High-dimensional nonconvex LASSO-type $M$-estimators

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April 14, 2022

Abstract

This paper proposes a theory for $\ell_1$-norm penalized high-dimensional $M$-estimators, with nonconvex risk and unrestricted domain. Under high-level conditions, the estimators are shown to attain the rate of convergence $s_0 \sqrt{\log(nd)/n}$, where $s_0$ is the number of nonzero coefficients of the parameter of interest. Sufficient conditions for our main assumptions are then developed and finally used in several examples including robust linear regression, binary classification and nonlinear least squares.

MSC 2020 subject classifications: Primary 62F12; 62J02.

Key Words: Lasso; High-dimensional regime; Nonconvexity; Unconstrained estimation.
1 Introduction

Consider the standard statistical problem of estimating a parameter $\theta_0 \in \Theta \subset \mathbb{R}^d$, $d \geq 1$, defined as minimizing the unknown true risk $R : \Theta \to \mathbb{R}_\geq 0$ which is estimated by the empirical risk $\hat{R} : \Theta \to \mathbb{R}_\geq 0$ that depends on a random sample of size $n \geq 1$. The set $\Theta$ is called the parameter space and assumed to be convex (for simplicity). Motivated by large scale learning applications, focus is on the high-dimensional case in which the number $d$ of parameters is large relative to the sample size $n$. To account for this situation, the asymptotic regime that shall be considered throughout the paper is the one of high-dimensional statistics given by

$$n \to \infty \quad \text{and} \quad d := d_n \to \infty,$$

in which case, standard approaches, that directly minimize the empirical risk, are known to be inconsistent. The quantities $\theta_0$, $\Theta$, $R$ and $\hat{R}$ implicitly depend on $n$, but we avoid to index them by $n$ to simplify the exposition. Reference textbooks dedicated to high-dimensional statistics include Bühlmann and Van De Geer (2011), Giraud (2015), Hastie et al. (2015).

A leading approach, that will be followed in this paper, is to regularize the empirical risk by the $\ell_1$-norm of the parameters vector. In that, the estimate of $\theta_0$ is given by

$$\hat{\theta} \in \arg \min_{\theta \in \Theta} \{\hat{R}(\theta) + \lambda_n |\theta|_1\}, \quad (1)$$

where $\lambda_n > 0$ is a penalty level that shall be chosen with respect to $n$.

Such a penalization approach, also referred to as the lasso, has been successful in many cases such as linear regression (Tibshirani, 1996; Bickel et al., 2009), logistic regression (Meier et al., 2008) and Cox regression (Tibshirani, 1997; Bradic et al., 2011; Huang et al., 2013; Kong and Nan, 2014). In presence of a sparsity structure for $\theta_0$, i.e., when the number of nonzero coordinates $s_0 := s_{0n}$ of $\theta_0$ is small, the previous papers show that the lasso method is reliable even in the challenging regime $s_0 \sqrt{\log(nd)/n} \to 0$. More specifically, results from the literature claim that the error $|\hat{\theta} - \theta_0|_1$ is of order $s_0 \sqrt{\log(nd)/n}$. While the results obtained for the three previous flagship examples, namely the linear, the logistic and the Cox regression, are strong evidence of lasso’s success, they all are developed for specific risk functions that are convex. Extending the results to - still convex - but more general risk functions is the subject of recent work such as (Van de Geer, 2008; Negahban et al., 2012). In both cases the true risk is globally (everywhere) convex and locally strongly convex. Their main differences arise because Van de Geer (2008) assumes that the true risk is strongly convex on some neighborhood of the true parameter, while Negahban et al. (2012) assumes that the empirical risk is strongly convex on a cone of approximately sparse vectors. The latter assumption is called restricted strong convexity.

More recently, the lasso has been shown to be powerful in several cases with non-convex risk functions, e.g. Yang et al. (2016) for nonlinear least squares, Städler et al. (2010) for mixture regression models, Loh (2017) for robust linear regression estimators and Genetay et al. (2021) for clustering. Even though these studies are carried out for specific estimates, only local convexity on a small $\ell_2$-ball around the true parameter is needed. Note that Städler et al. (2010) actually provides an oracle inequality for maximum likelihood estimators but does not obtain any rate of convergence on the estimation error. Several papers
Wang et al. (2014); Loh and Wainwright (2015); Mei et al. (2018) propose high-level theories for nonconvex regularized high-dimensional $M$-estimators. In contrast to Yang et al. (2016); Loh (2017) (on specific applications), they require the strong convexity of the empirical risk on some sparse directions Wang et al. (2014) or on a cone Loh and Wainwright (2015). The empirical gradient and Hessian’s behavior is investigated in Mei et al. (2018) but, concerning the asymptotic regime $s_0 \sqrt{\log(nd)/n} \to 0$, no high-level result on the convergence of the estimator is given.

The present paper establishes rates of convergence on $|\hat{\theta} - \theta_0|$ in the challenging regime $s_0 \sqrt{\log(nd)/n} \to 0$ without restrictive convexity assumptions. The contributions can be summarized as follows:

(i) (generality) The proposed results are valid under a fairly general setting in that the risk function is not convex but only locally strongly convex in an $\ell_2$-ball around the true parameter.

(ii) (interpretability and applicability) The results bear resemblance with well-known (low-dimensional) $M$-estimation theory Newey and McFadden, 1994; Van der Vaart, 2000; Geer et al., 2000 and can therefore be easily interpreted. We develop sufficient conditions for our high-level assumptions in order to simplify the application of the results.

(iii) (unrestricted domain) The proposed results do not require any restrictions on the parameter space.

As a secondary contribution, we apply our results to several examples including robust regression, binary regression, and nonlinear least squares. In each examples, the high-level results are easy to apply and the parameter space is $\Theta = \mathbb{R}^d$ illustrating the previous claims.

The fact that we allow the parameter space to be unrestricted may be surprising since the domain is restricted in Städler et al. (2010); Wang et al. (2014); Loh and Wainwright (2015); Loh (2017); Mei et al. (2018). This novel property is obtained through to a two-step technical argument. First, thanks to the penalization, we show that, regardless of $\Theta$ and with probability going to 1, $\hat{\theta}$ belongs to an $\ell_1$-ball $B$ with center $\theta_0$ and radius of order $\lambda^{-1} + |\theta_0|$. Second, the consistency of $\hat{\theta}$ is obtained under an identification assumption on $R$ and a uniform convergence condition of $\hat{R}$ on $B$. Because the radius of $B$ grows to infinity sufficiently slowly, the uniform convergence can be obtained in the applications of interest.

Note that a related but different problem is the one of computing the estimator (II). Gradient descents algorithms usually converge to local minima of the objective function. Hence, $\hat{\theta}$ may not be computable in practice. In the present paper, we do not consider this issue. Remark however that several papers Wang et al. (2014); Loh and Wainwright (2015); Yang et al. (2016); Loh (2017); Mei et al. (2018) treat both the optimization and statistical problems together by investigating the behavior of local minima of the function $\theta \in \Theta \mapsto \hat{R}(\theta) + \lambda_n |\theta|_1$.

Another related line of work studies lasso-type estimators in the low-dimensional context where $d$ is fixed. In this setting, lasso-type estimators can be used for variable selection. For instance, Fu and Knight (2000) develops an asymptotic theory for linear models in this framework and Wang et al. (2013) derives oracle properties in the more general case of possibly nonconvex semiparametric $M$-estimators.
Outline. In Section 2, we present the high-level results. Then, sufficient conditions for our high-level assumptions are stated in Section 3. Next, we apply the results to three examples in Section 4. Section 5 concludes the main text by discussing further research directions. The proofs of the high-level results and their sufficient conditions are in the Appendix. The results regarding the applications are proved in the supplement.

Notations. The notations \(| \cdot |_1\), \(| \cdot |_2\) and \(| \cdot |_\infty\) correspond to the \(\ell_1\), \(\ell_2\) and sup norms, respectively. For a twice differentiable function \(F : \mathbb{R}^K \rightarrow \mathbb{R}\), \(\nabla F\) is its gradient and \(\nabla^2 F\) its Hessian.

2 High-level results

Let \(\hat{R} : \Theta \rightarrow \mathbb{R}_{\geq 0}\) be a random function and \(R : \Theta \rightarrow \mathbb{R}_{\geq 0}\) be a function. Consider \(\hat{\theta}\) (resp. \(\theta_0\)) defined as a minimizer of \(\hat{R}(\theta) + \lambda_n|\theta|_1\) (resp. \(R\)) over \(\Theta\). In this section, the aim is to provide conditions on \(\hat{R}\), \(R\) and \(\Theta\) to ensure certain convergence properties of \(\hat{\theta}\) toward \(\theta_0\).

2.1 Reduction of the parameter space

Thanks to the penalty term, we can show that, with probability going to 1, \(\hat{\theta}\) belongs to an \(\ell_1\)-ball \(B\) defined as

\[ B = \{ \theta \in \Theta : |\theta|_1 \leq \lambda_n^{-1}(R(\theta_0) + 1) + |\theta_0|_1 \}. \]  

This is formally claimed in the following proposition.

Proposition 1. We have that \(\theta_0 \in B\) and if \(\hat{R}(\theta_0) \rightarrow R(\theta_0)\), in probability, then \(\hat{\theta} \in B\) with probability going to 1.

This result is important because it allows in the mathematical development to restrict the attention to a smaller set \(B\) included in the parameter set \(\Theta\). The set \(B\) has finite diameter (although its diameter can grow with \(n\)) while \(\Theta\) could have infinite width. Hence, assumptions on the behaviour of \(\hat{R}\) are less demanding when restricted to \(B\). This fact will be of good help when dealing with the applications.

2.2 Consistency

To obtain consistency, we make the following assumptions.

Assumption 1. For all \(\eta > 0\), there exists \(\epsilon > 0\) such that, for all \(n \geq 1\),

\[ \inf_{\theta \in \Theta, |\theta - \theta_0|_1 \geq \eta} \{ R(\theta) - R(\theta_0) \} \geq \epsilon. \]

This is an identification assumption restricting the shape of the true risk function. When \(n\) is fixed, this condition holds if the risk is continuous and \(\theta_0\) is its unique maximizer (the standard identification assumption in the literature of low-dimensional \(M\)-estimators).
specificity of the high-dimensional context is that we require this condition to be satisfied uniformly in $n$. In view of Proposition 1, this assumption could be weakened by replacing $\Theta$ by $B$ but this does not bring much simplification because the set $B$ is intended to grow to $\Theta$ whenever $n$ is getting large. It is also possible to relax Assumption 1 by letting $\epsilon$ go to 0 with $n$. This could however prevent consistency if $\epsilon$ were to go to 0 too quickly. This has not been further investigated since Assumption 1 is valid in the applications considered in Section 4.

