Invariant-based inverse engineering for fluctuation transfer between membranes in an optomechanical cavity system

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In this paper, by invariant-based inverse engineering, we design classical driving fields to transfer quantum fluctuations between two suspended membranes in an optomechanical cavity system. The transfer can be quickly attained through a non-adiabatic evolution path determined by a so-called dynamical invariant. Such an evolution path allows one to optimize the occupancies of the unstable “intermediate” states thus the influence of cavity decays can be suppressed. Numerical simulation demonstrates that a perfect fluctuation transfer between two membranes can be rapidly achieved in one step, and the transfer is robust to both the amplitude noises and cavity decays.

Keywords: Invariant-based inverse engineering; Fluctuation transfer; Optomechanical system

I. INTRODUCTION

The field of quantum optomechanics has been intensively investigated due to its fundamental aspects in quantum mechanics of macroscopic bodies and possible applications in quantum metrology and hybrid quantum systems\textsuperscript{[1, 2]}. It explores the interaction between light and mechanical motion by composing an optical (or microwave) cavity and a mechanical resonator. Over the past decades, tremendous progress has been made in the field of quantum optomechanics. On the one hand, non-trivial quantum phenomena have been realized in optomechanical systems, such as sideband\textsuperscript{[3, 4]} and near-ground-state cooling\textsuperscript{[5–8]}, strong coupling effects\textsuperscript{[9–11]}, squeezing of a mechanical oscillator\textsuperscript{[12–14]}, and so on.\textsuperscript{[15–17]} On the other hand, complementary setups have been realized, for example, cavities with an oscillating end mirror\textsuperscript{[18–19]}, evanescent-wave resonators\textsuperscript{[20]}, and membrane-in-the-middle (MIM) cavities\textsuperscript{[21–23]}. In recent years, because optomechanical systems have the ability to transfer a quantum state between photons with vastly differing wavelengths (quantum information and quantum fluctuations of an optical field can be reversibly mapped to a mechanical state), the direct utility of such systems in quantum information processing becomes attractive\textsuperscript{[23–32]}. For example, Tian and Wang have studied a quantum state conversion between cavity modes of distinctively different wavelengths by applying classical driving fields to swap the cavity and the mechanical states\textsuperscript{[30]}. Fiore et al. have experimentally reported the demonstration of storing optical information as a mechanical excitation in a silica optomechanical resonator\textsuperscript{[35]}. Noting that, in the process of exploiting the utility of optomechanical systems for quantum information processing, growing interest has been shown toward the adiabatic control of such systems because a dark-state evolution is usually robust against noise and dissipation\textsuperscript{[36–39]}. For example, Wang and Clerk performed the high-fidelity transfer of a quantum state between two electromagnetic cavities by adiabatic control\textsuperscript{[37]}. Soon after that, Dong et al. experimentally realized an adiabatic transfer of optical fields between two optical modes of a silica resonator\textsuperscript{[38]}. Recently, Garg et al. proposed a scheme to transfer quantum fluctuations between mechanical oscillators based on adiabatic control\textsuperscript{[39]}. These researches further open up the possibility of using optomechanical coupling in various applications without cooling the mechanical oscillator to its ground state. However, applying adiabatic control usually requires a long-time interaction to guarantee the adiabaticity of the system. To speed up the adiabatic schemes without losing the advantages of adiabatic control, various versions of approaches called shortcuts to adiabaticity (STA)\textsuperscript{[40–48]} have been proposed in recent years, including, e.g., transitionless driving algorithm\textsuperscript{[44–49]}, inverse engineering based on invariants\textsuperscript{[50–59]}, and so on.\textsuperscript{[60–67]} The basic idea of the STA approach is to design a suitable Hamiltonian to drive a quantum system to evolve along a non-adiabatic route to reproduce the same final state of an adiabatic process. Thus, the relevant adiabatic requirement can be removed and the evolution could be driven fast. Very recently, the STA approach has been applied to optomechanical systems for fast quantum state conversions\textsuperscript{[68–70]}.

In this paper, we proceed further to exploit the use of STA in optomechanical systems. Precisely speaking, we show how to use inverse engineering based on Lewis-Riesenfeld invariants to coherently and deterministically transfer the quantum fluctuations from one mechanical oscillator to the other. The invariant-based inverse engineering is an efficient non-adiabatic approach (usually based on the Lewis-Riesenfeld theory) to analytically obtain an exact dynamical evolution for an arbitrary quantum system. The Lewis-Riesenfeld theory\textsuperscript{[71]} tells that the quantum dynamics of a quantum system can be dictated by a dynamical invariant $I(t)$ obeying the von Neu-
The system can be divided into three independent subcavities. For convenience, we mark the three subcavities as: left (L), middle (M), and right (R), respectively.

