Discretization and superintegrability all rolled into one

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Received 11 February 2019, revised 17 January 2020
Accepted for publication 12 May 2020
Published 31 July 2020

Abstract
Abelian integrals appear in mathematical descriptions of various physical processes. According to Abel’s theorem these integrals are related to motion of a set of points along a plane curve around fixed points, which are rarely used in physical applications. We propose to interpret coordinates of the fixed points either as parameters of exact discretization or as additional first integrals for equations of motion reduced to Abelian quadratures on a symmetric product of algebraic curve.

Keywords: Abel theorem, exact discretization, superintegrable systems
Mathematics Subject Classification numbers: 14H70, 37J35, 70H06.

(Some figures may appear in colour only in the online journal)

1. Introduction

Most of the applications nowadays of Abel’s theorem use Riemannian ideas and, therefore, in current textbooks Abel’s theorem looks as follows:

Modern version of Abel’s theorem: let $X$ be a compact Riemann surface and $D$ be a divisor of degree zero on $X$. Then $D$ is the divisor of a meromorphic function on $X$ if and only if $\mu(D) = 0$ in the Jacobian of $X$.

Here $\mu$: Div $X \rightarrow$ Jac $X$ is an Abel–Jacobi map, so if $\mu(D) = 0$, there is a collection of paths $\lambda_k$ from base point $P_0 \in X$ to points $P_k$ in the divisor $D$ so that

$$\sum_k \int_{\lambda_k} \omega = 0.$$ 

This theorem has well-known roots in classical mechanics. Indeed, in 1694 James Bernoulli studied the curve for which time taken by an object sliding without friction in a uniform gravity to its lowest point is independent of its starting point, and introduced integrals which can not be expressed in terms of elementary functions. Similar integrals were discovered in attempts to
rectify elliptical orbits of planets, so such integrals became known as ‘elliptic integrals.’ Later, Euler and Lagrange provided an analytical solution to the so-called tautochrone problem and applied the addition law for elliptic integrals to search of the algebraic trajectories among transcendental ones in the two fixed center problem [4, 12].

Therefore, it is not surprising that Abel in his Mémoire [1] studies integrals of algebraic functions using rational time parametrisation of a plane curve and motion of variable points along this curve, see historical remarks in [6].

Original version of Abel’s theorem: a set of $k$ points $(x_j, y_j)$ moving along plane curve $X$ can be subjected to a finite number of algebraic constraints in such a way that a sum of indefinite integrals

$$\int \omega(x_1, y_1) \, dx_1 + \int \omega(x_2, y_2) \, dx_2 + \cdots + \int \omega(x_k, y_k) \, dx_k$$ (1.1)

can be expressed in terms of algebraic and logarithmic functions of coordinates $(x_j, y_j)$ of the moving points provided these coordinates satisfy the constraints.

Algebraic constraints are independent of differentials $\omega(x, y) \, dx$ and, according to Clebsch and Gordan, we can replace the ‘algebraic constraints’ with ‘coordinates of fixed points’. Movable and fixed points form a divisor which division into two types of points allows to describe the so-called canonical injections of $k$-fold symmetric products $X(k)$ of algebraic curve $X$

$$j_{mk} : X(k) \to X(m), \quad k > m,$$

which are compatible with the Abel–Jacobi map $\mu$. In [3] Chow proposed a projective construction of the Jacobian using these injections and the Riemann theorem. This exhibited basic character of the Jacobian in a new way. It was taken up by Matsusaka and later by Grothendieck in their works on the Picard variety, see discussion in textbook [7].

**Example 1.** As an illustration of this generic theory we take cubic curve $X$ defined by a short Weierstrass equation

$$X : \quad y^2 = x^3 + ax + b,$$ (1.2)

and consider variable points of intersection of $X$ with a family of straight lines all passing through the same fixed point and depending on parameter $t$

$$Y : \quad y = b_1(t)x + b_0(t).$$

In his proof of Euler’s results Lagrange identified $t$ with time and introduced equations of motion in the projective plane

$$\frac{dx_1}{dt} = y_1, \quad \frac{dy_1}{dt} = \frac{3x_1 + a}{2} \quad \text{and} \quad \frac{dx_2}{dt} = y_2, \quad \frac{dy_2}{dt} = \frac{3x_2 + a}{2},$$ (1.3)

associated with differential $\omega = dx/y$ of the first kind on $X$. These equations of motion in the projective plane have an integral of motion, i.e. fixed point $P_3$ in figure 1. All details of the Lagrange calculations can be found on page 144 of Greenhill’s textbook [5] and in [6].

