CENTRAL STABILITY FOR THE HOMOLOGY OF CONGRUENCE SUBGROUPS AND THE SECOND HOMOLOGY OF TORELLI GROUPS

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Abstract. We prove a representation stability result for the second homology groups of Torelli subgroups of mapping class groups and automorphism groups of free groups. This strengthens the results of Boldsen–Hauge Dollerup and Day–Putman. We also prove a new representation stability result for the homology of certain congruence subgroups, partially improving upon the work of Putman–Sam. These results follow from a general theorem on syzygies of certain modules with finite polynomial degree.

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1. Introduction

The purpose of this paper is to prove some cases of two conjectures from Church and Farb [CF13] as well as to answer a question posed by Putman [Put15]. Specifically, we prove a central stability result for the second homology groups of the Torelli subgroups of mapping class groups and automorphism groups of free groups, and a central stability result for the homology groups in all degrees of certain congruence subgroups of general linear groups. These results improve some of the results of Boldsen and Hauge Dollerup [BHD12], Day and Putman [DP14], and Putman and Sam [PS14]. The main technical lemma of this paper is a result showing higher central stability for modules over certain categories with finite polynomial degree, such as modules over the categories appearing in work of Putman and Sam [PS14]. The proof of this lemma involves verifying a general acyclicity criterion developed by the second author [Pat17] for complexes associated to these modules. From this result we deduce that if a module satisfies a polynomial condition—a condition often appearing in twisted homological stability theorems—then this module exhibits a form of representation stability. We begin by recalling various related definitions of representation stability.

1.1. Representation stability and central stability. Let $K$ be a commutative ring; all rings are assumed to have a unit unless otherwise stated. Let $G_0 \to G_1 \to G_2 \to \cdots$ be a sequence of groups and group homomorphisms, and let $A_0 \to A_1 \to A_2 \to \cdots$ be a sequence of $K$–modules equipped with a linear action of $G_n$ on $A_n$ such that the map $A_n \to A_{n+1}$ is $G_n$–equivariant. In this paper, the groups $G_n$ will be symmetric groups, general linear groups, subgroups of general linear groups with restricted determinant, or symplectic groups, and $K$ will be the integers. For $K = \mathbb{Q}$ and $\{G_n\}$ being the special linear groups $\{\text{SL}_n \mathbb{Q}\}$, the general linear groups $\{\text{GL}_n \mathbb{Q}\}$, the symplectic groups $\{\text{Sp}_{2n} \mathbb{Q}\}$, the symmetric groups $\{\mathfrak{S}_n\}$ and the hyperoctahedral groups $\{W_n\}$, Church and Farb [CF13] described a condition they called representation stability for these types of sequences $\{A_n\}$. In its original formulation, representation stability was defined for semisimple representations $A_n$ and for families of groups admitting natural identifications between irreducible $G_n$–representations for different $n$. Under these assumptions, the sequence $\{A_n\}$ is called representation stable if the multiplicities of the irreducible constituents of $\{A_n\}$ eventually stabilize as $n$ increases, in a mode compatible with the maps $A_n \to A_{n+1}$.

Putman [Put15] and Church, Ellenberg, and Farb [CEF15, CE15] each independently developed a more formal theory of representation stability. Putman introduced central stability, which we define below. Church–Ellenberg–Farb’s approach realized the sequences as modules over a certain category, and their corresponding notion of stability is presentability in finite degree; see Definition 3.8. In the case that the groups $G_n$ are the symmetric groups, these two definitions are equivalent. Moreover, Church–Ellenberg–Farb showed that for $K = \mathbb{Q}$, these two definitions imply representation stability in the sense of [CF13]. Both frameworks naturally generalize to other families of groups, and in Section 3.2 we discuss when these two generalizations are equivalent. An advantage of central stability and finite presentability is that these concepts are well-defined even without a complete classification of irreducible
We write $IA_n$–representations, and in contexts where the representation theory is not semisimple. We can apply them, for example, to sequences of $\mathbb{Z}[GL_n(\mathbb{Z})]$–modules.

For purposes of exposition, we will now specialize to the case that the groups $G_n$ are either the family of general linear groups $GL_n(\mathbb{Z})$ or the family of symmetric groups $S_n$. Let $\sigma_n \in G_n$ be the transposition $(n-1\ n)$ in the case $G_n = S_n$ and the associated permutation matrix in the case $G_n = GL_n(\mathbb{Z})$. Consider a sequence of $G_n$–representations $\{A_n\}$ with $G_n$–equivariant maps $\phi_n : A_n \to A_{n+1}$. We assume that $\sigma_n$ fixes the map $(\phi_{n-1} \circ \phi_n) : A_{n-2} \to A_n$. There are two natural maps

$$\text{Ind}_{G_{n-2}}^{G_n} A_{n-2} \to \text{Ind}_{G_{n-1}}^{G_n} A_{n-1},$$

the first induced by $\phi_{n-2}$ and the second by postcomposing this induced map by $\sigma_n$. We say that the sequence $\{A_n\}$ has \textit{central stability degree} $\leq d$ if for all $n > d$ the map

$$\text{coeq} \left( \text{Ind}_{G_{n-2}}^{G_n} A_{n-2} \to \text{Ind}_{G_{n-1}}^{G_n} A_{n-1} \right) \to A_n$$

induced by $\phi_{n-1}$ is an isomorphism. We say that $\{A_n\}$ is \textit{centrally stable} if it has finite central stability degree. If $\{A_n\}$ has central stability degree $\leq d$, then the entire sequence is determined by the finite sequence $A_0 \to A_1 \to \cdots \to A_d$. Analogous definitions exist for symplectic and general linear groups with restricted determinant, which we review in Section 3. The main result of this paper is to prove that certain homology groups of Torelli groups and congruence subgroups are centrally stable.

\textbf{Remark 1.1.} Putman–Sam [PS14] used a slightly different definition of central stability than the one used here and the one used by Putman [Put15]. Work of Djament [Dja16, Proposition 2.14] imply that central stability degree in the sense of Putman–Sam [PS14] agrees with our definition of presentation degree; see also Gan–Li [GL15a, Theorem 3.2]. See results of the second author [Pat17, Proposition 6.1 and 6.2] for a comparison of these different forms of stability.

### 1.2. Central stability for Torelli groups and congruence subgroups.

\textit{Automorphisms of free groups.} Let $F_n$ denote the free group on $n$ letters. A group automorphism $f : F_n \to F_n$ induces a linear map $f^{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ on the abelianization of $F_n$. This construction defines a surjective map

$$\text{Aut}(F_n) \to GL_n(\mathbb{Z}).$$

We write $IA_n$ to denote the kernel of this map, which is often called the \textit{Torelli subgroup} of $\text{Aut}(F_n)$. If we fix an inclusion of $\text{Aut}(F_n)$ into $\text{Aut}(F_{n+1})$ as the stabilizer of a free generator, the Torelli group $IA_n$ maps into $IA_{n+1}$. The induced maps $H_i(IA_n) \to H_i(IA_{n+1})$ are equivariant with respect to the induced actions of the groups $GL_n(\mathbb{Z})$, and so we may ask whether the sequences of homology groups $\{H_i(IA_n)\}$ exhibit central stability as $GL_n(\mathbb{Z})$–representations.

The first homology group of $IA_n$ was computed independently in work of Andreadakis [And65], Cohen–Pakianathan (unpublished), Farb (unpublished) and Kawazumi [Kaw06]. Very little is known about the higher homology groups of $IA_n$. Bestvina–Bux–Margalit [BBM07] proved that $H_2(IA_3)$ is not finitely generated as an abelian group which implies that $IA_3$ is not finitely presented, a result originally due to Krsti´c–McCool [KM97]. However, for $n > 3$, it is unknown if $H_2(IA_n)$ is finitely generated or if $IA_n$ is finitely presented.

Church and Farb [CF13, Conjecture 6.3] conjectured “mixed representation stability” for a certain summand of the rational homology of $IA_n$. Given Putman’s subsequent work on central stability, it
seems natural to modify Church and Farb’s Conjecture 6.3 to also ask if the integral homology groups of \( \text{IA}_n \) exhibit central stability as representations of the general linear groups. We prove this result for the second homology groups.

**Theorem A.** The sequence \( H_2(\text{IA}_n) \) has central stability degree \( \leq 38 \) as \( \text{GL}_n(\mathbb{Z}) \)-representations.

This theorem partially improves upon a result of Day and Putman [DP14, Theorem B], which established surjectivity of the maps

\[
\text{coeq} \left( \text{Ind}_{\text{GL}_{n-2}(\mathbb{Z})}^{\text{GL}_n(\mathbb{Z})} H_2(\text{IA}_{n-2}) \Rightarrow \text{Ind}_{\text{GL}_{n-1}(\mathbb{Z})}^{\text{GL}_n(\mathbb{Z})} H_2(\text{IA}_{n-1}) \right) \to H_2(\text{IA}_n)
\]

for \( n > 6 \). Our techniques only show these maps surject for \( n > 18 \), but additionally prove the maps are injective for \( n > 38 \). We note that our method is substantially different from that of Day and Putman: our proof centers on properties of general linear groups, while their proof focused on properties of automorphism groups of free groups. Our proof strategy applies to Torelli subgroups of mapping class groups with little modification. In contrast, Day and Putman [DP14, Remark 1.3] noted that their techniques for Torelli subgroups of \( \text{Aut}(F_n) \) do not easily generalize to Torelli subgroups of mapping class groups, and that the techniques used by Boldsen and Hauge Dollerup [BHD12] for Torelli subgroups of mapping class groups do not easily generalize to prove results about \( \text{IA}_n \).

**Mapping class groups.** Let \( \Sigma_{g,1} \) denote an oriented genus-\( g \) surface with one boundary component, and let \( \text{Mod}_g \) denote the group of connected components of the group of orientation-preserving diffeomorphism of \( \Sigma_{g,1} \) that fix the boundary pointwise. The induced action of these diffeomorphisms on the first homology group of \( \Sigma_{g,1} \) preserves the intersection form, which happens to be symplectic. This action therefore induces a surjective map

\[
\text{Mod}_g \to \text{Sp}_{2g}(\mathbb{Z}),
\]

where \( \text{Sp}_{2g}(R) \) denotes the group of linear automorphisms of \( R^{2g} \) that preserve the standard symplectic form. The kernel \( I_g \) of this map is called the Torelli subgroup of \( \text{Mod}_g \). Just as with the groups \( \text{IA}_n \), very little is known about the homology groups \( H_i(I_g) \) for \( i > 1 \). Church and Farb conjectured a form of representation stability for \( H_i(I_g) \) [CF13, Conjecture 6.1], motivated by their earlier work [CF12] constructing nontrivial classes in \( H_i(I_g; \mathbb{Q}) \). In this paper we prove a central stability result for the homology groups \( H_2(I_g) \) as symplectic group representations.

**Theorem B.** The sequence \( H_2(I_g) \) has central stability degree \( \leq 45 \) as \( \text{Sp}_{2g}(\mathbb{Z}) \)-representations.

This partially improves upon a result of Boldsen and Hauge Dollerup [BHD12, Theorem 1.0.1] proving that

\[
\text{coeq} \left( \text{Ind}_{\text{Sp}_{2g-4}(\mathbb{Z})}^{\text{Sp}_{2g}(\mathbb{Z})} H_2(I_g; \mathbb{Q}) \Rightarrow \text{Ind}_{\text{Sp}_{2g-2}(\mathbb{Z})}^{\text{Sp}_{2g}(\mathbb{Z})} H_2(I_{g-1}; \mathbb{Q}) \right) \to H_2(I_g; \mathbb{Q})
\]

is surjective for \( g > 6 \); our techniques only show the map is surjective for \( g > 22 \) but also establish injectivity in this range, and our result holds with integral coefficients. In work in progress, Kassabov–Putman [KP] independently proved an integral version of [BHD12, Theorem 1.0.1].
**Congruence subgroups of $\text{GL}_n(R)$**. We also investigate representation stability for congruence subgroups of general linear groups. Homological and representation stability properties of congruence subgroups have had applications in homotopy theory – for example, in work of Charney [Cha84] on excision for algebraic $K$-theory – and applications in number theory; see Calegari–Emerton [CE16]. Let $I$ be a two-sided ideal in a ring $R$, and let $\text{GL}_n(R, I)$ denote the kernel of the “reduction modulo $I$” map

$$\text{GL}_n(R) \to \text{GL}_n(R/I).$$

We call $\text{GL}_n(R, I)$ the **level $I$ congruence subgroup** of $\text{GL}_n(R)$.

We note that, up to isomorphism, the group $\text{GL}_n(R, I)$ only depends on the ideal $I$, viewed as a non-unital ring, and not the ambient ring $R$.

For $H$ a subgroup of the group of units $R^\times$ of $R$, let $\text{GL}_n^H(R)$ denote the subgroup of matrices with determinant in $H$. Let $\mathfrak{I}$ denote the image of $R^\times$ in $R/I$ and observe that $\text{GL}_n^\mathfrak{I}(R/I)$ is the image of $\text{GL}_n(R, I)$ in $\text{GL}_n(R/I)$. The homology groups $H_i(\text{GL}_n(R, I))$ have natural linear $\text{GL}_n^\mathfrak{I}(R/I)$–actions and there are equivariant maps $H_i(\text{GL}_n(R, I)) \to H_i(\text{GL}_{n+1}(R, I))$ induced by the inclusions $\text{GL}_n(R, I) \hookrightarrow \text{GL}_{n+1}(R, I)$.

