Central limit theorem for linear spectral statistics of deformed Wigner matrices

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Abstract

We consider large-dimensional Hermitian random matrices of the form \( W = M + \varrho V \) where \( M \) is a Wigner matrix and \( V \) is a random or deterministic, real, diagonal matrix whose entries are independent of \( M \). For a large class of diagonal matrices \( V \), we prove that the fluctuations of linear spectral statistics of \( W \) for analytic test function can be decomposed into that of \( M \) and of \( V \), and that each of those weakly converges to a Gaussian distribution. We also calculate the formulae for the means and variances of the limiting distribution.

1 Introduction

Ever since its discovery, the central limit theorem has been considered as one of the most fundamental concepts in probability theory. Corresponding to its motivation of studying the fluctuation of a sum of independent random variables, analogous objects in the random matrix theory stand out: for random matrices with large size \( N \) and eigenvalues \( \lambda_1, \cdots, \lambda_N \), their linear eigenvalue statistics (LES) or linear spectral statistics (LSS), which is defined for an appropriate test function \( \varphi \) as

\[
\sum_{i=1}^{N} \varphi(\lambda_i).
\]  

The fluctuations of LES have been studied by many different authors for various random matrix models including Wigner matrices [4, 12, 13, 14, 19, 20, 26, 31], sample covariance matrices [3, 18], spiked Wigner matrices [16, 5], Wishart ensembles [12, 13, 14], random band matrices [1, 16, 32], and elliptic random matrices [27]. Let us consider the example of Wigner random matrices. The celebrated Wigner semi-circle law states the convergence

\[
\frac{1}{N} \sum_{i=1}^{N} \varphi(\lambda_i) \to \int_{\mathbb{R}} \varphi(\lambda) \rho_{sc}(d\lambda) = : \langle \varphi, \rho_{sc} \rangle
\]

where \( \rho_{sc} \) is the semi-circular distribution, establishing an analogue of the law of large numbers. From this convergence one can immediately predict that the center of the fluctuation of the linear eigenvalue statistics...
is $\langle \varphi, \rho_{sc} \rangle$. With respect to the center, Bai and Yao proved in [4] that for analytic test function $\varphi$, the random variable

$$\sum_{i=1}^{N} \varphi(\lambda_i) - N \langle \varphi, \rho_{sc} \rangle$$

(1.3)

converges in distribution to a Gaussian random variable, giving also the explicit mean and variance of the limiting distribution.

In this paper we consider deformed Wigner matrices, given by $W = N^{-\frac{1}{2}}A + \vartheta V$ where $N^{-\frac{1}{2}}A$ is a real symmetric Wigner matrix and $V$ is a real, diagonal, random or deterministic matrix independent of $A$. Two matrices $N^{-\frac{1}{2}}A$ and $V$ are normalized so that each of them has eigenvalues of order one, and $\vartheta$ is a parameter that controls the order of deformation by $V$. Since the case of $\vartheta \ll N^{-\frac{1}{2}}$ or $\vartheta \sim N^{-\frac{1}{2}}$ results in another Wigner matrix, which has been studied widely by many authors, we focus on the case where $\vartheta \sim 1$ or $N^{-\frac{1}{2}} \ll \vartheta \ll 1$.

Assuming that the empirical eigenvalue distribution of $V$

$$\hat{\nu} := \frac{1}{N} \sum_{i} \delta_{\nu_{i}}$$

(1.4)

converges weakly (weakly in probability if $V$ is random) to a deterministic distribution $\nu$, it was proved in [30] that for $\lambda \sim 1$ the eigenvalues of $W$ converges weakly in probability to a deterministic measure, $\rho_{fc}$. In the present paper, the limiting distribution is called the \textit{deformed semicircle law}, following [25]. Even though it is now known the explicit formula concerning $\rho_{fc}$ (e.g., its density function) in terms of $\nu$, its Stieltjes transform was characterized in [30] as the solution of an integral equation concerning $\nu$. Equivalently, the limiting measure $\rho_{fc}$ can be considered as the free additive convolution of the semicircle law and $\nu$. In [7], it was shown that $\rho_{fc}$ admits a density, which may be supported on multiple disjoint intervals. For simplicity, we impose some conditions on $\nu$ so that the limiting density is supported on a single compact interval. We also assume $\hat{\nu}$ is such that there is no outlying eigenvalues of $W$, i.e. the eigenvalues of $W$ stay close to the support of $\rho_{fc}$. As mentioned above, the results of [30] implies that whenever we are given a continuous bounded function $\varphi$, we immediately get the convergence

$$\langle \varphi, \rho \rangle := \int_{R} \varphi \, d\rho = \frac{1}{N} \sum_{i} \varphi(\lambda_{i}) \rightarrow \int_{R} \varphi \, d\rho_{fc}$$

(1.5)

where $\lambda_{i}$ are the eigenvalues and $\rho$ is the empirical spectral distribution of $W$.

Given the convergence of empirical distribution, the fluctuations of various statistics of the deformed Wigner matrices have been studied. The deformed Gaussian unitary ensemble (GUE) for a special case, where the eigenvalues of $V$ are $\pm a$ with the equal multiplicity, was considered in [10] using the Deift/Zhou steepest descent method for the Riemann-Hilbert problem. For a general deformed Gaussian unitary ensemble (GUE), the maximal eigenvalue of $W$ was studied in [17], where it was proved that for the limiting distribution of the maximal eigenvalue of $W$ converges weakly to a Tracy-Widom or a Gaussian distribution, depending on the parameter $\vartheta$. In particular, the transition occurs at $\vartheta \sim N^{-\frac{1}{2}}$, so that the limiting distribution is Gaussian for $\vartheta \gg N^{-\frac{1}{2}}$ and the Tracy-Widom for $\vartheta \ll N^{-\frac{1}{2}}$. Also for non-Gaussian generic Wigner ensemble $W$, many other statistics has been studied, including the eigenvectors in [10] [22] [24], ‘four-moment theorem’ in [28], extremal eigenvalues in [23] [24] [25], and bulk universality in [25].

As mentioned above, the Stieltjes transform and the corresponding resolvent of $W$ are extensively used in the analysis of $\rho_{fc}$. In particular, a “local law” was first established in [22], asserting that the normalized trace of the resolvent of $W$ is almost of order $N^{-1}$ away from that of the limiting distribution $\rho_{fc}$ and the non-diagonal entries of the resolvent of $W$ cannot be much larger than $N^{-\frac{1}{2}}$ under the macroscopic scale
of $\eta \sim 1$. In the same paper, it was also observed that the fluctuation of the Green function of $W$ can be separated by two parts, one coming from the fluctuation of the Wigner matrix $A$ and the other from that of the diagonal matrix $V$. For the diagonal entries of the resolvent, it was proved in [25] that they also are about $N^{-\frac{1}{2}}$ away from its center, coming from the free additive convolution of the empirical spectral measure of $\nu$ and the semicircle distribution. Finally, [21] established an analogue of the local law for the deformed Wigner matrix with a non-diagonal matrix $V$, by proving a sufficient condition concerning the corresponding local law of deformed Gaussian ensembles whose deformation is the diagonalization of $V$. The rigidity of eigenvalues and the edge universality is proved under the same condition, which, together with the local law, are the central estimates of this paper. The local law for deformed Wigner matrices was a crucial input in the universality results in [22, 24] and it was also used in [28] to establish a ‘four-moment theorem’ by Lindeberg’s replacement strategy.

As addressed above, we are concerned about the fluctuation of $\langle \varphi, \rho \rangle$ in (1.5) under proper normalization. Heuristically, the fluctuation of $\langle \varphi, \rho \rangle$ can be separated into two components, one coming from the fluctuation of Wigner matrix $\hat{V}A$, and the other coming from that of the diagonal matrix $V$. To be specific, as shown in [22], the fluctuation of Stieltjes transform of $\mu$ can be analyzed with respect to two different centers: Stieltjes transform of the limiting distribution $\rho_{fc}$ and that of the free convolution of the semicircle law with the empirical spectral distribution of $V$, i.e. $\hat{\rho}_{fc} := \rho_{sc} \boxplus \hat{\nu}$. Under the macroscopic scale of $\eta = \Im z \sim 1$ where $z$ is the argument of Stieltjes transform, it is shown that the fluctuation with respect to the former center has order $N^{-1}$, and that with respect to the latter center has order $\eta N^{-\frac{1}{2}}$. For analytic test function $\varphi$, we prove the convergence of LES utilizing the Stieltjes transform and the fluctuation of $\langle \varphi, \rho \rangle$ is analyzed in following two settings:

- With center $\langle \varphi, \hat{\rho}_{fc} \rangle$, normalized by $N$,
- With center $\langle \varphi, \rho_{fc} \rangle$, normalized by $\sqrt{N}$.

By adapting the proof in [4], we are able to prove the corresponding central limit theorem for $\langle \varphi, \rho \rangle - \langle \varphi, \hat{\rho}_{fc} \rangle$ when the entries of $V$ are deterministic. Along the proof, the main difficulty is the absence of symmetry in $\rho_{fc}$ or $\hat{\rho}_{fc}$, whereas $\rho_{sc}$ has many exploitable features that make explicit calculations possible. Moreover, even if several applicable properties of $\rho_{fc}$ were obtained, upon dealing with $\hat{\rho}_{fc}$, one still needs to extend those properties to $\hat{\rho}_{fc}$. The difficulties are handled using the detailed analysis of the Stieltjes transforms of $\hat{\rho}_{fc}$ and $\rho_{fc}$, given in [25]. For example, a stability bound, which corresponds to $|m_{sc}(z)|$ for the Stieltjes transform of $\rho_{sc}$ and is proved therein, is used widely throughout the paper. To overcome the second difficulty, following [25], we introduce an event where the behavior of $\hat{\nu}$ resembles that of $\nu$ and prove that $\hat{\rho}_{fc}$ also behaves similarly as $\rho_{fc}$ on the event.

The analysis on the behavior of $\langle \varphi, \rho \rangle - \langle \varphi, \hat{\rho}_{fc} \rangle$ in the proof of the first part is then applied to annihilate its contribution on the fluctuation of $\langle \varphi, \rho \rangle - \langle \varphi, \rho_{fc} \rangle$. In particular, such annihilation procedure applies also for coupling parameters following asymptotic of the form $\vartheta \sim N^{-\frac{1}{4}} \sqrt{\log N}$, since the proof of the first part enables us to be free of so-called high-probability bounds. Therefore, in the second part, the problem reduces to analyzing the fluctuation of $\hat{\rho}_{fc}$ with respect to the center $\rho_{fc}$. As easily seen, we may conceive it as rising from the fluctuation of $\hat{\nu}$, which results in the classical central limit type behavior. The proof heavily depends on the analysis of the Stieltjes transforms of $\hat{\rho}_{fc}$ and $\rho_{fc}$ using self-comparison.

The paper consists of 5 sections and 4 appendices, including the introduction. Section 2 is dedicated to preliminary materials such as definitions, our model and assumptions on it, and the precise statements of our results. Section 3, where we provide the strategy of our proof that uses mainly probabilistic and complex analytic methods, contains the statements of Propositions 3.7 and 3.8 that are the central parts of the proof. In the same section we also collect some lemmas to be used in the rest of the paper, including the local deformed semicircle laws, whose proofs are given in the Appendix E.
Section 4 and 5 are devoted respectively to the proofs of Proposition 3.1 and Propositions 3.2, 3.3 whose ingredients are stated therein and proved in the following sections. In Appendices A and B, we obtain the convergence of the mean and variance of \( m_N - \hat{m}_{fc} \), respectively. Appendix C gives the proof of the tightness of the processes given in the statements of Propositions 3.1, 3.2, which also is a part of the proof of the propositions. Appendix D is dedicated to the proof of Lemma 3.5, the final ingredient of the proof of the main theorems. Finally, Appendix E provides the proof of the lemmas given in Section 3.2.

Notational Remark 1.1. Throughout the paper, we use \( C \) or \( c \) to denote a constant that is independent of \( N \). Even if the constant is different from one place to another, we may use the same notation \( C \) or \( c \) as long as it does not depend on \( N \) for the convenience of presentation.

Notational Remark 1.2. For positive numbers \( a \equiv a_N \) and \( b \equiv b_N \) depending on \( N \), we write \( a_N \ll b_N \) or \( a_N = o(b_N) \) to indicate that \( a_N b_N \to 0 \) as \( N \to \infty \). We write \( a_N = O(b_N) \) when there exists a constant \( C > 1 \) independent of \( N \) such that \( a_N \leq b C N \) and \( a_N \sim b N \) when \( C^{-1} a_N \leq b_N \leq C a_N \).

2 Definitions, assumptions and main results

2.1 Definitions

Definition 2.1. For a probability distribution \( \rho \) on \( \mathbb{R} \), the Stieltjes transform of \( \rho \) is defined by

\[
m_{\rho}(z) := \int_{\mathbb{R}} \frac{1}{x - z} d\rho(x), \quad z \in \mathbb{C}^+.
\]

Assumption 2.2. Let \( \{ A_{ij} : i \leq j \in \mathbb{N} \} \) be a collection of independent real random variables satisfying the following:

1. \( \mathbb{E}[A_{ij}] = 0 \).
2. For \( i < j \), \( \mathbb{E}[A_{ij}^2] = 1 \), \( \mathbb{E}[A_{ij}^3] = W_3 \) and \( \mathbb{E}[A_{ij}^4] = W_4 \) for some constant \( W_3 \in \mathbb{R} \) and \( W_4 > 0 \).
3. For all \( i \in \mathbb{N} \), \( \mathbb{E}[A_{ii}^2] = w_2 \), for some constant \( w_2 \geq 0 \).
4. For any \( k \geq 3 \) there is a constant \( c_k \) such that,

\[
\sup_{1 \leq i, j \leq N, N \in \mathbb{N}} \mathbb{E}[|A_{i,j}|^2] \leq c_k.
\]

(2.2)

Define \( A_{ji} := A_{ij} \) for \( i < j \) and \( A = A_N = (A_{ij})_{i,j=1}^N \) be the random matrix with entries \( A_{ij} \).

Definition 2.3. Let \( V = V_N = \{v_i\}_{1 \leq i \leq N} \) be an \( N \times N \) real diagonal, random or deterministic matrix, with empirical spectral distribution \( \hat{\nu} \), i.e.

\[
\hat{\nu} = \frac{1}{N} \sum_i \delta_{v_i}.
\]

(2.3)

Assumption 2.4. There exists a deterministic, centered, compactly supported probability measure \( \nu \) on \( \mathbb{R} \) satisfying the following:

(i) If \( V \) is a random matrix, we assume that the collection \( \{v_i\} \) are i.i.d. random variables with law \( \nu \), independent of \( A \).
(ii) If $V$ is a deterministic matrix, we assume that there exists $\alpha_0 > 0$ such that for any fixed compact set $\mathcal{D} \subset \mathbb{C}^+$ with $\mathcal{D} \cap \text{supp} \nu = \emptyset$, there exists $C > 0$ such that

$$\max_{z \in \mathcal{D}} \left| \int \frac{1}{x - z} \, d\tilde{\nu}(x) - \int \frac{1}{x - z} \, d\nu(x) \right| \leq CN^{-\alpha_0}, \quad (2.4)$$

for sufficiently large $N$.

**Assumption 2.5.** Let $I_\nu$ be the smallest interval such that supp $\nu \subset I_\nu$. Then there exists $\omega > 0$ such that

$$\inf_{x \in I_\nu} \int \frac{1}{(v-x)^2} \, d\nu(v) \geq 1 + \omega. \quad (2.5)$$

Letting $I_{\tilde{\nu}}$ to be the smallest interval such that supp $\tilde{\nu} \subset I_{\tilde{\nu}}$, we also assume the similar condition to $\tilde{\nu}$:

(i) For random $\{v_i\}$, there exists a constant $t > 0$ such that

$$\mathbf{P} \left[ \inf_{x \in I_{\tilde{\nu}}} \int \frac{1}{(v-x)^2} \, d\tilde{\nu}(x) \geq 1 + \omega \right] \geq 1 - N^{-t}. \quad (2.6)$$

(ii) For deterministic $\{v_i\}$,

$$\inf_{x \in I_{\tilde{\nu}}} \int \frac{1}{(v-x)^2} \, d\tilde{\nu}(x) \geq 1 + \omega \quad (2.7)$$

for sufficiently large $N$.

**Assumption 2.6.** Let $\{\vartheta \equiv \vartheta_N\} \subset \Theta_\vartheta := [0, 1 + \omega]$ be a sequence of parameters such that $\vartheta_N \gg N^{-\frac{1}{2}}$ and $\lim_{N \to \infty} \vartheta_N = \vartheta_\infty \in \Theta_\vartheta$.

**Definition 2.7.** We define

$$W = \frac{1}{\sqrt{N}} A + \vartheta V \quad (2.8)$$

to be the deformed Wigner matrix. We denote its resolvent $(W - z I)^{-1}$ by $R^\vartheta(z) \equiv R_N^\vartheta(z)$, and its normalized Green function $\frac{1}{N} \text{Tr} R_N^\vartheta(z) = \text{tr} R_N^\vartheta(z)$ by $m^\vartheta(z) \equiv m_N^\vartheta(z)$.

**Remark 2.8.** For $\vartheta \sim N^{-\frac{1}{2}}$, $W$ itself is another Wigner matrix with the variance of its diagonal entries increased. On the other hand for $\vartheta \ll N^{-\frac{1}{2}}$, the fluctuation is dominated by that of $A$ henceforth the fluctuation of LES is identical to that of the Wigner matrices, which is thoroughly studied in [4]. Therefore, we here focus on the case $\vartheta \gg N^{-\frac{1}{2}}$.

**Notational Remark 2.9.** For convenience, we denote the distribution of $\vartheta v_1$ and the empirical spectral distribution of $\vartheta V$ by $\nu^\vartheta$ and $\tilde{\nu}^\vartheta$, respectively. Similarly, we define $m^\vartheta_\nu := m_{\nu,^\vartheta}$ and $m^\vartheta_\tilde{\nu} := m_{\tilde{\nu},^\vartheta}$.

**Definition 2.10.** For $\vartheta \in \Theta_\vartheta$, we define $m_{fc}^\vartheta$ to be the analytic function on $\mathbb{C}^+$ with positive imaginary part, which is the unique solution of the self-consistent equation

$$m_{fc}^\vartheta(z) = \int \frac{1}{x - z - m_{fc}^\vartheta(z)} \, d\nu^\vartheta(x) = \int \frac{1}{\vartheta x - z - m_{fc}^\vartheta(z)} \, d\nu(x). \quad (2.9)$$

Also we define $\rho_{fc}^\vartheta$ to be the corresponding (i.e., $m_{fc}^\vartheta(z) = m_{fc}^\vartheta(z)$) probability measure on $\mathbb{R}$, which is referred to as the deformed semi-circle law.

In particular, $\rho_{fc}^0 = \rho_{sc} \boxplus \nu^0$, the free additive convolution of the semi-circular distribution $\rho_{sc}$ with $\nu^0$. 


Lemma 2.11 (Proposition 3 and Corollary 4 of [7]). The free convolution measure \( \rho_f^\theta = \rho_{sc} \boxplus \nu^\theta \) satisfies \( \limsup_{n \to \infty} \Im \rho_f^\theta(E + in) < \infty \) for \( E \in \mathbb{R} \), hence is absolutely continuous. Its density, which also is denoted by \( \rho_f^\theta \), is analytic on \( \{ E \in \mathbb{R} : \rho_f^\theta(E) > 0 \} \).

Theorem 2.12 ([30], Deformed semi-circle law). The empirical spectral distribution of \( W \) converges in probability to \( \rho_f^\theta \).

In [30], it turned out that \( \rho_f^\theta \) being supported on a single interval is crucial for the Gaussian convergence of the LES. Although stated below, we here state the fact here as it is need for the statement of our results.

Lemma 2.13. Suppose that \( \nu \) satisfies Assumption 2.4 Then for any \( \theta \in \Theta_{\infty} \), there exists \( L^\theta_-, L^\theta_+ \in \mathbb{R} \) with \( L^\theta_- < 0 < L^\theta_+ \) such that \( \text{supp} \rho_f^\theta = [L^\theta_-, L^\theta_+] \).

Remark 2.14. For \( \theta = 0 \), one immediately gets \( \rho_f^\theta = \rho_{sc}, \rho_{fc}^\theta = \rho_{sc}, \) and \( L^\theta_\pm = \pm 2 \).

Though rigorously defined and justified below in Definition 3.10 and Lemma 7.10, we define the measure \( \hat{\rho}_f \) corresponding to the empirical measure \( \nu \), as it is needed for the statement of our result:

Definition 2.15. We define \( \hat{\rho}_f \) to be the random measure, given by \( \rho_f := \rho_{sc} \boxplus \nu^\theta \).

2.2 Statement of the result

Theorem 2.16. Suppose that \( W \) is the deformed Wigner matrix with deterministic \( V \) satisfying the Assumption 2.4, \( \nu \) and \( \nu \) satisfy Assumptions 2.4 and 2.5 and \( \hat{\theta} \) satisfies the Assumption 2.6. Then for each \( \varphi \in C(\mathbb{R}) \) with compact support that is analytic on an open neighborhood of \([L^\infty_-, L^\infty_+]\), the random variable

\[
T_N^\infty(\varphi) := \sum_{i=1}^{N} \varphi(\lambda_i) - N \int_{\mathbb{R}} \varphi(x) d\hat{\rho}_f(x) \tag{2.10}
\]

converges in distribution to the Gaussian random variable \( T(\varphi) \) with mean \( M^\infty(\varphi) \) and variance \( V^\infty(\varphi) \) given as follows:

\[
M^\infty(\varphi) = -\frac{1}{2\pi i} \oint_{\Gamma} \varphi(z) b^\infty(z) dz, \tag{2.11}
\]

\[
V^\infty(\varphi) = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma} \varphi(z_1) \varphi(z_2) \Gamma^\infty(z_1, z_2) dz_1 dz_2, \tag{2.12}
\]

where

\[
b^\infty(z) = \frac{\left(m^\infty_{fc}\right)'(z)}{2(1 + (m^\infty_{fc})'(z))^2} \left[ (w_2 - 1) + (m^\infty_{fc})'(z) + (W_4 - 3) \frac{\left(m^\infty_{fc}\right)'(z)}{1 + (m^\infty_{fc})'(z)} \right], \tag{2.13}
\]

\[
\Gamma^\infty(z_1, z_2) = (w_2 - 2 \frac{\partial^2 I}{\partial z_1 \partial z_2} + (W_4 - 3) \left( I + \frac{\partial^2 I}{\partial z_1 \partial z_2} + \frac{\partial I}{\partial z_1} \frac{\partial I}{\partial z_2} \right)
+ \frac{2}{(1 - I)^2} \left( \frac{\partial I}{\partial z_2} \frac{\partial I}{\partial z_1} + (1 - I) \frac{\partial^2 I}{\partial z_1 \partial z_2} \right), \tag{2.14}
\]

\[
I(z_1, z_2) := \Gamma^\infty(z_1, z_2) = \int_{\mathbb{R}} \frac{1}{(\varphi_{\infty}(x) - z_1 - m^\infty_{fc}(z_1))(\varphi_{\infty}(x) - z_2 - m^\infty_{fc}(z_2))} d\nu(x), \tag{2.15}
\]

and \( \Gamma \) is a rectangular contour with vertices \((a_\pm, \pm i\nu_0)\) so that \( \pm(a_\pm - L^\infty_\pm) > 0 \) and \( \Gamma \) lies within the analytic domain of \( \varphi \).
Remark 2.17. For $\vartheta_\infty = 0$, we have $m_{f_c}^{\vartheta_\infty}(z) = m_{sc}(z)$ and $I^{\vartheta_\infty}(z_1, z_2) = m_{sc}(z_1)m_{sc}(z_2)$, so that

\begin{equation}
\begin{align*}
b(z) &= m_{sc}(z)^3((w_2 - 1) + m_{sc}'(z)) + (W_4 - 3)m_{sc}(z)^2, \\
\Gamma(z_1, z_2) &= m_{sc}'(z_1)m_{sc}'(z_2)\left((w_2 - 2) + 2(W_4 - 3)m_{sc}(z_1)m_{sc}(z_2) + \frac{2}{1 - m_{sc}(z_1)m_{sc}(z_2)}\right),
\end{align*}
\end{equation}

which coincides with the limiting formulae given in [4] and [5]. Therefore $M(\varphi)$ and $V(\varphi)$ are given by

\begin{equation}
M(\varphi) = \frac{1}{4}(\varphi(2) + \varphi(-2)) - \frac{1}{2}\tau_0(\varphi) + (w_2 - 2)\tau_2(\varphi) + (W_4 - 3)\tau_4(\varphi) \tag{2.18}
\end{equation}

and

\begin{equation}
V(\varphi) = (w_2 - 2)\tau_1(\varphi)^2 + (W_4 - 3)\tau_2(\varphi)^2 + 2\sum_{\ell=1}^\infty \ell\tau_\ell(\varphi)^2, \tag{2.19}
\end{equation}

where

\begin{equation}
\tau_\ell(\varphi) = \frac{1}{2\pi} \int_{-\pi}^\pi \varphi(2\cos \theta) \cos(\ell\theta) d\theta. \tag{2.20}
\end{equation}

Remark 2.18. For complex Hermitian $A$, with the additional assumption $\mathbb{E}[A_{ij}^2] = 0$, the same result holds with $(w_2 - 2)$ and $(W_4 - 3)$ replaced by $(w_2 - 1)$ and $(W_4 - 2)$, respectively.

Theorem 2.19. Suppose that $W$ is the deformed Wigner matrix with random $V$ satisfies the assumptions in Theorem 2.16 Then for any test function $\varphi$ satisfying the conditions in Theorem 2.16, the random variable

\begin{equation}
S_N^0(\varphi) := \frac{1}{\sqrt{N\vartheta}} \sum_{i=1}^N \left[ \varphi(\lambda_i) - \int_\mathbb{R} \varphi(x) d\rho_{f_c}^0(x) \right] \tag{2.21}
\end{equation}

converges in distribution to the Gaussian random variable $S(\varphi)$ with mean zero and variance $\tilde{V}^{\vartheta_\infty}(\varphi)$ given by

- For $\vartheta_\infty \neq 0$,

\begin{equation}
\tilde{V}^{\vartheta_\infty}(\varphi) = -\frac{1}{4\pi^2 \vartheta_\infty^2} \oint_{\Gamma} \oint_{\Gamma} \left[ \varphi(z_1)\varphi(z_2)(1 + (m_{f_c}^\vartheta)'(z_1))(1 + (m_{f_c}^\vartheta)'(z_2)) \right.
\left. \left\{ I(z_1, z_2) - m_{f_c}^\vartheta(z_1)m_{f_c}^\vartheta(z_2) \right\} \right] dz_1 dz_2 \tag{2.22}
\end{equation}

- For $\vartheta_\infty = 0$,

\begin{equation}
\tilde{V}^0 = \tilde{V}(\varphi) = \text{Var}[v_1] \frac{1}{(2\pi)^2} \oint_{\Gamma} \oint_{\Gamma} \varphi(z_1)\varphi(z_2)m_{sc}'(z_1)m_{sc}'(z_2) dz_1 dz_2 = \tau_1(\varphi)^2, \tag{2.23}
\end{equation}

where $\Gamma$ and $I$ are given as in Theorem 2.16.

