h_{122}\beta_{12}[\gamma_{12}])
\end{pmatrix}
\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\geq
\begin{pmatrix}
\beta_{11}[F_1(u, v)] + \mu \beta_{11}[e] \\
\beta_{12}[F_1(u, v)] + \mu \beta_{12}[e]
\end{pmatrix}.$$

The matrix

$$M_1 = \begin{pmatrix}
1 - h_{111}\beta_{11}[\gamma_{11}] & -h_{122}\beta_{12}[\gamma_{12}] \\
-h_{111}\beta_{12}[\gamma_{11}] & 1 - h_{122}\beta_{12}[\gamma_{12}]
\end{pmatrix}$$

satisfies the hypotheses of Lemma 2.2; thus, $(M_1)^{-1}$ is order preserving. If we apply $(M_1)^{-1}$ to both sides of the last inequality, then we obtain

$$\begin{pmatrix}
\beta_{11}[u] \\
\beta_{12}[u]
\end{pmatrix}
\leq
\frac{1}{M_1}
\begin{pmatrix}
1 - h_{122}\beta_{12}[\gamma_{12}] & h_{111}\beta_{12}[\gamma_{11}] \\
-h_{111}\beta_{11}[\gamma_{11}] & 1 - h_{111}\beta_{11}[\gamma_{11}]
\end{pmatrix}
\begin{pmatrix}
\beta_{11}[F_1(u, v)] \\
\beta_{12}[F_1(u, v)]
\end{pmatrix},$$

and therefore,

$$u(t) \geq \frac{\gamma_{11}(t)}{D_1}h_{111}(1 - h_{122}\beta_{12}[\gamma_{12}]) + \frac{\gamma_{12}(t)}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{11}] \times \int_{t_0}^{1} K_{11}(s)g_1(s)f_1(s, u(s), v(s)) ds$$

$$+ \frac{\gamma_{11}(t)}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{12}] + \frac{\gamma_{12}(t)}{D_1}h_{111}h_{111}\beta_{11}[\gamma_{11}]h_{122} \times \int_{t_0}^{1} K_{12}(s)g_1(s)f_1(s, u(s), v(s)) ds$$

$$+ \int_{t_0}^{1} k_1(t, s)g_1(s)f_1(s, u(s), v(s)) ds + \mu.$$

Then, we have, for $t \in [a_1, b_1],

$$u(t) \geq \frac{c_{11}[\gamma_{11}]}{D_1}h_{111}(1 - h_{122}\beta_{12}[\gamma_{12}]) + \frac{c_{12}[\gamma_{12}]}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{11}] \times \int_{a_1}^{b_1} K_{11}(s)g_1(s)f_1(s, u(s), v(s)) ds$$

$$+ \frac{c_{11}[\gamma_{11}]}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{12}] + \frac{c_{12}[\gamma_{12}]}{D_1}h_{111}h_{111}\beta_{11}[\gamma_{11}]h_{122} \times \int_{a_1}^{b_1} K_{12}(s)g_1(s)f_1(s, u(s), v(s)) ds$$

$$+ \int_{a_1}^{b_1} k_1(t, s)g_1(s)f_1(s, u(s), v(s)) ds + \mu.$$

Taking the minimum over $[a_1, b_1]$ gives

$$\rho = \min_{t \in [a_1, b_1]} u(t) \geq \rho f_1, \rho, \rho/c \left( \frac{c_{11}[\gamma_{11}]}{D_1}h_{111}(1 - h_{122}\beta_{12}[\gamma_{12}]) + \frac{c_{12}[\gamma_{12}]}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{11}] \right) \times \int_{a_1}^{b_1} K_{11}(s)g_1(s) ds$$

$$+ \rho f_1, \rho, \rho/c \left( \frac{c_{11}[\gamma_{11}]}{D_1}h_{111}h_{122}\beta_{12}[\gamma_{12}] + \frac{c_{12}[\gamma_{12}]}{D_1}(1 - h_{111}\beta_{11}[\gamma_{11}]h_{122}) \times \int_{a_1}^{b_1} K_{12}(s)g_1(s) ds$$

$$+ \rho f_1, \rho, \rho/c \frac{1}{M_1} \mu,$$

Using the hypothesis (12), we obtain $\rho > \rho + \mu$, a contradiction. \qed
The following lemma provides a result of index 0 on $V_\rho$ of a different flavor; the idea is to control the growth of just one nonlinearity $f_1$ at the cost of having to deal with a larger domain. The proof is omitted as it follows from the previous proof (for details, see [13, 36]). We mention that nonlinearities with different growth were studied also in [38, 44, 45].

