An Implementation of the Polynomial Lie Algebra Methods for Solving a Class of Nonlinear Models in Quantum Optics

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Abstract

We develop some calculation schemes to determine dynamics of a wide class of integrable quantum-optical models using their symmetry adapted reformulation in terms of polynomial Lie algebras $su_{pd}(2)$. These schemes, based on "diagonal" representations of model evolution operators (via diagonalizing Hamiltonians with the help of the $su_{pd}(2)$ defining relations), are implemented in the form adapted for numerical calculations. Their efficiency is demonstrated on the example of the second-harmonic-generation model.

1 Introduction

During last decades a great attention is being paid to examine different quantum-optical models with Hamiltonians given by nonlinear functions in Lie algebra generators since they enable to reveal new physical effects and phenomena (see, e.g., [1-10] and references therein). To analyze such models one uses mainly numerical calculations [10] because standard Lie-algebraic techniques well adapted for solving problems with linear (in Lie algebra generators) Hamiltonians [11], are non-efficient here, and most of other analytical techniques (e.g., the algebraic Bethe ansatz [12]) require in general cases tedious calculations and do not yield simple analytical expressions for physical quantities. On other hand, standard numerical calculation schemes dealing with initial formulations of models [10] are limited by computer powers and are not adapted to reveal many peculiarities of model dynamics [8,9].

However, recently a new universal Lie-algebraic approach, essentially improving both analytical and numerical solutions of physical problems, has been suggested in [4,5] and developed in [5-9] for some nonlinear quantum models whose Hamiltonians $H$ have invariance groups $G_i : [G_i, H] = 0$. It is based on reformulations of models under study in terms of (introduced in [4,5,13]) polynomial Lie algebras (PLA) $g_{pd}$ as dynamic symmetry algebras $g^D : g^D = g_{pd}$ completely describing model dynamics. Specifically, two analytical approximations of evolution operators $U_H(t)$ were found in [8] for a wide class of quantum-optical models with $g^D = g_{pd} = su_{pd}(2)$. Furthermore, appropriate Lie-algebraic path-integral schemes for solving physical problems were developed in [9]. They enable to examine model dynamics at quasiclassical levels described by "large" values of certain characteristic parameters [8]. However, up to now systematic examinations of their accuracy and efficiency were absent because $su_{pd}(2)$-techniques of getting exact solutions [5-7] yield only difference and difference-differential equations and fairly complicated (for practical calculations) algorithms rather than closed analytical expressions.
required for this aim. In the present work we cancel in part this lacune. Specifically, after some preliminaries (Section 2) we develop an algorithm and routines to implement an exact calculation scheme for determining diagonal representations of model evolution operators via solving a spectral problem by means of the $su_{pd}(2)$-techniques [5] (Section 3). Their efficiency is demonstrated with the help of the computer experiments for the second-harmonic-generation model which are also used to investigate the accuracy of the appropriate quasiclassical approximation obtained in [8] (Section 4).

2 Models and their symmetry adapted $su_{pd}(2)$-formulations

We consider a wide-spread class of quantum-optical models with Hamiltonians [1,3]

$$H^{mps}(m; n) = \hbar \left( \sum_{i=1}^{m} \omega_i a_i^+ a_i + \omega_0 a_0^+ a_0 + ga_i^+ a_i^+ a_0^+ a_0 + g^* a_i a_i a_0 a_0^+ \right),$$

$$1 \leq i_1 \leq i_2 \ldots \leq i_n \leq m, \quad n \geq 2 \quad (2.1a)$$

$$H^D = \hbar \left\{ \omega_1 a_1^+ a_1 + \sum_{i=1}^{M} \left[ \sigma_0(i) \epsilon/2 + g\sigma_+(i)(a_1^+ a_1 a_i a_i^+) + g^* \sigma_-(i)(a_1^+ a_1 a_i a_i^+) \right] \right\} \quad (2.1b)$$

