GEVREY REGULARITY OF SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT CUTOFF

TENG-FEI ZHANG AND ZHAOYANG YIN

ABSTRACT. In this paper, we study the Gevrey regularity of spatially homogeneous Boltzmann equation without angular cutoff. We prove the propagation of Gevrey regularity for $C^\infty$ solutions with the Maxwellian decay to the Cauchy problem of spatially homogeneous Boltzmann equation. The idea we use here is based on the framework of Morimoto’s recent paper (See Morimoto: J. Pseudo-Differ. Oper. Appl. (2010) 1: 139-159, DOI:10.1007/s11868-010-0008-z), but we extend the range of the index $\gamma$ satisfying $\gamma + 2s \in (-1, 1)$, $s \in (0, 1/2)$ and in this case we consider the kinetic factor in the form of $\Phi(v) = |v|^\gamma$ instead of $\langle v \rangle^\gamma$ as Morimoto did before.

1. INTRODUCTION

We first introduce the Boltzmann equation for the spatially inhomogeneous case:

$$f_t(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f,f)(v), \quad t \in \mathbb{R}^+, \quad x, v \in \mathbb{R}^3,$$

where $f = f(t,x,v)$ is the density distribution function of particles located around position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The right-hand side of the above equation is the so called Boltzmann bilinear collision operator acting only on the velocity variable $v$:

$$Q(g,f) = \int_{\mathbb{R}^3} \int_{S^2} B(v-v_*,\sigma) \{ g'_* f' - g_* f \} \, d\sigma dv_*.$$

Hereafter we use the notation $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$, and for convenience we choose the $\sigma$-representation to describe the relations between the post- and pre-collisional velocities, that is, for $\sigma \in S^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

We note that the collision process satisfies the conservation of momentum and kinetic energy, that is,

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The collision cross section $B(z,\sigma)$ is a given non-negative function, and depends only on the interaction law between particles. By a mathematical language, that means $B(z,\sigma)$ depends only on the relative velocity $|z| = |v - v_*|$ and the deviation angle $\theta$ through the scalar product $\cos \theta = \frac{z}{|z|} \cdot \sigma$.
In what follows, we consider the case in which the cross section $B$ can be assumed to be of the form:

$$B(v - v^*, \cos \theta) = \Phi(|v - v^*|) |\cos \theta|,$$

where the kinetic factor $\Phi$ is given by

$$\Phi(|v - v^*|) = |v - v^*|^\gamma,$$

and the angular part $b$ with singularity satisfies,

$$\sin \theta b(\cos \theta) \sim K\theta^{-1-2s}, \quad \text{as } \theta \to 0^+,$$

for some positive constant $K$ and $0 < s < 1$.

We remark here if the inter-molecule potential satisfies the inverse-power law potential $U(\rho) = \rho^{-(p-1)}$, $p > 2$, it holds that $\gamma = \frac{p-5}{p-1}$, $s = \frac{1}{p-1}$. Generally, the case $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ correspond to so called hard, Maxwellian, and soft potentials respectively. And the case $0 < s < 1/2$, $1/2 \leq s < 1$ correspond to so called mild singularity and strong singularity respectively.

In the paper we consider the Cauchy problem for the spatially homogeneous Boltzmann equation without cutoff, with a $T > 0$, (1.1)

$$\begin{cases}
    f_t(t, v) = Q(f, f)(v), & t \in (0, T], v \in \mathbb{R}^n, \\
    f(0, v) = f_0(v),
\end{cases}$$

where “spatially homogeneous” means that $f$ depends only on $t$ and $v$.

Let us give a brief review about the study for Boltzmann equation. Since Grad introduced a cutoff assumption for cross section $B$ due to the difficulties coming from the singularity of the angular part $b$, there have been a lot of results. And as for the non-cutoff theory, great progress has been made in recent years, for that we refer the reader to Alexandre’s review paper [6]. And when considering non-cutoff Boltzmann equation in Gevrey spaces (see Definition 1.2 below), Ukai in [18] shows that the Cauchy problem for the Boltzmann equation admits a unique local solution in Gevrey classes for both spatially homogeneous and inhomogeneous cases, under the assumption on the cross section:

$$|B(|z|, \cos \theta)| \leq K(1 + |z|^{-\gamma'} + |z|^{\gamma}) \theta^{-n+1-2s}, \quad n \text{ is dimensionality,}$$

$$(0 \leq \gamma' < n, 0 \leq \gamma < 2, 0 \leq s < 1/2, \gamma + 6s < 2).$$

By introducing the norm of Gevrey space

$$\|f\|_{U^\rho,\nu} = \sum_\alpha \frac{|\alpha|!}{\rho^{|\alpha|} \nu^{2|\alpha|}} \|e^{\delta(v)^2} \partial_v^\alpha f\|_{L^\infty(\mathbb{R}^n)},$$

it was proved that in the spatially homogeneous case, for instance, under some assumptions for $\nu$ and the initial datum $f_0(v)$, the Cauchy problem (1.1) has a unique solution $f(t, v)$ for $t \in (0, T]$.

We then turn to the work of Devillettes, he firstly studied in [16] the $C^\infty$ smoothing effect for solutions of Cauchy problem for spatially homogeneous non-cutoff case, and conjectured Gevrey smoothing effect. Some years later, he proved in [15] the propagation of Gevrey regularity for solutions without any assumption on the decay at infinity in $v$ variables.

So far there have been extensive studies on the Gevrey regularity of solutions. We remark that in [17], Morimoto et al. considered the Gevrey regularity for the
linearized Boltzmann equation around the absolute Maxwellian distribution, by using a mollifier as follows:

$$G_{\delta}(t, D_v) = \frac{e^{t(D_v)\gamma/\nu}}{1 + \delta e^{t(D_v)\gamma/\nu}}, \quad 0 < \delta < 1.$$  

The same method was used later for many related research. We refer the reader to [7, 9, 10] which were about the ultra-analytic smoothing effect for spatially homogeneous nonlinear Landau equation in the Maxwellian case and the linear and non-linear Fokker-Planck equations, and [8] which studied the Kac’s equation (a simplification of Boltzmann equation to one dimension case).

Recently Morimoto considered in [1] the Gevrey regularity of $C^\infty$ solutions with the Maxwellian decay to the Cauchy problem of spatially homogeneous Boltzmann equation. We here consider the general kinetic factor $\Phi(|v - v_*|) = |v - v_*|^{\gamma}$ taking the place of the modified kinetic factor $\Phi(|v - v_*|) = (1 + |v - v_*|^2)^{\gamma/2}$ used in [1], and extend his results for the range of $\gamma$ by using some new estimates.

In the paper we consider the mild singularity case $0 < s < 1/2$, and we assume that

$$-1 < \gamma + 2s < 1.$$  

In the case $0 < s < 1/2$, Huo et al. proved in [14] that any weak solution $f(t, v)$ to (1.1) satisfying the natural boundedness on mass, energy and entropy, namely,

$$(1.2) \quad \int_{\mathbb{R}^n} f(v)[1 + |v|^2 + \log(1 + f(v))]dv < +\infty,$$

belongs to $H^{+\infty}(\mathbb{R}^n)$ for any $0 < t \leq T$, and moreover,

$$(1.3) \quad f \in L^{\infty}([t_0, T]; H^{+\infty}(\mathbb{R}^n)),$$

for any $T > 0$ and $t_0 \in (0, T)$. And in the paper [3] Alexandre et al. considered a kind of solution having the Maxwellian decay, which means that,

$$(1.4) \quad \exists \delta_0 > 0 \text{ such that } e^{\delta_0(v)^2} f \in L^{\infty}([t_0, T]; H^{+\infty}(\mathbb{R}^n)).$$

We mention that we could assume $t_0 = 0$ in above equation by translation when considering the Gevrey regularity. Then we introduce the following definition:

**Definition 1.1.** We say that $f(t, v)$ is a smooth Maxwellian decay solution to the Cauchy problem (1.1) if

$$\begin{cases} f \geq 0, \neq 0, \\ \exists \delta_0 > 0 \text{ such that } e^{\delta_0(v)^2} f \in L^{\infty}([0, T]; H^{+\infty}(\mathbb{R}^n)). \end{cases}$$

It should be noted that the same arguments as in the proof of Theorem 1.2 of [3] shows the uniqueness of the smooth Maxwellian decay solution to the Cauchy problem (1.1).

