Dominated splittings and the spectrum of quasi-periodic Jacobi operators

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Abstract
We prove that the resolvent set of any, possibly singular, quasi-periodic Jacobi operator is characterized as the set of all energies whose associated Jacobi cocycles induce a dominated splitting. This extends a well-known result by Johnson for Schrödinger operators.

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1. Introduction

The purpose of this paper is to give a dynamical characterization of the spectrum of quasi-periodic Jacobi operators (QPJ). To this end, let $\alpha \in \mathbb{R}^d$ be fixed with components linearly independent over $\mathbb{Q}$ and let $c, v : \mathbb{T}^d \to \mathbb{C}$ be continuous satisfying $v(\mathbb{T}^d) \subseteq \mathbb{R}$. A quasi-periodic Jacobi operator is a family of bounded self-adjoint operators on $L^2(\mathbb{Z})$ of the form,

$$[H_x\psi]_n = c(T^{n-1}x)\psi_{n-1} + c(T^n x)\psi_{n+1} + v(T^n x)\psi_n,$$

for $x \in \mathbb{T}^d$, (1.1)

generated upon evaluation of $c, v$ along the trajectories of $T : \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto x + \alpha$. Let $\mu$ denote the Haar probability measure on $\mathbb{T}^d$. We are assuming $\log |c| \in L^1(\mathbb{T}^d, d\mu)$, as is commonly done. To simplify notation, we will also set $X := \mathbb{T}^d$.

Motivated by the now famous almost Mathieu operator where for $d = 1$, $v(x) = 2\lambda \cos(2\pi x)$, $\lambda \in \mathbb{R}$, and $c \equiv 1$, most of the literature on QPJ has so far focussed on the special case $c \equiv 1$, commonly known as quasi-periodic Schrödinger operators (QPS). In recent years, a dynamical system approach to the spectral theory of the latter proved to be extremely fruitful. In particular, this approach allowed for a more global picture of the spectral properties of QPS [1, 2].
The relevant dynamical system for QPS is Schrödinger cocycles, quasi-periodic $SL_2(\mathbb{R})$-cocycles whose iteration generates solutions to the finite difference equation, $H_\lambda \psi = E \psi$. More generally, a (continuous) $M_2(\mathbb{C})$-cocycle is a dynamical system on $X \times \mathbb{C}^2$ induced by $T$ and a matrix-valued function $D \in \mathcal{C}(X, M_2(\mathbb{C}))$, defined by $(x, \psi) \mapsto (Tx, D(x)\psi)$. We will denote this cocycle map by the pair $(T, D)$.

One fundamental ingredient of above-mentioned dynamical approach to the spectral theory of QPS is a theorem due to Johnson [21] which characterizes the spectrum in terms of Schrödinger cocycles. More specifically, it is shown in [21] that the resolvent set of a QPS is determined by uniformly hyperbolic dynamics of the Schrödinger cocycles. We mention that Johnson’s theorem has recently been generalized to discrete long-range Schrödinger operators [17]. The goal of this paper is to find an appropriate extension of Johnson’s theorem to QPJ.

The main problem in extending Johnson’s theorem to the more general Jacobi setting is that, whereas Schrödinger cocycles are $SL(2, \mathbb{R})$, the relevant cocycles for QPJ are in general not even invertible. In this context, we call a cocycle $(T, D)$ singular if $\det D(x_0) = 0$ for some $x_0 \in X$. For QPJ, singular cocycles automatically arise once the sampling function $c$ has zeros, in which case (1.1) is called a singular QPJ.

We mention that QPJ originated in solid states physics, where both $c$, $v$ are trigonometric polynomials, hence the possibility of $c$ having zeros cannot be excluded in general. For instance, one prominent example of a QPJ relevant in physics is given by extended Harper’s model, where for $d = 1$ the sampling functions are given by $v(x) = 2 \cos(2\pi x)$ and $c(x) = \lambda_1 e^{-\pi i x + \pi i \nu/2} + \lambda_2 + \lambda_3 e^{\pi i x - \pi i \nu/2}$ with $\lambda_j \in \mathbb{R}$, $1 \leq j \leq 3$. Proposed by Thouless in context with the integer quantum Hall effect [34], extended Harper’s model constitutes a singular QPJ for a large set of coupling parameters $(\lambda_1, \lambda_2, \lambda_3)$. Even though several recent works on the spectral theory of QPJ have started to account for singularity (many of which motivated by extended Harper’s model) [8, 11, 12, 16, 20, 32, 33], an extension of Johnson’s theorem to (possibly singular) QPJ has so far been missing.

The presence of singular cocycles obviously requires a dynamical framework different from uniform hyperbolicity. Recent work on the continuity and positivity of the Lyapunov exponent for Jacobi operators [8, 14, 15, 18] indicated the notion ‘dominated splitting’ as an appropriate analogue of uniform hyperbolicity, suitable when passing from the Schrödinger to the general Jacobi setting.

A cocycle $(T, D)$ is said to induce a dominated splitting (write $(T, D) \in \mathcal{DS}$ if there exists $N \in \mathbb{N}$ and a continuous, non-trivial splitting of $\mathbb{C}^2 = S^{(1)} \oplus S^{(2)}$ satisfying $D_N(x) S^{(j)} \subset S^{(j)} \oplus x \subseteq S^{(j)} \oplus x$, $1 \leq j \leq 2$, which exhibits uniform domination in the sense,\)
\[
\frac{\|D_N(x)v_1\|}{\|v_1\|} > \frac{\|D_N(x)v_2\|}{\|v_2\|}, \quad \text{all } x \in X, \tag{1.2}
\]
for all $v_j \in S^{(j)} \setminus \{0\}$. Here, for $n \in \mathbb{N}$, $D_n(x) := D(T^{n-1}x) \ldots D(x)$ denotes the $n$-th iterate of $(T, D)$ on the fibres, where $D_0(x) := 1$. $\mathcal{DS}$ is a very ‘robust property’, e.g. it is well known [25] that cocycles inducing a dominated splitting are open in $\mathcal{C}(X, M_2(\mathbb{C}))$, accompanied by continuity, even real analyticity of the (top) Lyapunov exponent (LE),\)
\[
L(T, D) := \lim_{n \to \infty} \frac{1}{n} \int \log \|D_n(x)\| \, d\mu(x). \tag{1.3}
\]
$\mathcal{DS}$ specializes to uniform hyperbolicity ($\mathcal{UH}$) if the matrix cocycles are unimodular. More generally, for any non-singular cocycle, $(T, D) \in \mathcal{DS}$ if and only if $(T, \frac{D}{\sqrt{\det D}}) \in \mathcal{UH}$. The advantage of the notion $\mathcal{DS}$ is, however, that it makes sense for both singular and non-singular cocycles.
Another feature not present for QPS is that the relevant cocycles for QPJ (Jacobi cocycles) are not unique. For instance, one possible choice for Jacobi cocycles is given by

