GLOBAL EXISTENCE AND GEVREY REGULARITY TO THE NAVIER-STOKES-NERNST-PLANCK-POISSON SYSTEM IN CRITICAL BESOV-MORREY SPACES

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Abstract. The paper is concerned with the Navier-Stokes-Nernst-Planck-
Poison system arising from electrohydrodynamics in $\mathbb{R}^d$. By means of the implicit function theorem, we prove the global existence of mild solutions for Cauchy problem of this system with small initial data in critical Besov-Morrey spaces. In comparison to the previous works, our existence result provides a new class of initial data, for which the problem is global solvability. Meanwhile, based on the so-called Gevrey estimates, we verify that the obtained mild solutions are analytic in the spatial variables. As a byproduct, we show the asymptotic stability of solutions as the time goes to infinity. Furthermore, decay estimates of higher-order derivatives of solutions are deduced in Morrey spaces.

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1. Introduction and main results. In this paper, we study the following Cauchy problem of the Navier-Stokes-Nernst-Planck-Poisson system

$$
\begin{aligned}
\begin{cases}
  u_t + u \cdot \nabla u - \Delta u + \nabla \Pi = \Delta \phi \nabla \phi, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  n_t + u \cdot \nabla n - \Delta n = -\nabla \cdot (n \nabla \phi), & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  p_t + u \cdot \nabla p - \Delta p = \nabla \cdot (p \nabla \phi), & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  \Delta \phi = n - p, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  \nabla \cdot u = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  (u, n, p)|_{t=0} = (u_0, n_0, p_0), & \text{in } \mathbb{R}^d,
\end{cases}
\end{aligned}
$$

(1)

where unknown functions $u$, $\Pi$, $n$ and $p$ denote the velocity field, the pressure, the densities of the negative and positive charged particles, respectively, and $\phi$ denotes the electrostatic potential caused by the net charged particles. The Navier-Stokes-Nernst-Planck-Poisson system arises from electrohydrodynamics, which describes the dynamic coupling between incompressible flows and diffuse charge systems, and has very important applications in biology, chemistry and pharmacology, see [4, 13, 26] for more details.

Ryham [28] and Schmuck [29] obtained the global existence of weak solutions in a bounded domain with Neumann and Dirichlet boundary conditions, respectively. Li [24] studied the quasineutral limit in periodic domain. Joseph [13] established the existence of a unique smooth local solution by the Fujita-Kato approach. Deng, Zhao and Cui [7, 8, 35, 36] studied the local and global well-posedness in the Lebesgue spaces, modulation spaces, Triebel-Lizorkin spaces and Besov spaces for $d = 3$. In [33], the authors of present paper proved the global well-posedness of (1) for $d = 3$ with large initial vertical velocity component in the critical Besov spaces.

In the case that the flow is charge-free, i.e., $n \equiv 0$ and $p \equiv 0$, (1) reduces to the problem related to the classical incompressible Navier-Stokes equations

$$
\begin{aligned}
\begin{cases}
  u_t + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  \nabla \cdot u = 0, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
  u|_{t=0} = u_0, & \text{in } \mathbb{R}^d.
\end{cases}
\end{aligned}
$$

(2)

The mathematical theory of problem (2) originates with the celebrated paper [21], in which Leray proved the global existence of weak solutions for $d = 2, 3$, but the uniqueness and regularity of weak solutions are still open in $\mathbb{R}^3$. In 1964, Fujita and Kato [10] obtained the first well-posedness result of problem (2) and proved that it is locally well-posed in $H^s(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$ and globally well-posed in $H^\frac{1}{2}(\mathbb{R}^3)$ with small initial data. These results were later extended to various other function spaces including particularly the following critical ones: $L^4(\mathbb{R}^d)$ by Kato [14], $B_{p,\infty}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ by Cannone, Meyer and Planchon [5], BMO$^{-1}(\mathbb{R}^d)$ by Koch and Tataru [16], Morrey spaces $M_{p,d-p}(\mathbb{R}^d)$ by Kato [15] and Taylor [31], Besov-Morrey spaces $\dot{N}^{-1+\frac{d}{p}}_{r,\lambda,\infty}(\mathbb{R}^d)$ by Kozono and Yamazaki [18], Fourier-Besov spaces $F\dot{B}_{p,\infty}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ (1 < $p$ < $\infty$) by Konieczny and Yoneda [17], Fourier-Herz spaces $\dot{B}^{-1}_{q} = F\dot{B}_{1,q}^{-1}(1 \leq q \leq 2)$ by Lei and Lin [19], Iwabuchi and Takada [12] and Cannone and Wu [6]. For an extensive study of the Navier-Stokes equations by means of Fourier analysis techniques, the reader may refer to the monographs [3] by Bahouri et al. and [20] by Lemarié-Rieusset.
The goals of the present paper are to establish the global existence and the asymptotic stability of mild solutions to problem (1) in the critical Besov-Morrey spaces by applying the implicit function theorem, and prove the spatial analyticity of the obtained solutions by the Gevrey class approach pioneered by Foias and Temam [9] through estimating space analyticity radius as a function of time and subsequently developed by [2, 11, 27]. Let $\mathcal{X}$ be a Banach space and $BC_w([0, \infty); \mathcal{X})$ be the set of bonded weakly-star continuous functions on $(0, \infty)$ with values in the Banach space $\mathcal{X}$. For an operator $T : \mathcal{X} \to \mathcal{X}$, we denote

$$T\mathcal{X} := \{ Tf : f \in \mathcal{X} \} \quad \text{with} \quad \|f\|_{T\mathcal{X}} := \|Tf\|_{\mathcal{X}},$$

and a solution $u(t, \cdot) \in \mathcal{X}$ is said to be Gevrey regular if

$$\sup_{0 < t \leq T} \|u\|_{e^{\sqrt{\lambda_1}} \mathcal{X}} < \infty \quad \text{for arbitrary } T > 0,$$

(3)

where $\Lambda_1$ is the Fourier multiplier whose symbol is given by $|\xi|_1 := \sum_{i=1}^d |\xi_i|$. We mention that the finiteness of the corresponding Gevrey norm implies that the functions are space analytic. This approach enables one to avoid cumbersome recursive estimation of higher order derivatives. For the studies on the space analyticity for other models, we refer to [22, 23, 30, 34, 37].

Before stating our main results, we introduce the following assumptions A1, A2 and A3 on the exponents $\lambda, q_i, r_i, \alpha_i, \beta_i$ for $i = 1, 2$:

A1: $d \geq 2, 0 \leq \lambda < d, 1 \leq r_i < q_i < \infty, \ i = 1, 2$;

A2: $d - \lambda < r_1, \frac{d - \lambda}{2} < r_2 < d - \lambda, \frac{2(d - \lambda)}{d - 4} < q_2 < d - \lambda, \frac{1}{r_i} - \frac{1}{q_i} < \frac{1}{q_2} - \frac{3}{2(d - \lambda)}$;

A3: $\alpha_1 = 1 - \frac{d - \lambda}{r_1}, \alpha_2 = 2 - \frac{d - \lambda}{r_2}, \beta_1 = 1 - \frac{d - \lambda}{q_1}, \beta_2 = 2 - \frac{d - \lambda}{q_2}$.

