Error correcting code using tree-like multilayer perceptron

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Abstract. An error correcting code using tree-like multilayer perceptron is proposed. An original message $s^0$ is encoded into a codeword $y_0$ using tree-like committee machine (committee tree) or tree-like parity machine (parity tree) whose transfer functions are non-monotonic. The codeword $y_0$ is then transmitted via a Binary Asymmetric Channel (BAC) where it is corrupted by noise. The analytical performance of these schemes is investigated using the replica method of statistical mechanics. Under some specific conditions, all the schemes are shown to saturate the Shannon bound at the infinite codeword length limit.
1. Introduction

Reliability in communication as always been one of the major concerns when dealing with digital data. Especially in today’s information dependent society, it is now vital to design efficient way of preventing data corruption when transmitting information. Error correcting codes have been developed for this purpose since the birth of the information theory field following the work of Shannon [1].

Until 1989, no theoretical nor practical code capable of reaching the Shannon bound were found. In his 1989 paper [2], Sourlas was the first to derive a set of error correcting codes, the so called Sourlas codes, which theoretically saturate the Shannon bound. However these codes are unpractical.

Nonetheless, his paper shows the possibility to use methods from statistical physics to investigate error correcting code schemes. Following this paper, the tools of statistical mechanics have been then successfully applied in a wide range of problems of information theory in recent years. In the field of error correcting codes itself [3, 4, 5], as well as in spreading codes [6, 7], and compression codes [8, 9, 10, 11, 12, 13], statistical mechanical techniques have shown great potential.

The present paper uses similar techniques to investigate an error correcting code where the codeword is encoded using tree-like multilayer perceptron neural networks. For this purpose, the perceptron network makes use of a non-monotonic transfer function. This rather uncommon feature should make such error correcting codes able to deal with asymmetric channels, like the Binary Asymmetric Channel (BAC) we investigate in this paper.

The majority of popular error correcting codes like turbo codes [14] and low density parity check codes [15, 16] which gives near Shannon performance in practical time frames have been widely studied but this was generally restricted to symmetric channels. On the other hand, apart from a few studies [17, 18], little is known when dealing with asymmetric channels.

It has been shown analytically that perceptron-like neural networks can be used to construct a Shannon optimal lossy compression scheme [10, 12, 13]. Since compression codes and error correcting codes are closely linked, we want to investigate perceptron-like neural networks as part of an error correcting code scheme. This was first briefly studied by Shinzato et al. [19] for the simple perceptron. Here we make use of a much more general multilayer network and discuss more deeply the necessary conditions to retrieve optimal performance.

The paper is organized as follows. Section 2 introduces the framework of error correcting codes. Section 3 exposes our model. Section 4 deals with the BAC capacity. Section 5 presents the mathematical tools used to evaluate the performance of the present scheme. Section 6 states the results and elucidates the location of the phase transition which characterizes the best achievable performance of the model. Section 7 is devoted to conclusion and discussion.
2. Error correcting codes

In a general scheme, an original message $s^0$ of size $N$ is encoded into a codeword $y_0$ of size $M$ by some encoding device. The aim of this stage is to add redundancy into the original data. Therefore, we necessarily have $M > N$. Based on this redundancy, a proper decoder device should be able to recover the original data even if it were corrupted by some noise in the transmission channel. The quantity $R = N/M$ is called the code rate and evaluates the trade-off between redundancy and codeword size. The codeword $y_0$ is then fed into a channel where the bits are subject to some noise. The received corrupted message $y$ (which is also $M$ dimensional) is then decoded using its redundancy to infer the original $N$ dimensional message $s^0$. In other words, in a Bayesian framework, one tries to maximize the following posterior probability,

$$P(s|y) \propto P(y|s)P(s). \quad (1)$$

As data transmission is costly, generally one wants to be able to ensure error free transmission while transmitting the less possible bits. In other words, one wants to ensure error free transmission keeping the code rate as large as possible. For this purpose, the well known Shannon bound [1] gives a way to compute the optimal code rate which allows error free recovery. However, while it gives us the value of such an optimal code rate, it does not give any clue on how to construct such an optimal code. Therefore, several codes have been proposed throughout the years in a permanent quest to find a code which can reach this theoretical bound. Sourlas was the first to find a code which saturates the Shannon bound theoretically at the infinite code length limit [2].

3. Error correcting codes using non-monotonic multilayer perceptrons

In this paper, since we make use of techniques derived from statistical mechanics, we will use Ising variables rather than Boolean ones. The Boolean 0 is mapped onto 1 in the Ising framework while the Boolean 1 is mapped to $-1$. This mapping can be used without any loss of generality.