\[ A^E(x) = \begin{pmatrix} E - v(x) & -c(T^{-1}x) \\ c(x) & 0 \end{pmatrix}, \tag{1.4} \]

where \( E \in \mathbb{C} \) is the spectral parameter. There are, however, alternative choices, which, depending on the problem in mind, may be more advantageous. We emphasize that all these choices share that they are singular precisely if \( c \) has zeros. We will comment more on the flexibility in the choice of Jacobi cocycles in section 2. Our results account for this flexibility, and give a dynamical characterization of the spectrum of QPJ applicable for the different choices (see theorem 2.1).

To formulate our main result, we recall that for a QPJ, minimality of \( T \) implies that the spectrum of the operators \( H_x \) is constant in \( x \in X \); we denote this set by \( \Sigma \), a compact subset of \( \mathbb{R} \).

**Theorem 1.1.** Let \( \alpha \in \mathbb{R}^d \) be fixed with components linearly independent over \( \mathbb{Q} \). Consider the (possibly singular) quasi-periodic Jacobi operator with frequency \( \alpha \), defined in (1.1), i.e. \( T \) is the translation by \( \alpha \) on \( \mathbb{T}^d \), and \( c, v \in C(\mathbb{T}^d, \mathbb{C}) \) with \( v \) real valued and \( c \in L^1(\mathbb{T}^d, d\mu) \). Then, for the Jacobi cocycle defined in (1.4), one has

\[ \Sigma = \{ E : (T, A^E) \notin DS \}. \tag{1.5} \]

**Remark 1.2.**

(i) An analogous statement holds for the Jacobi cocycles alternative to (1.4) which will be described in section 2, see theorem 2.1.

(ii) Our proof of theorem 1.1 does in fact not depend on the specifics of the background dynamics induced by translations on \( \mathbb{T}^d \). Even though our main motivation for this work is QPJ, the argument we present applies to any uniquely ergodic, invertible map \( T \) on a compact and connected Hausdorff space \( X \), where the \( T \)-invariant probability measure \( \mu \) is topological, i.e. positive on open sets. Connectedness of \( X \) ensures that minimality of \( T \) implies minimality of all its iterates (‘total minimality’) [9, 23, 36], which enters in the proof of the ‘\( \subseteq \)’-statement of theorem 1.1 for singular Jacobi operators (see (5.5) in the proof of proposition 5.1). In particular, theorem 1.1 holds for all almost periodic Jacobi operators, i.e. operators of the form (1.1) where \( X \) is a compact topological group, \( \mu \) is its Haar probability measure, and \( T \) is translation by a fixed element in \( X \) whose orbit is dense.

(iii) We mention that Johnson’s original result extends to Schrödinger operators with background dynamics given by a minimal transformation \( T \) on a compact space \( X \). Our proof of theorem 1.1 requires unique ergodicity of \( T \) (due to proposition 4.1), we however believe that the result should also hold true for the case of merely minimal \( T \).

From a dynamical point of view the most interesting aspect of theorem 1.1 is that it implies domination outside the spectrum. In particular, complexifying the energy generates \( DS \) with all the ‘nice’ properties such dynamics entails. This leads to a dynamical point of view of Kotani theory, which expressed from the point of view of theorem 1.1, studies the limiting properties of the invariant sections as \( \text{Im} E \to 0^+ \). We mention that such a dynamical reformulation of Kotani theory has played an important roll in a dynamical description of the absolutely continuous spectrum of QPS [4–6], see also [7] for an even more general perspective (‘monotonic cocycles’).

We structure the paper as follows. Section 2 briefly recalls the relation of the Jacobi cocycles to the solutions of the finite difference equations, \( H_\psi = E\psi \). We will also describe alternative choices for Jacobi cocycles appearing in the literature, for which theorem 1.1 holds in an analogous form (see theorem 2.1).
As mentioned earlier, the main point of this paper is to account for singular Jacobi operators. For non-singular operators, theorem 1.1 could be obtained by simple adaptions of the proof for Schrödinger operators, which we present in section 3.

Section 4 forms the main part of the paper, and is devoted to proving $DS$ outside the spectrum. The latter is done by verifying a cone condition. We outline the strategy in section 4.1, the proof is carried out in section 4.2. One noteworthy aspect of our proof is that it explicitly reveals the spectral theoretic meaning of the dynamical quantities involved. The invariant sections of the splitting are shown to be given in terms of the Weyl $m$-functions ($m_{\pm}$), with $m_{-}$ giving rise to the dominating section, see (4.7). The key estimate, which verifies the cone condition, is obtained in proposition 4.1. It shows that the derivative of iterates of the Jacobi cocycle along the dominating section decays exponentially in the number of iterates, with a decay rate given by the Lyapunov exponent of the QPJ. Moreover, the angle between the invariant sections of the splitting is shown to be determined by the inverse of the Green’s function of the QPJ, see (4.33)–(4.34).

We conclude with section 5 where we show that $DS$ cannot occur on the spectrum.

2. Jacobi cocycles

Fixing the spectral parameter $E \in \mathbb{C}$, solving $H_{x}\psi = E\psi$ over $\mathbb{C}^{\mathbb{Z}}$ can be formulated as iteration of the measurable cocycle of $(T, B^{E})$ applied to a given initial condition $(\psi_{0}, \psi_{-1})$ for $\psi$,

$$B_{n}^{E}(x) \begin{pmatrix} \psi_{0} \\ \psi_{-1} \end{pmatrix} = \begin{pmatrix} \psi_{n} \\ \psi_{n-1} \end{pmatrix}, \quad (2.1)$$

where $B^{E}(x) := \frac{1}{c(x)} A^{E}(x)$ and $A^{E}$ is given in (1.4). This iterative procedure is a consequence of the second order difference nature of Jacobi operators. In spectral theory, it is better known as transfer matrix formalism, where the transfer matrix is given by $B^{E}(x)$.