The first result in this paper is the following one concerning the Gevrey class regularity result of problem (1) in the critical Besov-Morrey spaces:

**Theorem 1.1.** Suppose that Assumptions A1, A2 and A3 hold. Let $\theta \in \{0, 1\}$. Then there exists $\epsilon > 0$ such that for every

$$(u_0, n_0, p_0) \in \tilde{N}^{-\alpha_1}_{r_1, \lambda, \infty}(\mathbb{R}^d) \times \tilde{N}^{-\alpha_2}_{r_2, \lambda, \infty}(\mathbb{R}^d) \times \tilde{N}^{-\alpha_2}_{r_2, \lambda, \infty}(\mathbb{R}^d)$$

(4)

with $\nabla \cdot u_0 = 0$ satisfying

$$\|(u_0, n_0, p_0)\|_{\tilde{N}^{-\alpha_1}_{r_1, \lambda, \infty} \times \tilde{N}^{-\alpha_2}_{r_2, \lambda, \infty} \times \tilde{N}^{-\alpha_2}_{r_2, \lambda, \infty}} < \epsilon,$$

(5)

problem (1) admits a unique global solution $(u, n, p) \in Y$, where

$$Y := \left\{ \begin{array}{l}
t^\frac{\partial}{\partial t} u \in BC_w\left([0, \infty); e^{\theta \sqrt{\lambda}} M_{q_1, \lambda}(\mathbb{R}^d)\right), \\
t^\frac{\partial}{\partial t} p \in BC_w\left([0, \infty); e^{\theta \sqrt{\lambda}} M_{q_2, \lambda}(\mathbb{R}^d)\right), \\
t^\frac{\partial}{\partial t} n \in BC_w\left([0, \infty); e^{\theta \sqrt{\lambda}} M_{q_2, \lambda}(\mathbb{R}^d)\right) : \|(u, n, p)\|_Y < \epsilon \end{array} \right\}$$

(6)

with the norm

$$\|(u, n, p)\|_Y := \sup_{t > 0} t^\frac{\partial}{\partial t} \left\| e^{\theta \sqrt{\lambda}} u \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^\frac{\partial}{\partial t} \left\| e^{\theta \sqrt{\lambda}} n \right\|_{M_{q_2, \lambda}}$$

$$+ \sup_{t > 0} t^\frac{\partial}{\partial t} \left\| e^{\theta \sqrt{\lambda}} p \right\|_{M_{q_2, \lambda}}.$$

(7)
Remarks.

(i) In the case of $\theta = 0$, for the exponents $\lambda, q_i, r_i, \alpha_i, \beta_i, i = 1, 2$ prescribed in Assumptions A1-A3, Theorem 1.1 shows that there exists a $\epsilon > 0$ such that if $(u_0, n_0, p_0) \in \mathcal{N}_{r_1, \lambda, \infty}^\gamma(\mathbb{R}^d) \times \mathcal{N}^\gamma_{r_2, \lambda, \infty}(\mathbb{R}^d) \times \mathcal{N}^\gamma_{r_0, \lambda, \infty}(\mathbb{R}^d)$ satisfying (5), then problem (1) admits a unique global mild solution. In the case of $\theta = 1$, Theorem 1.1 indicates that the obtained solutions satisfies the Gevrey estimate (3) with
\[
\mathcal{Y} = M_{q_1, \lambda}(\mathbb{R}^d) \times M_{q_2, \lambda}(\mathbb{R}^d) \times M_{q_2, \lambda}(\mathbb{R}^d),
\]
which implies that the solutions are analytic in the spatial variables, i.e.,
\[
(u(t, \cdot), n(t, \cdot), p(t, \cdot)) \in C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d), \text{ for any } t \in (0, \infty).
\]
(ii) For all $x \in \mathbb{R}^d$ and $t \geq 0$, then, if $(u, n, p)$ solves problem (1) with initial data $(u_0, n_0, p_0)$, so does $(u_\mu, n_\mu, p_\mu)$ with initial data
\[
(u_\mu, n_\mu, p_\mu)(x, t) = (\mu u_0, \mu^2 n_0, \mu^2 p_0)(\mu x, \mu^2 t)
\]
fors all $x \in \mathbb{R}^d$. The critical Besov-Morrey spaces \(\mathcal{N}^{\gamma_{r_1, \lambda, \infty}}(\mathbb{R}^d) \times \mathcal{N}^{\gamma_{r_2, \lambda, \infty}}(\mathbb{R}^d) \times \mathcal{N}^{\gamma_{r_0, \lambda, \infty}}(\mathbb{R}^d)\) are the ones whose norms are invariant under the scaling.

(iii) In [35] and [36], Zhao, Deng and Cui proved the global existence of problem (1) with the small initial data $(u_0, n_0, p_0)$ in $L^d(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ and in $\dot{B}^{-1+\epsilon}_{q, \infty}(\mathbb{R}^d) \times \dot{B}^{-1+\epsilon}_{p, \infty}(\mathbb{R}^d) \times \dot{B}^{-\epsilon}_{p, \infty}(\mathbb{R}^d)$ for $d < q < \infty, \frac{2}{d} < p < d$ and $\frac{1}{p} - \frac{1}{2q} < \frac{3}{2d}$, respectively. While, it follows [18] that
\[
L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{-1+\epsilon}_{q, \infty}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{\gamma_{r_1, \lambda, \infty}}(\mathbb{R}^d), \quad r_1 > q,
\]
and
\[
L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{-2+\epsilon}_{p, \infty}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{\gamma_{r_2, \lambda, \infty}}(\mathbb{R}^d), \quad r_2 > p,
\]
which imply that Theorem 1.1 improves the results in [35] and [36].

(iv) In [8], Deng, Zhao and Cui verified that for
\[
(u_0, n_0, p_0) \in \text{BMO}^{-1}(\mathbb{R}^d) \times \dot{B}^{-2+\epsilon}_{p, \infty}(\mathbb{R}^d) \times \dot{B}^{-\epsilon}_{p, \infty}(\mathbb{R}^d)
\]
with small norm, problem (1) exists a global-in-time solution. From [25, Page. 1314], it is easy to see that
\[
\mathcal{N}^{\gamma_{r, \lambda, \infty}}(\mathbb{R}^d) \subset \text{BMO}^{-1}(\mathbb{R}^d), \quad r \geq 2, \quad d \geq 2, \quad 0 \leq \lambda < d, \quad r > d - \lambda
\]
and
\[
\mathcal{N}^{\gamma_{r, \lambda, \infty}}(\mathbb{R}^d) \not\subset \text{BMO}^{-1}(\mathbb{R}^d), \quad \text{BMO}^{-1}(\mathbb{R}^d) \not\subset \mathcal{N}^{\gamma_{r, \lambda, \infty}}(\mathbb{R}^d), \quad 0 \leq \lambda < d, \quad d \geq 2,
\]
from which, the initial data class of us
\[
\mathcal{N}^{\gamma_{r_1, \lambda, \infty}}(\mathbb{R}^d) \times \mathcal{N}^{\gamma_{r_2, \lambda, \infty}}(\mathbb{R}^d) \times \mathcal{N}^{\gamma_{r_0, \lambda, \infty}}(\mathbb{R}^d)
\]
is not included in the space
\[
\text{BMO}^{-1}(\mathbb{R}^d) \times \dot{B}^{-2+\epsilon}_{p, \infty}(\mathbb{R}^d) \times \dot{B}^{-\epsilon}_{p, \infty}(\mathbb{R}^d).
\]
Hence, compared with [8], Theorem 1.1 provides a new class of initial data for which problem (1) is global solvability, and may be regarded as a new global existence result on problem (1).

