Effective anisotropic stresses of the relic gravitons

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Abstract

The effective anisotropic stresses induced by the scalar modes of the geometry depend on the coordinate system so that the comparison of the competing results is ultimately determined by the evolution of the pivotal variables in each particular gauge. After arguing that the only reasonable physical coordinate systems for this problem are the ones where the gauge freedom is completely fixed (like the longitudinal and the uniform curvature gauges), we propose a novel gauge-invariant strategy for the comparison of gauge-dependent results. Instead of employing the pivotal variables of a given coordinate system, the effective anisotropic stress is solely expressed in terms of the gravitating normal modes of the plasma and in terms of their conformal time derivatives. The new approach is explicitly gauge-invariant and when the wavelengths of the normal modes are either shorter or larger than the sound horizon, the physical limits of the anisotropic stresses are determined without relying on the specific details of the background evolution. The relevance of the proposed strategy is discussed in the general situation where the scalar anisotropic stress and the non-adiabatic pressure fluctuations are simultaneously present. We finally argue that the anisotropic stress can be most efficiently obtained from the second-order effective action of the curvature inhomogeneities.
1 Introduction

The effective energy densities and the pressures of the relic gravitons are neither unique nor gauge-invariant. This inevitable feature is ultimately caused by the equivalence principle that forbids the localization of the energy-momentum tensor of the gravitational field [1]. The effective anisotropic stresses induced by the scalar inhomogeneities of the geometry also affect the evolution of the relic gravitons and they are customarily assessed always by using the Landau-Lifshitz approach [2] with the proviso that besides the second-order tensor modes (leading to the energy-momentum pseudo-tensor) also the second-order scalar modes should be consistently taken into account. Within the concordance paradigm the spectral energy density of the relic gravitons scales linearly with the amplitude of the tensor power spectrum (i.e. $A_T$) while the correction due to the effective anisotropic stresses coming from the scalar modes is quadratic in the amplitude of the scalar power spectra $A_R$. The corrections coming from the scalar anisotropic stresses are therefore smaller than the leading-order results by a factor where $A_R/r_T$ where $r_T$ denotes the tensor to scalar ratio (see, for instance, [3] for a recent review). There are two related aspects that make this problem often confusing. First the Landau-Lifshitz approach is not unique; second the effective anisotropic stresses are, by construction, gauge dependent. In this investigation we shall address both issues with the aim of proposing a novel gauge-invariant approach to the analysis of the effective anisotropic stresses induced by the scalar modes of the geometry.

It is actually well known that the Landau-Lifshitz strategy [2] is not unique: the Brill-Hartle averaging [4], the Isaacson approach [5, 6] and the Ford-Parker proposal [7] are the main suggestions put forward through the years for a proper definition of the energy-momentum pseudo-tensor of the gravitational field. As recently argued these approaches are not always equivalent: if applied in a cosmological context different proposals lead to sharply different forms of the energy density and of the pressure of the relic gravitons [8]. If the frequencies of the gravitons are larger than the rate of variation of the background the different pseudo-tensors lead to coincident results but the conclusions are sharply different in the opposite limit.

While some other suggestions have been presented through the years, they can all be related, either directly or indirectly, to the original ideas mentioned in the previous paragraph. So for instance the proposal of Ref. [9] coincides with the Landau-Lifshitz approach while the results of Refs. [10, 11] follow from the strategies of Refs. [4, 5, 7]. The suggestion of Refs. [12] coincide with the approach of the effective action [7] (see also [6]). The authors of Ref. [13] claimed a result with all the necessary properties of a true energy-momentum tensor of the gravitational field itself (i.e. symmetry, uniqueness, gauge-invariance and covariant conservation). While this result has been subsequently challenged by Refs. [14, 15], the geometrical object most closely related to the suggestion [13] is the Landau-Lifshitz pseudo-tensor [2]. The ambiguity of the competing definitions may be solved by imposing a number of physical requirements (e.g. the positivity of the energy density both inside and outside the Hubble radius) [8]. These criteria pin down the Ford-Parker proposal [7] where the energy-momentum pseudo-tensor follows from the variation of the effective action of the relic gravitons with respect to the background metric.

The possibility of higher-order processes makes the problem more acute and, in a sense, even less gauge-invariant. For instance the long-wavelength gravitons induce curvature inhomogeneities both during inflation and in the subsequent radiation-dominated phase [16]. Similarly curvature inhomogeneities may cause higher-order corrections to the stochastic backgrounds of relic gravitons and this second effect involves an effective anisotropic stress [17]. The gauge-dependence of the effective anisotropic stresses has been originally suggested in Ref. [18]. Even if the description of
the longitudinal gauge is considered more computable and hence more reliable (see e. g. [19]), there are no reasons why this should be the case so that the scheme of Ref. [18] has been subsequently replicated with different and sometimes contradictory conclusions [20, 21, 22]. Reference [20] attributes the difference of the results to the evolutionary features of each gauge. Reference [21] suggests that the effective anisotropic stress is gauge-independent but the authors also imply, in their conclusions, that the observational sensitivities for the tensor perturbations induced from the effective anisotropic stress will be different from those for conventional gravitational waves. In this sense the observation of this tensor perturbation might require a discussion about the suitable gauge for the observation because of its gauge dependence. This last statement is at odds with the claimed gauge-independence. Finally Ref. [22] overlaps significantly with previous works and, by admission of the authors, it just revisits the gauge dependence of gravitational waves generated at second order from scalar perturbations. This analysis suggests that the various backgrounds affect the gauge-invariance of the final results and claims that the obtained conclusions are not really gauge-independent and, to some extent, even background dependent.

We propose here a method that is simultaneously gauge-invariant and background independent. The idea is to obtain the effective anisotropic stresses in a particular gauge and then to express the obtained result solely in terms of $R$ and $R'$ that will denote throughout the curvature inhomogeneities (defined on comoving orthogonal hypersurfaces) and their corresponding (conformal) time derivatives. Since the new variables coincide with the gravitating normal modes of the system their evolution is the same in any gauge. Therefore, within the present approach, the effective anisotropic stresses in different gauges will depend on the same set of pivotal variables obeying the same master equation: unlike the strategies pursued so far the comparison between the gauge-dependent results will therefore be immediate. In Refs. [23] and [24] the main aspects of this approach have been outlined in the simplest possible situation, namely the one where the non-adiabatic pressure fluctuations are absent and the total anisotropic stress vanishes.

To avoid potential confusions we stress that three different quantities shall be repeatedly mentioned hereunder namely:

- the anisotropic stress induced by free-streaming particles ($\Pi_t$ in what follows) and affecting the evolution of the scalar modes of the geometry;
- the effective anisotropic stress induced by the (second-order) scalar inhomogeneities and affecting the evolution of the tensor modes;
- the non-adiabatic pressure fluctuations ($\delta p_{nad}$ in what follows) depend on the composition of the plasma and it vanishes in the case of a single fluid.

The anisotropic stress caused by the free-streaming particles (for short the scalar anisotropic stress) in the concordance paradigm is mainly due to neutrinos and it affects the initial conditions of the Einstein-Boltzmann hierarchy necessary for the calculation of the temperature and polarization anisotropies of the Cosmic Microwave Background [25]. The neutrinos free-stream after electron-positron annihilation and their anisotropic stress also affects directly the relic graviton background by suppressing its spectral energy density [26, 27]. The second-order scalar modes of the geometry induce instead an effective anisotropic stress which is the one considered more directly here. The non-adiabatic pressure fluctuations vanish in the case of the concordance paradigm but may contribute to more general scenarios both at early and at late times.

Also in the presence of $\delta p_{nad}$ and $\Pi_t$ the effective anisotropic can be solely expressed in terms $R$ and $R'$ but the non-adiabatic pressure fluctuations and the scalar anisotropic stress will however
introduce a source term in the evolution equation for the gravitating normal modes of the system. An important technical advantage of the present approach concerns the approximate solutions of the evolution that can be analyzed in a background-independent manner. Indeed the single master equation obeyed by \( R \) and \( R' \) can be analyzed within the Wentzel-Kramers-Brillouin (WKB) approximation.

All in all the layout of this investigation is the following. In section 2 we shall present the gauge-invariant evolution of the gravitating normal modes of the plasma when the non-adiabatic pressure fluctuations and the total anisotropic stress are present. The general properties of the effective anisotropic stresses will also be outlined with particular attention to the coordinate systems where the gauge freedom is completely fixed. In sections 3 and 4 the main idea will be illustrated by explicitly deriving the expressions of the effective anisotropic stress in terms of the gauge-invariant normal modes. In particular the longitudinal gauge will be discussed in section 3 while section 4 will be instead focussed on the coordinate system where the spatial curvature is uniform. It will be shown that different gauge-invariant descriptions (like the one following from the density contrast on uniform curvature hypersurfaces) cannot be traded for the one based on \( R \) and \( R' \). In section 5 the gauge-dependent results will be compared in gauge-invariant terms with particular attention to the limits of the effective anisotropic stress for typical wavelengths larger or shorter than the sound horizon. In section 6 the spectral energy density of relic gravitons will be computed in the case of the concordance paradigm and for a radiation-dominated plasma. In section 7 we shall clarify how the effective anisotropic stress could be derived from the second-order action of the scalar modes in full analogy with the procedure leading to the effective energy density of the relic gravitons. Section 8 contains the concluding considerations.

## 2 General gauge-invariant evolution

### 2.1 Gravitating normal modes

In a conformally flat and homogeneous background geometry the fluctuations of a gravitating, irrotational and relativistic fluid admit a normal mode that shall be conventionally denoted hereunder by \( R \). This quantity has been originally discussed by Lukash [28] even prior to the actual formulation of the conventional inflationary paradigm and in the context of the pioneering analyses of the relativistic theory of large-scale inhomogeneities [29, 30]. There are different situations where the evolution of \( R \) can be studied. In the simplest case the non-adiabatic pressure fluctuations and the scalar anisotropic stress are absent. The evolution of \( R \) obeys then the following decoupled equation:

\[
R'' + 2 \frac{z_t'}{z_t} R' - c_{st}^2 \nabla^2 R = 0, \tag{2.1}
\]

where the prime denotes a derivation with respect to the conformal time coordinate \( \tau \) which is related to the cosmic time as \( a(\tau) d\tau = dt \); in Eq. (2.1) \( c_{st}^2 \) and \( z_t \) are defined as:

\[
c_{st}^2 = \frac{p_t'}{\rho_t'}, \quad z_t = \frac{a^2 \sqrt{\rho_t + p_t}}{H c_{st}}. \tag{2.2}
\]

From Eq. (2.2) \( z_t \) and \( c_{st}^2 \) depend on the total energy density \( \rho_t \) and on the total pressure \( p_t \); moreover, using standard notations, \( \mathcal{H} = a'/a = aH \) and \( H \) denotes the Hubble expansion rate. In
the absence of further sources $\mathcal{H}$, $p_t$ and $\rho_t$ will obey the conventional Friedmann-Lemaître equations

$$3H^2 = \ell_p^2 a^2 \rho_t, \quad 2(\mathcal{H}^2 - \mathcal{H}') = \ell_p^2 a^2 (\rho_t + p_t),$$

(2.3)

where $\ell_p = \sqrt{8\pi G}$. The variable $R$ deduced in Ref. [28] and obeying Eq. (2.1) coincides with the curvature perturbation on comoving orthogonal hypersurfaces and it is invariant under infinitesimal coordinate transformations as required in the context of the Bardeen formalism [31]. Subsequent analyses [32, 33] followed the same logic of [28] but in the case of scalar field matter. All the normal modes identified in Refs. [28, 32, 33] can be related to the (rescaled) curvature perturbations on comoving orthogonal hypersurfaces [34].

### 2.2 Non-adiabatic pressure fluctuations

Equation (2.1) is obtained by assuming the absence of the non-adiabatic pressure fluctuations and the absence of any source of anisotropic stress due to free-streaming particles. We are now going to relax both hypotheses. The non-adiabatic pressure fluctuations [35, 36, 37] arise for several reasons even if in the context of the concordance paradigm they are bound to vanish. In general terms the pressure fluctuations may not be only caused by the inhomogeneities of the energy density of the plasma, as it happens in the concordance paradigm, so that the pressure perturbation shall be written as the sum of two different contributions:

$$\delta_s p_t = c^2_{st} \delta_s \rho + \delta p_{nad}, \quad \delta p_{nad}(\vec{x}, \tau) = \sum_{ab} \frac{\partial p_t}{\partial \varsigma_{ab}} \delta \varsigma_{ab}(\vec{x}, \tau).$$

(2.4)

While the first term of Eq. (2.4) accounts for the fluctuations of the pressure coming from the inhomogeneity of the energy density, the explicit expression of $\delta p_{nad}$ depends on the composition of the plasma and it vanishes in the case of a single fluid. In Eq. (2.4) $\varsigma_{ab}$ denotes the specific entropy, i.e. the ratio between the entropy density and the concentration of the given species; the indices $a$ and $b$ denote instead the various species of the plasma. The entropy fluctuation is therefore defined as the relative fluctuation of the specific entropy for a given pair of species in the plasma:

$$\varsigma_{ab}(\vec{x}, \tau) = \frac{\delta \varsigma_{ab}(\vec{x}, \tau)}{\varsigma_{ab}} = \frac{\delta_b}{w_b + 1} - \frac{\delta_a}{w_a + 1}, \quad \varsigma_{ab} = -\varsigma_{ba}$$

(2.5)

While the first expression in Eq. (2.5) follows from the definition, the second equality holds when the different species with the density contrasts $\delta_a = \delta \rho_a / \rho_a$ and $\delta_b = \delta \rho_b / \rho_b$ are characterized by the constant barotropic indices $w_a$ and $w_b$. For a collection of fluids with different equations of state and different sound speeds the explicit form of $\delta p_{nad}$ is

$$\delta p_{nad}(\vec{x}, \tau) = \frac{1}{6H\rho_t} \sum_{ab} \rho'_a \rho'_b (c^2_{sa} - c^2_{sb}) S_{ab}(\vec{x}, \tau), \quad S_{ab}(\vec{x}, \tau) = \frac{\delta \varsigma_{ab}(\vec{x}, \tau)}{\varsigma_{ab}},$$

