SPECTRAL ASYMPTOTICS OF ONE-DIMENSIONAL FRACTAL LAPLACIANS IN THE ABSENCE OF SECOND-ORDER IDENTITIES

SZE-MAN NGAI
College of Mathematics and Computer, Hunan Normal University
Changsha, Hunan 410081, China
and
Department of Mathematical Sciences, Georgia Southern University
Statesboro, GA 30460-8093, USA

WEI TANG AND YUANYUAN XIE
Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China)
College of Mathematics and Computer Science, Hunan Normal University
Changsha, Hunan 410081, China

(Communicated by Robert L. Jerrard)

Abstract. We observe that some self-similar measures defined by finite or infinite iterated function systems with overlaps are in certain sense essentially of finite type, which allows us to extract useful measure-theoretic properties of iterates of the measure. We develop a technique to obtain a closed formula for the spectral dimension of the Laplacian defined by a self-similar measure satisfying this condition. For Laplacians defined by fractal measures with overlaps, spectral dimension has been obtained earlier only for a small class of one-dimensional self-similar measures satisfying Strichartz second-order self-similar identities. The main technique we use relies on the vector-valued renewal theorem proved by Lau, Wang and Chu [24].

1. Introduction. The origin of spectral asymptotics can be traced back to the work of Weyl. Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and with volume $\text{vol}(U)$, $\Delta$ be the Dirichlet Laplacian on $U$, $\{\lambda_n\}$ be the eigenvalues of $-\Delta$, and $N(\lambda, -\Delta)$ be the number of eigenvalues less than or equal to $\lambda$. In a seminal work started in 1911, Weyl [40] proved the following asymptotic formula, known as the Weyl law:

$$N(\lambda, -\Delta) \sim \frac{\omega_d}{(4\pi)^{d/2}} \text{vol}(U) \lambda^{d/2} + o(\lambda^{d/2}),$$

(1.1)
where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \). Using this formula he proved a conjecture posed independently by A. Sommerfeld and physicist H. A. Lorentz, which states that the density of standing electromagnetic waves in a bounded cavity \( U \) is, at high frequencies, independent of the shape of \( U \). This formula has generated an enormous amount of work, both in Euclidean domains and manifolds, most notably the work concerning Weyl’s conjecture on the remainder estimate [3, 7, 15, 25, 37]. For domains with fractal boundaries, remainder estimate, in terms of the Minkowski dimension of the boundary, was obtained by Lapidus [20].

For Dirichlet Laplacians \(-\Delta_\mu\) on domains defined by a measure, we would like to obtain a crude analogue of (1.1) of the form

\[
C_1 \lambda^{d_s/2} \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^{d_s/2}
\]

for all sufficiently large \( \lambda \), where \( d_s = d_s(-\Delta_\mu) \) is the spectral dimension of \(-\Delta_\mu\) (or simply of \( \mu \)) (see definition below). Spectral dimension has been computed by McKean and Ray [29] for the Cantor measure, by Fujita [10] and Naimark and M. Solomyak [30] for self-similar measures satisfying the open set condition (OSC) (see [14] and definition below), and by Freiberg [9] for generalized measure geometric Laplacians on Cantor like sets. Kigami and Lapidus [19] computed the spectral dimension of Laplacians on postcritically finite self-similar sets with a harmonic structure. Hambly and Nyberg [12] obtained spectral dimension for a finitely ramified graph-directed fractal that admits a Laplacian. Spectral dimension on random fractals have also been studied by Croyden and Hambly [4, 11]. Eigenvalue asymptotics of the Hanoi attractors have been obtained recently by Alonso-Ruiz and Freiberg [1]. Asymptotics of spectral partition functions associated with self-similar sets have been studied by Kajino [17,18].

Throughout this paper, unless stated otherwise, an iterated function system (IFS) refers to a finite or countably infinite family of contractive similitudes \( \{S_i\}_{i \in \Lambda} \) defined on a compact subset \( X \) of \( \mathbb{R}^d \). To avoid triviality, we assume throughout this paper that the cardinality of the limit set (see Section 2) is at least 2. If necessary, we use FIFS and IIFS respectively to distinguish between finite and infinite IFSs. IFSs are used to generate fractal sets and measures. It is well-known that an FIFS and a probability vector together determine a unique probability measure, called a self-similar measure. However, for an IIFS, the existence of self-similar measures needs some additional assumptions (see [27] and Proposition 6.14). Throughout this paper, we assume that to each IIFS and each probability vector, there corresponds a unique self-similar measure. An IFS \( \{S_i\}_{i \in \Lambda} \) is said to satisfy the open set condition (OSC) if there exists a nonempty bounded open set \( O \subseteq \mathbb{R}^d \) such that \( \bigcup_{i \in \Lambda} S_i(O) \subseteq O \) and \( S_i(O) \cap S_j(O) = \emptyset \) for all \( i, j \in \Lambda \) with \( i \neq j \). We call \( O \) an OSC-set.

(OSC) is a separation condition. If it fails, we say that the IFS, as well as any associate self-similar measure, has overlaps. In this case, it is much harder to compute the spectral dimension. The first author [31] computed the spectral dimension of a class of one-dimensional self-similar measures satisfying second-order identities. These identities were first introduced by Strichartz and are used in [38] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio. However, very few self-similar measures are known to satisfy second-order identities. In fact, for the class of symmetric infinite Bernoulli convolutions with overlaps, only the one associated with the golden ratio has been verified rigorously to satisfy this condition. Other examples are all defined by IFSs with contraction
ratios equal to the reciprocal of an integer. This includes a class of convolutions of the Cantor measure. To the best of the authors’ knowledge, in the absence of second-order identities, the spectral dimension of Laplacians defined by IFSs with overlaps has not been obtained before, and this is a main motivation of this paper.

For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [13]. Let \( U \subseteq \mathbb{R}^d \) be a bounded open subset and \( \mu \) be a positive finite Borel measure with \( \text{supp}(\mu) \subseteq \overline{U} \) and \( \mu(U) > 0 \). We assume that \( \mu \) satisfies the Poincaré inequality (PI) for measures: There exists a constant \( C > 0 \) such that
\[
\int_U |u|^2 \, d\mu \leq C \int_U |\nabla u|^2 \, dx \quad \text{for all } u \in C_c^\infty(U) \tag{1.2}
\]
(see, e.g., [13, 26, 30]). (PI) implies that each equivalence class \( u \in H^1_0(U) \) contains a unique (in the \( L^2(U,\mu) \) sense) member \( \hat{u} \) that belongs to \( L^2(U,\mu) \) and satisfies both conditions below:

1. there exists a sequence \( \{u_n\} \) in \( C_c^\infty(U) \) such that \( u_n \to \hat{u} \) in \( H^1_0(U) \) and \( u_n \to \hat{u} \) in \( L^2(U,\mu) \);
2. \( \hat{u} \) satisfies inequality (1.2).

We call \( \hat{u} \) the \( L^2(U,\mu) \)-representative of \( u \). Define a mapping \( \iota : H^1_0(U) \to L^2(U,\mu) \) by
\[
\iota(u) = \hat{u}.
\]
\( \iota \) is a bounded linear operator, but not necessarily injective. Consider the subspace \( \mathcal{N} \) of \( H^1_0(U) \) defined as
\[
\mathcal{N} := \{ u \in H^1_0(U) : \|\iota(u)\|_{L^2(U,\mu)} = 0 \}.
\]
Now let \( \mathcal{N}^\perp \) be the orthogonal complement of \( \mathcal{N} \) in \( H^1_0(U) \). Then \( \iota : \mathcal{N}^\perp \to L^2(U,\mu) \) is injective. Unless explicitly stated otherwise, we will denote the \( L^2(U,\mu) \)-representative \( \hat{u} \) simply by \( u \).

Consider a non-negative bilinear form \( \mathcal{E}(\cdot, \cdot) \) in \( L^2(U,\mu) \) given by
\[
\mathcal{E}(u,v) := \int_U \nabla u \cdot \nabla v \, dx \tag{1.3}
\]
with domain \( \text{dom} \mathcal{E} = \mathcal{N}^\perp \), or more precisely, \( \iota(\mathcal{N}^\perp) \). (PI) implies that \( (\mathcal{E}, \text{dom} \mathcal{E}) \) is a closed quadratic form on \( L^2(U,\mu) \). Hence there exists a non-negative self-adjoint operator in \( L^2(U,\mu) \), which we denote by \( -\Delta_\mu \) and call the (Dirichlet) \( L^2(U,\mu) \)-representative \( \hat{u} \) simply by \( u \).

We assume \( L^2(U,\mu) \) is infinite dimensional. It is known that there exists an orthonormal basis \( \{\varphi_n\}_{n=1}^\infty \) of \( L^2(U,\mu) \) consisting of the eigenfunctions of \( -\Delta_\mu \). The eigenvalues \( \lambda_n = \lambda_n(-\Delta_\mu) \) satisfy \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \lim_{n \to \infty} \lambda_n = \infty \). The eigenvalue counting function for \( -\Delta_\mu \) is defined as
\[
N(\lambda, -\Delta_\mu) := \# \{ n : \lambda_n \leq \lambda \}.
\]
where \( \#A \) denotes the cardinality of a set \( A \). Define lower and upper spectral dimensions of \( -\Delta_\mu \), respectively, as
\[ d_s(-\Delta_\mu) := \lim_{\lambda \to \infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln \lambda} \quad \text{and} \quad \overline{d}_s(-\Delta_\mu) := \lim_{\lambda \to \infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln \lambda}. \]

If \( d_s(-\Delta_\mu) = \overline{d}_s(-\Delta_\mu) \), the common value, denoted \( d_s(-\Delta_\mu) \) (or simply \( d_s \) if no confusion is possible), is called the \textit{spectral dimension} of \( -\Delta_\mu \) (or simply of \( \mu \)); it measures the asymptotic growth rate of the eigenvalue counting function as well as the magnitude of the \( n \)-th eigenvalue.

This paper studies measures that are essentially of finite type (EFT), a condition that we introduce to describe the finiteness of basic measure types, as defined below. (EFT) is a key assumption in computing spectral dimension and is formulated in Section 2.2.

Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded open subset and \( \mu \) be a positive finite Borel measure with \( \text{supp}(\mu) \subseteq \overline{\Omega} \) and \( \mu(\Omega) > 0 \). Roughly speaking, two cells (that is, subsets of \( \Omega \) with positive \( \mu \) measure), \( U \) and \( V \) are \( \mu \)-equivalent if \( \mu|_V = w\mu|_U \circ \tau^{-1} \) for some \( w > 0 \) and some surjective similitude \( \tau : U \to V \), where \( \mu|_F \) denotes the restriction of the measure \( \mu \) to \( F \subseteq \mathbb{R}^d \). A \( \mu \)-\textit{partition} \( P \) of \( U \) is a finite family of measure disjoint sub-cells of \( U \) such that \( \mu(U) = \sum_{V \in P} \mu(V) \). A sequence of \( \mu \)-partitions \( (P_k)_{k \geq 1} \) is \textit{refining} if each member of \( P_{k+1} \) is a subset of some member of \( P_k \).

Intuitively, \( \mu \) satisfies (EFT) if there exist some bounded open set \( \Omega \subseteq \mathbb{R}^d \) with \( \text{supp}(\mu) \subseteq \overline{\Omega} \) and \( \mu(\Omega) > 0 \), together with a finite family \( \mathbf{B} := \{ B_{1,\ell} : \ell \in \Gamma \} \) of measure disjoint cells in \( \Omega \) such that for each \( \ell \in \Gamma \), there is a family of refining \( \mu \)-partitions \( (P_{k,\ell})_{k \geq 1} \) of \( B_{1,\ell} \) satisfying the following conditions: (1) for all \( k \geq 2 \), \( P_{k+1,\ell} \) contains all cells in \( P_{k,\ell} \) that are \( \mu \)-equivalent to some cell in \( \mathbf{B} \); (2) the sum of the \( \mu \)-measures of those cells \( B \in P_{k,\ell} \) that are not \( \mu \)-equivalent to any cell in \( \mathbf{B} \) tends to 0 as \( k \to \infty \). In this case, we call \( \Omega \) an EFT-set, \( \mathbf{B} \) a basic family of cells in \( \Omega \), and \( (\mathbf{B}, \mathbf{P}) := (\{ B_{1,\ell} \}, (P_{k,\ell})_{k \geq 1})_{\ell \in \Gamma} \) a basic pair with respect to \( \Omega \). The precise statements are given in Definition 2.11. In particular, we say that \( (\mathbf{B}, \mathbf{P}) \) is \textit{regular} if each cell \( B \subseteq \bigcup_{k \geq 1, \ell \in \Gamma} P_{k,\ell} \) is connected, and for any \( \ell \in \Gamma \), there exist some similitude \( \tau_\ell \) and some constant \( w(\ell) > 0 \) such that \( \tau_\ell(\Omega) \subseteq B_{1,\ell} \) and \( \mu \geq w(\ell)\mu \circ \tau^{-1} \) on \( \tau_\ell(\Omega) \).

Let \( \mu \) be a continuous positive finite Borel measure on \( \mathbb{R} \). Assume that \( \mu \) satisfies (EFT) with \( \Omega \subseteq \mathbb{R} \) being an EFT-set and that there exists a regular basic pair \( (\mathbf{B}, \mathbf{P}) := (\{ B_{1,\ell} \}, (P_{k,\ell})_{k \geq 1})_{\ell \in \Gamma} \) with respect to \( \Omega \). Then we can derive renewal equations for the eigenvalue counting functions, and express them in vector form as:

\[ f = f \ast M_\alpha + z, \quad (1.4) \]

where \( \alpha \geq 0 \), and

\[ f := [f^{(\alpha)}(t)]_{\ell \in \Gamma}, \quad t \in \mathbb{R}; \]

\[ M_\alpha := [\mu^{(\alpha)}_{\ell,m}]_{\ell,m \in \Gamma} \quad \text{is some finite matrix of Borel measures on } \mathbb{R}; \quad (1.5) \]

\[ z := [z^{(\alpha)}(t)]_{\ell \in \Gamma} \quad \text{is an error function.} \]

Let

\[ M_\alpha(\infty) := [\mu^{(\alpha)}_{\ell,m}(\mathbb{R})]_{\ell,m \in \Gamma}. \quad (1.6) \]

For each \( \ell \in \Gamma \) and \( \alpha \geq 0 \), define

\[ F_\ell(\alpha) := \sum_{m \in \Gamma} \mu^{(\alpha)}_{\ell,m}(\mathbb{R}), \quad D_\ell := \{ \alpha \geq 0 : F_\ell(\alpha) < \infty \}, \quad \bar{\alpha}_\ell := \inf D_\ell. \quad (1.7) \]
If the error functions decay exponentially to 0 as \( t \to \infty \), then \( d_\ell(\Delta_\mu) \) is given by the unique \( \alpha \) such that the spectral radius of \( M_\alpha(\infty) \) is equal to 1. The following is the main result.

**Theorem 1.1.** Let \( \mu \) be a continuous positive finite Borel measure on \( \mathbb{R} \). Assume that \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set on which \( \Delta_\mu \) is defined, and assume that there exists a regular pair with respect to \( \Omega \). Let \( M_\alpha(\infty) \), \( F_\ell(\alpha) \), and \( \tilde{\alpha}_\ell \) be defined as in (1.6) and (1.7). Assume that for each \( \ell \in \Gamma \), \( \lim_{\alpha \to \tilde{\alpha}_\ell^+} F_\ell(\alpha) > 1 \).

(a) There exists a unique \( \alpha > 0 \) such that the spectral radius of \( M_\alpha(\infty) \) is equal to 1.

(b) If we assume, in addition, that for the unique \( \alpha \) in (a), there exists \( \sigma > 0 \) such that for all \( \ell \in \Gamma \), \( z_\ell^{(\alpha)}(t) = o(e^{-\sigma t}) \) as \( t \to \infty \), then we have

(i) \( d_\alpha = 2\alpha \);

(ii) if \( M_\alpha(\infty) \) is irreducible, then there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^\alpha \quad \text{for all sufficiently large } \lambda.
\]

Let \( \mu \) be a self-similar measure defined by a finite type IFS on \( \mathbb{R}^d \). In Section 2, we define the set of all level-\( k \) islands (see Definition 2.6). Roughly speaking, a level-\( k \) island corresponds to a connected component of the level-\( k \) iterates of some fixed connected open set \( \Omega \). Two islands are of the same measure type (with respect to \( \mu \)) if their corresponding connected components are \( \mu \)-equivalent. For FIFSs, we give a sufficient condition for (EFT) with a regular basic family of cells (see Proposition 2.15 for details). In Section 5, we illustrate Theorem 1.1 by the following family of FIFSs:

\[
S_1(x) = r_4 x, \quad S_2(x) = r_2 x + r_1 (1 - r_2), \quad S_3(x) = r_2 x + 1 - r_2,
\]

where the contraction ratios \( r_1, r_2 \in (0, 1) \) satisfy \( r_1 + 2r_2 - r_1 r_2 \leq 1 \), i.e., \( S_2(1) \leq S_3(0) \) (see Figure 1). The Hausdorff dimension of each self-similar set in the family is computed in [23]. This family is also used as basic examples of IFSs of general finite type [16, 22]. The multifractal properties of the corresponding self-similar measures are recently studied by Deng and two of the authors [6, 34].

**Theorem 1.2.** Let \( (S_i)_{i=1}^3 \) be an FIFS in (1.9), \( (p_i)_{i=1}^3 \) be a probability vector, \( \mu \) be the associated self-similar measure, and \( \Delta_\mu \) be defined on \( \Omega = (0, 1) \). Then there exists a unique positive real number \( \alpha \) satisfying

\[
(1 - (p_2 r_2)^\alpha)(1 - (p_3 r_2)^\alpha) \sum_{k=0}^{\infty} (w_1(k) r_1 p_2^k)^\alpha + (p_2^\alpha + p_3^\alpha) r_2^\alpha = 1,
\]

where \( w_1(k) := p_1 \sum_{i=0}^{k} p_2^{k-i} p_3^i \). Moreover, \( d_\mu = 2\alpha \), and there exist positive constants \( C_1, C_2 \) such that \( C_1 \lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^\alpha \) for all sufficiently large \( \lambda \).

Numerical approximations by taking \( p_1 = 1/4, p_2 = 1/2, p_3 = 1/4, r_1 = 1/3, r_2 = 2/7 \), and taking \( k \) up to 500 yield \( d_\mu = 0.871431 \ldots \).

In Section 6, we study (EFT) for IIFSs, which is more complicated because of the presence of the so-called “tails” (see Definition 6.1). We give an analogous sufficient condition for a self-similar measure defined by an IIFS on \( \mathbb{R} \) to satisfy (EFT) with a regular basic family of cells (see Proposition 6.4).
We illustrate Theorem 1.1 by the following family of IIFSs, which is studied in [32]:

\[
S_1(x) = rx + 1 - r, \quad S_{2k}(x) = r^k x + s(1 - r^{k-1}), \quad S_{2k+1}(x) = r^k x + s(1 - r^{k-1}) + r^{k}(1 - r), \quad \text{for } k \geq 1,
\]

where \(0 < r < (2 - \sqrt{2})/2 = 0.292893 \ldots, r(2 - r)/(1 - r) < s < 1 - r\) (see Figure 6).

**Theorem 1.3.** Let \((S_i)_{i=1}^\infty\) be an IIFS as in (1.11), \((p_i)_{i=1}^\infty\) be a probability vector, \(\mu\) be the associated self-similar measure, and \(\Delta_\mu\) be defined on \(\Omega = (0, 1)\). Assume that there exists some integer \(L \geq 2\), which is chosen to be the minimal one, such that

\[
\left\{ \frac{p_{2i}}{p_{2i+1}} : i \geq L \right\} \subseteq \left\{ \frac{p_{2j}}{p_{2j+1}} : 1 \leq j \leq L - 1 \right\}. \tag{1.12}
\]

Let \(M_\alpha(\infty)\) be defined as in (1.6).

(a) Then there exists a unique positive real number \(\alpha\) such that the spectral radius of \(M_\alpha(\infty)\) is equal to 1. Moreover, if

\[
\frac{p_{2k}}{p_{2L}} \geq \frac{p_{2(k+1)}}{p_{2(L+1)}} \quad \text{for all } k \geq L \text{ and } i \geq 0, \tag{1.13}
\]

then \(d_s = 2\alpha\).

(b) If, in addition, \(M_\alpha(\infty)\) is irreducible, then there exist positive constants \(C_1, C_2\) such that \(C_1 \lambda^n \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^n\) for all sufficiently large \(\lambda\).

Numerical approximations by taking \(r = 1/4, s = 2/3, p_1 = 1/3, \) and \(p_{2k} = p_{2k+1} = 1/4^k\) for all \(k \geq 1\) yield \(d_s = 0.93168\ldots\) (see Example 6.20).

We state some open problems in Section 7. Finally, we include a vector-valued renewal theorem proved by Lau, Wang and Chu [24] in the Appendix for convenience.

2. **Self-similar measures and measures that are essentially of finite type.**

2.1. **Finite type condition and measure type.** Let \(X\) be a compact subset of \(\mathbb{R}^d\) with nonempty interior, and \(\{S_i\}_{i \in \Lambda}\) be an IFS of contractive similitudes on \(X\) with limit set \(K \subseteq \mathbb{R}^d\). If \(\Lambda\) is finite, \(K\) is the unique compact subset satisfying

\[
K = \bigcup_{i \in \Lambda} S_i(K),
\]

called the attractor or self-similar set of the IFS. For IIFSs, \(K\) need not be compact (see [27]).

If \(\Lambda\) is finite, then to each probability vector \((p_i)_{i \in \Lambda}\) (i.e., \(p_i > 0\) and \(\sum_{i \in \Lambda} p_i = 1\)), there corresponds a unique probability measure, called a self-similar measure, satisfying the self-similar identity

\[
\mu = \sum_{i \in \Lambda} p_i \mu \circ S_i^{-1}. \tag{2.1}
\]

Moreover, \(\text{supp}(\mu) = K\). An analogous result, with \(\text{supp}(\mu) = \overline{K}\), holds for IIFSs under additional assumptions (see [27] for IIFSs satisfying (OSC) and Proposition 6.14 for IIFSs studied in this paper). We assume that \(#K \geq 2\). It is well-known that in this case \(\mu\) is continuous, and is in fact of pure type (see, e.g., [35]).

