On the existence of turning points in D-dimensional Schwarzschild-de Sitter and anti-de Sitter spacetimes

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Abstract

We investigate the motion of a test particle in a d-dimensional, spherically symmetric and static space-time supported by a mass $M$ plus a $\Lambda$-term. The motion is strongly dependent on the sign of $\Lambda$. In Schwarzschild-de Sitter (SdS) space-time ($\Lambda > 0$), besides the physical singularity at $r = 0$ there are cases with two horizons and two turning points, one horizon and one turning point and the complete absence of horizon and turning points. For Schwarzschild-Anti de Sitter (SAdS) space-time ($\Lambda < 0$) the horizon coordinate is associated to a unique turning point.

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I. INTRODUCTION

The best theory for describing gravitational interaction is the classical general relativity, and the prediction of black holes is usually considered one of its major triumphs. Many observational aspects of black holes concern the properties of time-like and null geodesics. In particular, the study of time-like geodesics in spherically symmetric and static spacetimes (Schwarzschild and Reissner-Nordström (RN) solutions) has been discussed by several authors [1,2]. Such studies lead to a reasonable comprehension about the motion of a test particle near or inside the horizon of uncharged (charged) black holes.

On the other hand, Einstein introduced the $\Lambda$-term in 1917, and soon after de Sitter found a well behaved vacuum static spherically symmetric solution for the modified equations. The $\Lambda$-term alters considerably the solutions of the field equations. In Schwarzschild and RN solutions, for example, the resulting spacetimes have an extra event horizon. More recently, in the cosmological context, the $\Lambda$-term has been interpreted as a net vacuum energy density of all the existing quantum fields, and, as suggested by the recent observations, may be responsible by the present accelerating stage of the Universe [3].

In this work we analyze some properties of time-like geodesics in spherically static spacetimes. More precisely, we discuss the existence of turning points in the Schwarzschild spacetime with positive and negative $\Lambda$. Such spacetimes are usually referred to as Schwarzschild-de Sitter (SdS) and Schwarzschild-anti de Sitter (SAdS), and for the sake of generality we consider the d-dimensional extension of both cases.

The geometry of a d-dimensional spherically symmetric and static spacetime with a $\Lambda$-term reads

\[
\text{ds}^2 = f(r)c^2dt^2 - f^{-1}(r)dr^2 - r^2d\Omega_{d-2}^2
\]  

where the function $f(r)$ is given by

\[
f(r) = 1 - \frac{2m}{r^{d-3}} - \frac{\Lambda r^2}{3}.
\]
In the above expression, the constant $m$ is defined by the black hole mass $M$ ($m = 2MG/c^2$).

If $\Lambda$ is positive, the spacetime is asymptotically de Sitter in $d$-dimensions. In the limit $\Lambda \to 0$ the metric (1) goes to the Schwarzschild $d$-dimensional spacetime. In addition, if $r \to \infty$ we have the Minkowski flat manifold in $d$ dimensions ($d \geq 4$). The quantity $d\Omega_{d-2}^2$ is the standard metric for a $(d - 2)$-dimensional unit sphere: $d\Omega_{d-2}^2 = (d\theta_1)^2 + \sin^2 \theta_1 (d\theta_1)^2 + \ldots + \sin^2 \theta^1 + \ldots + \sin^2 \theta^{d-2} d(\theta^{d-2})^2$. For $d = 4$ it reduces to $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

II. SCHWARZSCHILD DE-SITTER SPACETIME

In this geometry, the cosmological constant is positive and can be written as $\Lambda = 3/a^2$ where $a$ is the “cosmological radius”. In order to discuss the existence of turning points we consider the classification scheme of possible horizons for a SdS spacetime as recently proposed by Molina [4]. The number of horizons are easily determined by the real roots of the function $f(r) = 1 - 2m/r^{d-3} - r^2/a^2$. Assuming $d \geq 4, a^2 > 0$ and $m > 0$ one has:

- The spacetime has two horizons if and only if the condition
  \[
  \frac{m^2}{a^{2(d-3)}} < \frac{(d - 3)^{d-3}}{(d - 1)^{d-1}}
  \]
  (3)

  is satisfied.

- The spacetime has one horizon if and only if the condition
  \[
  \frac{m^2}{a^{2(d-3)}} = \frac{(d - 3)^{d-3}}{(d - 1)^{d-1}}
  \]
  (4)

  is satisfied. This case is the extreme Schwarzschild de-Sitter black hole.

- The spacetime has no horizon if and only if the condition
  \[
  \frac{m^2}{a^{2(d-3)}} > \frac{(d - 3)^{d-3}}{(d - 1)^{d-1}}
  \]
  (5)

  is satisfied. Such spacetime has a naked singularity at $r = 0$.

