Design of Joint Source-Channel Codes Based on a Generic Protograph

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Abstract

In this paper, we propose using a generic protograph to design joint source-channel codes (JSCCs). We present a generalized algorithm, called protograph extrinsic information transfer for JSCC algorithm (PEXIT-JSCC algorithm), for analyzing the channel threshold of the proposed JSCC. We also propose a source generic protograph EXIT (SGP-EXIT) algorithm, which is more appropriate than the extended source protograph extrinsic information transfer (ESP-EXIT) algorithm for evaluating the source threshold of a generic protograph. Moreover, a collaborative optimization method based on the SGP-EXIT and PEXIT-JSCC algorithms is proposed to construct generic-protograph JSCCs with good source and channel thresholds. Finally, we construct generic-protograph JSCCs, analyze their decoding thresholds, and compare their theoretical and error performance with JSCC systems based on optimized double-protographs. Results show that our proposed codes can outperform double-protograph-based JSCCs.

Index Terms

Asymptotic weight distribution, double protograph, joint source-channel coding, protograph-based extrinsic information transfer analysis, protograph-based low-density-parity-check codes, generic protograph.

I. INTRODUCTION

In a digital communication system, source coding is employed to reduce the redundancy in the original information by compression while channel coding is used to protect the compressed data during transmission by adding redundant information (parity check bits). Traditionally, these two types of coding are studied and optimized separately. In particular, when the code length is very large (approaching infinity), the separate design of source and channel coding can theoretically achieve the optimal error performance over an additive white Gaussian noise (AWGN) channel according to the Shannon information theory [2]. However, most real application scenarios cannot afford very long code lengths. They require low encoding/decoding latency and hence prefer short to moderate code lengths. With the recent development of Internet of Things, there has been a growing interest in combining source coding and channel coding with an aim to simplifying the system and/or further optimizing the transmission efficiency and effectiveness.
The concept of joint source-channel coding (JSCC) was first conceived more than 40 years ago [3]. It has been further investigated since 1990’s [4]–[6]. The main idea is to allow the residual redundancy left by the source encoder to be utilized in the tandem joint source-channel decoding algorithms. For example, it is shown that considerable coding gains can be obtained by providing the prior probability of the source bits to the channel decoder [5]. Also when the redundancy of images is utilized in the decoding process, errors can be reduced [7]. In [8], the use of JSCC in JPEG2000 transmission over a two-way relay network is studied; while in [9], [10], the applications of JSCC schemes to wireless video transmissions are investigated. In [11], an iterative joint source-channel decoder is proposed, where messages are exchanged between the decoder of variable length (source) codes (VLC) and the decoder of recursive systematic convolutional codes or low-density parity-check (LDPC) codes. With the use of this iterative joint source-channel decoder, significant error performance gains are observed compared with tandem decoding.

Low-density parity-check (LDPC) codes form a type of capacity-approaching channel codes and have been successfully deployed in many communication systems [12]–[14]. In [15], [16], two LDPC codes are concatenated in series to form a new type of joint source-channel code. Moreover, an iterative decoding algorithm is presented to illustrate the extrinsic information exchange between the source-LDPC decoder and the channel-LDPC decoder.

Protograph LDPC (P-LDPC) codes [17]–[20] are a subset of LDPC codes and they offer fast encoding and decoding structures as well as the linear minimum Hamming distance property. They have been proposed to replace the regular LDPC codes in [16], forming the double protograph-based LDPC JSCC (DP-LDPC JSCC) system [21]. The protograph representation of this JSCC scheme is shown in Fig. 1. Variable nodes and check nodes are represented by circles and squares, respectively; and gray circles denote punctured variable nodes (VNs). The source and channel protographs are shown in the left and right dotted frames, respectively. Firstly, the source symbols are compressed based on the source (left) protograph. Then, the compressed symbols are regarded as the inputs to the channel encoder. The solid green lines, which connect check nodes (CNs) in the source protograph and VNs in the channel protograph in a one-to-one manner, reflecting the cascading relationship between the source encoder and the channel encoder. Finally, the codeword is generated based on the channel (right) protograph.
Fig. 1 can be represented by a joint protomatrix $B_{J0}$, i.e.,

$$B_{J0} = \begin{pmatrix}
B_s & I_{m_s 	imes n_s} \\
0_{m_c 	imes n_s} & B_c
\end{pmatrix}$$

(1)

where $B_s$ indicates the source protomatrix with size $m_s \times n_s$, $B_c$ indicates the channel protomatrix with size $m_c \times n_c$, $I_{m_s}$ is an identity matrix of size $m_s \times m_s$, and $0$ denotes a zero matrix with size indicated by its subscript. Optimizations on the DP-LDPC JSCC system has been performed under different scenarios. In [22] and [23], respectively, an unequal error protection (UEP) technique and an unequal power allocation scheme have been proposed and applied to the DP-LDPC JSCC system. In [24] and [25], the source protomatrix $B_s$ and the channel protomatrix $B_c$, respectively, have been redesigned in the JSCC system to improve the error performance. In [26], both source and channel protomatrices are redesigned at the same time to achieve good error-correction capability; while in [27], the optimal distribution of degree-2 VNs in both source and channel protomatrices is studied. In the above studies, error floors are observed at high SNR for DP-LDPC JSCC systems with relatively good performance in the waterfall region.

In [28], it is shown that the aforementioned error floors can be lowered by adding new edges between VNs in the source protograph and CNs in the channel protograph. At the same time, the error performance in the waterfall region will be moderately sacrificed. With the additional edges between VNs in the source protograph and CNs in the channel protograph, Fig. 1 is modified to Fig. 2 and the joint protomatrix in (1) is modified to $B_J$, i.e.,

$$B_J = \begin{pmatrix}
B_s & B_{SSCV} \\
B_{SSCV} & B_c
\end{pmatrix}$$

(2)

where

$$B_{SSCV} = \begin{pmatrix}
I_{m_s} & 0
\end{pmatrix}$$

(3)

indicates the source-check-channel-variable linking protomatrix with size $m_s \times n_c$ and $B_{SSCV}$ indicates the source-variable-channel-check linking protomatrix with size $m_c \times n_s$ [29], [30]. Subsequently, the effect of $B_{SSCV}$ on the code performance is studied. In [29], it has been discovered that connecting high-weight columns in $B_s$ to the rows in $B_c$ via $B_{SSCV}$ can obtain better error performance. Moreover, if the identity matrix $I_{m_s}$ in $B_{SSCV}$ is aligned with the high-weight columns in $B_c$, the error performance can be improved. In other words, CNs in $B_s$ should be connected to VNs with high degrees in $B_c$ via $I_{m_s}$.

In [31], several design principles for optimizing $B_{SSCV}$ are proposed to improve the waterfall performance when the source entropy is relatively high. In [32], a search algorithm is proposed to find the best column permutation of $B_{SSCV}$ given $B_s$ and $B_c$ are fixed. (Note that permuting the columns of $B_{SSCV}$ while keeping $B_c$ fixed is equivalent to permuting the columns of $B_c$ while keeping $B_{SSCV}$ fixed.) In [33], a joint optimization algorithm is provided to construct a joint protograph by taking the error floor and waterfall performance into account at the same time.

Some theoretical analysis tools for the DP-LDPC JSCC system have also been proposed. A joint protograph extrinsic information transfer (JPEXIT) algorithm is proposed in [25] for calculating the channel threshold of the joint protograph shown in Fig. 2. Lowering the channel threshold improves the waterfall performance. To evaluate the error-floor level, we also
need to calculate the source threshold of the double protographs. In [34], a source protograph extrinsic information transfer (SPEXIT) algorithm is proposed to calculate the source thresholds of DP-LDPC codes in the JSCC system when $B_{svcc}$ is a zero matrix. When $B_{svcc}$ is a non-zero matrix, an extended source protograph EXIT (ESP-EXIT) algorithm is proposed in [30] to calculate the source thresholds of DP-LDPC codes.

