On the Complexity of Query Containment and Computing Certain Answers in the Presence of ACs

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Abstract

We often add arithmetic to extend the expressiveness of query languages and study the complexity of problems such as testing query containment and finding certain answers in the framework of answering queries using views. When adding arithmetic comparisons, the complexity of such problems is higher than the complexity of their counterparts without them. It has been observed that we can achieve lower complexity if we restrict some of the comparisons in the containing query to be closed or open semi-interval comparisons. Here, focusing a) on the problem of containment for conjunctive queries with arithmetic comparisons (CQAC queries, for short), we prove upper bounds on its computational complexity and b) on the problem of computing certain answers, we find large classes of CQAC queries and views where this problem is polynomial.

Keywords: query containment, query rewriting, conjunctive queries with arithmetic comparisons

1. Introduction

For conjunctive queries, the query containment problem is NP-complete [1]. When we have constants that are numbers (e.g., they may represent prices, dates, weights, lengths, heights) then, often, we want to compare them by checking, e.g., whether two numbers are equal or whether one is greater than

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the other, etc. To reason about numbers we want to have a more expressive language than conjunctive queries and, thus, we add arithmetic comparisons to the definition of the query. We know that the query containment problem for conjunctive queries with arithmetic comparisons is $\Pi_2^p$-complete \cite{2, 3, 4}. In previous literature \cite{5, 6, 7, 8}, it has been noticed that there are classes of CQACs for which the query containment problem remains in NP and these classes can be syntactically characterized.

In the framework of answering queries using views, we want to find all certain answers of the query on a given view instance, i.e., all the answers that are provable “correct.” A popular way for answering queries using views is by finding rewritings of the query in terms of the views that are contained in the query. There may exist many contained rewritings in a certain query language. We want to find the maximal contained rewriting (MCR for short) that contains all the rewritings, if there exists such a rewriting. Query containment and finding rewritings when we use the language of CQAC or unions of CQAC are closely related.

In this paper, we present the following results:

**Query Containment** We solve an open problem mentioned in \cite{9} by extending significantly the class of CQAC queries that admit an NP containment test. As concerns closed arithmetic comparisons, we think we are close to the boundary between the problem being in NP and being in $\Pi_2^p$. The class of queries we consider includes the following case: The contained query is allowed to have any closed arithmetic comparisons and the containing query is allowed to have any closed arithmetic comparisons that involve the head variables (but not between a head variable and a body variable) and the comparisons that are allowed in the body variables are the following: Several left semi-interval arithmetic comparisons and at most one right semi-interval arithmetic comparison.

This result is proven via a transformation of the queries to a Datalog query (for the containing query) and a conjunctive query (for the contained query) and reducing checking containment between these two. This result captures all results in \cite{9} but in a new way that allows us to further use the transformation
to compute MCRs and certain answers in the framework of the problem of answering queries using views.

**MCRs** We extend the results in [6] and prove that we can find an MCR in the language of Datalog with arithmetic comparisons in the case where the query has the restrictions of the containing query above and the views use any closed ACs, except ACs between the head and non-head variables. In [6], only semi-interval arithmetic comparisons were allowed in the query and in the views.

**Computing certain answers** We show for the first time how to compute certain answers in polynomial time using MCRs for the case the conjunctive queries have arithmetic comparisons.

1.1. Related work

The homomorphism property for query containment was studied in [2,10] [7]. Recent work can be found in [11], where the authors propose to extend graph functional dependencies with linear arithmetic expressions and arithmetic comparisons. They study the problems of testing satisfiability and related problems over integers (i.e., for non-dense orders). A thorough study of the complexity of the problem of evaluating conjunctive queries with inequalities ($\neq$) is done in [12]. In [13] the complexity of evaluating conjunctive queries with arithmetic comparisons is investigated for acyclic queries, while query containment for acyclic conjunctive queries was investigated in [14]. Recent works [15,16] have added arithmetic to extend the expressiveness of tuple generating dependencies and data exchange mappings, and studied the complexity of related problems. We use monadic Datalog containment to prove our results. Among recent work on containment problem among modanic Datalog and conjunctive queries is [17].

2. Preliminaries

We will state our results in detail first by referring to conjunctive queries with arithmetic comparisons and query containment. Thus we will define here
these queries and later in the paper we will define what is a query rewriting using views. Then, we will extend the results about query containment to query rewriting and computing certain answers. We will discuss briefly the chase algorithm based on dependencies with arithmetic comparisons.

A relation schema is a named relation defined by its name (called relation name or relational symbol) and a vector of attributes. An instance of a relation schema is a collection of tuples with values over its attribute set. The schemas of the relations in a database constitute its database schema. A relational database instance (database, for short) is a collection of stored relation instances.

A conjunctive query (CQ in short) $Q$ over a database schema $S$ is a query of the form:

$$h(\overline{X}) : e_1(\overline{X}_1), \ldots, e_k(\overline{X}_k)$$

where $h(\overline{X})$ and $e_i(\overline{X}_i)$ are atoms, i.e., they contain a relational symbol (also called predicates - here, $h$ and $e_i$ are predicates) and a vector of variables and constants. The head $h(\overline{X})$, denoted $\text{head}(Q)$, represents the results of the query, and $e_1 \ldots e_k$ represent database relations (also called base relations) in $S$. The part of the conjunctive query on the right of symbol $:$ is called the body of the query and is denoted $\text{body}(Q)$. Each atom in the body of a conjunctive query is said to be a subgoal. Every argument in the subgoal is either a variable or a constant. The variables in $\overline{X}$ are called head or distinguished variables, while the variables in $\overline{X}_i$ are called body or nondistinguished variables of the query. We say that a CQ has self-joins if there are at least two subgoals in its body with the same relational symbol. A conjunctive query is said to be safe if all its distinguished variables also occur in its body. We only consider safe queries here.

The result (or answer), denoted $Q(D)$, of a CQ $Q$ when it is applied on a database instance (i.e., when applied on the base relations) $D$ is the set of atoms such that for each assignment $h$ of variables of $Q$ that makes all the atoms in the body of $Q$ true (i.e., the produced atoms in the body represent tuples in
the atom \( h(\text{head}(Q)) \) is in \( Q(D) \). The atoms produced by replacing their variables with constants are also called \textit{ground} atoms (or simply \textit{facts}).

\textit{Conjunctive queries with arithmetic comparisons (CQAC for short)} are conjunctive queries that, besides the \textit{ordinary} relational subgoals use also builtin subgoals that are arithmetic comparisons (AC for short), i.e., of the form \( X \theta Y \) where \( \theta \) is one of the following: \(<, >, \leq, \geq, =, \neq\). Also, \( X \) is a variable and \( Y \) is either a variable or constant. If \( \theta \) is either \(<\) or \(>\) we say that it is an open arithmetic comparison and if \( \theta \) is either \(\leq\) or \(\geq\) we say that it is a closed AC. If the AC is either of the form \( X < c \) or \( X \leq c \) (either \( X > c \) or \( X \geq c \), resp.) then it is called \textit{left semi-interval}, LSI for short (\textit{right semi-interval}, RSI for short, resp.), where \( X \) is a variable and \( c \) is a constant.

In the following, we use the notation \( Q = Q_0 + \beta \) to describe a CQAC query \( Q \), where \( Q_0 \) are the relational subgoals of \( Q \) and \( \beta \) are the arithmetic comparison subgoals of \( Q \). We define the \textit{closure} of a set of ACs to be all the ACs that are implied by this set of ACs. The result \( Q(D) \) of a CQAC \( Q \), when it is applied on a database \( D \), is given by taking all the assignments of variables (similar to CQs) such that the produced atoms are included in \( D \) and the ACs are true.

Moreover, the following assumptions must hold:

1. Values for the arguments in the arithmetic comparisons are chosen from an infinite, totally densely ordered set, such as the rationals or reals.

2. The arithmetic comparisons are not contradictory (or, otherwise, we say that they are consistent); that is, there exists an instantiation of the variables such that all the arithmetic comparisons are true.

3. All the comparisons are safe, i.e., each variable in the comparisons also appears in some ordinary subgoal.

A \textit{union of CQs} (resp. CQACs) is defined by a set \( Q \) of CQs (resp. CQACs) whose heads have the same arity, and its answer \( Q(D) \) is given by the union
of the answers of the queries in $\mathcal{Q}$ over the same database instance $D$; i.e.,

$$\mathcal{Q}(D) = \bigcup_{Q_i \in \mathcal{Q}} Q_i(D).$$

A **view** is a named query which can be treated as a regular relation. The query defines the view is called **definition** of the view. A view is said to be materialized if its answer is stored in the database.

A query $Q_1$ is **contained** in a query $Q_2$, denoted $Q_1 \subseteq Q_2$, if for any database $D$ of the base relations, the answer computed by $Q_1$ is a subset of the answer computed by $Q_2$, i.e., $Q_1(D) \subseteq Q_2(D)$. The two queries are **equivalent**, denoted $Q_1 \equiv Q_2$, if $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$.

A **homomorphism** $h$ from a set of relational atoms $\mathcal{A}$ to another set of relational atoms $\mathcal{B}$ is a mapping of variables and constants from one set to variables or constants of the other set that maps each variable to a single variable or constant and each constant to the same constant. Each atom of the former set should map to an atom of the latter set with the same relational symbol. We also say that the homomorphism $h'$ from a set $\mathcal{A}' \supseteq \mathcal{A}$ is an **extension** of $h$ if for each variable or constant $x$ in $\mathcal{A}' \cap \mathcal{A}$ we have $h'(x) = h(x)$.

A **containment mapping** from a conjunctive query $Q_1$ to a conjunctive query $Q_2$ is a homomorphism from the atoms in the body of $Q_1$ to the atoms in the body of $Q_2$ that maps the head of $Q_1$ to the head of $Q_2$. All the mappings we refer to in this paper are containment mappings unless we say otherwise. Chandra and Merlin [1] show that a conjunctive query $Q_1$ is contained in another conjunctive query $Q_2$ if and only if there is a containment mapping from $Q_2$ to $Q_1$.

Let $I$ be a set consisting of atoms over a schema $\mathcal{S}$, having both constants and variables, such that $I$ is associated with a total order over its constants and variables. We call $I$ a *t-instance*. Let $J_1, J_2$ be sets of atoms over the schema $\mathcal{S}$ such that $J_1$ is associated with a partial order and $J_2$ is a t-instance. An **order-homomorphism** $h : J_1 \rightarrow J_2$ is a mapping from the atoms in $J_1$ to the atoms in $J_2$ with the following properties:

1. For every constant $c$ in $J_1$, we have $h(c) = c$. 

2. For every atom \( r(X_1, \ldots, X_m) \) in \( J_1 \), we have that \( r(h(X_1), \ldots, h(X_m)) \) is an atom in \( J_2 \), where \( X_1, \ldots, X_m \) are either variables or constants.

3. If \( (X_1 \theta X_2) \) is true in \( J_1 \), where \( \theta \) is \(<, >, \leq, \geq, =\), then \( (h(X_1) \theta h(X_2)) \) is implied by the partial order of \( J_2 \).

### 2.1. Testing query containment

In this section, we describe two popular tests for CQAC query containment; using containment mappings and using canonical databases. Both extend the corresponding approaches used for CQ query containment in order to properly handle the presence of ACs.

First, we present the test using containment mappings. Although finding a single containment mapping suffices to test query containment for CQs (see the previous section), it is not enough in the case of CQACs. In fact, as we will see in the following, in the general case of CQACs, all the containment mappings from the containing query to the contained one are taken into consideration.

Before we describe how containment mappings could be used in order to test query containment between two CQACs, we define the concept of normalization of a CQAC.

**Definition 2.1.** Let \( Q_1 \) and \( Q_2 \) be two conjunctive queries with arithmetic comparisons (CQACs). We want to test whether \( Q_2 \sqsubseteq Q_1 \). To do the testing, we first normalize each of \( Q_1 \) and \( Q_2 \) to \( Q'_1 \) and \( Q'_2 \), respectively. We normalize a CQAC query as follows:

- For each occurrence of a shared variable \( X \) in a normal (i.e., relational) subgoal, except for the first occurrence, replace the occurrence of \( X \) by a fresh variable \( X_i \), and add \( X = X_i \) to the comparisons of the query; and

- For each constant \( c \) in a normal subgoal, replace the constant by a fresh variable \( Z \), and add \( Z = c \) to the comparisons of the query.

In essence, the normalization of a CQAC moves all the filtering (i.e., constants in relational subgoals) and joining (represented by shared variables) conditions included in the relational part of the body into the ACs-part. Such a
conversion ensures that all the conditions are taken into consideration when we test the ACs, as we will see in the following. Theorem 2.2 \[18, 10\] describes how we can test the query containment of two CQACs using containment mappings. In particular, Theorem 2.2 says that $Q_2 \sqsubseteq Q_1$ if and only if the comparisons in the normalized version $Q'_2$ of $Q_2$ logically imply (denoted by “⇒”) the disjunction of the images of the comparisons of the normalized version $Q'_1$ of $Q_1$ under each containment mapping from the ordinary subgoals of $Q'_1$ to the ordinary subgoals of $Q'_2$. The proof of the theorem is included in the Appendix B.

**Theorem 2.2.** Let $Q_1, Q_2$ be CQACs, and $Q'_1 = Q'_{10} + \beta'_1, Q'_2 = Q'_{20} + \beta'_2$ be the respective queries after normalization. Suppose there is at least one containment mapping from $Q'_{10}$ to $Q'_{20}$. Let $\mu_1, \ldots, \mu_k$ be all the containment mappings from $Q'_{10}$ to $Q'_{20}$. Then $Q_2 \sqsubseteq Q_1$ if and only if the following logical implication $\phi$ is true:

$$\phi : \beta'_2 \Rightarrow \mu_1(\beta'_1) \lor \cdots \lor \mu_k(\beta'_1).$$

(We refer to $\phi$ as the containment entailment in the rest of this paper.)

**Example 2.3.** Consider the following normalized CQACs.

$$Q_1 : q() \quad : - \quad a(X_1, Y_1, Z_1), X_1 = Y_1, Z_1 < 5$$
$$Q_2 : q() \quad : - \quad a(X, Y, Z'), a(X', Y', Z), X \leq 5, Y \leq X, Z \leq Y, X' = Y', Z' < 5$$

Testing the containment $Q_2 \sqsubseteq Q_1$, it is easy to see that there are the following containment mappings:

- $\mu_1 : X_1 \rightarrow X, Y_1 \rightarrow Y, Z_1 \rightarrow Z'$
- $\mu_2 : X_1 \rightarrow X', Y_1 \rightarrow Y', Z_1 \rightarrow Z$

Hence, the containment entailment is given as follows:

$$X \leq 5 \land Y \leq X \land Z \leq Y \land X' = Y' \land Z' < 5 \Rightarrow$$

$$\left( \mu_1(X_1) = \mu_1(Y_1) \land \mu_1(Z_1) < 5 \right) \lor$$

$$\left( \mu_2(X_1) = \mu_2(Y_1) \land \mu_2(Z_1) < 5 \right)$$
which is equivalently written:

\[
X \leq 5 \land Y \leq X \land Z \leq Y \land X' = Y' \land Z' < 5 \Rightarrow \\
(X = Y \land Z' < 5) \lor (X' = Y' \land Z < 5)
\]

It is easy to verify that the above implication is true (due to the second part of the disjunction in the right-hand side which is also included in the antecedent).

As the following theorem \[9\] shows, if the CQACs have only closed ACs, then normalization is not necessary. See for the proof in \[9\].

**Theorem 2.4.** Consider two CQAC queries, \(Q_1 = Q_{10} + \beta_1\) and \(Q_2 = Q_{20} + \beta_2\) over densely totally ordered domains. Suppose \(\beta_1\) contains only \(\leq\) and \(\geq\), and each of \(\beta_1\) and \(\beta_2\) does not imply any “=” restrictions. Then \(Q_2 \sqsubseteq Q_1\) if and only if

\[
\phi : \beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_l(\beta_1),
\]

where \(\mu_1, \ldots, \mu_l\) are all the containment mappings from \(Q_{10}\) to \(Q_{20}\).

Summarizing the containment test for checking \(Q_2 \sqsubseteq Q_1\) using containment mappings, once we normalize both \(Q_1\) and \(Q_2\) (if we have at least one open AC), we initially find all the containment mappings from \(Q_1\) to \(Q_2\). Then, we construct the containment entailment and check whether containment entailment is true.

As mentioned in the beginning of this section, there is another containment test for CQACs, which uses canonical databases (see, e.g., in \[19\]). Considering a CQ \(Q\), a canonical database is a database instance constructed as follows. We consider an assignment of the variables in \(Q\) such that a distinct constant which is not included in any query subgoal is assigned to each variable. Then, the ground subgoals produced through this assignment define a canonical database of \(Q\). Note that although there is an infinite number of assignments and canonical databases, depending on the constants selection, all the canonical databases are isomorphic; hence, we refer to such a database instance as the canonical database of \(Q\). To test, now, the containment \(Q_2 \sqsubseteq Q_1\) of the CQs \(Q_1, Q_2\) \[19\], we compute the canonical database \(D\) of \(Q_2\) and check if \(Q_2(D) \subseteq Q_1(D)\).
Extending the test using canonical databases to CQACs, a single canonical database does not suffice. Let us initially construct a canonical database of a CQAC $Q_2$ with respect to a CQAC $Q_1$ as follows. Considering the set $S$ including the variables of $Q_2$, and the constants of both $Q_1$ and $Q_2$. Then, we partition the elements of $S$ into blocks such that no two constants are included in the same block. Let $\mathcal{P}$ be such a partition, $\mathcal{P}_C$ be the set of all the blocks in $\mathcal{P}$ including constants, and $S_C$ be the set including all the constants in $S$. Considering a totally ordered set $C$ of distinct constants such that $S_C \subseteq C$ and $|C|$ equals the number of blocks in $\mathcal{P}$, we assign a distinct constant in $(C - S_C)$ to each block in $\mathcal{P}$ that does not include any constant and the constant of the block for each block in $\mathcal{P}$ including a constant. Let $\phi$ be such an assignment and $D$ be the database instance including all the ground atoms produced by applying the assignment $\phi$ on the subgoals of $Q_2$. The database $D$ is a canonical database of $Q_2$ with respect to $Q_1$.

Although there is an infinite number of canonical databases, depending on the constants selected, there is a bounded set of canonical databases such that every other canonical database is isomorphic to one in this set. Such a set is referred as the set of canonical databases of $Q_2$ w.r.t. $Q_1$. To test now the containment $Q_2 \subseteq Q_1$ of the CQACs $Q_1$, $Q_2$ [19], we construct all the canonical databases of $Q_2$ w.r.t. $Q_1$ and, for each canonical database $D$, we check if $Q_2(D) \subseteq Q_1(D)$.

**Theorem 2.5.** A CQAC query $Q_2$ is contained into a CQAC query $Q_1$ if and only if, for each database belonging to the set of canonical databases of $Q_2$ with respect to $Q_1$, the query $Q_1$ computes all the tuples that $Q_2$ computes if applied on it.

### 2.2. Rewriting queries using views

In this section, we describe the problem of answering queries using views through query rewritings [20, 21, 19]. In particular, considering a set of views $\mathcal{V}$ and a query $Q$ over a database schema $\mathcal{S}$, we want to answer $Q$ by accessing only the instances of views. To answer the query $Q$ using $\mathcal{V}$ we could rewrite...
Q into a new query R such that R is defined in terms of views in V (i.e., the predicates of the subgoals of R are view names in V). If, now, for every database instance D, we have R(V(D)) = Q(D) we say that R is an equivalent rewriting of Q using V. In addition, if R(V(D)) ⊆ Q(D), then R is a contained rewriting of Q using V. To find and check query rewritings we use the concept of rewriting expansion (expansion, for short), which is defined as follows.

