BLOW-UPS OF CALORIC MEASURE IN TIME VARYING DOMAINS AND APPLICATIONS TO TWO-PHASE PROBLEMS

MIHALIS MOURGOGLOU AND CARMELO PULIATTI

In memory of Professor Michel Marias (1953–2020)

ABSTRACT. We develop a method to study the structure of the common part of the boundaries of disjoint and possibly non-complementary time-varying domains in \( \mathbb{R}^{n+1} \), \( n \geq 2 \), at the points of mutual absolute continuity of their respective caloric measures. Our set of techniques, which is based on parabolic tangent measures, allows us to tackle the following problems:

1. Let \( \Omega_1 \) and \( \Omega_2 \) be disjoint domains in \( \mathbb{R}^{n+1} \), \( n \geq 2 \), which are quasi-regular for the heat equation and regular for the adjoint heat equation, and their complements satisfy a mild non-degeneracy hypothesis on the set \( E \subset \partial \Omega_1 \cap \partial \Omega_2 \) of mutual absolute continuity of the associated caloric measures \( \omega_i \) with poles at \( \bar{p}_i = (p_i, t_i) \in \Omega_i \), \( i = 1, 2 \). Then, we obtain a parabolic analogue of the results of Kenig, Preiss, and Toro, i.e., we show that the parabolic Hausdorff dimension of \( \omega_1|_E \) is \( n + 1 \) and the tangent measures of \( \omega_1 \) at \( \omega_1 \)-a.e. point of \( E \) are equal to a constant multiple of the parabolic \( (n + 1) \)-Hausdorff measure restricted to hyperplanes containing a line parallel to the time-axis.

2. If, additionally, \( \omega_1 \) and \( \omega_2 \) are doubling, \( \log \frac{d\omega_2|_E}{d\omega_1|_E} \in VMO(\omega_1|_E) \), and \( E \) is relatively open in the support of \( \omega_1 \), then their tangent measures at every point of \( E \) are caloric measures associated with adjoint caloric polynomials. As a corollary we obtain that if \( \Omega_1 \) is a \( \delta \)-Reifenberg flat domain for \( \delta \) small enough and \( \Omega_2 = \mathbb{R}^{n+1} \setminus \overline{\Omega_1} \), and \( \log \frac{d\omega_2}{d\omega_1} \in VMO(\omega_1) \), then \( \Omega_1 \cap \{t < t_2\} \) is vanishing Reifenberg flat. This generalizes results of Kenig and Toro for the Laplacian.

3. Finally, we establish a parabolic version of a theorem of Tsirelson about triple-points for harmonic measure. Assuming that \( \Omega_i \), \( 1 \leq i \leq 3 \), are quasi-regular domains for both the heat and the adjoint heat equations, the set of points on \( \bigcap_{i=1}^3 \partial \Omega_i \), where the three caloric measures are mutually absolutely continuous has null caloric measure. In the course of proving our main theorems, we obtain new results on heat potential theory, parabolic geometric measure theory, and nodal sets of caloric functions, that may be of independent interest.

CONTENTS

1. Introduction ................................................................. 2
2. Discussion of the proofs .................................................. 10

2010 Mathematics Subject Classification. 35K05, 35B44, 28A75, 35B05, 31C45, 28A78, 28A33.
Key words and phrases. Heat equation, caloric measure, Green function, time-varying domains, free boundary problems, capacity density condition, tangent measures, absolute continuity, triple points, heat potential theory, parabolic Reifenberg flat domains.

M.M. was supported by IKERBASQUE and partially supported by the grant MTM-2017-82160-C2-2-P of the Ministerio de Economía y Competitividad (Spain), and by IT-1247-19 (Basque Government). C.P. was supported by IT-1247-19 (Basque Government).
1. Introduction

In this paper, we study two-phase problems for harmonic measure associated with the heat equation, which is traditionally called \textit{caloric measure}. We consider two disjoint open sets $\Omega^+$ and $\Omega^-$ in $\mathbb{R}^{n+1}$, and $\omega^+$ and $\omega^-$ the associated caloric measures, so that their common boundary $\partial \Omega^+ \cap \partial \Omega^-$ has positive $\omega^+$ measure. Our main goal is to study how mutual absolute continuity of the respective caloric measures provides information about the infinitesimal structure of $\partial \Omega^+ \cap \partial \Omega^-$. 

In the elliptic case, Bishop, Carleson, Garnett, and Jones studied in [BCGJ89] a two-phase problem for simply connected planar disjoint domains: they showed that the two harmonic measures are mutually singular if and only if the intersection of the respective sets of inner tangent points has zero one-dimensional Hausdorff measure. More recently, in [KPT09], Kenig, Preiss, and Toro studied the higher-dimensional case, assuming that $\Omega^+$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \bar{\Omega}^+$ are both NT-A-domains (as defined by Jerison and Kenig in [JK82]). Combining the blow-up analysis at points of mutual absolute continuity in [KT06] with the theory of tangent measures along with the monotonicity formula of Alt, Caffarelli and Friedman (see [ACF84]), they showed that mutual absolute continuity of the interior and exterior harmonic measure $\omega^+$ and $\omega^-$ implies that the harmonic measure has Hausdorff dimension $n$ and that as we zoom in at $\omega^+$-a.e. point of the common boundary, $\partial \Omega^\pm$ looks flatter and flatter. This blow-up technique was further improved by Azzam, the first named author, and Tolsa [AMT17] and the same authors along with Volberg in [AMTV19], who eventually showed that, without any further assumption on the domains, the harmonic measure restricted to the set of mutual absolute continuity is $n$-rectifiable. The connection between the Riesz transform and $n$-rectifiability was an element of major importance in the proofs of [AMT17] and [AMTV19] that allowed to improve on the results of [KPT09] from this perspective as well (since $n$-rectifiable measures have Hausdorff dimension $n$).

All the aforementioned results describe \textit{a.e.} phenomena, while it is interesting to investigate conditions that ensure some nice limiting behavior of our blow-ups at \textit{every} point. In [KT06], Kenig and Toro considered $\Omega^+$ and $\Omega^-$ to be complementary 2-sided NT-A domains and $\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+)$, and they showed that, in a sequence of arbitrarily small scales, the boundary around any point starts resembling the zero set of a harmonic polynomial (instead of a hyperplane). If the domains are $\delta$-Reifenberg flat, for $\delta$ small enough,
then the conclusion improves to a hyperplane implying that the domains are Reifenberg flat with vanishing constant. They also proved that the same conclusion holds without the $\delta$-Reifenberg flatness assumption if we assume that $\Omega^+$ is two-sided NTA, $\partial\Omega^+$ is Ahlfors-David regular and the logarithms of the interior and exterior Poisson kernels are in VMO with respect to the surface measure on $\partial\Omega^+$. Badger went a step further in [Bad11], showing that those harmonic polynomials are always homogeneous, while later, in [Bad13], he explored the topological properties of sets where the boundary is approximated by zero sets of harmonic polynomials in this way. In [AM19], Azzam and the first named author, among others, extended the results in [KPT09], [KT06], and [Bad11] to general domains and elliptic measures associated with uniformly elliptic operators in divergence form with merely bounded coefficients that are also in the Sarason’s space of vanishing mean oscillation with respect to $\omega^+$. For relevant results for elliptic operators with $W^{1,1}$-coefficients see [TZ17].

It is interesting to understand if similar phenomena occur also in the context of parabolic PDEs and, in particular, the heat equation. The implementation of blow-up methods in the context of one-phase free boundary problems for the heat equation was initiated by Hofmann, Lewis, and Nyström in [HLN04]. They extended the work of Kenig and Toro showing that, in parabolic chord arc domains with vanishing constant, the logarithm of the density of caloric measure with respect to a certain projective Lebesgue measure (or else the Poisson kernel) is of vanishing mean oscillation, and also obtained a partial converse, which amounts to a one-phase problem in Reifenberg flat domains with parabolic uniformly rectifiable boundaries. The full converse was proved by Engelstein in [Eng17A], where a key fact of his proof was a delicate classification of “flat” blow-ups, which was an open problem in the parabolic setting. He also examined free boundary problems with conditions above the continuous threshold. This problem was already solved by Nyström (see [Nys06B] and [Nys12]) in graph domains under the assumption that that the Green function is comparable with the distance function from the boundary.

Results analogous to those in [KT06] for the heat equation were considered by Nyström in [Nys06A]. He proved that, if $\Omega$ and $\mathbb{R}^{n+1} \setminus \Omega$ are parabolic NTA domains with parabolic Ahlfors regular boundary and the logarithms of the associated Poisson kernels are of vanishing mean oscillation, then a portion of $\partial \Omega$ suitably away from the poles of the caloric measures is Reifenberg flat with vanishing constant. Furthermore, it was shown in [Nys06C] that if one drops the Ahlfors regularity hypothesis and, instead of parabolic NTA, they ask for the domains to be $\delta$-Reifenberg flat for $\delta > 0$ small enough, then the same conclusion holds if one substitutes the assumption on the Poisson kernels with a vanishing mean oscillation hypothesis on the logarithm of the density $d\omega^-/d\omega^+$ of the caloric measures associated with the two domains. We remark that the proof uses the construction of the Green function and caloric measure with pole at infinity. Finally, we refer to [Eng17B] for an interesting geometric result for planar NTA domains with an application to the previously discussed one-phase problem.

To the best of our knowledge, the present paper is the first one that studies free boundary problems in so general domains.

A proficient way to address the kind of questions we referred above is to analyze the fine structure of the boundary by “zooming-in” on boundary points. There are two ways to do that. The first one is by looking at the Hausdorff convergence of rescaled copies of
the support of a measure as we zoom in, for which we follow the framework of Badger and Lewis [BL15].

We define the ball with respect to the parabolic norm \( \| \bar{y} \| := \max \{|y|, |s|^{1/2}\} \) centered at \( \bar{x} \in \mathbb{R}^{n+1} \) with radius \( r > 0 \) as
\[
C_r(\bar{x}) = \{ \bar{y} \in \mathbb{R}^{n+1} : \|\bar{y} - \bar{x}\| < r \}.
\]
Note that \( C_r(\bar{x}) \) is a euclidean right circular cylinder centered at \( \bar{x} \) of height \( 2r^2 \) and radius \( r \). If \( \bar{x} = 0 \), we simply write \( C_r(\bar{x}) = C_r \) unless stated otherwise. We also need the following time-backwards and time-forwards versions of the parabolic ball:
\[
\begin{align*}
C_r^{-}(x,t) &= \{(y,s) \in \mathbb{R}^{n+1} : y \in B_r(x), \ t - r^2 < s < t \} \\
C_r^{+}(x,t) &= \{(y,s) \in \mathbb{R}^{n+1} : y \in B_r(x), \ t < s < t + r^2 \}.
\end{align*}
\]

Let \( A \subset \mathbb{R}^{n+1} \) be a set and let \( \mathcal{S} \) be a collection of subsets of \( \mathbb{R}^{n+1} \). Given \( \bar{x} \in A \) and \( r > 0 \) we define
\[
\Theta_{\mathcal{S}}(\bar{x},r) = \inf_{S \in \mathcal{S}} \max \left\{ \sup_{a \in a \in \bar{c}(\bar{x})} \frac{\text{dist}(\bar{a}, \bar{x} + S)}{r}, \sup_{\bar{z} \in \bar{z}(\bar{x})} \frac{\text{dist}(\bar{z}, A)}{r} \right\},
\]
where “\( \text{dist} \)” denotes the distance with respect to the parabolic norm \( \| \cdot \| \). We say that \( A \) is \textit{locally bilaterally well approximated by} \( \mathcal{S} \), also abbreviated \( \text{LBWA}(\mathcal{S}) \), if for all \( \varepsilon > 0 \) and all \( K \subset A \) compact, there is \( r_{\varepsilon,K} > 0 \) such that \( \Theta_{\mathcal{S}}(\bar{x},r) < \varepsilon \) for all \( \bar{x} \in K \) and \( 0 < r < r_{\varepsilon,K} \). In other words, for \( \bar{x} \in A \) to be a \( \mathcal{S} \) point means that, as we zoom in on \( A \) at the point \( \bar{x} \), the set \( A \) resembles more and more an element of \( \mathcal{S} \). Remark that this element may change as we move from one scale to the next.

The second way is by investigating the weak convergence of rescaled copies of the measure itself. In [Pr87], Preiss developed the theory of \textit{tangent measures}, which turned out to be vital in the study of two-phase problems for harmonic and elliptic measure; his results are at the core of [KPT09], [Bad11], [AMT17], [AMTV19], and [AM19]. The definition of a tangent measure can be readily adapted to the parabolic context, in which \( \mathbb{R}^{n+1} \) is naturally endowed with a set of non-isotropic dilations.

A point \( \bar{x} \in \mathbb{R}^{n+1} \) is denoted by \( \bar{x} = (x,t) = (x',x_n,t) := (x_1,\ldots,x_n,t) \) and we also use the notation \( \bar{0} = (0,\ldots,0) \). For \( r > 0 \) and \( \bar{x}, \bar{y} \in \mathbb{R}^{n+1} \), we set
\[
\delta_r(\bar{x}) := (rx, r^2t), \quad \text{and} \quad T_{\bar{y},r}(\bar{x}) := \delta_{1/r}(\bar{x} - \bar{y}).
\]
If \( \mu \) and \( \nu \) are Radon measures in \( \mathbb{R}^{n+1} \), we define
\[
T_{\bar{y},r}[\mu](A) := \mu(\delta_r(A) + \bar{y}) = \mu(T_{\bar{y},r}^{-1}(A)), \quad A \subset \mathbb{R}^{n+1}.
\]
We say that \( \nu \) is a \textit{tangent measure} of \( \mu \) at a point \( \bar{x} \in \mathbb{R}^{n+1} \) and write \( \nu \in \text{Tan}(\mu, \bar{x}) \) if \( \nu \) is a non-zero Radon measure on \( \mathbb{R}^{n+1} \) and there are sequences \( c_i > 0 \) and \( r_i \searrow 0 \) so that \( c_i T_{\bar{y},r_i}[\mu] \) converges weakly to \( \nu \) as \( i \to \infty \).

Let us record here that a parabolic version of Besicovitch covering theorem is available for parabolic balls (see [It18]), which allows for the basic properties of tangent measures to hold in the parabolic setting by the same proofs as in the Euclidean case. For further details, we refer to Section 4.

The geometry of the blow-ups at a point of mutual absolute continuity of caloric measures is intimately related with that of nodal sets of caloric functions. We define the heat and
adjoint heat equations by

\[(1.1) \quad Hu := \Delta u - \partial_t u = 0, \quad \text{and} \quad H^* u := \Delta u + \partial_t u = 0,\]

and a $C^{2,1}$ function satisfying $Hu = 0$ (resp. $H^* u = 0$) is called (resp. adjoint) caloric function.

Let $\Theta$ denote the set of caloric functions vanishing at $\bar{0}$, $P(d)$ the set of the caloric polynomials vanishing at $\bar{0}$, and $F(d)$ the set of homogeneous caloric polynomials of degree $d$. In Lemma 6.1, we show that, for any caloric function $h$ on $\mathbb{R}^{n+1}$, there exists a unique Radon measure $\omega_h$ such that

$$\int \varphi \, d\omega_h = \frac{1}{2} \int |h| (\Delta + \partial_t) \varphi, \quad \text{for all} \varphi \in C^\infty_c (\mathbb{R}^{n+1}).$$

The measure $\omega_h$ is called caloric measure $\omega_h$ associated to $h$. In fact, this is the unique adjoint caloric measure with pole at infinity in $\Omega^\pm = \{ h^\pm \neq 0 \}$ and $h^\pm$ is the Green function with pole at infinity. If $h$ is an adjoint caloric function in $\mathbb{R}^{n+1}$, we define the adjoint caloric measure analogously.

Thus, we define

$$\mathcal{H}_\Theta := \{ \omega_h : h \in \Theta \}, \quad \mathcal{P}(d) := \{ \omega_h : h \in P(d) \} \quad \text{and} \quad \mathcal{F}(d) := \{ \omega_h : h \in F(d) \}.$$

Given a caloric function $h$, we use the notation $\Sigma^h := \{ h = 0 \}$. Furthermore, we indicate

$$\mathcal{H}_\Sigma := \{ \Sigma^h : h \in \Theta \},$$

$$\mathcal{P}_\Sigma(d) := \{ \Sigma^h : h \in P(1) \cup \cdots \cup P(d) \} \quad \text{and} \quad \mathcal{F}_\Sigma(d) := \{ \Sigma^h : h \in F(d) \}.$$

The families $\Theta^*, P^*(d), F^*(d), \mathcal{P}^*(d), \mathcal{F}^*(d), \mathcal{F}^*_\Sigma(d), \mathcal{P}^*_\Sigma(d)$, and $\mathcal{F}^*_\Sigma(d)$ are defined analogously for adjoint caloric functions, polynomials, and homogeneous polynomials. Set, moreover,

$$\mathcal{F}_\Sigma := \{ V \subset \mathbb{R}^{n+1} : V \text{ is an } n\text{-plane through } \bar{0} \text{ containing a line parallel to the } t\text{-axis} \}$$

and

$$\mathcal{F} := \mathcal{H}^{n+1}_P | V \in \mathcal{F}_\Sigma,$$

and observe that $\mathcal{F} = \mathcal{F}(1) = \mathcal{F}^*(1)$ and $\mathcal{F}_\Sigma = \mathcal{F}_\Sigma(1) = \mathcal{F}^*_\Sigma(1)$.

Potential theory is also very important in problems relating the metric properties of caloric measure to the geometry of the boundary but, in order to take full advantage of it, we need to ensure that the boundary is regular enough so that the continuous Dirichlet problem is well-posed in that domain. One can find several geometric conditions in the literature that imply the desired regularity of the boundary; e.g., the parabolic Reifenberg flatness (see the definition before Corollary 1.3), the non-tangential accessibility (see e.g. [Nys06A, pp. 263-264]), and the time-backwards Ahlfors-David regularity (see [GH20]).

A well-known condition which implies regularity of the boundary in the Euclidean setting and has been used in the harmonic measure framework is the so-called Capacity-Density Condition (see e.g. [AMT17]). The CDC ensures that the capacity of the complement of $\Omega$ in each ball centered at the boundary is large in a scale invariant way. We will now introduce two similar notions of thickness that, in the parabolic context, serve the same purpose as the CDC.
Given \( \rho > 0 \) and \( \bar{x} = (x, t) \in \mathbb{R}^{n+1} \) we define the heat ball centered at \( \bar{x} \) with radius \( \rho \) to be the set
\[
E(\bar{x}; \rho) := \{ (y, s) \in \mathbb{R}^{n+1} : |x - y| < \sqrt{2n(t - s) \log \left( \frac{\rho}{t - s} \right)}, t - \rho < s < t \}
\]
(1.2)
\[
= \{ \bar{y} \in \mathbb{R}^{n+1} : \Gamma(\bar{x} - \bar{y}) > (4\pi \rho)^{-n/2} \},
\]
where \( \Gamma(\cdot, \cdot) \) stands for the fundamental solution of the heat equation (see Section 3). We indicate by \( E^*(\bar{x}; r) \) the adjoint heat ball.

We say that \( \bar{x} \in \partial \Omega \) belongs to the parabolic boundary of \( \Omega \) (resp. adjoint parabolic boundary) and write \( \bar{x} \in \mathcal{P} \Omega \) (resp. \( \mathcal{P}^+ \Omega \)) if \( C^-_c(\bar{x}) \cap \Omega^c \neq \emptyset \) (resp. \( C^+_c(\bar{x}) \cap \Omega^c \neq \emptyset \)) for every \( r > 0 \). If \( \bar{x} \in \mathcal{P} \Omega \) (resp. \( \mathcal{P}^+ \Omega \)), we say that \( \bar{x} \) is in the bottom boundary of \( \Omega \) (resp. adjoint bottom boundary) and write \( \bar{x} \in B \Omega \) (resp. \( B^+ \Omega \)) if there exists \( \varepsilon > 0 \) such that \( C^+_c(\bar{x}) \subset \Omega \) (resp. \( C^-_c(\bar{x}) \subset \Omega \)). Moreover, we define the lateral boundary (resp. adjoint lateral boundary) as \( \Omega^l = \mathcal{P} \Omega \setminus B \Omega \) (resp. \( \Omega^{l*} = \mathcal{P}^+ \Omega \setminus B^+ \Omega \)) of \( \Omega \) and denote by \( \Omega^q \) the quasi-lateral boundary of \( \Omega \) (for the precise definition see Section 2).

We also refer to Section 3 for the definition of \( \text{Cap}(\cdot, \cdot) \), the thermal capacity of a set.

**Definition 1.1.** Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open set and \( F \subset \mathcal{P} \Omega \) be compact. We say that a \( \Omega \) satisfies the Time-Backwards Capacity Porosity Condition (TBCPC) at \( F \) if there exists a constant \( c > 0 \) and two sequences \( \{\xi_j\}_{j \geq 1} \subset F \) and \( r_j \to 0 \), such that
\[
\text{Cap}(E(\xi_j, r_j^2) \cap \Omega^c) \geq c r_j^n.
\]
(1.3)
If \( F = \{\xi\} \), then we say that \( \Omega \) satisfies the TBCPC at \( \xi \). We remark that \( c, \tilde{\xi}_j \), and \( r_j \) may depend on \( F \).

Let \( \Omega^+ \) and \( \Omega^- \) be two disjoint open sets in \( \mathbb{R}^{n+1} \) with \( \mathcal{S}' \Omega^+ \cap \mathcal{S}' \Omega^- \neq \emptyset \) and assume that \( F \subset \mathcal{S}' \Omega^+ \cap \mathcal{S}' \Omega^- \) is compact. Then we say that \( \Omega^+ \) and \( \Omega^- \) satisfy the joint TBCPC at \( F \) if there exists a sequence \( \{\xi_j, r_j\}_{j \geq 1} \subset F \times (0, 1) \) so that \( r_j \to 0 \) and (1.3) holds for \( \Omega^\pm \), then there exists a subsequence of \( \{\xi_j, r_j\}_{j \geq 1} \) so that (1.3) holds for \( \Omega^\pm \) (with possibly different constant \( c \)). We may define the time-forwards CPC (TFCPC) by replacing the heat balls by adjacent heat balls in the definitions above.

Note that since \( \Omega^+ \) and \( \Omega^- \) are disjoint, it holds that \( \mathcal{P} \Omega^\pm \cap B \Omega^\mp = B \Omega^+ \cap B \Omega^- = \emptyset \) and thus, we could have assumed \( \mathcal{P} \Omega^+ \cap \mathcal{P} \Omega^- \neq \emptyset \) in the definition above. For \( \Omega \subset \mathbb{R}^{n+1} \), we denote
\[
T_{\text{min}} = \inf \{ T \in \mathbb{R} : \Omega \cap \{ t = T \} \neq \emptyset \},
\]
\[
T_{\text{max}} = \sup \{ T \in \mathbb{R} : \Omega \cap \{ t = T \} \neq \emptyset \}.
\]
Note that it is possible that \( T_{\text{min}} = -\infty \) and \( T_{\text{max}} = +\infty \). We also define
\[
E_s = \{(x, t) \in E : t = s\}
\]
to be the time-slice of \( E \) for \( t = s \).

**Definition 1.2.** We say that \( \Omega \) satisfies the Time-Backwards Capacity Density Condition (TBCDC) at \( \xi_0 = (\xi_0, t_0) \in \mathcal{S} \Omega \) if there exists \( \tilde{c} > 0 \) such that
\[
\text{Cap}(E(\xi_0, r^2) \cap \Omega^c) \geq \tilde{c} r^n, \quad \text{for all } 0 < r < \sqrt{t_0 - T_{\text{min}}/2}.
\]
(1.4)
Analogously, we say that $\Omega$ satisfies the Time-Forwards Capacity Density Condition (TFCDC) if, for some $\tilde{c} > 0$,
\begin{equation}
\text{Cap}(E^*(\xi_0; r^2) \cap \Omega^c) \geq \tilde{c} r^n, \quad \text{for all } 0 < r < \sqrt{T_{\text{max}} - t_0/2}.
\end{equation}

The joint TBCPC guarantees regularity of the parabolic boundary for the heat equation and is a much weaker condition than TBCDC, which, in turn, is a natural setting for our problems (see e.g. [AMT17]). The joint TBCPC assumption is a sufficient condition for our blow-up arguments to work and, at the moment, it is not clear how to remove it from our hypotheses. In the case of harmonic and elliptic measures, it has been proved in [AMTV19, Lemma 5.3] and [AM19, Lemma 4.13] that mutual absolute continuity of the interior and the exterior measures implies an even stronger version of (1.3) in terms of the $(n+1)$-dimensional Lebesgue measure of euclidean balls. This method, however, is based on a dichotomy argument and cannot be generalized directly to the caloric measure because it is not enough to obtain that the exterior of the domain is “large” in the whole parabolic ball; we need to know that this is true in the time-backwards cylinder, which complicates things significantly.

Criteria for parabolic Wiener regularity have been extensively studied in the literature (see Section 3 for the definition and more references). This property is particularly important to our purposes because we need to extend the Green function by zero to the complement of the domain in a continuous way. In order to work in more generals domains, the intrinsic difficulties of heat potential theory constitute a challenging obstacle to overcome. For instance, we remark that Harnack inequality is time-directed (see e.g. [Wa12, Corollary 1.33]) and that it is possible to construct domains so that the capacity of the set of the irregular points for the heat equation is positive (see [TW85, p.336]). The most general class of domains in which (some of) our arguments work seem to be the so-called quasi-regular domains for the heat operator (resp. adjoint heat operator), which amounts to domains for which the set of irregular points of the essential boundary for the heat equation (resp. adjoint heat equation) is polar (see Section 3). This is in accordance to the elliptic theory where, indeed, the set of irregular points is always polar (which is not the case in the parabolic context).

Before we state our first main theorem, which is a generalization of [KPT09] and [AM19, Theorem I], we need the notion of parabolic Hausdorff dimension of a Borel measure $\omega$. This is defined as
\begin{equation}
\text{dim}(\omega) := \inf \{ \text{dim}(Z) : \omega(\mathbb{R}^{n+1} \setminus Z) = 0 \},
\end{equation}
where “dim” stands for the parabolic Hausdorff dimension (see Section 4).

**Theorem I.** Let $\Omega^+$ and $\Omega^-$ be two disjoint domains in $\mathbb{R}^{n+1}$ which are quasi-regular for $H$ and regular for $H^*$ and let $\omega^\pm$ be the caloric measures associated with $\Omega^\pm$ with poles $\bar{p}_\pm \in \Omega^\pm$. Let also $E \subset \mathcal{P}^*\Omega^+ \cap \mathcal{P}^*\Omega^- \cap \text{supp}\omega^+$ be such that $\omega^+(E) > 0$ and $\omega^+|_E \ll \omega^-|_E \ll \omega^+|_E$, and assume that $\Omega^+$ and $\Omega^-$ satisfy the joint TBCPC at all points of $E$. Then, $\tan(\omega^\pm, \xi) \subset \mathcal{F}$ for $\omega^\pm$-a.e. $\xi \in E$ and $\dim \omega^\pm|_E = n + 1$.

Moreover, if $\Omega^\pm$ also satisfy the TFCDC,
\begin{equation}
\lim_{r \to 0} \Theta_{\partial \Omega^\pm}(\bar{\xi}, r) = 0 \text{ for } \omega^+\text{-a.e. } \bar{\xi} \in E.
\end{equation}
The theorem above (and similarly all the theorems we prove) can be formulated for ad-
joint caloric measures if we assume the joint TFCPC and the TBCDC in place of the joint
TBCPC and TFCD and let \( E \) be in \( \mathcal{P} \Omega^+ \cap \mathcal{P} \Omega^- \). We point out that caloric measure is not
necessarily doubling in domains such as the ones considered in Theorem I and also that it is
natural to study mutual absolute continuity of \( \omega^+ \) and \( \omega^- \) on the lateral part of the boundary
because it cannot occur elsewhere. For more details on this matter, we refer to Section 9.

Secondly, we prove the caloric equivalent of Tsirelson’s theorem [Ts97] about triple-
points for harmonic measure. Tsirelson’s method is based on the fine analysis of filtrations
for Brownian and Walsh-Brownian motions, although, more recently, Tolsa and Volberg
[TV18] obtained the same result using analytical tools and, in particular, blow-up argu-
ments, which is exactly the method we follow as well.

**Theorem II.** Let \( \Omega_i \subset \mathbb{R}^{n+1}, 1 \leq i \leq 3 \), be three disjoint open sets which are quasi-regular
for both \( H \) and \( H^* \). Let also \( \omega^j \) be the caloric measure in \( \Omega^j \) with pole at \( \tilde{p}_j \in \Omega \) and assume
that \( E \subset \cap_{i=1}^3 \mathcal{P}^* \Omega_i \cap \text{supp} \omega^1 \) is such that \( \omega^i|_E \ll \omega^j|_E \ll \omega^i|_E \), for \( 1 \leq i, j \leq 3 \). Then
\( \omega^i(E) = 0 \) for \( i = 1, 2, 3 \).

Note that Theorem II does not need neither the joint TBCPC assumption on the domains
nor regularity for \( H^* \). Quasi-regularity is all we need to assume since, by an interior approx-
imation argument that we prove in Section 3, we show that it suffices to study the problem
in regular domains.

Under a pointwise VMO-type condition on \( d\omega^-/d\omega^+ \) on a particular subset of the boundary
\( E \), we prove a local analogue of [AM19, Theorem II].

**Theorem III.** Let \( \Omega^+ \) and \( \Omega^- \) be two disjoint domains in \( \mathbb{R}^{n+1} \) which are quasi-regular
for \( H \) and regular for \( H^* \) and let \( \omega^\pm \) be the caloric measures associated with \( \Omega^\pm \) with poles
\( \tilde{p}_\pm \in \Omega^\pm \). Let also \( E \subset \mathcal{P}^\Omega \cap \mathcal{P}^\Omega^- \cap \text{supp} \omega^+ \) be a relatively open set in \( \supp \omega^+ \) such
that \( \omega^+(E) > 0 \) and \( \omega^-|_E \ll \omega^+|_E \). Assume that \( \Omega^+ \) and \( \Omega^- \) satisfy the joint TBCPC at
all points of \( E \) and, if we set \( f = \frac{\omega^-}{\omega^+} \) to be the Radon-Nikodym derivative of \( \omega^-|_E \)
with respect to \( \omega^+|_E \), we assume that for \( \xi \in E \),

\[
\lim_{r \to 0} \left( \int_{C_r(\xi)} f \, d\omega^+_E \right) \exp \left( -\int_{C_r(\xi)} \log f \, d\omega^+_E \right) = 1,
\]

and \( \text{Tan}(\omega^+, \xi) \neq \emptyset \), then there is \( k \geq 1 \) such that \( \text{Tan}(\omega^+, \xi) \subset \mathcal{F}(k) \) and

\[
\limsup_{r \to 0} \frac{\omega^+(C_{2r}(\xi))}{\omega^+(C_r(\xi))} < \infty.
\]

Furthermore, if \( \Omega^+ \) and \( \Omega^- \) also satisfy the TFCD, then

\[
\Theta_{\frac{\partial}{\partial\Omega^\pm}} (\xi, r) \to 0, \quad \text{as } r \to 0.
\]

Before we go any further, let us introduce the space of Vanishing Mean Oscillation. Given a
Radon measure \( \omega \) in \( \mathbb{R}^{n+1} \), we denote

\[
f_{C_r(\tilde{x})} := \int_{C_r(\tilde{x})} f \, d\omega := \frac{1}{\omega(C_r(\tilde{x}))} \int_{C_r(\tilde{x})} f \, d\omega, \quad \tilde{x} \in \text{supp} \omega,
\]
and we say that \( f \in \text{VMO}(\omega) \) if \( f \in L^1_{\text{loc}}(\omega) \), and
\[
\lim_{r \to 0} \sup_{\bar{x} \in \text{supp} \omega} \int_{C_r(\bar{x})} \left| f(\bar{y}) - f_{C_r(\bar{x})} \right| d\omega(\bar{y}) = 0.
\]

The condition (1.7) implies a vanishing mean oscillation assumption on \( d\omega^- / d\omega^+ \) on \( E \). However, in general, these assumptions are not equivalent. For more details, we refer to [AM19, Section 7].

The next theorem is a global version of Theorem III under the additional assumption that \( \omega^\pm \) is doubling. This is the analogue of [AM19, Theorem III], which, in turn generalized, the main results of [KT06] and [Bad11]. We prove that at sufficiently small scales, the support of \( \omega^+ \) resembles the zero set of an adjoint caloric polynomial uniformly on compact subsets of the set of mutual absolute continuity. This assertion can be formulated both in terms of \( \Theta_{\partial \Omega^\pm}^{d_r(d)} \) and the functional \( d_1(\cdot, \mathcal{P}(d)) \), which is essentially a distance between measures and the set \( \mathcal{P}(d) \) (see (4.4) for the exact definition).

