REAL AND COMPLEX OPERATOR IDEALS

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Abstract. The powerful concept of an operator ideal on the class of all Banach spaces makes sense in the real and in the complex case. In both settings we may, for example, consider compact, nuclear, or 2–summing operators, where the definitions are adapted to each other in a natural way. This paper deals with the question whether or not that fact is based on a general philosophy. Does there exists a one–to–one correspondence between “real properties” and “complex properties” defining an operator ideal? In other words, does there exist for every real operator ideal a uniquely determined corresponding complex ideal and vice versa?

Unfortunately, we are not able to give a final answer. Nevertheless, some preliminary results are obtained. In particular, we construct for every real operator ideal a corresponding complex operator ideal and for every complex operator ideal a corresponding real one. However, we conjecture that there exists a complex operator ideal which can not be obtained from a real one by this construction.

The following approach is based on the observation that every complex Banach space can be viewed as a real Banach space with an isometry acting on it like the scalar multiplication by the imaginary unit $i$.

1. Preliminaries

Let $X$ always denote a Banach space over the field of real numbers. The letters $A$, $B$, $T$ and $S$ refer to linear operators between real Banach spaces. The identity map of $X$ is denoted by $I_X$.

For $x \in X$ the canonical injections

$$X \rightarrow X \oplus X : x \mapsto (x, o) \quad \text{and} \quad X \rightarrow X \oplus X : x \mapsto (o, x)$$

are denoted by $J_1^X$ and $J_2^X$, respectively.

For $x_1, x_2 \in X$ the canonical surjections

$$X \oplus X \rightarrow X : (x_1, x_2) \mapsto x_1 \quad \text{and} \quad X \oplus X \rightarrow X : (x_1, x_2) \mapsto x_2$$

are denoted by $Q_1^X$ and $Q_2^X$, respectively.

Let $\mathcal{L}$ always denote the ideal of all (real or complex) bounded linear operators.

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For the theory of operator ideals we refer to the monographs of Pietsch, [3] and [4].

**Definition.** The Banach space $X \oplus X$ becomes a complex Banach space under the operations

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2)$$

$$(\alpha + i\beta)(x_1, x_2) := (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2),$$

and the norm

$$\| (x_1, x_2) \| := \left( \frac{1}{2\pi} \int_{-\pi}^{+\pi} \| x_1 \cos \phi + x_2 \sin \phi \|^2 \, d\phi \right)^{1/2},$$

where $x_1, x_2, y_1, y_2 \in X$.

The obtained complex Banach space is called the complexification of $X$.

2. **Banach spaces with an $i$–operator**

Note that the scalar multiplication by the imaginary unit $i$ in a complex Banach space yields an isometric operator on this Banach space. This observation leads to the following definition.

**Definition.** A **Banach space with an $i$–operator** is a pair $[X, A]$ consisting of a real Banach space $X$ and a linear operator $A : X \to X$ such that $A^2 = -I_X$ and

$$\| \alpha x + \beta Ax \| = \| x \| \quad \text{for } |\alpha|^2 + |\beta|^2 = 1.$$  \hspace{1cm} (1)

We refer to the operator $A$ as an **$i$–operator** on the Banach space $X$.

If $[X, A]$ is a Banach space with an $i$–operator, then we obviously get

$$(-A)^2 = -I_X$$

and

$$\| \alpha x - \beta Ax \| = \| x \| \quad \text{for } |\alpha|^2 + |\beta|^2 = 1,$$

and hence $[X, -A]$ is a Banach space with an $i$–operator, too.

**Definition.** We denote by $[\overline{X}, A] := [X, -A]$ the **complex conjugate Banach space** of $[X, A]$.

A linear operator between two complex Banach spaces obviously commutes with the scalar multiplication by $i$. The next definition is due to this fact.

