Covariant Isotropy of Grothendieck Toposes

Jason Parker

Brandon University, Manitoba, Canada

Toposes Online
July 2, 2021
Introduction

- Covariant isotropy is a (recent) categorical construction that can be seen as providing a generalized notion of *conjugation* or *inner automorphism* for an arbitrary category.

- In previous work (cf. [3, 6, 4]), we used techniques from categorical logic to characterize the covariant isotropy of any locally finitely presentable category $\mathbb{C}$, and in particular of any *presheaf category*.

- In this talk, we will provide an overview of covariant isotropy and show that its characterization for any presheaf category (essentially) extends to any Grothendieck topos. This is based on my recent preprint [7] *Covariant isotropy of Grothendieck toposes*: https://arxiv.org/abs/2104.13487
Motivation for Covariant Isotropy

George Bergman proved in [1] that the inner automorphisms of groups can be characterized purely categorically as the group automorphisms that extend naturally along any group homomorphism.

To see this, observe first that if $\alpha$ is an inner automorphism of a group $G$ (induced by $s \in G$), then for each group morphism $f : G \to H$ we can ‘push forward’ $\alpha$ to define an inner automorphism $\tilde{\alpha}_f : H \tilde{\rightarrow} H$ by conjugation with $f(s) \in H$ (so that $\alpha_{\text{id}_G} = \alpha$).
Motivation

This family of automorphisms \((\alpha_f)_f\) is coherent, in the sense that it satisfies the following naturality property: if \(f : G \to G'\) and \(f' : G' \to G''\) are group homomorphisms, then the following diagram commutes:

\[
\begin{array}{ccc}
G' & \xrightarrow{\alpha_f} & G' \\
\downarrow f' & & \downarrow f' \\
G'' & \xrightarrow{\alpha_{f' \circ f}} & G'' \\
\end{array}
\]
Bergman’s Theorem

For a group $G$, let us call an arbitrary family of automorphisms

$$
\left( \alpha_f : \text{cod}(f) \xrightarrow{\sim} \text{cod}(f) \right)_{\text{dom}(f) = G}
$$

with the above naturality property an extended inner automorphism of $G$. Such a family is a natural automorphism of $G/\text{Group} \to \text{Group}$.

**Theorem (Bergman [1])**

Let $G$ be a group and $\alpha : G \xrightarrow{\sim} G$ an automorphism of $G$. Then $\alpha$ is an inner automorphism of $G$ iff there is an extended inner automorphism $(\alpha_f)_f$ of $G$ with $\alpha = \alpha_{\text{id}_G}$.

This provides a completely categorical characterization of inner automorphisms of groups: they are exactly those group automorphisms that are ‘coherently extendible’ along morphisms out of the domain.
Covariant Isotropy

We have a functor $\mathcal{Z} : \text{Group} \to \text{Group}$ that sends any group $G$ to its group of extended inner automorphisms $\mathcal{Z}(G)$. We refer to $\mathcal{Z}$ as the covariant isotropy group (functor) of the category $\text{Group}$. (Bergman’s theorem actually entails that $\mathcal{Z} \cong \text{Id} : \text{Group} \to \text{Group}$.)

In fact, any category $\mathcal{C}$ has a covariant isotropy group (functor)

$$\mathcal{Z}_\mathcal{C} : \mathcal{C} \to \text{Group}$$

that sends each object $C \in \mathcal{C}$ to the group of extended inner automorphisms of $C$, i.e. families of automorphisms

$$\left( \alpha_f : \text{cod}(f) \xrightarrow{\sim} \text{cod}(f) \right)_{\text{dom}(f) = C}$$

in $\mathcal{C}$ with the same naturality property as before, i.e. natural automorphisms of the projection functor $\mathcal{C}/\mathcal{C} \to \mathcal{C}$.
We can also turn Bergman’s characterization of inner automorphisms in Group into a definition of inner automorphisms in an arbitrary category $\mathcal{C}$: if $C \in \mathcal{C}$, we say that an automorphism $\alpha : C \overset{\sim}{\rightarrow} C$ is inner if there is an extended inner automorphism $(\alpha_f)_f \in Z_C(C)$ with $\alpha \text{id}_C = \alpha$.

Notice that Group is the category of (set-based) models of an algebraic theory, i.e. a set of equational axioms between terms.

