Subleading terms in asymptotic Passarino-Veltman functions

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Abstract

We write explicit and self-contained asymptotic expressions for the tensorial $B$, $C$ and $D$ Passarino-Veltman functions. These include quadratic and linear logarithmic terms, as well as subleading constant terms. Only mass-suppressed $O(m^2/s)$ contributions are neglected. We discuss the usefulness of such expressions, particularly for studying one-loop effects in 2-to-2 body processes at high energy.

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1 Introduction

The relevance of high energies for simplifying the parameter independent tests of the Standard Model (SM) and its supersymmetric extensions, like e.g. the Minimal Supersymmetric Standard Model (MSSM), has been stressed during the last few years with applications to lepton and hadron colliders. Indeed, at the one-loop leading logarithmic level, the high energy behavior of the various 2-body helicity amplitudes in $e^+e^−, \gamma\gamma, q\bar{q}, qg, gg,...$ processes reflects in a direct way the gauge and the Yukawa structures of the basic Lagrangian. For reviews in SM see [1]; while the MSSM case has been studied in e.g. [2], where the leading 1-loop SUSY and standard virtual effects have been identified for processes containing any kind of external particles. In particular, simple rules have been established giving the coefficients of the leading $\ln$ and $\ln^2$ contributions.

These rules can be checked explicitly by computing the one loop diagrams in terms of Passarino-Veltman (PV) functions [3], and then using their asymptotic expansion at the leading logarithmic level (LL). Such calculations have already appear in [1, 2].

However, things are less simple if one wants to keep the subleading no-logarithmic asymptotic contributions, described by the so called constant terms, which are independent of the invariant c.m. energy of any pair of external legs. Such terms include a priori true constants (numbers), but also angular dependent contributions, or terms involving ratios of external and/or internal masses. The later are particularly relevant for SUSY cases, involving diagrams containing many different internal masses. In such cases, the constant subleading terms may be used to identify tests of model-parameters at high energies, which may be simpler than whatever is possible at lower energies.

Another possibility is to explicitly check the details of the remarkable total helicity conservation (HC) property, which has been established to all orders, for any 2-to-2 body process at high energy $(s, |t|, |u|)$, in any supersymmetric extension of the standard model [4]. According to HC, in such supersymmetric extensions, only the amplitudes where the sum of the helicities of the two incoming particles equals the sum of the helicities of the two outgoing ones, could be non-vanishing at high energies and fixed angles.

For these purposes it should be convenient to have at our disposal asymptotic expressions of the PV functions, which include also the subleading terms contributing to the constants of the physical asymptotic amplitudes discussed above. These go beyond the expressions presented in [5], which only include the logarithmic structure. In achieving this we only need the expressions for the $B_j$ functions and the tensorial decomposition of $C_j$ and $D_j$ functions partly given in [6], combined with the asymptotic expressions of the $C_0$ and $D_0$ functions established by Denner and Roth in [5].

In the present work we use the same notation as in Hagiwara et al [6]. For convenience, we thought that it would be worthwhile to include in the paper all aforementioned formulae. We hope this will be useful for future analyses of the kind mentioned above.

The contents of the paper are as follows. In Sect. 2 we write the definitions for the $C_j$ and $D_j$ functions, as well as the exact expressions for the and $B_j$ functions. In Sect.3
we give the explicit analytical results for the asymptotic quadratic and linear logarithmic
terms involving the correct mass-scales, as well as the subleading constant terms of \( B, C \)
and \( D \) functions. In Sect. 4 we present some illustrations and discuss specific properties.
The conclusions are given in Section 5; while in the appendices we present the connections
between the the Hagiwara notation and the one adapted by LoopTools [6, 7], as well the
reduction formalism for the \( C_j \) and \( D_j \) functions.

2 Definitions and conventions

This writing is self-contained, meaning that all definitions and conventions have been
recalled in a uniform fashion in terms of external and internal masses and momenta. We
use the Hagiwara et al [6] definitions of the tensorial functions, but in Appendix we give
also the relations with LoopTools definitions.

\[
A(m_1) = \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2} \int d^n k \frac{m_1^2}{N_1} \left[ \Delta - \ln \frac{m_1^2}{\mu^2} + 1 \right] ,
\]

\[
B_{\mu}(12) = \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2} \int d^n k \frac{[1, k^\mu, k^\nu]}{N_1 N_2} ,
\]

\[
C_{\mu}(123) = \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2} \int d^n k \frac{[1, k^\mu, k^\nu, k^\rho]}{N_1 N_2 N_3} ,
\]

\[
D_{\mu}(1234) = \frac{(2\pi\mu)^{2\epsilon}}{i\pi^2} \int d^n k \frac{[1, k^\mu, k^\nu, k^\rho, k^\sigma]}{N_1 N_2 N_3 N_4} ,
\]

with

\[
n = 4 - 2\epsilon , \quad \Delta = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) ,
\]

\[
N_1 = k^2 - m_1^2 + i\epsilon ,
N_2 = (k + p_1)^2 - m_2^2 + i\epsilon ,
N_3 = (k + p_1 + p_2)^2 - m_3^2 + i\epsilon ,
N_4 = (k + p_1 + p_2 + p_3)^2 - m_4^2 + i\epsilon .
\]

The definitions of the above functions are completed by the Figures 1, 2, 3 where all
external momenta are incoming, and the arrowed internal line carries the momentum \( k \)
in the direction of the arrow; compare 1, 4. The internal masses are also indicated there.

Following Hagiwara et al [6], we expand the tensorials in 2, 3, 4 for the respective
\( B_j \), \( C_j \) and \( D_j \) functions using the definitions

\[
B^{\mu}(12) = p_1^{\mu} B_1(12) ,
B^{\mu\nu}(12) = p_1^{\mu} p_1^{\nu} B_{21}(12) + g^{\mu\nu} B_{22}(12) ,
B_j(12) = B_j(p_1^2; m_1, m_2) = B_j(p_2^2; m_1, m_2) ,
\]
\[
\begin{align*}
C^\mu(123) &= p_1^\mu C_{11}(123) + p_2^\mu C_{12}(123) ,
\quad C^{\mu\nu}(123) = p_1^\mu p_1^\nu C_{21}(123) + p_2^\mu p_2^\nu C_{22}(123) + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)C_{23}(123) + g^{\mu\nu}C_{24}(123) ,
\quad C^{\mu\nu\rho}(123) = \sum_{i=1,2} C_{00i}(123)(g^{\mu\nu}\rho_i + g^{\mu\rho}\nu_i + g^{\nu\rho}\mu_i) + \sum_{i,j,k=1,2} C_{ijk}(123)p_i^\mu p_j^\nu p_k^\rho ,
\quad C_j(123) = C_j(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) ,
\end{align*}
\]

