Orthogonality Spaces Associated with Posets

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Abstract
An orthogonality space is a set equipped with a symmetric, irreflexive relation called orthogonality. Every orthogonality space has an associated complete ortholattice, called the logic of the orthogonality space. To every poset, we associate an orthogonality space consisting of proper quotients (that means, nonsingleton closed intervals), equipped with a certain orthogonality relation. We prove that a finite bounded poset is a lattice if and only if the logic of its orthogonality space is an orthomodular lattice. We prove that that a bounded poset is a chain if and only if the logic of the associated orthogonality space is a Boolean algebra.

Keywords Orthogonality space · Dacey space · Poset · Lattice

1 Introduction
In his PhD. thesis [1], Dacey explored the notion of “abstract orthogonality”, by means of sets equipped with a symmetric, irreflexive relation \( \perp \). He named these structures orthogonality spaces. Every orthogonality space has an orthocomplementation operator \( X \mapsto X \perp \) defined on the set of all its subsets. Dacey proved that \( X \mapsto X \perp \) is a closure operator and that the set of all closed subsets of an orthogonality space forms a complete ortholattice, which we call the logic of an orthogonality space. Moreover, Dacey gave a characterization of orthogonality spaces such that their logic is an orthomodular lattice. The orthogonality spaces of this type are nowadays called Dacey spaces.

Let us remark that (as a special case of a polarity between two sets) the idea of an orthogonality space and its associated lattice of closed sets appears already in the classical monograph [2, Section V.7]. Nevertheless, Dacey was probably the first person that explored the orthomodular law in this context.

Since orthogonality spaces are nothing but (undirected, simple, loopless) graphs, it is perhaps not surprising that the \( X \mapsto X \perp \) mapping appears under the name neighbourhood operator in graph theory. Implicitly, the logic associated with the neighbourhood operator was used in the seminal paper [3], in which Lovász proved Kneser’s conjecture. Explicitly,
they were used by Walker in [4]. In that paper, the ideas from the [3] paper were generalized and reformulated in the language of category theory.

For an overview of the results on orthogonality spaces and more general test spaces, see [5]. See also [6, 7] for some more recent results on orthogonality spaces.

In this paper, we construct an orthogonality space \((Q^+(P), \perp)\) from every poset \(P\). The elements of \(Q^+(P)\) are pairs \((a, b)\) of elements of \(P\) with \(a < b\), which we call quotients. The orthogonality relation \(\perp\) on \(Q^+(P)\) can be loosely described as “the line segments representing the quotients are above/below each other in the Hasse diagram of \(P\)” (see Fig. 3).

Then we examine how are the properties of \(P\) reflected in the structure of the orthogonality space \((Q^+(P), \perp)\) and its logic. In the last section, we describe some straightforward connections between the logic of \(Q^+(P)\) and two constructions: the Kalmbach construction and the MacNeille completion.

### 2 Preliminaries

#### 2.1 Posets, Lattices

A poset \(P\) is lower bounded if it has the smallest element, which we denote by 0. Similarly, the greatest element of an upper bounded poset is denoted by 1. A poset that is both upper and lower bounded is called bounded.

If \(X\) is a subset of a poset \(P\), then \(a\) is a lower bound of \(X\) if and only if for all \(x \in X\), \(a \leq x\). For every set \(X \subseteq P\), we write \(X^\downarrow\) for the set of all lower bounds of \(X\). The notions of an upper bound and \(X^\uparrow\) are defined dually. For a singleton subset \(\{x\}\), we abbreviate \(
\{x\}^\downarrow = x^\downarrow \text{ and } \{x\}^\uparrow = x^\uparrow.\)

Note that \(\emptyset^\uparrow = \emptyset^\downarrow = P\).

A poset \(P\) is a complete lattice if and only if for all subsets \(X\) of \(P\), \(X^\uparrow\) has a smallest element, denoted by \(\bigvee X\) and called the smallest upper bound of \(X\). In every complete lattice \(P\), every subset \(X \subseteq P\) has a greatest lower bound, denoted by \(\bigwedge X\). We say that a poset \(P\) is a lattice if and only if for all \(x, y \in P\), \(\bigwedge\{x, y\}\) and \(\bigvee\{x, y\}\) exist. In a lattice, we write \(x \land y = \bigwedge\{x, y\}\) and \(x \lor y = \bigvee\{x, y\}\). A bounded lattice \(P\) is complemented if for every \(x \in P\) there is a \(x^* \in P\) such that \(x \land x^* = 0\) and \(x \lor x^* = 1\). The element \(x^*\) is then called a complement of \(x\).