**Definition.** Given two Banach spaces with $i$–operators $[X, A]$ and $[Y, B]$, we say that an operator $T : X \to Y$ respects $A$ and $B$ if

$$TA = BT.$$  \hspace{1cm} (2)

In this case, we write $[T, A, B]$ for the induced operator from $[X, A]$ to $[Y, B]$. 
If \( T \) respects the \( i \)-operators \( A \) and \( B \), then we obviously get
\[
T(-A) = (-B)T
\]
and hence \( T \) respects the \( i \)-operators \( -A \) and \( -B \), too.

**Definition.** We denote by
\[
[T, A, B] := [T, -A, -B] : \mathcal{X}, -A \to \mathcal{Y}, -B
\]
the complex conjugate operator of \([T, A, B]\).

The class of all Banach spaces with an \( i \)-operator \([X, A]\) together with the morphisms \([T, A, B]\) forms a category. Two Banach spaces with an \( i \)-operator \([X, A]\) and \([Y, B]\) are said to be isomorphic, if they are isomorphic in the sense of this category. We then write \([X, A] \simeq [Y, B]\).

The mappings
\[
[X, A] \to [X, A] \quad \text{and} \quad [T, A, B] \to [T, A, B]
\]
yield a covariant functor on this category.

### 3. Examples of Banach spaces with an \( i \)-operator

#### 3.1. Complex Banach spaces.
Of course the concept of a Banach space with an \( i \)-operator exactly simulates that of complex Banach spaces. This means if \( Y \) is a complex Banach space, then \([Y, A]\) is a Banach space with an \( i \)-operator, provided \( Y \) is considered as a real space and \( A \) is given by
\[
Ay := i \cdot y \quad \text{for all } y \in Y.
\]
In this way every complex Banach space can be considered as a Banach space with an \( i \)-operator.

Also, if \( Y \) and \( Z \) are complex Banach spaces and \( T : Y \to Z \) is a complex operator then \( T \) respects \( A \) and \( B \) defined as above on \( Y \) and \( Z \), respectively.

Conversely, given a Banach space with an \( i \)-operator \([X, A]\) we can introduce a complex scalar multiplication in \( X \) by
\[
i \cdot x := Ax
\]
or more generally
\[
(\alpha + i\beta)x := \alpha x + \beta Ax \quad \text{for } \alpha, \beta \in \mathbb{R}.
\]
In this way we obtain a complex Banach space.

Also, if \([T, A, B]\) is an operator respecting \( A \) and \( B \), then \( T \) yields a complex operator on the assigned complex spaces.

Thus the category of Banach spaces with an \( i \)-operator can be identified with the category of complex Banach spaces.
3.2. Complexification. For a real Banach space \( X \), we can define a natural \( i \)-operator \( N_X \) on the space \( X \oplus X \) by
\[
N_X(x_1, x_2) := (-x_2, x_1), \quad \text{for } x_1, x_2 \in X.
\]
It is easy to see that the space \([X \oplus X, N_X]\) equipped with the norm
\[
\|(x, y)\| := \left( \frac{1}{2\pi} \int_{-\pi}^{+\pi} \|x \cos \phi + y \sin \phi\|^2 \, d\phi \right)^{1/2}
\]
corresponds to the complexification of the space \( X \) as defined in Section 1.

Also, if \( T : X \to Y \) is a real operator, then
\[
T \oplus T : (x_1, x_2) \mapsto (Tx_1, Tx_2)
\]
respects the \( i \)-operators \( N_X \) and \( N_Y \). The corresponding operator \([T \oplus T, N_X, N_Y]\) is called the complexification of the operator \( T \).

It is natural to ask whether or not every complex Banach space can be obtained as the complexification of a real Banach space. The following proposition yields a characterization of complex Banach spaces with that property. Example 2 will then show that there are complex Banach spaces which are not a complexification.

**Proposition 1.** The following conditions are equivalent:
1. The complex Banach space \([X, A]\) is a complexification of a real Banach space \( Y \), i.e. there exists an isomorphism
\[
[S, A, N_Y] : [X, A] \to [Y \oplus Y, N_Y]
\]
of the spaces \([X, A]\) and \([Y \oplus Y, N_Y]\).
2. There exists an automorphism \( T \) of \( X \) such that
\[
TA = -AT \quad \text{and} \quad T^2 = I_X,
\]
in particular the spaces \([X, A]\) and \( \overline{[X, A]} \) are (complex) isomorphic.