\[
\begin{align*}
D^\mu(1234) &= p_1^\mu D_{11}(1234) + p_2^\mu D_{12}(1234) + p_3^\mu D_{13}(1234) ,
\quad D^{\mu\nu}(1234) = p_1^{\mu\nu} D_{21}(1234) + p_2^{\mu\nu} D_{22}(1234) + p_3^{\mu\nu} D_{23}(1234) + (p_1^{\mu\nu} p_2^\rho + p_2^{\mu\nu} p_1^\rho)D_{24}(1234) + (p_1^{\mu\rho} p_2^\nu + p_2^{\mu\rho} p_1^\nu)D_{25}(1234) + (p_1^{\mu\nu} p_3^\rho + p_3^{\mu\nu} p_1^\rho)D_{26}(1234) + g^{\mu\nu}D_{27}(1234) ,
\quad D^{\mu\nu\rho}(1234) = \sum_{i=1,2,3} D_{00i}(1234)(g^{\mu\nu}p_i^\rho + g^{\mu\rho}p_i^\nu + g^{\nu\rho}p_i^\mu) + \sum_{i,j,k=1,2,3} D_{ijk}(1234)p_i^\mu p_j^\nu p_k^\rho ,
\quad D_j(1234) = D_j(p_1^2, p_2^2, p_3^2, p_4^2; (p_1 + p_2)^2, (p_2 + p_3)^2; m_1, m_2, m_3, m_4) .
\end{align*}
\]

Particularly for the \(D\)-functions in (9), the notation

\[
t = (p_1 + p_2)^2 , \quad s = (p_2 + p_3)^2 , \quad u = (p_1 + p_3)^2 ,
\]

is also convenient; compare Fig.4. Since \(C_{ijk}\) and \(D_{ijk}\) in (8,9) do not depend on the permutation of their indices, they are traditionally defined with the indices in ascending order.

In the case of \(B\) functions, the exact expressions for any \(s = p_1^2 = p_2^2\), may be obtained by integrating (2), which gives

\[
\begin{align*}
B_0(q^2; m_1, m_2) &= \Delta - \ln \frac{m_1 m_2}{\mu^2} + \frac{1}{q^2} \left[ (m_2^2 - m_1^2) \ln \frac{m_1}{m_2} + \sqrt{\lambda(q^2 + i\epsilon, m_1^2, m_2^2)} \text{ArcCosh} \left( \frac{m_1^2 + m_2^2 - q^2 - i\epsilon}{2m_1 m_2} \right) \right] ,
\quad B_1(q^2; m_1, m_2) = \frac{1}{2q^2} \left[ A(m_1) - A(m_2) + (m_2^2 - m_1^2 - q^2)B_0(q^2; m_1, m_2) \right] ,
\quad B_21(q^2; m_1, m_2) = \frac{1}{3q^2} \left[ A(m_2) - m_1^2 B_0(q^2; m_1, m_2) - 2(q_1^2 + m_1^2 - m_2^2)B_1(q^2; m_1, m_2) - \frac{(m_1^2 + m_2^2)}{2} + \frac{q^2}{6} \right] ,
\quad B_{22}(q^2; m_1, m_2) = \frac{1}{6} \left[ A(m_2) + 2m_1^2 B_0(q^2; m_1, m_2) + (m_1^2 - m_2^2 + q^2)B_1(q^2; m_1, m_2) - \frac{q^2}{3} + m_1^2 + m_2^2 \right] .
\end{align*}
\]

(11)
where
\[ \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc . \] (12)

and
\[ \text{ArcCosh} \left( \frac{m_1^2 + m_2^2 - q^2 - i\epsilon}{2m_1m_2} \right) = \ln(z + \sqrt{z^2 - 1}) , \quad z = \left( \frac{m_1^2 + m_2^2 - q^2 - i\epsilon}{2m_1m_2} \right) . \] (13)

Separating out the divergent and \( \mu \)-dependent parts in (11), we obtain
\[ B_0(q^2; m_1, m_2) = D - \ln\left( \frac{m_1m_2}{\mu^2} \right) + b_0(q^2; m_1, m_2) , \]
\[ B_1(q^2; m_1, m_2) = -\frac{1}{2} \left[ A - \ln\left( \frac{m_1m_2}{\mu^2} \right) \right] + b_1(q^2; m_1, m_2) , \]
\[ B_{21}(q^2; m_1, m_2) = \frac{1}{3} \left[ A - \ln\left( \frac{m_1m_2}{\mu^2} \right) \right] + b_{21}(q^2; m_1, m_2) , \] (14)

with
\[ b_0(q^2; m_1, m_2) = 2 + \frac{1}{q^2} \left[ (m_2^2 - m_1^2) \ln \frac{m_1}{m_2} + \sqrt{\lambda(q^2 + i\epsilon, m_1^2, m_2^2)} \text{ArcCosh} \left( \frac{m_1^2 + m_2^2 - q^2 - i\epsilon}{2m_1m_2} \right) \right] , \]
\[ b_1(q^2; m_1, m_2) = \frac{1}{2q^2} \left[ m_1^2 - m_2^2 + (m_1^2 + m_2^2) \ln \frac{m_2}{m_1} + (m_2^2 - m_1^2 - q^2)b_0(q^2; m_1, m_2) \right] , \]
\[ b_{21}(q^2; m_1, m_2) = \frac{(q^2 + m_1^2 - m_2^2)^2 - q^2m_1^2}{3q^4} b_0(q^2; m_1, m_2) + \frac{(m_2^2 - m_1^2)}{6q^2} \left[ 3 + \frac{2(m_1^2 - m_2^2)}{q^2} \right] + \frac{1}{3q^2} \left[ 2m_2^2 + m_1^2 - \frac{(m_2^4 - m_1^4)}{q^2} \right] \ln \frac{m_1}{m_2} + \frac{1}{18} . \] (15)

Using these, we define
\[ b_j^{(12)} \equiv b_j(p_1^2; m_1, m_2) , \]
\[ b_j^{(13)} \equiv b_j((p_1 + p_2)^2; m_1, m_3) , \]
\[ b_j^{(14)} \equiv b_j((p_1 + p_2 + p_3)^2; m_1, m_4) , \]
\[ b_j^{(23)} \equiv b_j(p_2^2; m_2, m_3) , \]
\[ b_j^{(24)} \equiv b_j((p_2 + p_3)^2; m_2, m_4) , \]
\[ b_j^{(34)} \equiv b_j(p_3^2; m_3, m_4) , \] (16)

which are in the same sprit as the expressions (D.33) in [6]. These functions only depend on ratios of internal or external masses and contribute to the constant asymptotic terms discussed above.

In Appendix 1 we give the relation of the above expansion for the \( B_j \), \( C_j \) and \( D_j \) functions, with that of the LoopTools library. From this expansion one obtains the exact
expressions of the various PV functions in terms of the basic \( B_0, C_0, D_0 \) and of \( A(m_i) \) for various combinations of external and internal masses. Most of these relations have been written in Hagiwara’s Appendix D \[6\]; in our Appendix 2 we have added a few more, relevant for the \( C_{ijk} \) and \( D_{ijk} \) functions.

