ΛCDM epoch reconstruction from $F(R, G)$ and modified Gauss–Bonnet gravities

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Abstract

Dark energy cosmology is considered in a modified Gauss–Bonnet model of gravity with and without a scalar field. It is shown that these generalizations of general relativity endow it with a very rich cosmological structure: it may naturally lead to an effective cosmological constant, quintessence or phantom cosmic acceleration, with the possibility of describing the transition from a decelerating to an accelerating phase explicitly. It is demonstrated here that these modified GB and scalar-GB theories are perfectly viable as cosmological models. They can describe the ΛCDM cosmological era without any need for a cosmological constant. The specific properties of these theories of gravity in different particular cases, such as the de Sitter one, are studied.

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1. Introduction

Recent observational data indicate that our universe is accelerating. This acceleration is explained in terms of the so-called dark energy (DE), which may be expressed in modified gravity models (for a general review see [1]). DE could also result from a cosmological constant, from an ideal fluid with a different form of equation of state and negative pressure, a scalar field with quintessence-like or phantom-like behaviour (see [2] and references therein), etc. The choice of possibilities reflects the indisputable fact that the true nature and origin of DE has not been convincingly explained yet. It is not even clear what type of DE is more seemingly to explain the current epoch of the universe. Observational data point towards some kind of DE with an equation of state (EoS) parameter which is very close to $-1$, maybe even less than $-1$, the so-called phantom case (see [3]). A quite appealing possibility is the already
mentioned modification of general relativity (GR). Modifications of the Hilbert–Einstein action by introducing different functions of the Ricci scalar have been systematically explored, the so-called \( F(R) \) gravity models, whose reconstruction has been carried out in [4–9]. As is known, \( F(R) \) gravity can be written in terms of a scalar field (quintessence or phantom like) by redefining the function \( F(R) \) with the use of a scalar field, and then performing a conformal transformation. It has been shown that, in general, for any given \( F(R) \) the corresponding scalar–tensor theory can, in principle, be obtained, although the solution is going to be very different from one case to another. Also, attention has been paid to the reconstruction of \( F(R) \) gravity from a given scalar–tensor theory. It is known, too, that the phantom case in the scalar–tensor theory does not exist, in general, when starting from \( F(R) \) gravity. In fact, the conformal transformation becomes complex when the phantom barrier is crossed, and therefore the resulting \( F(R) \) function becomes complex. These situations where addressed in [9] in detail, where to avoid this hindrance, a dark fluid was used in order to produce the phantom behaviour in such a way that the \( F(R) \) function reconstructed from the scalar–tensor theory continues to be real.

On the other hand, it has also been suggested in the literature (see [10, 11]) to consider modified Gauss–Bonnet gravity, that is, a function of the GB invariant. Different cosmological properties of the modified gravity models of this kind have been studied in [10–26]. Both possibilities have, in principle, the capability to explain the accelerated expansion of the universe and even the primordial inflationary phase (see [4] and [9]), with no need to introduce a new form of energy. In this paper we will study some specific modified Gauss–Bonnet theories and, by using a technique developed in [6], the corresponding cosmological theory will be reconstructed for several cosmological solutions.

We will here work with the spatially flat FRW universe metric

\[
\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \sum_{i=1}^{3} (\mathrm{d}x^i)^2,
\]

(1)

where \( a(t) \) is the scale factor at cosmological time \( t \). In GR, the corresponding system of equations are the usual Friedmann equations, namely

\[
\frac{3}{k^2} H^2 = \rho, \quad \frac{1}{k^2}(2 \dot{H} + 3H^2) = -p, \quad \dot{\rho} = -3H(\rho + p).
\]

(2)

The last of them is the continuity equation for a perfect fluid. Usually, the Hubble rate \( H \) is defined by \( H \equiv \dot{a}/a \). In (2), \( \rho \) and \( p \) are the matter energy–density and pressure. The Gauss–Bonnet invariant is given by

\[
G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}
\]

(3)

The cosmological models coming from the different versions of modified GB gravity considered here will be carefully investigated with the help of several particular examples where calculations can be carried out explicitly.

