Abstract

In this paper, we study the $L^p$ boundedness and $L^p(w)$ boundedness ($1 < p < \infty$ and $w$ a Muckenhoupt $A_p$ weight) of fractional maximal singular integral operators $T^{\#}_{\Omega, \alpha}$ with homogeneous convolution kernel $\Omega(x)$ on an arbitrary homogeneous group $\mathbb{H}$ of dimension $Q$. We show that if $0 < \alpha < Q$, $\Omega \in L^1(\Sigma)$ and satisfies the cancellation condition of order $[\alpha]$, then for any $1 < p < \infty$,

$$\|T^{\#}_{\Omega, \alpha}f\|_{L^p(\mathbb{H})} \lesssim \|\Omega\|_{L^1(\Sigma)} \|f\|_{L^p_{\alpha}(\mathbb{H})}.$$ 

where for the case $\alpha = 0$, the $L^p$ boundedness of rough singular integral operator and its maximal operator were studied by Tao (Indiana Univ Math J 48:1547–1584, 1999) and Sato (J Math Anal Appl 400:311–330, 2013), respectively. We also obtain a quantitative weighted bound for these operators. To be specific, if $0 \leq \alpha < Q$ and $\Omega$ satisfies the same cancellation condition but a stronger condition that $\Omega \in L^q(\Sigma)$ for some $q > Q/\alpha$, then for any $1 < p < \infty$ and $w \in A_p$,

$$\|T^{\#}_{\Omega, \alpha}f\|_{L^p(w)} \lesssim \|\Omega\|_{L^q(\Sigma)} \{w\}_{A_p}(w) \|f\|_{L^p_{\alpha}(w)}, \quad 1 < p < \infty.$$ 

Keywords Quantitative weighted bounds · Singular integral operators · Maximal operators · Rough kernel · Homogeneous groups
1 Introduction

1.1 Background

Throughout this paper, we regard $\mathbb{H} = \mathbb{R}^n$ ($n \geq 2$) as a homogeneous group, which is a nilpotent Lie group. It has multiplication, inverse, dilation, and norm structures

$$(x, y) \mapsto xy, \quad x \mapsto x^{-1}, \quad (\lambda, x) \mapsto \lambda \circ x, \quad x \mapsto \rho(x)$$

for $x, y \in \mathbb{H}$, $\lambda > 0$. The multiplication and inverse operations are polynomials and form a group with identity 0, the dilation structure preserves the group operations and is given in coordinates by

$$\lambda \circ (x_1, \ldots, x_n) := (\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n)$$

for some positive numbers $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. Without loss of generality, we shall assume that $a_1 = 1$ (see [18]). Moreover, $\rho(x) := \max\{|x_j|^{1/a_j}\}$ is a norm associated to the dilation structure. We call $n$ the Euclidean dimension of $\mathbb{H}$, and the quantity $Q := \sum_{j=1}^{n} a_j$ the homogeneous dimension of $\mathbb{H}$, respectively.

Let $\Sigma := \{x \in \mathbb{H} : \rho(x) = 1\}$ be the unit sphere on $\mathbb{H}$ and $\sigma$ be the Radon measure on $\Sigma$ (see for example [18, Proposition 1.15]) such that for any $f \in \mathbb{H}$,

$$\int_{\mathbb{H}} f(x)dx = \int_{0}^{\infty} \int_{\Sigma} f(r \circ \theta)r^{Q-1}d\sigma(\theta)dr. \quad (1.1)$$

Let $\Omega$ be a locally integrable function on $\mathbb{H} \backslash \{0\}$ and it is homogeneous of degree 0 with respect to group dilation, that is, $\Omega(\lambda \circ x) = \Omega(x)$ for $x \neq 0$ and $\lambda > 0$. We say that $\Omega$ satisfies the cancellation condition of order $N$ if for any polynomial $P_m$ on $\mathbb{H}$ of homogeneous degree $m \leq N$, we have

$$\int_{\Sigma} \Omega(\theta)P_m(\theta)d\sigma(\theta) = 0.$$  

In this paper, we will study the singular integral operators $T_{\Omega, \alpha}$ ($\alpha \geq 0$), and the maximal singular integral operators $T^#_{\Omega, \alpha}$, which are formally defined by

$$T_{\Omega, \alpha}f(x) := \lim_{\varepsilon \to 0} T_{\Omega, \alpha, \varepsilon}f(x) := \lim_{\varepsilon \to 0} \int_{\rho(y^{-1}x) > \varepsilon} \frac{\Omega(y^{-1}x)}{\rho(y^{-1}x)^{Q+\alpha}} f(y)dy,$$

$$T^#_{\Omega, \alpha}f(x) := \sup_{\varepsilon > 0} |T_{\Omega, \alpha, \varepsilon}f(x)|,$$
where $\Omega$ satisfies the cancellation condition of order $[\alpha]$. Here we used the notation $[\alpha]$ to denote the integer part of $\alpha$.

It is well known that for the case $\alpha = 0$ and $\mathbb{H}$ is an isotropic Euclidean space, Calderón and Zygmund [4] used the method of rotations to show that if $\Omega \in L \log^+ L(S^{n-1})$, then $T_{\Omega,0}$ is bounded on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. Later, Ricci and Weiss [34] relaxed the condition to $\Omega \in H^1(S^{n-1})$, under which this result was extended to the maximal singular integral operators $T_{\Omega,0}^\#$ by Fan and Pan [17]. Next, the authors in [7, 8] considered the case of $\alpha \geq 0$, which includes a larger class of singular integrals which are of interest in harmonic analysis and partial differential equation, such as the composition of partial derivative and Riesz transform. They established the $(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ boundedness when $\Omega \in H^q(S^{n-1})$ with $q = \frac{n-1}{n-1+\alpha}$. There are also many other significant progress on rough singular integral operators in the setting of Euclidean space (see for example [3, 6, 9, 10, 13, 15, 17, 19, 20, 30, 36]). However, due to the lack of fundamental tools, the parallel results on homogeneous groups are extremely limited. Among these results, we would like to highlight that by studying the left-invariant differentiation structures of homogeneous groups and applying the iterated $TT^*$ method, Tao [39] has created a pioneering work which illustrates that if $\Omega \in L \log^+ L(S^{n-1})$, then $T_{\Omega,0}$ is bounded on $L^p$ for any $1 < p < \infty$. Inspired by his work, Sato [35] obtained the $L^p$ boundedness of $T_{\Omega,0}^\#$ under the same condition. Thus, it is natural to ask whether one can obtain $(L^p_0(\mathbb{H}), L^p(\mathbb{H}))$ boundedness of the operators $T_{\Omega,\alpha}$ and $T_{\Omega,\alpha}^\#$ for $\alpha > 0$, where $L^p_0(\mathbb{H})$ is the Sobolev space on $\mathbb{H}$ defined in (1.3) with $w = 1$.

Another motivation of this research comes from the investigation of fractional-order partial differential equation. Caffarelli and Silvestre [2] constructed a fractional differentiation from an extension problem to the upper half space for a specific elliptic partial differential equation. As a continuation of their work, Chamorro and Jarrín [5] further investigated this problem on general nilpotent Lie group. In order to investigate fractional-order partial differential equations more deeply on homogeneous groups, the first aim of our paper is to establish the $(L^p(\mathbb{H}), L^p(\mathbb{H}))$-boundedness of the operators $T_{\Omega,\alpha}$ since it connects closely to the fractional Laplacian operator and the composition of fractional derivative with Riesz transform.

**Theorem 1.1** Let $0 < \alpha < Q$. Suppose that $\Omega \in L^1(\Sigma)$ and satisfies the cancellation condition of order $[\alpha]$, then for any $1 < p < \infty$,

$$\|T_{\Omega,\alpha}^\# f\|_{L^p(\mathbb{H})} \leq C_{Q,\alpha, p} \|\Omega\|_{L^1(\Sigma)} \|f\|_{L^p_0(\mathbb{H})}$$

for some constant $C_{Q,\alpha, p}$ independent of $\Omega$.

The second aim of our paper is to establish the quantitative weight inequalities for $T_{\Omega,\alpha}^\#$.

In the Euclidean setting, after the establishment of the sharp weight inequalities for Ahlfors–Beurling operator by Petermichl and Volberg [33], for Hilbert transform and Riesz transform by Petermichl [31, 32] and for general Calderón–Zygmund operators by Hytönen [21] (see also [24–26]), Hytönen–Roncal–Tapiola [23] first quantitatively obtained the weighted bounds for $T_{\Omega,0}$ (see also [11, 27]). Later, this result
was extended to the maximal singular integrals \( T^\#_{\Omega,0} \) by Di Plinio, Hytönen and Li [12] and Lerner [28] via sparse domination, which gives

\[
\| T^\#_{\Omega,0} f \|_{L^p(w)} \leq C_{n,p} \| \Omega \|_{L^\infty(\mathbb{R}^{n-1})} \{ w \}_A \| f \|_{L^p(w)},
\]

where \( w \) is an \( A_p \) weight, \( \{ w \}_A \), and \( \{ w \}_A \) are the quantitative constants with respect to \( w \) and \( L^p(w) \) is the weighted \( L^p \) space (all definitions are provided in Sect. 2.2). However, it is still unclear that whether a quantitative weight bound for \( T^\#_{\Omega,0} \) can be obtained on homogeneous groups. Moreover, there is no weighted result for the case \( \alpha > 0 \) even non-quantitative one in the special case of \( \mathbb{R}^n \). To fill in these gaps, we conclude the following result.

**Theorem 1.2** Let \( 0 \leq \alpha < Q \) and \( q > Q/\alpha \). Suppose that \( \Omega \in L^q(\Sigma) \) and satisfies the cancellation condition of order \( [\alpha] \), then for any \( 1 < p < \infty \) and \( w \in A_p \),

\[
\| T^\#_{\Omega,\alpha} f \|_{L^p(w)} \leq C_{Q,\alpha,p,q} \| \Omega \|_{L^q(\Sigma)} \{ w \}_A \| f \|_{L^p(w)}
\]

for some constant \( C_{Q,\alpha,p,q} \) independent of \( \Omega \) and \( w \).

where \( L^p_\alpha(w) \) \( (1 < p < \infty, \alpha \geq 0, w \in A_p) \) is the homogeneous weighted Sobolev space defined by

\[
\| f \|_{L^p_\alpha(w)} := \left( \int_{\mathbb{H}} \left| (-\Delta_{\mathbb{H}})^{\alpha/2} f(x) \right|^p w(x) \, dx \right)^{1/p},
\]

where the definition of \( \Delta_{\mathbb{H}} \) is defined in Sect. 2.1 and \( (-\Delta_{\mathbb{H}})^{\alpha/2} \) is defined spectrally.

### 1.2 Comparisons with Previous Results

Table I highlights our contributions in the \( L^p \) boundedness and the quantitative \( L^p(w) \) boundedness for the operators \( T_{\Omega,\alpha} \) and \( T^\#_{\Omega,\alpha} \) in \( \mathbb{R}^n \) and \( \mathbb{H} \) via a comparison of known results under different conditions. Among these results, we would like to mention that the \( L^p \) boundedness of \( T_{\Omega,\alpha} \) and \( T^\#_{\Omega,\alpha} \) \( (0 < \alpha < Q) \) are new even when \( \Omega \in L^\infty \).

Unfortunately, whether one can weaken the size condition of \( \Omega \) to certain Hardy space is still open even when \( \alpha = 0 \). Furthermore, for the weighted setting, the weight bound \( \{ w \}_A(w)_A \) we obtained is consistent with that obtained in [12]. It is still open that whether this is sharp, but it is the best known quantitative result for this class of operators.
Table 1  Highlight of our contributions

| Operator | Setting | Quantitative $L^p(w)$ boundedness |
|----------|---------|----------------------------------|
| $T_{\Omega, \alpha}$ | $\mathbb{R}^n$ | $\mathbb{H}$ |
| $\alpha = 0$ | $[34]: \Omega \in H^1$ | $[39]: \Omega \in L^{\log + L}$ |
| $0 < \alpha < Q$ | $[23]: \Omega \in L^\infty$ | Theorem 1.2: $\Omega \in L^\infty$ |
| $T_{\Omega, \alpha}^\#$ | $\mathbb{R}^n$ | $\mathbb{H}$ |
| $\alpha = 0$ | $[17]: \Omega \in H^1$ | $[35]: \Omega \in L^{\log + L}$ |
| $0 < \alpha < Q$ | $[12]: \Omega \in L^\infty$ | Theorem 1.2: $\Omega \in L^\infty$ |

1.3 Difficulties and Strategy of Our Proof

Due to the generality of the underlying space, many techniques in the Euclidean setting cannot be applied directly. In particular, some difficulties occur:

- We cannot apply Fourier transform and Plancherel’s theorem as effectively as in [15];
- The order of convolution cannot be exchanged on non-Abelian homogeneous groups;
- Generally, the topology degree and homogeneous degree in Taylor’s inequality are not equal;
- $L^p$ boundedness of the directional maximal function on homogeneous groups is not at hand;
- Due to the non-isotropic property of homogeneous groups, the method of rotation cannot be applied directly.

To overcome these difficulties, our main ideas are the following:

1. Either in the unweighted setting or weighted setting, the first step is to decompose our maximal operator into a non-dyadic maximal one and a dyadic one.
2. The weighted estimate of the non-dyadic maximal operator can be deduced from that of Hardy–Littlewood maximal function directly, whereas under the weaker assumption $\Omega \in L^1(\Sigma)$, the $L^p$ estimate of this operator is technical. Our strategy is to reduce the problem to a kind of mixed $L^2$ norm estimates of certain maximal singular integral operators with smooth kernels by applying Gagliardo–Nirenberg inequality.
3. We apply Cotlar–Knapp–Stein Lemma and some tricks of geometric means instead of Fourier transform, Plancherel’s theorem and the exchange of convolution order to show an abstract $L^2$ decay lemma: suppose that $\{\mu^\alpha_j\}_{j \in \mathbb{Z}}$ is a family of Borel measures on $\mathbb{H}$ such that its behavior like $\frac{\Omega(x)}{\rho(x)^{2+\alpha}} \chi_{\rho(x) \sim 2^j}$, then...
there exists a constant $\tau > 0$ such that for any $j \in \mathbb{Z}$, $0 < \alpha < Q$ and $q > 1$,

$$\| G_{\alpha}^{j}(t)f \|_2 \lesssim 2^{-\tau|j|}\|\Omega\|_{L^1(\Sigma)}\|f\|_2, \quad \| G_{j}^{0}(t)f \|_2 \lesssim 2^{-\tau|j|}\|\Omega\|_{L^q(\Sigma)}\|f\|_2,$$

where $G_{\alpha}^{j}(t)f$ is defined in (2.36) and (2.37).

(4) We use Cotlar’s decomposition together with Littlewood–Paley theory to decompose operators with rough kernel into summation of operators with smooth kernel. Then we adapt the above $L^2$ decay Lemma together with Khinchin’s inequality to establish the $L^2$ decay estimates of decomposed parts.

For the convenience of readers who are interested in our proof of framework, we provide two figures in the following. Here in Fig. 1, the definitions of $M_{\Omega,\alpha}, T_{\Omega,\alpha}^{k}, V_{k_j,t}, I_1, I_2, I_3, G_{k,s,j}^{\alpha}, T^{\alpha}_{j}$ can be found in (3.9), (3.10), (3.14), (3.36), (3.36), (3.41), (3.2), respectively. In Fig. 2, the definitions of $\tilde{T}_{k}^{\alpha}_{\Omega,\alpha}, I, II, III, \tilde{T}_{j}^{\alpha,N}, R^{\alpha}, A_{k+s}^{\alpha}, K_{0}^{\alpha}, \Delta[2^{k}]\phi$ can be found in (4.33), (4.35), (4.35), (4.35), (4.1), (2.11), (2.7), (2.6), respectively.

**Frame of proof of Theorem 1.1:**

![Diagram](https://via.placeholder.com/150)

**Fig. 1** Unweighted setting
Frame of proof of Theorem 1.2:

![Diagram](image-url)

Fig. 2 Weighted setting

1.4 Notation and Structure of the Paper

For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(\mathbb{H})$ by $\|f\|_p$. If $T$ is a bounded linear operator on $L^p(\mathbb{H})$, $1 \leq p \leq +\infty$, we write $\|T\|_{p \to p}$ for the operator norm of $T$. The indicator function of a subset $E \subseteq X$ is denoted by $\chi_E$. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some constant $C > 0$, and $A \sim B$ to denote the statement that $A \lesssim B$ and $B \lesssim A$.

This paper is organized as follows. In Sect. 2 we provide the preliminaries, including some auxiliary lemmas on homogeneous groups, the definitions of $A_p$ weights and Calderón–Zygmund operators with Dini-continuous kernel on homogeneous group $\mathbb{H}$, two types of decompositions and their corresponding $L^2$ decay estimates. In Sects. 3 and 4, we give the proofs of Theorems 1.1 and 1.2, respectively.

2 Preliminaries and Fundamental Tools on Homogeneous Lie Groups

2.1 Auxiliary Lemmas on Homogeneous Group $\mathbb{H}$

In this subsection, we recall some basic definitions on homogeneous groups and then show some auxiliary lemmas. To begin with, we recall the multiplication structure on $\mathbb{H}$ (see [1, Theorem 1.3.15]): if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{H}$, then for any $2 \leq j \leq n$,
\[(xy)_1 = x_1 + y_1, \quad (xy)_j = x_j + y_j + Q_j(x, y), \tag{2.1}\]

where \(Q_j(x, y)\) is a sum of mixed monomials in \(x, y\), only depends on \(x_i, y_i, i = 1, 2, \ldots, j - 1\), and satisfies \(Q_j(\lambda \circ x, \lambda \circ y) = \lambda^{a_j} Q_j(x, y)\) for all \(\lambda > 0\). As a corollary (see \([1, \text{Corollary 1.3.16}]\)), for any \(a_j\) homogeneous degree \(q_j\) follows.

\[Q_j = \sum_{i} a_{i} x_{i}^{n_{i}} y_{i}^{n_{i}}, \tag{2.2}\]

where \(q_j(x)\) only depends on \(x_1, x_2, \ldots, x_{j-1}\), and is a polynomial function in \(x\) of homogeneous degree \(a_j\), that is, \(q_j(\lambda \circ x) = \lambda^{a_j} q_j(x)\) for all \(\lambda > 0\). The following simple observation is useful to exploit the \([\alpha]\)-order cancellation property of \(\Omega\) in what follows.

\textbf{Lemma 2.1} \hspace{1em} (1) Let \(N\) be a positive integer and \(P(x)\) be a polynomial of homogeneous degree \(N\), then \(P(x^{-1})\) is a polynomial in \(x\) of homogeneous degree \(N\).

(2) Let \(N\) be a positive integer and \(P(x)\) be a polynomial of homogeneous degree \(N\), then for any \(y \in \mathbb{H}\), \(P(xy)\) and \(P(yx)\) are summations of polynomials in \(x\), with coefficients depending on \(y\), of homogeneous degree less than or equal to \(N\).

\textbf{Proof} \hspace{1em} (1) According to the hypothesis, \(P(x)\) is of the form: \(P(x) = \sum_{a_1 b_1 + \cdots + a_n b_n = N} c_{b_1}^{b_1} \cdots x_{n}^{b_n}\). Therefore, it follows from (2.2) that for any \(\lambda > 0\),

\[P(\lambda \circ x^{-1}) = \sum_{a_1 b_1 + \cdots + a_n b_n = N} c_{b_1}^{b_1} (-\lambda^{a_1} x_1)^{b_1} (-\lambda^{a_2} x_2 + q_2(\lambda \circ x))^{b_2} \cdots \]

\[\times (-\lambda^{a_n} x_n + q_n(\lambda \circ x))^{b_n} = \sum_{a_1 b_1 + \cdots + a_n b_n = N} c_{b_1}^{b_1} (-\lambda^{a_1} x_1)^{b_1} (-\lambda^{a_2} x_2 + \lambda^{a_2} q_2(x))^{b_2} \cdots \]

\[\times (-\lambda^{a_n} x_n + \lambda^{a_n} q_n(x))^{b_n} = \lambda^{N} P(x^{-1}).\]

This implies the first statement of Lemma 2.1.

