A Stationary Accumulated Projection Method for Linear System of Equations

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Abstract

It is shown in this paper that, almost all current prevalent iterative methods for solving linear system of equations can be classified as what we called extended Krylov subspace methods. In this paper a new type of iterative methods are introduced which do not depend on any Krylov subspaces. This type of methods are based on the so-called accumulated projection technique proposed by authors. It overcomes some shortcomings of classical Row-Projection technique and takes full advantages of the linear system. Comparing with traditional Krylov subspace methods which always depend on the matrix-vector multiplication with some fixed matrix, the newly introduced method (SAP) uses different projection matrices which differ in each step in the iteration process to form an approximate solution. More importantly some particular accelerative schemes (named as MSAP1 and MSAP2) are introduced to improve the convergence of the SAP method. Numerical experiments show some surprisingly improved convergence behavior; some superior experimental behavior of MSAP methods over GMRES and block-Jacobi are demonstrated in some situations.

Keywords: Iterative method; Accumulated projection; Krylov subspace

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1. Introduction

Linear systems of the form

$$Ax = b$$  \hspace{1cm} (1)

where $A \in \mathbb{R}^{n \times n}$ is nonsingular arise from tremendous mathematical applications and are the fundamental objects of almost every computational process. From the very ancient Gaussian elimination to state-of-the-art methods like GMRES, PCG, Bicgstab \([1, 2, 15]\) as well as Multigrid method \([8, 9]\), numerous solvers of linear systems have been introduced and studied in extreme detail. Basically all solvers fall into two categories: Direct methods and iterative methods.

Although some state-of-the-art direct methods can be applied to solve systems with pretty large amount of unknowns \([3, 6]\) in some situations, for even larger scale sparse systems (say, with unknowns up to a few millions) one can resort to the LGO-based solver \([10, 11]\) recently introduced by authors, iterative methods are the only option available for many practical problems. For example, detailed three-dimensional multiphysics simulations lead to linear systems comprising hundreds of millions or even billions of unknowns, systems with several millions of unknowns are now routinely encountered in many applications, making the use of iterative methods virtually mandatory.

Traditional iterative methods are classified as stationary and non-stationary methods. Stationary methods usually take the form:

$$x^{k+1} = Gx^k + v, \hspace{1cm} k = 0, 1, 2, \ldots$$  \hspace{1cm} (2)

where $v$ is a fixed vector and $x^0$ as the first guess.

Excellent books covering the detailed analysis of error and convergence of these methods include works by Varga \([14]\) and David Young \([17]\), etc.

More recent iterative methods can be classified as Krylov subspace methods (or non-stationary methods in some literature)\([13]\). Krylov subspace methods take the following form

$$x_k = x_0 + y_k, \hspace{1cm} k = 1, 2, \ldots$$  \hspace{1cm} (3)

where $x^0$ is an initial guess and the correction vector $y_k$ belongs to a so-called Krylov subspace

$$K_m(G, r_0) \equiv \text{span}\{r_0, Gr_0, G^2r_0, \ldots, G^{m-1}r_0\}.$$
By assuming different strategies for seeking $y_k$ from $K_m(G, r_0)$ with $G$ usually taken as $A$ or $A'$, one gets a variety of iterative methods such as CG, GMRES, BiCG, FOM, MNRES, SYMMLQ, etc. ([7, 13, 15]).

As a matter of fact, if we would refer extended Krylov subspace methods as those at each stage of iteration either the approximate solution or the correction vectors always come from Krylov subspaces with a few (one or two) fixed generator matrices (By a “generator” matrix to Krylov subspace $K_m(A, v)$ we mean matrix $A$ here), then the traditional stationary iterative methods such as Jacobi, Gauss-Seidal and SOR can also be classified as extended Krylov subspace methods. Since for example one can easily see from (2) that

$$x_{k+1} = v + Gv + G^2v + G^3v + \cdots + G^k v + G^{k+1} x_0 \equiv y_{k+1} + z^{k+1}$$

where $y_{k+1} = v + Gv + G^2v + G^3v + \cdots + G^k v \in K_{k+1}(G, v)$ and $z^{k+1} = G^{k+1} x_0 \in K_{k+2}(G, x_0)$ and $x_0$ is the initial guess to the system. It has been shown in [3] that another well-known type of methods, the row projection method (or ART method called in CT-related techniques, a generalization of RP can also be seen in [12]), can also be put into the form of (2), and thus they still belong to the category of extended Krylov subspace methods.

Krylov subspace methods can be very effective when used in case the condition number of the coefficient matrix $A$ is relatively small. However in case of $A$ having large condition numbers, they are not effective any more or even fail to converge.

Problems with current Krylov subspace methods lie on the fact that the successive corrections to previous approximation come from Krylov subspace with a fixed “generator” matrix and usually some fixed starting vector (say vector $v$ in above). If we take a look at the structure of Krylov subspace $K_m(A, v) = \text{span}\{v, Av, A^2v, \ldots, A^{m-1}v\}$, we can see that the base vectors of this subspace always have the form $A^k v$, which are increasingly closer to the subspaces formed by the eigenvectors corresponding to the eigenvalues with the largest magnitudes. For the sake of simplicity, we will refer them as generalized eigenspace denoted by $L_s(A)$, i.e,

$$L_s(A) = \text{span}\{v_1, v_2, \ldots, v_s\}$$

where $Av_i = \lambda_i$ and $|\lambda_i| \geq |\lambda_{i+1}|$, for $i = 1, 2, \ldots, n$. It is thus inefficient to find a good “approximation” to the error vector $e_m \equiv x - x_0$ in such a subspace when $e_m$ contains rich eigenvector components corresponding to
the smallest eigenvalues in magnitude. Especially when vector $e_m$ is almost perpendicular to the Krylov subspace $K_m(A,v)$.

It is thus always desirable for us to use some types of preconditioning when we apply Krylov subspace iterative methods to solve linear system of equations, especially for large scale computing. Though numerous preconditioning techniques are exploited in recent decades and some of them turn out to be extremely efficient in some special situations, there does not exist a simple preconditioning technique which can be applied in general cases. Another important factor is, all preconditioning techniques can be traced back to certain algebraic iterative schemes ([4, 15, 16]).

Our motivation here is to develop a set of purely algebraic algorithms that can in someway overcome the difficulties arising in the Krylov subspace methods. Instead of seeking corrections from certain Krylov subspaces when solving system (1), our new approach always tries to get a sequence of projection vectors $\{v_k\}$ of the solution $x$ with each $v_k$ guaranteed to be closer to $x$ than its predecessor $v_{k-1}$ by the so-called “accumulated projection” technique. More importantly We will also develop some accelerating techniques to improve the convergence of our iterative methods.

2. An Accumulated Projection Idea

Let’s start from a simple projection idea. Assume $x$ is the solution to (1) and we can get the projection vector $p$ of $x$ onto some subspace easily. Of course $p$ can be used as an approximation to $x$ and the error vector $e = x - p$ satisfies

$$Ae = r$$

where $r = b - Ap$.