Note that since $\log |c| \in L^{1}(X, d\mu)$, the set $Z(c) := \{ x \in X : c(x) = 0 \}$ is necessarily of $\mu$-measure zero. In particular, positivity of $\mu$ on open sets implies that $(T, B^{E})$ is well defined and invertible on the full measure, and therefore dense $G_{\delta}$-set,

$$X_{0} := X \setminus \left( \bigcup_{n \in \mathbb{Z}} T^{n} Z(c) \right). \quad (2.2)$$

As $B^{E}(x)$ is only defined for $\mu$-a.e. $x$, it is often more convenient to work with $(T, A^{E})$, which inherits the continuity of the sampling functions $c, v$. We reiterate that presence of zeros in $c(x)$ translates to singularity of $(T, A^{E})$.

An alternative choice for the transfer matrix $B^{E}$ is given by

$$\tilde{B}^{E}(x) = \frac{1}{c(T^{-1}x)} \begin{pmatrix} E - v(x) & -|c(T^{-1}x)|^2 \\ 1 & 0 \end{pmatrix}, \quad (2.3)$$

which induces a measurable cocycle satisfying

$$| \det \tilde{B}^{E}(x) | = 1, \quad \mu\text{-a.e.} \quad (2.4)$$

We mention that for $E \in \mathbb{R}$, (2.3) even induces a complex symplectic, measurable cocycle.

Thus, for non-singular QPJ where $(T, \tilde{B}^{E}(x))$ is continuous, dynamical considerations reduce directly to the more familiar notion of uniform hyperbolicity. The latter is explored in section 3.

1 One can weaken the definition of a (continuous) cocycle, requiring the matrix valued function $D : X \rightarrow M_{2}(\mathbb{C})$ to only be measurable with $\log_{+} \|D(.\|) \in L^{1}(X, d\mu)$, in which case $(T, D)$ is called a measurable cocycle.

2 In view of the extension of theorem 1.1 to general uniquely ergodic systems as discussed in remark 1.2(i), density of $X_{0}$ forms the reason for requiring $\mu$ to be strictly positive on non-empty open sets of $X$. 

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The definition of \( \tilde{B}(x) \) is suggested by the ‘scaled’ discrete Laplacian in (1.1) [13, 24]; more specifically, \( \psi \in C^Z \) satisfies \( H \psi = E \psi \) if and only if
\[
\tilde{B} \psi_n(x) = \psi_{n-1} \left( c(T^{-1}x) \psi_0 - |c(T^{-1}x)|^2 \right),
\]
(2.5)
We mention that \( \tilde{B}(x) \) is particularly natural in view of the Weyl m-function \( m(x, E) \), see (4.8)³.

For singular QPJ, however, as \( \tilde{B}(x) \) is only defined for \( \mu \)-a.e. \( x \), one introduces in analogy to \( A^E \) above,
\[
\tilde{A}^E(x) := \begin{pmatrix} E - v(x) & -|c(T^{-1}x)|^2 \\ 1 & 0 \end{pmatrix},
\]
(2.6)
which induces the (continuous) cocycle derived from \( (T, \tilde{B}) \). \( (T, \tilde{A}) \) thus yields an alternative Jacobi cocycle, which, like \( (T, A^E) \), is singular precisely if \( c \) exhibits zeros.

Referring to (2.5), observe that the dynamics of the cocycles \( (T, A^E) \) and \( (T, \tilde{A}) \) is related by the measurable conjugacy⁴,
\[
M(Tx)^{-1} \tilde{A}^E(x) M(x) = A^E(x), \quad M(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
(2.7)
which in particular yields equality of the top Lyapunov exponents,
\[
L(T, A^E) = L(T, \tilde{A}), \quad L(T, B^E) = L(T, A^E) - \int \log |c|d\mu = L(T, \tilde{B}).
\]
(2.8)
We mention that in spectral theory, \( L(T, B^E) = L(T, \tilde{B}) \) is usually called the Lyapunov exponent of the QPJ.

For non-singular Jacobi operators, (2.7) becomes a continuous conjugacy, whence \( (T, A^E) \in DS \) if and only if \( (T, \tilde{A}) \in DS \).

In conclusion, we account for the flexibility in the choice of cocycles associated with the spectral theory of QPJ, formulating our main result also for \( (T, \tilde{A}) \).

**Theorem 2.1 (Theorem 1.1 amended).** Under the same hypotheses as in theorem 1.1, one has
\[
\Sigma = \{ E : (T, A^E) \notin DS \}.
\]
(2.9)
The statement also holds when replacing \( (T, A^E) \) by \( (T, \tilde{A}) \).

### 3. Non-singular Jacobi operators

As mentioned earlier, the spectral theory for non-singular Jacobi operators can be described by continuous cocycles with unimodular determinant similar to the Schrödinger case. In particular, since \( (T, A^E) \) are \( (T, \tilde{A}) \) are continuously conjugate, theorem 1.1 is equivalently formulated as
\[
\mathbb{C} \setminus \Sigma = \{ E : (T, \tilde{B}) \notin UH \}.
\]
(3.1)

We mention that the transfer matrix proposed in [13, 24] is in fact adapted to the Weyl m-function \( m(x, E) \), hence differs from (2.3). In view of proving presence of a dominated splitting, it is, however, more natural to adapt the cocycle to \( m(x, E) \), as the latter will be shown to give rise to the dominating section (see section 4). We mention that all arguments in this note carry over to the cocycles considered in [13, 24], in particular theorem 1.1 also applies to those cocycles.

⁴ As usual, a measurable conjugacy is a conjugacy between measurable cocycles where the mediating coordinate change in (2.7) is measurable with \( \log \| M(.) \|, \log \| M(.)^{-1} \| \in L^1(X, d\mu) \). The latter condition guarantees preservation of the LE. Note that log-integrability of the coordinate change under consideration in (2.7) follows from \( \log |c| \in L^1(X, d\mu) \).
Thus, theorem 1.1 can be concluded from arguments along the lines of [21]. We briefly outline these straightforward adaptations.

To prove the ‘⊆’ statement in (3.1), let $E \in \mathbb{C} \setminus \Sigma$ be given. It is well known that [26–28, 30] (see also, [37] for a more recent proof) a (continuous) cocycle $(T, D)$ is not $U\mathcal{H}$ if and only if for some $x_0 \in X$ and $v \in \mathbb{C}^2 \setminus \{0\}$,

$$\sup_{n \in \mathbb{Z}} \|D_n(x_0)v\| < \infty.$$  

(3.2)

Thus, using (2.5), $(T, \tilde{B}E) \notin U\mathcal{H}$ would imply that for some $x_0 \in X$, $H_{x_0} \psi = E \psi$ admits a bounded, non-trivial solution over $\mathbb{C}^2$, whence (see also 5.9) $E \in \Sigma$—a contradiction.