where $a_i, a_i^+$ are operators of field modes with frequencies $\omega_i$, $\sigma_\alpha(i)$ are Pauli matrices, $\epsilon$ is an energy difference of two-level atoms, $g$ is a coupling constant, and non-quadartic parts of Eqs. (2.1a) describe different processes of multiphoton scattering including non-degenerated frequency conversion ($H^{mps}(n; n)$) and higher harmonics generation ($H^{mps}(1; n)$) while Eq. (2.1b) yields $n$-photon point-like Dicke models of matter-radiation interactions [1,8]. Hilbert spaces $L(H^{mps})$ are Fock spaces $L_F(k \leq m) = \operatorname{Span}\{\{|n_i\rangle\} \propto \prod_{i=1}^{k}(a_i^+ a_i^+ a_0^+ a_0^+)\}$ and $L(H^D) = L_F(1) \otimes L_a, L_a = \operatorname{Span}\{\prod_{i=1}^{M}|\pm\rangle(i)\}$ where $|\pm\rangle(i)$ are eigenstates of $i$-th atom [3,4].

According to [5,9] Hamiltonians (2.1) have invariance groups

$$G_i = C_n \otimes \prod_j U_j(1), \quad C_n = \{a_{i_n}^+ \rightarrow \exp(i2\pi k/n)a_{i_n}\}^\dagger; \quad U_j(1) = \{\exp(i\beta_j R_j)\}^\dagger \quad (2.2)$$

where for $H^{mps}(m; n)$ $R_j \in \operatorname{Span}\{a_i^+ a_i = N_i\}$ and for $H^D R_j$ are functions in $\sum_{i=1}^{M} \sigma_\alpha(i)$ and, besides, $G_i$ has the permutation group $S_M = \{\sigma_\alpha(i) \rightarrow \sigma_\alpha(j)\}$ as an extra factor. Therefore, one can introduce (via the $G_i$-invariant polynomial Jordan mappings [5,9]) two sets of collective operators: integrals of motion $R_j = R_j(a_i^+ a_i, a_i, \ldots, j = 1, \ldots$ and $G_i$-invariant dynamic variables $V_{\alpha=0, \pm} = V_{\alpha=0, \pm}(a_i, a_i^+, \ldots)$ obeying the commutation relations

$$[V_0, V_\pm] = \pm V_\pm, \quad [V_-, V_+] = \phi(V_0; \{R_j\}) = \psi(V_0 + 1; \{R_j\}) - \psi(V_0; \{R_j\}),$$

$$\psi(V_0; \{R_j\}) = A_{n} \prod_{i=1}^{n} (V_0 + B^\psi_i(\{R_j\})), \quad n \psi \geq 3 \quad (2.3a)$$
which resemble those for the $su(2)$ generators $Y_\alpha$ but with polynomial structural functions

$$\psi(V_0; \{ R_j \})$$

depending additionally on invariant operators $R_i : [V_\alpha, R_j] = 0$ unlike the quadratic function

$$\psi(Y_0; J) = (Y_0 + J) (J - Y_0 + 1) \quad (2.3b)$$

for $su(2)$ [9]. Therefore, $V_\alpha$ can be considered as generators of PLA $su_{pd}(2)$ acting on

$L(H)$ complementarily to $G_i$ [5], i.e., $[G_i, su_{pd}(2)] = 0$ and $L(H)$ are decomposed into direct sums

$L(H) = \sum L([l_i]), L([l_i]) = \text{Span}\{[[l_i=0,1,...]; f] = N(f; [l_i])V_0^f[[l_i]]\}, V_0[[l_i]; f] = (l_0 + f) [[l_i]; f],

\[ R_i[[l_i]; f] = l_i[[l_i]; f], \quad N(f; [l_i]) = \prod_{r=0}^{f-1} [\psi(l_0 + f - r)]^{-1/2}, \quad N(0; [l_i]) = 1, \quad V_- [[l_i]] = 0, \]

\[ \langle [l_i]; f|[l'_i]; f' \rangle = \delta_{[l_i][l'_i]} \delta_{f f'}, \quad I = \sum_{[l_i]} |l_i; f \rangle \langle [l_i]; f| \]  

(2.4a)