(2.6)

where, as in Eq. (2.5) the indices $a$ and $b$ are not tensor indices but denote two generic species of the pre-equality plasma; $c^2_{sa} = p'_a / \rho_a$ and $c^2_{sb} = p'_b / \rho_b$ are the corresponding sound speeds. In the case of a fluid made of two different components (e.g. radiation and matter) the corresponding total energy density is $\rho_t = (\rho_M + \rho_R)$ with $\rho'_M = -3H\rho_M$ and $\rho'_R = -4H\rho_R$. From Eq. (2.6) the explicit expression of $\delta p_{nad}$ will be given by:

$$\delta p_{nad} = \frac{4}{3} \frac{\rho_M \rho_R}{3\rho_R + 3\rho_M} S_s, \quad S_s = S_{MR} = -S_{RM}.$$  

(2.7)
Note that $S_*$ can also be expressed as the fractional variation of the specific entropy $\varsigma = T^3_R/n_M$ where $T_R$ is the temperature of the radiation background and $n_m$ is concentration of matter species; indeed we have $\delta \varsigma/\varsigma = (3\delta R/4 - \delta M)$ where $\delta R = \delta \rho_R/\rho_R$ and $\delta M = \delta \rho_M/\rho_M$; exactly the same result follows from Eq. (2.5) From the expressions of $\rho_M$ and $\rho_R$ in terms of the scale factor Eq. (2.7) becomes

$$\delta p_{nad} = \rho_M c^2_{st} S_*, \quad c^2_{st} = \frac{p_t}{\rho_t} = \frac{4}{3[(a/a_*) + 4]},$$

where it is understood that the plasma is dominated by radiation for $a > a_*$ and by matter for $a < a_*$. In the conventional terminology [35, 36, 37] Eqs. (2.7) and (2.8) describe either the CDM-radiation mode (if $\rho_M = \rho_{cdm}$) or the baryon-radiation mode (provided $\rho_M = \rho_{baryon}$). In the concordance paradigm, when the dark energy does not fluctuate, there are, overall five different sets of Cauchy data: one adiabatic and four non-adiabatic [35, 36, 37] initial conditions.

### 2.3 Quasi-normal modes

Equation (2.1) also neglects the scalar anisotropic stress due to free-streaming particles (e.g. neutrinos in the case of the concordance paradigm). The total anisotropic stress associated with the scalar modes will therefore be expressed in one of the following equivalent ways:

$$\partial_i \partial_j \Pi^{ij} = \nabla^2 \Pi_t, \quad \Pi_t = (\rho_t + p_t) \sigma_t. \quad (2.9)$$

If the plasma contains a total anisotropic stress and non-adiabatic pressure fluctuations Eq. (2.1) gets modified by inheriting a source term $S_R(\vec{x}, \tau)$:

$$\mathcal{R}'' + 2 \frac{z_t'}{z_t} \mathcal{R}' - c^2_{st} \nabla^2 \mathcal{R} = S_R(\vec{x}, \tau). \quad (2.10)$$

If $\delta p_{nad} \neq 0$ and $\Pi_t \neq 0$ the source term $S_R$ can be written in the following matter:

$$S_R(\vec{x}, \tau) = \Sigma'_R + 2 \frac{z'_t}{z_t} \Sigma_R + 3a^4 \frac{1}{z_t^2} \Pi_t, \quad (2.11)$$

$$\Sigma_R(\vec{x}, \tau) = - \frac{\mathcal{H}}{\rho_t + \rho_t} \delta p_{nad} + \frac{\mathcal{H}}{\rho_t + \rho_t} \Pi_t. \quad (2.12)$$

In Fourier space Eq. (2.10) becomes therefore:

$$\left( \mathcal{R}'_k - \Sigma'_k \right)' + 2 \frac{z'_t}{z_t} \left( \mathcal{R}'_k - \Sigma'_k \right) + k^2 c^2_{st} \mathcal{R}_k = \frac{3a^4}{z_t^2} \Pi_k. \quad (2.13)$$

where $\Pi_k$ is the Fourier transform of $\Pi_t$; $\Sigma_k$ and $\Gamma_k$ are instead the Fourier transforms of $\Sigma_R$ and $\delta p_{nad}$:

$$\Sigma_k = \frac{\mathcal{H}}{\rho_t + \rho_t} \left( \Pi_k - \Gamma_k \right) \quad (2.14)$$

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\(^2\)On top of the CDM-radiation mode and of the baryon-entropy mode the remaining two non-adiabatic modes are the neutrino entropy mode and the neutrino isocurvature velocity mode. The considerations discussed hereunder are not bound to the case of the illustrative examples of Eqs. (2.7) and (2.8) but apply for all the non-adiabatic solutions.
While non-adiabatic pressure fluctuations and the anisotropic stress have been given in general terms, in the context of the concordance paradigm $\delta p_{nad} = 0$ and the total anisotropic stress is only due to neutrinos:

$$\Pi_t = (p_\nu + \rho_\nu)\sigma_\nu, \quad \sigma_\nu = \frac{F_\nu}{2}. \quad (2.15)$$

In Eq. (2.15) $F_\nu$ is the quadrupole of the neutrino phase-space distribution. The lower moments of $F_{\nu\ell}$ (i.e. with $\ell = 0, 1$) are related with the density contrast and with the peculiar velocity of the neutrinos while for $\ell \geq 3$ the evolution of $F_{\nu\ell}$ is given by:

$$F_{\nu\ell}' = \frac{k}{2\ell + 1} \left[ \ell F_{\nu(\ell-1)} - (\ell + 1)F_{\nu(\ell+1)} \right], \quad \ell \geq 3. \quad (2.16)$$

The evolution of the anisotropic stress of the neutrinos can be obtained by cutting the Boltzmann hierarchy of Eq. (2.16) and by requiring, for instance, $F_{\nu 3} = 0$ (but, according to Eq. (2.16), $F_{\nu 3}' \neq 0$). To get a decoupled equation we have to pay the price of higher derivatives of $\sigma_\nu$ and the result is:

$$\sigma_\nu'' + \frac{8}{5} \mathcal{H}^2 R_\nu \Omega_R' + \frac{6}{5} k^2 \sigma_\nu' - \frac{32}{5} \mathcal{H}^3 R_\nu \Omega_R \sigma_\nu = \frac{8}{15 \sqrt{c_\text{st}}^2} \left( \mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}} \right) \left( \frac{\mathcal{H}'}{\mathcal{H}} - 2 \mathcal{H} \right) (\Sigma_R - \Sigma_\mathcal{R}) + \frac{8}{15 \sqrt{c_\text{st}}^2} \left( \frac{\mathcal{H}'}{\mathcal{H}} - \mathcal{H} \right) k^2 \mathcal{R}, \quad (2.17)$$

where $\Omega_R = \rho_R/\rho_t$ is the critical fraction of radiation; as anticipated $R_\nu$ and $R_\gamma$ count the fraction of neutrinos and photons in the radiation plasma.

### 2.4 Effective anisotropic stresses of the relic gravitons

The effective anisotropic stress of the relic gravitons follows by perturbing the Einstein equations as

$$\delta_1^{(1)} G_i^j + \delta_2^{(2)} G_i^j = \ell_P^2 \delta_2^{(2)} T_i^j. \quad (2.18)$$

Equation (2.18) follows directly from the Landau-Lifshitz strategy [2] and few notational comments are in order:

- $G_\mu^\nu$ will denote throughout the Einstein tensor while $T_\mu^\nu$ is a generic energy-momentum tensor of the matter sources; for the present discussion we shall be mostly concerned with the case of hydrodynamical matter where $T_\mu^\nu = (\rho_t + p_t)u_\mu u^\nu - p_t \delta_\mu^\nu$;

- in Eq. (2.18) $\delta_1^{(1)}$ denotes the first-order tensor fluctuation while $\delta_2^{(2)}$ denotes the second-order scalar of the corresponding quantities;

- at the left-hand-side of Eq. (2.18) there is a further contribution coming from the second-order tensor fluctuations $G_\mu^\nu$, i.e. $\delta_2^{(2)} G_\mu^\nu$: note in fact that $-\delta_2^{(2)} G_\mu^\nu/\ell_P^2$ is nothing but the Landau-Lifshitz pseudo-tensor [2] (see also [8] for different ways of assigning the energy density and pressure of the relic gravitons).

While $\delta_1^{(1)} G_i^j$ is gauge-invariant to first-order, the second-order contributions are both gauge-dependent. With this proviso, since the explicit expression of $\delta_1^{(1)} G_i^j$ is:

$$\delta_1^{(1)} G_i^j = -\frac{1}{2a^2} \left( h_i^j'' + 2 \mathcal{H} h_i^j' - \nabla^2 h_i^j \right), \quad (2.19)$$
Eq. (2.18) can also be expressed as
\[ h_i^{jj''} + 2\mathcal{H}h_i^{jj'} - \nabla^2 h_i^{jj} = -2\ell_P^2 a^2 \Pi_i^{(X)j}, \] (2.20)
where \( \Pi_i^{(X)j} \) now defines the effective anisotropic stress determined from the scalar fluctuations of the geometry and computed in the gauge \( X \):
\[ \Pi_i^{(X)j} = \delta_s^{(2)} T_i^{(X)j} - \frac{1}{\ell_P^2} \delta_s^{(2)} G_i^{(X)j}. \] (2.21)

Equation (2.20) is ambiguous: while at the left-hand side the tensor part is formally gauge-invariant, the effective anisotropic stress is instead gauge-dependent so different anisotropic stresses, computed in diverse coordinate systems will determine different tensor amplitudes which should be instead coordinate-independent. This is, in a nutshell, one of the motivations of the present analysis: to avoid manifest contradictions it is important to find a gauge-invariant method to compare various gauge-dependent results.

The effective anisotropic stress \( \Pi_i^{(X)j} \) appearing in Eqs. (2.20) and (2.21) is determined up to total spatial derivatives involving quadratic combinations of the pivotal variables of a given gauge. This property is a direct consequence of the Landau-Lifshitz approach leading to Eqs. (2.18) and (2.19). Therefore, given a quadratic combination of two first-order fluctuations (e.g. \( \mathcal{Q} \) and \( P \)) in a specific gauge, the identity
\[ \partial_i \mathcal{Q} \partial^j P = -\mathcal{Q} \partial_i \partial^j P + \partial_i (\mathcal{Q} \partial^j P), \] (2.22)
can always be used with the aim of neglecting the second term at the right-hand side. This is possible since the effective anisotropic stress must be always projected along the two tensor polarizations and, in this process, the total derivative of Eq. (2.22) carries a comoving three-momentum \( q^i \) which is orthogonal to both tensor polarizations. To clarify this point we recall that the Fourier transforms of \( h_i^{jj}(\vec{x}, \tau) \) and \( \Pi_i^{(X)j}(\vec{x}, \tau) \) are defined as:
\[ h_i^{jj}(\vec{q}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3x \ h_i^{jj}(\vec{x}, \tau), \quad \Pi_i^{(X)j}(\vec{q}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3x \ \Pi_i^{(X)j}(\vec{x}, \tau). \] (2.23)

If the Fourier amplitude is expanded in the basis of the tensor polarizations we obtain:
\[ h_i^{jj}(\vec{q}, \tau) = \sum_{\lambda=\oplus, \otimes} e_i^{(\lambda)}(\vec{q}) \ h_\lambda(\vec{q}, \tau), \quad \Pi_i^{(X)j}(\vec{q}, \tau) = \sum_{\lambda=\oplus, \otimes} e_i^{(\lambda)}(\vec{q}) \ \Pi_\lambda^{(X)}(\vec{q}, \tau). \] (2.24)

In Eq. (2.24) \( e_i^{(\oplus)}(\vec{q}) \) and \( e_i^{(\otimes)}(\vec{q}) \) are given by:
\[ e_i^{(\oplus)}(\vec{q}) = \hat{m}_i \hat{m}_j - \hat{n}_i \hat{n}_j, \quad e_i^{(\otimes)}(\vec{q}) = \hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j, \] (2.25)
where \( \hat{m} \), \( \hat{n} \) and \( \hat{q} \) are three mutually orthogonal unit vectors. Using Eq. (2.24) Eq. (2.20) becomes, in Fourier space,
\[ h_i^{jj''} + 2\mathcal{H}h_i^{jj'} + q^2 h_i^{jj} = -2\ell_P^2 a^2(\tau) \Pi_\lambda^{(X)}. \] (2.26)

Equations (2.24) and (2.26) imply then that total spatial derivatives [like the second term at the right hand side of Eq. (2.22)] will not contribute to \( \Pi_\lambda^{(X)}(\vec{q}, \tau) \) since they will always be orthogonal both to \( e_i^{(\oplus)}(\vec{q}) \) and to \( e_i^{(\otimes)}(\vec{q}) \) [i.e. \( \hat{q} e_i^{(\oplus)}(\vec{q}) = \hat{q} e_i^{(\otimes)}(\vec{q}) = 0 \)].
2.5 Gauges for the effective anisotropic stresses and their drawbacks

Since the effective anisotropic stress of the relic gravitons must be evaluated in a particular gauge the potential presence of spurious gauge modes should be avoided. These unwanted modes arise when the gauge freedom is not completely removed and they mix with the evolution of the physical modes by often making unphysical the obtained expressions of the effective anisotropic stresses. This drawback is already present to first-order (see e.g. [38]) but it becomes even more acute when dealing with quadratic combinations of the perturbations variables in a given gauge, as it happens for the explicit evaluation of the effective anisotropic stresses. For this purpose we recall that the scalar fluctuations of the (3+1)-dimensional metric are parametrized by four independent functions which can be eventually reduced by specifying (either completely or partially) the coordinate system:

\[ \delta^{(1)}_{s} g_{00}(\vec{x}, \tau) = 2a^2 \phi, \quad \delta^{(1)}_{s} g_{ij}(\vec{x}, \tau) = 2a^2 (\psi \delta_{ij} - E_{ij}), \quad \delta^{(1)}_{s} g_{0i}(\vec{x}, \tau) = -a^2 V_i, \]  