Remark 2.20. The result holds also for complex Wigner matrix $A$ without any modification, since the fluctuation of $\hat{\rho}_{f_c}$ dominates and that of $A$ is neglected, as shown in Section 6.
3 Strategy of the proof

3.1 Strategy of the proof

For each $x \in [L_1^{0,∞}, L_2^{0,∞}]$, the Cauchy integral formula gives

$$\varphi(x) = \frac{1}{2\pi i} \oint_{Γ} \frac{\varphi(z)}{z-x} dz,$$  \hspace{1cm} (3.1)

where $Γ$ is the rectangular contour with vertices given by $(a_± \pm iv_0)$ where $a_+ - L_1^{0,∞}, L_2^{0,∞} - a_-$ and $v_0$ are small enough so that $Γ$ lies in the analytic domain of $\varphi$ but fixed positive real numbers. Then denoting by $ρ_N$ the empirical spectral distribution of $W_N$, we have the equality

$$T_N^θ(\varphi) = N \int_R \varphi(x)(ρ_N - \hat{ρ}^θ_{f,c})(dx) = N \oint_{Γ} \frac{\varphi(z)}{z-x}(ρ_N - \hat{ρ}^θ_{f,c})(dx) dz = - \oint_{Γ} \varphi(z) ξ_N^θ(z) dz$$ \hspace{1cm} (3.2)

and

$$S_N^θ(\varphi) = - \oint_{Γ} \varphi(z) \bar{ξ}_N^θ(z) dz$$ \hspace{1cm} (3.3)

where

$$ξ_N^θ(z) := N(m_N^θ(z) - \hat{m}^θ_{f,c}(z)) \quad \text{and} \quad \bar{ξ}_N^θ(z) := \frac{\sqrt{N}}{θ}(m_N^θ(z) - m_{f,c}(z)),$$ \hspace{1cm} (3.4)

whenever the eigenvalues of $ρ_N$ and $\hat{ρ}^θ_{f,c}$ are contained in $[a_-, a_+]$, so that the equality holds with probability greater than $1 - cN^{-1}$ for sufficiently large $N$. We then decompose the contour $Γ$ into $Γ_u \cup Γ_d \cup Γ_l \cup Γ_r \cup Γ_0$, where

$$Γ_u := \{z = x + iv_0 : x \in [a_-, a_+]\},$$
$$Γ_d := \{z = x - iv_0 : x \in [a_-, a_+]\},$$
$$Γ_l := \{z = a_+ + iv : N^{-δ} \leq v \leq v_0\},$$
$$Γ_r := \{z = a_+ + iv : N^{-δ} \leq v \leq v_0\},$$
$$Γ_0 := \{z = a_+ + iv : |v| \leq N^{-δ}\}.$$ \hspace{1cm} (3.5)

Each path is given the linear parametrization $[0,1] \rightarrow Γ$, which is also denoted by $Γ$. In sections below, we prove the following propositions and lemmas:

**Proposition 3.1.** Suppose that $V$ is deterministic. For a fixed constant $c > 0$ and a path $Κ \subset \{ζ \geq c\}$, the process $\{ξ_N^θ(z) : z \in Κ\}$ converges weakly to the Gaussian process $\{ξ^θ(z) : z \in Κ\}$ with the mean $b^θ(z)$ and the covariance $Γ^θ(z_1, z_2)$ given by

$$b^θ(z) = \frac{1}{2} \frac{(m_{f,c}^θ)'(z)}{1 + (m_{f,c}^θ)'(z)} \left[ (w_2 - 1) + (m_{f,c}^θ)'(z) + (W_4 - 3) \frac{(m_{f,c}^θ)'(z)}{1 + (m_{f,c}^θ)'(z)} \right]$$ \hspace{1cm} (3.6)

and

$$Γ^θ(z_1, z_2) = (w_2 - 2) \frac{∂^2 I}{∂z_1∂z_2} + (W_4 - 3) \left( I \frac{∂^2 I}{∂z_1∂z_2} + \frac{∂I}{∂z_1} \frac{∂I}{∂z_2} + \frac{2}{(I-1)^2} \frac{∂I}{∂z_2} \frac{∂I}{∂z_1} + (I-1) \frac{∂^2 I}{∂z_1∂z_2} \right),$$ \hspace{1cm} (3.7)

where

$$I(z_1, z_2) = \int_R \frac{1}{(θ^∞x - z_1 - m_{f,c}^θ(z_1))(θ^∞x - z_2 - m_{f,c}^θ(z_2))} dv(x).$$ \hspace{1cm} (3.8)

**Proposition 3.2.** Suppose that $V$ is random and $θ^∞ > 0$. For fixed constant $c > 0$ and a path $Κ \subset \{ζ > c\}$, the process $\{ξ_N^θ(z) : z \in Κ\}$ converges weakly to a Gaussian process $\{ξ^θ(z) : z \in Κ\}$ with zero mean.
and the covariance

$$
\varphi^2(z) = (1 + (m^\infty_{f_e})(z_1))(1 + (m^\infty_{f_e})(z_2)) \left[I(z_1, z_2) - m^\infty_{f_e}(z_1)m^\infty_{f_e}(z_2)\right], \tag{3.9}
$$

where $I(z_1, z_2)$ is given as above.

**Proposition 3.3.** Suppose that $V$ is random and $\varphi^2 = 0$. For fixed constant $c > 0$ and a path $K \subset \{z \geq c\}$, the process $\{\xi^\alpha(z) : z \in K\}$ converges weakly to a Gaussian process $\{\xi(z) : z \in K\}$ with zero mean and the covariance $\text{Var}[v_1] m^\infty_{f_e}(z_1)m^\infty_{f_e}(z_2)$.

**Notational Remark 3.4.** For notational simplicity, we denote $\tilde{\xi}^\alpha(z)$ by $\tilde{\xi}(z)$.

**Lemma 3.5.** For any sufficiently small $\delta > 0$ and a sequence of events $\{\Omega_n\}_{n \in \mathbb{N}}$ with $P[\Omega_n] \to 1$,

$$
\lim_{v_0 \to 0^+} \limsup_{N \to \infty} \int_0^1 E \left[ |\xi_N(\Gamma_\#(t))|^2 \chi_{\Omega_n}(t) \right] |\Gamma'_\#(t)| dt = 0, \tag{3.10}
$$

and

$$
\lim_{v_0 \to 0^+} \int_0^1 E \left[ |\xi^\alpha(\Gamma_\#(t))|^2 \right] |\Gamma'_\#(t)| dt = 0 \tag{3.11}
$$

where $\Gamma_\#$ can be $\Gamma_i, \Gamma_r$ or $\Gamma_0$.

**Lemma 3.6.** For any sufficiently small $\delta > 0$ and a sequence of events $\{\Omega_n\}_{n \in \mathbb{N}}$ with $P[\Omega_n] \to 1$,

$$
\lim_{v_0 \to 0^+} \limsup_{N \to \infty} \int_0^1 E \left[ |\xi_N(\Gamma_\#(t))|^2 \chi_{\Omega_n}(t) \right] |\Gamma'_\#(t)| dt = 0, \tag{3.12}
$$

and

$$
\lim_{v_0 \to 0^+} \int_0^1 E \left[ |\tilde{\xi}^\alpha(\Gamma_\#(t))|^2 \right] |\Gamma'_\#(t)| dt = 0 \tag{3.13}
$$

where $\Gamma_\#$ can be $\Gamma_i, \Gamma_r$ or $\Gamma_0$.

Given these results, we deduce that

$$
E \left[ \chi_{\Omega_n(z)} |\xi_N(z)\varphi(z)|^2 dz \right] \leq \left( \sup_{\mathbb{R}_+ \ni a, z \leq v_0} |\varphi(z)|^2 \right) \int_0^1 E \left[ |\xi_N(\Gamma_\#(t))|^2 |\Gamma'_\#(t)|^2 dt \right] \leq C \int_0^1 E \left[ |\xi_N(\Gamma_\#(t))|^2 |\Gamma'_\#(t)|^2 dt \right], \tag{3.14}
$$

which together with the fact $P[\Omega_n] \to 1$ gives the in probability convergence $\int_{\Gamma_\#} \xi_N(z)\varphi(z)dz \to 0$. Similarly, $\int_{\Gamma_\#} \xi^\alpha(\Gamma_\#(t))\varphi(z)dz$, $\int_{\Gamma_\#} \tilde{\xi}^\alpha(\Gamma_\#(t))\varphi(z)dz$ also converge in probability to 0 as $v_0 \to 0^+$, for $\Gamma_\# = \Gamma_i, \Gamma_r, \Gamma_0$. Since $T_N(\varphi)$ and $S_N(\varphi)$ do not depend on $v_0$ as long as it is strictly positive, combining **Proposition 3.1** and **Lemma 3.6**, we get the convergence

$$
T_N^\alpha(\varphi) \Rightarrow T(\varphi) \quad \text{and} \quad S_N^\alpha(\varphi) \Rightarrow S(\varphi) \tag{3.15}
$$
in distribution, where $T(\varphi)$ and $S(\varphi)$ are the Gaussian random variables in **Theorems 2.10** and **2.11**.
Remark 3.7. In [33], where the deformed Gaussian orthogonal ensembles were analyzed in depth, it was proved that for $W = N^{-\frac{1}{2}} A + N^{-\frac{1}{2}} V$ where $\alpha \in (0, 1)$, $A$ is a GOE matrix, and $V$ is a random diagonal matrix as in Assumption [2,4] the mean and variance of $m_N(z) = \text{tr}(W - zI)^{-1}$ are given by
\begin{align}
\mathbb{E}[m_N(z)] &= m_{sc}(z) + \frac{m_{sc}(z)^3}{1 - m_{sc}(z)^2} \cdot \frac{\text{Var}[v_1]}{N^\alpha} + O(N^{-\min(2, 3\alpha)}) , \\
\text{Var}[m_N(z)] &= \left| \frac{m_{sc}(z)^2}{1 - m_{sc}(z)^2} \right| ^2 \cdot \frac{\text{Var}[v_1]}{N^{1+\alpha}} + O(N^{-\min(2+3\alpha, 3+\alpha)}) ,
\end{align}
which coincide with our results given in Proposition [26].

Remark 3.8. For the case where $\vartheta = \sigma N^{-\frac{1}{2}}$ for some constant $\sigma > 0$, $W$ itself is another Wigner matrix with $w_2$ replaced by $w_2 + \sigma^2 \text{Var}[v_1]$, and hence it is known (see [3], for instance) that $\{N(m_N(z) - m_{sc}(z)) : z \in \mathbb{K}\}$ converges in distribution to the Gaussian process with mean
\begin{align}
m_{sc}(z)^3 (1 + m'_{sc}(z))((w_2 + \sigma^2 \text{Var}[v_1] - 2) + m'_{sc}(z) + (W_4 - 3)m_{sc}(z)^2)
\end{align}
and covariance
\begin{align}
m'_{sc}(z_1)m'_{sc}(z_2) \left( (w_2 + \sigma^2 \text{Var}[v_1] - 2) + 2(W_4 - 3)m_{sc}(z_1)m_{sc}(z_2) + \frac{2}{1 - m_{sc}(z_1)m_{sc}(z_2)^2} \right).
\end{align}
After normalizing by $N^\frac{1}{2}\vartheta^{-1}$, the mean is
\begin{align}
\frac{1}{\sigma} \mathbb{E}[m_N(z) - m_{sc}(z)] &= \sigma m_{sc}(z)^3 (1 + m'_{sc}(z)) \text{Var}[v_1] + O(\sigma^{-1})
\end{align}
and the covariance is
\begin{align}
m'_{sc}(z_1)m'_{sc}(z_2) \text{Var}[v_1] + O(\sigma^{-2}).
\end{align}
The difference between our results stems from the deterministic factor $m_{fc} - m_{sc}$. Considering the self-consistent equation [24], for $\vartheta = \sigma N^{-\frac{1}{2}}$, $\nu$ being centered implies
\begin{align}
\Lambda := \frac{m_{fc}(z) - m_{sc}(z)}{\vartheta^2} &= \sigma^2 m_{sc}(z) \Lambda^2 + m_{sc}(z)^2 \Lambda - \vartheta^{-1} m_{sc}(z) \int_x^\infty \frac{x}{\vartheta x - z - m^2_{fc}(z)} \text{d} \nu(x) \\
&= m_{sc}(z)^2 \Lambda + \vartheta^{-1} m_{sc}(z) \int_x^\infty \left( \frac{x}{z - m_{sc}(z)} - \frac{x}{\vartheta x - z - m^2_{fc}(z)} \right) \text{d} \nu(x) + O(\vartheta^2 \Lambda^2) \\
&= m_{sc}(z)^2 \Lambda + m_{sc}(z) \int_x^\infty \frac{x^2}{(\vartheta x - z - m^2_{fc}(z))(z - m_{sc}(z))} \text{d} \nu(x) + O(\vartheta \Lambda + \vartheta^2 \Lambda^2) \\
&= m_{sc}(z)^2 \Lambda + m_{sc}(z)^3 \text{Var}[v_1] + O(\vartheta + \vartheta^2 \Lambda + \vartheta \Lambda + \vartheta^2 \Lambda^2),
\end{align}
where we add an auxiliary factor
\begin{align}
\vartheta^{-1} m_{sc}(z)^2 \int_x x \text{d} \nu(x) = \vartheta^{-1} m_{sc}(z) \int_x^\infty \frac{x}{z - m_{sc}(z)} \text{d} \nu(x) = 0
\end{align}
in the second equality. Then by assuming $\Lambda = O(1)$ we get
\begin{align}
\Lambda \to \frac{m_{sc}(z)^3}{1 - m_{sc}(z)^2} \text{Var}[v_1] = m_{sc}(z)^3 (1 + m'_{sc}(z)) \text{Var}[v_1],
\end{align}
and indeed
\[ \frac{N}{C} (m_{fc} - m_{sc}) = C \frac{m_{sc}(z)^3}{1 - m_{sc}(z)^2} \text{Var} [v_1] + O(C^2 N^{-\frac{1}{2}}). \quad (3.25) \]

It can be easily seen that this term precisely compensates \( C m_{sc}(z)^3 (1 + m'_{sc}(z)) \text{Var} [v_1] \) in (3.24), and the same holds for the covariance.

### 3.2 Preliminary results

Here we collect some preliminary results concerning the behavior of \( m_{fc}^{\varphi} \) and \( \hat{m}_{fc}^{\varphi} \). Some of the lemmas are not cited, and their proofs are addressed in the Appendix.

#### 3.2.1 Deformed semicircle laws

**Notational Remark 3.9.** For random variables \( X \equiv X_N \) and \( Y \equiv Y_N \) depending on \( N \), we use the notations \( X \prec Y \) and \( X = O(Y) \) to indicate that for any \( \epsilon, D > 0 \),
\[ P \left[ |X| > N^\epsilon |Y| \right] \leq N^{-D}. \quad (3.26) \]
for any sufficiently large \( N \). Similarly, for a given event \( \Omega \equiv \Omega_N \), we write \( X \prec Y \) on \( \Omega \) to indicate that for any \( \epsilon, D > 0 \),
\[ P \left[ |X| > N^\epsilon |Y| \cap \Omega \right] \leq N^{-D}. \quad (3.27) \]
Also we write \( X = O_p(Y) \) when \( X \) is bounded by \(|Y| N^\epsilon \) in probability, for any \( \epsilon > 0 \).

**Definition 3.10.** We define \( \hat{m}_{fc}^{\varphi}(z) \) to be the solution of the equation
\[ \hat{m}_{fc}^{\varphi}(z) = \int \frac{1}{x - z - \hat{m}_{fc}^{\varphi}(z)} d\nu^\varphi(x), \quad \Im \hat{m}_{fc}^{\varphi}(z) \geq 0, \quad \text{for } z \in \mathbb{C}^+. \quad (3.28) \]

**Definition 3.11.** Let \( \Omega \equiv \Omega_N(\alpha) \) be the event on which the following holds:

(i) We have
\[ \inf_{x \in I_\nu} \int \frac{1}{(v - x)^2} d\nu(x) \geq 1 + \varphi. \quad (3.29) \]

(ii) For any fixed compact set \( D \subset \mathbb{C}^+ \) with \( D \cap \text{supp } \nu = \emptyset \), there exists a constant \( C > 0 \) such that for any sufficiently large \( N \),
\[ \sup_{z \in D} |m_\varphi(z) - m_\nu(z)| \leq CN^{-\alpha} \quad (3.30) \]

(iii) If \( V \) is random, we impose another condition: for any fixed compact set \( D \subset \Theta_\varphi \times \mathbb{C}^+ \) satisfying
\[ \inf_{(\varphi, z) \in D, x \in I_\nu} |\varphi x - z| > 0, \quad (3.31) \]
there exists a constant \( C > 0 \) such that for any sufficiently large \( N \),
\[ \sup_{(\varphi, z) \in D, \varphi \neq 0} |m_\varphi'(z) - m_\nu'(z)| \leq CN^{-\alpha} \quad (3.32) \]

**Remark 3.12.** We remark that for random \( V \), our conditions of \( \Omega_N \) is stronger than the conditions in Definition 3.1 of [25]. Therefore the bounds on \( \Omega_N \) given in [25] applies without modification.
The following lemma controls the probability of the complementary event $\Omega_N$, allowing us to focus on the analysis within $\Omega_N$:

**Lemma 3.13.** For random $V$ and any fixed $\epsilon_0 > 0$, there exists $c > 0$ such that

$$
P \left[ \Omega_N \left( \frac{1}{2} - \epsilon_0 \right) \right] \geq 1 - cN^{-t},$$

(3.33)

where $t$ is given in Assumption 2.6.

**Remark 3.14.** If $V$ is deterministic, $\Omega_N(\alpha_0)$ holds with probability 1 for $\alpha_0$ given in (2.4) and $N$ sufficiently large, by Assumptions 2.4 and 2.6. Thus, in Appendices 4, 7 whenever we assume $V$ is deterministic, we let $N$ be large enough so that the assumptions on $\Omega_N(\alpha_0)$ holds with probability 1.

**Notational Remark 3.15.** We write $\Omega_N \equiv \Omega_N(\alpha_0)$ if $V$ is deterministic, and $\Omega_N \equiv \Omega_N(\frac{1}{2} - \epsilon_0)$ if $V$ is random, where $\epsilon_0$ is small enough but fixed positive real number.

**Lemma 3.16** (Lemma 3.2 of [25]). Suppose that $\vartheta$ and $\nu$ satisfy Assumptions 2.4 and 2.5. Then for any $N \in \mathbb{N}$ and $\vartheta \in \Theta_\infty$, the inversion formulae

$$
\rho_{fc}^\vartheta := \lim_{\eta \to 0^+} \frac{1}{\pi} \Im m_{fc}^\vartheta(E + i\eta), \quad E \in \mathbb{R}
$$

(3.34)

and

$$
\tilde{\rho}_{fc}^\vartheta := \lim_{\eta \to 0^+} \frac{1}{\pi} \Im \tilde{m}_{fc}^\vartheta(E + i\eta), \quad E \in \mathbb{R}
$$

(3.35)

define absolutely continuous measures $\rho_{fc}^\vartheta$ and $\tilde{\rho}_{fc}^\vartheta$. Moreover, $\rho_{fc}^\vartheta$ is supported on a single interval with strictly positive density inside the interval and the same conclusion holds for $\tilde{\rho}_{fc}^\vartheta$ on $\Omega_N$ for any sufficiently large $N$.

**Notational Remark 3.17.** For simplicity, densities of $\rho_{fc}^\vartheta$ and $\tilde{\rho}_{fc}^\vartheta$ are also denoted by the same symbol.

**Definition 3.18.** We denote the supporting interval of $\tilde{\rho}_{fc}^\vartheta$ by $[\tilde{L}_{-\vartheta}, \tilde{L}_{+\vartheta}]$, and let $E_0 \geq 1 + \max\{|L_{-\vartheta}^1|, |L_{+\vartheta}^1|\}$. We also define domains

$$
\mathcal{D} := \{ z \in E + i\eta \in \mathbb{C}^+ : |E| \leq E_0, N^{-\delta} \leq \eta \leq 3 \}
$$

(3.36)

and

$$
\mathcal{D}' := \{ z \in E + i\eta \in \mathbb{C}^+ : |E| \leq E_0, 0 < \eta < 3 \},
$$

(3.37)

where $0 < \delta < 1$ is a sufficiently small but fixed constant (independent of $N$), to be determined.

**Lemma 3.19** (Theorem 3.3 of [25], Strong local deformed semicircle law). Under Assumptions 2.2, 2.4 and 2.7, the following hold on $\Omega_N$:

For any $z \in \mathcal{D}$ and $\vartheta \in \Theta_\infty$ (both of which possibly vary with $N$),

$$
|m_{N}^\vartheta(z) - \tilde{m}_{fc}^\vartheta(z)| \lesssim \frac{1}{N^\eta}
$$

(3.38)

and

$$
|R_{ij}^\vartheta(z) - \delta_{ij}\tilde{g}_{i}^\vartheta(z)| \lesssim \sqrt{\frac{\Im m_{fc}^\vartheta(z)}{N^\eta}} + \frac{1}{N^\eta}, \quad \text{for any } 1 \leq i, j \leq N,
$$

(3.39)

where we defined

$$
\tilde{g}_{i}^\vartheta(z) := \frac{1}{\vartheta_{ij} - z - \tilde{m}_{fc}^\vartheta(z)}.
$$

(3.40)
Definition 3.20. The eigenvalues of $W$ are denoted by $\lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_N^0$, and we use $\tilde{\gamma}_i^0$ and $\gamma_i^0$ to denote the respective classical locations of laws $\tilde{\rho}_f^0$ and $\rho_f^0$, i.e.,

$$
\int_{-\infty}^{\gamma_i^0} \tilde{\rho}_f^0(x)dx = \frac{i - \frac{1}{2}}{N} \quad \text{and} \quad \int_{-\infty}^{\tilde{\gamma}_i^0} \rho_f^0(x)dx = \frac{i - \frac{1}{2}}{N} \quad \text{for } 1 \leq i \leq N. \tag{3.41}
$$

Notational Remark 3.21. We also define $\tilde{\gamma}_0 := \tilde{L}_-$ and $\gamma_0 := L_-.$

Lemma 3.22 (Corollary 3.4 of [25], Rigidity estimates). For a deterministic $V$, under Assumptions 2.2, 2.4 and 2.5, the following hold on $\Omega_N$:

$$
|\lambda_i^0 - \tilde{\gamma}_i^0| \prec N^{-\frac{1}{2}} \tilde{\alpha}_i^{-\frac{1}{2}} \quad \text{for } 1 \leq i \leq N \quad \text{and} \quad \sum_{i=1}^{N} |\lambda_i^0 - \tilde{\gamma}_i^0|^2 \prec \frac{1}{N}, \tag{3.42}
$$

uniformly for $\vartheta \in \Theta_v$, where $\tilde{\alpha}_i := \min\{i, N - i + 1\}$.

Lemma 3.23 (Theorem 2.22 of [22], Rigidity estimates). For a random $V$, under Assumptions 2.2, 2.4 and 2.5, the following hold on $\Omega_N$:

$$
|\lambda_i^0 - \gamma_i^0| \prec N^{-\frac{1}{2}} \tilde{\alpha}_i^{-\frac{1}{2}} \left( \Theta_{\gamma_i}^{N^{\frac{1}{2}}(1 + \vartheta^2 N^4)} \right) + \vartheta^2 N^{-\frac{1}{2}} \tilde{\alpha}_i^{-\frac{1}{2}} + \vartheta N^{-\frac{1}{2}}, \tag{3.43}
$$

uniformly for $\vartheta \in \Theta_v$, where $\tilde{\alpha}_i := \min\{i, N - i + 1\}$.

In order to control $m_N(z) - m_{f,c}(z)$ for $z \in \Gamma_0$ and sample path outside of $\Omega$, we propose a similar bound following from 3.33:

Corollary 3.24. For any $z \in \Gamma$ and $\vartheta \in \Theta_v$, we have

$$
|m_N^0(z) - m_{f,c}^0(z)| \prec \frac{\vartheta}{\sqrt{N}}. \tag{3.44}
$$

Lemma 3.25 (Lemmas 3.5, 3.6, and A.1 of [25], Square-root behavior). Let $\nu$ and $\tilde{\nu}$ satisfy Assumptions 2.4 and 2.7, $\kappa_E = \min\{|E - L_-|, |E - L_+|\}$. Then the following hold for any $\vartheta \in \Theta_v$:

(i) For any $z = E + i\eta \in \mathcal{D}'$,

$$
\Im m_{f,c}^0(z) \sim \begin{cases} \sqrt{\kappa_E + \eta}, & E \in [L_-, L_+], \\
\frac{\eta}{\sqrt{\kappa_E + \eta}}, & E \notin [L_-, L_+], \end{cases} \tag{3.45}
$$

where

$$
\kappa_E := \min\{|E - L_-^0|, |E - L_+^0|\}. \tag{3.46}
$$

(ii) There exists a constant $C > 1$ such that for any $z \in \mathcal{D}'$ and $x \in I_\nu$,

$$
C^{-1} \leq |\vartheta x - z - m_{f,c}^0(z)| \leq C. \tag{3.47}
$$

(iii) There exists a constant $C > 1$ such that for any $z = E + i\eta \in \mathcal{D}'$,

$$
C^{-1} \sqrt{\kappa_E + \eta} \leq \left| 1 - \int_{\mathbb{R}} \frac{1}{(\vartheta x - z - m_{f,c}^0(z))^2} d\nu(x) \right| \leq C \sqrt{\kappa_E + \eta}. \tag{3.48}
$$
The constants in \[3.35\], \[3.47\], \[3.48\] can be chosen uniformly in \(\vartheta \in \Theta_\infty\).

Furthermore, on \(\Omega_N\), the following hold for sufficiently large \(N\):

(i) There exists constant \(c > 0\) such that for any \(z \in \mathcal{D}'\),
\[
|\hat{m}_{f,c}(z) - m_{f,c}(z)| \leq N^{-\frac{1}{2}} \quad \text{and} \quad |\hat{L}_\pm^\theta - L_\pm^\theta| \leq N^{-c}\alpha \tag{3.49}
\]
for sufficiently large \(N\).

(ii) For any \(z = E + i\eta \in \mathcal{D}'\),
\[
\Im m_{f,c}(z) \sim \begin{cases} \frac{\sqrt{\kappa_E + \eta}}{\eta}, & E \in [\hat{L}_-^\theta, \hat{L}_+^\theta], \\ \frac{\sqrt{\kappa_E}}{\eta}, & E \notin [\hat{L}_-^\theta, \hat{L}_+^\theta], \end{cases} \tag{3.50}
\]

(iii) There exists constant \(C > 1\) such that for any \(z \in \mathcal{D}'\) and \(x \in I_\vartheta\),
\[
C^{-1} \leq |\vartheta x - z - \hat{m}_{f,c}(z)| \leq C. \tag{3.51}
\]

The constants in \[3.50\] and \[3.51\] can be chosen uniformly in \(\vartheta \in \Theta_\infty\) and \(N \in \mathbb{N}\) for \(N\) sufficiently large.

The square-root behavior of \(\Im m_{f,c}^\theta\) and \(\Im \hat{m}_{f,c}^\theta\) implies the following fact, to be used later on.

