$P_\infty$ algebra of KP, free fermions and 2-cocycle in the Lie algebra of pseudodifferential operators

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The symmetry algebra $P_\infty = W_\infty \oplus H \oplus I_\infty$ of integrable systems is defined. As an example the classical Sophus Lie point symmetries of all higher KP equations are obtained. It is shown that one (“positive”) half of the point symmetries belongs to the $W_\infty$ symmetries while the other (“negative”) part belongs to the $I_\infty$ ones. The corresponding action on the tau-function is obtained for the positive part of the symmetries. The negative part cannot be obtained from the free fermion algebra. A new embedding of the Virasoro algebra into $gl(\infty)$ describes conformal transformations of the KP time variables. A free fermion algebra cocycle is described as a PDO Lie algebra cocycle.

1. INTRODUCTION

In this paper we obtain and investigate the symmetries of integrable hierarchies based on the use of the algebra of pseudodifferential operators (PDO). It is a certain generalization of the famous Kadomtsev-Petviashvili (KP) hierarchy of partial differential equations (PDE). We shall use the word “symmetry” in a very general sense: a “symmetry” of an equation, or a set of equations, will be any further equation, compatible with the studied one. In the context of integrable evolution equations we shall understand “symmetries” as flows, commuting with the considered flow.

The KP equation [1] was introduced in the context of waves propagating in shallow water, or in a plasma, and corresponds to small transverse perturbations of solutions of the Korteweg-de Vries equation. Later it became clear that applications of the KP equation go far beyond its first ones. It is currently under intense study in the context of conformal quantum field theory and string theory, a partial review of this topic can be found in Ref. [2].

The KP equation is known to be integrable, in the sense that there is a Lax pair associated with it [3], it allows infinitely many conservation laws, multisolitons solutions and has all the usual attributes of integrability [4]–[6].

The KP equation is the first nontrivial member of an infinite hierarchy of mutually compatible equations [7,8], each representing a flow with respect to a different “time” $t_n$.

We shall use papers [7] and papers [9]–[12]. The famous $\hat{gl}(\infty)$ group symmetry transformations acting on the tau – function of the KP hierarchy were presented in Ref. [7]. Here we make the most use of paper [12], where symmetries which we shall refer as “PDO (pseudodifferential operators) symmetries” were found. We shall use the notation $P_\infty$ for them in the present paper. The PDO symmetry algebra is a direct sum of three subspaces: $P_\infty = W_\infty \oplus H \oplus I_\infty$. The subalgebra $H$ is the well-known Abelian algebra of higher KP flows. The “positive part” of $P_\infty$, now known now as $W_\infty$ algebra symmetries was exhaustively studied and its embedding into the $\hat{gl}(\infty)$ symmetries is explicitly described in [9]. The $W_\infty \oplus H$ subalgebra of KP symmetries results from the $\hat{W}_{1+\infty}$ algebra action on the KP tau-function. The $W_{1+\infty}$ symmetries have numerous applications in matrix models [14]–[16], [2], [16]. For new applications see Ref. [17]. For the representation theory of the $\hat{W}_{1+\infty}$ algebra without connections with the soliton theory see [18] (for applications in soliton theory only special weights are available).

The “negative part” of $P_\infty$, which we call $I_\infty$ here, has so far not been adequately studied in [10], [12].

The other infinite dimensional Lie group associated with the KP equation is the Lie group of local “point transformations” taking solutions of the equation into solutions. The Lie algebra of point symmetries forms a subalgebra in the algebra of all symmetries; historically it is this group that is usually called the “symmetry group” of an equation. Point symmetries form classical object in the theory of partial differential equations, arising from the work of Sophus Lie, see [14] for review. The calculation of the “symmetry group” of a differential equation is entirely algorithmic [14] and can be done using various computer packages [20]. It turns out that for a number of equations the corresponding

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symmetry algebra has a certain typical structure: it is a semidirect sum of the Virasoro algebra and a certain nilpotent part of a Kac-Moody one (both are without central charge) \([21,22]\). This result was obtained for the KP and Davey-Stewartson equations, for the 2D Toda lattice, for 3D 3-wave resonant interaction system and for some other equations \([23,24]\). The Virasoro algebra corresponds to the reparametrization of the time variable. We remark that it is not the Virasoro algebra corresponding to the reparametrization of the \(\textit{spectral parameter}\) in the soliton theory \([11,12,9,30,31]\) which was applied to the matrix models \([3,15]\) and which completely belongs to the \(W_\infty\) part of \(P_\infty\). We shall show that point symmetries belong to the whole \(P_\infty\) and not only to its \(W_\infty\) part.

This paper has the following goals: (1) To introduce the \(P_\infty\) algebra. (2) To get all point symmetries of the KP and the higher KP equations from the \(P_\infty\) ones and to get the conformal algebra of reparametrizations of higher KP time variables. (3) To compare the free fermion algebra \([8]\) and PDO Lie algebra cocycles \([32]\). (4) To get a nonstandard Virasoro algebra from free fermions and to show that only the positive part of point symmetries results from free fermion algebra. (5) To calculate the difference between free fermion Virasoro algebra flows and point algebra flows and to show that this difference corresponds to “Liouville equation flows” \([33]\). (6) To consider the compatibility of flows and constraints and to explain why only a finite part of the infinite-dimensional algebra of conformal symmetries survives when one reduces \((3D)\) higher KP equations to any two-dimensional \(KdV - \text{type}\) equation.

We note two facts about point symmetries: 1. Generally they do not preserve solutions described by the Segal-Wilson Grassmannian \([34]\). 2. They create rational solutions which do not vanish at infinity \([35]\).

We mention the following. The KP hierarchy and higher KP equations have different representations. The notion of “point symmetry” depends on the choice of the representation. Here we choose the KP higher equation in its most traditional form as evolutionary equation written on one function \(w(x,y,t_N)\) in two space variables \(x, y\) and one time variable \(t_N\), where \(N\) is the number of the equation in the hierarchy.

In Section II we review some results on the KP hierarchy and its symmetries, making use of the Gel’fand-Dickey approach via the algebra of pseudodifferential operators \(\mathcal{A} \mathbb{S}\) and a space of formal Zakharov-Shabat dressings \(K\). We present the results in a unified manner and include some known, but not easily accessible results \([4,11]\). We introduce a space of generalized KP flows \(V\) and \(P_\infty\) symmetries, \(P_\infty \subset V\).

Section III is devoted to the Lie point symmetries of the first two equations in the KP hierarchy.

The relation between the \(P_\infty\) (and its subalgebra \(W_\infty\)) and Lie point symmetries are derived in Section IV. Two known 2-cocycles are compared: the free fermion \(\mathfrak{g}(\infty)\) one \([8]\) and the two-dimensional PDO Lie algebra ones \([22]\). We pose the compatibility problem for pairs of \(P_\infty\) constraints and for a constraint and flows, and prove that only a finite set of point symmetries passes through the \(KdV\)-type reduction. Examples of conditional symmetries are given.

For group times we shall use the following notations: \(z, \{t_k\}, \{t_{mn}\}\). Except the results in Section IV B this paper is mainly based on Ref. \([41]\).

II. THE GENERALIZED KP HIERARCHY AND THE \(P_\infty\) SYMMETRIES

A. The space of formal Zakharov-Shabat dressings \(K\)

We shall make use of the associative algebra of pseudodifferential operators (PDO) in one variable on the line \(a(x, \partial)\) satisfying the permutation rule

\[
a(\partial)b(x) = \sum_{k=0}^{\infty} \frac{b^{(k)}(x)a^{(k)}(\partial)}{k!},
\]

where \((k)\) denotes the \(k\)-th derivative with respect to the argument. For a detailed exposition see e.g. \([8]\). There is a natural Lie algebra structure on the PDOs given by the commutator \([A, B] = AB - BA\), we denote this algebra by \(\mathcal{A}\).

An operation of conjugation (\(\ast\)) is introduced, defined by the rules

\[
x^* = x, \quad \partial^* = -\partial, \quad (AB)^* = B^*A^*.
\]

We shall also make use of the splitting of the space of PDO into the direct sum of two linear spaces, \(\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-\), with

\[
(\Sigma a_n \partial^n)_+ = \sum_{n \geq 0} a_n \partial^n, \quad (\Sigma a_n \partial^n)_- = \sum_{n < 0} a_n \partial^n.
\]
The highest power of the operator $\partial$ in a given PDO is called the order of this PDO. Let us introduce a space of formal Zakharov–Shabat dressing operators:

$$\mathcal{K} = \{ K = 1 + \sum_{j=1}^{\infty} K_j(x) \partial^{-j} \}.$$  \hspace{1cm} (2.4)

We call $K$ the formal Zakharov-Shabat dressing operator, by analogy with the analytical dressing operator of [4]. Points of the space of the formal Zakharov-Shabat dressings are parametrized by a semiinfinite set of arbitrary functions of one variable $x$, and sometimes we shall write:

$$\mathcal{K} = \{ K_j(x), \; j = 1, 2, 3, ... \infty \}.$$  \hspace{1cm} (2.5)

Remark. Under certain restrictions one can identify the space $\mathcal{K}$ with the “rotated Segal-Wilson Grassmannian” $g^{-1}W$ [34]. We do not consider the Grassmannian approach in the present paper.

We consider only sufficiently smooth functions $K_j(x)$, then an inverse integral operator exists, namely

$$K^{-1} = 1 + \sum_{j=1}^{\infty} \tilde{K}_j(x) \partial^{-j},$$  \hspace{1cm} (2.6)

with coefficients

$$\tilde{K}_j(x) = -K_j(x) + P(K_1, K_2, \ldots, K_{j-1}),$$  \hspace{1cm} (2.7)

where $P$ is a differential polynomial in the indicated arguments.

We shall also need the $*$-conjugate operator

$$K^{*^{-1}} = 1 + \sum_{j=1}^{\infty} K^*_j(x) \partial^{-j},$$  \hspace{1cm} (2.8)

(where $K^*_j(x)$ is simply a notation for new coefficients, not $*$-conjugates of $K_j(x)$).

In the next subsection we consider certain vector fields on the space $\mathcal{K}$.