(2.4b)

of $G_i \otimes su_{pd}(2)$-irreducible subspaces $L([l_i])$ with finite dimensions $d([l_i])$ where $[[l_i]] \in L(H)$ are ”lowest” weight basic vectors, quantum numbers $l_i, i = 0, 1, \ldots$ are eigenvalues of operators $R_i$ and ”lowest” weight operators $R_0$ are determined by the $su_{pd}(2)$ Casimir operators

$$\Psi(R_0) \equiv \psi(V_0; \{ R_j \}) - V_+ V_- = \psi(V_0 + 1; \{ R_j \}) - V_- V_+, \quad [V_\alpha, \Psi(R_0)] = 0 \quad (2.4c)$$

which, by construction (due to the invariant theory [5]), satisfy the characteristic identities

$$\Psi(R_0)|_{L(H)} \equiv 0 \Rightarrow \Psi(l_0) = \psi(l_0; \{ l_j \}) = 0 = \psi(l_0 + d([l_i]); \{ l_j \}) = \Psi(l_0 + d([l_i])) \quad (2.4d)$$

which are useful in applications [5-9].

In the general cases the $G_i$-invariant Jordan mappings can be given as follows [9]:

$$V_+ = a_i^+ \ldots a_i^+ a_0, \quad V_- = a_i \ldots a_i a_0^+, \quad V_0 = (\sum_{i=1}^{m} N_i - N_0)/(n + 1),$$

$$R_1 = (\sum_{i=1}^{m} N_i + n N_0)/(n + 1), \quad R_{1<j<m} = \mu_j N_{j-1} - \mu_{j-1} N_j \quad (2.5a)$$

for models (2.1a) ($\mu_j$ is the multiplicity of the $a_j^+$ appearance in $V_{i_1,i_2,i_3}^+$) and

$$V_0 = J_0, \quad V_+ = J_+ (a_1)^n, \quad V_- = J_- (a_1)^n, \quad J_0 = \frac{1}{2} \sum_{i=1}^{M} \sigma_0(i), \quad J_\pm = \sum_{i=1}^{M} \sigma_\pm(i),$$

$$R_1 = J_0 + (a_1^+ a_1)/n, \quad R_2 = J, \quad J(J + 1) = J_0^2 + (J_+ J_- + J_- J_+)/2 \quad (2.5b)$$

for the model (2.1b).

Appropriate structural polynomials (2.3a) are determined from Eqs. (2.5), (2.4c)-(2.4d). Specifically, they are given by expressions [9]

$$\psi_D(V_0; R_1, R_2) = (R_2 + V_0)(R_2 + 1 - V_0)(nR_1 + n - nV_0)^{(n)}, \quad (2.6a)$$
for the model (2.1b) and spread specifications of the model (2.1a) with $H^{m_{ps}}(1; n) \equiv H^{hg}(n)$ (high-harmonics-generation models) respectively; here $(A)^{(m)} \equiv A(A - 1) \ldots (A - m + 1)$. In the general case for the model (2.1a) we get fairly complex expressions (see, e.g., them for $H^{m_{ps}}(n; n) \equiv H^{fc}(n)$ in [9]), and, therefore, a special (based on the Symbolic Computer Algebra System [14]) routine was developed to get such expressions automatically.

The introduction of the collective operators above enables us to express all Hamiltonians (2.1) as follows [5,9]:

$$H = \hbar[\Delta V_0 + gV_+ + g^*V_- + C(\{R_i\})], \quad [V_\alpha, C] = 0, \quad (2.7)$$

where coefficients $\Delta$ and functions $C = C(\{R_i\})$ are found from Eq. (2.1) with the help of Eqs. (2.5) for $R_\alpha, V_\alpha$. In such a manner we get [9]:

$$C^{hg}(R_1) = (\omega_1 + \omega_0)R_1, \quad \Delta^{hg} = n\omega_1 - \omega_0 \quad \text{for } H^{hg}(n), \quad (2.8a)$$

$$C^{fc}(\{R_i\}) = \omega_0R_1 + \sum_{j=1}^n \frac{\omega_j}{n} \left(R_1 - \sum_{i=1}^{j-1} iR_{i+1}\right) + \sum_{j=1}^{n-1} \omega_j \sum_{i=j+1}^n R_i, \quad \Delta^{fc} = \sum_{i=1}^n \omega_i - \omega_0, \quad (2.8b)$$