We assume that the original message $s_0$ is generated from the uniform distribution and that all the bits are independently generated so that we have

$$P(s^0) = \frac{1}{2^N}. \quad (2)$$

The channel considered in this study is the Binary Asymmetric Channel (BAC) where each bit is flipped independently of the others with asymmetric probabilities. If the original bit fed into the channel is 1, then it is flipped with probability $r$. Conversely, if the original bit is $-1$, it is flipped with probability $p$. Figure 1 shows the BAC properties in detail. The well known Binary Symmetric Channel (BSC) corresponds to the particular case where $r = p$.

Finally, the corrupted message $y$ is received at the output of the channel. The goal is then to find back $s^0$ using $y$. The state of the estimated message is denoted by the vector $s$. The general outline of the scheme is shown in Figure 2. From Figure 1 we can
easily derive the following conditional probability,
\[
P(y^\mu|y_0^\mu) = \frac{1}{2} + \frac{y^\mu}{2}[(1 - r - p)y_0^\mu + (r - p)],
\]  
where we make use of the notations \(y_0 = (y_0^1, \ldots, y_0^M), y = (y^1, \ldots, y^\mu, \ldots, y^M)\).

Since we assume that the bits are flipped independently, we deduce
\[
P(y|y_0) = \prod_{\mu=1}^{M} \left\{ \frac{1}{2} + \frac{y^\mu}{2}[(1 - r - p)y_0^\mu + (r - p)] \right\}.
\]

To encode the original message \(s^0\) into a codeword \(y_0\), we make use of non-monotonic tree-like parity machine or committee machine neural networks. The original message \(s^0\) is split down into \(N/K\)-dimensional \(K\) disjoint vectors so that \(s^0\) can be written \(s^0 = (s_0^1, \ldots, s_K^0)\). In this paper, we will focus on three different architectures for the neural network. There are the followings:

(I) Multilayer parity tree with non-monotonic hidden units (PTH).
\[
y_0^\mu(s^0) \equiv \prod_{l=1}^{K} f_k \left( \sqrt{\frac{K}{N}} s_l^0 \cdot x_0^\mu \right).
\]

(II) Multilayer committee tree with non-monotonic hidden units (CTH).
\[
y_0^\mu(s^0) \equiv \text{sgn} \left( \sum_{l=1}^{K} f_k \left[ \sqrt{\frac{K}{N}} s_l^0 \cdot x_0^\mu \right] \right).
\]

Note that in this case, if the number of hidden units \(K\) is even, then there is a possibility to get 0 for the argument of the sign function. We avoid this uncertainty by considering only an odd number of hidden units for the committee tree with non-monotonic hidden units in the sequel.
(III) Multilayer committee tree with a non-monotonic output unit (CTO).

\[ y_0^\mu(s^0) \equiv f_k \left( \sqrt{\frac{1}{K}} \sum_{l=1}^{K} \text{sgn} \left[ \sqrt{\frac{K}{N}} s_l^0 \cdot x_l^\mu \right] \right). \]  

In each of these structure, \( f_k \) is a non-monotonic function of a real parameter \( k \) of the form

\[ f_k(x) = \begin{cases} 
1 & \text{if } |x| \leq k \\
-1 & \text{if } |x| > k,
\end{cases} \]

and the vectors \( x_l^\mu \) are fixed \( N/K \)-dimensional independent vectors uniformly distributed on \( \{-1, 1\} \). The sgn function denotes the sign function taking 1 for \( x \geq 0 \) and \(-1\) for \( x < 0 \). Each of this architecture applies a different non-linear transformation to the original data \( s^0 \). The general architecture of these perceptron based encoders is shown in Figure 3. Note that we can also consider an encoder based on a committee-

Figure 3. General architecture of the treelike multilayer perceptrons with \( N \) input units and \( K \) hidden units.

tree where both the hidden-units and the output unit are non-monotonic. However, this introduces an extra-parameter (we will have one threshold parameter for the hidden-units, and one for the output unit) to tune and the performance should not change drastically. For simplicity, we restrict our study to the above three cases only.

The use of random input vectors is known to maximize the storage capacity of perceptron networks and since each \( y_0^\mu \) is computed using the whole set of original bits, redundancy is added into the codeword. This makes such kind of scheme promising for error correcting tasks.