To prove the ‘⊇’ statement in (3.1), let $E$ such that $(T, \tilde{B}E) \in U\mathcal{H}$. Then, all non-trivial solutions of $H_{x} \psi = E \psi$ over $\mathbb{C}^2$ increase exponentially on at least one of $\mathbb{Z}^\pm$. Thus, the Sch’nol–Berezanskii theorem [10, 29] (see (5.9)) and openness of $U\mathcal{H}$ in the (continuous) cocycles with unimodular determinant implies $E \in \mathbb{C} \setminus \Sigma$ (see section 5).

4. Domination outside the spectrum

In this section we prove the ‘⊇’-statement in (2.9). We start with some remarks on dominated splittings.

4.1. Dominated splittings and cone conditions

For $1 \leq j \leq 2$, let $e_j$ denote the standard basis of $\mathbb{C}^2$ and set $E_j := \text{Span}\{e_j\}$. Taking advantage of the manifold structure of $\mathbb{PC}_2$ induced by the charts, $\phi_1 : \mathbb{PC}_2 \setminus E_1 \to \mathbb{C}$, $\phi_1(\text{Span}\{v_1\}) = \frac{v_1}{v_2}$,

$$\phi_2 : \mathbb{PC}_2 \setminus E_2 \to \mathbb{C}, \phi_2(\text{Span}\{v_1\}) = \frac{v_2}{v_1},$$

(4.1)

in local coordinates a given matrix $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ acts on $\mathbb{PC}_2 \setminus \ker D$ as a linear fractional transformation.

We will denote the coordinate-free action of $D$ on $\mathbb{PC}_2 \setminus \ker D$ by $D \cdot z$ and its derivative, respectively, by $\partial D \cdot z$. Moreover, we will find it convenient to identify $\mathbb{PC}_2$ with $\mathbb{C}$ extending $\phi_2$ in (4.1) to all of $\mathbb{PC}_2$.

First, observe that a cocycle $(T, D) \in \mathcal{D}\mathcal{S}$ if and only if some iterate is continuously conjugate to a diagonal cocycle, i.e. there exists $N \in \mathbb{N}$ and a coordinate change $M \in \mathcal{C}(X, GL(2, \mathbb{C}))$ such that

$$M(T^N x)^{-1} D_N(x) M(x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix},$$

(4.2)

where $\lambda_j \in \mathcal{C}(X, \mathbb{C})$, $1 \leq j \leq 2$, satisfy

$$|\lambda_1(x)| > |\lambda_2(x)|, \quad \forall x \in X.$$  

(4.3)

In particular, a necessary condition for $(T, D) \in \mathcal{D}\mathcal{S}$ is a non-degenerate Lyapunov spectrum, i.e.

$$L(T, D) > \frac{1}{2} \int \log |\det D(x)| d\mu(x).$$

(4.4)

A well-known technique to detect $\mathcal{D}\mathcal{S}$ is to verify a cone condition: given an $M_2(\mathbb{C})$-cocycle $(T, D)$, a conefield for $(T, D)$ is an open subset $U \subset X \times \mathbb{PC}_2$ of the form $\cup_{x \in X} \{x\} \times U_x$.
such that, for all $x \in X$, $\mathcal{U}_x$ is non-empty, properly contained in $\mathbb{P} \mathbb{C}^2$, and $\mathcal{U}_x \cap \ker D(x) = \emptyset$.

A cone field $U = \bigcup_{x \in X} \{x\} \times \mathcal{U}_x$ for $(T, D)$ is said to satisfy a cone condition if there exists $N \in \mathbb{N}$ such that for every $x \in X$, one can show that $D_N(x) \cdot \mathcal{U}_x \subseteq U_{T^N}$. It is known (see, e.g. [3], or [8] for singular matrix cocycles) that verifying a cone condition implies $DS$. In particular, note that if $s_1(x)$ denotes the dominating section for $(T, D)$ and $N \in \mathbb{N}$ is as in the cone condition, then one necessarily has

$$\sup_{x \in X} |\delta D_N(x) \cdot s_1(x)| < 1.$$  \hfill (4.5)

Conversely, suppose for some $N \in \mathbb{N}$ a given invariant section $s(.) \in C(X, \overline{\mathbb{C}})$, transverse to $\ker D$, satisfies the uniform contraction condition (4.5), then $(T, D)$ admits a cone field satisfying a cone condition, whence $(T, D) \in DS$ with $s(x)$ determining the dominating section.

4.2. Proof of theorem 2.1, "$\Sigma$"-statement

Set $\rho := \mathbb{C} \setminus \Sigma$. Throughout this section, let $E \in \rho$ be fixed. We will prove the "$\Sigma$"-statement in (2.9) by showing that if $E \in \rho$, then both $(T, A^\ell)$ and $(T, \tilde{A}^\ell)$ admit an invariant section such that (4.5) holds for some $N \in \mathbb{N}$. As discussed in the end of section 4.1, the latter implies that both Jacobi cocycles induce a $DS$. Here, as we will argue, the dominating, invariant section is provided from the spectral theory of Jacobi operators.

To this end, let $m_{\pm}(x, E)$ denote the standard Weyl $m$-functions [35],

$$m_{\pm}(x, E) := \langle \delta_{\pm 1}, (H_{x, \pm} - E)^{-1} \delta_{\pm 1} \rangle,$$ \hfill (4.6)

defined in terms of the positive (negative) half-line operator $H_{x, \pm} := P_{\pm} H_x P_{\pm}$, where $P_{\pm}$ is the orthogonal projection onto the closure of $\text{Span}\{\delta_n, n \in \mathbb{Z}_{\pm}\}$ and $\delta_n, n \in \mathbb{Z}$, are the elements of the standard basis of $F^2(\mathbb{Z})$, i.e. $[\delta_n]_m = \delta_{n,m}$.

In view of $(T, A^\ell)$, we consider

$$s_-(x, E) := -c(T^{-1}x) m_-(x, E),$$
$$s_+(x, E) := -[c(T^{-1}x)m_+(T^{-1}x, E)]^{-1},$$ \hfill (4.7)

and for $(T, \tilde{A}^\ell)$, respectively,

$$\tilde{s}_-(x, E) := -m_-(x, E),$$
$$\tilde{s}_+(x, E) := -[|c(T^{-1}x)|^2m_+(T^{-1}x, E)]^{-1}.$$ \hfill (4.8)

Clearly, for all $E \in \mathbb{C} \setminus \mathbb{R}$,

$$0 < |m_{\pm}(x, E)| \leq \frac{1}{|\text{Im} E|}.$$ \hfill (4.9)

thus $s_{\pm}(x, E), \tilde{s}_{\pm}(x, E)$ are well defined with values in $\overline{\mathbb{C}}$.

