Co-Gorenstein Algebras

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Abstract
We review the theory of Co-Gorenstein algebras, which was introduced in Beligiannis (Commun Algebra 28(10):4547–4596, 2000). We show a connection between Co-Gorenstein algebras and the Nakayama and Generalized Nakayama conjecture.

Keywords  Homological algebra · Nakayama conjecture · Generalized Nakayama conjecture

Mathematics Subject Classification 16G10 · 16E65

Fix a commutative artinian ring $R$ and an artin $R$-algebra $\Lambda$. Let $\text{mod} \ -\Lambda$ be the category of finitely generated right $\Lambda$-modules, and let

$$D(\_):= \text{Hom}_R(\_ , I): (\text{mod} \ -\Lambda)^{\text{op}} \to \text{mod} \ -\Lambda^{\text{op}}$$

denote the equivalence where $I$ is the injective envelope of $S_1 \oplus S_2 \oplus \ldots \oplus S_n$ and $S_1, S_2, \ldots, S_n$ is a complete set of representatives of isomorphism classes of simple $R$-modules. Let

$$\cdots \to P_1(D\Lambda) \to P_0(D\Lambda) \to D\Lambda \to 0$$

be a minimal projective resolution of right $\Lambda$-modules and let

$$0 \to \Lambda \to I_0(\Lambda) \to I_1(\Lambda) \to \cdots$$

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be a minimal injective resolution of $\Lambda$ as a right module. Recall that the dominant dimension $\text{dom} \dim \Lambda$ of $\Lambda$ is the smallest integer $d$ such that $I_d(\Lambda)$ is not projective. We write $\text{dom} \dim \Lambda = \infty$ if no such integer exists. The following conjectures are important in the representation theory of artin algebras.

(i) **Generalized Nakayama Conjecture (GNC)** If $P$ is an indecomposable projective right $\Lambda$-module, then $P$ is a summand of $P_n(D\Lambda)$ for some $n$;

(ii) **Nakayama Conjecture (NC)** If $\text{dom} \dim \Lambda = \infty$, then $\Lambda$ is selfinjective.

Since $\text{dom} \dim \Lambda = \infty$ if and only if $P_i(D\Lambda)$ is injective for all $i \geq 0$ by [8], it follows that GNC implies NC.

In this note we show a relation between these conjectures and the notion of a Co-Gorenstein algebra, which was introduced by Beligiannis in [4]. More precisely, we show that there exist implications

$$
\text{GNC} \overset{\text{Proposition 16}}{\Rightarrow} \text{Conjecture 1} \overset{\text{Proposition 15}}{\Rightarrow} \text{NC}.
$$

where Conjecture 1 is as follows:

**Conjecture 1** If $\Omega^n(\text{mod} - \Lambda)$ is extension closed for all $n \geq 1$, then $\Lambda$ is right Co-Gorenstein.

We start by reviewing the construction and properties of Co-Gorenstein categories. In particular, we give some equivalent properties for an algebra to be right Co-Gorenstein, see Corollary 10. In Sect. 2 we show the implications above.

Throughout the note $R$ denotes a commutative artinian ring, $\Lambda$ an artin $R$-algebra. Also, we fix the notation

$$
\Omega^n(\text{mod} - \Lambda) := \{ M \in \text{mod} - \Lambda \mid \text{there exists an exact sequence } 0 \to M \to P_1 \to \cdots \to P_n \text{ with } P_i \in \text{mod} - \Lambda \text{ projective for } 1 \leq i \leq n \}
$$

### 1 Co-Gorenstein Categories

Let $\mathcal{A}$ be an abelian category with enough projectives, and let $\mathcal{P} := \text{Proj}(\mathcal{A})$ denote the full subcategory of projective objects in $\mathcal{A}$. The projectively stable category $\mathcal{A} := \mathcal{A}/\mathcal{P}$ of $\mathcal{A}$ consists of the same objects as $\mathcal{A}$, and with morphisms

$$
\mathcal{A}(A_1, A_2) := \mathcal{A}(A_1, A_2)/\sim
$$

where $f \sim g$ if $f - g$ factors through a projective object. For a morphism $f : A_1 \to A_2$ in $\mathcal{A}$ we let $f : A_1 \to A_2$ denote the corresponding morphism in $\mathcal{A}$. For each object $A \in \mathcal{A}$ choose an exact sequence $0 \to \Omega A \to P \to A \to 0$ where $P$ is projective. The association $A \mapsto \Omega A$ induces a functor $\Omega : \mathcal{A} \to \mathcal{A}$ [6, Sect. 3]. Furthermore, if $0 \to K \to Q \to A \to 0$ is any exact sequence with $Q$ projective, then there exists a unique isomorphism

$$
K \cong \Omega A
$$

1 This is Lemma 6.19 part (3) in [4]. Beligiannis claims that it follows immediately from results in [2]. However, this is not clear to the authors, so we state it as a conjecture.
in \( \mathcal{A} \) which is induced from a morphism \( K \to \Omega A \) in \( \mathcal{A} \) such that there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow & & \downarrow \\
0 & \to & \Omega A \\
\end{array}
\begin{array}{ccc}
& & Q \\
& & \downarrow 1_A \\
& & A \\
\end{array}
\begin{array}{ccc}
& & 0 \\
\end{array}
\]

for some morphism \( Q \to P \).