In Clebsch and Gordan’s interpretation of Abel’s result points $P_1, P_2$ and $P_3$ form an intersection divisor of plane curves $X$ and $Y$

$$D = P_1 + P_2 + P_3 = 0,$$ (1.4)

where $+$ and $=$ are addition and linear equivalence of divisors on $X$. There are two well-known interpretations of intersection divisor $D$:
Figure 1. Motion of points $P_1$ and $P_2$ around fixed point $P_3$ on $X$

(a) $P_1$ and $P_2$ form an effective divisor or the point of $X(2)$, whereas $P_3$ is a point of $X(1)$. In this case equation (1.4) and figure 1 describe canonical injection $j_{12} : X(2) \to X(1)$ so that

$$j_{12}(P_1, P_2) = P_3.$$ 

(b) $P_1$, $P_2$ and $P_3$ are elements of the Jacobian of $X$. In this case equation (1.4) and figure 1 describe a group law of algebraic group $\text{Jac}(X)$

$$P_1 + P_3 = -P_2.$$

According to Jacobi we can identify $X(k)$ with Lagrangian submanifolds in phase space $M \simeq \mathbb{R}^{2k}$ with respect to a family of compatible Poisson brackets [10]. In this case two mathematical interpretations of intersection divisor $D$ generates two physical interpretations of the corresponding Poisson maps:

(a) $P_1$ and $P_2$ describe evolution of some dynamical system with two degrees of freedom with respect to time $t$, whereas coordinates of $P_3$ are integrals of motion (superintegrable systems);

(b) $P_1$, $P_2$ are states of some dynamical system with one degree of freedom at $t$ and $t + \Delta t$, whereas fixed point $P_3$ plays the role of discretization step (integrable discrete maps).

The first interpretation appeared in the Euler and Lagrange investigations of the two-center problem. The second interpretation is closely related with so-called Bäcklund transformations of the Hamilton–Jacoby equation.

There is one algebraic group $\text{Jac}(X)$ associated with curve $X$ and only two families of Poisson maps generated by addition and multiplication on the Jacobian. Canonical injections

$$j_{mn} : X(n) \to X(m), \quad n > m.$$ 

generate many other Poisson maps which properties have not been studied till now. Nevertheless, we have some examples of application of these maps for studying relations between various integrable system [19–21] and constructing new integrable systems [22–25].

There are also other relations between symmetric products $X(k)$ of algebraic curve $X$

$$\tilde{j}_{mn} : X(n) \to X(m)$$

without restriction $n > m$, which can be used in classical mechanics.

**Example 2.** Let us take a family of parabolas $Y'$

$$Y' : y - c = x (b_1(t)x + b_0(t)).$$
all passing through the same fixed point and depending on parameter \( t \), see figure 2. The equation for \( Y' \) is obtained by multiplying the equations for straight line \( Y \) by \( x \) and shifting the ordinate by an arbitrary parameter \( c \).

At any five points \( P_1, \ldots, P_5 \) lie on parabola \( Y' \) and movable points satisfy

\[
D' = P_1 + P_2 + P_4 + P_5 = 0, \quad D' \in \text{Div} X'.
\]

(1.5)

Fixed point \( P_3 \) does not belong to \( X \) and, therefore, equation (1.5) does not include this point in contrast with equation (1.4). This equation and figure 2 determine a mapping \( j_{22} : X(2) \to X(2) \) so that

\[
j_{22}(P_1, P_2) \to (P_4, P_5).
\]

In [24] we applied this map to construction of a new integrable system on a plane with two integrals of motion which are polynomials of second and six order in momenta.

In this note we continue to discuss applications of equations (1.4) and (1.5) in classical mechanics. Our main aim is to draw attention to the possibilities of using well-known and no-so-well-known relations between symmetric products of the algebraic curves in classical mechanics that are not inferior to the possibilities of using group operations on Jacobian, torsion subgroup actions on Jacobian, isogenies of Jacobians, etc. All the examples below will be related to cubic curve \( X \) in the Weierstrass form (1.2) in order to discuss the most simplest integrable systems.

2. Abel’s sums with holomorphic differentials

Let us rewrite equation (1.4) in its expanded form. At any time coordinates of two points \( P_i \) and \( P_j \) determine coefficients

\[
\begin{align*}
b_1(t) &= \frac{y_i - y_j}{x_i - x_j}, & b_0(t) &= \frac{x_i y_j - x_j y_i}{x_i - x_j}.
\end{align*}
\]

and coordinates of third point \( P_k \)

\[
x_k = b_1(t)(x_i + x_j), \quad y_k = b_1(t)x_k + b_0(t).
\]

(2.6)

In classical mechanics coordinates of movable points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) can be identified:
for dynamical system with one degree of freedom with coordinates \((q, p)\) on phase space \(M\) at two different times

\[
x_1 = q(t_n), \quad y_1 = p(t_n) \quad \text{and} \quad x_2 = q(t_{n+1}), \quad y_1 = p(t_{n+1}).
\] (2.7)

for dynamical system with two degrees of freedom with coordinates \((u_1, p_{u_1})\) and \((u_2, p_{u_2})\) on phase space \(M\) at the same time

\[
x_1 = u_1(t), \quad y_1 = p_{u_1}(t) \quad \text{and} \quad x_2 = u_2(t), \quad y_1 = p_{u_2}(t).
\] (2.8)

In the first case we rewrite equation (1.4) in the form

\[
P_2(t_{n+1}) = -P_1(t_n) - P_3
\]

and interpret it as a discrete map \(P(t_n) \rightarrow P(t_{n+1})\) depending on some fixed parameter \(P_3\).

In the second case we rewrite equation (1.4) in the form

\[
P_3 = -P_1(t) - P_2(t)
\]

and interpret it as a definition of the additional first integral \(P_3\).

Thus, discretization and superintegrability have all combined into one arithmetic equation in \(\text{Div} X\).