In [3, 6, 4] we generalized ideas from the proof of Bergman’s Theorem to give a logical characterization of the (extended) inner automorphisms of $\mathbb{T}\text{mod}$, i.e. of the covariant isotropy group of $\mathbb{T}\text{mod}$, for any finitary quasi-equational theory $\mathbb{T}$. 

Covariant Isotropy
Quasi-Equational Theories

- A finitary quasi-equational theory $\mathcal{T}$ over a multi-sorted finitary equational signature $\Sigma$ is a set of implications (the axioms of $\mathcal{T}$) of the form $\varphi \Rightarrow \psi$, with $\varphi, \psi$ finitary Horn formulas (see [5]).

- The operation symbols of a quasi-equational theory are only required to be partially defined. If $t$ is a term, we write $t \downarrow$ as an abbreviation for $t = t$, meaning ‘$t$ is defined’.

- If $\lambda$ is a regular cardinal, a $\lambda$-ary quasi-equational theory $\mathcal{T}$ allows for $\lambda$-ary operations and $\lambda$-ary conjunctions.
Examples

- Any algebraic theory is a finitary quasi-equational theory, as are the theories of categories, groupoids, strict monoidal categories, and presheaves on any small category.

- If \((\mathcal{C}, \mathcal{J})\) is a small site, then the Grothendieck topos \(\text{Sh}(\mathcal{C}, \mathcal{J})\) is the category of models for a \(\lambda\)-ary quasi-equational theory \(T(\mathcal{C}, \mathcal{J})\). The sorts are the objects of \(\mathcal{C}\), for any arrow \(f : C \to D\) of \(\mathcal{C}\) there is a unary operation symbol \(\hat{f} : D \to C\), and for any covering sieve \(J \in \mathcal{J}(\mathcal{C})\) there is an (infinitary) operation symbol \(\sigma_J : \prod_{f \in J} \text{dom}(f) \to C\).

- One then has axioms expressing functoriality and the fact that any matching family has a unique amalgamation, so that the models of \(T(\mathcal{C}, \mathcal{J})\) are exactly the sheaves on \((\mathcal{C}, \mathcal{J})\).
The Isotropy Group of a Quasi-Equational Theory

- Fix a $\lambda$-ary quasi-equational theory $\mathcal{T}$ over a $\lambda$-ary signature $\Sigma$, and let $\mathcal{T}_{\text{mod}}$ be its category of (set-based) models.

- We will now review the *logical/syntactic* characterization of the covariant isotropy group

  $$Z_\mathcal{T} : \mathcal{T}_{\text{mod}} \to \text{Group}$$

  of $\mathcal{T}_{\text{mod}}$. This was achieved for finitary $\mathcal{T}$ in [3, 6, 4] and extended to general $\lambda$-ary $\mathcal{T}$ in [7].

- Using the quasi-equational syntax of $\mathcal{T}$, we can define a notion of *definable automorphism* for a model $M$ of $\mathcal{T}$, and the definable automorphisms of any $M \in \mathcal{T}_{\text{mod}}$ form a group $\text{DefInn}(M)$. 
Definable Automorphisms

- If $T$ is single-sorted, then given $M \in T \text{mod}$, one can form the $T$-model $M \langle x \rangle$ obtained from $M$ by freely adjoining an indeterminate element $x$. Elements of $M \langle x \rangle$ are congruence classes $[t]$ of terms $t$ involving $x$ and constants from $M$, where two terms $s, t$ are congruent if they are provably equal in the diagram theory $T(M, x)$ of $M$ extended by the axiom $T \vdash x \downarrow$.

- An element $[t] \in M \langle x \rangle$ is (substitutionally) invertible if there is some $[s] \in M \langle x \rangle$ with

$$T(M, x) \vdash t[s/x] = x = s[t/x].$$

- If $f$ is an $n$-ary operation symbol of $\Sigma$, then $[t] \in M \langle x \rangle$ commutes generically with $f$ if $T(M, x_1, \ldots, x_n)$ proves the sequent

$$f(x_1, \ldots, x_n) \downarrow \vdash t[f(x_1, \ldots, x_n)/x] = f(t(x_1), \ldots, t(x_n)).$$
Definable Automorphisms

- If $f$ is an $n$-ary operation symbol of $\Sigma$, then $[t] \in M\langle x \rangle$ reflects definedness of $f$ if $T(M, x_1, \ldots, x_n)$ proves the sequent

$$t[f(x_1, \ldots, x_n)/x] \Downarrow \vdash f(x_1, \ldots, x_n) \Downarrow.$$ 

- We define $\text{DefInn}(M)$ to be the group of all elements $[t] \in M\langle x \rangle$ that are substitutionally invertible and commute generically with and reflect definedness of every operation symbol of $\Sigma$.