Then before ending our review of Boltzmann equation, we recall the definition of Gevrey spaces $G^s(\Omega)$ where $\Omega$ is an open subset of $\mathbb{R}^n$. (It could be found in [7].)

**Definition 1.2.** For $0 < s < +\infty$, we say that $f \in G^s(\Omega)$, if $f \in C^{\infty}(\Omega)$, and there exist $C > 0, N_0 > 0$ such that

$$\|D^\alpha f\|_{L^2(\Omega)} \leq C^{\lceil|\alpha|\rceil + 1} \{\alpha!\}^s, \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \geq N_0.$$
If the boundary of $\Omega$ is smooth, by using the Sobolev embedding theorem, we have the same type estimate with $L^2$ norm replaced by any $L^p$ norm for $2 < p \leq +\infty$. On the whole space $\Omega = \mathbb{R}^n$, it is also equivalent to

$$e^{c_0(-\Delta)^{1/(2s)}} (\partial^\beta_0 f) \in L^2(\mathbb{R}^n),$$

for some $c_0 > 0$ and $\beta_0 \in \mathbb{N}^n$, where $e^{c_0(-\Delta)^{1/(2s)}}$ is the Fourier multiplier defined by

$$e^{c_0(-\Delta)^{1/(2s)}} u(x) = \mathcal{F}^{-1}(e^{c_0|\xi|^{1/s}} \hat{u}(\xi)).$$

If $s = 1$, it is usual analytic function. If $s > 1$, it is Gevrey class function. For $0 < s < 1$, it is called ultra-analytic function.

In the paper, we consider the smooth solution satisfying the following assumptions:

$\mathbf{P}_1$: The solution satisfying (1.2) exists and satisfies (1.3), moreover, has the same Maxwellian decay as (1.4).

$\mathbf{P}_2$: There is a unique smooth Maxwellian decay solution to the Cauchy problem (1.1) (similar as the case considered in [3]).

Now we give our main result in the paper:

**Theorem 1.3.** Let $\nu > 1$ (which is indepedent of $s$) and assume that $0 < s < 1/2$, $-1 < \gamma + 2s < 1$. Let $f(t, v)$ be a smooth Maxwellian decay solution to the Cauchy problem (1.1). If there exist $\rho'$, $\delta'$ such that

$$\sup_{\alpha} \frac{\rho'^{|\alpha|} \|e^{\delta' (v)^2} \partial^\alpha f(0)\|_{L^2}}{\{|\alpha|\}^{\nu}} < +\infty,$$

then there exist $\rho > 0$ and $\delta, \kappa > 0$ with $\delta > \kappa T$ such that

$$\sup_{t \in [0, T]} \sup_{\alpha} \frac{\rho^{\alpha} \|e^{(\delta - \kappa T) (v)^2} \partial^\alpha f(t)\|_{L^2}}{\{|\alpha|\}^{\nu}} < +\infty.$$

**Remark 1.4.** It should be noted that the above theorem is similar as Theorem 1.2 in [1], but we here extend the range of $\gamma$ and consider $\Phi = |v|^\gamma$.

By using the similar arguments as Section 4 in [1], we obtain the Gevrey smoothing effect of order $1/(2s)$ as follows:

**Theorem 1.5.** Assume that $0 < s < 1/2$, $-1 < \gamma + 2s < 1$. Let $\nu = 1/(2s)$ and let $f(t, v)$ be a smooth Maxwellian decay solution to the Cauchy problem (1.1), then for any $t_0 \in (0, T)$, there exist $\rho > 0$ and $\delta, \kappa > 0$ with $\delta > \kappa T$ such that

$$\sup_{t \in [t_0, T]} \sup_{\alpha} \frac{\rho^{\alpha} \|e^{(\delta - \kappa T) (v)^2} \partial^\alpha f(t)\|_{L^2}}{\{|\alpha|\}^{\nu}} < +\infty.$$

The rest of the paper will be organized as follows. In Section 2, we will cite some definitions and main lemma as the authors of [1] did, then we could complete immediately the proof of Theorem 1.3. The proof of one main lemma will be given in Section 3, where we use some new estimates on the collision operator in the framework of [1], and obtain the new result improving the range of $\gamma$. 


2. Proof of Theorem 1.3

Firstly we introduce some basic definitions (see [1] for details).
Let \( l, r \in \mathbb{Z}_+ \) which will be fixed later. For \( \delta, \rho > 0 \) we set:

\[
\|f\|_{\delta,l,\rho,\alpha,r} = \rho^{\alpha_1} \| (v) e^{\delta (v)^2} \|_{L^2} \frac{\partial^\alpha f}{\{\alpha - r\}!}\nu,
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_n^+ \), and we denote \( (\alpha - r)! = (\alpha_1 - r)! \cdots (\alpha_n - r)! \).

Now we give the following definition:

\[
\|f\|_{l,\rho,r,N}(t) = \sup_{rn \leq |\alpha| \leq N} \|f\|_{\delta - \kappa t, l, \rho, \alpha, r},
\]

with fixed \( \delta, \kappa \) satisfying \( \delta > \kappa T \). Here \( N \) satisfying \( rn \leq |\alpha| \leq N \) is a fixed large number. Then for \( h > 1 \) we could obtain

\[
\|f\|_{l,\rho(1+h)-\nu,r,N}(t) \leq \left( \frac{r!}{h}\right)^\nu \|f\|_{l,\rho,0,N}(t).
\]

Now let \( \rho = \rho' \) in the above inequality and take a large enough \( h \). Then it follows from the initial assumption (1.5) that \( \|f\|_{l,\rho(1+h)-\nu,r,N}(0) \) is as small as we want, where \( \delta \) can be chosen any positive less than \( \delta' > 0 \) in (1.5).

So (1.6) could be proved if only we could prove that (with \( \rho = \rho'(1+h)^{-\nu} \))

\[
\sup_{t \in (0,T]} \|f\|_{l,\rho,r,N}(t) < \infty,
\]

under the assumption that \( \|f\|_{l,\rho,r,N}(0) \) is sufficiently small.

We mention that we will consider the Cauchy problem in \( \mathbb{R}^3 \) in the paper.

Lemma 2.1. If \( l \geq 4 \) and \( r > 1 + \nu/(\nu - 1) \) then for any \( \alpha \) satisfying \( 3r \leq |\alpha| \leq N \) we have

\[
\|f(t)\|_{l,\rho,0,N}(t) + 2\kappa \int_0^t \|f(\tau)\|_{l,\rho,0,N}(\tau) d\tau
\]

\[
\leq \|f(0)\|_{l,\rho,0,N} + C_\kappa \int_0^t \left( \|f(\tau)\|_{l,\rho,0,N}(\tau) + \|f(\tau)\|_{l,\rho,0,N}(\tau) \right) d\tau
\]

\[
+ \kappa \sup_{3r \leq |\alpha| \leq N} \int_0^t \|f(\tau)\|_{l,\rho,0,N}(\tau) d\tau,
\]

where \( \beta = 1 - (\gamma + 2s) \).

Then we could prove Theorem 1.3 with the same arguments as in Section 2 in [1], and we omit the proof here. The proof of this lemma will be given in the next section.

