Abstract. We construct an algebra morphism from the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ to a certain elliptic version of the “quantum loop groups in higher genus” studied by V. Rubtsov and the first author. This provides an embedding of $E_{\tau,\eta}(\mathfrak{sl}_2)$ in an algebra “with central extension”. In particular we construct $L^\pm$-operators obeying a dynamical version of the Reshetikhin–Semenov-Tian-Shansky relations. To do that, we construct the factorization of a certain twist of the quantum loop algebra, that automatically satisfies the “twisted cocycle equation” of O. Babelon, D. Bernard and E. Billey, and therefore provides a solution of the dynamical Yang-Baxter equation.

Introduction. The aim of this paper is to compare the $\mathfrak{sl}_2$-version of the elliptic quantum groups introduced by the second author ([14]) with quasi-Hopf algebras introduced by V. Rubtsov and the first author ([11, 12]). Elliptic quantum groups are presented by exchange (or “$RLL$”) relations, whereas the algebras of [11] are “quantum loop algebras”. Our result can be viewed as an elliptic version of the results of J. Ding and I. Frenkel ([1]) and of S. Khoroshkin ([21]), where Drinfeld’s quantum current algebra ([3]) was shown to be isomorphic with the Reshetikhin-Semenov $L$-operator algebra of [22, 13], in the trigonometric and rational case respectively.

Elliptic quantum groups are based on a matrix solution $R(z,\lambda)$ of the dynamical Yang-Baxter equation (YBE). Here “dynamical” means that in addition to the spectral parameter $z$ (belonging to an elliptic curve $E$), $R$ depends on a parameter $\lambda$, which undergo certain shifts in the Yang-Baxter equation. The $RLL$ relations defining the elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ are then an algebraic variant of the dynamical YBE.

In [3], O. Babelon, D. Bernard and E. Billey studied the relation between the dynamical and quasi-Hopf Yang-Baxter equations. They showed that given a family of twists of a quasi-triangular Hopf algebra, satisfying a certain “twisted cocycle equation”, the quasi-Hopf...
YBE satisfied by the twisted $R$-matrices was indeed equivalent to the dynamical YBE.

The quantum loop algebras of $[11]$ are generally associated with complex curves and rational differentials. As it was shown in $[12]$, they can be endowed with a quasi-Hopf structure, quantizing Drinfeld’s “higher genus Manin pairs” ($[7]$). To precise which quantum loop algebra should be associated with elliptic quantum groups, we first make a quasi-classical study (section 1.1). The classical $\tau$-matrix $r_\lambda(z, w)$ as-sociated with $R(z-w, \lambda)$ corresponds to what we may call a “dynamical Manin triple”, that is to a family $g_\lambda$ of maximal isotropic complements of a fixed maximal isotropic subalgebra $g_O$ in a Lie algebra $g$, endowed with a nondegenerate inner product. Here $g$ is a double extension of the Lie algebra $\mathfrak{sl}_2 \otimes k$, where $k$ is the local field at the origin of the elliptic curve $E$ of modulus $\tau$, $g_O$ is a cocentral extension of $\mathfrak{sl}_2 \otimes O$, where $O$ is the local ring of $E$ at the same point, and $g_\lambda$ is an extension of the sum $(n_+ \otimes L_\lambda) \oplus (h \otimes L_0) \oplus (n_- \otimes L_{-\lambda})$, where $L_\lambda$ are the sets of expansions at the origin of $E$ of functions on its universal cover with certain transformation properties.

According to $[12]$, this Manin pair $(g, g_O)$ defines quantum loop algebras $U_\hbar g_O \subset U_\hbar g(\tau)$; $U_\hbar g(\tau)$ is endowed with coproducts $\Delta$ and $\bar{\Delta}$, which are conjugated by a certain twist $F$. These algebras are presented in section 1.2 (analogous relations can be found in $[5]$). Our aim is to find a solution of the dynamical YBE in this algebra, quantizing $r_\lambda(z, w)$.

To do that, we will construct twisted cocycles (in the sense of $[3]$). For that, we follow the method of $[9]$. In that paper, we gave the construction of a Hopf algebra cocycle in the double Yangian algebra $DY(\mathfrak{sl}_2)$, by factorizing the Yangian analogue $F_{Yg}$ of $F$ as a product $F_{Yg} = F_2 F_1$, with $F_1 \in A^{<0} \otimes A^{\geq 0}$ and $F_2 \in A^{\geq 0} \otimes A^{<0}$, with $A^{\geq 0}$ and $A^{<0}$ the subalgebras of $DY(\mathfrak{sl}_2)$ generated by the nonnegative and negative Fourier modes of the quantum currents.

This method does not apply directly here: we look for a family of twists $(F_\lambda^1, F_\lambda^2)$. Moreover, we indeed have an analogue of $A^{\geq 0}$, that is $U_\hbar g_O$, but no analogue of $A^{<0}$. Our idea is to construct subalgebras of the algebra of families of elements of $U_\hbar g(\tau)^{\otimes 2}$ depending on $\lambda$, which play the role of $A^{<0} \otimes A^{\geq 0}$ and $A^{\geq 0} \otimes A^{<0}$, and to which $F_\lambda^1$ and $F_\lambda^2$ should belong (section 3). The properties of these algebras $A^{++}$ and $A^{+-}$ are based on properties of relations between “half-currents” (generating series for elements of deformations of $n_+ \otimes L_{\pm \lambda}$ and $n_+ \otimes O$), following from the vertex relations for $U_\hbar g(\tau)$ (sections 1.4, 1.5).

The decomposition $F$ as a product $F_2^2 F_1^1$ is carried out in section 4. It imitates the similar decomposition in $[3]$; applying certain projections
to the decomposition identity leads us to guess the values of $F^1_\lambda$. We then show that the decomposition identity is indeed satisfied. The proof uses Hopf algebra duality results of section 2, and results on coproducts of section 1.7.

Next we prove the the $F^1_\lambda$ obtained that way indeed satisfies a twisted cocycle equation. For that, we introduce subalgebras of $U_h\mathfrak{g}(\tau)$ of $A\geq 0 \otimes A\geq 0 \otimes A\leq 0$, and show that they contain images by $\Delta$ and $\bar{\Delta}$ of $A^- \otimes A^-$, $A^- \otimes A^-$, and show that they contain images by $\Delta$ and $\bar{\Delta}$ of $A^+ \otimes A^-$, $A^+ \otimes A^-$. This shows that the ratio $\Phi_\lambda$ between the two sides of the twisted cocycle equation has a special form ($\sum_i a_i \otimes h_i$, where $h$ is the “Cartan element” of $U_h\mathfrak{g}(\tau)$). We then prove a “twisted pentagon equation” for $\Phi_\lambda$ (prop. 5.3), which in fact shows that it is equal to 1.

After that we can apply the result of [3], and obtain a solution $R_\lambda$ of the dynamical YBE in $U_h\mathfrak{g}(\tau)$ (Thm. 6.1). We next study level zero representations of $U_h\mathfrak{g}(\tau)$; this study was led in the general case in [11]. We obtain a family $\pi_\zeta$ of 2-dimensional evaluation modules (Prop. 7.1), indexed by a point $\zeta$ of the formal neighborhood of the origin in $E$; we compute the image of $R_\lambda$ by these representations, and find an answer closely connected to $R(z,\lambda)$. This enables us to prove that the $L$-operators $(\pi_\zeta \otimes 1)(R_\lambda)$ and $(1 \otimes \pi_\zeta)(R_\lambda)$ satisfy the following dynamical version of the Reshetikhin–Semenov-Tian-Shansky relations (Thm. 7.1):

$$ R^\pm(\zeta - \zeta', \lambda) L^{\pm(1)}_{\lambda - \gamma h(1)}(\zeta) L^{\pm(2)}_{\lambda}(\zeta') = L^{\pm(2)}_{\lambda}(\zeta') L^{\pm(1)}_{\lambda}(\zeta) R^\pm(\zeta - \zeta', \lambda - \gamma h) $$

$$ L^{-(1)}_{\lambda}(\zeta) R^-(\zeta - \zeta', \lambda - \gamma h) = L_{\lambda - \gamma h(1)}^{+(2)}(\zeta') R^+(\zeta - \zeta' + K\gamma, \lambda) L^{-(1)}_{\lambda - \gamma h(2)}(\zeta) A(\zeta, \zeta' - K\gamma) A(\zeta, \zeta'), $$

where $R^\pm(z, \lambda)$ are elliptic $R$-matrices and $A$ is a certain elliptic version of the usual ratio of gamma-functions. These relations extend the $RLL$-relations of the elliptic quantum group, which we recover in Thm. 9.1.

Let us now say some words about problems connected to the present work. In [14], [15], quantum Knizhnik-Zamolodchikov-Bernard equations were defined as difference equations involving the dynamical $R$-matrix $R(z, \lambda)$. It would be interesting to derive these equations from considerations involving coinvariants of $U_h\mathfrak{g}(\tau)$-modules. This may also shed light on the question what the equation for dependence in the moduli should be in the quantum situation. There is some indication that this equation is the Ruijsenaars-Schneider (RS) equation at the critical level. At that level, one may apply the Reshetikhin-Semenov
method for expressing elements of the center of $U_{\hbar}g(\tau)$, and then explicitly compute their actions on coinvariant spaces. Recent work of A. Varchenko and the second author leads to the impression that the situation is more complicated outside the critical level. It might be interesting to connect such an approach to the RS models with those of \cite{1,2}.

Finally, it would be interesting to find an analogue of the theory developed in the present work, for the situations of higher genus (\cite{11}). Recall that the classical $r$-matrices underlying the elliptic quantum groups are dynamical $r$-matrices for the Hitchin system associated with an elliptic curves. Dynamical $r$-matrices for Hitchin systems in higher genus have been introduced in \cite{18,8}. In this respect, it seems that a dynamical version of the Poisson-Lie theory would be of interest.

Let us also mention here the work \cite{20}, where a dynamical approach to other “elliptic quantum groups” (those of \cite{14}) is presented.

This work was done during our stay at the “Semestre systèmes intégrables” organized at the Centre Emile Borel, Paris, UMS 839, CNRS/UPMC. We would like to express our thanks to its organizers for their invitation to this very stimulating meeting. We also would like to acknowledge discussions with O. Babelon, D. Bernard, C. Fronsdal, V. Rubtsov and A. Varchenko on the subject of this work.

1. Quantum loop algebras associated with elliptic curves

1.1. The classical situation. In \cite{12}, we constructed quasi-Hopf algebras, associated to the general data of a Frobenius algebra, a maximal isotropic subalgebra of it, and an invariant derivation. An example of such data is the following.

Let us fix a complex number $\tau$, with $\text{Im}(\tau) > 0$. Let $L \subset \mathbb{C}$ be the lattice $\mathbb{Z} + \tau \mathbb{Z}$; call $E$ the elliptic curve $\mathbb{C}/L$. Let $z$ be the coordinate on $\mathbb{C}$, and let $\omega$ be the differential form on $E$, equal to $dz$. Let $k = \mathbb{C}((z))$ be the completed local field of $E$ at its origin 0, and $\mathcal{O} = \mathbb{C}[[z]]$ the completed local ring at the same point. Endow $k$ with the scalar product $\langle \cdot, \cdot \rangle_k$ defined by

$$\langle f, g \rangle_k = \text{res}_0(fg\omega).$$

Define on $k$ the derivation $\partial$ to be equal to $d/dz$. Then $\partial$ is invariant w.r.t $\langle \cdot, \cdot \rangle_k$, and $\mathcal{O}$ is a maximal isotropic subring of $k$.

Let us set $a = \mathfrak{sl}_2(\mathbb{C})$, and denote by $\langle \cdot, \cdot \rangle_a$ an invariant scalar product on $a$. Let us set $g = (a \otimes k) \oplus CD \oplus CK$; let us define on $g$ the Lie algebra structure defined by the central extension of $a \otimes k$

$$c(x \otimes f, y \otimes g) = \langle x, y \rangle_a \langle f, \partial g \rangle_k K.$$
and by the derivation \([D, x \otimes f] = x \otimes \partial f\).

Let \(g_\Omega\) be the Lie subalgebra of \(g\) equal to \((\mathfrak{a} \otimes \mathcal{O}) \oplus \mathbb{C} D\). Define \(\langle , \rangle_{\mathfrak{a} \otimes k}\) as the tensor product of \(\langle , \rangle_{\mathfrak{a}}\) and \(\langle , \rangle_k\), and \(\langle , \rangle_g\) as the scalar product on \(g\) defined by \(\langle f, a \otimes k \rangle_g = \langle K, a \otimes k \rangle_g = 0\), and \(\langle D, K \rangle_g = 1\). Then \(g_\Omega\) is a maximal isotropic Lie subalgebra of \(g\).

To define maximal isotropic supplementaries of \(g_\Omega\) in \(g\), we first define certain subspaces of \(k\).

For \(\lambda \in \mathbb{C}\), define \(L_\lambda\) as follows. If \(\lambda\) does not belong to \(L\), define \(L_\lambda\) to be the set of expansions near 0 of all holomorphic functions on \(\mathbb{C} - L\), \(1\)-periodic and such that \(f(z + \tau) = e^{-2i\pi\lambda} f(z)\). For \(\lambda = 0\), define \(L_\lambda\) as the maximal isotropic subspace of \(k\) containing all holomorphic functions \(f\) on \(\mathbb{C} - L\), \(L\)-periodic, such that \(\oint_a f(z) dz = 0\), where \(a\) is the cycle \((i\epsilon, i\epsilon + 1)\) (with \(\epsilon\) small and > 0). Finally, define \(L_\lambda = e^{-2i\pi m z} L_0\) for \(\lambda = n + m\tau\).

Let \(\theta\) be the holomorphic function defined on \(\mathbb{C}\) by the conditions that \(\theta'(0) = 1\), the only zeroes of \(\theta\) are the points of \(L\), \(\theta(z + 1) = -\theta(z)\), and \(\theta(z + \tau) = -e^{i\pi\tau} e^{-2i\pi z} \theta(z)\). \(\theta\) is then odd.

We then have

\[
L_\lambda = \bigoplus_{j \geq 0} \mathbb{C} \left( \frac{\theta'}{\theta} \right)^{(j)} e^{-2i\pi mz}, \quad \text{if} \quad \lambda = n + m\tau, \tag{1}
\]

\[
L_\lambda = \bigoplus_{i \geq 0} \mathbb{C} \left( \frac{\theta(\lambda + z)}{\theta(z)} \right)^{(i)} , \quad \text{if} \quad \lambda \in \mathbb{C} - L, \tag{2}
\]

where for \(f \in k\), we let \(f' = \partial f\) and \(f^{(i)} = \partial^i f\).

Moreover, the orthogonal of \(L_\lambda\) for the scalar product \(\langle , \rangle_k\) is equal to \(L_{-\lambda}\).

Consider now the decomposition

\[
g = g_\Omega \oplus g_\lambda, \tag{3}
\]

where

\[
g_\lambda = (\mathfrak{h} \otimes L_0) \oplus (\mathfrak{n}_+ \otimes L_\lambda) \oplus (\mathfrak{n}_- \otimes L_{-\lambda}) \oplus \mathbb{C} K. \tag{4}
\]

\(g_\Omega\) is a maximal isotropic subalgebra of \(g\), and \(g_\lambda\) is a maximal isotropic subspace of it. Therefore, (3) defines a Lie quasi-bialgebra structure on \(g_\Omega\), and (as in [12]), of double Lie quasi-bialgebra on \(g\). Its classical \(r\)-matrix is given by the formula

\[
r_\lambda = D \otimes K + \sum_i \frac{1}{2} h[e^i] \wedge h[e_{i|0}] + e[e^i] \wedge f[e_{i|\lambda}] + f[e^i] \wedge e[e_{i|-\lambda}],
\]
for \((e^i)_{i \geq 0}, (e_{i;\lambda})_{i \geq 0}\) dual bases of \(O\) and \(L_\lambda\), with the valuation of \(e^i\) tending to infinity with \(i\), and we denote \(x \otimes f\) by \(x|f|\); in other terms,
\[
r_\lambda(z, w) = \frac{1}{2}(h \otimes h) \frac{\theta'}{\theta}(z - w) + (e \otimes f) \frac{\theta(z - u + \lambda)}{\theta(z - u)} + (f \otimes e) \frac{\theta(z - u - \lambda)}{\theta(z - u)\theta(\lambda)} + D \otimes K;
\]
this formula (without \(D \otimes K\)) coincides with that of the classical \(r\)-matrix arising in the elliptic versions of the KZB equations (see [17]) and of the Hitchin system (see [10]).

Remark 1. \(O\) plays the role of the ring \(R\), in the notation of [12].

1.2. Relations for \(U_\hbar g(\tau)\). The quasi-Hopf algebra associated in [12] to the Lie quasi-bialgebras \(g\) and \(g_\hbar O\), are twists of the Hopf algebra \((U_\hbar g(\tau), \Delta)\) and of its subalgebra \(U_\hbar g_\hbar O(\tau)\), that we now present. We will sometimes denote \(U_\hbar g(\tau)\) by \(A(\tau)\) or simply by \(A\), and \(U_\hbar g_\hbar O\) by \(A^+\).