As can be observed, all the aforementioned JSCC systems with DP-LDPC codes have a structural constraint on $B_{svcc}$, which is shown in (3). In this paper, we remove the above constraint. In other words, $B_{svcc}$ can be made up of arbitrary non-zero and zero entries. As a result, we view the so-called “joint protomatrix $B_J$” as a generic protomatrix, and name the corresponding JSCC as “generic protograph-based JSCC (GP-JSCC)” [1]. We consider the performance of the GP-JSCC in both the waterfall region and high signal-to-noise-ratio (SNR) region. Since a lower channel threshold implies a better performance in the waterfall region, we present a protograph EXIT for JSCC (PEEXIT-JSCC) algorithm for calculating the channel threshold of a generic P-LDPC (GP-LDPC) code in the JSCC system. It is also known that the source threshold affects the error floor in the high-SNR region. We therefore propose a source generic protograph EXIT (SGP-EXIT) algorithm, which is more generic than the ESP-EXIT algorithm [30], for calculating the source threshold of a GP-LDPC code in the JSCC system. In addition to the source threshold, the linear minimum distance property of a generic protograph can affect the error floor. Given a GP-LDPC code, we will apply the asymptotic weight distribution (AWD) tool [17], [19], [35] to calculate its typical minimum distance ratio (TMDR). If the code has a TMDR, there is a high probability that it possesses the linear minimum distance property, i.e., the minimum distance increases linearly with the codeword length. We illustrate our P-JSCC system by using ‘accumulate-repeat-4-jagged-accumulate” (AR4JA) and “accumulate-repeat-3-and-accumulate” (AR3A) codes as examples and forming AR4JA-JSCC and AR3A-JSCC codes [1].

To design a P-JSCC with both good waterfall and error floor performance, we propose a joint optimization method which aims to achieve targeted source threshold and channel threshold. Using AR4JA-JSCC and AR3A-JSCC as benchmarks, we further search for generic protographs in the JSCC schemes using the proposed joint optimization method. Finally, we compare the theoretical (source and channel) thresholds and simulated error performance of the generic protographs found with those of AR4JA-JSCC and AR3A-JSCC. We also compare the results with those from optimized JSCC based on double protographs.
The main contributions of the paper are as follows.

1) We propose a JSCC scheme based on a generic protograph, namely the “protograph-based JSCC (P-JSCC)”.

2) We present a generalized algorithm, namely protograph EXIT for JSCC (PEXIT-JSCC) algorithm for calculating the channel threshold of a GP-LDPC code in the JSCC system.

3) We propose a source generic protograph EXIT (SGP-EXIT) algorithm for evaluating the source threshold of a generic protograph. The proposed technique can be used to calculate the source threshold of a double protograph, and the threshold value obtained is found to be the same as that calculated by the ESP-EXIT algorithm in [30]. The ESP-EXIT algorithm, on the other hand, cannot be utilized to calculate the source threshold of a generic protograph in the JSCC system. Thus the proposed SGP-EXIT algorithm is more generic.

4) We propose a first-source-then-channel-thresholds (FSTCT) joint optimization method based on the SGP-EXIT and PEXIT-JSCC algorithms. The objective is to obtain a generic protograph with a high source threshold and a low channel threshold at the same time. This method is implemented in two steps. The first step is to design a sub-protomatrix (related to the connections between the untransmitted VNs and the connected CNs) in a generic protomatrix to achieve a high source threshold based on the SGP-EXIT algorithm. The second step is to design the remaining part of the generic protomatrix to achieve a low channel threshold based on the PEXIT-JSCC algorithm. Finally, we need to use the asymptotic weight distribution (AWD) tool to analyze the linear minimum distance property of the generic protomatrix/protograph.

5) By using the proposed joint optimization method, we construct some generic protomatrices (P-JSCCs). Both theoretical analysis and error simulations reveal they can outperform AR3A-JSCC, AR4JA-JSCC and the optimized double protographs in [33].

6) Based on the simulation results, we have found that the waterfall performance becomes worse as the source entropy increases. Moreover, a generic protograph designed for a given source entropy does not guarantee its optimality for other given source entropies.

We organize this work as follows. Section II shows the details the proposed JSCC based on a generic protograph. In particular, we provide the structure of the P-JSCC, describe its encoding and decoding method, present a PEXIT-JSCC algorithm for analyzing its channel threshold, propose a SGP-EXIT algorithm for evaluating its source threshold, and propose a FSTCT joint optimization method for constructing good P-JSCCs. In Section III, we present some optimized generic protographs and their theoretical thresholds. We further compare their error rate performance with JSCCs based on optimized double protographs with relevant discussions. Finally, we give some concluding remarks and some future directions in Section IV.

II. PROPOSED JSCC SYSTEM WITH A GENERIC PROTOGRAPH

Referring to Fig. 3, we illustrate the system model of the proposed P-JSCC system using a generic protograph that has no structural constraints. As shown in the figure, VNs and CNs are depicted by circles and squares, respectively. Punctured VNs are denoted by gray circles. VNs corresponding to source symbols are denoted by red circles. This generic protograph can
alternatively be written as a generic protomatrix $B_{sp}$, i.e.,
\[
B_{sp} = \begin{pmatrix}
  e_{1,1} & \cdots & e_{1,n_r} & \cdots & e_{1,n_r+n_p} & \cdots & e_{1,n} \\
  e_{2,1} & \cdots & e_{2,n_r} & \cdots & e_{2,n_r+n_p} & \cdots & e_{2,n} \\
  \vdots & & & & \vdots & & \vdots \\
  e_{m,1} & \cdots & e_{m,n_r} & \cdots & e_{m,n_r+n_p} & \cdots & e_{m,n}
\end{pmatrix}
\]
(4)
where $e_{i,j}$ denotes the $(i,j)$-th element ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n_r, \ldots, n_r + n_p, \ldots, n$). Here, $m$ denotes the total number of CNs; $n$ denotes the total number of VNs; $n_r$ denotes the number of VNs corresponding to source symbols; $n_p$ denotes the number of punctured VNs. The overall symbol code rate of the P-JSCC system is given by
\[
R = n_r/(n - n_r - n_p).
\]
(5)

Example: We adopt the conventional AR3A code as our proposed P-JSCC for illustration. Referring to Fig. 4(a), our proposed P-JSCC, namely AR3A-JSCC, consists of
- $m = 3$ CNs ($C_1$ to $C_3$);
- $n = 5$ VNs ($V_1$ to $V_5$);
- $n_r = 2$ VNs corresponding to source symbols (indicated by red filled circles $V_1$ and $V_2$);
- $n_p = 1$ punctured VNs (indicated by the gray circle $V_3$);
- and an overall symbol code rate $R = 2/(5 - 2 - 1) = 1$.

The corresponding protomatrix of the AR3A-JSCC code is shown in Fig. 5(a) and its size is $3 \times 5$. When the conventional AR4JA code is used in our proposed P-JSCC, Fig. 4(b) and Fig. 5(b) show, respectively, the protograph of the AR4JA-JSCC and its corresponding protomatrix.
Fig. 4. The protograph of (a) an AR3A-JSCC; (b) an AR4JA-JSCC. Variable nodes and check nodes are, respectively, represented by circles and squares, with gray circles indicating punctured variable nodes and red circles signifying variable nodes corresponding to source symbols.

(a)

(b)

Fig. 5. The protomatrix of (a) an AR3A-JSCC code; (b) an AR4JA-JSCC code.

A. Encoder

To generate an overall parity-check matrix, $B_{\text{sp}}$ can be lifted by the progressive-edge-growth (PEG) algorithm [36] which can maximize the girth (i.e., smallest cycle) of the resultant Tanner graph. Assume a lifting factor of $z$, the size of the lifted parity-check matrix, denoted by $H_{\text{sp}}$, equals $mz \times nz$. We assume a binary independent and identically distributed (i.i.d.) Bernoulli source. Denoting the probability of “1” in the source sequence by $p_1$, the probability of “0” in the source equals $1 - p_1$. Therefore, the source entropy is given by

$$H_p = -p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1),$$  \hspace{1cm} (6)$$

where $p_1 \neq 0.5$. For example, when the source probability $p_1 = 0.04$, the source entropy $H_p = 0.242$ bit/symbol.

