THE SWIFT-HOHENBERG EQUATION UNDER DIRECTIONAL-QUENCHING:
FINDING HETEROCLINIC CONNECTIONS USING SPATIAL AND SPECTRAL DECOMPOSITIONS

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Abstract. We study the existence of patterns (nontrivial, stationary solutions) for one-dimensional Swift-Hohenberg Equation in a directional quenching scenario, that is, on $x \leq 0$ the energy potential associated to the equation is bistable, whereas on $x \geq 0$ it is monostable. This heterogeneity in the medium induces a symmetry break that makes the existence of heteroclinic orbits of the type point-to-periodic not only plausible but, as we prove here, true. In this search, we use an interesting result of [FLT17] in order to understand the multiscale structure of the problem, namely, how fast/slow scales interact with each other. In passing, we advocate for a new approach in finding connecting orbits, using what we call “far/near decompositions”, relying both on information about the spatial behavior of the solutions and on Fourier analysis. Our method is functional analytic and PDE based, relying minimally on dynamical system techniques and making no use of comparison principles whatsoever.

1. Introduction
In this paper we study the one-dimensional Swift-Hohenberg Equation (SHE),
$$\partial_t u(z,t) = -(1 + \partial^2_z)^2 u(z,t) + \delta^2 \mu(z) u(z,t) - u^3(z,t), \quad z \in \mathbb{R}, \quad t \in \mathbb{R}^+,$$
(1)
a model originally derived in the study of hydrodynamic instability due to thermal convection [SH77]. Here, $u(z,t)$ denotes the concentration of fluid at the point $(z,t)$ of space and time, and $\mu(\cdot)$ is a control parameter that may vary in space, representing the difference in temperature between the bottom and the top of the fluid. Whenever $\mu(\cdot) \equiv 1$ it is known that (1) supports a 3 parameter family $(\delta, \gamma, \omega)$ of $\frac{2\pi}{\omega}$-periodic solutions,
$$u^{(\delta, \omega, \gamma)}(\omega z) = \varepsilon \cos(\omega z + \gamma) + \mathcal{O}(\varepsilon^2),$$
(2)
where $\gamma \in \mathbb{R}, |1 - \omega^2| < \delta, \varepsilon(\delta, \omega) = \sqrt{\frac{4}{3} (\delta^2 - (1 - \omega^2)^2) + \mathcal{O}(\delta^2 - (1 - \omega^2)^2)}$ for all $\delta$ sufficiently small (cf. [CF90], Chapter 17; [Mic95], §4; see also Corollary 3.3). We contemplate the particular case of a control parameter $\mu(\cdot)$ that varies spatially due to inhomogeneities in the media and given by
$$\mu(x) = \left\{ \begin{array}{ll} 1, & \text{for } x \leq 0, \\ -1, & \text{for } x > 0. \end{array} \right.$$  
(3)

From the mathematical point of view, consequences of the assumption are seen on the energy potential associated with the equations, which jumps from bistable on $x \leq 0$ to monostable on $x > 0$, a scenario that describes what is known as directional quenching; we name quenching front the boundary point $x = 0$ across which the system changes its stability. Physically, the jump in (3) closely emulates physically interesting experiments where heterogeneities are introduced in the media aiming control of micro-phase separation. Such a technique has been applied in block copolymers [HBFK99], dewetting and colloidal
deposition [BHD+12], patterning of surfaces [KGFC10]; in a similar spirit, directional quenching has been studied in macro-phase separation models, cf. [FW12].

Due to the dissimilarity in the media induced by the parameter $\mu(\cdot)$ it is reasonable to expect phase separated states connected to homogeneous states, as one can see in numerical simulations; see Fig. 1. In the present study, we aim to understand a horizontal cross section of the pattern seen on the wake of the quenching front of Figure 1; we shall constrain the analysis to 1D, therefore looking for point-to-periodic, time-independent solutions, as sketched in Figure 2.

Before stepping into the discussion, we introduce a change variables $z \mapsto x/\omega$ which, allied to the degree zero homogeneity property $\mu(x) = \mu(\omega x)$, allow us to rewrite (1) as

$$\frac{\partial}{\partial t} u(x, t) = -(1 + \omega^2 \partial_x^2)^2 u(x, t) + \delta^2 \mu(x) u(x, t) - u^3(x, t);$$

we shall look for are 1D, time-independent solutions $u(\cdot)$ satisfying

$$\lim_{x \to -\infty} u(x) = u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) = 0, \quad \lim_{x \to +\infty} u(x) = 0, \quad 0 = -(1 + \omega^2 \partial_x^2)^2 u(x) + \delta^2 \mu(x) u(x) - u^3(x).$$

Evidently, what we mean by a solution is still to be discussed, otherwise, we could consider

$$u(x) = \begin{cases} u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x + x_0), & \text{on } x \in (-\infty, 0); \\ 0, & \text{on } x \in (0, +\infty), \end{cases}$$

which not only satisfies the Ordinary Differential Equation (ODE) in $\mathbb{R} \setminus \{0\}$, but can also be adjusted for $x_0$ to be continuous on the whole line. A few remarks are in hand before we carefully characterize the type of solutions we are looking for.

**Remark 1.1.** With regards to the ODE (5), the note that:

(i) Classical ODE theory shows that any solution to (5) must be smooth on $\mathbb{R} \setminus \{0\}$. Classical elliptic regularity theory shows that any weak solution must be locally $C^3(\mathbb{R})$; the latter condition immediately rules out the “solution” (6).

(ii) (Phase constraint condition) We note that the parameter $\mu(\cdot)$ breaks the reversible symmetry ($x \mapsto -x$) of the equation. Nevertheless, on the intervals $(-\infty, 0)$ and $(0, +\infty)$ there exists associated Hamiltonians which are conserved quantities. Indeed, the Ordinary Differential Equation (ODE)

$$-(1 + \omega^2 \partial_x^2)^2 V(x) + \eta V(x) - V^3(x) = 0,$$
has a Hamiltonian

\[ H[\mathcal{V}; \eta] = -\frac{\omega^4}{2} (\partial_x^2 \mathcal{V})^2 + \omega^2 (\partial_x \mathcal{V})^2 + \omega^4 \partial_x \mathcal{V} \partial_x^3 \mathcal{V} + \frac{1}{4} (1 - \eta + \mathcal{V}^2)^2. \]

(cf. [Pis06] §4.5.3, page 261). Thus, define the quantities \( H^{(i)}[\mathcal{V}] := H[\mathcal{V}; \delta^2], \mathcal{V} \mapsto H^{(r)}[\mathcal{V}] := H[\mathcal{V}, -\delta^2]. \) Then, for any solution \( u(\cdot) \in \mathcal{C}^3(\mathbb{R} \setminus \{0\}) \cup \mathcal{C}^3(\mathbb{R}) \) to the ODE \((5)\), we must have

\[ (-\infty, 0) \ni x \mapsto H^{(i)}[u(x)] \equiv H^{(i)}[u_{\text{rolls}}(\delta, \omega, \gamma)] \quad \text{and} \quad (0, +\infty) \ni x \mapsto H^{(r)}[u(x)] \equiv H^{(r)}[0]. \]

where \( H^{(i)}[u_{\text{rolls}}(\delta, \omega, \gamma)] \) and \( H^{(r)}[0] \) are constants that depend independently of \( x \), but depend on \((\delta, \omega, \gamma)\). Furthermore, since \( u(\cdot) \in \mathcal{C}^3(\mathbb{R}) \), the following phase constraint must be satisfied:

\[ -\delta^2 u_x^2|_{x=0} - \delta^2 = H^{(i)}[u_{\text{rolls}}(\delta, \omega, \gamma)] - H^{(r)}[0]. \]

In this case, we say that the parameters \((\delta, \omega, \gamma)\) are admissible.

(iii) Notice that the change of variables \( z \mapsto \frac{z}{\omega} \) fixes the period of the mapping \( x \mapsto u_{\text{rolls}}(\delta, \omega, \gamma)(x) \), which is now \( 2\pi \)-periodic, a fact that should not be overlooked due to its important consequences; see for instance Lemma [7.3]. We remark that \( x \mapsto u_{\text{rolls}}(\delta, \omega, \gamma)(x) \neq u_{\text{rolls}}^{(1)}(x) \), because the amplitude of the rolls depend nonlinearly on \( \omega \); see \([2]\).

(iv) Problem \((5)\) has one unknown \( u(\cdot) \) and 3 parameters, \((\delta, \omega, \gamma)\). Since \( \gamma \) is a translation parameter, throughout the analysis it will be fixed.

In this manner, we have the following

**Definition 1.2 (Heteroclinic orbits).** We say that a solution \((x, \delta, \omega, \gamma) \mapsto u_{\text{rolls}}(\delta, \omega, \gamma)(x) \in \mathcal{C}(\mathbb{R} \setminus \{0\}) \cap \mathcal{C}^3(\mathbb{R})\), is a heteroclinic orbit to problem \((5)\) whenever the latter is satisfied and the phase constraint condition \((8)\) holds.

In order to find these solutions we shall explore a path that relies minimally on the theory of dynamical systems. We advocate for a different perspective, functional-analytic based. Our first step benefits from the asymptotic conditions in \((5)\), based on which we decompose the space where solutions are sought for; we shall refer to this step as a far/near (spatial) decomposition. Now turning to the mathematical construction, we shall consider a time-independent Ansatz of the form

\[ U(x) = v(x) + \chi(\varepsilon \beta x)u_{\text{rolls}}(\delta, \omega, \gamma)(x), \]

where a few unknowns are introduced: one of them is the parameter \( \beta > 0 \), to be found later using matched asymptotics (see Lemma [5.6]). \( \chi(\cdot) \geq 0 \) is a fixed smooth function also to be chosen later and such that

\[ \lim_{x \to -\infty} \chi(x) = 1, \quad \lim_{x \to +\infty} \chi(x) = 0. \]

In fact, we shall see that \( \chi(\cdot) \) can be chosen to be a heteroclinic orbit satisfying a second order ODE (see Lemma [5.5] and Section [8.4]). Note that the Ansatz \((9)\) behaves as a non-compact perturbation, namely, it imposes a perturbation outside any given compact set, a region referred to as far field. With regards to \((9)\), whenever \( \delta = 0 \) we have \( u_{\text{rolls}}^{(0, \omega, \gamma)} \equiv 0 \), consequently taking \( v(\cdot) \equiv 0 \) leads to \( U(\cdot) \equiv 0 \), i.e., a trivial solution to \((1)\). From this point of view, one can consider the parameter \( \delta \) as bifurcation parameter that, roughly speaking, turns on a spatial-periodic perturbation in the far field as \( \delta > 0 \); keeping this adage in mind, the function \( v(\cdot) \) now plays the role of a corrector, and must be chosen in an appropriate functional space.

![Figure 2. Sketch of heteroclinic connection between rolls and homogeneous states for 1D SHE; this sketch can be seen as a horizontal profile of the patterns seen in the wake of the quenching front in Figure |](image-url)
In virtue of the phase constraint (8), we must also have the following phase constraint

$$-\delta^2 \left( v(0) + \chi(0) u^{(\delta, \gamma)}_{rolls}(0) \right)^2 - \delta^2 = \mathcal{H}^{(\ell)}(u^{(\delta, \gamma)}_{rolls}(\cdot)) - \mathcal{H}^{(r)}[0].$$  \hfill (12)

Here we write $\chi = \chi(e^\beta x)$ and $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ to denote the commutator of the operators $\mathcal{A}$ and $\mathcal{B}$. The functional space that contains $v(\cdot)$ consists of the domain of the operator $\mathcal{L}$, that is, $H^4(\mathbb{R})$.