(2.27)

where, in the scalar case, \( V_i = \partial_i B \) and \( E_{ij} = \partial_i \partial_j E \). For infinitesimal coordinate shifts of the type:

\[ \tau \rightarrow \tilde{\tau} = \tau + \epsilon_0, \quad x^i \rightarrow \tilde{x}^i = x^i + \partial^i \epsilon, \]  

(2.28)

the functions \( \phi(\tilde{x}, \tau), B(\tilde{x}, \tau), \psi(\tilde{x}, \tau) \) and \( E(\tilde{x}, \tau) \) introduced in Eq. (2.27) transform as:

\[ \phi \rightarrow \tilde{\phi} = \phi - \mathcal{H} \epsilon_0 - \epsilon', \quad \psi \rightarrow \tilde{\psi} = \psi + \mathcal{H} \epsilon_0, \]  

(2.29)

\[ B \rightarrow \tilde{B} = B + \epsilon_0 - \epsilon', \quad E \rightarrow \tilde{E} = E - \epsilon. \]  

(2.30)

Two commonly employed coordinate systems where the gauge freedom is completely fixed are the conformally Newtonian (or longitudinal) gauge where \( E = 0 \) and \( B = 0 \) and the off-diagonal (or uniform curvature) gauge where \( E = 0 \) and \( \psi = 0 \). In fact if we start from the situation where \( E \neq 0 \) and \( B \neq 0 \) the longitudinal condition \( \tilde{E} = \tilde{B} = 0 \) can be always recovered by setting

\[ \epsilon(\tilde{x}, \tau) = E(\tilde{x}, \tau), \quad \epsilon_0(\tilde{x}, \tau) = E'(\tilde{x}, \tau) - B(\tilde{x}, \tau). \]  

(2.31)

Similarly if we start from the situation where \( E \neq 0 \) and \( \psi \neq 0 \) the off-diagonal coordinate system \( \tilde{E} = \tilde{\psi} = 0 \) follows by setting

\[ \epsilon(\tilde{x}, \tau) = E(\tilde{x}, \tau), \quad \epsilon_0(\tilde{x}, \tau) = -\frac{\psi(\tilde{x}, \tau)}{\mathcal{H}}. \]  

(2.32)

In the case of Eqs. (2.31) and (2.32) the coordinate system is completely fixed. Conversely there are gauges where the gauge freedom can only be fixed up to arbitrary (space-dependent) constants. For instance the synchronous coordinate system is defined by \( \phi = 0 \) and \( B = 0 \) and if we start from a physical situation where the synchronous condition is not verified (i.e. \( \phi \neq 0 \) and \( B \neq 0 \)) the condition \( \tilde{\phi} = \tilde{B} = 0 \) can only be satisfied up to two arbitrary constants. Indeed from Eqs. (2.29) and (2.30) we see that the condition \( \tilde{\phi} = \tilde{B} = 0 \) is recovered provided:

\[ \epsilon_0(\tilde{x}, \tau) = \frac{C_1(\tilde{x})}{a(\tau)} + \frac{1}{a(\tau)} \int_{0}^{\tau} \phi(\tilde{x}, \tau_1) d\tau_1, \]

\[ \epsilon(\tilde{x}, \tau) = C_2(\tilde{x}) + C_1(\tilde{x}) \int_{0}^{\tau} \frac{d\tau_1}{a(\tau_1)} + \int_{0}^{\tau} B(\tilde{x}, \tau_1) d\tau_1 + \int_{0}^{\tau} \frac{d\tau_1}{a(\tau_1)} \int_{0}^{\tau_1} \phi(\tilde{x}, \tau_2) d\tau_2, \]  

(2.33)

From Eq. (2.33) it is apparent that the synchronous gauge condition is not completely fixed unless \( C_1(\tilde{x}) \) and \( C_2(\tilde{x}) \) are specified. This overall ambiguity causes the presence of spurious gauge modes.
This problem is potentially even more acute in the case of the effective anisotropic stresses and, for this reason, the illustrative considerations of the following two sections shall mainly involve those coordinate systems where the gauge freedom is completely fixed.

When the coordinate system is completely fixed the individual linear order variables used in one gauge cannot immediately compared to the ones of another gauge and this is especially true in the case of the effective anisotropic stresses containing quadratic combinations of the metric inhomogeneities. The variables $\phi$ and $\psi$ in the $L$-gauge (or $\phi$ and $B$ in the $U$-gauge) are not gauge-invariant since they take a different form when the coordinate system changes. Conversely $\mathcal{R}$ and $\mathcal{R}'$ obey the same equation in any coordinate system. This is, in a nutshell, the advantage of working directly with the gravitating normal modes of the plasma.

### 3 The longitudinal gauge picture

The standard approach to the analysis of the effective anisotropic stresses of the relic gravitons relies on gauge-dependent treatments. By this we mean that not only the anisotropic stress is computed in a specific gauge but that also the evolution of the various variables is followed in that specific coordinate system. The idea pursued here is different: instead of studying and evolving the effective anisotropic stresses in terms of the pivotal variables of a specific coordinate system we express the pivotal variables of that gauge in terms of the curvature perturbations and of their first-order derivatives with respect to the conformal time coordinate. Among the possible gauges where the effective anisotropic stresses could be computed, only the ones where the gauge freedom is completely fixed guarantee the absence of spurious gauge modes. For this reason in the present section we shall first examine the longitudinal picture while in the following section the uniform curvature gauge will be more specifically analyzed. Recalling Eqs. (2.27), (2.29)–(2.30) and (2.31), in the longitudinal gauge the metric fluctuations are expressed as:

\[
\delta_s^{(1)} g_{00}(\vec{x}, \tau) = 2a^2 \phi, \quad \delta_s^{(1)} g_{ij}(\vec{x}, \tau) = 2a^2 \psi \delta_{ij}. 
\]

(3.1)

In the standard approach the effective anisotropic stress is computed in terms of $\phi$ and $\psi$ so that effective anisotropic stress depends on the evolutionary features of the longitudinal gauge. Since our aim is to compare the effective anisotropic stresses in different gauges the idea is to trade the pivotal variables of a given gauge for the curvature inhomogeneities. So, for instance, the relation between the curvature perturbations on comoving orthogonal hypersurfaces and the longitudinal degrees of freedom (3.1) in Fourier space is given by:

\[
\mathcal{R}_k = -\psi_k - \frac{\mathcal{H}(\phi_k + \psi'_k)}{\mathcal{H}^2 - \mathcal{H}'}, 
\]

(3.2)

\[
\mathcal{R}'_k = \Sigma_k + \frac{2a^2 k^2 \psi_k}{\ell_P^2 \mathcal{H} z^2 k}, 
\]

(3.3)

The accuracy of Eqs. (3.2) and (3.3) can be immediately verified by checking that they lead to the equation of the the quasinormal modes already discussed in Eq. (2.10). Since Eq. (2.10) is gauge-invariant it can be derived in any gauge and, in particular, in the gauge (3.1). Let us therefore derive once both sides of Eq. (3.2) with respect to the conformal time coordinate $\tau$; if we then use,

\footnote{Note that $\phi$ in the $U$-gauge and in the $L$-gauge is different insofar as it obeys different equations.}
in the obtained expression, Eqs. (3.1) and (3.2) we arrive at the following expression

\[ R''_{k} + 2 \frac{z'_t}{z_t} R'_{k} = \Sigma'_{k} + 2 \frac{z'_t}{z_t} \Sigma_{k} - \frac{a^2 k^2 (H^2 - H')}{4 \pi G H z_t^2} R + \frac{k^2 a^2 H}{4 \pi G H z_t^2} (\psi_k - \phi_k), \] (3.4)

where the only dependence on the longitudinal fluctuations of the metric is in the last term. We then recall that, in the gauge (3.1), the scalar anisotropic stress discussed in Eq. (2.9) accounts for the mismatch between the two longitudinal fluctuations of the metric. In Fourier space we then have

\[ k^2 (\phi_k - \psi_k) = \Delta_k, \quad \Delta_k = -\frac{3}{2} \ell_P^2 a^2 \Pi_k. \] (3.5)

Inserting Eq. (3.5) into Eq. (3.4) the obtained result coincides, as expected, with Eq. (2.13). We stress that in Eq. (3.5) we introduced, for the sake of conciseness, \( \Delta_k \) which is only a convenient auxiliary quantity.

### 3.1 The effective anisotropic stress in terms of \( \phi \) and \( \psi \)

In the \( L \)-gauge of Eq. (3.1) the effective anisotropic stress given in Eq. (2.21) follows from the standard Landau-Lifshitz approach and it is formally expressed as:

\[ \Pi_i^{(L)} = \delta_s^{(2)} T_i^{(L)} - \frac{1}{\ell_P^2} \delta_s^{(2)} G_i^{(L)}. \] (3.6)

The second-order fluctuation of the sources appearing in the first term at the right-hand side of Eq. (3.6) is easily computed by recalling that, in the \( L \)-gauge,

\[ \delta_s u_i = \frac{2}{a \ell_P^2 (p_t + \rho_t)} \partial_i \left( \psi + H \phi \right). \] (3.7)

Equation (3.7) follows from the first-order fluctuation of the Einstein equations with mixed indices in the longitudinal gauge. Neglecting the trace we therefore have that \( \delta_s^{(2)} T_i^{(L)} j \) is given by:

\[ \delta_s^{(2)} T_i^{(L)} j = (\rho_t + p_t) \delta_s u_i \delta_s u^j = -\frac{2}{a^2 \ell_P^2 (H^2 - H')} \partial_i \left( H \phi + \psi' \right) \partial^j \left( H \phi + \psi' \right). \] (3.8)

Similarly, always neglecting the trace, \( \delta_s^{(2)} G_i^{(L)} j \) is given by:

\[ \delta_s^{(2)} G_i^{(L)} j = \frac{1}{a^2} \left[ \partial_i \phi \partial^j \phi - \partial_i \psi \partial^j \psi - 2 \psi \partial_i \partial^j \left( \phi - \psi \right) + \partial_i \phi \partial^j \psi + \partial_i \psi \partial^j \phi \right]. \] (3.9)

Putting together the obtained results the effective anisotropic stress of Eq. (3.6) becomes

\[
\Pi_i^{(L)} (x, \tau) = -\frac{1}{a^2 \ell_P^2} \left[ \partial_i \phi \partial^j \phi - \partial_i \psi \partial^j \psi + \partial_i \phi \partial^j \psi + \partial_i \psi \partial^j \phi \right] \\
- 2 \psi \partial_i \partial^j (\phi - \psi) + \frac{2}{(H^2 - H')} \partial_i \left( H \phi + \psi' \right) \partial^j \left( H \phi + \psi' \right). \] (3.10)

The result of Eq. (3.10) follows by recalling that the enthalpy density of the background (i.e. \( p_t + \rho_t \)) can always eliminated thanks to Eq. (2.3). Equation (3.10) is then further simplified thanks to Eq. (2.22): since the effective anisotropic stress will be eventually projected along the two tensor
polarizations the total spatial derivatives do not contribute to the final expression. In particular in the $L$-gauge Eq. (2.22) implies:

$$\partial_i \phi \partial^i \psi = -\phi \partial_i \partial^i \psi + \partial_i \left( \phi \partial^i \psi \right).$$  \hspace{1cm} (3.11)

Thanks to Eq. (3.11) the effective anisotropic stress of Eq. (3.10) becomes:

$$\Pi^{(L)}_i(x, \tau) = \frac{1}{a^2 \ell_p^2} \left[ \phi \partial_i \phi \partial^i - \partial_i \psi \partial^i \psi + \phi \partial_i \partial^i \psi + \psi \partial_i \partial^i \phi \right] + 2\psi \partial_i \partial^i (\phi - \psi) + \frac{2}{(\mathcal{H}^2 - \mathcal{H}')}(\mathcal{H} \phi + \psi') \partial_i \partial^i \left( \mathcal{H} \phi + \psi' \right).$$  \hspace{1cm} (3.12)

Finally, in Fourier space Eq. (3.12) reads:

$$\Pi^{(L)}_i(\vec{q}, \tau) = \frac{1}{(2\pi)^{3/2}} \int e^{i \vec{q} \cdot \vec{x}} \Pi^{(L)}_i(x, \tau) = -\frac{1}{(2\pi)^{3/2} a^2 \ell_p^2} \int d^3k k_i k_j \left[ \phi_{\vec{q}-\vec{k}} \phi_{\vec{k}} - \psi_{\vec{q}-\vec{k}} \psi_{\vec{k}} + \phi_{\vec{q}-\vec{k}} \psi_{\vec{k}} + \psi_{\vec{q}-\vec{k}} \phi_{\vec{k}} \right] + 2\psi_{\vec{q}-\vec{k}} (\phi_{\vec{k}} - \psi_{\vec{k}}) + \frac{2}{\mathcal{H}^2 - \mathcal{H}'} (\mathcal{H} \phi_{\vec{q}-\vec{k}} + \psi'_{\vec{q}-\vec{k}}) (\mathcal{H} \phi_{\vec{k}} + \psi'_{\vec{k}}).$$ \hspace{1cm} (3.13)

### 3.2 The effective anisotropic stress in terms of $\mathcal{R}$ and $\mathcal{R}'$

Equation (3.13) can be directly studied in terms of $\phi_{\vec{k}}$ and $\psi_{\vec{k}}$ which are the pivotal variables of the $L$-gauge. This is what has been done in previous studies but this approach is not ideal for a sound physical comparison of the results obtained in different gauges. If this strategy is strictly followed the anisotropic stresses derived in different gauges can only be compared at the very end and also for specific classes of background evolutions. The aim of this analysis is opposite: we would like to compare the different results before specifying the evolution of the background. The idea is therefore to use Eqs. (3.2)–(3.3) by trading $\phi_{\vec{k}}$ and $\psi_{\vec{k}}$ for $\mathcal{R}_{\vec{k}}$ and $\mathcal{R}'_{\vec{k}}$:

$$\phi_{\vec{k}} = \psi_{\vec{k}} + \Delta_{\vec{k}}, \quad \psi_{\vec{k}} = \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H} k^2 c_{st}^2} (\mathcal{R}'_{\vec{k}} - \Sigma_{\vec{k}}).$$ \hspace{1cm} (3.14)