The roots of $f(r)$ in the above quoted cases are known as pseudo-singularities because the metric (1) diverges when $f = 0$, although considering that all the scalars of curvature
are finite. Hence, as happens in the 4-dimensional case, the physical singularity is located at \( r = 0 \).

Now, since the turning points are determined by the condition \( \dot{r} = 0 \), we need to consider only the radial equation of motion \([5]\)

\[
\left( \frac{dr}{dt} \right)^2 = E^2 - f(r) \tag{6}
\]

where \( E \) is an integration constant. Inserting \( f(r) \) and the value of \( \Lambda \) one has

\[
\left( \frac{dr}{dt} \right)^2 = E^2 - 1 + \frac{2m}{r^{d-3}} + \frac{r^2}{a^2}. \tag{7}
\]

Let us now consider the condition \( \dot{r} = 0 \), or equivalently,

\[
g(r) = E^2 - 1 + \frac{2m}{r^{d-3}} + \frac{r^2}{a^2} = 0. \tag{8}
\]

Comparing the above expression with equation (2), we see that the only difference between \( g(r) \) and the function \( f(r) \) is the term \( E^2 \). This means that the existence of turning points can also be classified adopting the scheme applied to the horizons. In this case, assuming \( d \geq 4, a^2 > 0, m > 0 \) and \( E^2 < 1 \) the following statements are true:

- The function \( g(r) \) has two real and positive zeros if and only if the condition

  \[
  \frac{m^2}{a^{2(d-3)}} < \frac{(d-3)^{d-3}(1 - E^2)^{d-1}}{(d-1)^{d-1}} \tag{9}
  \]

  is satisfied.

- The function \( g(r) \) has one real and positive zero if and only if the condition

  \[
  \frac{m^2}{a^{2(d-3)}} = \frac{(d-3)^{d-3}(1 - E^2)^{d-1}}{(d-1)^{d-1}} \tag{10}
  \]

  is satisfied.

- The function \( g(r) \) has no real and positive zero if and only if the condition

  \[
  \frac{m^2}{a^{2(d-3)}} > \frac{(d-3)^{d-3}(1 - E^2)^{d-1}}{(d-1)^{d-1}} \tag{11}
  \]

  is satisfied.
These real and positive roots are the turning points because the radial component of the 3-velocity vanishes. Naturally, the case where \( E^2 > 1 \) should also be considered. However, as one may show, in this case there are no turning points regardless of the value of \( \Lambda \) (the roots of \( g(r) \) are complex or negative).

**III. SCHWARZSCHILD ANTI-DE SITTER SPACETIME**

Following the same procedure of the previous section one can study the turning points in the SAdS spacetime. The only difference here is that the cosmological constant must assume only negative values, thereby leading to a change in the basic propositions. Since the cosmological constant is negative, it is convenient to introduce a “pseudo radius” \( b > 0 \) such that \( \Lambda = -3/b^2 \). In terms of \( b \), the metric element \( f(r) \) can be written as:

\[
f(r) = 1 - \frac{2m}{r^{d-3}} + \frac{r^2}{b^2}.
\]  

(12)

Let us now consider the Molina [6] statement for a SAdS spacetime: Assuming \( m > 0 \), \( d > 4 \) and \( \Lambda < 0 \), one has:

- If \( d \) is pair or odd, the spacetime has just one horizon.

In principle, the function \( f(r) \) could have more than one real and positive root. However, the above proposition determines the existence of just one real and positive root which means that SAdS spacetime has only one event horizon regardless of the number of spatial dimensions.

On the other hand, the time-like geodesic equations for SAdS spacetime take the same form as the one for a SdS spacetime (see equations (6) and (7)). The unique difference appearing in \( f(r) \) is due to the negative value of the cosmological constant, as well as the constant \( E^2 \) which now can assume any positive value.

In this way, we can enunciate the following statement: Assuming \( m > 0 \), \( d > 4 \) and \( \Lambda < 0 \), one has:

- If \( d \) is pair or odd, the function \( g(r) \) has one real and positive root. This zero of \( g(r) \) defines a turning point.
IV. CONCLUSION

We have discussed some aspects of the time-like geodesics for d-dimensional S\(d\)S and S\(d\)\(\Lambda\)S spacetimes. In the first case (\(\Lambda > 0\)) there are many possibilities. Firstly, if \(E^2 > 1\) there are no turning points. But if \(E^2 < 1\), the number of turning points is equal to the number of horizons. For S\(d\)\(\Lambda\)S in d dimensions, the black hole manifold has only one horizon and one turning point. This result holds regardless of the value of \(E^2\). Therefore, for all these cases, the number of possible turning points (when they exist!) is just the same number of horizons.

It should be also interesting to extend these results for a d-dimensional charged black hole with \(\Lambda\). We recall that in the Reissner-Nordstöm spacetime (4-dimensions) there is a turning point inside the internal radius (see, for instance, [5] and [7]). In general, one expects a number of horizons different from the number of turning points.

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