Non-Analytic Non-Equilibrium Field Theory: Stochastic Reheating of the Ising Model

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Abstract

Many-body non-equilibrium steady states can still be described by a Landau-Ginzburg theory if one allows non-analytic terms in the potential. We substantiate this claim by working out the case of the Ising magnet in contact with a thermal bath and undergoing stochastic reheating: it is reset to a paramagnet at random times. By a combination of stochastic field theory and Monte Carlo simulations, we unveil how the usual $\varphi^4$ potential is deformed by non-analytic operators of intrinsic non-equilibrium nature. We demonstrate their infrared relevance at low temperatures by a renormalization-group analysis of the non-equilibrium steady state. The equilibrium ferromagnetic fixed point is thus destabilized by stochastic reheating, and we identify the new non-equilibrium fixed point.

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1 Introduction

The Landau-Ginzburg theory is a monumental cornerstone of modern physics that unifies the various equilibrium phase transitions of matter in a common framework. It relies on the effective description of many-body systems in terms of a local order-parameter field, say $\phi(x)$. The probability distribution at equilibrium (EQ) is given by $P_{\text{EQ}}[\phi] \sim \exp(-F_{\text{EQ}}[\phi])$, where the free-energy functional $F_{\text{EQ}}[\phi]$ is built on simple principles: locality, symmetry, stability, and analyticity. Subsequently, $F_{\text{EQ}}[\phi]$ can be fed to a renormalization group (RG) analysis in order to access the universal infrared features of the many-body system.

Away from thermal equilibrium, non-equilibrium phase transitions can be studied in the framework of non-equilibrium extensions to the dynamical field theories classified by Hohenberg and Halperin [1]. The theories that perhaps received the most attention are those for which the dynamics conserve a global quantity such as the particle number, i.e. whose field-theoretic description revolves around the Model B. Examples include driven-diffusive systems [2, 3] such as the driven lattice gas [4, 5], but also active matter systems [6] displaying, e.g., phase separation or pattern formation [7, 8]. Another class of non-equilibrium systems are the so-called driven-dissipative systems, with no conserved quantity. This includes growth processes, such as the directed percolation [9, 10] or the Kardar-Parisi-Zhang problems [11, 12].

While the quest for a unified principled-based approach to all those non-equilibrium theories still seems out of reach, a more reasonable ambition is to restrict the focus to non-equilibrium steady states (NESS), i.e. time-translational invariant non-thermal states described by a stationary probability measure $P_{\text{NESS}}[\phi]$. Recent developments have led to propose the following claim.

**Claim.** A Landau-Ginzburg theory can be developed for NESS at the cost of abandoning the principle of analyticity of the Landau-Ginzburg functional $F_{\text{NESS}}[\phi] \equiv -\log P_{\text{NESS}}[\phi]$. 

This idea emerged out of a heuristic observation in the context of a driven-dissipative quantum antiferromagnet with a $\mathbb{Z}_2$-symmetric staggered order parameter $\varphi$ [13]. The idea was then sharpened in Ref. [14] where the claim was formulated in the framework of the Ising magnet undergoing driven-dissipative dynamics. The explicit computations were made possible by a mean-field approximation: the magnetization $\varphi$ was assumed to be homogeneous throughout the magnet. The static steady state $P_{\text{NESS}}(\varphi) \sim \exp(-V_{\text{NESS}}(\varphi))$ was computed in a couple of simple concrete examples, such as a two-bath Ising model. They unveiled non-analytic Landau potentials of the form

$$V_{\text{NESS}}(\varphi) = a_2 \varphi^2 + a_4 \varphi^4 + c_\alpha |\varphi|^{2+\alpha} + \ldots$$

where the exponent $\alpha$, with $0 < \alpha < 2$, depends on the low-energy features of the environment and can be non-integer valued. The coefficients $a_2$, $a_4$, and $c_\alpha$ are smooth functions of the external parameters, and $c_\alpha$ vanishes at equilibrium. The non-analytic terms are therefore of intrinsic non-equilibrium nature.

The results of Ref. [14] definitely brought an important conceptual milestone to the construction of a field theory for non-equilibrium steady states, specifically the loss of analyticity of the Landau potential. This heralds a possible extension of the concept of universality class to non-equilibrium steady states: critical exponents are not only determined by the dimensions of the system and the symmetries of its order parameter, but also by the low-energy features of its environment.

However, questions remain to ascertain that these findings are not artifacts of the mean-field approximation that was used, but are indeed robust non-equilibrium field-theoretic hallmarks present in finite dimensions:

1. Is there a finite upper critical dimension $d_{\text{uc}}$ above which fluctuations are irrelevant to the non-equilibrium steady state and the mean-field approximation is exact?

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*Exception being topological phase transitions.
†Conversely, the absence of such non-analytic terms does not imply equilibrium.
2. Do the non-analytic terms of the non-equilibrium Landau potential survive in the infrared (IR)? Or are they washed out when the order parameter is coarse-grained on larger and larger scales?

3. Do the non-analytic terms survive fluctuations below the upper critical dimension?

In this paper, we address these general questions in the specific framework of the driven-dissipative dynamics of the Ising model undergoing stochastic reheating. We introduce the non-equilibrium model and its field-theoretic description in Sec. 2. In Sec. 3 we present analytical arguments that give positive and unambiguous answers to the first two questions above. Importantly, these positive answers validate the claim stated above beyond the scope of the specific non-equilibrium dynamics studied here. We close in Sec. 4 by presenting solid numerical arguments in favor of a positive answer to the third question, and by discussing possible routes to complete the field-theoretical picture initiated in this work.

