QSD IV:
2+1 Euclidean Quantum Gravity as a model to test 3+1 Lorentzian Quantum Gravity

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Abstract

The quantization of Lorentzian or Euclidean 2+1 gravity by canonical methods is a well-studied problem. However, the constraints of 2+1 gravity are those of a topological field theory and therefore resemble very little those of the corresponding Lorentzian 3+1 constraints.

In this paper we canonically quantize Euclidean 2+1 gravity for arbitrary genus of the spacelike hypersurface with new, classically equivalent constraints that maximally probe the Lorentzian 3+1 situation. We choose the signature to be Euclidean because this implies that the gauge group is, as in the 3+1 case, $SU(2)$ rather than $SU(1,1)$. We employ, and carry out to full completion, the new quantization method introduced in preceding papers of this series which resulted in a finite 3+1 Lorentzian quantum field theory for gravity.

The space of solutions to all constraints turns out to be much larger than the one as obtained by traditional approaches, however, it is fully included. Thus, by suitable restriction of the solution space, we can recover all former results which gives confidence in the new quantization methods. The meaning of the remaining “spurious solutions” is discussed.

1 Introduction

The canonical quantization of 2+1 (pure) gravity is a well studied problem and the literature on this subject is extremely rich (see [1] and references therein). It may appear awkward to write yet another paper on this subject.

The point of this paper is to quantize 2+1 gravity by starting with a new Hamiltonian (constraint) rather than the one that imposes flatness of the connection (see, for instance,
Therefore, we are actually dealing with a new field theory. The reason why we still can call this theory 2+1 gravity (although in the Euclidean signature) is because classically both theories are equivalent, it is in the quantum theory only where discrepancies arise.

The motivation to study this model comes from 3+1 gravity: In [3] a new method is introduced to quantize the Wheeler-DeWitt constraint for 3+1 Lorentzian gravity and one arrives at a finite quantum field theory. It is therefore of interest to check whether that quantum theory describes a physically interesting phase of the full theory of quantum gravity. One way to do that is to apply the formalism to a model system which maximally tests the 3+1 theory while being completely solvable.

It is often said that 2+1 gravity in its usual treatment as for instance in [2] is such a model which tests the 3+1 theory in various technical and conceptual ways. The author disagrees with such statements for a simple reason: The constraints of usual 2+1 gravity and of 3+1 gravity are not even algebraically similar. Thus, one has to expect that the resulting quantum theories are mutually singular in a certain sense. We will find that this expectation turns out to be correct.

One can partially fix this by studying Euclidean 2+1 gravity to test Lorentzian 3+1 gravity because then the two gauge groups (SU(2)) coincide, in the Lorentzian signature the gauge group of 2+1 gravity would be SU(1,1). However, this is not enough: while now the Gauss constraints of both theories generate the same motions the rest of the constraints are still very different with respect to each other. More precisely, the 2+1 remaining constraint says that the connection $A$ is flat, that is, its curvature $F$ vanishes. Thus it does not involve the momenta $E$ conjugate to the connection at all. The situation in the 3+1 theory is very different: here we have as the remaining constraints a constraint that generates diffeomorphisms and the famous Wheeler-DeWitt constraint. Both constraints depend on the momenta, the Wheeler-DeWitt constraint even non-analytically. In [4] the authors propose to quantize the constraints $FE = FEE = 0$. However, this has never been done in the literature, one reason being that the $FEE$ constraint is as difficult to quantize as in the the 3+1 case. Moreover, the two constraints $FE = FEE = 0$ are equivalent to the $F = 0$ constraint only when the two-metric $q$ is non-singular, that is, $\det(q) > 0$ and therefore it is no surprise that the two theories are not even classically equivalent as was shown in [5] (for the theory defined by the $F = 0$ constraint the condition $\det(q) > 0$ is put in by hand in order to have Euclidean signature).

In this paper we are using the constraints $FE = FEE/\sqrt{\det(q)} = 0$. There are several reasons that speak for this choice: First of all these constraints are at least classically completely equivalent to the $F = 0$ constraints because clearly they make sense only when $\det(q) > 0$. In fact we will show that there is a field dependent non-singular map between the Lagrange multipliers of the two theories which map the two sets of constraints into each other. Secondly, they are just as in the 3+1 theory non-analytic in $E$ (because $\det(q)$ is a function of $E$) and so will test this feature of the 3+1 theory as well. In particular, both constraints are densities of weight one and only constraints of this type have a chance to result in densely-defined diffeomorphism covariant operators as argued in [3]. Thirdly, these constraints are maximally in analogy to all the 3+1 constraints.

The plan of the present paper is as follows: In section 2 we review the classical theory of Euclidean 2+1 gravity and outline our main strategy of how to arrive at a well-defined Hamiltonian constraint operator.
In section 3 we review the necessary background information on the mathematical tools that have been developed for diffeomorphism invariant theories of connections. Those Hilbert space techniques are identical for the 2+1 and 3+1 theory so that we have one more reason to say that the model under consideration tests the 3+1 situation. Also we need to construct a volume operator which as in the 3+1 theory plays a key role in the regularization of the (analog of the) Wheeler-DeWitt constraint operator. The 2+1 volume operator turns out to be much less singular than the 3+1 operator which has some important impact on the regularization of the constraint operators.

In section 4 we regularize the Wheeler-DeWitt operator. Many of the details are exactly as in the 3+1 theory although there are some crucial differences coming from the lower dimensionality of spacetime and also from the different singularity structure of the volume operator.

In section 5 we perform various consistency checks on the 2+1 Wheeler-DeWitt operator obtained, in particular whether it is a linear, covariant and anomaly-free operator.

In section 6 we construct the full set of solutions to all constraints. It is here where we encounter, besides reassuring results that give faith in the programme started in [3], several surprises:

- The quantum theory admits solutions which correspond to degenerate metrics. This happens although classically such solutions do not exist given our constraints. This should not be confused with the situation in [5] because there degenerate metrics are allowed even at the classical level.

- We find an uncountable number of rigorous distributional solutions to all constraints which reveal an uncountable number of quantum degrees of freedom just as in any field theory with local degrees of freedom. This is in complete contrast to the usual treatment via the $F = 0$ constraints which results in a topological quantum field theory with only a finite number of degrees of freedom.

- The space of solutions contains the solutions to the quantum $F = 0$ constraints as a tiny subspace. This subspace of solutions can be equipped with an inner product which is precisely the one that one obtains in traditional approaches. This is reassuring that our methods lead to well-established results and do not describe some unphysical phase of the theory.

- The huge rest of the solutions cannot be equipped with the inner product appropriate for the $F = 0$ constraints because they do not correspond to measurable functions with respect to the corresponding measure. However, there is another natural inner product available with respect to which they are normalizable. This inner product is likely to be the one that is appropriate also for the physically interesting solutions of the 3+1 constraints. The solutions to the $F = 0$ constraint in turn are not normalizable with respect to this second inner product. Thus as expected, the two sets of constraints have solution spaces which lie in the same space of distributions but they cannot be given the same Hilbert space topology. It is in this sense that the quantum theories are mutually singular.

In section 7 we conclude with some speculations of what the present paper teaches us for the 3+1 theory with regard to the solutions that are spurious from the point of view of the $F = 0$ constraint.
In the appendix we compute the spectrum of the 2+1 volume operator for the simplest states.

Throughout the paper we mean by the wording “2+1 or two-dimensional” always 2+1 Euclidean gravity while by “3+1 or three-dimensional” we always mean 3+1 Lorentzian gravity.

2 Classical Theory

Let us start by reviewing the notation (see, for instance, [2]). We assume that the three-dimensional spacetime is of the form \( M = \mathbb{R} \times \Sigma \) where \( \Sigma \) is a two-dimensional manifold of arbitrary topology, for instance, a compact, connected two-dimensional smooth manifold, that is, a Riemann surface of genus \( g \) or an asymptotically flat manifold. Let \( e^a_i \) be the co-dyad on \( \Sigma \) where \( a, b, c, \ldots = 1, 2 \) denote tensor indices and \( i, j, k, \ldots = 1, 2, 3 \) denote \( su(2) \) indices. The fact that we are dealing with \( su(2) \) rather than \( su(1,1) \) implies that the two-metric \( q_{ab} := e^a_i e^b_i \) has Euclidean signature. Moreover, let \( A^a_i \) be an \( su(2) \) connection and define the field \( E^a_i := \epsilon^{ab} e^b_i \) where \( \epsilon_{ab} \) is the metric-independent totally skew tensor of density weight \(-1\). Then it turns out that the pair \((A^a_i, E^a_i)\) is a canonical one for the Hamiltonian formulation of 2+1 gravity based on the Einstein Hilbert action \( S = \int_M d^3x \sqrt{|\text{det}(g)|} R(3) \) where \( g \) is the three-metric and \( R(3) \) its scalar curvature. In other words, \( E^a_i \) is the momentum conjugate to \( A^a_i \) so that the symplectic structure is given by

\[
\{ A^a_i(x), E^b_j(y) \} = \delta^b_a \delta^j_i \delta(x,y) .
\] (2.1)

The Hamiltonian of the theory is a linear combination of constraints, \( \int d^2x (\Lambda^i G_i + N^i C_i) \) for some Lagrange multipliers \( \Lambda^i, N^i \) where

\[
G_i := D_a E^a_i = \partial_a E^a_i + \epsilon_{ijk} A^a_j E^a_k : \text{Gauss constraint},
\]

\[
C_i := \frac{1}{2} \epsilon^{ab} F^a_{ab} : \text{Curvature constraint}
\] (2.2)

where \( F_{ab} \) denotes the curvature of \( A_a \). The Gauss constraint appears also in 3+1 gravity, however, the curvature constraint is completely different from the constraints that govern 3+1 gravity, [3]. The equivalent of \( C_i \) in 3+1 gravity are two types of constraints, one of them, \( V_a \), generates diffeomorphisms, the other one, \( H \), generates dynamics. The curvature constraints \( C_i \) on the other hand do not generate any such gauge transformations, in fact, the connection Poisson-commutes with \( C_i \) and shows that it is a Dirac observable with respect to \( C_i \). The constraint \( C_i = 0 \) imposes that the connection should be flat and thus the classically reduced phase space becomes the cotangent bundle over the moduli space of flat \( su(2) \) connections which is finite-dimensional.

It is obvious that the quantization of the model as defined by (2.2) will not give too much insight into the 3+1 situation. In the following we will reformulate (2.2) in such a way that it brings us in connection with 3+1 gravity.

It will turn out that the following compound field, called the degeneracy vector; for reasons that will become obvious soon

\[
E^i := \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} E^a_j E^b_k
\] (2.3)
is a crucial one. Let us compute the square of this density of weight one:

\[ E_i E^i = \frac{1}{2} \epsilon_{ab} \epsilon_{cd} [E^a_i E^c_j][E^b_j E^d_j] = \frac{1}{2} q_{bd} [E^b_j E^d_j] = \frac{1}{2} q_{bd} q_{ac} \epsilon_{ba} \epsilon_{dc} = \det(q), \]  

(2.4)

that is, the two-metric is degenerate if and only if \( E_i = 0 \) is identically zero. We also see that \( \det(q) \) is manifestly non-negative. Notice that \( E_i^a E^i = 0 \).

Whenever the degeneracy vector is non-vanishing we can perform the following non-singular transformation \((N^i) \leftrightarrow (N^a, N)\) for a vector field \( N^a \), called the shift, and a scalar function \( N \), called the lapse:

\[ N^i = N^a \epsilon_{ab} E^b_i + N \frac{E^i}{\sqrt{\det(q)}} \iff N^a = \epsilon_{ijk} \frac{E^i E^a_j}{\det(q)} N^k, \quad N = \frac{N^i E^i}{\sqrt{\det(q)}}. \]  

(2.5)

Notice that formula (2.3) respects that \( N^i, N^a, N \) have density weight zero. Using (2.5), we can now write the curvature constraint in the form

\[ N^i C_i = N^a V_a + NH \quad \text{where} \quad V_a := F^b_{ai} E^b_i : \text{Diffeormorphism constraint} \]

\[ H := \frac{1}{2} F^i_{ab} \epsilon_{ijk} \frac{E^i E^b_k}{\sqrt{\det(q)}} : \text{Hamiltonian constraint}. \]  

(2.6)

Apart from the fact that we are in two rather than three space dimensions these are precisely the constraints of Euclidean 3+1 gravity \([3]\). Since the 3+1 Euclidean Hamiltonian constraint operator plays a key role in the quantization of the 3+1 Lorentzian Hamiltonian constraint \([1]\), we claim that the set of constraints (2.6) bring us in maximal contact with the 3+1 theory. Notice that unlike in \([4]\), we have a factor of \( 1/\sqrt{\det(q)} \) in the definition of the Hamiltonian constraint. This difference has two important consequences:

1) Classical:

The denominator in \( H = F_i E^i / \sqrt{\det(q)} \) where \( F_i := \epsilon^{ab} F^i_{ab}/2 \) (or \( F^i_{ab} = \epsilon_{ab} F^i \)) blows up as \( E^i \) vanishes. Since the limit \( \lim_{E \to 0} E^i / \|E\| \) depends on the details of the limiting procedure we must exclude degenerate metrics classically. This is in contrast to \([3]\) where the authors exploit the possible classical degeneracy of the metric when one discards the denominator to demonstrate that one has already an infinite number of degrees of freedom at the classical level (notice, however, that their solutions, where \( F^a \) or \( E^i \) become null, do not apply since we are dealing with \( su(2) \)).

2) Quantum:

It is by now known that one of the reasons for why \( \tilde{H} := \sqrt{\det(q)} H \) suffers from

\[ \text{1In fact, once one has densely defined the Euclidean operator in 3+1 dimensions the fact that the Lorentzian operator in 3+1 is densely defined is a simple Corollary [3].} \]
huge problems upon quantizing it is due to the fact that $\tilde{H}$ has density weight two rather than one. As argued in [3], only densities of weight one have a chance to be promoted into densely defined, covariant operators. This is why we must keep the denominator $1/\sqrt{\det(q)}$ in (2.3) at the quantum level.

Just like in 3+1 gravity we wish to work in a connection representation, that is, states are going to be functions of connections. Then an immediate problem with $H$ is that one has to give a meaning to the denominator $1/\sqrt{\det(q)}$. In [3] that was achieved for Lorentzian 3+1 gravity by noting that the denominator could be absorbed into a Poisson bracket with respect to a functional $V$ of $q_{ab}$. The idea was then to use the quantization rule that Poisson brackets should be replaced by commutators times $1/(i\hbar)$ and to replace $V$ by an appropriate operator $\hat{V}$. Such an operator indeed exists and it is densely defined.

