DIFFERENCE DISCRETE VARIATIONAL PRINCIPLE, EULER-LAGRANGE COHOMOLOGY AND SYMPLECTIC, MULTISYMPLECTIC STRUCTURES

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Abstract
We study the difference discrete variational principle in the framework of multi-parameter differential approach by regarding the forward difference as an entire geometric object in view of noncommutative differential geometry. By virtue of this variational principle, we get the difference discrete Euler-Lagrange equations and canonical ones for the difference discrete versions of the classical mechanics and classical field theory. We also explore the difference discrete versions for the Euler-Lagrange cohomology and apply them to the symplectic or multisymplectic geometry and their preserving properties in both Lagrangian and Hamiltonian formalism. In terms of the difference discrete Euler-Lagrange cohomological concepts, we show that the symplectic or multisymplectic geometry and their difference discrete structure preserving properties can always be established not only in the solution spaces of the discrete Euler-Lagrange/canonical equations derived by the difference discrete variational principle but also in the function space in each case if and only if the relevant closed Euler-Lagrange cohomological conditions are satisfied. We also apply the difference discrete variational principle and cohomological approach directly to the symplectic and multisymplectic algorithms.

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1 Introduction

It is well known that the symplectic structure plays crucially important role in the both Lagrangian and Hamiltonian formalism for classical mechanics [1][2]. On the other hand, the multisymplectic structure plays also very important role in the Lagrangian and Hamiltonian formalism for classical field theories [3][4][5][6][7]. Specially, in the computational science, they are extremely important in the symplectic and multisymplectic algorithms for the finite dimensional Hamiltonian systems [8][9] and infinite-dimensional [4][7] respectively. These algorithms are quite powerful and successful in numerical calculations of the relevant systems in comparison with other various non-symplectic/multisymplectic numerical schemes since the symplectic and multisymplectic schemes preserve the symplectic structure and multisymplectic structure of the systems respectively.

Very recently, it has been found [10][11] that there exist what is called the Euler-Lagrange cohomology in either classical mechanics or classical field theory and it plays very important role for the symplectic or multisymplectic structure preserving property in each case. It has also been studied the difference discrete version for classical mechanics and field theory mainly in Lagrangian formalism. For this purpose, it has been proposed a difference discrete variational principle by regarding the forward (or backward) difference as an entire geometric object to deal with variation of the difference discrete classical mechanics and field theory [10][11]. In [2][12] and [13][14][15], special investigation has been made for the symplectic algorithm as well as the symplectic and multisymplectic structure preserving in simple element method respectively from the cohomological point of view. In [16], the multi-parameter differential approach has been introduced in order to deal with in the same framework the variation of functional and the exterior differential calculus in the function space, it has been further studied the Euler-Lagrange cohomology and its relation with symplectic and multisymplectic structure preserving properties for classical mechanics and field theory in both the Lagrangian and Hamiltonian formalism. The cohomological approach has also been applied to what are called Hamiltonian-like ODEs and PDEs respectively.

In this paper, we further study in some details the difference discrete variational
principle and apply it to action functional not only in the Lagrangian formalism but also in the Hamiltonian formalism. We also study the difference discrete versions for the Euler-Lagrange cohomology, symplectic and multisymplectic structures with their structure preserving properties in the classical mechanics and field theory. We generalize the multi-parameter differential approach for the both variational principle and exterior differential calculus in the function space to the difference discrete variational principle and exterior differential calculus in the function space on difference discrete base space in the Lagrangian and Hamiltonian formalism. It is shown that the difference discrete variational principle gives rise to the difference discrete version of the Euler-Lagrange equations and that of the canonical equations of motion that preserve the symplectic or multisymplectic structures in the Lagrangian and Hamiltonian formalism for the difference discrete mechanics and field theory, respectively. It is also shown that the difference discrete version of the Euler-Lagrange cohomology in each case is nontrivial and that of symplectic and multisymplectic structures are preserved if and only if relevant closed Euler-Lagrange conditions are satisfied without making use of the discrete Euler-Lagrange equations or the canonical ones in general. Although both the difference discrete Euler-Lagrange equations or the canonical ones do satisfy the difference discrete closed Euler-Lagrange conditions. Therefore, it is important that these difference discrete version for the symplectic and multisymplectic structure-preserving properties hold in the configuration space and its tangent space in the Lagrangian formalism or on the phase space in the Hamiltonian formalism with the relevant closed Euler-Lagrange conditions in general rather than in the solution space of the difference discrete Euler-Lagrange equations or that of the canonical ones only.

One of the key issues of this paper that is different from the others is the difference discrete variational principle is first proposed in [10][11] to get difference discrete Euler-Lagrange equations. As was emphasized in [10][11], in view of noncommutative differential calculus, the difference is defined as the (discrete) derivative so that it should be regarded as an entire geometric object. Furthermore, it can also combine together in certain manner as a geometric object to construct the numerical schemes (see section 5). In the difference discrete variational principle approach, this point of view has been carried out. In this paper, this approach is applied not only to the Lagrangian formalism but also the Hamiltonian formalism for the both difference discrete mechanics and field theory. Together with suitable Leibniz law for the differences, it is also directly applied to the derivation of the numerical schemes in symplectic and multisymplectic algorithms.

Second key issue of this paper is about the difference discrete version of the Euler-Lagrange cohomological concepts and content and their role-played in the symplectic and multisymplectic structure preserving properties. in each case. As a matter of fact, the nontriviality of the difference discrete version for the Euler-Lagrange cohomology plays a crucial role and is directly related to the symplectic and multisymplectic structure preserving properties.

In the course of numerical calculation, the “time” $t \in \mathbb{R}$ is always discretized, say, with equal spacing $\tau = \Delta t$ and the space coordinates are also discretized in many cases, especially, for the classical field theory. In addition to these computational approach, there also exist various discrete physical systems with discrete or difference discrete Lagrangian and Hamiltonian functional. It is well known that the differences of functions do not obey the ordinary Leibniz law. In order to explore that the difference discrete symplectic and multisymplectic structures in these difference discrete systems and their structure-preserving properties, some noncommutative differential calculus should be employed,
Even for the well-established symplectic algorithm. This is the third key point of this paper. Recently, the noncommutative differential calculus in regular lattice has been employed to deal with the difference discrete phase space for finite dimensional systems with separable Hamiltonian \[17\] \[18\] \[19\]. Similar noncommutative differential calculus will be employed in the present paper.

Another key point of this paper is the multi-parameter differential approach to the difference discrete variational principle and to deal with the exterior differential calculus in the function space. This approach provides the same framework for the both DDVP and EL cohomological approach. It was employed in \[16\] for the continuous cases. In the present paper we will employ the multi-parameter differential approach for the cases on the difference discrete version of the base space, i.e. the “time” in the difference discrete mechanics and the “spacetime/space” in the difference discrete field theory.

The plan of this paper is as follows. We first explore, in the framework of the multi-parameter differential approach, the difference discrete variational principle in the both Lagrangian and Hamiltonian formalism for classical mechanics and classical field theory respectively in section 2. It is shown that difference discrete variational principle with simply modified Leibniz law for the differences offers the difference discrete version for both Euler-Lagrange equations and the canonical equations of motion. In section 3 and 4, also in the framework of the multi-parameter differential approach and using the exterior differential calculus in the function space in the case of difference discrete base space, we deal with such kind of the difference discrete versions of the Euler-Lagrange cohomology as well as the symplectic structure preserving and multisymplectic structure preserving properties in the both Lagrangian and Hamiltonian formalism for classical mechanics and field theory respectively. It is shown that the relevant difference discrete versions of the Euler-Lagrange cohomology in each case is nontrivial and it is directly linked with the difference discrete symplectic and multisymplectic structure preserving properties. We explore in some details the difference discrete variational principle approach and the difference discrete Euler-Lagrange cohomological approach to the symplectic and multisymplectic algorithms in section 5. It is pointed out that the difference discrete the Euler-Lagrange equations, the canonical equations of motion for the classical mechanics and field theory present themselves certain symplectic and multisymplectic schemes respectively. We also show that the Euler midpoint scheme in the symplectic algorithm, the midpoint box scheme for a type of PDEs and the midpoint box scheme for the Hamiltonian field theory in the multisymplectic algorithm can be derived by the difference discrete variational principle with a suitable difference Leibniz law. And the difference discrete Euler-Lagrange cohomology and its relation with the difference discrete symplectic and multisymplectic structure preserving properties offer a cohomological scenario to show whether numerical schemes are symplectic or multisymplectic. Finally, we end with some concluding remarks in section 6. In the appendix, some simple relevant noncommutative differential calculus on regular lattice with equal step-length on each direction are given. For the sake of self-containing in relevant sections, the content for the continuous case is briefly recalled and then the approach is generalized to deal with the difference discrete case.
2 Variational and difference discrete variational principle in multi-parameter differential approach

In order to consider certain difference discrete versions of the simplicial and multisymplectic structures and their structure-preserving properties in both Lagrangian and Hamiltonian formalism for classical mechanics and field theory, we study the variational principle and difference discrete variational principle in both Lagrangian and Hamiltonian formalism for classical mechanics and field theory and their difference discrete versions, in the framework of the multi-parameter differential approach in this section. We consider the cases in classical mechanics in the subsection 2.1, and that in classical field theory in the subsection 2.2.

The difference discrete variational principle approach was first proposed in [10][11] with vanishing condition at $t_k = \pm \infty$ for the infinitesimal variations of coordinates in the configuration space, $\delta q^i(t_k), k \in \mathbb{Z}$, in the difference discrete classical mechanics and the corresponding vanishing condition at infinity in 1+1 dimensional or 2 dimensional cases for the infinitesimal variations of a set of generic field variables, $\delta u^{\alpha(i,j)}, \alpha = 1, \cdots, r, (i, j) \in \mathbb{Z} \times \mathbb{Z}$, in the difference discrete classical field theory. As was emphasized, the most important point of the approach is regarding the forward difference or its certain combination in each difference discrete case as an entire geometric object in the sense of noncommutative differential calculus. Of course, if the backward difference is preferred rather than the forward one, the framework is almost the same. The framework of the multi-parameter differential approach has been employed for the continuous case in [16]. We review this approach and generalize it to deal with the difference discrete variational principle.

2.1 Variational and difference discrete variational principle in continuous and difference discrete classical mechanics

We begin with recall some content of the multi-parameter differential approach to variational principle for classical Lagrangian mechanics and transfer it to the Hamiltonian formalism. Then we generalize it to deal with the difference discrete variational principle in the both Lagrangian and Hamiltonian formalism for the difference discrete classical mechanics.

2.1.1 Variational principle in multi-parameter differential approach for classical mechanics

Let time $t \in \mathbb{R}^1$ be the base manifold, $M$ the $n$-dimensional configuration space on $t$ with coordinates $q^i(t), (i = 1, \cdots, n)$, $TM$ the tangent bundle of $M$ with coordinates $(q^i(t), \dot{q}^i(t))$, where $\dot{q}^i(t)$ is the time derivative of $q^i$, $F(TM)$ the function space on $TM$.

2.1.1.1 Variational principle in Lagrangian formalism

The Lagrangian of the systems is denoted by $L(q^i, \dot{q}^i)$. For simplicity, we suppose that the Lagrangian does not manifestly depend on $t$. The action functional along a curve $q(t)$ in $M$, $C^b_a$ with two endpoints $a$ and $b$, can be constructed by integrating of $L$ along the tangent of the curve

$$S(q(t)) := \int_a^b dt L(q^i(t), \dot{q}^i(t)).$$

(1)
Let us consider the case that at the moment \( t \) both \( q^i(t) \) and \( \dot{q}^i(t) \) variate by an infinitesimal increments and the curve \( C_a^b \) becomes a congruence of curves \( C_{ca}^b \). The infinitesimal variations of \( q^i \) and \( \dot{q}^i \) in the congruence can be described as follows

\[
q^i(t) \rightarrow q^i_c(t) = q^i(t) + \epsilon^k \delta_k q^i(t), \quad \dot{q}^i(t) \rightarrow \dot{q}^i_c(t) = \dot{q}^i(t) + \epsilon^k \delta_k \dot{q}^i,
\]

where \( \epsilon^k, k = 1, \ldots, n \), are \( n \) free parameters that each of them corresponds one direction in the configuration space \( M \), \( \delta_k q^i(t) \) and \( \delta_k \dot{q}^i(t) \) infinitesimal increments of \( q^i(t) \) and \( \dot{q}^i(t) \) at the moment \( t \) along the direction \( k \) in the congruence of curves \( C_{ca}^b \):

\[
\delta_k q^i(t) := \frac{\partial}{\partial \epsilon^k} |_{\epsilon^k=0} q^i(t), \quad \delta_k \dot{q}^i(t) := \frac{\partial}{\partial \epsilon^k} |_{\epsilon^k=0} \dot{q}^i(t).
\]

Here the differentials of \( q^i_c(t) \) and \( \dot{q}^i_c(t) \) with respect to \( \epsilon^k \) in the function space \( F(TM) \) are manipulated. Namely, the differentials of \( q^i_c(t) \) and \( \dot{q}^i_c(t) \) at the moment \( t \) in the congruence of curves \( C_{ca}^b \) can be calculated by:

\[
dq^i_c := \frac{\partial q^i}{\partial \epsilon^k} de^l = de^k \delta_k q^i, \quad d\dot{q}^i_c := \frac{\partial \dot{q}^i}{\partial \epsilon^k} de^l = de^k \delta_k \dot{q}^i.
\]

This framework is called the multi-parameter differential approach. Furthermore, the exterior differential calculus in the function space can also be well established in this framework. It should be mentioned that in the standard parameter-differential approach to the variation calculation is usually to introduce only one free parameter along the curve. For the variation calculation it is enough, but it should have more degree of freedom for the exterior differential calculation for the functions and functionals. The multi-parameter differential setting, in fact, offers the same framework to deal with both variation and exterior differential calculation for the functions and functionals.

In the congruence of curves \( C_{ca}^b \), the Lagrangian now becomes a family of Lagrangian and the same for the action functional:

\[
S(q(t)) \rightarrow S_\epsilon(q_\epsilon(t)) = \int_a^b dt L_\epsilon(q_\epsilon^i(t), \dot{q}_\epsilon^i(t)),
\]

where the upper-index \( k \) of \( \epsilon^k \) is omitted.

Hamilton’s principle, i.e. the (least) variational principle, seeks the curve \( C_a^b \in C_{ca}^b \) along which the action \( S \) is stationary against all variations of \( q^i(t) \) along any directions. In the multi-parameter differential approach, similar to (3), this can be manipulated by taking differentiation with respect to \( \epsilon^k \) and setting \( \epsilon^k = 0 \) afterwards:

\[
\delta_k S(q(t)) := \frac{\partial}{\partial \epsilon^k} |_{\epsilon^k=0} S_\epsilon(q_\epsilon(t)) = 0, \quad \forall k = 1, \ldots, n,
\]

for all \( \delta_k q^i(k) = \delta_k q^i(t) \) with \( \delta_k q^i(a) = \delta_k q^i(b) = 0 \).

It is straightforward to get the differentiation of the action with respect to \( \epsilon^k \)

\[
dS_\epsilon(q_\epsilon(t)) = \int_a^b dt \frac{\partial}{\partial \epsilon^k} L(q_\epsilon^i(t), \dot{q}_\epsilon^i(t)) \cdot de^k = \int_a^b dt dq^i \{ \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\epsilon^i} \} + \frac{\partial L}{\partial \dot{q}_\epsilon^i} d\dot{q}_\epsilon^i |_a \cdot \]

Therefore, the variation of the action along the direction \( k \) is given by

\[
\delta_k S(t) := \int_a^b dt \delta_k q^i (\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\epsilon^i}) + \frac{\partial L}{\partial \dot{q}_\epsilon^i} \delta_k \dot{q}_\epsilon^i |_a \cdot \]

The last term in the above equation vanishes due to \( \delta_k q^i(a) = \delta_k q^i(b) = 0 \), hence the stationary requirement for \( S \), i.e. the variations of \( S \) along any direction should be vanish, yields the Euler-Lagrange equations

\[
\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\epsilon^i} = 0.
\]
2.1.1.2 Variational principle in Hamiltonian formalism

The action principle can also be carried out on the phase space in the Hamiltonian formalism. In order to transfer to the Hamiltonian formalism, we introduce a family of conjugate momenta from the family of Lagrangian \( L \)

\[
p_{je} = \frac{\partial L_{\varepsilon}}{\partial \dot{q}_i^j},
\]

and take a Legendre transformation to get the Hamiltonian in the family \( H_{\varepsilon} := H(q_{\varepsilon}^i, p_{je}) = p_{\varepsilon} q_{\varepsilon}^i - L(q_{\varepsilon}^i, \dot{q}_{\varepsilon}^i) \).

Now the family of the action functionals can be expressed as

\[
S_{\varepsilon} = \int_{t_1}^{t_2} dt \{ p_{\varepsilon} \dot{q}_{\varepsilon}^i - H(q_{\varepsilon}^i, p_{je}) \}
\]

The variation of the action functional along the direction \( \varepsilon \) can be calculated also in terms of differentiation with respect to the parameter \( \varepsilon \) and setting \( \varepsilon = 0 \) afterwards

\[
\delta_k S = \frac{\partial}{\partial \varepsilon} S_{\varepsilon} \bigg|_{\varepsilon=0} = \int_{t_1}^{t_2} dt \{ -\left( \frac{\partial H_{\varepsilon}}{\partial p_{\varepsilon}} \right) \delta_k p_{\varepsilon} - \left( \frac{\partial H_{\varepsilon}}{\partial q_{\varepsilon}^i} \right) \delta_k q_{\varepsilon}^i + \dot{p}_{\varepsilon} \delta_k q_{\varepsilon}^i + \frac{d}{dt} (p_{\varepsilon} \delta_k q_{\varepsilon}^i) \} \bigg|_{\varepsilon=0}.
\]

Thus, the stationary requirement for the action against all variations along any direction, i.e. \( \delta_k S = 0, \forall k = 1, \ldots, n \) together with the fixed endpoint condition lead to the canonical equations

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.
\]

2.1.2 Difference discrete variational principle in multi-parameter differential approach for discrete classical mechanics

Let us now introduce the difference discrete variational principle \([10] [11]\) for the difference discrete version of the classical mechanics. Here we generalize the multi-parameter differential approach and employ it to deal with the difference discrete variational principle for the difference discrete classical mechanics.