The second assumption ensures that the empirical risk converges uniformly to the true risk on $B$.

**Assumption 2.** $\sup_{\theta \in B} \left| \hat{R}(\theta) - R(\theta) \right| = o_P(1)$.

In the low-dimensional context, a similar condition is usually required on a compact set which does not depend on $n$. The main difference in the present context is that the radius of $B$ grows with $n$.

The following theorem states that the estimator is consistent in $\ell_2$-norm.

**Theorem 1.** Under Assumptions 1 and 2, if $\lambda_n |\theta_0|_1 \to 0$, we have $|\hat{\theta} - \theta_0|_2 = o_P(1)$.

The result relies on the additional condition $\lambda_n |\theta_0|_1 \to 0$, which, roughly speaking, means the added penalty term has only a negligible effect on the objective function evaluated at $\theta_0$.

### 2.3 Rate of convergence

The following conditions are required to obtain a bound on the convergence rate of $\hat{\theta}$ toward $\theta_0$.

**Assumption 3.** There exist constants $\rho_*, \eta_* > 0$ such that for all $n \geq 1$ and $\theta \in B$, $|\theta - \theta_0|_2 \leq \eta_*$

$$R(\theta) - R(\theta_0) \geq \frac{\rho_*}{2} |\theta - \theta_0|_2^2.$$  

This is a local strong convexity assumption also imposed in the literature on nonconvex low-dimensional $M$-estimators. We stress that this condition is only imposed on an $\ell_2$-ball with radius fixed with $n$ (although the $\ell_2$-ball itself can change with $n$ since $\Theta$ and $\theta_0$ depends on $n$). This condition does not require global convexity. Let $\mathcal{V} = \{\theta \in B, |\theta - \theta_0|_2 \leq \eta_*\}$. A sufficient condition to obtain the previous assumption is to ask that $R$ is twice differentiable, $\theta_0$ is an interior point of $\Theta$, and the following eigenvalue property that for all $n \geq 1$,

$$\inf_{\theta \in \mathcal{V}} \rho_{\min}(\nabla^2 R(\theta)) \geq \rho_*,$$

where $\rho_{\min}(\cdot)$ is the minimal eigenvalue. Indeed, as $\nabla R(\theta_0) = 0$, by the second-order mean-value theorem, for all $n \geq 1$ and $\theta \in \mathcal{V}$, there exists $\tilde{\theta} \in \mathcal{V}$ such that

$$R(\theta) - R(\theta_0) = (\theta - \theta_0)^\top \frac{\nabla^2 R(\tilde{\theta})}{2} (\theta - \theta_0).$$
The last of our high-level conditions considers the difference between the empirical and the true risk

\[ \hat{\Delta}(\theta) = \hat{R}(\theta) - R(\theta), \]

and requires a certain convergence rate, \( r_n \), for its increments.

**Assumption 4.** There exist positive sequences \((r_n)_{n \geq 1}\) and \((\delta_n)_{n \geq 1}\) such that

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{\theta \in \mathcal{V}} \left| \hat{\Delta}(\theta) - \hat{\Delta}(\theta_0) \right| \leq r_n \right) = 1
\]

A similar condition is also imposed in Städler et al. (2010) for maximum likelihood estimators. In applications, \( r_n \) and \( \delta_n \) are typically of order \( \sqrt{\log(nd)/n} \) and \( \sqrt{\log(d)/n} \), respectively.

When the risk function is differentiable, the previous condition holds true as soon as the gradient satisfies \( \sup_{\theta \in \mathcal{V}} |\nabla \hat{\Delta}(\theta)|_{\infty} \leq r_n \). Indeed in virtue of the mean value theorem, there exists \( \bar{\theta} \in \mathcal{V} \) such that

\[ \left| \hat{\Delta}(\theta) - \hat{\Delta}(\theta_0) \right| = \left| \nabla \hat{\Delta}(\bar{\theta})^T (\theta - \theta_0) \right| \leq r_n \|\theta - \theta_0\|_1. \]

As a result, Assumption 4 cares about the closeness (expressed through \( r_n \)) between the derivatives of the empirical risk and the ones of the true risk.

Remark also that the condition in Assumption 4 depends on \( \mathcal{V} \) which is itself defined through Assumption 3. However, since \( \mathcal{V} \subset B \), a stronger version of Assumption 4 simply assumes

\[
\mathbb{P} \left( \sup_{\theta \in B} \left| \hat{\Delta}(\theta) - \hat{\Delta}(\theta_0) \right| \leq r_n \right) \to 1,
\]

where we stress that the supremum is taken on \( B \) rather than on \( \mathcal{V} \). In Proposition 3 (see Section 3), we provide sufficient conditions for the stronger result (3). The proof of Proposition 3 leverages empirical process theory. It avoids using the differentiability of the risk as outlined before.

Recall that \( s_0 \) is the number of non-zero coordinates of \( \theta_0 \). We have the following Theorem.

**Theorem 2.** Under Assumption 1, 2, 3, 4, if \( \lambda_n \|\theta_0\|_1 \to 0 \) and \( \lambda_n \geq 2r_n \), with probability going to 1, we have

\[ \|\hat{\theta} - \theta_0\|_1 \leq \left( \frac{24}{\rho^*} s_0 r_n \right) \lor \delta_n. \]

Since in the applications, \( r_n \) and \( \delta_n \) are of order \( \sqrt{\log(nd)/n} \) and \( \sqrt{\log(d)/n} \), respectively, Theorem 2 gives us a rate of convergence of order \( s_0 \sqrt{\log(nd)/n} \), which is standard in high-dimensional statistics.
3 Sufficient conditions

In this section, we develop sufficient conditions for our high-level assumptions. They are leveraged to illustrate our theory with applications in Section 4.

3.1 Conditions on the true risk

Two conditions are dealing with the function $R$, namely Conditions 1 and 3. We here provide sufficient conditions, (i) and (ii) below, on the gradient $\nabla R$, under which Conditions 1 and 3 are valid. They are based on the following proposition.

**Proposition 2.** Let $R : \Theta \to \mathbb{R}$ be differentiable and such that

(i) For all $\theta \in \Theta$, we have $\nabla R(\theta)^\top(\theta - \theta_0) \geq 0$.

(ii) For all $\gamma > 0$, there exists $c(\gamma) > 0$, decreasing in $\gamma$, such that, for all $n \geq 1$,

$$\inf_{\theta \in \Theta : |\theta - \theta_0|_2 \leq \gamma} \frac{\nabla R(\theta)^\top(\theta - \theta_0)}{|\theta - \theta_0|_2^2} \geq c(\gamma).$$

then for all $\theta \in \Theta$ and $\eta > 0$ such that $|\theta - \theta_0|_2 \geq \eta$,

$$R(\theta) - R(\theta_0) \geq c(\eta) \frac{\eta^2}{2}.$$  \hspace{1cm} (4)

This implies also that Assumptions 1 and 3 hold.

Thanks to this proposition, only working on the function $\theta \mapsto \nabla R(\theta)^\top(\theta - \theta_0)$ is enough to obtain Assumptions 1 and 3.

3.2 Conditions on the empirical risk

Consider the standard regression setup where the goal is to predict $Y$, the response variable, with support $Y \subset \mathbb{R}$, based on a random vector $X$ with support $X \subset \mathbb{R}^d$. Let us use the notation

$$\mathcal{T} = \{ x^\top \theta : x \in X, \theta \in \Theta \}.$$

Interest is devoted to single index types of risk defined as

$$R(\theta) = E \left[ \ell(X^\top \theta, Y) \right],$$  \hspace{1cm} (5)

for all $\theta \in \Theta$, where $\ell : (s,y) \in \mathcal{T} \times \mathcal{Y} \mapsto \mathbb{R}$.

Let $\{(Y_i, X_i)\}_{i=1}^n$ be an independent and identically distributed (i.i.d.) collection of random variables distributed as $(Y, X)$. The estimate of $R$ is defined as

$$\hat{R}(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(X_i^\top \theta, Y_i).$$  \hspace{1cm} (6)

We make the following assumption.
Assumption 5.

(i) There exists a constant $M_X > 0$ such that $|x|_\infty \leq M_X$ for all $x \in \mathcal{X}$.

(ii) There exists a constant $M_\ell > 0$ such that

$$
\sup_{t, y \in \mathcal{Y}} |\ell(t, y)| \leq M_\ell.
$$

(iii) There exists a constant $L > 0$ such that

$$
\sup_{t, t', y \in \mathcal{Y}} \frac{|\ell(t, y) - \ell(t', y)|}{|t - t'|} \leq L.
$$

The first two conditions stipulate that the features and the loss are bounded. The third condition imposes that the loss $\ell$ is Lipschitz with respect to its first argument uniformly in its second argument.

Let $|B|_1$ be the $\ell_1$-diameter of the set $B$ defined in (2), that is

$$
|B|_1 = \sup_{\theta \in B} |\theta|_1.
$$

We have the following proposition.

**Proposition 3.** Let $R$ and $\hat{R}$ be defined as in (5) and (6). Under Assumption 4, if

$$
\log(d)|B|_1^2 n^{-1} \rightarrow 0,
$$

then Assumptions 4 holds and property (3) is satisfied with

$$
\delta_n = \sqrt{\frac{\log(2d)}{n}}, \ r_n = 16LM_X \sqrt{\log(4nd)/n}.
$$

The fact that (3) is satisfied directly implies that Assumption 4 holds since $\mathcal{V} \subset B$. By definition of $B$, the assumption that $\log(d)|B|_1^2 n^{-1} \rightarrow 0$ is both a condition on the rate of convergence to 0 of $\lambda_n$ (which should not be too fast) and on the size of $|\theta_0|_1$ (which shall not be too large). Since $|\theta_0|_1$ and $s_0$ (the number of nonzero components of $\theta_0$) are strongly related, the latter can be interpreted as an assumption on the parsimony level $s_0$ which, roughly speaking, shall not exceed $\sqrt{n}$.