We shall see in Lemma 7.1 that the constraint (12) is an essential ingredient in the selection mechanism, i.e., in the parametrization of $\omega$ by $\delta$.

1.1. Properties of the model, parameter choices and main results. Throughout this paper some assumptions on the model and on the parameters are made:

(H1) The partition function $\chi(\cdot) \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ satisfies (10), and the convergence takes place in an exponential fashion, that is, there exists a $C_*>0$ and a $S_*>0$ for which

$$|\chi(x) - 1| \lesssim e^{-C_*|x|}, \quad |\chi(x)| \lesssim e^{-C_*|x|},$$

for all $x < -S_*$ and $x > S_*$, respectively.

(H2) We shall further constrain $\omega > 0$ so that $|1 - \omega^2| \leq \frac{1}{4}\delta$, a range in which the existence of roll solutions is guaranteed (cf. [CE90, Chapter 17]; see also Figure 3). With this in mind, we introduce a parameter $\Omega$ such that

$$\omega^2 = 1 + \delta \Omega; \quad \Omega \in \left( -\frac{1}{3}, \frac{1}{3} \right).$$

This readily implies that, for $\delta > 0$,

$$\varepsilon(\delta, \omega) = \varepsilon(\delta, \Omega) = \delta \sqrt{\frac{4}{3}(1 - \Omega^2)} + O(\delta^2 \left[ 1 - \Omega^2 \right])$$

Consequently, there exists\footnote{Interestingly, the role played by $\delta > 0$ is equivalent to the one played by $\varepsilon > 0$. This assertion is proved later, in Corollary 3.3} a $\delta_0 > 0$ such that $\frac{1}{4} \delta \leq \varepsilon \leq 4\delta$, for $\delta \in [0, \delta_0)$.

Remark 1.3 (Parameter region blow-up). Property [H2] is nothing but a parameter blow-up, for $(\delta, \Omega)$ belongs to an open neighborhood of the zero in $\mathbb{R}^2$, whereas $(\delta, \omega)$ belongs to a cusp-type domain (see Figure 3). This is going to be a crucial ingredient in section 7 when we apply an Implicit Function Theorem based result.

Remark 1.4 (On the role of the conserved quantities and Hamiltonian structure). We will see at the end of [6] that, fixed an appropriate $\chi(\cdot)$ and $\gamma$, solutions to problem (11) are parametrized by $(\delta, \omega)$ sufficiently small; in fact, problem (11) is meaningful per se, regardless of the phase constraint (12). Hence, ignored the latter constraint, then no selection mechanism as the one described below in Theorem 1.5(ii) happens. Hence, the constraints imposed by the Hamiltonian structure impose a severe restriction in parameter region, and this where the selection mechanism comes from. Therefore problem (11), although derived from (5), can be considered as a separated problem which, only in conjunction with the phase-constraint condition (12) gives an equivalent formulation of the problem in (5).

Our main result is the following:

Theorem 1.5 (Existence of 1-parameter family of heteroclinic connections). For every $\gamma \in \mathbb{R}$ fixed there exists a $\delta_* > 0$ and a one parameter family of stationary solutions of the form

$$|0, \delta_*| \ni \delta \mapsto u^{(\delta)}(z) = v^{(\delta)}(\omega^{(\delta)} z) + \chi(e^{\omega^{(\delta)} z}) u^{(\delta, \omega^{(\delta)} \gamma)}_{rolls}(\omega^{(\delta)} z), \quad z \in \mathbb{R},$$  \hfill (13)

to (1) with the following properties:

(i) (Asymptotic properties and regularity) The function $z \mapsto u^{(\delta)}(z) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\}) \cap \mathcal{C}^3(\mathbb{R})$; furthermore, it satisfies

$$\lim_{z \to -\infty} \left| u^{(\delta)}(z) - u^{(\delta, \omega^{(\delta)} \gamma)}_{rolls}(z) \right|, \quad \lim_{z \to +\infty} \left| u^{(\delta)}(z) \right| = 0,$$

where $\delta \mapsto \omega^{(\delta)}$ is a continuous mapping, as defined below. Moreover, the mapping in (13) is continuous in the sup norm if and only if $\omega^{(\delta)} \equiv 1$.\footnote{Interestingly, the role played by $\delta > 0$ is equivalent to the one played by $\varepsilon > 0$. This assertion is proved later, in Corollary 3.3}
neither been used simultaneously.

Both techniques had been seen separately in earlier works, although to the authors' knowledge they have not been exploited before. Interestingly, the introduction of a matching parameter $\beta$ exploited in combination with homogenization techniques in the study and simulation of micro-structures field-core decomposition and defects [BLBL12, BLBL15].

In terms of extending it to 2D phenomena, like line defects or amplitude walls. Nevertheless, focusing on a dimensional dynamical systems theory; our proof, functional-analytic in nature, seems to be more promising than $[SW18]$ with regards to the selection mechanism presented in Theorem 1.5 (see also §8.7). It should be said that the latter paper's analysis is shorter; on the other hand, the analysis is highly dependent of 1-dimensional dynamical systems theory; our proof, functional-analytic in nature, seems to be more promising in terms of extending it to 2D phenomena, like line defects or amplitude walls. Nevertheless, focusing on a mathematical problem that is now rigorously understood give us a fair ground for comparison between the dynamical-systems approach we advocate for and the dynamical-systems approach.

The heart of the paper lies in two decompositions:

(i) the first has a spatial nature, intrinsically contained in $[9]$. It takes into account the far field spatial behavior of solution. We shall call it the far/near (spatial) decomposition;

(ii) the second has a spectral nature, relying on the Fourier representation of the operator $L[\cdot]$ in $[11]$ as a Fourier multiplier. We shall call it the far/near (spectral) decomposition.

Both techniques had been seen separately in earlier works, although to the authors' knowledge they have never been used simultaneously.

The far/near (spatial) decomposition induced by the Ansatz $[9]$ has been called by some authors far field-core decomposition: this type of decomposition has been a building block in the construction of multidimensional patterns in extended domains [dPKPW10] and in the study of perturbation effects on the far field of multidimensional patterns [MS18, MMon18], it has also been exploited before in the context of pattern formation in the work of Scheel and collaborators (see for instance [MS17, CS16], specially [LS17] and [MS15, §5]) in combination with bifurcation techniques. Similar types of decompositions have also been exploited in combination with homogenization techniques in the study and simulation of micro-structures and defects [BLBL12, BLBL15]. Interestingly, the introduction of a matching parameter $\beta$ in $[9]$ seems to be a new feature that has not been exploited before.
The far/near (spectral) decomposition has a different origin, being deeply motivated and inspired by the work of [FLTW17] on topological insulators. Therein, the authors consider perturbed Hamiltonian problems of the form
\[ \partial_x^2 \mathcal{U}(x) + V_{\text{per}}(x) \mathcal{U}(x) + \delta \kappa(\delta x) W_0(x) \mathcal{U}(x) = 0, \quad x \in \mathbb{R}, \]
where \( V_{\text{per}}(\cdot) \) is a known L-periodic potential, \( W_0(\cdot) \) denotes an L-periodic perturbation and \( \kappa(\cdot) \) is a smooth function with the property that \( \lim_{x \to \pm \infty} \kappa(x) = \kappa_{\pm} \). In spite of the nonlinear nature of \( \mathcal{H} \), our problem can be seen as a far field perturbation whose nature is similar to the problem studied in [FLTW17]. The idea of what we call far/near (spectral) decomposition has its roots on the Bloch-Floquet analysis of the spectrum of the periodic coefficients operator \( \partial_x^2 \mathcal{U}(x) + V_{\text{per}}(x) \mathcal{U}(x) \) (i.e., the linear operator at \( \delta = 0 \)), whose multiplier/eigenvalues are “lifted” to the case \( \delta \neq 0 \) using perturbation techniques. The next step, exploited in an ingenious way in [FLTW17], consists of decomposing the Brillouin zone for certain modes into a near and far region based on which one can derive a decomposition of the function \( \mathcal{U} \) as
\[ \mathcal{U}(\cdot) = \mathcal{U}_{\text{near}}(\cdot) + \mathcal{U}_{\text{flat}}(\cdot), \]
where \( \mathcal{U}_{\text{near}}(\cdot) \) is a band limited function that parametrizes (or, say, dominates) the far components \( \mathcal{U}_{\text{flat}}(\cdot) \); this step is proved using a Lyapunov-Schmidt reduction. Surprisingly, as a stationary counterpart, the far/near (spectral) decomposition has similar features to active/passive modes discussed in [NPL93] [1], where wavenumbers are divided in a group that saturates the asymptotic dynamic behavior, while passive modes are damped; when focused on stability issues, finite wavenumber instability studies dates from even earlier, cf. [NW69].

A full use of the results in [FLTW17] stumbles upon the nonlinear nature of SHE, which brings new features not seen in their work; in dealing with these matters our analysis is closer to the works of Schneider [Sch96a, Sch94, Sch96b], where many crucial ideas of how to deal with the nonlinear terms are seen. Overall, Schneider implements a similar far/near (spectral) decomposition, using what he named mode filters; although his goals were mostly on the long time dynamic behavior of initial value problems, we can somehow say that the far/near (spectral) decomposition is essentially a stationary counterpart to the mode filters introduced by him.

Last, an important feature should be emphasized: the memoir [FLTW17] makes no use of comparison principles. Even though this comment seems out of context upon discussion of a 4th order problem as we study here, this fact should not be overlooked, in special in regards to [FLTW17], a second order problem where elliptic theory plays an important role. Overall, another extra remark is worth of being made: neither the approach in [FLTW17] nor ours rely on Center Manifold theory.

We now give a brief outline of the paper: as we designed a far/near (spatial) decomposition in (9), two new unknowns are introduced: a constant \( \beta \) and a function \( \chi(\cdot) \), both to be fully characterized later on (in Observation 5.7 and Lemma 6.5, respectively). In §2 we introduce a decomposition of the Fourier wavenumbers using a far/near (spectral) decomposition, which implies that the corrector \( v(\cdot) \) can be written as
\[ v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{flat}}(\cdot). \]
Thus, one can define a coupled system of nonlinear equations for \( v_{\text{near}}(\cdot) \) and \( v_{\text{flat}}(\cdot) \). This sets the ground for a Lyapunov-Schmidt reduction in §3 where we show that under the appropriate conditions \( v_{\text{flat}}(\cdot) \) is parametrized by \( v_{\text{near}}(\cdot) \), \( \delta \) and \( \Omega \) (see Proposition 3.1). Afterwards, in Section §4, a derivation that is close to weakly nonlinear theory (cf. NW69 NPL93): we focus our study in the equation that describes \( v_{\text{near}}(\cdot) \), which in Fourier space satisfies
\[ \text{supp} (\hat{v}_{\text{near}}) \subset \{ -1 + \varepsilon^\beta B \} \cup \{ 1 + \varepsilon^\beta B \}, \quad B = \{ x \in \mathbb{R} ||x| \leq 1 \}. \]
that is, its support is contained in disjoint intervals that shrink to points as \( \varepsilon \downarrow 0 \). We overcome this issue by desingularizing the limit, following the approach of [FLTW17] [6.4]. In our case the consequences of this desingularization are quite interesting, giving a surprising interpretation of the near component \( v_{\text{near}}(\cdot) \): the latter can be seen as
\[ v_{\text{near}}(x) = \varepsilon^\beta e^{-i\varepsilon x} g_1(\varepsilon^\beta x) + \varepsilon^\beta e^{i\varepsilon x} g_1(\varepsilon^\beta x), \quad g_{\pm 1}(\cdot) \in H^2(\mathbb{R}); \]
which takes us to the initial steps of the Ginzburg-Landau formalism, commonly seen in modulation theory (cf. vH91, Sch96a [3]; see also Proposition 4.1).