**Definition 2** The **costabilization** \( \mathcal{R}(\mathcal{A}) \) of \( \mathcal{A} \) is a category with objects consisting of sequences \((A_n, \alpha_n)_{n \in \mathbb{Z}}\) where \( A_n \in \mathcal{A} \) and \( \alpha_n : A_n \xrightarrow{\cong} \Omega A_{n+1} \) is an isomorphism in \( \mathcal{A} \). A morphism

\[
(A_n, \alpha_n) \to (B_n, \beta_n)
\]

in \( \mathcal{R}(\mathcal{A}) \) consists of a sequence \((f_n)_{n \in \mathbb{Z}}\) of morphisms \( f_n : A_n \to B_n \) in \( \mathcal{A} \) satisfying \( \beta_n \circ f_n = \Omega(f_{n+1}) \circ \alpha_n \).

**Remark 3** Here we explain the name and the universal property of the costabilization. We follow the conventions in [7]. A **category with suspension** is a pair \((\mathcal{C}, T)\) where \( \mathcal{C} \) is a category and \( T : \mathcal{C} \to \mathcal{C} \) is a functor. A **weakly stable morphism**

\[
(F, \phi) : (C_1, T_1) \to (C_2, T_2)
\]

between two categories with suspension is given by a functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) together with an isomorphism \( \phi : F \circ T_1 \xrightarrow{\cong} T_2 \circ F \). It is called **stable** if \( \phi \) is the identity morphism. Composition of weakly stable morphisms is given by

\[
(G, \psi) \circ (F, \phi) = (G \circ F, \psi F \circ G(\phi)).
\]

This gives a category where the objects are categories with suspensions, and where the morphisms are the weakly stable morphisms. If \( (\mathcal{C}, T) \) is a category with suspension, then we say that \( (\mathcal{C}, T) \) is **stable** if \( T : \mathcal{C} \to \mathcal{C} \) is an autoequivalence. Given a category with suspension \((\mathcal{C}, T)\), we can associate a stable category \((\mathcal{R}(\mathcal{C}, T), \hat{T})\), called its **costabilization**, as follows:

An object of \( \mathcal{R}(\mathcal{C}, T) \) is a sequence \((C_n, \alpha_n)_{n \in \mathbb{Z}}\) where \( C_n \in \mathcal{C} \) and \( \alpha_n : C_n \xrightarrow{\cong} TC_{n+1} \) is an isomorphism in \( \mathcal{C} \), and a morphism \((C_n, \alpha_n) \to (C_n', \beta_n)\) in \( \mathcal{R}(\mathcal{C}, T) \) is a sequence \((f_n)_{n \in \mathbb{Z}}\) of morphisms \( f_n : C_n \to C_n' \) in \( \mathcal{C} \) satisfying \( \beta_n \circ f_n = T(f_{n+1}) \circ \alpha_n \). The autoequivalence \( \hat{T} : \mathcal{R}(\mathcal{C}, T) \to \mathcal{R}(\mathcal{C}, T) \) is given by

\[
\hat{T}(C_n, \alpha_n) = (C_{n-1}, \alpha_{n-1}).
\]

Note that if we consider \((\mathcal{A}, \Omega)\) as a category with suspension, then we have that \( \mathcal{R}(\mathcal{A}, \Omega) = \mathcal{R}(\mathcal{A}) \) where \( \mathcal{R}(\mathcal{A}) \) is as in Definition 2. Now for a category with suspension \((\mathcal{C}, T)\) there exists a weakly stable morphism

\[
(R, \gamma) : (\mathcal{R}(\mathcal{C}, T), \hat{T}) \to (\mathcal{C}, T)
\]

where \( R : \mathcal{R}(\mathcal{C}, T) \to \mathcal{C} \) is the forgetful functor sending \((C_n, \alpha_n)\) to \( C_0 \), and \( \gamma : R \circ \hat{T} \xrightarrow{\cong} T \circ R \) is the isomorphism given by

\[
R\hat{T}(C_n, \alpha_n) = C_{-1} \xrightarrow{\alpha_{-1}} T(C_0) = TR(C_n, \alpha_n)
\]