To get a projection vector $p$ of $x$ onto any subspace $W$, we need the information of all inner products $v_i^t x$ where $v_i (i = 1, \cdots, m)$ are the base vectors of $W$, i.e., $W = \text{span}\{v_1, v_2, \cdots, v_m\}$. From system (1) we see that actually any groups of row vectors $\{A_{i_1}, A_{i_2}, A_{i_3}, \cdots, A_{i_m}\}$ can be used as the base vectors where $A_{i_k}$ stands for the $i_k$th row vector of matrix $A$. To get a better approximation $x_{l+1}$ to $x$ provided that $x_l$ is given, people usually use the residual equation

$$Ae_l = r_l$$

(5)

to find a correction vector $y$ so that $x_{l+1} = x_l + y$ is “closer” to $x$ in some measurement while $y$ is obtained in the same way as $x_l$ is calculated, as done in Row Projection Methods [5].
Our idea here goes like the following. Instead of using the residual equations to get a correction vector, we use the given approximation $p_l$ (on subspace $W_l$) and use it as a base vector to form another subspace $W_{l+1}$ together with a group of different row vectors selected from matrix $A$ since $p^k x$ can also be calculated when $p_l$ is formed. The projection vector $p_{l+1}$ of $x$ onto $W_{l+1}$ usually have a larger length and is thus “closer” to $x$ than $p_l$, which is proved in Lemma 2.1 and Lemma 2.2. This process can be repeated until $p_l$ reaches its limit position $x$ (actually $\bar{x}$, the projection of $x$ onto $\text{ran}(A')$).

The major features of this approach are: First it takes full advantage of all given information about the exact solution $x$ (the projection information of $x$ on each base vector $A_i$ ($A_i x = b_i$)) as well as every approximating vector $x_l$ ($x' x_l$ is recorded and used in later calculations); secondly the whole process uses only the original data of the system to reach a sequence of approximation $x_l$ which approaches to $x$ steadily, i.e., the correction vector (which is implicitly calculated) does not rely on the residual equations. It is by this reason we name it as a “stationary” method. By this way we can avoid the negative impact of successive matrix-vector multiplication between a few fixed generator matrices and some starting vector (i.e. the term $A^k v$ in Krylov subspace construction); Thirdly, there does not exist any constant iterative matrix (like matrix $G$ in (2)) between adjacent approximations $x_l$ and $x_{l+1}$.

Assume $x' v_i = b_i$, $(i = 1, 2)$ with $b_1 \neq 0$ and $||v_i|| = 1 (i = 1, 2)$, we wish to find a real number $t$ such that the function $f(t)$ defined by

$$f(t) = \frac{|x' v|}{||v||}$$

is maximized among all possible vectors in the form $v = v_1 + tv_2$. As a matter of fact, this optimization problem is equivalent to searching a vector from subspace $\text{span}\{v_1, v_2\}$ so that it is as close to $x$ as possible.

It is easy to see from analysis that the answer to the above optimization problem lies on the following conclusion.

**Lemma 2.1.** Let $x' v_i = b_i$, $(i = 1, 2)$ with $|b_1| \geq |b_2|$ and $||v_i|| = 1 (i = 1, 2)$, and $\alpha = v'_1 v_2$. Let $s = \frac{b_2 - ab_1}{b_1 - ab_2}$. Then

$$f(s) = \frac{|x'(v_1 + sv_2)|}{||v_1 + sv_2||} = \max_{t \in R} \frac{|x'(v_1 + tv_2)|}{||v_1 + tv_2||}. \quad (7)$$
Furthermore
\[ f(s) \geq \max\{|b_1|, |b_2|\} \quad (8) \]

Proof. Let
\[ g(t) = \frac{x'(v_1 + tv_2)}{||v_1 + tv_2||}. \]
We have
\[ g(t) = \frac{b_1 + tb_2}{\sqrt{1 + 2\alpha t + t^2}}. \]
Thus
\[ g'(t) = \frac{b_2(1 + 2\alpha t + t^2) - (b_1 + tb_2)(\alpha + t)}{(1 + 2\alpha t + t^2)^{3/2}} \]
\[ = \frac{b_2 - \alpha b_1 - (b_1 - \alpha b_2)t}{(1 + 2\alpha t + t^2)^{3/2}} \]
\[ = \frac{(b_1 - \alpha b_2)(s - t)}{(1 + 2\alpha t + t^2)^{3/2}} \]

Let \( g'(t) = 0 \) we have the solution as \( t = \frac{b_2 - \alpha b_1}{b_1 - \alpha b_2} \equiv s \), i.e., \( s \) is an extreme point for function \( f(t) \).

case 1. \( b_1 > \alpha b_2 \), we have \( g'(t) > 0 \) if \( t < s \) and \( g'(t) < 0 \) if \( t > s \). That means \( g(t) \) reaches the maximal value at \( s \). Since \( g(t) \to -b_2 \) when \( t \to -\infty \) and \( g(t) \to b_2 \) when \( t \to +\infty \), we have \( g(s) \geq g(t) \geq -b_2 \) for all \( t < s \) and \( b_2 < g(t) \leq g(s) \) for all \( t > s \), thus function \( f(t) = |g(t)| \) reaches its maximal value at \( s \).

case 2. \( b_1 < \alpha b_2 \), we have \( g'(t) < 0 \) if \( t < s \) and \( g'(t) > 0 \) if \( t > s \). That means \( g(t) \) reaches the minimal value at \( s \). Since \( g(t) \to -b_2 \) when \( t \to -\infty \) and \( g(t) \to b_2 \) when \( t \to +\infty \), we have \( g(s) \leq g(t) \leq -b_2 \) for all \( t < s \) and \( b_2 < g(t) \geq g(s) \) for all \( t > s \), thus we have \( f(t) = |g(t)| \) reaches its maximal value at \( s \).

Thus in both cases we have \( f(s) > |b_2| \). Since \( f(0) = |g(0)| = |b_1| \) and \( f(s) \) is the maximal value of \( f(t) \), thus we also have \( f(s) > |b_1| \). See figure 1. \( \square \)

Remark: Assuming \( b_1 \neq 0 \), \( f(s) \) can be rewritten as following (by replacing \( s \) as \( s = \frac{b_2 - \alpha b_1}{b_1 - \alpha b_2} \))
\[ f(s) = \frac{|b_1 + sb_2|}{\sqrt{1 + 2\alpha s + s^2}} = \frac{\sqrt{1 - 2\alpha r + r^2}}{\sqrt{1 - \alpha^2}}|b_1| = |b_1|\sqrt{1 + \frac{(r - \alpha)^2}{1 - \alpha^2}} \quad (9) \]
where \( r = \frac{b_2}{b_1} \).
In view of (9), \( f(s) \to \infty \) when \( \alpha \to 1 \) (assuming that \( r \) is independent of \( \alpha \)). It is thus attempting for us to increase the length of \( p_1(x) \) based on previous projection direction \( p_0 \) with \( p'_0x = b_1 \) by carefully selecting suitable vector \( d \) with \( d'x = b_2 \) easily obtained so that \( \alpha = p'_0p_1 \) is as close as possible to 1 (i.e., the angle between \( p_0 \) and \( p_1 \) should be very small). However this seems to be very hard and thus we turn to an easier scheme to fulfill our task—we will use subspaces on which projections of \( x \) are easily available. For this purpose we now generalize our conclusion in Lemma 2.1 into the following statement.