**B. A space of vector fields $V$. Generalized KP flows**

Let us now consider the linear space of pseudodifferential operators of the form

$$A_i = \sum_n a_n^i(x, z_1, z_2, z_3, \ldots) \partial^n \in \mathcal{A},$$  \hspace{1cm} (2.9)

where $a_n^i$ are some fixed functions of their arguments. The “time” $z_i$ plays a privileged role in the operator $A_i$ and in the coefficients $a_n^i$. Note that the summation can be over both positive and negative values of $n$. Let us now introduce a mapping

$$A_i \rightarrow V_{A_i},$$  \hspace{1cm} (2.10)

from the PDOs $A_i$ onto vector fields $V_{A_i}$ on the space $\mathcal{K}$ according to the rule

$$V_{A_i}K \equiv \left[ \frac{\partial}{\partial z_i}, K \right] = -(KA_iK^{-1})_+ K.$$  \hspace{1cm} (2.11)

We shall denote the space of the vector fields $V_{A_i}$ by $\mathcal{V}$. One can check that these vector fields keep the chosen form of $K$ given by eq. (2.4). Each vector field $V_{A_i}$ induces a flow of all of the coefficients $K_n(x)$ of $K$, once we allow the coefficients $K_n(x, z_i)$ in eq. (2.4) to depend on $z_i$, as well as on $x$. We call these flows the generalized KP flows and denote the space of the vector fields $V_{A_i}$ by $\mathcal{V}$.

A result that will be used a great deal below is the following.
Theorem 1 The commutator of \( V_{A_i}, V_{A_j} \in \mathcal{V} \) satisfies
\[
[V_{A_i}, V_{A_j}]K = -(KF_{ij}K^{-1})_1 K , \tag{2.12}
\]
where we have
\[
F_{ij} = [A_i, A_j] + \frac{\partial A_i}{\partial z_j} \frac{\partial A_j}{\partial z_i}. \tag{2.13}
\]

The proof is a direct calculation, see Ref. [14].

Lemma 1 Let \( A_i \) be a PDO of the order \( p \). The constraint
\[
(KA_iK^{-1})_1 = 0 , \quad K \in \mathcal{K} \tag{2.14}
\]
restricts the space \( \mathcal{K} \) to a subspace parametrized by \( p - 1 \) arbitrary functions \( \{K_j(x), j = 1, 2, ..., p - 1\} \) for \( p > 0 \). If \( p < 0 \), (2.14) has no solutions.

Remark. One can treat (2.14) as the condition of invariance of a certain subspace of \( \mathcal{K} \) with respect to the flow \( V_i \). This condition allows us to express \( \{K_j(x), j \geq p\} \) via \( \{K_j(x), j = 1, 2, ..., p - 1\} \) which remains a set of arbitrary functions of one variable.

Corollary 1 Two flows with respect to \( z_i \) and to \( z_j \) commute iff
\[
(KF_{ij}K^{-1})_1 = 0 . \tag{2.15}
\]
If \( F_{ij}(x) \) does not vanish identically this condition restricts the space \( \mathcal{K} \) to a subspace parametrized by \( p - 1 \) arbitrary functions \( \{K_j(x), j = 1, 2, ..., p - 1\} \) for \( p > 0 \), where \( p \) is the order of \( F_{ij} \). The flows do not commute if \( p < 0 \).

When \( F_{ij}(x) \) does not vanish identically, but (2.13) is valid, then we are in the situation known as conditional symmetries [29], namely, when two flows commute only under some restriction on the space of solutions.

Remark 1. The operators \( A_i = A_i^{ij}(z)x^i\partial^m \) can be viewed as connections corresponding to the algebra of PDO and eq. (2.13) defines a curvature. Eq. (2.13) shows that the mapping (2.10) is not a mapping of the Lie algebra of PDO \( \mathcal{A} \) to the Lie algebra of vector fields \( \mathcal{V} \). The zero curvature condition \( F_{ij} = 0 \) guarantees that the flows with respect to \( z_i \) and \( z_j \) are compatible.

2. A supersymmetric version of eq. (2.13) has been used to construct supersymmetries for the super KP hierarchy [1]. A discrete version of eq. (2.13) was also used in this paper.

C. The KP hierarchy

Let us choose \( A_n = \partial^n \), where \( n \) is any nonzero integer. Let us again consider the integral operator \( K \) of eq. (2.4) and this time interpret the coefficients \( K_j \) as depending on an infinite sequence of times \( t_n \). We shall identify the first two as space variables \( t_1 = x, t_2 = y \), the third will be \( t_3 = t \). We keep the notation \( \partial \equiv \partial_x \equiv \partial_{t_1} \).

We can write an infinite hierarchy of partial differential equations, the Kadomtsev-Petviashvili hierarchy, in the following compact form
\[
\frac{\partial K}{\partial t_n} = -(K\partial^n K^{-1})_1 K . \tag{2.16}
\]

It follows from Theorem 1 that the flows (2.16) all commute, i.e. \( [\partial_n, \partial_m]K = 0 \), \( \partial_n = \partial/\partial_{t_n} \). For \( n = 0 \) we shall use the different notation \( t_{00} \). As a rule we shall omit the dependence on the nonpositive times and shall put \( t_{00} = 0, t_n = 0, n < 0 \). We shall use the notation \( \hat{t} \) for the collection of “KP higher times” \( t_1, t_2, t_3, \ldots \). We put standard notations \( t_1 = x, t_2 = y \).

Each operator equation (2.16) (for a fixed \( n > 0 \)) gives rise to an infinite coupled set of PDEs for the coefficients \( K_j \) of eq. (2.4).

Each equation can be labeled by \( (n;i) \):
\[
\partial_{n}K_{i} - \sum_{\ell=1}^{n} \binom{n}{\ell-1} K_{i}^{(n-\ell+1)} = \text{Pol}(K_{1}, K_{2}, \ldots, K_{n+i-2}) , \quad i = 1, 2, 3, \ldots , \tag{2.17}
\]

where Pol is a differential polynomial in \( K_{j} , \quad j < n + i - 1 \) and their \( x \)-derivatives upto order \( n - 1 \). The linear part of the polynomials has been separated out in the left hand side of eq. (2.17), so all terms on the right hand side are quadratic or higher. Note that each equation in the system (2.17) involves just one time \( t_{n} \) and the variable \( x \).

**How does one get the three dimensional KP and higher KP equations from (2.16)?** These equations are written for a set \( \{ K_{i} , \quad 1 \leq i \leq r \} \) as unknown functions of \( d \) variables \( t_{nk} \), \( k = 1, 2, \ldots , \), which play the role of space (time) variables in \( d \)-dimensional space. They are obtained by taking the set of equations (2.17) with labels \( (n_{k}; i) \). To get equations in the three dimensional space spanned by \( x, t_{m}, t_{n} \) one should consider a system of \( r \) equations for \( \{ K_{i} , \quad 1 \leq i \leq r \} , \quad r = m + n - 2 \). In the present paper we shall consider KP higher equations in 3-dimensional space spanned by \( x, y, t_{N} \) and we shall call this equation KPN.

For example, to obtain the Kadomtsev-Petviashvili equation itself, or \( KP_{3} \), take \( n = 3 \) and \( n = 2 \) and hence \( (n; i) = (3; 1), (2; 1) \) and \( (2; 2) \). The corresponding equations are

\[
\partial_{1}K_{1} - (K_{1}'' + 3K_{2}'' + 3K_{1}'') = 3K_{1}'K_{1}' - 3K_{1}K_{1}'' - 3K_{2}'K_{1} - 3K_{1}'K_{2} , \tag{2.18}
\]

\[
\partial_{2}K_{1} - (K_{1}'' + 2K_{2}'') = -2K_{1}'K_{1}' , \quad \partial_{2}K_{2} - (K_{2}'' + 2K_{2}'') = -2K_{1}'K_{2} . \tag{2.19}
\]

All three are evolution equations (in \( t_{3}, t_{2} \) and \( t_{2} \), respectively) for the functions \( K_{1}, K_{2} \) and \( K_{3} \). Using (2.19) to eliminate \( K_{3} \) and \( K_{2} \) from eq. (2.18), we obtain the KP equation

\[
w_{xx} + \frac{1}{4}w_{xxx} + \frac{3}{2}w_{xw} + \frac{3}{4}w_{yy} , \quad w(\hat{\tau}) = -2K_{1}(\hat{\tau}) \tag{2.20}
\]

(the usual KP equation is satisfied by \( u \equiv w_{x} \)).

Next example is \( KP_{4} \). It is obtained by taking \( n = 4 \) and hence \( (n; i) = (4; 1), (3; 1), (3; 2), (2; 1), (2; 2), (2; 3) \). One gets:

\[
w_{xxxx} + 6w_{xxw} + 4w_{xw} + 2w_{xxxw} + 2w_{xw} + 2w_{xxw} + w_{yy} = 0 . \tag{2.21}
\]

In what follows we shall use the evolutionary equation which describes KP flow (2.16) on the function \( w = -2K_{1}(x, y, t_{3}, \ldots) \)

\[
\frac{\partial w}{\partial t_{n}} = 2 \text{res}_{0} K \partial^{n-1} K^{-1} \tag{2.22}
\]

as the \( KP_{n} \) equation.

**D. Formal Baker-Akhiezer functions**

We shall make use of the formal Baker-Akhiezer functions \[8\]. They can be defined in terms of the PDO \( K \) of eq. (2.4) by putting

\[
\varphi(\lambda, t_{0}, \hat{\tau}) = Ke^{\lambda} = e^{\lambda \left( 1 + \sum_{n=1}^{N} \lambda^{-n}K_{n}(t_{0}, \hat{\tau}) \right)} , \tag{2.23a}
\]

\[
\varphi^{*}(\lambda, t_{0}, \hat{\tau}) = K^{*}e^{-\lambda} = e^{-\lambda \left( 1 + \sum_{n=1}^{N} \lambda^{-n}K_{n}^{*}(t_{0}, \hat{\tau}) \right)} , \tag{2.23b}
\]

\[
\zeta(\lambda, t_{0}, \hat{\tau}) = \sum_{k=1}^{N} \lambda^{k}t_{k} + t_{0} \ln \lambda , \quad \hat{\tau} = t_{1}, t_{2}, \ldots , \quad t_{1} = x , \tag{2.23c}
\]

where \( \lambda \) is a formal complex parameter. We introduce here the additional parameter \( t_{0} \) as in \[30\], \[31\] (\( t_{0} \) is not \( t_{0}! \)). It is responsible for the Schlesinger discrete transformations - special cases of Darboux transformations of the \( KP \) hierarchy. It was not used in \[8\].

We need the following lemma, due to Dickey \[8\].
Lemma Let \( P = \sum_k p_k \partial^k \) and \( Q = \sum_k q_k \partial^{-k} \) be any two PDOs and let
\[
\text{res}_\partial \sum a_n \partial^{-n} = a_1 .
\] (2.24)

Then we have
\[
\text{res}_\lambda (P e^{\lambda x})(Q e^{-\lambda x}) = \text{res}_\partial PQ^*. \tag{2.25}
\]

Using the above lemma, eq. (2.22) and (2.23) we obtain a formula summing up the KP hierarchy in terms of the formal Baker-Akhiezer functions:
\[
\frac{\partial w(t_0, \vec{t})}{\partial t_m} = 2 \text{res}_\lambda \lambda^m \varphi(\lambda, t_0, \vec{t}) \varphi^*(\lambda, t_0, \vec{t}) ,
\] (2.26)

which can be rewritten as
\[
2 \varphi \varphi^* = 2 + \frac{w_x}{\lambda^2} + \frac{w_y}{\lambda^3} + \frac{w_t}{\lambda^4} + \ldots .
\] (2.27)

The asterisk in eq. (2.27) has the same meaning as in eq. (2.28), i.e. it does not indicate conjugation. The term \( \lambda^{-1} \) is absent in eq. (2.27), since we have \( K_1 + K_1^* = 0 \). The \( \lambda^m \) term on the right hand side of eq. (2.27) follows from the asymptotic behavior of the Baker functions for \( \lambda \to +\infty \).

