Autonomous Brownian gyrators: a study on gyrating characteristics

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We study the nonequilibrium steady-state (NESS) dynamics of two-dimensional Brownian gyrators under harmonic and nonharmonic potentials via computer simulations and Fokker-Planck-equation analyses, while our nonharmonic cases feature a double-well potential and an isotropic quartic potential. In particular, we report two simple methods that can help understand gyrating patterns. For harmonic potentials, we use the Fokker-Planck equation to survey the NESS dynamical characteristics, i.e., the NESS currents gyrate along the equiprobability contours and the stationary point coincides with the potential minimum. As a contrast, the NESS results in our nonharmonic potentials show that these properties are largely absent, as the gyrating patterns are much distinct from those of corresponding probability distributions. Furthermore, we observe a critical case of the double-well potential, where the harmonic contribution to the gyrating pattern becomes absent, and the NESS currents do not circulate about the equipotential contours nearby the potential minima even at low temperatures.

I. INTRODUCTION

The recent discussions over autonomous Brownian gyrators[1–7] have shed light on the development of autonomous Brownian engines[9–17]. In the simplest description of such systems, a Brownian particle performs two-dimensional random walks under the influence of some conservative potential, while thermal fluctuations of unequal strength are supplied along the two cartesian coordinates, respectively. The resulting dynamics is signified by an average nonzero, circulating flow field. The autonomous gyrating behavior makes it a promising candidate for a microscopic heat engine. Experimentally, realization of a Brownian gyrorator was demonstrated through a colloidal system using optical tweezers along with artificial noises[6]. Meanwhile, it has been noticed that a coupled RC circuit system with thermal noises can serve as a complete analog to a Brownian gyrorator under a harmonic potential[7,8]. Similar mechanical and electrical realizations of these autonomous engines with the use of artificial noises were reported and often featured with less trivial interactions[13,16].

Theory-wise, the nonequilibrium steady-state (NESS) dynamics for harmonic potentials have been studied quite thoroughly[2,4,5,7,12,18]. In particular, these studies brought about the important features that the NESS currents gyrate about the equiprobability contours, and the stationary points coincide with the potential minima. These successes in harmonic systems also lead to a keen desire in the investigation over a broader range of potentials. On the one hand, one should ask whether these traits from harmonic results still persist regarding more general potential cases. On the other hand, the nonlinearity in potentials often brings novelty to the NESS dynamics. For example, it has been mentioned in Ref. [18] that the anharmonicity in potentials can act as “extranormal” currents that shifts the stationary point of the correspondingly harmonic problems. And therefore, unlike the results of harmonic potentials, the overall currents do not necessarily gyrate about the potential contours. Furthermore, new gyrating patterns may emerge due to the existence of anharmonicity[19].

While there have been numerous theoretical efforts towards the general treatment in NESS dynamics[21–27], we choose to focus our current work on a two-dimensional Brownian system under some conservative potential, due to the desire to understand the gyrorator characteristics. The paper is organized as follows. First we introduce in Sec. III our system of interest, and its corresponding Fokker-Planck equation. In Sec. III we introduce two simple criteria that can examine the gyrating directions given a general potential. To study the gyrorator characteristics, we first re-derive in Sec. IV the NESS dynamics for the harmonic potential, starting from its major characteristic that the NESS currents gyrate about the equiprobability contours. As for nonharmonic potentials, we consider two specific examples: a tilted double-well potential is studied in Sec. VI and an isotropic quartic potential is studied in Sec. VII. In Sec. VII we briefly discuss about the entropy change along the autonomous trajectories.

II. SYSTEM

We consider a Brownian particle in two dimensions. The corresponding Langevin equation is

\[
\gamma \ddot{\vec{r}} = -\nabla U + \vec{\xi},
\]

(1)

where \(\vec{r} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) is the position of the particle, \(U\) is the potential energy, and \(\vec{\xi}\) is the random force. The components of the random force are Gaussian white and uncorrelated, namely, \(\langle \xi_i(t)\xi_j(t') \rangle = 2k_B T \delta_{ij} \delta(t-t')\), where \(k_B\) is the Boltzmann constant. Throughout this
or equivalently, the system does not achieve thermal equilibrium, and our focus is on its NESS characteristics.