**Lemma 2.2.** Let \( x, v_i \in \mathbb{R}^n \) \( (i = 1, 2, \ldots, m) \), and \( W = \text{span}\{v_1, v_2, \ldots, v_m\} \). Let \( \bar{x} \) be the projection of \( x \) onto space \( W \). Then

\[
\frac{\bar{x}'x}{||x||} = \max_{v \in W} \frac{|x'v|}{||v||}.
\]

**Proof.** Without loss of generality we can assume \( ||x|| = 1 \). By the definition of angles between vectors we have

\[
f(v) = \frac{|x'v|}{||v||} = \frac{|x'v|}{||v|| ||x||} = |\cos < x, v > |\]

where \( < x, v > \) denotes the angle between vector \( x \) and \( v \). Obviously \( f(v) \) reaches its maximum value if and only if \( < x, v > \) is minimized, which is true only when \( v \) lies on the projection of \( x \) onto subspace \( W \). \( \square \)
By using this result, one can always expect a searching direction \( d \) on which vector \( x \) has a projection vector with larger length than any given base vectors of subspace \( W = \text{span}\{v_1, v_2, \cdots, v_m\} \) with \( x'v_i (i = 1, 2, \cdots, m) \) given. Since we have \( n \) vectors \( A_i (i = 1, 2, 3, \cdots, n) \) to form subspaces of \( \mathbb{R}^n \), this give us plenty of choices when it comes to construct subspaces. More importantly we can use parallel process to construct these subspaces and figure out projections of \( x \) on each of them. Instead of using successive “partial” projections which did not adequately make use of current system information, all these projections of \( x \) can be used to construct a better approximation to the current system.

### 2.1. An Accumulated Projection Algorithm

In this subsection we present a basic algorithm for calculating a projection vector \( p \) of \( x \) to the system (1) based on current system data, i.e., the coefficient matrix \( A \) and the right-hand side vector \( b \).

In preparation, we begin with the division of all row vectors of \( A \) into groups of vectors \( \{G_i\}_1^k \), with each group \( G_i \) contains \( m_i \) vectors, where \( m_i (i = 1, \cdots, k) \) are relatively small integers satisfying \( m_i < m, \forall 1 \leq i \leq k \). \( m \) is a suitable integer so that the QR factorization of matrix \( A_i \) (a submatrix of \( A \)) formed by all vectors in group \( G_i \) is applicable; in case of sparse coefficient matrix, QS factorization process based on LGO method [11] can be used and thus \( m \) can be relatively large (say, up to \( O(10^5) \) or even larger). The right-hand side vector \( b \) is divided correspondingly into vectors \( b_i (i = 1, \cdots, k) \). One thing needs to be mentioned here is that we assume two adjacent groups \( G_i \) and \( G_{i+1} \) contain about half of their vectors in common and any row vector in \( A \) must lie in at least one of the groups, we will refer this group \( \{G_i\} \) as an overlapped division of \( A \). A non-overlapped division of \( A \) means the intersection of any two groups in the division is empty.

Our approach is to use a sequential projection process to get a final projection vector \( p \) of \( x \). We begin with an initial projection vector \( p_0 \) of \( x \) and let it combine with all row vectors in the first group \( G_1 \) to form a subspace \( W_1 \) of \( \mathbb{R}^n \), and then find the projection vector \( p_1 \) of \( x \) in \( W_1 \). \( p_1 \) is then used to combine with all row vectors in the next group \( G_2 \) to form a subspace \( W_2 \) so that a projection vector \( p_2 \) of \( x \) in \( W_2 \) can be obtained. The above process is repeated until all groups are handled so that the final projection vector \( p_k \) are available. The following algorithm gives the details.
Algorithm 1. (An accumulated projection method-AP) The following procedure produces an approximate vector \( p \) to the solution vector \( x \) which satisfies \( Ax = b \).

Step 1. Divide matrix \( A \) into \( k \) blocks: \( A_1, A_2, \cdots, A_k \), divide \( b \) correspondingly: \( b = b_1, b_2, \cdots, b_k \) (blocks \( A_i \) and \( A_{i+1} \) may contain common row vectors).

Step 2. Initialize \( p_0 \) as

\[
p_0 = \alpha A' b \quad \text{and} \quad c_0 = \alpha \frac{||b||^2}{||A'b||^2}, \]

where \( \alpha = \frac{||b||^2}{||A'b||^2} \).

Step 3. For \( i = 1 \) to \( k \)

Step 3.1 Construct matrix \( W_i = [p_{i-1}, A'_i] \) and vector \( l = [c_{i-1}, b'_i]' \).

Step 3.2 Compute the projection vector \( p_i \) of \( x \) onto subspace \( \text{ran}(W_i) \) and the scalar \( c_i = x' p_i \) as

\[
p_i = W_i(W_i' W_i)^{-1} W_i' x \quad \text{and} \quad c_i = l' (W_i' W_i)^{-1} l.
\]

Step 3.3 Go to next \( i \).

Step 4. Output \( p (= p_k) \) and \( c (= c_k) \).

It should be mentioned here that the AP algorithm depicts a successive projection process over subspace \( \text{ran}(W_i) = \text{span}\{p_{i-1}, v_1, v_2, \cdots, v_{m_i}\} \) (\( i = 1, \cdots, k \)), where \( v_1, v_2, \cdots, v_{m_i} \) denotes the row vectors of submatrix \( A_i \) of \( A \), and \( p_i \) is the projection of \( x \) over subspace \( \text{ran}(W_i) \) with \( p_0 \) stands for the initial searching direction (usually a projection vector of \( x \)). Obviously we have \( ||p_{i+1}|| \geq ||p_i|| \) for \( 1 \leq i \leq k \) by Lemma 2.2.

Hence the whole AP process can be written in the matrix form as

\[
p = P_k x \quad \text{where} \quad P_i (i = 1, \cdots, k) \quad \text{represents the projection matrix over subspace} \quad \text{ran}(W_i).
\]

It is easy to see that \( P_i \) depends on vector \( x \). As a matter of fact, \( P_k \) has the form

\[
P_k = W_k (W_k' W_k)^{-1} W_k'
\]

where \( W_k = [p_{k-1}, A'_k] \), assuming \( p_{k-1} \notin \text{ran}(A'_k) \). Here and after we always use \( \text{ran}(A) \) to denote the range of matrix \( A \), i.e, the subspace formed by all column vectors of matrix \( A \).

As a straightforward application, Algorithm 1 can be used to solve the linear system \( Ax = b \) as stated in the next algorithm.

Algorithm 2. (Stationary Accumulated Projection Method-SAP). Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^n \) with \( m \leq n \), \( \epsilon \) be a given tolerance. The following procedure produces an approximation \( p \) to the vector \( x \in \mathbb{R}^n \) satisfying \( Ax = b \).
Step1. Initialize $s$ as $s = 0$, vector $x_0$ as $x_0 = \alpha A'b$, $c_0 = \alpha b'b$, $tol = ||b - Ax_0||/||b||$, where $\alpha = ||b||^2/||A'b||^2$.

Step2. While $tol > \epsilon$

Step2.1 Use Algorithm 1 to get a projection vector $x_{s+1}$ of $x$ and $c_{s+1}(= x'x_{s+1})$ with $p$ and $c$ in step 2 replaced by $x_s$ and $c_s$ respectively.

Step2.2 Calculate $tol = ||b - Ax_s||/||b||$;

Step2.3 $s = s + 1$;

Step3. Output $p(= x_s)$ and $c(= c_s)$.

Remark: In actual implementation of SAP algorithm, in order to effectively obtain the projection of $x$ over each subspace $ran(W_i)$ through Algorithm 1, one can store the resulting QR or LGO factors ( $Q_i$ and $R_i$ for QR, or $Q_i$ and $S_i$ for LGO respectively) of all submatrix $A'_i (i = 1, 2, \cdots, k)$ once in advance and reuse them in later projections. Although the projection matrix $P_s$ varies constantly, the projection vector can always be obtained in an economic count of flops, as it can be seen in later sections.

