Nonlinear Supersymmetry on the Plane in Magnetic Field and Quasi-Exactly Solvable Systems

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Abstract

The nonlinear $n$-supersymmetry with holomorphic supercharges is investigated for the 2D system describing the motion of a charged spin-$1/2$ particle in an external magnetic field. The universal algebraic structure underlying the holomorphic $n$-supersymmetry is found. It is shown that the essential difference of the 2D realization of the holomorphic $n$-supersymmetry from the 1D case recently analysed by us consists in appearance of the central charge entering non-trivially into the superalgebra. The relation of the 2D holomorphic $n$-supersymmetry to the 1D quasi-exactly solvable (QES) problems is demonstrated by means of the reduction of the systems with hyperbolic or trigonometric form of the magnetic field. The reduction of the $n$-supersymmetric system with the polynomial magnetic field results in the family of the one-dimensional QES systems with the sextic potential. Unlike the original 2D holomorphic supersymmetry, the reduced 1D supersymmetry associated with $x^6 + ...$ family is characterized by the non-holomorphic supercharges of the special form found by Aoyama et al.

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1 Introduction

The nonlinear supersymmetry is one of the new developments of quantum mechanics revealing itself variously in such different systems as the parabosonic and parafermionic oscillator models, the fermion-monopole system, and the $P,T$-invariant systems of planar fermions and Chern-Simons fields. Being a natural generalization of the usual supersymmetry, it is characterized by the polynomial superalgebra resembling the nonlinear finite $W$-algebras.

A simple universal algebraic structure with oscillator-like bosonic and oscillator fermionic variables underlies the usual (linear) supersymmetry in classical and quantum mechanics. The classical nonlinear supersymmetry with holomorphic supercharges has also the transparent algebraic structure related to the $n$-fold mapping of the complex plane associated with the oscillator-like bosonic variables. In what follows we will refer to the nonlinear supersymmetry generated by the holomorphic supercharges with Poisson bracket (anticommutator) being proportional to the $n$-th order polynomial in the Hamiltonian as to the holomorphic $n$-supersymmetry. However, the attempt to quantize the nonlinear supersymmetry immediately faces the problem of the quantum anomaly. The quantization of the one-dimensional systems was investigated by us in detail in Ref. where we showed that the anomaly-free quantum systems with holomorphic $n$-supersymmetry turn out to be closely related to the quasi-exactly solvable (QES) systems.

This paper is devoted to generalization of the analysis of Ref. for the case of two-dimensional systems. It will allow us not only to find a universal algebraic structure underlying the holomorphic $n$-supersymmetry at the quantum level, but also to demonstrate a nontrivial relation of the holomorphic $n$-supersymmetry to the non-holomorphic nonlinear supersymmetry of Aoyama et al. and to establish the relationship of the 2D holomorphic $n$-supersymmetry with the family of QES systems with sextic potential not comprised by the 1D holomorphic $n$-supersymmetry. Nowadays, this special class of QES systems attracts attention in the context of the $P,T$-invariant quantum mechanics.

The paper is organized as follows. In Section we consider the holomorphic $n$-supersymmetry realized in the classical 2D system describing the motion of a charged spin-1/2 particle in an external magnetic field. Section illustrates the simplest anomaly-free quantum realization of the holomorphic $n$-supersymmetry in the case of the constant magnetic field. Section is devoted to investigation of the general aspects of the anomaly-free quantization of the holomorphic $n$-supersymmetry. We show that the quantum mechanical $n$-supersymmetry can be realized only for magnetic field of special configurations of the exponential and quadratic form. Here we find the universal algebraic structure underlying the holomorphic $n$-supersymmetry. The nonlinear superalgebra with the central charge is discussed in Section where we consider also the reduction of the 2D systems with the exponential magnetic field to the 1D systems with the holomorphic $n$-supersymmetry. In Section we show that the spectral problem of the 2D system with the quadratic magnetic field is equivalent to that of the 1D QES systems with the sextic potential, and observe the relation of the 2D holomorphic $n$-supersymmetry to the non-holomorphic $N$-fold supersymmetry. In Section the brief summary of the obtained results is presented and some open problems to be interesting for further investigation are discussed.
2 Classical $n$-supersymmetry

The classical Hamiltonian of a charged spin-$1/2$ particle ($e = m = 1$) with gyromagnetic ratio $g$ moving in a plane and subjected to a magnetic field $B(x)$ is given by

$$H = \frac{1}{2} \mathbf{P}^2 + gB(x)\theta^+\theta^-,$$

where $\mathbf{P} = p + \mathbf{A}(x)$, $\mathbf{A}(x)$ is a 2D gauge potential, $B(x) = \partial_1 A_2 - \partial_2 A_1$. The variables $x_i, p_i, i = 1, 2$, and complex Grassman variables $\theta^\pm, (\theta^+)^* = \theta^-$, are canonically conjugate with respect to the Poisson brackets, $\{x_i, p_j\}_{PB} = \delta_{ij}, \{\theta^-, \theta^+\}_{PB} = -i$. For even values of the gyromagnetic ratio $g = 2n, n \in \mathbb{N}$, the system (2.1) is endowed with the nonlinear $n$-supersymmetry. In this case the Hamiltonian (2.1) takes the form

$$H_n = \frac{1}{2} Z^+ Z^- + \frac{n}{2} \{Z^-, Z^+\}_{PB} \theta^+\theta^-,$$

which admits the existence of the odd integrals of motion

$$Q_n^\pm = 2^{-\frac{n}{2}} (Z^\pm)^n \theta^\pm$$

(2.4)

generating the nonlinear $n$-superalgebra [3]

$$\{Q_n^-, Q_n^+\}_{PB} = -i(H_n)^n, \quad \{Q_n^+, H_n\}_{PB} = 0.$$  

(2.5)

This $n$-superalgebra does not depend on the explicit form of the even complex conjugate variables $Z^\pm$. Therefore, in principle, $Z^\pm$ in generators (2.2) and (2.4) can be arbitrary functions of the bosonic dynamical variables of the system.

