A cubical flat torus theorem and some of its applications

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Abstract

The article is dedicated to the proof of the following cubical version of the flat torus theorem. Let $G$ be a group acting on a CAT(0) cube complex $X$ and $A \leq G$ a normal finitely generated abelian subgroup. Then there exists a median subalgebra $Y \subset X$ which is $G$-invariant and which decomposes as a product of median algebras $T \times F \times Q$ such that: (1) the action $G \curvearrowright Y$ decomposes as a product of actions $G \curvearrowright T, F, Q$; (2) $F$ is a flat; (3) $Q$ is a finite-dimensional cube; (4) $A$ acts trivially on $T$. Some applications are included. For instance, a splitting theorem is proved and we show that a polycyclic group acting properly on a CAT(0) cube complex must be virtually abelian.

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1 Introduction

In the last decades, CAT(0) cube complexes have become fruitful tools in the study of groups. The reason of this success is twofold. First, many groups of interest turn out to act non-trivially on CAT(0) cube complexes, including many Artin groups, many 3-manifold groups, Coxeter groups, many small cancellation groups, one-relator groups with torsion, many free-by-cyclic groups, random groups and some Burnside groups. And second, powerful tools are now available to answer various questions about cube complexes. As a consequence, looking for an action on a CAT(0) cube complex in order to study a given group turns out to be a good strategy in order to find valuable information on it.

In this article, we are interested in the structure of subgroups of groups acting on CAT(0) cube complexes. In finite dimension, generalising the Tits alternative proved in [SW05], it follows from the combination of [CS11] and [CFI16] that, if a group acts properly on a finite-dimensional CAT(0) cube complex, then any of its subgroups either contains a non-abelian free subgroup or is virtually (locally finite)-by-(free abelian). Examples of groups in the second category include for instance wreath products $F \wr \mathbb{Z}^n$ where $F$ is a finite group and $n \geq 1$ [Gen17a, Proposition 9.33] and Houghton groups [Lee12], [FH14, Example 4.3]. However, if we allow infinite-dimensional cube complexes, the dichotomy no longer holds. For instance, Thompson’s group $F$ acts on an infinite-dimensional CAT(0) cube complex [Far03] but it does not contain any non-abelian free group and it is not virtually free abelian (in fact, it is even not virtually solvable).

There is still a lot to understand about groups acting on infinite-dimensional CAT(0) cube complexes. Here, we are mainly concerned with the following question: which solvable groups act properly on (infinite-dimensional) CAT(0) cube complexes? Partial answers can be found from CAT(0) geometry, in particular thanks to the flat torus theorem [BH99, Theorem II.7.1]. However, this theorem only applies for actions by semi-simple isometries, so avoiding parabolic isometries. And it turns out that groups may act on infinite-dimensional CAT(0) cube complexes with parabolic isometries, like Thompson’s group $F$ [AKWW13, Sal12].

The main goal of this article is to state and prove a cubical version of the flat torus theorem, and deduce some information about solvable groups acting on cube complexes. Here is our main result:

**Theorem 1.1.** Let $G$ be a group acting on a CAT(0) cube complex $X$ and $A \leq G$ a normal, finitely generated, abelian subgroup. Then there exists a median subalgebra $Y \subset X$ which is $G$-invariant and which decomposes as a product $T \times F \times Q$ of median algebras $T, F, Q$ such that:

- $G \curvearrowright Y$ decomposes as a product of actions $G \curvearrowright T, F, Q$;
- $F$ is a flat;
- $Q$ is a finite-dimensional cube;
- $A$ acts trivially on $T$.

This statement requires some explanation. There exists a natural ternary operation on the vertices of a CAT(0) cube complex. Namely, for any triple of vertices $x, y, z \in X$, there exists a unique vertex $m \in X$, called the *median point*, such that

$$
\begin{align*}
  d(x, y) &= d(x, m) + d(m, y) \\
  d(x, z) &= d(x, m) + d(m, z) \\
  d(y, z) &= d(y, m) + d(m, z)
\end{align*}
$$
where \( d \) denotes the metric in the one-skeleton. Otherwise saying, \( m \) belongs to the intersection of three geodesics between \( x, y, z \). A median subalgebra of \( X \) referred to a set of vertices which is stable under the median operation. In comparison to the traditional flat torus theorem, median subalgebras play the role of convex subspaces. A flat can be thought of as a median subalgebra of a cube complex \( \mathbb{R}^n, n \geq 1 \). We refer to Section 3 for a precise definition.

The main interest of our cubical flat torus theorem is that it provides an action on a flat. Finding such an action is interesting because the isometry group of (the cubulation of) a flat is quite specific, imposing severe restrictions on the action we are looking at. Our main application is the following statement:

**Theorem 1.2.** Let \( G \) be a solvable group acting on a CAT(0) cube complex. Assume that \( G \) does not contain an infinite torsion subgroup and that it does not contain a free abelian group of arbitrary large rank. Then \( G \) contains a finite-index subgroup which is (locally elliptic)-by-(free abelian).

We emphasize that we are not imposing any restriction on the action and that the cube complex may be infinite-dimensional. As a consequence of Theorem 1.2, it turns out that many solvable groups do not act properly on CAT(0) cube complexes:

**Corollary 1.3.** Let \( G \) be a solvable group acting properly on a CAT(0) cube complex. Assume that \( G \) does not contain an infinite torsion subgroup and that it does not contain a free abelian group of arbitrary large rank. Then \( G \) must be virtually free abelian.

The statement is essentially sharp. Indeed, the wreath products \( \mathbb{Z}_2 \wr \mathbb{Z} \), which does not contain \( \mathbb{Z}^2 \) but which contains an infinite torsion subgroup, and \( \mathbb{Z} \wr \mathbb{Z} \), which is torsion-free but which contains a free abelian group of infinite rank, act properly on CAT(0) cube complexes. See [CSV12, Gen17b] for more details.

**Corollary 1.3** covers [Cor13, Corollary 6.C.8], which proves the same statement for polycyclic groups. Notice also that the combination of [Con00] and [Hag07] proves the statement for finitely generated solvable groups of finite virtual cohomological dimension. However, we expect that Theorem 1.2 will be useful in the study of groups acting on (infinite-dimensional) CAT(0) cube complexes with infinite vertex-stabilisers. In a forthcoming article, we plan to apply this strategy in order to study solvable groups in braided Thompson’s group \( \mathbb{br}V \).

Other interesting results follow from our cubical flat torus theorem, including the following one, which proves that centralisers in cubulable groups are cubulable as well, improving a statement of [Hae15]:

**Theorem 1.4.** Let \( G \) be a group acting geometrically on a CAT(0) cube complex \( X \). For every infinite-order element \( g \in G \), the centraliser \( C_G(g) \) also acts geometrically on a CAT(0) cube complex.

Also, we deduce a splitting theorem (compare to [BH99, Theorem II.6.12]), namely:

**Theorem 1.5.** Let \( G \) be a group acting on a CAT(0) cube complex \( X \) and \( A \leq G \) a central, torsion-free, finitely generated subgroup. Assume that a non-trivial element of \( A \) is never elliptic. Then \( A \) is a direct factor of some finite-index subgroup of \( G \).

As a classical application, it follows that most mapping class groups of surfaces do not act properly on CAT(0) cube complexes. See Section 6.2 for a precise statement.

**Comparison to other cubical flat torus theorems.** Theorem 1.1 is not the first cubical version of the flat torus theorem which appears in the literature. The first one is [WW17, Theorem 3.6], proving that, if a group acts geometrically on a CAT(0)
cube complex, then its highest virtually abelian subgroups act geometrically on convex subcomplexes isomorphic to products of quasi-lines. So this statement (whose proof is fundamentally based on the traditional flat torus theorem) deals with geometric actions and only with specific virtually abelian subgroups. Next, the theorem was generalised in [Woo17], proving that, if a virtually abelian group acts properly on a CAT(0) cube complex, then it has to preserve an isometrically embedded subcomplex isomorphic to a product of quasi-lines. Compared to this statement, Theorem 1.1 does not make any assumption on the action, and it applies to groups containing normal abelian subgroups. The latter difference is the key of the article: it explains why we are able to study solvable groups instead of virtually abelian groups only. Another difference compared to [WW17, Woo17] is that we find median subalgebras instead of isometrically embedded subcomplexes. The reason is that, if we want to replace median subalgebras with isometrically embedded subcomplexes in the statement of Theorem 1.1, then the conclusion may no longer hold. For instance, let $Y$ be the cube complex obtained by gluing two squares along a single vertex $p$ and let $d$ denote the union of the two diagonals of the squares which meet at $p$. If we consider the cube complex $Y \times \mathbb{R}$ and the isometry $g = (\text{reflection along } d, \text{translation})$, then the only reasonable isometrically embedded subcomplex left invariant by $\langle g, \text{Isom}(Y) \rangle$ is the whole complex. But $g$ does not act trivially on $Y$ (which would play the role of $T$ in Theorem 1.1). The fact that $g$ acts trivially on $T$ is fundamental in our arguments: as a consequence, we can look at the action on $T$ of the quotient of the group by $\langle g \rangle$, which allows an induction. Nevertheless, it turns out that a well-chosen subdivision of (the cubulation of) a median subalgebra naturally embeds isometrically in the initial complex. As a consequence, the main theorem of [Woo17] can be recovered from Theorem 1.1. However, we did not write the construction because it was not necessary to have subcomplexes for our applications.

Organisation of the article. Section 2 is dedicated to the definitions, basic properties and preliminary lemmas which will be used in the rest of the article. Next, in Section 3, we show that isometry groups of (cubulations of) flats are quite specific and we show how to associate a flat to any loxodromic isometry. In Section 4, we prove our cubical flat torus theorem for infinite cyclic groups, i.e., the section is dedicated to the dynamics of isometries. The proof of Theorem 1.1 is achieved in Section 5. Finally, Section 6 contains various applications, including Theorems 1.2, 1.4 and 1.5.

2 Preliminaries

2.1 Cube complexes, hyperplanes, projections

A cube complex is a CW complex constructed by gluing together cubes of arbitrary (finite) dimension by isometries along their faces. It is nonpositively curved if the link of any of its vertices is a simplicial flag complex (i.e., $n + 1$ vertices span a $n$-simplex if and only if they are pairwise adjacent), and CAT(0) if it is nonpositively curved and simply-connected. See [BH99] page 111] for more information.

Fundamental tools when studying CAT(0) cube complexes are hyperplanes. Formally, a hyperplane $J$ is an equivalence class of edges with respect to the transitive closure of the relation identifying two parallel edges of a square. Notice that a hyperplane is uniquely determined by one of its edges, so if $e \in J$ we say that $J$ is the hyperplane dual to $e$. Geometrically, a hyperplane $J$ is rather thought of as the union of the midcubes transverse to the edges belonging to $J$ (sometimes referred to as its geometric realisation). See Figure 1. The carrier $N(J)$ of a hyperplane $J$ is the union of the cubes intersecting (the geometric realisation of) $J$. 
There exist several metrics naturally defined on a CAT(0) cube complex. In this article, we are only interested in the graph metric defined on its one-skeleton, referred to as its \textit{combinatorial metric}. In fact, from now on, we will identify a CAT(0) cube complex with its one-skeleton, thought of as a collection of vertices endowed with a relation of adjacency. In particular, when writing \( x \in X \), we always mean that \( x \) is a vertex of \( X \).

The following theorem will be often used along the article without mentioning it.

**Theorem 2.1.** \cite{Sag95} Let \( X \) be a CAT(0) cube complex.

- If \( J \) is a hyperplane of \( X \), the graph \( X \setminus J \) obtained from \( X \) by removing the (interiors of the) edges of \( J \) contains two connected components. They are convex subgraphs of \( X \), referred to as the halfspaces delimited by \( J \).

- A path in \( X \) is a geodesic if and only if it crosses each hyperplane at most once.

- For every \( x, y \in X \), the distance between \( x \) and \( y \) coincides with the cardinality of the set \( W(x, y) \) of the hyperplanes separating them.

Another useful tool when studying CAT(0) cube complexes is the notion of projection onto a convex subcomplex, which is defined by the following proposition (see \cite[Lemma 13.8]{HW08}):

**Proposition 2.2.** Let \( X \) be a CAT(0) cube complex, \( C \subset X \) a convex subcomplex and \( x \in X \setminus C \) a vertex. There exists a unique vertex \( y \in C \) minimizing the distance to \( x \). Moreover, for any vertex of \( C \), there exists a geodesic from it to \( x \) passing through \( y \).

Below, we record a couple of statements related to projections for future use. Proofs can be found in \cite[Lemma 13.8]{HW08} and \cite[Proposition 2.7]{Gen16} respectively.

**Lemma 2.3.** Let \( X \) be a CAT(0) cube complex, \( Y \subset X \) a convex subcomplex and \( x \in X \) a vertex. Any hyperplane separating \( x \) from its projection onto \( Y \) separates \( x \) from \( Y \).

**Lemma 2.4.** Let \( X \) be a CAT(0) cube complex and \( Y \subset X \) a convex subcomplex. For every vertices \( x, y \in X \), the hyperplanes separating the projections of \( x \) and \( y \) onto \( Y \) are precisely the hyperplanes separating \( x \) and \( y \) which cross \( Y \). As a consequence the projection onto \( Y \) is 1-Lipschitz.
2.2 Classification of isometries

Following [Hag07], isometries of CAT(0) cube complexes can be classified into three families. Namely, given a CAT(0) cube complex $X$ and an isometry $g \in \text{Isom}(X)$,

- $g$ is **elliptic** if its orbits are bounded;
- $g$ is **inverting** if its orbits are unbounded and if one of its powers inverts a hyperplane, i.e., it stabilises it and swap the halfspaces it delimits;
- $g$ is **loxodromic** if there exists a bi-infinite geodesic on which $g$ acts by translation of length $\|g\| := \min\{d(x, gx) \mid x \in X\}$, referred to as an **axis** of $g$.