The convergence of this algorithm is put forward to the next section. We need to point out that each sweep in step 2 is a projection process with projection matrix $P_s$ ($s = 1, 2, \cdots k$) varies. Figure 2 shows the comparison between approximate solutions at different iterations by this algorithm, and

Figure 2: Comparison of approx. solns at different iteration numbers

Table 1 gives the needed iteration for a convergent solution under given
tolerance, where the coefficient matrix $A$ is chosen as $A = \text{diag}(-1, 2, -1)$ with $A \in \mathbb{R}^{100 \times 100}$ and the block size is chosen as 20 when applying Algorithm 2 in this case.

Table 1: SAP–iteration numbers needed for convergence

| tolerance | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| iter#     | 724       | 872       | 1020      | 1169      | 1317      |

3. Error Analysis

In this section we present some analysis results for AP process and the SAP algorithms. We need to mention here that unlike classical Krylov subspace methods, the SAP method proposed here can actually be used to solve any under-determined systems.

3.1. AP analysis

We first present some analysis about the AP process described in Algorithm 1.

Lemma 3.1. Assume that matrix $A \in \mathbb{R}^{m \times n}$ ($m \leq n$) has full row rank, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ where $m \leq n$ satisfy $Ax = b$. Let $A$ be divided into $k$ submatrices by its rows: $A = (A'_1, A'_2, \ldots, A'_k)'$ with $A_i \in \mathbb{R}^{m_i \times n}$, and $b$ is divided as $b = (b'_1, b'_2, \ldots, b'_k)'$ correspondingly. Let $\{p_i\}_1^k$ be the vector sequence produced by AP process (Algorithm 1).

1. There holds for every $i = 1, 2, \ldots, k$

   \[(x - p_i, p_s) = 0, \quad (s = i, i - 1). \quad (10)\]

2. Vector $p_{i+1} - p_i$ ($i = 0, 1, \cdots, k - 1$) is orthogonal to $p_i$, i.e.

   \[(p_{i+1} - p_i, p_i) = 0 \quad (11)\]

3. There holds for $i = 1, 2, \cdots, k$

   \[||p_i||^2 + ||p_{i+1} - p_i||^2 = ||p_{i+1}||^2 \quad (12)\]
(4) For every $s(1 \leq s \leq k)$, there holds

$$||p_s||^2 = ||p_0||^2 + \sum_{i=1}^{s} ||p_i - p_{i-1}||^2$$  \hspace{1cm} (13)$$

**Proof.** (1) We first show that $(x - p_0, p_0) = 0$. As a matter of fact, since 
$$\alpha = b'b/(b'AA'b),$$
we have

$$(x - p_0, p_0) = (x - \alpha A'_1 b_1, \alpha A'b) = \alpha x' A'_1 b - \alpha^2 b'A_1 A'b = \alpha b'b - \alpha b'b = 0$$

From the fact that $p_i$ is the projection of $x$ over subspace $\text{ran}(W_i)$ with $W_i = [p_{i-1}, A'_i]$, for any $i(1 \leq i \leq k)$ we must have

$$(x - p_i, p_i) = 0 \text{ and } (x - p_i, p_{i-1}) = 0$$

since both $p_i$ and $p_{i-1}$ belong to $W_i$.

(2) Note that from (10) we have

$$(p_{i+1} - p_i, p_i) = ((x - p_i) - (x - p_{i+1}), p_i) = (x - p_i, p_i) - (x - p_{i+1}, p_i) = 0,$$

which yields (11).

(3) From (11) we have

$$||p_{i+1} - p_i||^2 = (p_{i+1} - p_i, p_{i+1} - p_i)$$
$$= (p_{i+1} - p_i, p_{i+1})$$
$$= (p_{i+1}, p_{i+1}) - (p_i, p_{i+1})$$
$$= (p_{i+1}, p_{i+1}) - (p_i, (p_{i+1} - p_i) + p_i)$$
$$= (p_{i+1}, p_{i+1}) - (p_i, p_i)$$
$$= ||p_{i+1}||^2 - ||p_i||^2$$

from which (12) comes immediately.

(4) Equation (13) follows from the recursive application of (12):

$$||p_s||^2 = ||p_{s-1}||^2 + ||p_s - p_{s-1}||^2$$
$$= ||p_{s-2}||^2 + ||p_{s-1} - p_{s-2}||^2 + ||p_s - p_{s-1}||^2$$
$$\cdots$$
$$= ||p_0||^2 + ||p_1 - p_0||^2 + ||p_2 - p_1||^2 + \cdots + ||p_s - p_{s-1}||^2.$$
sequence, and obviously $\|x\|$ is actually one of its upper bounds. In order to find out how fast this sequence is increasing, we need to figure out the detailed information of each $\|p_i\|$ ($i = 1, 2, \cdots, k$). The following conclusion answers this question.

**Lemma 3.2.** Assume the same assumption in Lemma 3.1. Then $p_{i+1}$ has the following expression

$$p_{i+1} = \alpha_ip_i + A'_{i+1}u$$  \hspace{1cm} (14)

where $u$ is

$$u = \tilde{A}_{i+1}(b_{i+1} - \alpha_iA_{i+1}p_i)$$ \hspace{1cm} (15)

and

$$\alpha_i = \frac{x'p_i - (A_{i+1}p_i)'\tilde{A}_{i+1}b_{i+1}}{p'_ip_i - p'_iA'_{i+1}A_{i+1}A_{i+1}p_i}$$ \hspace{1cm} (16)

and

$$\tilde{A}_{i+1} = (A_{i+1}A'_{i+1})^{-1}.$$  

Furthermore

$$\|p_{i+1}\|^2 = \alpha_i\|p_i\|^2 + b'_{i+1}\tilde{A}_{i+1}b_{i+1} - \alpha^2_i(A_{i+1}p_i)'\tilde{A}_{i+1}(A_{i+1}p_i)$$ \hspace{1cm} (17)

**Proof.**

It is valid to express $p_{i+1}$ in the form like (14) for some $u \in R^{m_{i+1}}$ since $p_{i+1} \in \text{ran}(W_{i+1})$, where $m_i$ is the number of rows in submatrix $A_i$.

Since $p_{i+1}$ is the projection of $x$ over subspace $W_{i+1}$, we have

$$A_{i+1}(x - p_{i+1}) = 0$$

which leads to

$$b_{i+1} - \alpha_iA_{i+1}p_i - A_{i+1}A'_{i+1}u = 0$$

from which comes (15).

Similarly, by $p'_i(x - p_{i+1}) = 0$ we have

$$x'p_i - \alpha_ip'_ip_i + u'A_{i+1}p_i = 0,$$

replacing $u$ by (15) yields (16).

