THE NEF DIMENSION OF LOG MINIMAL MODELS

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Abstract. This is the resume of the talk delivered by the author at the Symposium on Hodge theory, Degeneration and Complex surfaces, Tagajo, Miyagi, March 2004. It is based on the papers [2, 3], with some improvements.

0. Introduction

The aim of this note is to discuss the Log Abundance Conjecture for log minimal models in terms of the nef dimension of their log canonical divisor.

A log minimal model is a complex projective log variety $(X, B)$ with Kawamata log terminal singularities, such that the log canonical divisor $K + B$ is nef. The Log Abundance Conjecture [17, 10] predicts that the pluricanonical linear system $|m(K + B)|$ of a log minimal model is base point free for some positive integer $m$. This conjecture is closely related to the Log Minimal Model Program Conjecture. A special case of Log Abundance, the Base Point Free Theorem [10], played a key role in establishing the known steps of the Log Minimal Model Program (see [10]). Conversely, the 3-dimensional case of the Log Minimal Model Program was one of the key ingredients in the proof of Log Abundance in dimension 3 ([9, 14, 15, 11, 12, 13]).

The traditional approach towards the abundance problem is due to Kawamata [9]. Modulo the Log Minimal Model Program and Log Abundance in dimension at most $d - 1$, Log Abundance in dimension $d$ is equivalent to the following non-vanishing statement: given a $d$-dimensional log minimal model $(X, B)$, if there exists a curve $C$ in $X$ such that $(K + B) \cdot C > 0$ then $\dim H^0(X, m(K + B)) \geq 2$ for some positive integer $m$. This non-vanishing statement was established by Miyaoka and Kawamata in the case of minimal 3-folds, and it was extended to the logarithmic case by Keel, Matsuki and McKernan. Their approach is based on a delicate analysis of the coefficients appearing in the Riemann-Roch formula for the pluricanonical divisors, and it

\footnote{This work was supported through a Twenty-First Century COE Kyoto Mathematics Fellowship.}
is unclear whether a similar approach could work in dimension 4 or higher.

In the following, we present a numerical approach towards the abundance problem, based on the nef dimension of a nef divisor, an invariant recently introduced by Tsuji [21] and Bauer et al [5]. Given a nef divisor $D$ on a normal projective variety $X$, there exists a rational dominant map $f : X \to Y$ such that $f$ is regular over the generic point of $Y$ and a very general curve $C$ is contracted by $f$ if and only if $D \cdot C = 0$. This rational map is called the nef reduction of $D$, and $n(X, D) := \dim(Y)$ is called the nef dimension of $D$. Our main result is the following

**Theorem 0.1.** Let $(X, B)$ be a log minimal model such that $K + B$ has nef dimension at most 3. Then the linear system $|m(K + B)|$ is base point free for some positive integer $m$.

As a corollary of Theorem 0.1 and the Base Point Free Theorem [10], the 4-dimensional case of the Log Abundance Conjecture is equivalent to the following

**Conjecture 0.2.** Let $(X, B)$ be a log minimal model with $\dim(X) = 4$. If $(K + B) \cdot C > 0$ for very general curves $C \subset X$, then $(K + B)^4 > 0$.

In fact, Log Abundance and Conjecture 0.2 are equivalent in dimension $d$ if the Log Minimal Model Program and Log Abundance hold for log varieties of dimension at most $d - 1$ (Theorem 5.1). As an application, we give a new proof of Abundance for smooth minimal surfaces.

Finally, it should be mentioned that the largest class of singularities for which Log Abundance is expected to hold is that of log varieties with log canonical singularities. We expect that Theorem 4.2, which is the key technical tool behind the equivalence of Log Abundance and Conjecture 0.2, extends to the case when the general fibre has log canonical singularities. Therefore the above mentioned equivalence should be also valid in the case of log minimal models with log canonical singularities.

1. Preliminary

A *variety* is a reduced and irreducible separable scheme of finite type, defined over an algebraically closed field of characteristic zero. A *contraction* is a proper morphism $f : X \to Y$ such that $O_Y = f_* O_X$.