**Theorem IV.** Let \( \Omega^+, \Omega^-, \omega^+, \omega^-, f \) and \( E \) be as in the statement of Theorem III. If \( \log f \in \text{VMO}(\omega^+|_E) \) and \( \omega^+|_E \) is \( C \)-doubling, i.e.
\[
\omega^+|_E(C_{2r}(\bar{x})) \leq C \omega^+|_E(C_r(\bar{x})), \quad \bar{x} \in \mathbb{R}^{n+1}, r > 0,
\]
then, there exists an integer \( d = d(n, C) > 0 \), so that for every compact set \( F \subset E \), it holds
\[
\lim_{r \to 0} \sup_{\xi \in F} d_1(T_{\xi,r}[\omega^+], \mathcal{P}(d)) = 0.
\]

If \( \Omega^+ \) and \( \Omega^- \) also satisfy the TFCDC, then
\[
\lim_{r \to 0} \Theta_{\partial \Omega^\pm}^{d_r(d)}(\bar{\xi}, r) = 0.
\]

Theorem IV also applies directly to the study of (parabolic) Reifenberg flat domains, and gives an alternative proof of a result that Nyström proved with different techniques in [Nys06C]. We recall that \( \Omega \subset \mathbb{R}^{n+1} \) is \( \delta \)-Reifenberg flat, \( \delta > 0 \), if for \( R > 0 \) and \( \xi \in \partial \Omega \) there exists a \( n \)-plane \( L_{\xi,R} \) through \( \xi \) containing a line parallel to the time-axis and such that
\[
\begin{align*}
\{ \bar{y} + r\hat{n} \in C_R(\bar{\xi}) : r > \delta R, \bar{y} \in L_{\xi,R} \} &\subset \Omega, \\
\{ \bar{y} - r\hat{n} \in C_R(\bar{\xi}) : r > \delta R, \bar{y} \in L_{\xi,R} \} &\subset \mathbb{R}^{n+1} \setminus \overline{\Omega},
\end{align*}
\]
where \( \hat{n} \) is the normal vector to \( L_{\xi,R} \) pointing into \( \Omega \) at \( \bar{\xi} \).

For \( t_0 \in \mathbb{R} \), we denote \( \Omega^{t_0} := \Omega \cap \{ t < t_0 \} \). The domain \( \Omega \) (resp. \( \Omega^{t_0} \)) is vanishing Reifenberg flat if it is \( \delta \)-Reifenberg flat for some \( \delta > 0 \) and, for any compact set \( K \) (resp. \( K \subset \subset \{ t < t_0 \} \)),
\[
\lim_{r \to 0} \sup_{\xi \in K \cap \partial \Omega} \Theta_{\partial \Omega^\pm}^{d_r(d)}(\bar{\xi}, r) = 0.
\]

Let us also observe that, if \( \Omega \) is \( \delta \)-Reifenberg flat, it readily follows that \( \partial_a \Omega = \partial_a^* \Omega = \emptyset \).
Additionally, it is easy to see that a \( \delta \)-Reifenberg flat domain satisfies the TBCDC and the TFCDC (with parameters depending on \( \delta \)) and, thus, is regular for \( H \) and \( H^* \).
Corollary 1.3. Let also $\Omega^+ \subset \mathbb{R}^{n+1}$ be a $\delta$-Reifenberg flat domain and set $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$. Let $\omega^\pm$ be the caloric measures of $\Omega^\pm$ with poles at $\hat{p}_\pm = (p_\pm, t_\pm) \in \Omega^\pm$. If $\omega^- \ll \omega^+$, $\log f = \log \frac{\omega^+}{\omega^-} \in \text{VMO}(\omega^+)$, and $\delta$ is small enough (depending on $n$), then $(\Omega^+)^{\pm}$ is vanishing Reifenberg flat.

The analogous result for harmonic measure can be found in [KT06, Corollary 4.1], while for the original theorem for caloric measure we refer to [Nys06C, Theorem 1].

Discussion of the proofs. In the current paper, we follow the same strategy as in [AM19], although there are many significant challenges along the way that we overcome to adapt this method successfully to the parabolic setting. In Section 3, we exhibit various properties of thermal capacity, the most important of which is the backwards in time self-improvement of a pointwise time-backwards capacity density condition on a particular scale. This is the building block of the proofs of several PDE estimates around the boundary that are absolutely necessary for our blow-up arguments in Section 8. Those results can find applications in future works and are new and interesting on their own. In Section 4, we develop the required parabolic GMT framework and confirm that, due to a parabolic Besicovitch covering theorem, the theory of tangent measures translates almost unchanged to the parabolic setting. Moreover, we show that the blow-ups of the “parabolic surface measure” on euclidean rectifiable sets are parabolic flat, meaning that there is a plane that contains a line parallel to the time axis such that the blow-up measure is the parabolic Hausdorff measure restricted to that plane. Han and Lin [HL94] had proven that if $h$ is caloric and $\Sigma^h = \{h = 0\}$, then in any ball centered on $\Sigma^h$, it holds that $\Sigma^h$ can be written as the union of its regular set, which is a smooth $n$-submanifold and its singular set, which is an $(n-1)$-rectifiable set. Unlike the harmonic case, this is not enough for our purposes and so, in Section 5, we investigate the finer structure of the regular set separating space and time-singularities. We demonstrate that any $\bar{x} \in \mathcal{R}_x = \Sigma^h \cap \{|\nabla h| \neq 0\}$ has a neighborhood so that $\Sigma^h$ is given by an admissible $n$-dimensional smooth graph, while for any $\bar{x} \in \mathcal{R}_t = \Sigma^h \cap \{|\nabla h| = 0\} \cap \{\partial_t h \neq 0\}$, we can find a neighborhood around it in which $\mathcal{R}_t$ is contained in a smooth $(n-1)$-graph. Those results are fundamental to several measure-theoretic arguments and are used repeatedly in the rest of the paper. To the best of our knowledge, this description of the time-regular set is novel in the literature. In Section 6, we prove the existence and uniqueness of the caloric measure associated with a caloric function of the form $d\omega_h = -\partial_{\nu_h} h d\sigma_h$, where $\sigma_h$ is the “surface measure” on $\Sigma^h$. In the elliptic setting, this is a straightforward application of the Gauss-Green formula on sets of locally finite perimeter, which is not the case here. The aim of Section 7 is to prove that, if the parabolic tangent measures $\text{Tan}(\omega, \xi)$ to a given Radon measure $\omega$ are caloric measures associated with caloric polynomials and there is a measure corresponding to a homogeneous caloric polynomial of degree $k$, then all the elements of $\text{Tan}(\omega, \xi)$ are caloric measures associated with a homogeneous caloric polynomial of the same degree. The connectivity arguments for tangent measures translate unchanged from [AM19], although, this is not the case for the analogues of the main lemmas from [Bad11]. In fact, it is not clear to us how to adapt Badger’s proofs directly to our case, although, his ideas inspired us to come up with new ones, which we find simpler even for harmonic functions. Section 8 groups the main one and two-phase blow-up lemmas, for which we follow the approach in [AM17], [AMVT19], and [AM19]. However, there are substantial obstacles such as the compact embedding of a “parabolic” Sobolev space $W^{1,2}$.
in $L^2$, the lack of analyticity in time, the unique continuation arguments, and the locality of Lemma 8.4. Finally, in Section 9 we provide the proofs of the four main theorems using the results from the previous sections.

**Acknowledgements.** We would like to thank Jonas Azzam, Max Engelstein, Luis Escauriaza, Steve Hofmann, Tuomo Kuusi, Andrea Merlo, and Xavier Tolsa for several fruitful discussions on topics related to the current paper. The first named author would like to dedicate this paper to the memory of his teacher, mentor, and dear friend, Professor Michel Maris.

2. Preliminaries and notation

We denote points in $\mathbb{R}^{n+1}$ by $\bar{x} = (x, t) = (x', x_n, t) := (x_1, \ldots, x_n, t)$ and define the (parabolic) norm

$$||\bar{x}|| := \max\{|x|, |t|^{1/2}\},$$

where $|\bar{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$ stands for the standard euclidean norm. We also use the notation $\bar{0} = (0, \ldots, 0)$.

If $E \subset \mathbb{R}^{n+1}$, we define the parabolic distance to $E$ as $\text{dist}_p(\bar{x}, E) = \inf\{||\bar{x} - \bar{y}|| : \bar{y} \in E\}$ and the parabolic diameter as $\text{diam}_p(E) = \sup\{||\bar{x} - \bar{y}|| : (\bar{x}, \bar{y}) \in E \times E\}$.

Given an open set $\Omega$, and a point $\xi \in \Omega$, we denote by $\Lambda(\bar{\xi}, \Omega)$ (resp. $\Lambda^\ast(\bar{\xi}, \Omega)$) the set of points $\bar{x} \in \Omega$ that are lower (resp. higher) than $\xi$ relative to $\partial \Omega$, in the sense that there is a polygonal path $\gamma \subset \Omega$ joining $\xi$ to $\bar{x}$, along which the time variable is strictly decreasing (resp. increasing). By a polygonal path, we mean a path which is a union of finitely many line segments.

Following [Wa12, Definition 8.1], we define the *normal boundary*

$$\partial_n \Omega = \begin{cases} \mathcal{P}_\Omega, & \text{if } \Omega \text{ is bounded} \\ \mathcal{P}_\Omega \cup \\{\infty\}, & \text{if } \Omega \text{ is unbounded} \end{cases}$$

and the *abnormal boundary* $\partial_a \Omega = \{\bar{x} \in \partial \Omega : \exists \varepsilon > 0 \text{ such that } C^-_\varepsilon(\bar{x}) \subset \Omega\}$. The abnormal boundary is further decomposed into $\partial_a \Omega = \partial_a \Omega \cup \partial_{ss} \Omega$, where $\partial_a \Omega$ stands for the *singular boundary* and $\partial_{ss} \Omega$ for the *semi-singular boundary*, which are defined respectively by

$$\partial_a \Omega = \{\bar{x} \in \partial_a \Omega : \exists \varepsilon > 0 \text{ such that } C^+_\varepsilon(\bar{x}) \cap \Omega = \emptyset\}$$

and

$$\partial_{ss} \Omega = \{\bar{x} \in \partial_a \Omega : C^+_r(\bar{x}) \cap \Omega \neq \emptyset, \text{ for all } r > 0\}.$$ 

By [Wa12, Theorem 8.40], $\partial_a \Omega$ is contained in a sequence of hyperplanes of the form $\mathbb{R}^n \times \{t\}$. The *essential boundary* is defined as $\partial_e \Omega = \partial_a \Omega \cup \partial_{ss} \Omega = \partial \Omega \setminus \partial_a \Omega$, replacing $\partial \Omega$ by $\partial \Omega \cup \{\infty\}$ if $\Omega$ is unbounded. Finally, following [GH20], we define the *quasi-lateral boundary* to be

$$\mathcal{S}_\Omega = \begin{cases} \partial \Omega, & \text{if } T_{\min} = -\infty \text{ and } T_{\max} = \infty \\ \partial \Omega \setminus (B \Omega)_{T_{\min}}, & \text{if } T_{\min} > -\infty \text{ and } T_{\max} = \infty \\ \partial \Omega \setminus (\partial_a \Omega)_{T_{\max}}, & \text{if } T_{\min} = -\infty \text{ and } T_{\max} < \infty \\ \partial \Omega \setminus ((B \Omega)_{T_{\min}} \cup (\partial_a \Omega)_{T_{\max}}), & \text{if } -\infty < T_{\min} < T_{\max} < \infty, \end{cases}$$
where \((B\Omega)_{T_{\min}}\) is the time-slice of \(B\Omega\) with \(t = T_{\min}\) and \((\partial_s \Omega)_{T_{\max}}\) is the time-slice of \(\partial_s \Omega\) with \(t = T_{\max}\). Observe that for a cylindrical domain \(U \times (T_{\min}, T_{\max})\), where \(U \subset \mathbb{R}^n\) is a domain in the spatial variables, the quasi-lateral boundary coincides with the lateral boundary. By [GH20, Lemma 1.14], both \(\partial_c \Omega\) and \(S\Omega\) are closed sets.

Given \(f: \mathbb{R}^{n+1} \to \mathbb{R}\) we write \(Df = (\nabla f, \partial_t f)\) for the (full) gradient of the function \(f\) and, \(D^{\alpha,\ell} f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \partial_t^{\ell} f\) for higher order derivatives, where \(\alpha \in \mathbb{Z}^n_+\) and \(\ell \in \mathbb{Z}_+\). If \(E \subset \mathbb{R}^{n+1}\) and \(f\) is a continuous function with compact support in \(E \subset \mathbb{R}^{n+1}\), then we write \(f \in C_c(E)\). If \(\Omega\) is an open set, we denote by \(C^{m, \frac{m}{2}}(\Omega)\) the class of functions such that \(f(\cdot, t) \in C^{m}(\Omega_t)\) for any fixed \(t \in (T_{\min}, T_{\max})\) and \(f(x, \cdot) \in C_{\text{loc}}^{m}(T_{\min}, T_{\max})\) for any fixed \(x \in \mathbb{R}^{n+1}\) such that \(\bar{x} \in \Omega\). If \(f\) is \(C^m\) in both space and time variables, we will simply write that \(f \in C^m(\Omega)\). Finally, we say that \(f \in C^{m, \frac{m}{2}}(E)\) if \(f\) has compact support in \(E \subset \mathbb{R}^{n+1}\) and \(f \in C^{m, \frac{m}{2}}(\mathbb{R}^{n+1})\).

We write \(a \preceq b\) if there is \(C > 0\) so that \(a \leq Cb\) and \(a \preceq_t b\) if the constant \(C\) depends on the parameter \(t\). We write \(a \approx b\) to mean \(a \preceq b \preceq a\) and define \(a \approx_t b\) similarly. Sometimes we also use the notation \(\bar{f}_E \mu\) for the average \(\mu(F)^{-1} \int_F d\mu\) over a set \(F \subset \mathbb{R}^{n+1}\) with respect to the measure \(\mu\).

If \(E \subset \mathbb{R}^{n+1}\) is a Borel set and \(\mathcal{H}^d\) stands for the Euclidean \(d\)-Hausdorff measure in \(\mathbb{R}^{n+1}\) for \(d \leq n - 1\), we define the \(d\)-“surface measure” on \(E\) as the measure \(\sigma_d = d\sigma_d dt\), where \(\sigma_d = \mathcal{H}^d|_{E_t}\).

If \(s \in [2, n + 2]\) and \(0 < \delta \leq \infty\), we set
\[
\mathcal{H}^s_{p, \delta}(E) = \inf \left\{ \sum \text{diam}_p(E_j)^s : E \subset \bigcup_j E_j, \text{diam}_p(E_i) \leq \delta \right\},
\]
for \(E \subset \mathbb{R}^{n+1}\), and, as in the Euclidean case, define the parabolic \(s\)-Hausdorff measure by
\[
\mathcal{H}^s_p(A) = \lim_{\delta \to 0} \mathcal{H}^s_{p, \delta}(E),
\]
which is a Borel measure by the Carathéodory criterion.

3. Heat Potential Theory and PDE

Let \(\Omega \subset \mathbb{R}^{n+1}\) be an open set. Define the parabolic operator
\[
H_a u := a\Delta - \partial_t, \quad \text{and} \quad H_a^* u := a\Delta + \partial_t, \quad \text{for} \ a > 0.
\]
When \(a = 1\), we simply write \(H_1 = H\) and \(H_1^* = H^*\) for the heat and the adjoint heat operator respectively.

For \(\bar{x} = (x, t) \in \mathbb{R}^{n+1}\), we denote by
\[
\Gamma_a(\bar{x}) = (4\pi at)^{-n/2} \exp\left(-|x|^2 / 4at\right) \chi_{(t > 0)}(t)
\]
the Gaussian kernel. Note that \(\Gamma_a\) is the fundamental solution associated with \(H_a\), i.e., it satisfies
\begin{enumerate}
  \item \(H_a \Gamma_a(x, t) = \delta_0(x, t)\), in the distributional sense, for \(\{t > 0\}\), where \(\delta_0\) stands for the Dirac mass at 0;
  \item \(\Gamma_a(x, t) = \delta_0(x)\), for \(t = 0\);
\end{enumerate}
Moreover, $\Gamma_a \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ and
\begin{equation}
\Gamma_a(x) \leq C_h \pi^{-n/2} \|x\|^{-n},
\end{equation}
for some constant $C_h > 0$ depending on $n$ and $a$.

We say that a function $u \in C^{2,1}(\Omega)$ is caloric (resp. adjoint caloric) if it satisfies the (adjoint) heat equation $Hu = 0$ (resp. $H^*u = 0$) in a pointwise sense. If $u \in C^{2,1}(\Omega)$ and $Hu \geq 0$ (resp. $Hu \leq 0$) in $\Omega$, we say that $u$ is subcaloric (resp. supercaloric).

A distribution $u \in \mathcal{D}'(\Omega)$ is called weakly caloric (resp. weakly subcaloric or weakly supercaloric), if it satisfies
\begin{equation}
\int_{\Omega} u H^* \varphi = 0, \quad (\text{resp. } \geq 0 \text{ or } \leq 0),
\end{equation}
for all $\varphi \in C^{2,1}(\Omega)$ (resp. $0 \leq \varphi \in C^{2,1}(\Omega)$). If $u$ is a supertemperature (resp. subttemperature), then it is weakly supercaloric (resp. weakly subcaloric) (see [Wa12, Definition 3.7, Theorem 6.28]). Moreover, by [Wa12, Theorem 6.28], if $u$ is a subcaloric in $\Omega$, then there exists a unique non-negative Radon measure $\mu_u$, which is called the Riesz measure associated with $u$, such that
\begin{equation}
\int_{\Omega} u H^* \varphi = \int_{\Omega} \varphi d\mu_u, \quad \text{for all } \varphi \in C^{2,1}(\Omega).
\end{equation}

By making the obvious adjustments we obtain the dual definitions for $H^*$.

**Lemma 3.1.** If $\Omega \subset \mathbb{R}^{n+1}$ is an open set, then the following are equivalent:

1. $u \in C^{2,1}(\Omega)$ is caloric in $\Omega$,
2. $u \in \mathcal{D}'(\Omega)$ is weakly caloric in $\Omega$,
3. $u \in C^\infty(\Omega)$ and $Hu = 0$ in $\Omega$.

**Proof.** That (1) implies (2) follows from integration by parts, while (3) implies (1) is trivial. Remark that the heat operator is hypoelliptic (see [Vl02, 15.6, p. 210]) since the heat kernel is the fundamental solution for the heat equation. Thus, the remaining implication is true because of [Vl02, Theorem 1, p. 210]. \qed

Let $f$ be an extended real-valued function defined in $\partial_e \Omega$. The upper class $\mathcal{L}_f$ of $f$ consists of lower bounded hypertemperatures $w$ in $\Omega$ (see [Wa12, Definition 3.10]) such that $w \geq f$ on $\partial_e \Omega$ in the sense that it holds
\begin{equation}
\liminf_{(y,s) \to (\xi,t)} w(y,s) \geq f(\xi,t), \quad \text{for all } (\xi,t) \in \partial_n \Omega,
\end{equation}
and
\begin{equation}
\liminf_{(y,s) \to (\xi,t^+)} w(y,s) \geq f(\xi,t), \quad \text{for all } (\xi,t) \in \partial_{ss} \Omega.
\end{equation}

The lower class $\mathcal{L}_f$ of $f$ consists of upper bounded hypotemperatures $v$ in $\Omega$ such that $v \leq f$ on $\partial_e \Omega$ in the sense that it holds
\begin{equation}
\limsup_{(y,s) \to (\xi,t)} v(y,s) \leq f(\xi,t), \quad \text{for all } (\xi,t) \in \partial_n \Omega,
\end{equation}
and
\begin{equation}
\limsup_{(y,s) \to (\xi,t^+)} v(y,s) \leq f(\xi,t), \quad \text{for all } (\xi,t) \in \partial_{ss} \Omega.
\end{equation}
The function $U_f(x) = \inf\{w \in \Omega : w \in \Omega_f\}$ is called the upper solution for $f$ in $\Omega$, and $L_f(x) = \sup\{v \in \Omega : v \in \Omega_f\}$ is called the lower solution for $f$ in $\Omega$. We say that $f$ is resolutive if $U_f = L_f =: S_f$ and $S_f$ is caloric in $\Omega$. In this case, $S_f$ is called the PWB solution for $f$ in $\Omega$. By [Wa12, Theorem 8.26], any $f \in C(\partial \Omega)$ is resolutive. Therefore, for any $x \in \Omega$, the map $f \mapsto u_f(x)$ is linear and, by the Riesz representation theorem, there exists a unique probability measure $\omega^x$ on $\partial \Omega$, which is called caloric measure, such that

$$u_f(x) = \int_{\partial \Omega} f \, d\omega^x.$$  

More generally, if $f$ is an extended real-valued function and $x \in \Omega$ and $\int_{\partial \Omega} f \, d\omega^x$ exists, then

$$U_f(x) = L_f(x) = \int_{\partial \Omega} f \, d\omega^x.$$

Conversely, if $U_f(x) = L_f(x)$ and is finite, then $f$ is $\omega^x$-integrable and (3.4) holds (see [Wa12, Theorem 8.32]). In the last statement, if we also assume that $f$ is Borel, then we obtain that $f$ is resolutive. In particular, if $E \subset \partial \Omega$ is $\omega^x$-measurable for all $x \in \Omega$, then $\chi_E$ is resolutive and $S_{\chi_E}(x) = \omega^x(E)$ (see [Wa12, Corollary 8.33]). Therefore, if $\omega^x(E) = 0$ for some $x \in \Omega$, then, by minimum principle, $\omega^x(E) = 0$ for all $y \in \Lambda(x, \Omega)$.

If $f \in C(\partial \Omega)$, we say that $u$ is a solution of the classical Dirichlet problem with data $f$ if $u$ is caloric and it holds that

$$\lim_{(y,s) \to (\xi,t)} u(y, s) = f(\xi, t), \quad \text{for all } (\xi, t) \in \partial_n \Omega,$$

and

$$\lim_{(y,s) \to (\xi,t^+)} u(y, s) = f(\xi, t), \quad \text{for all } (\xi, t) \in \partial_s \Omega.$$

If there exists a solution to the classical Dirichlet problem in $\Omega$ with data $f$, then $f$ is resolutive and $u_f$ is the PWB solution for $f$ in $\Omega$ (see [Wa12, Theorem 8.26]).

**Definition 3.2.** A point $\xi_0 \in \partial_n \Omega$ (resp. $\partial_s \Omega$) is called regular for $H$ or $H$-regular if for any $f \in C(\partial \Omega)$, the PWB solution $S_f$ satisfies (3.5) (resp. (3.6)). A point $\xi_0 \in \partial \Omega$ that is not $H$-regular is called $H$-irregular.

Let $G_\Omega(\cdot, \cdot)$ be the non-negative real-valued function defined in $\Omega \times \Omega$ as

$$G_\Omega(x, \bar{p}) = \Gamma(x - \bar{p}) - h_\Omega(x, \bar{p}),$$

where $h_\Omega(\cdot, \bar{p})$ is the greatest thermic minorant of $\Gamma(\cdot - \bar{p})$, which is non-negative (see [Wa12, Definition 3.65] and the paragraph before this definition). We call $G(\cdot, \bar{p})$ the Green function in $\Omega$ with pole at $\bar{p} \in \Omega$ and, by [Wa12, Theorem 8.53], if $\psi_\bar{p} = \Gamma(\cdot - \bar{p})|_{\partial \Omega}$, we also have the representation

$$G_\Omega(x, \bar{p}) = \Gamma(x - \bar{p}) - S_{\psi_\bar{p}}(x) = \Gamma(x - \bar{p}) - \int_{\partial \Omega} \Gamma(\bar{y} - \bar{p}) \, d\omega^x(\bar{y}),$$

where the last equality follows from the fact that $\Gamma(\cdot, \bar{p})$ is continuous on $\partial \Omega$ and thus resolutive. It is straightforward to see that

$$G_\Omega(x, \bar{y}) \leq 2C \pi^{-n/2} \|x - \bar{y}\|^{-n}.$$
In fact, more is true: if \( C_r := B_r \times (0, r) \) is a cylinder of radius \( r > 0 \), then there exists \( c_1 > 0 \) and \( c_2 > 0 \) (independent of \( r \)) such that
\[
(3.8) \quad c_1 m(\bar{x}, \bar{y}) \Gamma_{c_2}(\bar{x} - \bar{y}) \leq G_{C_r}(\bar{x}, \bar{y}) \leq c_1^{-1} m(\bar{x}, \bar{y}) \Gamma_1(\bar{x} - \bar{y}), \quad \text{for } \bar{x} \neq \bar{y} \in C_r,
\]
where
\[
m(\bar{x}, \bar{y}) := \min \left( 1, \frac{\dist(x, B_r)}{\sqrt{t-s}} \right) \min \left( 1, \frac{\dist(y, B_r)}{\sqrt{t-s}} \right).
\]
See e.g. [Zh02, Theorem 1.2] and [Cho06, Theorem 3.3].

Note that \( G_\Omega(\cdot, \cdot) \) is lower semi-continuous on the diagonal \( \{(\bar{p}, \bar{p}) : p \in \Omega \} \) and continuous everywhere else in \( \Omega \). In fact, it is a supertemperature in \( \Omega \) and a temperature in \( \Omega \setminus \{\bar{p}\} \). Recall that \( \Lambda^*(\bar{p}, \Omega) \) is the set of points \( \bar{x} \in \Omega \) for which there is a polygonal path \( \gamma \subset \Omega \) joining \( \bar{p} \) to \( \bar{x} \), along which the time variable is strictly increasing. By [Wa12, Theorem 6.7], it holds that \( G_\Omega(\cdot, \bar{p}) > 0 \) in \( \Lambda^*(\bar{p}, \Omega) \) and \( G_\Omega(\cdot, \bar{p}) = 0 \) in \( \Omega \setminus \Lambda^*(\bar{p}, \Omega) \), for any \( \bar{p} \in \Omega \). If \( \bar{x} \in \partial_n \Omega \) (resp. \( \partial_{ss} \Omega \)) is a regular point for \( H \), we have that \( \lim_{\bar{x} \to \bar{p}} G_\Omega(\bar{x}, \bar{p}) = 0 \) (resp. \( \lim_{\bar{x} \to \bar{p}} G_\Omega(\bar{x}, \bar{p}) = 0 \)). When the domain \( \Omega \) where the Green function is defined is clear from the context, we will drop the subscript and just write \( G(\cdot, \cdot) \).

Let us remark that the Riesz measure associated with the Green function \( G_\Omega(\bar{x}, \bar{y}) \) in \( \Omega \) is the caloric measure \( \omega^{\bar{x}}_{\bar{y}} \) in \( \Omega \) (see [Do84]), i.e.,
\[
(3.9) \quad \int G_\Omega(\bar{x}, \bar{y}) H \varphi(\bar{y}) \, d\bar{y} = \int \varphi \, d\omega^{\bar{x}}_{\bar{y}}, \quad \text{for any } \varphi \in C^\infty_c(\mathbb{R}^{n+1}).
\]

Given a Borel measure \( \mu \) on \( \mathbb{R}^{n+1} \) and an open set \( \Omega \subset \mathbb{R}^{n+1} \),
\[
G_\Omega \mu(\bar{x}) := \int G_\Omega(\bar{x}, \bar{y}) \, d\mu(\bar{y})
\]
defines a non-negative supertemperature in \( \Omega \) provided the integral is finite in a dense subset of \( \Omega \) and is called the heat potential of \( \mu \) in \( \Omega \). If \( \Omega = \mathbb{R}^{n+1} \), we replace \( G_\Omega \) by \( \Gamma \).

Given a compact set \( K \subset \mathbb{R}^{n+1} \), we define its thermal capacity as
\[
(3.10) \quad \text{Cap}(K, \Omega) := \sup \{ \mu(K) : G_\Omega \mu \leq 1 \text{ in } \Omega, \mu \geq 0 \text{ and } \text{supp} \mu \subset K \}.
\]
Equivalently, if \( u = \tilde{R}_1(K, \Omega) \) is the smoothed reduction 1 over \( K \) in \( \Omega \) (see [Wa12, Definition 7.23]) and \( \mu_u \) is the associated Riesz measure, then \( \text{Cap}(K, \Omega) = \mu_u(K) \) (see [Wa12, Definition 7.33]). By [Wa12, Theorem 7.28],
\[
u = \tilde{R}_1(K, \Omega) = G_\Omega \mu_u.
\]
It is easy to see that if \( K \subset \Omega_1 \subset \Omega_2 \subset \mathbb{R}^{n+1} \) are open sets, then
\[
(3.11) \quad \text{Cap}(K, \mathbb{R}^{n+1}) \leq \text{Cap}(K, \Omega_2) \leq \text{Cap}(K, \Omega_1).
\]

This definition can be extended to arbitrary sets \( S \subset \mathbb{R}^{n+1} \): the inner thermal capacity of \( S \) is given by
\[
(3.12) \quad \text{Cap}_-(S, \Omega) = \sup \{ \text{Cap}(K, \Omega) : S \supset K \text{ compact} \},
\]
and its outer thermal capacity by
\[
(3.13) \quad \text{Cap}_+(S, \Omega) = \inf \{ \text{Cap}_-(D, \Omega) : S \subset D \text{ open} \}.
\]

\(^1\)The constants depend on the geometry but not the diameter of the domain. For cylinders, the constants are only dimensional.
If the inner and the outer thermal capacities of $S$ coincide, we say that $S$ is capacitable and we denote $\text{Cap}(S, \Omega) := \text{Cap}_-(S, \Omega) = \text{Cap}_+(S, \Omega)$. In the case $\Omega = \mathbb{R}^{n+1}$, we simply write $\text{Cap}(S)$. If $E$ is a Borel set, then, by [Wa12, Theorems 7.15 and 7.49], it is capacitable.

A set $Z \subset \mathbb{R}^{n+1}$ is called polar if there is an open set $\Omega \supset Z$ and a supertemperature $w$ on $\Omega$ such that $Z \subset \{ \tilde{x} \in \Omega : w(\tilde{x}) = +\infty \}$. Observe that if $Z_1 \subset Z$ and $Z$ is polar, then $Z_1$ is polar itself, and a set $S \subset \mathbb{R}^{n+1}$ is polar if and only if $\text{Cap}(S) = 0$ (see [Wa12, Theorem 7.46]). Moreover, if a set is contained in a horizontal hyperplane its capacity is equal to its $n$-dimensional Lebesgue measure (see [Wa12, Theorem 7.55]). For more properties of polar sets we refer the reader to [Wa12, Chapter 7].

An open set $\Omega$ is quasi-regular (resp. regular) if the set of irregular points of $\partial_e \Omega$ is polar (resp. empty). We remark here that there are domains so that the capacity of the set of the irregular points of $\partial_e \Omega$ for the heat equation is positive (see [TW85, p. 336]). This comes in contrast to the corresponding result in elliptic potential theory, where the set of irregular boundary points has zero capacity and thus, is polar.

**Lemma 3.3.** If $C_r$ is a cylinder of radius $r$, there exists a dimensional constant $c_1 > 0$ such that

\[
\text{Cap}(\overline{C_r}, C_{2r}) \leq c_1 r^n.
\]

**Proof.** Let $\varphi \in C^\infty_c(C_{2r})$ such that $\varphi = 1$ in $\overline{C_r}$, $\varphi \geq 0$, and $|H^s \varphi| \leq c'_n r^{-2}$. Then, if $\mu_u$ is the Riesz measure associated with $u = \tilde{R}_1(\overline{C_r}, C_{2r})$, since $0 \leq u \leq 1$, it holds

\[
\text{Cap}(\overline{C_r}, C_{2r}) = \mu_u(\overline{C_r}) \leq \int \varphi \, d\mu_u = \int u \, H^s \varphi \leq c'_n r^{-2} |C_{2r}| = c_n r^n. \quad \square
\]

**Lemma 3.4.** Let $C_r$ be a cylinder of radius $r$ centered at $\xi_0 \in \mathbb{R}^{n+1}$ and let $K \subset \overline{C_r}$ be a compact set such that $\xi_0 \in K$. If $s \in (0, 2]$, then there exists $c_2 > 0$ (independent of $K$) such that

\[
\text{Cap}(K, C_{2r}) \geq c_2 \frac{\mathcal{H}^{n+s}_{p, \infty}(K)}{\min(\text{diam}(K), r)^s}.
\]

**Proof.** If we set $\rho = \min(\text{diam}(K), r)$, it holds that $K \subset C_\rho \cap C_r$. By the parabolic version of Frostman’s lemma, whose proof is analogous to the Euclidean one and is omitted (see e.g. [Mat95, Lemma 8.8]), we can find a Borel measure $\mu'$ supported on $K$, such that

- $\mu'(C_r(\tilde{y})) \lesssim r^{n+s}$, for any $r > 0$ and $\tilde{y} \in \mathbb{R}^{n+1}$, and
- $\mathcal{H}^{n+s}_{p, \infty}(K) \geq \mu'(K) \geq c \mathcal{H}^{n+s}_{p, \infty}(K),$

where $c_1$ depends on $n$.