The paper is organized as follows. In section 2, the general technique which will be used to reconstruct a gravity theory from a given cosmological evolution setup will be explained in detail. Several specific examples will be worked through, for the simple case where one considers a Hilbert–Einstein action plus a function of the GB invariant. In section 3, a general model whose action depends on a function of the Ricci scalar and of the GB invariant will be studied. With the help of several explicit examples we will show how the reconstruction of the cosmological solutions is carried out. In section 4, the more involved case of a scalar–tensor theory where the GB invariant is coupled to the scalar field will be considered. It will be shown there that any cosmological solution can be reproduced by a specific scalar potential
and a convenient coupling term. Finally, in the concluding section we will provide a summary of
the main results obtained in the paper.

2. The \([R + f(G)]\) model

Consider the following action, which describes GR plus a function of the Gauss–Bonnet term
(see [10] and [11]),

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + f(G) + L_m \right],
\]

(4)

where \(\kappa^2 = 8\pi G_N\), \(G_N\) being the Newton constant. By varying the action over \(g_{\mu\nu}\),
the following field equations are obtained:

\[
0 = 12 \kappa^2 \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} f(G) - 2 f_G R R^{\mu\nu} + 4 f_G R_{\rho\sigma} R^{\rho\sigma}

- 2 f G R^{\rho\sigma\tau} R_{\rho\sigma\tau} - 4 f_G R^{\mu\nu\rho\sigma} R_{\rho\sigma} + 2 (\nabla^{\mu} \nabla^{\nu} f_G) R - 2 g^{\mu\nu} (\nabla^2 f_G) R

- 4 (\nabla_p \nabla^p f_G) R^{\rho\sigma} - 4 (\nabla_p \nabla^p f_G) R^{\rho\sigma} - 4 (\nabla_p \nabla^p f_G) R^{\rho\sigma}.
\]

(5)

For the metric (1), these equations give the first FRW equation which has the form

\[
0 = -\frac{3}{\kappa^2} H^2 + G f_G - f - 24 G H^3 f_{GG} + \rho_m.
\]

(6)

The Hubble rate \(H\) is here defined by \(H = \dot{a}/a\), while \(G\) and \(R\) are given by

\[
G = 24 (H H^2 + H^4), \quad R = 6 (H + 2 H^2).
\]

(7)

The matter energy density \(\rho_m\) satisfies the standard continuity equation

\[
\dot{\rho}_m + 3 H (1 + w) \rho_m = 0.
\]

(8)

Let us now rewrite equation (6) by using a new variable \(N = \ln \frac{a}{a_0} = -\ln (1 + z)\), that is, the
number of e-foldings, instead of the cosmological time \(t\), where \(z\) is the redshift (this method
has been implemented in [6] for \(f(R)\) gravity). The following expressions are then easily
obtained:

\[
a = a_0 e^N, \quad H = N = \frac{dN}{dt}, \quad \frac{d}{dt} = H \frac{d}{dN},

\]

(9)

Equation (6) can thus be expressed as follows:

\[
0 = -\frac{3}{\kappa^2} H^2 + 24 H^3 (H' + H) f_G - f - 576 H^6 (H H'' + 3 H^2 + 4 H H') f_{GG} + \rho_m.
\]

(10)

where \(G\) and \(R\) are now

\[
G = 24 (H^3 H' + H^4), \quad G = 24 (H^3 H'' + 3 H^3 H^2 + 4 H^4 H'),

R = 6 (H H' + 2 H^2).
\]

(11)

By introducing a new function \(g\) as \(g = H^2\), we have

\[
H = \sqrt{g}, \quad H' = \frac{1}{2} g^{-1/2} g', \quad H'' = -\frac{1}{4} g^{-3/2} g'' + \frac{1}{2} g^{-1/2} g'''.
\]

(12)
Hence, equation (10) takes the form
\[ 0 = -\frac{3}{k^2} g + 12g(g' + 2g) f_G - f - 24^2 g \left[ \frac{1}{2} g^2 g'' + \frac{1}{2} g^2 g^2 g' \right] f_{GG} + \rho_m. \] (13)
where we have used the expressions
\[ G = 12g g' + 24g^2, \quad \dot{G} = 12g^{-1/2}[g^2 g'' + gg' + 4g^2 g']. \] (14)

Finally, we can write the FRW equations in a slightly different form [13]
\[ \frac{3}{k^2} H^2 = \rho_{\text{eff}}, \quad \frac{1}{k^2} (2\dot{H} + 3H^2) = -p_{\text{eff}}, \] (15)
where the effective energy and pressure densities are
\[ \rho_{\text{eff}} = w_{\text{eff}} \rho_{\text{eff}}, \quad \dot{\rho}_{\text{eff}} = -3H(\rho_{\text{eff}} + p_{\text{eff}}), \] (16)
with
\[ \dot{\rho}_{\text{eff}} = \rho_G + \rho_m, \quad p_{\text{eff}} = p_G + p_m. \] (17)