Next, we present the global stability of solutions.
Theorem 1.2. Suppose that Assumptions A1, A2 and A3 hold. Let \((u_0, n_0, p_0)\) and \((u'_0, n'_0, p'_0)\) satisfy
\[
\| (u_0, n_0, p_0) \|_{\dot{N}^{-\alpha_1}_{r_1, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty}} < \epsilon
\]
and
\[
\| (u'_0, n'_0, p'_0) \|_{\dot{N}^{-\alpha_1}_{r_1, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty}} < \epsilon
\]
with \(\epsilon > 0\) obtained in Theorem 1.1, and let \((u, n, p)\) and \((u', n', p')\) be the solutions of (1) in \(Y_\epsilon\) with \((u, n, p)_{t=0} = (u_0, n_0, p_0)\) and \((u', n', p')_{t=0} = (u'_0, n'_0, p'_0)\), respectively. Then, for any given \(\eta > 0\), there is a constant \(\delta > 0\), it holds true that if
\[
\| (u_0 - u'_0, n_0 - n'_0, p_0 - p'_0) \|_{\dot{N}^{-\alpha_1}_{r_1, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty} \times \dot{N}^{-\alpha_2}_{r_2, \lambda, \infty}} \leq \delta,
\]
we have
\[
\sup_{t>0} t^{\frac{\alpha}{2}} \| e^{\theta \sqrt{\Lambda}} (u(t) - u'(t)) \|_{M_{q_1, \lambda}} + \sup_{t>0} t^{\frac{\alpha}{2}} \| e^{\theta \sqrt{\Lambda}} (n(t) - n'(t)) \|_{M_{q_2, \lambda}}
\]
\[
+ \sup_{t>0} t^{\frac{\alpha}{2}} \| e^{\theta \sqrt{\Lambda}} (p(t) - p'(t)) \|_{M_{q_2, \lambda}} \leq \eta.
\]

Moreover, as a consequence of working with Gevrey norms, we also obtain the higher-order derivatives and the asymptotic behavior of the solutions.

Theorem 1.3. Let \(m \geq 0\) and \(D^m\) be the Fourier multiplier whose symbol is given by \(|\xi|^m\), where \(\xi = (\xi_1, \xi_2, \cdots, \xi_d) \in \mathbb{R}^d\). There exist a positive constant \(C\) such that
\[
\|D^m u\|_{M_{q_1, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_1}{2}} \text{ and } \|D^m n\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_2}{2}},
\]
\[
\|D^m p\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_1}{2}} \text{ and } \|D^m u - D^m e^{t\Delta} u_0\|_{M_{q_1, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_1}{2}},
\]
\[
\|D^m p - D^m e^{t\Delta} p_0\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_1}{2}} \text{ and } \|D^m n - D^m e^{t\Delta} n_0\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{\alpha}{2} - \frac{\beta_1}{2}},
\]
where \(\beta_1 = 1 - \frac{d\lambda}{q_1}\) and \(\beta_2 = 2 - \frac{d\lambda}{q_2}\).

The layout of the paper is as follows. In Section 2, we collect and proof some technical Lemmas. Our main results are proved in Section 3.

2. Preliminaries. We first give the definition of the Morrey spaces \(M_{p, \lambda}(\mathbb{R}^d)\). For \(1 \leq p < \infty\) and \(0 \leq \lambda < d\), Morrey spaces \(M_{p, \lambda}(\mathbb{R}^d)\) are defined by
\[
M_{p, \lambda}(\mathbb{R}^d) := \left\{ f \in L^p_{loc}(\mathbb{R}^d), \| f \|_{p, \lambda} < \infty \right\}
\]
with norm given by
\[
\| f \|_{p, \lambda} := \sup_{x_0 \in \mathbb{R}^d} \sup_{r > 0} r^\lambda \left( \int_{B(x_0, r)} |f(y)|^p \, dy \right)^{\frac{1}{p}},
\]
where \(B(x_0, r)\) denotes the ball in \(\mathbb{R}^d\) with center \(x_0\) and radius \(r\). It is easy to see that \(M_{p, 0}(\mathbb{R}^d) = L^p(\mathbb{R}^d)\) for \(p > 1\), and \(M_{1, 0}(\mathbb{R}^d)\) is the set of finite Radon measures on \(\mathbb{R}^d\). Whereas \(p = \infty\), we can include \(M_{\infty, \lambda}(\mathbb{R}^d)\) means \(L^\infty(\mathbb{R}^d)\). The space \(M_{p, \lambda}(\mathbb{R}^d)\) endowed with the norm \(\| \cdot \|_{p, \lambda}\) is a Banach space and has the following nice scaling property
\[
\| f(\mu x) \|_{p, \lambda} = \mu^{-\frac{d\lambda}{p}} \| f(x) \|_{p, \lambda} \text{ for } \mu > 0.
\]

For the definition of Besov-Morrey space \(\dot{N}^{s}_{p, \lambda, q}(\mathbb{R}^d)\), we recall the homogeneous Littlewood-Paley decomposition. As usual, we denote by \(S(\mathbb{R}^d)\) the Schwartz class.
of rapidly decreasing functions and by \( \mathcal{S}(\mathbb{R}^d) \) the space of tempered distributions on \( \mathbb{R}^d \). Let \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote Fourier and inverse Fourier transforms of \( L^1(\mathbb{R}^d) \) functions, respectively, which are defined by

\[
\mathcal{F} f = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx
\]

and

\[
\mathcal{F}^{-1} f = \hat{f}(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\xi x} f(\xi) d\xi.
\]

More generally, Fourier transform of \( f \in \mathcal{S}(\mathbb{R}^d) \) is given by \( (\mathcal{F} f, g) = (f, \mathcal{F} g) \), for any \( g \in \mathcal{S}(\mathbb{R}^d) \). Choose two radial functions \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \) such that their Fourier transforms \( \hat{\varphi} \) and \( \hat{\psi} \) satisfy the following properties:

\[
\text{supp} \ \hat{\varphi} \subset B := \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \},
\]

\[
\text{supp} \ \hat{\psi} \subset C := \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \},
\]

and

\[
\sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j} \xi) = 1 \quad \text{for all} \ \xi \in \mathbb{R}^3 \setminus \{0\}.
\]

Let \( \varphi_j(x) := 2^{dj}\varphi(2^jx) \) and \( \psi_j(x) := 2^{dj}\psi(2^jx) \) for all \( j \in \mathbb{Z} \). The homogeneous dyadic blocks \( \Delta_j \) and the homogeneous low-frequency cutoff operators \( S_j \) are defined by

\[
\Delta_j f := \psi_j * f \quad \text{and} \quad S_j f := \varphi_j * f, \quad \text{for} \ j \in \mathbb{Z} \quad \text{and} \ f \in \mathcal{S}(\mathbb{R}^d).
\]

Define \( \mathcal{S}'(\mathbb{R}^3) := \mathcal{S}(\mathbb{R}^d)/\mathcal{P}[\mathbb{R}^d] \), where \( \mathcal{P}[\mathbb{R}^d] \) denotes the linear space of polynomials on \( \mathbb{R}^d \). Then we have the formal decomposition

\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{and} \quad S_j f = \sum_{j' \leq j-1} \Delta_{j'} f \text{ in } \mathcal{S}'(\mathbb{R}^3).
\]

Moreover, Littlewood-Paley decomposition satisfies the property of almost orthogonality:

\[
\Delta_j \Delta_k f = 0, \quad \text{if} \ |j - k| \geq 2, \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0, \quad \text{if} \ |j - k| \geq 5.
\]

For the more facts on Littlewood-Paley theory, one can see [3] for instance.

