Transcendence of Values of the Iterated Exponential Function at Algebraic Points

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Abstract

We say that the limit of a sequence of functions

\[ x, \ x^x, \ x^{x^x}, \ldots \]

is the iterated exponential function, denoted by \( h(x) \). By a result of Barrow, this limit is convergent for every \( x \in [e^{-e}, e^{1/e}] \). In this paper, we prove that, for each fixed integer \( k \geq 2 \), the limit \( h(A) \) is transcendental for all but finitely many algebraic numbers \( A \in [e^{-e}, e^{1/e}] \) with \( k = \min\{n \in \mathbb{N} \mid A^n \in \mathbb{Q}\} \). Furthermore, let \( Q(k) \) be the cardinality of exceptional points \( A \). We prove that the ratio \( Q(k)/\varphi(k) \) approaches \( e - 1/e \) as \( k \to \infty \), where \( \varphi(k) \) denotes Euler’s totient function.

1 Introduction

We say that a complex number \( \alpha \) is algebraic if there exists a non-zero polynomial \( f(X) \) with rational coefficients such that \( f(\alpha) = 0 \), and \( \alpha \) is transcendental if \( \alpha \) is not algebraic. Let \( \mathbb{A} \) and \( \mathbb{T} \) denote the set of all algebraic numbers and transcendental numbers, respectively. A fundamental problem in transcendental number theory is to determine the transcendence (or algebraicity) of a given number.

In 1934, Gelfond and Schneider (independently) solved one of the big problems in the area, called Hilbert’s 7th problem.

**Theorem 1** (Gelfond-Schneider [6, 7, 12, 13]). If \( \alpha \in \mathbb{A} \setminus \{0, 1\} \) and \( \beta \in \mathbb{A} \setminus \mathbb{Q} \), then \( \alpha^\beta \) is transcendental.

By using this result, we study the transcendence of the limit of a sequence

\[ x, \ x^x, \ x^{x^x}, \ldots \]  \hspace{1cm} (1)

This limit is denoted by \( h(x) \), called the iterated exponential function. Formally, the limit \( h(x) \) can be written as

\[ h(x) = x^{x^x} \].

The limit of a sequence (1) is convergent for every \( e^{-e} \leq x \leq e^{1/e} \) from a result of Barrow [3, Theorem 5], and he also proved that

\[ h(x) = x^{h(x)}, \text{ and } \ 1/e \leq h(x) \leq e. \]  \hspace{1cm} (2)

for every \( e^{-e} \leq x \leq e^{1/e} \). We propose the following question:

**Question 2.** Suppose \( A \) is algebraic and \( h(A) \) is convergent. Is \( h(A) \) transcendental?

For some algebraic numbers \( A \), the transcendence of \( h(A) \) is already known from the following result of Sondow and Marques:
Proposition 3 ([14, Corollary 4.2]). Let $A \in [e^{-e}, e^{1/e}]$. If either

(i) $A^n \in \mathbb{A} \setminus \mathbb{Q}$ for all $n \in \mathbb{N}$, or

(ii) $A \in \mathbb{Q} \setminus \{1/4, 1\}$,

then $h(A)$ is transcendental.

However, they did not study the case when there exists an integer $n \geq 2$ such that $A^n \in \mathbb{Q}$. This paper gives new results in this unknown case.

To state our main theorems, we now define the function $\text{ord} : \mathbb{A} \to \mathbb{N} \cup \{\infty\}$ to be

$$\text{ord}(A) = \min\{n \in \mathbb{N} : A^n \in \mathbb{Q}\}$$

if there exists $n \in \mathbb{N}$ such that $A^n \in \mathbb{Q}$, and define $\text{ord}(A) = \infty$ otherwise. We say that $\text{ord}(A)$ is the order of an algebraic number $A$. The first goal of this paper is to prove the following theorem.

Theorem 4. Fix an integer $k \geq 2$. For every $A \in \mathbb{A} \cap [e^{-e}, e^{1/e}]$ with $\text{ord}(A) = k$, the limit $h(A)$ is transcendental, except possibly for $A \in \mathcal{E}(k)$, where

$$\theta := (\log 2 - 1/e)^{-1} = 3.074390 \cdots,$$

and

$$\mathcal{E}(k) := \left\{ \left( \frac{kt}{s} \right)^{1/t} : 1 \leq t \leq \theta \log k, \ kt/e \leq s \leq kte, \ s, t \in \mathbb{N}, \ gcd(kt, s) = 1 \right\}.$$

We prove Theorem 4 in Subsection 3.1. Moreover, for the case that $k$ is a square-free integer, we can characterize the set of all algebraic numbers $A$ of order $k$ such that the limit $h(A)$ is algebraic.

Theorem 5. If $k \geq 3$ is square-free, then

$$\left\{ A \in \mathbb{A} \cap [e^{-e}, e^{1/e}] : \begin{array}{l} h(A) \text{ is algebraic,} \\
\text{ord}(A) = k \end{array} \right\} = \left\{ \left( \frac{k}{s} \right)^{1/t} : k/e \leq s \leq ke, \ gcd(s, k) = 1 \right\}.$$

Remark 6. We also get the result for $k = 2$. The explicit form is stated after the proof of Theorem 5.

We do not know whether the set $\mathcal{E}(k)$ is equal to the set of all algebraic numbers $A$ with order $k$ such that the limit $h(A)$ is algebraic. As we discussed previously, the case $k = 1$ or $\infty$ was already proven by Sondow and Marques (Proposition 3). We define $\mathcal{E}(1) = \{1/4, 1\}$ and $\mathcal{E}(\infty) = \emptyset$. From Theorem 4 and Proposition 3, the limit $h(A)$ is transcendental except possibly for $A \in \mathcal{E}(k)$ for every $k \in \mathbb{N} \cup \{\infty\}$.
It is clear that $\mathcal{E}(k)$ is a finite set for every $k \geq 1$. Thus we can define the arithmetic function $Q(k)$ to be

$$Q(k) = \# \{ A \in \mathbb{A} \cap [e^{-e}, e^{1/e}] : h(A) \text{ is algebraic, and } \text{ord}(A) = k \},$$

where $\#X$ denotes the cardinality of $X$ for every finite set $X$.