More precisely, Haglund showed that, if $g$ is neither elliptic nor inverting, then, for every vertex $z$ of the **minimising set**

$$\text{Min}(g) := \left\{ x \in X \mid d(x, gx) = \min_{y \in X} d(y, gy) \right\}$$

and for every geodesic $[z, gz]$ between $z$ and $gz$, the concatenation $\bigcup_{k \in \mathbb{Z}} g^k \cdot [z, gz]$ is a geodesic on which $g$ acts by translations of length $\|g\|$.

We emphasize that, as shown by [Hag07, Proposition 5.2], if $g$ stabilises a geodesic and acts on it by translations of positive lengths, then it is necessarily an axis of $g$, that is to say, $g$ automatically acts on this geodesic by translations of length $\|g\|$.

Finally, let us mention that an isometry is elliptic if and only if it stabilises a (finite-dimensional) cube (see for instance [Rol98, Theorem 11.7] for a proof). As a consequence, another characterisation is that an isometry is elliptic if and only if one of its orbits is finite.

2.3 Wallspaces, cubical quotients

Given a set $X$, a **wall** $\{A, B\}$ is a partition of $X$ into two non-empty subsets $A, B$, referred to as **halfspaces**. Two points of $X$ are **separated** by a wall if they belong to two distinct subsets of the partition.

A **wallspace** $(X, W)$ is the data of a set $X$ and a collection of walls $W$ such that any two points are separated by only finitely many walls. Such a space is naturally endowed with the pseudo-metric

$$d : (x, y) \mapsto \text{number of walls separating } x \text{ and } y.$$

As shown in [CN05, Nic04], there is a natural CAT(0) cube complex associated to any wallspace. More precisely, given a wallspace $(X, W)$, define an orientation $\sigma$ as a collection of halfspaces such that:

- for every $\{A, B\} \in W$, $\sigma$ contains exactly one subset among $\{A, B\}$;
- if $A$ and $B$ are two halfspaces satisfying $A \subset B$, then $A \in \sigma$ implies $B \in \sigma$.

Roughly speaking, an orientation is a coherent choice of a halfspace in each wall. As an example, if $x \in X$, then the set of halfspaces containing $x$ defines an orientation. Such an orientation is referred to as a **principal orientation**. Notice that, because any two points of $X$ are separated by only finitely many walls, two principal orientations are always **commensurable**, i.e., their symmetric difference is finite.

The **cubulation** of $(X, W)$ is the cube complex
Figure 2: A CAT(0) cube complex and one of its cubical quotients.

- whose vertices are the orientations within the commensurability class of principal orientations;
- whose edges link two orientations if their symmetric difference has cardinality two;
- whose $n$-cubes fill in all the subgraphs isomorphic to one-skeleta of $n$-cubes.

See Figure 2 for an example.

Definition 2.5. Let $X$ be a CAT(0) cube complex and $\mathcal{J}$ be a collection of hyperplanes. Let $\mathcal{W}(\mathcal{J})$ denote the set of partitions of $X$ induced by the hyperplanes of $\mathcal{J}$. The cubical quotient $X/\mathcal{J}$ of $X$ by $\mathcal{J}$ is the cubulation of the wallspace $(X, \mathcal{W}(\mathcal{J}))$.

See Figure 2 for an example. It can be shown that $X/\mathcal{J}$ can be obtained from $X$ in the following way. Given a hyperplane $J \in \mathcal{J}$, cut $X$ along $J$ to obtain $X\setminus J$. Each component of $X\setminus J$ contains a component of $N(J)\setminus J = N_1 \sqcup N_2$. Notice that $N_1$ and $N_2$ are naturally isometric: associate to each vertex of $N_1$ the vertex of $N_2$ which is adjacent to it in $N(J)$. Now, glue the two components of $X\setminus J$ together by identifying $N_1$ and $N_2$. The cube complex obtained is still CAT(0) and its set of hyperplanes corresponds naturally to $\mathcal{H}(X\setminus J)$ if $\mathcal{H}(X)$ denotes the set of hyperplanes of $X$. Thus, the same construction can repeated with a hyperplane of $\mathcal{J}\setminus \{J\}$, and so on. The cube complex which is finally obtained is the cubical quotient $X/\mathcal{J}$.

Notice that a quotient map is naturally associated to a cubical quotient, namely:

\[
\begin{align*}
X & \rightarrow X/\mathcal{J} \\
x & \mapsto \text{principal orientation defined by } x
\end{align*}
\]

The next lemma relates the distance between two vertices of $X$ to the distance between their images in $X/\mathcal{J}$.

Lemma 2.6. Let $X$ be a CAT(0) cube complex and $\mathcal{J}$ a collection of hyperplanes. If $\pi : X \rightarrow X/\mathcal{J}$ denotes the canonical map, then

\[d_{X/\mathcal{J}}(\pi(x), \pi(y)) = \#(\mathcal{W}(x, y)\setminus \mathcal{J})\]

for every vertices $x, y \in X$.

Recall that, for every vertices $x, y \in X$, the set $\mathcal{W}(x, y)$ denotes the collection of the hyperplanes of $X$ separating $x$ and $y$. 

\[\square\]
2.4 Roller boundary

Let $X$ be a CAT(0) cube complex. An orientation of $X$ is an orientation of the wallspace $(X, \mathcal{W}(J))$, as defined in the previous section, where $J$ is the set of all the hyperplanes of $X$. The Roller compactification $\overline{X}$ of $X$ is the set of the orientations of $X$. Usually, we identify $X$ with the image of the embedding $\{ X \rightarrow \mathcal{X} \}$ principal orientation defined by $x$ and we define the Roller boundary of $X$ by $\mathcal{R}X := \overline{X} \setminus X$.

The Roller compactification is naturally endowed with a topology via the inclusion $\overline{X} \subset 2^{\{\text{halfspaces}\}}$ where $2^{\{\text{halfspaces}\}}$ is endowed with the product topology. Otherwise saying, a sequence of orientations $(\sigma_n)$ converges to $\sigma \in \overline{X}$ if, for every finite collection of halfspaces $D$, there exists some $N \geq 0$ such that $\sigma_n \cap D = \sigma \cap D$ for every $n \geq N$.

The Roller compactification is also naturally a cube complex. Indeed, if we declare that two orientations are linked by an edge if their symmetric difference has cardinality two and if we declare that any subgraph isomorphic to the one-skeleton of an $n$-cube is filled in by an $n$-cube for every $n \geq 2$, then $\overline{X}$ is a disjoint union of CAT(0) cube complexes. Each such component is referred to as a cubical component of $\overline{X}$. See Figure 3 for an example. Notice that the distance (possibly infinite) between two vertices of $\overline{X}$ coincides with the number of hyperplanes which separate them, if we say that a hyperplane $J$ separates two orientations when they contain different halfspaces delimited by $J$.

An alternative description of the Roller boundary is the following. Let $X$ be a CAT(0) cube complex and $x \in X$ a basepoint. Denote by $\mathcal{S}_x X$ the collection of the geodesic rays of $X$ starting from $x$ up to the equivalence relation which identifies two rays if they cross exactly the same hyperplanes. According to [Gen16, Proposition A.2], the map

$$
\begin{align*}
\mathcal{S}_x X & \rightarrow \mathcal{R}X \\
 r & \mapsto \alpha(r)
\end{align*}
$$

is a bijection, where $\alpha(r)$ denotes the orientation containing the halfspaces in which the ray $r$ is eventually included. Notice that the metric in the cube complex $\overline{X}$ corresponds
to the metric \((r_1, r_2) \mapsto \#(\mathcal{H}(r_1) \Delta \mathcal{H}(r_2))\) in \(\mathcal{G}_2 X\), where \(\mathcal{H}(\cdot)\) denotes the collection of the hyperplanes which are crossed by the ray we are looking at and where \(\Delta\) denotes the symmetric difference.

We conclude this subsection by proving a lemma which will be useful later.

**Lemma 2.7.** Let \(X\) be a CAT(0) cube complex which is isomorphic to an isometrically embedded subcomplex of \(\mathbb{R}^n\) for some \(n \geq 1\). Then \(X\) has finitely many cubical components and one of them is bounded.

**Proof.** If \(X\) is bounded, there is nothing to prove, so we suppose that \(X\) is unbounded. Without loss of generality, suppose that \(X\) is an isometrically embedded subcomplex of \(\mathbb{R}^n\), where \(n \geq 1\). As a consequence of the description of \(\mathfrak{R}X\) in terms of geodesic rays, if we choose \((0, \ldots, 0)\) as our basepoint then any point of \(\mathfrak{R}X\) can be represented by an element of \(\mathbb{Z}^n\) where \(\mathbb{Z} = \mathbb{Z} \cup \{\pm \infty\}\). Moreover, the distance (possibly infinite) in the cube complex \(\mathfrak{R}X\) between two points represented by \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) coincides with

\[
\sum_{i=1}^n |a_i - b_i|, \text{ where by convention } \begin{cases} 
\infty - \infty = -\infty + \infty = 0 \\
\infty + \infty = \infty \\
-\infty - \infty = -\infty
\end{cases}.
\]

As a consequence, two distinct points of \(\mathfrak{R}X\) cannot be represented by the same element of \(\mathbb{Z}^n\). From this description, it is clear that \(\mathfrak{R}X\) has at most \(3^n - 1\) cubical components, proving the first assertion of our lemma.

Now fix a point of \(\alpha \in \mathfrak{R}X\) whose representation in \(\overline{\pi} \in \overline{\mathbb{Z}}^n\) contains a maximal number of infinite coordinates. Up to permuting and inverting the coordinates of our representation, we may suppose without loss of generality that \(\overline{\pi} = (a_1, \ldots, a_r, +\infty, \ldots, +\infty)\) where \(r \geq 0\) and \(a_1, \ldots, a_r \in \mathbb{Z}\). Suppose by contradiction that the cubical component of \(\mathfrak{R}X\) containing \(\alpha\) is unbounded. It implies that, once again up to permuting and inverting coordinates, there exist some \(s \geq 1\), some \(b_1, \ldots, b_{r-s-1} \in \mathbb{Z}\) and increasing sequences \((a_{r-s}(k)), \ldots, (a_r(k))\) such that

\[
(b_1, \ldots, b_{r-s-1}, a_{r-s}(k), \ldots, a_r(k), +\infty, \ldots, +\infty)
\]

represents a point of \(\mathfrak{R}X\) for every \(k \geq 0\). By construction of our representation, there exist increasing sequences \((c_{n-r}(p)), \ldots, (c_n(p))\) such that

\[
(b_1, \ldots, b_{r-s-1}, a_{r-s}(k), \ldots, a_r(k), c_{n-r}(p), \ldots, c_n(p))
\]

represents a point of \(X\) for every \(k \geq 0\) and \(p \geq 0\). Set

\[
q(k) := (b_1, \ldots, b_{r-s-1}, a_{r-s}(k), \ldots, a_r(k), c_{n-r}(k), \ldots, c_n(k))
\]

for every \(k \geq 0\). Notice that the fact that our sequences are increasing implies that the points \(q(0), q(1), \ldots \) all belong to a common geodesic ray starting from \((0, \ldots, 0)\). As a consequence, the point \((b_1, \ldots, b_{r-s-1}, +\infty, \ldots, +\infty)\) represents a point of \(\mathfrak{R}X\), contradicting the definition of \(\alpha\).

Thus, we have proved that the cubical component of \(\mathfrak{R}X\) containing \(\alpha\) is bounded, concluding the proof of our lemma.

\[\square\]

### 2.5 Median algebras

A **median algebra** \((X, \mu)\) is the data of a set \(X\) and a map \(\mu : X \times X \times X \to X\) satisfying the following conditions:
• $\mu(x, y, y) = y$ for every $x, y \in X$;
• $\mu(x, y, z) = \mu(z, x, y) = \mu(x, z, y)$ for every $x, y, z \in X$;
• $\mu((\mu(x, w, y), w, z)) = \mu(x, w, \mu(y, w, z))$ for every $x, y, z, w \in X$.

The interval between two points $x, y \in X$ is
\[
I(x, y) = \{ z \in X \mid \mu(x, y, z) = z \};
\]
and a subset $Y \subset X$ is convex if $I(x, y) \subset Y$ for every $x, y \in Y$. In this article, we are only interested in median algebras whose interval are finite; they are referred to as discrete median algebras.

As proved in [Nic04], a discrete median algebra is naturally a wallspace. Indeed, let us say that $Y \subset X$ is a halfspace if $Y$ and $Y^c$ are both convex. Then a wall of $X$ is the data of halfspace and its complement, and it turns out that only finitely many walls separate two given point of $X$. The cubulation of a discrete median algebra refers to the cubulation of this wallspace. In this specific case, it turns out that any orientation commensurable to a principal orientation must be a principal orientation itself. Consequently, the cubulation of a discrete median algebra $X$ coincides with the cube complex
• whose vertex-set is $X$;
• whose edges link two points of $X$ if they are separated by a single wall;
• whose $n$-cubes fill in every subgraph of the one-skeleton isomorphic to the one-skeleton of an $n$-cube, for every $n \geq 2$.

Therefore, a discrete median algebra may be naturally identified with its cubulation, and so may be thought of as a CAT(0) cube complex. The dimension and the Roller compactification of a discrete median algebra coincides with the dimension and the Roller compactification of its cubulation.

