FOLDING MAPS ON A CROSS-CAP

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Abstract. We study the reflectional symmetry of a surface in the Euclidean 3-dimensional space with a cross-cap singularity with respect to planes. This symmetry is picked up by the singularities of folding maps on the cross-cap. We give a list of the generic singularities that appear in the members of the family of folding maps on a cross-cap and characterise them geometrically.

1. Introduction

Given a plane $W$ in the Euclidean 3-space $\mathbb{R}^3$ with normal vector $\eta$, the folding map with respect to $W$ is the map $f_W : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f_W(p) = q + \lambda^2 \eta,$$

where $q$ is the projection of $p$ into $W$ along $\eta$ and $\lambda$ is the distance between $p$ and $W$. Thus, $p$ and its reflection with respect to $W$ have the same image under $f_W$. Varying the plane $W$ gives the family of folding maps.

Bruce and Wilkinson studied in [7, 21] the family of folding maps restricted to a smooth surface $M$ in $\mathbb{R}^3$. A member of the family is locally a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$. They showed that the singularities types of the folding map, when allowing smooth changes of coordinates in the source and target, capture some aspect of the extrinsic geometry of the surface $M$. For instance, the folding map is singular at $p \in M$ if and only if $W$ is a member of the pencil of planes containing the normal line of $M$ at $p$. The singularity is more degenerate than a cross-cap if and only if $W$ is orthogonal to a principal direction of $M$ at $p$. A certain type of singularities of the folding maps $(S_2)$ occurs along a curve on the surface called the sub-parabolic curve. It turns out that the sub-parabolic curve is the locus of points where a principal curvature is extremal along the lines of the other principal curvature. Another singularity type $(B_2)$ occurs on the ridge curve which is the locus of points where a principal curvature is extremal along its own lines of principal curvature. Bruce and Wilkinson also proved that the bifurcation set of the family if folding maps is the dual of the union of the focal and symmetry sets of $M$. This captures, for instance, the parabolic set on the
focal surface of $M$. An analogous work was carried out in [12] for surfaces in the hyperbolic or the Sitter 3-dimensional spaces.

The cross-cap singularity occurs stably on parametrized surfaces in $\mathbb{R}^3$. The extrinsic differential geometry of the cross-cap was initiated by Bruce and West in [6, 19]. Since then, several work was carried out on the subject; see for example [2, 3, 8, 9, 10, 18]. This paper is part of this ongoing work on the geometry of the cross-cap. We consider here the reflectional symmetry of a cross-cap with respect to planes and consider folding maps on the cross-cap. We obtain the list of generic singularities of these folding maps (Theorem 3.3) and describe their associated geometry (Theorem 4.3). We give some preliminaries in Section 1 and 2.

2. Preliminaries

2.1. Geometric cross-cap. Whitney showed that maps $\mathbb{R}^2 \to \mathbb{R}^3$ can have a stable local singularity under smooth changes of coordinates in the source and target. A model of this singularity is given by $f(x, y) = (x, xy, y^2)$. A map germ $g : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ is said to have a cross-cap singularity if it is $A$-equivalent to $f$ (we write $f \sim_A g$), that is, there exist germs of diffeomorphisms $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and $k : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$, such that $k \circ f = g \circ h$. The image of the map-germ $g$ is a germ of a singular surface called cross-cap.

West showed in [19] that a parametrization of a cross-cap can be taken, by a suitable choice of a coordinate system in the source and isometries in the target, in the form

$$
\phi(x, y) = (x, xy + p(y), ax^2 + bxy + y^2 + q(x, y)),
$$

where $a$ and $b$ are constant real numbers. We write

$$
p(y) = p_3 y^3 + p_4 y^4 + O(5),
q(x, y) = \sum_{k=0}^3 q_{3k} x^{3-k} y^k + \sum_{k=0}^4 q_{4k} x^{4-k} y^k + O(5),
$$

where $O(l)$ denotes a reminder in $(x, y)$ of order $l$.

A cross-cap with a parametrization as in (2) is called geometric cross-cap (in contrast to the $A$-model $f$ above where the geometry is destroyed by the diffeomorphisms in the target).

