HOW SINGULAR ARE MOMENT GENERATING FUNCTIONS?

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Abstract. This short note concerns the possible singular behaviour of moment generating functions of finite measures at the boundary of their domain of existence. We look closer at Example 7.3 in O. Barndorff-Nielsen’s book Information and Exponential Families in Statistical Theory (1978) and elaborate on the type of exhibited singularity. Finally, another regularity problem is discussed and it is solved through tensorizing two Barndorff-Nielsen’s distributions.

1. Introduction

Let \( \mu \) be a finite positive measure on \( \mathbb{R}^d (d \geq 1) \), and denote by
\[
G : \mathbb{R}^d \rightarrow \mathbb{R}_+, \quad G(u) = \int_{\mathbb{R}^d} e^{\langle u, \xi \rangle} \mu(d\xi)
\]
its moment generating function. (Here \( \langle \cdot, \cdot \rangle \) denotes the standard Euclidean scalar product on \( \mathbb{R}^d \) ) That is defined on the domain
\[
V := \{ u \in \mathbb{R}^d | G(u) < \infty \}.
\]
It is well known that \( V \) is convex, and \( u \mapsto G(u) \) is convex thereon. Also, \( G \) is analytic on the interior \( V^\circ \) of \( V \).

We only talk about measures for which \( V^\circ \neq \emptyset \). Accordingly, we may assume without loss of generality that \( 0 \in V^\circ \) (use exponential tilting). Let us first have a look at the behaviour of \( G \) along half rays through the origin. Fix \( u \in \mathbb{R}^d \) and define
\[
\vartheta^* := \sup\{ \vartheta > 0 | G(\vartheta u) < \infty \}.
\]
Clearly we have
\[
\vartheta^* = \sup\{ \vartheta > 0 | \vartheta u \in V \} = \sup\{ \vartheta > 0 | \vartheta u \in V^\circ \} \in (0, \infty].
\]

Remark 1.1. Either \( \vartheta^* = \infty \) (not exciting), or \( \vartheta^* < \infty \), in which case the analytic function \( G \) must exhibit a singularity at \( \vartheta^* u \in \partial V^\circ \). Along the ray \( [0, \vartheta^* u] \), two situations may occur:

(i) Either \( \lim_{\vartheta \uparrow \vartheta^*} g(\vartheta u) = +\infty \), in which case \( \vartheta^* u \notin V \).
(ii) Or \( \lim_{\vartheta \uparrow \vartheta^*} g(\vartheta u) = g(\vartheta^* u) < \infty \) where the equality follows from Lebesgue’s monotone convergence theorem. By definition, \( \vartheta^* u \in V \).

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In the one-dimensional situation \((d = 1)\) it is clear that no other limits then those along straight lines can be considered, hence all possible singularities are of the above kind. However, the situation is different in \(\mathbb{R}^d, d \geq 2\). In the following section we shall formulate a related non-trivial problem, and we recall a partial answer from Barndorff-Nielsen’s book \([1]\) in section 2 which involves a bivariate distribution; that we then elaborate in greater detail. The final section 3 poses a problem about singularities of moment generating functions along the boundary of their existence domains. That is solved by tensorizing Barndorff-Nielsen’s distribution with itself—hence it involves a four dimensional distribution.

1.1. Problem. Does there exist a probability measure \(\mu\) on \(\mathbb{R}^d\) whose moment generating function \(G\) has the following properties:

(i) There exists a continuous curve \(c : [0, 1] \to V, t \mapsto c(t)\) such that

(ii) \(G(c(1)) \neq \lim_{t \to 1} G(c(t)) < \infty\), or, more generally: There exists a sequence \(t_k \uparrow 1\) such that the sequence \((G(c(t_k)))_{k=1}^\infty\) has an accumulation point \(p\) in \(\mathbb{R}_+ \cup \{\infty\}\) different from \(G(c(1))\).

Since moment generating functions are continuous along rays through the origin, a solution \(c\) of this problem necessarily needs to be crooked; a solution is provided by Lemma 2.2 below.

1.2. Motivational background. In 2008, I had the pleasure to jointly elaborate with Damir Filipović on a moment problem involving affine processes. It concerned the question, for which real \(u \in \mathbb{R}^d\) a stochastic process \(X\) on \(D = \mathbb{R}_+^m \times \mathbb{R}^n\) satisfies the affine property

\[
\mathbb{E}^x[e^{\langle u, X_t \rangle} \mid X_0 = x] = e^{\phi(t,u)+\langle \psi(t,u), x \rangle}, \quad x \in D.
\]

We managed to completely characterize the validity of this affine transform formula for affine diffusion processes\(^1\). Either side is well defined, if the other is. In any case, the exponents \(\phi, \psi\) solve a \((d + 1)\) dimensional system of Riccati differential equations (with initial values \((0,u))\) with blow up strictly beyond \(t^2\).

