Inverted Hierarchy Models of Neutrino Masses

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Abstract

We study models of neutrino masses which naturally give rise to an inverted mass hierarchy and bi-maximal mixing. The models are based on the see-saw mechanism with three right-handed neutrinos, which generates a single mass term of the form $\nu_e(\nu_\mu + \nu_\tau)$ corresponding to two degenerate neutrinos $\nu_e$ and $\nu_\mu + \nu_\tau$, and one massless neutrino $\nu_\mu - \nu_\tau$. Atmospheric neutrino oscillations are accounted for if the degenerate mass term is about $5 \times 10^{-2}$ eV. Solar neutrino oscillations of the Large Mixing Angle MSW type arise when small perturbations are included leading to a mass splitting between the degenerate pair of about $(1.7 - 2.0) \times 10^{-4}$ eV for the successful cases. We study the conditions that such models must satisfy in the framework of a $U(1)$ family symmetry broken by vector singlets, and catalogue the simplest examples. We then perform a renormalisation group analysis of the neutrino masses and mixing angles, assuming the supersymmetric standard model, and find modest radiative corrections of a few per cent, showing that the model is stable. At low energies we find $\sin^2 2\theta_{23} \approx 0.93 - 0.96$ and $\sin^2 2\theta_{12} \approx 0.9 - 1.0$.

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1 Introduction

The latest atmospheric neutrino results based on 1117 days of data from Super Kamiokande are still consistent with a standard two neutrino oscillation $\nu_\mu \rightarrow \nu_\tau$ with a near maximal mixing angle $\sin^2 2\theta_{23} > 0.88$ and a mass square splitting $\Delta m^2_{23}$ from $1.5 \times 10^{-3}$ to $5 \times 10^{-3}$ $eV^2$ at 90% CL [1]. The sterile neutrino oscillation hypothesis $\nu_\mu \rightarrow \nu_s$ is excluded at 99% CL.

Super Kamiokande is also beginning to provide important clues concerning the correct solution to the solar neutrino problem. The latest results from 1117 days of data from Super Kamiokande [2] sees a one sigma day-night asymmetry, and a flat energy spectrum, which together disfavour the small mixing angle (SMA) MSW solution [3], the just-so vacuum oscillation hypothesis [4] and the sterile neutrino hypotheses. All three possibilities are now excluded at 95% CL. The results allow much of the large mixing angle (LMA) MSW region [3], which now looks like the leading candidate for the solution to the solar neutrino problem. For example a typical point in the LMA MSW region is $\sin^2 2\theta_{12} \approx 0.75$ and $\Delta m^2_{12} \approx 2.5 \times 10^{-5} eV^2$ [5].

Once the CHOOZ constraint [6] is taken into account, the latest results seem to imply an approximate bi-maximal mixing scenario $\tan \theta_{23} \approx 1$, $\tan \theta_{12} \approx 1$, $\theta_{13} \ll 1$ relating the three standard neutrinos $\nu_e, \nu_\mu, \nu_\tau$, to the three mass eigenvalues $m_{\nu_1}, m_{\nu_2}, m_{\nu_3}$, where only the values of $|\Delta m^2_{23}| = |m_{\nu_3}^2 - m_{\nu_2}^2|$ and $|\Delta m^2_{12}| = |m_{\nu_2}^2 - m_{\nu_1}^2|$ are determined by current experiments. We do not know if the neutrino masses are hierarchical or approximately degenerate with small mass splittings. Another possibility is that the neutrinos have an inverted mass hierarchy, i.e. two of them are approximately degenerate with a small mass splitting, while the third is much lighter. The conventional and inverted neutrino mass hierarchies are depicted schematically in Figures 1 and 2. Neutrino factories will be able to determine the sign of $\Delta m^2_{23}$ and hence distinguish between the two cases [7].
The see-saw mechanism \cite{8} implies that the three light neutrino masses arise from some large mass scales corresponding to the Majorana masses of some heavy "right-handed neutrinos" $N^p_R M^{pq}_{RR}$ ($p, q = 1, \ldots, Z$) whose entries take values which extend from say $\sim 10^{16}$ GeV down to perhaps several orders of magnitude lower. The presence of electroweak scale Dirac mass terms $m^{ip}_{LR}$ (a $3 \times Z$ matrix) connecting the left-handed neutrinos $\nu^i_L$ ($i = 1, \ldots, 3$) to the right-handed neutrinos $N^p_R$ then results in a very light see-saw suppressed effective $3 \times 3$ Majorana mass matrix

$$m_{LL} = m_{LR} M^{-1}_{RR} m^T_{LR} \quad (1)$$

Figure 1: A conventional neutrino mass hierarchy $m_{\nu_1} \ll m_{\nu_2} \ll m_{\nu_3}$.

Figure 2: An inverted neutrino mass hierarchy $m_{\nu_1} \ll m_{\nu_2} \approx m_{\nu_3}$. Note that the mass splitting $|m_{\nu_1} - m_{\nu_2}|$ must be much smaller than in the hierarchical case (as discussed later in the text.)
for the left-handed neutrinos $\nu^i_L$, which are the light physical degrees of freedom observed by experiment. If the neutrino masses arise from the see-saw mechanism then it is natural to assume the existence of a physical neutrino mass hierarchy $m_{\nu_1} \ll m_{\nu_2} \ll m_{\nu_3}$, which implies $\Delta m^2_{23} \approx m^2_{\nu_3}$, and $\Delta m^2_{12} \approx m^2_{\nu_2}$, which fixes $m_{\nu_3} \approx 5 \times 10^{-2} eV$, and (assuming the LMA MSW solution) $m_{\nu_2} \approx 5 \times 10^{-3} eV$, with rather large errors. Thus $m_{\nu_2}/m_{\nu_3} \sim 0.1$.

