RESEARCH ARTICLE

Bounds on bilinear forms with Kloosterman sums

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Abstract
We prove new bounds on bilinear forms with Kloosterman sums, complementing and improving a series of results by É. Fouvry, E. Kowalski and Ph. Michel (2014), V. Blomer, É. Fouvry, E. Kowalski, Ph. Michel and D. Miličević (2017), E. Kowalski, Ph. Michel and W. Sawin (2019, 2020) and I. E. Shparlinski (2019). These improvements rely on new estimates for Type II bilinear forms with incomplete Kloosterman sums. We also establish new estimates for bilinear forms with one variable from an arbitrary set by introducing techniques from additive combinatorics over prime fields. Some of these bounds have found a crucial application in the recent work of Wu (2020) on asymptotic formulas for the fourth moments of Dirichlet $L$-functions. As new applications, an estimate for higher moments of averages of Kloosterman sums and the distribution of divisor function in a family of arithmetic progressions are also given.

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1 | INTRODUCTION

1.1 | Backgrounds

This paper concerns bilinear forms of the shape
\[ \sum_{m \in I} \sum_{n \in J} \alpha_m \beta_n \phi(m, n) \]
with \( \phi \) coming from complete or incomplete Kloosterman sums, where \( I, J \) are two intervals given by
\[
I = \{M_0 + 1, \ldots, M_0 + M\}, \quad J = \{N_0 + 1, \ldots, N_0 + N\} \subseteq \mathbb{Z}
\] (1.1)
for \( M, N \geq 1 \), and \( \alpha = \{\alpha_m\}, \beta = \{\beta_n\} \) are arbitrary complex coefficients supported on \( I \) and \( J \), respectively. According to that \( \beta \) is the indicator function of \( J \) or not, we call the above bilinear forms Type I or Type II sums, respectively.

As usual we define the Kloosterman sum
\[
\mathcal{K}_q(m, n) = \sum_{x \in \mathbb{Z}_q^*} e_q(mx + nx^{-1})
\]
for \( q \in \mathbb{Z}^+ \) and \( m, n \in \mathbb{Z} \), where \( \mathbb{Z}_q^* \) denotes the group of units in the residue ring \( \mathbb{Z}_q \) modulo \( q \), \( x^{-1} \) is the multiplicative inverse of \( x \) in \( \mathbb{Z}_q^* \) and \( e_q(z) = \exp(2\pi iz/q) \). Put
\[
S_{q,a}(\alpha, \beta) = \sum_{m \in I} \sum_{n \in J} \alpha_m \beta_n \mathcal{K}_q(m, an),
\]
where \( a \in \mathbb{Z} \), and we also write \( S_{q,a}(\alpha) \) for abbreviation if \( \beta \) is taken as the indicator function of \( J \). Moreover, we write \( S_q(\alpha, \beta) \) and \( S_q(\alpha) \), respectively, in the case \( a = 1 \).

The celebrated Weil bound (see [27, Corollary 11.12]) gives
\[
\mathcal{K}_q(m, n) \ll \gcd(m, n, q)^{1/2} q^{1/2+o(1)},
\] (1.2)
from which it follows, in the particular case \( |\alpha_m|, |\beta_n| \leq 1 \), for instance, that
\[
S_{q,a}(\alpha, \beta) \ll M N q^{1/2+o(1)}
\] (1.3)
uniformly over \( a \), see Section 1.3 for the meaning of \( A \ll B \). We refer this as the trivial bound and our focus is to go beyond this boundary as a fundamental problem of independent interest.

On the other hand, the study of \( S_{q,a}(\alpha, \beta) \), as well as diverse variations, is highly motivated by applications to analytic number theory, including, for example:

- Asymptotic formulas for moments of \( L \)-functions in families; see, for instance, a series of papers by Blomer, Fouvry, Kowalski, Michel, Miličević and Sawin [7, 8, 18, 34], Shparlinski [46] and Wu [49]. In particular, some bounds of this work are used in the work of Wu [49], which gives the currently best bound on the error term in such formulas, see Section 3.1 for more details.
• Results on the equidistribution of divisor functions in arithmetic progressions, see Kerr and Shparlinski [31], Liu, Shparlinski and Zhang [36], Wu and Xi [48] and Xi [51], for instance, see Section 3.3 for new applications.

• Sums of Kloosterman sums with arithmetic weights such as the von Mangoldt function \( \Lambda(n) \), Möbius function \( \mu(n) \) and divisor function \( \tau(n) \), that is,

\[
\sum_{n \leq N} w(n) K_q(n, a),
\]

where \( w = \Lambda, \mu, \tau \). See, for example, Fouvry, Kowalski and Michel [18], Korolev and Shparlinski [33], Kowalski, Michel and Sawin [35] and Liu, Shparlinski and Zhang [38].

A large amount of investigations into \( S_{q,a}(\alpha, \beta) \) are deeply influenced by the work of Deligne [14] on the Riemann Hypothesis for algebraic varieties over finite fields, as well as the subsequent developments thanks to Laumon, Katz, et al. In particular, Kowalski, Michel and Sawin [34, 35] introduced the 'shift by \( ab \)' trick of Vinogradov, Karatsuba and Friedlander–Iwaniec, and transformed \( S_{q,a}(\alpha, \beta) \) with prime \( q \), as well as its extensions, to a certain sum of products of Kloosterman sums with suitable shifts. The task then reduces to proving that the resultant function comes from some \( \ell \)-adic sheaves satisfying reasonable purity and irreducibility conditions. The novelty in [34, 35] allows one to go beyond the so-called Pólya–Vinogradov range when \( M, N = q^{1/2+o(1)} \), which turns out to be very crucial in many applications (see [9, 34] on the analytic theory of \( GL_2 \) \( L \)-functions).

Note that the use of Deligne’s work forces one to deal with prime moduli \( q \), or square-free moduli with some extra effort. The input from \( \ell \)-adic cohomology also applies to bilinear forms with a very wide class of kernel functions \( \phi \), which are not necessarily Kloosterman sums. This generality admits the diversity of applications of such bilinear forms. On the other hand, Shparlinski [46] developed an alternative device which suits very well for bilinear forms with complete and incomplete Kloosterman sums. The argument therein utilises the exact shape of Kloosterman sums, is completely elementary and thus works for arbitrary composite moduli.

In this paper, we enhance the argument of Shparlinski [46] by developing elementary point counting methods to obtain effective control on the counting function \( J_q(a, K) \) defined by (5.1). We also introduce input from additive combinatorics which allows new estimates for bilinear forms where one variable may come from an arbitrary set. A consequence of such estimates is a new bound for higher moments of Kloosterman sums as given in Theorem 3.1 below. However, these results only apply to prime moduli.

In the background of our results is a series of new bounds on Type II bilinear forms with incomplete Kloosterman sums, see Section 2.2. Such sums are of independent interest as they appear in a number of applications which include distributions, in earlier approaches by Friedlander and Iwaniec [21] and Heath-Brown [24], of the ternary divisor function

\[
\tau_3(n) = \# \{(n_1, n_2, n_3) \in \mathbb{N}^3 : n_1 n_2 n_3 = n \}
\]

in arithmetic progressions, and a new level of distribution could be guaranteed by some more stronger estimates for bilinear forms with incomplete Kloosterman sums.

One of the most important applications of our results can be found in the recent work of Wu [49], which gives new asymptotic formulas on the fourth moments of Dirichlet \( L \)-functions,
improving and generalising those of Young [52] and then by Blomer, Fouvry, Kowalski, Michel and Milićević [7, 8]. We give more details in Section 3.1.

1.2 | Previous results

We now collect previous bounds for $S_q(\alpha)$ and $S_q(\alpha, \beta)$, which can be compared with our new bounds in Section 2. This aids in applications which require selecting the strongest bound in various ranges of parameters. We note that for many applications, the range $M, N \sim q^{1/2}$ is critical and this is where our new bounds improve on all previous results.

We also refer to Section 2.4 on some comment concerning on our approach and its on new features. In Section 2.5, we outline further possible generalisations and applications of our ideas.

To ease the comparisons, we now assume that the coefficients $\alpha, \beta$ are both bounded and $\alpha = 1$.

- Besides the trivial bound (1.3), we also have

$$S_q(\alpha) \ll M q^{1+o(1)}$$

(1.4)

by applying Poisson summation or equivalently invoking the Pólya–Vinogradov method. This classical approach also yields

$$S_q(\alpha, \beta) \ll M N q^{1/2+o(1)} \left( q^{1/4} M^{-1/2} + q^{-1/4} + N^{-1/2} \right)$$

(1.5)

for the Type II sums, see [18, Theorem 1.17], which is non-trivial as long as $M > q^{1/2+\varepsilon}$ and $N > q^{\varepsilon}$ for any $\varepsilon > 0$. Improving the exponent $1/2$ is critical for most interesting applications.

- Kowalski, Michel and Sawin [34, 35] employed the ‘shift by $ab$’ trick to deal with both of $S_p(\alpha)$ and $S_p(\alpha, \beta)$. Driven by $\ell$-adic cohomology, the arguments in [34, 35] work very well with all (hyper-)Kloosterman sums and general hyper-geometric sums without necessary obstructions. In the special case of Kloosterman sums, it is proven in [34] that

$$S_p(\alpha) \ll M N \frac{p^{3/2+o(1)}}{M^2 N^5}$$

and

$$S_p(\alpha, \beta) \ll M N \frac{p^{1/2+o(1)}}{M^{-1/2} + (MN)^{-3/16} p^{11/64}}$$

with some mild restrictions on the sizes of $M$ and $N$. In particular, they succeed in obtaining non-trivial bounds in the Pólya–Vinogradov range when $M, N = p^{1/2+o(1)}$. Similar to [35], some variants of $S_p(\alpha)$ and $S_p(\alpha, \beta)$ with an interval $I$ replaced by an arbitrary set $M$ have been estimated in [3] and [2], respectively.

It is worth mentioning that Kowalski, Michel and Sawin [35] are able to improve the trivial bound (1.3) as

$$S_p(\alpha, \beta) \ll M N p^{1/2-\eta}$$
for some $\eta > 0$, provided that $M, N > p^\delta$ and $MN > p^{3/4+\delta}$ for any $\delta > 0$. The novelty here is that they do reach the exponent $3/4$, which is believed to be a classical barrier analogous to the $1/4$ exponent which occurs in Burgess’ bound for short character sums.

- Shparlinski and Zhang [47] proved
  \[
  S_q(1_I) \ll (MN + q)q^{o(1)} \quad (1.6)
  \]
  for all primes $q$, as long as $\alpha = 1_I$ is the indicator function of $I$. This was shortly extended by Blomer, Fouvry, Kowalski, Michel and Milkic’evi´c [8] to non-correlations among Kloosterman sums and Fourier coefficients of modular forms. For an arbitrary $\alpha$, Shparlinski and Zhang [47] proved
  \[
  S_q(\alpha) \ll (MN)^{1/2}q^{1+o(1)} \quad (1.7)
  \]
  for all primes $q$, and since the arguments in [47] are completely elementary, one can easily check that the bounds (1.6) and (1.7) can be identically extended to composite $q$.

- Shparlinski [46] developed an elementary argument to show that
  \[
  S_q(\alpha) \ll M^{3/4} \left( N^{1/8}q + N^{1/2}q^{3/4} \right)q^{o(1)}. \quad (1.8)
  \]
  This saves $q^{1/16}$ against the trivial bound in the Pólya–Vinogradov range $M, N = q^{1/2+o(1)}$, which is also utilised in [46] to produce an asymptotic formula with a very sharp error term for second moments of twisted modular $L$-functions. Note that (1.8) also applies without any changes to more general bilinear forms (2.4).

- Xi [50] developed an iteration process to produce large sieve inequalities of general trace functions. As a special consequence, it is proven that
  \[
  S_q(\alpha, \beta) \ll MNq^{1/2+o(1)} \left( M^{-1} + N^{-1} + q^{-1} + (MN)^{-1}q \right)
  \]
  for all squarefree $q$. This gives square-root cancellations among Type II sums in the ‘complete’ case $M = N = q$.

- In order to obtain applications to $\tau_3$ in arithmetic progressions, Xi [51] combined the Pólya–Vinogradov method with arithmetic exponent pairs developed in [48], and for any squarefree $q$ with all prime factors at most $q^\epsilon$ for any $\epsilon > 0$, it is proven that
  \[
  S_q(\alpha, \beta) \ll MNq^{1/2+o(1)} \left( q^{\kappa/2}M^{(\lambda-\kappa-1)/2} + q^{-1/4} + N^{-1/2} \right),
  \]
  where $(\kappa, \lambda)$ is (essentially) an arithmetic exponent pair defined as in [48]. In particular, the choice $(\kappa, \lambda) = (1/2, 1/2)$ reproduces the above bound (1.5).

We emphasise that most of the above results can also be extended to the sums $S_{q,a}(\alpha, \beta)$ and $S_{q,a}(\alpha)$.

We also note a new approach of Shkredov [43], which applies only to prime moduli $q = p$ but works for very general sums. However, in the case of the sums $S_p(\alpha)$ and $S_p(\alpha, \beta)$, it produces results which so far have been weaker than the best known. Furthermore, we stress that our approach seems to be the only known way to obtain results for composite moduli $q$. 
1.3  Notation and conventions

We adopt the Vinogradov symbol ≪, that is,

\[ f \ll g \iff f = O(g) \iff |f| \leq cg \]

for some absolute constant \( c > 0 \). We also adopt \( o \) notation

\[ f = o(g) \iff |f| \leq \varepsilon g, \]

for any \( \varepsilon > 0 \) and sufficiently large values of parameters. Sometimes, we will combine \( \ll \) and \( o \) notation and allow certain dependence on implied constants. In particular, an expression of the form

\[ f \ll q^{o(1)} g \]

will mean that for all \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that

\[ |f| \leq C_\varepsilon q^\varepsilon g, \]

for sufficiently large values of \( q \) (and some other parameters).

It is convenient to write \( n \sim N \) to indicate \( N < n \leq 2N \).