4 Applications

In this section, we show how to use the general results given previously to derive consistency results in specific applications, namely robust regression, binary classification and nonlinear least squares.
4.1 General regression setup

Let us now introduce a regression framework that is similar to the one considered in Section 3.2 but with some additional assumptions. This framework will be adopted in the three examples that follows. The response $Y$ has support $Y \subset \mathbb{R}$, the covariates vector $X$ has support $X \subset \mathbb{R}^d$. Let $\{(Y_i, X_i)\}_{i=1}^n$ be an i.i.d. collection of random variables with the same distribution as $(Y, X)$. The parameter space $\Theta$ is equal to $\mathbb{R}^d$. We further assume that $\theta_0 \neq 0$ for simplicity. This condition ensures that the number $s_0$ of nonzero components of $\theta_0$ is strictly positive, which allows to simplify the statements of the rates of convergence.

The following assumption is made on the covariates vector.

Assumption 6. There exists a constant $M_X$ such that for all $n \geq 1$, $|x|_\infty \leq M_X$ for all $x \in X$. The random vector $X$ has mean zero and is $M_X^2$ sub-Gaussian, that is $E[X] = 0$ and $E[e^{X^T v}] \leq e^{M_X^2 |v|^2}$, for all $v \in \mathbb{R}^d$. There exists also $\rho_X > 0$ such that for all $n \geq 1$, $\rho_{\min}(E[XX^T]) \geq \rho_X$.

The fact that $X$ has bounded support allows to bound $X^T \theta$ when $\theta$ lies in an $\ell_1$-ball. The sub-Gaussianity assumption ensures that $X^T \theta$ remains small when $\theta$ lies in an $\ell_2$-ball. These two facts prevent $X^T \theta$ to take too large values which can have undesirable consequences on the estimation of the true risk and its shape. Then, the condition that $\rho_{\min}(E[XX^T]) \geq \rho_X$ is a classic identification assumption. The condition $E[X] = 0$ can be easily avoided (at the cost of additional derivations) but is imposed for simplicity. Note that similar assumptions on the regressors are also imposed in Mei et al. (2018).

4.2 Robust regression

We consider the following model:

$$\epsilon = Y - X^T \theta_0$$

is such that $\epsilon \perp X$ and $E[\epsilon] = 0$

and study robust estimators of the form

$$\hat{\theta} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \rho(Y_i - X_i^T \theta) + \lambda_n |\theta|_1,$$

where $\rho : \mathbb{R} \mapsto \mathbb{R}_+$ is some loss function and $\lambda_n > 0$ is the penalty term. This type of estimator falls into the class studied in Section 3.2 with $\ell(t, y) = \rho(y - t)$. The distribution of $\epsilon$ is assumed to be independent of $n$ (unlike that of $X$, since the dimension $d$ of $X$ is allowed to grow with the sample size). We make the following assumption.

Assumption 7.

(i) The mapping $\rho$ is continuously differentiable, $\rho'$ is odd, $\rho'(t) \geq 0$ for all $t \geq 0$ and there exists $M_\rho > 0$, for all $t \in \mathbb{R}$,

$$|\rho(t)| \vee |\rho'(t)| \leq M_\rho.$$
(ii) The error term $\epsilon$ has a symmetric distribution, moreover, defining

$$g : t \in \mathbb{R} \mapsto E[\rho'(t + \epsilon)],$$

we have $g(t) > 0$ for all $t > 0$, $g$ is differentiable at 0 and $g'(0) > 0$.

This type of conditions can also be found in the robust regression example developed in Mei et al. (2018). Condition (ii) is satisfied by Tukey’s bisquare loss, which is usual in robust regression and given by

$$\rho_{\text{Tukey}}(t) = \begin{cases} 1 - \left(1 - \left(\frac{t}{t_0}\right)^2\right)^3 & \text{for } |t| \leq t_0 \\ 1 & \text{for } |t| > t_0. \end{cases}$$

Given that $\rho$ is odd and $\epsilon$ has a symmetric distribution, the condition that $g(t) > 0$ for all $t > 0$ holds when $\epsilon$ has a density which is strictly positive and decreasing on $\mathbb{R}_+$. We set

$$\lambda_n = 32M_pM_X\sqrt{\log(4nd)/n}.$$ 

We have the following Theorem.

**Proposition 4.** Under Assumptions 6 and 7, if $\log(nd)|\theta_0|_1^2/n^{-1} \to 0$, we have

$$\left| \hat{\theta} - \theta_0 \right|_1 = O_P\left(s_0\sqrt{\log(nd)/n}\right).$$

The additional condition $|\theta_0|_1^2 \log(nd)/n \to 0$ is a sparsity condition, guaranteeing that $\lambda_n|\theta_1| \to 0$ and $\log(n)|B|_2^2/n^{-1} \to 0$. In Loh (2017) and Mei et al. (2018), comparable results are obtained on robust regression estimators. On the one hand, in our case the parameter space is unrestricted while in Loh (2017) (respectively, Mei et al. 2018) it is limited to be an $\ell_1$-ball (respectively, $\ell_2$-ball). On the other hand, the rate of convergence derived by Loh (2017) and Mei et al. (2018) applies to any local minimum of the estimation criterion while our rate only holds for the global minimum $\hat{\theta}$.

### 4.3 Binary classification

Suppose here that $Y$ is binary, i.e., $\mathcal{Y} = \{0, 1\}$ and consider the following model

$$\mathbb{P}(Y = 1|X) = \sigma(X^\top \theta_0),$$

where $\sigma : \mathbb{R} \to [0, 1]$. We define $R(\theta) = \mathbb{E}[(Y - \sigma(X^\top \theta))^2]$ and let

$$\hat{\theta} \in \arg \min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (Y_i - \sigma(X_i^\top \theta))^2 + \lambda_n|\theta|_1.$$

This estimator estimator belongs to the class studied in Section 3.2 with $\ell(t, y) = (y - f(t))^2$. 

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Assumption 8.

(i) The mapping $\sigma$ is differentiable, $\inf_{|t| \leq s} \sigma'(t) > 0$ for all $s > 0$ and there exists a constant $M_\sigma > 0$ such that $\sup_{t \in \mathbb{R}} \sigma'(t) \leq M_\sigma$.

(ii) There exists a constant $M_0 > 0$ such that, for all $n \geq 1, \|\theta_0\|_2 \leq M_0$.

Assumption 8 (i) means that $\sigma$ is strictly increasing and bounded. It imposes $\sigma'(t) \to 0$ as $t \to \pm \infty$. Such an assumption is, for instance, satisfied by the usual logistic function $\sigma(t) = (1 + e^{-t})^{-1}$. We set

$$\lambda_n = 96M_\sigma M_X \sqrt{\frac{\log(4nd)}{n}}.$$ 

We have the following proposition.

**Proposition 5.** Under Assumptions 6 and 8, if $\log(nd)\|\theta_0\|_2^2 / n \to 0$, we have

$$\|\hat{\theta} - \theta_0\|_1 = O_P\left(s_0\sqrt{\frac{\log(nd)}{n}}\right).$$

Remark that Mei et al. (2018) obtains a similar result under close assumptions. As for robust regression, their rate of convergence holds for local minima of the objective function, but the parameter space is restricted to an $\ell_2$-ball. Note also that, thanks to Assumption 8 (ii) and the Cauchy-Schwarz inequality, one can show that $\log(nd)\|\theta_0\|_2^2 / n \to 0$ holds if $\log(nd)s_0n^{-1} \to 0$.

4.4 Nonlinear least squares

The last application studies the following model:

$$\epsilon = Y - f(X^\top \theta_0) \quad \text{is such that} \quad \epsilon |X \sim \mathcal{N}(0, \sigma^2),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function and $\sigma \geq 0$ is a constant. Define $R(\theta) = \mathbb{E}[(Y - f(X^\top \theta))^2]$ and let

$$\hat{\theta} \in \arg \min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} (Y_i - f(X_i^\top \theta))^2 + \lambda_n \|\theta\|_1.$$

This case and that of binary classification are different because, here, the loss function may not be bounded nor Lipschitz since the support of $Y$ is no longer bounded. This prevents us from using Proposition 3. To overcome this issue, we remark that

$$(Y - \sigma(X^\top \theta))^2 = (f(X^\top \theta) - f(X^\top \theta_0))^2 + \epsilon^2 - 2\epsilon[f(X^\top \theta) - f(X^\top \theta_0)],$$

where the term $(f(X^\top \theta) - f(X^\top \theta_0))^2$ is bounded and satisfies some Lipschitz property. The other terms can easily be handled using the Gaussian property on the distribution of $\epsilon$.

We make the following Assumption:
Assumption 9.

(i) The mapping \( f \) is differentiable, \( \inf_{|t| \leq s} f'(t) > 0 \) for all \( s > 0 \) and there exists a constant \( M_f > 0 \), such that \( \sup_{t \in \mathbb{R}} |f(t)| < M_f \) and \( \sup_{t \in \mathbb{R}} f'(t) < M_f \).

(ii) There exists a constant \( M_0 > 0 \) such that, for all \( n \geq 1 \), \( |\theta_0|_2 \leq M_0 \).

Assumption 9 is similar to Assumption 8 with \( \sigma \) replaced by \( f \). We have the following proposition.

Proposition 6. Under Assumptions 6 and 9, if \( \log(nd)|\theta_0|_1 n^{-1} \to 0 \), there exists a constant \( K > 0 \) such that, when

\[
\lambda_n \geq K \sqrt{\frac{\log(4nd)}{n}},
\]

we have

\[
|\hat{\theta} - \theta_0|_1 = O_P \left( s_0 \sqrt{\frac{\log(nd)}{n}} \right).
\]

5 Conclusion

Some additional research directions are of interest. First, we could examine the variable selection properties of \( \hat{\theta} \). When a \( \ell_1 \)-penalty is used, obtaining support recovery guarantees usually requires an incoherence assumption (see e.g. Wainwright (2009) and references therein). Meanwhile, Loh and Wainwright (2017) have shown that the incoherence condition can be avoided when nonconvex penalization schemes are used. Hence, attractive variable selections properties may be obtained in our general setup but with nonconvex regularizers (such as SCAD or MCP). Second, one may seek to study the prediction error in the present framework. In this case, the identification assumption may not be necessary and it should be possible to obtain oracle inequalities on the risk. Städler et al. (2010) have obtained such results for maximum likelihood estimators. Finally, the behavior of semiparametric high-dimensional \( M \)-estimators could also be investigated.