For every pair of functions $f(k), g(k)$ and for every non-negative function $h(k)$, we write $f(k) = g(k) + O(h(k))$ if there exists some constant $C > 0$ such that $|f(k) - g(k)| \leq Ch(k)$. Let $\varphi(k)$ be the number of positive integers up to a given integer $k$ that are relatively prime to $k$; this is called Euler’s totient function. We find an asymptotic formula for $Q(k)$, where the main term is $(e - 1/e)\varphi(k)$; furthermore, the ratio $Q(k)/\varphi(k)$ approaches $e - 1/e$ as $k \to \infty$. More precisely, we get the following result:

**Theorem 7.** For every $k \geq 3$, we have

$$Q(k)/\varphi(k) = e - \frac{1}{e} + O \left( \frac{1}{k^{1/2} \log \log k} \right).$$ (3)

In particular, we have

$$\lim_{k \to \infty} \frac{Q(k)}{\varphi(k)} = e - \frac{1}{e}.$$

**Remark 8.** We know that the limit $h(A)$ is transcendental for every $A \in \mathbb{A} \cap [e^{-e}, e^{1/e}]$ with $\text{ord}(A) = \infty$ from Proposition 3. Thus we might guess that $\lim_{k \to \infty} Q(k) = 0$. However, Theorem 7 implies that $\lim_{k \to \infty} Q(k) = \infty$.

**Theorem 9.** The exceptional set

$$\{ A \in \mathbb{A} \cap [e^{-e}, e^{1/e}] : h(A) \text{ is algebraic} \}$$

is dense in $[e^{-e}, e^{1/e}]$.

If we fix the order of algebraic numbers, then we can find that the exceptional set is finite by Theorem 4. On the other hand, we see that the union

$$\bigcup_{k=1}^\infty \{ A \in \mathbb{A} \cap [e^{-e}, e^{1/e}] : h(A) \text{ is algebraic, and } \text{ord}(A) = k \}$$

is dense in $[e^{-e}, e^{1/e}]$ from Theorem 9.

In the above four results, we consider only the case where $x$ is positive. We can extend the iterated exponential function (1) to $\mathbb{C}$, and it is known that there exists a non-real number $x$ such that (1) converges. Let

$$\mathcal{R} = \{ e^{t e^{-t}} \mid |t| < 1, \text{ or } t \text{ is a root of unity} \}.$$

In 1983, Baker and Rippon [2] showed that if $x \in \mathcal{R}$, the sequence (1) converges to $e^t$. 

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Theorem 10. Let $x \in \mathbb{A} \cap \mathbb{R}$. Then if $h(x)$ is an algebraic number, then $x$ is real and positive.

This theorem makes our results valid for all $\mathbb{A}$ with the condition that the sequence (1) converges. We show this result in Section 5.

As one of the generalizations of these results, we also consider the case $x = \alpha^\beta$, where both $\alpha$ and $\beta$ are algebraic numbers with $\alpha \not\in \{0, 1\}$. If $\beta$ is not a rational number, then $\alpha^\beta$ is a transcendence number by Theorem 1. We characterize the pairs $(\alpha, \beta)$ such that $h(\alpha^\beta)$ is an algebraic number in Section 5.

A complex-valued function $f(x)$ is called transcendental, if there exists no non-zero polynomial $P(y)$ with $\mathbb{C}(x)$ coefficients such that $P(f(x)) \equiv 0$. It is known that there are entire transcendental functions $f$ such that $f(\alpha)$ is an algebraic number for every algebraic number $\alpha$ [11].

In this paper, we also consider the exceptional set for the iterated exponential function.

We give some notation. In this paper, the expression $a \mid b$ denotes that $b$ can be divided by $a$, and $p^k \parallel a$ denotes that $p^k \mid a$ and $p^{k+1} \nmid a$.

2 Preliminary discussion

To prove Theorem 4, 5, and 7, we show the following lemmas.

Lemma 11. Let $x \geq 2$ be an integer, and let $a$ and $b$ be relatively prime positive integers.
If $x^{a/b}$ is a positive integer, then $x^{1/b}$ is also a positive integer.

Proof. Let $y = x^{a/b}$. Note that $y \in \mathbb{N}$. From the prime factorization, it follows that $x = \prod_{i=1}^n p_i^{a_i}$ and $y = \prod_{i=1}^n p_i^{b_i}$ for some prime numbers $p_1, \ldots, p_n$ and positive integers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This yields that $\beta_j b = \alpha_j a$ for every $1 \leq j \leq n$. From $\gcd(a, b) = 1$, it is obtained that $b \mid \alpha_j$ for every $1 \leq j \leq n$. Thus we conclude $x^{1/b} \in \mathbb{N}$.

Lemma 12. Let $A \in \mathbb{A} \setminus \{0\}$ and $k \geq 1$. If $A^k \in \mathbb{Q}$, then $\text{ord}(A) \mid k$.

Proof. We define $\mathbb{A}^\times$ and $\mathbb{Q}^\times$ to be the multiplicative group $\mathbb{A}$ and $\mathbb{Q}$, respectively. Let $A \in \mathbb{A}^\times / \mathbb{Q}^\times$ be the equivalent class of $A \in \mathbb{A}$. Then the cardinality of the cyclic group $\langle A \rangle = \{A^n \in \mathbb{A}^\times/\mathbb{Q}^\times : n \in \mathbb{Z}\}$ is equal to $\text{ord}(A)$, and $(\langle A \rangle)^k = \mathbb{I}$. By the theory of groups, we obtain $\text{ord}(A) \mid k$.