A subset \(I\) of a poset \(P\) is an order ideal of \(P\) if for every \(x \in I\), \(x^\downarrow \subseteq I\). The dual notion is order filter. An order ideal \(I\) of a poset \(P\) is closed if and only if \(I = I^\downarrow\). It is well known [8] that every poset \(P\) embeds into a complete lattice, called the Dedekind-MacNeille completion of \(P\), denoted by \(\hat{P}\). The poset \(\hat{P}\) consists of all closed order ideals of \(P\), ordered by inclusion. There is a canonical embedding \(\eta_P : P \to \hat{P}\), given by the rule \(\eta_P(x) = x^\downarrow\).

It is well-known that \(P\) is a complete lattice if and only if \(\eta_P\) is surjective. Since a finite poset is a lattice if and only if it is a complete lattice, this gives us a characterization of finite lower bounded posets that are not lattices, as follows.

**Lemma 2.1** A finite lower bounded poset \(P\) is not a lattice if and only if there is a closed order ideal \(I\) with at least two maximal elements.

**Proof** Suppose that there is a closed order ideal \(I\) of \(P\) with two distinct maximal elements \(a, b\). We shall prove that \(a \lor b\) does not exist in \(P\). Assume that \(a \lor b\) exists in \(P\). Since \(a, b\) are distinct and incomparable, \(a \lor b > a\) and \(a \lor b > b\). Clearly,

\[
a \lor b \in \{a, b\}^\downarrow \subseteq I^\downarrow = I,
\]

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however, this contradicts the maximality of \(a, b\) in \(I\).

Suppose that \(P\) is not a lattice. Then there is a closed order ideal \(I\) that is not of the form \(x^\downarrow\), that means, the number of maximal elements of \(I\) is not equal to 1. Since \(P\) is lower bounded, \(\emptyset^\uparrow = P^\uparrow = \{0\}\), hence the empty order ideal is not closed. Therefore, the number of maximal elements of \(I\) must be at least 2.

\[\square\]

### 2.2 Ortholattices, Orthomodular Lattices

An ortholattice is a bounded lattice \((L, \lor, \land, 0, 1, \perp)\) equipped with an antitone mapping called orthocomplementation \(\perp: L \to L\) such that

- \(0^\perp = 1, 1^\perp = 0\)
- \(x^{\perp\perp} = x\)
- \((x \lor y)^\perp = x^\perp \land y^\perp\)
- \((x \land y)^\perp = x^\perp \lor y^\perp\)
- \(x \land x^\perp = 0\)
- \(x \lor x^\perp = 1\)

An ortholattice is \(L\) an orthomodular lattice if it satisfies the orthomodular law

\[
x \leq y \implies y = x \lor (y \land x^\perp),
\]

for all \(x, y \in L\).

Similarly as for modular or distributive lattices, orthomodular lattices can be characterized among ortholattices by absence of a “forbidden sublattice”.

**Proposition 2.2** (see for example [9, Theorem 1.3.2]) An ortholattice \(L\) is orthomodular if and only if it does not contain a sub-ortholattice isomorphic to the benzene ring (see Fig. 1).

A Boolean algebra is a distributive ortholattice. We shall use the following well-known characterization.

**Proposition 2.3** [10] An ortholattice \(L\) is a Boolean algebra if and only if, for all \(a, b \in L\),

\[
a \land b = 0 \implies a \leq b^\perp. \quad (1)
\]

**Proof** Clearly, every Boolean algebra satisfies (1).

![Fig. 1 The benzene ring](image)
By [9, Proposition 1.3.7] an ortholattice is a Boolean algebra iff it is uniquely complemented. We shall prove that Eq. 1 implies that \( L \) is uniquely complemented. Let \( y \) be a complement of \( x \). Since \( x \land y = 0 \), \( y \leq x^\perp \) by Eq. 1. Since \( x \lor y = 1 \),

\[
0 = 1^\perp = (x \lor y)^\perp = x^\perp \land y^\perp.
\]

By Eq. 1 \( x^\perp \land y^\perp = 0 \) implies that \( x^\perp \leq y^\perp \perp = y \). Hence \( y = x^\perp \) and \( y \) is unique. \( \square \)

3 Orthogonality Spaces

Definition 3.1 Let \( V \) be a set, write \( \mathcal{P}(V) \) for the powerset of \( V \). A mapping \( \text{cl} : \mathcal{P}(V) \to \mathcal{P}(V) \) is a closure operator if it satisfies the conditions, for all \( X, Y \in \mathcal{P}(V) \)

1. \( X \subseteq \text{cl}(X) \),
2. \( X \subseteq Y \implies \text{cl}(X) \subseteq \text{cl}(Y) \),
3. \( \text{cl}(\text{cl}(X)) = \text{cl}(X) \).

The pair \((V, \text{cl})\) is then called a closure space.

In a closure space \((V, \text{cl})\), a set \( X \subseteq V \) is called closed if \( \text{cl}(X) = X \). It is easy to see that the set of all closed subsets of a closure space forms a complete lattice under inclusion.