**Lemma 2.6**

Assume that

$$(f_0^*)^* \text{ there exists } \rho > 0 \text{ such that for some } i = 1, 2,$$

$$f^*_{i(0, \rho/c)} \left( \left( c_1 \| y_1 \|_{h_{11}} \right) (1 - h_{12} \beta_1 \| y_1 \|) + c_2 \| y_2 \|_{h_{21}} \beta_2 \| y_1 \|_1 \right) \int_0^f K_{i1}(s) g_i(s) \, ds + \left( \frac{c_1 \| y_1 \|_{h_{11}}}{D} h_{12} \beta_1 \| y_1 \| \right)$$

$$+ \left( \frac{c_2 \| y_2 \|_{h_{21}}}{D} (1 - h_{12} \beta_1 \| y_1 \|) \right) \int_0^f K_{i2}(s) g_i(s) \, ds + \frac{1}{M_i} > 1,$$

where

$$f^*_{i(0, \rho/c)} = \inf \left\{ \frac{f(t, u, v)}{\rho} : (t, u, v) \in [a_i, b_i] \times [0, \rho/c] \times [0, \rho/c] \right\}.$$

Then, $\imath K(T, V_\rho) = 0$.

The aforementioned lemmas can be combined to prove the following theorem, and here, we deal with the existence of at least one, two, or three solutions. We stress that, by expanding the lists in conditions (S_5), (S_6), it is possible, in a similar way as in [46], to state the results for four or more positive solutions. We omit the proof that follows from the properties of fixed point index.

**Theorem 2.7**

The system (8) has at least one positive solution in $K$ if either of the following conditions holds.

**S_1** There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1/c < \rho_2$ such that $\left( \frac{\rho_0}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$ hold.

**S_2** There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < \rho_2$ such that $\left( \frac{\rho_1}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$ hold.

The system (8) has at least two positive solutions in $K$ if one of the following conditions holds.

**S_3** There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1/c < \rho_2 < \rho_3$ such that $\left( \frac{\rho_0}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$, $\left( \frac{\rho_0}{\rho_3} \right)$ hold.

**S_4** There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \rho_2$, and $\rho_2/c < \rho_3$ such that $\left( \frac{\rho_1}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$, $\left( \frac{\rho_0}{\rho_3} \right)$ hold.

The system (8) has at least three positive solutions in $K$ if one of the following conditions holds.

**S_5** There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ with $\rho_1/c < \rho_2 < \rho_3$ and $\rho_3/c < \rho_4$ such that $\left( \frac{\rho_0}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$, $\left( \frac{\rho_0}{\rho_3} \right)$, and $\left( \frac{\rho_0}{\rho_4} \right)$ hold.

**S_6** There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3 < \rho_4$ such that $\left( \frac{\rho_1}{\rho_1} \right)$, $\left( \frac{\rho_0}{\rho_2} \right)$, $\left( \frac{\rho_0}{\rho_3} \right)$, and $\left( \frac{\rho_0}{\rho_4} \right)$ hold.

**Remark 2.8**

If the nonlinearities $f_1$ and $f_2$ have some extra positivity properties, for example, if the condition (S_1) holds and moreover we assume that $f_1(t, 0, v) > 0$ in $[a_1, b_1] \times \{0\} \times [0, \rho_2]$ and $f_2(t, u, 0) > 0$ in $[a_2, b_2] \times [0, \rho_2] \times \{0\}$, then the solution $(u, v)$ of the system (8) is such that $\|u\|_\infty$ and $\|v\|_\infty$ are strictly positive.

