A polynomial algorithm for minimizing discrete convic functions in fixed dimension

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Abstract
In [3], classes of conic and discrete conic functions were introduced. In this paper we use the term convic instead conic. The class of convic functions properly includes the classes of convex functions, strictly quasiconvex functions and the class of quasiconvex polynomials. On the other hand, the class of convic functions is properly included in the class of quasiconvex functions. The discrete convic function is a discrete analogue of the convic function. In [3], the lower bound $3^{n-1} \log(2\rho-1)$ for the number of calls to the comparison oracle needed to find the minimum of the discrete convic function defined on integer points of some $n$-dimensional ball with radius $\rho$ was obtained. But the problem of the existence of a polynomial (in $\log \rho$ for fixed $n$) algorithm for minimizing such functions has remained open. In this paper, we answer positively the question of the existence of such an algorithm. Namely, we propose an algorithm for minimizing discrete convic functions that uses $2^{O(n^2 \log n)} \log \rho$ calls to the comparison oracle and has $2^{O(n^2 \log n)} \text{poly}(\log \rho)$ bit complexity.

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1. Introduction

A well-known and intensively studied generalization of the integer linear programming problem is the problem of integer minimization of (quasi-) convex functions subject to (quasi-) convex constraints \[2, 5, 4, 7, 8, 9, 10, 14, 16\]. The objective function and the constraints in the problem can be specified explicitly or using an oracle. In \[5, 16\], a polynomial algorithm in terms of \(\log \rho\) (when the dimension \(n\) is fixed) is proposed, if the domain of the function is contained in a ball of radius \(\rho\) and the function is specified by the separation hyperplane oracle. In \[16\], it is established that a polynomial algorithm in terms of \(\log \rho\) (when \(n\) is fixed) can be obtained for the next three oracles: the feasibility oracle, the linear integer optimization oracle and the separation hyperplane oracle. A new approach in integer convex optimization, based on the concept of a centerpoint, is proposed in \[1\]. In those papers, the lower bound \(\Omega(2^n \log \rho)\) for the complexity of the algorithms of integer convex minimization using the separating hyperplane oracle is also established.

The main disadvantage of the oracles mentioned above is the complexity of their implementation. In many situations, it is more convenient to use the comparison oracle and the 0-order oracle. For any two points \(x, y\) in the domain of \(f\), the comparison oracle allows us to determine which of the two inequalities is satisfied: \(f(x) \leq f(y)\) or \(f(x) > f(y)\). Given \(x\), the 0-order oracle returns \(f(x)\). Note that, as shown in \[3\], the separating hyperplane oracle for the problems under consideration cannot be polynomially reduced to the comparison oracle.

The problems of integer minimization of convex (and close to them) functions defined by the comparison oracle or/and by the 0-order oracle are considered in \[2, 3, 13, 20\]. In \[2\], an algorithm for integer minimization of symmetric strictly quasiconvex functions with \(n = 2\) with the number of calls to the comparison oracle at most \(2 \log_2 \rho + 22 \log_2 \rho\) is proposed. In \[19\], a similar problem with the 0-order oracle was considered and an algorithm for integer minimization of such functions was proposed with the number of calls to the oracle at most \(4 \log_2 \rho\). In addition, the lower bound \(1.44 \log_2 \rho - 2\) for this problem was obtained in \[19\].

The paper \[3\] discusses the possibility of extending the class of functions for which the integer optimization problem with the comparison oracle can still be solved in polynomial in \(\log \rho\) time for any fixed \(n\). In particular, in \[3\], the classes of conic and discrete conic functions were introduced. The term conic in this context led to confusion and misunderstanding, so here,
instead of conic, we will use the invented word convic (derived from conic and convex). The definitions of convic and discrete convic functions see in Section 2.

The class of convic functions contains, as proper subclasses, the following important classes of functions: convex functions, strictly quasiconvex functions, and quasiconvex polynomials. The class of discrete convic functions is a discrete analogue of the class of convic functions.