3. Proof of Lemma 2.1

Let \( \mu = \mu_{\delta,\kappa}(t) = e^{-(\delta - \kappa t)(v)^2} \) with \( \delta > \kappa T \). Since the translation invariance of the collision operator with respect to the variable \( v \) implies that (see [16] [18]), for the translation operator \( \tau_h \) in \( v \) by \( h \), we have

\[
\tau_h Q(f, g) = Q(\tau_h f, \tau_h g).
\]
Thus we have
\[ \partial_t^\alpha Q(f, g) = \sum_{\alpha = \alpha' + \alpha''} \frac{\alpha!}{\alpha'!\alpha''!} Q(f^{(\alpha')}, f^{(\alpha'')}). \]

Then we could obtain from Eq. (3.1) that
\[ \partial_t (\partial_t^\alpha f) = Q(f, f^{(\alpha)}) + \sum_{\alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} Q(f^{(\alpha')}, f^{(\alpha'')}). \]

Multiplying both sides of the above equation by \( \mu^{-1} \), we obtain
\[ (\partial_t + \kappa \langle v \rangle^2)(\mu^{-1} \partial_t^\alpha f) = \mu^{-1} Q(f, f^{(\alpha)}) + \sum_{\alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} \mu^{-1} Q(f^{(\alpha')}, f^{(\alpha'')}). \]

Set \( F = \mu^{-1} f \) and denote \( F^{(\alpha)} = \mu^{-1} f^{(\alpha)} \) for \( \alpha \in \mathbb{Z}_+^n \). Noticing that \( \mu \mu_* = \mu' \mu_*' \), we get the following formula
\[ \mu^{-1} Q(f, g) = Q(\mu F, G) + \int \int B(\mu_* - \mu_*') F_* G' d\nu_* d\sigma. \]

Then it follows from (3.1) that
\[ (\partial_t + \kappa \langle v \rangle^2) F^{(\alpha)} = Q(\mu F, F^{(\alpha)}) + \sum_{\alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} Q(\mu F^{(\alpha')}, F^{(\alpha'')}) \]
\[ + \sum_{\alpha = \alpha' + \alpha''} \frac{\alpha!}{\alpha'!\alpha''!} \int \int \int B(\mu_* - \mu_*') (F^{(\alpha')}') (F^{(\alpha'')})' d\nu_* d\sigma. \]

Hereafter we denote \( W_l = \langle v \rangle^l \). Multiplying by \( W^2_* F^{(\alpha)} \) both sides of the above equation and integrating with respect to \( v \), we have
\[ \frac{1}{2} \frac{d}{dt} \| W_l F^{(\alpha)} \|^2 + \kappa \| W_{l+1} F^{(\alpha)} \|^2 \]
\[ = (Q(\mu F, F^{(\alpha)}), W^2_* F^{(\alpha)}) + \sum_{\alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} (Q(\mu F^{(\alpha')}, F^{(\alpha'')}), W^2_* F^{(\alpha)}) \]
\[ + \sum_{\alpha = \alpha' + \alpha''} \frac{\alpha!}{\alpha'!\alpha''!} \int \int \int B(\mu_* - \mu_*') (F^{(\alpha')}') (F^{(\alpha'')})' W^2_* F^{(\alpha)} d\nu_* d\sigma \]
\[ = \Psi^{(0,\alpha)}(t) + \sum_{\alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} \Psi^{(\alpha',\alpha'')}(t) + \sum_{\alpha = \alpha' + \alpha''} \frac{\alpha!}{\alpha'!\alpha''!} \Psi^{(\alpha',\alpha'')}(t) \]
\[ = \Psi^{(0,\alpha)}(t) + J^\alpha(t) + K^\alpha(t). \]

Then multiplying by \( \frac{\rho^{2|\alpha|}}{((\alpha - \tau))^2|\alpha|} \) both sides of the above equation, and integrating from 0 to \( t \in (0, T) \), we obtain
\[ \| f(t) \|^2_{\delta - \kappa t, l, p, \alpha, r} + 2 \kappa \int_0^t \| f(\tau) \|^2_{\delta - \kappa t, l + 1, p, \alpha, r} d\tau \]
\[ \leq \| f(0) \|^2_{\delta, l, p, \alpha, r} + 2 \int_0^t \frac{\rho^{2|\alpha|}}{((\alpha - \tau))^2|\alpha|} (\Psi^{(0,\alpha)}(\tau) + J^\alpha(\tau) + K^\alpha(\tau)) d\tau. \]
Firstly we consider the estimate on $\Psi_2^{(\alpha', \alpha'')} (t)$.

By (3.2), we have

$$
\Psi_2^{(\alpha', \alpha'')} (t)
= \iint \iint B(\mu_\ast - \mu_\ast')(F^{(\alpha')}_\ast)'(W_{l-1}F^{(\alpha'')}_\ast)'W_{l+1}F^{(\alpha)} d\nu d\sigma
+ \iint \iint B(\mu_\ast - \mu_\ast')(F^{(\alpha')}_\ast)'(W_{l-1} - W_{l-1})'(F^{(\alpha'')}_\ast)'W_{l+1}F^{(\alpha)} d\nu d\sigma
= \Psi_2^{(\alpha', \alpha'')} (t) + \Psi_2^{(\alpha', \alpha'')} (t).
$$

Notice that, see Lemma 2.3 in [3],

$$
\text{by using the change of variables } v \mapsto \gamma v, \quad \text{for fixed } \mu_\ast, \quad \text{we have}
$$

$$
\left| W_{l-1} - W_{l-1}' \right| \leq C \sin(\theta/2)(W_{l-1} + W_{l-1}')W_{l-2} + \sin((l-2)(\theta/2))W_{l-1,s}
\leq C(\theta W_{l-1} + \theta^{l-1}W_{l-1,s}).
$$

We split $\Psi_2^{(\alpha', \alpha'')} (t)$ into $G_1 + G_2$ corresponding to the two terms of the right-hand side. On the other hand, we could get

$$
\left| \mu_\ast - \mu_\ast' \right| \leq C \theta^\lambda |v' - v'_\ast|^{\lambda}, \quad \lambda \in [0, 1], \quad t \in [0, T].
$$

In the case of $0 < \gamma + 2s < 1$, there exists a $\lambda \in (0, 1)$ such that $\lambda > 2s, \gamma + \lambda < 1$. So we have $\gamma + \lambda > 0$ and $|v' - v'_\ast|^{\gamma + \lambda} \leq (v')^{\gamma + \lambda}(v'_\ast)^{\gamma + \lambda}$ immediately. Then we have

$$
G_1 \leq \iint \iint b(\cos \theta)\theta^{\lambda + 1}|v' - v'_\ast|^{\gamma + \lambda}|W_{l-1}W_{l-1}'|W_{l-1}'(F^{(\alpha')}_\ast)'(F^{(\alpha'')}_\ast)')|(W_{l+1}F^{(\alpha)})| d\nu d\sigma
\leq \iint \iint b(\cos \theta)\lambda^{\lambda + 1}|W_{l+1}F^{(\alpha')}_\ast||W_{l-1+\gamma + \lambda}F^{(\alpha'')}_\ast'||(W_{l+1}F^{(\alpha)})| d\nu d\sigma
= \iint \iint b(\cos \theta)\theta^{\lambda + 1}|W_{l-1+\gamma + \lambda}F^{(\alpha')}_\ast||W_{l-1+\gamma + \lambda}F^{(\alpha'')}_\ast'||(W_{l+1}F^{(\alpha)})| d\nu d\sigma
\leq \left( \iint \iint b(\cos \theta)\theta^{\lambda + 1}|W_{l+1}F^{(\alpha')}_\ast||W_{l-1+\gamma + \lambda}F^{(\alpha'')}_\ast'||(W_{l+1}F^{(\alpha)})|^2 d\nu d\sigma \right)^{1/2}
\times \left( \iint \iint b(\cos \theta)\theta^{\lambda + 1}|W_{l+1}F^{(\alpha')}_\ast||W_{l+1}F^{(\alpha)}|^2 d\nu d\sigma \right)^{1/2}
= G_{1,1}^{1/2} \times G_{1,2}^{1/2}.
$$