The nilpotent quantity $N = \theta^+\theta^-$ is another obvious even integral of motion. When the gauge potential $\mathbf{A}(x)$ is a 2D vector, the system (2.3) possesses the additional even integral of motion $L = \varepsilon_{ij} x_i p_j$. The integrals $N$ and $L$ generate the $U(1)$ rotations of the odd, $\theta^\pm$, and even, $Z^\pm$, variables, respectively. Their linear combination

$$J_n = L + nN$$

(2.6)

is in involution with the supercharges, $\{J_n, Q_n^+\}_{PB} = 0$, and plays the role of the central charge of the classical $n$-superalgebra. As we shall see, at the quantum level the form of the nonlinear $n$-superalgebra (2.3) is modified generically by the appearance of the nontrivial central charge in the anticommutator of the supercharges.

3 Quantum $n$-supersymmetry: constant magnetic field

We start our investigation of the quantum two-dimensional nonlinear supersymmetry with considering the simplest case of the constant magnetic field. The quantum $n$-supersymmetric Hamiltonian for such a system is

$$H_n = \frac{1}{4} \{Z^+, Z^-\} + \frac{n}{2} B\sigma_3,$$

(3.1)
where \( Z^\pm \) are the quantum analogues of the variables (2.3) with \( \mathcal{P}_i = -i\partial_i - \frac{1}{2}\varepsilon_{ij} x_j B, \) \([\mathcal{P}_1, \mathcal{P}_2] = -iB\), corresponding to the choice of the symmetric gauge. Here and in what follows we put \( \hbar = 1 \). The quantum Hamiltonian (3.1) is related to the classical analogue (2.2) via the quantization prescription \( \theta^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), Z^+ Z^- \to \frac{1}{2}\{Z^+, Z^-\}, \theta^+ \theta^- \to \frac{1}{2}\{\theta^+, \theta^-\} = \frac{1}{2}\sigma_3 \).

The system (3.1) has the integrals of motion \( \tilde{\mathcal{P}}_i = -i\partial_i + \frac{1}{2}\varepsilon_{ij} x_j B, \) which are in involution with \( \mathcal{P}_i \) and satisfy the relation \([\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2] = iB\). In terms of the creation-annihilation operators \( a^\pm = \sqrt{2}|B|\mathcal{P}_1 \pm i\varepsilon\mathcal{P}_2, b^\pm = \sqrt{2}|B|\tilde{\mathcal{P}}_1 \mp i\varepsilon\tilde{\mathcal{P}}_2, \) \([a^-, a^+] = 1, [b^-, b^+] = 1, \varepsilon = \text{sign} B\), the Hamiltonian (3.1) is represented in the form

\[
H_n = |B|\left( a^+a^- + \frac{1}{2} + \frac{n}{2}\varepsilon\sigma_3 \right).
\]

Since the Hamiltonian does not depend on \( b^\pm \), the energy levels of the system are infinitely degenerate. The \( n \)-supersymmetry of the system (3.1) is generated by the supercharges

\[
Q_n^\pm = |B|^{\frac{1}{2}} \begin{cases} (a^\pm)^n\theta^\pm, & \text{for } B > 0, \\ (a^\pm)^n\theta^\pm, & \text{for } B < 0, \end{cases}
\]

\[
\{Q_n^-, Q_n^+\} = (H_n + \frac{n-1}{2}B) (H_n + \frac{n-3}{2}B) \cdots (H_n - \frac{n-3}{2}B) (H_n - \frac{n-1}{2}B), \quad [Q_n^\pm, H_n] = 0.
\]

Therefore, the present 2D system corresponds to the \( n \)-supersymmetric 1D oscillator [3]. Due to the axial symmetry of the system, the operator

\[
J_n = \frac{1}{B}H_n - \varepsilon b^+b^-
\]

is (up to an inessential additive constant) the quantum analogue of the classical integral (2.4). However, here the quantum central charge \( J_n \) of the \( n \)-superalgebra plays a secondary role since it is represented in terms of \( H_n \) and integrals \( b^\pm \).

As in the case of the one-dimensional theory [3], the attempt to generalize the \( n \)-supersymmetry of the system (3.1) to the case of the magnetic field of general form faces the problem of quantum anomaly. In the next section we show that the generalization is nevertheless possible for the magnetic field of special form. Such a phenomenon has an algebraic foundation and is similar to that taking place in the 1D theory.

4 Quantum \( n \)-supersymmetry: general magnetic field

Here we investigate the general case of the \( n \)-supersymmetry with holomorphic supercharges for the 2D charged particle in an external magnetic field.

A priori, the quantization prescription that respects the classical \( n \)-supersymmetry is not known. We take the quantum Hamiltonian of the general form

\[
H = \frac{1}{2} \mathcal{P}^2 + V(x) + L(x)N
\]
with $P_i = -i\partial_i + A_i(x)$, $i = 1, 2$, $N = \theta^+\theta^- = \frac{1}{2}(\sigma_3 + 1)$, and fix the unknown functions from the condition of existence of the $n$-supersymmetry. To begin with, we analyse the $n = 2$ supersymmetry, and then will generalize the construction for arbitrary natural $n$. By analogy with the one-dimensional $n = 2$ supersymmetry, we consider the second order odd holomorphic operators \[ Q^\pm_2 = \frac{1}{2}((Z^\mp)^2 - q)\theta^\pm, \tag{4.2} \]
where $q \in \mathbb{C}$ and $Z^\pm$ are the quantum analogues of (2.3). These odd operators commute with the Hamiltonian when $L(x) = 2B(x)$, $V(x) = -B(x)$, and magnetic field obeys the equations
\[ (\partial_1\partial_2 + 2\Im q)B(x) = 0, \quad (\partial_1^2 - \partial_2^2 - 4\Re q)B(x) = 0. \tag{4.3} \]

Let us note that the expression (4.2) is the most general form of the second order holomorphic supercharges since the term linear in $Z^\pm$ is excluded from them by the condition $[H_2, Q^\pm_2] = 0$.