We first extend the finite type condition [16, 22, 33] to IIFSs and then introduce the concept of measure type of an island. The finite type condition for FIFSs is a weaker notion of separation under which the dimension of the attractor can be computed in terms of the spectral radius of some weighted incidence matrix. We remark that the term island is adopted from [2].
We extend the finite type condition to include IIFSs. Define the following sets of indices
\[ \Lambda^k := \{(i_1, \ldots, i_k) : i_j \in \Lambda \text{ for } j = 1, \ldots, k\}, \quad k \geq 1 \]
and \[ \Lambda^* := \bigcup_{k \geq 0} \Lambda^k \]
(with \( \Lambda^0 := \{\emptyset\} \)). We call \( i = (i_1, \ldots, i_k) \in \Lambda^k \) a word of length \( k \), and denote its length by \( |i| \). If no confusion is possible, we will denote \( i = (i_1, \ldots, i_k) \) simply by \( i := i_1 \cdots i_k \); in particular, if \( i_j = i_1 \) for all \( j = 1, \ldots, k \), we write \( i := i_1^k \). For \( k \geq 0 \) and \( i = (i_1, \ldots, i_k) \in \Lambda^k \), we use the standard notation
\[ S_i := S_{i_1} \circ \cdots \circ S_{i_k}, \quad r_i := r_{i_1} \cdots r_{i_k}, \quad p_i := p_{i_1} \cdots p_{i_k} \]
with \( S_0 := \text{id}, r_0 = p_0 := 1 \), where \( \text{id} \) is the identity map on \( \mathbb{R}^d \).

For two indices \( i, j \in \Lambda^* \), we write \( i \preceq j \) if \( i \) is a prefix of \( j \) or \( i = j \), and denote by \( i \not\preceq j \) if \( i \prec j \) does not hold. Let \( \{\mathcal{M}_k\}_{k=1}^\infty \) be a sequence of index sets, where \( \mathcal{M}_k \subseteq \Lambda^* \). Let\[ m_k = m_k(\mathcal{M}_k) := \min\{|i| : i \in \mathcal{M}_k\} \quad \text{and} \quad m_k = m_k(\mathcal{M}_k) := \max\{|i| : i \in \mathcal{M}_k\}. \]
We also let \( \mathcal{M}_0 := \{\emptyset\} \).

**Definition 2.1.** We say that \( \{\mathcal{M}_k\}_{k=0}^\infty \) is a sequence of nested index sets if it satisfies the following conditions:

1. both \( \{m_k\} \) and \( \{m_k\} \) are nondecreasing, and \( \lim_{k \to \infty} m_k = \lim_{k \to \infty} m_k = \infty \);
2. for each \( k \geq 1 \), \( \mathcal{M}_k \) is an antichain in \( \Lambda^* \);
3. for each \( j \in \Lambda^* \) with \( |j| > m_k \) or \( j \in \mathcal{M}_{k+1} \), there exists \( i \in \mathcal{M}_k \) such that \( i \preceq j \);
4. for each \( j \in \Lambda^* \) with \( |j| < m_k \) or \( j \in \mathcal{M}_{k-1} \), there exists \( i \in \mathcal{M}_k \) such that \( j \preceq i \);
5. there exists a positive integer \( L_0 \), independent of \( k \), such that for all \( i \in \mathcal{M}_k \) and \( j \in \mathcal{M}_{k+1} \) with \( i \not\preceq j \), we have \(|j| - |i| \leq L_0 \).

Condition (2) means that the indices in \( \mathcal{M}_k \) are incomparable. We also remark that (4) actually follows from (3). Clearly, by letting \( \mathcal{M}_k = \Lambda^k \) for all \( k \geq 0 \), we obtain an example of a sequence of nested index sets.

To define neighborhood types, we fix a sequence of nested index sets \( \{\mathcal{M}_k\}_{k=0}^\infty \). For each integer \( k \geq 0 \), let \( \mathcal{V}_k \) be the set of level-\( k \) vertices (with respect to \( \{\mathcal{M}_k\} \)) defined as\[ \mathcal{V}_0 := \{(id, 0)\} \quad \text{and} \quad \mathcal{V}_k := \{(S_i, k) : i \in \mathcal{M}_k\} \quad \text{for all } k \geq 1. \]
We call \((id, 0)\) the root vertex and denote it by \( \nu_{\text{root}} \). Let \( \mathcal{V} := \bigcup_{k \geq 0} \mathcal{V}_k \) be the set of all vertices. For \( \nu = (S_i, k) \in \mathcal{V}_k \), we use the convenient notation \( S_{\nu} := S_i \) and \( r_{\nu} := r_i \). Note that it is possible to have \( \nu = (S_i, k) = (S_j, k) \) with \( i \neq j \). More generally, for any \( k \geq 0 \) and any subset \( A \subset \mathcal{V}_k \), we use the notation\[ S_A(\Omega) := \bigcup_{\nu \in A} S_{\nu}(\Omega). \]

Let \( \Omega \subseteq X \) be a nonempty open set which is invariant under \( \{S_i\}_{i \in \Lambda} \), i.e., \( \bigcup_{i \in \Lambda} S_i(\Omega) \subseteq \Omega \). Such an \( \Omega \) exists by our assumption; in particular, \( X^0 \) is such a set. Two level-\( k \) vertices \( \nu, \nu' \in \mathcal{V}_k \) (allowing \( \nu = \nu' \)) are said to be neighbors (with respect to \( \Omega \) and \( \{\mathcal{M}_k\} \)) if \( S_{\nu}(\Omega) \cap S_{\nu'}(\Omega) \neq \emptyset \). We call the set of vertices \( \mathcal{N}(\nu) = \mathcal{N}_\Omega(\nu) := \{\nu' : \nu' \in \mathcal{V}_k \text{ is a neighbor of } \nu \} \)
the neighborhood of \( v \) (with respect to \( \Omega \) and \( \{ M_k \} \)). Note that \( v \in \mathcal{R}_\Omega(v) \) by definition.

Define an equivalence relation on the set of vertices \( V \). Let \( \mathcal{S} := \{ S_j S_i^{-1} : i, j \in \Lambda^* \} \). Two vertices \( v \in V_k \) and \( v' \in V_{k'} \) are said to be equivalent, denoted \( v \sim v' \) (or simply \( v \sim v' \)), if for \( \tau := S_v S_{v'}^{-1} (\in \mathcal{S}) : \bigcup_{u \in \mathcal{R}(v)} S_u(X) \to X \), the following conditions hold:

1. \( \{ S_u : u \in \mathcal{R}(v') \} = \{ \tau S_u : u \in \mathcal{R}(v) \} \); in particular, \( \tau S_u \) is defined for all \( u \in \mathcal{R}(v) \).
2. for \( u \in \mathcal{R}(v) \) and \( u' \in \mathcal{R}(v') \) such that \( S_{u'} = \tau S_u \), and for any positive integer \( \ell \geq 1 \), an index \( i \in \Lambda^* \) satisfies \( (S_u S_i, k + \ell) \in V_{k+\ell} \) if and only if it satisfies \( (S_{u'} S_i, k' + \ell) \in V_{k'+\ell} \).

It is straightforward to show that \( \sim \) is an equivalence relation. We denote the equivalence class containing \( v \) by \([v]\) and call it the \((\text{neighborhood})\) type of \( v \) (with respect to \( \Omega \) and \( \{ M_k \} \)). Condition (1) is needed in showing that equivalent vertices generate the same number of offspring of each neighborhood type, as shown in Proposition 2.3.

We define an infinite graph \( G \) with vertex set \( V \) and directed edges as follows. Let \( v \in V_k \) and \( u \in V_{k+1} \). Suppose there exists \( i \in M_k \), \( j \in M_{k+1} \), and \( l \in \Lambda^* \) such that

\[
\begin{align*}
v &= (S_i, k), \\
u &= (S_j, k+1), \\
j &= (i, l).
\end{align*}
\]

Then we connect a directed edge \( l : v \to u \). We call \( v \) a parent of \( u \) and \( u \) an offspring of \( v \). We write \( G = (V, E) \), where \( E \) is the set of all directed edges defined above. We call \( v = (S_i, k) \) a predecessor of \( u = (S_j, k') \), and \( u \) a descendant of \( v \), if \( i \preceq j \) and \( k' \geq k + 1 \).

**Remark 2.2.** Only vertices in \( \mathcal{R}(v) \) can be parents of any offspring of \( v \) in \( G \) (see [22, Remark 2.3]).

**Proposition 2.3.** For two equivalent vertices \( v \in V_k \) and \( v' \in V_{k'} \), let \( \{ u_i \}_{i \in A_1} \) and \( \{ u'_i \}_{i \in A'_1} \) be the offspring of \( v \) and \( v' \) in \( G \), respectively. Then, counting multiplicity,

\[
\{ u_i : i \in A_1 \} = \{ u'_i : i \in A'_1 \}.
\]

In particular, \( \# A_1 = \# A'_1 \).

**Proof.** The proof is similar to that of [14, Proposition 2.4(b)]; we give an outline as it is needed for the proof of Proposition 2.7. Let \( \mathcal{R}(v) := \{ v_j : j \in A_2 \} \) and \( \mathcal{R}(v') := \{ v'_j : j \in A'_2 \} \), with \( A'_2 = A_2 \) as \( v \) and \( v' \) are equivalent vertices. Assume that \( S_{v'_j} = \tau \circ S_{v_j} \) for all \( j \in A_2 \).

**Step 1.** Let \( i, j \in A_1 \) and assume that \( i_1, i_2 \in E \) such that \( v_i \xrightarrow{i_1} u, \ v_j \xrightarrow{i_2} w, \ v'_i \xrightarrow{i_1} u', \ v'_j \xrightarrow{i_2} w' \). One can show that \( u = w \) if and only if \( u' = w' \), and \( u, w \) are neighbors if and only if \( u', w' \) are.

**Step 2.** Let \( U \) and \( U' \) be the sets of offspring of the vertices in \( \mathcal{R}(v) \) and \( \mathcal{R}(v') \) respectively. Define a map \( \hat{\tau} : U \to U' \) as follows. Suppose \( u \) is an offspring of \( v_j \) in \( G \) by an edge \( i \) for some \( j \in A_2 \). Then let \( \hat{\tau}(u) \) be the offspring of \( v'_j \) by the edge \( i \). The definition of \( \sim \) and Step 1 imply that \( \hat{\tau} \) is well defined and bijective.

Moreover, Step 1 also shows that \( u \) is an offspring of \( v \) in \( G \) if and only if \( \hat{\tau}(u) \) is an offspring of \( v' \) in \( G \). Combining this with Remark 2.2, we have \([u_i] = [\hat{\tau}(u_i)]\), which completes the proof. \( \square \)
Example 2.5. If $\{S_i\}_{i \in \Lambda}$ satisfies (OSC), then it is of finite type. In fact, $V/\sim$ consists of just one element (see [22, Example 2.5]).

Definition 2.6. A subset $I \subseteq V_k$ is called a level-$k$ island (with respect to $\Omega$ and \{M_k\}) if the following conditions hold:

1. for any two vertices $v, v' \in I$ (allowing $v = v'$), there exists a finite family of vertices $v_0, v_1, \ldots, v_n$ such that $v_0 = v$, $v_n = v'$, and $S_{v_i}(\Omega) \cap S_{v_{i+1}}(\Omega) \neq \emptyset$ for all $i = 0, \ldots, n - 1$;
2. for any $u \in V_k \setminus I$ and any $v \in I$, $S_u(\Omega) \cap S_v(\Omega) = \emptyset$.

Intuitively, if $\Omega$ is connected, then for each level-$k$ island $I$, $S_\Omega(\Omega)$ is a connected component of $S_{V_k}(\Omega)$. Note that for each $v \in V_k$, there exists a unique island, denoted $I(v)$, containing $\Omega(v)$. Clearly, if $\{S_i\}_{i \in \Lambda}$ satisfies (OSC) with $\Omega$ being an OSC-set, then $I(v) = \{v\}$ for all $v \in V$. Let

$$I_k := \{I : I \text{ is a level-$k$ island}\} \quad \text{and} \quad \mathcal{I} := \bigcup_{k \geq 0} I_k$$

be the collection of all level-$k$ islands and the collection of all islands, respectively.

Generalizing (2.2), for any $k \geq 0$ and any subset $B \subseteq I_k$, we use the notation

$$S_B(\Omega) := \bigcup_{I \in B} S_I(\Omega). \quad (2.3)$$

We say that two islands $I \in \mathcal{I}_k$ and $I' \in \mathcal{I}_{k'}$ are equivalent, and denote it by $I \approx_{\tau} I'$ (or simply $I \approx I'$), if there exists some $\tau \in \mathcal{I}$ such that $\{S_{v'} : v' \in I'\} = \{\tau S_v : v \in I\}$ and, moreover, $v \sim_{\tau} v'$ for any $v \in I$ and $v' \in I'$ satisfying $S_{v'} = \tau S_v$. We denote the equivalence class of $I$ by $[I]$ and we call $[I]$ the (island) type of $I$.

An island $I \in \mathcal{I}_k$ is said to be a parent of an island $I' \in \mathcal{I}_{k+1}$ and $I'$ an offspring of $I$, if each $v' \in I'$ has a parent in $I$. For any $k \geq 0$ and $I \in \mathcal{I}_k$, let

$$O(I) := \{J : J \text{ is an offspring of } I\} \quad (2.4)$$

be the collection of all offspring of $I$. Analogously, we define predecessor and descendant of an island.

Let $\mu$ be a self-similar measure defined by an IFS $\{S_i\}_{i \in \Lambda}$ of finite type with $\Omega$ being an FTC-set. Two equivalent vertices $v \in V_k$ and $v' \in V_{k'}$ are $\mu$-equivalent, denoted $v \sim_{\mu, \tau, w} v'$ (or simply $v \sim_{\mu} v'$), if for $\tau = S_{v'} \circ S_v^{-1}$, there exists a number $w > 0$ such that

$$\mu_{S_{\Omega(v)}(\Omega)} = w \cdot \mu_{S_{\Omega(v')}(\Omega)} \circ \tau^{-1}.$$ 

As $\sim$ is an equivalence relation, so is $\sim_{\mu}$. Denote the $\mu$-equivalence class of $v$ by $[v]_\mu$ and call it the (neighborhood) measure type of $v$ (with respect to $\Omega$, $\{M_k\}$ and $\mu$). Intuitively, $v \sim_{\mu} v'$ means that the measures $\mu_{S_{\Omega(v)}(\Omega)}$ and $\mu_{S_{\Omega(v')} (\Omega)}$ have the same structure. The following proposition shows that $\mu$-equivalent vertices generate the same number of offspring of each neighborhood measure type.
Proposition 2.7. Assume the hypotheses of Proposition 2.9. If \([v]_\mu = [v']_\mu\), then, counting multiplicity, \(\{[u]_\mu : i \in \Lambda_1\} = \{[u']_\mu : i \in \Lambda_1\}\).

Proof. Let \(u\) and \(u'\) be offspring of \(v\) and \(v'\) in \(\mathcal{G}\) by an edge \(i\), respectively. By the proof of Proposition 2.3, we have \(u \sim_\tau u'\), where \(\tau := S_{v'} \circ S_{u}^{-1} = S_{v'} \circ S_{v}^{-1} \in \mathcal{S}\). Thus it suffices to show that \(u \sim_\mu u'\). Since \([v]_\mu = [v']_\mu\), there exists \(w > 0\) such that \(\mu|_{S_{\tau}(\omega)}(\Omega) = w \cdot \mu|_{S_{\tau}(\omega)}(\Omega) \circ \tau^{-1}\). It follows that

\[
\mu|_{S_{\tau}(\omega)}(\Omega) = w \cdot \mu|_{S_{\tau}(\omega)}(\Omega) \circ \tau^{-1},
\]

since \(S_{\tau}(\omega)\subseteq S_{\tau'}(\omega)\) and \(u \sim_\tau u'\). Hence, \([u]_\mu = [u']_\mu\), completing the proof. \(\square\)

Definition 2.8. Let \(\mu\) be a self-similar measure defined by a finite type IFS \(\{S_i\}_{i \in \Lambda}\) on \(\mathbb{R}^d\) with \(\Omega\) being an FTC-set. Two islands \(I, I' \in \mathbb{I}_k\) are said to be \(\mu\)-equivalent, denoted \(I \approx_{\mu, \tau, w} I'\) (or simply \(I \approx_{\mu} I'\)), if \(I \approx_{\tau} I'\) for some \(\tau \in \mathcal{S}\) and there exists some \(w > 0\) such that

\[
\mu|_{S_{\tau}(\omega)}(\Omega) = w \cdot \mu|_{S_{\tau}(\omega)}(\Omega) \circ \tau^{-1}.
\]

We remark that (2.5) holds if and only if \(v \sim_{\mu, \tau, w} v'\) for any \(v \in I\) and \(v' \in I'\) satisfying \(S_{v'} = \tau S_{v}\). We note that \(\approx_{\mu}\) is an equivalence relation. We denote the \(\mu\)-equivalence class of \(I\) by \([I]_\mu\), and call \([I]_\mu\) the (island) measure type of \(I\) (with respect to \(\Omega\), \(\{\mathbb{I}_k\}\) and \(\mu\)). From the definition of \(\approx_{\mu}\), we obtain an analog of Proposition 2.7 concerning \(\approx_{\mu}\). That is, \(\mu\)-equivalent islands generate the same number of offspring of each island measure type.

Definition 2.9. Let \(\mu\) be a self-similar measure defined by a finite type IFS \(\{S_i\}_{i \in \Lambda}\) on \(\mathbb{R}^d\) with \(\Omega\) being an FTC-set. Let \(\mathbb{B} \subseteq \mathbb{I}_k\) for some \(k \geq 0\) and \(\mathbb{B}_\mu := \{[I]_\mu : I \in \mathbb{B}\}\). We call \(\mathbb{I}\) a level-2 nonbasic island with respect to \(\mathbb{B}\) if \(I \in O(\mathcal{J})\) for some \(\mathcal{J} \in \mathbb{B}\) and \([I]_\mu \notin \mathbb{B}_\mu\). Inductively, for \(\ell \geq 3\), we call \(\mathbb{I}\) a level-\(\ell\) nonbasic island with respect to \(\mathbb{B}\) if \(\mathbb{I}\) is an offspring of some level-\((\ell - 1)\) nonbasic island with respect to \(\mathbb{B}\) and \([I]_\mu \notin \mathbb{B}_\mu\).

We remark that, by definition, for any \(\ell \geq 2\), \(\mathbb{I}\) is a level-\(\ell\) nonbasic island with respect to \(\mathbb{B}\) if and only if \([I]_\mu \notin \mathbb{B}_\mu\), and there exists a finite sequence of islands \(I_1, \ldots, I_{\ell-1}\) such that \(I_i \in \mathbb{B}\), \(I_i \in \mathcal{O}(I_{i-1})\) is a level-\(i\) nonbasic island with respect to \(\mathbb{B}\) for all \(i = 2, \ldots, \ell - 1\), and \(I \in \mathcal{O}(I_{\ell-1})\).

Analogously, we define the equivalence and \(\mu\)-equivalence of two subsets \(\mathbb{B} \subseteq \mathbb{I}_k\) and \(\mathbb{B}' \subseteq \mathbb{I}_{k'}\), denoted \(\mathbb{B} \approx_{\tau} \mathbb{B}'\) (or simply \(\mathbb{B} \approx \mathbb{B}'\)) and \(\mathbb{B} \approx_{\mu, \tau, w} \mathbb{B}'\) (or simply \(\mathbb{B} \approx_{\mu} \mathbb{B}'\)), respectively. Moreover, we also denote the equivalence and \(\mu\)-equivalence class of \(\mathbb{B}\) by \([\mathbb{B}]\) and \([\mathbb{B}]_\mu\), respectively.

2.2. Measures essentially of finite type. Let \(\Omega \subseteq \mathbb{R}^d\) be a bounded open subset and \(\mu\) a positive finite Borel measure with \(\text{supp}(\mu) \subseteq \overline{\Omega}\) and \(\mu(\Omega) > 0\). We call a \(\mu\)-measurable subset \(U \subseteq \Omega\) a cell (in \(\Omega\)) if \(\mu(U) > 0\). Clearly, \(\Omega\) itself is a cell. Two cells \(U, V \subseteq \Omega\) are measure disjoint with respect to \(\mu\) if \(\mu(U \cap V) = 0\).

We say that two cells \(U\) and \(V\) are \(\mu\)-equivalent, denoted \(U \approx_{\mu, \tau, w} V\) (or simply \(U \approx_{\mu} V\)), if there exist some similitude \(\tau : U \to V\) and some constant \(w > 0\) such that \(\tau(U) = V\) and

\[
\mu|_V = w \mu|_U \circ \tau^{-1}.
\]

It is easy to check that \(\approx_{\mu}\) is an equivalence relation.

Let \(U \subseteq \Omega\) be a cell. We call a finite family \(\mathbf{P}\) of measure disjoint cells a \(\mu\)-partition of \(U\) if \(V \subseteq U\) for all \(V \in \mathbf{P}\), and \(\mu(U) = \sum_{V \in \mathbf{P}} \mu(V)\). A sequence of \(\mu\)-partitions \((\mathbf{P}_k)_{k \geq 1}\) is refining if for any \(V \in \mathbf{P}_k\) and any \(W \in \mathbf{P}_{k+1}\), either
Remark 2.10. Let $\mu$ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i \in \Lambda}$ on $\mathbb{R}^d$ with $\Omega$ being an FTC-set. The following can be verified directly.