Let $G_{2r}$ be the Green function in $C_{2r}$ and define $\mu = \mu' \rho^{-s}$. We claim that the corresponding heat potential $G_{2r, \mu}(\tilde{x}) \lesssim 1$ for any $\tilde{x} \in C_{2r}$. Indeed, notice that

\[
G_{\mu}(\tilde{x}) = \int_{|\tilde{y} - \tilde{x}| \leq \rho} G_{2r}(\tilde{x}, \tilde{y}) \, d\mu(\tilde{y}) + \int_{|\tilde{y} - \tilde{x}| > \rho} G_{2r}(\tilde{x}, \tilde{y}) \, d\mu(\tilde{y}) =: I_1 + I_2.
\]
Using (3.8) along with that \( \mu' \) is supported on \( K \) and has \((n+s)\)-growth, we have that
\[
I_2 \lesssim \int_{\|\vec{y} - \vec{x}\| > \rho} \|\vec{y} - \vec{x}\|^{-n} d\mu(\vec{y}) 
\lesssim \rho^{-n} \mu'(K) \rho^{-s} \leq \mu'(C_\rho) \rho^{-n-s} \lesssim 1.
\]
If \( A_k(\vec{x}) := C_{2-k\rho}(\vec{x}) \setminus C_{2-k-1\rho}(\vec{x}) \), arguing as above, we infer that
\[
I_1 \lesssim \sum_{k \geq 0} \int_{A_k(\vec{x})} (2^{-k-1}\rho)^{-n} d\mu(\vec{y}) \lesssim \sum_{k \geq 0} (2^{-k-1}\rho)^{-n} \rho^{-s} (2^{-k}\rho)^{n+s} \lesssim 1,
\]
concluding the proof of the claim.

If we normalize \( \mu \) so that \( G_{2r}\mu(\vec{x}) \leq 1 \), we deduce that \( \mu \) is an admissible measure for the definition of thermal capacity on compact sets. Therefore,
\[
\text{Cap}(K, C_{2r}) \geq \mu(K) \gtrsim \rho^{-s} \mathcal{H}^{n+s}_{p,\infty}(K),
\]
and the proof of the lemma is now complete. \( \square \)

**Corollary 3.5.** Let \( 0 < b < a \leq 1 \) and \( r > 0 \), and for \( \vec{\xi}_0 = (\xi_0, t_0) \in \mathbb{R}^{n+1} \) we set
\[
R_{a,b}(\vec{\xi}_0; r) = B(\vec{\xi}_0, r) \times (t_0 - (ar)^2, t_0 - (br)^2).
\]
Then, we have that
\[
(3.16) \quad \text{Cap}(\overline{R_{a,b}(\vec{\xi}_0; r)}, C_{2r}(\vec{\xi}_0)) \approx r^n,
\]
where the implicit constant depends on \( n, a, \) and \( b \).

**Proof.** The upper bound follows from Lemma 3.3. For the lower bound, if \( s = n + 2 \) and \( K = \overline{R_{a,b}(\vec{\xi}_0; r)} \), the proof of Lemma 3.4 goes through unchanged if instead of the Frostman’s measure we use \( \mathcal{H}^{n+2}_{p} \), which, in turn, is equal to a constant multiple of the \((n+1)\)-dimensional Lebesgue measure \( \mathcal{L}^{n+1} \), showing that
\[
(3.17) \quad \text{Cap}(K, C_{2r}(\vec{\xi}_0)) \gtrsim \mathcal{L}^{n+1}(K)/r^2 \approx_{a,b,n} r^n.
\]
As \( a \) and \( b \) are arbitrary, we may approximate \( R_{a,b}(\vec{\xi}_0; r) \) by a sequence of compact subsets of the form \( \overline{R_{a_k,b_k}(\vec{\xi}_0; r)} \) and obtain our result. \( \square \)

For \( \vec{x} = (x, t) \in \mathbb{R}^{n+1} \), we define the heat ball centered at \( \vec{x} \) with radius \( \rho > 0 \) to be the set
\[
(3.18) \quad E(x, t; \rho) := \{ \vec{y} \in \mathbb{R}^{n+1} : \Gamma(\vec{x} - \vec{y}) > (4\pi \rho)^{-n/2} \}
= \{ (y, s) \in \mathbb{R}^{n+1} : |x - y| < \sqrt{2n(t - s) \log \left( \frac{\rho}{t - s} \right)}, t - \rho < s < t \}.
\]
It is not hard to see that \( E(x, t; \rho) \) is a convex body, axially symmetric about the line \( \{x\} \times \mathbb{R} \), and
\[
E(\vec{x}; \rho) \subset B\left(x, \sqrt{\frac{2n\rho}{e}} \right) \times (t - \rho, t).
\]
Let us set
\[
A(\vec{x}; \rho/2, \rho) = E(\vec{x}; \rho) \setminus E(\vec{x}; \rho/2)
\]
to be the closed heat annulus of radius $\rho$. It is clear that if $\bar{y} \in A(\bar{\xi}_0; \rho/2, \rho)$ satisfies $0 < t - s < \rho/2$, then it holds that

$$(3.19) \quad 2n(t - s) \log \left( \frac{\rho}{t - s} \right) - [2n \log 2](t - s) \leq |x - y|^2 \leq 2n(t - s) \log \left( \frac{\rho}{t - s} \right).$$

**Remark 3.6.** Note that if $\rho = r^2$, $\alpha \in (0, 1)$, and $t - s = \alpha r^2$, then the region given by (3.19) is an annulus in the spatial variable which can be covered by at most $c'_{n/2}$ many (spatial) cubes of sidelenath $\sqrt{\alpha r}$.

There are parabolic analogues of the Wiener criterion that determine whether a point $\bar{\xi} \in \partial_n \Omega$ is regular in terms of the capacity of the complement of $\Omega$ that lies either in heat balls centered at $\bar{\xi}_0$ or in time-backwards parabolic cylinders $R_{ar}(\bar{\xi}_0)$. To be precise, $\bar{\xi}_0 \in \partial_n \Omega$ is regular if and only if

$$(3.20) \quad \int_0^1 \frac{\text{Cap}(A(\bar{\xi}_0; \rho/2, \rho) \cap \Omega^c)}{\rho^{n/2}} \frac{d\rho}{\rho} = \infty.$$

The necessity was proved by Evans and Gariepy [EG82, Theorem 1] and the sufficiency by Lanconelli [Lanc73].

**Remark 3.7.** In [EG82] and [Lanc73], the authors do not specify that this criterion only works for points on $\partial_n \Omega$, although it is clear this is the case. Observe that, by definition, if $\bar{\xi}_0 \in \partial_n \Omega$, there exists $\rho > 0$ small enough so that $E(\bar{\xi}_0; \rho) \cap \Omega^c = \emptyset$ and thus, the integral in (3.21) clearly converges. Nevertheless, Watson [Wa14, Theorem 4.1] proved that a point $\bar{\xi}_0 \in \partial_s \Omega$ is regular if and only if there exists $r_0 > 0$ such that $H(\bar{\xi}_0, r_0)$ is a connected component of $\Omega \cap B(\bar{\xi}_0, r_0)$, where $B(\bar{\xi}_0, r_0) = \{(x, t) : |x - \bar{\xi}_0| + |t - t_0|^2 < r_0^2\}$ and $H(\bar{\xi}_0, r_0) = B(\bar{\xi}_0, r_0) \cap \{t < t_0\}$.

$$(3.21) \quad \int_0^1 \frac{\text{Cap}(E(\bar{\xi}_0; \rho) \cap \Omega^c)}{\rho^{n/2}} \frac{d\rho}{\rho} = \infty.$$

This follows from (3.20) and [Br90, Theorem 1.13]. Alternatively, $\bar{\xi}_0$ is regular if and only if

$$(3.22) \quad \sum_{k \geq 0} \lambda^{-kn} \text{Cap}(A(\bar{\xi}_0; \lambda^{-k-1}, \lambda^{-k}) \cap \Omega^c) = \infty,$$ for some $\lambda > 1$. In fact if $\bar{\xi}_0$ is regular, then (3.22) holds for all $\lambda > 1$.

$\text{Lemma 3.8.}$ Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and $\bar{\xi}_0 = (\xi_0, t_0) \in \partial_n \Omega$. If

$$\text{Cap}(E(\bar{\xi}_0; r^2) \cap \Omega^c) \geq 2cr^{n},$$

then there exists $a \in (0, 1/2)$ depending on $n$ such that

$$(3.23) \quad \text{Cap}(E(\bar{\xi}_0; r^2) \cap \Omega^c \cap \{t < t_0 - (ar)^2\}) \geq cr^n.$$

**Proof.** If we set

$$E_k := \{ \bar{x} = (x, t) \in E(\bar{\xi}_0; r^2) : 2^{-2(k+1)}r^2 \leq t_0 - t \leq 2^{-2(k+1)}r^2 \},$$

A different necessary and sufficient condition is given by [Land69].
then, for any $\tilde{x} \in E_k$, it holds
\[
|x - \xi_0| < 2^{-k} r \sqrt{2n \log \left( \frac{r^2}{2 - (2k+1)^2} \right)} \leq \sqrt{2n \log 2} \frac{\sqrt{k}}{2^k} r.
\]
Thus, we can cover $E_k$ by at most $c(n)k^{n/2}$ number of cylinders $C_j(k)$ of radius $2^{-k} r$, which, in view of the subadditivity of capacity and Lemma 3.3, infers that
\[
\text{Cap}(E_k) \leq \sum_{j \geq 1} \text{Cap}(C_j(k)) \leq c(n)c_1 k^{n/2} 2^{-kn} r^n.
\]
Therefore, if we set $E_M := \bigcup_{k \geq M} E_k$, we have that
\[
\text{Cap}(E_M) \leq \sum_{k \geq M} \text{Cap}(E_k) \leq \tilde{c}(n) \sum_{k \geq M} k^{n/2} 2^{-kn} r^n.
\]
Since $\sum_{k \geq 1} k^{n/2} 2^{-kn}$ converges, we can find $M > 0$ depending on $n$ so that $\text{Cap}(E_M) \leq cr^n$, which implies (3.23) for $a = 2^{-M}$. \hspace{1cm} \Box

Similarly we can prove the following lemma.

**Lemma 3.9.** If $\Omega \in \mathbb{R}^{n+1}$ is an open set and $\bar{\xi}_0 = (\xi_0, t_0) \in \partial_n \Omega$ is a regular point, then there exists a sequence $M_k > 1$ such that
\[
\sum_{k \geq 0} 2^{-kn} \text{Cap}(A(\bar{x}_0, 2^{-2(k+1)^2}, 2^{-2k}) \cap \{ t < t_0 - (2^{-M_k} 2^{-k})^2 \cap \Omega^c \}) = \infty.
\]

Given $a \in (0, 1)$, we define the translated time-backwards cylinder
\[
R_a^-(\bar{x}; r) := C_r^-(x, t - (ar)^2) = B(x, r) \times (t - r^2, t - (ar)^2).
\]

**Lemma 3.10.** If $a \in (0, 1)$ and $\bar{\xi}_0 = (\xi_0, t_0) \in \mathbb{R}^{n+1}$, it holds that
\[
R_a^-(\bar{\xi}_0; r) \subset E(\bar{\xi}_0; \rho), \quad \text{for } \rho \geq e^{\frac{\lambda}{2n}} r^2.
\]

**Proof.** Let $R' := B(\xi_0, r) \times \{ t_0 - (ar)^2 \}$ and $R'' := B(\xi_0, r) \times \{ t_0 - r^2 \}$. By the definition (3.18), we have that $R' \subset E(\xi_0; \rho)$ if and only if
\[
\rho \geq a^2 \exp \left( \frac{1}{2na^2} \right) r^2 =: \tilde{c}_1 r^2,
\]
and $R'' \subset E(\xi_0; \rho)$ if and only if
\[
\rho \geq \exp \left( \frac{1}{2n} \right) r^2 =: \tilde{c}_2 r^2.
\]
Remark that, for any fixed $\lambda > 0$, the function $ye^{\lambda y}$ is increasing for $y > -\lambda^{-1}$, and thus, $\tilde{c}_2 > \tilde{c}_1$. Hence, if $\rho \geq \tilde{c}_2 r^2$, we have that $R' \cup R'' \subset E(\bar{\xi}_0; \rho)$ and, by the convexity of $E(\bar{\xi}_0; \rho)$, we get (3.26) for $\tilde{c} = \tilde{c}_2$. \hspace{1cm} \Box

As backwards cylinders $R_a^-(\bar{\xi}_0; r)$ appear more naturally in applications, it is interesting to know a version of the Wiener’s criterion with $R_a^-(\bar{\xi}_0; r)$ instead of heat balls. This can be obtained using the following lemma (compare it with [GZ82, Theorem 3.1]).
Lemma 3.11. Let $\Omega$ be an open set and $\xi_0 \in \partial_n \Omega$. If there exists a constant $a > 0$ such that
\[
\int_0^1 \frac{\text{Cap}(R_a(\xi_0; r) \cap \Omega^c)}{\text{Cap}(R_a(\xi_0; r))} \frac{dr}{r} = \infty,
\]
then $\xi_0$ is regular. Conversely, if $\xi_0 \in \partial_n \Omega$ is regular, then there exists a function $a : (0, 1) \to (0, \frac{1}{2})$ so that (3.27) holds for $R_a(\xi_0; r)$.

\[\text{Proof.}\] For $\xi_0 \in \partial_n \Omega$, we have that
\[
\int_0^\varepsilon \frac{\text{Cap}(E(\xi_0; \rho) \cap \Omega^c)}{\rho^{n/2}} \frac{d\rho}{\rho} = \frac{2}{\tilde{c}^{n/2}} \int_0^1 \frac{1}{r} \text{Cap}(E(\xi_0; \tilde{c}r^2) \cap \Omega^c) \frac{dr}{r}
\]
where $\tilde{c}$ is given in the proof of Lemma 3.10. The converse direction follows from Lemma 3.9.

Lemma 3.12. If $\Omega$ has the TBCPC at $\xi_0 \in \partial_n \Omega$, then (3.27) is satisfied and $\xi_0$ is regular.

\[\text{Proof.}\] If $\Omega$ satisfies the TBCPC at $\xi_0$, by Lemma 3.8, there exists $a \in (0, 1/2)$ so that $a \approx 1$ and
\[
\int_0^1 \frac{\text{Cap}(R_a(\xi_0; r) \cap \Omega^c)}{r^n} \frac{dr}{r} \geq \sum_{j \geq 1} \int_{r_j}^{2r_j} \frac{\text{Cap}(R_a(\xi_0; r) \cap \Omega^c)}{r^n} \geq \sum_{j \geq 1} 1 = \infty,
\]
where $r_j \to 0$ is the sequence in the definition of TBCPC. Thus, (3.27) holds and $\xi_0$ is regular.

We will now introduce a class of regular domains that has played an important role in (free) boundary value problems for harmonic and elliptic measure.

Let $\xi_0 \in \mathcal{S}\Omega$. If there exists $a \in (0, 1]$ and $c > 0$ such that
\[
\frac{\text{Cap}(R_a(\xi_0; r) \cap \Omega^c)}{\text{Cap}(R_a(\xi_0; r))} \geq c, \text{ for all } 0 < r < \sqrt{(t_0 - T_{min})/2},
\]
then we say that $\Omega$ has the time backwards cylindrical capacity density condition at the point $\xi_0$. Although the TBCDC looks more general, because of Lemmas 3.10 and 3.8, we can see that the two conditions are in fact equivalent. Note that, by Lemma 3.11 and Remark 3.13, if $\Omega$ has the TBCDC at the point $\xi_0 \in \mathcal{S}\Omega$, then $\xi_0$ is a regular point and belongs to $\partial_n \Omega$.

Remark 3.13. If $\Omega$ satisfies the TBCDC, then it clearly holds that $\partial_s \Omega \cap \mathcal{S}\Omega = \partial_s \Omega \cap \mathcal{S}\Omega = \emptyset$ and
\[
\mathcal{S}\Omega = \partial_n \Omega \cap \mathcal{S}\Omega = \partial_s \Omega \cap \mathcal{S}\Omega.
\]
Moreover, the range of $r$ in (3.28) is chosen so that, if $\bar{x} = (x, t) \in \mathcal{S}\Omega$ and $\bar{y} \in C_r(\bar{x}) \cap \Omega$, then
\[
\text{dist}_p(\bar{y}, \mathcal{S}\Omega) = \text{dist}_p(\bar{y}, \partial_s \Omega).
\]
Indeed, if we choose $r > 0$ so that $r < \sqrt{t - T_{\min}} - r^2$, we have
\[
\text{dist}_p(\bar{y}, \mathcal{S} \Omega) \leq \text{dist}_p(\bar{y}, \bar{x}) < r < \sqrt{t - T_{\min}} - r^2
\]
\[
= \text{dist}_p(\partial C_r(\bar{x}), \mathcal{B}\Omega|T_{\min}) \leq \text{dist}_p(\bar{y}, \mathcal{B}\Omega|T_{\min}).
\]
Recalling that $R_{ar}^{-}(\xi_0) = C_r(\xi_0, t_0 - (ar)^2)$, we obtain the range of $r$ in (3.28).

A crucial property of our definitions of TBCDC is their invariance under parabolic scaling.

**Lemma 3.14.** Let $\Omega$ be a domain that satisfies the TBCDC with constants $a, c$ as in (3.28). Let $\xi = (\xi, \tau) \in \mathcal{S} \Omega$, $\rho > 0$ and denote $\tilde{\Omega} := T_{\xi, \rho}[\Omega]$. Then $\tilde{\Omega}$ satisfies the TBCDC with the same parameters.

**Proof.** Let us observe that for $\tilde{\zeta} \in \mathcal{S} \tilde{\Omega}$ and every $r > 0$ we have that
\[
C_{r/\rho}^{-}(\tilde{\zeta}) \cap \tilde{\Omega}^c = T_{\xi, \rho}(C_r^{-}(\tilde{\zeta}) \cap \Omega^c)
\]
and $\Gamma(\delta_{\rho} \bar{x}) = \rho^{-n}\Gamma(\bar{x})$. We claim that
\[
\text{Cap}(C_{r/\rho}^{-}(\tilde{\zeta}) \cap \tilde{\Omega}^c) = \rho^{-n}\text{Cap}(C_r^{-}(\tilde{\zeta}) \cap \Omega^c).
\]
Set $K = C_r^{-}(T_{\xi, \rho}^{-1}(\tilde{\zeta})) \cap \Omega^c$ and $\tilde{K} = C_{r/\rho}^{-}(\tilde{\zeta}) \cap \tilde{\Omega}^c$, and let $\mu$ be the unique Radon measure supported in $\tilde{K}$ such that $\mu(\tilde{K}) = \text{Cap}(\tilde{K})$. If $\tilde{\mu} = T_{\xi, \rho}^{-1}\mu$, then it is clear that $\text{supp} \; \tilde{\mu} \subset K$.
Moreover, since $G\mu \leq 1$ in $\mathbb{R}^{n+1}$, it holds that
\[
G\tilde{\mu}(\bar{x}) = \int \Gamma(\bar{x} - T_{\xi, \rho}^{-1}(\bar{y})) \, d\mu(\bar{y}) = \rho^{-n} \int \Gamma(T_{\xi, \rho}\bar{x} - \bar{y}) \, d\mu(\bar{y}) \leq \rho^{-n}.
\]
Therefore, the measure $\rho^n \tilde{\mu}$ is an admissible measure for $\text{Cap}(K)$ and thus,
\[
\rho^n \text{Cap}(\tilde{K}) = \rho^n \mu(\tilde{K}) = \rho^n \tilde{\mu}(K) \leq \text{Cap}(K).
\]
The proof of the converse inequality is similar and we omit it. This proves (3.31), which, in turn, is valid if we replace the backwards in time cylinders $C_{r/\rho}^{-}$ by $R_{ar}^{-}(\zeta; r/\rho)$, the truncated cylinders defined in (3.25). Thus, the TBCDC for $\tilde{\Omega}$ readily follows from the TBCDC for $\Omega$. \hfill \Box

The next lemma was proved in [GH20, Lemma 2.2] in the particular case of domains with Ahlfors-David regular boundaries that, in addition, satisfy the time-backwards Ahlfors-David regular condition. In light of (3.15), those domains satisfy the TBCDC.

**Lemma 3.15.** Let $\omega^\x$ be the caloric measure in $\Omega$ with pole at $\bar{x} = (x, t) \in \Omega$. If $\tilde{\xi}_0 = (\xi_0, t_0) \in \mathcal{P} \Omega \cap \mathcal{S} \Omega$, $r > 0$, and $a \in (0, 1/2)$, we define
\[
\tilde{R}_a^+ (\xi_0; r) = B(\xi_0, r) \times (t_0 - (ar)^2/2, t_0 + r^2).
\]
Then there exists $M_0 > 1$ depending on $n$ and $a$, such that for any $\bar{x} \in \tilde{R}_a^+ (\xi_0; r) \cap \Omega$,
\[
\omega^\x\left(C_{M_0^a}(\bar{x}_0) \cap \{|t - t_0| < r^2\}\right) \geq \frac{\text{Cap}(\tilde{R}_a^+ (\xi_0; r) \cap \Omega^c)}{r^n},
\]
where the implicit constant depends only on \(n\) and \(a\). If \(\Omega\) satisfies the TBCDC at \(\xi_0\), then there exists \(a \in (0, 1/2)\) and \(c = c(n, a) > 0\) such that
\[
\omega^2(C_r(\xi_0)) \geq c, \quad \text{for } \bar{x} \in \overline{R_a^+(\xi_0; r/M_0)} \cap \Omega, \quad \text{and } 0 < r < M_0\sqrt{(t_0 - T_{\min})/2}.
\]

Proof. Without loss of generality, we may assume that \(\xi_0 = \bar{0} \in S\Omega\) and set \(K = \overline{R_a^-(0; r)} \cap \Omega^c\). For notational convenience, we will just write \(R_{a,r}^-\) and \(\widehat{R}_{a,r}^+\). We also assume that \(\text{Cap}(\overline{R_{a,r}^-} \cap \Omega^c) > 0\) since otherwise (3.33) is trivially true.

By the definition of capacity on compact sets, there exists a unique positive Radon measure \(\mu\) supported on \(K\) such that \(\|G\mu\|_{L^\infty} \leq 1\) and \(\|\mu\| = \text{Cap}(K)\). Since for any \(\bar{x} \in \overline{R_{a,r}^-}\) and \(\bar{y} = (y, s) \in \widehat{R}_{a,r}^+\), it holds that
\[
a^2r^2/2 \leq t - s \leq 2r^2 \quad \text{and} \quad |x - y| \leq 2r,
\]
we infer that
\[
\Gamma(x - y) \geq (8\pi r^2)^{-n/2}e^{-2a^2} =: c_1 r^{-n},
\]
which, in turn, implies that
\[
G\mu(\bar{x}) \geq c_1 \|\mu\| r^{-n}, \quad \text{for any } \bar{x} \in \widehat{R}_{a,r}^+.
\]
Moreover, if \(\bar{x} \in \mathbb{R}^{n+1} \setminus C_{M_0}\) and \((y, s) \in R_{a,r}^-\), we have that
\[
\|\bar{x} - \bar{y}\| \geq \|\bar{x}\| - \|\bar{y}\| \geq M_0r - r =: (M_0 - 1)r,
\]
and thus, by (3.2),
\[
G\mu(\bar{x}) \leq ((M_0 - 1)^\sqrt{\pi} r^{-n}\|\mu\| r^{-n}, \quad \text{for any } \bar{x} \in \mathbb{R}^{n+1} \setminus C_{M_0}.
\]
Define \(u = G\mu - ((M_0 - 1)^\sqrt{\pi} r^{-n}\|\mu\| r^{-n}\) and choose \(M_0\) so that \(2((M_0 - 1)^\sqrt{\pi})^{-n} \leq c_1\).

Then, since \(\|G\mu\|_{L^\infty} \leq 1\), the following hold:

1. \(u\) is continuous in \(\mathbb{R}^{n+1}\) and \(Hu = 0\) in \(\Omega\),
2. \(u \leq 1\) in \(\mathbb{R}^{n+1}\),
3. \(u \leq 0\) in \(\mathbb{R}^{n+1} \setminus C_{M_0} \cup [\mathbb{R}^n \times \{t \leq r^2\}]\),
4. \(u \geq \frac{c_1}{2}\|\mu\| r^{-n}\) in \(\overline{R}_{a,r}^-\),

Since \(F := C_{M_0} \cap [\mathbb{R}^n \times \{t > -r^2\}] \cap \partial_\Omega\) is Borel, we have that \(\chi_F\) is resolutive, \(u \in \mathcal{L}_{\chi_F}\), and \(\omega^2(F) = S_{\chi_F}(\bar{x})\). Thus, \(u(\bar{x}) \leq \omega^2(F)\) for all \(\bar{x} \in \Omega\) which, by the item (4) above and the fact that \(\text{supp} \omega^2 \subset \partial_\Omega \cap [\mathbb{R}^n \times \{t < r^2\}]\) for any \(\bar{x} \in \overline{R}_{a,r}^-\), proves (3.33).

It is straightforward to see that, by the definition of TBCDC, (3.34) follows from the latter estimate.

Remark 3.16. Lemma 3.15 holds for the adjoint caloric measure if we replace the cylinders \(R_a^-(\xi_0, r)\) and \(\widehat{R}_a^+(\xi_0, r)\) with their reflections across the hyperplane passing through their centers orthogonal to the time axis.

If \(\Omega\) is regular there is the above lemma has a much easier proof based on the reduction function. Also, the dilation factor \(M_0\) need not be taken large enough and just \(M_0 = 2\) does the job.
Lemma 3.17. Let \( \Omega \subset \mathbb{R}^{n+1} \) be open, \( \xi_0 \in \mathcal{P} \Omega \cap \partial \Omega \), \( a \in (0,1/2) \), \( \beta \geq 2 \), and \( K = \overline{R_a(\xi_0; r)} \cap \Omega^c \). If we set \( u = \tilde{R}_1(K, C_{\beta r}(\xi_0)) \), then for any \( \tilde{x} \in \overline{R_a^+(\xi_0; r)} \cap \Omega^c \)
\[ u(\tilde{x}) \gtrsim \text{Cap}(K, C_{\beta r}(\xi_0)) / r^n \gtrsim \text{Cap}(K) / r^n. \] (3.35)

If \( w \) is a non-negative subtemperature in \( C_{\beta r}(\xi_0) \) so that \( w = 0 \) on \( K \) and \( w \leq 1 \) in \( C_{\beta r}(\xi_0) \), there exists \( \kappa \in (0,1) \) such that
\[ w(\tilde{x}) \leq 1 - \kappa \text{Cap}(K, C_{\beta r}(\xi_0)) / r^n \leq 1 - \kappa \text{Cap}(K) / r^n, \] for \( \tilde{x} \in \overline{R_a^+(\xi_0; r)} \cap \Omega^c \).

If \( S\Omega \cap C_{\beta r}(\xi_0) \subset \mathcal{P}\Omega \) consists of regular points, then (3.34) holds with \( M_0 = 2 \).

Proof. As in the proof of the previous lemma, we assume that \( \xi_0 = 0 \) and use the notation \( R_{a,r} \) and \( \tilde{R}_{a,r} \). Moreover, we assume that \( \text{Cap}(K) > 0 \) since otherwise (3.35) is trivial. We also denote
\[ \tilde{R}_{a,r} = B(0, (1 + 0.25a)r) \times (-2.25a^2r^2, -0.75a^2r^2) \]
and let \( \varphi \in C_c^\infty(\tilde{R}_{a,r}) \) such that \( \varphi = 1 \) in \( R_{a,r} \), and \( |H^* \varphi| \lesssim a_n r^{-2} \). Then, if \( \mu \) is the unique Radon measure such that \( \mu(K) = \text{Cap}(K) \), it holds that
\[ r^{-n} \text{Cap}(K) = r^{-n} \mu(K) \leq r^{-n} \int \varphi \, d\mu = r^{-n} \int u H^* \varphi \lesssim r^{-(n+2)} \int_{\tilde{R}_{a,r}} u \lesssim a_n \inf_{\tilde{R}_{a,r}} u, \]
where in the last step we used the weak Harnack inequality since \( u \) is a non-negative supertemperature and thus, a non-negative supercaloric function (see [Lieb96, Corollary 6.24, p. 128]).

By the definition of \( u \), we have that for any non-negative supertemperature \( v \) in \( C_{\beta r}(\xi_0) \) such that \( v \geq 1 \) on \( K \), it holds that \( v \geq u \) and thus, (3.35) is true for \( v \) as well. Now, if \( w \) is a subtemperature in \( C_{\beta r}(\xi_0) \) so that \( w = 0 \) on \( K \) and \( w \leq 1 \) in \( C_{2r}(\xi_0) \), then \( 1 - w \) is a non-negative supertemperature in \( C_{\beta r}(\xi_0) \) that is identically 1 on \( K \) and (3.36) readily follows. Finally, by the regularity of \( \partial h \Omega \cap C_{\beta r}(\xi_0) \), we may extend \( \omega^\beta(C_{\beta r}(\xi_0)) \) by 1 in \( C_{\beta r}(\xi_0) \setminus \Omega \) and use [Wa12, Theorem 7.20] to show that it is a non-negative supertemperature in \( C_{\beta r}(\xi_0) \) that is 1 on \( K \) and non-negative in \( C_{\beta r}(\xi_0) \) and use (3.35) to show (3.34) for \( M_0 = 2 \). \( \square \)

Remark 3.18. One can show that (3.36) holds for upper semicontinuous weakly subcaloric functions as well. Indeed, if we follow the proof of Lemma 3.15 substituting \( \Gamma(\cdot, \cdot) \) with the Green function \( G_{C_{\beta r}(\xi_0)}(\cdot, \cdot) \) and using (3.8) along with the weak minimum principle\(^3\) (instead of the resolutivity of characteristic functions of Borel sets) we can show that if \( v \) is a non-negative supercaloric function in \( \Omega \cap C_{\beta M^\beta}(\xi_0) \) such that \( \liminf_{x \to \xi} v \geq 1 \) for every \( \xi \in \partial_h \Omega \cap C_{\beta M^\beta}(\xi_0) \), then \( v \) satisfies (3.35). The rest of the proof is the same as before and we skip the details.

\(^3\)The weak minimum principle still holds for lower semicontinuous (instead of continuous in \( \overline{\Omega} \)) supercaloric functions in the form \( \liminf_{x \to \xi \in \partial \Omega} v \leq \inf_{\Omega} v \). See e.g. the proof of [Lieb96, Corollary 6.26].
Lemma 3.19. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, $\xi_0 \in \mathcal{P}\Omega \cap \partial \Omega$, $a \in (0, 1/2)$, $a_1 = a/\sqrt{2}$, and $\beta = a_1^{-1}$. Let $w$ be a non-negative subtemperature in $C_{\beta r}(\xi_0)$ so that $w = 0$ on $R_a^-(\xi_0; r) \cap \Omega^c$ and $w \leq M_1$ in $C_{\beta r}(\xi_0)$. If $\bar{x} \in C_p(\xi_0) \cap \Omega$ for some $0 < \rho < r$, then

$$
(3.37) \quad w(\bar{x}) \leq \exp \left( -c \int_{\rho}^{r} \frac{\text{Cap}(R_a^-(\xi_0; s) \cap \Omega^c)}{s} \, ds \right) \sup_{\Omega \cap C_{\beta r}(\xi_0)} w,
$$

for some constant $c > 0$ depending on $n$ and $a$.

Proof. We assume that $\text{Cap}(R_a^-(\xi_0; \rho) \cap \Omega^c) > 0$, since otherwise (3.38) is trivial. We set $a_k = a_1^k$ for $k \geq 0$, and

$$
\theta_k := \frac{\text{Cap}(R_a^-(\xi_0; a_{2k}r) \cap \Omega^c)}{(a_{2k}r)^n},
$$

and note that $a_{k-1} = \beta a_k$. By (3.36), it holds that

$$
w(\bar{x}) \leq (1 - \kappa \theta_1) \sup_{C_{a_1r}(\xi_0)} w \leq \exp(-\kappa \theta_1) \sup_{C_{a_1r}(\xi_0)} w, \quad \text{for } \bar{x} \in \overline{R_a^+(\xi_0; r)}.
$$

In particular, the latter inequality holds in $C_{a_1r}(\xi_0)$. We apply (3.36) once again in $C_{a_2r}(\xi_0)$ and get that

$$
w(\bar{x}) \leq \exp(-\kappa \theta_2) \sup_{C_{a_2r}(\xi_0)} w \leq \exp(-\kappa(\theta_1 + \theta_2)) \sup_{C_{a_2r}(\xi_0)} w, \quad \text{for } \bar{x} \in \overline{R_a^+(\xi_0; a_2r)}.
$$

The latter inequality holds in $C_{a_3r}(\xi_0)$ and we may apply (3.36) in $C_{a_3r}(\xi_0)$. By iteration, if $M$ is an integer such that $a_{2M+2} \leq \rho \leq a_{2M}$, we get that

$$
w(\bar{x}) \leq \exp \left( -\kappa \sum_{k=0}^{M} \theta_k \right) \sup_{C_{a_r}(\xi_0)} w, \quad \text{for } \bar{x} \in \overline{R_a^+(\xi_0; a_{2M}r)}.
$$

Therefore, since

$$
\int_{\rho}^{r} \frac{\text{Cap}(R_a^-(\xi_0; s) \cap \Omega^c)}{s} \, ds \leq \sum_{k=0}^{M} \int_{a_{2k+2}}^{a_{2k+2}} \frac{\text{Cap}(R_a^-(\xi_0; s) \cap \Omega^c)}{s} \, ds \leq \beta^{3n+2} \sum_{k=0}^{M} \theta_k,
$$

if $c = a_1^{3n+2} \kappa$, we obtain that

$$
(3.38) \quad w(\bar{x}) \leq \exp \left( -c \int_{\rho}^{r} \frac{\text{Cap}(R_a^-(\xi_0; s) \cap \Omega^c)}{s} \, ds \right) \sup_{C_{a_r}(\xi_0)} w, \quad \text{for } \bar{x} \in C_{\rho}(\xi_0).
$$

Given $T > 0$, we define $E(T) := \{(x, t) \in \mathbb{R}^{n+1} : t < T\}$. Moreover, for an open set $\Omega, \bar{x} \in \partial \Omega$ and $r > 0$ we define $\Omega_r := \Omega \cap C_r(\bar{x})$ and $\Omega_r(T) := \Omega_r \cap E(T)$. 