In section §5 we use deep results in [FLTW17] [6] that allow for a better understanding of the periodic structure of the far field (“fast scale”) and its interaction with the “slow scale” structure of the correctors; this allows for crucial simplifications in the equations. In the end we use matched asymptotic to finally

\[ \text{For a thorough careful exposition of center Manifold Theory, see [HI11]. A nice discussion more tailored to the topics we are discussing, as lack of spectral gap and finite band instabilities, see NPL93 [1].} \]
obtain the value for $\beta$: we have to choose $\beta = 1$. Once this part is cleared out, we condense the results of the previous section right at the beginning of Section 4 where we write (48), a reduced equation that contains the dominant features of the problem, a reduced nonlinear equation of the form

$$\hat{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}] - \|\varepsilon_{-\varepsilon}^{-1}\| h_\ast(x, \xi) = \Omega(\delta, \Omega) \hat{g}_{-1, \Omega}^{-1}(g_{-1}, g_{+1});$$

where $(g_{-1}, g_{+1}) \mapsto \hat{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}]$ is a mapping from $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ to $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and $\varepsilon_{-\varepsilon}^{-1}$ is a localized term that can be made as small as we want upon choosing $\chi(\cdot)$ “nicely” (see Proposition 6.5). In order to prepare the ground for the resolution of this nonlinear problem, the inverse of $\hat{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}]$ is found using an approximate inverse which, roughly speaking, corresponds to this operator in the limit regime $\delta \downarrow 0$. This analysis culminates in another application of the Contraction Mapping Theorem (Proposition 6.1), showing that problem (15) has a family of solutions $(0, \delta_\ast) \times (-\delta_\ast, \delta_\ast) \ni (\delta, \Omega) \mapsto \left(\hat{g}_{-1}^{(\delta, \Omega)}(\cdot), g_{+1}^{(\delta, \Omega)}(\cdot)\right),$

where $g_{\pm 1}^{(\delta, \Omega)}(\cdot) \in H^2(\mathbb{R})$, that are band-limited, that is, supp $\hat{g}_{\pm 1}^{(\delta, \Omega)} \subset e^{\mp \varepsilon_{-\varepsilon}^{-1}B}$, with $B = \{x \in \mathbb{R} \mid |x| \leq 1\}$. At this point we reach section 7, in which the Hamiltonian structure of the equations (1) is exploited and where Theorem 1.5 is finally proved. An important step in its derivation comes from the study of the phase constraint (12) (see Lemma 7.3), where we show that in fact, fixed $\gamma \in \mathbb{R}$, only one parameter is necessary in the parametrization of the heteroclinic orbits to (1). Namely, we show that

$$[0, \delta_\ast) \ni \delta \mapsto \omega^{(\delta)} = \sqrt{1 + \delta \Omega^{(\delta)}},$$

where the mapping $\delta \mapsto \Omega^{(\delta)}$ is shown to be continuous and so that $\lim_{\delta \downarrow 0} \Omega^{(\delta)} = 0$; this results consists of a selection mechanism, showing that the wavenumber $\omega$ gets parametrized by $\delta$; see Figure 3. In conjunction with the results of Section 4 it implies that we can find a solution $\delta \mapsto u^{(\delta)}(\cdot, x)$ to (11) and (12). Undoing the change of variables $x \mapsto \omega^{(\delta)}x$ and plugging this solution in the Ansatz (9) we obtain a stationary solution

$$[0, \delta_\ast) \ni \delta \mapsto u^{(\delta)}(x) = v^{(\delta)}(\omega^{(\delta)}x) + \chi(\varepsilon \omega^{(\delta)}x)u_{\text{rolls}}^{(\delta, \Omega^{(\delta)})}(\omega^{(\delta)}x), \quad x \in \mathbb{R},$$

as described in Theorem 1.5. Further properties of this mapping are studied in Lemma 7.5.

In a quick summary, the construction of the pattern goes along the steps below:

- Bifurcation equation setup: “far/near” spatial decomposition: Equation (11)
- “Far/near” spectral decomposition: (2)
- Enslaving of far components by near components: (3)
- Blow up of the Fourier parameter, nonlinear interaction and approximation Lemmas (4)
- Simplifications using a lemma of Fefferman, Thorpe and Weinstein, and matched asymptotics: (5)
- Approximation and solvability of the reduced equation: (6)
- Wavenumber selection – proof of Theorem 1.5 (7)

Once the main result is proved we discuss many open problems in 8 where some of the techniques we use are compared to previous methods and also put in a broader context. An appendix containing some important calculations then closes the paper.

### 1.3 Notation and a functional-analytic settings

Throughout this work we define the Fourier transform (resp., inverse Fourier transform) of a function $f(\cdot) \in L^2(\mathbb{R})$ (resp., $\hat{f}(\cdot) \in L^2(\mathbb{R})$) by

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}dx,$$

resp.,

$$\mathcal{F}[f](\xi) = \mathcal{F}^{-1}[\hat{f}^{-1}](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}d\xi.$$  

A few properties of the Fourier transform shall be used, in special

$$\mathcal{F}[f](\varepsilon \xi) = \frac{1}{\varepsilon} \mathcal{F}[f](\varepsilon^{-1}\xi),$$

(17a)

$$\mathcal{F}[f](\alpha + \xi) = \mathcal{F}[e^{-i\alpha}(f)](\xi),$$

(17b)

$$\|\mathcal{F}[f]\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}.$$  

(17c)
The pairing in Sobolev spaces $H^s(\mathbb{R})$, $s \geq 0$, is defined as

$$\langle f, g \rangle_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\eta|^2)^s \hat{f}(\eta) \overline{\hat{g}(\eta)} d\eta,$$

where $\overline{(\cdot)}$ denoted complex conjugation in $\mathbb{C}$; thanks to Plancherel Theorem (cf. [Tay11] §3, Proposition 3.2]), we have that $\|f\|_{L^2(\mathbb{R})} = \frac{1}{2\pi} \|f\|_{H^0(\mathbb{R})}$. We make repeated use of the following Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$:

$$\|v\|_{L^\infty(\mathbb{R})} \lesssim \int_{\mathbb{R}} |\hat{v}(\xi)| d\xi \leq \|v\|_{H^1(\mathbb{R})} \sqrt{\int_{\mathbb{R}} \frac{1}{(1 + \xi^2)} d\xi} \lesssim \|v\|_{H^1(\mathbb{R})},$$

(18)

and of the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow \mathcal{C}^{(s-1)}(\mathbb{R})$ (cf. [Bre11] Theorem 8.2]). The unit ball in $\mathbb{R}$ is denoted by $B = \{x \in \mathbb{R}||x| \leq 1\}$, while translated balls with radius $\theta$ and centered at a point $\alpha$ is written $\alpha + \theta B = \{x \in \mathbb{R}|x - \alpha| \leq \theta\}$.

The characteristic function of a Lebesgue measurable set $A$ is written $1_A(\tilde{\kappa}) = 1_{(\tilde{\kappa} \in A)}(\tilde{\kappa})$, where $1_A(x) = 1$, whenever $x \in A$, and $1_A(x) = 0$ whenever $x \not\in A$. The support of a Lebesgue measurable function $f(\cdot)$ is denoted by supp$(f)$.

We write $H^s_{\text{near}, A}(\mathbb{R}) \subset H^s(\mathbb{R})$ to refer to the space of band limited functions with Fourier transform supported in $\eta B$, that is

$$H^s_{\text{near}, A}(\mathbb{R}) := \{g(\cdot) \in H^s(\mathbb{R})|\text{supp}(\hat{g}) \subset \eta B\}.$$ 

Given Banach spaces $X$ and $Y$, an unbounded operator $\mathcal{H} : X \to Y$ will have its domain written $\mathcal{D}(\mathcal{H})$. Thus, given $\mathcal{H} : \mathcal{D}(\mathcal{H}) \subset X \to Y$, we write $\text{Ker}(\mathcal{H}) := \{v \in X|\mathcal{H}v = 0\}$ and $\text{Rg}(\mathcal{H}) := \{f \in Y|\exists v \in \mathcal{D}(\mathcal{H}), \text{such that } \mathcal{H}v = f\}$.

**Remark 1.6** (Embedding for band limited functions). As pointed out in [FLT17] Remark 2.2, whenever $f \in L^2(\mathbb{R})$ is band limited equation then $f(\cdot) \in H^s(\mathbb{R})$ for all $s \geq 1$ and

$$\|f\|_{H^s(\mathbb{R})} \lesssim \|f(\cdot)\|_{L^2(\mathbb{R})}.$$

In particular, $f(\cdot) \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$, thanks to the Sobolev Embedding Lemma (cf. [Tay11] §4-Corollary 1.4]).

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2. The far/near (spectral) decomposition: the role of multipliers

To begin with, we shall represent equation (11) in a more concise form,

$$\mathcal{L}[v] := - (1 + \omega^2 \partial_x^2)^2 v = \sum_{j=1}^{4} \mathcal{A}^{(j)}[v, u^{(\delta, \omega, \gamma)}],$$

where $v(\cdot) \in \mathcal{D} (\mathcal{L}) = H^4(\mathbb{R})$, the domain of the operator $\mathcal{L}$. The nonlinearities are

$$\begin{align*}
\mathcal{A}^{(1)}[v, u^{(\delta, \omega, \gamma)}] &= - \left( \delta^2 \mu(x) - 3(\delta x)^2 u^{(\delta, \omega, \gamma)}(x) \right) v(x), \\
\mathcal{A}^{(2)}[v, u^{(\delta, \omega, \gamma)}] &= 3(\delta x)^2 u^{(\delta, \omega, \gamma)}(x) v^2(x), \\
\mathcal{A}^{(3)}[v, u^{(\delta, \omega, \gamma)}] &= v^3(x), \\
\mathcal{A}^{(4)}[v, u^{(\delta, \omega, \gamma)}] &= \chi(\delta x)(\chi^2(x) - 1) u^{(\delta, \omega, \gamma)}(x)^3 \\
&\quad + \left[1 + \omega^2 \partial_x^2 \right] \chi(\delta x) u^{(\delta, \omega, \gamma)}(x) - \delta^2 \chi(\delta x)(\mu(x) - 1) u^{(\delta, \omega, \gamma)}(x).
\end{align*}$$

(19)

In Fourier space, the operator $\mathcal{L}$ admits a multiplier representation, i.e.,

$$\mathcal{F}[\mathcal{L}[v]](\kappa) = m(\kappa; \mathcal{L}) \mathcal{F}[v](\kappa) = - (1 - \omega^2 \kappa^2)^2 \hat{v}(\kappa), \quad v(\cdot) \in H^4(\mathbb{R}).$$

(20)

Taking into account the properties of the mapping $\kappa \mapsto \frac{1}{m(\kappa; \mathcal{L})}$, we decompose the frequency space in two disjoint sets:

(i) Near frequency region: this is the part around the zeros of $\kappa \mapsto m(\kappa; \mathcal{L})$ (i.e., $\kappa = \pm \frac{1}{\sqrt{2}}$), where the mapping $\kappa \mapsto \frac{1}{m(\kappa; \mathcal{L})}$ has a “bad behavior”;

(ii) Far frequency region: the complement to the near frequency region, where we have a better behavior for $\kappa \mapsto \frac{1}{m(\kappa; \mathcal{L})}$. 

which leads us to the following splitting:

Near frequencies = \{-1 + \varepsilon^2 B\} \cup \{1 + \varepsilon^2 B\}, \quad \text{Far frequencies} = \mathbb{R} \setminus \{\text{near frequencies}\}, \quad \text{(21)}

where we introduce a new parameter \(\tau > 0\) to be chosen later. As we can see, the linearized operator \(v \mapsto \mathcal{L}[v]\) has continuum spectrum up to the imaginary axis in the complex plane. The decomposition (21) splits the spectrum in modes that are close to the imaginary axis (near frequencies) and those that are far, or relatively far, from it (far frequencies); as we shall prove in [3] the near frequencies are the most relevant to this problem. We remark that the choice for a parametrization in terms of \(\varepsilon\) is somewhat a matter of convenience: our choice is due to the fact that nonlinearities scale in \(\varepsilon\); in principle, a similar analysis could be done in terms of \(\delta\), thanks to the equivalence between them asserted in [H2] (see also Corollary 3.3).