**Corollary 3.26.** If \(\mathcal{D} \subset \mathbb{C}\) is a compact subset with a constant \(c > 0\) satisfying \(\inf_{z \in \mathcal{D}} |\vartheta| > c\) and
\[
\mathcal{D} \cap \{z = E + i\eta \in \mathbb{C} : E \in [\hat{L}_-^\theta, \hat{L}_+^\theta]\} \subset \{z \in \mathbb{C} : |\eta| > c\}, \tag{3.52}
\]
then there exists a constant \(c' > 0\) such that \(\Im m_{f,c}^\theta(z) > c'\eta\) and also \(\Im \hat{m}_{f,c}^\theta(z) > c'\eta\) on \(\Omega_N\) for sufficiently large \(N\). The constant \(c'\) can be chosen uniformly in \(\vartheta \in \Theta_\infty\) and \(N \in \mathbb{N}\) for \(N\) sufficiently large.

Using the rigidity estimates, we can bound \(m_{f,c} - \hat{m}_{f,c}\) for \(\Im z < 1\), in particular, for \(z \notin \Gamma_u\). Also, the square-root behavior enables us to enlarge the domain of \(z\) in the entrywise local law in **Lemma 3.16**.

**Corollary 3.27.** For fixed constant \(c > 0\), define \(\mathcal{D}_c := \{z = E + i\eta : \eta \in (c, 3), |E| \leq E_0\}\). Then on \(\Omega_N\), for any \(z \in \mathcal{D}_c\) and \(\vartheta \in \Theta_\infty\),
\[
|m_{N}^\theta(z) - \hat{m}_{f,c}^\theta(z)| \sim \frac{1}{N} \tag{3.53}
\]
and
\[
|R_{ij}^\theta(z) - \hat{R}_{ij}^\theta(z)| \sim \frac{1}{\sqrt{N}}. \tag{3.54}
\]
Moreover, the estimate \[3.53\] holds on \(\Omega_N\) for \(z \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_r\) and \[3.54\] holds on \(\Omega_N\) for \(z \in \Gamma_1 \cup \Gamma_r\).

In fact, the bound of \(\hat{m}_{f,c}^\theta(z) - m_{f,c}^\theta(z)\) can be improved with a stronger assumption on the domain, following the proof of **Lemma 3.6** of [25]:

**Lemma 3.28.** For random \(V\), if \(\mathcal{D}_0 \subset \mathcal{D}'\) is a compact subset with \(\inf\{|\kappa_E + \eta| : z = E + i\eta \in \mathcal{D}_0\} \sim 1\), there exists a constant \(C > 0\) such that
\[
|\hat{m}_{f,c}^\theta(z) - m_{f,c}^\theta(z)| \leq C\vartheta N^{-\frac{1}{4} + \epsilon_0} \tag{3.55}
\]
for all \(z \in \mathcal{D}_0\) on \(\Omega_N\), and the constant \(C\) can be chosen uniformly in \(\vartheta \in \Theta_\infty\), \(z \in \mathcal{D}_0\), \(N \in \mathbb{N}\) for sufficiently large \(N\).
Corollary 3.29. Let $\nu$ and $\hat{\nu}$ satisfy Assumptions 3.2 and 3.5, $\kappa_E = \min\{|E - L_-|, |E - L_+|\}$. Define
\[
I^\theta(z_1, z_2) := \int_{\mathbb{R}} \frac{1}{(x - z_1 - m_f^\theta_c(z_1))(x - z_2 - m_f^\theta_c(z_2))} d\nu(x), \quad \hat{I}_k^\theta(z_1, z_2) := \frac{1}{N} \sum_{p > k} \hat{g}_p(z_1)\hat{g}_p(z_2) \quad (3.56)
\]
for $0 \leq k \leq N - 1$. Then for each fixed compact subset $D \subset \mathbb{C}$ with a constant $c > 0$ satisfying $\inf_{z \in D} \sqrt{\kappa_E + \eta} > c$ and
\[
D \cap \{z = E + i\eta \in \mathbb{C} : E \in [L_-, L_+^2]\} \subset \{z \in \mathbb{C} : |\eta| > c\}, \quad (3.57)
\]
there exists a constant $r \in (0, 1)$ such that
\[
\sup \left\{ |I^\theta(z_1, z_2)| : z_1, z_2 \in D_c, \vartheta \in \Theta_w \right\} < r \quad (3.58)
\]
and on $\Omega_N$,
\[
\sup \left\{ |\hat{I}_k^\theta(z_1, z_2)| : z_1, z_2 \in D_c, 0 \leq k \leq N, \vartheta \in \Theta_w \right\} < r, \quad (3.59)
\]
for any sufficiently large $N$.

Using the bound for $|\hat{m}_f^\theta_c(z) - m_f^\theta_c(z)|$ in Lemma 3.30, we prove another estimate concerning the covariance of $(\vartheta v_i^1 - z - \hat{m}_f^\theta(z))^{-1}$. To this end, we prove another lemma used along its proof.

Lemma 3.30. Let $G^\theta(z) = z + m_f^\theta(z)$ on $\mathbb{C}^+$ and $c \in (0, 3)$ be a constant. Then for each compact subset $D \subset \mathbb{C}$ satisfying the assumptions of Corollary 3.29, there exists a constant $d > 0$ such that for any $z_1, z_2 \in D$,
\[
|G^\theta(z_1) - G^\theta(z_2)| \geq d |z_1 - z_2|. \quad (3.60)
\]
The constant $d$ can be chosen uniformly in $z_1, z_2 \in D_c$ and $\vartheta \in \Theta_w$.

Proof. The lemma directly follows from (2.9):
\[
\left| \frac{G^\theta(z_1) - G^\theta(z_2)}{z_1 - z_2} \right| = \left| 1 - \frac{1}{\int_{\mathbb{R}} (x - z_1 - m_f^\theta_c(z_1))(x - z_2 - m_f^\theta_c(z_2)) d\nu(x)} \right| \geq 1 - |I^\theta(z_1, z_2)| \geq 1 - r, \quad (3.61)
\]
where $r$ is given in Corollary 3.29.\]

Now with the aid of Lemma 3.30, we state and prove the desired result.

Corollary 3.31. Suppose that $V$ is random. Let $c > 0$ be given, $D$ satisfy the assumptions of Corollary 3.29, and $I^\theta(z_1, z_2)$ and $\hat{I}_k^\theta(z_1, z_2)$ be defined as in Corollary 3.29. For any compact subset $D_1 \subset D$, there exists a constant $C > 0$ such that
\[
|I^\theta(z_1, z_2) - \hat{I}_0^\theta(z_1, z_2)| \leq C \vartheta N^{-\frac{1}{4} + \varepsilon_0} \quad (3.62)
\]
for $z_1, z_2 \in D_1$, on $\Omega_N$. The constant $C$ can be chosen uniformly in $z_1, z_2 \in D_1, \vartheta \in \Theta_w$, and $N \in \mathbb{N}$ for sufficiently large $N$.

Since we are assuming that $\vartheta$ varies with $N$, with the limit $\vartheta_\infty$, the deterministic function $m_{f_c}^\theta(z)$ converges to $m_{f_c}^\theta(z)$, which is the Stieltjes transform of the semicircle law if $\vartheta_\infty = 0$, with rate $O(|\vartheta - \vartheta_\infty|)$. Even though this fact have been addressed by many authors previously, we here propose another proof following a method that will be used frequently throughout this paper: the self-comparison. The following lemma, which describes the behavior of $m_{sc}$, is used along the proof for the case where $\vartheta_\infty = 0$.\]

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Lemma 3.32 (Lemma 4.2 of [14]. Square-root behavior of $m_{sc}(z)$). Suppose that $z = E + i\eta \in \mathbb{C}^+$ with $|E| \leq 5$. Then
\[
|m_{sc}(z)| = |m_{sc}(z)^{-1} + z|^{-1} \leq 1. \tag{3.63}
\]
If in addition we have $\eta \leq 10$, then
\[
|m_{sc}(z)| \sim 1, \quad |1 - m_{sc}(z)| \sim \sqrt{\kappa + \eta}, \tag{3.64}
\]
where $\kappa \equiv \kappa_E := \min\{|E - 2|, |E + 2|\}$.

Using the lemma, we now prove the required estimate for $|m_{fc}^\theta(z) - m_{fc}^{\theta\infty}(z)|$ using the self-comparison.

Lemma 3.33. For any fixed compact set $D \subseteq \mathbb{C}$ with $\text{dist}(D, [E^\theta, L^\theta]) \sim 1$, there exists a constant $C > 0$ such that
\[
\sup_{z \in D} |m_{fc}^\theta(z) - m_{fc}^{\theta\infty}(z)| \leq C |\vartheta - \vartheta_{\infty}| \tag{3.65}
\]
for any sufficiently large $N \in \mathbb{N}$.

Given the bound of $m_{fc}^\theta(z) - m_{sc}(z)$, that of the covariance term can be deduced easily.

Corollary 3.34. Under the assumptions of Lemma 3.33, there exists a constant $C > 0$ such that
\[
\sup_{z_1, z_2 \in D} |I^\vartheta(z_1, z_2) - I^{\vartheta\infty}(z_1, z_2)| \leq C |\vartheta - \vartheta_{\infty}|. \tag{3.66}
\]

Proof. Given the Lemma 3.33 the lemma is a direct consequence of the self-consistent equation (2.9) and the stability bound (3.4).

\[
I^\vartheta(z_1, z_2) - I^{\vartheta\infty}(z_1, z_2)
= \int_{\mathbb{R}} \left[ \frac{1}{(\vartheta x - z_1 - m_{fc}^\vartheta(z_1))(\vartheta x - z_2 - m_{fc}^\vartheta(z_2))} - \frac{1}{(\vartheta_{\infty} x - z_1 - m_{fc}^{\vartheta\infty}(z_1))(\vartheta_{\infty} x - z_2 - m_{fc}^{\vartheta\infty}(z_2))} \right] d\nu(x)
= O(|\vartheta - \vartheta_{\infty}|), \tag{3.67}
\]
where the uniformity follows from (3.47) and Lemma 3.33. \hfill \Box

Remark 3.35. Note that for $\vartheta_{\infty} = 0$, we have
\[
I^{\vartheta\infty}(z_1, z_2) = \frac{1}{(-z - m_{sc}(z_1))(-z - m_{sc}(z_2))} = m_{sc}(z_1)m_{sc}(z_2). \tag{3.68}
\]

Recalling the definition of $W$, we observe that the off-diagonal terms are identical to that of Lemma 5.3 in [5] with $J = 0$, so that we have the following lemma holds:

Lemma 3.36 (Lemma 5.3 of [5]. Large deviation estimates). Let $S$ be an $(N-1) \times (N-1)$ matrix independent of $\{W_{ai} : 1 \leq a \leq N, a \neq i\}$ with matrix norm $\|S\|$. Then for $n = 1, 2$, there exists a constant $C_n$ depending only on $W_4$ in Assumption 2.2 such that
\[
\mathbb{E}\left[ \sum_{p,q}^{(i)} W_{ip} S_{pq} W_{qi} - \frac{1}{N} \sum_{p}^{(i)} S_{pp} \right]^{2n} \leq \frac{C_n \|S\|^{2n}}{N^n}. \tag{3.69}
\]

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Moreover,
\[
\sum_{p,q}^{(i)} W_{ip} S_{pq} W_{qj} - \frac{1}{N} \sum_p^{(i)} S_{pp} - \frac{\|S\|}{\sqrt{N}}
\]  
(3.70)

### 3.2.2 Matrix identities

**Lemma 3.37** (Lemma 3.1 of [23], Matrix identities). Let \(X\) be an \(N \times N\), symmetric matrix and \(R(z) := (X - zI)^{-1}, z \in \mathbb{C}\). Then for \(i, j, k \in \{1, \ldots, N\}\), the following identities hold:

1. **Schur complement formula:**
   \[
   R_{ii} = \left( X_{ii} - z - \sum_{m,n}^{(i)} X_{im} R_{mn}^{(i)} X_{ni} \right)^{-1}.
   \]  
   (3.71)

2. For \(i \neq j\),
   \[
   R_{ij} = -R_{ii} \sum_m^{(i)} X_{im} R_{mj}^{(i)} = -R_{ii} R_{jj}^{(i)} \left( X_{ij} - \sum_{m,n}^{(i,j)} X_{im} R_{mn}^{(i,j)} X_{nj} \right). 
   \]  
   (3.72)

3. For \(i, j \neq k\),
   \[
   R_{ij} = R_{ij}^{(k)} + \frac{R_{ik} R_{kj}}{R_{kk}}.
   \]  
   (3.73)

### 4 Proof of Proposition 3.1

To serve our purpose of proving Proposition 3.1, we assume that \(V\) is deterministic in this section. Therefore, as addressed in Remark 3.14, we may assume the conditions of \(\Omega_N\).

As both of propositions asserts convergence of processes, it suffices to prove the finite-dimensional convergence and the tightness of processes, using Theorem 8.1 of [8]. For the finite-dimensional convergence, following [4] and [5], we express \(\xi_{\vartheta N}(z)\) as a martingale and use the following theorem concerning the central limit theorem of martingales:

**Lemma 4.1** (Theorem 35.12 of [9]). Suppose that for each fixed \(n \in \mathbb{N}\), \(\{X_{n,k}\}_{k \in \mathbb{N}}\) is a martingale with respect to a filtration \(\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \cdots\). Let \(Y_{n,k} := X_{n,k} - X_{n,k-1}\) where \(X_{n,0} = 0\), and suppose that \(\sum_k Y_{n,k}\) converges a.s. and \(Y_{n,k} \in L^2\) for each \(k\). Denote \(\sigma^2_{n,k} := \mathbb{E} \left[ Y_{n,k}^2 \bigg| \mathcal{F}_{n,k-1} \right]\) where \(\mathcal{F}_{n,0}\) is the trivial \(\sigma\)-algebra. If

\[
\sum_k \sigma_{n,k}^2 \to \sigma^2
\]  
(4.1)

in probability where \(\sigma \in \mathbb{R}_+\) is a constant and

\[
\sum_k \mathbb{E} \left[ Y_{n,k}^2 \mathbb{I}[\|Y_{n,k}\| \geq \epsilon] \right] \to 0 
\]  
(4.2)

for each \(\epsilon > 0\), then \(\sum_{k=1}^{\infty} Y_{n,k}\) converges weakly to the normal distribution with zero mean and variance \(\sigma^2\).

**Lemma 4.1** is used to conclude that the finite dimensional distribution of \(\xi_{\vartheta N}(z) - \mathbb{E} \left[ \xi_{\vartheta N}(z) \right]\) converges weakly to the centered Gaussian distribution with designated covariance, and the convergence of mean is dealt separately. In particular, the filtration to which our martingale is adapted is defined as follows:
**Definition 4.2.** We define the (decreasing) filtration \( \{ \mathcal{F}_k : 0 \leq k \leq N \} \) as

\[
\mathcal{F}_k := \sigma(W_{ij} : k < i, j \leq N), \quad k = 0, 1, \cdots, N \tag{4.3}
\]

and denote the conditional expectation \( \mathbb{E}[\cdot|\mathcal{F}_k] \) by \( \mathbb{E}_k[\cdot] \). We also define

\[
\mathcal{G}_k := \sigma(W_{ij} : k < i, j \leq N) \vee \sigma(v_i : i > k), \quad k = 0, 1, \cdots, N. \tag{4.4}
\]

**Notational Remark 4.3.** For random \( V \), we also define

\[
\mathcal{F}_k := \sigma(W_{ij} : k < i, j \leq N) \vee \sigma(v_m : 1 \leq m \leq N), \quad k = 0, 1, \cdots, N \tag{4.5}
\]

and \( \mathbb{E}_k \) similarly. Note that for deterministic \( V \), \( \sigma(v_m : 1 \leq m \leq N) \) is the trivial \( \sigma \)-algebra, so that the definition is still consistent.

The convergence of mean \( \mathbb{E} \left[ \xi_N^\theta(z) \right] \) is contained in the following lemma, which is proved in *Appendix A*.

**Lemma 4.4.** Define

\[
b_N^\theta(z) := \mathbb{E} \left[ \xi_N^\theta(z) \right] = \mathbb{E}_N \left[ \xi_N^\theta(z) \right] = N\mathbb{E} \left[ m_N(z) - \hat{m}_f(z) \right]. \tag{4.6}
\]

For \( z \in \mathcal{D}_c \) or \( z \in \Gamma_r \cup \Gamma_l \),

\[
b_N^\theta(z) = \frac{1}{2} \frac{(m_f^\theta(z))^\prime(z)}{(1 + (m_f^\theta(z))^\prime(z))^2} \left[ (w_2 - 1) + (m_f^\theta(z))^\prime(z) + (W_4 - 3) \frac{(m_f^\theta(z))^\prime(z)}{1 + (m_f^\theta(z))^\prime(z)} \right] + O(\vartheta N^{-\alpha} + N^{-1/\beta + \epsilon}) \tag{4.7}
\]

if \( \vartheta_\infty > 0 \), and

\[
b_N^\theta(z) = m_{sc}(z)^3(1 + m_{sc}^\prime(z))(w_2 - 1) + m_{sc}^\prime(z) + (W_4 - 3)m_{sc}(z)^2 + O(\vartheta + N^{-1/\beta + \epsilon}) \tag{4.8}
\]

if \( \vartheta_\infty = 0 \).

**Remark 4.5.** For random \( V \), the same proof with \( \mathbb{E}[\cdot] \) replaced by \( \mathbb{E}_N[\cdot] \) gives us the absolute bound

\[
| \mathbb{E}_N \left[ m_N(z) - \hat{m}_f(z) \right] |_{\chi_{\Omega_N}} = O(1/N). \tag{4.9}
\]

Now given the convergence of means, to use Lemma 4.4 as addressed above, we express \( \xi_N^\theta(z) - \mathbb{E} \left[ \xi_N^\theta(z) \right] \) as a martingale. Letting

\[
\zeta_N := \xi_N - \mathbb{E} \left[ \xi_N \right] = \text{Tr} R - \mathbb{E} \left[ \text{Tr} R \right], \tag{4.10}
\]

one can rewrite \( \zeta_N \) as sum of a martingale difference sequence as follows:

\[
\zeta_N = \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} \left[ \text{Tr} R \right] - \mathbb{E}_k \left[ \text{Tr} R \right] \right) = \sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k) \text{Tr} R R = \sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k)(\text{Tr} R R - \text{Tr} R^{(k)}). \tag{4.11}
\]

In *Appendix B* we further simplify the martingale decomposition to get

\[
\zeta_N = \sum_{k=1}^{N} \mathbb{E}_{k-1} \left[ \phi_N^{\theta_k} \right] + \mathcal{O}_p(N^{-1/2}) \tag{4.12}
\]

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where

\[
\phi_k^\vartheta = \tilde{\phi}_k^\vartheta \left( \sum_{p,q}^{(k)} W_{kp}(R^{(k)})_{pq} W_{qk} - (\hat{m}_f^\vartheta)\right)^2 + (\tilde{\phi}_k^\vartheta)^2 \left( -W_{kk} + \vartheta v_k + \sum_{p,q}^{(k)} W_{kp} R^{(k)}_{pq} W_{qk} - \hat{m}_f^\vartheta \right) \left( 1 + (\hat{m}_f^\vartheta)\right).
\]

(4.13)

In Appendix B we also prove the conditions (4.1) and (4.2) of Lemma 4.7.

**Lemma 4.6.** For distinct points \(z_1, z_2 \in \mathcal{K}\), we let

\[
\Gamma_k^\vartheta(z_1, z_2) = \sum_{k=1}^{N} \mathbb{E}_k \left[ \mathbb{E}_{k-1} \left[ \phi_k^\vartheta(z_1) \right] \cdot \mathbb{E}_{k-1} \left[ \phi_k^\vartheta(z_2) \right] \right].
\]

(4.14)

Then

\[
\Gamma_k^\vartheta(z_1, z_2) = (w_2 - 2) \frac{\partial^2 I}{\partial z_1 \partial z_2} + (W_4 - 3) \left( I \frac{\partial^2 I}{\partial z_1 \partial z_2} + \frac{\partial I}{\partial z_1} \frac{\partial I}{\partial z_2} \right)
\]

\[
+ \frac{2}{(1-I)^2} \left( \frac{\partial I}{\partial z_1} \frac{\partial I}{\partial z_2} + (1-I) \frac{\partial^2 I}{\partial z_1 \partial z_2} \right) + O(N^{-\frac{1}{2}}) + O(\vartheta N^{-\alpha_0})
\]

(4.15)

if \(\vartheta > 0\) where

\[
I(z_1, z_2) = I^\vartheta(z_1, z_2) := \int_{\mathbb{R}} \frac{1}{(\vartheta \infty x - z_1 - m_{f_c}^{\vartheta}(z_1))(\vartheta \infty x - z_2 - m_{f_c}^{\vartheta}(z_2))} \, d\nu(x),
\]

(4.16)

and

\[
\Gamma(z_1, z_2) = m_{sc}^\vartheta(z_1)m_{sc}^\vartheta(z_2) \left( (w_2 - 2) + 2(W_4 - 3)m_{sc}(z_1)m_{sc}(z_2) + \frac{2}{(1-m_{sc}(z_1)m_{sc}(z_2))^2} \right)
\]

\[
\quad + O(\vartheta) + O(N^{-\frac{1}{2}})
\]

(4.17)

if \(\vartheta = 0\).

**Lemma 4.7.** For any \(z \in \mathcal{K}\) and \(\epsilon > 0\),

\[
\sum_{k} \mathbb{E} \left[ \mathbb{E}_{k-1} \left[ \phi_k^\vartheta \right] \right] \chi(\mathbb{E}_{k-1} \left[ \phi_k^\vartheta \right] > \epsilon) \rightarrow 0.
\]

(4.18)

**Remark 4.8.** As in Remark 4.3 for random \(V\), the proofs of Lemma 4.6 and Lemma 4.7 imply the fact that

\[
\mathbb{E} \left[ |m_N(z) - \mathbb{E}_N \left[ m_N(z) \right]|^2 \chi_{\Omega_N} \right] = O(N^{-2+\epsilon}).
\]

(4.19)

Now given the finite-dimensional convergence, it remains to prove the tightness. Since the mean \(b_N(z)\) converges, the tightness of \(\zeta_N^\vartheta(z)\) implies that of \(\zeta_N^\vartheta(z)\). Following [4], by Theorem 12.3 of [8], it suffices to check the tightness for a fixed \(z \in \mathcal{K}\) and prove a Hölder condition given below. The tightness for fixed \(z \in \mathcal{K}\) follows directly from the finite-dimensional convergence and hence the tightness reduces to the following Hölder condition:

\[
\mathbb{E} \left[ |\zeta_N^\vartheta(z_1) - \zeta_N^\vartheta(z_2)|^2 \right] \leq K |z_1 - z_2|^2, \quad \forall z_1, z_2 \in \mathcal{K},
\]

(4.20)

for some constant \(K\) independent of \(N \in \mathbb{N}\) and \(z_1, z_2 \in \mathcal{D}_c\).
The proof starts with an application of the resolvent equation \( R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2) \), to get
\[
E \left[ \left| \zeta_N^\vartheta(z_1) - \zeta_N^\vartheta(z_2) \right|^2 \right] = E \left[ \left| \text{Tr} R(z_1) - E \left[ \text{Tr} R(z_2) \right] \right| - (\text{Tr} R(z_2) - E \left[ \text{Tr} R(z_2) \right]) \right]^2 \\
= \left| z_1 - z_2 \right|^2 E \left[ \left| \text{Tr} R(z_1)R(z_2) - E \left[ \text{Tr} R(z_1)R(z_2) \right] \right|^2 \right].
\] (4.21)

Therefore the following lemma completes the proof of Proposition 3.1, which is proved in Appendix A:

**Lemma 4.9.** For \( z_1, z_2 \in K \) and sufficiently large \( N \in \mathbb{N} \), we have
\[
E \left[ \left| \text{Tr} R(z_1)R(z_2) - E \left[ \text{Tr} R(z_1)R(z_2) \right] \right|^2 \right] \leq K,
\] (4.22)
where \( K \) is a constant independent of \( z_1, z_2 \) and \( N \in \mathbb{N} \).

## 5 Proof of Propositions 3.2 and 3.3

As in the statements of the propositions to be proved, we assume that \( V \) is random in this section. Since \( P[\Omega_N] \to 1 \), assuming \( \Omega_N \) for any sample paths below will do no harm to our proof.

### 5.1 Primary reduction

Since \( \{\tilde{\xi}_N^\vartheta(z) : z \in K\} \) defines a continuous process for each \( n \in \mathbb{N} \), we again use Theorem 8.1 of [8] to prove Proposition 3.3, hence it suffices to prove the finite-dimensional convergence and the tightness. To this end, we start the proof by reducing \( \tilde{\xi}_N^\vartheta(z) \) into a simplified form which looks pleasant to apply the classical central limit theorem.