Throughout the paper, \( p \) always denotes a prime number. For a finite set \( S \), we use \( \# S \) to denote its cardinality.

We also write

\[ e(z) = \exp(2\pi i z) \quad \text{and} \quad e_q(z) = e(z/q) \quad (\text{for } q \in \mathbb{Z}^+). \]

We identify \( \mathbb{Z}_q \) by the set \{0, 1, ..., q - 1\} and \( \mathbb{Z}_q^* \) the subset consisting of all elements coprime to \( q \), and by \{0, 1, ..., p - 1\} the finite field \( \mathbb{F}_p \). Hence, an interval in \( \mathbb{F}_p \) is understood as the set of the form

\[ \{N_0 + 1 \mod p, ..., N_0 + N \mod p\} \subseteq \{0, 1, ..., p - 1\}, \]

that is, the set of residues modulo \( p \) of some sequence of \( N \) consecutive integers \( N_0 + 1, ..., N_0 + N \).

For the complex weight \( \alpha = \{\alpha_m\} \) and \( \sigma \geq 1 \), we define the norms

\[ \|\alpha\|_\infty = \max_m |\alpha_m| \quad \text{and} \quad \|\alpha\|_\sigma = \left( \sum_m |\alpha_m|^\sigma \right)^{1/\sigma}. \]

For \( g \in L^1(\mathbb{R}) \), define the Fourier transform

\[ \widehat{g}(\lambda) = \int_\mathbb{R} g(x) e(-\lambda x) dx. \]

In what follows, by a test function \( \Phi \), we always mean a non-negative \( C^\infty \)-function (i.e. a function having derivatives of all orders) which dominates the indicator function of \([-1, 1]\). Using that
$\Phi \in C^\infty$, applying integration by parts, we have

$$\hat{\Phi}(y) \ll_A (1 + |y|)^{-A}$$  \hfill (1.9)

for any $A > 0$, where in the above $\ll_A$ indicates that the implied constant depends on $A$.

\section{NEW RESULTS}

\subsection{Type I sums of complete Kloosterman sums}

We first state our main results on upper bounds for $S_{q,a}(\alpha)$ with general $q$ and $a$.

\textbf{Theorem 2.1.} Let $q$ be a positive integer and let $I$, $J$ be two intervals as in (1.1). For any $M, N \geq 1$ and $a \in \mathbb{Z}$ with $d = \gcd(a, q)$, we have

$$S_{q,a}(\alpha) \ll \|\alpha\|_2 M^{1/2} N q^{1/2+o(1)} \Delta_1(M, N, q, d),$$

where we may take $\Delta_1(M, N, q, d)$ freely among

\begin{align*}
M^{-1/4} N^{-1} q^{1/2} d^{-1/4} + q^{1/2} N^{-1} M^{-1/2} + N^{-1/2}, \\
M^{-1/2} (N^{-3/4} q^{1/2} + d^{1/2}) + N^{-1/2}, \\
M^{-1/2} (N^{-1} q^{1/2} + (qd)^{1/4}) + N^{-1/2}.
\end{align*}

We note that around the ‘diagonal’, that is, for $M = Nq^{o(1)}$ the bounds (2.1a) and (2.1b) coincide. In particular, we note that in the Pólya–Vinogradov range $M, N = q^{1/2+o(1)}$ and $\|\alpha\|_\infty = q^{o(1)}$, the choice of the bound (2.1a) in Theorem 2.1 yields $|S_{q,a}(\alpha)| \ll q^{11/8+o(1)}$ for $\gcd(a, q) = 1$, and thus saves $1/8$ against the trivial bound $q^{3/2+o(1)}$, which is significantly better than the saving $1/24$ from [7] and $1/16$ from [46]. On the other hand, the bound (2.1c) is better than (2.1a) and (2.1b) for some skewed choices of $M$ and $N$, namely when

$$MN^2 \geq q^{1+\delta} \quad \text{and} \quad M \geq q^{1/2+\delta}$$

with some fixed $\delta > 0$.

By virtue of the Selberg–Kuznetsov identity (see (4.2) below) and Möbius inversion, one may see the above three upper bounds also work for Type I sums with $K_q(mn,1)$.

\textbf{Corollary 2.2.} Let $q$ be a positive integer and let $I$, $J$ be two intervals as in (1.1). For any $M, N \geq 1$, we have

$$\sum_{m \in I} \sum_{n \in J} \alpha_m K_q(mn, a) \ll \|\alpha\|_2 M^{1/2} N q^{1/2+o(1)} \Delta_1(M, N, q, 1)$$

uniformly in $a \in Z_q^*$, where $\Delta_1(M, N, q, 1)$ can be taken as in Theorem 2.1.
Figure 1 gives a plot of a polygon in the \((\mu, \nu)\)-plane, with

\[ M = q^{\mu + o(1)} \quad \text{and} \quad N = q^{\nu + o(1)}, \]

where Theorem 2.1 improves the trivial bound (1.3), as well as (1.4), (1.7) and (1.8). Simple calculations show that the polygon on Figure 1 has vertices

\[(\mu, \nu) = (0, 1/2), (0, 1), (1, 1), (1, 0), (1/2, 1/4), (1/2, 1/3), (2/5, 2/5).\]

Remark 2.3. Examining the proof of Theorem 2.1 in Section 6.3, one can easily see that it can be generalised identically to estimate Type I sums of the weighted Kloosterman sum

\[
\sum_{x \in \mathbb{Z}_q^*} \xi(x) e_q(mx + nx^{-1})
\]

with an arbitrary complex weight \(\xi\) such that \(\|\xi\|_\infty \leq 1\), to which the algebraic geometry method in [7, 18, 34, 35] does not apply. In particular, our bound holds for Salié sums

\[
S_q(m, n) = \sum_{x \in \mathbb{Z}_q^*} \left( \frac{x}{q} \right) e_q(mx + nx^{-1})
\]
with all odd $q \geq 1$, where $(\cdot_q)$ denotes the Jacobi symbol mod $q$. Bilinear forms with Salie sums have been studied extensively in [15, 16, 30, 44, 45] due to their applications to the moments of $L$-functions of half-integral weight modular forms and to the distribution of modular square roots of primes. In particular, the bounds in Theorem 2.1 applied to Salie sums improve [30, Theorem 2.2] for a wide range of parameters.

### 2.2 Type II sums of incomplete Kloosterman sums

The above estimates for Type I sums $S_q(\alpha)$ are essentially based on estimates for the following bilinear form with incomplete Kloosterman sums

$$W_{q,a}(\alpha, \gamma; r, c) = \sum_{m \in I} \sum_{k \in \mathbb{Z}^*_q} \alpha_m \gamma_k e_q(amk^{-1}),$$

where $\gamma$ is an arbitrary weight, and we define $\langle n \rangle_r$ to be the (unique) integer $y \in [1, r]$ such that $n \equiv y \pmod{r}$. Hence, a natural restriction is that $K \leq r$. Note that there are at most $Kq/r$ values of $k$ in the above average; thus, a trivial bound could be

$$|W_{q,a}(\alpha, \gamma; r, c)| \leq \|\alpha\|_2 \|\gamma\|_{\infty} M^{1/2} K q r^{-1}$$

for all $c \in \mathbb{Z}^*_r$ and $M, K \geq 1$ with $K \leq r$.

**Theorem 2.4.** Let $q, r$ be positive integers with $r | q$, and $I$ an interval as in (1.1). For any $M, K \geq 1$ with $K \leq r$, we have

$$W_{q,a}(\alpha, \gamma; r, c) \ll \|\alpha\|_2 \|\gamma\|_{\infty} M^{1/2} K q^{1+o(1)} r^{-1} \Delta_2(M, K, q, r)$$

uniformly in $a \in \mathbb{Z}^*_q$ and $c \in \mathbb{Z}^*_r$, where we may take $\Delta_2(M, K, q, r)$ freely among

$$
\begin{align*}
(Mq/r)^{-1/4} + M^{-1/2} + (Kq/r)^{-1/2}, & \quad (2.2a) \\
M^{-1/2}(1 + (K/r)^{-1/4} + K^{-1} r^{1/2}) + (Kq/r)^{-1/2}, & \quad (2.2b) \\
M^{-1/2}(1 + K^{-1/2} r^{1/4} + K^{-1} r^{3/4}) + (Kq/r)^{-1/2}. & \quad (2.2c)
\end{align*}
$$

In the typical case $(r, c) = (q, 1)$, for which we denote by $W_{q,a}(\alpha, \gamma)$ the corresponding bilinear sum, (2.2a) produces a non-trivial bound essentially in the full range $M, K \geq q^\varepsilon$ and (2.2b) saves more than (2.2a) if $MK \gg q$. In turn, (2.2c) is better than (2.2a) and (2.2b) when $K$ is suitably large.

Besides applications to concluding Theorem 2.1, Theorem 2.4 should be of independent interests and may indicate many other applications. In particular, Wu [49] utilised such estimates for $W_{q,a}(\alpha, \gamma)$ to evaluate the fourth moment of Dirichlet $L$-functions with arbitrary moduli; see Section 3.1 for more details.
2.3 | New bilinear forms with arbitrary support

We now formulate some bilinear forms which generalise $S_q(\alpha)$ or $W_{q,a}(\alpha, \gamma)$ to the case that one variable is supported on an arbitrary set. Since additive combinatorics in finite fields is employed, we now restrict our moduli to primes.

Let $\mathcal{K}$ be an interval given by

$$\mathcal{K} = \{K_0 + 1, \ldots, K_0 + K\}$$

(2.3)

for some integers $K_0$ and $K \geq 1$. For each prime $p$ and an arbitrary set $\mathcal{M} \subseteq \mathbb{F}_p$ of cardinality $M$, define

$$W_{p,a}^\#(\alpha, \gamma) = \sum_{m \in \mathcal{M}} \sum_{k \in \mathcal{K}} \alpha_m \gamma_k e_p(\alpha m k^{-1}),$$

where we also keep in mind the underlying restriction $k \in \mathbb{F}_p^*$. In fact, it is contained implicitly in Shparlinski [46, Theorem 2.1] that

$$W_{p,a}^\#(\alpha, \gamma) \ll \|\alpha\|_\infty \|\gamma\|_\infty M^{3/4} (K^{7/8} p^{1/8} + K^{1/2} p^{1/4}) p^{o(1)},$$

which can also be generalised identically to the situation in $\mathbb{Z}_q$.

By combining the ‘shift by $a\beta$’ trick of Vinogradov, Karatsuba and Friedlander–Iwaniec with techniques related to the Balog–Szemerédi–Gowers theorem, we obtain the following result.

**Theorem 2.5.** Let $p$ be a prime and let $r \in \mathbb{Z}^+$ be fixed. For an arbitrary set $\mathcal{M} \subseteq \mathbb{F}_p$ of cardinality $M$ and any interval $\mathcal{K} \subseteq \mathbb{F}_p^*$ as in (2.3) of length $K \geq p^{1/r}$, we have

$$W_{p,a}^\#(\alpha, \gamma) \ll \|\alpha\|_\infty \|\gamma\|_\infty MK^{p^{o(1)}} \left( \frac{1}{M} + \frac{p^{1+1/r}}{MK^2} \right)^{7/24r}$$

uniformly in $a \in \mathbb{F}_p^*$.

We now present another variant of Theorem 2.5.

**Theorem 2.6.** Let $p$ be a prime and let $r \in \mathbb{Z}^+$ be fixed. For an arbitrary set $\mathcal{M} \subseteq \mathbb{F}_p$ of cardinality $M \leq p^{1/2}$ and any interval $\mathcal{K} \subseteq \mathbb{F}_p^*$ as in (2.3) of length $K \geq p^{1/r}$, we have

$$W_{p,a}^\#(\alpha, \gamma) \ll \|\alpha\|_\infty \|\gamma\|_\infty MK^{p^{o(1)}} \left( \frac{1}{M^{2r}} + \frac{1}{K} + \frac{p^{1+1/r}}{M^{10/13} K^2} \right)^{1/4r}$$

uniformly in $a \in \mathbb{F}_p^*$.

The upper bounds for $S_{q,a}(\alpha)$ benefit from those for $W_{q,a}(\alpha, \gamma; r, c)$ as mentioned above. Following this spirit, one can imagine that Theorems 2.5 and 2.6 can be utilised to study the following bilinear form of Kloosterman sums

$$S_{p,\alpha}(\alpha) = \sum_{m \in \mathcal{M}} \sum_{n \in J} \alpha_m \mathcal{K}_p(m, n),$$

(2.4)
where $\mathcal{M} \subseteq \mathbb{F}_p$ is an arbitrary set and $J$ is an interval given by (1.1). The sums $S_p^\#(\alpha)$ have also been estimated in [3, Theorem 2.4] (also with higher dimensional Kloosterman sums). Our new bounds are stronger and extend the range in which nontrivial bounds of such sums are available.

**Theorem 2.7.** Let $p$ be a prime and let $r \in \mathbb{Z}^+$ be fixed. For an arbitrary set $\mathcal{M} \subseteq \mathbb{F}_p$ of cardinality $M$ and any interval $J \subseteq \mathbb{F}_p^*$ as in (1.1) of length $N \leq p^{1-1/r}$, we have

$$S_p^\#(\alpha) \ll \|\alpha\|_\infty MNp^{1/2+o(1)} \left( \frac{p^{1/2}}{M^{7/24r}N} + \frac{p^{1/2-7(r-1)/24r^2}}{M^{7/24r}N^{1-7/12r}} \right).$$

Slightly modifying the argument of the proof of Theorem 2.7, we also obtain the following bound.

**Theorem 2.8.** With the notation as in Theorem 2.7 and assuming $M \leq p^{1/2}$, we have

$$S_p^\#(\alpha) \ll \|\alpha\|_\infty MNp^{1/2+o(1)} \times \left( \frac{p^{1/2}}{M^{1/2}N} + \frac{p^{1/2-1/4r}}{N^{1-1/4r}} + \frac{p^{1/2-(r-1)/4r^2}}{M^{13/40r}N^{1-1/2r}} \right).$$

Hence, we beat the Pólya–Vinogradov barrier in the following sense.