**Figure 4.** Near frequencies region and far frequencies region. The decomposition depends on the nature of the multiplier \(m(\cdot; \mathcal{L})\) and its behavior.

**Lemma 2.1** (Multiplier behavior over frequency regions). The linear operator \(v \mapsto \mathcal{L}[v]\) can be represented as a multiplier in Fourier space, that is, we have the equivalence

\[
\mathcal{L}[v] = \mathcal{F}^{-1} [m(\cdot; \mathcal{L})\hat{v}(\cdot)](x), \quad \mathcal{D}(\mathcal{L}) = H^4(\mathbb{R}),
\]

where \(m(\kappa; \mathcal{L}) = -1 \cdot (1 - \omega^2 \kappa^2)^2\). Whenever \(\omega\) satisfies [H2], we have that \(\omega \in \text{near frequencies}\). Furthermore, whenever \(0 < \tau < 1\) there exists a \(C = C(\varepsilon_0) > 0\) such that

\[
|m(\kappa; \mathcal{L})|_{\{\kappa \in \text{Far frequencies}\}} \geq C \varepsilon^{2\tau}, \quad \forall \varepsilon \in [0, \varepsilon_0).
\]

**Proof.** The first statement is a consequence of standard Fourier analysis. We shall prove the lower bound of the multiplier. Recall from [H2] that \(\omega(\delta) \in \left[\sqrt{1 - \frac{\delta}{3}}, \sqrt{1 + \frac{\delta}{3}}\right]\). Initially we prove that the zeros of \(m(\cdot; \mathcal{L})\) are always in the Near Frequency region. Thanks to property [H2] and \(0 < \tau < 1\), it suffices to show that the inequalities

\[
1 - \varepsilon^\tau \leq 1 - \frac{\delta^\tau}{4\tau} \leq \frac{1}{\sqrt{1 + \frac{\delta}{3}}} \leq \frac{1}{\omega} \leq \frac{1}{\sqrt{1 - \frac{\delta}{3}}} \leq 1 + \frac{\delta^\tau}{4\tau} \leq 1 + \varepsilon^\tau.
\]

(22)

hold. The first and the last inequalities are straightforward consequences of [H2]. For the other inequalities, we just need to show that

\[
1 \leq \left(1 + \frac{\delta^\tau}{4\tau}\right)^2 \left(1 - \frac{\delta}{3}\right), \quad \text{and} \quad \left(1 - \frac{\delta^\tau}{4\tau}\right)^2 \left(1 + \frac{\delta}{3}\right) \leq 1.
\]

holds true whenever \(0 < \tau < 1\) and \(\delta\) is sufficiently small. Expanding the inequality on the left hand side (resp. right hand side) and rearranging it, one can show that it is equivalent to

\[
0 \leq \delta^\tau \left[\frac{2}{4\tau} + \frac{\delta^\tau}{4\tau} - \frac{\delta^{1 - \tau}}{3} \left(1 + \frac{\delta^\tau}{4\tau}\right)^2\right], \quad \text{(resp.} \quad \delta^\tau \left[-\frac{2}{4\tau} + \frac{\delta^\tau}{4\tau} + \frac{\delta^{1 - \tau}}{3} \left(1 - \frac{\delta^\tau}{4\tau}\right)^2\right] \leq 0\).
\]

hence it holds for all \(\delta \geq 0\) sufficiently small. Our second step consists in finding

\[
\min_{|\varepsilon| \geq 1} \left(1 - \omega^2 \varepsilon^2\right)^2, \quad \text{for} \quad \omega(\delta) \in \left[\sqrt{1 - \frac{\delta}{3}}, \sqrt{1 + \frac{\delta}{3}}\right].
\]

By symmetry, it suffices to consider \(\xi \in \{\xi \geq 0\} \cap \{\xi - 1 \geq \varepsilon\}^c\). Calculating the critical points of this function in \(\xi\), we can see from (22) that the minimum is taken for \(\xi \in \pm \frac{1}{2}\) in Near frequency region. Hence, one can reduce the minimization problem to evaluating \(\xi\) in \(\{1 \pm \varepsilon\}^c\). Let’s work with the case \(\xi = 1 + \varepsilon\); the other case is similar. Writing \(\omega^2 = 1 + \delta\Omega\), we are facing the problem of minimizing

\[
\left(1 - (1 + \Omega \varepsilon)(1 + \varepsilon^\tau)^2\right)^2, \quad \text{for} \quad \Omega \in \left[-\frac{1}{3}, \frac{1}{3}\right] = \left[-\frac{1}{3}, \frac{1}{3}\right].
\]

It suffices to check the minimum at \(\Omega = \pm \frac{1}{3}\). At \(\xi = 1 + \varepsilon\) we have

\[
\left[1 - \left(1 + \frac{\varepsilon}{3}\right)(1 + \varepsilon^\tau)^2\right]^2 = -\varepsilon^\tau \left(2 + \varepsilon^\tau\right) - \frac{\varepsilon}{3} (1 + \varepsilon^\tau)^2 = \varepsilon^{2\tau} \left(2 + \varepsilon^\tau\right) + \frac{\varepsilon^{1 - \tau}}{3} (1 + \varepsilon^\tau)^2.
\]
Likewise,
\[ \left[ 1 - \left(1 - \frac{\varepsilon}{3} \right) (1 + \varepsilon^T)^2 \right] = \left[ -\varepsilon^T (2 + \varepsilon^T) + \frac{\varepsilon}{3} (1 + \varepsilon^T)^2 \right] = \varepsilon^T \left[ (2 + \varepsilon^T) - \frac{\varepsilon - 1}{3} (1 + \varepsilon^T)^2 \right]. \]

A similar analysis can be applied in the case \( \xi = 1 - \varepsilon^T \) and to the cases \( \xi = -1 \pm \varepsilon^T \), showing that there exists a constant \( C > 0 \) such that \( |m(\kappa; \mathcal{L})|_{[\kappa \in \text{Far frequencies}]} \geq C\varepsilon^{2T} \), whenever \( \varepsilon \in (0, \varepsilon_0) \). This finishes the proof.

### 2.1. The multiplier structure and the far/near decomposition.

In the next part we split the spectrum in order to excise the zeros of the multiplier \( m(\cdot; \mathcal{L}) \). The reasoning follows [ELTW17] §6.2 [see also Sch96a §2]. In the discussion below, we assume that \( v(\cdot) \in H^s(\mathbb{R}) \). We define cut-offs in Fourier space, which parametrize the excised region around the zeros of the multiplier \( m(\cdot; \mathcal{L}) \):

\[ \hat{v}(\cdot) \mapsto \hat{v}_{\text{near}}[v](\kappa) = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) + \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \hat{v}(\kappa) =: \hat{v}_{\text{near}}(\kappa) ; \]

this operator naturally induces the following projection in physical space,

\[ v(\cdot) \mapsto P_{\text{near}}[v](x) = \mathcal{F}^{-1} \circ \hat{P}_{\text{near}} \circ \mathcal{F}[v](x) =: v_{\text{near}}(x), \]

or, in other words, \( v_{\text{near}}(x) = \mathcal{F}^{-1}[\hat{v}_{\text{near}}(\cdot)](x) \). Similarly, we define the space of far frequencies and its associated physical space projection as

\[ \hat{v}(\cdot) \mapsto \hat{v}_{\text{far}}[v](\kappa) = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) - \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \hat{v}(\kappa) =: \hat{v}_{\text{far}}(\kappa) , \]

and its associated physical space projection as

\[ v(\cdot) \mapsto P_{\text{far}}[v](x) = \mathcal{F}^{-1} \circ \hat{P}_{\text{far}} \circ \mathcal{F}[v](x) =: v_{\text{far}}(x). \]

It is clear that \( \hat{v}_{\text{near}}[v](\kappa) \), \( \hat{v}_{\text{far}}[v](\kappa) \) and associated physical space projections all depend on \( (\varepsilon, \tau) \); by abuse (or lack) of notation, we shall omit this dependence. Several properties of these mappings are readily available:

\[ \hat{v}_{\text{near}}(\kappa) = \hat{P}_{\text{near}}[\hat{v}_{\text{near}}](\kappa) = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) + \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \hat{v}_{\text{near}}(\kappa), \]

\[ \hat{v}_{\text{far}}(\kappa) = \hat{P}_{\text{far}}[\hat{v}_{\text{far}}](\kappa) = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) - \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \hat{v}_{\text{far}}(\kappa), \]

\[ v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) , \]

\[ \left\langle v_{\text{near}}, v_{\text{far}} \right\rangle_{H^s(\mathbb{R})} = 0 , \]

\[ \|v\|^2_{H^s(\mathbb{R})} = \|v_{\text{near}}\|^2_{H^s(\mathbb{R})} + \|v_{\text{far}}\|^2_{H^s(\mathbb{R})} . \]

Thanks to these properties, we define

\[ H^s(\mathbb{R}) = X^s_{\text{near}, \varepsilon^T} \oplus X^s_{\text{far}, \varepsilon^T} \]

as subspaces, we shall adopt the norm induced by the \( H^s(\mathbb{R}) \) space.

### 2.2. A Lyapunov-Schmidt reduction.

As we have seen, the near and far frequency regions were defined taking into account the behavior of the multiplier \( m(\cdot; \mathcal{L}) \) on them. In particular, we were able to construct projections onto \( X^s_{\text{near}, \varepsilon^T} \) and \( X^s_{\text{far}, \varepsilon^T} \), which are put in use in the implementation of a Lyapunov-Schmidt reduction; our approach closely follows the ideas in [ELTW17] §6.