A key feature of the cross-cap is its double point curve. It is a regular curve in the source given by $x = -p_3 y^2 + \text{h.o.t.}$ for a cross-cap parametrized as in (2) (see [19]). This curve is mapped by $\phi$ to a segment of a curve ending at the cross-cap point, see Figure 1.

The tangent space to the cross-cap at the cross-cap point is one dimensional. We called this space the tangential line. The orthogonal complement to the tangential line is called the normal plane. The tangent cone at the cross-cap point is also an important object for the study of the differential
geometry of the cross-cap. For a cross-cap with a parametrization as in (2) the tangential line is parallel to the $x$-axis, the tangent cone coincides with the $xz$-plane and the normal plane is the $yz$-plane (see Figure 1).

![Figure 1. The tangential line, tangent cone and normal plane to an elliptic cross-cap.](image)

The parabolic curve on a geometric cross-cap is studied in [6, 19]. When $a \neq 0$, the parabolic set in the source has a Morse singularity. If $a < 0$, the parabolic set is an isolated point (the cross-cap point) and every regular point on the surface is hyperbolic. The cross-cap is called a hyperbolic cross-cap (Figure 2, right). If $a > 0$, the parabolic set in the source is the union of two transverse curves and the cross-cap is called an elliptic cross-cap (Figure 2, left). When $a = 0$, the cross-cap is called a parabolic cross-cap (Figure 2, center). For a generic parabolic cross-cap, the parabolic set in the source has a cusp singularity. We say that a geometric cross-cap is generic if $a \neq 0$.

Let $p$ be a regular point on the cross-cap surface, and suppose that it is not an umbilic point. Let $v_i$ be a principal direction and $\kappa_i$ its associated principal curvature at $p$, for $i = 1, 2$. Denote by $v_i(\kappa_j)$ the directional derivative of $\kappa_j$ along $v_i$, $i, j = 1, 2$. The point $p$ is a sub-parabolic point relative to $v_i$ if $v_i(\kappa_j)(p) = 0$, $i \neq j$. Similarly, $p$ is a ridge point relative to $v_i$ if $v_i(\kappa_i)(p) = 0$.

**Proposition 2.1 ([19])**. Suppose that $C$ is a regular curve in the source that passes through the origin, parametrised by $\gamma(t) = (\alpha t + O(2), \beta t + O(2))$. Then as we approach the cross-cap point along the curve parametrised by...
Figure 2. Geometric cross-caps ([6] [19]).

\[ \phi \circ \gamma \text{ (for } \phi \text{ as in [2]) one principal curvature tends to} \]
\[ \frac{2(\alpha \alpha^2 - \beta^2)}{\alpha (\alpha^2 + (2\beta + \alpha b)^2)^{\frac{3}{2}}} \]
\[ \text{and the other tends to infinity.} \]

We denote by \( \kappa_1 \) the principal curvature that tends to (3) and by \( \kappa_2 \) the other one. The limiting principal vectors \( \nu_i \) associate to \( \kappa_i \) are defined in [9] by a natural limiting process. The ridge curve on a cross-cap is studied in [9] and [19] and the sub-parabolic curve in [9].

It is shown in [9] that there is at least one and at most three regular sub-parabolic curves relative to \( \nu_2 \) and there are no sub-parabolic points relative to \( \nu_1 \) at the cross-cap. Denote by \( (w_1, w_2) \) the tangent directions of these curves in the source, then
\[ abw_1^3 + (b^2 + 2a + 1)w_1^2w_2 + 3bw_1w_2^2 + 2w_2^3 = 0, \]
where \( a, b \) are the coefficients in the parametrisation (2). There are two ridge curves relative to \( \nu_2 \) at the cross-cap and their tangent directions \( (w_1, w_2) \) in the source satisfy
\[ (bw_1 - 2w_2)w_1 = 0. \]

In [18] the author studied the lines of principal curvatures on a cross-cap and showed that there is generically one single topological model. In particular, there are three separatrices of the foliations, one is given by \( y = -\frac{1}{2}q_1x^2 + \text{h.o.t.} \) and the other two are given by \( x = \lambda_i y^2 + \text{h.o.t.}, \) \( i = 1, 2, \) with
\[ \lambda_1^2 + 3p_3\lambda_i - 2 = 0. \]
3. The singularities of the folding maps on a cross-cap