1-A. **Divisors.** Let $X$ a normal variety, and let $L \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. An $L$-Weil divisor is an element of $Z^1(X) \otimes \mathbb{Z} L$. Two $\mathbb{R}$-Weil divisors $D_1, D_2$ are *$L$-linearly equivalent*, denoted $D_1 \sim_L D_2$, if there exist $q_i \in L$ and
rational functions $\varphi_i \in k(X)^\times$ such that $D_1 - D_2 = \sum_i q_i(\varphi_i)$. An $\mathbb{R}$-Weil divisor $D$ is called

(i) **L-Cartier** if $D \sim_L 0$ in a neighborhood of each point of $X$.
(ii) **nef** if $D$ is $\mathbb{R}$-Cartier and $D \cdot C \geq 0$ for every curve $C \subset X$.
(iii) **ample** if $X$ is projective and the numerical class of $D$ belongs to the real cone generated by the numerical classes of ample Cartier divisors.
(iv) **semi-ample** if there exists a contraction $\Phi: X \to Y$ and an ample $\mathbb{R}$-divisor $H$ on $Y$ such that $D \sim_{\mathbb{R}} \Phi^* H$. If $D$ is rational, this is equivalent to the linear system $|kD|$ being base point free for some $k$.
(v) **big** if there exists $C > 0$ such that $\dim H^0(X, kD) \geq C k^{\dim(X)}$ for $k$ sufficiently large and divisible. By definition,

$$H^0(X, kD) = \{ a \in k(X)^\times; (a) + kD \geq 0 \} \cup \{ 0 \}.$$ 

The **Iitaka dimension** of $D$ is $\kappa(X, D) = \max_{k \geq 1} \dim \Phi_{kD}(X)$, where $\Phi_{kD}: X \dashrightarrow \mathbb{P}(kD)$ is the rational map associated to the linear system $|kD|$. If all the linear systems $|kD|$ are empty, $\kappa(X, D) = -\infty$. If $D$ is nef, the **numerical dimension** $\nu(X, D)$ is the largest non-negative integer $k$ such that there exists a $k$-dimensional cycle $C \subset X$ with $D^k \cdot C \neq 0$.

1-B. **B-divisors.** (V.V. Shokurov [19, 20].) An **L-b-divisor** $D$ of $X$ is a family $\{D_{X'}\}_{X'}$ of $L$-Weil divisors indexed by all birational models of $X$, such that $\mu_*(D_{X''}) = D_{X'}$, if $\mu: X'' \to X'$ is a birational contraction.

Equivalently, $D = \sum E \cdot \mult_E(D) E$ is a $L$-valued function on the set of all geometric valuations of the field of rational functions $k(X)$, having finite support on some (hence any) birational model of $X$.

**Example 1.** (1) Let $\omega$ be a top rational differential form of $X$. The associated family of divisors $K = \{(\omega)_{X'}\}_{X'}$ is called the **canonical b-divisor** of $X$.

(2) A rational function $\varphi \in k(X)^\times$ defines a b-divisor $\overline{D}_{\varphi} = \{(\varphi)_{X'}\}_{X'}$.

(3) An $\mathbb{R}$-Cartier divisor $D$ on a birational model $X'$ of $X$ defines an $\mathbb{R}$-b-divisor $\overline{D}$ such that $\overline{D}_{X'} = \mu^* D$ for every birational contraction $\mu: X'' \to X'$.

An $\mathbb{R}$-b-divisor $D$ is called **L-b-Cartier** if there exists a birational model $X'$ of $X$ such that $D_{X'}$ is $L$-Cartier and $D = \overline{D}_{X'}$. In this case, we say that $D$ **descends to** $X'$. An $\mathbb{R}$-b-divisor $D$ is **b-nef** (b-semi-ample, b-big, b-nef and good) if there exists a birational contraction $X' \to X$ such that $D = \overline{D}_{X'}$ and $D_{X'}$ is nef (semi-ample, big, nef and good).
1-C. **Log pairs.** A *log pair* \((X, B)\) is a normal variety \(X\) endowed with a \(\mathbb{Q}\)-Weil divisor \(B\) such that \(K + B\) is \(\mathbb{Q}\)-Cartier. A *log variety* is a log pair \((X, B)\) such that \(B\) is effective. The *discrepancy* \(\mathbb{Q}\)-b-divisor of a log pair \((X, B)\) is the \(\mathbb{Q}\)-b-divisor of \(X\) defined by
\[
\mathbf{A}(X, B) = K - K + B.
\]