Let \( v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) \in X^s_{\text{near}, \varepsilon^T} \oplus X^s_{\text{far}, \varepsilon^T} \). Thanks to (24), we can re-write (11) in an equivalent form by first applying \( \hat{P}_{\text{near}} \circ \mathcal{F} \) to it,

\[ -(1 - \omega^2 \kappa^2)^2 \hat{v}_{\text{near}}(\kappa) = \hat{P}_{\text{near}} \circ \mathcal{F} \left[ \sum_{j=1}^{4} \mathcal{A}(j) [v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}^{(\delta, \omega, \gamma)}], \varepsilon \right](\kappa) \]

\[ = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) + \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \mathcal{F} \left[ \sum_{j=1}^{4} \mathcal{A}(j) [v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}^{(\delta, \omega, \gamma)}], \varepsilon \right](\kappa) , \]

and then applying \( \hat{P}_{\text{far}} \circ \mathcal{F} \), obtaining a complementary equation:

\[ -(1 - \omega^2 \kappa^2)^2 \hat{v}_{\text{far}}(\kappa) = \hat{P}_{\text{far}} \circ \mathcal{F} \left[ \sum_{j=1}^{4} \mathcal{A}(j) [v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}^{(\delta, \omega, \gamma)}], \varepsilon \right](\kappa) \]

\[ = \left( 1 - \mathbb{1}_{[-1+\varepsilon^T \mathbb{B}]}(\kappa) - \mathbb{1}_{[1+\varepsilon^T \mathbb{B}]}(\kappa) \right) \mathcal{F} \left[ \sum_{j=1}^{4} \mathcal{A}(j) [v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}^{(\delta, \omega, \gamma)}], \varepsilon \right](\kappa) . \]
for all $\kappa \in \mathbb{R}$. Our goal is to show that the near field in fact dominate the far filed components. With this in mind, we recall that $\varepsilon(\cdot, \cdot)$ is a function of both $\delta$ and $\Omega$ (see [2]), invoke Lemma 2.1 to then rewrite (25b) as a fixed point equation
\[
v_{\text{far}}(x) = \mathcal{B}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega](x),
\]
defined as a mapping in $\mathcal{E} : X_{\text{near}, \varepsilon}^4 \times X_{\text{far}, \varepsilon}^4 \times (0, +\infty) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \to X_{\text{far}, \varepsilon}^4$. Using (20), this equation also reads as
\[
\mathcal{B}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega](x) := \mathcal{F}^{-1} \left[ \left( \frac{1}{m(\cdot, \mathcal{Z})} \right)^2 \mathcal{F}_{\text{far}} \circ \mathcal{F} \left[ \sum_{j=1}^{4} \mathcal{N}^{(j)} [v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}^{(\delta, \omega, \gamma)}], \varepsilon \right] \right](x).
\]

3. Near field components dominate far energy terms: A Lyapunov-Schmidt reduction.

The main result of this section gives the parametrization of the far energy terms to the near field terms. For the sake of notation, throughout this section we write $\| \cdot \|_{H^s}$ and $\| \cdot \|_{L^p}$ to denote the norms $\| \cdot \|_{H^s(\mathbb{R})}$ and $\| \cdot \|_{L^p(\mathbb{R})}$, respectively.

**Proposition 3.1 (Near field domination).** Assume $[H1] - [H3]$. Let $R$, $\tau$, $\delta$, $\beta$ be fixed positive numbers, the latter two of which were introduced in [21] and [6], respectively. Assume the following constraints on $\tau$ and $\beta$:
\[
0 < \tau < \frac{1}{16}, \quad \text{and} \quad 1 \leq \beta < 1 + \tau,
\]
( $\beta = 1$ and $\tau = \frac{1}{16}$ will do). Recall from $[H1]$ that $\varepsilon(\delta, \Omega)$ is a smooth function of its arguments. Then, there exists an $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the system (26) has a fixed point, namely, there exists a mapping
\[
(v_{\text{near}}(\cdot), \delta, \Omega) \mapsto v_{\text{far}}[v_{\text{near}}, \delta, \Omega](\cdot)
\]
from $\{u(\cdot) \in X_{\text{near}, \varepsilon}^4 \| \| \|_{H^s} \leq R \varepsilon^\beta \} \times (0, \delta_0) \times \left(-\frac{1}{3}, \frac{1}{3}\right)$ to $\{v(\cdot) \in X_{\text{far}, \varepsilon}^4 \| \| \|_{H^s} \leq R \varepsilon^\beta \}$ and satisfying
\[
v_{\text{far}}[v_{\text{near}}, \delta, \Omega](\cdot) = \mathcal{B}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega], u_{\text{rolls}}, \delta, \Omega](\cdot).
\]
Furthermore, we have that
(i) the function $v_{\text{far}}(\cdot)$ satisfies the bound
\[
\|v_{\text{far}}[v_{\text{near}}, \delta, \Omega]\|_{H^s} \leq \Theta(\varepsilon; \tau) \|v_{\text{near}}\|_{H^s} + (\varepsilon^{3-\beta} + \varepsilon^{1+\beta} + \varepsilon^{2+\beta}),
\]
where $\Theta(\varepsilon; \tau) = O \left( \sum_{j=1}^{3} \varepsilon^{3-j-2\tau+j(1-j)} \right).
\]
(ii) the mapping $v_{\text{far}}[\delta, \Omega, \cdot](\cdot)$ can be extended continuously as $v_{\text{far}}[\delta, \Omega, \cdot](\cdot)_{\| \cdot \| = 0}$.
(iii) the mapping $v_{\text{near}, \delta, \Omega} \mapsto v_{\text{far}}[\delta, \Omega, v_{\text{near}}](\cdot)$ is continuous from $X_{\text{near}, \varepsilon}^4 \times (0, \delta_0) \times \left(-\frac{1}{3}, \frac{1}{3}\right)$ to $X_{\text{far}, \varepsilon}^4$.

An auxiliary Lemma concerning the scaling of $u_{\text{rolls}}^{(\delta, \omega, \gamma)}(\cdot)$ with respect to its amplitude $\varepsilon$ is presented next; it shows in particular how the amplitude of the rolls scale as $\varepsilon$ decreases. The result is essentially given in [Mic95] Section 4 and Theorem 4.1.

**Lemma 3.2 (Scaling of $u_{\text{rolls}}^{(\delta, \omega, \gamma)}(\cdot)$ in $\varepsilon = \varepsilon(\delta, \Omega)$).** Fix $\gamma \in \mathbb{R}$. Given the equation (1) with $\mu \equiv 1$, there exists an $\varepsilon = \varepsilon(\delta, \Omega) > 0$ and a mapping $(-\delta_0, \delta_0) \ni \delta \mapsto u_{\text{rolls}}^{(\delta, \omega, \gamma)}(\cdot) \in H^\infty_{\text{per}}([0, 2\pi])$, and so that $u_{\text{rolls}}^{(\delta, \omega, \gamma)}(\cdot)|_{\varepsilon = 0} = 0$.

Furthermore, we can define $u_{\text{rolls}}^{(\delta, \omega, \gamma)} = \frac{u_{\text{rolls}}^{(\delta, \omega, \gamma)}}{\varepsilon} \in L^\infty([0, 2\pi])$ and find a $\varepsilon_0 > 0$ such that $\varepsilon_0 \lesssim \delta_0 \lesssim \varepsilon_0$.

**Corollary 3.3 (Reparametrization of rolls by their amplitudes).** In the region $\omega \in (\sqrt{1 - \delta}, \sqrt{1 + \delta})$ the rolls described in [2] can be reparametrized as functions of $\varepsilon(\omega, \gamma)$, where
\[
\delta = \delta(\varepsilon, \omega) = \sqrt{\frac{3\varepsilon^2}{4} + (1 - \omega^2)^2} + O \left( \frac{3\varepsilon^2}{4} + (1 - \omega^2)^2 \right);
\]
moreover, this mapping is a homeomorphism whenever $\varepsilon > 0$ and $\omega \in (\sqrt{1 - \delta}, \sqrt{1 + \delta})$. 

Proof. This result is a consequence of Mielke’s derivation of the rolls using a Lyapunov-Schmidt reduction. Inspecting his proof, on note that the Implicit Function Theorem can be applied\(^7\) either in \(\varepsilon^2\) or in \(\omega^2\), from which one can obtain either a function \(\varepsilon^2(\delta^2, \omega)\), or \(\delta^2(\varepsilon, \omega)\). The homeomorphism is derived from the homeomorphism of the mapping \(x \mapsto \sqrt{x}\) on \((0, \infty)\).

A few useful consequences of this result are readily available, thanks to the fact that the nonlinear terms in \([H]\) scale in \(\varepsilon\). Indeed, we can write

\[
\mathcal{N}(j)[v, u(\varepsilon \omega, \gamma), \varepsilon] = \mathcal{N}(j)[v, u(\varepsilon \omega, \gamma), \varepsilon], \quad j \in \{1, 2, 3, 4\}. \tag{29}
\]

We point out that \(\mathcal{N}(1)[\cdot]\) depends explicitly on \(\delta^2\); however, thanks to the choice of parameters in \([H]\) any upper bound in terms of \(\delta\) can be rewritten as an upper bound in terms of \(\varepsilon\).

Lemma 3.4. Assume \([H1]-[H3]\). Let \(u(\cdot) \in X^{4}_{\text{near}, \varepsilon}\) be fixed and parameters \(\beta\) and \(\tau\) satisfying the constraints in \([27]\). Given the nonlinear mappings \(X^4_{\text{far}, \varepsilon} \ni v(\cdot) \mapsto \mathcal{N}(j)[u + v, u(\delta \omega, \gamma), \varepsilon]\) for \(j \in \{1, 2, 3, 4\}\), defined in \([11]\), the following properties hold:

(i) Let \(\|v\|_{H^s} < 1\), and \(\|v^{(1)}\|_{H^s} < 1\). For every \(j \in \{1, 2, 3, 4\}\) there exists quantities \(M_j > 0\) and \(E_j[v^{(1)}, u, \varepsilon] \geq 0\) such that

\[
\|\mathcal{N}(j)[u + v^{(1)}, u(\delta \omega, \gamma), \varepsilon]\|_{L^2} \leq E_j[u, v^{(1)}, \varepsilon],
\]

where

\[
E_j[u, v^{(1)}, \varepsilon] \leq \begin{cases} M_1 \varepsilon^{3-j} \left( \|u\|^2_{H^2} + \|v^{(1)}\|^2_{H^s} \right) & \text{for } j = 1, 2, 3; \\ M_1 \varepsilon^{3-j} \left( \varepsilon^2 + \varepsilon^{1+\frac{3}{2}} + \varepsilon^{2+\frac{3}{2}} \right) & \text{for } j = 4. \end{cases}
\]

(ii) For every \(j \in \{1, 2, 3, 4\}\) there exists quantities \(M_j > 0\) and \(E_j[v^{(2)}, v^{(3)}, \varepsilon] \geq 0\) such that

\[
\|\mathcal{N}(j)[u + v^{(2)}, u(\delta \omega, \gamma), \varepsilon] - \mathcal{N}(j)[u + v^{(3)}, u(\delta \omega, \gamma), \varepsilon]\|_{L^2} \leq D_j[v^{(2)}, v^{(3)}, u, \varepsilon]\|v^{(2)} - v^{(3)}\|_{H^s},
\]

where

\[
D_j[v^{(2)}, v^{(3)}, u, \varepsilon] \leq \begin{cases} M_2 \varepsilon^{3-j} \left( \|v^{(2)}\|^2_{H^{-1}} + \|v^{(3)}\|^2_{H^{-1}} + \|u\|^2_{H^2} \right) & \text{for } j = 1, 2, 3; \\ 0 & \text{for } j = 4. \end{cases}
\]

Proof. For the sake of notation, we write \(u_{\text{rolls}}(\cdot)\) to denote \(u(\delta \omega, \gamma)(\cdot)\). Whenever \(j \in \{1, 2, 3\}\) all the inequalities are a consequence of the scaling \([29]\), of the polynomial nature of the nonlinearity, and of the Sobolev embedding \(H^4(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\) given in \([18]\); indeed, for all \(p \geq 1\), \(w \in H^4(\mathbb{R})\), we have

\[
\|w\|^p_{L^2} \leq \|w\|_{L^p} \|w\|_{H^4}^{p-1} \lesssim \|w\|_{H^4}^{p-1},
\]

in the particular case \(j = 2\) we resort to property \([H2]\), using the similarity \(\delta \approx \varepsilon\).