Definition 3.2 An orthogonality space \((O, \perp)\) is a set \( O \) equipped with an irreflexive symmetric binary relation \( \perp \subseteq O \times O \).

Let \((O, \perp)\) be an orthogonality space. We say that two elements \( x, y \in O \) with \( x \perp y \) are orthogonal. A subset \( B \) of \( O \) is pairwise orthogonal if every distinct pair of elements of \( B \) is orthogonal. Two subsets \( X, Y \) of \( O \) are orthogonal (in symbols \( X \perp Y \)) if \( x \perp y \), for all \( x \in X \) and \( y \in Y \). We write \( y \perp X \) for \( \{y\} \perp X \). For every subset \( X \) of \( O \), we write

\[
X^\perp = \{y \in O \mid y \perp X\}
\]

so that \( X^\perp \) is the greatest subset of \( O \) orthogonal to \( X \). Note that the mapping \( X \mapsto X^\perp \) on the \((\mathcal{P}(O), \subseteq)\) is antitone. Clearly, for every family \((X_i)_{i \in H}\) of subsets of \( O \),

\[
(\bigcup_{i \in H} X_i)^\perp = \bigcap_{i \in H} X_i^\perp.
\]

Since the orthocomplementation is antitone, the mapping \( X \mapsto X^\perp \) is isotone. Moreover, for every \( X \subseteq O \) we have \( X \subseteq X^{\perp \perp} \) and \( X^\perp = X^{\perp \perp \perp} \). Consequently, \( X \mapsto X^\perp \) is a closure operator. Note that \( s \in X^{\perp \perp} \) if and only if, for all \( a \in O \),

\[
a \perp X \implies s \perp a.
\]

We say that a subset \( X \) of \( O \) is orthoclosed if \( X = X^{\perp \perp} \). The set of all orthoclosed subsets of an orthogonality space forms a complete ortholattice \( L(O, \perp) \), with meets given by intersection and joins given by

\[
X \lor Y = (X \cup Y)^{\perp \perp}
\]

Note that \( X \lor Y = (X^\perp \cap Y^\perp)^\perp \). The smallest element of \( L(O, \perp) \) is \( \emptyset \) and the greatest element of \( L(O, \perp) \) is \( O \). We say that \( L(O, \perp) \) is the logic of \( O \).

Let \((O, \perp)\) be an orthogonality space, let \( X \) be an orthoclosed subset of \( O \). A maximal pairwise orthogonal subset of \( X \) is called a basis of \( X \).

In his PhD. thesis, J.C. Dacey proved the following theorem.
Theorem 3.3 [1] Let \((O, \perp)\) be an orthogonality space. Then \(L(O, \perp)\) is an orthomodular lattice if and only if for every orthoclosed subset \(X\) of \(O\) and every basis \(B\) of \(X\), \(X = B^\perp\).

An orthogonality space satisfying the condition of Theorem 3.3 is called a Dacey space. A orthoclosed subset \(X\) in an orthogonality space such that \(X = B^\perp\), for every basis \(B\) of \(X\) is called a Dacey set.

Example 3.4 On Fig. 2, there are two orthogonality spaces \(G_1, G_2\) depicted as graphs. We have \(G_1 = (\{a, b, c\}, \perp)\), where the orthogonal pairs are \(a \perp b\) and \(b \perp c\). Similarly, \(G_2 = (\{a, b, c, d\}, \perp)\) and the orthogonal pairs are \(a \perp b\), \(b \perp c\) and \(c \perp d\).

The logic of \(G_1\) is a 4-element Boolean algebra, hence an orthomodular lattice, so \(G_1\) is Dacey. The logic of \(G_2\) is a benzene ring, which is not an orthomodular lattice. Therefore, by Theorem 3.3, \(G_2\) is not Dacey. To see this directly, consider the orthoclosed subset \(\{b, d\}\). Then \(\{b\}\) is a basis of \(\{b, d\}\). However,

\[
\{b\}^\perp = \{a, c\}^\perp = \{b\} \neq \{b, c\},
\]

so \(\{b, d\}\) is a non-Dacey orthoclosed set.

Lemma 3.5 Let \(X\) be an orthoclosed subset of an orthogonality space. The following are equivalent.

(a) \(X\) is Dacey.
(b) For every basis \(B\) of \(X\), \(B^\perp = X^\perp\).
(c) For every basis \(B\) of \(X\), \(B^\perp \subseteq X^\perp\).