In [3], an algorithm for minimizing convic functions using $2^{O(n \log n)} \log \rho$ calls to the comparison oracle is proposed. But the problem of the existence of a polynomial (in $\log \rho$ for fixed $n$) algorithm for minimizing discrete convic functions has remained open. We note that the algorithm in [3] is not applied to discrete convic functions because it may appeal to the oracle at fractional points. Now we deal with discrete convic functions and have to call to the oracle at only integer points. On the other hand, in [3], generalizing results of [20], a lower bound $3^{n-1} \log(2\rho - 1)$ for this problem was obtained.

In this paper, we answer positively to the question of the existence of a polynomial algorithm for minimizing discrete convic functions using the comparison oracle. Namely, we propose such a minimization algorithm with the number of calls to the oracle at most $2^{O(n^2 \log n)} \log \rho$ and bit complexity $2^{O(n^2 \log n)} \text{poly}(\log \rho)$.

As in [3, 10, 16] when constructing our algorithm, we use the modified Lenstra method (β-rounding algorithm) for solving the feasibility problem described in general form in [6, 18]. It uses two main ideas: the ellipsoid method [15] and the lattice basis reduction [13]. Since, in the problems under consideration, there is no function defined analytically, these problems cannot be reduced to the problem of satisfiability. Nevertheless, due to the properties of discrete convic functions, the β-rounding procedure can be applied here.

Also note that in the papers cited above, the separation oracle is used, and the opportunity to ask questions to the oracle at any rational points of the domain is essentially used. The difference of our work, among others, is that it is enough for our oracle to ask queries only at integer points. This allows one to work with partially defined functions, as well as with functions for which the separation oracle is complex. We achieve this due to a more subtle use of the properties of the reduced basis and the β-rounding algorithm.

The paper is organized as follows. Section 2 introduces the necessary definitions and notation. Section 3 contains a general description of the β-rounding algorithm. In Section 4, the equations of the new ellipsoid and the cut-off hyperplane at the current iteration of the β-rounding algorithm are
derived. Section 5 contains a description of the main algorithm for minimizing discrete convex functions and a bound for the algorithm complexity.

2. Main definitions and notations

Let $X$ be a subset of $\mathbb{R}^n$. Denote by $\text{span} X$, $\text{cone} X$ and $\text{conv} X$ the linear, conic and convex hulls of $X$, respectively.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex, if for any $c$ the set $M_c = \{ x \in \mathbb{R}^n : f(x) \leq c \}$ is convex. Let $D$ be a discrete set in $\mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is called discrete quasiconvex, if for any $c$ the set $M_c = \{ x \in D : f(x) \leq c \}$ is discrete convex, i.e. $\text{conv}(M_c) \cap D = M_c$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called conic, if for any $y \in \mathbb{R}^n$ and for any $z \in \mathbb{R}^n$, belonging to the set $y + \text{cone}\{ y - x : f(x) \leq f(y), x \in \mathbb{R}^n \}$, it holds that $f(y) \leq f(z)$ [3].

A function $f : D \to \mathbb{R}$ is called discrete conic, if for any $y \in D$ and any $z \in D$, belonging to the set $y + \text{cone}\{ y - x : f(x) \leq f(y), x \in D \}$, it holds that $f(y) \leq f(z)$ [3]; see Fig. 2.

In this paper, we consider the problem of minimization of a discrete conic function defined on a set $D = \mathbb{Z}^n \cap B_\rho$, where $B_\rho$ is the ball of radius $\rho$, centered at the origin.

Note that the classes of conic functions and discrete conic functions are quite wide and natural. Here we give a few arguments in support of this thesis [3].

As already mentioned, the class of (discrete) conic functions contains the class of (discrete) convex functions and the class of (discrete) strictly quasiconvex functions. On the other hand, there exist conic functions that are not convex nor strictly quasiconvex, e.g., $\max\{1, \sqrt{x}\}$. Note that the maximum of conic functions is conic. If $f$ is a non-decreasing conic function and $g$ is conic then $f(g(x))$ is conic. Any affine transformation of coordinates transforms a conic function into a conic one.

