Synchronization of Chaotic Oscillators due to Common Delay Time Modulation

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We have found a synchronization behavior between two identical chaotic systems when their delay times are modulated by a common irregular signal. This phenomenon is demonstrated both in two identical chaotic maps whose delay times are driven by a common chaotic or random signal and in two identical chaotic oscillators whose delay times are driven by a signal of another chaotic oscillator. We analyze the phenomenon by using the Lyapunov exponents and discuss it in relation with generalized synchronization.

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Synchronization in chaotic oscillators [1, 2, 3, 4], which is characterized by the loss of exponential instability or neutrality in the transverse direction due to the interaction, has given rise to much attention for its application to diverse disciplines of science such as biology [5], chemistry [6], and physics [7, 8, 9, 10]. And the extensive investigations have been taken to understand its underlying mechanism [1, 2, 3, 4]. Synchronization can be classified depending on the characteristics of coupled systems. Complete synchronization (CS) [4] is observed in identical systems while phase synchronization [7] and lag synchronization [8] is in slightly detuned systems. Recently the more general type of synchronization has been described in coupled systems with different dynamics which is called generalized synchronization (GS) [9, 10]. GS is characterized by the appearance of functional relationship between the master, \( y = G(y) \), and the slave systems, \( x = F(x, h(y)) \), where \( h(y) \) is the function that describes the coupling between the master and the slave. Equivalently, existence of the functional relationship implies that CS has been established between the slave and its replica, \( \dot{x} = F(x, h(y)) \) such that \( |x - x'| \to 0 \) as \( t \to \infty \). Generally, it is discussed that this type of synchronization phenomenon is also established even if the driving signal is noisy [11].

In a real situation, time delay is inevitable, since the propagation speed of the information signal is finite [12, 13, 14, 15, 16, 17, 18]. Since the Volterra’s predator-prey model [19], the delay time has been considered as various forms to incorporate the realistic effects, e.g., distributed, state-dependent, and time-dependent delay times. Up to now, the effects of these forms of delay in dynamical systems have been extensively studied in many fields of physics [12], biology [13], and economy [14]. Also it was reported that synchronized behavior is enhanced in neural systems by the discrete time delays [15]. Recently, the effect of delay time modulation on the characteristic of chaotic signal was reported [16]. In the report, the delayed system transits to a complex state not to simplify the chaotic attractor into low-dimensional manifold. In this regard, it is important to understand the effect of delay time modulation on the synchronization of chaotic oscillators, because it is one of the fundamental phenomena in dynamical systems.

In this paper, we report a synchronization phenomenon between two identical chaotic systems when their delay times are driven by a common irregular signal. We analyze the synchronous behaviors in two identical logistic maps by studying the conditional and maximal Lyapunov exponents, and demonstrate them in the two Rössler oscillators whose delay times are driven by the Lorenz oscillator.

Our system can be described as follows:

\[
\begin{align*}
\dot{y} &= f(y), \quad \text{master,} \\
\dot{x} &= g(x, x(t - \tau(y))), \quad \text{slave 1,} \\
\dot{x}' &= g(x', x'(t - \tau(y))), \quad \text{slave 2,}
\end{align*}
\]

where the signal \( y \) of the master system drives the two slaves. The model describes that two identical chaotic systems are influenced by the common delay time modulation (DTM) [22]. If the delay time is a constant, the two slaves become two independent time-delayed systems with fixed delay. To emphasize, when the delay times are modulated in time, what we observed is that the two slaves transit to the synchronization state above the threshold.