Lemma 3.20. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the TBCDC, and let $\xi_0 \in S\Omega$ and $0 < r < \sqrt{(t_0 - T_{\text{min}})/2}$. Then for any non-negative function $u$, which is either weakly subcaloric or subtemperature in $\Omega_{2r}(T_1)$ vanishing continuously on $C_{2r}(\xi_0) \cap \partial_\Omega \cap E(T_1)$, it holds that

$$(3.39) \quad u(\bar{y}) \lesssim (\text{dist}(\bar{y}, \partial \Omega)/r)^\alpha \sup_{\Omega_{3r/2}(T_1)} u, \quad \text{for every } \bar{y} \in \Omega_r(T_1),$$

where $T_1 = T_{\text{max}}(C_r(\xi_0)) = t_0 + r^2$.

**Proof.** For subtemperatures, this is a direct consequence of Lemma 3.19 (with a slightly larger cylinder on the right hand-side of (3.39)), while for weakly subcaloric functions one can follow the proof of [GH20, Lemma B.2] (which still works for TBCDC domains). □

Lemma 3.21. Let $\Omega \subset \mathbb{R}^{n+1}$ be a quasi-regular open set for $H$ and let $M > 1$. If $\xi_0 \in S\Omega$ and $r > 0$, for any $\bar{x} \in \Omega \setminus C_r(\xi_0)$ and $\bar{y} \in \Omega \cap C_{r/4}(\xi_0)$ we have that

$$(3.40) \quad \inf_{\bar{z} \in C_r(\xi_0) \cap \Omega} \omega^2(C_{Mr}(\xi_0)) \frac{G(\bar{x}, \bar{y})}{|r|^n} \lesssim \omega^2(C_{Mr}(\xi_0)).$$

Moreover, if $\Omega$ satisfies either the TBCDC at $\xi_0$, then if $M_0 > 1$ is the constant from Lemma 3.15, we have that for $\bar{x} \in \Omega \setminus C_r(\xi_0)$ and $\bar{y} \in \Omega \cap C_{r/4}(\xi_0)$,

$$(3.41) \quad G(\bar{x}, \bar{y}) r^n \lesssim \omega^2(C_{Mr}(\xi_0)), \quad 0 < r < M_0 \sqrt{(t_0 - T_{\text{min}})/2}.$$

**Proof.** Fix $\bar{y} = (y, s) \in \Omega \cap C_{r/4}(\xi_0)$ and recall that $G(\bar{x}, \bar{y}) = 0$ for any $\bar{x} \not\in \Lambda^s(\bar{y}, \Omega)$. If $\bar{x} = (x, t) \in \partial \Omega \cap C_r(\xi_0) \cap \Lambda^s(\bar{y}, \Omega)$, then there exists $c > 0$ such that

$$G(\bar{x}, \bar{y}) r^n \lesssim \frac{r^n}{\|\bar{x} - \bar{y}\|^n} \leq c,$$

and so, for any $\bar{x} \in \partial \Omega \cap C_r(\xi_0) \cap \Lambda^s(\bar{y}, \Omega)$,

$$(3.42) \quad \inf_{\bar{z} \in C_r(\xi_0) \cap \Omega} \omega^2(C_{Mr}(\xi_0)) \frac{G(\bar{x}, \bar{y})}{|r|^n} \leq c \inf_{\bar{z} \in C_r(\xi_0) \cap \Omega} \omega^2(C_{Mr}(\xi_0)) \leq c \omega^2(C_{Mr}(\xi_0)).$$

Define now

$$u(\bar{x}) := c^{-1} \inf_{\bar{z} \in C_r(\xi_0)} \omega^2(C_{Mr}(\xi_0)) G(\bar{x}, \bar{y}) r^n, \quad v(\bar{x}) := \omega^2(C_{Mr}(\xi_0)).$$

If we set $w(\bar{x}) = u(\bar{x}) - v(\bar{x})$, then it is clear that $w$ is a caloric function, hence continuous, in $\Omega \setminus C_{r/2}(\xi_0)$. Let $E(\xi_0, r/2)$ be the ellipsoid that circumscribes $C_{r/2}(\xi_0)$ lying inside $C_r(\xi_0)$. If $\bar{z}_* = (z_*, t_*)$ is the point on the boundary of the ellipsoid such that $t_* < t$ for every $\bar{z} = (z, t) \in \partial E(\xi_0, r/2) \setminus \{\bar{z}_*\}$, then

$$\partial E(\xi_0, r/2) \setminus \{\bar{z}_*\} \subset \partial_\Omega(\mathbb{R}^{n+1} \setminus E(\xi_0, r/2)) \quad \text{and} \quad \bar{z}_* \in \partial_{ss}(\mathbb{R}^{n+1} \setminus E(\xi_0, r/2)),$$

while it is clear by (3.21) and Remark 3.7, all the points of $\partial E(\xi_0, r/2)$ are regular points of $\partial_\star(\mathbb{R}^{n+1} \setminus E(\xi_0, r/2))$ for both $H$ and $H^\star$. Using (3.42) we obtain

$$\lim_{\Omega, E(\xi_0, r/2) \ni \bar{y} \to \bar{z}_*} w(\bar{y}) = w(\bar{z}_*) \leq 0,$$

for any $\bar{z}_* \in \partial E(\xi_0, r/2) \cap \Omega$. 
We set $Z$ to be the set of irregular points of $\partial_x \Omega$, which is polar because of the quasi-regularity assumption on $\Omega$. Thus, if $\zeta \in \partial_x \Omega \setminus Z$ (resp. $\partial_{x\alpha} \Omega \setminus Z$), it holds that $u(\tilde{x}) \to 0$ as $\Omega \ni \tilde{x} \to \zeta$ (resp. $\Omega \ni (x, t) \to (\zeta, \tau^t)$). Moreover, it is clear that

$$\limsup_{\tilde{x} \to \zeta \in Z} w(\tilde{x}) < \infty.$$ 

Hence, by the maximum principle [Wa12, Theorem 8.2] in $\Omega \setminus \bar{\mathcal{E}}(\xi_0, r/2)$, we infer that $w \leq 0$ in $\Omega \setminus \mathcal{E}(\xi_0, r/2)$ concluding the proof of the lemma.

If we assume the TBDC, (3.41) readily follows from (3.40) using (3.34). $\square$

A pretty standard result in elliptic theory is that polar sets have zero harmonic measure. The same is true for caloric measure as well. As we were not able to find an appropriate reference, we will write the proof for completeness.

**Proposition 3.22.** Let $\Omega$ be an open set, $\bar{x} \in \Omega$, and $\omega^\bar{x}$ be the caloric measure in $\Omega$. If $E \subset \mathbb{R}^{n+1}$ is polar, then $\omega^\bar{x}(E) = 0$.

**Proof.** Fix $\bar{x} \in \Omega$. Since $E \cap \partial \Omega$ is polar as a subset of the polar set $E$, without loss of generality, we may assume that $E \subset \partial \Omega$. As we have that $\text{Cap}(E) = 0$, by (3.12), there exists a sequence of open sets $S_j \supset E$ such that $\lim_{j \to \infty} \text{Cap}_\omega(S_j) = 0$. Set now $\bar{E} := \bigcap_j S_j$, and note that $\bar{E}$ is Borel, $E \subset \bar{E}$, and $\text{Cap}(\bar{E}) = 0$. Therefore, without loss of generality, we may assume that $E$ is Borel.

Let us assume that $\text{Cap}(\bar{\Omega}^c) > 0$. Then, there exists an open set $\bar{\Omega}$ such that $\Omega \cup E \subset \bar{\Omega}$ and $\text{Cap}(\bar{\Omega}^c) > 0$. Indeed, as $\Omega^c \setminus E$ is Borel, by (3.12), there exists a compact set $K \subset \Omega^c \setminus E$ so that $\text{Cap}(K) > \text{Cap}(\Omega^c \setminus E)/2$. We set $\bar{\Omega} = \Omega^c$, which is an open set such that $\Omega \cup E \subset \bar{\Omega}$, and, since $\text{Cap}(E) = 0$, it clearly satisfies $\text{Cap}(\bar{\Omega}^c) > 0$. If $\text{Cap}(\Omega^c) = 0$, we simply assume that $\bar{\Omega} = \mathbb{R}^{n+1}$.

Now, we apply [Wa12, Theorem 7.3] to find $u \geq 0$, a supertemperature in $\bar{\Omega}$, such that $u = \infty$ on $\bar{E}$ and $u(\bar{x}) < +\infty$. Hence, $\lambda u|_{\bar{\Omega}}$ belongs to the upper class $\mathcal{U}_E$ for every $\lambda > 0$, which implies that $0 \leq \omega^\bar{x}(E) \leq \lambda u(\bar{x})$. We conclude the proof by taking $\lambda \to 0$. $\square$

**Lemma 3.23.** Let $\Omega_1$ and $\Omega_2$ be two disjoint quasi-regular open sets in $\mathbb{R}^{n+1}$ for $H$ and $H^*$ with caloric measures $\omega_i = \omega_{\Omega_i}^\bar{x}_i$ for some $\bar{x}_i \in \Omega_i$ and suppose $\omega_1 \ll \omega_2 \ll \omega_1$ on a Borel set $E \subset \mathcal{P}\Omega_1 \cap \mathcal{P}\Omega_2$ and $\omega_{\Omega_1}^\bar{x}(E) > 0$. Then there are open subsets $\bar{\Omega}_i \subset \Omega_i$ which are regular for $H$ and $H^*$ and contain $\bar{x}_i$ for $i = 1, 2$ so that there exists $G_0 \subset E \setminus \bar{\Omega}_i$ with $\bar{\omega}_i(G_0) > 0$ upon which $\bar{\omega}_2 \ll \bar{\omega}_1 \ll \bar{\omega}_2$.

**Proof.** We follow closely the proof of [AMTV19, Lemma 2.3]. Let $F_i$ and $F_i^*$ be the sets of irregular points for $\bar{\Omega}_i$ for $H$ and $H^*$ respectively, which are polar and co-polar sets in $\mathbb{R}^{n+1}$. Although, since by [Wa12, Theorem 7.46] polar and co-polar sets coincide, by [Wa12, Definition 7.1], there is a positive supertemperature $v_i$ on $\bar{\Omega}_i$ so that

$$\lim_{\bar{y} \to \bar{x}} v_i(\bar{y}) = \infty \text{ for all } \bar{x} \in F := F_1 \cup F_2 \cup F_1^* \cup F_2^*.$$ 

Let $\lambda > 0$. Since $v_i$ is supertemperature on $\Omega_i$, it is lower semicontinuous, and remains so when we extend it by zero to $\Omega_i^c$. Thus, for each $\bar{x} \in F$ there is a closed cylinder $C_i'(x)$ centered at $\bar{x}$ (not containing either $\bar{x}_i$) such that $v_i \geq \lambda$ on $C_i'(x) \cap \Omega_i$. For each $C_i'(x)$
let \( C_i(x) \) be the cylinder of the same center and half the radius of \( C_i'(x) \). Let \( \{C_j\} \) be a Besicovitch subcovering (see Lemma 4.8) and if \( E_j \) is the closed ellipsoid of revolution around the axis of the cylinder \( C_i' \) that is inscribed in \( C_i' \), we define
\[
\bar{\Omega}_i = \Omega_i \setminus H \subset \Omega_i, \quad \text{where} \quad H = \bigcup_{j \geq 1} E_j.
\]

Note that \( \bar{\Omega}_i \) is open. Indeed, to show that \( \bar{\Omega}_i \) is closed consider \( \bar{x}_k \in \bar{\Omega}_i \), \( k \geq 1 \) and \( \bar{x}_k \to \bar{x} \). Then we need to show \( \bar{x} \in \bar{\Omega}_i \). If there is a subsequence contained in \( \bar{\Omega}_i \), we are done. Otherwise, assume that \( \bar{x}_k \in H' \cap \Omega_i \). If \( \bar{x}_k \in E_j \) for infinitely many \( k \), then \( \bar{x} \in E_j \) and we are done since \( E_j \) is closed and \( E_j \subset \bar{\Omega}_i \). Otherwise, suppose \( \bar{x}_k \) is not in any \( E_j \) more than finitely many times. By the bounded overlap property, if \( j(\bar{x}_k) \) is such that \( \bar{x}_k \in E_j(\bar{x}_k) \), then \( \text{diam}(E_j(\bar{x}_k)) \downarrow 0 \) as \( k \to \infty \), and since the ellipsoids are centered on \( F \subset \Omega_i \), \( \bar{x} \in \Omega_i \) and we are done. Thus, \( \Omega_i \) is open.

Set \( V_j = \mathbb{R}^{n+1} \setminus E_j \) and note that if \( z^1_j = (z_1, t_1) \) and \( z^2_j = (z_2, t_2) \), are the points on \( \partial E_j \cap \ell_j \), where \( \ell_j \) is the line containing the axis of \( C_j \), so that \( t_1 < t_2 \), it holds that \( z_1 \in \partial NS V_j \) and every \( x \in \partial E_j \setminus \{z^1_j\} \subset \partial^V_j \) is in \( \partial_n V_j \). Moreover, it is clear by (3.21) and Remark 3.7 that every \( x \in \partial E_j \) is regular for \( V_j \). Therefore, by [Wa12, Corollary 8.47], it holds that \( \partial E_j \cap \Omega_i \) consists of regular points of \( \partial \bar{\Omega}_i \). As the points of \( Z \cap \partial \bar{\Omega}_i \) are regular for \( \Omega_i \), then, by another application of [Wa12, Corollary 8.47], they are regular for \( \bar{\Omega}_i \) as well. Therefore, \( \bar{\Omega}_i \) is regular for \( \Omega_i \) and \( E \subset \partial \bar{\Omega}_i \).

Let \( \tilde{\omega}_i = \omega_i^{\mathcal{P}_i} \) and \( G = E \setminus H \). Since \( v_i \) is positive in \( \bar{\Omega}_i \) and \( \omega_i^{\mathcal{P}_i}(H) \leq 1 \leq v_i(\tilde{x})/\lambda \) for all \( \tilde{x} \in H \), by the maximum principle on \( \bar{\Omega}_i \), we have that \( \omega_i(H) \leq \lambda^{-1} v_i(\tilde{p}_i) \). Picking
\[
\lambda^{-1} < \frac{1}{2} \min_{i=1,2} \{\omega_i(E)/v_i(\tilde{p}_i)\},
\]
this gives \( \omega_i(H) \leq \frac{1}{2} \omega_i(E) \) and hence \( \omega_i(G) > 0 \). Similarly, by the maximum principle, since \( v_i / \lambda \geq 1 \) on \( H', \tilde{\omega}_i(H) \leq \lambda^{-1} v_i(\tilde{p}_i) \). Picking
\[
\lambda^{-1} < \frac{1}{2} \min_{i=1,2} \{\omega_i(G)/v_i(\tilde{p}_i)\},
\]
this gives
\[
\tilde{\omega}_i(H) \leq \frac{1}{2} \omega_i(G).
\]

Moreover, by the maximum principle, and since \( \bar{\Omega}_i \) is a regular domain,
\[
\tilde{\omega}_i(H^c \cap G^c) \leq \omega_i(H^c \cap G^c).
\]

Thus,
\[
\tilde{\omega}_i(G) = 1 - \tilde{\omega}_i(G^c) = 1 - \omega_i(H \cap G^c) - \omega_i(H^c \cap G^c) \\
\geq 1 - \frac{1}{2} \omega_i(G) - \omega_i(H^c \cap G^c) \geq \omega_i(G) - \frac{1}{2} \omega_i(G) = \frac{1}{2} \omega_i(G) > 0.
\]

Note that \( \tilde{\omega}_1 \ll \omega_1 \) on \( G \) by the maximum principle and since \( \tilde{\omega}_1(G) > 0 \), it is not hard to show using the Lebesgue decomposition theorem that there is \( G_1 \subset G \) of full \( \tilde{\omega}_1 \)-measure upon which we also have \( \omega_1 \ll \tilde{\omega}_1 \). Hence \( \omega_1(G_1) > 0 \), which implies \( \omega_2(G_1) > 0 \). The same reasoning gives us a set \( G_2 \subset G_1 \) upon which \( \tilde{\omega}_2 \ll \omega_2 \ll \tilde{\omega}_2 \). Thus, \( \tilde{\omega}_2 \ll \tilde{\omega}_1 \ll \tilde{\omega}_2 \) on \( G_2 \).
The notion of halving metric space was first introduced by Korey [Kor98] and plays an important role in the proofs of our theorems.

**Definition 3.24.** A probability space \((X, \omega)\) is called halving if for every subset \(E \subset X\) such that \(\omega(E) > 0\), there exists \(F \subset E\) such that \(\omega(F) = \omega(E)/2\).

We conclude this section by proving the caloric analogue of [AM19, Lemma 7.9].

**Lemma 3.25.** Let \(\Omega \subset \mathbb{R}^{n+1}\) be an open set and \(\bar{x} \in \Omega\). Then, the probability space \((\partial_\varepsilon \Omega, \omega^\varepsilon)\) is halving.

**Proof.** Set \(\omega := \omega^\varepsilon_{\Omega}\) and let us assume that there exists \(E \subset \partial \Omega\) such that \(\omega(E) > 0\) and \(\omega(F) \neq \omega(E)/2\) for every \(F \subset E\). Given \(s \in \mathbb{R}\) and \(v \in \mathbb{S}^n\), we define the half-space \(H_{s,v} := \{\xi \in \mathbb{R}^{n+1} : \xi \cdot v \geq s\}\). By the mean value theorem and our assumption, the map \(s \mapsto \omega(H_{s,v} \cap \Omega)\) is not continuous for any \(v \in \mathbb{S}^n\). In particular, for any \(v \in \mathbb{S}^n\), there exists \(s_v\), such that \(\omega(\partial H_{s_v,v} \cap \Omega) > 0\). Set now \(V_v := \partial H_{s_v,v}\), which is an \(n\)-dimensional plane. Define
\[
S := \{(y', y_n, \tau) \in \mathbb{S}^n : y' = 0\},
\]
and observe that, since \(S\) is uncountable, there is \(\varepsilon > 0\) such that \(\omega(V_v \cap \Omega) > \varepsilon\) for all \(v\) in some uncountable subset \(A\) of \(S\). Let us consider \(A' \subset A\) countable. For every \(v, v' \in A'\) we have that \(V_v \cap V_{v'}\) is an \((n-1)\)-plane orthogonal to the \(t\) axis, which is a polar set since it is contained in a horizontal hyperplane and has \(n\)-dimensional Lebesgue measure zero (see [Wa12, Theorem 7.55]). Therefore, by Proposition 3.22, it holds that \(\omega(V_v \cap V_{v'}) = 0\). Set now
\[
W_u := V_u \setminus \bigcup_{v \in A', v \neq u} V_v,
\]
and notice that \(\omega(W_u \cap \Omega) = \omega(V_u \cap \Omega) > \varepsilon\) and \(W_u \cap W_{u'} = \emptyset\) for any \(u \neq u' \in A'\). Thus, since \(A'\) is countable,
\[
1 \geq \omega(E) \geq \sum_{u \in A'} \omega(W_u \cap \Omega) = +\infty,
\]
which is a contradiction. \(\square\)

Every result related to the heat equation we have stated so far has a dual for the adjoint heat equation \(H^* u = 0\) obtained by reversing the sign of the time variable. Therefore, we can define the associated fundamental solution \(\Gamma^*\), the Green function \(G^*_{\Omega}\), the adjoint caloric measure denoted by \(\omega^*_{\Omega}\), the associated parabolic capacity \(\text{Cap}_{\omega}^*\) and so forth. A solution of \(H^* u = 0\) is called adjoint caloric. Remark that, by [Wa12, Theorem 6.10], we have that
\[
\Gamma(\bar{x}, y) = \Gamma^*(\bar{y}, \bar{x}) \quad \text{and} \quad G_{\Omega}(\bar{x}, y) = G_{\Omega}^*(\bar{y}, \bar{x}).
\]
For the regularity of the \(\partial_\varepsilon \Omega\) for \(H^*\) and the corresponding capacity density conditions, it is pretty clear that we should just take time-forwards cylinders and the so-called co-heat balls, which are defined as the heat balls using the adjoint heat kernel.
4. HAUSDORFF AND TANGENT MEASURES

If \( d \leq n + 1 \) is an integer and \( \mathcal{H}^d(E) = 0 \) (resp. \( \mathcal{H}^d(E) < \infty \)), then \( \mathcal{H}^{d+1}_p(E) = 0 \) (resp. \( \mathcal{H}^{d+1}_p(E) < \infty \)). For a proof see [He17, Lemma 3.2]. Note that this was originally stated under the additional hypothesis that \( E \) is Euclidean \( d \)-rectifiable but it is easy to see it is redundant. Moreover, by [He17, Lemma 3.8], it holds that there exists \( c_1 > 0 \) and \( c_2 > 0 \) depending only on \( d \), such that on \( \mathbb{R}^{d-1} \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} \),

\[
\mathcal{H}^{d+1}_p = c_2 \mathcal{L}^d = c_1 \mathcal{H}^{d-1}_p \times \mathcal{H}^2_p.
\]

The Hausdorff dimension of a Borel measure \( \omega \) is defined by

\[
\dim(\omega) := \inf \{ \dim(Z) : \omega(\mathbb{R}^{n+1} \setminus Z) = 0 \}.
\]

This definition is related with the concept of pointwise dimension of \( \omega \) at \( \bar{x} \in \text{supp} \omega \). More specifically, if

\[
d_{\mu}(\bar{x}) = \liminf_{r \to 0} \frac{\log \mu(C_r(\bar{x}))}{\log r}
\]

and

\[
n_{\mu}(\bar{x}) = \limsup_{r \to 0} \frac{\log \mu(C_r(\bar{x}))}{\log r}
\]

denote the lower and upper pointwise dimension respectively, we can argue as in [BW06] to show that

\[
\dim(\omega) = \text{ess sup} \{ d_{\mu}(\bar{x}) : \bar{x} \in \text{supp} \mu \}.
\]

If \( d_{\mu}(\bar{x}) = n_{\mu}(\bar{x}) \), we denote the common value by \( d_\mu(\bar{x}) \).

**Definition 4.1.** If \( \ell \in [\frac{1}{2}, 1] \), we say that a function \( \psi : \mathbb{R}^d \to \mathbb{R} \) is \((1, \ell)\)-Lipschitz and write \( f \in \text{Lip}_{1, \ell}(\mathbb{R}^d) \) if there exists a constant \( L > 0 \) such that

\[
|\psi(x', t) - \psi(y', s)| \leq L \max \left( |x' - y'|, |t - s|^{\frac{\ell}{2}} \right), \quad \text{for} \quad (x', t) \neq (y', s) \in \mathbb{R}^{d-1} \times \mathbb{R}.
\]

We call \( L \) the Lipschitz constant of \( f \) and write \( \text{Lip}(f) = L \).

Observe that for \( \ell = 1/2 \) these functions correspond to Lipschitz functions with respect to the parabolic norm \( \| \cdot \| \) or the equivalent norm \( |x' - y'| + |t - s|^{1/2} \). For \( \ell = 1 \), those are just the usual Lipschitz functions with respect to the Euclidean norm.

**Definition 4.2.** We say that \( \Gamma \subset \mathbb{R}^{n+1} \) is an admissible \( d \)-dimensional graph if there exists a vector field \( f : \mathbb{R}^d \to \mathbb{R}^{n-d+1} \) such that, possibly after a rotation in space and a translation,

\[
\Gamma = \Gamma_f = \{ (x', f(x', t), t) : x \in \mathbb{R}^{d-1}, t \in \mathbb{R} \}.
\]

If \( f \in C^m(\mathbb{R}^d; \mathbb{R}^{n-d+1}) \), \( 1 \leq m \leq \infty \), we say that \( \Gamma_f \) is an admissible \( d \)-dimensional \( C^m \)-graph. If \( f \) is an affine map, then \( \Gamma_f = V \) is a plane that contains a line parallel to the time-axis and we call it an admissible \( d \)-dimensional plane. In fact, after rotation in space and a translation, we can always assume \( V = \mathbb{R}^{d-1} \times \{ 0 \} \times \mathbb{R} \), where \( 0 = (0, \ldots, 0) \in \mathbb{R}^{n-d+1} \).

**Definition 4.3.** A closed set \( E \subset \mathbb{R}^{n+1} \) is Euclidean \( d \)-rectifiable if there exists \( E_i \subset \mathbb{R}^d \) and \( \text{Lip}_{(1,1)} \)-functions \( f_i : E_i \to \mathbb{R}^{n+1} \) such that \( \mathcal{H}^d(E \setminus \bigcup_{i=1}^{\infty} f_i(E_i)) = 0 \). Equivalently, \( E \) is Euclidean \( d \)-rectifiable if there exists a countable collection of \( d \)-dimensional \( \text{Lip}_{(1,1)} \)-graphs (or \( d \)-dimensional \( C^1 \)-manifolds) \( \{ \Gamma_i \}_{i \geq 1} \) such that \( \mathcal{H}^d(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0 \).
By the co-area formula [He17, Theorem D], if $E$ is Euclidean $d$-rectifiable, then

$$\sigma = c\mathcal{H}_{d+1}^d|_E.$$  

In fact, using the Rademacher theorem in [Or19], one can show that this theorem is true for parabolically rectifiable sets as well, that is, for sets that are exhausted, up to a set of $\mathcal{H}_{d+1}^d$-measure zero, by admissible $Lip_{1,\frac{1}{2}}$-graphs with $\frac{1}{2}$-derivative in time in BMO.\(^4\)

If $\mu$ and $\nu$ are Radon measures on $\mathbb{R}^{n+1}$, we define the distance between $\mu$ and $\nu$ in the parabolic ball $C_r$ by

$$d_{C_r}(\mu, \nu) = \sup f \int f d(\mu - \nu),$$

where the supremum is taken over all functions $f \in Lip_{(1,\frac{1}{2})}(\mathbb{R}^{n+1})$ which are supported in $C_r$ and satisfy $\text{Lip}(f) \leq 1$. If a sequence of Radon measures $\mu_j$ converges weakly to a Radon measure $\mu$, we use the notation $\mu_j \rightharpoonup \mu$.

For $r > 0$ and $\bar{x}, \bar{y} \in \mathbb{R}^{n+1}$, set

$$\delta_r(\bar{x}) := (rx, r^2t), \quad \text{and} \quad T_{\bar{y},r}(\bar{x}) := \delta_{1/r}(\bar{x} - \bar{y}).$$

If $\mu$ and $\nu$ are Radon measures in $\mathbb{R}^{n+1}$, we define

$$T_{\bar{y},r}[\mu](A) := \mu(\delta_r(A) + \bar{y}) = \mu(T_{\bar{y},r}^{-1}(A)), \quad A \subset \mathbb{R}^{n+1}.$$  

and

$$F_r(\mu, \nu) := \sup f \int f d(\mu - \nu), \quad \text{for} \quad r > 0,$$

where the supremum is taken over all functions $f \in Lip_{(1,\frac{1}{2})}(\mathbb{R}^{n+1})$ which are supported in $C_r$ and satisfy $\text{Lip}(f) \leq 1$. By density, it is enough to consider the supremum in the class of $C_0^\infty(C_r)$ functions such that $\text{Lip}(f) \leq 1$.

We also define $F_r(\mu) := F_r(\mu, 0)$. A standard argument shows that

$$F_r(\mu) = \int \text{dist}_p(\bar{x}, \mathbb{R}^{n+1} \setminus C_r) \, d\mu(\bar{x}) = \int_0^r \mu(C_s) \, ds.$$  

As it is easy to see that

$$\text{dist}_p(\bar{x}, \mathbb{R}^{n+1} \setminus C_r) = (r - \|\bar{x}\|)_+,$$

where $(\cdot)_+$ stands for the positive part of a function, we infer that

$$F_r(\mu) = \int (r - \|\bar{x}\|)_+ \, d\mu(\bar{x}).$$

**Definition 4.4 (d-cone).** A set of Radon measures $\mathcal{M}$ is a $d$-cone if $cT_{0,r}[\mu] \in \mathcal{M}$ for all $\mu \in \mathcal{M}$, $c, r > 0$. Given a $d$-cone $\mathcal{M}$, the set $\{ \mu \in \mathcal{M} : F_1(\mu) = 1 \}$ is referred to as its basis. We say that $\mathcal{M}$ has closed (resp. compact) basis if its basis is closed (resp. compact) with respect to the weak topology of the space of Radon measures.

\(^4\)We omit the detailed definitions and the proof since we will not be dealing with such general sets in the present paper.
For a $d$-cone $M$, $r > 0$, and a Radon measure $\mu$ such that $F_r(\mu) \in (0, \infty)$, we define

$$\tag{4.4} d_r(\mu, M) := \inf \left\{ F_r \left( \frac{\mu}{F_r(\mu)} \right) : \nu \in M, \ F_r(\nu) = 1 \right\}. $$

The next lemma collects some of the relevant properties of $F_r$ and $d_r(\cdot, M)$. For more details, see [KPT09, Section 2] and the references therein.

**Lemma 4.5.** Let $\mu, \nu$ be Radon measures in $\mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^{n+1}$ and $r > 0$. The following properties hold:

1. $F_r(\mu) = r F_1(T_{0,r}[\mu])$.
2. $\frac{2}{r} \mu(C_{r/2}) \leq F_r(\mu) \leq r \mu(C_r)$.
3. $\mu_j \rightharpoonup \mu$ if and only if $F_r(\mu_j, \mu) \to 0$ for all $r > 0$.
4. $d_r(\mu, M) \leq 1$ and $d_r(\mu, M) = d_1(T_{0,r}[\mu], M)$.
5. If $\mu_j \rightharpoonup \mu$ and $F_r(\mu) > 0$, then $d_r(\mu_j, M) \to d_r(\mu, M)$.

**Definition 4.6.** We say that $\nu$ is a tangent measure of $\mu$ at a point $\bar{x} \in \mathbb{R}^{n+1}$ if $\nu$ is a non-zero Radon measure on $\mathbb{R}^{n+1}$ and there are sequences $c_i > 0$ and $r_i \searrow 0$ so that $c_i T_{\bar{x}, r_i}[\mu]$ converges weakly to $\nu$ as $i \to \infty$ and write $\nu \in \text{Tan}(\mu, \bar{x})$.

**Remark 4.7.** A Besicovitch covering theorem for parabolic balls in $\mathbb{R}^{n+1}$ was proved in [It18, Theorem 1.1]. This is an important tool for parabolic geometric measure theory. In particular, one can show that Radon measures in $\mathbb{R}^{n+1}$ satisfy the Lebesgue density theorems and the Lebesgue differentiation theorems with respect to parabolic balls, as reported in the next lemma.

**Lemma 4.8.** Let $\mu$ be a Radon measure on $\mathbb{R}^{n+1}$. If $f : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{\infty\}$ is locally $\mu$-integrable, then

$$f(\bar{x}) = \lim_{r \to 0} \frac{1}{\mu(C_r(\bar{x}))} \int_{C_r(\bar{x})} f \, d\mu \quad \text{for } \mu\text{-a.e } \bar{x} \in \mathbb{R}^{n+1}. $$

Furthermore, if $E \subset \mathbb{R}^{n+1}$ is $\mu$-measurable, then the limit

$$\lim_{r \to 0} \frac{\mu(E \cap C_r(\bar{x}))}{\mu(C_r(\bar{x}))}$$

exists and equals $1$ for $\mu$-a.e $\bar{x} \in E$ and $0$ for $\mu$-a.e $\bar{x} \in \mathbb{R}^{n+1} \setminus E$.

**Proof.** The proof follows from the argument in [Mat95, Corollary 2.14] using the Besicovitch theorem for parabolic balls in [It18].

**Remark 4.9.** Once we have made the appropriate modifications in the definitions of the blow-up mappings to reflect the parabolic dilation, all the theorems related to tangent measures that are required to obtain our results hold with the same proofs as in the Euclidean setting.