### 2.1. Integrable discrete map

Let us identify \(X\) with Lagrangian submanifold in phase space \(M = T^* \mathbb{R}\) and consider Lagrange equation of motion (1.3) on a cubic curve (1.2)

\[
\frac{dq}{dt} = p \quad \text{where} \quad p^2 = q^3 + aq + b.
\]

This equation appears when we take the Hamilton function

\[
H = p^2 - q^3 - aq,
\] (2.9)

and canonical Poisson brackets \(\{q, p\} = 1\), which define Hamiltonian equations of motion

\[
\dot{q} = \frac{\partial H}{\partial p} = 2p, \quad \dot{p} = -\frac{\partial H}{\partial q} = 3q^2 + a.
\] (2.10)

At \(H = b\) these equations are reduced to (1.3).

The same Lagrange equation (1.3) appears when we consider the motion of the symmetric heavy top. In the Lagrange case equation for nutation \(\gamma_3(t)\)

\[
\left(\frac{d\gamma_3}{dt}\right)^2 = (1 - \gamma_3)(h - k^2 - 2\gamma_3) - (c - k\gamma_3)^2
\] (2.11)

is also reduced to (1.3), see [9, 12]. Here \(c, h, k\) are the values of the corresponding integrals of motion.

According [26, 27] equation (1.4) is a finite-difference equation, which determine exact two-point discretization of equations of motion (2.10) or (2.11). Indeed, substituting (2.7) and \(i = 1, j = 3, k = 2\) into (2.6) one gets

\[
q(t_{n+1}) = b_1^2 - (q(t_n) + x_3), \quad p(t_{n+1}) = p(t_n) + b_1 (x_3 - q(t_n)), \quad b_1 = \frac{y_3 - p(t_n)}{x_3 - q(t_n)}.
\]
If we also put in \( x_3 = \lambda_n \) and \( y_3 = \sqrt{\lambda_n^3 + a\lambda_n + b} \), we obtain an iterative system of 2-point invertible mappings \( M \to M \) depending on a family of parameters \( \lambda_k \):

\[
\ldots \longrightarrow (q(t_{k-1})) \longrightarrow \left( \begin{array}{c} q(t_k) \\ p(t_k) \end{array} \right) \longrightarrow (q(t_{k+1})) \longrightarrow \left( \begin{array}{c} q(t_{k+2}) \\ p(t_{k+2}) \end{array} \right) \longrightarrow \ldots
\]

It is the so-called exact discretization of the equations of motion (2.10) preserving the form of integrals of motion and Poisson bracket, see e.g. [23, 24, 26, 27, 29].

**Proposition 1.** Equation (1.4) in Div \( X \) yields an integrable discrete map on phase space \( M \to M \), \( \dim M = 1 \), preserving the form of integrals of motion and the canonical Poisson bracket.

Here we explicitly present the exact discretization of motion in cubic potential and of motion of the Lagrange top associated with elliptic curve in the Weierstrass form (1.2). In similar way we can take an elliptic curve in the Jacobi form

\[
X : \quad y^2 = ax^4 + bx^2 + c
\]

and obtain exact discretization of the Duffing oscillator [26, 27] and of the Euler top [28].

### 2.2. Superintegrable system with two degrees of freedom

Let symmetric product \( X(2) \) be a Lagrangian submanifold in phase space \( M = T^*\mathbb{R}^2 \). If we identify abscissas and ordinates of points \( P_1 \) and \( P_2 \) with canonical coordinates (2.8) on phase space \( M \) and solve a pair of equation (1.2)

\[
\begin{align*}
p_{u_1}^2 &= u_1^3 + au_1 + b, \\
p_{u_2}^2 &= u_2^3 + au_2 + b
\end{align*}
\]  

with respect to \( a \) and \( b \), we obtain two functions on the phase space

\[
\begin{align*}
a &= \frac{p_{u_1}^2}{u_1 - u_2} + \frac{p_{u_2}^2}{u_2 - u_1} - u_1^2 - u_1u_2 - u_2^2, \\
b &= \frac{u_2 p_{u_2}^2}{u_2 - u_1} + \frac{u_1 p_{u_1}^2}{u_1 - u_2} + (u_1 + u_2)u_1u_2,
\end{align*}
\]  

which are in involution with respect to the canonical Poisson bracket

\[
\{u_1, u_2\} = 0, \quad \{p_{u_1}, p_{u_2}\} = 0, \quad \{u_i, p_{u_j}\} = \delta_{ij},
\]

Taking \( H = a \) as a Hamiltonian, one gets integrable system on phase space \( M = T^*\mathbb{R}^2 \) with Hamiltonian equations of motion

\[
\dot{u}_i = \frac{\partial H}{\partial p_{u_i}}, \quad \dot{p}_{u_i} = -\frac{\partial H}{\partial u_i},
\]  

which are reduced to quadratures

\[
\int \frac{u_1 du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{u_2 du_2}{\sqrt{u_2^3 + au_2 + b}} = -2t
\]
and
\[ \int \frac{du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{du_2}{\sqrt{u_2^3 + au_2 + b}} = \text{const}. \] 
(2.16)

According Euler and Lagrange [4, 12], the first quadrature determines parameterization of trajectories, whereas the second quadrature determines the form of trajectories. Thus, we can use first quadrature for discretization of time variable and second quadrature to the search of algebraic trajectories associated with additional algebraic integral of motion.