To keep notation as general as possible, as long as explicit use of the encoder is not necessary in computations, we will denote the transformation performed on the vector \( s \) by the respective encoders using the following notation

\[ \mathcal{F}_k \left( \sqrt{\frac{K}{N}} s_l \cdot x_l^\mu \right). \]

\( \mathcal{F}_k \) takes a different expression for the three different types of network and \( k \) denotes the fact that all the encoders depends on a real threshold parameter \( k \). Furthermore,
it should also be noted that $F_k$ contains all the term depending on index $l$ (i.e.: $F_k(u_l)$ contains all the terms $u_1, \ldots, u_l, \ldots, u_K$).

4. Binary Asymmetric Channel (BAC) capacity

In this section, we compute the capacity of the BAC. According to Shannon’s channel coding theorem, the optimal code rate is given by the capacity of the channel. Any code rate bigger than the capacity of the channel will lead to inevitable loss of information. The definition of the capacity $C$ of a channel is

$$C = \max_{\text{input probability}} \{I(X,Y)\},$$

(10)

where $I$ denotes mutual information, $X$ denotes the channel input distribution, and $Y$ denotes the channel output distribution. Computation of the capacity of such binary channel requires only simple algebra and calculations are straightforward, giving finally

$$C_{BAC} = H_2(\gamma_C) - \frac{1 + \Omega_C}{2} H_2(p) - \frac{1 - \Omega_C}{2} H_2(r),$$

(11)

where

$$H_2(x) = -x \log_2(x) - (1-x) \log_2(1-x),$$

(12)

$$\gamma_C = \frac{1}{1 + \Delta_C} = \frac{1}{2} \left[(1-p)(1+\Omega_C) + r(1-\Omega_C)\right],$$

(13)

$$\Delta_C = \left[\frac{r^p(1-r)^{1-r}}{p^p(1-p)^{1-p}}\right]^{1/1-r-p},$$

(14)

$$\Omega_C = \frac{2\gamma_C - 1 - r + p}{1 - r - p}.$$  

(15)

In the special case where $r = p$, the capacity simplifies to

$$C_{BSC} = 1 - H_2(p),$$

(16)

which corresponds to the capacity of the BSC.

5. Analytical Evaluation

As said in section II, our goal is to maximize the posterior $P(s|y)$. To do so, let us define the following Hamiltonian

$$\mathcal{H}(y, s) = -\ln[P(s|y)P(s)] = -\ln P(y, s).$$

(17)

The ground state of the above Hamiltonian trivially corresponds to the maximum a posteriori (MAP) estimator of the posterior $P(s|y)$. Then, let us compute the joint probability of $y$ and $s$. We have

$$P(y, s) = P(y|s)P(s).$$

(18)
Since the relation between an arbitrary message \( s \) and the codeword fed into the channel is deterministic, for any \( s \), we can write

\[
P(y|s) = P\left( y \mid \mathcal{F}_k \left( \frac{\sqrt{K}}{N} s_l \cdot \mathbf{x}_l^\mu \right) \right),
\]

\[
P(y|s) = \prod_{\mu=1}^{M} \left\{ \frac{1}{2} + \frac{y^\mu}{2} \left[ (1 - r - p) \mathcal{F}_k \left( \sqrt{\frac{K}{N}} s_l \cdot \mathbf{x}_l^\mu \right) + (r - p) \right] \right\}.
\]

We finally get the explicit expression of the Hamiltonian,

\[
\mathcal{H}(y, s) = -\ln P(y, s)
\]

\[
\mathcal{H}(y, s) = -\ln \left[ \frac{1}{2N} \prod_{\mu=1}^{M} \left\{ \frac{1}{2} + \frac{y^\mu}{2} \left[ (1 - r - p) \mathcal{F}_k \left( \sqrt{\frac{K}{N}} s_l \cdot \mathbf{x}_l^\mu \right) + (r - p) \right] \right\} \right].
\]

Using this Hamiltonian, we can define the following partition function

\[
Z(\beta, y, x) = \sum_s \exp \left[ -\beta \mathcal{H}(y, s) \right],
\]

where the sum over \( s \) represents the sum over all the possible states for the vector \( s \). \( \beta \) denotes the inverse temperature parameter. Such a partition function can be identified with the partition function of a spin glass system with dynamical variables \( s \) and quenched variables \( x \). The average of this partition function over \( y \) and \( x \) naturally contains all the interesting typical properties of the scheme such as the free energy. However, evaluating this average is hard and we need some technique to investigate it. In this paper we use the so-called \textit{Replica Method} in order to calculate the average of the partition function. Once the free energy is obtained, one can compute the critical code rate at which a phase transition between the ferromagnetic phase (error recovery possible) and the paramagnetic phase (decoding impossible) occurs. This gives us the best code rate the scheme can achieve. A code rate bigger than this critical value makes decoding impossible. The replica method’s calculations to obtain the average of the partition function \( \langle Z(\beta, y, x) \rangle_{y,x} \) are detailed in Appendix A.