Let \(h\) be a formal parameter. Generators of \(U_\hbar g(\tau)\) are \(D, K\) and the \(x[e], x = e, f, h, \epsilon \in k\); they are subject to the relations
\[
x[\alpha e] = \alpha x[e], \quad x[\epsilon + e'] = x[\epsilon] + x[e'], \quad \alpha \in \mathbb{C}, \epsilon, e' \in k.
\]
They serve to define the generating series
\[
x(z) = \sum_{i \in \mathbb{Z}} x[e^i]e_i(z), \quad x = e, f, h,
\]
\((e^i)_{i \in \mathbb{Z}}, (e_i)_{i \in \mathbb{Z}}\) dual bases of \(k\); recall that \((e^i)_{i \in \mathbb{N}}, (e_{i;0})_{i \in \mathbb{N}}\) are dual bases of \(O\) and \(L_0\) and set
\[
h^+(z) = \sum_{i \in \mathbb{N}} h[e^i]e_{i;0}(z), \quad h^-(z) = \sum_{i \in \mathbb{N}} h[e_{i;0}]e^i(z).
\]
We will also use the series
\[
K^+(z) = e^{\sum_{i \neq 0} \hbar\theta^{-1}(z + \hbar i)}, \quad K^-(z) = q^{h^-(z)},
\]
where \(q = e^{\hbar}\). The relations presenting \(U_\hbar g(\tau)\) are then
\[
[K^+(z), K^+(w)] = [K^-(z), K^-(w)] = 0, \quad \theta(z - w - \hbar)\theta(z - w + \hbar + hK)K^+(z)K^-(w) \quad (7)
\]
\[
= \theta(z - w + \hbar)\theta(z - w - \hbar + hK)K^-(w)K^+(z),
\]
\[
K^+(z)e(w)K^+(z)^{-1} = \frac{\theta(z - w + \hbar)}{\theta(z - w - \hbar)}e(w) \quad (8)
\]

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moreover, we will have
\(K^+(z)e(w)K^+(z)^{-1} = \frac{\theta(w-z + hK + \hbar)}{\theta(w-z + hK - \hbar)}e(w),\) (9)

\(K^+(z)f(w)K^+(z)^{-1} = \frac{\theta(w-z + \hbar)}{\theta(w-z - \hbar)}f(w),\) (10)

\(K^-(z)f(w)K^-(z)^{-1} = \frac{\theta(z-w + \hbar)}{\theta(z-w - \hbar)}f(w),\) (10)

\(\theta(z-w - \hbar)e(z)e(w) = \theta(z-w + \hbar)e(w)e(z),\) (11)

\(\theta(w-z - \hbar)e(z)f(w) = \theta(w-z + \hbar)f(w)f(z),\) (12)

\([e(z), f(w)] = \frac{1}{\hbar}\left(\delta(z,w)K^+(z) - \delta(z,w - \hbar K)K^-(w)^{-1}\right).\) (13)

Let us introduce the generating series \(k^+(z)\) and \(k^-(z)\), defined by
\(k^+(z) = e(\frac{\theta}{2\hbar}h^+(z)), \quad k^-(z) = q^{(1+q^{-1})h^-}(z);\) (14)

they satisfy the relations
\(K^+(z) = k^+(z)k^+(z - \hbar), \quad K^-(z) = k^-(z)k^-(z - \hbar).\) (15)

Equations (8), (9) and (10) may be replaced by
\(k^+(z)e(w)k^+(z)^{-1} = \frac{\theta(w-z + \hbar)}{\theta(z-w)}e(w),\) (16)

\(k^-(z)e(w)k^-(z)^{-1} = \frac{\theta(w-z + hK)}{\theta(w-z + hK - \hbar)}e(w),\)

and
\(k^+(z)f(w)k^+(z)^{-1} = \frac{\theta(w-z)}{\theta(w-z - \hbar)}f(w),\) (17)

\(k^-(z)f(w)k^-(z)^{-1} = \frac{\theta(z-w + \hbar)}{\theta(z-w)}f(w);\) (17)

moreover, we will have
\((k^+(z), k^-(w)) = 1, \quad (k^+(z), k^-(w)) = f_K(z-w),\) (18)

where we use the group commutator notation \((x, y) = xyx^{-1}y^{-1}\) and \(f_K(\zeta)\) is the formal series of \(1 + \hbar C[[\zeta]][[\hbar]]\) defined by the functional equation
\(f_K(\zeta)f_K(\zeta - \hbar) = \frac{\theta(\zeta + h)}{\theta(\zeta)}\frac{\theta(\zeta + hK)}{\theta(\zeta + hK)};\) (19)

if \(K = -2p,\) with \(p\) integer, we have
\(f_K(\zeta) = \frac{\theta(\zeta)}{(\prod_{k=0}^{p-1} \theta(\zeta - (2k+1)\hbar))} \frac{(\prod_{k=0}^{p-1} \theta(\zeta - 2k\hbar))}{(\prod_{k=1}^{p-1} \theta(\zeta - 2k\hbar))}.\) (20)
The algebra $U_\hbar g(\tau)$ is endowed with a Hopf structure given by the coproduct $\Delta$ defined by

$$\Delta(k^+(z)) = k^+(z) \otimes k^+(z), \quad \Delta(K^-(z)) = K^-(z) \otimes K^-(z + hK_1),$$

$$\Delta(e(z)) = e(z) \otimes K^+(z) + 1 \otimes e(z), \quad \Delta(f(z)) = f(z) \otimes 1 + K^-(z)^{-1} \otimes f(z + hK_1),$$

$$\Delta(D) = D \otimes 1 + 1 \otimes D, \quad \Delta(K) = K \otimes 1 + 1 \otimes K,$$

the counit $\varepsilon$, and the antipode $S$ defined by them; we set $K_1 = K \otimes 1, K_2 = 1 \otimes K$.

$U_\hbar g(\tau)$ is also endowed with another Hopf structure given by the coproduct $\bar{\Delta}$ defined by

$$\bar{\Delta}(k^+(z)) = k^+(z) \otimes k^+(z), \quad \bar{\Delta}(K^-(z)) = K^-(z) \otimes K^-(z + hK_1),$$

$$\bar{\Delta}(e(z)) = e(z - hK_2) \otimes K^-(z - hK_2)^{-1} + 1 \otimes e(z), \quad \bar{\Delta}(f(z)) = f(z) \otimes 1 + K^+(z) \otimes f(z),$$

$$\bar{\Delta}(D) = D \otimes 1 + 1 \otimes D, \quad \bar{\Delta}(K) = K \otimes 1 + 1 \otimes K,$$

the counit $\varepsilon$, and the antipode $\bar{S}$ defined by them.

The Hopf structures associated with $\Delta$ and $\bar{\Delta}$ are connected by a twist

$$F = \exp \left( \hbar \sum_{i \in \mathbb{Z}} \epsilon_i \otimes \epsilon^i \right),$$

where $(\epsilon^i)_{i \in \mathbb{Z}}$ is the basis of $k$ dual to $(\epsilon_i)_{i \in \mathbb{Z}}$ w.r.t. $\langle , \rangle_k$; that is, we have $\bar{\Delta} = \text{Ad}(F) \circ \Delta$.

Then $F$ satisfies the cocycle equation

$$(F \otimes 1)(\Delta \otimes 1)(F) = (1 \otimes F)(1 \otimes \Delta)(F)$$

(see [12], Prop. 3.1).

**Remark 2.** In the notation of [12], 8.2, we have

$$q(z, w) = \frac{\theta(z - w + h)}{\theta(z - w - h)},$$

and $\sigma(z) = z - hK$. We also have the relation

$$[h^+(z), h^-(w)] = \frac{1}{h} \left( \frac{\theta'(z - w)}{\theta(z - w)} - \frac{\theta'(z - w + hK)}{\theta(z - w + hK)} \right).$$
1.3. **Completions.** The algebras and vector spaces introduced above possess natural topologies: the field \( k \) and the ring \( O \) are given the formal series topology; on the other hand, the spaces \( L_\lambda \) are given the discrete topology. For \( V, W \) two topological vector spaces, with basis of neighborhoods of the origin \( (V_a)_{a \in \mathbb{Z}} \) and \( (W_b)_{b \in \mathbb{Z}} \), define \( V \otimes W \) as the inverse limit of the \( V \otimes W / V_a \otimes W_b \), and \( V \bar{\otimes} W \) as the inverse limit of the \( (V \otimes W) / (V \otimes W + V \otimes W_b) \).

Define then the completed tensor algebra \( \hat{T}(V) \) of \( V \) as the direct sum \( \bigoplus_{i \geq 0} V^\otimes i \). Then \( \mathcal{U}_\hbar \mathfrak{g}(\tau) \) is viewed as a quotient of \( \hat{T}(\mathfrak{g}) \), and is endowed with the corresponding topology. \( \mathcal{U}_\hbar \mathfrak{g} \bar{\otimes} O \) is then a closed subspace of \( \mathcal{U}_\hbar \mathfrak{g}(\tau) \). On the other hand, the fields \( x(z) \) belong to the completed tensor product \( \mathcal{U}_\hbar \mathfrak{g}(\tau) \bar{\otimes} k \). The coproduct \( \Delta \) maps \( \mathcal{U}_\hbar \mathfrak{g}(\tau) \) to the completion of its tensor square defined as the suitable quotient of \( \hat{T}(\mathfrak{g} \oplus \mathfrak{g}) \). In the sequel we will consider tensor product of subspaces of \( \mathcal{U}_\hbar \mathfrak{g}(\tau) \) to be completed w.r.t. the topology of \( \hat{T}(\mathfrak{g} \oplus \mathfrak{g}) \), \( \hat{T}(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}) \), etc.

1.4. **Relations for half-currents.** Fix a complex number \( \lambda \) and set for \( x = e, f, K^+ \),

\[ x^+_{\lambda}(z) = \sum_i x[e^i]e_{i;\lambda}(z), \quad (31) \]

and for \( x = e, f, K^- \),

\[ x^-_{\lambda}(z) = \sum_i x[e_{i;\lambda}]e_i(z); \quad (32) \]

recall that \( (e^i), (e_{i;\lambda}) \) are dual bases of \( O \) and \( L_\lambda \).

The fields \( e(z) \) and \( f(z) \) are then split according to

\[ e(z) = e^+_{\lambda}(z) + e^-_{\lambda}(z), \quad f(z) = f^+_{-\lambda}(z) + f^-_{-\lambda}(z); \quad (33) \]

we call the expression \( x^+_{\lambda}(z) \) “half-currents”. In the above equality, we made use of the continuous inclusions of \( \mathcal{U}_\hbar \mathfrak{g} \bar{\otimes} L_\lambda \) and of \( \mathcal{U}_\hbar \mathfrak{g} \bar{\otimes} O \) into \( \mathcal{U}_\hbar \mathfrak{g} \bar{\otimes} k \).

For \( x = e, f \), we have \( x^-_{\lambda}(z) \in \mathcal{U}_\hbar \mathfrak{g}(\tau) \bar{\otimes} O \), so that \( x^-(z) \) can be viewed as a formal series in \( z \), regular at 0, and \( x^+_{\lambda}(z) \in \mathcal{U}_\hbar \mathfrak{g}(\tau) \bar{\otimes} L_\lambda \), so that \( x^+_{\lambda}(z) \) can be viewed as a function of \( z \), satisfying

\[ x^+_{\lambda}(z + 1) = x^+_{\lambda}(z), \quad x^+_{\lambda}(z + \tau) = e^{-2i\pi \lambda} x^+_{\lambda}(z). \quad (34) \]

We then have:
Proposition 1.1. The generating series $e_{\lambda}^{\pm}(z), f_{\lambda}^{\pm}(z)$ satisfy the following relations:

$$
\frac{\theta(z-w-h)}{\theta(z-w)} e_{\lambda+h}^{\epsilon}(z)e_{\lambda-h}^{\epsilon'}(w) + \epsilon \epsilon' \frac{\theta(w-z-\lambda)\theta(-h)}{\theta(w-z)} e_{\lambda+h}^{\epsilon'}(w)e_{\lambda-h}^{\epsilon'}(w) = \frac{\theta(z-w+h)}{\theta(z-w)} e_{\lambda-h}^{\epsilon}(z)e_{\lambda-h}^{\epsilon'}(w) + \epsilon \epsilon' \frac{\theta(z-w-\lambda)\theta(h)}{\theta(z-w)} e_{\lambda-h}^{\epsilon'}(w)e_{\lambda-h}^{\epsilon'}(w)
$$

where $\epsilon, \epsilon'$ take the values $+, -$. In these relations, the expressions of the form $\frac{1}{z-w}(f'(z,w)x_{\lambda}^{\pm}(z)-f''(z,w)x_{\lambda}^{\pm}(w))$, resp. $\frac{1}{z-w}(f'(z,w)x_{\lambda}^{\pm}(z)-f''(z,w)x_{\lambda}^{\pm}(w))$, where $f', f''$ are formal series in $z, w$ coinciding for $z = w$, and $x = e, f, K^+$, should be understood as the sums

$$
\sum_{i \geq 0} x^i e \frac{f'(z,w)e_{i,\lambda}(z)-f''(z,w)e_{i,\lambda}(w)}{z-w}, \quad \text{resp.} \quad \sum_{i < 0} x^i e \frac{f'(z,w)e^i(z)-f''(z,w)e^i(w)}{z-w},
$$

which belong to $U_{\hbar}g(\tau) \otimes (k \otimes k)$.

Proof. Let us show relation (35) in the case $\epsilon = \epsilon' = +$. Let us denote by $Z_{++}(z,w)$ the difference of the left and right hand sides of this equation, and by $\ell$ any continuous linear form on $A(\tau)$. Clearly, $\ell(Z_{++}(z,w))$ belongs to $L_\lambda \otimes L_\lambda$ (recall that $\ell[e_i e_j]$ is equal to zero when $i$ or $j$ are large enough), and is antisymmetric. (We attach coordinates $z, w$ to the first and second factor of the tensor product.) On the other hand, the difference of $\ell(\theta(z-w)Z_{++}(z,w))$ and of $\ell(\theta(z-w-h)\theta(z-w+h)e(w) e(z))$ can be expressed as a sum of quadratic monomials in the $e_{\lambda+h}^{\pm}(z,w)$ using at least one $e_{\lambda-h}^{\pm}(z,w)$, and therefore belongs to $k \otimes \Omega + \Omega \otimes k$. Therefore, the same is true for $(z-w)\ell(Z_{++}(z,w))$. Let us set $(z-w)Z_{++}(z,w) = Y_1 + Y_2$, with $Y_1 \in k \otimes \Omega, Y_2 \in \Omega \otimes k$. Let us denote by a tilde the exchange of arguments $z$ and $w$, and set $Y = (Y_1 + \tilde{Y}_2)/2$. We have $Y \in k \otimes \Omega$ and

$$(z-w)\ell(Z_{++}(z,w)) = Y + \tilde{Y}.$$

Set now $Z = Y \sum_{i \geq 0} z^{-i-1}w^i$; since $Y$ belongs to $k \otimes \Omega = k[[w]]$, so does $Z$. Since we have

$$(z-w)\ell(Z_{++}(z,w)) - Z + \tilde{Z} = 0,$$

it follows that for some $f$ in $k$, we have

$$\ell(Z_{++}(z,w)) - Z + \tilde{Z} = f(z)\delta(z,w).$$

(37)
Consider both sides of (37) as the kernel of some operator, defined by $(Tf_0)(z) = \text{res}_0(f_0(z)\delta(z,w)dw$. Since the l.h.s. of (37) is antisymmetric in $z$ and $w$, this operator should be antisymmetric, i.e. satisfy $\langle Tf_0, g_0 \rangle_k + \langle f_0, Tg_0 \rangle_k = 0$, for $f_0, g_0 \in k$. On the other hand, $T$ coincides with the multiplication operator by $f$. It follows that $f = 0$. Therefore $\ell(Z_{++}(z,w)) = Z - \bar{Z}$. Since the left and right hand sides of this equality belongs to $L_\lambda \otimes L_\lambda$ and to $\mathcal{O} \otimes k + k \otimes \mathcal{O}$ respectively, $\ell(Z_{++}(z,w)) = 0$. This proves (35) in the case where $\epsilon = \epsilon' = +$.

Let us now show (35) in the case where $\epsilon = \epsilon' = -$. Let us denote by $Z_{--}(z,w)$ the difference of the left and right hand sides of this equation. Clearly, $\ell(Z_{--}(z,w))$ belongs to $(\mathcal{O} \otimes \mathcal{O})[[\hbar]]$, and is antisymmetric. On the other hand, the difference of $\theta(z-w)\ell(Z_{--}(z,w))$ with $\ell(\theta(z-w-h)\epsilon(z)e(w) - \theta(z-w-h)\epsilon(e)(w)\epsilon(z))$ belongs to $\mathcal{F}_{\lambda,*} + \mathcal{F}_{\lambda,*}$, where $\mathcal{F}_{\lambda,*}$ is the subspace of $\text{Hol}(\mathbb{C} - L)((w))$ formed by the functions $f(z,w)$ such that

$$f(z+1,w) = f(z,w), \quad f(z+\tau,w) = -e^{-i\pi\tau}e^{-2i\pi\lambda}e^{-2i\pi(z-w)}f(z,w),$$

any $\mathcal{F}_{\lambda,*}$ is the subspace of $\text{Hol}(\mathbb{C} - L)((z))$ formed by the functions $g(z,w)$ such that

$$g(z,w+1) = g(z,w), \quad g(z,w+\tau) = -e^{-i\pi\tau}e^{-2i\pi\lambda}e^{-2i\pi(z-w)}g(z,w);$$

here $\text{Hol}(\mathbb{C} - L)$ is the space of holomorphic functions defined on $\mathbb{C} - L$.

Set $\theta(z-w)\ell(Z_{--}(z,w)) = Y_1' + Y_2'$, with $Y_1' \in \mathcal{F}_{\lambda,*}$ and $Y_2' \in \mathcal{F}_{\lambda,*}$. Set $Y' = (Y_1' + Y_2')/2$. We have $Y' \in \mathcal{F}_{\lambda,*}$ and

$$\theta(z-w)\ell(Z_{++}(z,w)) = Y' + \bar{Y}'$$

Set now

$$Z' = Y' \sum_{i\geq 0} (\theta^{-1})(i)(z)(-w)^i/i!$$

then $Z'$ belongs to $\text{Hol}(\mathbb{C} - L)((w))$, and we have as before

$$\ell(Z_{--}(z,w)) = Z' - \bar{Z}' + f'(z)\delta(z,w),$$

for some $f' \in k$. The same reasoning as above shows that $f' = 0$ and then that $\ell(Z_{--}(z,w)) = 0$.

Let us now prove (35) in the case where $\epsilon = +, \epsilon' = -$. Let us denote by $Z_{+-}(z,w)$ the difference of the left and right hand sides of this equality. Let us substract from (14), the sum of equalities (35) with $\epsilon = \epsilon' = +$ and $\epsilon = \epsilon' = -$. We obtain that $Z_{+-}(z,w) + Z_{+-}(w,z) = 0$. Therefore $Z_{+-}(z,w)$ is antisymmetric in $z$ and $w$. On the other hand, we have for any linear functional $\ell$ on $A(\tau)$, $\ell(Z_{+-}(z,w)) \in L_\lambda \otimes \mathcal{O}$. Since the intersection of $L_\lambda$ and $\mathcal{O}$ is zero, $Z_{+-}(z,w)$ is equal to zero. Therefore (35) is valid in the case $\epsilon = +, \epsilon' = -$. 

ELLITIC QUANTUM GROUPS $E_{r,n}(sl_2)$ AND QUASI-HOPF ALGEBRAS
The case \( \epsilon = -, \epsilon' = + \) is obtained from \( \epsilon = +, \epsilon' = - \), by exchanging \( z \) and \( w \).

Relations (36) can be obtained in a similar way. \( \square \)

Let us define \( U_h n_+(\tau) \) and \( U_h n_-(\tau) \) as the subalgebras of \( A(\tau) \) generated by the \( e[\epsilon], \epsilon \in k \) and the \( f[\epsilon], \epsilon \in k \).

Let us denote by \( U^{(e)}_+(\tau) \), resp. \( U^{(f)}_+(\tau) \) the subalgebras of \( U_h n_+(\tau) \) generated by the \( e[r], r \in O \), resp. \( f[r], r \in O \).

For \( \mu \in \mathbb{C} - L \), and \( x = e, f, h \), denote by \( x[\mu] \) the element \( x[\mu(e)] \), where \( p_{-\mu} \) is the projection on \( L_{-\mu} \) parallel to \( O \). For \( \beta \in \mathbb{C} \), define also \( x[\mu+\beta n][\epsilon] = \sum_{i \geq 0} \partial^i x[\mu][\epsilon]/\partial \mu^i(\beta n)^i/\beta! \).

Let us denote by \( U^{(e)}_-(\tau) \), \( U^{(f)}_-(\tau) \) the subspaces of \( U_h n_-(\tau) \) linearly spanned by the products \( e^{-\lambda+2\eta_1}[\eta_0] \cdots e^{-\lambda}[\eta_n] \), resp. by the products \( f^{-\lambda}[\eta_0] \cdots f^{-\lambda+2\eta_1}[\eta_n], n \geq 0, \eta_i \in k \).

Let \( \epsilon_i \in \mathbb{Z} \) be a basis of \( k \) such that \( \epsilon_i = e^i \) for \( i \geq 0 \). We can now formulate a Poincaré-Birkhoff-Witt result for \( U_h n_\pm(\tau) \).

**Proposition 1.2.** 1) Bases of \( U^{(e)}_+(\tau) \) and of \( U^{(f)}_+(\tau) \) are respectively given by the monomials \( e[\epsilon_{i_0}] \cdots e[\epsilon_{i_p}] \), \( f[\epsilon_{i_0}] \cdots f[\epsilon_{i_p}] \), \( i_0 \leq i_1 \leq \cdots \leq i_p \); bases of \( U^{(e)}_-(\tau) \) and \( U^{(f)}_-(\tau) \) are respectively given by the \( e^{-\lambda+2\eta_1}[\epsilon_{i_0}] \cdots e^{-\lambda}[\epsilon_{i_n}], i_0 \leq \cdots \leq i_n < 0 \), resp. by the \( f^{-\lambda}[\epsilon_{i_0}] \cdots f^{-\lambda+2\eta_1}[\epsilon_{i_n}], i_0 \leq \cdots \leq i_n < 0 \).

