NON-ISOMETRIC PAIRS OF RIEMANNIAN MANIFOLDS WITH THE SAME GUilleMIN-RUELLE ZETA FUNCTION

Hy P.G. Lam

ABSTRACT. In [Su85], T. Sunada constructed a vast collection of non-isometric Laplace-isospectral pairs \((M_1, g_1)\), resp. \((M_2, g_2)\) of Riemannian manifolds. He further proves that the Ruelle zeta functions

\[ Z_g(s) := \prod_{\gamma}(1 - e^{-sL(\gamma)})^{-1} \]

of \((M_1, g_1)\), resp. \((M_2, g_2)\) coincide, where \(\{\gamma\}\) runs over the primitive closed geodesics of \((M, g)\) and \(L(\gamma)\) is the length of \(\gamma\). In this article, we use the method of intertwining operators on the unit cosphere bundle to prove that the same Sunada pairs have identical Guillemin-Ruelle dynamical L-functions

\[ L_G(s) = \sum_{\gamma}\frac{L^#_{\gamma}}{\det(I - P_{\gamma})}, \]

where the sum runs over all closed geodesics.

1 INTRODUCTION

The purpose of this article is to use a modified Sunada construction to produce non-isometric pairs with the same Guillemin-Ruelle dynamical zeta function. To state the result, we need to introduce some background and notation. Assume that a connected compact manifold \(M_0\) fits as the base of a diagram of finite normal covers,

\[ \pi_1 \quad (M, \pi^* g_0) \quad \pi_2 \]

\[ \begin{aligned}
(M_1 := H_1 \setminus M, p_1^* g_0) \\
(M_2 := H_2 \setminus M, p_2^* g_0)
\end{aligned} \]

\[ (M_0 = G \setminus M, g_0) \]

where the triplet \((G, H_1, H_2)\) satisfies \(L^2(G/H_1) \simeq L^2(G/H_2)\) as unitarily equivalent \(G\)-modules. Suppose that \(G\) acts freely on \(M\). Sunada then proves that for any metric \(g_0\) on \(M_0\), the induced metrics \(g_i := p_i^* g_0\) \((i = 1, 2)\) are Laplace-isospectral [Su85]. In addition, the pairs have the same Ruelle length L-function \(L_g(s) = \sum_{\gamma} e^{-sL(\gamma)}\), where the sum runs over all closed geodesics (see [Su94] for more geometric descriptions of \(L\)-functions).

Diagram (1.0.1) induces a diagram of finite covers of cosphere bundles,

\[ \begin{aligned}
S^* M & \quad \tilde{\pi}_1 \quad \tilde{\pi}_2 \\
S^* M_1 & \quad \tilde{\pi} \\
S^* M_0 & \quad \tilde{p}_1 \quad \tilde{p}_2 \\
S^* M_2 &
\end{aligned} \]

For each cosphere bundle of (1.0.2), we denote the Hamiltonian flow by \(G^t_g\) (§2.1), and the set of its periodic orbits \(\gamma = \gamma(t)\) by \(\mathcal{G}\). We denote \(L_\gamma\) and \(L^#_\gamma\) respectively the length and

*Research partially supported by NSF RTG grant DMS-1502632.
prime period of \( \gamma \). Let \( \mathbf{P}_\gamma \) be the associated linearized Poincaré map (§2.2). \( G^\nu_g \) is said to be Lefschetz if all of its periodic orbits are non-degenerate, that is, the maps \( I - \mathbf{P}_\gamma \) are invertible. Under this hypothesis, we define the Guillemin-Ruelle zeta function with respect to a given Riemannian metric \( g \) by

\[
L_G(s) = \sum_{\gamma \in \mathcal{D}} \frac{L^\#_g e^{-sL_\gamma}}{|\det(I - \mathbf{P}_\gamma)|}
\]  

(1.0.3)

More generally, a geodesic flow is said to be clean if its fixed point sets are Bott-Morse non-degenerate (§2.3). Let \( \text{Fix}(G^t_g) := \{ \zeta \in S^*M : G^t_g\zeta = \zeta \} \) be the fixed points set of \( G^t_g \). Under the clean fixed point condition, \( \text{Fix}(G^t_g) = Z_1 \cup \ldots \cup Z_d \), each \( Z_j \) is a submanifold of \( S^*_gM \) and \( T_\gamma Z_j = \text{Fix}((dG^t_g)_\gamma) \). The Poincaré map is replaced by the normal Poincaré map \( P^\#_\gamma \) in (1.0.3) (see §3.4), which is simply the bijective mapping induced by \( dG^t_g \)

\[
(I - P^\#_Z) := (I - dG^t_g)^\#: TS^*M/TZ_j \to TS^*M/TZ_j.
\]  

(1.0.4)

Furthermore, on each \( Z_j \), there is a Gelfand-Leray form

\[
\int_{Z_j} dv_{Z_j}(\zeta) = \int_{Z_j} \frac{\mu_L}{|\det(I - P^\#_Z)|} \beta
\]

where \( \mu_L \) is the normalized Liouville measure on \( S^*_gM \) and \( \beta \) is the volume density on the normal space \( NZ_j \) determined by the defining equation \( dG^t_g\zeta = \zeta \) (§6.2, see also §3.5).

Let \( \mathcal{D} \) be the union of all component of all fixed point sets. Let \( L_Z \) be the common length of all closed geodesics in \( Z \) so that \( G^t_g \) fixes \( Z \).

\[
L_G(s) = \sum_{Z \in \mathcal{D}} e^{-sL_Z} \int_Z \left| \det(I - P^\#_Z) \right|^{-1} d\mu_Z
\]  

(1.0.5)

**Theorem 1.1.** Let \( \pi : (M, \pi^*g_0) \to (M_0, g_0) \) be as in the Sunada diagram (1.0.1). Let \( G^t_{gi} \) be Hamiltonian flows on each cosphere bundle \( (S^*M_i, g_i) \), \( i = 1, 2 \). Let \( L_G \) be the Guillemin-Ruelle L-function associated to \( g_i \), \( i = 1, 2 \). If the flows \( G^t_{gi} \) are clean and the triplet \( (G, H_1, H_2) \) satisfies \( L^2(G/H_1) \simeq L^2(G/H_2) \) as unitarily equivalent \( G \)-modules (†), then \( L_G = L_G \).

To prove Theorem 1.1, we first show that the Koopman operators associated to the (co-)geodesic flows \( G^t_{gi} \) on the pair \( S^*M_i \), \( i = 1, 2 \) from (1.0.1) are unitarily equivalent by a unitary Fourier integral operator. The Koopman operators on \( L^2(S^*M_i, d\mu_{L_i}) \) are defined by \( V_{gi}f = f \circ G^t_{gi} \) (§3.1). The unitary Fourier integral operators are constructed via a generalization of the technique developed by S. Zelditch in [Z92]. In the article, the author provides an explicit intertwining operator defined by a finite Radon transform associated to the diagram of covers in (1.0.1). By definition, a finite Radon transform is a Fourier integral operator associated to a finitely multi-valued symplectic correspondence. In our case, we define a finite Radon transform on the unit cosphere bundle \( \tilde{U}_A : L^2(S^*M_1, dx_1) \to L^2(S^*M_2, dx_2) \) associated to the induced covering diagram (1.0.2).

Here, we identify the sphere bundle to the copshere bundle via the metric. \( \tilde{U}_A \) has the form

\[
\tilde{U}_A = \frac{1}{\#H_1} \sum_{a \in H} A(a) T_a \tilde{\pi} g_2 \tilde{\pi}^*_1 : L^2(S^*_gM, d\mu_{L_1}) \to L^2(S^*_gM, d\mu_{L_2})
\]  

(1.0.6)

where \( A \) is the function on \( H_2 \setminus G/H_1 \) inducing the intertwining operator \( A : L^2(G/H_1) \to L^2(G/H_2) \) (§5.2).
Theorem 1.2. Let \( \pi : (M, \pi^* g_0) \rightarrow (M_0, g_0) \) be a normal finite Riemannian covering with covering transformation group \( G \) as in (1.0.1). If the triplet \((G, H_1, H_2)\) satisfies (1), then the Koopman operators \( V^t_{g_i} : C^0_\infty S^* M_i \rightarrow C^0_\infty S^* M_i \) associated with the subcovers \((M_i, g_i)\) for each \( i = 1, 2 \), are unitarily equivalent via \( \hat{U}_A \). Namely,

\[
\tilde{U}_A V^t_{g_1} \tilde{U}_A^* = V^t_{g_2}
\]

The Schwartz kernel \( k^t(\xi, \eta) \) of the Koopman operator \( V^t_{g_i} \) is a \( \delta \)-function \( \delta(\xi - G^t_g \eta) \). We refer to [G77] for the use of the Schwartz kernel theorem in this statement. Guillemin further defined the flat trace of \( V^t_g \) formally by

\[
\text{Tr}^\flat V^t_g = \int_{S^* M} \delta(\xi - G^t_g \xi) d\mu_L(\xi)
\]

More precisely, the trace is defined as a pushforward-pullback operation on distributions by \( \Pi, \Delta^* \delta(\xi - G^t_g \xi) \) where \( \Delta : S^* M \rightarrow S^* M \times S^* M \) is the diagonal embedding, and \( \Pi : S^* M \times \mathbb{R}^+ \rightarrow S^* M \) is the natural projection.