Consider a plane in $\mathbb{R}^3$ whose points $p$ satisfy

$$W_{(\eta, \delta)} : \langle p, \eta \rangle = \delta,$$

with $\eta \in S^2$ and $\delta \in \mathbb{R}$. Varying $\eta$ and $\delta$ gives all the planes in $\mathbb{R}^3$. For a fixed geometric cross-cap $\phi$ as in [2], the family $F : \mathbb{R}^2 \times S^2 \times \mathbb{R} \to \mathbb{R}^3$ of folding maps on the cross-cap is given by

$$F(x, y, \eta, \delta) = \phi(x, y) + ((\eta, \phi(x, y)) - \delta)((\eta, \phi(x, y)) - \delta - 1)\eta.$$

We write $f_{(\eta, \delta)}$ for the restriction of $F$ to the plane $W_{(\eta, \delta)}$. This is locally a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$. The singularities of such map-germs under the action of the group $A$ of smooth changes of coordinates in the source and target are classified by Mond in [14, 15]. We determine here the $A$-singularities of the map-germs $f_{(\eta, \delta)}$. Before that, we need some notation.

Let $\mathcal{E}_n$ denote the local ring of germs of functions $\mathbb{R}^n, 0 \to \mathbb{R}$ and $\mathcal{M}_n$ its maximal ideal. Denote by $\mathcal{E}(2, 3)$ the 3-tuples of elements in $\mathcal{E}_2$. The tangent space to the $A$-orbit of $f : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ at the germ $f$ is given by

$$TA \cdot f = \mathcal{M}_2 \cdot \{f_x, f_y\} + f^* (\mathcal{M}_3) \cdot \{e_1, e_2, e_3\},$$

where $f_x$ and $f_y$ are the partial derivatives of $f$, $\{e_1, e_2, e_3\}$ denotes the standard basis vectors of $\mathbb{R}^3$ considered as elements of $\mathcal{E}(2, 3)$ and $f^* (\mathcal{M}_3)$ is the pull-back of the maximal ideal in $\mathcal{E}_3$.

The extended tangent space is defined as

$$TeA \cdot f = \mathcal{E}_2 \cdot \{f_x, f_y\} + f^* (\mathcal{E}_3) \cdot \{e_1, e_2, e_3\},$$

and the $A_e$-codimension of the germ $f$ is

$$A_e \text{-codim } (f) = \dim_{\mathbb{R}} \frac{\mathcal{E}(2, 3)}{TeA \cdot f}.$$

Let $f \in \mathcal{M}_n \cdot \mathcal{E}(n, p)$. A $q$-parameter unfolding $(q, F)$ of $f$ is a map-germ $F : \mathbb{R}^n \times \mathbb{R}^q, (0, 0) \to \mathbb{R}^p \times \mathbb{R}^q, (0, 0)$ in the form $F(x, u) = (\bar{f}(x, u), u)$, with $\bar{f}(x, 0) = f(x)$. The family $\bar{f} : \mathbb{R}^n \times \mathbb{R}^q, (0, 0) \to \mathbb{R}^p, 0$ is called a $q$-parameter deformation of $f$. Let $I$ be the identity element in $A$. A morphism between two unfoldings $(q_1, F)$ and $(q_2, G)$ is a pair $(\alpha, \psi) : (q_1, F) \to (q_2, G)$ with $\alpha : \mathbb{R}^{q_1}, 0 \to A, I, \psi : \mathbb{R}^{q_1}, 0 \to \mathbb{R}^{q_2}, 0$, such that $\bar{f}_u = \alpha(u) \cdot \bar{g}_{\psi(u)}$. The unfolding $(q_1, F)$ is then said to be induced from $(q_2, G)$ by $(\alpha, \psi)$. An unfolding $(q_1, F)$ of a map-germ $f$ is said to be $A_e$-versal if any unfolding $(q_2, G)$ of $f$ can be induced from $(q_1, F)$. 
Now we state a fundamental theorem on unfoldings. Given an unfolding \( F(x,u) = (f(x,u),u) \), the initial speeds, \( \dot{F}_i \in \mathcal{E}(n,p) \), of \( F \) are defined by
\[
\dot{F}_i(x) = \frac{\partial f}{\partial u_i}(x,0), \quad \text{for} \quad i = 1, \ldots, q.
\]

