ON THE INTERPRETATION OF RELATIVISTIC SPIN NETWORKS AND THE BALANCED STATE SUM

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1. THE GEOMETRIC INTERPRETATION

In [1], a state sum model was proposed which had a natural relationship to the quantum theory of general relativity in 3+1 dimensions. In order to establish an actual connection to general relativity, however, it will be necessary to show that the stationary phase terms in the sum can be interpreted as discrete approximations to solutions of the vacuum Einstein equations.

The purpose of this note is to show how terms in the state sum may be interpreted as geometries, i.e., as discretized metrics on the underlying triangulated manifold. We still do not have a proof that steepest descent corresponds to Einstein, but a plausible argument can be made.

The geometric interpretation we propose makes use of some operators introduced by Barbieri in [2] for tetrahedra in three dimensions, then extended to relativistic or 4d tetrahedra in [3]. See also the discussion in [4].

The operator Barbieri constructs is a quantum mechanical analog of the determinant which computed the volume of a tetrahedron. Classically if \( V_i, i = 1, 2, 3; V_i = (x_i, y_i, z_i) \) are the three vectors connecting one vertex of a tetrahedron in \( R^3 \) (thought of as possessing a natural inner product) to the other 3 vertices, then the volume of the tetrahedron is given by

\[
V = \frac{1}{6} \det[x_i, y_i, z_i]. \tag{1}
\]

As Barbieri points out, this volume can instead be computed from the determinant

\[
\det[V_2 \times V_3, V_3 \times V_1, V_1 \times V_2]; \tag{2}
\]

which is exactly the matrix of cofactors of the first determinant, and consequently has determinant equal to its square. The second determinant is formed from the bivectors associated to three of the faces of the tetrahedron. Since we are in 3 dimensions, bivectors can be naturally interpreted as vectors. This is, of course, the mathematical meaning of the cross product of vectors. If we make any other choice of three of the four faces of a tetrahedron, the determinant is the same.

The picture both in [1] and [2] is to regard the bivectors on faces of a tetrahedron as fundamental variables rather than the lengths or directions of edges, to reinterpret the bivectors as angular momenta, and then to quantize them by replacing them with representations of \( su(2) \) as is usual in the quantization of the spin.
Thus it is important to replace an expression for the volume from edges by one from the bivectors on the faces. Barbieri then writes an operator for the quantum theory which is the quantum analog of determinant [2], which he denotes $U$:

$$U = J_1 \cdot J_2 \times J_3$$

which is the obvious quantum analog of determinant (2).

In [3], Barbieri writes a similar expression to his $U$ operator for a four dimensional quantum tetrahedron as defined in [1], i.e. for a four-valent vertex for a relativistic spin net.

For a tetrahedron in $R^4$, however, a scalar volume does not contain all possible information. The volume form of a tetrahedron in $R^4$, obtained analogously to our determinant (1) by wedging together the three 1-forms dual to the three edge vectors connecting any vertex to the other three; is a 3-form whose magnitude gives the volume of the tetrahedron (times 6) and whose “direction” gives the orientation of the hyperplane containing the tetrahedron. This 3-form has four components, which can be expressed as determinants analogous to four copies of determinant (1). They are the determinants of the four $3 \times 3$ minors of the $4 \times 3$ matrix formed by the 3 4-vectors.

Using definitions analogous to Barbieri’s, it is possible to assign to a “relativistic quantum tetrahedron” i.e. to a tetravalent vertex for a relativistic spin net, a vector of four operators, analogous to the four components of the 3-form in the classical situation. The definition is straightforward. One begins with the expressions for the two copies of su(2) in the splitting of so(4) or so(3,1):

$$J^x_\pm = J_{yz}^\pm \pm J_{xt}$$

or respectively

$$J^x_\pm = J_{yz}^\pm \pm i J_{xt}$$

The i in these formulas is the only thing to distinguish the Euclidean and Minkowski signature cases in this approach to quantum gravity.

We can simply invert these and obtain our six bivector operators for the spin on each face of a quantum tetrahedron. We can then form our four 4d analogs of the $U$ operator as $3 \times 3$ matrices of operators. Let us call them $U_x, U_y, U_z, U_t$ or $U^{\rightarrow}$.

For example, the $dy \wedge dz \wedge dt$ component of the 3-form associated to the tetrahedron generated by the edges numbered 1,2,3 is given by the determinant of the $3 \times 3$ matrix of operators whose ith row is $J_{yz}^i, J_{zt}^i, J_{tx}^i$.

Now let us see what we have in the case of a “quantum 4-simplex”, i.e. a 15J symbol in the category of relativistic spin nets. This is what we get for one labelling of one 4-simplex in the state sum of [1]. For each tetrahedron in
the boundary of the labelled 4-simplex, we now have a 4-vector of operators $U_x, U_y, U_z, U_t$ as above.

It is now possible to form invariant combinations of these vector operators, namely their inner products with themselves and each other. This gives us the quantum volumes of the tetrahedra and their hyperdihedral angles. These suffice to specify a geometry corresponding to definite values of the $U$ operator products.

(The reader familiar with [1] will doubtless note the similarity between the proof of the classical geometrical theorem there and this construction).

It is very important for us that we now have operators corresponding to the hyperdihedral angles, since in a four dimensional triangulated geometry it is the sums of the hyperdihedral angles around faces which specify the curvature.

We close with conjectures about how the quantum dihedral angles we have defined will behave when we combine 15J symbols into the state sum in [1].

**Conjecture 1:** The evaluation of relativistic 15j symbol has a rapid oscillation of phase with respect to the variation of a quantum hypergeometric angle as defined above.

**Conjecture 2:** If the state sum of [1] is considered without the balance constraints, stationary phase will impose that the sum of the hyperdihedral angles around a face be $2\pi$ in the classical limit; the effect of the constraints is to reduce the stationary phase condition to a combination of sums of hyperdihedral angles converging to Ricci flatness in the classical limit.

The verification of these will require a good deal of work. Research on them is under way.

2. ON THE "ULTRAVIOLET" LIMIT

The suggestion was made in [1] that the states to be considered in the quantum theory of gravity would be asymptotically self dual, so that the result of the balanced state sum at some particular triangulation would be a good approximation to an exact result. We want to make a more physical version of this proposal here.

Let us make the hypothesis that the universe began in a self dual state. This is plausible because the boundary conditions at the event horizon of a black hole are self dual [5,6]. This would mean that the state of the whole universe remained self dual. The state of the whole universe, however, is not what an observer experiences. The wavefunction of the universe has gone through many collapses or bifurcations, depending on whether one likes the Copenhagen or many worlds picture. In any event, the history of the universe has in effect measured a very large number of areas, giving rise to non self dual bivectors in many places. However, if one takes a triangulation $T$ fine enough to contain everything that any macroscopic physical subsystem of the universe has measured, then the states in any tetrahedron strictly finer than the triangulation would be self dual. Since the self dual states are in effect governed by a topological state
sum, it is unnecessary to make a finer triangulation, and the state sum on $T$ will give an exact answer.

It appears that this interpretation of the model may lead to testable consequences, in the form of history dependence of some physical property of the universe. This requires further thought.

CONCLUSIONS

It is, of course, much too soon to speak of any real success for this model. What can be fairly said is that many deep issues of quantum gravity, ranging from the ultraviolet behavior to the Minkowskian signature difficulties, can be attacked within it by well defined computations. Given the difficult history of quantum gravity, however, this is at least noteworthy.

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