If all periodic trajectories of the geodesics flows \( G^t_i \), are non-degenerate, then the righthand side of (1.0.7) becomes \( G77 \)

\[
\text{Tr}^\flat V^t_g = \sum_{\gamma \in \mathcal{G}} \frac{L^\#_\gamma \delta(t - L_\gamma)}{|\det(I - P_\gamma)|}, \quad t > 0
\]

In \( \S 3.7 \) we show that (1.0.8) has a natural extension to geodesic flows with clean fixed point sets.

Proposition 1.1. Assume all of the fixed points of \( G^t_i \) are clean, then

\[
\text{Tr}^\flat V^t_g = \sum_{Z \in \mathcal{Z}} \delta(t - L_Z) \int_{\mathcal{Z}} \frac{d\mu_Z}{|\det(I - P^\#_Z)|}
\]

We show in \( \S 4 \) that the Riemannian pairs resulting from Sunada’s construction have the same Guillemin flat trace, namely, \( \text{Tr}^\flat V^t_{g_1} = \text{Tr}^\flat V^t_{g_2} \). The flat trace is not an \( L^2 \)-trace and therefore it does not follow from Theorem 1.2 that \( V^t_{g_1}, V^t_{g_2} \) have the same flat trace.

Theorem 1.3. Let \( S^a_i M, i = 1, 2 \) be the pair of cosphere bundles of the Sunada pair as in (1.0.2). Let \( G^t_i \) be Hamiltonian flows on \( S^a_i M \). For each \( i = 1, 2 \), if all fixed points of \( G^t_i \) are clean, then \( \text{Tr}^\flat V^t_{g_1} = \text{Tr}^\flat V^t_{g_2} \). Hence,

\[
L_{G_1}(s) = L_{G_2}(s).
\]

Remark. Proposition 1.1 is the analogue for the Koopman operator of the Duistermaat-Guillemin wave trace formula. In the case of Lefschetz flows, the formula states [GuDu]

\[
\text{Tr} \left( \exp \left( it \sqrt{\Delta_g} \right) \right) = \sum_{j=1}^{\infty} e^{i\lambda_j t} \sum_{\gamma \in \mathcal{G}} \frac{i^{\sigma_\gamma} L^\#_\gamma \delta(t - L_\gamma)}{|\det(I - P_\gamma)|^2} + L^1_{\text{loc}}, \quad t \in \mathbb{R}
\]

with respect to the Laplace-Beltrami operator \( \Delta_g : C^\infty M \rightarrow C^\infty M \), where \( \lambda_j \) are the eigenvalues of \( \sqrt{\Delta_g} \), and \( \sigma_\gamma \) is the Maslov index of \( \gamma \). Laplace isospectrality is equivalent to

\[
\text{Tr} \left( \exp \left( it \sqrt{\Delta_{g_1}} \right) \right) = \text{Tr} \left( \exp \left( it \sqrt{\Delta_{g_2}} \right) \right),
\]

3
and by (1.0.9),
\[ \sum_{\gamma \in \mathcal{G}_{g_1}} i^{\sigma_{\gamma} L^\#} \delta(t - L_{\gamma}) \frac{1}{|\det(I - P_{\gamma})|^2} = \sum_{\gamma \in \mathcal{G}_{g_2}} i^{\sigma_{\gamma} L^\#} \delta(t - L_{\gamma}) \frac{1}{|\det(I - P_{\gamma})|^2} \] (1.0.11)

S.Chen and A.Manning [ChMa] studied the associated L-function
\[ \log Z_g(s) = \sum_{\gamma \in \mathcal{G}} \sum_{k=1}^{\infty} e^{-skL^\#_{\gamma}} \frac{|\det(I - P_{\gamma}^k)|^{1/2}}{k} \]
where \( P_{\gamma}^k \) being the linearized Poincaré map corresponding to the \( k \)-fold iterate of \( \gamma \). It follows from (1.0.10) and (1.0.11) that isospectrality implies \( \log Z_{g_1} = \log Z_{g_2} \). However, if there are multiplicities in the length spectrum of \( S^*M \), one cannot determine the individual term of each length from the sum, and therefore one cannot determine \( L_G \) from (1.0.10), (1.0.11).

1.1 Convergence of the L-function. For compact negatively curved manifolds, Chen-Manning [ChMa] proved that the zeta function \( Z_g(s) \) converges absolutely for \( \Re s > P(\alpha/2) \), where \( \alpha \) is the volume growth rate in the orientable unstable foliation on the cotangent bundle of \( S^*M \), and \( P \) is the pressure of Sinai’s function on \( S^*M \). It follows from their theorem that \( L_G(s) \) converges absolutely on the right half-plane \( \{ \Re s > P(\alpha) \} \) under the same assumption.

It is proven by Dyatlov-Zworski that \( Z_g(s) \) has a meromorphic continuation to the whole complex plane [DyZw]. There are many articles concerned with the meromorphic continuation of the Ruelle zeta function; we refer to [DyZw] for references and background.

A natural problem is to show that there is a half-plane of convergence for \( L_G(s) \) for any compact Riemannian manifold and to study its meromorphic continuation. In the forthcoming article [H22], we will use a heat kernel approach to flat traces to prove that \( L_G(s) \) converges in \( \{ \Re(s) > 0 \} \) at least for the cases of flat tori, spheres, compact hyperbolic manifolds and for Heisenberg nilmanifolds. In work in progress, we extend the result to general non-positively curved manifolds.

1.2 Acknowledgment. This article is part of the author’s PhD thesis at Northwestern University under the guidance of Steve Zelditch.

2 Preliminaries

2.1 Hamiltonian flows. Consider \( G^l_g : S^*_g M \to S^*_g M \) to be the geodesic flow on the unit cosphere bundle generated by the quadratic form
\[ H(x, \xi) = \frac{1}{2} |\xi|_g^2 = \frac{1}{2} g^{ab}(x)\xi_a\xi_b \]
with \( g^{ab}(x) \) is the inverse metric tensor via \( g^{ab}(x)g_{bc}(x) = \delta^a_c \), and \( \xi_x \) is a covector in \( S^*_x M \) realized under the local trivialization of \( S^*_M \) restricted to a coordinate neighborhood \( U \), in which a 1-form \( \zeta = \xi_a dx^a \) in \( S^*_x M|_U \) is identified with the point \( (x, \xi_a) \in U \times S^{n-1} \). In local (Darboux) coordinates, \( G^l_g \) satisfies the geodesic equations
\[ \frac{dx^a}{\partial \xi_a} = g^{ab}(x)\xi_b \quad \dot{\xi}_a = -\frac{\partial H}{\partial x^a} = -\frac{1}{2} \frac{\partial g^{bc}(x)}{\partial x^a} \xi_a \xi_b \]
and is a Hamiltonian flow on \( S^*_g M \), which projects to a geodesic line on \( M \).
The Hamiltonian vector field $\Xi_H$ is defined by
$$\Xi_H \cdot \omega = -dH$$
where $\omega$ is the canonical symplectic form on $T^*M$.

### 2.2 Poincaré map

Let $\gamma = \gamma(t)$ be a periodic trajectory of the Hamiltonian flow $G^t_g$ with period $\tau$ on $S^*M$ starting at $\zeta$. Let $\Sigma$ be a transversal local hypersurface to the flow at $\zeta$. Let $T$ be the first return time of an orbit starting from $\Sigma$ to $\Sigma$. The first-return map is the following
$$\Phi_\Sigma : \Sigma \rightarrow \Sigma; \quad \eta \mapsto G^T_g(\eta).$$

At the periodic point $\zeta$, the linear Poincare map is defined by
$$P_\gamma = (d\Phi_\Sigma)_\zeta : T^*_\zeta \Sigma \rightarrow T^*_\zeta \Sigma$$
where $U_i(t), V_i(t)$ are the basis of normal Jacobi fields along $\gamma$ with $U_i(0) = v_i, V_i(0) = 0$.

### 2.3 Clean fixed point set of $G^t_g$

We denote the fixed point set of $dG^t_g : TS^*_g M \rightarrow TS^*_g M$ by $\text{Fix}(dG^t_g)$ note that $T\text{Fix}(G^t_g) \subset \text{Fix}(dG^t_g)$ as $G^t_g$ is the identity map on $\text{Fix}(G^t_g)$. In general, the fixed point set of $dG^t_g$ can be larger than $T\text{Fix}(G^t_g)$, or equivalently, there can exist normal Jacobi fields that do not come from varying the periodic geodesics in the fixed point set.

As defined above, $G^t_g$ is said to have a clean fixed point set for period $\tau$ if $T\text{Fix}(G^t_g) = \text{Fix}(dG^t_g)$.

Assume $Z$ is a clean component of period $\tau$ where all points in $Z$ share the same periodic trajectory, then a local transversal region to the geodesic flow restricted to a neighborhood about a point $\zeta \in Z$ is a hyperspace $\Sigma$ with codimension strictly greater than 1.

The manifold $M$ is said to satisfy the Clean Intersection Hypothesis if $G^t_g$ has a clean fixed point set for all period $\tau \in \text{Lsp}(S^*_g M)$.