### 3. An application to coupled systems of boundary value problems

We study the existence of positive solutions for the system of second order ODEs

$$u''(t) + g_1(t)f_1(t, u(t), v(t)) = 0, \quad t \in (0, 1),$$

$$v''(t) = g_2(t)f_2(t, u(t), v(t)), \quad t \in (0, 1),$$

with the nonlinear, nonlocal BCS

$$u(0) = H_{11}(\beta_{11}[u]) + L_{11}(\delta_{11}[v]), \quad u(1) = H_{12}(\beta_{12}[u]) + L_{12}(\delta_{12}[v]),$$

$$v(0) = H_{21}(\beta_{21}[v]) + L_{21}(\delta_{21}[u]), \quad v''(0) = 0, \quad v(1) = 0,$$

$$v''(1) + H_{22}(\beta_{22}[v]) + L_{22}(\delta_{22}[u]) = 0.$$
This differential system can be rewritten in the integral form

\[
\begin{align*}
u(t) &= (1 - t)(H_{11}(\beta_{11}[u]) + L_{11}(\delta_{11}[v])) + t(H_{12}(\beta_{12}[u]) + L_{12}(\delta_{12}[v])) + \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds, \\
v(t) &= (1 - t)(H_{21}(\beta_{21}[v]) + L_{21}(\delta_{21}[u])) + \frac{1}{6} t(1 - t^2)(H_{22}(\beta_{22}[v]) + L_{22}(\delta_{22}[u])) + \int_0^1 k_2(t, s)g_2(s)f_2(s, u(s), v(s)) \, ds,
\end{align*}
\]

where

\[
k_1(t, s) = \begin{cases} s(1 - t), & s \leq t, \\ t(1 - s), & s > t, \end{cases} \quad \text{and} \quad k_2(t, s) = \begin{cases} \frac{1}{2}s(1 - t)(2t - s^2 - t^2), & s \leq t, \\ \frac{1}{8} t(1 - s)(2s - t^2), & s > t, \end{cases}
\]

are nonnegative continuous functions on \([0, 1] \times [0, 1]\).

The intervals \([a_1, b_1]\) and \([a_2, b_2]\) may be chosen arbitrarily in \((0, 1)\). It is easy to check that

\[
k_1(t, s) \leq s(1 - s) := \Phi_1(s), \quad \min_{t \in [a_1, b_1]} k_1(t, s) = c_1 s(1 - s),
\]

where \(c_1 = \min\{1 - b_1, a_1\}\). Furthermore, see [47], we have that

\[
k_2(t, s) \leq \Phi_2(s) := \begin{cases} \frac{\sqrt{3}}{27} s(1 - s^2)^{\frac{3}{2}}, & 0 \leq s \leq \frac{1}{2}, \\ \frac{\sqrt{3}}{27} (1 - s) s^{\frac{3}{2}} (2 - s)^{\frac{3}{2}}, & \frac{1}{2} < s \leq 1, \end{cases}
\]

and

\[
k_2(t, s) \geq c_2(t) \Phi_2(s),
\]

where

\[
c_2(t) = \begin{cases} \frac{3\sqrt{3}}{27} t(1 - t^2), & t \in [0, 1/2], \\ \frac{3\sqrt{3}}{27} t(1 - t)(2 - t), & t \in (1/2, 1], \end{cases}
\]

so that

\[
c_2 = \min_{t \in [a_2, b_2]} c_2(t) > 0.
\]

The existence of multiple solutions of the systems (14)–(15) follows from Theorem 2.7.

The nonlinearities that occur in the next example, taken from [13], are used to illustrate, under a mathematical point of view, the constants that occur in our theory.

**Example 3.1**

Consider the system

\[
\begin{align*}
u'' + (1/8) \left( u^3 + r^3 v^3 \right) + 2 &= 0, \quad t \in (0, 1), \\
u(0) &= H_{11}(u(1/4)) + L_{11}(v(1/4)), \quad u(1) = H_{12}(u(3/4)) + L_{12}(v(3/4)), \\
v(0) &= H_{21}(v(1/3)) + L_{21}(u(1/3)), \quad v'(0) = 0, \quad v(1) = 0, \\
v''(1) + H_{22}(v(2/3)) + L_{22}(u(2/3)) &= 0,
\end{align*}
\]

where the nonlocal conditions are given by the functionals \(\beta_j[w] = \delta_j[w] = w(n_j)\) and the functions \(H_j\) and \(L_j\) satisfy the condition

\[
h_{i1} w \leq H_j(w) \leq h_{i2} w, \quad L_j(w) \leq l_{i2} w,
\]