By applying the inverse engineering based on Lewis-Riesenfeld invariants, we show how quantum fluctuations can be transferred from one mechanical oscillator to the other in a short time with a high fidelity. Specifically, we consider a weak optomechanical coupling to apply the rotating-wave approximation and quantum Zeno dynamics to obtain an effective Hamiltonian. Then, in absence of decays of the cavity modes and the membranes, we design parameters by invariant-based inverse engineering to complete the fluctuation transfer. This is different from the previous reported works wherein such a transfer occurs between two cavity modes. The exchange of the energy fluctuations between two mechanical systems may pose as a possible quantum communication protocol. In this scheme, the occupancies of the unstable “intermediate” states can be optimized by suitably choosing the parameters that suppress the cavity decays. The sensitivity with respect to amplitude-noise errors in the driving fields is also analyzed by numerical simulation. The result shows the system is robust against amplitude noises with suitable parameters.

The paper is organized as follows. In Sec. II, we present the model and the corresponding effective Hamiltonian. In Sec. III, we show how to design the Hamiltonian by invariant-based inverse engineering to transfer quantum fluctuations. In Sec. IV, we analyze the robustness of the scheme against noises and decays by numerical simulation. The conclusion is given in Sec. V.

II. MODEL

As shown in Fig. 1 we consider an optomechanical cavity system with two suspended membranes. Assuming that the membranes are fully reflecting at their surfaces, the system can be divided into three independent cavities: the system is Robust against amplitude-noise errors in the driving fields is also analyzed by numerical simulation. The result shows the system is robust against amplitude noises with suitable parameters.

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II. MODEL

As shown in Fig. 1 we consider an optomechanical cavity system with two suspended membranes. Assuming that the membranes are fully reflecting at their surfaces, the system can be divided into three independent cavities.
determines the optomechanical coupling for the first (second) membrane with left (right) and middle modes, while \( J_1 (J_2) \) represents the transmission coefficient between the left (right) and middle cavity modes through first (second) membrane. Beware that \( J_k = 0.5 \omega_{m,k} \) is necessary to ensure that the coupling is linear in the displacement quadrature \( X \equiv (b + b^\dagger) \) in a MIM setup. In the rotating frame of laser frequencies, the Hamiltonian \( H \) takes the following form,

\[
H = \sum_{j=L,M,R} [\Delta_j \alpha_j^\dagger \alpha_j + \Omega_j (\alpha_j + H.c.)] + \sum_{k=1,2} \omega_{m,k} b_k^\dagger b_k
\]

\[-g_1 (\alpha_L^\dagger \alpha_L - \alpha_M^\dagger \alpha_M) (b_1 + b_1^\dagger)
+ g_2 (\alpha_M^\dagger \alpha_M - \alpha_R^\dagger \alpha_R) (b_2 + b_2^\dagger)
- (J_1 \delta a_L^\dagger + J_2 \delta a_R^\dagger) H.c.,
\]

(3)

where \( \Delta_j = \omega_{c,j} - \omega_l \) and \( \Delta_M = \omega_{c,M} - \omega_L \). By using the standard linearization procedure \([2]\), all the bosonic operators can be expanded as a sum of the average values and the zero-mean fluctuation as follows: \( \alpha_j \rightarrow \alpha_j + \delta \alpha_j \), and \( b_k \rightarrow \delta b_k \). Then, \( \delta \alpha_j \) and \( \delta b_k \) are generally complex and denote the steady-state values of the respective annihilation operators. Then, an effective Hamiltonian for the operator fluctuations (derivation is given in Appendix B) is given as

\[
H = -g_1 (\alpha_L^\dagger \delta \alpha_L - \alpha_M^\dagger \delta \alpha_M)
+ g_2 (\alpha_M^\dagger \delta \alpha_M - \alpha_R^\dagger \delta \alpha_R)
- (J_1 \delta a_L^\dagger \delta a_M + J_2 \delta a_R^\dagger \delta a_M) + H.c.,
\]

(4)

where

\[
\alpha_L = \frac{\Omega_L - J_1 \alpha_M}{-\Delta_0 + i\gamma_L/2},
\alpha_R = \frac{-\Delta_0 + i\gamma_R/2}{-\Delta_0 + i\gamma_R/2},
\alpha_M = \frac{-\Delta_0 + i\gamma_M/2}{-\Delta_0 + i\gamma_M/2}.
\]

(5)

Here \( \Delta_0 \) is given according to Eq. (3B9) and \( \gamma_j \) is the decay rate of the \( j \)th cavity mode. For the sake of simplification, we choose \( \Omega_M = J_1 \alpha_L + J_2 \alpha_R \) so that

\[
\alpha_M = 0,
\alpha_L = \frac{\Omega_L}{-\Delta_0 + i\gamma_L/2},
\alpha_R = \frac{-\Delta_0 + i\gamma_R/2}{-\Delta_0 + i\gamma_R/2}.
\]

(6)