Noticing $0 < s < 1/2$, we have $\int_{S^2} b(\cos \theta)\theta^{\lambda + 1} d\sigma \leq C$. Thus

$$
G_{1,1} \leq C ||W_{l+1+\gamma + \lambda}F^{(\alpha')}||_{L^1} ||W_{l-1+\gamma + \lambda}F^{(\alpha'')}||_{L^2}.
$$

By using the change of variables $v \mapsto v' = \frac{v + v_\ast}{2} + \frac{|v - v_\ast|}{2} \sigma$ for fixed $\sigma$ and $v_\ast$ whose Jacobian satisfying:

$$
\left| \frac{dv'}{dv} \right| = \frac{\cos^2(\theta/2)}{4},
$$

we have then

$$
G_{1,2} \leq \iint \iint b(\cos \theta)\theta^{\lambda + 1} \frac{4}{\cos^2(\theta/2)} |W_{l+1+\gamma + \lambda}F^{(\alpha')}_\ast||W_{l+1}F^{(\alpha)}|^2 d\nu d\sigma
\leq C ||W_{l+1+\gamma + \lambda}F^{(\alpha')}||_{L^1} ||W_{l+1}F^{(\alpha)}||_{L^2}^2.
$$
So we could obtain in the case of $0 < \gamma + 2s < 1$:

$$G_1 \leq C ||W_{1+\gamma+\lambda}F^{(\alpha')}||_{L^1} ||W_{l-1+\gamma+\lambda}F^{(\alpha'')}||_{L^2} ||W_{l+1}F^{(\alpha)}||_{L^2}$$

$$\leq C ||W_lF^{(\alpha')}||_{L^2} ||W_lF^{(\alpha'')}||_{L^2} ||W_{l+1}F^{(\alpha)}||_{L^2},$$

if $l \geq 4 > 5/2 + \gamma + \lambda$ by using the embedding

$$L^2_{\lambda/2+\varepsilon}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3), \varepsilon > 0.$$

And on the other hand,

$$G_2 \leq \iiint b(\cos \theta)\theta^{\lambda+l-1}|v' - v_*|^\gamma|\gamma|^\lambda |(W_{l-1}F^{(\alpha')})(v')'||(F^{(\alpha'')})'|(W_{l+1}F^{(\alpha)})|dvdv_*d\sigma$$

$$\leq \iiint b(\cos \theta)\theta^{\lambda+l-1}|(W_{l-1+\gamma+\lambda}F^{(\alpha')})_*||[(W_{l+1}F^{(\alpha')})]'dvdv_*d\sigma$$

$$\leq \left( \iiint b(\cos \theta)\theta^\lambda \gamma \theta |W_{\gamma+\lambda}F^{(\alpha'')}||W_{l-1+\gamma+\lambda}F^{(\alpha')}||^2dvdv_*d\sigma \right)^{1/2}$$

$$\times \left( \iiint b(\cos \theta)\theta^{\lambda+2l-2}|(W_{\gamma+\lambda}F^{(\alpha'')})(v')'||(W_{l+1}F^{(\alpha)})'|^2dvdv_*d\sigma \right)^{1/2}$$

$$= G_{2,1}^{1/2} \times G_{2,2}^{1/2}.$$

Firstly we have

$$G_{2,1} \leq C ||W_{\gamma+\lambda}F^{(\alpha'')}||_{L^1} ||W_{l-1+\gamma+\lambda}F^{(\alpha')}||_{L^2}^2.$$

As for $G_{2,2}$, we use the change of variables $v_* \mapsto v' = \frac{\sin \theta v_*}{\sin^2(\theta/2)} + \frac{\sin \theta - \gamma}{\sin \theta} \sigma$ whose Jacobian satisfying:

$$\left| \frac{dv'}{dv_*} \right| = \frac{\sin^2(\theta/2)}{4}.$$

Then we have

$$G_{2,2} \leq \iiint b(\cos \theta)\theta^{\lambda+2l-2} \frac{4}{\sin^2(\theta/2)} |(W_{\gamma+\lambda}F^{(\alpha'')})(v')'||(W_{l+1}F^{(\alpha)})'|^2dvdv'd\sigma$$

$$\leq C ||W_{\gamma+\lambda}F^{(\alpha'')}||_{L^1} ||W_{l+1}F^{(\alpha)}||_{L^2}^2.$$

if $l \geq 4 > 2 + s - \lambda/2$. Hence we obtain

$$G_2 \leq C ||W_{\gamma+\lambda}F^{(\alpha'')}||_{L^1} ||W_{l-1+\gamma+\lambda}F^{(\alpha')}||_{L^2} ||W_{l+1}F^{(\alpha)}||_{L^2}$$

$$\leq C ||W_lF^{(\alpha')}||_{L^2} ||W_lF^{(\alpha'')}||_{L^2} ||W_{l+1}F^{(\alpha)}||_{L^2}.$$

On the other hand if $-1 < \gamma + 2s \leq 0$, we choose $\lambda = 2s$. Noticing that

$$\langle v - v_* \rangle^r \leq C \langle v \rangle^{r_1} \langle v_* \rangle^{r_2}, r \in \mathbb{R},$$
then we have
\[
G_1 \leq \iint b(\cos \theta) \theta^{2s+1} \frac{|v' - v_s|^{\gamma+2s}}{|v' - v_s|^{(\gamma+2s)}} (v')^{\gamma+2s} W_{t-1} W_1 (F(\alpha))'_s \times |(F'(\alpha)'_s)\|(W_{t+1} F(\alpha))|dvdu_d\sigma
\]
\[
\leq \iint b(\cos \theta) \theta^{2s+1} (1 + |v' - v_s|^{\gamma+2s} 1_{|v' - v_s| \leq 1}) (W_{t-1} \gamma - 2s F(\alpha)_s) \times |(W_{t-1} \gamma + 2s F(\alpha)')_s\|(W_{t+1} F(\alpha))|dvdu_d\sigma
\]
\[
= \left( \iint b(\cos \theta) \theta^{2s+1} (1 + |v - v_s|^{\gamma+2s} 1_{|v - v_s| \leq 1}) (W_{t-1} \gamma - 2s F(\alpha)_s) \right)
\times |(W_{t-1} \gamma + 2s F(\alpha)')_s\|(W_{t+1} F(\alpha))|dvdu_d\sigma
\]
\[
ge^{G_{1,3}/2} \times G_{1,4}^{1/2}
\]
We mention that there exists a positive \(\delta\) such that \(\gamma + 2s = \delta - 1\), which implies that
\[
\chi(v) = |v|^{\gamma+2s} 1_{|v| \leq 1} \in L^{\frac{3-s}{2s}}.
\]
Choosing \(\varepsilon = 3\delta\), we have \(\chi(v) \in L^3\). And since \(0 < s < 1/2\), we have
\[
G_{1,3} \leq C \left( \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^1} + \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^{\frac{3}{2}}} \right) \|W_{t-1} \gamma + 2s F(\alpha)\|_{L^2}.
\]
By using the change of variables \(v \mapsto v'\), we get
\[
G_{1,4} \leq \iint b(\cos \theta) \theta^{2s+1} \frac{4}{\cos^2(\theta/2)} (1 + |v - v_s|^{\gamma+2s} 1_{|v - v_s| \leq 1})
\times |(W_{t-1} \gamma + 2s F(\alpha)'_s)\|(W_{t+1} F(\alpha))|dvdu_d\sigma
\]
\[
\leq C \left( \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^1} + \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^{\frac{3}{2}}} \right) \|W_{t+1} F(\alpha)\|_{L^2}.
\]
So we obtain in the case of \(-1 < \gamma + 2s \leq 0:\)
\[
G_1 \leq C \left( \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^1} + \|W_{t-1} \gamma - 2s F(\alpha)\|_{L^{\frac{3}{2}}} \right) \|W_{t-1} \gamma + 2s F(\alpha)\|_{L^2}
\times \|W_{t+1} F(\alpha)\|_{L^2}
\]
\[
\leq C \|W_t F(\alpha)\|_{L^2} \|W_t F(\alpha)'\|_{L^2} \|W_{t+1} F(\alpha)\|_{L^2},
\]
if \(l \geq 4 > \max \{5/2 - (\gamma + 2s), 3/2 - (\gamma + 2s)\}\) by using the embedding
\[
L^2_{3/2+\varepsilon}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3), \quad L^2_{1/2+\varepsilon}(\mathbb{R}^3) \subset L^2_{\frac{5}{2}}(\mathbb{R}^3), \quad \varepsilon > 0.
\]
And as for $G_2$, we have