For simplicity, we adopt the dimensionless convention that $k_B = 1$ and $\gamma = 1$. The latter can be achieved through a rescale of time by the factor $1/\gamma$. The probability distribution function of this system $P$ is described by the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \nabla \cdot \hat{D} \nabla P + \nabla \cdot (P \nabla U), \quad (2)$$

where $\hat{D} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ under our dimensionless description. Since we are interested in the NESS only, the left-hand side of Eq. 2 drops to zero. The flux density $\vec{J}$ is defined as

$$\vec{J} \equiv -\hat{D} \nabla P - P \nabla U. \quad (3)$$

For convenience let us denote $\phi \equiv -\log P$. Then one can rewrite the flux density as

$$\vec{J} = P(\hat{D} \nabla \phi - \nabla U) \equiv P\vec{v}, \quad (4)$$

where $\vec{v}$ is the average velocity about the steady-state current. The equation of continuity at NESS then gives

$$\nabla \cdot \vec{J} = P\nabla \cdot \vec{v} + \vec{v} \cdot \nabla P = 0, \quad (5)$$

or equivalently,

$$\nabla \cdot \vec{v} - \vec{v} \cdot \nabla \phi = 0. \quad (6)$$

Let us assume that the NESS currents gyrate about the equiprobability contours, i.e., $\vec{v} \cdot \nabla \phi = 0$. Then Eq. 5 shows that this is equivalent to the statement $\nabla \cdot \vec{v} = 0$. Therefore, the NESS flow field can be compared to that of an ideal fluid, and $d\vec{v}/dt = 0$ along any NESS trajectory. Now if we consider the point where $U$ is at its extremum. Then Eq. 4 implies that $\vec{v} = \hat{D} \nabla \phi$. Since $\hat{D}$ is positive, the assumption that $\vec{v} \cdot \nabla \phi = 0$ requires that $\phi$ is also an extremum, and therefore $\vec{v}$ must vanish at this position. The inverse statement is also true: nearby an extremum of $\phi$, since the flux is circulating about the extremum, there must be a stationary point within the circulating area. By this examination one concludes that $\vec{v} = 0$ at this extremum, and from Eq. 4 one deduces that $U$ is also an extremum. Hence if the property $\vec{v} \cdot \nabla \phi = 0$ holds, then the potential extremum, the extremum of probability distribution, and the stationary point must coincide with each other.

While the aforementioned properties are valid in systems of harmonic potentials, they are largely absent when the potential becomes nonharmonic. In this study we investigate their gyrating patterns through simulations and analytical arguments. To facilitate our discussion, we choose to study the special cases such that the potential profiles possess inversion symmetry, i.e., $U(-\vec{r}) = U(\vec{r})$.

Subsequently, one has $P(-\vec{r}) = P(\vec{r})$, and therefore $\phi(-\vec{r}) = \phi(\vec{r})$ since the random walker cannot sense any physical difference between any opposite pair of points $\vec{r}$ and $-\vec{r}$. From Eqs. 3 and 4 one also learns that $\vec{J}(-\vec{r}) = -\vec{J}(\vec{r})$ and $\vec{v}(-\vec{r}) = -\vec{v}(\vec{r})$, and $\vec{v} = 0$ at the origin. From Eq. 5 one finds that $\nabla \cdot \vec{v} = 0$ at the origin.

III. DIRECTION OF CIRCULATION

In this work we employ two methods to study the direction of circulation in the NESS flow field. First, we consider the curl of the average velocity, as from Eq. 4 one can derive

$$\nabla \times \vec{v} = \nabla \times \hat{D} \nabla \phi - \nabla \times \nabla U = -(T_1 - T_2) \partial_1 \partial_2 \phi. \quad (7)$$

Let us define the tilted axes $x'_1 \equiv \frac{x_1 + x_2}{\sqrt{2}}$ and $x'_2 \equiv \frac{x_1 - x_2}{\sqrt{2}}$, while $\partial'_1 = \frac{\partial}{\partial x'_1} = \frac{1}{\sqrt{2}} \partial_1 + \frac{1}{\sqrt{2}} \partial_2$ and $\partial'_2 = \frac{\partial}{\partial x'_2} = \frac{1}{\sqrt{2}} \partial_1 - \frac{1}{\sqrt{2}} \partial_2$. Then Eq. 7 becomes

$$\nabla \times \vec{v} = -\frac{1}{2}(T_1 - T_2)(\partial'_1^2 - \partial'_2^2) \phi. \quad (8)$$

Equation 8 tells us that the circulation of the NESS currents is correlated with the difference in curvature of $\phi$ along the directions $x'_1$ and $x'_2$. Since $T_1 > T_2$ in all our studied cases, one can observe a counterclockwise gyration ($\nabla \times \vec{v}$ being positive) if $\partial_1 \partial_2 \phi > 0$ (i.e., $\partial'_2^2 \phi > \partial'_1^2 \phi$) and vice versa.