The costabilization satisfies the following universal lifting property: If

\[
(F, \mu) : (\mathcal{B}, \Sigma) \to (\mathcal{C}, T)
\]
is a weakly stable morphism and \((\mathcal{B}, \Sigma)\) is stable, then there exists a unique stable morphism 
\((G, 1): (\mathcal{B}, \Sigma) \to (\mathcal{R}(\mathcal{C}, T), \hat{T})\) satisfying
\[(R, \gamma) \circ (G, 1) = (F, \mu).\]

Explicitly, \(G\) is given by \(G(B) = (B_n, \beta_n)\) where \(B_n = F\Sigma^{-n}(B)\) and \(\beta_n: B_n \xrightarrow{\cong} TB_{n+1}\) is given by
\[B_n = F\Sigma^{-n}(B) \xrightarrow{\mu\Sigma^{-n-1}(B)} TF\Sigma^{-n-1}(B) = TB_{n+1}.\]

This lifting property is dual to the universal extension property for the stabilization [7, Proposition 1.1], whence the name costabilization.

We fix some notation. Let \(\mathcal{C}(\mathcal{A})\) be the category of complexes in \(\mathcal{A}\). An object in \(\mathcal{C}(\mathcal{A})\) is denoted by
\[(P_\bullet, d_\bullet) := \cdots \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \cdots\]

For each integer \(n \in \mathbb{Z}\) we have functors
\[Z_n(-): \mathcal{C}(\mathcal{A}) \to \mathcal{A} \text{ and } H_n(-) : \mathcal{C}(\mathcal{A}) \to \mathcal{A}\]
given by taking the \(n\)th cycles \(Z_n(P_\bullet, d_\bullet) = \text{Ker} \, d_n\) and the \(n\)th homology \(H_n(P_\bullet, d_\bullet) := \text{Ker} \, d_n/\text{im} \, d_{n-1}\). We say that \((P_\bullet, d_\bullet)\) is acyclic if \(H_n(P_\bullet, d_\bullet) = 0\) for all \(n \in \mathbb{Z}\). We call a morphism \((P_\bullet, d_\bullet) \xrightarrow{f_\bullet} (Q_\bullet, d'_\bullet)\) of complexes null-homotopic if there exists morphisms \(h_i: P_i \to Q_{i-1}\) in \(\mathcal{A}\) such that \(f_i = d'_{i-1} \circ h_i + h_{i+1} \circ d_i\) for all \(i \in \mathbb{Z}\). Let \(\mathcal{C}_{ac}(\mathcal{P})\) denote the full subcategory of \(\mathcal{C}(\mathcal{A})\) consisting of acyclic complexes with projective components, and let \(\mathcal{K}_{ac}(\mathcal{P})\) denote the homotopy category of \(\mathcal{C}_{ac}(\mathcal{P})\). Explicitly, \(\mathcal{K}_{ac}(\mathcal{P})\) has the same objects as \(\mathcal{C}_{ac}(\mathcal{P})\), and with morphism spaces
\[\mathcal{K}_{ac}(\mathcal{P})((P_\bullet, d_\bullet), (Q_\bullet, d'_\bullet)) = \mathcal{C}_{ac}(\mathcal{P})((P_\bullet, d_\bullet), (Q_\bullet, d'_\bullet))/\sim\]

where \(f_\bullet \sim g_\bullet\) if the difference \(f_\bullet - g_\bullet\) is null-homotopic.

Given \((P_\bullet, d_\bullet)\) in \(\mathcal{C}_{ac}(\mathcal{P})\), we obtain an object \((Z_n(P_\bullet, d_\bullet), \alpha_n)\) in \(\mathcal{R}(\mathcal{A})\) where
\[\alpha_n : Z_n(P_\bullet, d_\bullet) \to \Omega Z_{n+1}(P_\bullet, d_\bullet)\]
is the induced isomorphism as in (1). Furthermore, given a morphism \(f_\bullet : (P_\bullet, d_\bullet) \to (Q_\bullet, d'_\bullet)\) in \(\mathcal{C}_{ac}(\mathcal{P})\) we obtain morphisms
\[Z_n(f_\bullet) : Z_n(P_\bullet, d_\bullet) \to Z_n(Q_\bullet, d'_\bullet)\]
in \(\mathcal{A}\) for each \(n \in \mathbb{Z}\), and it is easy to see that they make the diagram