After long calculations, the replica symmetric (RS) free energy is obtained,

\[
- f_{RS}(q, \hat{q}, m, \hat{m}) = \text{extr}_{q, \hat{q}, m, \hat{m}} \left\{ \sum_y \int_{-\infty}^{\infty} \left[ \prod_{l=1}^{K} DR_l \right] \int_{-\infty}^{\infty} \left[ \prod_{l=1}^{K}Dt_l \right] \times \ln [I(y, R_l, t_l, m, q)] \times \left( \frac{1}{2} + \frac{y}{2} \left[ (1 - r - p) \mathcal{F}_k (R_l) + (r - p) \right] \right) \right. \\
+ R \int_{-\infty}^{\infty} DU \ln \left( 2 \cosh \left[ \sqrt{\hat{q}U + \hat{m}} \right] \right) - R \ln 2 \\
\left. - Rm\hat{m} - R\hat{q}(1 - q) \right\},
\]

where

\[
I(y, R_l, t_l, m, q) = \int_{-\infty}^{\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times \left[ \frac{1}{2} + \frac{y}{2} (r - p) \right].
\]
\[ + \frac{y}{2} (1 - r - p) k \left( \sqrt{1 - q z_l + \sqrt{q - m^2 t_l + m R_l}} \right) \]

\[ D_x = e^{-\frac{x^2}{2}} \sqrt{2\pi} dx. \]

(26)

and where extr denotes extremization. The sum denotes the sum other all the possible states for the variable \( y \), that is \( \pm 1 \).

Also note that we put \( \beta = 1 \). This choice of finite temperature decoding (in opposition with \( \beta \to \infty \) which corresponds to the zero temperature limit) corresponds to the \textit{maximizer of posterior marginals} (MPM) estimator while the zero temperature decoding corresponds to the MAP estimator \([20, 21]\). It is known that the MPM estimator is optimal for the purpose of decoding \([20, 21]\). On top of that, in this paper, we suppose that all the channel properties (i.e.: the true values of \((p, r)\)) are known to the decoder which implies that the system’s state we consider is located on the Nishimori line \([20]\).

To retrieve the free energy one has to extremize (24) with respect to the order parameters \( q, \hat{q}, m, \hat{m} \). This is done by solving the following saddle point equations

\[ \frac{\partial f_{RS}}{\partial q} = 0 \Leftrightarrow \hat{q} = -2R^{-1} \sum_{y=\pm 1}^{\infty} \left[ \prod_{l=1}^{K} D R_l \right] \left[ \prod_{l=1}^{K} D t_l \right] \times \frac{I'_q(y, R_l, t_l, m, q)}{I(y, R_l, t_l, m, q)} \times \left( \frac{1}{2} + \frac{y}{2} \left( (1 - r - p) k \left( R_l \right) + (r - p) \right) \right), \]

(27)

\[ \frac{\partial f_{RS}}{\partial m} = 0 \Leftrightarrow \hat{m} = R^{-1} \sum_{y=\pm 1}^{\infty} \left[ \prod_{l=1}^{K} D R_l \right] \left[ \prod_{l=1}^{K} D t_l \right] \times \frac{I'_m(y, R_l, t_l, m, q)}{I(y, R_l, t_l, m, q)} \times \left( \frac{1}{2} + \frac{y}{2} \left( (1 - r - p) k \left( R_l \right) + (r - p) \right) \right), \]

(28)

\[ \frac{\partial f_{RS}}{\partial \hat{q}} = 0 \Leftrightarrow q = \int_{-\infty}^{\infty} DU \tanh^2(\sqrt{q U} + \hat{m}), \]

(29)

\[ \frac{\partial f_{RS}}{\partial \hat{m}} = 0 \Leftrightarrow m = \int_{-\infty}^{\infty} DU \tanh(\sqrt{q U} + \hat{m}), \]

(30)

where

\[ I'_q(y, R_l, t_l, m, q) = \frac{\partial I(y, R_l, t_l, m, q)}{\partial q}, \]

(31)

\[ I'_m(y, R_l, t_l, m, q) = \frac{\partial I(y, R_l, t_l, m, q)}{\partial m}. \]

(32)

An error correcting code scheme typically admits two solutions, one where \( m = q = 1 \) called the ferromagnetic solution, and one where \( m = q = 0 \) called the paramagnetic solution. As the names indicate, these solutions come from the physical ferromagnet and correspond to the case where the spins are all ordered (\( m = q = 1 \)) and to the case where the spins take completely random states (\( m = q = 0 \)). As we can deduced from equation (A.3) and (A.6), the ferromagnetic solution corresponds to decoding success.
since \( m = 1 \) implies perfect overlap. On the opposite, the paramagnetic phase implies failure in the decoding process (overlap \( m \) is 0).