**Definition 2.6.** The expansion of a query P defined in terms of views in V, denoted by P_{exp}, is obtained from P as follows. For each subgoal v_i of P and the corresponding view definition V_i in V, if µ_i is the mapping from the head of V_i to v_i we replace v_i in P with the body of µ_i(V_i). The non-distinguished variables in each view are replaced with fresh variables in P_{exp}.

To test now whether a query R defined in terms of views set V is a contained (resp. equivalent) rewriting of another query Q defined in terms of the base relations, we check the query containment (resp. equivalence) of the expansion of the first one with the second one [19]; i.e., check P_{exp} ⊑ Q (resp. P_{exp} ≡ Q).

In the language of CQs, we can find either an equivalent or a contained rewriting (if there is any) with at most the number of subgoals of the query [19]. In the presence of ACs the picture changes though. We may have arbitrarily long rewritings as the following example shows.

**Example 2.7.** Let Q be a CQAC and V = {V_1, V_2, V_3} be a set of CQ views with the following definition.

\[
\begin{align*}
Q & : q() : - a(X, Y), X = 5, Y < 5. \\
V_1 & : v_1(X, Y) : - a(X, Y), X = 5, Y \leq 5. \\
V_2 & : v_2(X, Y) : - a(X, Y), X \leq 5, Y \leq 5. \\
V_3 & : v_3(X, Y) : - a(X, Y), Y < 5.
\end{align*}
\]

Consider now the rewriting R of the query Q using the three views with the following definition.

\[
R : q() : - v_1(X_1, X_2), v_2(X_2, X_3), v_2(X_3, X_4), \ldots, v_2(X_{n-2}, X_{n-1}), v_3(X_{n-1}, X_n)
\]
Notice that $R$ is a contained rewriting of $Q$ using $V$. To verify this, let us check the containment entailment (after finding the $n-1$ containment mappings and replacing with $R^{exp}$ variable) below constructed by the ACs of $P^{exp}$ and $Q$.

$$(X_1 = 5 \land X_2 \leq 5) \land (X_2 \leq 5 \land X_3 \leq 5) \land \cdots$$

$$\land (X_{n-2} \leq 5 \land X_{n-1} \leq 5) \land (X_n \leq 5) \Rightarrow \bigvee_{i=1}^{n-1} (X_i = 5 \land X_{i+1} < 5)$$

It is easy to see that the containment entailment is equivalently rewritten into the following:

$$X_1 = 5 \land X_2 \leq 5 \land \cdots \land X_{n-1} \leq 5 \land X_n < 5 \Rightarrow \bigvee_{i=1}^{n-1} (X_i = 5 \land X_{i+1} < 5)$$

The aforementioned implication is true. To see this, we sequentially check the variables $X_2, \ldots, X_{n-1}$, from left to right, and stop at the first variable $X_k$ which is not equal to 5 (if there is any). If such a $k$ exists then we have $(X_{k-1} = 5 \land X_k < 5)$; which corresponds to the $(k-1)$-th disjunct of the right-hand side of the implication. Otherwise, we have $X_2 = X_3 = \cdots = X_{n-1} = 5$; hence, the last disjunct of the right-hand side of the implication is true. Consequently, there is always a mapping ensuring that the containment entailment is true.

Considering now a CQ $Q$, a set of CQ views $V$ and rewritings in the language of CQs, we describe how we find an equivalent rewriting of $Q$ using $V$. In particular, we compute the canonical rewriting $R$ of $Q$ using the views in $V$ as follows. First, we freeze the variables of $Q$ to distinct constants, constructing a database $D$, and evaluate the views on the database $D$. Then, we de-freeze the constants back to their corresponding variables, constructing a set of atoms (also called view-tuples). Those atoms are the subgoals of $R$, while the head of $R$ is identical to the head of $Q$. Technically, computing the views on the database with the frozen variables is equivalent to finding a homomorphism from the view’s subgoals to the query subgoals. Hence, we can derive the following theorem.

**Theorem 2.8.** Suppose query and views are CQs. Then, there is an equivalent rewriting in the language of CQs iff the canonical rewriting is such a rewriting.

Let us now focus on contained rewritings. There are settings where there is no equivalent rewriting of the query using the views. In such a case, finding a
containing rewriting returning as many answers of the query as possible matters. In this context, we define a contained rewriting, called maximally contained rewriting (MCR, for short), that returns most of the answers of the query.

**Definition 2.9.** A rewriting \( R \) is called a maximally contained rewriting (MCR) of query \( Q \) using views \( V \) with respect to query language \( \mathcal{L} \) if

1. \( R \) is a contained rewriting of \( Q \) using \( V \) in \( \mathcal{L} \), and
2. every contained rewriting of \( Q \) using \( V \) in language \( \mathcal{L} \) is contained in \( R \).

### 2.3. Datalog queries

In this section, we define the Datalog query \[22\] and describe how we can test the containment between a Datalog query and a CQ. In particular, a Datalog query (or Datalog query) is a finite set of Datalog rules, where a rule is a CQ whose predicates in the body could either refer to a base relation or to a head of a rule in the query (either the same rule or other rule). Furthermore, there is a designated predicate, which is called query predicate, and returns the result of the query.

The atoms in the body of each rule in a Datalog query are of two types; the ones referring to base relations and the ones referring to a head of a rule. The predicates of former type are called extensional (EDB, for short) while the predicates of the latter are called intensional (IDB, for short). The atom whose predicate is an EDB (resp. IDB) is called base atom (resp. derived atom). A Datalog query is called monadic if all the IDBs are unary.

The evaluation of a Datalog query is performed by applying the rules on the database until no more facts (i.e., ground head atoms) are added to the set of the derived atoms. The answer of a Datalog query on a database is a set of facts derived on the database for the query predicate of the query. Namely, the evaluation follows the fixpoint semantics. Initially, the rules having only base atoms are computed over the database, and then we recursively apply the remaining rules over both the base relations and the derived ones. Note that we consider safe Datalog queries where each variable in the head of each rule is
also appeared in the body of the rule. Extending Datalog query such as each rule is a CQAC, denoted $Datalog^{AC}$ query, the evaluation process remains the same.

**Example 2.10.** Let $\mathcal{D}$ be a Datalog query with rules $r_1$, $r_2$ defined over the base relation $\text{edge}$ as follows.

$$
\begin{align*}
  r_1 : \text{tc}(X,Y) & : - \text{edge}(X,Y). \\
  r_2 : \text{tc}(X,Y) & : - \text{edge}(X,Z), \text{tc}(Z,Y).
\end{align*}
$$

It is easy to verify that the query $\mathcal{D}$ computes the transitive closure of the graph defined by the relation $\text{edge}$; where $r_1$ is a non-recursive rule and $r_2$ is a recursive rule. To evaluate this query, we initially evaluate $r_1$ over the relational instance of $\text{edge}$ and the recursively evaluate the rule $r_2$ over both the instance of $\text{edge}$ and the derive instance of $\text{tc}$. Finally, we end up with a fixpoint instance of $\text{tc}$ which does not change (i.e., no additional facts are computed). This final instance of $\text{tc}$ is the answer of query.

A partial expansion of a Datalog query is a conjunctive query that results from unfolding the rules one or more times; the partial expansion may contain IDB predicates. A datalog-expansion of a Datalog query is a partial expansion that contains only EDB predicates. Note that a Datalog query might not have a unique datalog-expansion (due to recursive rules).

**Example 2.11.** Considering the Datalog query in Example 2.10, it is easy to see that the following CQ is a partial expansion of $\mathcal{D}$.

$$
\text{tc}(X,Y) : - \text{edge}(X,Z), \text{edge}(Z,W), \text{tc}(W,Y).
$$

Furthermore, the following CQ is one of the datalog-expansions of $\mathcal{D}$.

$$
\text{tc}(X,Y) : - \text{edge}(X,Z), \text{edge}(Z,W), \text{edge}(W,V), \text{edge}(V,Y).
$$

A derivation tree depicts a computation. Considering a fact $e$ in the answer of the Datalog query, we construct a derivation tree as follows. Each node in this tree, which is rooted at $e$, is a ground fact. For each non-leaf node $n$ in this
tree, there is a rule in the query which has been applied to compute the atom node \( n \) using its children facts. The leaves are facts of the base relations. Such a tree is called \textit{derivation tree} of the fact \( e \).

Although to test the containment of two Datalog queries is undecidable [23], the containment of a Datalog query in a CQ is decidable and could be tested as follows. Consider a Datalog query \( Q_D \) and a CQ \( Q_{CQ} \) over the same database schema. To test the containment \( Q_D \subseteq Q_{CQ} \), we compute the canonical database \( D \) of \( Q \) and apply \( Q_D \) on \( D \). If \( Q_D \) computes the head of \( Q_{CQ} \) over \( D \), the containment holds; otherwise, \( Q_{CQ} \) is not contained in \( Q_D \). During the computation we use an \textit{instantiated rule}, which is a rule where all variables have been replaced by constants. We say that a rule is \textit{fired} if there is an instantiation of this rule where all the atoms in the body of the rule are in the currently computed intermediate database.

In the general case of non-monadic Datalog query, the problem of containment of a CQ in a recursive Datalog query is EXPTIME-complete [24, 25, 26]. The containment of a CQ in a linear monadic Datalog (i.e., each rule has at most one IDB) is NP-complete [27]. As [28] shows, the containment between monadic Datalog queries is decidable.

3. The algorithm to check satisfaction of a collection of ACs

We will present \textbf{algorithm AC-sat} which, on input a collection of ACs, checks whether there is a satisfying assignment, i.e., an assignment of real numbers to the variables that makes all ACs in the collection true. If there is not then we say that the conjunction of ACs is false or that the collection of ACs is \textit{contradictory} or is not \textit{consistent}.

We define the \textit{induced directed graph} of a collection \( C \) of ACs of the form \( X\theta Y \) where \( \theta \) is one of the \( <,>,\leq,\geq,=,\neq \). We consider that this collection is divided into two sub-collections, the collection \( C_A \) including all the ACs where \( \theta \) is one of the \( <,>,\leq,\geq,= \) and the collection \( C_B \) including all the ACs where \( \theta \) is \( \neq \). The induced directed graph is built using the ACs in \( C_A \) and has nodes
that are variables or constants. There is an edge labeled \( \leq \) between two nodes \( n_1, n_2 \) if there is an AC in the collection \( C_A \) which is \( n_1 \leq n_2 \). There is an edge labeled \( < \) between two nodes \( n_1, n_2 \) if there is an AC in the collection \( C_A \) which is \( n_1 < n_2 \). (We only label edges \( < \) or \( \leq \) since the other direction, \( > \) or \( \geq \) is indicated by the direction of the edge.) We treat each equation \( X = Y \) in \( C_A \) as two ACs of the form \( X \leq Y \) and \( X \geq Y \) and we add edges accordingly. Finally we add edges labeled \( < \) between all the pairs of constants depending on their order.

**Algorithm AC-sat:** We consider the induced directed graph \( G \) of the collection \( C \) of ACs. We then find all strongly connected components of \( G \). We say that an edge belongs to a strongly connected component if it joins two nodes in this strongly connected component.

The collection \( C \) of ACs is contradictory if either of the following cases is true.

**Case 1.** There is a strongly connected component with two distinct constants belonging to it.

**Case 2.** There is a strongly connected component with an edge labeled \( < \).

**Case 3.** There is a \( A_1 \neq A_2 \) AC in \( C_B \) such that \( A_1 \) and \( A_2 \) belong to the same strongly connected component and this component has only \( \leq \) edges on it.

**Lemma 3.1.** The algorithm AC-sat is a complete and sound procedure to check that a conjunction of ACs is contradictory.

**Proof.** First we prove that this procedure is complete; i.e., we prove that if the procedure shows that the conjunction is not false then we can assign constants to variables to make all ACs true.

Since neither Case 1 nor Case 2 happens, all strongly connected components have \( \leq \) labels and at most one constant. Thus, we assign to each of the elements of a strongly connected component the same constant, which is either a new constant or the constant of the component, as follows: We collapse each strongly
connected component to one node and the induced directed graph is reduced to an acyclic directed graph. We consider a topological sorting of this acyclic graph into a number of levels. We assign constants following this topological sorting, so that constants in the next level are greater than the constants in the previous levels. This makes all ACs true.

Now we prove that this procedure is sound. Whenever the procedure stops in Cases 1 and 2 then there is no assignment that satisfies all ACs in this strongly connected component because there is a cycle with either two distinct constants on it or with an edge labeled \(<\). This cycle means that all variables on it should be the same. The existence of two distinct constants on it or of an edge labeled \(<\) means that two variables on the cycle should be distinct. Whenever the procedure stops in Case 3, then \(A_1\) and \(A_2\) should be equal according to the strongly connected component they belong. Thus we cannot find an assignment that satisfies also the AC \(A_1 \neq A_2\).

\[\square\]

4. Analysing the containment entailment

In this section, we develop tools for the proofs we provide later. Consider the containment entailment that is used in Theorem 2.2 or in Theorem 2.4:

\[\beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_k(\beta_1).\]

4.1. Containment Implications

The right hand side of the containment entailment is a disjunction of disjuncts, where each disjunct is a conjunction of ACs. We can turn this, equivalently, to a conjunction of conjuncts, where each conjunct is a disjunction of ACs. We call each of these last conjuncts a rhs-conjunct (from right hand side conjunct). Now we can turn the containment entailment, equivalently, into a number of implications. In each implication, we keep the left hand side of the containment entailment the same and have the right hand side be one of the rhs-conjuncts. We call each such implication a containment implication.

We illustrate on an example.
Example 4.1. We continue from Example 2.3. We repeat the queries considered:

\[ Q_1 : q() : -a(X_1, Y_1, Z_1), X_1 = Y_1, Z_1 < 5. \]

\[ Q_2 : q() : -a(X, Y, Z'), a(X', Y', Z), X \leq 5, Y \leq X, Z \leq Y, X' = Y', Z' < 5. \]

Now we consider the containment entailment we built in Example 2.3. According to what we analyzed in this section, we can rewrite this containment entailment equivalently by transforming its right hand side into a conjunction, where each conjunct is a disjunction of ACs. The transformed entailment is the following, where \( \beta = X \leq 5 \land Y \leq X \land Z \leq Y \land X' = Y' \land Z' < 5 \):

\[ \beta \Rightarrow (X = Y \lor X' = Y') \land (X = Y \lor Z < 5) \land (Z' < 5 \lor X' = Y') \land (Z' < 5 \lor Z < 5) \]

The following two theorems are proved in [9] (referring to a more general framework that concerns pairs of queries that have the homomorphism property) and serve as an introduction to the results in the present paper (the second theorem is proven based on the first theorem):

Theorem 4.2. The containment entailment has one disjunct in the rhs if and only if each containment implication does.

Theorem 4.3. If the containing query contains only closed LSIs and the contained query any closed AC then the containment problem is in NP.

4.2. ACs over single-mapping variables

Now, we consider a special class of variables in the containing query, the single-mapping variables. We start our analysis with head variables and then extend our approach to a wider class of variables. In particular, we show how the containment entailment can be decomposed, in certain cases, in two implications, one of which has only a conjunction of ACs on the rhs.

Suppose two CQACs \( Q_1, Q_2 \) with closed ACs. We want to check the containment \( Q_1 \sqsubseteq Q_2 \). The contained query \( Q_2 \) has any closed ACs. The containing
query $Q_1$ has closed ACs, such that there are no ACs joining distinguished to nondistinguished variables. Let us analyse, now, the containment entailment:

$$\beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_k(\beta_1)$$  \hspace{1cm} (1)

where $\mu_1, \ldots, \mu_k$ are all the containment mappings from $Q_1$ to $Q_2$\footnote{we will always mean containment mapping from $Q_{10}$ to $Q_{20}$, all through this paper} and $\beta_1$, $\beta_2$ are the conjunctions of ACs of $Q_1$, $Q_2$, respectively. Considering that $\beta_1$ has at least one AC over a distinguished variable, we write $\beta_1 = \beta_{11} \land \beta_{12}$ where $\beta_{11}$ is the conjunction of ACs among and on the distinguished variables and $\beta_{12}$ the ACs on the nondistinguished variables. Now, we observe that in the containment entailment, each term on the right hand side becomes:

$$\mu_i(\beta_1) = \mu_i(\beta_{11}) \land \mu_i(\beta_{12}).$$

However, $\mu_i(\beta_{11})$ is the same for every term on the right hand side of the entailment because all the containment mappings $\mu_i$ are the same as concerns the distinguished variables, by definition. Thus, applying the distributive law, we write the containment entailment:

$$\beta_2 \Rightarrow \mu_1(\beta_{11}) \land [\mu_1(\beta_{12}) \lor \cdots \lor \mu_k(\beta_{12})].$$

Consequently, the containment entailment is equivalent to the following to the conjunction of the following two entailments:

$$\beta_2 \Rightarrow \mu_1(\beta_{11}).$$

$$\beta_2 \Rightarrow \mu_1(\beta_{12}) \lor \cdots \lor \mu_k(\beta_{12}).$$

Hereon, we will call the second entailment the \textit{containment body entailment} (or simply containment entailment when confusion does not arise) and the first the \textit{head entailment}.

This analysis is mainly valid because of the fact that the variables in a part of the ACs in $Q_1$ always map to the same variable in $Q_2$, independently of the
containment mapping from \( Q_1 \) to \( Q_2 \). Such a property could be straightforwardly extended to other cases where the conjunction of ACs of the containing query could be decomposed in two parts; the first one includes all the ACs whose variables are always mapped on the same variables of the contained query, for any containment mapping. We call these variables single-mapping variables. The other part does not use any single-mapping variables. The single-mapping variables are formally defined as follows.

**Definition 4.4.** Let \( Q_1 = Q_{10} + \beta_1 \), \( Q_2 = Q_{20} + \beta_2 \) be two CQACs, such that there is at least one containment mapping from \( Q_{10} \) to \( Q_{20} \). Consider the set \( M \) of all the containment mappings from \( Q_{10} \) to \( Q_{20} \). Each variable \( X \) of \( Q_1 \) which is always mapped on the same variable of \( Q_2 \) (i.e., for each \( \mu \in M \) the \( \mu(X) \) always equals the same variable) is called a single-mapping variable with respect to \( Q_2 \).

Notice that the head variables of \( Q_1 \) are single-mapping variables with respect to any query. Consider now that there is a predicate \( r \) such that \( Q_1 \) has \( g_{11}, g_{12}, \ldots, g_{1n} \) subgoals with predicate \( r \) and \( Q_2 \) has a single subgoal \( g_2 \) with predicate \( r \). Since each of the \( g_{11}, g_{12}, \ldots, g_{1n} \) subgoals maps on \( g_2 \), for every containment mapping from \( Q_1 \) to \( Q_2 \), the variables (also called non-self-join variables) in \( g_{11}, g_{12}, \ldots, g_{1n} \) subgoals are single-mapping variables.