A key finding which led to our characterization was that for any \(u \in \mathbb{R}^d\)

\[
\mathbb{E}^x[e^{\langle \theta u, X_t \rangle} \mid X_0 = x] \uparrow \infty
\]

when

\[
\theta \uparrow \theta^* := \sup\{\theta > 0 \mid \mathbb{E}^x[e^{\langle \theta u, X_t \rangle} \mid X_0 = x] < \infty\}.
\]

That is, the moment generating function of an affine diffusion does not exhibit any exotic singularities, but only the one described in Remark 1.1. If \(D = \mathbb{R}_+\), then any affine diffusion equals a square Bessel processes \(X\), and therefore for each \(t\), \(X_t\) is chi-square distributed (when appropriately scaled), and in that case it is even obvious that the moment generating function of the transition law of \(X_t\) has a blow up singularity at the boundary of its domain\(^3\).

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\(^1\)Note that a-priori it is only clear that affine processes have a characteristic function which is exponentially affine in the state-variable. This is the key defining property of affine processes.

\(^2\)For Fourier-pricing applications/implications, see [2, Theorem 3.3].

\(^3\)Strictly speaking, one needs to exclude deterministic motion, that is, one should require that the diffusion coefficient of \(X\) does not vanish.
The problems of this paper have arisen naturally in the context of our work. However, it turned out that a classification of singularities of the moment generating function of processes which exhibit jumps was not helpful for providing a characterization of the affine property beyond the pure diffusion case. Also, such a characterization seems to be not feasible (this is a subject believe of the author). Ongoing work with Martin Keller-Ressel (Berlin) provides a deeper understanding of the existence and non-existence issues of exponential moments of affine jump-diffusions.

2. A bivariate distribution

In the following the notation for points in $\mathbb{R}^2$ with coordinates $x, y$, namely $(x, y)$, should not be confused with open intervals of the form $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$; the reader will distinguish the notation easy from the context.

Barndorff-Nielsen used the following bivariate distribution $\mu$ on $\mathbb{R}^2$ [1, Example 7.3] defined by its density $f(\xi) := \mu(d\xi)/d\xi$ (henceforth called Barndorff-Nielsen’s example or -function):

**Example 2.1.**

\[
f(\xi) = \frac{1}{2\sqrt{\pi}}(1 + \xi_1^2)^{-3/2}e^{-\xi_1^2-\xi_2^2/[4(1+\xi_1^2)]},
\]

According to [1] page 105, for any $u = (u_1, u_2) \in \mathbb{R}^2$ one has

\[
\int_{\mathbb{R}} e^{\langle u, \xi \rangle} f(\xi) d\xi_2 = (1 + \xi_1^2)^{-1} e^{u_2^2 + u_1 \xi_1 - (1 - u_2^2) \xi_1^2}
\]

whence

\[
V = (\mathbb{R} \times (-1, 1)) \cup [(0, 1), (0, -1)].
\]

That is, the domain of the moment generating functions consists of an infinite strip of width 2 parallel to the $\xi_1$-axis, and of two isolated points on the $\xi_2$-axis.

For $u_2 \in (-1, 1)$ one has

\[
G(u) = I_1 \times I_2 \times I_3, \quad \text{where}
\]

\[
I_1(u) = e^{u_2^2 + u_1^2(1 - u_2^2)},
\]

\[
I_2(u) = e^{-u_1^2}, \quad \text{and}
\]

\[
I_3(u) = \int_{\mathbb{R}} (1 + \xi_1^2)^{-1} \exp(-(1 - u_2^2)\xi_1) d\xi_1.
\]

Barndorff-Nielsen shows that there exists a curve $c : [0, 1] \to V$ such that $c(0) = (0, 0), c(1) = (0, 1)$ but

\[
\pi = G(c(1)) \neq \lim_{t \to 1} G(c(t)) = \infty
\]

Here $e$ denotes the Euler number $e = 2.71828\ldots$. But much more can be said about $G$. In a moment we reveal the following facts

\footnote{The question was posed to me by Damir Filipović in early 2009.}
Lemma 2.2. Let $C$ be the set of continuous curves $c : [0, t] \to V$ such that $c(1) = (0, 1)$. Then $G$ exhibits the following (interrelated) properties:

(i) There exists $c \in C$ such that for each $p \in [e\pi, \infty]$ there exists a sequence $(t_k)_{k=1}^{\infty}$ for which
\[
\lim_{k \to \infty} G(c(t_k)) = p
\]

(ii) For each $p \in [e\pi, \infty]$ there exists $c \in C$ such that
\[
\lim_{t \uparrow 1} G(c(t)) = p
\]

(iii) Let $(g^k)_{k=1}^{\infty}$ be any sequence in $[e\pi, \infty]$. Then there exists some $c \in C$ and a sequence $t_j \uparrow t$ such that $(g^k)_{k=1}^{\infty}$ equals the accumulation points of the sequence $(G(c(t)))_{j=1}^{\infty}$.