Proof
(a) \(\implies\) (b): \(B^\perp = B^{\perp\perp\perp} = X^\perp\).
(b) \(\implies\) (c): Trivial.
(c) \(\implies\) (a): Since \(B^\perp \subseteq X^\perp\), \(X = X^{\perp\perp} \subseteq B^{\perp\perp}\). Since \(B \subseteq X\), \(B^{\perp\perp} \subseteq X^{\perp\perp} = X\). Since \(X \subseteq B^{\perp\perp}\) and \(B^{\perp\perp} \subseteq X\), \(B^{\perp\perp} = X\) and \(X\) is Dacey. \(\square\)

Lemma 3.6 Let \(X_1, X_2\) be two Dacey subsets of an orthogonality space \((O, \perp)\) with \(X_1 \perp X_2\). If \(X_1 \cup X_2\) is orthoclosed, then \(X_1 \cup X_2\) is Dacey.

Proof Let \(B\) be a basis of the orthoclosed set \(X_1 \cup X_2\). We claim that \(B \cap X_1\) is a basis of \(X_1\). Since \(B\) is pairwise orthogonal, \(B \cap X_1\) is pairwise orthogonal. Suppose that \(B \cap X_1\) is
not a basis of $X_1$, and let $x \in (X_1 \setminus B)$ be such that $x \perp (B \cap X_1)$. Then $X_1 \perp X_2$ implies that $x \perp X_2 \supseteq B \cap X_2$ and thus

$$x \perp (B \cap X_1) \cup (B \cap X_2) = B \cap (X_1 \cup X_2) = B.$$ 

This contradicts the maximality of $B$ in $X_1 \cup X_2$, hence $B \cap X_1$ is a basis of $X_1$. Similarly, $B \cap X_2$ is a basis of $X_2$.

Since $X_1$ and $X_2$ are Dacey, we may use Lemma 3.5 (b) to compute

$$B^\perp = (B \cap (X_1 \cup X_2))^\perp = ((B \cap X_1) \cup (B \cap X_2))^\perp = (B \cap X_1)^\perp \cap (B \cap X_2)^\perp = (X_1 \cup X_2)^\perp$$

and by Lemma 3.5 (b), we see that $X_1 \cup X_2$ is Dacey.

\[\square\]

### 4 Orthogonality Spaces Associated with Posets

In this section, we will associate to every poset $P$ an orthogonality space. For a finite bounded poset, we will prove that this orthogonality space is Dacey if and only if the poset is a lattice.

For a poset $P$, we write $Q^+(P)$ for the set of all pairs $(a, b) \in P \times P$ with $a < b$. In lattice theory, the elements of $Q^+(A)$ are called proper quotients.

An element $(a, b) \in Q^+(P)$ is denoted by $[a < b]$. A quotient $(a, b)$ is usually denoted by $b/a$ but $[a < b]$ will be more handy for our purposes. For $[a < b], [c < d] \in Q^+(P)$, we write $[a < b] \leq [c < d]$ if $c \leq a < b \leq d$. Note that $(Q^+(P), \leq)$ is a subposet of the poset $P^* \times P$, where $P^*$ is the dual poset of $P$ and that $(Q^+(P), \leq)$ is isomorphic to the poset of non-singleton closed intervals in $P$, ordered by inclusion.

For $[a < b], [c < d] \in Q^+(P)$ we write $[a < b] \perp [c < d]$ if $b \leq c$ or $d \leq a$, see Fig. 3. Clearly, $\perp$ is symmetric and irreflexive, so $(Q^+(P), \perp)$ is an orthogonality space.

We equipped $Q^+(P)$ with two relations: the orthogonality $\perp$ and the partial order $\leq$. Various classes of posets equipped with an orthogonality relation were considered in [11].

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**Fig. 3** The orthogonality relation in $Q^+(P)$
In the terminology of that paper, every \((Q^+(P), \perp, \leq)\) is a poset with a strong degenerate orthogonality.

**Lemma 4.1** Let \(P\) be a poset, let
\[
B = \{[a_1 < a_2], \ldots, [a_{2n-1} < a_{2n}]\}
\]
be a finite pairwise orthogonal subset of \(Q^+(P)\). Then \(a_1, a_2, \ldots, a_{2n}\) are pairwise comparable.

**Proof** Whenever \([a < b] \perp [c < d]\), the elements in the set \(\{a, b, c, d\}\) are pairwise comparable.

Observe that \([u < v] \leq [a < b] \perp [c < d]\) implies that \([u < v] \perp [c < d]\).

**Lemma 4.2** Let \(P\) be a poset. Every orthoclosed subset of \((Q^+(P), \perp)\) is a lower set in \((Q^+(P), \leq)\).

**Proof** Let \(X\) be an orthoclosed subset of \((Q^+(P), \perp)\). Let \([a < b] \in X\) and \([u < v] \leq [a < b]\). We need to prove that \([u < v] \notin X\), that means for all \([c < d] \in Q^+(P), [c < d] \perp X\) implies \([c < d] \perp [u < v]\). However, this is easy: if \([c < d] \perp X\), then \([c < d] \perp [a < b]\) and since \([u < v] \leq [a < b], [c < d] \perp [u < v]\).