Consider the case that “time” \( t \) is difference discretized while the \( n \)-dimensional configuration space \( M_k \) at each moment \( t_k, k \in Z \), is still continuous and smooth enough. Let us assume, without loss generality, that the “time” \( t \in R \) be discretized as a set of nodes and links with equal step-length \( \tau = \Delta t \):

\[
t \in R \rightarrow t \in T = \{(t_k, t_{k+1} = t_k + \tau, \ k \in Z)\}.
\]

Let \( N \) and \( L \) be the set all nodes and links with index set \( Ind(N) = Ind(L) = Z \), \( M = \bigcup_{k \in Z} M_k \) the configuration space on \( T \) that is still continuous and at least piece wisely smooth enough. At the moment \( t_k \), \( N_k \) and \( L_k \) be the set of nodes and links neighboring \( t_k \) respectively. For example, \( L_k \) includes two links \([t_{k-1}, t_k] \text{ and } [t_k, t_{k+1}]\) with endpoints \((t_{k-1}, t_k, t_{k+1})\). Let \( T_k \) the index set of nodes of \( N_k \) including \( t_k, N_k = \bigcup_{Ind(N) \in T_k} N \) etc. The coordinates of \( M_k \) are denoted by \( q^i(t_k) = q^{i(k)}, i = 1, \ldots, n \).

\( T(M_k) \) the tangent bundle of \( M_k \) in the sense that difference at \( t_k \) is its base, \( T^*(M_k) \) its dual. Let \( M_k = \bigcup_{l \in T_k} M_l \) be the union of configuration spaces \( M_l \) at \( t_l, l \in T_k \) on
$N_k, T\mathcal{M}_k = \bigcup_{l \in L_k} TM_l$ the union of tangent bundles on $\mathcal{M}_k$, $F(TM_k)$ and $F(T\mathcal{M}_k)$ the function spaces on each of them respectively, etc.. In the difference discrete variational principle, we will use these notions.

### 2.1.2.1 Difference discrete variational principle in Lagrangian formalism

We first study the difference discrete version of the Lagrangian formalism. It is clear that the difference discrete Lagrangian written as

$$L_D^{(k)} = L_D(q^{i(k)}, q_t^{i(k)})$$

is a functional on $F(TM_k)$, since $q_t^{i(k)}$ is the forward difference of $q^{i(k)}$ at $t_k$ defined by

$$\Delta_t q^{i(k)} := \frac{d}{dt} q^{i(k)} = q_t^{i(k)} = \frac{1}{\tau} \{ q^{i(k+1)} - q^{i(k)} \}.$$

It is the (discrete) derivative and the base of $T(TM_k)$ in the sense of noncommutative differential calculus on a regular lattice $L^1$ with equal step-length $\tau$ [17] (see also the appendix) and the same notation for it as in the continuous case may be employed if it does not cause any ambiguity.

As was emphasized, in what follows the forward difference is viewed as an entire geometric object and its dual $d_T t$ is the base of $T^*(T)$ in the sense

$$d_T t(\Delta_t) = 1.$$  

It is well known that the (forward) difference as the discrete derivative does not obey the Leibniz law but the modified one

$$\Delta_t (f \cdot g)^{(k)} = \Delta_t f^{(k)} \cdot g^{(k)} + f^{(k+1)} \cdot \Delta_t g^{(k)}, \quad f, g \in FM = \Omega^0_T.$$  

On the other hand, however, it is important to note (see the appendix) that in the space $T^*(T)$ dual to $T(T)$, an exterior differential operator $d_T$ exists such that

$$d_T : \Omega^l_T \rightarrow \Omega^{l+1}_T, \quad d^2_T = 0,$$

where $\Omega^l_T$ the space of l-forms, $l = 0, 1$, on $T^*(T)$ and $d_T$ does satisfy the Leibniz law:

$$d(\omega \wedge \tau)^{(k)} = d\omega^{(k)} \wedge \tau^{(k)} + (-1)^{\deg(\omega)} \omega^{(k)} \wedge \tau^{(k)}.$$  

The action functional in the continuous case (1) now becomes

$$S_D = \sum_{k \in Z} L_D(q^{i(k)}, q_t^{i(k)}),$$

where the summation is taken over $k \in Z$.

We now consider how to calculate the variation of the action functional $S_D$ in this case. Since only the “time” is discretized while either the configuration space at each moment $t_k$, i.e. at the node $k$, or at its neighboring union are still continuous and the variational calculation that will be carried out is mainly local, therefore, the difference discrete variations may still be manipulated in the framework of the multi-parameter differential approach. In addition, as in the continuous case of the classical mechanics, the differential and exterior differential calculus in the function space can also be carried out in either $F(TM_k)$ and $F(T\mathcal{M}_k)$, etc..
In order to make use of the multi-parameter differential approach the variations of \( q^{i(k)} \) and \( q_t i^{i(k)} \) with the multi-parameter \( e^l \) should be introduced. At the moment \( t_k \), we have
\[
q^i(k) = q^{i(k)} + e^l \delta q^{i(k)}, \quad \Delta_t q^i(k) = \Delta_t q^{i(k)} + e^l \delta_l (\Delta_t q^{i(k)}),
\]
and
\[
\delta l q^{i(k)} := \frac{\partial \Delta_t q^{i(k)}}{\partial e^l} |_{e^l=0} = \delta q^{i(k)}, \quad \delta l q_t i^{i(k)} := \frac{\partial \Delta_t q_t i^{i(k)}}{\partial e^l} |_{e^l=0} = \delta q_t i^{i(k)}.
\]
Then the action functional in (22) becomes a family of action functionals
\[
S_D \rightarrow S_{D e} = \sum_{k \in Z} L_{D e}^{(k)}
\]
and the variation of the action functional along the direction \( l \)
\[
\delta_l S_{D e} = \frac{\partial \delta S_{D e}}{\partial e^l} |_{e^l=0}.
\]
i.e.
\[
\delta_l S_{D e} = \sum_{k \in Z} \left\{ \frac{\partial L_{D e}^{(k)}}{\partial q^{i(k)}} \delta_l q^{i(k)} + \frac{\partial L_{D e}^{(k)}}{\partial (\Delta_t q^{i(k)})} \delta_l (\Delta_t q^{i(k)}) \right\} |_{e^l=0}.
\]
By virtue of the modified Leibniz law [19] for \( \Delta_t = \partial_t \), we have
\[
\Delta_t \left( \frac{\partial L_{D e}^{(k-1)}}{\partial (\Delta_t q^{i(k-1)})} \right) \delta_l q^{i(k)} = \frac{\partial L_{D e}^{(k-1)}}{\partial q^{i(k-1)}} \delta_l q^{i(k)} + \Delta_t \left( \frac{\partial L_{D e}^{(k-1)}}{\partial (\Delta_t q^{i(k-1)})} \right) \delta_l q^{i(k)}.
\]
Therefore,
\[
\delta_l S_D = \sum_{k \in Z} \left\{ \left( \frac{\partial L_{D e}^{(k)}}{\partial q^{i(k)}} - \Delta_t \left( \frac{\partial L_{D e}^{(k-1)}}{\partial (\Delta_t q^{i(k-1)})} \right) \right) \delta_l q^{i(k)} + \sum_{k \in Z} \Delta_t \left( \frac{\partial L_{D e}^{(k-1)}}{\partial (\Delta_t q^{i(k-1)})} \right) \delta_l q^{i(k)} \right\} |_{e^l=0}.
\]
Using the properties (see the appendix)
\[
\sum_{k \in Z} \Delta_t f(t_k) = f(t_{k=+\infty}) - f(t_{k=-\infty}),
\]
and assuming \( \delta_l q^{i(k)} |_{k \pm \infty} = 0 \), it follows the discrete Euler-Lagrange equations
\[
\frac{\partial L_{D e}^{(k)}}{\partial q^{i(k)}} - \Delta_t \left( \frac{\partial L_{D e}^{(k-1)}}{\partial (\Delta_t q^{i(k-1)})} \right) = 0.
\]
It should be mentioned here in general, for the forward difference calculation more general Leibniz law can be adopted and it will lead to more general difference discrete version of the Euler-Lagrange equations. We will explore this issue mainly in the Hamiltonian formalism in the section 5.

Let us consider an example.

**Example 2.1. A difference discrete classical mechanics**

Consider a difference discrete version of classical mechanics with following difference discrete Lagrangian:
\[
L_{D e}(q^{i(k)}, \Delta_t q^{i(k)}) = \frac{1}{2}(\Delta_t q^{i(k)})^2 - V(q^{i(k)}).
\]
The difference discrete variational principle gives the discrete Euler-Lagrange equation
\[
\Delta_t (\Delta_t q^{i(k-1)}) - \frac{\partial}{\partial q} V(q^{i(k)}) = 0,
\]
i.e.
\[
\frac{1}{t^2}(q^{i(k+1)} - 2q^{i(k)} + q^{i(k-1)}) = \frac{\partial}{\partial q} V(q^{i(k)}).
\]
This is the difference discrete counterpart of the equation in the continuous case. It has correct continuous limit.

2.1.2.2 Difference discrete variational principle in Hamiltonian formalism

Now we consider the difference discrete variational principle on the phase space in the difference discrete (“time”) Hamiltonian formalism.

To transfer to the difference discrete Hamiltonian formalism, we first define a family of the discrete canonical conjugate momenta

\[ p_{\epsilon}(k) = \partial L_{D_{\epsilon}}(k-1) \partial (\Delta t q_{\epsilon}(k-1)). \] (34)

Then a family of the difference discrete Hamiltonian can be introduced through the discrete Legendre transformation in the family

\[ H_{D_{\epsilon}}(k) = p_{\epsilon}(k+1) \Delta t q_{\epsilon}(k) - L_{D_{\epsilon}}(k). \] (35)

Now the difference discrete version of the action functional in (22) becomes a family of action functionals:

\[ S_{D_{\epsilon}} = \sum_{k \in \mathbb{Z}} \{ p_{\epsilon}(k+1) \Delta t q_{\epsilon}(k) - H_{D_{\epsilon}}(k) \}. \] (36)

Then the variation of the action along the direction \( l \) in (36) can be calculated as

\[ \delta l S = \frac{\partial}{\partial \epsilon} S_{D_{\epsilon}} |_{\epsilon = 0}. \] (37)

For the differential of \( S_{D_{\epsilon}} \) in the above equation, we have

\[ dS_{D_{\epsilon}} = \sum_{k \in \mathbb{Z}} \{ dp_{\epsilon}(k+1) (\Delta t q_{\epsilon} - \partial H_{D_{\epsilon}}(k) \partial p_{\epsilon}(k+1)) - (\Delta t p_{\epsilon}(k) + \partial H_{D_{\epsilon}}(k) \partial q_{\epsilon}(k)) dq_{\epsilon}(k) + \Delta t (p_{\epsilon}(k) dq_{\epsilon}(k)) \}. \] (38)

Here, the modified Leibniz law (19) has been used.

Now, the difference discrete variational principle gives rise to the difference discrete version of the canonical equations of motion (14)

\[ \Delta t q_{\epsilon}(k) = p_{\epsilon}(k+1), \quad \Delta t p_{\epsilon}(k) = -\partial \partial q_{\epsilon}(k+1). \] (39)

In fact, the first set of equations above can directly be derived from the Legendre transformation (35) and the second set can be gotten from the Legendre transformation (35) and the Euler-Lagrange equations (30). This indicates that the difference discrete variational principle approach to the difference discrete version of classical mechanics is self-consistent.

Let us consider the example 2.1 in the discrete Hamiltonian formalism.

**Example 2.2.** Hamiltonian formalism for the example 2.1.

First, the difference discrete conjugate momentum is introduced

\[ p_{\epsilon}(k) = \frac{\partial L_{D_{\epsilon}}(k-1)}{\partial (\Delta t q_{\epsilon}(k-1))} = \Delta t q_{\epsilon}(k-1). \] (40)

The Hamiltonian is introduced through the discrete Legendre transformation

\[ H_{D_{\epsilon}}(k) = p_{\epsilon}(k+1) \Delta t q_{\epsilon}(k) - L_{D_{\epsilon}}(k) = \frac{1}{2} p_{\epsilon}(k+1)^2 + V(q_{\epsilon}(k)). \] (41)

And a pair of difference discrete canonical equations read now

\[ \Delta t q_{\epsilon}(k) = p_{\epsilon}(k+1), \quad \Delta t p_{\epsilon}(k) = -\partial \partial q_{\epsilon}(k+1). \] (42)

In fact, the time difference discrete derivative of \( p_{\epsilon}(k) \) can also be derived from the difference discrete Lagrangian and the discrete Euler-Lagrange equation (32).
2.2 Variational and difference discrete variational principle in continuous and difference discrete classical field theory

We now study the difference discrete variational principle in Lagrangian and Hamiltonian formalism for the difference discrete classical field theory. For the sake of simplicity, let us consider the 1+1-d and 2-d cases in discrete classical field theory for a set of generic fields $u^\alpha, \alpha = 1, \cdots, r$. We first recall the multi-parameter differential approach to the variation of functional in Lagrangian formalism [16] and deal with the Hamiltonian formalism, then generalize it to the difference discrete variational principle for the difference discrete classical field theory in both Lagrangian and Hamiltonian formalism.

2.2.1 Variational principle in multi-parameter differential approach for classical field theory

For the sake of simplicity, let $X^{(1,n-1)}$ be an $n$-dimensional Minkowskian space as base manifold with coordinates $x^\mu, (\mu = 0, \cdots, n-1)$, $M$ the configuration space on $X^{(1,n-1)}$ with a set of generic fields $u^\alpha(x), (\alpha = 1, \cdots, r)$, $TM$ the tangent bundle of $M$ with coordinates $(u^\alpha, u^\alpha_\mu)$, where $u^\alpha_\mu = \frac{\partial u^\alpha}{\partial x^\mu}$, $F(TM)$ the function space on $TM$ etc. We also assume these fields to be free of constraints. In fact, the approach here can easily be applied to other cases.

2.2.1.1 Variational principle in Lagrangian formalism

The Lagrangian of the fields now is a functional of the set of generic fields under consideration:

$$L(u^\alpha, \dot{u}^\alpha) = \int d^{n-1}x L(u^\alpha(x,t), u^\alpha_\mu(x,t)), \quad u^\alpha(x) = u^\alpha(\mathbf{x},t), \quad \text{etc.},$$

and the action is given by

$$S(u^\alpha) = \int dt L(u^\alpha, \dot{u}^\alpha) = \int d^n x L(u^\alpha(x), u^\alpha_\mu(x)),$$

where $L(u^\alpha(x), u^\alpha_\mu(x))$ is the Lagrangian density.

In order to apply Hamilton’s principle we first consider how to define the variation of the action functional $S(u^\alpha)$ in a manner analog to the case of classical mechanics. In order to achieve this purpose, let us suppose that both $u^\alpha(x)$ and $u^\alpha_\mu(x)$ variate by an infinitesimal increments such that at a spacetime point of $x$ the infinitesimal variations of $u^\alpha$ and $u^\alpha_\mu$ can be described as follows

$$u^\alpha_e(x) = u^\alpha(x) + \epsilon^\beta \delta_\beta u^\alpha(x), \quad u^\alpha_\mu(x) = u^\alpha_\mu(x) + \epsilon^\beta \delta_\beta u^\alpha_\mu(x),$$

where $\epsilon^\beta, \beta = 1, \cdots, r$ are free parameters, each of which corresponds one direction in the configuration space $M$, and

$$\delta_\beta u^\alpha_e(x) := \frac{\partial}{\partial \epsilon^\beta} |_{\epsilon^\beta = 0} u^\alpha_e(x), \quad \delta_\beta u^\alpha_\mu(x) := \frac{\partial}{\partial \epsilon^\beta} |_{\epsilon^\beta = 0} u^\alpha_\mu(x),$$

the infinitesimal increments of $u^\alpha(x)$ and $u^\alpha_\mu(x)$ along the direction $\beta$ at the spacetime point $x$ respectively.

On the other hand, similar to the case of classical mechanics, the differential calculus of $u^\alpha(x)$ and $u^\alpha_\mu(x)$ in the function space $F(TM)$ can also be manipulated in the framework of
the multi-parameter differential approach. Furthermore, the exterior differential calculus in this framework can also be well established.