Finally from (14) we have

$$\|p_{i+1}\|^2 = (\alpha_ip_i + A'_{i+1}u)'(\alpha_ip_i + A'_{i+1}u)$$

$$= \alpha^2_ip'_ip_i + 2\alpha_ip'_iA'_{i+1}u + u'A_{i+1}A'_{i+1}u.$$  \hspace{1cm} (18)
Since
\[ 2\alpha_i p_i' A_{i+1}' u = 2\alpha_i p_i' A_{i+1}' A_{i+1} (b_{i+1} - \alpha_i A_{i+1}' p_i) \]
\[ = 2\alpha_i p_i' A_{i+1}' A_{i+1} b_{i+1} - 2\alpha_i^2 (A_{i+1}' p_i)' \tilde{A}_{i+1} (A_{i+1}' p_i) \] (19)
and
\[ u' A_{i+1} A_{i+1}' u = (b_{i+1} - \alpha_i A_{i+1}' p_i)' \tilde{A}_{i+1} (b_{i+1} - \alpha_i A_{i+1}' p_i) \]
\[ = b_{i+1}' \tilde{A}_{i+1} b_{i+1} - 2\alpha_i b_{i+1}' \tilde{A}_{i+1} A_{i+1} p_i \]
\[ + \alpha_i^2 (A_{i+1}' p_i)' \tilde{A}_{i+1} (A_{i+1}' p_i), \] (20)
equation (17) comes from (18) (19) (20) combined. □

Lemma 3.3. Assume the same assumption in lemma 3.1. Let \( I \) be the identity matrix in \( \mathbb{R}^n \). Then \( p_{i+1} \) has the following expression
\[ p_{i+1} = p_i + \tilde{A}_{i+1}' v \] (21)
and
\[ ||p_{i+1}||^2 - ||p_i||^2 = (b_{i+1} - (p_i' x) d)' (\tilde{A}_{i+1} \tilde{A}_{i+1}')^{-1} (b_{i+1} - (p_i' x) d) \] (22)
where \( \tilde{A}_{i+1} \) is a rank-one modification of submatrix \( A_{i+1} \) as
\[ \tilde{A}_{i+1} = A_{i+1} - dp_i' = A_{i+1} (I - u_i u_i') \] (23)
with \( u_i = p_i / ||p_i||, \) \( d \in \mathbb{R}^{m_i+1} \) a vector taken as \( d = A_{i+1} p_{i+1} / ||p_i||^2 \) and \( v \) is defined as
\[ v = (\tilde{A}_{i+1} \tilde{A}_{i+1}')^{-1} (b_{i+1} - (x' p_i) d) \]
assuming the related inverse exists.

Proof.
Since \( p_{i+1} \) is the projection of \( x \) over subspace \( W_{i+1} = \text{ran}(p_i, A_{i+1}') \), it can be constructed as follows.

First we modify row vectors in \( A_{i+1} \) so that they are orthogonal to vector \( p_i \), this can be depicted as a rank-one modification to \( A_{i+1} \) as
\[ \tilde{A}_{i+1} = A_{i+1} - dp_i', \]

\[ 14 \]
where $d$ can be obtained from the fact that

\[ \bar{A}_{i+1} p_i = 0 \]

which leads to

\[ A_{i+1} p_i - d p'_i p_i = 0, \]

hence

\[ d = A_{i+1} p_i / (p'_i p_i), \]

and

\[ \bar{A}_{i+1} = A_{i+1} - d p'_i = A_{i+1} - A_{i+1} p_i p'_i / (p'_i p_i) = A_{i+1} (I - u_i u'_i), \]

where $u_i = p_i / ||p_i||$.

Next we calculate the projection vector $\tilde{p}_{i+1}$ of $x$ over $\text{ran}(\bar{A}_{i+1})$ as

\[ \tilde{p}_{i+1} = \bar{A}_{i+1} v, \]

where $v$ can be derived from the fact that

\[ \bar{A}_{i+1} (x - \tilde{p}_{i+1}) = 0, \]

which leads to

\[ v = (\bar{A}_{i+1} \bar{A}'_{i+1})^{-1} (b_{i+1} - (p'_i x) d) \]

assuming $(\bar{A}_{i+1} \bar{A}'_{i+1})^{-1}$ exists.

Since $\tilde{p}_{i+1} = \bar{A}_{i+1} v$ is the projection of $x$ over $\text{ran}(\bar{A}'_{i+1})$ and $\bar{A}_{i+1} p_i = 0$, we must have $(p_i, \tilde{p}_{i+1}) = 0$. Therefore

\[ ||p_{i+1}||^2 - ||p_i||^2 = ||\tilde{p}_{i+1}||^2 \]

\[ = v' \bar{A}_{i+1} \bar{A}'_{i+1} v \]

\[ = (b_{i+1} - (p'_i x) d)'(\bar{A}_{i+1} \bar{A}'_{i+1})^{-1} (b_{i+1} - (p'_i x) d). \]

noting that matrix $(\bar{A}_{i+1} \bar{A}'_{i+1})^{-1}$ is symmetric (actually positive definite symmetric).

Remark: It can be shown that the length difference between $p_{i+1}$ and $p_i$ can also be written as

\[ ||p_{i+1}||^2 - ||p_i||^2 = \tilde{x}' G \tilde{x} \]

(24)

where $G = (\bar{A}_{i+1} \bar{A}'_{i+1})^{-1}$ and $\tilde{x} = \bar{x} - (x' u) u$, where $\bar{x}$ denotes the projection of $x$ on $\text{ran}(\bar{A}')$ and $(x' u) u$ is the projection of $x$ (as well as $\bar{x}$) on the direction of $u = p_i / ||p_i||$.

Note that in the above lemma, we need to assume the existence of each matrix $(\bar{A}_{i+1} \bar{A}'_{i+1})^{-1}$. The following conclusion gives the sufficient and necessary conditions for these to hold true.
Lemma 3.4. Let $A \in \mathbb{R}^{m \times n} (m \leq n)$ and $\text{rank}(A) = m$, $u \in \mathbb{R}^n$ be a unit vector in $\mathbb{R}^n$. Let $\bar{A} = A(I - uu')$ and $G = \bar{A}\bar{A}'$, where $I$ denote the identity matrix in $\mathbb{R}^n$. Then $G$ is nonsingular if and only if $u \notin \text{ran}(A')$.

Proof. 

Note that $G = \bar{A}\bar{A}'$ is invertible if and only if $\bar{A}$ is of full row rank. 

(Necessity) Assume $G$ is invertible, we need to show that $u \notin \text{ran}(A')$. If this is not the case, i.e., $u \in \text{ran}(A')$, then there is a $v \in \mathbb{R}^n$ ($v \neq 0$) such that $u = A'v$. Thus

$$\bar{A}'v = (A(I - uu'))'v = (A - Auu')'u = A'v - uu'A'v = u - u(u'u) = 0$$

since $||u|| = 1$. This means $\bar{A}$ is not of full rank, hence $G$ is singular, a contradiction with our assumption.

(Sufficiency). Assume $u \notin \text{ran}(A')$, we need to show that $G$ is invertible. As a matter of fact, if $G$ is not invertible, then $\bar{A}$ is not of full-row rank. Therefore there exists a nonzero vector $v \in \mathbb{R}^m$ such that $\bar{A}'v = 0$. That means

$$0 = (A(I - uu'))'v = A'v - uu'A'v = A'v - \alpha u$$

where $\alpha = u'(A'v)$ is a scalar. It is easy to see from here that $\alpha \neq 0$, otherwise we would have $A'v = 0$ which means $A$ is not of full row rank. Hence $u = A'v/\alpha$, i.e., $u \in \text{ran}(A')$, this is contradictory with the assumption. \qed

Lemma 3.5. Assume the same assumption in Lemma 3.4. Vector sequence $p_0, p_1, \cdots, p_k$ are produced in one AP process, then

$$||p_i|| \leq ||p_{i+1}|| \quad (i = 0, 1, 2, \cdots, k) \quad (25)$$

and the equal sign holds if and only if

$$A_{i+1}p_i = b_{i+1}$$

Proof. Inequality (25) comes from (12) directly. We now prove the necessary condition for $||p_{i+1}|| = ||p_i||$. 