\[
\begin{array}{ccc}
Z_n(P_\bullet, d_\bullet) & \xrightarrow{\cong} & \Omega Z_{n+1}(P_\bullet, d_\bullet) \\
\downarrow Z_n(f_\bullet) & & \downarrow \Omega Z_{n+1}(f_\bullet) \\
Z_n(Q_\bullet, d'_\bullet) & \xrightarrow{\cong} & \Omega Z_{n+1}(Q_\bullet, d'_\bullet)
\end{array}
\]

commute where the horizontal isomorphisms are as in (1). Hence, we obtain a morphism
\[
(Z_n(f_\bullet) : Z_n(P_\bullet, d_\bullet) \to Z_n(Q_\bullet, d'_\bullet))_{n \in \mathbb{Z}} \in \mathcal{R}(\mathcal{A}), \text{ and therefore, we have a functor}
\]
\[
\mathcal{C}_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A}).
\]
If $f_\bullet$ is null-homotopic, then the morphism $Z_n(f_\bullet) : Z_n(P_\bullet, d_\bullet) \to Z_n(Q_\bullet, d'_\bullet)$ factors through $P_n$, and hence $Z_n(f_\bullet) = 0$. Therefore, we get an induced functor

$$K_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A})$$

**Proposition 4** The functor $K_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A})$ is dense and full.  

**Proof** Let $(A_n, \alpha_n)$ be an arbitrary object in $\mathcal{R}(\mathcal{A})$. By assumption, for all $n \in \mathbb{Z}$ there exists an object $P_{n-1}' \in \mathcal{P}$ and an exact sequence

$$0 \to \Omega A_n \to P_{n-1} \to A_n \to 0$$

Since $A_n \cong \Omega A_{n+1}$ in $\mathcal{A}$, there exists objects $P_{n-1}', P_{n-1}'' \in \mathcal{P}$ and an isomorphism $A_n \oplus P_{n-1}' \cong \Omega A_{n+1} \oplus P_{n-1}''$ in $\mathcal{A}$. This implies that there also exists an exact sequence

$$0 \to \Omega A_n \to Q_{n-1} \to \Omega A_{n+1} \to 0$$

in $\mathcal{A}$ where $Q_{n-1} \in \mathcal{P}$. Hence, we obtain a complex $(Q_\bullet, d'_\bullet)$ in $K_{ac}(\mathcal{P})$ with differential $d'_n$ given by the composite $Q_n \to \Omega A_{n+2} \to Q_{n+1}$. Furthermore, by construction the image of the complex $(Q_\bullet, d'_\bullet)$ under the functor $K_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A})$ is the object $(\Omega A_{n+1}, \Omega(\alpha_{n+1}))$. Since we have an isomorphism

$$(\Omega A_{n+1}, \Omega(\alpha_{n+1})) \cong (A_n, \alpha_n)$$

in $\mathcal{R}(\mathcal{A})$ it follows that the functor $K_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A})$ is dense.

Let $(Q_\bullet, d'_\bullet)$ and $(Q'_\bullet, d''_\bullet)$ be complexes in $K_{ac}(\mathcal{P})$ and let $A_n = Z_n(Q_\bullet, d_\bullet)$ and $A'_n = Z_n(Q'_\bullet, d'_\bullet)$ so that we have short exact sequences

$$0 \to A_n \xrightarrow{i_n} Q_n \xrightarrow{p_n} A_{n+1} \to 0$$

$$0 \to A'_n \xrightarrow{i'_n} Q'_n \xrightarrow{p'_n} A'_{n+1} \to 0$$

where $d_n = i_{n+1} \circ p_n$ and $d'_n = i'_{n+1} \circ p'_n$. Under the functor $K_{ac}(\mathcal{P}) \to \mathcal{R}(\mathcal{A})$ these complexes correspond to objects $(A_n, \alpha_n)$ and $(A'_n, \alpha'_n)$ in $\mathcal{R}(\mathcal{A})$. Let

$$(f_n) : (A_n, \alpha_n) \to (A'_n, \alpha'_n)$$

be an arbitrary morphism between these objects in $\mathcal{R}(\mathcal{A})$. For each $n \in \mathbb{Z}$ choose a lifting $g_n : Q_n \to Q'_n$ of $f_{n+1} : A_{n+1} \to A'_{n+1}$. Since $A_n = \text{Ker } p_n$ and $A'_n = \text{Ker } p'_n$, we get a unique morphism $k_n : A_n \to A'_n$ satisfying $i'_n \circ k_n = g_n \circ i_n$. It is easy to see that $k_n = f_n$, and hence there exists a morphism $h_n : A_n \to Q'_{n-1}$ such that

$$p'_{n-1} \circ h_n = f_n - k_n.$$  

Now since

$$d'_n \circ (g_n - h_{n+1} \circ p_n) = d'_n \circ g_n - d'_n \circ h_{n+1} \circ p_n$$

