Combining Probability Forecasts and Understanding Probability Extremizing through Information Diversity

Ville A. Satopää¹, Robin Pemantle², and Lyle H. Ungar³

¹Department of Statistics, The Wharton School of the University of Pennsylvania, 400 Jon M. Huntsman Hall, 3730 Walnut Street, Philadelphia, PA 19104-6340
²Department of Mathematics, University of Pennsylvania, David Rittenhouse Laboratories, 209 S. 33rd Street, Philadelphia, PA 19104-6395
³Department of Computer and Information Science, University of Pennsylvania, 504 Levine, 200 S. 33rd Street, Philadelphia, PA 19104-6309

Summary. Randomness in scientific estimation is generally assumed to arise from unmeasured or uncontrolled factors. However, when combining subjective probability estimates, heterogeneity stemming from people's cognitive or information diversity is often more important than measurement noise. This paper presents a novel framework that models the heterogeneity arising from experts that use partially overlapping information sources, and applies that model to the task of aggregating the probabilities given by a group of experts who forecast whether an event will occur or not. Our model describes the distribution of information across experts in terms of easily interpretable parameters and shows how the optimal amount of extremizing of the average probability forecast (shifting it closer to its nearest extreme) varies as a function of the experts' information overlap. Our model thus gives a more principled understanding of the historically ad hoc practice of extremizing average forecasts.

Keywords: Expert Beliefs; Information Aggregation; Model Averaging; Probability Forecasting; Stochastic Process; Wisdom of Crowds

1 Introduction

There is strong empirical evidence that bringing together the strengths of different experts by aggregating their probability forecasts into a single consensus improves predictive performance (Armstrong (2001); Clemen (1989)). Prompted by the many applications including medical diagnosis (Pepe (2003); Wilson et al. (1998)), political and socio-economic foresight (Tetlock (2005)), and meteorology (Baars and Mass (2005); Sanders (1963); Vislocky and Fritsch (1995)), researchers have proposed many different probability aggregators. These aggregators rely on a wide range of modeling assumptions. Choosing appropriate assumptions to explain the process of subjective...
probability forecasting is of outmost importance because this links theory to practice and largely
determines the predictive performance of the resulting probability aggregator.

One possibility is to adopt the interpreted signal framework introduced by Hong and Page
(2009). They consider a probability forecast “interpreted” if the expert forms it based on a personal
interpretation of (a subset of) factors or cues that influence the future event to be predicted. For
instance, consider two experts following a presidential campaign speech. One expert may choose
to focus on the content of the speech while the other pays closer attention to the way the candidate
interacts with the audience. Even though they both receive the same information, they choose
to interpret it differently. Consequently, their probability forecasts of the candidate winning the
election are likely to be different. Therefore, under the interpreted signal framework, forecast het-
erogeneity is assumed to stem from cognitive diversity. This is a very realistic framework that has
been analyzed and extended to many other settings. For instance, Parunak et al. (2013) demonstrate
that interpreted forecasts require aggregation techniques that can leave the convex hull of the indi-
vidual probability forecasts. Broomell and Budescu (2009) analyze inter-expert correlation under
the assumption that the cues can be mapped to the experts’ forecasts via different linear regression
functions. Unfortunately, any previous work on interpreted forecasts has only produced abstract
concepts and still lacks a formal model with quantitative predictions. The main problem is that it
is not clear how “cognitive diversity” can be measured in complex real-world situations.

In practice it is often easier and more common to explain subjective forecasts with the classical
measurement error framework. This framework assumes a “true” probability (interpreted as the
probability forecast made by an ideal expert) for the future event. The experts are able to mea-
sure this probability but with some mean-zero idiosyncratic error. Each forecast is then modeled
as a draw from a common probability distribution that is typically centered at the “true” probabil-
ity. Therefore the resulting aggregators are often different types of averages such as the average
probability or the average log-odds. Unfortunately, these “average aggregators” are confined to the
convex hull of the individual forecasts and therefore cannot be optimal for aggregating interpreted
forecasts. Ranjan and Gneiting (2010) provide a similar result for experts whose long-term prob-
ability forecasts agree with the empirical distribution of the events. They show that any convex
combination of such probability forecasts agree with the empirical distribution of the events. They show that any convex
combination of such probability forecasts is necessarily under-confident, i.e. too close to 0.5.

The predictive performance of the average aggregates is often improved by a post-transformation
that can take them outside the convex hull. One popular choice is extremizing that shifts the aver-
age aggregate closer to its nearest extreme (0.0 or 1.0). For instance, Ranjan and Gneiting (2010)
propose transforming the (weighted) average probability with the cumulative distribution function
(CDF) of a beta distribution. If both the shape and scale of this beta distribution are equal and
constrained to be at least 1.0, the aggregator extremizes and has some attractive theoretical prop-
erties (Wallsten and Diederich (2001)). Satopää et al. (2014) use a logistic regression model to
derive an aggregator that extremizes the average log-odds. Baron et al. (2014) give two intuitive
justifications for extremizing and discuss an extremizing technique that has been previously used by
Erev et al. (1994), Shlomi and Wallsten (2010), and even Karmarkar (1978). In an empirical study,
Mellers et al. (2014) show that extremizing improves the quality of average aggregates of interna-
tional events. Turner et al. (2013) and Ariely et al. (2000) also mention the benefits of extremizing.
Therefore, given that predictive performance of the average aggregates can be improved by allowing them to leave the convex hull of the individual forecasts, probability forecasts are likely to be more similar to interpreted forecasts than to draws from a probability distribution.

This leaves the current state-of-the-arts aggregators to an unwieldy position: they first compute an average based on a model that is likely to be at odds with the actual process of probability forecasting, and then aim to correct this misalignment via ad hoc extremizing techniques. A more natural approach would be to develop a new model which aligns with the process of probability forecasting enough to yield aggregators that can intrinsically leave the convex hull of the individual forecasts. This would provide a more direct approach to probability aggregation and also address some of the other drawbacks of the current aggregators. For instance, the current extremizing techniques typically provide very little insight beyond the amount of extremizing itself. This is one major reason why it is still not well-understood what the main factors are and how they affect the amount of extremizing in different forecasting setups. Furthermore, the current aggregators often learn the amount of extremizing by optimizing a scoring rule over a separate training set (see Gneiting and Raftery (2007) for a discussion on scoring rules). Given that such a training set requires repeated realizations of a single event, it is not clear how these aggregators can be applied for a one-time event.

Our first contribution is a concrete model for probability forecasts made by a group of experts. The model is based on the partial information framework under which the experts first collect information from different sources such as newspapers, other people, websites, photographs, and even languages. For instance, one expert may only skim over a local newspaper while another expert actively follows news both in English and Russian. The experts then make forecasts only based on their own information. Therefore, as different information is considered to lead to different beliefs, forecast heterogeneity stems from information diversity. This maintains a close link with interpreted forecasts while permitting models that can be potentially estimated in practice. For instance, our model for probability forecasts describes the distribution of information among the experts with a covariance matrix that can be constrained and estimated in different ways to suit the available resources. Unlike the classical measurement error framework, the partial information framework does not assume a “true” probability. Instead, the target event either happens or does not, and each probability forecast is a reflection of the expert’s personal amount of information. This guides the discussion away from the meaning of a “true” probability and avoids potential philosophical debate.