Is a similar trick also available for 2+1 gravity? At first sight the answer seems to be in the negative because the underlying reason for why such a trick worked for 3+1 gravity was that the co-triad $e^i_a$, the precise analogue of the degeneracy vector $E^i$, considered as a function of $E^a$ was integrable, the generating functional being given by the total volume $V$ of $\Sigma$. In other words, we had

$$e^i_a = \frac{1}{2} \epsilon^{ijk} \epsilon_{abc} E^b E^c_k / \sqrt{-\det(q)} = \frac{\delta V}{\delta E^a_i} \text{ with } V := \int_\Sigma d^3 x \sqrt{\det(q)} \quad (2.7)$$

However, if we take over the definition of $V$ (with $d^3 x$ replaced by $d^2 x$) then we find instead

$$\{A^i_a, V\} = \frac{\delta V}{\delta E^a_i} = \frac{q_{ab} E^b_i}{\sqrt{-\det(q)}} \text{ with } V := \int_\Sigma d^2 x \sqrt{-\det(q)} . \quad (2.8)$$

Thus, there seems not such a trick available in the 2+1 case. However, it is a matter of straightforward computation to verify that indeed

$$E^i = \frac{1}{2} \epsilon^{ab} \epsilon_{ijk} \{A^j_a, V\} \{A^b_k, V\} \quad (2.9)$$

which does not seem to help much because what we need is $E^i/\sqrt{-\det(q)}$ rather than $E^i$ itself.

The new input needed here as compared to the 3+1 case is as follows: Notice that if we could replace $\sqrt{-\det(q)}$ by $V$ then we could absorb it into the Poisson brackets by using the identity

$$\{A^j_a, V\} \{A^k_b, V\} = 4 \{A^j_a, \sqrt{V}\} \{A^k_b, \sqrt{V}\}$$

As we will see, $V$ can be promoted, just as in the 3+1 case, into a densely defined positive semi-definite operator. Therefore its square root exists and it would follow that the last equation with Poisson brackets replaced by commutators would make sense as an operator. In the next section we will define a Hilbert space and the corresponding operator.

What remains is to justify the replacement of $\sqrt{-\det(q)}$ by $V$. That this is possible we will show in the section after the following. It happens because the Poisson bracket gives a local quantity and therefore we may actually replace $V$ by $V(x, \epsilon)$ in

$$\{A^i_a(x), V\} \equiv \{A^i_a(x), V(x, \epsilon)\} \text{ where } V(B) = \int_B d^2 x \sqrt{-\det(q)}$$
is the volume of a compact region $B$ and $V(x, \epsilon)$ is the volume of an arbitrarily small open neighbourhood of the point $x$, the smallness governed by $\epsilon$. It is then easy to see that $\lim_{\epsilon \to 0} V(x, \epsilon)/\epsilon^2 = \sqrt{\det(q)(x)}$. Now, in the quantum theory we are going to point split the quantity $H$ and we will use a regularized $\delta$ distribution with point split parameter $\epsilon$. As we will see, that parameter can be absorbed into $V(x, \epsilon)$ to serve as a replacement for $\sqrt{\det(q)(x)}$. The details are displayed in the following sections.

3 Quantum Theory and Volume operator

In this section we will review the definition of a Hilbert space for diffeomorphism invariant theories of connections [6]. This will be our kinematical framework. On that Hilbert space we are going to construct a 2+1 volume operator which turns out to be actually more complicated than the one for the 3+1 theory [8, 9].

3.1 Quantum kinematics

In what follows we give an extract from [6, 7]. The reader interested in the details is urged to study those papers.

We will denote by $\gamma$ a finite piecewise analytic graph in $\Sigma$. That is, we have analytic edges $e$ which are joined in vertices $v$. We subdivide each edge into two parts and equip each part with an orientation that is outgoing from the vertex (the point where these two parts meet is a point of analyticity and therefore not a vertex of a graph, thus each edge from now on can be viewed to be incident at precisely one vertex). Given an $su(2)$ connection $A_i^a$ on $\Sigma$ we can compute its holonomy (or path-ordered exponential) $h_e(A)$ along an edge $e$ of the graph. Recall that all representations of $SU(2)$ are completely reducible and that the (equivalence class of equivalent) irreducible ones can be characterized by a half integral non-negative number $j$, the spin of the representation. We will denote the matrix elements of the $j$-representation at $g \in SU(2)$ by $\pi_j(g)$.

Consider now a vertex $v$ of the graph and the edges $e_1, ..., e_n$ incident at $v$, that is, the graph has valence $n$. Under a gauge transformation $g$ at $v$ the holonomy transforms as $h_{e_i} \to gh_{e_i}$, $i = 1, ..., n$. Now consider the transformation of the following function

$$\otimes^n_{i=1} \pi_{j_i}(h_{e_i}) \to [\otimes^n_{i=1} \pi_{j_i}(g)] \cdot [\otimes^n_{i=1} \pi_{j_i}(h_{e_i})].$$

We are interested in making this function gauge invariant at $v$. To that end we orthogonally decompose the tensor product of the $\pi_{j_i}(g)$ into irreducibles and look for the independent singlets in that decomposition. There is an orthogonal projector $c_v$ on each of these singlets, we say that it is compatible with the spins $j_1, ..., j_n$, and so we can make our function gauge invariant at $v$ by contracting : $c_v \cdot \otimes^n_{i=1} \pi_{j_i}(h_{e_i})$.

If we do that for each vertex we obtain a completely gauge invariant function called a spin-network function. Thus a spin-network function is labelled by a graph $\gamma$, a colouring of its edges $e$ with a spin $j_e$ and a dressing of each vertex $v$ with a gauge-invariant projector $c_v$. If we denote by $E(\gamma), V(\gamma)$ the set of edges and vertices of $\gamma$ respectively then we use the shorthand notation

$$T_{\gamma,j,\vec{c}},$$

where $j := \{j_e\}_{e \in E(\gamma)}$, $\vec{c} := \{c_v\}_{v \in V(\gamma)}$, for that spin-network function.
The Hilbert space $\mathcal{H}$ that we are going to use for gauge invariant functions of connections is most easily described by saying that the set of all spin-network functions is a complete orthonormal basis of $\mathcal{H}$ (so each spin-network function comes with a specific finite normalization factor). Notice that therefore $\mathcal{H}$ is not separable. Another characterization of $\mathcal{H}$ which is very useful is to display it as a certain $L^2$ space. To that end, consider the finite linear combinations $\Phi$ of spin-network functions. $\Phi$ can be turned into an Abelian $C^*$ algebra by saying that involution is just complex conjugation and by completing it with respect to the $sup$--norm over the space $A/G$ of smooth connections modulo gauge transformations. That $C^*$ algebra is isometric isomorphic by standard Gel'fand techniques to the $C^*$ algebra of continuous functions $C(A/G)$ where $A/G$ is the set of all homomorphisms from the original algebra into the complex numbers. The space $\overline{A/G}$, as the notation suggests, is a certain extension of $A/G$ and will be called the set of distributional connections. Indeed, it is the maximal extension such that (the Gel'fand transform of the) spin-network functions are continuous. By standard results, the resulting topology is such that $\overline{A/G}$ is a compact Hausdorff space and as such positive linear functionals $\Gamma$ on $C(\overline{A/G})$ are in one to one correspondence with regular Borel measures $\mu$ on $\overline{A/G}$ via $\Gamma(f) = \mu(f) = \int_{\overline{A/G}} d\mu f$.

Now the measure $\mu_0$ underlying $\mathcal{H}$ is completely characterized by the integral of spin-network functions and is given by $\mu_0(T_{\gamma,j,c}) = 1$ if $T_{\gamma,j,c} = 1$ and 0 otherwise. So we have $\mathcal{H} = L_2(\overline{A/G},d\mu_0)$ and spin-network functions play the same role for $\mu_0$ that Hermite functions play for Gaussian measures.

In the sequel we will topologize the space $\Phi$ of finite linear combinations of spin-network functions in a different way and we will call $\Phi$ henceforth the space of cylindrical functions. A function $f_\gamma$ is said to be cylindrical with respect to a graph $\gamma$ if it can be written as a finite linear combination of spin-network functions on that $\gamma$. The norm of $f_\gamma$ will be the $L_1$ norm $||f_\gamma||_1 = \sum_1 |T_1, f > |$ which equips $\Phi$ with the structure of a topological vector space. The distributional dual $\Phi'$ is the set of all continuous linear functionals on $\Phi$. Certainly every element of $\mathcal{H}$ is an element of $\Phi'$ by the Schwarz inequality and every element of $\Phi$ is trivially an element of $\mathcal{H}$. Thus we have the inclusion $\Phi \subset \mathcal{H} \subset \Phi'$ (this is not a Gel'fand triple in the strict sense because the topology on $\Phi$ is not nuclear).

This furnishes the quantum kinematics. Notice that we can take over the results from \cite{6} without change concerning the Diffeomorphism constraint : given an analyticity preserving diffeomorphism $\varphi$ we have a unitary operator on $\mathcal{H}$ which acts on a function cylindrical with respect to a graph as $\hat{U}(\varphi)f_\gamma = f_{\varphi(\gamma)}$, that is, the diffeomorphism group Diff($\Sigma$) is unitarily represented. This implies that one can group average with respect to the diffeomorphism group as in \cite{6}. We will return to this point in section 6.

3.2 The 2+1 volume operator

The plan of this subsection is as follows : Since $\bar{E}^i(x)$ is a density of weight one, it makes sense that it will give rise to a well-defined and diffeomorphism-covariantly defined operator valued distribution. In a second step we will point-split $\det(q) = E^iE^i$ and take the square root of the resulting operator. Again, since $\sqrt{\det(q)}$ is a density of weight one, it can be turned into a well-defined operator-valued distribution even in regulated form and the limit as the regulator is removed exists. Let us then begin with $E^i$. Let as in the previous section $f_\gamma$ denote a function cylindrical
with respect to a graph $\gamma$ and denote by $E(\gamma)$ its set of edges. Edges are, by suitably sub-dividing them into two halves, in the sequel always supposed to be oriented as outgoing at a vertex. We will compute the action of various operators first on functions of smooth connections and then extend the end result to all of $\overline{\mathcal{A}/G}$.

Let $\delta_\epsilon(x, y) = \delta_{\epsilon_1}(x^1, y^1)\delta_{\epsilon_2}(x^2, y^2)$ be any two-parameter family of smooth functions of compact support such that $\lim_{\epsilon, \epsilon_2 \to 0} \int_S d^2 y \delta_{\epsilon}(x, y)f(y) = f(x)$ for any, say smooth, function on $\Sigma$ where $\epsilon = (\epsilon_1, \epsilon_2)$ parametrizes the size of the support. Consider the point-split operator

$$
\hat{E}^a_{\epsilon, \sigma}(x) := \frac{1}{2} \epsilon_{ab} \epsilon^{ijk} \int_{\Sigma} d^2 y \int_{\Sigma} d^2 z \delta_\epsilon(x, y)\delta_\sigma(x, z) \hat{E}^a_j(\gamma) \hat{E}^b_k(z)
$$

(3.1)

and apply it to $f_\gamma$. Notice that upon replacing $\hat{E}^a_i(x) = -i\hbar \delta/\delta A^i_a(x)$

$$
\hat{E}^a_i(x)f_\gamma = -i\hbar \sum_{e \in E(\gamma)} \int_0^1 dt \delta(x, e(t)) \dot{e}^a(t) X^i_e(t)f_\gamma
$$

(3.2)

where $X^i_e(t) = \text{tr}([h_\epsilon(0, t)\tau_i h_\epsilon(t, 1)]T \partial/\partial h_\epsilon(0, 1))$, $h_\epsilon(a, b)$ is the holonomy from parameter value $a$ to $b$ and $\tau_i$ are generators of $su(2)$ with structure constants $\epsilon_{ijk}$. We also need the quantity $X^i_j(t, s) = \text{tr}([h_\epsilon(0, s)\tau_i h_\epsilon(s, t)\tau_j h_\epsilon(t, 1)]T \partial/\partial h_\epsilon(0, 1))$ for $s < t$ (modulo 1). Then it is easy to see that

$$
\hat{E}^a_{\epsilon, \sigma}(x)f_\gamma = -\hbar \frac{1}{2} \epsilon_{ab} \epsilon^{ijk} \sum_{e, e' \in E(\gamma)} \int_{\Sigma} d^2 y \int_{\Sigma} d^2 z \delta_\epsilon(x, y)\delta_\sigma(x, z) \times
$$

$$
\times \int_0^1 dt \delta(y, e(t)) \dot{e}^a(t) \int_0^1 dt' \delta(z, e'(t')) \dot{e}^b(t') \times
$$

$$
\times [X^k_{\epsilon}(t') X^j_{\epsilon}(t) + \delta_{e, e'}\{\theta(t, t') X^k_{\epsilon}(t, t') + \theta(t', t) X^j_{\epsilon}(t', t)\}]f_\gamma
$$

(3.3)

where $\theta(s, t) = 1$ if $s < t$ and 0 otherwise. We are now interested in the limit $\epsilon \to 0$ and proceed similar as in [3]. We must adapt the regularization to each pair $e, e'$ to get a well-defined result.

1) Case $e = e'$.

If $x$ does not lie on $e$ then for sufficiently small $\epsilon$ we must get $\delta_\epsilon(x, e(t)) = 0$ for any $t \in [0, 1]$. Thus in the limit we get a non-vanishing contribution if and only if there exists a value $t_x \in [0, 1]$ such that $e(t_x) = x$ (there is at most one such value $t_x$ because edges are not self-intersecting). Since $\dot{e}$ is nowhere vanishing we must have $\dot{e}'(t_x) \neq 0$ (switch 1 $\leftrightarrow$ 2 if necessary). We send $\epsilon_1, \epsilon_1' \to 0$ and find that $\delta_\epsilon(x, e(t)) \to \delta_\epsilon(x, t_x)\delta(t - t_x)/|\dot{e}'(t_x)|$ and similar for $\delta_\epsilon(x, e'(t'))$. Inserting this into (3.3) we find that there is no contribution for $e = e'$ because of the two zeroes 0 = $\epsilon_{ab}\dot{e}^b(t_x)\dot{e}^a(t_x)$ and 0 = $\epsilon_{ijk}[X^i_{\epsilon}(t_x, t_x) + X^j_{\epsilon}(t_x, t_x)]$. Notice that it was crucial to have $e_2, e_2'$ still finite as otherwise the appearing $\delta_\epsilon(0)\delta_\epsilon'(0)$ would be meaningless.
2) Case $e \neq e'$.
If again $x$ does not lie on both $e, e'$ then by choosing $\bar{e}, \bar{e}'$ sufficiently small we must get zero. Therefore $e, e'$ must intersect and as we have divided edges into two halves they can intersect at most in their common starting point corresponding to $t = t' = 0$ which is thus a vertex $v$ of the graph $\gamma$.

A) Subcase
Consider first the case that $e, e'$ have co-linear tangents at $t = 0$ and let us assume that $\dot{e}_1(0), \dot{e}'_1(0) \neq 0$ (switch 1 $\leftrightarrow$ 2 if necessary). Then we first send $e_1, e'_1 \to 0$ which results in

$$
\delta_\tau(x, e(t))\delta_\tau(x, e'(t')) \to \delta_{\epsilon_2}(x^2, e^2(t))\delta_{\epsilon'_2}(x^2, e'^2(t')) \frac{\delta(t)\delta(t')}{|\dot{e}_1(0)|\dot{e}'_1(0)|}
$$

and thus performing the two $t$ integrals we get zero as above because $0 = \epsilon_{ab}\dot{\epsilon}_a(0)\dot{\epsilon}'_b(0)$ by assumption.