$$G_2 \leq \iiint b(\cos \theta) b^{\gamma - 2s} |v^\prime - v_s^\prime|^2 \left| (W_{l-1} F^{(\alpha')} \right)_s \left| (W_{l+1} F^{(\alpha)}) \right| dv_d v_s d\sigma$$

$$\leq \iiint b(\cos \theta) b^{\gamma - 2s} (1 + |v - v_s|^2) \left| (W_{l-1} F^{(\alpha')} \right)_s \left| (W_{l+1} F^{(\alpha)}) \right| dv_d v_s d\sigma$$

$$\leq \left( \int \int \int b(\cos \theta) \theta (1 + |v - v_s|^2) \left| (W_{l-1} F^{(\alpha')} \right)_s \left| (W_{l+1} F^{(\alpha)}) \right| \right)^{1/2}$$

$$= G_{2,3}^{1/2} \times G_{2,4}^{1/2}.$$
And if \(-1 < \gamma + \lambda \leq 0\), we could obtain the similar estimate as on \(G_1\):

\[
\left| \Psi_{2,1}\left(\alpha', \alpha''\right) \right| \leq C \int \int b(\cos \theta) \theta^\lambda \left(1 + |v' - v'| \gamma^\lambda \right) \mathbf{1}_{|v' - v'| \leq 1} \left| (W - \gamma - \lambda F^{(\alpha')})' \right| \\
\times \left| (W_{1-1+\lambda} F^{(\alpha'')})' \right| \left| (W_{1+1} F^{(\alpha)}) \right| dv_{\nu} \, d\sigma \\
= C \int \int b(\cos \theta) \theta^\lambda \left(1 + |v - v| \gamma^\lambda \right) \mathbf{1}_{|v - v| \leq 1} \left| (W - \gamma - \lambda F^{(\alpha')}) \right| \\
\times \left| (W_{1-1+\lambda} F^{(\alpha'')}) \right| \left| (W_{1+1} F^{(\alpha)}) \right| dv_{\nu} \, d\sigma \\
\leq C \left( \left| (W - \gamma - \lambda F^{(\alpha')}) \right|_{L^1} + \left| (W - \gamma - \lambda F^{(\alpha')}) \right|_{L^2} \right) \left| W_{1-1+\lambda} F^{(\alpha'')} \right|_{L^2}
\]

So we have under the assumption \(-1 < \gamma + 2s < 1\):

\[
\left(3.7\right) \quad \left| \Psi_{2,1}\left(\alpha', \alpha''\right) \right| \leq C \left| W_1 F^{(\alpha')} \right|_{L^2} \left| W_1 F^{(\alpha'')} \right|_{L^2} \left| W_{1+1} F^{(\alpha)} \right|_{L^2},
\]

if \(l \geq 4\).

Since \(\Psi_{2,2}\left(\alpha', \alpha''\right)\) and \(\Psi_{2,1}\left(\alpha', \alpha''\right)\) have the same bound (see \(3.3\) and \(3.4\)), we obtain

\[
\left(3.8\right) \quad \frac{\rho^{|\alpha|} |\Psi_{2,2}\left(\alpha', \alpha''\right)|}{(\alpha - r)!} \leq C \left\{ \left(\alpha' - r\right)! \right\}^\nu \left\{ \left(\alpha'' - r\right)! \right\}^\nu \times \left| f(t) \right|_{L^1} \left| f(t) \right|_{L^1} \left| f(t) \right|_{L^1}
\]

As for \(\Psi_1^{0,\alpha}\) we decompose

\[
\Psi_1^{0,\alpha} = \left( Q(\mu F, W_1 F^{(\alpha)}), W_1 F^{(\alpha)} \right) + \left( W_1 Q(\mu F, F^{(\alpha)}), W_1 F^{(\alpha)} \right)
\]

\[
= \Psi_1^{0,\alpha} + \Psi_{1,2}^{0,\alpha}.
\]

In order to estimate the first term \(\Psi_1^{0,\alpha}\) we cite the following coercivity estimate given in \(4\) (we mention that the form here is slight different from \(4\) because it need only to consider \(0 < s < 1/2\)).

**Lemma 3.1.** Let \(0 < s < 1/2\) and assume that the nonnegative function \(g\) satisfies

\[
\|g\|_{L^1_\gamma(\mathbb{R}^2)} + \|g\|_{L^2_\gamma(\mathbb{R}^2)} < \infty.
\]

Then there exists a constant \(C_g > 0\) depending on \(B, \|g\|_{L^1_\gamma(\mathbb{R}^2)}\) and \(\|g\|_{L^2_\gamma(\mathbb{R}^2)}\) such that in the case of \(0 < \gamma + 2s < 1\), there holds

\[
\left(3.9\right) \quad \left| (Q(g, f), f) \right|_{L^2} \geq C_g \|f\|_{H^s_{\gamma/2}}^2 - C \|g\|_{L^1_\gamma} \|f\|_{H^\gamma_{\gamma/2}}^2 \\
- C \left( \|g\|_{L^1_\gamma} + C_g \|g\|_{L^2_\gamma} \right) \|f\|_{L^2_{\gamma/2}}^2,
\]

where \(\eta < s\) depends on \(\gamma\), \(s\) and \(\gamma = |\gamma + 2|_{\gamma \leq 0} + |\gamma - 2|_{\gamma \geq 0}\); And in the case of \(-1 < \gamma + 2s \leq 0\), there holds

\[
\left(3.10\right) \quad \left| (Q(g, f), f) \right|_{L^2} \geq C_g \|f\|_{H^s_{\gamma/2}}^2 - C \left( \|g\|_{L^1_\gamma} + \|g\|_{L^2_\gamma} \right) \|f\|_{H^\gamma_{\gamma/2}}^2 \\
- C \|g\|_{L^1_\gamma} \|f\|_{L^2_{\gamma/2}}^2,
\]