More precisely, fix a top rational differential form \(\omega \in \wedge^{\dim(X)} \Omega^1_{k(X)/k}\) with \(K = (\omega)_X\). For a birational contraction \(\mu : Y \to X\), the Weil divisor \((\omega)_Y\) is a canonical divisor of \(Y\). Then \(\mathbf{A}(X, B)_Y\) is the unique \(\mathbb{Q}\)-Weil divisor on \(Y\) such that the following adjunction formula holds:
\[
\mu^* ((\omega)_X + B) = (\omega)_Y - \mathbf{A}(X, B)_Y.
\]

It is easy to see that \(\mathbf{A}(X, B)\) is independent of the choice of \(\omega\) and in fact it is independent of the choice of the canonical divisor \(K\) in its linear equivalence class.

A log pair \((X, B)\) is said to have at most *Kawamata log terminal singularities* if \(\text{mult}_E(\mathbf{A}(X, B)) > -1\) for every geometric valuation \(E\).

2. **Nef reduction**

The existence of the nef reduction map is originally due to Tsuji \[21\]. An algebraic proof of the sharper statement below is due to Bauer, Campana, Eckl, Kebekus, Peternell, Rams, Szemberg and Wotzlaw \[5\].

**Theorem 2.1.** \[21, 5\] Let \(D\) be a nef \(\mathbb{R}\)-Cartier divisor on a normal projective variety \(X\). Then there exists a rational map \(f : X \dasharrow Y\) to a normal projective variety \(Y\), satisfying the following properties:

(i) \(f\) is a dominant rational map with connected fibers, which is a morphism over the general point of \(Y\).

(ii) There exists a countable intersection \(U\) of Zariski open dense subsets of \(X\) such that for every curve \(C\) with \(C \cap U \neq \emptyset\), \(f(C)\) is a point if and only if \(D \cdot C = 0\).

Moreover, \(D|_W \equiv 0\) for general fibers \(W\) of \(f\).

The rational map \(f\) is unique, and is called the *nef reduction* of \(D\). The dimension of \(Y\) is called the *nef dimension* of \(D\), denoted by \(n(X, D)\). In general, the following inequalities hold \[9, 5\]:
\[
\kappa(X, D) \leq \nu(X, D) \leq n(X, D) \leq \dim(X).
\]

**Definition 2.2.** A nef \(\mathbb{Q}\)-Cartier divisor \(D\) is called good if
\[
\kappa(X, D) = \nu(X, D) = n(X, D).
\]
Remark 2.3. This is equivalent to Kawamata’s definition \cite{Kawamata}. If
\[ \kappa(X, D) = \nu(X, D), \]
there exists a dominant rational map \( f: X \to Y \) and a nef and big \( \mathbb{Q} \)-divisor \( H \) on \( Y \) such that \( \overline{D} \sim_{\mathbb{Q}} f^*(\overline{H}) \), by \cite{Kawamata}. In particular, \( n(X, D) \) coincides with the Iitaka and the numerical dimension in this case.

Remark 2.4. \cite{Kawamata} The extremal values of the nef dimension are:
(i) \( n(X, D) = 0 \) if and only if \( D \) is numerically trivial (\( \nu(X, D) = 0 \)).
(ii) \( n(X, D) = \dim(X) \) if and only if there exists a countable intersection \( U \) of Zariski open dense subsets of \( X \) such that \( D \cdot C > 0 \) for every curve \( C \) with \( C \cap U \neq \emptyset \).

3. Fujita decomposition

Definition 3.1. \cite{Fujita} An \( \mathbb{R} \)-Cartier divisor \( D \) on a normal proper variety \( X \) has a Fujita decomposition if there exists a b-nef \( \mathbb{R} \)-b-divisor \( P \) of \( X \) with the following properties:
(i) \( P \leq \overline{D} \).
(ii) \( P = \sup\{ H; H \text{ b-nef } \mathbb{R} \text{-b-divisor, } H \leq \overline{D} \} \).
The \( \mathbb{R} \)-b-divisor \( P = P(D) \) is unique if it exists, and is called the semi-positive part of \( D \). The \( \mathbb{R} \)-b-divisor \( E = \overline{D} - P \) is called the negative part of \( D \), and \( \overline{D} = P + E \) is called the Fujita decomposition of \( D \).