**Proof.** First of all, note that the complex conjugation
\[
C : Y \oplus Y \to Y \oplus Y : (y_1, y_2) \mapsto (y_1, -y_2)
\]
respects the \( i \)-operators \( N_Y \) and \( -N_Y \) and hence yields an isomorphism \([C, N_Y, -N_Y]\) of the complex conjugate spaces \([Y \oplus Y, N_Y]\) and \([Y \oplus Y, N_Y]\).

Moreover, for a given isomorphism \([S, A, N_Y]\) the complex conjugate map
\[
[S, -A, -N_Y] = [S, A, N_Y] : [X, A] \to [Y \oplus Y, N_Y]
\]
yields an isomorphism of the spaces \( \overline{[X, A]} \) and \( \overline{[Y \oplus Y, N_Y]} \).

Putting together these isomorphisms, we get an isomorphism
\[
[S^{-1}CS, A, -A] : [X, A] \to \overline{[X, A]}
\]
of the spaces \([X, A]\) and \([X, \overline{A}]\) which moreover satisfies
\[(S^{-1}CS)^2 = I_X.\]
This proves that (1) implies (2).

To prove the converse, we consider the subspace \(Y \subseteq X\) defined by
\[Y := \{x + Tx : x \in X\}.\]
Then the map
\[S : X \to Y \oplus Y : x \mapsto (Ax + TAx, x + Tx)\]
respects the \(i\)-operators \(A\) and \(N_Y\). Moreover, by (1), we have
\[
\|Sx\| = \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} \|(I + T)(Ax \cos \phi + x \sin \phi)\|^2 d\phi\right)^{1/2}
\leq \|I + T\| \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} \|Ax \cos \phi + x \sin \phi\|^2 d\phi\right)^{1/2}
= \|I + T\| \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} \|x\|^2 d\phi\right)^{1/2} = \|I + T\| \cdot \|x\|
\]
and hence
\[
\|S\| \leq \|I + T\|.
\]
By \(T^2 = I_X\) and \(TA = -AT\), it can easily be verified that the inverse map of \(S\) is given by
\[S^{-1} : Y \oplus Y \to X : (x_1 + Tx_1, x_2 + Tx_2) \mapsto \frac{1}{2} \left((x_2 + Tx_2) - A(x_1 + Tx_1)\right).
\]
Consequently, \(S\) defines an isomorphism of the spaces \([X, A]\) and \([Y \oplus Y, N_Y]\).

3.3. Pathological examples. Note that on a finite dimensional space an \(i\)-operator can be defined if and only if the space is of even dimension. However, there are also examples of infinite dimensional Banach spaces not admitting any \(i\)-operator.

Example 1 (James [2]). The James space \(J\) satisfies
\[
\dim(J^{**}/J) = 1
\]
and hence cannot admit any \(i\)-operator.

Remark. In [5] Szarek gives an example of a superreflexive infinite dimensional Banach space that does not admit an \(i\)-operator.
At first sight, one could think that the spaces $[X, A]$ and $\overline{[X, A]}$ are quite similar. However, the following example shows that they can differ in the worst imaginable way.

**Example 2 (Bourgain [1]).** There exists a Banach space with an $i$–operator $[X, A]$ such that $[X, A]$ and $\overline{[X, A]}$ are not isomorphic.

In view of the considerations in the next section the following problem arises.

**Problem 1.** Does there exist a Banach space with an $i$–operator $[X, A]$ such that $X$ is not (real) isomorphic to a square of a real Banach space $Y$?

By Proposition 1 every complex Banach space $[X, A]$ which is not isomorphic to $[X, A]$ can not be (complex) isomorphic to a square $Y \oplus Y$ equipped with the natural $i$–operator $N_Y$. However, this problem only deals with real isomorphisms on real spaces.