Hierarchical neutrino masses are not guaranteed because a hierarchy in $m_{LR}$ does not necessarily imply a hierarchy in $m_{LL}$ since $M_{RR}$ is also expected to be hierarchical, and the hierarchies may cancel out in the see-saw mechanism. Nevertheless it would seem surprising if the resulting neutrino spectrum came out to be degenerate $m_{\nu_1} \approx m_{\nu_2} \approx m_{\nu_3}$ with three accurately equal masses of say an eV but with small mass splittings appropriate to the atmospheric and solar data especially taking into account radiative corrections [9], although such a scenario could be enforced by a symmetry [10]. The possibility of an inverted mass hierarchy $m_{\nu_3} \ll m_{\nu_2} \approx m_{\nu_1}$ with bi-maximal mixing however is more natural, since it follows from a simple form of mass matrix [11]

$$m_{LL} \sim \begin{pmatrix} 0 & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \quad (2)$$

which corresponds to neutrino masses of the form $\nu_e(b\nu_\mu + c\nu_\tau)$, which implies two degenerate neutrinos $\nu_e$ and $b\nu_\mu + c\nu_\tau$, and a massless neutrino $c\nu_\mu - b\nu_\tau$, with neutrino mixing angles $\tan \theta_{23} = c/b$, $\tan \theta_{12} = 1$, $\theta_{13} = 0$. It has also been pointed out that the mass matrix in Eq.4 can be generated from the see-saw mechanism using two right-handed neutrinos with an off-diagonal Majorana mass, using a chiral $U(1)$ family symmetry to give the desired textures [12].

Our discussion of the inverted hierarchy models differs from that already presented in the literature in the following three ways. Firstly we consider the see-saw mechanism in the framework of the supersymmetric standard model with three right-handed
neutrinos, and focus on a particular texture for the heavy right-handed neutrinos which is capable of approximately reproducing the mass matrix in Eq.3. Secondly we introduce a vector $U(1)$ family symmetry which is broken by vector-like singlets, which allows both signs of charges to contribute, and make a computer scan over all charges which can lead to the desired mass matrix, and tabulate the simplest cases. Thirdly we perform a renormalisation group (RG) analysis of some cases, in order to examine the radiative corrections in going from the GUT scale to low energy.

In section 2 we give an analytic discussion of the model, and in section 3 we introduce a $U(1)$ family symmetry and tabulate the simplest charges consistent with our scheme. Section 4 contains a renormalisation group analysis of some examples, and section 4 concludes the paper.

## 2 Analytic Discussion

We begin our discussion by returning to the case of a conventional neutrino mass hierarchy where the presence of a large 23 mixing angle looks a bit surprising at first sight, especially given our experience with small quark mixing angles. Several explanations have been proposed [13], but the simplest idea is that the contributions to the 23 block of the light effective Majorana matrix come predominantly from a single right-handed neutrino, which causes the 23 subdeterminant to approximately vanish. A dominant single right-handed neutrino then naturally leads to a hierarchical neutrino mass spectrum, with a single dominant physical neutrino mass and two much lighter neutrinos. This mechanism, called single right-handed neutrino dominance (SRHND), was proposed in [14], and developed for bi-maximal mixing in [15]. The effect of radiative corrections on the bi-maximal SRHND case was considered in [16]. We then show that for the off-diagonal heavy Majorana texture, a reversal of the SRHND conditions leads to an inverted mass hierarchy and bi-maximal mixing.
We first write the neutrino Yukawa matrix in general (in the LR basis) as

$$Y_\nu = \begin{pmatrix} a' & a & d \\ b' & b & e \\ c' & c & f \end{pmatrix}$$ (3)

There are now three distinct textures for the heavy Majorana neutrino matrix which maintain the isolation of the dominant right-handed neutrino $N_{R3}$, namely the diagonal, democratic and off-diagonal textures introduced previously [1,4,15]. Only one of these textures, namely the off-diagonal case, can lead to an inverted mass hierarchy. We therefore specialise to the off-diagonal heavy Majorana texture:

$$M_{RR} = \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & Y \end{pmatrix}$$ (4)

From Eqs.3 and 4, the see-saw formula Eq.1 implies

$$m_{LL} = \begin{pmatrix} \frac{d^2}{Y} + \frac{2aa'}{X} & \frac{de}{Y} + \frac{a'b}{X} + \frac{ab'}{X} & \frac{df}{Y} + \frac{a'c}{X} + \frac{ac'}{X} \\ \frac{e^2}{Y} + \frac{2bb'}{X} & \frac{ef}{Y} + \frac{bb'}{X} + \frac{bc'}{X} \\ \frac{f^2}{Y} + \frac{2cc'}{X} \end{pmatrix} v_2^2$$ (5)

where $v_2$ is the Higgs vacuum expectation value which relates the neutrino mass matrix to the neutrino Yukawa matrix $m_{LR} = Y_\nu v_2$. From this starting point we may obtain either a conventional hierarchy or an inverted hierarchy, depending on the conditions applied to the couplings.

- The conditions for a conventional hierarchy as in Fig.1 via SRHND are simply that the third right-handed neutrino dominates the 23 block of $m_{LL}$,

$$\frac{e^2}{Y} \sim \frac{ef}{Y} \sim \frac{f^2}{Y} \gg \frac{xx'}{X}$$ (6)

Actually this texture represents a set of off-diagonal Majorana textures related by a re-ordering of the right-handed neutrino fields. For example if we cyclically permute the right-handed neutrino fields as $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$ then we would have

$$Y_\nu = \begin{pmatrix} d & a' & a \\ e & b' & b \\ f & c' & c \end{pmatrix}, \quad M_{RR} = \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & X \\ 0 & X & 0 \end{pmatrix}$$

with $m_{LL}$ unchanged. The results in this paper therefore apply to this larger class of model, corresponding to all possible re-orderings of right-handed neutrino fields. For example the matrices $Y_\nu$ and $M_{RR}$ in Tables 3,4 may be permuted so that the large elements of order unity appear in the 23 and 33 positions of $Y_\nu$, which may be more natural from the perspective of unified theories where a large 33 element is expected.
where \( x \in a, b, c \) and \( x' \in a', b', c' \). For example in the limit that only \( e \approx f \) are non-zero the spectrum consists of a decoupled massless \( \nu_e \), plus another massless state \( \nu_\mu - \nu_\tau \) (because the 23 subdeterminant vanishes) giving maximal atmospheric mixing with the massive state \( \nu_\mu + \nu_\tau \). The milder conditions in Eq.6 allow the massless degeneracy to be broken, and \( \nu_e \to \nu_\mu - \nu_\tau \) solar oscillations. Assuming SRHND the contribution to the lepton 23 and 13 mixing angles from the neutrino sector are approximately

\[
\tan \theta_{23} \approx \frac{e}{f}, \quad \tan \theta_{13} \approx \frac{d}{\sqrt{e^2 + f^2}}. \tag{7}
\]

so that Super-Kamiokande and CHOOZ imply

\[
d \ll e \approx f \tag{8}
\]