Let us summarize the key results:

- the field theory for non-equilibrium steady states in finite dimensions is found to be a deformation of the equilibrium $\varphi^4$ theory by non-analytic operators
- the non-analytic operators are shown to be IR relevant, or IR irrelevant, depending on the bath temperature
- $d_{\text{uc}}$ is conjectured to be 4, i.e. the same as the equilibrium upper critical dimension
- above $d = 4$, the Landau potentials at the IR fixed points are computed exactly
- supporting evidence from Monte Carlo numerics is provided in both $d = 4$ and $d = 2$.

2 Kinetic Ising model with stochastic reheating

We address the questions listed above in a simple framework: a driven-dissipative many-body system with no conserved quantity and whose non-equilibrium steady state is homogeneous and isotropic. Specifically, we consider the infamous $\mathbb{Z}_2$-symmetric Ising magnet whose relaxation dynamics are generated by the coupling to an equilibrium bath at temperature $T$ – inducing thermal spin flips – and whose non-equilibrium drive is realized by a stochastic reheating protocol: the magnet is randomly reheated, with rate $r$, to an infinite-temperature paramagnetic state. Between two reheating events, the dynamics are the ones of a quench from infinite temperature to the bath temperature $T$.

The choice of stochastic reheating as the non-equilibrium drive is guided by the expectation that the mean-field description of our many-body problem reduces to a diffusive single particle undergoing so-called stochastic resetting: it is reset to a given position at random times with rate $r$ [15]. The generation of non-equilibrium states by stochastic resetting is a simple alternative compared to traditional setups where multiple reservoirs are attached to the system. Over the past decade, a body of exact results has already been obtained in different implementations ranging from space- or time-dependent resetting rate [16], resetting to a random position [17], generalisations to higher dimensions [18] and extended systems in the Kardar-Parisi-Zhang universality class [19]. The use of stochastic resetting as a non-equilibrium drive to many-body systems was pioneered very recently in Ref. [20] where the Ising model is reset to an ordered ferromagnetic state.

2.1 Instability of the ferromagnetic fixed point

Equilibrium The absence of stochastic reheating, when $r = 0$, corresponds to thermal equilibrium. The Ising model undergoes Glauber dynamics governed by the detailed balance at the temperature $T$ of the bath. This is often referred as the kinetic Ising model. The corresponding field-theory description is given by the well-known $O(n = 1)$-symmetric $\varphi^4$ theory [21], where the
Figure 1: Schematic of the renormalization of the non-equilibrium steady-state Landau potential $V_{NESS}(\varphi)$ of the Ising magnet undergoing stochastic reheating with rate $r$ and in $d \geq 4$. At bath temperatures $T > T_c$, the infrared fixed point is simply the equilibrium fixed point of Eq. (10). For $T \leq T_c$, a cusp develops at $\varphi = 0$ yielding a new non-equilibrium fixed point. The corresponding non-analytic fixed-point potentials are computed explicitly in Eqs. (13) and (15). The equilibrium scenario ($r = 0$) is recalled with dotted lines.

The real field $\varphi(x)$ is a coarse-grained measure of the magnetization, say on cells of linear size $l$ \cite{22}. The scale $l$ plays the role of the ultraviolet (UV) cutoff of the field theory. Cooling down the bath temperature across the critical temperature $T_c$, the phase transition from the paramagnetic phase to the ferromagnetic phase is described by the spontaneous symmetry breaking of the $\mathbb{Z}_2$ symmetry when the shape of the Landau potential changes from a parabola to a double-well Mexican hat.

Non-equilibrium steady state The case of interest, when the reheating rate $r > 0$, corresponds to non-equilibrium dynamics. After a transient regime which depends on the initial state preparation, the many-body dynamics are expected to reach a non-equilibrium steady state (NESS) described by a static probability distribution $P_{NESS}[\varphi]$.

Importantly, we can readily argue that a finite reheating rate, $r > 0$, is a singular non-equilibrium perturbation in the sense that it changes drastically the physics of the ordered phase that is found at equilibrium ($r = 0$). Indeed, the stochastic reheating protocol effectively restores the $\mathbb{Z}_2$ symmetry that is spontaneously broken in equilibrium at low temperatures. This simple observation has important consequences:

(i) at fixed $r > 0$, the average local magnetization vanishes identically in the steady state, irrespective of the bath temperature: $\langle \varphi(x) \rangle = 0$ where the average is taken with respect to $P_{NESS}$. However, the vanishing of the order parameter at all temperatures must not hide the presence of two distinct non-equilibrium phases:

(a) a paramagnetic phase at high temperatures, $T > T_c$, where the system is always and everywhere paramagnetic. There, the stochastic reheating protocol is expected to be mostly inconsequential since those long wavelength modes with relaxation timescales longer than the reheating timescale set by $1/r$ mostly live at infinite temperature, while the modes with timescales shorter than $1/r$ have the time to thermalize to the temperature $T$,

(b) an ordering phase at low temperatures, $T < T_c$, where after each reheating event the system undergoes the much-studied coarsening dynamics of a ferromagnet after a quench from infinite temperature and across the phase transition \cite{23,24,25}: ferromagnetic
domains of opposite magnetization compete and grow as $\xi(t) \sim \sqrt{t}$. The stochastic reheating halts this growth, yielding a typical correlation length $\xi_{\text{max}} \sim 1/\sqrt{r}$.

These phases are not distinguishable by the vanishing global order parameter, but they can be captured, e.g., by the fluctuations of local order parameter.