### 3.4. Some relations.

The previous considerations show that there is a one–to–one correspondence between the categories of complex Banach spaces and real Banach spaces with an $i$–operator. Under this correspondence, those complex Banach spaces that are a complexification of a real space are exactly the Cartesian squares of real Banach spaces $[X \oplus X, N_X]$ with the natural $i$–operator.

On the other hand, there is the forget functor acting from either the category of complex Banach spaces or the category of real Banach spaces with an $i$–operator into the category of real Banach spaces. This functor simply forgets about the complex structure on a Banach space. However, as the examples in the previous section show, this functor is neither injective nor surjective.

Namely, the spaces $[X, A]$ and $\overline{[X, A]}$ of Example 2 are both mapped to the same real Banach space $X$, which shows that the forget functor is not injective. The spaces of Example 1 show that the forget functor is not surjective.

Moreover, Proposition 1 shows that the space $[X, A]$ of Example 2 is not a complexification of a real space.
Problem 1 deals with the question whether there exists a real Banach space that admits an $i$–operator but is no real square.

4. Operator ideals

In the following let $\mathcal{R}$ and $\mathcal{C}$ denote ideals of real or complex operators, respectively.

**Definition.** We define the *complexification* $\mathcal{R}_C$ of the real operator ideal $\mathcal{R}$ as the ideal consisting of all complex operators $[T, A, B]$ such that $T$ belongs to $\mathcal{R}$, i.e.

\[
\mathcal{R}_C([X, A], [Y, B]) := \{ [T, A, B] \in \mathcal{L}([X, A], [Y, B]) : T \in \mathcal{R}(X, Y) \}.
\]

We define the *real form* $\mathcal{C}_R$ of the complex operator ideal $\mathcal{C}$ as the ideal consisting of all real operators $T$ such that its complexification belongs to $\mathcal{C}$, i.e.

\[
\mathcal{C}_R(X, Y) := \{ T \in \mathcal{L}(X, Y) : [T \oplus T, N_X, N_Y] \in \mathcal{C}([X \oplus X, N_X], [Y \oplus Y, N_Y]) \}.
\]
Easy computations show that for a given operator ideal \( \mathcal{C} \) the class of all conjugate operators \([T, A, B]\) of operators \([T, A, B] \in \mathcal{C}\) is an operator ideal, too.

**Definition.** For a complex operator ideal \( \mathcal{C} \), we denote by 
\[
\overline{\mathcal{C}} := \{[\overline{T}, A, B] : [T, A, B] \in \mathcal{C}\}
\]
the complex conjugate operator ideal of \( \mathcal{C} \).

We say that an operator ideal \( \mathcal{C} \) is self conjugate if \( \overline{\mathcal{C}} = \mathcal{C} \).

Of course one may ask whether or not the ideals \( \mathcal{R} \) and \( \mathcal{C} \) are uniquely determined by their complex or real analogues \( \mathcal{R}_C \) and \( \mathcal{C}_R \), respectively. This makes it necessary to examine the combination of the procedures above, namely under which conditions on \( \mathcal{R} \) and \( \mathcal{C} \) is it true that

\[(2) \quad (\mathcal{R}_C)_R = \mathcal{R}\]

and

\[(3) \quad (\mathcal{C}_R)_C = \mathcal{C}?\]

Theorem 1 below will give a satisfactory answer if we start from a real ideal \( \mathcal{R} \). In the complex case, to fulfill (3), we must additionally require that the ideal \( \mathcal{C} \) is self conjugate.

5. The real case

**Proposition 2.** A real operator ideal \( \mathcal{R} \) is uniquely determined by its restriction to Cartesian squares, i.e.
\[
T \oplus T \in \mathcal{R}(X \oplus X, Y \oplus Y) \iff T \in \mathcal{R}(X, Y).
\]

**Proof.** Looking at the the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow J_1^X & & \downarrow Q_1^Y \\
X \oplus X & \xrightarrow{T \oplus T} & Y \oplus Y
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow Q_1^X & & \downarrow J_2^Y \\
X \oplus X & \xrightarrow{T \oplus T} & Y \oplus Y
\end{array}
\]

we easily see that
\[
T = Q_1^Y (T \oplus T) J_1^X \quad \text{and} \quad T \oplus T = J_1^Y T Q_1^X + J_2^Y T Q_2^X.
\]

The assertion now follows from the ideal properties of \( \mathcal{R} \). \( \square \)
The following theorem means that real operator ideals are uniquely determined by their complexification.