Using the above expression the effective anisotropic stress can be expressed as:

$$\Pi^{(L)}_{i,j}(\vec{q}, \tau) = -\frac{2(\mathcal{H}^2 - \mathcal{H}')}{(2\pi)^{3/2} \ell_p^2 a^2 \mathcal{H}^2} \int d^3k k_i k_j \left[ \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{q}-\vec{k}} + \frac{\mathcal{H}^2}{2(\mathcal{H}^2 - \mathcal{H}')^2} \Delta_{\vec{k}} \Delta_{\vec{q}-\vec{k}} \right] + \frac{\mathcal{H}^2 - \mathcal{H}'}{k^2 \mathcal{H}^2 |\vec{q}-\vec{k}|^2 c_{st}^2} (\mathcal{R}'_{\vec{k}} - \Sigma_{\vec{k}}) (\mathcal{R}'_{\vec{q}-\vec{k}} - \Sigma_{\vec{q}-\vec{k}}) + \frac{3\mathcal{H}}{2 k^2 c_{st}^2} \Delta_{\vec{q}-\vec{k}} (\mathcal{R}'_{\vec{k}} - \Sigma_{\vec{k}}) + \frac{3\mathcal{H}}{2 |\vec{q}-\vec{k}|^2 c_{st}^2} \Delta_{\vec{k}} (\mathcal{R}'_{\vec{q}-\vec{k}} - \Sigma_{\vec{q}-\vec{k}}) + \frac{(\mathcal{H}^2 - \mathcal{H}')}{{\mathcal{H}} |\vec{q}-\vec{k}|^2 c_{st}^2} \mathcal{R}_{\vec{k}} (\mathcal{R}'_{\vec{q}-\vec{k}} - \Sigma_{\vec{q}-\vec{k}}) + \frac{(\mathcal{H}^2 - \mathcal{H}')}{\mathcal{H} k^2 c_{st}^2} \mathcal{R}_{\vec{k}} (\mathcal{R}'_{\vec{q}-\vec{k}} - \Sigma_{\vec{q}-\vec{k}}).$$ \hspace{1cm} (3.15)

The advantage of Eq. (3.15) in comparison with Eq. (3.13) is evident: while $\phi_{\vec{k}}$ and $\psi_{\vec{k}}$ obey the equations that are specific to the $L$-gauge, $\mathcal{R}_{\vec{k}}$ and $\mathcal{R}'_{\vec{k}}$ obey instead Eq. (2.13) that has the same
form in any coordinate system (i.e. it is gauge-invariant). In previous studies (see for instance Ref. [22]) the curvature inhomogeneities have been used to normalise the results obtained in different gauges. This procedure is, strictly speaking, background-dependent insofar as $\mathbf{R}_k$ is taken to be strictly constant. Some of these approaches (like the one of Ref. [22]) are only consistent in the case where the curvature inhomogeneities are time-independent and cannot be used in the general situation where, on the contrary, Eq. (3.15) applies without approximations.

We conclude this part of the discussion by remarking that the gauge-invariant evolution of the neutrino anisotropic stress of Eq. (2.17) can also be obtained directly in the $L$-gauge of Eq. (3.1). In other words, starting form the lowest multiples of the neutrino hierarchy we can easily deduce Eq. (2.17) directly in the longitudinal gauge. In short the derivation is the following. Recalling that the lowest multipoles of the neutrino hierarchy read, in the longitudinal gauge,

$$\delta'_k = -\frac{4}{3}\theta'_k + 4\psi'_k,$$  
(3.16)

$$\theta'_k = \frac{k^2}{4}\delta'_k - k^2\sigma_\nu + k^2\phi'_k,$$  
(3.17)

$$\sigma'_\nu = \frac{4}{15}\theta'_k - \frac{3}{10}k\mathcal{F}_{\nu 3},$$  
(3.18)

where $\delta'_k$ is the neutrino density contrast and $\theta'_k$ is the three-divergence of the corresponding peculiar velocity. If we take the conformal time derivative of both sides of Eq. (3.16); we thus obtain

$$\sigma''_\nu = \frac{k^2}{15}\delta'_k + \frac{4}{15}k^2\phi'_k - \frac{11}{21}k^2\sigma_\nu,$$  
(3.19)

where the neutrino hierarchy has been truncated, for illustration, to the octupole (notice, however, that $\mathcal{F}'_{\nu 3} \neq 0$). From Eq. (3.19) it also follows that:

$$\sigma'''_\nu + \frac{6}{7}k^2\sigma'_\nu = \frac{4k^2}{15}(\phi'_k - \psi'_k)' + \frac{8}{15}k^2\psi'_k,$$  
(3.20)

In Eq. (3.20) the term $k^2(\phi'_k - \psi'_k)'$ can be replaced by taking the derivative of both sides of Eq. (3.5); the other term appearing at the right hand side of Eq. (3.20) is instead replaced by taking the derivative of Eq. (3.3) and by inserting, in the obtained expression, the decoupled equation for $\mathbf{R}_k$, i.e. Eq. (2.10). The result in terms of $k^2\psi'_k$ becomes:

$$k^2\psi'_k = \frac{1}{c_{st}^2}\left(\mathcal{H} - \mathcal{H}'\frac{\mathcal{H}'}{\mathcal{H}} - 2\mathcal{H}\right)(\mathcal{R}'_k - \Sigma_k) + 6\mathcal{H}^3\Omega_\nu R_\nu\sigma_\nu - \left(\mathcal{H} - \mathcal{H}'\frac{\mathcal{H}'}{\mathcal{H}}\right)k^2\mathbf{R}_k.$$  
(3.21)

If Eq. (3.21) is now plugged into Eq. (3.20) we obtain the equation already reported in Eqs. (2.17) provided the term $k^2(\phi'_k - \psi'_k)'$ is eliminated by means of the derivative of Eq. (3.3).

### 3.3 Complementary gauge-invariant descriptions

Equation (3.15) demonstrates that the effective anisotropic stress can be expressed directly in terms of $\mathbf{R}_k$ and $\mathbf{R}'_k$ not only asymptotically (i.e. when $\mathbf{R}'_k \rightarrow 0$) but in general terms. There could be some suggesting that $\mathbf{R}$ should also be traded for another popular gauge-invariant variable conventionally denoted by $\zeta$. The gauge-invariant relation between the two variables is:

$$\zeta - \mathbf{R} = \frac{2\nabla^2\psi}{3c_p^2a^2(p_t + p_i)} = \frac{\Sigma_\mathbf{R} - \mathbf{R}'}{3\mathcal{H}c_{st}^2}.$$  
(3.22)
The first equality in Eq. (3.22) holds in the $L$-gauge while the second relation is gauge-invariant. Equation (3.22) shows that if we would trade $R$ for $\zeta$ we should also generate new terms proportional to $R'$. In the $L$-gauge the explicit definition of $\zeta$ is given by

$$\zeta = -\psi - H \frac{\delta p_t}{\rho_t}.$$ (3.23)

In the $U$-gauge $\zeta$ describes instead the curvature fluctuations in the hypersurfaces where the total energy density is uniform. In the limit of large length-scales there seem to be no difference between $\zeta$ and $R$. However, thanks to Eqs. (3.22)–(3.23) the second-order equation obeyed by $\zeta$ is far more involved than Eqs. (2.1) and (2.10) even if the two equations coincide in the $k\tau \ll 1$ limit. The decoupled equation for $\zeta$ is formally non-local since it contains the inverse of the function $1 + k^2/[3(H^2 - H')]$. To lowest order in $k\tau < 1$ we have that $f(k, \tau) \to 1$: in this limit $\zeta$ and $R$ evolve at the same rate. In Fourier space the evolution of $\zeta_k$ can then be written as:

$$\zeta''_k + H[1 + f_k + 3c_{st}^2(f_k - 1)]\zeta'_k + k^2 c_{st}^2 \left[1 - \frac{1 + 3c_{st}^2}{3} f_k \zeta_k \right] \zeta_k = \mathcal{S}_\zeta, \quad f_k(\tau) = \frac{1}{1 + \frac{k^2}{3(H^2 - H')}}.$$ (3.24)

where $\mathcal{S}_\zeta$ is given by

$$\mathcal{S}_\zeta = \left(\Sigma_k - \frac{H \Pi_t}{\rho_t + p_t}\right)' + H[1 + f_k + 3c_{st}^2(f_k - 1)] \left(\Sigma_k - \frac{H \Pi_t}{\rho_t + p_t}\right) - \frac{1}{H} \nabla^2 \left(\Sigma_R - \frac{H \Pi_t}{\rho_t + p_t}\right).$$ (3.25)

Given the expression of $f_k(\tau)$, Eq. (3.24) is non-local. In the limit $k^2 \ll (H^2 - H')$ Eqs. (3.24) and (3.25) are compatible with the evolution of $R$ as implied by Eq. (3.22). Non-local terms can therefore be avoided with specific approximations which are however unnecessary if the gauge-invariant evolution is studied and solved in terms of $R$ and $R'$. After having computed $R$ and $R'$ the value of $\zeta$ can always be obtained from Eq. (3.22). We therefore conclude that if the effective anisotropic stresses are expressed in terms of $\zeta_k$ the evolution will necessarily involve non-local terms that are however absent in the approach suggested in this paper.

4 Derivation in the uniform curvature gauge

Recalling Eqs. (2.27), (2.29)–(2.30) and (2.32) the gauge-freedom can also be completely removed in the coordinate system characterized by the following perturbed metric:

$$\delta^{(1)}_s g_{00}(\vec{x}, \tau) = 2a^2(\tau) \phi(\vec{x}, \tau), \quad \delta^{(1)}_s g_{ij}(\vec{x}, \tau) = -a^2 V_i(\vec{x}, \tau).$$ (4.1)

Even if we shall eventually set $V_i = \partial_i B$ it will be convenient, just for the notational convenience, to write the general formulas in terms of $V_i$. In this section we shall therefore repeat in the $U$-gauge all the steps leading to Eq. (3.15). The expression obtained in the $U$-gauge will still depend on $R_k$ and $R'_k$, but it will be sharply different from the expression of the $L$-gauge. This is what we meant in section 3 when introducing the concept of a gauge-invariant comparison of gauge-dependent results.

4.1 The effective anisotropic stress in terms of $\phi$ and $V_i$

In the $U$-gauge the effective anisotropic stress follows from Eq. (2.21) with $X = U$:

$$\Pi_i^{(U)j} = \delta^s_j T_i^{(U)j} - \frac{1}{\epsilon^2} \delta^s_j g_i^{(U)j}.$$ (4.2)
Neglecting, as usual, the terms proportional to the trace we will have that
\[
\delta^{(2)} G^{(U)}_{i} = - \frac{\phi}{a^2} \left( \partial_i V^j + \partial^j V_i \right) - \frac{\phi'}{2a^2} \left( \partial_i V^j + \partial^j V_i \right) - 2 \frac{\mathcal{H} \phi}{a^2} \left( \partial_i V^j + \partial^j V_i \right) + \frac{1}{a^2} \partial_i \phi \partial^j \phi - 2 \frac{\mathcal{H}}{a^2} V^j \partial_i \phi,
\]
(4.3)
\[
\delta^{(2)} T^{(U)}_{i} = - \frac{4 \mathcal{H}}{a^4 \ell_p^4 (\rho_t + \rho_i)} \partial_i \phi \left[ \mathcal{H} \partial^j \phi + \left( \mathcal{H}^2 - \mathcal{H}' \right) V^j \right].
\]
(4.4)

Inserting Eqs. (4.3) and (4.4) into Eq. (4.2) the explicit expression of the effective anisotropic stress is therefore given by:
\[
\Pi^{(U)}_{i} (\vec{x}, \tau) = \frac{1}{a^2 \ell_p^2} \left[ 2 \phi \partial_i \partial^j B' + \phi' \partial_i \partial^j B + 4 \mathcal{H} \phi \partial_i \partial^j B + \frac{3 \mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'} \phi \partial_i \partial^j \phi \right],
\]
(4.6)

where we used that \( V_i = \partial_i B \). In Fourier space the relation connecting \( B_{\vec{k}}, \phi_{\vec{k}} \) and the scalar anisotropic stress is given by:
\[
B'_{\vec{k}} + 2 \mathcal{H} B_{\vec{k}} = - \phi_{\vec{k}} + \Delta_{\vec{k}}.
\]
(4.7)

so that Eq. (4.6) becomes, in Fourier space,
\[
\Pi^{(U)}_{i} (\vec{q}, \tau) = - \frac{1}{(2\pi)^{3/2}} \int e^{i \vec{q} \cdot \vec{x}} \Pi^{(U)}_{i} (\vec{q}, \tau) \nonumber
\]
\[
= - \frac{1}{(2\pi)^{3/2} a^2 \ell_p^2} \int d^3 k \ k_i k_j \left[ \phi_{\vec{q}-\vec{k} \Delta_{\vec{k}}} + \phi_{\vec{k} \Delta_{\vec{q}-\vec{k}}} + \frac{\mathcal{H}^2 + \mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'} \phi_{\vec{k} \phi_{\vec{q}-\vec{k}}} \right] + \frac{1}{2} \left( \phi'_{\vec{q}-\vec{k} \cdot B_{\vec{k}}} + \phi'_{\vec{k} \cdot B_{\vec{q}-\vec{k}}} \right) \nonumber
\]
(4.8)

Equation (4.8) is the \( U \)-gauge analog of Eq. (3.13) which is instead valid in the \( L \)-gauge. It is however clear that Eqs. (4.8) and (3.13) are not comparable in any way since the pivotal variables of each gauge obey a different set of equations. Note, incidentally, that the variable \( \phi \) appearing in Eq. (3.13) is defined in the \( L \)-gauge whereas Eq. (4.8) holds in the \( U \)-gauge where the variable \( \phi \) evolves in a different way.