(2) By (2.1), \(P(xy)\) is of the form:

\[P(xy) = \sum_{a_1 b_1 + \cdots + a_n b_n = N} c_{b}^{b_1} (xy)^{b_1} (xy)^{b_2} \cdots (xy)^{b_n} \]

\[= \sum_{a_1 b_1 + \cdots + a_n b_n = N} c_{b}^{b_1} (x_1 + y_1)^{b_1} (x_2 + y_2 + Q_2(x_1, y_1))^{b_2} \cdots \]

\[\times (x_n + y_n + Q_n(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}))^{b_n},\]

where \(Q_j(x, y)\) is a sum of mixed monomials in \(x, y\), only depends on \(x_i, y_i, i = 1, 2, \ldots, j - 1\), and satisfies \(Q_j(\lambda \circ x, \lambda \circ y) = \lambda^{a_j} Q_j(x, y)\) for all \(\lambda > 0\). Hence, by expanding each term in the brackets and handling \(P(yx)\) similarly, we can easily conclude the second statement of Lemma 2.1. □
Now we denote the set of left-invariant vector fields on $\mathbb{H}$ by $\mathfrak{g}$, which is called the Lie algebra of $\mathbb{H}$. One identifies $\mathfrak{g}$ and $\mathbb{H}$ via the exponential map

$$\exp : \mathfrak{g} \rightarrow \mathbb{H},$$

which is a globally defined diffeomorphism. Let $X_j$ (resp. $Y_j$) be the left-invariant (resp. right-invariant) vector field that agrees with $\partial/\partial x_j$ (resp. $\partial/\partial y_j$) at the origin. Equivalently ([18]),

$$X_j f(x) = \frac{d}{dt} f(x \exp(t X_j))|_{t=0}, \quad Y_j f(x) = \frac{d}{dt} f(\exp(t X_j)x)|_{t=0}.$$

By Proposition 1.2.16 in [1], the family $\{X_j\}_{1 \leq j \leq n}$ (resp. $\{Y_j\}_{1 \leq j \leq n}$) forms a Jacobian basis of $\mathfrak{g}$. We adopt the following multi-index notation for higher order derivatives. For $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$, we denote $X^I := X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}$. Moreover, let $|I|$ be the order of the differential operator $X^I$, while $d(I)$ is the corresponding homogeneous degree, that is,

$$|I| := i_1 + i_2 + \cdots + i_n, \quad d(I) := a_1 i_1 + a_2 i_2 + \cdots + a_n i_n.$$

Moreover, let $\Delta_{\mathbb{H}} := \sum_{j=1}^n X_j^2$ be the sub-Laplacian operator on $\mathbb{H}$.

**Lemma 2.2** Assume that $f \in C_0^{N+1}(\mathbb{H})$ for some $N \in \mathbb{N}$, satisfying $\text{supp} f \subset B(0, 1)$. For any $x \in \mathbb{H}$, set $f_k(x) = f(2^{-k} \circ x)$. Moreover, assume that $\{m_k\}_{k \in \mathbb{Z}}$ be a family of Borel measure satisfying $N$-order cancellation and $\text{supp} m_k \subset B(0, 2^k)$. Then there are constants $C_N, \kappa_N > 0$ such that

$$|(f_k * m_j)(x)| \leq C_N 2^{-k(N+1)} \sup_{\rho(z) \leq \kappa_N 2^{j-k}} |(X^I f)((2^{-k} \circ x)z)| \int_{\mathbb{H}} \rho(y)^{N+1} d|m_j|(y),$$

and

$$|(m_j * f_k)(x)| \leq C_N 2^{-k(N+1)} \sup_{\rho(z) \leq \kappa_N 2^{j-k}} |(Y^I f)(z(2^{-k} \circ x))| \int_{\mathbb{H}} \rho(y)^{N+1} d|m_j|(y).$$

**Proof** It suffices to show the first inequality since the second one is similar. To this end, let $L^f_k$ be the left Taylor polynomial of $f$ at $x$ of homogeneous degree $N$. Then by the $N$-order cancellation condition of $m_k$ and Taylor’s inequality on homogeneous groups $\mathbb{H}$ (see [16, Corollary 1.44]),

$$|(f_k * m_j)(x)| = \left| \int_{\mathbb{H}} \left( f(2^{-k} \circ (xy^{-1})) - L^f_k \circ x (2^{-k} \circ y^{-1}) \right) dm_j(y) \right|$$

$$\leq C_N \int_{\mathbb{H}} \sup_{\rho(z) \leq \kappa_N \rho(2^{-k} \circ y)} |(X^I f)((2^{-k} \circ x)z)| \rho(2^{-k} \circ y)^{N+1} d|m_j|(y).$$
\[ \leq C_N 2^{-k(N+1)} \sup_{\rho(z) \leq \kappa_N 2^{l-k}} |(X^I f)((2^{-k} \circ x)z)| \int_{\mathbb{H}} \rho(y)^{N+1} d|m_j|(y) \]

for some constants \( C_N, \kappa_N > 0 \). This ends the proof of Lemma 2.2. \( \square \)

We have the following mean value theorem of integral type.

**Lemma 2.3** Let \( f \in C^1(\mathbb{H}) \) and \( j = 1, 2, \ldots, v \) be the indices such that \( a_j = 1 \), then

\[ \int_{\mathbb{H}} |f(yx) - f(x)| dx \lesssim \rho(y) \sum_{j=1}^v \int_{\mathbb{H}} |(Y_j f)(x)| dx. \]

**Proof** Let \( V_1 \) be the linear span of \( X_1, \ldots, X_v \). Since \( \mathbb{H} \) is a stratified group, by [18, Lemma 1.40], there exists a constant \( N \in \mathbb{N} \) such that each element \( y \in \mathbb{H} \) can be represented as \( y = y_1 \cdots y_N \) with \( y_k \in \exp(V_1) \) and \( \rho(y_k) \lesssim \rho(y) \) for all \( k \). Moreover, since \( V_1 \) is the linear span of \( X_1, \ldots, X_v \), each \( y_k \) can be further written as \( y_k = \exp(t_{k,1}X_1 + \cdots + t_{k,v}X_v) \).

Equivalently,

\[ \exp^{-1} y_k = t_{k,1}X_1 + \cdots + t_{k,v}X_v. \tag{2.3} \]

On the other hand, by [1, Theorem 1.3.28], the exponential map \( \exp \) and its inverse map \( \exp^{-1} \) are globally defined diffeomorphisms with polynomial component functions, which implies that if we write \( y_k = (y_{k,1}, \ldots, y_{k,n}) \), then (see [1, (1.75b)])

\[ \exp^{-1} y_k = y_{k,1}X_1 + (y_{k,2} + C_2(y_{k,1}))X_2 + \cdots + (y_{k,v} + C_v(y_{k,1}, \ldots, y_{k,v-1}))X_v, \tag{2.4} \]

where \( C_j \)'s are polynomial functions of homogeneous degree 1, completely determined by the multiplication law on \( \mathbb{H} \). Combining the equalities (2.3) and (2.4), we conclude that

\[ t_{k,j} = \begin{cases} y_{k,1}, & j = 1, \\ y_{k,j} + C_j(y_{k,1}, \ldots, y_{k,j-1}), & j = 2, \ldots, v. \end{cases} \]

Hence, \( |t_{k,j}| \lesssim \rho(y_k) \lesssim \rho(y) \). Therefore,

\[ \int_{\mathbb{H}} |f(yx) - f(x)| dx \leq \sum_{k=1}^N \int_{\mathbb{H}} |f(y_k y_{k+1} \cdots y_N x) - f(y_{k+1} \cdots y_N x)| dx \]

\[ = \sum_{k=1}^N \int_{\mathbb{H}} \left| \int_0^1 \left( (t_{k,1}Y_1 + \cdots + t_{k,v}Y_v)f \right) \times \left( \exp(s(t_{k,1}X_1 + \cdots + t_{k,v}X_v))y_{k+1} \cdots y_N x \right) ds \right| dx \]

\( \square \) Springer
\[
\sum_{k=1}^{N} \rho(z_k) \leq \rho(y_k) \int_{\mathbb{H}} \left| (t_{k,1} Y_1 + \cdots + t_{k,v} Y_v) f \right| (z_k y_{k+1} \cdots y_N x) \, dx
\]
\[
\sum_{k=1}^{v} |t_{k,j}| \sup_{\rho(z_k) \leq \rho(y)} \int_{\mathbb{H}} \left| Y_j f \right| (z_k y_{k+1} \cdots y_N x) \, dx
\]
\[
\rho(y) \sum_{j=1}^{v} \int_{\mathbb{H}} \left| Y_j f \right| (x) \, dx,
\]

where we note that the integrand in the right-hand side of the first inequality above becomes \( f(y_N x) - f(x) \) when \( k = N \) and similar notation happens in what follows. This ends the proof of Lemma 2.3. \( \Box \)

### 2.2 \( A_p \) Weights on Homogeneous Group \( \mathbb{H} \)

We next recall the definition and some properties of \( A_p \) weight on \( \mathbb{H} \). To begin with, we define a left-invariant quasi-distance \( d \) on \( \mathbb{H} \) by \( d(x, y) = \rho(x^{-1} y) \), which means that there exists a constant \( A_0 \geq 1 \) such that for any \( x, y, z \in \mathbb{H} \),

\[
d(x, y) \leq A_0[d(x, z) + d(z, y)].
\]

Next, let \( B(x, r) := \{ y \in \mathbb{H} : d(x, y) < r \} \) be the open ball with center \( x \in \mathbb{H} \) and radius \( r > 0 \).

For \( 1 < p < \infty \), we say that \( w \in A_p \) if there exists a constant \( C > 0 \) such that

\[
[w]_{A_p} := \sup_B \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{1-p'} \, dx \right)^{p-1} \leq C.
\] (2.5)

Moreover, let \( A_{\infty} := \bigcup_{1 \leq p < \infty} A_p \) and we have

\[
[w]_{A_{\infty}} := \sup_B \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \exp \left( \frac{1}{|B|} \int_B \log \left( \frac{1}{w} \right) \, dx \right) < \infty,
\]

where the supremum is taken over all balls \( B \subset \mathbb{H} \) with respect to the quasi-distance function \( d \). We recall the following variants of the weight characteristic (see for example [23]):

\[
[w]_{A_{p}} := \left[ w \right]_{A_{p}}^{1/p} \max \{ [w]_{A_{\infty}}^{1/p'}, [w^{1-p'}]_{A_{\infty}}^{1/p} \}, \quad (w)_{A_{p}} := \max \{ [w]_{A_{\infty}}, [w^{1-p'}]_{A_{\infty}} \}.
\]

### 2.3 Calderón–Zygmund Operators with Dini-Continuous Kernel

Let \( T \) be a bounded linear operator on \( L^2(\mathbb{H}) \) represented as

\[
T f(x) = \int_{\mathbb{H}} K(x, y) f(y) \, dy, \quad \forall x \notin \text{supp } f.
\]
A function $\omega : [0, 1] \rightarrow [0, \infty)$ is a modulus of continuity if it satisfies the following three properties: (1) $\omega(0) = 0$; (2) $\omega(s)$ is an increasing function; (3) For any $s_1, s_2 > 0$, $\omega(s_1 + s_2) \leq \omega(s_1) + \omega(s_2)$.

**Definition 2.4** We say that the operator $T$ is an $\omega$-Calderón–Zygmund operator if the kernel $K$ satisfies the following two conditions:

(1) (size condition): There exists a constant $C_T > 0$ such that

$$|K(x, y)| \leq \frac{C_T}{d(x, y)^Q}.$$ 

(2) (smoothness condition): Whenever $d(x, y) \geq 2A_0d(x, x') > 0$, we have

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega\left(\frac{d(x, x')}{d(x, y)}\right) \frac{1}{d(x, y)^Q}.$$ 

Furthermore, $K$ is said to be a Dini-continuous kernel if $\omega$ satisfies the Dini condition:

$$\|\omega\|_{\text{Dini}} := \int_0^1 \omega(s) \frac{ds}{s} < \infty.$$ 

### 2.4 Two Types of Decompositions

To begin with, we recall that for appropriate functions $f$ and $g$ defined on $\mathbb{H}$, the convolution $f \ast g$ is defined by

$$f \ast g(x) := \int_{\mathbb{H}} f(y)g(y^{-1}x)dy.$$ 

For simplicity, denote

$$K_\alpha(x) := \frac{\Omega(\rho(x)^{-1} \circ x)}{\rho(x)^{Q+\alpha}}.$$ 

Let $K_\alpha^0$ be the restriction of $K_\alpha$ to the annulus $\Sigma_0 := \{x \in \mathbb{H} : 1 \leq \rho(x) \leq 2\}$. Then we decompose the kernel $K_\alpha$ into smooth dyadic parts in the following way:

$$K_\alpha(x) = \frac{1}{\ln 2} \int_0^\infty \Delta_\alpha[t]K_\alpha^0(x) \frac{dt}{t},$$ 

where for each $t$, we define the $\alpha$-scaling map by

$$\Delta_\alpha[t]f(x) := t^{-Q-\alpha} f(t^{-1} \circ x). \quad (2.6)$$
For simplicity, we set \( \Delta_1 f(x) := \Delta_0 f(x) \). Hence, we have the following decomposition

\[
K_\alpha(x) = \sum_{j \in \mathbb{Z}} 2^{-j} \int_0^\infty \varphi(2^{-j} t) \Delta_\alpha \Delta_0 f(t) dt =: \sum_{j \in \mathbb{Z}} A_j K_\alpha^0(x), \tag{2.7}
\]

where \( \varphi \) is a smooth cut-off function localized in \( \{ \frac{1}{2} \leq t \leq 2 \} \) such that \( \sum_{j \in \mathbb{Z}} 2^{-j} t \varphi(2^{-j} t) = \frac{1}{\ln 2} \). Hence,

\[
T_{\Omega, \alpha} f = \sum_{j \in \mathbb{Z}} f * A_j K_\alpha^0 =: \sum_{j \in \mathbb{Z}} T_j^\alpha f. \tag{2.8}
\]

Since \( \text{supp} K_\alpha^0 \subset \{ x \in \mathbb{H} : 1 \leq \rho(x) \leq 2 \} \), we see that for any \( q \geq 1 \),

\[
\| A_j^\alpha K_\alpha^0 \|_1 \lesssim 2^{-\alpha j} \| K_\alpha^0 \|_1 \lesssim 2^{-\alpha j} \| \Omega \|_{L^q(\Sigma)}. \tag{2.9}
\]

In addition, we introduce the following non-smooth dyadic decomposition of \( K_\alpha \), which plays a key role in obtaining the \( L^p \) estimate of \( T_{\Omega, \alpha} \) for the case \( \alpha > 0 \):

\[
K_\alpha(x) = \sum_{j \in \mathbb{Z}} \frac{\Omega(x)}{\rho(x)^{Q+\alpha}} \chi_{2^j < \rho(x) \leq 2^{j+1}} =: \sum_{j \in \mathbb{Z}} B_j^\alpha \Omega(x). \]

Therefore,

\[
T_{\Omega, \alpha} f = \sum_{j \in \mathbb{Z}} f * B_j^\alpha \Omega =: \sum_{j \in \mathbb{Z}} V_j^\alpha f. \]

Moreover, it is direct that for any \( q \geq 1 \),

\[
\| B_j^\alpha \Omega \|_1 \lesssim 2^{-\alpha j} \| \Omega \|_{L^q(\Sigma)}. \tag{2.10}
\]

**Remark 2.5** It is obvious that these two types of kernel truncations share many similar properties, including: \( A_j^\alpha K_\alpha^0 \) and \( B_j^\alpha \Omega \) are supported in the annulus \( \{ x \in \mathbb{H} : \rho(x) \sim 2^j \} \) and satisfy the cancellation condition of order \( [\alpha] \). The key difference lies in the \( L^2 \) decay estimate for \( \alpha = 0 \) (see Lemma 2.7): the integral in the definition of \( A_j^\alpha K_\alpha^0 \) plays a crucial role in implementing the iterated \( TT^* \) argument from [39], which relies on the integration by part with respect to \( t \). Moreover, the truncation \( B_j^\alpha \Omega \) is feasible to obtain the unweighted \( L^p \) estimate of \( T_{\Omega, \alpha} \) in the case \( \alpha > 0 \), while the feasibility of using \( A_j^\alpha K_\alpha^0 \) is unclear.

We now introduce a form of Littlewood–Paley theory without any explicit use of the Fourier transform. To this end, let \( \hat{\phi} \in C_c^\infty(\mathbb{H}) \) be a smooth cut-off function satisfying

1. \( \text{supp} \phi \subset \{ x \in \mathbb{H} : \frac{1}{200} \leq \rho(x) \leq \frac{1}{100} \} \);
2. \( \int_{\mathbb{H}} \phi(x)dx = 1 \);
3. \( \phi \geq 0 \);
4. \( \phi = \hat{\phi} \).

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where $\tilde{F}$ denotes the function $\tilde{F}(x) := F(x^{-1})$. In addition, for each integer $j$, denote

$$\Psi_j := \Delta[2^{j-1}]\phi - \Delta[2^j]\phi.$$  

With this choice of $\phi$, it follows that $\Psi_j$ is supported on the ball of radius $C2^j$, has mean zero, and $\tilde{\Psi}_j = \Psi_j$. Next we define the partial sum operators $S_j$ by

$$S_j f := f * \Delta[2^{j-1}]\phi.$$  

Their differences are given by

$$S_j f - S_{j+1} f = f * \Psi_j.$$  

### 2.5 $L^2$ Decay Estimates

Let $p(t, x, y) \ (t > 0, x, y \in \mathbb{H})$ be the heat kernel associated to the sub-Laplacian operator $-\Delta_{\mathbb{H}}$ (that is, the integral kernel of $e^{-t\Delta_{\mathbb{H}}}$). For convenience, we set $p(t, x) = p(t, x, o)$ (see [18, Chap. 1, Sect. G]) and $p(x) = p(1, x)$. Moreover, for $0 < \alpha < Q$, let $R^\alpha$ be the kernel of Riesz potential operator $(-\Delta_{\mathbb{H}})^{-\alpha/2}$ of order $\alpha$, that is,

$$(-\Delta_{\mathbb{H}})^{-\alpha/2} f = f * R^\alpha. \quad (2.11)$$

Note that for any $t > 0, x, y \in \mathbb{H}$, we have the following representation formula:

$$R^\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} p(t, x) dt, \quad (2.12)$$

where $p(t, x)$ satisfies the following Gaussian estimate (see [40, Theorem 4.2]): for any multi-index $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$,

$$|Y^I p(t, x)| \lesssim t^{-\frac{Q+|I|}{2}} \exp \left( -c \frac{\rho^2(x)}{t} \right).$$

It follows from this formula that for $0 < \alpha < Q$,

$$|R^\alpha(x)| \lesssim \rho(x)^{-Q+\alpha}. \quad (2.13)$$

For convenience, we extend the definition of $R^\alpha$ to $\alpha = 0$ by setting $R^0 = \delta_0$, where $\delta_0$ is a dirac measure.

Moreover, let $0 < \alpha < Q$. We suppose that $\{\mu_j^\alpha\}_{j \in \mathbb{Z}}$ is a family of Borel measures on $\mathbb{H}$ and for any $j \in \mathbb{N}$,

1. $\text{supp } \mu_j^\alpha \subset \{ x \in \mathbb{H} : \rho(x) \sim 2^j \}$ and $\text{supp } \tilde{\mu}_j^\alpha \subset \{ x \in \mathbb{H} : \rho(x) \sim 2^j \};$
(2) \( \mu_j^\alpha \) and \( \tilde{\mu}_j^\alpha \) satisfy the cancellation condition of order \( \lfloor \alpha \rfloor \), i.e., for any polynomial \( P \) on \( \mathbb{H} \) of homogeneous degree \( \leq \lfloor \alpha \rfloor \),
\[
\int_{\mathbb{H}} P(x) d\mu_j^\alpha(x) = \int_{\mathbb{H}} P(x) d\tilde{\mu}_j^\alpha(x) = 0;
\]

(3) \( \| \mu_j^\alpha \|_1 \lesssim 2^{-\alpha j} \| \Omega \|_{L^1(\Sigma)} \) and \( \| \tilde{\mu}_j^\alpha \|_1 \lesssim 2^{-\alpha j} \| \Omega \|_{L^1(\Sigma)} \), where \( \tilde{\mu}_j^\alpha \) denotes the reflection of \( \mu_j^\alpha \), i.e., \( \tilde{\mu}_j^\alpha(E) := \mu_j^\alpha(\{x^{-1}: x \in E\}) \), for any \( \mu_j^\alpha \)-measurable set \( E \subset \mathbb{H} \).

Then we have the following key \( L^2 \) decay estimate.

**Proposition 2.6** Let \( 0 < \alpha < Q \). Then there exist constants \( C_{Q, \alpha} > 0 \) and \( \tau > 0 \) such that for any \( j, k \in \mathbb{Z} \),
\[
\| f \ast \Psi_k \ast R^\alpha \ast \mu_j^\alpha \|_2 \leq C_{Q, \alpha} 2^{-\tau |j-k|} \| \Omega \|_{L^1(\Sigma)} \| f \|_2,
\]  
and
\[
\| f \ast R^\alpha \ast \mu_j^\alpha \ast \Psi_k \|_2 \leq C_{Q, \alpha} 2^{-\tau |j-k|} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]

**Proof** We first give the proof of the first statement. Let \( \eta \) be a standard smooth cut-off function supported on \( \frac{1}{2} \leq \rho(x) \leq 2 \) and satisfying \( \sum_{\ell \in \mathbb{Z}} \eta(2^{-\ell} \circ x) = 1 \). Then we can decompose the kernel \( R^\alpha \) of the Riesz potential \( (-\Delta_\mathbb{H})^{-\alpha/2} \) as follows.
\[
R^\alpha(x) = \sum_{\ell \in \mathbb{Z}} R^\alpha(\eta(2^{-\ell} \circ x)) =: \sum_{\ell \in \mathbb{Z}} R^\alpha_{\ell}(x).
\]

It can be verified from (2.13) that \( \| R^\alpha_{\ell} \|_1 \lesssim 2^{\alpha \ell} \). Moreover, it follows from the representation formula (2.12) that \( R^\alpha_{\ell}(x) = 2^{-(Q-\alpha)\ell} \zeta(2^{-\ell} \circ x) \), where
\[
\zeta(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1} \rho(t, x) \eta(x) dt
\]
is a smooth function supported in the region \( \{ \frac{1}{2} \leq \rho(x) \leq 2 \} \).