$$= i'_{n+1} \circ f_{n+1} \circ p_n - i'_{n+1} \circ (f_{n+1} - k_{n+1}) \circ p_n$$

$$= i'_{n+1} \circ k_{n+1} \circ p_n$$

---

2 This functor is claimed to be an equivalence in Theorem 3.11 in [4]. It is not clear to the authors why this is true.
and
\[(g_{n+1} - h_{n+2} \circ p_{n+1}) \circ d_n = g_{n+1} \circ d_n = g_{n+1} \circ i_{n+1} \circ p_n = i_{n+1} \circ k_{n+1} \circ p_n\]
it follows that the maps \(l_n = g_n - h_{n+1} \circ p_n : Q_n \to Q'_n\) for all \(n \in \mathbb{Z}\) induce a map of chain complexes \(l \cdot : (Q \cdot , d \cdot) \to (Q' \cdot , d' \cdot)\). Since \(Z_n(l_n) = k_n\) and \(k_n = f_n\), it follows that the functor \(K_{ac}(\mathcal{P}) \to \mathcal{R}(A)\) is full. \(\square\)

**Remark 5** It would be interesting to determine if the functor \(K_{ac}(\mathcal{P}) \to \mathcal{R}(A)\) is an equivalence in general, or to find a counterexample and to determine in which cases it induces an equivalence.

Let \(R : \mathcal{R}(A) \to A\) be the forgetful functor sending \((A_n, \alpha_n)\) to \(A_0\), and let
\[\text{im } R = \{ A \in A \mid A \cong R(X) \text{ for some } X \in \mathcal{R}(A)\}.\]
denote the essential image of \(R\).

**Definition 6** Let \(A\) be an abelian category with enough projectives. We say that \(A\) is \(\mathcal{P}\text{-Co-Gorenstein}\) if the following holds:

(i) The forgetful functor \(R : \mathcal{R}(A) \to A\) is full and faithful;
(ii) If \(0 \to A_1 \to A_2 \to A_3\) is an exact sequence in \(A\) with \(A_1, A_3 \in \text{im } R\), then \(A_2 \in \text{im } R\). \(^3\)

The notion of Co-Gorenstein category was defined more generally for left triangulated categories in [4, Definition 3.13] and for an exact category in [4, Definition 4.9]. However, we only consider the case above.

**Remark 7** We explain the name Co-Gorenstein: Let \(S(A, \Omega)\) be the stabilization of the pair \((A, \Omega)\), see [7]. By [7, Proposition 1.1] there exists a functor \(A \to S(A, \Omega)\) which satisfies a universal extension property dual to the universal lifting property stated in Remark 3 for \(\mathcal{R}(A)\). Following [4], the category \(A\) is called \(\mathcal{P}\text{-Gorenstein}\) if there exists a full left triangulated subcategory \(\mathcal{V} \subset A\) such that the composite \(\mathcal{V} \to A \to S(A, \Omega)\) is an equivalence of left triangulated categories, see [4, Definition 3.13]. This coincides well with the terminology in the literature, since if \(A = \text{Mod} - \Lambda\) where \(\Lambda\) is a noetherian ring, then \(A\) is \(\mathcal{P}\text{-Gorenstein}\) if and only if \(\Lambda\) is an Iwanaga-Gorenstein ring, i.e. if the left and right injective dimension of \(\Lambda\) as a module over itself is finite [4, Theorem 6.9 and Corollary 6.11]. Since the definition of \(\mathcal{P}\text{-Co-Gorenstein}\) is in terms of the costabilization rather than then the stabilization, this can explain the name.

Our goal in the remainder of this subsection is to give a different characterization of \(\mathcal{P}\text{-Co-Gorenstein}\) categories. To this end, let
\[\Omega^\infty(A) := \{ A \in A \mid \text{there exists an exact sequence } 0 \to A \to P_0 \to P_1 \to \cdots \text{ with } P_i \in \mathcal{P} \forall i \geq 0\}.

**Lemma 8** Let \(A\) be an abelian category with enough projectives. Then \(X \in \text{im } R\) if and only if there exists \(A \in \Omega^\infty(A)\) such that \(A \cong X\).