with \(\eta < s\).
We omit the proof here. Using this lemma with \( g = \mu F, f = W_1 F^{(\alpha)} \), then in the case of \( 0 < \gamma + 2s < 1 \) we have
\[
(3.11) \quad \Psi_{1,1}^{(0,\alpha)} + c_0 \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{H^s}^2 \leq C \left( \left\| \mu W_1 \right\|_{L^1} + \left\| \mu F \right\|_{L^1}^2 + \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2 \right) + C \left\| \mu W_{1+\gamma/2} F^{(\alpha)} \right\|_{H^s}^2,
\]
where \( c_0 \) is a constant depending only on the bounds of \( \| f \|_{L^1} \), \( \| f \|_{L^0 \log L} \).
We now need the following interpolation inequality, for \( 0 < \eta < s \) and any \( \varepsilon > 0 \),
\[
\| f \|_{H^\eta} \leq \| f \|_{H^s}^{\eta/s} \| f \|_{L^2}^{(s-\eta)/s} \leq \varepsilon \| f \|_{H^s} + \varepsilon^{-\frac{2\eta}{s-\eta}} \| f \|_{L^2}.
\]
Then
\[
\| \mu W_{1+\gamma/2} F^{(\alpha)} \|_{L^1} \| W_{1+\gamma/2} F^{(\alpha)} \|_{L^2}^2 \leq \varepsilon \| W_{1+\gamma/2} F^{(\alpha)} \|_{L^2}^2 + C \varepsilon \left\| \mu W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^1} \| W_{1+\gamma/2} F^{(\alpha)} \|_{L^2}^2.
\]
Now choosing \( \varepsilon = c_0/2 \), we have
\[
(3.12) \quad \Psi_{1,1}^{(0,\alpha)} + \frac{c_0}{2} \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{H^s}^2 \leq C \left( \left\| \mu W_1 \right\|_{L^1} + \left\| \mu F \right\|_{L^1}^2 + \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2 \right) + C \left\| \mu W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2 \leq C \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2} \| W_{1+\gamma} F^{(\alpha)} \|_{L^2},
\]
where we used the fact
\[
\left( \left\| \mu W_1 \right\|_{L^1} + \left\| \mu F \right\|_{L^1}^2 + \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2 \right) \leq C.
\]
We remark here when considering the above inequality, we could obtain, for example,
\[
\| \mu W_1 \|_{L^1} \leq C \| W_1 \|_{L^2} = C \| f \|_{\delta - \Delta, t, \rho, 0, r} \leq C,
\]
due to Definition 1.1 about the smooth Maxwellian decay solution. The rest two terms could be considered similarly.
In the case of \(-1 < \gamma + 2s \leq 0\), using the similar method we obtain
\[
(3.13) \quad \Psi_{1,1}^{(0,\alpha)} + c_0 \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{H^s}^2 \leq C \left( \left\| \mu W_1 \right\|_{L^1} + \left\| \mu W_1 \right\|_{L^2} \right) + C \left\| \mu W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2 \leq C \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{H^s}^2 + C \left\| W_{1+\gamma/2} F^{(\alpha)} \right\|_{L^2}^2.
\]
On the other hand, in order to estimate \( \Psi_{1,2}^{(0,\alpha)} \) with \( 0 < \gamma + 2s < 1 \) we need the following commutator estimate:

**Lemma 3.2.** There exists a \( C > 0 \) such that
\[
(3.14) \quad \left| \left( W_1 Q(g, f) - Q(g, W_1 f), h \right) \right|_{L^2} \leq C \| g \|_{L^1_{1+\gamma}} \| f \|_{L^2_{1+\gamma}} \| h \|_{L^2}.
\]

**Proof.** Firstly the Taylor formula yields
\[
| W_1 - W_1' | \leq C | v - v' | W_{1-1} (\tau v' + (1 - \tau) v) \leq C (1/2)^{|v - v'|} | W_{1-1} | W_{1-1}',
\]
\[
\leq C \theta | v - v' | (W_{1-1} + W_{1-1}), \leq C \theta | v - v' | W_{1-1} W_{1-1},
\]

where \( \tau \in [0, 1] \) and we used Lemma 2.3 in \cite{1}. Noticing that \( \gamma + 1 > \gamma + 2s > 0 \), we have

\[
\left| (W_i Q(g, f) - Q(g, W_if), \eta) \right|_{L^2} \\
= \int \int \int b(\cos \theta) |v - v_*|^{\gamma} g \cdot f (W_i - W_i') h' d\nu_\eta d\sigma \\
\leq C \int \int \int b(\cos \theta) |v - v_*|^{\gamma+1} |(W_i g)\cdot (W_i f)| h' d\nu_\eta d\sigma \\
\leq C \int \int \int b(\cos \theta) (W_i g)\cdot (W_i f) |h'| d\nu_\eta d\sigma \\
\leq C \left( \int \int \int b(\cos \theta) (W_i g)\cdot (W_i f) |h'|^2 d\nu_\eta d\sigma \right)^{1/2} \\
\times \left( \int \int \int b(\cos \theta) (W_i g)\cdot (W_i f) |h'|^2 d\nu_\eta d\sigma \right)^{1/2} \\
\leq C \| g \|_{L_{\theta \gamma}^1} \| f \|_{L_{\theta \gamma}^2} \| h \|_{L^2}.
\]

Now set \( g = \mu F, \ f = F(\alpha), \) and \( h = W_i F(\alpha) \). Then in the case of \( 0 < \gamma + 2s < 1 \) we have

\[
(3.15) \quad \left| \Psi_{1,2}^{(\alpha, \alpha)} \right| \leq C \| \mu W_i F \|_{L^1} \| W_i F(\alpha) \|_{L^2} \| W_i F(\alpha) \|_{L^2} \\
\leq C \| W_i F(\alpha) \|_{L^2} \| W_i F(\alpha) \|_{L^2}.
\]

In the case of \(-1 < \gamma + 2s \leq 0 \), we recall Corollary 2.1 in \cite{3}:

**Lemma 3.3.** Let \( N_1 = |N_2| + |N_3| + \max \{|m - 2|, |m - 1|\} \) and \( \tilde{N}_1 = N_2 + N_3 \) with \( N_2, N_3, m \in \mathbb{R} \). Then if \( \tilde{N}_1 \geq m + \gamma \) and \( 0 < s < 1/2 \), one has

\[
(3.16) \quad \left| (W_{m} Q(g, f) - Q(g, W_{m} f), \eta) \right|_{L^2} \leq C \| g \|_{L_{N_1}^1} \| f \|_{H_{N_2}^s} \| h \|_{H_{N_3}^s}.
\]

where \( \eta < s \).

We remark here the result of Lemma 3.2 is included in Lemma 3.3, but the process will be more simple if we use Lemma 3.2 in the case of \( 0 < \gamma + 2s < 1 \).

In order to estimate \( \Psi_{1,2}^{(\alpha, \alpha)} \) in the case of \(-1 < \gamma + 2s \leq 0 \), using Lemma 3.3 with \( g = \mu F, \ f = F(\alpha), \) and \( h = W_i F(\alpha), \) and setting \( m = l, \ N_2 = l + \gamma/2, \ N_3 = \gamma/2, \) then \( N_1 = 2l + \gamma - 1 \), and we have

\[
(3.17) \quad \left| \Psi_{1,2}^{(\alpha, \alpha)} \right| \leq C \| \mu W_{2l+\gamma/2}^{\alpha} F \|_{L^1} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \\
\leq C \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s}.
\]

Together with (3.13) and (3.17) we obtain

\[
(3.18) \quad \Psi_{1}^{(\alpha, \alpha)} + c_0 \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \\
\leq C \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{L^2} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{L^2} \leq C \varepsilon \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{H^s} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{L^2} \| W_{l+\gamma/2}^{\alpha} F(\alpha) \|_{L^2}.
\]
Choosing $\varepsilon = c_0/2$, we have

\begin{equation}
\Psi_{1}^{(0,\alpha)} + \frac{c_0}{2} \|W_{t+\gamma/2}F^{(\alpha)}\|_{H^s}^2 \leq C \|W_{t+\gamma/2}F^{(\alpha)}\|_{L^2}^2
\leq C \|W_{t+\gamma}F^{(\alpha)}\|_{L^2} \|W_tF^{(\alpha)}\|_{L^2}.
\end{equation}

We mention that in the case of $0 < \gamma + 2s < 1$ we will obtain the same estimate on $\Psi_{1}^{(0,\alpha)}$ as above, by means of (3.12) and (3.15).