We identify second quadrature (2.16) and the corresponding Abel’s sum with equation (1.4)

\[ P_1(t) + P_2(t) = P_3 = \text{const}. \]

According to Euler and Lagrange [4, 12], the first quadrature determines parameterization of trajectories, whereas the second quadrature determines the form of trajectories. Thus, we can use first quadrature for discretization of time variable and second quadrature to the search of algebraic trajectories associated with additional algebraic integral of motion.

We identify second quadrature (2.16) and the corresponding Abel’s sum with equation (1.4)

\[ P_1(t) + P_2(t) = P_3 = \text{const}. \]

Substituting (2.8) and \( i = 1, j = 2, k = 3 \) into (2.6) one gets additional first integrals of equations of motion (2.14)

\[ x_3 = \left( \frac{p_{u_1} - p_{u_2}}{u_1 - u_2} \right)^2 - u_1 - u_2, \quad y_3 = p_{u_1} + \left( \frac{p_{u_1} - p_{u_2}}{u_1 - u_2} \right)(x_3 - u_1). \] 
(2.17)

Functions \( a, b (2.13) \) and functions \( x_3, y_3 (2.17) \) on phase space \( M = T^*\mathbb{R}^2 \) form an algebra of integrals

\[ \{a, b\} = 0, \quad \{a, x_3\} = 0, \quad \{a, y_3\} = 0, \]
\[ \{b, x_3\} = 2y_3, \quad \{b, y_3\} = 3x_3^2 + a, \quad \{x_3, y_3\} = -1, \] 
(2.18)

in which Weierstrass equation (1.2) plays the role of syzygy

\[ y_3^2 = x_3^3 + ax_3 + b. \]

**Proposition 2.** Equation (1.4) in \( \text{Div} X \) describes a representation of the algebra of integrals (2.18), i.e. superintegrable system on phase space \( M, \dim M = 2 \).

First equation in (2.17) is nothing more than an additional law for the Weierstrass function

\[ \varphi(u_1 + u_2) = \frac{1}{4} \left( \frac{\varphi'(u_1) - \varphi'(u_2)}{\varphi(u_1) - \varphi(u_2)} \right)^2 - \varphi(u_1) - \varphi(u_2), \]

and, therefore, algebra of the first integrals coincides with well-know relations between Weierstrass \( \varphi \)-function and its derivatives, see [2, 5, 8].

After canonical transformation of variables

\[ u_1 = q_1 - \sqrt{q_2}, \quad u_2 = q_1 + \sqrt{q_2}, \quad p_{u_1} = \frac{p_1}{2} + \frac{(u_1 - u_2)p_2}{2}, \quad p_{u_2} = \frac{p_1}{2} - \frac{(u_1 - u_2)p_2}{2} \]

these first integrals look like

\[ a = H = p_1p_2 - 3q_1^2 - q_2, \quad b = \frac{p_1^2}{4} - q_1p_1p_2 + q_2p_2^2 + 2q_1(q_1^2 - q_2), \]
\[ x_3 = p_2^2 - 2q_1, \quad y_3 = \frac{p_1}{2} + p_3^2 - 3p_2q_1. \]

Similar superintegrable systems on the plane with quadratic Hamiltonians

\[ H = p_1p_2 + V(q_1, q_2) \]
and cubic first integrals are discussed in [13, 16, 17].

2.3. Other representations of the algebra of first integrals

Below we consider the arithmetic equation

\[ D = \sum_{i=1}^{k} n_i P_i = 0, \quad n_i \in \mathbb{Z} \]

and Abel’s sum with holomorphic differentials involving more than three terms

\[ \sum_{i=1}^{k} n_i \int \omega(x_i, y_i)dx_i = \text{const}. \]

Different exact discretizations of Hamiltonian and non-Hamiltonian equations of motion associated with such arithmetic equations in Div X are discussed in [23, 24, 26–29].

Superintegrable systems associated with the same arithmetic equations are discussed in [30, 31]. These superintegrable systems can be considered as different representations of the algebra of integrals (2.18)

\[
\begin{align*}
\{a, b\} &= 0, \\
\{a, x_3\} &= 0, \\
\{a, y_3\} &= 0, \\
\{b, x_3\} &= 1, \\
\{b, y_3\} &= 0, \\
\{x_3, y_3\} &= -1,
\end{align*}
\]

labelled by two integers \(n\) and \(m\). Here \(f'\) is a derivative of function \(f\) from the definition of elliptic curve \(X\):

\[ y^2 = f(x). \]

Indeed, let us make a trivial non-canonical transformation

\[ p_{u_1} \rightarrow p_{u_1} n, \quad p_{u_2} \rightarrow p_{u_2} m \]

in the separated relations (2.12) [30]. Solving the new separated relations

\[
\begin{align*}
\left( \frac{p_{u_1}}{n} \right)^2 &= u_1^3 + au_1 + b, \\
\left( \frac{p_{u_2}}{m} \right)^2 &= u_2^3 + au_2 + b
\end{align*}
\]