### 3 Proof of Proposition 3.1

#### 3.1 Koopman operator

Let $\{G^t_g\}_{t \geq 0}$ be a contact Hamiltonian geodesic flow on the unit copshere bundle $S^*M$ of a compact Riemannian manifold $(M,g)$ ($\S 2.1$). The Koopman operator, denoted by $\{V^t_g\}_{t \geq 0}$, is defined as the one-parameter group of unitary operators on $L^2(S^*M)$ given by
$$V^t_g : L^2(S^*M, d\mu_L) \rightarrow L^2(S^*M, d\mu_L); \quad V^t_g(f) = f \circ G^t_g.$$

$V^t_g$ is unitary because $\mu_L$ is preserved by $G^t_g$. 

5
3.2 Unitary intertwining operator. For a Sunada pair \((M_i, g_i)\) as in the diagram of covers \([1.0.1]\), there exists an unitary intertwining kernel \(A : L^2(G/H_1) \to L^2(G/H_2)\) between the isomorphic \(G\)-modules. The intertwining operator \(A\) is identified as a convolution kernel \(A \in \Delta(H_2 \backslash G / H_1)\), which is a function on the double coset space satisfying the identity \(A(b_2 a b_1) = \chi_2(b_2) A(a) \chi_1(b_1)\) for all \(b_2 \in H_2, b_1 \in H_1\) \([pg. 365, [Ma78]\]), where \(\chi_1, \chi_2\) are respectively the 1-dimensional characters of \(H_1, H_2\).

For each A, S. Zelditch constructed a unitary operator given by the averaged weighted sum of Radon transforms

\[
U_A = \frac{1}{\#H_1 \#H_2} \sum_{a,b \in G} \tilde{A}(b^{-1}) A(a) \iota (\pi_{1 \ast} \tilde{T}_{a} \iota \chi_{1 \ast}(b)) \iota (\pi_{2 \ast} \tilde{T}_{a} \iota \chi_{2 \ast}(b))
\]

where \(T_a\) is the translation by a group element \(a\) on the finite covering space \(M\) \([pg. 709, [Z92]\]).

We show that \(\tilde{U}_A\) lifts to a unitary operator \(\tilde{U}_A\) between the space of \(L^2\)-distributions on \(S^*_{g_i} M\) associated with the diagram of finite covers \([1.0.2]\) which we repeat for clarity

\[
\begin{aligned}
S^*(H_1 \backslash M) &\quad \overset{\pi_1}{\longrightarrow} \quad S^* M \\
\tilde{\pi}_1 &\quad \overset{\tilde{\pi}_2}{\longrightarrow} \quad S^*(H_2 \backslash M)
\end{aligned}
\]

**Proposition 3.1.** The finite Radon transform \(\tilde{U}_A : L^2(S^*_g M, d \mu_{L_1}) \to L^2(S^*_g M, d \mu_{L_2})\) given in \([1.0.6]\) is a unitary operator mapping the space of \(L^2\)-distributions on \(S^*_g M\) into the space of \(L^2\)-distributions on \(S^*_g M\).

**Proof.** It is clear that \(\pi_i : M \to M_i = H_i \backslash M\) determines the lifted cover \(\tilde{\pi}_i : S^* M \to S^* M_i\) given by the dual of the differentials \(d \pi_i\) restricted to the unit tangent bundle. Thus, the \(\tilde{\pi}_i\)'s are basewise covers and cover the base maps. The translations \(\{T_a\}_{a \in G}\) are local isometries that lift to a map \(T_a : S^* M \to S^* M\) where \(T_a(x, \xi) = (a \cdot x, (a^{-1})^\ast \xi)\).

For each \(i\), we regard \(L^2(S^*_g M)\) as the space \(L^2(S^*_g M)^{H_i}\) consisting of \(H_i\)-invariants elements of \(L^2(S^*_g M)\) since \(S^*_g M\) is obtained via quotient the induced action of \(H_i\) on \(S^*_g M\). For \(f \in L^2(S^*_g M)\),

\[
\tilde{\pi}_i f(x', \xi') = \sum_{h \in H_i} f(T_h(x), T_{h^{-1}}(\xi)) = \sum_{h \in H_i} (\tilde{T}_h)^\ast f(x', \xi').
\]

Therefore, \(\tilde{\pi}_i = \sum_{h \in H_i} \tilde{T}_h^\ast\). Let us write \(\tilde{\pi}_i = \sum_{h \in H_i} \tilde{T}_h\). Consider the product of \(\tilde{U}_A\) with its adjoint, by change of variables,

\[
\tilde{U}_A^\ast \tilde{U}_A = \frac{1}{\#H_1 \#H_2} \sum_{a,b \in G} \tilde{A}(b^{-1}) A(a) (\pi_{1 \ast} \tilde{T}_{b^{-1}} \iota \chi_{1 \ast})(\pi_{2 \ast} \tilde{T}_{a} \iota \chi_{2 \ast})
\]

\[
= \frac{1}{\#H_1 \#H_2} \sum_{a,b \in G} \tilde{A}(b^{-1}) A(a) \iota (\sum_{b_1 \in H_1} \tilde{T}_{b_1} \iota \chi_{1 \ast}) \iota (\sum_{b_2 \in H_2} \tilde{T}_{b_2} \iota)\]

\[
= \frac{1}{\#H_1 \#H_2} \sum_{g \in G, b_1 \in H_1, b_2 \in H_2} \tilde{A}(b^{-1}) A(a) \iota \tilde{T}_{b_1 b^{-1} b_2} \tilde{T}_a
\]

\[
= \frac{1}{\#H_1 \#H_2} \sum_{b_1, b_2, a, b} \tilde{A}(b^{-1}) A(a) \tilde{T}_{b_1 b^{-1}} \tilde{T}_a \quad (b^{-1} := b_1 b^{-1} b_2)
\]
\[
\begin{align*}
&= \frac{1}{\#H_1 \#H_2} \sum_{b_1, b_2} \sum_{a, b} A(b^{-1}) A(b^{-1} a) \tilde{T}_{b^{-1} b^{-1} a} \\
&= \frac{1}{\#H_1 \#H_2} \sum_{b_2} \sum_{b_1} A^*(a) \tilde{T}_a \quad (a \mapsto b^{-1} b^{-1} a) \\
&= \frac{1}{\#H_2} \sum_{b_2} \frac{1}{\#H_1} \sum_{a} \chi_{H_1}(a) \tilde{T}_a \quad (A \text{ is unitary}) \\
&= \frac{1}{\#H_1} \sum_{b_1} \tilde{T}_{b_1} \text{Id}_{L^2(S^* M)^{H_1}}.
\end{align*}
\]

3.3 Guillemin flat trace. The Schwartz kernel of the Koopman operator is given by
\[
V_g^* f(\xi) = \int_{S^* M} f(\eta) \delta(\eta - G_g^* \xi) d\mu_L(\eta),
\]
that is, the kernel is a \(\delta\)-distribution in the sense of [G77] with support on the graph of \(G_g^*\). (see Appendix §6.2). Recall the diagonal map
\[
\Delta : S^*_g M \times \mathbb{R}^+ \to S^*_g M \times S^*_g M \times \mathbb{R}^+
\]
\((\zeta, t) \mapsto (\zeta, \zeta, t)\) and \(\Pi : S^*_g M \times \mathbb{R}^+ \to S^*_g M\) the projection map \((\zeta, t) \mapsto \zeta\). Assuming the diagonal map intersects the graph of \(G_g^*\) transversally, Guillemin defines the flat trace
\[
\text{Tr}^\flat V_g^* = \Pi_* \Delta^* k^* \tag{3.3.1}
\]
in Theorem 6, [G77] and shows that it is a well-defined \(\delta\)-distribution on \(\mathbb{R}^+\). The transversality hypothesis is equivalent to the Lefschetz condition imposed on the flow \(G_t^g\) on \(S^*_g M\) under which, for each \(t > 0\), the left-hand side of (3.3.1) is the formula (1.0.8).

Under the general condition of clean intersection (§2.3), (3.3.1) is still a well-defined \(\delta\)-distribution on \(\mathbb{R}^+\) (§6.2).

3.4 Canonical volume density on a clean submanifold and the normal Poincare map. Suppose \(Z_j^{(\tau)}\) is a clean component of \(\text{Fix}(G_g^*)\) consisting of periodic orbits of period \(\tau \in \text{Lsp}(S^*_g M)\). A periodic orbit in \(Z_j^{(\tau)}\) projects under \(\pi\) to a closed geodesic \(\bar{\gamma}(t)\) on \(M\).

The geodesic variations coming from the family of closed geodesics on \(Z_j^{(\tau)}\) form the total space of Jacobi fields on \(\pi(Z_j^{(\tau)})\). Given a Jacobi field \(\bar{J}\) in this space, \(\bar{J}\) lifts to a flow-invariant field on \(S^* M\) along \(\bar{\gamma}\) via the correspondence
\[
\bar{J} \mapsto (\bar{J}^h, \nabla_{\bar{\gamma}} \bar{J}^v)
\]
where \(\bar{J}^h, \nabla_{\bar{\gamma}} \bar{J}^v\) are respectively the horizontal and vertical lifts of \(\bar{J}\) and \(\nabla_{\bar{\gamma}} \bar{J}\) [Lemma 3.1.6, [Kl]].