A Technical reminders

The results of this section are useful technical lemmas. They already appear in the supplementary material of Beyhum et al. (2022) but are recalled to keep the paper self-contained.
A.1 A Bound on the expectation

Lemma 1. Let \( \{Z_i\}_{i=1}^n \) be i.i.d. mean zero \( d \)-dimensional random vectors such that \( |Z_i|_\infty \leq M \) almost surely for some constant \( M > 0 \). Then, we have

\[
E \left[ \frac{1}{n} \sum_{i=1}^n Z_i \right] \leq M \sqrt{\frac{2 \log(2d)}{n}}.
\]

Proof. Take \( v \in \mathbb{R} \) and \( k \in \{1, \ldots, d\} \). By Hoeffding’s Lemma, we have \( E[\exp(vZ_{ki})] \leq \exp(v^2M^2/2) \). By independence of the \( \{Z_i\}_{i=1}^n \), we obtain \( E[\exp((v/n) \sum_{i=1}^n Z_{ki})] \leq \exp(v^2M^2/(2n)) \).

For all \( v > 0 \), this implies

\[
E \left[ \frac{1}{n} \sum_{i=1}^n Z_i \right] = vE \left[ \frac{1}{n} \sum_{i=1}^n Z_i /v \right]
\]
\[
= vE \left[ \log \left( \exp \left( \frac{1}{n} \sum_{i=1}^n Z_i /v \right) \right) \right]
\]
\[
\leq v \log \left( E \left[ \exp \left( \frac{1}{n} \sum_{i=1}^n Z_i /v \right) \right] \right)
\]
\[
\leq v \log \left( \sum_{k=1}^d E \left[ \exp \left( \frac{1}{n} \sum_{i=1}^n Z_{ki}/v \right) \right] + E \left[ \exp \left( \frac{1}{n} \sum_{i=1}^n Z_{ki}/v \right) \right] \right)
\]
\[
\leq v \log \left( 2d \exp \left( \frac{M^2}{2nv^2} \right) \right) = v \left( \log(2d) + \frac{M^2}{2nv^2} \right),
\]

where the first inequality is due to Jensen’s inequality. Choosing \( v = \sqrt{M^2/(2n \log(2d))} \) yields the result. \( \square \)

A.2 Contraction theorem

The following contraction theorem (Theorem 16.2 in Van de Geer (2016)) will play an important role in our proofs. We now recall its statement for the sake of clarity.

Theorem 3. Let \( \{Z_i\}_{i=1}^n \) be a collection of random variables valued in \( Z \) and \( \{\epsilon_i\}_{i=1}^n \) be a collection of independent Rademacher variables independent of \( \{Z_i\}_{i=1}^n \). Let \( \mathcal{F} \) be a class of functions defined on \( Z \) and valued in \( \mathbb{R} \). Let \( \rho : \mathbb{R} \times Z \to \mathbb{R} \) be such that for all \( (t, t') \in \mathbb{R}^2 \) and all \( z \in Z \),

\[
|\rho(t, z) - \rho(t', z)| \leq L_* |t - t'|,
\]

for some \( L_* > 0 \). Then, for all \( f_* \in \mathcal{F} \), we have

\[
E \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^n \epsilon_i (\rho(f(Z_i), Z_i) - \rho(f^*(Z_i), Z_i)) \right] \leq 2L_* E \left[ \sup_{f \in \mathcal{F}} \sum_{i=1}^n \epsilon_i (f(Z_i) - f^*(Z_i)) \right].
\]
Proof. Denote by \( E_n \) the conditional expectation given \( \{Z_i\}_{i=1}^n \). Applying Theorem 16.2 in Van de Geer (2016), we get
\[
E_n \left[ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \epsilon_i (\rho(f(Z_i), Z_i) - \rho(f^*(Z_i), Z_i)) \right| \right] \leq 2L_* E_n \left[ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \epsilon_i (f(Z_i) - f^*(Z_i)) \right| \right].
\]
We conclude using the law of iterated expectations. \( \square \)

B Proof of the results of Section 2

B.1 Proof of Proposition 1

The first statement is obvious because \( R(\theta_0) \) is positive. Let us prove that \( \hat{\theta} \in B \) with probability approaching 1. We have \( \hat{R}(\hat{\theta}) + \lambda_n|\hat{\theta}|_1 \leq \hat{R}(\theta_0) + \lambda_n|\theta_0|_1 \), which implies \( \lambda_n|\hat{\theta}|_1 \leq \hat{R}(\theta_0) + \lambda_n|\theta_0|_1 \) because \( \hat{R} \) is positive. Therefore, we obtain \( \lambda_n|\hat{\theta}|_1 = R(\theta_0) + o_P(1) + \lambda_n|\theta_0|_1 \). Hence with probability going to 1, \( \lambda_n|\hat{\theta}|_1 \leq (R(\theta_0) + 1) + \lambda_n|\theta_0|_1 \) which yields the desired result.

B.2 Proof of Theorem 1

Let \( \eta > 0 \). By Assumption 2, there is \( \epsilon > 0 \) such that \( \theta \in \Theta \) and \( |\theta - \theta_0|_2 \geq \eta \) implies that \( R(\theta) - R(\theta_0) \geq \epsilon \). As a result, by the union bound,
\[
P(|\hat{\theta} - \theta_0|_2 \geq \eta) \leq P \left( \{R(\hat{\theta}) - R(\theta_0) \geq \epsilon \} \cap \{\hat{\theta} \in B\} \right) + P \left( \hat{\theta} \notin B \right).
\]
By Proposition 1, \( \theta_0 \in B \) and so (by Assumption 2), \( \hat{R}(\theta_0) \to R(\theta_0) \), in probability. Then, invoking Proposition 1 again, the second term in the right-hand side goes to 0. It remains to show that the first term goes to 0 as well.

We have \( R(\hat{\theta}) \geq R(\theta_0) \) and \( \hat{R}(\hat{\theta}) + \lambda_n|\hat{\theta}|_1 \leq \hat{R}(\theta_0) + \lambda_n|\theta_0|_1 \). It follows that, on the event \( \{\hat{\theta} \in B\} \),
\[
0 \leq R(\hat{\theta}) - R(\theta_0) = [R(\hat{\theta}) - \hat{R}(\hat{\theta})] + [\hat{R}(\hat{\theta}) - \hat{R}(\theta_0)] + [\hat{R}(\theta_0) - R(\theta_0)] \\
\leq 2 \sup_{\theta \in B} \left| \hat{R}(\theta) - R(\theta) \right| + \lambda_n(|\theta_0|_1 - |\hat{\theta}|_1) \\
\leq 2 \sup_{\theta \in B} \left| \hat{R}(\theta) - R(\theta) \right| + \lambda_n|\theta_0|_1,
\]
where the second inequality is due to the fact that \( \theta_0 \in B \). As a result
\[
P \left( \{R(\hat{\theta}) - R(\theta_0) \geq \epsilon \} \cap \{\hat{\theta} \in B\} \right) \\
\leq P \left( \{2 \sup_{\theta \in B} \left| \hat{R}(\theta) - R(\theta) \right| + \lambda_n|\theta_0|_1 \geq \epsilon \} \cap \{\hat{\theta} \in B\} \right) \\
\leq P \left( 2 \sup_{\theta \in B} \left| \hat{R}(\theta) - R(\theta) \right| + \lambda_n|\theta_0|_1 \geq \epsilon \right).
\]
In virtue of Assumption 2 and the fact that \( \lambda_n|\theta_0|_1 \to 0 \), the above term goes to 0.
B.3 Proof of Theorem 2

For any set $\mathcal{E} \subset \Omega$, where $\Omega$ is the sample space of the probability space, define $\mathcal{E}^c = \Omega \setminus \mathcal{E}$. Let

$$A = \left\{ |\hat{\theta} - \theta_0|_1 > \left( \frac{24}{\rho_s s_0 r_n} \right) \vee \delta_n \right\},$$

$$B = \{ \hat{\theta} \in V \}$$

and $C = \left\{ |\hat{\Delta}(\hat{\theta}) - \hat{\Delta}(\theta_0)| \leq r_n |\hat{\theta} - \theta_0|_1 \}$. We have

$$P(A) = P(A \cap B \cap C) + P(A \cap (B \cap C)^c)$$

$$= P(A \cap B \cap C) + P(A \cap (B^c \cup C^c))$$

$$\leq P(A \cap B \cap C) + P(A \cap B^c) + P(A \cap C^c)$$

$$:= P_1 + P_2 + P_3,$$

where the second inequality is due to the union bound. It holds that $P_2 \to 0$ because of Theorem 1. Concerning $P_3$, since on $A \cap C^c$, it holds that $|\hat{\theta} - \theta_0|_1 \vee \delta_n = |\hat{\theta} - \theta_0|_1$, we find that $A \cap C^c$ implies that

$$|\hat{\Delta}(\hat{\theta}) - \hat{\Delta}(\theta_0)| > r_n (|\hat{\theta} - \theta_0|_1 \vee \delta_n),$$

which, by Assumption 4, has probability going to 0. Hence, it suffices to show that $P_1 \to 0$. We will show the even stronger result $P_1 = 0$.