Let \(U\) be a subset of a poset \(P\). We denote
\[
\tau(U) = \{[a < b] \in Q^+(P) \mid b \in U\}
\]
\[
\beta(U) = \{[a < b] \in Q^+(P) \mid a \in U\}.
\]

Note that, for all \([x < y] \in Q^+(P), [x < y] = \tau(x^\downarrow) \cup \beta(y^\uparrow)
\]
\[
[x < y] = \beta(x^\uparrow) \cap \tau(y^\downarrow).
\]

**Lemma 4.3**

(a) For every lower set \(I\) of a lower-bounded poset \(P\), \(\tau(I) = \beta(I^\uparrow)\).

(b) For every upper set \(F\) of a upper-bounded poset \(P\), \(\beta(F) = \tau(F^\downarrow)\).

**Proof**

(a) Let us prove that \(\tau(I) \subseteq \beta(I^\uparrow)\). Let \([x < y] \in \tau(I)\). We need to prove that \([x < y] \in \beta(I^\uparrow)\), that means that, for all \(b \in I, b \leq x\). If \(b = 0\), there is nothing to prove, so let us assume that \(0 < b\). Then \([0 < b] \in \tau(I)\), so \([0 < b] \perp [x < y]\) and this clearly implies \(b \leq x\).

For the opposite inclusion, let \([x < y] \in \beta(I^\uparrow)\), meaning that \(x \in I^\uparrow\). For all \([a < b] \in \tau(I), b \in I\) and thus \(b \leq x\). Therefore, \([a < b] \perp [x < y]\) and hence \([x < y] \in \tau(I)^\perp\).

(b) Dually.

**Corollary 4.4** For every lower set \(I\) of a bounded poset \(P\), \(\tau(I)^\perp = \tau(I^\uparrow)\)

**Proof** Using (a) and (b) of Lemma 4.3,
\[
\tau(I)^\perp = \beta(I^\uparrow)^\perp = \tau(I^\downarrow)\]
Corollary 4.5 Let $P$ be a poset. For every $[x < y] \in Q^+(P)$,

$$[x < y] \perp = [x < y] \downarrow$$

Proof By Lemma 4.3,

$$[x < y] \perp = (\tau(x) \cup \beta(y)) = (\tau(x) \downarrow) \cap (\beta(y) \downarrow) = (\beta(x) \cap \tau(y)) = [x < y] \downarrow$$

Lemma 4.6 Let $P$ be a finite poset. For every $[x < y] \in Q^+(P)$, $[x < y] \downarrow$ is Dacey.

Proof By Corollary 4.5, $[x < y] \downarrow$ is orthoclosed.

Let $B$ be a basis of $[x < y] \downarrow$. Since $B$ is pairwise orthogonal, we may write

$$B = \{[c_1 < d_1], \ldots, [c_k < d_k]\}$$

and by Lemma 4.1 we may assume without loss of generality that $d_i \leq c_{i+1}$, for $i \in \{1, \ldots, k - 1\}$. If $x < c_1$, then $[x < c_1] \perp B$ and $B$ is not maximal pairwise orthogonal, so $x = c_1$. Similarly, $d_k = y$ and, for all $i \in \{1, \ldots, k - 1\}, d_i = c_{i+1}$. It is now obvious that $[p < q] \in B^\perp$ implies either $q \leq c_1 = x$ or $y = d_k \leq p$, so $[p < q] \in [x < y] \perp = ([x < y] \downarrow \perp)$. Therefore, $B^\perp \subseteq ([x < y] \downarrow \perp)$ and by Lemma 3.5 (c), this implies that $[x < y] \downarrow$ is Dacey.

Lemma 4.7 Let $P$ be a lattice and let $X$ be an orthoclosed subset of $Q^+(P)$. Let $[a < b], [c < d] \in X$ be such that $[a < b] \not\perp [c < d]$. Then

(a) $([a < b], [c < d]) \perp = \beta((b \vee d) \uparrow) \cup \tau((a \wedge c) \downarrow)$

(b) $[a \wedge c < b \vee d] \in X$

Proof

(a) The inclusion $([a < b], [c < d]) \perp \supseteq \beta((b \vee d) \uparrow) \cup \tau((a \wedge c) \downarrow)$ is clear, let us prove the other one. Let $[x < y] \in ([a < b], [c < d]) \perp$. Since $[x < y] \perp [a < b]$, either $b \leq x$ or $y \leq a$.

Suppose that $b \leq x$. Since $[x < y] \perp [c < d]$, either $y \leq c$ or $d \leq x$. If $y \leq c$ then $a < b \leq x < y \leq c < d$, which contradicts $[a < b] \not\perp [c < d]$. Therefore, $d \leq x$, so $b \vee d \leq x$ meaning that $[x < y] \in \beta((b \vee d) \uparrow)$.

