The Primary Spin-4 Casimir Operators in the Holographic \( SO(N) \) Coset Minimal Models

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Abstract

Starting from \( SO(N) \) current algebra, we construct two lowest primary higher spin-4 Casimir operators which are quartic in spin-1 fields. For \( N \) is odd, one of them corresponds to the current in the \( WB_{N-1} \) minimal model. For \( N \) is even, the other corresponds to the current in the \( WD_{N} \) minimal model. These primary higher spin currents, the generators of wedge subalgebra, are obtained from the operator product expansion of fermionic (or bosonic) primary spin-\( \frac{N}{2} \) field with itself in each minimal model respectively. We obtain, indirectly, the three-point functions with two real scalars, in the large \( N \) ’t Hooft limit, for all values of the ’t Hooft coupling which should be dual to the three-point functions in the higher spin \( AdS_{3} \) gravity with matter.
1 Introduction

Gaberdiel and Gopakumar have conjectured in [1] that the large $N$ 't Hooft limit of the $WA_{N-1} = W_N$ minimal model [2] is dual to a particular $AdS_3$ higher spin theory of Vasiliev [3, 4, 5]. The boundary theory is an $A_{N-1}$ coset model which has a higher spin $WA_{N-1}$ symmetry generated by currents of spins $s = 2, 3, \cdots, N$ [6]. See the work of [7] for the $W$ symmetry in two-dimensional conformal field theory. The theory is labeled by two positive integers $(N, k)$ where $k$ is the level of the current algebra and the 't Hooft coupling $\lambda = \frac{N}{N+k}$ is fixed in the large $N$ 't Hooft limit. The bulk theory has an infinite tower of massless fields with spins $s = 2, 3, \cdots$ coupled to two complex scalars. The higher spin Lie algebra describes interactions between the higher spin fields and the scalars. The scalars have equal mass determined by the algebra, $M^2 = -(1 - \lambda^2)$. The quantization with opposite boundary conditions leads to their conformal dimensions $h_\pm = \frac{1}{2}(1 \pm \lambda)$.

The partition function of the $WA_{N-1}$ minimal model was obtained in [8]. Since certain states become null and decouple from correlation functions, the resulting states that survive exactly match the gravity prediction for all values of the 't Hooft coupling. The strict infinite $N$ limit, where the sum of the number of boxes and antiboxes in the Young tableau has maximum value in the conformal field theory partition function, is used. The three point functions with scalars at tree level in the undeformed bulk theory were computed [9]. In particular, they have checked for spin-3 current and made predictions for the three-point functions of spin $s \geq 4$, at fixed 't Hooft coupling $\lambda = \frac{1}{2}$, in the $WA_{N-1}$ minimal model. In [10], the three-point functions for spin-4 with scalars for all values of 't Hooft coupling were found in the large $N$ 't Hooft limit of the $WA_{N-1}$ minimal model. The three-point functions with scalars in the deformed bulk theory was found in [11] and they were given by scalar-scalar two-point functions. The result from conformal field theory, along the line of [12], agrees with the correlators from the bulk.

There exist other types of minimal models in [13, 14]. It is natural to ask what the three-point functions with two scalars in these minimal models are, as described in [10] briefly.

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\footnote{For $WA_{N-1}$ coset model we have described in [10], the coefficient functions in the coset primary spin-4 field were determined by the fact that it should commute with the diagonal spin-1 current and should transform as a primary field of dimension 4 under the coset stress energy tensor. However, there remain two unknown and undetermined coefficient functions. This implies that the above two requirements in the $WA_{N-1}$ coset minimal model are not enough to determine all the coefficients in the coset primary spin-4 field explicitly. As we described above, the field contents for the $W_N$ minimal model for finite $N$ are given by the fields with spins $s = 2, 3, 4, \cdots, N$. By using the operator product expansion of coset primary spin-3 field with itself and reading off the particular singular term $\frac{1}{(z-w)^2}$, the above two unknown coefficients are fixed completely.}
In this paper, we construct spin-4 primary Casimir operators in $WB_{N-1}$ and $WD_{N}$ minimal models [13]. Then we compute the three-point functions with two real scalars in the large $N$ 't Hooft limit for all values of 't Hooft coupling. The way in which the coset spin-4 currents in these minimal models are obtained is rather different from the procedure that one uses for the $WA_{N-1}$ minimal model. In [10], we have constructed the coset primary spin-4 field by considering the operator product expansion of coset primary spin-3 field with itself (footnote 1). These (spins 3 and 4) are two lowest higher spin currents in the $W_N$ minimal model. If one continues to compute the operator product expansions between these lower spin currents, one obtains the higher spin currents successively. In other words, one determines the spin-3 current using the above two requirements and then fixes the spin-4 current. Then the spin-5 current can be fixed from the operator product expansion between the spin-3 current and the spin-4 current and so on.

However, there exists an extra field of spin-$\frac{N}{2}$ for each case in the present minimal models. Depending on the $N$, this field is either fermionic or bosonic. Then all the field contents of each minimal model are located at the singular terms in the operator product expansion between this spin-$\frac{N}{2}$ field and itself. Then the possible terms for primary spin-4 field can be read off from the operator product expansion of this extra field with itself. It is straightforward to apply the above two requirements for the coset primary spin-4 field. It turns out that all the coefficient functions are fixed except two unknown coefficient functions. So far, the story looks similar to the one for $WA_{N-1}$ minimal model.

Recall that the spin-4 field is the lowest higher spin field in each minimal model. In other words, the lowest spin greater than 2 is 4. In order to fix these two constants in terms of $(N,k)$, one should compute the operator product expansion of primary spin-4 field with itself directly but it is almost impossible, by hand (or other method), to compute these quantities because the number of operator product expansions of various spin-4 fields is 324 for $WB_{N-1}$ minimal model and 452 for $WD_{N}$ minimal model. Instead of doing this, what one can do, at the moment, is to resorts to the higher spin Lie algebra. From the eigenvalues for spin-4 zero mode in the higher spin Lie algebra, along the line of [12], one can find the above two undetermined coefficient functions (therefore all the coefficient functions) in terms of $(N,k)$ in the large $N$ 't Hooft limit. Of course, this is indirect approach but so far there is no direct approach, in practice, to fix the above two constants.\footnote{This feature is different from the behavior for the $W_N$ minimal model where the eigenvalue equations for primary spin-4 zero mode satisfy the higher spin Lie algebra automatically. By assuming the higher spin algebra for our primary spin-4 field (i.e., the zero mode of primary spin-4 field should satisfy the wedge subalgebra described in [12]), one obtains the complete structure for primary spin-4 field and the three-point functions arise automatically.}
In section 2, we review the Goddard-Kent-Olive (GKO) coset construction. We are interested in the specific minimal model characterized by the coset central charge $\tilde{c}$ that can be obtained from the highest singular term in the operator product expansion of the above spin-2 coset Virasoro current.

In section 3, we consider the $WB_{N-1/2}$ minimal model. The fundamental (generating) field content is given by the fermionic spin-$\frac{N}{2}$ field where $N$ is odd. All the field contents $s = 2, 4, 6, \cdots, (N-1)$ are obtained from the singular terms in the operator product expansion of this fundamental field with itself. It turns out that there exist 18 independent spin-4 fields which are written in terms of two arbitrary coefficients. From the eigenvalue equations of primary spin-4 zero mode acting on the two primaries, we construct the three-point functions of the spin-4 coset primary field with two real scalar fields in terms of 't Hooft coupling constant under the assumption of higher spin Lie algebra.

In section 4, we move on the $WD_{N/2}$ minimal model. The generating field content is the bosonic spin-$\frac{N}{2}$ field where $N$ is even. In this case, there exist 21 independent spin-4 fields where there are 3 more spin-4 fields, compared to the previous case.