The problem of minimization of $f(x)$, s.t. $g_i(x) \leq 0$ ($i = 1, 2, \ldots, m$), $x \in \mathbb{Z}^n$, where $f$, $g_i$ are conic (or discrete conic), can be reduced to the problem of integer minimization of the conic (or, respectively, discrete conic) function

$$\max\{0, f(x), M \cdot g_1(x), \ldots, M \cdot g_m(x)\},$$

where the positive constant $M$ is sufficiently large and in many cases can be simply computed from the input of the problem. In particular, the integer
Рис. 1: An illustration of the discrete convex function definition. Here, we have $f: \mathbb{Z}^2 \to \mathbb{R}$ and $\max_i f(x_i) \leq f(y)$. 

$z: f(z) \geq f(y)$
linear programming problem can be reduced to the problem of minimizing a discrete convex function. Note that using lexicography it is possible to do this reduction without using constant $M$.

Note that not every discrete convex function can be extended to a convex one. On the other hand, if it is even extended, we may not be able to find the extension, if the function is given by the oracle or in other cases. For example, the function $f(x) = 2\rho^3 x_2 - x_1^3$ in the ball $B_\rho$, $\rho > 1$, is not convex, but the induced function defined in integer points of the ball is discrete convex. Hence, the algorithm in [3] is not applied here and a new algorithm for minimizing discrete convex functions is required.

For convenience of our further exposition, we reduce the problem under consideration to the problem of finding the lattice vector which is minimum with respect to the linear order given by a discrete convex function.

Note that any function $f : D \to \mathbb{R}$ induces some linear order on the set $D$:

$$x \preceq y \iff f(x) \leq f(y).$$

Moreover, instead of considering the functions $f$, we can investigate special linear orders defined on the set $D$.

Let $L$ be a lattice in $\mathbb{Z}^n$. Consider a linear order $\preceq$ on the set $D = L \cap B_\rho$ with the following property: for every $z \in y + \text{cone}\{y - x : x \preceq y\}$, it holds that $y \preceq z$. There is an oracle capable for any $x, y \in D$ to verify the truth of the statement $x \preceq y$. It is required to find a minimum point with respect to this order.

3. The $\beta$-rounding algorithm

In this section, we give a general description of Lenstra's method (see [6, 18]). Its main idea is constructing $\beta$-rounding of the set $M$, i.e. constructing the ellipsoid

$$E(A, a) = \{x \in \mathbb{Q}^n : (x - a)^\top A^{-1} (x - a) \leq 1\},$$

such that

$$E(\beta^2 A, a) \subseteq M \subseteq E(A, a). \quad (1)$$

If $E(A, a)$ is sufficiently "large", then $E(\beta^2 A, a)$ contains an integer point belonging to $M$.

If $E(A, a)$ is "small", then one can specify a number $k(\beta)$ that does not depend on $M$, such that all integer points in $M$ are located on at most $k(\beta)$
hyperplanes, therefore, the \( n \)-dimensional problem reduces to \( k(\beta) \) problems in dimension \( n - 1 \), and the recursion can be applied.

Given a sequence of vectors

\[
c_0, c_1, \ldots
\]

and numbers \( R, \epsilon \), the algorithm for \( \beta \)-rounding of the set \( M \) finds the sequence

\[
(A_0, a_0) = (R^2 I, 0), (A_1, a_1), \ldots
\]
such that \( E(A_{k+1}, a_{k+1}) \) is the ellipsoid with minimum volume containing the set \( \{x \in E(A_k, a_k) : c_k^T x \leq c_k^T a_k + \beta \sqrt{c_k^T A_k c_k} \} \).

The algorithm finishes its work at the \( k \)-th step if at least one of the following conditions is fulfilled:

1) the sequence (2) is terminated;
2) the pair \( (A_k, a_k) \) satisfies (1);
3) \( \det A_k \leq \epsilon \).