Conversely, a CAT(0) cube complex $X$ naturally defines a discrete median algebra (see [Che00] and [Hag08, Proposition 2.21]). Indeed, for every triple of vertices $x, y, z \in X$, there exists a unique vertex $\mu(x, y, z) \in X$ satisfying
\[
\begin{align*}
  d(x, y) &= d(x, \mu(x, y, z)) + d(\mu(x, y, z), y) \\
  d(x, z) &= d(x, \mu(x, y, z)) + d(\mu(x, y, z), z) \\
  d(y, z) &= d(y, \mu(x, y, z)) + d(\mu(x, y, z), z)
\end{align*}
\]

Otherwise saying, $I(x, y) \cap I(y, z) \cap I(x, z) = \{\mu(x, y, z)\}$. The vertex $\mu(x, y, z)$ is referred to as the median point of $x, y, z$. Then $(X, \mu)$ is a discrete median algebra, motivating the following terminology:

**Definition 2.8.** Let $X$ be a CAT(0) cube complex. A median subalgebra $Y \subset X$ is a set of vertices stable under the median operation.

The median structure defined on $X$ can be extended continuously to the Roller compactification $\overline{X}$. Namely, given three orientations $\sigma_1, \sigma_2, \sigma_3$, define $\mu(\sigma_1, \sigma_2, \sigma_3)$ as the set of halfspaces which belong to at least two orientations among $\{\sigma_1, \sigma_2, \sigma_3\}$. Then the map $\mu : \overline{X} \times \overline{X} \times \overline{X} \to \overline{X}$ extends the previous median operation, making $(\overline{X}, \mu)$ a median algebra (not discrete in general). Moreover, the ternary operator turns out to be continuous when $\overline{X}$ is endowed with the topology defined in the previous section.

We conclude this subsection by stating and proving a few lemmas which will be useful later.
Lemma 2.9. Let \((X_1, \mu_1)\) and \((X_2, \mu_2)\) be two discrete median algebras of dimensions \(d_1\) and \(d_2\) respectively. Then \((X_1 \times X_2, \mu_1 \times \mu_2)\) is a discrete median algebra of dimension \(d_1 + d_2\).

The proof is straightforward and it is left to the reader.

Lemma 2.10. Let \(X\) be a CAT(0) cube complex and \(Y \subset X\) a median subalgebra. The cubulation of \(Y\) coincides with the cubulation of the wallspace obtained from cutting \(Y\) along the hyperplanes of \(X\).

Proof. It is sufficient to show that, for every \(D \subset Y\), \(D\) is a halfspace of the median algebra \(Y\) if and only if \(D\) is the trace of a halfspace of \(X\). It is clear that the trace of a halfspace of \(X\) provides a halfspace of the median algebra of \(Y\), so let \(D\) be a halfspace of the median algebra \(Y\).

Fix two points \(x \in D\) and \(y \in D^c\) minimising the distance between \(D\) and \(D^c\). Fixing a hyperplane \(J\) of \(X\) separating \(x\) and \(y\), we claim that \(J\) separates \(D\) and \(D^c\). For convenience, let \(J^-\), \(J^+\) denote the halfspaces delimited by \(J\) such that \(x \in J^-\) and \(y \in J^+\).

Let \(z \in Y\) be a point which is not separated from \(x\) by \(J\). Notice that the median point \(\mu(x, y, z)\) has to belong to \(J^-\) by convexity of \(J^-\). In particular, \(d(y, \mu(x, y, z)) > 0\). Because \(Y\) is median, it has to contain \(\mu(x, y, z)\). But, since \(d(x, \mu(x, y, z)) < d(x, y)\), \(\mu(x, y, z)\) cannot belong to \(D^c\), hence \(\mu(x, y, z) \in D\). Consequently,
\[
d(x, \mu(x, y, z)) = d(x, y) - d(\mu(x, y, z), y) = d(y, D) - d(\mu(x, y, z), y) \\
\leq d(y, \mu(x, y, z)) - d(\mu(x, y, z), y) = 0,
\]

hence \(\mu(x, y, z) = x\). Therefore, \(\mu(x, y, z)\) belongs to the interval between \(y\) and \(z\). As a consequence, it \(z\) belongs to \(D^c\), then \(\mu(x, y, z)\) has to belong to \(D^c\) by convexity, which is not the case, so \(z \in D\).

Thus, we have proved that \(J^- \cap Y \subset D\). We show similarly that \(J^+ \cap Y \subset D^c\), hence \(J^- \cap Y = D\) and \(J^+ \cap Y = D^c\). The proof of our lemma is complete. \(\square\)

Lemma 2.11. Let \(X\) be a CAT(0) cube complex, and let \(\{a, b, c\}\) and \(\{x, y, z\}\) be two triples of vertices of \(X\). If a hyperplane \(J\) separates the median points \(\mu(a, b, c)\) and \(\mu(x, y, z)\), then it has to separate \(a\) and \(x\), or \(b\) and \(y\), or \(c\) and \(z\).

Proof. Let \(J^-\) and \(J^+\) denote the halfspaces delimited by \(J\) which contain \(\mu(a, b, c)\) and \(\mu(x, y, z)\) respectively. By convexity of halfspaces, \(J^-\) must contain at least two vertices of \(\{a, b, c\}\) and similarly \(J^+\) must contain at least two vertices of \(\{x, y, z\}\). If \(J^-\) does not contain \(a\), then \(J^+\) has to contain either \(y\) or \(z\), so that \(J\) separates \(b\) and \(y\) or \(c\) and \(z\). One argues similarly if \(J^-\) does not contain \(b\) or \(c\), concluding the proof of our lemma. \(\square\)

3 Flats and their isometries

In the traditional flat torus theorem, the abelian subgroup acts on a product of geodesic lines. In our cubical version of the theorem, these subspaces are replaced with flats. This section is dedicated to the study of these median algebras.

Definition 3.1. A flat is a non-empty, finite-dimensional, discrete median algebra which is included into the interval of two points of its Roller boundary.
For instance, the Euclidean spaces $\mathbb{Z}^n$ are flats, as well as the chain of squares illustrated by Figure [4]. However, $[0,1] \times \mathbb{Z}$ or a line on which we glue an additional edge are not flats. It is worth noticing that being a flat is preserved under products:

**Lemma 3.2.** A product of finitely many flats is a flats.

**Proof.** Let $F_1, \ldots, F_n$ be flats. So, for every $1 \leq i \leq n$, there exist $\zeta_i, \xi_i \in \mathcal{F}F_i$ such that $F_i$ coincides with the interval $I(\zeta_i, \xi_i)$. For every $(a_1, \ldots, a_n) \in F_1 \times \cdots \times F_n$, we have

$$
\mu((\zeta_1, \ldots, \zeta_n), (a_1, \ldots, a_n), (\xi_1, \ldots, \xi_n)) = (\mu(\zeta_1, a_1, \xi_1), \ldots, \mu(\zeta_n, a_n, \xi_n)) = (a_1, \ldots, a_n).
$$

Therefore, $F_1 \times \cdots \times F_n$ coincides with the interval between $(\zeta_1, \ldots, \zeta_n)$ and $(\xi_1, \ldots, \xi_n)$.

In order to motivate the analogy between products of geodesic lines in CAT(0) spaces and flats in median algebras, let us mention that, as a consequence of [BCG+09] Theorem 1.14, a flat always embeds into a Euclidean space:

**Lemma 3.3.** Let $F$ be a flat of dimension $d$. Then its cubulation isometrically embeds into the cube complex $\mathbb{Z}^d$.

The main interest of our cubical flat torus theorem is that it provides an action on a flat. Finding such an action is interesting because the isometry group of (the cubulation of) a flat is quite specific, imposing severe restrictions on the action we are looking at. The following proposition motivates this idea:

**Proposition 3.4.** Let $X$ be the cubulation of a flat. There exist a finite-index subgroup $\text{Isom}_0(X) \leq \text{Isom}(X)$, an integer $n \geq 1$ and a morphism $\beta : \text{Isom}_0(X) \to \mathbb{R}^n$ such that $\ker(\beta)$ is locally elliptic.

A subgroup is called locally elliptic if every finitely generated subgroup is elliptic (or equivalently in CAT(0) cube complexes, has a finite orbit or stabilises a cube).

**Proof of Proposition 3.4.** Let $\text{Isom}^+(X)$ denote the kernel of the action of $\text{Isom}(X)$ on the set of all the cubical components of $\mathcal{F}X$. Notice that, by combining Lemmas 3.3 and 2.7, we know that $X$ has only finitely many cubical components, so that $\text{Isom}^+(X)$ has to be a finite-index subgroup of $\text{Isom}(X)$. We also know that $\mathcal{F}X$ contains a cubical component which is bounded. Therefore, there exists some $\alpha \in \mathcal{F}X$ such that $\text{Isom}^+(X) \cap \text{stab}(\alpha)$ has finite index in $\text{Isom}(X)$. According to [CFI16] Theorem B.1, there exists a finite-index subgroup $\text{stab}_0(\alpha) \leq \text{stab}_{\text{Isom}(X)}(\alpha)$, an integer $n \geq 1$ and a morphism $\beta : \text{stab}_0(\zeta) \to \mathbb{Z}^n$ such that $\ker(\beta)$ is locally elliptic. Therefore, the restriction of $\beta$ to $\text{Isom}_0(X) := \text{Isom}^+(X) \cap \text{stab}_0(\zeta)$ provides the desired morphism.

We emphasize that the isometry group of a flat is not necessarily virtually free abelian. For instance, the flat illustrated by Figure [4] has its isometry group isomorphic to the wreath product $\mathbb{Z}_2 \wr D_\infty$. Nevertheless, any group acting properly on a flat turns out to be virtually free abelian.

The notion of flat arises naturally when we study the dynamics of loxodromic isometries of CAT(0) cube complexes, as justified by the next statement:

**Proposition 3.5.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. Fix two points $\zeta, \xi \in \mathcal{F}X$ such that $g$ admits an axis having $\zeta$ and $\xi$ as points at infinity. The union of all the axes of $g$ having $\zeta$ and $\xi$ as points at infinity is a flat.
Figure 4: A flat whose isometry group is isomorphic to $\mathbb{Z}_2 \wr D_\infty$.

The first step is to show that such a union turns out to be a median subalgebra of $X$, which is essentially a consequence of the following lemma:

**Lemma 3.6.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. Fix three axes $\alpha, \beta, \gamma$ of $g$ and three vertices $x \in \alpha$, $y \in \beta$, $z \in \gamma$. If $m$ denotes the median point $\mu(x,y,z)$ and if $[m, gm]$ is an arbitrary geodesic between $m$ and $gm$, then $\bigcup_{k \in \mathbb{Z}} g^k[m, gm]$ is an axis of $g$.

*Proof.* Let $x, y, z \in \text{Med}(g)$ be three vertices, and let $m$ denote the median point of $\{x, y, z\}$. In order to show that $m$ belongs to $\text{Med}(g)$, it is sufficient to prove that $\bigcup_{k \in \mathbb{Z}} g^k[m, gm]$ is a geodesic.

Suppose by contradiction that $\bigcup_{k \in \mathbb{Z}} g^k[m, gm]$ is not a geodesic. So there exist an index $k \in \mathbb{Z} \setminus \{0\}$ and a hyperplane $J$ such that $J$ intersects both $[m, gm]$ and $g^k[m, gm]$. Let $J^-$ and $J^+$ denote the halfspaces delimited by $J$ such that $m \in J^-$ and $gm \in J^+$. By convexity of halfspaces, $J^-$ has to contain at least two vertices among $\{x, y, z\}$ and similarly $J^+$ has to contain at least two vertices among $\{gx, gy, gz\}$. Therefore, we must have $x \in J^-$ and $gx \in J^+$, or $y \in J^+$ and $gy \in J^+$, or $z \in J^-$ and $gz \in J^+$. Without loss of generality, suppose that $x \in J^-$ and $gx \in J^+$. We also know from Lemma 2.11 that $J$ has to separate $g^kx$ and $g^{k+1}x$, or $g^ky$ and $g^{k+1}y$, or $g^kz$ and $g^{k+1}z$. Notice that $J$ cannot separate $g^kx$ and $g^{k+1}x$ since it already separates $x$ and $gx$ and that $\bigcup_{k \in \mathbb{Z}} g^k[x, gx]$ is a geodesic (for any choice of a geodesic $[x, gx]$ between $x$ and $gx$). Without loss of generality, suppose that $J$ separates $g^ky$ and $g^{k+1}y$. Now, we distinguish two cases.

**Case 1:** $g^{k+1}y$ belongs to $J^-$ and $g^kx$ to $J^+$.

So $J$ separates $\{x, g^{k+1}y\}$ and $\{gx, g^ky\}$. Set $N = d(x, g^{k+1}y) + 1$. Because, for every $0 \leq j \leq N - 1$, the hyperplane $g^jJ$ separates $\{g^jx, g^{k+j}y\}$ and $\{g^{j+1}x, g^{k+j}y\}$, it follows that $J, gJ, \ldots, g^{N-1}J$ all separate $g^Nx$ and $g^{k+N+1}y$. Therefore,

$$d \left( x, g^{k+1}y \right) = d \left( g^N x, g^{k+N+1}y \right) \geq N = d \left( x, g^{k+1}y \right) + 1,$$

a contradiction. Let us record what we have just proved for future use:

**Fact 3.7.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. For every vertices $x, y \in \text{Med}(g)$ and integer $k \in \mathbb{Z}$, no hyperplane can separate $\{x, g^{k+1}y\}$ and $\{gx, g^ky\}$.

Now let us focus on the second case we have to consider.

**Case 2:** $g^{k+1}y$ belongs to $J^+$ and $g^kx$ to $J^-$.

So far, we know that $\{x, y, gz, g^ky\} \subset J^-$ and $\{gx, g^kx, g^{k+1}x, g^{k+1}y\} \subset J^+$. Because $gm \in J^+$, necessarily at least two vertices among $\{gx, gy, gz\}$ have to belong to $J^+$, so that $gz$ has to belong to $J^+$. Since $g^{k+1}x$ and $g^{k+1}y$ belong to $J^+$, necessarily $g^{k+1}m \in J^+$, so that $g^km \in J^-$. Therefore, at least two vertices among $\{g^kx, g^ky, g^kz\}$ have to belong to $J^-$, hence $g^kz \in J^-$. Thus, we have proved that $J$ separates $gz$ and
\[ g^k z \text{ with } g z \in J^+ \text{ and } g^k z \in J^- . \]

Necessarily, \( z \in J^+ \) and \( g^{k+1} z \in J^- \). By applying Fact 3.7 to \( \{x, g^{k+1} z\} \) and \( \{gx, gz\} \), we find a contradiction.

