We show how a large class of geometrical critical systems including dilute polymers, polymers at the theta point, percolation and to some extent brownian motion, are described by a twisted N=2 supersymmetric theory with $k=1$ (it is broken in the dense polymer phase that is described simply by a $\eta, \xi$ system). This allows us to give for the first time a consistent conformal field theory description of these problems. The fields that were described so far by formally allowing half integer labels in the Kac table are built and their four point functions studied. Geometrical operators are organized in a few representations of the twisted $N=2$ algebra. A noticeable feature is that in addition to Neveu Schwartz and Ramond, a sector with quarter twists has sometimes to be introduced. The algebra of geometrical operators is determined. Fermions boundary conditions are geometrically interpreted, and the partition functions that were so far defined formally as generating functions for the critical exponents are naturally understood, sector by sector.

Twisted $N=2$ provides moreover a very unified description of all these geometrical models, explaining for instance why the exponents of polymers and percolation coincide. It must be stressed that the physical states in these geometrical problems are not the physical states for string theory, which are usually extracted by the BRS cohomology. In polymers for instance, $Q_{BRS}$ is precisely the operator that creates polymers out of the vacuum, such that the topological sector is the sector without any polymers.

It seems that twisted $N=2$ is the correct continuum limit (in two dimensions) for models with Parisi Sourlas supersymmetry. Some possible explanation of this fact is advanced.

As an example of application of $N=2$ supersymmetry we discuss in the first appendix the still unsolved problem of backbone of percolation. We conjecture in particular the value $D=25/16$ for the fractal dimension of the backbone, in good agreement with numerical computations.

Finally some of these ideas are extended to the off critical case in the second appendix where it is shown how to give a meaning to the $n \to 0$ limit of the $O(n)$ model $S$ matrix recently introduced by Zamolodchikov by introducing fermions.
1 Introduction

Since the beginnings of conformal invariance in two dimensions, geometrical systems have been the subject of active studies. Besides their interest for computer simulations and experiments, they provide a very attractive playground for the theory, and have led to predictions of infinite hierarchies of critical exponents, universal ratios, winding angle distributions... Despite the remarkable success of these predictions, the corresponding conformal theories were not so far fully understood.

A good example is the dilute polymers where half of the critical exponents, including the exponent that counts configurations of an open chain, were obtained by using formally Kac formula with half integer indices for the theory with central charge $c = 0$. If one introduces the $L$ legs operators $\Phi_L$ (figure 1) then several distinct arguments suggest they are Virasoro primaries with dimension

$$h_L = \frac{9L^2 - 4}{96}$$

These coincide formally with the Kac dimensions $h_{L/2+2,3}$, but the labels are integer for $L$ even only. For $L$ odd, the labels are half integer, and four point functions cannot be built in the Dotsenko Fateev Coulomb gas, neither operator products calculated. Another difficulty is that dilute polymers, corresponding to the $n \to 0$ limit of the $O(n)$ model, have a (naive) partition function equal to one. Some generating functions of critical dimensions were introduced on formal grounds in [3], but their meaning remained so far obscure.

Similar difficulties are met in the study of polymers at the theta point, the percolation problem, or brownian motion.

Still another kind of problem occurs with dense polymers. Although it is known they correspond to the $n \to 0$ limit of the low temperature phase of the $O(n)$ model, the nature of the broken symmetry and the "$O(-1)$" Goldstone modes remain mysterious [4].

It is the purpose of this paper to solve the above difficulties. Our result is that twisted $N = 2$ supersymmetry [6] describes correctly all the above models but the dense polymers. For the latter this symmetry is precisely broken, and they are described by a $\eta, \xi$ system. We show how to determine the four point functions and operator algebra including the fields with half integer labels. We define various partition functions classified by the fermions boundary conditions. The trivial $Z = 1$ corresponds to doubly periodic sector, while the objects introduced in [3, 7] are obtained by summing over Ramond and Neveu Schwartz, and in some cases over a $Z_4$ sector also. The existence of this sector is a new practical application of the spectral flow [8].

Although describing geometrical systems by twisted $N = 2$ is (so far) merely a guess justified by results, the idea has its roots in the work of Parisi and Sourlas [9]. These authors showed how the $n \to 0$ limit of the $O(n)$ model could be replaced by considering a field theory with $2N$ real bosons $\phi^i, i = 1, \ldots, 2N$ and $2N$ real fermions $\psi^i, \bar{\psi}, i = 1, \ldots, N$ such that the Lagrangian is invariant under the supersymmetry transformations

$$\phi^j \to a_{ij} \phi^j + c_{ij} \psi^j, \psi^j \to d_{ij} \phi^j + b_{ij} \psi^j$$

that leave $\phi^2 + \bar{\psi}\psi$ invariant. These actually correspond to a global $Osp(2N, 2N)$ symmetry. Let us now restrict to two dimensions. In the case of an ordinary compact group $G$, one would
expect the continuum limit of a theory whose lagrangian is invariant under $G$ to be some Wess Zumino model on the group $G$ with local, left right, symmetry \cite{10}. This cannot be expected for supergroups. One reason is for instance that the polymers can be described by a $Osp(2N,2N)$ symmetric Lagrangian for any $N$, while the WZW models built on these groups have some features, like the spectrum of dimensions, that depend on $N$ (some other features like the central charge are independent of $N$, $c = 0$ here). A reason for the failure of the usual argument \cite{11} may be the non compactness. Also it seems that the introduction of fermions and bosons to simulate a $n = 0$ limit leads to somewhat artificial conserved currents that may not pertain to the correct continuum limit theory. In our case, comparison of the $Osp(2N,2N)$ WZW model spectra with known polymers dimensions shows that none of these is correct. Although there is no corresponding lattice model or field theory à la Parisi Sourlas, there are arguments (see section 4) which suggest that only one boson and one fermion can propagate along polymer loops. These degrees of freedom correspond to the fundamental representation of $Gl(1,1)$ or $Sl(1,1)$. Also these two algebras are the smallest non trivial subalgebras of the $Osp(2N,2N)$ sequence. However one can check that polymers are not described by a $Gl(1,1)$ WZW model (there is no $Sl(1,1)$ WZW model as discussed in \cite{11}).

We have in fact to look for a less naive argument, although the $Gl(1,1)$ WZW model is ”almost” the correct answer. Instead of considering $Gl(1,1)$ we could start from $Osp(2,2)$ and perform some reduction to keep only the expected physical currents. Recall the defining relations of $osp(2,2)$ algebra

$$[S_+, S_-] = 2H$$
$$[H, S_{\pm}] = \pm S_\pm$$
$$\left[H \pm \frac{I}{2}, V_\pm\right] = 0, \left[H + \frac{I}{2}, V_\pm\right] = \pm V_\pm$$
$$\left[H \mp \frac{I}{2}, \bar{V}_\pm\right] = 0, \left[H + \frac{I}{2}, \bar{V}_\pm\right] = \pm \bar{V}_\pm$$
$$\{V, \bar{V}\} = \{\bar{V}, V\} = 0$$
$$\{V_+, \bar{V}_-\} = -\frac{1}{2}(H + \frac{I}{2})$$
$$\{\bar{V}_+, V_-\} = -\frac{1}{2}(H - \frac{I}{2})$$
$$\{V_\pm, \bar{V}_\pm\} = \pm \frac{1}{2} J_\pm$$

(3)

For the WZW model on $Osp(2,2)$ all these operators give rise to currents, whose zero modes satisfy these commutation relations. The stress tensor is obtained in the usual Sugawara form, and the central charge vanishes. To keep only the subalgebra $gl(1,1)$ made out of say $V_+, \bar{V}_-, H \pm I/2$\footnote{In the following $gl(1,1)$ is defined by a slight change of notation $\{V_+, V_-\} = E, [F, V_+] = F, [F, V_-] = -V_-$} we must in particular twist the stress energy tensor such that $c$ remains equal to zero and these currents keep weight one. The only possible choice is $T \to T + \partial (T + H/2)$. Such a modification can be made in two steps, first twisting $T \to T + \partial T$, reducing to $N = 2$ \cite{12}, and twisting again
$T \rightarrow T + \partial(H/2)$ to get twisted $N = 2$. For some unknown reason this procedure is the right one (maybe it is the only way of getting in the end a theory with spectrum unbounded from below).

The paper is organized as follows. We first discuss dense polymers. It turns out that a lattice formulation, by means of generalizations of the Kirchoff theorem, allows to find out the geometrical meaning of fermionic degrees of freedom and of their boundary conditions (section two). The continuum limit of dense polymers is discussed in section three. They are described by an $\eta, \xi$ system, whose pathologies as a conformal field theory are compatible with the nature of the dense phase. For instance that $\xi$ has vanishing conformal weight is associated with the existence of the density operator with non vanishing expectation value. Partition functions are analyzed. The $L$ legs operators are classified in three sectors, Ramond, Neveu Schwartz, and a $Z_4$ sector for $L$ odd. Their operator algebra is determined, as well as some four point functions.

The structure of sectors and the geometrical interpretation of fermions boundary conditions holds also for dilute polymers, where the $\eta, \xi$ system has to be replaced by a twisted $N = 2$ theory with $k = 1$, whose degrees of freedom are $\eta, \xi$ and two bosonic fields $\phi, \tilde{\phi}$. All results for dense polymers are extended to the dilute case in section four.

Section five contains a discussion of the percolation problem, which can be mapped onto polymers at the theta point, the low temperature Ising model, or to some extent brownian motion. It turns out to be also described by twisted $N = 2$ with $k = 1$, which explains the coincidence of percolation and dilute polymers exponents. A discussion of the same nature as before is provided.

The conclusion contains some comments on the topological nature of some sectors of our theory, and the geometrical meaning of the BRS operator in twisted $N = 2$.

As an application of the $N = 2$ formalism, we discuss in the first appendix the problem of percolation backbone, and predict in particular the exact valued of the fractal dimension of the backbone to be $D = 25/16$. In the second appendix we show how the introduction of fermions allows also to give a meaning to the $n \rightarrow 0$ limit for the off critical $O(n)$ model.

2 Dense Polymers: Lattice Results

2.1 Dense polymers and Jordan Curves

Recall that dense polymers are obtained by considering a finite number of self avoiding and mutually avoiding loops or open chains that cover a finite fraction of the available volume. They are known to possess some partially solvable realizations. Consider for instance a square lattice $\mathcal{L}$ and its medial $\mathcal{M}$, that is the other square lattice obtained by putting a vertex on every edge of $\mathcal{L}$, and joigning the vertices of $\mathcal{M}$ by an edge if they belong to perpendicular edges of $\mathcal{L}$ with a common end point. Consider now a polygon decomposition of $\mathcal{M}$ obtained by splitting each vertex of $\mathcal{M}$ in one of the two possible ways shown in figure 2 and require that when connecting all the arcs only one connected polygon is obtained (this is usually called a Jordan Curve). This polygon is self avoiding and covers densely the medial lattice $\mathcal{M}$. Various studies have confirmed that such a polygon has critical properties similar to those of a dense polymer. Similarly if a finite number of polygons is formed by connecting the arcs, the critical properties are those of several
dense polymers.  

2.2 Jordan Curves, Spanning Trees, and the Discrete Laplacian

Suppose we color an edge of $\mathcal{L}$ every time the splitting of the corresponding vertex of $\mathcal{M}$ does not intersect it, as illustrated in figure 3. Then it is easy to see that for every Jordan Curve, the colored edges draw a Spanning Tree, that is a subgraph of $\mathcal{L}$ without loops that contain all its vertices. The number of spanning trees of any graph $\mathcal{G}$ is given by Kirchoff’s theorem:

**Theorem** (Kirchoff):

For any graph $\mathcal{G}$ with $V$ vertices denoted $i, j, \ldots$ define the discrete Laplacian $-\Delta$ to be the $V^{\otimes 2}$ matrix with entries $-\Delta_{ii}=$ number of edges incident to $i$, $-\Delta_{ij}=$ - number of edges with end points $i$ and $j$. Then every principal minor of $-\Delta$ equals the number of spanning trees of $\mathcal{G}$.

This result leads for instance to the computation of the exponent gamma for dense polymers. We are not going to use it as it stands. The reason is that the determinant of $\Delta$ vanishes, and to get a non trivial information one has to suppress the zero mode; this is precisely what we want to avoid here. We shall instead obtain results without taking any limit but using fermions.

2.3 XX Chain

Besides the discrete Laplacian, which is the Lagrangian aspect of the problem, it is also convenient to keep in mind the Hamiltonian point of view. This can easily be done by realizing the above polygon decomposition algebraically in the Temperley Lieb algebra with relations

\[ e_i^2 = \delta e_i \]
\[ e_i e_{i\pm 1} e_i = e_i \]
\[ [e_i, e_j] = 0, \ |i - j| > 1 \]

that can be realized graphically, ie by acting on polymers, as in figure 4. In our case we have to take $\delta = 0$.