Alternatively, the gyration direction of the NESS currents can be studied utilizing the second law of thermodynamics. Let us consider an arbitrary cycle on the $x_1 - x_2$ plane. In our overdamped system as described by Eq. 4 the net amount of heat dissipating into the reservoir $T_1$ during a cycle is $Q_1 = -\int \partial U dx_1$, and the corresponding heat dissipated into reservoir $T_2$ is $Q_2 = -\int \partial_2 U dx_2$. The fact that $\int \partial U \cdot d\vec{r} = 0$ reminds us that $Q_1 = -Q_2$. The total entropy change of an infinitesimal path is $dS_{\text{tot}} = dS_{\text{sys}} + dS_Q$, where $S_{\text{sys}} \equiv -\log P = \phi$ is the system entropy and $dS_Q = dQ_1/T_1 + dQ_2/T_2$ is the net entropy dissipation into the heat reservoirs. Since the system entropy is a state function, the total entropy change after an autonomous cycle is

$$\int dS_{\text{tot}} = \int dS_Q = - \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \int \partial_1 U dx_1 > 0, \quad (9)$$

as is required by the second law in thermodynamics. The sign in Eq. 9 indicates that only the cycles with $\int \partial_1 U dx_1 > 0$ can occur spontaneously.

Using the Stokes’ theorem, one can rewrite Eq. 9 as

$$\int dS_{\text{tot}} = -\int (\hat{D}^{-1} \nabla U) \cdot d\vec{l} = -\int \nabla \cdot (\hat{D}^{-1} \nabla U) \cdot dA \quad (10)$$
where the directions of the length and area differential elements $d\ell$ and $dA$ are chosen such that $\oint dS_{\text{tot}}$ is positive. Since $\nabla \times (\hat{\mathbf{D}}^{-1}\nabla U) = (T_1 - T_2)\partial_1\partial_2 U/(T_1T_2)$, one can conclude that the autonomous gyration is clockwise if $\partial_1\partial_2 U > 0$ (or equivalently, $\partial_1^2 U > \partial_2^2 U$) and vice versa.

The difference between these two methods is mainly attributed to the uses of $\phi$ and $U$, respectively. Despite this difference, their qualitative results are mostly similar due to the geometric resemblance between $U$ and $\phi$.

It is worth noting that one can rescale the variable $x_1$ such that the system behaved as the one with identical temperatures on both dimensions: let $y_1 \equiv x_1/c$ and $y_2 \equiv x_2$. Then one can rewrite Eq. 11 as

$$\dot{y} = \hat{\mathbf{D}}_y \nabla_y \phi + \hat{\mathbf{f}}_y,$$

where $\xi_{y,1} \equiv \xi_1/c$ and $\xi_{y,2} = \xi_2$, while $f_{y,1} = -\partial_1 U/c$ and $f_{y,2} = -\partial_2 U$. In the following we use the subscript $y$ to denote physical quantities in the $(y_1, y_2)$ coordinates. Through the choice $c = \sqrt{T_1/T_2}$, the random variables $\xi_{y,1}$ and $\xi_{y,2}$ can have identical distributions. Note that in this coordinate system the force $\hat{f}_y$ in Eq. 11 is not conservative. The NESS flow velocity under this rescaled coordinates is

$$\bar{\mathbf{v}}_y = \hat{\mathbf{D}}_y \nabla_y \phi + \hat{\mathbf{f}}_y.$$

Since the heat baths along the two dimensions are now identical, $\hat{\mathbf{D}}_y$ is proportional to the identity matrix, and therefore

$$\nabla_y \times \bar{\mathbf{v}}_y = \nabla_y \times \hat{\mathbf{f}}_y = \partial_1 f_{y,2} - \partial_2 f_{y,1} = -\frac{1}{c} (c^2 - 1) \partial_1^2 U = -\sqrt{T_1T_2} (\nabla \times \hat{\mathbf{D}}^{-1} \nabla U) = \sqrt{T_1T_2} (\nabla \times \hat{\mathbf{D}})^{-1} \nabla U).$$

Comparing Eq. 13 with Eq. 10, we find that our gyration argument via the positivity in total entropy change over a loop happens to coincide with the study on the curl of $\bar{\mathbf{v}}_y$ over the rescaled coordinates.