\(^3\) In [4, Definition 3.13] it is only required that \(R\) is full and faithful, and it is claimed that this implies assumption (ii), see [4, Proposition 2.13 part (1)]. This is not clear to the authors, so we include this assumption in the definition.
\[ \text{Proof} \quad \text{This follows immediately from Proposition 4.} \]
A complex \((P_\bullet, d_\bullet)\) in \(C_{ac}(\mathcal{P})\) is called totally acyclic if the complex
\[
\cdots \to A(P_1, Q) \xrightarrow{-o d_1} A(P_0, Q) \xrightarrow{-o d_0} A(P_n, Q) \xrightarrow{-o d_{n-1}} \cdots
\]
is acyclic for any \(Q \in \mathcal{P}\). An object \(A \in \mathcal{A}\) is called Gorenstein projective if \(A = Z_0(P_\bullet, d_\bullet)\) for some totally acyclic complex \((P_\bullet, d_\bullet)\). The subcategory of Gorenstein projective objects in \(\mathcal{A}\) is denoted by \(\mathcal{GP}(\mathcal{A})\).

**Corollary 10** (Theorem 4.10 in [4]) Let \(\mathcal{A}\) be an abelian category with enough projectives. The following statements are equivalent:

(i) \(\mathcal{A}\) is \(\mathcal{P}\)-Co-Gorenstein;
(ii) \(\Omega^\infty(\mathcal{A}) \subseteq 1^\perp \mathcal{P}\);
(iii) \(\Omega^\infty(\mathcal{A}) \subseteq \perp \mathcal{P}\);
(iv) \(\Omega^\infty(\mathcal{A}) = \mathcal{GP}(\mathcal{A})\).

**Proof** Let \(\hat{\Omega}: \mathcal{R}(\mathcal{A}) \to \mathcal{R}(\mathcal{A})\) be the autoequivalence given by \(\hat{\Omega}(A_n, \alpha_n) = (A_{n-1}, \alpha_{n-1})\). Then there exists an isomorphism \(R \circ \hat{\Omega} \cong \Omega \circ R\). Hence, if \(\mathcal{A}\) is \(\mathcal{P}\)-Co-Gorenstein, then \(R\) is an equivalence onto \(\mathcal{R}\), and therefore \(\Omega\): \(\im R \to \im R\) is also an equivalence. It follows that \(\Omega^\infty(\mathcal{A}) \subseteq 1^\perp \mathcal{P}\) by Lemmas 8 and 9, which proves (i) \(\implies\) (ii). The implications (ii) \(\implies\) (iii) and (iii) \(\implies\) (iv) are straightforward. For (iv) \(\implies\) (i), note first that \(\Omega\): \(\im R \to \im R\) is full and faithful by Lemmas 8 and 9. Let \((f_0): (A_n, \alpha_n) \to (A'_n, \alpha'_n)\) be a morphism in \(\mathcal{R}(\mathcal{A})\). For \(n < 0\) we can write \(f_0\) as a composite
\[
A_n \cong \Omega(A_{n+1}) \cong \cdots \cong \Omega^{-n}(A_0) \xrightarrow{\Omega^{-n}(f_0)} \Omega^{-n}(A'_0) \cong \cdots \cong A'_n
\]
and for \(n > 0\) we can write \(\Omega^n(f_0)\) as a composite
\[
\Omega^n(A_n) \cong \Omega^{n-1}(A_{n-1}) \cong \cdots \cong A_0 \xrightarrow{f_0} A'_0 \cong \cdots \cong \Omega^n(A'_n)
\]
Hence, if \(f_0 = 0\) then \(f_n = 0\) for \(n < 0\) and \(\Omega^n(f_n) = 0\) for \(n > 0\). Since \(\Omega\) is faithful, it follows that \(f_n = 0\) for all \(n \in \mathbb{Z}\), and therefore \(R\) is faithful. To show that \(R\) is full, we chose again two objects \((A_n, \alpha_n)\) and \((A'_n, \alpha'_n)\) in \(\mathcal{R}(\mathcal{A})\), and we let \(f_0: A_0 \to A'_0\) be an arbitrary morphism in \(\mathcal{A}\). Define morphisms \(f_n: A_n \to A'_n\) for \(n < 0\) and \(g_n: \Omega^n(A_n) \to \Omega^n(A'_n)\) for \(n > 0\) in \(\mathcal{A}\) by Eqs. (2) and (3), respectively. Since \(\Omega\) is full and faithful, there exists for each \(n > 0\) a unique morphism \(f_n: A_n \to A'_n\) satisfying \(\Omega^n(f_n) = g_n\). A straightforward computation then shows that \((f_n): (A_n, \alpha_n) \to (A'_n, \alpha'_n)\) is a morphism in \(\mathcal{R}(\mathcal{A})\), and hence \(R\) is full. Finally, part (ii) in the definition of \(\mathcal{P}\)-Co-Gorenstein holds since \(\mathcal{GP}(\mathcal{A})\) is closed under extensions and by Lemma 8. Hence, the claim follows.

\[
\square
\]

### 2 Co-Gorenstein Artin Algebras

We now restrict ourselves to the case where \(\mathcal{A} = \text{mod-}\Lambda\) and \(\mathcal{P} = \text{Proj(mod-}\Lambda)\) for an artin \(R\)-algebra \(\Lambda\).