In addition to (2.26) we consider ‘Schlesinger transformation’ as a discrete equation with respect to \( t_0 \):
\[
w(t_0 + 1, \vec{t}) - w(t_0, \vec{t}) = 2 \partial \ln \varphi(0, t_0, \vec{t}) .
\] (2.28)

E. \( P_\infty \) symmetries of the KP hierarchy

We call a “symmetry” of the KP equation any differential, or integrodifferential equation that is evolutionary in group time \( z \) i.e. of the form
\[
\frac{\partial w}{\partial z} = F[w, \vec{t}, z],
\] (2.29)

where \( F \) is a function of \( w \), its \( x \) and \( y \) derivatives and of integrals of the type \( \partial^{-k} w \), \( k > 0 \), that is compatible with the KP equation itself. Similarly, a symmetry of the KP hierarchy will be any equation of the form (2.29), compatible with the entire hierarchy. That is for each \( m \)
\[
\partial_z \partial_{t_m} w = \partial_{t_m} \partial_z w .
\] (2.30)

All higher equations in the KP hierarchy are Abelian symmetries of the KP equation itself. Other symmetries of the KP equation exist and, contrary to the KP hierarchy, typically the corresponding equations (2.29) have coefficients explicitly depending on the independent variables \( x, y, t \). Among them we mention the Lie points symmetries [21,22]. The most complete treatment of all symmetries of the KP hierarchy in the framework of integrability theory is given in Ref. [9, 10, 12]. To put those results into the present context, let us use the mapping of Section II.B from the space of pseudodifferential operators to vector fields.

First let us consider the following extension:
\[
x \to \hat{x} = x + t_0 \partial^{-1} + \sum_{k \neq 1} kt_k \partial^{k-1} .
\] (2.31)

The notation \( \Box \) was used in Ref. [8, 10, 12] rather than \( \hat{x} \), and there was \( t_0 \equiv 0 \). We choose the PDOs to be
\[
A_{mn} = \hat{x}^n \partial^m , \quad n, m \in \mathbb{Z}
\] (2.32)

and construct the mapping \( A_{mn} \to V_{mn} \) with
\[
V_{mn} K = [\partial_{mn}, K] = -(K \hat{x}^n \partial^m K^{-1})_+ K ,
\] (2.33)
where $\partial_{mn}$ is a derivative with respect to the group time $t_{mn}$. We have $t_{m0} \equiv t_m$, $m \neq 0$ ($t_{00}$ is not $t_0$). It is necessary to define negative powers of $\hat{x}$ to represent them as a power series in $\partial$. This can be done in many different ways. In the present paper we shall define them in the two ways. We shall consider that $KP$ higher times vanish starting from a certain number $N + 1$, where $N$ is the order of $\hat{x}$. Then we put

$$ (\hat{x})^{-1} = (Nt_{N})^{-1}(\partial)^{1-N}(1 + O(\partial^{-1})) ,$$ (2.34)

which is a purely integral operator of order $-N$. For other applications we take $t_k = 0$, $k < 0$ and put

$$ (\hat{x})^{-1} = (1 + O(\partial))(t_0 + 1)^{-1}\partial ,$$ (2.35)

which is a differential operator of infinite order, the value of $t_0$ is noninteger.

For any given choice of $\hat{x}^{-1}$ due to the extension (2.31) it follows from Theorem 1 that we have

$$ (\partial_{mn}\partial_k - \partial_k\partial_{mn})K = 0 , \quad n, m \in \mathbb{Z} ,$$ (2.36)

and hence for any values of $m$ and $n$ the flow of (2.32) is compatible with the flows of the KP hierarchy.

The Lie algebra of the vector fields $V_{mn}$, generated by $\hat{x}^n\partial^n$ for $n \neq 0$, $n \in \mathbb{Z}$ and $\partial_{lm} - \partial^n_l$ for $n = 0$, $m > 0$, is the algebra of “additional symmetries” of the KP hierarchy introduced in [12], [10]. Now we consider $m, n, m', n' \in \mathbb{Z}$. The commutation relations have the form:

$$ [V_{mn}, V_{m'n'}] \sim [x^n\partial^n, x^{n'}\partial^{n'}] , \quad nn' \neq 0 ,$$ (2.37a)

$$ [V_{mn}, V_{m'n'}] = 0 , \quad nn' = 0 .$$ (2.37b)

The algebra of vector fields $V_{mn}$ will be called the $P_{\infty}$ algebra. It is a direct sum of three subspaces $P_{\infty} = W_{\infty} \oplus H \oplus I_{\infty}$. The subalgebra of vector fields $V_{mn}$, $n > 0$ is a $W_{\infty}$ algebra. The case $n = 0$ describes the higher KP flows. Then “negative” subalgebra $n < 0$ will be referred to as $I_{\infty}$. Note that due to the last commutation relation (2.37) this algebra is different from the Lie algebra of PDOs in one variable $A$ described in Section II A.

Until a certain convention on what is $\hat{x}^{-n}$ is adopted, the symmetry action of the $I_{\infty}$ algebra on the KP solutions is undefined (it was undefined in (12)). We also note that to get $P_{\infty}$ symmetries from the free fermion algebra $\hat{gl}(\infty)$ action on tau-functions, we choose the convention (2.34) and keep the parameter $t_0$ as noninteger.

$P_{\infty}$ has infinitely many different infinite dimensional Abelian subalgebras. Any one of them can be chosen to generate a hierarchy of commuting flows in addition to higher KP flows (see Remark 7 in [3]). They could be called “second-level KP hierarchies”. Each hierarchy $KP[Q]$ is defined by a PDO $Q(\hat{x}, \partial)$, which produces the flows:

$$ V_{Q^n} \in P_{\infty} , \quad n \in \mathbb{Z} $$ (2.38)

with respect to the times $t_{n}^{(Q)}$. For $Q = \partial$ we get the KP hierarchy itself. Both hierarchies commute due to the extension $x \rightarrow \hat{x}$. As in the case of the KP hierarchy, it is possible to construct finite closed subsystems of $r$ partial differential equations for a subset of the functions $(K_i(x), i = 1, 2, ..., r < \infty)$ if ord $Q < \infty$. As in Section II C it is then possible to construct evolution equations for one function $w$, evolving in a 3-dimensional space, spanned by $(x, i^{(Q)}_m, i^{(Q)}_n)$ with any integers $m, n$.

Also the algebra $P_{\infty}$ of flows contains infinitely many different sets of Virasoro flows:

$$ V_{Q^{n+1}P} \in P_{\infty} , \quad Q = Q(\hat{x}, \partial) , \quad P = P(\hat{x}, \partial) , \quad [P, Q] = 1 .$$ (2.39)

**Remark 1** In the same way as the $P_{\infty}$ algebra for the KP hierarchy, one constructs the $P_{\infty}$ algebra for the $KP[Q]$ hierarchy if an invertible $\hat{P}$ exists. One considers flows $V_{P^{\infty}Q^{\infty}}$ similar to (2.33), (2.34) with the change $(\hat{x}, \partial) \rightarrow (\hat{P}, Q)$, where $\hat{P}$ is the extension of $P$:

$$ \hat{P} = P + t_0^{(Q)}Q^{-1} + \sum k_i^{(Q)}Q^{k-1} , \quad k \in \mathbb{Z} .$$ (2.40)

These second-level $P_{\infty}$ flows commute with both KP and KP[Q] hierarchies. This way one can produce the n-th level KP hierarchy and its $P_{\infty}$ flows.
We shall not discuss these higher-level $P_\infty$ in the present paper.

Amongst the Virasoro flows we mention one of particular interest. It corresponds to $V_x\partial m \equiv V_{m1} \in W_\infty$. It was introduced in Ref. [3]–[13], and corresponds to a reparametrization of the spectral parameter $\lambda$. It has been used for establishing relations between soliton theory and quantum field theory in [3]–[13]. These flows are completely in the $W_\infty$ part of $P_\infty$ symmetries.

Example. We shall consider below the following Virasoro flows $V_{Pn+1Q} \in W_\infty$, $H$, $I_\infty$ for $n > -1$, $n = -1$, $n < -1$ respectively. As we shall see for $Q = \partial^N$ these flows correspond to the reparametrizations of a higher KP time variable $t_N$. For $N = 3, 4$ these flows result in the point symmetries of $KP_3$ and $KP_4$ equations which first were obtained in [21]–[22] with the help of a computer program.

Following [4] we introduce the pair

$$L = K\partial K^{-1}, \quad M = \hat{K} - K^{-1}, \quad [L, M] = 1,$$

(2.41)

which act in the following way on the formal Baker function:

$$L\phi(\lambda) = \lambda\phi(\lambda), \quad M\phi(\lambda) = \frac{\partial\phi(\lambda)}{\partial\lambda}.$$

(2.42)

Let us define $M^{-1} = K\hat{K}^{-1}$ and the corresponding $\partial^{-1}_\lambda$. We write formally:

$$M^{-1}\phi(\lambda) = \partial^{-1}_\lambda\phi(\lambda).$$

(2.43)

In agreement with eq. (2.34), we treat $\partial^{-1}_\lambda$ in the following way: we expand $e^\xi$ in the Baker function (2.23) into a positive power series in $\lambda$ and formally multiply it by the rest $O(1)$ part of $\phi$. Then we integrate each term according to the following rule:

$$\partial^{-1}_\lambda\lambda^n + t_o = (n + t_o + 1)^{-1}\lambda^{n+t_o+1}, \quad n \in \mathbb{Z},$$

(2.44)

where we should take a noninteger value of $t_o$. This convention is available to embed $P_\infty$ symmetries into the Japanese fermionic $gl(\infty)$ ones, see below.

In agreement with (2.34), we treat $\partial^{-1}_\lambda$ as a path integral over $\lambda$ from the point $\lambda$ to $\lambda = \infty$:

$$\partial^{-1}_\lambda\phi(\lambda, t_o, \tilde{t}) = \int^\lambda_{-\infty}\phi(\lambda', t_o, \tilde{t})d\lambda', \quad |\tilde{t}| \neq 0,$$

(2.45)

where the path is so chosen that for $\lambda \to \infty$, the integrand vanishes. This convention will be used below to obtain point symmetries. For each convention (2.45), (2.44) $\partial^{-n}_\lambda = (\partial^{-1}_\lambda)^n$.