### 4.2 The effective anisotropic stress in terms of \( R \) and \( R' \)
As it happens in the \( L \)-gauge the pivotal variables of the \( U \)-gauge are univocally related to the curvature perturbations on comoving orthogonal hypersurfaces:
\[
\phi_{\vec{k}} = - \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2} R_{\vec{k}},
\]
(4.9)
\[
B_{\vec{k}} = - \frac{(\mathcal{H}^2 - \mathcal{H}')}{{\mathcal{H}^2}} \left( \frac{\mathcal{H}^2}{k^2 c_{st}^2} \right) \left( R'_{\vec{k}} - \Sigma_{\vec{k}} \right).
\]
(4.10)
Equations (4.9) and (4.10) are the $U$-gauge analog of Eqs. (3.2)–(3.3) and (3.14). Inserting Eqs. (4.9) and (4.10) into Eq. (4.8) the effective anisotropic stress becomes:

$$\Pi_{ij}^{(U)}(q, \tau) = -\frac{(H^2 - H')^2}{(2\pi)^{3/2} a^2 \ell_p^2 H^4} \int d^3 k k_i k_j \left\{ \frac{H^2 + H'}{H^2 - H'} R_{\bar{k}} R_{\bar{q} - \bar{k}} + \frac{k^2 + |\bar{q} - \bar{k}|^2}{2 c_s^2 k^2 |\bar{q} - \bar{k}|^2} R_{\bar{k}} R_{\bar{q} - \bar{k}}' + \frac{3}{2} H(w - c_s^2) \left[ \frac{R_{\bar{k}}' R_{\bar{q} - \bar{k}}}{c_s^2 k^2} + \frac{R_{\bar{q} - \bar{k}}' R_{\bar{k}}}{c_s^2 |\bar{q} - \bar{k}|^2} \right] - \frac{\Sigma_{\bar{q} - \bar{k}}[R_{\bar{k}}' + 3H(w - c_s^2)R_{\bar{k}}]}{2 c_s^2 k^2} - \frac{\Sigma_{\bar{k}}[R_{\bar{q} - \bar{k}}' + 3H(w - c_s^2)R_{\bar{q} - \bar{k}}]}{2 c_s^2 |\bar{q} - \bar{k}|^2} - \frac{H^2}{(H^2 + H')} (R_{\bar{q} - \bar{k}} \Delta_{\bar{k}} + R_{\bar{k}} \Delta_{\bar{q} - \bar{k}}) \right\}. \quad (4.11)$$

Equation (4.11) is the $U$-gauge analog of Eq. (3.15). Since $R_{\bar{k}}$ and $R_{\bar{k}}'$ obey Eq. (2.10) the results of Eqs. (3.15) and Eq. (4.11) can be directly compared since they are expressed in terms of the same set of gauge-invariant variables. This comparison will be explicitly illustrated in the following section. It is finally rather easy to verify that Eq. (2.10) can be directly derived in the $U$-gauge. This step will be omitted here since it mirrors exactly the analysis of the $L$-gauge. The procedure is to derive both sides of Eq. (4.10) and to eliminate $B_{\bar{k}}'$ by first using Eq. (4.7). This step will lead to a dependence on $\phi_{\bar{k}}$ and $B_{\bar{k}}$ that will be eliminated thanks to Eqs. (4.9) and (4.10); Eq. (2.10) will then be recovered. This proves that, unlike the evolution equations of a each particular gauge, both Eqs. (2.1) and (2.10) are invariant under infinitesimal coordinate transformations of the type introduced in Eq. (2.28).

5 Gauge-invariant comparison of gauge-dependent results

Equations (3.15) and (4.11) have been derived in two different gauges but are expressed in terms of the same gauge-invariant variables obeying Eq. (2.10). This is the operational definition of the strategy introduced in section 4 namely a gauge-invariant comparison of the gauge-dependent results. This apparent oxymoron emphasizes that the results obtained in different coordinate systems can be compared in a physically meaningful way only by expressing the gauge-dependent results in terms of the gravitating normal modes of the system. Since this logic has never been used before, we intend to illustrate the power of our method by studying the limits of the effective anisotropic stresses when the typical wavelengths are either smaller or larger than the sound horizon $r_s(\tau)$.

5.1 Wavelengths inside the sound horizon

Let us first consider the simplest physical case where the non-adiabatic pressure fluctuations vanish, the scalar anisotropic stress is absent and the wavelengths of the scalar phonons are sufficiently small; in formulas

$$\Sigma_{\bar{k}} \to 0, \quad \Delta_{\bar{k}} \to 0, \quad k^2 c_s^2 \gg \left| \frac{\rho''}{\rho'} \right| z_t. \quad (5.1)$$

In the situation of Eq. (5.1) we have that Eq. (2.10) becomes

$$q_k'' + k^2 c_s^2 q_k = 0, \quad q_k = z_t R_{\bar{k}} \quad (5.2)$$
It is relevant to stress that, in this regime, the solution of Eqs. (2.10) and (5.2) follows from Wentzel-Kramers-Brillouin (WKB) approximation without specifying the background evolution and it is given by:

$$\mathcal{R}_k(\tau) = \frac{C_k}{z_t \sqrt{2k c_{st}}} \cos [k r_s(\tau)] + \frac{\mathcal{D}_k}{z_t \sqrt{2k c_{st}}} \sin [k r_s(\tau)].$$  \hspace{1cm} (5.3)

Equation (5.3) is a WKB solution of Eq. (5.2) provided $c'_{st}/(2c_{st}) < k c_{st}$; in Eq. (5.3) $C_k$ and $D_k$ are two constants (possibly determined from the boundary conditions) and $r_s(\tau)$ defines the sound horizon:

$$r_s(\tau) = \int_{\tau_i}^{\tau} c_{st}(\tau') d\tau'.$$  \hspace{1cm} (5.4)

The wavelengths satisfying Eq. (5.3) will be said to be inside the sound horizon (i.e. $k r_s(\tau) \gg 1$).

We are now going to consider separately the limits of Eqs. (3.15) and (4.11) inside the sound horizon. For the sake of illustration we shall first consider the case of Eq. (5.3) and then comment on the main differences when $\Sigma_k \neq 0$ and $\Delta_k \neq 0$. From Eq. (3.15) the expression of the effective anisotropic stress in the L-gauge becomes:

$$\Pi^{(L)}_{ij}(\vec{q}, \tau) = -\frac{2(\mathcal{H}^2 - \mathcal{H}')}{{(2\pi)^{3/2}} \ell_p^2 a^2(\tau) \mathcal{H}^2} \int d^3k \; k_i k_j \left\{ \mathcal{R}_k \mathcal{R}_{\vec{q} - \vec{k}} + \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}} \left[ \frac{\mathcal{R}_k \mathcal{R}'_{\vec{q} - \vec{k}}}{c_{st}^2 |\vec{q} - \vec{k}|^2} + \frac{\mathcal{R}'_{\vec{q} - \vec{k}}}{c_{st}^2 k^2} + \frac{(2\mathcal{H}^2 - \mathcal{H}')(\mathcal{H}^2 - \mathcal{H}')}{\mathcal{H}^2 c_{st}^2 k^2 |\vec{q} - \vec{k}|^2} \mathcal{R}_k \mathcal{R}'_{\vec{q} - \vec{k}} \right] \right\}. \hspace{1cm} (5.5)

From Eq. (5.3) inside the sound horizon, the curvature perturbations and their derivatives are approximately related as

$$\mathcal{R}'_{\vec{k}} \simeq k c_{st} \mathcal{R}_k \left[ 1 + \mathcal{O}\left( \frac{a H}{k c_{st}} \right) \right] + \ldots, \hspace{1cm} (5.6)$$

and this relation holds in spite of the details of the underlying background geometry; the ellipses in Eq. (5.6) denote the higher-order corrections that are always negligible for $k c_{st} \gg H a$; an analog expansion holds in the case $\mathcal{R}'_{\vec{q} - \vec{k}}$ when $|\vec{q} - \vec{k}| c_{st} \gg a H$. Inserting Eq. (5.6) into Eq. (5.5) we get the following result:

$$\Pi^{(L)}_{ij}(\vec{q}, \tau) = -\frac{2(\mathcal{H}^2 - \mathcal{H}')}{{(2\pi)^{3/2}} \ell_p^2 a^2(\tau) \mathcal{H}^4} \int d^3k \; k_i k_j \mathcal{R}_k \mathcal{R}_{\vec{q} - \vec{k}} \left\{ 1 + \frac{(\mathcal{H}^2 - \mathcal{H}')}{c_{st} \mathcal{H} k |\vec{q} - \vec{k}|} \right. \left. \right\} \left\{ \mathcal{H}^2 + \frac{\mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'} \right\} \mathcal{R}_k \mathcal{R}'_{\vec{q} - \vec{k}} \left\{ 1 + \frac{(\mathcal{H}^2 - \mathcal{H}')}{c_{st}^2 |\vec{q} - \vec{k}|^2} \right\}, \hspace{1cm} (5.7)$$

where, as in Eq. (5.6), the ellipses stand for the higher-order contributions. The first term at the right hand side of Eq. (5.7) dominates in the limit $k c_{st} \gg H a$ while the two remaining contributions are of higher order. For short the range of validity of Eq. (5.7) can be dubbed as $k c_{st} \tau \gg 1$, $|\vec{q} - \vec{k}| c_{st} \tau \gg 1$ with $(k c_{st} \tau)/(|\vec{q} - \vec{k}| c_{st} \tau) \to 1$ since $\mathcal{H} = a H \sim 1/\tau$. Note, however, that this is just some kind of shorthand notation that does not imply the choice of a specific background as it happens in gauge-dependent and background-dependent studies.

The same analysis leading to Eq. (5.7) can be repeated in the U- gauge. More specifically we have that for $\Sigma_k \to 0$ and $\Delta_k \to 0$ Eq. (4.11) becomes:

$$\Pi^{(U)}_{ij}(\vec{q}, \tau) = -\frac{(\mathcal{H}^2 - \mathcal{H}')^2}{{(2\pi)^{3/2}} \ell_p^2 a^2(\tau) \mathcal{H}^4} \int d^3k \; k_i k_j \left\{ \frac{(\mathcal{H}^2 + \mathcal{H}')}{\mathcal{H}^2 - \mathcal{H}'} \right\} \mathcal{R}_k \mathcal{R}_{\vec{q} - \vec{k}} + \frac{3}{2} \mathcal{H} (w - c_{st}^2) \left[ \frac{\mathcal{R}'_{\vec{q} - \vec{k}}}{c_{st}^2 |\vec{q} - \vec{k}|^2} + \frac{\mathcal{R}_{\vec{q} - \vec{k}} \mathcal{R}'_{\vec{q} - \vec{k}}}{k^2 c_{st}^2} \right] + \frac{k^2 + |\vec{q} - \vec{k}|^2}{2 c_{st}^2 |\vec{q} - \vec{k}|^2 k^2} \mathcal{R}_k \mathcal{R}'_{\vec{q} - \vec{k}}. \hspace{1cm} (5.8)$$
At this point it is important to recall that Eqs. (2.10) and (5.2) are both gauge-invariant: they are therefore the same in any coordinate system. Equation (5.6) can then be inserted into Eq. (5.8) so that, in the limits \( k_{\text{ct}}\tau \gg 1 \), \(|\vec{q} - \vec{k}|\) \( c_{\text{st}}\tau \gg 1 \) with \((k_{\text{ct}}\tau)/(|\vec{q} - \vec{k}|) c_{\text{st}}\tau \to 1 \) the same steps leading to Eq. (5.7) leads, in the case of Eq. (5.8), to the following result:

\[
\Pi_{ij}^{(u)}(\vec{q},\tau) = -\frac{(\mathcal{H}^2 - \mathcal{H}')^2}{(2\pi)^{3/2} \ell_p^2 \mathcal{H}^4 a^2} \int d^3 k \ k_i k_j \mathcal{R}_{\vec{k}} \mathcal{R}_{-\vec{k}} \{ 1 + \frac{\mathcal{H}^2 + \mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'} \frac{3}{2} \mathcal{H} \left( w - c_{\text{ct}}^2 \right) (k + |\vec{q} - \vec{k}|) + \ldots \}.
\]

The direct comparison of Eqs. (5.7) and (5.9) demonstrates that the leading terms of both expansions are the same. Therefore, as long as the wavelengths of the gravitating normal modes are shorter than the sound horizon the anisotropic stresses will coincide up to subleading corrections:

\[
\Pi_{ij}^{(l)}(\vec{q},\tau) = \Pi_{ij}^{(u)}(\vec{q},\tau) + \mathcal{O}\left( \frac{aH}{k_{\text{ct}}} \right) + \mathcal{O}\left( \frac{aH}{|\vec{q} - \vec{k}| c_{\text{st}}} \right) + \mathcal{O}\left( \frac{a^2H^2}{k |\vec{q} - \vec{k}| c_{\text{st}}^2} \right) + \ldots
\]

So far we considered the case \( \Sigma_{\vec{k}} \to 0 \) and \( \Delta_{\vec{k}} \to 0 \). To avoid a repetitive discussion we shall only mention the main differences arising in the case \( \Sigma_{\vec{k}} \neq 0 \) and \( \Delta_{\vec{k}} \neq 0 \). Equation (5.3) must be replaced by the following equation

\[
\mathcal{R}_{\vec{k}}(\tau) = \mathcal{R}_{\vec{k}}^{(1)}(\tau) + \int_{\tau_i}^{\tau} d\xi \mathcal{G}_{\mathcal{R}}[q(\xi - \tau)] \mathcal{S}_{\mathcal{R}}(\xi),
\]

where \( \mathcal{R}_{\vec{k}}^{(1)}(\tau) \) is the solution of the homogeneous equation given in (5.3) while \( \mathcal{G}_{\mathcal{R}}[q(\xi - \tau)] \) and \( \mathcal{S}_{\mathcal{R}}(\xi) \) are defined as

\[
\mathcal{G}_{\mathcal{R}}[q(\xi - \tau)] = -\frac{z_t(\xi)}{q_{\text{ct}} z_t(\tau)} \sin[q(\xi - \tau)], \quad \mathcal{S}_{\mathcal{R}}(\xi) = \Sigma'_{\vec{k}} + 2\frac{z'_t}{z_t} \Sigma_k + 3a^4 z_t^3 \Pi_{\vec{k}}.
\]

Equation (5.11) must then be inserted into Eqs. (3.15) and (4.1). Using the analog of Eq. (5.6) and neglecting all the terms that are subheading inside the sound horizon the result of Eq. (5.10) can be recovered.