To see that (4.7) and (4.8) are actually well defined for all $E \in \rho$ (i.e. undefined expressions of the form '0 $\times \infty$' do not occur), first observe that $H_x$ and $H_{x, -} \oplus H_{x, +}$ only differ by a finite rank perturbation, hence their essential spectra must agree. In particular, any real $E$ in the resolvent set of $H_x$ is either in the resolvent sets of both $H_{x, \pm}$, i.e. $m_{\pm}(x, E) \in \mathbb{C}^5$, or in the discrete spectrum$^6$ of at least one of $H_{x, \pm}$, i.e. $m_{\pm}(x, E) = \infty$ corresponding to a pole at $E$. Thus, undefined expressions in (4.7)–(4.8) of the form '0 $\times \infty$' are excluded: indeed,

$^5$ In contrast to $H_x$, the resolvent sets for $H_{x, \pm}$ may in general depend on $x$.

$^6$ As usual, the discrete spectrum of a bounded operator is defined as the set of isolated eigenvalues of finite multiplicity; the remaining elements of the spectrum define the essential spectrum.
as \( c(T^{-1}x) = 0 \) implies \( H_x = H_{x,+} \oplus H_{T^{-1}x,+} \), any \( E \in \sigma_{\text{disc}}(H_{x,+}) \cup \sigma_{\text{disc}}(H_{T^{-1}x,+}) \) would automatically be an eigenvalue of \( H_x \), thereby violating our assumption that \( E \in \rho \).

We claim

**Lemma 4.1.**

\[
s \Delta (\cdot, \cdot), \tilde{s} \Delta (\cdot, \cdot) \in C(X \times \rho, \overline{\mathbb{C}}). \tag{4.10}
\]

**Proof.** It suffices to show joint continuity of \( m_{\pm}(x, E) \). Let \((x_0, E_0) \in X \times \rho \) be fixed and arbitrary. As outlined above, there are two possible situations to consider.

If \( E_0 \) is in the resolvent set of \( H_{x,0} \), basic resolvent estimates (see e.g. theorem 3.15 in [22]) imply that \((H_{x,0} - E)^{-1}\) exists and is jointly continuous in \((x, E)\) (w.r.t operator norm for the \( x \)-dependence) in some open neighbourhood of \((x_0, E_0)\); in particular, \( m_{\pm}(x, E) \) is jointly continuous at \((x_0, E_0)\).

Continuity for the case that \( E_0 \) is in the discrete spectrum of \( H_{x,0} \) follows from well-known facts on the continuity of a finite system of isolated eigenvalues of finite multiplicity for a norm-continuous family of bounded operators (see e.g. section IV.5 in [22]): specifically, let

\[
P_{\sigma(H_{x,0})}(x_0) := \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(E_0)} (H_{x,0} - z)^{-1} \, dz,
\]

be the spectral projection of \( H_{x,0} \) onto \( \{E_0\} \); where \( r > 0 \) is such that \( \mathcal{B}_r(E_0) \cap \sigma(H_{x,0}) = \{E_0\} \).

Then, from section IV.5 in [22] (see also [31], theorem 2.3.8), it is known that there exists \( \delta > 0 \) such that for all \( x \) with \(|x - x_0| < \delta\), one has

(i) \( \sigma(H_{x,0}) \) can be partitioned into the compact sets \( \sigma_1(H_{x,0}) \) and \( \sigma_2(H_{x,0}) \) such that \( \sigma_1(H_{x,0}) \subseteq B_r(E_0) \) and \( \sigma_2(H_{x,0}) \subseteq \mathbb{C} \setminus \overline{\mathcal{B}_r(E_0)} \).

(ii) The spectral projections

\[
P_{\sigma_1(H_{x,0})}(x) := \frac{1}{2\pi i} \int_{\partial \mathcal{B}_r(E_0)} (H_{x,0} - z)^{-1} \, dz,
\]

depend continuously (in norm topology) on \( x \) and are unitarily equivalent to \( P_{\sigma_1(H_{x,0})}(x_0) \).

In particular, the spectrum of \( H_{x,0} \) inside \( B_r(E_0) \) is discrete with dimension given by \( 1 \leq \dim \text{Ran} P_{\sigma_1(H_{x,0})}(x_0) < \infty \).

Thus, using (i)–(ii), for \(|x - x_0| < \delta\), decompose

\[
H_{x,0} = H_{x,0}^{(1)} \oplus H_{x,0}^{(2)},
\]

where \( H_{x,0}^{(1)} := H_{x,0} \cap \sigma_1(H_{x,0}) \) and \( H_{x,0}^{(2)} := H_{x,0} \setminus \sigma_2(H_{x,0}) \). Then, all \( E \in B_r(E_0) \) are in the resolvent set of \( H_{x,0}^{(2)} \), in particular, as above, \((H_{x,0}^{(2)} - E)^{-1}\) is jointly continuous in \((x, E) \in (x_0 - \delta, x_0 + \delta) \times B_r(E_0) \).

Therefore, joint continuity of \( m_{\pm}(x, E) \) at \((x_0, E_0)\) as \( \overline{\mathbb{C}} \)-valued functions simply reduces to the continuity of the \( \overline{\mathbb{C}} \)-valued functions,

\[
(\delta_{\pm}, (H_{x,0}^{(2)} - E)^{-1} \delta_{\pm}), \quad \text{for} \ E \in B_r(E_0), \ |x - x_0| < \delta,
\]

associated with the finite dimensional operators \( H_{x,0}^{(2)} \). Here, \( (4.14) \) has poles at the eigenvalues of \( H_{x,0}^{(2)} \) for \( E \in B_r(E_0) \). \( \square \)

Recall, that the definition of the full-measure set \( X_0 \subseteq X \) given in (2.2), guarantees that \( c(T^k x) \neq 0, \forall k \in \mathbb{Z} \), whenever \( x \in X_0 \). For all \( x \in X_0 \), this in turn implies existence of
solutions \( \psi_{\pm}(x, E) \) of \( H_x \psi = E \psi \) over \( \mathbb{C}^2 \) which are never zero, are \( l^2 \) at \( \pm \infty \), and from (2.1), respectively (2.3), one has

\[
s_{\pm}(x, E) = \frac{\psi_{\pm}(-1, x, E)}{\psi_{\pm}(0, x, E)}, \quad \tilde{s}_{\pm}(x, E) = \frac{\psi_{\pm}(-1, x, E)}{c(T^{-1}x)\psi_{\pm}(0, x, E)}, \quad x \in X_0, \tag{4.15}
\]

and

\[
-m_{\pm}(Tx, E)^{-1} = \begin{cases} (E - v(x)) - c(T^{-1}x)s_{\pm}(x, E), \\ (E - v(x)) - |c(T^{-1}x)|^2\tilde{s}_{\pm}(x, E). \end{cases} \tag{4.16}
\]