For simple example, we consider a system which consists of a logistic map for the master and two logistic maps for the slaves as follows:

\[
\begin{align*}
y_{n+1} &= 4y_n(1 - y_n), \quad \text{master,} \\
x_{n+1} &= \gamma x_n(\tau(1 - x_{n-\tau})) + \alpha x_{n-\tau}, \quad \text{slave 1,} \\
x'_{n+1} &= \gamma x'_n(\tau(1 - x'_{n-\tau})) + \alpha x'_{n-\tau}, \quad \text{slave 2,}
\end{align*}
\]

where \( x_n(\tau) = (1 - \alpha)x_n + \alpha x_{n-\tau} \) and \( \alpha \) is coupling strength. Here we take \( \tau = [\Lambda y_n] \) as the common delay time and \( \Lambda \) is scaling parameter of DTM. Here \( [\Lambda y_n] \) is the largest integer less than \( \Lambda y_n \) which is introduced to get the integer number for the iteration. Figure 1 shows temporal behaviors of two coupled logistic maps by common DTM. The modulated delay time as a function of time is presented in Fig. 1 (a) and the temporal
where ∆ we iterate the above equation with one master and two autonomous and has an unusual term of $K_n$, which describes the chaotic property of a system. In this case, we need one more replica of the master system with different initial condition such that: $\Delta X_{n+1} = J_n \Delta X_n + K_n$, where $J_n = (1 - \alpha) \gamma (1 - (\bar{x}_n(\tau) + \bar{x}'_n(\tau)))$ and $K_n = \alpha \gamma (1 - (\bar{x}_n(\tau) + \bar{x}'_n(\tau')))(x_{n-\tau} - x'_{n-\tau})$.

The conditional and maximal Lyapunov exponents are presented as functions of $(\alpha, \Lambda)$ in Fig. 2. The conditional Lyapunov exponent shows the synchronized regime (the gray region of Fig. 2 (a)) for the two slaves in the $(\alpha, \Lambda)$ space. In that regime the transverse variable $\Delta X_n$ converges to zero. That is the system becomes stable in the transverse direction, $\Delta X_n$ due to DTM. The maximal Lyapunov exponent which describes the chaotic property of the system is positive except in the narrow periodic regime (the dark gray region on the $(\alpha, \Lambda)$ plane of Fig. 2 (b)). Therefore one see synchronization of the two slave logistic maps in the regime where two slaves are chaotic.

The periodic regime corresponds to the imprint of the periodic behavior of slave systems when the delayed feedbacks are absent. However when the delayed feedback is turned on, system becomes to be chaotic. The slaves return to the independent logistic maps when $\alpha = 0$ or $\Lambda < 1$ (the results of Fig. 1 - 3 show the chaotic output of the slave systems and the wide synchronization regime depending on the modulation amplitude and the coupling strength, even though we took $\gamma = 3.5$. We also performed the same studies in other parameters $\gamma = 3.8, 3.9, 4.0$ and observed a synchronization regime).

By tracing the time series in the $(\alpha, \Lambda)$ space, we obtain the synchronization regime which we show in Fig. 3 (a). One can see that the synchronization regime in Fig. 3 (a) coincides with that of Fig. 2 (a). To confirm the numerical result of Fig. 3 (a), we redraw the conditional Lyapunov exponent as shown in Fig. 3 (b) when $\Lambda = 60$ and $\Lambda = 40$. Even when we replace the master system by a random signal $\xi_n$, we can observe the similar synchronization regime whose border is presented by a dashed line in Fig. 3 (a). One see that the synchronization is enhanced when the delay time modulation is noisy. This fact leads us to understand the synchronization phenomenon in the framework of the synchronization by a common signal which can be chaotic or noisy [11]. Specifically, in our system the driving common signal is fed into delay time implicitly, while in previous systems the driving signal is explicitly introduced [11].

To show the universal feature of this type of synchronization, we consider the Lorenz oscillator as master and the Rössler oscillators as slaves [11] as follows:

$$\epsilon \dot{p} = \sigma(q - p),$$

$$\epsilon \dot{q} = -pr + ap - q,$$
The slaves in the form of explicitly defined function $h$

two slaves are converged into a synchronization state,

on two slaves are quite different from the case of GS un-

effective forces acting

development between the master and the slave is not ex-

On the contrary, since the delay times of our slave sys-

In this model, the Lorenz oscillator plays a role of driving

above the threshold, the two slaves are syn-

In conclusion, we have investigated the synchroniza-

Regarding an experimental realization of our method,

Because the feedback signal is proportional to the value

where $\sigma = 10$, $a = 28$, $b = 8/3$ and $c = 0.15$, $d = 0.2$

where the delay time can be modulated by the vibrat-

In a chaotic state. As we analyzed in the logistic maps, we

It is worth discussing the relationship between this

by common DTM in chaotic maps and flows. And we

synchronization could be classified

In an electronic circuit, the delay time modulation can

by common DTM in chaotic maps and flows. And we

to round off error in simulation.