If $\mu$ is a Radon measure, $s \in [0, \infty)$, and $\bar{x} \in \mathbb{R}^{n+1}$, we define the lower and upper $s$-density of $\mu$ at $\bar{x}$ as

$$\Theta^s_{\mu,s}(\bar{x}) := \liminf_{r \to 0} \frac{\mu(C_r(\bar{x}))}{r^s} \quad \text{and} \quad \Theta^{s,s}_{\mu}(\bar{x}) := \limsup_{r \to 0} \frac{\mu(C_r(\bar{x}))}{r^s}. $$
A measure $\mu$ is asymptotically doubling at $\bar{x} \in \mathbb{R}^{n+1}$ if
\[
\limsup_{r \to 0} \frac{\mu(C_{2r}(\bar{x}))}{\mu(C_r(\bar{x}))} < \infty.
\]
Remark that if $0 < \Theta^{s}_{\mu,s}(\bar{x}) \leq \Theta^{s}_{\mu,s}(\bar{x}) < \infty$, then $\mu$ is asymptotically doubling at $\bar{x}$ since
\[
\limsup_{r \to 0} \frac{\mu(C_{2r}(\bar{x}))}{\mu(C_r(\bar{x}))} = 2^s \frac{\Theta^{s}_{\mu,s}(\bar{x})}{\Theta^{s}_{\mu,s}(\bar{x})} < \infty.
\]

**Lemma 4.10.** If $\mu$ is a Radon measure on $\mathbb{R}^{n+1}$ and $\bar{x} \in \mathbb{R}^{n+1}$, then the following hold:
1. If $\nu \in \text{Tan}(\mu, \bar{x})$, there exists $\{r_i\}_{i \geq 1}$ decreasing to 0 and $\rho, c > 0$ so that
   \[
   \frac{T_{\bar{x},r_i}[\mu]}{\mu(B(\bar{x}, r_i))} \to c T_{\hat{0},\rho} [\nu] \quad \text{and} \quad c T_{\hat{0},\rho} [\nu](C_1(0)) > 0.
   \]
   In this case, $0 \in \text{supp} \nu$ for all $\nu \in \text{Tan}(\mu, \bar{x})$.
2. If, additionally, $\mu$ is asymptotically doubling at $\bar{x} \in \mathbb{R}^{n+1}$, then for any $\nu \in \text{Tan}(\mu, \bar{x})$ there exists $c_1 > 0$ and a sequence $\{r_i\}_{i \geq 1}$ decreasing to 0 such that
   \[
   c_1 \frac{T_{\bar{x},r_i}[\mu]}{\mu(B(\bar{x}, r_i))} \to \nu.
   \]
   (3) If, additionally, $0 < \Theta^{s}_{\mu,s}(\bar{x}) \leq \Theta^{s}_{\mu,s}(\bar{x}) < \infty$, then for any $\nu \in \text{Tan}(\mu, \bar{x})$ there exists $c_1 > 0$ and a sequence $\{r_i\}_{i \geq 1}$ decreasing to 0 such that
   \[
   c_2 \frac{T_{\bar{x},r_i}[\mu]}{r_i^s} \to \nu.
   \]

**Proof.** For a proof see [Mat95, Theorem 14.3] and [Mat95, Remarks 14.4 (1)-(4)]. □

As a result of Lemma 4.8 we obtain the following localization property of tangent measures.

**Lemma 4.11.** Let $\mu$ be a Radon measure in $\mathbb{R}^{n+1}$ and $f \in L^1(\mu)$ a non-negative Borel function. Then, for $\mu$-a.e. $\bar{x} \in \mathbb{R}^{n+1}$, it holds that $\text{Tan}(f \mu, \bar{x}) = f(\bar{x}) \text{Tan}(\mu, \bar{x})$.

**Proof.** With Lemma 4.8 at our disposal, we just follow the proof of [DeL08, Lemma 3.12]. □

**Lemma 4.12** (see [Mat95], Theorem 14.16). Let $\mu$ be a Radon measure on $\mathbb{R}^{n+1}$. For $\mu$-a.e. $\bar{x} \in \mathbb{R}^{n+1}$, if $\nu \in \text{Tan}(\mu, \bar{x})$, then the following properties hold:
1. $T_{\bar{y},r}[\nu] \in \text{Tan}(\mu, \bar{x})$ for all $\bar{y} \in \text{supp} \nu$ and $r > 0$.
2. $\text{Tan}(\nu, \bar{y}) \subset \text{Tan}(\mu, \bar{x})$ for all $\bar{y} \in \text{supp} \nu$.

**Lemma 4.13.** Let $\Gamma \subset \mathbb{R}^{n+1}$ be an admissible $d$-dimensional Lip$_{(1,1)}$-graph\footnote{Lemma 4.13 can be proved also for parabolic Lipschitz graphs. Although, since we will not deal with such general sets in the present manuscript and the proof is more involved, we will present it in a future work.} and let $\mu = \mathcal{H}^{d+1}_p|_{\Gamma}$. Then, for $\mu$-a.e. $\bar{x} \in \Gamma$, there exists a positive constant $c_{\bar{x}}$ and an admissible $d$-plane $V_{\bar{x}}$ passing through the origin such that
\[
(4.5) \quad r^{-d-1} T_{\bar{x},r}[\mu] \to \mathcal{H}^{d+1}_p|_{V_{\bar{x}}} \quad \text{as} \quad r \to 0.
\]
Proof. By the parabolic Rademacher theorem in [Or19], one can show that a locally parabolic Lipschitz vector field \( \psi : \mathbb{R}^d \to \mathbb{R}^{n+1-d} \) satisfies
\[
(4.6) \quad \frac{|\psi(\bar{y}) - \psi(\bar{x}) - A_{\bar{x}}(y - x)|}{|\bar{y} - \bar{x}|} = \epsilon_{\bar{x}}(|\bar{y} - \bar{x}|),
\]
where \( \epsilon_{\bar{x}}(r) \to 0 \) as \( r \to 0 \). In fact, that theorem is only stated for \( d = n \) and globally parabolic Lipschitz functions (see e.g. [Or19, Definition 3.1]) but the local hypothesis is enough. Moreover, one can apply (4.6) to each component of \( \psi \) and obtain the result above. Note that \( A_{\bar{x}} : \mathbb{R}^d \to \mathbb{R}^{n+1-d} \) is a horizontal linear map. In fact, it is the horizontal differential for the Lipschitz vector field \( \psi(\cdot, t) : \mathbb{R}^{d-1} \to \mathbb{R}^{n+1-d} \) at \( \bar{x} \). Using this map we can construct the approximating admissible \( d \)-plane passing through the origin. Namely, since any admissible \( \text{Lip}_{(1,1)} \)-function is locally parabolic Lipschitz, if we set \( V_{\bar{x}} = (y', A_{\bar{x}}(y'), s) \) and follow the proof in [DeL08, pp. 38-39] using the parabolic area formula [He17, Theorem 3.64], we can prove (4.5). We skip the details. \( \square \)

Corollary 4.14. Let \( E \subset \mathbb{R}^{n+1} \) such that \( \sigma(E \setminus \bigcup_{j \geq 1} \Gamma_j) = 0 \), where \( \Gamma_j \) are admissible \( d \)-dimensional \( \text{Lip}_{(1,1)} \)-graphs and \( \sigma \) be the surface measure on \( E \). Then, for \( \sigma \)-a.e. \( \bar{x} \in \mathbb{R}^{n+1} \), there exists an admissible \( d \)-dimensional plane \( V_{\bar{x}} \) passing through the origin, such that
\[
\text{Tan}(\sigma, \bar{x}) = \{ c\mathcal{H}_{p}^{d+1}|_{V_{\bar{x}}}, c > 0 \}.
\]

Proof. By Lemma 4.11, we have that for \( \mathcal{H}_{p}^{d+1} \)-a.e. \( \bar{x} \in \Gamma_j \cap E, j \geq 1 \), it holds that
\[
\text{Tan}(\mathcal{H}_{p}^{d+1}|_{E, \bar{x}}) = \text{Tan}(\mathcal{H}_{p}^{d+1}|_{\Gamma_j \cap E, \bar{x}}) = \text{Tan}(\mathcal{H}_{p}^{d+1}|_{\Gamma_j, \bar{x}}).
\]
Observe that \( \mathcal{H}_{p}^{d+1}|_{\Gamma_j}(C_r(\bar{x})) \approx r^{-d-1} \) for any \( \bar{x} \in \Gamma_j \) and \( r > 0 \) (the implicit constant depends on the Lipschitz character of \( \Gamma_j \)). Thus, Lemma 4.13 implies that if \( \nu \) is \( \text{Tan}(\mathcal{H}_{p}^{d+1}|_{\Gamma_j, \bar{x}}) \), there exists a positive constant \( c_{\bar{x}} \) and an admissible \( d \)-dimensional plane \( V_{\bar{x}} \) passing through the origin, such that \( \nu = c_{\bar{x}}\mathcal{H}_{p}^{d+1}|_{V_{\bar{x}}} \). Thus, as \( E \) is Euclidean \( d \)-rectifiable, it holds that \( \sigma = c\mathcal{H}_{p}^{d+1}|_{E} \) and the result follows. \( \square \)

5. Nodal set of caloric functions

Given a caloric function \( h \), we denote by \( \Sigma^h := \{ h = 0 \} \) the nodal set of \( h \) and by \( \sigma_h \) the associated surface measure
\[
(5.1) \quad d\sigma_h = d\mathcal{H}^{n-1}|_{\Sigma^h} dt.
\]
In most of the paper the function \( h \) is clear from the context, in which case we simply understand \( \sigma = \sigma_h \).

If \( h \) is a non-zero caloric function on \( \mathbb{R}^{n+1} \), by unique continuation [Po96, Theorem 1.2] it cannot vanish in an open set. Hence, we may apply [HL94, Theorem 1.1] and [HL94, Proposition 1.2] to deduce that \( \Sigma^h \) has locally finite \( \mathcal{H}^n \)-measure and \( \Sigma^h \cap |Dh|^{-1}(0) \) is a (euclidean) \( (n-1) \)-rectifiable set. In particular, for any cylinder \( C_r \) centered on \( \Sigma^h \), the set \( \Sigma^h \cap C_r \) can be decomposed into a union of an \( n \)-dimensional \( C^1 \)-submanifold \( C_r \cap \Sigma^h \cap \{|Dh| > 0\} \) with finite \( n \)-dimensional Hausdorff measure and a closed set \( C_r \cap \Sigma^h \cap \{|Dh| = 0\} \) of Hausdorff dimension not larger than \( n-1 \).
For our purposes, we need a finer study of this decomposition in terms of admissible graphs and so we define the regular and the singular set of $\Sigma^h$ by
\[
\mathcal{R} := \{ \bar{x} \in \Sigma^h : |\partial_t h| + |\nabla h| > 0 \}
\]
\[
\mathcal{S} := \{ \bar{x} \in \Sigma^h : |\partial_t h| + |\nabla h| = 0 \}.
\]

Additionally, we set
\[
\mathcal{R}_x := \{ \bar{x} \in \mathcal{R} : |\nabla h| > 0 \}
\]
\[
\mathcal{R}_t := \{ \bar{x} \in \mathcal{R} : |\partial_t h| > 0 \} \cup \{ \bar{x} \in \mathcal{R} : |\nabla h| = 0 \} = \mathcal{R} \backslash \mathcal{R}_x,
\]
to be the space-regular and the time-regular sets respectively.

**Lemma 5.1** (Structure of the space-regular set). If $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is a non-zero caloric function, then for every $\bar{y} \in \mathcal{R}_x$ there exists $\rho = \rho(\bar{y}) > 0$ and an admissible $C^\infty$-graph $\Sigma^h_0$ such that $\Sigma^h \cap C_\rho(\bar{y}) = \Sigma^h_0 \cap C_\rho(\bar{y})$.

**Proof.** The proof is an easy application of the implicit function theorem. Indeed, if $|\nabla h(\bar{y})| > 0$, we can assume without loss of generality that $\partial_n h(\bar{y}) \neq 0$. Thus, we can find a cylinder $C_\rho(\bar{y})$ centered at $\bar{y}$ in which $\partial_n h \neq 0$ and a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that
\[
\bar{x} = (x', \varphi(x', t), t), \quad \text{for any } \bar{x} \in C_\rho(\bar{y}) \cap \Sigma^h.
\]
We remark that, despite the implicit function theorem defines $\varphi$ just locally, we can extend it to a $C^\infty$-function defined in the whole $\mathbb{R}^n$ (see e.g. [St70, Theorem 5, p. 181]).

In general, we cannot expect to express $\mathcal{R}_t$ locally as an admissible graphs (see also the examples after the next lemma). However, we can still make some general consideration about its geometry and we show that its dimension is lower than that of $\mathcal{R}_x$.

**Lemma 5.2** (Structure of the time-regular set). If $h : \mathbb{R}^{n+1} \to \mathbb{R}$ is a non-zero caloric function, then for every $\bar{y} \in \mathcal{R}_t$ there exists $\bar{\rho} = \bar{\rho}(\bar{y}) > 0$ and a smooth $(n-1)$-dimensional $C^\infty$-graph $\tilde{\Sigma}^\rho_0$ such that $\mathcal{R}_t \cap C_\rho(\bar{y}) \subset \tilde{\Sigma}^\rho_0 \cap C_\rho(\bar{y})$. In particular, $\sigma(\mathcal{R}_t) = 0$.

**Proof.** If $\mathcal{R}_t = \emptyset$ there is nothing to prove. Fix $\bar{y} = (y, s) \in \mathcal{R}_t$, and note that since $\partial_t h(\bar{y}) \neq 0$, by the implicit function theorem, we can find $\rho = \rho(\bar{y}) > 0$ such that $C_\rho(\bar{y}) \cap \Sigma^h$ agrees (possibly after a rotation in space) with the graph $(x, \varphi(x))$ of a $C^\infty$-function $\varphi : \mathbb{R}^n \to \mathbb{R}$ inside $C_\rho(\bar{y})$. Moreover, since $H h = 0$ the latter implies that $\Delta h(\bar{y}) \neq 0$. Without loss of generality, we assume that $\partial^2_{xx} h(\bar{y}) \neq 0$, where $\partial_n$ stands for the partial derivative in $x_n$ variable.

If we denote $g := \partial_n h$ and $\Sigma^g = \{ g = 0 \}$, then $g$ is smooth, $\bar{y} = (y', y_n, s) \in \Sigma^g$ and $\partial_n g(\bar{y}) \neq 0$. Thus, by the implicit function theorem, there exists $\rho' \leq \rho$ such that $C_{\rho'}(\bar{y}) \cap \Sigma^g$ is the (possibly rotated) graph $(x', \psi(x', t), t)$ of a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{y} = (y', \psi(y', s), s)$. Let us define
\[
\widetilde{h}(x', t) := h(x', \psi(x', t), t)
\]
and note that $\widetilde{h}(y', s) = h(y, s) = 0$. So, by the chain rule and the fact that $(y, s) \in \mathcal{R}_t$ (hence $\partial_n h(y, s) = 0$),
\[
\partial_t \widetilde{h}(y', s) = \partial_n h(y, s) \partial_t \psi(y', s) + \partial_t h(y, s) = \partial_t h(y, s) \neq 0.
\]
Hence, as \( \tilde{h} \) is clearly a smooth function, we can apply the implicit function theorem at \((y', s)\) and obtain that there exists a neighborhood of \((y', s)\) and a \(\mathcal{C}^\infty\) function \(\phi: \mathbb{R}^{n-1} \to \mathbb{R}\) such that \(\{h = 0\}\) coincides with the graph \((\cdot, \phi(\cdot))\) in that neighborhood. More specifically, there exists \(\rho'' \leq \rho'\) such that \(\overline{\Sigma}_{\rho''} := \overline{\Sigma}_{\rho} \cap \Sigma^\rho \cap C_{\rho'}(y, s)\) admits the parametrization

\[
(x', \psi(x', \varphi(x'))), \quad x' \in B_{\rho'}(y').
\]

In particular, \(\overline{\Sigma}_{\rho''}\) is euclidean \((n - 1)\)-rectifiable, so \(\mathcal{H}^{n-1}(\overline{\Sigma}_{\rho''}) = \mathcal{H}^n(\overline{\Sigma}_{\rho''}) = 0\) and, by the co-area formula [He17, Theorem D],

\[
\int \mathcal{H}^{n-1}(\overline{\Sigma}_{\rho''} \cap t) \, d\mathcal{H}^2(t) = 0,
\]

which, since \(\mathcal{H}^2 \approx L^1\), proves that \(\sigma(\overline{\Sigma}_{\rho''}) = 0\). It is clear that \(R_t \cap C_{\rho'}(\bar{y}) \subset \overline{\Sigma}_{\rho''}\) and, by a covering argument, we get that \(\sigma(R_t) = 0\), as wished. \(\square\)

**Example 5.3.** a) Let us consider the caloric polynomial \(h_1(x_1, x_2, t) = x_1^2 + x_2^2 + 4t\). The set \(\Sigma_{h_1}\) is a rotational paraboloid around the time-axis. Its time-regular set \(R_t\) is the singleton \(\{0\}\) and its space-regular set is \(\Sigma_{h_1} \setminus \{0\}\).

b) Let \(h_2(x_1, x_2, t) = x_1^2 + x_2^2 - 2x_1x_2 + 4t\), whose nodal set is a parabolic cylinder. We have that \(\mathcal{S} = \emptyset\), which gives that \(\Sigma_{h_2} = R_x \cup R_t\) and \(R_t = \{(x, x, 0) : x \in \mathbb{R}\}\). So \(R_t\) is a 1-dimensional manifold (in particular, a line), in contrast to what we have for the function \(h_1\) of the previous example.

6. **CALORIC MEASURE ASSOCIATED WITH A CALORIC FUNCTION**

We say that a function \(f(x, t)\) is (parabolic) homogeneous of degree \(m \in \mathbb{R}\) if for any \(\lambda > 0\) and \((x, t) \in \mathbb{R}^{n+1} \setminus \{0\}\), we have

\[
f(\delta_\lambda(x, t)) = f(\lambda x, \lambda^2 t) = \lambda^m f(x, t).
\]

We set \(\Theta\) to be the space of caloric functions such that \(h(0) = 0\). We denote by \(F(d)\) the space of homogeneous caloric polynomials of degree \(d\). We remark that every caloric polynomial can be written as the sum of homogeneous caloric polynomials. Indeed, assuming that \(h(x, t) = \sum_{\alpha, \ell} c_{\alpha, \ell} x^\alpha t^\ell\) for \(c_{\alpha, \ell} \in \mathbb{R}\) and that \(c_{\alpha, \ell} = 0\) if \(|\alpha| + 2\ell > d\), we can write

\[
h(x, t) = \sum_{j=0}^{d} \left( \sum_{|\alpha| + 2\ell = j} c_{\alpha, \ell} x^\alpha t^\ell \right) = \sum_{j=0}^{d} h_j(X, t).
\]

Observe that \(h_j\) is a homogeneous polynomial of degree \(j\), which implies that \(HH_j\) is a homogeneous polynomial of degree \(j - 2\). So

\[
HH = \sum_j HH_j = 0
\]

if and only if \(HH_j = 0\) for every \(j = 0, \ldots, d\). Hence, it is consistent to say that a caloric function \(h\) is a caloric polynomial of degree \(d\) if it can be written as the sum of homogeneous caloric polynomials of degree at most \(d\).
We denote by $P(d)$ the set of caloric polynomials of degree $d$ vanishing at the origin. Analogously, we define the set $\Theta^*$ of adjoint caloric functions such that $h(0) = 0$ and by $F^*(d)$ and $P^*(d)$ the associated spaces of polynomials.

**Lemma 6.1.** Let $h$ be a caloric function in $\mathbb{R}^{n+1}$ and let $h^+$ and $h^-$ indicate the positive and negative parts of $h$ respectively. There exists a unique Radon measure $\omega_h$ supported on $\{h = 0\}$ such that 

$$
\int \varphi \, d\omega_h = \int h^+ H^* \varphi = \int h^- H^* \varphi = \frac{1}{2} \int |h| H^* \varphi, \quad \text{for } \varphi \in C_c^\infty(\mathbb{R}^{n+1}).
$$

**Proof.** Let $\tilde{h}^\pm$ be the extension by zero of $h^\pm$ in the complement of $\{h^\pm > 0\}$. Then, since $h$ is continuous in $\mathbb{R}^{n+1}$, $h^\pm \to 0$ continuously on $\{h = 0\}$. Thus, as $h^\pm$ is caloric in $\{h^\pm > 0\}$, we have that $\tilde{h}^\pm$ is a subcaloric function in $\mathbb{R}^{n+1}$ and by [Wa12, Theorem 6.28], there exists a unique Radon measure $\tilde{\omega}_{h^\pm}$ such that 

$$
\int \varphi \, d\tilde{\omega}_{h^\pm} = \int h^\pm H^* \varphi = \int \{h^\pm > 0\} h^\pm H^* \varphi, \quad \text{for } \varphi \in C_c^\infty(\mathbb{R}^{n+1}).
$$

Now, since $\tilde{h}^\pm$ is caloric in $\{h^\pm > 0\}$ and zero in $\{h^\pm < 0\}$, we have that $\int \varphi \, d\tilde{\omega}_{h^\pm} = 0$ for $\varphi \in C_c^\infty(\{h^\pm > 0\})$ and $\varphi \in C_c^\infty(\{h^\pm < 0\})$, implying that $\text{supp} \tilde{\omega}_{h^\pm} \subset \{h = 0\}$. Therefore, since $L^{n+1}(\Sigma_h) = 0$, for any $\varphi \in C_c(\mathbb{R}^{n+1})$, we have that 

$$
\int \varphi \, d\omega_{h^+} - \int \varphi \, d\omega_{h^-} = \int_{\mathbb{R}^{n+1}} \tilde{h}^+ H^* \varphi - \int_{\mathbb{R}^{n+1}} \tilde{h}^- H^* \varphi = \int_{\mathbb{R}^{n+1}} h H^* \varphi = 0,
$$

and so, $\omega_{h^+} = \omega_{h^-}$. Thus, the first two equalities in (6.1) hold, while the last one follows by adding instead of subtracting the two identities above. \qed

Given a caloric function $h$, by the discussion in Section 5, $\Sigma^h$ is smooth away from a euclidean $(n - 1)$-rectifiable set, and so, the set $\Omega^\pm = \{h^\pm > 0\}$ is a set of locally finite perimeter in $\mathbb{R}^{n+1}$ (see Definition 5.1 and Theorem 5.23 in [EG15]). Hence, by [Mag12, Theorem 18.11], for a.e. $t \in \mathbb{R}$, its horizontal section $\Omega^\pm_t$ is a set of locally finite perimeter in $\mathbb{R}^n$. In fact, more is true. By Lemma 5.1, around $\sigma$-a.e. any point, we have that $\Sigma^h$ is given by an admissible Lipschitz graph, while the rest of the points lie on an $(n - 1)$-rectifiable set. So, for a.e. $t$, $\Sigma^h_t$ is also locally Lipschitz and thus, its measure theoretic boundary (see [EG15, Definition 5.7]) coincides with its topological boundary. Therefore, for a.e. $t$ there is a unique measure theoretic outward unit normal $\nu^\pm_t$ to $\partial \Omega^\pm_t$ such that we have the generalized Gauss-Green Theorem 

$$
\int_{\Omega^\pm_t} \text{div} \varphi \, dx = \int_{\Sigma^h_t} \varphi \cdot \nu^\pm_t \, d\mathcal{H}^{n-1}, \quad \text{for all } \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n).
$$

Moreover, $\nu^\pm_t$ coincides with the usual (geometric) outward unit normal on $\partial^* \Omega^\pm$ (see Definitions 5.4 and 5.6, Theorems 5.15 and 5.16, and Lemma 5.5 in [EG15]).

**Lemma 6.2.** If $h$ is a caloric function in $\mathbb{R}^{n+1}$ and let $\sigma$ be the surface measure on $\Sigma^h$ as defined in (5.1), then for any $\varphi \in C^2_c(\mathbb{R}^{n+1})$,

$$
\int \varphi \, d\omega_h = - \int_{\mathbb{R}^{n+1}} h \partial_t \varphi \, dx \, dt - \int_{\Sigma^h} \frac{\partial h}{\partial \nu^+_t} \varphi \, d\sigma,
$$

(6.3)
where \( \frac{\partial h}{\partial \nu_t^+} := \nu_t^+ \cdot \nabla h \neq 0 \) \( \sigma \)-a.e.

**Proof.** If we apply (6.2) first to \( h(\cdot, t) \nabla \varphi(\cdot, t) \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \) and then to \( \varphi(\cdot, t) \nabla h(\cdot, t) \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \) in \( \Omega_t^\pm \), for fixed \( t \), and integrate in \( t \), we obtain

\[
\int_{\Omega^\pm} hH^\ast \varphi \, dx \, dt - \int_{\Omega^\pm} h \partial_t \varphi \, dx \, dt = \int_{\Omega^\pm} h \Delta \varphi \, dx \, dt
\]

(6.4)

\[
= -\int_{\Omega^\pm} \nabla h \nabla \varphi \, dx \, dt + \int_{\Sigma_h} h \frac{\partial \varphi}{\partial \nu_t^+} \, d\sigma = \int_{\Omega^\pm} \Delta h \varphi \, dx \, dt - \int_{\Sigma_h} \frac{\partial h}{\partial \nu_t^+} \varphi \, d\sigma
\]

\[
= \int_{\Omega^\pm} \partial_t h \varphi \, dx \, dt - \int_{\Sigma_h} \frac{\partial h}{\partial \nu_t^+} \varphi \, d\sigma,
\]

where we used that \( h \) is caloric and \( h = 0 \) on \( \Sigma^h \). Thus, since \( h \chi_{\Omega^\pm} = h^\pm \chi_{\mathbb{R}^{n+1}} \) and \( \partial_t h \chi_{\Omega^\pm} = \partial_t h^\pm \chi_{\mathbb{R}^{n+1} \setminus \Sigma_h} \), we get that

\[
2 \int \varphi \, d\omega_h = \int_{\Omega^+} hH^\ast \varphi \, dx \, dt - \int_{\Omega^-} hH^\ast \varphi \, dx \, dt
\]

(6.1)

\[
= -\int_{\mathbb{R}^{n+1}} h \partial_t \varphi \, dx \, dt + \int_{\mathbb{R}^{n+1} \setminus \Sigma_h} \partial_t h \varphi \, dx \, dt - \int_{\Sigma_h} \frac{\partial h}{\partial \nu_t^+} \varphi \, d\sigma - \int_{\Sigma_h} \frac{\partial h}{\partial \nu_t^-} \varphi \, d\sigma
\]

\[
= -2 \int_{\mathbb{R}^{n+1}} h \partial_t \varphi \, dx \, dt - 2 \int_{\Sigma_h} \frac{\partial h}{\partial \nu_t^+} \varphi \, d\sigma,
\]

where in the last equality we used \( \nu_t^+ = -\nu_t^- \) and integrated by parts in \( t \) using that \( \mathcal{L}_{n+1}(\Sigma_h) = 0 \) and \( \partial_t h \) is continuous everywhere in \( \mathbb{R}^{n+1} \). Recall that the points of \( \Sigma^h \) where \( \nabla h = 0 \) are contained in an \( (n - 1) \)-rectifiable set and thus, have \( \sigma \)-measure zero. As the tangential component of \( \nabla h \) on each slice \( \Sigma_t^h \) is zero, we have that \( \nabla h = \partial_t \varphi \) and \( \partial_t \varphi = 0 \) \( \sigma \)-almost everywhere.

Let us recall the parabolic Cauchy estimates for caloric functions.

**Proposition 6.3** (see e.g. [HL94], Proposition 2.1). Let \( R > 0 \) and let \( h \) be a caloric function in \( C_R \). For \( r < R \) and \( \alpha \in \mathbb{Z}_+^n \) and any positive integer \( \ell \) with \( |\alpha| + 2\ell = m \), we have

\[
|D_\alpha^\ell h(x, t)| \lesssim_{n, m} (R - r)^{-m} \|h\|_{L^\infty(C_R)}, \quad \text{for all } (x, t) \in C_r.
\]

(6.5)

**Lemma 6.4.** If \( h \) is a caloric function in \( \mathbb{R}^{n+1} \) and \( \sigma \) is the surface measure on \( \Sigma^h \) as defined in (5.1), then we have that

\[
\omega_h(A) = -\int_A \frac{\partial h}{\partial \nu_t^+} \, d\sigma, \quad \text{for any Borel } A \subset \Sigma^h,
\]

(6.6)

where the associated Poisson kernel \( k_h = -\frac{\partial h}{\partial \nu_t^+} \) is positive \( \sigma \)-a.e. and in \( L^\infty_{\text{loc}}(\sigma) \). In particular, \( \omega_h \ll \sigma \).

**Proof.** Let \( A \) be a compact subset of \( \Sigma^h \). Note that since \( \sigma \) and \( \omega_h \) are Radon measures, it holds that \( \omega_h(A) < \infty \) and \( \sigma(A) < \infty \). By Urysohn’s lemma, we can find a sequence of decreasing functions \( \{ \varphi_j \}_{j=1}^\infty \subset C^\infty_c(\mathbb{R}^{n+1}) \), \( 0 \leq \varphi_j \leq 1 \), so that \( \varphi_j = 1 \) on \( A \) and
$\varphi_j \to \chi_A$ pointwisely. Let us denote $K_j := \text{supp } \varphi_j$ and observe that $K_{j+1} \subset K_j$ for all $j \geq 1$. Then, by (6.3), we have

$$\int \varphi_j \, d\omega_h = - \int_{\Sigma^h} \frac{\partial h}{\partial \nu^+} \varphi_j \, d\sigma - \int_{\mathbb{R}^{n+1}} h \, \partial_t \varphi_j.$$ 

Note that $\varphi_j$ has compact support $K_j$ and $h$ is smooth in $\mathbb{R}^{n+1}$ by caloricity, hence it is locally Lipschitz in $\mathbb{R}^{n+1}$ (in the euclidean norm). Integrating by parts in $t$ we get

$$\int \varphi_j \, d\omega_h = - \int_{\Sigma^h} \frac{\partial h}{\partial \nu^+} \varphi_j \, d\sigma + \int_{\mathbb{R}^{n+1}} \varphi_j \, \partial_t h.$$ 

Since $\partial_t h$ is bounded in $K_1$ and $K_j \subset K_1$ for any $j \geq 1$, by dominated convergence, we have

$$\omega_h(A) = - \lim_{j \to \infty} \int_{\Sigma^h} \frac{\partial h}{\partial \nu^+} \varphi_j \, d\sigma.$$ 

As $\sigma(A) < \infty$ and $\Sigma^h$ is Euclidean $n$-rectifiable, (4.2) entails $\mathcal{H}^{n+1}_p(A) < \infty$ and consequently $\mathcal{L}^{n+1}(A) \approx \mathcal{H}^{n+2}_p(A) = 0$. Thus, the second integral on the right-hand side of (6.8) is zero, inferring that

$$\omega_h(A) = - \lim_{j \to \infty} \int_{\Sigma^h} \frac{\partial h}{\partial \nu^+} \varphi_j \, d\sigma.$$ 

If $C_\rho$ is a cylinder of radius $\rho \approx \text{diam } K_1$ which is centered at $\Sigma^h$ and satisfies $K_1 \subset C_{\rho/2}$, by (6.5) we have that

$$\sup_{j \geq 1} \left| \frac{\partial h}{\partial \nu^+} \varphi_j \right| \lesssim_n \| \nabla h \|_{L^\infty(C_{\rho/2})} \chi_{K_1} \lesssim_{\rho} \| h \|_{L^\infty(C_\rho)} \chi_{K_1} \in L^1(\sigma),$$

since $h \in L^\infty(C_\rho)$ and $\sigma$ is locally finite. Thus, by (6.9) and the dominated convergence theorem, we conclude (6.6) for compact subsets of $\Sigma^h$. Recall that $\omega_h$ and $\sigma$ are Radon measures and so, the result for Borel sets follows from inner regularity (see [Mat95, Definition 1.5]).

## 7. Caloric Polynomial Measures

**Lemma 7.1.** If $h \in \Theta$ and $\omega_h$ the associated caloric measure, then

$$T_{0,r}^\omega[h] = r^n \omega_{h \circ T_{0,r}^{-1}}.$$ 

Moreover, if $h \in F(k)$,

$$T_{0,r}^\omega[h] = r^{n+k} \omega_h.$$ 

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. An application of the chain rule gives

$$H^*(\varphi \circ T_{0,r})(\vec{x}) = \frac{1}{r^s} ((H^*)^*(\varphi) \circ T_{0,r})(\vec{x}).$$
So, given \( h \in \Theta \), we have
\[
\int \varphi \, dT_{0,r}^{\varphi} = \int \varphi \circ T_{0,r} \, d\omega_h
= \frac{1}{2^n} \int |h| H^* (\varphi \circ T_{0,r}) \quad \text{(7.3)}
= \frac{1}{2^n} \int \left| H^* (\varphi \circ T_{0,r}) \right| \, d\omega_h,
\]
which proves the first part of the statement. If we further assume that \( h \in F(k) \), we obtain
\[
h \circ T_{0,r}^{-1}(\bar{x}) = h(\delta_r(\bar{x})) = r^k h(\bar{x}), \quad \bar{x} \in \mathbb{R}^{n+1},
\]
and (7.2) follows. \( \square \)

We recall that we use the notation \( C_r := C_r(\bar{0}) \) for cylinders centered at the origin. A consequence of the previous lemma is the following corollary.

**Corollary 7.2.** If \( h \in F(k) \) and \( r_1, r_2 > 0 \) it holds that
\[
F_{r_1}(\omega_h) = \left( \frac{r_1}{r_2} \right)^{n+k+1} F_{r_2}(\omega_h).
\]
In particular, for any \( r > 0 \) and \( M \geq 1 \),
\[
\frac{M}{2} \left( \frac{r}{M} \right)^{n+k+1} \omega_h(C_{M/2}) \leq F_{r}(\omega_h) \leq M \left( \frac{r}{M} \right)^{n+k+1} \omega_h(C_M).
\]

**Proof.** By Lemma 4.5 and Lemma 7.1 we have that
\[
F_{r}(\omega_h) = r F_1(T_{0,r}^{\omega_h}) = r^{n+1} F_1(\omega_h T_{0,r}^{-1}) = r^{n+k+1} F_1(\omega_h),
\]
and so, (7.4) readily follows. By Lemma 4.5-(2) and (7.4), it is straightforward to see that (7.5) holds. \( \square \)

**Lemma 7.3.** Let \( h_j \) be a sequence in \( \Theta \) which converges uniformly on compact sets to some \( h \in \Theta \). Then \( \omega_{h_j} \rightharpoonup \omega_h \).