(ii) at bath temperatures $T < T_c$, there is a discontinuous phase transition when turning on $r$ between the equilibrium ferromagnet (with $\langle \varphi(x) \rangle \neq 0$) and the non-equilibrium steady state (with $\langle \varphi(x) \rangle = 0$),

(iii) at finite $r$ and at scales larger than the correlation length, $l \gtrsim \xi_{\text{max}}$, the corresponding non-equilibrium Landau-Ginzburg functional can only have a single global minimum at $\varphi = 0$.

From an RG viewpoint, this means that stochastic reheating is a relevant perturbation that destabilizes the equilibrium low-temperature (Gaussian or Wilson-Fisher) fixed point, irrespective of the dimension.

2.2 Correspondence with quench dynamics

Renewal formula The task of computing the stationary measure $P_{\text{NESS}}[\varphi]$ of the stochastic reheating problem can be much simplified by the use of the so-called renewal formula \[15, 26\]

$$P_{\text{NESS}}[\varphi] = r \int_0^\infty dt \, e^{-rt} P_0([\varphi]; t),$$

where $P_0([\varphi]; t)$ is the time-dependent probability distribution after a quench from infinite temperature and in the absence of reheating. These purely dissipative dynamics will be briefly described below.

The renewal formula \[2\] has important implications. It connects the statics of a fully non-equilibrium problem to the relaxation dynamics of the field in contact with an equilibrium bath. As we shall see in Sec. 3, this relation is instrumental to performing an RG analysis in the non-equilibrium steady state (NESS) using the knowledge of what is known about the relaxation dynamics of magnetization. In particular, it tells us that field fluctuations can be neglected in the NESS whenever it is legitimate to neglect them in the relaxation dynamics: we therefore expect the upper critical dimension of the stochastic reheating problem to be $d_{\text{uc}} = 4$. From a numerical perspective, the renewal formula also implies that $P_{\text{NESS}}[\varphi]$ can be simply reconstructed from the data of a quench up to finite times on the order of a few $1/r$. It is worth noticing that although the integrand in Eq. \[(2)\] may be analytic in the field, the integral itself can be non-analytic.

Model A relaxation The dynamics of magnetization after a quench from infinite temperature are described by the initial condition $\varphi(x, t = 0) = 0$ and by the so-called Model A dynamics \[1\] given by the stochastic equation

$$\eta \partial_t \varphi(x, t) = -\frac{\delta F_{\text{EQ}}[\varphi]}{\delta \varphi(x, t)} + \xi(x, t),$$

where $\eta > 0$ is a dimensionless “friction” parameter, and $\xi$ is a Gaussian white noise with $\langle \xi(x, t) \rangle = 0$ and $\langle \xi(x, t) \xi(x', t') \rangle = 2\eta \delta(x - x') \delta(t - t')$. $F_{\text{EQ}}[\varphi]$ is the equilibrium Landau-Ginzburg free-energy functional ensuring that the stationary measure of the stochastic process is the equilibrium measure: $\lim_{t \to \infty} P_0([\varphi]; t) = F_{\text{EQ}}[\varphi] \sim \exp(-F_{\text{EQ}}[\varphi])$. Following the set of principles of the Landau-Ginzburg theory, the expression for $F_{\text{EQ}}[\varphi]$ is local (it involves a local free-energy density, function of $\varphi$ and $\nabla \varphi$), $O(n = 1)$ symmetric, stable (it involves a confining potential), and analytic in $\varphi$ and $\nabla \varphi$. This yields the usual $\varphi^4$ theory, also called the Ginzburg-Landau-Wilson model, reading

$$F_{\text{EQ}}[\varphi] = \int d^d x \left( \frac{1}{2} \mu \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \ldots + \frac{1}{2} (\nabla \varphi)^2 + \ldots \right).$$

\[4\]
The various static parameters $\mu$, $\lambda$, etc., as well as the dynamic parameter $\eta$ depend on the coarsegraining scale $l$. The “mass” $\mu$ corresponds to the distance to criticality, i.e. $\mu \sim T - T_c$, which diverges in the infrared away from criticality:

$$\mu \sim l^{-\alpha},$$

where $\alpha$ is the critical exponent. The infrared fixed point is the equilibrium (paramagnetic) fixed point, the non-analytic terms are relevant below the critical temperature. We shall compute explicitly the corresponding non-analytic fixed-point potential above the upper critical dimension.

2.3 Non-equilibrium steady-state Landau-Ginzburg functional

In the following sections, we shall argue that the non-equilibrium steady-state Landau-Ginzburg functional, $\mathcal{F}_{\text{NESS}}[\varphi] \equiv -\log \mathcal{F}_{\text{NESS}}[\varphi]$, is of the form

$$\mathcal{F}_{\text{NESS}}[\varphi] = \int d^d x \left( c_1 |\varphi|^2 + c_2 |\varphi|^4 + \ldots + \frac{1}{2} (\nabla \varphi)^2 + \ldots \right).$$

This form follows the usual Landau-Ginzburg set of principles recalled above in Sec. 2.2 except for the principle of analyticity of the potential that has been abandoned. Indeed, the terms in $|\varphi|^2$, $|\varphi|^4$, etc., are non-analytic at $\varphi = 0$. They are of intrinsic non-equilibrium nature and their coefficient, $c_1$, $c_3$, etc., must vanish at thermal equilibrium (when $r = 0$) in order to recover the equilibrium Landau-Ginzburg-Wilson theory. The possible presence, below the upper critical dimension, of non-conventional non-analytic terms in the gradient expansion, will be discussed in Sec. 4.