Lemma 13. Let $A \in \mathbb{A} \cap [e^{-e}, e^{1/e}]$. If $h(A) \in \mathbb{R} \setminus \mathbb{Q}$, then $h(A) \in \mathbb{T}$.

Proof. Assume that $h(A) \in \mathbb{A}$. Then $1/h(A) \in \mathbb{A} \setminus \mathbb{Q}$ since the same is true for $h(A)$. From Theorem 1, we see that $h(A)^{1/h(A)}$ is transcendental, but this is a contradiction by (2).
3 Proof of main theorems

3.1 Proof of Theorem 4

Proof of Theorem 4. Fix an integer $k \geq 2$. Assume that $h(A)$ is rational. The goal of this proof is to show that $A \in \mathcal{E}(k)$ from Lemma 13. It can be written as $h(A) = a/b$ for some $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Since $\ord(A) = k$, it also can be written as $A = (x/y)^{1/k}$ for some $x, y \in \mathbb{N}$ with $\gcd(x, y) = 1$. From (2), the equation

$$
\left(\frac{a}{b}\right)^{\frac{b}{a}} = \left(\frac{x}{y}\right)^{\frac{1}{k}}
$$

holds. From $\gcd(a, b) = \gcd(x, y) = 1$ and (4), it follows that

$$
a^{bk} = x^a, \quad b^{bk} = y^a. \quad (5)
$$

If $x = 1$, then it is easily seen that $a = 1$ from (5). This does not happen since $k | a$ by Lemma 12. Thus we may assume that $x \geq 2$. Let $t = a/k$. We next show that $1 \leq t \leq \theta \log k$. From $k | a$, the integer $t$ is a positive integer, and $\gcd(t, b) = 1$ holds. From (5), it is seen that $x^{t/b}/t = k$. From Lemma 11, we have $x^{1/b} \in \mathbb{N}$. Therefore $x$ can be written as $x = x_0^b$ for some positive integer $x_0$. We see that $x_0 \neq 1$ from $x \geq 2$. Thus $x^{1/b} = x_0 \geq 2$. Therefore, we have

$$
\frac{2^t}{t} \leq \frac{x^{t/b}}{t} = k,
$$

which implies that

$$(\log 2 - 1/e) t \leq \log \frac{2^t}{t} \leq \log k.$$

Thus we have $1 \leq t \leq \theta \log k$, where recall $\theta = (\log 2 - 1/e)^{-1}$.

Let $s = b$. We find that $h(A) = a/b = kt/s$. From (2), we have $kt/e \leq s \leq kte$.

From the above discussion, it follows that

$$
1 \leq t \leq \theta \log k, \quad kt/e \leq s \leq tke, \quad s, t \in \mathbb{N},
$$

$$
x = a^{bk/a} = (kt)^{s/t}, \quad y = b^{bk/a} = s^{s/t}, \quad \gcd(kt, s) = 1,
$$

$$
A = \left(\frac{x}{y}\right)^{\frac{1}{k}} = \left(\frac{kt}{s}\right)^{\frac{s}{t}}.
$$

Lemma 11 and $\gcd(kt, s) = 1$ imply that $(kt)^{1/t}$ and $s^{1/t}$ are positive integers. Therefore, we conclude $A \in \mathcal{E}(k)$. \qed

3.2 Proof of Theorem 5

Let $f(y) = y^{1/y}$ on $1/e \leq y \leq e$. The function $f$ is an injection. Indeed,

$$
f'(y) = y^{1/y-2}(1 - \log y)
$$
holds from taking the logarithmic derivative. Therefore \( f'(y) > 0 \) for every \( y \in (1/e, e) \), which means that \( f \) is an injection. Hence we immediately get the following lemma.

**Lemma 14.** Let \( A \in A \cap [e^{-e}, e^{1/e}] \). If there exists \( q \in Q \cap [1/e, e] \) such that \( A = q^{1/q} \), then \( h(A) = q \).

**Proof of Theorem 5.** First, we prove that the set on the left-hand side set contains the set on the right-hand side. Since \( k/e \leq s \leq ke \), we obtain \((k/s)^{s/k} \in [e^{-e}, e^{1/e}]\). By Lemma 14, we have \( h((k/s)^{s/k}) = k/s \). When we assume that \( \text{ord}(A) < k \), i.e. there is an integer \( l \) such that

\[
1 \leq l < k, \quad \text{and} \quad \left( \frac{k}{s} \right)^{\frac{l}{k}} = \frac{x}{y} \quad (\gcd(x, y) = 1),
\]

we see that

\[
k^l = x^k,
\]

because \( \gcd(s, k) = 1 \) and \( \gcd(x, y) = 1 \). Therefore \( k \) divides \( l \), because \( \gcd(s, k) = 1 \) and \( k \) is square-free. This is a contradiction. Hence we have \( \text{ord}(A) = k \).

Next, we prove that the set on the left-hand side is a subset of the set on the right-hand side. By Lemma 13, we can see that \( h(A) \in Q \). When we put

\[
h(A) = \frac{a}{b} \quad (\gcd(a, b) = 1),
\]

we can obtain

\[
A = \left( \frac{a}{b} \right)^{\frac{b}{a}} \quad (a/e \leq b \leq ae).
\]

Now we prove \( a = k \). It also can be written as \( A = (x/y)^{1/k} \) with \( \gcd(x, y) = 1 \) because \( \text{ord}(A) = k \). Therefore

\[
\left( \frac{a}{b} \right)^{\frac{b}{a}} = \left( \frac{x}{y} \right)^{\frac{1}{k}},
\]

and we have

\[
a^{kb} = x^a.
\]

We can write \( a = km \) (\( m \in \mathbb{N} \)) and the above equation can be rewritten as

\[
(km)^b = x^m.
\]