To begin with, we generate a source sequence with a size of $1 \times n_r z$ where the probability of “1” is $p_1$. The source is directly
Fig. 6. Decoding of the P-JSCC.

utilized as the input (referred to as $v_1$ to $v_{n_rz}$ VNs in Fig. 6) to the joint encoder and generate the codeword $(v_1, \ldots, v_{nz})$ based on the parity-check matrix $H_{sp}$. Finally, the source symbols ($v_1$ to $v_{n_rz}$) and the punctured bits (i.e., $v_{n_rz+1}$ to $v_{(n_r+n_p)z}$ in Fig. 6) are not transmitted while the code bits $v_{(n_r+n_p)z+1}$ to $v_{nz}$ are sent.

B. Decoder

Fig. 6 illustrates the decoding of the P-JSCC. We define the following.

- $I_{\text{max}}$ is the maximum number of decoding iterations.
- Binary phase-shift-keying modulation is used where bit “1” and “0” are mapped to “−1” and “+1”, respectively.
- The noise variance of the AWGN channel is given by $\sigma^2$.
- The received signal is given by $y = \pm 1 + \eta$ where $\eta \sim N(0, \sigma^2)$ denotes the AWGN.
- $M$ and $N$, respectively, represent the number of CNs and VNs in the parity-check matrix $H_{sp}$. They equal $mz$ and $nz$, respectively, i.e., $M = mz$ and $N = nz$.
- $LLR_{\text{init}}(j)$ represents the initial log-likelihood-ratio (LLR) of the $j$-th VN ($j = 1, 2, \ldots, N$).
- $\alpha_{ij}$ represents the LLR message sent from the $i$-th CN to the $j$-th VN.
- $\beta_{ij}$ denotes the LLR message sent from the $j$-th VN to the $i$-th CN.
- $L_{\text{APP},j}$ denotes the posterior LLR of the $j$-th VN.
- $\mathcal{M}(j)$ and $\mathcal{N}(i)$ represent the set of all CNs connected to the $j$-th VN and the set of all VNs connected to the $i$-th CN, respectively.
- $\mathcal{M}(j) \setminus i$ denotes the set of all CNs connected to the $j$-th VN excluding the $i$-th CN; and $\mathcal{N}(i) \setminus j$ denotes the set of all VNs connected to the $i$-th CN excluding the $j$-th VN.

The decoding process is described as follows. It is similar to that of decoding a traditional LDPC code except the initialization process.

Initialization:

- Set $\alpha_{ij} = \beta_{ij} = 0 \ \forall i, j$. 
As shown in Fig. 6, $LLR_{init}$ is given by

$$
LLR_{init}(j) = \begin{cases} 
LLR_s(j), & j = 1, 2, ..., n_r z \\
0, & j = n_r z + 1, n_r z + 2, \\
& ..., (n_r + n_p) z \\
LLR_{ch}(j), & j = (n_r + n_p) z + 1, \\
& (n_r + n_p) z + 2, ..., n z
\end{cases}
$$

(7)

where $LLR_s(j) = \ln((1 - p_1)/p_1)$ represents the initial LLR information of the source symbols; and $LLR_{ch}(j) = 2y_j/\sigma^2$ represents the initial LLR information from the channel.

Iterative process:

- Start: Set the iteration counter $r = 1$.
- Step 1) Updating LLRs from the VNs to the CNs:

$$
\beta_{ij} = LLR_{init}(j) + \sum_{i' \in \mathcal{M}(j) \setminus i} \alpha_{i'j}, \forall i, j
$$

(8)

- Step 2) Update LLRs from the CNs to the VNs

$$
\alpha_{ij} = 2 \tanh^{-1} \left( \prod_{j' \in \mathcal{N}(i) \setminus j} \tanh(\beta_{ij'}/2) \right), \forall i, j.
$$

(9)

- Step 3) Calculate the posterior LLRs by

$$
L_{APP,j} = LLR_{init}(j) + \sum_{i \in \mathcal{M}(j)} \alpha_{ij}, \forall j.
$$

(10)

- Step 4) Estimate $v_j$ by

$$
\hat{v}_j = 0 \text{ if } L_{APP,j} \geq 0, \text{ otherwise } \hat{v}_j = 1, \forall j.
$$

(11)

If $\hat{v} \cdot H_{sp}^T = 0$ is satisfied where $\hat{v} = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_{nz}\}$, or $r = I_{max}$, stop the iteration and output $\hat{v}$ as the joint source-channel codeword; otherwise increase the iteration counter $r$ by 1, and repeat Step 1) to Step 4).

C. PEXIT-JSCC algorithm

Denoting $E_s$ as the average transmitted energy per source symbol and $N_0$ as the noise power spectral density, the AWGN noise variance $\sigma^2$ and $E_s/N_0$ is related by

$$
\frac{E_s}{N_0} = 10 \log_{10} \frac{1}{2\sigma^2 R} \text{ dB}.
$$

(12)

We further define the following.

- $I_{A_{VC}}(i,j)$ denotes the a priori mutual information (AMI) from the $j$-th VN to the $i$-th CN (see Fig. 7).
- $I_{A_{CV}}(i,j)$ denotes the AMI from the $i$-th CN to the $j$-th VN (see Fig. 7).
- $I_{E_{CV}}(i,j)$ denotes the extrinsic mutual information (EMI) from the $i$-th CN to the $j$-th VN.
TABLE I
THE PARAMETER SETTINGS OF THE PEXIT-JSCC AND SGP-EXIT ALGORITHMS

| Algorithm      | $\Delta$ (dB) | $t_{\text{max}}$ | $\delta$ |
|----------------|---------------|------------------|----------|
| PEXIT-JSCC     | 0.001         | 200              | $10^{-6}$|
| SSP-JSCC       | $\bar{p}_1$   | $t_{\text{max}}$ | $\theta$ |

- $I_{E,VC}(i,j)$ denotes the EMI from the $j$-th VN to $i$-th CN.
- $I_{\text{APP}}(j)$ denotes the mutual information (MI) between the a posterior log-likelihood-ratio (APP-LLR) of the $j$-th VN and its corresponding symbol.
- $(E_s/N_0)^*$ denotes the channel threshold.
- The mutual information (MI) between the VN $V_s$ corresponding to the source symbol and its corresponding $LLR_s$ is defined by [16]

$$J_{\text{BSC}}(\mu, p_1) = p_1 \times I(V_s; \omega^{(p_1)}) + (1 - p_1) \times I(V_s; \omega^{(1-p_1)})$$

where $\mu$ signifies the average LLR value of the VN $V_s$, $\omega^{(p_1)} \sim N(\mu - LLR_s, 2\mu)$, $\omega^{(1-p_1)} \sim N(\mu + LLR_s, 2\mu)$. $I(a; b)$ denotes the MI between $a$ and $b$.

- An indicator function $\psi(x)$ is defined as

$$\psi(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise.} \end{cases}$$

In Algorithm 1, we present our generalized algorithm, namely protograph EXIT for JSCC algorithm (PEXIT-JSCC algorithm), for analyzing the channel threshold of the proposed P-JSCC. Note that our generalized algorithm is similar to those used in analyzing DP-LDPC codes in the literature, e.g., those in [25]. However, our algorithm is generalized in the sense that the protograph does not need to satisfy any specific constraint, i.e., the requirement given by (3) does not exist. The maximum number of iterations $t_{\text{max}}$, step size $\Delta$, and tolerance value $\delta$ used in Algorithm 1 are listed in Table I. Using the PEXIT-JSCC algorithm, the channel thresholds of AR3A-JSCC and AR4JA-JSCC under $p_1 = 0.04$ are found and listed in Table II.

TABLE II
THE CHANNEL THRESHOLDS AND TMDR VALUES OF DIFFERENT PROTOMATRICES GIVEN $p_1 = 0.04$. THE SHANNON LIMIT IS $-7.00$ dB

| Protograph       | $B_{\text{sp, opt1}}$ | $B_{\text{sp, opt2}}$ | $B_{\text{sp, opt3}}$ | $B_1$ | $B_2$ |
|------------------|------------------------|------------------------|------------------------|-------|-------|
| $(E_s/N_0)^*$    | $-5.918$               | $-5.767$               | $-6.102$               | $-5.810$ | $-5.782$ |
| TMDR value       | None                   | 0.017                  | None                   | None  | 0.003 |

D. SGP-EXIT chart

The algorithms in [30], [34] are not suitable for calculating the source thresholds of the P-JSCC system in Fig. 3. In [34], the SPEXIT algorithm is applied to calculate the source threshold of a double protograph with no connections between VNs in the source P-LDPC code and CNs in the channel P-LDPC code. In [30], the ESP-EXIT algorithm does not consider the case with punctured variable nodes. Also, there is a constraint on the structure of $B_{\text{src}}$, i.e., each non-zero column only allows
Fig. 7. The protograph of the PEXIT-JSCC algorithm.