To continue, we divide our proof into six cases.

**Case 1:** \( k \geq j \).

**Case 1.1:** If \( \ell \leq j \), then we apply Lemma 2.2 to see that
\[
\| \Psi_{k} \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_1 \lesssim 2^{-k(\lfloor \alpha \rfloor + 1)} \int_{\mathbb{H}} |R^\alpha_{\ell} \ast \mu_j^\alpha(y)| \rho(y)^{\lfloor \alpha \rfloor + 1} dy \lesssim 2^{-(\lfloor \alpha \rfloor + 1)(k-j)} \| R^\alpha_{\ell} \ast \mu_j^\alpha \|_1.
\]
This, in combination with Young’s inequality, yields
\[
\| f \ast \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_2 \lesssim 2^{-(|\alpha|+1)(k-j)} \| R^\alpha_{\ell} \|_1 \| \mu_j^\alpha \|_1 \| f \|_2 \\
\lesssim 2^{-(|\alpha|+1)(k-j)} 2^{-\alpha(j-\ell)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]  
(2.18)

**Case 1.2:** If \( \ell \geq j \), then we apply Lemma 2.2 to see that
\[
\| R^\alpha_{\ell} \ast \mu_j^\alpha \|_1 \lesssim 2^{-(|\alpha|+1-\alpha)\ell} \int_\mathbb{H} \rho(y)^{|\alpha|+1} d|\mu_j^\alpha|(y) \lesssim 2^{-(|\alpha|+1-\alpha)(\ell-j)} \| \Omega \|_{L^1(\Sigma)}.
\]  
(2.19)

This, in combination with Young’s inequality, yields
\[
\| f \ast \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_2 \leq \| R^\alpha_{\ell} \ast \mu_j^\alpha \|_1 \| f \|_2 \lesssim 2^{-(|\alpha|+1-\alpha)(\ell-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]  
(2.20)

**Case 1.2.1:** If \( \ell \geq k \), then the estimate (2.20) is enough.

**Case 1.2.2:** If \( j \leq \ell \leq k \), then we apply Lemma 2.2 to see that
\[
\| \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_1 \lesssim 2^{-(|\alpha|+1)k} \int_\mathbb{H} |R^\alpha_{\ell} \ast \mu_j^\alpha(y)| \rho(y)^{|\alpha|+1} dy \\
\lesssim 2^{-(|\alpha|+1)(k-\ell)} \| R^\alpha_{\ell} \ast \mu_j^\alpha \|_1.
\]

This, in combination with the estimate (2.19), implies
\[
\| f \ast \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_2 \lesssim 2^{-(|\alpha|+1)(k-\ell)} \| R^\alpha_{\ell} \ast \mu_j^\alpha \|_1 \\
\lesssim 2^{-(|\alpha|+1)(k-j)} 2^\alpha(\ell-j) \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]  
(2.21)

Back to the proof of case 1, we combine the estimates (2.18), (2.20) and (2.21) to obtain that
\[
\| f \ast \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_2 \lesssim \sum_{\ell \leq j} 2^{-(|\alpha|+1)(k-j)} 2^{-\alpha(j-\ell)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2 \\
+ \sum_{j \leq \ell \leq k} 2^{-(|\alpha|+1)(k-j)} 2^\alpha(\ell-j) \| \Omega \|_{L^1(\Sigma)} \| f \|_2 \\
+ \sum_{\ell \geq k} 2^{-(|\alpha|+1-\alpha)(\ell-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2 \lesssim 2^{-\tau(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]
for some constant \( \tau > 0 \).

**Case 2:** If \( k \leq j \), then by Young’s inequality,
\[
\| f \ast \Psi_k \ast R^\alpha_{\ell} \ast \mu_j^\alpha \|_2 \lesssim \| \Psi_k \|_1 \| R^\alpha_{\ell} \|_1 \| \mu_j^\alpha \|_1 \| f \|_2 \lesssim 2^\alpha(\ell-j) \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]  
(2.22)
**Case 2.1:** If $\ell \leq k$, then the estimate \((2.22)\) is enough.

**Case 2.2:** If $\ell \geq k$, then by the cancellation property of $\Psi_k$ and the mean value theorem on homogeneous groups (see [18]), we see that $\|\Psi_k * R^{\alpha}_\ell \|_1 \lesssim 2^\alpha \gamma \gamma^{-(\ell-k)}$. Therefore,

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \leq \| \Psi_k * R^{\alpha}_\ell \|_1 \| \mu_j^\alpha \|_1 \| f \|_2 \lesssim 2^{\alpha(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2. \] (2.23)

**Case 2.2.1:** If $k \leq \ell \leq j$, then taking geometric means of \((2.22)\) and \((2.23)\), we see that for any $\gamma \in [0,1]$,

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \lesssim 2^{\gamma(\ell-j)} 2^{\alpha(1-\gamma)(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 \]

Choosing $\max\{1-\alpha, 0\} < \gamma < 1$, we obtain

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \lesssim 2^{\tau(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 \] (2.24)

for some constant $\tau > 0$.

**Case 2.2.2:** If $\ell \geq j$, then \((2.19)\) holds. This, in combination with Young’s inequality, yields

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \leq \| R^{\alpha}_\ell * \mu_j^\alpha \|_1 \| f \|_2 \lesssim 2^{-(\alpha+1-\alpha)(\ell-j)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2. \] (2.25)

Taking geometric means of the estimates \((2.23)\) and \((2.25)\), we obtain that for any $\gamma \in [0,1]$,

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \lesssim 2^{-\gamma(\alpha+1-\alpha)(\ell-j)} 2^{\alpha(1-\gamma)(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 \]

Choosing $\gamma = \frac{1}{2}$ when $\alpha \leq 1$ and $\gamma \in (\frac{\alpha-1}{|\alpha|}, 1)$ when $\alpha > 1$ such that $-\gamma(\alpha) + \alpha - 1 < 0$ for $0 < \alpha < \frac{\alpha}{Q}$, we obtain that

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \lesssim 2^{-\tau(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 \] (2.26)

for some constant $\tau > 0$.

Back to the proof of case 2, we combine the estimates \((2.22)\), \((2.24)\) and \((2.26)\) to conclude that

\[ \| f * \Psi_k * R^{\alpha}_\ell * \mu_j^\alpha \|_2 \lesssim \sum_{\ell \leq k} 2^{\alpha(\ell-j)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 + \sum_{k \leq \ell \leq j} 2^{\tau(\ell-j)} \gamma^{-(\ell-k)} \| \Omega \|_{\Lambda^1(\Sigma)} \| f \|_2 \]
\[
+ \sum_{\ell \geq j} 2^{-\tau(j-k)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]
\[
\lesssim 2^{-\tau(j-k)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]

This finishes the proof of the first statement.

Now we turn to the second statement. Before presenting its proof, we first point out that in the setting of Euclidean space, the second statement is clearly equivalent to the first one. However, generally speaking, the convolution operation cannot be exchanged in homogeneous groups, the proof of the second statement is slightly different from the first one and cannot be directly deduced from the first one.

We divide the proof of the second statement into four cases.

**Case 1:** If \( k \geq j \), then we apply Lemma 2.2 to see that
\[
\| \mu_{\alpha}^j * \psi_k \|_1 \lesssim 2^{-([\alpha]+1)k} \int \rho(y)^{[\alpha]+1} d|\mu_{\alpha}^j|_1(y) \lesssim 2^{-([\alpha]+1)(k-j)} 2^{-\alpha j} \| \Omega \|_{L^1(\Sigma)}.
\]
(2.27)

**Case 1.1:** If \( \ell \leq k \), then we apply Young’s inequality and the estimate (2.27) to obtain
\[
\| f * R_{\alpha}^{\ell} * \mu_{\alpha}^j * \psi_k \|_2 \lesssim 2^{\alpha \ell} \| \mu_{\alpha}^j * \psi_k \|_1 \| f \|_2
\]
\[
\lesssim 2^{-\alpha(k-\ell)} 2^{-([\alpha]+1-\alpha)(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]
(2.28)

**Case 1.2:** If \( \ell \geq k \), then we apply Lemma 2.2 to see that
\[
\| R_{\alpha}^{\ell} * \mu_{\alpha}^j * \psi_k \|_1 \lesssim 2^{-(\alpha+1-\alpha)\ell} \int \mu_{\alpha}^j * \psi_k(y) \rho(y)^{\alpha} \]d\(y\)
\[
\lesssim 2^{-(\alpha+1-\alpha)\ell} 2^{([\alpha]+1)k} \| \mu_{\alpha}^j * \psi_k \|_1.
\]
(2.29)

This, in combination with the estimate (2.27) and Young’s inequality, yields
\[
\| f * R_{\alpha}^{\ell} * \mu_{\alpha}^j * \psi_k \|_2 \lesssim 2^{-([\alpha]+1-\alpha)\ell-k} 2^{-(\alpha+1-\alpha)(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]
(2.30)

Back to the proof of case 1, we combine the estimates (2.28) and (2.30) to get that
\[
\| f * R_{\alpha}^{\ell} * \mu_{\alpha}^j * \psi_k \|_2 \lesssim \sum_{\ell \leq k} 2^{-\alpha(k-\ell)} 2^{-(\alpha+1-\alpha)(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]
\[
+ \sum_{\ell \geq k} 2^{-(\alpha+1-\alpha)(\ell-k)} 2^{-(\alpha+1-\alpha)(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]
\[
\lesssim 2^{-(\alpha+1-\alpha)(k-j)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.
\]
By taking geometric means of the estimates (2.22) and (2.31), we obtain that for any \( \gamma \in [0, 1] \), we have

\[
|Y_i R_\ell^\alpha(x)| \lesssim 2^{-(Q-\alpha+1)\ell} 2^{-\alpha j} \|\Omega\|_{L^1(\Sigma)}, \quad \text{uniformly in } x \in \mathbb{H}.
\]

**Case 2:** If \( \ell \leq j \), then it follows from (2.12) that for any \( i = 1, 2, \ldots, n \), we have

\[
\|Y_i R_\ell^\alpha \mu_j^\alpha(x)\| \lesssim 2^{-(Q-\alpha+1)\ell} 2^{-\alpha j} \|\Omega\|_{L^1(\Sigma)}.
\]

This, in combination with Young’s inequality, implies

\[
\|f * R_\ell^\alpha \mu_j^\alpha \Psi_k\|_2 \lesssim 2^{-(Q-\alpha+1)\ell} 2^{-\alpha j} j^2 \|\Omega\|_{L^1(\Sigma)} \|f\|_2.
\]

By taking geometric means of the estimates (2.22) and (2.31), we obtain that for any \( \gamma_1 \in [0, 1] \),

\[
\|f * R_\ell^\alpha \mu_j^\alpha \Psi_k\|_2 \lesssim 2^{\gamma_1(\ell-j)} 2^{-(1-\gamma_1)(Q-\alpha+1)\ell} 2^{(1-\gamma_1)(Q-\alpha) j} 2^{(1-\gamma_1) k}
\]

\[
\times \|\Omega\|_{L^1(\Sigma)} \|f\|_2.
\]

**Case 2.1:** If \( \ell \leq j \), then we apply Lemma 2.2 to see that

\[
\|R_\ell^\alpha \mu_j^\alpha \Psi_k\|_1 \lesssim 2^{-(Q-\alpha+1)\ell} 2^{(Q-\alpha) j} \|\Omega\|_{L^1(\Sigma)}
\]

\[
\times \int_{\mathbb{H}} \rho(y) \|\Psi_k(y)\| \, dy \lesssim 2^{-(Q-\alpha+1)\ell} 2^{(Q-\alpha) j} 2^k \|\Omega\|_{L^1(\Sigma)}.
\]

**Case 2.2:** If \( \ell \geq j \), then we apply Lemma 2.2 to see that

\[
\|R_\ell^\alpha \mu_j^\alpha \Psi_k\|_1 \lesssim 2^{-(1-\alpha)\ell} 2^{-\alpha j} \|\Omega\|_{L^1(\Sigma)}
\]

\[
\int_{\rho(x) \lesssim 2^k} \rho(y) \|\Psi_k(y)\| \, dy \lesssim 2^{-(1-\alpha)\ell} 2^{-\alpha j} 2^k \|\Omega\|_{L^1(\Sigma)}.
\]

This, in combination with Young’s inequality, implies

\[
\|f \ast R_\ell^\alpha \mu_j^\alpha \Psi_k\|_2 \lesssim 2^{-(1-\alpha)\ell} 2^{-\alpha j} 2^k \|\Omega\|_{L^1(\Sigma)} \|f\|_2.
\]

Moreover, from the proof of (2.29) and the fact \( \supp \mu_j^\alpha \Psi_k \subset \{x \in \mathbb{H} : \rho(x) \lesssim 2^j\} \) we see that

\[
\|R_\ell^\alpha \mu_j^\alpha \Psi_k\|_1 \lesssim 2^{\alpha \ell} \int_{\mathbb{H}} |\mu_j^\alpha \Psi_k(y)| \rho(2^{-\ell} \circ y)^{[\alpha]+1} \, dy
\]

\[
\lesssim 2^{-(\lfloor \alpha \rfloor+1-\alpha)\ell} 2^{(\lfloor \alpha \rfloor+1) j} \|\mu_j^\alpha \Psi_k\|_1.
\]

This, in combination with Young’s inequality \( \|\mu_j^\alpha \Psi_k\|_1 \lesssim 2^{-\alpha j} \|\Omega\|_{L^1(\Sigma)} \), yields

\[
\|f \ast R_\ell^\alpha \mu_j^\alpha \Psi_k\|_2 \lesssim 2^{((\lfloor \alpha \rfloor+1-\alpha)(j-\ell))} \|\Omega\|_{L^1(\Sigma)} \|f\|_2.
\]
Taking geometric means of the estimates (2.33) and (2.34), we obtain that for any \(\gamma_2 \in [0, 1]\),
\[
\| f \ast R^\alpha_k \ast \mu_j^\alpha \ast \Psi_k \|_2 \\
\lesssim 2^{\gamma_2([\alpha]+1-\alpha)(j-\ell)} 2^{-(1-\gamma_2)(-\alpha+1)} 2^{-(1-\gamma_2)\alpha j} 2^{(1-\gamma_2)k} \| \Omega \|_{L^1(\Sigma)} \| f \|_2. \tag{2.35}
\]

Back to the proof of case 2, choosing \(\gamma_1 \in (\frac{Q-1}{Q+1}, 1)\) and \(\gamma_2 = \frac{1}{2}\) when \(\alpha < 1\), and \(\frac{Q-1}{|\alpha|} < \gamma_2 < 1\) when \(\alpha \geq 1\), respectively, such that \(\gamma_1 \alpha - (1 - \gamma_1)(Q - \alpha + 1) > 0\) and \(\gamma_2([\alpha] + 1 - \alpha) + (1 - \gamma_2)(-\alpha + 1) > 0\), we combine the estimates (2.32) and (2.35) to obtain that
\[
\| f \ast R^\alpha \ast \mu_j^\alpha \ast \Psi_k \|_2 \lesssim \sum_{\ell \leq j} 2^{\gamma_1 \alpha (\ell-j)} 2^{-(1-\gamma_1)(Q-\alpha+1)} 2^{(1-\gamma_1)(Q-\alpha)} 2^{(1-\gamma_1)k} \\
\times \| \Omega \|_{L^1(\Sigma)} \| f \|_2 \\
+ \sum_{\ell \geq j} 2^{\gamma_2([\alpha]+1-\alpha)(j-\ell)} 2^{-(1-\gamma_2)(-\alpha+1)} 2^{-(1-\gamma_2)\alpha j} 2^{(1-\gamma_2)k} \\
\times \| \Omega \|_{L^1(\Sigma)} \| f \|_2 \\
\lesssim 2^{-\tau(j-k)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]
for some constant \(\tau > 0\). This finishes the proof of Proposition 2.6. \(\square\)

Let \(0 < \alpha < Q\) and \(\mu_j^\alpha\) be a family of Borel measures defined as above. In addition, we let \(\{r_{k,j}(t)\}_{k,j \in \mathbb{Z}}\) be a family of functions such that for any \(k, j \in \mathbb{Z}\) and \(t \in [0, 1]\), \(r_{k,j}(t) = 1\) or \(r_{k,j}(t) = -1\). We consider the operators \(G^\alpha_j(t)\), associated with \(\mu_k^\alpha\) and \(r_{k,j}(t)\), defined by
\[
G^\alpha_j(t) f := \sum_{k \in \mathbb{Z}} r_{k,j}(t) f \ast \Psi_{k-j} \ast R^\alpha_k \ast \mu_k^\alpha =: \sum_{k \in \mathbb{Z}} G^\alpha_{j,k}(t) f. \tag{2.36}
\]
For simplicity, we extend this definition to the critical case \(\alpha = 0\) by setting \(d\mu_j^0 = A^0 j K^0_0 dx\) and
\[
G^0_j(t) f := \sum_{k \in \mathbb{Z}} r_{k,j}(t) f \ast \Psi_{k-j} \ast A^0_k K^0_0 =: \sum_{k \in \mathbb{Z}} G^0_{j,k}(t) f. \tag{2.37}
\]
We have the following \(L^2\) decay lemma.

**Lemma 2.7** Let \(0 < \alpha < Q\) and \( q > 1\), then there exist constants \(C_{Q,\alpha}, C_{Q,q} > 0\) and \(\tau > 0\) (independent of the value of \(r_{k,j}(t)\)) such that for any \(j \in \mathbb{Z}\),
\[
\| G^\alpha_j(t) f \|_2 \leq C_{Q,\alpha} 2^{-\tau j} \| \Omega \|_{L^1(\Sigma)} \| f \|_2, \tag{2.38}
\]
\(\square\) Springer
and

\[\|G_j^0(t) f\|_2 \leq C_{Q,q} 2^{-|j|} \|\Omega\|_{L^q(\Sigma)} \|f\|_2.\]  \quad (2.39)

**Proof** We first give the proof of estimate (2.38). By Cotlar–Knapp–Stein Lemma (see [38]), it suffices to show that:

\[\|(G_{j,k}^\alpha(t))^* G_{j,k'}^\alpha(t)\|_{2 \to 2} + \|G_{j,k}^\alpha(t) (G_{j,k}^\alpha(t))^*\|_{2 \to 2} \leq C_Q 2^{-2|j|} 2^{-|k-k'|} \|\Omega\|^2_{L^1(\Sigma)}.\]  \quad (2.40)

We only estimate the first term, since the estimate of the second one is similar. Note that

\[(G_{j,k}^\alpha(t))^* G_{j,k'}^\alpha(t) f = r_{k',j}(t) r_{k,j}(t) f * \Psi_{k'-j} * R_{\alpha} * \mu_{k'}^\alpha * \tilde{\mu}_k^\alpha * R_{\alpha} * \Psi_{k-j}
\]

\[= \sum_{\ell \in \mathbb{Z}} \sum_{\ell' \in \mathbb{Z}} r_{k',j}(t) r_{k,j}(t) f * \Psi_{k'-j} * R_{\alpha} * \mu_{k'}^\alpha * \Psi_{\ell} * \Psi_{\ell'}
\]

\[\times * \tilde{\mu}_k^\alpha * R_{\alpha} * \Psi_{k-j}.\]  \quad (2.41)

On the one hand, it follows from Young’s inequality, Proposition 2.6 and its dual version that

\[\|(f * \Psi_{k'-j}) * (R_{\alpha} * \mu_{k'}^\alpha * \Psi_{\ell}) * (\Psi_{\ell'} * \tilde{\mu}_k^\alpha * R_{\alpha}) * \Psi_{k-j}\|_2
\]

\[\lesssim \|(f * \Psi_{k'-j}) * (R_{\alpha} * \mu_{k'}^\alpha * \Psi_{\ell}) * (\Psi_{\ell'} * \tilde{\mu}_k^\alpha * R_{\alpha})\|_2
\]

\[\lesssim 2^{-|k-k'|} \|\Omega\|_{L^1(\Sigma)} \|(f * \Psi_{k'-j}) * (R_{\alpha} * \mu_{k'}^\alpha * \Psi_{\ell})\|_2
\]

\[\lesssim 2^{-|k-k'|} 2^{-|\ell-\ell'|} \|\Omega\|^2_{L^1(\Sigma)} \|f * \Psi_{k'-j}\|_2
\]

\[\lesssim 2^{-|k-k'|} 2^{-|\ell-\ell'|} \|\Omega\|^2_{L^1(\Sigma)} \|f\|_2.\]  \quad (2.42)

On the other hand, using the cancellation and smoothness properties of \(\Psi_{\ell}\) and \(\Psi_{\ell'}\) to get that

\[\|\Psi_{\ell} * \Psi_{\ell'}\|_1 \lesssim 2^{-|\ell-\ell'|}.\]  \quad (2.43)

This, in combination with Young’s inequality, Proposition 2.6 and its dual version, indicates that

\[\|f * (\Psi_{k'-j} * R_{\alpha} * \mu_{k'}^\alpha) * (\Psi_{\ell} * \Psi_{\ell'}) * (\tilde{\mu}_k^\alpha * R_{\alpha} * \Psi_{k-j})\|_2
\]

\[\lesssim 2^{-|j|} \|\Omega\|_{L^1(\Sigma)} \|f * (\Psi_{k'-j} * R_{\alpha} * \mu_{k'}^\alpha)\|_2
\]

\[\lesssim 2^{-|j|} 2^{-|\ell-\ell'|} \|\Omega\|_{L^1(\Sigma)} \|f * \Psi_{k'-j} * R_{\alpha} * \mu_{k'}^\alpha\|_2
\]

\[\lesssim 2^{-2|j|} 2^{-|\ell-\ell'|} \|\Omega\|^2_{L^1(\Sigma)} \|f\|_2.\]  \quad (2.44)
Taking geometric mean of (2.42) with (2.44), we see that
\[ \|f \ast (\Psi_{k-j}^* \ast R^\alpha \ast \mu_k^0) \ast (\Psi_{k} \ast \Psi_{k'}) \ast (\tilde{\mu}_k^0 \ast R^\alpha \ast \Psi_{k-j})\|_2 \lesssim 2^{-\tau|j|} 2^{-\tau|k-k'|} 2^{-\tau|k'-\ell|} \left\| \Omega \right\|_{L^1}(\Sigma) \|f\|_2. \]
Combining this inequality with the equality (2.41), we obtain the estimate (2.40) and therefore the first statement of Lemma 2.7.