**Proof.** The proof is a minor variant of the one of \cite[Lemma 5.4]{AM19} and we omit it. \( \square \)

**Lemma 7.4.** Let \( h \in \Theta \) and let \( \omega_h \) be the associated caloric measure. Then \( \bar{0} \in \text{supp} \omega_h \).

**Proof.** Let us recall that \( \Sigma^h = \mathcal{R}_x \cup \mathcal{R}_t \cup \mathcal{S} \) (see section 5) and that the Poisson kernel is given by \(-\partial_n h\). Therefore, by Lemma 5.1, for any \( \bar{x} \in \mathcal{R}_x \) there is a sufficiently small neighborhood of \( \bar{x} \) in which \( \Sigma^h \) agrees with an admissible smooth graph. Since \( h \) vanishes on \( \Sigma^h \), the component of \( \nabla h \) which is tangential to \( \Sigma^h \) is the zero vector and we have \( \nabla h = \partial_n h \). So, \( \partial_n h(\bar{x}) \neq 0 \) for any \( \bar{x} \in \mathcal{R}_x \) and thus \( \bar{x} \in \text{supp} \omega_h \) showing that \( \mathcal{R}_x \subset \text{supp} \omega_h \subset \Sigma^h \).

In light of \cite[Theorem 1.1]{HL94} and Lemma 5.2, for \( \rho > 0 \) small enough, it holds that \( \mathcal{H}^{n-1}(\mathcal{R}_t \cup \mathcal{S}) \cap C_\rho < \infty \), and so, \( \mathcal{R}_t \cup \mathcal{S} \) has empty interior in the relative topology of \( \Sigma^h \). Hence,
\[
\bar{0} \in \Sigma^h \cap C_\rho = \overline{\Sigma^h \setminus (\mathcal{R}_t \cup \mathcal{S})} \cap C_\rho = \overline{\mathcal{R}_x \cap C_\rho} \subset \text{supp} \omega_h,
\]
which finishes the proof. \hfill \Box

Lemma 7.5. The $d$-cones $\mathcal{F}(k)$ and $\mathcal{P}(k)$ have compact basis for all $k$. Moreover, for $h \in \mathcal{P}(k)$ and $r > 0$ we have

$$\|h\|_{L^\infty(C_r)} \approx_k r^{-n-1} F_r(\omega_h).$$

Proof. In light of Lemma 7.4, the proof is a routine adaptation of Lemmas 5.5 and 5.6, and Corollary 5.7 in [AM19]. \hfill \Box

In our argument we need the following formula for the expansion of caloric functions by Han and Lin, which we report for the reader’s convenience.

Theorem 7.6 (see [HL94], Theorem 2.2). Let $R > 0$ and let $h$ be a caloric function in $C_R$. Then, for any positive integer $d$, $0 < r < R/2$, and $\bar{x} \in C_r$, we have that

$$h(x,t) = \sum_{j=1}^{d} \sum_{|\alpha| + 2\ell = j} \frac{D^\alpha \partial_t^\ell h(\bar{0})}{\alpha! \ell!} x^\alpha t^\ell + R_d(x,t)$$

where

$$|R_d(x,t)| \lesssim_{n,d} \frac{r^{d+1}}{(R-r)^{d+1}} \|h\|_{L^\infty(C_R)},$$

for $(x,t) \in C_r$.

Lemma 7.7. Let $h \in \Theta$ and $h_j$ be as in (7.6). If $m \geq 1$ is the smallest integer such that $h_m \not\equiv 0$, then there exists $p_0 > 0$ and $c_0 > 0$ such that

$$c_0^{-1} F_r(\omega_{h_m}) \leq F_r(\omega_h) \leq c_0 F_r(\omega_{h_m})$$

for all $r \leq p_0$.

Proof. Let $r < 1/100$. By Theorem 7.6 and the fact that $h_j = 0$ for $j < m$, there exists a function $R_m: \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$h(x) = h_m(x) + R_m(x), \quad x \in C_{2r},$$

and

$$|R_m(x)| \lesssim_{n,m} \frac{r^{m+1}}{(1-2r)^{m+1}} \|h\|_{L^\infty(C_1)}, \quad x \in C_{2r}.$$

Let $\psi_r \in C^{2,1}(C_r)$ be so that $\psi_r = 1$ in $C_{r/2}$ and $0 \leq \psi_r \leq 1$, satisfying

$$|\nabla \psi_r| \leq C/r, \quad \text{and} \quad |\partial_i \psi_r| + |\partial^2_{xixj} \psi_r| \leq C/r^2, \quad \text{for } 1 \leq i,j \leq n.$$

One way to construct such a cut-off is the following: let $\eta \in C^\infty([0,\infty))$ be the standard cut-off so that $\eta = 1$ in $[0,1/2]$, $\eta = 0$ in $(1,\infty)$, $0 \leq \eta \leq 1$, $|\eta'| \leq C$ and $|\eta''| \leq C$. If $\zeta_r(x) = \eta(|x|/r)$ and $\xi_r(t) := \eta(|t|/r^2)$, we define $\psi_r(x,t) = \zeta_r(x) \xi_r(t)$. It is easy to see that the function $\phi = (2C)^{-1}r \psi_r$ is an admissible function for $F_r$. With this choice of $\phi$ we have

$$\int \phi d\omega_h - \int \phi d\omega_{h_m} = \int_{\mathbb{R}^{n+1}} (|h| - |h_m|) H^* \phi \leq \frac{1}{r} \int_{C_{2r}} |R_m(x,t)|$$

and by (7.14),

$$\frac{1}{r} \int_{C_{2r}} |R_m(x,t)| \lesssim_{n,m} \frac{r^{m+n+2}}{(1-2r)^{m+1}} \|h\|_{L^\infty(C_1)}.$$
Hence, since \( r < 1/100 \), there is \( c(n,m) > 0 \) such that

\[
\left| \int \phi \, d\omega_h - \int \phi \, d\omega_{h_m} \right| \leq c(n,m) r^{n+m+2} \| h \|_{L^\infty(C_1)}
\]

and so, if for a fixed \( \varepsilon > 0 \) we further assume

\[
r \leq \rho_0 := \varepsilon F_1(\omega_{h_m}) c(n,m)^{-1} \| h \|_{L^\infty(C_1)}^{-1},
\]

the inequality (7.10) reads

\[
\left| \int \phi \, d\omega_h - \int \phi \, d\omega_{h_m} \right| \leq \varepsilon r^{n+m+1} F_1(\omega_{h_m}) = \varepsilon F_r(\omega_{h_m}),
\]

where in the last step we used (7.4). Let us observe that, by Lemma 7.4, \( 0 \in \operatorname{supp} \omega_{h_m} \), which, in turn, by (7.5), implies that \( 0 < F_r(\omega_{h_m}) < \infty \) for all \( r > 0 \).

Therefore, since \( \phi \) is admissible for \( F_r \), we obtain

\[
\frac{1}{C^{2m+n+4} F_r(\omega_{h_m})} \leq \frac{r}{4C \omega_{h_m}(C_r/2)} \leq \int \phi \, d\omega_{h_m} \leq F_r(\omega_h) + \varepsilon F_r(\omega_{h_m})
\]

and

\[
(4C)^{-1} F_r/2(\omega_h) \leq \frac{r}{4C \omega_h(C_r/2)} \leq \int \phi \, d\omega_h \leq (1 + \varepsilon) F_r(\omega_{h_m}) = 2^{n+m+1}(1 + \varepsilon) F_r/2(\omega_{h_m}),
\]

which conclude (7.7) by choosing \( \varepsilon = C^{-1} 2^{-n-m-4} \). \( \square \)

We apply the previous lemma to show that the first non-zero term in the expansion \( h_m \) of \( h \) determines the density of \( \omega_h \) at 0.

**Lemma 7.8.** Let \( h, h_m \) and \( c_0 \) be as in Lemma 7.7. Then

\[
\lim_{r \to 0} \frac{\omega_h(C_r)}{r^{n+m}} \approx \lim_{r \to 0} \frac{\omega_{h_m}(C_r)}{r^{n+m}} \approx \omega_{h_m}(C_1) \in (0, \infty),
\]

where the implicit constants depend on \( n, m \) and \( c_0 \).

**Proof.** Let \( \rho_0 \) and \( c_0 \) be as in Lemma 7.7 and let \( r < \rho_0/2 \). We apply Lemma 4.5-(2), (7.8), and (7.5), and we have that

\[
\frac{\omega_h(C_r)}{r^{n+m}} \leq r^{-n-m-1} F_{2r}(\omega_h) \leq c_0 r^{-n-m-1} F_{2r}(\omega_{h_m}) = c_0 2^{n+m+1} F_1(\omega_{h_m}) \lesssim \omega_{h_m}(C_1).
\]

Arguing similarly but using the converse inequalities of the ones we used above, we infer that

\[
\frac{\omega_{h_m}(C_r)}{r^{n+m}} \geq r^{-n-m-1} F_r(\omega_h) \geq c_0^{-1} r^{-n-m-1} F_r(\omega_{h_m}) = c_0^{-1} 2^{n+m+1} F_1(\omega_{h_m}) \gtrsim \omega_{h_m}(C_1) > 0,
\]

where the last inequality holds because \( \bar{0} \in \operatorname{supp} \omega_{h_m} \). \( \square \)

**Lemma 7.9.** Let \( h \in \Theta \) and \( h_j \) be as in (7.6). If \( m \geq 1 \) is the smallest integer such that \( h_m \neq 0 \), then \( \operatorname{Tan}(\omega_h, \bar{0}) = \{ c \omega_{h_m} : c > 0 \} \).
Proof. Let $R > 0$ and $r < 1/2$. By Theorem 7.6 and $h_j = 0$ for $j < m$, we have that
\[ h(\bar{x}) = h_m(\bar{x}) + R_m(\bar{x}), \quad \bar{x} \in C_{rR}, \]
where
\begin{equation}
|R_m(\bar{x})| \lesssim_{n,m} \frac{r^{m+1}}{R^{m+1}(1 - r)^{m+1}} \|h\|_{L^\infty(C_{rR})}, \quad \bar{x} \in C_{rR}.
\end{equation}

Given $\bar{y} \in C_R$, we have that $\delta_r(\bar{y}) \in C_{rR}$, so we combine (7.14) with the homogeneity of $h_m$ in order to get
\[
|r^{-m}h \circ T_{0,r}^{-1}(\bar{y}) - h_m(\bar{y})| = |r^{-m}h(ry, r^2s) - h_m(y, s)| = r^{-m}|h(ry, r^2s) - h_m(ry, r^2s)| \lesssim_{n,m} \frac{r}{R^{m+1}} \|h\|_{L^\infty(C_{rR})}.
\]
where the last term converges to 0 as $r \to 0$. In particular, this shows that $r^{-m}h \circ T_{0,r}^{-1}$ converges to $h_m$ uniformly on compact subsets of $\mathbb{R}^{n+1}$.

The definition of caloric polynomial measure together with Lemma (7.1), gives us that
\begin{equation}
\omega_{r^{-m}h \circ T_{0,r}^{-1}} = r^{-m}\omega_{h \circ T_{0,r}^{-1}} = r^{-m-1}\omega_{T_{0,r}[\omega_h]}.
\end{equation}
In order to finish the proof of the lemma, it suffices to use (7.15) and the fact that $\omega_{r^{-m}h \circ T_{0,r}^{-1}}$ converges weakly to $\omega_{h_m}$ by Lemma 7.3. Indeed, $\omega_h$ has positive lower and finite upper $(n + m)$-density at 0 by Lemma 7.8, so we can apply Lemma 4.10 to conclude that every measure in $\Tan(\omega_h, 0)$ is of the form $c\omega_{h_m}$ for some $c > 0$. \hfill \Box

The next result is an important application of the previous lemma.

**Lemma 7.10.** Let $\omega$ be a Radon measure on $\mathbb{R}^{n+1}$, $\bar{x} \in \text{supp } \omega$ and let $d$ be the minimal integer such that $\Tan(\omega, \bar{x}) \cap \mathcal{P}(d) \neq \emptyset$. Then $\Tan(\omega, \bar{x}) \cap \mathcal{P}(d) \subset \mathcal{F}(d)$.

**Proof.** It is enough to argue as in [AM19, Lemma 5.9] and invoke Lemma 7.9. \hfill \Box

**Lemma 7.11.** Let $h \in P(d)$ and assume that
\[ h(x, t) = \sum_{j=1}^d h_j(x, t) := \sum_{j=1}^d \sum_{k|k|+2\ell=j} a_{k\ell}x^k t^\ell. \]
Then, there exists $r_0 > 0$ and $C_0 \geq 1$ such that
\begin{equation}
C_0^{-1}F_r(\omega_{h_j}) \leq F_r(\omega_h) \leq C_0 F_r(\omega_{h_d}), \quad \text{for all } r \geq r_0,
\end{equation}
where $r_0$ depends on $d, n$, $F_1(\omega_{h_d})$, $\max_{1 \leq j \leq d-1} \sum_{|k|+2\ell=j} |a_{k\ell}|$, and $C_0$ depends on $d$ and $n$.

**Proof.** Let $\psi_r$ be a cut-off function as in Lemma 7.7. In particular, $\psi_r \in C^{2,1}(C_r)$, $0 \leq \psi_r \leq 1$, $\psi_r = 1$ in $C_{r/2}$ and
\[
|\nabla \psi_r| \leq C/r, \quad \text{and} \quad |\partial_t \psi_r| + |\partial^2_{xxj} \psi_r| \leq C/r^2, \quad \text{for } 1 \leq i, j \leq n.
\]
Hence, the function $\phi := (2C)^{-1}r\psi$ is admissible for the functional $F_r, |H^*\phi| \leq r^{-1}$, and we have
\[
\left| \int \phi \, d\omega_h - \int \phi \, d\omega_{h_d} \right| = \left| \int_{\mathbb{R}^{n+1}} (|h| - |h_d|) H^* \phi \right| \\
\leq \frac{1}{r} \int_{C_{2r}} \left| |h| - |h_d| \right| \leq \frac{1}{r} \sum_{j=1}^{d-1} \int_{C_{2r}} |h_j|.
\]

For any $\bar{x} \in C_{2r}$, it holds that
\[
|h_j(\bar{x})| \leq (2r)^j \sum_{|k|+2|l|=j} |a_{k\ell}| =: c_j r^j,
\]
which, in turn, implies that
\[
r^{-1} \sum_{j=1}^{d-1} \int_{C_{2r}} |h_j| \leq r^{-1} (2r)^{n+2} \left( \max_{1 \leq j \leq d-1} c_j \right) \sum_{j=1}^{d-1} r^j \leq 2^{n+2} \frac{r^{n+d+1}}{r-1} \max_{1 \leq j \leq d-1} c_j.
\]

Therefore, if we let
\[
r_0 > 1 + (\varepsilon F_1(\omega_{h_d}))^{-1} 2^{n+2} \max_{1 \leq j \leq d-1} c_j,
\]
for some $\varepsilon > 0$ small enough, we infer that
\[
\left| \int \phi \, d\omega_h - \int \phi \, d\omega_{h_d} \right| \leq \varepsilon r^{d+n+1} F_1(\omega_d) = \varepsilon F_r(\omega_d),
\]
where in the last equality we used (7.4). Note that $0 \in \text{supp}(\omega_{h_d})$ and thus, by (7.5), we have that $0 < F_1(\omega_{h_d}) < \infty$. We conclude the proof arguing as in (7.12) and (7.13). \(\square\)

**Lemma 7.12.** Let $h, r_0$ and $C_0$ be as in Lemma 7.11. There exists $\varepsilon_0 > 0$ and $r_1 > 0$ such that if $d_r(\omega_h, \mathcal{F}(k)) < \varepsilon_0$ for all $r \geq r_1$, then $k = d$.

**Proof.** Fix $\tau \geq 2$ to be chosen and pick $r \geq r_1$ such that $d_{\tau r}(\omega_h, \mathcal{F}(k)) < \varepsilon_0$. In particular, there exists $\psi \in \mathcal{F}(k)$ such that $F_{\tau r}(\psi) = 1$ and
\[
F_r\left( \frac{\omega}{F_{\tau r}(\omega_d)}, \psi \right) \leq F_{\tau r}\left( \frac{\omega_h}{F_{\tau r}(\omega_d)}, \psi \right) < \varepsilon_0.
\]
\[
\text{(7.17)} \quad F_r(\psi) - \varepsilon_0 < \frac{F_r(\omega_h)}{F_{\tau r}(\omega_d)} < F_r(\psi) + \varepsilon_0.
\]

Also, since $\psi$ is homogeneous, (7.4) gives
\[
\text{(7.18)} \quad F_r(\psi) = r^{n+k+1} F_r(\psi) = \tau^{-n-k-1} F_1(\psi) = \tau^{-n-k-1}.
\]
Assuming that $r > r_2 := \max\{r_0, r_1\}$, by Lemma 7.11 and (7.4) we have
\[
\text{(7.19)} \quad \frac{F_r(\omega_h)}{F_{\tau r}(\omega_d)} \leq C_0^2 \frac{F_r(\omega_d)}{F_{\tau r}(\omega_d)} \leq C_0^2 \tau^{-n-d-1}.
\]
Therefore, (7.17), (7.18), and (7.19) infer that
\[
\tau^{-n-k-1} - \varepsilon_0 < C_0^2 \tau^{-n-d-1},
\]
or equivalently,
\[ \tau^{d-k}(1 - \varepsilon_0 \tau^{n+d+1}) < C_0^2. \]
If we pick \( \tau/2 = C_0^2 \) and \( \varepsilon_0 = \tau^{-n-d-2} \), the latter inequality implies that
\[ \tau^{d-k}(\tau - 1) < \frac{\tau^2}{2}, \]
which can only hold if \( d = k \), since \( \tau \geq 2 \).

The next lemma is crucial to our purposes: it provides a connectivity result for parabolic cones of Radon measures. The proof translates unchanged to the parabolic setting and we skip it.

**Lemma 7.13** (see Lemma 3.10, [AM19]). Let \( \mathcal{F} \) and \( \mathcal{M} \) be parabolic \( d \)-cones and assume that \( \mathcal{F} \) has compact basis. Moreover, suppose that there is \( \varepsilon_0 > 0 \) such that the following property holds: if \( \mu \in \mathcal{M} \) and there exists \( r_0 > 0 \) such that \( d_r(\mu, \mathcal{F}) \leq \varepsilon_0 \) for all \( r \geq r_0 \), then \( \mu \in \mathcal{F} \). If \( \eta \) is a Radon measure and \( \bar{x} \in \text{supp} \eta \) are such that \( \text{Tan}(\eta, \bar{x}) \subset \mathcal{M} \) and \( \text{Tan}(\eta, \bar{x}) \cap \mathcal{F} \neq \emptyset \), then \( \text{Tan}(\eta, \bar{x}) \subset \mathcal{F} \).

The following proposition gathers all the results of this section. After proving all the previous lemmas, the proof is analogous to that of [AM19, Proposition II]. We report it anyways, in order to give the reader the precise references inside this section.

**Proposition 7.14.** Let \( \omega \) be a Radon measure in \( \mathbb{R}^{n+1} \) and let \( \bar{x} \in \mathbb{R}^{n+1} \) be such that \( \text{Tan}(\omega, \bar{x}) \subset \mathcal{P}(k) \) for some \( k \). If \( \text{Tan}(\omega, \bar{x}) \cap \mathcal{F}(k) \neq \emptyset \) for some integer \( k \), then \( \text{Tan}(\omega, \bar{x}) \subset \mathcal{F}(k) \).

**Proof.** Let us assume that \( \text{Tan}(\omega, \bar{x}) \subset \mathcal{P}(k) \) and let \( m \leq k \) be the smallest integer for which \( \text{Tan}(\omega, \bar{x}) \cap \mathcal{P}(m) \neq \emptyset \). In particular \( \text{Tan}(\omega, \bar{x}) \cap \mathcal{P}(m) \subset \mathcal{F}(m) \) by Lemma 7.10, which gives that \( \text{Tan}(\omega, \bar{x}) \cap \mathcal{F}(m) \neq \emptyset \). Furthermore, \( \mathcal{F}(k) \) has compact basis by Lemma 7.5 and, in light of Lemma 7.12, we can apply Lemma 7.13 with \( \mathcal{M} = \mathcal{P}(k) \), \( \mathcal{F} = \mathcal{F}(k) \) and \( \eta = \omega \). Thus, \( \text{Tan}(\omega, \bar{x}) \subset \mathcal{F}(k) \).

\[ \square \]

8. **BLOW-UPS IN TIME VARYING DOMAINS**

The following lemma studies the blow-up of caloric measure and Green’s function in a domain that is quasi-regular for \( H \) and regular for \( H^* \). The main difference compared to the elliptic case (see [AMTV19], [AM19, Lemma 4.12]) is that due to the lack of a proper bound for the time-derivative of the Green’s function, one cannot use Rellich-Kondrachov theorem, and so we have to identify the right relative compactness theorem in mixed-norm Sobolev spaces and apply it in our setting.

**Lemma 8.1.** Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open set which is quasi-regular for \( H \) and regular for \( H^* \). Let also \( \omega = \omega^{\Omega}_{\tilde{p}} \) be the associated caloric measure with pole at \( \tilde{p} \in \Omega \). Assume that \( F \subset \mathcal{P}^* \Omega \cap \mathcal{S}' \Omega \) is compact, \( \xi_j \) is a sequence in \( F \), and there exists \( r_j \to 0 \) and \( c_j > 0 \) such that
\[ \omega_j := c_j T_{\xi_j, r_j}[\omega] \rightharpoonup \omega_{\infty} \]
for some non-zero Radon measure $\omega_\infty$. Let us also assume that there exists a subsequence of $r_j$ and a constant $c > 0$ so that

$$ \text{Cap}(\overline{E(\xi_0; r_j^2)} \setminus \Omega) \geq c r_j^d. $$

(8.1)

If $u = G_\Omega(\bar{p}, \cdot)$ in $\Omega$ and $u = 0$ in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, let us denote $u_j := G(\bar{p}, T_{\xi_j}^{-1} \cdot)$ and

$$ C_r := C_{r}(0) = B_r(0) \times (-r^2, r^2) =: B_r \times \mathcal{I}_r, \quad r > 0. $$

Then, if $a \in (0, 1/2)$ is as in Lemma 3.8 and $\alpha = a/16 \in (0, 1/32)$, there exists $R > 0$, a subsequence of $\{r_j\}$ and a non-negative function $u_\infty \in L^2(C_{\alpha R}) \cap L^2(\mathcal{I}_{\alpha R}; W^{1,2}(B_{\alpha R}))$ such that $u_j \to u_\infty$ in $L^2(C_{\alpha R})$-norm and weakly in $L^2(\mathcal{I}_{\alpha R}; W^{1,2}(B_{\alpha R}))$. Moreover $u_\infty$ is adjoint caloric in $C_{\alpha R} \cap \{u_\infty > 0\}$,

$$ \|u_\infty\|_{L^2(C_{\alpha R})} \lesssim_{a, R} \omega_\infty(C_{M R}) $$

and

$$ \int \varphi \, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty \, H\varphi, \quad \text{for all } \varphi \in C_c(\mathcal{I}_{\alpha R}). $$

Proof. For simplicity, we will only prove the lemma assuming $\bar{\xi}_j \equiv \bar{\xi}$ for some $\bar{\xi} \in \mathcal{P}^{s} \Omega \cap \partial S\Omega$, since the proof of the general case is analogous. Let us recall that for $\xi_0 \in \mathbb{R}^{n+1}$, $r > 0$, and $a > 0$ we denote

$$ \hat{R}_a^+ (\xi_0; r) = B_r(\xi_0) \times (t_0 - (ar)^2/2, t_0 + r^2). $$

and it holds that $C_{ar/2}(\xi_0) \subset \hat{R}_a^+ (\xi_0; r)$.

As (8.1) is satisfied, by Lemma 3.8 we have that there exists $a \in (0, 1/2]$ such that

$$ \text{Cap}(\hat{R}_a (\xi_0; r_j^2) \setminus \Omega) \gtrsim c r_j^d, $$

and so, by (3.33), it holds that

$$ \omega^\ast (C_{M R_j}(\xi)) \gtrsim c > 0, $$

for all $\bar{\xi} \in C_{ar_j}(\xi) \subset \hat{R}_a^+ (\xi_0; r_j)$. Since $\omega_\infty \neq 0$, there exists $R > 0$ such that $\omega_\infty(C_{R}) > 0$. Without loss of generality we may assume that $R = a/8$. By passing to a subsequence, if necessary, we may assume that $\bar{p} \in \Omega \setminus C_{ar_j}(\xi)$ and combining the latter bound with Lemma 3.21 for $ar_j/2$ and $M' = 2M/a$, we obtain

$$ \omega^\ast(C_{M R_j}(\xi)) \gtrsim (ar_j)^n u(\bar{y}), \quad \text{for } \bar{y} \in \Omega \cap C_{ar_j/8}(\xi). $$

(8.3)

In particular, if we set $\Omega_j := T_{\xi, r_j}(\Omega)$, the previous inequality implies

$$ a^{-n} \omega_j(C_M(\bar{0})) \gtrsim u_j(\bar{y}), \quad \text{for } \bar{y} \in \Omega_j \cap C_{a/8}(\bar{0}). $$

Thus

$$ \limsup_{j \to \infty} \|u_j\|_{L^2(C_{a/8}(\bar{0}))} \leq \limsup_{j \to \infty} \|u_j\|_{L^\infty(C_{a/8}(\bar{0}))(\mathcal{L}^{n+1}(C_{a/8}(\bar{0})))^{1/2}} \lesssim \limsup_{j \to \infty} a^{1 - \frac{n}{2}} \omega_j(C_M(\bar{0})) \leq a^{1 - \frac{n}{2}} \omega_\infty(C_{M}(\bar{0})). $$

(8.4)

Since $G(\bar{p}, \cdot)$ is in $C^{2,1}(\Omega \setminus \{\bar{p}\}) \cap C(\Omega \cup \partial^* \Omega \setminus \{\bar{p}\})$ and $\Omega$ is regular for $H^s$, then, for $j$ large, its extension by zero is continuous in $C_{ar_j/8}(\xi)$ and in the vector-valued Sobolev
space \(L^2(\mathcal{I}_{ar_j/8}(t); W^{1,2}(B_{ar_j/8}(\xi)))\). Moreover, by [Wa12, Theorem 7.20], it is a subtemperature of \(H^*\) and thus adjoint subcaloric in \(C_{ar_j/8}(\xi)\). By a simple rescaling argument, the latter implies that \(u_j \in L^2(\mathcal{I}_{a/8}; W^{1,2}(B_{a/8}))\) is a continuous non-negative adjoint subcaloric function in \(C_{a/8}\) and so, if we apply Caccioppoli’s inequality and use the bound \(8.4\), there exists \(j_0 \geq 1\) such that for \(j \geq j_0\),

\[
\int_{C_{a/16}} |\nabla u_j(x, t)|^2 \, dx \, dt \lesssim a^{-2} \int_{C_{a/8}} |u_j(x, t)|^2 \, dx \, dt \lesssim_n a^{-n} \omega(c(M(0)))^2.
\]

Moreover, once again by \(8.4\),

\[
\sup_{j \geq j_0} I_j(h) := \sup_{j \geq j_0} \int_{\mathcal{I}_{a/16} \cap \{t < (a/16) - h\}} \int_{B_{a/16}} |u_j(x, t + h) - u_j(x, t)|^2 \, dx \, dt 
\lesssim_n a^{2-n} \omega(\overline{C_{M(0)}})^2.
\]

Therefore, there exists \(j_1 \geq j_0\) such that \(\sup_{j \geq j_1} I_j(h) \leq 2I_{j_1}(h)\), which, in turn, by dominated convergence and the continuity of \(u_j\) in \(C_{a/16}\), implies that

\[
\|u_j(x, t + h) - u_j(x, t)\|_{L^2(\mathcal{I}_{a/16} \cap \{t < (a/16) - h\}; L^2(B_{a/16}))} \to 0, \quad \text{as} \ h \to 0,
\]

uniformly in \(j \geq j_0\). Consequently, since \(W^{1,2}(B_{a/16})\) embeds compactly in \(L^2(B_{a/16})\) and, by \(8.5\) we have that \(\{u_j\}_{j \geq j_0}\) is a bounded subset of \(L^2(\mathcal{I}_{a/16}; W^{1,2}(B_{a/16}))\), in light of \(8.6\), we can apply [Sim87, Theorem 5] (see also Theorems 3 and 6 in [Sim87]), to deduce that \(\{u_j\}_{j \geq j_0}\) is relatively compact in \(L^2(C_{a/16})\). Hence, there exists a subsequence of \(\{u_j\}_{j \geq j_0}\) and a function \(u_\infty \in L^2(C_{a/16}) \cap L^2(\mathcal{I}_{a/16}; W^{1,2}(B_{a/16}))\) such that

\[
u_j \to u_\infty \quad \text{strongly in} \ L^2(C_{a/16}) \quad \text{and weakly in} \ L^2(\mathcal{I}_{a/16}; W^{1,2}(B_{a/16})).
\]

A change of variables gives

\[
\int \varphi \, d\omega_j = \int u_j H \varphi, \quad \varphi \in C_c^\infty(C_{a/16})
\]

which, if we take limits as \(j \to \infty\), entails

\[
\int \varphi \, d\omega_\infty = \int u_\infty H \varphi, \quad \varphi \in C_c^\infty(C_{a/16}).
\]

In order to complete the proof of the lemma, it suffices to choose \(\alpha = a/16\). \(\square\)

To prove Theorem 1, we need the following “two-phase” blow-up lemma.

**Lemma 8.2.** Let \(\Omega^\pm\) be two disjoint open sets in \(\mathbb{R}^{n+1}\) which are quasi-regular for \(H\) and regular for \(H^*\), and let \(\omega^\pm\) be the respective caloric measures with poles \(\bar{p}_\pm \in \Omega^\pm\). Assume that \(F \subset \mathcal{P}^*\Omega^+ \cap \mathcal{P}^*\Omega^- \cap S'\Omega^+ \cap S'\Omega^-\) is compact, \(\xi_j\) is a sequence in \(F\) and that \(\Omega^+\) and \(\Omega^-\) satisfy the joint TBCPC. Let us also assume that there exists \(r_j \to 0\), \(c_j > 0\), and a constant \(c > 0\), such that

\[
\omega^+_j := c_j T_{\xi_j, r_j} [\omega^+] \to \omega^+_\infty
\]

and

\[
\omega^-_j := c_j T_{\xi_j, r_j} [\omega^-] \to c \omega^+_\infty =: \omega^-_\infty.
\]
Let $u_\pm := G_{\Omega^\pm}(\bar{p}_{\pm}, \cdot)$ on $\Omega^\pm$, $u_\pm \equiv 0$ on $\mathbb{R}^{n+1} \setminus \Omega$ and denote
\[ u_j^\pm(x) := c_j u_\pm(\delta_j x + \xi_j) r_j^n. \]

The following properties hold:

(a) There exists $\alpha \in (0, 1/16)$, $R > 0$, a subsequence of $r_j$, $u_{\infty}^\pm \in L^2(C_{\alpha R})$, and $u_{\infty} \in L^2(C_{\alpha R})$, a non-zero adjoint caloric function in $C_{\alpha R}$, such that $u_j^\pm$ converge in $L^2(C_{\alpha R})$-norm to $u_{\infty}^\pm$ and $u_j := u_j^+ - c^{-1} u_j^-$ converges in $L^2(C_{\alpha R})$-norm to $u_{\infty} = u_{\infty}^+ - c^{-1} u_{\infty}^-$.  

(b) The function $u_{\infty}$ extends to an adjoint caloric function in $\mathbb{R}^{n+1}$ and it holds that
\[
\left\| u_{\infty} \right\|_{L^2(C_{\eta})} \lesssim \alpha^{-n} \omega_{\infty}^\pm \left( \frac{C_{\alpha R}}{a} \right)
\]
for all $\eta > 0$, and for any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,
\[
\int \varphi \, d\omega_{\infty}^+ = \int u_{\infty}^+ H \varphi = \frac{1}{2} \int_{\mathbb{R}^{n+1}} |u_{\infty}| H \varphi.
\]

(c) We have that
\[
\supp \omega_{\infty}^+ = -\partial \{u_{\infty}^+ > 0\} = \{u_{\infty} = 0\} =: \Sigma_{\infty},
\]
and $d\omega_{\infty}^+ = -\frac{\partial u_{\infty}}{\partial \nu} d\sigma_{\Sigma_{\infty}}$.