Fig. 8. (a) Passing of AMI in the protograph representing a P-JSCC. (b) The protomatrix of a P-JSCC is split into two sub-protomatrices. $B_p$ contains the untransmitted VNs and $B_t$ contains the transmitted VNs.

First, we refer to Fig. 8(a) and define the following.

- $\mathcal{V}_s = \{V_1, V_2, ..., V_{n_r}\}$ denotes the set of VNs corresponding to source symbols.
- $\mathcal{V}_p = \{V_1, V_2, ..., V_{n_r+n_p}\}$ denotes the set of untransmitted VNs of the generic protograph.
- $\mathcal{V}_t = \{V_{n_r+n_p+1}, V_{n_r+n_p+2}, ..., V_n\}$ denotes the set of transmitted VNs.
- $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ denotes the set of CNs.
- $I_{A \cdot CV}(i,j)$ denotes the AMI from the $j$-th untransmitted VN $\in \mathcal{V}_p$ to the $i$-th CN.
- $I_{A \cdot VtC}(i,j)$ denotes the AMI from the $j$-th transmitted VN $\in \mathcal{V}_t$ to the $i$-th CN.
- $I_{A \cdot CVp}(i,j)$ denotes the AMI from the $i$-th CN $\in \mathcal{C}$ to the $j$-th untransmitted VN $\in \mathcal{V}_p$.

A weight of 2. Here, we propose a more generic algorithm for calculating the source threshold of a JSCC system, called the source generic protograph EXIT (SGP-EXIT) algorithm.

First, we refer to Fig. 8(a) and define the following.

- $\mathcal{V}_s = \{V_1, V_2, ..., V_{n_r}\}$ denotes the set of VNs corresponding to source symbols.
- $\mathcal{V}_p = \{V_1, V_2, ..., V_{n_r+n_p}\}$ denotes the set of untransmitted VNs of the generic protograph.
- $\mathcal{V}_t = \{V_{n_r+n_p+1}, V_{n_r+n_p+2}, ..., V_n\}$ denotes the set of transmitted VNs.
- $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ denotes the set of CNs.
- $I_{A \cdot CV}(i,j)$ denotes the AMI from the $j$-th untransmitted VN $\in \mathcal{V}_p$ to the $i$-th CN.
- $I_{A \cdot VtC}(i,j)$ denotes the AMI from the $j$-th transmitted VN $\in \mathcal{V}_t$ to the $i$-th CN.
- $I_{A \cdot CVp}(i,j)$ denotes the AMI from the $i$-th CN $\in \mathcal{C}$ to the $j$-th untransmitted VN $\in \mathcal{V}_p$.
Algorithm 1: The PEXIT-JSCC algorithm. The definitions $J(\cdot)$ and $J^{-1}(\cdot)$ are given in [14], [37].

Set the maximum number of iterations $t_{\text{max}}$, step size $\Delta$, and tolerance value $\delta$.
Set a sufficiently small $E_s/N_0$.
for a given $E_s/N_0$ do
  Set $t = 1$, $I_{E_{VC}}(i,j) = I_{A_{VC}}(i,j) = I_{E_{CV}}(i,j) = I_{A_{CV}}(i,j) = 0$ and $I_{APP}(j) = 0$, $\forall i, j$
  while $\sum_{j=1}^{n} (1 - I_{APP}(j)) > \delta$ and $t \leq t_{\text{max}}$
    for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ do
      $I_{E_{VC}}(i,j) = \begin{cases} 
      I_{BSC}(e_{i,j}) \left( \sum_{i' \neq i} e_{i',j}[J^{-1}(I_{A_{VC}}(i',j))]^2 + (e_{i,j} - 1)[J^{-1}(I_{A_{VC}}(i,j))]^2, p_1 \right), & j = 1, 2, ..., r \\
      I_{BSC}(e_{i,j}) \left( \sqrt{\sum_{i' \neq i} e_{i',j}[J^{-1}(I_{A_{VC}}(i',j))]^2 + (e_{i,j} - 1)[J^{-1}(I_{A_{VC}}(i,j))]^2 + \sigma^2_{ch}(j)} \right), & j = n_r + 1, n_r + 2, ..., n, 
      \end{cases}$
      where $\sigma^2_{ch}(j) = \begin{cases} 
      0 & j = n_r + 1, n_r + 2, ..., n_r + p, \\
      4/\sigma^2 & j = n_r + p + 1, n_r + p + 2, ..., n, 
      \end{cases}$
    end for
  Set $I_{A_{VC}}(i,j) = I_{E_{VC}}(i,j), \forall i, j$
  for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ do
    $I_{E_{CV}}(i,j) = \psi(e_{i,j}) \left( 1 - J \left( \sum_{j \neq j'} e_{i,j'} [J^{-1}(1 - I_{A_{VC}}(i,j'))]^2 + (e_{i,j} - 1)[J^{-1}(1 - I_{A_{VC}}(i,j))]^2 \right) \right)$
  end for
  Set $I_{A_{CV}}(i,j) = I_{E_{CV}}(i,j), \forall i, j$
  for $j = 1, 2, ..., n$ do
    $I_{APP}(j) = \begin{cases} 
    J_{BSC} \left( \sum_{i} e_{i,j} [J^{-1}(I_{A_{VC}}(i,j))]^2, p_1 \right), & j = 1, 2, ..., r \\
    J \left( \sum_{i} e_{i,j} [J^{-1}(I_{A_{VC}}(i,j))]^2 + \sigma^2_{ch}(j) \right), & j = n_r + 1, n_r + 2, ..., n, 
    \end{cases}$
    where $\sigma^2_{ch}(j) = \begin{cases} 
    0 & j = n_r + 1, n_r + 2, ..., n_r + p, \\
    4/\sigma^2 & j = n_r + p + 1, n_r + p + 2, ..., n, 
    \end{cases}$
  end for
  Set $t = t + 1$
  end while
  if $\sum_{j=1}^{n} (1 - I_{APP}(j)) < \delta$ then
    $(E_s/N_0)^* = E_s/N_0$
    break
  else
    $E_s/N_0 = E_s/N_0 + \Delta$
  end if
end for

- $I_{E_{CV}}(i,j)$ denotes the EMI from the $i$-th CN $\in C$ to the $j$-th untransmitted VN $\in V_p$.
- $I_{E_{VC}}(i,j)$ denotes the EMI from the $j$-th untransmitted VN $\in V_p$ to the $i$-th CN.
- $I_{APP}(j)$ denotes the MI between the APP-LLR of the $j$-th untransmitted VN $\in V_p$ and its corresponding symbol.

Next, we can derive the SGP-EXIT curves (i.e., inner-code curve and outer-code curve) of a given P-JSCC using Algorithm 2.

Note that our objective is to investigate the effect of the source probability $p_1$ on the source symbol error performance at the high SNR region, i.e., when the noise power is very small. Thus we assume that the average AMI from the transmitted VNs to CNs is equal to 1, i.e., $I_{A_{VC}}(i,j^*) = 1$ and $J^{-1}(1 - I_{A_{VC}}(i,j^*)) = 0$ for $i = 1, 2, ..., m$, $j^* = n_r + n_p + 1, n_r + n_p + 2, ..., n$, and apply it in (17) in Algorithm 2. As can be observed in Algorithm 2 and Fig. 8(a), we only need to consider the
Algorithm 2 Inner-code curve and outer-code curve.