**Corollary 2.9.** Let $\mathcal{M} \subseteq \mathbb{F}_p$ be a set of cardinality $M$ and let $J \subseteq \mathbb{F}_p^*$ be an interval as in (1.1) of length $N$. Then for any $\varepsilon > 0$, there exist some $\delta, \eta > 0$ such that

$$S_p^\#(\alpha) \ll \|\alpha\|_\infty MNp^{1/2-\eta},$$

provided that $M > p^{\varepsilon}$ and $N > p^{1/2-\delta}$.

### 2.4 General comments about the methods used

In closing, we condense two new ingredients in this paper.

On the one hand, all starting points have roots in analysing the Type II sum $W_{q,q}(\alpha,\gamma;r,c)$, for which we reduce the problem to bounding $J_q(a,K)$ from above, individually and on average (see Section 5 for details). Regarding the averaged bound for $J_q(a,K)$, our tools include a variant of counting device due to Heath-Brown [23, 24] and Cilleruelo and Garaev [12]. A new individual bound for $J_q(a,K)$ is based on square-root cancellations among the products of two Kloosterman sums; see $T(x,t,z;q)$ as defined by (4.4) and Lemma 4.1 for an upper bound. These treatments of $J_q(a,K)$ form a vital part in this paper, and the relevant bounds should admit many applications on other occasions. We note that as in [46, Theorem 2.2], our method allows to obtain stronger bounds for almost all $q$. In fact, the first application of bounds on $J_q(a,K)$ to estimating bilinear forms with incomplete Kloosterman sums has been given by Heath-Brown [23, Section 3]. In turn, such estimates have found a new application in the very recent work of Zhao [54] on products of primes in arithmetic progressions.

On the other hand, we employ a recent result by Rudnev, Shkredov and Stevens [41] on the decomposition of subsets of $\mathbb{F}_p$, with which we combine the ‘shift by $ab’ trick of Vinogradov, Karatsuba and Friedlander–Iwaniec (see, e.g. [20, 22, 34]) to study the new bilinear forms.
Additive combinatorics then enters the picture and all details can be found in Section 7. Here, we should also mention a pioneering work by Bourgain [10], who has employed tools from additive combinatorics (sum-product estimates) to study bilinear forms with incomplete Kloosterman sums, as well as a very recent work by Shkredov [43] concerning bilinear forms with complete Kloosterman sums.

Comparing with algebro-geometric methods, additive combinatorics could be an alternative approach, which sometimes can also provide very strong estimates. Note that although the results of Shkredov [43] do not seem to improve the best known estimates, there is certainly a strong potential for further improvements. More precisely, the method of Shkredov [43] is based on the so-called incidence bounds for hyperbolas. It is possible that the recent progress on such bounds due to Rudnev and Wheeler [42] can be used to improve the results within this approach. On the other hand, due to the nature of the underlying technique, it is not likely to work in residue rings modulo a composite number.

### 2.5 Possible generalisations within our method

To show its flexibility and without getting into details, we note that at the cost of only simple typographical changes, our bounds on the sums $S_p(\alpha)$ extend without any changes to Type I bilinear forms

$$\sum_{m \in I} \sum_{n \in J} \alpha_m \mathcal{K}_q(\xi, m, an), \quad a \in \mathbb{Z}_q,$$

with more general sums

$$\mathcal{K}_q(\xi, m, n) = \sum_{x \in \mathbb{Z}_q^*} \xi_x \mathcal{e}_q(mx + nx^{-1})$$

with arbitrary complex weights $\xi = \{\xi_x\}$, supported on $\mathbb{Z}_q^*$ and satisfying $\|\xi\| \leq 1$.

Another generalisation can involve the sum with more general binomials in the exponents, that is, with $mx^s + nx^{-t}$ with some integers $s, t \in \mathbb{Z}$ instead of $mx + nx^{-1}$ as in [37]. Such sums (with $(s, t) = (1, -2)$) appear in the investigation of the distribution of square-free numbers in arithmetic progressions, see [40].

### 3 Applications

#### 3.1 Moments of Dirichlet $L$-functions

In a recent work of Wu [49], our results have already been applied in deducing an asymptotic formula for the fourth moment of central values of Dirichlet $L$-functions. More precisely, he proved that

$$\frac{1}{\varphi^*(q)} \sum_{\chi \pmod{q}} |L\left(\frac{1}{2}, \chi\right)|^4 = \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} P_4(\log q) + O(q^{-\delta + o(1)})$$

with $\delta = (1 - 6\delta)/4$, where $P_4$ is an explicit polynomial of degree 4 and $\delta$ denotes the exponent towards the Ramanujan–Petersson conjecture (RPC). Here, $q$ is a large positive integer with $q \neq$
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2 (mod 4), and $\chi$ runs over all primitive characters mod $q$ with $\varphi^*(q)$ denoting the number of all such characters. Note that one may take $\delta = 7/64$ thanks to Kim and Sarnak [32].

Based on their deep observations in random matrix theory, Conrey, Farmer, Keating, Rubinstein and Snaith [11, §3.1 & §4.3] formulated the exact shape of $P_4$ and conjectured that one may take $\delta = 1/2$ in (3.1), which coincides with the square-root cancellation philosophy. The first instance establishing the existence of $\delta > 0$ is due to Young [52], who proved that $\delta = (1 - 2\delta)/80$ is admissible for all large prime $q$. In more recent works by Blomer, Fouvry, Kowalski, Michel and Miličević [7, 8], the dependence on the RPC can be removed for prime moduli $q$, and one can even take a much better choice $\delta = 1/20$. Note that new bounds on bilinear forms with Kloosterman sums [18, 34, 47] play an important role in their approach.

One of the ingredients in Wu [49] is the application of our estimates for bilinear forms with incomplete Kloosterman sums as stated in Theorem 2.4, which play a crucial role in bounding off-diagonal terms with very unbalanced lengths of summations. Ignoring the effect of the RPC, one notes that $\delta = (1 - 6\delta)/14$ in [49] indicates the same power saving as in [8], which shows $\delta = 1/14$ for prime moduli under the RPC. Note that in [8], a weaker result with $\delta = 1/16$ is claimed under the RPC, but it turns out to be a mistake in calculations, which has been observed in [53, Theorem 1.1]. This means that, for general moduli, Theorem 2.4 gains the same saving as the previous work for prime moduli. Power saving in [7, 8] relies on an estimate of the following sum of Kloosterman sums:

$$\sum_{m_1, m_2, m_3, m_4} W_1 \left( \frac{m_1}{M_1} \right) W_2 \left( \frac{m_2}{M_2} \right) W_3 \left( \frac{m_3}{M_3} \right) W_4 \left( \frac{m_4}{M_4} \right) \kappa_p(m_1 m_2, m_3 m_4)$$

with some compactly supported smooth functions $W_i(x), i = 1, \ldots, 4$. The estimate of this sum is based on both Type I (see [8, Equation (1.3)]) and Type II (see [7, Equation (5.1)]) sums of complete Kloosterman sums, which turns out to be quite different from Theorem 2.4. It is interesting to see that two different approaches lead to the same saving, while Theorem 2.4 is available for general moduli and does not impose any smoothness conditions on the weights.

Furthermore, Bettin [5, Lemma 25] has essentially used the bound of Young [52, Proposition 4.3] on the sums $W_{q,a}(\alpha, \gamma; r, c)$ from Section 2.2 to obtain an asymptotic formula on some mixed moments for Dirichlet $L$-functions. One may obtain an improved error term in [5, Theorem 1] using Theorem 2.4 since it improves Young [52, Proposition 4.3].

3.2 Moments of short sums of Kloosterman sums

A direct consequence of Theorem 2.7 is the following bound on moments of short sums of Kloosterman sums.

**Theorem 3.1.** Let $r$ be a positive integer and let $J \subseteq \mathbb{F}_p^*$ be an interval of length $N \leq p^{1-1/r}$. For any fixed $\alpha \in [1, 12r/7]$, we have

$$\sum_{\lambda \in \mathbb{F}_p^*} \left| \sum_{n \in J} \kappa_p(\lambda, n) \right|^{2\alpha} \ll p^{2\alpha + o(1)} N^{(12r - 7\alpha)/(12r - 7)} \left( 1 + \frac{N^2}{p^{1-1/r}} \right)^{7(\alpha - 1)/(12r - 7)}.$$
In particular, taking $\alpha = 12r/7$ in Theorem 3.1, we have

$$\sum_{\lambda \in \mathbb{F}_p^*} \left| \sum_{n \in J} K_p(\lambda, n) \right|^{24r/7} \ll p^{24r/7 + o(1)} \left( 1 + \frac{N^2}{p^{1 - 1/r}} \right).$$

(3.2)

In fact, we show in the proof of Theorem 3.1 that this is essentially the only case which has to be established.

As we have mentioned, the sums considered in Theorem 2.5 are related to work of Bourgain [10] on short bilinear sums of incomplete Kloosterman sums. In particular [10, Theorem A.1] is based on [10, Lemma A.2] which states that for an interval $I$ of length $M < p^{1/2}$, we have

$$\# \left\{ \lambda \in \mathbb{F}_p : \left| \sum_{x \in I} e_p(\lambda x^{-1}) \right| \geq p^{-\varepsilon} M \right\} \leq p^{1+\delta} M^{-2}.$$  

(3.3)

Analysing the proof of [10, Lemma A.2], one sees that Bourgain’s argument [10] allows the choice

$$\delta = (4 + o(1)) \sqrt{\varepsilon}.$$  

It is possible to use Theorem 2.5, with the set $\mathcal{M}$ which appears on the left-hand side of (3.3), to show that one may take

$$\delta = \left( 2 \sqrt{\frac{24}{7}} + o(1) \right) \sqrt{\varepsilon} \approx 3.7032 \sqrt{\varepsilon}$$

in (3.3).

### 3.3 Divisor function in a family of arithmetic progressions

For integers $a$ and $q \geq 2$ with $\gcd(a, q) = 1$, consider the divisor sum

$$S(X; a, q) = \sum_{\substack{n \leq X \\text{mod } q}} \tau(n).$$

By standard heuristic arguments, the expected main term for $S(X; a, q)$ should be

$$M(X; a, q) = \frac{\varphi(q)}{q^2} X (\log X + 2\gamma - 1) - \frac{2}{q} X \sum_{d|q} \frac{\mu(d) \log d}{d},$$

where $\varphi(q)$ is the Euler function of $q$. The basic problem is to prove that the error term

$$R(X; a, q) = S(X; a, q) - M(X; a, q)$$

can be bounded by $O(X^{1-\delta}/q)$ with some constant $\delta > 0$ and $q$ being as large as possible compared to $X$. This was realised independently by Selberg and Hooley [25], as long as $q \leq X^{2/3-\varepsilon}$ for any $\varepsilon > 0$.

Banks, Heath-Brown and Shparlinski [4] and then Blomer [6, Theorem 1.1] established bounds on the second moment of error term $R(X; a, q)$ which are nontrivial in the essentially optimal range $q \leq X^{1-\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$. 
Kerr and Shparlinski [31] have considered a mixed scenario between pointwise and average bounds on \(R(X; a, q)\). Namely, given a set \(A \subseteq \mathbb{Z}_q^*\), we define

\[
\mathcal{E}(X; A, q) = \sum_{a \in A} R(X; a, q).
\]

Then by [31, Theorem 1.2], for any integers \(A, X\) and \(q\) with

\[
A \leq q \quad \text{and} \quad X \geq q \geq X^{19/31},
\]

and any interval \(A \subseteq \mathbb{Z}_q^*\) of length \(A\), we have

\[
|\mathcal{E}(X; A, q)| \leq (AX^{1/2}q^{-1/2} + A^{1/8}X^{1/4}q^{1/2} + A^{1/2}X^{1/4}q^{1/4})q^o(1).
\]

We refer to [31] for more details.

Theorem 2.1 allows us to derive the following bound for \(\mathcal{E}(X; A, q)\).

**Theorem 3.2.** Let \(q\) be a positive integer and \(I\) an interval of length \(A < q\). For all \(X \geq q\) and any set \(A = I \cap \mathbb{Z}_q^*\) with \(q^3 < AX^2/8\), we have

\[
|\mathcal{E}(X; A, q)| \leq (E(A, N, q) + A^{2/3}X^{1/3})X^o(1),
\]

where

\[
E(A, N, q) = \min\{q^{1/2}X^{1/4} + X^{1/2}, X^{1/2}(A^{1/4} + Aq^{-1/2}), X^{1/2}(1 + Aq^{-1/4})\}.
\]

Theorem 3.2 is non-trivial only for \(q^3 \ll AX^2\) and improves the results of [31] for this range of parameters. If we could ignore the term \(A^{2/3}X^{1/3}\), then Theorem 3.2 would improve [31] in the full range due to the appearance of \(q^{1/2}X^{1/4}\). In the crucial range \(q \approx X^{2/3}\), Theorem 3.2 beats the Selberg–Hooley barrier on average for \(A \gg q^{3/8+\varepsilon}\) for any fixed \(\varepsilon > 0\), while one requires \(A \gg q^{3/7+\varepsilon}\) in [31].

4 | **EXPONENTIAL SUMS**

4.1 | Basic properties of Kloosterman sums

We first present some materials for Kloosterman sums, some of which can be found in [26, Section 4.3]. The following elementary properties are well known and trivial to verify:

\[
\mathcal{K}_q(m, n) = \mathcal{K}_q(n, m),
\]

and

\[
\mathcal{K}_q(cm, n) = \mathcal{K}_q(m, cn) \quad \text{if} \quad \gcd(c, q) = 1. \tag{4.1}
\]

We also need the following Selberg–Kuznetsov identity

\[
\mathcal{K}_q(m, n) = \sum_{d|\gcd(m, n, q)} d \mathcal{K}_{qd^{-1}}(mnd^{-2}, 1). \tag{4.2}
\]
Kloosterman sums also enjoy the twisted multiplicativity
\[ \mathcal{K}_q(m,n) = \mathcal{K}_{q_1}(mq^{-1}_2,nq^{-1}_2)\mathcal{K}_{q_2}(mq^{-1}_1,nq^{-1}_1) \]
for any \( q_1, q_2 \) satisfying \( q = q_1 q_2 \) and \( \gcd(q_1, q_2) = 1 \).