$$C^D(R_1, R_2) = n R_1 \omega_1, \quad \Delta^D = \epsilon - \omega_1 \quad (2.8c)$$

for $H^{hg}(n), H^{fc}(n) \equiv H^{m_{ps}}(n; n)$ and $H^D$ respectively. At the same time decompositions (2.3) of $L(H)$ are specified by determining quantum numbers $l_i$ (integral of motions) and ”lowest” weight vectors $|l_i\rangle \in L(H)$ as solutions of defining relations in (2.4a) [4,5]. Specifically, in such a manner one gets [9]:

$$L^{hg}(|l_i\rangle \equiv L^{hf} \left(l_0 = \frac{\kappa - s}{1 + n}, l_1 = \frac{\kappa + ns}{1 + n}\right), \quad |l_i(\kappa, s)\rangle = \frac{(a_i^+)^s(a_0^+)^s}{[\kappa!s!]^{1/2}}|0\rangle, \quad 0 \leq \kappa \leq n - 1, s \geq 0, \quad (2.9a)$$

$$L^{fc}(|l_i\rangle \equiv L^{fc} \left(l_0 = \sum_{i=1}^{n} \kappa_i - s)/(1 + n), l_1 = \sum_{i=1}^{n} \kappa_i + ns)/(1 + n), l_{2 \leq i \leq n} = \kappa_{i-1} - \kappa_i\right), \quad |l_i(\kappa_i, s)\rangle = \frac{\prod_{i=1}^{n}(a_i^+)^{\kappa_i}(a_0^+)^s}{[\prod_{i=1}^{n} \kappa_i!s!]^{1/2}}|0\rangle, \quad \kappa_i \geq 0, s \geq 0, \quad \prod_{i=1}^{n} \kappa_i = 0, \quad (2.9b)$$

$L^D(|l_i\rangle \equiv L^D(l_0 = -j = -l_2, l_1 = -j + \frac{\kappa}{n}; l_{r \geq 2+r} = j_r), \quad \kappa \geq 0, j = \frac{N}{2} - \left[ \frac{N}{2} \right], \ldots, \frac{N}{2}$,

$$|l_{s \leq 2}(\kappa, j); \{j_r\}; \{j_r\}; -j\rangle_a = |\kappa\rangle_f \otimes |j; \{j_r\}; -j\rangle_a, \quad |\kappa\rangle_f = [\kappa!]^{-1/2}(a_i^+)^{\kappa}|0\rangle \in L_F(1) \quad (2.9c)$$

where $|j; \{j_r\}; -j\rangle_a$ are ”lowest” weight vectors of the $su(2)$-irreducible ”atomic” subspaces $L(j; \{j_r\}) \subset L_a$ and extra integral of motions (“intermediate atomic quasispins” [3,9]) $j_r$ are due to the factor $S_M$ in $G^{D}_n$; in all these cases $d(|l_i\rangle) = s + 1$, where $s = \min\{2j, \kappa/n\}$ for $H^D$.

So, Eqs. (2.4)-(2.9) yield the $"su_{ps}(2)\)-cluster” formulation of models (2.1) entailing a complete description of model dynamics with the help of the $su_{ps}(2)$ algebra formalism. In particular, it provides adequate representations [5-9] for model evolution operators $U_H(t)$ to calculate quantum expectations $\langle O(t) \rangle$ of any operators $O = O(a_i^+, \ldots)$ acting on $L(H)$:

$$\langle O(t) \rangle = \text{Tr}[U_H(t) \rho U_H^+(t) O] = \sum \langle |l_i; f| \rho U_H^+(t) O U_H(t) |l_i; f\rangle. \quad (2.10)$$
3 Diagonal representations of model evolution operators and algebraic schemes to determine them