2) The maps \( U^{(e)}_+(\tau) \otimes U^{(e)}_-(\tau) \to U_h n_+(\tau) \), \( U^{(e)}_+(\tau) \otimes U^{(f)}_-(\tau) \to U_h n_+(\tau) \), \( U^{(f)}_+(\tau) \otimes U^{(e)}_-(\tau) \to U_h n_-(-\tau) \), \( U^{(f)}_+(\tau) \otimes U^{(f)}_-(\tau) \to U_h n_-(-\tau) \), induced by the multiplication, define vector spaces isomorphisms.

**Proof.** We should first derive identities expressing the \( e[\epsilon_i] e[\epsilon_j], i > j \) in terms of combinations of the \( e[\epsilon_k] e[\epsilon_l], k \leq l \), for \( i, j, k, l \geq 0 \); the \( e^{-\lambda+2\eta_1}[\epsilon_k] e^{-\lambda}[\epsilon_j], i > j \) in terms of the \( e^{-\lambda+2\eta_1}[\epsilon_k] e^{-\lambda}[\epsilon_l], k \leq l \), for \( i, j, k, l < 0 \); the \( e^{-\lambda}[\epsilon_i] e^{-\lambda}[\epsilon_j], i < 0 \) in terms of the \( e[\epsilon_k] e^{-\lambda}[\epsilon_l], l < 0 \leq k \), and the \( e[\epsilon_i] e^{-\lambda}[\epsilon_j], j < 0 \) in terms of the \( e[\epsilon_k] e^{-\lambda}[\epsilon_l], k < 0 \leq l \). For this, we may first assume that \( \epsilon_i = z^i \). Then we multiply (35) by \( z - w \) and combine the Fourier coefficients as in [12], sect. 4. This proves that the families of 1) generate \( U^{(e)}_+ \) and \( U^{(e)}_- \), and that the two first maps of 2) are surjective.

The facts that these families are free, and that these maps are injective, follow from [12], Lemma 4.4. \( \square \)

**Remark 3.** An informal way to derive (35) is the following one. For example, if \( \epsilon = \epsilon' = + \), we have

\[
e^{+}_\lambda(z) = \int_{C_\theta} \frac{\theta(\zeta - z - \lambda)}{\theta(\zeta - z)\theta(-\lambda)} e(\zeta) d\zeta, \quad e^{-}_\lambda(z) = \int_{C_{0,z}} \frac{\theta(\zeta - z - \lambda)}{\theta(\zeta - z)\theta(-\lambda)} e(\zeta) d\zeta,
\]

(38)
that is
\[ e\in \cdot \] in the r.h.s. by the resulting identity, replace in the l.h.s., e.e by θ

Clockwise (resp. clockwise), and

The contribution of the terms in e.e λ, z, ℏ, ζ, τ, η, (39) being a contour encircling 0 and \( z \) (resp. 0) counter-clockwise (resp. clockwise), and

Multiply the identity

\[
\frac{\theta(\zeta - \zeta' - h)}{\theta(\zeta - \zeta')} e(\zeta) e(\zeta') = \frac{\theta(\zeta - \zeta' + h)}{\theta(\zeta - \zeta')} e(\zeta') e(\zeta)
\]

by \( \frac{\theta(\zeta - \zeta' - h)}{\theta(\zeta - \zeta')} \frac{\theta(\zeta' - w - \lambda)}{\theta(\zeta' - w)} \), and integrate it over the cycles \( C_0^- \) for \( \zeta \), and \( C_{0'}^- \) for \( \zeta' \), where \( C_{0'}^- \) is a deformation of \( C_0^- \), such that \(|\zeta| < |\zeta'|\). In the resulting identity, replace in the l.h.s., \( e(\zeta) \) by \( e_+^{\pm}(\zeta') + e_-^{\pm}(\zeta) \) and in the r.h.s. by \( e_+^{\pm}(\zeta') + e_-^{\pm}(\zeta') \). The contributions to the integral of the terms in \( e^- \) vanish, because these terms are regular at 0. We then obtain

\[
\oint_{C_0^-} \frac{\theta(\zeta' - w - \lambda)}{\theta(\zeta' - w) \theta(\zeta' - z) \theta(\zeta' - -\lambda)} \left\{ \frac{\theta(\zeta - \zeta' - h)}{\theta(\zeta - \zeta')} e_+^{\pm}(\zeta') e_+^{\pm}(\zeta) d\zeta'ight\} \\
+ \oint_{C_{0'}^-} \frac{\theta(\zeta' - z - \lambda) \theta(\zeta - \zeta')}{\theta(\zeta' - z) \theta(\zeta' - -\lambda)} e_+^{\pm}(\zeta') e_+^{\pm}(\zeta) d\zeta'
\]

that is

\[
e_+^{\pm}(\zeta') e(\zeta') + e(\zeta') e_+^{\pm}(\zeta') - e_+^{\pm}(\zeta') e_-^{\pm}(\zeta')
\]

Specializing (\( \mathbb{P} \)) for \( \zeta = \zeta' \), we find \( e(\zeta')^2 = 0 \). Therefore

\[
e_+^{\pm}(\zeta') e(\zeta') + e(\zeta') e_+^{\pm}(\zeta') = e_+^{\pm}(\zeta') e_+^{\pm}(\zeta') - e_+^{\pm}(\zeta') e_-^{\pm}(\zeta')
\]

The contribution of the terms in \( e^- \) to the the r.h.s. of (40) is zero, since these terms are regular at 0, and the contribution of the terms in \( e^+ \) is evaluated by the residues formula.

(Note that this method apparently cannot be adapted to derive relations between \( e_+^z(z) \) and \( e_+^w(w) \) when \( \lambda \neq \lambda' \pm 2\hbar \), because then the
term in $e^+$ in the r.h.s. of (40) would no longer be a meromorphic function on $E$.)

As we will see from Thm. \[1\] it is also possible to derive relations between fields $k^+(z)$ and $e^+_\lambda(w), f^+_\lambda(w)$; for example, we have

$$k^+(z)e^+_\lambda(w)k^+(z)^{-1} = \frac{\theta(z-w+h)}{\theta(z-w)}e^+_\lambda+\theta(h)-\frac{\theta(z-w-\lambda)\theta(h)}{\theta(z-w)\theta(-\lambda)}e^+_{\lambda+h}(z),$$  \hfill (41)

etc.

### 1.5. Shifts in $h$. In what follows, we will simply denote $h[1]$ by $h$. We will also use the following notation. Let us define in each tensor power \(A^{{\otimes}_n}, a^{(j)}\) as the element \(1^{{\otimes}(i-1)} \otimes a \otimes 1^{{\otimes}(n-i)}\) for \(a \in A\), and for \(\beta \in \mathbb{C}\), \(x_{\mu+h\beta h^{(i)}}[\epsilon]\) as

\[
\frac{\partial^n x_{\mu+h\beta h^{(i)}}[\epsilon]}{\partial \mu^n} \frac{(h\beta h)^n}{\alpha!}.
\]

If $n = 1$, let us denote also $x_{\mu+h\beta h^{(i)}}^{-1}[\epsilon]$ simply by $x_{\mu+h\beta h}[\epsilon]$. Let us also set for $\epsilon = +, -$,

\[
x_{\mu+h\beta h}^+(z) = \sum_{\alpha \geq 0} (\partial/\partial \mu)^\alpha (x_{\mu}^+(z)) (h\beta h)^\alpha/\alpha!;
\]

we have then

\[
x(z) = x_{\mu+h\beta h}^+(z) + x_{\mu+h\beta h}^-(z), \quad x_{\mu+h\beta h}^-(z) = \sum_{i \in \mathbb{Z}} x_{\mu+h\beta h}^{-1}[\epsilon_i] \epsilon_i(z).
\]

#### Lemma 1.1. We have the relations

\[
\frac{\theta(z-w+h)}{\theta(z-w)}f^+_{\lambda+h}(z)f^+_{\lambda+h}(w) + \epsilon \frac{\theta(w-z-\lambda-hh-3h)\theta(h)}{\theta(w-z)\theta(-\lambda-hh-3h)}f^+_{\lambda+h}(w)f^+_{\lambda+h}(w)
\]

\[
\frac{\theta(z-w-h)}{\theta(z-w)}f^+_h(w)f^+_h(z) + \epsilon \frac{\theta(z-w-\lambda-hh-3h)\theta(h)}{\theta(z-w)\theta(-\lambda-hh-3h)}f^+_h(w)f^+_h(z),
\]

where $\epsilon, \epsilon'$ take the values $+$ and $-$. \hfill (42)

**Proof.** The identity

\[
\sum_{n \geq 0} \phi(\lambda) (\partial/\partial \lambda)^n \left(f^+_{\lambda}(z_1)f^+_{\lambda}(z_2)\right) \frac{(hh)^n}{n!} \rho(\lambda) \left(h(h+4)\right) r!
\]

\[
f^+_{\lambda}(z_1) \frac{(hh+2)}{q!} f^+_{\lambda}(z_2) \frac{(hh)^q}{p!}
\]

implies that

\[
\sum_{n \geq 0} \phi(\lambda) f^+_{\lambda}(z_1)f^+_{\lambda}(z_2) \frac{(hh)^n}{n!} = \phi(\lambda + h(h+4)) f^+_{\lambda+h+h}(z_1)f^+_{\lambda+h+h}(z_2).
\]

(42) then follows from (36), after the change of $\lambda$ into $\lambda - h$. \hfill \Box
In what follows we will use the notation

\[ \gamma = -\hbar; \]

as we will see, \( \gamma \) is connected to the \( \eta \) of elliptic quantum groups by the relation \( \gamma = 2\eta \).

1.6. **Properties of** \( K^+(z) \). Since

\[ h^+(z) = \frac{\theta'}{\theta}(z)h^+[1] + \sum_{i>0} h^+[\epsilon_i]e^i(z), \]

where the \( e^i \) are elliptic functions, we have

\[
\left( \frac{q^\partial - q^{-\partial}}{2\partial} h^+ \right)(z) = \frac{1}{2} \ln \left( \frac{\theta(z + \hbar)}{\theta(z - \hbar)} \right) h^+[1] + \sum_{i>0} h^+[\epsilon_i]e^i(z),
\]

with \( e^i \) again elliptic functions. Therefore,

\[ K^+(z) = \left( \frac{\theta(z - \gamma)}{\theta(z + \gamma)} \right)^{h/2} \sum_{i \geq 0} \varphi^{(i)}(z)\alpha_i, \quad \alpha_i \in U_{\hbar\mathfrak{h}}(\tau); \quad (43)\]

here \( \left( \frac{\theta(z - \gamma)}{\theta(z + \gamma)} \right)^{h/2} \) is defined as \( \exp \left( \frac{h}{2} \ln \frac{\theta(z + \gamma)}{\theta(z - \gamma)} \right) \), where the argument of the exponential is considered as a formal power series in \( \gamma \), and we define \( \varphi \) by \( \varphi = -\partial^2(\ln \theta) \).

**Remark 4.** \( (43) \) implies that \( K^+(z) \) has the properties

\[ K^+(z + 1) = K^+(z), \quad K^+(z + \tau) = e^{-2\pi\hbar h} K^+(z); \quad (44)\]

informally, we can write

\[ K^+(z) = K^+_{-\gamma h}(z). \] \[ \square \]

1.7. **Properties of the coproducts.**

**Lemma 1.2.** Let us fix \( \lambda \in \mathbb{C} - L \). For \( \epsilon \in k \), we have

\[ \Delta(e^{-\lambda}[\epsilon]) = \sum_{i \geq 0} e_{-\lambda + \gamma h(i)}^{(-1)}[\epsilon_i] (1 \otimes a^i_\lambda(\epsilon)) + e_{-\lambda}^{(-2)}[\epsilon], \quad (45)\]

and

\[ (\Delta \otimes 1)(e_{-\lambda + \gamma h(i)}^{(-1)}[\epsilon]) = \sum_{i \in \mathbb{Z}} e_{-\lambda + \gamma h(i)}^{(-1)}[\epsilon_i] \sum_{\alpha \geq 0} (1 \otimes \frac{\partial^\alpha a^i_\lambda(\epsilon)}{\partial \lambda^\alpha} \otimes (\gamma h)^\alpha) + e_{-\lambda + \gamma h(i)}^{(-2)}[\epsilon], \quad (46)\]

where \( a^i_\lambda \) are linear maps from \( k \) to the subalgebra \( U_{\hbar\mathfrak{h}^+}(\tau) \) of \( A(\tau) \), generated by the \( h^+[r], r \in \mathcal{O} \), depending holomorphically on \( \lambda \in \mathbb{C} - L \).
Proof. We have \( \Delta(e(z)) = e(z) \otimes K^+(z) + 1 \otimes e(z) \), so that
\[
\Delta(e^{-\lambda}[e]) = \langle e(z) \otimes K^+(z), p_\lambda^-(\epsilon) \rangle_k + 1 \otimes e^{-\lambda}[e].
\]
By (43), we have
\[
\langle e(z) \otimes K^+(z), p_\lambda^-(\epsilon) \rangle_k = \langle e(z) \otimes \left( \frac{\theta(z - \gamma)}{\theta(z + \gamma)} \right)^{h/2} \sum_{i \geq 0} \varphi^{(i)}(z)_{\alpha_i}, p_\lambda^-(\epsilon) \rangle_k.
\]
Note now that the map associating to \( \epsilon \in k \), the series
\[
A_\lambda(\epsilon) = \left( \frac{\theta(z - \gamma)}{\theta(z + \gamma)} \right)^{h/2} \sum_{i \geq 0} \frac{\partial^i p_\lambda^-(\epsilon)}{\partial \lambda^i} \frac{(\gamma h)^i}{i!},
\]
is a linear map from \( k \) to \( L_\lambda[[\gamma h]] \). Therefore
\[
\left( \frac{\theta(z - \gamma)}{\theta(z + \gamma)} \right)^{h/2} p_\lambda^-(\epsilon) = \sum_{i \geq 0} \frac{(-\gamma h)^i}{i!} \frac{\partial^i A_\lambda(\epsilon)}{\partial \lambda^i},
\]
and
\[
\langle e(z) \otimes K^+(z), p_\lambda^-(\epsilon) \rangle_k = \sum_{i \geq 0} \epsilon^{(i)}_{-\lambda+\gamma h(\alpha)} \varphi^{(i)} A_\lambda(\epsilon) (1 \otimes \alpha_i).
\]
(45) follows, if we set \( a_i^\lambda(\epsilon) = \sum_{j \geq 0} \varphi^{(j)} A_\lambda(\epsilon, \rho^j) \alpha_j \), where \( (\rho^j)_{j \geq 0} \) is the dual basis to \( (\epsilon_j)_{j \geq 0} \).
(46) then follows directly from (45). \( \square \)

Lemma 1.3. There exists a family of linear maps \( (b_i)_{i \geq 0} \) from \( \mathcal{O} \) to the subalgebra \( U_h \mathfrak{h}(\tau) \) of \( A(\tau) \) generated by the \( h[e], \epsilon \in k, \) and \( K \), such that
\[
\Delta(f[r]) = f[r] \otimes 1 + \sum_{i \geq 0} b_i(r) \otimes f[\epsilon_i],
\]
for \( r \in \mathcal{O} \); recall that \( (\epsilon_i)_{i \geq 0} \) is a basis of \( \mathcal{O} \).

Proof. We have \( \Delta(f(z)) = f(z) \otimes 1 + q^{-h^{-}(z)} \otimes f(z + hK_1) \), so that
\[
\Delta(f[r]) = f[r] \otimes 1 + \langle q^{-h^{-}(z)} \otimes f(z + hK_1), r \rangle_k;
\]
since \( q^{-h^{-}(z)} = \sum_{i \geq 0} \beta_i \epsilon_i(z) \), for certain \( \beta_i \in U_h \mathfrak{h}(\tau) \), we have
\[
\langle q^{-h^{-}(z)} \otimes f(z + hK_1), r \rangle_k = \langle \sum_{i \geq 0} \beta_i \epsilon_i(z) \otimes f(z + hK_1), r \rangle_k = \sum_{i \geq 0} \beta_i \otimes f[q^{K\beta}(r \epsilon_i)].
\]
We then set for \( i \geq 0, b_i(r) = \sum_{j \geq 0} \beta_j \langle q^{K\beta}(r \epsilon_i), \rho^j \rangle \), where \( (\rho^j)_{i \geq 0} \) is the dual basis to \( (\epsilon_i)_{i \geq 0} \). \( \square \)
Lemma 1.4. There exist families of linear maps \((c_i^j)_{i \geq 0}\) from \(\mathcal{O}\) to \(U(h_\tau)\) and \((d_i)_{i \in \mathbb{Z}}\) from \(k\) to \(U(h_\tau)\), such that the dependence of \(c^j_i\) in \(\lambda \in \mathbb{C} - L\) is holomorphic, and

\[
\Delta(c^j_i) = 1 \otimes c^j_i + \sum_{i \geq 0} c^j_i \otimes c^j_i(r), \quad (48)
\]

for \(r \in \mathcal{O}\),

\[
\Delta(f^{-}_\lambda) = f^{-}_\lambda \otimes 1 + \sum_{i \in \mathbb{Z}} (d_i(\epsilon) \otimes 1) f^{-}(\lambda, \epsilon + h) \otimes 1, \quad (49)
\]

and

\[
\Delta(f^{-}(\lambda, \epsilon + h)) = f^{-}(\lambda, \epsilon + h) \otimes 1 + \sum_{i \in \mathbb{Z}} (d_i(\epsilon) \otimes 1) f^{-}(\lambda, \epsilon + h) \otimes 1. \quad (50)
\]

Proof. \((48)\) is proved in the same way as \((47)\). Let us prove \((49)\). We have for \(\epsilon \in k\),

\[
\Delta(f^{-}_\lambda) = f^{-}_\lambda \otimes 1 + \langle K^+(z) \otimes f(z), \varphi^j_\lambda(\epsilon) \rangle_k;
\]

recall that \(K^+(z) = \left(\frac{\theta(z - \gamma)}{\theta(z + \gamma)}\right)^{h/2} \sum_{i \geq 0} \alpha_i \psi^{(i)}_\lambda(\epsilon)\), with \(\alpha_i \in U(h_\tau)\), therefore

\[
\langle K^+(z) \otimes f(z), \varphi^j_\lambda(\epsilon) \rangle_k = \sum_{i \geq 0} \alpha_i \otimes \langle f(z), \psi^{(i)}_\lambda(\epsilon) \rangle_k \left(\frac{\theta(z - \gamma)}{\theta(z + \gamma)}\right)^{h/2};
\]

since each \(\psi^{(i)}_\lambda(\epsilon) \varphi^j_\lambda(\epsilon) \left(\frac{\theta(z - \gamma)}{\theta(z + \gamma)}\right)^{h/2}\) can be expressed as an expansion

\[
\sum_{j \geq 0} \frac{\partial^j B_\lambda(z)}{\partial \lambda^j} \frac{(-\gamma h(1))^j}{j!},
\]

with \(B_\lambda(z) \in L_{-\lambda}[[\gamma h(1)]]\), we have

\[
1 \otimes \langle f(z), \psi^{(i)}_\lambda(\epsilon) \varphi^j_\lambda(\epsilon) \theta(z - \gamma h(1)) \rangle_k = f^{-}(\lambda, \epsilon)_i(\lambda, \epsilon),
\]

where \(\lambda_i\) are certain linear endomorphisms of \(k\). This shows \((43)\). \(\Box\) can be deduced from \((43)\) by using the expansion

\[
f^{-}(\lambda, \epsilon)_i = \sum_{j \geq 0} \frac{\partial^j f^{-}(\lambda, \epsilon)_i}{\partial \lambda^j} \frac{(-\gamma h(1))^j}{j!},
\]

the identity \(\Delta(h) = h(1) + h(2)\) and by replacing \(\lambda\) by \(\lambda + 2 \gamma\). \(\Box\)
2. Duality

Let $U_h \mathfrak{g}_+(\tau)$ be the subalgebra of $A(\tau)$ generated by $D$, the $h[r], r \in \mathcal{O}$, and $U_h \mathfrak{n}_+(\tau)$, and let $U_h \mathfrak{g}_-(\tau)$ be the subalgebra of $A(\tau)$ generated by $K$, the $h[\lambda], \lambda \in L_0$, and $U_h \mathfrak{n}_-\tau$.