Here
\[ \rho_G = Gf_G - f - 24H^3 \dot{G} f_{GG}, \]
\[ p_G = -\rho_G + 8H^2 G^2 f_{GG} - 192 f_{GG}(4H^6 H - 8H^3 H H - 6H^2 H^3 - H^4 \dot{H} - 3H^3 H - 18H^2 H^2). \] (18)
As was pointed out in [13], for de Sitter space \( p_G = w_G \rho_G = -\rho_G \), that is \( w_G = -1 \).

2.1. Example 1

As a first example we consider the \( \Lambda \)CDM model. As will be shown below, for the gravity theory described by the action (4), it is possible to reconstruct such evolution with no need of a cosmological constant term, and the transition from a decelerated to an accelerated epoch is achieved (note that this can also be performed in the spinor field scenario, see [27]). For the \( \Lambda \)CDM case, we have
\[ g = H^2 = H_0^2 + \frac{k^2 \rho_0}{3} a^{-3} = H_0^2 + \frac{k^2 \rho_0}{3} a^{-3} e^{-3N} = H_0^2 + l z, \] (19)
where \( l = \frac{k^2 \rho_0}{3} e^{-3N} \), \( z = e^{-3N} \). Hence, we get
\[ z' = -3z, \quad g' = -3lz, \quad g'' = 9lz. \] (20)

Finally, for \( G \) we obtain the expressions
\[ G = 24H_0^3 + 12H_0^2 l z - 12f^2 z^2, \quad \dot{G} = -3z HW = -3Hz(12H_0^2 l - 24f^2 z), \] (21)
where \( W = 12H_0^2 l - 24f^2 z \). We can reverse equation (21) and the number of e-foldings can be written as a function of the Gauss–Bonnet term. This yields
\[ z = \frac{3H_0^2 - \sqrt{81H_0^4 - 3G}}{6l} = e^{-3N}, \] (22)
and, from here
\[ N = \frac{1}{3} \ln \frac{6l}{3H_0^2 - \sqrt{81H_0^4 - 3G}}, \] (23)
Hence, it follows that
\[ g = H^2 = H_0^2 + \frac{3H_0^2 - \sqrt{81H_0^4 - 3G}}{6} = \frac{9H_0^2 - \sqrt{81H_0^4 - 3G}}{6}. \] (24)

Finally, for the function \( f(z) \) we get the equation
\[ a_2 f_{zz} + a_1 f_z + a_0 f + b = 0, \] (25)
where
\[ a_0 = -w^2 = -242l^4z^2 + 242H_0^2l^3z - 144H_0^4l^2, \]
\[ a_1 = 12(H_0^2 + lz)[144H_0^2z(H_0^2 + lz) + (2H_0^2 - lz)w], \]
\[ a_2 = 72zw(H_0^2 + lz)^2, \quad b = \left[ \rho_m - \frac{3}{\kappa^2}(H_0^2 + lz) \right] w. \] (26)

The energy density can be expressed as
\[ \rho_m = -Dz - E, \] (27)
being
\[ D = 144H_0^2l^3 \left[ \frac{83}{5\kappa^2} - \frac{16}{9} + \frac{20\delta}{9} \right], \quad E = 144H_0^2l^2 \left[ \frac{4}{9} - \frac{13}{5\kappa^2} - \frac{5\delta}{9} \right]. \] (28)

Here \( \delta = \text{const.} \) As a consequence, equation (25) has the following particular solution:
\[ f(z) = \theta z^2 + \vartheta z + H_0^2\delta, \] (29)
where
\[ \theta = \frac{l^2}{H_0} \left[ \kappa^{-2} - \frac{2}{9}(\delta + 1) \right], \quad \vartheta = l \left[ \frac{1}{5}\kappa^{-2} + \frac{2}{9}(\delta + 1) \right]. \] (30)

Thus, we have
\[ f(G) = \theta \left[ \frac{3H_0^2 - \sqrt{81H_0^4 - 3G}}{6l} \right]^2 + \vartheta \left[ \frac{3H_0^2 - \sqrt{81H_0^4 - 3G}}{6l} \right] + H_0^2\delta. \] (31)

We observe that this function reproduces exactly the same behaviour as the \( \Lambda \text{CDM} \) model in the context of Gauss–Bonnet gravity.