Remark 3.2. Allowing divisors with real coefficients is necessary: there exist Cartier divisors (in dimension at least 3) which have a Fujita decomposition with irrational semi-positive part \cite{Iitaka}.

Clearly, a nef \( \mathbb{R} \)-Cartier divisor \( D \) has a Fujita decomposition, with semi-positive part \( \overline{D} \). More examples can be constructed using the following property:

Proposition 3.3. \cite{Fujita} Let \( f: X \to Y \) be a proper contraction, let \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( Y \) and let \( E \) be an effective \( \mathbb{R} \)-Cartier divisor on \( X \) such that \( E \) is vertical and supports no fibers over codimension one points of \( Y \).

Then \( D \) has a Fujita decomposition if and only if \( f^*D + E \) has a Fujita decomposition, and moreover, \( P(f^*D + E) = f^*(P(D)) \).

Lemma 3.4. Assume LMMP and Log Abundance. Let \( (X, B) \) be a log variety with log canonical singularities. Then \( K + B \) has a Fujita decomposition if and only if \( \kappa(X, K + B) \geq 0 \), and the semi-positive part is semi-ample. Moreover,
\[ P(K + B) = \overline{K_Y + B_Y}, \]
for a log minimal model \((Y, B_Y)\).

**Lemma 3.5.** \([9, 7]\) Let \(f : X \to Y\) be a contraction of normal proper varieties, and let \(D\) be a nef \(\mathbb{R}\)-divisor on \(X\) which is vertical on \(Y\). Then there exists a \(b\)-nef \(\mathbb{R}\)-\(b\)-divisor \(\overline{D}\) of \(Y\) such that \(\overline{D} = f^*D\).

### 4. LC-trivial fibrations

This section is a brief introduction to lc-trivial fibrations. We refer the reader to \([1, 3]\) for more details. As we shall see in a while, these type of fibrations appear naturally in inductive arguments in the Log Minimal Model Program.

**Definition 4.1.** An lc-trivial fibration \(f : (X, B) \to Y\) consists of a contraction of normal varieties \(f : X \to Y\) and a log pair \((X, B)\), satisfying the following properties:

1. \((X, B)\) has Kawamata log terminal singularities over the generic point of \(Y\).
2. \(\text{rank} \, f_*\mathcal{O}_X([A(X, B)]) = 1\).
3. There exist a positive integer \(r\), a rational function \(\varphi \in k(X)^\times\) and a \(\mathbb{Q}\)-Cartier divisor \(D\) on \(Y\) such that
   \[
   K + B + \frac{1}{r}(\varphi) = f^*D.
   \]

For an lc-trivial fibration \(f : (X, B) \to Y\), we define \(B_Y = \sum_{P \in Y} b_P P\), where the sum runs after all prime divisors of \(Y\), and
\[
1 - b_P = \sup \{t \in \mathbb{R}; \exists U \ni \eta_P, (X, B + tf^*(P)) \text{ lc sing}/U\}.
\]
It is easy to see that \(B_Y\) is a well defined \(\mathbb{Q}\)-Weil divisor on \(Y\). By (3), there exists a unique \(\mathbb{Q}\)-Weil divisor \(M_Y\) such that the following adjunction formula holds:
\[
K + B + \frac{1}{r}(\varphi) = f^*(K_Y + B_Y + M_Y).
\]
The \(\mathbb{Q}\)-Weil divisors \(B_Y\) and \(M_Y\) are called the discriminant and moduli part of the lc-trivial fibration \(f : (X, B) \to Y\).