In this section, we assume \( z \in K \). Note first that
\[
\tilde{\xi}_N^\vartheta(z) = \sqrt{N} \left( m_N^\vartheta(z) - \hat{m}_N^\vartheta(z) \right) = \frac{1}{\sqrt{N}} \xi_N^\vartheta(z) + \frac{1}{\sqrt{N}} \left( \hat{m}_N^\vartheta(z) - m_N^\vartheta(z) \right).
\] (5.1)

Considering the first term, we decompose it as
\[
\frac{1}{\sqrt{N}} \xi_N^\vartheta(z) = \frac{\sqrt{N}}{\vartheta} \left( m_N(z) - E_N \left[ m_N(z) \right] \right) + \frac{\sqrt{N}}{\vartheta} \left( E_N \left[ m_N(z) \right] - \hat{m}_f(z) \right).
\] (5.2)

Then by Remarks 4.5 and 4.8
\[
E \left[ \frac{1}{\sqrt{N}} \xi_N^\vartheta(z) \right]^2 \chi_{\Omega_N} = O(N^{-1} \vartheta^{-2}),
\] (5.3)

which, together with the fact that \( P[\Omega_N] \to 1 \), implies the in probability convergence \( N^{-1} \vartheta^{-1} \xi_N^\vartheta(z) \to 0 \). In this sense, we let \( \xi_N^\vartheta(z) := \vartheta^{-1} \sqrt{N} \left( \hat{m}_N^\vartheta(z) - m_N^\vartheta(z) \right) \) and try to estimate it.
Given the estimate above, we first rewrite \( \tilde{\zeta}^\theta_N(z) \) as follows:

\[
\tilde{\zeta}^\theta_N(z) = \frac{\sqrt{N}}{\vartheta} \left( \frac{1}{N} \sum_{i=1}^N \tilde{\vartheta}^\theta_i(z) - m_f^\theta(z) \right) = \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - \tilde{m}_f^\theta(z)} - \int \frac{1}{\vartheta x - z - \tilde{m}_f^\theta(z)} \vartheta \, dx \right]
\]

\[
= \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - \tilde{m}_f^\theta(z)} - \int \frac{1}{\vartheta x - z - m_f^\theta(z)} \vartheta \, dx \right] + \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \frac{\tilde{m}_f^\theta(z) - m_f^\theta(z)}{(\vartheta v_i - z - m_f^\theta(z))(\vartheta v_i - z - \tilde{m}_f^\theta(z))}
\]

\[
= \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - \tilde{m}_f^\theta(z)} - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z - m_f^\theta(z)} \right] \right] + \tilde{\zeta}^\theta_N(z) \frac{1}{N} \sum_{i=1}^N \frac{1}{(\vartheta v_i - z - \tilde{m}_f^\theta(z))(\vartheta v_i - z - m_f^\theta(z))}.
\] (5.4)

To estimate the second summand in the right-hand side of (5.4), we expand it in terms of \( \tilde{m}_f(z) - m_f(z) \):

\[
\frac{1}{N} \sum_{i=1}^N \frac{1}{(\vartheta v_i - z - \tilde{m}_f(z))(\vartheta v_i - z - m_f(z))}
\]

\[
= \int \left[ \frac{1}{(\vartheta x - z - m_f(z))^2} + \frac{\tilde{m}_f(z) - m_f(z)}{(\vartheta x - z - m_f(z))^2(\vartheta x - z - \tilde{m}_f(z))} \right] \vartheta \, dx
\]

\[
= \int \frac{1}{(\vartheta x - z - m_f(z))^2} \vartheta \, dx + \frac{\vartheta \tilde{\zeta}^\theta_N(z)}{\sqrt{N}} \int \frac{1}{(\vartheta x - z - m_f(z))^2(\vartheta x - z - \tilde{m}_f(z))} \vartheta \, dx. \quad (5.5)
\]

Substituting, we get

\[
\left( 1 - \int \frac{1}{(\vartheta x - z - m_f(z))^2} \vartheta \, dx \right) \tilde{\zeta}_N(z) = \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - m_f(z)} - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z - m_f(z)} \right] \right]
\]

\[
+ \tilde{\zeta}^\theta_N(z) \left( \int \frac{1}{(\vartheta x - z - m_f(z))^2} \left( \vartheta \tilde{\vartheta}^\theta(x) - \vartheta \vartheta^\theta(x) \right) \right) + \frac{\vartheta \tilde{\zeta}^\theta_N(z)^2}{\sqrt{N}} \int \frac{1}{(\vartheta x - z - m_f(z))^2(\vartheta x - z - \tilde{m}_f(z))} \vartheta \, dx,
\] (5.6)

so that

\[
\left( 1 - \int \frac{1}{(\vartheta x - z - m_f(z))^2} \vartheta \, dx \right) \tilde{\zeta}_N(z) = \frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - m_f(z)} - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z - m_f(z)} \right] \right]
\]

\[
+ \tilde{\zeta}^\theta_N(z) \left( (m_f^\theta(z) + m_f^\theta(z)) - (m_f^\theta(z) + m_f(z)) \right) + \frac{\vartheta \tilde{\zeta}^\theta_N(z)^2}{\sqrt{N}} \int \frac{1}{(\vartheta x - z - m_f(z))^2(\vartheta x - z - \tilde{m}_f(z))} \vartheta \, dx
\] (5.7)

First, we recall the existence of a constant \( C \) such that

\[
\left| 1 - \int \frac{1}{(\vartheta x - z - m_f(z))^2} \vartheta \, dx \right| \geq C \sqrt{\kappa + \eta} \quad \text{for } z = E + i \eta \in \mathcal{D}'
\] (5.8)

uniformly in \( \vartheta \in \Theta_m \). By a standard continuity argument, the bound can be extended to \( z = a \) without changing the constant, so that we may divide the equality (5.6) by the quantity above.
Then \ref{3.32} together with Lemma \ref{3.28} and Cauchy integral formula implies
\[
\left| \tilde{\zeta}_N^\theta (z) \left( \frac{d}{dz} m_\theta^\theta (z + m_\theta^\theta (z)) - \frac{d}{dz} m_\theta^\nu (z + m_\theta^\nu (z)) \right) \right| = O(\vartheta N^{-\frac{1}{2} + 2\epsilon_0 + \epsilon_1})
\] (5.9)
on \Omega.

Again recalling \ref{3.47} and other bounds following it, we have
\[
\left| \int \frac{1}{(\vartheta x - z - m_\theta^\nu (z))^2 (\vartheta x - z - m_\theta^\nu (z))} d\tilde{\nu}(x) \right| \leq C
\] (5.10)
on \Omega, giving
\[
\tilde{\zeta}_N^\theta (z)^2 \sqrt{N} \int \frac{1}{(\vartheta x - z - m_\theta^\nu (z))^2 (\vartheta x - z - m_\theta^\nu (z))} d\tilde{\nu}(x) = O(\vartheta N^{-\frac{1}{2} + 2\epsilon_0 + 2\epsilon_1})
\] (5.11)
on \Omega.

Finally, recalling that
\[
1 - \int \frac{1}{(\vartheta x - z - m_\theta^\nu (z))^2} d\nu(x) = \frac{1}{1 + (m_\theta^\nu)'(z)}
\] (5.12)
we can conclude that on \( \Omega \),
\[
\tilde{\zeta}_N^\theta (z) = (1 + (m_\theta^\nu)'(z)) \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ \frac{1}{\vartheta v_i - z - m_\theta^\nu (z)} - \frac{1}{\vartheta v_i - z - m_\theta^\nu (z)} \right] + O(\vartheta N^{-\frac{1}{2} + 2\epsilon_0 + \epsilon_1}).
\] (5.13)

**Remark 5.1.** Noting that the expression of \( \tilde{\zeta}_N^\theta \) holds also for \( z \in \Gamma_r \cup \Gamma_l \), we remark that
\[
\mathbb{E} \left[ \tilde{\zeta}_N^\theta (z) \chi_\Omega \right] \to 0 \quad \text{for } z \in \Gamma \setminus \Gamma_0,
\] (5.14)
since we have **Remark 4.5** and from Cauchy-Schwarz inequality,
\[
\mathbb{E} \left[ \tilde{\zeta}_N^\theta (z) \chi_\Omega \right] \leq \left( P \left[ \Omega^c \right] \frac{1}{N \vartheta^2} \sum_{i} \mathbb{E} \left[ \left| \vartheta v_i - z - m_\theta^\nu (z) - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z - m_\theta^\nu (z)} \right] \right|^2 \right] \right)^{\frac{1}{2}} + o(1) \to 0.
\] (5.15)

### 5.2 Finite-dimensional convergence

To serve our purpose of proving the finite-dimensional convergence, we assume in this section that we have a fixed number of points \( z_1, \cdots, z_p \) in \( K \).
5.2.1 The case $\vartheta_\infty > 0$

As easily seen in (5.13), the term

$$\frac{1}{\sqrt{N}} \sum_i \left[ \frac{1}{\vartheta} \frac{1}{\vartheta} \right] \left( \frac{1}{\vartheta} \frac{1}{\vartheta} \right) - \mathbb{E} \left[ \frac{1}{\vartheta} \frac{1}{\vartheta} \right] \right]$$

(5.16)

results in the Gaussian convergence. Indeed, if we consider the sum corresponding to (5.16) with $\vartheta$ replaced by $\vartheta_\infty$, from Lemma 3.33 we have

$$\left| \frac{1}{\vartheta} \frac{1}{\vartheta} \right| \left( \frac{1}{\vartheta} \frac{1}{\vartheta} \right) \leq \left| \frac{1}{\vartheta} \frac{1}{\vartheta} \right| \left( \frac{1}{\vartheta} \frac{1}{\vartheta} \right) \leq C \vartheta \vartheta_\infty \left| \vartheta - \vartheta_\infty \right|,$$

(5.17)

where the constant $C$ is chosen uniformly in $x \in \text{supp} \nu$ and $N \in \mathbb{N}$ sufficiently large. Then the central limit theorem together with the fact that $\mathbb{P} \left( \Omega_N \cap \Omega_N' \right) \to 1$ gives the weak convergence of the random vector

$$\left( \tilde{\zeta}_N(z_1), \cdots, \tilde{\zeta}_N(z_p) \right)$$

(5.18)

to a Gaussian random vector with mean zero and covariance matrix

$$\Gamma^0(z_i, z_j) = \vartheta^2 \left( 1 + \frac{d}{dz} m_{\vartheta_\infty} (z_i) \right) \left( 1 + \frac{d}{dz} m_{\vartheta_\infty} (z_j) \right) \left( \Gamma_{\vartheta_\infty} (z_i, z_j) - m_{\vartheta_\infty} (z_i) m_{\vartheta_\infty} (z_j) \right)$$

(5.19)

for $1 \leq i, j \leq p$. Recalling that $\tilde{\zeta}_N^0(z) - \tilde{\zeta}_N^0 = O_p(N^{-\frac{1}{2}} \vartheta^{-1})$, we obtain the same convergence for $(\tilde{\zeta}_N^0(z_1), \cdots, \tilde{\zeta}_N^0(z_p))$.

5.2.2 The case $\vartheta_\infty = 0$

For $\vartheta = 0$, we reduce each summand to as follows:

$$\frac{1}{\vartheta} \frac{1}{\vartheta} = \frac{\vartheta x}{z + m_{\vartheta_\infty}(z)} + \frac{\vartheta^2 x^2}{(\vartheta x - z - m_{\vartheta_\infty}(z))^2}.$$  

Since the first term is a constant, it vanishes after subtracting its expectation. The contribution of the last term is negligible, using the bound $|z + m_{\vartheta_\infty}(z)| \geq C$ obtained from (5.17), together with the following bound of variance:

$$\mathbb{E} \left[ \left( \frac{1}{\sqrt{N}} \sum_i \left( \frac{\vartheta^2 x^2}{(\vartheta x - z - m_{\vartheta_\infty}(z))^2} \right) \right)^2 \right] \leq \vartheta^2 \int \left| \frac{x^2}{\vartheta x - z - m_{\vartheta_\infty}(z)} \right|^2 d\nu(x) \leq C \vartheta^2.$$  

(5.21)

To annihilate the dependence on $\vartheta$, with the aid of Lemma 3.33 we reduce the summand further to

$$\frac{x}{(z + m_{\vartheta_\infty}(z))^2} - \frac{x}{(z + m_{\vartheta_\infty}(z))^2} = \frac{x}{(z + m_{\vartheta_\infty}(z))^2} \frac{(2z + m_{sc}(z) + m_{\vartheta_\infty}(z))}{(z + m_{\vartheta_\infty}(z))^2} = O(\vartheta),$$

(5.22)
so that
\[
\mathbb{E} \left[ \frac{2z + m_{sc}(z) + mf_c(z)}{(z + m_{sc}(z))^2} (m_{sc}(z) - m_{sc}(z)) \right] \leq \frac{1}{\sqrt{N}} \sum_i v_i^2 \leq C \vartheta^2
\] (5.23)

Noting that the terms of variance \( O(\vartheta^2) \) vanishes for \( \vartheta_\infty = 0 \), we may rewrite
\[
\frac{1}{\sqrt{N} \vartheta} \sum_{i=1}^N \left[ \frac{1}{\vartheta v_i - z - m_{sc}^\prime(z)} - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z - m_{sc}^\prime(z)} \right] \right] = - \frac{1}{\vartheta} \left[ \frac{1}{\sqrt{N} \sum_i v_i} + X_N \right] = -m_{sc}(z)^2 \left[ \frac{1}{\sqrt{N} \sum_i v_i} + X_N \right] (5.24)
\]
where \( X \) is a random variable with \( \mathbb{E} \left[ |X|^2 \right] = O(\vartheta^2) \).

As above, by the central limit theorem, the random vector
\[
(\tilde{\xi}_N(z_1), \cdots, \tilde{\xi}_N(z_p)) \tag{5.25}
\]
converges weakly to the Gaussian random vector with mean zero and covariance
\[
\Gamma(z_i, z_j) = (1 + m_{sc}^\prime(z_i))(1 + m_{sc}^\prime(z_j))m_{sc}(z_i)^2 m_{sc}(z_j)^2 \text{Var } [v_1] = m_{sc}(z_i)m_{sc}^\prime(z_j) \text{Var } [v_1]. \tag{5.26}
\]
Finally, by the same reasoning as above, the convergence can be extended to that of \((\xi_N^\vartheta(z_1), \cdots, \xi_N^\vartheta(z_p))\).

### 5.3 Tightness of \( \tilde{\xi}_N^\vartheta(z) \)

In this section, we prove the tightness of \( \{\xi_N(z) : z \in K\} \), which completes the proof of Propositions 3.2 and 5.2. Given the finite-dimensional convergence of \( \xi_N(z) \), the tightness for a fixed point \( z \in K \) directly follows, so that it remains to prove the Hölder condition, as given in Section 4

\[
\frac{1}{N \vartheta^2} \mathbb{E} \left[ |(\text{Tr } R_N(z_1)) - \mathbb{E}[\text{Tr } R_N(z_1)] - (\text{Tr } R_N(z_2)) - \mathbb{E}[\text{Tr } R_N(z_2)]|^2 \right] \leq C |z_1 - z_2|^2 \tag{5.27}
\]
for \( z_1, z_2 \in K \) where \( K \) is a fixed constant independent of \( z_1, z_2 \in K \) and \( N \in \mathbb{N} \).

As in Section 4, we start with an application of the resolvent equation to get

\[
\frac{1}{N \vartheta^2} \mathbb{E} \left[ |(\text{Tr } R_N(z_1)) - \mathbb{E}[\text{Tr } R_N(z_1)] - (\text{Tr } R_N(z_2)) - \mathbb{E}[\text{Tr } R_N(z_2)]|^2 \right] = \frac{1}{N \vartheta^2} |z_1 - z_2|^2 \mathbb{E} \left[ |\text{Tr } R_N(z_1) R_N(z_2) - \mathbb{E}[\text{Tr } R_N(z_1) \text{Tr } R_N(z_2)]|^2 \right]. \tag{5.28}
\]

The proof of the corresponding bound is also given in Appendix C.

**Lemma 5.2.** For \( z_1, z_2 \in K \) and \( N \in \mathbb{N} \) sufficiently large, we have
\[
\frac{1}{N \vartheta^2} \mathbb{E} \left[ |\text{Tr } R_N(z_1) R_N(z_2) - \mathbb{E}[\text{Tr } R_N(z_1) R_N(z_2)]|^2 \right] \leq K, \tag{5.29}
\]
where \( K > 0 \) is a constant independent of \( z_1, z_2 \in K \) and \( N \in \mathbb{N} \) sufficiently large.
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A Mean function $b(z)$

As mentioned in Section 4 in this section we suppose that $V$ is deterministic and calculate the limiting formula of the mean $\mathbb{E}[\xi_N(z)]$. First, in Section A.2–A.3 we reduce the mean of $\xi_N(z)$ to a form depending on $\vartheta$. Then, we repeatedly use Lemma 3.33 to replace $\vartheta$ with its limit $\vartheta_\infty$. Throughout the primary simplification, all the bounds we require are irrelevant of $\vartheta$, therefore we drop the superscript $\vartheta$ in Sections A.1–A.3. Note that we are assuming that $N$ is sufficient large so that $\mathbb{P}[\Omega_N] = 1$.

Set

$$b_N(z) := \mathbb{E}[\xi_N(z)] = N\mathbb{E}[m_N(z) - \hat{m}_{fc}(z)].$$

Then, letting

$$Q_i := -W_{ii} + \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi},$$

we get

$$R_{ii} = \frac{1}{-z - Q_i} = \frac{1}{-z - (\hat{m}_{fc}(z) - v_i)} + \frac{Q_i - (\hat{m}_{fc}(z) - v_i)}{(-z - (\hat{m}_{fc}(z) - v_i))^2} \left(\frac{Q_i - (\hat{m}_{fc}(z) - v_i)}{(-z - (\hat{m}_{fc}(z) - v_i))^3}ight)^3 + \frac{1}{-z - Q_i} \left(\frac{Q_i - (\hat{m}_{fc}(z) - v_i)}{(-z - (\hat{m}_{fc}(z) - v_i))^3}ight)^3.$$

By the local law $|R_{ii} - \hat{g}_i(z)| = O(N^{-\frac{1}{2}})$,

$$Q_i - (\hat{m}_{fc}(z) - v_i) = -\frac{1}{R_{ii}} - z - \hat{m}_{fc}(z) + v_i = -\frac{1}{\hat{g}_i(z)} + 1 = O(N^{-\frac{1}{2}}),$$

which implies

$$b_N(z) = \mathbb{E} \left[\sum_{i=1}^{N} R_{ii} - \hat{g}_i(z)\right] = \sum_{i=1}^{N} \mathbb{E} \left[\hat{g}_i(z)^2(Q_i - (\hat{m}_{fc}(z) - v_i)) + \hat{g}_i(z)^3(Q_i - (\hat{m}_{fc}(z) - v_i))^2\right] + O(N^{-\frac{1}{2} + \epsilon})$$

(A.5)

A.1 $\mathbb{E}[Q_i - (\hat{m}_{fc}(z) - v_i)]$

By the definition of $Q_i$,

$$\mathbb{E}[Q_i - \hat{m}_{fc}(z) + v_i] = \mathbb{E} \left[-W_{ii} + \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi}\right] - \hat{m}_{fc}(z) + v_i = \frac{1}{N} \mathbb{E} \left[\sum_{p} R_{pp}^{(i)}\right] - \hat{m}_{fc}(z) \quad (A.6)$$
On the other hand, \( (3.37) \) together with the local law gives

\[
\sum_p \left( R^{(i)}_{pp} - R_{pp} \right) = - \sum_p \frac{R_{pi} R_{ip}}{R_{ii}} = - \sum_p \left( \tilde{g}_i(z)^{-1} R_{pi} R_{ip} + \frac{R_{pi} R_{ip} (\tilde{g}_i(z) - R_{ii})}{R_{ii} \tilde{g}_i(z)} \right) = - \tilde{g}_i(z)^{-1} \sum_p R_{pi} R_{ip} + O(N^{-\frac{3}{4}}),
\]

which in turn implies

\[
\sum_p R^{(i)}_{pp} = Nm_N(z) - R_{ii} - \frac{1}{\tilde{g}_i(z)} ((R^2)_{ii} - R_{ii}^2) + O(N^{-\frac{3}{4}}) = Nm_N(z) - \frac{1}{\tilde{g}_i} (R^2)_{ii} + O(N^{-\frac{3}{4}}),
\]

so that

\[
\sum_i \tilde{g}_i(z)^2 \mathbb{E} [Q_i - \tilde{m}_{fc}(z) + v_i] = \sum_i \tilde{g}_i(z)^2 \left( \mathbb{E} \left[ \frac{1}{N} \sum_p R_{pp} \right] - \tilde{m}_{fc}(z) \right)
= \frac{1}{N} \sum_i \tilde{g}_i(z)^2 \left[ Nm_N(z) - \frac{1}{\tilde{g}_i} (R^2)_{ii} \right] - \tilde{m}_{fc}(z) \sum_i \tilde{g}_i(z)^2 + O(N^{-\frac{3}{4} + \epsilon})
= b_N(z) \frac{1}{N} \left( \sum_i \tilde{g}_i(z)^2 \right) - \frac{1}{N} \mathbb{E} \left[ \sum_i \tilde{g}_i(z) (R^2)_{ii} \right] + O(N^{-\frac{3}{4} + \epsilon}).
\]

Considering \( (R^2)_{ii} \), we have

\[
R_{ij} = \langle e_i, Re_j \rangle = \sum_\alpha \frac{1}{\lambda_\alpha - z} \langle e_i, v_\alpha \rangle \langle v_\alpha, e_j \rangle
\]

where \( v_\alpha \) denotes the eigenvector of \( W \) corresponding to \( \lambda_\alpha \). Thus

\[
(R^2)_{ij} = \sum_k R_{ik} R_{kj} = \sum_k \left( \sum_\alpha \frac{1}{\lambda_\alpha - z} \langle e_i, v_\alpha \rangle \langle v_\alpha, e_k \rangle \right) \left( \sum_\beta \frac{1}{\lambda_\beta - z} \langle e_k, v_\beta \rangle \langle v_\beta, e_j \rangle \right)
= \sum_{\alpha, \beta} \frac{\langle e_i, v_\alpha \rangle \langle v_\beta, e_j \rangle}{(\lambda_\alpha - z)(\lambda_\beta - z)} \sum_k \langle v_\alpha, e_k \rangle \langle e_k, v_\beta \rangle = \sum_{\alpha, \beta} \frac{\langle e_i, v_\alpha \rangle \langle v_\beta, e_j \rangle}{(\lambda_\alpha - z)(\lambda_\beta - z)} \delta_{\alpha, \beta} = \sum_\alpha \frac{\langle e_i, v_\alpha \rangle \langle v_\alpha, e_j \rangle}{(\lambda_\alpha - z)^2} = \frac{d}{dz} R_{ij}(z)
\]

Then the local law \( |R_{ii} - \tilde{g}_i| \lesssim N^{-\frac{3}{4}} \) together with the Cauchy integral formula (applied on a small contour of length \( N^{-\delta} \) enclosing \( z \)) gives

\[
| (R^2)_{ii} - \frac{d}{dz} \tilde{g}_i | \lesssim N^{-\frac{3}{4} + \delta}.
\]

Plugging this in, we get

\[
\frac{1}{N} \mathbb{E} \left[ \sum_i \tilde{g}_i (R^2)_{ii} \right] = \frac{1}{N} \sum_i \tilde{g}_i \frac{d}{dz} \tilde{g}_i + O(N^{-\frac{3}{4} + \delta + \epsilon}).
\]
On the other hand,

\[
\frac{1}{N} \sum_i \hat{g}_i \frac{d}{dz} \hat{g}_i = \frac{1}{2} \frac{d}{dz} \left( \frac{1}{N} \sum_i \hat{g}_i^2 \right) = \frac{1}{2} \frac{d}{dz} \int_{\mathbb{R}} \frac{1}{(x-z-\hat{m}_{fc}(z))^2} d\tilde{\nu}(x) = \frac{1}{2} \frac{d}{dz} \left( \frac{\hat{m}_{fc}'(z)}{1+\hat{m}_{fc}'(z)} \right) = \frac{1}{2} \frac{\hat{m}_{fc}''(z)}{(1+\hat{m}_{fc}'(z))^2} = \frac{1}{2} \frac{m_{fc}''(z)}{(1+m_{fc}'(z))^2} + O(\vartheta N^{-\alpha_0}). \tag{A.14}
\]

Combining the results above, we get

\[
\sum_i \mathbb{E} \left[ \hat{g}_i(z)^2 (Q_i - \hat{m}_{fc}(z) + v_i) \right] = b_N(z) \frac{m_{fc}'(z)}{1+m_{fc}'(z)} - \frac{1}{2} \frac{m_{fc}''(z)}{(1+m_{fc}'(z))^2} + O(\vartheta N^{-\alpha_0} + N^{-\frac{1}{2} + \delta + \varepsilon}). \tag{A.15}
\]

\section*{A.2 \( \mathbb{E} \left[ (Q_i - (\hat{m}_{fc}(z) - v_i))^2 \right] \)}

\[
\mathbb{E} \left[ (Q_i - (\hat{m}_{fc}(z) - v_i))^2 \right] = \mathbb{E} \left[ \left( -\frac{1}{\sqrt{N}} A_{it} + \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi} - \hat{m}_{fc}(z) \right)^2 \right]
\]

\[
= \hat{m}_{fc}(z)^2 + \frac{w_2}{N} + \mathbb{E} \left[ \left( \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi} \right)^2 \right] - 2\hat{m}_{fc}(z) \mathbb{E} \left[ \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi} \right]
\]

\[
= \hat{m}_{fc}^2 - 2\hat{m}_{fc} \mathbb{E} \left[ m_{N}^{(i)} \right] + \frac{w_2}{N} + \mathbb{E} \left[ \left( \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi} \right)^2 \right]. \tag{A.16}
\]

Expanding the last term,

\[
\mathbb{E} \left[ \left( \sum_{p,q} W_{ip} R_{pq}^{(i)} W_{qi} \right)^2 \right] = \sum_{p,q,r,t} E \left[ W_{ip} W_{qi} W_{ir} W_{ti} R_{pq}^{(i)} R_{rt}^{(i)} \right] = \sum_{p,q,r,t} E \left[ W_{ip} W_{qi} W_{ir} W_{ti} \right] E \left[ R_{pq}^{(i)} R_{rt}^{(i)} \right] \tag{A.17}
\]

As \( E \left[ W_{ip} W_{qi} W_{ir} W_{ti} \right] \neq 0 \) implies \( |\{p, q, r, t\}| = 1 \) or each index is repeated twice, we separate into two cases.

(i) \( |\{p, q, r, t\}| = 1 \).

\[
\sum_{p} E \left[ W_{ip}^4 \right] E \left[ (R_{pp}^{(i)})^2 \right] = \frac{W_4}{N^2} \sum_{p} \sum_{p} E \left[ (R_{pp}^{(i)})^2 \right] = \frac{W_4}{N^2} \sum_{p} \hat{g}_p(z)^2 + O(N^{-\frac{1}{2} + \varepsilon}) \tag{A.18}
\]

(ii) \( |\{p, q, r, t\}| = 2 \).

(a) For \( p = q \neq r = t \): Since

\[
R_{pp}^{(i)} - \hat{g}_p(z) = R_{pp} - \hat{g}_p(z) + \frac{R_{pi} R_{ip}}{R_{ii}} = O(N^{-\frac{1}{2}}), \tag{A.19}
\]
we get

\[
\sum_{p \neq r}^{(i)} \mathbb{E} \left[ W_{ip}^2 W_{ir}^2 \right] \mathbb{E} \left[ R_{pp}^{(i)} R_{rr}^{(i)} \right] = \frac{1}{N^2} \mathbb{E} \left[ \sum_{p \neq r}^{(i)} R_{pp} R_{rr} \right] = \frac{1}{N^2} \mathbb{E} \left[ \sum_{p}^{(i)} R_{pp} (N m_N^{(i)} - R_{pp}) \right] = \mathbb{E} \left[ (m_N^{(i)})^2 \right] - \frac{1}{N^2} \sum_p (\hat{g}_p(z))^2 + O(N^{-\frac{2}{3} + \epsilon}). \quad (A.20)
\]

(b) For \( p = t \neq q = r \):

\[
\sum_{p \neq r}^{(i)} \mathbb{E} \left[ W_{ip}^2 W_{ir}^2 \right] \mathbb{E} \left[ (R_{pp}^{(i)})^2 \right] = \frac{1}{N^2} \mathbb{E} \left[ \sum_{p \neq r}^{(i)} (R_{pp}^{(i)})^2 \right] = \frac{1}{N^2} \left( \mathbb{E} \left[ \text{Tr}(R^{(i)})^2 \right] - \mathbb{E} \left[ \sum_{p}^{(i)} (R_{pp}^{(i)})^2 \right] \right) = \frac{1}{N} (m_N^{(i)})^2 - \frac{1}{N^2} \sum_p \hat{g}_p(z)^2 + O(N^{-\frac{2}{3} + \epsilon}), \quad (A.21)
\]

where we have used (A.19) in the third equality and

\[
m_N^{(i)} - \hat{m}_{fc} = \frac{1}{N} \sum_p^{(i)} (R_{pp}^{(i)} - R_{pp}) - \frac{1}{N} R_{ii} + (m_N - \hat{m}_{fc}) = O(N^{-1}) \quad (A.22)
\]

together with the Cauchy integral formula in the last equality.

(c) For \( p = r \neq q = t \):

By symmetry, the result is same as above.