3 Asymptotic expansion of the B, C and D functions

We consider non zero external and internal masses. For what concerns very light leptons and quarks, although logarithmic singularities \( \ln(s/m^2) \) may appear temporarily inside certain PV functions, they finally disappear in physical amplitudes. This can be checked explicitly in each particular process, as it is just the consequence of the theorem in \[8\].

For what concerns the photon, we keep a fictitious mass \( m_\gamma \). It can be used as an infrared regulator which will finally disappear when adding the soft photon radiation.

3.1 Asymptotic B functions

Using (14, 15) and Fig.1, at asymptotic energies \( p_1^2 \equiv s \gg m_i^2 \), one gets

\[
\begin{align*}
B_0(s; m_1, m_2) &\approx \Delta + 2 - \ln s \mu \\
B_1(s; m_1, m_2) &\approx \frac{\Delta}{2} - 1 + \frac{\ln s \mu}{2} \\
B_{21}(s; m_1, m_2) &\approx -\frac{\ln s \mu}{3} + \frac{\Delta}{3} + \frac{13}{18}, \\
B_{22}(s; m_1, m_2) &\approx \frac{s \ln s \mu}{12} - \frac{s \Delta}{12} - \frac{2s}{9},
\end{align*}
\]

where the definition

\[
s \mu \equiv \frac{-s - i\epsilon}{\mu^2},
\]

correctly describes the real and imaginary parts at asymptotic (positive or negative) \( s \). We note that the asymptotic expressions (17) contain only the leading logarithmic and subleading constant contributions, while the neglected terms are \( \mathcal{O}(m_i^2/s) \). The divergent quantity \( \Delta \) has been defined in \[5\].

3.2 Asymptotic C functions

Based on Fig.2, we consider the case in which the square of only one of the external momenta is large, the other two, as well as the internal masses, being much smaller; i.e.

\[
p_3^2 \equiv s = (p_1 + p_2)^2 \gg (m_i^2, |p_1^2|, |p_2^2|).
\]
Defining also
\[ s_\mu = \frac{-s - i\epsilon}{\mu^2}, \quad s_2 = \frac{-s - i\epsilon}{m_2^2}, \quad s_{ij} = \frac{-s - i\epsilon}{m_im_j}, \]  
and neglecting terms like \((1/s)O(m_i^2/s)\), the asymptotic expression of the \(C_0\) function established by Denner and Roth\,[5]\ is written as
\[ C_0(p_1^1, p_2^2, s; m_1, m_2, m_3) \simeq \frac{(\ln s_2)^2}{2s} + \frac{L_{223} + L_{121}}{s}, \]  
where
\[ L(p_a, m_b, m_c) \equiv L_{abc} = \text{Li}_2\left(\frac{2p_a^2 + i\epsilon}{m_b^2 - m_c^2 + p_a^2 + i\epsilon + \sqrt{\lambda(p_a^2 + i\epsilon, m_b^2, m_c^2)}}\right) + \text{Li}_2\left(\frac{2p_a^2 + i\epsilon}{m_b^2 - m_c^2 + p_a^2 + i\epsilon - \sqrt{\lambda(p_a^2 + i\epsilon, m_b^2, m_c^2)}}\right) \]  
describes contributions involving ratios of internal and external masses.

Using (21, 16, 22) and Appendix 2 and the results in [6], the implied asymptotic results for \(C_i\) are
\[ C_{11} \simeq -\frac{(\ln s_2)^2}{2s} + \frac{\ln s_{12}}{s} - \frac{L_{222} + L_{121} - b_0^{(12)}}{s} + 2, \]  
\[ C_{12} \simeq -\frac{(\ln s_2)^2}{s} + \frac{2 - b_0^{(23)}}{s}, \]  
\[ C_{24} \simeq -\frac{(\ln s_2)^2}{4} + \frac{\Delta + 3}{4}, \]  
\[ C_{21} \simeq -\frac{L_{222} + L_{121} - b_0^{(12)} + b_1^{(12)}}{s} + 3, \]  
\[ C_{23} \simeq \frac{\ln s_{23}}{s} + \frac{2b_0^{(23)} - 5}{2s}, \]  
\[ C_{22} \simeq -\frac{\ln s_{23}}{s} + \frac{1 + b_1^{(23)}}{s}, \]  
\[ C_{001} \simeq -\frac{\ln s_{12}}{6} + \frac{\Delta}{6} - \frac{19}{36}, \]  
\[ C_{002} \simeq -\frac{\ln s_{12}}{12} + \frac{\Delta}{12} - \frac{2}{9}, \]  
\[ C_{111} \simeq -\frac{(\ln s_2)^2}{2s} + \frac{11\ln s_{12}}{6s} - \frac{L_{222} + L_{121}}{s} + \frac{b_0^{(12)} - b_1^{(12)} + b_2^{(12)}}{s} - \frac{67}{18s} + \frac{11\ln s_{12}}{6s}, \]  
\[ C_{112} \simeq -\frac{(\ln s_2)^2}{s} + \frac{b_0^{(23)}}{s} + 17, \]  
\[ C_{122} \simeq -\frac{(\ln s_2)^2}{2s} + \frac{b_1^{(23)}}{s} + 7, \]  
\[ C_{222} \simeq -\frac{(\ln s_2)^2}{3s} + \frac{b_2^{(23)}}{s} + 13, \]
where terms of $O(m_i^2/s)$ relative to those kept, have been neglected. Among the PV functions listed above, we note that $B_1$, $C_{24}$, $C_{001}$ and $C_{002}$, are the only divergent ones.

### 3.3 Asymptotic D-functions

Based on Fig.3 and the definition (40), we are interested in the asymptotic D functions in the domain

$$\left(|s|, |t|, |u|\right) \gg \left(|p_i^2|, m_i^2\right),$$

which means large energy and momentum transfer squares, and fixed angles different from 0 or $\pi$. Defining also

$$r_{ts} = \frac{-t - ie}{s - ie},$$

$$t_1 = \frac{-t - ie}{m_1^2}, \quad t_2 = \frac{-t - ie}{m_2^2}, \quad t_3 = \frac{-t - ie}{m_3^2}, \quad t_4 = \frac{-t - ie}{m_4^2},$$

$$s_1 = \frac{-s - ie}{m_1^2}, \quad s_2 = \frac{-s - ie}{m_2^2}, \quad s_3 = \frac{-s - ie}{m_3^2}, \quad s_4 = \frac{-s - ie}{m_4^2},$$