**Theorem 3.1** ([13]). An unfolding \((q,F)\) of \( f \in \mathcal{M}_n \cdot \mathcal{E}(n,p) \) is \( A_e\)-versal if and only if
\[
T_eA \cdot f + R \cdot \{\dot{F}_1, \ldots, \dot{F}_q\} = \mathcal{E}(n,p).
\]

Our goal is to classify germs of folding maps on a cross-cap and to analyze the deformations in the members of the family of folding maps. We start with the following result.

**Proposition 3.2.** For any \((\eta,\delta) \in S^2 \times \mathbb{R}\), the folding map \( f(\eta,\delta) \) on a cross-cap as in (2) is singular at the origin. The singularity is more degenerate than a cross-cap if, and only if, the plane \( W(\eta,\delta) \) contains the origin, that is, \( \delta = 0 \).

**Proof.** The first part follows by observing that \( \frac{\partial f(\eta,\delta)}{\partial y}(0,0) = (0, 0, 0) \) for all \((\eta,\delta) \in S^2 \times \mathbb{R}\), thus rank \( df(\eta,\delta)(0,0) \leq 1 \). For the second part we use Whitney’s criteria for recognition of the cross-cap singularity (see [20]).

We list below the non-stable \( A \)-singularities that occur at the origin in the members of the family of folding maps on a cross-cap. By Proposition 3.2 above, \( \delta = 0 \). We denote \( f(\eta,0) \) by \( f_\eta \) and write \( \eta = (\alpha, \beta, \gamma) \).

**Theorem 3.3.** For a generic cross-cap \( \phi \) as in (2), the folding map \( f_\eta \) has a singularity \( A \)-equivalent to one in Table 1 when \( \eta \) is transverse to the tangent cone, and to one in Table 2 when \( \eta \) is in the tangent cone but is not parallel to the tangential direction.

When \( \eta \) is parallel to the tangential direction, \( f_\eta \) has corank 2 and is \( A \)-equivalent to
\[
(x^2, xy + y^3, y^2 + Ax^3 + Bx^2y + Cxy^2 + y^3),
\]
with \( \Theta(A,B,C) \neq 0 \), where
\[
\Theta(A,B,C) = -10240A^3 + (-2560C^2 - 3840B - 1440C - 135)A^2 +
+ (-160C^4 + 2880B^2C + 800BC^2 - 20C^3 + 60B^2 +
+ 90BC)A + 40B^2C^3 - 540B^4 - 180B^3C + 5B^2C^2 - 20B^3.
\]
Proposition 3.4. The non-stable singularities of \( f_\eta \) of the calculations can be found in [2]. The conditions for the folding map \( \eta \) cross-caps.

where \( \Phi(\beta, \gamma) = -2b\beta^3 + (4a - b^2 + 2)\beta^2\gamma + \gamma^3 \) and \( F_{1,0} \) is a unimodal singularity (see [14, page 29]).

\((*)\) The singularity \( B_1^\pm \) occurs for generic \( \eta \) on the plane curve \( \gamma = p_3\alpha \) in \( S^2 \).

\((**)*\) The singularity \( B_1^\pm \) occurs for special values of \( \eta \) on the curve \( \gamma = p_3\alpha \) whose expression too lengthy to reproduce here, see [2] for details.