Let $J = \text{Supp}(\theta_0)$. For a vector $v \in \mathbb{R}^d$, we denote by $v_J$ the vector in $\mathbb{R}^d$ such that $(v_J)_k = v_k$ for all $k \in J$ and $(v_J)_k = 0$ otherwise. We also define $v_{J^c} = v - v_J$. Throughout the rest of the proof, we work on the event $B \cap C$. By definition of $\hat{\theta}$, we have

$$\hat{R}(\hat{\theta}) + \lambda_n |\hat{\theta}|_1 \leq \hat{R}(\theta_0) + \lambda_n |\theta_0|_1.$$

Next, remark that

$$|\theta_0|_1 - |\hat{\theta}|_1 = |\theta_0|_1 - |\hat{\theta} - \theta_0 + \theta_0|_1$$

$$= |\theta_0|_1 - |(\hat{\theta} - \theta_0)_J + \theta_0|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1$$

$$\leq |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1,$$

where we have just used that $|a|_1 - |b|_1 \leq |a - b|_1$. Therefore, it holds that

$$R(\hat{\theta}) - R(\theta_0) = \left\{ \hat{\Delta}(\theta_0) - \hat{\Delta}(\hat{\theta}) \right\} + (\hat{R}(\hat{\theta}) - \hat{R}(\theta_0))$$

$$\leq r_n |\hat{\theta} - \theta_0|_1 + \lambda_n \left( |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1 \right).$$

By Assumption 3, we have

$$R(\hat{\theta}) - R(\theta_0) \geq \frac{\rho_s}{2} |(\hat{\theta} - \theta_0)|_2^2 \geq \frac{\rho_s}{2s_0} |(\hat{\theta} - \theta_0)_J|_1^2,$$

which implies

$$\frac{\rho_s}{2s_0} |(\hat{\theta} - \theta_0)_J|_1^2 \leq r_n |\hat{\theta} - \theta_0|_1 + \lambda_n \left( |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1 \right)$$

$$= r_n \left( 3 |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1 \right)$$

$$= r_n \left( 3 |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1 \right)$$

$$= r_n \left( 2 |(\hat{\theta} - \theta_0)_J|_1 - |(\hat{\theta} - \theta_0)_{J^c}|_1 \right).$$
which yields that $3|\hat{\theta} - \theta_0|_1 \geq |\hat{\theta} - \theta_0|_C$ and $|\hat{\theta} - \theta_0|_1 \leq 6s_0r_n/\rho_*$. Hence, on the event $B \cap C$, we have $\left\{ |\hat{\theta} - \theta_0|_1 \leq \frac{24}{\rho_*}s_0r_n \right\}$, which implies $P_1 = 0$.

C Proof of the results of Section 3

C.1 Proof of Proposition 2

Take $\eta > 0$ and $\theta \in \Theta$, $|\theta - \theta_0|_2 \geq \eta$. Let us consider the mapping $t \in [0, r] \mapsto R(\theta_t)$, where $r = |\theta - \theta_0|_2$ and $\theta_t = \theta_0 + tr^{-1}(\theta - \theta_0)$. This mapping is differentiable, with derivative $r^{-1}\nabla R(\theta_t)^{\top}(\theta - \theta_0)$. As a result, we have

$$R(\theta) - R(\theta_0) = \int_0^r r^{-1}\nabla R(\theta_t)^{\top}(\theta - \theta_0)dt = \int_0^r t^{-1}\nabla R(\theta_t)^{\top}(\theta_t - \theta_0)dt \geq \int_0^\eta t^{-1}\nabla R(\theta_t)^{\top}(\theta_t - \theta_0)dt,$$

where the inequality results from (i). By (ii) we have, for all $t \in [0, \eta]$,

$$\nabla R(\theta_t)^{\top}(\theta_t - \theta_0) \geq c(\eta)|\theta_t - \theta_0|^2 = c(\eta)t^2r^{-2}|\theta - \theta_0|^2 = c(\eta)t^2.$$

Taking the integral yields (4).

Let $\eta > 0$. By taking the infimum over $|\theta - \theta_0|_2 \geq \eta$ in (4), we obtain Condition 4 To obtain Condition 5 remark that (4) applied with $\eta = |\theta - \theta_0|$, leads to $R(\theta) - R(\theta_0) \geq c(|\theta - \theta_0|)|\theta - \theta_0|^2/2$ for all $\theta \in S$. Pick $\eta_0 > 0$. Since $c(\cdot)$ is decreasing, for any $\theta \in \Theta$, $|\theta - \theta_0|_2 \leq \eta_0$, we get

$$R(\theta) - R(\theta_0) \geq c(\eta_0)|\theta - \theta_0|^2/2,$$

which proves that Assumption 5 is satisfied too.

C.2 Proof of Proposition 3

C.2.1 An auxiliary result

Recall that, for all $\theta \in \Theta$, we have $\hat{\Delta}(\theta) = \hat{R}(\theta) - R(\theta)$. For a bounded subset $C$ of $\Theta$, define $\mu_C = \sup_{\theta \in C} |\theta - \theta_0|_1$.

Lemma 2. Under the assumptions of Proposition 5 for all bounded subsets $C$ of $\Theta$, we have

$$\mathbb{P} \left( D \geq 2K\sqrt{\frac{2\log(2d)}{n}}\mu_C + K\mu_Ct \right) \leq \exp \left( -\frac{nt^2}{8} \right),$$

where $K = 2LM_X$ and $D = \sup_{\theta \in C} |\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|$. 

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Second, applying Lemma 2, we obtain

\[ E[D] \leq 2E \left[ \sup_{\theta \in C} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i [\ell(X_i^\top \theta, Y_i) - \ell(X_i^\top \theta_0, Y_i)] \right| \right]. \]

The function \( \ell(\cdot, y) \) is Lipschitz on \( T \) by Assumption \( \Box \)(iii). It can be extended on \( \mathbb{R} \) taking instead \( \ell(p(s), y) \) where \( p(s) \) is the projection on \( T_n \). This new function coincides with \( \ell(\cdot, y) \) and has the same Lipschitz constant. Hence, we can apply Theorem 3 to obtain

\[ E[D] \leq 4LE \left[ \sup_{\theta \in C} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i^\top (\theta - \theta_0) \right| \right]. \]

By Hölder’s inequality, Lemma 1 and the definition of \( \mu_C \), we get

\[
E[D] \leq 4LE \left[ \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \right|_\infty \sup_{\theta \in C} |\theta - \theta_0|_1 \right] \\
\leq 2K \sqrt{\frac{2\log(2d)}{n}} \mu_C. \tag{8}
\]

Concentrating \( D \) around \( E[D] \). Because

\[
\{ \ell(X_i^\top \theta, Y_i) - \ell(X_i^\top \theta_0, Y_i) - \{R(\theta) - R(\theta_0)\} \}_{i=1}^{n}
\]

are i.i.d., bounded by \( K\mu_C \) and mean zero, we can apply Massart’s inequality (Theorem 16.4 in Van de Geer (2016)). It leads to

\[
P(D \geq KE[D] + K\mu_C t) \leq \exp \left(-\frac{nt^2}{8} \right),
\]

for all \( t > 0 \). Using (8), we obtain the result. \( \square \)

C.2.2 Proof that Assumption \( \Box \) holds

Define \( A = \sup_{\theta \in B} |\widehat{\Delta}(\theta)| \). Remark that \(|A| \leq D + |\widehat{\Delta}(\theta_0)|\), where \( D = \sup_{\theta \in B} |\widehat{\Delta}(\theta) - \widehat{\Delta}(\theta_0)| \). We now show that both terms go to 0 in probability. First, notice that the random variables \( \{\ell(X_i^\top \theta_0, Y_i) - R(\theta_0)\}_{i=1}^{n} \) are unidimensional i.i.d. random variables with mean zero and bounded almost surely by \( 2M \). Hence, by Hoeffding’s inequality, we have \( |\widehat{\Delta}(\theta_0)| = o_P(1) \). Second, applying Lemma 2 we obtain

\[
P \left(D \geq 2K \sqrt{\frac{2\log(2d)}{n}} |B|_1 + K|B|_1 t\right) \leq \exp \left(-\frac{nt^2}{8} \right),
\]

which yields (choosing \( t = M/\sqrt{n} \) with \( M \) large) that \( D = o_P(1) \) (since \( \log(d)|B|_1^2/n \to 0 \). As a result, \( |A| = o_P(1) \).
Define the collection of rings $C_k = \{ \theta \in B : 2^k \leq |\theta - \theta_0|_1 \leq 2^{k+1} \}$, $k \in \mathbb{Z}$. Define

$$A = \sup_{|\theta - \theta_0|_1 \leq \delta_n} |\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|$$

$$A_k = \sup_{\theta \in C_k} |\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|.$$  

Define $u_n = \left\lceil \log(\delta_n) \log(2) \right\rceil$ and $v_n = \left\lceil \log(B) \log(2) \right\rceil$ and further assume that $\delta_n \leq B$ (the case $\delta_n > B$ is simpler as only $A$ needs to be bounded). Since $\{ |\theta - \theta_0|_1 \leq \delta_n \} \cup \{ \cup_{k=u_n}^{v_n} C_k \}$ covers the set $B$ and because $|\theta - \theta_0|_1 \geq 2^k$ on $C_k$, it holds that

$$\frac{|\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|}{|\theta - \theta_0|_1 \lor \delta_n} \leq (A\delta_n^{-1}) \lor (\max_{u_n \leq k \leq v_n} A_k 2^{-k}).$$

We now focus separately on each of the two terms appearing in the above upper bound. For the term in the left, by Lemma 2, it holds that

$$\mathbb{P} \left( A_k \geq 2K\sqrt{\frac{2\log(2d)}{n}} 2^{k+1} + K 2^{k+1} t \right) \leq \exp \left( -\frac{nt^2}{8} \right),$$

and the union bound yields

$$\mathbb{P} \left( \max_{u_n \leq k \leq v_n} 2^{-k} A_k \geq 4K \sqrt{\frac{2\log(2d)}{n}} + 2K t \right) \leq (v_n - u_n + 1) \exp \left( -\frac{nt^2}{8} \right) \quad (9)$$

For the term in the left, by Lemma 2 it holds that

$$\mathbb{P} \left( A\delta_n^{-1} \geq 2K \sqrt{\frac{2\log(2d)}{n}} + K t \right) \leq \exp \left( -\frac{nt^2}{8} \right). \quad (10)$$

The union bound and the two inequalities (9) and (10) together gives that for all $t > 0$,

$$\mathbb{P} \left( \sup_{\theta \in B} \frac{|\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|}{|\theta - \theta_0|_1 \lor \delta_n} \geq 4K \sqrt{\frac{2\log(2d)}{n}} + 2K t \right) \leq (v_n - u_n + 2) \exp \left( -\frac{nt^2}{8} \right)$$

Choosing $t_n = \sqrt{8 \log(2n)/n}$ and using that $\lceil x \rceil \leq x + 1$, we obtain that

$$\mathbb{P} \left( \sup_{\theta \in B} \frac{|\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|}{|\theta - \theta_0|_1 \lor \delta_n} \geq 2K \left( \sqrt{\frac{8\log(2d)}{n}} + t_n \right) \right) \leq \frac{v_n - u_n + 2}{2n} \leq \frac{\log_2(B) + 3}{2n}.$$  

By assumption, it holds that $B^2 = o(n)$ which implies that the previous probability goes to 0. Hence, taking $R = 8K\sqrt{\frac{\log(4nd)}{n}}$ we get

$$\mathbb{P} \left( \sup_{\theta \in B} \frac{|\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|}{|\theta - \theta_0|_1 \lor \delta_n} \geq R \right) \leq \mathbb{P} \left( \sup_{\theta \in B} \frac{|\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)|}{|\theta - \theta_0|_1 \lor \delta_n} \geq 2K \left( \sqrt{\frac{8\log(2d)}{n}} + t_n \right) \right)$$
where the inequality follows from the fact that $2K \left( \sqrt{\frac{8 \log(2d)}{n}} + t_n \right) \leq r_n$ which comes from \( \sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)} \) for all \( a, b > 0 \). Since the previous upper bound goes to 0 we have just obtained (3).