The symmetric group $\mathfrak{S}_n$ is naturally a subgroup of $\text{GL}_n^\mathfrak{I}(R/I)$, so we may view the the groups $H_i(\text{GL}_n(R, I))$ as representations of the symmetric group. Putman [Put15] proved that when $R$ has finite stable rank (see Definition 2.19 and Bass [Bas64]), these homology groups have central stability as $\mathfrak{S}_n$–representations. Putman’s result gave explicit stable ranges for the homology groups $H_i(\text{GL}_n(R, I); K)$ over certain fields $K$, and later Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem D] proved a finite presentation result for the homology of certain congruence subgroups with coefficients in a general Noetherian ring $K$ but without explicit bounds. Church and Ellenberg [CE15, Theorem D'] generalized both theorems with a result for integral homology with explicit stable ranges.

Putman [Put15, fifth Remark] comments that it would be ideal to understand stability properties of $\text{GL}_n(R, I)$ as $\text{GL}_n^\mathfrak{I}(R/I)$–representations instead of just $\mathfrak{S}_n$–representations. We provide the following partial solution to this problem.

**Theorem C.** Let $I$ be a two-sided ideal of a ring $R$ and let $t$ be the minimal stable rank of all rings containing $I$ as a two-sided ideal. If $R/I$ is a PID of stable rank $s$, then the central stability degree of the sequence $H_i(\text{GL}_n(R, I))$ is

$$
\begin{align*}
\leq s + 1 & \quad \text{for } i = 0 \\
\leq \max(5 + t, 5 + s) & \quad \text{for } i = 1 \\
\leq (2^{i-1})(6t + 21) - 10 + s & \quad \text{for } i \geq 2
\end{align*}
$$

as $\text{GL}_n^\mathfrak{I}(R/I)$–representations.

**Remark 1.2.** Note that the quotient $R/I$ need not be a PID in order to define an action on $H_*(\text{GL}_n(R, I))$ by a group $\text{GL}_n(Q)$ with $Q$ a PID. For example, given an non-unital ring $I$, we can realize $I$ as a two-sided ideal in its unitalization $I_+$, its image under the left adjoint to the forgetful functor from the category of (unital) rings to the category of non-unital rings. Then $I_+/I \cong \mathbb{Z}$, and $\text{GL}_n(I_+, I) \cong \text{GL}_n(R, I)$. Hence in particular $H_*(\text{GL}_n(R, I))$ is always a $\text{GL}_n(\mathbb{Z})$–representation.
It follows from the work of Putman and Sam [PS14] that $H_i(\text{GL}_n(R, I))$ has finite central stability degree when $R$ is the ring of integers in an algebraic number field. However, their techniques do not give an explicit central stability range or address congruence subgroups with infinite quotients $R/I$.

### 1.3. Polynomial degree.

The homology groups of the Torelli groups $\text{IA}_n$ have the structure of modules over the category $\text{VIC}(\mathbb{Z})$, an algebraic construction introduced by Putman–Sam [PS14].

**Definition 1.3.** For a commutative unital ring $R$, let $\text{VIC}(R)$ denote the category whose objects are finite rank free $R$–modules and whose morphisms from $V$ to $W$ are given by the set of pairs $(f, C)$ of an injective homomorphism $f: V \to W$ and a free submodule $C \subseteq W$ such that $\text{im}(f) \oplus C = W$. A $\text{VIC}(R)$–module is a functor from $\text{VIC}(R)$ to the category of abelian groups.

Evaluating a $\text{VIC}(R)$–module $A$ on the modules $R^n$ gives a sequence of abelian groups $\{A_n\}$. The inclusions $R^n \to R^{n+1}$ give maps $A_n \to A_{n+1}$. Since the endomorphisms of $R^n$ in the category $\text{VIC}(R)$ is $\text{GL}_n(R)$, the abelian group $A_n$ has the structure of a $\text{GL}_n(R)$–representation such that the maps $A_n \to A_{n+1}$ are $\text{GL}_n(R)$–equivariant and postcomposition by the involution $\sigma_n$ fixes the map $A_n \to A_{n+2}$. Thus the question of central stability is well-posed for a $\text{VIC}(R)$–module, and this $\text{VIC}(R)$–module formalism helps clarify many central stability proofs.

In addition to being relevant for central stability, $\text{VIC}(R)$–modules are useful for stating twisted homological stability theorems for general linear groups; see for example van der Kallen [vdK80], Dwyer [Dwy80], or Randal-Williams–Wahl [RWW14]. If $A$ is a $\text{VIC}(R)$–module, then the map $\text{GL}_n(R) \to \text{GL}_{n+1}(R)$ induces a map

$$H_i(\text{GL}_n(R); A_n) \to H_i(\text{GL}_{n+1}(R); A_{n+1}).$$

The usual conditions on $A$ to make these maps isomorphisms in a range are called polynomial degree conditions (Definition 3.19), and analogous conditions exist for other families of groups like $\mathfrak{S}_n$ and symplectic groups.

A sequence of symmetric group representations is centrally stable if and only if it has finite polynomial degree; see Theorem 3.25 as well as Randal-Williams–Wahl [RWW14, Example and Proposition 4.18]. In general, however, these two notions of stability are quite different, as is illustrated in the following example adapted from Putman–Sam [PS14, Remark 1.28].

**Example 1.4.** Suppose that $R$ is the finite ring $R = \mathbb{Z}/m\mathbb{Z}$, and let $A$ be the $\text{VIC}$–module that maps a free $R$–module $R^n$ to the free abelian group on the set $\{(v, C)\}$ with $v \in R^n$ a nonzero vector and $C$ a direct complement of $\text{span}(v)$. Then $A$ has central stability degree 3. However, the ranks of the groups $A_n$ grow exponentially in $n$, while the definition of finite polynomial degree implies that ranks for such functors grow at most polynomially.

The primary ingredient in the proofs of the theorems of this paper is the following result comparing this polynomial condition to central stability. When $R$ is a principal ideal domain, we prove that $\text{VIC}(R)$–modules with finite polynomial degree exhibit higher centrally stability (Definition 3.11). Higher central stability is a vanishing condition on the central stability homology groups $\tilde{H}_*(A)$ of $A$ defined in Definition 3.10. It implies central stability and also controls degrees of higher syzygies of $A$. 
Corollary 3.29 Assume $R$ is a PID of stable rank $s$ and let $H \leq R^\times$. If $A$ is a $\mathcal{VIC}^H(R)$–module of polynomial degree $\leq 0$ in ranks $>d$ for some $d \geq -1$, then

$$\tilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d + i + 2, 2i + s + 1).$$

If $A$ is a $\mathcal{VIC}^H(R)$–module of polynomial degree $\leq r$ in ranks $>d$ for some $r \geq 1$ and $d \geq -1$, then

$$\tilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > 2^{i+1}(d + r + s + 1) - s.$$

In particular, $A$ has central stability degree $\leq \max(d + 2, s + 1, 2d + 2r + s + 2)$.

The category $\mathcal{VIC}^H(R)$ is a generalization of $\mathcal{VIC}(R)$ with the constraint that determinants of automorphisms must lie in $H$. Our proof of Corollary 3.29 uses a result of the second author [Pat17] which gives a homological condition on categories such as $\mathcal{VIC}(R)$ that ensures that polynomial modules over these categories have higher central stability. In Corollary 3.33, we prove an analogous result for symplectic group representations.

1.4. Outline. In Section 2, we prove that some relevant semisimplicial sets are highly connected. We use these connectivity results in Section 3 to prove that certain modules with finite polynomial degree exhibit higher central stability. In Section 4, we apply our results on polynomial degree to prove Theorem A, Theorem B, and Theorem C.

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2. High connectivity results

As is common in stability arguments, our proofs will involve establishing high connectivity for certain spaces with actions of our families of groups. We start out in Section 2.1 with a review of simplicial complexes and semisimplicial sets, and an overview of techniques to prove that their geometric realizations are highly connected. Then in Section 2.2 we review elementary properties of free modules and symplectic structures over PIDs. Finally, in Sections 2.3 and 2.4, we prove that the relevant spaces are highly connected.

2.1. Review of simplicial techniques. Recall that the data of a semisimplicial set is the same as the data of a simplicial set without degeneracy maps. See for example Randal-Williams–Wahl [RWW14, Section 2] for a precise definition of semisimplicial objects. We will also consider simplicial complexes in this paper. Simplicial complexes differ from semisimplicial sets in several ways. For example, each simplex of a semisimplicial set comes equipped with an order on its set of faces, while the faces of a simplicial complex are not ordered. Further, a collection of vertices can be the set of vertices of at most one simplex of a simplicial complex, but there are no such restrictions for semisimplicial sets. See Randal-Williams–Wahl [RWW14, Section 2.1] for a discussion of the differences between semisimplicial sets and simplicial complexes.
We say that a semisimplicial set or simplicial complex is $n$–connected if its geometric realization is $n$–connected. If $\sigma$ is a simplex of a simplicial complex $X_\circ$, we write $Lk^X(\sigma)$ to denote the link of $\sigma$ in $X$, or simply $Lk(\sigma)$ when the ambient complex is clear from context. We now recall the definition of weakly Cohen–Macaulay simplicial complexes.

**Definition 2.1.** A simplicial complex $X_\circ$ is called *weakly Cohen–Macaulay* (abbreviated wCM) of dimension $n$ if it satisfies the following two conditions.

- $X_\circ$ is $(n - 1)$–connected.
- If $\sigma$ is a $p$–simplex of $X_\circ$, then $Lk^X(\sigma)$ is $(n - 2 - p)$–connected.

**Definition 2.2.** If $X_\circ$ is a simplicial complex, let $X_\bullet = X_\circ^{ord}$ be the associated semisimplicial set formed by taking all simplices of $X_\circ$ with all choices of orderings on their vertices.

**Convention 2.3.** We adopt the following convention on subscript notation: we always write $X_\bullet$ for a semisimplicial set and $X_\circ$ for a simplicial complex. If we denote a semisimplicial set and a simplicial complex by the same letter, then they are related by $X_\circ^{ord} = X_\bullet$. By $X_p$ we will always mean the set of $p$–simplices of the semisimplicial set $X_\bullet$, not the set of $p$–simplices of the simplicial complex $X_\circ$, which is then $X_p/\mathcal{S}_{p+1}$.

The following proposition is well known. See for example Kupers–Miller [KM16, Lemma 3.16].

**Proposition 2.4.** If $X_\bullet = X_\circ^{ord}$ is $n$–connected, then so is $X_\circ$.

Proposition 2.4 has the following partial converse. See Randal-Williams–Wahl [RWW14, Proposition 2.14] for a proof.

**Theorem 2.5.** If $X_\circ$ is weakly Cohen–Macaulay of dimension $n$, then $X_\bullet = X_\circ^{ord}$ is $(n - 1)$–connected.

These results will allow us to pass between high connectivity results for semisimplicial sets and simplicial complexes. The following definition and theorem will be our main tool for proving simplicial complexes are highly connected.

**Definition 2.6.** A map of simplicial complexes $\pi : Y_\circ \to X_\circ$ is said to exhibit $Y_\circ$ as a *join complex* over $X_\circ$ if it satisfies all of the following:

- $\pi$ is surjective
- $\pi$ is simplexwise injective
- a collection of vertices $(y_0, \ldots, y_p)$ spans a simplex of $Y$ whenever there exists simplices $\theta_0, \ldots, \theta_p$ such that for all $i$, $y_i$ is a vertex of $\theta_i$ and the simplex $\pi(\theta_i)$ has vertices $\pi(y_0), \ldots, \pi(y_p)$.

This definition is illustrated in Figure 1.

The following result is due to Hatcher–Wahl [HW10].

**Theorem 2.7** (Hatcher–Wahl [HW10, Theorem 3.6]). Let $\pi : Y_\circ \to X_\circ$ be a map of simplicial complexes exhibiting $Y_\circ$ as a join complex over $X_\circ$. Assume $X_\circ$ is wCM of dimension $n$. Further assume that for all $p$–simplices $\tau$ of $Y_\circ$, the image of the link $\pi(Lk^Y(\tau))$ is wCM of dimension $(n - p - 2)$. Then $Y_\circ$ is $\left(\frac{n - 2}{2}\right)$–connected.
Definition 2.8. Let $X_\circ$ be a simplicial complex and $X_\bullet = X_\circ^{ord}$. Let $\tilde{\sigma} \in X_p$ and $\sigma \subseteq X_\circ$ the corresponding simplex. Then we define the link
\[
\text{Lk}_{X_\bullet} \tilde{\sigma} = (\text{Lk}_{X_\circ} \sigma)^{ord}
\]
as a sub-semisimplicial set of $X_\bullet$.

2.2. Algebraic preliminaries. In this subsection, we recall some basic facts and definitions concerning free modules over PIDs as well as symplectic structures.

Let $R$ be a ring. Throughout this paper we write $R[S]$ to denote the free $R$–module with basis $S$. Notably, this does not denote the polynomial algebra with variables $S$.

Given a submodule $W$ of a free module $V$, we say $W$ is splittable or has a complement if there exists a submodule $U$ with $V = W \oplus U$. Given a submodule $W$ of a free module $V$, let $\text{sat}(W)$ denote the intersection of all splittable submodules of $V$ which contain $W$, equivalently, $\text{sat}(W)$ is the preimage of the torsion submodule of $V/W$ under the quotient map $V \rightarrow V/W$. We call $\text{sat}(W)$ the saturation of $W$. We write $\text{rk}(W)$ for the rank of a free module. The following proposition collects some elementary facts concerning free modules over PIDs and their submodules; see for example Kaplansky [Kap54].