Anticipating on the results that follow, we sketch the renormalization of the Landau potential $\mathcal{F}_{\text{NESS}}[\varphi]$. This form follows the usual Landau-Ginzburg set of principles recalled above in Sec. 2.2 except for the principle of analyticity of the potential that has been abandoned. Thus, making use of the renewal formula (2), solving our non-equilibrium many-body problem reduces to solving a single-particle problem: a diffusive particle undergoing Langevin dynamics

$$\eta \partial_t \varphi(t) = -\partial_x \mathcal{V}_{\text{EQ}}(\varphi) + \xi(t),$$

with the initial condition $\varphi(0) = 0$, and where $\eta > 0$ is a “friction” parameter, $\xi$ is a Gaussian white noise with correlator $\langle \xi(t)\xi(t') \rangle = 2 \eta \delta(t - t')$, and $\mathcal{V}_{\text{EQ}}(\varphi) = \frac{\lambda}{2} \varphi^2 + \frac{\mu}{4} \varphi^4 + \ldots$ is the equilibrium potential of a coarsegraining cell. Here, we redefined the coefficients $\eta$, $\mu$, $\lambda$, etc., that were introduced in Eqs. (3) and (4) to include the volume of the coarsegraining cell. Furthermore, at $d_{\text{uc}} = 4$ and above, all the infrared equilibrium fixed points of the RG are Gaussian.

$$\mathcal{V}_{\text{EQ}}(\Phi) \overset{IR}{\longrightarrow} \mathcal{V}_{\text{EQ}}^*(\Phi) = \begin{cases} +\frac{1}{2} \Phi^2 & \text{for } T > T_c \text{ (paramagnet)} \\ 0 & \text{at } T = T_c \text{ (critical)} \\ -\frac{1}{2} \Phi^2 + 0^+ \Phi^4 & \text{for } T < T_c \text{ (ferromagnet)} \end{cases},$$

where we introduced the rescaled fields $\Phi \equiv \sqrt{\eta} \varphi$ for the critical case $T = T_c$ and $\Phi \equiv \sqrt{|\mu|} \varphi$ for the non-critical cases $T \neq T_c$. In the latter cases, it will be also useful to introduce the reduced reheating rate $\tilde{R} \equiv r \eta/|\mu|$ which flows to 0 in the infrared limit. Owing to the linearity of the resulting Langevin equation, the infrared limit of the non-equilibrium steady state, $\mathcal{P}_{\text{NESS}}^*(\Phi)$, can be computed exactly. We refer the reader to Ref. [27] for details of such computations. As it was already predicted in Sec. 2.4 using general arguments, the form of the resulting effective Landau potential $\mathcal{V}_{\text{NESS}}^*(\Phi)$ will markedly depend on whether the bath temperature $T$ is above or below $T_c$.

Noteworthy, even before reaching the IR fixed point, the equilibrium potential $\mathcal{V}_{\text{EQ}}$ may still be truncated to the quadratic $\mathcal{V}_{\text{EQ}}^*$ in Eq. (7) when the reheating rate is large enough, i.e. when
the diffusive particle does not have time to experience the effect of the non-linear terms before being reset to \( \varphi = 0 \). We may roughly estimate the minimum reheating rate above which one may neglect the \( \varphi^4 \) term in the potential as \( R_{\text{min}} \sim 1/\log(\mu^2/\lambda) \).

3.1 High-temperature phase

For bath temperatures above the critical temperature, \( T > T_c \), and at scales for which the equilibrium system would be in the vicinity of its paramagnetic fixed point, we find the non-equilibrium steady state potential

\[
\mathcal{V}_\text{NESS}(\Phi) = \frac{1}{2} \Phi^2 - \log \left[ H \left( -R, \frac{|\Phi|}{\sqrt{2}} \right) \right] + \text{cst} \\
= \text{cst} + c_1 |\Phi| + a_2 |\Phi|^2 + c_3 |\Phi|^4 + \ldots,
\]

with \( \Phi \equiv \sqrt{\mu} \varphi \) and where \( H(n, x) \) is the Hermite polynomial of degree \( n \). The coefficients \( c_1 = \sqrt{2^{n-1} (R/2)^n} \geq 0 \), \( a_2 = \frac{1-R}{2} + \frac{c_1^2}{2} > 1/4 \), and \( c_3 = \frac{c_1^4}{4} (c_1^2 + 1/2 - R) \geq 0 \). The terms in \(|\Phi|\), \(|\Phi|^3\), etc., are non-analytic around \( \Phi = 0 \). They are intrinsically non-equilibrium in nature and one can check that their coefficients \( c_1, c_3, \text{etc.} \), vanish in the equilibrium limit (i.e., in the limit \( r \to 0 \)). The expression in Eq. (9) justifies the form of the non-equilibrium steady-state Landau-Ginzburg functional \( \mathcal{F}_\text{NESS}[\varphi] \) that was anticipated in Sec. 2.3.