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Supplement to “High-dimensional nonconvex lasso-type \( M \)-estimators”

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The supplement contains some technical reminders (Section A) and the proofs of Propositions 4, 5 and 6 of the main text (respectively in Sections B, C, D).

A  Technical reminders on (sub-)Gaussian random variables

Lemma A.1. Let \( Z \) and \( Z' \) be two sub-Gaussian random variables with variance parameter \( v \) and \( v' \), respectively. Then for all \( \delta > 0 \), it holds

\[
\mathbb{E}\left[ Z^2 \mid |Z'| > 2\sqrt{v' \log(4\sqrt{2}/\delta)} \right] \leq v\delta.
\]

Proof. Use Cauchy-Schwarz inequality to obtain

\[
\mathbb{E}\left[ Z^2 \mid |Z'| > t \right] \leq \mathbb{E}[Z^4]^{1/2} \mathbb{P}(|Z'| > t)^{1/2}.
\]

Use Theorem 2.1 in Boucheron et al. (2013) to get

\[
\mathbb{E}[Z^4]^{1/2} \leq 4v.
\]

By definition of sub-Gaussian variables, \( \mathbb{P}(|Z'| > t) \leq 2 \exp(-t^2/2v') \). It follows that

\[
\mathbb{E}[Z^2 \mid |Z'| > t] \leq 4v\sqrt{2} \exp(-t^2/2v') = 4v\sqrt{2} \exp(-t^2/4v'),
\]

and using \( t = 2\sqrt{v' \log(4\sqrt{2}/\delta)} \) we obtain the statement of the lemma. \( \square \)

Lemma A.2. Let \( \{\epsilon_i\}_{i=1}^n \) be i.i.d. \( \mathcal{N}(0,\sigma^2) \) random variables for some \( \sigma \geq 0 \) and \( \{X_i\}_{i=1}^n \) be i.i.d. \( d \)-dimensional random vectors independent of \( \{\epsilon_i\}_{i=1}^n \). Assume that there exists a constant \( M_X > 0 \) such that such that \( |X_i|_\infty \leq M_X \) almost surely. Then, we have

\[
\mathbb{E}\left[ \left\| \frac{1}{n} \sum_{i=1}^n \epsilon_i X_i \right\|_\infty \mid \{X_i\}_{i=1}^n \right] \leq \sigma M_X \sqrt{\frac{2\log(2d)}{n}}.
\]

Proof. Take \( v \in \mathbb{R} \) and \( k \in \{1,\ldots,d\} \). The random variable \( n^{-1} \sum_{i=1}^n \epsilon_i X_{ki} \) follows a \( \mathcal{N}(0,\sigma^2n^{-2}\sum_{i=1}^n X_{ki}^2) \) distribution conditional on \( \{X_i\}_{i=1}^n \). Hence, by a standard property
of Gaussian variables,
\[
E \left[ \exp \left( \frac{v}{n} \sum_{i=1}^{n} \epsilon_i X_{ki} \right) \left| \{X_i\}_{i=1}^{n} \right. \right] \leq \exp \left( v^2 \sigma^2 \sum_{i=1}^{n} X_{ki}^2 / (2n^2) \right) \leq \exp \left( v^2 \sigma^2 M_X^2 / (2n) \right).
\]

For all \( v > 0 \), this implies
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \right] \leq v E \left[ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \right] / v \left| \{X_i\}_{i=1}^{n} \right. \right] \leq v \log \left( E \left[ \exp \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \right) / v \right] \right) \left| \{X_i\}_{i=1}^{n} \right. \right] \leq v \log \left( E \left[ \exp \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \right) / v \right] \right) \left| \{X_i\}_{i=1}^{n} \right. \right] \leq v \log \left( 2d \exp \left( \frac{\sigma^2 M_X^2}{2nv^2} \right) \right) = v \log(2d) + \frac{\sigma^2 M_X^2}{2nv^2},
\]
where the first inequality is due to Jensen’s inequality. Choosing \( v = \sqrt{\sigma^2 M_X^2 / (2n \log(2d))} \),
we obtain the result.

B Proof of Proposition 4

To prove Proposition 4, it suffices to show that the assumptions of Theorem 2 hold and apply the latter theorem. To prove that Assumptions 2 and 4 are satisfied, we use Proposition 3, of which the conditions are shown to hold in the next subsection. Then, we prove that Assumptions 1 and 3 are satisfied in Section B.2, thanks to Proposition 2. Finally, remark that \( \lambda_1 |\theta_0|_1 \to 0 \) since \( \log(nd)|\theta_0|^2 n^{-1} \to 0 \) and \( \lambda_n = 2r_n \), where \( r_n \) is defined in Proposition 3.

B.1 Showing that the assumptions of Proposition 3 hold

First, we show that Assumption 5 holds. Remark that Assumptions 5 (i) and (ii) are implied by Assumptions 6 and 7 (i), respectively. The Lipschitz property of \( \ell \) is a direct consequence of \( |\rho'(t)| \leq M_\rho \) which is included in Assumption 7 (i).

It remains to prove that \( \log(d)|B|^2 n^{-1} \to 0 \). Since \( R_n(\theta_0) \leq M_\rho \), we have \( |B|_1 \leq \lambda_n^{-1} M_\rho + |\theta_0|_1 \). Hence,
\[
\log(d)|B|_1^2 n^{-1} \leq \frac{\log(d)}{(32M_X)^2 M_\rho \log(4nd)} + 2 \log(d)|\theta_0|_1^2 n^{-1},
\]
which goes to 0 by assumption.

### B.2 Showing that the assumptions of Proposition 2 hold

To obtain both Assumptions 1 and 2 we only need to show that Condition (i) and (ii) in Proposition 2 are satisfied under Assumptions 6 and 7.

**Proof of (i) in Proposition 2.** Take $\gamma > 0$ and $\theta \in \Theta$, $\theta \neq \theta_0$, $|\theta - \theta_0|^2 \leq \gamma$. For $s \geq 0$, define the event

$$E_s = \{|X^\top(\theta - \theta_0)| \leq s\}.
$$

For $s > 0$, let $L(s) = \inf_{0 < |t| \leq s} \frac{g(t)}{t}$. Let us show that $L(s) > 0$ for all $s > 0$. Since, $\rho'$ is odd and $\epsilon$ has a symmetric distribution, $g$ is also odd and as a result $L(s) = \inf_{0 < t \leq s} \frac{g(t)}{t}$. Recall that we assumed that $g'(0) > 0$. Hence, since

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{g(t)}{t},$$

there exists $\eta > 0$ such that $\inf_{0 < t \leq \eta} \frac{g(t)}{t} > 0$. Next, since $t \mapsto \frac{g(t)}{t}$ is continuous and strictly positive on $[\eta, \infty)$, we have $\inf_{\eta < t \leq s} \frac{g(t)}{t} > 0$ for all $s > 0$. This yields that $L(s) > 0$ for all $s > 0$. As a result, for all $s > 0$, we have

$$\nabla R(\theta)^\top(\theta - \theta_0) = E[E[\rho'(X^\top(\theta - \theta_0) + \epsilon)|X] X^\top(\theta - \theta_0)]$$

$$= E[g(X^\top(\theta - \theta_0))X^\top(\theta - \theta_0)]$$

$$\geq E[g(X^\top(\theta - \theta_0))X^\top(\theta - \theta_0)1_{E_s}]$$

$$\geq L(s)E[\{X^\top(\theta - \theta_0)\}^2 1_{E_s}],$$

where, in the first inequality, we used the fact that $g(X^\top(\theta - \theta_0))X^\top(\theta - \theta_0)$ is positive almost surely since $g$ is odd. This shows that $\nabla R(\theta)^\top(\theta - \theta_0) \geq 0$.

**Proof of (ii) in Proposition 2.** Write

$$E[\{X^\top(\theta - \theta_0)\}^2 1_{E_s}] = E[\{X^\top(\theta - \theta_0)\}^2] - E[\{X^\top(\theta - \theta_0)\}^2 1_{X^\top(\theta - \theta_0)\geq s}].$$

Noting that $E[\{X^\top(\theta - \theta_0)\}^2] \geq \rho_X|\theta - \theta_0|^2$, we now provide an upper bound on $E[\{X^\top(\theta - \theta_0)\}^2 1_{X^\top(\theta - \theta_0)\geq s}]$ when $s = s_\gamma$ with

$$s_\gamma = 2\sqrt{M_X^2 \gamma^2 \log(8\sqrt{2}M_X^2/\rho_X)}.$$

Let $s_\theta = 2\sqrt{M_X^2 |\theta - \theta_0|^2 \log(8\sqrt{2}M_X^2/\rho_X)}$. Since $X^\top(\theta - \theta_0)$ is $\nu = M_X^2 |\theta - \theta_0|^2$-sub-Gaussian, in virtue of Lemma A.1 (with $\delta = \rho_X/(2M_X^2)$) and the fact that $s_\theta \leq s_\gamma$, we obtain

$$E[\{X^\top(\theta - \theta_0)\}^2 1_{X^\top(\theta - \theta_0)\geq s_\gamma}] \leq E[\{X^\top(\theta - \theta_0)\}^2 1_{X^\top(\theta - \theta_0)\geq s_\theta}]$$

$$\leq |\theta - \theta_0|^2 \rho_X/2$$
Back to the initial decomposition, it follows that
\[ E[\{X^\top (\theta - \theta_0)\}^2 1_{\varepsilon, n}] \geq |\theta - \theta_0|^2 \rho X/2. \]
and finally we showed that
\[ \nabla R(\theta)^\top (\theta - \theta_0) \geq L(s) \frac{\rho X}{2} |\theta - \theta_0|^2. \]
Hence, (ii) holds true with \( c(\gamma) = L(s) \frac{\rho X}{2} \). The mapping \( c(\cdot) \) is decreasing because \( L(\cdot) \) is decreasing (by definition) and \( s, \gamma \) is increasing in \( \gamma \).