Using the Hamiltonian in Eq. (3) and according to the input-output formalism \([7] \), we obtain Langevin equations for the operator fluctuations as [30]

\[
\dot{\delta \alpha}_L = i J_1 \delta \alpha_M + i g_1 \alpha_L \delta b_1 - \frac{\gamma_L}{2} \delta \alpha_L + \sqrt{\gamma_L} \delta a_{in},
\]

\[
\dot{\delta \alpha}_R = i J_2 \delta \alpha_M - i g_2 \alpha_R \delta b_2 - \frac{\gamma_R}{2} \delta \alpha_R + \sqrt{\gamma_R} \delta a_{in},
\]

\[
\dot{\delta \alpha}_M = i J_1 \delta \alpha_L + i J_2 \delta \alpha_R - \frac{\gamma_M}{2} \delta \alpha_M + \sqrt{\gamma_M} \delta a_{in},
\]

\[
\dot{\delta b}_1 = i g_1 \alpha_L \delta \alpha_L - \frac{\gamma_m}{2} \delta b_1 + \sqrt{\gamma_m} \delta a_{in},
\]

\[
\dot{\delta b}_2 = -i g_2 \alpha_R \delta \alpha_R - \frac{\gamma_m}{2} \delta b_2 + \sqrt{\gamma_m} \delta a_{in},
\]

(7)

where the overdot stands for a time derivative and \( \delta a_{in} \) and \( \delta b_{in} \) denote the fluctuations of the noise operators. We can rewrite the Langevin equations in Eqs. (7) in matrix form as \( i \Psi(t) = M(t) \Psi(t) \), where

\[
|\Psi(t)\rangle = \begin{bmatrix} \delta \alpha_L \\ \delta \alpha_R \\ \delta \alpha_M \\ \delta b_1 \\ \delta b_2 \end{bmatrix},
\]

(8)

\[
M(t) = \begin{bmatrix} -i\gamma_L/2 & -J_1 & 0 & -g_1 \alpha_L & 0 \\ -J_1 & -i\gamma_M/2 & -J_2 & 0 & 0 \\ 0 & -J_2 & -i\gamma_R/2 & 0 & g_2 \alpha_R \\ -g_1 \alpha_L & 0 & 0 & -i\gamma_m/2 & 0 \\ 0 & g_2 \alpha_R & 0 & 0 & -i\gamma_m/2 \end{bmatrix}.
\]

(9)

Assuming that the decay rates of the subcavities and membranes are much smaller than their fundamental frequencies such that there is no significant decay of the photons during the transfer process, the non-Hermitian terms in Eqs. (7) can be neglected.

Referring to the formula of quantum Zeno dynamics \([72], \) we write the matrix \( M(t) \) in the absence of the decay terms (\( \gamma_j = \gamma_{m,k} = 0 \)) as \( M(t) = \Omega(M_p + K M_q) \), where \( \Omega = \sqrt{(g_1 \alpha_L)^2 + (g_2 \alpha_R)^2} \), \( K = \sqrt{2} J/\Omega \), and

\[
M_p = \frac{1}{\Omega} \begin{bmatrix} 0 & 0 & 0 & -g_1 \alpha_L & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_2 \alpha_R \\ -g_1 \alpha_L & 0 & 0 & 0 & 0 \\ 0 & g_2 \alpha_R & 0 & 0 & 0 \end{bmatrix},
\]

(10)

\[
M_q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(11)

Here we have chosen \( J_1 = J_2 = J \). When the strong coupling limit \( K \rightarrow \infty (\sqrt{2} J \gg \Omega) \) is satisfied, we obtain the effective interaction matrix \( M_{eff} = \sum_{i} P_i M_p P_l + K e_i P_l \), where \( P_l \) is the \( l \)th eigenprojection and \( e_i \) is the
corresponding eigenvalue of $M_2$: $M_2 = \sum_i \epsilon_i P_i$. Thus, in the Zeno dark subspace ($\epsilon_1 = 0$) spanned by

$$
|\psi_1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},
|\psi_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
|\psi_3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},
$$

(12)

the effective interaction matrix $M_{eff}(t)$ reads

$$
M_{eff}(t) = -\frac{1}{\sqrt{2}} (g_1\alpha_L|\psi_1\rangle\langle\psi_3| + g_1\alpha_L|\psi_3\rangle\langle\psi_1|
+ g_2\alpha_R|\psi_2\rangle\langle\psi_3| + g_2\alpha_R|\psi_3\rangle\langle\psi_2|).
$$

(13)

**III. INVARIANT-BASED INVERSE ENGINEERING FOR FAST ENERGY FLUCTUATION TRANSFER**

We show in this section how to transfer fluctuation excitation from mode $b_1$ to mode $b_2$ in a fast and robust way. For the time-dependent matrix $M_{eff}(t)$ in Eq. (13), the dynamical invariant $I(t)$ satisfying the von Neumann-like equation $\partial_t I(t) - i[M_{eff}(t), I(t)] = 0$ is given by

$$
I(t) = \cos \varphi \sin \theta|\psi_1\rangle\langle\psi_3| + \cos \varphi \cos \theta|\psi_2\rangle\langle\psi_3|
- \sin \varphi|\psi_1\rangle\langle\psi_2| + \cos \varphi \sin \theta|\psi_3\rangle\langle\psi_1|
+ \cos \varphi \cos \theta|\psi_3\rangle\langle\psi_2| + \sin \varphi|\psi_2\rangle\langle\psi_1|.
$$