Our second contribution is an improved understanding of probability extremizing. The first step is to introduce an “oracle” expert who knows everything that the group of experts knows. The oracle’s probability forecast utilizes all the information among the experts and provides the single best aggregate forecast under the partial information model. Even though this aggregate typically cannot be attained in practice, it provides a theoretical “gold-standard” for determining how much extremizing is necessary for different average aggregates. The amount of extremizing is summarized with a single number that follows a Cauchy distribution. This distribution depends on a structure of information among the experts, and therefore establishes a link between the experts’ information and the amount of extremizing of average aggregates. The analysis gives two main
results. First, the modal amount of extremizing is increasing in the level of information diversity and the total amount of information among experts. Second, no matter how information is distributed among the experts, extremizing the average probit forecast is more likely to be beneficial than harmful as long as all experts’ forecasts are not the same.

Our third and final contribution is a partial information aggregator that behaves similarly to the oracle aggregate but can be estimated in practice. Furthermore, it a) depends only on two intuitive parameters, namely the average amount of information known by an expert and the average amount of information shared between any two experts, whose values can be estimated directly from the probability forecasts, b) can leave the convex hull of the individual forecasts, and c) extremizes the the average probit forecast as long as all the experts’ forecasts are not the same. Therefore it is likely to be appropriate for combining probability forecasts of a one-time event.

The rest of the paper is organized as follows. Section 2 briefly reviews several commonly used probability aggregators that are based on the classical measurement error model. Section 3 presents the partial information model for probability forecasts. Section 4 introduces the oracle expert, derives the distribution for the amount of extremizing, and studies extremizing under unstructured and compound symmetric information. Section 5 develops a partial information aggregate and studies some of its important properties. Section 6 links the discussion to our past experience with real-world forecasting data and describes model limitations along with potential future directions.

2 Classical Aggregation Procedures

The classical measurement error framework assumes a “true” probability $\theta$ for a binary event $A$. Each of the $N$ experts is then able to measure $\theta$ but with some error. This can be formalized by regarding the $j$th expert’s probability forecast $p_j$ as a draw from a probability density function (PDF) $f(\cdot|\theta)$. The PDF $f(\cdot|\theta)$ is supported on the unit interval and depends on $\theta$. Implicit in the assumption of $f(\cdot|\theta)$ is that the forecasts are conditionally independent given $\theta$. In other words, each forecast is a fresh draw from the corresponding probability distribution.

To introduce the first classical aggregator, consider a noise-added model in which the experts share an identical PDF $f(\cdot|\theta)$ and all forecasts are conditionally independent given $\theta$. If $f(\cdot|\theta)$ has mean $\theta$, the average of the forecasts $\{p_j\}_{j=1}^N$ converges to $\theta$ as $N \to \infty$. Therefore the simple average (a.k.a. the (equally weighted) linear opinion pool)

$$p_N^{\text{ave}} = \frac{1}{N} \sum_{j=1}^N p_j$$

is a reasonable candidate for an aggregate forecast when $N$ is large.

It may not, however, be appropriate to assume an additive error. For instance, if $\theta$ is near 0.0 or 1.0, the density $f(\cdot|\theta)$ is likely to skewed towards 0.5. Consequently, $p_N^{\text{ave}}$ is biased towards 0.5. This drawback is typically circumvented by modeling the log-odds of $p_j$ instead with a probability distribution whose mean is $\log\{\theta/(1-\theta)\}$. Therefore, under this model, the average of the experts’
log-odds converges to \( \log \left\{ \theta / (1 - \theta) \right\} \) as \( N \to \infty \). Transforming the average log-odds back to the probability space gives the (equally weighted) logarithmic opinion pool

\[
P_{\text{ave log}} = \exp \left[ \frac{1}{N} \sum_{j=1}^{N} \log \left\{ \frac{p_j}{1 - p_j} \right\} \right] \bigg/ \left[ 1 + \exp \left[ \frac{1}{N} \sum_{j=1}^{N} \log \left\{ \frac{p_j}{1 - p_j} \right\} \right] \right]
\]

This aggregator has been analyzed and utilized previously by many other investigators (see, e.g., Bacharach (1975); Dawid et al. (1995); Genest and Zidek (1986)).

It is possible to consider other transformations of \( p_j \) besides the log-odds. For instance, if \( \Phi(\cdot) \) denotes the CDF of a standard Gaussian distribution, the probit of \( p_j \) is defined as \( \Phi^{-1}(p_j) \). This transformation is common in economics while researchers in other disciplines tend to prefer the log-odds (Bryan and Jenkins (2013)). The choice is typically made based on computational or interpretation reasons: the log-odds are often considered more interpretable while the probit can be computationally more convenient. Analytically, however, these transformations are very similar. Therefore, following a similar argument as given for the log-odds, another reasonable aggregator, namely the probit opinion pool, is

\[
P_{\text{ave prb}} = \Phi \left\{ \frac{1}{N} \sum_{j=1}^{N} \Phi^{-1}(p_j) \right\}
\]

Given that these aggregators are different types of averages, they are confined to the convex hull of the individual forecasts and hence cannot be optimal for combining interpreted forecasts (Parunak et al. (2013)). However, their predictive performance can be often improved by post-transforming them via an extremizing function. This takes them closer to their nearest extremes (0.0 or 1.0) and potentially outside the convex hull of the individual forecasts. Understanding how much the average aggregates should be extremized is one of the main research objectives in this paper. Our analysis focuses on the extremizing of the probit opinion pool because a) it is likely to be a more reasonable aggregator than simple average, and b) it, as opposed to the very similar logarithmic opinion pool, yields an overall cleaner discussion of extremizing.

### 3 Partial Information Model for Probability Forecasts

This section derives a *partial information model* for probability forecasts made by a group of experts who aim to forecast whether a binary event will occur or not. The first step is to consider a probability space \((\Omega, \mathcal{F}, P)\), where the set \(\Omega\) contains all possible states of the world, \(\mathcal{F}\) is a \(\sigma\)-field of all subsets of \(\Omega\), and \(P\) is a probability measure. Let \(S = [0, 1]\) denote the unit interval and, on the probability space \((\Omega, \mathcal{F}, P)\), define a Gaussian process \(\{X_B\}\) that is indexed by Borel measurable subsets \(B \in S\). Endow the unit interval \(S\) with the uniform measure \(\mu\) such that the Gaussian process has a covariance structure \(\text{cov}(X_B, X_{B'}) = \mu(B \cap B') = |B \cap B'|\), i.e. the length of the intersection. This process provides the pool of information that is central to the partial information
model. The target event is defined as $A = \{ X_S > 0 \}$. Even though the pool of information is indexed by the unit interval, it is important to emphasize that there is no sense of time or ranking of information. Instead the pool is a collection of information, where each piece of information has \textit{a priori} an equal chance to contribute to the final outcome of the event. It is not necessary to assume anything about the source or form of the information. For instance, the information may stem from photographs, survey research, books, or even interviews. All these details have been abstracted away. Instead, any single piece of information is completely characterized by its effect on the target event.