Thus we obtain with $-1 < \gamma + 2s < 1$:

\begin{equation}
\frac{\rho^2|\Psi_{1}^{(0,\alpha)}(t)|}{\{(\alpha - r)!\}^{2\nu}} + \frac{c_0 \rho^2|\Psi_{1}^{(0,\alpha)}(t)|}{\{(\alpha - r)!\}^{2\nu}} \\
\leq C \|f(t)\|_{\delta - \kappa t, l, \rho, \alpha, r} \|f(t)\|_{\delta - \kappa t, l + 1, \rho, \alpha, r}.
\end{equation}

In order to estimate the term $\Psi_{1}^{(\alpha', \alpha'')} (\alpha' \neq 0)$, we decompose in the case of $0 < \gamma + 2s < 1$:

$$
\Psi_{1}^{(\alpha', \alpha'')} = \left( Q(\mu F^{(\alpha')}, F^{(\alpha'')}), W_{t} F^{(\alpha)} \right)
= \left( Q(\mu F^{(\alpha')}, W_{t} F^{(\alpha'')}), W_{t} F^{(\alpha)} \right)
+ \left( W_{t} Q(\mu F^{(\alpha')}, F^{(\alpha'')}), W_{t} F^{(\alpha)} \right)
= \Psi_{1,1}^{(\alpha', \alpha'')} + \Psi_{1,2}^{(\alpha', \alpha'')}.
$$

As for the estimate on $\Psi_{1,2}^{(\alpha', \alpha'')}$, setting $g = \mu F^{(\alpha')}$, $f = F^{(\alpha'')}$, and $h = W_{t} F^{(\alpha)}$ in the Lemma 3.2 then we have

\begin{align*}
\left| \Psi_{1,2}^{(\alpha', \alpha'')} \right| &\leq C \|\mu F^{(\alpha')}\|_{L^{1}_{\gamma + 1}} \|F^{(\alpha'')}\|_{L^{2}_{\gamma + s}} \|W_{t} F^{(\alpha)}\|_{L^2} \\
&\leq C \|W_{t} F^{(\alpha')}\|_{L^2} \|W_{t+1} F^{(\alpha'')}\|_{L^2} \|W_{t} F^{(\alpha)}\|_{L^2}.
\end{align*}

In order to estimate $\Psi_{1,1}^{(\alpha', \alpha'')}$, we need the following upper bound estimate (see Proposition 3.6 of [2]):

**Lemma 3.4.** Let $\gamma + 2s > 0$ and $0 < s < 1$. For any $\sigma \in [2s - 1, 2s]$ and $p \in [0, \gamma + 2s]$ we have

$$
\left| \left( Q(f, g), h \right)_{L^2(\mathbb{R}^3)} \right| \leq C \|f\|_{L^{1}_{\gamma + 2s}} \|g\|_{H^{\sigma+2s-\rho}_\gamma} \|h\|_{H^{\sigma}_\rho}.
$$

Next, we use this lemma with $f = \mu F^{(\alpha')}$, $g = W_{t} F^{(\alpha'')}$, $h = W_{t} F^{(\alpha)}$, $p = \gamma + 2s$, and $\sigma = 2s$. Then by setting $\beta = 1 - \gamma - 2s > 0$, we have

\begin{align*}
\left| \Psi_{1,1}^{(\alpha', \alpha'')} \right| &\leq C \|\mu F^{(\alpha')}\|_{L^{1}_{\gamma + 2s}} \|W_{t} F^{(\alpha'')}\|_{H^{2s}} \|W_{t} F^{(\alpha)}\|_{L^2_{\gamma + 2s}} \\
&\leq C \|W_{t} F^{(\alpha')}\|_{L^2} \|W_{t+1} F^{(\alpha'')}\|_{L^2} \|W_{t+\beta} F^{(\alpha)}\|_{L^2} \\
&\quad + C \|W_{t} F^{(\alpha')}\|_{L^2} \|W_{t} F^{(\alpha')}\|_{L^2} \|W_{t+1} F^{(\alpha+1)}\|_{L^2} \|W_{t+1} F^{(\alpha)}\|_{L^2},
\end{align*}

in view of $2s < 1$ and

$$
\partial_{\nu}(W_{t+1} f^{(\alpha')} - f^{(\alpha')} + W_{t} f^{(\alpha')} + W_{t}^{(\alpha+1)} f^{(\alpha+1)}).
$$
Then
\[
\left| \Psi_{1,a',\alpha''}(t) \right| \leq \left| \Psi_{1,1+a''}(t) \right| + \left| \Psi_{1,2}(t) \right|
\]
\[
\leq C \left\| W_{1}^{1} \alpha(t) \right\|_{L^{2}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1+\beta}(t) \right\|_{L^{2}}
\]
\[
+ C \left\| W_{1}^{1} \alpha(t) \right\|_{L^{2}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1+\beta}(t) \right\|_{L^{2}}
\].

On the other hand, if \(-1 < \gamma + 2s \leq 0\), we will estimate \(\Psi_{1,a',\alpha''}(t)\) by using the following result:

**Lemma 3.5.** Let \(0 < s < 1\) and \(-1 < \gamma + 2s \leq 0\). For any \(p \in \mathbb{R}\) and \(m \in [s-1, s]\), there exists a \(C > 0\) such that
\[
\left| Q(f, g, h) \right|_{L^2(\mathbb{R}^3)} \leq C \left( \left\| f \right\|_{L^{p} \left\{ (x) \right\}^{(2s+\gamma_0)} + \left\| g \right\|_{L^{p} \left\{ (x) \right\}^{\gamma_0}} \right) \left\| W_{1}^{1} \alpha(t) \right\|_{H^{2s}}
\]
\[
\times \left\| W_{1}^{1+\beta} \alpha(t) \right\|_{L^{2} \left\{ (x) \right\}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1+\beta}(t) \right\|_{L^{2}}
\]
\[
+ C \left\| W_{1}^{1} \alpha(t) \right\|_{L^{2}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1}\alpha(t) \right\|_{L^{2}}
\].

Notice that we have same estimate on \(\Psi_{1,a',\alpha''}(t)\) in the two cases of \(0 < \gamma + 2s < 1\) and \(-1 < \gamma + 2s \leq 0\).

Since the Hölder inequality yields
\[
\left\| W_{1}^{1+\beta} \alpha(t) \right\|_{L^{2}} \leq \left\| G \right\|_{L^{2}} \left\| W_{1}^{1+\beta} \alpha(t) \right\|_{L^{2}},
\]
we have
\[
(3.21) \quad \left| \Psi_{1,a',\alpha''}(t) \right| \leq C \left\| W_{1}^{1} \alpha(t) \right\|_{L^{2}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1+\beta}(t) \right\|_{L^{2}}
\]
\[
+ C \left\| W_{1}^{1} \alpha(t) \right\|_{L^{2}} \left\| W_{1}^{1+a''}(t) \right\|_{L^{2}} \left\| W_{1}^{1+\beta}(t) \right\|_{L^{2}}
\]
\[
= J_{1}(a',\alpha')(t) + J_{2}(a',\alpha')(t).
\]

Thus under the assumption \(-1 < \gamma + 2s < 1\) we obtain the estimate for \(\Psi_{1,a',\alpha''}(t)\) with \(\alpha' \neq 0\) as follows:
\[
(3.22) \quad \left\| J_{1}(a',\alpha')(t) \right\|_{\{(x) \}} \leq C \left\{ (\alpha'-r)! \right\}^{\nu} \left\{ (\alpha''-r)! \right\}^{\nu}
\]
\[
\times \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}} \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}} \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}}
\]
\[
(3.23) \quad \left\| J_{2}(a',\alpha')(t) \right\|_{\{(x) \}} \leq C \left\{ (\alpha'-r)! \right\}^{\nu} \left\{ (\alpha''-r)! \right\}^{\nu}
\]
\[
\times \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}} \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}} \left\| f(t) \right\|_{\delta^{-2s+1,l,p,\alpha',\nu}}.
\]
Now we have completed the estimates for $\Psi_{1}^{(0,\alpha)}(t)$, $\Psi_{1}^{(\alpha',\alpha'')}(t)$, and $\Psi_{2}^{(\alpha',\alpha'')}(t)$. The rest proof of Lemma 2.1 is similar as in [1]. We give a brief outline here for the self-containedness.