INNER-CODE CURVE
Given $I_{A_{CV_p}(i, j)} \in [0, 1], i = 1, 2, ..., m, j = 1, 2, ..., n_r + n_p$
for $i = 1, 2, ..., m, \ j = 1, 2, ..., n_r$
do
\[
I_{E_{CV_p}(i, j)} = \psi(e_{i,j})J_{BSC}\left(\sum_{i' \neq i} e_{i',j}[J^{-1}(I_{A_{CV_p}(i', j)})]^2 + (e_{i,j} - 1)[J^{-1}(I_{A_{CV_p}(i, j)})]^2, p_1\right)
\]
end for
for $i = 1, 2, ..., m, \ j = n_r + 1, n_r + 2, ..., n_r + n_p$
do
\[
I_{E_{CV_p}(i, j)} = \psi(e_{i,j})J\left(\sum_{i' \neq i} e_{i',j}[J^{-1}(I_{A_{CV_p}(i', j)})]^2 + (e_{i,j} - 1)[J^{-1}(I_{A_{CV_p}(i, j)})]^2\right)
\]
end for

OUTER-CODE CURVE
Given $I_{A_{CV_p}C(i, j)} \in [0, 1]$ and $I_{A_{CV_p}C(i, j^*)} = 1, i = 1, 2, ..., m, j = 1, 2, ..., n_r + n_p, j^* = n_r + n_p + 1, ..., n$
for $i = 1, 2, ..., m, \ j = 1, 2, ..., n_r + n_p$
do
\[
I_{E_{CV_p}(i, j)} = \psi(e_{i,j})\left(1 - J\left(\sum_{j' \neq j} e_{i,j'}[J^{-1}(1 - I_{A_{CV_p}C(i, j^*)})]^2 + \sum_{j^* = n_r + n_p + 1}^{n} e_{i,j^*}[J^{-1}(1 - I_{A_{CV_p}C(i, j^*)})]^2\right) + (e_{i,j} - 1)[J^{-1}(1 - I_{A_{CV_p}C(i, j)})]^2\right)
\]
end for

untransmitted VNs $\mathcal{V}_p$ and their connected CNs in deriving the SGP-EXIT curves. In other words, we only need to consider the sub-protomatrix

\[
\mathbf{B}_p = \begin{pmatrix}
  e_{1,1} & ... & e_{1,n_r} & ... & e_{1,n_r+n_p} \\
  e_{2,1} & ... & e_{2,n_r} & ... & e_{2,n_r+n_p} \\
  \vdots & & \ddots & & \vdots \\
  e_{m,1} & ... & e_{m,n_r} & ... & e_{m,n_r+n_p}
\end{pmatrix}
\]

(18)

which is also shown in Fig. 8(b).

Considering the protomatrix

\[
\mathbf{B}_{sp_{opt}} = \begin{pmatrix}
  1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 2 & 0 & 1 \\
  3 & 2 & 2 & 0 & 0
\end{pmatrix}
\]

(19)

and assuming $n_r = 2$ and $n_p = 1$, we form the sub-protomatrix

\[
\mathbf{B}_p = \begin{pmatrix}
  1 & 1 & 1 \\
  0 & 0 & 2 \\
  3 & 2 & 2
\end{pmatrix}
\]

(20)

and plot the corresponding SGP-EXIT curves in Fig. 9. We can see that the gap between the inner-coder curve and the outer-
TABLE III
THE SOURCE THRESHOLDS OF DIFFERENT PROTOMATRICES.

| $p_{1 \text{th}}$ | AR3A-JSCC | AR4JA-JSCC | $B_{sp, opt1}$ | $B_{sp, opt2}$ | $B_{sp, opt3}$ | $B_{sp, opt1}$ | $B_{sp, opt2}$ |
|-----------------|-----------|-------------|----------------|---------------|---------------|---------------|---------------|
| 0.228           | 0.212     | 0.25        | 0.275          | 0.242         |               | 0.290         | 0.276         |

Fig. 9. The SGP-EXIT chart of the sub-protomatrix $B_p$ in (20) under different $p_1$ values. Estimated source threshold is 0.25.

Fig. 10. The SGP-EXIT chart of the sub-protomatrix $B_p$ in (21) under different $p_1$ values. Estimated source threshold is 0.12.

code curve becomes smaller as $p_1$ increases. When $p_1$ increases beyond a certain value, these two curves will cross each other. The decoding can be performed successfully only when the inner-coder curve is above the outer-coder curve. The larger the gap between the outer-code curve and the inner-code curve, the faster the decoder converges. The maximum $p_1$ value that makes these two curves closest without crossing is the source threshold $p_{1 \text{th}}$. We can see from Fig. 10 that when $p_1 = 0.25$, the two curves are closest. Then the source threshold of the sub-protomatrix $B_p$ in (20) is estimated to be $p_{1 \text{th}} = 0.25$. To achieve a more precise source threshold, we propose the SGP-EXIT algorithm shown in Algorithm 3. The maximum number of iterations $l_{\text{max}}$, step size $\tilde{p}_1$ and tolerance value $\theta$ used in the algorithm are listed in Table I. Using these parameters, we obtain also $p_{1 \text{th}} = 0.25$ for the sub-protomatrix $B_p$ in (20). Similarly, we apply Algorithm 3 to obtain the source thresholds of the AR3A-JSCC code and AR4JA-JSCC code in Fig. 5. The source thresholds of the above codes are listed in Table III.

We further consider a DP-LDPC code given by [30, Eq.(26)] where $m = 8, n = 16, n_r = 8, n_p = 0$. Since there is no
Algorithm 3 SGP-EXIT algorithm.

Set the maximum number of iterations \( l_{\text{max}} \), step size \( \hat{p}_1 \) and tolerance value \( \theta \).

Set a sufficiently large \( p_1 < 0.5 \).

\[ \textbf{for} \text{ a given } p_1 < 0.5 \text{ do} \]

\[ \text{Set } l = 1, I_{E_{V_p}C}(i, j) = I_{A_{CV_p}}(i, j) = I_{E_{CV_p}}(i, j) = I_{A_{VCV_p}}(i, j) = 0, I_{APP_p}(j) = 0, i = 1, 2, ..., m, j = 1, 2, ..., n_r + n_p. \]

\[ \text{while } \sum_{j=1}^{n_r+n_p} (1 - I_{APP_p}(j)) > \theta \text{ and } l \leq l_{\text{max}} \]

\[ \text{Compute } I_{E_{V_p}C}(i, j) \text{ using } (15) \text{ and } (16) \forall i, j. \]

\[ \text{Set } I_{A_{CV_p}}(i, j) = I_{E_{CV_p}}(i, j) \forall i, j. \]

\[ \text{Compute } I_{E_{CV_p}}(i, j) \text{ using } (17) \forall i, j. \]

\[ \text{Calculate } I_{APP_p}(j) = \begin{cases} J_{\text{BSC}} \left( \sum_{i} e_{i,j} [J^{-1}(I_{A_{CV_p}}(i, j))]^2, p_1 \right), & j = 1, 2, ..., n_r \\ J \left( \sum_{i} e_{i,j} [J^{-1}(I_{A_{CV_p}}(i, j))]^2 \right), & j = n_r + 1, n_r + 2, ..., n_r + n_p \end{cases}. \]

\[ \text{Set } l = l + 1. \]

\[ \textbf{end while} \]

\[ \text{if } \sum_{j=1}^{n_r+n_p} (1 - I_{APP_p}(j)) < \theta \text{ then} \]

\[ \hat{p}_{1,\text{th}} = p_1 \]

\[ \text{break} \]

\[ \text{else} \]

\[ p_1 = p_1 - \hat{p}_1 \]

\[ \text{end if} \]

\[ \textbf{end for} \]

punctured VN, the sub-protomatrix needs to be considered when deriving the source threshold is simply the sub-protomatrix

\[
\begin{pmatrix}
B_s \\
B_{\text{succ}}
\end{pmatrix}
\].

The sub-protomatrix being studied is given by [30, Eq.(26)]

\[
B_p = \begin{pmatrix}
B_s \\
B_{\text{succ}}
\end{pmatrix}
= \begin{pmatrix}
3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\
2 & 3 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\
3 & 0 & 1 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\). \quad (21)

Fig. 10 plots the SGP-EXIT chart using Algorithm 2 under different \( p_1 \) values. The estimated source threshold is 0.12. Using Algorithm 3, we obtain a more precise threshold, i.e., \( p_{1,\text{th}} = 0.1156 \). This result is the same as that calculated by the ESP-EXIT algorithm in [30].
The first-source-then-channel-thresholds (FSTCT) joint optimization method

Designing $B_p$ to obtain a high source threshold → Designing $B_t$ → Having a low channel threshold?