It is helpful to begin the introduction by considering only two experts. Assume that experts $E_1$ and $E_2$ see respective $\delta_1$ and $\delta_2$ portions of the Gaussian process. These portions form their information sets. The overlap in their information sets is a fixed share $\rho$ of what is seen by either expert. Therefore, if $I_1, I_2 \subseteq S$ denote the information sets observed by $E_1$ and $E_2$, respectively, then

$$
\mu(I_1) = |I_1| = \delta_1 \\
\mu(I_2) = |I_2| = \delta_2 \\
\mu(I_1 \cap I_2) = |I_1 \cap I_2| = \rho
$$

Figure 1 illustrates this setup. In this diagram the Gaussian process has been partitioned into four parts based on the experts’ information sets:

$$
U = X_{I_1/I_2} \\
V = X_{I_2/I_1} \\
M = X_{I_1 \cap I_2} \\
W = X_{(I_1 \cup I_2)^c}
$$

Figure 1: Illustration of the partial information model with 2 experts.
Then,

\[ X_{I_1} = U + M \]
\[ X_{I_2} = M + V \]
\[ X_S = U + M + V + W, \]

where \( U, V, M, W \) are independent Gaussians with respective variances \( \delta_1 - \rho, \delta_2 - \rho, \rho, 1 + \rho - \delta_1 - \delta_2 \). The random variable \( X_{I_j} \) can be interpreted as the information known by expert \( E_j \). The joint distribution of \( X_S, X_{I_1}, \) and \( X_{I_2} \) is a multivariate Gaussian distribution. That is,

\[
\begin{pmatrix}
X_S \\
X_{I_1} \\
X_{I_2}
\end{pmatrix}
\sim \mathcal{N}
\begin{pmatrix}
0 \\
\begin{pmatrix}
1 & \delta_1 & \delta_2 \\
\delta_1 & \delta_1 & \rho \\
\delta_2 & \rho & \delta_2
\end{pmatrix}
\end{pmatrix}
\]

Given that \( X_S \) has mean zero, the prior probability for the event is \( \mathbb{P}(X_S > 0) = \mathbb{P}(A) = 0.5 \). This can be easily adjusted to any probability \( \tilde{p} \) by letting \( A = \{ X_S > \Phi^{-1}(1-\tilde{p}) \} \). For instance, if the event to be predicted is one of a sequence of repeated events, it may be reasonable to set \( \tilde{p} \) to the empirical rate of occurrence of the already realized events. Given that this paper is not concerned with any particular event, the prior probability is fixed at \( \tilde{p} = 0.5 \).

Consider now \( N \) experts. Let \( |I_j| = \delta_j \) be the amount of information known by expert \( E_j \) for \( j = 1, \ldots, N \), and \( |I_i \cap I_j| = \rho_{ij} \) be the information overlap between experts \( E_i \) and \( E_j \) with...
Expression (1) generalizes to the vector \((X_S, X_{I_1}, X_{I_2}, \ldots, X_{I_N})\) as follows.

\[
\begin{pmatrix}
X_S \\
X_{I_1} \\
\vdots \\
X_{I_N}
\end{pmatrix}
\sim 
\mathcal{N}
\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix}
= 
\begin{pmatrix}
1 & \delta_1 & \delta_2 & \ldots & \delta_N \\
\delta_1 & \rho_{1,2} & \rho_{1,3} & \ldots & \rho_{1,N} \\
\delta_2 & \rho_{2,1} & \rho_{2,3} & \ldots & \rho_{2,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta_N & \rho_{N,1} & \rho_{N,2} & \ldots & \delta_N
\end{pmatrix}
\]

(2)

This case is illustrated in Figure 2. It is important to notice that \(I_j\) does not have to be a contiguous segment of the unit interval. Instead, each expert can know any Borel measurable subset of the full information. Given that the information structure is described by the sub-matrix \(\Sigma_{22}\), learning about the information among the \(N\) experts is equivalent to estimating a covariance matrix under several restrictions. First, each element of \(\Sigma_{22}\) must be in the unit interval, and no off-diagonal element can be larger than the corresponding diagonal element in the same row. Second, \(\Sigma_{22}\) must be symmetric, non-singular, and coherent. The matrix \(\Sigma_{22}\) is coherent if and only if its information structure can be described by a diagram such as the one given in Figure 2.

The next step is to link this model with the probability forecasts. If \(P_{I_j} = X_{I_j}/\sqrt{1 - \delta_j}\) represents \(E_j\)'s probit forecast, the corresponding probability forecast is given by

\[
p_j = P(A|\mathcal{F}_{I_j}) = \Phi \left( P_{I_j} \right)
\]

(3)

Let \(A_1, A_2, \ldots\) be an infinite sequence of events, each defined similarly to \(A\). If the \(j\)th expert’s probability forecast for \(A_i\) is given by (3), then the expert’s forecasts align with the long-term frequency of the events. That is, when considering only those events for which \(p_j\) takes on some preassigned value \(p'_j \in [0, 1]\), the long term frequency of occurrence of those events is \(p'_j\). Such an expert is deemed well-calibrated (see, e.g., [DeGroot and Fienberg (1983)]). Given that several experiments have shown that experts are often poorly calibrated (see, e.g., [Cooke (1991); Shlyakhter et al. (1994)]), assuming calibrated forecasts can be unrealistic. This could be remedied by either including an additive error term in (3) such that the experts are on average calibrated, or by extending the total information by an amount unknown to the experts. Furthermore, if the experts make repeated forecasts for a sequence of events (e.g., whether it rains or not over a series of days), it may be possible to calibrate their forecasts (see, e.g., [Brier (1950); Foster and Vohra (1998)]). All of these extensions, however, are considered beyond the scope of this paper and hence deferred to future work.

The marginal distribution of \(p_j\) is

\[
m(p_j|\delta_j) = \sqrt{\frac{1 - \delta_j}{\delta_j}} \exp \left\{ \Phi^{-1}(p_j)^2 \left( 1 - \frac{1}{2\delta_j} \right) \right\}
\]

This is uniform on \([0, 1]\) if the expert knows half of the information, i.e. \(\delta_j = 0.5\). If the expert knows less than half of the information, i.e. \(\delta_j < 0.5\), the marginal distribution is unimodal at 0.5 with the variance decreasing to 0 as \(\delta_j \to 0\). Therefore an expert with no information always
Figure 3: The marginal distribution of $p_j$ under different levels of $\delta_j$. The more the expert knows, i.e. the higher $\delta_j$ is, the more the probability forecasts are concentrated around the extreme points 0.0 and 1.0.

reports a “non-informative” forecast of 0.5. On other hand, if the expert knows more than half of the information, i.e. $\delta_j > 0.5$, the marginal distribution is more heavily concentrated around extreme probabilities 0.0 and 1.0. More specifically, the marginal distribution becomes uniform over the set $\{0.0, 1.0\}$ as $\delta_j \to 1$. Therefore an expert who knows all the information always reports the correct outcome. Figure 3 illustrates the marginal distribution for $\delta_j$ equal to 0.3, 0.5, and 0.7.

4 Probability Extremizing

The best in-principle forecast given the knowledge of $N$ experts is $P(X_S > 0 | \mathcal{F}')$, where $\mathcal{F}' = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_N$. In this paper such an aggregate forecast is called the crowd belief. Even though in practice knowledge of $\mathcal{F}'$ is almost always unattainable due to company confidentiality or the experts’ inability to specify the knowledge that ultimately leads to their opinions (Dawid et al. (1995)), in theory the crowd belief can be obtained by introducing an “oracle” expert whose information set is $I' = I_1 \cup \cdots \cup I_N$. Therefore the oracle knows everything that the group of experts
knows. The can be appended to the multivariate Gaussian distribution (2). More specifically,

\[
\begin{pmatrix}
X_S \\
X_I \\
\vdots \\
X_{1N}
\end{pmatrix}
\sim N
\begin{pmatrix}
0, \\
\Sigma_1' \\
\Sigma_2'
\end{pmatrix}
\]

where \(X_I\) is the information known by the oracle and \(\delta' = |I'|\) is the amount of this information.