**IV. HARMONIC POTENTIAL**

We first revisit the harmonic potential cases using the Fokker-Planck-equation approach. In our previous study\[7\] of an autonomous Brownian gyrorator demonstrated by a coupled RC system, the potential profile is quadratic, and the resulting $\phi$ also possesses a quadratic form. For general random walks over two dimensions under harmonic potentials, one can also use the quadratic ansatz in $\phi$. Following Eq. 3 one finds that $\bar{\mathbf{v}}$ is linear in $x_1$ and $x_2$, and therefore $\nabla \cdot \bar{\mathbf{v}}$ is a constant. From our discussion at the end of Sec. II, one concludes that this constant is zero, and the NESS currents gyrate around the equiprobability contour lines.

Since $\bar{\mathbf{v}}$ is perpendicular to $\nabla \phi$ and therefore $\nabla \phi$, one can rewrite $\bar{\mathbf{v}} = -\epsilon \hat{\mathbf{Y}} \nabla \phi$, where

$$\hat{\mathbf{Y}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a $2 \times 2$ matrix that rotates $\nabla \phi$ clockwise by the angle $\pi/2$, and $\epsilon$ is a scalar function to be determined. Then Eq. 14 becomes

$$\bar{\mathbf{v}} = \hat{\mathbf{D}} \nabla \phi - \nabla U = -\epsilon \hat{\mathbf{Y}} \nabla \phi$$

or

$$(\hat{\mathbf{D}} + \epsilon \hat{\mathbf{Y}}) \nabla \phi = \nabla U.$$

Note that $\hat{\mathbf{D}} + \epsilon \hat{\mathbf{Y}}$ is equivalent to the matrix $\hat{\mathbf{G}}$ as described in Refs. 15, 16.

Substituting Eq. 15 into the relation $\nabla \cdot \bar{\mathbf{v}} = 0$, one gets (using the fact that $\hat{a} \cdot \hat{Y} \hat{b} = \hat{a} \times \hat{b} \equiv a_1b_2 - a_2b_1$ for any $\hat{a}$ and $\hat{b}$)

$$\nabla \epsilon \times \nabla \phi + \nabla \epsilon \nabla \phi = \nabla \epsilon \times \nabla \phi = 0.$$

Therefore either $\nabla \epsilon$ is parallel to $\nabla \phi$, or $\epsilon$ is simply a constant.

From Eq. 16 one has

$$0 = \nabla \times (\hat{\mathbf{D}}_a \nabla U) = \nabla (\hat{\mathbf{Y}} (\hat{\mathbf{D}}_a - \epsilon \hat{\mathbf{Y}}) \nabla U) = \frac{\nabla \hat{\mathbf{Y}} (\hat{\mathbf{D}}_a - \epsilon \hat{\mathbf{Y}}) \nabla U}{\det \hat{\mathbf{G}}} = \frac{-2 \epsilon \nabla \epsilon \times (\hat{\mathbf{D}}_a - \epsilon \hat{\mathbf{Y}}) \nabla U}{\det \hat{\mathbf{G}}} + \frac{\nabla \times (\hat{\mathbf{D}}_a \nabla U) + \nabla \cdot (\epsilon \nabla U)}{\det \hat{\mathbf{G}}},$$

where $\hat{\mathbf{D}}_a$ is the adjugate matrix of $\hat{\mathbf{D}}$. The first term of Eq. 18 is 0, and therefore one has

$$-\nabla \times (\hat{\mathbf{D}}_a \nabla U) = \nabla \epsilon \cdot \nabla + \epsilon \nabla \cdot \nabla U.$$

Since the potential is quadratic, the solution that $\epsilon$ is a constant serves as a legitimate answer for the above equation, as

$$\epsilon = \frac{-\nabla \times (\hat{\mathbf{D}}_a \nabla U)}{\nabla \epsilon \cdot \nabla} = -(T_1 - T_2) \frac{\partial_1 \partial_2 U}{\nabla \epsilon \cdot \nabla}. $$