The Kloosterman sum \( \mathcal{K}_q(m,n) \) reduces to the Ramanujan sum
\[ c_q(m) = \sum_{x \in \mathbb{Z}^*} e_q(mx) \]
if \( n \equiv 0 \pmod{q} \), see [27, §3.2] for a background on Ramanujan sums. In particular, we frequently use the inequality
\[ |c_q(m)| \leq \gcd(q,m). \quad (4.3) \]

### 4.2 Fourier transforms of products of Kloosterman sums

By virtue of Kloosterman sums, we define
\[ T(x,y,z; q) = \frac{1}{q} \sum_{t \in \mathbb{Z}_q} \mathcal{K}_q(x,t)\mathcal{K}_q(y,t)e_q(-zt). \quad (4.4) \]

This can be regarded as the (normalised) Fourier transform of products of two Kloosterman sums. For the purpose of estimating bilinear forms in this paper, we would like to explore an upper bound for \( T(x,y,z; q) \) which exhibits square-root cancellations up to some harmless factors.

The Chinese Remainder Theorem, together with the twisted multiplicativity of Kloosterman sums yields the following twisted multiplicativity:
\[ T(x,y,z; q) = T(xq^{-1}_2,yq^{-1}_2,z;q_1)T(xq^{-1}_1,yq^{-1}_1,z;q_2) \quad (4.5) \]
for any \( q_1, q_2 \) satisfying \( q_1 q_2 = q \) and \( \gcd(q_1, q_2) = 1 \). Hence, the evaluation of \( T(x,y,z; q) \) reduces to the situation of prime power moduli.

Here is our upper bound for \( T(x,y,z; q) \).

**Lemma 4.1.** Let \( q \) be a positive integer. For \( x, y, z \in \mathbb{Z} \), we have
\[ T(x,y,z; q) \ll \gcd(x,y,q)^{1/2} \gcd\left(x - y, z, \frac{q}{(x,y,q)}\right)^{1/2} q^{1/2 + o(1)}. \]

The proof of Lemma 4.1 is given in the Appendix, since the arguments are based on vanishings and exact expressions of Kloosterman sums, which might be of independent interests.

### 5 COUNTING WITH RECIPROCALS

#### 5.1 Notation

For integers \( a \) and \( q \), we are interested in the counting function
\[ J_q(a,K) = \# \{(k_1, k_2) \in [1, K]^2 : k^{-1}_1 - k^{-1}_2 \equiv a \pmod{q}\}, \quad (5.1) \]
which turns out to be an important object and occupies a central role in our later arguments. There is a trivial bound

\[ J_a(a, K) \leq K(K/q + 1) \]

for all \( K > 1 \), which reaches the true order of magnitude as \( K \geq q \). The situation now becomes subtle as long as \( K < q \), in which case we need, for the sake of later applications, to beat the above bound in as wide a range of parameters as possible.

### 5.2 Pointwise bounds

The following lemma gives a uniform bound for \( J_q(a, K) \), which is, in fact, contained implicitly in a series of papers of Heath-Brown; see [23, p. 367] or [24, p. 46], for instance.

**Lemma 5.1.** Let \( q \) be a positive integer and let \( a \in \mathbb{Z} \). For any \( K \leq q \), we have

\[ J_q(a, K) \leq \left( \frac{K^{3/2}}{q^{1/2}} + \frac{K^2 \gcd(a, q)}{q} + 1 \right)^{o(1)}. \]

By virtue of explicit evaluations of Kloosterman sums, we also have the following alternative upper bound for \( J_q(a, K) \), which is sharper than Lemma 5.1 as long as \( K > q^{2/3} \).

**Lemma 5.2.** Let \( q \) be a positive integer and let \( a \in \mathbb{Z} \). For any \( K \leq q \), we have

\[ J_q(a, K) \leq \left( \frac{K^2}{q} + \frac{K \gcd(a, q)}{q^{1/2}} + q^{1/2} \right)^{o(1)}. \]

**Proof.** Let \( \Phi \) be a non-negative smooth function which dominates the indicator function of \([-1, 1]\). Trivially, we write

\[ J_q(a, K) \ll \sum_{k_1^{-1} - k_2^{-1} \equiv a \pmod{q}} \Phi \left( \frac{k_1}{K} \right) \Phi \left( \frac{k_2}{K} \right). \]

Applying Poisson summation to each of \( k_1, k_2 \pmod{q} \), we find

\[ J_q(a, K) \ll \frac{K^2}{q^2} \sum_{m, n \in \mathbb{Z}} T(-m, n, a; q) \hat{\Phi} \left( \frac{mK}{q} \right) \hat{\Phi} \left( \frac{nK}{q} \right), \]

where \( T(-m, n, a; q) \) is defined by (4.4). In view of the decay (1.9), we may truncate the \( m \) and \( n \)-sums at \( H = q^{1+\varepsilon}/K \) with an arbitrary \( \varepsilon > 0 \) at a cost of a negligible error term. Hence,

\[ J_q(a, K) \ll \frac{K^2}{q^2} \sum_{0 \leq |m|, |n| \leq H} |T(m, n, a; q)| + 1. \]

The result now follows immediately from Lemma 4.1. \( \square \)
5.3 | Bounds on average

We may also improve Lemmas 5.1 and 5.2 on average, provided that the length of averaging is not too short.

**Lemma 5.3.** Let $q$ be a positive integer and let $\gcd(a, q) = 1$. For any positive integers $N, K \leq q$, we have

$$\sum_{1 \leq n \leq N} J_q(an, K) \leq \left( \frac{K^2 N^{1/2}}{q^{1/2}} + K \right) q^{o(1)}.$$  

**Proof.** Denote by $J$ the sum in question, which counts the number of solutions to

$$k_1^{-1} - k_2^{-1} \equiv an \pmod{q}$$

with $1 \leq k_1, k_2 \leq K$ and $1 \leq n \leq N$. Moreover, denote by $I$ the number of solutions to

$$k_1^{-1} - k_2^{-1} \equiv a(n_1 - n_2) \pmod{q}$$

in $1 \leq k_1, k_2 \leq K$ and $1 \leq n_1, n_2 \leq 2N$. Clearly, we have

$$J \ll I/N \quad (5.2)$$

since for each $1 \leq n \leq N$ we have

$$\#\{(n_1, n_2) \in [1,2N]^2 : n = n_1 - n_2\} \gg N.$$  

The problem now reduces to bounding $I$ from above. For each $\lambda \in \mathbb{Z}_q$, define

$$I(\lambda) = \#\{(k, n) \in [1, K] \times [1, 2N] : k^{-1} + an \equiv \lambda \pmod{q}\}, \quad (5.3)$$

so that

$$I = \sum_{\lambda \in \mathbb{Z}_q} I(\lambda)^2. \quad (5.4)$$

Since

$$\sum_{\lambda \in \mathbb{Z}_q} I(\lambda) \ll KN, \quad (5.5)$$

it suffices to produce an upper bound for $I(\lambda)$ uniformly in $\lambda \in \mathbb{Z}_q$.

Recall the definition (5.3). It is useful to note that $I(\lambda)$ counts the number of solutions to the congruence

$$(ak)^{-1} + n \equiv a^{-1}\lambda \pmod{q}, \quad 1 \leq k \leq K, \quad 1 \leq n \leq 2N. \quad (5.6)$$

We now adopt some ideas of Cilleruelo and Garaev [12, Theorem 1]. By the Dirichlet pigeonhole principle, we can choose integers $t$ and $u$ satisfying

$$a^{-1}\lambda u \equiv t \pmod{q}, \quad 0 \leq |t| \leq T, \quad 0 < |u| \leq \frac{2q}{T},$$

where $T$ is a fixed integer. By evaluating the quadratic polynomial

$$a^{-1}\lambda u \equiv t \pmod{q}, \quad 0 \leq |t| \leq T, \quad 0 < |u| \leq \frac{2q}{T},$$

we obtain

$$\sum_{\lambda \in \mathbb{Z}_q} I(\lambda)^2 \ll \frac{K^2}{q^{1/2}} + K.$$
where $T > 2$ is a parameter to be determined later. In this way, we infer from (5.6) that

$$k(t - nu) \equiv a^{-1}u \pmod{q}.$$  

Denote by $a^{-1}u \equiv b \pmod{q}$ with $1 \leq b \leq q - 1$. We conclude that there exists some integer $r$ such that

$$0 \neq k(t - nu) = b + qr, \quad |r| \leq 2R$$

with

$$R = \frac{KT + KNq/T + q}{q} = 1 + \frac{KT}{q} + \frac{KN}{T}.$$  

Collecting the above arguments, we derive that

$$I(\lambda) \leq \sum_{k \leq K} \sum_{n \leq 2N} \sum_{|r| \leq 2R} 1 \ll \sum_{|r| \leq 2R} \tau(|b + qr|) \leq Rq^{o(1)}.$$  

Taking $T = (qN)^{1/2}$, we find

$$I(\lambda) \leq (1 + K(N/q)^{1/2})q^{o(1)},$$

from which and (5.5) it follows that

$$I \leq KN(1 + K(N/q)^{1/2})q^{o(1)}.$$  

This readily proves the lemma in view of (5.2).

We note that Karatsuba [28, Lemma 8] has previously estimated the quantity $I$ defined by (5.4) as

$$I \leq KN(1 + N/K + K^{1/2}N^{3/2}q^{-1/2})q^{o(1)},$$  

(5.7)

which implies a sharper bound compared to Lemma 5.3 for smaller values of $N$. However, for such ranges of parameters, the bound (5.7) is comparable to an application of Lemma 5.1.

### 5.4 A slight extension

Let $q$ be a positive integer and $r \mid q$. Define

$$J_q(a, K; r, c) = \# \{(k_1, k_2) \in (\mathbb{Z}_q^*)^2 : k_1^{-1} - k_2^{-1} \equiv a \pmod{q}, \langle ck_1 \rangle_r, \langle ck_2 \rangle_r \leq K\}.$$  

(5.8)

Clearly, we have $J_q(a, K; q, 1) = J_q(a, K)$ for all $K \leq q$. In general, we may conclude the following inequality, illustrating an upper bound for $J_q(a, K; r, c)$ in terms of the original counting function (5.1).
**Lemma 5.4.** Let \( q \) be a positive integer and \( r \mid q \). For all \( K \geq 1 \) and \( c \in \mathbb{Z}_r^* \), we have

\[
J_q(a, K; r, c) \leq \frac{q}{r} J_r(c^{-1}a, K).
\]

**Proof.** Note that \( J_q(a, K; r, c) \) is exactly the number of quadruples \((u_1, u_2, v_1, v_2)\) with

\[
u_1, u_2 \pmod{r}, \quad v_1, v_2 \pmod{q/r}, \quad \gcd((u_1 + rv_1)(u_2 + rv_2), q) = 1,
\]

\[
(u_1 + rv_1)^{-1} - (u_2 + rv_2)^{-1} \equiv a \pmod{q},
\]

\[
\langle cu_1 \rangle_r, \langle cu_2 \rangle_r \leq K.
\]

The congruence condition also indicates that \( u_1^{-1} - u_2^{-1} \equiv a \pmod{r} \). For each fixed \( u_2, u_2 \pmod{r} \) and \( v_1 \pmod{q/r} \), there should exist at most one \( v_2 \pmod{q/r} \) with

\[
\gcd(u_2 + rv_2, q) = 1, \quad (u_1 + rv_1)^{-1} - (u_2 + rv_2)^{-1} \equiv a \pmod{q}.
\]

Therefore, we infer

\[
J_q(a, K; r, c) \leq \frac{q}{r} \# \{(u_1, u_2) \in (\mathbb{Z}_r^*)^2 : u_1^{-1} - u_2^{-1} \equiv a \pmod{r}, \quad \langle cu_1 \rangle_r, \langle cu_2 \rangle_r \leq K \}.
\]

Since \( c \in \mathbb{Z}_r^* \), we complete the proof after a change of variable. \( \square \)

## 6 | BILINEAR FORMS WITH KLOOSTERMAN SUMS OVER INTERVALS

### 6.1 | Preliminary comments

Following the approach of [46], we relate Type I bilinear forms with complete Kloosterman sums to Type II bilinear forms with incomplete Kloosterman sums. Hence, we first prove Theorem 2.4, from which Theorem 2.1 is derived.