Proposition 2.9. Let $R$ be a PID and $A, B, C$ submodules of a finitely generated free $R$–module $V$.

i) If $A$ and $B$ have complements in $V$, then so does $A \cap B$.

ii) Let $B$ have a complement. Then $A$ has a complement containing $B$ if and only if there is a submodule $D$ with $V = A \oplus B \oplus D$.

iii) $\text{rk}(A) = \text{rk}(\text{sat}(A))$.

iv) $\left( (A \cap C) \oplus (B \cap C) \right) \subseteq (A \oplus B) \cap C$ but equality does not hold in general.

v) If $V = A \oplus B$ and $C \supseteq A$, then $C = A \oplus (B \cap C)$.

vi) If $V = A \oplus B$ and $C \subseteq A$, then $C = A \cap (B \oplus C)$.

vii) $A$ has a complement in $V$ if and only if $V/A$ is torsion free.

viii) If $A \subseteq B$ and $A$ has a complement in $V$, then $A$ has complement in $B$.

Recall that a symplectic form on an $R$–module $V$ is a perfect alternating bilinear pairing $\langle \ , \ \rangle : V \times V \rightarrow R$ such that $\langle v, v \rangle = 0$ for all $v \in V$. A submodule $W \subseteq V$ is called isotropic if $\langle w_1, w_2 \rangle = 0$ for all $w_1, w_2 \in W$. The following proposition collects some elementary facts concerning free symplectic modules over PIDs and their submodules.

Proposition 2.10. Let $R$ be a PID and let $A, B, C$ be submodules of a finitely generated free symplectic module $V$. 
i) The rank of \( V \) is even and there is a basis \( v_1, w_1, \ldots, v_n, w_n \) of \( V \) such that
\[
\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0, \quad \langle v_i, w_j \rangle = \delta_{ij} \quad \text{for all } i, j = 1, \ldots, n.
\]

ii) If \( A \) has a complement in \( V \), every maximal symplectic submodule \( U \) of \( A \) has the same rank and is splittable. In particular
\[
A = U \oplus (U^\perp \cap A),
\]
and \( U^\perp \cap A \) is isotropic.

2.3. Generalized partial basis complexes. In this subsection, we define several simplicial complexes and semisimplicial sets involving partial bases and complements. These connectivity results will imply that \( \text{VIC}(R) \)-modules satisfying a polynomial condition exhibit higher central stability. Let \( R \) be a PID and \( V \) be a finite rank \( R \)-module.

**Definition 2.11.** A *partial basis* of a free module \( V \) is a linearly independent set \( \{v_0, \ldots, v_p\} \subseteq V \) such that there is a free (possibly zero) submodule \( C \) with \( \text{span}(v_0, \ldots, v_p) \oplus C = V \). The set \( \{v_0, \ldots, v_p\} \) is also called *unimodular*, and the submodule \( C \) is called a *complement* for the partial basis. An *ordered partial basis* is a partial basis with a choice of bijection with a set of the form \( \{0, \ldots, p\} \).

**Definition 2.12.** For \( p \geq 0 \), let \( \text{PB}_p(V) \) be the set of ordered partial bases of size \( p + 1 \). For \( 0 \leq i \leq p \), there are maps \( d_i : \text{PB}_p(V) \to \text{PB}_{p-1}(V) \) given by forgetting the \( i \)th basis element. With these maps, the sets \( \text{PB}_p(V) \) assemble into a semisimplicial set \( \text{PB}_*(V) \). Let \( \text{PB}_*(V) \) to be the simplicial complex formed by quotienting by the action of \( \mathfrak{S}_{p+1} \) on \( \text{PB}_p(V) \).

The link of a simplex \( \sigma = (v_0, v_1, \ldots, v_p) \in \text{PB}_p(V) \) is the subcomplex of ordered partial bases \((u_0, u_1, \ldots, u_q)\) such that \( \{v_0, v_1, \ldots, v_p, u_0, u_1, \ldots, u_q\} \) is a partial basis of \( V \). Notably, this link only depends on the submodule \( W = \text{span}(v_0, \ldots, v_p) \). By abuse of notation, we will often denote \( \text{Lk}_*(\sigma) \) by \( \text{Lk}_*(W) \). By convention if \( W = 0 \), we let \( \text{Lk}_*(W) \) be the entire complex of ordered partial bases \( \text{PB}_* \). If \( U \subseteq V \) is a splittable submodule, then there is a canonical inclusion of \( \text{PB}_*(U) \subseteq \text{PB}_*(V) \).

**Definition 2.13.** Let \( U, W \subseteq V \) be splittable submodules. Define
\[
\text{PB}_*(U, W) = \text{PB}_*(U) \cap \text{Lk}_*(W) \subseteq \text{PB}_*(V)
\]
and \( \text{PB}_*(U, W) \) to be the simplicial complex formed by quotienting by the symmetric group actions.

Concretely, \( \text{PB}_*(U, W) \) is the sub-semisimplicial set of \( \text{PB}_*(U) \) consisting of ordered nonempty partial bases of \( U \) contained in a complement of \( W \). The complex \( \text{PB}_*(U, W) \) depends only on the submodule \( \text{sat}(U + W) \) and not on \( V \). We note that
\[
\text{PB}_*(U, W) = \text{Lk}_*^{PB(U)}(W) \quad \text{if} \ W \subseteq U,
\]
and in particular that
\[
\text{PB}_*(V, V) = \emptyset \quad \text{and} \quad \text{PB}_*(V, 0) = \text{PB}_*(V).
\]
More generally, whenever \( W \) is contained in any complement of \( U \), then \( \text{PB}(U, W) \cong \text{PB}(U) \).
Remark 2.14. For vector spaces,
\[ \text{PB}_* (U, W) = \text{PB}_* (U, W \cap U) = \text{Lk}^{\text{PB} (U)} (W \cap U) \quad (R \text{ a field}). \]

We caution, however, that this identification does not hold in general. For example, when \( R = \mathbb{Z} \) and \( V = \mathbb{Z}^3 \), consider the submodules
\[ U = \text{span} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad W = \text{span} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \]

Since the determinant of the matrix
\[ \begin{pmatrix} 1 & 1 & a \\ 1 & -1 & b \\ 0 & 0 & c \end{pmatrix} \]
is a multiple of 2 for any \( a, b, c \in \mathbb{Z} \), there is no basis for \( V \) that contains both a basis for \( U \) and a basis for \( W \), and \( \text{PB}_* (U, W) \) is empty. In contrast, the complex \( \text{PB}_* (U, W \cap U) = \text{PB}_* (U, 0) = \text{PB}_* (U) \) is nonempty.

We next define a variation of the semisimplicial set \( \text{PB}_* (V) \) consisting of ordered partial bases with distinguished choices of complements.

Definition 2.15. Let \( \text{PBC}_p (V) \) be the set of ordered partial bases \((v_0, \ldots, v_p)\) of \( V \) as well as a choice of complement \( C \) such that
\[ C \oplus \text{span}(v_0, \ldots, v_p) = V. \]

Let \( d_i : \text{PBC}_p (V) \to \text{PBC}_{p-1} (V) \) be given by the formula
\[ d_i (v_0, \ldots, v_p, C) = (v_0, \ldots, \hat{v}_i, \ldots, v_p, C \oplus R v_i). \]

Here the hat indicates omission. These sets assemble to form a semisimplicial set \( \text{PBC}_* (V) \). Let \( \text{PBC}_* (V) \) be the simplicial complex whose set of \( p \) simplices is given by the quotient of \( \text{PBC}_p (V) \) by the natural action of \( \mathcal{S}_{p+1} \).

In terms of vertices, we can describe \( \text{PBC}_* (V) \) as follows.

Proposition 2.16. Let \((v_0, C_0), \ldots, (v_p, C_p)\) be vertices of \( \text{PBC}_* (V) \). They span a \( p \)-simplex if and only if \( v_i \in C_j \) for all \( i \neq j \).

Proof. The vertices of a simplex \((v_0, \ldots, v_p, C)\) in \( \text{PBC}_p (V) \) are
\[ (v_j, C \oplus \text{span}(v_0, \ldots, \hat{v}_j, \ldots, v_p)). \]

This proves one direction.

Conversely, let \((v_0, C_0), \ldots, (v_p, C_p)\) in \( \text{PBC}_0 (V) \) such that \( v_i \in C_j \) for \( i \neq j \). We will prove that
\[ (v_0, \ldots, v_p, C) \in \text{PBC}_p (V), \quad \text{where} \quad C = \bigcap_{i=0}^{p} C_i. \]

To do so, we define
\[ D_j = \bigcap_{i=0}^{j} C_i, \]
and we will prove by induction that
\[ \text{span}(v_0, \ldots, v_j) \oplus D_j = V. \]
The base case is the statement that \( \text{span}(v_0) \oplus C_0 = V \). Since \( \text{span}(v_0, \ldots, v_{j-1}) \subseteq C_j \) by assumption, Proposition 2.9 Part v) and the inductive hypothesis imply that
\[ \text{span}(v_0, \ldots, v_{j-1}) \oplus (D_{j-1} \cap C_j) = C_j. \]
By taking the direct sum of both sides of this equation with \( \text{span}(v_j) \), we conclude the inductive step.
Applying this result when \( j = p \) yields the desired decomposition
\[ \text{span}(v_0, \ldots, v_p) \oplus C = \text{span}(v_0, \ldots, v_p) \oplus D_p = V. \]

We remark in particular that the \( j \)th vertex of a simplex \((v_0, \ldots, v_p, C)\) in \( \text{PBC}_\bullet(V) \) is
\[ (v_j, C_j) = (v_j, \text{span}(v_0, \ldots, \hat{v}_j, \ldots, v_p) \oplus C). \]

It will be convenient for us to realize the complexes \( \text{PBC}_\bullet(V) \) and their links as special cases of the following more general construction. In the following, we will often identify an ordered partial basis \((v_0, \ldots, v_p)\) of \( V \) with an \( R \)-linear monomorphism \( f: R^{p+1} \to V \).

**Definition 2.17.** Let \( U, W \subseteq V \) be splittable submodules. Define the sub-semisimplicial set \( \text{PBC}_\bullet(V, U, W) \subseteq \text{PBC}_\bullet(V) \) by
\[ \text{PBC}_p(V, U, W) = \{(f, C) \in \text{PBC}_p(V) | \text{im} f \subseteq U, W \subseteq C\}. \]
Define \( \text{PBC}_\circ(V, U, W) \) to be the simplicial complex formed by quotienting by the symmetric group actions.

In particular,
\[ \text{PBC}_\bullet(V) \cong \text{PBC}_\bullet(V, V, 0). \]
Given a simplex \( \sigma = (f, C) \in \text{PBC}_p(V) \), we can identify its link
\[ \text{Lk}_\bullet(\sigma) = \text{PBC}_\bullet(V, C, \text{im} f). \]
This link is isomorphic to \( \text{PBC}_\bullet(C) \), which we will see is a special case of Lemma 2.18 below. The embedding of \( \text{PBC}_\bullet(C) \) into \( \text{PBC}_\bullet(V) \) depends on the image of \( f \):
\[ \text{PBC}_p(C) \to \text{Lk}_\bullet(\sigma) \subseteq \text{PBC}_p(V) \]
\[ (c_0, \ldots, c_p, D) \mapsto (c_0, \ldots, c_p, \text{im} f \oplus D) \]
In general \( \text{PBC}_\bullet(V, U, W) \) is not isomorphic to \( \text{PBC}_\bullet(U) \), but the following lemma gives a more general picture.

**Lemma 2.18.** Let \( U, W \subseteq V \) be splittable and \( A \oplus B = V \) such that \( U \subseteq A \) and \( B \subseteq W \). Then
\[ \text{PBC}_\bullet(V, U, W) \xrightarrow{\cong} \text{PBC}_\bullet(A, U, W \cap A) \]
\[ (f, C) \mapsto (f, C \cap A) \]
is an isomorphism. In particular, if \( U \oplus W = V \), then \( \text{PBC}_\bullet(V, U, W) \cong \text{PBC}_\bullet(U) \).
Proof. Let \((f, C) \in \text{PBC}_\bullet(V, U, W)\). Proposition 2.9 Part v) implies
\[ \text{im } f \oplus (C \cap A) = A \]
and so we conclude that \((f, C \cap A) \in \text{PBC}_\bullet(A, U, W \cap A)\). If \((g, D) \in \text{PBC}_\bullet(A, U, W \cap A)\), then \((g, D \oplus B) \in \text{PBC}_\bullet(V, U, W)\). These two maps are inverses because
\[ (C \cap A) \oplus B = C \]
by Proposition 2.9 Part v), and by Proposition 2.9 Part vi),
\[ (D \oplus B) \cap A = D. \]
When \(U \oplus W = V\), taking \(A = U\) and \(B = W\) gives the special case
\[ \text{PBC}_\bullet(V, U, W) \cong \text{PBC}_\bullet(U, U, 0) = \text{PBC}_\bullet(U). \]

The main theorem of this subsection is Theorem 2.20. To state this theorem, we will use Bass’ stable range condition for rings.