From Eq. 16 one sees that the result that $\epsilon$ is a constant leads to a quadratic $\phi$. This is consistent with our initial conjecture. Using the expressions $\phi \equiv \hat{r} \cdot \hat{O} \cdot \hat{r}$ and $U \equiv \hat{r} \cdot \hat{U} \cdot \hat{r}$, one can derive from Eq. 16 that $(\hat{\mathbf{D}} + \epsilon \hat{\mathbf{Y}}) \hat{\mathbf{O}} = \hat{\mathbf{U}}$, and therefore

$$\hat{\mathbf{O}} = (\hat{\mathbf{D}} + \epsilon \hat{\mathbf{Y}})^{-1} \hat{\mathbf{U}}.$$
If one substitutes the result of $\phi$ into Eq. 7, it is straightforward to show that
\[ \nabla \times \vec{v} = -\frac{2(T_1 - T_2)}{T_1 T_2 + \epsilon^2} U_{12}, \tag{22} \]
where $U_{12}$ is the off-diagonal element of the matrix $\hat{U}$. Because $T_1 > T_2$, the autonomous gyration is counterclockwise if $U_{12} < 0$ and vice versa. Therefore, for the case of harmonic potential, both of our methods regarding the circulating direction are in full agreement.

![Image](image-url)

**FIG. 1.** (a) Theoretical prediction and (b) numerical simulation result of a Brownian gyrator under a harmonic potential $U = \frac{1}{2} k(x_1^2 + x_2^2) + k'x_1 x_2$. We use the setting $k = 1.5$ and $k' = 0.5$, while $T_1 = 0.8$ and $T_2 = 0.2$. The closed loops represent equipotential contours, while the color maps and vectors designate the NESS distribution of $\phi = -\log P$ and flux densities $\hat{J}$, respectively.

We use the harmonic potential $U(\vec{r}) = \frac{k}{2}(x_1^2 + x_2^2) + k'x_1 x_2$ as a specific example, and we apply the parameter settings $k = 1.5$ and $k' = -0.5$, while $T_1 = 0.8$ and $T_2 = 0.2$. Our analytical and numerical results are presented in Fig. 1(a) and (b), respectively. Our simulation result again exhibits the feature that the NESS currents gyrate along the equiprobability contours. Moreover, the counterclockwise gyration direction echoed the prediction from Eq. 22. Alternatively, the circulating direction can be hinted through the comparison of the curvatures of $U$ along the $x_1'$ and $x_2'$ axes. One can observe in Fig. 1 that the curvature along the $x_1'$ direction is milder than that along the $x_2'$ direction, i.e., $\partial^2_{\vec{r}} U < \partial^2_{x_2'} U$. Follow the argument in Eqs. 9 and 10, one also reaches the conclusion of a counterclockwise gyration.

**V. RESULT OF A NONHARMONIC DOUBLE-WELL POTENTIAL**

Next we study the autonomous Brownian gyrators under nonharmonic potentials. We first consider a tilted double-well potential
\[ U(\vec{r}) = x_1'^4 - 2x_1'^2 + \frac{1}{2} k_2 x_2'^2. \tag{23} \]
The two wells are lined up in the $x_1'$ axis. The potential is harmonic along the $x_2'$ axis and asymptotically quartic along the $x_1'$ axis. Our simulation results of the NESS probability and flux distributions are presented in Fig. 2 using various values of $k_2$. In the first and second rows of Fig. 2 we consider the temperature setting $T_1 = 0.8$ and $T_2 = 0.2$, while in the third row we use $T_1 = 0.16$ and $T_2 = 0.04$. First, one can find significant areas in the first and second rows of Fig. 2 that the NESS currents do not circulate about the equiprobability contours, i.e., $\vec{v} \cdot \nabla P \neq 0$. Therefore, the system entropy is not a constant of motion along the NESS trajectories. Instead, it exhibits some sort of oscillatory pattern during a cycle. Moreover, one can observe in the first second rows of Fig. 2 nonvanishing NESS currents at the probability maxima, and the probability and potential extrema do not coincide (the second row of Fig. 2 provides a closer look).