### 6.2 | Proof of Theorem 2.4

Without loss of generality, we assume that \( \gamma \) is bounded. By the Cauchy–Schwarz inequality

\[
|W_{q,a}(\alpha, \gamma; r, c)|^2 \leq \|\alpha\|_2^2 \sum_{m \in \mathbb{Z}} \Phi \left( \frac{m - M_0}{M} \right) \sum_{\langle ck \rangle_r \leq K} \gamma_k e_q(amk^{-1})^2,
\]

where \( \Phi \) is a non-negative smooth function which dominates the indicator function of \([-1, 1]\) (and \( M_0 \) is as in (1.1)). Squaring out and interchanging summations, we get

\[
|W_{q,a}(\alpha, \gamma; r, c)|^2 \leq \|\alpha\|_2^2 \sum_{\langle ck_1 \rangle_r, \langle ck_2 \rangle_r \leq K} \gamma_{k_1} \overline{\gamma}_{k_2} \Sigma(k_1, k_2)
\]

\[
\Sigma(k_1, k_2) = \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - M_0}{M} \right) \sum_{\langle cm \rangle_r \leq K} e_q(amk_1^{-1}) e_q(amk_2^{-1})^*.
\]
with
\[ \Sigma(k_1, k_2) = \sum_{m \in \mathbb{Z}} \Phi\left( \frac{m - M_0}{M} \right) e_q(\alpha m(k_1^{-1} - k_2^{-1})). \]

From Poisson summation, it follows that
\[ \Sigma(k_1, k_2) = M \sum_{n \equiv a(k_1^{-1} - k_2^{-1}) \pmod{q}} \hat{\Phi}\left( \frac{Mn}{q} \right) e_q(-nM_0). \]

In view of the decay (1.9), the contributions from those \( n \) with \( |n| > q^{1+\varepsilon}/M \) are negligible, so that
\[ \Sigma(k_1, k_2) = M \sum_{0 \leq |n| \leq q^{1+\varepsilon}/M} \hat{\Phi}\left( \frac{Mn}{q} \right) e_q(-nM_0) + O(K^{-2}), \]

giving
\[ W_{q,a}(\alpha, \gamma; r, c) \lesssim \|\alpha\|_2^2 Mq/r \sum_{1 \leq |n| \leq q^{1+\varepsilon}/M} J_r(\alpha^{-1}n, K) + \|\alpha\|_2^2 MKq/r, \]

where \( J_r(\alpha^{-1}n, K) \) is defined as (5.8). Note that the first term does not appear unless \( M \leq q^{1+2\varepsilon} \), which we henceforth assume. In view of Lemma 5.4, we further have
\[ W_{q,a}(\alpha, \gamma; r, c) \lesssim \|\alpha\|_2^2 MKq/r \sum_{1 \leq |n| \leq q^{1+\varepsilon}/M} J_r((\alpha c)^{-1}n, K) + \|\alpha\|_2^2 MKq/r. \]

To conclude upper bounds for \( W_{q,a}(\alpha, \gamma; r, c) \), it suffices to invoke individual or averaged estimates for \( J_r(\cdot, K) \). We proceed on a case by case basis depending on if \( q^{1+\varepsilon}/M \leq r \) or not.

Suppose first that \( q^{1+\varepsilon}/M > r \). We have
\[ W_{q,a}(\alpha, \gamma; r, c) \lesssim \|\alpha\|_2^2 \frac{q^{2+\varepsilon}r^2}{r^2} \sum_{n=1}^r J_r((\alpha c)^{-1}n, K) + \|\alpha\|_2^2 MKq/r. \]

From the trivial observation
\[ \sum_{n=1}^r J_r((\alpha c)^{-1}n, K) = \sum_{n=1}^r J_r(n, K) \leq K^2, \]

we conclude that
\[ W_{q,a}(\alpha, \gamma; r, c) \lesssim \|\alpha\|_2^2 \frac{q^{2+\varepsilon}K^2}{r^2} + \|\alpha\|_2^2 MKq/r. \]

Due to the presence of terms \( M^{-1/2} \) in (2.2a), (2.2b) and (2.2c), we see that Theorem 2.4 is satisfied in this case.

Suppose next that \( q^{1+\varepsilon}/M \leq r \) and note \( K \leq r \). Since \( \varepsilon > 0 \) is arbitrary, it follows from Lemmas 5.3 and 5.4 that
\[ |W_{q,a}(\alpha, \gamma; r, c)|^2 \lesssim \|\alpha\|_2^2 (K^2 M^{1/2}(q/r)^{3/2} + MKq/r) q^{o(1)}, \]
which gives the choice (2.2a). The remaining two choices (2.2b) and (2.2c) are produced by applying Lemmas 5.1 and 5.2 in place of Lemma 5.3, respectively.

We now complete the proof of Theorem 2.4.

### 6.3 Proof of Theorem 2.1

We now prove Theorem 2.1 as a consequence of Theorem 2.4. We assume \( N \leq q \); otherwise, we may choose \( \Delta_1(M, N, q, d) = q^{1/2} / N \), which is better than \( N^{-1/2} \).

Following the approach in [46, Theorem 2.1], we put \( I = \lceil \log(N/2) \rceil \) and consider \( 2(I + 1) \) sets

\[
\begin{align*}
\mathcal{X}_0 &= \{ x \in \mathbb{Z}_q^n : a_1 x \equiv y \pmod{q_1} \text{ for some } y \in [1, q_1/N], \\
\mathcal{X}_i &= \{ x \in \mathbb{Z}_q^n : a_1 x \equiv y \pmod{q_1} \text{ for some } y \in (e^{i-1}q_1/N, e^iq_1/N] \},
\end{align*}
\]

for \( i = 1, \ldots, I \), where \( a_1 = a/d, q_1 = q/d \) with \( d = \gcd(a, q) \). By [46, Equation (5.3)], we have

\[
S_{q,a}(\alpha) \ll \sum_{\pm} \sum_{0 \leq i \leq I} |S_i^\pm|,
\]

where for \( i = 0, \ldots, I \),

\[
S_i^\pm = \sum_{m \in I} \sum_{x \in \mathcal{X}_i} \alpha_m \gamma_x e_q(\pm mx^{-1})
\]

with some complex weights \( \gamma_x \) satisfying

\[
\gamma_x \ll e^{-i}N,
\]

and \( \sum_{\pm} \) means that sum includes both \( S_i^+ \) and \( S_i^- \). Each sum \( S_i^\pm \) is of the type \( W_{q,a}(\alpha, \gamma; r, c) \) with \( K = e^iq_1/N \) and \( r = q_1 \). We are then in a position to apply Theorem 2.4, and the first choice in (2.2a) yields

\[
S_i^\pm \ll \|\alpha\|_2 M^{1/2} q((dM)^{-1/4} + (N/q)^{1/2})q^{o(1)}.
\]

Summing over \( i \), we obtain the choice (2.1a) of \( \Delta_1(M, N, q, d) \) for Theorem 2.1. The other two choices (2.1b) and (2.1c) of \( \Delta_1(M, N, q, d) \) correspond to applications of Theorem 2.4 with the remaining choices (2.2b) and (2.2c) of \( \Delta_2(M, K, q, r) \).

### 7 SOME RESULTS FROM ADDITIVE COMBINATORICS

#### 7.1 Preliminary comments

We next present some preliminaries for the proof of Theorems 2.5 and 2.6. The results of this section are specialised to \( F_p \) instead of \( \mathbb{Z}_q \) for two reasons. The first is our use of various results from additive combinatorics (see Lemma 7.3) and the second is that we require uniform estimates for counting zeros of polynomials in one variable over finite fields (see (8.8) below).
To begin with, we state a series of results on the decomposition of subsets and counting problems in $\mathbb{F}_p$.

### 7.2 Energy and special partitions of sets

For $A, B \subseteq \mathbb{F}_p$, as usual we define the **difference set**

$$A - B = \{a - b : (a, b) \in A \times B\}$$

and the **additive energy**

$$E(A, B) = \# \{(a_1, a_2, b_1, b_2) \in A^2 \times B^2 : a_1 - a_2 = b_1 - b_2\}.$$ 

We also write $E(A, A) = E(A)$ for abbreviation.

We quote a special case of Rudnev, Shkredov and Stevens [41, Theorem 2.15] with $k = 2$ therein.

**Lemma 7.1.** Let $L \geq 1$. For any subset $A \subseteq \mathbb{F}_p$ with

$$E(A) \geq \frac{(\#A)^3}{L},$$

there must exist some $B \subseteq A$ and $D \subseteq A - A$ satisfying

$$\#B \geq \frac{\#A}{16L} \quad \text{and} \quad \#D \leq 16L\#A,$$

such that for each $b \in B$, we have

$$\#\{(a, d) \in A \times D : b = a - d\} \geq \frac{\#A}{4L}.$$

Lemma 7.1 guarantees the existence of $B \subseteq A$ and $D \subseteq A - A$ such that $A \cap (b + D)$ is suitably large for any $b \in B$, provided that the energy $E(A)$ can be bounded from below. This means that a certain number of elements in $A$ can be represented by $b + d$ with $d \in B$ and $d \in D$, which creates more room for possible cancellations in bilinear forms to be studied later.

Iterating Lemma 7.1 gives the following decomposition of an arbitrary set $A \subseteq \mathbb{F}_p$.

**Lemma 7.2.** For any $L \geq 1$ and $A \subseteq \mathbb{F}_p$, there exist disjoint sets $A_0, A_1, \ldots, A_J \subseteq \mathbb{F}_p$ satisfying

$$A = \bigcup_{0 \leq j \leq J} A_j$$

with $J \ll L$, such that both of the following hold:

(i)

$$E(A_0) \ll \frac{(\#A)^3}{L}, \quad \min_{1 \leq j \leq J} \#A_j \gg \frac{\#A}{L};$$
(ii) for each \(1 \leq j \leq J\), there exists \(D_j \subseteq A - A\) satisfying
\[
#D_j \ll L#A,
\]
such that for each \(a_j \in A_j\), we have
\[
\#\{(a, d_j) \in A \times D_j : a_j = a - d_j\} \gg \#A/L.
\] (7.1)

Proof. Given an arbitrary \(L \geq 1\), we assume
\[
E(A) \geq \frac{(#A)^3}{L},
\]
since otherwise the proposition holds trivially with \(J = 0\) and \(A_0 = A\). By Lemma 7.1, there exist \(A_1 \subseteq A\) and \(D_1 \subseteq A - A\) such that
\[
#A_1 \gg \frac{#A}{L}
\]
and (ii) also holds for \(j = 1\).

We now write \(B_1 = A \setminus A_1\), and assume that
\[
E(B_1) \geq \frac{(#A)^3}{L},
\]
since otherwise the proposition holds with \(J = 1\) and \(A_0 = B_1\). We now apply Lemma 7.1 with
\[
A \leftarrow B_1, \quad L \leftarrow L \left(\frac{#B_1}{#A}\right)^3,
\]
getting that there exist some \(A_2 \subseteq B_1\) and \(D_2 \subseteq B_1 - B_1\) satisfying
\[
#A_2 \gg \frac{#B_1}{L} \left(\frac{#A}{#B_1}\right)^3 \gg \frac{#A}{L}
\]
and
\[
#D \ll L \left(\frac{#B_1}{#A}\right)^3 #B_1 \leq L#A,
\]
such that for each \(a_2 \in A_2\), we have
\[
\#\{(a, d_2) \in B_1 \times D_2 : a_2 = a - d_2\} \gg \frac{#B_1}{L} \left(\frac{#A}{#B_1}\right)^3 \gg \frac{#A}{L},
\]
which yields (7.1) trivially. Further iterating, if \(A_2 \neq B_1\), completes the proof.

\[
\square
\]

7.3 Counting with products

For a set \(A \subseteq \mathbb{F}_p\), define
\[
D_p(A) = \#\{(a_1, \ldots, a_8) \in A^8 : (a_1 - a_2)(a_3 - a_4) = (a_5 - a_6)(a_7 - a_8)\}.
\]
This quantity mixes the addition and multiplication in $\mathbb{F}_p$, and has been studied widely in additive combinatorics. Trivially, we have

$$D_p(A) \leq (\#A)^7.$$ 

The best known bound up to now is due to Macourt, Petridis, Shkredov and Shparlinski [39, Theorem 4.3].

**Lemma 7.3.** For any subset $A \subseteq \mathbb{F}_p$ of cardinality $A \leq p^{1/2}$, we have

$$D_p(A) \ll A^{84/13}p^{o(1)}.$$ 

We recall the following bound on the number of solutions of multiplicative congruences, which follows from a result of Ayyad, Cochrane and Zheng [1, Theorem 1]; see also Kerr [29] for a stronger statement.

**Lemma 7.4.** Suppose $A, B \subseteq \mathbb{F}_p^*$ are two intervals with $\#A = A$ and $\#B = B$. Then we have

$$\sum\sum\sum\sum 1 \leq \left(\frac{AB}{p} + 1\right)ABp^{o(1)}.$$ 

We remark that using the result of Cochrane and Shi [13, Theorem 2], one can extend Lemma 7.4 to congruences with arbitrary moduli.

**Lemma 7.5.** Suppose $A, B \subseteq \mathbb{F}_p^*$ are two intervals with $\#A = A$ and $\#B = B$. For any subset $C \subseteq \mathbb{F}_p$, we have

$$\sum\sum\sum\sum E(a_1 C, a_2 C) \leq E(C) \left(\frac{AB}{p} + 1\right)ABp^{o(1)}.$$ 

**Proof.** The Cauchy–Schwarz inequality implies

$$E(a_1 C, a_2 C)^2 \leq E(a_1 C)E(a_2 C) = E(C)^2$$

for any $a_1, a_2 \in \mathbb{F}_p^*$. The lemma follows directly from Lemma 7.4. \qed

We now derive a bound for hybrid counting problems as an alternative of Lemmas 7.3 and 7.4.

**Lemma 7.6.** Suppose $A \subseteq \mathbb{F}_p^*$ is an interval with $\#A = A$, and $C \subseteq \mathbb{F}_p$ is an arbitrary subset with $\#C = C$. Denote by $N$ the number of solutions $(a_1, a_2, c_1, c_2, c_3, c_4) \in A^2 \times C^4$ to

$$a_1(c_1 - c_2) = a_2(c_3 - c_4) \in \mathbb{F}_p^*.$$ 

For $C \leq p^{1/2}$, we have

$$N = \frac{A^2C^4}{p} + O(AC^{42/13}p^{o(1)}).$$  

(7.2)
Proof. Denote by $\mathcal{X}$ be the set of all multiplicative characters in $\mathbb{F}_p^*$ and by $\mathcal{X}^*$ the set of all non-trivial ones. Orthogonality yields

$$N = \frac{1}{p-1} \sum_{\chi \in \mathcal{X}} \left| \sum_{a \in A} \chi(a) \right|^2 \left| \sum_{c_1, c_2 \in \mathcal{C}} \chi(c_1 - c_2) \right|^2.$$ 

The contribution from the trivial character is

$$\frac{A^2C^2(C-1)^2}{p-1} = \frac{A^2C^4}{p} + O\left( \frac{A^2C^3}{p} \right),$$

which gives the main term in (7.2). Denote by $N_1$ the remaining contribution. The Cauchy–Schwarz inequality implies

$$N_1^2 \leq \frac{1}{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in A} \chi(a) \right|^4 \times \frac{1}{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{c_1, c_2 \in \mathcal{C}} \chi(c_1 - c_2) \right|^4.$$ 

The estimate of Ayyad, Cochrane and Zheng [1, Theorem 2] yields

$$\frac{1}{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in A} \chi(a) \right|^4 \leq A^2 p^{o(1)},$$

from which we obtain

$$N_1^2 \leq D_p(C)A^2 p^{o(1)}.$$ 

Then the error term in (7.2) readily follows from Lemma 7.3. $\square$

8 | BILINEAR FORMS WITH KLOOSTERMAN SUMS OVER ARBITRARY SETS

8.1 | Proof of Theorem 2.5

Without loss of generality, we assume that $\alpha, \gamma$ are both bounded by 1. Define

$$\delta = \frac{1}{M} + \frac{p^{1+1/r}}{MK^2},$$

and note that we may assume $\delta < 1$ since otherwise Theorem 2.5 becomes trivial.