Similarly to the individual experts, the oracle’s probability forecast, i.e. the crowd belief, is given by

\[
p_{crowd}^N = P(X_S > 0|F') = \Phi \left( \frac{X_I'}{\sqrt{1 - \delta'}} \right)
\]

This forecast is calibrated and efficiently utilizes all the information among the experts. Therefore it provides a “gold standard” for analyzing probability extremizing, which in this paper is understood as an increase in the strength of the belief indicated by the probability forecast. More specifically, a probability \(p\) is extremized by another probability \(p'\) if and only if \(p'\) is closer to 0 when \(p \leq 0.5\) and closer to 1 when \(p \geq 0.5\). This can be expressed more succinctly in terms of probits as follows: if \(P = \Phi^{-1}(p), P' = \Phi^{-1}(p')\), and \(\alpha\) is a real-valued constant such that \(\alpha P = P'\), then \(p'\) extremizes \(p\) when \(\alpha > 1\). Therefore \(\alpha\) is a univariate quantity that represents the multiplicative amount of extremizing that \(p'\) performs for \(p\). Letting \(p = p_{ave}^{prob}\) and \(p' = p_{crowd}^N\) gives

\[
\alpha = \frac{P_I'}{\frac{1}{N} \sum_{j=1}^{N} P_{I_j}}
\]

This is a random quantity that spans the entire real line. That is, it is possible to find a set of probability forecasts and a distribution of information for any possible value of \(\alpha \in \mathbb{R}\). Therefore extremizing is not guaranteed to always improve the probit opinion pool. To understand when extremizing is likely to be beneficial, it is necessary to derive the probability distribution of \(\alpha\). First, given that

\[
P_{I'} \sim N \left( 0, \frac{\sigma^2}{1 - \delta'} \right)
\]

\[
\frac{1}{N} \sum_{j=1}^{N} P_{I_j} \sim N \left( 0, \sigma^2 = \frac{1}{N^2} \left( \sum_{j=1}^{N} \frac{\delta_j}{1 - \delta_j} + 2 \sum_{i,j: i < j} \frac{\rho_{ij}}{\sqrt{(1 - \delta_j)(1 - \delta_i)}} \right) \right)
\]

the amount of extremizing \(\alpha\) is a ratio of two correlated Gaussian random variables. The Pearson
product-moment correlation coefficient for them is

$$\kappa = \frac{\sum_{j=1}^{N} \delta_j}{\sqrt{1 - \delta_j}}$$

$$\sqrt{\delta' \left( \sum_{j=1}^{N} \delta_j \frac{1}{1 - \delta_j} + 2 \sum_{i,j:i<j}^{} \rho_{ij} \sqrt{\frac{1}{1 - \delta_j}(1 - \delta_i)} \right)}$$

The amount of extremizing $\alpha$ then follows a Cauchy distribution as long as $\sigma_1 \neq 1$, $\sigma_2 \neq 1$, or $\kappa \neq 1$ (see, e.g., Cedilnik et al. (2004) for this well-known result). These conditions are very mild under the partial information model. For instance, if no expert knows as much as the oracle, the conditions are satisfied. Consequently, the probability density function of $\alpha$ is

$$f(\alpha|x_0, \gamma) = \frac{1}{\pi} \frac{\gamma}{(\alpha - x_0)^2 + \gamma^2},$$

where

$$x_0 = \frac{\kappa \sigma_1}{\sigma_2} \quad \quad \gamma = \frac{\sigma_1}{\sigma_2} \sqrt{1 - \kappa^2}$$

The parameter $x_0$ represents the location (the median and mode) and $\gamma$ specifies the scale (half the interquartile range) of the Cauchy distribution. This leads to the following proposition whose proof is deferred to Appendix A.

**Proposition 4.1.** The amount of extremizing $\alpha \sim \text{Cauchy}(x_0, \gamma)$, where the location parameter $x_0 \geq 1$. The equality $x_0 = 1$ occurs only when $\delta_j = \delta'$ for all $j = 1, \ldots, N$. Consequently, if $\delta_j \neq \delta'$ for some $j = 1, \ldots, N$, then the probability that the probit opinion pool requires extremizing is $\mathbb{P}(\alpha > 1|\Sigma_{22}, \delta') \in (0.5, 1]$, where the bounds are sharp.

This proposition shows that, on any non-trivial problem, extremizing is more likely to be improve the probit opinion pool than to degrade it. This can partially explain why extremizing aggregators perform well on large sets of real-world prediction problems. To understand how the probability $\mathbb{P}(\alpha > 1|\Sigma_{22}, \delta')$ changes across different information structures, it is helpful, if not necessary, to make simplifying assumptions. A natural choice is to consider all experts exchangeable and develop intuition at a higher level. This is done in the next subsection that analyzes extremizing under compound symmetric information.

### 4.1 Extremizing under Compound Symmetric Information

This section assumes that the experts’ information sets have the same size and that the amount of overlap between any two information sets is constant, i.e. $|I_1| = \cdots = |I_N|$ and $|I_i \cap I_j| = |I_h \cap I_k|$ for all $i \neq j$ and $h \neq k$. Therefore each expert knows and shares the same amount of information with every other expert. The corresponding information structure $\Sigma_{22}$ is compound symmetric.
That is,
\[
\begin{pmatrix}
X_S \\
X_I' \\
X_{I_1} \\
\vdots \\
X_{I_N}
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
0 \\
\Sigma_{11}' \\
\Sigma_{21}' \\
\Sigma_{22}'
\end{pmatrix} = \begin{pmatrix}
1 & \delta' & \delta & \ldots & \delta \\
\delta' & \delta & \delta & \ldots & \delta \\
\delta & \delta & \lambda\delta & \ldots & \lambda\delta \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\delta & \delta & \lambda\delta & \lambda\delta & \ldots & \delta
\end{pmatrix}
\]

The amount of information known by each expert is denoted with \( \delta \in [0, 1] \). The value of \( \lambda \) is the proportion of the known information that is shared between any two experts. This minor change of parametrization was made for the sake of simplifying some of the following expressions. To ensure that \( \Sigma_{22} \) is coherent, a domain restriction must be placed on \( \lambda \). First, given that under any combination of \( \delta \) and \( N \) all experts may know the exact same information, the value of \( \lambda \) is upper bounded by 1. Second, observe that information overlap is unavoidable when \( \delta > 1/N \). The minimum sharing occurs when all information is either shared or private. In other words, if \( \delta > 1/N \) and \( I_i \cap I_j = I \) with \( |I| = \lambda\delta \) for all \( i \neq j \), the value of \( \lambda \) is minimized when \( \lambda\delta + N(\delta - \delta\lambda) = 1 \). Therefore the lower bound for \( \lambda \) is \( \max \left\{ \frac{N - \delta^{-1}}{N - 1}, 0 \right\} \), and \( \Sigma_{22} \) is coherent if and only if
\[
\delta \in [0, 1] \quad \lambda \in \left[ \max \left\{ \frac{N - \delta^{-1}}{N - 1}, 0 \right\}, 1 \right], \quad (5)
\]

where the open upper bound on \( \lambda \) ensures a non-singular \( \Sigma_{22} \). In the following discussion, however, this technical restriction is ignored and the case \( \lambda = 1.0 \) is analyzed among others.