The potential $V(x)$ has the pure quantum nature being proportional to $\hbar$. Since the resulting Hamiltonian has the form
\[ H_2 = \frac{1}{2}P^2 + B(x)\sigma_3, \tag{4.4} \]
one can treat $V(x)$ as a quantum correction term providing the same quantization prescription as in the case of the constant magnetic field.

In the complex variables
\[ z = \frac{1}{2}(x_1 + ix_2), \quad \bar{z} = \frac{1}{2}(x_1 - ix_2), \tag{4.5} \]
Eqs. (4.3) can be rewritten equivalently as
\[ (\partial^2 - \omega^2)B(z, \bar{z}) = 0, \tag{4.6} \]
where $B^* = B$, $\omega^2 = 4q$, and the notation $\partial = \partial_z$ ($\bar{\partial} = \partial_{\bar{z}}$) is introduced. Below we shall see that the holomorphic nonlinear $n = 2$ supersymmetry given by the supercharges (4.2) and Hamiltonian (4.4) with magnetic field $B$ defined by Eq. (4.6) admits the generalization for the case of arbitrary $n \in \mathbb{N}$ with magnetic field $B$ of the same structure.

The general solution to Eq. (4.6) is
\[ B(z, \bar{z}) = w_+e^{\omega z + \bar{\omega}z} + w_-e^{-(\omega z + \bar{\omega}z)} + we^{\omega z - \bar{\omega}z} + \bar{w}e^{-(\omega z - \bar{\omega}z)}, \tag{4.7} \]
where $w_+ \in \mathbb{R}$, $w \in \mathbb{C}$, $\bar{w} = w^*$. On the other hand, for $\omega = 0$ the solution to Eq. (4.6) is the polynomial
\[ B(x) = c ((x_1 - x_{10})^2 + (x_2 - x_{20})^2) + c_0, \tag{4.8} \]
with $c$, $c_0$, $x_{10}$, $x_{20}$ being some real constants.
Though the latter solution can be obtained formally from (4.7) in the limit \( \omega \to 0 \) by rescaling appropriately the parameters \( w_\pm, w \), the corresponding limit procedure is singular and the cases (4.7) and (4.8) have to be treated separately.

Rewriting the magnetic field (4.7) in terms of the real variables \( x_{1,2} \), we have

\[
B(x) = w_+ \exp(x \omega) + w_- \exp(-x \omega) + w \exp(i x \times \omega) + \bar{w} \exp(-i x \times \omega),
\]

where \( x \times \omega = \epsilon_{ij} x_i \omega_j \), and we have introduced the constant two-dimensional vector \( \omega = (\text{Re}\omega, -\text{Im}\omega) \). The vector \( \omega \) defines the preferable coordinate system,

\[
u = \frac{i}{|\omega|} (\omega z - \bar{\omega} \bar{z}), \quad \bar{v} = \frac{i}{|\omega|} (\omega \bar{z} + \bar{\omega} z),
\]

related to the initial one by the rotation. In the new coordinates the magnetic field is represented in the form

\[
B(u, v) = B_u + B_v,
\]

where

\[
B_u = w_+ e^{\omega |u|} + w_- e^{-|\omega|u}, \quad B_v = w e^{i|\omega|v} + \bar{w} e^{-i|\omega|v}.
\]

Thus, the magnetic field is hyperbolic in the \( u \) direction and periodic in the \( v \) direction. A typical example of the magnetic field with \( \text{sign} B_u(-\infty) = \text{sign} B_u(+\infty) \) is depicted on Fig. 1.

Figure 1: Magnetic field with \( \text{sign} B_u(-\infty) = \text{sign} B_u(+\infty) \).

For analysing the nonlinear \( n \)-supersymmetry for arbitrary \( n \in \mathbb{N} \), it is convenient to introduce the complex oscillator-like operators

\[
Z = \partial + W(z, \bar{z}), \quad \bar{Z} = -\bar{\partial} + \bar{W}(z, \bar{z}),
\]

(4.11)
where the complex superpotential is defined by $\text{Re} W = A_2(x)$, $\text{Im} W = A_1(x)$. The operators $Z, \bar{Z}$ obey the relation

$$[Z, \bar{Z}] = 2B(z, \bar{z}).$$

(4.12)

The $n$-supersymmetric Hamiltonian has the form

$$H_n = \frac{1}{4} \{ \bar{Z}, Z \} + \frac{n}{4} [Z, \bar{Z}] \sigma_3$$

(4.13)

generalizing (4.4) to the case of arbitrary $n$. Eq. (4.13) can be rewritten as the algebraic relation

$$[Z, [Z, [Z, \bar{Z}]]] = \omega^2 [Z, \bar{Z}],$$

(4.14)

which can be treated as an "integrability condition" of the nonlinear holomorphic supersymmetry. Eqs. (4.13) and (4.14) allow us to prove algebraically by the mathematical induction that the supercharges defined by the relations

$$Q_{n+2}^+ = \frac{1}{2} \left(Z^2 - \left(\frac{n+1}{2}\right)^2 \omega^2\right) Q_n^+, \quad Q_0^+ = \theta^+, \quad Q_1^+ = 2^{-\frac{1}{2}} Z \theta^+,$$

(4.15)

are preserved. This recurrent relation reproduces correctly the supercharges $Q_2^\pm$ constructed above.

Since the conservation of the supercharges is proved algebraically, the operators $Z, \bar{Z}$ can have any nature (the action of $Z, \bar{Z}$ is supposed to be associative). For example, they can have a matrix structure. With this observation the nonlinear supersymmetry can be applied to the case of matrix Hamiltonians [20, 29, 30, 31, 32].