Table 1. Singularities of the folding map \( f_\eta \) when \( \eta \) transverse to the tangent cone.

| Type  | Normal form                                  | \( A_\alpha \)-cod. | Conditions                  |
|-------|----------------------------------------------|---------------------|-----------------------------|
| \( B_1^\pm \) | \((x, y^2, x^2y \pm y^3)\)                   | 2                   | \( \alpha \neq 0, \gamma \neq p_3\alpha \) |
| \( B_3^\pm \) | \((x, y^2, x^2y \pm y^7)\)                   | 3                   | \( \alpha \neq 0, \gamma = p_3\alpha \), (*) |
| \( B_3^\pm \) | \((x, y^2, x^2y \pm y^9)\)                   | 4                   | \( \alpha \neq 0, \gamma = p_3\alpha \), (**) |
| \( C_3^\pm \) | \((x, y^2, xy^3 \pm x^3y)\)                  | 3                   | \( \alpha = 0, \Phi(\beta, \gamma) \neq 0 \) |
| \( C_4^\pm \) | \((x, y^2, xy^3 \pm x^4y)\)                  | 4                   | \( \alpha = 0, \Phi(\beta, \gamma) = 0 \) |
| \( F_{1,0} \) | \((x, y^2, x^3y + A_1xy^3 + B_1y^7)\)       | 4                   | \( \beta = 1, 4A_1^3 + 27B_1^2 \neq 0 \) |

Table 2. Singularities of the folding map \( f_\eta \) when \( \eta \) is in the tangent cone but is not parallel to the tangential direction.

| Type  | Normal form                                  | \( A_\alpha \)-cod. | Conditions                  |
|-------|----------------------------------------------|---------------------|-----------------------------|
| \( P_1 \) | \((x, xy + y^3, xy^2 + ky^4)\)               | 3                   | \( \alpha \neq -p_3\gamma, k \neq \frac{\gamma}{2}, 1, \frac{\gamma}{2} \) |
| \( P_4(\frac{1}{6}) \) | \((x, xy + y^6, xy^2 + \frac{1}{3}y^6)\)   | 4                   | \( \alpha \neq -p_3\gamma, \Phi(\frac{1}{6}) = 0 \) |
| \( P_4(\frac{1}{6}) \) | \((x, xy + y^6, x^2y + \frac{1}{3}y^4)\)   | 4                   | \( \alpha \neq -p_3\gamma, \Phi(\frac{1}{6}) = 0 \) |
| \( P_4(1) \) | \((x, xy + y^6, xy^2 + y^4)\)               | 4                   | \( \alpha \neq -p_3\gamma, \Psi(1) = 0 \) |
| \( R_4 \) | \((x, xy + y^6 + A_2y^7, xy^2 + y^4 + B_2y^6)\) | 4                   | \( \alpha = -p_3\gamma \) |
| \( T_4 \) | \((x, xy + y^3, y^4)\)                      | 4                   | \( \gamma = 1 \) |

where \( \Psi(k) = \Psi(k, \alpha, \gamma) = -2k\alpha(\alpha + p_3\gamma) + 1 \).

Proof. The proof follows by making successive changes of coordinates on the jet level and using the conditions for a map-germ to be \( A \)-equivalent to one in Mond’s list. When \( \eta \) is parallel to the tangential direction, the condition \( \Theta(A, B, C) \neq 0 \) is for the germ [7] to be \( 3\cdot A_1 \)-determined. Details of the calculations can be found in [2]. The conditions for the folding map \( f_\eta \) to have the different types of singularities listed are given on the vector \( \eta \), therefore the result is true for an open set in the space of geometric cross-caps.

Proposition 3.4. The non-stable singularities of \( f_\eta \) are not \( A_\alpha \)-versally unfolded by the family \( F \).
Proposition 3.2, the map germ \( f \)

It follows that \( \xi = (u, v, w) \in \mathbb{R}^3 \) and consider a real number \( \delta \neq 0, |\delta| < \varepsilon \) (\( \varepsilon \) small). Thus, \( f(\eta, \delta) \) with \( \bar{\eta} = \frac{\eta_0 + \xi}{|\eta_0 + \xi|} \in \mathbb{S}^2 \) is a deformation of \( (\eta_0, 0) \). Let \( F(x, y, u, v, w, \delta) = (f(x, y, u, v, w, \delta), u, v, w, \delta) \), where \( f(x, y, u, v, w, \delta) = f(\eta, \delta)(x, y) \) and 

\[
  f(x, y, 0, 0, 0, 0) = f(\eta_0, 0)(x, y).
\]