(14)

The parameters satisfy

$$
g_1\alpha_L = -\sqrt{2}(\dot{\theta} \cot \varphi \sin \theta + \dot{\varphi} \cos \theta)
= -\Omega \sin \vartheta,
g_2\alpha_R = -\sqrt{2}(\dot{\theta} \cot \varphi \cos \theta - \dot{\varphi} \sin \theta)
= -\Omega \cos \vartheta,
$$

(15)

where

$$
\Omega = \sqrt{\frac{2(\dot{\theta} \cot \varphi)^2 + \dot{\varphi}^2)}},
\vartheta = \theta + \arctan \left( \frac{\varphi}{\dot{\theta} \cot \varphi} \right).
$$

(16)

The driving fields are thus designed according to Eqs. (6) and (15).

$$
\Omega_L = \sqrt{\frac{2\Delta_0}{g_1}}(\dot{\theta} \cot \varphi \sin \theta + \dot{\varphi} \cos \theta),
\Omega_R = \sqrt{\frac{2\Delta_0}{g_2}}(\dot{\theta} \cot \varphi \cos \theta - \dot{\varphi} \sin \theta),
\Omega_M = -\frac{J}{\Delta_0} (\Omega_L + \Omega_R).
$$

(17)

According to Lewis-Riesenfeld theory [71], the solution of the differential equation $i|\dot{\Psi}(t)\rangle = M(t)|\Psi(t)\rangle$, is $|\Psi(t)\rangle = \sum_n C_n e^{i\epsilon_n t} |\phi_n\rangle$. We choose the eigenvector $|\phi_0\rangle$ with zero-eigenvalue as the evolution path, which means $C_0 \neq 0$ and $C_{n \neq 0} = 0$. For the invariant given in Eq. (14),

$$
|\phi_0(t)\rangle = \cos \varphi \cos \theta|\psi_1\rangle - \cos \varphi \cos \theta|\psi_2\rangle
- i \sin \varphi|\psi_3\rangle.
$$

(18)

Thus, the solution of the differential equation $i|\dot{\Psi}(t)\rangle = M(t)|\Psi(t)\rangle$ is obtained:

$$
|\Psi(t)\rangle = C_0[(\cos \varphi \cos \theta) \delta b_1 - (\cos \varphi \sin \theta) \delta b_2
- \frac{i}{\sqrt{2}} \sin \varphi (\delta a_L - \delta a_R)].
$$

(19)

By choosing the initial average excitation fluctuation of the first membrane as $\langle \delta b_1^* \delta b_1 \rangle = 1$, we have $C_0 = 1$. Meanwhile, to transfer the fluctuation excitation from mode $b_1$ to mode $b_2$ and to simulate classical driving fields with a finite duration, the parameters $\theta$ and $\varphi$ should satisfy the boundaries

$$
\theta(t_i) = 0, \theta(t_f) = \pi/2,
\varphi(t_i) = 0, \varphi(t_f) = 0,
\varphi(t_i) = 0, \varphi(t_f) = 0.
$$

(20)

We choose a Vitanov shape [81] for $\theta$ and a symmetric Gaussian shape for $\varphi$,

$$
\theta = \frac{\pi}{2[1 + \exp(-t/t^*)]}, \varphi = \pi \varphi_0 \exp(-t^2/r_c^2),
$$

(21)

where the time scale $t^*$ controls the effective duration of the classical driving fields, $r_c$ is a related coefficient, and $0 < \varphi_0 < \pi/2$ is the maximum value of $\varphi$. When the Zeno requirement is satisfied, the occupancy of the unstable “intermediate” state $|\phi_0\rangle$ is controllable according to Eqs. (13) and (21). We will show in the following that there is a trade-off between the occupancy of state $|\phi_0\rangle$ and the total interaction time $T$ to obtain a high-fidelity fluctuation transfer. With suitable parameters,
we display the classical driving fields versus time in Fig. 2 (a). Applying the classical driving fields in Fig. 2 (a), complete fluctuation transfer can be achieved as shown in Fig. 2 (b) which displays the time evolution of the average excitation fluctuations of the two membranes by solving numerically the Langevin equations (in the absence of the decay terms) in Eq. (7).

For the sake of convenience, we define the final (when \( t = t_f \)) average excitation fluctuation of the second membrane \( \langle \delta b_1 \delta b_2 \rangle = |\langle \Psi(t_f)| \psi_2 \rangle|^2 = F \) as the fidelity of the transfer process. Then, we plot Fig. 3 (a) to show the fidelity \( F \) versus parameters \( T \) and \( \varphi_0 \). As shown in the figure, to achieve high-fidelity fluctuation transfer with \( F \geq 0.999 \), the shortest time required for the scheme is only \( T \approx 0.32/g \) when \( \varphi_0 \approx 0.15 \). For \( \varphi_0 \approx 0.15 \), according to Eqs. (18), the maximal occupancy of the “intermediate” state \( |\psi_3 \rangle \) is only \( |\sin (0.15\pi)|^2 \approx 0.2 \). Moreover, we find from the figure that the fidelity decreases with the increase of \( \varphi_0 \). This is because we have chosen a relatively large \( \tau_c \) to plot the figure so that the boundaries given in Eq. (20) are hard to satisfy when \( \varphi_0 \) is large. For example, when \( \varphi_0 = 0.4 \), we have \( \varphi(t_f) = \varphi(t_f) \approx 0.03\pi \) leading to \( C_0 \approx 0.9956 \) and \( F \leq |C_0 \cos [\varphi(t_f)]| \approx 0.98 \).