$\epsilon \dot{x} = pq - br$, \hspace{1cm} \text{master, (3)}$

$\dot{x} = y - z,$

$\dot{y} = \bar{x} + cy,$

$\dot{z} = d + z + \bar{x}$, \hspace{1cm} \text{slave, (4)}$

where $\lambda = 10$, $a = 28$, $b = 8/3$ and $c = 0.15$, $d = 0.2$

$\dot{x} = (1 - \alpha)x + \alpha x(t - \tau(p))$. Here $\epsilon$ is the time

In this model, the Lorenz oscillator plays a role of driving

Maximal Lyapunov exponent $\lambda_m$ as function of $(\Lambda, \alpha)$. The gray region (or green region)

shows the regime where the exponent is positive and the dark gray region (or the red region) shows the regime in which the exponent is negative. We added a small noise of order $10^{-8}$ to one of the slaves in order to avoid abrupt synchronization due to round off error in simulation.

$\lambda_c$ as function of $(\Lambda, \alpha)$; The gray region (or green region) indicates the regime in which

phenomenon extends the concept of GS, and that

maximal Lyapunov exponents. We expect that the ob-

demonstrated that the synchronization can be achieved

we can consider the laser system with optical feedback,

where the delay time can be modulated by the vibrat-

We have confirmed the observed synchronization could be classified

an extended type of GS.

Regarding an experimental realization of our method,

we can consider the laser system with optical feedback,

where the delay time can be modulated by the vibrat-

in an electronic circuit, the delay time modulation can

be implemented by using digital delay line or computer

In conclusion, we have investigated the synchroniza-

behavior between two chaotic systems whose delay
time is modulated by a common irregular signal. We have demonstrated that the synchronization can be achieved by common DTM in chaotic maps and flows. And we have clarified that a common DTM alters the stability of two chaotic oscillators and it leads the systems to be synchronized. We have confirmed the observed phenomenon through the analysis of the conditional and maximal Lyapunov exponents. We expect that the observed phenomenon extends the concept of GS, and that the introduced systems will be useful for understanding

FIG. 2: (a) Conditional Lyapunov exponent $\lambda_c$ as function of $(\Lambda, \alpha)$; The gray region (or green region) indicates the regime in which the conditional Lyapunov exponent is negative (i.e., the synchronization regime) and the white region shows the regime in which the exponent is positive. (b) Maximal Lyapunov exponent $\lambda_m$ as function of $(\Lambda, \alpha)$. The gray region (or green region) shows the regime where the exponent is positive and the dark gray region (or the red region) shows the regime in which the exponent is negative. We added a small noise of order $10^{-8}$ to one of the slaves in order to avoid abrupt synchronization due to round off error in simulation.
FIG. 3: (a) Synchronization regime determined by the time series. The gray region (or green region) show the synchronization regime which coincides with that of the contour plot of Fig. 2 (a). 'A' and 'B' indicate the reference points to present the time series in Fig. 1. The dashed lines are the border of synchronization when the driving signal is replaced by a random signal $\xi_n$. One sees that the synchronization regime is extended in this case. (b) The conditional Lyapunov exponents on the two reference lines of (a) as a function of $\alpha$. The upper line is at $\Lambda = 60$ and the below at $\Lambda = 40$.

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FIG. 4: Temporal behaviors of the Rössler oscillators of Eqs. (3) and (4). (a) $\tau(t)$ as a function of time when $\epsilon = 0.012, \beta = 0.0067$, and $\tau_0 = 0.9$; and the difference motion between two slaves when (b) the coupling strength $\alpha = 0.238$ and (c) $\alpha = 0.240$.

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