Thus the (exterior) differentials of \( u^\alpha(x) \) and \( u^\alpha_\mu(x) \) in the function space \( F(TM) \) at the spacetime point \( x \) can be defined as:

\[
\text{du}_\alpha^\beta(x) := \frac{\partial u^\alpha(x)}{\partial x^\beta} \, dx^\beta = \delta^\beta_\beta u^\alpha, \quad \text{du}_\mu^\alpha(x) := \frac{\partial u^\alpha_\mu(x)}{\partial x^\beta} \, dx^\beta = \delta^\beta_\beta u^\alpha_\mu. \tag{47}
\]

Now, the Lagrangian also becomes a family of Lagrangian functionals

\[
L_\epsilon(u^\alpha_\epsilon, \dot{u}^\alpha_\epsilon) = \int \! d^{n-1}x \mathcal{L}(u^\alpha_\epsilon(x, t), u^\alpha_\mu\epsilon(x, t)), \tag{48}
\]

and the action \( S(u^\alpha) \) becomes a family of functionals as well

\[
S \rightarrow S_\epsilon = S(u^\alpha_\epsilon). \tag{49}
\]

Then the variation of the action along the direction \( \beta \) can be manipulated as the derivative of \( S_\epsilon \) with respect to \( \epsilon^\beta \) and setting \( \epsilon^\beta = 0 \) afterwards. Namely,

\[
\delta_\beta S := \frac{\partial}{\partial \epsilon^\beta} \bigg|_{\epsilon^\beta=0} S_\epsilon. \tag{50}
\]

Manipulating the variation of the action functional in this manner and integrating by parts, it follows that

\[
\delta_\beta S = \int \! d^n x \{ (\frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha_\epsilon} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial u^\alpha_\mu\epsilon})) \delta_\beta u^\alpha_\epsilon + \partial_\mu (\frac{\partial \mathcal{L}}{\partial u^\alpha_\mu\epsilon} \delta_\beta u^\alpha_\epsilon) \} \bigg|_{\epsilon^\beta=0}.
\]

Assuming \( \delta_\beta u^\alpha_\epsilon \big|_{\pm \infty} = 0 \), and requiring \( \delta_\beta S = 0 \) along all directions according to Hamilton’s principle, then the Euler-Lagrange equations follow:

\[
\frac{\partial \mathcal{L}}{\partial u^\alpha} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial u^\alpha_\mu}) = 0. \tag{52}
\]

### 2.2.1.2 Variational principle in Hamiltonian formalism

In order to use the multi-parameter differential approach for the variational principle in Hamiltonian formalism for the classical field theory, we first define a family of a set of “momenta” that are canonically conjugate to the family of field variables

\[
\pi_{\beta \epsilon}(x) = \frac{\partial \mathcal{L}_\epsilon}{\partial \dot{u}^\alpha_\epsilon}, \tag{53}
\]

and take a Legendre transformation to get the Hamiltonian density in the family

\[
\mathcal{H}_\epsilon(u^\alpha_\epsilon, \pi_{\alpha \epsilon}, \nabla_a u^\alpha_\epsilon) = \pi_{\alpha \epsilon}(x) \dot{u}^\alpha_\epsilon(x) - \mathcal{L}_\epsilon(u^\alpha_\epsilon, \dot{u}^\alpha_\epsilon, \nabla_a u^\alpha_\epsilon), \tag{54}
\]

where \( \nabla_a = \frac{\partial}{\partial x^a}, a = 1, \ldots, n - 1 \). A family of the Hamiltonian then is given by

\[
H_\epsilon(t) = \int \! d^{n-1}x \mathcal{H}_\epsilon(x), \tag{55}
\]

with the Legendre transformation

\[
H_\epsilon(t) = \int \! d^{n-1}x \pi_{\alpha \epsilon}(x) \dot{u}^\alpha_\epsilon(x) - \mathcal{L}_\epsilon(u^\alpha_\epsilon, \dot{u}^\alpha_\epsilon, \nabla_a u^\alpha_\epsilon). \tag{56}
\]

The action \( S(u^\alpha) \) becomes a family of functionals as well

\[
S \rightarrow S_\epsilon = \int \! d^n x \{ \pi_{\alpha \epsilon}(x) \dot{u}^\alpha_\epsilon(x) - \mathcal{H}_\epsilon(u^\alpha_\epsilon, \dot{u}^\alpha_\epsilon, \nabla_a u^\alpha_\epsilon) \}. \tag{57}
\]
Then the variation of the action along the direction $\beta$ can be manipulated as the derivative of $S$ with respect to $\epsilon^\beta$ and setting $\epsilon^\beta = 0$ afterwards as was shown in (50). Namely,

$$\delta_\beta S = \int d^n x \{ \frac{\partial}{\partial \epsilon^\alpha} \pi_{\alpha c}(\dot{u}_c^\alpha - \nabla_a \frac{\partial H_a}{\partial u^c}) - \frac{\partial}{\partial \epsilon^\alpha} \dot{u}_c^\alpha(x) (\dot{\pi}_{\alpha c} + \frac{\partial H_a}{\partial u^c} \frac{\partial}{\partial \epsilon^\alpha} \dot{u}_c^\alpha) \} \bigg|_{\epsilon^\beta = 0}. \quad (58)$$

Thus, the canonical equations of motion follow form the stationary requirement of the action principle

$$\dot{u}_c^\alpha(x) = \frac{\partial H_a}{\partial u^c} + \nabla_a \frac{\partial H_a}{\partial u^c}, \quad \pi_{\alpha c}(x) = -\nabla_a \frac{\partial H_a}{\partial u^c}. \quad (59)$$

### 2.2.2 Difference discrete variational principle in multi-parameter differential approach for discrete classical field theory

For the sake of simplicity, we consider the case of $1+1$-dimensional spacetime or 2-dimensional space. It is straightforward to generalize for higher dimensional case.

Let $X^{(1,1)}$ or $X^{(2)}$ with suitable signature of the metrics be the base manifold, $L^2 = X$ a regular lattice with 2-directions $x_\mu, (\mu = 1, 2)$ on $X^{(1,1)}$ or $X^{(2)}$, $N$ the all nodes on $L^2$ that are coordinated by $x_{(i,j)}, (i, j) \in Z \times Z$ with index set $Ind(N)$, $M_D := M_{(i,j)}$ the piece of configuration space with a set of generic field variables $u^\alpha(x_{(i,j)}) = u^{(i,j)} \in M_D$ at the node $x_{(i,j)}$, $TM_{(i,j)}$ the tangent bundle of $M_{(i,j)}$ with the set of field variables and their differences $(u^{(i,j)} \mu, u^{(i,j)} \mu) \in T(M_{(i,j)}), F(TM_{(i,j)})$ the function space on $TM_{(i,j)}$, etc.

For a given node with coordinates $x_{(i,j)}$, let $N_{(i,j)}$ be the set of nodes neighboring to $x_{(i,j)}$ with index set $I_{(i,j)} = Ind(N)_{(i,j)}$, $X_{(i,j)} = \bigcup_{Ind(N) \in I_{(i,j)}} N$ a set of nodes that is related to $x_{(i,j)}$ by the differences, $M_D := M_{X_{(i,j)}} = \bigcup_{Ind(N) \in I_{(i,j)}} M_N$ the union of the pieces of configuration space on $X_{(i,j)}$.

It is known that the forward differences along each direction in $F(TM_{X_{(i,j)}})$ are defined by $\Delta_\mu u^{(i,j)} = u^{(i,j)} \mu$:

$$\Delta_1 u^{(i,j)} = \frac{1}{h_1} (u^{(i+1,j)} - u^{(i,j)}), \quad \Delta_2 u^{(i,j)} = \frac{1}{h_2} (u^{(i,j+1)} - u^{(i,j)}). \quad (60)$$

They are the bases of $T(X)$ and the upper-indexes reflect the corresponding coordinates of nodes on $X$. And their dual $d x^\mu = d X x^\mu$ are the bases of $T^*(X)$

$$d X x^\mu (\Delta_\nu) = \delta_\mu^\nu. \quad (61)$$

As in the previous subsection, the (forward) differences as the discrete derivatives do not obey the Leibniz law but the modified one along each direction. While in the space $T^*(X)$ dual to $T(X)$, an exterior differential calculus can be introduced (see the appendix) such that there exists an operator $d X$ with the following properties

$$d X : \Omega^l_{(i,j)} \to \Omega^{l+1}_{(i,j)}, \quad d X^2 = 0, \quad (62)$$

where $\Omega^l_{(i,j)}$ is the space of all $l$-forms in $T^*(X)$ and $d X$ does satisfy the Leibniz law:

$$d X (\omega \wedge \tau)_{(i,j)} = d X \omega_{(i,j)} \wedge \tau_{(i,j)} + (-1)^{\text{deg}(\omega)} \omega_{(i,j)} \wedge d X \tau_{(i,j)} \quad (63)$$

It is important to note that although the base manifold is discretized either the configuration space at each node or its neighboring union is still continuous. In addition, the variational calculation that will be carried out is mainly local. Therefore, similar
to the case of difference discrete classical mechanics, in what follows the difference discrete variations will be manipulated on the framework of the multi-parameter differential approach.

In addition, as was pointed out in the difference discrete version of classical mechanics, the differential and the exterior differential calculus in the function space can also be carried out in either \( F(TM_{i,j}) \) or \( F(TM_{X(i,j)}) := \bigcup_{\alpha \in \mathbb{I}} F(TN) \), etc. on the framework of the multi-parameter differential approach.

### 2.2.2.1 Difference discrete variational principle in Lagrangian formalism

For the difference discrete version of the classical field theory, the difference discrete Lagrangian denoted as

\[
\mathcal{L}_D^{(i,j)} = \mathcal{L}_D(u^{\alpha(i,j)}, u^{\alpha(i,j)}_\mu)
\]

is a functional in \( F(TM_{X(i,j)}) \). The action functional is given by

\[
S_D = \sum_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{L}_D(u^{\alpha(i,j)}, u^{\alpha(i,j)}_\mu).
\]

Taking the variation of \( S_D \) by the multi-parameter differential approach, the variation along the direction \( \beta \) is given by

\[
\delta_\beta S_D = \frac{\partial}{\partial \varepsilon^\beta} S_D|_{\varepsilon^\beta = 0}.
\]

For the sake of simplicity, in what follows we omit the multi-parameters \( \varepsilon^\beta \) in the course of calculation. Thus, we have

\[
\delta_\beta S_D = \sum_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \left\{ \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(i,j)}} \delta_\beta u^{\alpha(i,j)} + \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(i,j)}_\mu} \delta_\beta u^{\alpha(i,j)}_\mu \right\}.
\]

Employing the modified Leibniz law \([19]\) for the forward difference, we have

\[
\Delta_1 \left( \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial u^{\alpha(k-1,l)}} \delta_\beta u^{\alpha(k,l)} \right) = \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(k,l)}} \delta_\beta u^{\alpha(k,l)} + \Delta_1 \left( \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial u^{\alpha(k-1,l)}} \delta_\beta u^{\alpha(k,l)} \right),
\]

\[
\Delta_2 \left( \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial u^{\alpha(k,l-1)}} \delta_\beta u^{\alpha(k,l)} \right) = \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(k,l)}} \delta_\beta u^{\alpha(k,l)} + \Delta_2 \left( \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial u^{\alpha(k,l)}} \delta_\beta u^{\alpha(k,l)} \right).
\]

Assuming that \( \delta_\beta u^{\alpha(k,l)} \)’s vanish at infinity, it follows the discrete Euler-Lagrange equations

\[
\frac{\partial \mathcal{L}_D^{(i.j)}}{\partial u^{\alpha(i,j)}} - \Delta_1 \left( \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial \Delta_1 u^{\alpha(i,j-1)}} \right) - \Delta_2 \left( \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial \Delta_2 u^{\alpha(i,j-1)}} \right) = 0.
\]

Let us consider an example to show that the difference discrete variational principle gives right result.

**Example 2.3. A discrete classical field theory with difference discrete Lagrangian**

Consider the following discrete classical field theory with difference discrete Lagrangian:

\[
\mathcal{L}_D(u^{\alpha(i,j)}, u^{\alpha(i,j)}_\mu) = \frac{1}{2}(\Delta_1 u^{\alpha(i,j)})^2 - V(u^{\alpha(i,j)}).
\]

The difference discrete variational principle gives the discrete Euler-Lagrange equation \((67)\) as follows

\[
\Delta_1 (\Delta_1 u^{\alpha(i-1,j)}) + \Delta_2 (\Delta_2 u^{\alpha(i,j-1)}) - \frac{\partial}{\partial u^{\alpha(i,j)}} V(u^{\alpha(i,j)}) = 0,
\]

\[
(69)
\]
Then the Hamiltonian is introduced through the discrete Legendre transformation. The action functional (65) now is given by

\[ \frac{1}{h_1}(u^{\alpha(i+1,j)} - 2u^{\alpha(i,j)} + u^{\alpha(i-1,j)}) + \frac{1}{h_2}(u^{\alpha(i,j+1)} - 2u^{\alpha(i,j)} + u^{\alpha(i,j-1)}) = \frac{\partial}{\partial u^\alpha} V(u^{\alpha(i,j)}). \]  

(70)

This is also what is wanted for the difference discrete counterpart of the relevant Euler-Lagrange equation in the continuous limit.

2.2.2.2 Difference discrete variational principle in Hamiltonian formalism

Let \( X^{(1,1)} \) be the base space. We first define a set of the discrete canonical conjugate momenta on the tangent space of the set of nodes neighboring to the node \( x(i, j) \), i.e. on \( T(X_{(i,j)}) \):

\[ \pi^\alpha_{(i,j)} = \frac{\partial L^\mu}{\partial (\dot{u}^\mu_{(i,j)})}. \]  

(71)

The difference discrete Hamiltonian is introduced through the discrete Legendre transformation

\[ H_D^{(i,j)} = \pi^\alpha_{(i+1,j)} \Delta_t u^\alpha_{(i,j)} - L_D^{(i,j)}. \]  

(72)

The action functional (63) now is given by

\[ S_D = \sum_{(i,j) \in Z \times Z} (\pi^\alpha_{(i+1,j)} \Delta_t u^\alpha_{(i,j)} - H_D^{(i,j)}). \]  

(73)

Taking the variation of \( S_D \) by the multi-parameter differential approach and using the modified Leibniz law, the variation along the direction \( \beta \) is given by

\[ \delta_\beta S_D = \sum_{(i,j) \in Z \times Z} \left\{ \delta_\beta \pi^\alpha_{(i+1,j)} (\Delta_t u^\alpha_{(i,j)} - \frac{\partial H_D^{(i,j)}}{\partial \pi^\alpha_{(i+1,j)}}) - (\Delta_t \pi^\alpha_{(i,j)} + \frac{\partial H_D^{(i,j)}}{\partial u^\alpha_{(i,j)}}) - \Delta_x (\frac{\partial H_D^{(i,j-1)}}{\partial \Delta_x u^\alpha_{(i,j-1)}}) \delta_\beta u^\alpha_{(i,j)} + \Delta_t (\pi^\alpha_{(i,j)} \delta_\beta u^\alpha_{(i,j)}) - \Delta_x (\frac{\partial H_D^{(i,j-1)}}{\partial \Delta_x u^\alpha_{(i,j-1)}}) \delta_\beta u^\alpha_{(i,j)}) \right\}. \]  

(74)

The stationary requirement for difference discrete version of action functional against all variations along any direction gives rise to the difference discrete version of the canonical equations of motion:

\[ \Delta_t u^\alpha_{(i,j)} = \frac{\partial H_D^{(i,j)}}{\partial \pi^\alpha_{(i+1,j)}}, \]

\[ \Delta_t \pi^\alpha_{(i,j)} = -\frac{\partial H_D^{(i,j)}}{\partial u^\alpha_{(i,j)}} + \Delta_x (\frac{\partial H_D^{(i,j-1)}}{\partial \Delta_x u^\alpha_{(i,j-1)}}). \]  

(75)

It should be noted that the first set of equations above may directly follow from the difference discrete Legendre transformation (72), while the second set from the transformation (72) and the difference discrete Euler-Lagrange equations (67). As in the case of difference discrete version of the classical mechanics, this also indicates that the difference discrete variational principle approach is self-consistent.

Let us consider the example 2.3 in the discrete Hamiltonian formalism.

Example 2.4: Difference discrete Hamiltonian formalism for the example 2.3.

First, a set of the difference discrete conjugate momenta are introduced

\[ \pi^\alpha_{(i,j)} = \frac{\partial L_D^{(i-1,j)}}{\partial \dot{u}^\alpha_{(i,j)}} = \Delta_t u^\alpha_{(i-1,j)}. \]  

(76)

Then the Hamiltonian is introduced through the discrete Legendre transformation

\[ H_D^{(i,j)} = \pi^\alpha_{(i+1,j)} \Delta_t u^\alpha_{(i,j)} - L_D^{(i,j)} = \frac{1}{2} \pi^\alpha_{(i+1,j)}^2 + V(u^\alpha_{(i,j)}). \]  

(77)
And a pair sets of difference discrete canonical equations read now
\[ \Delta_t u^{\alpha(i,j)} = \pi^{\alpha(i+1,j)}, \quad \Delta_t \pi^{\alpha(i,j)} = -\frac{\partial V(u^{\alpha(i,j)})}{\partial u^{\alpha(i,j)}}. \] (78)

In fact, the time discrete derivative of \( \pi^{\alpha(i,j)} \) follows from the difference discrete Lagrangian and the discrete Euler-Lagrange equations \( (69) \).

3 Euler-Lagrange cohomology, symplectic structure preserving property in continuous and discrete classical mechanics

Now we are ready to study certain difference discrete version of the Euler-Lagrange cohomology, its relations to the simplectic and multisymplectic structures as well as their preserving properties in the classical mechanics and field theory in both Lagrangian and Hamiltonian formalism. In this and the next section, we deal with the case of classical mechanics and field theory respectively. We first recall some content on the issues in the continuous case for the mechanics in subsection 3.1. Then we consider these issues in difference discrete classical mechanics in the Lagrangian and Hamiltonian formalism in subsection 3.2 and 3.3 respectively.

3.1 Euler-Lagrange cohomology, symplectic structure preserving property in continuous classical mechanics

By virtue of the multi-parameter differential approach to exterior derivatives in the function space, the Euler-Lagrange cohomology and symplectic structure preserving property in Lagrangian formalism for the classical mechanics has been studied in [16]. We recall the content in 3.1.1 and deal with the the relevant issues in the Hamiltonian formalism in 3.1.2.

3.1.1 Lagrangian formalism

It is obvious but important to note that in manipulating the variation of the action functional for the classical mechanics in Lagrangian formalism in the subsection 2.1, the variation of the action is very closely linked with the differential of \( S_\epsilon \) with respect to the free parameters \( \epsilon^k \) in the congruence of curves, i.e. \( dS_\epsilon \).

Furthermore, the exterior differential calculus in the framework of the multi-parameter space can be well established. Namely, in the free multi-parameter space, the standard exterior differential calculus can be introduced. And this multi-parameter exterior differential calculus can also be employed to deal with the exterior differential calculus in the function spaces on \( M \) and \( TM \), i.e. \( FM \) and \( F(TM) \).

Therefore, the integrand in (7) for \( dS_\epsilon \) and the boundary term may be views as 1-forms. And the differential of the family of Lagrangian functions \( L_\epsilon := L(q^i(t), \dot{q}^i(t)) \) with respect to \( \epsilon^k \) is given by
\[ dL_\epsilon \mid_{\epsilon^k=0} = \left\{ \frac{\partial L}{\partial q^j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} \right\} dq^j + \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}^j} dq^j \right\}. \] (79)
Let us define a family of the Euler-Lagrange 1-forms and that of the canonical 1-forms \( \theta \) on \( T^*M \),

\[
E_\epsilon(q^i_\epsilon, \dot{q}^i_\epsilon) := \left\{ \frac{\partial L_\epsilon}{\partial q^i_\epsilon} \right\} \frac{dq^i_\epsilon}{dt},
\]

\[
\theta_{L\epsilon} := \frac{\partial L_\epsilon}{\partial q^i_\epsilon} dq^i_\epsilon,
\]

we have

\[
dL_\epsilon(q^i_\epsilon, \dot{q}^i_\epsilon) |_{\epsilon=k=0} = E(q^i_\epsilon, \dot{q}^i_\epsilon) + \frac{d}{dt} \theta_L.
\]

Furthermore, owing to the nilpotency of \( d \) with respect to \( \epsilon_k \) in the cotangent space of the congruence of curves on \( F(T^*M) \),

\[
d^2 L_\epsilon(q^i_\epsilon, \dot{q}^i_\epsilon) |_{\epsilon=k=0} = 0,
\]

it follows that

\[
dE(q^i_\epsilon, \dot{q}^i_\epsilon) + \frac{d}{dt} \omega_L = 0,
\]

where \( \omega_L \) is the symplectic 2-form in the Lagrangian formalism defined by

\[
\omega_L = d\theta_L = \frac{\partial^2 L_\epsilon}{\partial q^i_\epsilon \partial q^j_\epsilon} dq^i_\epsilon \wedge dq^j_\epsilon + \frac{\partial^2 L_\epsilon}{\partial \dot{q}^i_\epsilon \partial \dot{q}^j_\epsilon} d\dot{q}^i_\epsilon \wedge d\dot{q}^j_\epsilon.
\]

And it does not change if the canonical 1-form transforms as

\[
\theta \rightarrow \theta_L' = \theta_L + d\beta(q^i_\epsilon, \dot{q}^i_\epsilon),
\]

where \( \beta(q^i_\epsilon, \dot{q}^i_\epsilon) \) is an arbitrary function of \((q^i_\epsilon, \dot{q}^i_\epsilon)\).