To check if the strong coupling limit \( K \to \infty \) is satisfied with the current parameters, we plot \( K_{\text{min}} \) (the minimum value of \( K \)) versus \( T \) and \( \varphi_0 \) in Fig. 3 (b). It is interesting to find from Fig. 3 that, a high fidelity \( F \approx 0.99 \) is achievable even when the strong coupling limit \( K \to \infty \) is not satisfied: \( K_{\text{min}} \approx 2 \) when \( F \approx 0.99 \). In fact, this phenomenon has been discussed in detail in Ref. [53]. It is not so necessary to ideally satisfy the condition \( K \to \infty \) in applying invariant-based inverse engineering and quantum Zeno dynamics for a time-dependent system. We know that when the Zeno requirement is not well satisfied, the “intermediate” states in the Zeno subspaces with \( \epsilon \neq 0 \) are no longer negligible. However, according to Ref. [53], under certain conditions, the intermediate states can help to speed up the population transfer. \( K \geq 2 \) is enough for the current system to obtain high-fidelity fluctuation transfer.

### IV. Robustness Against Noises and Decays

In real experiment, there is usually a stochastic kind of noise on the parameters that should be considered. We assume that the interaction matrix \( M(t) \) is perturbed by a stochastic part \( \mu M_s(t) \) describing amplitude noise. The Langevin equations in form of \( i\dot{\Psi}(t) = M(t)|\Psi(t)\rangle \) thus become \( i\dot{\Psi}(t) = [M(t) + \mu M_s(t)|\xi(t)\rangle |\Psi(t)\rangle \), where \( \xi(t) = \delta W_t \) is heuristically the time derivative of the Brownian motion \( W_t \). \( \xi(t) \) satisfies \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)|\xi(t')\rangle = \delta(t - t') \) because the noise should have zero mean and the noise at different times should be uncorrelated. Then, we define \( \rho(t) = |\Psi(t)\rangle \langle \Psi(t)| \) and the dynamical equation for \( \rho(t) \) is thus given as

\[
\dot{\rho}(t) = -i[M(t), \rho(t)] - i\mu[M_s(t), \xi(t)|\Psi(t)\rangle \langle \Psi(t)|]. \tag{22}
\]

After averaging over the noise, Eq. (22) becomes

\[
\dot{\rho}(t) \simeq -i[M(t), \rho(t)] - i\mu[M_s(t), \langle \xi(t)\rangle \rho(t)], \tag{23}
\]

where \( \rho(0) = |\psi_2\rangle \langle \psi_2| \).

In the current scheme, we consider independent amplitude noise in Rabi frequency \( \Omega(t) \) as well as in \( \Omega(t) \) with the same intensity \( \mu \). The amplitude noise in other parameters affect the scheme very weakly thus can be ignored. In this case, the master equation is

\[
\dot{\rho}(t) = -i[M(t), \rho(t)] - \mu^2[M_{\text{aL}}(t), [M_{\text{aL}}(t), \rho(t)]]
- \mu^2[M_{\text{aR}}(t), [M_{\text{aR}}(t), \rho(t)]], \tag{24}
\]

where

\[
M_{\text{aL}} = \frac{g_1 \Omega_L}{\Delta_0} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
M_{\text{aR}} = \frac{g_2 \Omega_R}{\Delta_0} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{25}
\]
Assuming the intensity of noise is $\mu = 0.05$, the sensitivity with respect to amplitude-noise error of the population transfer is shown in Fig. 4 (a). The sensitivity with respect to amplitude-noise error decreases with the increases of the $T$ and $\varphi_0$. As we know, the adiabatic condition for the current system is satisfied better with a longer interaction time, and an adiabatic process is generally much more robust against noises than a nonadiabatic process. This is why we find from the figure that shortening the interaction time increases the sensitivity with respect to amplitude-noise error.

In the following, we would like to analyze the influence of the decays on the scheme. The non-Hermitian terms in Eq. (7) should be taken into account in the numerical simulation in this case. Without loss of generality, we set $\gamma_j = j$ and $\gamma_{m,k} = \kappa$. Then, with Eq. (8), we plot Fig. 4 (b) to show the fidelity $F$ versus $J$ and $T$ when decays are considered. The influence of decays increase with the increase of $T$ and $\varphi_0$ as shown in the figure. This result is contradictory to that of Fig. 4 (a) as large $T$ and $\varphi_0$ are required for the robustness against noises. In practice, if the experimental facility allows for weak noises in the driving fields, one can achieve the energy fluctuation transfer between membranes with a high fidelity $\geq 0.95$ by choosing $T \leq 0.5/g$ and $\varphi_0 \leq 0.15$.