Denote the total space of periodic Jacobi fields on \(Z_j\) along \(\gamma\) by \(\mathcal{J}_{Z_j^{(\tau)}}(\gamma)\). Given \(X, Y \in \mathcal{J}_{Z_j^{(\tau)}}(\gamma)\), the Riemannian inner product is
\[
(X, Y) = \int_\gamma g(\pi(X_h(s)), \pi(Y_h(s))) \, ds \tag{3.4.1}
\]
where $X_h$, $Y_h$ are the horizontal components of $X$ and $Y$ via the splitting induced from the Riemannian connection on $S^*_gM$.

Denote further $\mathcal{J}_j^{\|}(\gamma)$ to be the subspace of periodic $J$-fields that are $Z_j^{\tau}$-tangential along $\gamma$. This subspace spans $T|_\gamma Z_j^{\tau}$. Also, we have an orthonormal basis of $\mathcal{J}_j^{\|}(\gamma)$ including $\dot{\gamma}$, and hence obtain a collection of 1-forms $\beta^1, \ldots, \beta^k$ on $Z_j^{\tau}$ corresponding to the dual basis, which yields

$$|d\text{Vol}_{\text{can}}(Z_j^{\tau})(\zeta)| = \left|\beta^1 \wedge \ldots \wedge \beta^k\right|$$

(3.4.2)
as a natural volume element on $Z_j^{\tau}$. We should like to point out that there is a symplectic structure on the space of all Jacobi fields along $\gamma$ given by the Wronskian

$$\omega(J, J') = g(J, \nabla \dot{\gamma} J') - g(\nabla \dot{\gamma}, J').$$

Now denote $\mathcal{J}_j^{\perp}(\gamma) := \mathcal{J}_j^{\|}(\gamma)/\mathcal{J}_j^{\|}(\gamma)$ to be the space of periodic Jacobi fields along $\gamma$ that are transversal to $Z_j^{\tau}$. This space spans $T|_\gamma S^*_gM/T|_\gamma Z_j^{\tau}$. The normal Poincare map is realized as the linear symplectic transformation on $\mathcal{J}_j^{\perp}(\gamma)$, which is given by

$$(P^{\#}_{Z_j^{\tau}})\zeta : T\xi S^*_gM/T\xi Z_j^{\tau} \oplus T\xi S^*_gM/T\xi Z_j^{\tau} \otimes \text{translate by } t=\tau$$

$$(J(0), \nabla \dot{\gamma} J(0)) \mapsto (J(\tau), \nabla \dot{\gamma} J(\tau))$$

where $J(t) \in \mathcal{J}_j^{\perp}(\gamma)$. Let $\nu_1, \ldots, \nu_{2n-1-k}$ ($k = k(j, \tau) = \dim Z_j^{\tau}$) be an orthonormal basis of $T\xi S^*_gM/T\xi Z_j^{\tau}$ along $\gamma$ with respect to the metric (3.4.1). Let $U_i(t), V_i(t)$ be an orthonormal basis of $\mathcal{J}_j^{\perp}(\gamma)$ with the initial conditions

$$\begin{cases}
U_i(0) = \nu_i , \quad V_i(0) = 0 \\
\nabla \dot{\gamma} U_i(0) = 0 , \quad \nabla \dot{\gamma} V_i(0) = \nu_i.
\end{cases}$$

(3.4.3)

In block-matrix form, the transformation of $U_i, V_i$ by $(P^{\#}_{Z_j^{\tau}})\zeta$ is

$$(P^{\#}_{Z_j^{\tau}})\zeta = \begin{pmatrix}
(\langle \nu_i, U_i(\tau) \rangle)_{i,l} & (\langle \nu_i, V_i(\tau) \rangle)_{i,l} \\
(\langle \nu_i, \nabla \dot{\gamma} U_i(\tau) \rangle)_{i,l} & (\langle \nu_i, \nabla \dot{\gamma} V_i(\tau) \rangle)_{i,l}
\end{pmatrix}. $$

(3.4.4)

3.5 Stationary phase with Bott-Morse nondegenerate critical submanifold. We will need the clean version of the stationary phase formula and recall its statement in this section. For further background and proofs of the formula, we refer to [GuSc13], [GuSc90], [Hor].

Let $(\mathcal{X}, \mu)$ be an $N$-dimensional manifold equipped with a positive density measure. Let $a, \phi \in C^\infty(\mathcal{X}; \mathbb{R})$ and denote

$$C_\phi := \{x \in \mathcal{X} : (X\phi)(x) = 0, \forall X \in \Gamma(T\mathcal{X})\}$$
to be the set of critical points of $\phi$. The function $\phi$ is said to be Bott-Morse if $C_\phi$ is a submanifold of $\mathcal{X}$ and the transverse Hessian of $\phi$ denoted by $\text{Hess}^\perp \phi$, is non-degenerate.
Consider the integral
\[ I(h) = \int_N e^{ih\phi} d\mu \]
with large parameter \( h \). Let \( Z \) be a \( k(Z) \)-dimensional connected component of \( C_\phi \), and let \( y_1, \ldots, y_N \) be a system of coordinates on the normal bundle associated with \( \phi \) defined by \( y_1 = \ldots = y_{k(Z)} = 0 \). The coordinates about a fixed point \( x \) in \( Z \) are given by
\[ y_1 = x_1 - \phi(x), \ldots, y_{k(Z)} = x_{k(Z)} - \phi(x), y_{k(Z)+1} = x_{k(Z)+1}, \ldots, y_N = x_N. \]
With respect to the basis \( \nu_1 = \partial_{y_{k(Z)+1}}|_x, \ldots, \nu_{N-k(Z)} = \partial_{y_{N}}|_x \) of the fibre \( T_xN/T_xZ \), the determinant of the transverse Hessian is
\[ |\det \text{Hess}^{\bot}|_x|_{\phi} = |d^2\phi(\nu_1 \wedge \ldots \wedge \nu_{N-k(Z)}, \nu_1 \wedge \ldots \wedge \nu_{N-k(Z)})| = |\det(\delta_{y_i, y_j}|_{y=x})|. \]
The density measure on \( Z \) is given by the quotient
\[ d\nu_Z = \frac{d\mu}{|\det \text{Hess}^{\bot}|_x|_{\phi}^{\frac{1}{2}}|dy_{k(Z)+1} \ldots dy_N|}. \]

**Lemma 3.1.** If \( \phi \) is a Bott-Morse function, then
\[ I(h) = \sum_{Z \subset C_\phi} (2\pi h)^{-\frac{N-k(Z)}{2}} \left( e^{2\pi i \text{sgn}(Z)} e^{ih\phi(Z)} \int_Z d\nu_Z + O(h^{-1}) \right), \quad (3.5.1) \]
where \( \text{sgn}(Z) \) is the signature of the symmetric bi-linear form \( d^2\phi|_Z \).

### 3.6 Mollification of the flat-trace kernel
Recall the kernel of the Guillemin flat trace is the \( \delta \)-function \( \delta(x - G_\phi^t \xi) \) on \( S^*M \). Denote \( \chi^{(\tau)}(\xi) := \delta(x - G_\phi^t \xi) \) to be the indicating function on the clean submanifold \( \text{Fix}(G_\phi^t) \). Let us fix a finite smooth atlas on \( S^*M \) with coordinate charts \( (\Omega_i, \varphi_i) \) where the \( \varphi_i \)'s are local diffeomorphisms. We regularize \( \chi^{(\tau)} \) as follows.

First let \( \{ \rho_i \in \mathcal{C}_c^\infty(\Omega_i; [0, 1]) \} \) be a corresponding partition of unity on \( S^*M \) satisfying \( \sum_i \rho_i = 1 \). Next we define the \( C^\infty \)-function \( \psi : \mathbb{R}^{2n-1} \rightarrow \mathbb{R} \)
\[ \psi(\theta) := \begin{cases} \frac{C e^{-|\theta|^2-1}}{\theta}, & \text{if } |\theta| < 1 \\ 0, & \text{if } |\theta| \geq 1 \end{cases} \]
where the constant \( C \) is chosen so that \( \int_{\mathbb{R}^{2n-1}} \psi(\theta) d\theta = 1 \). Denote
\[ \psi_{h^{-1}}(\theta) := h^{2n-1} \psi(h\theta); \quad h > 0. \]
The convolution of \( \chi^{(\tau)} \) by \( \psi \) is defined by
\[ \chi^{(\tau)} * \psi_{h^{-1}} := \sum_i (\rho_i \chi^{(\tau)}) * \psi_{h^{-1}}. \]