Thus, we have proved that \( \gamma := \bigcup_{k \in \mathbb{Z}} g^k[m, gm] \) is a geodesic, concluding the proof of our lemma. \( \square \)

**Proof of Proposition 3.5.** Let \( F \) denote the union of all the axes of \( g \) having \( \zeta \) and \( \xi \) as points at infinity. We first verify that \( F \) is a median subalgebra of \( X \). So let \( x, y, z \in F \) be three vertices. We know from Lemma 4.10 that \( \mu(x, y, z) \) belongs to a geodesic \( \gamma \) on which \( g \) acts by translations. We have

\[
\gamma(+\infty) = \lim_{n \to +\infty} g^n \mu(x, y, z) = \mu \left( \lim_{n \to +\infty} g^n x, \lim_{n \to +\infty} g^n y, \lim_{n \to +\infty} g^n z \right) = \mu(\xi, \xi, \xi) = \xi.
\]

Similarly, one shows that \( \gamma(-\infty) = \zeta \). Thus, we have proved that \( \mu(x, y, z) \) belongs to an axis of \( g \) having \( \zeta \) and \( \xi \) as points at infinity, ie., \( \mu(x, y, z) \in F \).

It remains to show that that \( F \) is finite-dimensional. We begin by proving two preliminary claims.

**Claim 3.8.** If \( x, y \in F \) are adjacent in \( X \), then there exists an axis of \( g \) passing through both of them.

Fix two axes \( \alpha \) and \( \beta \) passing through \( x \) and \( y \) respectively. Let \( J \) denote the hyperplane separating \( x \) and \( y \). Notice that, for every \( n \neq 0 \), the hyperplane \( J \) does not separate \( g^n x \) and \( g^n y \). Indeed, otherwise \( g^n \) would stabilise \( J \), as well as the halfspaces it delimits because \( g^n \) does not invert any hyperplane, so that \( J \) would separate \( \{g^{nk} x \mid k \in \mathbb{Z}\} \) and \( \{g^{nk} y \mid k \in \mathbb{Z}\} \), contradicting the fact that \( \alpha \) and \( \beta \) have the same points at infinity.

On the other hand, once again because \( \alpha \) and \( \beta \) have the same points at infinity, \( J \) necessarily crosses \( \alpha \) and \( \beta \). The conclusion is that \( J \) has to separate \( \{g^{-1} x, g^{-1} y\} \) and \( \{gx, gy\} \). Now, two cases may happen: either \( J \) separates \( x \) from \( \{gx, gy\} \) or it separates \( y \) from \( \{gx, gy\} \). Without loss of generality, we suppose that we are in the former case.

Let \( a, b \in \beta \) denote the endpoints of the edge of \( \beta \) crossed by \( J \) (so that \( b \) is between \( a \) and \( y \) along \( \beta \)). Assume that \( b \neq y \). Let \( c \in \beta \) denote the neighbor of \( b \) which is distinct from \( a \). See Figure 5. Notice that the hyperplane \( H \) separating \( b \) and \( c \) is transverse to \( J \). Indeed, because \( H \) does not separate \( x \) and \( y \) and that \( H \) has to cross \( \alpha \), it follows that \( H \) separates \( x \) from some \( g^{-k} x \). So \( H \) separates \( b \) and \( c \), and \( x \) and \( g^{-k} x \), but \( J \) separates \( \{b, c\} \) and \( \{x, g^{-k} x\} \). Therefore, \( J \) and \( H \) must be transverse. As a consequence, the two edges \( [b, a] \) and \( [b, c] \) have to span a square. Let \( d \) denote the fourth vertex of this square, and let \( \delta \) denote the geodesic between \( g^{-1} y \) and \( y \) obtained from the subsegment \( [g^{-1} y, y] \subset \beta \) by replacing \( [a, b] \cup [b, c] \) with \( [a, d] \cup [d, c] \). The concatenation \( \gamma := \bigcup_{k \in \mathbb{Z}} g^k \delta \) defines an axis of \( g \) having the same points at infinity as \( \alpha \) and \( \beta \). Notice that, by construction, the vertex \( y \) still belongs to \( \gamma \). Moreover, the edge of \( \gamma \) crossed by \( J \) is now closer to \( y \).
By iterating this argument to $\beta$, and next to $\alpha$, it follows that there exist two axes $\alpha'$ and $\beta'$ passing through $x$ and $y$ respectively and such that the edges of $\alpha'$ and $\beta'$ crossed by $J$ share an endpoint with $[x,y]$. Because two adjacent edges cannot be crossed by the same hyperplane, it follows that $\alpha'$ and $\beta'$ both pass through $x$ and $y$, concluding the proof of our claim.

**Claim 3.9.** Two distinct hyperplanes of $X$ separating at least two vertices of $F$ induce different partitions of $F$.

Let $J$ and $H$ be two hyperplanes of $X$ separating at least two vertices of $F$. Necessarily, $J$ and $H$ separates $\zeta$ and $\xi$. Therefore, if $\alpha$ denotes an axis of $g$ having $\zeta$ and $\xi$ as points at infinity, then $J$ and $H$ have to cross $\alpha$. Let $x \in \alpha$ be a vertex which is between $J$ and $H$ along $\alpha$. Say that $J$ separates $x$ from $\zeta$. Then there exists some sufficiently large $n \geq 1$ such that $H$ separates $x$ from $g^nx$ but $J$ does not separate these two vertices. Consequently, $J$ and $H$ induce distinct partitions of $F$, concluding the proof of our claim.

We are now ready to show that $F$ is finite-dimensional. In fact, we will prove that the cubulation of $F$ is uniformly locally finite. By combining Lemma 2.10 with Claim 3.9, it follows that, for every $N \geq 1$, if the cubulation of $F$ contains a vertex with $N$ neighbors, then so does $F$ in $X$. So fix a vertex $x \in F$ and $N$ of its neighbors $x_1, \ldots, x_N$. According to Claim 3.8 for every $1 \leq i \leq N$, there exists an axis of $g$ passing through $x$; as a consequence, the hyperplane separating $x$ and $x_i$ has to separate $x$ from $gx$. Because two adjacent edges have to be dual to distinct hyperplanes, it follows that $N \leq d(x,gx) = \|g\|$. Thus, we have proved that:

**Fact 3.10.** Every vertex of the cubulation of $F$ admits at most $\|g\|$ neighbors.

A fortiori, the cubulation of $F$ has dimension at most $\|g\|$, concluding the proof of our proposition.

### 4 Median sets of isometries

Our goal in this section is to associate to any isometry a median subalgebra of the cube complex which behaves similarly to minimising sets of isometries in CAT(0) spaces. Our sets are:

**Definition 4.1.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ an isometry. The median set of $g$, denoted by $\text{Med}(g)$, is

- the union of all the $d$-dimensional cubes stabilised by $\langle g \rangle$ if $g$ elliptic, where $d$ is the minimal dimension of a cube stabilised by $\langle g \rangle$;
- the union of all the axes of $g$ if $g$ is loxodromic;
- the pre-image under $\pi$ of the union of all the axes of $g$ in $X/J$ if $g$ is inverting, where $J$ is the collection of the hyperplanes inverted by powers of $g$ and where $\pi : X \rightarrow X/J$ is the canonical map to the cubical quotient $X/J$.

The fact that the inverting isometry $g$ of $X$ induces a loxodromic isometry of $X/J$ is proved by Lemma 4.20 below. So the median set of an inverting isometry is well-defined. Its definition may seem to be technical, but it turns out to be a very natural set. The reader can keep in mind the example given by Figure 6.

Then, the main result of this section is:
Theorem 4.2. Let $G$ be a group acting on a CAT(0) cube complex $X$ and $g \in G$ an isometry such that $\langle g \rangle$ is a normal subgroup of $G$. Then $\text{Med}(g)$ is a median subalgebra of $X$ which is $G$-invariant and which decomposes as a product $T \times F \times Q$ of three median algebras $T, F, Q$ such that:

- the action $G \curvearrowright T \times F \times Q$ decomposes as a product of three actions $G \curvearrowright T, F, Q$;
- $F$ is a single vertex if $g$ is elliptic, and otherwise it is a flat on which $g$ acts by translations of length $\lim_{n \to +\infty} \frac{1}{n} d(x, g^n x) > 0$;
- $Q$ is a finite-dimensional cube, possibly reduced to a single vertex;
- $g$ acts trivially on $T$.

Moreover, the dimension of $Q$ is zero if $g$ is loxodromic, it coincides with the minimal dimension of a cube of $X$ stabilised by $g$ if $g$ is elliptic, and otherwise it coincides with the number of hyperplanes inverted by $g$.

In order to prove Theorem 4.2, we distinguish three cases, depending on whether the isometry is elliptic, reversing or loxodromic. So Theorem 4.2 is the combination of Propositions 4.3, 4.9 and 4.18 below. Before turning to the proofs, we emphasize that, although the median set of a loxodromic isometry $g$ coincides with its minimising set $\text{Med}(g) = \left\{ x \in X \mid d(x, gx) = \min_{y \in X} d(y, gy) \right\}$ (see Lemma 6.2), it turns out that the minimising set of an elliptic or inverting isometry may not be median. See Remarks 4.8 and 4.24 below.

4.1 Elliptic groups of isometries

In this subsection, we prove Theorem 4.2 for elliptic isometries. In fact, we prove a more general statement dealing with arbitrary elliptic subgroups:

Proposition 4.3. Let $G$ be a group acting on a CAT(0) cube complex $X$ and $E$ a normal subgroup which has a bounded orbit. If $d$ denotes the minimal dimension of a cube stabilised by $E$, then the union of the $d$-dimensional cubes stabilised by $E$ is a $G$-invariant median subalgebra which decomposes as a product $Q \times Z$ of two median algebras $Q, Z$ such that:

- the action $G \curvearrowright Q \times Z$ decomposes as a product of two actions $G \curvearrowright Q, Z$;
- $Q$ is a $d$-dimensional cube;
- $E$ acts trivially on $Z$.

Proof. Let $Y$ denote the union of the $d$-dimensional cubes stabilised by $E$. Our first goal is to show that $Y$ is median. We begin by proving an easy but useful observation.
Claim 4.4. Two d-dimensional cubes stabilised by E are crossed exactly by the same hyperplanes of X.

Indeed, if $C_1$ and $C_2$ are two such cubes, then the projection of $C_1$ onto $C_2$ provides a subcube $P \subset C_2$, which has to be $E$-invariant. By minimality of the dimension $d$, it follows that $P = C_2$. It follows from Lemma 2.4 that $C_1$ and $C_2$ are crossed by the same hyperplanes of $X$, concluding the proof of our claim.

We are now ready to show that $Y$ is median. Let $x_1, x_2, x_3 \in Y$ be three vertices. For every $i = 1, 2, 3$, there exists a $d$-dimensional cube $C_i$ which is stabilised by $E$ and which contains $x_i$. According to Claim 4.4 these cubes are crossed by the same hyperplanes of $X$; let $H$ denote the collection of these hyperplanes. It follows from Lemma 2.11 that any hyperplane separating two vertices of $\mu(C_1, C_2, C_3)$ has to cross one the cubes $C_1, C_2, C_3$, i.e., it has to belong to $H$. As a consequence, because $H$ is a collection of $d$ pairwise transverse hyperplanes, the convex hull of $\mu(C_1, C_2, C_3)$ has to be a $d$-dimensional cube. As $\mu(C_1, C_2, C_3)$ is clearly $E$-invariant, so is this cube. Thus, we have proved that $\mu(x_1, x_2, x_3)$ belongs to a $d$-dimensional cube stabilised by $E$, i.e., $\mu(x_1, x_2, x_3) \in Y$.

Now, our goal is to decompose $Y$ as a product. For this purpose, we fix a $d$-dimensional cube $Q$ stabilised by $E$ and a vertex $v \in Q$, and we set

$$Z = \{ \text{proj}_Q(v) \mid C \text{ $d$-dimensional cube stabilised by $E$} \}.$$ 

Notice that any two distinct $d$-dimensional cubes stabilised by $E$ have an empty intersection. Indeed, otherwise their intersection would define a lower dimensional cube stabilised by $E$, contradicting the minimality of $d$. Consequently, for every vertex $x \in Y$, we can denote by $C(x)$ the unique $d$-dimensional cube stabilised by $E$ which contains $x$. We are interested in the map

$$\Phi : \begin{cases} 
Y & \rightarrow & Q \times Z \\
 x & \mapsto & (\text{proj}_Q(x), \text{proj}_{C(x)}(v)) \end{cases}.$$ 

First of all, let us verify that $Z$ is a median subalgebra of $X$. Once again, we denote by $H$ the collection of all the hyperplanes crossing the $d$-dimensional cubes stabilised by $E$.

Claim 4.5. No hyperplane of $H$ separates two vertices of $Z$.