Similarly $y \leq a$ implies that $y \leq a \wedge c$, hence $[x < y] \in \tau((a \wedge c) \downarrow)$.

(b) Clearly,

$$\beta((b \vee d) \uparrow) \cup \tau((a \wedge c) \downarrow) = [a \wedge c < b \vee d] \perp$$

By (a), this implies that

$$[a \wedge c < b \vee d] \perp = ([a < b], [c < d]) \perp \supseteq X \perp.$$  \hspace{1cm} (2)

If we apply the orthocomplementation on Eq. 2 and use Corollary 4.5 we obtain

$$[a \wedge c < b \vee d] \in [a \wedge c < b \vee d] \perp = [a \wedge c < b \vee d] \perp \subseteq X \perp = X.$$  \hspace{1cm} \square

Lemma 4.8 Let $P$ be a poset, let $X$ be an orthoclosed subset of $Q^+(P)$. If $[p < q], [q_1 < r] \in X$ and $q_1 \leq q_2$ and $p < r$, then $[p < r] \in X$.  \hspace{1cm} \square
Proof Let \([x < y] \in Q^+(P)\) be such that \([x < y] \in [(p < q_2), [q_1 < r]]\). We shall prove that this implies \(y \leq p\) or \(r \leq x\). Indeed, assume that \(y \not\leq p\) and \(r \not\leq x\). Then \([x < y] \perp [p < q_2]\) implies \(q_2 \leq x\) and \([x < y] \perp [q_1 < r]\) implies \(y \leq q_1\). Hence \(q_2 \leq x < y \leq q_1\) and this contradicts \(q_1 \leq q_2\).

We have proved that \(y \leq p\) or \(r \leq x\) and each of these implies \([x < y] \perp [p < r]\). Therefore,

\[ [p < r] \in [(p < q_2), [q_1 < r]] \perp \subseteq X \perp \subseteq X \]

\qed

Let \(P\) be a finite poset. We say that a lower subset \(X\) of \((Q^+(P), \leq)\) is of chain type if for all pairs \([a < b], [c < d]\) of distinct maximal elements of \(X\) of \(Q^+(P)\), we have \([a < b] \perp [c < d]\) and either \(b < c\) or \(d < a\).

Lemma 4.9 Let \(P\) be a finite poset. Every \(X \subset Q^+(P)\) of chain type is Dacey.

Proof Let us prove that \(X\) is orthoclosed. Write

\[ \text{max}(X) = \{[a_1 < a_2], \ldots, [a_{2n-1} < a_{2n}]\} \]

for the set of maximal elements of \(X\). Since \(X\) is of chain type, we may assume without loss of generality that \(a_1 < a_2 < a_3 < \cdots < a_{2n-1} < a_{2n}\). Obviously,

\[ X = [a_1 < a_2] \cup \cdots \cup [a_{2n-1} < a_{2n}] \]

Note that

\[ \text{max}(X) \perp = \tau(a_1) \cup [a_2 < a_3] \cup [a_4 < a_5] \cup \cdots \cup [a_{2n-2} < a_{2n-1}] \cup \beta(a_{2n}) \]

It is easy to see that \(X = \text{max}(X) \perp \perp\). In particular, \(X\) is orthoclosed.

To prove that \(X\) is Dacey, we shall use induction with respect to \(n\).

If \(n = 0\), then \(X = \emptyset\) and \(X\) is Dacey.

Let \(n > 0\).

Put \(X_1 = [a_{2n-1} < a_{2n}] \uparrow\), \(X_2 = X \setminus ([a_{2n-1} < a_{2n}] \uparrow)\) and note that \(X_1 \perp X_2\). By Lemma 4.6, \([a_{2n-1} < a_{2n}] \uparrow\) is Dacey. The set \(X_2\) is of chain type, hence \(X_2\) is Dacey by induction hypothesis. The conditions of Lemma 3.6 are satisfied, hence \(X = X_1 \cup X_2\) is Dacey.

\qed

Theorem 4.10 Let \(P\) be a finite bounded poset. The following are equivalent.

(a) \(P\) is a lattice.
(b) Every orthoclosed subset of \(Q^+(P)\) is of chain type.
(c) \(Q^+(P)\) is Dacey.
(d) \(L(Q^+(P), \perp)\) is an orthomodular lattice.