Thus, we can extend the previous analysis and show that the containment entailment could be decomposed into two parts where the first part is an implication including only the ACs of the single-mapping variables, i.e., show the following proposition:

**Proposition 4.5.** Let \( Q_1 = Q_{10} + \beta_1 \), \( Q_2 = Q_{20} + \beta_2 \) be two CQACs with closed ACs, such that there is at least one containment mapping from \( Q_{10} \) to \( Q_{20} \). We assume that the set \( X_1 \) of variables of \( Q_1 \) can be partitioned into the sets \( X_1^{sv} \), \( X_1^{nsv} \), s.t. \( X_1^{sw} \cap X_1^{nsv} = \emptyset \), \( X_1^{sw} \) contains only single-mapping variables of \( Q_1 \) and there are no ACs of \( Q_1 \) joining a variable in \( X_1^{sw} \) with a variable in \( X_1^{nsv} \). Then, the containment entailment \( \beta_2 \Rightarrow \mu_1(\beta_1) \land \cdots \land \mu_k(\beta_1) \) is true if and only if both the following two are true:
\[ \beta_2 \Rightarrow \mu_1(\beta_{11}), \text{ head entailment and} \]
\[ \beta_2 \Rightarrow \mu_1(\beta_{12}) \lor \cdots \lor \mu_k(\beta_{12}), \text{ body entailment} \]

where \( \mu_1, \ldots, \mu_k \) are all the containment mappings from \( Q_{10} \) to \( Q_{20} \) and \( \beta_1 = \beta_{11} \land \beta_{12} \), where \( \beta_{11} \) includes all the ACs of \( \beta_1 \) over the variables in \( X_{1}^{sv} \), and \( \beta_{12} \) includes all the ACs of \( \beta_1 \) over the variables in \( X_{1}^{nsv} \).

### 4.3. The classes of queries

We call the pair of queries \((Q_1, Q_2)\) that have the properties as in Proposition 4.5 disjoint-AC pair.

We define a CQAC CRSII+ query, or simply RSI1+ query hereon, to be a query that:

1. It has only closed ACs.
2. There are no ACs between a head variable and a nondistinguished variable.
3. The ACs on nondistinguished variables are semi-interval ACs and there is a single right semi-interval AC.

When there are no ACs on the head variables, then we say that this is a RSII query.

Notice that, given a query \( Q_1 \) which is a RSII+ and any CQAC query \( Q_2 \) then the pair \((Q_1, Q_2)\) is a disjoint-AC pair.

The following definitions formally describes the RSI1 disjoint-AC pair.

**Definition 4.6.** Let \( V_{sm} \) be a set of single-mapping variables in \( Q_1 \) and \( V_{Q_1} \) be the set of variables of \( Q_1 \). A pair of CQACs \((Q_1, Q_2)\) is called RSI1 disjoint-AC pair with respect to \( V_{sm} \) if the following is true:

1. Both \( Q_1 \) and \( Q_2 \) have only closed ACs.

\( ^2 \)We retain the same names as in the simple case above, they are actually single-mapping entailment and non-single-mapping entailment.
2. There are no ACs between the variables in $V_{sm}$ and the variables in $V_{Q_1} - V_{sm}$.

3. The ACs in $Q_1$ are such that the following is true:

(a) The ACs on variables in $V_{Q_1} - V_{sm}$ are semi-interval (SI, for short), and

(b) there is a single right semi-interval (RSI) AC, among the ACs on variables in $V_{Q_1} - V_{sm}$.

Again, notice that, given a query $Q_1$ which is a RSI1+ and any CQAC query $Q_2$ then the pair $(Q_1, Q_2)$ is an RSI1 disjoint-AC pair.

We say that a body containment entailment is an RSI1 entailment if the ACs in each disjunct on the right hand side include only one RSI AC and the others are LSI ACs.

- For any RSI1 disjoint-AC pair, the body containment entailment is an RSI1 entailment.

- In the next section, we consider an RSI1 disjoint-AC pair of queries.

Naturally, because of symmetry, we can define LSI1 disjoint-AC pairs of queries where now only one LSI is allowed and all the results are also valid for this class.

5. CQAC Query Containment Using Datalog

The main result of this section is the following theorem:

Theorem 5.1. Consider a pair $(Q_1, Q_2)$ which is a RSI1 disjoint-AC pair of queries. Then testing containment of $Q_2$ to $Q_1$ is NP-complete.

A byproduct of the proof of this theorem is a reduction of the CQAC containment problem, in this special case, to a containment problem where we check containment of a CQ to a Datalog query, where both these queries have
no ACs, i.e., their definitions use only relational atoms. This reduction is also important in other sections of this paper where we use it to construct MCRs for CQAC queries and views and prove that certain answers can be computed in polynomial time.

As already discussed, in the rest of this section, we consider the body containment entailment of the two CQAC queries and we ignore the ACs of the containing query that are on the single-mapping variables. After ignoring such ACs, we call the resulting query, the reduced containing query. For the rest of this section, we will only refer to the reduced containing query, so, sometimes, we will say simple containing query. Note, here, that we do not ignore any AC from the contained query, since all the ACs of the contained query are required in order to check body containment entailment (see the body entailment in Proposition 4.5).

Thus this section has two large parts:

- Transformation of the reduced containing query $Q_1$ to a Datalog query and transformation of the contained query $Q_2$ into a CQ query.

- Proving that $Q_2$ is contained in $Q_1$ if and only if their transformed CQ and Datalog queries, respectively, are contained in each other.

Theorem 5.1 extends significantly the corresponding result in [29]. The transformations and the proof are similar to the transformations and the proof explained in [29] with many modifications to capture the new features. Algorithm AC-sat presented in Section 3 is missing from [29]. This algorithm is used to prove the lemmas in the beginning of the appendix.

From here on, in this section, we discuss only the body entailment, except if explicitly mentioned. We may imagine that we discuss only RSI1 containing queries. Of course, we put all together in the main theorem of this section.

5.1. The tree-like structure of the containment entailment

First, as Theorem 2.4 shows, the query normalization is not needed for testing containment into this setting. The following proposition is where the
class of RSI1s comes useful.

**Proposition 5.2.** Let $\beta$ be a conjunction of closed ACs which is consistent, and each $\beta_1, \beta_2, \ldots, \beta_k$ be a conjunction of closed RSI1s. Suppose the following is true:

$$\beta \Rightarrow \beta_1 \vee \beta_2 \vee \ldots \vee \beta_k.$$ 

Then there is a $\beta_i$ (w.l.o.g. suppose it is $\beta_1$) such that either of the following two happens:

1. $\beta \Rightarrow \beta_1$, or
2. There is an AC $e$ in $\beta_1$ such that the following are true:
   - $\beta \Rightarrow \beta_2 \vee \ldots \vee \beta_k$ (or equivalently, $\beta \Rightarrow \beta_2 \vee \ldots \vee \beta_k \vee e$),
   - $\beta \Rightarrow \beta_1 \vee \neg e$, and
   - All the other ACs, besides $e$, in $\beta_1$ are directly implied by $\beta$.

**Proof.** Suppose there is no $\beta_i$ such that $\beta \Rightarrow \beta_i$.

Then we claim that there is a $\beta_i$ (w.l.o.g. suppose it is $\beta_1$) such that all the ACs in $\beta_1$ are directly implied by $\beta$ (i.e., $\beta \Rightarrow e_i$ if $e_i$ is an AC in $\beta_1$), except for one AC $e_i$, i.e., we claim that also the following is true:

$$\beta \Rightarrow \beta_1 \vee \neg e$$

Towards contradiction, suppose that for all the $\beta_i$’s there are at least two ACs that are not directly implied by $\beta$. Since all the $\beta_i$’s are RSI1s, each $\beta_i$ has at least one LSI that is not directly implied. If we take all these LSI’s after applying the distributive law and converting the right-hand side from a disjunction of conjunctions to a conjunction of disjunctions, then we will have a conjunct that contains only LSIs, none of which is directly implied by $\beta$. We prove now that this is impossible — i.e., it is not true that $\beta \Rightarrow ac_1 \vee ac_2 \cdots$ if none of the LSI $ac_i$ is directly implied by $\beta$. This is proved in Lemma [Appendix A.1](#).
Now we write equivalently the implication in the statement of the proposition as:

\[ \beta \land \neg \beta_1 \Rightarrow \beta_2 \lor \beta_3 \cdots \lor \beta_k, \]

or equivalently (assuming \( \beta_1 = e_1 \land \cdots \land e_t \), where the \( e_i \)s are ACs)

\[ (\beta \land \neg e_1) \lor (\beta \land \neg e_2) \lor \cdots \lor (\beta \land \neg e_t) \Rightarrow \beta_2 \lor \beta_3 \cdots \lor \beta_k. \]

Assume w.l.o.g. that \( e = e_1 \). Since each \( e_i \), with the exception of \( e_1 \), is entailed by \( \beta \), each disjunct with the exception of the first one in the left-hand side is always false. Hence, the latter entailment yields:

\[ \beta \land \neg e \Rightarrow \beta_2 \lor \beta_3 \cdots \lor \beta_k. \quad \square \]

Proposition 5.2 informally says that, considering a containment entailment in the special case of RSI1 containing queries and CQAC contained queries\(^3\), then, in the right hand side of the containment entailment, there is a disjunct that has the property that all its ACs but one are directly implied by the left hand side of the containment entailment.

The above proposition begins to show a tree-like structure of the containment entailment and it gives the first intuition for constructing a Datalog query from the containing query that will help in deciding query containment. The following example gives an illustration of this intuition.

**Example 5.3.** Let us consider the following two Boolean queries.

\[ Q_1 : q() : - a(X,Y,Z), X \leq 8, Y \leq 7, Z \geq 6. \]
\[ Q_2 : q() : - a(X,Y,Z), a(U_1,U_2,X), a(V_1,V_2,Y), \]
\[ a(Z,Z_1,Z_2), a(U'_1,U'_2,U_1), a(V'_1,V'_2,V_1), \]
\[ U'_1 \leq 8, U'_2 \leq 7, U_2 \leq 7, V'_1 \leq 8, \]
\[ V'_2 \leq 7, V_2 \leq 7, Z_1 \leq 7, Z_2 \geq 6. \]

\(^3\)we will always mean that we have closed ACs in the queries although sometimes we may not say so
The query $Q_2$ is contained in the query $Q_1$. To verify this, notice that there are 6 containment mappings from $Q_1$ to $Q_2$. These mappings are given as follows: $\mu_1 : (X, Y, Z) \rightarrow (X, Y, Z)$, $\mu_2 : (X, Y, Z) \rightarrow (U_1, U_2, X)$, $\mu_3 : (X, Y, Z) \rightarrow (V_1, V_2, Y)$, $\mu_4 : (X, Y, Z) \rightarrow (Z, Z_1, Z_2)$, $\mu_5 : (X, Y, Z) \rightarrow (U_1', U_2', U_1)$, and $\mu_6 : (X, Y, Z) \rightarrow (V_1', V_2', V_1)$. After replacing the variables as specified by the containment mappings, the query entailment is $\beta \Rightarrow \beta_1 \lor \beta_2 \lor \beta_3 \lor \beta_4 \lor \beta_5 \lor \beta_6$, where:

$\beta : U_1' \leq 8 \land U_2' \leq 7 \land U_2 \leq 7 \land V_1' \leq 8 \land V_2' \leq 7 \land V_2 \leq 7 \land Z_1 \leq 7 \land Z_2 \geq 6.$

$\beta_1 : X \leq 8 \land Y \leq 7 \land Z \geq 6.$

$\beta_2 : U_1 \leq 8 \land U_2 \leq 7 \land X \geq 6.$

$\beta_3 : V_1 \leq 8 \land V_2 \leq 7 \land Y \geq 6.$

$\beta_4 : Z \leq 8 \land Z_1 \leq 7 \land Z_2 \geq 6.$

$\beta_5 : U_1' \leq 8 \land U_2' \leq 7 \land U_1 \geq 6.$

$\beta_6 : V_1' \leq 8 \land V_2' \leq 7 \land V_1 \geq 6.$

Note that the entailment is true since the variables $X$, $Y$, $Z$, $V_1$ and $U_1$ (the variables that are in the intersections of the circles) could take any value without affecting the output of the implication.

We now refer to Figure 1 to offer some intuition about and visualization on Proposition 5.2, using the above queries. The circles in the figure represent

\footnote{we always mean containment mappings from the relational subgoals of $Q_1$ to the relational subgoals of $Q_2$}
the mappings $\mu_1, \ldots, \mu_6$, and the dots are the variables of $Q_2$. Notice now the intersections between the circles. Proposition 5.2 refers to these intersections, such as the one between $\mu_3$ and $\mu_6$ (or, the one between $\mu_2$ and $\mu_5$).

The $AC V_1 \geq 6$ ($V_1$ is included in the intersection between $\mu_3$ and $\mu_6$) is the one that is not directly implied by $\beta$, as stated in the case (ii) of the Proposition 5.2. In particular, it is easy to verify that the following are true:

- $\beta \wedge \neg (V_1 \geq 6) \Rightarrow \beta \vee \beta_2 \vee \beta_3 \vee \beta_4 \vee \beta_5$.
- $\beta \Rightarrow \beta_6 \vee \neg (V_1 \geq 6)$ (i.e., $\beta \Rightarrow (V'_1 \leq 8 \wedge V'_2 \leq 7 \wedge V_1 \geq 6) \vee \neg (V_1 \geq 6)$).
- $\beta \Rightarrow (V'_1 \leq 8)$ and $\beta \Rightarrow (V'_2 \leq 7)$.

We will use the Proposition 5.4 to see how the $AC$ on this variable is related to other $AC$s on the same variable in another mapping (here it is the mapping $\mu_4$).

Proposition 5.4 is a generalization of Proposition 5.2.

**Proposition 5.4.** Let $\beta$ be a conjunction of closed SI $AC$s which is consistent, and $\beta_1, \beta_2, \ldots, \beta_k$ each be a conjunction of closed RSIs (i.e., in each conjunct there is only one RSI and the rest are LSI ACs). Suppose the following is true:

$$\beta \Rightarrow \beta_1 \vee \beta_2 \vee \cdots \vee \beta_k \vee e_1 \vee e_2 \vee \cdots$$

where $e_i$s are closed SIs such that the following implication is not true: $\beta \Rightarrow e_1 \vee e_2 \vee \cdots$. Then there is a $\beta_i$ (w.l.o.g. suppose it is $\beta_1$) such that either of the following two happen:

(i) $\beta \Rightarrow \beta_1 \vee e_1 \vee e_2 \vee \cdots$, or

(ii) there is an $AC$ $e$, called special for this mapping, in $\beta_1$ such that the following are true:

(a) $\beta \wedge \neg e \Rightarrow \beta_2 \vee \cdots \vee \beta_k \vee e_1 \vee e_2 \vee \cdots$, or equivalently,

$$\beta \Rightarrow \beta_2 \vee \cdots \vee \beta_k \vee e \vee e_1 \vee e_2 \vee \cdots,$$
(b) $\beta \Rightarrow \beta_1 \lor e \lor e_1 \lor e_2 \lor \cdots$ and

c) all the other ACs $ac_j$ in $\beta_1$, with $j = 1, 2, \ldots$, besides $e$, are either
directly implied by $\beta$ or coupled with one of the $e_i$s for $\beta$ i.e., either
$\beta \Rightarrow ac_j$ or $\beta \Rightarrow e_i \lor ac_j$.

Proof. Suppose there is no $\beta_i$ such that

$$\beta \Rightarrow \beta_i \lor e_1 \lor e_2 \lor \cdots$$

Then we claim that there is a $\beta_i$ (w.l.o.g. suppose it is $\beta_1$) such that all the ACs
$a_i$ in $\beta_1$ are such that $a_i \lor e_1 \lor \cdots$ is directly implied by $\beta$ (i.e., $\beta \Rightarrow a_i \lor e_1 \lor \cdots$
if $a_i$ is an AC in $\beta_1$), except for one AC $a_1 = e$ (wlog suppose this is $a_1$), i.e.,
we claim that the following is true for $e$:

$$\beta \Rightarrow \beta_1 \lor \neg e \lor e_1 \lor e_2 \lor \cdots$$

Towards contradiction, suppose that for all the $\beta_i$s there are at least two ACs
(say AC $a_{i12}$ is such an AC) such that the following does not happen:

$$\beta \Rightarrow a_{i12} \lor e_1 \lor e_2 \lor \cdots$$ \hfill (2)

Since all the $\beta_i$’s are RSI1s, each $\beta_i$ has at least one LSI for which the implication [2] is not true. If we take all these LSI’s (after applying the distributive
law and converting the right-hand side from a disjunction of conjunctions to a
conjunction of disjunctions), then we will have a conjunct that contains only
LSIs, none of which is such that the implication [2] is true. Then we will have a
case like in Lemma [Appendix A.2] According to Lemma [Appendix A.2] there
are two cases: a) There is a single SI on the rhs which is implied by $\beta$ or b) there are two SI in the rhs whose disjunction is implied, of which one is LSI and
one is RSI. Thus, in both cases, we have only one LSI, say it is $a_{LSI}$ such that

$$\beta \Rightarrow a_{LSI} \lor e_1 \lor e_2 \lor \cdots.$$

This is a contradiction to our assumption.
We write equivalently the implication in the statement of the proposition as:

$$\beta \land \neg[\beta_1 \lor e_1 \lor e_2 \lor \cdots] \Rightarrow \beta_2 \lor \beta_3 \lor \cdots \lor \beta_k$$

or equivalently (assuming $$\beta_1 = a_1 \land \cdots \land a_t$$, where the $$e_i$$s are ACs)

$$(\beta \land \neg a_1 \land \neg e_1 \land \neg e_2 \land \cdots) \lor (\beta \land \neg a_2 \land \neg e_1 \land \neg e_2 \land \cdots) \lor \cdots \lor (\beta \land \neg a_t \land \neg e_1 \land \neg e_2 \land \cdots)$$

$$\Rightarrow \beta_2 \lor \beta_3 \lor \cdots \lor \beta_k$$

Assume w.l.o.g. that $$e = a_1$$. Since each $$a_i \lor e_1 \lor \cdots$$, with the exception of $$a_1$$, is entailed by $$\beta$$, each disjunct with the exception of the first one in the left-hand side is always false. Hence, the latter entailment yields:

$$\beta \land \neg e \Rightarrow \beta_2 \lor \beta_3 \lor \cdots \lor \beta_k \lor e_1 \lor e_2 \lor \cdots \lor \beta_k$$

Example 5.5. Continuing Example 5.3, we will use the Proposition 5.4 to see how the AC on the variable $$V_1$$ is related to other ACs on the same variable in another mapping (here it is the mapping $$\mu_4$$). To see that, notice that the AC $$V_1 \geq 6$$ is the special AC for $$\mu_6$$ and it is coupled with the AC $$V_1 \leq 8$$ in $$\beta_3$$ (i.e., $$\beta \Rightarrow (V_1 \geq 6) \lor (V_1 \leq 8)$$). In particular, as we saw in Example 5.3, the following is true.

$$\beta \Rightarrow \beta_1 \lor \beta_2 \lor \beta_3 \lor \beta_4 \lor \beta_5 \lor (V_1 \geq 6)$$

Then, according to the Proposition 5.4 (where $$e_1 = V_1 \geq 6$$), there is $$\beta_i$$ (in this case, $$\beta_3$$ is such a $$\beta_i$$) such that the following are true (case (ii) in the proposition):

- $$\beta \land \neg (Y \geq 6) \Rightarrow \beta_1 \lor \beta_2 \lor \beta_4 \lor \beta_5 \lor (V_1 \geq 6)$$.
- $$\beta \Rightarrow \beta_3 \lor \neg (Y \geq 6) \lor (V_1 \geq 6)$$; i.e.,

$$\beta \Rightarrow (V_1 \leq 8 \land V_2 \leq 7 \land Y \geq 6) \lor \neg (Y \geq 6) \lor (V_1 \geq 6)$$.