It follows that \( F \) is a 4-parameter unfolding of the map germ \( f(\eta_0, 0) \). By Proposition 3.2, the map germ \( f(\eta, \delta) \) has singularity type cross-cap at origin. By changes of coordinates in the source and target, the map germ \( j^2 f(\eta_0, 0) \) is \( \mathcal{A} \)-equivalent to

\[
  (8) \quad j^2 f(\eta_0, 0) \sim_{\mathcal{A}} (x, -\gamma_0 (\gamma_0 - b\beta_0) x y + \beta_0 \gamma_0 y^2, \beta_0 (\gamma_0 - b\beta_0) x y - \beta_0^2 y^2).
\]

We consider the 2-jet of \( f(\eta_0, 0) \) since the map germ \( f(\eta, \delta) \), with \( \delta \neq 0 \), has a cross-cap singularity. We take, without loss of generality, \( f(\eta_0, 0) \) as \( \Theta \).

Thus, the initial speeds of \( F \) are given by

\[
  \begin{align*}
    \dot{F}_u(x, y) &= (0, 2\alpha_0 xy, 2b\alpha_0 xy + 2\alpha_0 y^2), \\
    \dot{F}_v(x, y) &= (0, (b\gamma_0 + 2\beta_0) xy + \gamma_0 y^2, \gamma_0 xy), \\
    \dot{F}_w(x, y) &= (0, b\beta_0 xy + \beta_0 y^2, (2b\gamma_0 + \beta_0) xy + 2\gamma_0 y^2), \\
    \dot{F}_\delta(x, y) &= (0, (2b\beta_0 \gamma_0 + 2(1 - \gamma_0^2)) xy + 2\gamma_0 \beta_0 y^2, \\
                      & \quad 2(\beta_0 \gamma_0 + b(1 - \beta_0^2)) xy + 2(1 - \beta_0^2) y^2).
  \end{align*}
\]

Note that the Jacobian ideal of \( j^2 f(\eta_0, 0) \) is generated by

\[
  (j^2 f(\eta_0, 0))_x = (1, -\gamma_0 (\gamma_0 - b\beta_0) y, \beta_0 (\gamma_0 - b\beta_0) y), \\
  (j^2 f(\eta_0, 0))_y = (0, -\gamma_0 (\gamma_0 - b\beta_0) x + 2\beta_0 \gamma_0 y, \beta_0 (\gamma_0 - b\beta_0) x - 2\beta_0^2 y).
\]

It follows that

\[
  \begin{align*}
    (j^2 f(\eta_0, 0))_x - (j^2 f(\eta_0, 0))^* (1) \cdot e_1 &= (0, -\gamma_0 (\gamma_0 - b\beta_0) y, \beta_0 (\gamma_0 - b\beta_0) y), \\
    (j^2 f(\eta_0, 0))_y + (\gamma_0 - b\beta_0) (j^2 f(\eta_0, 0))^* (x) \cdot (\gamma_0 e_2 - \beta_0 e_3) &= (0, 2\beta_0 \gamma_0 y - 2\beta_0^2 y).
  \end{align*}
\]

It follows that it is not possible to simultaneously obtain the vectors \((0, y, 0)\) and \((0, 0, y)\) in the extended tangent space \( T_e \mathcal{A}(j^2 f(\eta_0, 0)) \). Therefore \( T_e \mathcal{A}(j^2 f(\eta_0, 0)) + \mathbb{R} \{ \dot{F}_u, \dot{F}_v, \dot{F}_w, \dot{F}_\delta \} \neq \mathcal{E}(2, 3) \), which concludes the proof.

\[\square\]

4. The geometry of the folding maps on a cross-cap

In this section we characterize geometrically some singularities that occur in the members of the family of folding maps on a cross-cap. We use the concepts in \( \Theta \) and the maps which we define below.

Let \( \theta(t) = (t, \alpha(t)) \) be a germ of a regular curve with \( \alpha'(0) = a_1 \neq 0 \) and consider its image \( \mu(t) = \phi \circ \theta(t) \) on the cross-cap. Note that \( \mu \) is a
regular curve and \( \mu'(0) \) is parallel to the tangential line. When \( \alpha'(0) = 0 \) the image on the cross-cap is a singular curve. Curves with this condition will be considered later.