Let now \(\sigma : Y' \to Y\) be a birational contraction from a normal variety \(Y'\). Let \(X'\) be a resolution of the main component of \(X \times_Y Y'\) which dominates \(Y'\). The induced morphism \(\mu : X' \to X\) is birational, and let \((X', B_{X'})\) be the crepant log structure on \(X'\), i.e. \(\mu^*(K + B) = K_{X'} + B_{X'}\):

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\mu} & (X', B_{X'}) \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\sigma} & Y'
\end{array}
\]
We say that the lc-trivial fibration $f': (X', B_{X'}) \to Y'$ is induced by base change. Let $B_{Y'}$ be the discriminant of $K_{X'} + B_{X'}$ on $Y'$. Since the definition of the discriminant is divisorial and $\sigma$ is an isomorphism over codimension one points of $Y$, we have $B_Y = \sigma_*(B_{Y'})$. This means that there exists a unique $\mathbb{Q}$-b-divisor $B$ of $Y$ such that $B_{Y'}$ is the discriminant on $Y'$ of the induced fibre space $f': (X', B_{X'}) \to Y'$, for every birational model $Y'$ of $Y$. We call $B$ the discriminant $\mathbb{Q}$-b-divisor of the lc-trivial fibration $f': (X, B)$ on the birational class of $Y$. Accordingly, there exists a unique $\mathbb{Q}$-b_divisor $M$ of $Y$ such that

$$K_{X'} + B_{X'} + \frac{1}{r}(\varphi) = f'^*(K_{Y'} + B_{Y'} + M_{Y'})$$

for every lc-trivial fibration $f': (X', B_{X'}) \to Y'$ induced by base change on a birational model $Y'$ of $Y$. We call $M$ the moduli $\mathbb{Q}$-b-divisor of the lc-trivial fibration $f: (X, B) \to Y$.

The positivity properties of the moduli $\mathbb{Q}$-b_divisor of an lc-trivial fibration play a key role in applications. By [1], Theorem 0.2, the moduli $\mathbb{Q}$-b-divisor $M$ is b-nef. However, it is expected that $M$ is in fact b-semiample. This is true, for instance, if $Y$ is a curve [1], Theorem 0.1. Under an extra assumption, we can prove that $M$ is “almost” b-semiample (cf. [3], Theorem 3.3):

**Theorem 4.2.** Let $f: (X, B) \to Y$ be an lc-trivial fibration such that the geometric generic fibre $X_\eta = X \times_Y \text{Spec}(k(Y))$ is projective and $B_\eta$ is effective.

Then the moduli $\mathbb{Q}$-b-divisor $M$ is b-nef and good.

5. Reduction argument

**Theorem 5.1.** Let $(X, B)$ be a projective log variety with Kawamata log terminal singularities such that the log canonical divisor $K + B$ is nef, of positive nef dimension $n = n(X, K + B)$.

If the Log Minimal Model Program and Log Adjunction hold in dimension $n(X, K + B)$, then $K + B$ is a semi-ample $\mathbb{Q}$-divisor.

**Proof.** Let $\Phi: X \dashrightarrow Y$ be the quasi-fibration associated to the nef divisor $K + B$, and let $\Gamma$ be the normalization of the graph of $\Phi$:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{f} & Y \\
\mu \downarrow & & \\
X & \xleftarrow{\mu} & \\
\end{array}
$$

Since $\Phi$ is a quasi-fibration, $\mu$ is birational, $f$ is a contraction and $\text{Exc}(\mu) \subset \Gamma$ is vertical over $Y$. Let $W$ be a general fiber of $f$. Let $K_\Gamma + B_\Gamma = \mu^*(K + B)$ and let $B_W = B_\Gamma|_W$. 


Step 1: $(W, B_W)$ is a projective log variety with Kawamata log terminal singularities, and $K_W + B_W \sim_{\mathbb{Q}} 0$. Since $\mu$ is an isomorphism in a neighborhood of $W$, we infer that $B_W$ is effective, $(W, B_W)$ has Kawamata log terminal singularities and $K_W + B_W = \mu^*(K + B)|_W$. The definition of $\Phi$ implies that $K_W + B_W$ is numerically trivial. From [3], Theorem 0.1, we conclude that $K_W + B_W \sim_{\mathbb{Q}} 0$.

Step 2: There exist a diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\mu} & X' \\
\downarrow{f'} & & \downarrow{f'} \\
Y' & & Y''
\end{array}
$$

satisfying the following properties:

(a) $\mu$ is a birational contraction and $X', Y'$ are nonsingular.
(b) $f': (X', B_{X'}) \to Y'$ is an lc-trivial fibration, where $K_{X'} + B_{X'} = \mu^*(K + B)$.
(c) The moduli $\mathbb{Q}$-b-divisor $M$ of the lc-trivial fibration $f': (X', B_{X'}) \to Y'$ descends to $Y'$ and there exists a contraction $h: Y' \to Z$ and a nef and big $\mathbb{Q}$-divisor $N$ on $Z$ such that $M_{Y'} \sim_{\mathbb{Q}} h^*N$.
(d) Let $E$ be any prime divisor on $X'$. If $E$ is exceptional over $Y'$, then $E$ is exceptional over $X$.