One of adequate representations of $U_H(t)$ has a diagonal form [8,9]

$$U_H(t) = \sum_{[l_i];v} e^{-it\epsilon([l_i];v)} |E_v([l_i])\rangle \langle E_v([l_i])|,$$

$$E_v([l_i]) = \hbar \epsilon([l_i];v) = \hbar[C\{\{l_i]\} + \lambda_v([l_i])]$$

where $E_v([l_i])$ and $\{|E_v([l_i])\rangle\}$ are, respectively, eigenvalues and complete sets of orthonormalized eigenvectors of the Hamiltonian (2.7):

$$H|E_f([l_i])\rangle = E_f([l_i])|E_f([l_i])\rangle, \langle E_f(\delta_{ff'}, I = \sum_{[l_i],v} |E_v([l_i])\rangle \langle E_v([l_i])|$$

In the ”linear” case, when $n_\psi = 2$ in (2.3a), $\psi(V_0; \{R_i\}) = \psi(Y_0; J)$ and PLA $su_{pd}(2)$ are reduced to the familiar $su(2)$ algebra, the eigenproblem (3.2) is solved exactly [8] with the help of the $SU(2)$ displacement operators $S_Y(\xi = r g/|g|) = \exp(\xi Y_+ - \xi^* Y_-)$ [11]; herewith solutions are given by simple analytical expressions [5-8].

However, it is not the case when $n_\psi \geq 3$ in (2.3a) in view of the absence of explicit expressions for matrix elements $\langle [l_i]; f | \exp(\sum_i a_i V_i) | [l_i]; v \rangle$ [8]. In fact, in [6] an algorithm has been developed to solve eigenproblem (3.2) with $n_\psi \geq 3$ via ”dressing” an auxiliary ”linear” (with $n_\psi = 2$ one; but it is unsuitable for practical calculations, and, really nowadays it is known only an approximate analytical solution [8] of the problem (3.2) given by approximate $SU(2)$-quasiclassical eigenfunctions

$$|E^{qc}_v([l_i]; \xi)\rangle \equiv |E^{SU(2)}_v([l_i]; \xi = r g/|g|)\rangle = \exp(\xi Y_+ - \xi^* Y_-)|[l_i]; v\rangle = \sum_f S^f_{l_v}(\xi) |[l_i]; f\rangle,$$

$$S^f_{l_v}(\xi) = (g/|g|)^{j - v} d^i_{j - f, -j + v}(2r), \quad 2j = s = d[l_i] - 1$$

and eigenenergies

$$E_v^{qc}([l_i]; \xi) = \langle E^{qc}_v([l_i]; \xi)|H|E^{qc}_v([l_i]; \xi)\rangle = \hbar[C\{\{l_i]\} + \lambda_v^{qc}([l_i]; r)], \quad \lambda_v^{qc}([l_i]; r) = \Delta(j + l_0)$$

$$-\Delta(j - v) \cos 2r + 2|g| \sum_{f=0}^{s} \sqrt{(s - f)(f + 1)} \phi(-j + f) d^{i}_{j - f, -j + v}(2r) d^{j}_{j - f + 1, -j + v}(2r)$$

$$\approx \lambda_v^{cm}(\{[l_i]\}; r) = \Delta(j + l_0) - \Delta(j - v) \cos 2r - 2|g| (j - v) \sin 2r \sqrt{\phi(-j + v) \cos 2r},$$

$$\phi(-j + f) \equiv \langle [l_i]; f | \psi(V_0 + 1; \{R_i\}) / \psi(Y_0 + 1; j)|[l_i]; f\rangle$$

where generators $Y_\alpha$ of the $su(2)$ algebra are connected with those of PLA $su_{pd}(2)$ via generalized Holstein-Primakoff mappings [5,8], $d^{i}_{m,n}(2r)$ is the $SU(2) d$-function expressed in terms of the Gauss hypergeometric function [15] and approximate values $\lambda_v^{cm}(\{[l_i]\}; r)$ are calculated in the cluster mean-field approximation: $\langle [l_i]; f | F(V_\alpha)[l_i]; f \rangle = F(\{[l_i]; f | V_\alpha[l_i]; f\}$). Values of the parameter $r$ in (3.3b) are found from energy-stationarity-conditions and/or from minimizing a proximity measure of Hamiltonians $H$ and $H^{qc}(\xi) = \sum_v\langle l_i; \xi | E^{qc}_v([l_i]; \xi)\rangle\langle E^{qc}_v([l_i]; \xi)\rangle$: a standard measure for such estimates on the subspaces $L([l_i])$ is defined with the help of the euclidean operator norm as follows [5,8]