(2.19)

with respect to \(a\) and \(b\), we obtain two functions on the phase space

\[
\begin{align*}
\alpha &= \frac{2p_{u_1}}{n^2(u_1 - u_2)} + \frac{p_{u_2}^2}{m^2(u_2 - u_1)} - u_1^2 - u_1u_2 - u_2^2, \\
\beta &= \frac{u_2p_{u_1}^2}{n^2(u_2 - u_1)} + \frac{u_1p_{u_2}^2}{m^2(u_1 - u_2)} + (u_1 + u_2)u_1u_2,
\end{align*}
\]

(2.20)

which are in involution with respect to the canonical Poisson bracket. Taking \(H = \alpha\) as a Hamiltonian, one gets Hamiltonian equations of motion (2.14)

\[
\frac{du_1}{dt} = \frac{2p_{u_1}}{n^2(u_1 - u_2)} \quad \text{and} \quad \frac{du_2}{dt} = \frac{2p_{u_2}}{m^2(u_2 - u_1)}.
\]

(2.21)

Substituting \(p_{u_1}\) and \(p_{u_2}\) from (2.19) into (2.21) we obtain equations

\[
\frac{ndu_1}{u_1^3 + au_1 + b} = \frac{2dt}{u_1 - u_2} \quad \text{and} \quad \frac{mdu_2}{u_2^3 + au_2 + b} = \frac{2dt}{u_2 - u_1}.
\]
which are reduced to quadratures
\[
n \int \frac{u_1 \, du_1}{\sqrt{u_1^2 + au_1 + b}} + m \int \frac{u_2 \, du_2}{\sqrt{u_2^2 + au_2 + b}} = -2t
\]
and
\[
n \int \frac{du_1}{\sqrt{u_1^2 + au_1 + b}} + m \int \frac{du_2}{\sqrt{u_2^2 + au_2 + b}} = \text{const.}
\]

Second quadrature is the homogeneous Abel’s sum associated with an arithmetic equation in \(\text{Div } X\)
\[
[n]P_1 + [m]P_2 + P_3 = 0. \tag{2.22}
\]
Coordinates of the fixed point \(P_3 = (x_3, y_3)\)
\[
x_3 = \left(\frac{[n]y_1 - [m]y_2}{[n]x_1 - [m]x_2}\right)^2 - ([n]x_1 + [m]x_2), \quad y_3^2 = x_3^3 + ax_3 + b \tag{2.23}
\]
are functions on the phase space commuting with Hamiltonian \(H = a \tag{2.20}\). Here \([k]x\) and \([k]y\)
are affine coordinates of point \([k]P\) on the projective plane defined by well-known equation
\[
[k]P \equiv ([k]x, [k]y) = \left(x - \frac{\psi_{k-1} \psi_{k+1}}{\psi_k}, \frac{\psi_{2k}}{2\psi_k^2}\right)
\]
where \(\psi_k(x, y)\) are the so-called division or torsion polynomials in a ring \(\mathbb{Z}[x, y, a, b]\), see \([11, 32]\). The first four polynomials are defined explicitly as
\[
\psi_1 = 1, \quad \psi_2 = 2y, \quad \psi_3 = 3x^4 + 6ax^2 + 12bx - a^2, \\
\psi_4 = 4y(x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - 8b^2 - a^3),
\]
the subsequent polynomials are defined inductively as
\[
\psi_{2k+1} = \psi_{k+1} \psi_k^3 - \psi_{k-1} \psi_k^3, \quad k \geq 2, \\
\psi_{2k} = (2y)^{-1} \psi_k (\psi_{k+1} \psi_k - 1) - \psi_{k-2} \psi_k^2, \quad k \geq 3.
\]
Using these division polynomials we can easily calculate integrals of motion \(2.23\). For instance, at \(n = 2\) and \(m = 1\) additional first integral \(x_3\) is a rational function of the form
\[
x_3 = 4u_1 - 3u_2 - \frac{8(u_1 - u_2) (6u_1^3 - 12u_1^2 u_2 + 6u_1 u_2^2 + p_{u_1} p_{u_2} - 2p_{u_1}^2)}{(8u_1^3 - 12u_1^2 u_2 + 4u_1^2 - p_{u_1}^2 + 4p_{u_1} p_{u_2} - 4p_{u_1}^2)} + \frac{64^2 (u_1 - u_2)^4 (2u_1^3 - 3u_1^2 u_2 + u_1 u_2^2 + p_{u_1} p_{u_2} - p_{u_1}^2)}{(8u_1^3 - 12u_1^2 u_2 + 4u_1^2 - p_{u_1}^2 + 4p_{u_1} p_{u_2} - 4p_{u_1}^2)^2}.
\]
At \(n = 3\) and \(m = 1\) the additional first integral is equal to
\[
x_3 = -(u_1 + u_2) + \frac{(p_{u_1} - 9p_{u_1}^2)^2}{81(p_{u_1} + p_{u_2})^2(u_1 - u_2)^2} + \frac{8p_{u_1} A_1}{9(p_{u_1} + p_{u_2})^2 B} - \frac{64p_{u_1}^2 A_2}{9(p_{u_1} + p_{u_2}) B^2},
\]
\[
4932
\]
where
\[ B = p_{11}^1 - 8p_{11}^0p_{02} + 18 \left( p_{12}^1 - u_1^0u_2 + 2u_1u_2^2 - u_2^3 \right) p_{11}^0 - 27 \left( p_{12}^2 - 2u_1^1 + 3u_1^0u_2 - u_2^3 \right)^2, \]
and
\[ A_1 = (15u_1 + 19u_2)p_{11}^1 - 6(13u_1 + 5u_2)p_{11}^3p_{02} - 3(11u_1 + 19u_2)(2u_1 + u_2)(u_1 - u_2)^2 p_{11}^2 \]
\[ + 54 \left( (u_1 + u_2)p_{12}^2 + (2u_1 + u_2)(u_1 - u_2)^3 \right) p_{11}p_{02} \]
\[ - 27p_{12}^1(5u_1 + u_2) \left( p_{12}^1 - (2u_1 + u_2)(u_1 - u_2)^2 \right), \]
\[ A_2 = 2(u_1 + u_2)p_{11}^1 - 4(7u_1 + 5u_2)p_{11}^3p_{02} + 54p_{12}^1(5u_1 + u_2) \left( p_{12}^1 - (2u_1 + u_2)(u_1 - u_2)^2 \right) \]
\[ + 3(24(2u_1 + u_2)p_{12}^3 - (7u_1^2 + 16u_1u_2 + 13u_2^2)(u_1 - u_2)^2) p_{11}^2 \]
\[ - 9(12(3u_1 + u_2)p_{12}^2 - (23u_1^2 + 38u_1u_2 + 11u_2^2)(u_1 - u_2)^2) p_{01}p_{02}. \]