Proof

(a) \(\Rightarrow\) (b): Let \(X\) be an orthoclosed subset of \(Q^+(P)\), let \([a < b], [c < d]\) be distinct maximal elements of \(X\). Let us prove that \([a < b] \perp [c < d]\). Assume the contrary; by Lemma 4.7 (b), \([a \wedge c < b \lor d] \in X\). As \([a < b] \leq [a \wedge c < b \lor d]\) and \([a < b]\) is maximal in \(X\), \([a < b] = [a \wedge c < b \lor d]\). This implies \(a = a \wedge c\) and \(b = b \lor d\), in other words, \(a \leq c\) and \(d \leq b\) so \([c < d] \leq [a < b]\), hence \([c < d] = [a < b]\) because \([c < d]\) is maximal in \(Q^+(P)\) and we have reached the desired contradiction.
From \([a < b] \perp [c < d]\) it follows that either \(b \leq c\) or \(d \leq a\). Assume that \(b = c\). Then, by Lemma 4.8, \([a < d] \in X\) and this contradicts the maximality of \([a < b]\) and \([c < d]\) in \(X\).

(b) \(\implies\) (a): By Lemma 4.9.

(c) \(\implies\) (a): Suppose that \(P\) is not a lattice. We need to prove that \(Q^+(P)\) is not Dacey. By Lemma 2.1, since \(P\) is not a lattice, there exists a lower set \(I\) of \(P\) with \(I = I^{↑↓}\) such that \(I\) has at least two maximal elements. Let \(a, b\) be two distinct maximal elements of \(I\). By Corollary 4.4,
\[
\tau(I)^{↑↓} = \tau(I) = \tau(I),
\]
hence \(\tau(I)\) is orthoclosed. Clearly, \([0 < a]\) is a basis of \(\tau(I)\) and we may apply Lemma 4.3 to compute
\[
([0 < a])^{↑↓} = (\beta(a^↑)) = \tau(a^↑) = \tau(a^↓).
\]
However, as \([0 < b] \notin \tau(a^↓), \tau(a^↓) \neq \tau(I)\). Therefore, \((Q^+(P), \bot)\) is not Dacey.

(c) \(\iff\) (d): By Theorem 3.3.

**Example 4.11** Consider the bounded poset \(P\) on the left side of Fig. 4. Since it is not a lattice, Theorem 4.10 tells us that \((Q^+(P), \bot)\) is not a Dacey space or, equivalently, that \(L(Q^+(P), \bot)\) is not orthomodular. Hence \(L(Q^+(P), \bot)\) must contain a benzene ring as its sub-ortholattice. One of such benzene rings is depicted on the right side of Fig. 4. To see more directly that \((Q^+(P), \bot)\) is not a Dacey space, just consider the orthoclosed set \([0 < a], [0 < b]\) and its basis \([0 < a]\).

**Theorem 4.12** Let \(P\) be a bounded poset. Then \(P\) is a chain if and only if \(L(Q^+(P), \bot)\) is a Boolean algebra.

**Proof** Suppose that \(P\) is a chain. By Proposition 2.3, \(L(Q^+(P), \bot)\) is a Boolean algebra if and only if, for all orthoclosed subsets \(X, Y\),
\[
X \cap Y = \emptyset \implies X \perp Y
\]
or, equivalently,
\[
X \nleq Y \implies X \cap Y \neq \emptyset.
\]
Assume that \(X, Y\) are not orthogonal. We need to prove that \(X, Y\) are not disjoint.

Since \(X, Y\) are not orthogonal, there are \([x_1 < x_2] \in X\) and \([y_1 < y_2] \in Y\) such that \([x_1 < x_2] \nleq [y_1 < y_2]\). As \([x_1 < x_2] \nleq [y_1 < y_2]\), we see that \(x_2 \not\leq y_1\) and \(y_2 \not\leq x_1\). Since \(P\) is a chain, this is equivalent to \(y_1 < x_2\) and \(x_1 < y_2\).

\[
\begin{align*}
\text{Fig. 4} & \quad \text{A finite bounded poset that is not a lattice and a sub-ortholattice of its logic}
\end{align*}
\]
For every finite lattice $P$, $K(P)$ is an orthomodular lattice.

Theorem 5.2 For every finite lattice $P$, $K(P) ≃ L(Q^+(P), \bot)$.

Proof For $C = [x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}] \in K(P)$, write $f(C) = [x_1 < x_2] \cup \cdots \cup [x_{2n-1} < x_{2n}]$. Then $f(C)$ is a lower subset of $Q^+(P)$ of the chain type. By Lemma 4.9, $f(C)$ is Dacey and hence orthoclosed. Obviously, $f$ is isotone, $f(\emptyset) = \emptyset$ and $f(P) = f([0 < 1]) = [0 < 1] = Q^+(P)$, hence $f : K(P) \rightarrow L(Q^+(P), \bot)$ is an isotone mapping preserving the bounds. Moreover, $f$ preserves the complementation. There are

5 Connections to Other Constructions

Let us extend our $[x < y]$ notation to an arbitrary finite number of elements: if $C = \{x_1, \ldots, x_n\}$ is a finite chain in a poset $P$, we write $C = [x_1 < \cdots < x_n]$ to indicate the partial order.