In addition to the fact that the potential energy minima are not stationary points, Fig. 2 also reveals that there exist two stationary points near each potential minimum. Furthermore, the gyration directions around the two stationary points are opposite to each other. While this novel result is due to nonharmonic effects, it can be understood using Eq. 8. First, the probability distribution shows that $\phi$ exhibits a similar double-well shape compared with the potential profile. If $\phi$ has a geometric structure that is similar to the potential, then along the $x_1'$ axis, the curl of $\vec{v}$ must be negative at large $x_1'$. This is because the quartic-like behavior along the $x_1'$ axis results in a larger curvature that outweighs the rather harmonic behavior along the $x_2'$ axis. Thus this argument leads to a clockwise gyration behavior at large $x_1'$. Moreover, for this double potential one has $\partial^2_{x_1'} U - \partial^2_{x_2'} U = 12 x_1'^2 - 4 - k_2$. If we follow the argument from the total entropy production following Eqs. 9 and 10 we find the argument predicts a clockwise gyration for $|x_1'| > \sqrt{(k_2 + 4)/12}$ and a counterclockwise one for $|x_1'| < \sqrt{(k_2 + 4)/12}$. In particular, the potential energy minimum is located at the clockwise gyration region for $k_2 = 4$ and counterclockwise gyration region for $k_2 = 12$, which are evidenced by our simulation result in Fig. 2.

Note that the setting $k_2 = 8$ stands as a special case in our discussion. First, the above analysis on the gyration behavior holds for all temperatures that meet the criterion $T_1 > T_2$. And since for $k_2 = 8$ the potential minima occur at the positions $(x_1' = \pm 1, x_2' = 0)$, one can always anticipate two oppositely gyrating regions neighboring a potential minimum. Our simulation result also confirms this gyration signature, while this behavior can still be observed even with our low-temperature setup ($T_1 = 0.16$ and $T_2 = 0.04$; please refer to the third row of Fig. 2). At a first look, this result appears perplexing, since unlike the results for other values of $k_2$, it does not approach the harmonic behavior at low temperatures. This can be understood knowing that for the case $k_2 = 8$, the shape of the potential nearby each minimum is harmonic but circular (the latter fact can be shown by $\partial^2_{x_1'} U - \partial^2_{x_2'} U = 0$.
FIG. 2. NESS simulation results of the double-well potential \( U = x_1^4 - 2x_1^2 + \frac{1}{2}k_2x_2^2 \). The values \( k_2 = 4, 8, 12 \) are applied in the figures of the first, second, and third columns, respectively. The temperatures \( T_1 = 0.8 \) and \( T_2 = 0.2 \) are applied in the first and second rows, while the third row provides the result of our low-temperature setup (\( T_1 = 0.16 \) and \( T_2 = 0.04 \)). (The notational designation follows that of Fig. 1).

at the potential minimum). As a result, the leading harmonic approximation gives no NESS currents, while the nonharmonic part of the potential makes the dominant contribution.

There is a similar puzzle regarding the NESS probability distribution that needs to be addressed. On the one hand, nearby an extremum of probability distribution, the quadratic behavior dominates in \( \phi \). Then according to Eq. 7, \( \nabla \times \vec{v} \) is approximately constant, which implies a uniform gyrating direction in this region. On the other hand, at the special case \( k_2 = 8 \), our analysis in the last paragraph gives the speculation that the quadratic contribution to the NESS currents is absent (though the quadratic contribution here means that from the probability distribution), and there exist oppositely gyrating regions nearby the extremum at all temperatures. This paradox can be resolved noting that for the case \( k_2 = 8 \), the behavior of \( \phi \) near its extremum is quadratic but unshifted, as can be observed in the second column of Fig. 2. Therefore, according to Eq. 7 the quadratic part in \( \phi \) be-
comes irrelevant in \( \nabla \times \vec{v} \), and the non-quadratic part in \( \phi \) serves as the major contributor in the gyrating pattern.

From the above discussions, we find the setting \( k_2 = 8 \) gives a special scenario in that the harmonic contribution to the NESS currents vanishes. From Fig. 2 one can observe that when \( k_2 < 8 \), the gyration nearby the potential minimum is clockwise, and the probability distribution at low temperatures is approximately harmonic, while the semi-major axis is slightly tilted in the clockwise direction. And when \( k_2 > 8 \), the gyration and the tilting of the semi-major axis just exhibit the opposite trend.