Hence we get

\[
\mathbb{E} \left[ \left( \sum_{p \neq q}^{(i)} W_{ip} R_{pq}^{(i)} W_{qt} \right)^2 \right] = \frac{W_4}{N^2} \sum_p \hat{g}_p^2 + \frac{2}{N} \hat{m}_{fc}' - \frac{3}{N^2} \sum_p \hat{g}_p^2 + \mathbb{E} \left[ (m_N^{(i)})^2 \right] + O(N^{-\frac{2}{3} + \epsilon})
\]

\[
= \mathbb{E} \left[ (m_N^{(i)})^2 \right] + \frac{W_4 - 3}{N^2} \sum_p \hat{g}_p^2 + \frac{2}{N} \hat{m}_{fc}' + O(N^{-\frac{2}{3} + \epsilon}) \quad (A.23)
\]

\textit{Remark A.1.} For complex Hermitian \( A \) where we assume the independence of the real and the complex entries and \( \mathbb{E} \left[ A_{ij}^2 \right] = 0 \) for \( i \neq j \), (ii)-(c) leads to 0.

Summing up, we get from \( |m_N^{(i)} - \hat{m}_{fc}| \ll N^{-1} \)

\[
\mathbb{E} \left[ (Q_i - \hat{m}_{fc}(z) + v_i)^2 \right] = \hat{m}_{fc}'^2 - 2 \hat{m}_{fc} \mathbb{E} \left[ m_N^{(i)} \right] + \frac{w_2}{N} + \mathbb{E} \left[ (m_N^{(i)})^2 \right] + \frac{W_4 - 3}{N^2} \sum_p \hat{g}_p^2 + 2 \hat{m}_{fc}' + O(N^{-\frac{2}{3} + \epsilon})
\]

\[
= \frac{1}{N} (w_2 + 2 \hat{m}_{fc}') + \frac{W_4 - 3}{N^2} \sum_p \hat{g}_p^2 + O(N^{-\frac{2}{3} + \epsilon}). \quad (A.24)
\]

Therefore
\[
\sum_i \hat{g}_i^3 \mathbb{E} [(Q_i - \hat{m}_{fc} + v_i)^2] = \left( \frac{1}{N} \sum_i \hat{g}_i^3 \right) (w_2 + 2\hat{m}_{fc}) + (W_4 - 3) \left( \frac{1}{N} \sum_i \hat{g}_i^3 \right) \left( \frac{1}{N} \sum_p \hat{g}_p^3 \right). \quad (A.25)
\]

On the other hand, Corollary A.3.1 implies
\[
\frac{1}{N} \sum_i \hat{g}_i^3 = \int_{\mathbb{R}} \frac{1}{(x - z - \hat{m}_{fc}(z))^3} \hat{\nu}(x) = \frac{\hat{m}_{fc}'(z)}{1 + \hat{m}_{fc}(z)} = \frac{m_{fc}'(z)}{1 + m_{fc}(z)} + O(\partial N^{-\alpha_0}). \quad (A.26)
\]

Similarly, differentiating (A.26) we have
\[
\frac{1}{N} \sum_i \hat{g}_i^3 = \int_{\mathbb{R}} \frac{1}{(x - z - \hat{m}_{fc}(z))^3} \hat{\nu}(x) = \frac{1}{2} \frac{\hat{m}_{fc}''(z)}{(1 + \hat{m}_{fc}(z))^3} = \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^3} + O(\partial N^{-\alpha_0}). \quad (A.27)
\]

A.3 The mean \( b(z) \)

Summing up the results, we get
\[
\begin{align*}
b_N(z) &= \sum_{i=1}^{N} \mathbb{E} [\hat{g}_i(z)^2 (Q_i - (\hat{m}_{fc}(z) - v_i))] + \hat{g}_i(z)^3 (Q_i - (\hat{m}_{fc}(z) - v_i))^2] + O(N^{-\frac{1}{2} + \epsilon}) \\
&= b_N(z) \frac{m_{fc}'(z)}{1 + m_{fc}(z)} - \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^2} + \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^3} (w_2 + 2m_{fc}(z)) \\
&\quad + \frac{W_4 - 3}{2} \frac{m_{fc}(z)}{1 + m_{fc}(z)} + O(\partial N^{-\alpha_0} + N^{-\frac{1}{2} + \epsilon}), \quad (A.28)
\end{align*}
\]

hence
\[
\begin{align*}
b_N(z) &= \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^2} (w_2 - 1) + \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^3} (w_2 + 2m_{fc}(z)) + \frac{W_4 - 3}{2} \frac{m_{fc}(z)m_{fc}'(z)}{(1 + m_{fc}(z))^3} + O(\partial N^{-\alpha_0} + N^{-\frac{1}{2} + \epsilon}) \\
&= \frac{1}{2} \frac{m_{fc}''(z)}{(1 + m_{fc}(z))^2} \left[ (w_2 - 1) + (m_{fc}'(z) + (W_4 - 3) \frac{m_{fc}'(z)}{1 + m_{fc}(z)} \right] + O(\partial N^{-\alpha_0} + N^{-\frac{1}{2} + \epsilon}). \quad (A.29)
\end{align*}
\]

A.3.1 \( \partial_{\infty} > 0 \)

Now re-writing the dependence on \( \partial \), we have
\[
\begin{align*}
b_N^*(z) &= \frac{1}{2} \frac{(m_{fc}''(z))'}{(1 + (m_{fc}')^2(z))^2} \left[ (w_2 - 1) + (m_{fc}')'(z) + (W_4 - 3) \frac{(m_{fc}')'(z)}{1 + (m_{fc})'z} \right] + O(\partial N^{-\alpha_0} + N^{-\frac{1}{2} + \epsilon}). \quad (A.30)
\end{align*}
\]

Using Lemma 3.33 it suffices to prove that \( 1 + \frac{d}{dz} m_{fc}(z) \) is lower bounded, uniformly in \( \partial \). Recalling the self-consistent equation (2.49), we write
\[
\begin{align*}
\frac{d}{dz} m_{fc}(z) &= \int_{\mathbb{R}} \frac{1 + \frac{d}{dz} m_{fc}(z)}{(\partial x - z - m_{fc}(z))^2} \hat{\nu}(x), \quad (A.31)
\end{align*}
\]

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so that
\[
|1 + \frac{d}{dz} m_{fc}^\vartheta(z)| = \left| 1 + \left(1 - \int_\mathbb{R} \frac{1}{(\vartheta x - z - m_{fc}^\vartheta(z))^2} d\nu(x)\right)^{-1} \int_\mathbb{R} \frac{1}{(\vartheta x - z - m_{fc}^\vartheta(z))^2} d\nu(x) \right| = \left| 1 - \int_\mathbb{R} \frac{1}{(\vartheta x - z - m_{fc}^\vartheta(z))^2} d\nu(x) \right|^{-1}. \tag{A.32}
\]

Then (3.48) implies the required lower bound, which together with several applications of Cauchy integral proves that
\[
b_N^\vartheta(z) = b^{\vartheta_\infty}(z) + O(|\vartheta - \vartheta_\infty| + N^{-\alpha_0} + N^{-\frac{1}{4} + \epsilon}), \tag{A.33}
\]
where
\[
b^{\vartheta_\infty}(z) := \frac{(m_{fc}^{\vartheta_\infty})'(z)}{2(1 + (m_{fc}^{\vartheta_\infty})'(z))^2} \left[(w_2 - 1) + (m_{fc}^{\vartheta_\infty})'(z) + (W_4 - 3)\frac{(m_{fc}^{\vartheta_\infty})'(z)}{1 + (m_{fc}^{\vartheta_\infty})'(z)}\right]. \tag{A.34}
\]

**Remark A.2.** If we consider $V = 0$, then we have $\rho_{fc} = \hat{\rho}_{fc} = \rho_{sc}$ and $m_{fc} = m_{fc} = g_i = m_{sc}$. In this case, the self-consistent equation $z + m_{sc}(z) = -\frac{1}{m_{sc}(z)}$ gives
\[
m_{sc}(z)^2 = \frac{1}{(z + m_{sc}(z))^2} = \frac{m_{sc}'(z)}{1 + m_{sc}'(z)} \tag{A.35}
\]
and similarly differentiating the second equality in (A.33) implies
\[
m_{sc}(z)^3 = \frac{1}{2} \frac{m_{sc}''(z)}{(1 + m_{sc}(z))^3} \tag{A.36}
\]
so that
\[
b_N(z) \to m_{sc}(z)^3(1 + m_{sc}''(z))((w_2 - 1) + m_{sc}'(z) + (W_4 - 3)m_{sc}(z)^2), \tag{A.37}
\]
which is given in Proposition 3.1 of [4].

**A.3.2 $\vartheta = o(1)$**

To handle with the case $\vartheta = o(1)$, we rewrite the result above with the superscript $\vartheta$ recaptured:
\[
b_N^\vartheta(z) = \frac{1}{2} \frac{(m_{fc}^\vartheta)'(z)}{(1 + (m_{fc}^\vartheta)'(z))^2} \left[(w_2 - 1) + (m_{fc}^\vartheta)'(z) + (W_4 - 3)\frac{(m_{fc}^\vartheta)'(z)}{1 + (m_{fc}^\vartheta)'(z)}\right] + O(\vartheta N^{-\alpha_0} + N^{-\frac{1}{4} + \epsilon}), \tag{A.38}
\]
uniformly for $z \in \Gamma \setminus \Gamma_0$.

Recalling Lemma 5.5.3 again the Cauchy integral formula gives the same bound for $|m_{fc}' - m_{sc}'|$ and $|(m_{fc})'' - m_{sc}''|$. Then from the stability bound 3.48 gives the uniform convergence
\[
b_N^\vartheta(z) = \frac{1}{2} \frac{m_{sc}''(z)}{(1 + m_{sc}(z))^2} \left[(w_2 - 1) + m_{sc}'(z) + (W_4 - 3)\frac{m_{sc}'(z)}{1 + m_{sc}(z)}\right] + O(\vartheta + N^{-\frac{1}{4} + \epsilon})
\]
\[
= m_{sc}(z)^3(1 + m_{sc}'(z))((w_2 - 1) + m_{sc}'(z) + (W_4 - 3)m_{sc}(z)^2) + O(\vartheta + N^{-\frac{1}{4} + \epsilon}), \tag{A.39}
\]
by the preceding remark.
B Covariance function

As indicated in Section 3, this section is devoted to the proof of Lemma 4.6 and 4.7. Similar to the preceding section, we simplify the covariance in Sections B.1–B.4 without tracking the superscript $\vartheta$, as the bounds used along the simplification is independent of $\vartheta$. Also, throughout this section, we assume $z \in \Gamma_u \cup \Gamma_d$.

B.1 Martingale decomposition

The matrix identities (3.37) implies

\[
\text{Tr} \ R - \text{Tr} \ R^{(k)} = R_{kk} + \sum_{i}^{(k)} R_{ik} R_{ki} R_{kk} = R_{kk} + \sum_{i}^{(k)} R_{kk} R_{ik} R_{ki} R_{kk}
\]

\[
= R_{kk} \left( 1 + \sum_{i}^{(k)} \left( - \sum_{p} W_{kp} R_{pq}^{(k)} \right) \right) R_{kk} \left( 1 + \sum_{i}^{(k)} \left( \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} R_{iq}^{(k)} \right) \right)
\]

so that

\[
\zeta_N = \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left( R_{kk} \left( 1 + \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} \right) \right).
\]

Recalling the expansion (A.3), we take

\[
X_k := \left( \hat{g}_k + \hat{g}_k^2 (Q_k - \hat{m}_fc + v_k) \right) \left( 1 + \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} \right)
\]

and

\[
Y_k := R_{kk} \left( \frac{Q_k - \hat{m}_fc + v_k}{z - \hat{m}_fc + v_k} \right) \left( 1 + \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} \right),
\]

so that $X_k + Y_k = \text{Tr} \ R - \text{Tr} \ R^{(k)}$. Using Lemma 3.36 we have

\[
\sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} = \mathcal{O}(1),
\]

so that $Y_k = \mathcal{O}(N^{-1})$.

On the other hand, we observe that for $k > l$,

\[
\mathbb{E} \left[ (\mathbb{E}_{k-1} - \mathbb{E}_k) Y_k \cdot (\mathbb{E}_{l-1} - \mathbb{E}_l) Y_l \right] = \mathbb{E} \left[ (\mathbb{E}_{k-1} - \mathbb{E}_k) Y_k \cdot \mathbb{E}_{k-1} \left[ (\mathbb{E}_{l-1} - \mathbb{E}_l) Y_l \right] \right] = 0,
\]

which in turn gives

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k) Y_k \right)^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{N} \left( (\mathbb{E}_{k-1} - \mathbb{E}_k) Y_k \right)^2 \right] = \mathcal{O}(N^{-1+2c}).
\]
Hence a typical application of Markov inequality implies

\[
\zeta_N = \sum_{k=1}^{N} (E_{k-1} - E_k)X_k + \Theta_p(N^{-\frac{1}{2}}) = \sum_{k=1}^{N} \bar{g}_k(E_{k-1} - E_k) \left[ 1 + \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] + \sum_{k=1}^{N} \bar{g}_k^2(E_{k-1} - E_k) \left( Q_k - \hat{m}_{fc} + \nu_k \right) \left[ 1 + \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] + \Theta_p(N^{-\frac{1}{2}}). \quad \text{(B.8)}
\]

where \( \Theta_p(N^{-\frac{1}{2}}) \) stands for the terms bounded by \( N^{-\frac{1}{2}+\epsilon} \) in probability.

**B.1.1 The first term**

We consider the first term of \( \text{(B.8)} \):

\[
E_k \left[ \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] = \frac{1}{N} \sum_{p} E_k \left[ (R^{(k)})_{pp}^2 \right] = \frac{1}{N} \sum_{p} W_{kp}(R^{(k)})_{pq}^2 W_{kq} = \Theta(N^{-1}). \quad \text{(B.9)}
\]

Hence the first term is given by

\[
(E_{k-1} - E_k) \left[ \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] = E_{k-1} \left[ \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} - \hat{m}_{fc}' \right] + \Theta(N^{-1}). \quad \text{(B.10)}
\]

**B.1.2 The second term**

To calculate the second term, we first observe that the \textit{Lemma B.36} and \textit{A.3} imply

\[
(Q_k - \hat{m}_{fc} + \nu_k) \left[ \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} - \frac{1}{N} \sum_{p} (R^{(k)})_{pp}^2 \right] \sim N^{-1}. \quad \text{(B.11)}
\]

Therefore

\[
(E_{k-1} - E_k) \left[ (Q_k - \hat{m}_{fc} + \nu_k) \left[ 1 + \sum_{p,q} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] \right]
\]

\[
= (E_{k-1} - E_k) \left[ (Q_k - \hat{m}_{fc} + \nu_k) \left[ 1 + \sum_{p} W_{kp}(R^{(k)})_{pq}^2 W_{kq} \right] \right] + \Theta(N^{-1})
\]

\[
= (E_{k-1} - E_k) \left[ (Q_k - \hat{m}_{fc} + \nu_k) \left[ 1 + \hat{m}_{fc}' \right] + \Theta(N^{-1}) \right]. \quad \text{(B.12)}
\]
As above, we first reduce the term concerning $E_k$:

$$
E_k \left[ Q_k - \hat{m}_{fc} + v_k \right] = E_k \left[ -W_{kk} + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} + v_k \right] = E_k \left[ \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} \right]
$$

$$
= \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)} - \hat{m}_{fc} = E_k \left[ m_N^{(k)} - \hat{m}_{fc} \right] = \mathcal{O}(N^{-1}). \quad \text{(B.13)}
$$

Hence the second term is given by

$$
(E_{k-1} - E_k) \left[ (Q_k - \hat{m}_{fc} + v_k) \left( 1 + \sum_{p,q}^{(k)} W_{kp} (R^{(k)})_{pq}^2 W_{qk} \right) \right] = E_{k-1} \left[ Q_k - \hat{m}_{fc} + v_k \right] (1 + \hat{m}'_{fc}) + \mathcal{O}(N^{-1})
$$

$$
= E_{k-1} \left[ -W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} \right] (1 + \hat{m}'_{fc}) + \mathcal{O}(N^{-1}). \quad \text{(B.14)}
$$

### B.2 Simplification

Combining the results above and using the argument of (B.7), we get

$$
\zeta_N = \sum_{k=1}^{N} E_{k-1} \left[ \phi_k \right] + \mathcal{O}_p(N^{-\frac{1}{2}}) \quad \text{(B.15)}
$$

where

$$
\phi_k = \tilde{g}_k \left( \sum_{p,q}^{(k)} W_{kp} (R^{(k)})_{pq}^2 W_{qk} - \hat{m}'_{fc} \right) + \tilde{g}_k^2 \left( -W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} \right) (1 + \hat{m}'_{fc}). \quad \text{(B.16)}
$$

Using the identities $\frac{d}{dz} R_{p,q}^{(k)} = (R^{(k)})_{pq}^2$ and $\tilde{g}_k = (1 + \hat{m}'_{fc}) \tilde{g}_k$, we get

$$
\phi_k = \frac{d}{dz} \left[ \tilde{g}_k \left( -W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} \right) \right]. \quad \text{(B.17)}
$$

Since

$$
- W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \hat{m}_{fc} = Q_k + v_k - \hat{m}_{fc} = \mathcal{O}(N^{-\frac{1}{2}}) \quad \text{(B.18)}
$$

and $\tilde{g}_k(z) \sim 1$ as $z \in \Gamma_u \cup \Gamma_d$, we have $\phi_k \sim N^{-\frac{1}{2}}$.

### B.3 Covariance

Let $z_1, \cdots, z_p \in \Gamma_u$ be distinct points. (Note that by symmetry $\xi_N(z) = \xi_N(-z)$, it suffices to consider points in $\Gamma_u$.) Using the martingale convergence theorem, we prove that the distribution of random vector $(\xi_N(z_1), \cdots, \xi_N(z_p))$ converges weakly to the $p$-dimensional centered Gaussian of covariance matrix given in Proposition 3.1.
For distinct points \( z_1, z_2 \in \Gamma_u \), we let

\[
\Gamma_N(z_1, z_2) = \sum_{k=1}^N E_k \left[ E_{k-1} \left[ \phi_k(z_1) \right] \cdot E_{k-1} \left[ \phi_k(z_2) \right] \right] \tag{B.19}
\]

and

\[
\tilde{\Gamma}_N(z_1, z_2) = \sum_{k=1}^N \hat{g}_k(z_1) \hat{g}_k(z_2) E_k \left[ E_{k-1} \left[ -W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)}(z_1) W_{qk} - \hat{m}_{fc}(z_1) \right] \cdot E_{k-1} \left[ -W_{kk} + v_k + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)}(z_2) W_{qk} - \hat{m}_{fc}(z_2) \right] \right] \tag{B.20}
\]

so that

\[
\Gamma_N(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \tilde{\Gamma}_N(z_1, z_2). \tag{B.21}
\]

### B.4 Reduction of \( \tilde{\Gamma}_N \)

For simplicity, we define

\[
S_k(z) := \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)}(z) W_{qk} \quad \text{and} \quad T_k := -W_{kk} + v_k. \tag{B.22}
\]

Then each summand of (B.20) is

\[
\hat{g}_k(z_1) \hat{g}_k(z_2) E_k \left[ E_{k-1} \left[ T_k + S_k(z_1) - \hat{m}_{fc}(z_1) \right] \cdot E \left[ T_k + S_k(z_2) - \hat{m}_{fc}(z_2) \right] \right]. \tag{B.23}
\]

Using Lemma 3.36, we get

\[
| S_k(z) - \hat{m}_{fc}(z) | \leq \left| \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)}(z) W_{qk} - \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)}(z) \right| + \left| \hat{m}_{fc}(z) - m^{(k)}_N(z) \right| \lesssim N^{-\frac{1}{2}}. \tag{B.24}
\]

On the other hand, we have

\[
E_k \left[ E_{k-1} \left[ T_k \right] E_{k-1} \left[ T_k \right] \right] = \frac{1}{N} E_k \left[ A_{kk}^2 \right] = \frac{\mu_2}{N}, \tag{B.25}
\]

\[
E_k \left[ E_{k-1} \left[ T_k \right] E_{k-1} \left[ S_k(z) - \hat{m}_{fc}(z) \right] \right] = E_k \left[ T_k (S_k(z) - \hat{m}_{fc}(z)) \right] = 0, \tag{B.26}
\]

\[
E_k \left[ E_{k-1} \left[ S_k(z_1) \right] E_{k-1} \left[ \hat{m}_{fc}(z_2) \right] \right] = \hat{m}_{fc}(z_2) E_k \left[ S_k(z_1) \right] = \hat{m}_{fc}(z_2) E_k \left[ m^{(k)}_N(z_1) \right]. \tag{B.27}
\]

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Hence

\[ E_k \left[ E_{k-1} \left[ T_k + S_k(z_1) - \tilde{m}_{fc}(z_1) \right] E_{k-1} \left[ T_k + S_k(z_2) - \tilde{m}_{fc}(z_2) \right] \right] \]

\[ = \frac{w_2}{N} + E_k \left[ E_{k-1} \left[ S_k(z_1) \right] E_{k-1} \left[ S_k(z_2) \right] - \tilde{m}_{fc}(z_1) E_k \left[ m_N^{(k)}(z_2) \right] - \tilde{m}_{fc}(z_2) E_k \left[ m_N^{(k)}(z_1) \right] + \tilde{m}_{fc}(z_1) \tilde{m}_{fc}(z_2) \right] \]

\[ + E_k \left[ E_{k-1} \left[ m_N^{(k)}(z_1) - \tilde{m}_{fc}(z_1) \right] E_{k-1} \left[ m_N^{(k)}(z_2) - \tilde{m}_{fc}(z_2) \right] \right]. \] (B.28)

**B.4.1** \[ E_k \left[ E_{k-1} \left[ S_k(z_1) \right] E_{k-1} \left[ S_k(z_2) \right] \right] \] By the definition of \( S_k \), we get

\[ E_k \left[ E_{k-1} \left[ S_k(z_1) \right] E_{k-1} \left[ S_k(z_2) \right] \right] = \sum_{p, q, r, t}^{(k)} E_k \left[ E_{k-1} \left[ W_{kp} R_{pq}^{(k)}(z_1) W_{qk} \right] E_{k-1} \left[ W_{kr} R_{rt}^{(k)}(z_2) W_{tk} \right] \right]. \] (B.29)

Note that

\[ E_{k-1} \left[ W_{kp} R_{pq}^{(k)} W_{qk} \right] = \begin{cases} W_{kp} W_{qk} E_{k-1} \left[ R_{pq}^{(k)} \right] & \text{if } p, q > k, \\ W_{kp} E \left[ W_{qk} \right] E_{k-1} \left[ R_{pq}^{(k)} \right] = 0 & \text{if } p > k, q < k, \\ E \left[ W_{kp} \right] W_{qk} E_{k-1} \left[ R_{pq}^{(k)} \right] = 0 & \text{if } p < k, q > k, \\ E \left[ W_{kp} W_{qk} \right] E_{k-1} \left[ R_{pq}^{(k)} \right] & \text{if } p, q < k, \end{cases} \] (B.30)

Thus

\[ E_k \left[ E_{k-1} \left[ W_{kp} R_{pq}^{(k)}(z_1) W_{qk} \right] E_{k-1} \left[ W_{kr} R_{rt}^{(k)}(z_2) W_{tk} \right] \right] = \begin{cases} E \left[ W_{kp} W_{qk} W_{kr} W_{tk} \right] E_k \left[ E_{k-1} \left[ R_{pq}^{(k)}(z_1) \right] E_{k-1} \left[ R_{rt}^{(k)}(z_2) \right] \right] & \text{if } p, q, r, t > k \\ 0 & \text{if } (p-k)(q-k) < 0 \text{ or } (r-k)(t-k) < 0, \end{cases} \] (B.31)

Using the result above, we deduce that each summand vanishes if there exists an index repeated only once. Hence, we divide the sum into following:

(i) If \( p = q = r = t \),

\[ \sum_{p}^{(k)} E_k \left[ E_{k-1} \left[ W_{kp} R_{pp}^{(k)}(z_1) W_{pk} \right] E_{k-1} \left[ W_{kp} R_{pp}^{(k)}(z_2) W_{pk} \right] \right] \]

\[ = \frac{W_4}{N^2} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \right] E_{k-1} \left[ R_{pp}^{(k)}(z_2) \right] \right] + \frac{1}{N^2} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \right] E_{k-1} \left[ R_{pp}^{(k)}(z_2) \right] \right] \]

\[ = \sum_{p > k}^{(k)} \frac{W_4}{N^2} \hat{g}_p(z_1) \hat{g}_p(z_2) + \sum_{p < k}^{(k)} \frac{1}{N^2} \hat{g}_p(z_1) \hat{g}_p(z_2) + O(N^{-2}). \] (B.32)
(ii) If \( p = q \neq r = t \),

\[
\frac{1}{N^2} \sum_{p \neq r}^{(k)} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \right] E_{k-1} \left[ R_{rr}^{(k)}(z_2) \right] \right]
\]

\[
= E_k \left[ E_{k-1} \left[ m_N^{(k)}(z_1) \right] E_{k-1} \left[ m_N^{(k)}(z_2) \right] \right] - \frac{1}{N^2} \sum_{p} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \right] E_{k-1} \left[ R_{pp}^{(k)}(z_2) \right] \right]
\]

\[
= E_k \left[ E_{k-1} \left[ m_N^{(k)}(z_1) \right] E_{k-1} \left[ m_N^{(k)}(z_2) \right] \right] - \frac{1}{N^2} \sum_{p} \hat{g}_p(z_1) \hat{g}_p(z_2) + O(N^{-2}). \quad (B.33)
\]

(iii) If \( p = t \neq q = r \),

\[
\sum_{p \neq q}^{(k)} E_k \left[ E_{k-1} \left[ W_{pq} R_{pq}^{(k)}(z_1) W_{qk} \right] E_{k-1} \left[ W_{kq} R_{qp}^{(k)}(z_2) W_{pk} \right] \right]
\]

\[
= \frac{1}{N^2} \sum_{p \neq q, p, q > k} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \sum_a W_{pa} R_{aq}^{(k,p)}(z_1) \right] E_{k-1} \left[ R_{pp}^{(k)}(z_2) \sum_b R_{qb}^{(k,p)}(z_2) W_{bp} \right] \right] =: Z_k. \quad (B.34)
\]

We again use \((3.37)\) to expand

\[
Z_k = \frac{1}{N^2} \sum_{p \neq q, p, q > k} E_k \left[ E_{k-1} \left[ R_{pp}^{(k)}(z_1) \sum_{a} W_{pa} R_{aq}^{(k,p)}(z_1) \right] E_{k-1} \left[ R_{pp}^{(k)}(z_2) \sum_{b} R_{qb}^{(k,p)}(z_2) W_{bp} \right] \right] + O(N^{-\frac{3}{2}})
\]

As \( R_{pp}^{(k)}(z) = \hat{g}_p(z) + O(N^{-\frac{3}{2}}) \) and \( R_{pq}^{(k)}(z) = O(N^{-\frac{3}{2}}) \), we get

\[
Z_k = \frac{1}{N^2} \sum_{p \neq q, p, q > k} \hat{g}_p(z_1) \hat{g}_p(z_2) \sum_{a,b} E_k \left[ E_{k-1} \left[ W_{pa} R_{aq}^{(k,p)}(z_1) \right] E_{k-1} \left[ R_{qb}^{(k,p)}(z_2) W_{bp} \right] \right] + O(N^{-\frac{3}{2}})
\]

\[
= \frac{1}{N^3} \sum_{p \neq q, p, q > k} \hat{g}_p(z_1) \hat{g}_p(z_2) \sum_{a,k} E_k \left[ E_{k-1} \left[ R_{aq}^{(k)}(z_1) \right] E_{k-1} \left[ R_{aq}^{(k)}(z_2) \right] \right] + O(N^{-\frac{3}{2}})
\]

\[
= \frac{1}{N^3} \sum_{p \neq q, p, q > k} \hat{g}_p(z_1) \hat{g}_p(z_2) \sum_{a,k} E_k \left[ E_{k-1} \left[ R_{aq}^{(k)}(z_1) \right] E_{k-1} \left[ R_{aq}^{(k)}(z_2) \right] \right] + O(N^{-\frac{3}{2}})
\]

\[
= \frac{1}{N^3} \sum_{q,a,k} \hat{g}_p(z_1) \hat{g}_p(z_2) \sum_{q,a,k} E_k \left[ E_{k-1} \left[ R_{aq}^{(k)}(z_1) \right] E_{k-1} \left[ R_{aq}^{(k)}(z_2) \right] \right] + O(N^{-\frac{3}{2}})
\]

\[
= \frac{1}{N} \left( \sum_{p \neq k} \hat{g}_p(z_1) \hat{g}_p(z_2) \right) \left( Z_k + \frac{1}{N^2} \sum_{q > k} E_k \left[ E_{k-1} \left[ R_{qq}^{(k)}(z_1) \right] E_{k-1} \left[ R_{qq}^{(k)}(z_2) \right] \right] \right) + O(N^{-\frac{3}{2}})
\]

\[
= \frac{1}{N} \left( \sum_{p \neq k} \hat{g}_p(z_1) \hat{g}_p(z_2) \right) \left( Z_k + \frac{1}{N^2} \sum_{q > k} \hat{g}_q(z_1) \hat{g}_q(z_2) \right) + O(N^{-\frac{3}{2}}). \quad (B.36)
\]
Therefore by denoting \( \frac{1}{N} \sum_{p>k} \hat{g}_p(z_1)\hat{g}_p(z_2) = \hat{I}_k \), we get

\[
(1 - \hat{I}_k)Z_k = \frac{1}{N} \hat{I}_k^2 + O(N^{-\frac{3}{2}}).
\]  

(B.37)

Also Lemma B.3(329) below implies \((1 - \hat{I}_k)^{-1} = O(1)\), hence

\[
Z_k = \frac{1}{N} \frac{\hat{I}_k^2}{1 - \hat{I}_k} + O(N^{-\frac{3}{2}}).
\]  

(B.38)

(iv) If \( p = r \neq q = t \), the symmetry gives

\[
\sum_{p \neq q}^k \mathbb{E}_k \left[ \mathbb{E}_{k-1} \left[ W_{kp} R_{pq}^{(k)}(z_1) W_{qk} \right] \mathbb{E}_{k-1} \left[ W_{kp} R_{pq}^{(k)}(z_2) W_{qk} \right] \right] = Z_k.
\]  

(B.39)

\textbf{Remark B.1.} If \( W \) is complex, then \( \text{iv} \) vanishes.