We have established the important and significant issues on the Euler-Lagrange cohomology and symplectic structure preserving law in the Lagrangian formalism. Before enumerating them and exploring their significance, let us investigate the relevant issues in the Hamiltonian formalism.

### 3.1.2 Hamiltonian mechanics

The above cohomological and other issues in the Lagrangian formalism of the classical mechanics can also be well established in the Hamiltonian formalism. Again the multiparameter differential approach will be employed.

In order to transfer to the phase space of the Hamiltonian formalism, in the subsection 2.1, we introduce a family of conjugate momenta (10) from the family of Lagrangian \( L_\epsilon \), i.e.,

\[
p_{j\epsilon} = \frac{\partial L_\epsilon}{\partial \dot{q}^j_\epsilon},
\]

and taken a Legendre transformation to get a set of the Hamiltonian functions in the family in (11), i.e.

\[
H_k := H(q^i_\epsilon, p_{j\epsilon}) = p_{k\epsilon} \dot{q}^k_\epsilon - L(q^i_\epsilon, \dot{q}^i_\epsilon).
\]

We have also calculated the variation of action functional along the direction \( k \) in (13) where the differentiation of a family of action functionals have been taken as follows

\[
dS_\epsilon = \int_{t_1}^{t_2} dt \{ (\dot{q}^i_\epsilon - \frac{\partial H_\epsilon}{\partial p_{i\epsilon}}) dp_{i\epsilon} - \left( \frac{\partial H_\epsilon}{\partial q^i_\epsilon} + \dot{p}_{i\epsilon} \right) dq^i_\epsilon + \frac{d}{dt}(p_{i\epsilon} dq^i_\epsilon) \}.
\]

Let us introduce a pair sets of canonical Euler-Lagrange 1-form families in the Hamiltonian formalism as follows:

\[
E_{pe}(q^i_\epsilon, p_{j\epsilon}) = -(\dot{p}_{j\epsilon} + \frac{\partial H_\epsilon}{\partial q^j_\epsilon}) dq^i_\epsilon, \quad E_{eq}(q^i_\epsilon, p_{j\epsilon}) = (\dot{q}^i_\epsilon - \frac{\partial H_\epsilon}{\partial p_{i\epsilon}}) dp_{j\epsilon}.
\]
And a family of the canonical 1-forms
\[ \theta_{H^e} = p_i dq^i. \] (88)

Thus, the equation (86) can be expressed as
\[ dS_e = \int_{t_1}^{t_2} dt \{ E_{pe} + E_{qe} + \frac{d}{dt} \theta_{H^e} \}. \] (89)

This equation can be viewed as the equation for 1-forms.

Furthermore, due to the nilpotency of \( d \) in the multi-parameter differential approach, it is straightforward from
\[ 0 = d^2 S_e \mid_{\epsilon=0} = 0 \] (90)
to get
\[ d(E_p + E_q) + \frac{d}{dt} \omega_H = 0. \] (91)

Here \( \omega_H \) is the symplectic 2-form in the Hamiltonian formalism
\[ d\omega_H = d\theta_H = dp_i \wedge dq^i. \] (92)

We may introduce a family of \( z^T_{\epsilon} = (p^T_{\epsilon}, q^T_{\epsilon}) \), where \( p^T_{\epsilon} = (p_{1\epsilon}, \cdots, p_{n\epsilon}), q^T_{\epsilon} = (q^1_{\epsilon}, \cdots, q^n_{\epsilon}) \), defined by
\[ z_{\epsilon}(t) := z(t) + \epsilon \delta_l z(t), \] (93)
where
\[ \delta_l z(t) = \left. \frac{d}{dt} \right|_{\epsilon=0} z_{\epsilon}(t) \] (94)
infinite variation of \( z(t) \) along the direction \( l \) in the configuration space.

Then the Euler-Lagrange 1-forms in (87) become
\[ E_{z\epsilon}(z_{\epsilon}, \dot{z}_{\epsilon}) = dz^T_{\epsilon}(\nabla_{z_{\epsilon}} H_{\epsilon} - J \dot{z}_{\epsilon}), \] (95)
where \( J \) is a \( 2n \times 2n \) symplectic matrix.

Taking the exterior differential of \( E_{z\epsilon} \) and setting \( \epsilon^k = 0 \) afterwards, it follows that
\[ dE_z + \frac{1}{2} dz^T \wedge J dz = 0. \] (96)

This is equivalent to the equation (91).

3.1.3 Remarks on Euler-Lagrange cohomology and symplectic structure preserving property in classical mechanics

It is easy to check that under the Legendre transformation (11) all relevant issues in the Lagrangian and Hamiltonian formalism are in one-to-one correspondence. For instance, we have the following equivalent relations:

| Lagrangian formalism          | Hamiltonian formalism                  |
|-------------------------------|----------------------------------------|
| \( L(q^i, \dot{q}^i) \leftrightarrow p_i \dot{q}^i - H(p_i, q^i) \) | \( E(q^i, \dot{q}^i) \leftrightarrow E_p(p_i, q^i) + E_q(p_i, q^i) \) |
| \( \theta_L \leftrightarrow \theta_H \) | \( \omega_L \leftrightarrow \omega_H \) |
| \( dL = E + \frac{d}{dt} \theta_L \leftrightarrow d(p_i \dot{q}^i - H) = E_p + E_q + \frac{d}{dt} \theta_H \) |
| \( dE + \frac{d}{dt} \omega_L = 0 \leftrightarrow d(E_p + E_q) + \frac{d}{dt} \omega_H = 0 \) |
Therefore, these important issues on the Euler-Lagrange cohomology and the symplectic structure preserving property in two formalisms are corresponding to each other. Let us enumerate these issues in the Hamiltonian formalism.

First, the null canonical Euler-Lagrange 1-forms with $\varepsilon^l = 0$ give rise to a pair sets of the canonical equations of motion (14). In terms of $z$, the null canonical Euler-Lagrange 1-form (95) with $\varepsilon^l = 0$ gives rise to the canonical equations in $z(t)$:

$$\dot{z} = J^{-1} \nabla_z H.$$ (98)

Secondly, the above null forms are the special case of the coboundary canonical Euler-Lagrange 1-forms, say,

$$E(z, \dot{z}) = d\alpha(z, \dot{z}),$$ (99)

where $\alpha(z, \dot{z})$ is an arbitrary function of $(z, \dot{z})$ on the phase space.

Thirdly, from either the expression of $d(p_i \dot{q}^i - H)$ or the definitions of the canonical Euler-Lagrange 1-forms it is easy to see that the canonical Euler-Lagrange 1-forms are not exact in general. Therefore, there exists a nontrivial Euler-Lagrange cohomology in the Hamiltonian formalism for classical mechanics defined as

$$H_{CM} := \{\text{Closed Euler-Lagrange forms}\} / \{\text{Exact Euler-Lagrange forms}\}.$$

Fourthly, from the equation (91) or (96) it follows a theorem on the symplectic structure preserving law in the Hamiltonian mechanics.

*Theorem 3.1:*

On the phase space of the Hamiltonian mechanics, the symplectic structure $\omega_H$ preserving law

$$\frac{d}{dt} \omega_H = 0$$ (100)

holds if and only if the canonical Euler-Lagrange forms are closed:

$$dE(z, \dot{z}) = 0, \quad \text{i.e.} \quad d(E_p(q^i, p_j) + E_q(q^i, p_j)) = 0.$$ (101)

This means that the symplectic conservation law holds not only in the solution space of the equations of motion as shown in the standard approach [1][2] but also in the function space with the closed Euler-Lagrange condition [10][11].

Finally, it should be mentioned that the Euler-Lagrange cohomological scenario in the Hamiltonian formalism may be performed in two slightly different processes. Namely, it may either start from the exterior derivative of the action functional or begin with the canonical equations. In the second process, the families of the canonical Euler-Lagrange 1-forms (87) and (95) may be introduced directly referring to the corresponding canonical equations (14) and (98) but releasing first all canonical variables from the solution space of the canonical equations (14) and (98) respectively. Then by taking the exterior differential of the families of the canonical Euler-Lagrange 1-forms (87) and (95) and setting the free parameters being vanish afterwards it also follows the theorem 3.1 as was shown in (100).

### 3.2 Discrete Euler-Lagrange cohomology and symplectic structure preserving property in difference discrete classical mechanics
3.2.1 Difference discrete Lagrangian formalism

Let us first consider the discrete Euler-Lagrange cohomology, its relation to the symplectic structure and its preserving property for the difference discrete classical mechanics in the Lagrangian formalism.

The difference discrete Lagrangian at the moment $t_k$ is written as in $\textbf{(16)}$ on $F(T\mathcal{M}_k)$. That is

\[ L_D^{(k)} = L_D(q_i^{(k)}, \dot{q}^{(k)}), \]

where $q_i^{(k)}$ is the forward difference of $q^{(k)}$ defined in $\textbf{(17)}$.

Taking the exterior differential $d$ of $L_D^{(k)}$ in the function space $F(T\mathcal{M}_k)$ as has been done in the subsection 2.1 in the framework of multi-parameter differential approach, it follows

\[ dL_D^{(k)} = \frac{\partial L_D^{(k)}}{\partial q_i^{(k)}} dq_i^{(k)} + \frac{\partial L_D^{(k)}}{\partial \dot{q}^{(k)}} d\dot{q}^{(k)}. \]

Using the modified Leibniz law $\textbf{(28)}$ with respect to the forward difference $\Delta_t$ defined in $\textbf{(17)}$ and introducing the discrete Euler-Lagrange 1-form as well as the discrete canonical 1-form $\theta_{DL}^{(k)}$

\[ E_D^{(k)}(q_i^{(k)}, \dot{q}^{(k)}) := \{ \frac{\partial L_D^{(k)}}{\partial \dot{q}^{(k)}} - \Delta_t \left( \frac{\partial L_D^{(k-1)}}{\partial (\Delta_t q^{(k-1)})} \right) \} dq^{(k)} , \quad \theta_{DL}^{(k)} = \frac{\partial L_D^{(k-1)}}{\partial (\Delta_t q^{(k-1)})} dq^{(k)}, \] (102)

we have

\[ dL^D^{(k)} = E_D^{(k)} + \Delta_t \theta_{DL}^{(k)}. \] (103)

Due to the nilpotency of $d$ on $T^*(\mathcal{M}_k)$, $d^2 L_D^{(k)} = 0$, we get

\[ dE_D^{(k)} + \Delta_t \omega_{DL}^{(k)} = 0, \] (104)

where $\omega_{DL}^{(k)}$ is a discrete symplectic 2-form on $T^*(\mathcal{M}_k)$

\[ \omega_{DL}^{(k)} = dq_i^{(k)} \wedge dq_j^{(k)} = \frac{\partial^2 L_D^{(k-1)}}{\partial q_i^{(k)} \partial (\Delta_t q^{(k-1)})} dq_i^{(k)} \wedge dq_j^{(k)} + \frac{\partial^2 L_D^{(k-1)}}{\partial (\Delta_t q^{(k)}) \partial (\Delta_t q^{(k-1)})} d\dot{q}_i^{(k)} \wedge d\dot{q}_j^{(k)}. \] (105)

3.2.2 Difference discrete Hamiltonian mechanics

Let us now study the symplectic structure preserving property in Hamiltonian formalism for the difference discrete mechanics when the time $t$ is discretized in the manner of last subsection. We start with what has been constructed for the difference discrete mechanics in Lagrangian formalism.

Let us first define the discrete canonical conjugate momentum

\[ p_i^{(k)} = \frac{\partial L_D^{(k-1)}}{\partial (\Delta_t q^{(k-1)})}. \] (106)

Now the difference discrete Hamiltonian can be introduced through the discrete Legendre transformation

\[ H_D^{(k)} = p_i^{(k+1)} \Delta_t q_i^{(k)} - L_D^{(k)}. \] (107)

A set of canonical equations for the time difference discrete derivative of $p_i^{(k)}$ follow from the difference discrete Euler-Lagrange equations $\textbf{(17)}$ and the above discrete Legendre transformation $\textbf{(108)}$

\[ \Delta_t p_i^{(k)} = -\frac{\partial H_D^{(k)}}{\partial q_i^{(k)}}. \] (108)
From the discrete Legendre transformation, another set of canonical equations for the time difference discrete derivative of \( q^{(k)} \) follow
\[
\Delta_t q^{(k)} = \frac{\partial H^{(k)}}{\partial p^{(k+1)t}}, \tag{110}
\]
In terms of \( z^{(k)} \), a pair of the canonical equations \((122)\) become
\[
\Delta_t z^{(k)} = J^{-1}\nabla_z H_D^{(k)}(z^{(k)}). \tag{111}
\]
We now consider the relevant cohomological issues in the difference discrete version of the Hamiltonian mechanics. This time we start from introducing the discrete Euler-Lagrange 1-forms. To this end, all time discrete canonical variables \((q^{(k)}, p_j^{(k)})\) should be released from the solution space of the difference discrete canonical equations \((111)\) or \((113)\). This can easily be realized by means of the multi-parameter differential approach. Thus a pair of discrete Euler-Lagrange 1-forms can be introduced:
\[
E_{Dp}^{(k)}(q^{(k)}, p_j^{(k)}) = (\Delta_t p_j^{(k)} + \frac{\partial H^{(k)}}{\partial q^{(k)}(t)}) dq^{(k)},
\]
\[
E_{Dq}^{(k)}(q^{(k)}, p_j^{(k)}) = (\Delta_t q^{(k)} - \frac{\partial H^{(k)}}{\partial p_j^{(k)}}) dp_j^{(k)}, \tag{112}
\]
or in term of \( z^{(k)} \)
\[
E_{Dz}^{(k)}(z^{(k)}) = dz^{(k)} T (J \Delta_t z^{(k)} - \nabla_z H_D^{(k)}(z^{(k)})). \tag{113}
\]
By taking the exterior differential of the discrete Euler-Lagrange 1-forms \((112)\) and \((113)\), it is straightforward to prove the following formula
\[
dE_{Dz}^{(k)}(z^{(k)}) + \Delta_t \omega_{DH}^{(k)} = 0, \tag{114}
\]
where \( \omega_{DH}^{(k)} \) is the difference discrete version of the symplectic 2-form at the moment \( t_k \) given by
\[
\omega_{DH}^{(k)} = \frac{1}{2} dz^{(k)} T \wedge J dz^{(k)}. \tag{115}
\]
We can also start from the exterior differential of \( L_D^{(k)} = p_i^{(k+1)} \Delta_t q^{(k)} - H_D^{(k)} \), introduce the discrete Euler-Lagrange 1-forms and the canonical 1-form then take the second exterior differential to get above equations.

### 3.2.3 Remarks on discrete Euler-Lagrange cohomology and symplectic structure preserving property in classical mechanics

As in the case of continuous classical mechanics, it is easy to check that under the difference discrete version of the Legendre transformation \((108)\) all relevant issues in the difference discrete Lagrangian and difference discrete Hamiltonian formalism are in one-to-one correspondence. In fact, we have the following equivalent relations:

\[
\begin{array}{ll}
\text{Discrete Lagrangian formalism} & \text{Discrete Hamiltonian formalism} \\
L_D^{(k)}(q^{(k)}, \dot{q}^{(k)}) & p_i^{(k+1)} \dot{q}^{(k)}(t) - H_D^{(k)}(p_i^{(k)}, q^{(k)}) \\
E_D^{(k)}(q^{(k)}, \dot{q}^{(k)}) & E_{Dp}^{(k)}(p_i^{(k)}, q^{(k)}) + E_{Dq}^{(k)}(p_i^{(k)}, q^{(k)}) \\
\theta_{DL}^{(k)} & \theta_{DH}^{(k)} \\
\frac{dL_D^{(k)}}{dt} + d\theta_{DL}^{(k)} & d(p_i^{(k+1)} \dot{q}^{(k)}(t) - H_D^{(k)}) = E_{Dp}^{(k)} + E_{Dq}^{(k)} + \frac{d}{dt} \theta_{DH}^{(k)} \\
\omega_{DL}^{(k)} & \omega_{DH}^{(k)} \\
dE_D^{(k)} + \frac{d}{dt} \omega_{DL}^{(k)} = 0 & d(E_{Dp}^{(k)} + E_{Dq}^{(k)}) + \frac{d}{dt} \omega_{DH}^{(k)} = 0
\end{array} \tag{116}
\]
Therefore, for enumerating the important issues on the difference discrete Euler-Lagrange cohomology and the difference discrete symplectic structure preserving property, we may also work with one formalism, the corresponding issues in another formalism can easily be established. Let us also enumerate these issues in the difference discrete Hamiltonian formalism as follows.

First, the null discrete canonical Euler-Lagrange forms give rise to the canonical equations and they are the special case of the coboundary discrete canonical Euler-Lagrange forms.

Secondly, since the first terms in the definitions (112) and (113) are not exact in general so that the discrete canonical Euler-Lagrange forms are not always exact. Therefore, there exists a nontrivial difference discrete version of the Euler-Lagrange cohomology in discrete Hamiltonian mechanics:

\[ H_{DCM} := \{ \text{Closed discrete Euler-Lagrange forms} \} / \{ \text{Exact discrete Euler-Lagrange forms} \}. \]

Thirdly, from the equation (114) it follows straightforwardly the following theorem for the necessary and sufficient condition of the difference discrete symplectic structure preserving law.

**Theorem 3.2:**

The difference discrete symplectic structure preserving equation

\[ \Delta t \omega_{DH}^{(k)} = 0, \quad i.e. \quad \omega_{DH}^{(k+1)} = \omega_{DH}^{(k)} \] (117)

holds if and only if the discrete Euler-Lagrange forms are closed:

\[ dE_{D}^{(k)}(z^{(k)}) = 0. \] (118)

Fourthly, the difference discrete symplectic structure preserving law holds in the function space associated with the difference discrete version of the closed Euler-Lagrange condition in general rather than in the solution space of the canonical equations only.