- $$\beta \Rightarrow (V_2 \leq 7)$$, while $$V_1 \leq 8$$ is coupled with $$V_1 \geq 6$$.

We give a first glance of what is going to happen in the rest of this section. In particular, we do the following:
1. We transform the containing query $Q_1$ into a Datalog query $Q_{Q_1}^{Datalog}$.

2. We transform the contained query into a CQ, $Q_{Q_2}^{CQ}$.

3. The above two transformations are done by keeping the relational subgoals of $Q_1$ ($Q_2$, respectively) and encoding the arithmetic comparisons into relational predicates.

4. We prove (Theorem 5.9) that $Q_2$ is contained in $Q_1$ if and only if $Q_{Q_2}^{CQ}$ is contained in $Q_{Q_1}^{Datalog}$.

Intuitively, using those transformations we aim to replace the ACs with relations; hence, transform the problem of CQAC containment to a containment problem of a Datalog query in a CQ. One might wonder why the transformation of the containing query to a Datalog query is required. The answer to this question is based on the containment entailment. The disjunction in the right-hand-side implies arbitrary combinations of the ACs.

5.2. Construction of Datalog Query for Containing Query

In this subsection, we describe the construction of a Datalog query for a given RSI1 query $Q$. When we consider the ACs in $Q_1$, we consider all the SI ACs in the closure of the ACs on the non-single-mapping variables.

The Datalog query has two kinds of rules: The rules that depend only on the containing query, and we call them basic rules, and the rules that also take into account the contained query, and we call them dependant rules.

In various places, in order to illustrate the construction, we will use the query in the following running example.

Example 5.6. The following query $Q_1$ is an RSI1 query:

$$Q_1(W_1, W_2) : - a(W_1, W_2, Y), e(X, Y), e(Y, Z), X \geq 5, Z \leq 8.$$  

For simplicity in the notation we will denote by $\overline{W}$ the vector $W_1, W_2$ of head variables. Thus, we are writing the query as:

$$Q_1(\overline{W}) : - a(\overline{W}, Y), e(X, Y), e(Y, Z), X \geq 5, Z \leq 8.$$
Construction of the basic rules $Q_1^{\text{Datalog}}$: We construct three kinds of rules, mapping rules, coupling rules, and a single query rule.

First, we introduce the EDB predicates and the IDB predicates that we use and how we construct them. The EDB predicates are all the predicates from the relational subgoals of $Q_1$ and an extra binary predicate $U$. Intuitively, $U(X,Y)$ encodes the inequality $X \leq Y$. Now, the IDB predicates are as follows:

1. We introduce new semi-unary IDBs\(^a\) two pairs for each constant $c$ in $Q_1$ that compares a non-single-mapping variable to this constant, namely $I_{\geq c}$, $I_{\leq c}$ and $J_{\geq c}$, $J_{\leq c}$. These predicates have as arguments the vector $W$ of variables in the head of the query $Q_1$ and another variable $X$.

2. For each inequality $X \theta c$, we construct the IDB predicate atoms $I_{\theta c}(X,W)$ and $J_{\theta c}(X,W)$, where $\theta$ is either $\leq$ or $\geq$.

3. For each inequality $X \theta c$, considering the IDB predicate atom $I_{\theta c}(X,W)$ ($J_{\theta c}(X,W)$, respectively), we refer to $J_{\theta c}(X,W)$ ($I_{\theta c}(X,W)$, respectively), as the associated $I$-atom (associated $J$-atom respectively) of $X \theta c$, and we refer to $X \theta c$ as the associated AC of $I_{\theta c}(X,W)$ ($J_{\theta c}(X,W)$, respectively). We also refer to $I_{\theta c}(X,W)$ as the associated $I$-atom of $J_{\theta c}(X,W)$ and vice versa.

4. We have also a query IDB predicate which is denoted $Q_1^{\text{Datalog}}(W)$

Now, we describe the construction of the basic rules of the Datalog query which use the EDB predicates of the containing query and are as follows. Notice, however, that the basic rules depend on the definition of the single-mapping variables, which, in turn, depends on the contained query as well. One case where the basic rules do not have to consider the contained query is when the single-mapping variables coincide with the head variables. We call them basic because they do not depend on the ACs of the contained query.

\(^a\)We call them semi-unary for reasons that will become apparent later during the proof.
1. The query rule copies into its body all the relational subgoals of $Q_1$, and replaces each AC subgoal of $Q_1$ that compares a non-single-mapping variable to a constant by its associated $I$-atom. The head of this rule is the same as the head of the query $Q_1$.

2. We get one mapping rule for each SI arithmetic comparison $e$ in $Q_1$ which is on a non-single-mapping variable. The body of each mapping rule is a copy of the body of the query rule, except that the $I$ atom associated with $e$ is deleted. The head is the $J$ atom associated with $e$.

3. For every pair of constants $c_1 \leq c_2$ used in $Q_1$, we construct two coupling rules. One rule is $I_{\leq c_2}(X, W) : - J_{\geq c_1}(X, W)$, and the other rule is $I_{\geq c_1}(X, W) : - J_{\leq c_2}(X, W)$.

4. We also construct coupling rules that use the binary EDB predicates. In particular, one coupling rule for each combination of an $I$ predicate with a $J$ predicate with variables $X, Y$, where their ACs $\theta$ and $\theta'$ are such that $X \leq Y \Rightarrow c\theta X \lor c'\theta' Y$.

   We add a new coupling rule of the form:

   $I_{\leq c_1}(X, W) : - J_{\geq c_2}(Y, W), U(X, Y)$.

   whenever $c_1 \geq c_2$.

Example 5.7. For the query $Q_1$ of Example 5.6, the construction we described yields the following basic rules of the Datalog query $Q_1^{Datalog}$:

\[
Q_1^{Datalog}(W) : - e(X, Y), e(Y, Z), a(W, Y), I_{\geq 5}(X, W), I_{\leq 8}(Z, W). \quad \text{(query rule)}
\]

\[
J_{\leq 8}(Z, W) : - e(X, Y), e(Y, Z), a(W, Y), I_{\geq 5}(X, W). \quad \text{(mapping rule)}
\]

\[
J_{\geq 5}(X, W) : - e(X, Y), e(Y, Z), a(W, Y), I_{\leq 8}(Z, W). \quad \text{(mapping rule)}
\]

\[
I_{\leq 8}(X, W) : - J_{\geq 5}(X, W). \quad \text{(coupling rule)}
\]

\[
I_{\geq 5}(X, W) : - J_{\leq 8}(X, W). \quad \text{(coupling rule)}
\]

\[
I_{\leq 8}(X, W) : - J_{\geq 5}(Y, W), U(X, Y). \quad \text{(coupling rule)}
\]

\[
I_{\geq 5}(X, W) : - J_{\leq 8}(Y, W), U(X, Y). \quad \text{(coupling rule)}
\]
The intuition of a coupling rule is that it denotes that a formula $AC_1 \lor AC_2$ is true for two SI comparisons $AC_1$ and $AC_2$. Thus, the first coupling rule in the above query says that $X \leq 8 \lor X \geq 5$ is true and the second coupling rule says the same but referring to different $I$ and $J$-atoms.

**Construction of the dependant rules $Q_1^{Datalog}$:**

First, we describe the EDB predicates that we introduce (they all depend on the ACs of the contained query):

- A unary predicate $U_{\theta c}(X, W)$, where $\theta$ is either $\leq$ or $\geq$, and $W$ is the vector of head variables, for each SI AC $X\theta c$ in the closure of the ACs in the contained query. Note that although $U_{\theta c}$ typically includes the vector of head variables, in the following, we could ignore it, for simplicity.

We have one kind of dependant rules, the *link rules*:

- So far we have constructed the recursive rules. Now we add a number of base rules, one rule of the form $I_e(X, W) : \neg U_e(X, W)$ for each combination of unary EDB predicate and semi-unary IDB predicate for each $e \Rightarrow e'$. These rules are called *link rules*.

More specifically, we do as follows: For each pair of constants $(c_1, c_2)$, one in SIs of $Q_1$ and the other in an SI in the closure of ACs of $Q_2$ and for which the AC in $Q_1$ is $X \geq c_1$ and the AC in $Q_2$ is $X \geq c_2$ then, if $c_1 \leq c_2$, we add the non-recursive link rule:

$$I_{\geq c_1}(X, W) : \neg U_{\geq c_2}(X, W).$$

Similarly, we perform in a symmetric way for the $\leq$ ACs in $Q_1$ and $Q_2$.

Thus, each link rule encodes an entailment of the form $X \leq 7 \Rightarrow X \leq 8$, i.e., it encodes, in general, an entailment $X \leq c_1 \Rightarrow X \leq c_2$ where $c_1 \leq c_2$.

For an example of dependant rules see next subsections.
5.3. Construction of CQ for Contained Query

When we consider the ACs in \( Q_2 \), we consider all the ACs in the closure of the ACs.

We now describe the construction of the contained query \( Q_2 = Q'_2 + \beta_2 \) turned into a CQ \( Q_{2,CQ} \), illustrated by an example.

**Construction of \( Q_{2,CQ} \):** We introduce new unary EDBs, specifically two of them, by the names \( U_{\geq c} \) and \( U_{\leq c} \), for each constant \( c \) in \( Q_2 \). In addition, we introduce a new binary predicate \( U \) which represents the closed SI ACs between two variables. Let us now construct the CQ \( Q_{2,CQ} \) from \( Q_2 \). We initially copy the regular subgoals of \( Q_2 \), and for each SI \( X_i \theta c_i \) in the closure of \( \beta_2 \) we add a unary predicate subgoal \( U_{\theta c_i}(X_i) \). Then, for each AC \( X \leq Y \) we add the unary subgoal \( U(X,Y) \) in the body of the rule.

For example, considering the CQAC \( Q_2 \) with the following definition:

\[
Q_2(W_1, W_2) : - e(A, B), e(B, C), e(C, D), e(D, E), A \geq 6, E \leq 7, a(W_1, W_2, B), a(W_1, W_2, D).
\]

we construct the \( Q_{2,CQ} \) whose definition is:

\[
Q_{2,CQ}(W_1, W_2) : - e(A, B), e(B, C), e(C, D), e(D, E), U_{\geq 6}(A), U_{\leq 7}(E), a(W_1, W_2, B), a(W_1, W_2, D).
\]

Thus the dependant rules for our running example, query \( Q_1 \), and the above contained query \( Q_2 \) are:

\[
I_{\geq 5}(X, W) : - U_{\geq 6}(X, W). \quad \text{link rule}
\]

\[
I_{\leq 8}(X, W) : - U_{\leq 7}(X, W). \quad \text{link rule}
\]

Now, we have completed the description of the construction of both \( Q_1^{Datalog} \) from \( Q_1 \) and \( Q_{2,CQ} \) from \( Q_2 \). We go back to our examples and put all together.

**Example 5.8.** Our contained query is the one in Subsection 5.3. Our containing query is the one in Example 5.6. The transformation of the contained query is shown in Subsection 5.3. The transformation of the contained query is shown in Example 5.7, where we see the basic rules. To complete the Datalog query, we add the following link rules:
\[ I_{\geq 5}(X, W) : - U_{\geq 6}(X, W). \text{ link rule} \]
\[ I_{\leq 8}(X, W) : - U_{\geq 7}(X, W). \text{ link rule} \]

I.e., we constructed the two new link rules in the Datalog query for \( Q_1 \). One rule links the constant 6 from the ACs of \( Q_2 \) to the constant 5 from the ACs of \( Q_1 \). The other link rule links constants 7 and 8 from queries \( Q_1 \) and \( Q_2 \) respectively.

5.4. Proving the main theorem and the complexity

The constructions of the Datalog query and the CQ presented in Sections 5.2 and 5.3 respectively, lead to the following theorem.

**Theorem 5.9.** Consider two conjunctive queries with arithmetic comparisons, \( Q_1 \) and \( Q_2 \) such that \((Q_1, Q_2)\) is an RSI1 disjoint-AC pair. Then, \( Q_1 \) contains \( Q_2 \) if and only if the following two happen a) \( Q_1^{\text{Datalog}} \) contains \( Q_2^{\text{CQ}} \) and b) the head entailment is true.

The challenging part of the Theorem 5.9 concerns the part (a) which is restated in the Theorem 5.10. The part (b) of Theorem 5.9 is a straightforward consequence of Proposition 4.5.

**Theorem 5.10.** Consider two conjunctive queries with arithmetic comparisons, \( Q_1 \) and \( Q_2 \) such that \((Q_1, Q_2)\) is an RSI1 disjoint-AC pair. Let \( Q_1^{\text{Datalog}} \) be the transformed Datalog query of \( Q_1 \). Let \( Q_2^{\text{CQ}} \) be the transformed CQ query of \( Q_2 \). Then, the body containment entailment for containment of \( Q_2 \) to \( Q_1 \) is true if and only if \( Q_1^{\text{Datalog}} \) contains \( Q_2^{\text{CQ}} \).

The proof of Theorem 5.10 is in the Appendix C. The following theorem proves that checking body containment entailment is NP-complete.

**Theorem 5.11.** Consider two conjunctive queries with arithmetic comparisons, \( Q_1 \) and \( Q_2 \) such that \((Q_1, Q_2)\) is an RSI1 disjoint-AC pair. Let \( Q_1^{\text{Datalog}} \) be the transformed Datalog query of \( Q_1 \). Let \( Q_2^{\text{CQ}} \) be the transformed CQ query of \( Q_2 \). Checking whether \( Q_2^{\text{CQ}} \) is contained in \( Q_1^{\text{Datalog}} \) is NP-complete.
Theorem 5.11 can be generalized to a stronger result, which is presented in Section 5.7. Hence, it is a corollary of the Theorem 5.16. Theorem 5.1 is a straightforward consequence of Theorem 5.12.

**Theorem 5.12.** Consider two conjunctive queries with arithmetic comparisons, $Q_1$ and $Q_2$ such that $(Q_1, Q_2)$ is an RSI1 disjoint-AC pair. Let $\phi_h$ and $\phi_b$ be the head and body entailments, respectively. Then, checking $\phi_h$ is polynomial and checking $\phi_b$ is NP-complete.

Consider two conjunctive queries with arithmetic comparisons, $Q_1$ and $Q_2$ such that $Q_1$ is an RSI1+ query and $Q_2$ is a CQAC with closed ACs. It is straightforward that $(Q_1, Q_2)$ is a RSI1 disjoint-AC pair with respect to the set of head variables of $Q_1$.

**Corollary 5.13.** Consider two conjunctive queries with arithmetic comparisons, $Q_1$ and $Q_2$ such that $Q_1$ is an RSI1+ query and $Q_2$ is a CQAC with closed ACs. Let $\phi_h$ and $\phi_b$ be the head and body entailments, respectively. Then, checking $\phi_h$ is polynomial and checking $\phi_b$ is NP-complete.

### 5.5. More examples to illustrate the technique

Another example to use later to illustrate the functionality of the second kind of coupling rules.

**Example 5.14.** Consider a relational schema with the binary relations $e$ and $a$, as well as the following two CQACs over this schema.

\[
Q_1 : q(W_1, W_2) : - a(W_1, W_2, Y), e(X, Y), e(Y, Z), X \geq 5, Z \leq 5
\]

\[
Q_2 : q(W_1, W_2) : - e(A, B), e(B, C_1), e(C_2, D), e(D, E), a(W_1, W_2, B),
\]

\[
a(W_1, W_2, D), C_1 \leq C_2, A \geq 5, E \leq 5
\]

Checking the containment $Q_2 \subseteq Q_1$, note that there are two containment mappings $\mu_1, \mu_2$ from $Q_{10}$ to $Q_{20}$ such that $\mu_1(W_i) = \mu_2(W_i) = W_i$, and

- $\mu_1 : Y \rightarrow B, X \rightarrow A, Z \rightarrow C_1$.
- $\mu_2 : Y \rightarrow D, X \rightarrow C_2, Z \rightarrow E$. 36
Then, applying the mappings on the query entailment we conclude the following implication:

\[(C_1 \leq C_2) \land (A \geq 5) \land (E \leq 5)) \Rightarrow ((A \geq 5) \land (C_1 \leq 5)) \lor ((C_2 \geq 5) \land (E \leq 5))\]

Analyzing the aforementioned entailment, it is easy to verify that it is true, since \((C_1 \leq C_2) \Rightarrow (C_1 \leq c) \lor (C_2 \geq c)\) is true for every constant \(c\); hence, \(Q_2 \sqsubseteq Q_1\).

Let us now construct \(Q_{\text{Datalog}}^1\) from \(Q_1\) and \(Q_{\text{CQ}}^2\) from \(Q_2\). To construct \(Q_{\text{Datalog}}^1\) from \(Q_1\) we follow the algorithm in Section 5.2. In particular, we initially construct the query rule, which is given as follows. For simplicity in the notation, we will denote by \(W\) the vector of head variables \(W_1, W_2\). Note that the subgoals \(I_\geq 5(X), I_\leq 5(Z)\) correspond to the ACs \(X \geq 5\) and \(Z \leq 5\), respectively.

\[Q_{\text{Datalog}}^1 : q(W) : - e(X,Y), e(Y,Z), a(W,Y), I_\geq 5(X), I_\leq 5(Z)\]

Then, we construct the basic mapping and coupling rules, which are given by the following rules:

\[J_\geq 5(X, W) : - e(X,Y), e(Y,Z), a(W,Y), I_\leq 5(Z, W) \quad \text{(mapping rule)}\]
\[J_\leq 5(Z, W) : - e(X,Y), e(Y,Z), a(W,Y), I_\geq 5(X, W) \quad \text{(mapping rule)}\]
\[I_\leq 5(X, W) : - J_\geq 5(X, W) \quad \text{(coupling rule)}\]
\[I_\geq 5(X, W) : - J_\leq 5(X, W) \quad \text{(coupling rule)}\]
\[I_\leq 5(X, W) : - J_\geq 5(Y, W), U(X,Y) \quad \text{(coupling rule)}\]
\[I_\geq 5(X, W) : - J_\leq 5(Y, W), U(X,Y) \quad \text{(coupling rule)}\]

To find the \(Q_{\text{CQ}}^2\), we initially copy the head \(Q_2\), along with its relational subgoals. Then, we consider the subgoal \(U(C_1, C_2)\) representing the AC \(C_1 \leq C_2\), as well as the unary subgoals \(U_\geq 5(A)\) and \(U_\leq 5(E)\) to represent the ACs \(A \geq 5\) and \(E \leq 5\), respectively. Consequently, we end up with the following CQ definition:

\[Q_{\text{CQ}}^2 : q(W_1, W_2) : - e(A,B), e(B,C_1), e(C_2, D), e(D, E), a(W_1, W_2, B), a(W_1, W_2, D), U(C_1, C_2), U_\geq 5(A), U_\leq 5(E)\]
Finally, the link rules included in the Datalog query $Q_1^{\text{Datalog}}$ are constructed as follows:

\[
I_{\leq 5}(X, W) : \neg U_{\leq 5}(X, W)
\]
\[
I_{\geq 5}(X, W) : \neg U_{\geq 5}(X, W)
\]

Useful observation: Notice that, because of the restrictions we have assumed on our queries, $W$ as it appears in the construction of the Datalog query does not contain any of the variables in the first position of a semi-unary predicate.