No → Get $B_s$ with a high source threshold and a low channel threshold

Yes

Fig. 11. The flow of the optimization algorithm.

E. Optimization method

We aim to design P-JSCCs with good error performance in both waterfall and high-SNR regions. Our objective is therefore to construct generic protographs with a high $p_{1\text{-}th}$ based on the SGP-EXIT algorithm and a low channel decoding threshold $(E_s/N_0)^*$ based on the PEXIT-JSCC algorithm. Moreover, we use the AWD tool to analyze the linear minimum distance properties of the constructed generic protographs.

We propose a first-source-then-channel-thresholds (FSTCT) joint optimization method to design generic protographs. The method is implemented in two steps. The first one is to optimize the connections between the untransmitted VNs and the CNs, i.e., $B_p$ shown in Fig. 8(b), to obtain a high source threshold (based on the SGP-EXIT algorithm). The second one is to design the remaining part of $B_s$, i.e., $B_t$ shown in Fig. 8(b), for a given $p_1$ to achieve a low channel threshold (based on the PEXIT-JSCC algorithm). The search scope is significantly reduced because $B_s$ is divided into two parts and are to be optimized one after another. The flow of this method is shown in Fig. 11. By using this optimization method, we can guarantee that the obtained generic protographs have good source and channel thresholds.

As an illustration, we assume a symbol code rate of $R = 1$ and $n_p = 1$ punctured VN. In order to reduce the search space and based on some existing code design rules, we further assume the following conditions.

(i) The maximum value of each entry in $B_s$ is $e_{max} = 3$.

(ii) The minimum size of the generic protomatrix is $3 \times 5$. For $k \in Z^+$, the number of VNs corresponding to source symbols is $n_r = k+1$. According to (5), the total number of VNs is $n = 2k+3$ and the total number of CNs is $m = n - n_r = k+2$.

(iii) The degree of each CN in $B_s$ is at least 3.

(iv) The maximum number of degree-2 VNs in $B_s$ is $n - m - n_p = k$.

(v) The maximum number of degree-1 VNs in $B_s$ is 1 and $n_p$ VNs with the highest degrees in $B_s$ are punctured.

(vi) The maximum degree of each VN in $B_s$ is $D_{max}$. (In this paper, we set $D_{max} = 8$ when $k = 1$; and $D_{max} = 11$ when $k = 2$.)

(vii) $p_{1-th} > \bar{p}_1$ where $\bar{p}_1$ is a preset source threshold benchmark.

(viii) $(E_s/N_0)^* < (E_s/N_0)'$ where $(E_s/N_0)'$ is a preset channel threshold benchmark.

When the search scope is small, a brute-force approach can be used to search for good generic protographs. For example when $k = 1$, the size of $B_p$ is $3 \times 3$ and the size of $B_t$ is $3 \times 2$. In this case, a brute-force search is an option. When the protomatrix is large, the brute-force approach will be very time consuming and even not feasible. Then a differential evolution
(DE) strategy can be used. The following is a more in-depth explanation of these two search methods.

1) Brute-force search: To begin with, we do an exhaustive search for all entries in $B_p$ that fulfill Conditions (i) to (vii) in accordance with the SGP-EXIT algorithm. Among the $B_p$’s having high source thresholds, we exhaustively search through all entries in $B_t$ to make sure $B_{sp}$ satisfies the conditions (i) to (vi) and (viii) according to the PEXIT-JSCC algorithm.

2) DE approach: We define
- $G$ as the number of generations;
- $S$ as the number of candidate matrices;
- $p_c$ as the crossover probability.

Step a) Initialization: For the $0$-th generation, randomly generate candidate matrices $B_{bp,1}^0, B_{bp,2}^0, ..., B_{bp,S}^0$, the size of which equals that of $B_p$ or $B_t$, depending on what matrix is to be constructed. Moreover, these matrices should satisfy Conditions (i) to (vi). Set $g = 0$.

Step b) Mutation: Generate $S$ mutation matrices from Generation $g$ using

$$B_{M,s}^g = \Phi(B_{bp,r1}^g + 0.5 \ast (B_{bp,r2}^g - B_{bp,r3}^g)), \quad s = 1, 2, ..., S$$  \hfill (22)

where $r1, r2$ and $r3$ are distinct random positive integers in the range $[1, S]$, and $\Phi(\cdot)$ is a function that converts each entry in a matrix to an integer closest to its absolute value.

Step c) Crossover: Create the matrix $B_{cr,s}^g (s = 1, 2, ..., S)$, in which the $(i,j)$-th element is set as the $(i,j)$-th element in $B_{M,s}^g$ with probability $p_c$, or as the $(i,j)$-th element in $B_{bp,s}^g$ with probability $1 - p_c$.

Step d) Selection: Generate the $(g + 1)$-th generation candidate matrices $B_{bp,s}^{g+1} (s = 1, 2, ..., S)$. Two cases are to be discussed.

a) Case One: When the goal is to find a $B_p$ with a high source threshold, $B_{bp,s}^{g+1} (s = 1, 2, ..., S)$ is generated as

$$B_{bp,s}^{g+1} = \begin{cases} B_{cr,s}^g, & \text{if } \Theta(B_{cr,s}^g) \Psi(B_{cr,s}^g) > \Theta(B_{bp,s}^g) \\ B_{bp,s}^g, & \text{otherwise} \end{cases}$$  \hfill (23)

where $\Theta(B_p)$ returns the source threshold $p_{1_{th}}$ of $B_p$ according to the SGP-EXIT algorithm; and

$$\Psi(B_{cr,s}^g) = \begin{cases} 1, & \text{if Conditions (i) to (vi) are} \\ 0, & \text{otherwise.} \end{cases}$$  \hfill (24)

b) Case Two: When the goal is to find a $B_t$ such that together with a pre-determined $B_p$ the generic protomatrix $B_{sp} = (B_p B_t)$ can achieve a low channel threshold $(E_s/N_0)^*$, $B_{bp,s}^{g+1} (s = 1, 2, ..., S)$ is generated as

$$B_{bp,s}^{g+1} = \begin{cases} B_{cr,s}^g, & \text{if } \Upsilon\left((B_p B_{cr,s}^g) \Psi(B_{cr,s}^g) \right) < \Upsilon\left((B_p B_{bp,s}^g) \right) \\ B_{bp,s}^g, & \text{otherwise} \end{cases}$$  \hfill (25)

where $\Upsilon(B_{sp})$ returns the channel threshold value of $B_{sp}$ according to the PEXIT-JSCC algorithm.
Step e) Termination: Set \( g = g + 1 \). Stop if \( g = G \); otherwise go to Step b).

### III. RESULTS AND DISCUSSIONS

In this section, we present some optimized generic protomatrices, and their theoretical thresholds and error rate simulation results. As discussed in the previous section, the channel thresholds and source thresholds of the AR3A-JSCC and AR4JA-JSCC codes (see Fig. 5) have been derived by the PEXIT-JSCC algorithm under \( p_1 = 0.04 \) and the SGP-EXIT algorithm, respectively. Referring to Tables II and III, the channel thresholds of AR3A-JSCC and AR4JA-JSCC codes are, respectively, \(-5.918\) dB and \(-5.767\) dB; and the source thresholds are, respectively, 0.228 and 0.212. Based on the results of the AR3A-JSCC and AR4JA-JSCC codes, we construct two sets of benchmarks, i.e.,

- \( B_1 : (\bar{p}_1 = 0.228, (E_s/N_0)^\prime = -5.918\) dB); and
- \( B_2 : (\bar{p}_1 = 0.212, (E_s/N_0)^\prime = -5.767\) dB).