**Lemma 3.2.** There exists a constant \( C = C(n) \) such that
\[ \left| \chi^{(\tau)} - \chi^{(\tau)} * \psi_{h^{-1}} \right| \leq Ch^{-2} \sum_i \rho_i \chi^{(\tau)} |d\text{Vol}|_{\Omega_i}. \quad (3.6.1) \]
Proof. Let $\xi$ be a point on $S^*_gM$, $\xi = \varphi_j^{-1}(\theta)$ for some $j$ and $\theta \in \mathbb{R}^{2n-1}$. Applying the Taylor expansion to $\rho_j S \varphi_j^{-1}$ about $\theta$ gives

$$
\rho_j \chi^{(r)} \varphi_j^{-1}(\theta - \theta') = \rho_j \chi^{(r)} \varphi_j^{-1}(\theta) + d(\rho_j \chi^{(r)} \varphi_j^{-1})_\theta (\partial \theta') + R_\theta (\theta')
$$

where there is a constant $C'$ such that

$$
R_\theta (\theta') \leq C' |\theta'|^2 \int_{\mathbb{R}^{2n-1}} |\rho_j \chi^{(r)} \varphi_j^{-1}|^2(\theta'') d\theta''.
$$

By (3.6.2) and a change of variable, we have

$$
\left| \chi^{(r)} - \chi^{(r)} \ast \psi_{h^{-1}} \right|(\xi) \leq \int_{\mathbb{R}^{2n-1}} |\psi_{h^{-1}}(\theta') \left( \rho_j \chi^{(r)} \varphi_j^{-1} - \rho_j \chi^{(r)} \varphi_j^{-1}(\theta) \right) | d\theta'
$$

$$
\leq C' \int_{\mathbb{R}^{2n-1}} |\rho_j \chi^{(r)} \varphi_j^{-1} |^2 (\theta'') d\theta'' \int_{\mathbb{R}^{2n-1}} h_{2n-1} |\psi_{h\theta}'| |\theta''|^2 d\theta'
$$

$$
\leq C h^{-2} \int_{\mathbb{R}^{2n-1}} |\rho_j \chi^{(r)} |^2 (\theta'') d\theta''
$$

hence the desired inequality (3.6.1).

Let $\epsilon = h^{-1}$ be arbitrarily small. Lemma 3.5.2 shows that $\chi^{(r)}_{\epsilon} := \chi^{(r)} \ast \psi_{\epsilon}$ approximates the flat-trace kernel. In addition, we may take its Fourier transform

$$
\hat{\chi}^{(r)}_{\epsilon}(\theta) := \mathcal{F} \chi^{(r)}_{\epsilon}(\theta) = \sum_j \mathcal{F} \rho_j \chi^{(r)}(\theta)
$$

$$
= \sum_j (2\pi)^{-\frac{2n-1}{2}} \int_{\mathbb{R}^{2n-1}} \rho_j \chi^{(r)} \varphi_j^{-1}(\theta') e^{-i\theta' \cdot \theta} d\theta'.
$$

Without loss of generality, we assume the flow-image of a neighborhood about $\xi$ at time $\tau$ is contained in the same coordinate chart in which we have a local parametrization $G_{\tau}^* \xi = \varphi_j^{-1}(\tilde{\theta})$. By the Fourier integral theorem,

$$
\chi^{(r)}(\xi) = (2\pi)^{-\frac{2n-1}{2}} \int_{\mathbb{R}^{2n-1}} \chi^{(r)}_{\epsilon}(\varphi_j^{-1}(\tilde{\theta})) \int_{\mathbb{R}^{2n-1}} e^{i(\theta' \cdot \theta')} d\theta' d\tilde{\theta}.
$$

Let us write $(\xi - G_{\tau}^* \xi)^j = \theta - \tilde{\theta}$. As $\epsilon \downarrow 0+$, we obtain

$$
\chi^{(r)}(\xi) = (2\pi)^{-\frac{2n-1}{2}} \int_{\mathbb{R}^{2n-1}} e^{i(\xi - G_{\tau}^* \xi)^j \theta'} d\theta'.
$$

(3.6.3)

3.7 Proof of Proposition 1.1 We want to prove

$$
\text{Tr}^g V_{\tau}^g = \sum_{\tau \in \text{Lap}(S^*_gM)} \sum_{1 \leq j \leq d(\tau)} \delta(t - \tau) \int_{Z_j^{(\tau)}} |d\text{Vol}_{\text{can}}(Z^{(\tau)})|(\xi) |(I - \mathbf{P}_Z^{(\tau)})_\xi|.
$$

We provide a proof of (3.7.1) using the method of stationary phase in [§3.5.3]. The justification of Proposition 1.1 following the same outline as in [Theorem 8, [G77]] is given in [§6.2].

Proof. Let $\gamma$ be a $\tau$-periodic trajectory starting at a point $\zeta$ on a clean component $Z_j^{(\tau)}$. Suppose $\zeta_1, \ldots, \zeta_{2n-1}$ are coordinates about $\zeta$ on $S^*_gM$ in which $Z_j^{(\tau)}$ is described by $\zeta_{k+1} = \ldots = \zeta_{2n-1} = 0$ where $\zeta_j$’s are defined by

$$
\xi_1 = \zeta_1, \ldots, \xi_k = \zeta_k, \xi_{k+1} = (G_{\tau}^* \xi)_{k+1} = \zeta_{k+1}, \ldots, \xi_{2n-1} = (G_{\tau}^* \xi)_{2n-1} = \zeta_{2n-1}.
$$
The orthonormal basis for $T_\xi S^*_g M/T_\xi Z^{(\tau)}_j$ is given by \{\partial_{\xi_{k+1}}|_{\xi}, \ldots, \partial_{\xi_{2n-1}}|_{\xi}\} (*).

Consider the function $\phi(\xi, \theta) = \sum_{1 \leq i \leq 2n-1} \theta_i(\xi_i - (G^*_g \xi)_i)$. The critical submanifold of $\phi$ in $S^*_g M \times \mathbb{R}^{2n-1}$ is the union of product manifolds $\bigcup_{j \leq d(\tau)}(Z^{(\tau)}_j \times \mathbb{R}^{k=k(j,\tau)}).$ In light of lemma 3.2, we let $\{\delta_\epsilon\}_{\epsilon > 0}$ be a net of smooth functions with compact support containing $Z^{(\tau)}_j$ and approximating the flat-trace kernel to which we have

$$\delta_\epsilon(\xi - G^*_g \xi) = (2\pi \epsilon)^{-n} \int_{\mathbb{R}^{2n-1}} \hat{\delta}_\epsilon(\theta)e^{\frac{i\theta(\xi,\bar{\theta})}{\epsilon}} d\theta.$$ 

Combining (3.6.3) with lemma 3.1 gives

$$\int_{S^* M} \delta(\xi - G^*_g \xi) d\mu_L(\xi) = \lim_{\epsilon \downarrow 0^+} \sum_{j \leq d(\tau)} \int_{Z^{(\tau)}_j \times \mathbb{R}^{k(j,\tau)}} \hat{\delta}_\epsilon(\theta)d\mu_L(\xi) \wedge d\theta$$

$$= \lim_{\epsilon \downarrow 0^+} \sum_{j \leq d(\tau)} \int_{Z^{(\tau)}_j \times \mathbb{R}^{k(j,\tau)}} \hat{\delta}_\epsilon(\theta)d\mu_L(\xi) \wedge d\theta.$$ 

(3.7.2)

(3.7.3)

In terms of the basis (*), the transverse Hessian of $\phi$ is the block-matrix

$$\text{Hess}^{-1}\left|_{(\zeta, \bar{\theta})}(\phi(\xi, \theta))\right| = \begin{pmatrix}
\text{d}^2_{\xi\xi} \phi\left|_{(\zeta, \bar{\theta})}\right.
& \text{d}^2_{\xi\theta} \phi\left|_{(\zeta, \bar{\theta})}\right.

\text{d}^2_{\theta\xi} \phi\left|_{(\zeta, \bar{\theta})}\right.
& \text{d}^2_{\theta\theta} \phi\left|_{(\zeta, \bar{\theta})}\right.
\end{pmatrix},$$

where

$$\text{d}^2_{\xi\xi} \phi\left|_{(\zeta, \bar{\theta})}\right. = \begin{pmatrix}
\partial_{\xi_{k+1}}|_{\xi=\zeta}(\xi - G^*_g \xi)_{k+1} & \ldots & \partial_{\xi_{k+1}}|_{\xi=\zeta}(\xi - G^*_g \xi)_{2n-1}

\cdots

\partial_{\xi_{2n-1}}|_{\xi=\zeta}(\xi - G^*_g \xi)_{k+1} & \ldots & \partial_{\xi_{2n-1}}|_{\xi=\zeta}(\xi - G^*_g \xi)_{2n-1}
\end{pmatrix},$$

$$\text{d}^2_{\theta\theta} \phi\left|_{(\zeta, \bar{\theta})}\right. = \begin{pmatrix}
1 - \partial_{\xi_{k+1}}|_{\xi=\zeta}(G^*_g \xi)_{k+1} & \ldots & -\partial_{\xi_{k+1}}|_{\xi=\zeta}(G^*_g \xi)_{2n-1}

\cdots

-\partial_{\xi_{2n-1}}|_{\xi=\zeta}(G^*_g \xi)_{k+1} & \ldots & 1 - \partial_{\xi_{2n-1}}|_{\xi=\zeta}(G^*_g \xi)_{2n-1}
\end{pmatrix},$$

$$\text{d}^2_{\xi\theta} \phi\left|_{(\zeta, \bar{\theta})}\right. = \begin{pmatrix}
-\sum_i \theta_i \partial^2_{\xi_{k+1}}|_{\xi=\zeta}(G^*_g \xi)_i & \ldots & -\sum_i \theta_i \partial^2_{\xi_{k+1}}|_{\xi=\zeta}(G^*_g \xi)_i

\cdots

-\sum_i \theta_i \partial^2_{\xi_{2n-1}}|_{\xi=\zeta}(G^*_g \xi)_i & \ldots & -\sum_i \theta_i \partial^2_{\xi_{2n-1}}|_{\xi=\zeta}(G^*_g \xi)_i
\end{pmatrix} = O,$$

$$\text{d}^2_{\theta\xi} \phi\left|_{(\zeta, \bar{\theta})}\right. = O.$$
arbitrary Schwartz distribution on $S^\ast$. Consequently, let $\tilde{\eta}$.