Since \( k \) is square-free, if a prime \( p \) divides \( k \) but not \( m \), then \( m \) divides \( b \). However we put \( \gcd(a, b) = 1 \). Hence this is a contradiction except possibly for the case of \( m = 1 \). If there is a prime \( p \) such that

\[
p \nmid k \quad \text{and} \quad p^\alpha \parallel m \ (\alpha \geq 1),
\]

then \( m \) divides \( \alpha \) since \( \gcd(b, m) = 1 \). Therefore \( p^\alpha \) divides \( \alpha \), but that is impossible. Hence \( k \) and \( m \) are not co-prime or \( m = 1 \). We assume that \( k \) and \( m \) are not co-prime and \( k \geq 3 \). Then there is a prime \( p > 2 \) such that

\[
p^{\alpha+1} \parallel km \ (\alpha \geq 1).
\]
Since \( \gcd(b, m) = 1 \), the integer \( m \) divides \( \alpha + 1 \), and therefore \( p^\alpha \) divides \( \alpha + 1 \). But this is impossible. Therefore \( m = 1 \).

For the case \( k = 2 \), the triple \((m, p, \alpha) = (2, 2, 1)\) satisfies \( p^{\alpha+1} \parallel km \). Then, we have \( a = 4 \). By the assumption, an odd integer \( b \) satisfies \( 4/e \leq b \leq 4e \) and \( b^p = y^2 \). Therefore, \( b = 9 \). In this case, we confirm that \( h((4/9)^{9/4}) = 4/9 \) and \( (4/9)^{9/4} \in Q(2) \). Thus, there is only one exceptional element \((2/3)^{9/2} \in Q(2)\).

### 3.3 Proof of Theorem 7

In order to estimate the value of \( Q(k) \), we need some evaluations of arithmetic functions. Let \( d(n) \) be the number of divisors of \( n \), \( \omega(n) \) be the number of distinct prime factors of \( n \), and \( \gamma \) be the Euler-Mascheroni constant.

**Lemma 15.** We have the following facts.

- [9, Theorem 2.9] For every \( n \geq 3 \), we have
  \[
  \varphi(n) \geq \frac{n}{\log \log n} \left( e^{-\gamma} + O \left( \frac{1}{\log \log n} \right) \right).
  \]

- [9, Theorem 2.11] For every \( n \geq 3 \),
  \[
  \log d(n) \leq \frac{\log n}{\log \log n} \left( \log 2 + O \left( \frac{1}{\log \log n} \right) \right),
  \]

- [9, Theorem 3.1] Let \( P \) be a positive integer. For every \( x \in \mathbb{R} \), and every \( y \geq 0 \),
  \[
  \sum_{\substack{x < n \leq x + y \\ \gcd(n, P) = 1}} 1 = \frac{\varphi(P)}{P} y + O \left( 2^{\omega(P)} \right).
  \]

**Remark 16.** From (7), there exists \( C > 0 \) such that for every \( n \geq 3 \), we have

\[
\begin{align*}
d(n) &\leq \exp \left( \frac{C \log n}{\log \log n} \right) \\
    &\leq \exp \left( \frac{C \log n}{\log \log n} \right).
\end{align*}
\]

We can take \( C = 1.5379 \) from the result of Nicolas and Robin [10], but we do not use this explicit value.

Since \( 2^{\omega(P)} \leq d(P) \) holds, by (8) we have

\[
\begin{align*}
\sum_{\substack{x < n \leq x + y \\ \gcd(n, P) = 1}} 1 &= \frac{\varphi(P)}{P} y + O \left( d(P) \right) .
\end{align*}
\]

For every function \( f(k) \) and for every non-negative function \( g(k) \), we define \( f(k) \ll g(k) \) to mean \( f(k) = O(g(k)) \).
Proof of Theorem 7. Let $E(k)$ be the set in Theorem 4. From Theorem 4 and Eq. (10), it follows that

$$Q(k) \leq \#E(k)$$

$$\leq \#\{(u, t) \in \mathbb{N}^2 : 1 \leq t \leq \theta \log k, (kt/e)^{1/t} \leq u \leq (kte)^{1/t}, \gcd(k, u) = 1\}$$

$$= \sum_{1 \leq t \leq \theta \log k} \sum_{(tk/e)^{1/t} \leq u \leq (kte)^{1/t}, \gcd(k, u) = 1} 1$$

$$= \sum_{1 \leq t \leq \theta \log k} \left( (tk/e)^{1/t} - \left(\frac{tk}{e}\right)^{1/t} \right) \frac{\varphi(k)}{k} + O(d(k))$$

$$= \left( e - \frac{1}{e} \right) \varphi(k) + \sum_{2 \leq t \leq \theta \log k} \left( (tk/e)^{1/t} - \left(\frac{tk}{e}\right)^{1/t} \right) \frac{\varphi(k)}{k} + O(d(k) \log k).$$

By the mean value theorem and the fact $t^{1/t}$ is bounded, the middle term is dominated by

$$\frac{\varphi(k)}{k} \sum_{2 \leq t \leq \theta \log k} \left( ke - \frac{k}{e} \right) \frac{(k/e)^{1/t-1}}{t} \ll \varphi(k)k^{-1/2} \log \log k.$$

By (6) and (9), we have

$$\varphi(k)k^{-1/2} \log \log k + d(k) \log k \ll \varphi(k)k^{-1/2} \log \log k.$$

Therefore there exists a constant $C_1 > 0$ such that

$$Q(k)/\varphi(k) - \left( e - \frac{1}{e} \right) \leq C_1 k^{-1/2} \log \log k.$$