Thus, the introduction of the operators $Z, \bar{Z}$ allows us to reduce the two-dimensional nonlinear $n$-supersymmetry to the pure algebraic construction.

5 Superalgebra for $\omega \neq 0$

Unlike the case of linear supersymmetry, the form of nonlinear superalgebra generated by the operators $Q_n^\pm$ and $H_n$ defined via relations (4.13)-(4.15) depends essentially on the concrete representation of the operators $Z, \bar{Z}$ satisfying the relation (4.14). We will use only the representation (4.14). In this case the $n$-supersymmetric system (4.13) with $\omega \neq 0$ has the central charge

$$J_n = -\frac{1}{4} (\omega^2 \bar{Z}^2 + \bar{\omega}^2 Z^2) + \partial B \bar{Z} + \bar{\partial} B Z - B^2 + \frac{n}{2} \bar{\partial} \partial B \sigma_3.$$  

(5.1)

The anticommutator of the supercharges contains it for any $n > 1$. For example, the $n = 2, 3, 4$ nonlinear superalgebras are

$$\{Q_2^-, Q_2^+\} = H_2^2 + \frac{1}{4} J_2 + \frac{|\omega|^4}{64},$$

$$\{Q_3^-, Q_3^+\} = H_3^2 + H_3 J_3 + \frac{1}{4} |\omega|^4 H_3 + 2|\omega|^2 (|w|^2 - w_+ w_-),$$

$$\{Q_4^-, Q_4^+\} = H_4^2 + \frac{5}{2} H_4 J_4 + \frac{41}{25} |\omega|^4 H_4^2 + \frac{9}{25} J_4^2 + 12 |\omega|^2 (|w|^2 - w_+ w_-) H_4$$

$$+ \frac{45}{27} |\omega|^4 J_4 + \frac{33}{212} |\omega|^8 - \frac{9}{2} |\omega|^4 (|w|^2 + w_+ w_-).$$

(5.2)
Let us discuss the eigenvalue problem for the Hamiltonian (4.13) with complex superpotential \( W(z, \bar{z}) \) (see the definition (1.11)). The superpotential corresponding to the magnetic field (1.7) is

\[
W(z, \bar{z}) = \frac{1}{\omega} \left( w_e^{\omega z + \bar{\omega} \bar{z}} - w_e^{-(\omega z - \bar{\omega} \bar{z})} - w_0^{\omega z - \bar{\omega} \bar{z}} + \bar{w}_e^{-(\omega z - \bar{\omega} \bar{z})} \right) + f(z) + i \int_{\tilde{\zeta}}^{\bar{\zeta}} F(z, \tilde{\zeta}) d\tilde{\zeta},
\]

where \( F(z, \tilde{\zeta}) = F(\zeta, \bar{z}) \) and \( f(z) \) is a holomorphic function. These arbitrary functions are associated with the gauge freedom of the system.

In general, the zero modes of the supercharge \( Q_n^+ \) can be found. For the sake of simplicity we consider the case \( n = 2 \) with the following zero modes of \( Q^+_2 \):

\[
\psi = \left( c_+ (\tilde{z}) e^{\frac{1}{2} \omega \tilde{z}} + c_- (\tilde{z}) e^{-\frac{1}{2} \omega \tilde{z}} \right) e^{\tilde{f}(\tilde{z}) \bar{d} \tilde{\zeta} - \tilde{f}(\zeta) \bar{d} \zeta} = 0.
\] (5.3)

where \( \tilde{f}(\tilde{z}) = f(z)^* \). Now, let us look for the eigenfunctions of the Hamiltonian associated with the zero modes. Substitution of (5.3) into the corresponding stationary Schrödinger equation gives the following coupled equations for \( c_\pm (\tilde{z}) \):

\[
4 \left( w_- e^{-\bar{\omega} \bar{z}} + \bar{w}_e^{\bar{\omega} \bar{z}} \right) c_+ (\tilde{z}) + 4 E c_- (\tilde{z}) - \omega c_- (\tilde{z}) = 0,
4 \left( w_+ e^{\bar{\omega} \bar{z}} + \bar{w}_e^{\bar{\omega} \bar{z}} \right) c_- (\tilde{z}) + 4 E c_+ (\tilde{z}) + \omega c_+ (\tilde{z}) = 0.
\]

This system of differential equations can be reduced to the Riccati equation for the function \( y(\tilde{z}) = c_+ (\tilde{z}) / c_- (\tilde{z}) \):

\[
\omega y' + 4 \left( e^{-\bar{\omega} \bar{z}} w_- + e^{\bar{\omega} \bar{z}} \bar{w}_e \right) y^2 + 8 E y + 4 \left( e^{\bar{\omega} \bar{z}} w_+ + e^{-\bar{\omega} \bar{z}} \bar{w}_e \right) = 0.
\] (5.4)

Hence, the holomorphic supersymmetry allows ones to reduce the two-dimensional spectral problem associated with the zero modes to the one-dimensional differential equation of the first order. Unfortunately, we have not succeeded in finding of the general solution to the equation (5.4). Therefore, in what follows we consider some special cases of the exponential magnetic field.