Similarly, since $\tilde{P}$ is a Schwartz distribution on $S^\ast$, $f$ is an arbitrary Schwartz distribution on $S^\ast M$, then

$$ (\tilde{U}^* \chi_{S^\ast M} | f)_{L^2(S^\ast M, d\mu_{L_2})} = \left( \chi_{S^\ast M} | \tilde{U} f \right)_{L^2(S^\ast M, d\mu_{L_2})} = \left( \frac{\# H_1}{\# H_2} \chi_{S^\ast M} \right)_{L^2(S^\ast M, d\mu_{L_2})}. $$

4 \hspace{1cm} PROOF OF THEOREM 1.1

**Proof of Theorem 1.3** Let $u, v$ be Schwartz distributions on $S^\ast_{g_2} M$, and $k_1, k_2$ be respectively the distributional kernel of the transfer operators $V^\ast_{g_1}, V^\ast_{g_2}$. From theorem 1.2,

$$ (k_1 | u \otimes v)_{L^2(S^\ast_{g_2} M \times S^\ast_{g_2} M, d\mu_{L_2} \times d\mu_{L_2})} = \left( V^\ast_{g_2} v | u \right)_{L^2(S^\ast_{g_2} M, d\mu_{L_2})} = \left( U V^\ast_{g_1} U^* v | u \right)_{L^2(S^\ast_{g_2} M, d\mu_{L_2})} = \left( V^\ast_{g_1} U^* v | U^* u \right)_{L^2(S^\ast_{g_1} M, d\mu_{L_1})} = (k_1 | U^* u \otimes v)_{L^2(S^\ast_{g_1} M \times S^\ast_{g_1} M, d\mu_{L_1} \times d\mu_{L_1})}. $$

In integral form, this reads

$$ \int_{S^\ast_{g_2} M \times S^\ast_{g_2} M} k_1^1(\xi, \eta) u(\xi, \eta) v(\eta) d\mu_{L_2}(\xi, \eta) = \int_{S^\ast_{g_1} M \times S^\ast_{g_1} M} k_1^1(\xi', \eta') U^* u(\xi', \eta') U^* v(\eta, \eta') d\mu_{L_1}(\xi', \eta') $$

for every $u, v \in S(S^\ast_{g_2} M)$. Assuming the free action of $H_i$ on $S^\ast M$, the following holds

$$ \chi_{\left\{ (\xi', \eta') \in S^\ast_{g_2} M \mid |\xi' - \eta'\rangle \right\}}(\xi', \eta') = \chi_{\left\{ (\tilde{\xi}, \tilde{\eta}) \in S^\ast M \mid |\tilde{\xi} - \eta\rangle \right\}}(\tilde{\xi}, \tilde{\eta}) = \frac{\# H_1^1}{\# H_2^1} \chi_{\left\{ (\tilde{\xi}, \tilde{\eta}) \in S^\ast M \mid |\tilde{\xi} = \eta\rangle \right\}}(\tilde{\xi}, \tilde{\eta}) $$

$$ = \frac{\# H_1^2}{\# H_2^2} \chi_{\left\{ (\tilde{\xi}, \tilde{\eta}) \in S^\ast M \mid |\tilde{\xi} = \eta\rangle \right\}}(\tilde{\xi}, \tilde{\eta}) $$

Similarly, since $\tilde{U}$ is a unitary operator mapping $L^2(S^\ast_{g_1} M, d\mu_{L_1})$ to $L^2(S^\ast_{g_2} M, d\mu_{L_2})$, $f$ is an arbitrary Schwartz distribution on $S^\ast_{g_1} M$, then

$$ (\tilde{U}^* \chi_{S^\ast_{g_2} M} | f)_{L^2(S^\ast_{g_1} M, d\mu_{L_1})} = \left( \chi_{S^\ast_{g_2} M} | \tilde{U} f \right)_{L^2(S^\ast_{g_2} M, d\mu_{L_2})} = \left( \frac{\# H_1}{\# H_2} \chi_{S^\ast_{g_1} M} \right)_{L^2(S^\ast_{g_2} M, d\mu_{L_2})}. $$

12
By density, \( \tilde{U}^*S_{g_2} = \frac{\#H_1}{\#H_2} \chi S_{g_1} \). Applying (1.0.1) and (1.0.2) gives

\[
\text{Tr}^L V_{g_2}^t = \int_{S_{g_2}^* \times S_{g_2}^*} k^L_2(f_2, \eta_2) \chi S_{g_2}^* (f_2) \chi S_{g_2}^* (\eta_2) d\mu_{L_2} (f_2, \eta_2)
\]

\[
= \frac{\#H_2}{\#H_1} \int_{S_{g_1}^* \times S_{g_1}^*} k^L_1(\xi_1, \eta_1) \tilde{U}^* \chi S_{g_2}^* (\xi_1) \tilde{U}^* \chi S_{g_2}^* (\eta_1) d\mu_{L_1} (\xi_1, \eta_1)
\]

\[
= \int_{\Delta(S_{g_1}^*)} k^L_1(\xi, \eta) d\mu_{L_1} (\xi, \eta) = \text{Tr}^L V_{g_1}^t.
\]

Theorem 1.1 follows as a direct corollary of Theorem 1.3.

\[\square\]

5 PROOF OF THEOREM 1.2

To establish Theorem 1.2, our goal is to show that \( \tilde{U}_A \) satisfies \( \tilde{U}_A V_{g_1}^t = V_{g_2}^t \tilde{U}_A \).

**Proof.** Let \( \Xi_i, i = 1, 2 \), respectively, \( \Xi_h \) be the Hamiltonian fields on each \( S_{g_2}^* M \) and \( S_{g_1}^* M \). Observe that \( V_{g_2}^t \) can be obtained via conjugating \( V_{g_1}^t \) with the pull-back \( \pi_i^* \). Indeed, given any point \( \tilde{\xi} \) on \( S_{g_2}^* M \), by uniqueness of the local geodesic flow, \( f(\exp(t\Xi_H(\tilde{\xi}))) = f(\exp(t\Xi_H(\tilde{\xi}))) \).

Hence,

\[
V_{g_2}^t(\pi_i^* f) (\tilde{\xi}) = (\pi_i^* f) \circ G_{g_2}^t(\tilde{\xi}) = f(\exp(t\Xi_H(\tilde{\xi}))) = \pi_i^* V_{g_1}^t f(\tilde{\xi})
\]

for any \( f \in L^2(S_{g_1}^* M) \).

Similarly, the conjugation applies with the push-forward operator \( \pi_i^* \). Indeed, let \( \zeta' \) be a point on \( S_{g_2}^* M \). For \( f \) an \( L^2 \)-function on \( S_{g_1}^* M \), we have

\[
\pi_i^* V_{g_2}^t f(\zeta') = \sum_{\eta \in \pi_i^{-1}(\zeta')} f \circ G_{g_2}^t(\eta) = \sum_{h_i \in H_i} f \circ G_{g_2}^t(\tilde{T}_i h_i, \eta),
\]

and the equality is independent of the choice of \( \eta \) along the fibre of \( \zeta' \) by transitivity. Furthermore, for any preimage \( \tilde{\zeta} \) of \( \zeta' \), the orbit of \( \tilde{\zeta} \) is the preimage of the orbit of \( \zeta' \), which gives

\[
V_{g_2}^t \pi_i^* f(\zeta') = \sum_{h_i \in H_i} f \circ G_{g_2}^t(\tilde{T}_i h_i, \eta') = \pi_i^* V_{g_2}^t f(\zeta')
\]

Combining the equalities and the fact that \( \tilde{T}_a \) commutes with the Koopman operator as the deck group transformations \( G \) acts on the cotangent bundle as isometries to attain

\[
\tilde{U}_A V_{g_1}^t = \frac{1}{\#H_1} \sum_{a \in G} A(a) \tilde{U}_a V_{g_2}^t \pi_i^* = \frac{1}{\#H_1} \sum_{a \in G} A(a) \tilde{U}_a \pi_i^* V_{g_2}^t = \frac{1}{\#H_1} \sum_{a \in G} A(a) V_{g_2}^t \tilde{U}_a \pi_i^* V_{g_2}^t = V_{g_2}^t \tilde{U}_A.
\]

\[\square\]
6.1 Metric on the cosphere bundle. Instead of defining the normal space as a quotient as in [1.0.4], we can define it using as the normal bundle with respect to the Kaluza-Klein metric \( S^* M \). The Kaluza-Klein metric is defined by equipping a connection \( \nabla \) on the cosphere bundle \( S^* M \), which then determines a splitting of the tangent bundle \( TS^* M \) into horizontal and vertical component

\[
TS^* M = T^h S^* M \oplus T^v S^* M
\]

where \( T^v S^* M \cong S^{n-1} \) is given the standard metric, and the natural projection \( d\pi : T^h S^* M \to TM \) is an isometry, and \( T^h S^* M \) is the kernel of the connection. That is, let \( x \) be a point on \( M \), for each fibre element \( \zeta \in \pi^{-1}(x) \subset S_x^* M \), the components are identified via \( T^h \zeta S^* M = \ker(d\pi : T^h \zeta S^* M \to T_x M) \) and \( T^v \zeta S^* M = \ker(\nabla_\zeta : T^h \zeta S^* M \to S_x^* M) \). The metric on \( T^h S^* M \) is the pullback metric induced by \( g \) on \( M \).