Let $a, b \in Z$ be two vertices. By definition of $Z$, one has $a = \text{proj}_{C(a)}(v)$ and $b = \text{proj}_{C(b)}(v)$. As a consequence of Lemma 2.3, the hyperplanes separating $v$ from $a$ or $b$ do not cross $C(a)$ or $C(b)$ respectively, i.e., they do not belong to $H$. Because any hyperplane separating $a$ and $b$ necessarily separates $v$ from either $a$ or $b$, our claim follows.
Now, let $a, b, c \in Z$ be three vertices and let $m$ denote their median point in $Y$. If there exists a hyperplane $J$ separating $m$ from $\text{proj}_{C(m)}(v)$, then it has to belong to $\mathcal{H}$ since the two vertices belong to the cube $C(m)$. But, as a consequence of Claim 4.5, the vertices $a, b, c$ necessarily belong to the halfspace delimited by $J$ which contains $\text{proj}_{C(m)}(v)$, so that $J$ separates $m$ from $a, b, c$, contradicting the convexity of halfspaces. Therefore, no hyperplane separates $m$ and $\text{proj}_{C(m)}(v)$, hence $m = \text{proj}_{C(m)}(v) \in Z$. Thus, we have proved that $Z$ is a median subalgebra of $X$.

The next step is to show that our map $\Phi$ is an isomorphism of median algebras. We begin by showing that $\Phi$ is surjective. So let $q \in Q$ and $z \in Z$ be two vertices. Notice that

$$\Phi \left( \text{proj}_{C(z)}(q) \right) = \left( \text{proj}_Q \left( \text{proj}_{C(z)}(q) \right), \text{proj}_{C(z)}(v) \right).$$

It follows from Claim 4.5 that $\text{proj}_{C(z)}(v) = z$. Also, by applying Lemma 2.3 twice, we know that no hyperplane separating $q$ and $\text{proj}_{C(z)}(q)$ crosses $C(z)$ and that no hyperplane of $\mathcal{H}$ separates $q$ and $\text{proj}_{Q}(\text{proj}_{C(z)}(q))$. Because these two points belong to $Q$ and that the collection of the hyperplanes crossing $Q$ coincides with $\mathcal{H}$, it follows that $\text{proj}_{Q}(\text{proj}_{C(z)}(q)) = q$. Thus, $\Phi(\text{proj}_{C(z)}(q)) = (q, z)$, proving the surjectivity of $\Phi$. We record this equality for future use:

**Fact 4.6.** The map

$$\Psi : \begin{cases} Q \times Z \to Y \\ (q, z) \mapsto \text{proj}_{C(z)}(q) \end{cases}$$

satisfies $\Phi \circ \Psi = \text{Id}_{Q \times Z}$.

Now, let $x, y \in Y$ be two vertices. As a consequence of Lemma 2.4,

$$\mathcal{W}(x, y) \cap \mathcal{H} = \mathcal{W} \left( \text{proj}_Q(x), \text{proj}_Q(y) \right).$$

We also have

$$\mathcal{W}(x, y)\mathcal{H} = \mathcal{W}(C(x), C(y)) = \mathcal{W} \left( \text{proj}_{C(x)}(v), \text{proj}_{C(y)}(v) \right)$$

where the last equality is justified by Claim 4.5. Therefore,

$$d(x, y) = d \left( \text{proj}_{C(x)}(v), \text{proj}_{C(y)}(v) \right) + d \left( \text{proj}_{C(x)}(v), \text{proj}_{C(y)}(v) \right).$$

Otherwise saying, if $Y, Q, Z$ are endowed with the metrics induced by $X$, then $\Phi$ is an isometry. It implies that $\Phi$ is injective, but also that $\Phi$ is an isomorphism of median algebras since the median structures of $Y, Q, Z$ are induced by the metric of $X$.

It remains to study the action of $G$ on $Y$ (which is clearly $G$-invariant because $E$ is a normal subgroup). By using the expression of $\Phi^{-1}$ given by Fact 4.6, the action of $G$ on $Q \times Z$ is given by

$$g \cdot (q, z) = \left( \text{proj}_Q \left( g \cdot \text{proj}_{C(z)}(q) \right), \text{proj}_{g \cdot C(z)}(v) \right)$$

for every $(q, z) \in Q \times Z$ and $g \in G$. Let us simplify this description.

**Claim 4.7.** The equality $\text{proj}_Q \left( g \cdot \text{proj}_{C(q)}(q) \right) = \text{proj}_Q(g \cdot q)$ holds for every $d$-dimensional cube $C$ stabilised by $E$, for every $(q, z) \in Q \times Z$, and for every $g \in G$.

It follows from Lemma 2.3 that no hyperplane separating $q$ from $\text{proj}_{C(q)}(q)$ crosses $C$, so that no hyperplane of $\mathcal{H}$ separates $q$ and $\text{proj}_{C(q)}$. As $\mathcal{H}$ is $G$-invariant, no hyperplane of $\mathcal{H}$ separates $g \cdot q$ and $g \cdot \text{proj}_{C(q)}$ either. We deduce from Lemma 2.4 that $g \cdot q$ and $g \cdot \text{proj}_{C(q)}$ have the same projection onto $Q$, proving our claim.
An isometry $g$ of a cube such that $\text{Med}(g)$ is not median and $\text{Med}(g) \subseteq \text{Med}(g)$.

Therefore, the action of $G$ on $Q \times Z$ simplifies as

$$g \cdot (q, z) = \left( \text{proj}_Q (g \cdot q), \text{proj}_{g \cdot C(z)}(v) \right)$$

for every $(q, z) \in Q \times Z$ and $g \in G$. It is clear that

$$g \cdot z = \text{proj}_{g \cdot C(z)}(v), \ g \in G, z \in Z$$

defines an action $G \acts Z$, and the fact that

$$g \cdot q = \text{proj}_Q (g \cdot q), \ g \in G, q \in Q$$

defines an action $G \acts Q$, and the fact that $E$ stabilises the cube $C(z)$ for every $z \in Z$, necessarily $E$ is included into the kernel of the action $G \acts Z$.

\[ \square \]

Remark 4.8. It is worth noticing that the median set of an elliptic isometry $g$ cannot be defined as $\text{Min}(g)$. Although it can be proved that $\text{Min}(g)$ is always included into $\text{Med}(g)$, the inclusion can be proper and in fact $\text{Min}(g)$ may not be median, as illustrated by Figure 8.

4.2 Loxodromic isometries

In this subsection, we prove Theorem 4.2 for loxodromic isometries, namely:

**Proposition 4.9.** Let $G$ be a group acting on a CAT(0) cube complex $X$ and $g \in G$ a loxodromic isometry such that $\langle g \rangle$ is normal in $G$. Then $\text{Med}(g)$ is a median subalgebra of $X$ which is $G$-invariant and which decomposes as a product $T \times F$ of two median algebras $T, F$ such that:

- the action $G \acts T \times F$ decomposes as a product of two actions $G \acts T, F$;
- $F$ is a flat on which $g$ acts by translations of length $\|g\|$;
- $g$ acts trivially on $T$.

In the rest of the section, we will occasionally use the following notation. If $g$ is a loxodromic isometry and $x$ a vertex which belongs to the minimising set of $g$, then the limit $\lim_{k \to \pm \infty} g^k x$ exists in $X$ and coincides with the point at infinity $\gamma(\pm \infty)$ if $\gamma$ is an axis of $g$ passing through $x$. For convenience, we may denote this limit by $g^{\pm \infty} x$.

The first step towards the proof of Proposition 4.9 is to show that the median set of a loxodromic isometry is median, which is a direct consequence of Lemma 3.6.
Lemma 4.10. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. Then $\text{Med}(g)$ is median.

The next step towards the proof of Proposition 4.9 is to show that the median set of a loxodromic isometry decomposes as a product. Before proving this assertion, we need the following preliminary lemma:

Lemma 4.11. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. A hyperplane of $X$ crossing an axis of $g$ crosses all the axes of $g$. As a consequence, such a hyperplane separates \( \left\{ \lim_{n \to +\infty} g^n x \mid x \in \text{Med}(g) \right\} \) and \( \left\{ \lim_{n \to +\infty} g^{-n} x \mid x \in \text{Med}(g) \right\} \).

Proof. Let $\gamma_1, \gamma_2$ be two axes of $g$ and $J$ a hyperplane intersecting $\gamma_1$. Assume by contradiction that $J$ does not intersect $\gamma_2$. As a consequence, $\gamma_1$ contains a subray $r$ such that $J$ separates $r$ from $\gamma_2$. There exists some $x \in \gamma_1$ such that $r = \{g^k x \mid k \geq 0\}$ or $r = \{g^{-k} x \mid k \leq 0\}$. Up to replacing $k$ with $-k$, we may suppose without loss of generality that $r = \{g^k x \mid k \geq 0\}$. Fix some $y \in \gamma_2$ and set $N = d(x,y)$. Notice that, for every $0 \leq j \leq N$, $g^j J$ separates $\{g^k x \mid k \geq j\}$ from $\gamma_2$ so that, in particular, it has to separate $g^N x$ from $g^N y$. As $J, gJ, \ldots, g^N J$ all separate $g^n x$ and $g^N y$, it follows that

\[
d(x,y) = d\left(g^N x, g^N y\right) \geq N + 1 = d(x,y) + 1,
\]
a contradiction. Thus, we have shown that $J$ must intersect $\gamma_2$ as well, proving the first assertion of our lemma.

In order to prove the second assertion, fix two vertices $x, y \in \text{Med}(g)$ and a hyperplane $J$ intersecting the axes of $g$. The only possibility for $J$ not to separate $g^{-N} y$ and $g^N x$ is that $J$ separates $\{g^\infty x, g^{-\infty} y\}$ and $\{g^{-\infty} x, g^\infty y\}$. But this is impossible according to Fact 3.7

Corollary 4.12. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. If $\gamma_1$ and $\gamma_2$ are two axes of $g$ satisfying $\gamma_1(\infty) = \gamma_2(\infty)$, then necessarily $\gamma_1(-\infty) = \gamma_2(-\infty)$.

Proof. If there exists a hyperplane $J$ separating $\gamma_1(-\infty)$ and $\gamma_2(\infty)$, then either $J$ separates $\gamma_1(-\infty)$ from the three points $\gamma_1(\infty), \gamma_2(-\infty), \gamma_2(\infty)$ or it separates $\gamma_2(-\infty)$ from the three points $\gamma_2(\infty), \gamma_1(-\infty), \gamma_1(\infty)$. In either case, $J$ intersects only one axis among $\gamma_1$ and $\gamma_2$, contradicting Lemma 4.11. Consequently, no hyperplane separates $\gamma_1(-\infty)$ and $\gamma_2(-\infty)$, hence $\gamma_1(-\infty) = \gamma_2(-\infty)$ as desired.

We are now ready to show that the median set of a loxodromic isometry naturally decomposes as a product.

Lemma 4.13. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. Let $T \subset \mathfrak{X}$ denote \( \left\{ \lim_{n \to +\infty} g^n x \mid x \in \text{Med}(g) \right\} \) and let $F$ be the union of all the axes of $g$ having a fixed pair of points at infinity $(\zeta, \xi) \in \mathfrak{X} \times \mathfrak{X}$. Then $T$ and $F$ are median subalgebras of $X$, and the map

\[
\Phi : \text{Med}(g) \to T \times F \\
x \mapsto (\lim_{n \to +\infty} g^n x, \mu(x, \zeta, \xi))
\]
defines an isomorphism of median algebras.
Indeed, if \( J \) is a hyperplane separating these two points, it follows from Lemma 2.11 that \( J \) separates \( g^p y \) and \( g^q y \) or \( g^{-p} y \) and \( g^{-q} y \). Suppose that \( J \) separates \( g^p y \) and \( g^q y \), the other case being similar. If \( J^+, J^- \) denote the halfspaces delimited by \( J \) such that \( \mu(x, g^p y, g^{-p} y) \in J^+ \) and \( \mu(x, g^q y, g^{-q} y) \in J^- \), then two cases may happen:

- either \( g^p y \in J^- \) and \( g^q y \in J^+ \), which is impossible because \( g^q y \) and \( g^{-q} y \) cannot both belong to \( J^+ \) as soon as \( \mu(x, g^q y, g^{-q} y) \) belongs to \( J^- \);
- or \( g^p y \in J^+ \) and \( g^q y \in J^- \), so that \( y \) and \( g^{-p} y \) must belong to \( J^- \) (since \( J \) is a geodesic) and \( x \) has to belong to \( J^+ \) (since \( x \) and \( g^{-p} y \) cannot be on the same side of \( J \)), which implies that \( J \) separates \( x \) from \( \{y, g^q y\} \), a contradiction.

Consequently, no hyperplane separates \( \mu(x, g^p y, g^{-p} y) \) and \( \mu(x, g^q y, g^{-q} y) \), whence the equality \( \mu(x, g^p y, g^{-p} y) = \mu(x, g^q y, g^{-q} y) \).
We conclude that
\[ \mu(x, \gamma(-\infty), \gamma(+\infty)) = \lim_{n \to +\infty} \mu(x, g^{-n}y, g^{n}y) = \mu(x, g^{-N}y, g^{N}y) \]

belongs to \( Y \) because \( Y \) is median according to Lemma 4.10 and because \( x, g^{-N}y, g^{N}y \) also belong to \( Y \). The proof of Claim 4.14 is complete.

So we know that \( \mu(x, \zeta, \xi) \) belongs to \( Y \). As a consequence, there exists a geodesic passing through \( \mu(x, \zeta, \xi) \) on which \( g \) acts by translations. We have
\[ \gamma(+\infty) = \lim_{n \to +\infty} g^{n}\mu(x, \zeta, \xi) = \lim_{n \to +\infty} \mu(g^{n}x, \zeta, \xi) = \mu(g^{\infty}x, \zeta, \xi). \]
The equality \( \xi = \mu(g^{\infty}x, \zeta, \xi) \) follows from the observation that a hyperplane separating \( \zeta \) and \( \xi \) does not separate \( \xi \) and \( g^{\infty}x \), as implied by the following immediate consequence of Lemma 4.11:

Fact 4.15. Any hyperplane crossing \( F \) separates \( T \) and \( T^{-} := \{ \lim_{n \to +\infty} g^{-n}x \ | \ x \in \text{Med}(g) \} \).