B) Subcase
We are left with the case that the tangents of $e, e'$ are linearly independent at $x = v$. We replace $\delta_\tau(x, e(t))\delta_\tau(x, e'(t'))$ by $\delta_\tau(e'(t'), e(t))\delta_\tau(x, v)$ and send first $\bar{e} \to 0$. Then

$$
\delta_\tau(e'(t'), e(t)) \to \frac{\delta(t)\delta(t')}{|\epsilon_{ab}\dot{\epsilon}_a(0)\dot{\epsilon}'_b(0)|}
$$

and we can perform the integral. Since we are integrating over a square $[0, 1]^2$ and the two-dimensional delta-distribution is supported at a corner we pick up a factor of $1/4$ upon setting $t = t' = 0$ and dropping the integral. At last we send $\bar{e}' \to 0$.

Summarizing, we find ($V(\gamma)$ denotes the set of vertices of $\gamma$)

$$
\hat{E}^i(x) f_\gamma = -\frac{\hbar^2}{4 \cdot 2} \sum_{v \in V(\gamma)} \delta(x, v) \sum_{e, e' \in E(\gamma), e \equiv v} \text{sgn}(e, e')\epsilon^{ijk}X_e^jX_{e'}^k f_\gamma
$$

(3.4)

where $X^i_e := X^i_e(0)$ is easily recognized as the right invariant vector field on $SU(2)$ evaluated at $g = h_e(0, 1)$ and $\text{sgn}(e, e')$ is the sign of $\epsilon_{ab}\dot{\epsilon}_a(0)\dot{\epsilon}'_b(0)$ and so is an orientation factor. This furnishes the definition of the operator corresponding to the degeneracy vector.

We now will define the volume operator for any compact region $B \subset \Sigma$. Our first task is to define an operator corresponding to $\text{det}(q)$ and then to take its square root. Since $\text{det}(q)$ is a density of weight two we expect this to be quite singular, in fact the naive definition $\text{det}(q)(x) := \hat{E}^i(x)\hat{E}^i(x)$ does not make any sense given the expression (3.4) which involves a factor of $\delta(x, v)$. Thus we are lead to point-split the two degeneracy vector operators and to hope that 1) the regulated operator is positive so that it makes sense to take its square root and 2) that one can remove the regulator from the square root. Let us then define similar as above

$$
\text{det}(q)_{\epsilon_\tau, \tau}(x) := \int_\Sigma d^2y \int d^2z\delta_\tau(x, y)\delta_\tau(x, z)\hat{E}^i(y)\hat{E}^i(z)
$$

(3.5)
and apply it to a function cylindrical with respect to a graph \( \gamma \). Given (3.4) the result is easily seen to be

\[
\text{det}(q)_{\varepsilon, \varepsilon'}(x) f_{\gamma} = \frac{\hbar^4}{64} \sum_{v, v' \in V(\gamma)} \delta_{\varepsilon}(x, v) \delta_{\varepsilon'}(x, v') \times \\
\times \sum_{e_1, e_2 \in E(\gamma)} \sum_{e_1 \cap e_2 = v} \sgn(e_1, e_2) \sgn(e_1', e_2') \times \\
\times [\varepsilon_{ijk} X_{e_1}^i X_{e_2}^j] [\varepsilon_{ilm} X_{e_1}^m X_{e_2}^n] f_{\gamma} .
\] (3.6)

We now will accomplish both hopes 1), 2) stated above by appropriately choosing the regulators.

1) Choose \( \varepsilon' =: \varepsilon' \), then we are able to display (3.5) as a square of an operator

\[
\text{det}(q)_{\varepsilon, \varepsilon}(x) f_{\gamma} := \frac{\hbar^2}{8} \sum_{v \in V(\gamma)} \delta_{\varepsilon}(x, v) \sum_{e_1, e_2 \in E(\gamma)} \sum_{e_1 \cap e_2 = v} \sgn(e_1, e_2) \sgn(e_1', e_2') \times \\
\times [\varepsilon_{ijk} X_{e_1}^i X_{e_2}^j] [\varepsilon_{ilm} X_{e_1}^m X_{e_2}^n] f_{\gamma} .
\] (3.7)

Since \( X_{e_1}^i X_{e_2}^j \) commute for \( e_1 \neq e_2 \) and because \( iX_{e_1}^i \) is essentially self-adjoint with range in its domain, so is \( X_{e_1}^i X_{e_2}^j \) and therefore the whole operator corresponding to one factor in (3.7). Thus, (3.7) is a square of essentially self-adjoint operators with range in its domain and so it is positive semi-definite. Therefore its square root is well defined.

2) Choose \( \varepsilon \) small enough such that \( \delta_{\varepsilon}(x, v) \delta_{\varepsilon}(x, v') = \delta_{v, v'} [\delta_{\varepsilon}(x, v)]^2 \), that is, given \( \gamma, x \) we must choose \( \varepsilon \) so small that for \( v \neq v' \) not both of them can be in the support of the function \( \delta_{\varepsilon}(x, \cdot) \) which is always possible. Then we may write (3.7) as

\[
\sqrt{\text{det}(q)}_{\varepsilon, \varepsilon}(x) f_{\gamma} = \frac{\hbar^4}{64} \sum_{v \in V(\gamma)} \delta_{\varepsilon}(x, v) \sum_{e_1, e_2 \in E(\gamma)} \sum_{e_1 \cap e_2 = v} \sgn(e_1, e_2) \sgn(e_1', e_2') \times \\
\times [\varepsilon_{ijk} X_{e_1}^i X_{e_2}^j] [\varepsilon_{ilm} X_{e_1}^m X_{e_2}^n] f_{\gamma} .
\] (3.8)

take its square root and define this to be the regulated operator corresponding to \( \sqrt{\text{det}(q)} \):

\[
\sqrt{\text{det}(q)}_{\varepsilon, \varepsilon}(x) f_{\gamma} := \sqrt{\text{det}(q)}_{\varepsilon, \varepsilon}(x) f_{\gamma} .
\] (3.9)

In considering the limit \( \varepsilon \to 0 \) notice that for small enough \( \varepsilon \) at most one vertex of \( \gamma \) lies in the support of \( \delta_{\varepsilon}(x, \cdot) \). Therefore we can take the sum over vertices and the factor \( [\delta_{\varepsilon}(x, v)]^2 \) out of the square root and find that

\[
\sqrt{\text{det}(q)}_{\varepsilon}(x) f_{\gamma} = \frac{\hbar^2}{8} \sum_{v \in V(\gamma)} \delta_{\varepsilon}(x, v) \sqrt{\sum_{e_1, e_2 \in E(\gamma)} \sum_{e_1 \cap e_2 = v} \sgn(e_1, e_2) [\varepsilon_{ijk} X_{e_1}^i X_{e_2}^j] [\varepsilon_{ilm} X_{e_1}^m X_{e_2}^n] f_{\gamma} .
\] (3.10)

But now the limit \( \varepsilon \to 0 \) is trivial to take, we finally find that

\[
\sqrt{\text{det}(q)}(x) f_{\gamma} = \frac{\hbar^2}{8} \sum_{v \in V(\gamma)} \delta(x, v) \sqrt{\sum_{e_1, e_2 \in E(\gamma)} \sum_{e_1 \cap e_2 = v} \sgn(e_1, e_2) [\varepsilon_{ijk} X_{e_1}^i X_{e_2}^j] [\varepsilon_{ilm} X_{e_1}^m X_{e_2}^n] f_{\gamma} .
\] (3.11)
or in integrated form

\[
\hat{V}(B)f_\gamma := \int_B d^2x \sqrt{\det(q)(x)} f_\gamma \\
= \frac{\hbar^2}{8} \sum_{v \in V(\gamma) \cap B} \sqrt{\sum_{e_1, e_2 \in E(\gamma), e_1 \cap e_2 = v} \text{sgn}(e_1, e_2) [\epsilon^{ijk} X^i_{e_1} X^k_{e_2}]}^2 f_\gamma.
\] (3.12)

Formula (3.12) motivates to introduce the “volume operator at a point” \(\hat{V}_v\): For each integer \(n \geq 2\) define \(\{[v, n]\}\) to be the set of germs of \(n\) analytical edges incident at \(v\) (a germ of an analytical edge at a point \(v\) is a complete set of analytical data available at \(v\) that are necessary to reconstruct it, that is, essentially the coefficients of its Taylor series). For a germ \(\vec{e}_n := (e_1, \ldots, e_n) \in \{[v, n]\}\) define

\[
\hat{V}_{\vec{e}_n} := \sqrt{\sum_{e_1, e_2 \in \vec{e}} \text{sgn}(e_1, e_2) [\epsilon^{ijk} X^i_{e_1} X^k_{e_2}]}^2.
\]

where the right invariant vector field \(X^i_e(g) = X^i_e(gh)\forall h \in SU(2)\), due to right invariance, depends really only on the germ of the edge \(e\) because it acts on a function in the same way no matter how “short” the segment of \(e\) is on which that function actually depends, as long as that segment starts at \(v = e(0)\). In particular all \(X^i_e, e\) incident at \(v\), commute as long as their germs are different. Then

\[
\hat{V}(B) = \sum_{v \in B} \hat{V}_v, \text{ where } \hat{V}_v = \sum_{n=2}^\infty \sum_{\vec{e}_n \in \{[v, n]\}} \hat{V}_{\vec{e}_n}.
\] (3.13)

We see that \(\hat{V}(B)\) is a densely defined, essentially self-adjoint, positive semi-definite operator on \(H\) for each bounded region \(B \subset \Sigma\). Its most interesting property is that it acts non-trivially only at vertices of the graph underlying a cylindrical function, moreover, that vertex has to be such that at least two edges incident at it have linearly independent tangents there. This is in complete analogy with the volume operator of the 3+1 theory just that we need to replace everywhere valence three by valence two. Unlike the the three-dimensional volume operator, however, its two-dimensional ”brother” does not vanish at two-valent and three-valent vertices at all as long as there are at least two edges with linearly independent tangents at the vertex under consideration. As we will see in the appendix, the two-dimensional volume operator is even positive definite on gauge invariant functions with two-and three valent vertices while the three-dimensional volume operator annihilates such functions identically. This is to be expected because by inspection of (3.13) the principal symbol of that operator is non-singular on two-and three valent vertices while in the three-dimensional case it is singular.

The fact that the volume operator acts only at vertices of the graph will enable us to take the infra-red limit in case we are dealing with asymptotically flat topologies and also ensure that the ultra-violet limit exists. Thus, the volume operator acts both as an IR and as an UV dynamical regulator, a point of view emphasized in [10].

Remark:
Notice that \(q_{ab} = \{A^i_a, V\} \{A^i_b, V\}\) just as in the three-dimensional case. This observation lead in the three-dimensional case to the construction of a length operator [11]. The only crucial property that was necessary to construct this operator was that the volume...
operator acts only at vertices. Since that is true for the two-dimensional operator as well we can therefore take over all the results and formulae from [7] to the two-dimensional case, except for the obvious differences which are due to different dimension and algebraic expressions in terms of right invariant vector fields of the volume operators. In particular, although the eigenvalues of the length operators are certainly different, qualitatively the spectrum is still discrete, the operator is positive semi-definite and essentially self-adjoint and the length of a curve as measured by a spin-network state is different from zero only if at least one edge of the graph crosses the curve, though not necessarily in a vertex. Thus we automatically have a two-dimensional length operator as well.

The fact that the two-dimensional length operator is less degenerate than the three-dimensional one can be traced back to the observation that what is length in two dimensions is what is area in three dimensions.

4 Regularization

This section is divided into three parts: in the first part we will derive a regulated Wheeler-DeWitt operator. The regularization consists in a triangulation of $\Sigma$ which is kept arbitrary at this stage. In the next part we will specify the properties that we wish to impose on the triangulation and then make a particular choice which satisfies those properties. Finally in the last part we complete the regularization by employing that triangulation and take the continuum limit which then equips us with a densely defined family of operators, one for each graph.

The presentation will be kept largely parallel to the one in [3] in order to facilitate comparison.

4.1 Derivation of the regulated operator

We wish to define an operator corresponding to

$$H(N) := \int_{\Sigma} d^2 x N F_i \frac{E^i}{\sqrt{\det(q)}}$$

$$= \frac{1}{2} \int d^2 x N \epsilon^{ab} \epsilon_{ijk} F_i \{A^j, V\} \{A^k, V\} \sqrt{\det(q)}$$

$$= \frac{1}{2} \int N \epsilon_{ijk} F_i \{A^j, V\} \wedge \{A^k, V\} \sqrt{\det(q)}$$

$$= - \int N \text{tr}(F \{A, V\} \wedge \{A, V\}) \sqrt{\det(q)} \quad (4.1)$$

where we have used that $\text{tr}(\tau_i \tau_j \tau_k) = -\epsilon_{ijk}/2$ and (2.3). Following the idea outlined in section 2 consider now a point splitting of the above expression as follows: Let $\epsilon$ be a small number and $\chi_{\epsilon}(x, y) := \theta(\frac{\epsilon}{2} - |x^1 - y^1|)\theta(\frac{\epsilon}{2} - |x^2 - y^2|)$ where $\theta(t) = 1$ if $t > 0$ and 0 otherwise, that is, $\chi_{\epsilon}$ is the characteristic function of square of coordinate volume $\epsilon^2$. Moreover, it is just true that $\{A^i_{\alpha}(x), V\} = \{A^i_{\alpha}(x), V(x, \epsilon)\}$ where

$$V(x, \epsilon) := \int_{\Sigma} d^2 y \chi_{\epsilon}(x, y) \sqrt{\det(q)(y)}$$
is the volume of the square around $x$ as measured by $q_{ab}$. Notice that trivially $\lim_{\epsilon \to 0} V(x, \epsilon)/\epsilon^2 = \sqrt{\det(q)(x)}$. Therefore we have the identity (we write the density $F^i$ as a 2-form)
\[
\lim_{\epsilon \to 0} \int N(x) \text{tr}(F(x)) \int \chi_{\epsilon}(x, y) \frac{\{A(y), V\} \wedge \{A(y), V\}}{V(y, \epsilon)} = -\frac{1}{2} H(N),
\]
that is, the point splitting singularity $1/\epsilon^2$ was absorbed into $V(y, \epsilon)$. The limit identity (4.2) motivates to define a point split expression
\[
H_\epsilon(N) := -\int N(x) \text{tr}(F(x)) \int \chi_{\epsilon}(x, y) \frac{\{A(y), V\} \wedge \{A(y), V\}}{V(y, \epsilon)}
\]
\[
= -\int N(x) \text{tr}(F(x)) \int \chi_{\epsilon}(x, y) \frac{\{A(y), V(y, \epsilon)\} \wedge \{A(y), V(y, \epsilon)\}}{\sqrt{V(y, \epsilon)}}
\]
\[
= -4 \int N(x) \text{tr}(F(x)) \int \chi_{\epsilon}(x, y) \{A(y), \sqrt{V(y, \epsilon)}\} \wedge \{A(y), \sqrt{V(y, \epsilon)}\},
\]
that is, the simple formula $\{\cdot, \sqrt{V(y, \epsilon)}\} = \{\cdot, V(y, \epsilon)\}/(2V(y, \epsilon))$ enabled us to bring the volume functional from the denominator into the nominator, of course, inside the Poisson bracket.