For the second statement, we recall that Tao [39, Proposition 4.1] applied iterated \((TT^*)^N\) method to obtain the following inequality with \(q = \infty\) and then Sato [35, Lemma 1] extended it to general \(q > 1\): there exist constants \(C_{Q,q} > 0\) and \(\tau > 0\) such that for any \(j, k \in \mathbb{Z}\),
\[ \|f \ast A_j^0 K_0^0 \ast \Psi_k\|_2 \leq C_{Q,q} 2^{-\tau|j-k|} \|f\|_2 \left\| \Omega \right\|_{L^q}(\Sigma). \] (2.45)
Thus, the second statement can be shown by repeating the above argument and with Proposition 2.6 replaced by (2.45). This finishes the proof of Lemma 2.7. \(\Box\)

**Remark 2.8** In what follows, \(\mu_j^\alpha\) will be endowed with several exact values. On the one hand, we will choose \(d\mu_j^\alpha := A_j^\alpha K_0^0 dx, B_j^\alpha \Omega dx\) and \(B_j^\alpha, t_\Omega dx\), where \(B_{j,t}^\alpha, \Omega\) is defined in Sect. 3.3. On the other hand, we also choose \(d\mu_j^\alpha := 2^{-j((Q-1+\alpha) - Q - \alpha)} \Omega dx\), where \(\sigma_r\) is the unique Radon measure on \(r \Sigma := \{x \in \mathbb{H} : \rho(x) = r\}\) satisfying for any \(f \in L^1(\Sigma)\),
\[ \int_\Sigma f(\theta) d\sigma(\theta) = \int_{r \Sigma} f(r^{-1} \circ \theta) d\sigma_r(\theta) r^{-(Q-1)}. \]
In both cases, it can be seen from the proof of Proposition 2.6 and Lemma 2.7 that the bounds on the right-hand side of (2.14), (2.15) and (2.38) are independent of \(t \in [1, 2]\). We would like to mention that in the Euclidean setting, for the particular choice \(d\mu_j^0 = B_j^0 \Omega dx\), the corresponding \(L^2\) decay estimate was obtained in [15] (see also [23]) via a standard argument of Fourier transform and Plancherel’s theorem. Actually, in this setting, one can also apply this argument to obtain the corresponding \(L^2\) decay estimates for the other choices of \(\mu_j^\alpha\) when \(0 \leq \alpha < Q\).

## 3 Unweighted \(L^p\) Estimate

Throughout this section, unless we mention the contrary, we suppose that \(\Omega\) satisfies the following assumption.

**Assumption** Let \(0 < \alpha < Q\). Suppose that \(\Omega \in L^1(\Sigma)\) and satisfies the cancellation condition of order \([\alpha]\).

### 3.1 Boundedness of \(T_{\Omega,\alpha}\)

In this section, we show the \((L^p(\mathbb{H}), L^p(\mathbb{H}))\) estimate of \(T_{\Omega,\alpha}\).
Proposition 3.1  Let $0 < \alpha < \mathbb{Q}$. Suppose that $\Omega \in L^1(\Sigma)$ and satisfies the cancellation condition of order $[\alpha]$, then for any $1 < p < \infty$,

$$\|T_{\Omega,\alpha} f\|_{L^p(\mathbb{H})} \leq C_{\mathbb{Q},\alpha,p} \|\Omega\|_{L^1(\Sigma)} \|f\|_{L^p(\mathbb{H})}$$

for some constant $C_{\mathbb{Q},\alpha,p}$ independent of $\Omega$.

To show Proposition 3.1, we first note that $S_j f \to f$ as $j \to -\infty$, which implies that:

$$V_\alpha^\alpha (-\Delta_\mathbb{H})^{-\alpha/2} = V_k^\alpha (-\Delta_\mathbb{H})^{-\alpha/2} S_k + \sum_{j=1}^{\infty} V_k^\alpha (-\Delta_\mathbb{H})^{-\alpha/2} (S_{k-j} - S_{k-(j-1)}).$$

(3.1)

In this way, $T_{\Omega,\alpha} (-\Delta_\mathbb{H})^{-\alpha/2} = \sum_{j=0}^{\infty} \tilde{T}_j^\alpha$, where

$$\tilde{T}_j^\alpha := \sum_{k \in \mathbb{Z}} V_k^\alpha (-\Delta_\mathbb{H})^{-\alpha/2} (S_{k-j} - S_{k-(j-1)}).$$

(3.2)

Lemma 3.2 There exist constants $C_{\mathbb{Q},\alpha} > 0$ and $\tau > 0$, such that for any $j \geq 0$,

$$\|\tilde{T}_j^\alpha f\|_2 \leq C_{\mathbb{Q},\alpha} 2^{-\tau j} \|\Omega\|_{L^1(\Sigma)} \|f\|_2.$$  

(3.3)

Proof Note that $\tilde{T}_0^\alpha = \sum_{j=-\infty}^0 G_j^\alpha\mu_j^\alpha$ and $r_{k,j}(t)$ chosen to be $B_j^\alpha \Omega$ and the constant function 1, respectively. This, together with Lemma 2.7, ends the proof of Lemma 3.2. 

Next we study the kernel of the operator $\tilde{T}_j^\alpha$. To this end, we denote

$$K_j^\alpha := \sum_{k \in \mathbb{Z}} [2^{k-j}] \Delta [2^{k-j}] \phi \ast R^\alpha \ast B_k^\alpha \Omega.$$ 

Lemma 3.3 The kernel $K_j^\alpha$ satisfies the following Hörmander condition: there exists a constant $C_{\mathbb{Q},\alpha} > 0$ such that for any $j \geq 0$,

$$\int_{\rho(x) \geq 2A_0 \rho(y)} |K_j^\alpha(y^{-1}x) - K_j^\alpha(x)| \, dx \leq C_{\mathbb{Q},\alpha} (1 + j) \|\Omega\|_{L^1(\Sigma)}.$$

Proof We will postpone the proof of this lemma to Lemma 3.8, in which we will show a stronger statement.
Lemma 3.4 For any $1 < p < \infty$, there exist constants $C_{Q,\alpha}$, $C_{Q,\alpha,p} > 0$ and $\tau > 0$ such that for any $j \geq 0$,

\[
\| \tilde{T}_j^\alpha f \|_{L^1,\infty} \leq C_{Q,\alpha}(1 + j)\| \Omega \|_{L^1(\Sigma)} \| f \|_1,
\]

(3.4)

and

\[
\| \tilde{T}_j^\alpha f \|_p \leq C_{Q,\alpha,p}2^{-\tau j} \| \Omega \|_{L^1(\Sigma)} \| f \|_p.
\]

Proof By Lemmas 3.2 and 3.3, the first statement is the consequence of a standard argument of Calderón–Zygmund decomposition (see for example [14, 38]). Then, the second one can be obtained by interpolating (3.3) with (3.4), and by a standard duality argument.

\[\Box\]

Proof of Proposition 3.1 By the equality (3.1) and Lemma 3.4,

\[
\| T_{\Omega,\alpha}(-\Delta^\Sigma)^{-\alpha/2} f \|_p \leq \sum_{j=0}^\infty \| \tilde{T}_j^\alpha f \|_p \lesssim \sum_{j=0}^\infty 2^{-\tau j} \| \Omega \|_{L^1(\Sigma)} \| f \|_p \\
\lesssim \| \Omega \|_{L^1(\Sigma)} \| f \|_p.
\]

Equivalently,

\[
\| T_{\Omega,\alpha} f \|_{L^p} \lesssim \| \Omega \|_{L^1(\Sigma)} \| f \|_{L^p}.
\]

This completes the proof of Proposition 3.1.

\[\Box\]

Remark 3.5 With a slight modification, the above arguments can be also applied to give an alternative proof of the $L^p$ boundedness of $T_{\Omega,0}$, which was shown in [39] and [35] by two different approaches, provided $\Omega \in L \log^+ L(\Sigma)$ and $\int_{\Sigma} \Omega(\theta)d\sigma(\theta) = 0$. For the convenience of the readers, we sketch the proof below.

Proof of Remark 3.5. Let $E_0 := \{ x \in \Sigma : |\Omega(x)| \leq 2 \}$ and $E_m := \{ x \in \Sigma : 2^{2m-1} < |\Omega(x)| \leq 2^{2m} \}$ for $m \geq 1$. In addition, for any $m \geq 0$, set

\[
\Omega_m(x) := \Omega(x)\chi_{E_m}(x) - \frac{1}{\sigma(\Sigma)} \int_{E_m} \Omega(\theta)d\sigma(\theta).
\]

Then $\int_{\Sigma} \Omega_m(x)d\sigma(x) = 0$ and $\Omega(x) = \sum_{m=0}^\infty \Omega_m(x)$. Let $T_{m,j}^0$ be the operator defined in (2.8) with $\alpha = 0$ and $\Omega$ replaced by $\Omega_m$. In addition, for any $j \geq 1$, let

\[
\tilde{T}_{m,j}^+ := \sum_{k \in \mathbb{Z}} T_{m,k}(S_{k+2j-1} - S_{k+2j}) \quad \text{and} \quad \tilde{T}_{m,j}^- := \sum_{k \in \mathbb{Z}} T_{m,k}(S_{k-2j} - S_{k-2j-1}).
\]

Then similar to the proof of Lemma 3.4, it can be verified that for any $j \geq 1$,

\[
\| \tilde{T}_{m,j}^\pm f \|_{L^1,\infty} \lesssim 2^j \| \Omega_m \|_{L^1(\Sigma)} \| f \|_1.
\]

\[\Box\]
This, along with (2.39) (where $\Omega$ is replaced by $\Omega_m$) and interpolation, shows that for any $1 < p \leq 2$,

$$
\| \tilde{T}_{m,j}^\pm f \|_p \lesssim 2^{-\tau 2^j} \| \Omega_m \|_{L^\infty(\Sigma)} \| f \|_p
$$

(3.6)

for some constant $\tau > 0$.

Moreover, by Young's inequality, for any $j, k \in \mathbb{Z}$, we have

$$
\| T_{m,k}^0 (S_{k-j} - S_{k-(j-1)}) f \|_2 \lesssim \| \Omega_m \|_{L^1(\Sigma)} \| f \|_2
$$

Next, similar to the proof of Lemma 2.7, it can be verified by Cotlar–Knapp–Stein Lemma that

$$
\| \tilde{T}_{m,j}^\pm f \|_2 \lesssim 2^j \| \Omega_m \|_{L^1(\Sigma)} \| f \|_2
$$

(3.7)

which interpolating with (3.5), in which $\Omega$ is replaced by $\Omega_m$, yields that for any $1 < p \leq 2$,

$$
\| \tilde{T}_{m,j}^\pm f \|_p \lesssim 2^j \| \Omega_m \|_{L^1(\Sigma)} \| f \|_p
$$

(3.8)

Combining (3.6) and (3.8), we obtain that

$$
\| T_{0,0} f \|_p \leq \sum_{m=0}^{\infty} \sum_{j \geq 1} \| \tilde{T}_{m,j}^+ f \|_p + \| \tilde{T}_{m,j}^- f \|_p
$$

$$
\lesssim \sum_{m=0}^{\infty} \sum_{2^j \leq \lambda 2^m} 2^j \| \Omega_m \|_{L^1(\Sigma)} \| f \|_p + \sum_{m=0}^{\infty} \sum_{2^j > \lambda 2^m} 2^{-\tau 2^j} \| \Omega_m \|_{L^\infty(\Sigma)} \| f \|_p
$$

$$
\lesssim \sum_{m=0}^{\infty} \lambda 2^m \| \Omega_m \|_{L^1(\Sigma)} \| f \|_p + \sum_{m=0}^{\infty} 2^{(-\tau \lambda + 1)2^m} \| f \|_p
$$

$$
\lesssim \left( \int_{\Sigma} \| \Omega(\theta) \|_{L^1(\Sigma)} \| f \|_p \right)
$$

where we choose $\lambda > 1/\tau$. This, along with a duality argument, shows the proof of Remark 3.5. \(\square\)

### 3.2 Fundamental Reduction of Theorem 1.1

To begin with, denote

$$
M_{\Omega, \alpha} f(x) := \sup_{k > 0, \varepsilon \in [2^k, 2^{k+1})} \left| \int_{\varepsilon < \rho(y^{-1}x) \leq 2^{k+1}} \frac{\Omega(y^{-1}x)}{\rho(y^{-1}x)^{Q+\alpha}} f(y) dy \right|
$$

(3.9)
and
\[
T^k_{\Omega,\alpha} f(x) := \int_{\rho(y^{-1}x) > 2^{k+1}} \frac{\Omega(y^{-1}x)}{\rho(y^{-1}x)^{Q+\alpha}} f(y) dy. \tag{3.10}
\]

Since there exists a unique \( k \in \mathbb{Z} \) such that \( 2^k \leq \varepsilon < 2^{k+1} \), we have
\[
T^\#_{\Omega,\alpha} f(x) \leq \sup_{\varepsilon > 0} \left| \int_{\rho(y^{-1}x) \leq 2^{\lfloor \log_2 \varepsilon \rfloor+1}} \frac{\Omega(y^{-1}x)}{\rho(y^{-1}x)^{Q+\alpha}} f(y) dy \right|
+ \sup_{\varepsilon > 0} \left| \int_{\rho(y^{-1}x) > 2^{\lfloor \log_2 \varepsilon \rfloor+1}} \frac{\Omega(y^{-1}x)}{\rho(y^{-1}x)^{Q+\alpha}} f(y) dy \right|
= M_{\Omega,\alpha} f(x) + \sup_{k \in \mathbb{Z}} |T^k_{\Omega,\alpha} f(x)|. \tag{3.11}
\]

Hence, Theorem 1.1 can be reduced to showing the following two propositions.

**Proposition 3.6** For any \( 1 < p < \infty \), there exists a constant \( C_{Q,\alpha,p} > 0 \) such that for any \( j \geq 0 \),
\[
\| M_{\Omega,\alpha} (-\Delta_{\mathbb{H}})^{-\alpha/2} f \|_p \leq C_{Q,\alpha,p} \| \Omega \|_{L^1(\Sigma)} f \|_p. \tag{3.12}
\]

**Proposition 3.7** For any \( 1 < p < \infty \), there exists a constant \( C_{Q,\alpha,p} > 0 \) such that for any \( j \geq 0 \),
\[
\left\| \sup_{k \in \mathbb{Z}} |T^k_{\Omega,\alpha} (-\Delta_{\mathbb{H}})^{-\alpha/2} f| \right\|_p \leq C_{Q,\alpha,p} \| \Omega \|_{L^1(\Sigma)} f \|_p. \tag{3.13}
\]

We will give the proof of Propositions 3.6 and 3.7 in Sects. 3.3 and 3.4, respectively.

### 3.3 \( L^p \) Estimate of Non-Dyadic Maximal Function

In this subsection, we give the proof of the Proposition 3.6. To begin with, for any \( t \in [1, 2) \) and \( j \in \mathbb{N} \), denote
\[
B^\alpha_{j,t,\Omega}(x) := \frac{\Omega(x)}{\rho(x)^{Q+\alpha}} \chi_{t2^j < \rho(x) \leq 2^{j+1}},
\]
\[ V^\alpha_{j,t} := f \ast B^\alpha_{j,t,\Omega}. \]

Noting that \( S_j f \to f \) as \( j \to -\infty \), we have the following identity:
\[
V^\alpha_{k,t}(-\Delta_{\mathbb{H}})^{-\alpha/2} = V^\alpha_{k,t}(-\Delta_{\mathbb{H}})^{-\alpha/2} S_k + \sum_{j=1}^{\infty} V^\alpha_{k,t}(-\Delta_{\mathbb{H}})^{-\alpha/2} (S_k - j - S_k - (j-1)) \]
\[
= \sum_{j=0}^{\infty} V^\alpha_{k,j,t}. \tag{3.14}
\]
where $V_{k,0}^\alpha := V_{k,t}^\alpha (-\Delta_H)^{-\alpha/2} S_k$ and $V_{k,j,t}^\alpha := V_{k,t}^\alpha (-\Delta_H)^{-\alpha/2} (S_{k-j} - S_{k-(j-1)})$ for $j \geq 1$. Denote

$$K_{k,j,t}^\alpha := \Delta[2^{k-j}] \phi \ast R^\alpha \ast B_{k,s}^\alpha \Omega. \quad (3.15)$$

**Lemma 3.8** The kernel $K_{k,j,t}^\alpha$ satisfies the following uniform Hörmander condition: there exists a constant $C_{Q,\alpha} > 0$ such that for any $j \geq 0$,

$$\int_{\rho(x) \geq 2 \Lambda_0 + \rho(y)} \sup_{t \in [1,2]} \sum_{k \in \mathbb{Z}} |K_{k,j,t}^\alpha (y^{-1}x) - K_{k,j,t}^\alpha (x)| \, dx \leq C_{Q,\alpha} (1 + j) \| \Omega \|_{L^1(\Sigma)}.$$ 

**Proof** By the fundamental theorem of calculus, we note that

$$K_{k,j,t}^\alpha (x) = - \int_t^2 \frac{d}{ds} K_{k,j,s}^\alpha (x) \, ds.$$ 

Hence,

$$\int_{\rho(x) \geq 2 \Lambda_0 + \rho(y)} \sup_{t \in [1,2]} \sum_{k \in \mathbb{Z}} |K_{k,j,t}^\alpha (y^{-1}x) - K_{k,j,t}^\alpha (x)| \, dx$$

$$\leq \int_1^2 \int_{\rho(x) \geq 2 \Lambda_0 + \rho(y)} \sum_{k \in \mathbb{Z}} \left| \frac{d}{ds} \Delta[2^{k-j}] \phi \ast R^\alpha \ast B_{k,s}^\alpha \Omega (y^{-1}x) \right| \, dx \, ds.$$

A direct calculation yields

$$\frac{d}{ds} R^\alpha \ast B_{k,s}^\alpha \Omega (x) = \frac{d}{ds} \left( \int_{s/2^k}^{2^{k+1}} \int_{\Sigma} R^\alpha (x (r \circ \theta)^{-1}) \Omega (\theta) d\sigma (\theta) \frac{dr}{r^{\alpha+1}} \right)$$

$$= - \frac{2^{-\alpha k}}{s^{\alpha+1}} \int_{\Sigma} R^\alpha (x ((s/2^k) \circ \theta)^{-1}) \Omega (\theta) d\sigma (\theta)$$

$$\times \left( 1 - \eta_0 \left( \frac{x}{A_0^2 \kappa_{[\alpha]} 2^{k+4}} \right) \right)$$

$$- \frac{2^{-\alpha k}}{s^{\alpha+1}} \int_{\Sigma} R^\alpha (x ((s/2^k) \circ \theta)^{-1}) \Omega (\theta) d\sigma (\theta) \eta_0 \left( \frac{x}{A_0^2 \kappa_{[\alpha]} 2^{k+4}} \right)$$

$$=: I_{1,\alpha,k,s}^1 (x) + I_{1,\alpha,k,s}^2 (x). \quad (3.16)$$

Therefore, it remains to show that there exists a constant $C_{Q,\alpha} > 0$ such that for any $j \geq 0$,

$$\int_{\rho(x) \geq 2 \Lambda_0 + \rho(y)} \sum_{k \in \mathbb{Z}} \left| \Delta[2^{k-j}] \phi \ast I_{1,\alpha,k,s}^1 (y^{-1}x) - \Delta[2^{k-j}] \phi \ast I_{1,\alpha,k,s}^1 (x) \right| \, dx$$

$$\leq C_{Q,\alpha} \| \Omega \|_{L^1(\Sigma)}, \quad (3.17)$$
and that
\[
\int_{\rho(x) \geq 2A_0 \rho(y)} \sum_{k \in \mathbb{Z}} \left| \Delta [2^{k-j-j}] \phi \ast I_{\alpha,k,s}^2 (y^{-1} x) - \Delta [2^{k-j}] \phi \ast I_{\alpha,k,s}^2 (x) \right| \, dx
\leq C_{\Omega, \alpha} (1 + j) \| \Omega \|_{L^1(\Sigma)}.
\] (3.18)