We next find a lower bound for $Q(k)$. Let

$$E_0(k) = \left\{ \left(\frac{k}{s}\right)^{\frac{1}{s}} \bigg| \frac{k}{e} \leq s \leq ek, s \in \mathbb{N}, \gcd(k, s) = 1, r \mid k \text{ and } r \neq 1 \Rightarrow s^{1/r} \notin \mathbb{N} \right\}$$

for every $k \geq 3$. Then $E_0(k) \subset [e^{-e}, e^{1/e}]$ holds for every $k \geq 3$. Indeed, since $f(x) = x^{1/x}$ is increasing on $x \in [1/e, e]$, we have

$$e^{-e} \leq \left( \frac{k}{s} \right)^{\frac{1}{s}} \leq e^{1/e}$$

for every $1/e \leq s/k \leq 1$. Therefore $h(A)$ can be defined for every $A \in E_0$. Fix $A \in E_0$ and write $A = (k/s)^{x/k}$. We next show that $\text{ord}(A) = k$. It follows that

$$\left( \frac{k}{s} \right)^{x/\text{ord}(A)} = A^{\text{ord}(A)} = \frac{x}{y}.$$
for some relatively prime positive integers $x$ and $y$. From Lemma 12, we obtain $\text{ord}(A) \mid k$. Since $\gcd(x, y) = \gcd(k, s) = 1$ implies that 

$$s^{k/\text{ord}(A)} = y,$$

it follows that $s^{k/\text{ord}(A)} \in \mathbb{N}$ from Lemma 11 and the fact that $\gcd(k, s) = 1$. Therefore, the definition of $\mathcal{E}_0(k)$ leads to $\text{ord}(A) = k$. Furthermore, the limit $h(A)$ is rational from Lemma 14. Hence we get the evaluation

$$\# \mathcal{E}_0(k) \leq Q(k).$$

We now find a lower bound for $\# \mathcal{E}_0(k)$. It is obtained that

$$\# \mathcal{E}_0(k) \geq \sum_{\substack{k/e \leq s \leq ek \\ \gcd(k,s)=1}} 1 - \sum_{\substack{r|k \\ r \neq 1}} \sum_{\substack{(k/e)^{1/r} \leq u \leq (ek)^{1/r} \\ \gcd(k,u)=1}} 1.$$

From (10), the first sum is equal to

$$\left( e - \frac{1}{e} \right) \varphi(k) + O(d(k)), \quad (12)$$

and the second sum is equal to

$$\sum_{\substack{r|k \\ r \neq 1}} \left( (ek)^{1/r} - \left( \frac{k}{e} \right)^{1/r} \right) \frac{\varphi(k)}{k} + O(d(k)^2). \quad (13)$$

By the mean value theorem and the estimate (9), this sum is dominated by

$$\frac{\varphi(k)}{k} \sum_{\substack{r|k \\ r \neq 1}} \frac{1}{r} \left( \frac{k}{e} \right)^{1/r} \leq \frac{\varphi(k)}{k} k^{1/2} \sum_{\substack{r|k \\ r \neq 1}} 1/r \leq \frac{\varphi(k)}{k} k^{1/2} \frac{k}{\varphi(k)} = k^{1/2}. \quad (14)$$

Therefore, by combining (12), (13), and (14), we have

$$\# \mathcal{E}_0(k)/\varphi(k) = e - \frac{1}{e} + O(d(k)^2/\varphi(k) + k^{1/2}/\varphi(k)).$$

Hence, by (6) and (9), there exists $C_2 > 0$ such that

$$-C_2 k^{-1/2} \log \log k \leq Q(k)/\varphi(k) - \left( e - \frac{1}{e} \right)$$

for every $k \geq 3$. Therefore we obtain

$$Q(k)/\varphi(k) = e - \frac{1}{e} + O \left( k^{-1/2} \log \log k \right).$$

Furthermore, we find that $Q(k)/\varphi(k) \to e - \frac{1}{e}$ as $k \to \infty$ from (6). \qed
Proof of Theorem 9. Let $\mathcal{E} = \{ A \in \mathbb{A} \cap [e^{-e}, e^{1/e}] : h(A) \text{ is algebraic} \}$, and let $f(x) = 1/x^x$. By the definition (11), we have

$$\{ f(p/2^k) : k \geq 2, \ p \text{ is odd prime}, 1/e \leq p/2^k \leq e \} \subseteq \bigcup_{k=3}^{\infty} \mathcal{E}_0(k) \subseteq \mathcal{E}. $$

Note that the function $f(x) = 1/x^x$ is a homeomorphism from $[1/e, e]$ into $[e^{-e}, e^{1/e}]$. Thus it is sufficient to show that the set

$$\mathcal{F} := \{ p/2^k \in \mathbb{Q} : \ k \geq 2, \ p \text{ is odd prime} \}$$

is dense in $(0, \infty)$. Here fix real numbers $x > 0$ and $\epsilon > 0$. It is clear from [9, Theorem 6.9] that if $y$ is a sufficiently large real number, then there exists an odd prime number $p$ such that $p \in [y, y + y/\log y]$. Therefore if we choose a sufficiently large integer $k = k(x, \epsilon)$, then we can find an odd prime number $p$ such that

$$(x - \epsilon)2^k < p < (x + \epsilon)2^k.$$ 

Then the following inequality holds:

$$|x - p/2^k| < \epsilon,$$

which implies that $\mathcal{F}$ is dense in $(0, \infty)$. \qed

4 Iterated exponential on $(0, e^{-e})$

Barrow [3] showed that $h(x)$ does not converge on the interval $(0, e^{-e})$, but he proved that sequences of the functions

$$x, \ x^x, \ x^{x^x}, \cdots \quad (15)$$

and

$$x^x, \ x^{x^x}, \ x^{x^{x^x}}, \cdots \quad (16)$$

are convergent for every $x \in (0, e^{-e})$. We define $h_o(x)$ and $h_e(x)$ to be the limits of the above sequences (15) and (16), respectively. We say that $h_o(x)$ is the odd iterated exponential function and $h_e(x)$ is the even iterated exponential function. Note that these functions can be defined on $(0, e^{-e})$. Barrow proved that

$$h_o(x) = x^{h_e(x)}, \ h_e(x) = x^{h_o(x)}, \ 0 < h_o(x) < 1/e < h_e(x) < 1 \quad (17)$$

for every $x \in (0, e^{-e})$. We define

$$R(k) = \# \{ A \in \mathbb{A} \cap (0, e^{-e}) : h_o(A) \text{ and } h_e(A) \text{ are algebraic, and ord}(A) = k \}. $$

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Question 17. Is $R(k)$ finite? If so, can we find an asymptotic formula of $R(k)$?