Finally, all these issues can be reached by taking the exterior differential of \( p_i^{(k+1)} \dot{q}^{(k)} - H_{D}^{(k)} \) first, then introducing the discrete Euler-Lagrange 1-forms and discrete canonical 1-form. The theorem 3.2 follows from \( d^2(p_i^{(k+1)} \dot{q}^{(k)} - H_{D}^{(k)}) = 0 \). In fact, in the difference discrete version of the Lagrangian formalism, it has started and progressed in this manner.

## 4 Euler-Lagrange cohomology, multisymplectic structure preserving property in continuous and difference discrete classical field theory

We now study the discrete Euler-Lagrange cohomology and its relation to the discrete multisymplectic structure in discrete classical field theory. In order to self-contained, we first recall the the content on Euler-Lagrange cohomology and multisymplectic structure preserving property in continuous classical field theory \[10\] as well as the multi-parameter differential approach to the exterior differential in the function spacein the subsection 4.1 for the both Lagrangian and Hamiltonian formalism. For the sake of simplicity, for the difference discrete cases, let us consider the 1+1-d and 2-d cases in discrete classical field theory for the a set of generic fields \( u^a \) without constraints in the Lagrangian and Hamiltonian formalism respectively in the subsection 4.2.
4.1 Euler-Lagrange cohomology, multisymplectic structure preserving property in continuous classical field theory

We recall some content on the Euler-Lagrange cohomology and its relation to the multisymplectic structure preserving property in both Lagrangian and Hamiltonian formalism for continuous classical field theory by means of the multi-parameter differential approach.

4.1.1 Classical field theory in Lagrangian formalism

In the multi-parameter differential approach to the variation of the action of the classical field theory in subsection 2.2.1, the variation along the direction $\beta$ of the action has been given by the equation (50), i.e.

$$\delta_\beta S := \frac{\partial}{\partial \epsilon_\beta} \bigg|_{\epsilon_\beta=0} S_\epsilon.$$  

It ia shown that the differentiation of the action functional with respect to the free parameters $\epsilon_\beta$ is given by

$$dS_\epsilon = \int d^m x \{ \frac{\partial L_\epsilon}{\partial u^\alpha_\epsilon} \partial u^\alpha_\epsilon + \partial_\mu \left( \frac{\partial L_\epsilon}{\partial u^\alpha_{\mu \epsilon}} du^\alpha_\epsilon \right) \}.$$  

Note that the integrand in the above equation is an equation for 1-forms:

$$dL_\epsilon \equiv \left( \frac{\partial L_\epsilon}{\partial u^\alpha_\epsilon} - \partial_\mu \left( \frac{\partial L_\epsilon}{\partial u^\alpha_{\mu \epsilon}} \right) \right) du^\alpha_\epsilon.$$  

(119)

Let us define a family of the Euler-Lagrange 1-forms

$$E_\epsilon(u^\alpha_\epsilon, u^\alpha_{\mu \epsilon}) := \left( \frac{\partial L_\epsilon}{\partial u^\alpha_\epsilon} - \partial_\mu \left( \frac{\partial L_\epsilon}{\partial u^\alpha_{\mu \epsilon}} \right) \right) du^\alpha_\epsilon,$$  

(120)

and $n$ sets of 1-forms that each set corresponds to a family of canonical 1-forms

$$\theta^\mu_\epsilon := \frac{\partial L_\epsilon}{\partial u^\alpha_{\mu \epsilon}} du^\alpha_\epsilon.$$  

(121)

Then the equation (119) becomes

$$dL_\epsilon = E_\epsilon(u^\alpha_\epsilon, u^\alpha_{\mu \epsilon}) + \partial_\mu \theta^\mu_\epsilon.$$  

(122)

Furthermore, due to the nilpotency of $d$ with respect to $\epsilon^k$, taking the second exterior differential of $L_\epsilon(u^\alpha_\epsilon, u^\alpha_{\mu \epsilon})$ and setting $\epsilon^\beta = 0$ afterwards $d^2 L_\epsilon(u^\alpha_\epsilon, u^\alpha_{\mu \epsilon}) |_{\epsilon^\beta=0} = 0$, it follows that

$$dE(u^\alpha_\epsilon, u^\alpha_{\mu \epsilon}) + \partial_\mu \omega_L^\mu = 0,$$  

(123)

where $\omega_L^\mu$ are $n$ symplectic structures defined by

$$\omega_L^\mu = d\theta^\mu = \frac{\partial^2 L}{\partial u^\alpha \partial u^\alpha_\mu} du^\alpha \wedge du^\beta + \frac{\partial^2 L}{\partial u^\alpha_\mu \partial u^\alpha_\nu} du^\alpha_\mu \wedge du^\beta.$$  

(124)

And they do not change if the set of $n$ canonical 1-forms transform as

$$\theta^\mu \rightarrow \theta'^\mu = \theta^\mu + d\beta(u^\alpha, u^\alpha_{\mu}),$$  

(125)

where $\beta(u^\alpha, u^\alpha_{\mu})$ is an arbitrary function of $(u^\alpha, u^\alpha_{\mu})$. 

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4.1.2 Classical field theory in Hamiltonian formalism

All steps that have been progressed in the Lagrangian field theory can also be well progressed very similarly in the Hamiltonian formalism. In order to do so, we first define a set of “momenta” that are canonically conjugate to the field variables

\[ \pi_\beta(x) = \frac{\partial L}{\partial \dot{u}^\beta}, \]

and take a Legendre transformation (54) to get the Hamiltonian density

\[ H(u^\alpha, \pi_\alpha, \nabla_a u^\alpha) = \pi_\alpha(x) \dot{u}^\alpha(x) - \mathcal{L}(u^\alpha, \dot{u}^\alpha, \nabla_a u^\alpha), \quad a = 1, \ldots, n - 1. \]

Let us consider the action \( S(u^\alpha) \) becoming a family of functionals as follows

\[ S \to S_\epsilon = \int d^n x \{ \pi_\alpha(x) \dot{u}^\alpha(x) - H_\epsilon(u^\alpha, \pi_\alpha, \nabla_a u^\alpha) \}. \tag{126} \]

The differential of \( S_\epsilon \) can be calculated in the multi-parameter differential approach and the integrand’s differential \( d\mathcal{L}_\epsilon \) reads

\[ d\mathcal{L}_\epsilon|_{\epsilon=0} = d\pi_\alpha(\dot{u}^\alpha - \frac{\partial H}{\partial u^\alpha} - (\ddot{\pi}_\alpha + \frac{\partial H}{\partial \dot{u}^\alpha} - \nabla_a(\frac{\partial H}{\partial (\nabla_a u^\alpha(x, j)})du^\alpha(i,j) \]

\[ + \nabla_t(\pi_\alpha d\alpha^\alpha) - \nabla_a(\frac{\partial H}{\partial (\nabla_a u^\alpha)} d\alpha^\alpha). \tag{127} \]

Introducing a pair sets of the canonical Euler-Lagrange 1-forms in the Hamiltonian formalism

\[ E_u = d\pi_\alpha(\dot{u}^\alpha(x) - \frac{\partial H}{\partial u^\alpha(x)}), \quad E_\pi = du^\alpha\{ -\ddot{\pi}_\alpha(x) - \frac{\partial H}{\partial \dot{u}^\alpha(x)} + \nabla_a \frac{\partial H}{\partial (\nabla_a u^\alpha)} \}, \tag{128} \]

and \( n \)-canonical 1-forms

\[ \theta^0 = \pi_\alpha du^\alpha, \quad \theta^\alpha = \frac{\partial H}{\partial (\nabla_a u^\alpha)} du^\alpha, \tag{129} \]

then we have

\[ d\mathcal{L}_\epsilon|_{\epsilon=0} = E_u + E_\pi + \nabla_t \theta^0 - \nabla_a \theta^\alpha. \tag{130} \]

Due to the nilpotency of \( d \), \( d^2 \mathcal{L}_\epsilon|_{\epsilon=0} = 0 \), it is straightforward to get the following formula:

\[ d(E_u + E_\pi) + \nabla_t \omega^0 - \nabla_a \omega^\alpha = 0, \tag{131} \]

where \( \omega^0 \) and \( \omega^\alpha \) a set of \( n \) symplectic 2-forms

\[ \omega^0 = d\pi_\alpha \wedge du^\alpha, \quad \omega^\alpha = d\left( \frac{\partial H}{\partial (\nabla_a u^\alpha)} \right) \wedge du^\alpha. \tag{132} \]

4.1.3 Remarks on Euler-Lagrange cohomology and multisymplectic structure preserving for classical field theory

It is easy to find that, similar to the case of classical mechanics, under the Legendre transformation (54) all relevant issues in the Lagrangian and Hamiltonian formalism are in one-to-one correspondent equivalent. In fact, we have also the following equivalent relations:

| Lagrangian formalism | Hamiltonian formalism |
|----------------------|----------------------|
| \( \mathcal{L}(u^\alpha, u_\mu^\alpha) \) | \( \pi_\alpha \dot{u}^\alpha - H \) |
| \( E(u^\alpha, u_\mu^\alpha) \) | \( E_u + E_\pi \) |
| \( \theta^\mu_L \) | \( \theta^0_H, \theta^\alpha_H \) |
| \( d\mathcal{L} = E + \partial_\mu \theta^\mu_L \) | \( d(\pi_\alpha \dot{u}^\alpha - H) = E_u + E_\pi + \partial_\mu \theta^\mu_H \) |
| \( \omega^\mu_L \) | \( \omega^0_H, \omega^\alpha_H \) |
| \( dE + \partial_\mu \omega^\mu_L = 0 \) | \( d(E_u + E_\pi) + \partial_\mu \omega^\mu_H = 0 \) |
Therefore, for enumerating the important issues on the Euler-Lagrange cohomology and the multisymplectic structure preserving property in classical field theory, we may work with one formalism, then the corresponding issues in another formalism are indicated automatically. Let us this time enumerate and verify these relevant issues in the Lagrangian formalism.

First, if the Euler-Lagrange 1-form in (120) is null with $\epsilon = 0$, i.e.
\[ E_{\epsilon}(u^\alpha, u_{\mu}^\alpha) |_{\epsilon=0}= 0, \]
(134)
it gives rise to the Euler-Lagrange equations (52).

Secondly, $E_{\epsilon}(u^\alpha, u_{\mu}^\alpha) = 0$ is a special case of the coboundary Euler-Lagrange 1-forms
\[ E_{\epsilon}(u^\alpha, u_{\mu}^\alpha) = d\alpha_{\epsilon}(u^\alpha, u_{\mu}^\alpha), \]
(135)
where $\alpha_{\epsilon}(u^\alpha, u_{\mu}^\alpha)$ a family of arbitrary functions of $(u^\alpha, u_{\mu}^\alpha)$. Although they are cohomologically trivial but it can already be seen that in the Euler-Lagrange 1-forms, $(u^\alpha, u_{\mu}^\alpha)$ are already not in the solution space of the Euler-Lagrange equations only rather they are in the function space with corresponding closed Euler-Lagrange condition (see below) in general.

Thirdly, if the Lagrangian density $\mathcal{L}$ in (18) changes to $\mathcal{L}'$ by adding certain term
\[ \mathcal{L}(u^\alpha, u_{\mu}^\alpha) \rightarrow \mathcal{L}'(u^\alpha, u_{\mu}^\alpha) = \mathcal{L}(u^\alpha, u_{\mu}^\alpha) + \mathcal{V}(u^\alpha), \]
(136)
where $\mathcal{V}(u^\alpha)$ is an arbitrary function of $u^\alpha$, the equation (119) changes to
\[ d\mathcal{L}'_{\epsilon} |_{\epsilon=0} = E'(u^\alpha, u_{\mu}^\alpha) + \frac{\partial}{\partial x^\mu}\theta^\mu, \]
(137)
where $E'(u^\alpha, u_{\mu}^\alpha)$ differs from $E(u^\alpha, u_{\mu}^\alpha)$ by changing $\mathcal{L}$ to $\mathcal{L}'$ in the expressions, while a set of $n$ canonical 1-forms $\theta^\mu$ have not been changed because $\mathcal{V}(u^\alpha)$ does not depend on $u_{\mu}^\alpha$. In fact, the Euler-Lagrange equations have been changed by adding a potential-like term that does not depend on $u_{\mu}^\alpha$. This means that even if by adding a coboundary term, the Euler-Lagrange equations do change and the set of canonical forms may still be the same as before. Furthermore, the canonical transformations or the multisymplectic mappings that preserve the Euler-Lagrange equations, correspondingly the canonical equations in the Hamiltonian formalism, and Euler-Lagrange 1-forms as well as the multi-symplectic structures form invariant. This will lead to the issues on the generating functions, its relation to the Euler-Lagrange cohomology and so on. We will explore these issues elsewhere.

Fourthly, from the equation (119) it is easy to see that $E_{\epsilon}(u^\alpha, u_{\mu}^\alpha)$ in general are not cohomologically trivial because the families of canonical 1-forms are not trivial. Therefore, There exists a nontrivial Euler-Lagrange cohomology in the classical field theory for the set of generic fields $u^\alpha(x)$:
\[ H_{CF\mathcal{T}} := \{ \text{closed Euler-Lagrange forms} \}/\{ \text{exact Euler-Lagrange forms} \}. \]

Furthermore, From the equations (123) and (131) in the Hamiltonian formalism it follows straightforwardly an important theorem in the classical Lagrangian field theory.

**Theorem 4.1:**
For a given Lagrangian field theory, there exists a set of $n$ symplectic structures $\omega_{L}^\mu$ and the multisymplectic preserving property, i.e. the conservation or divergence free law of the multisymplectic structures
\[ \frac{\partial}{\partial x^\mu}\omega_{L}^\mu = 0 \]
(138)
holds if and only if the relevant Euler-Lagrange 1-form is closed

\[ dE(u^\alpha, u^\alpha_\mu) = 0. \]  

(139)

It is interesting to see that if we introduce a new 2-form

\[ \Omega(u^\alpha, u^\alpha_\mu) = dE(u^\alpha, u^\alpha_\mu). \]  

(140)

Then \( \Omega \) may be viewed as a \( U(1) \)-like curvature 2-form while the Euler-Lagrange 1-form the \( U(1) \)-like connection 1-form. Therefore, the closed Euler-Lagrange condition is nothing but the flat connection condition. On the other hand, if for some reason that the multisymplectic conservation law is broken then the broken pattern may be described by the curvature 2-form \( \Omega \). There is a similar issue in the finite dimensional case as well.

It is also important to notice that the multisymplectic structure preserving property is directly linked with the closed Euler-Lagrange condition. And although the null Euler-Lagrange 1-form, the coboundary Euler-Lagrange 1-forms satisfy the Euler-Lagrange condition, it does not mean that the closed Euler-Lagrange 1-forms can always be exact as was pointed out above. In addition, \( u^\alpha(x) \)'s in the Euler-Lagrange condition are not in the solution space of the Euler-Lagrange equations only in general. Therefore, the multisymplectic structure preserving property, i.e. the conservation law of the set of \( n \) symplectic 2-forms \( \omega^\mu \), holds not only in the solution space of the field equations but also in the function space with the closed Euler-Lagrange condition in general.

Analog to the case of classical mechanics, there is also an another slightly different way to deal with the issues on the Euler-Lagrange cohomology and the multisymplectic structure preserving property. Namely, either taking the exterior differential of the Lagrangian first, or directly starting from the Euler-Lagrange equations (52). By means of the multi-parameter differential approach, it is easy to release all field variables form the solution space of the Euler-Lagrange equations to the function space \( F(TM) \) and introduce a family of the Euler-Lagrange 1-forms (120) associated with the equations (52). Then by taking the exterior differential of the Euler-Lagrange 1-forms and setting \( \epsilon^\alpha = 0 \), it is straightforwardly to re-derive the theorem on the necessary and sufficient condition for the multisymplectic structure preserving law.

4.2 Discrete Euler-Lagrange cohomology and multisymplectic structure preserving property in discrete field theory

For the sake of simplicity, we consider the cases of 1+1 or 2 dimensional base manifold. Let \( X^{1,1} \) or \( X^2 \) with suitable signature of the metrics be the base manifold, \( L^2 \) a regular lattice with 2-directions \( x_\mu, (\mu = 1, 2) \) on \( X^{1,1} \) or \( X^2 \), \( M_D \) the configuration space with \( u^\alpha(i,j) \in M_D \) and so forth as before.

4.2.1 Discrete classical field theory in Lagrangian formalism

The difference discrete Lagrangian for a set of the generic fields \( u^\alpha, \alpha = 1, \cdots, r \), is a functional in \( F(T(M_{x(i,j)}) ) \)

\[ \mathcal{L}_D^{(i,j)} = \mathcal{L}_D(u^\alpha(i,j), \Delta_\mu u^\alpha(i,j)). \]  

(141)
Taking exterior differential \( d \in T^*(\mathcal{M}_{\mathcal{X}_{(i,j)}}) \) of \( \mathcal{L}_D^{(i,j)} \) and making use of the modified Leibniz law\(^{(28)}\), in the framework of multi-parameter differential approach, we get

\[
\delta \mathcal{L}_D^{(i,j)} = E_D(u^{\alpha(i,j)}, \Delta \mu u^{\alpha(i,j)}) + \Delta \mu \theta_{DL}^{\mu(i,j)},
\]

(142)

where \( E_D^{(i,j)} \) are the discrete Euler-Lagrange 1-forms defined by

\[
E_D(u^{\alpha(i,j)}, \Delta \mu u^{\alpha(i,j)}) := \{ \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(i,j)}} - \Delta \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial (\Delta u^{\alpha(i-1,j)})} - \Delta \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial (\Delta u^{\alpha(i,j-1)})} \} du^{\alpha(k,l)},
\]

(143)

and \( \theta_{DL}^{\mu(i,j)} \) are two canonical 1-forms:

\[
\theta_{DL}^{1(i,j)} = \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial (\Delta u^{\alpha(i-1,j)})} du^{\alpha(k,l)}, \quad \theta_{DL}^{2(i,j)} = \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial (\Delta u^{\alpha(i,j-1)})} du^{\alpha(k,l)}.
\]

(144)

It is easy to see that there exist two symplectic 2-forms on \( T^*(\mathcal{M}_{\mathcal{X}_{(i,j)}}) \):

\[
\omega_{DL}^{\mu(i,j)} = d\theta_{DL}^{\mu(i,j)}, \quad \mu = 1, 2.
\]

(145)

The equation \( d^2 \mathcal{L}_D^{(i,j)} = 0 \), on \( T^*(\mathcal{M}_{\mathcal{X}_{(i,j)}}) \) leads to the discrete multisymplectic structure preserving property, i.e. the conservation law or the divergence free equation of \( \omega^{\mu(i,j)} \):

\[
dE_D(u^{\alpha(i,j)}, \Delta \mu u^{\alpha(i,j)}) + \Delta \mu \omega_{DL}^{\mu(i,j)} = 0.
\]

(146)

### 4.2.2 Discrete classical field theory in Hamiltonian formalism

Let us now study the difference discrete classical field theory in Hamiltonian formalism in the case of the spacetime/space \( x^\mu, \mu = 1, 2 \), are discretized in the manner of last subsection while the configuration space at each node of the relevant lattice \( L^2 \) with coordinates \( x(i, j), (i, j) \in Z \times Z \) and the ones on its neighboring are still continuous.