Definition 2.19. Let \(s\) be a positive integer. A ring \(R\) has stable rank \(s\) if \(s\) is the smallest positive integer \(m\) for which the following Condition \((B_m)\) holds: whenever
\[ a_0 R + a_1 R + \cdots + a_m R = R, \quad a_i \in R, \]
there exist elements \(x_1, x_2, \ldots, x_m \in R\) such that
\[ (a_1 + a_0 x_1) R + \cdots + (a_m + a_0 x_m) R = R. \]

The condition was first formulated by Bass to characterize when matrices in \(\text{GL}_{m+1}(R)\) can be row-reduced to matrices in the image of \(\text{GL}_m(R)\). The indexing in Definition 2.19 is a modern standard, though it differs from Bass’ original convention; he called it the stable range condition \(SR_{s+1}\). Bass [Bas64, Section 4] states that semi-local rings have stable rank 1, and Dedekind domains have stable rank at most 2. So, for example, \(R\) has stable rank 1 if \(R\) is a field, a direct sum of fields, or the ring \(\mathbb{Z}/m\mathbb{Z}\) for \(m \in \mathbb{Z}\), and \(R\) has stable rank at most 2 when \(R\) is a PID, or when \(R\) is the ring of integers of a number field. The integers \(R = \mathbb{Z}\) have stable rank 2. More generally, a commutative ring of Krull dimension \(d\) has stable rank at most \((d + 1)\).

Since the next result concerns PIDs, the stable rank \(s\) must be equal to 1 or 2.

Theorem 2.20. Let \(R\) be a PID. Let \(s\) be the stable rank of \(R\). If \(U\) and \(W\) be splittable submodules of \(V\), then \(\text{PBC}_\bullet(V, U, W)\) is \(\left(\frac{\text{rk}U - \text{rk}W - s - 2}{2}\right)\)–connected.

Theorem 2.20 is a partial generalization of the following theorem, [RWW14, Lemma 5.9].

Theorem 2.21 (Randal-Williams–Wahl [RWW14, Lemma 5.9]). If \(R\) is a ring with stable rank \(s\), then \(\text{PBC}_\circ(V)\) is \(\left(\frac{\text{rk} V - s - 2}{2}\right)\)–connected.

Following the proof of [RWW14, Lemma 5.9], we will prove Theorem 2.20 by comparing \(\text{PBC}_\circ(V, U, W)\) with its image in \(\text{PB}_\circ(V)\). The following is due to van der Kallen [vdK80, Theorem 2.6 (i) and (ii)].
Theorem 2.22 (van der Kallen [vdK80, Theorem 2.6 (i)–(ii)]). Assume $R$ is a ring with stable rank $s$. Let $U$ and $W$ be splittable subspaces of $V$. Then $\text{PB}_\bullet(U, W)$ is $(\text{rk} U - \text{rk} W - 1 - s)$–connected. In particular, $\text{PB}_\bullet(U)$ is $(\text{rk} U - 1 - s)$–connected.

This gives the following corollary.

Corollary 2.23. The simplicial complex $\text{PB}_0(U, W)$ is wCM of dimension $\text{rk} U - \text{rk} W - s$. In particular, $\text{PB}_0(U)$ is wCM of dimension $\text{rk} U - s$.

Proof. By Theorem 2.22 and Proposition 2.4, the complex $\text{PB}_\bullet(U, W)$ is $(\text{rk} U - \text{rk} W - 1 - s)$–connected. The link of $\{v_0, \ldots, v_p\}$ in $\text{PB}_\bullet(U, W)$ is isomorphic to $\text{PB}_\bullet(U, W \oplus \text{span}(v_0, \ldots, v_p))$ and so the links are $(\text{rk} U - \text{rk} W - s - 2 - p)$–connected as required. □

To prove $\text{PBC}_\bullet(V, U, W)$ is highly connected, we will show that $\text{PBC}_0(V, U, W)$ is wCM. To do this, we need the following lemma.

Lemma 2.24. Let $U, W, X, Y, Z$ be splittable submodules of the free $R$–module $V$.

i) $\text{PBC}_\bullet(V, X, Y) \cap \text{PBC}_\bullet(V, Z, W) = \text{PBC}_\bullet(V, X \cap Z, \text{sat}(Y + W))$.

ii) Any simplex $\sigma = (f, C) \in \text{PBC}_\bullet(V, U, W)$ has link $\text{Lk}_\bullet(\sigma) = \text{PBC}_\bullet(C, U \cap C, W)$.

Proof. i) Both sides of the equation describe the following semisimplicial set:

$$\{(f, C) \in \text{PBC}_\bullet(V) \mid \text{im} f \subseteq X, \text{im} f \subseteq Z, Y \subseteq C, W \subseteq C\}$$

$$= \{(f, C) \in \text{PBC}_\bullet(V) \mid \text{im} f \subseteq X \cap Z, \text{sat}(Y + W) \subseteq C\}$$

ii) Every simplex in $\text{PBC}_\bullet(V, U, W)$ contains every simplex in $\text{PBC}_\bullet(V)$ spanned by vertices in $\text{PBC}_\bullet(V, U, W)$ (we say that the inclusion $\text{PBC}_\bullet(V, U, W) \hookrightarrow \text{PBC}_\bullet(V)$ is full). Hence,

$$\text{Lk}^{\text{PBC}_\bullet(V, U, W)}(\sigma) = \text{Lk}^{\text{PBC}_\bullet(V)}(\sigma) \cap \text{PBC}_\bullet(V, U, W)$$

$$= \text{PBC}_\bullet(V, C, \text{im} f) \cap \text{PBC}_\bullet(V, U, W) = \text{PBC}_\bullet(V, U \cap C, W \oplus \text{im} f).$$

The last step uses Part i) the observation that, since $W$ is contained in the complement $C$ of $\text{im} f$,

$$\text{sat}(W + \text{im} f) = W \oplus \text{im} f.$$ 

By applying Lemma 2.18 with $A = C$ and $B = \text{im} f$, we find

$$\text{PBC}_\bullet(V, U \cap C, W \oplus \text{im} f) = \text{PBC}_\bullet(C, U \cap C, (W \oplus \text{im} f) \cap C).$$

Then Proposition 2.9 Part vi) implies that

$$(W \oplus \text{im} f) \cap C = W$$

and the result follows. □
Proposition 2.25. Let $U,W$ be splittable submodules of $V$. Then the map
\[ \theta: \text{PBC}_\bullet(V,U,W) \to \text{PB}_\bullet(U,W) \]
\[ (f,C) \mapsto f \]
is surjective.

Proof. Let $f: R^{p+1} \to U$ be a $p$-simplex in $\text{PB}_p(U,W)$. Because $f \in \text{Lk}_p(W)$, there is a submodule of $D \subseteq V$ such that
\[ D \oplus \text{im } f \oplus W = V. \]
Then the simplex $(f, D \oplus W)$ in $\text{PBC}_p(V,U,W)$ is a preimage of $f$ under $\theta$, and $\theta$ is surjective. \hfill \Box

Proposition 2.26. The map $\theta: \text{PBC}_o(V,U,W) \to \text{PB}_o(U,W)$ exhibits $\text{PBC}_o(V,U,W)$ as a join complex over $\text{PB}_o(U,W)$.

Proof. Proposition 2.25 establishes surjectivity, and simplex-wise injectivity is clear. It remains to verify the third condition of Definition 2.6. Let $(v_0,C_0),\ldots,(v_p,C_p)$ be vertices of $\text{PBC}_o(V,U,W)$ and let $(\beta_0,D_0),\ldots,(\beta_p,D_p)$ be simplices of $\text{PBC}_o(V,U,W)$ such that for each $i = 0,\ldots,p$, the vertex $(v_i,C_i)$ is a vertex of $(\beta_i,D_i)$, and $\theta(\beta_i,D_i)$ has vertices $\theta(v_0,C_0),\ldots,\theta(v_p,C_p)$. We wish to show that the vertices $(v_0,C_0),\ldots,(v_p,C_p)$ span a simplex in $\text{PBC}_o(V,U,W)$.

By Proposition 2.16, it suffices to check that $v_i \in C_j$ for all $i \neq j$. By assumption, $\theta(v_i,C_i) = v_i$ is a vertex of $\theta(\beta_j,D_j) = \beta_j$, so the element $v_i$ must be contained in the partial basis $\beta_j$. But $(v_j,C_j)$ is a vertex of the simplex $(\beta_j,D_j)$, so a second application of Proposition 2.16 implies that $v_i \in C_j$ as required. \hfill \Box

We now prove that $\text{PBC}_o(V,U,W)$ is highly connected.

Proposition 2.27. Let $R$ be a PID with stable rank $s$. Then the simplicial complex $\text{PBC}_o(V,U,W)$ is \( \left(\frac{\text{rk } U - \text{rk } W - s - 2}{2}\right) \)-connected.

Proof. The proposition will follow by applying Theorem 2.7 to the map $\theta: \text{PBC}_o(U,W) \to \text{PB}_o(U,W)$. Given the results of Proposition 2.26 and Corollary 2.23, it only remains to check that the images of links of $p$-simplices are wCM of dimension $(\text{rk } U - \text{rk } W - s - p - 2)$.

Let $(\beta,C)$ be a $p$-simplex of $\text{PBC}_o(V,U,W)$. Lemma 2.24 Part ii) implies that
\[ \text{Lk}_o(\beta,C) = \text{PBC}_o(C,U \cap C,W). \]
By Proposition 2.25,
\[ \theta(\text{Lk}_o(\beta,C)) = \text{PB}_o(U \cap C,W) \subseteq \text{PB}_o(U,W). \]
Since $\text{rk } (U \cap C) \geq \text{rk } U - p - 1$, Corollary 2.23 implies that $\theta(\text{Lk}_o(\beta,C))$ is wCM of dimension $\text{rk } U - \text{rk } W - (p + 1) - s$, in particular wCM of dimension $(\text{rk } U - \text{rk } W - s - p - 2)$. \hfill \Box

Combining Lemma 2.24 and Proposition 2.27 gives the following,

Corollary 2.28. If $R$ is a PID of stable rank $s$, the complex $\text{PBC}_o(V,U,W)$ is wCM of dimension \( \left(\frac{\text{rk } U - \text{rk } W - s}{2}\right) \).

Corollary 2.28 and Theorem 2.5 together establish Theorem 2.20.
2.4. Symplectic partial bases complexes. In this subsection, we consider the symplectic group analogues of the complexes from the previous section. The connectivity results that we establish will later be used to show that $Sl(R)$–modules satisfying a polynomial condition exhibit higher central stability.

**Definition 2.29.** A *symplectic partial basis* of a free symplectic module $V$ is a set of pairs

$$\{(v_0, w_0), \ldots, (v_p, w_p)\} \subseteq V \times V$$

such that $\{v_0, w_0, \ldots, v_p, w_p\}$ is a partial basis of $V$ with

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0 \quad \text{and} \quad \langle v_i, w_j \rangle = \delta_{ij}.$$  

An *ordered symplectic partial basis* is a symplectic partial basis with a choice of bijection between $\{(v_0, w_0), \ldots, (v_p, w_p)\}$ and $\{0, \ldots, p\}$.

**Definition 2.30.** For $p \geq 0$, let $V$ be a free symplectic $R$–module. Let $\text{SPB}_p(V)$ be the set of ordered symplectic partial bases of size $p + 1$. There are maps $d_i : \text{SPB}_p(V) \rightarrow \text{SPB}_{p-1}(V)$ given by forgetting the $i$th pair. With these maps, the sets assemble into a semisimplicial set $\text{SPB}_\bullet(V)$.

Mirzaii–van der Kallen [MvdK02, Definition 6.3] defined a broad concept of the *unitary stable rank* of a ring $R$, which specializes to apply to the study of symplectic, orthogonal, and unitary groups over $R$. The following definition is the version of their definition relevant to symplectic forms, specifically, we assume $R$ to be commutative and (in the notation of the paper [MvdK02]) we define unitary stable rank associated to the identity involution on $R$, $\epsilon = -1$, $\Lambda = R$, and the symplectic form

$$R^{2m} \times R^{2m} \rightarrow R,$$

$$\langle x, y \rangle = \sum_{i=1}^{m} (x_{2i-1}y_{2i} - x_{2i}y_{2i-1}).$$

**Definition 2.31.** A commutative ring $R$ has *(symplectic)* unitary stable rank $s$ if $s$ is the smallest value of $m$ for which $R$ satisfies Condition (Bm) of Definition 2.19, and Condition $(T_{m+1})$. To state Condition $(T_m)$ we define the group of elementary symplectic matrices $ES_{2m}(R)$ as follows. Let $e_{i,j}(r)$ denote the $(2m \times 2m)$ matrix with $r \in R$ in position $(i, j)$ and 0 elsewhere. Then $ES_{2m}(R)$ is the group generated by the matrices of the form:

$$I_{2m} + e_{2i-1,2i}(r), \ I_{2m} + e_{2i,2i-1}(r), \ I_{2m} + e_{2i-1,2j-1}(r) + e_{2j,2i}(-r), \ I_{2m} + e_{2i-1,2j}(r) + e_{2j-1,2i}(r), \ I_{2m} + e_{2i,2j-1}(r) + e_{2j,2i-1}(-r), \ 1 \leq i, j \leq m, \ i \neq j.$$  

With this notation, Condition $(T_m)$ is the statement that the group $ES_{2m}(R)$ acts transitively on the set of unimodular elements $v$ in $R^{2m}$.

Results of Mirzaii–van der Kallen [MvdK02, Remark 6.4] and Magurn–van der Kallen–Vaserstein [MVdKV88, Theorems 1.3 and 2.4] together imply that commutative semi-local rings (including fields) have symplectic unitary rank 1, and PIDs have symplectic unitary rank at most 2. Most relevant to this paper are the integers $R = \mathbb{Z}$, which have symplectic unitary stable rank 2.

Mirzaii and van der Kallen proved that $\text{SPB}_\bullet(V)$ is highly connected.
Theorem 2.32 (Mirzaii–van der Kallen [MvdK02, Thm 7.4]). If $R$ is a PID with unitary stable rank $s$ and $V$ is a free symplectic $R$–module of rank $2n$, then $\text{SPB}_\bullet(V)$ is $\left(\frac{n - s - 3}{2}\right)$–connected.