In the preferable coordinate system (1.9) the Hamiltonian takes the form

\[
H_n = \frac{1}{2} P^2_u + \frac{1}{2} P^2_v + \frac{n}{2} B(u, v) \sigma_3,
\] (5.5)

where \( P_u = -i \partial_u + A_u (u, v), P_v = -i \partial_v + A_v (u, v) \). For the magnetic field (4.10) the gauge potential can be chosen in the form

\[
A_u (u, v) = \frac{1}{|\omega|^2} B'_v, \quad A_v (u, v) = \frac{1}{|\omega|^2} B'_u.
\] (5.6)

Let us consider the system (5.3) with the reduced magnetic field: \( B_v = 0 \) \((w = 0)\). Then the gauge (5.6) is asymmetric and the central charge takes the form

\[
J_n = -\frac{1}{4} |\omega|^2 \bar{p}_v^2 + |\omega|^2 H_n - 4 w_+ w_-.
\]
where \( \hat{p}_v = -i\partial_v \). Therefore, in this case instead of \( J_n \), the integral \( \hat{p}_v \) can be considered as independent central charge. It generates translations in the \( v \)-direction. Then, e.g., in the case \( n = 2 \) the superalgebra is reduced to the form

\[
\{Q_2, Q_2^+\} = \left( H_2 + \frac{|\omega|^2}{8}\right)^2 - \frac{1}{16}|\omega|^2 \hat{p}_v^2 - w_+ w_-.
\]

In the gauge (5.6) the Hamiltonian can be written as

\[
H_n = -\frac{1}{2} \partial_u^2 + \frac{1}{2} (|\omega|^{-2} B_u' - i \partial_v)^2 + \frac{n}{2} B_u \sigma_3.
\]  

(5.7)

The coordinate \( v \) is cyclic. Representing the wave functions in the factorised form \( \psi(u, v) = e^{ivp_v} \psi(u) \), we reduce the Hamiltonian (5.7) to the one-dimensional QES Hamiltonian acting on the functions \( \psi(u) \):

\[
H_n = -\frac{1}{2} \partial_u^2 + \frac{1}{2} W(u)^2 + \frac{n}{2} W'(u) \sigma_3,
\]  

(5.8)

where

\[
W(u) = \frac{1}{|\omega|} \left( w_+ e^{i|\omega| u} - w_- e^{-i|\omega| u} \right) + p_v.
\]  

(5.9)

Hence, the 2D \( n \)-supersymmetric system with the **reduced** magnetic field is equivalent to the 1D \( n \)-supersymmetric system. In particular, using the results of Ref. [6] on the QES nature of the \( n \)-supersymmetric system (5.8) with the superpotential (5.9), one can calculate explicitly \( n \) “Landau levels” and find the corresponding wave functions in the described reduced case. One has also to note that, on the other hand, for some choice of the parameters of the superpotential (5.8), the well-known exactly solvable system with the Morse potential can be reproduced [6]. Hence, in this case one can find all the corresponding eigenstates and eigenvalues. The term “Landau levels” is justified here by the analogy with the case of the constant magnetic field in which the Hamiltonian eigenstates are bounded only in one of two directions corresponding to the continuous variables.

The reduced magnetic field with \( B_u = 0 \) (\( w_+ = 0 \)) can be considered exactly in the same way. In this case the resulting 1D \( n \)-supersymmetric system is characterized by the trigonometric superpotential. If \( v \in \mathbb{R} \), the corresponding \( n \) wave functions which can be found algebraically are not normalizable. The normalizability can be achieved by considering \( v \in [0, \frac{2\pi}{|\omega|} k] \), \( k \in \mathbb{N} \), i.e. by identifying the initial configuration space as a cylinder. However, the detailed consideration of such a problem lies out of the scope of the present paper.

### 6 Polynomial magnetic field and \( x^6 + \ldots \) family of quasi-exactly solvable potentials

Let us turn now to the case of the polynomial magnetic field. The operator

\[
J_n = \frac{1}{4c} \left( \partial B(z, \bar{z}) \bar{Z} + \bar{\partial} B(z, \bar{z}) Z - B^2(z, \bar{z}) + \frac{n}{2} \bar{\partial} \partial B(z, \bar{z}) \sigma_3 \right)
\]  

(6.1)
is the integral of motion of the system (4.13) with the magnetic field (4.8). It can be obtained from the operator $J_n$ (5.1) in the limit $\omega \to 0$ via the same rescaling of the parameters of the exponential magnetic field which transforms (4.7) into (4.8). The essential feature of this integral is its linearity in derivatives.

The polynomial magnetic field (4.8) is invariant under rotations about the point $(x_{10}, x_{20})$. Therefore, one can expect that the operator (6.1) should be related to a generator of the axial symmetry. To use the benefit of this symmetry, we pass over to the polar coordinate system with the center at the point $(x_{10}, x_{20})$. Then the magnetic field is radial,

$$B(r) = cr^2 + c_0,$$

(6.2)

and the Hamiltonian reads as

$$H_n = -\frac{1}{2} \left( \mathcal{D}_r^2 + r^{-2} \mathcal{D}_\varphi^2 + r^{-4} \mathcal{D}_r \right) + \frac{n}{2} B(r) \sigma_3.$$  

(6.3)

Here $\mathcal{D}_r = \partial_r + iA_r(r, \varphi), \mathcal{D}_\varphi = \partial_\varphi + iA_\varphi(r, \varphi)$, the magnetic field is given by

$$B = r^{-1} (\partial_r A_\varphi - \partial_\varphi A_r),$$

(6.4)

and the supercharges have the simple structure (cf. with Eq. (3.2)):

$$Q_n^+ = 2^{-\frac{n}{2}} Z^n \theta^+, \quad Q_n^- = 2^{-\frac{n}{2}} \bar{Z}^n \theta^-.$$  

(6.5)

As in the case $\omega \neq 0$, the anticommutator of the supercharges (6.3) is a polynomial of the $n$-th degree in $H_n$, $\{Q_n^-, Q_n^+\} = H_n^n + P(H_n, J_n)$, where $P(H_n, J_n)$ denotes a polynomial of the $(n - 1)$-th degree. For example, for $n = 2, 3, 4$ we have

$$\{Q_2^-, Q_2^+\} = H_2^2 + cJ_2, \\
\{Q_3^-, Q_3^+\} = H_3^3 + 4cH_3J_3 - 2c_0c, \\
\{Q_4^-, Q_4^+\} = H_4^4 + 10cH_4^2J_4 + 9c^2J_4^2 - 12c_0cH_4 - 9c^2.$$  

(6.6)

These expressions can be obtained from (5.2) via the limiting procedure discussed above.