For \( \beta = 1/(n + 1) \), the number of iterations of the algorithm is at most

\[
N_\epsilon = 5n(n + 1)^2|\log(\epsilon)| + 5n^2(n + 1)^2|\log(2R)| + \log(n + 1) + 1
\]
(see Theorem 3.3.9 in [6]).

4. The cutting hyperplane construction

To use the \( \beta \)-rounding procedure one must be able to find the cutting plane for any “sufficiently large” ellipsoid

\[
E(A, a) = \{x \in \mathbb{Q}^n : (x - a)^T A^{-1}(x - a) \leq 1 \}.
\]

We show how to do this.

Here we use the ideas of shallow cuts introduced by Nemirovsky and Yudin [15].

We will use the dot product \( (x, y) = x^T A^{-1} y \) and the induced norm \( \|x\| = \sqrt{(x, x)} \). In such a metric, the ellipsoid \( x^T A^{-1} x \leq r^2 \) is a ball \( B_r \).

**Lemma 1.** Suppose that \( r < R < 1 \), a point \( x' \) belongs to the boundary of \( B_R \), and \( K' = x' + \text{cone}\{x' - x : x \in B_r\} \). Then for any point \( x \in B_1 \setminus K' \) it holds that

\[
(x', x) \leq r^2 + \sqrt{(1 - r^2)(R^2 - r^2)}.
\]
Proof. Consider the set $S$ of all points $y \in B_r$, for which the ray $\ell = \{x' + t \cdot (x' - y) : t \geq 0\}$ is boundary for the cone $K'$. Since the extensions of all such rays touch the ball $B_r$, $(y, x' - y) = 0$ for all $y \in S$.

Let us verify that the ray $\ell$ intersects the boundary of the ball $B_1$ at a point $z = x' + t \cdot (x' - y)$ with

$$t = \sqrt{\frac{1 - r^2}{R^2 - r^2}} - 1.$$ 

Indeed, for $y \in S$ we have

$$\|z\|^2 = \|x' + t (x' - y)\|^2 = \|(t + 1) (x' - y) + y\|^2 =
= (t + 1)^2\|y - x'\|^2 + 2(t + 1) (x' - y, y) + \|y\|^2 =
= \frac{1 - r^2}{R^2 - r^2} (\|x'\|^2 - \|y\|^2) + r^2 = \frac{1 - r^2}{R^2 - r^2} (R^2 - r^2) + r^2 = 1.$$ 

Since $z$ lies on the boundary ray of the cone $K'$, for any point $x \in B_1 \setminus K'$ we have

$$(x', x) \leq (x', z) = (x', x' + t (x' - y)) = \|x'\| + (x', t (x' - y)) =
= R^2 + t (x' - y, x' - y) = r^2 + \sqrt{(1 - r^2)(R^2 - r^2)}.$$ 

□

Lemma 2. Let $g_1, \ldots, g_n$ be a basis of a lattice $L$, $\theta = \max_{i=1,\ldots,n} \|g_i\|$ and $R \geq r + \theta n$, then $B_r \subseteq \text{conv}(B_R \cap L)$.

Proof. For $x = x_1 g_1 + \cdots + x_n g_n$ we denote

$$[x] = \{x_1 g_1 + \cdots + x_n g_n : \lfloor x_j \rfloor \leq y_j \leq \lceil x_j \rceil (j = 1, \ldots, n)\}.$$ 

If $x \in B_r$, then

$$[x] \subseteq B_r + \{\alpha_1 g_1 + \cdots + \alpha_n g_n : |\alpha_j| < 1 (j = 1, \ldots, n)\} \subseteq B_{r + \theta n} \subseteq B_R.$$ 

Since $x \in [x], x \in \text{conv}(B_R \cap L)$.

\[\square\]