5.1. Replica symmetric solution using a parity tree with non-monotonic hidden units

Using a parity tree with non-monotonic hidden units (5), the encoder function becomes

\[
\mathcal{F}_k(u_l) = \prod_{l=1}^{K} f_k(u_l). \tag{33}
\]

Using this encoder function and substituting \( m = q = 0 \) in the saddle point equations, one can find a consistent solution where \( q = m = \hat{q} = \hat{m} = 0 \). This corresponds to the paramagnetic solution, where decoding of the received message fails. Using these conditions, one can retrieve the free energy in the paramagnetic phase,

\[
-f_{\text{para}} = - H_2 \left( \frac{1}{2} [(1 - p)(1 + \Omega) + r(1 - \Omega)] \right) \times \ln 2, \tag{34}
\]

where

\[
\Omega = \Omega_{PT} \equiv \prod_{l=1}^{K} \int_{-\infty}^{+\infty} Dz_l f_k(z_l). \tag{35}
\]

In the same way, substituting \( m = q = 1 \) in the saddle point equations, we numerically found \( \hat{m} \to \infty \) and \( \hat{q} \to \infty \) when \( K = 1 \), \( K = 2 \) and \( K = 3 \). For larger value of \( K \) we did not found any other solution.

Using \( m = q = 1, \hat{m} \to \infty \) and \( \hat{q} \to \infty \) in (24), one can finally get the expression of the free energy in the ferromagnetic phase,

\[
-f_{\text{erro}} = - \frac{\ln 2}{2} [(1 + \Omega)H_2(p) + (1 - \Omega)H_2(r)] - R \ln 2, \tag{36}
\]

where

\[
\Omega = \Omega_{PT}. \tag{37}
\]

5.2. Replica symmetric solution using a committee tree with non-monotonic hidden units

Using a committee tree with non-monotonic hidden units (6), the encoder function becomes

\[
\mathcal{F}_k(u_l) = \text{sgn} \left[ \sum_{l=1}^{K} f_k(u_l) \right]. \tag{38}
\]

Using this encoder function and substituting \( m = q = 0 \) in the saddle point equations, one can find a consistent solution where \( q = m = \hat{q} = \hat{m} = 0 \). This corresponds to the paramagnetic solution, where decoding of the received message fails.

In the same way, substituting \( m = q = 1 \) in the saddle point equations, we numerically found \( \hat{m} \to \infty \) and \( \hat{q} \to \infty \) when \( K = 3 \) (for \( K = 1 \) the present scheme is
equivalent to the parity tree case). For larger value of $K$ we did not found any other solution.

Using $m = q = 1$, $\hat{m} \to \infty$ and $\hat{q} \to \infty$ in (24) one can finally get the same results as in (34) and (36) but $\Omega$ which depends on the encoder is given by

$$
\Omega = \Omega_{CTH} \equiv \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times \text{sgn} \left[ \sum_{l=1}^{K} f_k(z_l) \right].
$$

(39)

5.3. Replica symmetric solution using a committee tree with a non-monotonic output unit

Using a committee tree with a non-monotonic output unit (11), the encoder function becomes

$$
\mathcal{F}_k(u_l) = f_k \left[ \sqrt{\frac{1}{K}} \sum_{l=1}^{K} \text{sgn}(u_l) \right].
$$

(40)

Using this encoder function and substituting $m = q = 0$ in the saddle point equations, one can find a consistent solution where $q = m = \hat{q} = \hat{m} = 0$ but only when considering an infinite number of hidden units, that is $K \to \infty$. This corresponds to the paramagnetic solution, where decoding of the received message fails. For finite $K$, $m = q = 0$ does not imply $\hat{m} = \hat{q} = 0$ and a non trivial solution is found making the free energy too complex to be investigated. The scheme is likely to give non-optimal performance in such case and will not be considered in the sequel.