In [18], Kozono and Yamazaki introduced the homogeneous Besov-Morrey spaces \( \dot{N}^s_{p,\lambda}(\mathbb{R}^d) \). Recall that the space \( \dot{N}^s_{p,\lambda}(\mathbb{R}^d) \) is defined by

\[
\dot{N}^s_{p,\lambda}(\mathbb{R}^d) = \left\{ f \in S'_d(\mathbb{R}^d) : \|f\|_{\dot{N}^s_{p,\lambda}(\mathbb{R}^d)} < \infty \right\},
\]

where

\[
\|f\|_{\dot{N}^s_{p,\lambda}} = \begin{cases} \left( \sum_{k \in \mathbb{Z}} \left( 2^{ks} \left\| \Delta_k f \right\|_{p,\lambda} \right)^q \right)^{\frac{1}{q}} , & \text{if} \ 1 \leq p \leq \infty, 1 \leq q < \infty, s \in \mathbb{R}, \\ \sup_{k \in \mathbb{Z}} \left( 2^{ks} \left\| \Delta_k f \right\|_{p,\lambda} \right) , & \text{if} \ 1 \leq p \leq \infty, q = \infty, s \in \mathbb{R}. \end{cases}
\]
Lemma 2.1. If $1 \leq p, q \leq \infty$, $s > 0$ and $0 \leq \lambda < d$, then $f \in \dot{N}^{-2s}_{p,\lambda,q}({\mathbb R}^d)$ if and only if
\[
\left\{ \left\{ \int_0^\infty \left( \frac{t}{t+1} \int_{t+1}^\infty \right)^q \frac{dt}{t} \right\}^\frac{1}{q} \right\}^\frac{1}{r}, \quad \text{if} \quad 1 \leq q < \infty,
\sup_{t>0} \left[ t^s \| e^{t\Delta} f \|_{M_{p,\lambda}} \right], \quad \text{if} \quad q = \infty.
\]

The above lemma 2.1 can be found in [25]. The following two Lemmas can be found in [18, 25, 31].

Lemma 2.2. Let $s_1, s_2 \in {\mathbb R}$, $1 \leq p_i, q_i \leq \infty$ and $0 \leq \lambda_i < d$ when $i = 1, 2, 3$.

(i) If $p_1 > p_2$, $s_1 = \frac{d-\lambda_1}{p_1} = s_2 - \frac{d-\lambda_2}{p_2}$, then
\[
\dot{N}^{s_2}_{p_2,\lambda_2,q_2}({\mathbb R}^d) \hookrightarrow \dot{N}^{s_1}_{p_1,\lambda_1,q_2}({\mathbb R}^d) \quad \text{and} \quad \dot{N}^{0}_{p_1,\lambda_1,1}({\mathbb R}^d) \hookrightarrow \dot{N}^{0}_{p_1,\lambda_1,\infty}({\mathbb R}^d).
\]

(ii) If $1 \leq q_1 \leq q_2 < \infty$, then
\[
\dot{N}^{s_1}_{p_1,\lambda_1,q_1}({\mathbb R}^d) \hookrightarrow \dot{N}^{s_2}_{p_1,\lambda_1,q_2}({\mathbb R}^d).
\]

(iii) If $1 \leq p_1 \leq p_2 \leq \infty$ and $\frac{d-\lambda_1}{p_1} = \frac{d-\lambda_2}{p_2}$, then
\[
\dot{M}^{s_1}_{p_1,\lambda_1}({\mathbb R}^d) \hookrightarrow \dot{M}^{s_2}_{p_2,\lambda_2}({\mathbb R}^d).
\]

(iv) If $\lambda_1 = \lambda_2 = \lambda_3$ and $h_1 \in \dot{M}^{p_1,\lambda_1}({\mathbb R}^d)$ when $i = 1, 2$, then
\[
\| h_1 h_2 \|_{\dot{M}^{p_3,\lambda_3}} \leq \| h_1 \|_{\dot{M}^{p_1,\lambda_1}} \| h_2 \|_{\dot{M}^{p_2,\lambda_2}}.
\]

Lemma 2.3. Let $s_1 \leq s_2, 1 \leq q \leq \infty, 1 \leq p_1 \leq p_2 \leq \infty$ and $0 \leq \lambda < d$. If $f \in \mathcal{S}'({\mathbb R}^d)$, then there exists a constant $c$ depending only on $d$ such that
\[
\| e^{t\Delta} f \|_{M^{s_2}_{p_2,\lambda}} \leq c t^{-\frac{1}{2} \frac{(d-\lambda_2)}{p_2}}, \quad \| \partial^\gamma e^{t\Delta} f \|_{M^{s_2}_{p_2,\lambda}} \leq c t^{-\frac{1}{2} \frac{|\gamma|}{p_2}},
\]
where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d) \in {\mathbb N}^d$ be multi-index with $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_d$.

Lemma 2.4. Let $d \geq 2$, $0 \leq \lambda < d$ and $1 < p < d - \lambda$. Then there exists a positive constant $C = C(d, p)$ such that
\[
\| \nabla (-\Delta)^{-1} f \|_{M^{p_1,\lambda}_{p_2}} \leq C \| f \|_{M^{p_1,\lambda}_{p_2}}
\]
for all $f \in M^{p_1,\lambda}_{p_2}({\mathbb R}^d)$, where $1/p_* = 1/p - 1/(d - \lambda)$.

Proof. Adams [1] showed the boundedness of fractional integral operators.

The following two Lemmas are useful to obtain the Gevrey estimates.

Lemma 2.5. If the operator $E := e^{-[(1-s)+\sqrt{s} - \sqrt{s}]}$ for $0 \leq s \leq t$, then $E$ is either the identity operator or an $L^1({\mathbb R}^d)$ kernel whose $L^1({\mathbb R}^d)$ norm is bounded independent of $s, t$.

Proof. For the proof of Lemma 2.5, we refer the reader to [2, 32].
Lemma 2.6. The operator \( E = e^{\frac{1}{2}a\Delta + \sqrt{\tau}A} \) is a Fourier multiplier which maps boundedly \( M_{p,\lambda}(\mathbb{R}^d) \to M_{p,\lambda}(\mathbb{R}^d), 1 < p < \infty \), and its operator norm is uniformly bounded with respect to \( a \geq 0 \).

Proof. For the proof of Lemma 2.6, we refer the reader to [1].

Finally, we introduce the following bounded estimation involving with the bilinear operator \( B^p_t(f, g) \) of the form

\[
B_t^p(f, g) := e^{\theta \sqrt{\tau}A} \left( e^{-\theta \sqrt{\tau}A} f e^{-\theta \sqrt{\tau}A} g \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot e^{\theta \sqrt{\tau}A} \xi_1 - \xi \cdot |\eta|_1 - |\eta|_i} f(\xi - \eta) \hat{g}(\eta) d\eta d\xi
\]

for \( \theta \in \{0, 1\} \).

Lemma 2.7. Let \( 1/p_1 + 1/p_2 = 1/p \) and \( 1 < p < \infty \). Then we have

\[
\| B_t^p(f, g) \|_{M_{p,\lambda}(\mathbb{R}^d)} \leq C \| f \|_{M_{p_1,\lambda}(\mathbb{R}^d)} \| g \|_{M_{p_2,\lambda}(\mathbb{R}^d)}
\]

where \( C \) is a positive constant independent of \( f \) and \( g \).