$$\delta^2_{l_v}(l_i) = \frac{T_{l_v,l}[H - H^{qc}(\xi)]^2}{\sum_v \{(\lambda_v^{cm}(l_i))^2 - (\lambda_v^{qc}(l_i); r)^2\}}$$
However, according to the general quasiclassicality theory [16] all approximations (3.3) are valid only for large values of $d([l_i])$, and, besides, the measure (3.4a) gives only a global rather than local characteristic of the approximate energy spectra $\{E^{qc}_v([l_i]; \xi)\}$ that does not allow to feel their important symmetry properties and local peculiarities related to "energy errors"

$$\Delta E_v([l_i]) = \hbar [\lambda_v([l_i]) - \lambda_v^{qc}([l_i]; r)] \equiv \delta E_v([l_i]) \cdot E_v([l_i])$$

(3.4b)

Therefore, it is useful to have convenient algorithms to get exact solutions of the eigenproblem (3.2) using $su_{pd}(2)$ defining relations.

Specifically, taking into account Eqs. (2.4a), one can look for eigenfunctions $|E_f([l_i])|$ on each $su_{pd}(2)$-irreducible space $L([l_i])$ in the form [5,8]

$$|E_f([l_i])| = \sum_{v=0}^{s} Q^f_v([l_i])|l_i; v\rangle = \sum_{v=0}^{s} \tilde{Q}^f_v([l_i]) V^+[l_i], \quad \tilde{Q}^f_v([l_i]) = N(v; [l_i]) Q^f_v([l_i])$$

(3.5a)

where $s = d[l_i] - 1$ and, in view of Eqs. (3.2) amplitudes $Q^f_v([l_i])$ satisfy the following orthonormalization and completeness conditions:

$$\sum_{v=0}^{s} Q^f_v([l_i]) Q^{f'}_v([l_i]) = \delta_{f'f}, \quad \sum_{f} Q^f_v([l_i]) Q^{f'}_v([l_i]) = \delta_{vv'}$$

(3.5b)

Then, inserting Eq. (3.5a) for $|E_f([l_i])|$ and Eq. (2.7) for $H$ in the first equation of (3.2) and using Eqs. (2.3a), (2.4c)-(2.4d), one gets a set of recurrence relations

$$P_{v+1}(\lambda) = [\lambda - \Delta(v + l_0)]P_v(\lambda) - |g|^2 \psi(l_0 + v; [l_i])P_{v-1}(\lambda), \quad v = 0, \ldots, s \tag{3.7a}$$

$$P_0(\lambda) = 1, \quad P_{-1}(\lambda) = 0 = [\lambda - \Delta(s + l_0)]P_s(\lambda) - |g|^2 \psi(l_0 + s; [l_i])P_{s-1}(\lambda)$$

(3.7b)

for finding non-classical orthogonal (in view of (3.5b)) polynomials

$$P_v(\lambda) = (g^*)^vN^{-2}(v; [l_i])\tilde{Q}_v([l_i]; \lambda)/\tilde{Q}_0([l_i]; \lambda) = (g^*)^vN^{-1}(v; [l_i])Q_v([l_i]; \lambda)/Q_0([l_i]; \lambda)$$

(3.8)

for finding non-classical orthogonal (in view of Eqs. (3.5b) polynomials in the discrete variable $\lambda$ on the non-uniform lattice $\{\lambda_f([l_i])\}_{\lambda=0}^{s}$ [5]. Indeed, Eqs. (3.7), (3.8) provide the following (easily realized by means of FORTRAN programs) algorithm for solving the eigenproblem (3.2).

i) Using the recursive formula (3.7a) with initial values from Eq. (3.7b) one calculates the polynomial sequence $\{P_v(\lambda)\}_{\lambda=0}^{s}$.