In section 5, we summarize what we have obtained in this paper and we make some comments on the future direction.

There are some (partial and incomplete) related works in [16]-[34], along the line of [1].

## 2 The GKO coset construction

Let us consider the diagonal coset model \cite{35,36}

$$\frac{\tilde{SO}(N)_{k} \oplus \tilde{SO}(N)_{1}}{SO(N)_{k+1}}.$$ (2.1)

The spin-1 fields $J^{ab}(z)$ and $K^{ab}(z)$, of level $k_1 = 1$ and $k_2 = k$, generate the affine Lie algebra $\tilde{SO}(N)_{k} \oplus \tilde{SO}(N)_{1}$. The indices $a, b$ take the values $a, b = 1, 2, \cdots, N$ in the representation of finite dimensional Lie algebra $SO(N)$. Due to the antisymmetric property of these fields ($J^{ab}(z) = -J^{ba}(z)$ and $K^{ab}(z) = -K^{ba}(z)$), the number of independent fields is given by $\frac{1}{2}N(N-1)$ respectively. The standard operator product expansions for these fields are given by

\begin{align}
J^{ab}(z)J^{cd}(w) &= -\frac{1}{(z-w)^2}k_1(-\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}) \\
&\quad + \frac{1}{(z-w)} \left[\delta^{bc}J^{ad}(w) + \delta^{ad}J^{bc}(w) - \delta^{ac}J^{bd}(w) - \delta^{bd}J^{ac}(w)\right] + \cdots, (2.2)
\end{align}
and
\[
K^{ab}(z)K^{cd}(w) = - \frac{1}{(z-w)^2} k_2 (-\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}) \\
+ \frac{1}{(z-w)} \left[ \delta^{bc}K^{ad}(w) + \delta^{ad}K^{bc}(w) - \delta^{ac}K^{bd}(w) - \delta^{bd}K^{ac}(w) \right] + \cdots. \tag{2.3}
\]

When the pair of integers in terms of a single indices \(A = (ab)\) and \(B = (cd)\) is used, then the Kronecker delta is given by \(\delta^{AB} = -\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}\). From this, \(\delta^{AB}\) is symmetric under the interchange of \(A\) and \(B\) (that is, \(a \leftrightarrow c\) and \(b \leftrightarrow d\)). However, under the change \(a \leftrightarrow b\) (or under the change \(c \leftrightarrow d\), it is antisymmetric. This is consistent with the operator product expansions (2.2) and (2.3). Similarly, the structure constant with single index notation \(f^{ABC}\) can be written as \(f^{ABC} = f^{(ab)(cd)(ef)} = \delta^{ae}(\delta^{bc}\delta^{df} - \delta^{bd}\delta^{cf}) + \delta^{be}(\delta^{ad}\delta^{cf} - \delta^{ac}\delta^{df}) \tag{3}\).

The spin-1 field of the diagonal affine Lie subalgebra \(\widetilde{SO}(N)_{k+1}\) in the coset model (2.1) can be viewed as perturbations of the \(\hat{K}^{(ab)}\) and \(\hat{K}^{ab}(z)\)
\[J^{ab}(z) = J^{ab}(z) + K^{ab}(z). \tag{2.4}\]

The operator product expansion of (2.4) can be obtained from the defining equations (2.2) and (2.3) and the fact that there are no singular terms in the operator product expansion \(J^{ab}(z)K^{cd}(w)\):
\[
J^{ab}(z)J^{cd}(w) = - \frac{1}{(z-w)^2} (k_1 + k_2)(-\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}) \\
+ \frac{1}{(z-w)} \left[ \delta^{bc}J^{ad}(w) + \delta^{ad}J^{bc}(w) - \delta^{ac}J^{bd}(w) - \delta^{bd}J^{ac}(w) \right] + \cdots. \tag{2.5}
\]

The level of the field \(J^{ab}(z)\) is the sum of \(k_1\) and \(k_2\), \(k' = k_1 + k_2 = 1 + k\). In a sense, the coset model (2.1) can be viewed as perturbations of the \(k \to \infty\) model.

The above GKO construction looks very similar to those for the coset model \(SU(N)_{k+1}/SU(N)_k\). The only difference appears in both the dual Coxeter number and the dimension of group if we use a single index notation. In next sections, we would like to construct the two lowest higher spin generators (extending the spin-2 coset construction in the Appendix A to the higher spin currents) from the spin-1 fields \(J^{ab}(z)\) and \(K^{ab}(z)\).
3 The fourth-order Casimir operator of $B_{N-1 \over 2} = SO(N)$ where $N$ is odd

In this section, we construct the spin-4 primary field, after that we take the large $N$ limit, and compute the three-point functions with scalars.

3.1 Primary spin-4 current

The $WB_{N-1 \over 2}$ algebra by Fateev and Lukyanov [15] is generated by the fields of spins

$$2, 4, \ldots, (N-1), {N \over 2}, \quad N : \text{odd}. \quad (3.1)$$

The orders of the independent Casimir operators for the non-simply-laced simple Lie algebra $B_{N-1 \over 2}$ are given by $2, 4, \ldots, (N-1)$. The spin contents of $WB_{N-1 \over 2}$ algebra are related to the exponent of the Lie superalgebra $B(0, {N-1 \over 2}) = OSp(1, N-1)$. The operator product expansion of fermionic spin-$N \over 2$ field with itself provides the structure of the remaining bosonic fields of spin $2, 4, \ldots, (N-1)$:

$$\tilde{U}_{WB}(z)\tilde{U}_{WB}(w) = \frac{1}{(z-w)^N} \frac{2\tilde{c}}{N} + \frac{1}{(z-w)^{N-2}} 2\tilde{T}(w) + \frac{1}{(z-w)^{N-3}} \partial \tilde{T}(w)$$

$$+ \frac{1}{(z-w)^{N-4}} \left[ \tilde{T}\tilde{T}(w), \partial^2 \tilde{T}(w), \tilde{V}_{WB}(w) \right] + O((z-w)^{-N+5}). \quad (3.2)$$

The bosonic currents can be read off from (3.2). The coset central charge $\tilde{c}$ is given by (A.4). The spin-2 field appears in the singular term $1 \over (z-w)^{N-2}$ and its descendant spin-3 field is located at the next singular term $1 \over (z-w)^{N-3}$. The terms in $1 \over (z-w)^{N-4}$ of (3.2) have spin-4 fields. We would like to find the spin-4 primary field $\tilde{V}_{WB}(z)$ explicitly. One should also consider the spin-4 fields $\tilde{T}\tilde{T}(z)$ and $\partial^2 \tilde{T}(z)$ coming from the stress energy tensor $\tilde{T}(z)$. In principle, the higher spin fields of spin $s$ greater than 4 arise in the lower singular terms in (3.2) but its exact structure is not known explicitly so far. The highest spin field with spin-$(-N-1)$ in (3.1) appears in the $1 \over (z-w)$ term which is the lowest singular term. Since there is no spin-1 field in this minimal model, there is no $1 \over (z-w)^{N-1}$ term in the operator product expansion (3.2). For $N = 3$, the $WB_3$ algebra coincides with the $N = 1$ super Virasoro algebra.