V. CONCLUSION

We have presented a scheme in an optomechanical cavity system to transfer the average fluctuation of excitation from one membrane to the other. The cavity is divided into three subcavities by placing two membranes, and the subcavities couple to each other via tunneling through the membranes. Each of the subcavities is considered to decay independently. By invariant-based inverse engineering, we have designed time-dependent classical driving fields in counterintuitive sequences to guide the system to evolve along a non-adiabatic path to realize the transfer. The evolution path is parametrized so that one can control the occupancies of the unstable intermediate states to suppress the influence of decays. We have numerically shown the choices of the optimal parameters to achieve a high-fidelity fluctuation transfer. Moreover, by numerical simulation, we have further investigated the influence of noises and decays; the result shows the present scheme is robust against the amplitude noises and decays of the membranes and the cavity modes. Thus, we hope the present scheme might benefit quantum information science based on quantum optomechanics.

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APPENDIX A: Derivation of system Hamiltonian in Eq. (2)

The dynamical Hamiltonian for the system in Fig. 1 without applying the classical driving fields is ($\hbar = 1$)

$$H = H_0 + H_J,$$

$$H_0 = \sum_{j=L,M,R} \omega_{c,j} a_j^\dagger a_j,$$

$$H_J = -(J_1 a_L^\dagger a_M + J_2 a_M^\dagger a_R + H.c.),$$

(A1)

Then, we send three monochromatic waves $\Omega_L$, $\Omega_R$, and $\Omega_M$ along the cavity axis to drive the modes $a_L$, $a_R$, and $a_M$, respectively. The driving Hamiltonian reads

$$H_D = \Omega_L a_L e^{i\omega_L t} + \Omega_R a_R e^{i\omega_R t} + \Omega_M a_M e^{i\omega_M t} + H.c.,$$

(A2)

Caused by the optical radiation pressure, the membranes 1 and 2 will move with displacements $q_1$ and $q_2$, respectively (as shown in Fig. 1). Thus, the resonance frequencies of the cavity modes will be changed [22, 77, 79] and $H_0$ in Eq. (A1) should be modified as

$$H'_0 = \frac{\omega_{c,L}}{1 + \frac{q_1}{\zeta_L}} a_L^\dagger a_L + \frac{\omega_{c,R}}{1 - \frac{q_2}{\zeta_R}} a_R^\dagger a_R + \frac{\omega_{c,M}}{1 + \frac{q_3}{\zeta_M}} a_M^\dagger a_M,$$

(A3)
Then, considering that \( q_1, q_2 < \zeta_j \mathcal{L} \) \((j = L, M, R)\), we have

\[
\begin{align*}
\frac{\omega_{e,L}}{1 + \frac{q_1}{\zeta_j \mathcal{L}}} &\approx (1 - \frac{q_1}{\zeta_j \mathcal{L}}) \omega_{e,L} - \frac{q_1}{\zeta_j \mathcal{L}} \frac{\omega_{e,L}}{1 + \frac{q_1}{\zeta_j \mathcal{L}}} \\
\frac{\omega_{e,R}}{1 - \frac{q_2}{\zeta_j \mathcal{L}}} &\approx (1 + \frac{q_2}{\zeta_j \mathcal{L}}) \omega_{e,R} + \frac{q_2}{\zeta_j \mathcal{L}} \frac{\omega_{e,R}}{1 - \frac{q_2}{\zeta_j \mathcal{L}}} \\
\frac{\omega_{e,M}}{1 + \frac{q_2 - q_1}{\zeta_j \mathcal{L}}} &\approx (1 - \frac{q_2 - q_1}{\zeta_j \mathcal{L}}) \omega_{e,M} - \frac{q_2 - q_1}{\zeta_j \mathcal{L}} \frac{\omega_{e,M}}{1 + \frac{q_2 - q_1}{\zeta_j \mathcal{L}}} \\
\end{align*}
\]

(A4)

Substituting Eq. (A3) into Eq. (A3), we obtain

\[
H'_0 \approx \sum_{j=L,M,R} \omega_{e,j} a_j^\dagger a_j - \left( \frac{\zeta_j \mathcal{L}}{\omega_{e,L}} a_L^\dagger a_L - \frac{\zeta_j \mathcal{L}}{\omega_{e,M}} a_M^\dagger a_M \right) q_1 + \left( \frac{\zeta_j \mathcal{L}}{\omega_{e,R}} a_R^\dagger a_R - \frac{\zeta_j \mathcal{L}}{\omega_{e,M}} a_M^\dagger a_M \right) q_2.
\]

(A6)

Meanwhile, the free Hamiltonian for the oscillators is

\[
H_{00} = \sum_{k=1,2} \frac{p_k^2}{2\mu_k} + \frac{1}{2} \mu_k \omega_{m,k}^2 q_k^2.
\]

(A7)

the Hamiltonian in Eq. (2) is obtained.