Denote

$$\beta = \frac{1}{n + 1}, \quad \sigma = \frac{2\beta^3}{27n}, \quad R = \frac{\beta}{3}, \quad r = \frac{R\sqrt{1 - \beta^2}}{\sqrt{1 - 2\beta R + R^2}}.$$
Lemma 3. \( R \geq r + \sigma n. \)

Proof.
\[
R - r = \frac{\beta}{3} - \frac{\beta \sqrt{1 - \beta^2}}{3\sqrt{1 - 2\beta^2/3 + \beta^2/9}} = \\
= \frac{\beta}{3} - \frac{\beta \sqrt{1 - \beta^2}}{\sqrt{9 - 5\beta^2}} = \beta \cdot \frac{\sqrt{9 - 5\beta^2} - 3\sqrt{1 - \beta^2}}{3\sqrt{9 - 5\beta^2}} = \\
= \beta \cdot \frac{4\beta^2}{3\sqrt{9 - 5\beta^2}(\sqrt{9 - 5\beta^2} + 3\sqrt{1 - \beta^2})} \geq \frac{2\beta^3}{27} \geq \sigma n.
\]

□

The following lemma describes how to construct a shallow cut.

Lemma 4. Let \( g_1, \ldots, g_n \) be a basis of a lattice \( L \subseteq \mathbb{Z}^n \), \( \theta = \max_{i=1,\ldots,n} \|g_i\| \), \( \theta \leq \sigma \). Then \( B_r \subseteq \text{conv}(B_R \cap L) \). Further, let \( x^* \) be the maximum point in \( B_R \cap L \) with respect to \( \preceq \), a point \( x^0 \) be the minimum point in \( B_1 \cap \mathbb{Z}^n \) with respect to \( \preceq \), and \( E \) be the minimum volume ellipsoid containing the set \( \{x \in B_1 : (x^*, x) \leq \beta \|x^*\| \} \). Then \( x^0 \in E \).

Proof. The inclusion \( B_r \subseteq \text{conv}(B_R \cap L) \) follows from lemmas 2 and 3.

Let \( x' \) be the intersection point of the ray \( \{tx^* : t \geq 0\} \) and the boundary of the ball \( B_R \).

Consider the following three cones (see Fig. 4):

\[
K_0 = x^* + \text{cone}\{x^* - x : x \in B_R \cap L\}, \\
K^* = x^* + \text{cone}\{x^* - x : x \in B_r\}, \\
K' = x' + \text{cone}\{x' - x : x \in B_r\}.
\]

It is obvious that \( K' \subseteq K^* \) and \( B_r \subseteq \text{conv}(B_R \cap L) \subseteq \text{conv}(B_R \cap \mathbb{Z}^n) \), hence \( K^* \subseteq K_0 \).

Since \( x^* \) is the maximum point in \( B_R \cap L \) with respect to \( \preceq \), we derive \( x^* \preceq x \) for all \( x \in K_0 \). But the cone \( K_0 \) contains \( K' \), hence the minimum point in \( B_1 \cap L \) with respect to \( \preceq \) belongs to \( B_1 \setminus K' \). By Lemma 1 for any
point \( x \in B_1 \setminus K' \) we have \((x', x) \leq r^2 + \sqrt{(1 - r^2)(R^2 - r^2)}\), hence

\[
\frac{(x^*, x)}{\|x^*\|} = \frac{(x', x)}{\|x'\|} \leq \frac{r^2 + \sqrt{(1 - r^2)(R^2 - r^2)}}{R} = \frac{R(1 - \beta^2) + \sqrt{(1 - 2\beta R + R^2 \beta^2)(R^4 + R^2 \beta^2 - 2\beta R^3)}}{R(1 - 2\beta R + R^2)} = \frac{R(1 - \beta^2) + (1 - \beta R)(\beta - R)}{1 - 2\beta R + R^2} = \frac{\beta(1 - 2\beta R + R^2)}{1 - 2\beta R + R^2} = \beta.
\]

\[\square\]

Рис. 2: An illustration for Lemma 4. Here, we have \( f : \mathbb{Z}^2 \to \mathbb{R} \) and the lattice \( L \) with the basis \( \left( \frac{2}{2} \right) \) and \( \left( \frac{4}{2} \right) \). For integral points from the blue cone we have \( f(x) \geq f(x^*) \), the shallow cut is denoted by the orange line.
Let $G = (g_1, g_2, \ldots, g_n)$ be a basis of a lattice $L \subseteq \mathbb{Z}^n$. Denote by
\[
def G = \frac{1}{\det L} \prod_{i=1}^{n} \|g_i\|_{A^{-1}}
\]
the orthogonality defect of $G$.