Proof. When \( \theta = 0 \), it is obvious to get

\[
\| B_t^0(f, g) \|_{M_{p,\lambda}} \lesssim \| f \|_{M_{p_1,\lambda}} \| g \|_{M_{p_2,\lambda}}, 1/p_1 + 1/p_2 = 1/p, 1 < p < \infty.
\]

When \( \theta = 1 \), we borrow ideas from [11] completely. For the reader's convenience, we shall prove (10) here. Let \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_d), \mu = (\mu_1, \mu_2, ..., \mu_d), \nu = (\nu_1, \nu_2, ..., \nu_d) \) with \( \lambda_i, \mu_i, \nu_i \in \{1, -1\} \) for \( i = 1, 2, ..., d \), we denote that

\[
D_{\lambda} := \{ \eta : \lambda_i\eta_i \geq 0, i = 1, 2, ..., d \},
\]

\[
D_{\mu} := \{ \xi - \eta : \mu_i(\xi_i - \eta_i) \geq 0, i = 1, 2, ..., d \},
\]

\[
D_{\nu} := \{ \xi : \nu_i\xi_i \geq 0, i = 1, 2, ..., d \}.
\]

Let \( \chi_D \) be the characteristic function on domain \( D \). Then we can rewrite \( B_t^1(f, g) \) as

\[
\sum_{\lambda_i, \mu_i, \nu_i \in \{1, -1\}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot \chi_{D_\lambda}(\xi) e^{\sqrt{(\xi_i - |\eta|_i)^2}} e^{\sqrt{(\eta_i - |\eta|_i)^2}}} \chi_{D_\mu}(\xi - \eta) f(\xi - \eta) \hat{g}(\eta) d\eta d\xi
\]

When \( \eta \in D_\lambda, \xi - \eta \in D_\mu \) and \( \xi \in D_\nu \), we obtain

\[
e^{\sqrt{\tau}((\xi_i - |\xi|_i)^2 + (\eta_i - |\eta|_i)^2)} \in \mathfrak{M} := \left\{ 1, e^{-2\sqrt{\tau}|\xi|_i}, e^{-2\sqrt{\tau}|\eta|_i}, e^{-2\sqrt{\tau}|\eta_i|} \right\}
\]

for \( i = 1, 2, \cdots, d \). It is easy to see that \( \chi_D \) and each element of \( \mathfrak{M} \) are the multipliers on \( M_{p,\lambda}(\mathbb{R}^d) \) with \( 1 < p < \infty \), and it follows from [18] that

\[
\| B_t^1(f, g) \|_{M_{p,\lambda}(\mathbb{R}^d)} \lesssim \| fg \|_{M_{p,\lambda}(\mathbb{R}^d)}.
\]

Thus, we have

\[
\| B_t^1(f, g) \|_{M_{p,\lambda}(\mathbb{R}^d)} \lesssim \| f \|_{M_{p_1,\lambda}(\mathbb{R}^d)} \| g \|_{M_{p_2,\lambda}(\mathbb{R}^d)} \text{ for } 1/p_1 + 1/p_2 = 1/p.
\]

This completes our proof. 

\[\square\]
3. **Proofs of main results.** To solve problem (1) for the given initial data \((u_0, n_0, p_0)\), we make use of the implicit function theorem.

Let us introduce two Banach spaces \(X\) and \(Y\) defined by

\[
X := \mathcal{N}_{r_1, \lambda, \infty}(\mathbb{R}^d) \times \mathcal{N}_{r_2, \lambda, \infty}(\mathbb{R}^d) \times \mathcal{N}_{r_2, \lambda, \infty}(\mathbb{R}^d)
\]  

and

\[
Y := \left\{ (u, n, p) : \int_{\mathbb{T}^d} u \in BC_w \left([0, \infty); e^{\sqrt{\lambda}M_{q_1, \lambda}(\mathbb{R}^d)} \right), \right. \\
\left. \int_{\mathbb{T}^d} p \in BC_w \left([0, \infty); e^{\sqrt{\lambda}M_{q_2, \lambda}(\mathbb{R}^d)} \right), \int_{\mathbb{T}^d} n \in BC_w \left([0, \infty); e^{\sqrt{\lambda}M_{q_2, \lambda}(\mathbb{R}^d)} \right) \right\}
\]  

for \(\theta \in \{0, 1\}\) with the norm

\[
\|(u, n, p)\|_Y := \sup_{t > 0} \left\| e^{\sqrt{\lambda}t} u \right\|_{M_{q_1, \lambda}(\mathbb{R}^d)} + \sup_{t > 0} \left\| e^{\sqrt{\lambda}t} n \right\|_{M_{q_2, \lambda}(\mathbb{R}^d)}
\]

where the exponents \(\lambda, q_1, r, \alpha_1, \beta, i = 1, 2\) prescribed in Assumptions A1-A3 above.

For \((u_0, n_0, p_0) \in X\) and \((u, n, p) \in Y\), we define the map

\[
F(u_0, n_0, p_0, u, n, p) \equiv (U, N, P)
\]

by

\[
\begin{align*}
U(t) &= u(t) - e^{t \Delta} u_0 + \int_0^t e^{(t-s) \Delta} \mathbb{P}(u(s) \cdot \nabla u(s))ds \\
&\quad - \int_0^t e^{(t-s) \Delta} \mathbb{P}(\Delta \phi(s) \nabla \phi(s))ds,
\\
N(t) &= n(t) - e^{t \Delta} n_0 + \int_0^t e^{(t-s) \Delta} (u(s) \cdot \nabla n(s))ds \\
&\quad + \int_0^t e^{(t-s) \Delta} \nabla \cdot (n(s) \nabla \phi(s))ds,
\\
P(t) &= p(t) - e^{t \Delta} p_0 + \int_0^t e^{(t-s) \Delta} (u(s) \cdot \nabla p(s))ds \\
&\quad - \int_0^t e^{(t-s) \Delta} \nabla \cdot (p(s) \nabla \phi(s))ds,
\\
\phi(t) &= (-\Delta)^{-1}(p(t) - n(t))
\end{align*}
\]

where \(\mathbb{P}\) is the Leray projection operator, which can be expressed as an \(d \times d\) matrix:

\[
\mathbb{P} = \{ \mathbb{P}_{j,k} \}_{1 \leq j, k \leq d} = \{ \delta_{j,k} + \mathbb{R}_j \mathbb{R}_k \}_{1 \leq j, k \leq d}
\]

with \(\delta_{j,k}\) being the Kronecker symbol and \(\mathbb{R}_j = \partial_j (-\Delta)^{-1}\) being the Riesz transform.