(Necessity) Note that if $||p_{i+1}|| = ||p_i||$ holds, by (12) we must have $p_{i+1} = p_i$. Also from (11) we know that

$$p_{i+1} = \alpha_i p_i + A_{i+1}' u,$$
thus

\[ A_{i+1}'u = p_{i+1} - \alpha_ip_i = (1 - \alpha_i)p_i. \]  \hfill (26)

Multiplying both sides of (26) by \( A_{i+1} \) we have

\[ A_{i+1}A_{i+1}'u = (1 - \alpha_i)Ap_i. \]  \hfill (27)

Note that from (15) we have

\[ A_{i+1}A_{i+1}'u = b_{i+1} - \alpha_iAp_i. \]  \hfill (28)

Combining (27) and (28) yields

\[ Ap_i = b_{i+1}. \]

(Sufficiency) Now we prove \( p_{i+1} = p_i \) under the assumption \( Ap_i = b_{i+1} \).

As a matter of fact, in view of (15) and (16) we only need to show that

\[ \alpha_i = 1 \]

in this case.

Since \((x - p_i, p_i) = 0\), we have

\[ x'p_i = p'_ip_i. \]  \hfill (29)

By using \( A_{i+1}p_i = b_{i+1} \) we obtain

\[ p'_iA_{i+1}'\tilde{A}_{i+1}A_{i+1}p_i = b_{i+1}\tilde{A}_{i+1}b_{i+1} \]  \hfill (30)

Hence from (16) we have

\[ \alpha_i = \frac{x'p_i - (A_{i+1}p_i)'\tilde{A}_{i+1}b_{i+1}}{p'_ip_i - p'_iA_{i+1}'\tilde{A}_{i+1}A_{i+1}p_i} = \frac{p'_ip_i - b'_ip_i - b'_{i+1}\tilde{A}_{i+1}b_{i+1}}{p'_ip_i - b'_{i+1}\tilde{A}_{i+1}b_{i+1}} = 1 \]

This completes the proof of the sufficient condition. \( \square \)

Based on the above error analysis about AP process, a practical AP algorithm can be implemented as follows.

**Algorithm 3.** (An accumulated projection method-AP) The following procedure produces an approximate vector \( p \) to the solution vector \( x \) which satisfies \( Ax = b \).
Step 1. Divide matrix $A$ into $k$ blocks by its row vectors: $A_1, A_2, \cdots, A_k$, divide $b$ correspondingly: $b_1, b_2, \cdots, b_k$ (Note, $A_i$ and $A_{i+1}$ may contain some common row vectors).

Step 2. Initialize $p_0$ as $p_0 = \alpha A' b$ and $c_0 = \alpha ||b||^2$, where $\alpha = ||b||^2 / ||A'b||^2$.

Step 3. For $i = 1$ to $k$

Step 3.1 Do QR factorization on submatrix $A_i : A'_i = Q_i R_i$.

Step 3.2 Compute vector $\tilde{b}_i = (R_i')^{-1} b_i$.

Step 3.3 Compute projection vector $x_i$ of $x$ over $\text{ran}(A'_i)$: $x_i = Q'_i \tilde{b}_i$.

Step 3.4 Store orthogonal matric $Q_i$ and vector $\tilde{b}_i$.

Step 3.5 Go to next $i$.

Step 4. For $i = 1$ to $k$

Step 4.1 Compute projection vector $\tilde{p}_{i-1}$ of $p_{i-1}$ on $\text{ran}(A'_i)$:

$$\tilde{p}_{i-1} = Q_i (Q'_i p_{i-1}).$$

Step 4.2 Compute vector $\tilde{p}_{i-1} = p_{i-1} - \tilde{p}_{i-1}$.

Step 4.3 Compute the projection vector $\tilde{x}_i$ of $x$ on direction $\tilde{p}_{i-1}$:

$$\tilde{x}_i = \beta \tilde{p}_{i-1}, \text{ where } \beta = (c_{i-1} - \tilde{b}'_i Q'_i p_{i-1})/(\tilde{p}'_{i-1} \tilde{p}_{i-1}).$$

Step 4.4 Compute the projection vector $p_i$ of $x$ over $\text{ran}([p_{i-1}, A'_i])$:

$$p_i = x_i + \tilde{x}_i$$

and the inner product between $p_i$ and $x$:

$$c_i = \tilde{b}'_i \tilde{b}_i + \beta (c_{i-1} - \tilde{b}'_i Q'_i p_{i-1})$$

Step 4.5 Go to next $i$.

Step 5. Output $p(= p_k)$ and $c(= c_k)$.

3.2. Convergence analysis of SAP

We now turn to the convergence of SAP (Algorithm 2). We first have the following conclusion.

**Theorem 3.6.** Let $\{x_s\}_t$ be any approximating vector sequence produced by the SAP process before convergence reached. Then there exists

$$||x_1|| < ||x_2|| < \cdots < ||x_t|| < ||\bar{x}||$$  \hspace{1cm} (31)
Proof. Let \( \{p_j^s\}_{j=1}^k \) be the projection vector in the \( s \)-th AP iteration in SAP algorithm. Then we have \( x_i = p_j^s \) \((i = 1, 2, \cdots, \infty)\). Since \( x_i \neq \bar{x} \) for any \( i < t \), from Lemma 3.3 we can always find some integer \( j \) such that \( A_j x_i \neq b_j \). Assuming \( j \) is the first of this kind, then start from the \( s+1 \)-th AP iteration in SAP algorithm, we have \( p_j^{i+1} = x_i \) \((i = 1, 2, \cdots, j-1)\) and by Lemma 3.1 we have
\[
||x_i|| < ||p_j^{i+1}||
\]
Note that we always have \( ||p_j^{i+1}|| \leq ||p_j^{i+1}|| = ||x_i+1||, \) thus
\[
||x_i|| < ||p_j^{i+1}|| \leq ||x_i+1||, \quad (i = 1, 2, \cdots, t)
\]
and since \( \bar{x} \) is the projection of \( x \) over \( \text{ran}(A') \) while \( x_i \) is the projection of some subspace \( W_k \) of \( \text{ran}(A') \), by Lemma 3.2 we always have \( ||x_i|| < ||\bar{x}|| \) for any \( i \) \((1 \leq i \leq t)\). These complete the proof. \( \square \)

In order to prove the convergence of SAP method, we need the following conclusion.

Lemma 3.7. Let \( \{y_s\}_1^\infty \) be any convergent subsequence of \( \{x_s\}_1^\infty \), and suppose \( \lim_{s \to \infty} y_s = y \). Then \( y = \bar{x} \), where \( \bar{x} \) is the projection of \( x \) satisfying \( Ax = b \).