Corresponding to the above conventions the transfer matrix takes the form (for a general system with anisotropy parameter $x$)

\[ \tau = \prod_{i=1}^L (1 + x e_{2i}) \prod_{i=1}^{L-1} (x + e_{2i+1}) \]

\[ \text{If a variable number of polygons is allowed, and controlled by a fugacity $\sqrt{Q}$, the critical properties are those of the clusters boundaries in the $Q$ state Potts model.} \]

\[ \text{In the Q state Potts model (resp. the O(n) model in the dense phase) approach one takes a Q(resp.n) derivative at Q(resp.n) = 0.} \]
Instead of the loop realization we can use the vertex model realization. $\tau$ describes now a 6 vertex model on the square lattice with a copy of $C^2$ per edge, diagonal propagation, and

$$e_i = 1 \otimes 1 \ldots \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 \\ 0 & -1 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \ldots$$ (8)

or

$$e_i = -\frac{1}{2} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + i(\sigma_i^z - \sigma_{i+1}^z) \right]$$ (9)

where $\sigma$'s are Pauli matrices. This is the XX Hamiltonian with some boundary term. The role of the later is to provide a zero mode. Introducing

$$V_{\pm} = \sum_{j=1}^{L} \prod_{k<j} \sigma_j^\pm$$ (10)

one has

$$[e_i, V_{\pm}] = 0, \text{ for any } i$$ (11)

while

$$(V_{\pm})^2 = 0, \{V_+, V_-\} = 0$$ (12)

The matrices $e_i$ commute also with the fermion number operator

$$F = \sum_{j=1}^{L} \frac{\sigma_j^z + 1}{2}$$ (13)

such that

$$[F, V_+] = V_+, [F, V_-] = -V_-$$ (14)

This symmetry is $gl(1,1)/u(1)_E$. The casimir operator is simply $V_+ V_-$. Due to the presence of the zero mode, all levels of $\tau$ are twice degenerate, and occur once with odd fermion number, once with even fermion number. Therefore

$$\text{Tr} \left[ (-)^F \tau^T \right] = 0, \text{ for any } T$$ (15)

This trace corresponds to computing the determinant of the discrete Laplacian on a cylinder of size $T \times L$, that is with periodic boundary conditions in the time direction. In this case it is easy to see that the contribution of non contractible loops vanishes because they are counted once with a factor of +1, and once with a factor of −1. Therefore instead of suppressing the zero mode ”by hand” or by taking some appropriate limit, one can get non trivial partition functions, and hence information about the dense polymers, by changing the boundary conditions. Let us discuss this in more details.
2.4  Frustrating Some Edges

We first establish the result

Lemma 2:

Consider a graph $G$ and some marked edge $[i,j]$. Consider the modified discrete Laplacian $-\Delta[i,j]$ obtained by changing from $-1$ to $+1$ the $ij$ and $ji$ matrix elements of $-\Delta$. The edge $[i,j]$ is said to be frustrated. One has then:

$$\det (-\Delta[i,j]) = 4 \times \text{number of subgraphs of } G \text{ containing all its vertices and made of one loop passing through } [i,j] \text{ with dangling arms (trees) attached to it (see figure 5).}$$

Proof:

The proof of this statement follows closely the proof of Kirchoff’s theorem. Suppose the graph $G$ has $V$ vertices and $E$ edges ($E \geq V - 1$). Give an arbitrary orientation to every edge and introduce the matrix $X_{\alpha i}$ of size $E \times V$ where $\alpha$ runs over the oriented edges and $i$ over the vertices, with $X_{\alpha i} = \pm 1$ if the edge $\alpha$ has origin (extremity) in $i$ and $X_{\alpha i} = 0$ if the vertex $i$ is not one extremity of the edge $\alpha$. Similarly introduce the matrix $Y$ deduced from $X$ by setting $Y_{[i,j],i} = Y_{[i,j],j} = 1$.

It is then easy to check that $-\Delta = X^t X$, $-\Delta[i,j] = Y^t Y$.

The modification of the discrete Laplacian suppresses the zero mode and $\det (-\Delta[i,j]) \neq 0$. We can now write

$$\det (Y^t Y) = \sum_{\alpha_1, \ldots, \alpha_V} \det (Y_{\alpha_1 \ldots \alpha_V}^t Y_{\alpha_1 \ldots \alpha_V})$$

where $Y_{\alpha_1 \ldots \alpha_V}$ is the square matrix obtained from $Y$ by keeping only the lines $\alpha_1 \ldots \alpha_V$. The corresponding edges draw a subgraph of $G$, $G_{\alpha_1 \ldots \alpha_V}$. Of course $\det (Y_{\alpha_1 \ldots \alpha_V})$ is non zero if and only if every vertex of $G$ is in $G_{\alpha_1 \ldots \alpha_V}$. It is also easy to see that $\det (Y_{\alpha_1 \ldots \alpha_V})$ vanishes unless the marked edge $[i,j]$ is one of the labels $\alpha_1 \ldots \alpha_V$, since it is the modification of the $\alpha = [i,j]$ line that makes $Y$ of rank $V$, while $X$ is of rank $V - 1$. Using Euler’s relation one finds that the subgraphs $G_{\alpha_1 \ldots \alpha_V}$ contributing to the summation (17) have as many loops as components. Expand now $\det (Y_{\alpha_1 \ldots \alpha_V})$ along the $[i,j]$ line. This line has two non trivial entries that are equal to 1, while for the matrix $X$ it contains a one and a minus one. In the latter case $\det (X_{\alpha_1 \ldots \alpha_V})$ vanishes because $X$ is of rank $V - 1$. Therefore the two corresponding minors have to be equal. Suppose first we are dealing with a one component subgraph, and hence one loop. In the case where the loop does not pass through the marked edge, the subgraphs associated with these minors contain one loop. Since these minors are the same as the ones of $X$, they must vanish. In the case where the loop does pass through the marked edge, the subgraphs associated with these minors are trees, and they take value $\pm 1$. The corresponding $\det (Y_{\alpha_1 \ldots \alpha_V})$ equals then $\pm 2$. If we were dealing with several components subgraphs, and hence several loops, the minors obviously would vanish in any case. Using (17) the result is therefore established.

Example:

\footnote{which we suppose without loss of generality to have no multiple edges}
Let us consider the graph in figure 6a with some choice of edges orientation. We have
\[
L = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}; \quad L' = \begin{pmatrix}
1 & 0 & 0 & -1 & 0 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]
and therefore
\[
-\Delta[3|4] = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 2 & 1 \\
-1 & -1 & 1 & 3
\end{pmatrix}
\]
such that
\[
\det (-\Delta[3|4]) = 12
\]
corresponding to the three subgraphs shown in figure 6b.

We shall use also the generalization of Lemma 2 to the case of several marked edges:

**Lemma 3:**
Consider a graph \( G \) and a set of marked edges \([i_1|j_1], \ldots, [i_k|j_k] \). Then one has:
\[
\det (-\Delta[i_1,j_1], \ldots, [i_k,j_k]) = \text{sum over subgraphs of } G \text{ containing all its vertices, that are made of several loops passing through an odd number of marked edges, and attached dangling arms (that may also pass through some marked edges) of number of components}
\]

**Proof:**
Let us start by a graph with two marked edges, say \([i|j] \) and \([k|l] \). Let us introduce the matrix \( Y \) and expand \( \det (-\Delta[i,j]|k,l] \) as above (17). The only subgraphs that may contribute are those where each vertex is extremity of at least one of the edges \( \alpha_1, \ldots, \alpha_V \), and where at least one of the marked edges appears. Also by Euler’s relation there are as many loops as components in each subgraph. If only one marked edge appears, we get the same submatrix \( Y_{\alpha_1,\ldots,\alpha_V} \) as above, hence the same result about allowed subgraphs. Suppose now the two marked edges appear. If they belong to two separate components, we can repeat the argument for each separately. If they appear in the same component and none of them belongs to the loop, the contribution vanishes. If one of them only belongs to the loop the determinant is still equal to \( \pm 2 \). If both of them belong to the loop, let us expand \( \det (Y_{\alpha_1,\ldots,\alpha_V}) \) with respect to say the \([i,j] \) line. This line as above contains two ones. If it contained a one and a minus one we would be dealing with the conditions of the preceding theorem, and the determinant would be equal to \( \pm 2 \). Now opening \([i,j] \) edge leaves a tree, that is the two minors of these two non vanishing entries have to be themselves equal to \( \pm 1 \). Therefore they are in fact opposite, and \( \det (Y_{\alpha_1,\ldots,\alpha_V}) \) vanishes.
Suppose now there is a larger number of marked edges, and we consider some subgraph of \( G \) with one loop that passes through \( n \) of them. Suppose we have proven that for \( m \) odd the resulting determinant is \( \pm 2 \), and for \( m \) even it vanishes, \( m < n \). Then expand with respect to say the \([i,j] \) line. This line as above contains two ones. If it contained a one and a minus one we would be dealing with a loop with \( n - 1 \) marked edges. Hence by the induction assumption if \( n - 1 \) is even the determinant would vanish, if \( n - 1 \) is odd the determinant would equal \( \pm 2 \). Opening the \([i,j] \)
edge leaves a tree, therefore the two minors are equal to \pm 1. Hence if \( n - 1 \) is even they take equal values, if \( n - 1 \) is odd they take opposite values. Therefore \( \det (Y_{\alpha_1...\alpha_V}) \) equal \( \pm 2 \) if \( n \) is odd and vanishes if \( n \) is even. This establishes the result.

**Example:**
Consider the graph of the preceding example where we mark the third and fifth edge. Therefore

\[
Y = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

One has

\[
-\Delta[2|4][3|4] = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 3 & -1 & 1 \\
0 & -1 & 2 & 1 \\
-1 & 1 & 1 & 3
\end{pmatrix}
\]

with determinant equal to twelve. We give in the following some of the determinants of the \( Y_{\alpha_1...\alpha_V} \) corresponding to the subgraphs of figure 7

\[
a) \det \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} = 0, \quad b) \det \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 1
\end{pmatrix} = 2
\]

\[
c) \det \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} = -2, \quad d) \det \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} = -2
\]

### 2.5 Non Contractible Polymers and the Laplacian with Antiperiodic Boundary Conditions

As an application of this last lemma, consider a rectangle of the square lattice with doubly periodic boundary conditions, that is wrapped on a torus. Suppose we frustrate all the horizontal edges along a given column. We refer to this column as a frustration line. The corresponding modified Laplacian is now what may be called the Laplacian with antiperiodic boundary conditions in the time direction, and still periodic boundary conditions in the space direction. We denote it by \( -\Delta_{AP} \). Due to the position of frustrated edges, the only loops that can occur in the expansion of the corresponding determinant according to the rules of Lemma 3 are non contractible, and moreover must intersect the frustration line an odd number of times. They contribute then by a factor of 4 to the summation. Now consider the medial lattice; to each such non contractible loop correspond two polymers which are as well non contractible and intersect the frustration line an odd number of times. This result immediately generalizes to the case where the frustration line is along the time axis. We can therefore write
Lemma 4:
\[ \det (-\Delta_{AP}(\text{resp.} PA)) = \text{Sum over dense coverings of the medial by an even number of (non contractible) polymers that cross } \omega_2(\text{resp.} \omega_1) \text{ an odd number of times of } 2^{\text{number of polymers}} \]

while if we put a pair of frustration lines along space and time axis, it is the total number of intersections with these axis that has to be odd:

Lemma 5:
\[ (\det - \Delta_{AA}) = \text{Sum over dense coverings of the medial by an even number of (non contractible) polymers that cross } (\omega_1, \omega_2) \text{ an odd number of times of } 2^{\text{number of polymers}} \]

Let us recall also that
\[ \det (-\Delta_{PP}) = 0 \] (18)
\[ \det' (-\Delta_{PP}) = \{ \text{number of configurations of a single contractible polymer densely covering the medial} \} \] (19)

As we remarked above, with periodic boundary conditions the contribution of contractible loops disappears because they are counted once with a weight 1, once with −1. This corresponds to the fact that they can be described either by a boson or by a fermion, with opposite contributions due to statistics. If we now impose \textbf{antiperiodic} boundary conditions for the fermion, the weight of a non contractible polymer becomes
\[ 1 + (-1)^{\text{number of intersections with the frustration line} + 1} \] (20)
thereby recovering the result of the above two lemmas.