For the radial magnetic field it is convenient to use the asymmetric gauge

$$A_\varphi = \frac{1}{4} cr^4 + \frac{1}{2} c_0 r^2, \quad A_r = 0.$$  

(6.7)

One could add an additive constant to $A_\varphi$ since this does not affect the magnetic field (6.4). But such a constant would lead to a singular gauge potential in the Cartesian coordinates owing to the singular at the coordinate origin nature of the polar system. By the same reason the constant cannot be removed by a gauge transformation. Therefore it has to vanish.

In the gauge (6.7), the Hamiltonian (6.3) is simplified:

$$H_n = -\frac{1}{2} \left( \partial_r^2 + r^{-1} \partial_r - r^{-2} \left( A_\varphi^2(r) - 2iA_\varphi(r) \partial_\varphi - \partial_\varphi^2 \right) \right) + \frac{n}{2} B(r) \sigma_3.$$  

(6.8)

The angle variable $\varphi$ is cyclic and the eigenfunctions of (6.8) can be represented as

$$\Psi(r, \varphi) = \begin{pmatrix} e^{im\varphi} \chi_m(r) \\ e^{im\varphi} \psi_{m'}(r) \end{pmatrix}, \quad m, m' \in \mathbb{N}.$$  

(6.9)
In the gauge (6.7) the integral \( J_n \) takes the form

\[
J_n = - i \partial_\varphi - \frac{c_0^2}{4c} + n \frac{\sigma_3}{2}.
\]

Up to a constant, this integral is equal to (3.3). Moreover, the system with the constant magnetic field is recovered in the limit \( c \to 0 \). In this case \( cJ_n \to -\frac{c_0^2}{4} \) that means that \( J_n \) disappears from (6.6) recovering the corresponding superalgebra for the system with the constant field.

The simultaneous eigenstates of the operators \( H_n \) and \( J_n \) have the structure

\[
\Psi_m(r, \varphi) = \begin{pmatrix} e^{i(m-n)\varphi} \chi_m(r) \\ e^{im\varphi} \psi_m(r) \end{pmatrix}
\]

(6.10)

and satisfy the equation

\[
J_n \Psi_m(r, \varphi) = \left( m - n \frac{\sigma_3}{2} + \frac{c_0^2}{4c} \right) \Psi_m(r, \varphi)
\]

(6.11)

Thus, the integral \( J_n \) is associated with the axial symmetry of the system under consideration and is (up to an additive constant) the exact quantum analogue of (2.6).

Since the angle variable \( \varphi \) is cyclic, the 2D Hamiltonian (6.8) can be reduced to the 1D Hamiltonian. The kinetic term of the Hamiltonian (6.8) is Hermitian with respect to the scalar product with the measure \( d\mu = rdrd\varphi \). In order to obtain a one-dimensional system with the usual scalar product defined by the measure \( d\mu = dr \), one has to perform the similarity transformation

\[
H_n \to UH_nU^{-1}, \quad \Psi \to U\Psi, \quad U = \sqrt{r}.
\]

(6.12)

Since the system obtained after such a transformation is originated from the two-dimensional system, one should always keep in mind that the variable \( r \) belongs to the half-line, \( r \in [0, \infty) \).

In what follows we refer to the Hamiltonian acting on the lower component of the state (6.10) as the bosonic Hamiltonian and to that acting on the upper component as the fermionic one. They form the \( n \)-supersymmetric system.

After transformation (6.12), the reduced bosonic one-dimensional Hamiltonian is

\[
H_n^{(2)} = - \frac{1}{2} \frac{d^2}{dr^2} + \frac{c_0^2}{32} r^6 + \frac{c_0c}{8} r^4 + \frac{1}{8} \left( c_0^2 - 2c(2n-m) \right) r^2 + \frac{m^2 - \frac{1}{2}}{2r^2} - \frac{1}{2} (n-m)c_0.
\]

(6.13)

This Hamiltonian gives (for \( c > 0 \)) the well-known family of the quasi-exactly solvable systems \([17, 18, 20, 21]\). According to the general theory of 1D QES systems, they are characterized by the weight \( j \) that defines the corresponding finite-dimensional non-unitary representation of the algebra \( sl(2, \mathbb{R}) \). In the case (6.13) the integer parameter \( n \) is related to the corresponding weight \( j \) as \( n = 2j + 1 \).

The superpartner \( H_n^{(1)} \) can be obtained from \( H_n^{(2)} \) by the substitution \( n \to -n, m \to m-n \). The supersymmetric pair of the 2D Hamiltonians is related to the corresponding pair of 1D Hamiltonians as

\[
e^{-i(m-n)\varphi} U H_n^{(1)} U^{-1} e^{i(m-n)\varphi} = H_n^{(1)}, \quad e^{-im\varphi} U H_n^{(2)} U^{-1} e^{im\varphi} = H_n^{(2)}.
\]

(6.14)
Here we imply that the operator $\partial$ on l.h.s. acts according to the rule $\partial e^{i\varphi} = e^{i\varphi} i k$.