It is ready to construct the above spin-$N \over 2$ field in terms of spin-1 fields $J^{ab}(z)$ and $K^{ab}(z)$. One defines the spin-1 field as composite of the $N$-free fermions [37]

$$J^{ab}(z) = \psi^a \psi^b(z). \quad (3.3)$$

The operator product expansion of fermions of spin $s = 1 \over 2$ is

$$\psi^a(z)\psi^b(w) = \frac{1}{(z-w)^{\delta^{ab}}} + \cdots. \quad (3.4)$$
The fermion fields anticommute and have the mode expansion with the Neveu-Schwarz sector or the Ramond sector. It is easy to check the operator product expansion (2.2) is satisfied with level \( k_1 = 1 \) by using (3.3) and (3.4). One also checks that this fermion is primary field of spin-\( \frac{1}{2} \) under the stress energy tensor \( T_{(1)}(z) = -\frac{1}{2}\hat{\psi}^a \partial \hat{\psi}_a(z) \) in (A.2). According to the observation of Watts [37], the spin-\( \frac{N}{2} \) field \( \hat{U}(z) \) consists of \( \frac{N+1}{2} \) independent terms with arbitrary coefficient functions \( \tilde{A} \)'s which depend on both \( N \) and \( k \) (the explicit expressions for these coefficients are given in [37]).

\[
\hat{U}_{WB}(z) = \epsilon^{a_1 a_2 \cdots a_N} [A_0(N, k) \psi^{a_1} J^{a_2 a_3} \cdots J^{a_{N-1} a_N}(z) + \cdots + A_{\frac{N-1}{2}}(N, k) \psi^{a_1} K^{a_2 a_3} \cdots K^{a_{N-1} a_N}(z)] .
\]

(3.5)

This is a singlet under the underlying \( SO(N) \) subalgebra of \( \tilde{SO}(N) \). The epsilon tensor of \( N \) indices is \( SO(N) \) group invariant. Now we substitute (3.5) into the operator product expansion (3.2) and look for \( \frac{1}{(z-w)^{N-1}} \) terms. Using the operator product expansions (2.2) and (2.3), the four indices in the left hand side will distribute to either Kronecker delta \( \delta^e_f \) or spin-1 fields \( J^e_f(z) \) or \( K^e_f(z) \) in the right hand side. At first, one sees the lowest singular term \( \frac{1}{(z-w)^{N-1}} \) in the operator product expansion \( \hat{U}_{WB}(z) \hat{U}_{WB}(w) \). The higher singular terms \( \frac{1}{(z-w)^n} \) where \( n = 2, 3, 4, \cdots, N \) can be obtained from the spin-(\( N-1 \)) field located at \( \frac{1}{(z-w)^N} \) term by contracting the remaining indices between the fields in the normal ordered product. For example, the operator product expansion between the first term of (3.5) with itself will lead to \( \epsilon^{a_1 a_2 \cdots a_N} \epsilon^{a_1 b_2 \cdots b_N} (J^{a_2 a_3} \cdots J^{a_{N-1} a_N})(J^{b_2 b_3} \cdots J^{b_{N-1} b_N})(w) \) after using the operator product expansion between the field \( \psi^{a_1}(z) \) and the field \( \psi^{b_1}(w) \). Further contractions between the remaining expression will give rise to the lower spin field of spin \( (N-2) \) by removing one current or spin \( (N-3) \) by removing two currents. Then the Kronecker delta's make a contraction between two \( SO(N) \) epsilon tensors and the order of the original spin-1 fields, \( (N-1) \), is reduced to \( (N-2), (N-3), (N-4) \cdots, 4, 3, 2, 0 \) depending on the location of singular terms where the descendant fields are added. The fields of spins \( 3, 5, \cdots, (N-2) \) will correspond to the descendant fields of bosonic fields of spins \( 2, 4, \cdots, (N-1) \) of \( WB_{\frac{N-1}{2}} \) minimal model.

How does one determine the nontrivial spin-4 field which has the lowest higher spin greater than 2 in the \( WB_{\frac{N-1}{2}} \) minimal model? It is easy to see, after completing the procedure in previous paragraph, that the spin-2 fields coming from \( \hat{U}_{WB}(z) \) are given by \( J^{cd} J^{e_f}(z) \), \( J^{cd} K^{e_f}(z) \) and \( K^{cd} K^{e_f}(z) \) with epsilon tensor \( \epsilon^{a_1 a_2 \cdots a_{N-4} cdef} \) and similarly those from \( \hat{U}_{WB}(w) \) are \( J^{gh} J^{ij}(w) \), \( J^{gh} K^{ij}(w) \) and \( K^{gh} K^{ij}(w) \) with epsilon tensor \( \epsilon^{b_1 b_2 \cdots b_{N-4} ghij} \). The normal ordered products of these fields with appropriate contracted two epsilon tensors arise
in the singular term \( \frac{1}{(z-w)^4} \) in (3.2). It turns out that the spin-4 fields have the following structure

\[
\epsilon^a_1 \epsilon^a_2 \cdots \epsilon^a_{2N-4} \epsilon^b_1 \epsilon^b_2 \epsilon^b_3 \epsilon^b_{4g} \epsilon^b_i \epsilon^b_j (J^{cd} J^{ef}) (J^{gh} J^{ij}) (z) \sim \delta^a_{[g} \delta^d_{h} \delta^e_{i} \delta^f_{j} (J^{cd} J^{ef}) (J^{gh} J^{ij}) (z),
\]

(3.6)

Then one writes down the possible various spin-4 fields, by simplifying the right hand side of (3.6), as follows:

\[
c_1 J^{cd} J^{ef} J^{cd} J^{ef} (z) + c_2 J^{cd} J^{ef} J^{cd} K^{ef} (z) + c_3 J^{cd} J^{ef} K^{cd} K^{ef} (z) + c_4 J^{cd} K^{ef} K^{cd} K^{ef} (z) +
\]

\[
c_5 K^{cd} K^{ef} K^{cd} K^{ef} (z) + c_6 J^{cd} J^{ef} J^{ce} J^{df} (z) + c_7 J^{cd} J^{ef} J^{ce} K^{df} (z) + c_8 J^{cd} J^{ef} K^{ce} K^{df} (z) +
\]

\[
c_9 J^{cd} K^{ef} K^{ce} K^{df} (z) + c_{10} J^{cd} K^{ef} K^{ce} K^{df} (z) + c_{11} J^{cd} J^{ef} J^{j} f (z) + c_{12} J^{cd} J^{ed} J^{ef} K^{ef} (z) +
\]

\[
c_{13} J^{cd} J^{cd} K^{ef} K^{ef} (z) + c_{14} J^{cd} J^{cd} K^{cd} K^{ef} (z) + c_{15} J^{cd} K^{cd} K^{ef} K^{ef} (z) + c_{16} J^{cd} J^{ce} J^{df} K^{ef} (z) +
\]

\[
c_{17} J^{cd} J^{ce} J^{df} K^{ef} (z) + c_{18} J^{cd} J^{ce} J^{df} K^{ef} (z) + c_{19} J^{cd} K^{ce} K^{ef} K^{df} (z) + c_{20} K^{cd} K^{ce} K^{df} K^{ef} (z) +
\]

\[
c_{21} J^{cd} J^{ce} J^{df} J^{ef} (z) + c_{22} J^{cd} J^{ce} J^{df} K^{ef} (z) + c_{23} J^{cd} J^{ce} J^{df} K^{ef} (z) + c_{24} J^{cd} K^{ce} K^{df} K^{ef} (z) +
\]

\[
c_{25} K^{cd} K^{ce} K^{df} K^{ef} (z).
\]

(3.7)

The spin-4 fields are quartic in the currents with appropriate index structure. Of course, one should expect that there exist some derivative terms between the spin-1 fields from the normal ordered products (3.6) to fully normal ordered products (3.7). Compared to the minimal model based on \( SU(N) \) group where the \( d \) symbols of different ranks are contracted with the currents, the symmetric \( SO(N) \) invariant tensor of rank 2, Kronecker delta, plays an important role.