VI. RESULT OF AN ISOTROPIC QUARTIC POTENTIAL

In the previous section, the double-well potential is locally harmonic nearby each energy minimum. As a contrast, our second nonharmonic case features an isotropic quartic potential which is entirely nonharmonic. The potential is defined by

\[
U(r) = r^4 = (x_1^2 + x_2^2)^2
\]

where \( r = \sqrt{x_1^2 + x_2^2} \) is the distance from the origin. The potential has a rather flat shape nearby the origin, and it starts to grow up drastically as \( r \) further increases (in our case the classification between flat and steep regions depends on the temperature scale).

Figure 3 shows the simulation results of the NESS probability and flux density distribution. Interestingly, the NESS currents exhibit four circulating regions, each residing in its own quadrant, and there is a stationary point within each circulating region. The origin behaves as a saddle point in terms of steady-state dynamics. Meanwhile, the shape of probability distribution is largely similar to that of the potential profile, and the distribution over the vertical direction is narrower due to the smaller temperature \( T_2 \). A more careful examination reveals a shallow double-well structure in \( \phi \) (see Fig. 3(b)), and thus the probability maxima do not occur at the potential minimum (i.e., origin). Instead, the origin again serves as a saddle point regarding \( \phi \) and therefore the probability distribution.

To understand the flowing behavior, we first note that the isotropic quartic potential is even in both \( x_1 \) and \( x_2 \), and the probability distribution must possess the same parities. From Eq. 3 one learns that

\[
J_1(-x_1, x_2) = -J_1(x_1, x_2) \\
J_2(-x_1, x_2) = J_2(x_1, x_2) \\
J_1(x_1, -x_2) = J_1(x_1, x_2) \\
J_2(x_1, -x_2) = -J_2(x_1, x_2)
\]

where \( J_1 \) and \( J_2 \) are the components of \( \vec{J} \). This property immediately leads to the fact that the NESS currents cannot circulate around the equiprobability contours all the time. In particular, the NESS currents on the \( x_1 \) axis cannot have a vertical component, whereas the currents on the \( x_2 \) axis cannot have a horizontal one.

The gyrating pattern for the NESS currents of this isotropic quartic potential can be further understood using our analysis in Sec. III. First, because \( \phi \) is even in both \( x_1 \) and \( x_2 \), Eq. 7 tells that \( \nabla \times \vec{v} \) must be odd in \( x_1 \) and \( x_2 \). Moreover, since the NESS profile of \( \phi \) has a structure which resembles the potential profile, one can represent it using the crude expression \( \phi \approx (a^2 x_1^4 + b^2 x_2^4)^2 \) (\( a \) and \( b \) are constants). Hence \( \nabla \times \vec{v} \approx 8a^2 b^2 x_1 x_2 \). Therefore, the direction of circulation, which is capitated through \( \nabla \times \vec{v} \), is identical in each quadrant, which agrees well with our observation in Fig. 3.

More precisely, the observed geometry of \( \phi \) differs with that of \( U \) and our crude approximation of \( \phi \) in the fact that it posess a very shallow double-well structure. We believe that such a mild difference does not result in a big property change in \( \nabla \times \vec{v} \). In fact, if one adds an harmonic bump upon the existing quartic geometrical shape, Eq. 7 shows that the harmonic term does not change \( \nabla \times \vec{v} \) as long as its shape is not tilted with respect to the \( x_1 \) axis. Alternatively, the total entropy analysis predicts that the gyrating direction is dependent on \( \partial_1 \partial_2 U = b x_1 x_2 \). The analysis provides a stronger evidence towards a clockwise gyration in the first and third quadrants and a counterclockwise one in the second and fourth quadrants.

The flow field can also help us get a better understanding in \( \phi \) and therefore \( P \). Equation 4 implies that

\[
d\phi = \frac{dx_1 \partial_1 U}{T_1} + \frac{dx_2 \partial_2 U}{T_2}
\]

along an autonomous path \( d\vec{\ell} = (dx_1, dx_2) \). For the isotropic quartic potential we consider, the NESS flux along the \( x_3 \) axis directs outward from the origin. Following this flowing direction along the \( x_2 \) axis, Eq. 26 reduces to \( d\phi > dU/T_2 \). On the other hand, if one follows the flowing direction on the \( x_1 \) axis towards the origin, the criterion gives \( d\phi > dU/T_1 \). Therefore, directing out from the origin, the increase of \( \phi \) is sharper than that of \( U/T_2 \) along the \( x_3 \) axis and milder than that of \( U/T_1 \) along the \( x_1 \) axis. The change of \( \phi \) along the \( x_1 \) axis
VII. TOTAL ENTROPY PRODUCTION