To reach the previous conclusion it is relevant to appreciate that inside the sound horizon the scalar anisotropic stress is more suppressed than the curvature inhomogeneities. Since this is a relevant aspect it seems appropriate to justify it in more detail in the simplest situation where the scalar anisotropic stress comes exclusively from the neutrino sector. Let us therefore write the coupled evolution of the curvature inhomogeneities and of the scalar anisotropic stress in the case of a radiation-dominated background. Equations (2.10) and (2.17) read

\[
\frac{d^2 \mathcal{R}_{\vec{k}}}{dy^2} + 2 \frac{d\mathcal{R}_{\vec{k}}}{y \ dy} + c_{\text{st}}^2 \mathcal{R}_{\vec{k}} = \frac{R_\nu}{y} \left( \frac{d\sigma_\nu}{dy} + \frac{2}{y} \sigma_\nu \right),
\]

\[
\frac{d^3 \sigma_\nu}{dy^3} + \left( \frac{6}{7} + \frac{8R_\nu}{5y^2} \right) \frac{d\sigma_\nu}{dy} - 16R_\nu \sigma_\nu + 16 \frac{3R_\nu}{y^2} \sigma_\nu + 3 \frac{d\mathcal{R}}{dy} + \frac{y R}{3} = 0,
\]

where \( c_{\text{st}} = 1/\sqrt{3} \) and \( y = k\tau; \) to derive Eqs. (5.13) and (5.14) we assumed \( \Pi_{\vec{k}} = (\rho_\nu + p_\nu)\sigma_\nu \). Inside the sound horizon the dominant solution of Eqs. (5.13) and (5.14) reads:

\[
\mathcal{R}_{\vec{k}}(\tau) = \mathcal{R}_{\vec{k}}(k) \frac{\sin c_{\text{st}} y \ y}{c_{\text{st}} y^2}, \quad \sigma_\nu(k,\tau) = \sigma(k) \frac{\sin c_{\text{st}} y \ y^2}{|c_{\text{st}} y^2|^2},
\]
where $\sigma(\vec{k}) = (112/55)\mathcal{R}(\vec{k})$. Therefore, as anticipated, inside the sound horizon (i.e. for $c_{st}y \gg 1$) the scalar anisotropic stress is always more suppressed in comparison with the curvature inhomogeneities.

### 5.2 Wavelengths outside the sound horizon

So far we investigated the effective anisotropic stresses when the corresponding wavelengths are shorter than the sound horizon. We shall now consider the opposite limit where the wavelengths are larger than the sound horizon i.e.

$$k r_s(\tau) \ll 1, \quad k^2 \ll \left| \frac{\dot{z}_f}{z_t} \right|.$$  \tag{5.16}$$

To investigate the limit of Eq. (5.16) we first rewrite Eq. (2.10) in the following form:

$$\partial_\tau \left[ z_t^2 \left( \mathcal{R}'_k - \Sigma_k \right) \right] = -k^2 c_{st}^2 z_t^2 \mathcal{R}_k + 3a^4 \Pi_k. \tag{5.17}$$

Equation (5.17) has the same content of Eq. (2.10) but it can be easily transformed into an integral equation which will be easier to handle in this situation:

$$\left( \mathcal{R}'_k - \Sigma_k \right) = \left( \frac{z_{ex}}{z_t} \right)^2 \left( \mathcal{R}'_k - \Sigma_k \right)_{ex} - k^2 c_{st}^2 \int_{\tau_{ex}}^\tau z_t^2(\tau_1)\mathcal{R}_k(\tau_1) + 3 \int_{\tau_{ex}}^\tau a^4(\tau_1) \Pi_k(\tau_1)d\tau_1. \tag{5.18}$$

In Eq. (5.18), $\tau_{ex}$ denotes the time at which the given scale exits the sound horizon (i.e. $k c_{st} \tau_{ex} \approx 1$); during inflation $z_t \rightarrow z_\varphi = a \varphi'/H$ (where $\varphi$ is the inflaton) so that $k \tau_{ex} \approx 1$ and the sound horizon coincides, in practice, with the Hubble radius. For wavelengths larger than the sound horizon the scalar anisotropic stress is negligible with respect to $\mathcal{R}_k$; this is what happens in the case of the concordance paradigm both in the case of the standard adiabatic mode and in the case of the other entropic modes \([25, 36, 37]\).

When the typical wavelengths are larger than the sound horizon the evolution of the curvature perturbations follows from Eq. (5.18). Now the idea will be to insert Eq. (5.18) both into Eq. (3.15) and into Eq. (4.11); at the very end the two expressions shall be compared. In general terms the effective anisotropic stresses in the $L$-gauge and in the $U$-gauge are expressible as:

$$\Pi^{(X)}_{ij}(\vec{q}, \tau) = -\frac{1}{(2\pi)^{3/2}a^2(\tau)\ell_p^2} \int d^3k \int_{k_j} A^{(X)}(\vec{k}, \vec{q}, \tau) \left[ 1 + \mathcal{O}\left( \frac{k c_{st}}{aH} \right) + \mathcal{O}\left( \frac{|\vec{q} - \vec{k}|}{aH} \right) + \ldots \right], \tag{5.19}$$

where $X = L, U$; the expansion of Eq. (5.19) holds when the corresponding wavelengths are larger than the sound horizon (i.e. $k c_{st} < aH$ and $|\vec{q} - \vec{k}| c_{st} < aH$) and $A^{(X)}(\vec{k}, \vec{q}, \tau)$ is the leading term in the expansion obtained after the insertion of Eq. (5.18) into Eqs. (3.15) and (4.11). The explicit form of the leading contribution in the $L$-gauge reads:

$$A^{(L)}(\vec{k}, \vec{q}, \tau) = 2 \frac{(\mathcal{H}'^2 - \mathcal{H}^2)(\mathcal{H}'^2 - \mathcal{H}_0'^2)}{\mathcal{H}'^4} \left( \frac{z_{ex}}{z_t} \right)^4 \left( \mathcal{R}'_k - \Sigma_k \right)_{ex} \left( \mathcal{R}'_{\vec{q} - \vec{k}} - \Sigma_{\vec{q} - \vec{k}} \right)_{ex}, \tag{5.20}$$

In the $U$-gauge the leading contribution implies instead:

$$A^{(U)}(\vec{k}, \vec{q}, \tau) = \frac{(\mathcal{H}'^2 - \mathcal{H}^2)^2}{2\mathcal{H}^4} \left( \frac{k^2 + |\vec{q} - \vec{k}|^2}{k^2 |\vec{q} - \vec{k}|^2 c_{st}^2} \right) \times \left[ \Sigma_k + \left( \frac{z_{ex}}{z_t} \right)^2 \left( \mathcal{R}'_k - \Sigma_k \right)_{ex} \right] \left[ \Sigma_{\vec{q} - \vec{k}} + \left( \frac{z_{ex}}{z_t} \right)^2 \left( \mathcal{R}'_{\vec{q} - \vec{k}} - \Sigma_{\vec{q} - \vec{k}} \right)_{ex} \right]. \tag{5.21}$$
The ratio between Eq. (5.21) and Eq. (5.20) is therefore the following:

$$\frac{A^{(U)}(\vec{k}, \vec{q}, \tau)}{A^{(L)}(\vec{k}, \vec{q}, \tau)} = \frac{(k^2 + |\vec{q} - \vec{k}|^2) c_{st}^2}{4(2\mathcal{H}^2 - \mathcal{H}')} \times \left[1 + \frac{\Sigma_{\vec{k}}}{(\mathcal{R}'_{\vec{k}} - \Sigma_{\vec{k}})_{ex}} \left(\frac{z_t}{z_{ex}}\right)^2\right] \left[1 + \frac{\Sigma_{\vec{q} - \vec{k}}}{(\mathcal{R}'_{\vec{q} - \vec{k}} - \Sigma_{\vec{q} - \vec{k}})_{ex}} \left(\frac{z_t}{z_{ex}}\right)^2\right]. \quad (5.22)$$

From Eq. (5.22) we see that $A^{(U)}(\vec{k}, \vec{q}, \tau)/A^{(L)}(\vec{k}, \vec{q}, \tau) = \mathcal{O}(k^2 c_{st}^2)$ which is always smaller than 1 when the corresponding wavelengths are larger than the sound horizon at the corresponding epoch.

Let us therefore summarize the main conclusions reached so far. We started by suggesting in section 4.4 a gauge-invariant comparison of gauge-dependent results. The novel idea of this comparison has been to express the effective anisotropic stresses directly in terms of the gravitating normal modes of the plasma which obey the same evolution equation in any coordinate system. Inside the sound horizon the effective anisotropic stresses are sharply different and that, in particular, the result in the $U$-gauge is much smaller than the one in the $L$-gauge. The obtained result suggest therefore the important conclusion that the effective anisotropic stresses are approximately gauge-invariant inside the sound horizon but sharply different outside of it.

### 5.3 Extensions to more general situations

The conclusion reached so far holds in a rather general situation and, in particular, when the evolution of the curvature inhomogeneities obeys Eq. (2.10). Even more general situations are described by a similar equation where however the source terms have a different expression. In particular further sources of anisotropic stress (besides the fluid component) and further sources of entropy perturbations can be always rephrased in a form similar to the one of Eqs. (2.11) and (2.12). To substantiate this statement we can consider the effect of the electric and magnetic fields on the scalar modes The fluctuations of the energy density and the anisotropic stresses are both quadratic in the electric and magnetic fields and are defined as

$$\delta_s \rho_B(\vec{x}, \tau) = \frac{B^2(\vec{x}, \tau)}{4\pi a^4}, \quad \delta_s \rho_E(\vec{x}, \tau) = \frac{E^2(\vec{x}, \tau)}{4\pi a^4}, \quad (5.23)$$

$$\Pi^{(B)}_j = \frac{1}{4\pi a^4} \left[ B_i B_j - \frac{B^2}{3} \delta^j_i \right], \quad \Pi^{(E)}_j = \frac{1}{4\pi a^4} \left[ E_i E^j - \frac{B E^2}{3} \delta^j_i \right], \quad (5.24)$$

where $\vec{E}$ and $\vec{B}$ denote the comoving electric and magnetic fields (see Ref. [39] and discussion therein). Using the standard notations for the scalar components of the magnetic and electric anisotropic stresses

$$\nabla^2 \Pi_B(\vec{x}, \tau) = \partial_i \partial_j \Pi^{ij}_{(B)}(\vec{x}, \tau), \quad \nabla^2 \Pi_E(\vec{x}, \tau) = \partial_i \partial_j \Pi^{ij}_{(E)}(\vec{x}, \tau), \quad (5.25)$$

the generalized expression for $\Sigma_{\mathcal{R}}(\vec{x}, \tau)$ now becomes\(^4\)

$$\Sigma_{\mathcal{R}}(\vec{x}, \tau) = -\frac{\mathcal{H}}{p_t + \rho_t} \delta p_{nad} + \frac{\mathcal{H}}{p_t + \rho_t} \left( c_{st}^2 - \frac{1}{3} \right) (\delta_s \rho_E + \delta_s \rho_B) + \Pi_t + E + B. \quad (5.26)$$

\(^4\)Note that we used $\Sigma_{\mathcal{R}}$ to distinguish it from $\Sigma_{\mathcal{R}}$ where the electromagnetic contribution is basent.
Equation (5.26) generalizes the results of Eqs. (2.10) and (2.11) and can be used to compute the evolution of the curvature inhomogeneities in the presence of electromagnetic disturbances. The main observation we ought to make is that the form of Eq. (2.10) is exactly the same but, this time $S_R$ is replaced by $\mathcal{S}_R$:

$$S_R(\vec{x}, \tau) = \Sigma' R + 2 \frac{z'}{z} \Sigma + \frac{3 \alpha^4}{z^2} \left( \Pi_t + \Pi_E + \Pi_B \right).$$

(5.27)

This also means, for instance, that the results of Eqs. (5.20), (5.21) and (5.22) can be easily deduced also in the presence of electromagnetic components by simply replacing $\Sigma' \mathbf{k}$ with $\Sigma' \mathbf{k}$ and by redefining $\Pi_t$ as $\Pi_t = \Pi_t + \Pi_E + \Pi_B$.