The idea is now to replace Poisson brackets by commutators and the volume functional by the volume operator and then take the limit $\epsilon \to 0$. In order to do that we must first write (4.2) in such quantities on which the volume operator knows how to act. Since, as obvious from the previous section, it only knows how to act on functions of holonomies along edges we must replace the connection field $A^i_a$ in (4.3) by holonomies. We are thus forced to introduce a triangulation of $\Sigma$.

Denote by $\Delta$ a solid triangle. Single out one of the corners of the triangle and call it $v(\Delta)$, the basepoint of $\Delta$. At $v(\Delta)$ there are incident two edges $s_1(\Delta)$, $s_2(\Delta)$ of $\partial \Delta$ which we equip with outgoing orientation, that is, they start at $v(\Delta)$. We fix the labelling as follows: let $s$ be the analytic extension of $s_1(\Delta)$ and $\bar{s}_1(\Delta)$ the half of $s$ starting at $v(\Delta)$ but not including $s_1(\Delta) - \{v\}$ with outgoing orientation at $v(\Delta)$. Let $U$ be a sufficiently small neighbourhood of $v(\Delta)$ which is split into two halves by $s$. Define the upper half $U^+$ of $U$ to be that half of $U$ which one intersects as one turns $s_1(\Delta)$ counterclockwise into $\bar{s}_1(\Delta)$. Now we require that there exists $U$ such that $U \cap s_2(\Delta) = U^+ \cap s_2(\Delta)$, that is, $s_2(\Delta)$ intersects the upper half of $U$.

**Definition 4.1** Two analytical edges $e_1, e_2$ incident and outgoing at $v = e_1 \cap e_2$ will be said to be right oriented iff there exists a neighbourhood $U$ of $v$, its upper half $U^+$ being defined by $e_1$, such that $e_2$ intersects $U^+$.

This prescription is obviously diffeomorphism invariant. Notice that we did not, as it is usually done for triangulations, require that the tangents of the edges bounding $\Delta$ must
have linearly independent tangents at their intersection. If they are linearly independent then our prescription is equivalent to saying that $\epsilon_{ab}s^a_1(0)s^b_2(0) > 0$

Finally, let $a(\Delta)$ denote the remaining edge of $\partial(\Delta)$, called the arc of $\Delta$, whose orientation we fix by requiring that it runs from the endpoint of $s_1(\Delta)$ to the endpoint of $s_2(\Delta)$. Then $\partial\Delta = a_{12}(\Delta) = s_1(\Delta) \circ a(\Delta) \circ s_2(\Delta)^{-1}$ is called the loop of $\Delta$ based at $v(\Delta)$. We define also $a_{21}(\Delta) := a_{12}(\Delta)^{-1}$.

Let us now write the integral over $\Sigma \times \Sigma$ in (4.3) as a double sum of integrals over $\Delta \times \Delta'$ where $\Delta, \Delta'$ are triangles of some triangulation $T$ of $\Sigma$

$$H_{T,\epsilon}(N) = -4 \sum_{\Delta, \Delta' \in T} \text{tr}(\int_{\partial\Delta'} N(x)F(x)\int_{\Delta} \chi_\epsilon(x, y)\{A(y), \sqrt{V(y, \epsilon)}\} \wedge \{A(y), \sqrt{V(y, \epsilon)}\}) \ .$$

The purpose of the notation just introduced is that we may approximate, for sufficiently fine triangulation, each of the integrals by a function of holonomies as follows : Let $\delta$ be a small parameter and $s_i(\Delta)$ be the image of $[0, \delta]$ under the path $s_i(\Delta, t)$. Then, using smoothness of the connection we find

$$N(v(\Delta'))\chi_\epsilon(v(\Delta'), y)e^{ij}h_{\alpha ij}(\Delta') = 2\delta^2 N(v(\Delta'))\chi_\epsilon(v(\Delta'), y)\frac{s^a_1(\Delta', 0)s^b_2(\Delta', 0)}{2}F_{ab}(v(\Delta')) + o(\delta^3)$$

$$= 2\int_{\Delta} \chi_\epsilon(x, y)N(x)F(x) + o(\delta^3)$$

$$\chi_\epsilon(x, v(\Delta))e^{ij}h_{\alpha ij}(\Delta)\{h^{-1}_{s i(\Delta)}; \sqrt{V(v(\Delta), \epsilon)}\}h_{s j(\Delta)}\{h^{-1}_{s j(\Delta)}; \sqrt{V(v(\Delta), \epsilon)}\}$$

$$\chi_\epsilon(x, v(\Delta))\delta^2 s^a_1(\Delta, 0)s^b_2(\Delta, 0)\{A_a(v(\Delta)), \sqrt{V(v(\Delta), \epsilon)}\}\{A_b(v(\Delta)), \sqrt{V(v(\Delta), \epsilon)}\}$$

$$= 2\int \chi_\epsilon(x, y)\{A(y), \sqrt{V(y, \epsilon)}\} \wedge \{A(y), \sqrt{V(y, \epsilon)}\} + o(\delta^3)$$

(4.5)

since the area of $\Delta$ is approximately $\delta^2\epsilon_{ab}s^a_1(\Delta, 0)s^b_2(\Delta, 0)/2$ so that both integrals are of order $\delta^2$ provided that the tangents of $\partial(\Delta)$ at $v(\Delta)$ are linearly independent. Thus, up to an error of order $\delta^2$ which vanishes in the limit as the we remove the triangulation we may substitute (4.4) by

$$H_{T,\epsilon}(N) = -2 \sum_{\Delta, \Delta' \in T} e^{ij}e^{kl}N(v(\Delta'))\chi_\epsilon(v(\Delta'), v(\Delta)) \times \text{tr}(h_{\alpha ij}(\Delta')h_{s k(\Delta)}\{h^{-1}_{s i(\Delta)}; \sqrt{V(v(\Delta), \epsilon)}\}h_{s j(\Delta)}\{h^{-1}_{s j(\Delta)}; \sqrt{V(v(\Delta), \epsilon)}\}) \ .$$

(4.6)

The result (4.9) is still purely classical and becomes $H(N)$ when taking

1) first the continuum limit (that is, refining the triangulation ad infinitum) and
2) taking $\epsilon \to 0$ on smooth connections $A^a_i$ and smooth momenta $E^a_i$.

A second way to guide the limit and that leads to $H(N)$ is by “synchronizing” $\epsilon \approx \delta$ and to take $\delta \to 0$ as follows : for each $\Delta$ define

$$\epsilon(\Delta) := \sqrt{\epsilon_{ab}s^a_1(\Delta, 0)s^b_2(\Delta, 0)}|\delta|,$$

replace for each $\Delta'$ :

1) $\chi_\epsilon(v(\Delta), v(\Delta'))$ by $\chi_\epsilon(\Delta')(v(\Delta), v(\Delta'))$ and
2) \( V(v(\Delta'), \epsilon) \) by \( V(v(\Delta'), \epsilon(\Delta')) \) and then take \( \delta \to 0 \). Notice that this corresponds to introducing \( \epsilon(y) = \rho(y)\delta \) instead of \( \epsilon \) in (4.2) where \( \rho(y) \) is an almost nowhere (with respect to \( d^2x \)) vanishing function such that \( \rho(v(\Delta))\delta = \epsilon(\Delta) \). Clearly \( \rho \) must be almost nowhere vanishing as otherwise we do not get a \( \delta \) distribution in the limit \( \delta \to 0 \). Notice that the set of \( v(\Delta') \)'s has \( d^2x \) measure zero so that a vanishing \( \rho(v(\Delta')) \) is not worrisome. It will be this latter limit which is meaningful in the quantum theory.

We have managed to write \( H(N) \) in terms of holonomies up to an error which vanishes in either of the limits that we have indicated.

The next step is to turn (4.6) into a quantum operator. This now just consists in replacing \( V(v(\Delta), \epsilon) \) by \( \hat{V}(v(\Delta), \epsilon) \) and Poisson brackets by commutators times \( 1/(i\hbar) \) because we work in a connection representation. The result is

\[
\hat{H}_{T,\epsilon}(N) = \frac{2}{\hbar^2} \sum_{\Delta, \Delta' \in T} e^{\epsilon_{ij} k^i} N(v(\Delta')) \chi_{\epsilon}(v(\Delta'), v(\Delta)) \times \tr(h_{\epsilon_{ij}}(\Delta'') h_{\epsilon_{kj}}(\Delta)) \left[ h_{s(\Delta)}^{-1}, \sqrt{\hat{V}(v(\Delta), \epsilon)} \right] h_{s(\Delta)} \left[ h_{s(\Delta)}^{-1}, \sqrt{\hat{V}(v(\Delta), \epsilon)} \right]. \tag{4.7}
\]

We wish to show that (4.7) is densely defined in the limit \( \epsilon \to 0 \) no matter how we choose the triangulation \( T \), as long as it is finite, thereby showing that the regulator \( \epsilon \) can be removed \textit{without encountering any singularity}. Thus, we prescribe the \( \epsilon \to 0 \) limit \textit{before} taking the limit of infinitely fine triangulation (continuum limit) and therefore have interchanged the order of limits as compared to the classical theory. However, as we will show shortly, one arrives at the same result when synchronizing \( \epsilon \approx \delta \) and taking \( \delta \) sufficiently small but finite for the moment being which corresponds to the second way to guide the classical limit indicated above and therefore interchanging the limits is allowed. For that purpose let \( f_{\gamma} \) be a function which is cylindrical with respect to a graph. Consider first some triangle \( \Delta \) which does not intersect \( \gamma \) at all. Then it is easy to see that

\[
[ h_{s(\Delta)}^{-1}, \sqrt{\hat{V}(v(\Delta), \epsilon)} ] f_{\gamma} = 0
\]

The reason for this is that the graphs \( \gamma \) and \( \gamma \cup s(\Delta) \) then do not have any two-valent vertex in the box around \( v(\Delta) \) parametrized by \( \epsilon \) other than the vertices of \( \gamma \) themselves. Thus the volume operator does not act on \( h_{s(\Delta)}^{-1} \) and the commutator vanishes. It follows that only tetrahedra which intersect the graph contribute in (4.7). So let \( \gamma \cap \Delta \neq \emptyset \). For the same reason as above we find a non-zero contribution only if \( s(\Delta) \) or \( s_2(\Delta) \) intersect \( \gamma \), that \( a_{12}(\Delta) \) alone intersects \( \gamma \) is not sufficient. Moreover, still for the same reason, if \( s(\Delta) \) intersects \( \gamma \) but not in the starting point of \( s(\Delta) \) then we still get zero upon choosing \( \epsilon \) sufficiently small so that the intersection point \( p \) lies outside the support of the characteristic function, that is, \( \chi_{\epsilon}(v(\Delta), p) = 0 \). Thus a triangle \( \Delta \) contributes to (4.7) if and only if \( v(\Delta) \in \gamma \). But if that is true then we may replace \( \hat{V}(v(\Delta), \epsilon) \) by the operator \( \hat{V}_{v(\Delta)} \) defined in (3.13) and so the \( \epsilon \)-dependence of \( \hat{V}(v(\Delta), \epsilon) \) has dropped out. The remaining \( \epsilon \)-dependence now just rests in the function \( \chi_{\epsilon}(v(\Delta'), v(\Delta)) \). Now, since we let \( \epsilon \to 0 \) first, at finite triangulation, we conclude altogether that the unrestricted double sum over triangles in (4.7) collapses to a double sum over triangles subject to the condition that their basepoints coincide and lies on the graph. In formulæ

\[
\hat{H}_T(N) f_{\gamma} := \lim_{\epsilon \to 0} \hat{H}_{T,\epsilon}(N) f_{\gamma}
\]
\[
\frac{2}{\hbar^2} \sum_{\Delta, \Delta' \in \mathcal{T}, \mathcal{v} : v(\Delta) = v(\Delta')} \epsilon^{ij} \epsilon^{kl} N(v) \times \text{tr}(h_{\alpha,ij}(\Delta') h_{s,i}(\Delta) h_{s,j}(\Delta')^{-1} \hat{V}_v) h_{s,i}(\Delta') \hat{V}_v)f_{\gamma} f_{\gamma}(4.8)
\]

which displays \( \hat{H}_T(N) \) as a densely defined operator which does not suffer from any singularities because at finite triangulation there are only a finite number of terms involved in (4.8), even if \( \Sigma \) is not compact.

Notice that in the \( \epsilon \to 0 \) limit we have recovered a gauge invariant operator as we should.

Let us now show that one arrives at the same result by synchronizing \( \epsilon \approx \delta \) as above and taking \( \delta \) sufficiently small but still finite: Namely, by choosing \( \epsilon(\Delta') \) as above we have arranged that only the starting points of the \( s_i(\Delta') \) are covered by the \( \epsilon(\Delta') \)-box around \( v(\Delta') \) that underlies the definition of \( \hat{V}(v(\Delta'), \epsilon(\delta')) \). This implies first of all that we need to sum only over \( v(\Delta) = v(\Delta') \). Next, as we will be forced to adapt the triangulation to the graph anyway, we can arrange that the \( \Delta \) intersect \( \gamma \) only either in whole edges or in vertices of \( \Delta \). If that is the case, then it follows that \( [h_{s,i}(\Delta'), \sqrt{\hat{V}(v(\Delta), \epsilon(\Delta'))}] f_{\gamma} \) is non-vanishing only if \( s_i(\Delta) \) intersects \( \gamma \) in \( v(\Delta) \) because the end-point is not covered by the \( \epsilon(\Delta) \)-box and if \( s_i(\Delta) \) is contained in \( \gamma \) but does not start in a vertex of \( \gamma \) then the commutator vanishes due to the properties of the volume operator. This is enough to see that we arrive at (4.8) again.

In either way of taking the limit we are now left with taking the continuum limit \( \delta \to 0 \) of refining the triangulation ad infinitum which we denote as \( T \to \infty \). Certainly that limit depends largely on the choice of the limit \( T \to \infty \). For instance, if we are not careful and refine \( T \) in such a way that the number of basepoints of triangles that intersect \( \gamma \) diverges we will not get a densely defined operator. We see that we must choose \( T \) according to \( \gamma \) so that we get actually a family of operators

\[
\hat{H}_{\gamma,T}(N) = \hat{H}_{\gamma}(N)
\]

where \( T(\gamma) \) is a triangulation adapted to \( \gamma \) together with a well-defined refinement procedure \( T \to \infty \). We will propose such a \( T(\gamma) \) in the next subsection guided by some physical principles. It will then be our task to verify that the family \( (\hat{H}_{\gamma,T}(N)) \) still defines a linear operator.

### 4.2 Choice of the triangulation

So far everything we said was in complete analogy with the three-dimensional case \[8\] except that there we did not even need a point-splitting. In particular, (4.8) is the precise counterpart of the three-dimensional Euclidean Hamiltonian constraint operator.

What is different now is that in the 3+1 case the volume operator was much more degenerate than in the 2+1 case, a result of which was that a basepoint of a simplex had to coincide with a vertex of the graph in order to contribute without further specification of the triangulation. Therefore, it was sufficient to adapt the triangulation to the graph in such a way that, among other things, the number of simplices intersecting a vertex stays constant as one refines the triangulation in order to arrive at well-defined continuum
limit. In the 2+1 case that is not true any longer and one must worry about the number of triangles intersecting the graph $\gamma$ off the vertices of $\gamma$.