On the other hand, one can think of the following derivatives

\[
d_1 \partial J^{ab} \partial J^{ab} (z) + d_2 \partial J^{ab} J^{ab} (z) + d_3 \partial J^{ab} K^{ab} (z) + d_4 \partial K^{ab} \partial K^{ab} (z) +
\]

\[
d_5 \partial J^{ab} K^{ab} (z) + d_6 \partial J^{ab} \partial K^{ab} (z) + d_7 J^{ab} \partial J^{ab} K^{ab} (z) + d_8 J^{ab} \partial J^{ac} K^{bc} (z) +
\]

\[
d_9 J^{ab} K^{ac} \partial K^{bc} (z),
\]

(3.8)

where some of these come from the derivative field of stress energy tensor \( \partial^2 \tilde{T} (z) \).
Therefore, the spin-4 candidate given by (3.7) and (3.8) can be further simplified and summarized by the following 21 (= 25 + 9 − 1) independent terms, via the detailed analysis in the Appendix B,

\[
\tilde{V}(z) = c_3 J^{cd} J^{ef} K^{cd} K^{ef}(z) + c_8 J^{cd} J^{ef} K^{ce} K^{df}(z) + c_9 J^{cd} K^{ef} K^{ce} K^{df}(z)
+ c_{10} K^{cd} K^{ef} K^{ce} K^{df}(z) + c_{11} J^{cd} J^{ef} J^{f}(z) + c_{12} J^{cd} J^{ef} K^{ef}(z) + c_{13} J^{cd} J^{ef} K^{ef}(z)
+ c_{14} J^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{15} K^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{16} J^{cd} J^{ef} K^{df}(z) + c_{17} J^{cd} J^{ce} J^{ef}(z)
+ c_{18} J^{cd} J^{ce} J^{ef}(z) + d_1 \partial J^{ab} \partial J^{ab}(z) + d_2 \partial^2 J^{ab} J^{ab}(z) + d_3 \partial K^{ab} \partial K^{ab}(z) + d_4 \partial^2 K^{ab} K^{ab}(z)
+ d_5 \partial^2 J^{ab} K^{ab}(z) + d_6 \partial J^{ab} \partial K^{ab}(z) + d_7 J^{ab} \partial^2 K^{ab}(z) + d_8 J^{ab} \partial J^{ac} K^{bc}(z)
+ d_9 J^{ab} K^{ac} \partial K^{bc}(z).
\]

(3.9)

Compared to the \(WB_2\) minimal model (i.e., \(N = 5\)) [38] (See also [39]), there exist three extra terms: \(c_8\)-term, \(c_{21}\)-term and \(c_{22}\)-term. We will see that these extra terms can be absorbed into the other independent terms for \(WB_{N-1}\) minimal model by using \(N\)-free fermion description. We also use these 21 independent terms for the \(WD_{\frac{N}{2}}\) minimal model.

It is ready to determine the coefficient functions in (3.9). At first, the primary spin-4 field should commute with the diagonal spin-1 field as follows [2]:

\[
J^{ab}(z)\tilde{V}(w) = \text{regular}.
\]

(3.10)

In other words, there are no singular terms (\(\frac{1}{(z-w)^n}\) terms where \(n = 5, 4, 3, 2, 1\)) in the operator product expansion (3.10). Secondly, the coset spin-4 primary field should transform as dimension 4 under the stress energy tensor (A.1) as follows [2]:

\[
\tilde{T}(z)\tilde{V}(w) = \frac{1}{(z-w)^2}4\tilde{V}(w) + \frac{1}{(z-w)}\partial\tilde{V}(w) + \cdots.
\]

(3.11)

That is, there should be no singular terms (\(\frac{1}{(z-w)^n}\) terms where \(n = 6, 5, 4, 3\)) in the operator product expansion (3.11). Sometimes, it is convenient to introduce the stress energy tensor in the affine Lie algebra \(\tilde{SO}(N)_k \oplus \tilde{SO}(N)_1\):

\[
T_{(1)}(z) + T_{(2)}(z) \equiv \tilde{T}(z).
\]

(3.12)

The equation (3.10) implies that there are no singular terms in the operator product expansion of \(T'(z)\tilde{V}(w)\) because \(T'(z)\) is quadratic in \(J^{ab}(z)\) from (A.2). Therefore, it is equivalent to compute the operator product expansion of \(\tilde{T}(z)\tilde{V}(w)\). In the Appendix C, we describe the operator product expansions (3.10) where we consider the spin-4 field in (3.7) and (3.8). The reason for why we do take these rather than (3.9) is that sometimes we want to express the
spin-4 field which is quartic in the currents without any derivative terms. In the Appendix D, we compute the operator product expansion $\hat{T}(z)\hat{V}(w)$ which should be equal to the equation (3.11) under the condition (3.10). We will describe some details in the Appendix B.

Therefore, we take the final correct spin-4 field as follows:

$$\tilde{V}_{WB}(z) = c_{33} J^{cd} J^{ef} K^{cd} K^{ef}(z) + c_{99} J^{cd} K^{ef} K^{ce} K^{df}(z) + c_{100} K^{cd} K^{ef} K^{ce} K^{df}(z)$$

$$+ c_{11} J^{cd} J^{ef} J^{ef}(z) + c_{12} J^{cd} J^{ef} K^{ef}(z) + c_{13} J^{cd} J^{ef} K^{ce} K^{df}(z) + c_{14} J^{cd} K^{cd} K^{ef} K^{ef}(z)$$

$$+ c_{15} K^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{16} J^{cd} J^{ef} K^{ef}(z) + c_{17} J^{cd} K^{cd} K^{cd} K^{ef}(z) + c_{18} J^{cd} J^{ef} K^{ef}(z) + d_{1} \partial J^{ab} \partial J^{ab}(z) + d_{2} \partial^{2} J^{ab} J^{ab}(z)$$

$$+ d_{3} \partial K^{ab} \partial K^{ab}(z) + d_{4} \partial^{2} K^{ab} \partial K^{ab}(z) + d_{5} \partial^{2} J^{ab} \partial K^{ab}(z) + d_{6} \partial J^{ab} \partial K^{ab}(z) + d_{7} J^{ab} \partial^{2} K^{ab}(z)$$

$$+ d_{8} J^{ab} \partial J^{ac} K^{bc}(z) + d_{9} J^{ab} K^{ac} \partial K^{bc}(z).$$

(3.13)

For $N = 5$, the field contents of (3.13) are exactly same as the ones in [38]. This is one of the reasons why we take the particular combination for the various spin-4 fields as in (3.13). In Appendix E, the requirements (3.10) and (3.11) are imposed and the coefficient functions appearing the spin-4 field in (3.13), in terms of finite $(N,k)$, are determined. However, the coefficient functions $c_{9}$ and $d_{8}$ are not fixed. All the coefficient functions are written in terms of these two coefficient functions. This common feature also occurs in the $W_{N}$ minimal model if one does not consider the operator product expansion between the primary spin-3 fields. According to the field contents of (3.11), there are no lower spin fields of spin less than 4, contrary to the $W_{A_{N-1}}$ minimal model. In order to fix these unknown coefficient functions, one should compute the operator product expansion of $\tilde{V}_{WB}(z)\tilde{V}_{WB}(w)$ explicitly. In the Appendix F, we present the field contents for the $W_{B_{2}}$ algebra corresponding to $N = 5$ case for convenience.

The primary spin-4 current which is fourth order Casimir operator of $SO(N)$ where $N$ is odd is given by (3.13) with the coefficient functions in (E.7). In next subsection, we describe this primary spin-4 current in the large $N$ limit and find three-point functions with scalars.