Following a steady-state trajectory, the rate of entropy dissipation into the environment is

$$\frac{dS_Q}{dt} = \frac{F_1 v_1}{T_1} + \frac{F_2 v_2}{T_2} = -\nabla U \cdot \hat{D}^{-1} \vec{v}, \quad (27)$$

where $F_i \equiv -\partial_i U$. Meanwhile, the time derivative of the system entropy is

$$\frac{dS_{sys}}{dt} = -\frac{d\log P}{dt} = \vec{v} \cdot \nabla \phi. \quad (28)$$

Thus the total entropy changing rate along a steady-state trajectory is

$$\frac{dS_{tot}}{dt} = \frac{dS_{sys}}{dt} + \frac{dS_Q}{dt} = \vec{v} \cdot (\nabla \phi - \hat{D}^{-1} \nabla U) = \vec{v} \cdot \hat{D}^{-1} \vec{v} \geq 0. \quad (29)$$

The last line in Eq. 29 is derived using Eq. 4. Therefore the total entropy changing rate is positive along a steady-state trajectory regardless of the type of conservative forces. Note that Eq. 29 implies that if one follows a NESS trajectory, the total entropy change is just equal to the contribution by the damping force.

If the potential is quadratic, the NESS currents circulate around the equiprobability contours, and the system entropy is a constant of time. Therefore, the net rate of heat dissipation towards the thermal baths is always positive at the NESS. On the other hand, regarding non-quadratic potentials, the system entropy becomes oscillatory along a NESS cycle (see Figs. 2 and 3 for examples). Moreover, at times one may encounter a region where the net entropy dissipation into the environment $\Delta S_Q$ turns negative. For example, from Fig. 3 one finds that along the $x_2$ axis, while the NESS current directs away from the origin, the conservative force is just anti-parallel to $\vec{v}$, and therefore the entropy dissipation into the surrounding is negative.

The above result may look counterintuitive, as during this stage, on average, the system absorbs heat from the low-temperature reservoir. Meanwhile, following the trajectory along the $x_1$ axis towards the origin, the system (again on average) dissipates heat into the high-temperature reservoir. The autonomous occurrence of these less intuitive trajectories can be understood by the corresponding changes in the system entropy, which is synonymous with the information possessed by the system, and the thermodynamic law dictates a positive change in total entropy. Moreover, noting that $\Delta S_Q = \Delta S_{tot}$ after one full cycle, one deduces that during the rest of the trajectory, behavior of opposite trends must occur. And after one full cycle, the system indeed absorbs a positive amount of heat from the high-temperature reservoir and dissipate it into the low-temperature reservoir.

VIII. CONCLUSION

In this work we apply two simple criteria to examine the gyration directions. The first one requires either the input of the NESS probability distribution, or the assumption that its corresponding $\phi$ resembles the potential geometry qualitatively. While the description of geometric resemblance is itself less well-defined, our second method provides a clearer recipe without a priori knowledge about the probability distribution. It is remarkable to note that the second law of thermodynamics plays a substantial role in understanding the gyration dynamics.

For the harmonic potential, the system possesses the remarkable feature that the NESS currents gyrate about the equiprobability contours. As for the double-well potential, oppositely gyrating regions exist nearby each potential minimum, and the gyration direction where the potential minima reside depends on the choice of $k_2$. At the critical value $k_2 = 8$, the equipotential contours nearby the potential minima become circular, and the harmonic contribution to gyration pattern vanishes completely. Finally, for the isotropic quartic potential, which is completely nonharmonic, the result reveals four circulating regions, and the flowing pattern can be comprehended using simple arguments in parity.

Our observations based on this work and other trial potentials lead us to speculate that, unlike harmonic potentials, the NESS currents do not faithfully follow the tangent of equiprobability contours for general non-harmonic potential cases. Nevertheless, in our nonharmonic results one can notice areas where this circulating-about-probability-contour feature is seemingly present (see Figs. 2 and 3 away from the interface between oppositely gyrating regions). Does the nonharmonic potential create extra vortices that locally break the gyration feature $\vec{v} \cdot \nabla \phi = 0$? This naive speculation is left to be further examined in future works. With a comprehensive understanding in the gyration patterns from non-harmonic potentials, one can anticipate more promising ideas in the advances of Brownian engines.

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