(a) For any island $I \in \mathbb{N}$, $S_I(\Omega)$ is a cell.
(b) Let $I$ and $I'$ be two islands. By definition, we have $I \approx \mu I'$ if and only if $I \approx \mu I$ and $S_I(\Omega) \approx \mu S_{I'}(\Omega)$.
(c) Let $k \geq m \geq 0$. Then for any $I \in \mathbb{N}_m$, $P := \{S_I(\Omega) : J \in \mathbb{N}_k$ is a descendent of $I\}$ is a $\mu$-partition of $S_I(\Omega)$.

Let $B := \{B_{1,\ell} : \ell \in \Gamma\}$ be a finite family of measure disjoint cells in $\Omega$, and for each $\ell \in \Gamma$, let $(P_{k,\ell})_{k \geq 1}$ be a family of refining $\mu$-partitions of $B_{1,\ell}$ with $P_{1,\ell} := \{B_{1,\ell}\}$. We divide each $P_{k,\ell}$, $k \geq 2$, into two (possibly empty) subcollections, $P_{k,\ell}^1$ and $P_{k,\ell}^2$, with respect to $B$, defined as follows:

$$
\begin{align*}
P_{k,\ell}^1 &:= \{B \in P_{k,\ell} : B \approx \mu B_{1,\ell} \text{ for some } i \in \Gamma\}, \\
P_{k,\ell}^2 &:= P_{k,\ell} \setminus P_{k,\ell}^1 = \{B \in P_{k,\ell} : B \notin P_{k,\ell}^1\}.
\end{align*}
$$

Definition 2.11. We say that a positive finite Borel measure $\mu$ on $\mathbb{R}^d$ is essentially of finite type (EFT) if there exist a bounded open subset $\Omega \subseteq \mathbb{R}^d$ with $\text{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$, and a finite family $B := \{B_{1,\ell} : \ell \in \Gamma\}$ of measure disjoint cells in $\Omega$ such that for any $\ell \in \Gamma$, there is a family of refining $\mu$-partitions $(P_{k,\ell})_{k \geq 1}$ of $B_{1,\ell}$ satisfying the following conditions:

1. $P_{1,\ell} = \{B_{1,\ell}\}$, and there exists some $B \in P_{2,\ell}^1$ such that $B \neq B_{1,\ell}$;
2. if for some $k \geq 2$, there exists some $B \in P_{k,\ell}^1$, then $B \in P_{k+1,\ell}$ and hence $B \in P_{m,\ell}$ for all $m \geq k$;
3. $\lim_{k \to \infty} \sum_{B \in P_{k,\ell}^2} \mu(B) = 0$.

Here $P_{k,\ell}^1$ and $P_{k,\ell}^2$ ($k \geq 2$) are defined as in (2.7). In this case, we call $\Omega$ an EFT-set, $B$ a basic family of cells (in $\Omega$), and $(B, P) := (\{B_{1,\ell}\}, (P_{k,\ell} : k \geq 1)_{\ell \in \Gamma})$ a basic pair (with respect to $\Omega$).

Remark 2.12. (a) By definition, we see that if $\mu$ satisfies (EFT) with $\Omega$ being an EFT-set and with $(B, P) = (\{B_{1,\ell}\}, (P_{k,\ell})_{k \geq 1})_{\ell \in \Gamma}$ being a basic pair (with respect to $\Omega$), then for any $\Omega' \subseteq \mathbb{R}^d$ with $\Omega \subseteq \Omega'$, $\mu$ satisfies (EFT) with $\Omega'$ being an EFT-set and $(B, P)$ a basic pair (with respect to $\Omega'$).
(b) We remark that conditions (1) and (2) are needed in Section 4 to derive the vector-valued renewal equation, and error estimate forces condition (3) to hold. In fact, in Section 4, we only need condition (2) as well as (1') the existence of some $B \in \bigcup_{k \geq 2} P_{k,\ell}^1$ such that $B \neq B_{1,\ell}$. Since condition (3) implies that $\bigcup_{k \geq 2} P_{k,\ell}^1 \neq \emptyset$, we have chosen to use the more convenient condition (1).
(c) Let $(B, P) := (\{B_{1,\ell}\}, (P_{k,\ell} : k \geq 1))_{\ell \in \Gamma}$ be a basic pair. Then for any $k \geq 2$, $P_{2,\ell}^2 = \emptyset$ if and only if $P_{m,\ell} = P_{k,\ell}$ for all $m \geq k$.
(d) For any $k \geq 2$ and any $B \in P_{k,\ell}^2$, $P_{k+1,\ell} \setminus P_{k,\ell}^1$ contains a unique $\mu$-partition of $B$.

Definition 2.13. Assume that $\mu$ satisfies (EFT) with $\Omega$ being an EFT-set and $(B, P) := (\{B_{1,\ell}\}, (P_{k,\ell} : k \geq 1))_{\ell \in \Gamma}$ being a basic pair with respect to $\Omega$. We say that $(B, P)$ is regular if each cell $B \in \bigcup_{k \geq 1, \ell \in \Gamma} P_{k,\ell}$ is connected, and for any $\ell \in \Gamma$, there exist some similitude $\tau_\ell$ and some constant $w(\ell) > 0$ such that $\tau_\ell(\Omega) \subseteq B_{1,\ell}$.
and $\mu \geq w(\ell)\mu \circ \tau_\ell^{-1}$ on $\tau_\ell(\Omega)$. In this case, we call $B$ a regular basic family of cells (in $\Omega$).

We remark that in one-dimension, the regularity of $B$ ensures that the eigenvalue counting function of the Laplacian on $\Omega$ behaves the same as that on each cell in $B$ (see Proposition 4.5).

**Proposition 2.14.** Let $\{S_i\}_{i \in \Lambda}$ be a finite type IFS on $\mathbb{R}^d$, $(p_i)_{i \in \Lambda}$ be a probability vector, and $\mu$ be the associated self-similar measure. Then for any $I \in I$, there exist some similitude $\tau$ and some constant $w > 0$ such that $\tau(\Omega) \subseteq S_I(\Omega)$ and $\mu \geq w\mu \circ \tau^{-1}$ on $\tau(\Omega)$.

**Proof.** Fix any $I \in I$. Let $v \in I$. Then $S_v(\Omega) \subseteq S_I(\Omega)$. Since $\mu = \sum_{i \in \Lambda} p_i \mu \circ S_i^{-1}$, we have $\mu|_{S_v(\Omega)} \geq p_v \mu|_{\Omega} \circ S_v^{-1}$. Thus the result follows.

As measures studied in this paper are mainly self-similar, we give a sufficient condition for FIFSs to satisfy (EFT). An analog for IIFSs will be given in Proposition 6.4.

**Proposition 2.15.** Let $\mu$ be a self-similar measure defined by a finite type FIFS on $\mathbb{R}^d$ with a connected FTC-set $\Omega$. Suppose there exists some $m \geq 0$ such that the following two conditions hold.

1. There exists a finite index set $\Gamma$ such that $\mathbb{I}_m = \{I_{I,\ell} : \ell \in \Gamma\}$; moreover, for each $\ell \in \Gamma$, there exist some constant $c(\ell) \geq 2$ (chosen to be the minimum) and descendant $J \in \mathbb{I}_{m+c(\ell)-1}$ of $I_{I,\ell}$ satisfying $S_J(\Omega) \neq S_{I_{I,\ell}}(\Omega)$ and $J \simeq I_{I,i}$ for some $i \in \Gamma$.

2. For $k \geq 2$, let $I_k$ be the collection of all level-$k$ nonbasic islands with respect to $\mathbb{I}_m$. Then $\lim_{k \to \infty} \sum_{I \in I_k} \mu(S_I(\Omega)) = 0$.

Then $\mu$ satisfies (EFT) with $\Omega$ being an EFT-set and with $B := \{S_{I_{I,\ell}}(\Omega) : \ell \in \Gamma\}$ being a regular basic family of cells in $\Omega$.

**Proof.** Let $B_{I,\ell} := S_{I_{I,\ell}}(\Omega)$ for all $\ell \in \Gamma$. Then $B = \{B_{I,\ell} : \ell \in \Gamma\}$. Definition 2.6 implies that any two distinct cells in $B$ are measure disjoint. Fix any $\ell \in \Gamma$. If for some $k \geq 2$, $P_{k,\ell}$ is a well-defined $\mu$-partition of $B_{I,\ell}$, then we let $P_{k,\ell}$ be defined as in (2.7) with respect to $B$. Define

$$P_{1,\ell} := \{B_{I,\ell}\} \quad \text{and} \quad P_{2,\ell} := \{S_J(\Omega) : J \in \mathbb{I}_{m+c(\ell)-1} \text{ is a descendant of } I_{I,\ell}\}. \quad (2.8)$$

Remark 2.10(c) implies that $P_{2,\ell}$ is a $\mu$-partition of $B_{I,\ell}$. Moreover, by assumption (1) above and Remark 2.10(b), we see that there exists some $B := S_J(\Omega) \in P_{2,\ell}$ such that $B \neq B_{I,\ell}$ and $B \simeq B_{I,i}$ for some $i \in \Gamma$. Thus $B \in P_{2,\ell}$, and hence condition (1) of (EFT) holds for $\ell$. For $k \geq 3$, if $P_{k-1,\ell} = \emptyset$, define $P_{k,\ell} := P_{k-1,\ell}$; otherwise, define

$$P_{k,\ell} := P_{k-1,\ell} \bigcup \{S_J(\Omega) : J \in O(\mathcal{I}) \text{ for some island } \mathcal{I} \text{ satisfying } S_J(\Omega) \in P_{2,\ell} \} \quad (2.9)$$

Similarly, Remark 2.10(c) implies that $(P_{k,\ell})_{k \geq 1}$ is a family of refining $\mu$-partitions of $B_{I,\ell}$. By (2.9), condition (2) of (EFT) holds for $\ell$. If $B \in P_{k,\ell}$ for some $k \geq 2$, then by the definition of $P_{k,\ell}$, there exists a sequence of islands $(\mathcal{I}_i)_{i=1}^{k}$ such that $S_{\mathcal{I}_1}(\Omega) \in P_{2,\ell}$ and $\mathcal{I}_{i+1} \in O(\mathcal{I}_i)$ for all $i = 2, \ldots, k-1$, and $B = S_{\mathcal{I}_k}(\Omega)$. Thus for any $i = 2, \ldots, k$, $\mathcal{I}_i$ is not $\mu$-equivalent to any island in $\mathbb{I}_m$. By the minimality of $c(\ell)$, $\mathcal{I}_2$ is a level-$c(\ell)$ nonbasic island with respect to $\mathbb{I}_m$. It follows that for all $i = 2, \ldots, k$, $\mathcal{I}_i$ is a level-$c(\ell)+i-2$ nonbasic island with respect to $\mathbb{I}_m$. 

Hence, \( P_{k,\ell}^2 \subseteq \{ S_I(\Omega) : I \in I_{(\ell+j)+k-2} \} \). Condition (3) of (EFT) now follows from assumption (2). Hence, \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set and \( B \) being a basic family of cells. Since \( \Omega \) is connected, each cell \( B \in \bigcup_{k,\ell \in I} P_{k,\ell} \) is connected. Thus the regularity of \( B \) follows from Proposition 2.14.

3. Examples for (EFT). We illustrate (EFT) by the following four classes of examples. It is well known that for any FIFS \( \{ S_i \}_{i \in \Lambda} \) on \( \mathbb{R}^d \) satisfying (OSC), there always exists an OSC-set \( \Omega \) that has nonempty intersection with the corresponding self-similar set (see [36]), and thus any associated self-similar measure \( \mu(\Omega) > 0 \). This is not true for IFSs (see [39]).

**Example 3.1.** Let \( \mu \) be a self-similar measure defined by an FIFS \( \{ S_i \}_{i \in \Lambda} \) on \( \mathbb{R}^d \) satisfying (OSC) with respect to an open set \( \Omega \) with \( \mu(\Omega) > 0 \). Then \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set and \( B := \{ B_{1,0} \} = \{ \Omega \} \) being a basic family of cells.

**Proof.** Define \( P_{1,0} := \{ B_{1,0} \} \) and \( P_{2,0} := \{ S_v(\Omega) : v \in V_1 \} \). It is obvious that \( S_v(\Omega) \subseteq \Omega \) for any \( v \in V_1 \). Combining this with (OSC), we see that \( P_{2,0} \) is a \( \mu \)-partition of \( \Omega \) and \( S_v(\Omega) \simeq \mu \Omega \) for all \( v \in V_1 \). Let \( \mu_{1,0} \) and \( \mu_{2,0} \) be defined as in (2.7). Then \( \mu_{2,0} = \mu_{2,0} \) and \( P_{2,0} = \emptyset \). Since \( \#V_1 = \#\Lambda \geq 2 \), condition (1) of (EFT) holds. Define \( P_{k,0} := P_{2,0} \) for all \( k > 2 \). Hence, \( \mu_{k,0} \) satisfies all the conditions of (EFT).

Our second example is the IFS defining the infinite Bernoulli convolution associated with the golden ratio, defined as:

\[
S_1(x) = \rho x, \quad S_2(x) = \rho x + (1 - \rho), \quad \rho = \frac{\sqrt{5} - 1}{2}.
\]

(3.10)

The corresponding self-similar identity is

\[
\mu = \frac{1}{2} \mu \circ S_1^{-1} + \frac{1}{2} \mu \circ S_2^{-1}.
\]

(3.11)

Strichartz et al. [38] showed that \( \mu \) satisfies a family of second-order identities with respect to the following auxiliary IFS:

\[
T_0(x) = \rho^2 x, \quad T_1(x) = \rho^3 x + \rho^2, \quad T_2(x) = \rho^2 x + \rho.
\]

(3.12)

The spectral dimension of \( \mu \) is computed in [31].

**Example 3.2.** Let \( \mu \) be the self-similar measure defined by the IFS in (3.10) and (3.11). Let \( \{ T_i \}_{i=0}^k \) be defined as in (3.12). Then \( \mu \) satisfies (EFT) with \( \Omega = (0, 1) \) being an EFT-set and with \( B := \{ T_1(\Omega) \} \) being a regular basic family of cells.

**Proof.** The result is a direct consequence of [31, Section 5]. Define \( P_{1,1} := \{ T_1(\Omega) \} \), \( P_{2,1} := \{ T_{11}(\Omega), i = 0, 1, 2 \} \), and

\[
P_{k,1} := \bigcup_{j=0}^{k-2} \{ T_{1i}(\Omega) : i \in \{0, 2\}^j \} \bigcup \{ T_{1i}(\Omega) : i \in \{0, 2\}^{k-1} \}
\]

for \( k \geq 3 \).

Thus, any two elements of \( P_{2,1} \) are measure disjoint, and \( V \subseteq T_1(\Omega) \) for any \( V \in P_{2,1} \). For \( k \geq 3 \), any \( V \in P_{k,1} \) is either of the form \( V = T_{1i}(\Omega) \) for some \( i \in \{0, 2\}^j \) and \( j \in \{0, 1, \ldots, k-2\} \), or of the form \( V = T_{1i}(\Omega) \) for some \( i \in \{0, 2\}^{k-1} \). Hence \( \mu \)-partitions of \( T_1(\Omega) \). If we let \( P_{k,1} \) be defined as
in (2.7) with respect to \( B \) for \( k \geq 2 \) and \( i = 1, 2 \), then it follows from [31, Section 5] that

\[
P_{k,1}^1 = \bigcup_{j=0}^{k-2} \{ T_{1i1}(\Omega) : i \in \{0,2\}^j \} \quad \text{and} \quad P_{k,1}^2 = \{ T_{1i}(\Omega) : i \in \{0,2\}^{k-1} \}.
\]

Conditions (1) and (2) of (EFT) are clearly satisfied. It follows from [21, Proposition 2.4(ii)] that \( \lim_{k \to \infty} \sum_{B \in P_{k,1}^2} \mu(B) = 0 \), and thus condition (3) of (EFT) holds. Consequently, \( \mu \) satisfies (EFT) with \( \Omega = (0, 1) \) being an EFT-set and with \( (B, P) := (\{ T_{1i}(\Omega) \}, (P_{k,1})_{k \geq 1}) \) being a basic pair. Finally, the self-similar identity implies that for all Borel subsets \( A \subseteq T_{1}(\Omega), \mu(A) \geq (1/8) \mu \circ T_{1}^{-1}(A) \). Thus \( B \) is regular.

We now consider the family of FIFSs in (1.9). It is known (see [16,22]) that each IFS in the family is of finite type with FTC-set \( \Omega = (0, 1) \) and \( \mathcal{M}_k = \Lambda_k \).

**Example 3.3.** Let \( (S_i)_{i=1}^3 \) be an FIFS in (1.9) (see Figure 1), \( (p_i)_{i=1}^3 \) be a probability vector, and \( \mu \) be the associated self-similar measure. Then \( \mu \) satisfies (EFT) with \( \Omega = (0, 1) \) being an EFT-set and there exists a regular basic pair with respect to \( \Omega \).

![Figure 1. First iteration of an IFS \{S_i\}_{i=1}^3 in (1.9), drawn by using \( r_1 = 1/3 \) and \( r_2 = 2/7 \).](image-url)

To prove Example 3.3, we need some propositions and lemmas. Define

\[
I_{1,0} := \{ (S_3, 1) \} \quad \text{and} \quad I_{1,1} := \{ (S_1, 1), (S_2, 1) \},
\]

and \( W_k := \{ 2^{k-i}13^i : i = 0, 1, \ldots, k \} \) for all \( k \geq 1 \) (see Figure 2). We remark that \( \mathbb{I}_1 = \{ I_{1,0}, I_{1,1} \} \). Proposition 3.4(a) below implies that all multi-indices in \( W_k \) correspond to the same vertex. We summarize without proof some elementary properties.

**Proposition 3.4.** Let \( (S_i)_{i=1}^3 \) be defined as in (1.9) and \( \Omega = (0, 1) \).

(a) Then \( S_{13} = S_{21} \). Moreover, \( S_i = S_j \) for all \( k \geq 1 \) and any \( i, j \in W_k \).

(b) \( S_1(\Omega) \cap S_{2k}(\Omega) = S_{2^k+1}(\Omega) \) for \( k \geq 1 \).

Let \( (S_i)_{i=1}^3 \) be an FIFS in (1.9), \( (p_i)_{i=1}^3 \) be a probability vector, and \( \mu \) be the associated self-similar measure. Define

\[
w_1(k) := p_1 \sum_{i=0}^{k} p_2^{k-i} p_3^i \quad \text{for} \quad k \geq 0.
\]

We remark that for \( k \geq 0 \),

\[
p_1 p_3^{k+1} + p_2 w_1(k) = p_1 p_2^{k+1} + p_3 w_1(k) = w_1(k+1) \quad \text{and} \quad w_1(k+1) \leq w_1(k) \leq p_1.
\]
For any $k \geq 1$, $w_1(k)$ denotes the sum of probability weights of all multi-indices in $W_k$, as can be seen from part (b) of the lemma below.

**Lemma 3.5.** Assume the hypotheses of Example 3.3 and let $\Omega = (0, 1)$. Define
\[
B_{1,i} := S_{I_{1,i}}(\Omega) \quad \text{for } i = 0, 1,
\]
where $I_{1,i}$ is defined in (3.13). Let $w_1(k)$ be defined as in (3.14). Then
\[
(a) \text{ for } k \geq 1, \mu|_{S_{2^k}(B_{1,0})} = p_2^k \cdot \mu|_{B_{1,0}} \circ S_{2^{k-1}}^{-1}; \\
(b) \text{ for } k \geq 0, \mu|_{S_{2^k}(B_{1,1})} = w_1(k) \cdot \mu|_{B_{1,1}} \circ S_{2^{k-1}}^{-1}; \\
(c) \text{ for } k \geq 1, \mu|_{S_{2^k}(B_{1,1})} = w_1(k-1)\mu|_{B_{1,0}} \circ S_{2^{k-1}}^{-1} + p_2^k \mu|_{B_{1,1}} \circ S_{2^{k-1}}^{-1}.
\]

**Proof.** (a) Let $A \subseteq S_{2^k}(B_{1,0})$ for some $k \geq 1$. Using Proposition 3.4(b), we have $S_{2^k}^{-1}(A) \subseteq S_{2^{k-1}}(B_{1,0}) \subseteq B_{1,1} \setminus S_1(\Omega)$ for any $0 \leq i \leq k - 1$. Hence
\[
\mu(A) = p_2\mu \circ S_{2}^{-1}(A) = \cdots = p_2^k \mu \circ S_{2^k}^{-1}(A).
\]
(b) Using Proposition 3.4(a), we have $\mu(A) = p_1\mu \circ S_{2}^{-1}(A)$ for any $A \subseteq S_{1}(B_{1,1})$, i.e., (3.17) holds for $k = 0$. Assume that (3.17) holds for $k = m$. For $k = m + 1$, let $A \subseteq S_{2^m+1}(B_{1,1})$. Then $S_{2}^{-1}(A) \subseteq S_{2^m+1}(B_{1,1})$. Using induction hypothesis, we get
\[
\mu(S_{2}^{-1}(A)) = w_1(m)\mu \circ S_{2^m+1}^{-1}(A).
\]
By Proposition 3.4(a,b), $S_{1}(\Omega) \cap S_{2^m+1}(B_{1,1}) = S_{2^m+1}(B_{1,1}) = S_{13^m+1}(B_{1,1})$. Thus $S_{1}^{-1}(A) \cap \Omega \subseteq S_{2^m+1}(\Omega)$. It follows that
\[
\mu(S_{1}^{-1}(A)) = p_3\mu \circ S_{3}^{-1}(S_{1}^{-1}(A)) = \cdots = p_3^{m+1}\mu \circ S_{3^{m+1}}^{-1}(A).
\]
Combining this with (3.18), we have
\[
\mu(A) = p_1\mu \circ S_{1}^{-1}(A) + p_2\mu \circ S_{2}^{-1}(A) \\
= p_1p_3^{m+1}\mu \circ S_{3^{m+1}}^{-1}(A) + p_2w_1(m)\mu \circ S_{2^m+1}^{-1}(A) \\
= w_1(m+1)\mu \circ S_{2^m+1}^{-1}(A),
\]
where $S_{13^{m+1}} = S_{2^m+1}$ and (3.15) are used in the last equality. This proves part (b).