Plugging these simplifications in (4) gives
\[
\alpha = \frac{X_I'}{\frac{1}{N} \sum_{j=1}^{N} X_{I_j}} \sqrt{\frac{1 - \delta}{1 - \delta'}}
\]

This follows a Cauchy \((x_0, \gamma)\) with
\[
x_0 = \frac{N}{1 + (N - 1)\lambda} \sqrt{\frac{1 - \delta}{1 - \delta'}}
\]
\[
\gamma = \sqrt{\frac{N(\delta' + \delta'\lambda(N - 1) - \delta N)}{\delta(\lambda(N - 1) + 1)^2}} \sqrt{\frac{1 - \delta}{1 - \delta'}}
\]

The total amount of information among the experts \( \delta' \) is uniquely determined by \( \delta \) and \( \rho \) when the group involves only two experts. More specifically, \( \delta' = \delta(2 - \lambda) \) when \( N = 2 \). Therefore, by assuming only two experts, both \( x_0 \) and \( \gamma \) can be analyzed graphically under different information structures. Figure 4 considers only \( N = 2 \) experts and shows \( \log(x_0), \gamma, \) and \( P(\alpha > 1|\Sigma_{22}, \delta') \)
Figure 4: The amount of extremizing follows a Cauchy($x_0, \gamma$), where $x_0$ is a location parameter and $\gamma$ is a scale parameter. This figure considers only $N=2$ experts and shows $\log(x_0)$, $\gamma$, and $P(\alpha > 1|\Sigma_{22}, \delta')$ under all plausible combinations of $\delta$ and $\lambda$.

under all plausible combinations of $\delta$ and $\rho$. High values have been censored to keep the scale manageable.

According to Figure 4, extremizing is required more often when $\delta$ is high and $\lambda$ is low. Given that $\delta'$ increases in $\delta$ but decreases in $\lambda$, the amount of extremizing can be expected to increase in the total amount of information among the experts $\delta'$. This, however, does not provide a full explanation of extremizing. Information diversity is also an important yet separate determinant of extremizing. To see this, observe that fixing $\delta'$ to some constant defines a curve $\lambda = 2 - \delta'/\delta$ on the plots in Figure 4. For instance, letting $\delta' = 1$ gives the boundary curve on the right side of each plot. This curve then shifts inwards and rotates slightly counterclockwise as $\delta'$ decreases. At the top end of each curve all experts know and share the total information, i.e. $\delta = \delta'$ and $\lambda = 1.0$. At the bottom end, on the other hand, the experts partition the total information, i.e. $\delta = \delta'/2$, and share nothing, i.e. $\lambda = 0.0$. Therefore moving down along the curve increases information diversity among the experts. Given that at the same time the modal amount of extremizing increases, information diversity can be considered as an important determinant of extremizing together with the group’s total information.

This relationship can be understood in broader terms as follows. Recall that the experts form their individual forecasts independently based on their own information. Given that this personal information cannot be larger than the group’s total information, the expert’s forecast is often less confident, i.e. closer to 0.5, than the crowd belief. Unfortunately, any average of the individual forecasts cannot be more confident than the most confident expert in the group. Therefore it cannot extend beyond convex hull of the individual forecasts even when this is suggested by the group’s total information. The crowd belief, on the other hand, is based on more information than any of the individual forecasts, and as a result, is likely to be more confident than any of them. The gap between the average aggregate and the crowd belief widens as the group’s total information
increases (the crowd belief becomes more confident) and the average amount information of a single expert decreases (the convex hull becomes more concentrated around 0.5 and the average aggregate becomes less confident). These two changes can happen simultaneously only if the information diversity among the experts increases.

In Figure 4, the location parameter $x_0 \geq 1$ and, consequently, $P(\alpha > 1|\Sigma_{22}, \delta') \in (0.5, 1.0]$. This follows from Proposition 4.1. When $\lambda = 1.0$, the location $x_0 = 1.0$ and scale $\gamma = 0.0$, which represents a point mass at $\alpha = 1.0$. In this case, all information is public, and the group of experts is as good as a single expert. Therefore, given that the forecast of a group with only one expert coincides with the crowd belief, $\lambda = 1.0$ always implies $\alpha = 1.0$ regardless of the value of $\delta$.

From the top of Figure 4a, the location $x_0$ increases indefinitely towards the border line defined by $\delta' = \delta(2 - \lambda) = 1$. At this point, the oracle knows everything and the crowd belief is always equal to the true outcome of the target event. Based on Figure 4c, the probit opinion pool is more likely to agree with the crowd belief at points closer to the end points of the border line, namely, when $(\delta, \lambda) = (0.5, 0.0)$ or $(\delta, \lambda) = (1.0, 1.0)$. At the former point, experts’ information sets form a partition of the full information. Therefore the experts know all the information. Such a group of experts can re-construct $X_S$ by simply adding up their information, and aggregation becomes deterministic voting: if the sum of the probit forecasts is above 0.0, the event $A$ materializes; else it does not. A similar observation has been made under the interpreted forecasts (Hong and Page (2009)). At the latter point, the outcome of the target event is public information and the experts can deterministically report the correct outcome. In between these two points, however, there is a chance that the probit opinion pool disagrees with the crowd belief on the more likely outcome of the target event. The correction requires a large but negative value of $\alpha$. Consequently, the scale $\gamma$ increases and $P(\alpha > 1|\Sigma_{22}, \delta')$ decreases slightly.

The amount of extremizing is positive only if the probit opinion pool and the crowd belief are on the same side of 0.5. The probability of this happening increases in $P(\alpha > 1|\Sigma_{22}, \delta')$, which according to Figure 4c decreases in $\lambda$. Given that increasing $\lambda$ results in more similar probability forecasts, the value of $\lambda$ is inversely proportional to the variance of the forecasts. Therefore, under the partial information model, higher variance can be considered helpful because it increases the chances of the probit opinion pool being on the same side 0.5 with the crowd belief. Contrast this with the classical measurement error model where increased variance is typically considered harmful. This reversal of the effect is an important property of the interpreted forecasts (Hong and Page (2009)).