**Theorem 1.** For any real operator ideal \( \mathfrak{R} \) we have

\[
(\mathfrak{R}_C)_R = \mathfrak{R}.
\]

**Proof.** Let \( T \in (\mathfrak{R}_C)_R \). By the definition of the real form of an operator ideal this is the case if and only if \( [T \oplus T, N_X, N_Y] \in \mathfrak{R}_C \). By the definition of \( \mathfrak{R}_C \) this is equivalent to \( T \oplus T \in \mathfrak{R} \). Now, using Proposition 2, the assertion follows. \( \square \)

### 6. The complex case

Let \([X, A]\) be a Banach space with an \( i \)-operator. Then \( A \oplus -A \) yields an \( i \)-operator on \( X \oplus X \). The following proposition shows the significance of this \( i \)-operator.

**Proposition 3.** Let \([X, A]\) be a Banach space with an \( i \)-operator. In the category of Banach spaces with an \( i \)-operator the space \([X \oplus X, A \oplus -A]\) is isomorphic to the complexification of the real space \([X \oplus X, N_X]\).

**Proof.** Let \( x_1, x_2 \in X \) and define \( T : X \oplus X \to X \oplus X \) by

\[
T(x_1, x_2) := (x_1 + Ax_2, x_1 - Ax_2).
\]

Then we have

\[
TN_X(x_1, x_2) = (Ax_1 - x_2, -Ax_1 - x_2),
\]
\[
(A \oplus -A)T(x_1, x_2) = (Ax_1 - x_2, -Ax_1 - x_2).
\]

Hence the operator \( T \) respects the \( i \)-operators \( N_X \) and \( A \oplus -A \). It is evident that \( T \) is a bounded bijection and consequently \( [T, N_X, A \oplus -A] \) defines an isomorphism of the spaces \([X \oplus X, N_X]\) and \([X \oplus X, A \oplus -A]\). \( \square \)

Note that, in order to make \( N_X \) an \( i \)-operator on \( X \oplus X \), we must modify the norm a little bit, introducing e.g.

\[
\| (x, y) \| := \left( \frac{1}{2\pi} \int_{-\pi}^{+\pi} \| x \cos \phi + y \sin \phi \|^2 d\phi \right)^{1/2}.
\]

On the other hand, in order to make \( A \oplus A \) an \( i \)-operator on \( X \oplus X \), we may use e.g.

\[
\| (x, y) \| := \| x \| + \| y \|.
\]

Nevertheless, the resulting spaces are still isomorphic.

In contrast to Proposition 2, in the complex case, we can only prove the following result.
Proposition 4. For a complex operator ideal $\mathcal{C}$, we have

$$[T \oplus T, A \oplus -A, B \oplus -B] \in \mathcal{C}([X \oplus X, A \oplus -A], [Y \oplus Y, B \oplus -B]) \implies [T, A, B] \in \mathcal{C}([X, A], [Y, B]).$$

Proof. First of all, note that for a Banach space with an $i$-operator $[X, A]$ the canonical injection $J^X_1 : X \to X \oplus X$ respects the $i$-operators $A$ and $A \oplus -A$ whereas the canonical injection $J^X_2 : X \to X \oplus X$ respects the $i$-operators $-A$ and $A \oplus -A$.

Similarly, the canonical surjection $Q^X_1 : X \oplus X \to X$ respects the $i$-operators $A \oplus -A$ and $A$ whereas the canonical surjection $Q^X_2 : X \oplus X \to X$ respects the $i$-operators $A \oplus -A$ and $-A$.