Let us adopt the physical principles listed in [3] which should guide one of how to choose the triangulation. In brief, they were:

1) The amount of ambiguity arising from the choice of the triangulation should be kept to a minimum.

2) The resulting operator should be non-trivial and not annihilate every state.

3) The choice of the contributing $\Delta$ should be diffeomorphism covariant as to interact well with the diffeomorphism invariance of the theory.

4) The choice of the $\Delta$ should be canonical and not single out one part of the graph as compared to the other or one graph as compared to another.

5) The family of operators $\hat{H}_\gamma(N)$ should define a linear operator $\hat{H}(N)$ (cylindrical consistency).

6) The resulting operator $\hat{H}_{\gamma,T}(N)$ should be densely defined with a well-defined continuum limit. That is, if $\Psi \in \Phi'$ is a diffeomorphism invariant distribution and $f_{\gamma}$ a function cylindrical with respect to a graph $\gamma$ then

$$\lim_{T \to \infty} \Psi(\hat{H}_{\gamma,T}(N)f_{\gamma}) =: \Psi(\hat{H}(N)f_{\gamma})$$

exists. The fact that $\Psi$ is diffeomorphism invariant is because we actually want to define $\hat{H}(N)$ on solutions to the diffeomorphism constraint which turn out to be distributions [4] and so the above limit is the precise sense in which $\hat{H}(N)$ is defined on distributions.

7) The operator $\hat{H}(N)$ should be free of anomalies, that is,

$$\Psi([\hat{H}(M), \hat{H}(N)]\phi) = 0$$

for each $\phi \in \Phi$ and every diffeomorphism invariant $\Psi \in \Phi'$.

Since we wish to obtain a densely defined operator no matter how fine the triangulation while keeping the extra structure coming from the triangulation to a minimum we are naturally lead to impose that the triangles that intersect $\gamma$ in its basepoint must be constant in number. There are only two diffeomorphism invariantly different possibilities: either $v(\Delta)$ is a vertex of $\gamma$ or it lies on an edge of $\gamma$ between its endpoints. Since we want to get a non-vanishing operator one of the two or both scenarios should happen.

Suppose first then that $v(\Delta)$ is an interior point of an edge $e$. Then there is no natural way how to choose the triangle $\Delta$ itself: the only structure available is the edge $e$ and one may therefore choose one of $s_1(\Delta)$, say $s_1(\Delta)$, to lie entirely in $e$. But then $s_2(\Delta)$ should certainly not lie in $e$ otherwise $v(\Delta)$ would be a vertex of $\gamma \cup \partial \Delta$ with only co-linear tangents of edges incident at it and the volume operator $\hat{V}_{s(\Delta)}$ would vanish. Thus there is at least a huge ambiguity in how to choose $s_2(\Delta)$.

If, on the other hand, $v(\Delta)$ is a vertex of the graph then there are at least two edges $e, e'$ of $\gamma$ incident at it and now it is a natural choice to assume that $s_i(\Delta)$ coincide with
segments of $e, e'$. In conclusion, guided by the principle of introducing as few ambiguous elements as possible into the triangulation we are motivated to exclude that a $v(\Delta)$ is an interior point of an edge or that it anyway does not contribute. The latter can be achieved by assuming that the edges $s_i(\Delta)$ have co-linear tangents at $v$ in this case.

Now we are left with those $v(\Delta)$ that are vertices of $\gamma$. Following the principle that our prescription should be canonical we must have that either each vertex of $\gamma$ is a basepoint of some $\Delta$ or none. Since the latter possibility is excluded by the principle of non-triviality we are now concerned with the issue of how many $\Delta$’s should have basepoint in each $v \in V(\gamma)$. A natural answer to this question is that there should be as many such $\Delta$’s as pairs of edges incident at $v$ because otherwise we would single out one pair to another. However, we still need to fulfill the requirement that the $\Delta$’s must come from a triangulation. Both observations motivate to define a whole family of triangulations adapted to $\gamma$ and to average over them.

Finally, we must fix in a diffeomorphism covariant way how to attach the arcs $a_{12}(\Delta)$ to $\gamma$. Notice that since $\Delta$ is a part of a triangulation with $v(\Delta)$ a vertex of $\gamma$ and with $s_i(\Delta)$ segments of edges of $\gamma$ incident at $v$, it is possible that the endpoints of $a_{12}(\Delta)$ are actually basepoints of other triangles $\Delta'$. This we either must avoid by choice (which is possible) or we must impose that the tangents of $s_i(\Delta)$ and $a_{12}(\Delta)$ are co-linear at the endpoints of $a_{12}(\Delta)$. As we will see, only the latter possibility leads to an anomaly-free theory. This furnishes our preliminary investigation of how to choose $T(\gamma)$.

We will now prescribe $T(\gamma)$. The prescription is simpler but very similar to the three-dimensional case.

Fix a vertex $v$ of $\gamma$ and let $n$ denote its valence. We can label the edges of $\gamma$ incident at $v$ in such a way that

1) the pairs $(e_1, e_2), (e_2, e_3), \ldots, (e_{n-1}, e_n)$ are right oriented and possibly also $(e_n, e_1)$ is right oriented according to definition (4.1) and

2) as one encircles $v$ counter-clockwise, one does not cross any other edge after one crosses $e_i$ and before one crosses $e_{i+1}$ where $e_{n+1} \equiv e_1$. We are going to construct a triangle $\Delta$ associated with each such right oriented pair which we will call $(e_1, e_2)$ from now on. We do not, in contrast to the 3+1 theory, construct a triangle associated with each pair because then $a_{12}$ in two dimensions would intersect not only $s_1, s_2$ but also other edges of the graph which we must avoid in order to have an anomaly-free theory as we will see. Moreover, in two dimensions the way we ordered the edges incident at $v$ is very natural and not available in three dimensions.

Finally, let $E(v)$ equal $n$ if $(e_n, e_1)$ is right oriented, otherwise let it equal $n - 1$. In particular, for $n = 2$ we must have $E(v) = 1$.

We choose now $s_i(\Delta)$ to be any segment of $e_i$ which does not include the other endpoint of $e_i$ different from $v$ and which starts at $v$. Furthermore, connect the endpoints of $s_1(\Delta)$ with the endpoint of $s_2(\Delta)$ by an arc $a_{12}(\Delta)$ with the special property that the tangent of $a_{12}(\Delta)$ is

1) parallel to the tangent of $s_1(\Delta)$ at the end-point of $s_1(\Delta)$ and

2) anti-parallel to the tangent of $s_2(\Delta)$ at the end-point of $s_2(\Delta)$.

Two remarks are in order:

a) Notice that we do not have to worry about any other edge of $\gamma$ intersecting $a_{12}(\Delta)$ because in two dimensions the topology of the routing of $a_{12}$ through the edges of $\gamma$ is very simple: there is no way that $a_{12}$ can intersect any other edges of $\gamma$ other than $s_1, s_2$ given
the labelling of $e_i$ made above. This is in contrast to the three-dimensional case where the topology of the routing was extremely complicated to prescribe in a diffeomorphism covariant way.

b) In contrast to the three-dimensional case we here prescribed the $C^1$ properties of the edges $s_1, s_2, a_{12}$ at their intersection points. The reason for this will become evident only later when we prove anomaly-freeness. We will see that the $C^1$ property of the intersection is crucial.

Whenever $(e_n, e_1)$ is a right oriented pair the $n$ triangles saturate $v$. Otherwise there are only $n-1$ triangles and they do not yet saturate $v$. We follow the approach proposed in \(3\) in order to achieve saturation. Namely, we take each of the $E(v)$ triangles and construct three more from it such that they altogether saturate $v$. Then we average over the $E(v)$ triangulations based on using only one such quadrupel of triangles. The details are as follows:

Let $s_i(t), a_{12}(t)$ be a parametrization of $s_i, a_{12}$ with $t \in [0,1]$. Let $s_i(t) := v - (s_i(t) - v) = 2v - s_i(t)$, $a_{12}(t) := 2v - a_{12}(t)$, $a_{21}(t) := s_2(1) + t(s_1(1) - s_2(1))$, $a_{21}(t) := s_2(1) + t(s_1(1) - s_2(1))$.

Then it is easy to see that $(s_1, s_2), (s_2, s_1), (s_2, s_1)$ are right oriented pairs and that the four triangles $\Delta_{12}, \Delta_{12}, \Delta_{12}, \Delta_{21}$ based on these triples of edges saturate $v$ (use $a_{12}(0) = s_1(1), a_{12}(1) = s_2(1)$ to see this).

Let now $S_i(v)$ denote the region in $\Sigma$ filled by these four triangles based on a pair of edges $(e_i, e_{i+1})$ incident at $v$. Also denote by $\Delta_i(v)$ the original triangle defined by $s_1, s_2, a_{12}$ for that pair from which we constructed the remaining three triangles as above. Let $S(v) := \cup_{i=1}^{E(v)} S_i(v)$ be the union of these regions given by all the $E(v)$ pairs and let $\bar{S}_i(v) = S(v) - S_i(v)$. We will choose all the triangles so small that the $S(v)$ are mutually disjoint. Finally, let $\bar{S} = \cup_{v \in V(\gamma)} S(v)$ and $\bar{S} = \Sigma - \bar{S}$. Then we can trivially decompose any integral over $\Sigma$ as follows

$$
\int_\Sigma = \int_{\bar{S}} + \sum_{v \in V(\gamma)} \int_{S(v)}
$$

$$
= \int_{\bar{S}} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{i=1}^{E(v)} \left[ \int_{\bar{S}_i(v)} + \int_{S_i(v)} \right]
$$

$$
= [\int_{\bar{S}} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{i=1}^{E(v)} \int_{\bar{S}_i(v)}] + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{i=1}^{E(v)} \int_{S_i(v)}
$$

$$
= \left[ \int_{\bar{S}} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{i=1}^{E(v)} \int_{\bar{S}_i(v)} \right] + \left[ \sum_{v \in V(\gamma)} \frac{4}{E(v)} \sum_{i=1}^{E(v)} \int_{\Delta_i(v)} + o(\delta^3) \right]. \tag{4.9}
$$

In the last line we have exploited that for smooth integrands and small triangles the integral over each of the four triangles constructed is the same up to higher order in the parameter $\delta$ introduced before equation \(4.5\). It is clear that the term in the first square bracket of the last line in \(4.3\) is a sum of integrals over regions of $\Sigma$ each of which does not contain vertices of $\gamma$.

We are now ready to specify the family of triangulations $T(\gamma)$ of $\Sigma$ which by \(4.3\) can actually be reduced to a family of triangulations of $\bar{S}, \bar{S}_i(v), \Delta_i(v)$ for $v \in V(\gamma), i = \cdots$
1, ..., E(v):
1) Triangulate $\Delta_i(v)$ by $\Delta_i(v)$
2) Triangulate $S$ and $S_i(v)$ arbitrarily subject to the condition that no basepoint of a triangle should lie on an edge of $\gamma$ or that all tangents at an intersection with an edge of $\gamma$ are co-linear
3) The triangles $\Delta_i(v)$ collapse to $v$ as $T \to \infty$ in such a way that all graphs $\gamma \cup \Delta_i(v)$ are diffeomorphic as $T \to \infty$. In fact as long as we keep the prescription of how to choose $s_i(\Delta), a_{12}(\Delta)$ specified above, all the graphs $\gamma \cup \Delta$ are related by an analyticity preserving smooth diffeomorphism no matter how “large” $\Delta$. Namely, such diffeomorphisms can leave the image of $\gamma$ invariant while putting $a_{12}$ in any diffeomorphic shape.

Notice that now we have a well-defined prescription for the continuum limit because by construction the triangles that triangulate $S, S_i(v)$ do not contribute to the operator (4.8). The fact that the number of triangles that have their basepoint in vertices of the graph (which are the only ones that contribute) stays constant (namely $E(v)$) indicates that the continuum operator will be densely defined.

### 4.3 Continuum Limit

Let us summarize: having specified the triangulation we have triangles $\Delta(\gamma, T)$ associated with the graph, more precisely $E(v)$ for each vertex $v$ of $\gamma$, the index $T$ indicating that the continuum limit has not been taken yet. Then the regulated operator (4.8) becomes

$$\hat{H}_T(N)f_\gamma := \hat{H}_{\gamma,T}(N)f_\gamma := \frac{1}{h^2} \sum_{v \in \mathcal{V}(\gamma)} N(v)(\frac{4}{E(v)})^2 \sum_{v(\Delta), v(\Delta') = v} \epsilon^{ij} \epsilon^{kl} \times$$
$$\times \text{tr}(h_{\alpha_i(\Delta)}h_{s_i(\Delta)}[h^{-1}_{s_k(\Delta)}; \sqrt{V_v}]h_{s_i(\Delta)}[h^{-1}_{s_k(\Delta)}; \sqrt{V_v}])f_\gamma$$

(4.10)

where we have dropped the dependence of the $\Delta$ on $\gamma, T$. Now, since as $T \to \infty$ all holonomies approach unity, the limit $T \to \infty$ does not have any meaning on the Hilbert space $\mathcal{H} = L_2(\mathcal{A}_G, d\mu_0)$. Indeed, on smooth connections we would get zero while on distributional connections the limit does not exist. Thus, the limit $T \to \infty$ must be understood in another way. Indeed, recall that we wanted to impose the Hamiltonian constraint actually on diffeomorphism invariant distributions $\Psi \in \Phi'$. Now, the operator $\hat{H}_T(N)$ defines for each $T$ an operator $\hat{H}_T(N)'$ on $\Phi'$ by the equation

$$[(\hat{H}_T(N)')\Psi](\phi) := \Psi(\hat{H}_T(N)\phi) \forall \Psi \in \Phi', \phi \in \Phi$$

(4.11)

because $\hat{H}_T(N)$ has domain and range in $\Phi$ which is dense in $\mathcal{H}$, for each $T$. Now, if $\Psi$ is diffeomorphism invariant then

$$[(\hat{H}(N)')\Psi](f_\gamma) := \lim_{T \to \infty} [(\hat{H}_T(N)')\Psi](f_\gamma) = \Psi(\hat{H}_{\gamma,T_0}(N)f_\gamma)$$

(4.12)

for each function $f_\gamma$ cylindrical with respect to a graph $\gamma$ and for each $\gamma$. In other words, the number $\Psi(\hat{H}_{\gamma,T}(N)f_\gamma)$ does not change under variation of $T$ which by prescription corresponded to a diffeomorphism and so on diffeomorphism invariant states we may evaluate it on any finite value $T_0$ and the $T \to \infty$ limit is trivial. It follows that on diffeomorphism invariant states the continuum limit is already taken for $(\hat{H}_T(N))'$. In fact, it is easy to see that this result can be extended to any product.
\[ \dot{H}_T(N_1) \dot{H}_T(N_2) \ldots \dot{H}_T(N_n) \]
because the triangles attached have, at each level, an unambiguously defined diffeomorphism covariant location. This observation is needed in order to give sense to commutator computations \[[3, 12]\].