We define the normal bundle \( \text{NFix}(G_g^t) \) along \( \text{Fix}(G_g^t) \) to be the orthogonal complement of \( T\text{Fix}(G_g^t) \). Identifying the normal bundle with the quotient induces a definition of \( P^# \) as a linear map

\[
P^#_N : \text{NFix}(G_g^t) \to \text{NFix}(G_g^t).
\]

We warn that the derivative of the geodesic flow \( dG_g^t \) does not preserve the splitting via the Kaluza-Klein metric. For this reason, we do not use the Kaluza-Klein metric in this paper.

6.2 \( \delta \)-distributions. The goal of this section is to describe the Schwartz kernel of the Koopman operator and its pullback \( \Delta^* k^\bullet \) as distributions. We refer to [G77], [GuSc90] for further background. Let us first recall the general settings of the so-called \( \delta \)-distributions.

Let \( X \) be a manifold. Denote \( |T|X \) to be the density bundle of \( X \) whose fibre elements at a point \( x \in X \) are the volume densities on \( X \). Let \( L \to X \) be a line bundle of \( X \) and \( L^* \) be its dual. A generalized section of the line bundle \( L \to X \) is a continuous linear functional on the space of \( C^\infty \)-sections of \( L^* \otimes |T|X \). These are volume forms with compact support in \( X \) taking values in the co-line bundle \( L^* \). In particular, if \( L = \mathbb{R} \), generalized sections of \( L \) are generalized functions on \( X \). Namely, these are elements in \( \text{Hom}(C^\infty_c(|T|X), \mathbb{R}) \). If \( L = |T|X \), generalized sections are now generalized densities on \( X \) or elements of \( \text{Hom}(C^\infty_c(X), \mathbb{R}) \) as \( L^* \otimes L \) is canonically trivial.

Now let \( Z \) be a closed submanifold of \( X \), the conormal bundle \( N^* Z \) is given by

\[
N^* Z = \{ (\zeta, \alpha) \in Z \times T^* S_g^* M | \alpha \in T^* : \alpha(u) = 0, \forall u \in T_\zeta Z \}.
\]

Given a vector bundle \( E \to X \), a \( \delta \)-section along \( Z \) is a smooth generalized section \( u \) of the tensor bundle

\[
E \mid Z \otimes |N^* Z|
\]

such that

\[
\langle u, \psi \rangle = \int_Z u\psi |_Z
\]

for any section \( \psi \) of \( E^* \otimes |T|X \). Let \( E = |T|X \). We have the short exact sequence

\[
0 \to T_\zeta Z \to T_\zeta X \to T_\zeta X/T_\zeta Z \to 0.
\]

Since \( |T|X \otimes |N^* Z| \cong |T|Z \), the pairing in (6.2.1) implies that \( u \) is a generalized density on \( X \) with support in \( Z \) taking values in the space of densities on \( Z \).
Next we recall pull-backs of $\delta$-sections under the condition of transversality. Let $Y$ be another manifold and $F : Y \to X$ a vector bundle. Consider a smooth map $h : X \to Y$. Let $W$ be a submanifold of $Y$ defined as a zero-set of a global chart on $Y$. If $h \cap W$, it follows that $h^{-1}W$ is a submanifold of $X$ and that $dh : T_xX \to T_yY$ maps $T_xh^{-1}(W)$ onto $T_yW$, which induces the mapping

$$N^*_xh^{-1}W \to N^*_yW; \beta \mapsto dh^*\beta.$$ (6.2.3)

If we let $\sigma$ to be a $\delta$-section along $W$, that is, a generalized section of $F|_W \otimes |N^*W|$, and $r(y) : F_y \to E_x$ be a fiber map, then $r \otimes dh^*| \sigma : F_y \otimes |N^*W| \to E_x \otimes |N^*_xh^{-1}W|$ maps $\sigma$ to its pullback $h^*\sigma$, which is a $\delta$-section along $h^{-1}W$.

Let $p_1, p_2$ be the projections of $X \times Y$ to $X$ and respectively, to $Y$. Let $k = k(x,y)dy$ be a generalized density of $p_2^*[T|Y$, that is a linear functional on $C_c^\infty Y$. We obtain from $k$ a linear operator

$$Ka = (p_1)_*p_2^*(a,k) = (p_1)_*(a,p_2k) \in C^\infty X; \quad a \in C_c^\infty Y.$$

**Theorem 6.1** (Schwartz kernel theorem). Every continuous linear operator is of this form.

In particular, if $f : X \to Y$ is a smooth map, the composition operator $f^*$ is

$$f^*a(x) = a(f(x)) = \int_Y \delta(y - f(x))a(y)dy.$$ $k = \delta(y - f(x))dy$ is then referred to as a $\delta$-distribution supported on the graph of $f$.

**Remark.** This description of $k$ is consistent with the definition of $k$ as a $\delta$-section of $p_2^*[T|\text{Graph}(f) Y \otimes |N^*\text{Graph}(f)|$.

A 1-form $\beta \in T^*_y$ pullbacks under $p_2$, which satisfies the condition

$$\beta(dI_y - df_x) = 0 \quad \text{or} \quad \beta_y + (-df_x)^*\beta_y = 0$$

induced by the defining equation $y = f(x)$ for the submanifold $\text{Graph}(f) \subset X \times Y$. Thus, if $(\alpha_x, \beta_y) \in p_2^*[\alpha_x|\text{Graph}(f) Y$, it is necessary that $\alpha_x = (-df_x)^*\beta_y$. This is precisely the condition for $(\alpha, \beta)$ to be an element in $N^*\text{Graph}(f)$, which is

$$(\alpha, \beta) : \left( \begin{array}{c} V_x \\ df_x(V) \end{array} \right) = \alpha(V) + \beta df_x(V) = 0.$$ 

Thereby,

$$p_2^*[T|\text{Graph}(f) Y \otimes |N^*\text{Graph}(f)| \cong 1,$$

which implies that $k$ is the canonically trivial distribution on $X \times Y$ supported on $\text{Graph}(f)$ taking values in the line of densities on $\text{Graph}(f)$.