Adding up, we get

\[
\mathbb{E}_k \left[ \mathbb{E}_{k-1} \left[ T_k + S_k(z_1) - \hat{m}_{fc}(z_1) \right] \mathbb{E}_{k-1} \left[ T_k + S_k(z_2) - \hat{m}_{fc}(z_2) \right] \right]
= \frac{w_2}{N} + \frac{W_4}{N^2} \sum_{p>k} \hat{g}_p(z_1)\hat{g}_p(z_2) + \frac{1}{N} \sum_{p<k} \hat{g}_p(z_1)\hat{g}_p(z_2) - \frac{1}{N} \sum_p \hat{g}_p(z_1)\hat{g}_p(z_2) + \frac{2}{N} \frac{\hat{I}_k^2}{1 - \hat{I}_k} + O(N^{-\frac{3}{2}})
\]

\[
= \frac{w_2}{N} + \frac{W_4}{N} \hat{I}_k + \frac{1}{N} (\hat{I}_0 - \hat{I}_k) - \frac{1}{N} \hat{I}_0 + \frac{2}{N} \frac{\hat{I}_k^2}{1 - \hat{I}_k} + O(N^{-\frac{3}{2}})
\]

\[
= \frac{w_2 - 2}{N} + \frac{W_4 - 3}{N} \hat{I}_k + \frac{2}{N} \frac{1}{1 - \hat{I}_k} + O(N^{-\frac{3}{2}}). \quad (B.40)
\]

\section{Conclusion for \( \Gamma(z_1, z_2) \)}

Now we retrieve the dependence on \( \vartheta \) and fully analyze the covariance. Summing over \( k \), we get

\[
\tilde{\Gamma}^\vartheta_N(z_1, z_2) = (w_2 - 2) \frac{1}{N} \sum_k \tilde{\vartheta}_k^\vartheta(z_1)\tilde{\vartheta}_k^\vartheta(z_2) + (W_4 - 3) \frac{1}{N^2} \sum_{k=1}^N \tilde{\vartheta}_k^\vartheta(z_1, z_2)\tilde{\vartheta}_k^\vartheta(z_1)\tilde{\vartheta}_k^\vartheta(z_2)
\]

\[
+ \frac{2}{N} \sum_{k=1}^N \frac{1}{1 - \tilde{I}_k^\vartheta(z_1, z_2)} \tilde{\vartheta}_k^\vartheta(z_1)\tilde{\vartheta}_k^\vartheta(z_2) + O(N^{-\frac{3}{2}}). \quad (B.41)
\]

For simplicity, we denote \( f_i(x) = (x - z_i - \hat{m}_{fc}(z_i))^{-1} \) and \( \hat{f}_i(x) = (x - z_i - \hat{m}_{fc}(z_i))^{-1} \) for \( x \in \mathbb{R} \).

\subsection{The first term}

The convergence of the first term follows directly from Lemma B.3(37)

\[
(w_2 - 2) \frac{1}{N} \sum_k \tilde{\vartheta}_k^\vartheta(z_1)\tilde{\vartheta}_k^\vartheta(z_2) = (w_2 - 2) \tilde{I}_0^\vartheta(z_1, z_2) = (w_2 - 2) I^\vartheta(z_1, z_2) + O(\vartheta N^{-\alpha_0}). \quad (B.42)
\]
B.5.2 The second term

By the definition of $\hat{I}_k$, again by Lemma 3.31 we get

\[
\frac{1}{N} \sum_{k=1}^{N} \hat{I}^0_k(z_1, z_2)\hat{g}^\theta_k(z_1)\hat{g}^\theta_k(z_2) = \frac{1}{N^2} \sum_{p>k} \hat{g}^\theta_k(z_1)\hat{g}^\theta_p(z_2)\hat{g}^\theta_p(z_1)\hat{g}^\theta_p(z_2)
\]

\[
= \frac{1}{2N^2} \sum_{p,k} \hat{g}^\theta_k(z_1)\hat{g}^\theta_p(z_2)\hat{g}^\theta_p(z_1)\hat{g}^\theta_p(z_2) - \frac{1}{2N^2} \sum_m (\hat{g}^\theta_m(z_1)\hat{g}^\theta_m(z_2))^2 = \frac{1}{2N^2} \left( \sum_k \hat{g}^\theta_k(z_1)\hat{g}^\theta_k(z_2) \right)^2 + O(N^{-1})
\]

\[
= \frac{1}{2} I^0(z_1, z_2) + O(N^{-1} + \vartheta N^{-\alpha_0}). \quad (B.43)
\]

B.5.3 The third term

To consider the third term, we define a polygonal path $C_N = \bigcup_k C_{k,N}$ connecting $\hat{I}^0_N, \hat{I}^0_{N-1}, \ldots, \hat{I}^0_0$ by

\[
C_{k,N} : [0, 1] \to \mathbb{C}, \quad C_{k,N}(t) = \hat{I}^0_k + \frac{t}{N} \hat{g}^\theta_k(z_1)\hat{g}^\theta_k(z_2).
\]

Then letting $F(z) = (1 - z)^{-1}$, we get

\[
\frac{1}{N} \left[ \frac{1}{1 - \hat{I}^0_k} \hat{g}^\theta_k(z_1)\hat{g}^\theta_k(z_2) \right] = \int_0^1 F(C_{k,N}(0))C_{k,N}'(t) dt = \int_{C_N} F(z) dz + \int_0^1 (F(C_{k,N}(0)) - F(C_{k,N}(t)))C_{k,N}(t) dt.
\]

By Lemma 3.24 we have $|F(C_{k,N}(0)) - F(C_{k,N}(t))| = O(N^{-1})$, so that

\[
\int_0^1 (F(C_{n,k}(t)) - F(C_{n,k}(0)))C_{n,k}(t) dt = O(N^{-2}), \quad (B.46)
\]

hence again by Lemma 3.24

\[
\frac{1}{N} \sum_k \frac{1}{1 - \hat{I}^0_k} \hat{g}^\theta_k(z_1)\hat{g}^\theta_k(z_2) = \int_{C_N} F(z) dz + O(N^{-1}) = -\log(1 - \hat{I}^0_k(z_1, z_2)) + O(N^{-1})
\]

\[
= -\log(1 - I^0(z_1, z_2)) + O(N^{-1} + \vartheta N^{-\alpha_0}). \quad (B.47)
\]

B.5.4 Summary

In summary, we have

\[
\tilde{\Gamma}_N(z_1, z_2) = (w_2 - 2)I^0(z_1, z_2) + \frac{1}{2}(W_4 - 3)I^0(z_1, z_2)^2 - 2\log(1 - I^0(z_1, z_2)) + O(N^{-1} + \vartheta N^{-\alpha_0}) + O(N^{-\frac{3}{2}}). \quad (B.48)
\]

B.6 Proof of Lemma 4.6

B.6.1 $\vartheta_\infty > 0$

For $\vartheta_\infty > 0$, recalling Corollaries 3.24 and 3.31 we have

\[
\tilde{\Gamma}_N(z_1, z_2) = \Gamma_{\vartheta_\infty} + O(N^{-1} + \vartheta N^{-\alpha_0} + |\vartheta - \vartheta_\infty|) + O(N^{-\frac{3}{2}}), \quad (B.49)
\]
where

\[ \tilde{\Gamma}^{0,\infty}(z_1, z_2) = (w_2 - 2)I^{0,\infty}(z_1, z_2) + \frac{1}{2}(W_4 - 3)I^{0,\infty}(z_1, z_2)^2 - 2 \log(1 - I^{0,\infty}(z_1, z_2)). \]  \hspace{1cm} (B.50)

Then by differentiating, again the Cauchy integral formula implies the in probability convergence of \( \Gamma_N \) to \( \Gamma \) defined in Proposition 4.1.

### B.6.2 \( \vartheta = o(1) \)

If \( \vartheta = o(1) \), similarly we reduce (B.48) further, using Corollaries 3.29 and 3.34 to

\[ \tilde{\Gamma}_N(z_1, z_2) = (w_2 - 2)I^\vartheta(z_1, z_2) + \frac{1}{2}(W_4 - 3)I^\vartheta(z_1, z_2)^2 - 2 \log(1 - I^\vartheta(z_1, z_2)) + O(N^{-1} + \vartheta N^{-\alpha_0}) + O(N^{-\frac{1}{2}}) \]

\[ = (w_2 - 2)m_{sc}(z_1)m_{sc}(z_2) + \frac{1}{2}(W_4 - 3)(m_{sc}(z_1)m_{sc}(z_2))^2 - 2 \log(1 - m_{sc}(z_1)m_{sc}(z_2)) + O(\vartheta) + O(N^{-\frac{1}{2}}). \]  \hspace{1cm} (B.51)

Differentiating, we get

\[ \Gamma_N(z_1, z_2) = m'_{sc}(z_1)m'_{sc}(z_2) \left( (w_2 - 2) + 2(W_4 - 3)m_{sc}(z_1)m_{sc}(z_2) + \frac{2}{(1 - m_{sc}(z_1)m_{sc}(z_2))^2} \right) + O(\vartheta) + O(N^{-\frac{1}{2}}), \]  \hspace{1cm} (B.52)

which converges to \( \Gamma(z_1, z_2) \) in probability.

**Remark B.2.** The result above and also coincides precisely with \( \Gamma(z_1, z_2) \) in Proposition 4.1 of [4].

### B.7 Proof of Lemma 4.7

Now it suffices to prove

\[ \sum_k \mathbb{E} \left[ |\mathbb{E}_{k-1}[\phi_k]|^2 \chi_{[\|\mathbb{E}_{k-1}[\phi_k]\| \geq \epsilon]} \right] \to 0 \]  \hspace{1cm} (B.53)

for any fixed \( \epsilon > 0 \). For

\[ \mathbb{E} \left[ |\mathbb{E}_{k-1}[\phi_k]| \chi_{[\|\mathbb{E}_{k-1}[\phi_k]\| \geq \epsilon]} \right] \leq \epsilon^{-2} \mathbb{E} \left[ |\mathbb{E}_{k-1}[\phi_k]|^4 \right], \]  \hspace{1cm} (B.54)

the bound \( \phi_k \prec N^{-\frac{1}{2}} \) gives

\[ \sum_k \mathbb{E} \left[ |\mathbb{E}_{k-1}[\phi_k]|^2 \chi_{[\|\mathbb{E}_{k-1}[\phi_k]\| \geq \epsilon]} \right] \leq \epsilon^{-2} \sum_k \mathbb{E} \left[ |\mathbb{E}_{k-1}[\phi_k]|^4 \right] = O(N^{-1+\epsilon'}) \to 0. \]  \hspace{1cm} (B.55)

### C Tightness of \( \xi_N \)

As mentioned in Section 5.4, this section provides the proofs of Lemmas 3.9 and 5.2 As in the sections above, we omit the superscript \( \vartheta \) to prevent unnecessary complication. Recalling the aim, in order to prove the Hölder conditions, we prove the existence of a constant \( K > 0 \) such that

\[ \mathbb{E} \left[ |\text{Tr} R(z_1)R(z_2) - \mathbb{E}[\text{Tr} R(z_1)R(z_2)]|^2 \right] \leq K, \quad \forall z_1, z_2 \in \mathcal{K} \]  \hspace{1cm} (C.1)
for deterministic $V$ and
\[
\frac{1}{N\vartheta^2} \mathbb{E} \left[ \left| \text{Tr} R_N(z_1) R_N(z_2) - \mathbb{E} \left[ \text{Tr} R_N(z_1) R_N(z_2) \right] \right|^2 \right] \leq K, \quad \forall z_1, z_2 \in \mathcal{K} \quad (C.2)
\]
for random $V$.

### C.1 Proof of Lemma 4.9

To prove the existence of the constant $K$ in (C.1), we again use the $\sigma$-algebra $\mathcal{F}_k$ introduced in Section 4:

\[
\mathbb{E} \left[ \left| \text{Tr} R(z_1) R(z_2) - \mathbb{E} \left[ \text{Tr} R(z_1) R(z_2) \right] \right|^2 \right] = \mathbb{E} \left[ \left| \sum_{k=1}^{N} (E_{k-1} - E_k) \left( \text{Tr} R(z_1) R(z_2) - \text{Tr} R^{(k)}(z_1) R^{(k)}(z_2) \right) \right|^2 \right]. \quad (C.3)
\]

In the following, we denote
\[
R := R(z_1), \quad S := R(z_2) \quad \text{and} \quad R^{(k)} := R^{(k)}(z_1), \quad S^{(k)} := R^{(k)}(z_2)
\]
for simplicity. Then one can observe that
\[
\| R \| \leq \frac{1}{\eta} \leq C \quad (C.5)
\]
for $C \geq c^{-1}$ and similarly
\[
\| S \|, \| R^{(k)} \|, \| S^{(k)} \| \leq C. \quad (C.6)
\]
Now for $i, j \neq k$, (3.37) gives
\[
R_{ij} S_{ji} - R^{(k)}_{ij} S^{(k)}_{ji} = (R_{ij} - R^{(k)}_{ij}) S^{(k)}_j + R^{(k)}(S_{ji} - S^{(k)}_{ji}) + (R_{ij} - R^{(k)}_{ij})(S_{ji} - S^{(k)}_{ji})
\]
\[
= \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{S_{kk}} + \frac{R_{ij} S_{jk} S_{ki}}{S_{kk}} + \frac{R_{ik} R_{kj} S_{jk} S_{ki}}{S_{kk}}. \quad (C.7)
\]

Hence we get
\[
\text{Tr} RS - \text{Tr} R^{(k)} S^{(k)} = \sum_{i,j}^{(k)} (R_{ij} S_{ji} - R^{(k)}_{ij} S^{(k)}_{ji}) + \sum_{j}^{(k)} R_{kj} S_{jk} + \sum_{i}^{(k)} R_{ik} S_{ki} + R_{kk} S_{kk}
\]
\[
= \sum_{i,j}^{(k)} \left( \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{S_{kk}} + \frac{R_{ij} S_{jk} S_{ki}}{S_{kk}} + \frac{R_{ik} R_{kj} S_{jk} S_{ki}}{S_{kk}} \right) + 2 \sum_{i}^{(k)} R_{ki} S_{ik} + R_{kk} S_{kk}, \quad (C.8)
\]
which together with (B.7) implies

\[
\mathbb{E} \left[ | \text{Tr} R(z_1) R(z_2) - \mathbb{E} \left[ R(z_1) R(z_2) \right] |^2 \right]
\]

\[
= \mathbb{E} \left[ \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \sum_{i,j}^{(k)} \left( R_{ik} R_{kj} S_{ji}^{(k)} \frac{R_{ji}}{R_{kk}} + R_{ij} S_{jk} S_{ki} \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{S_{kk}} \right) + \frac{2 (RS)_{kk} - R_{kk} S_{kk}}{S_{kk}} \right]^2
\]

\[
= \mathbb{E} \left[ \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \sum_{i,j}^{(k)} \left( R_{ik} R_{kj} S_{ji}^{(k)} + R_{ij} S_{jk} S_{ki} \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{S_{kk}} \right) + \frac{2 (RS)_{kk} - R_{kk} S_{kk}}{S_{kk}} \right]^2
\]

(C.9)

C.1.1 The first and the second term

We first rewrite

\[
\sum_{i,j}^{(k)} \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{R_{kk}} = \sum_{i,j}^{(k)} \frac{1}{R_{kk}} \left( R_{kk} \sum_{p}^{(k)} W_{pk} \right) \left( \sum_{q}^{(k)} W_{kq} R_{kq}^{(k)} \right) S_{ji}^{(k)}
\]

\[
= \sum_{i,j,p,q}^{(k)} R_{kk} W_{kq} R_{kq}^{(k)} S_{ji}^{(k)} R_{ip} W_{pk} = R_{kk} \sum_{p,q}^{(k)} W_{kq} S_{ji}^{(k)} R_{kq}^{(k)} W_{pk}.
\]

(C.10)

Noting that

\[
(\mathbb{E}_{k-1} - \mathbb{E}_k) \left[ \frac{\tilde{g}_k(z_1)}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] = 0,
\]

(C.11)

from \(|a + b|^2 \leq 2(|a|^2 + |b|^2)\) we get

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{R_{kk}} \right]^2
\]

\[
= \mathbb{E} \left[ \sum_{k=1}^{N} \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left[ R_{kk} \sum_{p,q}^{(k)} W_{kq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{\tilde{g}_k(z_1)}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] \right]^2
\]

\[
\leq 2 \sum_{k=1}^{N} \mathbb{E} \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left[ R_{kk} \sum_{p,q}^{(k)} W_{kq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{\tilde{g}_k(z_1)}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] \right]^2
\]

\[
+ 2 \sum_{k=1}^{N} \mathbb{E} \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left[ R_{kk} \sum_{p,q}^{(k)} W_{kq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{\tilde{g}_k(z_1)}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] \right]^2
\]

\[
\leq 4 \sum_{k=1}^{N} \mathbb{E} \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left[ R_{kk} \sum_{p,q}^{(k)} W_{kq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{\tilde{g}_k(z_1)}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] \right]^2
\]

(C.12)

where we have used the Jensen’s inequality in the third line.
Since \(| R_{kk} \), \(\| R^{(k)} \|, \| S^{(k)} \| \leq C\), from Lemma 3.36 we have

\[
\mathbb{E} \left[ R_{kk} \sum_{p,q}^{(k)} W_{pq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} W_{pk} - \frac{R_{kk}}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] \leq C \frac{\| R^{(k)} S^{(k)} R^{(k)} \|}{N} \leq \frac{C}{N}.
\]

(C.13)

On the other hand, we also have \(\text{tr}(R^{(k)} S^{(k)} R^{(k)}) \leq \| R^{(k)} S^{(k)} R^{(k)} \| \leq C\), so that

\[
\mathbb{E} \left[ \left( \frac{R_{kk}}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} - \frac{\tilde{g}_{k}(z_{1})}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right) \right] \leq C \mathbb{E} \left[ \left| R_{kk} - \tilde{g}_{k}(z_{1}) \right|^2 \right].
\]

(C.14)

Using the expansion (A.3), we get

\[
R_{kk} - \tilde{g}_{k}(z_{1}) = \tilde{g}_{k}(z_{1})^2 (Q_{k}(z_{1}) - \tilde{m}_{fc}(z_{1}) + v_{k}) + \mathcal{O}(N^{-1})
\]

\[
= \tilde{g}_{k}(z_{1})^2 (Q_{k}(z_{1}) - \tilde{m}_{N}^{(k)}(z_{1}) + v_{k}) + \mathcal{O}(N^{-1}).
\]

(C.15)

Now using Lemma 3.36 we get

\[
\mathbb{E} \left[ \left| R_{kk} - \tilde{g}_{k}(z_{1}) \right|^2 \right] \leq C \mathbb{E} \left[ \left| Q_{k}(z_{1}) - \tilde{m}_{N}^{(k)}(z_{1}) + v_{k} \right|^2 \right] = C \mathbb{E} \left[ -N^{-1} A_{kk} + \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)} \right]^2
\]

\[
\leq 2C \left( \frac{1}{N} + \mathbb{E} \left[ \sum_{p,q}^{(k)} W_{kp} R_{pq}^{(k)} W_{qk} - \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)} \right]^2 \right) \leq \frac{C}{N},
\]

(C.16)

so that

\[
\mathbb{E} \left[ \left| \frac{R_{kk}}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} - \frac{\tilde{g}_{k}(z_{1})}{N} \sum_{p}^{(k)} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right|^2 \right] \leq \frac{C}{N}.
\]

(C.17)

Altogether, we get a bound for the first term:

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \left( |\mathbb{E}_{k-1} - \mathbb{E}_{k}| \sum_{i,j}^{(k)} \frac{R_{kk} R_{ij} S_{ji}^{(k)}}{R_{kk}} \right|^2 \right] \leq C.
\]

(C.18)

By symmetry, we have the same bound for the second term:

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \left( |\mathbb{E}_{k-1} - \mathbb{E}_{k}| \sum_{i,j}^{(k)} R_{ij}^{(k)} S_{ij} \right|^2 \right] \leq C.
\]

(C.19)
C.1.2 The third term

We first expand the third term using (3.37):

\[
\sum_{i,j}^{(k)} \frac{R_{ik}R_{kj} S_{jk} S_{ki}}{S_{kk}} = R_{kk} S_{kk} \sum_{p,q,r,t}^{(k)} W_{pt} R_{ip} R_{qj} W_{kq} W_{rk} S_{ti} S_{jr} S_{ti} W_{kl}
\]

\[
= R_{kk} S_{kk} \sum_{p,q,r,t}^{(k)} \left( W_{pt} \left( S_{ti} R_{ip} \right) W_{pk} \right) \left( W_{kq} \left( R_{qj} S_{jr} \right) W_{rk} \right)
\]

\[
= R_{kk} S_{kk} \left( \sum_{t,p}^{(k)} W_{kt} \left( S^{(k)} R^{(k)} \right) W_{pk} \right) \left( \sum_{q,r}^{(k)} W_{kq} \left( R^{(k)} S^{(k)} \right) W_{rk} \right)
\]

\[
= R_{kk} S_{kk} \left( \sum_{p,q}^{(k)} W_{kp} \left( R^{(k)} S^{(k)} \right) W_{kq} \right)^2 , \quad (C.20)
\]

since \( R^{(k)} \) and \( S^{(k)} \) commutes. As above, we note that

\[
(\mathcal{E}_{k-1} - \mathcal{E}_k) \left[ \tilde{g}_k(z_1)\tilde{g}_k(z_2) \left( \frac{1}{N} \sum_p^{(k)} (R^{(k)} S^{(k)})_{pp} \right)^2 \right] = 0 , \quad (C.21)
\]

hence

\[
\mathbb{E} \left[ \left( \mathcal{E}_{k-1} - \mathcal{E}_k \right) \left[ \sum_{i,j}^{(k)} R_{ik} R_{kj} S_{jk} S_{ki} \right] \right]^2 
\]

\[
\leq C \mathbb{E} \left[ \left( \sum_{p,q}^{(k)} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{kq} \right)^2 - \left( \frac{1}{N} \sum_p^{(k)} (R^{(k)} S^{(k)})_{pp} \right)^2 \right] + C \mathbb{E} \left[ \left| R_{kk} S_{kk} - \tilde{g}_k(z_1)\tilde{g}_k(z_2) \right|^2 \right] . \quad (C.22)
\]

As above, \( |R_{kk} S_{kk}| , \left| \frac{1}{N} \sum_p^{(k)} (R^{(k)} S^{(k)})_{pp} \right| \leq \| R \| \| S \| \leq C \) gives

\[
\mathbb{E} \left[ \left( \mathcal{E}_{k-1} - \mathcal{E}_k \right) \left[ \sum_{i,j}^{(k)} R_{ik} R_{kj} S_{jk} S_{ki} \right] \right]^2 
\]

\[
\leq C \mathbb{E} \left[ \left( \sum_{p,q}^{(k)} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{kq} \right)^2 \right] + C \mathbb{E} \left[ \left| R_{kk} S_{kk} - \tilde{g}_k(z_1)\tilde{g}_k(z_2) \right|^2 \right] . \quad (C.23)
\]
Using $A^2 - B^2 = (A - B)^2 + 2B(A - B)$ together with Lemma 3.30 we get

\[
E \left[ \left( \sum_{p,q} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{qk} \right)^2 \right] - \left( \frac{1}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \right)^2
\]

\[
= E \left[ \left( \sum_{p,q} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{qk} - \frac{1}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right]
\]

\[
+ \frac{2}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \left( \sum_{p,q} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{qk} - \frac{1}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \right)^2
\]

\[
\leq C \left( E \left[ \sum_{p,q} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{qk} - \frac{1}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \right]^4 \right)
\]

\[
+ E \left[ \sum_{p,q} W_{kp} \left( R^{(k)} S^{(k)} \right)_{pq} W_{qk} - \frac{1}{N} \sum_p \left( R^{(k)} S^{(k)} \right)_{pp} \right]^2 \right) \leq C \left( \sum \sum_{i,j} \sum_{i,j} \frac{R_{ik} R_{kj} S_{jk} S_{kk}}{S_{kk}} \right)^2 \leq C \quad \text{(C.26)}
\]

On the other hand, by (C.11) we have

\[
E \left[ | R_{kk} S_{kk} - \hat{g}_k(z_1) \hat{g}_k(z_2) |^2 \right] = E \left[ | R_{kk} (S_{kk} - \hat{g}_k(z_2)) + \hat{g}_k(z_2) (R_{kk} - \hat{g}_k(z_1)) |^2 \right]
\]

\[
\leq C \left( E \left[ | S_{kk} - \hat{g}_k(z_2) |^2 \right] + E \left[ | R_{kk} - \hat{g}_k(z_1) |^2 \right] \right) \leq C \quad \text{(C.25)}
\]

Altogether, we have a bound of the third term:

\[
E \left[ \sum_{k=1}^N | E_{k-1} - E_k | \sum_{i,j} \frac{R_{ik} R_{kj} S_{jk} S_{kk}}{S_{kk}} \right]^2 \leq C
\]

\[
\text{C.1.3 Remainsders}
\]

We again expand the fourth term:

\[
\sum_{i}^{(k)} R_{ik} S_{ik} = R_{kk} S_{kk} \sum_{i}^{(k)} \left( \sum_{p} W_{kp} R_{pi}^{(k)} \right) \left( \sum_{q} S_{iq} W_{qk} \right)
\]

\[
= R_{kk} S_{kk} \sum_{p,q} W_{kp} \left( \sum_{i}^{(k)} R_{pi}^{(k)} S_{iq}^{(k)} \right) W_{qk} = R_{kk} S_{kk} \sum_{p,q} W_{kp} \left( R_{kk} S_{kk} \right)_{pq} W_{qk}. \quad \text{(C.27)}
\]
By the same argument as above, \(|R_{kk}S_{kk}|, \frac{1}{N} \sum_p (R^{(k)}S^{(k)})_{pp} | \leq C \) implies

\[
E \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) \left( \sum_i (R_{ki}S_{ik}) \right)^2 \right] \\
= E \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) R_{kk} \sum_{p,q} W_{kp} (R^{(k)}S^{(k)})_{pq} W_{qk} - \frac{1}{N} \sum_p (R^{(k)}S^{(k)})_{pp} \right] \\
\leq 2E \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) R_{kk} \sum_{p,q} (R^{(k)}S^{(k)})_{pq} W_{qk} - \frac{1}{N} \sum_p (R^{(k)}S^{(k)})_{pp} \right] \\
+ 2E \left[ \left( \mathbb{E}_{k-1} - \mathbb{E}_k \right) R_{kk} \sum_{p,q} (R^{(k)}S^{(k)})_{pq} W_{qk} - \frac{1}{N} \sum_p (R^{(k)}S^{(k)})_{pp} \right] \\
\leq CE \left[ \sum_{p,q} W_{kp} (R^{(k)}S^{(k)})_{pq} W_{qk} - \frac{1}{N} \sum_p (R^{(k)}S^{(k)})_{pp} \right]^2 + CE \left[ |R_{kk}S_{kk} - \hat{g}_k(z_1)\hat{g}_k(z_2)|^2 \right] \leq \frac{C}{N}.
\]

(C.28)

And similarly we have

\[
E \left[ |(\mathbb{E}_{k-1} - \mathbb{E}_k)[R_{kk}S_{kk}]|^2 \right] = E \left[ |(\mathbb{E}_{k-1} - \mathbb{E}_k)[R_{kk}S_{kk} - \hat{g}_k(z_1)\hat{g}_k(z_2)]|^2 \right] \leq \frac{C}{N}.
\]

(C.29)

Using (C.18), (C.19), (C.26), (C.28) and (C.29), we conclude that the Hölder condition holds, proving the tightness of \(\{\xi_N(z) : z \in \mathcal{K}\}\).