Definition 2.33. Let $\text{MPB}_p(V)$ be the set of $(p + 1)$–tuples $((v_0, w_0), \ldots, (v_p, w_p))$ such that

$$\{v_0, \ldots, v_p\} \cup \{w_i \mid w_i \neq 0\}$$

is a partial basis of $V$ with

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0 \quad \text{and} \quad \langle v_i, w_j \rangle = \delta_{ij} \quad \text{for } w_j \neq 0.$$

Thus here some of the $w_i$ may be zero. These sets form a semisimplicial set $\text{MPB}_\bullet(V)$, which we call the complex of mixed partial bases.

Note that there is a natural injection

$$\text{SPB}_\bullet(V) \hookrightarrow \text{MPB}_\bullet(V).$$

As with the semisimplicial set $\text{PB}_\bullet(V)$, the link of a $p$–simplex $\tau = ((v_0, w_0), \ldots, (v_p, w_p)) \in \text{SPB}_p(V)$ only depends on the symplectic submodule $W$ of $V$ generated by the partial basis $\tau$. Therefore we can speak of the link of the symplectic submodule. Specifically,

$$\text{Lk}^{\text{SPB}(V)}(W) \cong \text{SPB}_\bullet(W^\perp),$$

where

$$W^\perp = \{v \in V \mid \langle v, v_i \rangle = \langle v, w_i \rangle = 0 \text{ for all } i = 0, \ldots, p\}$$

is the symplectic complement $W$. The link of a simplex in $\text{MPB}_\bullet(V)$ also only depends on the submodule generated by its vectors, though this submodule may not be symplectic, and $\text{Lk}^{\text{MPB}(V)}(W)$ is well-defined for a general splittable submodule $W$ of $V$.

Observe that

$$\text{SPB}_p(V) \cap \text{Lk}^{\text{MPB}(V)}(W) = \{(v_0, w_0), \ldots, (v_p, w_p) \in \text{SPB}_p(V) \mid W \subseteq \text{span}(v_0, w_0, \ldots, v_p, w_p)^\perp\}.$$

Therefore

$$\text{Lk}^{\text{SPB}(V)}(W) = \text{SPB}_\bullet(V) \cap \text{Lk}^{\text{MPB}(V)}(W)$$

and

$$\text{SPB}_\bullet(V) \cap \text{Lk}^{\text{MPB}(V)}(W_1) \cap \text{Lk}^{\text{MPB}(V)}(W_2) = \text{SPB}_\bullet(V) \cap \text{Lk}^{\text{MPB}(V)}(\text{sat}(W_1 + W_2)).$$

Theorem 2.34. Let $W$ be a submodule of $V$, and let $U$ be a maximal symplectic submodule of $W$. Then

$$\text{SPB}_\bullet(V) \cap \text{Lk}^{\text{MPB}(V)}(W)$$

is $\left(\frac{\text{rk } V/2 + \text{rk } U/2 - \text{rk } W - s - 3}{2}\right)$–connected.

Proof. Let $U^\perp$ be the symplectic complement of $U$ in $V$ and $U' = U^\perp \cap W$. By Proposition 2.9 Part v), $U \oplus U' = W$, and $U'$ is isotropic. Observe that $U^\perp = W^\perp$ and $U' = W^\perp \cap W$ are independent of the choice of maximal symplectic subspace $U$. Mirzaii–van der Kallen [MvdK02, proof of Theorem 7.4] calculated that

$$\text{SPB}_\bullet(V) \cap \text{Lk}^{\text{MPB}(V)}(W) = \text{SPB}_\bullet(U^\perp) \cap \text{Lk}^{\text{MPB}(U^\perp)}(U').$$
is \( \left( \frac{\text{rk}(U^\perp)/2 - \text{rk}(U') - s - 3}{2} \right) \)-connected. The equality
\[
\left( \frac{\text{rk}(U^\perp)/2 - \text{rk}(U') - s - 3}{2} \right) = \left( \frac{\text{rk}(V)/2 + \text{rk}(U)/2 - \text{rk}(W) - s - 3}{2} \right)
\]
proves the assertion. \(\square\)

3. Modules over stability categories

In Section 3.1, we recall the definition of the categories \( \text{VIC}^H(R) \), \( \text{SI}(R) \), and \( \text{FI} \). Our approach to these topics uses the formalism of stability categories developed by the second author [Pat17]. We review the notions of central stability homology, presentation degree, and polynomial degree for modules over these categories in Sections 3.2, 3.3, and 3.4. The main result of Section 3 is that modules over \( \text{VIC}^H(R) \) and \( \text{SI}(R) \) with finite polynomial degree exhibit higher central stability, and we prove this result in Section 3.5. In Section 3.6, we describe a spectral sequence introduced by Putman–Sam [PS14], which we use in Section 4 to prove our representation stability results.

3.1. Preliminaries. In this subsection, we define the category \( \text{FI} \) of Church–Ellenberg–Farb [CEF15] and the categories \( \text{VIC}^H(R) \) and \( \text{SI}(R) \) of Putman–Sam [PS14, Section 1.2]. We will view these constructions as stability categories, as defined by the second author [Pat17]. Stability categories are homogeneous categories in the sense of Randal-Williams–Wahl [RWW14, Definition 1.3] and weakly complemented categories in the sense of Putman–Sam [PS14, Section 1.3]. We will state their definition using a related concept, stability groupoids.

Definition 3.1. Let \( (G, \oplus, 0) \) be a monoidal groupoid whose monoid of objects is the natural numbers \( \mathbb{N}_0 \). The automorphism group of the object \( n \in \mathbb{N}_0 \) is denoted \( G_n = \text{Aut}^G(n) \). Then \( G \) is called a stability groupoid if it satisfies the following properties.

i) The monoidal structure
\[
\oplus : G_m \times G_n \rightarrow G_{m+n}
\]
is injective for all \( m, n \in \mathbb{N}_0 \).

ii) The group \( G_0 \) is trivial.

iii) \( (G_{l+m} \times 1) \cap (1 \times G_{m+n}) = 1 \times G_m \times 1 \subseteq G_{l+m+n} \) for all \( l, m, n \in \mathbb{N}_0 \).

Definition 3.2. A homomorphism \( G \rightarrow \mathcal{H} \) of stability groupoids is a monoidal functor sending 1 to 1.

Stability categories are defined by the second author in [Pat17, Definition 3.5]. The construction of them was also used by Randal-Williams–Wahl [RWW14, Section 1.1] and is originally due to Quillen. We will only care about the following property, which can be found in [RWW14, Remark 1.4].

Proposition 3.3. Given a braided stability groupoid \( G \), there is a monoidal category \( U^G \mathcal{G} \) on the same objects, such that
\[
\text{Hom}^{U^G \mathcal{G}}(m, n) \cong G_n/G_{n-m}.
\]
as a \( G_n \)-set. The category \( U^G \mathcal{G} \) is called the stability category of \( G \).

Proposition 3.4 (Randal-Williams–Wahl [RWW14, Proposition 2.6]). If a stability groupoid \( G \) is symmetric monoidal, then so is \( U^G \mathcal{G} \).
Example 3.5. The following are examples of stability groupoids $\mathcal{G}$ and associated categories equivalent to $U\mathcal{G}$.

i) **Symmetric groups:** Let $\mathcal{S}$ be the stability groupoid of symmetric groups. Then $U\mathcal{S}$ is equivalent to the category $\text{FI}$ of finite sets and injections.

ii) **General linear groups:** Let $R$ be a commutative ring. Let $\text{GL}(R)$ be the stability groupoid of the general linear groups over $R$. Then $U\text{GL}(R)$ is equivalent to the category $\text{VIC}(R)$ whose objects are free $R$–modules of finite rank and whose morphisms are given by a monomorphism together with a choice of direct complement of the image. Concretely,

$$\text{Hom}_{\text{VIC}(R)}(V,W) = \left\{ (f,C) \mid f : V \hookrightarrow W \text{ an injective } R\text{-linear map, } C \subseteq W \text{ a free submodule with } \text{im } f \oplus C = W \right\}.$$  

Note that, since $R$ is assumed commutative, $C$ must have rank $\text{rk} W - \text{rk} V$. The composition law is defined by

$$(f,C) \circ (g,D) = (f \circ g, C \oplus f(D)).$$

iii) **General linear groups with restricted determinant:** Let $R$ be a commutative ring and $H$ a subgroup of the group of units $R^\times$. Let $\text{GL}^H(R)$ denote the stability groupoid of the subgroups

$$\text{GL}^H_n(R) = \{ A \in \text{GL}_n(R) \mid \det A \in H \}.$$  

Then $U\text{GL}^H(R)$ is equivalent to the category $\text{VIC}^H(R)$, defined as follows. Its objects are finite-rank free $R$–modules $V$ such that nonzero objects are assigned an $H$–orientation, that is, a generator of $\bigwedge^{\text{rk}(V)} V \cong R$ considered up to multiplication by $H$. If $\text{rk}(V) = \text{rk}(W)$, then the morphisms $\text{Hom}_{\text{VIC}^H(R)}(V,W)$ are linear isomorphisms that respect the chosen $H$–orientations. In particular, linear maps in $\text{Hom}_{\text{VIC}^H(R)}(V,V)$ must have determinants in $H$. For $\text{rk}(V) \neq \text{rk}(W)$, the endomorphisms $\text{Hom}_{\text{VIC}^H(R)}(V,W) \cong \text{Hom}_{\text{VIC}(R)}(V,W)$ are again injective complemented linear maps $(f,C)$, and we assign to $C$ the (unique) $H$–orientation that makes the $H$–orientations on $(\text{im } f \oplus C)$ and $W$ agree. See Putman–Sam [PS14, Section 1.2].

iv) **Symplectic groups:** Let $R$ be a commutative ring. Let $\text{Sp}(R)$ be the stability groupoid of the symplectic groups over $R$. Then $U\text{Sp}(R)$ is equivalent to the category $\text{SI}(R)$ of free finite-rank symplectic $R$–modules and isometric embeddings. Details are given in Putman–Sam [PS14, Section 1.2].

v) **Automorphisms of free groups:** Let $\text{Aut}F$ be the stability groupoid of the automorphism groups of free groups of finite rank. Then $U\text{Aut}F$ is equivalent to the category of finite-rank free groups and monomorphisms together with a choice of free complement, that is,

$$\text{Hom}(F,G) = \left\{ (f,C) \mid f : F \hookrightarrow G \text{ an injective group homomorphism, } C \subseteq G \text{ a free subgroup with } \text{im } f * C = G \right\}.$$  

vi) **Mapping class groups of compact oriented surfaces with one boundary component:** Let $\text{Mod}$ be the stability groupoid of mapping class groups of compact oriented surfaces with one boundary component. Its monoidal structure is induced by boundary connect sum. See Randal-Williams–Wahl [RWW14, Section 5.6] for more details.

We note that $\mathcal{S}$, $\text{Aut}F$, and $\text{Sp}(R)$ are symmetric monoidal. When $H$ contains $-1$, the groupoid $\text{GL}^H(R)$ is also symmetric monoidal. The groupoid $\text{Mod}$ is only braided monoidal.
**Definition 3.6.** We call functors from a category $\mathcal{C}$ to the category of abelian groups $\mathcal{C}$–modules and denote the category of $\mathcal{C}$–modules by Mod$_{\mathcal{C}}$. If $\mathcal{C}$ is a stability category or stability groupoid and $A$ is a $\mathcal{C}$–module, we let $A_n$ denote the value of $A$ on the object $n \in \mathbb{N}_0$.

If a category $\mathcal{C}$ is equivalent to the category $U\mathcal{G}$, then the categories of $\mathcal{C}$–modules and $U\mathcal{G}$–modules are equivalent. We therefore use the terms of $U\mathcal{G}$–module and FI–module interchangeably, and similarly for other items in Example 3.5.

If $\mathcal{G}$ is a stability groupoid, then the data of $\mathcal{G}$–module is precisely the data of a $\mathbb{Z}[G_n]$–module for every $n$. For $m$ fixed, we will often view a $\mathbb{Z}[G_m]$–module $W$ as a $\mathcal{G}$–module by putting $W$ in degree $m$ and the module 0 in all other degrees. We now define free $U\mathcal{G}$–modules.

**Definition 3.7.** Let $\mathcal{G}$ be a stability groupoid and let $M : \text{Mod}_{\mathcal{G}} \to \text{Mod}_{U\mathcal{G}}$ be the left adjoint to the forgetful functor Mod$_{U\mathcal{G}} \to \text{Mod}_{\mathcal{G}}$. We say that $M(W)$ is the free $U\mathcal{G}$–module on $W$. Concretely, given a $\mathbb{Z}[G_m]$–module $W$, the $\mathbb{Z}[G_n]$–module $M(W)_n$ is given by the formula

$$M(W)_n \cong \begin{cases} 0, & n < m \\ \mathbb{Z}[G_n/G_{n-m}] \otimes_{\mathbb{Z}[G_m]} W, & n \geq m. \end{cases}$$

For a general $\mathcal{G}$–module $W$,

$$M(W) = \bigoplus_{n=0}^{\infty} M(W_n).$$

We abbreviate $M(\mathbb{Z}[G_m])$ by $M(m)$.

### Section 3.2. Central stability homology and resolutions

We begin by defining generation and presentation degree and discuss how these concepts relate to central stability degree. We then review central stability homology and how it relates to the degrees of higher syzygies of $U\mathcal{G}$–modules.