Finally, it helps with the intuition to observe the following: Even if the query $Q_1$ was different but only as concerns AC that involve head variables, the Datalog query would be the same because we do the test for such ACs in the preliminary step. Thus the following CQAC would have been transformed to the same query as above:

\[
Q_1(W_1, W_2) : \neg a(W_1, W_2, Y), e(X, Y), e(Y, Z), X \geq 5, Z \leq 8, W_1 < W_2, W_1 < 4.
\]

5.6. Preliminary partial results and Intuition

Here we make several observations about $Q_2^{\text{CQ}}$ and $Q_1^{\text{Datalog}}$ that will be useful in the proof of the main result.

First, it is easy to see that the Datalog query $Q_1^{\text{Datalog}}$, for any CQAC $Q_1$, has only semi-unary recursive predicates and there are two kinds of them: the $J$ predicates that either appear in the head of a mapping rule or, as the single subgoal, in the body of a coupling rule and the $I$ predicates that either appear in the body of a mapping rule or in the head of a coupling rule. There are only two kinds of recursive rules, the coupling rules and the mapping rules. Finally, each computation of $Q_1^{\text{Datalog}}$ has a mapping round following a coupling round and vice versa. A mapping round applies all the mapping rules that are currently applicable and a coupling round applies all the coupling rules that are currently applicable.

**Definition 5.15.** For an $I$ fact and a $J$ fact with associated ACs $e_I$ and $e_J$ respectively, we say that the two facts (or the two ACs) are coupled if $\beta_2 \Rightarrow e_I \lor e_J$. 
Now we are turning our attention to the application of a mapping rule. The following remark starts discussing the interconnection between the application of a mapping rule and a mapping from the relational subgoals of query $Q_1$ to the relational subgoals of query $Q_2$ that satisfies certain implications for the ACs.

- When we fire a mapping rule we use a mapping, $\mu$, from its relational subgoals in the body on the canonical database of the transformed $Q_2$, $D$. Notice that mapping $\mu$ can be viewed also as a mapping from the relational subgoals of $Q_1$ to the relational subgoals of $Q_2$. We say that $\mu(\beta_1)$ is the *associated logical expression* for $\mu$. Notice that $\mu(\beta_1)$ is a conjunction of all associated ACs of the $I$ facts in the body of the mapping rule and the associated AC in the head of the mapping rule.

Now we need to consider the base rules too:

- In a similar way we define associated ACs for non-recursive $U$ facts. These ACs are all implied by $\beta_2$ (by construction).

**Convention:** Each application of a coupling rule has used a $J$ fact that couples with the $I$ fact computed by this application of the coupling rule. Thus we say (slightly abusively) that a mapping rule when fired uses a certain $J$ fact instead of saying that it uses its coupled $I$ fact.

Summarizing our remarks we have:

The link rules are nonrecursive rules. The query rule is also nonrecursive, it is fired only once in any computation. Moreover the recursive predicates are semi-unary predicates, in that, during the whole computation, the $W$ remains the same, while there is only one variable in the head of the recursive rules that takes several instantiations during the computation. Thus, hereon, we will refer only to this variable, ignoring the $W$.

Now, in our proof, we will apply the Datalog query on the canonical database of the CQ query constructed from the contained query $Q_2$. This canonical database uses constants (different from the constants in the ACs) that correspond one-to-one to variables of the query $Q_2$. Thus, as we compute facts, each
fact being either an $I$ fact or a $J$ fact, we do the following observations about the result of firings for each of the two kinds of recursive rules (i.e., the coupling rules and the mapping rules): (all the $\theta$s represent either $\leq$ or $\geq$ and the $c_i$s are constants from the ACs of the queries.

• We have two kinds of coupling rules. Consider a coupling rule of the first kind which is of the form:

$$I_{\theta_1c_1}(X, W) : \neg J_{\theta_2c_2}(X, W).$$

When this rule is fired, its variable $X$ is instantiated to a constant, $y$, in the canonical database, $D$, of $Q_{20}$. The constant $y$ corresponds to the variable $Y$ of $Q_2$ by convention. Then the following is true by construction:

$$\beta_2 \Rightarrow X\theta_1c_1 \lor X\theta_2c_2,$$

Now consider the other kind of coupling rule, which is of the form:

$$I_{\theta_1c_1}(X, W) : \neg J_{\theta_2c_2}(Y, W), U(X,Y).$$

By construction of the rule, the EDB $U(X,Y)$ is mapped in $D$ to two constants/variables (now we are allowed to use these two words in conjunction, after the explanations above) such that there in $Q_2$ an AC which is $X \leq Y$. Thus, by construction of the rule, the following is true again:

$$\beta_2 \Rightarrow X\theta_1c_1 \lor Y\theta_2c_2$$

We say in both cases of coupling rules that the facts in both sides of the rule are coupled and that the corresponding ACs are coupled.

• Consider a mapping rule

$$J_{\theta_1c_1}(Z, W) : \neg \text{Body}Q_1, I_{\theta_2c_2}(X, W), I_{\theta_3c_3}(X, W), \ldots.$$

The $\text{Body}Q_1$ denotes all the relational subgoals of $Q_1$. When a mapping rule is fired, then there is a containment mapping, $\mu$, from the relational
subgoals of $Q_1$ to the relational subgoals of $Q_2$ and, moreover, the $I$ facts in the body of the rule have been computed in previous rounds of the computation.

Since all the $I$ facts in the body of the rule (for the instantiation that fires the rule) are computed via coupling rules using $J$ facts, each such $I$ fact is coupled with a $J$ fact. Notice that each $I$ fact corresponds to an AC in $\mu(\beta_1)$ by construction of a mapping rule. Putting the implications we derived for coupling rules above together for all $I$ facts in the body of the mapping rule, we derive the implication:

$$\beta_2 \Rightarrow \mu(\beta_1) \lor e_1 \lor e_2 \lor \cdots \lor e_t$$

where $e_1, e_2, \ldots$ are the ACs corresponding to the $J$ facts from which each $I$ fact was computed. Finally, observe that by construction of the rule, one of the ACs in $\mu(\beta_1)$ is not represented in the body of the rule (it is represented in the head of the rule). This justifies the presence of $e_t$ in the implication, which represents this special AC in $\mu(\beta_1)$.

5.7. Semi-monadic Datalog - Containment

Here, we prove a stronger result than the one we need to prove Theorem 5.11. We define semi-monadic Datalog and prove that the problem of containment of a conjunctive query to a semi-monadic Datalog is in NP.

A binding pattern is a vector consisting of $b$ (for bound) and $f$ (for free) and its length is equal to the number of components in the vector, e.g., $bbfb$ is a binding pattern of length 4. An annotated predicate is a predicate atom together with a list of its variables and a binding pattern of length equal to the length of the list of the variables. E.g., $p^{bbf}_{X,Y, Z}(X,Y,X,Z)$ is an annotated predicate; the list of the variables has been put as subscript in the name of the predicate. Notice that $p^{bbf}_{X,Y, Z}(X,Z,X,Y)$ is a different annotated predicate. However $p^{bbf}_{X,W, Z}(X, W, X, Z)$ is the same as $p^{bbf}_{X,Y, Z}(X,Y,X,Z)$ because names of variables do not matter.
Now, in order to prove that a Datalog query is semi-monadic, we do as follows: We start with the query predicate and annotate it with the binding pattern with all b’s. For each already annotated IDB predicate and for each rule with head this predicate, we unify the arguments in the annotated predicate with the head of the rule. Those variables that are bound in the pattern are also bound in the IDB predicates in the body of the rule, all other variables are free. Thus, for each IDB predicate in the body of the rule, we create a new annotated IDB predicate by providing the binding pattern for its variables.

The above procedure will stop because there is only a finite number of distinct annotated predicates. If each annotated predicate constructed has a binding pattern with only one f, then we say that the Datalog query is semi-monadic.

Remark: A computation of a fact \( F \) for a semi-monadic Datalog query will use in the derivation tree only IDB facts that have the property: All the variables in the fact have values that are one of the constants of the fact \( F \), except one, which may have, in general, any value. To show this remark, consider, towards contradiction, that there is a fact in the derivation tree where this is not true. Then, by considering the path, in the derivation tree, from its root to this fact, it is easy to show that this succession of rules would have created an annotated IDB predicate with more than one f’s.

In the case of Theorem 5.11, all IDB predicates have binding patterns \( b \ldots bf \) of the same length.

**Theorem 5.16.** Consider a pair \((Q_1, Q_2)\) where \( Q_1 \) is a semi-monadic Datalog query and \( Q_2 \) is a CQ. Then testing containment of \( Q_2 \) to \( Q_1 \) is in NP.

See proof in Appendix D.

6. When U-CQAC MCRs compute certain answers

A view instance \( \mathcal{I} \), under the open world assumption (OWA), might be incomplete and only store some of the tuples that satisfy the view definitions
in $\mathcal{V}$, i.e. $\mathcal{I} \subseteq \mathcal{V}(D)$ where by $\mathcal{V}(D)$ we denote the result of computing all the views from $\mathcal{V}$ on $D$.

In this section we prove that, given CQAC query and views, if there is a maximally contained rewriting (MCR) in the language of (possibly infinite) union of CQACs then this MCR computes all the certain answers on any view instance.

Moreover, we prove this result in a more general setting, in that we also assume that there is a set of constraints $\mathcal{C}$ that the database ought to satisfy. In Appendix E we provide formal definitions of the context of query answering in the presence of dependencies. The set $\mathcal{C}$ contains tuple generating dependencies (tgds) and equality generating dependencies (egds). We assume that the chase algorithm (see description of chase algorithm in Appendix E) terminates on $\mathcal{C}$.

We give the definition of certain answers under constraints, as follows.

**Definition 6.1.** Suppose there exists a database instance $D$ such that $\mathcal{I} \subseteq \mathcal{V}(D)$. Then, we define the certain answers of $(Q, \mathcal{I})$ with respect to $\mathcal{V}$ as follows:

- **Under the Open World Assumption:**

  $$certain(Q, \mathcal{I}) = \bigcap \{Q(D) : D \text{ such that } \mathcal{I} \subseteq \mathcal{V}(D)\}$$

  In the presence of a set of constraints $\mathcal{C}$, we also require that all databases $D$ used for $certain(Q, \mathcal{I})$ satisfy $\mathcal{C}$ and denote it by $certain_\mathcal{C}(Q, \mathcal{I})$.

  *If there is no database instance $D$ such that $\mathcal{I} \subseteq \mathcal{V}(D)$, we say that the set $certain_\mathcal{C}(Q, \mathcal{I})$ is undefined.*

6.1. Contained CQAC rewritings under the OWA

We first define query containment under constraints:

**Definition 6.2.** Let $\mathcal{C}$ be a set of tgds and egds, and $Q_1, Q_2$ be two conjunctive queries. We say that $Q_1$ is contained in $Q_2$ under the dependencies $\mathcal{C}$, denoted $Q_1 \subseteq_c Q_2$, if for all databases $D$ that satisfy $\mathcal{C}$ we have that $Q_1(D) \subseteq Q_2(D)$.

\[\text{The Closed World Assumption assumes that } \mathcal{I} = \mathcal{V}(D).\]
We check containment under constraints by using the chase algorithm on the contained query and then checking containment as usual [19]. We define contained rewriting under constraints:

**Definition 6.3.** *(Contained rewriting)* Let $Q$ be a query defined on schema $S$, and $\mathcal{V}$ a set of views defined on $S$. Let $R$ be a query formulated in terms of the view relations in the set $\mathcal{V}$.

$R$ is a contained rewriting of $Q$ using $\mathcal{V}$ under the OWA and under the constraints $C$ if and only if for every view instance $I$ the following is true: For any database $D$ such that $I \subseteq \mathcal{V}(D)$ that satisfies the constraints in $C$, we have that $R(I) \subseteq Q(D)$.

We define the *expansion* of a rewriting CQAC using CQAC views: This is a query on the schema of the relations in the view definitions that results from the rewriting by replacing each view subgoal of it with the body of the definition of the corresponding view after obvious variable unification. The nondistinguished variables in the view definition are replaced with fresh variables that are not used anywhere else in the expansion of the rewriting. The AC subgoals are also included.

**Theorem 6.4.** Suppose query $Q$, views $\mathcal{V}$, and rewriting $R$ all belong to the language of CQACs. Then $R$ is a contained rewriting of $Q$ using views $\mathcal{V}$ if and only if $R^{exp} \sqsubseteq Q$.

*Proof.* If the expansion is not contained in the query, then it is easy to find a counterexample to prove that it is not a contained rewriting. If the expansion $R^{exp}$ is contained in the query then, since $I \subseteq \mathcal{V}(D)$ for any $D$ that satisfies the constraints, we have that

$$R(I) \subseteq R(\mathcal{V}(D))$$

However $R(\mathcal{V}(D))$ is equal to $R^{exp}(D)$ because to compute the former we first apply the mappings from the view definition to $D$ (to compute $\mathcal{V}(D)$) and then apply the mapping from $R$ to $\mathcal{V}(D)$ thus resulting in a mapping from $R^{exp}$ to
for each tuple that is computed. Consequently, the following is true:

\[ R(\mathcal{I}) \subseteq R(\forall(D)) \subseteq R^{exp}(D) \subseteq Q(D) \]

for any \( D \) that satisfies the constraints. Hence \( R \) is a contained rewriting under the constraints.

A \textit{maximally contained rewriting (MCR)} in a query language \( L \), is a contained rewriting in \( L \) that contains all contained rewritings in \( L \).

In this section, we will now prove that, for CQAC views, a maximally contained rewriting \( P \) with respect to U-CQAC\(^7\) of a CQAC query \( Q \) under the a set of given contraints computes the certain answers of \( Q \) under the OWA and under the given constraints, i.e., we prove the theorem:

\begin{theorem}
Let \( C \) be a set of constraints that are tgd and egds. Let \( Q \) be a CQAC query, \( V \) a set of CQAC views. Suppose there exists an MCR, \( R_{MCR} \), of \( Q \) with respect to U-CQAC and under the constraints \( C \). Let \( \mathcal{I} \) be a view instance such that the set \( certain_{C}(Q, \mathcal{I}) \) is defined. Then, under the open world assumption, \( R_{MCR} \) computes all the certain answers of \( Q \) on any view instance \( \mathcal{I} \) under the constraints \( C \), that is: \( R_{MCR}(\mathcal{I}) = certain_{C}(Q, \mathcal{I}) \).
\end{theorem}

6.2. \textit{Representative possible worlds (RPW)}

A \textit{database instance with ACs} is a database with domain a set of constants and a set of variables that we call \textit{labeled nulls} (the two sets are disjoint), i.e., it contains relational atoms that use labeled nulls and constants and possibly ACs among the labeled nulls or among labeled nulls and constants. From hereon, in this section, when confusion does not arise, we will call it simply database instance.

Given a view instance \( \mathcal{I} \), we define a set of \textit{representative possible worlds} (RPW, for short), \( P_{\mathcal{I}} \). A RPW is a database instance with ACs. The set \( P_{\mathcal{I}} \)

\( ^7\)In the literature, usually, by U-CQAC we define the class of finite unions of CQs, in this section we assume that it may be also infinite.
has the following properties: a) for all $D_I \in \mathcal{P}_I$ the following is true: $\mathcal{I} \subseteq \mathcal{V}(D_I)$, b) for each $D$ such that $\mathcal{I} \subseteq \mathcal{V}(D)$ there is a representative possible world, $D_I$, in $\mathcal{P}_I$ such that there is an order-homomorphism from $D_I$ to $D$.

The set $\mathcal{P}_I$ of RPWs is finite and we can construct it by the following algorithm:

Let $\mathcal{I}$ be a view instance. We use $\mathcal{I}$ to produce a Boolean CQAC rewriting as follows:

1. We turn all constants to variables so that distinct constants are turned into distinct variables.

2. We add on the variables the ACs that imply a total ordering, which is the ordering of the constants they came from (recall that constants are from a totally ordered domain).

Thus we have a Boolean rewriting, denoted by $R_\mathcal{I}$.

3. We consider the expansion, $R_\mathcal{I}^{exp}$, of $R_\mathcal{I}$. We consider the set $\mathcal{R}_\mathcal{I}$ of the canonical databases of $R_\mathcal{I}^{exp}$.

4. For each $D$ in $\mathcal{R}_\mathcal{I}$, we do as follows: We apply chase on $D$ with constraints $\mathcal{C}$. Thus, if the chase succeeds, we derive $D_{chased}$ and add it in $\mathcal{P}$ which is the set of representative possible worlds.

This finishes the construction of $\mathcal{P}_I$.

**Theorem 6.6.** The above procedure finds all representative possible worlds.

**Proof.** Let $D$ be a database instance such that $\mathcal{I} \subseteq \mathcal{V}(D)$. The tuples in $\mathcal{I} \cap \mathcal{V}(D)$ are produced by an order-homomorphism, $h_1$, from $R_\mathcal{I}^{exp}$ to $D$. To see that, imagine that we apply the view definitions on $D$ in one step since we know that $\mathcal{I} \subseteq \mathcal{V}(D)$.

This means that $D$ is contained (we can imagine that $D$ is a Boolean query with no variables, just constants) in $R_\mathcal{I}^{exp}$. Thus, by the containment test, and taking into account Theorem [Appendix E.7](#), there is a canonical database of

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8A rewriting is a CQAC query expressed in terms of the views; it stands alone, it does not have to be contained in a specific query.

9By canonical database here we mean only those on which $R_\mathcal{I}^{exp}$ computes to true.
$R_{Iz}^{exp}$ that maps isomorphically on $D$ by $h_2$ according to the following proposition.

**Proposition 6.7.** Suppose database instance $D$ which, viewed as a Boolean query, is contained in $CQAC Q$. Then, there is a canonical database of $Q$ that maps isomorphically on $D$.

The following is an example for how we construct $R_I$ and $R_{Iz}^{exp}$.

**Example 6.8.** Consider the query $Q$ and the views $V_1$, $V_2$ with the following definitions.

$$Q: q() : - a(X,Y,W), b(Y,Z,W), X \leq 14$$

$$V_1: v_1(X,Y) : - a(X,Y,Z), X \leq 9$$

$$V_2: v_2(X,Y) : - b(X,Y,Z)$$

Now, we consider the following view instance: $\mathcal{I} : \{v_1(1,2), v_2(2,3), v_1(5,6)\}$. We build a Boolean rewriting from $\mathcal{I}$ as we explained above, which, in this specific view instance is the following rewriting:

$$R_I: q() : - v_1(X_1, X_2), v_2(X_2, X_3), v_1(X_5, X_6),$$

$$X_1 < X_2, X_2 < X_3, X_5 < X_6, X_3 < X_5$$

This rewriting is a contained rewriting in the query $Q$. However this is not always the case, e.g., imagine a view instance that contained only $v_1(5,6)$; it is easy to verify that the rewriting built based on this view instance would not have been contained in $Q$.

The expansion of the rewriting $R_I$ is the following:

$$R_{Iz}^{exp}: q() : - a(X_1, X_2, Z_1), b(X_2, X_3, Z_2), a(X_5, X_6, Z_3),$$

$$X_1 < X_2, X_2 < X_3, X_5 < X_6, X_3 < X_5$$

The representative possible worlds for $\mathcal{I} : \{v_1(1,2), v_2(2,3), v_1(5,6)\}$ are obtained from the canonical databases of the expansion $R_{Iz}^{exp}$. Each RPW contains the relational atoms in $R_{Iz}^{exp}$ and the variables (labeled nulls) $X_i$ have the total
order shown in $R_{exp}^I$. However the variables $Z_i$ can have any ordering, thus all their orderings create more than one RPWs.

These representative possible worlds are isomorphic as concerns the relational part but with different total ordering on the labeled nulls.