The 12 mixing angle is controlled by the sub-dominant right-handed neutrinos, and the condition for a \( \tan \theta_{12} \sim 1 \) is [15]:

\[
\max \left( \frac{a'b}{X}, \frac{ab'}{X}, \frac{a'c}{X}, \frac{ac'}{X} \right) \sim \max \left( \frac{bb'}{X}, \frac{b'c}{X}, \frac{bc'}{X}, \frac{cc'}{X} \right) \tag{9}
\]

• In order to achieve an inverted hierarchy as in Fig.2 from the off-diagonal texture in Eq.4, we must require contributions to \( m_{LL} \) in Eq.5 such that Eq.2 is approximately reproduced. It is straightforward to show that only an off-diagonal texture such as Eq.4 allows this. This immediately implies that the first and second right-handed neutrinos with mass term \( X \) must give the dominant contribution to the matrix relative to the contributions from the third right-handed neutrino with mass \( Y \),

\[
\frac{(e + d + f)^2}{Y} \ll \max \left( \frac{(a' + b' + c')(a + b + c)}{X} \right) \tag{10}
\]

which is the opposite of the SRHND condition Eq.6. Furthermore we require one of the following conditions to be satisfied

\[
a', b, c \gg a, b', c', \quad \text{or} \quad a', b, c \ll a, b', c'. \tag{11}
\]
The simplest example which generates an inverted hierarchy is to take the limit that only $a', b, c$ are non-zero, so that $m_{LL}$ becomes

$$m_{LL} \sim \left( \begin{array}{ccc} 0 & b & c \\ b & 0 & 0 \\ c & 0 & 0 \end{array} \right) \frac{a' v^2}{X}$$

which is of the form in Eq. 2. In order to split the degeneracy, we must allow for small perturbations. Suppose to begin with that $d = e = f = 0$, then only two right-handed neutrinos contribute and

$$m_{LL} = \left( \begin{array}{ccc} \frac{2a'a'}{X} & \frac{a'b'}{X} + \frac{a'c'}{X} & \frac{a'c'}{X} + \frac{ac'}{X} \\ \frac{2b'b'}{X} & \frac{b'c'}{X} + \frac{bc'}{X} & \frac{bc'}{X} + \frac{cc'}{X} \end{array} \right) v^2$$

(13)

Allowing $a, b', c'$ to be non-zero but maintaining $a', b, c \gg a, b', c'$ we find neutrino masses

$$m_{\nu_3} = 0, m_{\nu_2} \approx m_{\nu_1} \approx \frac{a' \sqrt{b^2 + c^2} v^2}{X}, \quad (m_{\nu_1} - m_{\nu_2}) \approx \frac{2(a'a + b'b + c'c)v^2}{X}$$

(14)

and mixing angles

$$\tan \theta_{23} \approx \frac{c}{b}, \quad \theta_{13} \approx \frac{c'b - b'c}{a' \sqrt{b^2 + c^2}}, \quad \tan \theta_{12} \approx 1$$

(15)

Note that in the inverted hierarchy case $|\Delta m^2_{23}| \approx m^2_{\nu_1} \approx m^2_{\nu_2}$, which fixes $m_{\nu_1} \approx m_{\nu_2} \approx 5 \times 10^{-2}eV$. Since $|\Delta m^2_{12}| \approx m^2_{\nu_1} - m^2_{\nu_2}$, the LMA MSW solution implies $|m_{\nu_1} - m_{\nu_2}| \approx 2.5 \times 10^{-4}eV$, with rather large errors, which is much smaller than the corresponding mass splitting $5 \times 10^{-3}eV$ in the conventional hierarchy case. This implies that the perturbations in the inverted hierarchy case must be much smaller than in the conventional hierarchy case. A convenient way to describe such perturbations is in the framework of $U(1)$ family symmetry to which we now turn.

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3In fact the eigenvalues have the form $(|m_{\nu_1}|, -|m_{\nu_2}|, |m_{\nu_3}|)$, with $|m_{\nu_1}| \approx |m_{\nu_2}| \gg |m_{\nu_3}|$, but we can always redefine the masses to be positive, and this is assumed in Eq.14, and subsequently. Note that we take the convention $|m_{\nu_1}| > |m_{\nu_2}|$, which fixes the ordering of the 1st two columns of $V_{MNS}$. 
3 U(1) Family Symmetry

Introducing a $U(1)$ family symmetry [17], [18], [19], [20] provides a convenient way to organise the hierarchies within the various Yukawa matrices. For definiteness we shall focus on a particular class of model based on a single pseudo-anomalous $U(1)$ gauged family symmetry [19]. We assume that the $U(1)$ is broken by the equal VEVs of two singlets $\theta, \bar{\theta}$ which have vector-like charges $\pm 1$ [19]. The $U(1)$ breaking scale is set by $< \theta > = < \bar{\theta} >$ where the VEVs arise from a Green-Schwartz mechanism [21] with computable Fayet-Illiopoulos $D$-term which determines these VEVs to be one or two orders of magnitude below $M_U$. Additional exotic vector matter with mass $M_V$ allows the Wolfenstein parameter [22] to be generated by the ratio [19]

$$\frac{< \theta >}{M_V} = \frac{< \bar{\theta} >}{M_V} = \lambda \approx 0.22$$

(16)

The idea is that at tree-level the $U(1)$ family symmetry only permits third family Yukawa couplings (e.g. the top quark Yukawa coupling). Smaller Yukawa couplings are generated effectively from higher dimension non-renormalisable operators corresponding to insertions of $\theta$ and $\bar{\theta}$ fields and hence to powers of the expansion parameter in Eq.[16], which we have identified with the Wolfenstein parameter. The number of powers of the expansion parameter is controlled by the $U(1)$ charge of the particular operator. The fields relevant to neutrino masses are the lepton doublets $L_i$, the charge conjugated right-handed neutrinos and charged leptons $N_i^c$, $E_i^c$, up-type Higgs doublet $H_u$, and a singlet Higgs field whose VEV signals the heavy Majorana masses $\Sigma$. These are assigned $U(1)$ charges $l_i, n_i, e_i, h_u = 0, \sigma$, respectively. From Eq.[16], the neutrino Yukawa couplings and Majorana mass terms may then be expanded in powers of the Wolfenstein parameter,

$$M_{RR} \sim \left( \begin{array}{ccc} \lambda^{2n_1+\sigma} & \lambda^{n_1+n_2+\sigma} & \lambda^{n_1+n_2+\sigma} \\ \lambda^{n_1+n_2+\sigma} & \lambda^{2n_2+\sigma} & \lambda^{n_2+n_3+\sigma} \\ \lambda^{n_1+n_2+\sigma} & \lambda^{n_2+n_3+\sigma} & \lambda^{2n_3+\sigma} \end{array} \right) < \Sigma >$$