In the sequel we will drop the index \( T \) and understand that when finally evaluating everything on diffeomorphism invariant distributions the value of \( T \) is irrelevant.

## 5 Consistency

There are two kinds of consistencies to be discussed:

The first is the cylindrical consistency, that is, we have obtained a family of operators \( (\hat{H}_\gamma(N))_\gamma \) which should be projections to cylindrical subspaces of a “mother” \( \hat{H}(N) \). That such a \( \hat{H}(N) \) exists has to be proved.

The second is that we need to make sure that \( \hat{H}(N) \) does not suffer from quantum anomalies.

### 5.1 Cylindrical Consistency

In proving that a family of operators \( (\hat{O}_i, D_i)_{i \in I} \) on a Hilbert space \( \mathcal{H} \), where \( D_i \) is the domain of \( \hat{O}_i \) and where \( I \) is some partially ordered index set \( I \) with ordering relation \(<,\) is cylindrically consistent we need to reveal that whenever \( i < j \) that \( \hat{O}_j \) is an extension of \( \hat{O}_i \), that is

1) The domain of \( \hat{O}_i \) is contained in that of \( \hat{O}_j \), \( D_i \subset D_j \) and
2) The restriction of \( \hat{O}_j \) to \( D_i \) coincides with \( \hat{O}_i; \ (\hat{O}_j)_{|i} = \hat{O}_i \).

Let us check that this is the case for our operator family. Recall that a spin-network state depends on all of its edges non-trivially in the sense that all edges carry spin \( j > 0 \).

The space \( \Phi_\gamma \) is the set of finite linear combinations of spin-network states which depend on the graph \( \gamma \). Now, while the set of graphs can be partially ordered by the inclusion relation, the set of cylindrical functions cannot because a function which is defined on a smaller graph is defined also on any bigger graph that properly contains it, however, the additional edges in that graph automatically carry spin zero and so the cylindrical subspaces cannot be compared. Another way of saying this is that given a cylindrical function \( f \) we can uniquely decompose it as \( f = \sum f_\gamma \) where \( f_\gamma \in \Phi_\gamma \) and on \( f_\gamma \) we have unambiguously \( \hat{H}(N)f_\gamma = \hat{H}_\gamma(N)f_\gamma \). We cannot write \( \hat{H}(N)f_\gamma = \hat{H}_{\gamma'}(N)f_\gamma \) with \( \gamma < \gamma' \) because there is a condition on the spins of the edges of \( \gamma' \) involved when applying \( \hat{H}_{\gamma'} \) which is not satisfied for \( f_\gamma \). In other words, \( \Phi_\gamma \cap \Phi_{\gamma'} = \emptyset \) if \( \gamma \neq \gamma' \).

We conclude that the family \( (\hat{H}_\gamma(N)) \) is trivially cylindrically consistently defined and therefore defines a linear operator on all of \( \mathcal{H} \).

### 5.2 Anomaly-freeness

Recall that the classical Dirac algebra is given by

\[
\{H(M), H(N)\} = \int_\Sigma d^2 x(M, aN - MN_a)q^{ab}V_b
\]

where \( V_a \) is the vector constraint. That is, the Poisson bracket between two Hamiltonian constraints evaluated on the constraint surface defined by the diffeomorphism constraint vanishes.
In the quantum theory one would therefore like to verify that naively 
\[ [\hat{H}(M), \hat{H}(N)] f = 0 \] for any state \( f \) that satisfies \( \hat{V}_a f = 0 \). Several subtleties arise:

1) The solutions \( \hat{V}_a f \) are in general no elements of the Hilbert space but generalized eigenvectors (distributions). Indeed, in this context the solutions of the diffeomorphism constraint are not elements of \( \mathcal{H} \) but of \( \Phi' \) where we have the proper inclusion \( \Phi \subset \mathcal{H} \subset \Phi' \). Thus, since \( \hat{H}(N) \) is defined only on \( \Phi \), the only operator that is defined on \( \Phi' \) is the dual \( (\hat{H}(N))' \) via the pairing \( \Psi(\hat{H}(N)\phi) = [(\hat{H}(N))'\Psi](\phi) \).

2) Observe that the operator \( (\hat{H}(N))' \) was not defined on every distribution but actually only on those that are solutions of the diffeomorphism constraint. Now even if \( \Psi \) is diffeomorphism invariant, that is, \( \Psi(\hat{U}(\varphi)f_\gamma) = \Psi(f_\gamma) \), then \( (\hat{H}(N))'\Psi \) is not any longer as one can easily check. Thus we cannot verify that \( [(\hat{H}(M))', (\hat{H}(N))']\Psi = 0 \), this equation is simply not defined. However, what is well-defined is \( [(\hat{H}(M), \hat{H}(N))']\Psi = 0 \) and this is what we are going to verify. Indeed, there is no hope to make sense out of \( [(\hat{H}(M))', (\hat{H}(N))']\Psi \) since not even classically \( \hat{H}(M) \) is diffeomorphism covariant. On the other hand, one could proceed as in \[12\] and define \( \hat{H}'(M)\hat{H}'(N) := (\hat{H}(N)\hat{H}(N))' \) which makes sense again on diffeomorphism invariant states.

3) One might be even more ambitious and ask that 
\[
(\hat{H}(M), \hat{H}(N))' = (\int_\Sigma d^2x (M_\alpha N - MN_\alpha)q^{ab}V_b)' \tag{5.1}
\]
that is, the Dirac algebra is faithfully implemented in the quantum theory. However, there are several issues that prevent us from doing so. First of all, the generator of diffeomorphisms, \( V_a \), does not have a quantum analogue, the diffeomorphism group does not act strongly continuously on \( \mathcal{H} \). So the only thing that we can hope to obtain is something like \( \hat{O}'[\hat{U}(\varphi) - 1] \) for the right hand side of \[4.12\] for some \( \varphi \in \text{Diff}(\Sigma) \) and some dual operator \( \hat{O}' \) (notice that \( \hat{U}(\varphi)' = \hat{U}(\varphi^{-1}) \) can be extended to all of \( \Phi' \)). Secondly, the situation is even worse for \( q^{ab} \). Thirdly, since, as we said, \( (\hat{H}(M), \hat{H}(N))' \) is only well-defined on diffeomorphism invariant distributions, then either the dual of the commutator vanishes or it does not. In the latter case there is an anomaly even in the sense of \( \hat{U}(\varphi) - 1 \). In the former case we get just zero but then we can trivially make an equality of the form 
\[
(\hat{H}(M), \hat{H}(N))' = \hat{O}'[\hat{U}(\varphi) - 1]
\]
for any \( \hat{O}' \) that we like. It then remains to ask whether one can somehow make sense out of an operator corresponding to the combination \( f_\Sigma d^2x (M_\alpha N - MN_\alpha)q^{ab}V_b \) and that is actually the case: We will not prove this assertion here but refer the reader to \[12\] which treats the 3+1 case but from which it is obvious that the result can be extended to the 2+1 case.

Summarizing, we will check that \( (\hat{H}(M), \hat{H}(N))'\Psi = 0 \) on diffeomorphism invariant states. The key element of the proof is the following: as is obvious from \[4.10\], if \( f_\gamma \) is a function cylindrical with respect to a graph, then \( \hat{H}(N)f_\gamma \) is a linear combination of functions each of which depends on graphs with new vertices not contained in \( \gamma \). More
precisely, if \( v \in V(\gamma) \) then for each triangle \( \Delta \) based at \( v \) there is a term \( \frac{4N_v}{E(v)} \hat{H}_\Delta f_\gamma \) and this function is a linear combination of functions \( f' \) each of which depends on a graph \( \gamma' \) contained in the following list:

\[
\gamma \cup \Delta, (\gamma \cup \Delta) - s_1(\Delta), (\gamma \cup \Delta) - s_2(\Delta), (\gamma \cup \Delta) - (s_1(\Delta) \cup s_2(\Delta)).
\]

Whether they appear depends on the spins of the graph \( \gamma \). In any case these functions \( f' \) depend on two more vertices \( v_1, v_2 \) coming from the endpoints of the arc \( a_{12}(\Delta) \). They may not depend on the original vertex \( v \) if that vertex was two-valent with spins of the edges \( e_i \) corresponding to \( s_i(\Delta) \) being \( j = 1/2 \) for both \( i = 1, 2 \). In that case \( [h^{-1}_{s_i(\Delta(v))}, \hat{V}_v]f' = 0 \) because neither \( f' \) nor \( h^{-1}_{s_i(\Delta(v))}f' \) depend on graphs with more than one edge incident at \( v \).

The point is now that \( [h^{-1}_{s_1(\Delta(v_1))}, \hat{V}_{v_1}]f' = [h^{-1}_{s_1(\Delta(v_2))}, \hat{V}_{v_2}]f' = 0 \). The reason for this is that the vertices \( v_1, v_2 \) in the graphs on which \( f' \) and \( h^{-1}_{s_i(\Delta(v))}f' \) depend does not have edges with linearly independent tangents incident at \( v \) so that the volume operator annihilates these functions.

Let us now write (5.1) in the form

\[
\hat{H}_\gamma(N) = \frac{32}{h^2} \sum_{v \in V(\gamma)} N(v) \hat{H}_{\gamma,v} \\
\hat{H}_{\gamma,v} = \frac{1}{E(v)^2} \sum_{(\Delta,v,\sigma(\Delta)'=v)} \hat{H}_{\gamma,v,\Delta,\Delta'} \\
\hat{H}_{\gamma,v,\Delta,\Delta'} = e^{ij} e^{k\ell} \text{tr}(h_{\alpha ij}(\Delta) h_{sk(\Delta)} [h^{-1}_{s_k(\Delta)}, \sqrt{V_v}] h_{s_i(\Delta)} [h^{-1}_{s_i(\Delta)}, \sqrt{V_v}]) \tag{5.2}
\]

The function \( \hat{H}_{\gamma,v} f_\gamma \) now can be written as a linear combination of functions \( f'_{\gamma'} \) each of which depends on a graph \( \gamma' \) which is a proper subgraph of the graph \( \gamma(v) := \gamma \cup v(\Delta) = v(\Delta') = v(\Delta) \) and we will mean by \( \hat{H}_{\gamma(v),v'} \) the operator that reduces to \( \hat{H}_{\gamma',v'} \) on \( f_{\gamma'} \) for each \( v' \in V(\gamma') \) and is zero if \( v' \notin V(\gamma') \).

With this preparation we evaluate

\[
[\hat{H}(M), \hat{H}(N)]f_\gamma = \sum_{v \in V(\gamma)} [N_v \hat{H}(M) - M_v \hat{H}(N)] \hat{H}_{\gamma,v} f_\gamma \\
= \sum_{v \in V(\gamma)} \sum_{v' \in V(\gamma(v))} [N_v M_{v'} - M_v N_{v'}] \hat{H}_{\gamma(v),v'} \hat{H}_{\gamma,v} f_\gamma \\
= \sum_{v,v' \in V(\gamma)} [N_v M_{v'} - M_v N_{v'}] \hat{H}_{\gamma(v),v'} \hat{H}_{\gamma,v} f_\gamma \\
= \frac{1}{2} \sum_{v,v' \in V(\gamma)} [N_v M_{v'} - M_v N_{v'}] [\hat{H}_{\gamma(v),v'} \hat{H}_{\gamma,v} - \hat{H}_{\gamma(v'),v'} \hat{H}_{\gamma,v}] f_\gamma. \tag{5.3}
\]

Here we have used our notation to write the commutator as a double sum in \( v \in V(\gamma), v' \in V(\gamma(v)) \) in the second line, then in the third line we have used the important fact that the constraint does not act at the new vertices that it creates so that the sum over \( v' \in V(\gamma(v)) \) collapses to a sum over the original \( v' \in V(\gamma) \) and in the last step we have used the antisymmetry in the lapse functions to write the product of operators as their antisymmetrized sum of products. Clearly the term with \( v = v' \) vanishes trivially. If \( v \neq v' \) then, since \( \hat{H}_{\gamma,v} \) manipulates the graph only in a small neighbourhood of \( v \), we can commute the two operators in the last line of (5.3) to write both with the vertex \( v \) to the
right hand side as

\[
[\hat{H}(M), \hat{H}(N)]f_\gamma = \frac{1}{2} \sum_{v,v' \in V(\gamma)} \left[ N_v M_{v'} - M_v N_{v'} \right] \left[ \hat{H}_{\gamma(v),v} \hat{H}_{\gamma,v'} - \hat{H}_{\gamma,v'} \hat{H}_{\gamma(v'),v} \right] f_\gamma.
\]  

(5.4)

Now by inspection of (5.2) we see that the last square bracket is a linear combination of functions of the type \( f - f' \) where \( f, f' \) are related by an analyticity preserving diffeomorphism by construction of the triangulation which relates different choices for the loop attachment by such a diffeomorphism (this point is explained in more detail in [3]). Thus when evaluating (5.4) on a diffeomorphism invariant state we can remove those diffeomorphisms and obtain just zero. This suffices to show \( ([\hat{H}(M), \hat{H}(N)])' = 0. \)

Notice that it was essential in the argument that the additional vertices created by \( \hat{H}(N) \) when acting on \( f_\gamma \) do not contribute as we showed. If that was not the case the commutator would not vanish on diffeomorphism invariant distributions which is why we attached the loop in such a particular, \( C^1 \), way.

6 Solving the theory

This section is divided into two parts: In the first part we will describe the complete space of solutions to both the diffeomorphism and Hamiltonian constraint. In the second part this solution space is shown to contain the solutions to the curvature constraint which can be formulated in our language as well [13]. One can equip the solution space with at least two very natural inner products. One of them is the inner product appropriate for the curvature constraint, the other one arises from direct construction of the solutions in the first part this section. Neither of these inner products give all solutions a finite norm.

6.1 Complete set of solutions to all constraints

Let be given a spin-network state \( T_{\gamma, \vec{j}, \vec{c}} \) and let

\[
\{T_{\gamma, \vec{j}, \vec{c}}\} := \{\hat{U}(\varphi)T_{\gamma, \vec{j}, \vec{c}}, \varphi \in \text{Diff}(\Sigma)\}
\]

be its orbit under the diffeomorphism group of analyticity preserving smooth diffeomorphisms. We define a diffeomorphism invariant distribution on \( \Phi \) by

\[
[T_{\gamma, \vec{j}, \vec{c}}] := \sum_{T \in \{T_{\gamma, \vec{j}, \vec{c}}\}} T.
\]

That this is a continuous linear functional on \( \Phi \) follows from the fact that the spin-network states form an orthonormal basis by the argument given in [12]. Therefore it is also clear that every diffeomorphism invariant state is a linear combination of such \( [T_{\eta, \vec{j}, \vec{c}}] \)'s so that by this procedure we can claim to have found the general solution \( \Psi \) to the diffeomorphism constraint \( \Psi(\hat{U}(\varphi)f) = \Psi(f) \forall \varphi \in \text{Diff}(\Sigma)\). We will call this space \( \Phi_{\text{Diff}} \).

Given any \( f \in \Phi \) we can uniquely decompose it as \( f = \sum_I f_I T_I \) where \( f_I \) are some constants and \( T_I \) are spin-network states. We then define \( [f] := \eta_{\text{Diff}} f := \sum_I f_I [T_I] \).