The following construction was discovered by Kalmbach in [9]. Let $P$ be a bounded lattice, write

$$K(P) = \{C : C \text{ is a finite chain in } P \text{ with even number of elements}\}$$

Define a partial order on $K(P)$ by the following rule:

$$[x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}] \leq [y_1 < y_2 < \cdots < y_{2k-1} < y_{2k}]$$

if for every $1 \leq i \leq n$ there is $1 \leq j \leq k$ such that

$$y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}.$$

Define a unary operation $C \mapsto C^\perp$ on $K(P)$ to be the symmetric difference with the set $\{0, 1\}$.

Theorem 5.1 [9] For every bounded lattice $P$, $K(P)$ is an orthomodular lattice.

The lattice $K(P)$ is called the Kalmbach construction.
four cases to check, each of the cases consisting of a conjunction of one of \((0 \in C, 0 \not\in C)\) and one of \((1 \in C, 1 \not\in C)\). We will check one of the cases, the remaining three are very similar.

Suppose that \(0 \in C\), \(1 \not\in C\), that means, \(0 = x_1\) and \(x_{2n} < 1\):

\[
f(C) = f([0 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = [0 < x_2]^{\downarrow} \cup \cdots \cup [x_{2n-1} < x_{2n}]^{\downarrow}
\]

Then

\[
f(C^{\perp}) = f([x_2 < x_3 < \cdots < x_{2n} < 1]) = [x_2 < x_3]^{\downarrow} \cup \cdots \cup [x_{2n} < 1]^{\downarrow}
\]

It is easy to see that \(f(C)^{\perp} = f(C^{\perp})\).

Let us construct an isotone map inverse to \(f\). Let \(X\) be an orthoclosed subset of \(Q^+(P)\). By Theorem 4.10, \(X\) is of the chain type. Write \(\text{max}(X) = \{[x_1 < x_2], \ldots, [x_{2n-1} < x_{2n}]\}\) and assume, without loss of generality, that \(x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}\). This chain is an element of \(K(P)\), put \(g(X) = [x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]\). It is easy to check \(g\) is isotone, preserves the bounds and preserves the orthocomplementation.

For general infinite lattices, Theorem 5.2 is not true. For example, for the real unit interval \([0, 1]_\mathbb{R}\), the Kalmbach construction \(K([0, 1]_\mathbb{R})\) is not a complete lattice, whereas every \(L(Q^+(P), \perp)\) is a complete lattice.

For every poset \(P\), its Dedekind-MacNeille completion \(\hat{P}\) can be represented as the complete lattice of all lower sets \(I\) with \(I = I^{\uparrow\downarrow}\). For a family \((I_j)_{j \in H}\) of elements of \(\hat{P}\), their meet is their intersection and their join is the closure of their union:

\[
\bigvee_{j \in H} I_j = (\bigcup_{j \in H} I_j)^{\uparrow\downarrow}
\]

By Corollary 4.4, for every lower set \(I\) we have \(\tau(I)^{\perp\perp} = \tau(I^{\uparrow\downarrow})\). In particular, \(\tau(I^{\uparrow\downarrow})\) is orthoclosed. Therefore, for every \(I \in \hat{P}\), \(\tau(I)\) is orthoclosed.

Let \(L_1, L_2\) be complete lattices. A mapping \(f : L_1 \to L_2\) is a morphism of complete lattices if it preserves all suprema and infima:

\[
f\left(\bigvee_{i \in H} a_i\right) = \bigvee_{i \in H} f(a_i) \quad f\left(\bigwedge_{i \in H} a_i\right) = \bigwedge_{i \in H} f(a_i),
\]

for every family \((a_i)_{i \in H}\) of elements of \(L_1\).

**Theorem 5.3** For every bounded poset \(P\), \(\tau : \hat{P} \to L(Q^+(P), \perp)\) is an injective morphism of complete lattices.

**Proof** Clearly, \(\tau\) is injective and preserves the bounds of \(\hat{P}\). Let \((I_j)_{j \in H}\) be a family of elements of \(\hat{P}\). Clearly,

\[
\tau\left(\bigcap_{j \in H} I_j\right) = \bigcap_{j \in H} \tau(I_j)
\]
so $\tau$ preserves all meets. For joins, we may compute

$$\bigvee_{j \in H} \tau(I_j) = \left( \bigcup_{j \in H} \tau(I_j) \right)^\perp = \left( \bigcap_{j \in H} \tau(I_j) \right)^\perp = \left( \bigcap_{j \in H} \beta(I_j) \right)^\perp$$

$$= \beta\left( \bigcup_{j \in H} I_j \right)^\perp = \tau\left( \bigcup_{j \in H} I_j \right)$$

Let us close with a conjecture.

**Conjecture 1** There exists an infinite bounded poset $P$ such $P$ is not a lattice and $(Q^+\langle P \rangle, \perp)$ is Dacey.

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