The flow of the function $w = -2K_1$ with respect to the “time” $t_m$ of eq. (2.33) is given by the following theorem.

**Theorem 2** Let $K$ be the PDO of eq. (2.4) and $t_m$, where $m, n$ are any integers, the time defined in eq. (2.32), (2.33). Given the convention about $n < 0$, the $P_\infty$ flows of $w = -2K_1$ with respect to the times $t_m$ are given as

$$\partial_m w = 2\text{res}_\lambda(K\hat{x}^n\partial^m K^{-1}) = 2\text{res}_\lambda M^n L^m,$$

(2.46a)

or equivalently

$$\partial_m w = 2\text{res}_\lambda \lambda^n \frac{\partial^n\phi(\lambda)}{\partial\lambda^n} \phi^*(\lambda).$$

(2.46b)

**Proof.** Theorem 2 is a consequence of the Lemma of Section II.D. We set $P = K\hat{x}^m, Q = K^{* -1}$ and use the definition (2.23) of the formal Baker-Akhiezer functions. We then have

$$\text{res}_\lambda K\hat{x}^n\partial^m K^{-1} = \text{res}_\lambda(K\hat{x}^n\partial^m e^\xi)(K^{* -1}e^{-\xi}) = \text{res}_\lambda \lambda^n(K\hat{x}^n e^\xi)(K^{* -1}e^{-\xi}) = \text{res}_\lambda \lambda^n \frac{\partial^n\phi(\lambda)}{\partial\lambda^n} \phi^*(\lambda).$$

(2.47)
F. Symmetries via vertex operators

In order to link $\hat{\mathfrak{gl}}(\infty)$ symmetries \[7\] which act on $\tau$-function with “PDO” ones, \[9\], \ldots, \[12\] we need some results on vertex operators and $\tau$-functions. The vertex operators can be written as

$$X(\lambda, t_0, \vec{t}) = \exp(\sum_{k=1}^{\infty} \lambda^k t_k) \exp\left(-\partial_0 - \sum_{k=1}^{\infty} \frac{1}{k\lambda^k}\partial_k\right),$$

(2.48)

$$X^*(\lambda, t_0, \vec{t}) = \exp(-\sum_{k=1}^{\infty} \lambda^k t_k) \exp\left(\partial_0 + \sum_{k=1}^{\infty} \frac{1}{k\lambda^k}\partial_k\right).$$

(2.49)

The zero mode $t_0 \ln \lambda - \partial_0$ was added to the vertex operator in Ref. \[30\] to simplify calculations. It can be checked by a direct calculation that the vertex operators introduced above satisfy the fermion algebra anticommutation relations:

$$X(\lambda)X(\mu) + X(\mu)X(\lambda) = 0,$$

(2.50a)

$$X^*(\lambda)X^*(\mu) + X^*(\mu)X^*(\lambda) = 0,$$

(2.50b)

$$X(\mu)X^*(\lambda) + X^*(\lambda)X(\mu) = \delta(\lambda - \mu),$$

(2.50c)

where $\delta(\lambda - \mu)$ is the Dirac $\delta$-function with respect to integration about a circle $S^1$ (close to $\lambda \to \infty$)

$$\delta(\lambda - \mu) = \sum_{n=-\infty}^{\infty} \left(\frac{\mu}{\lambda}\right)^n \frac{1}{\lambda},$$

(2.51a)

$$\oint f(\lambda)\delta(\lambda - \mu) d\lambda = f(\mu).$$

(2.51b)

The “zero-time” $t_0$ as a discrete variable, was introduced in Ref. \[7\] and \[36\], though it was not used in these papers to add a zero mode to the vertex operators (2.48) and (2.49). One of the uses it was put to in Ref. \[30\] was to introduce the flag space of Grassmannians into KP theory.

A $\tau$-function is a function of all the times $\vec{t} = \{t_1, t_2, \ldots\}$ and also of the discrete “zero” time $t_0 \[7,36,31\]

$$\tau = \tau_n(\vec{t}), \quad n = t_0. \quad (2.52)$$

The formal Baker function near $\lambda = \infty$ can be expressed in terms of vertex operators and the $\tau$-function as

$$\varphi(\lambda, t_0, \vec{t}) = \frac{X(\lambda, t_0, \vec{t})\tau(t_0 + 1, \vec{t})}{\tau(t_0, \vec{t})},$$

(2.53)

(with a similar expression for $\varphi^*$).

For any sufficiently small shift $(\vec{t} - \vec{t}')$ we have the bilinear identity

$$\text{res}_{\lambda=\infty} \varphi(\lambda, t_0, \vec{t})\varphi^*(\lambda, t_0, \vec{t}') = 0.$$ \hspace{1cm} (2.54)

Let us now consider variations of the $\tau$-function due to the vector field

$$V_{\mu} \tau = X(\lambda)X^*(\mu)\tau.$$ \hspace{1cm} (2.55)

In the Kyoto school approach \[6\] this is an infinitesimal group transformation of the $\tau$-function, corresponding to an action of the algebra $\hat{\mathfrak{gl}}(\infty)$. Let us calculate the corresponding induced action on KP solutions $w_1$, following ref. \[6\]. We have

$$V_{\lambda} \varphi = -2 \text{res}_{k=\infty} (V_{\lambda} \varphi(k)) \varphi^*(k),$$

(2.56)
where $k$ is a spectral parameter and all times in $\varphi(k)$ and $\varphi^*(k)$ are set equal. In deriving \((2.57)\) use is made of the relations
\[
\varphi(k) = e^{\zeta(k)}\left(1 - \frac{w}{2k} + O\left(\frac{1}{k^2}\right)\right), \quad \varphi^*(k) = e^{-\zeta(k)}\left(1 + O\left(\frac{1}{k}\right)\right).
\] (2.57)

We recall that the relation between the $\tau$-functions and solutions is
\[
w(t_0, \bar{t}) = 2\frac{\partial}{\partial x}\ln \tau(t_0, \bar{t}),
\] (2.58)
but we do not use that here. For the Baker function we have
\[
V_{\lambda \mu} \varphi(k) = \frac{X(k)(V_{\lambda \mu} \tau)}{\tau} - \varphi(k) V_{\lambda \mu} \tau.
\] (2.59)
The last term in eq. \((2.59)\) does not contribute to the residue in eq. \((2.56)\) and we obtain
\[
V_{\lambda \mu} w = -2\text{res}_k \left(\frac{X(k)X(\lambda)X^*(\mu)\tau}{\tau}\right) \varphi^*(k).
\] (2.60)

From the fermion commutation relations \((2.54)\) we obtain
\[
[X(k), X(\lambda)X^*(\mu)] = -\delta(k - \mu)X(\lambda),
\] (2.61)
and hence
\[
-\text{res}_k \left(\frac{X(k)X(\lambda)X^*(\mu)\tau}{\tau}\right) \varphi^*(k) = \text{res}_k \delta(k - \mu) X(\lambda) \frac{\varphi^*(k)}{\tau} - \text{res}_k \frac{X(\lambda)X^*(\mu)X(k)\tau}{\tau} \varphi^*(k).
\] (2.62)

Making use of the bilinear identity \((2.54)\) we can see that the last term in eq. \((2.62)\) vanishes.

Using eq. \((2.60)\), \((2.61)\) and \((2.53)\), we obtain the final result
\[
V_{\lambda \mu} w = 2\varphi(\lambda)\varphi^*(\mu).
\] (2.63)

This formula was announced in Ref. [12] (with some obvious misprints) and a proof was given in Ref. [3]. Eq. \((2.63)\) plays a key role if we wish to relate the symmetries \((2.44)\) with the $\mathfrak{gl}(\infty)$ transforms \([6]\). Using eq. \((2.55)\) and \((2.63)\) we can write the following commutative diagram
\[
\begin{array}{c}
V_{\lambda \mu} \tau = X(\lambda)X^*(\mu) \tau \\
\downarrow \\
\partial_{mn} \tau = \text{res} \lambda^m \frac{\partial^n X(\lambda)X^*(\lambda)\tau}{\tau}
\end{array}
\] (2.64)
\[
\begin{array}{c}
V_{\lambda \mu} w = 2\varphi(\lambda)\varphi^*(\mu) \\
\downarrow \\
\partial_{mn} w = 2\text{res} \lambda^m \frac{\partial^n \varphi^* \varphi^*}{\partial x^i x^j}.
\end{array}
\]

Remark. In \((2.64)\) we have $n \geq 0$. The second formula defines the action of $\hat{W}_{1+\infty}$ generators acting on the tau-function. The case $n < 0$ is considered in detail in the forthcoming paper.

Below, in Section IV, we shall make use of the formalism of Sections II.E and II.F to obtain all point symmetries of the equations of the KP-hierarchy.

### III. POINT SYMMETRIES OF THE KP EQUATIONS

Point symmetry is the following particular case of general symmetry \((2.29)\):
\[
\frac{\partial w}{\partial \bar{z}} = (\bar{v}, \bar{\partial}) w - p, \quad \partial = (\partial_x, \partial_y, \partial_t), \quad \bar{v} = (v_1, v_2, v_N), \quad v_i = v_i(w, x, y, t_N), \quad p = p(w, x, y, t_N).
\] (3.1)

This equation looks like one which appears in the problem of a passive scalar (see the paper of I.V. Kolokolov in the present book).

We shall see that in our case $p, v_i$ are parametrized by five (or less) arbitrary functions $f, g, h, k, \ell$ of one variable. The point symmetry algebra of \((2.21)\) was calculated in Ref. [21], it is a sum of the vector fields
\[
\dot{V} = T(f) + Y(g) + X(h) + W(k) + U(\ell),
\] (3.2)
which act in four-dimensional space spanned by $x, y, t, w$. In terms of flows we have (we add $t_0$ term here):

$$w_z = f w_t + \left( \frac{2}{3} y f' + g \right) w_y + \left[ \frac{1}{3} x f' + \frac{2}{9} y^2 f'' + \frac{2}{3} y g' + h \right] w_x +$$

$$\left[ \frac{1}{3} w f' + \frac{1}{9} (x^2 + 4yt_0) f'' + \frac{4}{27} xy^2 f''' + \frac{4}{243} y^4 f'''' + \frac{2}{3} t_0 g' + \frac{4}{9} xyg'' - \frac{8}{81} y^2 g''' + \frac{2}{3} x h' - \frac{4}{9} y^2 h'' - yk - \ell \right].$$

The vector fields (3.3) can be integrated to yield point transformations, taking solutions of the PKP equation into solutions (2.21). The nonzero commutation relations of the symmetry algebra are

$$[T(f_1), T(f_2)] = T(f_1 f_2' - f_1' f_2), \quad (3.4)$$

$$[Y(g_1), Y(g_2)] = \frac{2}{3} X(g_1 g_2' - g_1' g_2), \quad (3.5a)$$

$$[Y(g), X(h)] = \frac{4}{9} W(hg'' - g'h') - \frac{8}{9} U(gh''), \quad (3.5b)$$

$$[Y(g), W(k)] = U(gk), \quad [X(h_1), X(h_2)] = \frac{2}{3} U(h_1 h_2' - h_1' h_2), \quad (3.5c)$$

$$[T(f), Y(g)] = Y \left( f\dot{g} - \frac{2}{3} g f \right), \quad [T(f), X(h)] = X \left( f\dot{h} - \frac{1}{3} h f \right), \quad (3.6a)$$

$$[T(f), W(k)] = W(fk + \dot{f} k), \quad [T(f), U(\ell)] = U \left( f\dot{\ell} + \frac{1}{3} \ell f \right). \quad (3.6b)$$

We see that the fields $T(f)$ form a centerless conformal algebra. The vector fields $\{Y(g), X(h), W(k), U(\ell)\}$ form a certain nilpotent subalgebra of centerless Kac-Moody algebra. This is a loop algebra with $t$ as the loop parameter.