6.3. When a view instance has at least one representative possible world

There is a broad class of views where the set certain$_C(Q,I)$ is always defined independently of the view instance $I$, as the following proposition says.

**Proposition 6.9.** Let $V$ be a set of CQAC views and $Q$ a CQAC query. If there are no egds in the set of constraints $C$ and, each view definition a) has no repeated variables in the head and b) has no ACs that contain head variables, then the set certain$_C(Q,I)$ is defined on any view instance $I$.

**Proof.** When we construct the RPWs, for each view tuple in the view instance $I$, we associate position-wise each variable in the head of the view definition with a constant in the view tuple. This should create a homomorphism from the head of the view definition to the view tuple which should also satisfy the ACs. This is always possible when there are no ACs on the head variables of the view definition and there are no repeated variables; the latter would cause a single variable having two distinct targets and this does not define a homomorphism.

However, even in the case where certain answers are not defined, an MCR can be used to produce results that make sense. We discuss it in the Appendix relating it to non-clean data.

6.4. Main result

We will now prove Theorem 6.5, which is the main result of this section and its main ingredients are the following Propositions 6.10 and 6.11. The first says that if we take the intersection of all answers computed by applying the query $Q$ on each of the representative possible worlds we produce all certain answers
of the query. The second one says that there is a CQAC contained rewriting that produces this intersection. We also need to use the fact that each CQAC contained rewriting computes only certain answers if applied on a view instance \( \mathcal{I} \); this is true by the definition of contained rewriting (Definition 6.3).

**Proposition 6.10.** Let \( C \) be a set of contraints that tgd and egds. Let \( V \) be a set of CQAC views and \( \mathcal{I} \) a view instance such that the set \( \text{certain}_{C}(Q, \mathcal{I}) \) is defined. Let \( Q \) be a CQAC query. Then \( \bigcap_{D_{I} \in P_{I}} Q(D_{I}) \) is equal to the certain answers of \( Q \) given \( V \) on view instance \( \mathcal{I} \) under the constraints \( C \).

**Proof.** Certainly, \( \bigcap_{D_{I} \in P_{I}} Q(D_{I}) \) is a superset of the set of certain answers. We want to prove that it is also a subset of the set of certain answers. By contradiction, suppose not. Then, there is a PW, \( D \), such that the answers of \( Q \) on \( D \) do not contain all tuples in \( \bigcap_{D_{I} \in P_{I}} Q(D_{I}) \). This means that there is a tuple \( t \) in \( \bigcap_{D_{I} \in P_{I}} Q(D_{I}) \) which is not in \( Q(D) \). However, according to the definition of RPW, there is a RPW, and taking into account Theorem Appendix E.7, \( D \), such that there is an order-homomorphism from \( D \) to \( D \), hence \( Q(D) \subseteq Q(D) \). Since \( t \) is in \( \bigcap_{D_{I} \in P_{I}} Q(D_{I}) \), \( t \) is also in \( Q(D) \). Hence contradiction.

**Proposition 6.11.** Let \( C \) be a set of contraints that tgd and egds. Let \( Q \) be CQAC query and \( V \) be a set of CQAC views. Let \( \mathcal{I} \) be a view instance such that the set \( \text{certain}_{C}(Q, \mathcal{I}) \) is defined. Then, given a tuple \( t_{0} \in \text{certain}(Q, \mathcal{I}) \), there is a contained CQAC rewriting \( R \) such that \( t_{0} \in R(\mathcal{I}) \).

**Proof.** We consider as \( R \) the Boolean query \( R_{\mathcal{I}} \) with the proper variables in the head that are the variables that represent the constants in \( t_{0} \).

Now we need to prove that \( R \) is a contained rewriting. \( R \) was created from \( R_{\mathcal{I}} \) which produces all RPWs. Since \( t_{0} \) is in the certain answers of the query \( Q \), there is an order-homomorphism from \( Q \) to every RPW and this homomorphism produces \( t_{0} \). All RPWs are all canonical databases of \( R_{\mathcal{I}}^{exp} \) chased with the constraints. Hence the aforementioned homomorphisms provide the proof for the containment test that proves containment of \( R_{\mathcal{I}}^{exp} \) to \( Q \) under the constraints \( C \). Since \( R \) only differs from \( R_{\mathcal{I}} \) as to the head, the same order-homomorphisms

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can be used to prove containment of $R^{exp}$ to $Q$ under the constraints $C$. Hence, we have proved that $R$ is a contained rewriting to $Q$ under the constraints $C$ which are used in the chase to construct $P_I$.

We now put all together to finish the proof of Theorem 6.5:

Proof. (Theorem 6.5) We will show the following:

1. $P(I) \subseteq \text{certain}(Q, I)$
2. $\text{certain}(Q, I) \subseteq P(I)$

Since $P$ is a contained rewriting of $Q$, the first is a direct consequence of the definition of a contained rewriting.

To prove the second, we use the above two propositions. One proposition says that we can compute all certain answers by considering only a finite number of possible worlds, $P(I)$. The other one uses $P(I)$ to prove that there is CQAC contained rewriting which computes a tuple $t_0$ if this tuple is in certain answers.

7. Finding MCR for CQAC-RSI1+ Query and CQAC Views

In this section, we show that for CQAC-RSI1+ query and a special case of CQAC views, we can find an MCR in the language of (possibly infinite) union of CQACs. We will show that this MCR is expressed in Datalog$^{AC}$ (to be defined shortly). In detail, we consider the following case of query and views:

- There are only closed arithmetic comparisons in both query and views.

- The views are CQAC queries which do not use ACs of the form $X \leq Y$ or $X \geq Y$ where $X$ is a head variable and $Y$ is a nondistinguished variable. We call this class of CQAC queries CQAC$^-$. 

- The query is a CQAC-RSI1+ query.
We define Datalog\textsuperscript{AC} to include rules that may have ACs in the body. The computation of the output of a Datalog\textsuperscript{AC} query on a database instance on a totally ordered dense domain is performed in the same fashion as in Datalog. As in Datalog, the number of facts computed are not more than polynomially many in terms of the number of tuples in the database instance, i.e., if the size of the program is fixed.

We think of an expansion of a rewriting as having three kinds of variables: a) the head variables of the query that the expansion represents, b) the view-head variables; these are all the variables that are present in the rewriting and c) the view-nondistinguished variables; these are all the other variables in the expansion of the rewriting. The head variables are also view-head variables.

7.1. ACs in rewritings

Consider a CQAC query and a set of CQAC views. When we have a rewriting, \( R \), the variables in the rewriting also satisfy some ACs that are in the closure of the ACs in the expansion of the rewriting. We include those ACs in the rewriting \( R \) and produce \( R' \), which we call the AC-rectified rewriting of \( R \). Thus, the expansions of \( R \) and \( R' \) are equivalent queries. Hence, we derive the following proposition:

**Proposition 7.1.** Given a set of CQAC views, a rewriting \( R \) and its rectified version \( R' \), the following is true: For any view instance \( \mathcal{I} \) such that there is a database instance \( D \) for which \( \mathcal{I} \subseteq V(D) \), we have that \( R(\mathcal{I}) = R'(\mathcal{I}) \).

**Definition 7.2.** We say that a rewriting \( R \) is AC-contained in a rewriting \( R_1 \) if the AC-rectified rewriting, \( R' \), of \( R \) is contained in \( R_1 \) as queries.

From hereon, when we refer to a rewriting, we mean the AC-rectified version of it and when we say that a rewriting is contained in another rewriting we mean that it is AC-contained. An example follows.

**Example 7.3.** Consider query \( Q \) and view \( V_2 \):

\[
Q(A) \quad :- \quad p(A), A < 4.
\]

\[
V_2(Y, Z) \quad :- \quad p(X), s(Y, Z), Y \leq X, X \leq Z.
\]
The following rewriting is a contained rewriting of the query in terms of the view in the language CQAC:

\[ R(Y_1) :- V_2(Y_1, Z_1), V_2(Y_2, Z_2), Z_1 \leq Y_2, Y_1 \geq Z_2, Y_1 < 4. \]

Now consider the following contained rewriting:

\[ R'(X) :- V_2(X, X), X < 4. \]

This rewriting uses only one copy of the view. We can show that \( R \) is not contained in \( R' \) and that \( R' \) is not contained in \( R \). However, they compute the same output on any view instance (to see that just include in \( R \) the ACs \( Y_1 \leq Z_1 \) and \( Y_2 \leq Z_2 \)).

7.2. Building MCRs

For the special case of query and views considered in this section, we show here that an MCR in the language of (possibly infinite) union of CQACs exists and is expressed by a Datalog\(^AC\) query. The algorithm for building such an MCR is the following:

**Algorithm for building MCR:**

1. We consider query \( Q' \) which results from the given query \( Q \) after we have removed the ACs that contain only head variables.

2. For the query \( Q' \), we construct the Datalog query \( Q^{\text{Datalog}} \). We use the construction in Subsection 5.2. The link rules will use the constants present in the views and in the query.

3. For each view \( v_i \), we construct a new view \( v_i^{\text{CQ}} \). We use the construction in Subsection 5.3.

4. Consider the EDB predicates introduced in Section 5 (and used in Steps 2 and 3 above). We use them to construct a new set of auxiliary views as follows: a) Views with head \( u_{\theta c} \), one for each semi-unary predicate \( U_{\theta c} \). The definition is \( u_{\theta c}(\overline{W}, X) : - U_{\theta c}(X) \). b) A single view \( u \), whose definition is \( u(X, Y) : - U(X, Y) \). We will refer to those EDB predicates (i.e.,
the \( U(X, Y) \) predicate and the semi-unary predicates) as \( AC\)-predicates or \( AC\)-subgoals.

5. We consider now the view set \( \mathcal{V}^{CQ} \) that contains the views as constructed in the two previous steps above.

6. We find an MCR, \( R_{MCR}^{CQ} \), for the Datalog query \( Q^{Datalog} \) using the views in \( \mathcal{V}^{CQ} \). For building the MCR we use the inverse rule algorithm from [30].

7. To obtain an MCR, \( R_{MCR} \), for \( Q' \), we replace in the found MCR \( R_{MCR}^{CQ} \), each \( v_i^{CQ} \) by \( v_i \), each \( u_{\theta c}(X) \) by \( AC X \theta c \) and each \( u(X, Y) \) by \( AC X \leq Y \).

8. We add a new rule in \( R_{MCR} \) (and obtain \( R_{MCR}' \)) to compute the query predicate \( Q \) as follows:

\[
Q'(W) : -Q'(W), ac_1, ac_2, \ldots
\]

where \( ac_1, ac_2, \ldots \) are the ACs that we removed in the first step of the present algorithm.

From hereon, in this section, we will study the query \( Q' \) and we will conveniently refer to it as \( Q \). Before we give the proof that the algorithm is correct, we give some examples of how the algorithm is applied.

7.3. Examples

The first example in this subsection does not use ACs that contain head variables (actually, the query is Boolean, hence, there are no head variables) in the query and, also, the reverse rule algorithm produces an MCR without including the auxiliary views, hence, we have not written these views, in order to keep things simple. The second example, again uses a Boolean query but there is an AC in the query that causes the inverse rule algorithm to use an auxiliary view, so we have written this auxiliary view; again other auxiliary views are not used.
Example 7.4. Consider the query $Q_1$ and the views:

\[ Q_1() : - e(X, Z), e(Z, Y), X \geq 5, Y \leq 8. \]
\[ V_1(X, Y) : - e(X, Z), e(Z, Y), Z \geq 5. \]
\[ V_2(X, Y) : - e(X, Z), e(Z, Y), Z \leq 8. \]
\[ V_3(X, Y) : - e(X, Z_1), e(Z_1, Z_2), e(Z_2, Z_3), e(Z_3, Y). \]

We have already built the Datalog program $Q_1^{\text{Datalog}}$ in Example 5.7 in a more general setting, where we assume that the query is not Boolean. Here, we use the same $Q_1^{\text{Datalog}}$ only that we delete $W$ from all the rules. We need add the link rules which will be with respect to constants 5 and 8 (these are the only constants that appear in the definitions).

The views that will be used to apply the inverse-rule algorithm are:

\[ V_1'(X, Y) : - e(X, Z), e(Z, Y), U \geq 5(Z). \]
\[ V_2'(X, Y) : - e(X, Z), e(Z, Y), U \leq 8(Z). \]
\[ V_3'(X, Y) : - e(X, Z_1), e(Z_1, Z_2), e(Z_2, Z_3), e(Z_3, Y). \]

Notice that we conveniently did not add any auxiliary views here because we guessed that they will not be needed.

In this example, it is relatively easy to anticipate the result of applying the inverse-rule algorithm, by observing the simple form of the expansions of $Q_1^{\text{Datalog}}$. Each expansion of $Q_1^{\text{Datalog}}$ is a simple path with two unary predicates, one at one end of the path and the other at the other end. Thus, the output of the inverse-rule algorithm is the following program. It is an MCR of $Q_1^{\text{Datalog}}$ using the views $V_1'(X, Y)$, $V_2'(X, Y)$, and $V_3'(X, Y)$.

\[ R'(\cdot) : - v_1'(X, W), T(W, Z), v_2'(Z, Y). \]
\[ T(W, W) : -. \]
\[ T(W, Z) : - T(W, U), v_3'(U, Z). \]

The following is an MCR of the input query $Q_1$ (rather than of $Q_1^{\text{Datalog}}$) using the views $V_1(X, Y)$, $V_2(X, Y)$ and $V_3(X, Y)$:

\[ R(\cdot) : - V_1(X, W), T(W, Z), V_2(Z, Y). \]
\[ T(W, W) : -. \]
\[ T(W, Z) : - T(W, U), V_3(U, Z). \]
Example 7.5. This is similar to Example 7.4 with slight alterations to make the point that we need to add more views in our view set before we compute the MCR, in order to compute the MCR correctly. The alterations are as follows: We have added a new relational subgoal $a(U)$ and a new AC on the variable of this relational subgoal in the query and we have added a relational subgoal on the same predicate on the first view.

Thus, we consider the query $Q_1$ and the views:

$Q_1() : e(X, Z), e(Z, Y), X \geq 5, Y \leq 8, a(U), U \geq 46.$

$V_1(X, Y) : e(X, Z), e(Z, Y), Z \geq 5, a(Y).$

$V_2(X, Y) : e(X, Z), e(Z, Y), Z \leq 8.$

$V_3(X, Y) : e(X, Z_1), e(Z_1, Z_2), e(Z_2, Z_3), e(Z_3, Y).$

The Datalog program $Q_1^{\text{Datalog}}$ is exactly the same as the one in Example 5.7.

The views that will be used to apply the inverse-rule algorithm are (now we have added one auxiliary view which we guessed will be needed):

$V_1'(X, Y) : e(X, Z), e(Z, Y), U_{\geq 5}(Z), a(Y), U_{\geq 46}(Y).$

$V_2'(X, Y) : e(X, Z), e(Z, Y), U_{\leq 8}(Z).$

$V_3'(X, Y) : e(X, Z_1), e(Z_1, Z_2), e(Z_2, Z_3), e(Z_3, Y).$

$V_4'(Y) : U_{\geq 46}(Y).$

The new view is view $V_4'$.

The Datalog program now for the query $Q$ is\(^{10}\).

\(^{10}\)Some link rules are ommitted since they are not used by the inverse rule algorithm to produce an MCR. For the same reason some coupling rules are ommitted.
\[
Q_1^{Datalog}(): = e(X,Y), e(Y,Z), a(U), U \geq 46(U).
\]
\[
I_{\geq 5}(X), I_{\leq 8}(Z).
\] (query rule)
\[
J_{\leq 8}(Z): = e(X,Y), e(Y,Z), a(W,Y),
\]
\[
I_{\geq 5}(X).
\] (mapping rule)
\[
J_{\geq 5}(X): = e(X,Y), e(Y,Z), a(W,Y),
\]
\[
I_{\leq 8}(Z).
\] (mapping rule)
\[
I_{\leq 8}(X): = J_{\geq 5}(X).
\] (coupling rule)
\[
I_{\geq 5}(X): = J_{\leq 8}(X).
\] (coupling rule)
\[
I_{\geq 5}(X): = U_{\geq 5}(X).
\] (link rule)
\[
I_{\leq 8}(X): = U_{\leq 8}(X).
\] (link rule)

This is the output of the inverse rule algorithm:

\[
R^\prime(): = v_1'(X,W), T(W,Z), v_2'(Z,Y), v_4'(W).
\]
\[
T(W,W): = .
\]
\[
T(W,Z): = T(W,U), v_3'(U,Z).
\]

The following is an MCR of the input query \(Q_1\) (rather than \(Q_1^{Datalog}\)) using the views \(V_1(X,Y), V_2(X,Y)\) and \(V_3(X,Y)\). Notice that we replace view \(v_4'(W)\) by \(W \geq 46\).

\[
R(): = V_1(X,W), T(W,Z), V_2(Z,Y), W \geq 46.
\]
\[
T(W,W): = .
\]
\[
T(W,Z): = T(W,U), V_3(U,Z).
\]

7.4. Proof

The proof of the following proposition is a straightforward consequence of the construction of \(R_{MCR}\) from \(R_{MCR}^{CQ}\).

**Proposition 7.6.** Consider the Datalog programs \(R_{MCR}^{CQ}\) and \(R_{MCR}\). For each CQAC Datalog-expansion, \(E\), of \(R_{MCR}\), there is a CQ Datalog-expansion, \(E^{CQ}\), of \(R_{MCR}^{CQ}\) (and vice versa), where the following is true: The relational subgoals of \(E\) are isomorphic to the purely relational subgoals of \(E^{CQ}\) (by “purely relational subgoals we mean those that do not encode ACs) and each ACs in \(E\) corresponds to a subgoal in \(E^{CQ}\) that encodes this AC.
Theorem 7.7. The found by the algorithm in Subsection 7.2 Datalog\(^AC\) program, \(R_{MCR}\), is a contained rewriting.

Proof. Consider a CQAC Datalog-expansion, \(R\), of the found Datalog\(^AC\) program, \(R_{MCR}\). Take the view-expansion, \(R_{\text{exp}}\), of \(R\). Transform \(R_{\text{exp}}\) into a CQ, \(R_{\text{exp}}^{\text{CQ}}\), using the construction in Subsection 5.3. Now, apply the observation in Proposition 7.6 to prove that \(R_{\text{exp}}^{\text{CQ}}\) is a Datalog-expansion of \(R_{MCR}^{\text{CQ}}\). According to the reverse-rule algorithm, \(R_{\text{exp}}^{\text{CQ}}\) is contained in the \(Q^{\text{Datalog}}\) program, hence according to the results in Section 5, \(R_{\text{exp}}\) is contained in the query \(Q\). Consequently, \(R\) is a contained rewriting of \(Q\). \(\square\)

Theorem 7.8. Let \(R\) be a CQAC contained rewriting. Then it is contained in the found by the algorithm in Subsection 7.2 Datalog\(^AC\) program, \(R_{MCR}\).