2.2. Example 2

Let us now consider a second example
\[ g = H^2 = l e^{-3N} = lz. \] (32)
This case describes a time evolution given by
\[ H(t) = \frac{2}{3(t - t_0)}, \] (33)
which is equivalent to the cosmological evolution of a pressureless fluid in GR, which gives a decelerated expansion. As in the example above, we find that \( z = \pm i\sqrt{3G} \). In the present case, we have the following differential equation:
\[ z^2 f_{zz} - \frac{7}{6}zf_z + \frac{1}{3}f - \frac{c}{3} = 0, \] (34)
where \( c = \rho_m - \frac{3}{\kappa^2}lz. \)
Let us assume that \( c = 0 \); then equation (34) takes the form
\[
z^2 f' - \frac{7}{6} z f + \frac{1}{3} f = 0. \tag{35}
\]
This equation admits the following exact solution:
\[
f_1(z) = C_1 z^2 + C_2 z^{\frac{1}{3}}. \tag{36}
\]
As a consequence, for this second example we obtain the following model:
\[
f(G) = -C_1 \frac{G}{12} + C_2 \left( \frac{\pm i \sqrt{3}}{6l} \right)^{\frac{1}{3}} G^{\frac{1}{2}} = C'_1 G + C'_2 G^{\frac{1}{2}}. \tag{37}
\]
It is now easy to see from here that the GB terms could actually contribute in the matter dominated epoch of the universe evolution, provided the \( f(G) \) function is taken properly, which is certainly an interesting result.

### 2.3. Example 3

As third and last example we will now consider the case when the Hubble parameter and the matter energy density behave as
\[
H = e^{mN}, \quad \rho_m = b_1 + \frac{3}{k^2} e^{2mN} + \frac{96(m+1)}{5} b_2 e^{5mN}, \tag{38}
\]
where \( b_j = \text{const.} \). This case corresponds to the so-called phantom behaviour, which means that we have an effective EoS parameter \( w_{\text{eff}} < -1 \), and produces a superaccelerated phase that could end in some kind of future singularity (see [26] and [28]). We first write the Hubble parameter as a function of time,
\[
H(t) = \frac{H_0}{t - t_s}, \tag{39}
\]
where \( t_s \), usually called the Rip time, defines the moment when the Big Rip singularity occurs. We can find the \( f(G) \) function that reproduces this model by solving equation (10). This yields the solution
\[
F(G) = F(N) = b_1 + \frac{96(m+1)}{5} b_2 e^{5mN} = b_1 + \frac{96(m+1)}{5} b_2 \left( \frac{G}{24(m+1)} \right)^{\frac{1}{2}}. \tag{40}
\]

### 3. The \( F(R, G) \) model

Let us now consider a more general model for a class of modified Gauss–Bonnet gravity. This can be described by the following action:
\[
S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2k^2} F(R, G) + L_m \right]. \tag{41}
\]

Varying over \( g_{\mu\nu} \), the gravity field equations are obtained [13],
\[
0 = \kappa^2 T^{\mu\nu} + \frac{1}{2} R_{\mu\nu} F(G) - 2 F_G R R^{\mu\nu} + 4 F_G R_{\rho\sigma} R^{\mu\nu} - 2 F_G R^{\rho\sigma\tau} R_{\mu\nu} - 4 F_G R^{\rho\sigma\tau\nu} R_{\mu\sigma} + 2 (\nabla^\rho \nabla^\nu F_G) R - 2 g^{\mu\nu} (\nabla^2 F_G) R - 4 (\nabla_\rho \nabla^\rho F_G) R^{\mu\nu} - 4 (\nabla_\rho \nabla^\nu F_G) R^{\mu\rho} + 4 (\nabla^2 F_G) R^{\mu\nu} + 4 g^{\mu\nu} (\nabla_\rho \nabla_\sigma F_G) R^{\rho\sigma} - 4 (\nabla_\rho \nabla_\sigma F_G) R^{\mu\rho\sigma} - F_G R^{\mu\nu} + \nabla^\nu \nabla^\rho F - g^{\mu\nu} \nabla^2 F. \tag{42}
\]
In the case of a flat FRW universe, described by the metric (1), the first FRW equation yields
\begin{equation}
0 = \frac{1}{2} (GF_G - F - 24 H^3 F_{Gt}) + 3(\dot{H} + H^2) F_R - 3H F_{Rt} + \kappa^2 \rho_m.
\end{equation}
And from here, using the techniques developed in the previous section, it is clear that explicit \( F(R, G) \) functions can be reconstructed for given cosmological solutions.