3 The Continuum Limit of Dense Polymers

3.1 Continuum Limit Partition Functions of Dense Polymers

3.1.1 \( Z_2 \) Sector

Let us define the lattice partition functions of dense polymers by the expressions of lemmas 4 and 5 that is
\[ Z_{AP} = \det (-\Delta_{AP}) \]
\[ Z_{PA} = \det (-\Delta_{PA}) \]
\[ Z_{AA} = \det (-\Delta_{AA}) \] (21)

The continuum limit of these partition functions is obtained using standard free field computations. In general for the continuum Laplacian \( D \) with twisted boundary conditions characterized by the phases \( \exp(2i\pi k/N) \) along \( \omega_1 \), \( \exp(2i\pi l/N) \) along \( \omega_2 \) one has (see e.g. [17])
\[ \det (-D_{1/N,k/N}) = |d_{1/N,k/N}|^2 \] (22)
where
\[ d_{l/N,k/N} = q^{[1-6k/N(1-k/N)]/12} \prod_{n=0}^{\infty} \left( 1 - e^{2i\pi l/N} q^{n+k/N} \right) \prod_{n=0}^{\infty} \left( 1 - e^{-2i\pi l/N} q^{n+1-k/N} \right) \] (23)

Therefore
\[ Z_{AP} = \left| d_{1/2,0} \right|^2 = 4 \left| q^{1/12} \prod_{n=1}^{\infty} (1+q^n) \right|^2 \]
\[ Z_{PA} = \left| d_{0,1/2} \right|^2 = \left| q^{-1/24} \prod_{n=0}^{\infty} \left( 1-q^{n+1/2} \right) \right|^2 \]
\[ Z_{AA} = \left| d_{1/2,1/2} \right|^2 = \left| q^{-1/24} \prod_{n=0}^{\infty} \left( 1+q^{n+1/2} \right) \right|^2 \] (24)

Define now the manifestly modular invariant quantity
\[ Z^e = \{ \text{Sum over dense coverings of the medial by an even number of non contractible polymers with weight 2 number of polymers} \} \] (25)

One checks easily that
\[ Z^e = \frac{1}{2} (Z_{AP} + Z_{PA} + Z_{AA}) \] (26)

and therefore in the continuum limit
\[ Z^e = \frac{1}{2} \det (-D_{1/2,0}) + \det (-D_{0,1/2}) + \det (-D_{1/2,1/2}) \] (27)
or
\[ Z^e = \frac{1}{2} \left[ |d_{0,1/2}|^2 + |d_{1/2,0}|^2 + |d_{1/2,1/2}|^2 \right] \] (28)

That the partition function with doubly periodic boundary conditions vanishes is characteristic of the dense phase. Although a naive examination of the loop expansion of say the $O(n)$ model gives a result one for this partition function (because all polymers disappear), as soon as some polymer is allowed by changing the boundary conditions, the partition function of the system grows exponentially with the volume due to the finite density. We thus have
\[ Z_{AP} \propto Z_{PA} \propto Z_{AA} \propto e^{\gamma x \text{area}}, \quad Z_{PP} \propto 1 \] (29)

When we write continuum limit partition functions we always discard such factors because they are non universal and generally do not depend on boundary conditions (this is not the case is this broken symmetry phase, and for instance $n \to 0$ limit does not commute with thermodynamic limit or changing the boundary conditions). For consistency we therefore have to set $Z_{PP} = 0$. 

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3.1.2 Comparison with Lattice Coulomb Gas Computations

Now the mapping of dense polymers, that is the low temperature phase of the $O(n)$ model for $n \to 0$, on a Coulomb Gas \cite{ref1, ref2}, allows another evaluation of the above partition functions. We will not recall here how this works but merely state some results that can easily be extracted from \cite{ref1, ref2}. Introduce the coupling constant $g$ related to $n$ by

$$n = -2 \cos \pi g, \quad g \in [0, 1] \text{(dense)}, \quad g \in [1, 2] \text{(dilute)}$$ \hspace{1cm} (30)

hence

$$g = \frac{1}{2}$$ \hspace{1cm} (31)

in the present case. Introduce also

$$Z_{mm'}(g) = \sqrt{\frac{g}{\Im \tau \eta \eta}} \exp \left( \frac{-\pi g}{\Im \tau} |m - m'|^2 \right)$$ \hspace{1cm} (32)

where $\eta(q)$ is the Dedekind function (not to be confused with the fermions introduced in the next section)

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2 \pi i \tau}$$ \hspace{1cm} (33)

Then one finds, using the above definitions in terms of non contractible loops and translating them in the Coulomb Gas language\cite{ref3, ref4}

\begin{align*}
Z_{PP} &= \frac{1}{2} \sum_{m, m' \in \mathbb{Z}} (-)^{m \wedge m'} Z_{mm'} \\
Z_{AP} &= \frac{1}{2} \sum_{m, m' \in \mathbb{Z}} (-)^m (-)^{m \wedge m'} Z_{mm'} \\
Z_{PA} &= \frac{1}{2} \sum_{m, m' \in \mathbb{Z}} (-)^{m'} (-)^{m \wedge m'} Z_{mm'} \\
Z_{AA} &= \frac{1}{2} \sum_{m, m' \in \mathbb{Z}} (-)^{m + m'} (-)^{m \wedge m'} Z_{mm'}
\end{align*} \hspace{1cm} (34)

where $m \wedge m'$ means the greatest common divisor of $m$ and $m'$. These expressions become after Poisson resummation

$$Z_{PP} = \frac{1}{2 \eta \eta} \left( \sum_{m \in 2\mathbb{Z}, e \in \mathbb{Z}/2} q^{h_{em} \bar{q}^{\bar{e}m}} - \sum_{m \in \mathbb{Z}, e \in \mathbb{Z}} q^{h_{em} \bar{q}^{\bar{e}m}} \right)$$

\footnote{In the following we refer to the mapping of the $O(n)$ model or $Q$ state Potts model onto a free field as lattice Coulomb gas. This is related, but not strictly equivalent to the usual Dotsenko Fateev representation holding for continuum theories.}

\footnote{The normalization of these continuum limit expressions is not fixed by the Coulomb gas mapping, and chosen for future convenience.}
\[ Z_{AP} = \frac{1}{\eta \bar{\eta}} \sum_{m \in 2Z, e \in Z+1/2} q^{h_{em}} \bar{q}^{\bar{h}_{em}} \]

\[ Z_{PA} = \frac{1}{2 \eta \bar{\eta}} \left( \sum_{m \in 2Z, e \in Z} - \sum_{m \in 2Z+1, e \in Z+1/2} \right) q^{h_{em}} \bar{q}^{\bar{h}_{em}} \]

\[ Z_{AA} = \frac{1}{2 \eta \bar{\eta}} \left( \sum_{m \in 2Z, e \in Z} + \sum_{m \in 2Z+1, e \in Z+1/2} \right) q^{h_{em}} \bar{q}^{\bar{h}_{em}} \]  

(35)

where

\[ h_{em} = \frac{1}{4g} (e + mg)^2, \quad \bar{h}_{em} = \frac{1}{4g} (e - mg)^2 \]  

(36)

Equality of the expressions (35) and (24) is readily checked using Jacobi identity. One finds also

\[ Z^{c} = \frac{1}{2 \eta \bar{\eta}} \sum_{m \in 2Z, e \in Z/2} q^{h_{em}} \bar{q}^{\bar{h}_{em}} = \frac{1}{2} Z_{c} [g = 1/2, f = 2] \]  

(37)

(A similar identity occurs when bosonizing a free complex fermion) where we defined the Coulomb partition functions

\[ Z_{c} [g, f] = \frac{1}{\eta \bar{\eta}} \sum_{m \in fZ, e \in Z/f} q^{h_{em}} \bar{q}^{\bar{h}_{em}} \]  

(38)

Hence Coulomb Gas computations are in complete agreement with the lattice analysis.

It must be stressed that the dense phase of polymers has critical properties independent of the density \( \rho > 0 \). Therefore the above expressions are expected to hold in general, while the lattice analysis based on the Laplacian occurs in the maximum density limit.

Now, from the general point of view of dense polymers it is natural to define still another partition function

\[ Z^{o} = \{ \text{Sum over dense coverings of the lattice by an odd number} \]

\[ \text{of non contractible polymers of an odd number of polymers} \} \]  

(39)

This object does not appear in consideration of Jordan Curves and the discrete Laplacian. Its continuum limit has been computed in the Coulomb Gas mapping

\[ Z^{o} = Z_{c} [1/2, 1/2] - \frac{1}{2} Z_{c} [1/2, 2] \]  

(40)

It is useful to recall that

\[ Z_{c} [1/2, 1] = Z_{c} [1/2, 2] \]  

(41)

Finally the total dense polymers partition function turns out to be

\[ Z^{c+o} = Z^{c} + Z^{o} \Rightarrow Z^{c+o} = Z_{c} [1/2, 1/2] \]  

(42)

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where the arrow indicates the continuum limit. Notice that the ground state with $h = 0$ is twice degenerate. This is expected in the $\eta, \xi$ system where both the identity and the $\xi$ field have vanishing conformal weight. This is also natural from the polymer point of view since, besides the identity, the density operator $\rho$ has a non-vanishing expectation value\(^{10}\) and therefore must have $h = 0$.

### 3.1.3 $Z_4$ Sector

One of the major obstacles in building the consistent theory of dense polymers has been the presence of the sector with an odd number of polymers that has no obvious lattice interpretation in terms of the discrete Laplacian. However in the continuum it seems a reasonable guess to expect that the description of the odd sector involves $Z_4$ twists\(^{11}\). It is an elementary exercise in applying the Jacobi identity to check the following nice formula

$$Z_o = \frac{1}{2} \left[ \det (-D_{0,1/4}) + \det (-D_{1/4,0}) + \det (-D_{1/4,1/4}) + \det (-D_{1/2,1/4}) + \det (-D_{1/4,1/2}) + \det (-D_{3/4,1/4}) \right]$$

or

$$Z_o = \frac{1}{2} \left[ |d_{0,1/4}|^2 + |d_{1/4,0}|^2 + |d_{1/4,1/4}|^2 + |d_{1/2,1/4}|^2 + |d_{1/4,1/2}|^2 + |d_{3/4,1/4}|^2 \right]$$

Having identified the correct object related to the Laplacian that reproduces $Z_o$ we shall assume that this is not a numerical coincidence but the correct answer in the continuum limit. We can now proceed and study the corresponding conformal field theory.

### 3.2 The Conformal Field Theory of Dense Polymers

#### 3.2.1 Generalities

To reproduce determinants of the Laplacian with various boundary conditions we introduce an $\eta, \xi$ system with action

$$S = \frac{1}{\pi} \int d^2 z \left( \eta \partial \xi + \overline{\eta} \partial \overline{\xi} \right)$$

where $\eta$ and $\xi$ denote conjugate Fermi fields of dimension 1 and 0 respectively. Short distance limit of the operator product is

$$\eta(z)\xi(w) \rightarrow \frac{1}{z-w}, \ \xi(z)\eta(w) \rightarrow \frac{1}{z-w} \text{ as } z \rightarrow w$$

and the stress energy tensor reads

$$T(z) = -\eta \partial \xi :$$

\(^{10}\)We remind the reader that the density is however not the order parameter from the point of view of the $O(n)$ model

\(^{11}\)There are actually some lattice arguments to justify this, based on considering the XX chain on a circle, with twisted boundary conditions
with central charge

\[ c = -2 \] (48)

This system is the free field representation for a current algebra based on \( gl(1,1)/u(1)_E \), the symmetry of dense polymers that was exhibited in the Hamiltonian point of view (XX chain). We consider now the properties of arbitrary twist fields in the \( \eta, \xi \) system. The twist field \( \sigma_{k/N} \) can be defined via the following short distance expansions \((k/N > 0)\)

\[ \eta(z) \sigma_{k/N} (w, \overline{w}) \propto (z - w)^{-k/N} \tau_{k/N} \] (49)
\[ \partial \xi(z) \sigma_{k/N} (w, \overline{w}) \propto (z - w)^{-1+k/N} \tau'_{k/N} \] (50)
\[ \overline{\eta}(z) \sigma_{k/N} (w, \overline{w}) \propto (z - w)^{-1+k/N} \tilde{\tau}_{k/N} \] (51)
\[ \overline{\partial} \xi(z) \sigma_{k/N} (w, \overline{w}) \propto (z - w)^{-k/N} \tilde{\tau}'_{k/N} \] (52)

In this formulas the non integer powers of the singularities are determined by the condition that \( \eta \) or \( \xi \) field picks up a phase \( e^{2i\pi k/N} \) when rotated around the origin (while the stress tensor has to remain periodic). The \( \tau \)'s are excited twist fields. The primes distinguish between two different types of excited twist fields with different values of \( h \) but the same \( h \) or vice versa. The tilde denoted fields related by complex conjugation \( h \leftrightarrow \tilde{h}, \ z \leftrightarrow \overline{z} \). The integer powers are determined as follows. Let us write the mode expansion

\[ \eta(z) = \sum_{m=-\infty}^{\infty} \eta_{m+k/N} z^{-m-1-k/N} \] (53)
\[ \xi(z) = \sum_{m=-\infty}^{\infty} \xi_{m-k/N} z^{-m+k/N} \] (54)

where, from the short distance singularity (46) one has the anticommutation relation

\[ \{ \xi_{m+k/N}, \eta_{n-k/N} \} = \delta_{m+n} \] (55)

while all other anticommutators vanish. The ground state of the \( k/N \) twisted sector is annihilated by all the positive frequency mode operators \( \eta_{m+1+k/N}, \xi_{m-k/N}, \ m > 0 \). The excited twist fields in the expression (52) can be expressed as descendents of \( | \sigma_{k/N} \rangle \) like for instance \( | \tau_{k/N} \rangle \). The conformal weight of the operator \( \sigma_{k/N} \) is easily computed following the technique of [18]. One introduces the object

\[ g(z, w) = \frac{\langle \eta(z) \partial \xi(w) \sigma_+(z_1) \sigma_-(z_2) \rangle}{\langle \sigma_+(z_1) \sigma_-(z_2) \rangle} \] (56)

where \( \sigma_+ \) stands for \( \sigma_{k/N} \), \( \sigma_- \) stands for its "anti twist", whose definition is similar to (52) but with \( k/N \) replaced by \( 1 - k/N \). \( g \) is uniquely determined by the various monodromy constraint, while letting \( z \) go to \( w \) allows to extract the stress energy tensor, and hence the required weight. One finds

\[ h = -\frac{1}{2} \frac{k}{N} \left( 1 - \frac{k}{N} \right) \] (57)
Owing to the local relation (47) we have

\[ L_n = \sum_{m=-\infty}^{\infty} (n - m - k/N) \eta_{m+k/N} \xi_{n-m-k/N}, \text{for } n \neq 0 \]

\[ L_0 = -\frac{1}{2 N} \sum_{m=1}^{\infty} \left( m - 1 + k/N \right) \xi_{m+1-k/N} \eta_{m-k/N} \]

\[ + (m - k/N) \eta_{m+k/N} \xi_{m-k/N} \] \hspace{2cm} (58)