**Lemma 3.1.** The map \(F\) defined by (13) is a continuous map from \(X \times Y\) into \(Y\).

**Proof.** *Step 1–To estimate*

\[
t^{\beta_1/2} U(\cdot) \in BC_w([0, \infty); e^{\sqrt{\lambda}t}M_{q_1, \lambda}(\mathbb{R}^d)).
\]
Due to $1 \leq r_1 < q_1 < \infty$, applying Lemmas 2.3 and 2.6 implies that there exists a positive constant $C$ such that
\begin{equation}
\left\| e^{\theta \sqrt{\Lambda}} e^t \Delta u_0 \right\|_{M_{q_1, \lambda}} = \left\| e^{\theta \sqrt{\Lambda} + \frac{1}{2} e^\frac{1}{2} \Delta} u_0 \right\|_{M_{q_1, \lambda}} \leq C \left\| e^\frac{1}{2} \Delta u_0 \right\|_{M_{q_1, \lambda}} \tag{15}
\end{equation}
for $\theta \in \{0, 1\}$. Moreover, thanks to $r_1 > d - \lambda$, $\alpha_1 = 1 - \frac{d - \lambda}{r_1}$, $\beta_1 = 1 - \frac{d - \lambda}{q_1}$, it follows from Lemma 2.1 and Lemma 2.2(i) that
\begin{align*}
\left\| e^\frac{1}{2} \Delta u_0 \right\|_{M_{q_1, \lambda}} &\leq C \left( \frac{t}{2} \right)^{-\beta_1/2} \sup_{t > 0} \left( \frac{t}{2} \right)^{\beta_1/2} \left\| e^\frac{1}{2} \Delta u_0 \right\|_{M_{q_1, \lambda}} \\
&\leq C \left( \frac{t}{2} \right)^{-\beta_1/2} \left\| u_0 \right\|_{N^{-\alpha_1}_{r_1, \lambda, \infty}}. \tag{16}
\end{align*}
Hence, we arrive at
\begin{equation}
\left\| e^{\theta \sqrt{\Lambda}} e^t \Delta u_0 \right\|_{M_{q_1, \lambda}} \leq C \left( \frac{t}{2} \right)^{-\beta_1/2} \left\| u_0 \right\|_{N^{-\alpha_1}_{r_1, \lambda, \infty}}. \tag{17}
\end{equation}
Due to $1 \leq r_1 < q_1 < \infty$, applying Lemmas 2.3, 2.5 and 2.6 yields that
\begin{align*}
&\left\| e^{\theta \sqrt{\Lambda}} \int_0^t e^{(t-s)\Delta \mathcal{P}} (u \cdot \nabla u) (s) ds \right\|_{M_{q_1, \lambda}} \\
&\leq \int_0^t \left\| e^{\theta \sqrt{\Lambda}} \int_0^s e^{(t-s)\Delta \mathcal{P}} (u \cdot \nabla u) (s) ds \right\|_{M_{q_1, \lambda}} ds \\
&\leq \int_0^t \left\| \nabla e^{\frac{1}{2} \Delta} e^{\theta \sqrt{\Lambda}} (u \otimes u) (s) \right\|_{M_{q_1, \lambda}} ds \tag{18}
\end{align*}
for $\theta \in \{0, 1\}$. Moreover, thanks to $\beta_1 = 1 - \frac{d - \lambda}{q_1}$, it follows Lemmas 2.3 and 2.7 that
\begin{align*}
&\int_0^t \left\| \nabla e^{\frac{1}{2} \Delta} e^{\theta \sqrt{\Lambda}} (u \otimes u) (s) \right\|_{M_{q_1, \lambda}} ds \\
&\leq \int_0^t (t-s)^{-\frac{1}{2} - \frac{d-\lambda}{q_1}} \left\| e^{\theta \sqrt{\Lambda}} (e^{-\theta \sqrt{\Lambda}} e^{\theta \sqrt{\Lambda}} u \otimes e^{-\theta \sqrt{\Lambda}} e^{\theta \sqrt{\Lambda}} u) (s) \right\|_{M_{q_1/2, \lambda}} ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{d-\lambda}{q_1}} \left\| e^{\theta \sqrt{\Lambda}} u (s) \right\|_{M_{q_1, \lambda}}^2 ds \\
&\leq C t^{-\frac{\beta_1}{2}} \left( \sup_{t > 0} \left\| e^{\theta \sqrt{\Lambda}} u (t) \right\|_{M_{q_1, \lambda}} \right)^2 B \left[ \frac{1}{2} - \frac{d - \lambda}{2 q_1}, \frac{d - \lambda}{q_1} \right], \tag{19}
\end{align*}
where $B[\cdot, \cdot]$ denotes the alpha function:
\begin{equation*}
B[x, y] := \int_0^1 (1 - \tau)^{x-1} \tau^{y-1} d\tau,
\end{equation*}
for $x > 0$ and $y > 0$. Hence, we arrive at
\begin{equation}
\left\| e^{\theta \sqrt{\Lambda}} \int_0^t e^{(t-s)\Delta \mathcal{P}} (u \cdot \nabla u) (s) ds \right\|_{M_{q_1, \lambda}} \leq C t^{-\frac{\beta_1}{2}} \left( \sup_{t > 0} \left\| e^{\theta \sqrt{\Lambda}} u (t) \right\|_{M_{q_1, \lambda}} \right)^2. \tag{20}
\end{equation}
Similarly to (20) and (21), we have
\[
\left\| e^{\theta \sqrt{T} \Lambda} \int_0^t e^{(t-s)\Delta} \mathbb{P} (\Delta \phi \nabla \phi) (s) ds \right\|_{M_{q_1,\lambda}} 
\leq C \int_0^t \left\| \nabla \cdot e^{\frac{t-s}{2} \Delta} e^{\theta \sqrt{T} \Lambda} \left( \nabla (-\Delta)^{-1} (p-n) \otimes \nabla (-\Delta)^{-1} (p-n) \right. \right. 
\left. \left. - \left| \nabla (-\Delta)^{-1} (p-n) \right|^2 I_{d\times d} \right) (s) \right\|_{M_{q_1,\lambda}} ds 
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( \frac{2}{q_1} - \frac{2}{q_2} - \frac{2}{q_1} - \frac{2}{q_2} \right) \left\| e^{\theta \sqrt{T} \Lambda} \left( \nabla (-\Delta)^{-1} (p-n) \otimes \nabla (-\Delta)^{-1} (p-n) \right. \right. 
\left. \left. - \left| \nabla (-\Delta)^{-1} (p-n) \right|^2 I_{d\times d} \right) (s) \right\|_{M_{q_2(q_1-q_2)\Lambda}} ds 
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \left( \frac{2}{q_1} - \frac{2}{q_2} - \frac{2}{q_1} - \frac{2}{q_2} \right) \left\| e^{\theta \sqrt{T} \Lambda} (p-n) \right\|_{M_{q_2,\lambda}}^2 ds 
\leq C t^{-\frac{1}{2}} B \left[ \frac{3}{2} + \frac{d-\lambda}{2q_1} - \frac{d-\lambda}{q_2}, -1 + \frac{d-\lambda}{q_2} \right] \left( \sup_{t>0} t^{\frac{d}{2}} \left\| e^{\theta \sqrt{T} \Lambda} (p-n) \right\|_{M_{q_2,\lambda}} \right)^2.
\]
Combining (17), (20) and (21) implies that
\[
\sup_{t>0} t^{\frac{d}{2}} \left\| e^{\theta \sqrt{T} \Lambda} U \right\|_{M_{q_1,\lambda}} 
\leq C \left\| u_0 \right\|_{X_{-\alpha_1,\infty}} + C \sup_{t>0} t^{\frac{d}{2}} \left\| e^{\theta \sqrt{T} \Lambda} u \right\|_{M_{q_1,\lambda}} + C \left( \sup_{t>0} t^{\frac{d}{2}} \left\| e^{\theta \sqrt{T} \Lambda} u(t) \right\|_{M_{q_1,\lambda}} \right)^2 
\Rightarrow \left( \sup_{t>0} t^{\frac{d}{2}} \left\| e^{\theta \sqrt{T} \Lambda} U \right\|_{M_{q_2,\lambda}} \right)^2.
\]
\[
\text{Step 2: To estimate} 
\quad t^{\frac{d}{2}} N(t) \in BC_w ([0, \infty) : M_{q_2,\lambda}(\mathbb{R}^d)).
\]
Similarly to (17), there exists a positive constant C such that
\[
\left\| e^{\theta \sqrt{T} \Lambda} e^{\Delta} n_0 \right\|_{M_{q_2,\lambda}} 
\leq C \left( \frac{d}{2} \right)^{-\beta_2/2} \sup_{t>0} \left( t^{\frac{d}{2}} \right)^{\beta_2/2} \left\| e^{\frac{d}{2} \Delta} n_0 \right\|_{M_{q_2,\lambda}} 
\leq C \left( \frac{d}{2} \right)^{-\beta_2/2} \left\| n_0 \right\|_{X_{-\alpha_2,\infty}}.
\]
Similarly to (20) and (21), one has
\[
\left\| e^{\theta \sqrt{T} \Lambda} \int_0^t e^{(t-s)\Delta} (u \cdot \nabla n)(s) ds \right\|_{M_{q_2,\lambda}} 
\leq \int_0^t \left\| e^{\theta \sqrt{T} \Lambda} \int_0^t e^{(t-s)\Delta} (u \cdot \nabla n)(s) ds \right\|_{M_{q_2,\lambda}} ds
\begin{align*}
& \leq C \int_0^t (t-s)^{-\frac{1}{3}} (s + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}) \frac{e^{-\lambda s}}{s^{\frac{1}{2}}}
& \quad \times \bigg\| e^{\theta \sqrt{\Lambda}} \left( e^{-\theta \sqrt{\Lambda}} e^{\theta \sqrt{\Lambda} u} \right) s \bigg\|_{M_{q_{12}, \lambda}} ds
& \leq C \int_0^t (t-s)^{-\frac{1}{3}} \left(\frac{d-\lambda}{2q_1}\right) e^{\theta \sqrt{\Lambda} u(s)} \left\| e^{\theta \sqrt{\Lambda} n(s)} \right\|_{M_{q_{1}, \lambda}} ds
& \leq CB \left[ 1 - \frac{d-\lambda}{2q_1}, 1 - \frac{\beta_1}{2} - \frac{\beta_2}{2} \right] t^{-\frac{\alpha_2}{2}} \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} u(t)} \right\|_{M_{q_{1}, \lambda}}
& \quad \times \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} n(t)} \right\|_{M_{q_{2}, \lambda}},
\end{align*}