(17)
The neutrino Yukawa matrix is explicitly
\[
Y_\nu \sim \begin{pmatrix}
\lambda|l_{1+n_1}| & \lambda|l_{1+n_2}| & \lambda|l_{1+n_3}|
\lambda|l_{2+n_1}| & \lambda|l_{2+n_2}| & \lambda|l_{2+n_3}|
\lambda|l_{3+n_1}| & \lambda|l_{3+n_2}| & \lambda|l_{3+n_3}|
\end{pmatrix}
\] (18)

which may be compared to the notation in Eq.3. The charged lepton Yukawa matrix is given by
\[
Y_e \sim \begin{pmatrix}
\lambda|l_{1+e_1}| & \lambda|l_{1+e_2}| & \lambda|l_{1+e_3}|
\lambda|l_{2+e_1}| & \lambda|l_{2+e_2}| & \lambda|l_{2+e_3}|
\lambda|l_{3+e_1}| & \lambda|l_{3+e_2}| & \lambda|l_{3+e_3}|
\end{pmatrix}
\] (19)

For the quarks we shall assume a common form for the textures of \(Y_u\) and \(Y_d\)
\[
Y_u \sim \begin{pmatrix}
\lambda^8 & \lambda^5 & \lambda^3
\lambda^7 & \lambda^4 & \lambda^2
\lambda^5 & \lambda^2 & 1
\end{pmatrix},
Y_d \sim \begin{pmatrix}
\lambda^4 & \lambda^3 & \lambda^3
\lambda^3 & \lambda^2 & \lambda^2
\lambda & 1 & 1
\end{pmatrix}
\] (20)

The conditions for achieving bi-maximal mixing with either a conventional neutrino hierarchy, or an inverted hierarchy then translate into conditions on the choice of \(U(1)\) charges for the different fields. We have already tabulated the simplest charges consistent with the SRHND conditions giving a hierarchical spectrum and the LMA MSW solution Eqs.3, 9 [15]. In Tables 1 and 2, we give the simplest \(U(1)\) charges consistent with an inverted neutrino mass hierarchy Eqs.10, 11.

The \(U(1)\) charges in Tables 1,2 can be arranged into four categories (referred to as cases I, II, III and IV) according to the perturbation terms present in \(m_{LL}\) which can be expressed in leading order, using Eqs.17, 18 together with Eqs.3, 4,5 as
\[
m_{LL} \sim \begin{pmatrix}
\delta & 1 & 1
1 & \epsilon & \epsilon
1 & \epsilon & \epsilon
\end{pmatrix}
\] (21)

where \(\delta = \epsilon = \lambda^4\) in cases I,II, \(\delta = \lambda^6, \epsilon = \lambda^4\) in case III, and \(\delta = \lambda^4, \epsilon = \lambda^6\) in case IV. Again one can differentiate cases I and II with respect to the textures of charged lepton Yukawa matrix as shown in Table 3. This may lead to different solar mixing angles at the end.
| Case | $l_1$ | $l_2$ | $l_3$ | $n_1$ | $n_2$ | $n_3$ | $\sigma$ |
|------|-------|-------|-------|-------|-------|-------|----------|
| I    | -3    | 3     | 3     | 2     | -3    | 0     | 1        |
|      | -3    | 3     | 3     | 2     | -3    | 1     | -1       |
|      | -3    | 3     | 3     | 3     | -2    | -1    | 1        |
|      | -3    | 3     | 3     | 2     | -2    | 0     | 0        |
|      | -3    | 3     | 3     | 3     | -2    | 0     | 0        |
|      | 3     | -3    | -3    | -3    | 2     | 0     | 1        |
|      | 3     | -3    | -3    | -3    | 2     | 1     | -1       |
|      | 3     | -3    | -3    | -2    | 2     | 0     | 0        |
|      | 3     | -3    | -3    | -2    | 3     | -1    | 1        |
|      | 3     | -3    | -3    | -2    | 3     | 0     | -1       |
|      | 3     | -3    | -3    | -2    | 3     | 1     | -2       |
| II   | -3    | 1     | 1     | 3     | -3    | 1     | -2       |
|      | -3    | 1     | 1     | 3     | -2    | 1     | -2       |
|      | -3    | 1     | 1     | 3     | -1    | 1     | -2       |
|      | -2    | 2     | 2     | 2     | -3    | 0     | 0        |
|      | -2    | 2     | 2     | 2     | -2    | 0     | 0        |
|      | -2    | 2     | 2     | 3     | -2    | 0     | 0        |
|      | -1    | 3     | 3     | 1     | -3    | -1    | 2        |
|      | -1    | 3     | 3     | 2     | -3    | -1    | 2        |
|      | -1    | 3     | 3     | 3     | -3    | -1    | 2        |
|      | 1     | -3    | -3    | -3    | 3     | 1     | -2       |
|      | 1     | -3    | -3    | -2    | 3     | 1     | -2       |
|      | 1     | -3    | -3    | -1    | 3     | 1     | -2       |
|      | 2     | -2    | -2    | -3    | 2     | 0     | 0        |
|      | 2     | -2    | -2    | -2    | 2     | 0     | 0        |
|      | 2     | -2    | -2    | -2    | 3     | 0     | 0        |
|      | 3     | -1    | -3    | -3    | 2     | -1    | 2        |
|      | 3     | -1    | -1    | -3    | 1     | -1    | 2        |
|      | 3     | -1    | -1    | -3    | 2     | -1    | 2        |
|      | 3     | -1    | -1    | -3    | 3     | -1    | 2        |

Table 1: Examples of charges (cases I and II) which satisfy the conditions in Eqs.10,11.

| Case | $l_1$ | $l_2$ | $l_3$ | $n_1$ | $n_2$ | $n_3$ | $\sigma$ |
|------|-------|-------|-------|-------|-------|-------|----------|
| III  | -3    | 3     | 3     | 2     | -3    | -1    | 2        |
|      | -3    | 3     | 3     | 2     | -3    | 0     | 0        |
|      | -3    | 3     | 3     | 3     | -3    | -1    | 1        |
|      | 3     | -3    | -3    | -3    | -3    | 3     | 1        |
|      | 3     | -3    | -3    | -2    | 3     | 0     | 0        |
| IV   | -3    | 3     | 3     | 3     | -2    | 0     | 0        |
|      | -3    | 3     | 3     | 3     | -2    | 1     | -2       |
|      | 3     | -3    | -3    | -3    | 2     | -1    | 2        |
|      | 3     | -3    | -3    | -3    | 2     | 0     | 0        |
|      | 3     | -3    | -3    | -3    | 3     | -1    | 1        |