**Estimate of (3.17):**
If \( \rho(x) \geq A_0^2 \kappa_{[\alpha]} 2^{k+4} \), then \( \rho((2^{-\ell} \circ x) z) \geq \frac{1}{A_0} \rho(2^{-\ell} \circ x) - \rho(z) \geq 2^{-\ell} \rho(x) \) whenever \( \rho(z) \leq 2^{k+2} \kappa_{[\alpha]} \). By the \([\alpha]\)-order cancellation condition of \( \Omega \) and Lemma 2.2, for any sufficient large constant \( N \),
\[
|I_{\alpha,k,s}^1 (x)| \\
\lesssim \frac{2^{-\alpha k}}{s^{\alpha+1}} \sum_{\ell \in \mathbb{Z}} 2^{-(Q-\alpha) \ell} \\
\times \int_{\Sigma} \rho\left((2^{(k-\ell)} \circ \theta) (\alpha) \right) \sup_{\rho(z) \leq \kappa_{[\alpha]} \rho((2^{(k-\ell)} \circ \theta))} |(X^I \xi)((2^{-\ell} \circ x) z)||\Omega(\theta)||d\sigma(\theta) \\
\lesssim \frac{2^{-\alpha k}}{s^{\alpha+1}} \sum_{\ell \in \mathbb{Z}} 2^{-(Q-\alpha) \ell} 2^{(\alpha+1)(k-\ell)} (1 + 2^{-\ell} \rho(x))^{-N} \| \Omega \|_{L^1(\Sigma)} \\
\lesssim \frac{2^{(\alpha+1)-\alpha} k}{\rho(x)^{(Q+1)[\alpha]-\alpha}} \| \Omega \|_{L^1(\Sigma)},
\] (3.19)

where \( \xi \) is the smooth function defined in (2.17). Since \( \text{supp} \phi \subset \{ x \in \mathbb{H} : \rho(x) \leq \frac{1}{100} \} \), we have
\[
|\Delta [2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (x)| \\
\lesssim \| \Omega \|_{L^1(\Sigma)} \int_{\rho(y) \leq 2^k} \frac{2^{(\alpha+1)+1-\alpha} k}{d(x,y)^{(\alpha+1)+1-\alpha}} \chi_{d(x,y) > A_0 \kappa_{[\alpha]} 2^{k+4}} |\Delta [2^{k-j}] \phi(y)| \, dy \\
\lesssim \| \Omega \|_{L^1(\Sigma)} \frac{2^{(\alpha+1)+1-\alpha} k}{\rho(x)^{(Q+1)[\alpha]-\alpha}} \chi_{\rho(x) \geq A_0 \kappa_{[\alpha]} 2^{k+3}} (x).
\] (3.20)

This also implies that whenever \( \rho(x) \geq 2A_0 \rho(y) \),
\[
|\Delta [2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (y^{-1} x)| \lesssim \| \Omega \|_{L^1(\Sigma)} \frac{2^{(\alpha+1)+1-\alpha} k}{\rho(y^{-1} x)^{(Q+1)[\alpha]-\alpha}} \chi_{\rho(y^{-1} x) \geq A_0 \kappa_{[\alpha]} 2^{k+3}} (y^{-1} x) \\
\lesssim \| \Omega \|_{L^1(\Sigma)} \frac{2^{(\alpha+1)+1-\alpha} k}{\rho(x)^{(Q+1)[\alpha]-\alpha}} \chi_{\rho(x) \geq \kappa_{[\alpha]} 2^{k+2}} (x).
\] (3.21)

Combining the estimates (3.20) and (3.21), we obtain that
\[
\sum_{k \in \mathbb{Z}} \left| \Delta [2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (y^{-1} x) - \Delta [2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (x) \right|
\]
\[ \|\Omega\|_{L^1(\Sigma)} \sum_{2^{k+2}k_{[\alpha]} \leq \rho(x)} 2([\alpha]+1-\alpha)k \frac{\rho(x)^Q}{\rho(x)^{\alpha+1}+\alpha} \leq \|\Omega\|_{L^1(\Sigma)} \rho(x)^Q. \] (3.22)

On the other hand, similar to the proof of (3.19), for any \( i = 1, 2, \ldots, n \),

\[ |X_i I^1_{\alpha,k,s}(x)| \lesssim \|\Omega\|_{L^1(\Sigma)} \frac{2([\alpha]+1-\alpha)k}{\rho(x)^{Q+2+\alpha-\alpha}}. \] (3.23)

Hence, for any \( i = 1, 2, \ldots, n \),

\[ |X_i \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(x)| \lesssim \|\Omega\|_{L^1(\Sigma)} \frac{2([\alpha]+1-\alpha)k}{\rho(x)^{Q+2+\alpha-\alpha}} X_{\rho(x) \geq A_0 \kappa_{[\alpha]} 2^{k+3}}(x). \] (3.24)

Therefore,

\[ \sum_{k \in \mathbb{Z}} \left| X_i \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(x) \right| \lesssim \|\Omega\|_{L^1(\Sigma)} \sum_{k \in \mathbb{Z}} \int_{\rho(y) \leq 2^k} \frac{2([\alpha]+1-\alpha)k}{d(x,y)^{Q+2+\alpha-\alpha}} X_{d(x,y) > A_0 \kappa_{[\alpha]} 2^{k+3}} \Delta[2^{k-j}] \phi(y) dy \]

\[ \lesssim \|\Omega\|_{L^1(\Sigma)} \sum_{k \in \mathbb{Z}} \frac{2([\alpha]+1-\alpha)k}{\rho(x)^{Q+2+\alpha-\alpha}} X_{\rho(x) \geq \kappa_{[\alpha]} 2^{k+2}}(x) \lesssim \|\Omega\|_{L^1(\Sigma)} \rho(x)^{-Q-1}. \] (3.25)

This, together with the mean value theorem on homogeneous groups (see [18]), shows

\[ \sum_{k \in \mathbb{Z}} \left| \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(y^{-1}x) - \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(x) \right| \lesssim \|\Omega\|_{L^1(\Sigma)} \frac{\rho(y)}{\rho(x)^{Q+1}}. \] (3.26)

Combining the estimates (3.22) and (3.26), we conclude that

\[ \sum_{k \in \mathbb{Z}} \left| \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(y^{-1}x) - \Delta[2^{k-j}] \phi * I^1_{\alpha,k,s}(x) \right| \lesssim \|\Omega\|_{L^1(\Sigma)} \omega\left(\frac{\rho(y)}{\rho(x)}\right). \] (3.27)
where $\omega(t) \leq \min\{1, t\}$ uniformly in $j \geq 0$. Thus,

\[
\int_{\rho(x) \geq 2A_0 \rho(y)} \sum_{k \in \mathbb{Z}} \left| \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (y^{-1}x) - \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^1 (x) \right| \, dx \\
\lesssim \|\Omega\|_{L^1(\Sigma)} \int_{\rho(x) \geq 2A_0 \rho(y)} \frac{1}{\rho(x)} \omega\left( \frac{\rho(y)}{\rho(x)} \right) \, dx \\
\lesssim \|\Omega\|_{L^1(\Sigma)} \sum_{k=0}^{\infty} \omega(2^{-k}) \lesssim \|\Omega\|_{L^1(\Sigma)}.
\]

(3.28)

**Estimate of (3.18):**
To begin with,

\[
\sum_{k \in \mathbb{Z}} \int_{\rho(x) \geq 2A_0 \rho(y)} \left| \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^2 (y^{-1}x) - \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^2 (x) \right| \, dx \\
\leq \sum_{k \in \mathbb{Z}} \int_{\rho(x) \geq 2A_0 \rho(y)} \int_{\mathbb{H}} \left| \Delta[2^{k-j}] \phi(y^{-1}xz^{-1}) - \Delta[2^{k-j}] \phi(xz^{-1}) \right| \, dx \, |I_{\alpha,k,s}^2(z)| \, dz.
\]

(3.29)

It can be verified directly that if $\rho(y) \geq 2^{k+7} A_0^{3/\alpha}$ and $\rho(x) \geq 2A_0 \rho(y)$, then

$$\rho(2^{-(k-j)} \circ y^{-1}xz^{-1}) \geq 1 \text{ and } \rho(2^{-(k-j)} \circ xz^{-1}) \geq 1.$$ 

This, in combination with a simple change of variable and Lemma 2.3, yields

\[
\sum_{k \in \mathbb{Z}} \int_{\rho(x) \geq 2A_0 \rho(y)} \left| \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^2 (y^{-1}x) - \Delta[2^{k-j}] \phi \ast I_{\alpha,k,s}^2 (x) \right| \, dx \\
\leq \sum_{2^{k+7} A_0^{3/\alpha} \geq \rho(y)} \int_{\mathbb{H}} \left| \Delta[2^{k-j}] \phi(y^{-1}xz^{-1}) - \Delta[2^{k-j}] \phi(xz^{-1}) \right| \, dx \, |I_{\alpha,k,s}^2(z)| \, dz \\
\times \int_{\rho(x) \geq 2A_0 \rho(y)} \left| \Delta[2^{k-j}] \phi(y^{-1}xz^{-1}) - \Delta[2^{k-j}] \phi(xz^{-1}) \right| \, dx \, |I_{\alpha,k,s}^2(z)| \, dz \\
\leq \sum_{2^{k+7} A_0^{3/\alpha} \geq \rho(y)} \int_{\mathbb{H}} \left| \phi \left( 2^{-(k-j)} \circ y^{-1}x (2^{-(k-j)} \circ xz^{-1}) \right) - \phi \left( x (2^{-(k-j)} \circ xz^{-1}) \right) \right| \, dx \, |I_{\alpha,k,s}^2(z)| \, dz \\
\times \int_{\mathbb{H}} \left| \phi \left( 2^{-(k-j)} \circ y^{-1}x (2^{-(k-j)} \circ xz^{-1}) \right) - \phi \left( x (2^{-(k-j)} \circ xz^{-1}) \right) \right| \, dx \, |I_{\alpha,k,s}^2(z)| \, dz \\
\lesssim \sum_{2^{k+7} A_0^{3/\alpha} \geq \rho(y)} \min \left\{ 1, \frac{\rho(y)}{2^{-j}} \right\} \|I_{\alpha,k,s}^2\|_1.
\]

(3.30)
Lemma 3.9

There exist constants $C_{Q, \alpha} > 0$ and $\tau > 0$ such that for any $j \geq 0$,

$$\left\| \sup_{t \in [1, 2]} \sup_{k \in \mathbb{Z}} |V_{k, j, t}^\alpha f| \right\|_2 \leq C_{Q, \alpha} 2^{-\tau j} \|\Omega\|_{L^1(\Sigma)} \|f\|_2. \quad (3.32)$$

**Proof** Using fundamental theorem of calculus to the function $|V_{k, j, t}^\alpha f(x)|^2$, we get the following type of Gagliardo–Nirenberg inequality:

$$\sup_{t \in [1, 2]} \sup_{k \in \mathbb{Z}} |V_{k, j, t}^\alpha f(x)| \lesssim \sum_{k \in \mathbb{Z}} \left( \int_1^2 |V_{k, j, s}^\alpha f(x)|^2 ds \right)^{1/4} \left( \int_1^2 \left| \frac{d}{ds} V_{k, j, s}^\alpha f(x) \right|^2 ds \right)^{1/4}. \quad (3.33)$$

By Cauchy–Schwartz’s inequality,

$$\left\| \sup_{t \in [1, 2]} \sup_{k \in \mathbb{Z}} |V_{k, j, t}^\alpha f| \right\|_2 \leq \left\| \sum_{k \in \mathbb{Z}} \left( \int_1^2 |V_{k, j, s}^\alpha f|^2 ds \right)^{1/2} \right\|_2^{1/2} \times \left\| \left( \sum_{k \in \mathbb{Z}} \int_1^2 \left| \frac{d}{ds} V_{k, j, s}^\alpha f \right|^2 ds \right)^{1/2} \right\|_2^{1/2}.$$ 

Thus, Lemma 3.9 can be reduced to showing that

$$\left\| \sum_{k \in \mathbb{Z}} \int_1^2 |H_{k, j, s}^\alpha f|^2 ds \right\|_2^{1/2} \lesssim 2^{-\tau j} \|\Omega\|_{L^1(\Sigma)} \|f\|_2 \quad (3.33)$$

for some $\tau > 0$, where $H_{k, j, s}^\alpha$ stands for the operator $V_{k, j, s}^\alpha$ or $\frac{d}{ds} V_{k, j, s}^\alpha$. 

□
Estimate of (3.33): To this end, let $r_k : [0, 1] \to \mathbb{R}$ be a collection of independent Rademacher random variables. From the calculation of (3.16) we see that

$$\frac{d}{ds} V_{k,j,s}^\alpha f = f * \Psi_{k-j} * R^\alpha * (-2^{-(Q-1+\alpha)k} - 2^{-\alpha} \Omega \delta_{s,2^k}).$$

Then by Khinchin’s inequality and Lemma 2.7,

$$\left\| \left( \sum_{k \in \mathbb{Z}} \int_1^2 |H_{k,j,s}^\alpha f|^2 ds \right)^{1/2} \right\|_2 \lesssim \sup_{s \in [1,2]} \left( \int_{\mathbb{R}} \left\| \sum_{k \in \mathbb{Z}} r_k(t) H_{k,j,s}^\alpha f \right\|_{L^2([0,1])}^2 dx \right)^{1/2} \lesssim \sup_{s \in [1,2]} \sup_{t \in [0,1]} \left\| \sum_{k \in \mathbb{Z}} r_k(t) H_{k,j,s}^\alpha f \right\|_2 \lesssim 2^{-\tau_j} \| \Omega \|_{L^1(\Sigma)} \| f \|_2.$$

This ends the proof of (3.33) and then Lemma 3.9.

\[\Box\]

Lemma 3.10 For any $1 < p < \infty$, there exist constants $C_{Q,\alpha}, C_{Q,\alpha,p} > 0$ and $\tau > 0$ such that for any $j \geq 0$,

$$\left\| \sup_{t \in [1,2]} \sup_{k \in \mathbb{Z}} \left| V_{k,j,t}^\alpha f \right| \right\|_{L^{1,\infty}} \leq C_{Q,\alpha} (1 + j) \| \Omega \|_{L^1(\Sigma)} \| f \|_1, \quad (3.34)$$

and

$$\left\| \sup_{t \in [1,2]} \sup_{k \in \mathbb{Z}} \left| V_{k,j,t}^\alpha f \right| \right\|_{L^p} \leq C_{Q,\alpha,p} 2^{-\tau_j} \| \Omega \|_{L^1(\Sigma)} \| f \|_p. \quad (3.35)$$

Proof By Lemmas 3.8 and 3.9, these inequalities can be obtained by a standard argument of Calderón–Zygmund decomposition (see for example [14, 38]), together with the interpolation theorem and a standard vector-valued duality argument (see for example [14, Theorem 5.17]).

\[\Box\]

Proof of Proposition 3.6. Note that $M_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2} f = \sup_{t \in [1,2]} \sup_{k \in \mathbb{Z}} \left| f * R^\alpha * B_{k,t}^\alpha \Omega \right|$. This, in combination with the equality (3.14) and inequality (3.35), yields

$$\| M_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2} f \|_p \leq \sum_{j=0}^{\infty} \left\| \sup_{t \in [1,2]} \sup_{k \in \mathbb{Z}} \left| V_{k,j,t}^\alpha f \right| \right\|_p \lesssim \| \Omega \|_{L^1(\Sigma)} \| f \|_p.$$

This ends the proof of Proposition 3.6.
3.4 $L^p$ Estimate of Dyadic Maximal Function

In this subsection, we give the proof of the Proposition 3.7. To this end, let $\phi$ be a smooth cut-off function defined in Sect. 2. Then

$$T_{\Omega, \alpha}^k f = T_{\Omega, \alpha} f \ast \Delta [2^k] \phi - \sum_{s = -\infty}^{0} f \ast B_{k+s}^{\alpha} \Omega \ast \Delta [2^k] \phi$$

$$+ \sum_{s = 1}^{\infty} f \ast B_{k+s}^{\alpha} \Omega \ast (\delta_0 - \Delta [2^k] \phi).$$

From this equality we see that

$$\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha}^k (\cdot - \Delta \mathbb{H})^{-\alpha/2} f| \right\|_p \leq \left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} (\cdot - \Delta \mathbb{H})^{-\alpha/2} f \ast \Delta [2^k] \phi| \right\|_p$$

$$+ \left\| \sup_{k \in \mathbb{Z}} \sum_{s = -\infty}^{0} f \ast R^{\alpha} \ast B_{k+s}^{\alpha} \Omega \ast \Delta [2^k] \phi \right\|_p$$

$$+ \left\| \sup_{k \in \mathbb{Z}} \sum_{s = 1}^{\infty} f \ast R^{\alpha} \ast B_{k+s}^{\alpha} \Omega \ast (\delta_0 - \Delta [2^k] \phi) \right\|_p \overset{}{=} \overset{}{=}: I_1 + I_2 + I_3. \quad (3.36)$$

**Estimate of $I_1$.** By Proposition 3.1 and the $L^p$ boundedness of the Hardy–Littlewood maximal function, we see that for any $1 < p \leq \infty$,

$$\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} (\cdot - \Delta \mathbb{H})^{-\alpha/2} f \ast \Delta [2^k] \phi| \right\|_p \lesssim \left\| MT_{\Omega, \alpha} (\cdot - \Delta \mathbb{H})^{-\alpha/2} f \right\|_p$$

$$\lesssim \left\| T_{\Omega, \alpha} (\cdot - \Delta \mathbb{H})^{-\alpha/2} f \right\|_p \lesssim \left\| \Omega \right\|_{L^1(\Sigma)} \left\| f \right\|_p.$$

**Estimate of $I_2$.** Observe that

$$\sum_{s = -\infty}^{0} B_{k+s}^{\alpha} \Omega = \frac{\Omega(x)}{\rho(x)^{Q+\alpha}} \chi_{\rho(x) \leq 2^{k+1}}.$$

Then using the cancellation condition of $\Omega$ and Lemma 2.2 to get that

$$\left\| \sum_{s = -\infty}^{0} B_{k+s}^{\alpha} \Omega \ast \Delta [2^k] \phi(x) \right\| \lesssim 2^{-Q+\omega+1} k$$

$$\times \int_{\rho(y) \leq 2^{k+2}} \frac{|\Omega(y)|}{\rho(y)^{Q+\omega+\alpha-1}} \text{d}y \chi_{\rho(x) \leq A_0 \omega} \lesssim \frac{\left\| \Omega \right\|_{L^1(\Sigma)}}{2^{(Q+\alpha)k}} \chi_{\rho(x) \leq A_0 \omega} \leq A_0 \omega \cdot 2^{k+2}, \quad (3.37)$$

$\rho$ Springer
where the last inequality follows by using polar coordinates (1.1).

**Case 1:** If \( \rho(x) \leq A_0^2 \kappa[\alpha] 2^{k+4} \), then

\[
\left| R^{\alpha} \ast \sum_{s=-\infty}^{0} B_{k+s}^{\alpha} \Omega \ast \Delta[2^k \phi](x) \right| = \left| \int_{\mathbb{R}} R^{\alpha}(xy^{-1}) \sum_{s=-\infty}^{0} B_{k+s}^{\alpha} \Omega \ast \Delta[2^k \phi](y) dy \right|
\]

\[
\lesssim \int_{d(x,y) \leq A_0^3 \kappa[\alpha] 2^{k+5}} \frac{\|\Omega\|_{L^1(\Sigma)}}{2^{(Q+\alpha)k}} \frac{1}{d(x,y)^{Q-\alpha}} dy
\]

\[
\lesssim 2^{-Qk} \|\Omega\|_{L^1(\Sigma)}.
\]

**Case 2:** If \( \rho(x) \geq A_0^2 \kappa[\alpha] 2^{k+4} \), then \( \rho((2^{-\ell} o x)z) \geq \frac{1}{A_0} \rho(2^{-\ell} o x) - \rho(z) \gtrsim 2^{-\ell} \rho(x) \) whenever \( \rho(z) \leq 2^{-\ell+2} \kappa[\alpha] \). Similar to the proof of (3.19), we use the [\alpha]-order cancellation condition of \( \Omega \) and Lemma 2.2 to get that for any sufficient large constant \( N \),

\[
\left| R^{\alpha} \ast \sum_{s=-\infty}^{0} B_{k+s}^{\alpha} \Omega \ast \Delta[2^k \phi](x) \right| \lesssim \sum_{\ell \in \mathbb{Z}} 2^{-(Q-\alpha)\ell} 2^{[(\alpha]+1)(k-\ell)} (1 + 2^{-\ell} \rho(x))^{-N} \|
\]

\[
\times \sum_{s=-\infty}^{0} B_{k+s}^{\alpha} \Omega \ast \Delta[2^k \phi] \|_1
\]

\[
\lesssim \|\Omega\|_{L^1(\Sigma)} \frac{2^{[(\alpha]+1-\alpha)k}}{\rho(x)^{Q+1+[\alpha]-\alpha}}.
\]

Combining the estimates (3.38) and (3.39) together, we conclude that

\[
\sup_{k \in \mathbb{Z}} \left| \sum_{s=-\infty}^{0} f \ast R^{\alpha} \ast B_{k+s}^{\alpha} \Omega \ast \Delta[2^k \phi](x) \right| \lesssim \|\Omega\|_{L^1(\Sigma)} Mf(x),
\]

where \( Mf \) is the Hardy–Littlewood maximal function.