Let \( N(p) \) denote the unit normal vector to the cross-cap surface away from the cross-cap point. At the cross-cap point, the surface has no well defined normal vector, that is, \( N(p) \) does not extend to the cross-cap point.

However, a simple calculation shows that

\[
\lim_{t \to 0} N(\mu(t)) = \left(0, -\frac{b+2a_1}{(b+2a_1)^2+1}^{\frac{3}{2}}, \frac{1}{(b+2a_1)^2+1}^{\frac{1}{2}}\right).
\]

Therefore, the normal vector has a limiting direction along the curve \( \mu \) at the cross-cap point, and this limiting direction is parallel to \((0, -(b+2a_1), 1)\).

Observe that the limiting direction depends only on \( \theta'(0) \) and not on the curve \( \mu \). Thus, we can define the following map.

**Definition 4.1.** Let \( X \) be a cross-cap as in (2) and let \( N_0X \) denote its normal plane at the origin. We call the limiting normal map of \( X \) the map

\[ LN: \mathbb{R}^2 \to N_0X \text{ given by} \]

\[ LN(v_1, v_2) = (0, -(bv_1 + 2v_2), v_1) . \]

The limiting normal map induces a bijection between \( S^1 \subset \mathbb{R}^2 \) and \( S^1 \subset N(0,0,0)X \).

Consider now the family of curves \( \varpi_\lambda(t) = (\beta_\lambda(t), t) \), with \( \beta_\lambda(0) = \beta'_\lambda(0) = 0 \) and \( \beta''_\lambda(0) = \lambda \). We have \( \varpi'_\lambda(0) = (0,1) \) and \( LN(0,1) = (0,-2,0) \).

(The direction \( LN(0,1) \) is orthogonal to the tangent cone of the cross-cap.)

Thus, the limiting normal direction to the cross-cap is the same for all the members of the family of curves \( \varpi_\lambda \). The image of \( \varpi_\lambda(t) \) by \( \phi \) is a singular curve on the cross-cap and its limiting tangent direction belongs to the tangent cone \( TC_0X \) of the cross-cap. More precisely,

\[
\lim_{t \to 0} (\phi \circ \varpi_\lambda)' (t) = (2\lambda, 0, 2) .
\]

**Definition 4.2.** We call the limiting tangent map of the cross-cap as in (2) the map \( LT: \mathbb{R}^2 \to TC_0X \) given by

\[ LT(\lambda, 1) = \frac{1}{2} \lim_{t \to 0} (\phi \circ \varpi_\lambda)' (t) = (\lambda, 0, 1) .\]

We use the limiting tangent and normal maps of the cross-cap to characterise the singularities of the folding maps.

**Theorem 4.3.** Let \( X \) be a cross-cap parametrised as in (2). We have the following characterization of the singularities of the folding maps \( f_\eta \) on \( X \).
(i) The $C_4$ singularity occurs when folding with respect to a plane generated by the tangential direction of the cross-cap and $LN(w)$, where $w$ is a tangent direction at the origin to a sub-parabolic curve.

(ii) The $F_{1,0}$ and $T_4$ singularities occur when folding with respect to the plane generated by the tangential direction of the cross-cap and $LN(w)$, where the $w$ is a tangent direction at the origin to a ridge curve relative to $v_2$. In particular $f_\eta$ has a singularity type $F_{1,0}$ when $\eta$ is orthogonal to the tangent cone.

(iii) The $P_{4}(\frac{1}{2})$ singularity occurs when folding with respect to the plane generated by $LN(0, 1)$ and the limiting tangent direction to the double point curve.

(iv) The $P_{4}(\frac{3}{2})$ singularity occurs when folding with respect to the plane generated by $LN(0, 1)$ and the limiting tangent direction to the separatrix of principal foliations.

(v) When $\eta$ is parallel to the limiting tangent direction of the double point curve, the singularity of $f_\eta$ is of type $R_4$.

(vi) The corank 2 map singularity of $f_\eta$ occur when $\eta$ is parallel to the tangential direction, i.e., the folding plane is the normal plane.