### 5 Probability Aggregation

This section uses the partial information model from Section 3 to derive a probability aggregator that can be applied in practice. Recall that assuming knowledge of the union of the experts’ information sets is hardly reasonable in practice. Therefore the best aggregate probability that can be realistically hoped for is $P(X_S > 0|p_1, \ldots, p_N)$. This can be constructed under the partial information model by first deriving the conditional distribution of $X_S$ given $X$. Referring back to (2), if
$\mathbf{X} = (X_1, X_2, \ldots, X_N)'$ is a column vector of length $N$ and $\Sigma_{22}$ is a coherent overlap structure such that $\Sigma_{22}^{-1}$ exists, then

$$X_S|\mathbf{X} \sim \mathcal{N} \left( \bar{\mu}, \bar{\Sigma} \right),$$

where

$$\bar{\mu} = \mu_1 + \Sigma_{12}(X - \mu_2) = \Sigma_{12}\Sigma_{22}^{-1}X$$

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

These expressions can be found directly from the formulas of the conditional multivariate Gaussian distribution (see, e.g., Result 5.2.10 on p. 156 in Ravishanker and Dey (2001)). This then leads to a probability aggregator that depends on $\Sigma_{22}$. More specifically, the partial information aggregator is

$$p_{N}^{info} = \mathbb{P}(A|\mathbf{X}) = \mathbb{P}(X_S > 0|\mathbf{X}) = \Phi \left( \frac{\Sigma_{12}\Sigma_{22}^{-1}X_1}{\sqrt{1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}} \right) \tag{6}$$

Furthermore, if $\mathbf{1}_N$ is a column vector of ones and $\mathbf{P} = (P_1, P_2, \ldots, P_N)'$, then the amount of extremizing that $p_{N}^{info}$ performs for $p_{N}^{ave\; prb}$ is

$$\alpha = \frac{N\Sigma_{12}\Sigma_{22}^{-1}X_1}{(\mathbf{1}_N'\mathbf{P}) \sqrt{1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}}$$

Given that accurate estimation of the fully general information structure $\Sigma_{22}$ based on the probability forecasts may not be feasible, it is necessary to constrain the information structure. The next section does this by assuming a compound symmetric information structure. This leads to an aggregator that can be appropriate for combining probability forecasts of a one-time event.

### 5.1 Aggregation under Compound Symmetric Information

The partial information aggregator (6) under the compound symmetric information structure simplifies to

$$\mathbb{P}(X_S > 0|\mathbf{X}) = \Phi \left( \frac{\sum_{j=1}^{N} \frac{1}{(N-1)\lambda+1} \frac{X_j}{N\delta}}{\sqrt{1 - \frac{(N-1)\lambda+1}{N\delta}}} \right), \tag{7}$$

where the values of $\delta$ and $\lambda$ can be estimated in practice via the maximum likelihood method. Technical details for estimation are provided in Appendix A. The domain restriction (5) ensures that the term under the square-root in (7) is always non-negative. Similarly to the aforementioned classical aggregators, the partial information aggregator (7) considers the experts exchangeable. However, it is based on widely different modeling assumptions. This makes it more appropriate for combining interpreted forecasts. This is stated more formally in the following proposition whose proof is deferred to Appendix A.
Figure 5: The amount of log-extremizing \( \log(\alpha) \) under different combinations of \( N \) (the number of experts), \( \delta \) (the average amount known by one expert), and \( \lambda \) (the average amount shared by any two experts).

**Proposition 5.1.** Under the compound symmetric information structure, (a) the partial information aggregator extremizes the probit opinion pool as long as \( p_j \neq p_i \) for some \( j \neq i \), and (b) the aggregator can leave the convex hull of the individual probability forecasts.

The amount of extremizing that (7) performs for the probit opinion pool is

\[
\alpha = \frac{N\sqrt{1-\delta}}{(N-1)\lambda+1} \sqrt{1 - \frac{N\delta}{(N-1)\lambda+1}}
\]  

(8)

This is a non-random quantity that depends only on three intuitive parameters. Therefore it can be conveniently analyzed graphically. Figure 5 considers \( N = 2 \) and \( N = 10 \) experts, and describes the amount of log-extremizing \( \log(\alpha) \) under all plausible combinations of \( \lambda \) and \( \delta \). High values have been censored to keep the scale manageable. Even though the patterns in Figure 5 are very similar to the ones shown earlier in Figure 4, comparing \( \alpha \) under \( N = 2 \) and \( N = 10 \) leads to new observations. First, holding both \( \delta \) and \( \lambda \) fixed, the amount of extremizing \( \alpha \) increases in \( N \). This can be considered as being caused by an increase in the total amount of information.
among the experts because $\delta'$ increases in $N$. Second, in general, having many experts, each with a considerable amount of information, simply leads to unavoidable information overlap. This is illustrated in Figure 5 where the set of possible values of $\lambda$ decreases very rapidly as $\delta$ increases. Furthermore, moving from Figure 5a to Figure 5b illustrates how the dependence between $\lambda$ and $\delta$ strengthens as $N$ increases. In fact, based on the domain restriction (5), the value of $\lambda \to 1$ as $N \to \infty$ under any given $\delta$. Therefore in the limit the group of experts is equivalent to a single expert. This restrictive limiting behavior is due to assuming that each pair of experts shares the same amount of information. The compound symmetric information structure, however, is almost fully general when $N = 2$. Therefore assuming compound symmetric information can be appropriate for small numbers of experts but becomes restrictive as more experts enter the group.

6 Discussion

This paper introduced a concrete model for probability forecasts made by a group of experts. The experts are assumed to make their forecasts based on different subsets of information that ultimately decides the outcome of the event. The model was used to derive an expression for the best in-principle forecast given the knowledge of $N$ experts. Even though this aggregate, called the crowd belief, may not be accessible in practice, it was used as a theoretical “gold standard” for linking extremizing of average aggregates with the information among the experts. The analysis gave two main results. First, the amount of extremizing is increasing in information diversity and the total amount of information among the experts. Second, no matter how information is distributed among the experts, extremizing the probit opinion pool is more likely to beneficial than harmful as long as the experts’ forecasts are not all the same. This was a partial motivation for developing an aggregator that always extremizes the average probit forecast. The partial information aggregator under compound symmetric information was shown to extremize the average probit forecast as long as the experts’ forecasts are not all the same. This aggregator depends only on two intuitive parameters, namely the average amount of information known by an expert and the average amount of information shared between any two experts. Given that these two quantities can be directly estimated from the experts’ forecasts, the aggregator can be appropriate for combining forecasts of a one-time event.

It is interesting to relate our discussion to the many empirical studies conducted by the Good Judgment Project (GJP) (Mellers et al. (2014); Ungar et al. (2012)). The GJP is a research study that has recruited thousands of forecasters from professional societies, research centers, and alumni associations. These forecasters are given questions about future international political events, such as who would win an election in Russia or the Congo. Individuals then estimate the probability of each event, and update their predictions when they feel the probabilities have changed. The forecasters know that their probability estimates are assessed for accuracy using Brier scores, i.e. the squared distance from the probability forecast to 1.0 or 0.0 depending on whether the event happened or not, respectively (Brier (1950)). This incentivizes the forecasters to report their true beliefs instead of attempting to game the system (Winkler and Murphy (1968)). In addition to
receiving $150 for meeting minimum participation requirements that do not depend on prediction accuracy, the forecasters receive status rewards for their performance via leader-boards displaying Brier scores for the top 20 experts. Every year the top 2% percent of the forecasters are selected to the elite group of “super-forecasters”. The super-forecasters work in groups to make highly accurate predictions on the same events as the rest of the forecasters.