Now let $X = Y = S^*_gM$ and $p_2 = \rho(\eta, \xi, t) \mapsto \xi$. 

```
| $\rho^*T^*S^*_gM$ | $T^*S^*_gM$ |
|---------------------|---------------------|
| $S^*_gM \times S^*_gM \times \mathbb{R}^+$ | $\rho$ |
| $S^*_gM$ |
```
From the previous paragraphs, the Schwartz kernel of \( V^\delta \) is the \( \delta \)-distribution - that is the \( \delta \)-section of the tensor bundle 
\[
\rho^*|T||_{\Graph(G^*_g)} S^*_g M \otimes |N^*\Graph(G^*_g)|
\]
given by
\[
k^\delta = k^\delta(\xi, \eta)d\mu_L(\eta)d\mu_L(\xi) = \delta(\eta - G^\delta_g \xi)d\mu_L(\eta)d\mu_L(\xi).
\]
Under transversality, \( \Delta^* k^\delta \) is a \( \delta \)-section of the tensor bundle
\[
\Delta^* \rho^*|T||_{\Fix(G^*_g)} S^*_g M \otimes |N^*\Fix(G^*_g)|.
\]
Indeed, we first note that \( \Delta^{-1}\Graph(G^*_g) = \Fix(G^*_g) = \{ (\zeta, \tau) : \zeta \in \Fix(G^*_g), \tau \in \Lsp(S^*_g M) \} \).
Similar to the discussion leading up to (16), letting \( h = \Delta \) yields the canonical isomorphism
\[
N^*\gamma \cong \Delta^* \rho^*|T|_{\Fix(G^*_g)} S^*_g M
\]
if \( \gamma \) is a non-degenerate \( \tau \)-periodic orbit of \( G^*_g \) whereby \( \Delta|_{\gamma} \cap \Graph G^*_g \).
More generally,
\[
N^*Z_j^{(\tau)} \cong \Delta^* \rho^*|T|_{\Fix(G^*_g)} S^*_g M
\]
if \( Z_j^{(\tau)} \) is a clean \( k \)-dimensional component of \( \Fix(G^*_g) \). To see this, notice for any 1-form \((\alpha, \beta)\) in \( \rho^*|T| S^*_g M \), we have
\[
\Delta^*(\alpha, \beta) = (d\Delta)^*(\alpha, \beta) = \alpha + \beta.
\]
Hence, since \( dG^\tau_g \) fixes the tangent space along \( \gamma \),
\[
\Delta^*(-dG^\tau_g)^*\beta = (I - dG^\tau_g)^*\beta
\]
is a 1-form in \( (\Delta^* \rho^*|T|_{\Fix(G^*_g)} S^*_g M)_{\zeta} \), which vanishes on \( T^\tau_c Z_j^{(\tau)} \) under cleanness, hence is in the conormal space \( N^*_\zeta Z_j^{(\tau)} \). Now let \( \alpha_\zeta \in N^*_\zeta Z_j^{(\tau)} \), it is obvious that \( \alpha|_{T^\tau_c Z_j^{(\tau)}} = (I - dG^\tau_g)^*\alpha_\zeta \).
Suppose \( J \in \ker(\alpha_\zeta - T^\tau_c Z_j^{(\tau)}) \), since \( (I - dG^\tau_g)^# \) is invertible on \( T^\tau_c S^* M/T^\tau_c Z_j^{(\tau)} \), there exists \( S \) in the cokernel of \( I - dG^\tau_g \) so that
\[
\alpha_\zeta(J) = (I - dG^\tau_g)^*\alpha_\zeta(S).
\]
This implies \( \alpha = (I - dG^\tau_g)^*\beta \) for some \( \beta \in (\Delta^* \rho^*|T|_{\Fix(G^*_g)} \zeta) \). We thus obtain an isomorphism of \( (\Delta^* \rho^*|T|_{Z_j^{(\tau)}}) \zeta \) with \( N^*_\zeta Z_j^{(\tau)} \). Consequently,
\[
|N^*Z_j^{(\tau)}| \otimes \Delta^* \rho^*|T|_{Z_j^{(\tau)}} = 1.
\]
Moreover, since \( d\Delta_\zeta + (I_\zeta, (dG^\tau_g)^#) \) is full-rank, this transversal condition implies that \( k^\tau \) pulls back to the canonical \( \delta \)-distribution \( \Delta^* k^\tau \) on \( S^*_g M \), which is supported on \( Z_j^{(\tau)} \) taking values to densities on \( Z_j^{(\tau)} \) (cf. [27], Theorem 6]). In both cases of Lefschetz and clean flows,
\[
\Delta^* k^\delta(\zeta, \tau) = \delta(\zeta - G^\tau_g \zeta)d\mu_L(\zeta) \cdot \delta(t - \tau)dt.
\]
We now compute \( \Pi_\zeta \Delta^* k^\delta \) for the general case of clean fixed point sets. Let \( \zeta \) be a point in \( Z_j^{(\tau)} \) and \( \gamma \in \mathcal{G} \) be a \( \tau \)-periodic orbit starting at \( \zeta \). Let \( \zeta_1, \ldots, \zeta_{2n-1} \) be coordinates about \( \zeta \).
With respect to the basis \(NZ_j^{(\tau)}\) is described by \(\zeta_1 = \ldots = \zeta_k(Z_j^{(\tau)}) = 0\) and the \(\zeta_j\)'s are defined by

\[
\xi_1 - (G^*_g \xi)_1 = \zeta_1, \ldots, \xi_{k(Z_j^{(\tau)})} - (G^*_g \xi)_{k(Z_j^{(\tau)})} = \zeta_k(Z_j^{(\tau)}), \xi_{k(Z_j^{(\tau)})+1} = \zeta_{k(Z_j^{(\tau)})+1}, \ldots, \xi_{2n-1} = \zeta_{2n-1}.
\]

Recall that the pullback under \(\rho\) of a 1-form \(\beta\) in \(T^*S^*_g M\) supported on the graph of \(G^*_g\) is

\[
\langle (-dG^*_g)^* \beta, \beta \rangle \in \rho^* T^*|_{\text{Graph}(G^*_g)} S^*_g M,
\]

which is further identified with

\[
\langle (-dG^*_g)^* \beta, \beta \rangle \overset{dA^*}{\rightarrow} \langle (I - dG^*_g)^* \beta \in \Delta^* \rho^* T^*|_{Z_j^{(\tau)}} \cong N^* Z_j^{(\tau)}.
\]

Since \(|NZ_j^{(\tau)}| \otimes |T|Z_j^{(\tau)} = |T|S^*_g M\), the induced \(\delta\)-distribution on \(Z_j^{(\tau)}\) is the Leray form

\[
d\nu_{Z_j^{(\tau)}}(\zeta) := \frac{d\mu_L(\zeta)}{|(I - dG^*_g)^* d\zeta_{k(Z_j^{(\tau)})+1} \ldots d\zeta_{2n-1}|(\zeta)}.
\]

With respect to the basis \(\{\partial_{k(Z_j^{(\tau)})+1} \zeta, \ldots, \partial_{2n-1} \zeta\}\), the transversal Jacobian is the matrix

\[
J^\perp|_{\zeta(I - G^*_g)} = \begin{pmatrix}
\partial_{k(Z_j^{(\tau)})+1} |_{\zeta(I - G^*_g)} & \ldots & \partial_{k(Z_j^{(\tau)})+1} |_{\zeta(I - G^*_g)} \\
\vdots & \ddots & \vdots \\
\partial_{2n-1} |_{\zeta(I - G^*_g)} & \ldots & \partial_{2n-1} |_{\zeta(I - G^*_g)}
\end{pmatrix}.
\]

Rewriting the row vectors in (6.2.4) in terms of an orthonormal basis of the space of periodic \(Z_j^{(\tau)}\)-transversal Jacobi fields along \(\gamma\) (§3.4) to obtain

\[
d\nu_{Z_j^{(\tau)}} = \frac{d\mu_L}{|\det(I - P^\#_{Z_j^{(\tau)}})| d\zeta_{k(Z_j^{(\tau)})+1} \ldots d\zeta_{2n-1}|}.
\]

By integrating along the fibre \(\Pi^{-1}(\zeta)\), it follows that

\[
\Pi^* \Delta^* k^i = \int_{\Pi^{-1}(\text{Fix}(G^*_g))} |\Delta^* k^i \wedge dt| = \sum_{\tau} \delta(t - \tau) \sum_j \int_{Z_j^{(\tau)}} d\nu_{Z_j^{(\tau)}}.
\]

**Remark.** If \(\gamma\) is a non-degenerate orbit, we note that \(\frac{d\mu_L}{|\gamma|}\) is a volume element on the local transversal hypersurface \(\Sigma\) to \(\gamma\). Equivalently, this is the symplectic form \(\gamma \cdot d\mu_L\). As a result,

\[
\Delta^* k^i_{\gamma} = \frac{d\mu_L}{(I - dG^*_g)^* (\gamma \cdot d\mu_L)} = \frac{|\gamma|}{|\det(I - P_{\gamma})|} \in |T|\gamma.
\]

In the presence of high-dimensional fixed point sets, the dimension of the normal space to the fixed point set decreases - hence the modified version of the Poincare map in the case of Bott-Morse non-degeneracy.
REFERENCES

[ChMa] Chen, Su-shing; Manning, Anthony. The convergence of zeta functions for certain geodesic flows depends on their pressure. Math. Z 176 (1981), 379-382.

[DyZw] Dyatlov, Semyon; Zworski, Maciej. Dynamical zeta functions for Anosov flows via microlocal analysis. Preprint, arXiv:1306.4203.

[GuDu] Guillemin, Victor; Duistermaat, Johannes. The Spectrum of positive elliptic operators and periodic bicharacteristics. Inventiones math. 29 (1975), 39-79.

[GuSc90] Guillemin, Victor; Sternberg, Schlomo. Geometric asymptotics. Mathematical surveys and monographs, no. 14. Amer. Math. Soc., Providence, RI, 1990.

[GuSc13] Guillemin, Victor; Sternberg, Shlomo. Semi-classical analysis. International Press, Boston, MA, 2013.

[G77] Guillemin, Victor. Lectures on spectral theory of elliptic operators. Duke Math. J. 44 (1977), no. 3, 485-517.

[Hor] Hörmander, Lars. The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Classics in Mathematics. Springer-Verlag, Berlin, 2003.

[H22] Lam, Hy. Flat trace of the Koopman operator of the geodesic flow. In preparation.

[Kl] Klingenberg, Wilhelm. Lectures on closed geodesics. Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[Ma78] Mackey, George. Unitary group representations. Benjamin/Cummings; Reading, MA 1978.

[Su85] Sunada, Toshikazu. Riemannian coverings and isospectral manifolds. Ann. of Math. (2) 121 (1985), no. 1, 169-186.

[Su94] Sunada, Toshikazu. On the number-theoretic method in geometry: geometric analogue of zeta and L-functions and its applications. Amer. Math. Soc. Transl. Ser. 2, 160, Amer. Math. Soc., Providence, RI, 1994.

[Z92] Zelditch, Steven. Isopectrality in the FIO category. J. Differential Geom. 35 (1992), no. 3, 689-710.

Northwestern University
Department of Mathematics
2033 Sheridan Road, Evanston, Illinois 60208 USA

E-mail address: hylam2023@u.northwestern.edu