The following lemma helps us to estimate the time need to construct the shallow cut.

**Lemma 5.** Let $G = (g_1, g_2, \ldots, g_n)$ be a basis of $\mathbb{Z}^n$, $\theta = \max_{i=1,\ldots,n} \|g_i\|$ and $\theta \leq \sigma$. Let $x^0$ be the minimum point in $B_1 \cap \mathbb{Z}^n$ with respect to $\preceq$. Then, the shallow cut $(x^*, x) \leq \beta \|x^*\|$ with the property
\[
x^0 \in \{x \in B_1 : (x^*, x) \leq \beta \|x^*\|\}
\]
can be computed by an algorithm with the oracle complexity $2^{O(n \log n)} \def G$ and the bit complexity $2^{O(n \log n)} \def(G) \poly(s)$, where $s = \size G + \size A$ is the input size.

**Proof.** Let $G^* = (g^*_1, g^*_2, \ldots, g^*_n)$ be the Gram–Schmidt basis corresponding to the lattice basis $G$. Let $T$ be the upper triangular matrix such that $G = G^* T$ and let
\[
d_i = \left\lfloor \frac{\sigma}{\|g_i\|} \right\rfloor \quad (i = 1, \ldots, n).
\]
It is clear that we have $\|d_i g_i\| \leq \sigma$.

Consider the lattice generated by the vectors $d_i h_i$ ($i = 1, \ldots, n$). To find the maximum point $x^*$ in $B_R \cap L$ with respect to $\preceq$ it is enough to enumerate all integer solutions to the inequality $x^* A^{-1} x \leq R^2$. Or, after replacement $x = G \diag(d_1, \ldots, d_n) y = G^* T \diag(d_1, \ldots, d_n) y$, it is enough to enumerate all integer solutions to the inequality
\[
\sum_{i=1}^{n} d_i^2 \|g_i^*\|^2 (y_i + t_{i,i+1} y_{i+1} + \cdots + t_{i,n} y_n)^2 \leq R^2.
\]
They are at most
\[
\prod_{i=1}^{n} \left( \frac{2R}{d_i \|g_i^*\|} + 1 \right) \leq \prod_{i=1}^{n} \left( \frac{4R}{d_i \|g_i^*\|} \right) = \\
= \prod_{i=1}^{n} \left( \frac{4R}{d_i \|g_i\|} \right) \prod_{i=1}^{n} \left( \frac{\|g_i\|}{\|g_i^*\|} \right) = \prod_{i=1}^{n} \left( \frac{4R}{d_i \|g_i\|} \right)
\]
def G \leq \\
\leq \left( \frac{8R}{\sigma} \right)^n \def G \leq (36n(n+1)^2 + 1)^n n^n = 2^{O(n \log n)} \def G \quad (4)

integer points. This procedure takes \( 2^{O(n \log n)} \def G \) calls to the oracle. In deriving the inequality in (4) we used that
\[
d_i \cdot \|g_i\| = \|g_i\| \cdot \left\lfloor \frac{\sigma}{\|g_i\|} \right\rfloor \geq \frac{\sigma}{2}.
\]
The last inequality here follows from the inequalities
\[
\left\lfloor \frac{u}{v} \right\rfloor \geq \left\{ \frac{u}{v} \right\} \Rightarrow 2 \left\lfloor \frac{u}{v} \right\rfloor \geq \frac{u}{v} \Rightarrow v \left\lfloor \frac{u}{v} \right\rfloor \geq \frac{u}{2},
\]
which are valid for \( u \geq v > 0 \).