ii) Inserting $P_{s-1}(\lambda), P_{s}(\lambda)$ in the last equality in (3.7b) one gets the algebraic equation with respect to $\lambda$; its solution yield the sequence $\{\lambda_f([l_i])\}_{\lambda=0}^{s}$ of admissible values...
iii) For each value $\lambda_f([l_i])$ using $\{P_v(\lambda)\}_{v=0}^s$ and Eq. (3.8) one finds the sequence $\{Q_v([l_i]) = Q_v([l_i]; \lambda_f)\}_{v=1}^s$ of all amplitudes as functions in the only undetermined quantity $Q_0(\lambda_f) = Q_0([l_i]; \lambda_f)$ which, in turn, is found from the normalization condition of Eqs. (3.5b).

In order to make numerical calculations we implemented this algorithm with the help of the REDUCE Package [14] similar to the algebraic construction in [17].

4 Numerical analysis for the second-harmonic generation model

In order to examine the efficiency of calculation schemes and the algorithm given above we tested them by means of computer experiments for the resonance second-harmonic-generation model widely examined in quantum optics [1,8,10] and determined by $H^{hg}(1; 2)$ with $\omega_0 = 2\omega_1$. In this case, according to Eqs. (2.6b), (2.8a), (2.9a), and (3.3b) we have

$$
\psi(l_0 + v + 1; [l_i]) = (k + 2v + 2)(k + 1 + 2v)(s - v), \quad k = 0, 1; s = 0, 1, \ldots, \quad (4.1a)
$$

$$
C(l_i = \frac{k + 2s}{3}) = \omega_1(k + 2s), \quad \Delta = 0, \quad \lambda_v^{cmf}([l_i]; r) = -2g|(j - v)\sin 2r\sqrt{\phi[-(j + v)\cos 2r]}
$$

$$
\phi(-j + f) = 2[s + 2k + 1 - (s - 2f)], \quad j = s/2 \quad (4.1b)
$$

We implemented calculations of exact values $\lambda_f([k, s])$ according to the algorithm of Section 3 and of their approximations $\lambda_v^{cmf}([k, s]; r_i)$ according to Eq. (4.1b) for $g = 1, k = 0, 1, s = 20, 100, 500, 1000, 10000$. Values of the fitting parameter $r$ were determined from energy-stationarity-conditions: $2r_1 = \arccos \frac{1}{3}$ (optimizing only the upper part of spectra) [8], $2r_3 = \arccos 0 = \frac{\pi}{2}$ (quasi-linear approximation) [5,8] and from minimizing the proximity measure (3.4a): $2r_2 = \arccos \frac{1}{\sqrt{3}}$ ("smooth" cluster mean-field approximation) [8]; herewith $\lambda_v^{cmf}([k, s]; \mp r_1)$ means that we take $r = -r_1$ in the first half of spectra $r = r_1$ the second one. To estimate the accuracy of approximations we used, besides Eq. (3.4a) the measures

$$
\delta_E^2([l_i]) = \frac{\sum_{v=0}^s [(\lambda_v([l_i]) - \lambda_v^{cmf}([l_i]; r)]^2}{\sum_{v=0}^s (\lambda_v([l_i]))^2}, \quad \delta_{E_{up}}^2([l_i]) = \frac{\sum_{v=s/2}^s [(\lambda_v([l_i]) - \lambda_v^{cmf}([l_i]; r)]^2}{\sum_{v=s/2}^s (\lambda_v([l_i]))^2} \quad (4.2)
$$

which characterize energy spectra more precisely in comparison with Eq. (3.4a).

Typical results of numerical calculations obtained are presented in two tables.