6 The example of the concordance paradigm

6.1 Basic considerations

The conclusions reached so far do not assume any specific background evolution. It is however useful to corroborate the results obtained so far with the illustrative example of a radiation-dominated plasma. Most of the discussion could be conducted in terms of a generic sound speed but for the sake of concreteness we shall consider the situation

$$c^2_{st} = w = \frac{1}{2}, \quad \mathcal{H} a = \mathcal{H}_1 a_1, \quad H a^2 = H_1 a_1^2,$$

(6.1)

where $\mathcal{H}a$ as well as $H a^2$ are constants throughout all the stages of the evolution. Furthermore we shall neglect both the non-adiabatic pressure fluctuations and the sources of scalar anisotropic stress (e.g. neutrinos). In the case (6.1) the scalar mode functions can be computed in a closed form:

$$\mathcal{R}_q(\tau) = \mathcal{R}(q) j_0(q c_{st} \tau), \quad \mathcal{R}'_q(\tau) = -q c_{st} \mathcal{R}(q) j_1(q c_{st} \tau),$$

(6.2)

where $j_0(q c_{st} \tau)$ and $j_1(q c_{st} \tau)$ are spherical Bessel functions of zeroth- and first-order [40, 41]. To identify more easily the various different contributions in the effective anisotropic stress the sound speed has been kept constant but generic in Eq. (6.2) (we shall eventually set $c_{st} \rightarrow 1/\sqrt{3}$ only at the very end). In Eq. (6.2) $\mathcal{R}(q)$ represents a scalar random field, whose correlation function and the associated power spectrum are:

$$\langle \mathcal{R}(q) \mathcal{R}(q') \rangle = \frac{2 \pi^2}{q^3} \mathcal{P}_R(q) \delta^3(q + q'), \quad \mathcal{P}_R(q) = A_R \left( \frac{q}{q_p} \right)^{n_s - 1}.$$

(6.3)

In Eq. (6.3) we used the standard normalizations where $A_R$ is the amplitude of the power spectrum at the pivot scale $q_p = 0.002$ Mpc$^{-1}$ corresponding to a frequency $\nu_p = 2\pi q_p = 3 \times 10^{-18}$ Hz; in Eq. (6.3) $0.9 < n_s < 1$ denotes the scalar spectral index (see e.g. [3] and discussions therein). With the same notation employed in Eq. (6.3) the two-point function of a (solenoidal and traceless) tensor random field will be written as:

$$\langle \mathcal{T}_{ij}(q) \mathcal{T}_{mn}(q') \rangle = \frac{2 \pi^2}{q^5} \mathcal{S}_{ijmn}(q) \mathcal{P}_T(q) \delta^3(q + q'), \quad \mathcal{P}_T(q) = A_T \left( \frac{q}{q_p} \right)^{n_T},$$

(6.4)
where $S_{ijmn}(\hat{q})$ is related to the sum over the two tensor polarizations defined in Eq. (2.26) and it is defined as:

$$S_{ijmn}(\hat{q}) = \frac{1}{4} \left[ p_{im}(\hat{q}) p_{jn}(\hat{q}) + p_{in}(\hat{q}) p_{jm}(\hat{q}) - p_{ij}(\hat{q}) p_{mn}(\hat{q}) \right], \quad p_{ij}(\hat{q}) = \delta_{ij} - \hat{q}_i \hat{q}_j. \quad (6.5)$$

According to the standard notations, $A_T = r_T A_R$ is the amplitude of the tensor power spectrum at the same pivot scale used for the scalars. The tensor to scalar ratio $r_T$ and the spectral index $n_T$ may be related by the so-called consistency relations (i.e. $n_T \approx r_T/8$) but this point is not central for the present discussion. In terms of the tensor random fields entering Eq. (6.4) the homogeneous solution of the equation of the tensor modes is

$$T_{ij}(\hat{q}, \tau) = T_{ij}(\hat{q}) \ j_0(q \tau), \quad H_{ij}(\hat{q}, \tau) = \partial_\tau T_{ij}(\hat{q}, \tau) = -q \ T_{ij}(\hat{q}) \ j_1(q \tau). \quad (6.6)$$

### 6.2 Explicit evaluation of the effective anisotropic stresses in a radiation plasma

Now the idea is, in short, the following:

- we are first going to insert Eqs. (6.2) and (6.3) into the exact expressions of the effective anisotropic stresses in the $L$-gauge and in the $U$-gauge obtained in Eqs. (3.15) and (4.11) respectively;
- then we shall compare the two exact expressions in the two physical limits when the wavelengths of the normal modes are either larger or smaller then the sound horizon;
- finally we will compute the spectral energy density of the relic gravitons and explicitly evaluate the correction induced by the effective anisotropic stress.

When Eqs. (6.2) and (6.3) are inserted into Eqs. (3.15) and (4.11) the resulting expression of the effective anisotropic stress becomes:

$$\Pi_{ij}^{(X)}(q, \tau) = -\frac{1}{(2\pi)^{3/2} \ell_P^2 a^2} \int d^3k \ k_i k_j \ \mathcal{R}(\vec{k}) \ \mathcal{R}(\vec{q} - \vec{k}) \ M^{(X)}(k c_{st} \tau, |\vec{q} - \vec{k}| c_{st} \tau). \quad (6.7)$$

The general expression of Eq. (6.7) is actually more general than the examples we are now describing; note, in particular, that $M^{(X)}(z, w)$ is symmetric for $w \rightarrow z$ and $z \rightarrow w$. In the particular case of the radiation-dominated plasma discussed in Eq. (6.1) the exact expressions of $M^{(X)}(z, w)$ (for $X = L, U$) are:

$$M^{(L)}(z, w) = 4 \left[ j_0(z) j_0(w) - 6 c_{st} \left( \frac{j_0(z) j_1(w)}{w} + \frac{j_1(z) j_0(w)}{z} \right) - \frac{54 c_{st}^2}{w z} j_1(w) j_1(z) \right], \quad (6.8)$$

$$M^{(U)}(z, w) = 6 c_{st}^2 \left( \frac{z}{w} + \frac{w}{z} \right) j_1(w) j_1(z). \quad (6.9)$$

The variable $z$ appearing in Eqs. (6.8) and (6.9) has nothing to do with the variable $z_t$ appearing, for instance, in Eqs. (2.1) and (2.10). In the limits $z = k c_{st} \xi \gg 1$, $w = |\vec{q} - \vec{k}| c_{st} \xi \gg 1$ and $q c_{st} \xi \gg 1$, Eqs. (6.8) and (6.9) become:

$$M^{(L)}(z, w) \rightarrow \frac{4 \sin z \sin w}{w z} + \ldots, \quad M^{(U)}(z, w) \rightarrow \frac{12 c_{st}^2 \cos z \cos w}{w z} + \ldots. \quad (6.10)$$
Equations (6.10) apply when the wavelengths are all inside sound horizon (i.e. $k c_{st} / (aH) > 1$); however since $c_{st} \leq 1$ (and $k/(aH) > c_{st}^{-1}$) the wavelengths are also inside the Hubble radius (i.e. $k/(aH) > 1$). It is important to appreciate that the results of Eq. (6.10) coincide exactly up to a phase and this because $c_{st} = 1/\sqrt{3}$. As we shall see later this phase will be immaterial for the final expression of the spectral energy density. When the corresponding wavelengths are outside the sound horizon the asymptotic forms of Eqs. (6.8) and (6.9) are

$$M^{(L)}(z, w) \rightarrow 4(6c_{st}^2 - 4c_{st} + 1) + \ldots, \quad M^{(U)}(z, w) \rightarrow \frac{2}{3}c_{st}^2[z^2 + w^2] + \ldots, \quad (6.11)$$

respectively.

6.3 The explicit expressions of the spectral energy density

The solution of Eq. (2.26) for $h_{\lambda}$ and $\partial_{\tau} h_{\lambda}$ is formally expressed in terms of the corresponding Green’s functions $G[q(\xi - \tau)]$ and $\widetilde{G}[q(\xi - \tau)]$:

$$h^{(X)}(q, \tau) = \overline{h}_{\lambda}(q, \tau) - 2\ell_P^2 \int_{\tau_1}^{\tau} d\xi a^2(\xi) G[q(\xi - \tau)] \Pi^{(X)}(q, \xi),$$

$$H^{(X)}(q, \tau) = \overline{H}_{\lambda}(q, \tau) - 2\ell_P^2 \int_{\tau_1}^{\tau} d\xi a^2(\xi) \widetilde{G}[q(\xi - \tau)] \Pi^{(X)}(q, \xi), \quad (6.12)$$

where $H^{(X)} = \partial_{\tau} h^{(X)}$ and $\overline{H}_{\lambda} = \partial_{\tau} \overline{h}_{\lambda}$; the overline distinguishes the (gauge-invariant) first-order contributions from their second-order (gauge-dependent) counterparts. After inserting Eq. (6.7) into Eqs. (6.12) the tensor amplitude $h_{\lambda}(q, \tau)$ follows by recalling the explicit expressions of the Green’s functions during the radiation-dominated stage i.e.

$$G[q(\xi - \tau)] = -\frac{a(\xi)}{qa(\tau)} \sin[q(\xi - \tau)], \quad \widetilde{G}[q(\xi - \tau)] = \frac{a(\xi)}{a(\tau)} \cos[q(\xi - \tau)]. \quad (6.13)$$

To compute the effective energy density of the relic gravitons we now need to estimate first their energy density which ultimately depends on the form of the energy-momentum pseudo-tensor of the relic gravitons. For instance the energy-momentum pseudo-tensor obtained from the variation of the effective action of the relic gravitons with respect to the background metric leads to the energy density firstly derived by Ford and Parker [7, 8]:

$$\rho_{gw} = \frac{1}{8\ell_P^2 a^2} \left[ \partial_{\tau} h_{k\ell} \partial_{\tau} h^{k\ell} - \partial_{m} h_{k\ell} \partial^{m} h^{k\ell} \right]. \quad (6.14)$$

Recalling now Eqs. (2.23), (6.4) and (6.12) the spectral energy density of the relic gravitons is obtained by taking the ratio between the average of Eq. (6.14) and the critical energy density according to a standard procedure[5] thus in our case the spectral energy density of the relic gravitons in critical units is given by:

$$\Omega^{(X)}_{gw}(q, \tau) = \frac{q^2 T_{\tau}(q)}{24 H^2 a^2 [q^2 \tau]^2} \left[ 1 + \sin q\tau - \sin 2q\tau \right] + \frac{q^3}{12} \left( \frac{a_1^4 H_0^2}{a^4 H^2} \right) \int_{-1}^{1} d\mu (1 - \mu^2)^2 \int dk k^6 \left( \frac{P_{R}(k)}{k^3} \right)^2 \left[ \frac{T^{(X)}(k, \xi)}{2} \right], \quad (6.15)$$

<sup>5</sup>Mutatis mutandis this analysis coincides with the results of an analog problem involving the spectrum of gravitational radiation induced by waterfall fields [42] (see also [43, 44, 45]).
where $\mathcal{T}^{(X)}(\vec{k}, \vec{q}, \tau)$ and $\mathcal{F}^{(X)}(\vec{k}, \vec{q}, \tau)$ are given by:

$$
\mathcal{T}^{(X)}(\vec{k}, \vec{q}, \tau) = \int_{\tau_i}^{\tau_f} \xi \sin[q(\xi - \tau)] M^{(X)}(k c_s \xi; |\vec{q} - \vec{k}| c_s \xi) d\xi,
$$
$$
\mathcal{F}^{(X)}(\vec{k}, \vec{q}, \tau) = \int_{\tau_i}^{\tau_f} \xi \cos[q(\xi - \tau)] M^{(X)}(k c_s \xi; |\vec{q} - \vec{k}| c_s \xi) d\xi.
$$

(6.16)

The spectral energy density of the relic gravitons inside the Hubble radius in its full form (i.e. including the second-order corrections) follows from Eqs. (6.15) and (6.16) by recalling the limit of Eq. (6.10). Thus the expressions of Eq. (6.15) (for $X = L$ and $X = U$) will eventually inherit a phase difference that however disappears after squaring and summing up the contributions of the two integrals (6.16) in each case. The common value of spectral energy density inside the sound horizon is therefore

$$
\Omega_{gw}^{(U)}(q, \tau_0) = \Omega_{gw}^{(L)}(q, \tau_0) = \frac{r_T A_R \Omega_{R0}}{12} \left( \frac{q}{q_p} \right)^{n_T} \left[ 1 + \frac{96 \pi^2 A_R}{5 r_T} f(n_s, q) \left( \frac{q}{q_p} \right)^{2(n_s-1)-n_T} \right],
$$

(6.17)

where $f(n_s, q) = a_1(n_s) + a_2(n_s) \left( \frac{q_{\text{max}}}{q} \right)^{2n_s-5}$,

(6.18)

where $\tau_0$ denotes the present value of the conformal time coordinate while $a_i(n_s)$ (with $i = 1, 2, 3$) are three numerical constants. The expressions of the coefficients $a_i(n_s)$ follow from the integration of Eq. (6.15) first over $\mu$ and then over $k$ between $q_p$ and $q_{\text{max}}$. The integration over $k$ can be approximated in two separate regions (i.e. $k < q$ and $k > q$); this way of approximating the integrals compares quite well with the numerical results as explicitly discussed in the case of waterfall fields where the power spectra appearing in the convolutions have larger slopes but similar analytical expressions. Since $\nu_p = 2\pi q_p$ is in the aHz region (see discussion after Eq. (6.3)) and $\nu_{\text{max}} = 2\pi q_{\text{max}} = 190$ MHz we have that $f(n_s, q) = O(10^{-2})$ for typical scalar spectral indices $0.9 < n_s < 1$.

Let us finally consider Eqs. (6.15) and (6.16) when the corresponding wavelengths are outside the sound horizon. Once again, with the help of these asymptotic expressions the integrals $\mathcal{T}^{(X)}(\vec{k}, \vec{q}, \tau)$ and $\mathcal{F}^{(X)}(\vec{k}, \vec{q}, \tau)$ of Eq. (6.16) can be estimated. The first-order contribution has the standard form valid during the radiation-dominated phase and it follows from the first term at the right-hand side of Eq. (6.15) for $q\tau \ll 1$; the second-order correction is however different in the two gauges so that the general form of $\Omega_{gw}^{(X)}(q, \tau)$ is:

$$
\Omega_{gw}^{(X)}(q, \tau) = \Omega_{gw}^{(X)}(q, \tau_0) \left[ 1 + \omega^{(X)}_{gw}(q, \tau) \right], \quad \omega^{(X)}_{gw}(q, \tau) = \frac{r_T A_R}{12} q^2 \tau^2 \left( \frac{q}{q_p} \right)^{n_T},
$$

(6.19)

where the two functions $\omega^{(L)}_{gw}(q, \tau_0)$ and $\omega^{(U)}_{gw}(q, \tau_0)$ are:

$$
\omega^{(L)}_{gw}(q, \tau) = \frac{64 A_R}{15} \Omega_{R0} q^2 \tau^2 \left[ 1 + \frac{q^2 \tau^2}{9} \right] \left( \frac{q}{q_p} \right)^{2(n_s-1)-n_T} \mathcal{T}^{(L)}(n_s, q),
$$
$$
\omega^{(U)}_{gw}(q, \tau) = \frac{4 A_R}{135} \Omega_{R0} q^6 \tau^6 \left[ 1 + \frac{q^2 \tau^2}{25} \right] \left( \frac{q}{q_p} \right)^{2(n_s-1)-n_T} \mathcal{F}^{(U)}(n_s, q).
$$

(6.20)

---

Even if the explicit expressions are immaterial for the present discussion we have that $a_1(n_s) = (n_s - 6)/(2n_s - 5)(n_s + 1)$, $a_2(n_s) = -1/(n_s + 1)$ and $a_3(n_s) = 1/(2n_s - 5)$. 