One shows similarly that \( \gamma(-\infty) = \lim_{n \to -\infty} g^{-n}\mu(x, \zeta, \xi) = \zeta \). Thus, we have shown that \( \mu(x, \zeta, \xi) \) belongs to \( F \), proving that our map \( \Phi \) is well-defined.

Now, we want to prove that \( \Phi \) is surjective. More precisely, we will prove:

Claim 4.16. For every \( \delta \in T \), let \( \delta^{-} \) denote the common endpoint in \( T^{-} \) of all the axes \( \gamma \) of \( g \) satisfying \( \gamma(+\infty) = \delta \). Then the map
\[ \Psi : \left\{ \begin{array}{c} T \times F \mapsto \text{Med}(g) \\ (\delta, q) \mapsto \mu(q, \delta, \delta^{-}) \end{array} \right. \]
satisfies \( \Phi \circ \Psi = \text{Id}_{T \times F} \).

The uniqueness of \( \delta^{-} \) is justified by Corollary 4.12. And the fact that \( \Psi \) is well-defined, ie., that \( \mu(q, \delta, \delta^{-}) \) belongs to \( \text{Med}(g) \) for every \( (\delta, q) \in T \times F \), follows from Claim 4.14.

Fix some \( (\delta, q) \in T \times F \). We have
\[ \Phi \circ \Psi(\delta, q) = (g^{\infty}\mu(q, \delta, \delta^{-}), \mu(q, \delta, \delta^{-}), \zeta, \xi). \]

Let \( J \) be a hyperplane separating \( \zeta \) and \( q \). Notice that \( J \) does not separate \( q \) and \( m := \mu(q, \delta, \delta^{-}) \), because otherwise it would not separate \( \delta \) and \( \delta^{-} \), contradicting Fact 4.15. Therefore, \( J \) separates \( \xi \) from \( \{m, \xi\} \). Similarly, one shows that any hyperplane separating \( \xi \) and \( q \) has to separate \( \xi \) from \( \{m, \zeta\} \). Finally, let \( J \) be a hyperplane separating \( m \) and \( q \). Then \( J \) separates \( q \) from \( \{\delta, \delta^{-}\} \), so that \( J \) does not separate \( \delta \) and \( \delta^{-} \). It follows from Fact 4.15 that \( J \) does not separate \( \zeta \) and \( \xi \) either. Therefore, \( J \) separates \( m \) from \( \{\zeta, \xi\} \). Thus, we have proved that \( q \) is the median point of \( \{m, \zeta, \xi\} \), ie.,
\[ q = \mu(q, \delta, \delta^{-}), \zeta, \xi). \]

Next, assume that there exists a hyperplane \( J \) separating \( \delta \) and \( g^{\infty}m \). As a consequence of Lemma 4.11, \( J \) has to separate \( \{\delta, \delta^{-}\} \) and \( \{g^{\infty}m, g^{\infty}m\} \). But this is impossible since \( m \) belongs to geodesics between \( g^{\infty}m \) and \( g^{\infty}m \), and \( \delta \) and \( \delta^{-} \). Therefore, no hyperplane separates \( \delta \) and \( g^{\infty}m \), proving that \( g^{\infty}m = \delta \).

Thus, we have proved that \( \Phi \circ \Psi(\delta, q) = (\delta, q) \) for every \( (\delta, q) \in T \times F \), ie., \( \Phi \circ \Psi = \text{Id}_{T \times F} \).

It is worth noticing that each of \( \text{Med}(g) \), \( T \) and \( F \) is included into a single cubical component of \( \overline{X} \), so that the median structures defined on these subalgebra all come from the metrics defined on the cubical components of \( \overline{X} \). As a consequence, in order to
show that \( \Phi \) defines an isomorphism of median algebras Med\((g) \rightarrow T \times F \), it is sufficient to show that it defines an isometry when Med\((g)\), T and F are all endowed with the metrics induced by \( X \).

So let \( x, y \in \text{Med}(g) \) be two vertices. Notice that, as a consequence of Lemma 4.11 we have
\[
W(x, y) \setminus W(\zeta, \xi) = W(g^\infty x, g^\infty y).
\]
Next, we claim that
\[
W(x, y) \cap W(\zeta, \xi) = W(\mu(x, \zeta, \xi), \mu(y, \zeta, \xi)).
\]
Indeed, let \( J \) be a hyperplane separating \( \mu(x, \zeta, \xi) \) and \( \mu(y, \zeta, \xi) \). Let \( J^- \) denote the halfspace delimited by \( J \) which contains the former median point and \( J^+ \) the halfspace containing the latter point. As \( \zeta, \mu(x, \zeta, \xi) \in J^- \) and \( \zeta \notin J^- \), necessarily \( x \in J^- \). Similarly, as \( \xi, \mu(y, \zeta, \xi) \in J^+ \) and \( \xi \notin J^+ \), necessarily \( y \in J^+ \). Therefore, \( J \) has to separate \( x \) and \( y \) (and it separates \( \zeta \) and \( \xi \) since \( \mu(x, \zeta, \xi) \) and \( \mu(y, \zeta, \xi) \) belong to \( F \)). Conversely, suppose that \( J \) is a hyperplane separating both \( x \) and \( y \) and \( \zeta \) and \( \xi \). Let \( J^-, J^+ \) be the halfspaces delimited by \( J \) such that \( \zeta \in J^- \) and \( \xi \in J^+ \). We may suppose without loss of generality that \( x \in J^- \) and \( y \in J^+ \). Because \( x \) and \( \zeta \) both belong to \( J^- \), necessarily \( \mu(x, \zeta, \xi) \in J^- \); and because \( y \) and \( \xi \) both belong to \( J^+ \), necessarily \( \mu(y, \zeta, \xi) \in J^+ \). Therefore, \( J \) separates \( \mu(x, \zeta, \xi) \) and \( \mu(y, \zeta, \xi) \), concluding the proof of our equality.

Consequently,
\[
d(x, y) = \#W(x, y) = \#(W(x, y) \cap W(\zeta, \xi)) + \#(W(x, y) \setminus W(\zeta, \xi))
\]
\[
= \#W(\mu(x, \zeta, \xi), \mu(y, \zeta, \xi)) + \#W(g^\infty x, g^\infty y)
\]
\[
= d(\mu(x, \zeta, \xi), \mu(y, \zeta, \xi)) + d(g^\infty x, g^\infty y)
\]
Thus, we have proved that \( \Phi \) is an isometry, concluding the proof of our lemma.

**Proof of Proposition 4.9** We know from Lemma 4.10 that Med\((g)\) is a median subalgebra. Moreover, if we fix some \( (\zeta, \xi) \in \mathfrak{R}X \times \mathfrak{R}X \) which are the endpoints of an axis of \( g \), if we denote by \( F \) the union of all the axes of \( g \) with endpoints \( (\zeta, \xi) \), and if we set \( T = \left\{ \lim_{n \rightarrow +\infty} g^n x \mid x \in \text{Med}(g) \right\} \), then we know from Lemma 4.13 that \( T \) and \( F \) are two median subalgebras of \( X \), that
\[
\Phi : \text{Med}(g) \rightarrow T \times F
\]
\[
x \mapsto \left( \lim_{n \rightarrow +\infty} g^n x, \mu(x, \zeta, \xi) \right)
\]
defines an isomorphism of median algebras, and we know from Proposition 3.5 that \( F \) is a flat. Notice that \( \text{Med}(g) \) is \( G \)-invariant. Indeed, let \( x \in \text{Med}(g) \) be a vertex and \( h \in G \) an element. As \( \langle g \rangle \) is normal in \( G \), there exists some \( k \in \mathbb{Z} \setminus \{0\} \) such that \( hgh^{-1} = g^k \). But
\[
\|g\| = \|hgh^{-1}\| = \|g^k\| = |k| \cdot \|g\|,
\]
hence \( k = \pm1 \). Therefore,
\[
d(g \cdot hx, hx) = d(h^{-1}gh \cdot x, x) = d(g^{\pm1} \cdot x, x) = d(gx, x) = \min\{d(y, gy) \mid y \in X\},
\]
hence \( hx \in \text{Med}(g) \).

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It remains to study the action \( G \acts T \times F \). By using the expression of \( \Phi^{-1} \) given by Claim 4.16, we deduce that 
\[
g \cdot (\delta, q) = (g^{\infty} h\mu(q, \delta, \delta^-), \mu(h\mu(q, \delta, \delta^-), \zeta, \xi))
\]
for every \((\delta, q) \in T \times F\). First, we want to simplify this expression. Let us begin by noticing that

**Claim 4.17.** The equality 
\[\mu(hq, \zeta, \xi) = \mu(h\mu(q, \delta, \delta^-), \zeta, \xi)\]
holds for every \( h \in G \) and \((\delta, q) \in T \times F\).

Indeed, if there exists a hyperplane \( J \) separating these two median points (which both belong to \( F\)), then \( J \) crosses \( F \) and it follows from Lemma 2.11 that \( J \) separates \( hq \) and \( h\mu(q, \delta, \delta^-) \). Since the collection of the hyperplanes crossing \( F \) is \( G \)-invariant, we deduce that there must exist a hyperplane crossing \( F \) which separates \( q \) and \( \mu(q, \delta, \delta^-) \). But such a hyperplane does not separate \( \delta \) and \( \delta^- \), so that it cannot cross \( F \) according to Fact 4.15. So our first equality is proved.

For our second equality, we need to introduce some notation. Recall that \( T^- = \left\{ \lim_{n \to \infty} g^{-n} x \mid x \in \text{Med}(g) \right\} \). For every \( \delta \in T^- \), we define \( \delta^+ \) as the common endpoint in \( T \) of all the axes \( \gamma \) of \( g \) satisfying \( \gamma(-\infty) = \delta \). Such an element is well-defined according to Corollary 4.12. For convenience, we set \( \delta^+ = \delta \) for every \( \delta \in T \). Now, we claim that 
\[
g^{\infty} h\mu(q, \delta, \delta^+) = (h\delta)^+ \text{ for every } h \in G \text{ and } (\delta, q) \in T \times F.
\]

Notice that 
\[
g^{\infty} h\mu(q, \delta, \delta^-) = \begin{cases} h g^{\infty} \mu(q, \delta, \delta^-) & \text{if } h^{-1} g h = g \\ h g^{-\infty} \mu(q, \delta, \delta^-) & \text{if } h^{-1} g h = g^{-1} \end{cases}
\]

and that 
\[
g^{\infty} \mu(q, \delta, \delta^-) = \delta \text{ and } g^{-\infty} \mu(q, \delta, \delta^-) = \delta^-.
\]

The proof of our second equality is complete.

Consequently, the action \( G \acts T \times F \) can be now described by 
\[
h \cdot (\delta, q) = \left( (h\delta)^+, \mu(hq, \zeta, \xi) \right) \text{ for every } h \in G \text{ and } (\delta, q) \in T \times F.
\]

It is clear that \( h \cdot \delta = (h\delta)^+ \), where \( h \in G \) and \( \delta \in T \), defines an action \( G \acts T \); and the fact that \( h \cdot q = \mu(hq, \zeta, \xi) \), where \( h \in G \) and \( q \in F \), defines an action follows from Claim 4.17. Thus, we have proved that the action \( G \acts T \times F \) decomposes as a product of actions \( G \acts T \) and \( G \acts F \). Moreover, it is clear that \( g \) acts trivially on \( T \) and by translations of length \( ||g|| \) on \( F \), concluding the proof of our proposition.

### 4.3 Inversing isometries

In this subsection, we prove Theorem 4.12 for inverting isometries, namely:

**Proposition 4.18.** Let \( G \) be a group acting on a CAT(0) cube complex \( X \) and \( g \in G \) an inverting isometry such that \( \langle g \rangle \) is normal in \( G \). Denote by \( k \) the number of hyperplanes inverted by powers of \( g \). Then \( \text{Med}(g) \) is a \( G \)-invariant median subalgebra which decomposes as a product \( T \times F \times Q \) of three median algebras \( T, F, Q \) such that:

- the action \( G \acts T \times F \times Q \) is a product of three actions \( G \acts T, F, Q \).
First of all, notice that

**Proof.**

Then \( g \in J \) defines a loxodromic isometry of the cubical quotient \( X/J \).

Now, let \( J, H \in \mathcal{H} \) be two hyperplanes. Let \( m, n \in \mathbb{Z} \) be two powers such that \( g^m \) and \( g^n \) invert \( J \) and \( H \) respectively. If \( \ell \) denotes the least common multiple of \( m \) and \( n \), then \( \ell \) cannot be a multiple of both \( 2m \) and \( 2n \), since otherwise \( \ell/2 \) would be a lower common multiple of \( m \) and \( n \). Say that \( \ell \) is not a multiple of \( 2m \). Because \( g^m \) and \( g^n \) stabilise \( J \) and \( H \) respectively, necessarily \( g^{\ell} \) stabilises both \( J \) and \( H \). Also, because \( \ell \) is not a multiple of \( 2m \), then \( g^{\ell} \) inverts \( H \). It follows that \( J \) and \( H \) must be transverse, since otherwise \( g^{\ell}J \) and \( J \) would be contained into two different halfspaces delimited by \( H \), contradicting \( g^{\ell}J = J \). Thus, we have proved that any two hyperplanes of \( J \) are transverse, concluding the proof of our lemma.

Now, we are ready to prove the following statement, which we claimed after Definition 4.1 and which is needed to justify that the median set of an inverting isometry is well-defined.

**Lemma 4.20.** Let \( X \) be a CAT(0) cube complex and \( g \in \text{Isom}(X) \) an inverting isometry. Let \( J \) denote the collection of the hyperplanes of \( X \) which are inverted by powers of \( g \). Then \( g \) defines a loxodromic isometry of the cubical quotient \( X/J \).