### 3.2 Primary spin-4 current in the large $N$ ’t Hooft limit and three-point functions with two scalars

The large $N$ ’t Hooft limit is described as [11]

$$N, k \rightarrow \infty, \quad \lambda \equiv \frac{N}{N + k} \text{ fixed.}$$

(3.14)

One should find the spin 4 zero mode on the vector representation. The spin-4 field is given by (3.13). Let us first consider the quartic terms. From the matrix representation in
the footnote \[ \footnote{3} \] one has
\[
\text{Tr}(T^{cd} T^{ef} T^{cd} T^{ef}) = i(\delta^c_i \delta^d_j - \delta^c_j \delta^d_i) i(\delta^e_k \delta^f_l - \delta^e_l \delta^f_k)i(\delta^c_i \delta^d_j - \delta^c_j \delta^d_i)
= 4N(N-1) \to 4N^2.
\]

(3.15)

Here we take the large \( N \) limit \((3.14)\). In order to obtain the eigenvalue, one should divide this \((3.15)\) by \( N \). Then, the zero mode (relevant to the \( c_1-c_5 \) terms) acting on the vector representation implies that

\[
J_0^c J_0^d J_0^e J_0^f |v >= 4N|v >.
\]

(3.22)

Combining the results in the Appendix \( E \), the leading contribution \( N^2 \) from \( d_8 \) factor comes from the coefficient functions, \( c_{18}, d_1, d_2, d_5, d_6, d_7, d_8 \) and \( d_9 \)
\[
N^3 c_{18} - 2Nd_1 - 4Nd_2 + 4Nd_5 + 2Nd_6 + 4Nd_7 - N^2 d_8 + N^2 d_9
\to \frac{N^2 (-12 - 16\lambda - 99\lambda^2 + 85\lambda^3)}{10(-2 + \lambda)\lambda(-6 + 5\lambda)} d_8.
\]

(3.23)

\[ \footnote{Similarly, from the identity \( \text{Tr}(T^{cd} T^{ef} T^{ce} T^{df}) = N^2(N-1) \to N^3 \) that can be obtained from the matrix representation for the generator, one obtains}
\[
J_0^c J_0^d J_0^e J_0^f |v >= 4N^2|v >.
\]

(3.16)

which are relevant to the \( c_6-c_{10} \) terms. It is straightforward to compute \( \text{Tr}(T^{cd} T^{cd} T^{ef} T^{ef}) = 4N(N-1)^2 \to 4N^3 \) which can be checked from the matrix representation, and the corresponding eigenvalue equation (relevant to \( c_{11}-c_{15} \) terms) leads to

\[
J_0^c J_0^d J_0^e J_0^f |v >= 4N^2|v >.
\]

(3.17)

Furthermore, from the relation \( \text{Tr}(T^{cd} T^{ce} T^{df} T^{ef}) = N(N-1)(N^2 - 3N + 4) \to N^4 \), one has

\[
J_0^c J_0^d J_0^e J_0^f |v >= N^3|v >.
\]

(3.18)

which are relevant to \( c_{16}-c_{20} \) terms. The last one should have is \( \text{Tr}(T^{cd} T^{ce} T^{df} T^{ef}) = N^2(N-1) \to N^3 \), and the eigenvalue equation (corresponding to \( c_{21}-c_{25} \) terms) is given by

\[
J_0^c J_0^d J_0^e J_0^f |v >= N^2|v >.
\]

(3.19)

Therefore, the power of \( N \) in (3.18) is higher than the ones in (3.22), (3.16), (3.17) or (3.19). The \( N^3 \) behavior of (3.18) is the same as the one in \( W A_{N-1} \) minimal model [10].

Let us consider the quadratic and cubic terms in (3.13). One needs to have

\[
\text{Tr}(T^{cd} T^{cd}) = -2N(N-1) \to -2N^2,
\]

(3.20)

which is relevant to the \( d_1-d_9 \) terms and dividing this (3.20) by \( N \), one obtains spin-2 zero mode on the vector representation in the large \( N \) ’t Hooft limit

\[
J_0^c J_0^d |v >= -2N|v >.
\]

(3.21)

From (3.21), one can obtain the spin-4 zero mode with two derivatives.
The leading contribution \( N^3 \) from \( c_9 \) factor comes from the coefficient functions \( d_7 \) and \( d_9 \).

\[
4Nd_7 + N^2d_9 \to - \left[ \frac{14N^3(-1 + \lambda)}{-6 + 5\lambda} \right] c_9.
\]  

(3.24)

Finally, by substituting the coefficient functions in the large \( N \) limit into (3.7) and (3.8) and evaluating the correct eigenvalues, one arrives at the final contributions acting on the representation \((v;0) \otimes (v;0)\), where \( J_{ab}^0 + K_{ab}^0 = 0 \), by combining (3.23) and (3.24),

\[
\tilde{V}_0|O_+ > = -\frac{N^2}{(-6 + 5\lambda)} \left[ \frac{d_8 (-12 - 16\lambda - 99\lambda^2 + 85\lambda^3)}{10(-2 + \lambda)\lambda} \right] + 14c_9N(-1 + \lambda)|O_+ >,
\]

(3.25)

where \( O_+ \equiv (v;0) \otimes (v;0) \) and this is equivalent to \((2,1 \frac{N-3}{2}|1 \frac{N-1}{2}) \otimes (2,1 \frac{N-3}{2}|1 \frac{N-1}{2})\) in the convention of [13].

Next let us consider the zero mode eigenvalue acting on \( |O_- > \equiv |(0;v) \otimes (0;v) >\). For the primary \((0;v) \otimes (0;v)\), the field \( K_{ab}^0 \) vanishes. Then there exist nonzero contributions from \( c_{11^-}, d_1^- \) and \( d_2^- \)-terms. The \( c_{11^-} \)-term has \( N^2 \times \frac{1}{N} = N \) dependence. The \( d_1^- \) and \( d_2^- \)-terms have \( N \times N = N^2 \) dependence. So one arrives at

\[
-2Nd_1 - 4Nd_2 = -\left[ \frac{N^2(-1 + \lambda)}{10\lambda} \right] d_8.
\]

In other words, one obtains

\[
\tilde{V}_0|O_- > = -N^2 \left[ \frac{(-1 + \lambda)}{10\lambda} \right] d_8|O_- >,
\]

(3.26)

where \( O_- \equiv (0;v) \otimes (0;v) \) which is equal to \((1 \frac{N-1}{2}|2,1 \frac{N-3}{2}) \otimes (1 \frac{N-1}{2}|2,1 \frac{N-3}{2})\) in the convention of [13]. The vector representation of \( SO(N) \) is self-conjugate and there is no separate conjugate representation, contrary to the fundamental representation of \( SU(N) \).

For the choice of

\[
c_9(N, \lambda) = -\frac{15(-2 + \lambda)\lambda^2(-5 + 3\lambda)}{7N^3(-1 + \lambda)}, \quad d_8(N, \lambda) = \frac{10(-3 + \lambda)(-2 + \lambda)\lambda}{N^2},
\]

(3.27)

one has the following eigenvalue equations, from (3.25) and (3.26),

\[
\tilde{V}_0|O_+ > = (1 + \lambda)(2 + \lambda)(3 + \lambda)|O_+ >, \quad O_+ \equiv (v;0) \otimes (v;0),
\]

\[
\tilde{V}_0|O_- > = (1 - \lambda)(2 - \lambda)(3 - \lambda)|O_- >, \quad O_- \equiv (0;v) \otimes (0;v).
\]

(3.28)

Recall that the zero mode eigenvalues for arbitrary spin \( s \) in the boundary theory are found in [11]. If one puts \( s = 4 \), then they are exactly the same as (3.28) up to unfixed \( \lambda \)-independent normalization which depends on the spin \( s \) explicitly. One expects that if one
computes the operator product expansion $\tilde{V}_{WB}(z)\tilde{V}_{WB}(w)$, the singular terms should behave as $\tilde{V}_{WB}(z)\tilde{V}_{WB}(w) = \frac{1}{(z-w)^2} + \frac{1}{(z-w)^4}2\tilde{T}(w) + \frac{1}{(z-w)^4}\partial\tilde{T}(w) + \mathcal{O}\left(\frac{1}{(z-w)^4}\right)$ where the stress energy tensor is given by (A.1) and (A.2). Then the undetermined two coefficient functions $c_9(N,k)$ and $d_8(N,k)$ occur in this operator product expansion. Only after this computation which will be very complicated (i.e., $18 \times 18 = 324$ operator product expansions one should compute) is done, they are fixed completely. Otherwise, one does not know what they are. They should take the form (3.27) as one takes the large $N$ limit.