Proof. Let \(R\) be a CQAC contained rewriting of \(Q\) using \(\mathcal{V}\) and let \(R_{\text{exp}}\) be the view-expansion of \(R\). We assume \(R\) is AC-rectified (see Subsection 7.1). Let \(Q^{\text{Datalog}}\) be the transformed query of \(Q\) as in Subsection 5.2

- \(R\) is a CQAC query which is a contained rewriting of \(Q\) using \(\mathcal{V}\).
- \(R_{\text{exp}}\) is the expansion of \(R\) with respect to \(\mathcal{V}\).
- \(R'\) is \(R\) with ACs in the closure of ACs in \(R\) turned into relational predicates.
- \(R'_{\text{exp}}\) is the expansion of \(R'\) with respect to \(\mathcal{V}^{\text{CQ}}\).
- \(R_{\text{exp}}^{\text{CQ}}\) is \(R_{\text{exp}}\) transformed into a CQ as in Subsection 5.3

We want to prove that \(R'_{\text{exp}}\) and \(R_{\text{exp}}^{\text{CQ}}\) are isomorphic. After that, it is easy to argue as follows: Since \(R_{\text{exp}}\) is contained in \(Q\), according to Section 5, \(R_{\text{exp}}^{\text{CQ}}\) is contained in \(Q^{\text{Datalog}}\). Hence, \(R'_{\text{exp}}\) too is contained in \(Q^{\text{Datalog}}\). Hence \(R'\) is a contained rewriting of \(Q^{\text{Datalog}}\) in terms of \(\mathcal{V}^{\text{CQ}}\), hence \(R'\) is contained in \(R_{MCR}^{\text{CQ}}\) (Remember \(R_{MCR}^{\text{CQ}}\) is an MCR of \(Q^{\text{Datalog}}\) in terms of \(\mathcal{V}^{\text{CQ}}\).). Now, \(R\) and \(R'\) differ only in that the ACs of one are AC-predicates of the other, in one to one fashion. Each Datalog-expansion of \(R_{MCR}\) and \(R_{MCR}^{\text{CQ}}\) differ in the same
way. Hence the Datalog-expansion of $R_{MC_{RM}}^{CQ}$ that proves $R'$ is in $R_{MC_{RM}}^{CQ}$ can be used to derive a Datalog-expansion of $R_{MC_{RM}}$ that proves $R$ is in $R_{MC_{RM}}$.

Now the key observation to prove that $R_{exp}'$ and $R_{exp}^{CQ}$ are isomorphic is the following:

- All ACs of the form $X \leq Y$ in $R_{exp}$ have both variables being view-head variables, since there is no possibility to form such an AC in the closure of ACs in $R_{exp}$, since there are, by statement in the present theorem, no ACs between a view-nondistinguished variable and a view-head variable. This would have been the only way to form an AC $X \leq Y$ where, e.g., $X$ appears in the view expansion of one view and $Y$ appears in the view expansion of another view.

Hence, in $R_{exp}^{CQ}$ all AC-predicates of the form $U(X,Y)$ have both variables being view-head variables. Hence, the $R_{exp}'$ has incorporated them since in $V_{CQ}$ there is an auxiliary view $v(W) : -U(X,Y)$.

The following theorem proves correctness of the algorithm and is a straightforward consequence of the above theorems and the following proposition which takes care of the ACs in the head variables that were removed in the first step of the algorithm.

**Proposition 7.9.** Given a query $Q$ which is CQAC-$SI_{I1}+$ and views $V$ which are CQACs and contained rewritings $R$ and $R'$ of $Q'$ using the views $V$ where $Q'$ is the one constructed in the first step of the algorithm, the following is true: Suppose $R$ is contained in $R'$. We add to $R'$ the ACs of the head variables that were removed and produce $R''$. Suppose $R_1$ is a contained rewriting to $Q$ and moreover, removing the head ACs from $R_1$ we produce $R$. Then $R_1$ is contained to $R''$.

**Theorem 7.10.** Given a query $Q$ which is CQAC-$SI_{I1}+$ and views $V$ which are CQAC$^{-}$s, the algorithm finds an MCR of $Q$ using $V$ in the language of (possibly infinite) union of CQACs.
A straightforward consequence of the above theorem and the main result in Section 6 is the following theorem:

**Theorem 7.11.** Given a query $Q$ which is CQAC-SI1+ and views $V$ which are CQAC−s, we can find all certain answers of $Q$ using $V$ on a given view instance $I$ in polynomial time on the size of $I$.

### 7.5. Extending the result

The following example shows that the result of Theorem 7.8 (hence the result of Theorem 7.10) cannot be extended to include views that are in the language of CQACs.

**Example 7.12.** Suppose we have the following query and views:

$q(Y) : ¬a(Y,X), b(X,Z), X \geq 5, Z \leq 6$

$V_1: v_1(Y) : ¬a(Y,X), b(X,X'), X \geq Y, X' \leq 6$

$V_2: v_2(Y) : ¬a(Y,Z'), b(Z', Z), Y \geq Z, Z' \geq 5$

The following is a contained rewriting:

$R: q(y) : ¬v_1(Y), v_2(Y)$

The view-expansion of $R$ is:

$R_{exp}: q(Y) : ¬a(Y,X), b(X,X'), X' \leq 6, a(Y,Z'), b(Z', Z), Z' \geq 5$

and after we transform it to a CQ it becomes:

$R_{exp}: q(Y) : ¬a(Y,X), b(X,X'), U \leq 6(X'), a(Y,Z'), b(Z', Z), U \geq 5(Z')$

Suppose we transform $Q$ to $Q_Datalog$, then the following is a Datalog-expansion of the program:

$E: a(Y,X), b(X,X'), U \leq 6(X'), a(Y,Z'), b(Z', Z), U \geq 5(Z'), U(Z, X)$

We see that the body of $R_{exp}$ and $E$ will be isomorphic if we delete $U(Z, X)$ from $E$. This remark highlights the reason for the failing of extending the result beyond Theorem 7.10.

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Appendix A. Technical lemmas

The lemmas in this appendix are all of the same flavor, in that they have the same proof technique, thus, they could be stated in a single lemma with a long statement. We state them separately for clarity. Lemma Appendix A.1 is more general than what we would need for our purposes in this paper.

Lemma Appendix A.1. Consider the following implication:

\[ c_1 \land c_2 \land \ldots \Rightarrow d_1 \lor d_2 \lor \ldots \]

where the \( c_i \)'s and \( d_i \)'s are ACs and the conjunction of ACs \( c_1 \land c_2 \land \ldots \) is consistent (i.e., it has a satisfying assignment from the set of real numbers). Then the following is true:

Suppose all \( c_i \)'s are any AC. Suppose \( d_i \)'s are ACs from the set \{var \leq const, var < const, var = const, var = var\} (i.e., besides equality, we use only
LSI ACs). Then the implication is true if and only if one of the following happens:

(i) there is a single \(d_i\) from the rhs such that

\[ c_1 \land c_2 \land ... \Rightarrow d_i \]

or

(ii) there are two ACs from the rhs, say \(d_i\) and \(d_j\) such that

\[ c_1 \land c_2 \land ... \Rightarrow d_i \lor d_j \]

The case (ii) happens only if i) there is a constant shared a) by \(d_i\), b) by one from the \(c_i\)'s and c) by \(d_j\) and ii) \(d_i\) is an open AC.

or

(iii) there are three ACs from the rhs, say \(d_i\), \(d_k\) and \(d_j\) such that

\[ c_1 \land c_2 \land ... \Rightarrow d_i \lor d_k \lor d_j \]

The case (iii) happens only if there are two LSI from the rhs and two LSI from the lhs, all four sharing the same constant and, in addition, there is a rhs AC of type \(\text{var} = \text{var}\).

One case where (i) happens always is when we do not have all of the following a) a rhs open LSI and a rhs closed LSI sharing the same variable and a b) a lhs equality of either type (i.e., either \(\text{var} = \text{var}\) or \(\text{var} = \text{const}\)). Another case where (i) happens always is when the lhs ACs do not include any closed LSIs.

Proof. Convention: We call the \(d_i\)'s the rhs ACs (for right hand side ACs) and the \(c_i\)'s the lhs ACs (for left hand side ACs).

In order to use the algorithm \textbf{AC-sat}, we write first the implication as \(\neg E\) where

\[ E = c_1 \land c_2 \land ... \land \neg d_1 \land \neg d_2 \land ... \]

We consider the induced graph of the ACs in \(E\) and we apply the algorithm to prove that \(E\) is false.

We consider the three cases of the algorithm \textbf{AC-sat}:
Case 1. Consider a strongly connected component with two distinct constants $c_1$ and $c_2$. Without loss of generality, suppose $c_1$ is adjacent to a rhs AC $d_i = c_1 \leq X$. From $X$, there is a path to a constant $c' \neq c_1$ (which is either $c_2$, which is $\neq c_1$ by our assumption or another one) such that $c'$ is the first constant on this path. Now, if $c' < c_1$ then the edge from $c'$ to $c_1$ forms a cycle with only one rhs AC on it (because all rhs ACs are related to a constant, since they are SIs). If $c' > c_1$ then the edge from $c_1$ to $c'$ forms a cycle that does not contain $d_i = c_1 \leq X$. Hence, we can proceed recursively until we find a cycle with only one rhs AC on it.

Case 2. Consider a strongly connected component with at least one edge (say $d_j = A_1 < A_2$) labeled by $\prec$. If this component has two distinct constants we argue as in case 1. Otherwise, it should have exactly one constant because the $c_i$ ACs are not contradictory, hence at least one rhs AC should be in this component. Consider arbitrarily one of those rhs ACs, say $d_i = c_1 \leq X$ (or $d_i = c_1 < X$ whichever is the case). There is a path from $X$ to $A_1$ and there is a path from $A_2$ to $c_1$; moreover these two paths do not contain any rhs AC edge because such edges are adjacent to constants (by definition) and we have assumed that there is only one constant on this strongly connected component. Hence we have created a cycle with an edge labeled by $\prec$ and with only one rhs AC on it.

Case 3. Consider a strongly connected component with exactly one constant $c$. All the rhs ACs/edges are adjacent to this constant. Let $c < X$ be such an rhs AC. For any node in this component there is a path to $c$ and a path from $X$ that and either path does not contain a rhs AC/edge. Hence a cycle is formed with only one rhs AC on it. Thus, if there is a $\neq$ AC between two nodes $A_1$ and $A_2$ of this strongly connected component, the set of contradictory ACs contain at most two rhs ACs (one for each $A_1$, $A_2$) and the $\neq$ AC (which is the negation of a $d_i$ which is an $= AC$).

Lemma Appendix A.2. Consider the following implication:

$$c_1 \wedge c_2 \wedge ... \Rightarrow d_1 \vee d_2 \vee ...$$
where the conjunction of ACs \( c_1 \land c_2 \land \ldots \) is consistent (i.e., it has a satisfying assignment from the set of real numbers) and the \( d_i \)'s are all closed SI (i.e., either LSI or RSI) comparisons. Then the implication is true if and only if one of the following happens:

(i) there is a single \( d_i \) from the rhs such that

\[
c_1 \land c_2 \land \ldots \Rightarrow d_i
\]

or

(ii) there are two ACs from the rhs from which one is LSI and one is RSI, say \( d_i \) and \( d_j \) (we call them coupling ACs for the conjunction \( c_1 \land c_2 \land \ldots \)) such that

\[
c_1 \land c_2 \land \ldots \Rightarrow d_i \lor d_j.
\]

Proof. We form the induced directed graph as we did in the first lemma in this appendix, and we reason on this graph further. We use the algorithm \texttt{AC-sat}. Here only Case 2 of the algorithm applies, i.e., there is a strongly connected component in the induced directed graph of the ACs with at least one rhs edge. We have two cases: Either this strongly connected component has only LSI or only RSI rhs ACs or it has of both kinds. In the first case, we argue as in the proof of Lemma Appendix A.1 only we have fewer cases since in the present lemma we only consider closed ACs.

For the second case, suppose a strongly connected component has two rhs ACs which are \( X < a \) and \( Y > b \) and they successive, i.e., there is a path from \( Y \) to \( X \) that uses only ACs from the left hand side of the implication. Then we consider the cycle that contains both. Then either of the following happens: a) the edge joining \( a \) and \( b \) forms a cycle which only contains \( X < a \) and \( Y > b \) from the rhs, and thus, we have proved our result or b) the edge joining \( a \) and \( b \) forms a cycle contains neither \( X < a \) nor \( Y > b \); so we proceed recursively considering now the new cycle that contains fewer rhs ACs on it. (Remember that the new cycle cannot contain only lhs ACs because we have assumed that they are consistent.)
Appendix B. Proof of Theorem 2.2

Proof. One of the directions is straightforward: If the containment entailment is true, then in any database that satisfies $\beta'_2$, one of the $\mu_i(\beta'_1)$ will be satisfied (because we deal with constants), and hence containment is proven.

For the “only-if” direction, suppose $Q_2$ is contained in $Q_1$, but the containment entailment is false. We assign constants to the variables that make this implication false. Then for all the containment mappings $\mu_i$ (for each of which $\mu_i(\beta'_1)$ does not hold), the query containment is false, because we have found a counterexample database $D$. Database $D$ is constructed by assigning the corresponding constants to the ordinary subgoals of $Q_2$. On this counterexample database $D$, $Q_2$ produces a tuple, but there is no $\mu_i$ that will make $Q_1$ produce the same tuple (because all $\mu_i(\beta'_1)$ fail). We need to remember that, using the $\mu_i$’s, we can produce all homomorphisms from $Q_1$ to any database where the relational atoms of $Q_2$ map via a homomorphism. This is because the $\mu_i$’s were produced using the normalized version of the queries – and, hence, $\mu_i$’s were not constrained by duplication of variables or by constants (recall that, in a homomorphism, a variable is allowed to map to a single target and a constant is allowed to map on the same constant).

Appendix C. Proof of Theorem 5.10

Proof. We consider the canonical database, $D$, of $Q^{CO}_2$. For convenience, the constants in the canonical database use the lower case letters of the variables they represent. Thus constant $x$ is used in the canonical database to represent the variable $X$. We will use the containment test that says that a Datalog query contains a conjunctive query $Q$ if and only if the Datalog query computes the head of $Q$ when applied on the canonical database of the conjunctive query $Q$.

“If” direction: By induction on the number of the times a mapping rule is fired during the computation of a $J$ fact in a computation that uses the shortest derivation tree.
**Inductive Hypothesis:** If, in the computation of a $J$ fact associated with AC $e$, we have used mappings $\mu_1, \mu_2, \ldots, \mu_k$ (via applications of mapping rules), where $k < n$ then the following holds:

$$\beta_2 \Rightarrow \mu_1(\beta_1) \lor \mu_2(\beta_1) \lor \cdots \lor \mu_k(\beta_1) \lor \neg e$$

**Proof of Inductive Hypothesis.**

The base case is straightforward, it is when a $J$ fact is computed after the application of one mapping rule, say by mapping $\mu_i$. This is enabled because each of the ACs in the $\mu_i(\beta_1)$ except one (the one associated with the computed $J$ fact) are directly implied by $\beta_2$.

Suppose we compute a fact via $n$ mappings. Then all its $I$ facts used in the computation are computed via at most $n - 1$ mappings, hence the inductive hypothesis holds for the corresponding $J$ facts that were used to compute the $I$ fact via a coupling rule.

I.e., according to the inductive hypothesis, each such $I$ fact that is computed via a $J$ fact, which in turn was computed via some mappings $\mu_{ij}, j = 1, 2, \ldots$ (i.e., these are the mappings for all mapping rules that were applied during the whole computation of $J$ fact), implies that the following is true:

$$\beta_2 \Rightarrow \mu_{i1}(\beta_1) \lor \mu_{i2}(\beta_1) \lor \cdots \lor \mu_{il}(\beta_1) \lor \neg e_i$$

or equivalently:

$$\beta_2 \land \neg \mu_{i1}(\beta_1) \land \neg \mu_{i2}(\beta_1) \land \cdots \land \neg \mu_{il}(\beta_1) \Rightarrow \neg e_i$$

Thus, we can combine the above implications for all $I$ facts used for the current application of a mapping rule and have that the following is true:

$$\beta_2 \land \bigwedge_{\text{for all } i} [\neg \mu_{i1}(\beta_1) \land \neg \mu_{i2}(\beta_1) \land \cdots \land \neg \mu_{il}(\beta_1)] \Rightarrow \neg e_1 \land \cdots$$

We write the above in the form:

$$\beta_2 \land \neg \mu_1(\beta_1) \land \neg \mu_2(\beta_1) \land \cdots \Rightarrow \neg e_1 \land \cdots \quad (2)$$
where for simplicity we have expressed the $\mu_{ij}, i = 1, 2, \ldots, j = 1, 2, \ldots$ as $\mu_1, \mu_2, \ldots$. Now $e_1, \ldots$ are the ACs each associated with the $J$ facts used for this mapping rule.

Suppose we apply mapping rule via mapping $\mu_{\text{current}}$ that uses $I$ facts computed in previous rounds using at most $n - 1$ mappings.

When a coupling rule of the first kind is fired then the two variables in the rules are such that their associated ACs $e_I$ and $e_J$ are such that $e_I \lor e_J$ is true.

When a coupling rule of the second kind is fired then the two variables $X, Y$ of the rule (which appear also in the binary EDB $U(X, Y)$ in the body of the rule) are instantiated to constants in the canonical database of $Q_2$ whose corresponding variables $X', Y'$ in $Q_2$ are such that $X' \leq Y'$. Hence we have for the associated ACs $e_I$ and $e_J$ of the IDB predicates of the rule that $\beta_2 \Rightarrow e_I \lor e_J$.

Thus, in any case, when a coupling rule is fired, the following is true: $\beta_2 \Rightarrow e_I \lor e_J$. We use this remark in the second implication (second arrow) of implications C.1

From (2), we have for each $e_i$:

$$\beta_2 \land \neg \mu_1(\beta_1) \land \neg \mu_2(\beta_1) \land \cdots \Rightarrow \neg e_i$$

which yields:

$$\beta_2 \land \neg \mu_1(\beta_1) \land \neg \mu_2(\beta_1) \land \cdots \Rightarrow \neg e_i \land \beta_2 \Rightarrow \neg e_i$$

Hence we have, combining all ACs in $\mu_{\text{current}}(\beta_1)$,

$$\beta_2 \land \neg \mu_1(\beta_1) \land \neg \mu_2(\beta_1) \land \cdots \Rightarrow ac_1 \land ac_2 \cdots \land \neg e_i$$

where $e_i$ is one of the $ac_i$’s and is the associated AC to the J facts computed by the mapping $\mu_{\text{current}}$ which is used to fire a mapping rule.

Thus, we get

$$\beta_2 \land \neg \mu_1(\beta_1) \land \neg \mu_2(\beta_1) \land \cdots \Rightarrow \mu_{\text{current}}(\beta_1) \lor \neg e_{\text{current}}$$
from which we get the implication in the inductive hypothesis.

To finish the proof of this direction, we need to argue about the application of the query rule via mapping $\mu$, whose only difference with a mapping rule is that all the ACs in the $\mu(\beta_1)$ are coupled, hence we derive finally the containment entailment.

**“Only-if” direction:** Assume the containment entailments holds, i.e., the following holds:

$$\beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_l(\beta_1)$$

We will prove this direction by induction on the number of $\mu_i$s in a containment entailment that is true and uses the minimal number of mappings.

**Inductive hypothesis** For $k \leq n$, there is a set of mappings among the ones in the containment entailment, i.e., let them be $\mu_{m+1}, \mu_{m+2}, \ldots, \mu_l$ (where $k=l-m$) such that the following two happen:

(i) The following is true:

$$\beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_{m}(\beta_1) \lor \mu_{m+1}(e_1^{\beta_1}) \lor \mu_{m+2}(e_2^{\beta_1}) \lor \cdots$$

where $m < l$ and $e_1^{\beta_1}, e_2^{\beta_1}, \ldots$ are ACs from $\beta_1$.