First we refer to Proposition 3.1 of [1]:

**Proposition 3.6.** If $\nu \geq 1$ and $2 \leq r \in \mathbb{N}$ then there exists a constant $B > 0$ depending only on $n$ and $r$ such that for any $\alpha \in \mathbb{Z}^{n}$

$$
\sum_{a=a' + \alpha} \frac{\alpha!}{\alpha! \alpha''!} \frac{((\alpha' - r)!)^\nu ((\alpha'' - r)!)^\nu}{((\alpha - r)!)^\nu} \leq B.
$$

Furthermore, if $\nu > 1$ and $r > 1 + \nu/(\nu - 1)$ then there exists a constant $B' > 0$ depending only on $n$, $\nu$ and $r$ such that for any $0 \neq \alpha \in \mathbb{Z}^{n}$

$$
\sum_{a=a' + \alpha', \alpha' \neq 0} \frac{\alpha!}{\alpha! \alpha''!} \frac{((\alpha' - r)!)^\nu ((\alpha'' + 1 - r)!)^\nu}{((\alpha - r)!)^\nu} \leq B'.
$$

Now we consider the second term of the right-hand side of (3.3). We consider firstly

$$
\int_{0}^{t} \rho^{2|\alpha|} K^\alpha(\tau) d\tau \leq C \sum \frac{\alpha!}{\alpha! \alpha''!} \frac{((\alpha' - r)!)^\nu ((\alpha'' - r)!)^\nu}{((\alpha - r)!)^\nu} \\
\times \left( \int_{0}^{t} ||f(\tau)||_{t,\kappa,l,p,\alpha',r} ||f(\tau)||_{t,\kappa,l,p,\alpha''-r} d\tau \right) \\
\leq C \left( \sum_{\min(|\alpha'|,|\alpha''|) < 3r} + \sum_{\min(|\alpha'|,|\alpha''|) \geq 3r} \right) \\
= M_1^\alpha + M_2^\alpha.
$$

We mention that $3r \leq |\alpha| \leq N$. Letting

$$
A = A_{r,n}(f) = \sup_{t \in [0,T]} \max_{|\alpha'| < 3r} ||f(t)||_{t,\kappa,l,p,\alpha',r},
$$

we obtain

$$
M_1^\alpha \leq CB \left( \frac{A^2}{4\epsilon} \int_{0}^{t} \left[ \int_{[0,T]} ||f||_{t,\kappa,l,p,\alpha',N}^2 \right] d\tau + \epsilon \sup_{3r \leq |\alpha| \leq N} \int_{0}^{t} ||f(\tau)||_{t,\kappa,l,p,\alpha',r}^2 d\tau \right),
$$

and

$$
M_2^\alpha \leq CB \left( \frac{1}{4\epsilon} \int_{0}^{t} \left[ \int_{[0,T]} ||f||_{t,\kappa,l,p,\alpha',N}^4 \right] d\tau + \epsilon \sup_{3r \leq |\alpha| \leq N} \int_{0}^{t} ||f(\tau)||_{t,\kappa,l,p,\alpha',r}^4 d\tau \right),
$$

for any $\epsilon > 0$. Taking a sufficiently smaller $\epsilon < \kappa$, we have

$$
\int_{0}^{t} \rho^{2|\alpha|} K^\alpha(\tau) d\tau \leq C_{\kappa} \int_{0}^{t} \left( ||f||_{t,\kappa,l,p,\alpha',N}^2 + ||f||_{t,\kappa,l,p,\alpha',N}^4 \right) d\tau \\
+ \frac{\kappa}{100} \sup_{3r \leq |\alpha| \leq N} \int_{0}^{t} ||f(\tau)||_{t,\kappa,l,p,\alpha',r}^2 d\tau.
$$
As for the integral including $J^\alpha$ in (3.23), we get
\[
\int_0^t \rho^{2|\alpha|} J^\alpha(\tau) \frac{d\tau}{(\alpha - \tau)!} \leq \int_0^t \sum_{\alpha = \alpha' + \alpha''} \frac{\alpha!}{\alpha'!\alpha''!} \rho^{2|\alpha|} |J^\alpha(\alpha',\alpha'')(\tau)| \frac{d\tau}{(\alpha - \tau)!}^{2\nu} \nonumber
\]
\[
+ \int_0^t \sum_{\alpha = \alpha' + \alpha'', \alpha' \neq 0} \frac{\alpha!}{\alpha'!\alpha''!} \rho^{2|\alpha|} |J^\alpha(\alpha',\alpha'')(\tau)| \frac{d\tau}{(\alpha - \tau)!}^{2\nu}.
\]
We notice that the integral including $J^\alpha_{2}(\alpha',\alpha'')$ has the same estimate as the estimate (3.26). On the other hand, the last factor of (3.22) is bounded by
\[
\varepsilon (\|f(t)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 + \|f(t)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2) + C_\varepsilon \|f(t)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 \nonumber
\]
for any small $\varepsilon > 0$. Then the integral including $J^\alpha_{1}(\alpha',\alpha'')$ is bounded by
\[
C_\kappa \int_0^t \left( \|f\|_{l,\rho,\ell,\kappa,N}^2 + \|f\|_{l,\rho,\ell,\kappa,N}^2 \right) d\tau \nonumber
\]
\[
+ \frac{\kappa}{100} \sup_{3r \leq |\alpha| \leq N} \int_0^t \|f(\tau)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 d\tau.
\]
So we have
\[
\int_0^t \rho^{2|\alpha|} J^\alpha(\tau) \frac{d\tau}{(\alpha - \tau)!} \leq C_\kappa \int_0^t \left( \|f\|_{l,\rho,\ell,\kappa,N}^2 + \|f\|_{l,\rho,\ell,\kappa,N}^2 \right) d\tau
\]
\[
+ \frac{\kappa}{100} \sup_{3r \leq |\alpha| \leq N} \int_0^t \|f(\tau)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 d\tau,
\]
where we used the fact $4 < 2(1 + \beta)/\beta$. Together with (3.23), (3.20), (3.27) and (3.29) we obtain finally
\[
\|f(t)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 + c_0 \int_0^t \rho^{2|\alpha|} \|W_{l+\gamma/2}^{-1} J^{\alpha} f(\tau)\|_{H^\epsilon}^2 + 2\kappa \int_0^t \|f(\tau)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 d\tau
\]
\[
\leq \|f(0)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 + C_\kappa \int_0^t \left( \|f\|_{l,\rho,\ell,\kappa,N}^2 + \|f\|_{l,\rho,\ell,\kappa,N}^2 \right) d\tau
\]
\[
+ \frac{\kappa}{100} \sup_{3r \leq |\alpha| \leq N} \int_0^t \|f(\tau)\|_{2-\kappa,\ell,1,\rho,\alpha,r}^2 d\tau.
\]
This leads to the desired estimate (2.14) including the extra second term of the left-hand side. This completes the proof of Lemma 2.1.

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