In order to transfer to the difference discrete version of the Hamiltonian formalism, we first define a set of the discrete canonical conjugate momenta on the tangent space of the set of nodes neighboring to the node \( x(i, j) \), i.e. on \( T(\mathcal{X}_{(i,j)}) \):

\[
\pi^{(i,j)} = \frac{\partial \mathcal{L}_D^{(i-1,j)}}{\partial (\Delta u^{\alpha(i-1,j)})}.
\]

(147)

The time difference discrete derivative of \( \pi^{\alpha(i,j)} \) follow from the discrete Euler-Lagrange equation\(^{(32)}\)

\[
\Delta_t \pi^{(i,j)} = \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial u^{\alpha(i,j)}} - \Delta_x \frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial (\Delta u^{\alpha(i,j-1)})}.
\]

(148)

Now the difference discrete Hamiltonian is introduced through the discrete Legendre transformation

\[
\mathcal{H}_D^{(i,j)} = \pi^{(i+1,j)} \Delta_t u^{\alpha(i,j)} - \mathcal{L}_D^{(i,j)}.
\]

(149)

The difference discrete version of action functional is given by

\[
S_{DH} = \sum_{(i,j)} \mathcal{L}_D^{(i,j)} = \sum_{(i,j)} \left\{ \pi^{(i+1,j)} \Delta_t u^{\alpha(i,j)} - \mathcal{H}_D^{(i,j)} \right\}.
\]

(150)

The differential of each term can be calculated in the multi-parameter differential approach

\[
\delta \mathcal{L}_D^{(i,j)}|_{\tau^2=0} = d\pi^{(i+1,j)} (\Delta_t u^{\alpha(i,j)} - \Delta_x(\frac{\partial \mathcal{H}_D^{(i,j)}}{\partial \pi^{\alpha(i,j)}})) - \Delta_x(\frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial (\Delta u^{\alpha(i,j-1)})}) du^{\alpha(i,j)} = \Delta_t (\pi^{(i,j)} du^{\alpha(i,j)}) - \Delta_x(\frac{\partial \mathcal{L}_D^{(i,j-1)}}{\partial (\Delta u^{\alpha(i,j-1)})}) du^{\alpha(i,j)}
\]

(151)
Now we introduce the difference discrete version of the canonical Euler-Lagrange 1-forms
\[ E_{Du}^{(i,j)} = d\pi_\alpha^{(i+1,j)}(\Delta_t u^\alpha(i,j) - \frac{\partial H}{\partial u^\alpha(i+1,j)}), \]
\[ E_{D\pi}^{(i,j)} = du^\alpha(i,j)(-\Delta_t \pi_\alpha(i,j) - \frac{\partial H}{\partial u^\alpha(i,j)} + \Delta_x(\frac{\partial H}{\partial(\Delta_x u^\alpha(i,j-1)})}, \]
and the difference discrete version of two canonical 1-forms
\[ \theta_{DH}^{0(i,j)} = \pi_\alpha(i,j)du^\alpha(i,j), \quad \theta_{DH}^{1(i,j)} = \frac{\partial H}{\partial(\Delta_x u^\alpha(i,j-1))}du^\alpha(i,j). \]
Thus the equation (151) becomes
\[ d\mathcal{L}_{De}^{(i,j)}|_{\epsilon = 0} = E_{Du}^{(i,j)} + E_{D\pi}^{(i,j)} + \Delta_\epsilon \theta_{DH}^{0(i,j)} - \Delta_\epsilon \theta_{DH}^{1(i,j)}. \]

Now due to the nilpotency of \( d \), \( d^2 \mathcal{L}_{De}^{(i,j)}|_{\epsilon = 0} = 0 \), it follows the equation for difference discrete multisymplectic structure preserving property:
\[ dE_{Du}^{(i,j)} + dE_{D\pi}^{(i,j)} + \Delta_\epsilon \omega_{DH}^{0(i,j)} - \Delta_\epsilon \omega_{DH}^{1(i,j)} = 0, \]
where \( \omega_{DH}^{0(i,j)} \) and \( \omega_{DH}^{1(i,j)} \) are two symplectic 2-forms:
\[ \omega_{DH}^{0(i,j)} = d\pi_\alpha(i,j) \wedge du^\alpha(i,j), \quad \omega_{DH}^{1(i,j)} = d\frac{\partial H}{\partial(\Delta_x u^\alpha(i,j-1))} \wedge du^\alpha(i,j). \]

### 4.2.3 Remarks on discrete Euler-Lagrange cohomology and multisymplectic structure preserving property in discrete field theory

It is clear that similar to the case for the classical field theory we have also the following equivalent relations between the difference discrete versions of the Lagrangian and Hamiltonian formalism for the difference discrete classical field theory:

\[ \text{Discrete Lagrangian formalism} \quad \L_D^{(i,j)} \iff \pi_\alpha^{(i+1,j)}(\Delta_t u^\alpha(i,j) - \mathcal{H}_D^{(i,j)}) \]
\[ \text{Discrete Hamiltonian formalism} \quad \mathcal{L}_D^{(i,j)} \iff E_D^{(i,j)} \iff E_{D\pi}^{(i,j)} + E_{Du}^{(i,j)} \]
\[ \theta_{LD}^{(i,j)} \iff \theta_{HD}^{0(i,j)} + \theta_{HD}^{1(i,j)} \]
\[ d\mathcal{L}_D^{(i,j)} = E_L^{(i,j)} + \Delta_\mu \theta_{LD}^{\mu(i,j)} = d(\pi_\alpha \dot{u}_\alpha - \mathcal{H})^{(i,j)} = (E_{Du}^{(i,j)} + E_{D\pi}^{(i,j)} + \Delta_\mu \theta_{DH}^{\mu(i,j)}) = 0 \]
\[ dE_D^{(i,j)} + \Delta_\mu \omega_{LD}^{\mu(i,j)} = 0 \]
\[ dE_{D\pi}^{(i,j)} + \omega_{HD}^{0(i,j)} + \omega_{HD}^{1(i,j)} = 0 \]
\[ dE_{Du}^{(i,j)} + \Delta_\mu \omega_{LD}^{\mu(i,j)} = 0 \quad dE_{Du} + E_{D\pi} + \Delta_\mu \theta_{DH}^{\mu(i,j)} = 0 \]
(157)

Let us enumerate and verify relevant important issues on the discrete Euler-Lagrange cohomology and difference discrete multisymplectic structure preserving property in the difference discrete Lagrangian formalism.

First, the null discrete Euler-Lagrange 1-form corresponds to the discrete Euler-Lagrange equations and it is a special case of coboundary discrete Euler-Lagrange 1-forms
\[ E_D^{(i,j)} = d\alpha_D^{(i,j)}, \]
where \( \alpha_D^{(i,j)} \) an arbitrary function on \( F(T^*\mathcal{M}_{(i,j)}) \).

Secondly, although they satisfy the discrete Euler-Lagrange condition, it does not mean that all closed discrete Euler-Lagrange 1-forms are exact. As a matter of fact, from the equation (142) it is easy to see that the Euler-Lagrange 1-forms are not exact in
general since the two canonical 1-forms $\theta_D^{\mu(i,j)}, (\mu = 1, 2)$ are not trivial. Therefore, for a given difference discrete field theory, there exists a nontrivial difference discrete version of the Euler-Lagrange cohomology:

$$H_{DCFT} := \{\text{closed Euler-Lagrange forms}\}/\{\text{exact Euler-Lagrange forms}\}.$$ 

Thirdly, from the equations (146) and (155) it follows the theorem on the necessary and sufficient condition for the difference discrete multisymplectic structure preserving law:

**Theorem 4.2:**

The difference discrete multisymplectic structure preserving law

$$\Delta_\mu \omega_D^{\mu(i,j)} = 0 \quad (159)$$

holds if and only if the discrete Euler-Lagrange 1-form satisfies the discrete Euler-Lagrange condition, i.e. it is closed:

$$dE_D^{(i,j)} = 0 \quad (160).$$

In addition, this also indicates that the variables $u^{\alpha(k,l)}$'s etc. in the cohomology are still in the function space rather than the ones in the solution space only. Consequently, this means that the difference discrete multisymplectic structure preserving law holds in the function space with the closed discrete Euler-Lagrange condition in general rather than in the solution space only.

Finally, it should be mentioned that all these issues can be straightforward to generalize to higher dimensional cases of spacetime $X^{1,n-1}$ or space $X^n$ as base manifold.

## 5 Difference discrete variational principle and discrete Euler-Lagrange cohomological approach to symplectic and multisymplectic algorithms

It is worthwhile to show that the difference discrete variational principle and the cohomological scenario described in the previous sections for the difference discrete mechanics as well as for the discrete classical field theory in the both Lagrangian and Hamiltonian formalism can be directly applied to the numerical schemes in the symplectic and multisymplectic algorithms respectively.

As a matter of fact, the difference discrete versions of the Euler-Lagrange equations and the canonical equations of motion in the both Lagrangian and Hamiltonian formalism offer themselves the numerical schemes in symplectic and multisymplectic algorithms for the difference discrete version of the classical mechanics and the difference discrete version of the classical field theory respectively.

In addition, as has been pointed out in [16] that the Euler-Lagrange-like cohomological approach may also be applied to the so-called Hamiltonian-like ODEs and PDEs respectively. Therefore, for the difference discrete versions of these Hamiltonian-like ODEs and PDEs the difference discrete variational principle and the difference discrete version of the cohomological approach may also be available.

We investigate the relevant issues for the symplectic algorithm first in the subsection 5.1, and the relevant issues in the multisymplectic algorithm first in the subsection 5.2.
5.1 Difference discrete variational principle and discrete cohomological approach to symplectic algorithm

As in the case of both Lagrangian and Hamiltonian formalism for classical mechanics, the above-established difference discrete variational principle and the discrete cohomological scenario in the difference discrete Lagrangian and Hamiltonian formalism for classical mechanics should be directly applied to the difference discrete versions of ODE’s with Lagrangian and/or Hamiltonian and to the numerical schemes for the symplectic algorithm. In this section we study this topic. We first consider how to apply the difference discrete variational principle to the numerical schemes in symplectic algorithm in the subsection 5.1.1. Then we show the difference discrete version of the Euler-Lagrange cohomological scenario offers how to justify a scheme is symplectic in subsection 5.1.2.

5.1.1 Difference discrete variational principle and symplectic algorithm

Let us first note that both the difference discrete versions of the Euler-Lagrange equations (30) in the Lagrangian formalism and the difference discrete versions of the canonical equations (42) or (111) in the Hamiltonian mechanics are derived from a relevant difference discrete variational principle in the framework of the multiparameter differential approach. These difference discrete equations may in fact offer certain numerical schemes and are automatically symplectic.

In the difference discrete Lagrangian formalism case, the discrete Euler-Lagrange equations (30) read

\[ \frac{\partial L_D^{(k)}}{\partial q^{(k)}} - \Delta_t \left( \frac{\partial L_D^{(k-1)}}{\partial (\Delta_t q^{(k-1)})} \right) = 0. \]

Their offering a numerical scheme that is symplectic can be seen manifestly from the difference discrete Euler-Lagrange equations (32) in the example 2.1:

\[ \Delta_t (\Delta_t q^{i(k-1)}) - \frac{\partial}{\partial q} V(q^{i(k)}) = 0, \]

i.e.

\[ \frac{1}{\tau^2} (q^{i(k+1)} - 2q^{i(k)} + q^{i(k-1)}) = \frac{\partial}{\partial q} V(q^{i(k)}). \]

For given initial values of \( q^{i(k=0)}, q^{i(k=1)} \), the equations give rise to \( q^{i(k=2)} \) and so on. For the Lagrangian (31) this is in fact a simplest scheme that preserves, in the sense of time difference discrete, the symplectic structure. This has been proved in the subsection 3.1.

For the case of difference discrete Hamiltonian mechanics, the difference discrete canonical equations in the (12) or (111) read

\[ \Delta_t q^{i(k)} = p^{(k+1)}, \quad \Delta_t p^{(k)} = -\frac{\partial}{\partial q} V(q^{i(k)}), \]

and

\[ \Delta_t z^{(k)} = J^{-1} \nabla_z H_D^{(k)}(z^{(k)}). \]

They also offer a set of numerical schemes in difference discrete Hamiltonian formalism and preserve the symplectic structure in the sense of time difference discrete. This has been proved in the subsection 3.1 as well.
In fact, in the multi-parameter differential approach to difference discrete variational principle, it is easy to see why the difference discrete variational principle offers the numerical schemes that are automatically symplectic. It has been shown that the variations of the difference discrete action functionals are calculated by taking the differential of the difference discrete action functionals and setting \( \varepsilon^k = 0 \) afterwards, i.e. \( dL_{Dt}|_{\varepsilon^k=0} \) in the Lagrangian formalism and \( d(p_i \Delta q^i - H)_{Dt}|_{\varepsilon^k=0} \) in the Hamiltonian formalism. As was shown in the subsections 2.1 and 3.1, the stationary requirement of difference discrete variational principle leads to either difference discrete Euler-Lagrange equations or difference discrete canonical equations of motion. Furthermore, the second exterior differentials of the difference discrete action functionals lead to the symplectic structure preserving law.

Let us consider the Euler mid-point scheme. We find that there exists a difference discrete action functional for the scheme.

The difference discrete Lagrangian of the scheme can be given by

\[
L_{mdpt}(k) = L(q^{i(k+\frac{1}{2})}, \Delta_t q^{i(k)}), \quad q^{i(k+\frac{1}{2})} := \frac{1}{2}(q^{i(k+1)} + q^{i(k)}).
\]

(161)

Note that at the moment \( t_k \) \( q^{i(k+\frac{1}{2})} \) and \( \Delta_t q^{i(k)} \) are the coordinates and the tangents for the scheme.

Then the discrete canonical momenta conjugated to the coordinates can be defined as

\[
p_i^{(k+\frac{1}{2})} = \frac{\partial L_{mdpt}(k)}{\partial (\Delta_t q^{i(k)})}, \quad p_i^{(k+\frac{1}{2})} := \frac{1}{2}(p_i^{(k+1)} + p_i^{(k)}).
\]

(162)

The discrete Legendre transformation is given by

\[
H(q^{i(k+\frac{1}{2})}, p_i^{(k+\frac{1}{2})}) = p_i^{(k+\frac{1}{2})} \Delta_t q^{i(k)} - L_{mdpt}(k).
\]

(163)

The difference discrete action functional of the scheme is given by

\[
S_{mdpt} = \sum_{k \in \mathbb{Z}} \{p_i^{(k+1/2)} \Delta_t q^{i(k)} - H(q^{i(k+1/2)}, p_i^{(k+1/2)})\}.
\]

(164)

Let us take the variation of this difference discrete action functional in the framework of the multi-parameter differential approach, i.e. the variation along the direction \( \ell \) is manipulated by

\[
\delta_\ell S_{mdpt} = \frac{\partial}{\partial \ell} S_{mdpt}|_{\ell=0}.
\]

(165)

Eventually, the differential of each term under the summation of \( S_{mdpt} \) is given by

\[
dL_{mdpt}(k) := dp_i^{(k+1/2)}(\Delta_t q^{i(k)} - \frac{\partial}{\partial p_i} H(q^{i(k+1/2)}, p_i^{(k+1/2)}))
- (\Delta_t p_i^{(k)} + \frac{\partial}{\partial q_i} H(q^{i(k+1/2)}, p_i^{(k+1/2)}))dq^{i(k+1/2)} + \Delta_t(p_i^{(k)}dq^{i(k)}).
\]

(166)

Note that here the following generalized modified Leibniz law \( [26] \) (see also the appendix) has been used:

\[
\Delta_t(f^{(k)} \cdot g^{(k)}) := \frac{1}{\tau}(f^{(k+1)} \cdot g^{(k+1)} - f^{(k)} \cdot g^{(k)})
= \frac{1}{\tau}(\Delta_t f^{(k)} \cdot g^{(k+1/2)} + f^{(k+1/2)} \cdot \Delta_t g^{(k)}).
\]

(167)

The first equality in the above equation is the definition for the forward difference, while the second can be easily proved.
It is now clear that the stationary requirement of the difference discrete variational principle leads to the Euler mid-point scheme as follows:

\[
\Delta_t q^{(k)} = \frac{\partial}{\partial p} H(q^{(k+1/2)}, p^{(k+1/2)}), \\
\Delta_t p^{(k)} = -\frac{\partial}{\partial q} H(q^{(k+1/2)}, p^{(k+1/2)}).
\]  

(168)

Thus we have shown that the Euler mid-point scheme is a difference discrete variational scheme with corresponding (dependent) difference discrete variables and suitable Leibniz law for the difference.

### 5.1.2 Discrete cohomological approach to symplectic algorithm

We now consider how to apply the difference discrete version of the Euler-Lagrange cohomological approach to the numerical schemes in symplectic algorithm.

In the standard approach, it is commonly accustomed to regarding a numerical scheme as a (time-discrete) mapping. In order to justify whether a given numerical scheme is symplectic, the standard approach is to see whether this mapping is symplectic preserving and the verification is always carried out in the solution space of the scheme [8][9]. In the difference discrete version of the Euler-Lagrange cohomological approach, however, instead of working on the solution space, it is working on the function space with the relevant cohomological issues. Analog to the case for the difference discrete classical mechanics, there are two slightly different ways to apply the cohomological approach. Namely, one is based upon the difference discrete variational principle for the schemes and taking second (exterior) differential of the action functional to get the necessary and sufficient condition for the symplectic structure preserving property of the scheme. Another is to release the scheme away from the solution space, even if the solution space does exist, and to introduce some suitable difference discrete Euler-Lagrange 1-forms associated with the scheme such that the null difference discrete Euler-Lagrange 1-forms give rise to the scheme. Then by taking the exterior derivative of the difference discrete Euler-Lagrange 1-forms to see whether follows a time-discrete symplectic structure preserving law.