$(U_h \mathfrak{g}_\pm(\tau), \Delta)$ are Hopf subalgebras of $(A(\tau), \Delta)$; $(U_h \mathfrak{g}_+(\tau), \Delta)$ and $(U_h \mathfrak{g}_-(\tau), \Delta')$ are dual to each other; the duality $\langle , \rangle$ is expressed by the rules

$$\langle e[\epsilon], f[\epsilon'] \rangle = \frac{1}{\hbar}\langle \epsilon, \epsilon' \rangle_k, \quad \langle h[r], h[\lambda] \rangle = \frac{2}{\hbar}\langle r, \lambda \rangle_k, \quad \langle D, K \rangle = \frac{1}{\hbar},$$

the other pairings between generators being trivial. We denote by $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$ the restriction of $\langle , \rangle$ to $U_h \mathfrak{n}_+(\tau) \times U_h \mathfrak{n}_-(\tau)$.

On the other hand, let $U_h \mathfrak{g}_+(\tau)$ and $U_h \mathfrak{g}_-(\tau)$ be the subalgebras of $A(\tau)$ respectively generated by $U_h \mathfrak{n}_+(\tau), K$ and the $h[\lambda], \lambda \in L_0$, and by $U_h \mathfrak{n}_-(\tau), D$ and the $h[r], r \in \mathcal{O}$. $(U_h \mathfrak{g}_\pm(\tau), \Delta)$ are Hopf subalgebras of $(A(\tau), \Delta)$; $(U_h \mathfrak{g}_+(\tau), \Delta')$ and $(U_h \mathfrak{g}_-(\tau), \Delta)$ are dual to each other; the duality $\langle , \rangle'$ is expressed by the rules

$$\langle e[\epsilon], f[\epsilon'] \rangle' = \frac{1}{\hbar}\langle \epsilon, \epsilon' \rangle_k, \quad \langle h[\lambda], h[r] \rangle' = \frac{2}{\hbar}\langle r, \lambda \rangle_k, \quad \langle K, D \rangle' = \frac{1}{\hbar},$$

the other pairings between generators being trivial. The restriction of $\langle , \rangle'$ to $U_h \mathfrak{n}_+(\tau) \times U_h \mathfrak{n}_-(\tau)$ coincides with $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$.

Let us also denote by $U_h \mathfrak{n}_\pm(\tau)[n]$ the homogeneous components of degree $n$ (in the $e[\epsilon]$ or $f[\epsilon]$) of $U_h \mathfrak{n}_\pm(\tau)$, and by $U^{(f)}_{\lambda-}$ the intersections $U^{(f)}_{\lambda-} \cap U_h \mathfrak{n}_\pm(\tau)[n]$.

Then

**Lemma 2.1.** 1) The annihilator of $U^{(f)}_{\lambda-}$ for $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$ is $\sum_{r \in \mathcal{O}} e[r] U_h \mathfrak{n}_+(\tau)$.

2) The annihilator of $U^{(f)}_{\lambda-}$ for $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$ is $\sum_{\epsilon \in \mathfrak{k}} f_{\lambda+2(\lambda-1)(\tau)-1}^r [\epsilon] U_h \mathfrak{n}_-(\tau)[n-1]$.

3) The annihilator of $U^{(f)}_{\lambda-}$ for $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$ is $\sum_{\tau \in \mathcal{O}} U_h \mathfrak{n}_-(\tau)[\tau][n-1]$.

4) The annihilator of $U^{(f)}_{\lambda-}$ for $\langle , \rangle_{U_h \mathfrak{n}_\pm(\tau)}$ is $\sum_{\epsilon \in \mathfrak{k}} U_h \mathfrak{n}_+(\tau)[\tau][n-1]$.

**Proof.** 1) and 3) are consequences of [12], Prop. 6.2.

Let us show 2). Let us first prove that $\sum_{\epsilon \in \mathfrak{k}} f_{\lambda+2\gamma}[\epsilon] U_h \mathfrak{n}_-(\tau)[n]$ is orthogonal to $U^{(f)}_{\lambda-}$.

Let

$$a = e_{\lambda-2\gamma}^r [\eta_1] \cdots e_{\lambda}^r [\eta_0], \quad \eta_i \in \mathfrak{k}$$

belong to $U^{(f)}_{\lambda-}[n+1];$ let $b$ belong to $U_h \mathfrak{n}_-(\tau)[n]$, $\epsilon$ belong to $\mathfrak{k}$ and let us compute $\langle a, f_{\lambda-}[\epsilon] b \rangle$. This is equal to

$$\sum_i \langle a_i, f_{\lambda-}[\epsilon] b \rangle \langle a'_i, b \rangle, \quad (51)$$
where $\Delta(a) = \sum_i a_i \otimes a'_i$.

From (45) follows that $\Delta(a)$ is the product of the terms

$$
\sum_{i \in \mathbb{Z}} e_{-\lambda - 2p \gamma + \gamma h(z)}^{(1)}[\epsilon_i] (1 \otimes a^{i}_{\lambda + 2p \gamma}) (\eta_p) + e_{-\lambda - 2p \gamma}^{(2)}[\eta_p],
$$

(52)

for $p = n, \ldots, 0$. This product belongs to $U_{\hbar}n_+(\tau) \otimes U_{\hbar}g_+(\tau)$. To evaluate (51), we may as well project the first factor of $\Delta(a)$ on $U_{\hbar}n_+(\tau)^{(1)}$ parallel to all other homogeneous components. The contribution of the $(n-p)$th term (54) is

$$
\sum_{i \in \mathbb{Z}} e_{-\lambda - 2n \gamma}^{(2)}[\eta_n] \cdots e_{-\lambda - 2(p+1) \gamma}^{(2)}[\eta_{p+1}] e_{-\lambda - 2p \gamma + \gamma h(z)}^{(1)}[\epsilon_i] (1 \otimes a^{i}_{\lambda + 2p \gamma}) (\eta_p)
$$

$$
+ e_{-\lambda - 2p \gamma}^{(2)}[\eta_p],
$$

that is, using the fact that

$$
(h - 2p) e_{-\lambda - 2n \gamma}^{(2)}[\eta_n] \cdots e_{-\lambda - 2(p+1) \gamma}^{(2)}[\eta_{p+1}] = e_{-\lambda - 2n \gamma}^{(2)}[\eta_n] \cdots e_{-\lambda - 2(p-1) \gamma}^{(2)}[\eta_{p-1}] (h - 2n),
$$

and

$$
\sum_{i \in \mathbb{Z}} \sum_{\alpha \geq 0} (-\partial/\partial \lambda)^\alpha (e_{-\lambda + 2n \gamma}^{(2)}[\epsilon_i])
$$

$$
\otimes \frac{h^\alpha}{\alpha!} e_{-\lambda - 2n \gamma}^{(2)}[\eta_n] \cdots e_{-\lambda - 2(p+1) \gamma}^{(2)}[\eta_{p+1}] a^{i}_{\lambda + 2p \gamma} (\eta_p) e_{-\lambda - 2(p-1) \gamma}^{(2)}[\eta_{p-1}] \cdots e_{-\lambda}^{(2)}[\eta_0].
$$

(53)

Note now that for any $x \in U_{\hbar}g_+(\tau)$ and $y \in U_{\hbar}n_-(\tau)$, we have

$$
\langle hx, y \rangle = 0.
$$

(54)

Indeed, $\langle hx, y \rangle = \langle h \otimes x, \Delta'(y) \rangle_{(2)}$ (denoting by $\langle , \rangle_{(2)}$ the tensor square of $\langle , \rangle$); but $\Delta'(y)$ belongs to $U_{\hbar}n_-(\tau) \otimes U_{\hbar}g_-(\tau)$, and $\langle h, U_{\hbar}n_-(\tau) \rangle = 0$, so that (54) holds.

Now the pairing of (53) with $f^-_{\lambda \chi}[\epsilon] \otimes b$ is equal to zero either (for $\alpha = 0$) because $\langle e_{-\lambda}^{(2)}[\epsilon], f^-_{\lambda \chi}[\epsilon] \rangle = 0$ for any $\epsilon, \eta \in k$ ($L_{\lambda}$ and $L_{-\lambda}$ being orthogonal to each other) or by (54) for $\alpha > 0$.

Then standard deformation arguments (see [12], proof of Prop. 6.2) show that the orthogonal of $U_{\hbar}^\epsilon n^{|n+1}$ is exactly $\sum_{\epsilon \in k} f^-_{\lambda + 2n \gamma}[\epsilon] U_{\hbar}n_-(\tau)^{|n]}$.

Let us now prove 4). Let us first show that for $\epsilon \in k$, $U_{\hbar}n_+(\tau)^{|n]} e_{-\lambda + 2n \gamma}^{(2)}[\epsilon]$ is orthogonal to $U_{\hbar}^\epsilon n^{|n+1}$. Let

$$
a = f^-_{\lambda \chi}[\eta_0] \cdots f^-_{\lambda - 2n \gamma}[\eta_n], \quad \eta_i \in k,
$$

belong to $U_{\hbar}n_+(\tau)^{|n]}$, let $b$ belong to $U_{\hbar}n_+(\tau)^{|n]}$, and let us compute $\langle be_{-\lambda + 2n \gamma}[\epsilon], a \rangle$. This is equal to

$$
\langle b \otimes e_{-\lambda + 2n \gamma}[\epsilon], \Delta(a) \rangle_{(2)}.
$$

(55)
From (49) follows that $\bar{\Delta}(\alpha)$ is the product of the terms
\[
f_{\lambda-2p\gamma}[\eta_p] \otimes 1 + \sum_{i \in \mathbb{Z}} (d_i(\eta_p) \otimes 1) f_{\lambda+\gamma h^{(1)}-2p\gamma}^{-(2)}[\epsilon_i],
\]
$p = 0, \ldots, n$. Assign degrees $-1$ to terms of the form $f[\epsilon], \epsilon \in k$, and zero to those belonging to $U_h(\tau)$; then in the expansion of the product of the terms (56), only those of degree $-1$ will contribute to (55). Therefore (55) is equal to
\[
\langle b \otimes e^{-\lambda+2n\gamma}[\epsilon], \sum_{\alpha \geq 0}^{n} \sum_{p=0}^{n-1} \prod_{k=0}^{p-1} (f_{\lambda-2k\gamma}[\eta_k] \otimes 1) d_i(\eta_p) \left( \prod_{k=p+1}^{n} f_{\lambda-2k\gamma}[\eta_k] \right) (\gamma h)^{\alpha} \rangle^{(2)}
\]
\[
\otimes (\partial/\partial \lambda)^{\alpha} f_{\lambda-2n\gamma}[\epsilon_i] \rangle^{(2)}.
\]
Using the identity
\[
(h - 2p) \prod_{k=p+1}^{n} f_{\lambda-2k\gamma}[\eta_k] = \left( \prod_{k=p+1}^{n} f_{\lambda-2k\gamma}[\eta_k] \right) (h - 2n),
\]
we rewrite (57) as
\[
\langle b \otimes e^{-\lambda+2n\gamma}[\epsilon], \sum_{\alpha \geq 0}^{n} \sum_{p=0}^{n-1} \prod_{k=0}^{p-1} (f_{\lambda-2k\gamma}[\eta_k] \otimes 1) d_i(\eta_p) \left( \prod_{k=p+1}^{n} f_{\lambda-2k\gamma}[\eta_k] \right) (\gamma h)^{\alpha} \rangle^{(2)}
\]
\[
\otimes (\partial/\partial \lambda)^{\alpha} f_{\lambda-2n\gamma}[\epsilon_i] \rangle^{(2)}.
\]
Note now that for $b \in U_h n_+(\tau), c \in U_h \tilde{g}_-(\tau),$
\[
\langle b, ch \rangle = 0.
\]
Indeed, $\langle b, ch \rangle = \langle \bar{\Delta}'(b), c \otimes h \rangle^{(2)} = 0$ because the second factors of the expansion of $\bar{\Delta}'(b)$ belong to $U_h n_+(\tau)$. Therefore (58) vanishes, either by (53) or because $\langle e^{-\lambda+2n\gamma}[\epsilon], f_{\lambda-2n\gamma}[\epsilon_i] \rangle = 0$.

It follows that $\sum_{\epsilon \in k} U_h n_+(\tau)[\eta] e^{-\lambda+2n\gamma}[\epsilon]$ is orthogonal to $U_{\lambda_+}^{(f)n+1}$. The same deformation arguments as above show that these spaces are in fact the orthogonals of each other.

We will also use the following lemma:

**Lemma 2.2.** 1) For $x \in U_h n_+(\tau)$, we have
\[
\langle F, id \otimes x \rangle = x;
\]
2) for $y \in U_h n_-(\tau)$, we have
\[
\langle F, y \otimes id \rangle = y;
3. Algebras $A^{+-}$ and $A^{-+}$ and Their Properties

For $X$ a vector space, we denote by $\text{Hol}(\mathbb{C} - L, X)$ the space of holomorphic functions from $\mathbb{C} - L$ to $X$ and set $\text{Hol}(\mathbb{C} - L) = \text{Hol}(\mathbb{C} - L, \mathbb{C})$.

**Definition 3.1.** Let us define $A^{-+}$ to be the subalgebra of $\text{Hol}(\mathbb{C} - L, A \otimes A)$ generated (over $\text{Hol}(\mathbb{C} - L)$) by $h^{(2)}$ and the $e^{-(1)}_{-\lambda + \gamma h^{(2)}}[\epsilon] f^{(2)}[r]$, with $\epsilon \in k$ and $r \in O$, and $A^{+-}$ as the subalgebra of $\text{Hol}(\mathbb{C} - L, A \otimes A)$ generated (over $\text{Hol}(\mathbb{C} - L)$) by $h^{(2)}$ and the $e^{(1)}[r] f^{(2)}_{-\gamma h^{(2)} + 2\gamma} [\epsilon]$, with $r \in O$, $\epsilon \in k$.

**Proposition 3.1.** The intersection of $A^{+-}$ and $A^{-+}$ is equal to $\text{Hol}(\mathbb{C} - L, 1 \otimes \mathbb{C}[h][[\gamma]])$.

**Proof.** Since we have $[h, f[r]] = -2f[r]$ for $r \in O$, we have the relations

$$f^{(2)}[r]e^{-1}_{-\lambda + \gamma h^{(2)}}[\epsilon] = e^{-1}_{-\lambda + \gamma h^{(2)}+2\gamma} [\epsilon] f^{(2)}[r],$$

for $\epsilon \in k, r \in O$; therefore $A^{+-}$ is linearly spanned by 1 and the

$$\xi = e^{-(1)}_{-\lambda + \gamma h^{(2)}}[\eta_0] \cdots e^{-(1)}_{-\lambda + \gamma h^{(2)}+2n\gamma} [\eta_n] f^{(2)}[r_0] \cdots f^{(2)}[r_n](h^{(2)})^p,$$

with $n, p \geq 0$, $\eta_i \in k$, $r_i \in O$.

On the other hand, $A^{-+}$ is linearly spanned by 1 and the

$$\eta = e^{(1)}[r_0] \cdots e^{(1)}[r_n] f^{(2)}_{-\gamma h^{(2)}+2\gamma} [\eta_0] \cdots f^{(2)}_{-\gamma h^{(2)}+2\gamma} [\eta_n](h^{(2)})^p,$$

$n, p \geq 0$, $\eta_i \in k$, $r_i \in O$.
Suppose that some combination of elements of the form (61) belongs to $A^+$. The image of this combination by $l \otimes 1$, $l$ any linear form on $A$, is some combination

$$\sum_{\lambda \geq 0} \lambda_{0,\mu} (h^{(2)})^{\mu} + \sum_{\nu \geq 0, 0} \sum_{i \in [\lambda + \gamma h + 2 u]} \lambda_{i,n,p} f_{\lambda - \gamma h + 2 u}^{(i)} \cdots f_{h^{(2)}}^{(i)} \cdots f_{h^{(2)}}^{(i)} (h^{(2)})^p,$$

$\lambda_{0,\mu}, \lambda_{i,n,p} \in \text{Hol}(\mathbb{C} - L)$, that should belong to $A^+$.

By (36) with $\epsilon = \epsilon' = -1$, a basis of the linear span of all elements of the form

$$f_{\lambda - \gamma h + 2 u}^{(i)} \cdots f_{h^{(2)}}^{(i)} (h^{(2)})^p,$$

with $\eta_i \in k$, is

$$f_{\lambda - \gamma h + 2 u}^{(i)} \cdots f_{h^{(2)}}^{(i)} (h^{(2)})^p, \quad i_0 \leq \cdots \leq i_n < 0.$$

On the other hand, from (42) follows that a basis of the linear span of all elements of the form $f_{\lambda}^{(i)} \cdots f_{\lambda}^{(i)} (h^{(2)})^p$, with $\epsilon_i \in k$, is given by

$$f_{\lambda - \gamma h + 2 u}^{(i)} \cdots f_{h^{(2)}}^{(i)} (h^{(2)})^p, \quad i_0 \leq \cdots \leq i_k < 0 \leq i_{i+k+1} \leq \cdots \leq i_n.$$

A basis of the intersection of $A^+$ with this linear span is

$$f_{\lambda}^{(i_0)} \cdots f_{\lambda}^{(i_n)} (h^{(2)})^p, \quad i_0 \geq \cdots \geq i_n \geq 0.$$

Therefore the only possibility that (62) belongs to $A^+$ is that $\lambda_{i,n,p} = 0$ for all $i, n, p$.

**Definition 3.2.** Let us define $A^\cdot \cdot \cdot$ as the subspace of the algebra $\text{Hol}(\mathbb{C} - L, A^{\otimes 3})$, linearly spanned (over $\text{Hol}(\mathbb{C} - L)$) by the elements of the form

$$\xi = e_{\lambda + \gamma (h^{(2)} + h^{(3)})}^{(1)} [\eta_1] \cdots e_{\lambda + \gamma (h^{(2)} + h^{(3)}) + 2(n-1) \gamma}^{(1)} [\eta_n] (1 \otimes a \otimes b),$$

$n \geq 0$ (recall that the empty product is equal to 1), where $\eta_i \in k$, and $a, b \in A$ are such that $[h^{(1)} + h^{(2)} + h^{(3)}, \xi'] = 0$; and $A^\cdot \cdot \cdot$ as the subspace of $\text{Hol}(\mathbb{C} - L, A^{\otimes 3})$ spanned (over $\text{Hol}(\mathbb{C} - L)$) by the elements of the form

$$\eta' = (a' \otimes b' \otimes 1) f^{(3)} [r_1] \cdots f^{(3)} [r_n] (h^{(3)})^s, \quad n, s \geq 0,$$

where $a', b' \in A, r_i \in \mathcal{O}$, and such that $[h^{(1)} + h^{(2)} + h^{(3)}, \eta'] = 0$.