- If $T$ is multi-sorted, one can extend the above definitions appropriately.
The Isotropy Group of a Quasi-Equational Theory

Theorem ([4, 7])

Let $\mathcal{T}$ be a $\lambda$-ary quasi-equational theory. For any $M \in \mathcal{T} \text{mod}$, the covariant isotropy group $\mathcal{Z}_{\mathcal{T}}(M)$, i.e. the group of extended inner automorphisms of $M$, is isomorphic to the group $\text{DefInn}(M)$ of definable automorphisms of $M$.

In [3] we used this result to show that the categorical inner automorphisms in many categories of algebraic structures (monoids, (abelian) groups, non-commutative unital rings, etc.) are precisely the conjugation-theoretic inner automorphisms.
Presheaf Categories

- In [4] we also characterized the covariant isotropy group of a presheaf category $\text{Set}^\mathcal{C}$ for a small category $\mathcal{C}$.

- If $F : \mathcal{C} \to \text{Set}$ is a presheaf, we showed that $\text{DefInn}(F)$ consists (up to isomorphism) of exactly the natural automorphisms $\alpha : F \sim F$ induced by some element $\psi \in \text{Aut}(\text{Id}_\mathcal{C})$, in the sense that

  $$(C \in \mathcal{C}) \quad \alpha_C = F(\psi_C) : F(C) \sim F(C).$$

- It then follows that the covariant isotropy group $\mathcal{Z} : \text{Set}^\mathcal{C} \to \text{Group}$ is *constant* on the automorphism group $\text{Aut}(\text{Id}_\mathcal{C})$ of $\text{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$. 


In [7], we wanted to determine if this result extended to arbitrary sheaf categories.

For convenient technical reasons, we first restricted our attention to subcanonical sites \((\mathbb{C}, J)\) where no object is covered by the empty sieve. If \(F\) is a sheaf over any such site, we showed that \(\text{DefInn}(F)\) consists (up to isomorphism) of precisely those natural automorphisms \(\alpha : F \xrightarrow{\sim} F\) induced by some element \(\psi \in \text{Aut}(\text{Id}_{\mathbb{C}})\), as above. The proof of this fact is the most non-trivial part of the overall result in [7].
Hence, as for presheaf categories, if $(\mathcal{C}, \mathcal{J})$ is any small subcanonical site in which no object is covered by the empty sieve, it follows that the covariant isotropy group $\mathcal{Z} : \text{Sh}(\mathcal{C}, \mathcal{J}) \to \text{Group}$ is constant on the automorphism group $\text{Aut}(\text{Id}_\mathcal{C})$ of $\text{Id}_\mathcal{C}$.

We now want to remove the assumptions of subcanonicity and no object being covered by the empty sieve. The second property is easier to remove: if $(\mathcal{C}, \mathcal{J})$ is any small subcanonical site, one can find another subcanonical site $(\mathcal{D}, \mathcal{K})$ in which no object is covered by the empty sieve, with $\text{Sh}(\mathcal{C}, \mathcal{J}) \simeq \text{Sh}(\mathcal{D}, \mathcal{K})$ and $\text{Aut}(\text{Id}_\mathcal{C}) \cong \text{Aut}(\text{Id}_\mathcal{D})$.

So if $(\mathcal{C}, \mathcal{J})$ is any small subcanonical site, then $\mathcal{Z} : \text{Sh}(\mathcal{C}, \mathcal{J}) \to \text{Group}$ is still constant on $\text{Aut}(\text{Id}_\mathcal{C})$. 
Grothendieck Toposes

- We now want to consider arbitrary small sites \((\mathbb{C}, J)\). First, if \(E\) is a (locally small) category with small full dense subcategory \(\mathbb{C} \hookrightarrow E\), then \(\text{Aut} (\text{Id}_E) \cong \text{Aut} (\text{Id}_\mathbb{C})\).

- Now if \((\mathbb{C}, J)\) is any small site, then there is a subcanonical topology \(K\) on the small full dense subcategory \(\text{ay}\mathbb{C} \hookrightarrow \text{Sh}(\mathbb{C}, J)\) for which \(\text{Sh}(\mathbb{C}, J) \cong \text{Sh}(\text{ay}\mathbb{C}, K)\).