Finally, we can identify the IR fixed point of the non-equilibrium steady state by sending \( \mu \to \infty \) in Eq. (8). Using the property \( H(0, x) = 1 \), this simply gives back the equilibrium potential

\[
\mathcal{V}_\text{NESS}(\Phi) \xrightarrow{\text{IR}} \mathcal{V}_\text{EQ}^*(\Phi) = \frac{1}{2} \Phi^2
\]

showing that the non-analytic terms in the potential are IR irrelevant above the critical temperature. We can therefore conclude that the paramagnetic equilibrium fixed point is robust against stochastic reheating.
3.2 Low-temperature phase

Let us now consider bath temperatures below the critical temperature, \( T < T_c \). At scales for which the equilibrium system would be in the vicinity of its ferromagnetic fixed point, we find

\[
V_{\text{NESS}}(\Phi) = -\log \left[ H \left( -1 - R, \frac{|\Phi|}{\sqrt{2}} \right) \right] + \text{cst} \\
= \text{cst} + c_1 |\Phi| + a_2 |\Phi|^2 + c_3 |\Phi|^3 + \ldots ,
\]

with \( \Phi \equiv \sqrt{-\mu} \varphi \) and the coefficients

\[
c_1 \equiv \sqrt{2} \Gamma \left( 1 + \frac{1}{4} \right) \geq 0, \quad a_2 \equiv \frac{3}{4} - \frac{1}{4}R < 0, \quad c_3 \equiv \frac{1}{4} \Gamma \left( 1 + \frac{1}{4} \right) \geq 0.
\]

Similarly to the high-temperature case, the effective Landau potential displays non-analytic features of intrinsic non-equilibrium nature around \( \Phi = 0 \), justifying the form of \( F_{\text{NESS}}[\varphi] \) proposed in Eq. (5).

The IR fixed point of the non-equilibrium steady state can be accessed by sending \( \mu \to -\infty \) in Eq. (11). This yields the non-equilibrium fixed-point potential

\[
V^*_{\text{NESS}}(\Phi) \underset{\text{IR}}{\to} V^*_{\text{NESS}}(\Phi) = -\log \left[ H \left( -1, \frac{|\Phi|}{\sqrt{2}} \right) \right] + \text{cst} \\
= \text{cst} + \sqrt{2} |\Phi| - (1/2 - 1/\pi)\Phi^2 + \frac{4/\pi - 1/\pi^2}{3\sqrt{2/\pi}} |\Phi|^3 + \ldots .
\]

This non-analytic infrared fixed-point potential is one of the main results of this paper. It is plotted in Fig. 2. It demonstrates that, contrary to the high-temperature phase discussed above, non-analyticities originally present in the microscopic Landau potential \( V_{\text{NESS}}(\Phi) \) survive in the infrared. In particular, the term in \( |\Phi| \) dominates \( V^*_{\text{NESS}}(\Phi) \) around the global minimum \( \Phi = 0 \), yielding a distinctive cusp. Notably, \( V^*_{\text{NESS}}(\Phi) \) does not depend on the reheating rate any longer.

3.3 Critical-temperature state

When the bath temperature is exactly at \( T = T_c \), following the single-particle computation in Ref. [15], we find the infrared non-equilibrium fixed-point potential

\[
V^*_{\text{NESS}}(\Phi) = \sqrt{\tau} |\Phi| + \text{cst},
\]

with \( \Phi \equiv \sqrt{T_c} \varphi \). Contrary to the high-temperature and low-temperature cases above, this infrared fixed-point potential still explicitly depends on the reheating rate \( r \). This was expected since the reheating protocol introduces a finite correlation length \( \xi_{\text{max}} \sim 1/\sqrt{T} \) which breaks the scale invariance found at equilibrium. The equilibrium fixed-point potential is recovered in the limit \( r \to 0 \).

3.4 Supporting Monte Carlo results

The theory developed so-far relied on a few assumptions. The main assumption was the validity of the mean-field approximation above the upper critical dimension. The latter was conjectured to be identical to the equilibrium upper critical dimension, i.e. \( d_{uc} = 4 \).

To support all the steps undertaken so far, we numerically solve for the many-body non-equilibrium steady state, constructing the field theory from a numerical standpoint, void of any assumption.

Details of the numerics We consider the Ising model

\[
H = - \sum_{(ij)} S_i S_j ,
\]

(16)
Figure 3: $d = 4$ Monte Carlo solution of the non-equilibrium steady-state potential $V_{\text{NESS}}(\varphi)$ for increasing coarsegraining lengths: $l = 2, 4, 8$ ($L = 8, r = 0.005$). (a) Above $T_c$ ($T = 1.1 T_c$), $V_{\text{NESS}}$ (solid lines) does not differ from the equilibrium potential $V_{\text{EQ}}$ (dotted lines). (b) Below $T_c$ ($T = 0.99 T_c$), $V_{\text{NESS}}$ deviates markedly from the equilibrium Mexican hat. A cusp develops at the new minimum $\varphi = 0$ becomes the global minimum in the infrared (IR).

where the $N = L^d$ classical spins $S_i = \pm 1$ live on a $d$-dimensional cubic lattice of linear length $L$ with periodic boundary conditions and interact ferromagnetically between nearest neighbors. Here, to directly compare with the theory developed in Sec. 3, we work in four dimensions: $d = 4$. In Sec. 4 we shall also present numerical results below the upper critical dimension, in $d = 2$. In the lack of a well-developed finite-size scaling theory for non-equilibrium steady states, we cannot afford to correct for finite-size effects. We therefore stay slightly away from the critical temperature $T_c(d = 4) \approx 6.68$ by working with $T = 0.99 T_c$ and $T = 1.1 T_c$.