We now prove the case \(j = 4\): the inequalities in (ii) are trivial because there is no dependence on \(\varepsilon\). In the case (i), first recall that

\[
\mathcal{N}(4)[u(\delta \omega, \gamma), \varepsilon](x) = \chi(\varepsilon^x) (\chi^2(\varepsilon^x) - 1) \left( u_{\text{rolls}}(\delta \omega, \gamma)(x) \right)^3 + [(1 + \partial_x^2)^2, \chi(\varepsilon^x)] u_{\text{rolls}}(\delta \omega, \gamma)(x) - \delta^2 \chi(\varepsilon^x)(\mu - 1) u_{\text{rolls}}(\delta \omega, \gamma)(x).
\]

The scaling of Lemma [3.2] implies that the first term is of the form

\[
\chi(\varepsilon^x) (\chi^2(\varepsilon^x) - 1) \left( u_{\text{rolls}}(\delta \omega, \gamma) \right)^3(x) = \varepsilon^3 \chi(\varepsilon^x) (\chi^2(\varepsilon^x) - 1) (u_{\text{rolls}}(\delta \omega, \gamma))^3(x),
\]

hence a change of variables gives

\[
\|\chi(\varepsilon^x) (\chi^2(\varepsilon^x) - 1) \left( u_{\text{rolls}}(\delta \omega, \gamma) \right)^3 \|_{L^2(\mathbb{R})} \leq \varepsilon^{-\frac{3}{2}} \|u_{\text{rolls}}(\delta \omega, \gamma)\|_{L^\infty} \sqrt{\int_{\mathbb{R}} |\chi(y)(\chi^2(y) - 1)|^2 dy},
\]

where the integrand in the latter integral is localized, thanks to the properties of \(\chi(\cdot)\). Similar reasoning shows that the \(L^2(\mathbb{R})\) norm of the last integral is \(O(\varepsilon^{-2})\). With regards to the second term, initially notice that \([(1 + \partial_x^2)^2, \chi(\varepsilon^x)] u_{\text{rolls}}(\delta \omega, \gamma) = \varepsilon^3 (1 + \partial_x^2)^2, \chi(\varepsilon^x) \left( u_{\text{rolls}}(\delta \omega, \gamma) \right)^3(x)\). Then notice that the lowest order terms in \(\varepsilon\) come from terms in the commutator that contain one derivative of \(\chi(\cdot)\), that is,

\[
\varepsilon^3 (1 + \partial_x^2)^2, \chi(\varepsilon^x) \left( u_{\text{rolls}}(\delta \omega, \gamma) \right)^3(x) = 4\varepsilon^{1+\beta} \chi(\varepsilon^x) \partial_x^3 (1 + \partial_x^2) u_{\text{rolls}}(\delta \omega, \gamma)(x) + O(\varepsilon^{2+1}).
\]

---

\(^3\)Our notation of \(\varepsilon, \omega\) translates to Mielke’s notation in [Mie95] Section 4 as \(\alpha\) and \(\varepsilon\), respectively.
However, \( \overline{u_{rolls}}(\cdot) = \cos(\cdot) + O(\varepsilon) \), hence \( 4\varepsilon^{1+\beta} x (\varepsilon^\beta x) \partial_x (1 + \partial_x^2) \overline{u_{rolls}}(\cdot) = O(\varepsilon^{\beta+2}) \). A change of variables then gives
\[
\int \varepsilon^{2+\beta} x (\varepsilon^\beta x)^2 dx = \varepsilon^{4+\beta} \int |\chi'(z)|^2 dz = O(\varepsilon^{4+\beta}),
\]
and this finishes the estimate.

We are now ready to prove the main result of this section.

**Proof.** [of Proposition 3.1] Throughout the proof we write \( \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \) to denote \( \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}, u_{\text{rolls}}, \varepsilon] \). We shall achieve the implicit parametrization of \( v_{\text{far}}(\cdot) \) by \( v_{\text{near}}(\cdot) \), \( \delta \) and \( \Omega \) by applying the Contraction Mapping Theorem [Ch82, Theorem 2.2]. Without loss of generality we assume that \( \varepsilon < 1 \).

To begin with, recall from (23) that \( v(\cdot) = v_{\text{far}}(\cdot) + v_{\text{near}}(\cdot) \). Writing
\[
\tilde{f}(\kappa) = (1 - \mathbb{1}_{\{+1+\varepsilon R\}}(\kappa) - \mathbb{1}_{\{-1-\varepsilon R\}}(\kappa)),
\]
we use Lemma 2.1 to estimate (26):
\[
\|\mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^1}^2 = \int \frac{(1 + |\kappa|^2)^4}{(1 - \omega^2|\kappa|^2)^4} \left( \tilde{f}(\kappa) \right)^2 \left| \mathcal{F} \left[ \sum_{j=1}^N \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \right] \right|^2 \kappa d\kappa
\]
\[
\leq \int \frac{(1 + |\kappa|^2)^4}{(1 - \omega^2|\kappa|^2)^4} \left( \tilde{f}(\kappa) \right)^2 \left| \mathcal{F} \left[ \sum_{j=1}^N \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \right] \right|^2 \kappa d\kappa
\]
\[
\leq 4 \int \frac{\varepsilon^2}{\varepsilon^2} \left| \mathcal{F} \left[ \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \right] \right|^2 \kappa d\kappa
\]
\[
\leq 4 \| \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \|_{L^2}^2.
\]

An application of Lemma 3.4 gives
\[
\|\mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^1}^2 \leq \sum_{j=1}^4 \frac{C}{\varepsilon^2} \| \mathcal{N}^{(j)}[v_{\text{near}} + v_{\text{far}}] \|_{L^2}
\]
\[
\leq \sum_{j=1}^4 \frac{C}{\varepsilon^2} E_j[v_{\text{near}}, v_{\text{far}}, u_{\text{rolls}}, \delta, \Omega]
\]
\[
\leq CM_1 \left\{ \sum_{j=1}^3 \epsilon^{3-j-2r} \left( \|v_{\text{near}}\|_{H^4}^2 + \|v_{\text{far}}\|_{H^4}^2 \right) + \epsilon^{3-\beta-2r} + \epsilon^{1+\frac{3\beta}{2}-2r} + \epsilon^{2+\frac{3}{2}-2r} \right\}.
\]

Now choosing \( v_{\text{near}} \in H^4 \leq R \varepsilon^\frac{4}{5} \), and \( \|v_{\text{far}}\|_{H^4} \leq R \varepsilon^\alpha \), where \( \alpha = \frac{4}{15} \), we get
\[
\|\mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^4} \leq CM_1 \left\{ \sum_{j=1}^3 \epsilon^{3-j-2r} R^j \left( \varepsilon^\frac{3}{2} + \varepsilon^{j\alpha} \right) + \epsilon^{3-\beta-2r} + \epsilon^{1+\frac{3\beta}{2}-2r} + \epsilon^{2+\frac{3}{2}-2r} \right\}.
\]

Hence,
\[
\|\mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^4} \leq CM_1 \left\{ \sum_{j=1}^3 \epsilon^{3-j-2r} R^j \left( \varepsilon^\frac{3}{2} + \varepsilon^{j\alpha} \right) + \epsilon^{3-\beta-2r} + \epsilon^{1+\frac{3\beta}{2}-2r} + \epsilon^{2+\frac{3}{2}-2r} \right\}
\]
\[
\leq 3CM_1 R \left\{ \varepsilon^{\frac{3}{2}-2r} + \sum_{j=1}^3 \epsilon^{3+j(\alpha-1)-2r} + \epsilon^{3-\beta-2r} + \epsilon^{1+\frac{3\beta}{2}-2r} + \epsilon^{2+\frac{3}{2}-2r} \right\},
\]
the last inequality being a consequence of the monotonic decay of the mapping \( x \mapsto \varepsilon^x \) in \( x > 0 \), for \( 0 < \varepsilon < 1 \) and \( \beta \geq 1 \); this property is used many times below. Indeed, after further simplification we get

\[
\| \mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega] \|_{H^4} \\
\leq 3CM_1 \sum_{j=1}^{3} \left( e^{3-j(\alpha-1)-2\tau} + e^{3-\frac{\beta}{2}-2\tau} + e^{1+\frac{\beta}{2}-2\tau} + e^{e+\frac{\beta}{2}-2\tau} \right) \\
\leq 3CM_1 \left( e^{3-2\tau-\alpha} + \sum_{j=1}^{3} \left( e^{3-j-2\tau+\alpha(j-1)} + e^{3-\frac{\beta}{2}-2\tau-\alpha} + e^{1+\frac{\beta}{2}-2\tau-\alpha} + e^{e+\frac{\beta}{2}-2\tau-\alpha} \right) \right) R \varepsilon^\alpha \\
=: \Theta_1(\alpha, \tau, \varepsilon, \beta) R \varepsilon^\alpha.
\]

In our second step, we derive the following estimate using similar arguments:

\[
\| \mathcal{E}[v_{\text{near}}, v_{\text{far}}, (1), \delta, \Omega] - \mathcal{E}[v_{\text{near}}, v_{\text{far}}, (2), \delta, \Omega] \|_{H^4} \\
\leq 3CM_2 \left( e^{3-j-2\tau} \left( e^{\alpha(j-1)} + e^{(j-1)\frac{\beta}{2}} \right) \right) \| v_{\text{far}}^{(1)} - v_{\text{far}}^{(2)} \|_{H^4} \\
\leq 2CM_2 \left( e^{3-j-2\tau} \left( e^{\alpha(j-1)} + e^{(j-1)\frac{\beta}{2}} \right) \right) \| v_{\text{far}}^{(1)} - v_{\text{far}}^{(2)} \|_{H^4} \\
\leq 2CM_2 \left( e^{3-j-2\tau+a(j-1)} + e^{\frac{\beta}{2}-2\tau} \right) \| v_{\text{far}}^{(1)} - v_{\text{far}}^{(2)} \|_{H^4}.
\]

Thus,

\[
\| \mathcal{E}[v_{\text{near}}, v_{\text{far}}, (1), \delta, \Omega] - \mathcal{E}[v_{\text{near}}, v_{\text{far}}, (2), \delta, \Omega] \|_{H^4} \\
\leq 6CM_2 \left( e^{3-j-2\tau+a(j-1)} + e^{1-2\tau} \right) \| v_{\text{far}}^{(1)} - v_{\text{far}}^{(2)} \|_{H^4} \\
=: \Theta_2(\alpha, \tau, \varepsilon, \beta) \| v_{\text{far}}^{(1)} - v_{\text{far}}^{(2)} \|_{H^4}.
\] (30)