(c) The proof is similar to that of (b). \qed

**Proof of Example 3.3.** We show that all assumptions in Proposition 2.15 are satisfied with $m = 1$. For each $k \geq 0$, let $M_k = \{1, 2, 3\}^k$. Let $I_{1,i}$ be defined as in (3.13). Then $I_1 = \{I_{1,0}, I_{1,1}\}$. Let $I_{0,1} := \{[I_{1,0}]_\mu, [I_{1,1}]_\mu\}$. We claim that $I_{k,1,2} := \{(S_{2^{k-1}}, k), (S_{2^k}, k)\}$ is the only level-$k$ nonbasic island with respect to $I_1$ for any $k \geq 2$ (see Figure 2). Since $\mathcal{I}(v_{\text{root}}) \approx \mu I_{1,0}$, $I$ is not a level-2 nonbasic island with respect to $I_1$ for any $I \in O(I_{1,0})$, and thus assumption (1) of Proposition 2.15 holds for $\ell = 0$ with $c(0) = 2$. Upon iterating the IFS once, $I_{1,1}$ generates three islands:
\[
I_{2,1,1} := \{(S_{11}, 2), (S_{12}, 2)\}, \quad I_{2,1,2} := \{(S_{21}, 2), (S_{22}, 2)\}, \quad I_{2,1,0} := \{(S_{23}, 2)\}.
\]
(3.19)

Lemma 3.5 implies that $I_{2,1,i} \in [I_{1,i}]_\mu$ for $i = 0, 1$, and $[I_{2,1,2}]_\mu \notin I_{1,\mu}$. Thus assumption (1) of Proposition 2.15 holds for $\ell = 1$ with $c(1) = 2$ and $I_{2,1,2}$ is the only level-2 nonbasic island with respect to $I_1$. Assume that $I_{k,1,2} :=
Figure 2. Level-$k$ islands $\mathbb{I}_k$ for $k = 0, 1, 2, 3$ in Example 3.3. $\mathbb{I}_1 = \{\mathbb{I}_{1,0}, \mathbb{I}_{1,1}\}$ corresponds to the basic family of cells and $\mathbb{I}_{k,1,2}$ is the unique level-$k$ nonbasic island with respect to $\mathbb{I}_1$ for $k \geq 2$. $W_k$ corresponds to those iterates in $S_{\mathbb{I}_{k+1,1,2}}(\Omega)$ that overlap exactly and hence give rise to the same vertex. Islands that are labeled consist of vertices enclosed by a box. The figure is drawn with $r_1 = 1/3$ and $r_2 = 2/7$.

\begin{align}
\{(S_{2^k-11}, k), (S_{2^k}, k)\} &\text{ is the only level-$k$ nonbasic island with respect to } \mathbb{I}_1. \\
\mathbb{I}_{k,1,2} &\text{ also generates three islands:} \\
\mathbb{I}_{k+1,1,1} &:= \{(S_{2^{k-11}}, k + 1), (S_{2^{k-12}}, k + 1)\}, \\
\mathbb{I}_{k+1,1,2} &:= \{(S_{2^{k+1}}, k + 1), (S_{2^{k+1}}, k + 1)\}, \\
\mathbb{I}_{k+1,1,0} &:= \{(S_{2^{k-3}}, k + 1)\}. \\
\end{align}

Lemma 3.5 again implies that $\mathbb{I}_{k+1,1,i} \in [I_{i,\mu}]$ for $i = 0, 1$, and $[\mathbb{I}_{k+1,1,2}]_{\mu} \notin \mathbb{I}_{i,\mu}$. Thus, $\mathbb{I}_{k+1,1,2}$ is the only level-$(k + 1)$ nonbasic island with respect to $\mathbb{I}_1$, completing the proof of the claim. Since the closure of $S_{\mathbb{I}_{k+1,1,2}}(\Omega)$ converges to a point, $\lim_{k \to \infty} \mu(S_{\mathbb{I}_{k+1,2}}(\Omega)) = 0$. Thus, assumption (2) in Proposition 2.15 holds.

We now give an example of a class of graph-directed self-similar measures with overlaps that satisfy (EFT).

A graph-directed iterated function system (GIFS) of contractive similitudes is an ordered pair $G = (V, E)$ described as follows (see [28]): $V := \{1, \ldots, q\}$ is the set of vertices and $E$ is the set of directed edges with each edge beginning and ending at a vertex. It is possible for any edge to begin and end at the same vertex and we allow more than one edge between two vertices. Let $E_{ij}$ denote the set of all edges that begin at vertex $i$ and end at vertex $j$. We call $e = e_1 \ldots e_k$ a path with length $k$, if the terminal vertex of each edge $e_i$ ($1 \leq i \leq k - 1$) equals the initial vertex of the edge $e_{i+1}$.

To each edge $e \in E$, we associate a contractive similitude $S_e$ with contraction ratio $r(e) \in (0, 1)$. It is known (see [8, 28]) that there exists a unique family of
non-empty compact sets $F_1, \ldots, F_q$ satisfying
\[
F_i = \bigcup_{j=1}^q \bigcup_{e \in E_{ij}} S_e(F_j), \quad i = 1, \ldots, q.
\]
Define $F := \bigcup_{i=1}^q F_i$. We call $F$ the graph-directed self-similar set defined by $G = (V, E)$. Suppose for each edge $e \in E$, there corresponds a transition probability $p(e)$; that is, $p(e) > 0$ and the weights of all edges leaving a given vertex $i$ sum to 1, namely, $\sum_{j \in V} \sum_{e \in E_{ij}} p(e) = 1$. Then for each $i \in V$, there exists a unique Borel probability measure $\mu_i$ such that
\[
\mu_i = \sum_{j=1}^q \sum_{e \in E_{ij}} p(e) \mu_j \circ S_e^{-1}.
\]

We note that $\text{supp}(\mu_i) = F_i$ for all $i \in V$. Finally, define $\mu(E) := \sum_{i=1}^q \mu_i(E \cap F_i)$ for all measurable set $E$, and we call $\mu$ the graph-directed self-similar measure. A GIFS, as well as any associated graph-directed self-similar measure, are said to have overlaps if the graph open set condition (see [28]) fails. We say that a family of GIFS, as well as any associated graph-directed self-similar measure, is said to have $E$ for all measurable set $F$. We note that $\text{supp}(\mu_i) = F_i$ for all $i \in V$.

Define $F := \bigcup_{i=1}^q F_i$. We call $F$ the graph-directed self-similar set defined by $G = (V, E)$. Suppose for each edge $e \in E$, there corresponds a transition probability $p(e)$; that is, $p(e) > 0$ and the weights of all edges leaving a given vertex $i$ sum to 1, namely, $\sum_{j \in V} \sum_{e \in E_{ij}} p(e) = 1$. Then for each $i \in V$, there exists a unique Borel probability measure $\mu_i$ such that
\[
\mu_i = \sum_{j=1}^q \sum_{e \in E_{ij}} p(e) \mu_j \circ S_e^{-1}.
\]

We note that $\text{supp}(\mu_i) = F_i$ for all $i \in V$. Finally, define $\mu(E) := \sum_{i=1}^q \mu_i(E \cap F_i)$ for all measurable set $E$, and we call $\mu$ the graph-directed self-similar measure. A GIFS, as well as any associated graph-directed self-similar measure, are said to have overlaps if the graph open set condition (see [28]) fails. We say that a family of non-empty bounded open subsets $\{\Omega_1, \ldots, \Omega_q\}$ of $\mathbb{R}^d$ is invariant under the GIFS $G = (V, E)$, if $\bigcup_{e \in E_{1i}} S_e(\Omega_j) \subseteq \Omega_i$ for $i = 1, \ldots, q$. For each path $e = e_1 \cdots e_k$ of length $k$, we use the notation
\[
p(e) := p(e_1) \cdots p(e_k), \quad S_e := S_{e_1} \circ \cdots \circ S_{e_k}.
\]

The GIFS $G = (V, E)$ below is used as basic example of the graph finite type condition [5].

**Example 3.6.** Consider the GIFS $G = (V, E)$ with $V = \{1, 2\}$ and $E = \{e_i : 1 \leq i \leq 5\}$, where $e_1, e_2 \in E_{11}, e_3 \in E_{12}, e_4 \in E_{21}, e_5 \in E_{22}$. The five similitudes associated with $E$ are defined by
\[
S_{e_1}(x) = x + \frac{3}{4}, \quad S_{e_2}(x) = x - \frac{5}{16},
\]
\[
S_{e_3}(x) = x + \frac{1}{4}, \quad S_{e_4}(x) = x + \frac{9}{4}, \quad S_{e_5}(x) = x + \frac{2}{4}
\]
(see Figure 3). Let $\mu$ be the graph-directed self-similar measure defined by $G = (V, E)$ and probability matrix $(p(e))_{e \in E}$. Then $\mu$ satisfies (EFT) with $\Omega := (0, 1) \cup (2, 3)$ being an EFT-set and there exists a regular basic pair with respect to $\Omega$.

| 0 | $\Omega_1$ | 1 | 2 | $\Omega_2$ | 3 |
|---|-----------|---|---|-----------|---|
| $S_{e_1}(\Omega_1)$ | $S_{e_2}(\Omega_1)$ | $S_{e_3}(\Omega_1)$ | $S_{e_4}(\Omega_1)$ | $S_{e_5}(\Omega_2)$ |

**Figure 3.** The first iteration of the GIFS $G = (V, E)$ defined in Example 3.6, where $\Omega_1 = (0, 1)$ and $\Omega_2 = (2, 3)$.

**Proof.** Let $\Omega_1 := (0, 1)$ and $\Omega_2 := (2, 3)$. Note that $\{\Omega_1, \Omega_2\}$ is invariant under the GIFS $G = (V, E)$. Let
\[
B_{1,0} := S_{e_1}(\Omega_1) \cup S_{e_3}(\Omega_2), \quad B_{1,1} := S_{e_2}(\Omega_1), \quad B_{1,2} := S_{e_4}(\Omega_1), \quad B_{1,3} := S_{e_5}(\Omega_2).
\]
Finally, the regularity of (2) and (3) of (EFT) hold with $P_{4.1}$. 

Renewal equation and proof of Theorem 1.1. In order to derive renewal equations and prove Theorem 1.1, we will only consider the one-dimensional case.

It is obvious that $B_{1,0}$ and $B_{1,1}$ are measure disjoint. Let $\Gamma := \{0,1\}$ and $B := \{B_{1,\ell} : \ell \in \Gamma\}$. Define $P_{1,\ell} := \{B_{1,\ell}\}$ for $\ell \in \Gamma$. If for some $k \geq 2$ and $\ell \in \Gamma$, $P_{k,\ell}$ is a well-defined $\mu$-partition of $B_{1,\ell}$, then we let $P_{1,\ell}$ and $P_{2,\ell}$ be defined as in (2.7) with respect to $B$. We note that $B_{1,3}$ is not $\mu$-equivalent to any cell in $B$ (see Figure 4). Define $P_{k,1} := \{S_{e_2}(B_{1,0}), S_{e_2}(B_{1,1})\}$ for $k \geq 2$. Since $\Omega_1 \simeq_{\mu} B_{1,1}$, $P_{k,1} = P_{k,1}$ for all $k \geq 2$. Thus $(P_{k,1})_{k \geq 1}$ is a sequence of refining $\mu$-partitions of $B_{1,1}$ satisfying all conditions of (EFT) with $\ell = 1$. We note that $S_{e_1 e_2} = S_{e_3 e_4}$ and $S_{e_1}(\Omega_1) \cap S_{e_3}(\Omega_2) = S_{e_1 e_2}(\Omega_1) = S_{e_3 e_4}(\Omega_1)$. Define

$$P_{2,0} := \{S_{e_1}(B_{1,0}), S_{e_1}(B_{1,1}) = S_{e_3}(B_{1,2}), S_{e_3}(B_{1,3})\}.$$  

Then the elements of $P_{2,0}$ are pairwise measure disjoint, with each being a subset of $B_{1,0}$. Hence $P_{2,0}$ is a $\mu$-partition of $B_{1,0}$. It follows from (3.21) and the equality $S_{e_1 e_2} = S_{e_3 e_4}$ that $P_{2,0} = \{S_{e_1}(B_{1,0}), S_{e_1}(B_{1,1})\}$ and $P_{2,0} = \{S_{e_3}(B_{1,3})\}$. Thus condition (1) of (EFT) holds with $\ell = 0$. For $k \geq 2$, define

$$P_{k+1,0} := P_{1,k,0} \cup \{S_{e_3 e_4}(B_{1,2}), S_{e_3 e_4}(B_{1,3})\}.$$  

Similarly, we can show that $B_{1,1} \simeq_{\mu} S_{e_3 e_4}(B_{1,2})$ and $B_{1,3} \simeq_{\mu} S_{e_3 e_4}(B_{1,3})$. Thus $P_{1,k,0} = P_{1,k,0} \cup \{S_{e_3 e_4}(B_{1,2})\}$ and $P_{2,k+1,0} = \{S_{e_3 e_4}(B_{1,3})\}$. Hence, conditions (2) and (3) of (EFT) hold with $\ell = 0$. Consequently, the first assertion follows. Finally, the regularity of $(\{B_{1,\ell}\}, (P_{k,\ell})_{k \geq 1})$ is obvious. 

4. Renewal equation and proof of Theorem 1.1. In order to derive renewal equations and prove Theorem 1.1, we will only consider the one-dimensional case.

4.1. Eigenvalue counting function. Let $(\mathcal{E}, \text{dom } \mathcal{E})$ be defined as in (1.3) with $U = (a,b)$ and let $-\Delta_\mu$ be the associated Dirichlet Laplacian in $L^2((a,b), \mu)$. Let $\mathcal{P} = \{a_i\}_{i=0}^{n+1}$ be a partition of $[a,b]$ satisfying $a_0 := a < a_1 < \cdots < a_{n+1} =: b$ for $i \in \{0, \ldots, n+1\}$.

Define $\mathcal{F} := \mathcal{F}(\mathcal{P}) = \{u \in \text{dom } \mathcal{E} : u(a_i) = 0 \text{ for all } i = 0, \ldots, n+1\}$. Then $\mathcal{F}$ is a closed subspace of $\text{dom } \mathcal{E}$. Define a relation $\sim_{\mathcal{E}}$ on $\text{dom } \mathcal{E}$, induced by $\mathcal{F}$, by $u \sim_{\mathcal{E}} v$
if and only if \( u - v \in \mathcal{F} \). Then \( \sim_{\mathcal{F}} \) is an equivalence relation on \( \text{dom } \mathcal{F} \). Define the quotient space

\[
\text{dom } \mathcal{F} / \mathcal{F} := \{ [u]_{\mathcal{F}} : u \in \text{dom } \mathcal{F} \},
\]

where \([u]_{\mathcal{F}}\) is the equivalence class of \( u \). Define addition and scalar multiplication on \( \text{dom } \mathcal{F} / \mathcal{F} \) as usual. For each \( i = 1, \ldots, n \), let \( f_i \) be a function in \( \text{dom } \mathcal{F} \) that satisfies

\[
f_i(a_j) = \delta_{ij}, \quad i, j = 1, \ldots, 2n,
\]

where \( \delta_{ij} \) is the Kronecker delta. Such an \( f_i \) clearly exists. It is easy to prove that

\[
\text{dom } \mathcal{F} / \mathcal{F} = \text{span } \{ [f_i]_{\mathcal{F}} : i = 1, \ldots, n \} \quad \text{and dim(dom } \mathcal{F} / \mathcal{F} \text{) = n}.
\]

Let \( -\Delta_{\mu,\eta}^{\mathcal{F}} \) be the Laplacian defined by the Dirichlet form (1.3) with \( \text{dom } \mathcal{E} = \mathcal{F} \), and \( N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) := \#\{ n : \lambda_n(-\Delta_{\mu,\eta}^{\mathcal{F}}) \leq \lambda \} \) be the associated eigenvalue counting function. If \( \mathcal{F} = \mathcal{N}^1 \), where \( \mathcal{N} \) is defined as in Section 1, then \( N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \) reduces to \( N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \). It follows from the variational formula

\[
N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \leq N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \leq N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) + \# \mathcal{P} - 2. \quad (4.1)
\]

If \( \text{supp}(\mu) = [a, b] \), then

\[
N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) = \sum_{i=0}^{n} N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}).
\]

We prove a similar formula.

**Proposition 4.1.** Let \( \mu \) be a continuous positive finite Borel measure on \([a, b]\) with \( \text{supp}(\mu) \subseteq [a, b] \). Suppose there exists a nonempty subset \( J \subseteq \{0, 1, \ldots, n\} \) such that \( \mu(a_i, a_{i+1}) > 0 \) for all \( i \in J \) and \( \mu(a_j, a_{j+1}) = 0 \) for any \( j \notin J \). Then

\[
N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) = \sum_{i \in J} N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}).
\]

**Proof.** Let \( \lambda \) be an eigenvalue of \( -\Delta_{\mu,\eta}^{\mathcal{F}} \) with an eigenfunction \( u \) for \( i \in J \). Define \( \tilde{u}_i := u_i \) on \((a_i, a_{i+1})\) and \( \tilde{u}_i := 0 \) otherwise. Then \( \tilde{u}_i \in \mathcal{F} \) and for all \( v \in \mathcal{F} \),

\[
\int_a^b \tilde{u}_i' v' \, dx = \int_{a_i}^{a_{i+1}} \tilde{u}_i' v' \, dx = \lambda \int_{a_i}^{a_{i+1}} u_i v \, d\mu = \lambda \int_a^b \tilde{u}_i v \, d\mu.
\]

It follows that \( \lambda \) is also an eigenvalue of \(-\Delta_{\mu,\eta}^{\mathcal{F}} \) with \( \tilde{u}_i \) being an eigenfunction. In particular, \( N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \geq \sum_{i \in J} N(\lambda, -\Delta_{\mu,\eta}^{\mathcal{F}}) \).

To prove the reverse inequality, let \( \lambda \) be an eigenvalue of \(-\Delta_{\mu,\eta}^{\mathcal{F}} \) with \( u \) being an eigenfunction. Define \( u_i := u_{(a_i, a_{i+1})} \) for all \( i \in J \). Then for all \( v \in \mathcal{F} \),

\[
\int_a^b u_i' v' \, dx = \lambda \int_a^b u_i v \, d\mu = \lambda \sum_{i \in J} \int_{a_i}^{a_{i+1}} u_i v \, d\mu.
\]

In particular, for all \( v \in C_c^\infty(a_i, a_{i+1}), \)

\[
\int_{a_i}^{a_{i+1}} u_i' \cdot v' \, dx = \lambda \int_{a_i}^{a_{i+1}} u_i v \, d\mu.
\]

Now let \( i \in J \) such that \( u_i \neq 0 \). Then \( \lambda \) is also an eigenvalue of \(-\Delta_{\mu,\eta}^{\mathcal{F}} \) with \( u_i \) being an eigenfunction, proving the reverse inequality. \( \square \)
4.2. Renewal equation. For the convenience of the reader, we first state a slightly modified version of [31, Propositions 2.2 and 2.3] below. We denote the Lipschitz constant of a similitude $\tau$ by $r$.

**Proposition 4.2.** ([31, Proposition 2.2]) Let $S : \mathbb{R} \to \mathbb{R}$ be a similitude, with Lipschitz constant $r$, such that $S([a, b]) = [c, d]$. Let $\nu$ be a continuous positive finite Borel measure on $[a, b]$ with $\text{supp}(\nu) \subseteq [a, b]$. Then

(a) $-\Delta_{\nu \circ S^{-1} \lfloor c,d \rfloor}$ and $r^{-1} \cdot (-\Delta_{\mu \lfloor a,b \rfloor})$ are unitarily equivalent.

(b) If, in addition, $\nu \lfloor (c, d) = w \nu \circ S^{-1}$ on $(c, d)$ for some constant $w > 0$, then $-\Delta_{\nu \lfloor (c,d) \rfloor}$ and $(rw)^{-1} \cdot (-\Delta_{\mu \lfloor (a,b) \rfloor})$ are unitarily equivalent.

Note that unitarily equivalent operators have the same set of eigenvalues.

**Proposition 4.3.** ([31, Proposition 2.3]) Let $\mu, \nu$ be continuous positive finite Borel measures on $[a, b]$ and assume that there exists some similitude $\tau$ with $\mu \leq c\nu$ on $[a, b]$. Then for any $n \geq 1$, $\lambda_n(-\Delta_{\mu}) \geq c^{-1} \cdot \lambda_n(-\Delta_{\nu})$.

The following result is a consequence of Propositions 4.2 and 4.3.

**Proposition 4.4.** Let $\mu$ be a continuous positive finite Borel measure on $\mathbb{R}$ and assume that there exist some similitude $\tau$ and some constant $w > 0$ such that $\tau([a, b]) = [c, d]$ and $\mu \geq w \mu \circ \tau^{-1}$ on $[c, d]$. Then $N(w \tau \lambda, -\Delta_{\mu \lfloor (a,b) \rfloor}) \leq N(\lambda, -\Delta_{\mu \lfloor (c,d) \rfloor})$.

Proof. Proposition 4.3 implies that $\lambda_n(-\Delta_{\mu \circ \tau^{-1} \lfloor c,d \rfloor}) \geq w \lambda_n(-\Delta_{\mu \lfloor (a,b) \rfloor})$ for all $n \geq 1$. Thus

\[
N(w \lambda, -\Delta_{\mu \circ \tau^{-1} \lfloor c,d \rfloor}) \leq N(\lambda, -\Delta_{\mu \lfloor (c,d) \rfloor}). \tag{4.2}
\]

Note that by Proposition 4.2(a), we have

\[
N(w \lambda, -\Delta_{\mu \circ \tau^{-1} \lfloor c,d \rfloor}) = N(w \lambda, r^{-1}(\Delta_{\mu \lfloor (a,b) \rfloor})) = N(w r \lambda, -\Delta_{\mu \lfloor (a,b) \rfloor}).
\]

The assertion follows by combining this with (4.2).