C.2 Proof of Lemma 5.2

The proof of (C.2) follows the same line as Section C.1 except we replace the conditional expectation \(\mathbb{E}_k \cdot \) with

\[
\hat{\mathbb{E}}_k \cdot := E \cdot |\hat{g}_k|,
\]

(C.30)

where \(\mathcal{G}_k\) is defined in Definition 4.2. To be specific, noting that \(R^{(k)}\) is independent of \(\{v_k, W_{k,i} : 1 \leq i \leq N\}\), we write

\[
E \left[ |\text{Tr} R(z_1)R(z_2) - E[\text{Tr} R(z_1)R(z_2)]|^2 \right] \\
= E \left[ \sum_{k=1}^{N} (\hat{\mathbb{E}}_{k-1} - \hat{\mathbb{E}}_k) \left( \text{Tr} R(z_1)R(z_2) - \text{Tr} R^{(k)}(z_1)R^{(k)}(z_2) \right) \right] \\
= E \left[ \sum_{k=1}^{N} (\hat{\mathbb{E}}_{k-1} - \hat{\mathbb{E}}_k) \sum_{i,j} \left( \frac{R_{ik}R_{kj}}{R_{kk}} S_{ij}^{(k)} + \frac{R_{ik}R_{kj} S_{kk}}{S_{kk}} + \frac{R_{ik}R_{kj} S_{jj}}{S_{kk}} + 2(RS)_{kk} - R_{kk} S_{kk} \right) \right]^2.
\]

(C.31)
To bound the first term, we replace the estimate
\[
E \left[ \left| R_{kk}(z) - \hat{g}_k(z) \right|^2 \right] \leq \frac{C}{N}
\] (C.32)
in Section C.4 with
\[
E \left[ \left| R_{kk}(z) + \frac{1}{z + m^{(k)}(z)} \right|^2 \right] \leq C\vartheta^2.
\] (C.33)

To prove the bound above, we first recall the Schur complement formula (3.71):
\[
R_{kk} = \frac{1}{\vartheta v_k + \frac{1}{\sqrt{N}} A_{kk} - z - \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk}}
\] (C.34)
so that
\[
R_{kk} + \frac{1}{z + m^{(k)}(z)} = \frac{R_{kk}}{z + m^{(k)}(z)} \left( \vartheta v_k + \frac{1}{\sqrt{N}} A_{kk} + \left( \frac{1}{N} \sum_{p} R_{pp}^{(k)} - \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} \right) \right).
\] (C.35)

On the other hand, \( z \in K \) implies
\[
\left| \frac{R_{kk}}{z + m^{(k)}(z)} \right| = \eta^{-2} = O(1),
\] (C.36)
and by Lemma 3.36 together with the assumption \( \vartheta \gg N^{-\frac{1}{2}} \),
\[
E \left[ \left( \vartheta v_k + \frac{1}{\sqrt{N}} A_{kk} + \left( \frac{1}{N} \sum_{p} R_{pp}^{(k)} - \sum_{p,q} W_{kp} R_{pq}^{(k)} W_{qk} \right) \right)^2 \right] = O(\vartheta^2).
\] (C.37)

By Hölder inequality, with two bounds above, we deduce (C.36).

After the replacement, using the fact that
\[
(\hat{E}_k - \hat{E}_{k-1}) \left[ \frac{1}{-z - m^{(k)}(z)} \cdot \frac{1}{N} \sum_{p} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right] = 0,
\] (C.38)
we obtain
\[
E \left[ \sum_{k=1}^{N} \left( \hat{E}_{k-1} - \hat{E}_k \right) \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj} S_{ji}^{(k)}}{R_{kk}} \right]^2
\]
\[
= E \left[ \sum_{k=1}^{N} \left( \hat{E}_{k-1} - \hat{E}_k \right) \left( R_{kk} \sum_{p,q} W_{pq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{1}{-z - m^{(k)}(z)} \cdot \frac{1}{N} \sum_{p} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right)^2 \right]
\]
\[
\leq 4 N E \left[ R_{kk} \sum_{p,q} W_{pq} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{qp} W_{pk} - \frac{1}{-z - m^{(k)}(z)} \cdot \frac{1}{N} \sum_{p} \left( R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right]^2 \leq C N \vartheta^2,
\] (C.39)
and a similar bound for the second term.

For the third term, we start by noting that

\[
(\hat{E}_k - \hat{E}_{k-1}) \left[ \frac{1}{(-z_2 - m_N^{(k)}(z_2))} \left( \frac{1}{N} \sum_{p} (R^{(k)} S^{(k)})_{pp} \right)^2 \right] = 0 \quad (C.40)
\]

and use the estimate

\[
\mathbb{E} \left[ \left| R_{kk} S_{kk} - \frac{1}{(-z_2 - m_N^{(k)}(z_2))} \right|^2 \right] \leq C \vartheta^2 \quad (C.41)
\]

following from (C.33), to conclude that

\[
\mathbb{E} \left\{ \sum_{k=1}^{N} \left( \hat{E}_{k-1} - \hat{E}_k \right) \sum_{i,j} \frac{R_{ik} R_{kj} S_{jk} S_{ki}}{R_{kk}} \right\}^2 \leq C N \vartheta^2. \quad (C.42)
\]

The corresponding bounds for the other terms follows similarly, essentially by (C.33).

### D Proof of Lemma 3.5

In this section, we omit the superscript \( \vartheta \) for simplicity, as the bounds are uniform over \( \vartheta \).

#### D.1 Proof of (3.11)

Recalling the formula of \( \Gamma(z_1, z_2) \) in Proposition 3.4, we observe from the symmetry \( \bar{\xi}(z) = \xi(\bar{z}) \) that

\[
\mathbb{E} \left[ |\xi - \mathbb{E}[\xi]|^2 \right] = \Gamma(z, \bar{z}) = (w_2 - 2) \int \frac{|1 + m'_{fc}(z)|^2}{|x - z - m_{fc}(z)|^4} d\nu(x)
\]

\[
+ (W_4 - 3) \left[ \left( \int \frac{1}{|x - z - m_{fc}(z)|^2} d\nu(x) \right)^2 \left( \int \frac{|1 + m'_{fc}(z)|^2}{|x - z - m_{fc}(z)|^4} d\nu(x) \right) \right]
\]

\[
\quad + \left( 1 + m'_{fc}(z) \right) \left( \int \frac{1}{|x - z - m_{fc}(z)|^2} d\nu(x) \right)^2 \left( \int \frac{1}{|x - z - m_{fc}(z)|^4} d\nu(x) \right) \right]^2
\]

\[
+ 2 \left( 1 - \int \frac{1}{|x - z - m_{fc}(z)|^2} d\nu(x) \right)^{-2} \left( 1 + m'_{fc}(z) \right) \left( \int \frac{1}{|x - z - m_{fc}(z)|^2} d\nu(x) \right) \left( \int \frac{1}{|x - z - m_{fc}(z)|^4} d\nu(x) \right)
\]

\[
+ 2 \left( 1 - \int \frac{1}{|x - z - m_{fc}(z)|^2} d\nu(x) \right)^{-1} \left( \int \frac{1}{|x - z - m_{fc}(z)|^4} d\nu(x) \right) \left( \int \frac{1 + m'_{fc}(z)}{|x - z - m_{fc}(z)|^4} d\nu(x) \right) \quad (D.1)
\]
Thus, applying Lemma 3.25 repeatedly, we find that var $\xi(z)$ is bounded for $z \in \Gamma_r \cup \Gamma_l \cup \Gamma_0$. By a similar argument, we also have

$$|E[\xi(z)]| = |b(z)| = \frac{1}{2} \left| \frac{m''_{fc}(z)}{(1 + m'_{fc}(z))^2} \right| (w_2 - 1) + m'_{fc}(z) + (W_4 - 3) \frac{m'_{fc}(z)}{1 + m'_{fc}(z)} = O(1). \quad (D.2)$$

Noting that $E \left[ |\xi(z)|^2 \right] = var(\xi(z))$ and $|\Gamma_r \cup \Gamma_l \cup \Gamma_0| \to 0$, we conclude

$$\int_{\Gamma#_r} E \left[ |\xi(\Gamma#(t))|^2 \right] |\Gamma#(t)| \, dt \leq C |\Gamma#| \to 0. \quad (D.3)$$

### D.2 Proof of (3.10)

We fix $\epsilon > 0$ and take $\Omega^{(1)}_N$ to be the event

$$\Omega^{(1)}_N := \left\{ \sup_{z \in \Gamma_0 \cup \Gamma_r \cup \Gamma_l} |m_N(z) - \hat{m}_{fc}(z)| \leq N^{-1+\epsilon} \right\} \quad (D.4)$$

For $z \in \Gamma_0$, (3.55) implies for any large (but fixed) $D > 0$,

$$\int_{\Gamma_0} E \left[ |\xi_N(z)|^2 \chi_{\Omega^{(1)}_N} \right] \, dz \leq N^{\epsilon} |\Gamma_0| \leq N^{\epsilon - \delta} \quad \text{and} \quad \mathbb{P} \left[ \Omega^{(1)}_N \right] \leq N^{-D} \quad (D.5)$$

for sufficiently large $N$, hence (3.10) follows for $\Gamma# = \Gamma_0$ letting $\epsilon = \frac{\delta}{2}$.

Now for $\Gamma_r$, as $\lim_{\infty \to 0^+} |\Gamma_r| = 0$, it suffices to prove $E \left[ |\xi_N(z)|^2 \right] \leq M$ uniformly on $\Gamma_r$, for some ($N$-independent) constant $M$. Recalling the results in Section 4, we had

$$b_N(z) = \frac{1}{2} \frac{m''_{fc}(z)}{(1 + m'_{fc}(z))^2} \left[ (w_2 - 1) + m'_{fc}(z) + (W_4 - 3) \frac{m'_{fc}(z)}{1 + m'_{fc}(z)} \right] + O(N^{-\frac{\epsilon\alpha}{2} + 2\delta \sqrt{k + \eta}}), \quad (D.6)$$

hence $|E \left[ |\xi_N(z)|^2 \right] \leq C$ for $z \in \Gamma_r$.

To bound Var $[\xi_N(z)] = E \left[ |\xi_N(z) - E[\xi_N(z)]|^2 \right] = E \left[ |\xi_N(z)|^2 \right]$ we again use the expansion

$$\zeta_N = \sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k) \left[ R_{kk} \left( 1 + \sum_{p,q} W_{kp} \left( R^{(k)}_{pq} \right)^2 W_{qp} \right) \right]. \quad (D.7)$$

Similar to proofs in Section 4, we use the fact

$$(\mathbb{E}_{k-1} - \mathbb{E}_k) \left[ \hat{g}_{kk} \left( 1 + \frac{1}{N} \sum_{p} \left( R^{(k)}_{pp} \right)^2 \right) \right] = 0, \quad (D.8)$$

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so that

\[
\mathbb{E}[|\zeta_N|]^2 = \mathbb{E}
\left[
\sum_{k=1}^{N} (E_{k-1} - E_k)
\right.
\left[
R_{kk} \left(1 + \sum_{p,q} W_{kp} \left(R^{(k)}_{pq}\right)^2 W_{qk}\right) - \hat{g}_k \left(1 + \frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right)\right]^2
\right]
\leq 4 \sum_{k=1}^{N} \mathbb{E}
\left[
R_{kk} \left(1 + \sum_{p,q} W_{kp} \left(R^{(k)}_{pq}\right)^2 W_{qk}\right) - \hat{g}_k \left(1 + \frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right)\right]^2
\]  

\text{(D.9)}

Defining the event

\[
\Lambda_N := \left[\lambda_N \leq \hat{\gamma}_N + N^{-\frac{1}{4}}\right],
\]  

\text{(D.10)}

we have \(\mathbb{P}[\Lambda_N^c] < N^{-D}\) for any large (but fixed) \(D > 0\) by Lemma 3.22. On the event \(\Lambda_N\), \(|\hat{L}_+ - L_+| \leq N^{-\epsilon_{\lambda_0}}\) gives the bound

\[
|R_{kk}| \leq \|R\| = \max_{1 \leq i \leq N} \left|\frac{1}{\lambda_i - z}\right| \leq \frac{1}{a_+ - \lambda_N} \leq \frac{1}{a_+ - \hat{\gamma}_N - N^{-\frac{1}{4}}} \leq \frac{1}{a_+ - \hat{L}_+ - N^{-\frac{1}{4}}} \leq C,
\]  

\text{(D.11)}

for any \(k = 1, \ldots, N\), uniformly for \(z \in \Gamma_r\). Also the Cauchy interlacing gives the same bound \(\|R^{(k)}\| \leq C\).

Therefore, from Lemma 3.36

\[
\mathbb{E}
\left[
\chi_{\Lambda_N} \left| R_{kk} \left(1 + \sum_{p,q} W_{kp} \left(R^{(k)}_{pq}\right)^2 W_{qk}\right) - R_{kk} \left(1 + \frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right)\right|^2
\right]
\leq C \mathbb{E}
\left[
\left| \sum_{p,q} W_{kp} \left(R^{(k)}_{pq}\right)^2 W_{qk} - \frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right|^2
\right]
\leq C \left\| R^{(k)} \right\|^4 \leq C \left(\frac{N}{N}\right),
\]  

\text{(D.12)}

Similarly, using

\[
\left|\frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right| \leq \left\| R^{(k)} \right\|^2 \leq C
\]  

\text{(D.13)}

and \(\text{(C.16)}\), we also have

\[
\mathbb{E}
\left[
\chi_{\Lambda_N} \left| (R_{kk} - \hat{g}_k) \left(1 + \frac{1}{N} \sum_{p} \left(R^{(k)}_{pp}\right)^2\right)\right|^2
\right]
\leq C \mathbb{E}
\left[
\left| R_{kk} - \hat{g}_k \right|^2
\right] \leq C \left(\frac{N}{N}\right).
\]  

\text{(D.14)}
We apply the trivial bound $\|R\|, \|R^{(k)}\| \leq N^{\delta}$ on $\Lambda_N^*$ so that the Cauchy-Schwarz inequality gives

\[
\begin{align*}
\mathbb{E} \left[ \chi_{\Lambda_N^*} \left| R_{kk} \left( \sum_{p,q} W_{kp} \left( R^{(k)}_{pq} \right)^2 W_{qk} - \frac{1}{N} \sum_{p} \left( R^{(k)}_{pp} \right)^2 \right) \right| ^2 \right] \\
\quad \leq \mathbb{E} \left[ \chi_{\Lambda_N^*} \left| R_{kk} \right|^4 \right]^{\frac{1}{2}} \mathbb{E} \left[ \left| \sum_{p,q} W_{kp} \left( R^{(k)}_{pq} \right)^2 W_{qk} - \frac{1}{N} \sum_{p} \left( R^{(k)}_{pp} \right)^2 \right| ^4 \right]^{\frac{1}{4}} \leq C \mathbb{P} \left[ \Lambda_N^* \right] \| R \|^2 \| R^{(k)} \|^4 N \leq C N^{6\delta - \frac{2}{3} - 1}. \quad (D.15)
\end{align*}
\]

where we used Lemma 6.36 in the second inequality. Similarly,

\[
\begin{align*}
\mathbb{E} \left[ \chi_{\Lambda_N^*} \left| (R_{kk} - \hat{g}_k) \left( 1 + \frac{1}{N} \sum_{p} \left( R^{(k)}_{pp} \right)^2 \right) \right| ^2 \right] \leq \left( 1 + \| R^{(k)} \|^2 \right)^2 \mathbb{E} \left[ \chi_{\Lambda_N^*} \left| R_{kk} - \hat{g}_k \right|^2 \right] \leq N^{4\delta} \mathbb{P} \left[ \Lambda_N^* \right] \| R_{kk} - \hat{g}_k \|^4. \quad (D.16)
\end{align*}
\]

Recalling the series expansion

\[
R_{kk} - \hat{g}_k(z) = \hat{g}_k(z)^2 (Q_k - m^{(k)}_N(z) + v_k) + \hat{g}_k(z) (m^{(k)}_N(z) - \tilde{m}_f(z)) + R_{kk} \left( Q_k - \tilde{m}_f(z) + v_k \right) \leq \hat{g}_k(z)^2 (Q_k - m^{(k)}_N(z) + v_k) + \mathcal{O}(N^{-1}), \quad (D.17)
\]

from the inequality $|a + b|^4 \leq 8(|a|^4 + |b|^4)$ we get

\[
\begin{align*}
\mathbb{E} \left[ \left| R_{kk} - \hat{g}_k \right|^4 \right] \leq C \mathbb{E} \left[ \left| Q_k - m^{(k)}_N(z) + v_k \right|^4 \right] = C \mathbb{E} \left[ -N^{-\frac{\delta}{2}} A_{kk} + \sum_{p,q} W_{kp} R^{(k)}_{pq} W_{qk} - \frac{1}{N} \sum_{p} R^{(k)}_{pp} \right] ^4 \leq C \left( \frac{\| R^{(k)} \|^4 + 1}{N^2} \right) \leq C N^{4\delta - 2}, \quad (D.18)
\end{align*}
\]

hence

\[
\begin{align*}
\mathbb{E} \left[ \chi_{\Lambda_N^*} \left| R_{kk} \left( 1 + \sum_{p,q} W_{kp} \left( R^{(k)}_{pq} \right)^2 W_{qk} \right) - \hat{g}_k \left( 1 + \frac{1}{N} \sum_{p} \left( R^{(k)}_{pp} \right)^2 \right) \right| ^2 \right] \leq C N^{6\delta - \frac{2}{3} - 1}. \quad (D.19)
\end{align*}
\]

Finally, we take $D > 12\delta$ to get

\[
\mathbb{E} \left[ |\xi_N|^2 \right] \leq C \quad \text{hence}
\]

\[
\mathbb{E} \left[ |\xi_N|^2 \right] \leq 2 \left( \mathbb{E} \left[ |\xi_N|^2 \right]^2 + \mathbb{E} \left[ |\zeta_N|^2 \right] \right) \leq C, \quad (D.20)
\]

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concluding the proof of lemma for \( \Gamma_r \). The proof for \( \Gamma_l \) is the same, except we take the event \( \Lambda_N \) as

\[
\Lambda_N := \left[ \lambda_1 \geq \gamma_1 - N^{-\frac{1}{2}} \right]
\]  
(D.21)

and take the bound for \( \Lambda_N \) as

\[
\| R \| \leq \frac{1}{a_- - \lambda_1} \leq \frac{1}{a_- - L_+ - N^{-\frac{1}{2}}}.
\]  
(D.22)

**D.3 Proof of (3.13)**

As we saw above, we have

\[
E \left[ \left| \tilde{\xi}(z) \right|^2 \right] \leq \left| 1 + m'_{fc}(z) \right|^2 \int \frac{1}{\left| v - z - m_{fc}(z) \right|^2} dv(v) \leq C,
\]

uniformly for \( z \in \Gamma \) from (3.37). As \( | \Gamma_0 \cup \Gamma_r \cup \Gamma_l | \to 0 \), (3.13) directly follows.

**D.4 Proof of (3.12)**

We take the event \( \Omega_N^{(2)} \) to be contained in

\[
\left[ \sup_{z \in \Gamma_0 \cup \Gamma_r \cup \Gamma_l} : \left| m_N(z) - m_{fc}(z) \right| \leq N^{-\frac{1}{2} + \epsilon} \right] \cap \max \{ \left| \lambda_1 - \gamma_1 \right|, \left| \lambda_N - \gamma_N \right| \} \leq cN^{-\frac{1}{2} + \epsilon} \right] \cap \Omega_N.
\]  
(D.24)

Then for \( \Gamma_0 \), by Corollary 3.24, we have

\[
\int E \left[ \left| \tilde{\xi}_N(\Gamma_0(t)) \right|^2 \right] | \Gamma_0'(t) | dt \leq N^{\epsilon - \delta},
\]

hence we get the result by taking \( \epsilon < \delta \).

For \( \Gamma_l \), we first observe that from Remark 5.1

\[
\left| E \left[ \tilde{\xi}_N(z) \chi_{\Omega_N^{(2)}} \right] \right| = O(N^{-\frac{1}{2} + \epsilon}),
\]

hence it suffices to prove

\[
E \left[ \left| \tilde{\xi}_N(z) - E \left[ \tilde{\xi}_N(z) \right] \right|^2 \chi_{\Omega_N^{(2)}} \right] \leq K
\]

for some \( z \), \( N \)-independent constant \( K \geq 0 \).

From the definition of \( \Omega_N^{(2)} \), we have

\[
\left| R_{kk} \right| \leq \left\| R \right\| = \max_i \frac{1}{\left| \lambda_i - z \right|} \leq \frac{1}{a_- - \lambda_1} \leq \frac{1}{a_- - \gamma_1 - cN^{-\frac{1}{2} + \epsilon}} \leq \frac{1}{L_+ - a_- - cN^{-\frac{1}{2} + \epsilon}} \leq C
\]

(D.28)
on $\Omega_N^{(2)}$ uniformly for $z \in \Gamma_1$. Then following lines of Section C.2

$$\mathbb{E}\left[ \left| \hat{\xi}_N(z) - \mathbb{E}[\hat{\xi}_N(z)] \right|^2 \right]$$

$$= \mathbb{E}\left[ \sum_k (\hat{\xi}_k - \hat{\xi}_k) \left[ R_{kk} \left( 1 + \sum_{p,q} W_{kp} \left( R^{(k)} \right)_{pq} W_{qk} \right) - \frac{1}{1 - z - m_N^{(k)}(z)} \left( 1 + \frac{1}{N} \sum_p \left( R^{(k)} \right)_{pp} \right) \right]^2 \right]$$

$$\leq C \sum_k \mathbb{E}\left[ R_{kk} \left( \sum_{p,q} W_{kp} \left( R^{(k)} \right)_{pq} W_{qk} - \frac{1}{N} \sum_p \left( R^{(k)} \right)_{pp} \right) + R_{kk} + \frac{1}{z + m_N^{(k)}(z)} \left( 1 + \frac{1}{N} \sum_p \left( R^{(k)} \right)_{pp} \right) \right]^2 \right].$$

(D.29)

Now using Lemma 3.3, 3.33, and (D.28), the result follows from the same argument as in Section D.2.

E Proofs of Lemmas in Section 3.2

In this appendix, we provide the proofs of lemmas that are stated but unproven in Section 3.2.

Proof of Lemma 3.13 Since Assumption 2.5 implies that (3.29) holds with probability $\geq 1 - N^{-2}$, we focus on (3.30) and (3.32).