Let $L \geq 1$ be some parameter and apply Lemma 7.2 to decompose $\mathcal{M}$ as the union of disjoint subsets $A_0, A_1, \ldots, A_J \subseteq \mathcal{M}$, which admit the same properties as in Lemma 7.2. In this way, we may write

$$W_{p,a}^\#(\alpha, \gamma) = \sum_{\sigma \in J \subseteq J} S_j,$$ 

where

$$S_j = \sum_{m \in A_j} \sum_{k \in \mathcal{K}} \alpha_m \gamma_k e_p(almk^{-1}).$$
We estimate $S_0$ and $S_j$ ($1 \leq j \leq J$) by different methods. The Cauchy–Schwarz inequality gives

$$|S_0|^2 \leq K \Sigma,$$

where

$$\Sigma = \sum_{m_1, m_2 \in A_0} \left| \sum_{k \in \mathcal{K}} e_p(a(m_1 - m_2)k) \right|^2.$$  

Note that $\mathcal{K}$ is an interval given by (2.3). We introduce the ‘shift by $ab$’ trick as in [20, 22, 34], which implies that for some $\xi \in \mathbb{R}$, we have

$$\Sigma \ll \log p \frac{U}{UV} \sum_{m_1, m_2 \in A_0} \sum_{k \in H \nu \sim V} \left| \sum_{v \sim V} e_p(a(m_1 - m_2)(k + uv)^{-1}) \right|^2,$$

where $H$ is another interval of length at most $2K$, and $U, V$ are to be optimised later subject to the constraint $UV \leq K$ (we recall the definition of $e(z) = \exp(2\pi iz)$ in Section 1.3).

To group variables, we put

$$R(\lambda, \mu) = \# \left\{ (k, m_1, m_2, v) : \begin{array}{l} a(m_1 - m_2) = v \lambda \in \mathbb{F}_p, \\ k = \nu \mu \in \mathbb{F}_p, \\ k \in H, m_1, m_2 \in A_0, v \sim V \end{array} \right\}$$

for $\lambda, \mu \in \mathbb{F}_p$. Now we can write

$$\Sigma \ll \log p \frac{U}{UV} \sum_{\lambda, \mu \in \mathbb{F}_p} R(\lambda, \mu) \left| \sum_{u \sim U} e_p(\lambda(\mu + u)^{-1}) \right|^2.$$

By Hölder’s inequality, we derive that

$$\Sigma \ll \log p \frac{1}{UV} \left( \Sigma_1^{1/2} \Sigma_2^{1/2} \Sigma_3^{1/2r} \right)^{2r},$$

where

$$\Sigma_1 = \sum_{\lambda, \mu \in \mathbb{F}_p} R(\lambda, \mu),$$

$$\Sigma_2 = \sum_{\lambda, \mu \in \mathbb{F}_p} R(\lambda, \mu)^2,$$

$$\Sigma_3 = \sum_{\lambda, \mu \in \mathbb{F}_p} \left| \sum_{u \sim U} e_p(\lambda(\mu + u)^{-1}) \right|^{2r}.$$  

Clearly,

$$\Sigma_1 \ll M^2 KV,$$

(8.6)
and \( \Sigma_2 \) counts all solutions to the system of congruences

\[
(m_1 - m_2)v_2 = (m_3 - m_4)v_1 \in \mathbb{F}_p,
\]

\[
k_1v_2 = k_2v_1 \in \mathbb{F}_p
\]

in \( k_1, k_2 \in \mathcal{H}, m_1, m_2, m_3, m_4 \in \mathcal{A}_0 \) and \( v_1, v_2 \sim V \). In the language of additive energies, we also have

\[
\Sigma_2 = \sum_{k_1, k_2 \in \mathcal{H}, v_1, v_2 \sim V} \sum_{k_1v_2 \equiv k_2v_1 \pmod{p}} E(v_1 \mathcal{A}_0, v_2 \mathcal{A}_0).
\]

From Lemma 7.5 and the prescribed bound

\[
E(\mathcal{A}_0) \ll \frac{M^3}{L}
\]
as in Lemma 7.2, we infer

\[
\Sigma_2 \leq M^3KV \left( \frac{KV}{p} + 1 \right) p^{o(1)}.
\]  

Appealing to the orthogonality of additive characters, we find

\[
\Sigma_3 \leq p\bar{\Sigma}_3,
\]

where \( \bar{\Sigma}_3 \) counts all solutions to the equation

\[
\sum_{1 \leq i \leq 2r} (-1)^i(\mu + u_i)^{-1} = 0
\]
in \( u_1, \ldots, u_{2r} \sim U \) and \( \mu \in \mathbb{F}_p \). If the set \( \{u_1, \ldots, u_{2r}\} \) can be partitioned into \( r \) pairs of equal elements \( u_i = u_j \), then there are \( p \) possible values for \( \mu \), and thus, the total contribution from such solutions is \( O(U^r p) \). For other choices of \( u_1, \ldots, u_{2r} \), there are obviously at most \( 2r - 1 \) possible values of \( \mu \) and thus to total contribution from such solutions is \( O(U^{2r}) \). Therefore, we find

\[
\Sigma_3 \leq p\bar{\Sigma}_3 \ll p(U^r p + U^{2r}).
\]  

Inserting the bounds (8.6), (8.7) and (8.9) to (8.5), we obtain

\[
\Sigma^{2r} \leq \frac{(M^2KV)^{2r-2}}{(UV)^{2r}} \frac{M^3KV}{L} \left( \frac{KV}{p} + 1 \right) (U^r p + U^{2r}) p^{1+o(1)}.
\]

Taking

\[
U = p^{1/r}, \quad V = K/U,
\]

we find \( V \geq 1 \) since we assume \( K \geq p^{1/r} \), and the above bound can be simplified to

\[
\Sigma^{2r} \leq \frac{M^{4r-1}K^{2r-2}}{L}(K^2 + p^{1+1/r}) p^{o(1)},
\]
which after the substitution in (8.3) implies that
\[ S_0 \ll MKL^{-1/4r} p^{o(1)} \delta^{1/4r}. \] (8.11)

We now turn to consider \( S_j \) for \( 1 \leq j \leq J \). Recalling (8.2), by the Cauchy–Schwarz inequality,
\[ |S_j|^2 \leq \#A_j \Pi_j, \] (8.12)
where
\[ \Pi_j = \sum_{m \in A_j} \left| \sum_{k \in \mathcal{K}} \gamma_k e_p(\alpha m k^{-1}) \right|^2. \]

In view of Lemma 7.2(ii), we obtain
\[ \Pi_j \ll \frac{L}{M} \sum_{m \in M} \sum_{d \in D_j} \left| \sum_{k \in \mathcal{K}} \gamma_k e_p(\alpha (m - d) k^{-1}) \right|^2. \]

Squaring out and switching summations, it follows that
\[ \Pi_j \ll \frac{L}{M} \sum_{k_1, k_2 \in \mathcal{K}} \left| \sum_{m \in M} e_p(\alpha m (k_1^{-1} - k_2^{-1})) \right|^2 \left| \sum_{d \in D_j} e_p(\alpha d (k_1^{-1} - k_2^{-1})) \right|. \]

Again, by the Cauchy–Schwarz inequality, we infer
\[ \Pi_j \ll \frac{L}{M} (\Pi_{j,1} \Pi_{j,2})^{1/2} \] (8.13)
with
\[ \Pi_{j,1} = \sum_{k_1, k_2 \in \mathcal{K}} \left| \sum_{m \in M} e_p(\alpha m (k_1^{-1} - k_2^{-1})) \right|^2, \]
\[ \Pi_{j,2} = \sum_{k_1, k_2 \in \mathcal{K}} \left| \sum_{d \in D_j} e_p(\alpha d (k_1^{-1} - k_2^{-1})) \right|^2. \]

The treatments to \( \Pi_{j,1} \) and \( \Pi_{j,2} \) are quite similar, and we only give the details for the former one. In fact,
\[ \Pi_{j,1} = \sum_{m_1, m_2 \in M} \left| \sum_{k \in \mathcal{K}} e_p(\alpha (m_1 - m_2) k^{-1}) \right|^2. \]

Following the above arguments, the ‘shift by ab’ trick in [20, 22, 34] yields
\[ \Pi_{j,1} \ll \frac{K \log p}{U^2 V} \sum_{m_1, m_2 \in M} \sum_{k \in \mathcal{K}} \sum_{\ell \sim V} \sum_{u \sim U} e(\xi u) e_p(\alpha (m_1 - m_2)(k + u)^{-1}) \left| \sum_{\xi \sim U} \sum_{\ell \sim V} e(\alpha (m_1 - m_2)(k + u)^{-1}) \right|^2 \]
for some \( \xi \in \mathbb{R} \) and \( U, V \geq \) with \( UV \leq K \), where \( H \) is an interval of length at most \( 2K \). Therefore,

\[
\Pi_{j,1} \ll \frac{K \log p}{U^2 V} \sum_{\lambda, \mu \in \mathbb{F}_p} T(\lambda, \mu) \left| \sum_{u \sim U} e(\xi u) e_p(\lambda(\mu + u)^{-1}) \right|^2 ,
\]

where \( T(\lambda, \mu) \) counts all solutions to the system of equations

\[
a(m_1 - m_2) = \nu \lambda \in \mathbb{F}_p, \quad k = \nu \mu \in \mathbb{F}_p
\]

in \( k \in H, m_1, m_2 \in M \) and \( \nu \sim V \). Following the previous arguments regarding \( S_0 \), we would like to apply Hölder’s inequality, for which the first and second moments of \( T(\lambda, \mu) \) need to be under control. In fact,

\[
\sum_{\lambda, \mu \in \mathbb{F}_p} T(\lambda, \mu) \ll M^2 KV
\]

and

\[
\sum_{\lambda, \mu \in \mathbb{F}_p} T(\lambda, \mu)^2 \leq E(M)KV(KV/p + 1) p^{o(1)} \leq M^3 KV(KV/p + 1) p^{o(1)}
\]

by Lemma 7.5 and the trivial bound \( E(M) \leq M^3 \). Therefore, Hölder’s inequality yields

\[
(\Pi_{j,1})^r \leq \left( \frac{MK^{2r-1}}{U^{2r} V} \left( \frac{KV}{p} + 1 \right) p^{o(1)} \times \sum_{\lambda, \mu \in \mathbb{F}_p} \left| \sum_{u \sim U} e(\xi u) e_p(\lambda(\mu + u)^{-1}) \right|^2 \right)^{2r}.
\]

As argued above, the last sums over \( \lambda, \mu, u \) contribute at most \( p(U^r p + U^{2r}) \), so that

\[
(\Pi_{j,1})^r \leq (MK)^{2r-1} \left( K + \frac{K^{1+1/r}}{K} \right) p^{o(1)}
\]

upon the choice of \( U \) and \( V \) as in (8.10). A similar argument shows

\[
(\Pi_{j,2})^r \leq (LMK)^{2r-1} \left( K + \frac{K^{1+1/r}}{K} \right) p^{o(1)},
\]

for which we use \( E(D) \leq (LM)^3 \) in view of Lemma 7.2 (ii).

Combining (8.14), (8.15) and (8.13), we obtain

\[
\Pi_j \leq MK^{2L^{2-1/2r}} p^{o(1)} \delta^{1/r}.
\]

Summing over \( j \), we derive from (8.12) that

\[
\sum_{1 \leq j \leq J} |S_j|^2 \leq M^2 K^2 L^{2-1/2r} p^{o(1)} \delta^{1/r}.
\]

Note that \( J \ll L \) as in Lemma 7.2, hence Cauchy–Schwarz yields

\[
\sum_{1 \leq j \leq J} |S_j| \leq MKL^{3/2-1/4r} p^{o(1)} \delta^{1/2r}.
\]
Inserting (8.11) and (8.16) into (8.1), we obtain

$$W_{p,a}^\#(\alpha, \gamma) \ll MKp^{o(1)}(L^{-1/4r} \delta^{1/4r} + L^{3/2-1/4r} \delta^{1/2r}).$$

Now Theorem 2.5 follows immediately by choosing $L = \delta^{-1/6r}$.

### 8.2 Proof of Theorem 2.6

We also assume $\alpha, \gamma$ are both bounded and proceed as in the proof of Theorem 2.5; however, we do not decompose the set $M$. Similarly, for any $U, V \geq 1$ with $UV \leq K/2$, we have

$$|W_{p,a}^\#(\alpha, \gamma)|^2 \leq K\Xi,$$

where

$$\Xi \ll \frac{\log p}{UV} \sum_{\lambda, \mu \in \mathbb{F}_p} R(\lambda, \mu) \left| \sum_{u \sim U} e(\xi u) e_p(\lambda(\mu + u)^{-1}) \right|$$

for some $\xi \in \mathbb{R}$, where $R(\lambda, \mu)$ is defined as in (8.4) with $A_0$ replaced by $M$.