**Proposition 4.5.** Let $\mu$ be a continuous positive finite Borel measure on $\mathbb{R}$. Assume that $\mu$ satisfies (EFT) with $\Omega$ being an EFT-set and with $\mathcal{B} := \{B_{1, \ell} : \ell \in \Gamma \}$ being a regular basic family of cells in $\Omega$. Then for each $\ell \in \Gamma$, there exists some constant $c_{\ell} > 0$ such that

\[
N(\lambda, -\Delta_{\mu | B_{1, \ell}}) \leq N(\lambda, -\Delta_{\mu | \Omega}) \leq N(c_{\ell} \lambda, -\Delta_{\mu | B_{1, \ell}}). \tag{4.3}
\]

Proof. Fix any $\ell \in \Gamma$. Let $(a, b)$ be the minimum open interval containing $\Omega$. The regularity of $\mathcal{B}$ implies that $B_{1, \ell}$ is an interval, and that there exist some similitude $\tau_{\ell}$ and some constant $w(\ell) > 0$ such that $\tau_{\ell}(\Omega) \subseteq B_{1, \ell}$ and $\mu \geq w(\ell) \mu \circ \tau_{\ell}^{-1}$ on $\tau_{\ell}(\Omega)$. We observe that $\tau_{\ell}(a, b)$ is the minimum open interval containing $\tau_{\ell}(\Omega)$. Thus $\tau_{\ell}(a, b) \subseteq B_{1, \ell}$. For any $\mu$-measurable $E \subseteq \tau_{\ell}(a, b)$, we see that

\[
\mu(E) \geq \mu(E \cap \tau_{\ell}(\Omega)) \geq w(\ell) \mu \circ \tau_{\ell}^{-1}(E \cap \tau_{\ell}(\Omega)) = w(\ell) \mu(\tau_{\ell}^{-1}(E) \cap \Omega) = w(\ell) \mu(\tau_{\ell}^{-1}(E)).
\]

That is, $\mu \geq w(\ell) \mu \circ \tau_{\ell}^{-1}$ on $\tau_{\ell}(a, b)$, which, together with Proposition 4.4, yields

\[
N(\lambda, -\Delta_{\mu | \tau_{\ell}(a, b)}) \leq N(\lambda, -\Delta_{\mu | \Omega}) \leq N(w(\ell)^{-1} r_{\tau_{\ell}}^{-1} \lambda, -\Delta_{\mu | B_{1, \ell}}). \tag{4.4}
\]

Since $\tau_{\ell}(a, b) \subseteq B_{1, \ell} \subseteq \Omega \subseteq (a, b)$, the variational formula implies that $\lambda_n(-\Delta_{\mu | \Omega}) \leq \lambda_n(-\Delta_{\mu | B_{1, \ell}}) \leq \lambda_n(-\Delta_{\mu | \tau_{\ell}(a, b)})$. Thus

\[
N(\lambda, -\Delta_{\mu | \tau_{\ell}(a, b)}) \leq N(\lambda, -\Delta_{\mu | B_{1, \ell}}) \leq N(\lambda, -\Delta_{\mu | \Omega}) \leq N(\lambda, -\Delta_{\mu | (a,b)}). \tag{4.5}
\]
The first inequality in (4.3) follows; the second one holds with \( c_\ell := w(\ell)^{-1}r_\ell^{-1} \) by combining (4.4) and (4.5).

Let \( \mu \) be a continuous positive finite Borel measure on \( \mathbb{R} \). In the rest of this section, assume that \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set on which \( \Delta_\mu \) is defined, with \( B = \{B_{1,\ell} : \ell \in \Gamma \} \) and \( (B, P) := (\{B_{1,\ell}\}, (P_{k,\ell})_{k \geq 1}) \) being a regular basic family of cells in \( \Omega \) and a regular basic pair with respect to \( \Omega \), respectively. Note that \( \mu(\partial \Omega) = 0 \). The regularity of \( (B, P) \) implies that in this case, each cell \( B \in \bigcup_{k \geq 1, \ell \in \Gamma} P_{k,\ell} \) is an interval. This allows us to apply Propositions 4.1–4.4. By Proposition 4.5, the regularity of \( B \) implies that the asymptotic behavior of \( N(\lambda, -\Delta_\mu) \) is controlled by that of \( N(\lambda, -\Delta_{\mu|f_{1,\ell}}) \) for \( \ell \in \Gamma \).

**Step 1.** Derivation of functional equations. For \( \ell \in \Gamma \) and \( k \geq 2 \), let \( P_{k,\ell}^1 \) and \( P_{k,\ell}^2 \) be defined as in (2.7) with respect to \( B \).

Without loss of generality, we may assume that \( \Gamma \) can be partitioned into two (possibly empty) sub-collections, \( \Gamma_1 \) and \( \Gamma_2 \), defined as follows. An index \( \ell \in \Gamma \) belongs to \( \Gamma_1 \) if there exists some integer \( k \) satisfying \( P_{k,\ell}^1 = \emptyset \). Let \( \kappa_\ell \geq 2 \) (depending on \( \ell \)) denote the smallest of such \( k \). Define \( \Gamma_2 := \Gamma \setminus \Gamma_1 \) and let \( \kappa_\ell := \infty \) for \( \ell \in \Gamma_2 \).

Fix any \( \ell \in \Gamma \). By the definition of (EFT), we see that for any \( 2 \leq k \leq \kappa_\ell \), there exist two finite disjoint index sets \( G_{k,\ell}, G_{k,\ell}' \subseteq \mathbb{N} \) such that

\[
P_{k,\ell}^1 = \bigcup_{m=2}^k \{B_{m,\ell,i} : i \in G_{m,\ell}\} \quad \text{and} \quad P_{k,\ell}^2 = \{B_{k,\ell,i} : i \in G_{k,\ell}'\}.
\]

Condition (1) of (EFT) implies that \( G_{2,\ell} \neq \emptyset \). If \( \ell \in \Gamma_2 \), condition (3) of (EFT) implies that \( \lim_{k \to \infty} \sum_{i \in G_{k,\ell}'} \mu(B_{k,\ell,i}) = 0 \).

**Proposition 4.6.** Assume that \( \mu \) satisfies (EFT). Let \( \ell \in \Gamma \) and \( 2 \leq k \leq \kappa_\ell \). If \( G_{k,\ell} \neq \emptyset \), then for each \( i \in G_{k,\ell} \), there exist some \( \xi(k, \ell, i) > 0 \) and \( c(k, \ell, i) \in \Gamma \) such that

\[
N(\lambda, -\Delta_{\mu|f_{k,\ell}}) = N(\xi(k, \ell, i) \lambda, -\Delta_{\mu|f_{1,\ell \circ (k, \ell, i)}}).
\]  

**Proof.** For any \( i \in G_{k,\ell} \), by the definition of \( P_{k,\ell}^1 \), there exist some similitude \( \tau(k, \ell, i) \), as well as constants \( w(k, \ell, i), c(k, \ell, i) \in \Gamma \) such that \( \mu_{B_{k,\ell,i}} = w(k, \ell, i) \cdot \mu_{B_{1,\ell \circ (k, \ell, i)}} \circ \tau^{-1}(k, \ell, i) \). Combining this with Proposition 4.2(b), we obtain (4.6) with \( \xi(k, \ell, i) := w(k, \ell, i) r_{\tau(k, \ell, i)} \).

For each \( \ell \in \Gamma \) and \( 1 \leq n \leq \kappa_\ell \), we define a partition \( \mathcal{P}_{n,\ell} \) of \( B_{1,\ell} \) as follows:

\[
\mathcal{P}_{n,\ell} := \{x : x \text{ is an end-point of some interval in } P_{n,\ell}\},
\]

and let \( \mathcal{F}_{n,\ell} := \mathcal{F}(\mathcal{P}_{n,\ell}) \). Note that for all \( \ell \in \Gamma \) and \( 2 \leq n \leq \kappa_\ell \), we have \( \# \mathcal{P}_{n,\ell} \leq 2 \# \mathcal{P}_{n,\ell} \). Proposition 4.1 implies that for \( \ell \in \Gamma \) and \( 2 \leq n \leq \kappa_\ell \),

\[
N(\lambda, -\Delta_{\mu|f_{n,\ell}}) = \sum_{j=2}^n \sum_{i \in G_{j,\ell}} N(\lambda, -\Delta_{\mu|f_{n,\ell}}) = \sum_{i \in G_{2,\ell}'} N(\lambda, -\Delta_{\mu|f_{n,\ell}}),
\]

which, together with (4.1) and (4.6), yields

\[
N(\lambda, -\Delta_{\mu|f_{1,\ell}}) = N(\lambda, -\Delta_{\mu|f_{1,\ell}}) + \varepsilon(\lambda, \kappa_\ell, \ell).
\]
\[
\begin{align*}
&= \sum_{j=2}^{\kappa_{\ell}} \sum_{i \in G_{j,\ell}} N(\xi(j, \ell, i) \lambda, -\Delta_{\mu_{B_{1, \ell}, i, j}}) + \varepsilon(\lambda, \kappa_{\ell}, \ell) \quad \text{for } \ell \in \Gamma_1,
\end{align*}
\]

where \(0 \leq \varepsilon(\lambda, \kappa_{\ell}, \ell) \leq 2\#P_{\kappa_{\ell}} - 2\). Similarly, we have

\[
N(\lambda, -\Delta_{\mu_{B_{1, \ell}}}) = \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} N(\xi(j, \ell, i) \lambda, -\Delta_{\mu_{B_{1, \ell}, i, j}}) + \sum_{i \in G_{n,\ell}} N(\lambda, -\Delta_{\mu_{B_{n, \ell}}}) + \varepsilon(\lambda, n, \ell) \quad \text{for } n \geq 2 \text{ and } \ell \in \Gamma_2,
\]

where \(0 \leq \varepsilon(\lambda, n, \ell) \leq 2\#P_{n,\ell} - 2\).

**Step 2.** Derivation of the vector-valued equation. For each \(\ell \in \Gamma \) and \(\alpha \geq 0\), define

\[
f_{\ell}(t) = f_{\ell}^{(\alpha)}(t) := e^{-\alpha t} N(e^{\ell}, -\Delta_{\mu_{B_{1, \ell}}}), \quad t \in \mathbb{R}.
\]

Let \(\lambda = e^{\ell}\). Then \(e^{-\alpha t} N(\beta \lambda, -\Delta_{\mu_{B_{1, \ell}}}) = \beta^{\alpha} f_{\ell}(t + \ln \beta)\) for any \(\beta > 0\). Now, multiply both sides of (4.7) and (4.8) by \(e^{-\alpha t}\). Then for \(\ell \in \Gamma_1\), we have

\[
f_{\ell}(t) = \sum_{j=2}^{\kappa_{\ell}} \sum_{i \in G_{j,\ell}} \xi(j, \ell, i)^{\alpha} f_{c(j, \ell, i)}(t + \ln(\xi(j, \ell, i))) + z_{\ell}^{(\alpha)}(t),
\]

where \(z_{\ell}^{(\alpha)}(t) := e^{-\alpha t} \varepsilon(e^{\ell}, \kappa_{\ell}, \ell)\). For \(\ell \in \Gamma_2\) and \(n \geq 2\), we obtain

\[
f_{\ell}(t) = \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j, \ell, i)^{\alpha} f_{c(j, \ell, i)}(t + \ln(\xi(j, \ell, i)))
\]

\[
- \sum_{j=n+1}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j, \ell, i)^{\alpha} f_{c(j, \ell, i)}(t + \ln(\xi(j, \ell, i))) + z_{\ell}^{(\alpha)}(t),
\]

where \(z_{\ell}^{(\alpha)}(t) := e^{-\alpha t} \left(\sum_{i \in G_{n,\ell}} N(e^{\ell}, -\Delta_{\mu_{B_{n, \ell}}}) + \varepsilon(e^{\ell}, n, \ell)\right)\). Since \(\lambda_{\ell}(-\Delta_{\mu_{B_{1, \ell}}}) > 0\) for all \(\ell \in \Gamma\), there exists \(t_0 \in \mathbb{R}\) such that \(f_{\ell}(t) = 0\) for any \(t < t_0\) and any \(\ell \in \Gamma\).

For each \(t \in \mathbb{R}\) and \(\ell \in \Gamma_2\), let \(n_{t} := n_{t}(\ell)\) be the smallest positive integer such that

\[
t + \max \{ \ln(\xi(j, \ell, i)) : i \in G_{j,\ell} \} < t_0 \quad \text{for all } j \geq n_{t}.
\]

Let \(n = n_{t}\) in (4.11). Then the second summation in (4.11) vanishes and thus we get

\[
f_{\ell}(t) = \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} \xi(j, \ell, i)^{\alpha} f_{c(j, \ell, i)}(t + \ln(\xi(j, \ell, i))) + z_{\ell}^{(\alpha)}(t) \quad \text{for } \ell \in \Gamma_2,
\]

where \(z_{\ell}^{(\alpha)}(t)\) is obtained from that in (4.11) by replacing \(n\) with \(n_{t}\).

For \(\ell, m \in \Gamma\), let \(\mu_{\alpha}^{(\alpha)}\) be the discrete measure such that

\[
\mu_{\alpha}^{(\alpha)}(-\ln(\xi(j, \ell, i))) := \xi(j, \ell, i)^{\alpha} \quad \text{for } 2 \leq j \leq \kappa_{\ell}, i \in G_{j,\ell}, c(j, \ell, i) = m.
\]

We summarize the above derivations in the following theorem.

**Theorem 4.7.** Let \(\mu\) be a continuous positive finite Borel measure on \(\mathbb{R}\). Assume that \(\mu\) satisfies (EFT) with \(\Omega\) being an EFT-set on which \(\Delta_{\mu}\) is defined and there exists a regular basic family of cells in \(\Omega\). Let \(f, M_{\alpha}\), and \(z\) be defined as in (1.5). Then \(f\) satisfies the vector-valued renewal equation \(f = f * M_{\alpha} + z\).
4.3. Proof of Theorem 1.1. (a) Similar to the proof of [31, Theorem 1.1(a)].

(b) For the rest of the proof we let \( \alpha \) be the unique number in part (a). Let \( m := \{ m_{ij}^{(\alpha)} \} = \{ \int_0^\infty x d \mu_{ij}^{(\alpha)} \} \) be the moment matrix. Following the proof of [31, Theorem 1.1(b)], we need to show that some moment condition holds, in order to apply Theorem A.1. It suffices to show that

\[
0 < \sum_{k \in \Gamma} m_{ik}^{(\alpha)} < \infty. \tag{4.15}
\]

It is easy to check that for \( \ell \in \Gamma \), \( \sum_{k \in \Gamma} m_{ik}^{(\alpha)} \) takes the following values:

\[
\sum_{j = 2}^{\kappa_{\ell}} \sum_{i \in \mathcal{G}_{j,\ell}} \xi(j, \ell, i, \alpha) |\ln(\xi(j, \ell, i))|.
\]

Hence, (4.15) follows from our assumption \( \lim_{\alpha \to -\alpha} F_\ell(\alpha) > 1 \). Moreover, it follows from the derivation of equations (4.10) and (4.13) that each column of \( M_\alpha \) is non-degenerate at 0. From Theorem 4.7, we have \( f = f \ast M_\alpha + z \), where, by assumption, \( z \) is directly Riemann integrable on \( \mathbb{R} \).

We first consider the case \( M_\alpha(\infty) \) is irreducible. Then all conditions of Theorem A.1 are satisfied. Hence there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
0 < c_1 \leq \lim_{t \to \infty} f_\ell(t) \leq \lim_{t \to \infty} f_\ell(t) \leq c_2 < \infty, \quad \ell \in \Gamma.
\]

It follows from the definition of \( f_\ell(t) \) in (4.9) that there exist positive constants \( C_1 \) and \( C_2 \) such that for all \( \ell \in \Gamma \), \( C_1 \lambda^{\alpha} \leq N(\lambda, -\Delta_{\mu_{\mathcal{G}_{B_{1,\ell}}}}) \leq C_2 \lambda^{\alpha} \) for all sufficiently large \( \lambda \). Thus (1.8) follows from (4.3). Combining (1.8), part (a), and the definition of \( d_s \), we get \( d_s = 2\alpha \).

It remains to consider the case \( M_\alpha(\infty) \) is reducible. As in the proof of [31, Theorem 1.1(b), Case 2], we have \( \lim_{t \to \infty} f_\ell^{(\beta)}(t) = 0 \) for all \( \ell \in \Gamma \) and all \( \beta < \alpha \). Moreover, there exists some \( \ell_0 \in \Gamma \) such that \( \lim_{t \to \infty} f_{\ell_0}^{(\alpha)}(t) > 0 \). Using the definition of \( f_\ell(t) \), we see that

\[
N(\lambda, -\Delta_{\mu_{\mathcal{G}_{B_{1,\ell}}}}) = o(\lambda^{\beta}) \quad \text{for } \ell \in \Gamma \text{ and } \beta < \alpha \text{ and } \lim_{\lambda \to \infty} \lambda^{-\alpha} N(\lambda, -\Delta_{\mu_{\mathcal{G}_{B_{1,\ell}}}}) > 0.
\]

Thus \( N(\lambda, -\Delta_{\mu}) = o(\lambda^{\beta}) \) for any \( \beta < \alpha \) and \( \lim_{\lambda \to \infty} \lambda^{-\alpha} N(\lambda, -\Delta_{\mu}) > 0 \). Hence, \( d_s(\Delta_{\mu}) \leq 2\alpha \) and \( d_s(\Delta_{\mu}) \geq 2\alpha \), which completes the proof.

5. The family of FIFSs with overlaps in (1.9). In this section, we derive renewal equations and compute the spectral dimension for the Laplacians defined by the family of self-similar measures associated with the IFSs in (1.9).

Let \( \{ S_i \}_{i=1}^3 \) be an FIFS in (1.9), \( \{ p_i \}_{i=1}^3 \) be a probability vector, and \( \mu \) be the associated self-similar measure. For \( \ell \in \Gamma := \{ 0, 1 \} \), let \( \mathcal{G}_{\ell,\Gamma} \) and \( B_{1,\ell} \) be defined as in (3.13) and (3.16), respectively. By the proof of Example 3.3, we see that \( \mu \) satisfies all conditions in Proposition 2.15 with \( m = 1 \) and \( c(\ell) = 2 \) for \( \ell \in \Gamma \). For each \( \ell \in \Gamma \) and \( k \geq 1 \), let \( \mathbf{P}_{k,\ell} \) be defined as in (2.8) and (2.9). Then \( \mu \) satisfies (EFT) with \( \Omega = (0, 1) \) being an EFT-set, \( \mathcal{B} := \{ B_{1,\ell} : \ell \in \Gamma \} \) being a regular basic family of cells in \( \Omega \), and \( (\mathcal{B}, \mathbf{P}) := \{ (B_{1,\ell}), (\mathbf{P}_{1,\ell})_{k \geq 1} \}_{\ell \in \Gamma} \) being a regular basic pair with respect to \( \Omega \).

In the rest of this section, we use the notation defined in Section 4.2. For \( \mathcal{I} \in \mathcal{I} \), let \( S_\mathcal{I}(\Omega) \) and \( O(\mathcal{I}) \) be defined as in (2.2) and (2.4), respectively. For \( \ell \in \Gamma \), \( i = 1, 2 \), and \( k \geq 2 \), let \( \mathbf{P}_{k,\ell} \) be defined as in (2.7) with respect to \( \mathcal{B} \). We first observe that
O(Ω, 0) = \{(Ω, 0), Ω, 1\}, where Ω, 0 := \{(S_{33}, 2)\} and Ω, 1 := \{(S_{31}, 2), (S_{32}, 2)\}. Since I(\nu_{root}) ∼\mu, S_{1}, p_{3}, I_1, 0, we have I_1, i ∼\mu, S_{1}, p_{3} I_2, 0, i for i = 0, 1. Define

\[ B_{2,0,i} := S_{I_2,0,i}(Ω) \quad \text{for } i = 0, 1. \tag{5.1} \]

Thus \( P_{2,0} = P_{2,0}^{1} = \{B_{2,0,0}, B_{2,0,1}\} \) and \( P_{2,0}^{2} = \emptyset \) (see Figure 5). It follows that \( P_{k,0} = P_{2,0} \) for all \( k \geq 2 \); in particular, \( 0 \in \Gamma_1, \kappa_0 = 2, G_{2,0} = \{0, 1\} \).

For \( k \geq 2 \) and \( i = 0, 1, 2 \), let \( I_{k,1,i} \) be defined as in (3.19) and (3.20). By the proof of Example 3.3, we see that \( O(I_{1,1}) = \{I_{2,1,0}, I_{2,1,1}, I_{2,1,2}\} \), \( O(I_{k,1,2}) = \{I_{k+1,1,0}, I_{k+1,1,1}, I_{k+1,1,2}\} \), and \( I_{k,1,2} \) is the only level-k nonbasic island with respect to \( I_1 \) (see Figure 2). Define

\[ B_{k,1,i} := S_{I_{k,1,i},(\Omega)} \quad \text{for } k \geq 2 \text{ and } i = 0, 1, 2. \tag{5.2} \]

Thus \( P_{k,1} = \bigcup_{m=1}^{k} B_{m,1,1} \) and \( P_{k,1}^{2} = \{B_{k,1,2}\} \) for all \( k \geq 2 \) (see Figure 5). Consequently, \( 1 \in \Gamma_2, \kappa_1 = \infty, G_{k,1} = \{0, 1\}, \) and \( G'_{k,1} = \{2\} \) for \( k \geq 2 \).

Now, we can use the method in Section 4.2 to derive renewal equations for \(-\Delta\mu\). In the rest of this section, let \( w_1(k) \) be defined as in (3.14).

**Proposition 5.1.** Let \( \xi(\cdot, \cdot, \cdot) \) and \( c(\cdot, \cdot, \cdot) \) be defined as in Proposition 4.6. Then

(a) \( \xi(2, 0, i) = p_{3} r_{2}, c(2, 0, i) = i \) for \( i = 0, 1 \), and \#\( P_{k,0,0} = \#\( P_{2,0} = 2 \). \)

(b) For \( k \geq 2 \) and \( i = 0, 1, \)

\[ \xi(k, 1, 0) = (p_{2} r_{2})^{k-1}, \quad \xi(k, 1, 1) = w_1(k-2) r_{1} r_{2}^{k-2}, \quad c(k, 1, i) = i, \tag{5.3} \]

and \#\( P_{k,1} = 2k - 1 \).