Hence we can form the composition of the operators $[J^X_1, A, A \oplus -A], [T \oplus T, A \oplus -A, B \oplus -B]$ and $[Q^Y_1, B \oplus -B, B]$ and we have

$$[T, A, B] = [Q^Y_1 (T \oplus T) J^X_1, A, B].$$

Now it follows from the ideal properties of $\mathcal{C}$ that $[T, A, B] \in \mathcal{C}$. □

To prove the converse of Proposition 4 an additional property of the operator ideal $\mathcal{C}$ is required.

Proposition 5. Let $\mathcal{C}$ be a complex operator ideal. The following conditions are equivalent:

1. The operator ideal $\mathcal{C}$ is self conjugate.
2. The operator ideal $\mathcal{C}$ is uniquely determined by its restriction to Cartesian squares, i.e.

$$[T \oplus T, A \oplus -A, B \oplus -B] \in \mathcal{C}([X \oplus X, A \oplus -A], [Y \oplus Y, B \oplus -B]) \iff [T, A, B] \in \mathcal{C}([X, A], [Y, B]).$$

Proof. If $\mathcal{C}$ is self conjugate then it follows from $[T, A, B] \in \mathcal{C}$ that $[T, -A, -B] \in \mathcal{C}$. Hence we may form the composition of the operators $[Q^X_1, A \oplus -A, A], [T, A, B]$ and $[J^Y_1, B, B \oplus -B]$ as well as of the operators $[Q^X_2, A \oplus -A, -A], [T, -A, -B]$ and $[J^Y_2, -B, B \oplus -B]$. The proof of condition (2) now follows from the formula

$$[T \oplus T, A \oplus -A, B \oplus -B] = [J^Y_1 T Q^X_1, A \oplus -A, B \oplus -B] + [J^Y_2 T Q^X_2, A \oplus -A, B \oplus -B]$$

and the ideal properties of $\mathcal{C}$.

If on the other hand condition (2) is satisfied, then it follows from $[T, A, B] \in \mathcal{C}$ that $[T \oplus T, A \oplus -A, B \oplus -B] \in \mathcal{C}$. Hence we may form the composition of $[J^X_2, -A, A \oplus -A], [T \oplus T, A \oplus -A, B \oplus -B]$ and $[Q^Y_2, B \oplus -B, -B]$. The identity

$$[T, -A, -B] = [Q^Y_2 (T \oplus T) J^X_2, -A, -B]$$
and the ideal properties of $\mathfrak{C}$ imply that $[T, -A, -B] \in \mathfrak{C}$ and hence $\mathfrak{C}$ is self conjugate.

Now we can prove the complex analogue to Theorem 1.

**Theorem 2.** For any complex operator ideal $\mathfrak{C}$ we have

$$(\mathfrak{C}_R)_C \subseteq \mathfrak{C}.$$ 

In addition, the following conditions are equivalent:

1. The operator ideal $\mathfrak{C}$ is self conjugate.
2. The operator ideal $\mathfrak{C}$ is uniquely determined by its real form, i.e. $$(\mathfrak{C}_R)_C = \mathfrak{C}.$$ 

**Proof.** Let $[T, A, B] \in (\mathfrak{C}_R)_C$. By the definition of the complexification of an operator ideal this is the case if and only if $T \in \mathfrak{C}_R$. By the definition of $\mathfrak{C}_R$ this is equivalent to $[T \oplus T, N_X, N_Y] \in \mathfrak{C}$. Next, using Proposition 3, this is the same as to say that $[T \oplus T, A \oplus -A, B \oplus -B] \in \mathfrak{C}$. Now, from Proposition 4 we see that $[T, A, B] \in \mathfrak{C}$ follows. This proves the first part of the theorem.

If the operator ideal $\mathfrak{C}$ is self conjugate, then by Proposition 5 it follows from $[T, A, B] \in \mathfrak{C}$ that $[T \oplus T, A \oplus -A, B \oplus -B] \in \mathfrak{C}$ and this implies that $[T, A, B] \in (\mathfrak{C}_R)_C$.