For $KP_4$ the algebra of point symmetries is again a semidirect sum of a Kac-Moody and Virasoro algebras:

$$V = T(f) + Y(g) + X(h) + U(\ell), \quad (3.7)$$

with

$$T(f) = f \partial_t + \frac{1}{4} f(x\partial_x + 2y\partial_y) - \frac{1}{4}(w f' + xy f) \partial_w, \quad (3.8a)$$

$$Y(g) = g \partial_y - \frac{1}{2} x\dot{g} \partial_w, \quad X(h) = h \partial_x - y \dot{h} \partial_w, \quad U(\ell) = \ell \partial_w, \quad (3.8b)$$

where $f(z), g(z), h(z)$ and $\ell(z)$ are all arbitrary functions of the time $t_4 = z$.

An earlier observation is that all known integrable PDEs in 3-dimensions have Kac-Moody-Virasoro algebras as Lie point symmetry algebras. To the KP, equation (21.22), the Davey-Stewartson equation (23), the 3 wave resonant interaction equations (24) and several others, we have just added the higher order KP equation (2.21).

In Section IV we shall show that the same is true for each equation in the KP hierarchy.
IV. LIE POINT SYMMETRIES, $P_\infty$ ONES AND $\hat{\mathfrak{gl}}(\infty)$ ALGEBRA

A. Extraction of point symmetries

Let us obtain the point symmetries for higher KP equations and show that the corresponding algebras have a Kac-Moody-Virasoro structure. The corresponding Lie point symmetries can be directly extracted from the symmetries generated by the pseudodifferential operators $A_{mn}$ of eq. (2.32) via Theorem 3. Moreover, we will show that all the symmetries given in eq. (2.46b) that are local (no integrals), are point symmetries. Below we shall call a symmetry trivial if it vanishes identically for any $w(x,y)$.

**Theorem 3**

a) Let $N$ be the number of an equation in the KP hierarchy.

b) $m - n \leq N(1 - n)$;

c) all $t_k \equiv 0$ except $t_0, t_1, t_2, t_N$.

1. Then, for $n \geq 0$ the $W_\infty \oplus H$ subalgebra of $P_\infty$ reduces to the positive part of the Lie point symmetries of the KP$_N$ equation.

2. For $n \leq 0$ the “negative” subalgebra $W_\infty \subset P_\infty$ reduces to the negative part of the Lie point symmetries of the KP$_N$ if we interpret $\hat{x}^{-1}$ as in eq. (2.33).

**Proof.** (shortened). First let us rewrite the symmetries (2.46) of the KP hierarchy using a different basis. Instead of the operators $A_{mn} = \hat{x}^m \partial^n$ of eq. (2.32), let us consider arbitrary functions $h_\alpha(x)$ (that can be expanded into a power series, or Laurent series). We shall consider the operators

$$h_{\alpha,N} = \lambda^\alpha h_\alpha(\partial_E), \quad E(\lambda) = \lambda^N, \quad N \neq 0; \quad E(\lambda) = \ln \lambda, \quad N = 0.$$  

Sometimes we shall omit the label $\alpha$ below. We have the splitting of $h$ into the differential and the integral parts:

$$h = h_+ + h_- = \sum_{n \geq 0} h_n \partial_E^n, \quad h_- = \sum_{n < 0} h_n \partial_E^n,$$

where $h_n$ are constants. We rewrite the symmetries (2.46b) in the form

$$V^N(\alpha, h)w(t_0, \vec{t}) = 2 \res_\lambda \lambda^\alpha (h(\partial_E(\lambda))) \varphi(\lambda, t_0, \vec{t})|\varphi^*(\lambda, t_0, \vec{t}),$$

where $V^N(\alpha, h)$ is a vector field acting on $w$ (a linear combination of the flows $V_{mn}$).

For $h(x) = (Nx)^n$, $\alpha = m + nN - n$, we recover the symmetries (2.46b) with $V^N(\alpha, h) = V_{mn} + \sum_{k>0} c_k V_{m-k,n-k}$.

The condition (a) of Theorem 3 is equivalent to the condition $N < N$.

Let us rewrite eq. (4.3) in a more convenient form, using eq. (2.23) for the formal Baker-Akhiezer functions. Our aim is to replace the power series in derivatives $\partial_\lambda$, implicit in eq. (4.3), by power series in $\lambda$ itself. The formula we are aiming at is

$$V^N(\alpha, h)w(t_0, \vec{t}) = 2 \res \lambda^\alpha f(\lambda, t_0, \vec{t}) \varphi(\lambda, t_0, \vec{t})|\varphi^*(\lambda, t_0, \vec{t}),$$

where $f(\lambda, t_0, \vec{t})$ is a function to be determined. We shall use the following relation, valid for differential operators and (in view of (c)) also for integral ones:

$$h(\partial)e^x = e^x h(\partial + \chi') \cdot 1 = -e^x \sum_{k=0}^{\infty} h^{(k)}(\chi')/(\chi')^{-1} \partial^k \cdot 1,$$

where $\chi' = \partial \chi/\partial x$.

Comparing eq. (4.3) and eq. (4.4) and using eq. (2.23) we have

$$f(\lambda) = \varphi^{-1}(\lambda) h(\partial_E)e^\Phi,$$

$$\Phi = \sum_{k=1}^{\infty} \chi^k t_k + \ln \left(1 + \sum_{n=1} \lambda^{-n} K_n(t)\right) + t_0 \ln \lambda.$$

The KP hierarchy is written in terms of the function $w = -2K_1$. All higher coefficients $K_n$ in eq. (4.6b) are expressed nonlocally in terms of $K_1$ (see eq. (2.17), . . . , (2.19)). Since we will be using eq. (4.4) to extract local symmetries, we shall drop all negative powers $\lambda^{-n}$ in eq. (4.6b), except for $\lambda^{-1}$. We obtain...
Another ingredient in eq. (4.4) is the expansion (2.27) which we rewrite as

\[ 2\varphi\varphi^* = 2 + \frac{w_x}{\lambda^2} + \frac{w_y}{\lambda^4} + \frac{w_t}{\lambda^4} + \ldots + \frac{w_{\lambda\lambda}}{\lambda^{N+1}} + O_2 \left( \frac{1}{\lambda^{N+2}} \right). \]  

Both \( O_1 \) and \( O_2 \) contain nonlocal terms as coefficients before each degree of \( \lambda^{N-1-m}, m > 0 \). By comparing linear nonlocal terms for each degree of \( \lambda \) in \( O_1 \) with similar terms in \( O_2 \) one can verify after some computation that they are different and therefore never cancel each other.

Before expanding the function \( h(\{ \} ) \) into a Taylor series we note that nonlocal terms (integrals of \( w \)) will be avoided if the terms \( O(\lambda^{N-2}) \) do not participate in the calculation of the residue in eq. (4.4). This imposes the necessary restriction

\[ \alpha \leq N. \]  

For the same reason, we must “freeze” all times in eq. (4.7) except \( t_0, t_1 = x, t_2 = y \) and \( t_N \), where \( N \) is the number of the chosen equation in the KP hierarchy that we are considering (\( N = 3 \) for the KP itself, \( N = 4 \) for \( KP_t \) eq. (2.21) and so on). Thus, we set

\[ t_k = 0, \quad k \geq N + 1, \quad 3 \leq k \leq N - 1 \]  

in eq. (4.7).

Finally, we expand \( f(\lambda) \) in a Taylor series about \( t_N \), keep only local terms and obtain for \( N > 2 \)

\[
V^N(\alpha, h_\alpha)x = \text{res}_{\lambda^0} \left( h_\alpha(t_N) + \frac{1}{N} h'_\alpha(t_N) \left[ 2\lambda^{2-N}y + \lambda^{1-N}x + \lambda^{-N}t_0 + \frac{1}{2} \lambda^{-1-N}w \right] \right)
\]

\[
+ \frac{1}{2!N^2} h''''_\alpha(t_N) \left[ 4\lambda^{1-2N}y^2 + 4\lambda^{3-2N}xy + \lambda^{2-2N}(x^2 + 2y(2 + 2t_0 - N)) \right] + \frac{1}{3!N^3} h'''_\alpha(t_N) \left[ 8\lambda^3\lambda^{6-3N} + 12x y^2 \lambda^{5-3N} \right]
\]

\[
+ \frac{1}{4!N^4} h''''''_\alpha(t_N) 16y^4 \lambda^{8-4N} + O_1(\lambda^{N-2}) \right) \left( 2 + \frac{w_x}{\lambda^2} + \frac{w_y}{\lambda^4} + \ldots + \frac{w_{\lambda\lambda}}{\lambda^{N+1}} + O_2(\lambda^{-1-N}) \right)
\]  

\[ \square \]

Remark. One can continue \[ (4.11) \] for \( N = 0, 1, 2 \).

**Theorem 4** (Under the conditions of Theorem 3) The nonvanishing Lie point symmetries \( (4.3) \) are: (a) Virasoro symmetries for \( \alpha = N \) (b) Kac-Moody symmetries \( \alpha = 2, 1, -1 \) (and also \( \alpha = 0 \) for \( N = 3 \)). This algebra is a nilpotent one (c) The algebra of point symmetries algebra is a semidirect sum of these two subalgebras.

**Proof.** From \[ (4.11) \] it follows that for \( N > \alpha > 2 \) symmetries vanish. The fact that for \( \alpha = N \) we get the Virasoro algebra (without central charge) follows from Theorem 1. This together with the vanishing of symmetries for \( N > \alpha > 2 \), completes the proof. \( \square \)

Note that the values \( N, 2, 1, 0, -1 \) are also the “dimensions” of \( t_N, y, x, t_0, w \). For five functions \( h_N, h_2, h_1, h_0, h_{-1} \) we shall use the notations \( f, g, h, k, \ell \) respectively.