and
\begin{align*}
& \left\| e^{\theta \sqrt{\Lambda}} \int_0^t e^{(t-s)\Delta} \nabla \cdot (n \nabla \phi)(\cdot, s) ds \right\|_{M_{q_{2}, \lambda}} \\
& \leq C \int_0^t \left\| \nabla e^{(-\frac{d-\lambda}{2q_1}) \Delta} e^{\theta \sqrt{\Lambda} \cdot (n \nabla (-\Delta)^{-1}(p-n))}(\cdot, s) \right\|_{M_{q_{2}, \lambda}} ds
\end{align*}

\begin{align*}
& \leq C \int_0^t (t-s)^{-\frac{1}{3}} \left(\frac{d-\lambda}{2q_1}\right) e^{\theta \sqrt{\Lambda} (n \nabla (-\Delta)^{-1}(p-n))(s)} \bigg\| M_{q_{2}(d-\lambda)} \frac{\lambda}{-\frac{d}{2} + 2}, \frac{\lambda}{2} \right\| \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} (p, n)} \right\|_{M_{q_{2}, \lambda}}
& \quad \times \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} n(t)} \right\|_{M_{q_{2}, \lambda}}.
\end{align*}

Thus it follows from (23)-(25) that
\begin{align*}
& \quad \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} \left(\frac{1}{0}, \infty\right) : e^{\theta \sqrt{\Lambda} M_{q_{2}, \lambda}(\mathbb{R}^d)} \right\|
& \quad \leq C \left\| n_0 \right\|_{M_{r_{2}, \infty}} + C \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} \left(\frac{1}{0}, \infty\right) : e^{\theta \sqrt{\Lambda} M_{q_{2}, \lambda}(\mathbb{R}^d)} \right\|
& \quad + \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} n(t)} \right\|_{M_{q_{2}, \lambda}} \left( \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} u(t)} \right\|_{M_{q_{1}, \lambda}}
& \quad + \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} n(t)} \right\|_{M_{q_{2}, \lambda}} \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} p(t)} \right\|_{M_{q_{2}, \lambda}} \right).
\end{align*}

\textbf{Step 3}

Similar as Step 2, it holds true that
\begin{align*}
& \quad \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} \left(\frac{1}{0}, \infty\right) : e^{\theta \sqrt{\Lambda} M_{q_{2}, \lambda}(\mathbb{R}^d)} \right\|
& \quad \leq C \left\| p_0 \right\|_{M_{r_{2}, \infty}} + C \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} \left(\frac{1}{0}, \infty\right) : e^{\theta \sqrt{\Lambda} M_{q_{2}, \lambda}(\mathbb{R}^d)} \right\|
& \quad + \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} p(t)} \right\|_{M_{q_{2}, \lambda}} \left( \sup_{t>0} t^{\frac{\alpha_2}{2}} \left\| e^{\theta \sqrt{\Lambda} u(t)} \right\|_{M_{q_{1}, \lambda}}
\end{align*}
Now, by (22), (26) and (3), we conclude that
\[
\left(\frac{\partial}{\partial t} + \theta \sqrt{T} A\right) n(t) + \sup_{t > 0} \left\| \left(\frac{\partial}{\partial t} + \theta \sqrt{T} A\right) p(t) \right\|_{M_{2\lambda, 1}} \right).
\]

Now, by (22), (26) and (3), we conclude that \((U, N, P) \in Y\) with
\[
\left\| (U, N, P) \right\|_Y \leq C \left(\left\| (u_0, n_0, p_0) \right\|_X + C \left(\left\| (u, n, p) \right\|_Y (1 + \left\| (u, n, p) \right\|_Y)\right) \right).
\]

\(\square\)

**Proposition 1.** For \((u_0, n_0, p_0) \in X\), the map \(F(u_0, n_0, p_0, \ldots)\) is of class \(C^1\) from \(Y\) into \(Y\).

**Proof.** For each \((u, n, p) \in X\), we define a linear map \(L_{(u, n, p)}(\tilde{u}, \tilde{n}, \tilde{p}) \equiv (\tilde{U}, \tilde{N}, \tilde{P})\) by
\[
\begin{align*}
\tilde{U} &= \tilde{u}(t) + \int_0^t e^{(t-s)\Delta} \nabla \tilde{u}(s) \nabla u(s) + u(s) \nabla \tilde{u}(s) ds \\
&\quad - \int_0^t e^{(t-s)\Delta} \nabla (\Delta \tilde{u}(s) \nabla \phi(s) + \Delta \phi(s) \nabla \tilde{u}(s) ds), \\
\tilde{N} &= \tilde{n}(t) + \int_0^t e^{(t-s)\Delta} (\tilde{u}(s) \nabla n(s) + u(s) \nabla \tilde{n}(s) ds \\
&\quad + \int_0^t e^{(t-s)\Delta} \nabla \cdot (\tilde{n}(s) \nabla \phi(s) + n(s) \nabla \tilde{u}(s)) ds, \\
\tilde{P} &= \tilde{p}(t) + \int_0^t e^{(t-s)\Delta} (\tilde{u}(s) \cdot \nabla p(s) + u(s) \cdot \nabla \tilde{p}(s) ds \\
&\quad - \int_0^t e^{(t-s)\Delta} \nabla \cdot (\tilde{p}(s) \nabla \phi(s) + p(s) \nabla \tilde{u}(s)) ds, \\
\tilde{\phi}(t) &= (-\Delta)^{-1}(\tilde{p}(t) - \tilde{n}(t)), \\
\phi(t) &= (-\Delta)^{-1}(p(t) - n(t)).
\end{align*}
\]

We shall show that for each fixed \((u_0, n_0, p_0) \in X\), \(L_{(u, n, p)}(\tilde{u}, \tilde{n}, \tilde{p})\) is the Fréchet derivative of \(F(u_0, n_0, p_0, u, n, p)\) at \((u, n, p) \in Y\).