The dynamics due to the local thermal baths are generated by the usual Monte Carlo Metropolis algorithm. The reheating dynamics are taken into account via the renewal formula in Eq. (2). This amounts in following the dynamics of the local magnetization

$$
\phi(x,t) = \frac{1}{l^d} \sum_{i \in v_l(x)} S_i(t),
$$

where the microscopic spin degrees of freedom are averaged on cubic coarsegraining cells $v_l(x)$ of linear size $l$ around $x$, after a quench from an infinite-temperature disordered initial state, $S_i(t = 0) = \pm 1$ with probability $1/2$. Here, the time $t$ is counted in units of Monte Carlo steps defined as a sequence of $N$ attempted spin flips. Rather than working with the magnetization which is bounded, $\phi \in [-1, 1]$, we perform a field redefinition and introduce the field

$$
\varphi \equiv \text{artanh}(\phi) \in \mathbb{R}.
$$

The non-equilibrium Landau potential $V_{\text{NESS}}(\varphi)$ is computed from the Monte Carlo data by first measuring the time-dependent probability distribution of the local magnetization after the quench, $P_0(\varphi; t)$. In practice, we improve the statistics by averaging $P_0(\varphi; t)$ over many realizations of the quench dynamics. Then, the renewal formula in Eq. (2) is used to compute the non-equilibrium steady-state distribution $P_{\text{NESS}}(\varphi)$, and finally $V_{\text{NESS}}(\varphi) \equiv - \log P_{\text{NESS}}(\varphi)$.

Note that, in order to reconstruct the full Landau-Ginzburg functional $\mathcal{F}_{\text{NESS}}[\varphi]$, one should in principle extract the joint probability distribution of the field and its gradients, $P_0(\varphi, \nabla \varphi; t)$. Here, operating above the upper critical dimension, we discard the information on the gradients and we concentrate on comparing $V_{\text{NESS}}(\varphi)$ with the theory developed in Sec. 3. We shall come back to the possibility of non-conventional gradient terms present in the Landau-Ginzburg functional below the upper critical dimension in the discussion of Sec. 4.
Figure 4: Validity of the mean-field approximation in $d = 4$: comparing non-equilibrium steady-state potentials $V_{\text{NESS}}(\varphi)$ computed from single-particle Langevin dynamics in Eq. (6) and $\eta = 1920$ (solid black lines) with those obtained from full-fledged Monte Carlo numerics (dashed lines), and for various reheating rates: $r = 0, 0.005, 0.05$. The $r = 0$ data corresponds to the equilibrium potential $V_{\text{EQ}}(\varphi)$. It was fitted with an (analytic) even potential and used as an input to the mean-field computation. $[L = 8, l = 4, T = 0.99 T_c]$

**Agreement with theory**  In Fig. 3 we plot the non-equilibrium steady-state potential $V_{\text{NESS}}(\varphi)$ for different values of the coarsegraining length $l$. We present separately the high-temperature phase when the bath temperature is above the critical temperature ($T > T_c$) from the low-temperature phase ($T < T_c$). For each value of $l$, the corresponding equilibrium potential $V_{\text{EQ}}(\varphi)$ is also plotted for comparison (dotted lines).

In the high-temperature phase, see Fig. 3 (a), the effect of reheating on the potential is found to be minimal. The equilibrium parabola is replaced by a potential with very similar features: a single global minimum at $\varphi = 0$, no other local minimum. This is all the more true when increasing the coarsegraining length $l$, i.e. deeper in the infrared. This numerically validates the claim made around Eq. (10) that above $T_c$ the infrared fixed point is the equilibrium fixed point.

In stark contrast, the effect of reheating is found to be much more disruptive on the low-temperature phase ($T < T_c$), see Fig. 3 (b). There, the equilibrium Mexican-hat potential with two degenerate global minima at $\varphi \neq 0$, remnant features of the Mexican hat, and a new minimum at $\varphi = 0$ that becomes the global minimum in the infrared. Moreover, contrary to $V_{\text{EQ}}(\varphi)$, we verify that $V_{\text{NESS}}(\varphi)$ cannot be satisfactorily fitted with an (analytic) even polynomial of the form $c_{\text{st}} + a_2\varphi^2 + a_4\varphi^4 + a_6\varphi^6 + a_8\varphi^8$. Instead, numerical fits to the non-analytic form $V_{\text{NESS}}(\varphi) = c_{\text{st}} + c_1|\varphi| + a_2\varphi^2 + c_3|\varphi|^3 + a_4\varphi^4$ give excellent results. Details of the fitting procedure are given in Appendix A. This numerically validates the claims made around Eq. (13) that below $T_c$ the Landau potential is non-analytic and flows towards a non-trivial non-equilibrium fixed point. As an example, we give the result of such a fitting procedure on the potential $V_{\text{NESS}}(\varphi)$ plotted in Fig. 3 (b): we find $c_1(l = 8)/c_1(l = 4) \approx 1.9 > 1$, illustrating that the non-analytic operator $|\varphi|$ is infrared relevant below $T_c$.

**Validating the mean-field approximation**  We also numerically check the mean-field approximation that was used when simplifying the Model A dynamics in Eq. (3) with the single-particle Langevin dynamics in Eq. (6). For that purpose, we compare the results of the full-fledged Monte Carlo numerics presented above with those obtained from solving the Langevin dynamics. In practice, the equilibrium potential $V_{\text{EQ}}(\varphi)$ is numerically extracted from *equilibrium* Monte Carlo dynamics and fitted with an (analytic) even polynomial. We illustrate the quality of such fits in
Figure 5: \( d = 2 \) Monte Carlo solution of the non-equilibrium steady-state potential \( V_{\text{NESS}}(\varphi) \) for increasing coarse-graining lengths: \( l = 8, 16, 32 \) (\( L = 32, r = 0.005 \)). (a) Above \( T_c \) (\( T = 1.1 T_c \)), \( V_{\text{NESS}} \) (solid lines) does not differ much from the equilibrium potential \( V_{\text{EQ}} \) (dotted lines). (b) Below \( T_c \) (\( T = 0.99 T_c \)), \( V_{\text{NESS}} \) deviates markedly from the equilibrium Mexican hat. A cusp develops at the new minimum \( \varphi = 0 \) becomes the global minimum in the infrared (IR).