Let us write down an analog of the Zakharov-Shabat representation for the point symmetry equations. Let us denote the group time corresponding to the flow \( V^N(\alpha, h) \) as \( z = z^N(\alpha, h) \). From the considerations in Section II one can obtain the symmetry equation as

\[
[\partial_y - \partial^2 - w_x, \partial_z + (L^\alpha h(\frac{1}{N}ML^{1-N}))_z] = 0, \quad N > 0, \]

where for \( L, M \) see (2.41), for \( M^{-1} \) see (2.34). The associated linear problem is the following one:

\[
(\partial_z - (L^\alpha h(\frac{1}{N}ML^{1-N}))_z)_{\varphi}(\lambda, t_0, \vec{t}) = \lambda^\alpha h(\partial_\lambda \varphi)(\lambda, t_0, \vec{t}), \quad (4.13)
\]

For \( N = 0 \) we take \( h(ML), h(\lambda \partial_\lambda) \).

From Ref. 3 (see section F in the present paper) and from a direct calculation it follows
Theorem 5 The point symmetries results from the following action on the \( \tau \)-function:

\[
V^N(\alpha, h)\tau(t_0, \vec{t}) = \text{res}_\lambda \lambda^\alpha [h(\partial_\lambda \omega)] X(\lambda, t_0, \vec{t}) X^\ast(\lambda, t_0, \vec{t}) \tau(t_0, \vec{t}), \tag{4.14}
\]

\( \alpha = N, 2, 1, 0, -1 \), which is considered at the point \( \vec{t} = t_1, t_2, 0, 0, ..., 0, t_N, 0, 0, ... \). For the integral part \( h_-(\partial_\lambda \omega) \) the convention \((4.13)\) is used.

Remark. In Ref \[9\] only the "\( \hat{W}_{1+\infty} \) algebra" action on the tau-function was considered.

B. The PDO Lie algebra 2-cocycle and the free fermion algebra 2-cocycle

Let us review the nice result of \[32\], where an explicit expression for independent nontrivial cocycles of the Lie algebra of the PDOs on the circle was obtained. In this subsection \( \partial \) is \( \partial_\lambda \). Given \( A(\lambda, \partial) \), \( B(\lambda, \partial) \), \( \lambda \in S^1 \) are PDOs on the circle, these cocycles are

\[
\omega_1(A, B) = \frac{1}{2\pi i} \oint \text{res}_\lambda A [\ln \partial, B] d\lambda, \quad \omega_2(A, B) = \frac{1}{2\pi i} \oint \text{res}_\lambda A [\ln \lambda, B] d\lambda. \tag{4.15}
\]

Now let us solve the following problem: how to embed the Lie algebra of the PDOs with a central extension into the free fermion algebra \( \mathfrak{gl}(\infty) \). Let us give a short review of some facts from \[7\]. One introduces free fermions

\[
\psi(\lambda, t_0) = \sum_{n \in \mathbb{Z}} \lambda^{n+t_0} \psi_n, \quad \psi^\ast(\lambda, t_0) = \sum_{n \in \mathbb{Z}} \lambda^{-n-t_0-1} \psi^\ast_n, \tag{4.16}
\]

where the fermion operators \( \psi_n, \psi_n^\ast \) and vacuum are defined as in \[7\]. One introduces the space \( V \) of quadratic operators:

\[
Q_A = \sum_{n, m = -\infty}^{+\infty} : \psi_n A_{nm} \psi_m^\ast : \tag{4.17}
\]

where :: denotes normal ordering and \( A_{nm} \) is a generalized Jacobian matrix, i.e. infinite matrix with finite number of nonzero diagonals. The corresponding Lie algebra with the following commutation relation

\[
[Q_A, Q_B] = Q_{[A, B]} + \omega_{\mathfrak{gl}(\infty)}(A, B), \quad \omega_{\mathfrak{gl}(\infty)}(A, B) = Tr[A, B]_{--} - Tr[A_{--}, B_{-}]. \tag{4.18}
\]

was called the \( \mathfrak{gl}(\infty) \) algebra. In \((4.17)\) the subscript -- means a projection of an infinite matrix \( A_{nm} \), \( n, m < 0 \). As in \[7\] we construct the embedding of differential operators into \( V \) via \((4.17)\) and:

\[
\lambda, \lambda^{-1} \rightarrow \Lambda, \Lambda^{-1}, \quad \partial_\lambda \rightarrow \Gamma, \quad (A)_{ik} = \delta_{i, k-1}, \Lambda^{-1} = \delta_{i, k+1}, \quad (\Gamma)_{ik} = (k + t_0) \delta_{i-1, k}. \tag{4.19}
\]

In \[7\] \( t_0 \equiv 0 \). Now we need noninteger \( t_0 \). Then we can also embed integral operators via:

\[
\partial_\lambda^{-1} \rightarrow \Gamma^{-1}, \tag{4.20}
\]

where \( \Gamma^{-1} \) is defined for \( t_0 \) noninteger. The calculations provide the following result:

**Theorem 6** For PDOs \( A(\lambda, \partial_\lambda) \), \( B(\lambda, \partial_\lambda) \) and embedding \((4.19), (4.20), (4.17)\) we have

\[
\omega_{\mathfrak{gl}(\infty)}(A(\Lambda, \Gamma), B(\Lambda, \Gamma)) \sim \omega_1(A(\lambda, \partial), B(\lambda, \partial)) + \omega_2(A(\lambda, \partial), B(\lambda, \partial)), \tag{4.21}
\]

where \( \omega_{1, 2} \) are given by \((4.17)\). Equivalently

\[
\omega_{\mathfrak{gl}(\infty)}(A(\Lambda, \Gamma), B(\Lambda, \Gamma)) \sim \text{res}_{\lambda = \infty} \text{res}_{\partial_\lambda} A(\lambda, \partial) [\text{sign}(\epsilon), B(\lambda, \partial)], \tag{4.22}
\]

where \( e = \lambda \partial_\lambda \) is an Euler operator and \( \text{sign}(\epsilon) \) is a primitive of \( \delta(\epsilon) \).
Let us note that sign due to the commutator in this formula. Let us note that sign(\(\alpha\)) immediately get from (4.17), (4.18) and Theorem 6 that for gl(\(\infty\)) space of formal Zakharov-Shabat dressings \(\omega\) Theorem 7 Each flow \(\phi\) these flows on tau functions produce flows on formal Baker-Akhiezer functions. The algebra of these flows is known as the Lie algebra of PDOs with the central extension given by Theorem 6: 
\[
\lambda \psi_n \psi_{n+k} ,
\]
where \(g\) is some element of Lie group \(\hat{GL}(\infty)\) of the Lie algebra \(\hat{gl}(\infty)\). Let us introduce the following \(\hat{gl}(\infty)\) flow on \(\hat{GL}(\infty)\) and hence the flow on the tau function:
\[
\hat{V}_{mn}g = Q_{\lambda=\Gamma_n}g , \quad \hat{V}_{mn}\tau = <0|\exp(\sum_{k>0} t_k H_k)Q_{\lambda=\Gamma_n}g(t_{mn})|0> , \quad n, m \in \mathbb{Z} .
\]

The Lie algebra of these flows is the Lie algebra of PDOs with the central extension given by Theorem 6: \(\hat{V}_{mn} \in \hat{A}_0\). These flows on tau functions produce flows on formal Baker-Akhiezer functions \(\varphi, \varphi^\prime\). Let us remember the definition of these functions given in the papers of the Kyoto school [7]. Then the formal Zakharov-Shabat dressings which correspond to Baker-Akhiezer functions are uniquely reconstructed, \(\tau \to K\), see Section II D.

**Theorem 7** Each flow \(\hat{V}_{mn}\tau\) (4.20) induces the \(P_\infty\) flow \(V_{mn}K\) (2.33), (2.37) with the convention (2.35) on the space of formal Zakharov-Shabat dressings \(K\).

For \(n \geq 0\) the flows (4.20) were introduced before in [3], where the notation \(\partial_{mn}\) was used rather than \(\hat{V}_{mn}\). The algebra of these flows is known as the \(\hat{W}_{1+\infty}\) algebra, and a nice formula for its central extension is given by \(\omega_{gl(\infty)} = \omega_1\) in [32]. We need \(\omega_2\) for \(I_\infty\) symmetries.

### C. The Virasoro subalgebra of point symmetries

It is only the part \(n \leq -1\) of Virasoro flows (4.23, 4.24) which reproduces the point symmetries. The other half of the Virasoro flows produces highly nonlinear expressions. We can show that the difference between negative half of the point symmetries and the flows resulting from this Virasoro algebra action on fermionic tau-function is a Liouville flow [33, 35]. They describe a special case of Darboux transformations.

Let us obtain the Virasoro point symmetry algebra for each chosen equation in the KP hierarchy. We have in mind the equations written for the function \(w(t_0, x, y, t_0, t_1, \ldots) = -2K_1(t_0, x, y, t, t_1, \ldots)\) in which all variables except \(t_0, x, y, t\ and t_N\ are “frozen”: \(\tilde{t} = 0\). To obtain the Virasoro symmetries, we set \(\alpha = N\) in eq. (4.11) and calculate the residue. The function \(f\) below is \(h_N\) of section A. Let us consider each value of \(N\) separately, starting formally from the cases \(N = 0, 1, 2:\)

0. \(N = \alpha = 0\), \(\tilde{t} = \tilde{c}\),
\[
T(f) = f'(t_0)\partial_w + O(\tilde{t}) .
\]

1. \(N = \alpha = 1\), \(\tilde{t} = x, \epsilon_2, \epsilon_3, \ldots\)
\[
T(f) = f'(x)\partial - (f'(x)w + f''(x)(t_0^2 - t_0))\partial_w + O(\tilde{t}) .
\]
This formula can be interpreted as the known transformation of a “current” \( w(x) \) under the conformal transformation of space variable \( x \). Then the term \( f''(t_0^2 - t_0) \) appears due to a central extension:

\[
w_x = \{w(x), T_f\}, \quad T_f = \int f(x') \left( \frac{1}{2}w^2(x') - (t_0^2 - t_0)w'(x') \right) dx', \quad \{w(x), w(x')\} = \delta'(x - x'). \tag{4.29}
\]

2. \( N = \alpha = 2 \), \( \vec{t} = x, y, \epsilon_3, \epsilon_4, \ldots \).

\[
T(f) = f(y)\partial_y + \frac{f'(y)}{2}x\partial - \left( \frac{f'(y)w}{2} \right) - x(1 - 2t_0)f''(y) + x^3f''(y)\partial_w + O(\epsilon). \tag{4.30}
\]

3. \( N = \alpha = 3 \), \( \vec{t} = x, y, t, \epsilon_4, \epsilon_5, \ldots \).