In the same way, substituting $m = q = 1$ in the saddle point equations, we numerically found $\hat{m} \to \infty$ and $\hat{q} \to \infty$ when $K = 2$ and $K = 3$ (the present scheme cannot be defined for $K = 1$). For larger value of $K$ we did not found any other solution.

So finally using $m = q = 1$, $\hat{m} \to \infty$, $\hat{q} \to \infty$ and $K \to \infty$ in (24) one can get the same results as in (34) and (36) but $\Omega$ which depends on the encoder is given by

$$
\Omega = \Omega_{CTO} \equiv \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times f_k \left[ \sqrt{\frac{1}{K}} \sum_{l=1}^{K} \text{sgn}(z_l) \right].
$$

(41)

6. Phase transition

For all the three schemes, we found a paramagnetic and a ferromagnetic solution of the form given by equations (34) and (36) (only the definition of $\Omega$ changes). It is then possible to calculate the critical value of the code rate $R$ for which a sharp phase transition occurs between the ferromagnetic and the paramagnetic phase. This indicates the boundary between possible decoding (ferromagnetic phase) and impossible decoding (paramagnetic phase). In other words, this enables us to calculate the optimal code rate for each scheme. At the phase transition point, we have

$$
f_{\text{para}} = f_{\text{ferro}}.
$$

(42)
Simple algebra leads to

\[ R = H_2(\gamma) - \frac{1 + \Omega}{2} H_2(p) - \frac{1 - \Omega}{2} H_2(r), \]  

where

\[ \gamma = \frac{1}{2} [(1 - p)(1 + \Omega) + r(1 - \Omega)] \]  

and where \( \Omega \) depends on the encoder considered. This equation has exactly the same form as the BAC capacity equation (11) and in fact is equivalent to the BAC capacity if and only if \( \Omega = \Omega_C \). Since \( \Omega \) depends on the encoder, we will treat each case in the following subsections.

6.1. Tuning of the parity tree with non monotonic hidden units

In the parity tree case, we have

\[ \Omega = \Omega_{PT} = \prod_{l=1}^{K} \int_{-\infty}^{+\infty} Dz_l f_k(z_l). \]  

For \( \Omega_{PT} \) to be equivalent to \( \Omega_C \) we have to solve the following equation, with respect to the threshold parameter \( k \),

\[ \Omega_{PT} = \Omega_C \iff \prod_{l=1}^{K} \int_{-\infty}^{+\infty} Dz_l f_k(z_l) = \Omega_C \]  

\[ \iff H(k) = \frac{1}{4} \left( 1 - k \sqrt{\Omega_C} \right), \]  

where

\[ H(x) = \int_{x}^{+\infty} Dz. \]  

If the threshold \( k \) is tuned to satisfy the above equation, then it means that the scheme achieve the Shannon limit. The only issue which remains is whether or not such an optimal threshold \( k \) exists. We solved (47) numerically with parameters \((p, r) \in \{0, 1\}^2\) and always found an optimal threshold parameter \( k \) until at least \( K = 11 \) (note that \( \Omega_C \) can be negative which causes problems for the \( K \)-th root when considering an even number of hidden units \( K \). However a simple permutation of the probability \( p \) and \( r \) changes the sign of \( \Omega_C \). Since the original messages are drawn from the uniform distribution, this permutation can be done without any loss of generality. Instead of using \( s_0 \), one uses \(-s_0\). We did not checked higher values of \( K \). This means that the parity tree with non monotonic hidden units saturates the Shannon bound in the large codeword length limit for any number of hidden units \( K \).

6.2. Tuning of the committee tree with non monotonic hidden units

In the committee tree with non monotonic hidden units case, we have

\[ \Omega = \Omega_{CTH} = \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times \text{sgn} \left[ \sum_{l=1}^{K} f_k(z_l) \right]. \]  

\[ \text{sgn}(x) \]  

is the sign function, which returns 1 if \( x > 0 \), -1 if \( x < 0 \), and 0 if \( x = 0 \).
For $\Omega_{CTH}$ to be equivalent to $\Omega_C$ we have to solve the following equation, with respect to the threshold parameter $k$,

$$\Omega_{CTH} = \Omega_C \Leftrightarrow \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times \text{sgn} \left[ \sum_{l=1}^{K} f_k(z_l) \right] = \Omega_C$$

(50)

$$\Leftrightarrow \Omega_C = \sum_{l=0}^{K-1} \binom{K-1}{l} \left( [2H(k)]^{l} [1 - 2H(k)]^{K-l} - [2H(k)]^{K-l} [1 - 2H(k)]^{l} \right),$$

(51)

where $\binom{K}{l}$ denotes the binomial coefficient. If the threshold $k$ is tuned to satisfy the above equation, then it means that the scheme achieve the Shannon limit. So we should check if such an optimal threshold $k$ exists. We solved (51) numerically with parameters $(p, r) \in \{0, 1\}^2$ and always found an optimal threshold parameter $k$ until at least $K = 11$. We did not checked higher values of $K$. Note also that as mentioned in the definition of this encoder, we considered only an odd number of hidden units $K$. So these results mean that the committee tree with non monotonic hidden units saturates the Shannon bound in the large codeword length limit for any odd number of hidden units $K$.