Proof. Since \( y_s = x_{t_s} \) for some integer \( t_s \) and \( x_s = p_k^s \) \((s = 1, 2, \cdots)\), by Lemma 3.1 we have
\[
||y_{s+1}||^2 - ||y_s||^2 = ||p_k^s_{t+1}||^2 - ||p_k^s||^2
\]
\[
= ||p_k^s_{t+1}||^2 - ||p_k^{s+1}||^2
\]
\[
> ||p_k^{s+1} - p_k^{s+1}||^2
\]
\[
= ||p_k^{s+1}||^2 - ||p_k^{s+1}||^2
\]
\[
= \tilde{x}^G \tilde{x}
\]
where \( \tilde{x} = \bar{x} - x'u_0 u_0', \ u_0 = p_k^s / ||p_k^s|| = y_s / ||y_s|| \) and \( G = (A_1 A_1')^{-1} \) with \( A_1 = A_1(I - u_0 u_0') \). i.e., we have
\[
||y_{s+1}||^2 - ||y_s||^2 > \tilde{x}^G \tilde{x} \quad \text{(32)}
\]
If \( y \neq \bar{x} \), taking the limits on both sides of (32) for \( s \) approaching to infinity we have
\[
0 = \lim_{s \to \infty} ||y_{s+1}||^2 - ||y_s||^2 > \lim_{s \to \infty} \tilde{x}^G \tilde{x} > 0
\]
since \( G \) is symmetric positive definite and \( \tilde{x} \neq 0 \). This is a contradiction and thus we must have \( y = \bar{x} \). \( \square \)
Theorem 3.8. Let \( \{x_s\}_{1}^{\infty} \) be the approximating vector sequence produced by the SAP process. Then

\[
\lim_{s \to \infty} x_s = \bar{x}
\]

where \( \bar{x} \) stands for the projection of \( x \) onto subspace \( \text{ran}(A') \) of \( \mathbb{R}^n \). Particularly if \( m = n \) and \( A \) is nonsingular, then we have \( \bar{x} = x \).

Proof. From Theorem 3.6 we see that sequence \( \{x_s\}_{1}^{\infty} \) is a bounded sequence. If \( \lim_{s \to \infty} x_s \) does not exist, then it has at least two different cluster points \( z_1 \) and \( z_2 \) such that there are two subsequences of \( \{x_s\}_{1}^{\infty} \) approaches to \( z_1 \) and \( z_2 \). However by Lemma 3.7 we have \( z_1 = \bar{x} \) and \( z_2 = \bar{x} \). This is a contradiction. Hence we must have

\[
\lim_{s \to \infty} x_s = \bar{x}.
\]

Proof is completed. \( \Box \)

4. Some Acceleration Strategies

We have observed from the preceding section that the convergence speed of the simple iterative algorithm may not be very satisfactory in general. In this section we are to design some accelerative approaches for the SAP algorithm.

4.1. Increase the Block Size

An apparent approach is to simply increase the size of each block. The following table (Table 2) shows the iteration numbers needed for a convergent solution when \( A = \text{diag}(-1, 2, -1) \in \mathbb{R}^{100 \times 100} \) and the tolerance is set at \( 10^{-5} \) for the relative residual error. One can see that the number of iterations may drastically decrease when the size of blocks is slightly increased. Unlike GMRES(\( m \)) with restarting where \( m \) stands for the inner iteration numbers for each outer iteration and \( m \) has to be very small comparing to the size of systems, this approach is viable since the size of each block can be selected much larger (in case of sparse systems, one can choose the size of each block as large as \( O(10^5) \) when LGO-based QS decomposition method [11] is used to orthogonalize the block submatrices.
Table 2: block-SAP– iteration numbers needed for convergence

| block size | 10  | 15  | 20  | 25  | 30  | 35  | 40  | 50  |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| iter#      | 11404 | 2994 | 1020 | 443 | 222 | 104 | 57  | 27  |

4.2. A Modified SAP Approach

Another option for accelerating the convergence is to add one simple step at the beginning of each loop in Algorithm 2 step 2. Specifically instead of using \( x_k \) as the initial approximate solution to start another AP process, we first get the projection vector \( p \) of \( x \) onto the subspace \( W = \text{span}\{x_{k-1}, x_k\} \) as well as \( c = x'p \) and then use it to replace \( x_k \) and \( c_k \) in Algorithm 2. The details come as follows.

Algorithm 4. (A Modified Stationary Accumulated Projection Method-MSAP version 1). Let \( A \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^n \). \( \epsilon \) be a given tolerance. The following procedure produces an approximation \( p \) to the solution \( x \) of system \( Ax = b \).

Step1. Initialize \( s \) as \( s = 0 \), vector \( x_0 \) as \( x_0 = \alpha A'b \), \( c_0 = \alpha b'b \), \( t = ||b - Ax_0||/||b|| \) where \( \alpha = ||b||^2/||A'b||^2 \). Let \( p = x_0 \) and \( c = c_0 \).

Step2. While \( t > \epsilon \)

   Step1. Use Algorithm 3 to get a projection vector \( x_{s+1} \) of \( x \) and \( c_{s+1} = x'x_{s+1} \) with \( p \) and \( c \) the initial projection vector and the corresponding scalar taken as \( x_s \) and \( c_s \) respectively;

   step 2.2 Calculate the projection \( p \) of \( x \) onto subspace \( W = \text{span}\{x_s, x_{s-1}\} \) and scalar \( c = x'p \). Rename \( p \) as \( x_{s+1} \) and \( c_{s+1} \) respectively;

   step 2.3 Calculate \( t = ||b - Ax_{s+1}||/||b|| \);

   step 2.4 \( s = s + 1 \);

Step3. Output \( p = (x_s) \) and \( c = (c_s) \).

The following table (Table 3) shows the needed iteration numbers when running the same example in subsection 4.1. It is easy to see that this simple acceleration technique works very well (comparing with the numbers in Table 2).

It is attempting to increase the dimension of the subspace \( W \) in Algorithm 4 step 2.2 to get a better convergence speed. Unfortunately this seems not work since the “distance” between \( \{x_s\}_1^n \) are not far enough and thus the
submatrix formed by these vectors tends to be very ill-conditioned, which eventually makes the idea not work well. The following subsection depicts an alternative option for this idea.

4.3. A Varying Subspace Method

It is attractive to use a subspace $W$ with larger dimension than that of subspace $W$ in Algorithm 4 step 2.2 to develop an accelerative method for SAP. However we have noted that as iteration goes on, the matrix formed by the successive SAP projection tends to be ill-conditioned. Hence we plan to use a more flexible strategy to handle the ill-conditioned systems. Our intension is to use a detector to check the conditioning of an intermediate matrix $H$, and then arrange the dimension of the subspace $W$ accordingly. The details are described in the following algorithm.

**Algorithm 5.** *(A Modified Stationary Accumulated Projection Method-MSAP version 2)*. Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$. $\epsilon$ be a given tolerance. The following procedure produces an approximation $p$ to the solution $x$ of system $Ax = b$.

**Step1.** Initialize $s$ as $s = 0$, vector $x_0$ as $x_0 = \alpha A'b$, $c_0 = \alpha b'b$, $t = ||b - Ax_0||/||b||$ where $\alpha = ||b||^2/||A'b||^2$. Let $p = x_0$ and $c = c_0$, and $m$ be a small predetermined integer.

**Step2.** While $t > \epsilon$

- **Step 2.1.** Use Algorithm 3 to get a projection vector $p_n$ of $x$ and $c_n(= x'x_{s+1})$ with $p$ and $c$ as the initial projection vector and the corresponding scalar.

- **Step 2.2.** Store $p_n$ as a row vector in matrix $H$ and $c_n$ into a column vector $L$.

- **Step 2.3.** If $H$ contains $m$ row vectors

  - **Step 2.3.1** if $H$ is well-conditioned,

    - **Step 2.3.1.1** update $x_{s+1}$ as the projection of $x$ on subspace $W = \text{ran}(H)$ and the scalar $c_{s+1} = x'x_{s+1}$. 