We obtain \( T(f) + O(\epsilon) \), where \( T(f) \) as in eq. (3.2), (3.3).

4. \( N = \alpha = 4 \), \( \vec{t} = x, y, \epsilon_3, \epsilon_4, \epsilon_5, \ldots \).

We obtain \( T(f) + O(\epsilon) \), where \( T(f) \) as in eq. (3.8a), \( f = f(t_4) \).

5. \( N = \alpha = 5 \), \( \vec{t} = x, y, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \ldots \).

\[
T(f) = f\partial_{t_5} + \frac{1}{5}f'(x\partial_x + 2y\partial_y) - \frac{1}{25}(5w f' + 4y^2f'')\partial_w + O(\epsilon), \quad f = f(t_5). \tag{4.31}
\]

6. \( N = \alpha \geq 6 \), \( \vec{t} = x, y, \epsilon_3, \ldots \epsilon_{N-1}, t_N, \epsilon_{N+1}, \ldots \).

\[
T(f) = f\partial_{t_N} + \frac{1}{N}f'(x\partial_x + 2y\partial_y) - \frac{1}{N}w f'\partial_w, \quad f = f(t_N) + O(\epsilon). \tag{4.32}
\]

The commutation relations are the same in all cases, namely those of eq. (3.4). \( O(\vec{0}) = 0 \).

We note that the generators which correspond to the reparametrization of \( y \) (4.30) and \( t_3 \) variables create a rational KP potential [55].

**D. The Kac-Moody subalgebras of point symmetries**

The functions \( g, h, k, \ell \) below are \( h_\alpha \) from (4.3) and correspond to \( \alpha = 2, 1, 0, -1 \) respectively.

We obtain the following results.

\( N = 0 \), \( \alpha = 1 \) yield the vector field

\[
U(\ell) = -2\ell(t_0)\partial_w + O(\vec{t}). \tag{4.33}
\]

\( N = 1 \), \( \alpha = 0 \), \( -1 \) yield the vector fields

\[
W(k) = -2t_0k'(x)\partial_w, \quad U(\ell) = -2\ell(x)\partial_w. \tag{4.34}
\]

\( N = 2 \), \( \alpha = 1, 0, -1 \) yield the vector fields

\[
X(h) = h(y)\partial_y - (t_0h'(y) + \frac{1}{4}h''(y)x^2)\partial_w, \quad W(k) = -k'(y)x\partial_w, \quad U(\ell) = -2\ell(y)\partial_w. \tag{4.35}
\]

\( N = 3 \), \( \alpha = 2, 1, 0 \) and \( -1 \) yield the vector fields \( Y, X, W, U \) of (3.2) respectively.

\( N = 4 \), \( \alpha = 2, 1 \) and \( -1 \) yield the vector fields \( Y, X, U \) of (3.8), respectively.

\( N = 5 \), \( \alpha = 2, 1 \) and \( -1 \) lead to

\[
Y(g) = g(t_5)\partial_y - \frac{4}{5}g' y\partial_w, \quad X(h) = h(t_5)\partial_x, \quad U(\ell) = -2\ell(t_5)\partial_w, \tag{4.36}
\]

respectively.

\( N \geq 6 \), \( \alpha = 2, 1 \) and \( -1 \) lead respectively to

\[
Y(g) = g(t_N)\partial_y, \quad X(h) = h(t_N)\partial_x, \quad U(\ell) = -2\ell(t_N)\partial_w. \tag{4.37}
\]

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E. The symmetry groups

The advantage of point symmetries is that they can be easily and explicitly be integrated to give the group transformations. Let us construct the corresponding transformations by integrating the one parameter subalgebras of the vector fields (3.1), obtained above (with $\varepsilon = 0$). For the sake of simplicity we shall use the same notation $z$ for different group times. The functions $f, g, h, k, \ell$ below are $h_\alpha$ from (3.4) for $\alpha = N, 2, 1, 0, -1$ respectively, see sections B,C.

\[
\frac{d\tilde{t}}{dz} = v_1(\tilde{x}, \tilde{y}, \tilde{t}_N, \tilde{w}) \quad , \quad \frac{d\tilde{y}}{dz} = v_2(\tilde{x}, \tilde{y}, \tilde{t}_N, \tilde{w}) \quad , \quad \frac{d\tilde{t}_N}{dz} = v_N(\tilde{x}, \tilde{y}, \tilde{t}_N, \tilde{w}) \quad , \quad \frac{d\tilde{w}}{dz} = p(\tilde{x}, \tilde{y}, \tilde{t}_N, \tilde{w}) ,
\]

(4.38a)

\[
\tilde{x}\big|_{z=0} = x \quad , \quad \tilde{y}\big|_{z=0} = y \quad , \quad \tilde{t}_N\big|_{z=0} = t_N \quad , \quad \tilde{w}\big|_{z=0} = w \quad , \quad \frac{\partial \tilde{t}_N}{\partial t_N} = \frac{f(\tilde{t}_N)}{f(t_N)} .
\]

(4.38b)

$N \geq 6$. Virasoro (eq. (4.32)):

\[
\tilde{t}_N = \phi^{-1}(z + \phi(t_N)) , \quad \tilde{x} = x \left( \frac{\partial \tilde{t}_N}{\partial t_N} \right)^{1/N} , \quad \tilde{y} = y \left( \frac{\partial \tilde{t}_N}{\partial t_N} \right)^{2/N} , \quad \tilde{w} = w \left( \frac{\partial \tilde{t}_N}{\partial t_N} \right)^{-1/N} , \quad \phi(t) = \int^t \frac{dt'}{f(t')} .
\]

(4.39a)

Thus we see that under the change of $t_N$ the variables $x, y, w$ transform as $\frac{1}{N}, \frac{2}{N},$ and $-\frac{1}{N}$ tensors respectively. The Kac-Moody algebra (eq.(4.37)) integrates to:

\[
\tilde{t}_N = t_N , \quad \tilde{x} = x + zh(t_N) \quad , \quad \tilde{y} = y + zg(t_N) \quad , \quad \tilde{w} = w + 2z\ell(t_N) .
\]

(4.39b)

$N = 5$. Virasoro (eq. (4.31)):

\[
\tilde{t}_5 = \phi^{-1}(z + \phi(t_5)) , \quad \tilde{x} = x \left( \frac{\partial \tilde{t}_5}{\partial t_5} \right)^{1/5} , \quad \tilde{y} = y \left( \frac{\partial \tilde{t}_5}{\partial t_5} \right)^{2/5} , \quad \tilde{w} = w \left( \frac{\partial \tilde{t}_5}{\partial t_5} \right)^{-1/5} \left[ w - \frac{4}{25} y^2 \hat{f}(\tilde{t}_5) - \frac{2}{5} g(t_5) g'(t_5) z^2 \right] .
\]

(4.40a)

Kac-Moody (eq.4.36)

\[
\tilde{t}_5 = t_5 , \quad \tilde{x} = x + zh(t_5) , \quad \tilde{y} = y + zg(t_5) , \quad \tilde{w} = w + 2\ell(t_5)z - \frac{4}{5} g'(t_5)yz - \frac{2}{5} g(t_5) g'(t_5) z^2 .
\]

(4.40b)

$N = 4$ Virasoro (eq. (3.8a)):

\[
\tilde{t}_4 = \phi^{-1}(z + \phi(t_4)) , \quad \tilde{x} = x \left( \frac{\partial \tilde{t}_4}{\partial t_4} \right)^{1/4} , \quad \tilde{y} = y \left( \frac{\partial \tilde{t}_4}{\partial t_4} \right)^{1/2} , \quad \tilde{w} = w \left( \frac{\partial \tilde{t}_4}{\partial t_4} \right)^{-1/4} \left[ w - \frac{1}{4} xy \hat{f}(\tilde{t}_4) - \frac{3}{2} \hat{f}(\tilde{t}_4) \right] .
\]

(4.41a)

Kac-Moody (eq. (3.8b)):

\[
\tilde{t}_4 = t_4 , \quad \tilde{x} = x + zh(t_4) , \quad \tilde{y} = y + zg(t_4) , \quad \tilde{w} = w + \left( 2\ell(t_4) - \frac{1}{2} x g(t_4) - y h(t_4) \right) z - \left( g(t_4) h(t_4) + 2g(t_4) \hat{h}(t_4) \right) \frac{z^2}{4} .
\]

(4.41b)

The formulas for the group transformations for $N = 3$ (the KP equation itself) are somewhat more complicated. They were given in Ref. [21,22] and we do not reproduce them here.
$N = 2$. Virasoro (eq. \[4.30\])

$$\dot{y} = \phi^{-1}(z + \phi(y)), \quad \dot{x} = x \left(\frac{\partial \dot{y}}{\partial y}\right)^{1/2}$$

$$\dot{w}(\dot{x}, \dot{y}) = \left(\frac{\partial \dot{y}}{\partial y}\right)^{-1/2}w(x, y) + \frac{\dot{x}(1 - 2t_{\omega})}{f(\dot{y})} [f'(\dot{y}) - f'(y)]$$

$$+ \frac{\dot{x}^3}{24f(y)^2} \{f(y)f''(y) - f(\dot{y})f''(\dot{y})\} + \frac{1}{2} [f'^2(\dot{y}) - f'^2(y)]$$

Kac Moody (eq. \[4.31\])

$$\dot{y} = y, \quad \dot{x} = x + zh(y)$$

$$\dot{w}(\dot{x}, \dot{y}) = w(x, y) - t_{\omega}h'(y)z - \frac{1}{4}h''(y)[x^2z + xh(y)z + \frac{1}{3}h^2(y)z^3]$$

The Virasoro algebra induces a reparametrization of time $t_N$, that is compensated for by a redefinition of the other variables.

**F. Restriction to integrable equations in (1+1)-dimensions. Compatible pairs and flows**

First we should mark that a large number of finite dimensional systems and integrable 1+1-dimensional systems can be obtained from the KP hierarchy or the vector fields considered in Chapter II by imposing constraints.

Let us impose the following constraint on the space of the formal Zakharov-Shabat dressings $K$: given a PDO $A_i$ of order $p_i$, we set that $(KA_iK^{-1})_\infty = L_i$ is a Volterra integral operator with degenerate kernel of rank $r_i < \infty$, i.e.:

$$L_i = KA_iK^{-1}, \quad K \in K : \quad \text{rank } (KA_iK^{-1})_\infty = r_i < \infty . \quad (4.42)$$

**Lemma 1** For $p_i > 0$ this constraint restricts the space $K$ to the subspace $K[A_i] \subset K$ which is parametrized by the set of $p_i + 2r_i - 1$ arbitrary functions of $x$.