**Proof.** First of all, notice that \( J \) is \( \langle g \rangle \)-invariant, so any element of \( \langle g \rangle \) naturally defines an isometry of \( X/J \). In order to show that \( g \) defines a loxodromic isometry of \( X/J \), it is sufficient to show that \( g \) has an unbounded orbit in \( X/J \) and that its powers do not invert any hyperplane of \( X/J \). The former assertion follows from the facts that \( g \) has an unbounded orbit in \( X \) and that \( J \) is finite (according to Lemma 4.19). And the latter assertion follows from the observation that, if a power of \( g \) inverts a hyperplane of \( X/J \), then it has to invert the corresponding hyperplane of \( X \) as well.

We need two last preliminary lemmas before turning to the proof of Proposition 4.18. The first one is the following:
Lemma 4.21. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ an inverting isometry. Denote by $\mathcal{J} = \{J_1, \ldots, J_n\}$ the collection of the hyperplanes which are inverted by powers of $g$. Then $\text{Med}(g) \subset \bigcap_{i=1}^n N(J_i)$.

Proof. Let $J \in \mathcal{J}$ be a hyperplane and let $x \in X$ be a vertex which does not belong to $N(J)$. As a consequence of Lemma 2.3, there exists a hyperplane $H$ separating $x$ from $N(J)$. If $k \in \mathbb{Z}$ is a power such that $g^k$ inverts $J$, then $g^kH$ separates $x$ and $g^kx$. Notice that, as $H$ separates $x$ and $g^kx$, necessarily $g^kH$ also separates $g^kx$ and $g^{2k}x$, so that $H$ crosses twice the path $\gamma := \bigcup_{k \in \mathbb{Z}} g^k[x,gx]$, where $[x,gx]$ is a geodesic between $x$ and $gx$.

Notice that, since $\mathcal{J}$ is $(g)$-invariant, the map $\pi$ is $(g)$-equivariant. Consequently, $\pi([x,gx])$ is a geodesic $[\pi(x), g\pi(x)]$ between $\pi(x)$ and $g\pi(x)$. We have

$$\pi(\gamma) = \pi \left( \bigcup_{k \in \mathbb{Z}} g^k[x,gx] \right) = \bigcup_{k \in \mathbb{Z}} g^k[\pi(x), g\pi(x)].$$

However, since $H$ is not transverse to $J$, we know from Lemma 4.19 that $H$ does not belong to $\mathcal{J}$, so that $H$ defines a hyperplane of $X/\mathcal{J}$ which crosses $\pi(\gamma)$ twice. As a consequence, $\pi(\gamma)$ is not a geodesic, which implies that $\pi(x)$ does not belong to $\text{Med}_{X/\mathcal{J}}(g)$, or equivalently, that $x$ does not belong to $\text{Med}(g)$.

Thus, we have proved that, if $x$ belongs to $\text{Med}(g)$, then it has to belong to $N(J)$ for every $J \in \mathcal{J}$. □

Finally, our last preliminary lemma is:

Lemma 4.22. Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ an inverting isometry. Let $\mathcal{J} = \{J_1, \ldots, J_n\}$ denote the collection of the hyperplanes of $X$ which are inverted by powers of $g$, and $\pi : X \to X/\mathcal{J}$ the canonical map from $X$ to the cubical quotient $X/\mathcal{J}$. If we fix an arbitrary halfspace $D_i$ delimited by $J_i$ for every $1 \leq i \leq n$, then $\pi$ induces an isometry $\text{Med}_X(g) \cap \bigcap_{i=1}^n D_i \to \text{Med}_{X/\mathcal{J}}(g)$ when $\text{Med}_X(g)$ and $\text{Med}_{X/\mathcal{J}}(g)$ are endowed with the metrics induced by $X$ and $X/\mathcal{J}$ respectively.

Proof. Because no two vertices of $\bigcap_{i=1}^n D_i$ are separated by a hyperplane of $\mathcal{J}$, it follows from Lemma 2.6 that $\pi$ induces an isometric embedding $\bigcap_{i=1}^n D_i \to X/\mathcal{J}$. It remains to show that, for every $z \in \text{Med}_{X/\mathcal{J}}(g)$, there exists some $x \in \text{Med}_X(g) \cap \bigcap_{i=1}^n D_i$ such that $\pi(x) = z$.

By definition of $\text{Med}_X(g)$, there exists some $a \in \text{Med}_X(g)$ such that $\pi(a) = z$. Let $x$ denote the projection of $a$ onto $\bigcap_{i=1}^n D_i$. According to Lemma 2.3, the hyperplanes separating $x$ and $a$ have to separate $a$ from $\bigcap_{i=1}^n D_i$. Since $a$ belongs to $\bigcap_{i=1}^n N(J_i)$ according to Lemma 4.21, it follows that $x$ and $a$ are only separated by hyperplanes of $\mathcal{J}$. Therefore, $\pi(x) = \pi(a) = z$. We record what we have just proved for future use:

Fact 4.23. Let $z \in \text{Med}_{X/\mathcal{J}}(g)$ be a vertex. For every $a \in \pi^{-1}(z)$, we have

$$\pi(\text{proj}_D(a)) = z$$

where $D := \bigcap_{i=1}^n D_i$. 

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The proof of our lemma is complete.

**Proof of Proposition 4.18.** Let $\mathcal{J} = \{J_1, \ldots, J_n\}$ be the collection of the hyperplanes of $X$ which are inverted by powers of $g$. Notice that $\text{Med}_X(g)$ is $G$-invariant. Indeed, the fact that $\langle g \rangle$ is a normal subgroup of $G$ implies that $\mathcal{J}$ is $G$-invariant, so that $G$ naturally acts on $X/\mathcal{J}$ and $\pi : X \to X/\mathcal{J}$ is $G$-equivariant. Moreover, we know from Proposition 4.9 that $\text{Med}_{X/\mathcal{J}}(g)$ must be $G$-invariant. Next, notice that $\text{Med}_X(g)$ is a median subalgebra of $X$. Indeed, let $x, y, z \in \text{Med}_X(g)$ be three vertices and let $m := \mu(x, y, z)$ denote their median point. Then $\pi(m)$ is the median point of $\pi(x)$, $\pi(y)$ and $\pi(z)$.

It follows from Lemmas 4.10 that $m \in \pi^{-1}(\pi(m)) \subset \pi^{-1}(\text{Med}_{X/\mathcal{J}}(g)) = \text{Med}_X(g)$, concluding the proof of our claim.

According to Lemma 4.19 the hyperplanes of $\mathcal{J}$ are pairwise transverse, so there exists a cube $Q \subset X$ such that $\mathcal{J}$ coincides with the collection of the hyperplanes crossing $Q$. Our goal now is to show that the map

$$\Phi : \{ \text{Med}_X(g) \to Q \times \text{Med}_{X/\mathcal{J}}(g) \}$$

is an isomorphism of median algebras. We begin by showing that $\Phi$ is surjective. For every $q \in Q$, let $D(q)$ denote the intersection of all the halfspaces delimited by the hyperplanes of $\mathcal{J}$ which contain $q$. Also, fix a section $s : X/\mathcal{J} \to X$ of $\pi$. Then we claim that

$$\Psi : \{ Q \times \text{Med}_{X/\mathcal{J}}(g) \to \text{Med}_X(g) \}$$

satisfies $\Phi \circ \Psi = \text{Id}_{Q \times \text{Med}_{X/\mathcal{J}}(g)}$. Indeed, for every $(q, z) \in Q \times \text{Med}_{X/\mathcal{J}}(g)$, we have $\pi(\text{proj}_{D(q)}(s(z))) = z$ according to Fact 4.23 and we also have $\text{proj}_Q(\text{proj}_{D(q)}(s(z))) = q$ because by construction $q$ and $\text{proj}_{D(q)}(s(z))$ are not separated by any hyperplane of $\mathcal{J}$.

Thus, the surjectivity of $\Phi$ is proved.

Notice that, because the median structures of $Q$ and $\text{Med}_{X/\mathcal{J}}(g)$ come from the metrics of $X$ and $X/\mathcal{J}$ respectively, it is sufficient to show that $\Phi$ is an isometry, when $\text{Med}_X(g)$, $Q$ and $\text{Med}_{X/\mathcal{J}}(g)$ are endowed with the metrics induced by $X$ and $X/\mathcal{J}$, in order to deduce that $\Phi$ defines an isomorphism of median algebras.

So let $x, y \in \text{Med}_X(g)$ be two vertices. For every $1 \leq i \leq n$, let $D_i$ denote the halfspace delimited by $J_i$ which contains $x$. Also, let $y'$ denote the projection of $y$ onto $D := \bigcap_{i=1}^n D_i$. Because no hyperplane of $\mathcal{J}$ separates $x$ and $y'$, it follows that $x$ and $y'$ have the same
projection onto $Q$. Moreover, it follows from Lemmas 2.3 and 4.21 that the hyperplanes separating $y$ and $y'$ all belong to $\mathcal{J}$. Consequently,

$$d(y, y') = d\left(\text{proj}_Q(y), \text{proj}_Q(y')\right) = d\left(\text{proj}_Q(y), \text{proj}_Q(x)\right).$$

Next, once again because the hyperplanes separating $y$ and $y'$ all belong to $\mathcal{J}$, we also know that $\pi(y) = \pi(y')$. Therefore, since $x$ and $y'$ both belong to $D$, it follows from Lemma 4.22 that

$$d(x, y') = d(\pi(x), \pi(y')) = d(\pi(x), \pi(y)).$$

We conclude that

$$d(x, y) = d(x, y') + d(y', y) = d\left(\text{proj}_Q(y), \text{proj}_Q(x)\right) + d(\pi(x), \pi(y)),$$

concluding the proof of our claim.

Now, let us focus on the action $G \curvearrowright Q \times \text{Med}_{X/\mathcal{J}}(g)$. By using the expression of $\Psi = \Phi^{-1}$ given above, this action can be described as

$$h \cdot (q, z) = \left(\text{proj}_Q\left(h \cdot \text{proj}_{D(q)}(s(z))\right), \pi\left(h \cdot \text{proj}_{D(q)}(s(z))\right)\right)$$

for every $(q, z) \in Q \times \text{Med}_{X/\mathcal{J}}(g)$. We would like to simplify this expression.

First, notice that, as a consequence of Lemmas 2.3 and 4.21, the vertices $s(z)$ and $\text{proj}_{D(q)}(s(z))$ are only separated by hyperplanes of $\mathcal{J}$, so they must have the same image under $\pi$. Consequently,

$$\pi\left(h \cdot \text{proj}_{D(q)}(s(z))\right) = h \cdot \pi\left(\text{proj}_{D(q)}(s(z))\right) = h \cdot \pi(s(z)) = h \cdot z.$$

Next, notice that no hyperplane of $\mathcal{J}$ separates $q$ and $\text{proj}_{D(q)}(s(z))$. Because $\mathcal{J}$ is $G$-invariant (since $g$ is a normal subgroup of $G$), it follows that no hyperplane of $\mathcal{J}$ separates $h \cdot q$ and $h \cdot \text{proj}_{D(q)}(s(z))$ either. Hence

$$\text{proj}_Q\left(h \cdot \text{proj}_{D(q)}(s(z))\right) = \text{proj}_Q(hq).$$

Thus, we have proved that the action $G \curvearrowright Q \times \text{Med}_{X/\mathcal{J}}(g)$ can be described as

$$h \cdot (q, z) = \left(\text{proj}_Q(h \cdot q), h \cdot z\right)$$

for every $(q, z) \in Q \times \text{Med}_{X/\mathcal{J}}(g)$.

Notice that

$$\text{proj}_Q\left(h_1 \cdot \text{proj}_Q(h_2 \cdot q)\right) = \text{proj}_Q(h_1h_2 \cdot q)$$

for every $h_1, h_2 \in G$ and $q \in Q$.

Indeed, $h_2q$ and $\text{proj}_Q(h_2q)$ are not separated by any hyperplane of $\mathcal{J}$. As $\mathcal{J}$ is $G$-invariant, it follows that $h_1h_2q$ and $h_1\text{proj}_Q(h_2q)$ are not separated by any hyperplane of $\mathcal{J}$ as well, showing that these two vertices have the same projection onto $Q$.

Thus, we have proved that the action $G \curvearrowright Q \times \text{Med}_{X/\mathcal{J}}(g)$ decomposes as a product of two actions $G \curvearrowright Q, \text{Med}_{X/\mathcal{J}}(g)$. It is worth noticing that our action $G \curvearrowright \text{Med}_{X/\mathcal{J}}(g)$ coincides with the action on $\text{Med}_{X/\mathcal{J}}(g)$ induced by $G \curvearrowright X/\mathcal{J}$.

By applying Proposition 4.9 to the action of $G$ on $X/\mathcal{J}$ and to the element $g$, we find that $\text{Med}_{X/\mathcal{J}}(g)$ decomposes as a product of median algebras $T \times F$ such that:

- the action $G \curvearrowright T \times F$ decomposes as a product of two actions $G \curvearrowright T, F$;
- $L$ is a flat on which $g$ acts by translations of length $\|g\|_{X/\mathcal{J}}$;
• $g$ acts trivially on $T$.

Notice that, as a consequence of Lemma 2.6, we have

$$d_{X/J} (\pi(x), g\pi(x)) \leq d_X (x, gx) \leq d_{X/J} (\pi(x), g\pi(x)) + |J|$$

for every vertices $x, y \in X$. Consequently,

$$\|g\|_{X/J} = \lim_{n \to +\infty} \frac{1}{n} d_{X/J} (x, g^n x) = \lim_{n \to +\infty} \frac{1}{n} d_X (x, g^n x).$$

Therefore, the decomposition $Q \times T \times F$ of the median set $\text{Med}_X(g)$ is the decomposition we were looking for.