Proof. Before we start the proof, let us note that for each $c \in C$ we have that
\[
\begin{align*}
\lim_{t \uparrow 1} I_1(c(t)) &= e, \\
\lim_{t \uparrow 1} I_3(c(t)) &= \int_{\mathbb{R}} (1 + \xi^2)^{-1} d\xi = \pi.
\end{align*}
\] (2.1) (2.2)

So all we have to control is the behaviour of $I_2(c(t))$ as $t \uparrow 1$.

Proof of (i) The function
\[
h : [0, 1) \to (0, \infty), \quad h(t) := \frac{\sin\left(\frac{1}{t}\right) + 2 - t}{1 - t}
\]
has the following properties: (a) $h(0, 1)) = (0, \infty)$, (b) For each $q \in [0, \infty]$ there exists a sequence $t_k \uparrow 1$ such that $(h(t_k))_{k=1}^{\infty}$ has accumulation point $q$.

Defining
\[
c_2(t) := \sqrt{1 - \frac{t}{4h(t)}}
\]
we introduce the curve $c(t) := (t, c_2(t))$. Then
\[
I_2(c(t)) = \exp(h(t))
\]
and therefore by Property (b) (with $q = \log(p)$) as well as (2.1)–(2.2) the claim (i) is proved.

Proof of (ii) $\log(p) > \log(e) = 1$, hence we may choose $c(t) = (t, c_2(t) = \sqrt{1 - t/(4 \log(p)))}$; then $I_2(c(t)) \equiv p$, and therefore $G(c(t)) \to I_1((0, 1)) \times I_2((0, 1)) = e\pi$.

Claim (iii) follows directly from (i) by taking a countable union $(t_j)_{j=1}^{\infty}$ of appropriate sequences $(t_k)_{k=1}^{\infty}$, which for fixed $j$ converge to $g^j$. □

3. Singularities along the boundary

In Barndorff-Nielsen’s example, the only boundary points contained in the domain of the moment generating function $V$ are (two) isolated points, namely $(0, \pm 1)$. Hence we were not allowed to look at regularity behaviour along the boundary. Here we pose a new problem. Let’s fix a ray (of flexible length) at the origin. With
its endpoint we let it strike along the boundary of $V$ to find jump regularities of finite height:

Does there exist a probability measure $\mu$ on $\mathbb{R}^d$ whose moment generating function $G$ has the following properties:

(i) There exists a continuous curve $c : [0, 1] \to \partial V, t \mapsto c(t)$ such that

(ii) $c([0, 1)) \subset V$.

(iii) $G(c(1)) \neq \lim_{t \uparrow 1} G(c(t)) < \infty$, or, more generally: There exists a sequence $t_k \uparrow 1$ such that the sequence $\{G(c(t_k))\}_{k=1}^\infty$ has an accumulation point $p$ in $\mathbb{R}_+ \cup \{\infty\}$ different from $G(c(1))$.

For a solution, we define a distribution $\mu$ on $\mathbb{R}^4$ by its distribution function

$$f(\xi) = f(\xi_1, \xi_2)f(\xi_3, \xi_4)$$

which is the product of two Barndorff-Nielsen functions. Then the boundary of the domain of the moment generating function for this distribution contains the set $V \times (0, 1)$, where $V$ is the domain of Barndorff-Nielsen m.g.f. $G$. On this set, $G(u)$ equal to $e^{\pi G(u)}$. Hence one may take any curve $\tilde{c}(t)$ from Lemma 2.2 in $V$ and define the new curve

$$c : c(t) = (t, \tilde{c}(t), 0, 1).$$

Then all kinds of singularities are exhibited as $t \uparrow 1$. Especially the above problem has a solution.

**References**

[1] O. Barndorff-Nielsen, *Information and Exponential Families in Statistical Theory*, Wiley series in probability & mathematical statistics), John Wiley & Sons Ltd., Chichester, 1978.

[2] D. Filipović and E. Mayerhofer, *Affine Diffusion Processes: Theory and Applications*, Radon Series Comp. Appl. Math 8, 1-40, 2009.

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