Proof. We will characterize, in each case, the normal direction $\eta$ to the folding plane.

(i) Consider the map germ $f_\eta$ such that $\eta = (0, \beta, \gamma)$ satisfying $\Phi(\beta, \gamma) = 0$ as in the Theorem 3.3. By direct calculations we have that

$$(LN)^{-1}((0, -\gamma, \beta)) = (w_1, w_2),$$

where $\lambda, \mu$ satisfies

$$abw_1^3 + (b^2 + 2a + 1)w_2w_1^2 + 3bw_1^2w_2^2 + 2w_2^3 = 0,$$

which is the same condition in the equation (4).

(ii) Take the map germ $f_\eta$ such that $\eta = (0, 1, 0)$, thus by the Theorem 3.3, $f_\eta$ has a $F_{1,0}$ singularity at the origin. Follows that

$$(LN)^{-1}(0, 0, 1) = \left(1, -\frac{b}{2}\right) = (w_1, w_2)$$

satisfying $bw_1 + 2w_2 = 0$,

which satisfies the equation (5). Now consider the map germ $f_\eta$ such that $\eta = (0, 0, 1)$ and again by the Theorem 3.3, $f_\eta$ has a $T_4$ singularity at the origin. Using the same idea

$$(LN)^{-1}(0, 1, 0) = \left(0, -\frac{1}{2}\right) = (w_1, w_2)$$

satisfying $w_1 = 0$,

which is the other one condition that satisfy the equation (5).
(iii) Consider the map germ \( f_\eta \) such that \( \eta = (\alpha, 0, \gamma) \) satisfying \( \Psi \left( \frac{1}{2} \right) = 0 \) and \( \alpha \neq -p_3 \gamma \). Note that

\[
\Psi \left( \frac{1}{2}, \alpha, \gamma \right) = -\alpha \gamma p_3 + (1 - \alpha^2) = \gamma (\gamma - p_3 \alpha).
\]

Again we consider the orthogonal \( \eta \) in the tangent cone and applying to the equation (9) and equaling to zero we obtain

\[
\Psi \left( \frac{1}{2}, -\gamma, \alpha \right) = \alpha (\alpha + p_3 \gamma) = 0.
\]

As \( \alpha \neq 0 \) follows that \( \alpha + p_3 \gamma = 0 \). Calculating \((LT)^{-1}(-p_3, 0, 1) = (-p_3, 1)\), which correspond with the 2-jet of curve of double points of the cross-cap \( x = -p_3 y^2 \).

(iv) Analogous as (iii) take a map germ \( f_\eta \) such that \( \eta = (\alpha, 0, \gamma) \) satisfying \( \Psi \left( \frac{3}{2} \right) = 0 \) and \( \alpha \neq -p_3 \gamma \). It follows that

\[
\Psi \left( \frac{3}{2}, \alpha, \gamma \right) = -3\alpha \gamma p_3 - 3\alpha^2 + 1 = -3\alpha \gamma p_3 - 2\alpha^2 + \gamma^2.
\]

As before we consider the orthogonal \( \eta \) in the tangent cone and applying to the equation (10) we obtain

\[
\Psi \left( \frac{3}{2}, -\gamma, \alpha \right) = 3\alpha \gamma p_3 + \alpha^2 - 2\gamma^2 = \lambda^2 + 3p_3 \lambda - 2 = 0,
\]

where \( \lambda = \frac{\alpha}{\gamma} \). Via \((LT)^{-1}\), the vectors \(( -\gamma, 0, \alpha ) \) in the tangent cone satisfying the equation (11) correspond with the curves \( x = \lambda y^2 \) where \( \lambda \) satisfies the same equation above. (Compare with the equation (6)).

(v) As \( \eta \) is tangent to \( (-p_3, 0, 1) \), in particular satisfies \( \alpha = -p_3 \gamma \), which is exactly the condition in the Theorem 3.3 for the germ \( f_\eta \) to have a singularity at the origin of type \( R_4 \).

(vi) In this case \( \eta = (1, 0, 0) \) and by definition, \( f_\eta \) is the folding map respect to the normal plane. The rest follows from Theorem 3.3. \( \Box \)

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