Table 2: Examples of charges (cases III and IV) which satisfy the conditions in Eqs.10,11.
4 Renormalisation Group Analysis

We now turn to a renormalisation group study for calculating radiative corrections to neutrino masses and mixing angles at low energies (see [16] and references therein). We pick up four representative examples, one each from each case listed in Tables 1,2. Our detailed procedure and methods follow closely that used in the case of SRHND in [16], which we summarise briefly. From [16] we make use of the renormalisation group equations (RGEs) to one-loop order for Yukawa matrices and three gauge couplings, in the minimal supersymmetric standard model with three right-handed neutrinos, including the effects of the heavy neutrino thresholds. Using the textures in Tables 3,4, and Eq.20 we first run the quantities $Y_f (f = u, d, e, \nu)$, $Y_{RR}$ and gauge couplings $g_{1,2,3}$ from the GUT scale $M_U = 2.0 \times 10^{16}$GeV down to the lightest right-handed heavy neutrino mass scale $M_{R1}$. We take the effects of the other heavy right-handed neutrino mass thresholds in successive steps in running the RGEs. Using the see-saw formula in the standard way, and inverting $M_{RR}$ numerically, the left-handed Majorana neutrino mass matrix $m_{LL}$ is calculated at the scale $M_{R1}$. The corresponding lepton mixing matrix $V_{MNS} = V_{eL}V_{\nu L}^\dagger$ is calculated after performing the diagonalisation of $m_{LL}^{diag} = V_{eL}m_{LL}V_{\nu L}^\dagger$ and $Y_e^{diag} = V_{eL}Y_eV_{eR}^\dagger$ at this scale. From $V_{MNS}$ mixing matrix we extract $S_{sol} = \sin^2 2\theta_{12}$ as well as $S_{ut} = \sin^2 2\theta_{23}$ as outlined in [16]. We also calculate the left-handed Majorana neutrino mass matrix and the same mixing parameters at the GUT scale for providing a meaningful comparison of radiative corrections at different energy scales though the see-saw mechanism in principle, does not operate at scales above the lightest right-handed neutrino mass $M_{R1}$. In the next step while moving from $M_{R1}$ scale to low energy scale $m_t$, we run the RGEs for the coefficient $\kappa$ of the dimension 5 neutrino mass operator in the diagonal charged lepton basis [16], which is interpreted as see-saw mass matrix.
$m'_{LL}(M_{R1}) = v_2^2 \kappa'(M_{R1})$ where $\kappa' = V_{eL} \kappa V^T_{eL}$. This in turn gives $m'_{LL}(m_t),$

$$m'_{LL}(m_t) = e^{\frac{3}{2}t_{21} e^{6t_{22}} e^{-6t}} 
\begin{pmatrix}
m'_{LL11}(M_{R1}) & m'_{LL12}(M_{R1}) & m'_{LL13}(M_{R1}) e^{-I_\tau} \\
m'_{LL21}(M_{R1}) & m'_{LL22}(M_{R1}) & m'_{LL23}(M_{R1}) e^{-I_\tau} \\
m'_{LL31}(M_{R1}) e^{-I_\tau} & m'_{LL32}(M_{R1}) e^{-I_\tau} & m'_{LL33}(M_{R1}) e^{-2I_\tau}
\end{pmatrix}$$  \tag{22}$$

and

$$I_g = \frac{1}{16\pi^2} \int_{m_t}^{M_{R1}} g^2_i(t) dt, \quad I_f = \frac{1}{16\pi^2} \int_{m_t}^{M_{R1}} h^2_i(t) dt$$ \tag{23}$$

where $f = t, \tau$. We also calculate $V_{MNS}(m_t) = V_{\nu L}'$ where $V_{\nu L}'$ is the matrix which diagonalises $m'_{LL}$. We then obtain the neutrino masses $m'_{LL}^{\text{diag}} = \text{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3})$ and mixing angle parameters $S_{sol}, S_{at}$ as before. The parameters $I_f$ along with the input textures of $Y^e, Y^\nu, Y_{RR}$ are given in Tables 3,4 which are needed in getting the results in Tables 5-8.