As argued above, we also need to bound the first and second moments of $R(\lambda, \mu)$ when applying Hölder’s inequality. In particular, we would like to bound the second moment by virtue of Lemma 7.6 in place of Lemma 7.5. To do so, we pick out the contribution from the terms with $\lambda = 0$, so that

$$\Xi \ll \Xi_1 + MKp^{o(1)}$$

with

$$\Xi_1 = \frac{\log p}{UV} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{\mu \in \mathbb{F}_p} R(\lambda, \mu) \left| \sum_{u \sim U} e(\xi u) e_p(\lambda(\mu + u)^{-1}) \right|.$$

Now Hölder’s inequality yields

$$\Xi_1^{2r} \leq \frac{(M^2KV)^{2r-2}}{(UV)^{2r}} (pU^r + U^{2r}) p^{1+o(1)} \Sigma^*,$$

where $\Sigma^*$ counts all solutions to the system of equations

$$(m_1 - m_2)v_2 = (m_3 - m_4)v_1 \in \mathbb{F}_p^*,

k_1v_2 = k_2v_1 \in \mathbb{F}_p$$

in $k_1, k_2 \in \mathcal{H}$ with $\#\mathcal{H} \leq 2K$, $m_1, m_2, m_3, m_4 \in M$, $v_1, v_2 \sim V$ with $m_1 \neq m_2$ and $m_3 \neq m_4$. From Lemma 7.6, it follows that

$$\Sigma^* \leq K \left( \frac{M^4V^2}{p} + M^{42/13}V \right)^{o(1)}.$$
After a substitution in (8.19), we infer
\[
\Xi_1^{2r} \leq M^{4r} K^{2r} \left( \frac{1}{K} + \frac{p^{1+1/r}}{M^{10/13} K^2} \right) p^{o(1)},
\]
and combined with (8.18), we obtain
\[
\Xi \leq M^2 K \left( \frac{1}{K} + \frac{p^{1+1/r}}{M^{10/13} K^2} \right)^{1/2r} p^{o(1)} + MKp^{o(1)}.
\]
Inserting this into (8.17), Theorem 2.6 follows readily.

### 8.3 Proofs of Theorems 2.7 and 2.8

We may argue as in the proof of Theorem 2.1, and it suffices to bound the Type II sum
\[
Y = \sum_{m \in \mathcal{M}} \sum_{x \in \mathcal{H}_{l,\pm}} \alpha_m y_x e_p(mx^{-1})
\]
for \(0 \leq i \leq \lceil \log(N/2) \rceil\), where \(\mathcal{H}_{l,\pm}\) is defined by (6.1) with \(q\) replaced by \(p\). Note that \(N \leq p^{1-1/r}\) guarantees \(e^i p/N \geq p^{1/r}\) for each \(i\), and we conclude from Theorem 2.5 that
\[
Y \ll \|\alpha\|_{\infty} M p^{1+o(1)} \left( \frac{1}{M} + \frac{N^2}{M p^{1-1/r}} \right)^{7/24r}.
\]
This proves Theorem 2.7, and the proof of Theorem 2.8 can also be completed if employing Theorem 2.6 instead of Theorem 2.5.

### 9 PROOFS OF APPLICATIONS

#### 9.1 Proof of Theorem 3.1

It suffices to prove (3.2), and the general case follows from Hölder’s inequality. To this end, we put
\[
m(\lambda) = \sum_{n \in J} \mathcal{K}_p(\lambda, n)
\]
and consider the \(2\alpha\)th moment
\[
\mathfrak{M}_\alpha = \sum_{\lambda \in \mathbb{F}_p^n} |m(\lambda)|^{2\alpha}.
\]
Hölder’s inequality yields
\[
\mathfrak{M}_\alpha \leq \mathfrak{M}_1^{\alpha \gamma_{12r/7}} \mathfrak{M}_{12r/7}^{7\alpha_2/12r}
\]
with \( \alpha_1 + \alpha_2 = \alpha, \frac{12r - 7\alpha}{12r - 7} \geq 0 \), \( \alpha_1 + \frac{7\alpha}{12r} = 1 \) and \( \alpha_1, \alpha_2 \geq 0 \). Equivalently, we have

\[
\alpha_1 = \frac{12r - 7\alpha}{12r - 7} \geq 0, \quad \text{and} \quad \alpha_2 = \frac{12r(\alpha - 1)}{12r - 7} \geq 0
\]
since \( 1 \leq \alpha \leq 12r/7 \). Note that the orthogonality of additive characters gives

\[
\sum_{\lambda \in \mathbb{F}_p^*} |\chi(\lambda)|^2 \ll p^2 N.
\]

Hence,

\[
\mathcal{M}_\alpha \leq (p^2 N)^{\frac{12r - 7\alpha}{12r - 7}} (\mathcal{M}_{12r/7})^{\frac{\alpha - 1}{12r - 7}},
\]

and the general case then follows from (3.2) readily.

We now turn to prove (3.2). Performing a dyadic decomposition and pigeonhole principle, there exist some \( V > 0 \) and a subset \( \Lambda \subseteq \mathbb{F}_p^* \) given by

\[
\Lambda = \{ \lambda \in \mathbb{F}_p^* : V \leq |\chi(\lambda)| < 2V \},
\]
such that

\[
\mathcal{M}_{12r/7} \ll p^{o(1)} V^{24r/7} \# \Lambda. \tag{9.1}
\]

On the other hand,

\[
\sum_{\lambda \in \Lambda} |m(\lambda)| \geq V \# \Lambda,
\]
from which and Theorem 2.7, it follows that

\[
V(\# \Lambda)^{7/24r} \ll p^{1+o(1)} \left( 1 + \frac{N^{7/12r}}{p^{(r-1)/24r^2}} \right).
\]

Alternatively, we have

\[
V^{24r/7} \# \Lambda \ll p^{24r/7+o(1)} \left( 1 + \frac{N^2}{p^{1-1/r}} \right),
\]
from which and (9.1) we obtain (3.2), completing the proof of Theorem 3.1.

### 9.2 Proof of Theorem 3.2

We essentially follow the proof of [31, Theorem 1.2] but apply Theorem 2.1 instead of (1.8) as used in [31].

We now sketch some details. Firstly, we fix some sufficiently small \( \varepsilon > 0 \) and for each positive divisor \( d \mid q \), we define

\[
U(d) = d^2 X^{-1} \quad \text{and} \quad V(d) = d^2 X Y^{-2} (qX)^{2\varepsilon},
\]
where \( Y \in [1, X/2] \) is to be chosen later. In particular, we have

\[
U(d) \leq V(d).
\]

Then, by [31, Equation (2.9)],

\[
|\mathcal{E}(X; A, q)| \leq \mathcal{E}^*(X; A, q) + O(A(Y/q + 1)(Yq)^\varepsilon),
\]

where

\[
\mathcal{E}^*(X; A, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm} \left| \sum_{a \in A} \sum_{1 \leq n \leq V(d)} w_d(n) \mathcal{K}_d(\pm n, a) \right|
\]

with some (explicit) weights \( w_d(n) \) satisfying

\[
|w_d(n)| \leq \begin{cases} 
    X^{1+\varepsilon+o(1)}d^{-1}, & \text{if } n \leq U(d), \\
    X^{1/4+\varepsilon+o(1)}d^{1/2}n^{-3/4}, & \text{if } U(d) < n \leq V(d).
\end{cases}
\]

Next, for each \( d \mid q \), we define the integer \( \ell(d) \) by the conditions

\[
2^{\ell(d)-1} U(d) \leq V(d) < U(d)2^{\ell(d)},
\]

and set

\[
V_i(d) = \min \{ 2^i U(d), V(d) \}, \quad i = 0, \ldots, \ell(d).
\]

Hence, we derive from (9.3) that

\[
\mathcal{E}^*(X; A, q) \leq \frac{1}{q} \sum_{d|q} \sum_{\pm} \left( |E_1^+(d)| + 2^{\ell(d)-1} \sum_{i=0}^{\ell(d)} |E_{2,i}^+(d)| \right),
\]

where

\[
E_1^+(d) = \sum_{a \in A} \sum_{n \leq U(d)} w_d(n) \mathcal{K}_d(\pm n, a)
\]

\[
E_{2,i}^+(d) = \sum_{a \in A} \sum_{V_i(d) \leq n < V_{i+1}(d)} w_d(n) \mathcal{K}_d(\pm n, a),
\]

see also [31, Equation (3.11)].

In order to apply Theorem 2.1, we have to remove the coprime condition, since \( \mathcal{A} = (B, B + A] \cap \mathbb{Z}_q^* \) is not exactly an interval. To do so, we appeal to the Möbius inversion, getting

\[
E_1^+(d) = \sum_{s|q} \mu(s) \sum_{B \leq a \leq B + A} \sum_{n \leq U(d)} w_d(n) \mathcal{K}_d(\pm n, a)
\]

\[
= \sum_{s|q} \mu(s) \sum_{B/s < a \leq (B + A)/s} \sum_{n \leq U(d)} w_d(n) \mathcal{K}_d(\pm n, sa).
\]

For \( s > A \), there is at most one element in the \( a \)-sum, which contributes to \( E_1^+(d) \) at most \( XU(d)d^{-1/2+o(1)} \leq d^{3/2}q^{o(1)} \). For \( s \leq A \), the above transformations allow us to apply Theorem 2.1
directly. We then arrive at

\[
E_1^\pm (d) \ll X^\varepsilon A d^{3/2} \sum_{s|q} s^{-1} \Delta_1(d^2/X, A/s, d, \gcd(s, d)) + d^{3/2} q^{o(1)}
\]

\[
\ll (qX)^\varepsilon A d^{3/2} \sum_{s|d} \frac{1}{s} \Delta_1(d^2/X, A/s, d, s) + d^{3/2} q^{o(1)} \tag{9.5}
\]

\[
\ll (qX)^\varepsilon A d^{3/2} \Delta_1(d^2/X, A, d, 1) + d^{3/2} q^{o(1)}.
\]

Similar arguments also work for \(E_{2,i}^\pm (d)\). In fact, we may obtain

\[
E_{2,i}^\pm (d) \ll X^{1/4} d^{1/2} V_i(d)^{-3/4} \left( A V_i(d) d^{1/2} \Delta_1(V_i(d), A, d, 1) + V_i(d) d^{1/2 + o(1)} \right)
\]

\[
\ll X^{1/4} d V_i(d)^{1/4} \left( A \Delta_1(V_i(d), A, d, 1) + 1 \right) q^{o(1)} \tag{9.6}
\]

\[
\ll A X^{1/4} d V_i(d)^{1/4} \Delta_1(V_i(d), A, d, 1) + X^{1/2 + \varepsilon} Y^{-1/2} d^{3/2}.
\]

Note that

\[
V_i(d)^{1/4} \Delta_1(V_i(d), A, d, 1) \leq (2^i U(d))^{1/4} \Delta_1(2^i U(d), A, d, 1) + V(d)^{1/4} A^{-1/2},
\]

from which it follows that

\[
E_{2,i}^\pm (d) \ll (qX)^\varepsilon \left( (2^i)^{1/4} A d^{3/2} \Delta_1(2^i d^2/X, A, d, 1) + (AX/Y)^{1/2} d^{3/2} \right) \tag{9.7}
\]

(in particular, we can drop the term \(X^{1/2 + \varepsilon} Y^{-1/2} d^{3/2}\) in (9.6)).

Clearly the bound (9.7) on \(E_{2,i}^\pm (d)\) dominates that of (9.5) on \(E_1^\pm (d)\), which after the substitution in (9.4) and using the well-known bound on the divisor function (see, e.g. [27, Equation (1.81)]), yields

\[
\mathcal{E}^*(X; A, q) \ll (qX)^\varepsilon \left( (qX)^{1/2} \Delta(X, A, q) + (qAX/Y)^{1/2} \right) \tag{9.8}
\]

where

\[
\Delta(X, A, q) = \min \left\{ X^{-1/4} + q^{-1/2}, A^{1/4} q^{-1/2} + Aq^{-1}, q^{-1/2} + Aq^{-3/4} \right\}.
\]

We note that \(\Delta(X, A, q)\) is derived from \(\Delta_i(q^2/X, A, q, 1)\) where we drop the last term \(A^{-1/2}\) (which is already incorporated in (9.7) and thus in (9.8)) and then we pull out the factor \(X^{1/2} A^{-1}\).

Therefore, we now infer from (9.2) and (9.4) that

\[
\mathcal{E}(X; A, q) \ll (qX)^\varepsilon \left( (qX)^{1/2} \Delta(X, A, q) + (qAX/Y)^{1/2} + A + AY/q \right).
\]

Taking \(Y = q(X/A)^{1/3}\) to balance the last two terms, we find

\[
\mathcal{E}(X; A, q) \ll (qX)^\varepsilon \left( (qX)^{1/2} \Delta(X, A, q) + A^{2/3} X^{1/3} \right).
\]

Note that the above choice of \(Y\) satisfies \(Y \leq X/2\) since we assume \(q^3 < AX^2/8\). This completes the proof of Theorem 3.2.
Our results can be used in the same problems as the results of previous works [7, 8, 18, 34, 35, 46, 47]. For example, using Theorem 2.1, one can improve some results of [31] on average values of the divisor function over some families of short arithmetic progressions.

Theorems 2.5 and 2.6 can be further improved as long as $K$ is not too large. In fact, in the corresponding proofs, the variable $\mu$ is supported on a set of cardinality at most $KV$. Therefore, the bound (8.9) can be improved as

$$\Sigma_3 \leq p\Sigma_3 \ll p(KU^r V + U^{2r})$$

for $KV \leq p$. This leads to the optimal choice

$$U = K^{2/(r+2)} \quad \text{and} \quad V = K^{(r-1)/(r+1)}.$$

Hence, under the condition,

$$K \leq p^{(r+1)/2r},$$

instead of those in Theorems 2.5 and 2.6, we now obtain better bounds (assuming $\|\alpha\|_{\infty}, \|\gamma\|_{\infty} \leq 1$)

$$W_{p,a}^\#(\alpha, \gamma) \ll KMP^{o(1)}\left(\frac{p}{K^{2r/(r+1)}}M\right)^{1/4r}$$

and

$$W_{p,a}^\#(\alpha, \gamma) \ll KMP^{o(1)}\left(\frac{1}{M^{2r}} + \frac{1}{K} + \frac{p}{K^{2r/(r+1)}M^{10/13}}\right)^{1/4r},$$

respectively. In turn, this leads to corresponding modifications of Theorems 2.7 and 2.8.