Moreover, uniqueness of the fundamental solutions \( \psi_{\pm}(x, E) \) up to scalar multiples implies that for all \( x \in X_0 \),

\[
A^E(x) \cdot s_{\pm}(x, E) = s_{\pm}(Tx, E), \quad \tilde{A}^E(x) \cdot \tilde{s}_{\pm}(x, E) = \tilde{s}_{\pm}(Tx, E). \tag{4.17}
\]

We emphasize that outside \( X_0 \) above fundamental solutions \( \psi_{\pm}(x, E) \) do not exist; indeed, \( c(T^kx) = 0 \) for some \( k \in \mathbb{Z} \) implies decoupling of \( H \), whence any solution \( \psi \) of \( H \psi = E \psi \) which is \( l^2 \) at \( \pm \infty \) has to vanish identically in a neighbourhood of \( \pm \infty \) otherwise \( E \) is an eigenvalue of \( H \).

In order to extend (4.16) and the invariance relations in (4.17) to all of \( X \), we use the following continuity arguments: since \( X_0 \) is dense in \( X \), continuity of both sides of (4.16) as \( \mathbb{C} \)-valued functions implies that (4.16) extends to all of \( X \). Moreover, one has the following.

**Lemma 4.2.** For all \( E \in \rho \) and all \( x \in X \), \( s_{\pm}(x, E) \) is transverse to \( \text{ker} \, A^E(x) \), similarly \( \tilde{s}_{\pm}(x, E) \) is transverse to \( \text{ker} \, \tilde{A}^E(x) \). Moreover, for all \( x \in X \),

\[
A^E(x) \cdot s_{\pm}(x, E) = s_{\pm}(Tx, E), \quad \tilde{A}^E(x) \cdot \tilde{s}_{\pm}(x, E) = \tilde{s}_{\pm}(Tx, E). \tag{4.18}
\]

**Proof.** We will focus on \( (T, A^E) \), the argument for \( (T, \tilde{A}^E) \) being similar.

First observe that for \( E \in \rho \), \( A^E(x) \neq 0 \) since \( c(x) = c(T^{-1}x) = 0 \) would require \( E - v(x) \neq 0 \) for \( E \) to be in the resolvent set of \( H_x \). In particular, \( \text{dim} \, A^E(x) \leq 1 \).

Based on (1.4), \( \ker \, A^E(x) \) is non-trivial if one of the following two situations applies.

If \( c(T^{-1}x) = 0 \), \( s_{\pm}(x, E) = 0 \) which is automatically transverse to \( \ker \, A^E(x) = E_2 \simeq \mathbb{C} \).

If \( c(x) = 0 \), we may assume \( c(T^{-1}x) \neq 0 \), otherwise consider above. Then, \( s_{\pm}(x, E) \simeq \text{Span}_{\nu_{\pm}(x, E)} = \ker \, A^E(x) \) if and only if

\[
(E - v(x)) - c(T^{-1}x)s_{\pm}(x, E) = 0, \tag{4.19}
\]

which as (4.16) holds on all of \( X \) would imply a pole of \( m_{\pm}(Tx, .) \) to \( E \), or equivalently, \( E \in \sigma_{\text{disc}}(H_{Tx, .}) \). The latter, however, is impossible for any \( E \in \rho \) as \( c(x) = 0 \) yields \( H_x = H_{Tx, .} \oplus H_{x, .} \).

Finally, transversality of \( s_{\pm}(x, E) \) to \( \ker \, A^E(x) \) for all \( x \in X \) implies \( A^E(\cdot) \cdot s_{\pm}(\cdot, E) \) is well defined and continuous with values in \( \mathbb{C} \), whence density of \( X_0 \) allows one to extend (4.17) to all of \( X \). \( \square \)

Based on the discussion in section 4.1, we show that \( (T, A^E), (T, \tilde{A}^E) \in \mathcal{D} \mathcal{S} \) verifying the contraction condition given in (4.5). Recall that by the Combes–Thomas estimate (see, e.g. [35], lemma 2.5, for a formulation for Jacobi operators)

\[
L(T, B^E) \geq \kappa \cdot \text{dist}(E; \Sigma) > 0, \quad E \in \rho, \tag{4.20}
\]

where, uniformly over any compact neighbourhood of \( \Sigma \), \( \kappa > 0 \) can be chosen to only depend on \( \|c\|_{\infty} \). In particular, using (4.4) and (2.8), (4.20) shows that for any \( E \in \rho \) both \( (T, A^E), (T, \tilde{A}^E) \) have a non-degenerate Lyapunov spectrum.

In view of the following, we let

\[
\rho_- := \mathbb{C} \setminus (\Sigma \cup \bigcup_{x \in \sigma_{\text{disc}}(H_{x, .})}). \tag{4.21}
\]

Clearly, \( \mathbb{C} \setminus \mathbb{R} \subseteq \rho_- \), moreover \( E \in \rho_- \) if \( |E| \geq 2\|c\|_{\infty} + \|v\|_{\infty} \).
Proposition 4.1. For any $E \in \rho_-\, one has

$$\sup_{x \in X} |\partial A^E_n(x) \cdot s_-(x, E)| \leq \|c\|_\infty^2 e^{-2(L(T, A^E) + o(1))} e^{-2(n-1)(L(T, B^E) + o(1))},$$  \tag{4.22}$$
as $n \to +\infty$. An analogous estimate holds for $\tilde{A}^E(x)$.

Remark 4.3. The proof also shows that the upper bound in (4.22) is optimal. Thus, as $L(T, A^E) \geq \int \log |c| d\mu, N$ in the definition of $DS$ (see (1.2)) will diverge whenever $L(T, B^E) \to 0$ as $\text{dist}(E; \Sigma) \to 0+$.