3.1. de Sitter solutions

As is well known, the de Sitter solution is one of the most important cosmological solutions nowadays, since the current epoch has been observed to have an expansion that behaves approximately as de Sitter. This solution is described by an exponential expansion of the scale factor, which gives a constant Hubble parameter \( H(t) = H_0 \). By inserting it in the Friedmann equation (43), one finds that any kind of \( F(R, G) \) function can possibly admit de Sitter solutions, with the proviso that the following algebraic equation has positive roots for \( H_0 \):
\begin{equation}
0 = \frac{1}{2} (G_0 F_G (G_0) - F(G_0, R_0)) + 3H_0^2 F_R (R_0),
\end{equation}
with \( R_0 = 12H_0^2 \) and \( G_0 = 24H_0^4 \); here we have neglected the contribution of matter for simplicity. As was pointed out in [8] and [9] for the case of modified \( F(R) \) gravity, the de Sitter points are critical points for the Friedmann equations, which could explain the current acceleration phase as well as the inflationary epoch. This explanation can be extended to the action (41), so that any kind of function \( F(R, G) \) with positive real roots for equation (44) could in fact explain the acceleration epochs of the universe in exactly the same way a cosmological constant does.

3.2. Phantom dark energy

Let us now explore the cosmic evolution described by (38) in the context of the action (41). This solution reproduces a phantom behaviour, i.e. a superaccelerated expansion that, according to recent observations, our universe could be either in, or close to cross the phantom barrier. We can now proceed with the reconstruction method, as explicitly shown in the section above, and the \( F(R, G) \) function for the Hubble parameter (38) will be obtained. For simplicity, we consider the following subfamily of functions:
\begin{equation}
F(R, G) = f_1(G) + f_2(R).
\end{equation}
Correspondingly, the Friedmann equation (43) can be split into two equations, as
\begin{align}
0 &= -24 H^3 F_{GG} f_1 GG + G f_{1G} - f_1, \\
0 &= -3 H R f_{2RR} + 3(\dot{H} + H^2) f_2 R - \frac{1}{2} f_2 + \kappa^2 \rho_m.
\end{align}

For the example (38), that is \( H = e^{mN} \), the Ricci scalar and the Gauss–Bonnet terms take the following form:
\begin{align}
G &= 24(m + 1) e^{4mN} = 24(m + 1)H^4, \\
R &= 6(m + 2) e^{2mN} = 6(m + 2)H^2.
\end{align}
Hence, the first equation in (46) can be written in terms of \( G \) as
\begin{equation}
G^2 f_{1GG} = \frac{m + 1}{4m} G f_{1G} + \frac{m + 1}{4m} f_1 = 0.
\end{equation}
This is an Euler equation, easy to solve, and yields
\begin{equation}
f_1(G) = C_1 G^{\frac{m}{4m}} + C_2,
\end{equation}
where \( C_1 \) and \( C_2 \) are integration constants.
where $C_1,2$ are integration constants. In the same way, for the case being considered here, the second equation in (46), for $R$, takes the form

$$\frac{R^2}{2m} f_{2,RR} - \frac{C_1}{2m} R f_{2, R} + \frac{m + 2}{2m} f_2 - \frac{\kappa^2 (m + 2)}{m} \rho_m = 0. \tag{50}$$

In the absence of matter (\(\rho_m = 0\)) this is also an Euler equation, with solution

$$f_2(R) = k_1 R^{\mu_+} + k_2 R^{\mu_-}, \quad \mu_{\pm} = \frac{1 + \frac{m+1}{2m} \pm \sqrt{1 + \frac{\frac{m+1}{2m} + \frac{m+3}{2m}}{2}}}{2}. \tag{51}$$

and $k_{1,2}$ are integration constants. Then, the complete function $F(R, G)$, given in (45), is reconstructed (in the absence of matter) yielding solutions (49) and (51). The theory (3.12) belongs to the class of models with positive and negative powers of the curvature introduced in [29].