The supercharges (6.3) are non-diagonal in $\varphi$ since in the gauge (6.4) the operators $Z, \bar{Z}$ have the form

$$Z = e^{-i\varphi} \left( \partial_r + \frac{1}{r} (A_\varphi(r) - i \partial_\varphi) \right), \quad \bar{Z} = e^{i\varphi} \left( - \partial_r + \frac{1}{r} (A_\varphi(r) - i \partial_\varphi) \right).$$

The operator $Z$ decreases the angular momentum of the state in 1 while $\bar{Z}$ increases it. Due to this property the supercharges (6.5) perform the proper mixing of the upper and lower states (6.10). The reduction of the corresponding 2D differential operators to 1D looks like

$$e^{-i(m-n)\varphi} U Z^n U^{-1} e^{i m \varphi} = \mathcal{Z}_n, \quad e^{-i(m-n)\varphi} U \bar{Z}^n U^{-1} e^{i m \varphi} = \mathcal{Z}_n^\dagger, \quad (6.15)$$

with $\mathcal{Z}_n$ given by

$$\mathcal{Z}_n = \left( A - \frac{n - 1}{r} \right) \left( A - \frac{n - 2}{r} \right) \ldots A,$$

where $A = \frac{d}{dr} + W(r)$ and the superpotential is

$$W(r) = \frac{1}{4} c r^3 + \frac{1}{2} c_0 r + \frac{m - \frac{1}{2}}{r}.$$ (6.16)

Using the relations (6.14), (6.15) and $[Q_n^\pm, H_n] = 0$, one can obtain the one-dimensional intertwining relations:

$$\mathcal{Z}_n \mathcal{H}_n^{(2)} = \mathcal{H}_n^{(1)} \mathcal{Z}_n, \quad \mathcal{Z}_n^\dagger \mathcal{H}_n^{(1)} = \mathcal{H}_n^{(2)} \mathcal{Z}_n^\dagger.$$ (6.18)

Hence, the one-dimensional odd operators

$$Q_n^+ = 2^{-\frac{n}{2}} Z_n \theta^+, \quad Q_n^- = (Q_n^+)\dagger$$ (6.17)

are the true supercharges of the 1D $n$-supersymmetric system,

$$[Q_n^\pm, H_n] = 0, \quad \text{where} \quad H_n = \begin{pmatrix} \mathcal{H}_n^{(1)} & 0 \\ 0 & \mathcal{H}_n^{(2)} \end{pmatrix}.$$ (6.18)

The form of the anticommutator of the supercharges can be obtained from the corresponding two-dimensional case via the formal substitution $J_n \rightarrow m - \frac{n}{2} - \frac{c^2}{4c}$. In the one-dimensional $n$-supersymmetric system (6.13), (6.17) the integer parameter $m$ can be formally extended to the whole real line ($m \in \mathbb{R}$). A similar prescription is used when the two-particle Calogero model is treated as that appearing from the reduction of the 3D oscillator [15]. Note that here the Calogero model can also be obtained from (6.13) in the case $c = 0$ corresponding to $B = \text{const}$. From the point of view of the 1D quasi-exactly solvable systems, the Hamiltonian (6.13) with $c > 0$ and $m < \frac{1}{2}$ has $n$ bound states which can be found algebraically. But from the viewpoint of the nonlinear supersymmetry, these $n$ algebraic states are the zero modes of the odd operator $Q_n^+$. The factorised form of the supercharges allows us to find the explicit form of the zero modes:

$$\tilde{\psi}_m(r) = P_{n-1}(r^2) r^{\frac{1}{2} - m} \exp \left( - \frac{c}{16} r^4 - \frac{c_0}{4} r^2 \right), \quad (6.19)$$
where $P_n$ is a polynomial of the $n$-th degree non-vanishing at zero. The same form for the algebraic states is given by the $\text{sl}(2, \mathbb{R})$ partial algebraization scheme [17, 18]. Substituting the combination (6.19) into the corresponding stationary Schrödinger equation, the algebraic system of equations for energy and coefficients in $P_{n-1}(r^2)$ can be obtained. On the other hand, the energies of the $n$ levels are the roots of the polynomial that the anticommutator of the supercharges is proportional to.

For $m \geq \frac{1}{2}$ the supercharge $Q^+_n$ has no zero modes, and hence, the 1D $n$-supersymmetry is spontaneously broken.

It is necessary to stress that for $m^2 = \frac{1}{4}$ the Hamiltonian (8.13) has the non-singular sextic potential and hence, in principle the system can be treated on the whole line, $r \in \mathbb{R}$. The approach based on the finite-dimensional representations of the algebra $\text{sl}(2, \mathbb{R})$ allows ones to find exactly $n$ even bound states for $m = \frac{1}{2}$ and $n$ odd states for $m = -\frac{1}{2}$. But this approach gives no explanation why the intermediate states of the opposite parity are omitted. Having in mind the tight relationship between the Lee-algebraic approach and the holomorphic $n$-supersymmetry (see also [6] for the details), one can say that the explanation lies in originating the system with the sextic potential from the two-dimensional $n$-supersymmetric system. Here it is necessary to note that even for the Hamiltonian (6.13) with the non-singular potential (at $m^2 = \frac{1}{4}$) the corresponding supercharges are singular at zero.

It is worth emphasizing that in the initial 2D supersymmetric system the supercharges are holomorphic whereas in the reduced 1D system they have a non-holomorphic form. The reduced 1D supersymmetric system belongs to the so called generalized $N$-fold supersymmetry of type A [7], being a generalization of the one-dimensional holomorphic supersymmetry [6].

We have obtained an interesting picture for the case of the polynomial magnetic field. The two-dimensional system with the nonlinear $n$-supersymmetry described in terms of the holomorphic supercharges (6.5) corresponds to an infinite number of the one-dimensional supersymmetric systems (6.18) with the non-holomorphic supercharges: for every (integer) $m$ satisfying the relation $m < 1$, the 1D $n$-supersymmetry is exact, while for $m \geq 1$ it is spontaneously broken. On the other hand, the 2D holomorphic $n$-supersymmetry is exact. It is characterized by the infinite-dimensional subspace of zero modes. However, there is essential difference between the cases $n = 1$ and $n \geq 2$. For the linear supersymmetry the basic relation $\{Q_1^-, Q_1^+\} = H_1$ means that all the zero modes have zero energy, and so, the ground state of such 2D system is infinitely degenerate in $m$. On the contrary, for the nonlinear supersymmetry with $n \geq 2$ the corresponding zero modes are non-degenerate in the energy by virtue of the nontrivial presence of the central charge $J_n$ in the superalgebra.