The numerical results are presented in Tables 5-8 for cases I-IV where the results in Table 8 is meant for relative comparison only. We define the measure of the splitting of neutrino masses by $\xi = (m_{\nu 1} - m_{\nu 2})/m_{\nu 2} \approx \frac{1}{2} \frac{\Delta m^2_{23}}{\Delta m^2_{12}}$ which is found to be decreasing from high energy scale $M_U$ to low energy scale $m_t$ by about (30, 43, 20)% for cases I,II and III respectively. But the decrease in moving from the scale $M_U$ to scale $M_{R1}$ is about (22, 36, 17)% which is a significant effect. We get at scale $m_t$ the values of the measure of the splitting, $\xi = (0.00238, 0.00281, 0.00266, 0.00004)$ which corresponds to $m_{\nu 1} - m_{\nu 2} = \xi m_{\nu 2} = (1.7, 2.0, 1.8, 0.03) \times 10^{-4}$ eV for cases I-IV, which are somewhat on the lower end but within the observational range except for case IV (which is completely ruled out). The low energy absolute values of neutrino masses are also estimated as $m_{\nu 2} = (0.0711, 0.0713, 0.0685, 0.0682) eV$ and $m_{\nu 3} \approx 0$ for cases I-IV respectively. There is almost a smooth decrease in $m_{\nu 2,3}$ by about 20% while moving from scale $M_U$ down to low energy scale $m_t$. There is a mild decrease of the solar mixing angle parameter $S_{sol}$ while moving from high scale $M_U$ to $m_t$ scale in all cases, and its low energy values are $S_{sol} = 0.9994, 0.8996, 0.9994, 0.9995$ for cases I-IV respectively. These values except for the case II, are on the higher side which
Table 3: For cases I and II: Textures of the Yukawa couplings of Dirac neutrino mass, right-handed Majorana neutrino mass, and also other relevant parameters needed for the numerical estimation of left-handed Majorana neutrino masses at low energies through seesaw mechanism. Here \(<\Sigma>\) is taken as a free parameter and \(M_{R1}\) is the lowest threshold scale in \(M_{RR}\). As noted earlier, the right-handed neutrinos may be permuted according to 1 \(\rightarrow\) 2, 2 \(\rightarrow\) 3, 3 \(\rightarrow\) 1 so that the large elements in \(Y^\nu\) appear in the 23 and 33 positions, which may be more natural from the viewpoint of unified theories where a large 33 element is expected.
| Parameter | Case III | Case IV |
|-----------|----------|--------|
| $U(1)$ charges | $l_{1,2,3} = -3, 3, 3$  
$n_{1,2,3} = 2, -3, -1$  
$e_{1,2,3} = -5, -1, -3$ | $l_{1,2,3} = 3, -3, -3$  
$n_{1,2,3} = -3, 3, -1$  
$e_{1,2,3} = 5, 1, 3$ |
| $\sigma$ | 2 | 1 |
| $n_{1,2,3}$ | $2, -3, -1$ | $-3, 3, -1$ |
| $e_{1,2,3}$ | $-5, -1, -3$ | $5, 1, 3$ |
| $\sigma$ | 2 | 1 |
| $Y^e$ | $\begin{pmatrix} a_{11} \lambda^8 & a_{12} \lambda^4 & a_{13} \lambda^6 \\ a_{21} \lambda^2 & a_{22} \lambda^2 & a_{23} \\ a_{31} \lambda^2 & a_{32} \lambda^2 & a_{33} \end{pmatrix}$ | $\begin{pmatrix} a_{11} \lambda^8 & a_{12} \lambda^4 & a_{13} \lambda^6 \\ a_{21} \lambda^2 & a_{22} \lambda^2 & a_{23} \\ a_{31} \lambda^2 & a_{32} \lambda^2 & a_{33} \end{pmatrix}$ |
| $a_{ij}$ | $\begin{pmatrix} 1.0 & 1.0 & 0.8 \\ 1.0 & 2.5 & 0.35 \\ 0.8 & 1.0 & 2.0 \end{pmatrix}$ | $\begin{pmatrix} 1.0 & 1.0 & 0.8 \\ 1.0 & 2.5 & 0.35 \\ 0.8 & 1.0 & 2.0 \end{pmatrix}$ |
| $Y^\nu$ | $\begin{pmatrix} \lambda & \lambda^6 & \lambda^4 \\ \lambda^5 & 1 & \lambda^2 \\ \lambda^5 & 1 & \lambda^2 \end{pmatrix}$ | $\begin{pmatrix} 1 & \lambda^6 & \lambda^2 \\ \lambda^6 & 1 & \lambda^4 \\ \lambda^6 & 1 & \lambda^4 \end{pmatrix}$ |
| $\begin{pmatrix} \lambda^6 & \lambda & \lambda^3 \\ \lambda & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$ | $\begin{pmatrix} \lambda^5 & \lambda & \lambda^3 \\ \lambda & \lambda^7 & \lambda^3 \\ \lambda^3 & \lambda^3 & \lambda \end{pmatrix}$ |
| $< \Sigma >$ | $2.592 \times 10^{14} GeV$ | $11.782 \times 10^{14} GeV$ |
| $M_{R_1}$ | $5.72 \times 10^{14} GeV$ | $11.782 \times 10^{14} GeV$ |
| $I_{g1,2}$ | 0.0470, 0.0728 | 0.0553, 0.0824 |
| $I_{1,\tau}$ | 0.1016, 0.0518 | 0.1125, 0.0620 |
| $I_{\mu,e}$ | $0.0002, 1.11 \times 10^{-8}$ | $0.0004, 3.64 \times 10^{-9}$ |

Table 4: For cases III and IV: Textures of the Yukawa couplings of Dirac neutrino mass, right-handed Majorana neutrino mass, and other relevant parameters needed for the numerical estimation of left-handed Majorana neutrino masses at low energies through see-saw mechanism. Here $< \Sigma >$ is taken as a free parameter and $M_{R_1}$ is the lowest threshold scale in $M_{RR}$. As noted earlier, the right-handed neutrinos may be permuted according to $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$ so that the large elements in $Y^\nu$ appear in the 23 and 33 positions, which may be more natural from the viewpoint of unified theories where a large 33 element is expected.
Table 5: Case I: Left-handed Majorana neutrino mass matrix and mixing matrix at different energy scales $M_{U}$, $M_{R_1}$ and $m_i$. Neutrino masses are expressed in eV. Note that we take the convention $|m_{
u_1}| > |m_{
u_2}|$, which fixes the ordering of the 1st two columns of $V_{MNS}$.

| Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ | Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ |
|-------------------------------------------|-------------------------------------------|
| $Y^e = \begin{pmatrix} 1.976.10^{-6} & 8.433.10^{-4} & 3.265.10^{-5} \\ 1.742.10^{-2} & 4.356.10^{-2} & 0.126 \\ 1.394.10^{-2} & 1.742.10^{-2} & 0.720 \end{pmatrix}$ | $V_{eL} = \begin{pmatrix} -0.999 & 0.018 & -0.003 \\ -0.019 & -0.984 & 0.175 \\ 0.000 & 0.175 & 0.985 \end{pmatrix}$ |
| $Y^e_{\text{diag}} = \text{diag}(2.9097.10^{-4}, 4.252.10^{-2}, 0.73155)$ | $m_\mu/m_\mu, m_\mu/m_\tau = 0.0068, 0.0581$ |
| $m_{LL}^{\nu} = \begin{pmatrix} 1.511.10^{-6} & 1.332.10^{-2} & 1.332.10^{-2} \\ 1.332.10^{-2} & 3.121.10^{-5} & 3.121.10^{-5} \\ 1.332.10^{-2} & 3.121.10^{-5} & 3.121.10^{-5} \end{pmatrix}$ | $S_{sol} = 0.9995, S_{at} = 0.8818$ |
| $m_{LL}^{\nu, \text{diag}} = \text{diag}(0.085782, -0.085492, -3.17.10^{-17})$ | |
| $Y^e_{\text{diag}} = \text{diag}(2.7605.10^{-4}, 3.9926.10^{-2}, 0.63712)$ | |
| $m_{LL}^{\nu} = \begin{pmatrix} 5.755.10^{-6} & 5.889.10^{-2} & 5.792.10^{-2} \\ 5.889.10^{-2} & 1.004.10^{-4} & 9.885.10^{-5} \\ 5.792.10^{-2} & 9.885.10^{-5} & 9.731.10^{-5} \end{pmatrix}$ | $V_{MNS} = V_{eL} V_{\nu L}^\dagger = \begin{pmatrix} -0.699 & -0.715 & -0.015 \\ -0.418 & 0.391 & 0.820 \\ 0.580 & -0.579 & 0.573 \end{pmatrix}$ |
| $m_{LL}^{\nu, \text{diag}} = \text{diag}(0.082704, -0.082500, 9.7.10^{-10})$ | $S_{sol} = 0.9994, S_{at} = 0.9107$ |
| $Y^e_{\text{diag}} = \text{diag}(2.391.10^{-4}, 3.456.10^{-2}, 0.4774)$ | |
| $m_{LL}^{\nu} = \begin{pmatrix} -1.633.10^{-3} & 4.340.10^{-2} & -5.634.10^{-2} \\ 4.340.10^{-2} & 1.691.10^{-3} & -1.140.10^{-3} \\ -5.634.10^{-2} & -1.140.10^{-3} & 1.119.10^{-4} \end{pmatrix}$ | |
| $m_{LL}^{\nu, \text{diag}} = \text{diag}(0.071229, -0.071059, -1.6.10^{-9})$ | $S_{sol} = 0.9994, S_{at} = 0.9352$ |
| $m_\mu/m_\mu, m_\mu/m_\tau = 0.0069, 0.0724$ | |
Table 6: Case II: Left-handed Majorana neutrino mass matrix and mixing matrix at different energy scales \( M_U, M_{R_1} \) and \( m_t \). Neutrino masses are expressed in eV. Note that we take the convention \( |m_{\nu_1}| > |m_{\nu_2}| \), which fixes the ordering of the 1st two columns of \( V_{MNS} \).
| Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ | Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ |
| --- | --- |
| $Y^e = \begin{pmatrix} 1.976.10^{-6} & 8.433.10^{-4} & 3.265.10^{-5} \\ 1.742.10^{-2} & 4.356.10^{-2} & 0.126 \\ 1.394.10^{-2} & 1.742.10^{-2} & 0.7200 \end{pmatrix}$ | $V_{eL} = \begin{pmatrix} -0.999 & 0.018 & -0.003 \\ -0.019 & -0.984 & 0.175 \\ 0.000 & 0.175 & 0.985 \end{pmatrix}$ |
| $m_{\nu}^{e} = \text{diag}(2.910.10^{-4}, 4.252.10^{-2}, 0.7316)$ | $m_e/m_\mu, m_\mu/m_\tau = 0.0068, 0.0581$ |
| $m_{\nu}^{\text{diag}} = \text{diag}(0.085779, -0.085495, -1.08.10^{-19})$ | $V_{\text{MNS}} = V_{eL}V_{\nu L}^\dagger = \begin{pmatrix} -0.699 & -0.715 & -0.015 \\ -0.418 & 0.391 & 0.820 \\ 0.580 & -0.579 & 0.573 \end{pmatrix}$ |
| Scale $\mu = M_{R1} = 5.72 \times 10^{13} GeV$ | Scale $\mu = M_{R1} = 5.72 \times 10^{13} GeV$ |
| $Y^e = \begin{pmatrix} 1.626.10^{-6} & 8.069.10^{-4} & 2.438.10^{-5} \\ 1.607.10^{-2} & 4.058.10^{-2} & 1.017.10^{-1} \\ 1.207.10^{-2} & 1.458.10^{-2} & 0.642 \end{pmatrix}$ | $V_{eL} = \begin{pmatrix} -0.999 & 0.019 & -0.003 \\ -0.019 & -0.987 & 0.159 \\ 0.000 & 0.159 & 0.987 \end{pmatrix}$ |
| $Y_{\text{diag}}^e = \text{diag}(2.7856.10^{-4}, 4.0277.10^{-2}, 0.6502)$ | $m_e/m_\mu, m_\mu/m_\tau = 0.0069, 0.0619$ |
| $m_{\nu}^{\text{diag}} = \text{diag}(0.079689, -0.07947, 3.27.10^{-10})$ | $V_{\text{MNS}} = V_{eL}V_{\nu L}^\dagger = \begin{pmatrix} -0.699 & -0.715 & -0.015 \\ -0.432 & 0.405 & 0.805 \\ 0.570 & -0.569 & 0.593 \end{pmatrix}$ |
| Scale $\mu = m_1 = 175 GeV$ | Scale $\mu = m_1 = 175 GeV$ |
| $Y_{\text{diag}}^e = \text{diag}(2.39.10^{-4}, 3.455.10^{-2}, 0.4771)$ | $m_e/m_\mu, m_\mu/m_\tau = 0.0069, 0.0724$ |
| $m_{\nu}^{\text{diag}} = \text{diag}(0.068639, -0.068457, 4.97.10^{-9})$ | $V_{\text{MNS}} = V_{eL}^\dagger = \begin{pmatrix} -0.698 & -0.716 & -0.015 \\ -0.446 & -0.419 & 0.791 \\ -0.560 & -0.559 & 0.612 \end{pmatrix}$ |
| $S_{\text{sol}} = 0.9994, S_{\text{at}} = 0.9374$ | $S_{\text{sol}} = 0.9994, S_{\text{at}} = 0.9114$ |

Table 7: Case III: Left-handed Majorana neutrino mass matrix and mixing matrix at different energy scales $M_U$, $M_{R1}$ and $m_1$. Neutrino masses are expressed in eV. Note that we take the convention $|m_{e1}| > |m_{e2}|$, which fixes the ordering of the 1st two columns of $V_{\text{MNS}}$. 
Table 8: Case IV: Left-handed Majorana neutrino mass matrix and mixing matrix at different energy scales $M_U$, $M_{R1}$ and $m_t$. Neutrino masses are expressed in eV. Note that we take the convention $|m_{\nu_i}| > |m_{\nu_j}|$, which fixes the ordering of the 1st two columns of $V_{MNS}$. 

| Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ | Scale $\mu = M_U = 2.0 \times 10^{16} GeV$ |
|---------------------------------------|---------------------------------------|
| $Y^e = \begin{pmatrix} 1.976.10^{-6} & 8.433.10^{-4} & 3.265.10^{-5} \\ 1.742.10^{-2} & 4.356.10^{-2} & 0.126 \\ 1.394.10^{-2} & 1.742.10^{-2} & 0.7200 \end{pmatrix}$ | $V_{eL} = \begin{pmatrix} -0.999 & 0.018 & -0.003 \\ -0.019 & -0.984 & 0.175 \\ 0.000 & 0.175 & 0.985 \end{pmatrix}$ |
| $Y^e_{\text{diag}} = \text{diag}(2.9097.10^{-4}, 4.252.10^{-2}, 0.7316)$ | $m_{e}/m_{\mu}, m_{\mu}/m_{\tau} = 0.0068, 0.0581$ |
| $m_{LL}^\nu = \begin{pmatrix} 2.9097.10^{-4} & 6.068.10^{-2} & 6.068.10^{-2} \\ 6.068.10^{-2} & 3.323.10^{-7} & 3.323.10^{-7} \\ 6.068.10^{-2} & 3.323.10^{-7} & 3.323.10^{-7} \end{pmatrix}$ | $V_{MNS} = V_{eL}V_{\nu L}^\dagger = \begin{pmatrix} 0.699 & 0.714 & -0.015 \\ 0.418 & -0.392 & 0.820 \\ -0.580 & 0.580 & 0.573 \end{pmatrix}$ |
| $m_{LL}^{\text{diag}} = \text{diag}(0.0858705, -0.0857475, 5.53.10^{-17})$ | $S_{sol} = 0.9996, S_{at} = 0.8818$ |

| Scale $\mu = M_{R1} = 11.782 \times 10^{14} GeV$ | Scale $\mu = M_{R1} = 11.782 \times 10^{14} GeV$ |
|---------------------------------------|---------------------------------------|
| $Y^e = \begin{pmatrix} 1.768.10^{-6} & 8.170.10^{-4} & 2.780.10^{-5} \\ 1.670.10^{-2} & 4.197.10^{-2} & 1.130.10^{-1} \\ 1.294.10^{-2} & 1.590.10^{-2} & 0.678 \end{pmatrix}$ | $V_{eL} = \begin{pmatrix} -0.999 & 0.018 & -0.003 \\ -0.019 & -0.986 & 0.167 \\ 0.000 & 0.167 & 0.986 \end{pmatrix}$ |
| $Y^e_{\text{diag}} = \text{diag}(2.8196.10^{-4}, 4.133.10^{-2}, 0.688)$ | $m_{e}/m_{\mu}, m_{\mu}/m_{\tau} = 0.0068, 0.0600$ |
| $m_{LL}^\nu = \begin{pmatrix} 4.520.10^{-6} & 5.644.10^{-2} & 5.596.10^{-2} \\ 5.644.10^{-2} & -2.070.10^{-7} & -2.011.10^{-7} \\ 5.596.10^{-2} & -2.011.10^{-7} & -1.953.10^{-7} \end{pmatrix}$ | $V_{MNS} = V_{eL}V_{\nu L}^\dagger = \begin{pmatrix} -0.699 & 0.715 & 0.015 \\ -0.425 & -0.399 & -0.812 \\ 0.575 & 0.575 & -0.583 \end{pmatrix}$ |
| $m_{LL}^{\text{diag}} = \text{diag}(0.0794824, -0.0794783, -4.0.10^{-13})$ | $S_{sol} = 0.9995, S_{at} = 0.8975$ |

| Scale $\mu = m_t = 175 GeV$ | Scale $\mu = m_t = 175 GeV$ |
|---------------------------------------|---------------------------------------|
| $Y^e_{\text{diag}} = \text{diag}(2.391.10^{-4}, 3.455.10^{-2}, 0.47714)$ | $m_{e}/m_{\mu}, m_{\mu}/m_{\tau} = 0.0069, 0.0724$ |
| $m_{LL}^\nu = \begin{pmatrix} -1.537.10^{-3} & 4.134.10^{-2} & -5.420.10^{-2} \\ 4.134.10^{-2} & 1.532.10^{-3} & -1.007.10^{-3} \\ -5.420.10^{-2} & -1.007.10^{-3} & 7.900.10^{-6} \end{pmatrix}$ | $V_{MNS} = V_{\nu L}^\dagger = \begin{pmatrix} -0.699 & 0.715 & -0.015 \\ -0.442 & -0.416 & 0.795 \\ 0.562 & 0.562 & 0.607 \end{pmatrix}$ |
| $m_{LL}^{\text{diag}} = \text{diag}(0.0681955, -0.0681929, -3.1.10^{-8})$ | $S_{sol} = 0.9995, S_{at} = 0.9305$ |
lie outside the allowed range. This depends on the texture of charged lepton Yukawa matrix given in Tables 3 and 4. However the atmospheric mixing angle parameter \( S_{at} \) is found to increase by about 6% from high scale \( M_U \) to low scale \( m_t \), of which 2% is in the energy range \( M_U \) to \( M_{R1} \). The low energy values are predicted as \( S_{at} = 0.935, 0.957, 0.937, 0.931 \) for cases I-IV respectively. These values are above the experimental lower bound \( S_{at} > 0.88 \). In our numerical analysis the charged lepton textures also predict almost consistent hierarchical ratios of charged lepton masses at low scale as shown in Tables 5-8.

5 Conclusion

In conclusion, we have studied models of neutrino masses which naturally give rise to an inverted mass hierarchy and bi-maximal mixing. The models are based on the see-saw mechanism with three right-handed neutrinos, which generates a single mass term of the form \( \nu_e (\nu_\mu + \nu_\tau) \) corresponding to two degenerate neutrinos \( \nu_e \) and \( \nu_\mu + \nu_\tau \), and one massless neutrino \( \nu_\mu - \nu_\tau \). Atmospheric neutrino oscillations are accounted for if the degenerate mass term is about \( 5 \times 10^{-2} \) eV. Solar neutrino oscillations of the Large Mixing Angle MSW type arise when small perturbations are included leading to a mass splitting between the degenerate pair of about \( (1.7 - 2.0) \times 10^{-4} \) eV. We have studied the conditions that such models must satisfy in the framework of a \( U(1) \) family symmetry broken by vector singlets, and catalogue the simplest examples. We distinguished four types of cases, and then performed a renormalisation group analysis of the neutrino masses mixing angles, assuming the supersymmetric standard model and large \( \tan \beta \), for one example from each case. Cases I,III,IV predict almost maximal solar mixing, and an atmospheric mixing angle which is near maximal, increasing by 6% due to RG running. However case IV predicts a splitting parameter \( \xi \) which is outside the allowed range, therefore this texture is not favoured for the
LMA MSW solution. Case II gives a somewhat smaller solar mixing angle, which decreases by about 3% due to RG running, while the atmospheric angle increases by about 5%. Although these examples predict a large solar mixing angle $\sin^2 2\theta_{12} > 0.9$, this prediction depends to some extent on the texture assumed for $Y_e$, since in the absence of charged lepton mixing angles, the 12 neutrino mixing is almost exactly maximal. Clearly all cases are stable under radiative corrections, leading to a natural explanation of bi-maximal mixing in terms of an experimentally testable inverted hierarchical spectrum.

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