$$s_{ij} = \frac{-s - ie}{m_i m_j}, \quad t_{ij} = \frac{-t - ie}{m_i m_j},$$

the basic expression of Denner and Roth [5] is

$$D_0 \approx \frac{1}{st} \left\{ - (\ln r_{ts})^2 - \pi^2 + \frac{1}{2} \left[ (\ln t_2)^2 + (\ln t_4)^2 + (\ln s_3)^2 + (\ln s_1)^2 \right] \\
+ L_{223} + L_{121} + L_{441} + L_{343} + L_{334} + L_{232} + L_{112} + L_{414} \right\},$$

where the neglected terms are suppressed by an additional factor of either $s$ or $t$ or $u$. To the same accuracy, the results of Appendix 2 imply

$$D_{11} \approx \frac{(u - t)(\ln r_{ts})^2}{2stu} - \frac{(\ln s_3)^2}{2st} - \frac{(\ln t_2)^2}{2st} - \frac{(\ln t_4)^2}{2st} - \frac{\pi^2(t - u)}{2stu},$$

$$D_{12} \approx \frac{(\ln r_{ts})^2}{2stu} - \frac{(\ln s_3)^2}{2st} - \frac{(\ln t_2)^2}{2st} - \frac{(\ln t_4)^2}{2st} + \frac{L_{232} + L_{334} + L_{343} + L_{441}}{st} + \frac{\pi^2}{2st},$$

$$D_{13} \approx \frac{(\ln r_{ts})^2}{2stu} - \frac{(\ln t_4)^2}{2stu} - \frac{\pi^2}{2stu} - \frac{L_{343} + L_{441}}{st},$$

$$D_{27} \approx \frac{(\ln r_{ts})^2}{2stu} - \frac{\pi^2}{4u},$$

$$D_{21} \approx \frac{(t^2 + u^2)(\ln r_{ts})^2}{2stu^2} + \frac{(\ln t_2)^2}{2stu} + \frac{(\ln t_4)^2}{2stu} + \frac{(\ln s_3)^2}{2stu} - \frac{\ln r_{ts}}{stu} - \frac{\ln s_{12} + \ln t_{14}}{st} \\
+ \frac{L_{121} + L_{223} + L_{232} + L_{334} + L_{343} + L_{441} - b_0^{(12)} - b_0^{(14)}}{st},$$

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\[
D_{22} \approx \frac{-\pi^2(t^2 + u^2)}{2stu^2} + \frac{4}{st}, \\
D_{23} \approx \frac{\ln(s_{14}^t t_{23}^s)}{st} + \frac{(\ln s_3^t)^2 + (\ln t_4^t)^2 - (\ln r_{ts})^2}{2st} + \frac{8 - \pi^2}{2stu^2} + \frac{L_{232} + L_{334} + L_{343} + L_{441} - b_0^{(14)} - b_0^{(23)}}{st}, \\
D_{24} \approx \frac{\ln s_3^t}{2st} + \frac{(\ln s_3^t)^2 + (\ln t_4^t)^2 - (\ln r_{ts})^2}{2st} - \frac{\ln s_{14}^t}{st} + \frac{4 - \pi^2}{2stu^2} + \frac{L_{232} + L_{334} + L_{343} + L_{441} - b_0^{(14)}}{st}, \\
D_{25} \approx \frac{-\ln r_{ts}}{su} - \frac{\ln t_{14}^t}{st} - \frac{t(\ln r_{ts})^2}{2stu^2} + \frac{(\ln t_4^t)^2}{2st} + \frac{L_{343} + L_{441} - b_0^{(14)}}{st} + \frac{4u^2 - \pi^2 t^2}{2stu^2}, \\
D_{26} \approx \frac{\ln r_{ts}}{2st} + \frac{\ln t_{14}^t}{st} + \frac{(\ln t_4^t)^2}{2st} - L_{343} + L_{441} - b_0^{(14)} + \frac{\pi^2 t + 4u}{2stu^2}, \\
D_{001} \approx \frac{(u - t)(\ln r_{ts})^2}{8u^2} - \frac{\ln r_{ts}}{4u} - \frac{\pi^2(t - u)}{8u^2}, \\
D_{002} \approx \frac{(\ln r_{ts})^2}{8u} - \frac{\pi^2}{8u^2}, \\
D_{003} \approx \frac{\ln r_{ts}}{4u} - \frac{\pi^2 t}{8u^2}, \\
D_{111} \approx \frac{-\ln r_{ts}}{2stu^2} + \frac{3}{2st} (\ln s_{12}^t + \ln t_{14}^t) + \frac{(u^3 - t^3)(\ln r_{ts})^2}{2stu^3} - \frac{(\ln s_3^t)^2 + (\ln t_4^t)^2}{2stu^3} - L_{121} + L_{223} + L_{232} + L_{334} + L_{343} + L_{441} - \frac{b_0^{(12)} + b_0^{(14)} - b_1^{(12)} - b_1^{(14)}}{st} + \frac{(t + 11u)u^2 + \pi^2(t^3 - u^3)}{2stu^3}, \\
D_{112} \approx \frac{3 \ln s_{14}^t + (\ln r_{ts})^2 - (\ln s_3^t)^2 - (\ln t_4^t)^2}{2stu} - \frac{L_{232} + L_{334} + L_{343} + L_{441}}{st} + \frac{b_0^{(14)} - b_1^{(14)}}{st} - \frac{\pi^2 - 5}{2st}, \\
D_{113} \approx \frac{3 \ln s_{14}^t + (\ln t_{14}^t)^2 - (\ln t_4^t)^2}{2stu^3} - \frac{(\ln t_4^t)^2}{2stu^3} - \frac{L_{343} + L_{441}}{st} + \frac{b_0^{(14)} - b_1^{(14)}}{st} - \frac{\pi^2 t^3 + tu^2 + 6u^3}{2stu^3},
\]
The main results of this paper are, apart from the simple case of (17) for the $B$ functions, (23,34) for the $C$ functions and (38-60) for the $D$ functions. In these analytic expressions one recognizes:

1. The true leading quadratic logarithms which in SM or SUSY only arise from gauge boson exchanges, and the linear logarithmic terms arising also from gauge boson exchanges, as well as from many other exchanges; see [1,2]. Note also that terms like

$$\ln^2 \frac{-s - i\epsilon}{m^2}, \quad \ln^2 \frac{-t - i\epsilon}{m^2}, \quad \ln^2 \frac{-u - i\epsilon}{m^2}, \quad (61)$$

4 Discussion of the results

The main results of this paper are, apart from the simple case of (17) for the $B$ functions, (23,34) for the $C$ functions and (38-60) for the $D$ functions. In these analytic expressions one recognizes:

1. The true leading quadratic logarithms which in SM or SUSY only arise from gauge boson exchanges, and the linear logarithmic terms arising also from gauge boson exchanges, as well as from many other exchanges; see [1,2]. Note also that terms like

$$\ln^2 \frac{-s - i\epsilon}{m^2}, \quad \ln^2 \frac{-t - i\epsilon}{m^2}, \quad \ln^2 \frac{-u - i\epsilon}{m^2}, \quad (61)$$
generate not only $\ln^2 s$ contributions, but also subleading, angular dependent and true constant terms, as seen in

\[
\begin{align*}
\ln^2 \frac{s - i\epsilon}{m^2} &= \left( \ln \frac{s}{m^2} - i\pi \right)^2 = \ln^2 \frac{s}{m^2} - 2i\pi \ln \frac{s}{m^2} - \pi^2, \\
\ln^2 \frac{t - i\epsilon}{m^2} &= \ln^2 \frac{t}{m^2} = \ln^2 s + 2 \ln s \ln \frac{1 - \cos \theta}{2} + \ln^2 \frac{1 - \cos \theta}{2} + O \left( \frac{m^2}{s} \right), \\
\ln^2 \frac{u - i\epsilon}{m^2} &= \ln^2 \frac{u}{m^2} = \ln^2 s + 2 \ln s \ln \frac{1 + \cos \theta}{2} + \ln^2 \frac{1 + \cos \theta}{2} + O \left( \frac{m^2}{s} \right).
\end{align*}
\]

Linear logarithms also appear as

\[
\begin{align*}
\ln \frac{m^2}{s} &= \ln \frac{s}{m^2} - i\pi, \\
\ln \frac{m^2}{s} &= \ln \frac{s}{m^2} + \ln \frac{1 - \cos \theta}{2} + O \left( \frac{m^2}{s} \right), \\
\ln \frac{m^2}{s} &= \ln \frac{s}{m^2} + \ln \frac{1 + \cos \theta}{2} + O \left( \frac{m^2}{s} \right),
\end{align*}
\]

in which mass suppressed terms have not been written explicitly.

2. The constant terms which consist, as one sees explicitly in Sect. (3.1, 3.2, 3.3), of true constant numbers (see for instance (23)), as well as of logarithmic or \( \text{Li}_2 \) functions involving ratios of masses or other kinematical quantities, as they appear in \( L_{ijk} \) and \( b_i \). They are called constant because they are indeed \( s \)-independent, but in some cases they may contain angular dependencies. A priori these \( L_{ijk} \) and \( b_i \) quantities contain all internal and external masses and mixings.

The omitted terms in all our asymptotic expression are mass-suppressed like \( m^2 / s \), relative to the retained ones and control the approach to asymptopia. In the remaining figures we illustrate this approach with a few examples, showing how the asymptotic PV functions match with the exact ones at high energies. This provides a useful insight about the properties of the various PV functions.

We begin with the basic \( C_0 \) function where, for illustration, we consider the simplest possible kinematical configuration with all internal masses put at a common scale, and the external squared momenta of two legs set also at a common mass scale; \( i.e. \)

\[
m_1^2 = m^2 , \quad p_1^2 = p_2^2 = M^2 , \quad p_3^2 = s .
\]

This way, the deviations between the exact and asymptotic results can be studied as a function of the dimensionless parameter \( \sqrt{s} / M \). In Fig. this is done for the real and imaginary parts of the dimensionless quantity \( sC_0 \), choosing also \( m = M \). As seen there, \( sC_0 \) becomes predominantly real at asymptotic \( s \), and the approximate expression (21) is quite accurate for \( \sqrt{s} / M \gtrsim 5. \)

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A similar analysis is done for $s^2 D_0$ in Fig. 5, where a common scale is again chosen as
\[ m_i^2 = m^2, \quad p_i^2 = M^2, \quad t = -\frac{s}{2}, \] (63)
and (10) is used. As seen from Fig. 5 (where we have again for simplicity chosen $m = M$), $s^2 D_0$ become predominantly imaginary at asymptotic $s$, and the exact and asymptotic results almost coincide for $\sqrt{s}/M \gtrsim 5$.

Another interesting application of the results in Sect. 3, concerns combinations of PV functions in which the asymptotic logarithmic contributions cancel out, and only mass-independent constants remain asymptotically. Examples of such combinations are
\[
\begin{align*}
&s(C_{23} + C_{12}), \quad sD_{27}, \\
&sD_{00}, \quad s^2(D_{112} - D_{123} + D_{214} - D_{206}), \\
&s^2(D_{113} - D_{112} + D_{122} - D_{123} + D_{25} - D_{24} + D_{22} - D_{26}),
\end{align*}
\] (64)
which often appear in some SUSY applications [9].

As a first example, we plot in the right panel of Fig. 6 the real part of $s(C_{23} + C_{12})$, as a function of $\sqrt{s}/M$; the other parameters chosen as in (62), while allowing for three ratios $M/m = 2, 1, 1/2$. As seen in the left panel of the same figure, the logarithmic terms strongly dominate the exact results for $\text{Re}[sC_{23}]$ and $\text{Re}[sC_{12}]$ at high $s$. But as the right panel indicates, these logarithmic contributions cancel out in $\text{Re}[s(C_{23} + C_{12})]$, and only a tiny constant contribution remains at high $s$, which seems independent of the mass ratio $M/m$. The prediction from our asymptotic expansion is also shown as an horizontal dash line, which agrees with the exact result for $\sqrt{s}/M \gtrsim 5$.

As a second example we present in Fig. 7 a similar analysis for $\text{Re}[sD_{27}]$ (upper panel) and $s^2(D_{113} - D_{112} + D_{122} - D_{123} + D_{25} - D_{24} + D_{22} - D_{26})$ (lower panel), plotted as functions of $\sqrt{s}/M$, with the other parameters chosen as in (63), using again $M/m = 2, 1, 1/2$. In both cases the asymptotic predictions from the results in Sect. 3 are indicated by the horizontal dash lines.

We now turn to PV combinations in which the constant terms also cancel out asymptotically, together with the logarithms. In such a case, only model dependent terms, suppressed by an extra power of $s$, remain at high energies. An example of such combinations is given by
\[
\begin{align*}
&s(C_{23} + C_{12} + 3C_{22} + 3C_{122}),
\end{align*}
\] (65)
which is plotted in the left panel of Fig. 8 as a function of $\sqrt{s}/M$, choosing again the internal and external masses as in (62). Its asymptotic vanishing is evident. The right panel of Fig. 8 shows what is obtained when (65) is multiplied by an additional $s$ factor, which allows to inspect its model dependence at high $\sqrt{s}/M$, where it can at most increase like a power of a logarithm. As the open circles show, a quadratic polynomial in $\ln s$ is accurately describing this rise, in the present case. Similar results are obtained for other such combinations.
5 Conclusions and Outlook

The above asymptotic expressions of $B$, $C$, $D$ functions should be useful for the analysis of many SM and MSSM (or NMSSM) 2-to-2 body processes at LHC and future high energy colliders. This is particularly true in situations where the energies may be much higher than all internal and external masses, and the scattering angles are kept fixed.

Particularly for MSSM (or NMSSM), the above expressions may be useful for exploiting the intriguing HC property, which induces logarithmically increasing 1-loop contributions to the total helicity conserving amplitudes; while striking cancellations appear for the amplitudes violating HC.