Fig. 4 Then, the Fokker-Planck equation corresponding to Eq. (6), namely

\[
\eta \partial_t P_0(\varphi; t) = \partial_\varphi \left[ \partial_\varphi V_{\text{EQ}}(\varphi) P_0(\varphi; t) \right] + \partial^2_\varphi P_0(\varphi; t),
\]

(19)

with the initial condition \( P(\varphi; t = 0) = \delta(\varphi) \), is solved numerically. The “friction” parameter \( \eta \) can be determined by requiring that the observable \( \langle \varphi(t) \rangle \) computed within this framework matches the one computed from full-fledged Monte Carlo numerics (see details in Appendix B). Finally, the non-equilibrium steady-state potential in the presence of a finite reheating rate, \( V_{\text{NESS}} \), is computed with the use of the renewal formula in Eq. (2).

In Fig. 4 we compare the resulting mean-field potentials to the ones obtained with full-fledged Monte Carlo numerics. The agreement is very good, thus validating the use of the mean-field approximation above the upper critical dimension and, therefore, the reduction of the non-equilibrium many-body problem to the non-equilibrium dynamics of a single particle in an ad-hoc environment.

If it were necessary, this also constitutes an a posteriori check that the Model A dynamics in Eq. (3), involving the equilibrium free energy \( F_{\text{EQ}}[\varphi] \) given in Eq. (4), are a faithful representation of the relaxation dynamics of the Ising model after a quench from infinite temperature.

4 Below the upper critical dimension and open questions

In dimensions below the upper critical dimension, \( d_{uc} = 4 \), the spatial fluctuations of the field cannot be neglected, signaling the breakdown of the validity of the mean-field approximation. At equilibrium, the \( O(n = 1) \)-symmetric phase is still Gaussian but the broken-symmetry phase is now characterized by the so-called Wilson-Fisher fixed point where \( \lambda \) is finite. For finite reheating rates, the general comments made in Sec. 2.1 still apply: we expect a destabilization of the Wilson-Fisher fixed point but the involvement of non-analytic operators down to the infrared limit has yet to be confirmed.

Monte Carlo numerics in \( d = 2 \) In the lack of proper analytic methodology to perform such an RG computation, we may seek answers from Monte Carlo numerics in \( d = 2 \). The details of the numerics are the same as for the \( d = 4 \) case presented above. We also stay slightly away from the critical temperature \( T_c(d = 2) = 2/\log(1 + \sqrt{2}) \approx 5.22 \) by working with \( T = 0.99 T_c \) and

\[\text{The match between Model A and Monte Carlo dynamics is only expected to hold at lengthscales larger than the microscopic lattice spacing and timescales larger than a Monte Carlo step.}\]
Figure 6: Monte Carlo solution of the non-equilibrium steady-state “gradient potential” $K_{NESS}(\nabla \phi) \equiv -\log P_{NESS}(\nabla \phi)$ for various reheating rates $r$ given in the key. (a) $d = 4$: $K_{NESS}$ does not appear to differ from equilibrium ($r = 0$) $[L = 8, l = 2, T = 0.99 T_c]$. (b) $d = 2$: $K_{NESS}$ undoubtedly depends on $r$ and could feature non-analyticities $[L = 64, l = 16, T = 0.99 T_c]$.

$T = 1.1 T_c$. Altogether, the results are qualitatively similar to those that were obtained in $d = 4$. In Fig. 5 the non-equilibrium steady-state potential $V_{NESS}(\phi)$ is plotted for different values of the coarsegraining length $l$.

Above $T_c$, see Fig. 5 (a), the potential is similar to the equilibrium potential, with a single minimum at $\phi = 0$, and an overall quadratic shape. Contrary to the $d = 4$ case in Fig. 3 (a), we may still note that the potential is slightly steeper in the presence of stochastic reheating. Additional investigations are needed to determine whether this difference subsists deeper in the infrared.

Below $T_c$, see Fig. 5 (b), the potential develops a new minimum at $\phi = 0$ with the same characteristic cusp as the one previously attributed to the non-analytic operator $|\phi|\nabla \phi$ in $d = 4$. As predicted in Sec. 2.1, and similarly to the $d = 4$ case, this minimum becomes the global minimum in the infrared. This strongly support the scenario of the equilibrium Wilson-Fisher fixed point being destabilized by non-analytic operators that are infrared relevant.

**Non-analytic gradient terms** In addition to the non-analytic terms in the Landau potential, it is fair to wonder whether the non-equilibrium steady-state field theory can also feature non-analytic gradient terms of the type, e.g., $|\nabla \phi|$ or $|\phi|(\nabla \phi)^2$. They would also be of intrinsic non-equilibrium nature and would depend on the reheating rate. Under RG, they could be generated by those non-analyticities already present in the potential.

Above the upper critical dimension, in the regime of validity of the mean-field approximation, gradient terms are expected to be inconsequential and the reheating protocol is not expected to alter this picture. To check this, we numerically compute the probability distribution of gradients on the non-equilibrium steady state, $P_{NESS}(\nabla \phi)$. We choose a relatively small coarsegraining length $l$ to operate before the infrared fixed point is reached. In Fig. 6 (a), we plot the “gradient potential” $K_{NESS}(\nabla \phi) \equiv -\log P_{NESS}(\nabla \phi)$ for various values of the reheating rate $r$ (including the equilibrium case $r = 0$), both below and above $T_c$. $K_{NESS}(\nabla \phi)$ clearly appears to be independent of $r$, supporting the scenario that non-equilibrium reheating dynamics do not affect the gradient terms of the equilibrium Landau-Ginzburg-Wilson theory. This justifies the form of the effective Landau-Ginzburg functional that was proposed in Eq. (5), with an analytic gradient expansion starting with a $(\nabla \phi)^2$ term.