#### A. Thresholds of Protomatrices Found

1) **Generic protomatrix of size 3 × 5 and \( p_1 = 0.04 \):** We assume a generic protomatrix of size 3 × 5 and use the proposed FSTCT joint optimization method under \( p_1 = 0.04 \) to search for protomatrices with better thresholds than the two sets of benchmarks separately. Since the protomatrix size is relatively small, we apply the brute force searching approach. With \( B_1 \) as the benchmark, the proposed FSTCT method finds \( B_{sp\_opt1} \) shown in (19) which is repeated below

\[
B_{sp\_opt1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 0 & 1 \\
3 & 2 & 2 & 0 & 0
\end{pmatrix}
\]  

(26)

and with \( B_2 \) as the benchmark, the proposed FSTCT method generates

\[
B_{sp\_opt2} = \begin{pmatrix}
3 & 2 & 2 & 0 & 0 \\
0 & 0 & 3 & 1 & 1 \\
0 & 1 & 1 & 0 & 1
\end{pmatrix}
\]  

(27)

and

\[
B_{sp\_opt3} = \begin{pmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 2 & 3 & 2 & 0 \\
0 & 1 & 1 & 0 & 3
\end{pmatrix}
\]  

(28)

The source and channel thresholds of these three codes and AR3A-JSCC and AR4JA-JSCC codes are shown in Table III and Table II, respectively. We can see that these three generic protographs have higher source thresholds than AR3A-JSCC and AR4JA-JSCC codes. (Note that Fig. 9 plots the SGP-EXIT chart of \( B_{sp\_opt1} \) under different \( p_1 \) values.) In addition, \( B_{sp\_opt1} \) has the lowest channel threshold \((E_s/N_0)^\prime = -6.1021\) dB) among these five codes, which is about 0.18 dB lower than the decoding threshold of the AR3A-JSCC code. \( B_{sp\_opt2} \) and \( B_{sp\_opt3} \) have lower channel thresholds compared with the AR4JA-JSCC code.
Using the AWD tool, we plot the asymptotic weight distribution curves of AR3A-JSCC code, AR4JA-JSCC code, \( B_{sp\ opt1} \), \( B_{sp\ opt2} \), and \( B_{sp\ opt3} \), in Fig. 12. As seen in the figure, AR4JA-JSCC code and \( B_{sp\ opt3} \) have TMDRs while AR3A-JSCC code, \( B_{sp\ opt1} \) and \( B_{sp\ opt2} \) do not have TMDRs. The TMDR values of AR4JA-JSCC code and \( B_{sp\ opt3} \) are around 0.017 and 0.003, respectively. A larger TMDR value implies a lower error-floor. When a P-LDPC code does not possess a TMDR value, its error-floor performance is hard to be predicted and can only be found out by simulations.

2) Generic protomatrix of size \( 4 \times 7 \) and \( p_1 = 0.04 \): Next, we increase the generic protomatrix size to \( 4 \times 7 \) and use the FSTCT method to search for protomatrices under Benchmarks \( B_1 \) and \( B_2 \) separately. Since the protomatrix size is not small, we apply the DE searching approach. The parameters \( G, S \) and \( p_c \) in the DE algorithm are set to 800, 800 and 0.88, respectively. With Benchmarks \( B_1 \) and \( B_2 \), the FSTCT method finds protomatrices shown in (29) and (30), respectively.

\[
B_{sp\ opt1}^{47} = \begin{pmatrix}
1 & 0 & 0 & 1 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 2 & 1 & 3 & 0 & 2 & 0 \\
1 & 0 & 0 & 2 & 0 & 0 & 0
\end{pmatrix}
\]

(29)

\[
B_{sp\ opt2}^{47} = \begin{pmatrix}
0 & 0 & 0 & 2 & 2 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 \\
1 & 2 & 0 & 3 & 1 & 1 & 0 \\
0 & 1 & 3 & 1 & 1 & 2 & 0
\end{pmatrix}
\]

(30)

The source and channel thresholds of \( B_{sp\ opt1}^{47} \) and \( B_{sp\ opt2}^{47} \) are shown in Table III and Table II, respectively. Both source thresholds are higher than those of the AR3A-JSCC and AR4JA-JSCC codes. Moreover, \( B_{sp\ opt1}^{47} \) has a lower channel threshold than AR3A-JSCC while \( B_{sp\ opt2}^{47} \) has a lower channel threshold than AR4JA-JSCC. Using the AWD tool, we further find that \( B_{sp\ opt1}^{47} \) has no TMDR and \( B_{sp\ opt2}^{47} \) has a TMDR of around 0.007.
3) Generic protomatrix of size $3 \times 5$ and $p_1 = 0.16$: We design a generic protograph with a size of $3 \times 5$ for a relatively large $p_1$, i.e. $p_1 = 0.16$. We use the source threshold and channel threshold of AR3A-JSCC at $p_1 = 0.16$ as the benchmark. Referring to Tables III and IV, we set Benchmark $B_3$: $(\hat{p}_1 = 0.228, (E_s/N_0)^\ast = -0.094)$ dB. By using the FSTCT joint optimization method and the brute force searching approach, we obtain

$$\mathbf{B}_{sp\_opt} = \begin{pmatrix}
0 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 2 & 2 & 0
\end{pmatrix}.$$  \hfill (31)

with a source threshold of 0.324 and a channel threshold of $-0.185$ dB at $p_1 = 0.16$. However, $\mathbf{B}_{sp\_opt4}$ has no TMDR.

4) Comparison of channel thresholds at different $p_1$ values: For the six $3 \times 5$ protomatrices discussed above, namely AR3A-JSCC, AR4JA-JSCC, and $\mathbf{B}_{sp\_opt1}$ to $\mathbf{B}_{sp\_opt4}$, we derive their channel thresholds at different $p_1$ values using the PEXIT-JSCC algorithm and list them in Table IV. We can see that among all codes, $\mathbf{B}_{sp\_opt1}$ has the lowest channel threshold ($-6.102$ dB) when $p_1 = 0.04$; AR3A-JSCC has the lowest channel thresholds ($-3.414$ dB and $-1.680$ dB, respectively) when $p_1 = 0.08$ and 0.12; $\mathbf{B}_{sp\_opt4}$ has the lowest channel thresholds ($-0.185$ dB and $0.816$ dB, respectively) when $p_1 = 0.16$ and 0.20. As shown in Table IV, these lowest channel thresholds are within 1 dB from the Shannon limits. Moreover, the six $3 \times 5$ protomatrices have different decoding-threshold rankings at different $p_1$ values. The results indicate that a generic protograph optimized at a given $p_1$ does not guarantee the best performance at other $p_1$ values.

5) Comparison of thresholds with those of DP-JSCCs: Table V shows the source thresholds and channel thresholds of the optimized double protographs in [33] (denoted by $\mathbf{B}_{j_1}^{sp\_opt-1}$ to $\mathbf{B}_{j_4}^{sp\_opt-4}$) when $p_1 = 0.04$. Comparing the results with those in Table II and Table III indicate that our constructed P-JSSCs can achieve better thresholds than $\mathbf{B}_{j_1}^{sp\_opt-1}$ to $\mathbf{B}_{j_4}^{sp\_opt-4}$.

### Table IV

| $p_1$ | Shannon limit | AR3A-JSCC | AR4JA-JSCC | $\mathbf{B}_{sp\_opt1}$ | $\mathbf{B}_{sp\_opt2}$ | $\mathbf{B}_{sp\_opt3}$ | $\mathbf{B}_{sp\_opt4}$ |
|---|---|---|---|---|---|---|---|
| 0.04 | -7.00 dB | -5.918 | -5.767 | -6.102 | -5.810 | -5.782 | -4.459 |
| 0.08 | -4.19 dB | -3.414 | -3.188 | -3.171 | -3.027 | -3.198 | -2.647 |
| 0.12 | -2.44 dB | -1.680 | -1.409 | -1.151 | -1.228 | -1.366 | -1.239 |
| 0.16 | -1.13 dB | -0.094 | 0.522 | 0.57 | 0.193 | 0.227 | -0.185 |
| 0.20 | -0.00 dB | 2.073 | 3.553 | 2.30 | 1.381 | 2.055 | 0.816 |

### Table V

| $p_1$ | $(E_s/N_0)^\ast$ (dB) | $\mathbf{B}_{j_1}^{sp\_opt-1}\_th$ | $\mathbf{B}_{j_1}^{sp\_opt-2}\_th$ | $\mathbf{B}_{j_4}^{sp\_opt-3}\_th$ | $\mathbf{B}_{j_4}^{sp\_opt-4}\_th$ |
|---|---|---|---|---|---|
| 0.04 | 0.082 | 0.082 | 0.144 | 0.136 |
| 0.16 | -5.130 | -5.398 | -5.267 | -5.571 |

### B. Error Performance

In addition to theoretical analysis, computer simulations are performed. The encoding and decoding processes have been described in Sect. II-A and Sect. II-B, respectively. We denote the number of source symbols in a frame by $N_s = n_{r,z}$ and set
the maximum number of decoding iterations to $I_{\text{max}} = 200$. Three types of error rates are recorded.