Let us now consider the case where matter is included. From the energy conservation equation (8) we have that, for a perfect fluid with constant EoS, $p_m = w_m \rho_m$, the solution is given by

$$\rho_m = \rho_{m0} e^{-\frac{3}{6}(1+w_m)} N. \tag{52}$$

By inserting the expression for $R$ (47) into this solution, we get

$$\rho_m = \frac{R}{6m + 2} e^{-\frac{3}{6}(1+w_m)} N. \tag{53}$$

Hence, we see that the solution for the Hubble parameter (38) can be easily recovered in the context of modified Gauss–Bonnet gravity. Nevertheless, it seems clear that, for more complex examples, one may not be able to solve the corresponding equations analytically and numerical analysis could be required.

4. Reconstruction of scalar-GB gravity

In this section we consider a four-dimensional action containing the Einstein–Hilbert part, a massless scalar field, and the Gauss–Bonnet term coupled to the scalar field. The corresponding action is

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - \xi(\phi) G + L_m \right]. \tag{54}$$

An action of this kind has been proposed as a model for DE in [12]. It can actually be related to modified GB gravity, as explained in [14] and [30]. The variation of this action over the metric $g_{\mu\nu}$ gives [5]

$$0 = \frac{1}{\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + \nabla^{\mu} \nabla_{\nu} \phi - \frac{1}{4} g^{\mu\nu} \partial_{\rho} \partial^{\rho} \phi \right. + \frac{1}{2} g^{\mu\nu} (-V + \xi G) - 2 \xi R^{\mu\nu} - 4 \xi R_{\mu}^{\nu} R_{\nu} - 2 \xi R_{\mu\rho\sigma\tau} R_{\rho\sigma\tau} \\
+ 4 \xi R_{\mu\nu}^{\rho\sigma\tau} R_{\rho\sigma\tau} + 2 (\nabla^\mu \nabla^\nu G) - 2 g^{\mu\nu} (V^2 + \xi G) R - 4 (\nabla_{\mu} V^\nu - 4 (\nabla_{\nu} V^\mu) R^{\mu\nu} \\
- 4 (\nabla_{\nu} V^\mu) R_{\mu\rho} + 4 (\nabla^{\mu} V^\nu) R^{\mu\nu} + 4 g^{\mu\nu} (\nabla_{\rho} \nabla_{\sigma} \xi) R_{\rho\sigma} + 4 (\nabla_{\rho} \nabla_{\sigma} \xi) R^{\mu\nu} \phi \right]. \tag{55}$$

In the FRW universe case, these equations and the equation of motion for the scalar become [14]
\[0 = -\frac{3}{\kappa^2} H^2 + \frac{1}{2} \dot{\phi}^2 + V(\phi) + 24H^3 \dot{\xi} + \rho_m, \quad (56)\]

\[0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) + \frac{1}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \dot{\xi} - 16H(\dot{H} + H^2) \dot{\xi} + \rho_m, \quad (57)\]

\[0 = \dot{\phi} + 3H\dot{\phi} + V'_\phi + \xi'_\phi G. \quad (58)\]

Note that, as a consequence of the Bianchi identity, equation (58) is satisfied automatically.

From equations (56–58), we get [14]

\[0 = \frac{2}{\kappa^2} \dot{H} + \dot{\phi}^2 - 8a \left( \frac{H^2}{a} \dot{\xi} \right). \quad (59)\]

For the variables \(\phi\) and \(\xi\), (56)–(58) constitute a system of second-order differential equations. It is useful to reduce this system to a first-order set of differential equations. This can be immediately achieved by introducing new variables, as

\[u = \dot{\phi}^2, \quad v = \dot{\xi}. \quad (60)\]

Then, we obtain the following first-order system:

\[0 = -\frac{3}{\kappa^2} H^2 + \frac{1}{2} u + V + 24H^3 v, \quad (61)\]

\[0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) + \frac{1}{2} u - V - 8H^2 \dot{v} - 16H(\dot{H} + H^2)v, \quad (62)\]

\[0 = \dot{u} + 6Hu + 2V + 2Gv. \quad (63)\]

From here it is easy to explore the stability and number of attractors of the physical model. To do so we rewrite the system (61)–(63) as

\[\dot{H} = \frac{1}{24} GH^{-2} - H^2, \quad (64)\]

\[\dot{u} = -(6Hu + 2V + 2Gv), \quad (65)\]

\[\dot{v} = \frac{1}{8H^2} \left[ \frac{2}{\kappa^2} \left( \frac{1}{24} GH^{-2} - H^2 \right) + u - (16H \dot{H} - 8H^3)v \right]. \quad (66)\]