**Proposition 3.2.** $A^\cdot \cdot \cdot$ and $A^\cdot \cdot \cdot$ are subalgebras of $\text{Hol}(\mathbb{C} - L, A^{\otimes 3})$.

We have

$$(\Delta \otimes 1)(A^+) \subset A^\cdot \cdot \cdot \cap A^\cdot \cdot \cdot,$$

$$(1 \otimes \Delta)(A^+) \subset A^\cdot \cdot \cdot \cap A^\cdot \cdot \cdot.$$

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\[\sum_{\lambda}^\infty \lambda \cdot e_{\lambda + \gamma h + 2 u}^{(i)} \cdots e_{h^{(2)}}^{(i)} (h^{(2)})^p, \quad i_0 \leq \cdots \leq i_n < 0.\]
ELLiptic Quantum Groups \(E_{r,\eta}(\mathfrak{sl}_2)\) and Quasi-Hopf algebras

Proof. That \(A^{\cdot ++}\) is a subalgebra of \(\text{Hol}(\mathbb{C} - L, A^{\otimes 3})\) follows easily from its definition. Let us show now that \(A^{\cdot \cdot \cdot}\) is a subalgebra of \(\text{Hol}(\mathbb{C} - L, A^{\otimes 3})\). Let \(\xi'\) and \(\xi''\) be elements of \(\text{Hol}(\mathbb{C} - L, A^{\otimes 3})\) of the form (63), that is

\[
\xi' = e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)})}[\eta_1] \cdots e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2(n - 1)\gamma}[\eta_n](1 \otimes a \otimes b),
\]

and

\[
\xi'' = e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)})}[\eta'_1] \cdots e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2(n' - 1)\gamma}[\eta'_{n'}](1 \otimes a' \otimes b'),
\]

with \(\eta_i, \eta'_i \in k, a, b, a', b' \in A\) are such that \(\xi'\) and \(\xi''\) commute with \(h^{(1)} + h^{(2)} + h^{(3)}\).

Since \([h^{(1)}, \xi'] = 2n\xi\), we have \([h^{(2)} + h^{(3)}, \xi'] = -2n\xi',\) so that \([h^{(2)} + h^{(3)}, 1 \otimes a \otimes b] = -2n(1 \otimes a \otimes b)\). It follows that for any \(p\),

\[
(1 \otimes a \otimes b)e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2(p - 1)\gamma}[\eta'_p] = e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2n\gamma + 2(p - 1)\gamma}[\eta'_p](1 \otimes a \otimes b).
\]

The product \(\xi'\xi''\) can then be written as

\[
e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)})}[\eta_1] \cdots e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2(n - 1)\gamma}[\eta_n]
e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)}) + 2n\gamma + 2(n' - 1)\gamma}[\eta'_{n'}](1 \otimes a a' \otimes b b'),
\]

which is of the form (63). Since we also have \([h^{(1)} + h^{(2)} + h^{(3)}, \xi'\xi''] = 0,\) \(\xi'\xi''\) belongs to \(A^{\cdot \cdot \cdot}\).

Let us now prove the first part of (64). From (66) follows that for \(\epsilon \in k, r \in \mathcal{O}, (\Delta \otimes 1)(e^{-1}_{-\lambda + \gamma h^{(2)}}[\epsilon]f^{(2)}[r])\) is equal to

\[
\sum_{i \geq 0} \sum_{\alpha \geq 0} e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)})}[\epsilon_i] \frac{\partial a_{\alpha}}{\partial \lambda^\alpha} \frac{\gamma(h^{(3)} \lambda)^\alpha}{\alpha!} f^{(3)}[r] + e^{-2}_{-\lambda + \gamma h^{(3)}}[\epsilon]f^{(3)}[r],
\]

and so belongs to \(A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\). We also have \((\Delta \otimes 1)(h^{(2)}) \in A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\). Since \(h^{(2)}\) and the \(e^{-1}_{-\lambda + \gamma h^{(2)}}f^{(2)}[r], \epsilon \in k, r \in \mathcal{O}, generate A^{\cdot +},\) and that \(A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\) is an algebra, we have \((\Delta \otimes 1)(A^{\cdot +}) \subset A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\).

Let us now prove the second part of (64). Clearly, \((1 \otimes \Delta)(h^{(2)})\) belongs to \(A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\). From Lemma 1.3 follows that for any \(\epsilon \in k, r \in \mathcal{O}, (1 \otimes \Delta)(e^{-1}_{-\lambda + \gamma h^{(2)}}[\epsilon] f^{(2)}[r])\) is equal to

\[
e^{-1}_{-\lambda + \gamma(h^{(2)} + h^{(3)})}[\epsilon]\left(f^{(2)}[r] + \sum_{i \geq 0} b_i(r)^{(2)} f^{(3)}[\epsilon_i]\right),
\]

and therefore belongs to \(A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\). Since \(h^{(2)}\) and the \(e^{-1}_{-\lambda + \gamma h^{(2)}}[\epsilon]f^{(2)}[r], \epsilon \in k, r \in \mathcal{O}, generate A^{\cdot +},\) and that \(A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\) is an algebra, this shows that \((1 \otimes \Delta)(A^{\cdot +}) \subset A^{\cdot \cdot \cdot} \cap A^{\cdot ++}\).
We now define analogues \( A^{+\cd} \) and \( A^{-\cd} \) of \( A^{-\cd\cd} \) and \( A^{\cd\cd\cd} \).

**Definition 3.3.** \( A^{+\cd} \) is the subspace of the algebra \( \text{Hol}(\mathbb{C} - L, A^{\otimes 3}) \) linearly spanned (over \( \text{Hol}(\mathbb{C} - L) \)) by the elements of the form

\[
\xi' = e^{(1)}[r_1] \ldots e^{(1)}[r_n](1 \otimes a \otimes b), \quad n \geq 0,
\]

where \( r_i \in \mathcal{O} \), and \( a, b \in A \) are such that \( [h^{(1)} + h^{(2)} + h^{(3)}, \xi'] = 0 \).

\( A^{-\cd} \) is the subspace of \( \text{Hol}(\mathbb{C} - L, A^{\otimes 3}) \) linearly spanned (over \( \text{Hol}(\mathbb{C} - L) \)) by the elements of the form

\[
\eta' = (a' \otimes b' \otimes 1) f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta_1] \ldots [\eta_n])(h^{(3)})^s, \quad n, s \geq 0,
\]

where \( \eta_i \in k \), and \( a', b' \in A \) are such that \( [h^{(1)} + h^{(2)} + h^{(3)}, \eta'] = 0 \).

We have then:

**Proposition 3.3.** \( A^{+\cd} \) and \( A^{-\cd} \) are subalgebras of \( \text{Hol}(\mathbb{C} - L, A^{\otimes 3}) \).
We have

\[
(\tilde{\Delta} \otimes 1)(A^{+\cd}) \subset A^{+\cd\cd} \cap A^{-\cd\cd}, \quad (1 \otimes \tilde{\Delta})(A^{-\cd}) \subset A^{+\cd\cd} \cap A^{-\cd\cd}.
\]

**Proof.** That \( A^{+\cd} \) is a subalgebra of \( \text{Hol}(\mathbb{C} - L, A^{\otimes 3}) \) follows easily from its definition.

Let us now prove that \( A^{-\cd\cd} \) is an algebra. Let us consider elements of the form (61),

\[
\eta = (a \otimes b \otimes 1) f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta_1] \ldots [\eta_n])(h^{(3)})^p,
\]

and

\[
\eta' = (a' \otimes b' \otimes 1) f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta'_1] \ldots [\eta'_n])(h^{(3)})^{p'},
\]

where \( \eta_i, \eta'_i \in k \), and \( a, b, a', b' \in A \) are such that \( [h^{(1)} + h^{(2)} + h^{(3)}, \eta] = [h^{(1)} + h^{(2)} + h^{(3)}, \eta'] = 0 \).

Since each \( f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta_p]) \) commutes with \( a' \otimes b' \otimes 1 \), \( \eta \eta' \) can be written as

\[
\eta \eta' = (aa' \otimes bb' \otimes 1) f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta_1] \ldots [\eta_n]) f_{\lambda - \gamma h^{(3)} + 2\gamma}^{-3}([\eta'_1] \ldots [\eta'_n])(h^{(3)} - 2n)^p(h^{(3)})^{p'},
\]

which is of the form (63); on the other hand, \( \eta \eta' \) clearly commutes with \( h^{(1)} + h^{(2)} + h^{(3)} \), which implies that it belongs to \( A^{-\cd\cd} \).
The proof of (67) is similar to that of Prop. 3.2. Lemma 1.4 implies that for $\epsilon \in k$, $r \in O$, $(\bar{\Delta} \otimes 1)(e^{(1)}[r]f^{-2}_{\lambda-h(h)+2}[\epsilon])$ is equal to

$$\left(e^{(2)}[r] + \sum_{i \geq 0} e^{(1)}[\epsilon_i]c^{(2)}_{\lambda}(r)\right)f^{-3}_{\lambda-h(h)+2}[\epsilon]$$

and therefore belongs to $A^{++} \cap A^{--}$. On the other hand, $(1 \otimes \bar{\Delta})(e^{(1)}[r]f^{-2}_{\lambda-h(h)+2}[\epsilon])$ is equal to

$$e^{(1)}[r]\left(f^{-2}_{\lambda-h(h)+2}[\epsilon] + \sum_{i \geq 0} d^{(2)}_{i}(\epsilon)f^{-3}_{\lambda-h(h)+2}[\epsilon_i]\right),$$

and so belongs to $A^{++} \cap A^{--}$. Finally, $(\bar{\Delta} \otimes 1)(h^{(2)})$ and $(1 \otimes \bar{\Delta})(h^{(2)})$ also belong to $A^{++} \cap A^{--}$.

Since the $e^{(1)}[r]f^{-2}_{\lambda-h(h)+2}[\epsilon]$ generate $A^{+-}$, and that $A^{++} \cap A^{--}$ is an algebra, this proves (67).

**Proposition 3.4.** We have

$$A^{++} \cap A^{--} = \text{Hol}(\mathbb{C}-L, 1 \otimes (A^{\otimes 2})^h), \quad A^{++} \cap A^{--} = \text{Hol}(\mathbb{C}-L, (A^{\otimes 2})^h \otimes \mathbb{C}[h]),$$

where $(A^{\otimes 2})^h$ are the elements of $A^{\otimes 2}$ commuting with $h^{(1)} + h^{(2)}$.

**Proof.** The proof is similar to that of Prop. 3.3. □

### 4. DECOMPOSITION OF $F$

#### 4.1. Notation

We will use the following general notation. For $(X_\lambda)_{\lambda \in \mathbb{C}-L}$ a family of maps from an algebra $A$ to $A^{\otimes m}$, depending on $\lambda \in \mathbb{C}-L$ in a holomorphic way, for $n \geq m$ and any injection $k \mapsto i_k$ of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$; for any complex numbers $\alpha_s, s = 1, \ldots, n$, we set

$$X^{(i_1, \ldots, i_m)}_{\lambda+\sum_{i=1}^n \gamma \alpha_i h^{(k)}}(a) = \sum_{i \geq 0} \frac{\partial^i X^{(i_1, \ldots, i_k)}_\lambda(a)}{\partial \lambda^i} \left(\sum_{i=1}^n \gamma \alpha_i h^{(k)}\right)^i, \quad a \in A,$$

where $X^{(i_1, \ldots, i_k)}_\lambda(a)$ denotes the image of $X_\lambda(a)$ in $A^{\otimes m}$ by the map sending the $k$th factor to the $i_k$th one. If the $X_\lambda$ are algebra morphisms, and that each $\alpha_{i_k}$ vanishes, then $X^{(i_1, \ldots, i_k)}_{\lambda+\sum_{i=1}^n \gamma \alpha_i h^{(k)}}$ is an algebra morphism.

This notation applies in particular if $A = \mathbb{C}$ (then $X_\lambda$ is a family of elements of $A^{\otimes m}$).
4.2. **Decomposition of** $F$. Let us denote by $U_{n}(\tau)^{[n]}$ the homogeneous part of of $U_{n}(\tau)$ of degree $n$. Let us set

$$F = \sum_{n \geq 0} F_{n}, \quad \text{with} \quad F_{n} \in U_{n}(\tau)^{[n]} \otimes U_{n}(\tau)^{[n]}.$$ 

**Proposition 4.1.** There exist families $(F_{\lambda}^{(i,p)})_{\lambda \in \mathbb{C} - L, p \geq 0}, i = 1, 2$ of elements of $A^{\otimes 2}$, where $F_{\lambda}^{1,p}$ is a linear combination with coefficients in \text{Hol}($\mathbb{C} - L$) of the

$$e_{-\lambda} e_{\lambda-2\gamma} \cdots e_{-\lambda-2\gamma} f_{\varepsilon_{j_{1}}} \cdots f_{\varepsilon_{j_{p}}}, \quad j_{\alpha} \geq 0,$$

$F_{\lambda}^{2,q}$ is a similar combination of the

$$e_{\varepsilon_{j_{1}}} \cdots e_{\varepsilon_{j_{q}}} \otimes f_{\lambda+2\gamma} \varepsilon_{j_{1}} \cdots f_{\lambda+2\gamma} \varepsilon_{j_{q}}, \quad i_{\alpha} \geq 0,$$

and $F_{\lambda}^{i,0} = 1, i = 1, 2$, such that

$$F_{n} = \sum_{p+q=n} F_{\lambda+2\gamma}^{2,q} F_{\lambda}^{1,p}. \quad (68)$$

**Proof.** Let us define the linear maps $\Pi_{+\lambda}^{(e)}$ and $\Pi_{-\lambda}^{(e)}$, from $U_{n}(\tau)$ to $U_{n}^{(e)}$, resp. to $U_{n}^{(e)}$, by

$$\Pi_{+\lambda}^{(e)}(ab) = a \varepsilon(b), \quad \Pi_{-\lambda}^{(e)}(ab) = \varepsilon(a)b, \quad a \in U_{n}^{(e)}, \quad b \in U_{n}^{(e)},$$

and $\Pi_{+\lambda}^{(f)}$ and $\Pi_{-\lambda}^{(f)}$, from $U_{n}(\tau)$ to $U_{n}^{(f)}$, resp. to $U_{n}^{(f)}$, by

$$\Pi_{+\lambda}^{(f)}(ab) = \varepsilon(a)b, \quad \Pi_{-\lambda}^{(f)}(ab) = a \varepsilon(b), \quad a \in U_{n}^{(f)}, \quad b \in U_{n}^{(f)}.$$ 

Note that $\Pi_{+\lambda}^{(e)}$ is a left $U_{n}^{(e)}$-module map, and $\Pi_{-\lambda}^{(f)}$ is a right $U_{n}^{(f)}$-module map. From Prop. 4.2 also follows that the kernels of $\Pi_{+\lambda}^{(e)}, \Pi_{-\lambda}^{(e)}$ $\Pi_{+\lambda}^{(f)}$ and of $\Pi_{+\lambda}^{(f)}$ are respectively $\sum_{e \in k} U_{n}(\tau)e_{-\lambda}[e], \sum_{r \in \mathcal{O}} e[r]U_{n}(\tau), \sum_{e \in k} f_{\lambda}[e]U_{n}(\tau)$ and $\sum_{r \in \mathcal{O}} U_{n}(\tau)f[r].$

**Lemma 4.1.** 1) For each $n \geq 0$, $(1 \otimes \Pi_{+\lambda+2\gamma}^{(f)})(F_{n})$ and $(\Pi_{-\lambda+2\gamma}^{(e)} \otimes 1)F_{n}$ both belong to $U_{\lambda+2\gamma}^{(e)} \otimes U_{n}^{(f)}$;

2) for each $n \geq 0$, $(1 \otimes \Pi_{-\lambda+2\gamma}^{(f)})(F_{n})$ and $(\Pi_{+\lambda+2\gamma}^{(e)} \otimes 1)F_{n}$ both belong to $U_{n}^{(e)} \otimes U_{\lambda+2\gamma}^{(f)}$;

3) we have the equalities

$$(1 \otimes \Pi_{+\lambda+2\gamma}^{(f)})(F_{n}) = (\Pi_{-\lambda+2\gamma}^{(e)} \otimes 1)F_{n}$$

and

$$(\Pi_{+\lambda+2\gamma}^{(e)} \otimes 1)F_{n} = (1 \otimes \Pi_{-\lambda+2\gamma}^{(f)})(F_{n}).$$
**Proof of the lemma.** Let us prove the first part of 1). \((1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n)\) clearly belongs to \(U_{\hbar}\mathfrak{n}_+(\tau) \otimes U^{(f)}_+\). Let now \(x\) belong to \(\sum_{e \in k} F_{+,\lambda+2n\gamma}[e][U_{\hbar}\mathfrak{n}_-(\tau)^{[n-1]}].\) Consider \(\langle (1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n), x \otimes \otimes \rangle;\) this is equal to \(\Pi^{(f)}_{+,\lambda+2n\gamma}(x)\) by Lemma 2.1, and therefore to zero. By Lemma 2.1, 2), it follows that \((1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n)\) also belongs to \(U^{(e)}_{\lambda+2\gamma,-} \otimes U_{\hbar}\mathfrak{n}_-(\tau).\)

The proof of the second part of 1) and of 2) is similar, and uses the other statements of Lemma 2.1, and the above description of the kernels of \(\Pi^{(e)}_{-,\lambda+2\gamma}, \Pi^{(f)}_{-,\lambda+2n\gamma}, \) and \(\Pi^{(e)}_{+,\lambda+2\gamma}.

Let us now prove the first part of 3). Let us fix \(a_+\) in \(U^{(f):n}_+\) and \(a_-\) in \(U^{(e):n}_{\lambda+2\gamma,-}\). Then

\[
\langle (1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n) - (\Pi^{(e)}_{-,\lambda+2\gamma} \otimes 1)(F_n), a_+ \otimes a_- \rangle = 0.
\]

Since \((1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n) - (\Pi^{(e)}_{-,\lambda+2\gamma} \otimes 1)(F_n)\) belongs to \(U^{(e):n}_{\lambda+2\gamma,-} \otimes U^{(f):n}_+\), and that the pairing \(\langle , \rangle\) induces injections from \(U^{(e):n}_{\lambda+2\gamma,-}\) in the dual of \(U^{(f):n}_+\) and from \(U^{(f):n}_+\) in the dual of \(U^{(e):n}_{\lambda+2\gamma,-}\), \((1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n) - (\Pi^{(e)}_{-,\lambda+2\gamma} \otimes 1)(F_n)\) is equal to zero.

The proof of the second part of 3) is similar. \(\square\)

Let us set now

\[
F^{1:n}_{\lambda} = (1 \otimes \Pi^{(f)}_{+,\lambda+2n\gamma})(F_n) = (\Pi^{(e)}_{-,\lambda+2\gamma} \otimes 1)(F_n) \quad (69)
\]

and

\[
F^{2:n}_{\lambda} = (\Pi^{(e)}_{+,\lambda+2\gamma} \otimes 1)(F_n) = (1 \otimes \Pi^{(f)}_{-,\lambda+2n\gamma})(F_n).
\]

To prove (78), let us consider the family (indexed by \(\lambda \in \mathbb{C} - L\)) of linear endomorphisms \(\ell_\lambda\) of \(U_{\hbar}\mathfrak{n}_+(\tau)\), defined by

\[
\ell_\lambda(x) = \langle \sum_{p+q=n} F^{2:q}_{\lambda+2p\gamma} F^{1:p}_{\lambda}, id \otimes x \rangle,
\]

for \(x\) in \(U_{\hbar}\mathfrak{n}_+(\tau)^{[n]}\).