The three-point functions with two real scalars, from (3.28), is summarized as

$$\langle O_+O_+\tilde{V} \rangle = (1 + \lambda)(2 + \lambda)(3 + \lambda),$$
$$\langle O_-O_-\tilde{V} \rangle = (1 - \lambda)(2 - \lambda)(3 - \lambda). \quad (3.29)$$

It would be interesting to find the three-point functions in the deformed $AdS_3$ bulk theory for all values of 't Hooft coupling constant and to compare to the three-point functions (3.29) in the $WB_N$ coset conformal field theory in the large $N$ limit. See, for example, [11].

We present the final spin-4 primary field with $(N,k)$ dependent coefficient functions in (E.14). In the large $N$ limit, this becomes further simple expression as follows:

$$\tilde{V}_{WB}(z) = -\left[\frac{10(-3 + \lambda)^2}{N^3}\right]J^{cd}J^{ce}K^{ef}K^{df}(z) - \left[\frac{3(-3 + \lambda)(-2 + \lambda)(-1 + \lambda)}{2N}\right]\partial J^{ab}\partial J^{ab}(z)$$
$$+ \left[\frac{(-3 + \lambda)(-2 + \lambda)(-1 + \lambda)}{N}\right]\partial^2 J^{ab}\partial J^{ab}(z) + \left[\frac{7(-3 + \lambda)(-2 + \lambda)\lambda}{2N}\right]\partial^2 J^{ab}K^{ab}(z)$$
$$- \left[\frac{(-3 + \lambda)\lambda(-1 + 3\lambda)}{N}\right]\partial J^{ab}\partial K^{ab}(z) + \left[\frac{(-2 + \lambda)\lambda(-7 + 29\lambda)}{14N}\right]J^{ab}\partial^2 K^{ab}(z)$$
$$+ \left[\frac{10(-3 + \lambda)(-2 + \lambda)\lambda}{N^2}\right]J^{ab}\partial J^{ac}K^{bc}(z) + \left[\frac{40(-2 + \lambda)\lambda^2}{7N^2}\right]J^{ab}K^{ac}\partial K^{bc}(z). \quad (3.30)$$

Of course, one can rewrite this (3.30) using the equations (B.2) in terms of quartic fields only. When one acts this spin-4 zero mode on the primary states, one sees that all the $N$-dependence disappears and it leads to the equation (3.28).

4 The fourth-order Casimir operator of $D_N = SO(N)$ where $N$ is even

In this section, the spin 4 primary field, its large $N$ limit and the three-point functions with scalars are constructed as previous section.
4.1 Primary spin-4 current

The $WD_{\frac{N}{2}}$ algebra \[13, 40\] is generated by the fields of spins

$$2, 4, \cdots, (N - 2), \frac{N}{2} \quad N : \text{even.} \quad (4.1)$$

The orders of the independent Casimir operator for the simple Lie algebra $D_{\frac{N}{2}}$ are $2, 4, \cdots, (N - 2)$ and $\frac{N}{2}$. Since $D_2 \simeq A_1 \times A_1$ is not simple, one shall restrict to $N \geq 3$. The operator product expansion of bosonic spin $\frac{N}{2}$ field with itself provides the bosonic fields of spin $2, 4, \cdots, (N-2)$:

$$\tilde{U}_{WD}(z)\tilde{U}_{WD}(w) = \frac{1}{(z-w)^N} \frac{2\tilde{e}}{N} + \frac{1}{(z-w)^{N-2}} 2\tilde{T}(w) + \frac{1}{(z-w)^{N-3}} \partial \tilde{T}(w) + \frac{1}{(z-w)^{N-4}} \left[ \tilde{T}(w), \partial^2 \tilde{T}(w), \tilde{V}_{WD}(w) \right] + O((z-w)^{-N+5}). \quad (4.2)$$

The highest spin field with spin-$(N - 2)$ in (4.1) appears in the $\frac{1}{(z-w)^{N-2}}$ term. We expect that the descendant of highest higher field of spin-$(N - 2)$ should appear in the singular term $\frac{1}{(z-w)^{N-4}}$ in (4.2). As in previous section, we would like to find the spin-4 primary field $\tilde{V}_{WD}(z)$ which will be present in the $\frac{1}{(z-w)^{N-4}}$ singular term of (4.2). There is no $\frac{1}{(z-w)^{N-3}}$ singular term.

The bosonic spin $\frac{N}{2}$ field $\tilde{U}_{WD}(z)$, that has $\frac{N+2}{2}$ terms, consists of

$$\tilde{U}_{WD}(z) = e^{a_1a_2\cdots a_N} [A_0(N, k) J^{a_1a_2} \cdots J^{a_{N-1}a_N}(z) + \cdots + A_{\frac{N+2}{2}}(N, k) K^{a_1a_2} \cdots K^{a_{N-1}a_N}(z) \right]. \quad (4.3)$$

The arbitrary coefficient functions depend on the two integers $(N, k)$. Then one can substitute (4.3) into (4.2). We would like to focus on the $\frac{1}{(z-w)^{N+4}}$ singular terms. From the singular term $\frac{1}{(z-w)^3}$ in the operator product expansion $\tilde{U}_{WD}(z)\tilde{U}_{WD}(w)$, the higher singular terms \( \frac{1}{(z-w)^n} \) where $n = 3, 4, \cdots, (N - 2)$) can be obtained from the spin-$(N - 2)$ field located at $\frac{1}{(z-w)^2}$ term, by contracting the remaining indices between the fields in the normal ordered product. The operator product expansion between the first term of (4.3) with itself will lead to $e^{a_1a_2\cdots a_N} e^{a_1a_2\cdots b_N} (J^{b_3b_4} \cdots J^{b_{N-1}b_N})(J^{b_{N-1}b_N})(w)$ after using the highest singular term in the operator product expansion between the field $J^{a_1a_2}(z)$ and the field $J^{b_1b_2}(w)$. Further contractions between the remaining expression will give rise to the lower spin field of spin $(N - 3)$ by removing one current (corresponding to the $\frac{1}{(z-w)^{n-1}}$ term of $J^{a_1a_2}(z) J^{b_1b_2}(w)$) or spin $(N - 4)$ by removing two currents (corresponding to the $\frac{1}{(z-w)^{n-2}}$ term of $J^{a_1a_2}(z) J^{b_1b_2}(w)$). Then the order of the original spin-1 fields, $(N - 2)$, is reduced to $(N - 3), (N - 4), (N - 5)\cdots, 4, 3, 2$ depending on the location of singular terms. The fields of spins $3, 5, \cdots, (N - 3)$ correspond to
the descendant fields of bosonic fields of spins $2, 4, \cdots, (N-2)$ of $WD_N$ minimal model. One expects that the descendant field for spin-$(N-2)$ field should appear in the lowest singular term $\frac{1}{(z-w)}$ in (4.3).