This, together with the \( L^p \) boundedness of the Hardy–Littlewood maximal function, yields

\[
I_2 \lesssim \|\Omega\|_{L^1(\Sigma)} \|f\|_p.
\]

**Estimate of \( I_3 \).**

To begin with, we write

\[
\sum_{s=1}^{\infty} f \ast R^{\alpha} \ast B_{k+s}^{\alpha} \Omega \ast (\delta_0 - \Delta[2^k \phi]) =: \sum_{s=1}^{\infty} G_{k,s}^{\alpha} f.
\]
Noting that \( S_j f \to f \) as \( j \to -\infty \), we have the following identity:

\[
G_{k,s}^\alpha = G_{k,s}^\alpha S_{k+s} + \sum_{j=1}^{\infty} G_{k,s}^\alpha (S_{k+s-j} - S_{k+s-(j-1)}) = \sum_{j=0}^{\infty} G_{k,s,j}^\alpha,
\]

(3.41)

where \( G_{k,s,0}^\alpha := G_{k,s}^\alpha S_{k+s} \) and \( G_{k,s,j}^\alpha := G_{k,s}^\alpha (S_{k+s-j} - S_{k+s-(j-1)}) \) for \( j \geq 1 \). Then

\[
I_3 = \left\| \sup_{k \in \mathbb{Z}} \left| \sum_{s=1}^{\infty} f * R_{k+s}^\alpha * B_{k+s}^\alpha \Omega \ast (\delta_0 - \Delta[2^k] \phi) \right|_p \right\|_{p} \leq \sum_{j=0}^{\infty} \sum_{s=1}^{\infty} \left\| \sup_{k \in \mathbb{Z}} |G_{k,s,j}^\alpha f| \right\|_p.
\]

To continue, we establish the following inequalities: for any \( j \geq 0 \),

\[
\left\| \sup_{k \in \mathbb{Z}} |G_{k,s,j}^\alpha f| \right\|_2 \lesssim 2^{-\tau(j+s)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]

(3.42)

for some \( \tau > 0 \), and

\[
\left\| \sup_{k \in \mathbb{Z}} |G_{k,s,j}^\alpha f| \right\|_{L^1,\infty} \lesssim (1 + j + s) \| \Omega \|_{L^1(\Sigma)} \| f \|_1.
\]

(3.43)

Applying the Marcinkiewicz interpolation theorem between (3.42) and (3.43) and a standard vector-valued duality argument (see for example [14, Theorem 5.17]), we see that for any \( 1 < p < \infty \),

\[
\left\| \sup_{k \in \mathbb{Z}} |G_{k,s,j}^\alpha f| \right\|_p \lesssim 2^{-\tau(j+s)} \| \Omega \|_{L^1(\Sigma)} \| f \|_p.
\]

Therefore,

\[
I_3 \leq \sum_{j=0}^{\infty} \sum_{s=1}^{\infty} 2^{-\tau(j+s)} \| \Omega \|_{L^1(\Sigma)} \| f \|_p \lesssim \| \Omega \|_{L^1(\Sigma)} \| f \|_p.
\]

We now return to verify the inequalities (3.42) and (3.43).

For the estimate (3.42), it follows from Khinchin’s inequality that for any \( s \geq 1 \) and \( i \in \mathbb{Z} \),

\[
\left\| \sup_{k \in \mathbb{Z}} |f * \Psi_{k+s-i} * R_{k+s}^\alpha * B_{k+s}^\alpha \Omega \ast (\delta_0 - \Delta[2^k] \phi)|_2 \right\| \leq \left( \left\| \sum_{k \in \mathbb{Z}} |f * \Psi_{k+s-i} * R_{k+s}^\alpha * B_{k+s}^\alpha \Omega \ast (\delta_0 - \Delta[2^k] \phi)|^2 \right\|_2 \right)^{1/2}
\]

\[
\lesssim \left\| \sum_{k \in \mathbb{Z}} r_k(t) f * \Psi_{k+s-i} * R_{k+s}^\alpha * B_{k+s}^\alpha \Omega \ast (\delta_0 - \Delta[2^k] \phi) \right\|_{L^2([0,1])}
\]
By Cotlar–Knapp–Stein Lemma (see [38]), it suffices to show that:

\[
\| (G_{k,s,i,\ell}^{\alpha}(t))^* G_{k',s,i,\ell}^{\alpha}(t) \|_2 \rightarrow 2 + \| G_{k',s,i,\ell}^{\alpha}(t)(G_{k,s,i,\ell}^{\alpha}(t))^* \|_2 \rightarrow 2
\]

\[
\lesssim 2^{-2\tau(\ell+s+|j|)} 2^{-|k-k'|} \| \Omega \|_{L^1(\Sigma)}^2. \tag{3.44}
\]

We only estimate the first term, since the second one can be estimated similarly.

On the one hand, by Proposition 2.6 with \(\mu_j^{\alpha} \) chosen to be \(B_j^\alpha \Omega\), Young’s inequality and the estimate (2.43),

\[
\| (G_{k,s,i,\ell}^{\alpha}(t))^* G_{k',s,i,\ell}^{\alpha}(t) f \|_2 = \| (f * \Psi_{k+s-i}^\alpha * R^\alpha * B_{k+s}^\alpha) \times (\Psi_{k-\ell}^\alpha * (B_{k+s}^\alpha \tilde{\Omega} * R^\alpha * \Psi_{k-s-i}^\alpha)) \|_2
\]

\[
\lesssim 2^{-2\tau(\ell+\ell)} 2^{-|k-k'|} \| \Omega \|_{L^1(\Sigma)}^2 \| f \|_2. \tag{3.45}
\]

On the other hand, we also have

\[
\| (G_{k,s,i,\ell}^{\alpha}(t))^* G_{k',s,i,\ell}^{\alpha}(t) f \|_2 = \| (f * \Psi_{k+s-i}^\alpha) * (R^\alpha * B_{k+s}^\alpha \Omega * \Psi_{k-\ell}) \times (\Psi_{k-s-i}^\alpha * B_{k+s}^\alpha \tilde{\Omega} * R^\alpha * \Psi_{k-s-i}^\alpha) \|_2
\]

\[
\lesssim 2^{-2\tau(s+\ell)} \| \Omega \|_{L^1(\Sigma)}^2 \| f \|_2. \tag{3.46}
\]

Taking geometric means of (3.45) and (3.46), we obtain (3.44) and therefore,

\[
\| \sup_{k \in \mathbb{Z}} | f * \Psi_{k+s-i}^\alpha * R^\alpha * B_{k+s}^\alpha \Omega * (\delta_0 - \Delta [2^k] \phi) | \|_2
\]

\[
\lesssim \sum_{\ell=0}^{\infty} \sup_{t \in [0,1]} \left\| \sum_{k \in \mathbb{Z}} G_{k,s,i,\ell}^{\alpha}(t) f \right\|_2
\]

\[
\lesssim \sum_{\ell=0}^{\infty} 2^{-\tau(\ell+s+i)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2
\]

\[
\lesssim 2^{-\tau(s+i)} \| \Omega \|_{L^1(\Sigma)} \| f \|_2. \tag{3.47}
\]

By replacing \(i\) with \(j\), it is direct that (3.47) implies (3.42) for \(j \geq 1\). While for \(j = 0\), the left-hand side of (3.42) is dominated by the summation of the left-hand side (3.47) over \(i\) from \(-\infty\) to 0, which gives the required estimate.

Next we verify (3.43), which, by a standard argument of Calderón–Zygmund decomposition (see for example [14, 38]), can be reduced to showing the following uniform Hörmander condition: there exists a constant \(C_{\Omega,\alpha} > 0\) such that for any
where \( K_{k+s,j_1}^\alpha \) is defined in (3.15). Note that the estimate (3.48) is a direct consequence of Lemma 3.8. Next, we borrow the argument in the proof of Lemma 3.3 to show (3.49). To this end, we first decompose \( R^\alpha \ast B_{k+s}^\alpha \Omega \ast \Delta [2^k] \phi \) as follows.

\[
R^\alpha \ast B_{k+s}^\alpha \Omega \ast \Delta [2^k] \phi 
= R^\alpha \ast B_{k+s}^\alpha \Omega \ast \Delta [2^k] \phi \left( \frac{x}{A_0^3 \kappa_{[\alpha]} 2^{k+s+5}} \right) + R^\alpha \ast B_{k+s}^\alpha \Omega \ast \Delta [2^k] \phi \eta_0
\times \left( \frac{x}{A_0^3 \kappa_{[\alpha]} 2^{k+s+5}} \right)
=: J_{\alpha,k,s}^1 + J_{\alpha,k,s}^2.
\]

Note that if \( \rho(x) \geq A_0^3 \kappa_{[\alpha]} 2^{k+s+5} \) and \( \rho(y) \leq 2^k \), then \( \rho(xy^{-1}) \geq A_0^2 \kappa_{[\alpha]} 2^{k+s+4} \). Hence, following the proof of inequalities (3.28) and (3.31), we deduce that there exists a constant \( C_{Q,\alpha} > 0 \) such that for any \( j \geq 0 \) and \( s \geq 1 \),

\[
\int_{\rho(x) \geq 2A_0 \rho(y)} \sum_{k \in \mathbb{Z}} \left| \Delta [2^{k+s-j}] \phi \ast J_{\alpha,k,s}^1 (y^{-1} x) - \Delta [2^{k+s-j}] \phi \ast J_{\alpha,k,s}^1 (x) \right| \, dx
\leq C_{Q,\alpha} \| \Omega \|_{L^1(\Sigma)},
\]

(3.50)

and that

\[
\int_{\rho(x) \geq 2A_0 \rho(y)} \sum_{k \in \mathbb{Z}} \left| \Delta [2^{k+s-j}] \phi \ast J_{\alpha,k,s}^2 (y^{-1} x) - \Delta [2^{k+s-j}] \phi \ast J_{\alpha,k,s}^2 (x) \right| \, dx
\leq C_{Q,\alpha} (1 + j) \| \Omega \|_{L^1(\Sigma)}.
\]

(3.51)

Combining the inequalities (3.50) and (3.51), we obtain (3.49).

Finally, by combining the estimates of I_1, I_2 and I_3, the proof of Theorem 1.1 is complete.

\( \square \)
4 weighted $L^p$ Estimate

Throughout this section, unless we mention the contrary, we suppose that $\Omega$ satisfies the following assumption.

**Assumption** Let $0 \leq \alpha < \mathbb{Q}$. Suppose that $\Omega \in L^q(\Sigma)$ for some $q > \mathbb{Q}/\alpha$, and satisfies the cancellation condition of order $[\alpha]$.

### 4.1 Kernel Truncation and Frequency Localization

To begin with, we borrow the ideas in [23] to modify the decomposition in Sect. 3.1. Since $S_j f \to f$ as $j \to -\infty$, for any sequence of integers $\{N(j)\}_{j=0}^{\infty}$ with $0 = N(0) < N(1) < \cdots < N(j) \to +\infty$, we have

$$T_k^\alpha(-\Delta_H)^{-\alpha/2} = T_k^\alpha(-\Delta_H)^{-\alpha/2}S_k + \sum_{j=1}^{\infty} T_k^\alpha(-\Delta_H)^{-\alpha/2}(S_{k-N(j)} - S_{k-N(j-1)}).$$

In this way, $T_{\Omega,\alpha}(-\Delta_H)^{-\alpha/2} = \sum_{j=0}^{\infty} \tilde{T}_j^\alpha N$, where $\tilde{T}_0^\alpha := \sum_{k \in \mathbb{Z}} T_k^\alpha(-\Delta_H)^{-\alpha/2}S_k$, and for $j \geq 1$,

$$\tilde{T}_j^\alpha N := \sum_{k \in \mathbb{Z}} T_k^\alpha(-\Delta_H)^{-\alpha/2}(S_{k-N(j)} - S_{k-N(j-1)}).$$

#### (4.1)

### 4.2 Calderón–Zygmund Theory of $\tilde{T}_j^\alpha N$

Let $K_j^\alpha N$ be the kernel of $\tilde{T}_j^\alpha N$, then we have the following lemma.

**Lemma 4.1** The operator $\tilde{T}_j^\alpha N$ is a Calderón–Zygmund operator satisfying: there exists a constant $C_{Q,\alpha,q} > 0$ such that for any $j \geq 0$,

$$|K_j^\alpha N(x)| \leq C_{Q,\alpha,q} \frac{||\Omega||_{L^q(\Sigma)}}{\rho(x)^{Q}},$$

and if $\rho(x) \geq 2A_0\rho(y)$,

$$|K_j^\alpha N(y^{-1}x) - K_j^\alpha N(x)| \leq C_{Q,\alpha,q} \frac{\omega_j(\rho(y)/\rho(x))}{\rho(x)^{Q}},$$

where $\omega_j(t) \leq ||\Omega||_{L^q(\Sigma)} \min\{1, 2^{N(j)}t\}$ and $\omega_j \in Dini \lesssim (1 + N(j))||\Omega||_{L^q(\Sigma)}$.

**Proof** We only give the proof of the case $0 < \alpha < \mathbb{Q}$ and $j \geq 1$ since the case $\alpha = 0$ and the case $j = 0$ are much more simpler. The $L^2$ boundedness of $\tilde{T}_j^\alpha N$ follows from Lemma 2.7 and the equality (2.36) directly. Next we estimate the expression $R^\alpha \ast A_k^\alpha K^0$. 

\[ \square \] Springer
**Case 1:** If $\rho(x) \leq A_0^3 k_{[\alpha]} 2^{k+5}$, then we first note that

$$|A_{\alpha}^k K_0^0(x)| = 2^{-k} \left| \int_{-\infty}^{+\infty} \phi(2^{-k} t) t^{-Q-\alpha} K_\alpha(t^{-1} \circ x) \chi_{1 \leq \rho(t^{-1} \circ x) \leq 2}(x) dt \right|$$

$$= \left| \int_{-\infty}^{+\infty} t \phi(t) (2^k t)^{-Q-\alpha} K_\alpha((2^k t)^{-1} \circ x) \chi_{2k+1 \leq \rho(x) \leq 2^{k+1}}(x) dt \right|$$

$$\lesssim \rho(x)^{-Q-\alpha} |\Omega(x)| \chi_{2k-1 \leq \rho(x) \leq 2^{k+2}}(x). \quad (4.4)$$

This, in combination with (2.13), yields that for $q > Q/\alpha$,

$$|R^\alpha * A_{\alpha}^k K_0^0(x)| \lesssim \int_{2^{k-1} \leq \rho(z) \leq 2^{k+2}} \frac{|\Omega(z)|}{\rho(z)^{Q+\alpha}} \frac{d(x,z)^{Q-\alpha}}{d(z)} dz$$

$$\lesssim \left( \int_{2^{k-1} \leq \rho(z) \leq 2^{k+2}} \frac{|\Omega(z)|^q}{\rho(z)^{(Q+\alpha)q}} dz \right)^{1/q} \times \left( \int_{d(x,z) \leq 2^k} \frac{1}{d(x,z)(Q-\alpha)^q} dz \right)^{1/q'} \lesssim 2^{-Qk} ||\Omega||_{L^q}(\Sigma). \quad (4.5)$$

**Case 2:** If $\rho(x) \geq A_0^3 k_{[\alpha]} 2^{k+5}$, then using the decomposition (2.16) and Lemma 2.2, we get

$$|R^\alpha * A_{\alpha}^k K_0^0(x)| \lesssim \frac{2^{(|\alpha|+1-\alpha)k}}{\rho(x)^{Q+1+[\alpha]-\alpha}} ||\Omega||_{L^1}(\Sigma). \quad (4.6)$$

Next, we combine the estimates (4.5) and (4.6) to estimate $\Delta[2^{k-Nj}] \phi * R^\alpha * A_{\alpha}^k K_0^0$. Since $\text{supp} \, \phi \subset \{ x \in \mathbb{H} : \rho(x) \leq \frac{1}{100} \}$, we have

$$|\Delta[2^{k-Nj}] \phi * R^\alpha * A_{\alpha}^k K_0^0(x)| \lesssim ||\Omega||_{L^q}(\Sigma)$$

$$\times \int_{\rho(y) \leq 2^k} 2^{-Qk} \chi_{d(x,y) \leq A_0^3 k_{[\alpha]} 2^{k+5}; \Delta[2^{k-Nj}] \phi(y)} dy$$

$$+ ||\Omega||_{L^1}(\Sigma) \int_{\rho(y) \leq 2^k} \frac{2^{(|\alpha|+1-\alpha)k}}{d(x,y)^{Q+1+[\alpha]-\alpha}} \chi_{d(x,y) > A_0^3 k_{[\alpha]} 2^{k+5}; \Delta[2^{k-Nj}] \phi(y)} dy$$

$$\lesssim ||\Omega||_{L^q}(\Sigma) 2^{-Qk} \chi_{\rho(x) \leq A_0^3 k_{[\alpha]} 2^{k+5}}(x) + ||\Omega||_{L^1}(\Sigma) \frac{2^{(|\alpha|+1-\alpha)k}}{\rho(x)^{Q+1+[\alpha]-\alpha}} \chi_{\rho(x) \geq A_0^3 k_{[\alpha]} 2^{k+4}}(x). \quad (4.7)$$

This, together with triangle’s inequality, implies

$$|K_{j, N}^\alpha(x)|$$

$$\lesssim \sum_{k \in \mathbb{Z}} \left( ||\Omega||_{L^q}(\Sigma) 2^{-Qk} \chi_{\rho(x) \leq A_0^3 k_{[\alpha]} 2^{k+5}}(x) + ||\Omega||_{L^1}(\Sigma) \frac{2^{(|\alpha|+1-\alpha)k}}{\rho(x)^{Q+1+[\alpha]-\alpha}} \chi_{\rho(x) \geq A_0^3 k_{[\alpha]} 2^{k+4}}(x) \right)$$

$$\lesssim ||\Omega||_{L^q}(\Sigma) \frac{2^{(|\alpha|+1-\alpha)k}}{\rho(x)^{Q}}. \quad (4.8)$$
To estimate (4.3), we first estimate $X_i \Delta [2^{k-N(j)}] \phi \ast R^\alpha \ast A_k^\alpha K_0^0(x)$ for any $i = 1, 2, \ldots, n$. We also consider it into two cases.