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The form of $\mathcal{F}^{(L)}(n_s, q)$ and $\mathcal{F}^{(U)}(n_s, q)$ is not central to the present discussion and it is anyway similar to $f(n_s, q)$ appearing in Eq. (6.18). What matters here is the parametric dependence of the correction upon $q \tau$, i.e. $\omega_{gw}^{(U)}(q, \tau)/\omega_{gw}^{(L)}(q, \tau) = \mathcal{O}(|q\tau|^4)$.

All in all we have that the results obtained in the case of are fully consistent with the ones obtained in Eqs. (5.10) and (5.22). In particular Eq. (6.17) corresponds to Eq. (5.10) and demonstrates that the spectral energy densities computed in different gauges coincide when the wavelengths of the scalar modes are inside the sound horizon. Equation (5.22) corresponds instead to Eq. (6.20) with the caveat that Eq. (5.22) applies for the anisotropic stresses while the spectral energy density of Eq. (6.20) is instead quadratic in the effective anisotropic stress. This is why the mismatch between the two expressions is not given by $|q\tau|^2$ (as in Eq. (5.22)) but by the square of it (i.e. $|q\tau|^4$). We conclude that the limit of the general expressions obtained without specifying the background geometry coincide, as expected, in the particular case of a radiation-dominated plasma once the corresponding expressions are evaluated either inside or outside the sound horizon.

7 Effective anisotropic stress from the second-order action

So far we suggested a gauge-invariant method to compare gauge-dependent results. The idea is to choose a coordinate system where the gauge freedom is completely fixed and to compute the effective anisotropic stress in that particular gauge. At the very end the results will then be expressed in terms of the gravitating normal modes of the system and of their conformal time derivatives. Since the gravitating normal modes obey the same evolution equations in different coordinate system, the results obtained in various gauges are most easily assessed. In general terms the gravitating normal modes will depend on the total anisotropic stress and on the non-adiabatic pressure fluctuations. The systematic use of the WKB approximation demonstrated that when the wavelengths are shorter than the sound horizon the results of different gauges are all consistent while in the opposite regime they are not. A similar problem arose in the past when discussing the effective energy density and pressure of the gravitational field: different strategies lead in fact to consistent results only inside the Hubble radius but not outside of it. In what follows we intend to suggest that probably the best way of defining the effective anisotropic stress is to start from the second-order action of the curvature inhomogeneities in the same way as the simplest way of defining the energy density of the relic gravitons is to start from their second-order action.

7.1 The effective energy density of the relic gravitons

Let us start by briefly examining the derivation of the energy-momentum pseudo-tensor of the relic gravitons [7, 8]. Since the effective action of the relic gravitons is:

$$S_t = \frac{1}{8\ell_P^2} \int d^4x \sqrt{-g} \, g^{\alpha\beta} \partial_\alpha h_{ij} \partial_\beta h^{ij},$$

(7.1)

the associated energy-momentum pseudo-tensor can be introduced from the functional derivative of $S_t$ with respect to $g_{\mu\nu}$ by considering $h_{ij}$ and $g_{\mu\nu}$ as independent variables [7, 8]. From Eq. (7.1) the explicit form of the energy-momentum pseudo-tensor is

$$T_{\mu\nu} = \frac{1}{4\ell_P^2} \left[ \partial_\mu h_{ij} \partial_\nu h^{ij} - \frac{1}{2} g_{\mu\nu} \left( g^{\alpha\beta} \partial_\alpha h_{ij} \partial_\beta h^{ij} \right) \right],$$

(7.2)
and it can be derived by computing the variation of $S_t$ with respect to $\delta g^{\mu \nu}$ i.e.
\[
\delta S_t = \frac{1}{2} \int d^4 x \sqrt{-g} T_{\mu \nu} \delta g^{\mu \nu}, \quad T^\nu_{\mu} = g^{\alpha \nu} T_{\alpha \mu}.
\]
(7.3)
Since the indices of $T_{\mu \nu}$ are raised and lowered with the help of the background metric, the energy density and the pressure are defined from the various components of the energy-momentum pseudo-tensor as:
\[
T^0_0 = \rho_{gw}, \quad T^0_i = \frac{1}{4\ell_P^2 a^2} \partial_t h_{k \ell} \partial_i h^{k \ell},
\]
\[
T_{ij} = -p_{gw} \delta^i_j + \Pi^j_{gwi},
\]
(7.4)
where $\rho_{gw}$ and $p_{gw}$ are the energy density and the pressure of the relic gravitons:
\[
\rho_{gw} = \frac{1}{8\ell_P^2 a^2} \left[ \partial_t h_{k \ell} \partial_t h^{k \ell} + \partial_m h_{k \ell} \partial^m h^{k \ell} \right],
\]
(7.5)
\[
p_{gw} = \frac{1}{8\ell_P^2 a^2} \left[ \partial_t h_{k \ell} \partial_t h^{k \ell} - \frac{1}{3} \partial_m h_{k \ell} \partial^m h^{k \ell} \right].
\]
(7.6)
The energy density obtained in Eq. (7.5) coincides in fact with the result already mentioned in Eq. (6.14). In Eq. (7.4) we have a further class of traceless anisotropic stresses (i.e. $\Pi^i_i = 0$), namely the anisotropic stress of the tensor modes:
\[
\Pi^j_{gwi} = \frac{1}{4\ell_P^2 a^2} \left[ -\partial_i h_{k \ell} \partial^j h^{k \ell} + \frac{1}{3} \delta^i_j \partial_m h_{k \ell} \partial_m h^{k \ell} \right].
\]
(7.7)
Equation (7.7) accounts for the anisotropic stress induced by the tensor modes. What we are looking for is the anisotropic stress induced by the scalar modes of the geometry.

### 7.2 The effective anisotropic stress in the case of an irrotational fluid

The evolution of the gravitating normal modes of Eq. (2.1) can be derived from an action that is very similar to the one of Eq. (7.1):
\[
S_R = \frac{1}{2} \int d^3 x \int d\tau z_t^2 \left[ \partial_i R \partial_i R - c_s^2 \partial_k R \partial^k R \right].
\]
(7.8)
Recalling the explicit form of $z_t$ (i.e. Eq. (2.2)) and using the background equations (2.3) the action $S_R$ can also be expressed as:
\[
S_R = \frac{1}{\ell_P^2} \int d^3 x \int d\tau a^2 \left( \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2} \right) \left[ \frac{1}{c_s^2} \partial_i R \partial_i R - \partial_k R \partial^k R \right],
\]
(7.9)
so that the effective anisotropic stress now becomes
\[
\Pi^i_j = -\frac{2(\mathcal{H}^2 - \mathcal{H}')}{a^2 \mathcal{H}^2 \ell_P^2} \left[ \partial_i R \partial^i R - \frac{1}{3} \partial_k R \partial^k R \delta^i_j \right].
\]
(7.10)
It is quite clear that the effective anisotropic stress of Eq. (7.10) gives exactly the leading-order contribution already deduced in the $L$-gauge and in the $U$-gauge. In particular if we Fourier transform Eq. (7.10) and project it along the tensor polarization we will have that

$$\Pi_\lambda(\vec{q}, \tau) = -\frac{(H^2 - \dot{H})}{(2\pi)^{3/2} \ell_p^2 a^2(\tau) H^2} \int d^3 k \ k^2 \ s_\lambda(\hat{k}, \hat{q}) \mathcal{R}_{-\hat{k}} \mathcal{R}_{\hat{q}-\hat{k}}$$  \hspace{1cm} (7.11)

where $s_\lambda(\hat{k}, \hat{q}) = \hat{k}_i \hat{k}_j e_\lambda^{ij}(\hat{q})$. While Eq. (7.11) coincides with the leading-order expression obtainable in specific gauges inside the sound horizon, outside of it this expression is the same. We therefore suggest that the second-order action of the scalar modes could be directly used to deduce the effective anisotropic stress of the relic gravitons.

### 7.3 The effective anisotropic stress in the case of scalar field matter

To corroborate even further the conclusions of the previous paragraph let us consider the case of scalar field matter. In this case the curvature perturbations obey the following effective action

$$S_R = \frac{1}{2} \int d^4 x \left( \frac{\varphi'}{H} \right)^2 \sqrt{-g} g^{\alpha\beta} \partial_\alpha \mathcal{R} \partial_\beta \mathcal{R}. \hspace{1cm} (7.12)$$

The background equation $\varphi'^2 = 2(H^2 - \dot{H})/\ell_p^2$ can be used in Eq. (7.12) and the action becomes

$$S_R = \frac{1}{\ell_p^2} \int d^4 x \left( \frac{H^2 - \dot{H}}{\ell_p^2} \right) \sqrt{-g} g^{\alpha\beta} \partial_\alpha \mathcal{R} \partial_\beta \mathcal{R}. \hspace{1cm} (7.13)$$

By taking the functional derivative with respect to $g^{\mu\nu}$ we have that the energy-momentum pseudo-tensor of the curvature inhomogeneities is:

$$\mathcal{T}^0_0 = \rho_R, \hspace{1cm} \mathcal{T}^j_i = -p_R \delta^j_i + \Pi^j_i, \hspace{1cm} (7.14)$$

where $\rho_R$, $p_R$ and $\Pi^j_i$ are given, respectively, by:

$$\rho_R = \frac{\varphi'^2}{2H^2 a^2} \left[ \partial_\tau \mathcal{R} \partial_\tau \mathcal{R} + \partial_k \mathcal{R} \partial^k \mathcal{R} \right], \hspace{1cm} (7.15)$$

$$p_R = \frac{\varphi'^2}{2H^2 a^2} \left[ \partial_\tau \mathcal{R} \partial_\tau \mathcal{R} - \frac{1}{3} \partial_k \mathcal{R} \partial^k \mathcal{R} \right],$$

$$\Pi^j_i = -\frac{2(H^2 - \dot{H})}{\ell_p^2 H^2 a^2} \left[ \partial_\tau \mathcal{R} \partial^j \mathcal{R} - \frac{\delta^j_i}{3} \partial_k \mathcal{R} \partial^k \mathcal{R} \right]. \hspace{1cm} (7.16)$$

By projecting Eq. (7.16) over the tensor polarizations we obtain the same result of Eq. (7.11).

\footnote{Note that the term proportional to $\delta_{ij}$ appearing in Eq. (7.10) does not contributed to $\Pi_\lambda$.}
8 Concluding remarks and general lessons

The starting point of this analysis has been the observation that the effective anisotropic stresses induced by the scalar modes of the geometry depends on the coordinate system where it is evaluated. Not all the coordinate systems are equally viable: the ones where the gauge freedom is completely eliminated guarantee the absence of spurious gauge modes and this is why the attention has been focussed on the longitudinal and on the uniform curvature gauges. In spite of this important difference the anisotropic stresses computed in different coordinate systems depend on the evolution of the pivotal variables of that particular gauge.

To avoid this drawback we suggested how the gauge-dependent results could be compared in a gauge-invariant manner. By this we simply stress that the results obtained in diverse coordinate systems can only be compared in a meaningful way by expressing the gauge-dependent results in terms of the gravitating normal modes of the system. This is the novel idea proposed and scrutinized in this paper. Since the gravitating normal modes of the plasma obey the same evolution equation in any coordinate system there will be a unique evolution equation determining the effective anisotropic stresses in different gauges. The results of this analysis are, in short, the following:

- the evolution of the gauge-invariant curvature inhomogeneities has been analyzed in general terms by including the non-adiabatic pressure fluctuations and the scalar anisotropic stress;

- inside the sound horizon the effective anisotropic stresses computed in the $L$-gauge and in the $U$-gauge coincide to leading order (i.e. they are gauge-invariant from the practical viewpoint);

- for typical wavelengths larger than the sound horizon the evolution of the normal modes imply instead that the anisotropic stresses are sharply different and that, in particular, the result in the $U$-gauge is much smaller than the one in the $L$-gauge;

- even if the present approach employs the WKB approximation (and does not assume any specific background evolution) the obtained results have been explicitly corroborated by the analysis of a radiation dominated plasma;

- we finally argued that the effective anisotropic stress of the curvature inhomogeneities can be obtained from the functional derivative of the second-order action of curvature inhomogeneities with respect to the background metric.

The obtained results suggest therefore that the effective anisotropic stresses are approximately gauge-invariant inside the sound horizon but sharply different outside of it. The same kind of spurious gauge-invariance examined here is also manifest when the energy density of the relic gravitons is derived from competing energy-momentum pseudo-tensors. To lowest order the ambiguity can be solved (or alleviated) by selecting an energy-momentum pseudo-tensor with reasonable physical properties such as the one obtained long ago by Ford and Parker. The present considerations show however that some ambiguities are likely to reappear from the higher-order processes as a direct consequence of the lack of localization of the energy-momentum of the gravitational field. Following the same logic that leads to the energy-momentum pseudo-tensor of the relic gravitons, we can use the second-order action of the scalar modes to obtain the effective anisotropic stress. It turns out that the results obtained in this way coincide (inside the sound horizon) with the expressions derived in different coordinate systems where the gauge freedom is completely fixed. When the wavelengths of the curvature inhomogeneities are larger than the sound horizon the gauge-dependent results are
sharply different; the second-order action leads instead to an expression that formally coincide with the result valid inside the sound horizon.

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