Generally extremizing has been found to improve the average aggregates (Mellers et al. (2014)). The average forecast of a team of super-forecasters, however, often requires very little or no extremizing. This can be explained by the partial information model as follows. Given that the super-forecasters are highly knowledgeable (i.e. they have a high $\delta$) individuals who work in groups (i.e. they have a high $\rho$ and $\lambda$), they are situated in Figures 4 and 5 around the upper-right corners where almost no extremizing is required. All forecasters were also asked to self-assess their level of expertise (on a 1-to-5 scale with 1 = Not At All Expert and 5 = Extremely Expert) on the events for which they provided forecasts. Given that expertise is largely defined by the forecaster’s personal knowledge, the value of $\delta$ can be considered positively associated with the self-assessed expertise. Marginally, this implies a positive relationship between the level of expertise and amount of extremizing. Satopää et al. (2014), however, suggest that the amount of extremizing is negatively associated with self-assessed expertise, i.e. lower expertise requires more extremizing. This can be explained by two observations. First, the average number of forecasters $N$ per expertise group across the 69 international events considered in Satopää et al. (2014) were around 68.3, 84.9, 61.6, 17.5, and 3.4 (from the lowest to the highest level of expertise). Second, as illustrated by Figure 5, the low experts had a chance of being highly diverse while the high experts were likely to experience considerable information overlap. These observations suggest that the low-expertise groups required more extremizing because they held more information in total than the high-expertise groups.

The partial information model offers many future research directions. For instance, new probability aggregators can be developed by finding different ways to estimate the information overlap among experts. Unfortunately, without any additional information besides the probability forecasts, it may not be possible to estimate the information structure accurately in full generality. Therefore the structure must be constrained in some manner. For instance, Appendix A provides explicit estimation instructions under the compound symmetric information structure. This structure, however, has restrictive limiting behavior. Therefore it will be necessary to develop a class of information structures that is both estimable and realistic for large numbers of experts. A different alternative is to construct a prior distribution for the information structure, update this prior to a posterior distribution via the multivariate Gaussian likelihood function, and then marginalize the information structure with respect to its posterior distribution.

Other future directions could remove some of the model limitations. For instance, assuming that the experts produce optimal probability forecasts given their private information sets may not be reasonable. The experts can believe in false information, hide their true beliefs, or be biased for many other reasons. This could be expressed in the model by introducing an error term that is added to the optimal forecasts before being assigned to the experts. Such an extension was not considered in this paper for the sake of providing a clean introduction to the partial information
model together with a clear illustration of our main results on probability extremizing.

Appendix A: Technical Details

A.1 Proofs

*Proof of Proposition 4.1*

\[ x_0 = \kappa \frac{\sigma_1}{\sigma_2} \]

\[ = \frac{\sum_{j=1}^{N} \frac{\delta_j}{\sqrt{1-\delta_j}}}{\sqrt{\delta^{'} \left\{ \sum_{j=1}^{N} \frac{\delta_j}{1-\delta_j} + 2 \sum_{i,j: i<j} \frac{\rho_{ij}}{\sqrt{(1-\delta_j)(1-\delta_i)}} \right\}}} \]

\[ = \frac{N \sum_{j=1}^{N} \frac{\delta_j}{\sqrt{(1-\delta_j)(1-\delta_j)}}}{\sum_{j=1}^{N} \frac{\delta_j}{1-\delta_j} + 2 \sum_{i,j: i<j} \frac{\rho_{ij}}{\sqrt{(1-\delta_j)(1-\delta_i)}}} \]

Given that all the remaining terms are positive, the location parameter \( x_0 \) is also positive. Next compare the \( N \) terms with a given subindex \( j \) in the numerator with the corresponding terms in the denominator. From \( \delta' \geq \delta_j \geq \rho_{ij} \) it follows that

\[ \frac{\delta_j}{1-\delta_j} = \frac{\sqrt{(1-\delta_j)(1-\delta_j)}}{\sqrt{(1-\delta_j)(1-\delta_j)}} \leq \frac{\delta_j}{\sqrt{(1-\delta_j)(1-\delta_j)}} \leq \frac{\delta_j}{\sqrt{(1-\delta_j)(1-\delta_j)}} \leq \frac{\delta_j}{\sqrt{(1-\delta_j)(1-\delta_j)}} \]

Therefore

\[ N \sum_{j=1}^{N} \frac{\delta_j}{\sqrt{(1-\delta_j)(1-\delta_j)}} \geq \sum_{j=1}^{N} \frac{\delta_j}{1-\delta_j} + 2 \sum_{i,j: i<j} \frac{\rho_{ij}}{\sqrt{(1-\delta_j)(1-\delta_i)}} \]

which gives that \( x_0 \geq 1 \). Given that the Cauchy distribution is symmetric around \( x_0 \), it must be the case that \( \mathbb{P}(\alpha \geq 1|\Sigma_{22}, \delta') \geq 0.5 \). Based on (9) and (10), the location \( x_0 = 1 \) only when all the experts know the same information, i.e. when \( \delta_j = \delta' \) for all \( j = 1, \ldots, N \). Under this particular setting, the amount of extremizing \( \alpha \) is non-random and always equal to 1.0. Therefore \( \mathbb{P}(\alpha \geq 1|\Sigma_{22}, \delta') = 1.0 \). Any deviation from this particular information structure makes \( \alpha \) stochastic, \( x_0 > 1 \), and hence \( \mathbb{P}(\alpha \geq 1|\Sigma_{22}, \delta') > 0.5 \). If the expert’s information sets partition the full information, the sum of their probits is always on the correct side of 0.0. At the same time the oracle deterministically outputs the correct outcome. Consequently, \( \alpha = +\infty \) and \( \mathbb{P}(\alpha \geq 1|\Sigma_{22}, \delta') = 1.0 \). Thus \( \mathbb{P}(\alpha > 1|\Sigma_{22}, \delta') \in (0.5, 1.0] \) when \( \delta_j \neq \delta' \) for some \( j = 1, \ldots, N \). \[\Box\]
Proof of Proposition 5.1. (a) For a given $\delta$, the amount of extremizing $\alpha$ is minimized when $(N-1)\lambda + 1$ is maximized. This happens as $\lambda \uparrow 1$. Plugging this into (8) gives

$$\alpha = \frac{\sqrt{1-\delta}}{\sqrt{1-(N-1)\lambda+1}} \downarrow \sqrt{1-\delta} \downarrow 1$$

(b) Assume without loss of generality that $\bar{P} > 0$. If $\max\{p_1, p_2, \ldots, p_N\} < 1.0$, then setting $\delta = 1/N$ and $\lambda = 0.0$ gives an aggregate probability $p_N^{inj} = 1.0$ that is outside the convex hull of the individual probabilities.

A.2 Estimation of $\delta$ and $\lambda$

The values of $\delta$ and $\lambda$ can be estimated via the maximum likelihood method. To make this more explicit, observe that the Jacobian for the map $P \rightarrow \Phi(P) = (\Phi(P_1), \Phi(P_2), \ldots, \Phi(P_N))'$ is

$$J(P) = (2\pi)^{-N/2} \exp \left(-\frac{P'P}{2}\right)$$

Let $J_{N \times N}$ represent a $N \times N$ matrix of ones. If $h(P)$ denotes the multivariate Gaussian density of $P \sim N_N(0, \Sigma_{22}(1-\delta)^{-1})$, the density for $p = (p_1, p_2, \ldots, p_N)'$ becomes

$$f(p|\delta, \lambda) = h(P)J(P)^{-1}
= \frac{(1-\delta)^{N/2}}{\sqrt{|\Sigma_{22}|}} \exp \left[-\frac{1}{2}\Phi^{-1}(p)'\left\{(1-\delta)\Sigma_{22}^{-1} - I_N\right\} \Phi^{-1}(p)\right],$$

where

$$|\Sigma_{22}| = (\delta(1-\delta))^{N/2} \left(1 + \frac{N\lambda}{1 - \lambda}\right)$$

$$\Sigma_{22}^{-1} = I_N \left(\frac{1}{\delta - \lambda \delta}\right) - J_{N \times N} \left(\frac{\lambda}{(1-\lambda)\delta(1 + (N-1)\lambda)}\right)$$