In both cases of \( m = 2 \) and \( m = 3 \) four integrals of motion \( a, b \) (2.20) and \( x_3, y_3 \) (2.23) form the algebra of integrals (2.18), same as in the previous case at \( n = m = 1 \).

We conjecture this is to be true for general \( n \) and \( m \).

**Proposition 3.** Equation (2.22) in \( \text{Div} X \) describes representation of the algebra of integrals (2.18) labelled by two integers \( n \) and \( m \), i.e. superintegrable system on phase space \( M, \dim M = 2 \).

Other examples of superintegrable systems associated with Abel’s sums including holomorphic differentials may be found in [16–18, 30, 31].

### 3. Abel’s sums with non-holomorphic differentials

Let us consider the motion of parabola \( Y' \) defined by an equation of the form
\[ Y' : \quad y - c = x \left( b_1(t)x + b_0(t) \right), \]
around fixed point \( P_3 = (0, c) \), see figure 2. If \( P_i(t) \) and \( P_j(t) \) are two movable intersection points of parabola \( Y' \) with cubic curve \( X \) (1.2), then
\[ b_i(t) = \frac{y_i^0x_j - y_j^0x_i}{x_jx_i(x_i - x_j)} \quad \text{and} \quad b_0(t) = -\frac{y_i^0x_j^2 - y_j^0x_i^2}{x_jx_i(x_i - x_j)} \quad (3.24) \]
due to Lagrange interpolation of parabola using three points \( P_i(t), P_j(t) \) and \( P_3 \).

Equation (1.5)
\[ D' = P_1 + P_2 + P_4 + P_5 = 0, \quad D' \in \text{Div} X, \]
can be considered as a discrete map in \( \text{Div} X \)
\[ (P_1, P_j) \rightarrow (P_i, P_m) \]
because coordinates of the remaining two movable points \( P_i \) and \( P_m \) are easily expressed via \( x_i, x_j \) and \( y_i, y_j \). Indeed, according to Abel [1] abscissas \( x_i \) and \( x_m \) are roots of the so-called Abel polynomial
\[ \Psi = x^3 + ax + b - (x(b_1x + b_0) + c) = b_1^2(x - x_i)(x - x_j)(x - x_3)(x - x_5), \quad (3.25) \]
whereas ordinates $y_\ell$ and $y_m$ are equal to
\[ y_\ell = x_\ell (b_1 x_\ell + b_0) + c, \quad y_m = x_m (b_1 x_m + b_0) + c. \tag{3.26} \]

As mentioned above, discrete map in $\text{Div} \ X$ generates an integrable discrete map on phase space $M$.

### 3.1. Integrable discrete map

Let us come back to the integrable system with two degrees of freedom defined by the following integrals of motion (2.13)
\[
\begin{align*}
    a &= \frac{\dot{p}_{u_1}^2}{u_1 - u_2} + \frac{\dot{p}_{u_2}^2}{u_2 - u_1} - u_1^2 - u_2^2, \\
    b &= \frac{u_2 p_{u_1}}{u_2 - u_1} + \frac{u_1 p_{u_2}}{u_1 - u_2} + (u_1 + u_2) u_1 u_2.
\end{align*}
\]

These integrals are in the involution with respect to the Poisson brackets
\[
\{u_1, u_2\}_\varphi = 0, \quad \{p_{u_1}, p_{u_2}\}_\varphi = 0, \quad \{u_i, p_{u_j}\}_\varphi = \delta_{ij} \hat{\varphi}(u_i, p_{u_i}),
\]
labelled by two arbitrary functions $\hat{\varphi}(u_i, p_{u_i})$. The corresponding Poisson bivector reads as
\[
\Pi = \begin{pmatrix}
    0 & 0 & \hat{\varphi}_1(u_1, p_{u_1}) & 0 \\
    0 & 0 & 0 & \hat{\varphi}_2(u_2, p_{u_2}) \\
    -\hat{\varphi}_1(u_1, p_{u_1}) & 0 & 0 & 0 \\
    0 & -\hat{\varphi}_2(u_2, p_{u_2}) & 0 & 0
\end{pmatrix}. \tag{3.27}
\]

Taking $H = a$ as a Hamiltonian, one gets an integrable system on the phase space $M = T^*\mathbb{R}^2$ with Hamiltonian equations of motion (2.14) which are reduced to quadratures (2.15) and (2.16).