**Note 1.** For the sake of simplicity, we considered only ellipsoids with centers in 0. However, the enumeration method, described in Lemma 6, can be applied for ellipsoids with arbitrary centers \((x - a)^\top A^{-1} (x - a) \leq R^2\).

5. The algorithm

Before we give the main minimization algorithm we describe a preprocessing procedure that will be very helpful, when we need to reduce the dimension of an initial problem and to find a short lattice basis in the reduced space. Here we fully follow Dadush’ IP-Preprocessing Algorithm [4, pp. 223–225].

**Lemma 6.** Let \( L \) be an \( n \)-dimensional lattice given by a basis \( B \in \mathbb{Q}^{n \times n} \), and \( H = \{ x \in \mathbb{R}^n : Ax = b \} \) be an affine subspace, where \( A \in \mathbb{Q}^{m \times n} \) and \( b \in \mathbb{Q}^m \). Let also \( E = a_0 + B \rho \), where \( a_0 \in \mathbb{Q}^n \) and \( \rho \in \mathbb{Q}_+^n \). Then, there is an algorithm Preprocessing with the bit complexity \( 2^{O(n \log n)} \poly(s) \), where \( s \) is input size, which either decides that \( E \cap L \cap H = \emptyset \) or returns
1) a shift $p \in L$,
2) a sublattice $L' \subseteq L$, $\dim L' = k \leq n$, given by a basis $b'_1, \ldots, b'_k$,
3) a vector $a'_0 \in \text{span} L'$ and a radius $\rho'$, $0 < \rho' \leq \rho$,

satisfying the following properties

1) $E \cap L \cap H = (E' \cap L') + p$, where $E' = \{ x \in \text{span} L' : \| x - a'_0 \| \leq \rho' \}$,
2) $\max_{1 \leq i \leq k} \| b'_i \| \leq 2\sqrt{kR'}$,
3) $a'_0, \rho', b'_1, \ldots, b'_k$ and $p$ have polynomial in $s$ encodings.

PROOF. See [4, pp. 223–225].

The algorithm for minimizing a discrete convex function is given on Fig. 5.

On the step 6 we construct the reduced Korkin–Zolotarev basis following [17].

Theorem 1. There exists an algorithm for minimizing a discrete convex function, given by the comparison oracle on $B_\rho \cap \mathbb{Z}^n$, using $2^O(n^2 \log n) \log \rho$ calls to the oracle. The bit complexity of the algorithm is $2^O(n^2 \log n) \text{poly}(\log \rho)$.

PROOF. Let $\preceq$ be an order on $\mathbb{Z}^n$ induced by the function $f$. To find a minimum point we need to call algorithm Min-DConic with input parameters $a_0 = 0, \rho, L = \mathbb{Z}^n$ and $H = \{ x \in \mathbb{R}^n : 0^\top x = 0 \}$.

The correctness of the algorithm follows directly from the fact that we enumerate all possible hyperplanes $H_\alpha := \{ x \in \text{span} L : d^\top x = \alpha \}$ in line 12, where $\alpha \in \{ d^\top x : x \in E \} \cap \mathbb{Z}$, while the ellipsoid $E$ localizes some minimum point.

Let us prove the claimed oracle complexity estimate. Firstly, let us estimate the number of iterations of the repeat-until loop in lines 4–10. Let $H = (h_1, h_2, \ldots, h_n)$, then

$$\det(A^{-1}) = \det(H^\top A^{-1} H) \leq \| h_1\|_{A^{-1}} \ldots \| h_n\|_{A^{-1}} \leq \tau^n.$$  

Hence, when $\det A$ becomes less then $\sigma^{-n}$, we will have $\tau > \sigma$, and the loop will be finished. Taking $\epsilon = \sigma^{-n}$, we obtain the bound $O(n^4 \log(n\rho))$ for the number of iterations in the $\beta$-rounding procedure.