Assign in \(U_{\hbar}\mathfrak{g}_+(\tau)\), the degree 0 to the elements of \(U_{\hbar}\mathfrak{g}_+(\tau)\), and the degree 1 to each \(e[\epsilon], \epsilon \in k\). Let us denote by \(U_{\hbar}\mathfrak{g}_+(\tau)^{[q]}\) the subspace of \(U_{\hbar}\mathfrak{g}_+(\tau)\) formed by the elements of degree \(q\). Then

\[
\Delta(U_{\hbar}\mathfrak{n}_+(\tau)^{[n]}) \subset \sum_{p+q=n} U_{\hbar}\mathfrak{n}_+(\tau)^{[p]} \otimes U_{\hbar}\mathfrak{g}_+(\tau)^{[q]}.
\]
Let us now fix \( x \) in \( U_h\mathfrak{n}_+(\tau)^{[n]} \), and let us set \( \Delta(x) = \sum_{i,p+q=n} x'_{p,i} \otimes x''_{q,i} \), with \( x'_{p,i} \in U_h\mathfrak{n}_+(\tau)^{[p]} \), \( x''_{q,i} \in U_h\mathfrak{g}_+(\tau)^{[q]} \). Expand \( \ell_\lambda(x) \) as

\[
\ell_\lambda(x) = \sum_{i,p+q=n} \langle F_{\lambda+2p,2q}^2, \text{id} \otimes x'_{q,i} \rangle \langle F_{\lambda,1}^p, \text{id} \otimes x''_{p,i} \rangle = \sum_{i,p+q=n} \Pi^{(e)}_{+,\lambda+2\gamma+2p\gamma}(x'_{q,i}) \Pi^{(e)}_{-,\lambda+2\gamma}(\langle F, \text{id} \otimes x''_{p,i} \rangle) = \sum_{i,p+q=n} \Pi^{(e)}_{+,\lambda+2\gamma+2p\gamma}(x'_{q,i}) \Pi^{(e)}_{-,\lambda+2\gamma}(\langle F, \text{id} \otimes x''_{p,i} \rangle) \quad (70)
\]

the first equality follows from the Hopf algebra pairing rules, the second one from Lemma 2.3, 1), 2), and the third one from the same Lemma, 3). This formula enables us to show:

**Lemma 4.2.** \( \ell_\lambda \) is a left \( U_+^{(e)}(\tau) \)-module map.

**Proof.** Recall that the product map defines a linear isomorphism from \( U_h\mathfrak{h}_+(\tau) \otimes U_+^{(e)} \otimes U_+^{(e)} \) onto \( U_h\mathfrak{g}_+(\tau) \). Define \( U_h\mathfrak{b}_+(\tau) \) as the image of \( U_h\mathfrak{h}_+(\tau) \otimes U_+^{(e)} \otimes 1 \) by this map. \( U_h\mathfrak{b}_+(\tau) \) is then a subalgebra of \( U_h\mathfrak{g}_+(\tau) \).

On the other hand, since for \( r \in \mathcal{O} \), \( \Delta(e[r]) \) is equal to \( \langle e(z) \otimes K^+(z), r \rangle_k + 1 \otimes e[r] \), it belongs to \( U_h\mathfrak{n}_+(\tau) \otimes U_h\mathfrak{b}_+(\tau) \). It follows that \( \Delta(U_+^{(e)}) \subseteq U_h\mathfrak{n}_+(\tau) \otimes U_h\mathfrak{b}_+(\tau) \).

\( \Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi \) is now defined as follows: to \( x \in U_h\mathfrak{g}_+(\tau) \) decomposed as

\[
\sum_i h_i x^+_i x^-_i, \quad \text{with} \quad h_i \in U_h\mathfrak{h}_+(\tau), x^+_i \in U_+^{(e)}, x^-_i \in U_-^{(e)} \lambda+2\gamma, \tag{71}
\]

it associates \( \Pi^{(e)}_{-,\lambda+2\gamma}(\sum_i \varepsilon(h_i) x^+_i x^-_i) \), that is \( \sum_i \varepsilon(h_i) x^+_i x^-_i \). Therefore, it satisfies

\[
(\Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi)(bx) = \varepsilon(b)(\Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi)(x),
\]

for \( x \in U_h\mathfrak{g}_+(\tau), b \in U_h\mathfrak{b}_+(\tau) \).

Let us fix now \( x \in U_h\mathfrak{n}_+(\tau)^{[n]} \), \( b \in U_+^{(e)}m \). Let us set \( \Delta(x) = \sum_{i,p+q=n} x'_{p,i} \otimes x''_{q,i}, x'_{p,i}, x''_{q,i} \) as above, and \( \Delta(b) = \sum_{i,j,p+q=m} b'_{p,j} \otimes b''_{q,j} \), \( b'_{p,j} \in U_h\mathfrak{n}_+(\tau)^{[p]} \), \( b''_{p,j} \in U_h\mathfrak{b}_+(\tau)^{[q]} \) (where \( U_h\mathfrak{b}_+(\tau)^{[q]} \) is the intersection of \( U_h\mathfrak{b}_+(\tau) \) and \( U_h\mathfrak{g}_+(\tau)^{[q]} \)).

Then

\[
\ell_\lambda(bx) = \sum_{i,j,p+q=n,p+r=n} \Pi^{(e)}_{+,\lambda+2\gamma+2(p+r)\gamma}(b'_{p,j} x''_{p,i}) \Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi(b''_{q,j} x''_{q,i}) = \sum_{i,j,p+q=n,p+r=n} \Pi^{(e)}_{+,\lambda+2\gamma+2(p+r)\gamma}(b'_{p,j} x''_{p,i}) \varepsilon(b''_{q,j} x''_{q,i}) (\Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi)(x''_{p,i}),
\]
where the first equality follows from (70), and the second one from (71). If \( p' \neq 0 \), \( \epsilon(b'_{p',j}) \) vanishes, so that \( \sum b'_{m,j} = \sum b'_{p',j} = b \). Therefore \( \ell_\lambda(bx) \) is equal to

\[
\sum_{i,j,p+q=n} \Pi^{(e)}_{+,\lambda+2\gamma+2p\gamma}(bx_{q,i})(\Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi)(x''_{p,i}) = \sum_{i,j,p+q=n} b\Pi^{(e)}_{+,\lambda+2\gamma+2p\gamma}(x'_{q,i})(\Pi^{(e)}_{-,\lambda+2\gamma} \circ \pi)(x''_{p,i}) = b\ell_\lambda(x),
\]

where the first equality follows from the fact that \( \Pi^{(e)}_{-,\lambda+2\gamma} \) is a left \( U_+^{(e)} \)-module map.

Set now for \( x \in U_hn_+(\tau)[q] \), \( \Delta(x) = \sum_{i,p+q=n} x'_{p,i} \otimes x''_{q,i} \), with \( x'_{p,i} \in U_hn_+(\tau)[p] \), \( x''_{q,i} \in U_h\tilde{g}_+(\tau)[q] \). Then

\[
\ell_\lambda(x) = \sum_{i,p+q=n} \langle F_{x_{q,i}}^{(e)} \otimes x''_{q,i}, F_{x'_{p,i}}^{(e)} \otimes x'_{p,i} \rangle
\]

\[
= \sum_{i,p+q=n} \Pi^{(e)}_{+,\lambda+2p\gamma+2\gamma}([F, id \otimes x''_{q,i}])\Pi^{(e)}_{-,\lambda+2\gamma}(x'_{p,i})
\]

\[
= \sum_{i,p+q=n} \Pi^{(e)}_{+,\lambda+2p\gamma+2\gamma}([F, id \otimes x''_{q,i}])\Pi^{(e)}_{-,\lambda+2\gamma}(x'_{p,i}),
\]

(72)

using the Hopf algebra pairing rules, and Lemma 2.2.3). Before we use this identity to derive a result analogous to Lemma 2.7 of [9], we will show the following results.

**Lemma 4.3.** For any \( \epsilon, \epsilon' \in k \), the product \( (q^{-h^{-}})[\epsilon]e_\lambda^{(-1)}[\epsilon'] \) is equal to some combination \( \sum_i \epsilon_{-\lambda_2}^{(-1)}[\epsilon'_i](q^{-h^{-}})[\eta_i] \), for certain \( \eta_i, \eta'_i \in k \).

**Proof.** Recall that in (9), the ratio of theta-functions should be expanded for the argument of \( K^{-} \) near 0; therefore, we have

\[
(q^{-h^{-}})(w)\theta(w - \gamma K - \gamma) = \sum_{\alpha \geq 0} (\partial/\partial w)^\alpha \left( \frac{\theta(w - \gamma K - \gamma)}{\theta(w - \gamma K + \gamma)} \right) \frac{(-z)^{\alpha}}{\alpha!} e(w).
\]

It follows that for \( \epsilon' \in k \), we have

\[
(q^{-h^{-}})(w)\theta(w - \gamma K - \gamma) = \sum_{\alpha \geq 0} \frac{(-z)^{\alpha}}{\alpha!} e \left( \epsilon'(w)(\partial/\partial w)^\alpha \left( \frac{\theta(w - \gamma K - \gamma)}{\theta(w - \gamma K + \gamma)} \right) \right),
\]

therefore we have for \( \epsilon \in k \),

\[
(q^{-h^{-}})[\epsilon]e_\lambda^{(-1)}[\epsilon] = \sum_{\alpha \geq 0} e \left( \epsilon'(w)(\partial/\partial w)^\alpha \left( \frac{\theta(w - \gamma K - \gamma)}{\theta(w - \gamma K + \gamma)} \right) \right) (q^{-h^{-}})[\epsilon(z)^{(-z)^{\alpha}}/\alpha!].
\]
Now if $\epsilon'$ belongs to $L_\lambda$, the products $\epsilon'(w)(\partial/\partial w)^\alpha\left(\frac{\partial^{(w-\gamma)K-\gamma}}{\partial^{(w-\gamma)K+\gamma}}\right)$ belong to $L_{\lambda-2\gamma}$. The lemma follows.

**Lemma 4.4.** For $x \in U_h n_+(\tau), \epsilon \in k$, we have:

1) $\Pi^{(e)}_{\pm,\lambda}(xe^{-\lambda}[\epsilon]) = \Pi^{(e)}_{\pm,\lambda}(x)e^{-\lambda}[\epsilon]$;

2) $\Pi^{(e)}_{\pm,\lambda}(xe^{-\lambda}[\epsilon]) = 0$.

**Proof.** This follows directly from the definitions of $\Pi^{(e)}_{\pm,\lambda}$.

**Lemma 4.5.** We have the identities

$$\ell_{\lambda-2\gamma}(xe^{-\lambda}[\epsilon]) = \ell_\lambda(x)e^{-\lambda}[\epsilon],$$

for $x \in U_h n_+(\tau), \epsilon \in k$.

**Proof.** Let us fix $x \in U_h n_+(\tau)^{[h]}$, and set as above $\bar{\Delta}(x) = \sum_{i,j,p+q=n} \bar{x}'_{p,i} \otimes \bar{x}''_{q,i}$, with $\bar{x}'_{p,i} \in U_h n_+(\tau)^{[p]}$, $\bar{x}''_{q,i} \in U_h \bar{u}_+(\tau)^{[q]}$. From formula (20) follows that $\bar{x}_q$ can be decomposed as a sum $\sum_j y_{q,i,j} h_{q,i,j}$, where $y_{q,i,j}$ belongs to $U_h n_+(\tau)^{[q]}$ and $h_{q,i,j}$ is a linear combination of products of the form $(q^{-h^-}[\epsilon_1] \cdots (q^{-h^-}[\epsilon_p], with $\epsilon_i \in k$ (where we denote $\langle q^{-h^-}(\gamma), \eta(z) \rangle_k$ by $\langle q^{-h^-}[\eta] \rangle$).

Let now $\epsilon$ belong to $k$; we have

$$\bar{\Delta}'(xe^{-\lambda}[\epsilon]) = \sum_{i,j,p+q=n} y_{q,i,j} h_{q,i,j} \otimes \bar{x}'_{p,i} \left(\langle q^{-h^-}(\gamma) \otimes e(z), p_{-\lambda}(\epsilon) \rangle_k + e^{-\lambda}[\epsilon] \otimes 1 \right).$$

Let us set

$$\langle q^{-h^-}(\gamma) \otimes e(z), p_{-\lambda}(\epsilon) \rangle_k = \sum_{s \in \mathbb{Z}} (q^{-h^-})[\epsilon_s] \otimes e[\rho_s(\epsilon)],$$

where $\rho_s$ is a family of linear endomorphisms of $k$.

The according to (72), we have

$$\ell_{\lambda-2\gamma}(xe^{-\lambda}[\epsilon]) = \sum_{s,i,j,p+q=n} \left(\Pi^{(e)}_{+,\lambda+2p\gamma+2\gamma \circ \pi'} \langle y_{q,i,j} h_{q,i,j}(q^{-h^-}[\epsilon_s]) \Pi^{(e)}_{-,\lambda}(x_{p,i} e[\rho_s(\epsilon)])\right)$$

$$+ \sum_{s,i,j,p+q=n} \left(\Pi^{(e)}_{+,\ell+2p\gamma \circ \pi'} \langle y_{q,i,j} h_{q,i,j} e^{-\lambda}[\epsilon] \Pi^{(e)}_{-,\lambda}(x_{p,i})\right).$$

(73)
ELLiptic QUantum GROUPS $E_{r,n}(sl_2)$ AND Quasi-HOPf ALGEBRAS 31

Using the property of $\pi'$ that $\pi'(xt) = \pi'(x)\varepsilon(t)$, for $x \in U_{h\mathfrak{z}_-}(\tau)$, $t \in U_{h\mathfrak{z}_-}(\tau)$, we write the first sum of the r.h.s. of (73) as

$$
\sum_{s,i,j,p+q=n} \left( \Pi_{+,\lambda+2p\gamma+2\gamma}^{(e)} \circ \pi' \right) (y_{q,i,j} h_{q,i,j}) \varepsilon((q^{-h^{-}})[\varepsilon_s]) \Pi_{-,\lambda}^{(e)}(\bar{x}'_{p,i} e[\rho_s(\varepsilon)])
$$

(74)

$$
= \sum_{i,j,p+q=n} \left( \Pi_{+,\lambda+2p\gamma+2\gamma}^{(e)} \circ \pi' \right) (y_{q,i,j} h_{q,i,j}) \Pi_{-,\lambda}^{(e)}(\bar{x}'_{p,i} e_{-\lambda}[\varepsilon])
$$

$$
= \sum_{i,j,p+q=n} \left( \Pi_{+,\lambda+2p\gamma+2\gamma}^{(e)} \circ \pi' \right) (y_{q,i,j} h_{q,i,j}) \Pi_{-,\lambda}^{(e)}(\bar{x}'_{p,i} e_{-\lambda}[\varepsilon])
$$

(75)

here the first equality follows from the properties of $\varepsilon$, and the second one from Lemma 4.4.1).

According to Lemma 4.3, each product $h_{q,i,j} e_{-\lambda}[\varepsilon]$ can be written as a sum

$$
\sum_{t \in \mathbb{Z}} e_{-\lambda-2p\gamma}[\varepsilon_t] h_{q,i,j,t},
$$

with $h_{q,i,j,t} \in U_{h\mathfrak{z}_-}(\tau)$. It follows that the second sum of the r.h.s. of (73) can be written as

$$
\sum_{t,s,i,j,p+q=n} \left( \Pi_{+,\lambda+2p\gamma}^{(e)} \circ \pi' \right) (y_{q,i,j} e_{-\lambda-2p\gamma}[\varepsilon_t] h_{q,i,j,t}) \Pi_{-,\lambda}^{(e)}(\bar{x}'_{p,i}).
$$

(76)

But

$$
\left( \Pi_{+,\lambda+2p\gamma}^{(e)} \circ \pi' \right) (y_{q,i,j} e_{-\lambda-2p\gamma}[\varepsilon_t] h_{q,i,j,t})
$$

$$
= \Pi_{+,\lambda+2p\gamma}^{(e)} (y_{q,i,j} e_{-\lambda-2p\gamma}[\varepsilon_t]) \varepsilon(h_{q,i,j,t}) = 0
$$

by Lemma 4.4.2). Therefore (70) vanishes. The lemma follows from this and (74).

We are now in position to show that for any $\lambda \in \mathbb{C} - L$, $x \in U_{h\mathfrak{n}_+}$,

$$
\ell_{\lambda}(x) = x.
$$

(77)

Using Prop. 4.2, decompose $x$ as a sum $\sum_{i,p,q} x^+_{p,i} x^-_{q,i}$, with $x^+_{p,i}$ in $U_{h\mathfrak{n}_+}^\prime(i)$, $x^-_{q,i}$ in $U_{h\mathfrak{n}_-}^\prime(i)$. By Lemma 4.2, $\ell_{\lambda}(x)$ is equal to $\sum_{i,p,q} x^+_{p,i} \ell_{\lambda}(x^-_{q,i})$; and by Lemma 4.5, this last expression is equal to $\sum_{i,p,q} x^+_{p,i} \ell_{\lambda+2p\gamma}(1)x^-_{q,i}$. We easily check that for any $\lambda \in \mathbb{C} - L$, $\ell_{\lambda}(1) = 1$. (77) follows.

The proposition now follows from the comparison of (77) and Lemma 2.2, and from the fact that $\langle \cdot, \cdot \rangle_{U_{h\mathfrak{n}_+(\tau)}}$ is non-degenerate.

We can now obtain another decomposition of $F$: 

\[\]
**Corollary 4.1.** There is a unique a decomposition of $F$ as
\[ F = F^2_\lambda F^1_\lambda, \quad \text{with } F^1_\lambda \in A^{--} \text{ and } F^2_\lambda \in A^{+-}, \] (78)
with $(\varepsilon \otimes 1)(F_i) = (1 \otimes \varepsilon)(F_i) = 1$, $i = 1, 2$.

**Proof.** Set
\[ F^2_\lambda = \sum_{q \geq 0} \sum_{\alpha \geq 0} (\partial/\partial \lambda)^\alpha (F^2_\lambda) \frac{(-\gamma h^{(2)})^\alpha}{\alpha!}, \]
and
\[ F^1_\lambda = \sum_{p \geq 0} \sum_{\alpha \geq 0} (\partial/\partial \lambda)^\alpha (F^1_\lambda) \frac{(-\gamma h^{(2)})^\alpha}{\alpha!}; \] (79)
$F^1_\lambda$ and $F^2_\lambda$ belong respectively to $A^{--}$ and $A^{+-}$. Since we have also
\[ F^2_\lambda = \sum_{q \geq 0} \sum_{\alpha \geq 0} \partial^\alpha (F^2_\lambda) \frac{(-\gamma h^{(2)} + 2p)^\alpha}{\alpha!}, \]
we can write
\[ F^2_\lambda F^1_\lambda = \sum_{p,q \geq 0} \sum_{\alpha \geq 0} \partial^\alpha (F^2_\lambda) \frac{(-\gamma h^{(2)})^p}{p!} = F. \]

Let us now prove the unicity of the decomposition (78). Let $(F^1_\lambda, F^2_\lambda)$ some other solution to (78). Then, by Prop. 3.1, we will have $F^2_\lambda = F^2_\lambda u$, $F^1_\lambda = u^{-1} F^1_\lambda$, with $u$ some invertible element of $\text{Hol}((\mathbb{C} - L, 1 \otimes \mathbb{C}[h][[\gamma]])$. On the other hand, $(\varepsilon \otimes 1)(F^2_\lambda) = (\varepsilon \otimes 1)(F^2_\lambda) = 1$ implies that $u = 1$. \qed

**Lemma 4.6.** We have an expansion
\[ F^1_\lambda = 1 + h \sum_{i \geq 0} e^{-(1)}_{-\lambda + \gamma h^{(2)}[e_i,-\lambda]} f^{(2)}[e^i] + U_h n_+ (\tau)^{\geq 2} \otimes U_h n_- (\tau)^{\geq 2} \mathbb{C}[h], \]
where $U_h n_+ (\tau)^{\geq 2} = \oplus_{i \geq 2} U_h n_+ (\tau)^{[i]}$.