| $v$ | $\lambda_f([0, s])$ | $\lambda_v^{cmf}([0, s]; r_1)$ | $\lambda_v^{cmf}([0, s]; \mp r_1)$ | $\lambda_v^{cmf}([0, s]; r_2)$ | $\lambda_v^{cmf}([0, s]; r_3)$ |
|-----|-------------------|-----------------|-----------------|-----------------|-----------------|
| 0   | -1536.9           | -1096.7         | -1545.3         | -1482.4         | -1421.2         |
| 10  | -1151.7           | -919.6          | -1205.2         | -1175.2         | -1137.0         |
| 20  | -798.1            | -720.0          | -880.0          | -873.3          | -852.7          |
| 30  | -480.3            | -499.3          | -570.2          | -576.7          | -568.5          |
| 40  | -205.5            | -259.0          | -276.7          | -285.6          | -284.2          |
| v  | $\lambda(0, s)$ | $\lambda_{cmf}(0, s; r_1)$ | $\lambda_{cmf}(0, s; 1)$ | $\lambda_{cmf}(0, s; 2)$ | $\lambda_{cmf}(0, s; 3)$ |
|----|----------------|-----------------|-----------------|-----------------|-----------------|
| 0  | -1539573       | -1088743        | -1539658        | -1421266        | -1414284        |
| 200| -1460154       | -1055592        | -1470663        | -1364145        | -1357712        |
| 600| -1304855       | -986414         | -1334422        | -1249971        | -1244570        |
| 1000| -1154419     | -913501         | -1200541        | -1135886        | -1131427        |
| 2000| -800913       | -715586         | -876392         | -851069         | -848570         |
| 3000| -483083       | -496354         | -567801         | -566815         | -565713         |
| 4000| -208106       | -257638         | -275425         | -283125         | -282856         |
| 4400| -113484       | -156775         | -163176         | -169807         | -169714         |
| 4800| -31728        | -52979          | -53690          | -56579          | -56571          |
| 5000| 0             | 0               | 0               | 0               | 0               |
| 5200| 31728         | 53690           | 53690           | 56557           | 56571           |
| 5600| 113484        | 163176          | 163176          | 169603          | 169714          |
| 6000| 208106        | 275425          | 275425          | 282559          | 282856          |
| 7000| 483083        | 567801          | 567801          | 564553          | 565713          |
| 8000| 800913        | 876392          | 876392          | 845978          | 848570          |
| 9000| 1154419       | 1200541         | 1200541         | 1126836         | 1131427         |
| 9400| 1304855       | 1334422         | 1334422         | 1239020         | 1244570         |
| 9800| 1460154       | 1470663         | 1470663         | 1351113         | 1357712         |
| 10000| 1539573     | 1539658         | 1539658         | 1407125         | 141284         |

As is seen from data given in Tables 1, 2, there is a satisfactory consent of exact results and those obtained with the help of the approximate formula (3.3b) at $s \gg 1$ in the most parts of energy spectra for $r = r_{i=2,3}$ and $r = \mp r_1$. Discrepancies between exact and approximate results in middle parts of spectra are explained by the availability of the square-root singularities in $H_{cmf}(\xi) = \sum_v \left| E^v_{cmf}(l_i; \xi) \right| \langle E^v_{cmf}(l_i; \xi) | E^v_{cmf}(l_i; \xi) \rangle$, and the symmetry breaking of spectra at $r = r_{i=1,2}$ is due to smoothing this singularities within the framework of the "smooth" $su(2)$ - quasiclassical approximation [8,9]. A more full analysis of results of computer experiments above will be given elsewhere.
5 Conclusion

So, we developed an algebraic scheme and routines (implemented with the help of the REDUCE Package [14]) to get exact solutions of both spectral and evolution problems for a wide class of integrable models of quantum optics having PLA $stu_{pd}(2)$ as their dynamic symmetry algebras. These results along with appropriate computer realizations of Eqs. (3.3) provide an efficient tool for examining models under study in appropriate (determined by computer powers) ranges of characteristic model parameters given by $l_i$. Numerical calculations given in Section 4 showed a good consent of exact and approximate results at $s \gg 1$ and at relevant choices of the fitting parameter $r$ in (3.3b). Furthermore, the approximate solutions of the eigenproblem (3.2) can be improved with the help of the algebraic perturbation algorithm like that developed in [17] for the hydrogen atom or by means of using modifications of algebraic schemes of the work [6]. Therefore, the exact calculation schemes developed above, when being completed by quasiclassical calculations [8] based on formulas like Eq. (3.3) as well as their improvements, can be used for analyzing models under consideration in all ranges of characteristic parameters $l_i$. The work along these lines is in progress. References

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