The goal of this section is to give the affirmative answer to Question 17. More precisely, we get the following results:

**Theorem 18.** Let $A$ be an algebraic number in the interval $(0, e^{-e})$. Then $h_o(A)$ and $h_e(A)$ are algebraic if and only if there exists a positive integer $v$ such that

$$A = \left(\frac{v}{v+1}\right)^{(v+1)(\frac{v+1}{v})^v}.$$  \hspace{1cm} (18)

From the above theorem, it follows that

$$R(k) = \# \left\{ v \in \mathbb{N} : \text{ord} \left( \left(\frac{v}{v+1}\right)^{(v+1)(\frac{v+1}{v})^v} \right) = k \right\}.$$  \hspace{1cm}

**Theorem 19** (the answer to Question 17). We have

$$R(k) = \begin{cases} 1, & \exists v \in \mathbb{N} \text{ s.t. } k = v^v; \\ 0, & \text{otherwise}. \end{cases}$$

In order to prove the results, we first show the following lemma:

**Lemma 20.** Let $A \in A \cap (0, e^{-e})$. If $h_o(A)$ or $h_e(A)$ is irrational, then $h_o(A)$ or $h_e(A)$ is transcendental.

**Proof.** If $h_o(A)$ is a transcendental number, then we immediately get this lemma. Thus we may assume that $h_o(A)$ is algebraic. It follows from (17) that $h_e(A) = A^{h_o(A)}$. Therefore $h_e(A)$ is transcendental from Theorem 1. \hfill \Box

By the result of Hurwitz [8], we obtain that

**Lemma 21.** All solutions of the Diophantine equation

$$x^y = y^x, \quad x, y \in \mathbb{Q}, \quad x > y > 0$$  \hspace{1cm} (19)

are

$$x = (1 + 1/v)^{1+v}, \quad y = (1 + 1/v)^v$$  \hspace{1cm} (20)

for all $v \in \mathbb{N}$.

We refer to the paper of Anderson [1] for readers who want to know the background of the equation (19).
Proof of Theorem 18. Assume that $h_o(A)$ and $h_e(A)$ are algebraic. From Lemma 20, the limits $h_o(A)$ and $h_e(A)$ are rational. From (17), we have

$$(1/h_o(A))^{1/h_e(A)} = (1/h_e(A))^{1/h_o(A)}.$$  

It follows from Lemma 21 that

$$h_o(A) = (1 + 1/v)^{-1-v}, \quad h_e(A) = (1 + 1/v)^{-v}$$

for some $v \in \mathbb{N}$. Thus the formula (18) is obtained from $A = h_o(A)^{1/h_e(A)}$. \hfill \Box

To prove the converse assertion, we shall prepare several lemmas.

Lemma 22 (cf. Lemma 21). If $(x, y) \in \mathbb{R}^2$ with $0 < y < x$ is a solution to

$$x^y = y^x,$$  \hspace{1cm} (21)

then there exists a positive $t > 0$ such that $y = (1 + 1/t)^t$, $x = (1 + 1/t)^{t+1}$.

Proof. Let $t = \frac{y}{x-y} > 0$. Then, we have $x = (1 + 1/t)y$. By (21), we compute as

$$y^{(1+1/t)y} = \left(\left(1 + \frac{1}{t}\right)y\right)^y \iff y^{1/t} = \left(1 + \frac{1}{t}\right) \iff y = \left(1 + \frac{1}{t}\right)^t,$$

which implies $x = (1 + 1/t)y = (1 + 1/t)^{t+1}$. \hfill \Box

Lemma 23. For every $t > 0$, we have

$$\frac{1}{t+1} - t \left(\log\left(1 + \frac{1}{t}\right)\right)^2 > 0.$$  

Proof. Let

$$G(t) = \frac{1}{t+1} - t \left(\log\left(1 + \frac{1}{t}\right)\right)^2.$$  

Then, we have

$$G'(t) = -\frac{1}{(t+1)^2} - \left(\log\left(1 + \frac{1}{t}\right)\right)^2 + 2 \log\left(1 + \frac{1}{t}\right) \frac{1}{t+1}$$

$$= -\left(\frac{1}{t+1} - \log\left(1 + \frac{1}{t}\right)\right)^2 < 0.$$  

Combining this with $\lim_{t \to 0} G(t) = 1$ and $\lim_{t \to \infty} G(t) = 0$, we confirm that $G(t) > 0$. \hfill \Box
Lemma 24. For every $t > 0$, let
\[ f(t) = \left( \frac{t}{t+1} \right)^{(t+1)\left(\frac{v+1}{v}\right)^t}. \]
Then, $f(t)$ is monotonically increasing on $t > 0$.

Proof. Let $g(t) = -\log f(t)$. It suffices to show that $g(t)$ decreases monotonically. The logarithmic derivative leads that
\[
\frac{g'(t)}{g(t)} = \log \left(1 + \frac{1}{t} \right) - \frac{1}{t(t+1)\log(1+1/t)}
= \frac{t(t+1)(\log(1+1/t))^2 - 1}{t(t+1)\log(1+1/t)} < 0.
\]
Since $g(t) > 0$, one confirms that $g'(t) < 0$ if and only if $G(t)$ as in Lemma 23 is positive. Lemma 23 ensures that $G(t) > 0$. Therefore $f(t)$ is monotonically increasing on $t > 0$. \qed

Proposition 25. If
\[ A = \left( \frac{v}{v+1} \right)^{(v+1)\left(\frac{v+1}{v}\right)^v} \]
for some $v \in \mathbb{N}$, then $h_o(A) = (1 + 1/v)^{-1-v}$ and $h_e(A) = (1 + 1/v)^{-v}$ which are algebraic.