As in [46], we can also apply our results to the double sums

$$R_q(I, J) = \sum_{m \in I} \sum_{n \in J} R_q(m, n, 1),$$

where $R_q(m, n, 1)$ is the double Kloosterman sum given by

$$R_q(m, n, \ell) = \sum_{x, y \in \mathbb{Z}^*_q} e_q(mx + ny + \ell x^{-1} y^{-1}).$$

The elementary arguments turn out to be very powerful in the studies on bilinear forms with Kloosterman sums, and the proof relies heavily on the exact shapes of such sums. The methods from $\ell$-adic cohomology employed in [18, 19, 34, 35] are quite deep and applicable to a large family of functions rather than Kloosterman sums only. It should be very meaningful and exhilarating to investigate if the above two approaches can be combined, and lead to stronger results than either of them separately.
APPENDIX: EVALUATIONS OF KLOOSTERMAN SUMS

A.1 Quadratic Gauss sums

Before presenting results related to the evaluation of Kloosterman sums, we require some facts about quadratic Gauss sums

\[ G(m, n; q) = \sum_{a \in \mathbb{Z}_q} e_q(ma^2 + na). \]

The following statements are well known (see, e.g. [17, §6]).

**Lemma A.1.** Suppose \( d = \gcd(m, q) \).

(i) If \( d = 1 \), then \( |G(m, n; q)| \leq 2\sqrt{q} \).

(ii) \( G(m, n; q) \) vanishes unless \( d \mid n \), in which case we have

\[ G(m, n; q) = dG\left(\frac{m}{d}, \frac{n}{d}; \frac{q}{d}\right). \]

In fact, we encounter the following modified quadratic Gauss sums

\[ G^*(m, n; q) = \sum_{a \in \mathbb{Z}_q^*} e_q(ma^2 + na) \tag{A.1} \]

in subsequent evaluations of Kloosterman sums. To associate the above two sums, we may appeal to the Möbius formula, getting

\[ G^*(m, n; q) = \sum_{d \mid q} \mu(d) \sum_{a \in \mathbb{Z}_q/d} e_q(m(ad)^2 + nad) \]

\[ = \sum_{d \mid q} \mu(d) G(md, n; q/d). \tag{A.2} \]

This, together with Lemma A.1 allows us to derive the following inequality.

**Lemma A.2.** Let \( q \) be a positive integer. For any \( m, n \in \mathbb{Z} \), we have

\[ G^*(m, n; q) \ll q^{1/2 + o(1)} \gcd(m, n, q)^{1/2}. \]

**Proof.** Note that \( G(md, n; q/d) \) in (A.2) vanishes unless \( \gcd(md, q/d) \mid n \) by Lemma A.1, in which case we have

\[ G(md, n; q/d) \ll \gcd(md, q/d)^{1/2}(q/d)^{1/2} \ll q^{1/2} \gcd(m, q/d)^{1/2}. \]

We then infer

\[ G^*(m, n; q) \ll q^{1/2} \sum_{d \mid q, \gcd(m, q/d) \mid n} \gcd(m, q/d)^{1/2} \]

\[ = q^{1/2} \sum_{d \mid q, \gcd(m, q/d) \mid n} \gcd(m, q/d, n)^{1/2}, \]

which readily gives Lemma A.2. \( \square \)
A.2 Vanishing of Kloosterman sums

We now turn to Kloosterman sums, and the following two lemmas characterise when such sums can vanish.

**Lemma A.3.** Suppose $q = p^j$ with a prime $p$ and an integer $j \geq 2$. For any $m \in \mathbb{Z}$ with $p \mid m$, we have

$$\mathcal{K}_q(m,1) = 0.$$  

**Proof.** Note that $x + y p^{j-1}$ presents all elements of $\mathbb{Z}_q^*$ exactly as long as $x$ runs over $\mathbb{Z}_{p^{j-1}}^*$ and $y$ runs over $\mathbb{Z}_p$, respectively. Therefore,

$$\mathcal{K}_q(m,1) = \sum_{x \in \mathbb{Z}_{p^{j-1}}^*} \sum_{y \in \mathbb{Z}_p} e_{p^j}(m(x + y p^{j-1})^{-1} + x + y p^{j-1})$$

$$= \sum_{x \in \mathbb{Z}_{p^{j-1}}^*} \sum_{y \in \mathbb{Z}_p} e_{p^j}(mx^{-1} + x + y p^{j-1})$$

since $p \mid m$. The inner sum over $y$ now vanishes due to the orthogonality of additive characters. This completes the proof. \[\square\]

**Lemma A.4.** Suppose $q = p^j$ with a prime $p$ and an integer $j \geq 2$. For any $m, n \in \mathbb{Z}$ with

$$q \nmid m, \quad q \nmid n, \quad \gcd(m,q) \neq \gcd(n,q),$$

we have

$$\mathcal{K}_q(m,n) = 0.$$  

**Proof.** Without loss of generality, we assume $m = p^\alpha m_1$, $n = p^\beta n_1$ with $\alpha < \beta < j$ and $p \nmid m_1 n_1$. From (4.2), it follows that

$$\mathcal{K}_q(m,n) = \sum_{0 \leq s \leq \alpha} p^s \mathcal{K}_{p^j}(m_1 n_1 p^{\alpha + \beta - 2s}, 1).$$

Note that $j - \alpha \geq 2$ and $\alpha + \beta - 2s \geq 1$ for any $0 \leq s \leq \alpha$. We are now in a position to apply Lemma A.3 to each Kloosterman sum on the right-hand side, and this proves the lemma immediately. \[\square\]

### A.3 Exact expressions of Kloosterman sums

The following lemma allows one to extract $\gcd(m,n,q)$ from $\mathcal{K}_q(m,n)$ if $q \nmid \gcd(m,n)$.

**Lemma A.5.** Suppose $q = p^j$ with a prime $p$ and an integer $j \geq 2$ and $m,n \in \mathbb{Z}$. Let $d = \gcd(m,n,q)$. If $d \neq q$, then we have

$$\mathcal{K}_q(m,n) = d \mathcal{K}_{q^*}(m^*,n^*),$$

where $m^* = m/d$, $n^* = n/d$ and $q^* = q/d$. 

Proof. From (4.2), it follows that

$$K_q(m, n) = \sum_{r \mid d} r K_{q/r}(m r^{-2}, 1).$$

Since $d \neq q$, we find $p^2 \mid q/r$ and $p \mid m r^{-2}$ unless $r = d$. By Lemma A.3, only the contribution from $r = d$ survives, and this proves Lemma A.5 immediately.

For the explicit evaluations of Kloosterman sums $K_q(m, n)$ with $q = p^j$ and $j \geq 2$, it suffices to consider the case $\gcd(m, n, q) = 1$ in view of Lemma A.5, which condition is equivalent to $p \nmid \gcd(m, n)$. Furthermore, the vanishing property in Lemma A.4 leads us to consider the case $\gcd(m n, q) = 1$.

We now formulate an exact expression of $K_q(m, n)$ modulo an odd prime power $q$ with $\gcd(m n, q) = 1$, and the case modulo $2^j$ can be derived in a similar manner. One may refer to Iwaniec [26, Proposition 4.3] for original resources.

Lemma A.6. Suppose $q = p^j$ with a prime $p$ and an integer $j \geq 2$ and $\gcd(q, 2mn) = 1$. We have $K_q(m, n) = 0$ unless $m \equiv l^2 n \pmod{q}$ for some $l \in \mathbb{Z}$, in which case there is

$$K_q(m, n) = \left(\frac{ln}{q}\right) q^{1/2} \text{Re} \{\varepsilon_q \varepsilon_q(2ln)\},$$

where $\left(\frac{\cdot}{q}\right)$ denotes the Legendre–Jacobi symbol $\pmod{q}$, and $\varepsilon_q = 1$ or $i$ according to $q \equiv 1$ or $-1 \pmod{4}$.

A.4 Estimates for $T(x, y, z; q)$

Our proof of Lemma 4.1 requires first considering some special cases.

Lemma A.7. Let $q = p^j$ with a prime $p$ and an integer $j \geq 2$. For $x, y, z \in \mathbb{Z}$ with $\gcd(xy, q) = 1$, we have

$$T(x, y, z; q) \ll \gcd(x - y, z, q)^{1/2} q^{1/2+o(1)}.$$
From Lemma A.6, it follows that
\[
T(x, y, z; q) = \frac{1}{2} \left( \frac{xy \gamma}{q} \right) \sum_{l \in \mathbb{Z}_{*}} \Re \{ \varepsilon_{q} e_{q}(2lx) \} \Re \{ \varepsilon_{q} e_{q}(2lc^{-1}x) \} e_{q}(-zl^{2}x).
\]

Expanding the real parts, \( T(x, y, z; q) \) bear four quadratic exponential sums of shapes \( G^{\ast}(\cdot, \cdot; q) \) as defined by \( (A.1) \). From Lemma A.2, it follows that
\[
T(x, y, z; q) \ll q^{1/2 + o(1)} \sum_{\pm} \gcd(zx, (1 \pm c^{-1})x, q)^{1/2}.
\]

Since \( c^{2}y \equiv x (\mod q) \) and \( \gcd(xy, q) = 1 \), we find
\[
\gcd((1 \pm c^{-1})x, q) = \gcd(x \pm cy, q),
\]
which divide
\[
\gcd(x^{2} - c^{2}y^{2}, q) = \gcd(x^{2} - xy, q) = \gcd(x - y, q).
\]
This completes the proof. \( \square \)

**Lemma A.8.** Let \( p \) be a prime. For any \( x, y, z \in \mathbb{Z} \), we have
\[
T(x, y, z; p) \ll \gcd(x, y, p)^{1/2} \gcd(\frac{p}{(x, y, p)})^{1/2} p^{1/2}.
\]

**Proof.** If \( p \mid z \), we find
\[
T(x, y, 0; p) = \frac{1}{p} \sum_{t \in \mathbb{Z}_{p}} \mathcal{K}_{q}(x, t) \mathcal{K}_{q}(y, t) = c_{p}(x - y)
\]
is a Ramanujan sum. The desired estimate then follows from the elementary inequality \( (4.3) \). For \( p \nmid z \), the orthogonality of additive characters yields
\[
T(x, y, z; p) = \sum_{a \in \mathbb{Z}_{p} \setminus \{z^{-1}\}} e_{p}(ax + a(az - 1)^{-1}y).
\]

With the change of variable \( a \to z^{-1}(a + 1) \), we derive that
\[
T(x, y, z; p) = \sum_{a \in \mathbb{Z}_{p} \setminus \{-1\}} e_{p}(xz^{-1}a + yz^{-1}a^{-1}) e_{p}((x + y)z^{-1})
\]
\[
= \mathcal{K}_{p}(xz^{-1}, yz^{-1}) e_{p}((x + y)z^{-1}) - 1.
\]
Now the results follows directly from the Weil bound \( (1.2) \). \( \square \)

**A.5  |  Proof of Lemma 4.1**

We are now ready to prove Lemma 4.1 for a general modulus \( q \). In view of the twisted multiplicativity \( (4.5) \) and Lemma A.8, it suffices to consider the situation of prime power moduli \( q = p^{j} \) with \( j \geq 2 \), and we split the performance to two cases according to the divisibility of \( x, y \) by \( q \).
Case I: $q \mid x$ or $q \mid y$. Without loss of generality, we may assume $q \mid x$. From (4.3), it follows that

$$|T(x, y, z; q)| \leq \frac{1}{q} \sum_{t \in \mathbb{Z}_q} \gcd(t, q) |\mathcal{K}_q(y, t)|$$

$$\leq \gcd(y, q) + q^{-1/2+o(1)} \gcd(y, q)^{1/2} \sum_{1 \leq t < q} \gcd(t, q)$$

$$\leq q^{1/2+o(1)} \gcd(y, q)^{1/2}$$

as desired.

Case II: $q \nmid x$ and $q \nmid y$. We first extract the zeroth frequency, and the bound (4.3) for Ramanujan sums gives

$$|T(x, y, z; q)| \leq |T_1(x, y, z; q)| + \frac{\gcd(x, q) \gcd(y, q)}{q}$$

with

$$T_1(x, y, z; q) = \frac{1}{q} \sum_{1 \leq t < q} \mathcal{K}(x, t) \mathcal{K}_q(y, t) e_q(-zt).$$

For each $1 \leq t < q$, Lemma A.4 yields that $\mathcal{K}_q(x, t) \mathcal{K}_q(y, t)$ vanishes unless

$$\gcd(x, q) = \gcd(y, q) = \gcd(t, q) = p^k$$

for some $k = k(t) < j$, in which case Lemma A.5 guarantees

$$\mathcal{K}_q(x, t) = p^k \mathcal{K}_q(x_0, t_0), \quad \mathcal{K}_q(y, t) = p^k \mathcal{K}_q(y_0, t_0)$$

with $x_0 = x/p^k, y_0 = y/p^k, t_0 = t/p^k, q_0 = q/p^k$. Hence, we have $\gcd(x_0, y_0, q_0) = 1$ and

$$T_1(x, y, z; q) = p^k T_1(x_0, y_0, z; q_0).$$

(A.4)

Note that

$$T(x_0, y_0, z; q_0) - T_1(x_0, y_0, z; q_0) \ll \frac{1}{q_0}$$

in view of (4.3), and Lemma A.7 yields

$$T_1(x_0, y_0, z; q_0) \ll \frac{1}{q_0} + \gcd(x_0 - y_0, z, q_0)^{1/2} q_0^{1/2+o(1)},$$

from which and (A.3), (A.4), Lemma 4.1 follows immediately.

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