(ii) The mappings $\mu_{m+1}, \mu_{m+2}, \ldots$ are used to compute the facts $I(x_{m+1}), I(x_{m+2}) \ldots$ where $x_{m+1}$ (similarly for $x_{m+2}$, etc) represents the variable in $\mu_{m+1}(e_1^{\beta_1})$.

More specifically when we compute the fact $I_{\theta c}(x_{m+1})$ then $\mu_{m+1}(e_1^{\beta_1})$ is the AC $X_{m+1}\theta c$. The same for $I(x_{m+2})$, etc.

Proof of the inductive hypothesis. To prove for $k = n + 1$, we argue as follows. We begin with the implication in the inductive hypothesis, and use Proposition [5.4]. According to this proposition there is a mapping, let it be $\mu_m$ such that the following is true:

$$\beta_2 \Rightarrow \mu_1(\beta_1) \lor \cdots \lor \mu_{m-1}(\beta_1) \lor \mu_m(e_1^{\beta_1}) \lor \mu_{m+1}(e_1^{\beta_1}) \lor \mu_{m+2}(e_1^{\beta_1}) \lor \cdots$$
This proves part (i) of the inductive hypothesis. Now we need to prove the (ii) of the inductive hypothesis, i.e., that we can compute the fact associated with $\mu_m(e_i^{\beta_1})$.

According to (b) in Proposition 5.3 and taking into account Lemma Appendix A.2 we have that for each $ac_i$ in $\mu_m(\beta_1)$ the following is true:

$$\beta_2 \Rightarrow ac_i \lor e_j$$

These (i.e., the corresponding variables of the constants in $ac_i$ and $e_j$) provide the instantiation for the firing of couplings rules that compute all the I facts necessary to fire a mapping rule by the instantiation provided by $\mu_m$.

\[
\square
\]

**Appendix D. Proof of Theorems 5.16**

We claim that, during a computation of the Datalog query on an input, only annotated IDB predicates are populated with facts (that are computed during the computation) that have in their pattern at most one f. Suppose there is a computation where an annotated IDB predicate fact appears with more than one f on its pattern. Take a path from the root of the derivation tree to this IDB fact. This path tells you that there is a sequence of rules, that, if taken, during the process of creating annotating IDBs, you will arrive in this annotated IDB. We need to prove that the length of this path is bounded. This is easy because the annotations are finite and if the path is long then annotations will appear more than once.

After the above observation, we assume wlog that the Datalog query is monadic.

Consider a shortest derivation tree, $T_F$, of the fact $F$. A shortest derivation tree is one with the shortest height, where the height of a tree is defined to be the length of the longest path from the root to a leaf. We define the level of node $u$ in $T_F$ to be the height of the subtree that is rooted in $u$. 71
Proposition Appendix D.1. Consider a Datalog query $P$ and a derivation tree $T_F$ for a fact $F$. Then, all identical facts in $T_F$ are at the same level.

Proof. If not then consider the fact residing on the node with the smallest level and replace all subtrees with the subtree rooted in this node. This does not increase the level of a node.

The proof of the Theorem 5.16 is given as follows.

Proof. We will prove that the following decision problem is in NP: We consider the canonical database, $D$, of the CQ $Q_2$ and we compute the Datalog query $Q_1$ on $D$ and derive the output $Q_1(D)$. Given a fact $F$, we ask the question whether $F$ is in $Q_1(D)$.

Let us now consider that the following certificate is given:

- The unary IDB facts computed (which are polynomially many).
- The derivation DAG $G$ that computes the IDB facts (polynomial in size).

Following the Proposition Appendix D.1 we prove that the following construction results in a Directed Acyclic Graph (DAG). We consider a derivation tree $T_F$ for fact $F$. We collapse all subsets of identical facts into a single node (the edges of the tree are retained). We call this a derivation DAG of the fact $F$ and denote $G_F$.

We have not proved yet that indeed $G_F$ is a DAG. Notice that each path on $G_F$ uses the same edges as the edges in $T_F$ and, thus, it corresponds to a path in $T_F$. However, an edge in $T_F$ leads joins two nodes in different levels. Hence a cycle in $G_F$, which is a path that corresponds in $T_F$ cannot exist because a path in $T_F$ joins nodes in descending levels and, for that to form a cycle two nodes from different levels have to be identified. By construction of $G_F$, this does not happen.

The derivation DAG $G$ that is considered above is depicted in Figure D.2. Each node in the $G$ is either a EDB fact in the canonical database of $Q_2$ or a tuple consisting of the following:
• the rule in $Q_1$ query that is fired to compute the IDB fact from the previously computed IDB facts and/or EDB facts,

• a topological order of the graph $G$, and

• the mapping from the variables of the rule to the constants in $D$ producing the IDB fact.

Each directed edge $(n_1, n_2)$ of $G$ describes that the fact $n_1$ is used to compute the fact $n_2$.

Considering now such a certificate. To test it, we perform the following:

1. we check whether the given graph is a DAG following the topological order,

2. for each non-EDB node, we apply the mapping on the rule, check whether the head of the rule equals the fact in the node and that all the facts used in the application of this rule are computed, which means they are on lower (in the topological ordering) nodes.

It is easy to verify that the aforementioned tests can be validated in polynomial time, which proves that the problem is in $NP$. □
Appendix E. Definition of tgds and the chase algorithm

In this section, we analyse the query answering in the presence of dependencies over the database instances. We consider that the queries are in the language of CQs and focus on two main types of dependencies, the tuple-generating dependencies (tgds, for short) and the equality-generating dependencies (egds, for short). Then, we describe the Chase algorithm, a significant tool for reasoning about dependencies.

The dependencies describe certain conditions that are defined over a database schema and are applied on each instance of the schema. In particular, we formally define the tgds and egds, as follows.

Definition Appendix E.1. Let $S$ be a database schema. A tuple-generating dependencies is a dependency that is defined by a formula of the following form:

$$d_t : \phi(\overline{X}) \rightarrow \psi(\overline{X}, \overline{Y}),$$

where $\phi$ and $\psi$ are conjunctions of atoms with predicates in $S$, and $\overline{X}, \overline{Y}$ are vectors of variables. A equality-generating dependencies is a dependency that is defined by a formula of the following form:

$$d_e : \phi(\overline{X}) \rightarrow (X_1 = X_2),$$

where $\phi$ is a conjunctions of atoms with predicates in $S$, and $\overline{X}$ is vector of variables and $X_1, X_2$ are included in $\overline{X}$. Considering a database instance $D$ of $S$, we say that $D$ satisfies $d_t$ if whenever there is a homomorphism $h$ from $\phi(\overline{X})$ to $D$, there exists an extension $h'$ of $h$ such that $h'$ is a homomorphism from $\phi(\overline{X}) \land \psi(\overline{X}, \overline{Y})$ to $D$. In addition, we say that $D$ satisfies $d_e$ if for each homomorphism $h$ from $\phi(\overline{X})$ to $D$, we have that $h(X_1) = h(X_2)$.

In essence, the tgd $d_t$ describes the following. If there are tuples in $D$ satisfying the conjunction $\phi$ and mapping the variables $\overline{X}$ into the vector of constants $\overline{x}$ then there is a vector of constants $\overline{y}$ such that the atoms in $\psi(\overline{x}, \overline{y})$ are tuples in $D$. As for the egd $d_e$, for every set of tuples in $D$ satisfying
the conjunction $\phi$ and mapping the variables $\overline{X}$ into the vector of constants $\overline{x}$, we have $x_1 = x_2$, where $x_1$, $x_2$ are constants that are mapped by $X_1$, $X_2$, respectively.

Consider now a database schema $\mathcal{S}$ and an instance $D$ of $\mathcal{S}$, as well as a set of views $\mathcal{V}$ and a query $Q$ over $\mathcal{S}$. Furthermore, suppose that we do not have access to the base relations in $D$, but only to an instance $\mathcal{I}$ of the views in $\mathcal{V}$. If we consider a view instance $\mathcal{I}$ under Closed World Assumption (CWA), then $\mathcal{I} = \mathcal{V}(D)$. On the other hand, $\mathcal{I}$ might be incomplete (i.e., it might not include all the tuples satisfying the view definitions in $\mathcal{V}$). In such a case, we say that the instance $\mathcal{I}$ is under Open-World Assumption (OWA); i.e. $\mathcal{I} \subseteq \mathcal{V}(D)$.

**Definition Appendix E.2.** We define the certain answers of $(Q, \mathcal{I})$ with respect to $\mathcal{V}$ as follows:

- **Under the Closed World Assumption:**

  $$\text{certain}(Q, \mathcal{I}) = \bigcap \{Q(D) : D \text{ such that } \mathcal{I} = \mathcal{V}(D)\}.$$ 

- **Under the Open World Assumption:**

  $$\text{certain}(Q, \mathcal{I}) = \bigcap \{Q(D) : D \text{ such that } \mathcal{I} \subseteq \mathcal{V}(D)\}.$$ 

  In the presence of a set $\mathcal{C}$ of constraints, we also require that all databases $D$ used for $\text{certain}(Q, \mathcal{I})$ satisfy $\mathcal{C}$ and denote it by $\text{certain}_\mathcal{C}(Q, \mathcal{I})$.

Let us now define the query containment in the presence of constraints. Considering a database schema $\mathcal{S}$, a set of constraints $\mathcal{C}$ over $\mathcal{S}$ and two CQs $Q_1$, $Q_2$ over $\mathcal{S}$, we say that $Q_2$ is contained in $Q_1$ under the constraints $\mathcal{C}$, denoted $Q_2 \subseteq_\mathcal{C} Q_1$, if for all databases $D$ that satisfy $\mathcal{C}$ we have that $Q_2(D) \subseteq Q_1(D)$.

Let us now describe the chase algorithm. Initially, we define the chase step, the building block of the chase algorithm, as follows.

**Definition Appendix E.3.** Let $\mathcal{S}$ be a database schema and $D$ be a database instance of $\mathcal{S}$. Consider also the following dependencies.
\[ d_t : \phi(X) \rightarrow \psi(X, Y), \]
\[ d_e : \phi(X) \rightarrow (X_1 = X_2), \]
where \( \phi, \psi \) are conjunction of atoms with predicate in \( S \). Then, the chase step for the dependencies \( d_t \) and \( d_e \) is defined as follows.

(tgd \( d_t \)) Let \( h \) be a homomorphism from \( \phi(X) \) to \( D \) such that there is no extension \( h' \) of \( h \) that maps \( \phi(X) \land \psi(X, Y) \) to \( D \). In such a case, we say that \( d_t \) can be applied to \( D \) and we define the database instance \( D' = D \cup F_\psi \), where \( F_\phi \) is the set of atoms of \( \psi \) obtained by substituting each variable \( x \) in \( X \) with \( h(x) \) and each variable in \( Y \) (not mapped through \( h \)) with a fresh variable; i.e., \( F_\phi = \{ \psi(h(X), Y) \} \). The fresh variables used to replace variables in \( Y \) are called labeled nulls. We say that the result of applying \( d_t \) to \( D \) with \( h \) is \( D' \) and write \( D \xrightarrow{d_t,h} D' \) to denote the chase step on \( D \) with the tgd \( d_t \).

(egd \( d_e \)) Let \( h \) be a homomorphism from \( \phi(X) \) to \( D \) such that \( h(X_1) \neq h(X_2) \). In such a case, we say that \( d_e \) can be applied to \( D \) and we define the database instance \( D' \) as follows:

- if there is a fact \( e \) in \( \{ \phi(h(X)) \} \cap D \) such that \( h(X_2), h(X_1) \) are constants and \( h(X_2) \neq h(X_1) \) then \( D' = \perp \); otherwise
- for each fact \( e \) in \( \{ \phi(h(X)) \} \cap D \), we replace \( h(X_2) \) with \( h(X_1) \) and add it into \( D' \).

We say that the result of applying \( d_e \) to \( D \) with \( h \) is \( D' \) and write \( D \xrightarrow{d_e,h} D' \) to denote the chase step on \( D \) with the egd \( d_e \). If \( D' = \perp \), we say that the step fails.

Then, the chase algorithm is defined as follows.

Definition Appendix E.4. Let \( C \) be a set of tgds and egds and \( D \) be a database instance. Then, we define the following.

- A chase sequence of \( D \) with \( C \) is a sequence of chase steps \( D_i \xrightarrow{d_i,h_i} D_{i+1} \), where \( i = 0, 1, \ldots, D_0 = D \) and \( d_i \in C \).
A finite chase of \( D \) with \( C \) is a finite chase sequence \( D_i \xrightarrow{d,h} D_{i+1} \), with 
\[ i = 0, 1, \ldots, n, \quad D_0 = D \text{ and } d \in C, \text{ such that either } n \text{-th step fails, or } \]
there is no dependency \( d \in C \) and there is no homomorphism \( h \) such that \( d \) can be applied to \( D_n \).

Considering a CQ \( Q \) over a database schema \( S \) and a set \( C \) of tgd \( s \) and egd \( s \) dependencies over \( S \), we construct the canonical database \( D \) of \( Q \) and apply the chase algorithm. Let \( D' \) be the database resulted by chase. We can now construct a CQ \( Q_C \) from \( Q \) and \( D' \), such that the head of \( Q \) equals the head of \( Q_C \) and \( Q_C \)’s body is constructed by de-freezing the constants back to their corresponding variables. If the chase of \( Q \) with \( C \) terminates then \( Q_C \) is the called the \textit{chased query} of \( Q \) with \( C \). In such a case, for all databases \( D \) that satisfy the constraints \( C \), we have that \( Q(D) \subseteq Q_C(D) \) (i.e., \( Q \sqsubseteq_C Q_C \)) [31].

One way to view the \textit{chase} algorithm is as generalizing the algorithm that computes the canonical rewriting. The chase algorithm considers tuple generating dependencies and equality generating dependencies. View definitions can be turned into tuple generating dependencies in a straightforward way. Thus, there is an alternative way to find the certain answers (for definitions of tuple generating dependencies and the chase algorithm see [19]). We turn the view definitions to tuple generating dependencies and apply the chase algorithm on the view instance. Then we compute the query on the result of the chase algorithm. Another problem where the chase algorithm is useful is when we check query containment under dependencies. However, if we add arithmetic comparisons to the tuple generating dependencies [16], then the chase algorithm does not work efficiently except in the case the homomorphism property holds for the tuple generating dependencies. We do not add details here, which can be found in [19]. However, we will explain informally on an example:

**Example Appendix E.5.** We use the three views of Example 2.7 and the query:

\[
Q : a(X,Y), X < 5, Y < 5
\]

The canonical database of \( Q \) is \( \{a(X,Y), X < 5, Y < 5\} \). We compute the views
on it and we construct the canonical rewriting, enhanced with ACs appropriately:

\[ R_{\text{can}} : \neg v_3(X,Y), v_2(X,Y), X < 5, Y < 5. \]

Notice that \( R_{\text{can}} \) is equivalent to \( Q \).

**Example Appendix E.6.** Consider the views and query in Example Appendix E.5. The views can be written as tuple generating dependencies (tgds for short) as follows:

\[ V_1 : a(X,Y), Y \leq 5, X = 5 \rightarrow v_1(X,Y) \]
\[ V_3 : a(X,Y), Y < 5 \rightarrow v_3(X,Y) \]
\[ V_2 : a(X,Y), X \leq 5, Y \leq 5 \rightarrow v_2(X,Y) \]

The canonical database of \( Q \) is \( \{a(X,Y), X < 5, Y < 5\} \). The chase algorithm applied on \( \{a(X,Y), X < 5, Y < 5\} \) will work as follows. For each tgd it will check whether there is a homomorphism from its left hand side on \( \{a(X,Y), X < 5, Y < 5\} \) that satisfies the ACs. If there is we add in \( \{a(X,Y), X < 5, Y < 5, v_3(X,Y), v_2(X,Y)\} \), which satisfies the given tgd's because: for any homomorphism from the left hand side of tgd on \( \{a(X,Y), X < 5, Y < 5, v_3(X,Y), v_2(X,Y)\} \) there is an extension of this homomorphism to a homomorphism from the atoms of both sides of the tgd on this instance. Now, the canonical rewriting can be formed by considering the view atoms in the result of the chase and it is the same as in Example Appendix E.5.

The following theorem states the property of chase that makes it useful:

**Theorem Appendix E.7.** Let \( C \) be a set of tgd's, and \( D \) a database instance that satisfies the dependencies in \( C \). Suppose \( K \) is a database instance, such that there exists a homomorphism \( h \) from \( K \) to \( D \). Let \( K_C \) be the result of a successful finite chase on \( K \) with the set of dependencies \( C \). Then the homomorphism \( h \) can be extended to a homomorphism \( h' \) from \( K_C \) to \( D \).
Appendix F. Certain answers, MCRs and unclean data

First we point out an interesting case which is essentially a technicality that we need to consider for the proof of Theorem 6.5 specifically the point in the statement of the theorem that requires the existence of a database instance $D$ such that $\mathcal{I} \subseteq \mathcal{V}(D)$. We illustrate with an example. Under the Open World Assumption, if $\mathcal{I} \not\subseteq \mathcal{V}(D)$ for all databases $D$, and $R$ is an MCR of a query $Q$ using views with respect to a query language $\mathcal{L}$, then there are cases where $\text{certain}(Q, \mathcal{I}) = \emptyset$ and $R(\mathcal{I}) \neq \emptyset$.

Example Appendix F.1. Consider the case where the query is $Q(x, y) :- a(x, y)$, we have only one view $v(x, x, y) :- a(x, y)$, and the view instance is $\mathcal{I} = \{v(1, 2, 3), v(4, 4, 5)\}$. Since $v(1, 2, 3) \in \mathcal{I}$ and $v(1, 2, 3) \not\in \mathcal{V}(D)$ for any database $D$, we have that $\mathcal{I} \not\subseteq \mathcal{V}(D)$ for all databases $D$.

There is only one rewriting $R(x, y) :- V(x, x, y)$. Then $R(\mathcal{I}) = \{(4, 5)\}$ and we have that

$$\text{certain}(Q, \mathcal{I}) = \bigcap_{D \text{ s.t. } \mathcal{I} \subseteq \mathcal{V}(D)} Q(D) = \emptyset \text{ because } \not\exists D \text{ such that } \mathcal{I} \subseteq \mathcal{V}(D).$$

We define, however, new semantics that can be useful in data cleaning, in the following way: We do not need to change (clean) the data and still get correct answers. Observe is our example, that the answer we got by applying the MCR to our data, $\mathcal{I}$, is correct in the following sense: The instance $\mathcal{I}$ in the example contained one tuple that could not have been produced by applying the view on any instance. Hence we can assume that this tuple is incorrect and define a new $\mathcal{I}'$ as follows:

- $\mathcal{I}'$ is maximal with respect to the property below.

- $\mathcal{I}'$ is a subset of $\mathcal{I}$ and there is a $D$ such that $\mathcal{I}' \subseteq \mathcal{V}(D)$.

It easy to see that we can produce $\mathcal{I}'$ by removing in any order facts from $\mathcal{I}$ until the property $\mathcal{I}' \subseteq \mathcal{V}(D)$ is satisfied. Formally, we have:
**Definition Appendix F.2.** We define the minimal consistent view instance of $I$ to be instance $I$ with the property: it is the maximal subset of $I$ such that there is a database $D$ such that $I' \subseteq \mathcal{V}(D)$.

**Definition Appendix F.3.** We define the correct certain answers of $I$ to be the certain answers of $I'$ which is the maximal consistent view instance of $I$.

We have the following theorem:

**Theorem Appendix F.4.** An MCR of the query using the views produces all correct certain answers of any view instance $I$. 