**Discussion and Outlook**

To conclude, let us summarize and discuss the obtained results and indicate some problems that deserve further attention.

- **The holomorphic $n$-supersymmetry of the 2D system describing the motion of a charged spin-$1/2$ particle in an external magnetic field provides the gyromagnetic ratio $g = 2n$ both at the classical and quantum levels.**

This is a natural generalization of the well-known restriction on the value of gyromagnetic ratio ($g = 2$) related to the linear supersymmetry [13]. Here it is necessary to have in mind
that saying about the spin-1/2 particle with gyromagnetic ratio \( g = 2n \), we proceed from the structure of the \( n \)-supersymmetric 2D Hamiltonian given by Eqs. (4.13), (4.12). However, the complete (3D) spin structure does not appear in the construction, and hence, the 2D system (4.13), (4.12) could also be interpreted, e.g., as the spin-\( n/2 \), \( g = 2 \) particle with separated polarizations \( s_z = \pm n/2 \). Such alternative interpretation is in correspondence with the relationship between the nonlinear supersymmetry and parasupersymmetry discussed in Ref. [3].

- **The algebraic foundation of the holomorphic \( n \)-supersymmetry is ascertained.**

The Hamiltonian (4.13) and the supercharges (4.15) of any holomorphic supersymmetric system are defined in terms of the operators \( Z \) and \( \bar{Z} \) only. The “integrability condition” (4.14) arises at the quantum level and guarantees the conservation of the supercharges in a pure algebraic way. Thus, the formulation of the nonlinear holomorphic supersymmetry does not depend on representation of the operators \( Z \) and \( \bar{Z} \). In this sense the holomorphic \( n \)-supersymmetry can be treated as a direct algebraic extension of the usual linear supersymmetry. The one-dimensional representation of the operators \( Z, \bar{Z} \) was explored in Ref. [6] while here we have investigated the realization of the holomorphic supersymmetry in the systems on the plane. We have found that there is an essential difference of the two-dimensional realization from the one-dimensional case.

- **In the 2D systems the additional integral of motion \( J_n \) has been found (see expressions (5.1) and (6.1)). This integral is a central charge that enters non-trivially into the nonlinear superalgebra (see equations (5.2) and (6.6)).**

Technically, the holomorphic \( n \)-supersymmetry facilitates finding the form of this additional integral. This is the important point since, in general, such a problem is rather laborious.

For the 2D systems the “integrability condition” (4.14) is represented as the differential equation (4.16) for the magnetic field. The general solution has the exponential form (4.7) for \( \omega \neq 0 \), or the polynomial form (4.8) for \( \omega = 0 \). The latter solution can be formally obtained from the former in the limit \( \omega \to 0 \). In the exponential case the magnetic field has the orthogonal hyperbolic and trigonometric directions. The oscillating behaviour of the field in one direction means that the eigenstates of the Hamiltonian are not normalizable. This situation is similar to the case of the constant magnetic field. However, for such exponential configuration of the magnetic field we have not succeeded in finding the energy levels of the Hamiltonian associated with the zero modes of the corresponding supercharge.

- **The systems with the magnetic field of the pure hyperbolic or pure trigonometric form have been reduced to the one-dimensional problems with the nonlinear holomorphic supersymmetry [4].**

In Ref. [13] a similar reduction of the 2D system with linear supersymmetry was considered in the context of application of the shape-invariant potentials to the 2D spectral problem.

- **The \( n \)-supersymmetric system with the polynomial magnetic field (4.8) has been reduced to the well-known family of one-dimensional QES systems with the sextic potential.**
This reduction confirms the intimate relationship between the nonlinear holomorphic supersymmetry and the QES systems observed in Ref. [6]. Moreover, it reveals the nontrivial relation of the holomorphic $n$-supersymmetry to the non-holomorphic $N$-fold supersymmetry discussed by Aoyama et al [7]. It is interesting to note that from the point of view of the reduction, the two-dimensional holomorphic supersymmetric system contains the infinite set of one-dimensional systems with the non-holomorphic supersymmetry in the exact and spontaneously broken phases. Having in mind the observed relationship between the 2D holomorphic and 1D non-holomorphic supersymmetries, it would be reasonable to clarify the following question: Is it possible to treat the 1D non-holomorphic $N$-fold supersymmetry of the general form [7] as a reduction of some holomorphic $n$-supersymmetry realized in the 2D Riemannian geometry?

The underlying algebraic structure of the nonlinear holomorphic supersymmetry allows ones to apply it for investigation of the wide class of quantum mechanical systems including the models described by the matrix Hamiltonians and the models on a non-commutative space. In Refs. [20, 29, 30, 31, 32] the matrix Hamiltonians were considered in the context of the QES systems. Therefore it would be interesting to investigate the possible relation of the matrix realization of the holomorphic supersymmetry to such systems. It is worth noting that the matrix extension of the two-dimensional system (4.1) corresponds to the non-relativistic particle in an external non-Abelian gauge field. In the case of the models on the non-commutative space [33, 34], the action of quantum mechanical operators is associative that, in principle, is enough for realizing the holomorphic $n$-supersymmetry (4.13)-(4.15).

At present time, the great attention is attracted by the so-called PT-invariant systems [22, 23, 24, 25, 26, 27, 28] described by the non-Hermitian Hamiltonians with a real spectrum. In Refs. [35, 36], an extension of the notion of usual supersymmetry was proposed for such systems. The QES systems have also found an application to this subject. Since the discussed holomorphic $n$-supersymmetry inherits the properties of the supersymmetric and QES systems, it would be interesting to extend the construction to the case of the PT-invariant systems.

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