In the previous section we use second quadrature (2.16), i.e. Abel’s sum with the holomorphic differential on $X$
\[
\int \frac{du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{du_2}{\sqrt{u_2^3 + au_2 + b}} = \text{const},
\]
to construct the additional first integrals. These integrals $x_3$ and $y_3$ coincide with coordinates of the fixed point $P_3$ on $X$, the existence of additional algebraic integral of motion guarantees an existence of algebraic trajectories, see Euler paper [4].

In order to construct a discrete integrable map $M \to M$ we have to take first quadrature (2.15), i.e. Abel’s sum with non-holomorphic differentials
\[
\int \frac{u_1 du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{u_2 du_2}{\sqrt{u_2^3 + au_2 + b}} = -2t,
\]
and interpret equation (1.5)
\[ P_1 + P_2 + P_3 + P_4 = 0 \]
as a discrete map relating pairs of movable points at $t_n$ and $t_{n+1}$, respectively:
\[ (P_i, P_j)(t_n) \to (P_i, P_m)(t_{n+1}). \]
For brevity, we omit dependence on time below.

Thus, let us identify variables on phase space $M$ with affine coordinates of two movable points $P_1$ and $P_2$ on a projective plane

$$x_1 = u_1, \quad y_1 = p_{u_1} \quad \text{and} \quad x_2 = u_2, \quad y_1 = p_{u_2}.$$ 

Coordinates of remaining movable points $P_4$ and $P_5$ are some other variables on $M$

$$x_4 = v_1, \quad -y_4 = p_{v_1} \quad \text{and} \quad x_5 = v_2, \quad -y_5 = p_{v_2}.$$ 

Here we change sign before ordinates $y_{4,5}$ in order to rewrite the equation (1.5) in the following form

$$P_1 + P_2 = P_4 + P_5.$$ 

At $c = 0$ in (3.25) and (3.26) one gets

$$(x - v_1)(x - v_2) = x^2 + \frac{(u_1 - u_2)\left(p_{u_2}u_1^2 - p_{u_1}u_2^2 - (u_1 - u_2)u_1^2u_2^2\right)}{(p_{u_2}u_2 - p_{u_1}u_1)^2} x$$

$$+ \frac{u_1u_2(u_1 - u_2)(p_{u_1}^2u_2^2 - u_1^2u_2^2 - u_1u_2^3)}{(p_{u_2}u_2 - p_{u_1}u_1)^2}$$

and

$$p_{v_1} = -v_1(b_1v_1 + b_0), \quad p_{v_2} = -v_2(b_1v_2 + b_0),$$

where

$$b_1 = \frac{p_{u_2}u_2 - p_{u_1}u_1}{u_1u_2(u_1 - u_2)}, \quad b_0 = \frac{(p_{u_1}u_2^2 - p_{u_2}u_1^2)}{u_1u_2(u_1 - u_2)}.$$

Thus, we have an integrable discrete map on the phase space $M$

$$\rho : \ (u_1, u_2, p_{u_1}, p_{u_2}) \to (v_1, v_2, p_{v_1}, p_{v_2})$$

preserving form of the integrals of motion (2.13) and form of the following Poisson bivector

$$\rho : \ \Pi = \begin{pmatrix} 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & u_2 \\ -u_1 & 0 & 0 & 0 \\ 0 & -u_2 & 0 & 0 \end{pmatrix} \to \Pi = \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & v_2 \\ -v_1 & 0 & 0 & 0 \\ 0 & -v_2 & 0 & 0 \end{pmatrix},$$

which belongs to a family of compatible Poisson bivectors (3.27).

**Proposition 4.** Equation (1.5) in $\text{Div} \ X$ yields an integrable discrete map $M \to M$, $\dim M = 2$, preserving the form of integrals of motion and one of the compatible Poisson bivectors (3.27).

Other examples of the Poisson maps associated with differentials $xdx/y$ and $x^2dxdy/y$ on an elliptic curve may be found in [22].
3.2. Construction of integrable systems with higher order polynomial integrals of motion

Let us identify symmetric product $\mathbb{X}(2)$ with Lagrangian submanifold in phase space $M = T^* \mathbb{R}^2$ such that affine coordinates of movable points are expressed via canonical variables on $M$ in the following way

\[
x_1 = u_1, \quad y_1 = u_1 p_{u_1} \quad \text{and} \quad x_2 = u_2, \quad y_1 = u_2 p_{u_2}
\]

and

\[
x_4 = v_1, \quad -y_4 = v_1 p_{v_1} \quad \text{and} \quad x_5 = v_2, \quad -y_5 = v_2 p_{v_2}.
\]

Then we determine canonical transformation $\rho' : M \to M$ preserving standard Poisson bivector

\[
\rho' : \quad \Pi = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \quad \to \quad \Pi = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

This canonical transformation is defined by the same relations (3.24)–(3.26).