In this case, the analysis of (3.6) also holds. Following the procedures in previous section, one obtains the possible spin-4 fields and the spin-4 field in $WD_N$ minimal model is given by (3.9):

$$\widetilde{V}_{WD}(z) = c_3 J^{cd} J^{ef} K^{cd} K^{ef}(z) + c_8 J^{cd} J^{ef} K^{ce} K^{df}(z) + c_9 J^{cd} K^{ef} K^{ce} K^{df}(z)$$

$$+ c_{10} K^{cd} K^{ef} K^{ce} K^{df}(z) + c_{11} J^{cd} J^{ef} J^{ef}(z) + c_{12} J^{cd} J^{ef} J^{ef}(z) + c_{13} J^{cd} J^{ef} K^{ef}(z)$$

$$+ c_{14} J^{cd} J^{ef} K^{ce} K^{df}(z) + c_{15} K^{cd} K^{ef} K^{ce} K^{df}(z) + c_{16} J^{cd} J^{ef} K^{df}(z) + c_{17} J^{cd} J^{ef} J^{df}(z)$$

$$+ c_{18} J^{cd} J^{ef} K^{ef}(z) + d_1 \partial J^{ab} \partial J^{ab}(z) + d_2 \partial^2 J^{ab} J^{ab}(z) + d_3 \partial K^{ab} \partial K^{ab}(z) + d_4 \partial^2 K^{ab} K^{ab}(z)$$

$$+ d_5 \partial^2 J^{ab} K^{ab}(z) + d_6 \partial J^{ab} \partial K^{ab}(z) + d_7 J^{ab} \partial^2 K^{ab}(z) + d_8 J^{ab} \partial J^{ac} K^{bc}(z)$$

$$+ d_9 J^{ab} K^{ac} \partial K^{bc}(z).$$  \hspace{1cm} (4.4)

One uses the relations (B.1), (3.8) and (B.2). Then one uses the two requirements (3.10) and (3.11) in order to determine the coefficient functions in (4.4). In Appendix G, the requirements (3.10) and (3.11) are imposed and the coefficient functions appearing the spin-4 field in (4.4), in terms of finite $(N,k)$, are determined. It turns out that they are written in terms of two unknown coefficient functions $c_8$ and $c_{10}$. Here we impose the following conditions for the coefficient functions

$$c_1 = c_2 = c_4 = c_5 = c_6 = c_7 = c_{16} = c_{17} = c_{19} = c_{20} = c_{23} = c_{24} = c_{25} = 0. \hspace{1cm} (4.5)$$

Note that $c_{10}$ are nonzero in this case. The whole independent terms consists of 12 quartic terms and 9 derivative terms. In order to determine the above coefficients $c_{10}$, one should compute the operator product expansion $\widetilde{V}_{WD}(z)\widetilde{V}_{WD}(w)$. In the Appendix H, we present the field contents for the $WD_3$ algebra corresponding to $N = 6$ case.

The primary spin-4 current which is fourth order Casimir operator of $SO(N)$ where $N$ is even is given by (4.4) with the coefficient functions in (G.3). In next subsection, we describe this primary spin-4 current in the large $N$ limit and find three-point functions with scalars, as in previous analysis for $WB_{N/4}$ minimal model.

### 4.2 Primary spin-4 current in the large $N$ ’t Hooft limit and three-point functions with two scalars

By substituting the coefficient functions (G.4), in the large $N$ limit, into (4.4) and evaluating the correct eigenvalues, one arrives at the final contributions for the spin-4 zero mode
eigenvalue acting on the representation \((v; 0) \otimes (v; 0)\), where \(J_0^{ab} + K_0^{ab} = 0\), from (G.5) and (G.6),
\[
\bar{V}_0|\mathcal{O}_+ > = \left[ -\frac{c_8 N^2 (-1 + 2\lambda + 39\lambda^2)}{4\lambda^2} + \frac{2c_{10} N^4 (6 + 11\lambda + 56\lambda^2 + 11\lambda^3)}{5\lambda^4}\right]|\mathcal{O}_+ >, (4.6)
\]
where \(\mathcal{O}_+ \equiv (v; 0) \otimes (v; 0)\) which is equivalent to \((2, 1)^{1\over 2} \otimes (1, 1)^{1\over 2}\) in the convention of \([13]\).

For the second primary, one has vanishing \(K_0^{ab}\) and this implies that there exist contributions from the \(c_{11}, c_{21}, d_1\) - and \(d_2\)-terms. The leading contribution \(N^4\) from \(c_{10}\) factor comes from the coefficient functions, \(d_1\) and \(d_2\), leads to
\[
-2Nd_1 - 4Nd_2 \rightarrow -\left[ \frac{2N^4(-3 + \lambda)(-2 + \lambda)(-1 + \lambda)}{5\lambda^4}\right] c_{10},
\]
and the leading contribution \(N^2\) from \(c_8\) factor comes from the coefficient functions, \(c_{11}, c_{12}, d_1\) and \(d_2\), leads to
\[
4N^2c_{11} + N^2c_{21} - 2Nd_1 - 4Nd_2 \rightarrow \left[ \frac{N^2(-1 + \lambda)^2}{4\lambda^2}\right] c_8.
\]
Using (3.17), (3.19) and (3.21) with correct multiplicities for the Fourier mode on the derivative terms, the following spin-4 zero mode eigenvalue equation reads, from (4.7) and (4.8),
\[
\bar{V}_0|\mathcal{O}_- > = \left[ \frac{c_8 N^2(-1 + \lambda)^2}{4\lambda^2} - \frac{2c_{10} N^4(-3 + \lambda)(-2 + \lambda)(-1 + \lambda)}{5\lambda^4}\right]|\mathcal{O}_- >, (4.9)
\]
where \(\mathcal{O}_- \equiv (0; v) \otimes (0; v)\) which is equivalent to \((1, 1)^{1\over 2} \otimes (1, 1)^{-1\over 2}\) in the convention of \([13]\).

In this case, the spin-2 Virasoro zero mode eigenvalues are fixed by the conformal dimension as before, (E.10) and (E.12). Moreover, the three-point functions with scalars are given by (E.13). Once again, from the observation of \([14]\), the eigenvalues are given by \((1 \pm \lambda)(2 \pm \lambda)(3 \pm \lambda)\) on the primaries \(\mathcal{O}_\pm >\). For the choice of
\[
c_8(N, \lambda) = \frac{20(-3 + \lambda)(-2 + \lambda)\lambda^3(5 + \lambda)}{N^2(11 + 134\lambda - 119\lambda^2 + 14\lambda^3)},
\]
\[
c_{10}(N, \lambda) = \frac{5\lambda^4(11 + 109\lambda - 99\lambda^2 + 19\lambda^3)}{2N^4(11 + 134\lambda - 119\lambda^2 + 14\lambda^3)}, (4.10)
\]
one obtains the eigenvalue equations given by (3.28) where the two primaries are given by the above \(\mathcal{O}_\pm >\), from (4.6) and (4.9). Finally, the three-point functions are summarized by (3.29). The undetermined two coefficient functions \(c_8(N, k)\) and \(c_{10}(N, k)\) occur in this operator product expansion \(\bar{V}_{WD}(z)\bar{V}_{WD}(w)\). Only after this computation which will be very complicated (i.e., \(21 \times 21 = 441\) operator product expansions one should compute) is done, they are fixed completely. They should take the form (4.10) as one takes the large \(N\) limit.
5 Conclusions and outlook

We have found the coset primary spin-4 field (3.13) with (E.7), where two coefficient functions are not fixed, in the \( WB_{N-1/2} \) minimal model. These coefficient functions can be fixed, in principle, only after the 324 operator product expansions are computed. With appropriate choice for these coefficient functions (recalling the higher spin Lie algebra), we have constructed the three-point functions with two scalars in (3.29) under the large \( N \) 't Hooft limit. Furthermore, we also have described the coset primary spin-4 field (4.4) with (G.3) in the \( WD_N \) minimal model (with two unknown coefficient functions) and found the three-point functions with scalars in the large \( N \) limit under the similar assumption on the higher spin Lie algebra. The explicit forms for the spin-4 fields in the large \( N \) limit are given in (3.30) and (G.8). For \( WA_{N-1} \) minimal model, since all the coefficient functions are fixed, the eigenvalue equations lead to those for higher spin algebra automatically. However, for \( WB_{N-1/2} \) and \( WD_N \) minimal models, we require that the eigenvalue equations should satisfy the higher spin Lie algebra in order to fix the undetermined coefficient functions and after that all the coefficient functions are determined completely. The complete expression for the primary spin-4 field with finite \((N,k)\) is known only for the \( WA_{N-1} \) minimal model so far. In order to obtain those for the \( WB_{N-1/2} \) and \( WD_N \) minimal models, one should compute the operator product expansions explicitly as one described before.