with

\[
h_{111} = \frac{1}{6}, h_{112} = \frac{1}{2}, h_{121} = \frac{1}{9}, h_{122} = \frac{1}{3}, h_{211} = \frac{1}{6}, h_{212} = \frac{1}{4}, h_{221} = \frac{1}{2}, h_{222} = \frac{2}{3},
\]
\[
h_{11} = \frac{1}{15}, h_{12} = \frac{1}{20}, l_{11} = \frac{1}{20}, l_{12} = \frac{1}{15}.
\]
The functions $H_l$ and $L_l$ can be built in a similar way as in [36] by choosing, for example,

$$H_{11}(w) = \begin{cases} \frac{1}{4} w, & 0 \leq w \leq 1, \\ \frac{1}{4} w + \frac{7}{4}, & w \geq 1, \end{cases} \quad L_{11}(w) = \frac{1}{11} \left( 1 + \sin \left( w - \frac{\pi}{2} \right) \right).$$

The choice $[a_1, b_1] = [a_2, b_2] = [1/4, 3/4]$ gives

$$c_1 = 1/4, \quad c_2 = 45\sqrt{3}/128, \quad c_{11} = c_{12} = c_{21} = 1/4, \quad c_{22} = 45\sqrt{3}/128,$$

$$m_1 = 8, \quad M_1 = 16, \quad m_2 = 384/5, \quad M_2 = 768/5.$$

We have that

$$\beta_{12} [\gamma_{12}] = \beta_{12} [\gamma_{11}] = \beta_{12} [\gamma_{11}] = \frac{3}{4}, \quad \beta_{12} [\gamma_{12}] = \frac{1}{4}, \quad \beta_{22} [\gamma_{21}] = \frac{2}{3}, \quad \beta_{22} [\gamma_{22}] = \frac{4}{81},$$

$$\beta_{22} [\gamma_{21}] = \frac{1}{3}, \quad \beta_{22} [\gamma_{22}] = \frac{5}{81}, \quad \delta_{11} [1] = \delta_{12} [1] = \delta_{21} [1] = \delta_{22} [1] = 1.$$

Because $K_{ij}(s) = k_i(\eta_j, s)$, we obtain

$$\int_0^1 K_{11}(s) \, ds = \int_0^1 K_{12}(s) \, ds = \frac{3}{32}, \quad \int_{1/4}^{3/4} K_{11}(s) \, ds = \int_{1/4}^{3/4} K_{12}(s) \, ds = \frac{1}{16},$$

$$\int_0^1 K_{21}(s) \, ds = \int_0^1 K_{22}(s) \, ds = \frac{11}{972}, \quad \int_{1/4}^{3/4} K_{21}(s) \, ds = \int_{1/4}^{3/4} K_{22}(s) \, ds = \frac{3985}{497664}.$$

Then, for $\rho_1 = 1/8, \rho_2 = 1, \rho_3 = 11$, we have the following constants have been rounded to two decimal places unless exact

$$\inf \{ f_1 (t, u, v) : (t, u, v) \in [1/4, 3/4] \times [0, 1/2] \times [0, 1/2] \} = f_1 (1/4, 0, 0) > 14.81 \rho_1,$$

$$\sup \{ f_1 (t, u, v) : (t, u, v) \in [0, 1] \times [0, 1] \times [0, 1] \} = f_1 (1, 1, 1) < 2.97 \rho_2,$$

$$\inf \{ f_2 (t, u, v) : (t, u, v) \in [0, 1] \times [0, 1] \times [0, 1] \} = f_2 (1, 1, 1) < 53.93 \rho_2,$$

$$\inf \{ f_3 (t, u, v) : (t, u, v) \in [1/4, 3/4] \times [11, 44] \times [0, 44] \} = f_1 (1/4, 11, 0) > 14.81 \rho_3,$$

$$\inf \{ f_4 (t, u, v) : (t, u, v) \in [1/4, 3/4] \times [0, 44] \times [11, 44] \} = f_2 (1/4, 0, 11) > 141.49 \rho_3;$$

that is, the conditions ($\mu_{p_l}^0$), ($\varphi_{p_l}^1$), and ($\mu_{p_l}^0$) are satisfied; therefore, the system (16) has at least two positive solutions in $K$.

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