**Remark 4.24.** It is worth noticing that the median set of an inverting isometry $g$ cannot be defined as $\text{Min}(g)$, since the minimising set may not be median. Indeed, let $g = (g_1, g_2)$ be the isometry of $[0, 1]^2 \times \mathbb{R}$ such that $g_1$ is the isometry of $[0, 1]^2$ given by Figure 8 and $g_2$ a translation of length one. Then $\text{Min}(g) = \text{Min}(g_1) \times \mathbb{R}$ is not median because $\text{Min}(g_1)$ is not a median subset of the square.

## 5 A cubical flat torus theorem

This section is dedicated to the proof of the following cubical version of the flat torus theorem (compare to [BH99, Theorem II.7.20] in the CAT(0) setting):

**Theorem 5.1.** Let $G$ be a group acting on a CAT(0) cube complex $X$ and $A \leq G$ a normal finitely generated abelian subgroup. Then there exists a median subalgebra $Y \subset X$ which is $G$-invariant and which decomposes as a product $T \times F \times Q$ of median algebras $T, F, Q$ such that:

• $G \acts Y$ decomposes as a product of actions $G \acts T, F, Q$;

• $F$ is a flat;

• $Q$ is a finite-dimensional cube;

• $A$ acts trivially on $T$.

**Proof.** We argue by induction over the rank of $A$. If $A$ is trivial then setting $F = Q = \{\text{vertex}\}$ and $T = X$ leads to the desired conclusion. Now, fixing some $r \geq 0$, assume that our theorem holds for abelian groups of rank $m$ and suppose that $A$ has rank $r + 1$. Write $A = A' \oplus \langle g \rangle$ for some $g \in A$ such that $A'$ has rank $r$.

According to Theorem 4.2, our group $G$ acts on the median set $\text{Med}(g) = T_0 \times F_0 \times Q_0$ such that:

• the action $G \acts T_0 \times F_0 \times Q_0$ decomposes as a product of three actions $G \acts T_0, F_0, Q_0$;

• $F_0$ is a single vertex if $g$ is elliptic, and otherwise it is a flat;

• $Q_0$ is a finite-dimensional cube, possibly reduced to a single vertex;

• $g$ acts trivially on $T_0$.

By applying our induction hypothesis to the subgroup $A' \leq G$ and to the action of $G$ on the cubulation of $T$, we get a $G$-invariant median subalgebra

$$Y = T \times F_1 \times Q_1 \subset T_0$$

such that:
• $G \rhd Y$ decomposes as a product of actions $G \rhd T, F_1, Q_1$;
• $F_1$ is a flat;
• $Q_1$ is a finite-dimensional cube;
• $A'$ acts trivially on $T$.

We claim that $Z = Y \times F_0 \times Q_0$ is the median subalgebra of $X$ we are looking for. Indeed, $Z$ is $G$-invariant and the product structure

$$T \times (F_0 \times F_1) \times (Q_0 \times Q_1).$$

provides the desired decomposition of $Z$. □

6 Applications

6.1 Cubulating centralisers

This section is dedicated to the proof of the following statement, which improves the cubulation of centralisers proved in [Hae15]:

**Theorem 6.1.** Let $G$ be a group acting geometrically on a CAT(0) cube complex $X$. For every infinite-order element $g \in G$, the centraliser $C_G(g)$ also acts geometrically on a CAT(0) cube complex, namely the cubulation of the median subalgebra $\text{Med}_{X'}(g)$ of the barycentric subdivision $X'$ of $X$.

Our theorem will be essentially a consequence of the following two preliminary lemmas:

**Lemma 6.2.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ a loxodromic isometry. Then $\text{Min}(g) = \text{Med}(g)$.

This lemma is a direct consequence of [Hag07, Corollary 6.2].

**Lemma 6.3.** Let $G$ be a group acting geometrically on a CAT(0) cube complex $X$ and $g \in G$ a loxodromic isometry. Then $C_G(g)$ acts on $\text{Min}(g)$ with finite stabilisers and with finitely many orbits.

**Proof.** Because $G$ acts properly on $X$, it is clear that $C_G(g)$ acts on $\text{Min}(g)$ with finite stabilisers. Now, suppose by contradiction that $C_G(g)$ acts on $\text{Min}(g)$ with infinitely many orbits. Fix a collection of vertices $x_0, x_1, \ldots \in \text{Min}(g)$ which belong to pairwise distinct $C_G(g)$-orbits. Because $G$ acts cocompactly on $X$, there exists some constant $C \geq 0$ such that, for every $i \geq 1$, there exists some $g_i \in G$ satisfying $d(x_0, g_ix_i) \leq C$. Because $X$ is locally finite, up to taking a subsequence, we may suppose without loss of generality that $g_ix_i = g_jx_j$ for every $i, j \geq 1$. Next, notice that

$$d(x_0, g_ig_j^{-1}x_0) = d(g_i^{-1}x_0, gg_j^{-1}x_0) \leq d(x_0, gx_0) + 2C = \|g\| + 2C.$$

Because $X$ is locally finite and since its vertex-stabilisers are finite, up to taking a subsequence, we may suppose without loss of generality that $g_ig_j^{-1} = g_jg_j^{-1}$ for every $i, j \geq 1$. Fixing two distinct indices $i, j \geq 1$, we have

$$C_G(g) \cdot x_j = C_G(g) \cdot g_i^{-1}g_jx_j = C_G(g) \cdot g_i^{-1}gx_i = C_G(g) \cdot x_i$$

because $g_i^{-1}g_j$ belongs to $C_G(g)$, contradicting the fact that $x_i$ and $x_j$ have distinct $G_C(g)$-orbits. □
Proof of Theorem 6.4: Up to taking the barycentric subdivision of $X$, we may suppose without loss of generality that $g$ is a loxodromic isometry of $X$. We know from Proposition 4.9 that $\text{Med}(g)$ is a median subalgebra of $X$ which is $C_G(g)$-invariant, so $C_G(g)$ naturally acts on the cubulation $C(\text{Med}(g))$. By combining Lemmas 6.2 and 6.3, it follows that $C_G(g)$ acts on $C(\text{Med}(g))$ with finite vertex-stabilisers and with finitely many orbits of vertices. Because $C(\text{Med}(g))$ is locally finite according to Fact 3.10, we conclude that $C_G(g)$ acts geometrically on the CAT(0) cube complex $C(\text{Med}(g))$. □

6.2 A splitting theorem

In this section, we show that the center of a group acting in a nice way on a CAT(0) cube complex must be a direct factor of some finite-index subgroup. Compare with [BH99, Theorem II.6.12] in the CAT(0) setting.

Theorem 6.4. Let $G$ be a group acting on a CAT(0) cube complex $X$ and $A \leq G$ a central, torsion-free, finitely generated subgroup. Assume that a non-trivial element of $A$ is never elliptic. Then $A$ is a direct factor of some finite-index subgroup of $G$.

Proof. Fix a non-trivial element $g \in A$. Up to subdividing $X$, we may suppose without loss of generality that $g$ is a loxodromic element. Because $g$ is central in $G$, it follows from Proposition 4.9 that $G$ acts on a median subalgebra $T \times F \subset X$ such that:

- the action $G \curvearrowright T \times F$ decomposes as a product of two actions $G \curvearrowright T, F$;
- $F$ is a flat on which $g$ acts by translations of positive length.

As a consequence of Proposition 3.4 there exist a finite-index subgroup $G_0 \leq G$ and a morphism $\varphi : G_0 \to \mathbb{Z}$ such that $\varphi(g) \neq 0$. Fix an element $a \in A$ such that $\varphi(a)$ generates the image $\varphi(A)$, and set $H := \varphi^{-1}(\varphi(A))$. Notice that $H$ is a finite-index subgroup of $G$. Because $a$ is central in $G$, it follows that $H = H_0 \oplus \langle a \rangle$ and $A = A_0 \oplus \langle a \rangle$ where $H_0 = \ker(\varphi) \cap H$ and $A_0 = \ker(\varphi) \cap A$. Notice that the rank of $A_0$ is smaller than the rank of $A$, so by arguing by induction we may suppose without loss of generality that $H_0$ contains a finite-index subgroup which decomposes as a direct sum $A_0 \oplus K$. Then $A_0 \oplus K \oplus \langle a \rangle = A \oplus K$ has finite index in $G$, concluding the proof of the theorem. □

Theorem 6.4 can be used to show that some groups do not act nicely on CAT(0) cube complexes. For instance, following [BH99], we show:

Corollary 6.5. Let $\Sigma$ be a closed surface of genus $\geq 3$. Then the mapping class group $\text{MCG}(\Sigma)$ does not act properly on a CAT(0) cube complex.

Proof. As shown during the proof of [BH99, Theorem II.7.26], $\text{MCG}(\Sigma)$ contains a subgroup $K$ whose center is not a direct factor in any finite-index subgroup of $K$. The desired conclusion now follows from Theorem 6.4. □

A nice application of Theorem 6.4 is the following result, which is a particular of a much more general statement [Got65]:

Corollary 6.6. Let $X$ be a compact nonpositively curved cube complex. If the Euler characteristic $\chi(X)$ is non-zero, then the center of $\pi_1(X)$ must be trivial.

Recall that the Euler characteristic of a group $G$ is

$$\chi(G) = \sum_{i \geq 0} (-1)^i \text{rank}_2 H_i(X)$$

where $X$ is a classifying space of $G$. This sum does not depend on the classifying space we choose, but it may not converge. A case where the Euler characteristic is clearly
well-defined is when the group admits a classifying space with only finitely many cells.
Because CAT(0) cube complexes are contractible, this case includes fundamental groups of compact nonpositively curved cube complexes.

Proof of Corollary 6.6. Assume that \( \pi_1(X) \) has a non-trivial center. According to Theorem 6.4 there exists a finite-index subgroup \( H \leq \pi_1(X) \) which decomposes as a product \( G = Z \times K \) where \( Z \) is the (infinite) center of \( \pi_1(X) \) and \( K \) some subgroup. Notice that it follows from Künneth theorem that \( \chi(K) \) is well-defined and that \( \chi(G) = \chi(Z) \cdot \chi(K) \). Because \( \chi(X) \) has to be a multiple of \( \chi(G) \) and that \( \chi(Z) = 0 \), it follows that \( \chi(X) \) has to be zero.

6.3 Solvable groups acting on CAT(0) cube complexes

Recall that the derived series of a group \( G \) is the decreasing sequence of normal subgroups

\[
G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright \cdots
\]

where \( G_i \) is the commutator subgroup \([G_{i-1}, G_{i-1}]\) of \( G_{i-1} \) for every \( i \geq 1 \). If the series is eventually constant to \( \{1\} \), then \( G \) is solvable, and the smallest \( k \geq 0 \) such that \( G_k = \{1\} \) is referred to as the derived length of \( G \).

The main result of this section is the following:

Theorem 6.7. Let \( G \) be a solvable group acting on a CAT(0) cube complex. Assume that \( G \) does not contain an infinite torsion subgroup and that it does not contain a free abelian group of arbitrary large rank. Then \( G \) contains a finite-index subgroup which is (locally elliptic)-by-(free abelian).

The theorem will be essentially a consequence of the following statement:

Proposition 6.8. Let \( G \) be a solvable group acting on a CAT(0) cube complex \( X \). Assume that \( G \) does not contain an infinite torsion subgroup and that it does not contain a free abelian group of arbitrary large rank. Then there exists a median subalgebra \( Y \subset X \) which is \( G \)-invariant and which decomposes as a product of a flat with a finite-dimensional cube.

Proof. We argue by induction over the derived length of \( G \). If \( G \) is trivial, there is nothing to prove; and if \( G \) is abelian, the desired conclusion follows from Theorem 5.1. So suppose that \( G \) has length at least two. Write its derived series

\[
\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G.
\]

Notice that \( G_1 \) is a normal abelian subgroup of \( G \). Moreover, it is clear that \( G_1 \) has to be finitely generated as a \( \mathbb{Z} \)-module, since it does not contain an infinite torsion subgroup nor a free abelian group of arbitrary large rank, so that \( G_1 \) has to be finitely generated as a group as well. It follows from Theorem 5.1 that \( X \) contains a median subalgebra \( Y \) which is \( G \)-invariant and which decomposes as a product \( T \times F \times Q \) such that

- the action \( G \rtimes Y \) decomposes as a product of actions \( G \rtimes T, F, Q \);
- \( F \) is a flat;
- \( Q \) is a finite-dimensional cube;
- \( G_1 \) acts trivially on \( T \).

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Now we apply our induction hypothesis to the action \( G/G_1 \curvearrowright T \) in order to find a median subalgebra \( M \subset T \) which is \( G/G_1 \)-invariant and which decomposes as a product of a flat with a finite-dimensional cube. Then \( M \times F \times Q \) provides the desired \( G \)-invariant median subalgebra of \( X \).

**Proof of Theorem 6.7.** The conclusion is a direct consequence of Propositions 6.8 and 3.4.

The motivation behind Theorem 6.7 was the following question: which solvable groups act properly on CAT(0) cube complexes? As a direct consequence of Theorem 6.7, we get:

**Corollary 6.9.** Let \( G \) be a solvable group acting properly on a CAT(0) cube complex. Assume that \( G \) does not contain an infinite torsion subgroup and that it does not contain a free abelian group of arbitrary large rank. Then \( G \) must be virtually free abelian.

It is worth noticing that the assumptions imposed on \( G \) cannot be removed. Indeed, the wreath products \( \mathbb{Z}_2 \wr \mathbb{Z} \), which does not contain \( \mathbb{Z}^2 \) but which contains an infinite torsion group, and \( \mathbb{Z} \wr \mathbb{Z} \), which is torsion-free but which contains a free abelian group of infinite rank, act properly on CAT(0) cube complexes. See [CSV12, Gen17b] for more details.

However, Theorem 6.7 can also be useful to study solvable subgroups of groups acting on CAT(0) cube complexes with infinite vertex-stabilisers. In a forthcoming article, we plan to apply this strategy to the braided Thompson group \( \text{brV} \).

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