- Source symbol error rate (SSER) is evaluated by comparing the original source symbols with the recovered source symbols.
- Transmitted bit error rate (TBER) is evaluated by comparing the code bits sent through the channel with the corresponding recovered bits.
- Frame error rate (FER) is evaluated by comparing the original JSCC codeword with the recovered codeword.

The error rate results are recorded if (i) the number of frames simulated exceeds $10^5$, or (ii) the number of error frames exceeds $50$ and the number of frames simulated is no smaller than $5000$.

Fig. 13 plots the SSER performance of all our constructed protographs optimized at $p_1 = 0.04$. We also plot the results of AR3A-JSCC, AR4JA-JSCC and the optimized double protographs in [33] (i.e., $B_{opt}^{10}$ to $B_{opt}^{7}$) for comparison. From Fig. 13(a) where $N_s$ is around $12800$, we can observe that

1) $B_{sp, opt}^1$ and $B_{sp, opt}^{47}$ outperform AR3A-JSCC in both the waterfall and high-SNR regions while all three codes have error floors;
2) $B_{sp, opt}^2$, $B_{sp, opt}^3$ and $B_{sp, opt}^{47}$ have, respectively, $0.20$ dB, $0.25$ dB and $0.25$ dB gains over AR4JA-JSCC at a BER of $10^{-6}$;
3) $B_{sp, opt}^2$ and $B_{sp, opt}^{47}$ has no error floor down to a BER of $10^{-6}$;
4) optimized double protographs $B_{opt}^2$ and $B_{opt}^4$ in [33] are outperformed by all the generic protographs in the waterfall region and they do not have error floors down to a BER of $10^{-6}$;
5) error floors exist or start to emerge for AR3A-JSCC, $B_{sp, opt}^1$, $B_{sp, opt}^2$ and $B_{sp, opt}^{47}$ which do not have TMDRs.

When $N_s$ is reduced to around $3200$, we can observe from Fig. 13(b) that the performance of these codes are degraded. Yet, $B_{sp, opt}^1$, $B_{sp, opt}^2$ and $B_{sp, opt}^{47}$ still outperform the optimized double protographs $B_{opt}^1$ and $B_{opt}^4$ in [33] down to a BER of around $10^{-6}$.

Fig. 14 depicts the SSER and TBER performance of $B_{sp, opt}^1$, $B_{sp, opt}^2$, $B_{sp, opt}^3$, $B_{sp, opt}^4$, AR3A-JSCC code, AR4JA-JSCC code when $N_s = 12800$ and $p_1 = 0.04, 0.08, 0.12, 0.16$ and $0.20$. We can observe that the error performance of all codes degrades in general as $p_1$ increases from $0.04$ to $0.20$. It is because the initial LLR information of the source symbols decreases as $p_1$ increases. Hence in order to achieve the same error performance, a larger $E_s/N_0$ is required to compensate the reduction in initial information as $p_1$ increases.

Referring to Fig. 14(a) where $p_1 = 0.04$, the initial LLR information of the source symbols is relatively large compared with the channel LLR information of the transmitted bits. The source symbols therefore have a higher chance of being decoded correctly even when the transmitted bits are decoded wrongly. Thus SSER is better (lower) than TBER for the same $E_s/N_0$ in the given SNR range. As $p_1$ increases, the initial LLR information of the source symbols decreases. The source symbols rely more heavily on the channel LLRs of the transmitted bits for correct decoding. When the transmitted bits cannot be decoded correctly and hence cannot pass reliable information to the source symbols, the source symbols become even less likely to be decoded correctly. Therefore when $p_1 = 0.20$, Fig. 14(e) shows that SSER is worse (higher) than TBER for the same $E_s/N_0$ in the given SNR range. Based on the same arguments, we can also conclude the following.
Moreover, at the waterfall region, we can observe

1) at $p_1 = 0.04$, $B_{sp\_opt1}$ performs the best while $B_{sp\_opt4}$ performs the worst;
2) at $p_1 = 0.08$, AR3A-JSCC performs the best while $B_{sp\_opt4}$ performs the worst;
3) at $p_1 = 0.12$, AR3A-JSCC performs the best while $B_{sp\_opt1}$ performs the worst;
4) at $p_1 = 0.16$, $B_{sp\_opt4}$ performs the best while $B_{sp\_opt1}$ performs the worst; and
5) at $p_1 = 0.20$, $B_{sp\_opt4}$ performs the best while AR4JA-JSCC performs the worst;

The relative error performances of the codes therefore match with the channel thresholds listed in Table IV.

Fig. 15 depicts the frame error rate (FER) performance of $B_{sp\_opt1}$, $B_{sp\_opt2}$, $B_{sp\_opt3}$, $B_{sp\_opt4}$, AR3A-JSCC code, AR4JA-JSCC code when $N_s = 12800$ and $p_1 = 0.04, 0.08, 0.12, 0.16$ and 0.20. The characteristics of the curves can be explained similarly using the aforementioned arguments.
Fig. 14. Source symbol error rate (SSER denoted by solid lines) and transmitted bit error rate (TBER denoted by dashed lines) performance of AR3A-JSCC, AR4JA-JSCC and the optimized codes constructed \(B_{sp\_opt1}\) to \(B_{sp\_opt4}\). \(N_s = 12800\). The source thresholds of AR3A-JSCC, AR4JA-JSCC, \(B_{sp\_opt1}\), \(B_{sp\_opt2}\), \(B_{sp\_opt3}\) and \(B_{sp\_opt4}\) are 0.228, 0.212, 0.25, 0.275, 0.242 and 0.324, respectively.

(f) The table lists the starting \(E_s/N_0\) values, from which the numbers of error frames recorded are below 50.

| \(p_1\) | \(E_s/N_0\) value of the point, starting from which the error frames does not reach 50. (dB) |
|-------|------------------------------------------------------------------|
| 0.04  | None \(-5.00\) None \(-5.25\) \(-4.00\) |
| 0.08  | None \(-2.50\) \(-2.625\) \(-2.50\) \(-2.625\) \(-2.20\) |
| 0.12  | \(-1.125\) \(-0.875\) \(-0.625\) \(-0.625\) \(-0.80\) \(-0.75\) |
| 0.16  | 0.50 1.25 0.75 0.75 0.875 0.30 |
| 0.20  | 3.25 None 3.25 2.15 3.00 1.375 |

\(p_1\) denotes the probability of error frames recorded.
Fig. 15. Frame error rate (FER) performance of AR3A-JSCC, AR4JA-JSCC and the optimized codes constructed (B\text{sp\_opt1} to B\text{sp\_opt4}). N_s = 12800. The source thresholds of AR3A-JSCC, AR4JA-JSCC, B\text{sp\_opt1}, B\text{sp\_opt2}, B\text{sp\_opt3} and B\text{sp\_opt4} are 0.228, 0.212, 0.25, 0.275, 0.242 and 0.324, respectively.
IV. CONCLUSIONS

In this paper, we propose a JSCC system based on a generic protograph, namely protograph-based JSCC (P-JSCC). We present a PEXIT-JSCC algorithm to evaluate the channel threshold and propose a source generic protograph EXIT (SGP-EXIT) algorithm to evaluate the source threshold of a P-JSCC. Based on the PEXIT-JSCC and SGP-EXIT algorithms, we further propose a joint optimization method to design P-JSCCs (i.e., generic protographs) with good channel and source thresholds. In terms of theoretical thresholds and error rates, the performance of the P-JSCCs constructed by the joint optimization method are found to outperform JSCCs based on double protographs.

We further find that the source symbol error rate (SSER) is better (lower) than the transmitted bit error rate (TBER) when the probability of “1” in the source sequence $p_1$ is small, and vice versa. Moreover, error floors are caused by (i) a lack of TMDRs in the protographs when $p_1$ is small, and (ii) small initial LLR information of the source symbols when $p_1$ is large and approaches the source thresholds.

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