Let
\[
(U, N, P) := F\left(u_0, n_0, p_0, u + \tilde{u}, n + \tilde{n}, p + \tilde{p}\right) - F\left(u_0, n_0, p_0, u, n, p\right) - L_{(u, n, p)}\left(\tilde{u}, \tilde{n}, \tilde{p}\right).
\]

It thus holds true that
\[
\begin{align*}
\mathcal{U}(t) &= \int_0^t e^{(t-s)\Delta} \nabla \tilde{u}(s) \nabla u(s) ds - \int_0^t e^{(t-s)\Delta} \nabla (\Delta \tilde{u}(s) \nabla \phi(s)) ds, \\
\mathcal{N}(t) &= \int_0^t e^{(t-s)\Delta} (\tilde{u}(s) \nabla n(s)) + \int_0^t e^{(t-s)\Delta} \nabla \cdot (\tilde{n}(s) \nabla \tilde{u}(s)) ds, \\
\mathcal{P}(t) &= \int_0^t e^{(t-s)\Delta} (\tilde{u}(s) \cdot \nabla p(s)) - \int_0^t e^{(t-s)\Delta} \nabla \cdot (\tilde{p}(s) \nabla \tilde{u}(s)) ds, \\
\tilde{\phi}(t) &= (-\Delta)^{-1}(\tilde{p}(t) - \tilde{n}(t)).
\end{align*}
\]

Similarly, we have by (20), (21), (24), (25) and (3) that
\[
\int_0^t e^{\theta \sqrt{T} A} \mathcal{U}(t) \left\|_{M_{q\lambda, 1}} \right\|^2 \leq C \left( \left( \sup_{t > 0} \left\| e^{\theta \sqrt{T} A} \tilde{u}(t) \right\|_{M_{q\lambda, 1}} \right)^2 \\
+ \left( \sup_{t > 0} \left\| e^{\theta \sqrt{T} A} \tilde{p}(t) \right\|_{M_{q\lambda, 1}} \right)^2 \right).
\]
Hence, we obtain by (31)-(33) that

\[
\begin{align*}
&\lim_{\|\tilde{u},\tilde{n},\tilde{p}\|_Y \to 0} \frac{1}{\|U, N, P\|_Y} \left\| \left( F(u_0, n_0, p_0, u, n + \tilde{n}, p + \tilde{p}) - F(u_0, n_0, p_0, u, n, p) \right. \right. \\
&\quad \left. \left. - L_{\{u,n,p\}}(\tilde{u}, \tilde{n}, \tilde{p}) \right) \right\|_Y \| \left( \tilde{u}, \tilde{n}, \tilde{p} \right) \|_Y \\
&\leq C \lim_{\|\tilde{u},\tilde{n},\tilde{p}\|_Y \to 0} \frac{\|\tilde{u},\tilde{n},\tilde{p}\|_Y^2}{\|\tilde{u},\tilde{n},\tilde{p}\|_Y} = 0
\end{align*}
\]

for each \((u_0, n_0, p_0) \in X \) and each \((u, n, p) \in Y\), which implies that the Fréchet derivative of \(F\) at point \((u_0, n_0, p_0, u, n, p) \in X \times Y\) in the direction to \((u, n, p)\) is equal to \(L_{\{u,n,p\}}(\tilde{u}, \tilde{n}, \tilde{p})\).

**Proof of Theorem 1.1** We shall show bijectivity of the Fréchet derivative \(L_{\{u,n,p\}}(\tilde{u}, \tilde{n}, \tilde{p})\) at \((u, n, p) = (0, 0, 0)\). We have an expression

\[
L_{\{0,0,0\}}(\tilde{u}, \tilde{n}, \tilde{p}) = (U_0, N_0, P_0)
\]

as \(U_0 = \tilde{u}, N_0 = \tilde{n}, P_0 = \tilde{p}\) for \((\tilde{u}, \tilde{n}, \tilde{p}) \in Y\). Hence it is easy to see that \((U_0, N_0, P_0) = (0, 0, 0)\) implies that \((\tilde{u}, \tilde{n}, \tilde{p}) = (0, 0, 0)\), which finally yields that \(L_{\{0,0,0\}}\) is injective.

For every \((U_0, N_0, P_0) \in Y\), we can take \((\tilde{u}, \tilde{n}, \tilde{p}) \in Y\) as \(\tilde{u} = U_0, \tilde{n} = N_0, \tilde{p} = P_0\), such that it holds true that \(L_{\{0,0,0\}}(\tilde{u}, \tilde{n}, \tilde{p}) = (U_0, N_0, P_0)\), which implies that \(L_{\{0,0,0\}}\) is surjective from \(Y\) onto itself.

Now, it follows from the Banach implicit function theorem that there exists a unique \(C^1\) map \(\Gamma:\)

\[
\Gamma : X_\epsilon = \{(u_0, n_0, p_0) \in X; \|(u_0, n_0, p_0)\|_X < \epsilon\} \to Y_\epsilon = \{(u, n, p) \in Y; \|(u, n, p)\|_Y < \epsilon\}
\]

for some \(\epsilon > 0\) such that

\[
\Gamma(0, 0, 0) = (0, 0, 0), \quad F(u_0, n_0, p_0, \Gamma(u_0, n_0, p_0)) = (0, 0, 0),
\]

where \((u_0, n_0, p_0) \in X_\epsilon\). It is easy to see that \(\Gamma(u_0, n_0, p_0)\) gives the unique solution of (1) provided that \((u_0, n_0, p_0)\) satisfies (4). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2** The stability (8) under the conditions of (7) is a consequence of continuity of the map \(\Gamma : X_\epsilon \to Y_\epsilon\). This proves Theorem 1.2.
Proof of Theorem 1.3 Theorem 1.1 tells us that the solution is globally in the Gevrey regular, that is, the solution \((u, n, p)\) of system (1) satisfies
\[
\sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} u} \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} n} \right\|_{M_{q_2, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} p} \right\|_{M_{q_2, \lambda}} < +\infty.
\]  
(36)

Moreover, it is easy to check that the operator \(D^m e^{-\sqrt{\Lambda}}\) is the convolution operator with a kernel \(K_m(t) \in L^1(\mathbb{R}^d)\) for all \(m \geq 0\) and \(t > 0\), and there exists a positive constant \(C = C(m)\) such that
\[
\|K_m(t)\|_{L^1} \leq C_m t^{\frac{m}{2}}.
\]
Then, for \(m \geq 0\), there exists positive constant \(C\) such that
\[
\sup_{t > 0} t^{\frac{3}{2}} \left\| D^m u \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| D^m n \right\|_{M_{q_2, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| D^m p \right\|_{M_{q_2, \lambda}}
\]
\[
= \sup_{t > 0} t^{\frac{3}{2}} \left\| D^m e^{-\sqrt{\Lambda}} e^{\sqrt{\Lambda} u} \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| D^m e^{-\sqrt{\Lambda}} e^{\sqrt{\Lambda} n} \right\|_{M_{q_2, \lambda}}
\]
\[
+ \sup_{t > 0} t^{\frac{3}{2}} \left\| D^m e^{-\sqrt{\Lambda}} e^{\sqrt{\Lambda} p} \right\|_{M_{q_2, \lambda}}
\]
\[
\leq Ct^{-\frac{m}{2}} \left( \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} u} \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} n} \right\|_{M_{q_2, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} p} \right\|_{M_{q_2, \lambda}} \right)
\]
\[
\leq Ct^{-\frac{m}{2}},
\]  
(37)

from which, we have
\[
\|D^m u\|_{M_{q_1, \lambda}} \leq Ct^{-\frac{m}{2}} - \frac{3}{2} \quad \text{and} \quad \|D^m n\|_{M_{q_2, \lambda}} + \|D^m p\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{m}{2}} - \frac{3}{2}.
\]

Furthermore, from (20), (21), (24), (25) and (3), we have
\[
\sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} (u - e^{t \Delta} u_0)} \right\|_{M_{q_1, \lambda}} + \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} (n - e^{t \Delta} n_0)} \right\|_{M_{q_2, \lambda}}
\]
\[
+ \sup_{t > 0} t^{\frac{3}{2}} \left\| e^{\sqrt{\Lambda} (p - e^{t \Delta} p_0)} \right\|_{M_{q_2, \lambda}} < +\infty.
\]

Similar as (37), we have
\[
\|D^m (u - e^{t \Delta} u_0)\|_{M_{q_1, \lambda}} \leq Ct^{-\frac{m}{2}} - \frac{3}{2}
\]
and
\[
\|D^m (n - e^{t \Delta} n_0)\|_{M_{q_2, \lambda}} + \|D^m (p - e^{t \Delta} p_0)\|_{M_{q_2, \lambda}} \leq Ct^{-\frac{m}{2}} - \frac{3}{2}.
\]
We complete the proof of Theorem 1.3.

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