Let \(q = u + 2V\). Then, this can be put as

\[0 = -\frac{3}{\kappa^2} H^2 + \frac{1}{2} q + 24H^3 v, \quad (67)\]

\[0 = \frac{1}{\kappa^2} (2\dot{H} + 3H^2) + \frac{1}{2} q - 2V + 8H^2 \dot{v} - 16H(\dot{H} + H^2)v, \quad (68)\]

\[0 = \dot{q} + 6Hq - 12HV + 2Gv, \quad (69)\]

and for (64)–(66)

\[\dot{H} = \frac{1}{24} GH^{-2} - H^2, \quad (70)\]

\[\dot{q} = 12HV - 6Hq - 2Gv, \quad (71)\]

\[\dot{v} = \frac{1}{8H^2} \left[ \frac{2}{\kappa^2} \left( \frac{1}{24} GH^{-2} - H^2 \right) + q - 2V - (16H \dot{H} - 8H^3)v \right]. \quad (72)\]
Let us now rewrite equations (56)–(58) in terms of the number of e-foldings $N$. We have
\begin{equation}
0 = -\frac{3}{k^2} H^2 + \frac{1}{2} H^2 \phi_N^2 + V(\phi) + 24 H^4 \xi_{1N},
\end{equation}
\begin{equation}
0 = \frac{1}{k^2} (2 H N + 3 H^2) + \frac{1}{2} H^2 \phi_N^2 - V(\phi) - 8 H^4 \xi_{1NN} - 8 H (2 H N + 3 H^2) H \xi_{1N},
\end{equation}
\begin{equation}
0 = H^2 \phi_N \phi_{NN} + (H_N + 3 H^2) \phi_N^2 + V_N + \xi_{1N} G.
\end{equation}

If we introduce new variables
\begin{equation}
0 = -\frac{3}{k^2} H^2 + \frac{1}{2} H^2 h + V(\phi) + 24 H^4 p,
\end{equation}
\begin{equation}
0 = \frac{1}{k^2} (2 H N + 3 H^2) + \frac{1}{2} H^2 h - V - 8 H^4 p_N - 8 H (2 H N + 3 H^2) H p,
\end{equation}
\begin{equation}
0 = H^2 h_N + 2 (H_N + 3 H^2) h + 2 V_N + 2 G p,
\end{equation}
the system (73)–(75) takes the form
\begin{equation}
0 = -\frac{3}{k^2} H^2 + \frac{1}{2} H^2 h + V(\phi) + 24 H^4 p,
\end{equation}
\begin{equation}
0 = \frac{1}{k^2} (2 H N + 3 H^2) + \frac{1}{2} H^2 h - V - 8 H^4 p_N - 8 H (2 H N + 3 H^2) H p,
\end{equation}
\begin{equation}
0 = H^2 h_N + 2 (H_N + 3 H^2) h + 2 V_N + 2 G p.
\end{equation}

To summarize, the original system of equations has been written in several ways that can be used to construct solutions. Needless to say, all the corresponding solutions are equally valid in every form of the system. Here we will find solutions for the one given by equations (61)–(63). To do that, we rewrite them in terms of the e-folding variable $N$
\begin{equation}
0 = -\frac{3}{k^2} H^2 + \frac{1}{2} H^2 u + V + 24 H^4 v,
\end{equation}
\begin{equation}
0 = \frac{1}{k^2} (2 H N + 3 H^2) + \frac{1}{2} u - V - 8 H^4 v_N - 16 H^2 (H_N + H) v,
\end{equation}
\begin{equation}
0 = H u_N + 6 H u + 2 H V_N + 2 G v.
\end{equation}
This has the following solution:
\begin{equation}
u = 8 H^4 W_{NN} + 8 H^3 (3 H_N - H) W_N - \frac{2}{k^2} H H_N ,\end{equation}
\begin{equation}
V = -4 H^4 W_{NN} - (12 H^2 H_N + 20 H^3) H W_N + \frac{3}{k^2} H^2 + \frac{1}{k^2} H H_N ,
\end{equation}
where $W = W(N)$ is some function of $N$. At the same time, $\phi$, $\xi$ are given by
\begin{equation}
\phi = \int dN \sqrt{8 H^2 W_{NN} + 8 H (3 H_N - H) W_N - \frac{2}{k^2} (\ln H)_N} ,
\end{equation}
\begin{equation}
\xi = W.
\end{equation}

Let us now consider several examples.