**Proof.** As in the proof of the previous lemma, we assume for simplicity that $\xi_j \equiv \xi \in \mathcal{P}^* \Omega^+ \cap \mathcal{P}^* \Omega^-$ for all $j$ and $\omega_{\infty}(C_{\alpha/16}) > 0$. Since the domains $\Omega^+$ and $\Omega^-$ are disjoint, the subadditivity property for the thermal capacity [Wa12, Theorem 7.45-(b)] entails
\[
\text{Cap}(E(\xi; r_j^2) \setminus \Omega^+) + \text{Cap}(E(\xi; r_j^2) \setminus \Omega^-) \geq \text{Cap}(E(\xi; r_j^2)^+), \quad j \geq 0,
\]
and so, possibly after passing to a subsequence, without loss of generality, we can assume that
\[
\text{Cap}(E(\xi; r_j^2) \setminus \Omega^+) \geq \frac{\text{Cap}(E(\xi; r_j^2))}{2} \approx r_j^n, \quad j \geq 0.
\]
Moreover, as $\Omega^+$ and $\Omega^-$ satisfy the joint TBCPC condition, there exists a subsequence such that
\[
\text{Cap}(E(\xi; r_j^2) \setminus \Omega^-) \gtrsim r_j^n, \quad j \geq 0.
\]

This provides us with a common sequence $r_j$ such that Lemma 8.1 applies to both $\omega^+$ and $\omega^-$. Thus, for $\alpha = a/16 \in (0, 1/32)$ there exist two adjoint subcaloric functions $u_{\infty}^\pm$ such that $u_j^\pm \to u_{\infty}^\pm$ in $L^2(C_{\alpha})$ and
\[
\int \varphi \, d\omega_{\infty}^\pm = \int u_{\infty}^\pm H \varphi, \quad \varphi \in C_c^\infty(C_{\alpha}).
\]

In particular, $u_j$ converges in $L^2(C_{\alpha})$-norm to the function $u_{\infty} := u_{\infty}^+ - c^{-1} u_{\infty}^-$ and, for $\varphi \in C_c^\infty(C_{\alpha})$,
\[
\int u_{\infty} H \varphi = \int u_{\infty}^+ H \varphi - c^{-1} \int u_{\infty}^- H \varphi \overset{(8.10)}{=} \int \varphi \, d\omega_{\infty}^+ - c^{-1} \int \varphi \, d\omega_{\infty}^-, \quad \int \varphi \, d\omega_{\infty}^+ - c^{-1} \int \varphi \, d\omega_{\infty}^- = 0.
\]

which proves that $u_{\infty}$ is adjoint caloric in $C_{\alpha}$. 

Furthermore, as \( \text{supp} \ u_j^+ \cap \text{supp} \ u_j^- = \emptyset \) for every \( j \), it holds that
\[
\int_{C_\alpha} u_j^+ u_j^- = \lim_{j \to \infty} \int_{C_\alpha} u_j^+ u_j^- = 0,
\]
which implies
\[
(8.12) \quad L^{n+1}(\text{supp} \ u_j^+ \cap \text{supp} \ u_j^- \cap C_\alpha) = 0.
\]
As a result of (8.10), \( \{ u_j^+ > 0 \} \cap C_\alpha \) is a set of positive \( L^{n+1} \)-measure and, thus, \( u_\infty \neq 0 \). Another consequence is that
\[
(8.13) \quad u_\infty^+ = u_\infty \chi_{\{ u_\infty > 0 \}} \quad \text{and} \quad u_\infty^- = -c^{-1} u_\infty \chi_{\{ u_\infty < 0 \}}.
\]
In particular, \( u_\infty^\pm \) are continuous in \( C_\alpha \) because \( u_\infty \) is an adjoint caloric function.

Let us prove (b). The same argument as the one in the proof Lemma 8.1 shows that, given \( K \geq 1 \),
\[
u^+_j(y) \lesssim a \omega^+_j(C_{KM}(0)), \quad \bar{y} \in C_{K^{1/8}}(0)
\]
with the implicit constant independent of \( j \). Moreover
\[
\limsup_j \omega^+_j(C_{KM}(0)) \leq \omega_\infty^+(C_{KM}(0)), \quad \text{for all } K \geq 1.
\]
So, arguing again as we did in the proof of Lemma 8.1, we get that for any fixed \( K = 1, 2, \ldots \), the sequence \( u_j \) admits a converging subsequence in \( L^2(C_{aK/4}) \). By a standard diagonalization argument we can show that \( u_\infty \) extends to a caloric function in \( \mathbb{R}^{n+1} \) satisfying (8.8). As it is easy to see that (8.9) holds, the proof of (b) is concluded.

Let us turn our attention to the proof of (c). Let \( \bar{x} \in \text{supp} \omega^+_\infty \) and assume that there exists a cylinder \( C_r(\bar{y}) \) such that \( \bar{x} \in C_r(\bar{y}) \subset \{ u^+_\infty > 0 \} \). In particular, (8.13) implies that \( C_r(\bar{y}) \subset \{ u^-_\infty = 0 \} \). Then, if \( \varphi \in C_0^\infty(C_r(\bar{y})) \) such that \( \varphi(\bar{x}) > 0 \), since \( u_\infty \) is globally adjoint caloric, we have the estimate
\[
(8.14) \quad 0 < \int \varphi \, d\omega^+_\infty = \int u^+_\infty H \varphi = \int u^-_\infty H \varphi = 0,
\]
so we get a contradiction. A similar argument shows that \( C_r(\bar{y}) \not\subset \{ u^+_\infty = 0 \} \) for any cylinder such that \( \bar{x} \in C_r(\bar{y}) \). So \( C_r(\bar{y}) \cap \{ u^+_\infty > 0 \} \neq \emptyset \) and \( C_r(\bar{y}) \cap \{ u^-_\infty = 0 \} \neq \emptyset \), which shows that
\[
(8.15) \quad \text{supp} \omega^+_\infty \subset \partial \{ u^+_\infty > 0 \}.
\]

We now claim that
\[
(8.16) \quad \partial \{ u^+_\infty > 0 \} \subset \text{supp} \omega^+_\infty.
\]
To prove this, we take \( \bar{x} = (x, t) \in \partial \{ u^+_\infty > 0 \} \) and assume, by contradiction, that \( \bar{x} \in C_{r}(\bar{y}) \subset (\mathbb{R}^{n+1} \setminus \text{supp} \omega^+_\infty) \) for some \( \bar{y} \in \{ \text{supp} \omega^+_\infty \}^c \) and \( r > 0 \). In particular,
\[
\int u^+_\infty H \varphi \overset{(8.9)}{=} \int \phi \, d\omega^+_\infty = 0 \quad \text{for all } \phi \in C_0^\infty(C_r(\bar{y})),
\]
which gives that \( u^+_\infty \) is adjoint caloric in \( C_r(\bar{y}) \). Since \( \bar{x} \in \partial \{ u^+ > 0 \} \cap C_r(\bar{y}) \), we have \( u^+_\infty(\bar{x}) = 0 \). Hence, the strong minimum principle for adjoint caloric functions (which readily follows from [Wal12, Theorem 3.11]) gives that \( u^-_\infty \equiv 0 \) on \( \{ (\zeta, \theta) \in C_r(\bar{y}) : \theta < t \} \). In particular, \( u^+_\infty = u^-_\infty = 0 \) in an open cylinder \( C_\theta(\bar{y}) \), where the first equality follows from (8.9). So, the fact that \( u_\infty \) extends to a globally adjoint caloric function and
3.28. Let \(\xi\in\mathcal{S}\Omega\) and \(\rho>0\), then, if we denote \(\tilde{\Omega}:=T_{\xi,\rho}[\Omega]\), we have that \(\mathcal{S}\Omega\subset\partial^*\tilde{\Omega}\).

The next lemma lists the main properties of the blow-ups of caloric measure and the adjoint Green’s function in a domain satisfying the assumptions of Lemma 8.1 that has the TFDC instead of just being regular for \(H^*\). Our proof is inspired from the one of the corresponding result for harmonic measure in [AMT17] and although it is the essentially the same for items (a)-(c), the proof of (d) is different.

**Lemma 8.3.** Let \(\Omega\subset\mathbb{R}^{n+1}\) be an open set which is quasi-regular for \(H\) and satisfies the TFDC, and let \(F\subset\mathcal{P}\Omega\cap\mathcal{S}\Omega\) be compact and \(\{\xi_j\}_{j}\subset F\). We denote by \(\omega\) the caloric measure for \(\Omega\) with pole at \(\tilde{p}\in\Omega\), and assume that there exists \(c_j>0\) and \(r_j\to0\) such that \(\omega_j:=c_jT_{\xi_j,r_j}[\omega]\to\omega_\infty\) for some non-zero Radon measure \(\omega_\infty\). If we denote \(\Omega_j:=T_{\xi_j,r_j}[\Omega]\) and assume that there is \(c>0\) so that

\[
\text{Cap}(E(\xi_j;r_j^2)\setminus\Omega)\geq cr_j^n, \quad j=1,2,\ldots
\]

then there is a subsequence of \(\{r_j\}_{j\geq1}\) and a closed set \(\Sigma\subset\mathbb{R}^{n+1}\) such that

(a) For all \(R>0\) sufficiently large, \(\partial\Omega_j\cap\overline{C_R(0)}\to\Sigma\cap\overline{C_R(0)}\) in the Hausdorff metric.
(b) \(\mathbb{R}^{n+1}\setminus\Sigma = \Omega_\infty\cup\widehat{\Omega}_\infty\) where \(\Omega_\infty\) is a nonempty open set and \(\widehat{\Omega}_\infty\) is also open but possibly empty. Moreover, there exists \(\lambda>1\) so that, if \(C_\lambda(\tilde{x})\) is a cylinder such that \(C_\lambda(\tilde{x})\subset\Omega_\infty\), then \(C_\lambda(\tilde{x})\) is contained in \(\Omega_j\) for all \(j\) large enough.
(c) \(\text{supp}\omega_\infty\subset\Sigma\).
(d) If we set \(u(\tilde{x})=G_{\xi\Omega}(\tilde{p}_+,\tilde{x})\) on \(\Omega\) and \(u\equiv0\) on \((\Omega)^c\), and define

\[
u_j(\tilde{x}) = c_j r_j^n u(\delta_{r_j} (\tilde{x}) + \tilde{\xi}_j),
\]

then the sequence \(u_j\) converges uniformly on compact subsets of \(\mathbb{R}^{n+1}\) to a non-zero function \(u_\infty\) which is continuous in \(\mathbb{R}^{n+1}\), adjoint caloric in \(\Omega_\infty\), and vanishes in \((\Omega_\infty)^c\). Moreover if \(a\in(0,1/2)\) and \(M>1\) are the constants obtained in Lemmas 3.8 and 3.15 respectively, then for \(\tilde{x}\in\Sigma\) and \(r>0\), it holds that

\[
u_\infty(\tilde{y}) \lesssim a^{-n} \omega_\infty(C_{4a^{-1}M}(\tilde{x})), \quad \tilde{y}\in C_\lambda(\tilde{x})\cap\Omega_\lambda^+,
\]
and
\begin{equation}
\int \varphi \, d\omega_\infty = \int_{\mathbb{R}^{n+1}} u_\infty \, H\varphi, \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^{n+1}).
\end{equation}

**Proof.** For simplicity, we assume \( K = \{\xi\} \), as the general case can be proved similarly.

**Proof of (a):** Let \( R > 0 \) and observe that \( \bar{0} \in \partial \Omega_j \) for all \( j \geq 1 \), which entails \( \partial \Omega_j \cap C_R(\bar{0}) \neq \emptyset \). The Hausdorff distance is a metric on the collection \( C(\overline{C_R(0)}) \) of all closed subsets of \( \overline{C_R(0)} \). Since \( \overline{C_R(0)} \) is compact, \( (C(\overline{C_R(0)}), d_H) \) is compact too. Thus, after passing to a subsequence, \( C_R(0) \cap \partial \Omega_j \rightarrow C_R(0) \cap \Sigma \) in the Hausdorff distance sense for some \( \Sigma \in C(\overline{C_R(0)}) \). A standard diagonalization argument concludes the proof of (a).

**Proof of (b):** We will first prove that there exists a parabolic cylinder \( C_\rho(\bar{x}') \) which is contained in all \( \Omega_j \), for \( j \) large enough. Arguing by contradiction we assume not. Let \( \phi \in C_c^\infty(\mathbb{R}^{n+1}) \) be a non-negative function such that \( \int \phi \, d\omega_\infty \neq 0 \) and \( \supp \phi \subset C_K(\bar{0}) \) for some \( K > 0 \). Hence, there exists \( \bar{x}_0 \) belonging to \( C_K(\bar{0}) \cap \supp \omega_\infty \), and our counterassumption implies that
\begin{equation}
\rho_j := \sup\{\dist_p(\bar{x}, (\Omega_j)^c) : \bar{x} \in C_{2K}(\bar{0})\} \rightarrow 0, \quad \text{as } j \rightarrow \infty.
\end{equation}

We denote by \( \bar{\zeta}_j(\bar{x}) \in (\Omega_j)^c \) a point which realizes \( \dist_p(\bar{x}, (\Omega_j)^c) \). In particular, since \( \rho_j \rightarrow 0 \) and \( 0 \in \partial \Omega_j \) for all \( j \), we have that \( \|\bar{x} - \bar{\zeta}_j(\bar{x})\| \leq \rho_j < 2K \), for \( j \) large enough and \( \bar{x} \in C_{2K}(\bar{0}) \). Moreover, \( \|\bar{x} - \bar{x}_0\| \leq \|\bar{x}\| + \|\bar{x}_0\| < 3K \).

Observe that \( \Omega_j \) satisfies the TFCDC with the same parameters as \( \Omega \) because of Lemma 3.14. Moreover, \( \bar{\zeta}_j(\bar{x}) \in \partial \Omega_j \) for \( j \) big enough because \( T_{\min}(\Omega_j) = (T_{\min}(\Omega) - \tau)/r_j^2 \rightarrow -\infty \) and \( T_{\max}(\Omega_j) = (T_{\max}(\Omega) - \tau)/r_j^2 \rightarrow +\infty \) as \( j \rightarrow \infty \). In fact, \( \bar{\zeta}_j(\bar{x}) \in \partial \Omega_j \) (see the discussion before the statement of this lemma).

Note that for \( \bar{x} \in C_{2K}(\bar{0}) \), we have \( C_{2K}(\bar{\zeta}_j(\bar{x})) \subset C_{4K}(\bar{0}) \cap C_{5K}(\bar{x}_0) \). If we consider \( j \) large enough so that \( \bar{p} \in C_{16K}(\bar{0}) \), then if we combine (3.40), (3.33), and (8.17), we have that for \( \bar{y} \in C_{4K}(\bar{0}) \),
\begin{equation}
u_j(\bar{y}) = c_j r_j^n u(\delta_r, \bar{y} + \xi) 
\leq c_j r_j^n (4K r_j)^{-n} \omega \left( \delta_r, C_{AKM}(\bar{0}) + \xi \right) \approx_K \omega_j(\bar{y}) (C_{AKM}(\bar{0})).
\end{equation}

Remark that \( M \) depends on \( a \) and is chosen so that (3.34) holds. Hence, using the \( \gamma \)-Hölder continuity at the boundary for \( u_j \) in \( \Omega_j \) (that holds with constants not depending on \( j \) because of Lemma 3.14), we infer that
\begin{equation}
0 < \int \phi \, d\omega_j = \int_{\Omega_j} u_j \, H(\phi) \left( \sup_{C_{AKM}(\bar{\zeta}_j(\bar{x}))(\bar{\zeta}_j(\bar{x}))} u_j^+ \right)^\gamma \frac{\|\bar{x} - \bar{\zeta}_j(\bar{x})\|}{2K} \, d\bar{x} \leq_K \omega_j(\bar{y}) (C_{AKM}(\bar{0}))(\frac{\rho_j}{2K})^\gamma \int |H\phi| \omega_j(\bar{y}) (C_{AKM}(\bar{0})) \rho_j^\gamma.
\end{equation}

So, taking the limit in the previous inequalities as \( j \rightarrow \infty \) we obtain
\begin{equation}
0 < \int \phi \, d\omega_\infty \leq \limsup_{j \rightarrow \infty} \omega_j(\bar{y}) (C_{AKM}(\bar{0})) \leq \omega_\infty(\bar{y}) \lim_{j \rightarrow \infty} \rho_j^\gamma \leq \omega_\infty(\bar{y}) \lim_{j \rightarrow \infty} \rho_j^\gamma \leq 0,
\end{equation}
which is a contradiction. Thus, after passing to a subsequence, if necessary, there exists a cylinder \( C_\rho(\bar{x}') \subset \Omega_j \) for \( j \) big enough.
Similarly, one can show that
\[
\|\bar{\gamma}\| = \max\{\|x - x\|, |u - t|^{1/2}\} \leq \max\{\|y - x\|, |s - t|^{1/2}\} = \|\bar{y} - \bar{x}\|.
\]
Similarly, one can show that \(\|\bar{y} - \bar{z}\| \leq \|\bar{y} - \bar{x}\|\).

If \(s < t\), we set \(\bar{\gamma}(\bar{x}, \bar{y}) = \gamma(\bar{y}, \bar{x})\), where \(\gamma(\cdot, \cdot)\) is defined above. The parabola \(\bar{\gamma}(\bar{x}, \bar{y})\) meets \(\partial\Omega_j\) at a point, say \(\bar{z}\). Since we choose \(j\) big enough, \((3.39)\) applies (with constant independent on \(j\)) because the TFCDC condition holds with uniform constants for \(\Omega_j\) and we have
\[
|u_j(\bar{x}) - u_j(\bar{y})| = u_j(\bar{x}) \leq \sup_{C_{2K}(\bar{x})} u_j \leq \|\bar{x} - \bar{z}\|^\alpha \|\bar{y} - \bar{x}\|^\beta \leq \|\bar{y} - \bar{x}\|^\alpha \|\bar{x} - \bar{z}\|^\beta \leq \|\bar{y} - \bar{x}\|^\alpha.
\]

Case 3. Let us assume that \(\bar{x}, \bar{y} \in \Omega_j \cap \bar{C}_K(0)\) and that there exists a point \(\bar{z} \in \mathbb{R}^{n+1} \setminus \Omega_j\) such that \(\text{dist}(\bar{z}, \bar{\gamma}(\bar{x}, \bar{y})) \leq \|\bar{x} - \bar{y}\|/3\). Without loss of generality, we assume \(u_j(\bar{x}) \leq u_j(\bar{y})\). If \(\bar{\zeta} \in \bar{\gamma}(\bar{x}, \bar{y})\) denotes a point such that \(\|\bar{z} - \bar{\zeta}\| = \text{dist}(\bar{z}, \bar{\gamma}(\bar{x}, \bar{y}))\), we have
\[
\|\bar{y} - \bar{z}\| \leq \|\bar{y} - \bar{\zeta}\| + \|\bar{\zeta} - \bar{z}\| \leq \|\bar{x} - \bar{y}\| + \frac{1}{3}\|\bar{x} - \bar{y}\| = \frac{4}{3}\|\bar{x} - \bar{y}\| \leq \frac{8}{3}K.
\]
Observe that \( \tilde{\gamma} \in C_{3K}(\tilde{z}) \subset C_{5K}(\tilde{0}) \). Since we have \( \bar{p} \in \Omega \setminus C_{400K}r_{j}(\tilde{\xi}) \), we can apply (3.39) and get

\[
|u_j(\bar{x}) - u_j(\bar{y})| \leq 2u_j(\bar{y}) \leq \left( \sup_{C_{3K}(\tilde{z})} u_j \right) \|\bar{y} - \bar{z}\|^\alpha \leq \left( \sup_{C_{5K}(\tilde{0})} u_j \right) \|\bar{y} - \bar{z}\|^\alpha \leq_{K} \omega_\infty(C_{5K}M) \|\bar{y} - \bar{z}\|^\alpha \leq_{K} \|\bar{x} - \bar{y}\|^\alpha .
\]

**Case 4.** Suppose that \( \bar{x}, \bar{y} \in \Omega_j \cap C_K \) and that dist\((\bar{z}, \gamma(\bar{x}, \bar{y})) > \|\bar{x} - \bar{y}\|/3 \) for all \( \bar{z} \in \mathbb{R}^{n+1} \setminus \Omega_j \). We denote \( \delta := \|\bar{x} - \bar{y}\|/100 \) and we cover \( \gamma(\bar{x}, \bar{y}) \) with \( N \) cylinders of the form \( C_{\delta}(\bar{y}_k) \), for \( \bar{y}_k \in \gamma(\bar{x}, \bar{y}) \), \( k = 1, \ldots, N \). The cylinders are chosen so that \( C_{\delta}(\bar{y}_{k+1}) \cap C_{\delta}(\bar{y}_k) \neq \emptyset \), \( \bar{x} \in C_{\delta}(\bar{y}_1) \) and \( \bar{y} \in C_{\delta}(\bar{y}_N) \). For \( i = 1, \ldots, N - 1 \) we choose \( \bar{x}_i \in C_{\delta}(\bar{y}_{i+1}) \cap C_{\delta}(\bar{y}_i) \) and define \( \bar{x}_0 = \bar{x} \) and \( \bar{x}_N = \bar{y} \). Remark that \( N \) is independent of \( \delta \) and \( j \). Since \( u_j \) is caloric in \( \Omega_j \cap C_K(\tilde{0}) \), we can use the interior Hölder continuity (see e.g. Lieb96, Theorem 6.29) and write

\[
|u_j(\bar{x}) - u_j(\bar{y})| \leq \sum_{i=1}^{N} |u_j(\bar{x}_{i+1}) - u_j(\bar{x}_i)| \leq \sum_{k=0}^{N} \|u_j\|_{L^\infty(C_{\delta}(\bar{y}_1))} \|\bar{x}_{k+1} - \bar{x}_k\|^\beta \leq \|u_j\|_{L^\infty(C_{2K}(\bar{0}))} \|\bar{x} - \bar{y}\|^\beta \lesssim K \omega_\infty(C_{2K}M) \|\bar{x} - \bar{y}\|^\beta ,
\]

where \( \beta \in (0,1) \) and the implicit constants are independent of \( j \).

Combining the four cases above, after passing to a subsequence, we obtain that \( u_j \) is equicontinuous in \( C_K \) for each \( K \geq 1 \). Hence, we can apply Ascoli-Arzelà’s Theorem and a standard diagonalization argument to get that \( u_j \) has a subsequence that converges uniformly on compact subsets of \( \mathbb{R}^{n+1} \) to a continuous function \( u_\infty \). Moreover, for \( \bar{x} \in \Sigma \), \( r > 0 \), and \( \bar{y} \in C_r(\bar{x}) \cap \Omega_\infty \), we have

\[
u_\infty(\bar{y}) = \lim_{j \to \infty} u_j(\bar{y}) \leq \limsup_j r^{-n} \omega_j(C_{4M}^{-1}(\bar{x})) \leq r^{-n} \omega_\infty(C_{4M}^{-1}(\bar{x})) < \infty ,
\]

which gives (8.18). It remains to prove (8.19). To this end, for \( \phi \in C_c(\mathbb{R}^{n+1}) \) it holds

\[
\int \phi \, d\omega_j = c_j \int \phi \, d\tau_{\bar{\xi}, r_j} \, d\omega = c_j \int H(\phi \, d\tau_{\bar{\xi}, r_j}) \, u = c_j r_j^{-2} \int ((\bar{H}\phi) \, d\tau_{\bar{\xi}, r_j}) \, u (8.26)
\]

Taking limits as \( j \to \infty \) in (8.26), in light of the fact that \( \text{supp}\, \omega_\infty \cap \Omega_\infty \subset \Sigma \cap \Omega_\infty \) by (c) and that \( u_j \to u_\infty \) locally uniformly in \( \mathbb{R}^{n+1} \), we have that (8.19) holds. It is straightforward to see that \( u_\infty \) is adjoint caloric in \( \Omega_\infty \).

The function \( u_\infty \) is not identically zero. Indeed, if \( \phi \in C_c(\mathbb{R}^{n+1}) \) is a non-negative function so that \( \int \phi \, d\omega_\infty > 0 \) and plug it in (8.19), we obtain

\[
0 < \int \phi \, d\omega_\infty = \int u_\infty \, H\phi .
\]

This readily implies that \( u_\infty \neq 0 \) in \( \mathbb{R}^{n+1} \).

In order to finish the proof of (d), we claim that \( u_\infty \equiv 0 \) on \( \mathbb{R}^{n+1} \setminus \Omega_\infty \). Assume that there is \( \bar{x} \in \mathbb{R}^{n+1} \setminus \Omega_\infty \) so that \( u_\infty(\bar{x}) > 0 \). By the continuity of \( u_\infty \), there exists \( \delta > 0 \) such that \( u_\infty(\bar{y}) > u_\infty(\bar{x})/2 \) for all \( \bar{y} \in C_\delta(\bar{x}) \). The local uniform convergence of \( u_j \) to \( u \)
implies that \( u_j > u_\infty(\bar{x})/4 \) on \( C_\delta(\bar{x}) \) for all \( j \) big enough. Thus, since \( u_j \) vanishes outside of \( \Omega_j \), we have that \( C_\delta(\bar{x}) \subset \Omega_j \) for \( j \) large. In particular, it holds that \( C_{\delta/2}(\bar{x}) \subset \mathbb{R}^{n+1} \setminus \Sigma \), so \( C_{\delta/2}(\bar{x}) \subset \Omega_\infty \) by the construction (8.22) and we can conclude that \( \bar{x} \in \Omega_\infty \), which is a contradiction. This proves our claim and finishes the proof of the lemma.

The next lemma studies the two-phase version of the previous result for domains that satisfy the joint TBCPC. Although, this proof is inspired by the one of [AMT17, Lemma 5.9], there are several differences as well. Let us remark that we do not assume the domains to be complementary, which gives us significant freedom in the applications in Section 9, but also requires additional care in the proof.

**Lemma 8.4.** Let \( \Omega^\pm \subset \mathbb{R}^{n+1} \) be two disjoint open sets that are quasi-regular for \( H \) and satisfy the joint TBCPC and the TFCDC. Let \( F \subset \mathcal{P}^+\Omega^+ \cap \mathcal{P}^+\Omega^- \cap \mathcal{S} \Omega^+ \cap \mathcal{S} \Omega^- \) be compact and \( \{\bar{\xi}_j\}_{j} \subset F \). We denote by \( \omega \) the caloric measure for \( \Omega^\pm \) with pole at \( \bar{p}_\pm \in \Omega^\pm \), and assume that there exists \( c_j > 0, c > 0 \) and \( r_j \to 0 \) such that

\[
\omega_j^+ := c_j T_{\bar{\xi}_j, r_j} [\omega^+] \to \omega_\infty^+ 
\]

and

\[
\omega_j^- := c_j T_{\bar{\xi}_j, r_j} [\omega^-] \to c \omega_\infty^- =: \omega_\infty^-.
\]

We set \( u_j^\pm(\bar{x}) = G_{\Omega^\pm}(\bar{p}_\pm, \bar{x}) \) on \( \Omega^\pm \) and \( u_j^\pm \equiv 0 \) on \( (\Omega^\pm)^c \), and define

\[
u_j^+ := c_j r_j^n u_j^+ (\delta_{r_j}(\bar{x}) + \bar{\xi}_j).
\]

Then there exists a subsequence of \( \{r_j\}_{j} \geq 1 \), \( u_{\infty}^+, \Omega_{\infty}^+, \hat{\Omega}_{\infty}^+, \) and \( \Sigma^+ \) such that the properties (a)-(d) in Lemma 8.3 hold; moreover, if \( u_j := u_j^+ + c^{-1} u_j^- \), then \( u_j \to u_\infty \) uniformly on compact subsets of \( \mathbb{R}^{n+1} \) and the following properties hold:

(a) \( u_\infty \) is an adjoint caloric function on \( \mathbb{R}^{n+1} \).
(b) \( \hat{\Omega}_{\infty}^+ = \Omega_{\infty}^+ \) and \( \Sigma^+ = \Sigma^- =: \Sigma \). Moreover \( \Sigma = \{u_\infty = 0\} \), with \( u_\infty > 0 \) on \( \Omega^+_{\infty} \) and \( u_\infty < 0 \) on \( \Omega^-_{\infty} \).
(c) \( \Sigma \) is a set of Euclidean Hausdorff dimension \( n \) and

\[
\partial \omega^+_{\infty} = -\partial_t u_\infty d\sigma_{\partial \Omega^+_{\infty}},
\]

where \( \sigma_{\partial \Omega^+_{\infty}} \) stands for the parabolic surface measure on \( \partial \Omega^+_{\infty} \) and \( \nu_t \) is the measure-theoretic outer unit normal in its time-slices.

**Proof.** Let us assume for brevity that \( \bar{\xi}_j = \bar{\xi} \). Given \( \Omega^\pm \) as in the statement, by the same pigeonholing argument as in Lemma 8.2 and in light of the joint TBCPC property, we can find a subsequence of \( r_j \) such that

\[
(8.27) \quad \text{Cap}(E(\bar{\xi}; r_j^2) \setminus \Omega^\pm) \geq r_j^n, \quad j \geq 0.
\]

So, (8.17) holds for both \( \Omega^+ \) and \( \Omega^- \) and we can apply Lemma 8.3 to find \( u_\infty^+, \Omega_{\infty}^+, \hat{\Omega}_{\infty}^+, \) and \( \Sigma^+ \) satisfying the properties (a)-(d) of Lemma 8.3.

**Proof of (a):** Let \( \phi \in C_0^\infty(\mathbb{R}^{n+1}) \). By hypothesis we have that \( \omega_j^+ \to \omega_\infty^+ \) and \( \omega_j^- \to c \omega_\infty^- \), and thus, by the local uniform convergence of \( u_j^\pm \) on \( \mathbb{R}^{n+1} \), we can find a subsequence
there exists $\Omega^- \subset \Omega^+$ that satisfies the continuation principle for globally adjoint caloric functions (Theorem 1.16). This gives that $u$ is a weakly adjoint caloric function in $\mathbb{R}^{n+1}$ and Lemma 3.1 applies.

**Proof of (b):** Let us first prove that $\mathbb{R}^{n+1} \setminus (\Omega^+ \cup \Omega^-) = \{ u_\infty = 0 \}$. By Lemma 8.3-(d) we have that the functions $u^\pm_\infty$ simultaneously vanish on $(\Omega^+_\infty)^c \cap (\Omega^-_\infty)^c = \mathbb{R}^{n+1} \setminus (\Omega^+_\infty \cup \Omega^-_\infty)$. Thus, $\mathbb{R}^{n+1} \setminus (\Omega^+_\infty \cup \Omega^-_\infty) \subset \{ u^+\infty = 0 \} \cap \{ u^-\infty = 0 \} \subset \{ u_\infty = 0 \}.$

In order to prove the converse inclusion, we first observe that $u_\infty$ is non-zero since by Lemma 8.3-(d) it holds that $u^\pm_\infty \neq 0$ and vanish outside $\Omega^\pm_\infty$. By construction, we have that $u^\pm_\infty \geq 0$ on $\Omega^\pm_\infty$. Let us assume by contradiction that $u^+(\hat{x}) = 0$ for some $\hat{x} \in \Omega^+_\infty$ and let

$$E^*(\hat{x}; r) := \{(y, s) \in \mathbb{R}^{n+1} : (y, t - s) \in E(\hat{x}, r)\}$$

be a reflected heat ball so that $E^*(\hat{x}; r) \subset \Omega^+_\infty$. The mean value theorem for adjoint caloric functions (that readily follows from the mean value theorem for caloric functions [Wa12, Theorem 1.16]) gives that

$$0 = u_\infty(\hat{x}) = (4\pi r)^{-n/2} \int_{E^*(\hat{x}; r)} \frac{|x - y|}{4(t - s)^2} u(x, t) \, dx \, dt = (4\pi r)^{-n/2} \int_{E^*(\hat{x}; r)} \frac{|x - y|}{4(t - s)^2} u^+(x, t) \, dx \, dt,$$

which implies that $u_\infty = u^+_\infty \equiv 0$ on $E^*(\hat{x}; r)$. In particular, $u_\infty$ has infinite order of vanishing at all the points of $E^*(\hat{x}; r)$ in the sense of [Po96, p. 522]. Hence, by the unique continuation principle for globally adjoint caloric functions [Po96, Theorem 1.2], we deduce that $u_\infty \equiv 0$ on $\mathbb{R}^{n+1}$, which is a contradiction. The same argument can be applied to show that $u^-_\infty > 0$ on $\Omega^-_\infty$. So, since $u^\pm_\infty = 0$ in $\mathbb{R}^{n+1} \setminus \Omega^\pm_\infty$, we get $u_\infty \neq 0$ in $\Omega^+_\infty \cup \Omega^-_\infty$ or equivalently,

$$\{ u_\infty = 0 \} \subset (\mathbb{R}^{n+1} \setminus \Omega^+_\infty) \cap (\mathbb{R}^{n+1} \setminus \Omega^-_\infty) = \mathbb{R}^{n+1} \setminus (\Omega^+_\infty \cup \Omega^-_\infty).$$

Let us remark that, by construction, $\mathbb{R}^{n+1} = \Omega^+_\infty \cup \Omega^-_\infty \cup \Sigma^\pm$, where the unions are disjoint. Hence

$$\hat{\Omega}^-_\infty = (\hat{\Omega}_\infty^- \cap \Omega^+_\infty) \cup (\hat{\Omega}_\infty^- \cap \Omega^-_\infty) \cup (\hat{\Omega}_\infty^- \cap \Sigma^+).$$

We claim that the second and third term under parenthesis in the right hand side of (8.29) are the empty set.