**Proof.** This follows from formulas (73), (83), and from the fact that $\Pi^{(p)}_{-\lambda+2\gamma}$ maps each $U_h n_+ (\tau)^{[i]}$ to itself. \qed

5. **Twisted cocycle property.**

Let us define for $\lambda \in \mathbb{C} - L$,
\[ \Phi_\lambda = F^{(1)}_{\lambda - \gamma h^{(3)}(\Delta \otimes 1)}(F^1_\lambda) \left( F^{(2)}_{\lambda} (1 \otimes \Delta)(F^1_\lambda) \right)^{-1}. \]

**Proposition 5.1.** The family $(\Phi_\lambda)_{\lambda \in \mathbb{C} - L}$ belongs to $A^{-\cdot} \cap A^{\cdot,-\cdot}$. 

Proof. First observe that if \((\phi_\lambda)_{\lambda \in \mathbb{C} - L}\) belongs to \(A^{+-}\), then \((\phi^{(12)}_{\lambda - \gamma_h(3)})_{\lambda \in \mathbb{C} - L}\) and \((\phi^{(23)}_{\lambda})_{\lambda \in \mathbb{C} - L}\) belong to \(A^{-\cdot\cdot} \cap A^{++}\); this follows easily from the definitions of these algebras. It follows that \((F^{(1)}_{\lambda - \gamma_h(3)})_{\lambda \in \mathbb{C} - L}\) and \((F^{(23)}_{\lambda})_{\lambda \in \mathbb{C} - L}\) also belong to \(A^{-\cdot\cdot} \cap A^{++}\).

By (64), the families \((\Delta \otimes 1)(F^{(1)}_\lambda)\) and \((1 \otimes \Delta)(F^{(1)}_\lambda)\) also belong to \(A^{-\cdot\cdot} \cap A^{++}\). Since \(A^{-\cdot\cdot} \cap A^{++}\) is an algebra, and has the following property: any \(x \in A^{-\cdot\cdot} \cap A^{++}\), invertible in \(\text{Hol}(\mathbb{C} - L, A^{33})\), is such that \(x^{-1}\) belongs to \(A^{-\cdot\cdot} \cap A^{++}\), \((\Phi_\lambda)_{\lambda \in \mathbb{C} - L}\) belongs to \(A^{-\cdot\cdot} \cap A^{++}\). \(\Box\)

Using (34), we may rewrite \(\Phi_\lambda\) as
\[
\Phi_\lambda = \left( (\Delta \otimes 1)(F^{(2)}_{\lambda - \gamma_a(h)}) \right)^{-1} (1 \otimes \bar{\Delta})(F^{(2)}_{\lambda})(1 \otimes F^{(23)}_{\lambda}).
\]

Proposition 5.2. \(\Phi_\lambda \in A^{+-} \cap A^{\cdot\cdot\cdot}\).

Proof. We now remark that if \((\psi_\lambda)_{\lambda \in \mathbb{C} - L}\) belongs to \(A^{+-}\), then \((\psi^{(12)}_{\lambda - \gamma_h(3)})_{\lambda \in \mathbb{C} - L}\) and \((\psi^{(23)}_{\lambda})_{\lambda \in \mathbb{C} - L}\) both belong to \(A^{++} \cap A^{\cdot\cdot\cdot}\). It follows that \((F^{(2)}_{\lambda - \gamma_h(3)})_{\lambda \in \mathbb{C} - L}\) and \((F^{(23)}_{\lambda})_{\lambda \in \mathbb{C} - L}\) also belong to \(A^{++} \cap A^{\cdot\cdot\cdot}\).

By (67), the families \((\Delta \otimes 1)(F^{(2)}_\lambda)\) and \((1 \otimes \Delta)(F^{(2)}_\lambda)\) also belong to \(A^{++} \cap A^{\cdot\cdot\cdot}\). Since \(A^{++} \cap A^{\cdot\cdot\cdot}\) also has the property that any \(x \in A^{++} \cap A^{\cdot\cdot\cdot}\), invertible in \(\text{Hol}(\mathbb{C} - L, A^{33})\), is such that \(x^{-1}\) belongs to \(A^{++} \cap A^{\cdot\cdot\cdot}\), \((\Phi_\lambda)_{\lambda \in \mathbb{C} - L}\) belongs to \(A^{++} \cap A^{\cdot\cdot\cdot}\). \(\Box\)

From the two above propositions follows that we have
\[
\Phi_\lambda = \sum_{i \geq 0} 1 \otimes a^{(i)}_\lambda \otimes h^i, \quad (80)
\]
for a certain family \(a^{(i)}_\lambda\) of elements of \(A(\tau)\), commuting with \(h\).

Let us define now
\[
\Delta_\lambda = \text{Ad}(F^{(1)}_\lambda) \circ \Delta;
\]
this is a family of algebra morphisms from \(A(\tau)\) to \(A(\tau)^{\otimes 2}\), depending on \(\lambda \in \mathbb{C} - L\) in a holomorphic way.

Then we have the twisted quasi-Hopf condition
\[
(\Delta_{\lambda - \gamma_h(3)} \otimes 1) \circ \Delta_\lambda = \text{Ad}(\Phi_\lambda) \circ (1 \otimes \Delta_\lambda) \circ \Delta_\lambda. \quad (81)
\]

Proposition 5.3. \((\Delta_\lambda)_{\lambda \in \mathbb{C} - L}\) and \((\Phi_\lambda)_{\lambda \in \mathbb{C} - L}\) satisfy the compatibility condition
\[
(\Delta_{\lambda - \gamma_h(3) + h(4)} \otimes 1 \otimes 1)(\Phi_\lambda)(1 \otimes 1 \otimes \Delta_\lambda)(\Phi_\lambda) = \Phi^{(123)}_{\lambda - \gamma_h(4)}(1 \otimes \Delta_{\lambda - \gamma_h(4)} \otimes 1)(\Phi_\lambda)\Phi^{(234)}_\lambda. \quad (82)
\]
Theorem 5.1. We have

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We are ready to conclude:

Theorem 5.1. We have Φ

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(83)

Proof. After substitution of (80), the l.h.s. of (82) becomes the product of

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(1 ⊗ h ⊗ 1 + 1 ⊗ h)

since h commutes with the a

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, it follows that these two terms commute. Therefore (82) simplifies to

(1 ⊗ 1 ⊗ Δλ)(Φ

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).

Apply now 1 ⊗ 1 ⊗ 1 ⊗ 1 to this identity. Since (1 ⊗ 1 ⊗ 1)(Φ

λ

) = 1, the first and second term map to 1. On the other hand, since (ε ⊗ 1) o Δλ = id, the last term maps to Φ

λ

. Therefore, Φ

λ

= 1. □
Remark 5. It would be interesting to find some analogue for the $M(\lambda)$ of [3], that is some family of elements of $A(\tau)$ whose twisted coboundary would be $F_\lambda$. This element should belong to some completion of $A(\tau)$, and for $\gamma = 0$ coincide with a “longest Weyl group element the affine algebra”.

6. Dynamical Yang-Baxter equation

The paper [3], sect. 2, contains the following result:

**Proposition 6.1.** (see [3]) Let $(\mathcal{A}, \Delta^A_\infty, \mathcal{R}^A_\infty)$ be a quasi-triangular Hopf algebra, with a fixed element $\widetilde{h}$. Let $F(\lambda)$ be a family of invertible elements of $\mathcal{A} \otimes \mathcal{A}$, parametrized by some subset $U \subset \mathbb{C}$. Set $\Delta(\lambda) = \text{Ad}(F(\lambda)) \circ \Delta^A_\infty$. Suppose that the identity

\[ F^{(12)}(\lambda - \gamma \widetilde{h}(3))(\Delta^A_\infty \otimes 1)(F(\lambda)) = F^{(23)}(\lambda)(1 \otimes \Delta^A_\infty)(F(\lambda)) \]

is satisfied. Then we have

\[ (\Delta(\lambda - \gamma \widetilde{h}(3)) \otimes 1) \circ \Delta(\lambda) = (1 \otimes \Delta(\lambda)) \circ \Delta(\lambda), \]

and if we set $\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}^A_\infty F(\lambda)^{-1}$, we have the identity

\[ \mathcal{R}^{(12)}(\lambda)\mathcal{R}^{(13)}(\lambda - \gamma \widetilde{h}(2))\mathcal{R}^{(23)}(\lambda) = \mathcal{R}^{(23)}(\lambda - \gamma \widetilde{h}(1))\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(12)}(\lambda - \gamma \widetilde{h}(3)). \]  

**Proof.** Define $\widetilde{\Delta}^{[12]}(\lambda)$ as the linear map from $\mathcal{A}^\otimes 2$ to $\mathcal{A}^\otimes 3$ defined by

\[ \widetilde{\Delta}^{[12]}(\lambda)(x \otimes y) = F^{(12)}(\lambda)(\Delta^A_\infty \otimes 1)(x \otimes y)F^{(12)}(\lambda - \gamma \widetilde{h}(3))^{-1}, \]

and $\widetilde{\Delta}^{[21]}(\lambda)(x \otimes y) = \widetilde{\Delta}^{[12]}(\lambda)(x \otimes y)^{(213)}$. Then we have

\[ \mathcal{R}^{(12)}(\lambda)\widetilde{\Delta}^{[12]}(\lambda)(x) = \widetilde{\Delta}^{[21]}(\lambda)(x)\mathcal{R}^{(12)}(\lambda - \gamma \widetilde{h}(3)), \quad x \in \mathcal{A}^\otimes 2 \]

and

\[ \widetilde{\Delta}^{[12]}(\lambda)(\mathcal{R}(\lambda)) = \mathcal{R}^{(13)}(\lambda - \gamma \widetilde{h}(2))\mathcal{R}^{(23)}(\lambda). \]

Applying (88) to $x = \mathcal{R}(\lambda)$ yields (86).

We could also define $\widetilde{\Delta}^{[23]}(\lambda)$ by

\[ \widetilde{\Delta}^{[23]}(x \otimes y) = F^{(32)}(\lambda - \gamma \widetilde{h}(1))(1 \otimes \Delta^A_\infty)(x \otimes y)F^{(32)}(\lambda)^{-1}, \]

$\widetilde{\Delta}^{[32]}(T) = \widetilde{\Delta}^{[23]}(T)^{(132)}$, for $T \in \mathcal{A}^\otimes 3$, then we have $\widetilde{\Delta}^{[23]}(\lambda)(x)\mathcal{R}^{(23)}(\lambda) = \mathcal{R}^{(23)}(\lambda - \gamma \widetilde{h}(1))\widetilde{\Delta}^{[32]}(T)$, and

\[ \widetilde{\Delta}^{[23]}(\lambda)(\mathcal{R}(\lambda)) = \mathcal{R}^{(12)}(\lambda)\mathcal{R}^{(13)}(\lambda - \gamma \widetilde{h}(2)). \]
Note also the identity \( \tilde{\Delta}^{[23]}(T) = (\tilde{\Delta}^{[12]}(T^{(21)}))^{(312)} \). \hfill \Box

Identities (84), (85) and (86) are respectively called the twisted cocycle condition for the family \( \tilde{F}(\lambda) \), the twisted coassociativity condition for \( \Delta(\lambda) \), and the dynamical Yang-Baxter equation for \( R(\lambda) \).

**Theorem 6.1.** Let us set in \( A(\tau)^{\otimes 2} \),
\[
R_{\infty} = q^{D \otimes K} \sum_{i \geq 0} h[\varepsilon] \otimes h[\varepsilon, \varepsilon] q^{\sum_{i \in \mathbb{Z}} \varepsilon[\varepsilon] \otimes f[\varepsilon, \varepsilon]}
\]
and for \( \lambda \in \mathbb{C} - L \), \( R_{\lambda} = (F_{\lambda}^{(12)})^{(21)} R_{\infty} (F_{\lambda}^{(12)})^{-1} \). Then the family \( (R_{\lambda})_{\lambda \in \mathbb{C} - L} \) satisfies the dynamical Yang-Baxter relation
\[
R_{\lambda \rightarrow \gamma h(2)} R_{\lambda}^{(12)} R_{\lambda \rightarrow \gamma h(1)} R_{\lambda}^{(23)} = R_{\lambda \rightarrow \gamma h(2)} R_{\lambda}^{(13)} R_{\lambda \rightarrow \gamma h(3)} R_{\lambda}^{(12)}.
\]

**Proof.** This follows directly from the above proposition and the fact that \( (A(\tau), \Delta, R_{\infty}) \) is a quasi-triangular Hopf algebra (see [12]). \hfill \Box

**Remark 6.** Let \( \tilde{\Delta}_{\lambda}^{[12]} \) be the linear map from \( A^{\otimes 2} \) to \( A^{\otimes 3} \) defined by (87), and \( \Delta_{\lambda}^{[12]} \) the map \( \tilde{\Delta}_{\lambda}^{[12]} \otimes id \) from \( A^{\otimes 2} \otimes \text{Diff}(\mathbb{C} - L) \) to \( A^{\otimes 3} \otimes \text{Diff}(\mathbb{C} - L) \). Here \( \text{Diff}(\mathbb{C} - L) \) is the ring of differential operators in \( \lambda \in \mathbb{C} - L \). Let us set
\[
R = e^{h[\partial]} \frac{\partial}{\partial \lambda} R_{\lambda}.
\]
Then we have the simple relation \( \Delta_{\lambda}^{[12]}(R) = R^{13} R^{23} \). The form of \( R \) indicates that the algebra element \( h[\partial] \) can be naturally added with the derivative \( \frac{\partial}{\partial \lambda} \). This indication could be useful for the study of twisted conformal blocks: in such a theory we need to add differential operators to the elements of \( g_{\lambda} \).

7. **Level 0 Representations of** \( A(\tau) \), **L-operators and RLL relations**

In [11], we studied the 2-dimensional representations, at level 0, of the quantum groups introduced there. In the case of the algebra \( A(\tau) \), these representations can be described as follows.

Let us denote by \( k_{\zeta} \) the local field \( \mathbb{C}(\zeta) \), by \( \partial_{\zeta} \) its derivation \( d/d\zeta \), and by \( k_{\zeta}[\partial_{\zeta}] \) the associated ring of differential operators. Let \( (v_1, v_{-1}) \) be the standard basis of \( \mathbb{C}^2 \), and \( E_{ij} \) the endomorphism of \( \mathbb{C}^2 \) defined by \( E_{ij}(v_a) = \delta_{i,j} v_i \).

**Proposition 7.1.** (see [11], Prop. 9) There is a morphism of algebras \( \pi_{\zeta} : A(\tau) \rightarrow \text{End}(\mathbb{C}^2) \otimes k_{\zeta}[\partial_{\zeta}][[\gamma]] \), defined by the formulas
\[
\pi_{\zeta}(K) = 0, \quad \pi_{\zeta}(D) = \text{Id}_{\mathbb{C}^2} \otimes \partial_{\zeta}, \quad \pi_{\zeta}(h[r]) = E_{11} \otimes \left( \frac{2}{1 + q^r} \right) (\zeta) - E_{-1-1} \otimes \left( \frac{2}{1 + q^{-r}} \right) (\zeta), \quad r \in \mathcal{O},
\]
\[ \pi_\zeta(h[\lambda]) = E_{11} \otimes \left( \frac{1 - q^{-\lambda}}{h \partial} \right) (\zeta) - E_{-1-1} \otimes \left( \frac{q^\lambda - 1}{h \partial} \right) (\zeta), \quad \lambda \in L_0, \]

\[ \pi_\zeta(e[\epsilon]) = \frac{\theta(h)}{h} E_{1-1} \otimes \epsilon(\zeta), \quad \pi_\zeta(f[\epsilon]) = E_{-1,1} \otimes \epsilon(\zeta), \quad \epsilon \in k. \]

**Lemma 7.1.** The image of \( R_{\lambda} \) by \( \pi_\zeta \otimes \pi_{\zeta'} \) is

\[ (\pi_\zeta \otimes \pi_{\zeta'})(R_{\lambda - \gamma}) = A(\zeta, \zeta') R^{-}(\zeta - \zeta', \lambda), \quad (91) \]

where

\[ R^{-}(z, \lambda) = E_{11} \otimes E_{11} + E_{-1-1} \otimes E_{-1-1} + \frac{\theta(z)}{\theta(z + \gamma)} E_{11} \otimes E_{-1-1} \]
\[ + \frac{\theta(\lambda - \gamma)\theta(\lambda + \gamma)}{\theta(\lambda)^2} \frac{\theta(z)}{\theta(z + \gamma)} E_{-1-1} \otimes E_{11} + \frac{\theta(z + \lambda)\theta(\gamma)}{\theta(z + \gamma)\theta(\lambda)} E_{-1-1} \otimes E_{-1-1} \]
\[ - \frac{\theta(z - \lambda)\theta(\gamma)}{\theta(z + \gamma)\theta(\lambda)} E_{-1,1} \otimes E_{1-1}, \]

and \( A(\zeta, \zeta') \) is equal to \( \exp(\sum_{i \geq 0} \left( \frac{1}{\theta q^{i+1}} \right) (\zeta) e_{i,0}(\zeta')). \)

**Proof.** Since the image by \( \pi_\zeta \) and \( \pi_{\zeta'} \) of \( U_{\hbar}n_{\pm}(\tau)_{\geq 2} \) is zero, and by Lemma 4.6, this image is the same as that of

\[ (1 + \hbar \sum_{i \in \mathbb{Z}} e_{-\lambda + \gamma h(i)}^{-2} \left[ e_i \right] f^{(1)}[\epsilon^i])q^{D \otimes R} q^\frac{1}{2} \sum_{i \in \mathbb{Z} h[\epsilon^i] \otimes h[\epsilon_{i,0}]} (1 + \hbar \sum_{i \geq 0} e_{i,0}(\zeta) e_i^i(w)). \]

After we use the expansions

\[ \sum_{i \geq 0} e_{i,\lambda}(z)e^i(w) = \frac{\theta(z - w + \lambda)}{\theta(z - w)\theta(\lambda)}, \quad \text{for} \quad \lambda \in \mathbb{C} - L, \quad \sum_{i \geq 0} e_{i,0}(z)e^i(w) = \frac{\theta'}{\theta}(z - w), \]

and the identities

\[ \sum_{i \geq 0} (f(\partial)e^i)(\zeta)e_{i,0}(\zeta') = \sum_{i \geq 0} e^i(\zeta)(f(-\partial)e_{i,0})(\zeta'), \]

for \( f \) any polynomial in \( \partial \), and

\[ \exp \left( \frac{q^{\theta z} - 1}{\partial_z} \frac{\theta'}{\theta}(z - w) \right) = \frac{\theta(z - w + h)}{\theta(z - w)}, \]
we find
\[(\pi_\zeta \otimes \pi_{\zeta'})(R_\lambda) = \]
\[A(\zeta, \zeta') \left( 1 + \theta(h)(E_{-1,1} \otimes E_{1,-1}) \frac{\theta(\zeta' - \zeta + \lambda + \gamma)}{\theta(\zeta' - \zeta) \theta(\lambda + \gamma)} \right) \cdot \]
\[(E_{1,1} \otimes E_{1,1} + E_{-1,-1} \otimes E_{-1,-1}) + \frac{\theta(\zeta' - \zeta)}{\theta(\zeta' - \zeta + \hbar)} E_{1,1} \otimes E_{1,-1} + \frac{\theta(\zeta' - \zeta - \hbar)}{\theta(\zeta' - \zeta)} E_{-1,-1} \otimes E_{1,1} \right) \cdot \]
\[\left( 1 - \theta(h)(E_{1,-1} \otimes E_{-1,1}) \frac{\theta(\zeta - \zeta' + \lambda + \gamma)}{\theta(\zeta - \zeta') \theta(\lambda + \gamma)} \right) ; \]
the lemma follows. \qed