Proof. By (17), we have $h_o(A)^{1/h_e(A)} = h_e(A)^{1/h_o(A)}$ and $h_o(A) < h_e(A)$. This yields that
\[
\left( \frac{1}{h_o(A)} \right)^{1/h_e(A)} = \left( \frac{1}{h_e(A)} \right)^{1/h_o(A)} = \frac{1}{A}, \quad 1/h_e(A) < 1/h_o(A)
\]
By combining this with Lemma 22, there exists a real number $t > 0$ such that
\[ 1/h_e(A) = (1 + 1/t)^t, \quad 1/h_o(A) = (1 + 1/t)^{t+1}. \]
Since $A = h_o(A)^{1/h_e(A)}$, we have
\[
\left( \frac{t}{t+1} \right)^{(t+1)\left(\frac{v+1}{v}\right)^t} = A = \left( \frac{v}{v+1} \right)^{(v+1)\left(\frac{v+1}{v}\right)^v}.
\]
Lemma 24 leads to $t = v$. Thus, we obtain that $h_o(A) = (1 + 1/v)^{-1-v}$ and $h_e(A) = (1 + 1/v)^{-v}$. As $v \in \mathbb{N}$, both $h_o(A)$ and $h_e(A)$ are algebraic. \qed

We have now completed the proof of Theorem 18.
Proof of Theorem 19. We find the solutions of the Diophantine equation

\[
\left( \frac{v}{v+1} \right)^{(v+1)(v+1)} = \left( \frac{x}{y} \right)^{1/k}, \quad v, x, y \in \mathbb{N}, \quad \gcd(x, y) = 1.
\] (22)

Let \( A \) be the left-hand side of (22). It follows that \( A^v \in \mathbb{Q} \). From Lemma 12, we have \( k \mid v \).

Let \( t = v^v / k \in \mathbb{N} \). It is seen that

\[
v^{(v+1)^{v+1}} = x^t, \quad (v+1)^{(v+1)^{v+1}} = y^t.
\]

There exists positive integers \( a \) and \( b \) such that

\[
v = a^t, \quad v + 1 = b^t
\]

from Lemma 11 and \( \gcd(v, v+1) = 1 \). Assume that \( t \geq 2 \). Then it follows from \( b > a \) that

\[
1 = (b - a)(b^{t-1} + b^{t-2}a + \cdots + a^{t-1}) \geq t,
\]

which is a contradiction. Therefore \( t = 1 \), which means that \( k = v^v \).

\[ \square \]

5 Generalized case

In this section, \( x \) denotes a complex number. First, we show Theorem 10. There are many results about convergence of iterated exponential (1). Carlsson [4] showed that convergence of (1) can occur only if \( x \in R = \{e^{te^{-t}} \mid |t| \leq 1 \} \) in 1907. In 1983, Baker and Rippon showed the following theorem.

Theorem 26 (Baker and Rippon [2]). Let

\[
\mathcal{R} = \{e^{te^{-t}} \mid |t| < 1, \text{ or } t \text{ is a root of unity}\}.
\]

If \( x \in \mathcal{R} \), the sequence (1) converges to \( e^t \). For almost all \( t \) on the unit circle \( |t| = 1 \) in the sense of the Lebesgue measure, the sequence (1) diverges.

An alternative proof of Theorem 26 using Lambert’s \( W \) function was given by Galidakis [5]. In the following, we denote by \( x = e^{te^{-t}} \) an element of \( \mathcal{R} \) and consider whether the value \( h(x) \) is a transcendental number. The Lindemann theorem states that if \( t \in \mathbb{A} \setminus \{0\} \), then \( h(x) = e^t \) is transcendental. Therefore, the limit \( h(x) \) can be an algebraic number only if \( t \) is either zero or a transcendental number. Moreover, we can show a similar lemma to Lemma 13 by the same argument of the proof of Lemma 13.
Lemma 27. Let \( x \in \mathbb{A} \cap \mathbb{R} \). If \( h(x) \not\in \mathbb{Q} \), then \( h(x) \in \mathbb{T} \).

Let \( x \in \mathbb{A} \cap \mathbb{R} \). Since \( |t| \leq 1 \), if we assume \( h(x) = e^t \in \mathbb{Q} \), then \( t \in \mathbb{R} \) and \( x \) is positive. Thus, there are no algebraic non-positive numbers \( x \in \mathbb{A} \cap \mathbb{R} \) such that \( h(x) \) is an algebraic number. This shows Theorem 10 and our results can be extended to all algebraic numbers \( A \in \mathbb{A} \cap \mathbb{R} \) such that \( h(A) \) converges.

Next, we consider the case \( x = \alpha^\beta \), where both \( \alpha \neq 1 \) and \( \beta \) are real algebraic numbers. Since \( \beta \) is transcendental then \( x \) becomes algebraic, this is one of the generalizations of our results. From Theorem 26, if \( \beta = \frac{e^{-t}}{\log \alpha} \) with \(-1 \leq t \leq 1\), then the sequence \((1)\) converges to \( e^t \). In the following, we specify the form of \( t \).

Lemma 28. Let \( \alpha \neq 1, \beta \) be real algebraic numbers. For \( \alpha^\beta \in \mathbb{R} \), we have \( \beta h(\alpha^\beta) \not\in \mathbb{Q} \) if and only if \( h(\alpha^\beta) \in \mathbb{T} \).