APPENDIX B: Derivation of effective Hamiltonian in Eq. (4)

According to Refs. [36, 71], by using the input-output formalism, we can obtain Langevin equations for the relevant operators in Eq. (3) as

\[
\begin{align*}
\dot{a}_L &= - \left( \frac{\gamma_L}{2} + i \Delta_L \right) a_L + i J_1 a_M + i g_1 a_L (b_1 + b_1^\dagger) - i \Omega_L - \sqrt{\gamma_L} a_L^{\dagger n} \\
\dot{a}_R &= - \left( \frac{\gamma_R}{2} + i \Delta_R \right) a_R + i J_2 a_M - i g_2 a_R (b_2 + b_2^\dagger) - i \Omega_R - \sqrt{\gamma_R} a_R^{\dagger n} \\
\dot{a}_M &= - \left( \frac{\gamma_M}{2} + i \Delta_M \right) a_M + i J_1 a_L + i J_2 a_R - i g_1 a_M (b_1 + b_1^\dagger) + i g_2 a_M (b_2 + b_2^\dagger) - i \Omega_M - \sqrt{\gamma_M} a_M^{\dagger n}, \\
\dot{b}_1 &= - \left( \frac{\gamma_m}{2} + i \omega_{m,1} \right) b_1 + i g_1 a_L^\dagger a_L - a_M^{\dagger n} a_M + \sqrt{\gamma_m} b_1^{\dagger n}, \\
\dot{b}_2 &= - \left( \frac{\gamma_m}{2} + i \omega_{m,2} \right) b_2 - i g_2 a_R^\dagger a_R - a_M^{\dagger n} a_M + \sqrt{\gamma_m} b_2^{\dagger n},
\end{align*}
\]

(B1)
where $\gamma_j$ is the decay rate of the $j$th mode of the cavity and $\gamma_{m,k}$ ($k = 1, 2$) is the dissipation rate of the $k$th membrane, $a_j^{in}$ and $b_k^{in}$ are noise operators satisfying

$$
\langle a_j^{in}(t)a_j^{in}(t') \rangle = \delta(t - t'),
\langle a_j^{in}(t)a_j^{in}(t') \rangle = 0,
\langle b_k^{in}(t)b_k^{in}(t') \rangle = (\bar{n}_{th} + 1)\delta(t - t'),
\langle b_k^{in}(t)b_k^{in}(t') \rangle = (\bar{n}_{th})\delta(t - t'),
$$

(B2)

where $\bar{n}_{th} = \{\exp[\hbar\omega_{m,k}/(k_B T)]\}^{-1}$ is the mean thermal excitation number in the bath, interacting with the mechanical oscillator with frequency $\omega_{m,k}$ at an equilibrium temperature $T$ and $k_B$ is the Boltzmann constant. Then, by using the standard linearization procedure to expand the boson operators as a sum of the average values and the zero-mean fluctuation as $a_j \rightarrow \alpha_j + \delta a_j$ and $b_k \rightarrow \beta_k + \delta b_k$, we obtain the following equations for the average of the operators

$$
\dot{\alpha}_L = - (\gamma_L/2 + i\Delta_L')\alpha_L + iJ_1\alpha_M - i\Omega_L,
\dot{\alpha}_R = - (\gamma_R/2 + i\Delta_R')\alpha_R + iJ_2\alpha_M - i\Omega_R,
\dot{\alpha}_M = - (\gamma_M/2 + i\Delta_M')\alpha_M + iJ_1\alpha_L + iJ_2\alpha_R - i\Omega_M,
\dot{\beta}_1 = - (\gamma_{m,1}/2 + i\omega_{m,1})\beta_1 + ig_1(\alpha_L|2 - |\alpha_M|^2),
\dot{\beta}_2 = - (\gamma_{m,2}/2 + i\omega_{m,2})\beta_2 - ig_2(|\alpha_R|^2 - |\alpha_M|^2),
$$

(B3)

with

$$
\Delta_L' = \Delta_L - g_1 \text{Re}(\beta_1),
\Delta_R' = \Delta_R + g_2 \text{Re}(\beta_2),
\Delta_M' = \Delta_M + g_1 \text{Re}(\beta_1) - g_2 \text{Re}(\beta_2).
$$

(B4)

Solving the equations $\dot{a}_j = 0$, the steady-state solutions for $\alpha_j$ are obtained as

$$
\alpha_L = \frac{\Omega_L - J_1\alpha_M}{-\Delta_L' + i\gamma_L/2},
\alpha_R = \frac{\Omega_R - J_2\alpha_M}{-\Delta_R' + i\gamma_R/2},
\alpha_M = \frac{\Omega_M - J_1\alpha_L - J_2\alpha_R}{-\Delta_M' + i\gamma_M/2}.
$$

(B5)