### 3.2.2 Operator Content of The Various Sectors

Let us start with the antiperiodic (Ramond) sector. One finds easily that

\[ \left| d_{1/2,1/2} \right|^2 = \frac{1}{\eta} \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^{(l+1)n} q^{n^2/2} \right) \times \text{cc} \] \hspace{2cm} (59)

We see that the ground state of the Ramond sector has conformal weight

\[ h = -\frac{1}{8} \] \hspace{2cm} (60)

in agreement with (57). The physical meaning of the operator \( \sigma_{1/2} \) in the polymer system is the following. Suppose we have two punctures in the plane, and two frustration lines going from the points to infinity. Following Lemma 3 the various configurations that contribute to this object are made of loops encircling one point, with dangling ends attached to them, counted with a factor of 4. The dominant contributions correspond to polymers that separate the two points by encircling one of them, with a factor two (figure 8). They are selected in the Coulomb gas by putting an electric charge \(-e_0 \) \( (e_0 = 1 - g) \) at both points, such that the polymer loops instead of having a weight \( e^{i\pi e_0} + e^{-i\pi e_0} = 0 \) get the weight \( 1 + 1 = 2 \). The associated dimension is \( h = -e_0^2/4g = -1/8 \) in agreement with the above result. The \( L_0 \) generator in that sector writes

\[ L_0 = \left( m - 1/2 \right) \left( \xi_{m+1/2} \eta_{m-1/2} + \eta_{m+1/2} \xi_{m-1/2} \right) \] \hspace{2cm} (61)

In addition to the ground state \( \left| \sigma_{1/2} \right> \), other primary fields appear with conformal weights given by

\[ h_{4l} = -\frac{1}{8} + \frac{l^2}{2} \] \hspace{2cm} (62)

They all have even multiplicity (for \( l > 0 \)) due to the \( \eta, \xi \) symmetry in (61). Their fermion number is equal to one (zero) for \( l \) odd (\( l \) even). The first one has \( h = 3/8 \) corresponding to \( \eta_{-1/2} \left| \sigma_{1/2} \right> \) and \( \xi_{-1/2} \left| \sigma_{1/2} \right> \). The weights can be reproduced by the Kac formula and one finds

\[ h_{4l} = h_{1,2+2l} \] \hspace{2cm} (63)
The exact decomposition of the sum (59) is provided by expansion on \( c = -2 \) characters of the various terms. One has, following the appendix of [3]

\[
\frac{1}{\eta} q^{2n^2} = \sum_{k=n}^{\infty} \chi_{1,2+4k}
\]

\[
\frac{1}{\eta} q^{(2n+1)^2/2} = \sum_{k=n}^{\infty} \chi_{1,4k+4}
\]

(64)

On the other hand, from the lattice point of view, a natural family of operators in the antiperiodic sector is obtained as follows. We can imagine selecting the configurations where say \( l \) of the loops pinch the two punctures as in figure 9. This is a 4\( l \) legs operator, with dimension from the Lattice Coulomb gas mapping given precisely by (63). As we observe these dimensions coincide also with those appearing in the expansion (59) as well as (64). An important observation is that the 4\( l \) legs operators appear for \( L = 0 \) mod 4 in the Ramond sector. This modulo 4 occurs because the fundamental objects are trees rooted to some loops, and on the medial graph these are associated to even number of polymer loops. Such modulo 4 is also observed in the periodic (Neveu Schwartz) sector to which we turn now.

We have

\[
|d_{l/2,0}|^2 = \frac{1}{\eta} \left( \sum_{n=-\infty}^{\infty} (-1)^{n} q^{(2n+1)^2/8} \right) \times cc
\]

(65)

while we can write

\[
L_{m} = -\sum_{-\infty}^{\infty} (m - n) \eta_{m} \xi_{n-m}
\]

(66)

\[
L_{0} = \sum_{1}^{\infty} m (\xi_{m} \eta_{m} + \eta_{m} \xi_{m})
\]

(67)

The \( SL_2 \) invariant vacuum is defined by \( \eta_{n} |0 >= 0, n \geq 0, \xi_{n} |0 >= 0, n > 0 \). The decomposition of (67) among irreducible representations of the Virasoro algebra is accomplished via the formulas

\[
\frac{1}{\eta} q^{(4n+1)^2/8} = \sum_{k=n}^{\infty} \chi_{1,3+4k} + \chi_{1,5+4k}, \ n \geq 0
\]

\[
\frac{1}{\eta} q^{(4n-1)^2/8} = \sum_{k=n}^{\infty} \chi_{1,1+4k} + \chi_{1,3+4k}, \ n > 0
\]

(68)

The ground state is twice degenerate, as discussed above. Due to the well known property that \( < 0 | \xi | 0 > \neq 0 \) we identify the density \( \rho \) with \( \chi + \xi \). The other operators have conformal weights given by \( h_{1,3+4k} \) and \( h_{1,5+4k} \) \((k \geq 0)\). These coincide with the dimensions of the 4\( l \) + 2 legs operators

\[
h_{4l+2} = -\frac{1}{8} + \frac{(l + 1/2)^2}{2}
\]

(69)

\[\text{\underline{12} However only the identity provides an } SL_2 \text{ invariant vacuum} \]
We turn finally to the $Z_4$ sector. We find

$$|d_{l/4,1/4}|^2 = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} (-1)^{(1-l/2)n} q^{(4n+1)^2/32} \times cc$$ \hspace{1cm} (70)

The ground state $|\sigma_{1/4}\rangle$ has conformal weight

$$h = -\frac{3}{32}$$ \hspace{1cm} (71)

In the polymer language this is the dimension of the one leg operator (its negative value was interpreted and discussed in [4]). In addition to this other primary fields appear in the expansion with weights

$$h_{4l+1} = -\frac{1}{8} + \frac{(l + 1/4)^2}{2}$$
$$h_{4l+3} = -\frac{1}{8} + \frac{(l + 3/4)^2}{2}$$ \hspace{1cm} (72)

They can be reproduced formally by the Kac formula with half integer labels

$$h_{4l+1} = h_{1/2l+5/2}$$
$$h_{4l+3} = h_{1/2l+7/2}$$ \hspace{1cm} (73)

The corresponding fields are therefore not degenerate with respect to the Virasoro algebra and the character expansion of (70) can be easily written as

$$|d_{l/4,1/4}|^2 = \sum_{n=0}^{\infty} (-1)^{(1-l/2)n} (\chi_{1,2n+5/2} + \chi_{1,2n+7/2})$$ \hspace{1cm} (74)

One has in particular that $\xi_{-1/4} |\sigma_{1/4}\rangle$ is primary with weight $h = 5/32$, $\eta_{-3/4} |\sigma_{1/4}\rangle$ is primary with weight $h = 21/32$. These operators are respectively the 3 and 5 legs operators.

The content in polymers operators of the various sectors is therefore the following

Neveu Schwartz : $L = 2 \mod 4$
Ramond : $L = 0 \mod 4$
$Z_4$ : $L = 1, 3 \mod 4$ \hspace{1cm} (75)

Moreover in each sector the fields are descendents of the one with lowest value of $L$ with respect to the $\eta, \xi$ algebra. So there are only three fundamental geometrical operators.

### 3.3 U(1) Current

The natural current of the $\eta, \xi$ system is given by

$$J = - : \eta \xi :$$ \hspace{1cm} (76)
The associated $U(1)$ charge $J_0$ is denoted $Q$ in the following (we consider only left $U(1)$ charge, as we considered only the left moving sector before). One has $Q = 1$ for $\xi$, $Q = -1$ for $\eta$. As was noticed in [18], when the vacuum is defined as the $SL_2$ invariant state $|0\rangle$, $J$ is not a primary field and one has

$$T(z)J(w) \to \frac{-1}{(z-w)^3} + \frac{J(z)}{(z-w)^2}$$  \hspace{1cm} (77)

and hence

$$[L_1, J_{-1}] = J_0 - 1$$

$$[L_1, J_{-1}]^\dagger = [L_{-1}, J_1] = -J_0$$  \hspace{1cm} (78)

The system has therefore charge asymmetry, $Q^I = 1 - Q$. For the polymer density operator $\rho$ we associate the value $Q_2 = 1$ in such a way that $<0|\rho|0> \neq 0$ holds. The other $4l + 2$ legs operators in the Neveu Schwartz sector therefore have charge $Q_{u+2} = 1 + l$. The $U(1)$ charge of the twist fields $\sigma_{k/N}$ is $Q = k/N$. To the ground state of the Ramond sector we can associate the charge $Q = 1/2$. Therefore the $L = 4l$ legs operators have $Q_{u} = 1/2 + l$. To the ground state of the $Z_4$ sector we associate the charge $Q = 3/4$, in such a way that the $L = 4l + 1$ operators have charge $Q_{u+1} = 3/4 + l$, the $L = 4l + 3$ operators have $Q_{u+3} = 5/4 + l$. In summary for the $L$ legs operator charges given by

$$\Phi_L \leftrightarrow Q = \frac{1}{2} \pm \frac{L}{4}$$  \hspace{1cm} (79)

Hence $Q$ counts essentially the number of legs, and one can think of $J$ as describing the correlations of orientation along the polymer chains [19].

### 3.3.1 Correlation Functions and Operator Algebra

In polymer theory one is mainly interested in correlations of $L$ legs operators. For $L$ even, these operators have dimensions in the Kac table. Their four point functions can therefore be computed in the Coulomb Gas formalism, that is by bosonizing the $\eta, \xi$ system and introducing screening charges. They can also be computed directly in the $\eta, \xi$ system. The consistency of the two approaches is expected, but not always straightforward to check. Let us consider first the Neveu Schwartz sector and the example of the 6 legs operator that has $h = 7$. These weights are reproduced by $h_{21}$ in the Kac table. The four point function needs therefore introduction of a single screening charge. One finds, using the formula (4.18) of [20]

$$\langle \Phi_6(1)\Phi_6(2)\Phi_6(3)\Phi_6(4) \rangle \propto \frac{1}{|z_{13}|^4} \frac{1}{|z_{24}|^4} \frac{1}{|x(1-x)|^4} \left[ 3 \left| x^3(2-x) \right|^2 + \left| 2 - 4x + 2x^3 - x^4 \right|^2 \right]$$  \hspace{1cm} (80)

where $x = z_{12}z_{34}/z_{13}z_{24}$. This can be recast as

$$\langle \Phi_6(1)\Phi_6(2)\Phi_6(3)\Phi_6(4) \rangle =$$

$$\frac{2}{|z_{13}|^2 |z_{24}|^2 |x(1-x)|^4} \left\{ 2 \left[ x^2x^2 + (1-x)^2(1-x) \right] + x^2(x^2(1-x)^2(1-x) \} (1-x) \right\}$$

$$- \left[ x^2x^2 + x^2(1-x)^2 + x^2(1-x)^2(1-x) \right]$$  \hspace{1cm} (81)
This is the correlator of the operator $\eta(z) + \partial \xi(z) \times cc$, computed easily in the free fermion theory.

For the other sectors, we need the knowledge of the four point functions of twist operators. They are determined using the general technique of \[18\]. One introduces the auxiliary function

$$g(z, w; z_i) = - \frac{\langle \eta(z) \partial \xi(w) \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle}{\langle \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle}$$

which is determined by the local and global monodromy conditions. Letting $z$ go to $w$ allows to extract the stress tensor in \[3\] and therefore, by letting further $z$ go to one of the $z_i$'s, a linear differential equation satisfied by the four point function, which can easily be integrated. One finds

$$\langle \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle \propto$$

where $F$ is the hypergeometric function

$$F_{k/N}(x) = F(k/N, 1 - k/N, 1; x)$$

Consider for instance the ground state of the Ramond sector. In that case twist and anti twist are identical. The conformal weights are $h = \frac{1}{2} = -1/8$ and coincide with the dimensions $h_{12}$ in the Kac table. Again the four point function could also be computed in the Coulomb Gas formalism. One screening charge is needed. The combination of conformal blocks entering the transformation formulas for $x$ from the transformation formulas for $x$.