Notice that one cannot define \( [f] \) as the sum of all states which are in its orbit under diffeomorphisms since spin-network states defined on different graphs have uncountably
infinite sets of diffeomorphisms that move one graph but not another. We have been here imprecise with the issue of graph symmetries which alter the above formulae somewhat. See [12] for more details.

This construction can be used to define an inner product on $\Phi'_{Diff}$ by

$$< [f], [g] >_{Diff} := [f](g)$$

which is clearly a positive definite sesquilinear form and equips $\Phi'_{Diff}$ with the structure of a pre-Hilbert space.

Remark:

It has been shown in [6] that if there are only strongly diffeomorphism invariant observables in the theory then those observables define a superselection rule, namely, they cannot map between spin-network states based on graphs which are in different diffeomorphism equivalence classes. As a result, the group average could be defined differently in every sector, that is, the inner product in every sector can be chosen individually which amounts to the ambiguity that the particular way of averaging given by $[f]$ is not selected by physical principles, meaning that for every diffeomorphism equivalence class of graphs there could be a different constant that multiplies $[T_{\gamma_1, \bar{\delta}}]$ in $[f]$.

However, as there are clearly weakly diffeomorphism invariant observables which, together with the Hamiltonian constraint map between those sectors, there is no superselection rule and the way we have averaged is selected by the requirement that averaged spin-network states remain orthonormal [12].

We now wish to employ this result to find the general solution to all constraints. To that end, consider the set

$$R := \{ \hat{H}(N) \phi, \ N \in S, \ \phi \in \Phi \},$$

the range of the Hamiltonian constraint on $\Phi$ where $S$ denotes the usual Schwartz space of test functions of rapid decrease. Consider its orthogonal complement in $\Phi$ denoted $S := R^\perp \subset \Phi$. Finally, consider the set $\{ [s], \ s \in S \}$. Then it is easy to see that every solution to all constraints is a linear combination of elements of this set and we will call the resulting span $\Phi'_{phys}$. Namely, let $s = \sum_I s_I T_I \in S$ then by definition $\sum_I \bar{s}_I f_I = 0$ for any $f = \sum_I f_I T_I \in R$. Thus $[s](f) = \sum_I \bar{s}_I [T_I](f) = \sum_I \bar{s}_I f_I = 0 \forall f \in R$.

A geometrical construction of the space $S$ was given for the three-dimensional theory in [3]. Here we could proceed similarly. However, since this is only a model we restrict ourselves to showing that the space $\Phi'_{phys}$ is uncountably infinite dimensional. Namely, a particular simple class of vectors in $S$ consists of those elements of $\Phi$ which are linear combinations of spin-network states whose underlying graph is not of the form $\gamma \cup \alpha(\Delta), [\gamma \cup \alpha(\Delta)] - s_1(\Delta), [\gamma \cup \alpha(\Delta)] - s_2(\Delta), [\gamma \cup \alpha(\Delta)] - [s_1(\Delta) \cup s_2(\Delta)]$ for any $\Delta = \Delta(\gamma), v(\Delta) \in V(\gamma)$ and $\gamma$ is a graph underlying the same restriction but is otherwise arbitrary. This particular class of solutions has the property that all of the resulting $[s]$ are normalizable with respect to $< . . . >_{Diff}$ while genuine elements of $\Phi'_{phys}$ will not be normalizable with respect to the kinematical inner product on $\Phi'_{Diff}$. On the other hand, since the 2+1 Hamiltonian constraint really resembles the 3+1 Euclidean Hamiltonian constraint it follows from the results on the kernel of the Euclidean Hamiltonian constraint given in [3, 12] that every solution is a (possibly infinite) linear combination of
basic solutions each of which is in fact normalizable with respect to \(< \ldots >_{\text{Diff}}\).
We will see that the solutions to the curvature constraint are not normalizable with respect
to \(< \ldots >_{\text{Diff}}\) and one needs to define another appropriate inner product \(< \ldots >_{\text{curv}}\) on
the subset of \(\Phi'_{\text{phys}}\) corresponding to the solutions of the curvature constraint. However,
it will turn out that the natural inner product \(< \cdot, \cdot >\) for our Hamiltonian constraint
as suggested by [12] is such that curvature constraint solutions are still not normalizable
and so \(< \ldots >_{\text{curv}}\) and \(< \ldots >_{\text{phys}}\) define genuinely non-isometric Hilbert spaces. We will
turn to that issue in the next subsection.

### 6.2 Comparison with the Topological Quantum Field Theory

As shown in [13], in our language a solution to the curvature constraint \(F_i = 0\) in the
quantum theory is a distribution \(\Psi_f \in \Phi'\) given by \(\Psi_f := \delta_{\mu_0}(F) f\) for any \(f \in \Phi\). Here
\(\delta_{\mu_0}(F)\) is a \(\delta\) distribution with respect to the inner product on \(\mathcal{H}\) which has support on
the space of flat connections modulo gauge transformation \(\mathcal{M}\). More precisely, we have the following:
Any function on \(\mathcal{M}\) is a gauge invariant function which depends on the
connection only through the holonomies along (representants of) the independent generators
\(\alpha_1, \ldots, \alpha_n\) of the fundamental group \(\pi_1(\Sigma)\), that is, \(f(A_0) = f_\alpha(h_{\alpha_1}(A_0), \ldots, h_{\alpha_n}(A_0))\)
for \(A_0 \in \mathcal{M}\). The measure \(\nu_0\) on \(\mathcal{M}\) for gauge group \(G\) is defined by

\[
\int_M d\nu_0(A_0) f(A_0) := \int_{G^n} d\mu_H(g_1) \ldots d\mu_0(g_n) f_n(g_1, \ldots, g_n)
\]

where \(\mu_H\) denotes the Haar measure on \(G\). Then the delta distribution for flat connections
is given by

\[
\delta_{\mu_0}(F(A)) := \int_M d\nu_0(A_0) \delta_{\mu_0}(A_0, A) \quad (6.1)
\]

where

\[
\delta_{\mu_0}(A_0, A) = \sum_{\gamma, \vec{c}} \overline{T_{\gamma, \vec{c}}(A)} T_{\gamma, \vec{c}}(A_0) \quad (6.2)
\]

and the sum runs over all possible spin-network states. It is possible to arrive at \(6.1\)
from first principles by following the group average proposal [13].

It is also possible to write \(6.1\) as a linear combination of distributions in \(\Phi'_{\text{Diff}}\). To that
end, denote by \(I\) the label of a spin-network state and define \(T_{[I]} := [T_I]\). Notice that the
integral \(k_I := \int_M d\nu_0(A_0) T_I(A_0) =: k_{[I]}\) is diffeomorphism invariant and thus only depends
only on \([I]\). Then we may write

\[
\delta_{\mu_0}(F(A)) = \sum_{[I]} k_{[I]} T_{[I]}(A) \quad (6.3)
\]

It is easy to see [13] that \(6.3\) is a distribution on \(\Phi'_{\text{Diff}}\) and certainly it is a distribution
on \(\Phi\). However, \(6.3\) is not normalizable with respect to \(< \ldots >_{\text{Diff}}\).

To see this we use \(6.3\) to notice that we can write \(\delta(F(A)) = \eta_{\text{Diff}} f_F\) where \(f_F := \sum_{[I]} \overline{T_{[I]} T_{I_0([I])}}(A)\) and \(I_0([I]) \in [I]\) is an arbitrary choice. Thus by definition of the inner
product between diffeomorphism invariant distributions we find

\[
||\delta_{\mu_0}(F)||^2_{\text{Diff}} = (\eta_{\text{Diff}} f_F)(f_F) = \sum_{[I]} |k_{[I]}|^2 \quad (6.4)
\]
where the sum is over diffeomorphism equivalence classes of spin-network labels. But quantity (5.4) is just plainly infinite.

Namely, it follows from the definition of \( k_I = k_{[I]} \) that \( k_{[I]} = T_I(A = 0) \) whenever the graph underlying \( I \) is contractable. There are at least countably infinite number of contractable, mutually non-diffeomorphic, non-trivial graphs \( \gamma_n, n = 1, 2, \ldots \) in any \( \Sigma \). An example is given by choosing \( \gamma_n \) to be an \( n \)-link, that is, a union of \( n \) mutually non-intersecting loops \( \alpha_1, \ldots, \alpha_n \) homeomorphic to a circle each of which is homotopically trivial (contractable). Choose \( I_n \) such that \( T_{I_n}(A) = \prod_{k=1}^n T_{\alpha_k}(A) \) where \( T_{\alpha}(A) := \text{tr}(h_{\alpha}(A)) \) is the Wilson-Loop function and \( h_{\alpha}(A) \) denotes the holonomy of \( A \) along the loop \( \alpha \). Using the basic integral \( \int_{SU(2)} d\mu_H(g) \delta_{AB} g_{CD} = \frac{1}{2} \delta_{AC} \delta_{BD} \) it is easy to see that \( T_{I_n} \) provide an orthonormal system of spin-network states. But \( T_{I_n}(A = 0) = 2^n \) and so (5.4) contains the meaningless sum \( \sum_{n=1}^\infty 2^n \).

We must check whether or not \( \Psi_f \) is also a solution to the constraint \( \hat{H}(N) \) (it obviously is diffeomorphism in variant). To that end we must compute

\[
\Psi_f(\hat{H}(N)f_\gamma) = \int_{A/G} d\mu_0 \delta_{\mu_0}(F(A))(\mathcal{T} \hat{H}_\gamma(N)f_\gamma)(A) = \int_{\mathcal{M}} d\nu_0(A_0)(\mathcal{T} \hat{H}_\gamma(N)f_\gamma)(A_0) = 0 \tag{6.5}
\]

because either \( \hat{H}_\gamma(N)f_\gamma \) is identically zero or it is a linear combination of the vectors (recall \( \mathfrak{g}_{2} \))

\[
\hat{H}_{\gamma,\nu,\Delta,\Delta'}f_\gamma = -2\epsilon^{ij} \epsilon^{kl} \text{tr}(h_{\alpha_{ij}(\Delta')} \tau_m) \text{tr}(\tau_m h_{s_k(\Delta)} [h_{s_k(\Delta)}^{-1}, \sqrt{V_c}] h_{s_l(\Delta)} [h_{s_l(\Delta)}^{-1}, \sqrt{V_c}]) f_\gamma
\]

which therefore are proportional to the matrix elements of \( [h_{\alpha_{12}} - h_{\alpha_{12}}^{-1}] \) for a contractable loop \( \alpha \) which vanishes on \( A_0 \in \mathcal{M} \). Here we have used the \( su(2) \) Fierz identity \( \text{tr}(\tau_i A) \text{tr}(\tau_i B) = \text{tr}(A) \text{tr}(B)/4 - \text{tr}(AB)/2 \) together with \( \epsilon^{ij} \text{tr}(h_{\alpha_{ij}}) = \text{tr}(h_{\alpha_{12}}) - \text{tr}(h_{\alpha_{12}}^{-1}) = 0 \), a particular property of \( SU(2) \) (we did not need to use this, the result holds for general \( G \)).

Thus, any solution to the curvature constraint is a solution to the Hamiltonian constraint. However, as we have demonstrated, there are an infinite number of more solutions to the Hamiltonian constraint, in particular those which are normalizable with respect to \( < \ldots >_{\text{Diff}} \) and no solution to the curvature constraint has this property. Notice that the inner product on the space of solutions to the curvature constraint comes from a group averaging map, it is just given by (13)

\[
< \Psi_f, \Psi_g >_{\text{Curv}} := \Psi_f(g) = \int_{\mathcal{M}} d\nu_0 f g.
\]

It is now tempting to view this result as the restriction to the special solutions of the curvature constraint of a more general inner product appropriate for the Hamiltonian constraint.

There seems to be an unsurmountable obstacle: the Hamiltonian constraint is not a self-adjoint operator on \( \mathcal{H} \) and so group averaging as defined in 3 cannot be employed. Moreover, group-averaging really means to exponentiate the Hamiltonian constraint and that in turn implies that we know the motions it generates and thus we would have to completely solve the theory. Thus, it seems that we cannot define a map \( \eta : \Phi \rightarrow \Phi'_{\text{phys}}; f \rightarrow \eta f \). However, in the case that we have self-adjoint constraint operator,
the group-average algorithm is nothing else than a sophisticated way to construct the projector onto the distributional kernel of the constraint operator (this is explained in more detail in [12]). We are therefore lead to define the map \( \eta \), in the case that we do not have a self-adjoint constraint operator, as a certain (generalized) projector on the kernel of the constraint operator. As in [13] we split the problem into two parts and proceed as follows:

Given \( f \in \Phi \) we have a group averaging map \( \eta_{Diff} : \Phi \to \Phi'_{Diff} \) defined by \( \eta_{Diff}(f) := [f] \) and an inner product defined by \( (f,g)_{Diff} := \langle f, g \rangle := \int d\mu_0 f \overline{g} \). We define now \( \Phi_{Ham} := \Phi'_{Diff} \) and would like to define a map \( \eta_{Ham} : \Phi_{Ham} \to \Phi'_{Ham} \). The space \( \Phi'_{Ham} \) coincides with \( \Phi'_{phys} \) when viewed as a space of distributions on \( \Phi \) via the map \( \eta := \eta_{Ham} \circ \eta_{Diff} \). It remains to construct \( \eta_{Ham} \).

As we have seen, the elements \([s] \in \Phi'_{phys}, s \in S \) span \( \Phi'_{phys} \). Moreover, by explicit construction (given for the 3+1 theory in [3]) we can orthonormalize them with respect to \( \langle . , . \rangle_{Diff} \), thus exploiting that in our case all these \([s]\) are normalizable with respect to \( \langle . , . \rangle_{Diff} \). We obtain particular elements \( \psi_\mu \in \Phi'_{Ham} \cap \Phi_{Ham} \) with the property that \( \langle \psi_\mu, \psi_\nu \rangle_{Diff} = \delta_{\mu,\nu} \). We are now ready to define the projector \( \eta_{Ham} : \Phi_{Ham} \to \Phi'_{Ham} \).

Notice that even if not all of the \([s]\) would be normalizable with respect to \( \langle . , . \rangle_{Diff} \) then one could still take (6.6) as the group average map, just the elements \( \psi_\mu \) now form a basis in the generalized sense that they are mutually orthogonal in the sense of generalized eigenvectors (similar to usual momentum generalized eigenfunctions of ordinary quantum mechanics which are not really orthonormal in the Hilbert space sense but only orthogonal in the sense of \( \delta \) distributions).

The fact that the \( \psi_\mu \) are normalizable with respect to \( \langle . , . \rangle_{Diff} \) displays \( \eta_{Ham} \) as a projector on a genuine subspace of \( \mathcal{H}_{Diff} \).

Observe the dual role of the \( \psi_\mu \) which we can view both as elements of \( \Phi'_{Ham} \) and as elements of \( \Phi_{Ham} = \Phi'_{Diff} \subset \mathcal{H}_{Diff} \). In particular, notice the peculiar identity \( \eta_{Ham} \psi_\mu = \psi_\mu \).