- So then \(Z : \text{Sh}(\mathbb{C}, J) \rightarrow \text{Group}\) is constant on \(\text{Aut} (\text{Id}_{\text{ay}\mathbb{C}}) \cong \text{Aut} (\text{Id}_{\text{Sh}(\mathbb{C}, J)})\).
In particular, since $\text{Set}^{\mathcal{C}^{\text{op}}} = \text{Sh}(\mathcal{C}, T)$ for the trivial (subcanonical) topology $T$ on $\mathcal{C}$ (only maximal sieves cover), we recover our earlier result for presheaf toposes.

If $(\mathcal{C}, \mathcal{J})$ is not subcanonical, there is in general no relation between $\text{Aut}(\text{Id}_\mathcal{C})$ and $\text{Aut}(\text{Id}_{\text{Sh}(\mathcal{C}, \mathcal{J})})$. E.g. if $\text{Aut}(\text{Id}_\mathcal{C})$ is non-trivial and $\mathcal{J}$ is such that every sieve covers, then $\mathcal{J}$ is not subcanonical and $\text{Sh}(\mathcal{C}, \mathcal{J})$ is trivial, so that $\text{Aut}(\text{Id}_{\text{Sh}(\mathcal{C}, \mathcal{J})}) \cong \text{Aut}(\text{Id}_{\text{Sh}(\mathcal{C}, \mathcal{J})})$ is trivial.

Our result illustrates a major difference between covariant isotropy $\text{Sh}(\mathcal{C}, \mathcal{J}) \to \text{Group}$ and contravariant isotropy $\text{Sh}(\mathcal{C}, \mathcal{J})^{\text{op}} \to \text{Group}$ (cf. [2]) for Grothendieck toposes: while the latter is always representable by a sheaf of groups $Z : \mathcal{C}^{\text{op}} \to \text{Group}$, the former is always constant (on the global sections of $Z$).
Conclusions

- Via Bergman's purely *categorical* characterization of the inner automorphisms of groups, covariant isotropy can be regarded as providing a notion of *conjugation* or *inner automorphism* for arbitrary categories.

- We have characterized the covariant isotropy group of $\mathcal{T}\text{mod}$ for any $\lambda$-ary quasi-equational theory $\mathcal{T}$: we have $\mathcal{Z}_\mathcal{T}(M) \cong \text{DefInn}(M)$ for any $M \in \mathcal{T}\text{mod}$.

- Using this result, we have shown that the characterization of covariant isotropy for presheaf toposes extends to all Grothendieck toposes: for a small subcanonical site $(\mathcal{C}, \mathcal{J})$, the covariant isotropy group $\mathcal{Z} : \text{Sh}(\mathcal{C}, \mathcal{J}) \to \text{Group}$ is constant on $\text{Aut}(\text{Id}_\mathcal{C})$. 
Conclusions

- Although this result shows that covariant isotropy (as opposed to contravariant isotropy) for Grothendieck toposes is in some sense ‘degenerate’, the proof of this fact is nevertheless non-trivial.

- We also intend to build on this result to characterize the covariant isotropy of categories of sheaves of algebraic structures, which will be non-constant in general (in [6] we showed that this is the case for categories of presheaves of algebraic structures).
Thank you!
References I

[1] G. Bergman. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. Publicacions Matematiques 56, 91-126, 2012.

[2] J. Funk, P. Hofstra, B. Steinberg. Isotropy and crossed toposes. Theory and Applications of Categories 26, 660-709, 2012.

[3] P. Hofstra, J. Parker, P. Scott. Isotropy of algebraic theories. Electronic Notes in Theoretical Computer Science 341, 201-217, 2018.

[4] P. Hofstra, J. Parker, P. Scott. Polymorphic automorphisms and the Picard group. 6th International Conference on Formal Structures for Computation and Deduction (FSCD 2021), N. Koyayashi, Ed. Dagstuhl Publications LIPIcs, Vol. 195, FSCD 2021 (to appear).

[5] E. Palmgren, S.J. Vickers. Partial Horn logic and cartesian categories. Annals of Pure and Applied Logic 145, 314-353, 2007.
[6] J. Parker. Isotropy groups of quasi-equational theories. PhD thesis, University of Ottawa, 2020.

[7] J. Parker. Covariant isotropy of Grothendieck toposes. Preprint, 2021. https://arxiv.org/abs/2104.13487