From Step 1, there exists a rational function $\varphi \in k(X)^\times$ and $r \in \mathbb{Q}$ such that $\mu^*(K + B) + r(\varphi)$ is an $f$-vertical nef $\mathbb{Q}$-divisor. By Lemma [**], there exists a b-nef $\mathbb{Q}$-b-divisor $D$ of $Y$ such that $K + B + r(\varphi) = f^*(D)$.

Let $Y'' \to Y'$ be a resolution of singularities such that $D$ descends to $Y''$, and let $X''$ be the normalization of the main component of $X \times_Y Y''$. We denote the induced birational contraction by $\mu: X'' \to X$, and let $\mu^*(K + B) = K_{X''} + B_{X''}$. It is clear that $f': (X'', B_{X''}) \to Y''$ is an lc-trivial fibration. By [1], Theorem 0.2, the corresponding moduli $\mathbb{Q}$-b-divisor $M$ is $\mathbb{Q}$-b-Cartier. Furthermore, the $f''$-horizontal part of $B_{X''}$ is effective since $\text{Exc}(X''/X)$ is $f''$-vertical and $B$ is effective. By [3], Theorem 3.3, we infer that the property (c) holds if we replace $Y'$ by a sufficiently high resolution.

Let $Y' \to Y''$ be a resolution of singularities such that (c) holds for the induced lc-trivial fibratation and such that $f': X' \to Y''$ dominates a flattening of $f$, where now $X'$ is a resolution of singularities of the main component of $X \times_Y Y'$. It is clear that the properties $(a) - (d)$ hold.

Step 3: There exists an effective $\mathbb{Q}$-divisor $\Delta'$ on $Y'$ such that $(Y', \Delta')$ is a log variety with Kawamata log terminal singularities, $K_{Y'} + \Delta'$ has
a Fujita decomposition and
\[ K + B \sim_Q f^*(\mathbb{P}(K_{Y'} + \Delta')). \]
Indeed, let \( B_{X'} = B_{X'}^+ - B_{X'}^- \) be the decomposition of \( B_{X'} \) into its positive and negative part. There exists a unique \( \mathbb{Q} \)-divisor \( F \) on \( Y' \) such that \( A := B_{X'}^- + f^*F \) is effective and supports no fibers of \( f' \), over a big open subset of \( Y' \). In particular, \( f' : (X', B_{X'}^+ - A) \rightarrow Y' \) is an lc-trivial fibration with the same moduli \( \mathbb{Q} \)-b-divisor \( M \). Let \( \Delta_{Y'} \) be the corresponding discriminant \( \mathbb{Q} \)-divisor on \( Y' \). Since \( A \) does not support fibers over codimension one points of \( Y' \), we infer that \( \Delta_{Y'} \) is effective. We have
\[ K_{X'} + B_{X'}^+ - A + \frac{1}{b}(\varphi) = f^*(K_{Y'} + \Delta_{Y'} + M_{Y'}). \]
It is clear that \((Y', \Delta_{Y'})\) is a log variety with Kawamata log terminal singularities. By (c), there exists an effective \( \mathbb{Q} \)-divisor \( \Delta' \) on \( Y' \) such that \((Y', \Delta')\) is a log variety with Kawamata log terminal singularities, and \( \Delta' \sim_Q B_{Y'} + M_{Y'} \). In particular,
\[ K_{X'} + B_{X'}^+ - A \sim_Q f^*(K_{Y'} + \Delta'). \]
Let \( A = A^+ - A^- \) be the decomposition of \( A \) into its positive and negative part. We have
\[ \mu^*(K + B) + B_{X'}^- + A^- \sim_Q f^*(K_{Y'} + \Delta') + A^+. \]
Both \( \mathbb{Q} \)-divisors \( B_{X'}^-, A^- \) are effective and \( \mu \)-exceptional. This is clear for \( B_{X'}^- \). As for \( A^- \), this follows from property (d) since \( A^- \) is \( f' \)-exceptional by construction. Finally, \( A^+ \) is effective, and it does not support fibers over codimension one points of \( Y' \).
Since \( K + B \) is nef, the left hand side has a Fujita decomposition, with semi-positive part \( K + B \). Proposition 3.3 applies, hence \( K_{Y'} + \Delta' \) has a Fujita decomposition as well and \( K + B \sim_Q f^*(\mathbb{P}(K_{Y'} + \Delta')) \).