**Definition 11** \(\Lambda\) is right Co-Gorenstein if mod-\(\Lambda\) is \(\mathcal{P}\)-Co-Gorenstein.

By the above results we know that \(\Lambda\) is right Co-Gorenstein if and only if one of the following equivalent conditions hold:
Co-Gorenstein Algebras 285

(i) $\Omega^\infty(\text{mod-}\Lambda) \subset 1^\perp \Lambda$;
(ii) $\Omega^\infty(\text{mod-}\Lambda) \subset \perp \Lambda$;
(iii) $\Omega^\infty(\text{mod-}\Lambda) = \mathcal{GP}(\text{mod-}\Lambda)$.

Note that any Iwanaga-Gorenstein algebra is Co-Gorenstein. The following example shows that the converse is not true.

Example 12 Let $\Lambda := k[x, y]/(x^2, xy, yx, y^2)$, and let $S$ be the unique simple $\Lambda$-module. $\Lambda$ is a 3-dimensional local algebra with a two dimensional socle, and therefore $\Lambda$ is not an Iwanaga-Gorenstein algebra as a local artin algebra is an Iwanaga-Gorenstein algebra if and only if it has simple socle. Note that

$$\Omega^1(\text{mod-}\Lambda) = \text{add } S \oplus \Lambda,$$

because $\Lambda$ is a radical square zero algebra and thus every kernel of a projective cover is semisimple. Hence, if $M \in \Omega^\infty(\text{mod-}\Lambda)$ then $M \cong \Lambda^n \oplus S^m$ for $m, n \geq 0$. Note that in a local algebra, every module has projective dimension zero or infinite and thus the cokernel of a monomorphism of the form $\Lambda^n \to \Lambda^r$ is projective. Therefore, any monomorphism $\Lambda^n \to \Lambda^r$ is split. It follows that $S^m \in \Omega^\infty(\text{mod-}\Lambda)$. On the other hand, if there exists an exact sequence

$$0 \to S^{m_1} \to \Lambda^{m_2} \to S^{m_3} \to 0$$

then we must have $m_1 = 2m_2 = 2m_3$. In particular, we have that $S^m \notin \Omega^2(\text{mod-}\Lambda)$ if $0 < m < 2^{2s-1}$. Since $S^m \in \Omega^\infty(\text{mod-}\Lambda)$ we must have that $m = 0$ and hence

$$\Omega^\infty(\text{mod-}\Lambda) = \text{add } \Lambda \subseteq \perp \Lambda.$$

Therefore, $\Lambda$ is right Co-Gorenstein. Finally, note that

$$\bigoplus_{i \in \mathbb{Z}} S \in \Omega^\infty(\text{Mod-}\Lambda).$$

where Mod-\Lambda is the category of all right $\Lambda$-modules, not necessarily finite dimensional. Since Gorenstein projective modules are closed under direct summands and $S$ is not Gorenstein projective, it follows that Mod-\Lambda is not $\mathcal{P}$-Co-Gorenstein.

Our goal now is to prove the implications between the conjectures. Fix a minimal projective resolution

$$\cdots \to P_1(\Lambda) \to P_0(\Lambda) \to \Lambda \to 0$$

and a minimal injective resolution

$$0 \to \Lambda \to I_0(\Lambda) \to I_1(\Lambda) \to \cdots$$

of $\Lambda$ as a right module. Let $\mathcal{X}_n := \text{add } \Omega^n(\text{mod-}\Lambda)$ denote the smallest additive subcategory of mod-\Lambda which contains $\Omega^n(\text{mod-}\Lambda)$ and is closed under direct summands. Note that $\mathcal{X}_n \neq \Omega^n(\text{mod-}\Lambda)$ in general, see the example after Proposition 3.5 in [2].

Theorem 13 The following are equivalent for $n \geq 1$:

(i) $\Omega^k(\text{mod-}\Lambda)$ is extension-closed for $1 \leq k \leq n$;
(ii) $\mathcal{X}_k$ is extension-closed for $1 \leq k \leq n$;
(iii) $\text{inj. dim } P_k(\Lambda) \leq k + 1$ for $0 \leq k < n$.

Proof This follows from [3, Theorem 4.7].

\[\square\]
Unfortunately, the conditions in Theorem 13 are not left-right symmetric, see the paragraph after Corollary 2.8 in [2]. However, the following result shows that after a small modification one obtains a symmetric condition.