We will consider some examples to show how the cohomological scenario works. Let us first retain to the mid-point scheme (168) and follow-up by the cohomological approach.

It is clear that by introducing the discrete Euler-Lagrange 1-forms

\[
E_q^{(k)} = dp^{(k+1/2)}(\Delta_t q^{(k)}) - \frac{\partial}{\partial p} H(q^{(k+1/2)}, p^{(k+1/2)}), \\
E_p^{(k)} = -(\Delta_t p^{(k)}) + \frac{\partial}{\partial q} H(q^{(k+1/2)}, p^{(k+1/2)})dq^{(k+1/2)},
\]  

(169)

and the difference discrete canonical 1-form for the scheme:

\[
\theta_{mdpt}^{(k)} = p^{(k)}dq^{(k)},
\]  

(170)

the equation (169) becomes

\[
dL_{mdpt}^{(k)} = E_q^{(k)} + E_p^{(k)} + \Delta_t \theta_{mdpt}^{(k)}.
\]  

(171)

By taking exterior differential of the above equation (171), due to \(d^2L_{mdpt} = 0\), it follows that

\[
d(E_q + E_p)^{(k)} + \Delta_t \omega_{mdpt}^{(k)} = 0,
\]  

(172)
where $\omega_{mdpt}^{(k)} = dp_i^{(k)} \wedge dq_i^{(k)}$. Since the null Euler-Lagrange 1-forms of (169) give rise to the scheme and automatically satisfy $d(E_q + E_p)^{(k)} = 0$, while the latter leads to the symplectic conservation law from (172):

$$\Delta_t \omega_{mdpt}^{(k)} := \frac{1}{\tau} (\omega_{mdpt}^{(k+1)} - \omega_{mdpt}^{(k)}) = 0.$$  

Therefore, the midpoint scheme is symplectic.

On the other hand, we can also start directly from the mid-point scheme in terms of $z^{(k)}$:

$$\Delta_t z^{(k)} = J^{-1} \nabla_z H \left( \frac{1}{2} (z^{(k+1)} + z^{(k)}) \right).$$  

This time we release the scheme form the solution space first, even if it does exist. This can be treated by the multi-parameter differential approach. For simplicity, we suppose this has been done already. Then we introduce a difference discrete Euler-Lagrange 1-form for the scheme

$$E_{zmdpt}^{(k)} = \frac{1}{2} d (z^{(k+1)} + z^{(k)})^T \left\{ J \Delta_t z^{(k)} - \nabla_z H \left( \frac{1}{2} (z^{(k+1)} + z^{(k)}) \right) \right\}$$  

such that the null discrete Euler-Lagrange form gives rise to the scheme. Then by taking the exterior differential of $E_{zmdpt}^{(k)}$ in the function space, it follows

$$dE_{zmdpt}^{(k)} = \frac{1}{2} d (z^{(k+1)} + z^{(k)})^T \wedge J \Delta_t dz^{(k)}.$$  

Therefore, the difference discrete symplectic structure preserving law

$$\Delta_t (dz^{(k)})^T \wedge J dz^{(k)} = 0$$  

holds if and only if the discrete Euler-Lagrange form is closed:

$$dE_{zmdpt}^{(k)} = 0.$$  

Furthermore, due to the exact forms do not change the closed condition, the null form may be redefined by adding certain exact forms so that the scheme may be generalized to a type of schemes while the difference discrete symplectic structure preserving law is the same.

It is also interesting that this issue offers a way to generalize the scheme to the high order ones while the difference discrete symplectic structure preserving law is the same.

Let us consider the 4-th order symplectic scheme as follows [22]:

$$\nabla_t z^{(n)} = J^{-1} \nabla_z H \left( \frac{1}{2} (z^{(n+1)} + z^{(n)}) \right)$$

$$-\frac{h^2}{24} J^{-1} \nabla_z \left( (\nabla_z H)^T J H_{zz} J \nabla_z H \right) \left( \frac{1}{2} (z^{(n+1)} + z^{(n)}) \right).$$

Introduce a new “Hamiltonian” $\mathcal{H}$

$$\mathcal{H} = H - \frac{h^2}{24} (\nabla_z H)^T J H_{zz} J \nabla_z H,$$

then this 4th-order symplectic scheme becomes

$$\nabla_t z^{(n)} = J^{-1} \nabla_z \mathcal{H} \left( \frac{1}{2} (z^{(n+1)} + z^{(n)}) \right).$$
The discrete Euler-Lagrange 1-form associated with this case now can be introduced:

\[ E_{4\text{th}}^{(k)} = \frac{1}{2} d(\mathbf{z}^{(k+1)} + \mathbf{z}^{(k)})^T \{ J \Delta_t \mathbf{z}^{(k)} - \nabla \mathbf{z} \mathcal{H}(\frac{1}{2}(\mathbf{z}^{(k+1)} + \mathbf{z}^{(k)})) \}. \] (180)

It is easy to check that these two discrete Euler-Lagrange forms differ by an exact form:

\[ E_{\text{mdpt}}^{(k)} - E_{4\text{th}}^{(k)} = \frac{h^2}{24} d\alpha, \] (181)

where \( \alpha = (\nabla \mathbf{z} \mathcal{H})^T J \mathcal{H} \nabla \mathbf{z} \mathcal{H} \) is a function of \( \frac{1}{2}(\mathbf{z}^{(k+1)} + \mathbf{z}^{(k)}) \).

This means that they are cohomologically equivalent. In addition, this also indicates that the 4-th order midpoint scheme is difference discrete variational as well.

5.2 Difference discrete variational principle and discrete cohomological approach to multisymplectic algorithm

As in the case of both Lagrangian and Hamiltonian formalism for classical field theory, the scenario of the above-established difference discrete variational principle and cohomological approach to the difference discrete Lagrangian and Hamiltonian formalism for classical field theory should be directly applied to the difference discrete versions of PDEs with Lagrangian and/or Hamiltonian and to the numerical schemes for the multisymplectic algorithm.

5.2.1 Difference discrete variational principle and multisymplectic algorithm

In this part of the subsection, we first point out that the difference discrete field equations derived by the difference discrete variational principle themselves offer numerical schemes for the multisymplectic algorithm. We also derive a new scheme for the difference discrete Hamiltonian formalism by the difference discrete variational principle with the generalized modified Leibniz law (167). Then we study how to apply the difference discrete variational principle approach to the numerical schemes for the Hamiltonian-like PDEs named in [16]. A type of so-called Hamiltonian PDEs proposed first in [4] (see also [21]) are in fact a type of the Hamiltonian-like PDEs.

As was just mentioned, the difference discrete Euler-Lagrange equations (67) derived via the difference discrete variational principle in the difference discrete Lagrangian formalism for classical field theory, i.e.

\[ \frac{\partial L_D^{(i,j)}}{\partial u^{\alpha(k,l)}} - \Delta_1 \left( \frac{\partial L_D^{(i-1,j)}}{\partial u^{\alpha(k-1,l)}} \right) - \Delta_2 \left( \frac{\partial L_D^{(i,j-1)}}{\partial u^{\alpha(k,l-1)}} \right) = 0 \]

offer themselves a numerical scheme that preserves the multisymplectic structures for the 2-dimensional spacetime/space in difference discrete version.

For the difference discrete Hamiltonian formalism of the field theory, the difference discrete canonical equations (73) derived via the difference discrete variational principle, i.e.

\[ \Delta_t u^{\alpha(i,j)} = \frac{\partial \mathcal{H}_D^{(i,j)}}{\partial \pi^{(i+1,j)}}, \]

\[ \Delta_t \pi^{(i,j)} = -\frac{\partial \mathcal{H}_D^{(i,j)}}{\partial u^{\alpha(i,j)}} + \Delta_x \left( \frac{\partial \mathcal{H}_D^{(i,j-1)}}{\partial (\Delta_x u^{\alpha(i,j-1)})} \right) \]
also offer themselves a set of numerical schemes for the multisymplectic algorithm. It has been proved in the subsection 4.2 that they preserve the multisymplectic structures as well for the difference discrete spacetime/space.

In the last subsection, it was shown that by the difference discrete variational principle with the generalized modified Leibniz law the Euler midpoint-scheme had been derived from a difference discrete action functional. Similarly, the same generalized modified Leibniz law \([167]\) can be employed to construct the numerical schemes for the multisymplectic algorithm. Let us consider the case in the difference discrete Hamiltonian formalism for the difference discrete field theory.

We introduce the difference discrete conjugate momenta

\[
\pi^{(i+\frac{1}{2},j)} = \frac{\partial \mathcal{L}_D^{(i,j)}}{\partial (\Delta t u^{(i,j)+\frac{1}{2}})}, \quad \pi^{(i+\frac{1}{2},j)} = \frac{1}{2} (\pi^{(i+1,j)} + \pi^{(i,j)}), \quad (182)
\]

where

\[
\mathcal{L}_D^{(i,j)} = \mathcal{L}_D(u^{(i+\frac{1}{2},j)+\frac{1}{2}}), \quad \Delta_t u^{(i,j+\frac{1}{2})}, \quad \Delta_x u^{(i+\frac{1}{2},j)}
\]

is the difference discrete Lagrangian density with suitable discrete variables \(u^{(i+\frac{1}{2},j)+\frac{1}{2}}\), \(\Delta_t u^{(i,j+\frac{1}{2})}\), \(\Delta_x u^{(i+\frac{1}{2},j)}\), and

\[
u^{(i+\frac{1}{2},j)} = \frac{1}{2} (u^{(i+1,j)} + u^{(i,j)}), \quad \nu^{(i,j+\frac{1}{2})} = \frac{1}{2} (u^{(i,j+1)} + u^{(i,j)}), \quad (183)
\]

The difference discrete Hamiltonian density via the difference discrete Legendre transformation

\[
\mathcal{H}_D^{(i,j)} = \mathcal{H}_D(u^{(i+\frac{1}{2},j)+\frac{1}{2}}), \quad \pi^{(i+\frac{1}{2},j)}, \Delta_x u^{(i+\frac{1}{2},j)}
\]

\[
= \pi^{(i+\frac{1}{2},j)} \Delta_t u^{(i,j+\frac{1}{2})} - \mathcal{L}_D(u^{(i+\frac{1}{2},j+\frac{1}{2}}), \Delta_t u^{(i,j+\frac{1}{2})}, \Delta_x u^{(i+\frac{1}{2},j)}), \quad (184)
\]

Now the difference discrete action functional is given by

\[
S_D = \Sigma_{i,j \in Z} \mathcal{L}_D^{(i,j)}. \quad (185)
\]

The variation of \(S_D\) can be manipulated in the framework of multiparameter differential approach and the differential of \(\mathcal{L}_D^{(i,j)}\) with respect to \(\varepsilon^{\beta}\) can be calculated to get

\[
d\mathcal{L}_D^{(i,j)} = (\Delta_t u^{\beta(i,j+\frac{1}{2})} - \frac{\partial \mathcal{H}_D}{\partial \pi^{\beta}})(i,j)d\pi^{\beta(i+\frac{1}{2},j)} \quad (186)
\]

\[
+ (-\Delta_t \pi^{(i,j)} - \frac{\partial \mathcal{H}_D}{\partial u^{\beta}})(i,j) + \Delta_x \pi^{(i,j)} d\nu^{\beta(i,j+\frac{1}{2})} + \Delta_t (\pi^{(i,j)} d\nu^{\beta(i,j+\frac{1}{2})}) - \Delta_x (\nu^{(i,j)} d\nu^{\beta(i,j+\frac{1}{2})})
\]

where the notation

\[
\pi^{(i,j+\frac{1}{2})} = \frac{1}{2} (\pi^{(i,j+1)} + \pi^{(i,j)}) = \frac{\partial \mathcal{H}_D}{\partial \nu_x^{\beta}}(i,j), \quad (187)
\]

and the generalized modified Leibniz law \([167]\) for the both \(\Delta_t\) and \(\Delta_x\) have been adopted.

Now the stationary requirement against all variations along any direction of difference discrete variational principle leads to a new scheme, which may be called the midpoint
box scheme, in the difference discrete Hamiltonian field theory, with assuming variations of both \( u^\beta(i,j+\frac{1}{2}) \) and \( u^\beta(i+\frac{1}{2},j) \) vanish at infinity, as follows:

\[
\Delta_t u^\beta(i,j+\frac{1}{2}) = \left( \frac{\partial H_D}{\partial \pi_{\beta}} \right)^{(i,j)}
\]

\[
\Delta_t \pi_{\beta}^{(i,j)} = -\left( \frac{\partial H_D}{\partial u^\beta} \right)^{(i,j)} + \Delta_x \pi'_{\beta}^{(i,j)}
\]

together with

\[
\pi'_{\beta}^{(i,j+\frac{1}{2})} = \left( \frac{\partial H_D}{\partial u^\beta} \right)^{(i,j)}.
\]

In fact, it is shown that this midpoint box scheme can be derived via the difference discrete variational principle with the generalized modified Leibniz law (167).

Finally, let us derive the midpoint box scheme for a type of PDEs (4) by means of the difference discrete variational principle with the generalized modified Leibniz law (167):

\[
M Z_t + \epsilon K Z_x = \nabla_z S(Z),
\]

where the same notations in (4) have been used and \( K^T = -K, L^T = -L, \epsilon = \pm \).

It can be shown (16) that there is an action functional for the PDEs (189)

\[
S = \int d^2 x \mathcal{L},
\]

\[
\mathcal{L} = \frac{1}{2} Z^T (MZ_t + \epsilon KZ_x) - S(Z),
\]

where \( \mathcal{L} \) is the Lagrangian density, and the PDEs (189) can be reached by the variational principle of action functional as the Euler-Lagrange equations.

Let us now consider its difference discrete version formulation. First, introduce the difference discrete Lagrangian density as follows

\[
\mathcal{L}_D^{(i,j)} = \mathcal{L}_D(Z^{(i+\frac{1}{2},j+\frac{1}{2})}, \Delta_t Z^{(i,j+\frac{1}{2})}, \Delta_x Z^{(i+\frac{1}{2},j)}),
\]

where \( Z^{(i+\frac{1}{2},j+\frac{1}{2})} \) is taken as the coordinates in the configuration space on the note \((i,j)\) in the difference discrete spacetime, \( \Delta_t Z^{(i,j+\frac{1}{2})} \) and \( \Delta_x Z^{(i+\frac{1}{2},j)} \) the ones on the tangent space of the configuration space. In the difference discrete variational principle, they are regarded as variables to be variated. Now the action functional becomes

\[
S_D = \sum_{(i,j) \in Z \times Z} \mathcal{L}_D^{(i,j)}
\]

The variation of \( S_D \) along the direction \( \beta \) in the configuration space of the system can be calculated in the framework of multi-parameter differential approach

\[
\delta_\beta S_D = \frac{\partial}{\partial \varepsilon^\beta} S_D |_{\varepsilon^\beta = 0}.
\]

It is straightforward to get the differential with respect to \( \varepsilon^\beta \) for each term under the summation

\[
d\mathcal{L}_D^{(i,j)} = dZ^{(i+\frac{1}{2}, j+\frac{1}{2})} T (M \Delta_t Z^{(i,j+\frac{1}{2})} + K \Delta_x Z^{(i+\frac{1}{2}, j)} - \nabla_z S(Z^{(i+\frac{1}{2}, j+\frac{1}{2})})) - \frac{1}{2} \Delta_t (dZ^{(i,j+\frac{1}{2})} T M Z^{(i,j+\frac{1}{2})}) - \frac{1}{2} \epsilon \Delta_x (dZ^{(i+\frac{1}{2}, j)} T K Z^{(i+\frac{1}{2}, j)}).
\]
Here the variables $Z^{(i+\frac{1}{2},j+\frac{1}{2})}$, $\Delta t Z^{(i,j+\frac{1}{2})}$ and $\Delta x Z^{(i+\frac{1}{2},j)}$ are regarded as the variational variables, the multi-parameter $\varepsilon^\beta$ are omitted and the generalized modified Leibniz law (167) for the differences are adopted.

Then the stationary requirement against all variations along any direction of difference discrete variational principle leads to the midpoint box scheme, with assuming variations of both $Z^{(i+\frac{1}{2},j)}$ and $Z^{(i+\frac{1}{2},j)}$ vanish at infinity, as follows:

$$
M \Delta t Z^{(i,j+\frac{1}{2})} + \epsilon K \Delta x Z^{(i+\frac{1}{2},j)} = \nabla_z S(Z^{(i+\frac{1}{2},j+\frac{1}{2})}).
$$

Thus, it is proved that the midpoint box scheme for the type of PDEs in [4] can also be derived via the difference discrete variational principle.

### 5.2.2 Cohomological approach to multisymplectic algorithm

Similar to the case for symplectic algorithm, it should be emphasized that in the conventional approach to the multisymplectic algorithm, it is commonly accustomed to regarding a numerical schemes as a set of (spacetime/space-discrete) mappings. In order to justify whether a set of given numerical schemes are multisymplectic, the usual approach to the multisymplectic algorithm is to verify this set of mappings are multisymplectic preserving and the verification is always carried out in the solution space of the schemes (see, for example, [4]). In the difference discrete version of the Euler-Lagrange cohomological approach, however, instead of working on the solution space, it is first to release the schemes away from the solution space, even if the solution space does exist, and introduce some suitable difference discrete Euler-Lagrange 1-forms associated with the schemes such that the null difference discrete Euler-Lagrange 1-forms give rise to the schemes. Then by taking the exterior derivative of the difference discrete Euler-Lagrange 1-forms to investigate whether it leads to a spacetime/space-discrete multisymplectic structure preserving law.

We first show the multisymplectic property of the midpoint box scheme (188) in the cohomological approach.