We now simply define an inner product on the elements \( \eta_{Ham} \psi \) by

\[
< \eta_{Ham} \psi, \eta_{Ham} \psi' >_{Ham} := (\eta_{Ham} \psi)(\psi') := \sum_\mu \langle \psi_\mu, \psi' >_{Diff} \langle \psi_\mu, \psi' >_{Diff} \quad (6.7)
\]

for each \( \psi, \psi' \in \Phi_{Ham} \). Expression (6.7) is clearly a positive semi-definite sesquilinear form with the property that the \( \psi_\mu \) remain orthonormal. It is also independent of the orthonormal system \( \psi_\mu \) (the label \( \mu \) is “nicely split” into a discrete piece and a continuous piece and \( \psi_\mu \)’s are orthonormal with respect to both pieces in the sense of Kronecker \( \delta \)'s, see [12] for details).

We now combine the two group average maps to obtain

\[
\eta := \eta_{phys} := \eta_{Ham} \circ \eta_{Diff} : \Phi := \Phi_{phys} \to \Phi'_{phys} := \Phi'_{Ham}, f \to \eta_{Ham}[f] = \sum_\mu \psi_\mu \psi_\mu(f) \quad (6.8)
\]

and the physical inner product for the elements \( \eta_{phys} f \) becomes

\[
< \eta_{phys} f, \eta_{phys} g >_{phys} = < \eta_{Ham}[f], \eta_{Ham}[g] >_{Ham}
\]
\[= \sum_{\mu} <\psi_{\mu}, [f] >_{Diff} <\psi_{\mu}, [g] >_{Diff}
\]
\[= \sum_{\mu} \psi_{\mu}(f) \psi_{\mu}(g) = (\eta_{\text{phys}} f)(g) \quad (6.9)\]

where in the second before the last equality \(\psi_{\mu}\) is viewed as an element of \(\Phi'\).

Notice that with this definition \(\mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{Diff}}\). That this makes sense is shown in [12]. In other words, infinite linear combinations of elements of \(\psi_{\mu}\) are allowed but only with suitably converging coefficients.

As we have already shown that \(\Psi_{f} \notin \mathcal{H}_{\text{diff}}\) it follows that \(\Psi_{f} \notin \mathcal{H}_{\text{phys}}\), no solution to the curvature constraint is normalizable with respect to \(<\ldots>_{\text{Ham}}\). Since what really determines the physical inner product is the Hamiltonian constraint, for instance via the group average approach, this result was expected given the totally different algebraic structure of the two sets of constraints. In particular, the scalar product \(<\ldots>_{\text{curv}}\) is rather unnatural from the point of view of the Hamiltonian constraint.

The reverse question, whether \(||\cdot||_{\text{Ham}}\) normalizable elements of \(\Phi'_{\text{Ham}}\) have finite norm with respect to \(||\cdot||_{\text{curv}}\) cannot even be asked in general because a general element of \(\Phi'_{\text{Ham}}\) cannot be written as \(\Psi_{f}, f \in \Phi\).

We conclude that the sectors of the theory described by either of the inner products \(<\ldots>_{\text{Curv}}\) and \(<\ldots>_{\text{phys}}\) are mutually singular (that is, the underlying measures of the scalar products are singular). On the other hand, as far as the space of solutions to the constraint is concerned we find that all solutions to the curvature constraint are annihilated by the Hamiltonian constraint. Moreover, if we choose \(<\ldots>_{\text{curv}}\) as the inner product then we find complete agreement with the results in [1, 2] although our set of constraints and our quantization approach was totally different from the outset. Since we copied step by step the quantization procedure of [3] to maximal extent, we conclude that this procedure does lead to the correct answer in the present model which is a small but non-trivial check whether the proposal of [3] is reliable or not.

### 7 Conclusions

The aim of the present paper was to check whether the method of quantizing 3+1 general relativity by the method proposed in [3] is reliable in the sense that when that procedure is applied to well-known models we get the known the results. When applied to 2+1 Euclidean gravity we find complete agreement thus giving faith in those methods, the more, as 2+1 Euclidean gravity is maximally similar to 3+1 Lorentzian gravity as far as the algebraic structure of the constraints and the gauge group are concerned (at least when we consider the Hamiltonian rather than the curvature constraint).

On the other hand, the quantum theory as obtained by our approach has a much bigger space of solutions to all constraints than the space as obtained by traditional approaches while the latter is properly included in our solution space. A natural question that arises is then what to do with those extra solutions and how to interpret them. In particular, there seems to be a clash between the number of classical and quantum degrees of freedom.

Now, a hint of how to interpret these solutions is that many of them are of the form \([s], s \in S\) and so \(s\) is a cylindrical function. Therefore the volume operator \(\hat{V}(B)\) vanishes on \(s\) for almost every \(B\). This suggests that \([s]\) is a spurious solution because if we want the classical theory defined by curvature and Hamiltonian and diffeomorphism
constraint to be equivalent (recall that we had to impose $\det(q) > 0$ classically) then we must really ask that the volume operator $\hat{V}(B)$ is strictly positive for any $B$, before taking the diffeomorphism constraint into account. We conclude that no cylindrical $s$ should give rise to an element of $\Phi_{\text{phys}}'$ via $s \to [s]$ (which can be achieved by superposition of an infinite number of the $[s]$ or by considering infinite graphs) which would presumably remove the clsh between numbers of degrees of freedom alluded to above. This latter observation gives rise to the speculation that also many of the solutions found in $\mathbb{R}$ for the 3+1 theory should be spurious because in the classical theory we have to impose the anholonomic condition $\det(q) > 0$ as well. On the other hand it may be desirable to allow for degenerate solutions at the quantum level because by passing through singularities of the metric one can describe changes in the topology of the hypersurface $\Sigma$. Therefore one may expect that some of the solutions actually carry topological information and, moreover, that although we have started from a fixed topology in the classical theory we end up describing all topologies at the quantum level. The complete answer to this puzzle is left to future investigations.

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A Spectral analysis of the two-dimensional volume operator

Since the gauge group is still $SU(2)$ we may copy the results from $\mathbb{R}$ to compute the full spectrum of the two-dimensional volume operator. In particular it follows immediately that this operator is essentially self-adjoint, positive semi-definite and that its spectrum is entirely discrete. This holds on either gauge invariant or non-gauge invariant functions. In this appendix we restrict ourselves to the part of the spectrum coming from graphs with vertices of valence not larger than three, that is, we display the eigenvalues of the operator $\hat{V}_v$ of (3.13) restricted to vertices $v$ of valence $n = 2, 3$ on gauge invariant functions. Indeed, as the volume operator cannot change the graph or the colouring of the edges of the graph with spin quantum numbers of a spin-network state it follows that it can change at most its vertex contractors. However, given the spins of the edges incident at $v$, the space of vertex contractors is one-dimensional for $n = 2, 3$ by elementary Clebsh-Gordon theory. Therefore spin-network states all of whose vertices have at most valence three must be eigenvectors of the volume operator (in any dimension). Notice also that all the $\hat{V}_v$ for different $v$’s are mutually commuting. In three dimensions these spin-network states are in the kernel of the volume operator, in two dimensions none of them is annihilated as we

2This speculation on the conceivably topological meaning of the solutions is due to Abhay Ashtekar.
will show (as long as the tangents of the edges at v span a plane).

For \( n = 3 \) there are only two generic non-trivial situations: Either (Case A) no two of \( e_1, e_2, e_3 \) have co-linear tangents at \( v \) or (Case B) two of them, say \( e_1, e_2 \) have co-linear tangents at \( v \) but not \( e_1, e_3 \) or \( e_2, e_3 \). Here \( e_1, e_2, e_3 \) are the three edges incident at \( v \) which are coloured with spins \( j_1, j_2, j_3 \in \{ j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| \} \) respectively. We can get the eigenvalue for the case \( n = 2 \) by taking the result for \( n = 3 \) and setting for instance \( j_3 = 0, j_1 = j_2 = j \neq 0 \).

In the calculations that follow we will use the following notation:

\[
\begin{align*}
\hat{q}_v & := \left( \frac{4}{\hbar^2} \hat{V}_v \right)^2 = \hat{E}_v^i \hat{E}_v^i \\
\hat{E}_v^i & := \frac{1}{2} \sum_{1 \leq I, J \leq 3} \text{sgn}(e_I, e_J) X^i_{I, J} & \text{where } X^i_{I, J} := \epsilon_{ijk} X^j_I X^k_J \\
X^i_I & := X^i_{e_i}, \quad \vec{X}_I := (X^i_I), \quad X_{I, J} = X^i_I X^i_J, \quad \Delta_I := X_{II} \quad (A.1)
\end{align*}
\]

and it is implied that \( I, J, K \in \{ 1, 2, 3 \} \) are mutually different so that \([X^i_I, X^j_J] = 0\). As the notation suggests, \( \Delta_I \) is the Laplacian on \( SU(2) \) with spectrum \(-j(j+1), 2j \geq 0\) integral. Notice that \( X^i_{I, J} = -X^i_{J, I} \) so that

\[
\hat{E}_v^i = \sum_{1 \leq I < J \leq 3} \text{sgn}(e_I, e_J) X^i_{I, J}.
\]

As in the main text we will use generators of \( su(2) \) with structure constants \(+\epsilon_{ijk}\) which implies that \([X^i_I, X^j_J] = -\epsilon_{ijk} X^k_J\) and so \( \epsilon_{ijk} X^j_I X^k_J = -X^k_J \) (the minus sign comes from the right rather then left invariance).

There are some identities among these quantities that we are going to use. The first one is the familiar spin recoupling identity

\[
2X_{I, J} = [\vec{X}_I + \vec{X}_J]^2 - \Delta_I - \Delta_J = \Delta_K - \Delta_I - \Delta_J \quad (A.2)
\]

where in the second equality we have used the fact that \( \vec{X}_1 + \vec{X}_2 + \vec{X}_3 = 0 \), that is, the total angular momentum operator vanishes of on gauge invariant functions. Then if \( f \) is gauge invariant

\[
[\vec{X}_I + \vec{X}_J]^2 f = -[\vec{X}_I + \vec{X}_J]\vec{X}_K f = -\vec{X}_K[\vec{X}_I + \vec{X}_J] f = [\vec{X}_K]^2 f = \Delta_K f
\]

and of course the \( \Delta_I \) commute with every \( X^i_I \). The next identity is, using basic \( \epsilon_{ijk} \) arithmetic

\[
\vec{X}_{I, J}^2 = X^i_I X^j_J (X^i_J X^j_I - X^j_I X^i_J) = \Delta_I \Delta_J - X^i_I [X^j_J, X^j_J] + X^j_J X^j_I X^i_J = \Delta_I \Delta_J + X_{I, J} - X_{I, J}^2 \quad (A.3)
\]

and by very similar arguments

\[
\vec{X}_{I, J} \vec{X}_{J, K} = -\Delta_J X_{I, K} + X_{I, J} X_{J, K} + \epsilon_{ijk} X^i_I X^j_J X^k_K \quad (A.4)
\]

The last term in (A.4) is essentially the basic operator from which the tree-dimensional volume operator is built and which vanishes in the three-valent case on gauge invariant functions. Indeed, replacing, say \( \vec{X}_J = -\vec{X}_I - \vec{X}_K \) and using the \( su(2) \) algebra for the \( \vec{X}^i_I \) we see that that term vanishes.
Remarkably, upon substituting for $X_{IJ}$ according to (A.2) we find
\[ \vec{X}_{IJ} \vec{X}_{JK} = \frac{1}{4} [2(\Delta_I \Delta_J + \Delta_J \Delta_K + \Delta_K \Delta_I) - (\Delta_J^2 + \Delta_K^2 + \Delta_I^2)] \] (A.5)
which is independent of the choice of the pairs $(IJ), (JK)$.

We have now all tools available to finish the calculation. We will treat cases A, B separately.

A) We may label edges without loss of generality such that $\text{sgn}(e_1, e_2) = \text{sgn}(e_2, e_3) = \text{sgn}(e_3, e_1) = 1$, that is, we cross $e_1, e_2, e_3$ in this sequence as we encircle $v$ counter-clockwise. Then $\vec{E}_v = \vec{X}_{12} + \vec{X}_{23} + \vec{X}_{31}$. We just need to use (A.2)-(A.5) and to be careful with the order of $I, J$ in $\vec{X}_{IJ}$ to find after tedious algebra that $\hat{q}_v = [\vec{E}_v]^2$ is just given by
\[
\hat{q}_v = \frac{9}{4} [2(\Delta_2 \Delta_3 + \Delta_1 \Delta_3) - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2)] - \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3). \tag{A.6}
\]
Thus, the eigenvalue is obtained by replacing $\Delta_I$ by $-j_I(j_I + 1)$. Expression (2.6) looks worrisome: is the eigenvalue going to be non-negative? A moment of reflection reveals that it is even strictly positive unless $j_1 = j_2 = j_3 = 0$ in which case it vanishes: It will be sufficient to show that the operator in the first line of (A.6) has non-negative eigenvalue. We just need to remember that $j_1, j_2, j_3$ are not arbitrary. We may assume without loss of generality that $j_2 \geq j_1$ such that $j_3 \in \{j_1 + j_2, j_1 + j_2 - 1, \ldots, j_2 - j_1\}$. We have
\[
f(\Delta_3) := 2(\Delta_1 \Delta_2 + \Delta_2 \Delta_3 + \Delta_3 \Delta_1) - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2) = 4\Delta_1 \Delta_2 - (\Delta_3 - \Delta_1 - \Delta_2)^2 \tag{A.7}
\]
which takes, in terms of eigenvalues, its lowest value at maximum value of the function $|\Delta_3 - \Delta_1 - \Delta_2|$. Given arbitrary $j_1 \leq j_2$, since $-\Delta_3$ is a strictly increasing function of $j_3$, we find that the extrema of that function are found for the extremal values $j_3 = j_2 \pm j_1$ and are given by $|2j_1j_2|$ and $| -2j_1(j_2 + 1)|$ respectively. Then (A.7) reveals that $f(\Delta_3) \geq 4j_1(j_2 + 1)(j_2 - j_1) \geq 0$ because $j_2 \geq j_1$.

In case that we consider a two-valent vertex, we may just set $\Delta_3 = 0$, $\Delta_1 = \Delta_2 = \Delta$ and find the extremely simple result
\[
\hat{q}_v = -\Delta. \tag{A.8}
\]

B) We may, without loss of generality, label edges such that $e_1, e_2$ have co-linear tangents at $v$ (that is, $\text{sgn}(e_1, e_2) = 0$) and such that $\text{sgn}(e_1, e_3) = \text{sgn}(e_2, e_3) = 1$. Then $\vec{E}_v = \vec{X}_{13} + \vec{X}_{32}$. The same algebraic manipulations show that we get now for $\hat{q}_v = [\vec{E}_v]^2$ the expression
\[
\hat{q}_v = [2(\Delta_1 \Delta_2 + \Delta_2 \Delta_3 + \Delta_3 \Delta_1) - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2)] - \Delta_3 \tag{A.9}
\]
which is positive unless, of course, $\Delta_3 = 0$ in which case it vanishes.

We conclude that the two-dimensional volume operator has a much smaller kernel than the three-dimensional one, in particular, two and three-valent vertices, whether gauge invariant or not, do not contribute to the kernel.
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