Proof. For brevity, we will focus on the cocycle $(T, A^E)$. First observe that for all $x \in X, s_-(x, E), m_-(T, E) \neq \infty$ as $E \in \rho_-$, whence from (4.16) we conclude

$$A^E(x) \cdot z = (\phi_2 \circ A^E \circ \phi_2^{-1})(z) = \frac{c(x)}{(E - v(x)) - c(T^{-1} x) z},$$  \tag{4.23}$$
locally about $z = s_-(x, E)$ for all $x \in X$.

Thus, again using (4.16), we compute

$$\partial A^E(x) \cdot s_-(x, E) = \frac{c(x)c(T^{-1} x)}{((E - v(x)) - c(T^{-1} x) s_-(x, E)) z}.$$  \tag{4.24}$$

From (4.17), (4.24), the chain rule implies

$$\partial A^E_n(x) \cdot s_-(x, E) = \prod_{j=0}^{n-1} \partial A^E(T^j x) \cdot \left( A^E_j(x) \cdot s_-(x, E) \right)$$

$$= \left( \prod_{j=0}^{n-1} c(T^j x) \right) \left( \prod_{j=1}^{n-2} c(T^j x) \right) \left( \prod_{j=1}^{n} m_2(T^j x, E) \right), \quad n \geq 2.$$  \tag{4.26}$$

Relating $m_-(x, E)$ to the fundamental solution $\psi_-(x, E)$, one determines, making use of ergodicity (see, e.g. [35], equation (5.40) therein), that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |m_-(T^j x, E)| = -L(T, A^E), \quad \text{a.e. } x \in X.$$  \tag{4.27}$$

Observe that since $m_-(x, E)$ is continuous with

$$0 \leq |m_-(x, E)| < \infty, \quad E \in \rho_-,$$

unique ergodicity of $T$ implies uniformity of the upper limit in (4.27) [16]\footnote{To obtain (4.28) and (4.29), we use theorem 1 in [16] which guarantees uniform convergence of upper limits in Caesaro means for continuous, sub-additive processes on a compact Hausdorff space $X$ equipped with a uniquely ergodic dynamical system. The proof in [16] carries over without changes to sub-additive processes $\{f_n\}$ which are only upper semi-continuous, thus in particular satisfy $\sup_{x \in X} f_n(x) < \infty$, which is used in the proof of [16]. We mention that recently, Furman\’s result has been extended to even encompass certain discontinuous process [19].}

$$\frac{1}{n} \sum_{j=0}^{n-1} \log |m_-(T^j x, E)| \leq -L(T, A^E) + o(1), \quad \text{uniformly for } x \in X.$$  \tag{4.29}$$
The same argument yields
\[ \frac{1}{n} \sum_{j=0}^{n-1} \log |c(T^j x)| \leq \int \log |c(x)| d\mu(x) + o(1), \] (4.30)
uniformly in \( x \in X \) as \( n \to +\infty \).

Thus, combination of (2.8), (4.29), (4.30), and (4.26) yields (4.22) as claimed. \( \square \)

Thus, letting \( S_\pm(., E) \) be continuous lifts of \( s_\pm(., E) \) to the subspaces of \( \mathbb{C}^2 \), proposition 4.1 and (4.17) implies a DS for all \( E \in \rho_- \), with \( S_-(x, E) \) corresponding to the dominating subspace, in particular
\[ A^E(x)S_-(x, E) = S_-(Tx, E). \] (4.31)
That \( S_+(x, E) \) determines the minorating subspace of the dominated splitting follows by first noting that from (4.17) one has for all \( x \in X_0 \),
\[ A^E(x)S_+(x, E) \subseteq S_+(Tx, E). \] (4.32)
Moreover, one has

Lemma 4.4. For all \( E \in \rho, S_\pm(., E) \) are uniformly transverse.

Proof. For any \( E \in \rho \) and \( x \in X_0 \), standard expressions for Green’s function of \( H_x \) in terms of \( \psi_\pm(x, E) \) yield
\[ \langle \delta_0, (H_x - E)^{-1}\delta_0 \rangle^{-1} = c(x) \begin{pmatrix} \psi_+(1, x, E) - \psi_+(0, x, E) \\ \psi_+(0, x, E) - \psi_+(0, x, E) \end{pmatrix} \]
\[ = c(T^{-1} x) \begin{pmatrix} \psi_-(1, x, E) - \psi_-(0, x, E) \\ \psi_-(0, x, E) - \psi_-(0, x, E) \end{pmatrix}, \] (4.33)
whence
\[ |s_+(x, E) - s_-(x, E)| \geq \frac{\text{dist}(E; \Sigma)}{\|c\|_\infty}, \quad \text{all } x \in X_0. \] (4.34)

By (4.9), \( s_+(x, E), s_-(x, E) \) can never both equal \( \infty \) for \( E \in \mathbb{C} \setminus \mathbb{R} \), whence (4.34) extends to all \( x \in X \) by continuity.

Given real \( E \in \rho \), using (4.34) there exists \( \eta > 0 \) only depending on \( \text{dist}(E; \Sigma) \) such that for all \( \epsilon > 0 \),
\[ \inf_{x \in X} \angle [S_+(x, E + i\epsilon), S_-(x, E + i\epsilon)] \geq \eta, \] (4.35)
which by continuity implies the claimed transversality taking \( \epsilon \to 0^+ \). \( \square \)

Proposition 4.1 was restricted to \( E \in \rho_- \) since this guaranteed boundedness of \( m_-(x, E) \), which was crucial to conclude uniformity of the upper limit in (4.29). Nonetheless, having shown \( (T, A^E), (T, \tilde{A}^E) \in DS \) for all \( E \in \mathbb{C} \setminus \mathbb{R} \), however, already implies the same for all of \( \rho \); we provide an argument for \((T, A^E)\).

From lemma 4.2, (4.31) already holds for all \( E \in \rho \). Moreover, using lemma 4.1, \( S_\pm(., E) \) extends continuously to all \( E \in \rho \), whence (4.32) holds for all \( x \in X \). In summary, taking \( M(x) \in M_2(\mathbb{C}) \) with the first (second) column vector in the direction of \( S_-(x, E) \) \( \langle S_+(x, E), M(x) \rangle \), lemma 4.4 implies that \( M(x) \in GL(2, \mathbb{C}) \) and
\[ M(Tx)^{-1} A^E(x) M(x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{pmatrix}, \] (4.36)
where $\lambda_j \in C(X, \mathbb{C})$, $1 \leq j \leq 2$. Since $S_-(x, E)$ is transversal from $\ker A^E(x)$, one has $\lambda_1(x) \neq 0$, thus by continuity of $\lambda_1$,

$$\inf_{x \in X} |\lambda_1(x)| > 0. \quad (4.37)$$

Finally, non-triviality of the Lyapunov spectrum of $(T, A^E)$ for all $E \in \mathbb{C} \setminus \Sigma$, the equations (4.36)–(4.37), and unique ergodicity of $T$ implies that for some $N \in \mathbb{N}$

$$\prod_{j=0}^{N} \lambda_1(T^jx) > \prod_{j=0}^{N} \lambda_2(T^jx), \quad \forall x \in X, \quad (4.38)$$

whence $(T, A^E) \in DS$ (see (4.2)–(4.3)).