To get a finite dimensional system (an ordinary differential equation), let us consider the pair of constraints $\[4.42\] i = 1, 2$. The question is: does any $K \in K$ solving both constraints exist? If it exists then we can consider the algebra of constraints spanned by $\{A_i, i = 1, 2\}$. Example 1: both $A_i$ do not depend on $x$. When the solution of $\[4.42\]$ exists it is the Burchnell-Chaundy commutative ring \[40\] spanned by $\{L_i, i = 1, 2\}$. Example 2: the string equations \[41\] \[40\], see \[2.39\], \[2.40\] for $A_i = Q, P, \dot{P}$. Example 3: the bispectral problem \[42\].

**Problem 1** To classify all pairs of PDOs $A_i$ with finite $r_i$ which can be simultaneously transformed by a formal Zakharov-Shabat dressing $K \in K$ to

$$\{A_1, A_2\} \rightarrow^K \{L_1, L_2\} : \quad \text{rank } L_i = r_i, \quad i = 1, 2, \quad (4.43)$$

and to describe the corresponding subspace of the formal Zakharov-Shabat dressings $K[A_1, A_2] \subset K$.

Let us denote the subalgebra of PDOs spanned by $A_i, i = 1, 2$ by $A[A_1, A_2]$. Let us make a simple proposition:

**Proposition 1** For $r_1, r_2 = 0$ the space $K[A_1, A_2]$ is empty if $A[A_1, A_2]$ contains purely integral operators.

Example. p-KdV equation is defined by

$$\{K_{\partial^p K^{-1}}\}_\infty = 0 . \quad (4.44)$$

It is impossible to impose the following Virasoro (“string-type”) constraint $\{K_{\dot{x} \partial^q K^{-1}}\}_\infty = 0$ for p-KdV equation if $p + q - 1 < 0$. 

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To get 1+1 dimensional integrable systems like the KdV or NLS equations one needs 1) To impose one constraint (4.42). 2) To choose the vector field (2.11). Again the question is whether the given flow (2.11) is compatible with the given constraint (4.42).

For $L_i = K A_i K^{-1}$ and the flow $V_j (2.11)$ we have:

$$V_j L_i = [ L_i , (K A_j K^{-1})_+ ] = [ (K A_j K^{-1})_+ , L_i ] + K F_{ij} K^{-1} , \quad F_{ij} = [ A_i , A_j ] + V_j A_i .$$

(4.45)

**Problem 2** To describe all flows (2.11) which preserve a given constraint (4.42).

If the curvature $F_{ij} = 0$ then the flow preserves the constraint (4.42). Then (4.45) is a Lax system of equations for $p - 1$ independent functions in the two dimensional space spanned by $x , z_i$. Examples: $K_0$, $K_1$, $K_2$, ... If we take $A_i = \partial_z , A_j = \partial^2 z$ we get the $r_i$-component nonlinear Schrödinger equation. By taking $x$-independent PDOs $A_i$ of higher order we get systems which were called “constrained KP hierarchies”. Certain pairs of such restrictions give rise to different types of restricted flows $\hat{A}$, which are either autonomous or nonautonomous finite-dimensional dynamical systems.

Let us consider the problem of the compatibility of the KdV-type constraints with any $KP$ time dependent symmetry. This includes point symmetries as a particular case.

By the KdV-type constraint we mean the constraint:

$$(K A_0 K^{-1})_+ = 0 ,$$

(4.46)

where $A_0$ is a PDO of order $p_0 > 0$ which does not depend on $x$.

As we know from the Lemma in Section II, if the order of $A_0 = p_0 < \infty$ then the constraint (4.46) restricts the space of formal Zakharov-Shabat dressings $K$ to the set of $p - 1$ functions $K = \{ K_j , j = 1, 2, ..., p - 1 \}$.

Then the condition of compatibility of this KdV-type constraint with $P_\infty$ symmetry action produced by $V_i$ with time $z_i$, where $z_i$ is the corresponding time due to the Theorem 1 is

$$(K F_{0i} K^{-1})_+ = 0 , \quad F_{0i} = [ A_0 , A_i ] + \frac{\partial A_0}{\partial z_i} = [ \nabla_i , A_0 ] .$$

(4.47)

and the same we is obtained for each polynomial of $A_0$. These are additional constraints. We continue this process and finally get some space of constraints that we denote by $A( A_0; \nabla_i )$ (where we put $\nabla_i = \partial_{z_i} - A_i$) resulting from original constraint and the given flow. Generally we shall denote by $A( \hat{A} ; \hat{\nabla} )$ the algebra of constraints which one gets by imposing a set of constraints (4.46) with $\hat{A} = A_0, A_1, ..., $ and vector fields $\hat{\nabla} = V_i, V_k, ...$ corresponding to $\hat{\nabla} = \nabla_i, V_k, ...$ by taking all products and sums of constraints and by commuting them with all “covariant derivatives”. As a result $A( \hat{A} ; \hat{\nabla} )$ is stable under multiplying by any constraint and under commuting with any $\nabla$. The corresponding subspace of $K$ which transform each element of $A( \hat{A} ; \hat{\nabla} )$ to a differential operator we denote by $K( A_0; A_i - \partial_{z_i} )$.

We obtain a simple proposition:

**Proposition 2** (a) If $A( A_0; A_i - \partial_{z_i} ) \subseteq A( A_0 )$ then the flow with respect to $z_i$ is compatible with the constraint. (b) If $A( A_0; A_i - \partial_{z_i} )$ contains a nonzero element which belongs to the purely integral operators, then $K( A_0; A_i - \partial_{z_i} )$ is empty.

For the case of general PDO $A_0$ and $\nabla$ we can formulate the following

**Conjecture 1** If $A( A_0; \nabla_i )$ does not satisfy neither condition (a) nor (b) of the previous Theorem then we get a further restriction of our space $K$ - it is finite dimensional.

Bellow in the case under consideration this Conjecture can be proved via Theorem 1.

Example. $p - KdV$ constraint:

$$(K \partial^p K^{-1})_+ = 0$$

(4.48)

We have $K = \{ K_j (x) , j = 1, 2, ..., p - 1 \}$. Let us treat the KP flow given by $A_i = \partial_i^j$. It is easy to see that $A( \partial^p ; \partial^j - \partial_{z_i} ) = A( \partial^p )$, it means the known fact that all $KP$ flows preserve the $p - KdV$ reduction. Another example is the compatibility of Virasoro flows from [3] with $p - KdV$ reduction:
\[ A(\partial^p; \partial_{q_1} - \hat{x}\partial^q) = A(\partial^{(pn+(q-1)m)}) , \quad n > 0, m \geq 0. \] (4.49)

Hence it is only for \( q = 1 - p + kp, k > 0 \) that the Virasoro flow is compatible with \( p - KdV \). For \( q < 1 - p \) Theorem 6 implies that there are no solutions. For different values of \( q \) we get the conditional symmetry for the finite-dimensional system corresponding to the rational curve finite-gap solution. This is in the accordance with the Conjecture.

From the above it follows that

Only linearly \( \hat{x} \) -dependent \( A_j \) are compatible with the KdV-type constraint (4.46). These are \( A_j = \hat{x}(A_0)^k(A_0')^{-1}. \)

The only possibility to get point symmetries is to treat flows corresponding to \( x \)-dependent \( A_j \) or corresponding to simple shifts in \( x \) and time variables. Thus we get simple shifts in \( x \) and time, and scaling and Galilei transformations only. This explains why only finite dimensional algebras of point symmetries are obtained by direct methods [21], [24].

V. CONCLUSIONS

We have described \( P_\infty \) symmetries of integrable systems. We found the classical symmetries of Sophus Lie (point symmetries) of all higher \( KP \) equations as an example. We proved that each time the symmetry algebra is a semidirect sum of the Virasoro and current algebras. Each Virasoro algebra corresponds to the reparametrization group of the corresponding \( KP \) higher time. The topic is embedded into the standard solitonic theory, like \( L - A \) pairs, tau-functions, \( W_\infty \) symmetries.

Let us mention some related problems which were not considered in this paper. We noted in the Introduction that the notion of point symmetries depends on the choice of variables and representations for \( KP \) flows. In soliton theory different variables can be connected in a highly nonlocal way. (Moreover, soliton theory may be viewed as a science about parametrizations and changes of variables, and in fact in this quality has a lot of applications in both mathematics and theoretical physics). The point transformations have the advantage of being easily integrated to the group transformations. Therefore for possible applications like Ward identities it may appear to be suitable to have a “point representation” for each symmetry of the system under investigation. The problem is: given a \( gl(\infty) \) or \( P_\infty \) symmetry flow, what are the variables, for which this flow corresponds to a point transformation? Does each element of a \( KP - gl(\infty) \) or \( KP - P_\infty \) algebra give rise to a certain point symmetry?

We can suggest some possible applications of these symmetries. First, the role of symmetries for the quantization problem is known. Let us note that the “spectral parameter reparametrization” \( [13] \) Virasoro algebra applications appeared nondirectly \([13,39,41,6] \). This algebra was not connected with reparametrizations of the space variable \( x \), which is of importance in conformal theories of phase transitions in 2D. One can expect the appearance of Virasoro algebra in the solvable models arising from space variables reparametrizations. We are sure that the “time reparametrization” Virasoro will be of use.

A different application is to use these reparametrization symmetries for considering integrable equations on Riemann surfaces, where one needs change of \( t \)-variable to glue different maps.

The other application is the standard one: the calculation of special solutions.

An interesting and very complicated problem is posed in the last part of the paper: to classify all compatible pairs \( A_i, A_j \) and corresponding subspaces of the formal Zakharov-Shabat dressings \( K[A_i, A_j] \). Along with [6] we can expect that all ordinary equations we get in such a way have the Painlevé property. Is the reverse statement true? What sort of special functions can be obtained in such a way? What sort of representation theory is connected with these special solutions? About applications of [13] see [17]. As for \( gl(\infty) \) part of symmetries we know each time the correspondence between free fermion algebra and differential equations. What to do with the \( I_\infty \) part which has no free fermion interpretation for the convention (2.34)?

The way of getting point symmetries described in this paper is available not only for the \( KP \) hierarchy but for the multicomponent \( KP \) hierarchy and therefore almost for all known integrable models, like the 3D 3 wave resonant system hierarchy, Toda lattice hierarchy, Davey-Stewartson one e.t.c. It is also known the continuous generalization of the multicomponent KP equation, where \( t \)-dependent symmetries were found via recursion operator method Ref [15]. The results of this article confirm a conjecture made earlier that the Lie point symmetries of integrable PDEs in 3-dimensions are infinite-dimensional and have “a characteristic Kac-Moody structure” [23]. Since Lie point symmetries can be found using simple algorithms and computer packages [20], they provide a tool for investigating any given equation, and excluding it from the list of integrable equations originating from the \( KP \) in case there is no infinite-dimensional symmetry group with the prescribed structure.
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