(11)

See Rao (2009) and the supplementary material of Dobbin and Simon (2005) for the derivations of the determinant and inverse of $\Sigma_{22}$, respectively. The maximum likelihood estimates of $\delta$ and $\lambda$ are then obtained from

$$\left(\hat{\delta}, \hat{\lambda}\right) = \arg \max_{\delta, \lambda} f(p|\delta, \lambda),$$

s.t. $\delta \in [0, 1]$ and $\lambda \in \left[\max\left\{\frac{N - \delta^{-1}}{N - 1}, 0\right\}, 1\right]$

Given that this cannot be solved analytically, numerical methods such as a simple grid-search must be used to find $\hat{\delta}$ and $\hat{\lambda}$. 
References

Ariely, D., W. Tung Au, R. H. Bender, D. V. Budescu, C. B. Dietz, H. Gu, T. S. Wallsten, and G. Zauberman (2000). The effects of averaging subjective probability estimates between and within judges. *Journal of Experimental Psychology: Applied* 6(2), 130.

Armstrong, J. S. (2001). Combining forecasts. In *Principles of forecasting*, pp. 417–439. Springer.

Baars, J. A. and C. F. Mass (2005). Performance of national weather service forecasts compared to operational, consensus, and weighted model output statistics. *Weather and Forecasting* 20(6), 1034–1047.

Bacharach, M. (1975). Group decisions in the face of differences of opinion. *Management Science* 22(2), 182–191.

Baron, J., B. A. Mellers, P. E. Tetlock, E. Stone, and L. H. Ungar (2014). Two reasons to make aggregated probability forecasts more extreme. *Decision Analysis* 11(2), 133–145.

Brier, G. W. (1950). Verification of forecasts expressed in terms of probability. *Monthly Weather Review* 78, 1–3.

Broomell, S. B. and D. V. Budescu (2009). Why are experts correlated? Decomposing correlations between judges. *Psychometrika* 74(3), 531–553.

Bryan, M. L. and S. P. Jenkins (2013). Regression analysis of country effects using multilevel data: A cautionary tale. Technical report, IZA Discussion Paper.

Cedilnik, A., K. Kosmelj, and A. Blejec (2004). The distribution of the ratio of jointly normal variables. *Metodoloski zvezki* 1(1), 99–108.

Clemen, R. T. (1989). Combining forecasts: A review and annotated bibliography. *International Journal of Forecasting* 5(4), 559–583.

Cooke, R. M. (1991). *Experts in Uncertainty: Opinion and Subjective Probability in Science*. New York, NY, USA: Oxford University Press.

Dawid, A., M. DeGroot, J. Morthera, R. Cooke, S. French, C. Genest, M. Schervish, D. Lindley, K. McConway, and R. Winkler (1995). Coherent combination of experts’ opinions. *TEST* 4(2), 263–313.

DeGroot, M. H. and S. E. Fienberg (1983). The comparison and evaluation of forecasters. *The Statistician* 32(1/2), 12–22.

Dobbin, K. and R. Simon (2005). Sample size determination in microarray experiments for class comparison and prognostic classification. *Biostatistics* 6(1), 27–38.
Erev, I., T. S. Wallsten, and D. V. Budescu (1994). Simultaneous over- and underconfidence: The role of error in judgment processes. *Psychological Review* 101(3), 519–527.

Foster, D. P. and R. V. Vohra (1998). Asymptotic calibration. *Biometrika* 85(2), 379–390.

Genest, C. and J. V. Zidek (1986). Combining probability distributions: A critique and an annotated bibliography. *Statistical Science* 1(1), 114–148.

Gneiting, T. and A. E. Raftery (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association* 102(477), 359–378.

Hong, L. and S. Page (2009). Interpreted and generated signals. *Journal of Economic Theory* 144(5), 2174–2196.

Karmarkar, U. S. (1978). Subjectively weighted utility: A descriptive extension of the expected utility model. *Organizational Behavior and Human Performance* 21(1), 61–72.

Mellers, B., L. Ungar, J. Baron, J. Ramos, B. Gurcay, K. Fincher, S. E. Scott, D. Moore, P. Atanasov, S. A. Swift, et al. (2014). Psychological strategies for winning a geopolitical forecasting tournament. *Psychological Science* 25(5), 1106–1115.

Parunak, H., S. A. Brueckner, L. Hong, S. E. Page, and R. Rohwer (2013). Characterizing and aggregating agent estimates. In *Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems*, pp. 1021–1028. International Foundation for Autonomous Agents and Multiagent Systems.

Pepe, M. S. (2003). *The Statistical Evaluation of Medical Tests for Classification and Prediction*. Oxford University Press Oxford.

Ranjan, R. and T. Gneiting (2010). Combining probability forecasts. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 72(1), 71–91.

Rao, C. R. (2009). *Linear statistical inference and its applications*, Volume 22. John Wiley & Sons.

Ravishanker, N. and D. K. Dey (2001). *A first course in linear model theory*. CRC Press.

Sanders, F. (1963). On subjective probability forecasting. *Journal of Applied Meteorology* 2(2), 191–201.

Satopää, V. A., J. Baron, D. P. Foster, B. A. Mellers, P. E. Tetlock, and L. H. Ungar (2014). Combining multiple probability predictions using a simple logit model. *International Journal of Forecasting* 30(2), 344–356.

Shlomi, Y. and T. S. Wallsten (2010). Subjective recalibration of advisors’ probability estimates. *Psychonomic Bulletin & Review* 17(4), 492–498.
Shlyakhter, A. I., D. M. Kammen, C. L. Broido, and R. Wilson (1994). Quantifying the credibility of energy projections from trends in past data: The US energy sector. *Energy Policy* 22(2), 119–130.

Tetlock, P. E. (2005). *Expert Political Judgment: How Good Is It? How Can We Know?* Princeton University Press.

Turner, B. M., M. Steyvers, E. C. Merkle, D. V. Budescu, and T. S. Wallsten (2013). Forecast aggregation via recalibration. *Machine Learning* 95(3), 1–29.

Ungar, L., B. Mellers, V. Satopää, P. Tetlock, and J. Baron (2012). The good judgment project: A large scale test of different methods of combining expert predictions. In *The Association for the Advancement of Artificial Intelligence 2012 Fall Symposium Series*.

Vislocky, R. L. and J. M. Fritsch (1995). Improved model output statistics forecasts through model consensus. *Bulletin of the American Meteorological Society* 76(7), 1157–1164.

Wallsten, T. S. and A. Diederich (2001). Understanding pooled subjective probability estimates. *Mathematical Social Sciences* 41(1), 1–18.

Wilson, P. W., R. B. D’Agostino, D. Levy, A. M. Belanger, H. Silbershatz, and W. B. Kannel (1998). Prediction of coronary heart disease using risk factor categories. *Circulation* 97(18), 1837–1847.

Winkler, R. L. and A. H. Murphy (1968). “Good” probability assessors. *Journal of Applied Meteorology* 7(5), 751–758.