They are determined using the general technique of \[18\]. One introduces the auxiliary function

$$\langle \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle \propto$$

where $F$ is the hypergeometric function

$$F_{k/N}(x) = F(k/N, 1 - k/N; 1, x)$$

Consider for instance the ground state of the Ramond sector, i.e. $k/N = 1/2$. In that case twist and anti twist are identical. The conformal weights are $h = \frac{1}{2} = -1/8$ and coincide with the dimensions $h_{12}$ in the Kac table. Again the four point function could also be computed in the Coulomb Gas formalism. One screening charge is needed. The combination of conformal blocks entering the result in formula \(4.18\) of \[20\] turns out however to vanish identically. To get a finite result one has that the part of (3.3.1) in bracket is identical with

$$k/N$$

This guarantees the identity of \(3\) (for $k/N = 1/2$) and \(3.3.1\).
Besides the advantage that correlators of fields in the Kac table can be computed without taking any limit, the main use of the fermionic formulation is to provide expressions for the polymer operators with an odd number of legs. The most important is the one leg operator, for which one finds
\[
\langle \Phi_1(1)\Phi_1(2)\Phi_1(3)\Phi_1(4) \rangle_{(12,34)+(13,24)} \propto |z_{12}z_{34}|^{3/8} |x(1-x)|^{3/8} [F_{1/4}(x)\overline{F}_{1/4}(1-x) + F_{1/4}(1-x)\overline{F}_{1/4}(x)]
\] (88)
where the subscript \((12,34) + (13,24)\) means that one of the polymers connects points 1 to 2 (resp. 1 to 3), the other one 3 to 4 (resp. 2 to 4) (see figure 10). The above correlator is not invariant in \(x \to 1/x\) since sources and sinks of polymers are distinguished. What one may call the ”full” one leg polymer operator is represented by \(|\sigma_{1/4}\rangle + |\sigma_{3/4}\rangle\) and its four point function is obtained by adding \([S]\) and the other function obtained by exchanging 1 and 2.

Knowledge of the four point functions gives now access to the polymer operator algebra. Consider first the \(L = 4l+2\) legs operators that all belong to the Neveu Schwartz sector. Short distance product of these must give rise to operators in the Neveu Schwartz sector again. This is confirmed by using the fusion rules of the \(\Phi_{1,3+2l}\) operators. One finds therefore
\[
\Phi_{4l+2,4l+2} \propto \sum_{l=0}^{l_1+l_2+1} \Phi_{4l+2}
\] (89)
In a similar fashion consider the \(L = 4l\) operators in the Ramond sector. Short distance product of these must give operators in the Neveu Schwartz sector. this is confirmed by the fusion rules of \(\Phi_{1,2+2l}\) operators
\[
\Phi_{4l_1,4l_2} \propto \sum_{l=0}^{l_1+l_2} \Phi_{4l+2}
\] (90)
As far as operators with an odd number of legs are concerned we must use the above analysis of the \(Z_4\) sector and correlators. Calling \(\Phi_{2l+1}\) the full (in the above sense) \(L = 4l + 1\) legs operators one finds that product of two fields on the \(Z_4\) sector gives rise to fields both in the Neveu Schwartz and Ramond sector that is
\[
\Phi_{2l_1+1,2l_2+1} \propto \sum_{l=1}^{l_1+l_2+2} \Phi_{2l}
\] (91)
The geometrical interpretation of these fusion rules is interesting and discussed in figure 11.

4 The Continuum Limit of Dilute Polymers

Recall first that dilute polymers are obtained by considering a finite number of self avoiding, mutually avoiding loops or open chains with a fugacity \(\mu^{-1}\) per unit length, where \(\mu\) is the inverse effective connectivity constant. We do not have an "almost" soluble model to describe the physics of dilute polymers. The study of the \(O(n), n \to 0\) phase transition \([4, 9]\) suggests that there is a boson fermion "supersymmetry" in the dilute phase such that \(n = 0\) is formally recovered by a cancellation between the two species, and that this symmetry is broken in the dense phase. We
know on the other hand from the above analysis that the dense phase is described by a \( \eta, \xi \) system. This corresponds as well to taking only the fermionic generators of the \( gl(1,1) \) algebra. Restoring the boson fermion symmetry should amount to using all the generators of this algebra. The most naive guess would then be to assume that the degrees of freedom of the continuum limit of dilute polymers are \( Gl(1,1) \) matrices, with a Wess Zumino action. As we commented in the introduction this is not true. It is nevertheless instructive to carry out the study of the \( Gl(1,1) \) WZW model, as the related free field representation will provide the right space to study dilute polymers, or twisted \( \mathcal{N} = 2 \) supersymmetry. The Gaussian decomposition of a \( Gl(1,1) \) element is

\[
g = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{\sqrt{\kappa}}(\tilde{\phi} - \frac{1}{2}\phi)} & 0 \\ 0 & e^{\frac{1}{\sqrt{\kappa}}(\tilde{\phi} + \frac{1}{2}\phi)} \end{pmatrix} \begin{pmatrix} 1 & -\xi^+ \\ 0 & 1 \end{pmatrix}
\]

(92)

Working out the expression of the currents \( \partial^\mu gg^{-1} \) one finds [11], after taking into account the various quantum corrections, the action (where \( \kappa \) is the level)

\[
S = \frac{1}{\pi} \int d^2 z \left[ \partial \tilde{\phi} \partial \phi - \frac{i}{8\sqrt{\kappa}} R\phi + \eta \partial \xi + \text{cc} \right]
\]

(93)

where the fields \( \eta \) and \( \tilde{\phi} \) are obtained from \( \partial \xi^+ \) and \( \tilde{\phi}' \) by including quantum corrections, \( R \) is the scalar curvature of the two dimensional metric. Corresponding to this action we have the following propagator

\[
<\phi(z)\phi(w)> = -\log(z-w)
\]

(94)

while for the fermions [46] still holds. The stress tensor reads

\[
T(z) = -: \eta \partial \xi : - : \partial \tilde{\phi} \partial \phi : + \frac{i}{2\sqrt{\kappa}} \partial^2 \phi
\]

(95)

In the following we set

\[
\beta = \frac{1}{4\sqrt{\kappa}}
\]

(96)

The \( Gl(1,1) \) WZW model [11] turns out to possess a number of features that are not compatible with our knowledge of dilute polymers physics, like a spectrum not bounded from below [13]. Nevertheless the above free field representation provides the correct fock space to represent the dilute polymers. Indeed one recognizes in the stress tensor [45] the stress tensor of a twisted \( \mathcal{N} = 2 \) superconformal theory. Recall that the Coulomb gas representation of ordinary \( \mathcal{N} = 2 \) superconformal theory as worked out for instance in [21, 22, 23] involves the degrees of freedom \( \psi, \tilde{\psi}, \phi, \tilde{\phi} \), with the stress tensor

\[
T_{N=2}(z) = -\frac{1}{2} : \left( \psi \partial \tilde{\psi} + \tilde{\psi} \partial \psi \right) : - : \partial \tilde{\phi} \partial \phi : + i\beta \left( \partial^2 \phi + \partial^2 \tilde{\phi} \right)
\]

(97)

where now \( \psi \) and \( \tilde{\psi} \) are both fermions of weight 1/2 and propagators as in (46). The \( U(1) \) current reads

\[
J(z) = - : \psi \tilde{\psi} : + 2i\beta \left( \partial \phi - \partial \tilde{\phi} \right)
\]

(98)

---

13It is interesting to notice that the \( Gl(1,1) \) WZW model was also introduced to give sense to a \( n \to 0 \) limit, but in the context of link invariants.
with normalization
\[ J(z)J(w) = \frac{c/3}{(z-w)^2} \]
and central charge
\[ c_{N=2} = 3 \left( 1 - \frac{1}{2\kappa} \right) \]
The twisting is obtained \[6\] by the substitution
\[ T \rightarrow T + \frac{1}{2} \partial J \]
leading to the result \[95\] where we have relabelled \( \psi \) by \( \eta \), \( \tilde{\psi} \) by \( \xi \) since these fields have now weights 1 and 0. The central charge of the twisted theory is \( c = 0 \) for any \( \kappa \). We shall now make the assumption that dilute polymers indeed are described by a twisted \( N = 2 \) theory, and investigate whether this is reasonable. A few computations of dimensions shows that the only possible level is
\[ \kappa = 3/4 \]
i.e. that the untwisted theory has \( c = 1, k = 1 \).

### 4.1 Continuum Limit Partition Functions of Dilute Polymers

#### 4.1.1 Spectral Flow and \( \zeta \) Algebras in twisted and untwisted \( N = 2 \)

The characters of the \( N = 2 \) theory with central charge \( c = 1 \) are given for instance in \[24, 25\]. One finds the following results, where the superscripts refer to the \( U(1) \) charge and dimension in the untwisted theory
\[ \Xi_{NS}^{(0,0)} = \frac{1}{\eta} \sum_n q^{(6n)^2/24} \]
\[ \Xi_{NS}^{(\pm 1/3,1/6)} = \frac{1}{\eta} \sum_n q^{(6n+2)^2/24} \]
and similarly
\[ \Xi_{R}^{(\pm 1/6,1/24)} = \frac{1}{\eta} \sum_n q^{(6n-1)^2/24} \]
\[ \Xi_{R}^{(\pm 1/2,3/8)} = \frac{1}{\eta} \sum_n q^{(6n+3)^2/24} \]

We recall that, as everywhere in this work, the names Ramond and Neveu Schwarz refer to boundary conditions in the plane. When going to the cylinder, periodic and antiperiodic boundary conditions are exchanged in the \( N = 2 \) theory where the fermions have half integer conformal weights, but they remain identical in the twisted theory where fermions acquire integer dimensions.
As emphasized in [26] there is actually a continuous set of sectors for the $N = 2$ theory that are related to each other by the spectral flow. For an arbitrary number $\zeta$ one can define the $\zeta$ algebra for which the supersymmetry generators have boundary conditions $G^\pm(z) = \exp(\pm 2i\pi \zeta)G^\pm(e^{2i\pi} z)$, with $\zeta = 0$ for NS, $\zeta = \pm 1/2$ for R. They all have three irreducible representations in our case. The corresponding transformation for weights and charges reads

$$ (Q = x/6, h = x^2/4) \rightarrow (Q = (x - 2\zeta)/6, h = (x - 2\zeta)^2/24) $$

To ensure modular invariance, $\zeta$ must be a rational number, $\zeta = k/N$.

Let us now consider the twisted theory. We first write the new relations satisfied by the generators. We expand

$$ G^+ (z) = \sum_m G^+_{m+1/2-k/N} z^{-m-2+k/N} $$
$$ G^- (z) = \sum_m G^-_{m+1/2+k/N} z^{-m-2-k/N} $$

and recall that $G^+$ acquires dimension 1, $G^-$ dimension 2. Then one has

$$ [L_m, L_n] = (m-n)L_{m+n} $$
$$ [L_n, J_m] = -mJ_{m+n} - \frac{n(n+1)c \delta_{m+n}}{6} $$
$$ [J_m, J_n] = \frac{c}{3m} \delta_{m+n} $$

$$ [L_n, G^+_{m+1/2-k/N}] = -(m+1-k/N)G^+_{n+m+1/2-k/N} $$
$$ [L_n, G^-_{m+1/2+k/N}] = (n-m-k/N)G^-_{n+m+1/2+k/N} $$

$$ \begin{align*}
\{G^+_{m+1/2-k/N}, G^-_{n+1/2+k/N}\} &= 2L_{m+n+1} + 2(n+1+k/N)J_{m+n+1} \\
&\quad + \frac{c}{3}[(m+1/2-k/N)^2 - 1/4] \delta_{m+n+1}
\end{align*} $$

$$ [J_n, G^+_{m+1/2-k/N}] = G^+_{n+m+1/2-k/N} $$
$$ [J_n, G^-_{m+1/2+k/N}] = -G^-_{n+m+1/2+k/N} $$

Notice that, as in the dense polymers case, the $U(1)$ current $J$ is not a (Virasoro) primary field anymore after twisting. The balance of charges is such that

$$ Q^f = 1/3 - Q $$

This balance can be decomposed in bosonic and fermionic balances (the fermionic balance is the same as before)

$$ Q^f_B = 1 - Q_F, Q^f_F = -2/3 - Q_B $$

The twisting affects charge and dimension as follows

$$ (Q = x/6, h = x^2/24) \rightarrow (Q = x/6, h = [(x-1)^2 - 1]/24) $$

23
A representation of the untwisted algebra is also a representation of the twisted one, possibly reducible. We denote in the following the characters by $\chi_{l/N,k/N}^{(Q,h)}$, where as in the dense case the labels $l/N, k/N$ stand for the fermions boundary conditions, with $k/N = 0$ for NS, $k/N = 1/2$ for R. One finds

$$\chi_{1/2,0}^{(0,0)} = \chi_{1/2,0}^{(1/3,0)} = \frac{1}{P} \sum_n q^{|(6n+1)^2-1|/24}$$

$$\chi_{1/2,0}^{(-1/3,1/3)} = \frac{1}{P} \sum_n q^{|(6n+3)^2-1|/24}$$

(111)

where $P = \prod_{n=1}^{\infty} (1 - q^n)$ and

$$\chi_{1/2,1/2}^{(-1/6,1/8)} = \frac{1}{P} \sum_n q^{|(6n+2)^2-1|/24}$$

$$\chi_{1/2,1/2}^{(1/2,1/8)} = \frac{1}{P} \sum_n q^{|(6n+2)/4-1|/24}$$

$$\chi_{1/2,1/2}^{(1/6, -1/24)} = \frac{1}{P} \sum_n q^{|(6n)^2-1|/24}$$

(112)

We notice that the above sums are also those that appear in the untwisted theory, but NS and R sectors are exchanged. We notice also that for $k/N = 0, 1/2$ two of the three characters are in fact identical. The characters $\chi_{0,1/2}$ are obtained by definition by $\chi_{1/2,1/2}^{(\tau + 1)}$. For the doubly periodic sector one finds $\chi_{0,0}^{(-1/3,1/3)} = \chi_{0,0}^{(2/3,1/3)} = 0$ and $\chi_{0,0}^{(1/3,0)} = 1$.