**Proof.** (a) For \( i = 0, 1 \), since \( I_{1,1} \approx\mu, S_{3}, p_{3}, I_{2,0,1}, \) Remark 2.10(b) implies \( B_{1,1} \approx\mu, S_{3}, p_{3}, B_{2,0}, i \). The result follows.

(b) Lemma 3.5 implies that for \( k \geq 2 \), \( B_{1,0} \approx\mu, S_{k-1, p_{2}}^{k-1} \) \( B_{k,1,0} \) and \( B_{1,1} \approx\mu, S_{k-1}, w_1(k-2) \) \( B_{k,1,1} \). Combining this with the proof of Proposition 4.6, we get (5.3). \[\square\]
Using Proposition 5.1 and the discussions preceding it, we can express the vector-valued renewal equations (4.10) and (4.13) precisely as

\[ f_0(t) = (p_3r_2)\alpha \sum_{i=0}^{1} f_i(t + \ln(p_3r_2)) + z_0^{(\alpha)}(t), \]

\[ f_1(t) = \sum_{j=1}^{\infty} (p_2r_2)^{\alpha} f_0(t + \ln(p_2r_2)^j) + \sum_{j=0}^{\infty} \left( w_1(j)r_1r_2^{\alpha} f_1(t + \ln(w_1(j)r_1r_2^{j})) + z_1^{(\alpha)}(t), \right), \]

where \( z_0^{(\alpha)}(t) = e^{-\alpha t}\varepsilon(e^t, 2, 0), z_1^{(\alpha)}(t) = e^{-\alpha t}N(e^t, -\Delta_{\mu|B_{nt,1,2}}) + e^{-\alpha t}\varepsilon(e^t, n_t, 1), \]

\( 0 \leq \varepsilon(e^t, 2, 0) \leq 2, \) and \( 0 \leq \varepsilon(e^t, n_t, 1) \leq 4(n_t - 1). \)

Let \( \{\mu_{\alpha}(n)\}_{l,m} \) be the discrete measures defined as in (4.14). Note that

\[ M_{\alpha}(\infty) = \left[ \frac{(p_3r_2)^\alpha}{(p_2r_2)^\alpha/(1 - (p_2r_2)^\alpha)} \sum_{j=0}^{\infty} (w_1(j)r_1r_2^{\alpha}) \right] \]

is irreducible.

**Proposition 5.2.** For \( \ell = 0, 1, \) let \( F_i(\alpha) \) and \( \tilde{\alpha}_\ell \) be defined as in (1.7). Then \( \tilde{\alpha}_\ell = 0 \) and \( F_i(0) > 1 \) for all \( \ell \in \{0, 1\}. \)

**Proof.** By the definition of \( F_i(\alpha), \ell = 0, 1, \) we see that

\[ F_0(\alpha) = 2(p_3r_2)^\alpha \quad \text{and} \quad F_1(\alpha) = \frac{(p_2r_2)^\alpha}{1 - (p_2r_2)^\alpha} + \sum_{k=0}^{\infty} (w_1(k)r_1r_2^{k})^\alpha. \]

We first note that the result holds for \( \ell = 0. \) For any \( \alpha > 0, \) since \( \sum_{k=0}^{\infty} (w_1(k)r_1r_2^{k})^\alpha \)

\[ \text{converges, } F_1(\alpha) < \infty. \]

By the definition of \( \tilde{\alpha}_\ell, \) we have \( \tilde{\alpha}_1 = 0 \) and \( F_1(0) = \infty. \]

Proposition 5.2 implies that there exists a unique \( \alpha > 0 \) such that the spectral radius of \( M_{\alpha}(\infty) \) is 1. That is, \( \alpha \) is the unique solution of \( |I_2 - M_{\alpha}(\infty)| = 0, \) where \( I_2 \) is the identity \( (2, 2)-\)matrix. More precisely, \( \alpha \) is the unique number satisfying (1.10).

Now we need to show that there exists some \( \sigma > 0 \) such that \( z_{\ell}^{(\alpha)}(t) = o(e^{-\alpha t}) \)

\[ \text{as } t \to \infty \text{ for } \ell = 0, 1. \]

To this end, we will first show that \( N(e^t, -\Delta_{\mu|B_{nt,1,2}}) \) is bounded.

**Proposition 5.3.** There exists \( C > 0 \) such that \( N(e^t, -\Delta_{\mu|B_{nt,1,2}}) \leq C. \)

**Proof.** Let \( A \subseteq B_{nt,1,2} = S_{2n_t - 1}(B_{1,1}), \) we have \( S_{2n_t - 2}^{-1}(A) \subseteq S_2(B_{1,1}). \) Thus

\[ \mu(S_{2n_t - 2}^{-1}(A)) = p_1\mu \circ S_{2n_t - 2}^{-1}(A) + p_2\mu \circ S_{2n_t - 1}^{-1}(A). \]

Multiplying both sides of (5.4) by \( w_1(n_t - 2)p_1^{-1} \) and using Lemma 3.5(c), we have

\[ w_1(n_t - 2)p_1^{-1} \mu(S_{2n_t - 2}^{-1}(A)) = w_1(n_t - 2)p_1^{-1} \mu(S_{2n_t - 2}^{-1}(A)) + w_1(n_t - 2)p_1^{-1} p_2\mu(S_{2n_t - 1}^{-1}(A)). \]

\[ \geq w_1(n_t - 2)p_1^{-1} \mu(S_{2n_t - 2}^{-1}(A)) + p_2^{-1} \mu(S_{2n_t - 1}^{-1}(A)) = \mu(A). \]
Thus \( \mu|_{B_{n_t,1.2}} \leq w_1 (n_t - 2) p_1^{-1} \mu \circ S_{2^{-n_t}}^{-1} \) on \( B_{n_t,1.2} \). Proposition 4.4 implies

\[
N(e^t, -\Delta_{\mu|_{B_{n_t,1.2}}}) \leq N\left(w_1 (n_t - 2) p_1^{-1} r_1^{-n_t-2} e^t, -\Delta_{\mu|_{S_{2}^{n_t}(\Omega)}}\right). \tag{5.5}
\]

By the definition of \( n_t \), i.e., (4.12), we have \( t + \ln (w_1 (n_t - 2) r_1^{n_t-2}) < \sigma \). Combining this with (5.5), we obtain

\[
N\left(e^t, -\Delta_{\mu|_{B_{n_t,1.2}}}\right) \leq N((p_1 r_1)^{-1} e^t_0, -\Delta_{\mu|_{S_{2}^{n_t}(\Omega)}}) := C.
\]

\( \square \)

**Proposition 5.4.** Let \( \alpha \) be defined as in (1.10). Then there exists some \( \sigma > 0 \) such that for \( i = 0, 1 \), \( z_i^{(\alpha)}(t) = o(e^{-\sigma t}) \) as \( t \to \infty \).

**Proof.** Proposition 5.3 implies that there exists some constant \( c > 0 \) such that

\[
z_0^{(\alpha)}(t) = e^{-\sigma t} N(e^t, -\Delta_{\mu|_{B_{n_t,1.2}}}) + e^{-\sigma t} e^t, n_t, 1) \leq (c + 4 n_t - 2) e^{-\sigma t}.
\]

Moreover, since \( z_0^{(\alpha)}(t) = e^{-\sigma t} e^t, 2, 0) \leq 2 e^{-\sigma t} \), it suffices to show that for any \( \sigma > 0 \), \( n_t e^{-\sigma t} = o(e^{-\sigma t}) \) as \( t \to \infty \). By the definition of \( n_t \), i.e., (4.12), we have

\[
t + \max \left\{ \ln (p_2^{-n_t-2} r_2^{-n_t-2}), \ln (w_1 (n_t - 3) r_1^{-n_t-3}) \right\} \geq \sigma t_0.
\]

Since \( w_1 (n_t - 3) \leq 1 \), we have \( t + \ln (r_2^{-n_t-3}) \geq \sigma t_0 \) and hence, for any \( \sigma < \alpha \), \( n_t e^{-\sigma t} = o(e^{-\sigma t}) \) as \( t \to \infty \), which completes the proof. \( \square \)

**Proof of Theorem 1.2.** Combine Propositions 5.2 and 5.4, and Theorem 1.1. \( \square \)

6. **IIFSs with overlaps.** In this section we study (EFT) for IIFSs and prove Theorem 1.3. We will only consider IIFSs on \( \mathbb{R} \). Let \( \mu \) be a self-similar measure defined by a finite type IIFS \( \{S_i\}_{i \in \Lambda} \) on \( \mathbb{R} \) with \( \Omega \) being an FTC-set. In this section, we use the notation introduced in Section 2.

6.1. **A sufficient condition for IIFSs to satisfy (EFT).** We first introduce the definition of a tail.

**Definition 6.1.** Let \( \mu \) be a self-similar measure defined by a finite type IIFS \( \{S_i\}_{i \in \Lambda} \) on \( \mathbb{R} \) with \( \Omega := (a, b) \) being an FTC-set. For \( k \geq 1 \), let \( \mathcal{T} \subseteq \mathbb{I}_k \) be a countably infinite sequence of islands, and let \( \mathcal{U} := (c, d) \subseteq \Omega \) be the minimal open interval containing \( S_{\mathcal{T}}(\Omega) \).

(a) We call \( \mathcal{T} \) a level-\( k \) semi-tail if it satisfies the following two conditions:

1. For any \( I \in \mathbb{I}_k \setminus \mathcal{T} \), \( S_{\mathcal{T}}(\Omega) \cap U = \emptyset \);  
2. For any \( c_1 > c \) and \( d_1 < d \), either \#\{\( I \in \mathcal{T} : S_{\mathcal{T}}(\Omega) \cap (c_1, d_1) \neq \emptyset \} < \infty \) or \#\{\( I \in \mathcal{T} : S_{\mathcal{T}}(\Omega) \cap (c, d_1) \neq \emptyset \} < \infty \).

(b) \( \mathcal{T} \) is called a level-\( k \) tail if it is a level-\( k \) semi-tail and the following two additional conditions hold:

1. There exists a finite subset \( \mathcal{B} \subseteq \mathbb{I}_k \setminus \mathcal{T} \) containing all island measure types in \( \mathcal{T} \) such that \( \mathcal{B} \cup \mathcal{T} \) is a maximal level-\( k \) semi-tail;  
2. \( \mathcal{T} \) is a maximal level-\( k \) semi-tail satisfying condition (3).

Intuitively, condition (1) means that the interior of the convex hull of \( S_{\mathcal{T}}(\Omega) \) does not contain any cell that corresponds to some level-\( k \) island \( I \notin \mathcal{T} \). Thus, \( \mu|_{U} \) and \( \mu|_{S_{\mathcal{T}}(\Omega)} \) have similar measure structures; in particular, the closures of their supports with respect to \( \Omega \) are the same. Condition (2) implies that \( \mathcal{T} \) can be expressed as a sequence of islands \( \{I_i\}_{i=1}^{\infty} \) such that \( \mathcal{T}_k := \{I_i\}_{i=k}^{\infty} \) is a semi-tail for all \( k \geq 2 \), and \( S_{\mathcal{T}_k}(\Omega) \) converging to either \( \{c\} \) or \( \{d\} \) in the Hausdorff metric as \( k \to \infty \). Conditions (3) and (4) imply the following properties of tails.
Remark 6.2. (a) Any two distinct level-$k$ tails are disjoint.

(b) For any level-$k$ tail $T$, there exists some $B \subseteq \mathbb{I}_k$, which is not contained in any level-$k$ tail, and contains all island measure types in $T$. In particular, $T$ contains only a finite number of island measure types.

We denote the collection of all level-$k$ tails by $T_k$ for $k \geq 1$, and define $T := \bigcup_{k \geq 1} T_k$. For $T \in T$, $|T|_\mu$ is said to be the (tail) measure type of $T$. Since any tail consists of countably infinitely many islands, if $\# I_k < \infty$ for some $k \geq 1$, then $T_k = \emptyset$.

Example 6.3. Assume that $\{S_i\}_{i=0}^\infty$ is an IIFS on $\mathbb{R}$ satisfying (OSC) with $\Omega = (a, b)$ being an OSC-set. Let $\mu$ be an associated self-similar measure. Define $v_i := (S_i, 1)$ for any $i \geq 0$. Assume that there exist some (possibly empty) finite subset $\Lambda_1 \subseteq \Lambda$ and some $x_0 \in \bar{\Omega}$ such that $\Lambda_1 = \{i : S_i(a) \geq x_0\}$, $A_2 := \Lambda \setminus \Lambda_1 = \{i : S_i(b) \leq x_0\}$, and $\lim_{s \to \infty} S_i(a) = \lim_{s \to \infty} S_i(b) = x_0$. Let $i_0 \in \Lambda_2$ such that $S_{i_0}(a) = \min\{S_i(a) : i \in \Lambda_2\}$. Then $T := \{T(v_i) : i \in \Lambda_2 \setminus \{i_0\}\}$ is the only level-1 tail.

Proof. Example 2.5 implies that $\{S_i\}_{i=0}^\infty$ is of finite type with $\Omega$ being an FTC-set. Let $i_1 \in \Lambda_2$ such that $S_{i_1}(a) = \min\{S_i(a) : i \in \Lambda_2 \setminus \{i_0\}\}$. By assumption, $(S_{i_1}(a), x_0)$ is the minimal open interval containing $S_T(\Omega)$ and thus $T$ is a semi-tail. Let $\mathbb{B} := \{T(v_{i_0})\}$. Then $\mathbb{B} \cup T$ is the maximal semi-tail containing $T$. Since $\mathbb{I}/\sim_\mu$ contains only one element, conditions (3) and (4) hold, which completes the proof.

We will give another example of a tail in Lemma 6.12.

Proposition 6.4. Let $\mu$ be a self-similar measure defined by a finite type IIFS $\{S_i\}_{i \in \Lambda}$ on $\mathbb{R}$ with $\Omega = (a, b)$ being an FTC-set. Suppose that there exists some $m \geq 1$ such that the following conditions hold:

1. both $T_m$ and $I_m := \mathbb{I}_m \setminus (\bigcup_{T \in T_m} T)$ are nonempty and finite;
2. for any $k \geq m$ and $\mathcal{I} \in \mathbb{I}_k$, $O(\mathcal{I})$ can be expressed as the disjoint union of a nonempty finite family $\{\mathcal{I}_i\}_{i \in \Lambda_i}$ of islands and a nonempty finite family $\{T_i\}_{i \in \Lambda_3}$ of semi-tails with the property that for any $i \in \Lambda_3$, there exists some $T \in T_m$ such that $|T|_\mu = |T_i|_\mu$;
3. the sum of the $\mu$-measures of all level-$k$ nonbasic islands with respect to $I_m$ tends to 0 as $k \to \infty$.

Then $\mu$ satisfies (EFT) with $\Omega$ being an EFT-set and there exists a regular basic pair with respect to $\Omega$.

Proof. Let $\Gamma_*$ and $\Gamma'_*$ be disjoint index sets such that $I_m = \{\mathcal{I}_{1, \ell} : \ell \in \Gamma_*\}$ and $T_m = \{T_{1, \ell} : \ell \in \Gamma'_*\}$. Define $B_{1, \ell} := S_{T_{1, \ell}}(\Omega)$ for $\ell \in \Gamma_*$. For $\ell \in \Gamma'_*$, let $B_{1, \ell}$ be the minimum open interval containing $S_{T_{1, \ell}}(\Omega)$. Let $\Gamma := \Gamma_* \cup \Gamma'_*$. We claim that $\mu$ satisfies (EFT) with $\mathcal{B} := \{B_{1, \ell} : \ell \in \Gamma\}$ being a basic family of cells. If for some $k \geq 2$ and $\ell \in \Gamma$, $P_{k, \ell}$ is a well-defined $\mu$-partition of $B_{1, \ell}$, then we let $P_{1, \ell}$ and $P_{k, \ell}$ be defined as in (2.7) with respect to $\mathcal{B}$.

Fix any $\ell \in \Gamma_*$. Assumption (2) above implies that there exist two disjoint non-empty finite index sets $\Lambda_{2, 1, \ell}$ and $\Lambda_{2, 2, \ell}$ such that

$$O(\mathcal{I}_{1, \ell}) = \left\{\mathcal{I}_{2, \ell, i} \in \mathbb{I}_{m+1} : i \in \Lambda_{2, 1, \ell}\right\} \bigcup \left(\bigcup_{i \in \Lambda_{2, 2, \ell}} T_{2, \ell, i}\right)$$
is a disjoint union, and for any $i \in \Lambda_2$, there is a $j \in \Gamma'$ such that $[T_{i,j}] = [T_{i,j}]$. If for some $k \geq 2$, $\mathcal{A}_{k,\ell,1}$ is given such that $\mathcal{I}_{k,\ell,1} \in \mathcal{I}_{m+k-1}$ for any $i \in \Lambda_{k,\ell,1}$, then we divide $\mathcal{A}_{k,\ell,1}$ into sub-collections, $\mathcal{A}_{k,\ell,1}^1$ and $\mathcal{A}_{k,\ell,1}^2$ as follows. An index $i \in \Lambda_{k,\ell,1}$ belongs to $\mathcal{A}_{k,\ell,1}^1$ if there exists some $j \in \Gamma'$, so that $B_{1,j} \approx \mathcal{I}_{k,\ell,1}(\Omega)$. Define $\mathcal{A}_{k,\ell,1}^2 := \Lambda_{k,\ell,1} \setminus \mathcal{A}_{k,\ell,1}^1$. For $k \geq 3$, if $\mathcal{A}_{k,\ell,1}^2 = \emptyset$, define $\Lambda_{n,\ell,2} := \emptyset$ for all $n \geq k$. Otherwise, in view of assumption (2) above again, there exist two disjoint non-empty finite index set $\Lambda_{k,\ell,1}$ and $\Lambda_{k,\ell,2}$ such that

$$\bigcup_{i \in \Lambda_{k,\ell,1}} O(\mathcal{I}_{k,\ell,i,1}) = \left\{ \mathcal{I}_{k,\ell,i,1} \in \mathcal{I}_{m+k-1} : j \in \Lambda_{k,\ell,1} \right\} \bigcup \left( \bigcup_{j \in \Lambda_{k,\ell,2}} \mathcal{T}_{k,\ell,j} \right)$$

is a disjoint union; moreover, for any $j \in \Lambda_{k,\ell,2}$, there is $n \in \Gamma$ such that $[T_{j,n}] = [T_{j,n}]$. Let $B_{k,\ell,i} := S_{\mathcal{T}_{k,\ell,i}}(\Omega)$ for $k \geq 2$ and $i \in \Lambda_{k,\ell,1}$. For $k \geq 2$ and $i \in \Lambda_{k,\ell,2}$, let $B_{k,\ell,i}$ be the minimum open interval containing $S_{\mathcal{T}_{k,\ell,i}}(\Omega)$. Define $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$, and for $k \geq 2$,

$$\mathbf{P}_{k,\ell} := \left( \bigcup_{j=2}^{k} \{B_{j,\ell,i} : i \in \Lambda_{j,\ell,1} \cup \Lambda_{j,\ell,2} \} \right) \bigcup \{B_{k,\ell,i} : i \in \Lambda_{k,\ell,1} \}$$

(6.1)

Then Remark 2.10(c) implies that $(\mathbf{P}_{k,\ell})_{k \geq 1}$ is a family of refining $\mu$-partitions of $B_{1,\ell}$. Moreover, by the definition of $\Lambda_{k,\ell,1}^1$, $i = 1, 2$, and $\Lambda_{k,\ell,2}$, we have $\mathbf{P}_{k,\ell}^1 = \bigcup_{j=2}^{k} \{B_{j,\ell,i} : i \in \Lambda_{j,\ell,1} \cup \Lambda_{j,\ell,2} \}$ and $\mathbf{P}_{k,\ell}^2 = \{B_{k,\ell,i} : i \in \Lambda_{k,\ell,1} \}$ for all $k \geq 2$, and hence conditions (1) and (2) of (EFT) hold for $\ell$. If $B \in \mathbf{P}_{k,\ell}^1$ for some $k \geq 2$, then there exists a sequence $(\mathcal{I}_{i})_{i=2}^{k}$ of islands such that $S_{\mathcal{I}_{i}}(\Omega) \in \mathbf{P}_{i,\ell}^2$ and $\mathcal{I}_{i+1} \subseteq O(\mathcal{I}_{i})$ for all $i = 2, \ldots, k-1$, and $B = S_{\mathcal{I}_{k}}(\Omega)$. Thus $\mathcal{I}_{k}$ is a level-$k$ nonbasic island with respect to $\mathcal{I}_{m}$. Hence, assumption (3) above implies that condition (3) of (EFT) holds for $\ell$.