5. Finishing up . . .

To complete the proof of theorem 2.1, it is left to show that $DS$ of $(T, A^E)$ or $(T, \tilde{A}^E)$ cannot occur on the spectrum.

Suppose, for some $E \in \mathbb{C}$ one has $(T, A^E) \in DS$, correspondingly giving rise to a conjugacy of the form (4.2), in particular

$$\det A^E_N(x) = \frac{\det M(T^Nx)}{\det M(x)} \lambda_1(x) \lambda_2(x). \quad (5.1)$$

Similarly, if $E \in \mathbb{C}$ such that $(T, \tilde{A}^E) \in DS$, (5.1) holds true with $A^E$ replaced by $\tilde{A}^E$.

**Proposition 5.1.** Suppose $E \in \mathbb{C}$ is such that $(T, A^E) \in DS$ or $(T, \tilde{A}^E) \in DS$. Then, there is $\gamma > 0$ such that for all $x \in X_0$ and $v \in \mathbb{C}^2 \setminus \{0\}$, there exists a subsequence $(k_l)_{l \in \mathbb{N}}$ of one of $Z^\pm$ satisfying

$$\|B^E_{k_N}(x)v\| \gtrsim e^{k \gamma} \|v\|, \quad \forall l \in \mathbb{N}. \quad (5.2)$$

**Proof.** For $E \in \mathbb{C}$ such that $(T, A^E) \in DS$, consider

$$(A^E)^2 := \frac{A^E(x)}{\sqrt{\det A^E(x)}}. \quad (5.3)$$

i.e. on $X_0 \times \mathbb{C}^2$, $(T, (A^E)^2)$ is well defined and invertible for all iterates $n \in \mathbb{Z}$ with $|\det (A^E)^2(x)| = 1$.

Fix $x \in X_0$. Given $v \in \mathbb{C}^2 \setminus \{0\}$, decompose $v = v_1 + v_2$ with $v_j \in S^{(j)}$. Possibly changing to inverse dynamics, we may assume $v_1$ to be non-zero. From (5.1), one concludes for $k \in \mathbb{N}$

$$\| (A^E_N)^2(x)v_1 \| \gtrsim \left| \frac{\det M(x)}{\det M(T^N x)} \right|^{1/2} \|v_1\| \prod_{j=0}^{k-1} \left| \frac{\lambda_1(T^j x)}{\lambda_2(T^j x)} \right|^{1/2} \gtrsim \|v_1\| e^{k \log \lambda}, \quad (5.4)$$

where $\lambda = \inf_{x \in X} \frac{\lambda_1(x)}{\lambda_2(x)} > 1$. Hence all non-trivial iterates of $(T, (A^E)^2)$ increase exponentially along a subsequence (depending both on $x$ and the initial condition).

To conclude the same for solutions of $H \psi = E \psi$, observe that for $v \in \mathbb{C}^2$ and $k \in \mathbb{N}$ one has

$$\| (A^E_N)^2(x)v \| = \frac{|c(T^{kN-1}x)|^{1/2}}{|c(T^{-1}x)|^{1/2}} \| B^E_{k_N}(x)v \|, \quad (5.5)$$
whence, by minimality of $T^N$ and openness of $X \setminus \mathcal{Z}(c \circ T^{-1})$, selecting a subsequence such that the scalar factor on the right-hand side of (5.5) stays bounded away from zero, we obtain the claim in (5.2).

The case when $(T, \tilde{A}^E) \in DS$ follows along the same line, noticing that for $x \in X_0$ and $v \in \mathbb{C}^2 \setminus \{0\}$ one has

$$\| (\tilde{A}^E_n)^{(x)} v \| = \| \tilde{B}^E_n (x)^{(x)} v \|, \quad n \in \mathbb{Z},$$

which shows in analogy to (5.4) that

$$\| \tilde{B}^E_{kN} (x)^{(x)} v \| \gtrsim e^{k\gamma} \| v \|,$$

for some $\gamma > 0$. Thus, for $x \in X_0$ and $v \in \mathbb{C}^2 \setminus \{0\}$, the conjugacy in (2.7) implies

$$e^{k\gamma} \| M(x)^{(x)} v \| \lesssim \| \tilde{B}^E_{kN}(x)^{(x)} M(x)^{(x)} v \| \leq \| M \|_{\infty} \| \frac{e(c(T^{N-1} - 1) x)}{|c(T^{-1} x)|} \| B^E_{kN}(x)^{(x)} v \|,$$

which, since $\| M(x)^{(x)} v \| \geq \frac{1}{\| M \|_{\infty} \| \frac{e(c(T^{N-1} - 1) x)}{|c(T^{-1} x)|} \|} \| v \|$, yields (5.2) by selecting, as before, a subsequence so that the rightmost side of (5.8) stays bounded away from zero. □

For fixed $x \in X$, denote by $\mathcal{E}_g(H_x)$ the set of generalized eigenvalues of $H_x$, i.e. all $E \in \mathbb{C}$ which admit a non-trivial polynomially bounded solution of $H_x \psi = E \psi$ over $\mathbb{C}^Z$. From the theorem of Sch’nol–Berezanskii [10, 29] it is well known that

$$\mathcal{E}_g(H_x) = \Sigma,$$

for all $x \in X$.

Proposition 5.1 shows that for all $x \in X_0$, all non-trivial solutions to $H_x \psi = E \psi$ over $\mathbb{C}^Z$ are not polynomially bounded, whence $E \not \in \mathcal{E}_g(H_x)$. But then, since $DS$ is an open property in $C(X, M_2(\mathbb{C}))$, we conclude that for all $x \in X_0$, $E$ cannot be a limit point of $\mathcal{E}_g(H_x)$ either. Thus, in summary, (5.9) implies the ‘⊆’-statement of theorem 1.1. Using proposition 5.1 for $\tilde{A}^E$, the same argument yields the ‘⊇’-statement for the alternative Jacobi cocycle $(T, \tilde{A}^E)$ in theorem 2.1.

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