By the Lemma 5, the oracle complexity of the step 9 is $2^{O(n \log n)} \text{def } H$.

It is known [12], that the upper bound for the orthogonality defect of a Korkin–Zolotarev reduced basis is $2^{O(n \log n)}$. Hence, the oracle complexity of the repeat-until loop in lines 4–10 become $2^{O(n \log n)}$.  

13
Input: The comparison oracle for the order \( \preceq \); a lattice \( L \subseteq \mathbb{Z}^n \); a point \( a_0 \in \mathbb{Q}^n \) and a radius \( \rho \in \mathbb{Q}_+ \); a rational affine subspace \( H \).

Output: Return EMPTY if the set \((a_0 + B_\rho) \cap L \cap H\) is empty. If it is not, return the minimum point \( x^\ast \) with respect to \( \preceq \) in the set \((a_0 + B_\rho) \cap L \cap H\).

1: \((a_0, \rho, L, p) := \text{Preprocessing}(a_0, \rho, L, H)\).

2: Set \( p \) as the origin, when we call the \( \preceq \) comparison oracle.

3: \( E = \{ x \in \text{span} L : \|x - a_0\|_2 \leq \rho \}, n := \dim L, \beta := \frac{1}{n+1}, \sigma := \frac{2^{\beta^3}}{27n} \).

4: repeat

5: Suppose that \( E = \{ x \in \text{span} L : (x - a)^\top A^{-1} (x - a) \leq 1 \}\).

6: Find the reduced Korkin–Zolotarev basis \( h_i (i = 1, \ldots, n) \) of the lattice \( L \) with respect to the norm \( \|x\|_{A^{-1}} = \sqrt{x^\top A^{-1} x} \), defined by the ellipsoid \( E \).

7: Find \( \tau = \max_{i=1,\ldots,n} \|h_i\|_{A^{-1}} \).

8: if \( \tau \leq \sigma \) then

9: With help of Lemma 5 construct cutting plain and perform one iteration of \( \beta \)-rounding algorithm from [6] for the ellipsoid \( E \) with shallow cut \( (x^\ast, x)_{A^{-1}} \leq \beta \|x^\ast\|_{A^{-1}} \). Let \( E \) be the resulting ellipsoid.

10: until \( \tau > \sigma \)

11: Let \( H = (h_1, \ldots, h_{n-1}) \). Find irreducible integer vector \( d \in \text{span} L \), such that \( d^\top H = 0 \).

12: for all \( \alpha \in \{d^\top x : x \in E\} \cap \mathbb{Z} \) do

13: \( H_\alpha := \{ x \in \text{span} L : d^\top x = \alpha \} \).

14: Call Min-DConic\((a_0, \rho, L, H_\alpha)\).

15: return If all recursive calls of the Min-DConic algorithm have returned EMPTY, then return EMPTY. In the opposite case, return \( p + y \), where \( y \) is a minimum point with respect to \( \preceq \) between all recursive calls of the Min-DConic algorithm.

Рис. 3: The algorithm Min-DConic for minimizing a discrete convex function.
Suppose that $\tau > \sigma$. As in [18, p. 258, formula (56)], it can be shown that $|d^T x| \leq \text{def } H/\tau$ for any $x \in E(A, a)$. Hence, the initial problem is reduced to solving at most

$$2 \text{def } H/\sigma + 1 = 27n(n + 1)^3 \text{def } H = 2^{O(n \log n)}$$

problems in the dimension $n - 1$.

Denoting by $\varphi(n, \rho)$ the total number of calls to the oracle, we obtain

$$\varphi(n, \rho) = 2^{O(n \log n)} \log \rho + 2^{O(n \log n)} \varphi(n - 1, \rho) = 2^{O(n^2 \log n)} \log \rho.$$ 

Now, to derive the claimed bound for the bit complexity, we could repeat the arguments from Dadush’ thesis [4, p. 231].

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