Fix any $\ell \in \Gamma$. By the definition of a tail, $\mathcal{T}_{1,\ell}$ can be uniquely expressed as $\mathcal{T}_{1,\ell} = \{\mathcal{I}_{k,\ell,0} : k \geq 2\}$ such that for each $k \geq 2$, $\mathcal{T}_{k,\ell,1} := \{\mathcal{I}_{k,\ell,0} : i \geq k+1\}$ is a semi-tail. Let $B_{k,\ell,0}$ be the minimal open interval containing $S_{\mathcal{T}_{k,\ell,1}}(\Omega)$, and $B_{k,\ell,0} := S_{\mathcal{T}_{k,\ell,0}}(\Omega)$ for all $k \geq 2$. Define $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$ and $\mathbf{P}_{2,\ell} := \{B_{2,\ell,0}, B_{2,\ell,1}\}$. For $k \geq 3$, if $B_{k-1,\ell,1} \in \mathbf{P}_{k-1,\ell,1}$, define

$$\mathbf{P}_{k,\ell} := \{B_{k,\ell,0} : 2 \leq i \leq k\} \bigcup \{B_{k,\ell,1}\}$$

(6.2)

otherwise, define $\mathbf{P}_{n,\ell} := \mathbf{P}_{k-1,\ell,1}$ for all $n \geq k$. We remark that $(\mathbf{P}_{k,\ell})_{k \geq 1}$ is a family of refining $\mu$-partitions of $B_{1,\ell}$. For any $k \geq 2$, by combining Remark 6.2(b) and the definition of $\mathcal{I}_{m}$, we have $\mathcal{I}_{1,i} \approx \mu \mathcal{I}_{k,\ell,0}$ for some $i \in \Gamma$. Hence Remark 2.10(b) implies that for any $k \geq 2$, $B_{k,\ell,0}$ is $\mu$-equivalent to some cell in $\mathbf{B}$. Thus $\{B_{k,\ell,0} : 2 \leq i \leq k\} \subseteq \mathbf{P}_{k,\ell}^1$ and $\mathbf{P}_{k,\ell}^2 \subseteq \{B_{k,\ell,1}\}$ for all $k \geq 2$. It follows that conditions (1) and (2) of (EFT) hold for $\ell$. By condition (2) of the definition of a tail, the closure of $B_{k,\ell,1}$ converges to a point. Hence, $\sum_{B \in \mathbf{P}_{k,\ell}^2} \mu(B) \leq \mu(B_{k,\ell,1}) \to 0$ as $k \to \infty$, i.e., condition (3) of (EFT) holds for $\ell$. Therefore, for each $\ell \in \Gamma$, all conditions of (EFT) hold. Since $\Omega = (\alpha, \beta)$, each cell $B \in \bigcup_{k \geq 1, \ell \in \Gamma} \mathbf{P}_{k,\ell}$ is connected, which, together with Proposition 2.14, yields the regularity of $(\mathbf{B}, \mathbf{P})$. 

We remark that in condition (2), it is possible that for some $i \in \Lambda_2$, $\mathcal{T}_{i} \not\in \mathcal{T}$. (See Figure 7; $\mathcal{T}_{2,1,2}$ is a semi-tail but not a tail.) Compared with that for FIFSs (Proposition 2.15), the above sufficient condition for (EFT) for IIFSs includes one additional assumption, namely, condition (2).
Example 6.6. Let \( \{I\} \subseteq \mathbb{I}_m \) for some \( m \geq 1 \). It follows that for any \( k > m \),

\[
\{[I]_\mu : I \in \mathbb{I}_k\} \subseteq \{[I]_\mu : I \in \mathbb{I}_m \text{ or } I \text{ is a level-}(k-m+1) \text{ nonbasic island w.r.t. } \mathbb{I}_m\}.
\]

Thus assumption (2) of Proposition 6.4 holds if for any \( I \in \mathbb{I}_m \), any \( n \geq 2 \), as well as any level-\( n \) nonbasic island \( I \) with respect to \( \mathbb{I}_m \), \( O(I) \) satisfies the property in assumption (2) of Proposition 6.4.

Using Proposition 6.4, we can prove that the measures in the following two classes of examples satisfy (EFT).

Example 6.7. Let \( \mu \) be a self-similar measure defined by an IIFS \( \{S_i\}_{i \in \Lambda} \) on \( \mathbb{R} \) satisfying (OSC) with \( \Omega = (a, b) \). If \( \mu(\Omega) > 0 \) and assumption (1) of Proposition 6.4 holds for \( m = 1 \), then \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set and, moreover, a regular basic pair exists.

**Proof.** Let \( I_1 := \mathbb{I}_1 \setminus (\bigcup_{T \in T_1} T) \). Then \( O(I(v_{\text{root}})) = I_1 \cup (\bigcup_{T \in T_1} T) \). By assumption, \( T_1 \) and \( I_1 \) are nonempty and finite. Moreover, (OSC) implies that for all \( k \geq 1 \) and all \( I \in \mathbb{I}_k \), \( I(v_{\text{root}}) \approx \mu I \). Thus assumptions (2) and (3) of Proposition 6.4 hold. Hence, the result follows from Proposition 6.4.

**Remark 6.5.** Assume that assumption (1) of Proposition 6.4 holds with \( \mathbb{I}_m \subseteq \mathbb{I}_m \) for some \( m \geq 1 \). Then \( \mu(\Omega) > 0 \) and assumption (1) of Proposition 6.4 holds. Then \( \mu \) satisfies (EFT) with \( \Omega \) being an EFT-set and there exists a regular basic pair with respect to \( \Omega \).

![Figure 6. First iteration of an IIFS \( \{S_i\}_{i=1}^\infty \) defined in (1.11). The figure is drawn with \( r = 1/4 \) and \( s = 2/3 \).](image)

In order to prove Example 6.7, we need to establish some preliminary results. We first summarize without proof some elementary properties. Define

\[
\mathcal{I}_{1,0} := \{(S_1, 1)\} \quad \text{and} \quad \mathcal{I}_{1,\ell} := \{(S_{2\ell, 1}, S_{2\ell+1, 1})\} \quad \text{for all } \ell \geq 1.
\]

We remark that \( I_1 = \mathcal{I}_{1,0} \). For \( k, \ell \geq 1 \), let \( W_{k, \ell} := \{(2\ell+1, 3^\ell, 2, 1^{k-1-i}) : i \in \{0, \ldots, k-1\}\} \cup \{(2\ell, 1^k)\} \) (see Figure 8). Proposition 6.8(a) below shows that all multi-indices in \( W_{k, \ell} \) correspond to the same vertex. We will see that for all \( i \in W_{k, \ell} \), \( S_i(\Omega) \subseteq S_{2\ell, \ell}(\Omega) \) and \( (S_i, k+1) \) belongs to a level-(\( k+1 \)) island of a new measure type with two vertices (see Figure 8).

**Proposition 6.8.** Let \( \{S_i\}_{i=1}^\infty \) be defined as in (1.11) and let \( \Omega = (0, 1) \).

(a) Then \( S_{2\ell+1, 1} = S_{2\ell+1, 2} \) for all \( \ell \geq 1 \). Moreover, \( S_i = S_j \) for any \( i, j \in W_{k, \ell} \).

(b) For \( k \geq 0 \) and \( \ell \geq 1 \), \( S_{2\ell}(\Omega) \cap S_{2\ell+1, 3^\ell}(\Omega) = S_{2\ell, 1}(\Omega) \).

**Lemma 6.9.** Each IIFS \( \{S_i\}_{i=1}^\infty \) in (1.11) is of finite type with \( \Omega = (0, 1) \) being an FTC-set and with \( M_k = \Lambda^k \), where \( \Lambda := \{i : i \in \mathbb{Z}_+\} \).
Proof. Using the method in [22, Example 2.8], one can show that \( \{S_i\}_{i=1}^\infty \) generates only three neighborhood types: \([S_1,1],[S_2,1],[S_3,1]\). We omit the details. □

**Proposition 6.10.** Let \( \{S_i\}_{i=1}^\infty \) be an IFS as in (1.11), \((p_i)_{i=1}^\infty\) be a probability vector, and \(\mu\) be the associated self-similar measure. Let \(I_{1,i}\) be defined as in (6.3) for \(i \geq 1\). Then for \(m > k \geq 1\), \(p_{2m}/p_{2m+1} = p_{2k}/p_{2k+1}\) if and only if \(I_{1,k} \approx \mu, \tau, w I_{1,m}\) with \(\tau(x) := r^{m-k}x + s(1-r^{m-k})\) and \(w := p_{2m}/p_{2k}\).

Proof. Fix any \(m > k \geq 1\). We first note that \(\tau \circ S_{2k} = S_{2m}\) and \(\tau \circ S_{2k+1} = S_{2m+1}\). It follows that \(I_{1,k} \approx \tau I_{1,m}\). For \(A \subseteq S_{I_{1,m}}(\Omega)\), we have \(\tau^{-1}(A) \subseteq S_{I_{1,k}}(\Omega)\). Thus
\[
\mu(A) = p_{2m} \cdot \mu \circ S_{2m}^{-1}(A) + p_{2m+1} \cdot \mu \circ S_{2m+1}^{-1}(A),
\]
and
\[
\mu(\tau^{-1}(A)) = p_{2k} \cdot \mu \circ S_{2m}^{-1}(A) + p_{2k+1} \cdot \mu \circ S_{2m+1}^{-1}(A).
\]
Hence \(\mu|_{S_{I_{1,m}}(\Omega)} = w \mu|_{S_{I_{1,k}}(\Omega)} \circ \tau^{-1}\) if and only if \(p_{2m}/p_{2m+1} = p_{2k}/p_{2k+1}\). The result now follows from the definition of \(\approx\).

For \(\ell \geq 1\), define
\[
\bar{w}_1(0, \ell) := p_{2\ell}, \quad \bar{w}_1(k, \ell) := p_{2\ell+1}p_2 \left( \sum_{i=0}^{k-1} p_3^{i}p_4^{1-i} \right) + p_{2\ell}p_4^k \quad \text{for} \quad k \geq 1,
\]
\[
\bar{w}_2(k, \ell) := p_{2\ell+1}p_4^k \quad \text{for} \quad k \geq 0.
\]
We remark that for \(k \geq 0\) and \(\ell \geq 1\),
\[
p_{2\ell}p_4^{k+1} + p_{2\ell+1} \bar{w}_1(k, 1) = \bar{w}_1(k+1, \ell) \quad \text{and} \quad p_{2\ell+1} \bar{w}_2(k, 1) = \bar{w}_2(k+1, \ell).
\]
Also, \(\bar{w}_1(k, \ell)\) denotes the sum of probability weights of all multi-indices in \(W_{k,\ell}\) for \(k, \ell \geq 1\). The following lemma shows how \(\bar{w}_i(k, \ell), i = 1, 2\), arise. Let \(L\) be given as in (1.12) and \(s\) be as in (1.11). Define
\[
B_{1,\ell} := S_{I_{1,\ell}}(\Omega) \quad \text{for} \quad 0 \leq \ell \leq L - 1, \quad \text{and} \quad B_{1,L} := (S_{2L}(0), s).
\]
We remark that \(B_{1,L}\) is the minimum open interval containing \(\bigcup_{\ell \geq L} S_{I_{1,\ell}}(\Omega)\).

**Proposition 6.11.** Assume the hypotheses of Proposition 6.10. Let \(\bar{w}_1(\cdot, \cdot)\) and \(\bar{w}_2(\cdot, \cdot)\) be defined as in (6.4), and \(B_{1,i}\) be defined as in (6.6) for \(0 \leq i \leq L\). Then for \(\ell \geq 1\) and \(k \geq 0\),
\[
(a) \mu|_{S_{|2\ell+1,3i|}}(B_{1,i}) = \bar{w}_1(k, \ell) \mu|_{B_{1,\ell}} \circ S_{|2\ell+1,3i|}^{-1};
\]
\[
(b) \mu|_{S_{|2\ell,i|}}(B_{1,i}) = \bar{w}_2(k, \ell) \mu|_{B_{1,\ell}} \circ S_{|2\ell,i|}^{-1} \quad \text{for} \quad 1 \leq i \leq L;
\]
\[
(c) \mu|_{S_{|2\ell+1,3i|}}(B_{1,i}) = \bar{w}_2(k, \ell) \mu|_{B_{1,\ell}} \circ S_{|2\ell+1,3i|}^{-1} \quad \text{for} \quad i = 0 \text{ or } 2 \leq i \leq L.
\]
Proof. Parts (a) and (b) can be proved by using Proposition 6.8, equation (6.5), and induction. Part (c) can be proved by using Proposition 6.8(c). We omit the details. □

**Lemma 6.12.** Assume the hypotheses of Example 6.7. Let \(I_{1,i}\) be defined as in (6.6) for \(i \in \mathbb{Z}_+\). Then \(T_{1,L} := \{I_{1,i} : i \geq L\}\) is the only level-1 tail (see Figure 7).

Proof. We first note that condition (1) holds with \((S_{2L}(0), s)\) being the minimal open interval. Also, note that \(\lim_{x \to \infty} S_{2k}(x) = \lim_{k \to \infty} S_{2k+1}(x) = s\) for any \(x \in (0, 1)\). Thus condition (2) of Definition 6.1 holds, and hence \(T_{1,L}\) is a semi-tail. Let \(B := \{I_{1,i} : 1 \leq i \leq L - 1\}\). Similarly, \(B \cup T_{1,L}\) is the maximal semi-tail containing \(T_{1,L}\). Moreover, by (1.12) and Proposition 6.10, we have \([\{I\}_\mu : I \in B] \supseteq [\{I\}_\mu : I \in T_{1,L}]\), and there exists some \(\bar{Z} \subseteq T_{1,L}\) such that \([\bar{Z}]_\mu \notin [\{I_{1,i}\}_\mu : 1 \leq i \leq L - 2]\). It follows that conditions (3) and (4) of Definition 6.1 hold. □
Figure 7. Islands, semi-tails, and tails for an IIFS in (1.11). The figure is drawn by using \( r = 1/4 \) and \( s = 2/3 \) and by assuming that (1.12) holds with \( L = 2 \). \( T_{1,2}, T_{2,1,1}, \) and \( T_{2,1,2} \) are defined in Lemma 6.12 and the proof of Example 6.7. They consist of islands enclosed by a box. \( T_{1,2} \) is the only level-1 tail (Lemma 6.12). One can verify directly that \( T_{2,1,1} \) is a tail with the set \( B \) in Definition 6.1 consisting of the island on its left. \( T_{2,1,2} \) is a semi-tail but not a tail; an analogous \( B \) cannot be found, and thus condition (3) of Definition 6.1 is not satisfied.

Proof of Example 6.7. For each \( k \geq 0 \), let \( \mathcal{M}_k = \mathbb{Z}^k_+ \). We show that all assumptions in Proposition 6.4 are satisfied with \( m = 1 \). Let \( I_{1,i} \) be defined as in (6.3) for \( i \geq 0 \). By assumption and Lemma 6.12, \( T_{1,L} := \{ I_{1,i} : i \geq L \} \) is the only level-1 tail. Let \( I_1 := I_1 \setminus T_{1,L} = \{ I_{1,i} : 0 \leq i \leq L-1 \} \). Thus assumption (1) in Proposition 6.4 holds with \( m = 1 \). Let \( I_{1,\mu} := \{ I_{1,i} \}_{i=0}^{L-1} \).

Since \( I(v_{\text{root}}) \approx \mu \) \( I_{1,0} \), \( I_{1,0} \) does not generate any level-2 nonbasic island with respect to \( I_1 \). It follows that assumption (2) in Proposition 6.4 holds for \( I = I_{1,0} \). Fix any \( \ell \in \{1, \ldots, L-1\} \). For \( i \geq 1 \), and \( j = 1, 2 \), define

\[
I^2_{2,\ell,i} := \{(S(2e,2i), 2), (S(2e+1,2i+1), 2)\}, \quad T^2_{2,\ell,i} := \{(S(2e,2i+1), 2), (S(2e+1,2i+1), 2)\}, \quad T^2_{2,\ell,0} := \{(S(2e+1,i), 2)\},
\]

(See Figure 8). Here for an island \( T^j_{2,\ell,i} \), the subscript 2 denotes the level \( V_2 \) that the island belongs, \( \ell \) indicates that \( T^j_{2,\ell,i} \) is an offspring of \( I_{1,\ell} \), and \( i \) indexes the islands according to the iterations of the IFS maps. The superscript \( j \) labels the parent of a vertex \( v \in T^j_{2,\ell,i} \). If \( j = 1 \), \( v \) is an offspring of \( (S_{2e}, 1) \); if \( j = 2 \), \( v \) is an offspring of \( (S_{2e+1}, 1) \). The set of all offspring of \( I_{1,\ell} \) is

\[
O(I_{1,\ell}) = \{ T^2_{2,\ell,i} : i = 1, \ldots, L-1 \} \cup \{ I^2_{2,\ell,0} \} \cup T^2_{\ell,1} \cup T^2_{2,\ell,2}.
\]

Let \( \tilde{w}_1(\cdot, \cdot) \) and \( \tilde{w}_2(\cdot, \cdot) \) be defined as in (6.4). Proposition 6.11 implies that \( I_{1,i} \approx_{\mu, S_{2e}, \tilde{w}_1(0, \ell)} T^2_{2,\ell,i} \) for \( i \geq 1 \), and \( I_{1,i} \approx_{\mu, S_{2e+1}, \tilde{w}_2(0, \ell)} T^2_{2,\ell,i} \) for \( i \geq 2 \) and \( i = 0 \); moreover, \( [I^2_{2,\ell,1}]_{\mu} \notin I_{1,\mu} \). Thus \( I_{1,\ell} \) generates only one level-2 nonbasic island, namely, \( T^2_{2,\ell,1} \), with respect to \( I_1 \). Also, \( T_{1,L} \approx_{\mu, S_{2e}, \tilde{w}_1(0, \ell)} T^2_{2,\ell,1} \) and \( T_{1,L} \approx_{\mu, S_{2e+1}, \tilde{w}_2(0, \ell)} T^2_{2,\ell,2} \). Similarly, for \( i \geq 1 \), \( j = 1, 2 \) and \( k \geq 3 \), define

\[
I^1_{k,\ell,i} := \{(S(2e+1,3^k-3,2i), k), (S(2e+1,3^k-3,2i+1), k)\}, \quad I^2_{k,\ell,i} := \{(S(2e+1,3^k-2,2i), k), (S(2e+1,3^k-2,2i+1), k)\}, \quad I^2_{k,\ell,0} := \{(S(2e+1,3^k-2), k)\}, \quad T_{k,\ell,j} := \{ I^j_{k,\ell,i} : i \geq L \}.
\]
Assume that \( S \) the collection of all level-\( k \) island with respect to \( I \). Using Proposition 6.8(a), we have \( I \) and \( m \). Corollary 6.13. Let \( I \) be a probability vector, and \( \Lambda \) for \( i \). Similarly, Proposition 6.11 implies that \( \mathcal{I}(v_{\text{root}}) \)

\[
\begin{align*}
\mathcal{I}_1 & \quad \mathcal{I}_{1,1} \quad \mathcal{I}_{1,2} \quad \mathcal{I}_{1,3} \quad \cdots \quad \mathcal{I}_{1,0} \\
\mathcal{I}_{2,1,1} & \quad \mathcal{I}_{2,1,1} \quad \mathcal{I}_{2,1,0} \\
\mathcal{I}_{3,1,1} & \quad \mathcal{I}_{3,1,1} \\
\vdots & \\
\end{align*}
\]

Figure showing some iterates of the IIFS

\[
\mathcal{I}(v_{\text{root}})
\]

\[
\begin{align*}
\mathcal{I}_1 & \quad \mathcal{I}_{1,1} \quad \mathcal{I}_{1,2} \quad \mathcal{I}_{1,3} \quad \cdots \quad \mathcal{I}_{1,0} \\
\mathcal{I}_{2,1,1} & \quad \mathcal{I}_{2,1,1} \quad \mathcal{I}_{2,1,0} \\
\mathcal{I}_{3,1,1} & \quad \mathcal{I}_{3,1,1} \\
\vdots & \\
\end{align*}
\]

that overlap exactly and hence give rise to the same vertex. All nonbasic islands are boxed.

Assume that \( \mathcal{I}_{m,\ell} \) \( \mathcal{I}_{m+1,\ell} \) for \( m \geq 2 \). Then the set of all offspring of \( \mathcal{I}_{m,\ell} \) is

\[
O(\mathcal{I}_{m,\ell}) = \{\mathcal{I}_{m+1,\ell,i} : i = 1, \ldots, L - 1, j = 1, 2\} \cup \{\mathcal{I}_{m+1,\ell,0}\} \cup \mathcal{T}_{m+1,\ell,1} \cup \mathcal{T}_{m+1,\ell,2}.
\]

Using Proposition 6.8(a), we have \( \mathcal{I}_{m+1,\ell,i} = \{(S_2(2l+1,3m-2), m), (S_2(2l+1,3m-1), m)\} \) is a level-\( m \) nonbasic island with respect to \( \mathcal{I}_1 \). Thus \( \mathcal{I}_{m+1,\ell,1} \) generates only one level-(\( m + 1 \)) nonbasic island, namely, \( \mathcal{I}_{m+1,\ell,1} \), with respect to \( \mathcal{I}_1 \), \( \mathcal{T}_{m+1,\ell,1} \approx \mu, S_2(2l+1,3m-1), \mathcal{T}_{m+1,\ell,1} \cup \mathcal{T}_{m+1,\ell,2} \). Hence, Remark 6.5 implies that assumption (2) in Proposition 6.4 holds. By induction, \( \{\mathcal{I}_{k,\ell} : \ell = 1, \ldots, L - 1\} \) is the collection of all level-\( k \) nonbasic islands with respect to \( \mathcal{I}_1 \). Since the closure of \( \mathcal{S}_{\mathcal{T}_{k,\ell,1}}(\mathcal{G}) \) converges to a point, assumption (3) in Proposition 6.4 holds.

The following corollary follows from Example 6.6 and Theorem 1.1.

**Corollary 6.13.** Let \( \{S_{i}\}_{i \in A} \) be an IIFS on \( \mathbb{R} \) satisfying (OSC) with \( \Omega = (a, b) \), \( (p_{i})_{i \in A} \) be a probability vector, and \( \mu \) be the associated self-similar measure. Assume that there exist two nonempty finite disjoint index sets \( \Gamma_{\ast} \) and \( \Gamma'_{\ast} \) such that \( \mathcal{T}_1 = \{\mathcal{T}_{1,\ell} : \ell \in \Gamma_{\ast}\} \) and \( \mathcal{I}_m = \mathcal{I}_1 \setminus (\bigcup_{\ell \in \Gamma_{\ast}} \mathcal{T}_{\ell}) = \{\mathcal{T}_{1,\ell} : \ell \in \Gamma'_{\ast}\} \). Let \( \Gamma := \Gamma_{\ast} \cup \Gamma'_{\ast} \). Let \( \mathcal{M}_{\mu}(\infty), \mathcal{F}_{\ell}(\alpha) \), and \( \tilde{\alpha}_{\ell} \) be defined as in (1.6) and (1.7) for \( \ell \in \Gamma \). For each \( \ell \in \Gamma_{\ast} \), let \( \mathcal{T}_{1,\ell} := \{\mathcal{I}_{1,\ell} : i \geq 0\} \). Assume that for each \( \ell \in \Gamma \), \( \lim_{\alpha \to \tilde{\alpha}_{\ell}} \mathcal{F}_{\ell}(\alpha) > 1 \). Also,
assume that there exist positive constants \( C, k_0 \) such that \( \mu|_{T_k,\ell} \leq C p_k \mu|_{T_1,\ell} \circ \tau_k^{-1} \) for any \( k \geq k_0 \geq 2 \), where \( I_k, T_{k,\ell} \) and \( T_{1,\ell} \) are the open intervals corresponding to \( I_k, T_{k,\ell} := \{\{1, i : i \geq k\} \) and \( T_{1,\ell}, \) respectively, and \( \tau_k \) is a similitude mapping \( I_0 \) onto \( I_k \). Then conclusions (a) and (b) of Theorem 1.1 hold.