Usually coordinates $u_1, u_2$ are standard curvilinear orthogonal coordinates on the plane, sphere or ellipsoid, whereas new canonical coordinates $v_{1,2}$ and $p_{v_{1,2}}$ are images of the curvilinear coordinates after some integrable Poisson maps. In [22–25, 29] we used these coordinates to construct new integrable systems in the framework of the Jacobi method.

For instance, let us take coordinates $v_{1,2}$ and momenta $p_{v_{1,2}}$ defined by (3.25) and (3.26)

\[
(x - v_1)(x - v_2) = x^2 + \frac{(u_1 - u_2)(p_{u_1}^2 - p_{u_2}^2 - u_1 + u_2)}{(p_{u_1} - p_{u_2})^2} x
\]

\[
+ \frac{(u_1 - u_2)(p_{u_1}^2 u_1 - p_{u_2}^2 u_2 - u_1^2 + u_2^2)}{(p_{u_1} - p_{u_2})^2},
\]

\[
v_j p_{v_j} = -v_j \left( p_{u_1} - p_{u_2} \right) p_{v_1,2} \frac{u_1 - u_2}{u_1 - u_2}, \quad j = 1, 2.
\]

Replacing these variables into the separated relations

\[
2 v_1 p_{v_1}^2 = 2 v_1^2 + H + \sqrt{K}, \quad \text{and} \quad 2 v_2 p_{v_2}^2 = 2 v_2^2 + H - \sqrt{K},
\]

one gets integrable systems with polynomial integrals of motion

\[
H = T + V = \frac{u_1 (2 u_1 + u_2) p_{u_1}^3}{u_1 - u_2} + \frac{u_2 (u_1 + 2 u_2) p_{u_2}^3}{u_2 - u_1} - 2 u_1^2 - 3 u_1 u_2 - 2 u_2^2,
\]

and

\[
K = \frac{u_1^2 u_2^2}{(u_2 - u_1)^2} \left( (3 u_1 + u_2) p_{u_1}^4 - (u_1 + 3 u_2) p_{u_2}^4 - 8 u_1 p_{u_1}^4 p_{u_2} + 8 u_2 p_{u_2}^4 + 6 (u_1 - u_2) p_{u_1}^2 p_{u_2}^2 - 2 (u_1 - u_2) (3 u_1 + u_2) p_{u_1}^2 - (u_1 - u_2)^2 \right),
\]

which are polynomials of second and fourth order in momenta. It is easy to prove that this
Hamiltonian has no integrals of motion which are polynomials of first, second and third order in momenta.

Integrable metric
\[
g = \begin{pmatrix}
  \frac{u_1(2u_1 + u_2)}{u_1 - u_2} & 0 \\
  0 & \frac{u_2(u_1 + 2u_2)}{u_3 - u_1}
\end{pmatrix}
\]

belongs to a family of integrable and superintegrable metrics from [22]. Here we add new potential \( V \) to the known kinetic energy \( T \).

Thus, we obtain a new non-trivial integrable system on a plane with natural quadratic Hamiltonian \( H = T + V \) and quartic second integral of motion. This systems belongs to a family of two-dimensional integrable systems with position-dependent mass (PDM), which has various applications in physics, see [14, 15] and references within. Using the proposed approach we can construct other new PDM systems with integrals of motion which are polynomials of second, third, fourth and even sixth order in momenta.

4. Conclusion

In modern textbooks Abel’s theorem provides the necessary and sufficient conditions for the existence of meromorphic functions with prescribed zeros and poles on a compact Riemann surface \( X \). As is well known, this problem is equivalent to existence of a parallel section for some complex connection in the holomorphic line bundle of the divisor. It is far from anything in Abel’s original works, although we continue to call it the Abel theorem.

If we come back to original Abel’s theorem we can find many applications of this theorem in physics. Indeed, many equations of mathematical physics are reduced to Abel’s quadratures using orthogonal curvilinear coordinates or more exotic variables, for instance, variables of separation for the Kowalevski top. Starting with these well-known systems we can get new integrable systems and discrete maps by using the proposed approach.

In classical mechanics there are integrable systems with a common level set of first integrals, which can be identified with a generalized Jacobian \( \text{Jac}(X) \), which is a commutative algebraic group. To study these systems we can use various properties of \( \text{Jac}(X) \) such as group operations, torsion subgroup actions, isogenies, etc.

There are also integrable systems with a common level set of first integrals, which can be identified with symmetric products \( X(n) \) of curve \( X \), which have no a group structure. Nevertheless, varieties \( X(n) \) and their canonical injections are classical objects of study in algebraic geometry, and it would be natural to expect to find various applications of these well-studied algebro-geometric tools in classical mechanics. Unfortunately, we could not find such applications in the current literature. In this note we try to fill this gap starting with the simplest cubic curve, its symmetric products and its Jacobian.

Acknowledgments

The work was supported by the Russian Foundation for Basic Research (project 18-01-00916).

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