C Proof of Proposition 5

The proof of Proposition 5 leverages Propositions 2 and 3 in the same manner as the proof of Proposition 4.

C.1 Showing that the assumptions of Proposition 3 hold

Let us show that Assumption 5 holds. Assumptions 5 (i) and (ii) are a direct consequence of Assumptions 6 and 8 (i) and the fact that \( Y \) is binary.

Now, we show the Lipschitz property, corresponding to Assumption 5 (iii). For \( t, t' \in \mathbb{R} \) and \( y \in \{0, 1\} \), we have
\[
|\ell(t, y) - \ell(t', y)| = |(y - \sigma(t))^2 - (y - \sigma(t'))^2| \\
= |2(y - \sigma(t))(\sigma(t') - \sigma(t)) + (\sigma(t') - \sigma(t))^2| \\
\leq 3|\sigma(t') - \sigma(t)| \\
\leq 3M|t - t'|,
\]
where we used the fact that the range of \( \sigma \) belongs to \([0, 1]\). This shows Assumption 5 (iii).

It remains to prove that \( \log(d)|B|^2 n^{-1} \to 0 \). Since \( R(\theta_0) \leq 1 \), we have \(|B|_1 \leq \lambda^{-1} + |\theta_0|_1\). By Assumption 8 (iii), we have \(|\theta_0|_1 \leq \sqrt{s_0}|\theta_0|_2 \leq \sqrt{s_0}M_0\). Hence,
\[
\log(d)|B|^2 n^{-1} \leq 2 \frac{\log(d)}{(96M^2 M_\sigma)^2 \log(4nd)} + 2 \log(d)|\theta_0|^2 n^{-1},
\]
which goes to 0 by assumption.

C.2 Showing that the assumptions of Proposition 2 hold

Proof of (i) in Proposition 2. Taking the derivative inside the expectation, we obtain
\[
\nabla R(\theta) = 2E[ -\sigma'(X^\top \theta) X(Y - \sigma(X^\top \theta))],
\]
and using that \( E[Y|X] = P(Y = 1|X) = \sigma(X^\top \theta_0) \), it follows that
\[
\nabla R(\theta) = 2E[\sigma'(X^\top \theta) X(\sigma(X^\top \theta) - \sigma(X^\top \theta_0))].
\]
Now introduce $\theta_t = t\theta + (1-t)\theta_0$ and $F(t) = \sigma(\theta_t^\top X)$, $0 \leq t \leq 1$, we can write
\[
\sigma(X^\top \theta) - \sigma(X^\top \theta_0) = F(1) - F(0) = \int_0^1 F'(t)dt = \{(\theta - \theta_0)^\top X\} \int_0^1 \sigma'(\theta_t^\top X)dt
\]
in order to obtain
\[
\nabla R(\theta)^\top (\theta - \theta_0) = 2\mathbb{E}[L(X, \theta)\{X^\top (\theta - \theta_0)\}^2]
\]
with $L(X, \theta) = \sigma'(X^\top \theta) \int_0^1 \sigma'(\theta_t^\top X)dt$. It implies that (i) holds by Assumption 7 (ii).

**Proof of (ii) in Proposition 2.** Introduce the two events
\[
\mathcal{E}_s = |X^\top (\theta - \theta_0)| \leq s \quad \text{and} \quad \mathcal{F}_u = |X^\top \theta_0| \leq u.
\]
If both are realized, then $|\theta_t X| \leq s + u$ for all $t \in [0,1]$. Consequently,
\[
L(X, \theta)1_{\mathcal{E}_s}1_{\mathcal{F}_u} \geq L(s, u) := \inf_{|t| \leq s+u} \sigma'(t)^2,
\]
and therefore we find
\[
\nabla R(\theta)^\top (\theta - \theta_0) \geq 2L(s, u)\mathbb{E}[1_{\mathcal{E}_s}1_{\mathcal{F}_u}\{X^\top (\theta - \theta_0)\}^2].
\]
Recalling that $\rho_X$ is a lower bound on the smallest eigenvalue of $\mathbb{E}[XX^\top]$, it follows that
\[
\nabla R(\theta)^\top (\theta - \theta_0) \geq 2L(s, u)(\rho_X |\theta - \theta_0|^2 - R(s, s_0))
\]
with $R(s, u) = \mathbb{E}[1_{\mathcal{E}_s \cup \mathcal{F}_u}\{X^\top (\theta - \theta_0)\}^2]$. The union bound gives
\[
R(s, u) \leq \mathbb{E}[(1_{\mathcal{E}_s} + 1_{\mathcal{F}_u})\{X^\top (\theta - \theta_0)\}^2].
\]
Let
\[
s_\theta = 2\sqrt{|\theta - \theta_0|^2 M_X^2 \log(16\sqrt{2}M_X^2/\rho_X)}, \quad u_0 = 2\sqrt{|\theta_0|^2 M_X^2 \log(16\sqrt{2}M_X^2/\rho_X)}.
\]
In virtue of Lemma A.1, we have
\[
R(s_\theta, u_0) \leq \frac{\rho_X}{4} |\theta - \theta_0|^2 + \frac{\rho_X}{4} |\theta - \theta_0|^2 = \frac{\rho_X}{2} |\theta - \theta_0|^2
\]
implying that for all $\theta \in \mathbb{R}^d$,
\[
\nabla R(\theta)^\top (\theta - \theta_0) \geq L(s_\theta, u_0)\rho_X |\theta - \theta_0|^2.
\]
Now take the infimum and use that $s \mapsto L(s, u_0)$ and $u \mapsto L(s, u)$ are nonincreasing functions to get
\[
\inf_{\theta \in \Theta, |\theta - \theta_0|^2 \leq \gamma} \frac{\nabla R(\theta)^\top (\theta - \theta_0)}{|\theta - \theta_0|^2} \geq L(s_\gamma, u_0)\rho_X \geq L(s_{\gamma,0}, s_{\gamma,0})\rho_X
\]
with $s_\gamma = 2\sqrt{\gamma^2 M_X^2 \log(16\sqrt{2}M_X^2/\rho_X)}$ and $s_{\gamma,0} = 2M_X(\gamma \vee M_0)\sqrt{\log(16\sqrt{2}M_X^2/\rho_X)}$. 5
D Proof of Proposition 6

To prove Proposition 6, we show that the assumptions of Theorem 2 hold and apply the latter theorem.

D.1 Proof that Assumptions 2 and 4 hold

Using \( \epsilon | X \sim \mathcal{N}(0, \sigma^2) \), we obtain

\[
\nabla R(\theta) = 2\mathbb{E}[f'(X^\top \theta)X(f(X^\top \theta) - f(X^\top \theta_0))].
\]

Hence, \( \nabla R(\theta) \) in the application of nonlinear least squares has the same shape as in the application of binary classification (albeit with \( \sigma \) replaced by \( f \)). Since we make similar assumptions on \( \sigma \) in Section 4.2 and on \( f \) in Section 4.3, Assumptions 1 and 3 can be proved to hold exactly in the same manner as in the proof of Proposition 5 in Section C.2. The proof is therefore omitted.

D.2 Proof that Assumptions 2 and 4 are satisfied

Using the decomposition of equation (7) in the main text, we have, for all \( \theta \in \Theta \),

\[
R(\theta) = R_1(\theta) - 2R_2(\theta) + E[\epsilon^2];
\]

\[
\hat{R}(\theta) = \hat{R}_1(\theta) - 2\hat{R}_2(\theta) + \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2,
\]

where

\[
R_1(\theta) = E[(f(X^\top \theta) - f(X^\top \theta_0))^2];
\]

\[
\hat{R}_1(\theta) = \frac{1}{n} \sum_{i=1}^{n} (f(X_i^\top \theta) - f(X_i^\top \theta_0))^2;
\]

\[
\hat{R}_2(\theta) = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i(f(X_i^\top \theta) - f(X_i^\top \theta_0)).
\]

For \( \theta \in \Theta \), let \( \hat{\Delta}(\theta) = \hat{R}(\theta) - R(\theta) \), \( \hat{\Delta}_1(\theta) = \hat{R}_1(\theta) - R_1(\theta) \) and \( \hat{\Delta}_2(\theta) = \hat{R}_2(\theta) \) (remark that \( E[\hat{R}_2(\theta)] = 0 \)). Lemmas D.1 and D.3 in the next two subsections show that Assumptions 2 and 4 hold for \( \hat{R}_1 \) and \( \hat{R}_2 \), respectively.

As a result, we have

\[
\sup_{\theta \in B} |\hat{\Delta}(\theta)| \leq \sup_{\theta \in B} |\hat{\Delta}_1(\theta)| + 2\sup_{\theta \in B} |\hat{\Delta}_2(\theta)| + \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i^2 - E[\epsilon^2] \right| = o_P(1),
\]

where we used the triangle inequality, Lemmas D.1 (i) and D.3 (i) and the fact that \( |n^{-1} \sum_{i=1}^{n} \epsilon_i^2 - E[\epsilon^2]| = o_P(1) \) by the law of large numbers. This shows that Assumption 2 is satisfied.
Next, let $\delta_n = \sqrt{\log(2d)/n}$ and $r_n = 5(K_2 \lor K_3)\sqrt{\log(4nd)/n}$, where $K_2$ and $K_3$ are defined in Lemmas D.1 (ii) and D.3 (ii). Remark that, for all $\theta \in B$, we have
\[
\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0) = \hat{\Delta}_1(\theta) - \hat{\Delta}_1(\theta_0) - 2(\hat{\Delta}_2(\theta) - \hat{\Delta}_2(\theta_0)). \tag{D.1}
\]
It holds that
\[
\mathbb{P}\left(\sup_{\theta \in B} \left| \frac{\hat{\Delta}(\theta) - \hat{\Delta}(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} \right| > r_n \right) \\
\leq \mathbb{P}\left(\sup_{\theta \in B} \frac{\hat{\Delta}_1(\theta) - \hat{\Delta}_1(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} + 2\sup_{\theta \in B} \frac{\hat{\Delta}_2(\theta) - \hat{\Delta}_2(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} > r_n \right) \\
\leq \mathbb{P}\left(\sup_{\theta \in B} \frac{\hat{\Delta}_1(\theta) - \hat{\Delta}_1(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} > \frac{r_n}{2} \right) + \mathbb{P}\left(\sup_{\theta \in B} \frac{\hat{\Delta}_2(\theta) - \hat{\Delta}_2(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} > \frac{r_n}{4} \right) \to 0,
\]
where in the first inequality, we used (D.1) and the triangle inequality, and, in the second inequality, we leveraged the union bound. The fact that the limit of the probability is 0 is a consequence of Lemmas D.1 (ii) and D.3 (ii). This proves that Assumption 4 is satisfied and concludes the proof.