By analogy with the dense polymer case we introduce also, in addition to the NS and R sectors a $Z_4$ sector, with the characters in the twisted theory

$$\chi_{1/2,1/4}^{(-1/12,5/96)} = \frac{1}{P} \sum_n q^{|(6n+3/2)^2-1|/24}$$

$$\chi_{1/2,1/4}^{(5/12,5/96)} = \frac{1}{P} \sum_n q^{|(6n+3)^2/2-1|/24}$$

$$\chi_{1/2,1/4}^{(1/4, -1/32)} = \frac{1}{P} \sum_n q^{|(6n+1)^2-1|/24}$$

$$\chi_{1/2,1/4}^{(-5/12,15/32)} = \frac{1}{P} \sum_n q^{|(6n+5/2)^2-1|/24}$$

(113)

Notice that the three characters are now all different. Characters for other values of $l/N$ are, by definition, obtained by successive modular transformations $\tau \rightarrow \tau + 1$. We also have

$$\chi_{1/4,1/2}^{(-1/6,1/8)} = \frac{-1}{P} \sum_n (-i)^n q^{|(6n-2)^2-1|/24}$$

$$\chi_{1/4,1/2}^{(1/2,1/8)} = \frac{i}{P} \sum_n (-i)^n q^{|(6n+2)^2-1|/24}$$

$$\chi_{1/4,1/2}^{(1/6, -1/24)} = \frac{1}{P} \sum_n (-i)^n q^{|(6n)^2-1|/24}$$

(114)

and similar expressions for the $\chi_{1/4,0}$.  

24
4.1.2 The Sector with an Even Number of Polymers

We now define, in a way similar to the dense case, the lattice partition functions

**Definition**

\[ Z_{AP}(\text{resp.} PA, \text{resp.} AA) = \text{Sum over configurations with an even number of non contractible dilute polymers, of total monomer length } L, \text{ that cross } \omega_2 \text{ (resp. } \omega_1, \text{ resp. } (\omega_1, \omega_2) \text{ an odd number of times, with weight } 2^{\text{number of polymers}} \mu^{-L}) \]

where \( \mu \) is the inverse connectivity constant. Contrarily to the dense case, because dilute polymers do not occupy a finite fraction of the available space, there is no free energy term as in (29), and one has to set

\[ Z_{PP} = 1 \]  

(115)

Let us define also the manifestly modular invariant quantity

\[ Z^c = \left\{ \text{Sum over configurations with an even number of non contractible polymers, of total monomer length } L, \text{ with weight } 2^{\text{number of polymers}} \mu^{-L} \right\} \]

(116)

One checks easily in that case that

\[ Z^c = \frac{1}{2}(Z_{AP} + Z_{PA} + Z_{AA} - Z_{PP}) \]  

(117)

The continuum limit of these expressions was worked out in [7]. The same expressions as the ones in subsection (3.1.2) apply, with the lattice Coulomb gas coupling constant \( g = 1/2 \) replaced by \( g = 3/2 \). The only difference is that the relation (41) is replaced by Euler identity

\[ Z_c[3/2, 2] - Z_c[3/2, 1] = 2 \]

(118)

and therefore

\[ Z_{PP} = 1 \]

(119)

Repeated use of the Jacobi identity provides the following results

\[ Z_{AP} = \frac{1}{2} \left( 2 \left| \chi_{1/2,0}^{(q,0)} \right|^2 + \left| \chi_{1/2,0}^{(q',1/3)} \right|^2 \right) \]

\[ Z_{PA} = \frac{1}{2} \left( 2 \left| \chi_{0,1/2}^{(q',1/8)} \right|^2 + \left| \chi_{0,1/2}^{(1/6,-1/24)} \right|^2 \right) \]

\[ Z_{AA} = \frac{1}{2} \left( 2 \left| \chi_{1/2,1/2}^{(q'',1/8)} \right|^2 + \left| \chi_{1/2,1/2}^{(1/6,-1/24)} \right|^2 \right) \]

(120)

Remarkably, this establishes that the boundary conditions of the continuum fermions in the twisted \( N = 2 \) theory are the same as those of lattice fermions (although no discrete action is known to explicitly reproduce this) such that a polymer loop is described either by a boson or a fermion,
giving a total weight zero for contractible ones, and zero or two for non contractible ones, depending on the boundary conditions and the winding number. Adding the above results we find

\[ Z_{AP} + Z_{PA} + Z_{AA} = Z_c[3/2,2] - 1 = Z_c[3/2,1] + 1 \]  

so that finally

\[ Z^c = \frac{1}{2} Z_c[3/2,1] \]  

Remarkably also, the combination of partition functions for \( Z^c \) includes projection on odd fermion number. That this special point of the Gaussian line was \( N = 2 \) supersymmetric was observed by Friedan and Shenker \[27\] and S.K. Yang \[28\]. Our point of view is of course a bit different since we consider the above as a twisted \( N = 2 \) partition function. From this point of view, \( Z^c \) is not a totally physical partition function because it does not contain any ground states for the twisted theory, since we have projection on odd fermion number. One can therefore decide to consider as a physical partition function the one with projection on even fermion number

\[ Z_{phys} = \frac{1}{2} (Z_{AP} + Z_{PA} + Z_{AA} + Z_{PP}) \]  

with

\[ Z^c \rightarrow \frac{1}{2} Z_c[3/2,2] \]  

In opposition to what happens in the case of say interacting around a face models, adding a constant to the partition function does not change the physics encoded in the fusion algebra and correlators. The combination of sectors in the geometrical models seems to be merely a matter of convention. Such an ambiguity did not occur in the dense case because \( Z_{PP} \) vanished. \[16\] Consider now, as in the dense case

\[ Z^o = \{ \text{Sum over configurations with an odd number of noncontractible} \]  

polymers of total monomer length \( L \), with weight \( 2^{\text{number of polymers}} \mu^{-L} \} \]  

whose continuum limit, evaluated by the Coulomb Gas mapping, is

\[ Z^o = Z_c[3/2,1/2] - \frac{1}{2} Z_c[3/2,1] \]  

After some algebra one can also identify it with a twisted \( N = 2 \) partition function provided one introduces \( Z_4 \) sectors

\[ 2 \left[ \chi_{1/4,1/2}^{(1/6,-1/24)} \right]^2 + \left[ \chi_{1/4,0}^{(1/2,1/8)} \right]^2 + \chi_{1/4,1/2}^{(1/6,1/8)} \left( \chi_{1/4,1/2}^{(-1/6,1/8)} \right)^* + \chi_{1/4,0}^{(1/3,0)} \left( \chi_{1/4,0}^{(0,0)} \right)^* + \text{cc} \]  

\[ 16\text{Besides the physical interpretation of this result that was given above, one can presumably explain it also by considering that the dense phase is in fact obtained by spontaneous breaking of supersymmetry of a theory with vanishing Witten index. The index is two for dilute polymers}
The character decomposition of
\[ Z^{c+o} \rightarrow Z^{c+o} = Z_c[3/2, 1/2] \] (128)
is finally obtained by adding the expressions (120) and (127).

It is also interesting to notice that the diagonal modular invariant, obtained by summing the modulus square of each of the characters, turns out to be equal to \( Z_c[3/2, 4] - 1/2Z_c[3/2, 2] \). Hence \( Z_c[3/2, 4] \) can also be expanded on twisted \( N = 2 \) characters.

### 4.1.3 Operator Content of the Various Sectors

We now discuss briefly the operator content of the various sectors. Let us start with the antiperiodic (Ramond) sector. The ground state \( |\Sigma_{1/2}\rangle \) has conformal weight
\[ h = -1/24 \] (129)
To understand the physical meaning of this operator, imagine a diluted version of the dense case. Hence consider again configurations where polymers separate two points by encircling one of them, with a factor of 2. They are selected in the lattice Coulomb gas by putting an electric charge \(-e_0 \) \((e_0 = 1 - g) \) at both points such that polymer loops encircling the punctures acquire weight 2. The conformal dimension is
\[ h = -e_0^2/4g = -1/(8 \times 3) = -1/24. \] The content of the R sector is easily found by decomposing each of the twisted \( N = 2 \) characters on the \( c = 0 \) Virasoro characters using formulas in the appendix of [3]. Consider first the representation \( h = -1/24, Q = 1/6 \). One has
\[ \frac{q^{(12n)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k+2,3}, \ n \geq 0 \]
\[ \frac{q^{(12n+6)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k+4,3}, \ n \geq 0 \] (130)
Therefore other Virasoro primary fields appear with weights
\[ h_{4l} = h_{2l+2,3} = -\frac{1}{24} + \frac{3}{2}l^2 \] (131)
They all have even degeneracy (except for \( l = 0 \)) due to the \( G^\pm \) symmetry. Their fermion number is equal to one (zero) for \( l \) odd (even). For instance the four polymers operator corresponds to \( G^+_2 |\Sigma_{1/2}\rangle \) and \( G^-_1 |\Sigma_{1/2}\rangle \). The operators with \( h = \bar{h} = h_{4l} \) \((l > 0)\) correspond to the \( L = 4l \) legs operators. Their \( U(1) \) charge is \( Q = 1/6 \pm L/4 \). For the representation \( h = 1/8, Q = -1/6 \) one uses
\[ \frac{q^{(12n-2)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k,1} + \chi_{4k+2,2}, \ n > 0 \]
\[ \frac{q^{(12n+4)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k+2,1} + \chi_{4k+4,2}, \ n \geq 0 \] (132)
These Virasoro primaries do not seem to correspond to any interesting geometrical correlators. Actually they can be considered as subdominant operators for the correlations of figure 4.

Consider now the NS sector. Notice that due to twisting we have now

\[
\left[ L_0, G^\pm_{\mp 1/2} \right] = 0
\]
\[
\{ G^+_{-1/2}, G^-_{1/2} \} = 2L_0
\]

(133)

Therefore every field but the ground state with \( h = 0 \) is twice degenerate. Let us discuss first the representation with \( h = 1/3, Q = -1/3 \). Using

\[
\frac{q^{(12n+3)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k+3,3} + \chi_{4k+5,3}
\]
\[
\frac{q^{(12n-3)^2/24}}{\eta} = \sum_{k=n}^{\infty} \chi_{4k+1,3} + \chi_{4k+3,3}
\]

(134)

We find that the new Virasoro highest weight operators have weights

\[
h_{4l+2} = h_{2l+3,3} = -\frac{1}{24} + \frac{3}{2} \left( l + \frac{1}{2} \right)^2
\]

(135)

They correspond to the \( L = 4l + 2 \) legs operators, with charge \( Q = 1/6 \pm L/4 \). The representations \( h = 0, Q = 0 \) and \( h = 0, Q = 1/3 \) both contain fields with integer dimensions only that do not seem to have any interesting meaning in terms of polymers.

Finally we consider the \( Z_4 \) sector. The Virasoro character expansion of the twisted \( N = 2 \) characters is straightforward since none of the dimensions belong to the Kac table. For the representation \( h = 5/96, Q = -1/12 \) one finds the set of conformal weights

\[
h_{4l+1} = -\frac{1}{24} + \frac{3}{2} \left( l + \frac{1}{4} \right)^2
\]
\[
h_{4l+3} = -\frac{1}{24} + \frac{3}{2} \left( l + \frac{3}{4} \right)^2
\]

(136)

which can formally be reproduced by the Kac formula

\[
h_{4l+1} = h_{2l+5/2,3}
\]
\[
h_{4l+3} = h_{2l+7/2,3}
\]

(137)

The ground state \( |\Sigma_{1/4}\rangle \) describes the one leg operator. The three leg operator for instance can be obtained by acting with \( G^+_{-5/4} \) or \( G^-_{-1/4} \). The two other representations are combined in the partition function. They correspond to Virasoro primary fields with non vanishing spin, and some non scalar polymer observables [31].
In summary the content in fuseau operators of the various sectors is once again given by

\[ L = \begin{cases} 2 \mod 4 \\
0 \mod 4 \\
1, 3 \mod 4 \end{cases} \]

and the \( U(1) \) charges are

\[ \Phi_L \leftrightarrow Q = 1/6 \pm L/4 \]

4.2 Correlators and Operator Algebra for Dilute Polymers

The new Coulomb Gas made of the \( \eta, \xi \) system and the bosonic fields \( \phi, \tilde{\phi} \) provides a much larger Fock space to represent the polymer theory than the usual Coulomb Gas a la Dotsenko Fateev, made of a single bosonic field. This Fock space is quite big, and can be restricted in many different ways (one of them corresponds for instance to the \( GL(1, 1) \) WZW model). Our task is to exhibit a subset of degrees of freedom for which all four point functions can be built, and that is closed under operator algebra.