**Definition 3.8.** Let $\mathcal{G}$ be a stability groupoid. We say a $\mathcal{G}$–module $W$ has degree $\leq d$ if $W_n \cong 0$ for $n > d$. A $U\mathcal{G}$–module $A$ is generated in degrees $\leq d$ if there is a short exact sequence of $U\mathcal{G}$–modules

$$M(W^0) \to A \to 0$$

with $W^0$ of degree $\leq d$. A $U\mathcal{G}$–module $A$ is related in degrees $\leq d$ if there is a short exact sequence of $U\mathcal{G}$–modules

$$M(W^1) \to M(W^0) \to A \to 0$$

with $W^1$ of degree $\leq d$. We say $A$ is presented in degrees $\leq d$ if it is generated and related in degrees $\leq d$.

For all categories considered in this paper, central stability is equivalent to presentability in finite degree. The statement concerning FI–modules is due to Church–Ellenberg [CE15] and the statement involving other categories is due to the second author [Pat17].

**Theorem 3.9.** A module $A$ over FI, VIC$^H(R)$ for $R$ of finite stable rank, or SI($R$) for $R$ of finite (symplectic) unitary stable rank is presented in finite degrees if and only if it has finite central stability degree. Specifically, we have the following bounds.

- **FI** (Church–Ellenberg [CE15, Proposition 4.2]): For $d \geq 0$, an FI–module $A$ is presented in degrees $\leq d$ if and only if $A$ has central stability degree $\leq d$. 
We define the theorem 3.9 with a proof that the vanishing of central stability homology controls the higher syzygies. The central stability homology controls the higher syzygies.

This chain complex is the tail of the central stability chain complex. Thus the central stability degree alone gives a bound on presentation degree, though this bound may be improved with the data of both central stability degree and the degree of generation.

Let $A$ be a $U_G$–module, central stability can be rephrased as acyclicity of the chain complex

$$\text{Ind}_{G_n}^{G_{n-2}} A_{n-2} \longrightarrow \text{Ind}_{G_n}^{G_{n-1}} A_{n-1} \longrightarrow A_n \longrightarrow 0.$$ 

This chain complex is the tail of the central stability chain complex of $A$, which we describe in Definition 3.10. We call its homology central stability homology. The second author generalized Theorem 3.9 with a proof that the vanishing of central stability homology controls the higher syzygies of $A$; see Theorem 3.15 quoting the second author [Pat17, Theorem 5.7].

**Definition 3.10.** Let $A$ be a $U_G$–module, and let $\phi_n : A_n \rightarrow A_{n+1}$ denote the map induced by the morphism

$$n \oplus 0 \rightarrow n \oplus 1.$$ 

We define $\tilde{C}_p^G(A)$ to be the semisimplicial $U_G$–module with $p$–simplices given by

$$\tilde{C}_p^G(A)_n = \mathbb{Z}[G_n] \otimes \mathbb{Z}[G_{n-(p+1)}] A_{n-(p+1)}$$

and the $i$th face map by

$$d_i : \mathbb{Z}[G_n] \otimes \mathbb{Z}[G_{n-(p+1)}] A_{n-(p+1)} \longrightarrow \mathbb{Z}[G_n] \otimes \mathbb{Z}[G_{n-p}] A_{n-p}$$

$$g \otimes a \longrightarrow g(id_{n-p} \oplus h_i) \otimes \phi_{n-p-1}(a)$$

where the coset $h_i G_1$ corresponds to the morphism

$$i \oplus 0 \oplus (p-i) \longrightarrow i \oplus 1 \oplus (p-i) \text{ in } \text{Hom}^{U_G}(p, p+1) \cong G_{p+1}/G_{(p+1)-p}.$$ 

Let $\tilde{C}_p^G(A)$ denote the chain complex associated to $\tilde{C}_p^G(A)$ and let $\tilde{H}_p^G(A)$ denote the homology of this chain complex.

We will drop the superscript $G$ when the groupoid is clear from context.

**Definition 3.11.** We say that a $U_G$–module $A$ has higher central stability if $\tilde{H}_i(A)_n \cong 0$ for $n$ sufficiently compared to $i$.

The central stability complex has appeared in the literature in varying degrees of generality. In the FI case, it is closely related to a complex introduced in [Put15] whose homology is called FI-homology by Church-Elkberg [CE15]. FI–homology has appeared in work of Church, Elkberg, Farb, and Nagpal [CEF15, CEFN14, CE15] and Gan and Li [GL15a, GL15b, Gan16, Li16]. For the category FI, the
complex $\tilde{C}_s(A)$ itself was denoted by $B_{s+1}(A)$ by Church–Ellenberg–Farb–Nagpal [CEFN14, Definition 2.16], by $\tilde{C}^F_{s+1}$ by Church–Ellenberg [CE15, Section 5.1], and by $\text{Inj}_s(A)$ by the first and third author [MW16, Section 2.2]. Putman–Sum [PS14, Section 3] defined the complex for modules $A$ over general “cyclically generated” complemented categories and used the notation $\Sigma_{s+1}(A)$. In this paper, we adopt notation used by the second author [Pat17].

When $A = M(0)$, the central stability homology is the reduced homology of a semisimplicial set that has been previously studied for many stability categories, often in the context of homological stability.

**Definition 3.12.** Let $(K_s\mathcal{G})_n$ be the semisimplicial set with set of $p$–simplices given by

$$(K_p\mathcal{G})_n = \text{Hom}^U\mathcal{G}(p + 1, n) \cong G_n/G_{n-(p+1)}$$

and the $i$th face map is induced by

$$i \oplus 0 \oplus (p-i) \longrightarrow i \oplus 1 \oplus (p-i).$$

By definition, the reduced homology groups $\tilde{H}_i([K_s\mathcal{G}]_n)$ agree with the central stability homology of $M(0)$. The semisimplicial set $(K_s\text{GL}_n(R))$ is isomorphic to $\text{PBC}_s(R^n)$, $(K_s\text{Sp}_n(R))$ is isomorphic to $\text{SPB}_s(R^n)$, and $(K_s\mathcal{G})_n$ is the complex of injective words introduced by Farmer [Far79]. Randal-Williams–Wahl [RWW14] proved that high connectivity of $(K_s\mathcal{G})_n$ implies homological stability for the groups $G_n$.

We will see in Theorem 3.15 that an analogue of Theorem 3.9 holds for any stability category satisfying the following condition.

**Definition 3.13.** Let $a, k \in \mathbb{N}_0$. We define the following condition for a stability category $U\mathcal{G}$.

$\bf{H3}(k,a)$: $\tilde{H}_i(M(0))_n = 0$ for all $i \geq -1$ and all $n > k \cdot i + a$.

In the following proposition, we compile information about this condition for the stability categories appearing in Example 3.5.

**Proposition 3.14.** $U\mathcal{G}$ satisfies $\bf{H3}(1, 1)$. $U\text{AutF}$ and $U\text{Mod}$ satisfy $\bf{H3}(2, 2)$. If $R$ is a ring with stable range $s$, then $U\text{GL}(R)$ and $U\text{GL}^R(R)$ satisfy $\bf{H3}(2, s + 1)$. If $R$ is a ring with unitary stable rank $s$, then $U\text{Sp}(R)$ satisfies $\bf{H3}(2, s + 2)$.

**Proof.** This due to Farmer [Far79] for $\mathcal{G} = \mathcal{G}$, due to Hatcher–Vogtmann for $\mathcal{G} = \text{Mod}$ [HV15], due to Randal-Williams–Wahl [RWW14, Lemma 5.9] in the case $\mathcal{G} = \text{GL}(R)$ (also see Charney [Cha84, Theorem 3.5]), due to Randal-Williams–Wahl [RWW14, Proposition 5.3] building heavily upon the work of Hatcher–Vogtmann [HV98, Proposition 6.4] in the case $\mathcal{G} = \text{AutF}$, and due to Mirzaii–van der Kallen [MvdK02, Thm 7.4] in the case $\mathcal{G} = \text{Sp}(R)$.

We now consider the case $\mathcal{G} = \text{GL}^H(R)$. Because

$$\text{Hom}^U\text{GL}^H(R)(m,n) = \text{Hom}^U\text{GL}(R)(m,n) \quad \text{for} \quad m < n,$$

it follows that

$$\tilde{C}^\text{GL}^H(R)(M(0))_n = \tilde{C}^\text{GL}(R)(M(0))_n \quad \text{for} \quad p \leq n - 2.$$
induces an isomorphism on homology groups for $* \leq (n - 3)$. Thus for $i \geq 0$,
\[
\tilde{H}_i^{GL^H(R)}(M(0))_n = \tilde{H}_i^{GL(R)}(M(0))_n = 0 \quad \text{when } n > 2i + s + 1
\]
since $U \text{GL}(R)$ satisfies $H_3(2, s + 1)$ and necessarily $(n - 3) \geq i$ in this range. Moreover, by inspection
\[
\tilde{H}_{i-1}^{GL^H(R)}(M(0))_n = 0 \quad \text{for } n > 0
\]
and we conclude $H_3(2, s + 1)$ for $U \text{GL}^H$. \hfill \Box

The following theorem generalizes Theorem 3.9 to general stability categories and higher central stability homology groups.

**Theorem 3.15** (Patzt [Pat17, Theorem 5.7]). Assume $H_3(k, a)$. Let $A$ be a $U \mathcal{G}$–module and \{dₙ\}ₙ∈ℕ₀ a sequence of integers with $d_{i+1} - d_i \geq \max(k, a)$, then the following statements are equivalent.

i) There is a resolution
\[
\cdots \rightarrow M(W^1) \rightarrow M(W^0) \rightarrow A \rightarrow 0
\]
with $W^i$ a $\mathcal{G}$–module of degree $\leq d_i$.

ii) The homology groups
\[
\tilde{H}_i(A)_n = 0
\]
for all $i \geq -1$ and all $n > d_{i+1}$.

An ingredient in the proof of Theorem 3.15 is the following propostion. In the case $\mathcal{G} = \mathfrak{S}$ the result is [MW16, Corollary 2.27] by the first and third author.

**Proposition 3.16** (Patzt [Pat17, Corollary 5.11]). Assume $H_3(k, a)$ and let $W$ be a $\mathcal{G}$–module of degree $\leq m$. Then $\tilde{H}_i(M(W))_n = 0$ for all $n > k \cdot i + m + a$.

### 3.3. Polynomial degree.

To prove that the second homology of Torelli subgroups are centrally stable, will first prove that the first homology group of these Torelli groups exhibit higher central stability.

The behavior of central stability homology groups is not yet well understood in general. For example, given a VIC($\mathbb{Z}$) or SI($\mathbb{Z}$)–module $A$ with finite central stability degree, it is currently unknown whether $A$ has higher central stability. Indeed, this result would imply central stability for the homology of the corresponding Torelli groups in every homological degree. We note that over a field of characteristic zero, it is true that central stability for VIC($\mathbb{Z}/p$) or SI($\mathbb{Z}/p$)–modules implies higher central stability [MW].

In this section, we do establish higher central stability for a different class of VIC($\mathbb{Z}$) and SI($\mathbb{Z}$)–modules, the polynomial modules. We can apply these results to the Torelli groups because their first homology groups are known to be polynomial. We now recall the definition of polynomial modules.

**Definition 3.17.** Define the endofunctor
\[
S : U \mathcal{G} \rightarrow U \mathcal{G}
\]
via the formula $S = 1 \oplus -$.

We will consider the natural transformation $\text{id} \rightarrow S$ given by
\[
n = 0 \oplus n \xrightarrow{\text{def}} 1 \oplus n.
\]
By abuse of notation, we denote the endofunctor of $UG$–modules given by precomposition by $S$ also by $S$. Again, there is an induced natural transformation $id \to S$ defined by precomposition with the above natural transformation.

Concretely, if $A$ is a module over Fl, VIC$(R)$, or $Sl(R)$, then there are isomorphisms of $G_n$–representations $(SA)_n \cong Res^{G_{n+1}}_{G_n} A_{n+1}$.

**Definition 3.18.** Given a $UG$–module $A$, we define $UG$–modules

$$\ker A := \ker(A \to SA)$$

and

$$\coker A := \coker(A \to SA).$$

**Definition 3.19.** We say that $A$ has **polynomial degree** $-\infty$ in ranks $> d$ if $A_n = 0$ for all $n > d$. For $r \geq 0$, we say $A$ has **polynomial degree** $\leq r$ in ranks $> d$ if $\ker A_n = 0$ for all $n > d$ and $\coker A$ has polynomial degree $\leq r - 1$ in ranks $> d$.

We say $A$ has **polynomial degree** $\leq r$ if it has polynomial degree $\leq r$ in all ranks $> -1$.

The second author [Pat17] proved that polynomial modules have higher central stability when the category satisfies the following condition.

**Definition 3.20.** Let $b, \ell \in \mathbb{N}_0$. Define the following condition on a stability category $UG$:

$$H_4(\ell, b): \widetilde{H}_i(\ker M(m))_n = 0 \text{ for all } m \geq 0, \text{ all } i \geq -1 \text{, and all } n > \ell \cdot (i + m) + b.$$

**Theorem 3.21** (Patzt [Pat17, Corollary 7.9]). Let $a, b, k, \ell \in \mathbb{N}_0$. Let $UG$ be a stability category satisfying $H_3(k,a)$ and $H_4(\ell,b)$ with $b \geq \max(k,a)$. If $A$ is a $UG$–module of polynomial degree $\leq 0$ in ranks $> d$ for some $d \geq 1$, then

$$\widetilde{H}_i(A)_n = 0 \text{ for all } i \geq -1 \text{ and } n > \max(d + i + 2, ki + a).$$

If $A$ is a $UG$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq 1$, then

$$\widetilde{H}_i(A)_n = 0 \text{ for all } i \geq -1 \text{ and } n > \ell^{i+1} (d+r) + (\ell^i + \ell^{i-1} + \cdots + \ell + 1)b + 1.$$ 

By specializing Theorem 3.21 to homological degrees $i = -1,0$ and invoking Theorem 3.15, we obtain the following consequences for functors of finite polynomial degree.