Below we illustrate this assertion with the case of the process $ug \rightarrow dW$, for which the analysis of [9] has revealed peculiar virtual SUSY effects in the helicity amplitudes. Particularly for the helicity violating amplitudes, spectacular high energy cancellations have been found. For these amplitudes, the one loop electroweak corrections have no leading logarithms, but they tend instead to a constant limit in SM, which in MSSM (or NMSSM) exactly vanishes, due to an opposite SUSY contribution. Thus, in MSSM, all helicity violating amplitudes are of order $O(m^2/s)$ and possibly negligible at LHC energies.

Since the general proof of the HC theorem in [4] neglected electroweak breaking, it is important to check in various cases how possible constant terms involving ratios of masses combine to assure the validity of the theorem. The above $B$, $C$, $D$ expressions should be useful for this. An example of this is seen in [9].

Beyond this though, the application in [9] also indicates that, although HC is an asymptotic theorem, it may be important at the LHC range, where it may strongly reduce the number of important amplitudes to just those respecting HC; thereby simplifying the analysis.

For this reason, the above high energy approximations of the $B$, $C$, $D$ functions should allow to make quantitative predictions for the physical amplitudes. This must be a valuable improvement with respect to the leading logarithmic level in two aspects. Firstly, the leading logarithms only test the gauge and Yukawa structures. Although this is an important step in SUSY checking, the constant terms should open the door to deeper tests of the SUSY structure. Second, the comparison with experimental results of the LL approximation, requires delicate experimental fits of logarithmic expressions, which are only realizable if several points in the high energy range are available. On the opposite, the more complete asymptotic expressions written in this paper are directly usable at any given high energy point. These expressions are analytically simple and can be easily put in a code allowing quick computations for any MSSM benchmark.
Appendix 1  Relation with the LoopTools conventions

In the LoopTools library \cite{looptools} the following momenta are defined
\[
\begin{align*}
k_1 &= p_1 , \\
k_2 &= p_1 + p_2 , \\
k_3 &= p_1 + p_2 + p_3 ,
\end{align*}
\]
leading to the tensorial decomposition
\[
\begin{align*}
B^\mu &= k_1^\mu B_1^L , \\
B^{\mu\nu} &= k_1^\mu k_1^\nu B_{11}^L + g^{\mu\nu} D_{00}^L ,
\end{align*}
\]
\[
\begin{align*}
C^\mu &= k_1^\mu C_1^L + k_2^\mu C_2^L , \\
C^{\mu\nu} &= \sum_{ij=1}^2 k_i^\mu k_j^\nu C_{ij}^L + g^{\mu\nu} C_{00}^L , \\
C^{\mu\nu\rho} &= \sum_{i,j,l=1}^2 k_i^\mu k_j^\nu k_l^\rho C_{ijl}^L + \sum_{i=1}^2 (g^{\mu\nu} k_i^\rho + g^{\mu\rho} k_i^\nu + g^{\nu\rho} k_i^\mu) C_{00i}^L ,
\end{align*}
\]
\[
\begin{align*}
D^\mu &= k_1^\mu D_1^L + k_2^\mu D_2^L + k_3^\mu D_3^L , \\
D^{\mu\nu} &= \sum_{i,j=1}^3 k_i^\mu k_j^\nu D_{ij}^L + g^{\mu\nu} D_{00}^L
\end{align*}
\]
where the superscript $L$ denotes the PV functions in the LoopTools notation. Comparing with \cite{looptools hagiwara}, their relations with the PV functions in the Hagiwara decomposition is
\[
\begin{align*}
B_1 &= B_1^L , \\
B_{21} &= B_{11}^L , \\
B_{22} &= B_{00}^L ,
\end{align*}
\]
\[
\begin{align*}
C_{11} &= C_1^L + C_2^L , \\
C_{12} &= C_2^L , \\
C_{21} &= C_{11}^L + 2C_{12}^L + C_{22}^L , \\
C_{22} &= C_{22}^L , \\
C_{23} &= C_{12}^L + C_{22}^L , \\
C_{24} &= C_{00}^L , \\
C_{001} &= C_{001}^L + C_{002}^L , \\
C_{002} &= C_{002}^L ,
\end{align*}
\]
\[ C_{111} = C_{111}^L + 3C_{112}^L + 3C_{122}^L + C_{222}^L, \]
\[ C_{222} = C_{222}^L, \]
\[ C_{112} = C_{112}^L + 2C_{122}^L + 3C_{222}^L, \]
\[ C_{122} = C_{122}^L + 3C_{222}^L, \tag{A.2} \]
\[ D_{11} = D_{11}^L + D_{2}^L + D_{3}^L, \]
\[ D_{12} = D_{2}^L + D_{3}^L, \]
\[ D_{13} = D_{3}^L, \]
\[ D_{21} = D_{11}^L + D_{2}^L + D_{3}^L + 2(D_{12}^L + D_{13}^L + D_{23}^L), \]
\[ D_{22} = D_{2}^L + 2D_{2}^L + D_{3}^L, \]
\[ D_{23} = D_{3}^L, \]
\[ D_{24} = D_{12}^L + D_{13}^L + D_{2}^L + 2D_{23}^L + D_{3}^L, \]
\[ D_{25} = D_{13}^L + D_{23}^L + D_{3}^L, \]
\[ D_{26} = D_{23}^L + D_{3}^L, \]
\[ D_{27} = D_{00}^L, \]
\[ D_{001} = D_{001}^L + D_{002}^L + D_{003}^L, \]
\[ D_{002} = D_{002}^L + D_{003}^L, \]
\[ D_{003} = D_{003}^L, \]
\[ D_{111} = D_{111}^L + 3D_{112}^L + 3D_{113}^L + 3D_{122}^L + 6D_{123}^L + 3D_{133}^L + D_{222}^L + 3D_{223}^L + 3D_{233}^L + D_{333}^L, \]
\[ D_{112} = D_{112}^L + D_{113}^L + 2D_{122}^L + 4D_{123}^L + 2D_{133}^L + D_{222}^L + 3D_{223}^L + 3D_{233}^L + D_{333}^L, \]
\[ D_{113} = D_{113}^L + 2D_{123}^L + 2D_{133}^L + D_{223}^L + 2D_{233}^L + D_{333}^L, \]
\[ D_{122} = D_{122}^L + 2D_{123}^L + D_{133}^L + D_{222}^L + 3D_{223}^L + 3D_{233}^L + D_{333}^L, \]
\[ D_{133} = D_{133}^L + D_{233}^L + D_{333}^L, \]
\[ D_{123} = D_{123}^L + D_{133}^L + D_{233}^L + 2D_{233}^L + D_{333}^L, \]
\[ D_{222} = D_{222}^L + 3D_{223}^L + 3D_{233}^L + D_{333}^L, \]
\[ D_{223} = D_{223}^L + 2D_{233}^L + D_{333}^L, \]
\[ D_{233} = D_{233}^L + D_{333}^L, \]
\[ D_{333} = D_{333}^L. \tag{A.3} \]