Hence, for any fixed set of parameters \( R, \alpha, \tau, \) and \( \beta \) satisfying conditions \((27)\), one can choose \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_1)} \Theta_1(\alpha, \tau, \varepsilon, \beta) \leq \frac{1}{2}, \quad \text{and} \quad \sup_{\varepsilon \in (0, \varepsilon_2)} \Theta_2(\alpha, \tau, \varepsilon, \beta) \leq \frac{1}{2}.
\]

Recall that \( \alpha = \frac{1}{\Pi} \); now, choosing \( \varepsilon_0 > 0 \) as in Lemma \( 3.2 \) and \( \varepsilon_* := \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\} \) we get that for all \( \varepsilon \in (0, \varepsilon_*) \) the mapping \( (v_{\text{near}}, v_{\text{far}}, u_{\text{rolls}}, \delta, \Omega) \mapsto \mathcal{E}[v_{\text{near}}, v_{\text{far}}, \delta, \Omega] \), maps the set

\[
(v_{\text{near}}, v_{\text{far}}, \delta, \Omega) \in \{ v \in X_{\text{near}, \varepsilon}^4 \| v \|_{H^4} \leq R \varepsilon^\delta \} \times \{ v \in X_{\text{far}, \varepsilon}^4 \| v \|_{H^4} \leq R \varepsilon^\tau \} \times (0, \delta_*),
\]

into \( \{ v \in H^4 \| v \|_{H^4} \leq R \varepsilon^\Pi \} \), therefore, we can apply the Contraction Mapping Theorem to obtain the existence of a fixed point \( v_{\text{far}} = v_{\text{far}}(\delta, \Omega, v_{\text{near}}) \) for all \( \varepsilon \in (0, \varepsilon_*) \), where \( v_{\text{far}}(\delta, \Omega, v_{\text{near}}) \varepsilon=0 = 0 \). This finishes the proof of the implicit parametrization of \( v_{\text{far}}(\cdot) \) by \( v_{\text{near}}(\cdot), \delta \) and \( \Omega \).
Now that \(v_{\text{far}}(\delta, \Omega, v_{\text{near}})\) is well defined, we can finally study its properties. The bounds in (i) are a consequence of

\[
\|v_{\text{far}}(\delta, \Omega, v_{\text{near}})\|_{H^4} = \|\delta[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^4}
\]

\[
\leq CM_1 \left\{ \sum_{j=1}^{3} \varepsilon^{3-j-2}\|v_{\text{near}}\|_{H^4} + \sum_{j=1}^{3} \varepsilon^{3-j-2}\|v_{\text{far}}\|_{H^4} + \left( \varepsilon^{3-\frac{3}{2}-2\varepsilon} + \varepsilon^{1+\frac{3}{2}-2\varepsilon} + \varepsilon^{2-\varepsilon} \right) \right\}
\]

\[
\leq CM_1 \sum_{j=1}^{3} \left( \varepsilon^{3-j-2\varepsilon+j(1)} \right) \|v_{\text{near}}\|_{H^4} + CM_1 \sum_{j=1}^{3} \left( \varepsilon^{3-j-2\varepsilon+j(1)} \right) \|v_{\text{far}}\|_{H^4}
\]

\[
+ CM_1 \left( \varepsilon^{3-\frac{3}{2}-2\varepsilon} + \varepsilon^{1+\frac{3}{2}-2\varepsilon} + \varepsilon^{2-\varepsilon} \right)
\]

Thanks to the parameter conditions in (27) and the definition of \(\varepsilon\), the term dependent of \(\|v_{\text{far}}\|_{H^4}\) on the right hand side can be absorbed, hence

\[
\|v_{\text{far}}(\delta, \Omega, v_{\text{near}})\|_{H^4} \lesssim \left( \sum_{j=1}^{3} \varepsilon^{3-j-2\varepsilon+j(1)} \right) \|v_{\text{near}}\|_{H^4} + \left( \varepsilon^{3-\frac{3}{2}-2\varepsilon} + \varepsilon^{1+\frac{3}{2}-2\varepsilon} + \varepsilon^{2-\varepsilon} \right),
\]

whenever \(\varepsilon \in (0, \varepsilon_0)\), and this finishes the proof of (i); item (ii) is a straightforward consequence of the bounds in (i). With regards to (iii), we must show that the mapping \((v_{\text{near}}, \delta, \Omega) \mapsto v_{\text{far}}(\delta, \Omega, v_{\text{near}})\) is continuous, this is also a consequence of the Contraction Mapping Principle, namely, it suffices to show that

\[
(v_{\text{near}}, \delta, \Omega) \mapsto \delta[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]
\]

is a continuous mapping from \(H^4(\mathbb{R}) \times (0, \delta_0) \times (-\frac{1}{2}, \frac{1}{2})\) to \(H^4(\mathbb{R})\); continuity with respect to \(v_{\text{near}}\) is easily obtained by exchanging the roles of \((v_{\text{near}}, v_{\text{far}})\) for \((v_{\text{far}}, v_{\text{near}})\) in the estimates (30). In this fashion, one obtains the similar bound

\[
\|\delta'[v_{\text{near}}(1), v_{\text{far}}, \delta, \Omega] - \delta'[v_{\text{near}}(2), v_{\text{far}}, \delta, \Omega]\|_{H^4} \leq \Theta_2 \left( \frac{1}{16}, \varepsilon, \beta \right) \|v_{\text{near}}(1) - v_{\text{near}}(2)\|_{H^4},
\]

Thanks to the change of variables \(z \mapsto x/\omega\) that lead to (5), we have that the mapping \(\delta \mapsto \mathcal{A}(\delta)(v, u_{\text{rolls}}, \omega, \gamma, \delta)\) is continuous in \(L^2(\mathbb{R})\). Exploiting the pointwise continuity of the mappings \((\delta, \Omega) \mapsto \delta'[v_{\text{near}}, v_{\text{far}}, \delta, \Omega](x)\) for any fixed \(x \in \mathbb{R}\) and the fact that the \(H^4(\mathbb{R})\) norm of \(\delta'[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\) is uniform in \(\varepsilon\), an application of the Lebesgue Dominated Convergence Theorem show that the other nonlinearities are also continuous with respect to \((\delta, \Omega)\). This finishes the proof.

**Remark 3.5** (The fine balance between the blow-up rate in \(\varepsilon\) and the scaling of the nonlinearities in \(\varepsilon\).) In [FLTW17] the near component domination relies on counterbalancing the blowup of the multiplier as \(\varepsilon \downarrow 0\) with appropriate rescaling of the solutions, a feature that strongly depends on the linear nature of the problem. In contrast, resorting to rescaling is not as effective in the nonlinear case, although other properties of the equations can be exploited; for instance, the scaling of \(\delta'[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\) in \(\varepsilon\) also relies on the nonlinearity, without which the term \(\varepsilon^{-2\varepsilon}\) coming from the multiplier would be too “harmful”.

4. Desingularization, nonlinear interaction estimates and approximation results

Once Proposition 3.4 establishes the parametrization by the near frequency components, we devote our concerns to \(v_{\text{near}}(\cdot)\), whose behavior is still to be understood. We brieﬂy recall that two unknown constants and an unknown function are still present: the constant \(\beta\), that was introduced in (9) and shall be deﬁned later using matched asymptotics; the constant \(\tau\), that was introduced in the far/near decomposition (20); and the function \(\chi(\cdot)\), that plays the role of an envelop function and was introduced in (9). A ﬁrst observation towards our next step is that

\[
\text{supp}(\tilde{v}_{\text{near}}) \subset \{-1 + \varepsilon^T B\} \cup \{1 + \varepsilon^T B\}.
\]

clearly, the set on the right hand side gets reduced to two points as \(\varepsilon \downarrow 0\). We extract the relevant properties of \(v_{\text{near}}(\cdot)\) doing a desingularization of this limit; that is, using blow-up variables in the frequency space, as done in [FLTW17] §6.4. As we shall see, this approach readily gives another representation of \(v_{\text{near}}(\cdot)\). To begin with, a slight modiﬁcation of the operator \(\hat{P}_{\text{near}}(\cdot)\) introduced in section 2.1 is necessary. Deﬁne the operators

\[
\tilde{v} \mapsto \hat{P}_{\text{near}}^\dagger[\tilde{v}](\hat{k}) = \mathbf{1}_{\{\hat{k} \in \varepsilon^T B\}}(\hat{k}) \tilde{v}(\pm 1 + \hat{k}) =: \hat{v}_{\text{near}}^\dagger(\hat{k}),
\]

and associated physical space action

\[
v \mapsto \hat{P}_{\text{near}}^\dagger[v](x) = \mathcal{F}^{-1} \left[ \hat{P}_{\text{near}}^\dagger \circ \mathcal{F}[v](\cdot) \right](x) = v_{\text{near}}^\dagger(x), \quad x \in \mathbb{R},
\]

(31)
Proposition 4.1 (Recentered projections). Let \( f(\cdot) \in L^2(\mathbb{R}) \), with decomposition \( f(\cdot) = f_{\text{near}}(\cdot) + f_{\text{far}}(\cdot) \in X_{\text{near}, \epsilon}^{(0)} \oplus X_{\text{far}, \epsilon}^{(0)} \). Consider the operators \( \mathcal{P}_{\text{near}}^{(\pm)}[\cdot] \) as defined in \(^{[31]}\). Then, the following properties hold:

(i) For any \( 0 < \epsilon < 1, \tau > 0 \), we have

\[
\mathcal{P}_{\text{near}}^{(\pm)}[f] (\hat{\kappa} + 1) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{P}_{\text{near}}^{(\pm)}[f] (\hat{\kappa}).
\]

(ii) \( \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]} = \mathcal{P}_{\text{near}}^{(-) \circ \mathcal{P}_{\text{near}}^{(+)}[f]} = 0. \)

(iii) Writing \( f_{\text{near}}(x) = \mathcal{F}^{-1} \left[ \mathcal{F}_{\text{near}}^{(\pm)}(x) \right], \) we have

\[
f_{\text{near}}(x) = e^{i \tau x} f_{\text{near}}^{(+)}(x) + e^{-i \tau x} f_{\text{near}}^{(-)}(x).
\]

(iv) For any given \( \alpha \in \mathbb{R} \) we have

\[
\text{supp} \left( \mathcal{F} \left[ e^{i \alpha} f_{\text{near}}^{-}(\cdot) \right] \right) \subset \alpha + \epsilon \tau B, \quad \text{and} \quad \text{supp} \left( \mathcal{F} \left[ e^{i \alpha} f_{\text{near}}^{(+)}(\cdot) \right] \right) \subset \alpha + \epsilon \tau B;
\]

We can say then that \( f \mapsto \mathcal{P}_{\text{near}}^{(\pm)}[f](x) : X_{\text{near}, \epsilon}^{(0)} \to H_{\text{near}, \epsilon}^{s}(\mathbb{R}) \);

(v) \( \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(\hat{\kappa}) = \mathcal{P}_{\text{near}}^{(-) \circ \mathcal{P}_{\text{near}}^{(+)}[f]}(\hat{\kappa}). \)

(vi) If \( f(\cdot) \) is real-valued, then \( f_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(x) = f_{\text{near}}^{(-) \circ \mathcal{P}_{\text{near}}^{(+)}[f]}(x). \)

