The Sectional Curvature Remains Positive When Taking Quotients by Certain Nonfree Actions

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Abstract—We study some cases in which the sectional curvature remains positive under the taking of quotients by certain nonfree isometric actions of Lie groups. We consider the actions of the groups $S^1$ and $S^3$ for which the quotient space can be endowed with a smooth structure by means of the fibrations $S^3/S^1 \simeq S^2$ and $S^7/S^3 \simeq S^4$. We prove that the quotient space possesses a metric of positive sectional curvature provided that the original metric has positive sectional curvature on all 2-planes orthogonal to the orbits of the action.

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INTRODUCTION

In this article we consider a certain method for constructing closed Riemannian manifolds of positive sectional curvature. There are few examples of such manifolds (the reader can find the whole list of all available examples, for example, in the introduction to [1]).

The main tool for constructing new examples is the taking of quotients by free isometric actions of Lie groups. More precisely, suppose that a Lie group $G$ acts on a closed smooth Riemannian manifold $M$ by isometries; i.e., the metric of $M$ is preserved by the action. Suppose further that this action is free; i.e., for every point $p \in M$ and every element $g \in G$ such that $g \neq 1$ we have $g \cdot p \neq p$. In this case the quotient space $M/G$ has the structure of a smooth manifold such that the projection map $\tau : M \to M/G$ is smooth.

Introduce a metric on $M/G$ as follows: Let $X$ and $Y$ be two vectors tangent to $M/G$ at some point $q$ and let $q = \tau(p)$ for some $p \in M$. Then there exist unique vectors $X'$ and $Y'$ tangent to $M$ at the point $p$ which are orthogonal to the orbit of $G$ and mapped by $\tau$ to $X$ and $Y$, respectively. Let $g$ be the metric of $M$. Put $g_*(X,Y) = g(X',Y')$. Since $G$ acts by isometries, this relation correctly defines some metric $g_*$ on $M/G$, which we call the quotient metric of the metric $g$.

To compute the curvature of the quotient metric, we use the curvature formulas for Riemannian submersions. A map $f : M \to N$ between manifolds is a submersion if and only if $f$ is surjective together with its differential at each point. Let $f$ be a submersion between Riemannian manifolds $(M,g_M)$ and $(N,g_N)$. Consider two vectors $X'$ and $Y'$ at a point $p \in M$ that are orthogonal to the submanifold $f^{-1}(f(p))$. Suppose that $f$ maps them to the respective vectors $X$ and $Y$. The map $f$ is a Riemannian submersion if and only if $g_M(X',Y') = g_N(X,Y)$ for all $X'$ and $Y'$. If $f$ is a Riemannian submersion and $R_M$ and $R_N$ are the curvature tensors of the manifolds $M$ and $N$ then the relations found in [5] for the curvature imply that

$$R_M(X',Y',Y',X') \leq R_N(X,Y,Y,X).$$

The projection map $\tau : M \to M/G$ is a Riemannian submersion. Therefore, we obtain the following fact: If a Lie group $G$ acts freely by isometries and the metric of $M$ has positive sectional curvature on all 2-planes orthogonal to the orbits of $G$ then the quotient metric on $M/G$ has positive sectional curvature.

In this article we prove a more general version of the above statement:

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Theorem 1. Let $M$ be a smooth manifold with an action of a group $G$ equal to either $S^1$ or $S^3 = \text{Sp}(1)$. Denote by $N$ the fixed point set of the action; and by $\tau : M \to M/G$, the projection map. Let $W \subset M$ be a neighborhood of $N$. Assume that $M$ is endowed with a metric $\langle \cdot, \cdot \rangle_0$ invariant under the action of $G$, and the following conditions hold:

1. The metric $\langle \cdot, \cdot \rangle_0$ has positive sectional curvature on all 2-planes orthogonal to the orbits of $G$.
2. For each point $p \in M$, its isotropy group is either trivial or the whole $G$.
3. The space $N$ is a compact submanifold of $M$ of codimension $2 \cdot (1 + \dim G)$.

Then the quotient space $M/G$ possesses a smooth structure such that $\tau$ is smooth at all free points of the action; moreover, this space possesses a metric of positive sectional curvature that coincides with the quotient metric of $\langle \cdot, \cdot \rangle_0$ outside $W$.

To construct the smooth structure on $M/G$, we use infinite cones over the Hopf fibration $S^3/S^1 \simeq S^2$ and the fibration $S^7/S^3 \simeq S^4$. Some problems arise from the fact that the quotient metric for the standard metric on the sphere in each of the fibrations is the standard metric on the sphere of radius $1/2$; therefore, we have a conic singularity in the fixed point set $N$.

In the first section we give a construction which enables us to smooth the quotient metric in a neighborhood of $N$. In the second and third sections we construct some particular metric on the normal bundle over $N$ and analyze its curvature. In the fourth section we show that, under certain conditions, one metric can be replaced by another in a neighborhood of a submanifold with positivity of sectional curvature preserved. Finally, in the fifth section we construct the smooth structure on $M/G$ and the sought metric of positive sectional curvature.

1. SMOOTHING THE CONIC METRIC

Consider the following metric on $\mathbb{R}^n$:

$$g = dr^2 + g^2(r)d\varphi^2. \quad (1.1)$$

Here $r(x)$ is the length of the vector $x$, $d\varphi^2$ is the standard metric of sectional curvature 1 on the sphere $rS^{n-1}$, and $g(r)$ is some positive function in $C^2(0, \infty)$ with the third derivative at zero.

The metric under consideration is $C^2$-smooth if and only if the following conditions hold:

$$g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 0. \quad (1.2)$$

By straightforward computations with use made of the formulas for curvature of Riemannian submersions, we can prove (however, this fact is not used in this paper directly) that the metric under consideration has positive sectional curvature if and only if the following conditions hold:

$$g'''(0) < 0, \quad g''(r) < 0, \quad |g'(r)| < 1 \text{ for } r > 0. \quad (1.3)$$

Now, we construct auxiliary functions $g_\varepsilon$ to be used later. Though the existence of such functions looks quite natural from the geometrical point of view (see the figure below), the formal proof encounters certain difficulties; therefore, the proof is described at length.