**Theorem 14** Let $n \geq 1$ be an integer. The following are equivalent:

(i) \( \text{inj. dim } P_k(D\Lambda) \leq k \text{ for all } 0 \leq k < n; \)

(ii) \( \text{proj. dim } I_k(\Lambda) \leq k \text{ for all } 0 \leq k < n. \)

**Proof** This follows from [5, Theorem 3.7]. \qed

We now show that Conjecture 1 implies NC

**Proposition 15** The following holds:

(i) If \( \text{dom. dim } \Lambda = \infty \text{ and Conjecture 1 holds, then } \Lambda \text{ is right Co-Gorenstein; } \)

(ii) If \( \text{dom. dim } \Lambda = \infty \text{ and } \Lambda \text{ is right Co-Gorenstein, then } \Lambda \text{ is selfinjective; } \)

(iii) Conjecture 1 implies NC.

**Proof** Part (i) follows from Theorems 13 and 14. We prove part (ii). Let \( i: \Lambda \to I_0(\Lambda) \) denote the injective envelope. We have exact sequences

\[
0 \to \Lambda \xrightarrow{i} I_0(\Lambda) \to \text{Coker } i \to 0
\]

and

\[
0 \to \text{Coker } i \to I_1(\Lambda) \to I_2(\Lambda) \to \cdots.
\]

Note that Coker \( i \in \Omega^{\infty}(\text{mod } -\Lambda) \) if \( \text{dom. dim } \Lambda = \infty \). Furthermore, \( \Lambda \) is selfinjective if and only if the sequence (4) is split, and this holds if Coker \( i \in \perp \Lambda \). By Corollary 10, this proves part (ii). Part (iii) follows from part (i) and (ii). \qed

We now show that GNC implies Conjecture 1.

**Proposition 16** The following holds:

(i) Suppose the GNC holds. If \( \Omega^n(\text{mod } -\Lambda) \) is extension closed for all \( n \geq 1 \), then \( \text{inj. dim } \Lambda < \infty \) as a right \( \Lambda \)-module;

(ii) If \( \text{inj. dim } \Lambda < \infty \) as right \( \Lambda \)-module, then \( \Lambda \) is right Co-Gorenstein;

(iii) GNC implies Conjecture 1.

**Proof** By Theorem 13 we have that inj. dim \( P_n(D\Lambda) \leq n + 1 \) for all \( n \geq 0 \). Now write \( \Lambda = P_0 \oplus \ldots \oplus P_m \) as a sum of indecomposable projective \( \Lambda \)-modules. Since GNC holds, there exists integers \( s_0, s_1, \ldots, s_m \) such that \( P_i \) is a direct summand of \( P_{s_i}(D\Lambda) \). Let \( s := \max\{s_0, \ldots, s_m\} + 1 \). Then inj. dim \( P_i \leq \text{inj. dim } P_{s_i}(D\Lambda) \leq s_i + 1 \leq s \) for all \( 0 \leq i \leq m \).

Hence, it follows that inj. dim \( \Lambda \leq s < \infty \).

For part (ii), assume inj. dim \( \Lambda = s \), and let \( M \in \Omega^{\infty}(\text{mod } -\Lambda) \). Then there exists an exact sequence

\[
0 \to M \to P_1 \to \cdots \to P_s \to K \to 0
\]

in mod -\( \Lambda \) with \( P_i \) projective. It follows by dimension shifting that

\[
\text{Ext}^i_\Lambda(M, \Lambda) \cong \text{Ext}^{i+s}_\Lambda(K, \Lambda).
\]

Since inj. dim \( \Lambda = s \), we have \( \text{Ext}^{i+s}_\Lambda(K, \Lambda) = 0 \) for all \( i \geq 1 \). This shows that \( M \in \perp \Lambda \), and hence \( \Lambda \) is Co-Gorenstein by Corollary 10. Part (iii) follows immediately from part (i) and (ii). \qed
**Remark 17** In Proposition 16 part (ii) we actually prove that

\[ \cap_{n \geq 1} \Omega^n (\mod - \Lambda) \subseteq \perp \Lambda. \]

However, under the assumption that \( \Omega^n (\mod - \Lambda) \) is extension closed for all \( n \geq 1 \), we have that

\[ \Omega^\infty (\mod - \Lambda) = \cap_{n \geq 1} \Omega^n (\mod - \Lambda). \]

In fact, by [3, Theorem 1.7 part b) and c)] we get that the modules in \( \cap_{n \geq 1} \Omega^n (\mod - \Lambda) \) can be identified with the modules which are \( n \)-torsion free for all \( n \), and it is easy to see that these modules are contained in \( \Omega^\infty (\mod - \Lambda) \).

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