First, let us show that $\hat{\Omega}_\infty^+ \cap \hat{\Omega}_\infty^- = \varnothing$. Indeed, since both sets are open, if $\hat{x} \in \hat{\Omega}_\infty^+ \cap \hat{\Omega}_\infty^-$, there exists $\rho > 0$ such that $C_\rho(\hat{x}) \subset \hat{\Omega}_\infty^+ \cap \hat{\Omega}_\infty^- \subset (\mathbb{R}^{n+1} \setminus \Omega^+_\infty) \cap (\mathbb{R}^{n+1} \setminus \Omega^-_\infty) \subset (\mathbb{R}^{n+1} \setminus (\Omega^+_\infty \cup \Omega^-_\infty)) \subset \{ u_\infty = 0 \}$. Hence, by [Po96, Theorem 1.2] we have that $u_\infty \equiv 0$, which is a contradiction.

Secondly, we prove that $\hat{\Omega}^-_\infty \cap \Sigma^+ = \varnothing$. If $\hat{x} \in \hat{\Omega}^-_\infty \subset \mathbb{R}^{n+1} \setminus \Omega^-_\infty$, Lemma 8.3-(d) gives that $u^-_\infty(\hat{x}) = 0$. Analogously, if $\hat{x} \in \Sigma^+ \subset \mathbb{R}^{n+1} \setminus \Omega^+_\infty$, we have $u^+_\infty(\hat{x}) = 0$. Hence, if $\hat{x} \in \hat{\Omega}_\infty^- \cap \Sigma^+$ we get $u_\infty(\hat{x}) = 0$. Moreover, $\hat{\Omega}_\infty^-$ is an open set, hence $C_\varepsilon(\hat{x}) \subset \hat{\Omega}_\infty^-$ for $\varepsilon$ small enough. In particular, $u_\infty = 0$ on $C_\varepsilon(\hat{x})$, which entails that $u_\infty \geq 0$ in this cylinder.
So, since \( u_\infty \) is globally adjoint caloric and \( u_\infty(\bar{x}) = 0 \), the strong minimum principle [Wa12, Theorem 3.11] entails \( u_\infty = 0 \) on \( C^+_\varepsilon(\bar{x}) \). Thus, by the unique continuation we get \( u_\infty \equiv 0 \) on \( \mathbb{R}^{n+1} \), which is again a contradiction.

Therefore, by (8.29), we obtain \( \Omega^- = \Omega^-_\infty \cap \Omega^+_\infty \subset \Omega^+_\infty \). Since \( \Omega^+ \) and \( \Omega^- \) are disjoint, we also have \( \Omega^+_\infty \subset \mathbb{R}^{n+1} \setminus \Omega^-_\infty = \Omega^-_\infty \), concluding that \( \hat{\Omega}^- = \Omega^+_\infty \) as desired. Symmetrically, we can show that \( \Omega^+_\infty = \hat{\Omega}^+ \).

In order to finish the proof of (b), it is enough to observe that the construction of \( \Sigma^\pm \) and the equality \( \Sigma = \Sigma \) imply
\[
\Sigma^+ = \mathbb{R}^{n+1} \setminus (\Omega^{+\infty} \cup \hat{\Omega}^+) = \mathbb{R}^{n+1} \setminus (\Omega^{+\infty} \cup \Omega^-_\infty)
\]
and
\[
\Sigma^- = \mathbb{R}^{n+1} \setminus (\Omega^{-\infty} \cup \hat{\Omega}^-) = \mathbb{R}^{n+1} \setminus (\Omega^{+\infty} \cup \Omega^-_\infty).
\]
Hence, the set \( \Sigma := \Sigma^+ = \Sigma^- \) is well-defined and it coincides with \( \mathbb{R}^{n+1} \setminus (\Omega^{+\infty} \cup \Omega^-_\infty) \).

Since we have already proved that \( \mathbb{R}^{n+1} \setminus (\Omega^{+\infty} \cup \Omega^-_\infty) = \{ u_\infty = 0 \} \), we obtain that \( \Sigma = \{ u_\infty = 0 \} \) as wished.

**Proof of (c):** The fact that \( \dim_H \Sigma \leq n \) is given by [HL94, Theorem 1.1]. To show that \( \dim_H \Sigma = n \), consider two cylinders \( C_+ \) and \( C_- \) contained in \( \Omega^+_\infty \) and \( \Omega^-_\infty \) respectively. By (b), \( \pm u_\infty > 0 \) on \( C_\pm \) and, by continuity of \( u_\infty \), any line connecting a point in \( C_+ \) to a point in \( C_- \) has to cross \( \Sigma \). Finally, the formula (8.27) holds because of Lemma 6.4.

\[ \square \]

9. **Proofs of the main theorems**

9.1. **Proofs of Theorems I and II.** Let us recall that we use the notation
\[
\mathcal{F} := \{ c\mathcal{H}^{n+1}_p|_{\Sigma_h} : c > 0, h \in F^*(1) \}
\]
\[
= \{ c\mathcal{H}^{n+1}_p|\mathcal{V} : c > 0, \mathcal{V} \text{ admissible } n \text{-plane passing through the origin} \},
\]
which is a \( d \)-cone of Radon measures.

Given \( \omega^+, \omega^- \), and a set \( E \) as in the statement of Theorem I, we denote
\[
E^* := \{ \xi \in E : \lim_{r \to 0} \frac{\omega^+(C_r(\xi) \cap E)}{\omega^+(C_r(\xi))} = \lim_{r \to 0} \frac{\omega^-(C_r(\xi) \cap E)}{\omega^-(C_r(\xi))} = 1 \},
\]
which, by Lemma 4.8 and the mutual absolute continuity assumption for \( \omega^\pm \), satisfies \( \omega^\pm(E \setminus E^*) = 0 \).

For \( \xi \in E^* \), we define the Radon-Nikodym derivative
\[
h(\xi) = \frac{d\omega^+}{d\omega^-}(\xi) = \lim_{r \to 0} \frac{\omega^+(C_r(\xi))}{\omega^-(C_r(\xi))} = \lim_{r \to 0} \frac{\omega^+(C_r(\xi) \cap E)}{\omega^-(C_r(\xi) \cap E)},
\]
the set
\[
\Lambda := \{ \xi \in E^* : 0 < h(\xi) < \infty \}
\]
and
\[
\Gamma := \{ \xi \in \Lambda : \xi \text{ is a Lebesgue point for } h \text{ with respect to } \omega^+ \}.
\]
Observe that \( \omega^+(E \setminus \Gamma) = 0 \) (see Lemma 4.8 and [Mat95, Remark 2.15 (3)]).

**Lemma 9.1.** Let \( \xi \in \Gamma, c_j > 0, \) and \( r_j \to 0 \) be such that \( \omega^+_j = c_j T_{\xi,r_j}[\omega^+] \to \omega^+_\infty. \) Then
\[
\omega^-_j = c_j T_{\xi,r_j}[\omega^-] \to h(\xi)\omega^-\infty.
\]
Proof. The proof is identical to the one of [AMT17, Lemma 5.8] and we omit it.

Lemma 9.2. Let $\omega^+$ be as in Theorem I. For $\omega^+$-a.e. $\xi \in \Gamma$ we have that

\[ \text{Tan}(\omega^+, \xi) \cap \mathcal{F} \neq \emptyset. \]

Proof. Under the hypotheses of Theorem I, we have that $E \subset \text{supp} \omega^+ \cap \text{supp} \omega^- \subset \partial \Omega^+ \cap \partial \Omega^-$ and that $\Omega^+$ is disjoint from $\Omega^-$. By the classification of boundary points in Section 2, this implies that $E \cap (B \Omega^+ \cup \partial \Omega^+) = \emptyset$ and thus, $E \subset S^\prime \Omega^+ \cap S^\prime \Omega^-$. It is not difficult to see that [Pr87, Theorem 2.5] translates to our context. In particular, the set $\text{Tan}(\omega^+, \xi)$ is nonempty for $\omega^+$-a.e. $\xi \in \Gamma$. Let $\xi \in \Gamma$ be such that $\text{Tan}(\omega^+, \xi) \neq \emptyset$ and consider $c_j > 0$ and $r_j \to 0$ such that $c_j T_{\xi,r_j}[\omega^+] \to \omega^\infty$ for some non-zero Radon measure $\omega^\infty \in \text{Tan}(\omega^+, \xi)$. Lemma 9.1 implies that $c_j T_{\xi,r_j}[\omega^-] \to h(\xi) \omega^\infty$ and so, we may apply Lemma 8.2 with $c = h(\xi)$. Thus, we obtain an adjoint caloric function $u_{\infty}$ in $\mathbb{R}^{n+1}$ such that

\[ \int \varphi \, d\omega^+_\infty = \int u^+_{\infty} H \varphi = \frac{1}{2} \int |u_{\infty}| \, H \varphi, \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^{n+1}), \]

and $\text{supp} \omega^+_\infty = \{u_{\infty} = 0\} =: \Sigma$. By Lemma 6.4 we get that $\omega^+_\infty$ is absolutely continuous with respect to the surface measure of $\Sigma = \partial \Omega^+_\infty$ and

\[ d\omega^+_\infty = -\partial \nu^+_{\infty} u^+_{\infty} \, d\sigma|\Sigma, \]

where $\partial \nu^+_{\infty}$ is the derivative along the measure theoretic outer unit normal to the time-slice $\Omega^+_\infty$. The set $\Sigma$ is the nodal set of an adjoint caloric function so, by Lemmas 5.1 and 5.2, for $\sigma|_{E}$-a.e. $\xi \in \Sigma$, there exists $\varepsilon > 0$ such that $\Sigma$ agrees with a smooth admissible graph in $C^\varepsilon(\xi)$. Hence, if we combine Lemmas 4.11 and 4.12 with Corollary 14.4, we can find a non-zero Radon measure $\mu \in \mathcal{F}^*(1) = \mathcal{F}$ such that $\mu \in \text{Tan}(\omega^+, \xi)$.

With the results of Section 7 at our disposal, the following lemma can be proved following the strategy of [AM19, Lemma 6.1]. Its proof relies on a corollary of a “connectivity” lemma (see [AM19, Corollary 3.12]) which also holds in the parabolic setting as it does not use the Euclidean structure.

Lemma 9.3. Let $\Omega$ be an open set in $\mathbb{R}^{n+1}$ and, given $\overline{p} \in \Omega$, let $\omega := \omega^\overline{p}$ be its associated caloric measure. Let $\xi \in \text{supp} \omega$, suppose that $\text{Tan}(\omega, \xi) \subset \mathcal{H}_{\overline{p}}$, and that there exists $\lambda > 0$ such that for all $\omega_h \in \text{Tan}(\omega, \xi)$ we have

\[ \|h\|_{L^\infty(C_r(\overline{0})))} \lesssim_\lambda r^{-n} \omega_h(C_{\lambda r}(\overline{0})), \quad \text{for } r > 0. \]

If $k$ is the smallest integer for which $\text{Tan}(\omega, \xi) \cap \mathcal{F}^*(k) \neq \emptyset$, then $\text{Tan}(\omega, \xi) \subset \mathcal{F}^*(k)$. Moreover,

\[ \lim_{r \to 0} \frac{\log \omega(C_r(\xi))}{\log r} = n + k. \]

We are now ready to gather all the results we have proved so far in order to prove the first main theorem of the paper.
Proof of Theorem I. We recall that, arguing as in Lemma 9.2, we can assume that assume that $E \subset S\Omega^+ \cap S\Omega^-$. Moreover, Lemma 9.2 infers that $\text{Tan}(\omega^+, \xi) \cap \mathcal{F} \neq \emptyset$ for $\omega^+$-a.e. $\xi \in \Gamma$, which, by Lemma 8.2, gives that $\text{Tan}(\omega^+, \xi) \subset \mathcal{H}_{\Theta^+}$. Therefore, we can apply Lemma 9.3 with $k = 1$ and obtain
\[
\lim_{r \to 0} \frac{\log \omega(C_r(\bar{\xi}))}{\log r} = n + 1, \quad \text{for } \omega^+\text{-a.e. } \bar{\xi} \in \Gamma.
\]
Since $\omega^+(E \setminus \Gamma) = 0$, the previous formula implies that $\dim \omega^+|_E = \dim \omega^-|_E = n + 1$. If, in addition, $\Omega^\pm$ satisfy the TFCDC assumption, we can apply Lemma 8.4 and use Lemma 8.3-(b) in order to conclude that $\lim_{r \to 0} \Theta_{\partial \Omega^\pm}(\bar{\xi}, r) = 0$ for $\omega^+$-a.e. $\bar{\xi} \in E$. \hfill \Box

We will now show Theorem II following the approach in [TV18].

Proof of Theorem II. The proof follows from the ones of Lemmas 8.1 and 8.2, and so, we will only sketch it. Let us assume that $\omega^1(E) > 0$, which by mutual absolute continuity implies that $\omega^i(E) > 0$ for $i = 2, 3$ as well. By Lemma 3.23 (or rather its proof), we can find open sets $\Omega^i \subset \Omega$, $i = 1, 2, 3$, which are regular for $H$ and $H^*$, and a set
\[
\bar{E} \subset E \cap \bigcap_{i=1}^3 \mathcal{P}^i \bar{\Omega}_i \cap \text{supp } \omega^\Omega_1,
\]
so that $\omega^i(\bar{E}) > 0$ for $i = 1, 2, 3$, and $\omega_{\bar{\Omega}_1}, \omega_{\bar{\Omega}_2},$ and $\omega_{\bar{\Omega}_3}$ are mutually absolutely continuous on $\bar{E}$. So, it is enough to prove the result assuming that $\Omega_i$ is regular for $H$ and $H^*$ for $i = 1, 2, 3$.

To this end, by [Pr87, Theorem 2.5], for $\omega^1$-a.e. $\bar{\xi} \in E$, we have that there exists $c_j > 0$ and $r_j \to 0$ such that $\omega^1_j \sim \omega^\Omega_\infty$, for some non-zero Radon measure $\omega^\Omega_\infty$. Let $\Gamma_2$ and $\Gamma_3$ be defined as $\Gamma$ with $\omega^+ = \omega^1$ and $\omega^- = \omega^1$, for $i = 2, 3$. By Lemma 9.1 and mutual absolute continuity, we obtain that $\omega^1_3 \to h^3(\bar{\xi}) \omega^\Omega_\infty$ and $\omega^1_3 \to h^3(\bar{\xi}) \omega^\Omega_\infty$ for $\xi \in \Gamma := \Gamma_2 \cap \Gamma_3$. Note that $\omega^1(\Gamma) = 0$ and so, $\omega^j(\Gamma) = 0$, for $i = 2, 3$. By a similar pigeonholing argument as in the proof of Lemma 8.2, after passing to a subsequence, we may assume without loss of generality that
\[
r^2_j \text{Cap}(E(\xi; r^2_j) \setminus \Omega^j) \overset{\text{(3.15)}}{\geq} H^{n+1}(E(\xi; r^2_j) \setminus \Omega^j) \overset{\text{[3.15]}}{\geq} r^{n+2}_j, \quad j \geq 0 \text{ and } i = 1, 2.
\]
In particular, there exists $F_j \subset E(\xi; r^2_j) \setminus (\Omega^1 \cap \Omega^2)$ such that $\mathcal{H}^{n+1}(F_j) \geq r^{n+2}_j$. If we set $G_j = T_{\bar{\xi}, r^2_j}(F_j) \subset \overline{E(0; 1)} \setminus (\Omega^1 \cap \Omega^2)$, then it is clear that $\mathcal{H}^{n+1}(G_j) \geq 1$ and we have that
\[
\int_{G_j} |u_j| = 0, \quad \text{for all } j \geq 1.
\]
Since $G_j$ is compact and $\mathcal{H}^{n+1}(G_j) \geq 1$ for all $j \geq 1$, there is a compact set $G$ such that, after passing to a subsequence, $G_j \to G$ in the Hausdorff metric and $\mathcal{H}^{n+1}(G) \geq 1$. It is easy to see that $\chi_{G_j} \to g$ weakly in $L^2 \left(\overline{E(0; 1)}\right)$ for some non-negative function $g \in L^2 \left(\overline{E(0; 1)}\right)$, and thus, we must have that $g = \chi_G$. Repeating the proof of Lemma 8.2 for $+ = 1$ and $- = 2$, we obtain that $u_j \to u_\infty$ in $L^2_{\text{loc}}(\mathbb{R}^{n+1})$ for some globally adjoint
caloric function $u_\infty \neq 0$ which vanishes only on a set of zero $H^{n+1}$-measure. Taking limits as $j \to \infty$ in (9.2), we obtain

$$
\int_G |u_\infty| = \lim_{j \to \infty} \int_{G_j} |u_j| = 0,
$$

which, in turn, implies that $u_\infty = 0$ on a set of positive Lebesgue measure reaching a contradiction. \qed

9.2. Proofs of Theorems III and IV.

**Lemma 9.4.** Let $\mu^\pm$ be two halving Radon measures such that $\mu^- \ll \mu^+$ and $\text{supp} \mu^+ = \text{supp} \mu^- \subset \partial_0 \Omega^+$. Let us define $f := \log \frac{d\mu^-}{d\mu^+}$ and assume that there is $r_j \to 0$ and $\xi_j \in \partial_0 \Omega^+$ so that $\mu_j^+ := \mu^+ (C_{r_j} (\xi_j))^{-1} T_{\xi_j, r_j} \mu^+ \to \mu$ for some Radon measure $\mu$ such that $\mu(C_1(\bar{0})) > 0$. Moreover, we assume that $\mu^+$ is so that for all $M > 0$

$$
\lim_j \left( \int_{C_{r_j}(\xi_j)} f d\mu^+ \right) \exp \left( - \int_{C_{r_j}(\xi_j)} \log f d\mu^+ \right) = 1.
$$

Then $\mu_j^- := \mu^- (C_{r_j} (\xi_j))^{-1} T_{\xi_j, r_j} \mu^- \to \mu$. \proof. The proof of [AM19, Lemma 8.1] does not use the Euclidean structure and can be repeated in the parabolic setting. \qed

**Proof of Theorem III.** Since $E$ is nonempty and relatively open in $\text{supp} \omega^+$, we have that $\omega^+(E) > 0$ and

$$
\lim_{r \to 0} \frac{\omega^+(C_r(\bar{\xi})) \setminus E}{\omega^+(C_r(\xi))} = 0 \quad \text{for all } \bar{\xi} \in E,
$$

so [Mat95, Lemma 14.5] gives $\text{Tan}(\omega^+, \bar{\xi}) = \text{Tan}(\omega^+|E, \bar{\xi})$. If $\omega \in \text{Tan}(\omega^+|E, \bar{\xi}) = \text{Tan}(\omega^+, \bar{\xi})$, then by Lemma 4.10-(1), $\omega = c T_{0_r} [\mu]$, for some constants $c > 0$ and $r > 0$, and some measure $\mu$ of the form

$$
\mu = \lim_{j \to \infty} \frac{1}{\omega^+(C_{r_j}(\bar{\xi}))} T_{\xi, r_j} [\omega^+] = \lim_{j \to \infty} \frac{1}{\omega(E(C_{r_j}(\bar{\xi})) \setminus E)} T_{\xi, r_j} [\omega^+|E],
$$

for some $r_j \to 0$, is such that $\mu(C_1(\bar{0})) > 0$ and the second equality holds because $E$ is relatively open in $\text{supp} \omega^+$. Let us observe that the hypothesis of Lemma 9.4 is satisfied for $\mu^\pm = \omega^\pm(E)^{-1} \omega^\pm|E$ (the normalization is chosen so that $\mu^\pm$ are probability measures) and for $\bar{\xi} = \xi$ for all $j$. Hence, we can conclude that

$$
\mu = \lim_{j \to \infty} \frac{1}{\omega^+(E(C_{r_j}(\bar{\xi})) \setminus E)} T_{\xi, r_j} [\omega^+|E] = \lim_{j \to \infty} \frac{1}{\omega^+(C_{r_j}(\bar{\xi}))} T_{\xi, r_j} [\omega^-].
$$

By Lemma 8.2 there exists $g \in \Theta^*$ such that $\mu = \omega\theta$ and since the caloric measures associated with adjoint caloric functions form a $d$-cone and $\omega = c T_{0_r} [\mu]$, we have that $\omega \in \mathcal{H}_\Theta^*$ too. More specifically there exists $h \in \Theta^*$ such that $\omega = \omega h$. Let us define

$$
h_j(\bar{x}) := \sum_{|\alpha|+2\ell=k} \frac{D^{\alpha,\ell} h(\bar{0})}{\alpha! \ell!} x^\alpha t^\ell, \quad j > 0,
$$

and let $k$ be the smallest number for which we have $h_k \neq 0$. By Lemmas 4.12 and 7.9, it follows that $\Tan(\omega_n, 0) \subset \Tan(\omega^+, \xi)$ and 

$$\Tan(\omega_n, 0) = \{ c\omega_n : c > 0 \} \subset F^*(k).$$

This shows that $\Tan(\omega^+, \xi) \cap F^*(k) \neq \emptyset$, which, in turn, by Proposition 7.14 implies $\Tan(\omega^+, \xi) \subset F^*(k)$. If we additionally assume that $\Omega^\pm$ have TFCDC, then we can argue as in the proof of Theorem I to show (1.8). □

Proof of Theorem IV. Since $F$ is compact, there exists $\rho > 0$ such that $C_\rho(\xi) \cap \supp \omega^+ \subset E$ for all $\xi \in F$. Arguing by contradiction, we assume that for every $d \in \mathbb{N}$ we can find sequences $\{\xi_j\}_{j \geq 1} \subset F$ and $r_j \to 0$ such that

$$(9.4) \quad d_1(T_{\xi_j, r_j}[\omega^+], \mathcal{P}^*(d)) \geq \varepsilon > 0.$$ 

By the doubling assumption on $\omega^+$, after passing to a subsequence, we have that

$$\lim_{j \to \infty} \frac{1}{\omega^+(C_{r_j}(\xi_j))} T_{\xi_j, r_j}[\omega^+] = \lim_{j \to \infty} \frac{1}{\omega^+|E|C_{r_j}(\xi_j)} T_{\xi_j, r_j}[\omega^+] = \omega$$

for some measure $\omega$, where the equality holds because $E$ is relatively open in $\supp \omega^+$. We claim that $\omega^-(C_{r_j}(\xi_j))^{-1} T_{\xi_j, r_j}[\omega^-] \to \omega$ as well. Indeed, since $\log f \in \text{VMO}(\omega^+|E)$ and $\omega^+|E$ is doubling by assumption, $f$ is an $A_p$-weight on small enough cylinders by the John-Nirenberg theorem (see [Gar07, Chapter 6.2] for the proof of this implication in the Euclidean setting), which still holds for spaces of homogeneous type. Hence, $\omega^-|E \in \text{VA}(\omega^+|E)$ by [AM19, Corollary 7.8] and the hypothesis of Lemma 9.4 is satisfied, so

$$\omega = \lim_{j \to \infty} \frac{1}{\omega^-|E(C_{r_j}(\xi_j))} T_{\xi_j, r_j}[\omega^-] = \lim_{j \to \infty} \frac{1}{\omega^-(C_{r_j}(\xi_j))} T_{\xi_j, r_j}[\omega^-].$$

Lemma 8.2 gives that $\omega = \omega_h$ for some $h \in \Theta^*$, i.e.

$$\int \varphi \, d\omega = \int h \, H\varphi, \quad \varphi \in C^\infty_c(\mathbb{R}^{n+1}).$$

The measure $\omega_h$ is doubling because $\omega^+|E$ is. Thus, there exists $\lambda \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and $N \in \mathbb{N}$,

$$(9.5) \quad \omega_h(C_{2^{k+N}}(0)) \lesssim 2^{-k\lambda} \omega_h(C_{2^N}(0)).$$

Let $a$ and $M$ be as in Lemma 8.2 and denote by $N$ a positive integer such that $4a^{-1}M \leq 2^N$. Inequality (9.5) together with the Cauchy estimates (6.5), gives that, for $\alpha \in \mathbb{N}^n$, $k \in \mathbb{N}$ and $\ell > 0$ such that $|\alpha| + 2\ell = m$,

$$|D^{\alpha, \ell} h(0)| \lesssim 2^{-mk} \|h\|_{L^\infty(C_{2^k}(0))} \lesssim_{\alpha, M} 2^{-k(m+n-\lambda)} \omega_h(C_{2^{k+n}}(0)).$$

where the second inequality follows from the proof of Lemma 8.2. Thus, for $m+n-\lambda > 0$ we have that the right hand side in the previous expression converges to 0 as $k \to \infty$, which implies that $D^{\alpha, \ell} h(0) = 0$. In particular, $h$ is a polynomial of degree at most $d = m+n-\lambda$, which contradicts (9.4). In order to prove (1.10), it suffices to use Lemma 8.3 and argue as in the proof of Theorem I. □
Proof of Corollary 1.3. One can modify the proof above according to the original proof in [KT06, pp.34-35] to conclude the corollary bearing in mind that \( \omega^- \ll \omega^+ \) implies that \( t_- \leq t_+ \). We leave the details to the interested reader. □

REFERENCES

[ACF84] H.W. Alt, L.A. Caffarelli, and A. Friedman. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.*, 282(2):431–461, 1984.

[AM19] J. Azzam and M. Mourgoglou. Tangent measures of elliptic measure and applications *Anal. PDE*, 12 (8):1891–1941, 2019.

[AMT17] J. Azzam, M. Mourgoglou, and X. Tolsa. Mutual absolute continuity of interior and exterior harmonic measure implies rectifiability. *Comm. Pure Appl. Math.*, 71(11):2121–2163, 2017.

[AMTV19] J. Azzam, M. Mourgoglou, X. Tolsa, and A. Volberg. On a two-phase problem for harmonic measure in general domains. *Amer. J. Math.*, 141(5):1259–1279, 2019.

[Bad11] M. Badger. Harmonic polynomials and tangent measures of harmonic measure. *Rev. Mat. Iberoam.*, 27(3):841–870, 2011.

[Bad13] M. Badger. Flat points in zero sets of harmonic polynomials and harmonic measure from two sides. *J. Lond. Math. Soc.*, 87(1):111–137, 2013.

[BL15] M. Badger, S. Lewis. Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets, and tangent sets. *Forum Math. Sigma*, 3, E24, 2015.

[BW06] L. Barreira and C. Wolf. Pointwise dimension and ergodic decompositions. *Ergodic Theory Dynam. Systems*, 26:653–671, 2006.

[BCGJ89] C. J. Bishop, L. Carleson, J. B. Garnett, and P. W. Jones. Harmonic measures supported on curves. *Pac. J. Math.*, 138(2):233–236, 1989.

[Br90] M. Brzezina. On the base and the essential base in parabolic potential theory *Czechoslovak Mathematical Journal* 40 (1): 87-103, 1990.

[Cho06] S. Cho. Two-sided Global Estimates of the Greens Function of Parabolic Equations. *Potential Anal.*, 25:387–398, 2006.

[DeL08] C. De Lellis. *Rectifiable sets, densities and tangent measures*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Do84] R. F. Gariepy and W. P. Ziemer. Thermal capacity and boundary regularity. *J. Differential Equations*, 45: 374–388, 1982.

[Do58] J. L. Doob. *Classical Potential Theory and its Probabilistic Counterpart*. Grundlehren der mathematischen Wissenschaften, Volume 262, Springer, New York, 1984.

[Eng17A] M. Engelstein. A free boundary problem for the parabolic Poisson kernel *Adv. Math.*, 314: 835–947, 2017.

[Eng17B] M. Engelstein. Parabolic NTA Domains in \( \mathbb{R}^2 \). *Communications in PDE*, 42: 1524–1536, 2017.

[EG82] L. C. Evans and R. F. Gariepy. Wiener's criterion for the heat equation. *Arch. Ration. Mech. Anal.*, 78: 293-314, 1982.

[EG15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*, Revised Edition. Textbooks in Mathematics. New York: Chapman and Hall/CRC, New York, 2015.

[GZ82] R. F. Gariepy and W. P. Ziemer. Thermal capacity and boundary regularity. *J. Differential Equations*, 45: 374–388, 1982.

[Gar07] J. B. Garnett. *Bounded analytic functions*, volume 236 of Graduate Texts in Mathematics. Springer, New York, first edition, 2007.

[GH20] A. Genschaw and S. Hofmann. A Weak Reverse Hölder Inequality for Caloric Measure *J. Geom. Anal.*, 30, 1530–1564, 2020.

[HL94] Q. Han and F. H. Lin. Nodal sets of solutions of parabolic equations: II. *Comm. Pure Appl. Math.*, 47(9):1219–1238, 1994.

[He17] O. Hershkovits. Isoperimetric properties of the mean curvature flow. *Trans. Amer. Math. Soc.*, 369(6): 4367–4383, 2017.

[Hof04] S. Hofmann, J.L. Lewis, K. Nyström. Caloric measure in parabolic flat domains. *Duke Math. J.*, 122: 281–346, 2004.
[It18] T. Itoh. The Besicovitch covering theorem for parabolic balls in Euclidean space. Hiroshima Math. J., 4:279–289, 2018.

[JK82] D. S. Jerison and C. E. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains. Adv. in Math., 46(1):80–147, 1982.

[KPT09] C. E. Kenig, D. Preiss, and T. Toro. Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions. J. Amer. Math. Soc., 22(3):771–796, 2009.

[KT06] C. E. Kenig and T. Toro. Free boundary regularity below the continuous threshold: 2-phase problems. J. Reine Angew. Math., 596:1–44, 2006.

[Kor98] M. B. Korey. Ideal weights: asymptotically optimal versions of doubling, absolute continuity, and bounded mean oscillation. J. Fourier Anal. Appl., 4(4-5):491–519, 1998.

[Lanc73] E. Lanconelli. Sul problema di Dirichlet per l’equazione del calore. Ann. Mat. Pura Appl., 97:83–114, 1973.

[Land69] E. M. Landis. Necessary and sufficient conditions for the regularity of a boundary point for the Dirichlet problem for the heat equation. Dokl. Akad. Nauk SSSR, 185:517–520, 1969; Soviet Math. Dokl., 10:380–384, 1969.

[Lieb96] G. Lieberman. Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[Mag12] F. Maggi. Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory. Cambridge University Press, Cambridge, 2012.

[Mat95] P. Mattila. Geometry of sets and measures in Euclidean spaces, volume 44 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.

[Nys06A] K. Nyström. Regularity below the continuous threshold in a two-phase free boundary problem. Math. Scand. 99: 257–286, 2006.

[Nys06B] K. Nyström. On blow-ups and the classification of global solutions to a parabolic free boundary problems. Indiana Univ. Math. J. 55: 1233–1290, 2006.

[Nys06C] K. Nyström. Caloric measure and Reifenberg flatness. Ann. Acad. Sci. Fenn. Math. 31: 405–436, 2006.

[Nys12] K. Nyström. On an inverse type problem for the heat equation in parabolic regular graph domains. Math. Z. 31: 197–222, 2012.

[Or19] T. Orponen. An Integralgeometric Approach to Dorronsoro Estimates International Mathematics Research Notices, rnz317. https://doi.org/10.1093/imrn/rnz317 30, 33.

[Po96] C. Poon. Unique continuation for parabolic equations Communications in Partial Differential Equations, 21(3–4): 521–539, 1996.

[Pr87] D. Preiss. Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. Ann. Math. (2), 125(3):537–643, 1987.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl., 146:65–96, 1987.

[St70] E. M. Stein. Singular Integrals and Differentiability Properties of Functions (PMS-30). Princeton University Press, 1970.

[TV18] X. Tolsa and Alexander Volberg. On Tsiirelson’s theorem about triple points for harmonic measure. Int. Math. Res. Notices (IMRN), Vol. 2018, No. 12, pp. 3671–3683, 2018.

[TW85] S. J. Taylor and N. A. Watson. A Hausdorff measure classification of polar sets for the heat equation. Math. Proc. Cambridge Phil. Soc., 97:325–344, 1985.

[TZ17] T. Toro and Z. Zhao. Boundary rectifiability and elliptic operators with $W^{1,1}$ coefficients. To appear in Adv. Calc. Var. doi:10.1515/acv-2017-0044, 17/10/2017.

[Ts97] B. Tsirelson. Triple points: from non-Brownian filtrations to harmonic measures. Geom. Funct. Anal. (GAF), Vol. 7:1096–1142, 1997.

[VI02] V. S. Vladimirov. Methods of the theory of generalized functions, Analytical Methods and Special Functions, Vol. 6, London-New York City: Taylor & Francis, pp. XII+353, 2002.

[Wa12] N.A. Watson. Introduction to Heat Potential Theory. Mathematical Surveys and Monographs, Volume 182, American Mathematical Society, 2012.
[Wa14] N. A. Watson. Regularity of boundary points in the Dirichlet problem for the heat equation. *B. Aust. Math. Soc.*, 90 (3):476–485, 2014.

[Zh02] Qi S. Zhang. The Boundary Behavior of Heat Kernels of Dirichlet Laplacians. *J. Differential Equations*, 182:416–430, 2002.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DEL PAÍS VASCO, BARRIO SARRIENA S/N 48940 LEIOA, SPAIN AND, IKERBASQUE, BASQUE FOUNDATION FOR SCIENCE, BILBAO, SPAIN.

E-mail address: michail.mourgoglou@ehu.eus

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DEL PAÍS VASCO, BARRIO SARRIENA S/N 48940 LEIOA, SPAIN.

E-mail address: carmelopuliatti@ehu.eus