Introducing the difference discrete Euler-Lagrange 1-forms

$$
E_u^{(i,j)} := (\Delta t u^\beta (i,j+\frac{1}{2})) - (\frac{\partial H_D}{\partial \pi^\beta}(i,j)) d\pi^\beta (i+\frac{1}{2},j),
$$

$$
E_\pi^{(i,j)} := (-\Delta t \pi^\beta (i,j) - (\frac{\partial H_D}{\partial u^\beta}(i,j) + \Delta x \pi^x (i,j))) du^\beta (i+\frac{1}{2},j+\frac{1}{2}),
$$

and difference discrete canonical 1-forms

$$
\theta^0(i,j) := \pi^\beta (i,j) du^\beta (i,j+\frac{1}{2}), \quad \theta^1(i,j) := \pi^x (i,j) du^\beta (i+\frac{1}{2},j),
$$

the equation (186) can be rewritten as

$$
dL_D^{(i,j)} = E_u^{(i,j)} + E_\pi^{(i,j)} + \Delta t \theta^0(i,j) - \Delta x \theta^1(i,j).
$$

Taking the exterior differential again, due to the nilpotency of $d$, $d^2 L_D^{(i,j)} = 0$, it follows that

$$
d(E_u + E_\pi)^{(i,j)} + \Delta t \omega^0(i,j) - \Delta x \omega^1(i,j) = 0,
$$

where

$$
\omega^0(i,j) := d\pi^\beta (i,j) \land du^\beta (i,j+\frac{1}{2}), \quad \omega^1(i,j) := d\pi^x (i,j) \land du^\beta (i+\frac{1}{2},j).
$$
Thus, the necessary and sufficient condition of the multisymplectic structure preserving law
\[ \Delta_t \omega^0(i,j) - \Delta_x \omega^1(i,j) = 0, \] (201)
for the midpoint box scheme is that the associated difference discrete Euler-Lagrange 1-forms are closed:
\[ d(E_u + E_\pi)^{(i,j)} = 0. \] (202)

Since the null difference discrete Euler-Lagrange 1-forms corresponding to the midpoint box scheme and they do satisfy the closed condition so that the midpoint box scheme is multisymplectic.

Let us now consider the midpoint box scheme for the type of PDEs in [4] and show that it is multisymplectic in the cohomological approach.

As before, we first define the difference discrete Euler-Lagrange 1-form and two difference discrete canonical 1-forms
\[ E_{\text{box}}^{(i,j)} := dZ^{(i+\frac{1}{2},j+\frac{1}{2})^T}(M\Delta_t Z^{(i,j+\frac{1}{2})} + \epsilon K \Delta_x Z^{(i+\frac{1}{2},j+\frac{1}{2})} - \nabla_z S(Z^{(i+\frac{1}{2},j+\frac{1}{2})})), \] (203)
\[ \theta^0(i,j) := \frac{1}{2}(dZ^{(i,j+\frac{1}{2})^T}M Z^{(i,j+\frac{1}{2})}) \quad \theta^1(i,j) := \frac{1}{2}(dZ^{(i+\frac{1}{2},j)^T}K Z^{(i+\frac{1}{2},j)}) \] (204)

Then the differential of \( L_D^{(i,j)} \) in the equation (194) can be rewritten as
\[ dL_D^{(i,j)} = E_{\text{box}}^{(i,j)} - \Delta_t \theta^0(i,j) - \epsilon \Delta_x \theta^1(i,j). \] (205)

Taking the second exterior differential of the above equation, due to the nilpotency of \( d \), \( d^2 L_D^{(i,j)} = 0 \), it follows that
\[ 0 = dE_{\text{box}}^{(i,j)} - \Delta_t \omega^0(i,j) - \epsilon \Delta_x \omega^1(i,j), \] (206)
where \( \omega^\mu(i,j), \mu = 0, 1 \) are two symplectic 2-forms
\[ \omega^0(i,j) := -\frac{1}{2}(dZ^{(i,j+\frac{1}{2})^T} \wedge M dZ^{(i,j+\frac{1}{2})}), \] (207)
\[ \omega^1(i,j) := -\frac{1}{2}(dZ^{(i+\frac{1}{2},j)^T} \wedge K dZ^{(i+\frac{1}{2},j)}). \]

Thus, the necessary and sufficient condition of the multisymplectic structure preserving law
\[ \Delta_t \omega^0(i,j) - \epsilon \Delta_x \omega^1(i,j) = 0, \] (208)
for midpoint box scheme of the type of PDEs in [4] is the associated difference discrete Euler-Lagrange 1-form is closed:
\[ dE_{\text{box}}^{(i,j)} = 0. \] (209)

Since the null difference discrete Euler-Lagrange 1-form corresponding to the box scheme and it does satisfy the closed condition so that the midpoint box scheme for the PDEs in [4] is multisymplectic.

It is clear that for the both midpoint box scheme for Hamiltonian field theory and for the type of Hamiltonian-like PDEs the discrete multisymplectic structure preserving law holds in function space with the discrete closed Euler-Lagrange condition in general and can also be required in the solution space in each case.

Finally, it should be mentioned that the both above difference discrete variational principle and cohomological scenario should also be applied to other numerical schemes in order to justify whether they are variational and multisymplectic.
6 Concluding remarks

A few remarks are in order:

1. The difference discrete variational formalism widely employed in this paper is different from the one of the Veselov type for the discrete classical mechanics 23 [24]. It has been emphasized that the difference as discrete derivative is an entire geometric object. The discrete integrals can also combine together in certain manner as a geometric object to construct some numerical schemes as was shown in the section 5. This is more obvious and natural from the viewpoint of noncommutative geometry. In the continuous limit, the results given here by the difference discrete variational principle lead to the correct continuous counterparts.

It is shown that the difference discrete variational principle works for the difference discrete version of classical mechanics and field theory in the both Lagrangian and Hamiltonian formalisms that present themselves as symplectic or multisymplectic numerical schemes and furthermore for other numerical schemes in both symplectic and multisymplectic algorithms respectively. And the role-played by the different Leibniz laws for the differences are quite important in constructing the numerical schemes. As a matter of fact, all numerical schemes in symplectic and multisymplectic algorithms should be derived by virtue of the difference discrete variational principle together with the suitable Leibniz law for differences. We will publish our further investigation on this issue elsewhere 26.

2. The cohomological approach adopted in this paper to the symplectic and multisymplectic geometry and their difference discrete versions in the both Lagrangian and Hamiltonian formalism for the classical mechanics and field theory had been missed in other approaches (see, for example, 1 2 3 4 5 6 7 8). The Euler-Lagrange cohomological concepts and their difference discrete versions, such as the Euler-Lagrange 1-forms, the null Euler-Lagrange 1-forms, the coboundary Euler-Lagrange 1-forms as well as the closed Euler-Lagrange conditions and their difference discrete versions, have been introduced and they have played very crucial roles in each case to show that the symplectic and multisymplectic structure preserving properties. It has been show that the necessary and sufficient condition for symplectic and multisymplectic structure preserving property in each case is the related closed Euler-Lagrange condition being satisfied. Therefore, these symplectic and multisymplectic structure preserving properties hold in the function space with the relevant Euler-Lagrange condition in general rather than in the solution space only. Although either the Euler-Lagrange equations and the canonical equations or the different difference discrete versions of them do preserve the relevant symplectic and multisymplectic structures.

It should pointed out that the content of the Euler-Lagrange cohomology and the roles played by the cohomology in each case should be further studied not only in classical level but also in quantum level. And needless to say, this cohomological scenario should also be tried to apply to other kinds of the mechanics and field theories such as the ones with different types of constraints and so on so forth.

3. As was shown in the text, the variational principle/difference discrete variational principle and the cohomological approach form a connecting link between the preceding and the following in either continuous or difference discrete case. And the multi-parameter differential approach provides a common framework for both of them.

It has been emphasized that both the variational principle including difference discrete variational principle and the Euler-Lagrange cohomological approach including its difference discrete version can be directly applied to the ODEs and PDEs and their discrete...
versions, which offer themselves certain numerical schemes in the symplectic and multisymplectic algorithms, no matter whether there are known Lagrangian and/or Hamiltonian associated with. In fact, the action functional may be constructed for certain types of ODEs and PDEs. Thus, the variational principle/difference discrete variational principle and the scenario of the cohomological approach are also available.

As was emphasized, in the cohomological approach it is always to release the ODEs, PDEs and numerical schemes away from their solution spaces and to work on the relevant function space rather than on the solution space even if it does exist. In the standard or conventional approaches to the numerical schemes in symplectic and multisymplectic algorithms, however, in order to show whether a given scheme is symplectic or multisymplectic, it is always working on the solution spaces. The implication of this difference is quite clear.

4. Some simple noncommutative differential calculus on the regular lattices are employed in our approach. Since the base space coordinates $t$ or $x$s are difference discretized and differences do not satisfy the ordinary commutative Leibniz law for the differential, in order to study the symplectic and multisymplectic geometry in these difference discrete systems it is natural and meaningful to make use of the noncommutative differential calculus.

5. The difference discrete version of the classical mechanics and field theory can be further generalized. What have been studied so far is the time discrete classical mechanics and spacetime/space discrete classical field theory, while the configuration spaces at each moment of $t_k$ in the discrete mechanics and the one at each node of the discretized spacetime/space on the lattice and so on are still continuous. As a matter of fact, these may be generalized to the case that the configuration spaces at each moment and/or node are also discretized. For the Hamiltonian mechanics, this is closely related to the case of difference discrete phase space approach to the systems with separable Hamiltonian.

6. Finally, it should be mentioned that there are lots of other problems to be further investigated.

Appendix

We have presented the noncommutative differential calculus on the regular lattice $L^n$ and its applications to the discrete symplectic algorithms with difference discrete phase space. In this appendix we briefly recall some content of the noncommutative differential calculus on $L^n$. General approach to the noncommutative differential geometry can be found in [20].

A.1. An noncommutative differential calculus on an Abelian discrete group

Let $G$ be an Abelian discrete group with a generator $t$, $A$ the algebra of complex valued functions on $G$. The left and/or right multiplication of a generator of $G$ on its element are commute to each other since $G$ is Abelian. Let us introduce right action on $A$ that is given by

\[ R_t f(a) = f(a \cdot t), \]

where $f \in A$, $a \in G$, $t$ the generator and $\cdot$ the group multiplication.

Let $V$ be the space of vector fields,

\[ V = span\{\partial_t\}, \]
where $\partial_t$ is the derivative with respect to the generator $t$ given by
\[(\partial_t f)(a) \equiv R_t f(a) - f(a) = f(a \cdot t) - f(a). \quad (211)\]
The dual space of $V$, the space of 1-form, is $\Omega^1 = \text{span}\{\chi^i\}$ that is dual to $V$:
\[\chi^i(\partial_t) = 1. \quad (212)\]
The whole differential algebra $\Omega^*$ can also be defined as $\Omega^* = \bigoplus_{n=0,1} \Omega^n$ with $A = \Omega^0$. Let us define the exterior differentiation in $\Omega^*$ such that $d : \Omega^0 \to \Omega^1$. It acts on a 0-form $f \in \omega^0 = A$ as follows
\[df = \partial_t f \chi^i \in \Omega^1. \quad (213)\]

Now, the following theorem can straightforwardly be proved.

**Theorem:** The exterior differential operator $d$ is nilpotent and satisfies
\[(a) \quad (df)(v) = v(f), v \in V, f \in \Omega^0, \\
(b) \quad d(\omega \otimes \omega') = d\omega \otimes \omega' + (-1)^{\text{deg}(\omega)} \omega \otimes d\omega', \quad \omega, \omega' \in \Omega^*, \quad (214)\]
if and only if
\[(1) \quad \chi^i \otimes \chi^j = (-1)\chi^j \otimes \chi^i, \\
(2) \quad d\chi^i = 0, \\
(3) \quad \chi^i f = (R_t f)\chi^i. \quad (215)\]

This theorem indicates that the $\otimes$-product should be defined as $\wedge$-product and $\chi^i$, the base of $\Omega^1$, can be denoted as $dt$.

As was shown here, in order to establish a well-defined differential algebra, it is necessary and sufficient to introduce the noncommutative property of the multiplication between function and 1-form.

The conjugation $\ast$ on the whole differential algebra $\Omega^*$ and metric on discrete Abelian group can also be defined.

In the case that the Abelian group is a discrete translation group with one generator on $R^1$, the action of the group generates a discrete chain $L^1$ with equal step-length. In the function space on $L^1$, the integrals can be defined (see, for example, [25]) as follows:
\[\int_{D_{-\infty}}^{+\infty} df(t) := \sum_{k \in Z} \Delta_t f(t_k) = f(t_{k=\infty}) - f(t_{k=-\infty}), \quad (216)\]

**A.2. An noncommutative differential calculus on Regular Lattice**

Let us consider the discrete translation group $G^m = \otimes_{i=1}^m G_i$ with $m$ generators, $A$ the function space on $G^m$ and a regular lattice with equal spacing in each direction of $L^m$ on an $m$-dimensional space $R^m$. Here $G_i$ the $i$-th discrete translation group with one generator acting on 1-dimensional space with coordinate $q$ in such a way:
\[R_{q^i} : q^i_n \to q^i_{n+1} = q^i_n + h^i, \quad h^i \in R_+, \quad (217)\]

$R_{q^i}$ the discrete translation operation of the group $G^i$ and it maps $q^i_n$ of $n$-th node of $q^i$ to the one $q^i_{n+1}$ at $n + 1$-th node, $h^i$ the discrete translation step-length along this direction and $R_+$ the positive real number. It is easy to see that the action of $G^i$ on $i$-th 1-dimensional space $R^1$ generates the $i$-th chain $L^i$, $i = 1, \cdots, m$, with equal spacing $h^i$. 

Similarly, the regular lattice $L^m$ with equal spacing $h^i$ on each direction is generated by $G^m$ acting on $R^m$. Since there is a one-to-one correspondence between nodes on $L^i$ and elements of $G^i$, one may simply regard $L^i$ as $G^i$. For the same reason, one may simply regard $L^m$ as $G^m$.

On the nodes of the regular lattice $L^m$, there are discrete coordinates $q^i_n$, $i = 1, \cdots, m$. There is a set of generators in the discrete translation group $G^m$ acting on $L^m$ in such a way:

$$R_{q^i} : q^i_n \rightarrow q^i_{n+1}, \quad i = 1, \cdots, m. \quad (218)$$

With respect to the generators there is a set of independent derivatives $\partial_{q^i}$ on $f_n(q^i) = f(q^i_n) \in A$. They should be defined as the correspondent forward differences of the functions valued at two nearest nodes, i.e.

$$\partial_{q^i} f(q^i_n) = \Delta_{q^i} f(q^i_n) = \frac{1}{h} [(R_{q^i} - id) f(q^i_n)] = \frac{1}{h} [f(q^i_{n+1}) - f(q^i_n)]. \quad (219)$$

The differential 1-form is defined by

$$df = \partial_{q^i} f dq^i = \Delta_{q^i} f dq^i, \quad f \in A. \quad (220)$$

The 2-forms and the whole differential algebra $\Omega^*$ can also be defined. Here $d$ is the exterior differential operator. Similarly, the following theorem can be proved for $d$.

**Theorem:** $d$ is nilpotent and satisfies the Leibniz rule, i.e.

$$d^2 = 0, \quad d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{deg(\omega)} \omega \wedge d\omega', \quad \omega, \omega' \in \Omega^*, \quad (221)$$

if and only if

$$f(q^i + h^i) dq^i = dq^i f(q^i),$$

$$q^i dq^i - dq^i q^i = -h^i dq^i. \quad (222)$$

The above two equations show the noncommutative properties between the functions (including the coordinates) and differential forms.

From these properties, it follows the modified Leibniz rule for derivatives (i.e. the forward differences):

$$\Delta_{q^i} (f \cdot g) = \Delta_{q^i} f \cdot g + \{R_{q^i} f\} \cdot \Delta_{q^i} g. \quad (223)$$

It should be noted that first from the definition of the forward difference, the more general Leibniz law may holds. For example, the generalized modified Leibniz law $[167]$ used in the section 5 is the special case of $a = 1/2$, while the above modified Leibniz law $\{223\}$ is corresponding to $a = 1$, in the following general Leibniz law with an arbitrary parameter $a \in [0, 1]$ $[20]$:

$$\Delta_a (f^{(k)} \cdot g^{(k)}) = \frac{1}{h} \{(af^{(k+1)} + (1 - a) f^{(k)}) \Delta_ag^{(k)} + \Delta_tf^{(k)} ((1 - a)g^{(k+1)} + ag^{(k)})\} |_{a=1/2}$$

$$= \frac{1}{h} (\Delta_tf^{(k)} \cdot g^{(k+1/2)} + f^{(k+1/2)} \cdot \Delta_tg^{(k)}), \quad f^{(k+1/2)} = 1/2(f^{(k+1)} + f^{(k)}). \quad (224)$$

It is straightforward to prove that this general one $\{224\}$ consists with the definition for the forward difference. Namely,

$$\Delta_t(f^{(k)} \cdot g^{(k)}) := \frac{1}{h} (f^{(k+1)} \cdot g^{(k+1)} - f^{(k)} \cdot g^{(k)}).$$

But the price has to paid is that the corresponding Leibniz law for the exterior differential operator $d$ no long holds (except for the case of $a = 1$).
Secondly, the definitions and relations given above for the noncommutative differential calculus on the regular lattice \( L^m \) are at least formally very similar to the ones in the ordinary commutative differential calculus on \( R^m \). The differences between the two cases are commutative or not.

Similarly, the contraction between forms and differences can be defined as the same as the one in \( R^m \):

\[
\langle dq^i \wedge dq^j, \Delta q^k \rangle = dq^i \delta^j_k, \\
\langle dq^i \wedge dq^j, \Delta q^i \Delta q^j \rangle = \delta^i_j \delta^j_i, \\
i \times f = \langle \alpha, X_f \rangle, \quad f \in A', \alpha \in \Omega^1,
\]

where \( X_f \) is the Hamiltonian vector field of \( f \).

The Hodge \( \ast \) operator and the co-differential operator

\[
\delta_L : \Omega^k \rightarrow \Omega^{k-1}
\]

on the regular lattice \( L^m \) can also be defined similarly as the ones on \( R^m \) (see, for example, [25]). Consequently, The Laplacian on the lattice \( L^m \) may also given by

\[
\Delta_L = d\delta_L + \delta_L d. \tag{226}
\]

It is in fact the discrete counterpart of the Laplacian \( \Delta \) on \( R^m \). For other objects and/or properties on \( R^m \), there may have the discrete counterparts on \( L^m \) as well. For example, the null-divergence equation of a form \( \omega \) on \( R^m \) reads

\[
\delta \omega = 0. \tag{227}
\]

Its counterpart on the lattice \( L^m \) is simply

\[
\delta_L \alpha_L = 0. \tag{228}
\]

This is the right forward difference version of the divergence-free equation.

The discrete version of the integral on \( L^m \) can also be defined (see, for example, [25]).

In the case of \( L^{1,m} \subset R^{1,m} \) with Lorentz signature, these equations become the conservation law of \( \alpha \) and its difference version of \( \alpha_L \). This is available not only for the symplectic geometry and symplectic algorithms but also the multisymplectic geometry and multisymplectic algorithms as well. It should be emphasized that for the difference discrete counterparts on the lattice, they obey the noncommutative differential calculus on the lattice \( L^m \) rather than the commutative differential calculus on \( R^m \). This is the most important point.

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