### 6.3. Tuning of the committee tree with a non monotonic output unit

In the committee tree with a non monotonic output unit case, we have

$$\Omega = \Omega_{CTO} = \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times f_k \left( \sqrt{\frac{1}{K} \sum_{l=1}^{K} \text{sgn}(z_l)} \right).$$

(52)

For $\Omega_{CTO}$ to be equivalent to $\Omega_C$ we have to solve the following equation, with respect to the threshold parameter $k$,

$$\Omega_{CTO} = \Omega_C \Leftrightarrow \int_{-\infty}^{+\infty} \left[ \prod_{l=1}^{K} Dz_l \right] \times f_k \left( \sqrt{\frac{1}{K} \sum_{l=1}^{K} \text{sgn}(z_l)} \right) = \Omega_C.$$  

(53)

Here, we will consider the case where the number of hidden units $K$ tends to infinity as already done when investigating the saddle point equations. In this case we can efficiently make use of the central limit theorem so that the above equation can be rewritten as

$$\Omega_{CTO} = \Omega_C \Leftrightarrow \int_{-\infty}^{+\infty} Dz \times f_k(z) = \Omega_C$$

(54)

$$\Leftrightarrow H(k) = \frac{1}{4} (1 - \Omega_C).$$

(55)

If the threshold $k$ is tuned to satisfy the above equation, then it means that the scheme achieve the Shannon limit. The only issue which remains is whether or not such an optimal threshold $k$ exists. We solved (55) numerically with parameters $(p, r) \in \{0, 1\}^2$ and always found an optimal threshold parameter $k$. This means that the committee tree with a non monotonic output unit and an infinite number of hidden units (i.e. $K \to \infty$) saturates the Shannon bound in the large codeword length limit. A closer
look to equation (55) and (47) even shows that in fact the committee tree with an infinite number of hidden units is equivalent to the parity tree with a single hidden unit.

7. Conclusion and Discussion

We investigated an error correcting code scheme for uniformly unbiased Boolean messages using non-monotonic parity tree and non-monotonic committee tree multilayer perceptrons. All the schemes were shown to saturate the Shannon bound under some specific conditions. The replica symmetric solution stability [22] was not checked because of the complexity of the equations, and because no replica symmetry breaking is expected on the Nishimori line [23].

The use of a non-monotonic transfer function enables the scheme to deal with asymmetric channel like the BAC while monotonic networks are expected to be able to deal only with symmetric channel like the BSC. This feature makes such scheme able to handle much more general situation.

However, the present paper discusses only the typical performance of the schemes at the infinite codeword length but does not provide with any explicit decoder. This issue remains an important one which will be presented in a future work. One promising algorithm is to use the popular belief propagation (BP) to calculate an approximation of the marginalized posterior probabilities. The BP algorithm is known to give good results when working in the ferromagnetic phase, where no frustration is present into the system.

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Appendix A. Analytical Evaluation using the replica method

The free energy can be evaluated by the replica method,

\[
f(\beta, R) = -\frac{1}{\beta N} \lim_{n \to 0} \frac{\langle Z(\beta, y, x)^n \rangle y, x - 1}{n} \tag{A.1}\]

where \( Z(\beta, y, x)^n \) denotes the \( n \)-times replicated partition function

\[
Z(\beta, y, x)^n = \sum_{s^1, \ldots, s^n} \prod_{a=1}^{n} \exp \left[ -\beta R(y, \hat{y}(s^a)) \right]. \tag{A.2}\]
The vector $s^a$ is given by $s^a = (s^a_1, \ldots, s^a_K)$ and the superscript $a$ denotes the replica index.