(i) As the first one consider the case when $W = 0$, which corresponds to the Einstein-scalar gravity. In this case, we have
\begin{equation}
0 = -\frac{2}{k^2} H H_N , \quad v = 0, \quad V = \frac{3}{k^2} H^2 + \frac{1}{k^2} H H_N .
\end{equation}
(ii) A second example is the model
\[ W = v, \]  
where \( v = \text{const} \). Here the solution of the system (83)–(85) is
\[ u = \frac{2}{\kappa^2} H H_N, \quad v = 0, \quad V = \frac{3}{\kappa^2} H^2 + \frac{1}{\kappa^2} H H_N. \]  
It corresponds to the so-called Einstein–Gauss–Bonnet gravity.

(iii) The third example will be
\[ W = \mu N + v \]  
where \( \mu, v \) are some constants. Then, the solution of the system (83)–(85) is given by
\[ u = 8 \mu H^3 (3H_N - H) - \frac{2}{\kappa^2} H H_N, \]  
\[ v = \mu H(= \mu N), \]  
\[ V = -\mu (12H^2 H_N + 20H^3) H + \frac{3}{\kappa^2} H^2 + \frac{1}{\kappa^2} H H_N. \]  
The next step is finding the explicit forms of \( \phi(t), H(t), \xi_1(t), V(t) \). To this end we consider the case when \( \phi \) has a kink form, that is, it obeys the (0+1)-dimensional Sine-Gordon equation
\[ \ddot{\phi} = \gamma^2 \sin \phi, \]  
where \( \gamma = \text{const} \)
\[ \phi = 4 \arctan[\exp(-\gamma(t-t_0))], \quad u = 2\gamma^2 (1 - \cos \phi) = \frac{4\gamma^2}{1 + \exp(\gamma(t-t_0))}. \]  
In this case \( W \) obeys the equation
\[ 8H^4 W_N + 8H^3 (3H_N - H) W_N - \frac{2}{\kappa^2} H H_N = 2\gamma^2 (1 - \cos \phi). \]  
We will solve this equation for the case (91) and \( H = \alpha + \beta \exp(N) \). There, \( y \) satisfies
\[ \epsilon y^2 + \delta y + \sigma = (12\mu \eta - 8\mu) y^2 + \left(12\mu \eta - 16\alpha \mu - \frac{\eta}{\kappa}\right) y - (u + 8\mu \alpha^2) = 0, \]  
which has the solution
\[ y(t) = y(\phi) = y_i = \frac{\delta \pm \sqrt{\delta^2 - 4\epsilon \sigma}}{2\epsilon}. \]  
Hence, we have
\[ N = N_i(t) = N_i(\phi) = \frac{1}{\eta} \ln \frac{y_i}{\beta} \]  
and
\[ H = H(t) = H(\phi) = \sqrt{\alpha + y_i}. \]  
We have thus seen that, from the action (54), any of the usual cosmologies can be obtained. The system of equations (83)–(85) provides a quite simple setup to reproduce any kind of cosmological solution, as has been clearly illustrated with the three examples above, where for a given Hubble parameter the scalar Gauss–Bonnet theory has been constructed.
5. Conclusions

In the present paper, several types of DE cosmologies in modified GB gravity—which can be viewed as being inspired by string considerations, [14]—have been investigated. We have studied different kinds of theories, in all of which the GB invariant plays an important role in the corresponding equations. First, we have shown that GR plus some function of the GB term provides a very powerful theory, where no sort of DE is actually needed to reproduce the standard ΛCDM cosmology.

More general theories have been considered too, as in section 3, where an action depending on a function of the Ricci scalar and GB invariant has been studied. In this case the DE behaviour can also be reproduced, while its extra degrees of freedom provide us with a powerful method to constraint the theory, in order to avoid the violation of the local gravity tests.

In the same way, the scalar GB theory studied in the last section adequately reproduces any kind of cosmological solution. As a follow-up, we have shown, with the help of several particular examples corresponding to explicit choices of the functions \( f(G) \), \( F(R, G) \) or the scalar field, that, in principle, any cosmic evolution can be obtained from these models, which includes the unification of early-time inflation with the late-time acceleration coming from astronomical observations. To finish, it has been indicated that DE cosmologies in a more general (and complicated) \( F(R, G) \) framework—generically requiring numerical analysis—can be reconstructed in a similar fashion, too.

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