**Corollary 3.22.** Let $a, b, k, \ell \geq 1$. Let $UG$ be a stability category satisfying $H_3(k,a)$ and $H_4(\ell,b)$ with $b \geq \max(k,a)$. Suppose $A$ is a $UG$–module of polynomial degree $\leq 0$ in ranks $> d$ for some $d \geq 1$. 

i) Then $A$ has central stability degree $\leq \max(a,d+2)$.

ii) Then $A$ is generated in degrees $\leq \max(a-k,d+1)$ and related in degrees $\leq \max(a-k+b,d+1+b)$.

Suppose $A$ is a $UG$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq 1$.

i) Then $A$ has central stability degree $\leq \ell(d+r) + b + 1$.

ii) Then $A$ is generated in degrees $\leq d + r + 1$ and $A$ is related in degrees $\leq \ell(d+r) + b + 1$. 
3.4. $\mathcal{F}I$–modules of finite polynomial degree. It is easy to see that $U\mathcal{S}$ is a subcategory of every symmetric stability category, in particular of $U\text{GL}(R)$, $U\text{Sp}(R)$, and $U\text{GL}^H(R)$ for any $H$ containing $-1$. It follows from the definition of polynomial degree that the polynomial degree of a $U\mathcal{G}$–module coincides with the polynomial degree of the underlying $\mathcal{F}I$–module. We now compute the polynomial degree of free $\mathcal{F}I$–modules.

**Proposition 3.23.** Let $W$ be a $\mathbb{Z}[\mathcal{S}_m]$–module, then $M(W)$ has polynomial degree $\leq m$.

**Proof.** We prove the assertion by induction over $m$. If $m = 0$, then

$$M(W) = M(0) \otimes_{\mathbb{Z}} W$$

and the map $M(W) \to SM(W)$ is an isomorphism. This implies that $M(W)$ has polynomial degree $\leq 0$. Now suppose $m > 0$. Church–Ellenberg [CE15, Lemma 4.4] showed that

$$\text{coker } M(W) = M(\text{Res}_{\mathcal{S}_m} \mathbb{Z}[\mathcal{S}_m] W).$$

Thus $\text{coker } M(W)$ has polynomial degree $\leq m - 1$ by induction. The proof of [CE15, Lemma 4.4] also shows that $\ker M(W) = 0$. This completes the induction. $\Box$

**Proposition 3.24.** $U\mathcal{S}$ satisfies $H4(1,1)$.

**Proof.** The isomorphisms

$$\text{coker } M(m) \cong \text{coker } M(\mathbb{Z}[\mathcal{S}_m]) \cong M\left(\text{Res}_{\mathcal{S}_{m-1}} \mathbb{Z}[\mathcal{S}_m]\right) \cong M(m - 1)^{\otimes m}$$

and Proposition 3.16 imply that $U\mathcal{S}$ in fact satisfies condition $H4(1,0)$. We will only use the weaker condition $H4(\ell,b) = H4(1,1)$ since the applications require that $b \geq \max(k,a)$ for some $k,a$ such that $U\mathcal{S}$ satisfies $H3(k,a)$. $\Box$

**Theorem 3.25.** An $\mathcal{F}I$–module $A$ is presented in finite degree if and only if it has finite polynomial degree. Specifically, if $A$ is generated in degrees $\leq r$ and presented in degrees $\leq d$, then $A$ has polynomial degree $\leq r$ in ranks $> d + \min(r,d) - 1$.

If $A$ has polynomial degree $\leq r$ in ranks $> d$ for some $d \geq -1$, it has

- generation degree $\leq d + r + 1$, and
- presentation degree $\leq d + r + 2$.

**Proof.** The first direction is given by Randal-Williams–Wahl [RWW14, 4.18]. Assume $A$ has polynomial degree $\leq r$ in ranks $> d$. Then by Proposition 3.14 and Proposition 3.24 we may apply Corollary 3.22 with the values $a = k = b = \ell = 1$. $\Box$

3.5. $\mathcal{V}IC$–, $\mathcal{V}IC^H$–, $\text{SI}$–modules of finite polynomial degree. In this subsection, we use the connectivity results of Section 2 to prove a vanishing result for the central stability homology of $\mathcal{V}IC^H(R)$–modules and $\text{SI}(R)$–modules with finite polynomial degree for a $R$ a PID. More specifically, we will show that the connectivity results of Section 2 establish the hypotheses of Theorem 3.21. The second author [Pat17] gave the following reformulation of $H4(\ell,b)$. 
Proposition 3.26 (Patzt [Pat17, Proposition 7.10]). Assume $H3(k,a)$ and let $b \geq \max(k,a)$. Then the condition $H4(\ell,b)$ holds if for every $m$–simplex $\tau \in (K_m G)_{n+1}$, the intersection

$$\text{Lk}_\bullet \tau \cap (K_\bullet G)_n$$

is \(\left( \frac{n-b}{\ell} - m - 2 \right)\)-connected.

Proposition 3.27. Let $R$ be a PID of stable rank $s$. Then $U_{\text{GL}}(R)$ satisfies $H4(2,s+1)$.

Proof. By Proposition 3.26, we must check that for every $m$–simplex $\tau \in (K_m G)_{n+1}$, the intersection

$$\text{Lk}_\bullet \tau \cap (K_\bullet G)_n$$

is \(\left( \frac{n-s-1}{2} - m - 2 \right)\)-connected. Recall that $\text{PBC}_\bullet(R^n) \cong (K_\bullet \text{GL}_n)$. Lemma 2.24 implies that the intersection appearing in Proposition 3.26 is isomorphic to

$$\text{PBC}_\bullet(R^{n+1}, C \cap R^n, \text{sat(im f + R)})$$

Since

$$\text{rk}(C \cap R^n) \geq n - m - 1$$

and

$$\text{rk sat(im f + R)} \leq m + 2,$$

Theorem 2.20 implies that the semisimplicial set is \(\left( \frac{n-m-1 - (m+2) - s - 2}{2} \right)\)-connected. But

$$\frac{n-m-1 - (m+2) - s - 2}{2} = \frac{n-1-s}{2} - m - 2,$$

and the result follows. \(\square\)

Proposition 3.28. Let $R$ be a PID of stable rank $s$, and let $H \leq R^\times$. Then $U_{\text{GL}}^H(R)$ satisfies $H4(2,s+1)$.

Proof. We must check that for every $m$–simplex $\tau \in (K_m \text{GL}_H^H)_{n+1}(R)$, the intersection

$$\text{Lk}_\bullet \tau \cap (K_\bullet \text{GL}_H^H)_n$$

is \(\left( \frac{n-s-1}{2} - m - 2 \right)\)-connected. Because

$$(K_p \text{GL}_{n+1}^H(R)) = (K_p \text{GL}_{n+1}(R))$$

for $p \leq n - 1$,

$$\text{Lk}_\bullet \tau \cap (K_\bullet \text{GL}_H^H(R)) \leftrightarrow \text{Lk}_\bullet \tau \cap (K_\bullet \text{GL}_n(R))$$

induces a bijection of the set of $p$-simplices for $m+p+2 < n+1$. Thus it induces an \(\left( n-m-2 \right)\)-connected map on geometric realizations and so the assertion follows from the connectivity of the codomain which was proved in Proposition 3.27. \(\square\)

Because $U_{\text{GL}}^H(R)$ satisfies $H3(2,s+1)$ and $H4(2,s+1)$, Theorem 3.21 implies the following corollary.
Corollary 3.29. Assume $R$ is a PID of stable rank $s$ and let $H \leq R^\times$. If $A$ is a $\text{VIC}^H(R)$–module of polynomial degree $\leq 0$ in ranks $> d$ for some $d \geq -1$, then

$$\widetilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d + i + 2, 2i + s + 1).$$

If $A$ is a $\text{VIC}^H(R)$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq -1$, then

$$\widetilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > 2^{i+1}(d + r + s + 1) - s.$$  

In particular, $A$ has central stability degree $\leq \max(d + 2, s + 1, 2d + 2r + s + 2)$.

Corollary 3.30. Let $R$ be a PID of stable rank $s$, and let $H \leq R^\times$ contain $-1$. Let $A$ be a $\text{VIC}^H(R)$–module such that the underlying $\text{FI}$–module is generated in degrees $\leq g$ and related in degrees $\leq r$. Then as a $\text{VIC}^H(R)$–module $A$ is generated in degrees $\leq g$ and presented in degree

$$\leq \max(2s, r + \min(g, r) + s + 1) \quad \text{if} \quad g = 0, \quad \text{and}$$
$$\leq 2r + 2g + 2\min(g, r) + s \quad \text{if} \quad g > 0.$$  

In particular, as a $\text{VIC}^H(R)$–module, $A$ has central stability degree

$$\leq \max(s + 1, r + 1) \quad \text{if} \quad g = 0, \quad \text{and}$$
$$\leq 2r + 2g + 2\min(g, r) + s \quad \text{if} \quad g > 0.$$

Proof. Since $\text{FI} \subseteq \text{VIC}^H(R)$, generation in degree $\leq g$ over $\text{FI}$ implies generation in degree $\leq g$ over the larger category $\text{VIC}^H(R)$. By Theorem 3.25, the sequence $A$ has polynomial degree $\leq g$ in ranks $> r + \min(g, r) - 1$, viewed either as a module over $\text{FI}$ or over $\text{VIC}^H(R)$. Because $\text{VIC}^H(R)$ satisfies conditions $\text{H}3(2, s + 1)$ and $\text{H}4(2, s + 1)$, by Propositions 3.14 and 3.28, respectively, we can use Corollary 3.22 to conclude the result. \qed

Remark 3.31. It is possible to show that central stability for $\text{VIC}(\mathbb{Z}/p)$–modules over a field of characteristic zero is quantitatively equivalent to uniform representation stability in the sense of Gan–Watterlond [GW16]. Corollary 3.30 can be rephrased in this context as saying that a sequence has uniform representation stability as a sequence of $\mathbb{Q}[\text{GL}_n(\mathbb{Z}/p)]$–modules if it has uniform representation stability as a sequence of $\mathbb{Q}[\mathfrak{S}_n]$–modules. Despite being a purely algebraic statement, the only proof we know relies on high connectivity of simplicial complexes.

Proposition 3.32. Let $R$ be a PID of unitary stable rank $s$. Then $\text{USp}(R)$ satisfies $\text{H}4(2, s + 2)$.

Proof. Once more we check the connectivity of the intersection

$$\text{Lk}_\bullet \tau \cap (K_\bullet \text{Sp}_n(R))$$

for every $m$–simplex $\tau \in (K_m \text{Sp}_{n+1}(R)) \cong \text{SPB}_m(R^{2n+2})$. Let $W'$ be the symplectic submodule of $R^{2n+2}$ generated by $\tau$ and

$$W = \text{sat}(W' + R^2).$$

Note that $W^\perp \subseteq R^{2n}$. Then the intersection is

$$\text{SPB}_\bullet(R^{2n+2}) \cap \text{Lk}^{\text{MPB}(R^{2n+2})}(W)$$
which is \( \left( \frac{n + 1 + \text{rk} U/2 - \text{rk} W - s - 3}{2} \right) \)-connected by Theorem 2.34, where \( U \) is a maximal symplectic submodule of \( W \). Because \( \text{rk} W \leq 2m + 4 \) and \( \text{rk} U \geq \text{rk} W' = 2m + 2 \),

\[
\frac{n + 1 + \text{rk} U/2 - \text{rk} W - s - 3}{2} \geq \frac{n - s - 1}{2} - m - 2.
\]

Proposition 3.26 then implies that \( U \text{Sp}(R) \) satisfies \( H_4(2, s + 1) \), though we will only apply the weaker condition \( H_4(2, s + 2) \).

Combining this result with Proposition 3.14 and Theorem 3.21 gives the following Corollary.

**Corollary 3.33.** Assume \( R \) is a PID of (symplectic) unitary stable rank \( s \). If \( A \) is a \( \text{Sl}(R) \)-module of polynomial degree \( \leq 0 \) in ranks \( > d \) for some \( d \geq -1 \), then

\[
\tilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d + i + 2, 2i + s + 2).
\]

If \( A \) is a \( \text{Sl}(R) \)-module of polynomial degree \( \leq r \) in ranks \( > d \) for some \( r \geq 1 \) and \( d \geq -1 \), then

\[
\tilde{H}_i(A)_n = 0 \quad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > 2^{i+1}(d + r + s + 2) - s - 1.
\]

**Remark 3.34.** All of the results in this section and the previous section apply equally well to orthogonal groups. We chose not to include these results because we do not know of any applications.

### 3.6. A spectral sequence

In this subsection, we summarize results about a spectral sequence introduced by Putman–Sam [PS14].

**Definition 3.35.** Let \( \mathcal{N}, \mathcal{G}, \mathcal{Q} \) be stability groupoids and \( \mathcal{N} \to \mathcal{G} \) and \( \mathcal{G} \to \mathcal{Q} \) homomorphisms of stability groupoids. We call this data a stability SES if

\[
1 \to \mathcal{N}_n \to \mathcal{G}_n \to \mathcal{Q}_n \to 1
\]

is a short exact sequence for every \( n \in \mathbb{N}_0 \).