We proceed to the calculation of the replicated partition function (A.2). Inserting the following two identities

$$1 = \prod_{a=1}^{n} \prod_{l=1}^{K} \int_{-\infty}^{+\infty} d\tilde{m}^a_l \delta \left( s^0_l \cdot s^a_l - \frac{N}{K} \tilde{m}^a_l \right)$$

$$1 = \left( \frac{1}{2\pi i} \right)^{nK} \int \left( \prod_{a} \prod_{l} d\tilde{m}^a_l d\hat{m}^a_l \right) \times \exp \left[ \sum_{a} \sum_{l} \hat{m}^a_l \left( s^0_l \cdot s^a_l - \frac{N}{K} \tilde{m}^a_l \right) \right]$$ (A.3)

and

$$1 = \prod_{a<b} \prod_{l=1}^{K} \int_{-\infty}^{+\infty} dq_{ab}^l \delta \left( s^a_l \cdot s^b_l - \frac{N}{K} q_{ab}^l \right)$$

$$1 = \left( \frac{1}{2\pi i} \right)^{(n-1)K/2} \int \left( \prod_{a<b} \prod_{l} dq_{ab}^l d\hat{q}_{ab}^l \right) \times \exp \left[ \sum_{a<b} \sum_{l} \hat{q}_{ab}^l \left( s^a_l \cdot s^b_l - \frac{N}{K} q_{ab}^l \right) \right]$$ (A.4)

into (A.2) enables us to separate the relevant order parameters, and to calculate the average moment $\langle Z(\beta, y, x)^n \rangle_{y, x}$ for natural numbers $n$ as,

$$\langle Z(\beta, y, x)^n \rangle_{y, x} \simeq \int \left( \prod_{a} \prod_{l} d\tilde{m}^a_l \frac{d\hat{m}^a_l}{2\pi i} \right) \times \int \left( \prod_{a<b} \prod_{l} dq_{ab}^l d\hat{q}_{ab}^l \right) \times \exp \left\{ N \left[ R^{-1} \ln \left( \sum_{y} \int \left( \prod_{l} du_l \frac{dv_l}{2\pi} dR_l \right) \frac{dW_l}{2\pi} \right) \right. \right.$$

$$\left. \left. \times \left( \frac{1}{2} + \frac{y}{2} \left[ (1 - r - p) F_k(R_l) + (r - p) \right] \right) \right. \right.$$

$$\left. \times \prod_{a} \left\{ \exp \left[ \beta \ln \left( \frac{1}{2} + \frac{y}{2} \left[ (1 - r - p) F_k(u^a_l) + (r - p) \right] \right) \right] \right. \right.$$

$$\left. \times \prod_{l} \left\{ \exp \left[ -\frac{1}{2} (W_l)^2 - \frac{1}{2} v_l \cdot Q_l \cdot v_l - W_l M_l \cdot v_l \right. \right. \right.$$

$$\left. \left. + iR_l W_l + i v_l \cdot u_l \right] \right\} \right\}$$

$$+ \frac{1}{K} \ln \left\{ \sum_{s^a} \exp \left[ \sum_{a,l} \hat{m}^a_l s^a_l + \sum_{a<b,l} \hat{q}_{ab}^l s^a_l s^b_l \right] \right\}$$

$$- \frac{1}{K} \sum_{a,l} m^a_l \hat{m}^a_l - \frac{1}{K} \sum_{a<b,l} q_{ab}^l \hat{q}_{ab}^l - \beta \sum_{a} \ln 2 \right\}, \quad (A.5)
where $Q_l$ is a $n \times n$ matrix having elements $\{q_{ab}^l\}$ and where $M_l$ is a $n$ dimensional vector having elements $\{m_a^l\}$. We analyze the scheme at the thermodynamic limit $N, M \to +\infty$, while the code rate $R$ is kept finite. In this limit, (A.5) can be evaluated using the saddle point method with respect to $m_a^l, \hat{m}_a^l, q_{ab}^l, \hat{q}_{ab}^l$ so that the free energy can be retrieved. To continue the calculation, we have to make some assumptions about the structure of these order parameters. In this paper, we use the so-called replica symmetric (RS) ansatz

\begin{align}
  m_a^l &= m, \quad q_{ab}^l = (1 - q)\delta_{ab} + q, \\
  \hat{m}_a^l &= \hat{m}, \quad \hat{q}_{ab}^l = (1 - \hat{q})\delta_{ab} + \hat{q},
\end{align}

(A.6)

where $\delta_{ab}$ denotes the Kronecker delta. This ansatz means that all the hidden units are equivalent after averaging over the disorder.

Also note that by definition, the order parameter $m$ is equivalent to the quantity $\frac{s^0 s}{N}$ which gives the overlap between the decoded message $s$ and the original message $s^0$. An overlap of 1 indicates perfect decoding while an overlap of 0 denotes complete failure.

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