Consider the vertex operators

\[ V_{\alpha, \tilde{\alpha}} = \exp \left( \tilde{\alpha} \phi + \alpha \tilde{\phi} \right) \]

In the twisted theory they have conformal weight

\[ h = \alpha (\tilde{\alpha} - 2 \beta) \]

where we have set

\[ \beta = \frac{1}{4 \sqrt{\kappa}} = \frac{1}{\sqrt{12}} \]

\( h \) is invariant in the following three transformations

\[ \alpha \rightarrow 2 \beta - \tilde{\alpha}, \tilde{\alpha} \rightarrow 2 \beta - \alpha \]
\[ \alpha \rightarrow -\alpha, \tilde{\alpha} \rightarrow 4 \beta - \tilde{\alpha} \]
\[ \tilde{\alpha} \rightarrow 2 \beta + \alpha, \alpha \rightarrow \tilde{\alpha} - 2 \beta \]

The balance of (bosonic) charges to obtain a non vanishing correlator must be

\[ \sum \alpha = 0, \sum \tilde{\alpha} = 4 \beta \]

Corresponding to a bosonic \( U(1) \) charge

\[ Q_B = 2 \beta (\alpha - \tilde{\alpha}) = -2/3 \]

The three screening operators [21] of the untwisted theory can be used as well in the twisted theory. Since they have vanishing total \( U(1) \) charge they are not affected by the twist. Introduce
the charges
\[
\alpha_+ = \tilde{\alpha}_+ = 2\beta \\
\alpha_- = \tilde{\alpha}_- = -\frac{1}{2\beta}
\] (146)
The even screening operator reads
\[
Q_{e,1} = \oint dz \left[ i\partial(\tilde{\alpha}_+ + \tilde{\phi} - \alpha_+ + \phi) + 2\alpha_+\tilde{\alpha}_+ + \xi\eta \right] e^{i(\alpha_+ + \tilde{\phi} + \alpha_+ + \phi)}
\] (147)
and the odd ones
\[
Q_{o,1} = \oint dz \xi(z) e^{i\alpha_+ - \tilde{\phi}} \\
Q_{o,2} = \oint dz \eta(z) e^{i\tilde{\alpha}_+ - \phi}
\] (148)
where the coefficients in the prefactors are chosen to insure the vanishing of the commutators of the \(Q\)'s with the modes of \(T, J, G^\pm\).

In the untwisted theory, the four point functions of the \(N = 2\) primary fields can be built when the following quantization rule is obeyed
\[
\alpha + \tilde{\alpha} = (1 - n)\alpha_+ - m\alpha_-
\] (149)
They involve \((n - 1)\) \(Q_{e,1}\) even screening operators, and \(m\) of each specie of odd screening operator \(Q_{o,1,2}\).

In the twisted theory we shall compute the four point functions using the representation deduced from the untwisted case, that is
\[
(V_{\alpha,\tilde{\alpha}}(1)V_{-\alpha,4\beta-\tilde{\alpha}}(2)V_{\alpha,\tilde{\alpha}}(3)V_{\tilde{\alpha}-2\beta,\alpha+2\beta}(4))
\] (150)
The balance of charge in this correlator is \(\sum \alpha = \alpha + \tilde{\alpha} - 2\beta\), \(\sum \alpha = \alpha + \tilde{\alpha} + 6\beta\). Comparing with the neutrality condition (143) we see that we have now to screen a net charge \((\alpha + \tilde{\alpha} - 2\beta, \alpha + \tilde{\alpha} + 2\beta)\). Let us therefore introduce the other even screening operator
\[
Q_{e,2} = \oint dz \partial \phi \ e^{-2i\tilde{\alpha}_+ \phi}
\] (151)
Recall the expressions
\[
G^+ =: -\sqrt{2}\xi \partial \phi + 2\sqrt{2i}\beta \partial \xi : \\
G^- =: -\sqrt{2}\eta \partial \tilde{\phi} + 2\sqrt{2i}\beta \partial \eta :
\] (152)
and for the \(U(1)\) current
\[
J =: -\eta \xi - 2i\beta \partial \tilde{\phi} + 2i\beta \partial \phi
\] (153)
One finds the following short distance expansions ($V$ is the vertex operator in the integrand of $Q_{e,2}$)

\[
G^{-}(z)V(w) = \sqrt{2}\partial_w \left( \frac{\eta e^{-2i\tilde{\alpha}_+\phi(w)}}{z-w} \right) + \text{reg. terms}
\]
\[
G^{+}(z)V(w) = 0 + \text{reg. terms}
\]
\[
J(z)V(w) = 2i\beta \partial_w \left( \frac{e^{-2i\tilde{\alpha}_+\phi(w)}}{z-w} \right)
\]
\[
T(z)V(w) = \partial_w \left( \frac{V(w)}{z-w} \right)
\]

(154)

Ensuring commutation of $Q_{e,2}$ with the twisted $N=2$ algebra. We emphasize that $Q_{e,2}$ is not a screening operator for the untwisted theory.

Operators in the Neveu Schwartz are represented by pure vertex operators. An interesting example is the two polymer legs operator that can be represented with

\[
\Phi_2 \leftrightarrow \left( -\frac{\alpha_+ + \alpha_-}{2}, \frac{\tilde{\alpha}_+ - \tilde{\alpha}_-}{2} \right)
\]

(155)

Its charge is $Q = Q_B = -1/3$. One has $\alpha + \tilde{\alpha} = -\alpha_-$. It therefore needs to be screened by the pair $Q_{o,1}, Q_{o,2}$ plus $Q_{e,2}$.

Operators in the Ramond sector are represented by the product of $\sigma_{1/2}$ and vertex operators. For the $h = -1/24$ operator that describes loops separating two punctures, as explained above, one has the representation

\[
\Phi_0 \leftrightarrow (-\alpha_+/2, \tilde{\alpha}_+/2)
\]

(156)

One has $\alpha + \tilde{\alpha} = 0$ such that screening by the operators $Q_{o,1}, Q_{o,2}$ is needed.

Finally operators in the $Z_4$ sector are represented by the product of a vertex operator by a $\sigma_{1/4}$ twist. For the one leg operator one finds

\[
\Phi_1 \leftrightarrow (\alpha_+/4 - \alpha_-/2, -\tilde{\alpha}_+/4 - \tilde{\alpha}_-/2)
\]

(157)

such that $\alpha + \tilde{\alpha} = -\alpha_-$ as for $\Phi_2$. We expect once again that the position of the cuts for the $\eta, \xi$ fields are in correspondence with the polymer lines.

Once these three basic four point functions are known, others can in principle be deduced by acting with the twisted $N = 2$ generators. For $L$ even they could also be obtained by the Dotsenko Fateev representation using only one bosonic field. For $L$ odd one needs the entire $N = 2$ free field representation.

We can finally consider the polymer operator algebra. Consider first the $L = 4l + 2$ operators that belong to the Neveu Schwartz sector. Upon fusion they must give rise to operators in NS again. This is confirmed by the fusion rules of $\Phi_{2l+3,3}$ operators. One finds

\[
\Phi_{4l_1+2} \Phi_{4l_2+2} \propto \sum_{l=0}^{l_1+l_2+1} \Phi_{4l+2} + \text{other operators with integer dimensions}
\]

(158)
For the $L = 4l$ operators that belong to Ramond sector, short distance expansion must produce fields in NS. This is confirmed by the fusion rules of $\Phi_{2l+2,3}$. One finds

$$\Phi_{4l_1} \Phi_{4l_2} \propto \sum_{l=0}^{l_1+l_2} \Phi_{4l+2} + \text{other operators with integer dimensions}$$

Finally for the odd number of legs operators one has to use the $N = 2$ representation and one finds that the $Z_4$ sector, upon fusion, branches to both R and NS.

$$\Phi_{2l_1+1} \Phi_{2l_2+1} \propto \sum_{l=1}^{l_1+l_2} \Phi_{2l} + \text{other operators with integer dimensions}$$

The closure of the full operator algebra for the polymer theory involving NS, R, and the $Z_4$ sector as coded in the partition functions $Z^e, Z^o$ can be checked as in [21].

5 Percolation and Related Problems

5.1 Percolation

Recall that the percolation problem on say the square lattice is obtained by choosing randomly edges to be occupied with a probability $p$ and empty with probability $1 - p$. The partition function is therefore one, but connected sets of occupied bonds (clusters) possess non trivial geometrical properties. Formally percolation is recovered by taking the $Q \to 1$ limit of the $Q$ state Potts model. Contrary to the polymer problem, we do not have at hand an almost soluble low temperature system to identify the degrees of freedom of the theory. Let us therefore spend some time discussing what they could be. A first possibility is to turn again to the medial graph, and make a polygon decomposition of it with the same rules as in the spanning tree case. Instead of a dense polymer, one gets a gas of polymer loops, each counted with a weight one. One may think of reproducing this factor one by allowing $2N + 1$ bosons and $2N$ fermions to propagate along each of these loops. In the simplest case of $N = 1$ this provides a Lagrangian a la Parisi Sourlas with $Sl(2,1)$ symmetry [30]. Since $sl(2,1)$ is isomorphic to $osp(2,2)$ we are entitled, as in the polymer case, to expect a continuum limit described by twisted $N = 2$ supersymmetry. However this introduction of fermions cannot be entirely correct. In particular, choosing antiperiodic boundary conditions for the fermions, and formally taking $N = 1/2$ as in polymers, would lead to weights 3 for non contractible loops, which cannot be obtained in the Coulomb gas description with real charges. Another alternative is to consider a field theory similar to the one of Isaacson and Lubensky [29] where the propagators draw directly the lattice clusters. In that case changing the fermions boundary conditions would give a weight 3 to the non contractible clusters, ie a weight $\sqrt{3}$ to the non contractible boundaries. This can be described in the lattice Coulomb Gas, and this is the route we shall follow.

Consider therefore the percolation problem and its medial polygons on a rectangle of the square lattice, and introduce as in the polymer case the partition functions

$$Z^{1}_{++} = \{\text{Sum over dense coverings of the medial lattice with a gas of loops}\}$$
each having weight one and an even number of non contractible loops\} \quad (161)

\[ Z^1_{++(\text{resp.}+,--)} = \{ \text{Same as above, but non contractible loops that cross } \omega_1 (\text{resp.} \omega_2, \text{resp.} (\omega_1, \omega_2) \text{ an odd number of times having a weight } 3 \} \quad (162) \]

One finds, based on the lattice Coulomb Gas analysis,

\[ Z^1_{++} = \sum_{mm' \in \mathbb{Z}} (e^{2i\pi/3})^{m \wedge m'} Z_{mm'} \]

\[ Z^1_{+-} = \sum_{mm' \in \mathbb{Z}} (-)^m (e^{2i\pi/3})^{m \wedge m'} Z_{mm'} \]

\[ Z^1_{-+} = \sum_{mm' \in \mathbb{Z}} (-)^{m'} (e^{2i\pi/3})^{m \wedge m'} Z_{mm'} \]

\[ Z^1_{--} = \sum_{mm' \in \mathbb{Z}} (-)^{m+m'} (e^{2i\pi/3})^{m \wedge m'} Z_{mm'} \quad (163) \]

After Poisson resummation one gets

\[ Z^1 = Z_c[2/3, 6] - Z_c[2/3, 2] \quad (164) \]

Notice that Jacobi identity is conveniently written in this case

\[ Z_c[2/3, 3] - Z_c[2/3, 1] = 2 \quad (165) \]

As was explained in 3 the \( Q \) state Potts model partition function contains in addition a sector where only \( Q - 1 \) colors are allowed on clusters with "cross topology". It is therefore natural to consider also expressions similar to the polymer case, where non contractible loops have weight zero or two depending on the way they intersect the axis and the boundary conditions. One finds

\[ Z^2 = Z_c[2/3, 2] \quad (166) \]

Summing both pieces gives rise to the result

\[ Z = Z^1 + Z^2 = Z_c[2/3, 6] = Z_c[3/2, 4] \quad (167) \]

This result is our proposal for the partition function of the percolation problem in the context of twisted \( N = 2 \) supersymmetry. As noticed in subsection 4.12, it is a diagonal twisted \( N = 2 \) modular invariant involving the Neveu Schwartz, Ramond and \( Z_4 \) sectors. The same analysis of the operator content as in the dilute polymer case can be carried out. In the percolation problem, a first set of interesting dimensions is provided by the analysis of boundaries of clusters. The exponents for a \( L \) hulls operator (see figure 12) are given by 32

\[ h_L = \frac{4L^2 - 1}{24} \quad (168) \]

33
For $L$ even they belong to the Ramond sector. For $L$ odd they belong to the Neveu Schwartz. Another set of interesting exponents are the thermal operators (without obvious general geometrical meaning), whose dimensions are

$$h_{Tn} = \frac{(3n + 1)^2 - 1}{24}$$

They belong to Ramond for $n$ odd and Neveu Schwartz for $n$ even. Finally one has the magnetic operators with dimensions

$$h_{Hn} = \frac{(6n - 3)^2 - 4}{96}$$

that belong to the $Z_4$ sector (see appendix A for more details on percolation).