**Proof.** Since we assume \( \beta \) is algebraic, if \( \beta h(\alpha^\beta) \) is a transcendental number then \( h(\alpha^\beta) \in \mathbb{T} \). In the following, we assume \( \beta h(\alpha^\beta) \in \mathbb{A} \setminus \mathbb{Q} \). From Theorem 1, it follows that \( h(\alpha^\beta) = \alpha^{\beta h(\alpha^\beta)} \) is transcendental. This proves the lemma. \( \square \)

Assume that \( \alpha^\beta \in [e^{-e}, e^{1/e}] \) with \( h(\alpha^\beta) \) being algebraic. Then we have \( \alpha^\beta = e^{e^{-t}} \), that is, \( \beta = \frac{e^{-t}}{\log \alpha} \) for some \(-1 \leq t \leq 1\) as in Theorem 26. Lemma 28 shows that \( h(\alpha^\beta) \) is algebraic if and only if

\[
\beta h(\alpha^\beta) = \frac{t}{\log \alpha} \in \mathbb{Q}.
\]

Therefore, there exists an \( a \in \mathbb{Q} \) such that \( t = \log \alpha^a \). One can check easily that \( \log \alpha^a \) is transcendental by the Lindemann theorem, so \( \log \alpha^a \) is not a root of unity. Thus, \( |t| < 1 \), that is,

\[
-|\log \alpha|^{-1} < a < |\log \alpha|^{-1}.
\]

We record it as a lemma.

Lemma 29. Let \( \alpha \neq 1, \beta \) be real algebraic numbers with \( \alpha^\beta \in \mathbb{R} \). Then the followings hold.

1. If \( h(\alpha^\beta) \in \mathbb{A} \) then \( \beta = \frac{a}{\alpha^a} \), where \( a \in \mathbb{Q} \cap (-|\log \alpha|^{-1}, |\log \alpha|^{-1}) \).

2. If there exists \( a \in \mathbb{Q} \cap (-|\log \alpha|^{-1}, |\log \alpha|^{-1}) \) such that \( \beta = \frac{a}{\alpha^a} \), then \( h(\alpha^\beta) = \alpha^a \).

Lemma 29 implies the following theorem.

Theorem 30. Let \( \alpha \neq 1, \beta \) be real algebraic numbers with \( \alpha^\beta \in \mathbb{R} \). If only one of \( \text{ord}(\alpha) \) and \( \text{ord}(\beta) \) is infinity then \( h(\alpha^\beta) \) is a transcendental number.

**Proof.** It suffices to show that when \( h(\alpha^\beta) \in \mathbb{A} \), the order of \( \alpha \) is infinity if and only if the order of \( \beta \) is so. First, we assume \( \text{ord}(\alpha) = k < \infty \) and \( \text{ord}(\beta) = \infty \). If \( h(\alpha^\beta) \in \mathbb{A} \) then Lemma 29 implies that there exists a rational number \( a = \frac{a_1}{a_2} \) such that \( \beta = \frac{a}{\alpha^a} \). Since \( \text{ord}(\alpha) = k \), \( \beta^{a_2k} = \frac{a_2^{a_2k}}{a_1^{a_2k}} \) is a rational number. This contradicts to \( \text{ord}(\beta) = \infty \). Next we assume \( \text{ord}(\alpha) = \infty \) and \( \text{ord}(\beta) = k < \infty \). As in the above, if \( h(\alpha^\beta) \in \mathbb{A} \) then \( \beta = \frac{a}{\alpha^a} \), that is, \( \alpha = \left( \frac{a}{\beta} \right)^{\frac{1}{a}} \) for some rational number \( a = \frac{a_1}{a_2} \). Then we have \( \alpha^{a_1k} = \left( \frac{a}{\beta} \right)^{a_2} \) is rational, but this contradicts to \( \text{ord}(\alpha) = \infty \). This proves the theorem. \( \square \)
A Transcendence of $h(1/\sqrt{n})$

We have not yet mentioned an example of $A \in \mathbb{A} \setminus \mathbb{Q}$ with $\text{ord}(A) < \infty$ such that $h(A)$ is transcendental. This appendix gives such an example.

**Proposition 31.** For every $n \geq 2$, the limit $h(1/\sqrt{n})$ is transcendental.

**Remark 32.** Let $f(x) = x^x$ on $(0, 1)$. From the logarithmic derivative, $f'(x) = x^x(\log x + 1)$ holds. Therefore $f(e^{-1}) = e^{-1/e}$ is the minimum value of $f$ on $(0, 1)$. It follows that $e^{-1/e} \leq f(x) \leq 1$, which implies that $h(f(x))$ is convergent for every $x \in (0, 1)$. Hence $h((1/n)^{1/n})$ can be defined for all $n \geq 2$.

**Proof of Proposition 31.** Fix $n \geq 2$. From Lemma 13, it is sufficient to show that $h(1/\sqrt{n})$ is not rational. Thus we assume that $h(1/\sqrt{n})$ is rational. It can be written as $h(1/\sqrt{n}) = a/b$ for some relatively prime positive integers $a, b$. From (2), it follows that

$$
\left(\frac{a}{b}\right)^{b/n} = \left(\frac{1}{n}\right)^{1/n},
$$

which implies that $a^{bn} = 1$ and $b^{bn} = n^a$. Thus $a = 1$ holds. Since $n \neq 1$, we have $b \neq 1$. Therefore it is obtained that

$$n < 2^n < 2^{2n} \leq b^{bn} = n.$$

This is a contradiction. \hfill $\Box$

It is well known that $h(\sqrt{2}) = 2$. Indeed we see that $\sqrt{2} \in [e^{-e}, e^{1/e}]$ from the calculation, and $h(\sqrt{2})^{1/h(\sqrt{2})} = \sqrt{2}$. Here $2^{1/2} = \sqrt{2}$ also holds. Therefore we have $h(\sqrt{2}) = 2$ from Lemma 14. On the other hand, $h(1/\sqrt{2})$ is transcendental from Proposition 31 with $n = 2$.

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