D.3 On the term $\hat{R}_1$

Recall that, for $\theta \in \Theta$, $\hat{\Delta}_1(\theta) = \hat{R}_1(\theta) - R_1(\theta)$.

Lemma D.1. Under Assumptions 6 and 9, we have
\[(i) \sup_{\theta \in B} \left| \hat{\Delta}_1(\theta) \right| = o_P(1). \]
\[(ii) \text{For all } \eta > 0, \]
\[
\mathbb{P}\left(\sup_{\theta \in B} \left| \frac{\hat{\Delta}_1(\theta) - \hat{\Delta}_1(\theta_0)}{|\theta - \theta_0|_1 \lor \delta_n} \right| \leq r_n \right) \to 1,
\]
where
\[
\delta_n = \sqrt{\log(2d)/n}, \quad r_n = K_1\sqrt{\log(4nd)/n},
\]
for some constant $K_1$.

Proof. Let $T$ be defined as in Section 3.2 and $\hat{Y} = \{x^\top \theta_0 : x \in \mathcal{X}_n\}$. For $\theta \in \Theta$, we have
\[
R_1(\theta) = E[\ell(X^\top \theta, \hat{Y})],
\]
where $\ell : (s, y) \in T \times \hat{Y} \mapsto (f(s) - f(y))^2$ and $\hat{Y} = X^\top \theta_0$. We also let
\[
\hat{R}_1(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(X_i^\top \theta, \hat{Y}_i),
\]
with $\tilde{Y}_i = X_i^\top \theta_0$. With these notations, $R_1$ belongs to the class of true risks $R$ considered in Section 3.2. It is clear that Assumption 5 (i) and (ii) holds thanks to Assumptions 6 and 9 (i), respectively. Moreover, Assumption 5 (iii) is satisfied because for $t, t' \in T$ and $y \in \tilde{Y}$, we have

$$|\ell(t, y) - \ell(t', y)| = |(f(t) - f(y))^2 - (f(t') - f(y))^2|$$

$$= |(f(t) - f(t'))(f(t) + f(t') - 2f(y))|$$

$$\leq 4M_f|f(t') - f(t)|$$

$$\leq 4M_f^2|t - t'|.$$

We can show that $\log(d)|B|^2n^{-1} \to 0$ as in the proof of Proposition 5. The result of the Lemma is then a direct consequence of Proposition 3 applied to $R_1$ and $\hat{R}_1$. \hfill \Box

### D.4 On the term $\hat{R}_2$

Recall that, for $\theta \in \Theta$, $\hat{\Delta}_2(\theta) = \hat{R}_2(\theta)$.

**Lemma D.2.** For a nonempty bounded set $C \subset \Theta$, define $\mu_C = \sup_{\theta \in C} |\theta - \theta_0|_1$ and 

$$D = \sup_{\theta \in C} \left| \hat{\Delta}_2(\theta) - \hat{\Delta}_2(\theta_0) \right|.$$ 

Under Assumptions 6 and 9, we have

$$\mathbb{P} \left( D \geq 2K_2\sqrt{\frac{2\log(2d)}{n}} \mu_C + K_2^2 \mu_C t \right) \leq 2\exp \left( -\frac{nt^2}{2} \right),$$

for some constant $K_2 > 0$.

**Proof.** Without loss of generality, we can assume that $\theta_0 \in C$. If not it suffices to replace $C$ by $C \cup \{\theta_0\}$ and check that nothing changes in the statement of the lemma. Notice that $\hat{R}_2(\theta_0) = 0$. Then, $\hat{\Delta}_2(\theta) - \hat{\Delta}_2(\theta_0) = \hat{R}_2(\theta)$ and $D = \sup_{\theta \in C} \left| \hat{R}_2(\theta) \right|$. The proof follows from the fact that $\{\hat{R}_2(\theta)\}_{\theta \in C}$ is a Gaussian process given $\{X_i\}_{i=1}^n$.

**Bounding $E[D]$.** Since $\epsilon|X \sim \mathcal{N}(0, \sigma^2)$, for $\theta, \gamma \in \Theta$, we have

$$\hat{R}_2(\theta) - \hat{R}_2(\gamma) \{X_i\}_{i=1}^n \sim \mathcal{N} \left( 0, \rho^2(\theta, \gamma) \right),$$

where $\rho(\theta, \gamma) = n^{-1}\sigma\sqrt{\sum_{i=1}^n (f(X_i^\top \theta) - f(X_i^\top \gamma))^2}$. Hence, by the Gaussian concentration inequality (Theorem 5.6 in *Boucheron et al. (2013)*), we have, for all $t \geq 0$,

$$\mathbb{P} \left( |\hat{R}_2(\theta) - \hat{R}_2(\gamma)| \geq t\rho(\theta, \gamma) \right) \{X_i\}_{i=1}^n \leq e^{-\frac{t^2}{2}}.$$ 

Remark that, by Assumption 9 (ii), $\rho^2(\theta, \gamma) \leq M_f^2\sigma^2n^{-2}\sum_{i=1}^n (X_i^\top (\theta - \gamma))^2$. This yields,

$$\mathbb{P} \left( |\hat{R}_2(\theta) - \hat{R}_2(\gamma)| \geq tM_f^2\sigma^2n^{-2}\sum_{i=1}^n (X_i^\top (\theta - \gamma))^2 \right) \{X_i\}_{i=1}^n \leq e^{-\frac{t^2}{2}}.$$
As a result, conditional on \{X_i\}_{i=1}^n, the process \{\tilde{R}_2(\theta)\}_{\theta \in C} has the same tail bound as the Gaussian process \{P_\theta = M_f \sigma_n^{-1} \sum_{i=1}^n \epsilon_i X_i^\top (\theta - \theta_0)\}_{\theta \in C}. Hence, using Theorem 2.1.5 in Talagrand (2005) and the fact that \theta_0 \in C, we get that there exists a constant \( L > 0 \) such that
\[
E[D\mid\{X_i\}_{i=1}^n] \leq E\left[ \sup_{\theta, \gamma \in C} |\tilde{R}_2(\theta) - \tilde{R}_2(\gamma)| \mid \{X_i\}_{i=1}^n \right] \leq LE\left[ \sup_{\theta \in C} |P_\theta| \mid \{X_i\}_{i=1}^n \right].
\]

Next, we have
\[
E\left[ \sup_{\theta \in C} |P_\theta| \mid \{X_i\}_{i=1}^n \right] \leq M_f \sigma E\left[ \frac{1}{n} \left( \sum_{i=1}^n \epsilon_i X_i \right) \right] \mu_C \leq \sigma M_f M_X \sqrt{\frac{2 \log(2d)}{n}} \mu_C,
\]
where the last inequality is due to Lemma A.2. By the law of iterated expectations, we get
\[
E[D] \leq \sigma LM_f M_X \sqrt{\frac{2 \log(2d)}{n}} \mu_C. \tag{D.2}
\]

Concentrating \( D \) around \( E[D] \). We have
\[
\tilde{R}_2(\theta)\mid\{X_i\}_{i=1}^n \sim \mathcal{N}(0, \rho^2(\theta, \theta_0)),
\]
By the Gaussian bound for processes corresponding to Theorem 5.8 in Boucheron et al. (2013), it holds that, for all \( t \geq 0 \),
\[
\mathbb{P}\left( |D - E[D]| \geq t\overline{\sigma} \right)\mid\{X_i\}_{i=1}^n \leq 2e^{-\frac{t^2}{2}}, \tag{D.3}
\]
where \( \overline{\sigma} = \sup_{\theta \in C} |\rho(\theta, \theta_0)| \). Remark that, for all \( \theta \in C \), it holds that
\[
\rho^2(\theta, \theta_0) \leq n^{-2} \sigma^2 M_f^2 \sum_{i=1}^n (X_i^\top (\theta - \theta_0))^2 \leq n^{-1} \sigma^2 M_f^2 M_X^2 \mu_C |\theta - \theta_0|_1 \leq n^{-1} \sigma^2 M_f^2 M_X^2 \mu_C,
\]
almost surely, where the second inequality leverages Assumption 6 and Hölder’s inequality. As a result, we have \( \overline{\sigma}^2 \leq \sigma^2 M_f^2 M_X^2 \mu_C^2 n^{-1} \) almost surely. By (D.3) and the bound on \( E[D] \) in (D.2), this yields the result the bound on the probability given \( \{X_i\}_{i=1}^n \). The unconditional bound can be directly obtained through the law of iterated expectations. \( \square \)

**Lemma D.3.** Under Assumptions 6 and 9, we have

(i) \( \sup_{\theta \in B} |\tilde{\Delta}_2(\theta)| = o_P(1) \).

(ii) For all \( \eta > 0 \),
\[
\mathbb{P}\left( \sup_{\theta \in B} \left| \tilde{\Delta}_2(\theta) - \tilde{\Delta}_2(\theta_0) \right| \leq \eta \right) \rightarrow 1,
\]

\[
\mathbb{P}\left( \sup_{\theta \in B} \left| \tilde{\Delta}_2(\theta) - \tilde{\Delta}_2(\theta_0) \right| \leq \eta \right) \rightarrow 1,
\]
where
\[ \delta_n = \sqrt{\frac{\log(2d)}{n}}, \quad r_n = K_3 \sqrt{\frac{\log(4nd)}{n}}, \]

where \( K_3 > 0 \) is a constant.

**Proof.** Lemma D.2 is a result similar to Lemma 2 in the main text. So, the proof can proceed similarly to the proof of Proposition 3 in Sections C.2.2 and C.2.3 of the main text, and is, therefore, omitted. \( \square \)

**References**

Boucheron, S., Lugosi, G., and Massart, P. (2013). *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press.

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