**Case 1:** If $\rho(x) \leq A_0^3 \kappa_{[\alpha]} 2^{k+5}$, by the estimates (4.5) and (4.6) and the fact $\text{supp } X_i \phi \subset \{x \in \mathbb{H} : \rho(x) \leq \frac{1}{100}\}$, we see that

$$
\left| X_i \Delta [2^{k-N(j)}] \phi \ast R^\alpha \ast A_k^\alpha K_0^0(x) \right| \lesssim \|\Omega\|_{L^q(\Sigma)} \int_{d(x, y) < 2^{k-N(j)}} \left( 2^{-\Omega k} \chi_{\rho(y) \leq A_0^3 \kappa_{[\alpha]} 2^{k+5}} + \frac{2^{[\alpha]+1-\alpha} k}{\rho(y)^{Q+1+[\alpha]-\alpha}} \chi_{\rho(y) \geq A_0^3 \kappa_{[\alpha]} 2^{k+5}} \right) 2^{-(Q+1)(k-N(j))} dy
\lesssim \|\Omega\|_{L^q(\Sigma)} 2^{N(j)} 2^{-(Q+1)k}.
$$

(4.9)

**Case 2:** If $\rho(x) \geq A_0^3 \kappa_{[\alpha]} 2^{k+5}$, we claim that

$$
\left| X_i \Delta [2^{k-N(j)}] \phi \ast R^\alpha \ast A_k^\alpha K_0^0(x) \right| \lesssim \frac{2^{[\alpha]+1-\alpha} k}{\rho(x)^{Q+2+[\alpha]-\alpha}} \|\Omega\|_{L^1(\Sigma)}.
$$

To show this inequality, we use the decomposition (2.16) and Lemma 2.2 to see that if $\rho(x) \geq A_0^3 \kappa_{[\alpha]} 2^{k+4}$, then

$$
\left| X_i R^\alpha \ast A_k^\alpha K_0^0(x) \right| \lesssim \frac{2^{[\alpha]+1-\alpha} k}{\rho(x)^{Q+2+[\alpha]-\alpha}} \|\Omega\|_{L^1(\Sigma)}.
$$

(4.10)

Next, we combine the estimates (4.9) and (4.10) to estimate $X_i \Delta [2^{k-N(j)}] \phi \ast R^\alpha \ast A_k^\alpha K_0^0$. Since $\text{supp } \phi \subset \{x \in \mathbb{H} : \rho(x) \leq \frac{1}{100}\}$ and $\rho(x) \geq A_0^3 \kappa_{[\alpha]} 2^{k+5}$, we have

$$
\left| X_i \Delta [2^{k-N(j)}] \phi \ast R^\alpha \ast A_k^\alpha K_0^0(x) \right| = \int_{d(x, z) \geq A_0^3 \kappa_{[\alpha]} 2^{k+4}} |\Delta [2^{k-N(j)}] \phi(z) || X_i R^\alpha \ast A_k^\alpha K_0^0(z^{-1} x)| dz
\lesssim \|\Omega\|_{L^1(\Sigma)} \int_{d(x, z) \geq A_0^3 \kappa_{[\alpha]} 2^{k+4}} |\Delta [2^{k-N(j)}] \phi(z)| \frac{2^{[\alpha]+1-\alpha} k}{d(x, z)^{Q+2+[\alpha]-\alpha}} dz
\lesssim \frac{2^{[\alpha]+1-\alpha} k}{\rho(x)^{Q+2+[\alpha]-\alpha}} \|\Omega\|_{L^1(\Sigma)}.
$$

This, together with triangle’s inequality and the fact that $N(j-1) \leq N(j)$, implies that for any $i = 1, 2, \ldots, n$,

$$
\left| X_i K_j^{\alpha, N}(x) \right| \lesssim \sum_{k \in \mathbb{Z}} \|\Omega\|_{L^q(\Sigma)} \frac{2^{N(j)}}{2^{(Q+1)k}} \chi_{\rho(x) \leq A_0^3 \kappa_{[\alpha]} 2^{k+5}} + \sum_{k \in \mathbb{Z}} \|\Omega\|_{L^q(\Sigma)} \frac{2^{[\alpha]+1-\alpha} k}{\rho(x)^{Q+2+[\alpha]-\alpha}} \chi_{\rho(x) > A_0^3 \kappa_{[\alpha]} 2^{k+5}}
\lesssim 2^{N(j)} \frac{\|\Omega\|_{L^q(\Sigma)}}{\rho(x)^{Q+1}}.
$$

(4.11)
This, in combination with the mean value theorem on homogeneous groups (see for example [18]), implies that if \( \rho(x) \geq 2A_0 \rho(y) \), then
\[
|K_j^{\alpha,N} (y^{-1}x) - K_j^{\alpha,N} (x)| \lesssim 2^{N(j)} \frac{\|\Omega\|_{L^q(\Sigma)} \rho(y)}{\rho(x)^{\Omega+1}} \rho(y).
\]
This, combined with (4.8), yields
\[
|K_j^{\alpha,N} (y^{-1}x) - K_j^{\alpha,N} (x)| \lesssim \frac{\omega_j(\rho(y)/\rho(x))}{\rho(x)^{\Omega}},
\]
where \( \omega_j(t) \leq \|\Omega\|_{L^q(\Sigma)} \min\{1, 2^{N(j)}t\} \). Then a direct calculation shows that
\[
\int_0^1 \omega_j(t) \frac{dt}{t} \lesssim (1 + N(j)) \|\Omega\|_q.
\]
This ends the proof of Lemma 4.1.

\[\square\]

### 4.3 Quantitative Weighted Bounds for \( \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} f * \Delta [2^k] \phi| \)

Let \( \phi \) be a cut-off function defined in Sect. 2. Then the following \((L^p_\alpha(w), L^p(w))\) boundedness of discrete maximal function holds.

**Proposition 4.2** For any \( 1 < p < \infty \) and \( w \in A_p \), there exists a constant \( C_{\Omega, \alpha, p, q} > 0 \) such that
\[
\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} f * \Delta [2^k] \phi| \right\|_{L^p(w)} \leq C_{\Omega, \alpha, p, q} \|\Omega\|_{L^q(\Sigma)} \{w\}_{A_p(w)} \|f\|_{L^p_\alpha(w)}.
\]

**Proof** It suffices to show
\[
\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} (-\Delta^H)^{-\alpha/2} f * \Delta [2^k] \phi| \right\|_{L^p(w)} \leq C_{\Omega, \alpha, p, q} \|\Omega\|_{L^q(\Sigma)} \{w\}_{A_p(w)} \|f\|_{L^p(w)}.
\]
(4.12)

To this end, we apply the decomposition in Sect. 4.1 to see that
\[
T_{\Omega, \alpha} (-\Delta^H)^{-\alpha/2} f * \Delta [2^k] \phi = \sum_{j=0}^{\infty} \tilde{T}^{\alpha,N}_j f * \Delta [2^k] \phi.
\]

Thus, for \( 1 < p < \infty \) and \( w \in A_p \),
\[
\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha} (-\Delta^H)^{-\alpha/2} f * \Delta [2^k] \phi| \right\|_{L^p(w)} \leq \sum_{j=0}^{\infty} \left\| \sup_{k \in \mathbb{Z}} |\tilde{T}^{\alpha,N}_j f * \Delta [2^k] \phi| \right\|_{L^p(w)}.
\]
It follows easily from Lemma 2.7 and the equality (2.36) that for \( j \geq 1, \)
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{0}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 \left\| \tilde{T}_{0}^{\alpha,N} f \right\|_{L^2} \lesssim_2 \| \Omega \|_{L^q(\Sigma)} \| f \|_{L^2},
\]
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 \left\| \tilde{T}_{j}^{\alpha,N} f \right\|_{L^2} \lesssim_2 2^{-\tau N(j-1)} \| \Omega \|_{L^q(\Sigma)} \| f \|_{L^2}.
\]
(4.13)

To continue, we claim that for any \( j \geq 0, \)
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 (1 + N(j)) \| \Omega \|_{L^q(\Sigma)} \{ w \} A_p \| f \|_{L^p(w)}. \tag{4.14}
\]

We assume (4.14) for the moment, whose proof will be given later. Taking \( w = 1 \) in (4.14) and then applying interpolation between (4.13) and (4.14), we see that for any \( j \geq 1, \)
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 \left\| \tilde{T}_{j}^{\alpha,N} f \right\|_{L^p} \lesssim_2 2^{-\tau_p N(j-1)} (1 + N(j)) \| \Omega \|_{L^q(\Sigma)} \| f \|_{L^p(w)}
\]
for some constant \( \tau_p > 0. \)

Next by choosing \( \epsilon = \frac{1}{2} c_{\Sigma} \{ w \} A_p, \) we see that the estimate (4.14) gives
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w^{1+\epsilon})} \lesssim (1 + N(j)) \| \Omega \|_{L^q(\Sigma)} \{ w \}^{1+\epsilon} \| f \|_{L^p(w^{1+\epsilon})}.
\]
(4.15)

Then applying interpolation with change of measure, we obtain that
\[
\left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 \| \Omega \|_{L^q(\Sigma)} (1 + N(j)) 2^{-\tau_p Q N(j-1) / (w)} \{ w \} A_p \| f \|_{L^p(w)}
\]
(4.16)
for some constant \( \tau_p, Q > 0. \)

If we choose \( N(j) = 2^j, \) then
\[
\left\| \sup_{k \in \mathbb{Z}} |T_{\Omega, \alpha}(\ast - \Delta)_{\mathbb{H}}^{\alpha/2} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)} \lesssim_2 \sum_{j=0}^{\infty} \left\| \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{\alpha,N} f \ast \Delta[2^k] \phi| \right\|_{L^p(w)}
\]
\[
\lesssim_2 \sum_{j=0}^{\infty} \| \Omega \|_{L^q(\Sigma)} (1 + N(j)) 2^{-\tau_p Q N(j-1) / (w)} \{ w \} A_p \| f \|_{L^p(w)}
\]
\[
\lesssim_2 \| \Omega \|_{L^q(\Sigma)} \{ w \} A_p \| f \|_{L^p(w)}.
\]

Now we return to give the proof of (4.14). To this end, we first give the kernel estimates of \( \{ K_{j}^{\alpha,N} \ast \Delta[2^k] \phi \}. \) We would like to establish the following inequalities. \( \square \)
Lemma 4.3 There exists a constant \( C_{Q,\alpha,q} > 0 \) such that for any \( j \geq 0 \),

\[
\sup_{k \in \mathbb{Z}} |K_j^{\alpha,N} \ast \Delta[2^k]\phi(x)| \leq C_{Q,\alpha,q} \frac{\|\Omega\|_{L^q(\Sigma)}}{\rho(x)^Q},
\]

(4.17)

and if \( \rho(x) \geq 2A_0 \rho(y) \),

\[
\sup_{k \in \mathbb{Z}} |K_j^{\alpha,N} \ast \Delta[2^k]\phi(y^{-1}x) - K_j^{\alpha,N} \ast \Delta[2^k]\phi(x)| \leq C_{Q,\alpha,q} \frac{\omega_j(\rho(y)/\rho(x))}{\rho(x)^Q},
\]

(4.18)

and

\[
\sup_{k \in \mathbb{Z}} |K_j^{\alpha,N} \ast \Delta[2^k]\phi(xy^{-1}) - K_j^{\alpha,N} \ast \Delta[2^k]\phi(x)| \leq C_{Q,\alpha,q} \frac{\omega_j(\rho(y)/\rho(x))}{\rho(x)^Q},
\]

(4.19)

where \( \omega_j(t) \leq \|\Omega\|_{L^q(\Sigma)} \min\{1, 2^{N(j)} t\} \).

**Proof** We first verify (4.17).

**Case 1:** If \( \rho(x) \leq A_0 \kappa_0 2^{k+4} \), then since \( \int_{\mathbb{H}} K_j^{\alpha,N}(z)dz = 0 \),

\[
K_j^{\alpha,N} \ast \Delta[2^k]\phi(x) = \int_{\mathbb{H}} \left( \Delta[2^k]\phi(z^{-1}x) - \Delta[2^k]\phi(x) \right) K_j^{\alpha,N}(z)dz.
\]

By the support of \( \Delta[2^k]\phi \), we see that \( \rho(z) \leq A_0^2 \kappa_0 2^{k+5} \). This, combined with (4.2) and the mean value theorem on homogeneous groups (see [18]) yields

\[
|K_j^{\alpha,N} \ast \Delta[2^k]\phi(x)| \leq \int_{\rho(z) \leq A_0^2 \kappa_0 2^{k+5}} |\Delta[2^k]\phi(z^{-1}x) - \Delta[2^k]\phi(x)| |K_j^{\alpha,N}(z)|dz
\]

\[
\lesssim \|\Omega\|_{L^q(\Sigma)} \int_{\rho(z) \leq A_0^2 \kappa_0 2^{k+5}} |\Delta[2^k]\phi(z^{-1}x) - \Delta[2^k]\phi(x)| \frac{1}{\rho(z)^Q}dz
\]

\[
\lesssim \|\Omega\|_{L^q(\Sigma)} \frac{2(Q+1)^k}{2^{Qk}} \int_{\rho(z) \leq A_0^2 \kappa_0 2^{k+5}} \frac{1}{\rho(z)^{Q-1}}dz \lesssim \|\Omega\|_{L^q(\Sigma)} \frac{\rho(x)^Q}{\rho(x)^Q},
\]

(4.20)

**Case 2:** If \( \rho(x) \geq A_0 \kappa_0 2^{k+4} \), then by the support of \( \Delta[2^k]\phi \), we see that \( \rho(z) \geq \frac{\rho(x)}{2} \). This, in combination with (4.2), implies

\[
|K_j^{\alpha,N} \ast \Delta[2^k]\phi(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} \int_{\rho(z) \geq \frac{\rho(x)}{2}} |\Delta[2^k]\phi(z^{-1}x)| \frac{1}{\rho(z)^Q}dz
\]

\[
\lesssim \|\Omega\|_{L^q(\Sigma)} \frac{\rho(x)^Q}{\rho(x)^Q}.
\]

Combining the estimates of two cases, we get (4.17).
Next, we verify (4.18). By the mean value theorem on homogeneous groups, it suffices to estimate $|X_i K_j^\alpha N \ast \Delta [2^k] \phi(x)|$ for any $i = 1, 2, \ldots, n$.

**Case 1:** If $\rho(x) \leq A_0 \kappa_0 2^{k+4}$, then (4.20) holds with $\phi$ replaced by $X_i \phi$. Thus,

$$
|X_i K_j^\alpha N \ast \Delta [2^k] \phi(x)| = \frac{1}{2^k} |K_j^\alpha N \ast \Delta [2^k](X_i \phi)(x)| \lesssim \frac{\|\Omega\|_{L^q(\Sigma)}}{2^{(Q+1)k}} \lesssim \frac{\|\Omega\|_{L^q(\Sigma)}}{\rho(x)^{Q+1}}.
$$

**Case 2:** If $\rho(x) \geq A_0 \kappa_0 2^{k+4}$, then it follows from the support property of $\phi$ and (4.11) that

$$
|X_i K_j^\alpha N \ast \Delta [2^k] \phi(x)| \lesssim \frac{2^{N(j)} \|\Omega\|_{L^q(\Sigma)}}{\rho(x)^{Q+1}} \int_{\rho(x) \leq \rho(z)} \left| \Delta [2^k] \phi(z^{-1} x) \right| \frac{2^N j \rho(y)}{\rho(x)^{Q+1}} dz.
$$

Combining the estimates of two cases, we see that for if $\rho(x) \geq 2A_0 \rho(y)$,

$$
\sup_{k \in \mathbb{Z}} |K_j^\alpha N \ast \Delta [2^k] \phi(y^{-1} x) - K_j^\alpha N \ast \Delta [2^k] \phi(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} 2^{N(j)} \frac{\rho(y)}{\rho(x)^{Q+1}}.
$$

(4.21)

This, in junction with (4.17), yields (4.18). Following a similar calculation, we obtain (4.19) as well. This ends the proof of Lemma 4.3.

In the following, we recall the grand maximal truncated operator $\mathcal{M}_T$ on homogeneous groups defined as follows.

$$
\mathcal{M}_T f(x) := \sup_{B \ni x} \sup_{\xi \in B} |T(f \chi_{\mathbb{H}\setminus C_{A_0} B})(\xi)|,
$$

where $C_{A_0}$ is a fixed constant depending only on $A_0$ (for the precise definition of $C_{A_0}$, we refer the readers to the notation $C \tilde{\zeta}$ in [16]), and the first supremum is taken over all balls $B \subset \mathbb{H}$ containing $x$.

**Lemma 4.4** Let $T$ be a sublinear operator. Assume that the operators $T$ and $\mathcal{M}_T$ are of weak type $(1, 1)$, then for every compactly supported $f \in L^1(\mathbb{H})$, we have

$$
|Tf(x)| \leq C_Q \|T\|_{L^1 \rightarrow L^{1,\infty}} + \|\mathcal{M}_T\|_{L^1 \rightarrow L^{1,\infty}} A_S(|f|)(x),
$$

where $A_S$ is the sparse operator (see for example [16]).

**Proof** The result in the Euclidean setting was shown in [27]. In our setting, by noting that the homogeneous group is a special case of space of homogeneous type in the sense of Coifman–Weiss, we can obtain this result by following the approach in [27] and using the dyadic grid and sparse operator on space of homogeneous type as developed in [16]. We leave the details to readers. □
For simplicity, we denote $T_{j}^{a,N} f := \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} f \ast \Delta[2^{k}]\phi| $ and $\mathcal{M}_{j}^{a,N} f := \mathcal{M}_{T_{j}^{a,N}} f $. It follows from Lemma 4.4 and the quantitative weighted $L^p$ boundedness of sparse operators that the estimate (4.14) is a direct consequence of the following lemma.

**Lemma 4.5** There exists a constant $C_{Q,a,q} > 0$ such that for any $j \geq 0$,

$$
\|T_{j}^{a,N} f\|_{L^{1,\infty}} \leq C_{Q,a,q} \|\Omega\|_{L^{q}(\Sigma)} (1 + N(j)) \|f\|_{1},
$$

(4.22)

and

$$
\|\mathcal{M}_{j}^{a,N} f \|_{L^{1,\infty}} \leq C_{Q,a,q} \|\Omega\|_{L^{q}(\Sigma)} (1 + N(j)) \|f\|_{1}.
$$

(4.23)

**Proof** To begin with, it follows from (4.18) that the following uniform Hörmander inequality holds:

$$
\int_{\rho(x)>2A_{0}\rho(y)} \sup_{k \in \mathbb{Z}} |K_{j}^{a,N} \ast \Delta[2^{k}]\phi(y^{-1}x) - K_{j}^{a,N} \ast \Delta[2^{k}]\phi(x)| \text{d}x
\lesssim \|\omega_{j}\|_{\text{Dini}} \lesssim (1 + N(j)) \|\omega\|_{L^{q}(\Sigma)}.
$$

Then the estimate (4.22) is a standard argument of Calderón–Zygmund decomposition (see for example [14, 38]).

Next, we verify (4.23). Let $x, \xi \in B := B(x_{0}, r)$. Let $B_{x}$ be the closed ball centered at $x$ with radius $4(A_{0}^{2} + C_{A_{0}})r$. Then $C_{A_{0}} B \subset B_{x}$, and we obtain

$$
\begin{align*}
|\sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} (f \chi_{\mathbb{H} \setminus C_{A_{0}} B}) \ast \Delta[2^{k}]\phi(\xi)| |
\leq \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} (f \chi_{\mathbb{H} \setminus B_{x}}) \ast \Delta[2^{k}]\phi(\xi)| - \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} (f \chi_{\mathbb{H} \setminus B_{x}}) \ast \Delta[2^{k}]\phi(x)| |
+ \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} (f \chi_{B_{x} \setminus C_{A_{0}} B}) \ast \Delta[2^{k}]\phi(\xi)| + \sup_{k \in \mathbb{Z}} |\tilde{T}_{j}^{a,N} (f \chi_{\mathbb{H} \setminus B_{x}}) \ast \Delta[2^{k}]\phi(x)| |
=: I + II + III.
\end{align*}
$$

**Estimate of I.** By (4.19),

$$
I \leq \int_{\mathbb{H} \setminus B_{x}} \sup_{k \in \mathbb{Z}} |K_{j}^{a,N} \ast \Delta[2^{k}]\phi(y^{-1}\xi) - K_{j}^{a,N} \ast \Delta[2^{k}]\phi(y^{-1}x)||f(y)|| \text{d}y
\lesssim \int_{d(x,y) > 4A_{0}r} \frac{\omega_{j}(d(x,\xi)/d(x,y))}{d(x,y)^{Q}} |f(y)|| \text{d}y
\lesssim \sum_{\ell=0}^{\infty} \omega_{j}(2^{-\ell})(2^{\ell}r)^{-Q} \int_{2^{\ell+2}A_{0}r \leq d(x,y) \leq 2^{\ell+3}A_{0}r} |f(y)|| \text{d}y
\lesssim \|\omega_{j}\|_{\text{Dini}}Mf(x).
$$

(4.24)
Estimate of II. It can be verified directly that \( d(x, y) \sim d(\xi, y) \sim r \) whenever \( y \in B_x \setminus C_0 B \). Then it follows from (4.17) that

\[
II \leq \int_{B_x \setminus C_0 B} \sup_{k \in \mathbb{Z}} |K_j^{\alpha,N} \ast \Delta[2^k] \phi(y^{-1} \xi)||f(y)||dy \\
\lesssim \|\Omega\|_{L^q(\Sigma)} \int_{d(x, y) \sim r} \frac{1}{d(x, y)^{\frac{Q}{2}}} |f(y)||dy \lesssim \|\Omega\|_{L^q(\Sigma)} Mf(x). \tag{4.25}
\]

Estimate of III. Note that

\[
III \leq \sup_{k \in \mathbb{Z}} \left| \int_{d(x, y) \leq A_0k0^2k+4} K_j^{\alpha,N} \ast \Delta[2^k] \phi(y^{-1} x)(f \chi_{\text{H} \setminus B_t})(y)dy \right| \\
+ \sup_{k \in \mathbb{Z}} \left| \int_{d(x, y) \geq A_0k0^2k+4} K_j^{\alpha,N} \ast \Delta[2^k] \phi(y^{-1} x)(f \chi_{\text{H} \setminus B_t})(y)dy \right| =: III_1 + III_2.
\]

For the term III_1, it follows from (4.20) that \( |K_j^{\alpha,N} \ast \Delta[2^k] \phi(x)| \lesssim 2^{-Qk}\|\Omega\|_{L^q(\Sigma)} \) whenever \( \rho(x) \leq A_0k0^2k+4 \). Hence,

\[
III_1 \lesssim \|\Omega\|_{L^q(\Sigma)} \sup_{k \in \mathbb{Z}} 2^{-Qk} \int_{d(x, y) \leq A_0k0^2k+4} |f(y)||dy \lesssim \|\Omega\|_{L^q(\Sigma)} Mf(x). \tag{4.26}
\]