6.2. The family of IIFSs with overlaps in (1.11). We first state a result concerning the existence of self-similar measures associated with an IIFS \( \{S_i\}_{i \in \Lambda} \). The proof is similar to that of [8, Theorem 2.8].

**Proposition 6.14.** Let \( \{S_i\}_{i \in \Lambda} \) be an IIFS of contractions (not necessarily similitudes) on \( \mathbb{R}^d \) and let \( r_i \) be the Lipschitz constant of \( S_i \). Assume there exists \( c > 0 \) such that \( r_i \leq c < 1 \) for all \( i \in \Lambda \). Then for any probability vector \( (p_i)_{i \in \Lambda} \), there exists a unique probability measure \( \mu \) satisfying the self-similar identity (2.1).

In this subsection, we consider the family of IIFSs defined as in (1.11) and fix the FTC-set \( \Omega = (0, 1) \). Proposition 6.14 implies the existence of a self-similar measure \( \mu \) for any probability vector \( (p_i)_{i \in \Lambda} \). Assume that (1.12) holds and \( L \) is given as in (1.12) in the rest of this section. Let \( I_1, i \) be defined as in (6.3) for \( i \geq 0 \), and \( T_{1,\ell} \) be defined as in Lemma 6.12. Then \( T_1 = \{T_{1,\ell}\} \) and \( I_1 := \mathbb{I} \setminus T_{1,\ell} = \{I_{1,i} : 0 \leq i \leq L - 1\} \). Let \( B_{1,\ell} \) be defined as in (6.6) for \( \ell \in \Gamma := \{0, \ldots, L\} \). Define \( B := \{B_{1,\ell} : \ell \in \Gamma\} \).

By the proof of Examples 6.7, we see that all assumptions in Proposition 6.4 hold with \( m = 1 \). Hence \( \mu \) satisfies (EFT) with \( \Omega = (0, 1) \) being an EFT-set. For each \( \ell \in \Gamma \), let \( (P_{k,\ell})_{k \geq 1} \) be the family of refining \( \mu \)-partitions of \( B_{1,\ell} \) defined as in the proof of Proposition 6.4. Then \( (B, P) := (\{B_{1,\ell}\}, (P_{k,\ell})_{k \geq 1})_{\ell \in \Gamma} \) is a regular basic pair.

In the rest of this section, we use the notation in Section 4.2. For \( k \geq 2 \) and \( \ell \in \Gamma \), let \( P_{k,\ell} \) and \( P_{k,\ell}^2 \) be defined as in (2.7) with respect to \( B \). Define

\[
\mathcal{I}_{2,0,0} := \{(S_{1(1,2)}, 2)\} \quad \text{and} \quad \mathcal{I}_{2,0,1} := \{(S_{1(1,2)}, 2), (S_{1(2,1)}, 2)\} \quad \text{for} \quad i \geq 1.
\]

Let \( \mathcal{T}_{2,0,0} := \{I_{2,i} : i \geq L\} \). We first observe that \( O(I_{1,0}) = \{I_{2,i} : 0 \leq i \leq L - 1\} \cup \{I_{2,0,0}\} \). Since \( I(v_{\text{root}}) = \mu_{S_{1,\ell}, p_{1,\ell}} \), \( I_{1,i} = \mu_{S_{1,\ell}, p_{1,\ell}} I_{2,0,0} \) for all \( 0 \leq i \leq L - 1 \) and \( T_{1,\ell} \approx \mu_{S_{1,\ell}, p_{1,\ell}} T_{2,0,0} \). Let

\[
B_{2,0,i} := S_{2,0,i}(\Omega) \quad \text{for} \quad 0 \leq i \leq L - 1 \quad \text{and} \quad B_{2,0,L} := S_{1}(B_{1,L}). \quad (6.9)
\]

Then \( B_{2,0,L} \) is the minimal open interval containing \( S_{2,0,0}(\Omega) \). Thus \( P_{2,0} := \{B_{1,0}\} \) and \( P_{k,0} := P_{2,0} = \{B_{2,i} : 0 \leq i \leq L\} \) for all \( k \geq 2 \).

For \( \ell \in \{1, \ldots, L - 1\} \), \( \pi \geq 1, j \geq 0, \) and \( k \geq 2 \), let \( I_{k,\ell,i}^1, I_{k,\ell,i}^2, T_{k,\ell,1}, T_{k,\ell,2} \) be defined as in (6.7) and (6.8). By the proof of Example 6.7, we see that

\[
O(I_{\ell,0}) = \{I_{k,\ell,i}^1 : i \in \pi\} \cup T_{2,1,1} \cup \{I_{2,\ell,i}^2 : i \in \pi\} \cup T_{2,2,2} \cup \{I_{2,\ell,0}\};
\]

moreover for \( k \geq 2 \), \( O(T_{k,\ell,1}) = \{I_{k+1,\ell,i} : i \in \pi\} \cup T_{k+1,\ell,1} \cup \{I_{k+1,\ell,0}\}, \) and \( T_{k,\ell,1} \) is the only level-\( k \) nonbasic island with respect to \( I_1 \) generated by \( I_1 \). For \( k \geq 2 \), define

\[
B_{k,\ell,i} := S_{k,\ell,i}(\Omega), \quad i \in \pi, \quad B_{k,\ell,L} := S_{2,1,3k-2}(B_{1,L}),
\]

\[
B_{k,\ell,2L} := S_{2,1,3k-2}(B_{1,L}), \quad B_{k,\ell,0} := S_{2,1,3k-2}(\Omega).
\]

Thus for all \( \ell \in \{1, \ldots, L - 1\}, \)

\[
P_{1,\ell} := \{B_{1,\ell}\}, \quad P_{k,\ell} := \bigcup_{m=2}^k \{B_{m,\ell,i} : i = 0, \ldots, L, L+2, \ldots, 2L\} \cup \{B_{k,\ell,\ell+1}\}, \quad k \geq 2.
\]
Moreover, \( P^1_{k,\ell} = \bigcup_{m=2}^k \{ B_{m,\ell,i} : i = 0, \ldots, L, L + 2, \ldots, 2L \} \) and \( P^2_{k,\ell} = \{ B_{k,\ell,L+1} \} \) for \( k \geq 2 \).

Define \( B_{k,L,0} := S_{I,k+L-2}(\Omega) \) and \( B_{k,L,1} := (S_{2(k+L-1)}(0), s) \) for all \( k \geq 2 \). Then \( B_{k,L,1} \) is the minimum open interval containing \( \bigcup_{i \geq k+1} B_{i,L,0} \). Define \( P_{1,L} := \{ B_{1,L} \} \). If there exists some \( k_0 \geq 2 \) such that \( B_{1,L} \simeq_{\mu} B_{k_0,L,1} \), then

\[
P_{k,L} := \{ B_{m,\ell,0} : 2 \leq m \leq k \} \cup \{ B_{k,L,1} \}, \quad 2 \leq k \leq k_0, \text{ and } P_{k,L} := P_{k_0,L} \text{ for } k > k_0.
\]

Otherwise, \( P_{k,L} := \{ B_{m,\ell,0} : 2 \leq m \leq k \} \cup \{ B_{k,L,1} \} \) for \( k \geq 2 \).

We use the method in Section 4.2 to derive renewal equations for \( -\Delta_{\mu} \). From the discussion above, we have the following results:

(a) \( 0 \in \Gamma_1, \kappa_0 = 2, G_{k,\ell} := \{ 0, \ldots, L \}, \) and \( \#P_{k_0,0} = L + 1 \).

(b) \( \Pi \subseteq \Pi_2 \). If \( \ell \in \Pi \), then \( G_{k,\ell} := \{ 0, \ldots, L, L + 2, 2L \}, \) and

\[
\#P_{k,\ell} = 2L(k-1) + 1 \text{ for } k \geq 2.
\]

(c) \( G_{k,L} = \{ 0 \} \) for all \( 2 \leq k \leq \kappa_\ell \) and \( G'_{k,L} = \{ 1 \} \) for all \( 2 \leq k < \kappa_\ell \); moreover, \( \#P_{k,L} = k \) for \( 1 \leq k \leq \kappa_L \).

Combining these results with the proof of Example 6.7, we obtain the following proposition.

**Proposition 6.15.** Let \( \xi(\cdot, \cdot, \cdot) \) and \( c(\cdot, \cdot, \cdot) \) be defined as in Proposition 4.6. Let \( \tilde{w}_1(\cdot, \cdot) \) and \( \tilde{w}_2(\cdot, \cdot) \) be defined as in (6.4). Then

(a) \( \xi(2,0,i) = p_{1r} \) and \( c(2,0,i) = i \) for all \( 0 \leq i \leq L \).

(b) for \( k \geq 2 \) and \( \ell \in \Pi \),

\[
\xi(k,\ell,i) = \tilde{w}_1(k-2,\ell)r^{k+\ell-2} \text{ and } c(k,\ell,i) = i \text{ for all } 1 \leq i \leq L;
\]

\[
\xi(k,\ell,i) = \tilde{w}_2(k-2,\ell)r^{k+\ell-2} \text{ and } c(k,\ell,i) = i + L \text{ for all } L + 2 \leq i \leq 2L \text{ or } i = 0.
\]

(c) \( 1 \leq c(k,L,0) \leq L - 1 \) and \( \xi(k,L,0) = p_{2k}/p_{2c(k,L,0)} \cdot r^{k-c(k,L,0)} \) for all \( 2 \leq k \leq \kappa_L \). If \( L \in \Gamma_1 \), then \( \xi(\kappa_L,L,1) = p_{2\kappa_L}/p_{2L} \cdot r^{\kappa_L-1} \) and \( c(\kappa_L,L,1) = L \).
Let \( f_\ell(t) \) be defined as in (4.9). Applying the vector-valued renewal equations (4.10) and (4.13), we have
\[
f_0(t) = (p_1 r)^\alpha \sum_{i=0}^L f_i(t + \ln(p_1 r)) + z_0(\alpha)(t),
\]
\[
f_\ell(t) = \sum_{k=0}^\infty (\overline{w}(k, \ell)r^{\ell+k})^\alpha \sum_{i=1}^L f_i(t + \ln(\overline{w}(k, \ell)r^{\ell+k}))
\]
\[
+ \sum_{k=0}^\infty (\overline{w}(k, \ell)r^{\ell+k})^\alpha \left( \sum_{j=2}^L f_j(t + \ln(\overline{w}(k, \ell)r^{\ell+k})) \right)
\]
\[
+ f_0(t + \ln(\overline{w}(k, \ell)r^{\ell+k})) + z_\ell(\alpha)(t), \quad \text{for } \ell \in \{1, \ldots, L - 1\},
\]
where \( z_0(\alpha)(t) := e^{-\alpha t}e^t(2, 0) \), \( z_\ell(\alpha)(t) := e^{-\alpha t}(N(e^t, -\Delta_\mu|_{B_{n,L}}) + e^t, nt, \ell) \) for \( 1 \leq \ell \leq L - 1 \). Moreover, if \( L \in \Gamma_1 \),
\[
f_L(t) = \sum_{i=2}^{\kappa_L} \left( (\xi(i, L, 0))^\alpha f_{\ell(i,L,0)}(t + \ln(\xi(i, L, 0))) \right)
\]
\[
+ (\xi(\kappa_L, L, 1))^\alpha f_L(t + \ln(\xi(\kappa_L, L, 1))) + z_L(\alpha)(t),
\]
where \( z_L(\alpha)(t) := e^{-\alpha t}e^t(\kappa_L, L) \). Otherwise, i.e., \( L \in \Gamma_2 \), we have
\[
f_L(t) = \sum_{i=2}^{\kappa_L} \left( (\xi(i, L, 0))^\alpha f_{\ell(i,L,0)}(t + \ln(\xi(i, L, 0))) \right) + z_L(\alpha)(t),
\]
where \( z_L(\alpha)(t) := e^{-\alpha t}(N(e^t, -\Delta_\mu|_{B_{n,L,1}}) + e^t, nt, L) \).

Let \( \mu_{\ell,m}^{\alpha} \), \( \ell, m \in \Gamma \), be the discrete measure defined in (4.14).

**Proposition 6.16.** Let \( F_\ell(\alpha) \) and \( \bar{\alpha}_\ell \) be defined as in (1.7) for \( \ell \in \Gamma \). Then \( \bar{\alpha}_\ell = 0 \) and \( F_\ell(0) > 1 \) for all \( \ell \in \Gamma \).

**Proof.** Similar to that of Proposition 5.2. \( \square \)

**Proposition 6.17.** Assume that (1.12) holds. For \( \ell \in \{1, \ldots, L - 1\} \), there exists a constant \( C > 0 \) such that \( N(e^t, -\Delta_\mu|_{B_{n,L,1}}) \leq C \).

**Proof.** We note that \( \mu|_{B_{n,L,1}} \leq (\bar{w}(n_{t-2, L})/p_2)\mu \circ S_{2t+3,3n_{t-2}}^{-1} \) on \( B_{n,L,1} \). The proof is similar to that of Proposition 5.3. \( \square \)

**Proposition 6.18.** Assume that (1.12) and (1.13) hold. If \( L \in \Gamma_2 \), then there exists a constant \( C > 0 \) such that \( N(e^t, -\Delta_\mu|_{B_{n,L,1}}) \leq C \).

**Proof.** Let \( \tau(x) = r^{n_{t-2, L}} + s(1 - r^{n_{t-2, L}}) \). Using (1.12), (1.13), and the proof of Proposition 6.10, we have \( \mu|_{B_{n,L,0}} \leq C(p_2 n_{t,L,0}) \mu \circ \tau^{-1} \) on \( B_{n,L,0} \) for any \( i \geq 0 \). Thus \( \mu(B_{n,L,1}) \leq (p_2 n_{t,L,0}) \mu \circ \tau^{-1} \) on \( B_{n,L,1} \). The result can now be deduced by using the method in Proposition 5.3. \( \square \)

**Proposition 6.19.** Assume that (1.12) and (1.13) holds. Let \( \alpha \) be the unique number such that the spectral radius of \( M_\alpha(\infty) \) is equal to 1. Then there exists some \( \sigma > 0 \) such that for all \( \ell \in \Gamma \), \( z_\ell(\alpha)(t) = o(e^{-\sigma t}) \) as \( t \to \infty \).

**Proof.** The proof is similar to that of Proposition 5.4. \( \square \)
Proof of Theorem 1.3. Combine Propositions 6.16 and 6.19, and Theorem 1.1.

Example 6.20. Let \((S_i)_{i=1}^\infty\) be an IIFS as in (1.11), \((p_i)_{i=1}^\infty\) be a probability vector, and \(\mu\) be the associated self-similar measure. Assume that \(r = 1/4\), \(s = 2/3\), \(p_1 = 1/3\), and \(p_{2k} = p_{2k+1} = 1/4^k\) for all \(k \geq 1\). Then \(d_s \approx 0.93168\).

Proof. We note that (1.12) and (1.13) hold with \(L = 2\); moreover \(L \in \Gamma_1\) and \(\kappa_L = 2\). Theorem 1.3 implies that \(\alpha\) is the unique positive number satisfying \(|I_3 - M_\alpha(\infty)| = 0\), where \(I_3\) is the \(3 \times 3\) identity matrix. Since

\[
M_\alpha(\infty) = \begin{bmatrix} (p_1r)^\alpha & (p_1r)^\alpha & (p_1r)^\alpha \\ a & b & a + b \\ 0 & (qr)^\alpha & (qr)^\alpha \end{bmatrix},
\]

where \(q = 1/4\), \(a := \sum_{k=0}^\infty \hat{w}_2(k, 1) r^{k+1}\alpha\) and \(b := \sum_{k=0}^\infty \hat{w}_3(k, 1) r^{k+1}\alpha\), \(\alpha\) is the unique positive number satisfying \((1 - (p_1r)^\alpha)((qr)^\alpha(a + 1) + b - 1) + a(p_1r)^\alpha = 0\). Hence, \(d_s = 2\alpha\), which can be easily approximated.

7. Comments and questions. We do not know whether the assumption concerning the error estimates in Theorem 1.1(b) can be removed.

It is of interest to express the eigenvalue counting function in terms of the properties of the measure and the domain, as in the original Weyl law. Also, in view of Weyl’s conjecture stated in the introduction, it is of interest to study the second order term in the asymptotic expansion of the eigenvalue counting function.

Appendix. Vector-valued renewal theorem. For convenience, we state the Appendix. Vector-valued renewal theorem. For convenience, we state the vector-valued renewal theorem by Lau et al. [24], which is used in the proof of Theorem 1.1. We first introduce some terminology and refer the reader to [24, 31] for unexplained terms. Let \(F\) be a matrix-valued Radon measure that vanishes on \((-\infty, 0)\), i.e.,

\[
F = \begin{bmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & \ddots & \vdots \\ F_{n1} & \cdots & F_{nn} \end{bmatrix},
\]

where \(F_{ij}(x) = \mu_{ij}(-\infty, x]\) and each \(\mu_{ij}\) is a Radon measure (i.e., positive Borel regular measure) on \(\mathbb{R}\) that vanishes on \((-\infty, 0)\). Define \(F(\infty) := [F_{ij}(\infty)]\) and let \(m = [m_{ij}] = \int_0^\infty xdF_{ij}\) be the moment matrix. If

\[
\sum_{i=1}^n F_{ij}(0) < \sum_{i=1}^n F_{ij}(\infty) \quad \text{for } 1 \leq j \leq n,
\]

each column of \(F\) is said to be nondegenerate at 0. For any path \(\gamma = (i_1, \ldots, i_k)\) with \(i_j \in \{1, \ldots, n\}\) for any \(j = 1, \ldots, k\), we define

\[
\mu_\gamma := \mu_{i_1 i_2} \ast \mu_{i_2 i_3} \ast \cdots \ast \mu_{i_{k-1} i_k}.
\]

In this case, \(\gamma\) is called a cycle if \(i_1 = i_k\); in particular, if it is a cycle and \(i_1, \ldots, i_{k-1}\) are distinct, then \(\gamma\) is said to be a simple cycle. We denote by \(\mathbb{R}_F\) the closed subgroup of \((\mathbb{R}, +)\) generated by

\[
\bigcup \{\text{supp}(\mu_\gamma) : \gamma \text{ is a simple cycle on } \{1, \ldots, n\}\}.
\]

The following theorem, stated in [31], is modified from [24, Theorem 4.3].
Theorem A.1. (Lau et al. [24]) Let $F$ be an $n \times n$ matrix-valued Radon measure defined on $\mathbb{R}$ that vanishes on $(-\infty, 0)$ and assume that each column of $F$ is non-degenerate at 0. Suppose $F(\infty)$ is irreducible and has maximal eigenvalue 1. Let $U = \sum_{k=0}^{\infty} F^k$ and let $z$ be a directly Riemann integrable function on $\mathbb{R}$ that vanishes on $(-\infty, x_0)$ for some $x_0 \in \mathbb{R}$. Then $f = z \ast U$ is a bounded Borel measurable solution of

$$f(x) = (f \ast F)(x) + z(x), \quad x \in \mathbb{R}, \quad (A.1)$$

and it is unique in the class of Borel measurable solutions that vanish on $(-\infty, x_0)$. Furthermore, the following hold:

(a) If $\mathbb{R}_F = \mathbb{R}$, then

$$\lim_{x \to \infty} f(x) = \left( \int_{-\infty}^{\infty} z(t) \, dt \right) A,$$

where

$$A = \frac{1}{\gamma} \begin{bmatrix} u_1 v_1 & \ldots & u_n v_n \\ \vdots & \ddots & \vdots \\ u_n v_1 & \ldots & u_n v_n \end{bmatrix}, \quad \gamma = [v_1, \ldots, v_n] \begin{bmatrix} m_{11} & \ldots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \ldots & m_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix},$$

and $u = [u_1, \ldots, u_n], v = [v_1, \ldots, v_n]$ are the unique normalized positive right and left 1-eigenvectors of $F(\infty)$, respectively. ($A = 0$ if one of the $m_{ij}$ is $\infty$.)

(b) If $\mathbb{R}_F = \langle \lambda \rangle$ for some $\lambda > 0$, then for each $x > 0$,

$$\lim_{\ell \to \infty} \left[ f_1(x + a_{11} + \ell \lambda), \ldots, f_n(x + a_{1n} + \ell \lambda) \right] = \left( \sum_{k=-\infty}^{\infty} z(x + k \lambda) \right) A,$$

where $a_{ij} \in \text{supp}(\mu_{\gamma(1,j)})$ and $\gamma(1,j)$ is any path from 1 to $j$ such that $\mu_{\gamma(1,j)} \neq 0$.

Acknowledgments. Part of this work was carried out when the first author was visiting the Center of Mathematical Sciences and Applications of Harvard University. He is indebted to Professor Shing-Tung Yau for making the visit possible, and thanks the center for its hospitality and support. The authors are grateful to the anonymous referee for some valuable suggestions and comments.

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Received November 2016; revised November 2017.

*E-mail address: smngai@georgiasouthern.edu*

*E-mail address: twmath2016@163.com*

*E-mail address: xieyuanyuan198767@163.com*