**Appendix 2  Reduction formalism for \( C_{ijk}, \ D_{ijk} \).**

The following relations have not been explicitly written in Hagiwara appendix. We write them below for completeness.
Appendix 2.1  \( C_{ijk} \) formulae

Same notation as in Hagiwara [6], with \((f_1, f_2, X)\) and \(B_{i}^{(jk)}\) taken respectively from (D.32), (D.31) and (D.33) of [6].

\[
X = \begin{pmatrix} 2p_1^2, 2p_1 p_2 \\ 2p_1 p_2, 2p_2^2 \end{pmatrix}, \quad (A.4)
\]

\[
\begin{pmatrix} C_{001} \\ C_{002} \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} B_{22}^{(13)} - B_{22}^{(23)} + f_1 C_{24} \\ B_{22}^{(12)} - B_{22}^{(13)} + f_2 C_{24} \end{pmatrix}, \quad (A.5)
\]

\[
\begin{pmatrix} C_{111} \\ C_{112} \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} B_{21}^{(13)} - B_{21}^{(23)} + f_1 C_{21} - 4C_{001} \\ B_{21}^{(12)} - B_{21}^{(13)} + f_2 C_{21} \end{pmatrix}, \quad (A.6)
\]

\[
\begin{pmatrix} C_{122} \\ C_{222} \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} B_{21}^{(13)} - B_{21}^{(23)} + f_1 C_{22} \\ -B_{21}^{(13)} + f_2 C_{22} - 4C_{002} \end{pmatrix}, \quad (A.7)
\]

\[
\begin{pmatrix} C_{112} \\ C_{122} \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} C_{21}^{(13)} + B_{21}^{(23)} + f_1 C_{23} - 2C_{002} \\ -B_{21}^{(13)} + f_2 C_{23} - 2C_{001} \end{pmatrix}, \quad (A.8)
\]

Appendix 2.2  \( D_{ijk} \) formulae

Expressions for \((f_1, f_2, f_3)\) and \(X\) are taken respectively from (D.37) and (D.36) of [6]. Using these we write

\[
D_{001} = \frac{1}{2} n_1^2 D_{11} - \frac{1}{4} \left[ f_1 D_{21} + f_2 D_{22} + f_3 D_{25} + C_{0}^{(234)} \right],
\]

\[
D_{002} = \frac{1}{2} n_1^2 D_{12} - \frac{1}{4} \left[ f_1 D_{24} + f_2 D_{22} + f_3 D_{26} + C_{11}^{(234)} \right],
\]

\[
D_{003} = \frac{1}{2} n_1^2 D_{13} - \frac{1}{4} \left[ f_1 D_{25} + f_2 D_{26} + f_3 D_{23} - C_{12}^{(234)} \right], \quad (A.9)
\]

and

\[
R_{40} = f_1 D_{21} - C_{0}^{(234)} + C_{21}^{(134)} - 4D_{001},
\]

\[
R_{41} = f_2 D_{21} - C_{21}^{(134)} + C_{21}^{(124)},
\]

\[
R_{42} = f_3 D_{21} - C_{21}^{(124)} + C_{21}^{(123)},
\]

\[
R_{44} = f_1 D_{24} + C_{21}^{(134)} + C_{21}^{(234)} - 2D_{002},
\]

\[
R_{50} = f_1 D_{22} - C_{21}^{(234)} + C_{21}^{(134)},
\]

\[
R_{56} = f_1 D_{23} - C_{22}^{(234)} + C_{22}^{(134)},
\]

\[
R_{45} = f_2 D_{24} - C_{21}^{(134)} + C_{23}^{(124)} - 2D_{001},
\]
\[ R_{51} = f_2 D_{22} - C_{21}^{(134)} + C_{22}^{(124)} - 4D_{002}, \]
\[ R_{57} = f_2 D_{23} - C_{22}^{(134)} + C_{22}^{(124)}, \]
\[ R_{46} = f_3 D_{24} - C_{23}^{(124)} + C_{23}^{(123)}, \]
\[ R_{52} = f_3 D_{22} - C_{22}^{(124)} + C_{22}^{(123)}, \]
\[ R_{58} = f_3 D_{23} - C_{22}^{(124)} - 4D_{003}. \]

(A.10)

\[ X_3 = \left( \begin{array}{c}
2p_1^2, 2p_1p_2, 2p_1p_3 \\
2p_1p_2, 2p_2^2, 2p_2p_3 \\
2p_1p_3, 2p_2p_3, 2p_3^2
\end{array} \right), \]
\[ \left( \begin{array}{c}
D_{111} \\
D_{112} \\
D_{113}
\end{array} \right) = X_3^{-1} \cdot \left( \begin{array}{c}
R_{40} \\
R_{41} \\
R_{42}
\end{array} \right), \]

(A.11)

\[ \left( \begin{array}{c}
D_{122} \\
D_{222} \\
D_{223}
\end{array} \right) = X_3^{-1} \cdot \left( \begin{array}{c}
R_{50} \\
R_{51} \\
R_{52}
\end{array} \right), \]
\[ \left( \begin{array}{c}
D_{133} \\
D_{233} \\
D_{333}
\end{array} \right) = X_3^{-1} \cdot \left( \begin{array}{c}
R_{56} \\
R_{57} \\
R_{58}
\end{array} \right), \]
\[ \left( \begin{array}{c}
D_{112} \\
D_{122} \\
D_{123}
\end{array} \right) = X_3^{-1} \cdot \left( \begin{array}{c}
R_{44} \\
R_{45} \\
R_{46}
\end{array} \right). \]

(A.12)

(A.13)

(A.14)

(A.15)

In them we need addition
\[ \begin{align*}
C_{24}^{(123)} &= \frac{1}{4} + \frac{1}{4}B_0^{(23)} + \frac{m_1^2}{2}C_0^{(123)} - \frac{f_1}{4}C_{11}^{(123)} - \frac{f_2}{4}C_{12}^{(123)}, \\
\left( \begin{array}{c}
C_{21}^{(123)} \\
C_{23}^{(123)}
\end{array} \right) &= \left( \begin{array}{c}
2p_1^2, 2p_1p_2, 2p_1p_3 \\
2p_1p_2, 2p_2^2, 2p_2p_3 \\
2p_1p_3, 2p_2p_3, 2p_3^2
\end{array} \right)^{-1} \left( \begin{array}{c}
B_1^{(13)} + B_0^{(23)} + f_1C_{11}^{(123)} - 2C_{24}^{(123)} \\
B_1^{(12)} - B_1^{(13)} + f_2C_{11}^{(123)}
\end{array} \right), \\
\left( \begin{array}{c}
C_{23}^{(123)} \\
C_{22}^{(123)}
\end{array} \right) &= \left( \begin{array}{c}
2p_1^2, 2p_1p_2, 2p_1p_3 \\
2p_1p_2, 2p_2^2, 2p_2p_3 \\
2p_1p_3, 2p_2p_3, 2p_3^2
\end{array} \right)^{-1} \left( \begin{array}{c}
B_1^{(13)} - B_1^{(23)} + f_1C_{12}^{(123)} \\
B_1^{(13)} - B_1^{(23)} + f_2C_{12}^{(123)} - 2C_{24}^{(123)}
\end{array} \right),
\end{align*} \]