To bound the probability of the event on which (3.30) does not hold, we first note that

$$|m'_\nu(z)| \leq \int \frac{1}{|x - z|^2} d\nu(x) \leq \frac{1}{C_1^2}$$

(E.1)

where $C_1 = \text{dist}(D, \text{supp} \nu)$ and similarly $|m'_\nu(z)| \leq C_1^{-2}$, so that

$$|m_\nu(z) - m_\nu(z')| \leq C_1^{-2} |z - z'|.$$

(E.2)

Now we let $Q_N$ be a lattice in $\mathbb{C}^+$ with $|Q_N| \leq C_2 N^{1-\epsilon_0}$ such that for any $z \in D$, $\inf_{z' \in Q_N} |z - z'| \leq C_1^2 N^{-1/2 + \epsilon_0}$ for sufficiently large $N$. Then,

$$\mathbb{P}\left[ \sup_{z \in D} |m_\nu(z) - m_\nu(z)| \geq 3N^{-1/2 + \epsilon_0} \right] \leq \sum_{z \in Q_N} \mathbb{P}\left[ |m_\nu(z) - m_\nu(z)| \geq N^{-1/2 + \epsilon_0} \right].$$

(E.3)

On the other hand, for each fixed $z \in D$, a typical application of the Chernoff inequality implies

$$\mathbb{P}\left[ |m_\nu(z) - m_\nu(z)| \geq N^{-1/2 + \epsilon_0} \right] = \mathbb{P}\left[ \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \frac{1}{v_i - z} - \mathbb{E} \left[ \frac{1}{v_i - z} \right] \right) \right| \geq N^{\epsilon_0} \right] \leq C \exp(-cN^{2\epsilon_0})$$

(E.4)

for some absolute constant $c, C > 0$, hence

$$\mathbb{P}\left[ \sup_{z \in D} |m_\nu(z) - m_\nu(z)| \geq 3N^{-1/2 + \epsilon_0} \right] \leq CN^{1-2\epsilon_0} \exp(cN^{-2\epsilon_0}) \leq N^{-1}.$$
Now we bound the probability of the event on which \(3.32\) fails. We first note that

\[
\vartheta^{-1}(m^\vartheta_p(z) - m^\vartheta_q(z)) = \frac{1}{\vartheta} \sum_{i=1}^{N} \left[ \frac{1}{\vartheta v_i - z} - \mathbb{E} \left[ \frac{1}{\vartheta v_i - z} \right] \right] = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{v_i}{z(\vartheta v_i - z)} - \mathbb{E} \left[ \frac{v_i}{z(\vartheta v_i - z)} \right] \right].
\] (E.6)

Considering the functions

\[
\tilde{F}(\vartheta, z) := \int \frac{x}{z(\vartheta x - z)} d\tilde{\nu}(x) \quad \text{and} \quad F(\vartheta, z) := \int \frac{x}{z(\vartheta x - z)} d\nu(x)
\] (E.7)
defined on \(D \subset \Theta_\infty \times \mathbb{C}^+\), \(F\) is jointly Lipschitz for

\[
|F(\vartheta_1, z_1) - F(\vartheta_2, z_2)| \leq \int \left| \frac{v_i(z_2 \vartheta_2 - z_1 \vartheta_1)}{z_1 z_2 (\vartheta x - z_1) (\vartheta x - z_2)} \right| d\nu(x)
\]

\[
\leq d^{-4} (C^2 |z_2 \vartheta_2 - z_1 \vartheta_1| + C |z_1 - z_2|),
\] (E.8)

where \(d = \inf_{(\vartheta, z) \in D} \text{dist}(z, \text{supp} \nu^\vartheta)\), and similarly \(\tilde{F}\) is also jointly Lipschitz with constant bounded uniformly in \(N\). Thus we conclude that there exists a constant \(C > 0\) independent of \((\vartheta, z) \in D\) and \(N \in \mathbb{N}\) satisfying

\[
\left| \frac{1}{\vartheta} (m^\vartheta_p(z_1) - m^\vartheta_q(z_1)) - \frac{1}{\vartheta_2} (m^\vartheta_p(z_2) - m^\vartheta_q(z_2)) \right| = |\tilde{F}(\vartheta_1, z_1) - \tilde{F}(\vartheta_2, z_2) + F(\vartheta_2, z_2) - F(\vartheta_1, z_1)| \leq C(|z_1 - z_2| + |\vartheta_1 - \vartheta_2|). \quad (E.9)
\]

Now we let \(Q_N\) be a lattice in \(D\) with \(|Q_N| \leq CN^6\) such that for any \((\vartheta, z) \in D, \inf_{(\vartheta', z') \in Q_N} |z - z'| + |\vartheta - \vartheta'| \leq N^{-2}\) for sufficiently large \(N\). Then,

\[
P \left[ \sup_{(\vartheta, z) \in D} \left| \frac{1}{\vartheta} (m^\vartheta_p(z) - m^\vartheta_q(z)) \right| \geq 2N^{-\frac{1}{2} + \epsilon_0} \right] \leq \sum_{(\vartheta, z) \in Q_N} P \left[ \frac{1}{\vartheta} \left| m^\vartheta_p(z) - m^\vartheta_q(z) \right| \geq N^{-\frac{1}{2} + \epsilon_0} \right]. \quad (E.10)
\]

On the other hand, for each fixed \((\vartheta, z) \in D\), a typical application of McDiarmid’s inequality implies

\[
P \left[ \frac{1}{\vartheta} \left| m^\vartheta_p(z) - m^\vartheta_q(z) \right| \geq N^{-\frac{1}{2} + \epsilon_0} \right] = P \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \frac{v_i}{z(\vartheta v_i - z)} - \mathbb{E} \left[ \frac{v_i}{z(\vartheta v_i - z)} \right] \right) \right. \geq \left. N^{\epsilon_0} \right] \leq C \exp(-cN^{2\epsilon_0}) \quad (E.11)
\]

for some absolute constant \(c, C > 0\), hence

\[
P \left[ \sup_{(\vartheta, z) \in D} \left| \frac{1}{\vartheta} (m^\vartheta_p(z) - m^\vartheta_q(z)) \right| \geq 2N^{-\frac{1}{2} + \epsilon_0} \right] \leq CN^6 \exp(cN^{-2\epsilon_0}) \leq N^{-4}. \quad (E.12)
\]

\underline{Proof of Corollary 3.22} We start with the bound

\[
\left| m^\vartheta_N(z) - m^\vartheta_f(z) \right| \leq \sum_i \frac{1}{N} \frac{1}{\vartheta^\vartheta_i - z} - \int_{\vartheta^\vartheta_{i-1}}^{\vartheta^\vartheta_i} \frac{1}{x - z} d\rho^\vartheta_f(x) + \frac{1}{N} \sum_i \left| \frac{1}{\vartheta^\vartheta_i - z} - \frac{1}{\vartheta^\vartheta_i - z} \right|,
\] (E.13)
where \( \tilde{\gamma}_0^\vartheta = L_0^\vartheta, \tilde{\gamma}_N^\vartheta = L_n^\vartheta \) and \( \int_{\mathbb{R}} \tilde{\gamma}_i^\vartheta d\rho_{fc}(x) = \frac{i}{N} \) for \( 1 \leq i \leq N - 1 \).

From \( \kappa_E + \eta \sim 1 \), we find that

\[
\left| \sum_i \frac{1}{N} \frac{1}{\gamma_i^\vartheta - z} - \int_{\gamma_i^\vartheta}^{\gamma_i^\vartheta} \frac{1}{\gamma_i^\vartheta - z} d\rho_{fc}(x) \right| = \left| \sum_i \int_{\gamma_i^\vartheta}^{\gamma_i^\vartheta} \left( \frac{1}{\gamma_i^\vartheta - z} - \frac{1}{x - z} \right) d\rho_{fc}(x) \right|
\leq \sum_i \int_{\gamma_i^\vartheta}^{\gamma_i^\vartheta} \left| \frac{\gamma_i^\vartheta - x}{(x - z)(\gamma_i^\vartheta - z)} \right| d\rho_{fc}(x) \leq \kappa_E^{-2} \sum_i \left| \gamma_i^\vartheta - \gamma_i^{-1} \right| \int_{\gamma_i^\vartheta}^{\gamma_i^\vartheta} d\rho_{fc}(x) = \frac{L_0^\vartheta - L_n^\vartheta}{\kappa_E^2 N} = O(N^{-1}).
\] (E.14)

On the other hand, using the rigidity estimate \( (3.43) \) we get the bound

\[
\frac{1}{N} \sum \left| \frac{1}{\gamma_i^\vartheta - z} - \frac{1}{\lambda_i^\vartheta - z} \right| \leq \kappa_E (\kappa_E - \max(\left| \gamma_i^\vartheta - \lambda_i^\vartheta \right|, \left| \gamma_i^\vartheta - \lambda_N^\vartheta \right|)) \leq \frac{1}{N} \sum \left| \gamma_i^\vartheta - \lambda_i^\vartheta \right| = \frac{2}{N} \sum_{i=1}^{N} \left( N^{-\frac{3}{4}} i^{-\frac{1}{4}} + cN^{-\frac{3}{4}} i^{-\frac{3}{4}} \right) + \vartheta N^{-\frac{3}{2}} + 2N^{-\frac{3}{2}} e (1 + cN^\frac{1}{2}) = \vartheta N^{-\frac{3}{2}} + \frac{2}{N} \int_0^{\frac{1}{N}} (x^{-\frac{1}{4}} + x^{-\frac{3}{4}}) dx + O(N^{-1}) = O(\vartheta N^{-\frac{3}{2}}),
\] (E.15)

since \( \vartheta \geq CN^{-\frac{1}{2}} \), where we let \( \varepsilon < \frac{1}{12} \).

\[ \square \]

**Proof of Corollary \( (3.22) \)** We omit the superscript \( \vartheta \) for simplicity. For \( z \in \mathcal{D}_c \), the bounds immediately follow from the strong local deformed semicircle laws \( \textbf{Lemma 3.11} \) as \( \eta \sim 1 \). For \( z \in \Gamma_r \cup \Gamma_i \cup \Gamma_0 \), we bound \( \left| m_N(z) - \tilde{m}_{fc}(z) \right| \) by

\[
| m_N(z) - \tilde{m}_{fc}(z) | = \left| \sum_i \left( \frac{1}{\Lambda_i - z} - \frac{1}{\gamma_i - z} \right) d\tilde{\gamma}_i \right| \leq \sum_i \left| \frac{1}{\Lambda_i - z} - \frac{1}{\gamma_i - z} \right| \leq \frac{1}{N} \sum \left( \frac{1}{\Lambda_i - z} - \frac{1}{\gamma_i - z} \right) = \frac{1}{N} \sum \left( \frac{1}{\Lambda_i - z} - \frac{1}{\gamma_i - z} \right),
\] (E.16)

where \( \tilde{\Lambda}_0 = \tilde{L}_-, \tilde{\Lambda}_N = \tilde{L}_+ \) and \( \int_{\mathbb{R}} \tilde{\gamma}_i d\tilde{\rho}_{fc} = \frac{i}{N} \) for \( 1 \leq i \leq N - 1 \).

Then \( \tilde{\kappa}_E \sim 1 \) and \( \left| \tilde{L}_+ - \tilde{L}_- \right| \leq N^{-c} \) implies

\[
\left| \sum_i \left( \int_{\tilde{\gamma}_{i-1}}^{\tilde{\gamma}_i} \frac{x - \tilde{\gamma}_i}{(x - z)(\tilde{\gamma}_i - z)} d\tilde{\gamma}_i \right) \right| \leq \tilde{\kappa}_E^{-2} \sum_i \int_{\tilde{\gamma}_{i-1}}^{\tilde{\gamma}_i} \left| \tilde{\gamma}_i - x \right| d\tilde{\gamma}_i \leq \frac{\tilde{\kappa}_E^{-2}}{N} \sum_i (\tilde{\Lambda}_i - \tilde{\Lambda}_{i-1}) = \frac{\tilde{\kappa}_E^{-2} (\tilde{L}_+ - \tilde{L}_-)}{N} = O(N^{-1})
\] (E.17)

and \( \textbf{Lemma 3.22} \) implies

\[
\frac{1}{N} \sum_i \left| \frac{1}{\Lambda_i - z} - \frac{1}{\gamma_i - z} \right| \leq \tilde{\kappa}_E^{-2} \frac{1}{N} \sum_i \left| \tilde{\gamma}_i - \Lambda_i \right| < N^{-\frac{3}{2}} \sum_i \tilde{\alpha}_i^{-\frac{3}{2}}
\leq 2N^{-1} \sum_{i \geq \frac{N}{2}} \left( \frac{i}{N} \right)^{-\frac{3}{2}} \sim N^{-1} \int_0^2 x^{-\frac{3}{2}} dx = O(N^{-1}).
\] (E.18)

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so that the bound $|z| \leq \eta$ for $z \in \Gamma_0 \cup \Gamma_r \cup 
abla \Gamma_r$ follows.

Now for $z \in \Gamma_0 \cup \Gamma_r$, (3.50) together with $\tilde{t}_E \sim 1$ implies
\[
\sqrt{\frac{3m_{fc}(z)}{N\eta}} \sim \sqrt{\frac{\eta}{\tilde{t}_E + \eta N\eta}} \sim \frac{1}{\sqrt{N}},
\] (E.19)
so that
\[
|R_{ij} - \delta_{ij} \hat{g}_i(z)| \leq \sqrt{\frac{3m_{fc}(z)}{N\eta}} + \frac{1}{N\eta} = O(N^{-\frac{1}{2}} + N^{-1+\delta}).
\] (E.20)
Taking $\delta < \frac{1}{2}$, we get the result.

\[\text{Proof of Corollary 3.31.}\] For each $N$ and $0 \leq k \leq N$ we have
\[
\left| I_k(z_1, z_2) \right| \leq \frac{1}{N} \sum_{p>k} \left| \tilde{g}_p(z_1) \tilde{g}_p(z_2) \right| \leq \frac{1}{N} \sum_p \left| \tilde{g}_p(z_1) \tilde{g}_p(z_2) \right| \leq \frac{1}{2N} \sum_p \left( \left| \tilde{g}_p(z_1) \right|^2 + \left| \tilde{g}_p(z_2) \right|^2 \right)
\leq \frac{1}{2} \int \frac{1}{|\varrho X - z_1 - \tilde{m}_{fc}^\varrho(z_1)|^2} + \frac{1}{|\varrho X - z - \tilde{m}_{fc}^\varrho(z_2)|^2} \, d\varrho(x) = \frac{1}{2} \left( \frac{\Im m_{fc}^\varrho(z_1)}{\Im (z_1 + \tilde{m}_{fc}^\varrho(z_1))} + \frac{\Im m_{fc}^\varrho(z_2)}{\Im (z_2 + \tilde{m}_{fc}^\varrho(z_2))} \right),
\] (E.21)
where we used (E.20) in the last equality. Similarly, from (E.24),
\[
\left| I^\varrho(z_1, z_2) \right| \leq \frac{1}{2} \left( \frac{\Im m_{fc}^\varrho(z_1)}{\Im (z_1 + \tilde{m}_{fc}^\varrho(z_1))} + \frac{\Im m_{fc}^\varrho(z_2)}{\Im (z_2 + \tilde{m}_{fc}^\varrho(z_2))} \right).
\] (E.22)
From the inequalities above together with the uniform lower bound of $\Im m_{fc}(z)$ and $\Im m_{fc}^\varrho(z)$ given in Corollary 3.29, it can be easily checked that there exists $r > 0$ satisfying the condition.

\[\text{Proof of Corollary 3.31.}\] Using (3.51) and Corollary 3.29, we first reduce $\tilde{I}_0^\varrho(z_1, z_2)$ as follows:
\[
\tilde{I}_0^\varrho(z_1, z_2) = \int \frac{1}{(\varrho X - z_1 - \tilde{m}_{fc}^\varrho(z_1))(\varrho X - z_2 - \tilde{m}_{fc}^\varrho(z_2))} \, d\varrho(x)
= \int \frac{1}{(\varrho X - z_1 - m_{fc}^\varrho(z_1))(\varrho X - z_2 - m_{fc}^\varrho(z_2))} \, d\varrho(x) + C\varrho N^{-\frac{1}{2} + \epsilon_o}. \quad (E.23)
\]
Note that the uniformity of the constant $C$ follows from that of (3.51) and Corollary 3.27. Then, letting $w_1^\varrho = z_1 + m_{fc}^\varrho(z_1)$ and similarly $w_2^\varrho$, we can rewrite the integrand as
\[
\frac{1}{(\varrho X - w_1^\varrho)(\varrho X - w_2^\varrho)} = \frac{1}{w_1^\varrho - w_2^\varrho} \left[ \frac{1}{\varrho X - w_1^\varrho} - \frac{1}{\varrho X - w_2^\varrho} \right], \quad (E.24)
\]
so that we have
\[
I^\varrho(z_1, z_2) - \tilde{I}_0^\varrho(z_1, z_2) = \frac{1}{w_1^\varrho - w_2^\varrho} \left[ (m_{fc}(w_1^\varrho) - m_{fc}(w_1^\varrho)) - (m_{fc}(w_2^\varrho) - m_{fc}(w_2^\varrho)) \right] + C\varrho N^{-\frac{1}{2} + \epsilon_o}. \quad (E.25)
\]
Now the Cauchy integral formula shows that
\[
\frac{(m^0_p(w_1^0) - m^0_p(w_2^0)) - (m^0_p(w_2^0) - m^0_p(w_2^0))}{w_1^0 - w_2^0} = \frac{1}{2\pi i} \oint_{\gamma} \frac{(m^0_p(w) - m^0_p(w)) - (m^0_p(w_2^0) - m^0_p(w_2^0))}{(w - w_1)(w - w_2)} \, dw
\]
for any contour \(\gamma\) in \(\mathbb{C}^+\) satisfying
\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w - w_1} \, dw = 1. \tag{E.27}
\]
For our particular choice of the contour, we first consider a positively oriented simple closed curve \(\gamma\) in \(D_c\) such that \(D_1\) lies inside \(\gamma\). If we set \(G(z) = G^0(z) := z + m^0_p(z)\), clearly the contour \(G(\gamma)\) is contained in \(\mathbb{C}^+\). Furthermore, the contour integral
\[
\frac{1}{2\pi i} \oint_{G(\gamma)} \frac{1}{w - w_1} \, dw \tag{E.28}
\]
is precisely the number of zeros of
\[
G(z) - w_1 = G(z) - G(z_1) = 0 \tag{E.29}
\]
within the region bounded by \(\gamma\). Now using Lemma E.30, which implies the injectivity of \(G\), together with the assumption on \(\gamma\), we see that the number is precisely 1, so that
\[
\frac{1}{2\pi i} \oint_{G(\gamma)} \frac{1}{w} \, dw = 1. \tag{E.30}
\]
Recalling the definition of \(\gamma\), we have a constant \(c_0 > 0\) such that \(|z - z_0| \geq c_0\) for any \(z \in \gamma\) and \(z_0 \in D_1\), so that whenever \(z \in \gamma\) and \(z_1, z_2 \in D_1\), from Lemma E.30 we have the lower bound
\[
| (G(z) - G(z_1))(G(z) - G(z_2)) | \geq d^2 |z - z_1| |z - z_2| \geq d^2 c_0^2. \tag{E.31}
\]
Therefore, we get the bound
\[
\left| \frac{1}{2\pi i} \oint_{G(\gamma)} \frac{(m^0_p(w) - m^0_p(w)) - (m^0_p(z_2^0) - m^0_p(z_2^0))}{(w - w_1)(w - w_2)} \, dw \right| \leq \frac{1}{c_0^2} \gamma \left[ \sup_{z \in \gamma} (|m^0_p(G(z)) - m^0_p(G(z))| + |m^0_p(G(z_2)) - m^0_p(G(z_2))|) |G'(z)| \right] \leq C\vartheta N^{-\frac{3}{2} + \epsilon_0}, \tag{E.32}
\]
where the last inequality follows from (3.32).

\textbf{Proof of Lemma E.33} We first note that by the self-consistent equation (2.30), together with the trivial bound
\[
|m^0_p(z)| \leq \int_{\mathbb{R}} \left| \frac{1}{x - z} \right| \, d\rho^0_p(x) \leq \text{dist}(z, [L^0_-, L^0_+])^{-1}, \tag{E.33}
\]
implies that any pointwise limit \(s(z)\) of \(m^0_p(z)\) must satisfy the self-consistent equation
\[
s(z) = \int_{\mathbb{R}} \frac{1}{\partial_{\infty} x - z - s(z)} \, d\nu(x), \quad \exists s(z) > 0 \text{ if } \exists z > 0, \tag{E.34}
\]
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which precisely coincides with that of $m_{\phi}^\alpha (z)$. Then, an application of Montel’s theorem gives the uniform convergence $m_{\phi}^\alpha (z) \rightarrow m_{\phi}^\alpha (z)$ on D.

Now given the uniform convergence, we again use (2.9) to get

$$m_{\phi}^\alpha (z) - m_{\phi}^\alpha (z) = \int_{\mathbb{R}} \frac{1}{x - z - m_{\phi}^\alpha (z)} - \frac{1}{x - z - m_{\phi}^\alpha (z)} \, d\nu(x)$$

$$= \int_{\mathbb{R}} \frac{(m_{\phi}^\alpha (z) - m_{\phi}^\alpha (z)) - (\vartheta - \Phi)}{(x - z - m_{\phi}^\alpha (z))(\vartheta - \Phi)} \, d\nu(x)$$

$$= (m_{\phi}^\alpha (z) - m_{\phi}^\alpha (z)) \int_{\mathbb{R}} \frac{1}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x)$$

$$- (\vartheta - \Phi) \int_{\mathbb{R}} \frac{x}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x). \quad (E.35)$$

so that

$$\left| 1 - \int_{\mathbb{R}} \frac{1}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x) \right| \leq |\vartheta - \Phi| \int_{\mathbb{R}} \frac{x}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x). \quad (E.36)$$

The stability bound (3.47) implies the existence of a constant $C > 0$ satisfying

$$\int_{\mathbb{R}} \frac{x}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x) \leq C, \quad (E.37)$$

uniformly for $z \in D$. On the other hand, our domain $D$ satisfies the assumptions of Corollary 3.26, so that there exists $c' > 0$ satisfying $\Im m_{\phi}^\alpha (z) > c' \Im z$ and similarly for $\Phi$. Therefore we have

$$\frac{\Im m_{\phi}^\alpha (z)}{\eta + \Im m_{\phi}^\alpha (z)} = 1 - \frac{\eta}{\eta + \Im m_{\phi}^\alpha (z)} > \frac{c'}{1 + c'} \quad (E.38)$$

and the same lower bound for $\Phi$. Then we deduce

$$1 - \int_{\mathbb{R}} \frac{1}{x - z - m_{\phi}^\alpha (z)}(\vartheta - \Phi) \, d\nu(x) \leq 1 - \left[ \frac{\Im m_{\phi}^\alpha (z)}{\eta + \Im m_{\phi}^\alpha (z)} \cdot \frac{\Im m_{\phi}^\alpha (z)}{\eta + \Im m_{\phi}^\alpha (z)} \right] \leq 1 - \frac{c'}{1 + c'} = \frac{1}{1 + c'}, \quad (E.39)$$

which completes the proof. \qed

**References**

[1] G. W. Anderson and O. Zeitouni. A CLT for a band matrix model. *Probab. Theory Related Fields*, 134(2):283–338, 2006.
[2] A. I. Aptekarev, P. M. Bleher, and A. B. J. Kuijlaars. Large $n$ limit of Gaussian random matrices with external source. II. *Comm. Math. Phys.*, 259(2):367–389, 2005.

[3] Z. D. Bai and J. W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *Ann. Probab.*, 32(1A):553–605, 2004.

[4] Z. D. Bai and J. Yao. On the convergence of the spectral empirical process of Wigner matrices. *Bernoulli*, 11(6):1059–1092, 2005.

[5] J. Baik and J. O. Lee. Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model with ferromagnetic interaction. *Ann. Henri Poincaré*, 18(6):1867–1917, 2017.

[6] L. Benigni. Eigenvectors distribution and quantum unique ergodicity for deformed Wigner matrices. *ArXiv e-prints*, Nov. 2017.

[7] P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.*, 46(3):705–718, 1997.

[8] P. Billingsley. *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[9] P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, third edition, 1995. A Wiley-Interscience Publication.

[10] P. Bleher and A. B. J. Kuijlaars. Large $n$ limit of Gaussian random matrices with external source. I. *Comm. Math. Phys.*, 252(1-3):43–76, 2004.

[11] P. M. Bleher and A. B. J. Kuijlaars. Large $n$ limit of Gaussian random matrices with external source. III. Double scaling limit. *Comm. Math. Phys.*, 270(2):481–517, 2007.

[12] T. Cabanal-Duvillard. Fluctuations de la loi empirique de grandes matrices aléatoires. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(3):373–402, 2001.

[13] S. Chatterjee. Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields*, 143(1-2):1–40, 2009.

[14] S. Chatterjee and A. Bose. A new method for bounding rates of convergence of empirical spectral distributions. *J. Theoret. Probab.*, 17(4):1003–1019, 2004.

[15] L. Erdős, H.-T. Yau, and J. Yin. Universality for generalized Wigner matrices with Bernoulli distribution. *J. Comb.*, 2(1):15–81, 2011.

[16] A. Guionnet. Large deviations upper bounds and central limit theorems for non-commutative functionals of Gaussian large random matrices. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(3):341–384, 2002.

[17] K. Johansson. From Gumbel to Tracy-Widom. *Probab. Theory Related Fields*, 138(1-2):75–112, 2007.

[18] D. Jonsson. Some limit theorems for the eigenvalues of a sample covariance matrix. *J. Multivariate Anal.*, 12(1):1–38, 1982.

[19] A. Khorunzhy, B. Khoruzhenko, and L. Pastur. On the $1/N$ corrections to the Green functions of random matrices with independent entries. *J. Phys. A*, 28(1):L31–L35, 1995.

[20] A. M. Khorunzhy, B. A. Khoruzhenko, and L. A. Pastur. Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.*, 37(10):5033–5060, 1996.
[21] A. Knowles and J. Yin. Anisotropic local laws for random matrices. *Probability Theory and Related Fields*, Aug 2016.

[22] J. O. Lee and K. Schnelli. Local deformed semicircle law and complete delocalization for Wigner matrices with random potential. *J. Math. Phys.*, 54(10):103504, 62, 2013.

[23] J. O. Lee and K. Schnelli. Edge universality for deformed Wigner matrices. *Rev. Math. Phys.*, 27(8):1550018, 94, 2015.

[24] J. O. Lee and K. Schnelli. Extremal eigenvalues and eigenvectors of deformed Wigner matrices. *Probab. Theory Related Fields*, 164(1-2):165–241, 2016.

[25] J. O. Lee, K. Schnelli, B. Stetler, and H.-T. Yau. Bulk universality for deformed Wigner matrices. *Ann. Probab.*, 44(3):2349–2425, 2016.

[26] A. Lytova and L. Pastur. Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Probab.*, 37(5):1778–1840, 2009.

[27] S. O’Rourke and D. Renfrew. Central limit theorem for linear eigenvalue statistics of elliptic random matrices. *J. Theoret. Probab.*, 29(3):1121–1191, 2016.

[28] S. O’Rourke and V. Vu. Universality of local eigenvalue statistics in random matrices with external source. *Random Matrices Theory Appl.*, 3(2):1450005, 37, 2014.

[29] L. Pastur. Limiting laws of linear eigenvalue statistics for Hermitian matrix models. *J. Math. Phys.*, 47(10):103303, 22, 2006.

[30] L. A. Pastur. On the spectrum of random matrices. *Theo. Math. Phys.*, 10(1):67–74, 1972.

[31] M. Shcherbina. Central limit theorem for linear eigenvalue statistics of the Wigner and sample covariance random matrices. *Zh. Mat. Fiz. Anal. Geom.*, 7(2):176–192, 197, 199, 2011.

[32] M. Shcherbina. On fluctuations of eigenvalues of random band matrices. *J. Stat. Phys.*, 161(1):73–90, 2015.

[33] Z. Su. Fluctuations of deformed wigner random matrices. *Frontiers of Mathematics in China*, 8(3):609–641, Jun 2013.