If on the other hand $$(\mathfrak{C}_R)_C = \mathfrak{C},$$
then $[T, A, B] \in \mathfrak{C}$ implies that $[T, A, B] \in (\mathfrak{C}_R)_C$ and hence $[T \oplus T, A \oplus -A, B \oplus -B] \in \mathfrak{C}$. Consequently, it follows from Proposition 5 that $\mathfrak{C}$ is self conjugate. $\square$

### 7. Self conjugate operator ideals

Theorem 2 gives a good criterion to check whether or not a complex operator ideal is uniquely determined by its real form.

However, if one examines known operator ideals, it turns out, that all of them are self conjugate. Hence the problem arises whether there exist operator ideals which are not self conjugate. No such example seems to be known.

**Problem 2.** Does there exist a complex operator ideal $\mathfrak{C}$ and an operator $[T, A, B] \in \mathfrak{C}$ such that $[T, A, B] = [T, -A, -B] \notin \mathfrak{C}$?

Proposition 6 will reduce this problem to the following problem of the existence of a certain Banach space.

**Problem 3.** Does there exist a complex Banach space $[X, A]$ such that the following two conditions are satisfied

1. $[X, A] \n\n\not\subseteq [X, -A] = [X, -A]$
2. $[X, A] \sim [X \oplus X, A \oplus A]$?
The examples given in Section 3.3 give an idea of how to construct such a space.

**Proposition 6.** Let \([X, A]\) be a Banach space with the properties (4) and (5). Then the operator ideal \(C\) defined by

\[
C := \{ [T, A, B] : [T, A, B] \text{ admits a factorization over the space } [X, A] \}
\]

is not self conjugate.

**Proof.** First of all, we see from condition (5) that \(C\) is indeed an operator ideal. Now, on the one hand, we surely have

\([I_X, A, A] \in C\).

Suppose that \([I_X, A, A] = [I_X, -A, -A] \in C\). Then there exists a factorization

\([I_X, -A, -A] = [R, A, -A][S, -A, A]\),

where \(R, S : X \to X\). Let \(Y := \text{Im}(S) = \text{Im}(SR)\) and \(Z := \text{Ker}(SR)\). Then \(Y\) is a complemented subspace in \(X\) and therefore we have

\[
\begin{align*}
(6) \quad [X, A] & \simeq [Y \oplus Z, A \oplus A] \\
(7) \quad [X, -A] & \simeq [Y \oplus Z, -A \oplus -A] \\
(8) \quad [Y, A] & \simeq [X, -A] \\
(9) \quad [Y, -A] & \simeq [X, A].
\end{align*}
\]

Moreover, it follows from (5) that

\[
(10) \quad [X, -A] \simeq [X \oplus X, -A \oplus -A].
\]

The following is also known as Pełczyński’s decomposition method. Using (6) through (9), we get

\[
[X, A] \simeq [Y \oplus Z, A \oplus A] \quad \text{by (6)}
\]
\[
\simeq [X \oplus Z, -A \oplus A] \quad \text{by (8)}
\]
\[
\simeq [X \oplus X \oplus Z, -A \oplus -A \oplus A] \quad \text{by (10)}
\]
\[
\simeq [X \oplus Y \oplus Z, -A \oplus A \oplus A] \quad \text{by (8)}
\]
\[
\simeq [X \oplus Y, -A \oplus A] \quad \text{by (6)}
\]
\[
\simeq [Y \oplus Z \oplus X, -A \oplus -A \oplus A] \quad \text{by (7)}
\]
\[
\simeq [X \oplus Z \oplus X, A \oplus -A \oplus A] \quad \text{by (9)}
\]
\[
\simeq [X \oplus Z, A \oplus -A] \quad \text{by (5)}
\]
\[
\simeq [Y \oplus Z, -A \oplus -A] \quad \text{by (9)}
\]
\[
\simeq [X, -A] \quad \text{by (7)}.
\]

This yields a contradiction to property 4 of the space \([X, A]\) and therefore completes the proof. \(\square\)
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