**Proposition 3.36** (Patzt [Pat17, Proposition 8.2]). Let \( \mathcal{G}, \mathcal{Q} \) be stability groupoids and \( \mathcal{G} \to \mathcal{Q} \) a homomorphism such that \( \mathcal{G}_n \to \mathcal{Q}_n \) is surjective for every \( n \in \mathbb{N}_0 \). Then there is a unique stability groupoid \( \mathcal{N} \) and a homomorphism \( \mathcal{N} \to \mathcal{G} \) such that

\[
\mathcal{N} \to \mathcal{G} \to \mathcal{Q}
\]

is a stability SES.

**Lemma 3.37** (Patzt [Pat17, Lemma 8.3]). Let

\[
1 \to \mathcal{N} \to \mathcal{G} \to \mathcal{Q} \to 1
\]

be a stability SES and \( A \) a \( U\mathcal{G} \)-module. Then for every \( i \geq 0 \) there is a \( U\mathcal{Q} \)-module that we denote by \( H_i(\mathcal{N}) \) with

\[
H_i(\mathcal{N})_n \cong H_i(\mathcal{N}_n).
\]

The following spectral sequence was constructed by Putman–Sam [PS14]. Also see Patzt [Pat17, Proposition 8.4].
Proposition 3.38 (Putman–Sam [PS14], Patzt [Pat17, Corollary 8.5]). Let

\[ 1 \to \mathcal{N} \to \mathcal{G} \to \mathcal{Q} \to 1 \]

be a stability SES. Assume that \( \mathcal{G} \) and \( \mathcal{Q} \) are braided and that \( \mathcal{G} \to \mathcal{Q} \) is a map of braided monoidal groupoids. For each \( n \), there is a spectral sequence

\[ E^2_{p,q} \approx \check{H}_p^Q(H_q(N))_n \]

that converges to zero for \( p + q \leq \left( \frac{n - a - 1}{k} \right) \) if \( U \mathcal{G} \) satisfies \( H_3(k,a) \).

![Figure 2. \( E^2_{p,q} \).](image)

4. Applications

In this section, we prove our central stability results for the second homology groups of Torelli subgroups of mapping class groups and automorphism groups of free groups as well as for the homology of congruence subgroups.

4.1. \( H_2(\text{IA}_n) \). We will recall a computation of \( H_1(\text{IA}_n) \). Then we will use a spectral sequence argument and our results about polynomial \( \text{VIC}(R) \)-modules to prove Theorem A, central stability for \( H_2(\text{IA}_n) \).

Recall that \( \text{AutF} \) is the stability groupoid given by \( \text{AutF}_n = \text{Aut}(F_n) \). Abelianization induces a monoidal functor

\[ \text{AutF} \to \text{GL} \]

that is surjective in every degree. By Proposition 3.36, the kernels form a stability groupoid which we denote by \( \text{IA} \).

Reinterpreting the work of Andreadakis [And65], Cohen–Pakianathan (unpublished), Farb (unpublished) and Kawazumi [Kaw06] in the language of \( \text{FI} \)-modules, Church, Ellenberg, and Farb [CEF15, Equation (25)] proved the following. See also Day–Putman [DP14, Page 5].

**Theorem 4.1** (Andreadakis, Farb, Kawazumi, Cohen–Pakianathan, Church–Ellenberg–Farb). Use the following notation for bases for the \( \text{FI} \)-module \( M(1) \):

\[ M(1)_n \cong \mathbb{Z} \langle e_1, e_2, \ldots, e_n \rangle. \]
Then
\[ H_1(\mathcal{I}) \cong \bigwedge^2 M(1) \otimes M(1) \]
\[ \cong M\left(\mathbb{Z}[e_1 \wedge e_2 \otimes e_1, e_1 \wedge e_2 \otimes e_2]\right) \oplus M\left(\mathbb{Z}[e_1 \wedge e_2 \otimes e_3, e_1 \wedge e_3 \otimes e_2, e_2 \wedge e_3 \otimes e_1]\right) \]
as an Fl–module.

We now prove Theorem A, central stability for \(H_2(\mathcal{I})\).

**Proof of Theorem A.** \(\text{AutF}\) is symmetric monoidal and thus so is \(U\text{AutF}\). This follows from Proposition 3.4. We can therefore apply the spectral sequences of Proposition 3.38 with \(N=\mathcal{I}\), \(\mathcal{G}=\text{AutF}\), \(Q=\text{GL}(\mathbb{Z})\). Proposition 3.14 says that \(U\text{AutF}\) satisfies \(H3(2,2)\), hence these spectral sequences converge to zero for \(p+q \leq \frac{n-3}{2}\).

We will show that
\[ \widetilde{H}^{\text{GL}(\mathbb{Z})}_{-1}(H_2(\mathcal{I}))_n \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_0(H_2(\mathcal{I}))_n = 0 \quad \text{for } n \geq 39 \]
by showing that there are no nontrivial differentials to or from the groups \(E^2_{-1,2}\) and \(E^2_{0,2}\) of the spectral sequence in this range, as in Figure 3. By Proposition 3.14, the central stability homology of the \(\text{VIC}(\mathbb{Z})\)–module \(H_0(\mathcal{I})\) \(\cong M(0)\) is zero in degree \(q\) for \(n \geq 2q + 4\). This implies the vanishing of the groups \(E^2_{-1,0} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_2(H_0(\mathcal{I}))_n\) and \(E^2_{3,0} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_4(H_0(\mathcal{I}))_n\) for \(n \geq 10\). Theorem 4.1 and Proposition 3.23 imply that \(H_1(\mathcal{I})\) has polynomial degree \(\leq 3\). Thus by Theorem 3.21 its central stability homology as a \(\text{VIC}(\mathbb{Z})\)–module \(\widetilde{H}_1(H_1(\mathcal{I}))_n\) vanishes for \(n \geq 2^{l+1}(5) - 1 = 19\). It follows that the groups \(E^2_{1,1} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_1(H_1(\mathcal{I}))_n\) vanish for \(n \geq 2^{l+1}(5) - 1 = 19\), and the groups \(E^2_{2,1} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_2(H_1(\mathcal{I}))_n\) vanish for \(n \geq 2^{l+1}(5) - 1 = 39\). Thus \(E^2_{-1,2} \cong E^2_{2,2} \cong 0\), and \(E^2_{0,2} \cong E^\infty_{0,2} \cong 0\) for in degrees > 38. Since \(E^2_{1,2} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_{-1}(H_2(\mathcal{I}))_n\) and \(E^2_{0,2} \cong \widetilde{H}^{\text{GL}(\mathbb{Z})}_0(H_2(\mathcal{I}))_n\), we deduce that \(H_2(\mathcal{I})\) has centrally stable degree \(\leq 38\) as claimed. \(\square\)

4.2. \(H_2(\mathcal{I})\). Using a similar proof strategy to that of the previous subsection, we will prove Theorem B, central stability for \(H_2(\mathcal{I})\).
Recall that Mod is the stability groupoid given by the mapping class groups of compact, connected, oriented surfaces with one boundary component. Its intersection form preserving action on the first homology group of the surface gives a monoidal functor

$$\text{Mod} \rightarrow \text{Sp}$$

that is surjective in every degree. Let $\mathcal{I}$ denote the stability groupoid given by assembling the kernels of this map using Proposition 3.36.

Similar to Theorem 4.1, we have the following description of $H_1(\mathcal{I})$.

**Theorem 4.2** (Johnson [Joh85b, Theorem 3]). As a module over $\mathfrak{fl} \subseteq \text{Sl}(\mathbb{Z})$, the first homology groups $H_1(\mathcal{I})$ decompose as follows. Let

$$H := H_1(\Sigma_{1,1}; \mathbb{Z}) \cong M(1)^{\oplus 2}, \quad H_g = \mathbb{Z}[a_1, b_1, b_2, \ldots, a_g, b_g],$$

Then

$$H_1(\mathcal{I}) \cong \bigwedge^3 H \oplus \left( \text{Sym}^0(H) \oplus \text{Sym}^1(H) \oplus \text{Sym}^2(H)/H \right) \otimes \mathbb{Z}/2\mathbb{Z}$$

$$\cong M \left( \mathbb{Z}[a_1 \wedge b_i \wedge b_j, a_i \wedge a_j \wedge b_i \mid \{i, j\} = \{1, 2\}] \right)$$

$$\oplus M \left( \mathbb{Z}[a_1 \wedge a_j \wedge a_k, a_i \wedge a_j \wedge b_k, a_i \wedge b_j \wedge b_k, b_i \wedge b_j \wedge b_k \mid \{i, j, k\} = \{1, 2, 3\}] \right)$$

$$\oplus M (\mathbb{Z}/2\mathbb{Z}[\mathcal{G}_0]) \oplus M (\mathbb{Z}/2\mathbb{Z}[a_1, b_1, a_1 b_1]) \oplus M (\mathbb{Z}/2\mathbb{Z}[a_1 a_2, a_1 b_1, a_2 b_1, b_1 b_2])$$

This description of the abelianization $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$ of the Torelli group $\mathcal{I}_{g,1}$ was first computed by Johnson in a series of papers [Joh80b, Joh83, Joh85a, Joh85b] building on work of Birman–Craggs [BC78]. The free part, $\bigwedge^3 H$, is the image of the Johnson homomorphism [Joh80a]. The torsion part can be identified with a certain space of Boolean polynomials of degree at most 2 with coefficients in $\mathbb{Z}/2\mathbb{Z}$; these polynomials represent part of the dual space to the space of mod 2 self-linking forms on $H$. See for example Farb–Margalit [FM11, Section 6.6] and Brendle–Farb [BF07, Section 2] for modern exposition.

We now prove Theorem B, central stability for $H_2(\mathcal{I})$.

**Proof of Theorem B.** We can apply Proposition 3.38 with $\mathcal{N} = \mathcal{I}, \mathcal{G} = \text{Mod}$, and $Q = \text{Sp}(\mathbb{Z})$.

Proposition 3.14 implies that $U \text{Mod}$ satisfies $H^3(2,2)$, so the spectral sequence converges to zero for $p + q \leq \frac{n-3}{2}$. By Theorem 4.2 and Proposition 3.23, $H_1(\mathcal{I})$ has polynomial degree $\leq 3$, thus by Theorem 3.21 its central stability homology $H^{\text{Sp}}_i(\mathcal{I})_n$ vanishes for $n \geq 2^{i+2} \cdot 3 - 2$. We also have that $H^{\text{Sp}}_i(\mathcal{I})_n \cong H^{\text{Sp}}_i(M(0))_n$ vanishes for $n > 2i + 4$. This implies that $E^{2}_{1,2} = 0$ for $n \geq 2^{i+2} \cdot 3 - 2 = 22$ and $E^{0}_{0,2} = 0$ for $n \geq 2^{i+2} \cdot 3 - 2 = 46$. This proves that $H_2(\mathcal{I})$ has central stability degree $\leq 45$. \qed

### 4.3. Congruence subgroups.

Let $R$ be a commutative ring and $I \subseteq R$ be a proper ideal. Let $\mathfrak{m}$ denote the image $R^\mathfrak{m}$ in $R/I$. Then the quotient map $R \rightarrow R/I$ induces a surjective homomorphism

$$\text{GL}_n(R) \rightarrow \text{GL}^\mathfrak{m}_n(R/I).$$

This gives a surjective homomorphism of stability groupoids

$$\text{GL}(R) \rightarrow \text{GL}^\mathfrak{m}(R/I).$$
We denote the kernel by $GL_n(R, I)$, which is called the level $I$ congruence subgroup of $GL_n(R)$. By Proposition 3.36, these groups assemble to form a stability groupoid which we denote by $GL(R, I)$. Improving upon the work of Putman [Put15] and Church–Ellenberg–Farb–Nagpal [CEFN14], Church–Ellenberg proved the following representation stability result for congruence subgroups viewed as $\mathcal{F}I$–modules.

**Theorem 4.3** (Church–Ellenberg [CE15, Theorem D']). If $R$ has stable rank $t$ and $i \geq 2$, then $H_i(GL(R, I))$ is generated in degrees

$$\leq 2^{i-2}(2t + 7) - 2$$

and presented in degrees

$$\leq 2^{i-2}(2t + 7) - 1$$

when viewed as an $\mathcal{F}I$–module.

Church–Ellenberg’s definition of presentation degree differs from ours. However, they are equivalent here by [CE15, Proposition 4.2].

**Proof of Theorem C.** For $i \geq 2$, the proof is immediate from Corollary 3.30, Theorem 4.3, and the fact that the $\mathcal{F}I$–module structure on $H_i(GL(R, I))$ does not depend on the choice of the ring $R$. For $i = 0$, $H_0(GL(R, I)) = M(0)$, and we use Proposition 3.14. For $i = 1$, we use the spectral sequence argument of the previous two subsections; this approach yields a better range than Corollary 3.30. □

**Remark 4.4.** We would be interested in knowing if a version of Theorem C were true with more mild assumptions on $R/I$. For example, perhaps it would suffice for $R/I$ to have finite stable rank.

**Remark 4.5.** Many of the central stability ranges in this paper can likely be improved. For example, Maazen [Maa79] proved a connectivity result for $PB(V)$ which is one better than that of van der Kallen [vdK80] when $R$ is a Euclidean domain of stable rank 2. A corresponding improvement for the connectivity of $PB(U,W)$ would imply better ranges in Theorem A and Theorem C for these rings. Church, the first author, Nagpal, and Reinhold [CMNR] have recently established a version of Theorem 4.3 with a quadratic stable range. These results may improve the range in Theorem C.

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