**Step 4:** From the Log Minimal Model Program and Log Abundance applied to the log variety \((Y', \Delta)\), the semi-positive part of \( K_{Y'} + \Delta' \) is \( b \)-semi-ample. Therefore \( K + B \)' is \( b \)-semi-ample, that is \( K + B \) is a semi-ample \( \mathbb{Q} \)-divisor.

**Proof.** (of Theorem 0.1) If \( K + B \) is numerically trivial, then Log Abundance holds by [3], Theorem 0.1.(1). If \( K + B \) has positive nef dimension, we may apply Theorem 5.1 since the Log Minimal Model Program and Log Abundance are known to hold up to dimension 3 ([18],[12]).

Finally, we show how Theorem 5.1 may be used to give another proof Abundance in dimension two, in characteristic zero.
Theorem 5.2. Let $X$ be a nonsingular projective complex surface whose canonical divisor $K$ is nef. Then there exists a positive integer $m$ such that the linear system $|mK|$ is base point free.

Proof. We may assume that $K$ has maximal nef dimension. Indeed, otherwise either $K$ is numerically trivial hence a torsion divisor by Kawamata [8], or $K$ has nef dimension one. The one dimensional case of Log Abundance is easy to check, hence Theorem 5.1 implies that $K$ is semi-ample if $n(X, K) = 1$.

If $K$ is big, the result follows from the Base Point Free Theorem [10]. In the following, we will assume by contradiction that $K$ has maximal nef dimension but $(K^2) = 0$. In particular, $K \cdot D = 0$ for every $D \in |mK|$ $(m \geq 1)$. Since $K$ has maximal nef dimension, we infer $\kappa(K) \leq 0$.

Since $(K^2) = 0$, the Riemann-Roch formula gives
\[
\chi(X, mK) = 1 - q(X) + p_a(X).
\]
Furthermore, $H^0(X, mK) = 0$ for $m \leq -1$, since $K$ has positive nef dimension. Therefore $H^2(X, mK) = 0$ for $m \geq 2$, by duality. In particular,
\[
h^0(X, mK) \geq 1 - q(X) + h^0(X, K) \text{ for } m \geq 2.
\]
Assume that $q(X) = 0$. In particular, $\kappa(K) \geq 0$, hence $\kappa(K) = 0$. Then there exists a positive integer $b$ such that
\[
\{ m \in \mathbb{Z}_{>0}; |mK| \neq \emptyset \} = b\mathbb{Z}_{>0}.
\]
But $|mK| \neq \emptyset$ for every $m \geq 2$, from above, hence $b = 1$. Therefore $h^0(X, K) \geq 1$, hence $h^0(X, 2K) \geq 2$, which contradicts $\kappa(K) = 0$.

Assume now that $q(X) > 0$. The Stein factorization of the Albanese map $X \to \text{Alb}(X)$ induces a nontrivial fiber space $f: X \to Y$. Assume first that $Y$ is a curve, and let $F$ be the general fibre of $f$. Since $K$ has maximal nef dimension, its restriction $K_F = K|_F$ has maximal nef dimension as well. Therefore $K_F$ is ample. Furthermore, $\kappa(Y) \geq 0$ ([22]) and Iitaka’s Addition Conjecture $C_{2,1}$ (see [4] for instance) implies
\[
\kappa(X) \geq \kappa(F) + \kappa(Y) \geq 1,
\]
a contradiction. Finally, assume that $f$ is birational. Since $\kappa(X) \leq 0$, we infer by [22], that $f$ is the Albanese map of $X$. Since $K_{\text{Alb}(X)} = 0$, there exists an effective $f$-exceptional divisor $E$ on $X$ such that $K = E$. Therefore $K$ does not have maximal nef dimension, a contradiction. \quad \Box

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