Preliminary results suggest the scenario may be different below the upper critical dimension. In $d = 2$, $K_{NESS}(\nabla \phi)$ is found to depend on the reheating rate $r$, and is quite different from the equilibrium case. We illustrate this point in Fig. 6 (b). Whether this will survive in the infrared, and whether this does involve non-analyticities in the gradient terms is left for future investigations.
**RG approach**  Ultimately, the discussion of the non-analytic non-equilibrium field theory below the upper critical dimension requires the construction of a computational framework to performing RG calculations. One route, specific to the reheating dynamics, is to use the renewal formula in Eq. (2). A time-dependent RG computation could, in principle, be performed at the level of the relaxation dynamics after a quench and the results translated to the case of reheating dynamics. A more ambitious route would be to start directly from a non-analytic Landau-Ginzburg functional such as the one in Eq. (5) and identify practical ways to perform the perturbative integration of high-energy modes in the presence of terms such as $|\phi|$.

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### A  Fitting the potential by a non-analytic function

![Figure 7: The potential $V_{NESS}(\varphi)$ obtained from Monte Carlo numerics in $d = 4$ for $T = 0.99T_c$, $r = 0.005$ and $L = l = 32$ [see the solid red curve in Fig. 3 (b)] is fitted to both a non-analytic form (triangles) and an analytic form (crosses).](image)

All the non-equilibrium steady states potentials $V_{NESS}$, at finite reheating rate $r > 0$, computed by means of Monte Carlo numerics and presented in this paper can be successfully fitted to an non analytic function of the form $c_1|\varphi| + a_2\varphi^2 + c_3|\varphi|^3 + a_4\varphi^4 + \ldots$. In Fig. 7, we provide the example of a fit of the potential $V_{NESS}(\varphi)$ presented in Fig. 3 (b) at $T = 0.99T_c$, $r = 0.005$ and $L = l = 32$ and reproduced in Fig. 7 with a solid line. After setting the overall additive constant such that $V_{NESS}(0) = 0$, it is fitted to a non-analytic function of the form

$$c_1|\varphi| + a_2\varphi^2 + c_3|\varphi|^3 + a_4\varphi^4,$$

(20)
where $c_1$, $a_2$, $c_3$ and $a_4$ are four fitting parameters determined by a nonlinear least-squares Marquardt-Levenberg algorithm. The resulting fit, represented by triangles, is excellent. For comparison, the crosses show the result of a fit to the analytic function

$$a_2\varphi^2 + a_4\varphi^4 + a_6\varphi^6 + a_8\varphi^8,$$

(21)

where $a_2$, $a_4$, $a_6$, and $a_8$ are also four fitting parameters. This fit fails to reproduce the data especially close to the cusp at $\varphi = 0$.

**B Determining the coefficient $\eta$ of Model A dynamics**

![Figure 8: Agreement of the time-dependent observable $\langle|\varphi|\rangle$ computed both from full-fledged Monte Carlo numerics in $d = 4$ after a quench from infinite temperature ($L = 8$, $l = 4$, $T = 0.99 T_c$), and from the mean-field Langevin approach with the initial condition $\varphi(t = 0) = 0$ and the equilibrium potential $V_{EQ}(\varphi)$ fitted in Fig. 4. The dynamic coefficient $\eta$ entering the Langevin equation (6) has been set to $\eta \approx 1920$ for the two curves to match.](image)

Figure 8: Agreement of the time-dependent observable $\langle|\varphi|\rangle$ computed both from full-fledged Monte Carlo numerics in $d = 4$ after a quench from infinite temperature ($L = 8$, $l = 4$, $T = 0.99 T_c$), and from the mean-field Langevin approach with the initial condition $\varphi(t = 0) = 0$ and the equilibrium potential $V_{EQ}(\varphi)$ fitted in Fig. 4. The dynamic coefficient $\eta$ entering the Langevin equation (6) has been set to $\eta \approx 1920$ for the two curves to match.

The dynamic “friction” parameter $\eta$ that enters the Model A dynamics in Eq. (3), or its mean-field version in the Langevin dynamics of Eq. (6), is a scale dependent parameter. We determine its value by requiring the relaxation dynamics of a simple observable, namely (the absolute value of) the coarsegrained magnetization after a quench from infinite temperature, to be the same when computed from the full-fledged Monte Carlo dynamics in $d = 4$ and when computed from the mean-field Langevin approach. In practice, this means finding the value of $\eta$ such that

$$\langle|\phi^{MF}(t)|\rangle_\xi = \int_{-\infty}^{\infty} d\varphi |\tanh \varphi| P_0(\varphi; t),$$

(22)

where $\langle \ldots \rangle_\xi$ indicates average with respect to the Langevin noise, matches

$$\langle|\phi^{MC}(x, t)|\rangle_{x, MC} = \langle|1/l \sum_{i \in n(x)} S_i(t)|\rangle_{x, MC},$$

(23)

where $\langle \ldots \rangle_{x, MC}$ indicates average with respect to the position and the Monte Carlo realizations and we recall the field redefinition $\varphi \equiv \text{artanh}(\phi)$.

We give an example of such a fit in Fig. 8. The excellent agreement between the two methods is another indication that the Model A dynamics in Eq. (3), involving the equilibrium free energy $F_{EQ}(\varphi)$ given in Eq. (4), are a faithful representation of the relaxation dynamics of the Ising model after a quench from infinite temperature.
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