Meanwhile, for the part of fluctuations, the Langevin equations are

$$
\dot{\delta a}_L = - (\gamma_L/2 + i\Delta_L')\delta a_L + iJ_1\delta a_M + ig_1\alpha_L(\delta b_1 + \delta b_1^\dagger) + \sqrt{\gamma_L}\delta a_L^{in},
\dot{\delta a}_R = - (\gamma_R/2 + i\Delta_R')\delta a_R + iJ_2\delta a_M + ig_2\alpha_R(\delta b_2 + \delta b_2^\dagger) + \sqrt{\gamma_R}\delta a_R^{in},
\dot{\delta a}_M = - (\gamma_M/2 + i\Delta_M')\delta a_M + iJ_1\delta a_L + iJ_2\delta a_R - ig_1\alpha_L(\delta b_1 + \delta b_1^\dagger) + ig_2\alpha_R(\delta b_2 + \delta b_2^\dagger) + \sqrt{\gamma_M}\delta a_M^{in},
\dot{\delta b}_1 = - (\gamma_{m,1}/2 + i\omega_{m,1})\delta b_1 + ig_1(\delta a_L + \delta a_L^\dagger - \alpha_M(\delta a_M + \delta a_M^\dagger)) + \sqrt{\gamma_{m,1}}\delta b_1^{in},
\dot{\delta b}_2 = - (\gamma_{m,2}/2 + i\omega_{m,2})\delta b_2 - ig_2(\delta a_R + \delta a_R^\dagger - \alpha_M(\delta a_M + \delta a_M^\dagger)) + \sqrt{\gamma_{m,2}}\delta b_2^{in}.
$$

(B6)

Equations (B6) can be derived from the following linearized Hamiltonian,

$$
H = \sum_{j=L,M,R} \Delta_j'\delta a_j^\dagger\delta a_j + \sum_{k=1,2} \omega_{m,k}\delta b_k^\dagger b_k
- g_1(\alpha_L^\dagger\delta a_L + \alpha_L\delta a_L^\dagger - \alpha_M^\dagger\delta a_M - \alpha_M\delta a_M^\dagger)(\delta b_1 + \delta b_1^\dagger)
+ g_2(\alpha_R^\dagger\delta a_R + \alpha_R\delta a_R^\dagger - \alpha_M^\dagger\delta a_M - \alpha_M\delta a_M^\dagger)(\delta b_2 + \delta b_2^\dagger)
- (J_1\delta a_L^\dagger\delta a_M + J_2\delta a_R^\dagger\delta a_M + H.c.).
$$

(B7)
Then, choosing $H_0 = \sum_{j=L,M,R} \Delta_j' \alpha_j^\dagger \alpha_j + \sum_{k=1,2} \omega_{m,k} \delta b_k^\dagger b_k$ as the unperturbed Hamiltonian, performing the unitary transformation $U = \exp(-i H_0 t)$, we obtain the Hamiltonian in the interaction picture as

$$H_i = U^\dagger H U = -g_1(\alpha_L^\dagger \delta a_L e^{-i\Delta'_L t} + \alpha \delta a_L^\dagger e^{i\Delta'_L t})(\delta b_1 e^{-i\omega_m t} + \delta b_1^\dagger e^{i\omega_m t})$$
$$+ g_1(\alpha_M^\dagger \delta a_M e^{-i\Delta'_M t} + \alpha \delta a_M^\dagger e^{i\Delta'_M t})(\delta b_2 e^{-i\omega_m t} + \delta b_2^\dagger e^{i\omega_m t})$$
$$+ g_2(\alpha_R^\dagger \delta a_R e^{-i\Delta'_R t} + \alpha \delta a_R^\dagger e^{i\Delta'_R t})(\delta b_2 e^{-i\omega_m t} + \delta b_2^\dagger e^{i\omega_m t})$$
$$- \{J_1 \delta a_L^\dagger \delta a_M e^{i(\Delta'_L-\Delta'_R) t} + J_2 \delta a_R^\dagger \delta a_M e^{i(\Delta'_L-\Delta'_R) t}\}. \quad \text{(B8)}$$

Considering the weak-coupling condition that $\Delta'_L, \omega_{m,k} \gg |g_1 \alpha_M|, |g_2 \alpha_R|$, we can perform the rotating-wave approximation, and we obtain the effective Hamiltonian

$$H_i \simeq -g_1(\alpha_L^\dagger \delta a_L \delta b_1^\dagger e^{i(\omega_m-\Delta'_L) t} + \alpha \delta a_L^\dagger \delta b_2^\dagger e^{i(\omega_m-\Delta'_L) t})$$
$$+ g_2(\alpha_R^\dagger \delta a_R \delta b_2^\dagger e^{i(\omega_m-\Delta'_R) t}) - \{J_1 \delta a_L^\dagger \delta a_M e^{i(\Delta'_L-\Delta'_R) t} + J_2 \delta a_R^\dagger \delta a_M e^{i(\Delta'_L-\Delta'_R) t}\} + H.c. \quad \text{(B9)}$$

Obviously, when we choose $\Delta'_L = \omega_{m,k} = \Delta_0$, the final effective Hamiltonian in Eq. (1) is obtained.

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