(A.16)

\[ C_{24}^{(124)} = \frac{1}{4} + \frac{1}{4}B_0^{(24)} + \frac{m_1^2}{2}C_0^{(124)} - \frac{f_1}{4}C_{11}^{(124)} - \frac{f_2 + f_3}{4}C_{12}^{(124)}, \]
\[ \left( \begin{array}{c}
C_{21}^{(124)} \\
C_{23}^{(124)}
\end{array} \right) = \left( \begin{array}{c}
2p_1^2, 2p_1(p_2 + p_3), 2(p_2 + p_3)^2 \\
2p_1(p_2 + p_3), 2(p_2 + p_3)^2
\end{array} \right)^{-1} \left( \begin{array}{c}
B_1^{(14)} + B_0^{(24)} + f_1C_{11}^{(124)} - 2C_{24}^{(124)} \\
B_1^{(12)} - B_1^{(14)} + (f_2 + f_3)C_{11}^{(124)}
\end{array} \right). \]

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\[
\begin{pmatrix}
C_{23}^{(124)} \\
C_{22}^{(124)}
\end{pmatrix} = \begin{pmatrix} 2p_1^2, 2p_1(p_2 + p_3) \\
2p_1(p_2 + p_3), 2(p_2 + p_3)^2
\end{pmatrix}^{-1} \begin{pmatrix} B_1^{(14)} - B_1^{(24)} + f_1C_{12}^{(124)} \\
- B_1^{(14)} + (f_2 + f_3)C_{12}^{(124)} - 2C_{24}^{(124)}
\end{pmatrix},
\]

(A.17)

\[
C_{24}^{(134)} = \frac{1}{4} + \frac{1}{4}B_0^{(34)} + \frac{m_1^2}{2}C_0^{(134)} - \frac{(f_1 + f_2)}{4}C_{11}^{(134)} - \frac{f_3}{4}C_{12}^{(134)},
\]

\[
\begin{pmatrix}
C_{21}^{(134)} \\
C_{23}^{(134)}
\end{pmatrix} = \begin{pmatrix} 2(p_1 + p_2)^2, 2(p_1 + p_2)p_3 \\
2(p_1 + p_2)p_3, 2p_3^2
\end{pmatrix}^{-1} \begin{pmatrix} B_1^{(14)} + B_0^{(34)} + (f_1 + f_2)C_{11}^{(134)} - 2C_{24}^{(134)} \\
B_1^{(13)} - B_1^{(14)} + f_3C_{11}^{(134)}
\end{pmatrix},
\]

(A.18)

\[
\begin{pmatrix}
C_{23}^{(134)} \\
C_{22}^{(134)}
\end{pmatrix} = \begin{pmatrix} 2(p_1 + p_2)^2, 2(p_1 + p_2)p_3 \\
2(p_1 + p_2)p_3, 2p_3^2
\end{pmatrix}^{-1} \begin{pmatrix} B_1^{(14)} - B_1^{(134)} + (f_1 + f_2)C_{12}^{(134)} \\
- B_1^{(14)} + f_3C_{12}^{(134)} - 2C_{24}^{(134)}
\end{pmatrix},
\]

(A.19)

\[
\begin{pmatrix}
C_{23}^{(234)} \\
C_{22}^{(234)}
\end{pmatrix} = \begin{pmatrix} 2p_2^2, 2p_2p_3 \\
2p_2p_3, 2p_3^2
\end{pmatrix}^{-1} \begin{pmatrix} B_1^{(24)} + B_0^{(34)} + (f_2 + 2p_1p_2)C_{11}^{(234)} - 2C_{24}^{(234)} \\
B_1^{(23)} - B_1^{(24)} + (f_3 + 2p_1p_3)C_{11}^{(234)}
\end{pmatrix},
\]

(A.19)
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[1] for a review and a rather complete set of references see e.g.
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Figure 1: Bubble graph for $B_j(12)$. Of course $p_1^2 = p_2^2$.

Figure 2: Triangular graph for $C_j(123)$.

Figure 3: Box graph for $D_j(1234)$.
Figure 4: Comparison of the exact and asymptotic results for \( sC_0 \), as a function of \( \sqrt{p_i^2/M} \equiv \sqrt{s}/M \), at \( m_i^2 = p_1^2 = p_2^2 = M^2 \). Real and Imaginary parts are studied in the left and right panels respectively.
Figure 5: Comparison of the exact and asymptotic results for $s^2D_0$, as a function of $\sqrt{s}/M$, at $m_i^2 = p_j^2 = M^2$ and $t = -s/2$. Real and Imaginary parts are studied in the left and right panels respectively.
Figure 6: The left panel shows the exact results for $\text{Re}[s C_{12}]$ and $\text{Re}[s C_{23}]$, as functions of $\sqrt{s}/M$, with the other parameters chosen as in (62). It clearly indicates the asymptotic logarithmic behavior. As seen in the right panel though, the logarithmic contribution cancels out for high $\sqrt{s}/M$ in the combination $\text{Re}[s(C_{12} + C_{23})]$, and only a universal constant remains. The asymptotic predictions from (24, 27), are described by the horizontal dashed line which agrees with the exact results at high $\sqrt{s}/M$.
Figure 7: In the upper panel, $\text{Re}[sD_{27}]$, which has no asymptotic logarithmic contribution, is plotted against $\sqrt{s}/M$, with the remaining parameters fixed as in (63). The exact results for $\text{Re}[sD_{27}]$ behave like a mass independent constant at high $\sqrt{s}/M$. This constant agrees with the asymptotic predictions from Sect. 3, described by the horizontal dashed line. In the lower channel a similar analysis is done for the combination $s^2 (D_{113} - D_{112} + D_{122} - D_{123} + D_{25} - D_{24} + D_{22} - D_{26})$, which has similar mathematical properties.
Figure 8: The left panel presents the asymptotically vanishing combination $s(C_{23} + C_{12} + 3C_{22} + 3C_{122})$, as a function of $\sqrt{s}/M$, with the remaining parameters chosen as in 62. The asymptotic vanishing is evident. The model dependence of the approach to this zero-value is shown in the right panel obtained by multiplying the whole expression by an additional $s$ factor, which leads to a quantity behaving like a quadratic polynomial in $\ln s$. 