5.2 Polymers at the Theta Point

Recall that polymers at the theta point are obtained by considering an attraction between monomers in addition to the self avoiding constraint. At a certain temperature, these two competing interactions reach some equilibrium, and the polymer is not as stretched as in the dilute case, not as compact as in the dense case. As was argued in [32] the statistics of such a polymer is the same as the one of percolation perimeters. Introduce therefore, by analogy with the dense and dilute cases, the partition functions $Z^e, Z^o$. One finds the continuum limits

$$Z^e = \frac{1}{2} Z_c[2/3, 1], Z^{e+o} = Z_c[2/3, 1/2] = Z_c[3/2, 2]$$

In that case only the Neveu Schwartz and Ramond sectors appear. The dimensions of the $L$ legs operators are given by

$$h_L = \frac{L^2 - 1}{24}$$

They belong to R for $L$ even, NS for $L$ odd. Notice that in this partition function the ground state is twice degenerate, since the one leg polymer operator has weight $h_1 = 0$. This is explained more naturally in the other version of the problem provided by the Ising model. Consider indeed an Ising model and its high temperature expansion. If we make analytic continuation to the low temperature phase in a finite system, then take the thermodynamic limit, we get again a model of dense contours with weight one. The one leg polymer operator becomes now the spin operator, and its vanishing dimension occurs because of the finite spontaneous magnetization. We shall reconsider the problem of the theta point and flow between various multicritical polymer points in [34].

5.3 Brownian Walks

Results on brownian motion are not very well established. Although the exponents $\nu = 1/2$ (associated with a field $h = 0$) and $\gamma = 1$ (associated with another $h = 0$) are easy to establish, the following exponents are non trivial. Consider $M$ brownian walks connecting two points, that cannot
intersect each other. This defines a $M$ legs operator whose dimension is known from numerical analysis [33] (figure 13)

$$h_M = \frac{4M^2 - 1}{24}$$  \hfill (173)

One notices that this exponent is the same as the one of the $L = 2M$ legs operator for tricritical polymers, or as well $M$ percolation hulls. This suggests that the interesting geometrical objects are boundaries of brownian walks, whose statistics would be the same as the one of polymers at the theta point, or as well percolation hulls. The reason for this is unknown, but the result fits again in the twisted $N = 2$ framework. Moreover by choosing the partition function of the doubly periodic sector to be the appropriate integer, the vanishing weights occurring from the values of $\nu$ and $\gamma$ can also be included in the theory.

6 Conclusion

In conclusion we would like first to comment on the "topological" nature of the geometrical models we have considered [33]. Let us start with dense polymers. In that case we could use the fermionic operator $G^+ = \eta$ to define a BRS charge in the periodic sector

$$Q_{BRS} = \oint \frac{dz}{2\pi} \eta(z)$$  \hfill (174)

It clearly satisfies $Q_{BRS}^2 = 0$. Moreover if we introduce $G^- = :\eta \xi \partial \xi :$, one has

$$\{Q_{BRS}, G^-\} = T$$  \hfill (175)

such that the stress tensor is itself a BRS commutator. In that case, what is usually considered the "physical" theory is trivial, ie $\text{Ker}Q_{BRS}/\text{Im}Q_{BRS}$ is empty. The partition function for this subset of states vanishes. Actually as is readily seen from the $XX$ chain formulation, $Q_{BRS}$ is the continuum limit of the operator that creates polymers (that its square vanishes occurs because two polymers cannot be created at the same place). Restricting to $\text{Ker}Q_{BRS}/\text{Im}Q_{BRS}$ means in lattice terms restricting to a sector without polymers, with vanishing partition function $Z_{PP} = 0$. Similarly if we consider dilute polymers in the periodic sector and the charge

$$Q_{BRS} = \oint G^+$$  \hfill (176)

one has, as is well known [33]

$$\{Q_{BRS}, G^-\} = T$$  \hfill (177)

and the "physical" theory contains only the identity. In that case again, it is not difficult to show how $Q_{BRS}$ is the continuum limit of the operator that creates polymers. Restricting to $\text{Ker}Q_{BRS}/\text{Im}Q_{BRS}$ leaves only states without polymers, with partition function $Z_{PP} = 1$ in agreement with $\text{Ker}Q_{BRS}/\text{Im}Q_{BRS} = \text{Id}$ in the conformal theory.

We therefore conclude that for statistical mechanics, physical states should not be identified with the $Q_{BRS}$ cohomology.
The main points of this paper, besides providing a practical tool to study the entire geometrical theories, are the connection between Parisi Sourlas supersymmetry and twisted $N = 2$, and the surprising unification of the standard geometrical models allowed by twisted $N = 2$. All these models are described by $k = 1$, except the dense polymers where the symmetry is broken to an $η, ξ$ system. Dilute polymers and percolation both involve NS, R and a $Z_4$ sector. One corresponds to a diagonal, the other to a non diagonal modular invariant. Tricritical polymers, contours in the low temperature Ising model, and to a certain extent brownian motion, involve only NS and R. Supersymmetry leads to non trivial predictions as well. As an example we give in appendix A a discussion of the backbone of percolation, together with the first conjecture of the exact value of its fractal dimension. A natural extension of this work is the study of higher values of $k$ in connection with multicritical geometrical problems. Another would be to try to describe more complicated systems with Parisi Sourlas Supersymmetry in the framework of twisted $N = 2$. Finally it is also mysterious how the Landau Ginzburg description of $N = 2$ theories can be applied directly to geometrical problems.

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Appendix A: Percolation Backbone

The purpose of this appendix is to demonstrate that, besides its unifying power, the recognition of $N = 2$ supersymmetry in two dimensional geometrical models leads to new predictions as well. We consider here the percolation problem. When an infinite cluster is formed at the percolation threshold, not all its pieces can contribute to say electricity transport. One makes the distinction between the useful part of the infinite cluster, the **backbone**, and the dangling ends. An important quantity is then the fractal dimension of the backbone, which is a measure of the amount of useful edges, and is smaller than the fractal dimension of the infinite cluster (the latter equals $91/48$ in two dimensions). A nice theoretical way of studying the backbone is to consider operators whose correlation function is defined as the sum over all clusters doubly connecting them. Let us call in general, much like the $L$ legs operators for polymers, $\Psi_L$ the $L$ connectedness operator. One has obviously

$$D = 2 - 2h_{L=2}$$

(178)

So far no exact prediction for $h_L$ based on lattice Coulomb gas mapping or conformal invariance had been made. Besides the difficulty of formulating the multiple connectedness locally in lattice variables, the too vague status of the conformal theory description did not allow a search among the possible exponents. However if we now assume that all static observables for percolation are described by a twisted $N = 2$ theory, the choice becomes quite small (recall that modular invariance constraint requires the sectors to have $\eta$ a rational number). Based on the order of magnitude of $D$ the only choice seems to be

$$h_2 = \frac{\eta(\eta + 1)}{6}, \quad \eta = \frac{3}{4}$$

(179)

ie

$$h_2 = \frac{21}{96}$$

(180)

This leads to

$$D = \frac{25}{16}$$

(181)

in good agreement with numerical computations ($D \in 1.55-1.62$). More generally our conjecture for the $L$ connectedness exponents is

$$h_L = \frac{\eta(\eta + 1)}{6}, \quad \eta = \frac{2L - 1}{4}$$

(182)

It reproduces the correct results for $L = 1, 2$. It also corresponds to a field in the $Z_4$ sector, as expected since the multiple connectedness operators all sit in the infinite cluster sector. Introducing the exponent $\beta = 2 \nu h$ with $\nu = 4/3$ one finds

$$\beta_L = \frac{(2L - 1)(2L + 3)}{36} = L\beta_1 + \nu \psi_L$$

(183)

with

$$\psi_L = \frac{(L - 1)(L + 3/4)}{12}$$

(184)
The result of $\epsilon$ expansion of [35] extended to two dimensions is $\psi_L = 16L(L - 1)/49$. Like in the polymers case, it has a form quite similar to the conjectured exact one in two dimensions.

With a little optimism one may consider the numerical or experimental verification of our $D = 25/16$ prediction as a check of $N = 2$ theory and associated non renormalization theorems.
Appendix B: S Matrices For Polymers

The purpose of this appendix is to comment briefly on the use of the supersymmetric formulation of polymers to study their off critical behaviour (when perturbed by the thermal operator) and the corresponding S matrix. Indeed Zamolodchikov \[37\] has written an S matrix for the O(n) model involving n particles, with the following ansatz

\[
S_{i_1 i_2}^{j_1 j_2} = S_1(\theta)\delta_{i_1 i_2}\delta_{j_1 j_2} + S_2(\theta)\delta_{i_1 i_2}\delta_{j_1 j_2} \tag{185}
\]

The factorization condition turned out to be equivalent to the following single equation

\[
S_1(\theta)S_1(\theta + \theta')S_2(\theta') + S_2(\theta)S_1(\theta + \theta')S_1(\theta')
+ S_2(\theta)S_2(\theta + \theta')S_2(\theta') + nS_2(\theta)S_1(\theta + \theta')S_2(\theta')
+ nS_2(\theta)S_2(\theta + \theta')S_2(\theta') + nS_2(\theta)S_1(\theta + \theta')S_2(\theta')
= S_1(\theta)S_2(\theta + \theta')S_1(\theta') \tag{186}
\]

The solution of (186) in the limit \(n \to 0\) can be used to compute some form factors. However in this limit the S matrix does not strictly make sense since in the point of view of [37] it has no degrees of freedom left onto which acting. The above analysis suggests to give a sense to the matrix elements extracted from (186) by introducing fermions. Indeed let us consider a S matrix acting in a graded space with N bosons, M fermions, and the ansatz

\[
S_{i_1 i_2}^{j_1 j_2} = S_1(\theta)(-)^{p(i_1)p(i_2)}\delta_{i_1 i_2}\delta_{j_1 j_2} + S_2(\theta)\delta_{i_1 i_2}\delta_{j_1 j_2} \tag{187}
\]

where \(p\) is the fermion number. One checks easily that, due to the signs generated by application of \(S_{13}\) when the state in the second space is a boson, the graded factorization equation is satisfied if and only if the equation (186) holds for

\[
n = N - M \tag{188}
\]

Therefore the results of [37] can actually be used to build a polymer S matrix with as many bosons as fermions. It is likely that the present formulation should allow application of the thermodynamic Bethe ansatz to extract new interesting informations. We also would like to comment that the S matrix in [37] is algebraically equivalent to the sum of an identity operator and a Temperley Lieb operator with parameter \(\delta = n\). As far as the computation of form factors is concerned, one can use as well the other representation of the Temperley Lieb algebra provided by the 6 vertex model. This means in particular that for the polymers, another alternative to using bosons and fermions is to work with the Sine Gordon S matrix. This was also noticed in [38]. This matrix exhibits precisely \(gl(1, 1)\) symmetry.
Figure Captions

Figure 1: The geometrical operators $\Phi_L(z)$ are are defined by asking that $L$ legs of polymers emanate from $z$. The correlators $\langle \Phi_L(z)\Phi_L(z') \rangle$ are then obtained by summing over all configurations where the $L$ legs describe self avoiding mutually avoiding configurations starting form $z$ and ending in $z'$.

Figure 2: A polygon decomposition of the medial lattice $M$ is obtained by splitting each vertex in one of the two possible ways indicated.

Figure 3: A polygon decomposition of $M$ is dual to a subgraph of $G$

Figure 4: The graphical representation of the Temperley Lieb algebra relations is precisely realized in polymer systems.

Figure 5: Schematic representation of the subgraphs encountered for the determinant of the Laplacian with one marked edge (indicated by a wiggly line).

Figure 6: An example of graph with a frustrated edge and the resulting configurations made of one loop passing through the marked edge and dangling ends attached to it.

Figure 7: The various subgraphs associated with the determinants given in section 2.4

Figure 8: The geometrical interpretation of the Ramond sector ground state. The frustration line connecting $z$ and $z'$ corresponds to antiperiodic boundary conditions for fermions. Loops that intersect it an odd number of times (and hence circle around one of the extremities) get a factor 2 instead of 0. Since loops are dual to trees, they always occur in pairs.

Figure 9: The next geometrical excitation in the Ramond sector occurs when a tree, ie a pair of loops, pinches the extremities $z$ and $z'$. This defines the 4 legs operator. In general the Ramond sector contains the $L = 4l$ legs operators.

Figure 10: Depending on the position of the cuts for the $Z_4$ twist in the $\eta, \xi$ system, different configurations with a pair of polymers connecting four points can be obtained.

Figure 11: The fusion rules for operators with an even number of legs do not correspond to naive addition of the number of legs. If $\Phi_{2l_1}$ and $\Phi_{2l_2}$ are multiplied, one gets fields with a number of legs $L = 2l_1 + 2l_2 + 2$ modulo four, corresponding to additional “foldings” of some of the legs.

Figure 12: Schematic representation of a 2 hulls operator for the percolation problem.

Figure 13: A 2 legs operator for brownian motion, with same exponent as a 2 hulls operator in percolation.

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