Define \(R^+(z, \lambda)\) as \(R^-(z, \lambda)^{-1}\). We then have
\[R^+(z, \lambda) = E_{11} \otimes E_{11} + E_{-1,-1} \otimes E_{-1,-1} + \frac{\theta(z)}{\theta(z - \gamma)} \frac{\theta(\lambda - \gamma) \theta(\lambda + \gamma)}{(\theta(\lambda))^2} E_{1,1} \otimes E_{-1,-1} \]
\[+ \frac{\theta(z + \lambda)}{\theta(z - \gamma)} \theta(\gamma) E_{1,-1} \otimes E_{1,1} \]
\[+ \frac{\theta(z - \lambda)}{\theta(z - \gamma)} \theta(\lambda) E_{-1,1} \otimes E_{1,-1}. \]

Let us define now the \(L\)-operators as follows. Set
\[L^+_\lambda(\zeta) = (1 \otimes \pi_\zeta)(R_{\lambda - \gamma}), \quad L^-_\lambda(\zeta) = (1 \otimes \pi_\zeta)(R_{\lambda + \gamma}^{(21)}). \]
Using again the fact that \(U_h \mathfrak{n}_\pm(\tau)^{\zeta_2}\) is mapped to zero by \(\pi_\zeta\), we compute
\[L^+_\lambda(\zeta) = \left( 1 + \theta(h)f^+_{\lambda - \gamma + \gamma}(\zeta) \otimes E_{1,-1} \right) \left( k^+(\zeta + \gamma) \otimes E_{1,1} + k^+(\zeta)^{-1} \otimes E_{-1,-1} \right) \]
\[\left( 1 + \hbar e^{+}_{-\lambda}(z) \otimes E_{-1,1} \right) \]
\[= \begin{pmatrix} 1 & \theta(h)f^+_{\lambda - \gamma + \gamma}(\zeta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^+(\zeta + \gamma) & 0 \\ 0 & k^+(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hbar e^+_{-\lambda}(\zeta) & 1 \end{pmatrix}, \tag{94} \]
and
\[L^-_\lambda(\zeta) = \left( 1 + \hbar e^-_{-\lambda}(\zeta) \otimes E_{-1,1} \right) q^{K_{0\lambda}} \left( k^- (\zeta - h) \otimes E_{1,1} + k^- (\zeta)^{-1} \otimes E_{-1,-1} \right) \]
\[\left( 1 + \theta(h)f^-_{\lambda - \gamma + \gamma}(\zeta) \otimes E_{1,-1} \right) \]
\[= q^{K_{0\lambda}} L^-_\lambda(\zeta), \]
where
\[L^-_\lambda(\zeta) = \begin{pmatrix} 1 & 0 \\ \hbar e^-_{\lambda}(\zeta - Kh) & 1 \end{pmatrix} \begin{pmatrix} k^- (\zeta - h) & 0 \\ 0 & k^- (\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \theta(h)f^-_{\lambda - \gamma + \gamma}(\zeta) \\ 0 & 1 \end{pmatrix}. \tag{95} \]
Theorem 7.1. The matrices $L^\pm_\lambda(\zeta)$ defined by (94) and (95) satisfy the relations

$$R^\pm(\zeta - \zeta', \lambda)L^\pm_\lambda(-\gamma h(z)) = L^\pm_\lambda(x_\gamma h(z))L^\pm_\lambda(x_\gamma h(z')) R^\pm(\zeta - \zeta', \lambda - \gamma h)$$(96)

$$L^\pm_\lambda(x_\gamma h(z))R^\pm(\zeta - \zeta', \lambda) = L^\pm_\lambda(x_\gamma h(z))L^\pm_\lambda(x_\gamma h(z')) A(\zeta, \zeta' - K\gamma).$$ (97)

Proof. It suffices to apply $id\otimes\pi_\zeta\otimes\pi_{\zeta'}$, $\pi_\zeta\otimes\pi_{\zeta'}\otimes id$ and $\pi_\zeta\otimes id\otimes\pi_{\zeta'}$ to (90), after the change of $\lambda$ into $\lambda - \gamma$, to simplify the coefficient $A(\zeta, \zeta')$ of Lemma (7.1) (which is independent of $\lambda$), and to transfer the factors $q^{K\delta_\zeta}$ and $q^{K\delta_{\zeta'}}$ to the left.

Remark 7. Connection of $A$ with $f_K$. The function $A(\zeta, \zeta')$ of Lemma (7.1) satisfies the functional equation

$$A(\zeta, \zeta')A(\zeta + h, \zeta') = \frac{\theta(\zeta - \zeta')}{\theta(\zeta - \zeta' + h)}.$$

after analytical prolongation, we see that $A$ only depends on $\zeta - \zeta'$. The ratio $A(\zeta, \zeta' - K\gamma)/A(\zeta, \zeta')$ of relation (97) is then simply connected with the function $f_K$ expressing the commutator $(k_+^+(z), k_-^-(w))$ (relation (18)) by

$$A(\zeta, \zeta' - K\gamma)/A(\zeta, \zeta') = f_K(\zeta' - \zeta - h)^{-1}.$$ 

8. Elliptic quantum group $E_{\tau, \eta}(sl_2)$.

8.1. Definition. Let us set $\eta = \gamma/2$ and define $E_{\tau, \eta}(sl_2)$ as the algebra generated by $h$ and the $a_i(\lambda), b_i(\lambda), c_i(\lambda), d_i(\lambda), i \geq 0, \lambda \in \mathbb{C} - L$, subject to the relations

$$[h, a_i(\lambda)] = [h, d_i(\lambda)] = 0, \quad [h, b_i(\lambda)] = -2b_i(\lambda), \quad [h, c_i(\lambda)] = 2c_i(\lambda),$$

and if we set

$$a(z, \lambda) = \sum_{i \geq 0} a_i(\lambda)e_{i, -\gamma h/2}(z), \quad b(z, \lambda) = \sum_{i \geq 0} b_i(\lambda)e_{i, -\gamma h - \gamma h - 2i}(z),$$

$$c(z, \lambda) = \sum_{i \geq 0} c_i(\lambda)e_{i, -\lambda + \gamma h}(z), \quad d(z, \lambda) = \sum_{i \geq 0} d_i(\lambda)e_{i, \gamma h/2}(z),$$

and

$$L(z, \lambda) = \begin{pmatrix} a(z, \lambda) & b(z, \lambda) \\ c(z, \lambda) & d(z, \lambda) \end{pmatrix},$$ (98)
the relations
\[
R^{+(12)}(z_1 - z_2, \lambda - \gamma h) L^{(1)}(z_1, \lambda) L^{(2)}(z_2, \lambda - \gamma h^{(2)}) = L^{(2)}(z_2, \lambda) L^{(1)}(z_1, \lambda - \gamma h^{(2)}) R^{+(12)}(\lambda, z_1 - z_2)
\]
and
\[
\det(z, \lambda) = d(z + \gamma, \lambda) a(z, \lambda + \gamma) - b(z + \gamma, \lambda) c(z, \lambda + \gamma) \frac{\theta(\lambda - \gamma h - \gamma)}{\theta(\lambda - \gamma h)} = 1.
\]

Here \(R^+(z, \lambda) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)\) is given by (93); we define \(h^{(1)}\) as \((E_{11} - E_{-1, -1}) \otimes 1\) and \(h^{(2)}\) as \(1 \otimes (E_{11} - E_{-1, -1})\). We also define as before, \(f(\lambda - \gamma h)\) as \(\sum_{\alpha \geq 0} (\partial^\alpha f)(\lambda) \frac{(-\gamma h)^\alpha}{\alpha!}\), and \(f(\lambda - \gamma h^{(i)})\) as \(\sum_{\alpha \geq 0} (\partial^\alpha f)(\lambda) (-\gamma h^{(i)})^\alpha / \alpha!\).

**Remark 8.** The \(L\)-operator defined by (98) is 1-periodic in the variables \(z\) and \(\lambda\), and satisfies
\[
L(z + \tau, \lambda) = t_{\lambda - \gamma h} L(z, \tau) t_{\lambda}^{-1},
\]
where \(t_{\lambda} = \left( \begin{array}{cc} e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{array} \right) \). On the other hand, the \(R\)-matrix (92) satisfies the conditions
\[
R^+(z + \tau, \lambda) = t_{\lambda - \gamma h^{(2)}}^{(1)} R^+(z, \lambda) t_{\lambda}^{-1(1)}.
\]

The fact that the periodicity conditions (101) and (102) seem compatible leads us to conjecture that the algebra \(E_{\tau, \eta}(\mathfrak{sl}_2)\) is a flat deformation of the function algebra of the group of holomorphic maps \(L^d\) from \((\mathbb{C} - L)^2\) to \(SL_2(\mathbb{C})\), such that
\[
L^d(z + \tau, \lambda) = \text{Ad}(t_{\lambda})(L^d(z, \lambda)).
\]
Since the morphism \(\Psi\) defined in Thm. 9 is obviously surjective, this algebra has “at least the size” of \(U_{\hbar}\mathfrak{o}(\tau)\).

8.2. **Connection with the usual formulas.** The formulas defining the elliptic quantum groups in [16] involve an \(R\)-matrix different from (92). Let us explain their connection with the above formalism.

Consider the ring \(\mathcal{F}[[\gamma]]\) of formal series in \(\gamma\), with coefficients meromorphic functions in \(\lambda\). Let us adjoin to it a square root \(\theta^{1/2}(\lambda)\) of \(\theta(\lambda)\). The new ring \(\mathcal{F}_{1/2}[[\gamma]]\) then contains a solution \(\varphi\) of the functional equation
\[
\frac{\varphi(\lambda + \gamma)}{\varphi(\lambda - \gamma)} = \frac{\theta(\lambda)}{\theta(\lambda - \gamma)};
\]
we have
\[
\varphi(\lambda) = \theta^{1/2}(\lambda) \exp \left( -\frac{1}{2\partial} \tanh(\gamma \partial/2) \frac{\theta'}{\theta} \right)(\lambda).
\]
Lemma 8.1. Let us set for \( \lambda \in \mathbb{C} - L, \)
\[
\bar{a}(z, \lambda) = \frac{\varphi(\lambda - \gamma h)}{\varphi(\lambda - \gamma)}a(z, \lambda), \quad \bar{b}(z, \lambda) = \frac{\varphi(\lambda - \gamma h)}{\varphi(\lambda + \gamma)}b(z, \lambda),
\]
\[
\bar{c}(z, \lambda) = \frac{\varphi(\lambda - \gamma h)}{\varphi(\lambda - \gamma)}c(z, \lambda), \quad \bar{d}(z, \lambda) = \frac{\varphi(\lambda - \gamma h)}{\varphi(\lambda + \gamma)}d(z, \lambda);
\]
let us set \( \bar{L}(z, \lambda) = \left( \bar{a}(z, \lambda) \quad \bar{b}(z, \lambda) \right), \) Define
\[
\bar{R}(z, \lambda) = E_{11} \otimes E_{11} + E_{-1, -1} \otimes E_{-1, -1} + \frac{\theta(\lambda + \gamma)\theta(z)}{\theta(\lambda)\theta(z - \gamma)}E_{1, 1} \otimes E_{-1, 1} \tag{105}
\]
\[
+ \frac{\theta(\lambda - \gamma)\theta(z)}{\theta(\lambda)\theta(z - \gamma)}E_{-1, 1} \otimes E_{11} - \frac{\theta(\lambda + z)\theta(\gamma)}{\theta(\lambda)\theta(z - \gamma)}E_{1, -1} \otimes E_{-1, 1}
\]
\[
- \frac{\theta(-\lambda + z)\theta(\gamma)}{\theta(-\lambda)\theta(z - \gamma)}E_{-1, -1} \otimes E_{1, 1};
\]
then we have the relations (see [10])
\[
h\bar{a}(z, \lambda) = \bar{a}(z, \lambda)h, \quad h\bar{d}(z, \lambda) = \bar{d}(z, \lambda)h, \tag{106}
\]
\[
h\bar{b}(z, \lambda) = \bar{b}(z, \lambda)(h - 2), \quad h\bar{c}(z, \lambda) = \bar{c}(z, \lambda)(h + 2), \tag{107}
\]
\[
\bar{R}^{(12)}(z_1 - z_2, \lambda - \gamma h)\bar{L}^{(1)}(z_1, \lambda)\bar{L}^{(2)}(z_2, \lambda - \gamma h^{(1)}) = \bar{L}^{(2)}(z_2, \lambda)\bar{L}^{(1)}(z_1, \lambda - \gamma h^{(2)})\bar{R}^{(12)}(\lambda, z_1 - z_2). \tag{108}
\]

Proof. We have
\[
\bar{R}(z, \lambda) = \varphi(\lambda - \gamma h^{(2)})\bar{R}^+(z, \lambda)\varphi(\lambda - \gamma h^{(1)})^{-1},
\]
and
\[
\bar{L}(z, \lambda) = \varphi(\lambda - \gamma h)L(z, \lambda)\varphi(\lambda - \gamma h^{(1)})^{-1};
\]
Substitute these expressions in (99); simplifications show that the \( \bar{R}(z, \lambda) \) and \( \bar{L}(z, \lambda) \) satisfy (108).
Remark 9. The formulas of Lemma 8.1 only use functions of $F[[\gamma]]$, although their proof uses the extension to $F_{1/2}[[\gamma]]$. 

Remark 10. In [16], the determinant is defined by the formula

$$\text{Det}(z, \lambda) = \frac{\theta(\lambda)}{\theta(\lambda - \gamma h)} (d(z + \gamma, \lambda)\bar{a}(z, \lambda + \gamma) - \bar{b}(z + \gamma, \lambda)\bar{c}(z, \lambda + \gamma)).$$

This formula is equivalent to the first equation of (100), as one sees by inserting the expressions for $\bar{a}, \ldots, \bar{d}$ in terms of $a, \ldots, d$ and using the identity

$$\varphi(\lambda - \gamma h)\varphi(\lambda - \gamma h + \gamma) = \theta(\lambda - \gamma h) \varphi(\lambda + \gamma).$$

Remark 11. By tensoring them with 1-dimensional representations, we can view the evaluation representations studied in [16] as representations of the factor algebra introduced in this paper by the relation $\text{Det}(z, \lambda) = 1$. After expansion in series in $\eta = \gamma/2$, the formulas defining the evaluation representations of [16] only have singularities for $\lambda \in L$ or $z - w \in L$. The effect of the tensoring with 1-dimensional representations is to multiply the matrix $L(w, \lambda)$ by a function $g_z(w)$ satisfying

$$g_z(w + \gamma)g_z(w) = \frac{\theta(z - w + (\Lambda - 1)\eta)}{\theta(z - w - (\Lambda + 1)\eta)}, \quad \eta = \gamma/2;$$

this equation can be solved in a similar way to that for $\varphi$, and we will find for $g_z(w)$ a formal series in $\gamma$ with coefficients functions of $z - w$ with only singularities for $z - w \in L$.

Therefore the final representations can be viewed as representations of the algebras $E_{\tau,\eta}(\mathfrak{sl}_2)$, provided $w$ is considered as a formal variable at the origin (as it is the case in [14]).

9. Quantum currents for $E_{\tau,\eta}(\mathfrak{sl}_2)$

Theorem 9.1. There is a morphism $\Psi$ from $E_{\tau,\eta}(\mathfrak{sl}_2)$ to $U_{\hbar}\mathfrak{g}_\mathcal{O}(\tau)$, defined by the formulas

$$\Psi(h) = h,$$

$$\Psi(a(z, \lambda)) = \hbar\theta(h)f_{\lambda - h - \gamma h}(z)k^+(z)^{-1}e^+_\lambda(z) + k^+(z - \hbar),$$

$$\Psi(b(z, \lambda)) = \theta(h)f_{\lambda - h - \gamma h}(z)k^+(z)^{-1}$$

$$\Psi(d(z, \lambda)) = k^+(z)^{-1}, \quad \Psi(c(z, \lambda)) = \hbar k^+(z)^{-1}e^+_\lambda(z),$$

(109) (110) (111)
ELLIPTIC QUANTUM GROUPS $E_{\tau,\eta}(\mathfrak{sl}_2)$ AND QUASI-HOPF ALGEBRAS

Proof. Let us first show that these formulas are generating series for images of the $a_i, b_i, c_i, d_i, i \geq 0$. For this, we note that their right-hand sides are holomorphic functions on $(\mathbb{C} - L)^2$ with values in $U_{\hbar\mathfrak{g}_{O,\tau}}$, 1-periodic in $z$ and $\lambda$ and with the quasi-periodicity properties in $z$ and $\lambda$ described by (101). For the periodicity properties in $z$, this follows directly from (34), the equations

$$k^+(z + 1) = k^+(z), \quad k^+(z + \tau) = e^{i\pi\gamma\hbar} k^+(z),$$

which are proved in the same way as (14), and the commutation relations between $h$ and the $e^+_\lambda(z), f^+_\lambda(z)$.

By Thm. 7.1, $\Psi(a(z, \lambda)), \Psi(b(z, \lambda)), \Psi(c(z, \lambda))$ and $\Psi(d(z, \lambda))$ satisfy the relations (99). Finally, one computes that the image by $\Psi$ of the middle term of equation (100) is equal to 1. This ends the proof of the theorem.

Remark 12. Since $\Psi(d(z, \lambda))$ is independent of $\lambda$, it should be clear that $\Psi$ is not injective.

Remark 13. There is an algebra morphism from the tensor product $E_{\tau,\eta}(\mathfrak{sl}_2) \otimes \text{Diff}(\mathbb{C} - L)$ to $E_{\tau,\eta}(\mathfrak{sl}_2)^{\otimes 2} \otimes \text{Diff}(\mathbb{C} - L)$; the formulas for it are $\Delta(L(z, \lambda)) = L^{(13)}(z - \gamma h^{(2)}, \lambda)L^{(23)}(z, \lambda)$. It would be interesting to understand better the relation of this formula with (89).

Remark 14. Relations (96) and (97) suggest to define double elliptic quantum groups generated by the matrices $L^\pm(z, \lambda)$, the derivation $D$ and the central element $K$, with the following functional properties: the $L^\pm(z, \lambda)$ are holomorphic functions in the variable $\lambda \in \mathbb{C} - L$; $L^+(z, \lambda)$ also depends holomorphically on $z \in \mathbb{C} - L$, and $L^-(z, \lambda)$ is a a regular power series in $z$; the periodicity conditions for $L^\pm(z, \lambda)$ in $\lambda$ are the same as those for $L(z, \lambda)$, and the periodicity conditions for $L^+(z, \lambda)$ in $z$ are the same as those for $L(z)$; and satisfying relations (90), (97) and $[D, L^\pm(z, \lambda)] = \partial L^\pm(z, \lambda)/\partial z$. This algebra should be, as it is the case for $E_{\tau,\eta}(\mathfrak{sl}_2)$ with respect to $U_{\hbar\mathfrak{g}_{O,\tau}}$, somewhat larger than $U_{\hbar\mathfrak{g}(\tau)}$.

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