---

Table 3: MSAP–iteration numbers needed for convergence

| block size | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 50 |
|------------|----|----|----|----|----|----|----|----|
| iter#      | 2134 | 403 | 134 | 69 | 38 | 34 | 18 | 15 |
Step 2.3.1.2 remove the first row vector of \( H \) and the first element in vector \( L \) correspondingly;

step 2.3.2 if \( H \) is ill-conditioned

Step 2.3.2.1 update \( x_{k+1} \) as the projection of \( x \) on subspace \( W = \text{span}(x_s, pn) \) and the scalar \( c_{s+1} = x'x_{s+1} \).

Step 2.3.2.2 remove all but the first row vectors of \( H \) and the elements in vector \( L \) correspondingly.

step 2.4 if \( H \) contains less than \( m \) row vectors, update \( x_{s+1} \) as the projection of \( x \) on subspace \( W = \text{span}(x_s, pn) \) and the scalar \( c_{s+1} = x'x_{s+1} \).

step 2.5 Calculate \( t = ||b - Ax_s||/||b|| \);

step 2.6 Set \( s = s + 1 \), \( p = x_{s+1} \) and \( c = c_{s+1} \).

Step3. Output \( p(= x_s) \) and \( c_s \).

The following table (Table 4) shows the astonishing acceleration speed of convergence when we use Algorithm 5 to solve the same problem in the preceding subsection.

Table 4: MSAP2—iteration numbers needed for convergence

| block size | 10  | 15  | 20  | 25  | 30  | 35  | 40  | 50  |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| iter#  | 185 | 102 | 42  | 30  | 16  | 14  | 10  | 7   |

5. Numerical Experiments

In this section we will show some application of the aforementioned SAP methods and we compare the results with those produced by GMRES—a benchmark Krylov subspace method. We use MSAP and GMRES to calculate the systems.

As the first example, we use the two-point boundary value problem

\[
\begin{align*}
(a(t)u'(t))' + b(t)u(t) &= f(t), \quad t \in (0,1) \\
u(0) &= u(1) = 0.
\end{align*}
\]  

This equation represents some important practical problems such as chord balancing, elastic beam problems, etc. We use finite element method to get
the numerical solution to the system, which ends up with a linear system of
equations in the form of \((1)\) with \(n\) unknowns, where \(n\) stands for the number
of grids which divide the interval \((0, 1)\) into \(n + 1\) equal-sized subintervals
\((x_i, x_{i+1})\) \((i = 1, 2, \cdots n)\). We use the linear interpolation function at each
grid point to construct the finite element space \(V_0^h\), test functions are also
from \(V_0^h\).

In our test we take \(a(t) = 1 + t, b(t) = t\) and \(f(t)\) is taken so that the exact
solution to the system is \(u(t) = t(1 - t)e^{2+t}\). By using the aforementioned
finite element space \(V_0^h\), we get a linear system of equation \(Ax = b\) with \(A\)
as a symmetric tridiagonal matrix in \(R^{n \times n}\), where \(n\) is taken as 200. The block
size for the MSAP method (version 2) is set to be the square root of the
restart number \(m\) of GMRES(m) multiplied by the number of unknowns \(n\)
so that submatrices of \(A\) have roughly the same number of non-zero elements
as those in Krylov subspace matrices formed in GMRES process.

Table 5 shows the comparison of iteration numbers needed for conver-
gence, the relative error in terms of \(|x - \text{approx. } x|/|x|\) (where \(\text{approx. } x\)
stands for the approximate solutions obtained by using GMRES and MSAP2
respectively) and the relative residual error in terms of \(|b - A \cdot \text{approx. } x|/|b|\|
is used for the convergence criteria. We also observed that the time cost in
these example also show some advantage of MSAP over GMRES as shown
in Table 5.

| settings | iter. # | time (in s) | rel. error |
|----------|---------|-------------|------------|
| blk_size | restart | m=5        |            |
|          |         | (out, in)   |            |
| 20       | 2       | 200         | 1.2690     | 0.6700    | 7.02e-7   | 6.96e-5   |
| 30       | 5       | 200         | 0.6340     | 1.6430    | 3.49e-7   | 5.68e-5   |
| 40       | 8       | 50          | 0.1320     | 1.8180    | 9.57e-7   | 4.92e-5   |
| 50       | 13      | 33          | 0.0800     | 1.3780    | 3.32e-7   | 4.4e-5    |
| 60       | 18      | 22          | 0.0510     | 1.1880    | 9.57e-7   | 4.01e-5   |
| 70       | 25      | 17          | 0.0380     | 1.1750    | 5.06e-7   | 3.71e-5   |
| 80       | 32      | 13          | 0.0380     | 0.9950    | 3.01e-8   | 3.48e-5   |

From the above table it seems that MSAP has a better relative error level
than that of GMRES at the same relative residual level. We have to point out
here that the construction of this test system is made so that the solution has rich eigenvector components corresponding to the smallest eigenvalues of the coefficient matrix \(A\). In case the condition number \(\text{cond}(A)\) is relatively small (say, less than \(O(10^3)\)), GMRES outperforms MSAP in terms of time costs and flops in our tests, while in case the condition number of the coefficient matrices are larger than \(O(10^3)\), MSAP generally outperforms GMRES in most of our test cases.

Table 7 is a comparison between MSAP and block Jacobi method for a system with coefficient matrix \(A \in \mathbb{R}^{200 \times 200}\). The block size for both methods are chosen as exactly the same. One can see a much less iteration number needed for MSAP than that of block Jacobi method in each case of block size, while the relative errors obtained by MSAP are much better than those obtained by block Jacobi method.

| blk_size | Jacobi time(in s) | Jerobi iteration# | rel. error | Jacobi msap iteration# | rel. error |
|----------|-------------------|------------------|------------|------------------------|------------|
| 10       | 5.666             | 7836             | 6.9553e-005| 1745                   | 7.0191e-007|
| 15       | 2.769             | 5347             | 5.6761e-005| 830                    | 3.4931e-007|
| 20       | 1.547             | 4082             | 4.921e-005 | 390                    | 9.5735e-007|
| 25       | 1.018             | 3316             | 4.3997e-005| 185                    | 3.3189e-007|
| 30       | 0.781             | 2806             | 4.0135e-005| 130                    | 2.4148e-007|
| 35       | 0.589             | 2440             | 3.7142e-005| 85                     | 5.0644e-007|
| 40       | 0.446             | 2159             | 3.4765e-005| 55                     | 3.0083e-008|
| 45       | 0.408             | 1946             | 3.2554e-005| 45                     | 3.132e-008 |

6. Comments and Summary

In this paper we present a new type of iterative methods for solving linear system of equations. This might be the first type of methods which do not belong to the category of extended Krylov subspace methods as we mentioned above. It can overcome some shortcomings of Krylov subspace methods and exhibit better performance in our test problems. We need to mention that convergence speed of these algorithms deteriorates when the number of subdividing blocks of the coefficient matrix exceeds 20, a remedy to this is
to embed an inner loop in the AP process, which will cause more flops but the obtained time efficiency payoff this costs in our tests. A relevant issue is the study of detailed error analysis for the SAP algorithm which may lead to a deep insight error estimation for each AP process in terms of subspace distance. We need to mention here that the SAP algorithm is nothing but a “horizontal” application of the AP process, i.e., the AP processes are always applied to the original linear system instead of residual equations. We find that a vertical application of AP process is also possible and the results are to appear in our later work.

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SAP solutions at different iterations with $A = \text{diag}(-1,2,-1)_n$, $n=100$