For the term III_2,

\[
III_2 \leq \sup_{k \in \mathbb{Z}} \left| \int_{d(x, y) \geq A_0k0^2k+4} K_j^{\alpha,N} \ast (\Delta[2^k] \phi - \delta_0)(y^{-1} x)(f \chi_{\text{H} \setminus B_t})(y)dy \right| \\
+ \sup_{k \in \mathbb{Z}} \left| \int_{d(x, y) \geq A_0k0^2k+4} K_j^{\alpha,N} (y^{-1} x)(f \chi_{\text{H} \setminus B_t})(y)dy \right|,
\]

where \( \delta_0 \) is the Dirac measure at 0. To continue, we first estimate \( K_j^{\alpha,N} \ast (\Delta[2^k] \phi - \delta_0)(x) \) for \( \rho(x) \geq A_0k0^2k+4 \). By the fact that \( \int_{\text{H}} \Delta[2^k] \phi dx = 1 \) and (4.3),

\[
|K_j^{\alpha,N} \ast (\Delta[2^k] \phi - \delta_0)(x)| = \left| \int_{\text{H}} \Delta[2^k] \phi(y^{-1} x)K_j^{\alpha,N}(y)dy - K_j^{\alpha,N}(x) \right| \\
= \left| \int_{\text{H}} \Delta[2^k] \phi(y^{-1} x)(K_j^{\alpha,N} - K_j^{\alpha,N})(y)dy \right| \\
\lesssim \int_{\text{H}} |\Delta[2^k] \phi(y^{-1} x)| \omega_j(2^k/\rho(x)) \frac{\rho(x)^Q}{\rho(x)^Q} dy \lesssim \frac{\omega_j(2^k/\rho(x))}{\rho(x)^Q}.
\]

Therefore,

\[
III_2 \lesssim \sup_{k \in \mathbb{Z}} \int_{d(x, y) \geq A_0k0^2k+4} \frac{\omega_j(2^k/d(x, y))}{d(x, y)^{\frac{Q}{2}}} |(f \chi_{\text{H} \setminus B_t})(y)||dy.
\]
where $\tilde{T}^{\alpha,N,n}_j$ is the truncated maximal operator of $\tilde{T}^{\alpha,N}_j$ defined by

$$\tilde{T}^{\alpha,N,n}_j f(x) = \sup_{\varepsilon > 0} \left| \int_{d(x,y) > \varepsilon} K^{\alpha,N}_j(x,y) f(y) dy \right|.$$  (4.27)

This, along with (4.26), implies

$$\text{III} \lessapprox (\|\tilde{\Omega}\|_{L^q(\Sigma)} + \|\omega_j\|_{\text{Dini}})M f(x) + \tilde{T}^{\alpha,n}_j f(x).$$  (4.28)

Combining the estimates (4.24), (4.25) and (4.28), we see that for any $x, \xi \in B$,

$$\sup_{k \in \mathbb{Z}} |\tilde{T}^{\alpha,N}_j (f \chi_{\mathbb{H}\setminus B_k}) * \Delta [2^k] \phi(\xi)| \lessapprox (\|\tilde{\Omega}\|_{L^q(\Sigma)} + \|\omega_j\|_{\text{Dini}})M f(x) + \tilde{T}^{\alpha,N,n}_j f(x),$$

which implies that

$$M^{\alpha,N}_j f(x) \lessapprox (\|\tilde{\Omega}\|_{L^q(\Sigma)} + \|\omega_j\|_{\text{Dini}})M f(x) + \tilde{T}^{\alpha,N,n}_j f(x).$$

By sparse domination theorem (see the arxiv version for more details),

$$|\tilde{T}^{\alpha,N,n}_j f(x)| \lessapprox (\|\tilde{\Omega}\|_{L^q(\Sigma)} + \|\omega_j\|_{\text{Dini}} + \|\tilde{T}^{\alpha,N}_j\|_{2 \to 2}) A_S(|f|)(x).$$

Moreover, it follows from Lemma 2.7 that $\|\tilde{T}^{\alpha,N}_j\|_{2 \to 2} \lessapprox 2^{-\alpha N(j-1)} \|\tilde{\Omega}\|_{L^q(\Sigma)} \|f\|_2$. Combining the above inequalities, the weak type $(1,1)$ boundedness of the Hardy–Littlewood maximal operators and of sparse operators (see for example [29]) together, we see that

$$\|M^{\alpha,N}_j f\|_{L^{1,\infty}} \lessapprox \|\tilde{\Omega}\|_{L^q(\Sigma)} (1 + N(j)) \|f\|_1,$$

which verifies (4.23). This finishes the proof of Lemma 4.5.  \(\square\)

### 4.4 Proof of Theorem 1.2

In this subsection, we modify the ideas in Sect. 3.2 to give the proof of Theorem 1.2. To begin with, recall from (3.11) that

$$T^{\#}_{\Omega,\alpha} f(x) \leq M_{\Omega,\alpha} f(x) + \sup_{k \in \mathbb{Z}} |T^{k}_{\Omega,\alpha} f(x)|.$$
Denote \( v_{\alpha, \epsilon}(x) := \frac{\Omega(x)}{\rho(x)Q + \alpha} \chi_{\rho(x) \leq 2^{[\log \epsilon] + 1}} \). If \( \alpha = 0 \), then it is direct that
\[
|v_{\alpha, \epsilon}(x)| \lesssim \|\Omega\|_{L^\infty(\Sigma)} e^{-Q} \chi_{\rho(x) \leq 2^{[\log \epsilon] + 1}} \lesssim \|\Omega\|_{L^\infty(\Sigma)} 2^{-Q[\log \epsilon]} \chi_{\rho(x) \leq 2^{[\log \epsilon] + 1}}.
\]
(4.29)

If \( 0 < \alpha < Q \), then similar to the proofs of (4.5) and (4.6), we get that
\[
|R^\alpha * v_{\alpha, \epsilon}(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} 2^{-Q[\log \epsilon]} \chi_{\rho(x) \leq \tfrac{3}{Q} \rho(x)^{1+\alpha} \leq 2^{[\log \epsilon] + 5}} + \frac{2([\alpha]+1-\alpha)[\log \epsilon]}{\rho(x)^Q+1+|\alpha|}\|\Omega\|_{L^1(\Sigma)} \chi_{\rho(x) \geq \tfrac{3}{Q} \rho(x)^{1+\alpha} \leq 2^{[\log \epsilon] + 5}}.
\]
(4.30)

In both cases, the above inequalities imply that for \( 0 \leq \alpha < Q \),
\[
|M_{\Omega, \alpha}(-\Delta_{\Pi})^{-\alpha/2} f(x)| = \sup_{\epsilon > 0} |f * R^\alpha * v_{\alpha, \epsilon}(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} Mf(x).
\]

Then by the sharp weighted boundedness of the Hardy–Littlewood maximal operator \( M \) (see for example [22, Corollary 1.10]), for any \( 1 < p < \infty \),
\[
\|Mf\|_{L^p(w)} \lesssim \|w\|_{A_p} \|f\|_{L^p(w)},
\]
(4.31)
and therefore,
\[
\|M_{\Omega, \alpha} f\|_{L^p(w)} \lesssim \|\Omega\|_{L^q(\Sigma)} \|w\|_{A_p} \|f\|_{L^p(w)},
\]
(4.32)

Hence, to prove Theorem 1.2, it suffices to show the \((L^p_\Omega(w), L^p(w))\) boundedness of \( \sup_{k \in \mathbb{Z}} |T^k_{\Omega, \alpha} f(x)| \). To this end, we define the smooth truncated kernel by
\[
K_{\alpha, k}(x) := K_{\alpha}(x) \int_{-\infty}^{+\infty} t \psi(t) \chi_{\rho(x) \geq 2^{k+1}}(x) dt,
\]
and the corresponding smooth truncated singular integral operator by
\[
\tilde{T}^k_{\Omega, \alpha} f(x) := f * K_{\alpha, k}(x).
\]
(4.33)

We next show that the weighted estimate of \( \sup_{k \in \mathbb{Z}} |T^k_{\Omega, \alpha} f(x)| \) is equivalent to that of \( \sup_{k \in \mathbb{Z}} |\tilde{T}^k_{\Omega, \alpha} f(x)| \) with the same bound. To this end, we note that
\[
K_{\alpha, k}(x) = c K_{\alpha}(x) \chi_{\rho(x) \geq 2^{k+1}}(x)
\]
when \( \rho(x) \geq 2^{k+2} \) or \( \rho(x) \leq 2^k \), where \( c := \int_{-\infty}^{+\infty} t \psi(t) dt \) and that \( K_{\alpha, k}(x) = c K_{\alpha}(x) \chi_{\rho(x) \geq 2^{k+1}}(x) \) satisfies the cancellation condition of order \([\alpha]\). Hence, if \( \alpha = 0 \), then it is direct that
\[
|K_{\alpha, k}(x) - c K_{\alpha}(x) \chi_{\rho(x) \geq 2^{k+1}}(x)| \lesssim \|\Omega\|_{L^\infty(\Sigma)} 2^{-Qk} \chi_{\rho(x) \leq 2^{k+2}}.
\]
(4.34)
If $0 < \alpha < \frac{Q}{2}$, then similar to the proofs of (4.5) and (4.6), we have

$$
|R^\alpha * (K_{\alpha,k}(x) - cK_{\alpha}(x)\chi_{\rho(x) \geq 2^{k+1}}(x))| \\
\lesssim \|\Omega\|_{L^q(\Sigma)} 2^{-Q} \chi_{\rho(x) \leq A_{0}_02^{k+5}} + \|\Omega\|_{L^1(\Sigma)} \frac{2(\alpha+1-\alpha)k}{\rho(x)2^{Q+1+\alpha} - \chi_{\rho(x) \geq 2^{k+5}}.}
$$

In both cases,

$$
|\tilde{T}^k_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2}f(x) - cT^k_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2}f(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} Mf(x).
$$

This, in combination with the inequality (4.31), implies that to prove Theorem 1.2, it suffices to obtain the $(L^p(w), L^p(w))$ boundedness of $\sup_{k \in \mathbb{Z}} |\tilde{T}^k_{\Omega,\alpha}f(x)|$. Observe that

$$
\tilde{T}^k_{\Omega,\alpha}f(x) = T^k_{\Omega,\alpha}f * \Delta[2^k]\phi - \sum_{s=-\infty}^{0} f * A^\alpha_{k+s}K^0_{\alpha} * \Delta[2^k]\phi \\
+ \sum_{s=1}^{\infty} f * A^\alpha_{k+s}K^0_{\alpha} * (\delta_0 - \Delta[2^k]\phi).
$$

From this equality we see that

$$
\left\|\sup_{k \in \mathbb{Z}} |\tilde{T}^k_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2}f(x)|\right\|_{L^p(w)} \\
\leq \left\|\sup_{k \in \mathbb{Z}} |T^k_{\Omega,\alpha}(-\Delta_{\mathbb{H}})^{-\alpha/2}f * \Delta[2^k]\phi|\right\|_{L^p(w)} \\
+ \left\|\sup_{k \in \mathbb{Z}} \sum_{s=-\infty}^{0} f * R^\alpha * A^\alpha_{k+s}K^0_{\alpha} * \Delta[2^k]\phi\right\|_{L^p(w)} \\
+ \left\|\sup_{k \in \mathbb{Z}} \sum_{s=1}^{\infty} f * R^\alpha * A^\alpha_{k+s}K^0_{\alpha} * (\delta_0 - \Delta[2^k]\phi)\right\|_{L^p(w)} =: I + II + III. \ (4.35)
$$

**Estimate of I.** By Proposition 4.2, we see that for $q > \frac{Q}{\alpha}$, $1 < p < \infty$ and $w \in A_p$,

$$
I \lesssim \|\Omega\|_{L^q(\Sigma)} \{w\}_{A_p(w)} \|f\|_{L^p(w)}.
$$

**Estimate of II.** Observe that

$$
\sum_{s=-\infty}^{0} A^\alpha_{k+s}K^0_{\alpha} = K_{\alpha}(x) \int_{-\infty}^{+\infty} \varphi(t) \chi_{\rho(x) \leq 2^{k+1}t}(x) dt.
$$
Note that (3.37)–(3.40) hold with $B_{k+s}^\alpha \Omega$ replaced by $A_{k+s}^\alpha K_\alpha^0$. In particular, if $0 < \alpha < Q$,

$$\sup_{k \in \mathbb{Z}} \left| \sum_{s=-\infty}^{0} f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast \Delta [2^{k}] \phi(x) \right| \lesssim \|\Omega\|_{L^1(\Sigma)} \|Mf(x)\|. \quad (4.36)$$

For the case $\alpha = 0$, by the 0-order cancellation property of $\Omega$ and the mean value theorem on homogeneous groups,

$$\left| \sum_{s=-\infty}^{0} A_{k+s}^0 K_0^0 \ast \Delta [2^{k}] \phi(x) \right| = 2^{-Qk} \left| \int \left[ \sum_{s=-\infty}^{0} A_{k+s}^0 K_0^0(y) \left( \phi(2^{-k} \circ (y^{-1}x)) - \phi(2^{-k} \circ x) \right) \right] dy \right| \lesssim 2^{-(Q+1)k} \int_{\rho(y) \leq 2^{k+2}} \Omega(y) \frac{\Omega(y)}{\rho(y)^Q} dy \chi_{\rho(y) \leq A_0 2^{k+3}} \lesssim 2^{-Qk} \|\Omega\|_{L^1(\Sigma)} \|X_{\rho(x)} \leq A_0 2^{k+3},$$

and therefore, (4.36) also holds in this case. This, together with (4.31), yields

$$\|\Omega\|_{L^1(\Sigma)} \{w\} A_\rho \|f\|_{L^p(w)}.$$

**Estimate of III.** We first claim that there exists a constant $\tau > 0$, such that for any $s \geq 1$,

$$\left\| \sup_{k \in \mathbb{Z}} |f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast (\delta_0 - \Delta [2^k] \phi)| \right\|_2 \lesssim 2^{-\tau s} \|\Omega\|_{L^q(\Sigma)} \|f\|_2. \quad (4.37)$$

To this end, by Khinchin’s inequality and Lemma 2.7,

$$\left\| \sup_{k \in \mathbb{Z}} |f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast (\delta_0 - \Delta [2^k] \phi)| \right\|_2 \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast (\delta_0 - \Delta [2^k] \phi)|^2 \right)^{1/2} \right\|_2 \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} r_k(t) f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast (\delta_0 - \Delta [2^k] \phi) \right) \right\|_{L^2(0,1)} \lesssim \sum_{\ell=0}^{\infty} \sup_{t \in [0,1]} \left\| \sum_{k \in \mathbb{Z}} r_{k-s}(t) f \ast R^\alpha \ast A_{k+s}^\alpha K_\alpha^0 \ast \Psi_{k-s} \right\|_2 \lesssim \sum_{\ell=0}^{\infty} 2^{-\tau(\ell+s)} \|\Omega\|_{L^q(\Sigma)} \|f\|_2 \lesssim 2^{-\tau s} \|\Omega\|_{L^q(\Sigma)} \|f\|_2.$
Next we show that
\[
\left\| \sup_{k \in \mathbb{Z}} |f * R^\alpha * A_{k+s}^0 K_\alpha^0 (\delta_0 - [2^k] \phi)| \right\|_{L^p(w)} \lesssim \|\Omega\|_{L^{q}(\Sigma)} \{w\} A_p \|f\|_{L^p(w)}. 
\]
(4.38)

To begin with, we note that
\[
\sup_{k \in \mathbb{Z}} |f * R^\alpha * A_{k+s}^0 K_\alpha^0 (\delta_0 - [2^k] \phi)(x)| \leq \sup_{k \in \mathbb{Z}} |f * R^\alpha * A_{k+s}^0 K_\alpha^0 (x)| \\
+ \sup_{k \in \mathbb{Z}} |f * R^\alpha * A_{k+s}^0 K_\alpha^0 * \Delta [2^k] \phi (x)|. 
\]
(4.39)

To continue, for the case \(\alpha = 0\), by (4.4) with \(k\) replaced by \(k + s\),
\[
|A_{k+s}^0 K_0^0 (x)| \lesssim 2^{-Q(k+s)} \|\Omega\|_{L^\infty(\Sigma)} \chi_{\rho(x) \leq 2^{k+s+2}}, 
\]
(4.40)

which implies that
\[
|A_{k+s}^0 K_0^0 * \Delta [2^k] \phi (x)| \lesssim \|\Omega\|_{L^\infty(\Sigma)} \\
\times \int_{\rho(y) \leq 2^k} 2^{-Q(k+s)} \chi_{\rho(x,y) \leq 2^{k+s+2}} |\Delta [2^k] \phi (y)| dy. 
\]
(4.41)

Moreover, if \(0 < \alpha < Q\), then by (4.5) and (4.6) with \(k\) replaced by \(k + s\), we have
\[
|R^\alpha * A_{k+s}^0 K_\alpha^0 (x)| \lesssim 2^{-Q(k+s)} \|\Omega\|_{L^q(\Sigma)} \chi_{\rho(x) \leq A_0^{3|\kappa_\alpha|} 2^{k+s+5}} \\
+ \frac{2(|\alpha| + 1 - \alpha)(k+s)}{\rho(x)^{Q+1+|\alpha| - \alpha}} \|\Omega\|_{L^1(\Sigma)} \chi_{\rho(x) \geq A_0^{3|\kappa_\alpha|} 2^{k+s+5}}, 
\]
(4.42)

which implies that
\[
|R^\alpha * A_{k+s}^0 K_\alpha^0 * \Delta [2^k] \phi (x)| \leq \|\Omega\|_{L^q(\Sigma)} \int_{\rho(y) \leq 2^k} 2^{-Q(k+s)} \chi_{\rho(x,y) \leq A_0^{3|\kappa_\alpha|} 2^{k+s+5}} |\Delta [2^k] \phi (y)| dy \\
+ \|\Omega\|_{L^1(\Sigma)} \int_{\rho(y) \leq 2^k} \frac{2(|\alpha| + 1 - \alpha)(k+s)}{d(x,y)^{Q+1+|\alpha| - \alpha}} \chi_{\rho(x,y) > A_0^{3|\kappa_\alpha|} 2^{k+s+5}} |\Delta [2^k] \phi (y)| dy \\
\lesssim \|\Omega\|_{L^q(\Sigma)} 2^{-Q(k+s)} \chi_{\rho(x) \leq A_0^{3|\kappa_\alpha|} 2^{k+s+6}} (x) \\
+ \|\Omega\|_{L^1(\Sigma)} \frac{2(|\alpha| + 1 - \alpha)(k+s)}{\rho(x)^{Q+1+|\alpha| - \alpha}} \chi_{\rho(x) \geq A_0^{2|\kappa_\alpha|} 2^{k+s+4}} (x). 
\]
(4.43)

In both cases, combining (4.40) and (4.42), we conclude that
\[
\sup_{k \in \mathbb{Z}} |f * R^\alpha * A_{k+s}^0 K_\alpha^0 (x)| \lesssim \|\Omega\|_{L^q(\Sigma)} Mf (x). 
\]
(4.44)
Moreover, combining (4.41) and (4.43), we conclude that

\[ \sup_{k \in \mathbb{Z}} |f \ast R^\alpha \ast A_{k+s}^{\alpha} K_0^\alpha \ast \Delta[2^k] \phi(x)| \lesssim \|\Omega\|_{L^q(\Sigma)} Mf(x). \tag{4.45} \]

Hence, it follows from (4.31), (4.39), (4.44) and (4.45) that (4.38) holds.

Now we back to the estimate of $I_3$. It follows from (4.38) with $w$ replaced by $w^{1+\epsilon}$, where $\epsilon = \frac{1}{2} c_Q/(w) A_p$, that

\[ \| \sup_{k \in \mathbb{Z}} \left| f \ast R^\alpha \ast A_{k+s}^{\alpha} K_0^\alpha \ast (\delta_0 - \Delta[2^k] \phi) \right| \|_{L^p(w^{1+\epsilon})} \lesssim \|\Omega\|_{L^q(\Sigma)} \{w\}^{1+\epsilon} A_p \|f\|_{L^p(w^{1+\epsilon})}. \tag{4.46} \]

Now interpolating between (4.37) and (4.46) with change of measures ([37, Theorem 2.11]), we obtain that there exists a constant $\tau > 0$ such that

\[ \| \sup_{k \in \mathbb{Z}} \left| f \ast R^\alpha \ast A_{k+s}^{\alpha} K_0^\alpha \ast (\delta_0 - \Delta[2^k] \phi) \right| \|_{L^p(w)} \lesssim \|\Omega\|_{L^q(\Sigma)} 2^{-\tau s/(w) A_p} \{w\} A_p \|f\|_{L^p(w)}. \]

Therefore,

\[ \begin{align*}
\text{III} \leq \sum_{s=1}^{\infty} & \left\| \sup_{k \in \mathbb{Z}} |f \ast R^\alpha \ast A_{k+s}^{\alpha} K_0^\alpha \ast (\delta_0 - \Delta[2^k] \phi)| \right\|_{L^p(w)} \\
\leq & \|\Omega\|_{L^q(\Sigma)} \sum_{s=1}^{\infty} 2^{-\tau s/(w) A_p} \{w\} A_p \|f\|_{L^p(w)} \leq \|\Omega\|_{L^q(\Sigma)} \{w\} A_p \|f\|_{L^p(w)}.
\end{align*} \]

Finally, by combining the estimates of I, II and III, the proof of Theorem 1.2 is complete. \( \Box \)

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**Declaration**

**Competing interests** We hereby certify that this paper consists of original, unpublished work which is not under consideration for publication elsewhere. There is no conflict of interest.


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