It is simple to ask what the corresponding three-point functions in three-dimensional higher spin gravity for the present minimal models are. Based on the works of [9] or more recently [11], it is an open problem to compute the three-point functions in the bulk for any deformation parameter \( \lambda \).

In this paper, we have considered only higher spin field of fixed spin \( s = 4 \). According to the observation of [11], the three-point functions are written for arbitrary spin \( s \). Via the AdS/CFT duality in [1], one should see those three-point functions in the \( W_N \) minimal model conformal field theory in the large \( N \) limit. This implies that the results of [10] and the present paper should be generalized to the construction of coset Casimir operators of arbitrary spin \( s \). It would be interesting to find the Casimir operators of spin \( s \) in the \( WA_{N-1}, WB_{N-1/2}, \) and \( WD_N \) minimal models.

The two undetermined coefficient functions in the present minimal models cannot be fixed by the requirements that it should be a primary field of spin-4 with respect to the spin-2 coset Virasoro field and that it should commute with the diagonal subalgebra. Without computing the operator product expansions of spin-4 field with itself, are there any ways to compute the unknown two coefficient functions explicitly? If one considers the extended
$\mathcal{N} = 1$ supersymmetric algebra which contains the field contents we have discussed in this paper and its superpartners, one can construct the spin-$\frac{3}{2}$ field $\tilde{G}(z)$ which is a fermionic partner of coset spin-2 Virasoro field $\tilde{T}(z)$ in the $WB_{\frac{\mathcal{N}+1}{2}}$ minimal model, along the line of [41, 7]. For the $WD_{\frac{\mathcal{N}}{2}}$ minimal model, it is not clear how to construct odd (fermionic) spin current. Then one can compute the operator product expansion between $\tilde{G}(z)$ and spin-4 field $\tilde{V}_{WB}(w)$. In the right hand side of this operator product expansion, one expects that the highest singular term $\frac{1}{(z-w)^4}$ should be proportional to $\tilde{G}(w)$. Then this will determine the unknown coefficient functions under the above assumption. It would be interesting to find whether the $WB_{\frac{\mathcal{N}+1}{2}}$ algebra can be extended to the extended $\mathcal{N} = 1$ superconformal algebra or not.

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References

[1] M. R. Gaberdiel and R. Gopakumar, Phys. Rev. D 83, 066007 (2011).
[2] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B 304, 371 (1988).
[3] S. F. Prokushkin and M. A. Vasiliev, Nucl. Phys. B 545, 385 (1999).
[4] S. Prokushkin and M. A. Vasiliev, arXiv:hep-th/9812242.
[5] M. A. Vasiliev, arXiv:hep-th/9910096.
[6] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. A 3, 507 (1988).
[7] P. Bouwknegt and K. Schoutens, Phys. Rept. 223, 183 (1993).
[8] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, JHEP 1108, 077 (2011).
[9] C. M. Chang and X. Yin, arXiv:1106.2580 [hep-th].
[10] C. Ahn, JHEP 1202, 027 (2012).
[11] M. Ammon, P. Kraus and E. Perlmutter, arXiv:1111.3926 [hep-th].
[12] M. R. Gaberdiel and T. Hartman, JHEP 1105, 031 (2011).
[13] C. Ahn, JHEP 1110, 125 (2011).
[14] M. R. Gaberdiel and C. Vollenweider, JHEP 1108, 104 (2011).
[15] S. L. Lukyanov and V. A. Fateev, Chur, Switzerland: Harwood (1990) 117 p. (Soviet Scientific Reviews A, Physics: 15.2).
[16] M. Gary, D. Grumiller and R. Rashkov, JHEP 1203, 022 (2012).
[17] C. M. Chang and X. Yin, arXiv:1112.5459 [hep-th].
[18] M. R. Gaberdiel and P. Suchanek, JHEP 1203, 104 (2012).
[19] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, JHEP 1202, 096 (2012).
[20] T. Creutzig, Y. Hikida and P. B. Ronne, JHEP 1202, 109 (2012).
[21] A. Castro, M. R. Gaberdiel, T. Hartman, A. Maloney and R. Volpato, Phys. Rev. D 85, 024032 (2012).
[22] B. Chen and J. Long, JHEP 1112, 114 (2011).

[23] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, arXiv:1110.4386 [hep-th].

[24] S. H. Shenker and X. Yin, arXiv:1109.3519 [hep-th].

[25] M. Vasilev, arXiv:1108.5921 [hep-th].

[26] K. Papadodimas and S. Raju, Nucl. Phys. B 856, 607 (2012).

[27] P. Kraus and E. Perlmutter, JHEP 1111, 061 (2011).

[28] A. Castro, T. Hartman and A. Maloney, Class. Quant. Grav. 28, 195012 (2011).

[29] A. Bagchi, S. Lal, A. Saha and B. Sahoo, JHEP 1112, 068 (2011).

[30] A. Bagchi, S. Lal, A. Saha and B. Sahoo, JHEP 1110, 150 (2011).

[31] A. Campoleoni, S. Fredenhagen and S. Pfenninger, JHEP 1109, 113 (2011).

[32] B. Chen, J. Long and J. -b. Wu, Phys. Lett. B 705, 513 (2011).

[33] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, JHEP 1110, 053 (2011).

[34] A. Jevicki, K. Jin and Q. Ye, J. Phys. A A 44, 465402 (2011).

[35] P. Goddard, A. Kent and D. I. Olive, Phys. Lett. B 152, 88 (1985).

[36] P. Goddard, A. Kent and D. I. Olive, Commun. Math. Phys. 103, 105 (1986).

[37] G. M. T. Watts, Nucl. Phys. B 339, 177 (1990).

[38] C. Ahn, J. Phys. A A 27, 231-238 (1994).

[39] C. Ahn, Int. J. Mod. Phys. A 7, 6799 (1992).

[40] S. L. Lukyanov and V. A. Fateev, Sov. J. Nucl. Phys. 49, 925 (1989) [Yad. Fiz. 49, 1491 (1989)].

[41] C. Ahn, K. Schoutens and A. Sevrin, Int. J. Mod. Phys. A 6, 3467 (1991).

[42] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B 304, 348 (1988).

[43] K. Thielemans, Int. J. Mod. Phys. C 2, 787 (1991).
[44] H. T. Ozer, Mod. Phys. Lett. A 14, 469 (1999).

[45] H. Lu, C. N. Pope, S. Schrans and X. J. Wang, Nucl. Phys. B 379, 47 (1992).

[46] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, Nucl. Phys. B 361, 255 (1991).

[47] H. G. Kausch and G. M. T. Watts, Nucl. Phys. B 354, 740 (1991).