(vii) Let \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be a mapping with a multiplier \( m(\cdot ; T) \), that is, \( \mathcal{F}(Tf)(\xi) = m(\xi ; T) \mathcal{F}(f)(\xi). \)

Then, \( \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(\hat{\kappa}) = m(\pm 1 + \epsilon \tau ; T) \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(\hat{\kappa}). \)

**Proof.** Property (i) is a simple consequence of the definition of \( \mathcal{P}_{\text{near}}^{(\pm)}[\cdot] \) given in section \(^{[2.1]}\). Property (ii) is a simple calculation: for \( f(\cdot) \in L^2(\mathbb{R}) \) we have

\[
\mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(\hat{\kappa}) = \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(-\hat{\kappa}) = \mathcal{P}_{\text{near}}^{(-) \circ \mathcal{P}_{\text{near}}^{(+)}[f]}(\hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{P}_{\text{near}}^{(+) \circ \mathcal{P}_{\text{near}}^{(-)}[f]}(\hat{\kappa}).
\]

And the result follows due to the conditions in \( \epsilon \) and \( \tau \), because \( \epsilon \tau B \cap \{ \pm 1 + \epsilon \tau B \} = \emptyset \). Property (iii) is a consequence of a simple computation: writing \( f_{\text{near}}^{(\pm)}(x) = \mathcal{F}^{-1} \left[ \mathcal{F}_{\text{near}}^{(\pm)}(x) \right] = \mathcal{F}^{-1} \mathcal{P}_{\text{near}}^{(\pm)} \mathcal{F}[f](x) \), we have

\[
f_{\text{near}}(x) = \frac{1}{2\pi} \int_{\hat{\kappa} \in \pm 1 + \epsilon \tau B} \mathcal{F}^{-1} (\hat{\kappa}) e^{i \tau x} \mathcal{F} \mathcal{F}_{\text{near}}^{(\pm)} (\hat{\kappa}) d\hat{\kappa} + \frac{1}{2\pi} \int_{\hat{\kappa} \in -1 + \epsilon \tau B} \mathcal{F}^{-1} (\hat{\kappa}) e^{i \tau x} d\hat{\kappa}.
\]

Property (iv) is a direct consequence of the definition in \(^{[31]}\), allied Fourier transform property \(^{[17b]}\); the fact that \( f \mapsto \mathcal{P}_{\text{near}}^{(\pm)}\mathcal{F}[f] : X_{\text{near}, \epsilon}^{(0)} \to H_{\text{near}, \epsilon}^{s}(\mathbb{R}) \) is a simple consequence of the definition of the spaces \( H_{\text{near}, \epsilon}^{s}(\mathbb{R}) \) (in Section \(^{[1.3]}\)). In order to prove (v) we study the action of the operators \( \mathcal{P}_{\text{near}}^{(\pm)} \circ \mathcal{F}[\cdot] \) on real-valued functions \( f(\cdot) \in L^2(\mathbb{R}; \mathbb{R}) \cap L^1(\mathbb{R}; \mathbb{R}) \) is the following:

\[
\mathcal{P}_{\text{near}}^{(+) \circ \mathcal{F}[f]}(\hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{F} \mathcal{F}[f] (1 + \hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{F}[f] (1 + \hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{F}[f] (1 + \hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} (\hat{\kappa}) \mathcal{F}[f] (1 - \hat{\kappa}).
\]

We now turn to (vii): thanks to definition of \( f_{\text{near}}(\cdot) \) given in section \(^{[2.1]}\), we can use item (i) to write

\[
\mathcal{F}_{\text{near}}^{(\pm)} (\hat{\kappa}) = \mathbb{1}_{\{ \hat{\kappa} \in \pm 1 + \epsilon \tau B \}} \mathcal{F}_{\text{near}} (\pm 1 + \hat{\kappa}).
\]
Thanks to (iv) we have \( \hat{f}^{(+)}(\tilde{\kappa}) = \hat{f}^{(-)}(\tilde{\kappa}) \) and consequently, in physical space, we have
\[
 f^{(+)}(x) = \mathcal{F}^{-1} \left[ \hat{f}^{(+)}(\tilde{\kappa}) \right] (x) = \mathcal{F}^{-1} \left[ \hat{f}^{(-)}(\tilde{\kappa}) \right] (x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}^{(-)}_\text{near}(\tilde{\kappa}) e^{i\tilde{\kappa}x} d\tilde{\kappa}
 = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}^{(-)}_\text{near}(-\tilde{\kappa}) e^{-ix\tilde{\kappa}} d\tilde{\kappa}
 = f^{(-)}(x).
\]

Last, the proof of (vii) is obtained after a direct computation:
\[
 p^{(\pm)}_\text{near}(\hat{T}f)(\tilde{\kappa}) = 1_{\{\epsilon > B\}}(\tilde{\kappa}) m(\pm 1 + \tilde{\kappa}; T) \hat{f}(\pm 1 + \tilde{\kappa}) = m(\pm 1 + \tilde{\kappa}; T) p^{(\pm)}(\hat{T}f)(\tilde{\kappa}) .
\]

### 4.1. Desingularization in Fourier space and the representation of \( v_{\text{near}}(\cdot) \) as a Ginzburg-Landau type approximation.

Now that we are able to center our parametrizations, we apply the results derived in the previous section to construct the functions
\[
v^{(\pm)}(\cdot) = p^{(\pm)}(v_{\text{near}}) \in H^1_{\text{near},\epsilon^\beta}(\mathbb{R}), \quad v^{(-)}(\cdot) = p^{(-)}(v_{\text{near}}) \in H^4_{\text{near},\epsilon^\beta}(\mathbb{R}),
\]
and their corresponding Fourier transforms, given respectively by
\[
\hat{v}^{(\pm)}(\cdot) = \hat{p}^{(\pm)}(\hat{v}_{\text{near}})(\cdot), \quad \hat{v}^{(-)}(\cdot) = \hat{p}^{(-)}(\hat{v}_{\text{near}})(\cdot).
\]

We then define functions \( g_{+1}(\cdot) \) and \( g_{-1}(\cdot) \) in the following manner,
\[
\hat{g}_{\pm 1}(\xi) = \hat{g}_{\pm 1}(\tilde{\kappa}) := \hat{v}^{(\pm)}_{\text{near}}(\epsilon^\beta \xi), \quad \text{where} \quad \xi = \frac{\kappa}{\epsilon^\beta}.
\]

We write then \( g_{\pm 1}(x) = \mathcal{F}^{-1} [\hat{g}_{\pm 1}](x) \); clearly, \( g_{\pm 1}(\cdot) \in H^4_{\text{near},\epsilon^\beta}(\mathbb{R}) \). Thanks to \((23)\) and to the identity \( \mathbb{1}(A)(x) = \mathbb{1}(\xi A)(x) \) (whenever \( \xi > 0 \)) we have
\[
\hat{g}_{\pm 1}(\xi)(\xi) = \mathbb{1}_{\{\epsilon > B\}}(\epsilon^\beta \xi) \hat{g}_{\pm 1}(\xi) = \mathbb{1}_{\{\epsilon > -B\}}(\xi) \hat{g}_{\pm 1}(\xi).
\]

The blow-up in Fourier space induces a rescaling in the physical space (see also [FLTW17, Equation 6.49]). Indeed, thanks to the identities \((176)\), we get that
\[
v^{(\pm)}_{\text{near}}(x) = \mathcal{F}^{-1} \left[ \hat{v}^{(\pm)}_{\text{near}}(\cdot) \right] (x) = \mathcal{F}^{-1} \left[ \hat{g}_{\pm 1}(\cdot) \right] (x) = \epsilon^\beta \mathcal{F}^{-1} \left[ \hat{g}_{\pm 1}(\cdot) \right] (\epsilon^\beta x) = \epsilon^\beta g_{\pm 1}(\epsilon^\beta x),
\]
which, in combination with Lemma 4.1(iii) gives
\[
v_{\text{near}}(x) = \epsilon^\beta e^{i\beta x} g_{+1}(\epsilon^\beta x) + \epsilon^\beta e^{-i\beta x} g_{-1}(\epsilon^\beta x).
\]

Following Lemma 4.1(vi) we also have that
\[
g_{+1}(\epsilon^\beta x) = g_{-1}(\epsilon^\beta x),
\]
whose structure goes back to modulation theory and the Ginzburg-Landau formalism (cf. Sch96a, §3, or the Ansatz in [KSM92, Equation (1.2)])

### 4.2. The functions \( g_{-1}(\cdot), g_{+1}(\cdot) \) and the topology of the space they are in.

According to Remark 1.6 we have
\[
\|v^{(\pm)}_{\text{near}}\|_{H^4(\mathbb{R})} \approx \|v_{\text{near}}\|_{L^2(\mathbb{R})} \approx \epsilon^\frac{3}{2} \|g_{-1}\|_{L^2(\mathbb{R})} + \epsilon^\frac{3}{2} \|\mathcal{G}_1\|_{L^2(\mathbb{R})},
\]
which shows that \( v_{\text{near}}(\cdot) \) respects the order in Proposition 3.1 which shows that \( v_{\text{far}} = v_{\text{far}}[\delta, \Omega, v_{\text{near}}] \) holds for \( \epsilon \) sufficiently small.

In fact, we have already derived the following result:

**Lemma 4.2** (The Ginzburg-Landau representation as a mapping). For any fixed \( \epsilon \geq 0 \), the representation ((33)) as a mapping
\[
(g_{+1}(\cdot), g_{-1}(\cdot)) \mapsto v_{\text{near}}(x) = v_{\text{near}}[g_{+1}(\cdot), g_{-1}(\cdot)](x) = \epsilon^\beta e^{i\beta x} g_{+1}(\epsilon^\beta x) + \epsilon^\beta e^{-i\beta x} g_{-1}(\epsilon^\beta x),
\]
is a continuous mapping from \( H^0_{\text{near},\epsilon^\beta}(\mathbb{R}) \times H^4_{\text{near},\epsilon^\beta}(\mathbb{R}) \) to \( H^4_{\text{near},\epsilon^\beta}(\mathbb{R}) \), and from \( H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) to \( H^s(\mathbb{R}) \), for all \( s \geq 1 \). Furthermore, using Continuity of the mapping \( H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), we conclude that this mapping is also continuous from \( H^2_{\text{near},\epsilon^\beta}(\mathbb{R}) \times H^2_{\text{near},\epsilon^\beta}(\mathbb{R}) \) to \( H^4_{\text{near},\epsilon^\beta}(\mathbb{R}) \).
We must highlight a few things concerning this result. First, the inequality (34) is misleading, for it give the impression that controlling the $L^2(\mathbb{R})$ norms of $g_{-1}(\cdot)$ and $g_{+1}(\cdot)$ is enough to simplify the problem. Second, the discussion of the mapping $(g_{+1}(\varepsilon^3 x), g_{-1}(\varepsilon^3 x)) \mapsto v_{\text{near}}(x)$ in the topology $H^2_{\text{near}, \varepsilon^{-\beta}}(\mathbb{R}) \times H^2_{\text{near}, \varepsilon^{-\beta}}(\mathbb{R})$ to $H^2_{\text{near}, \varepsilon^{-\beta}}(\mathbb{R})$ seems a bit far-fetched. This bias is a matter of convenience: among the many advantages of this norm, we show already in the next section that it is easier to control some “higher order” terms in this norm, and in section 6 we shall we see that we can approximate and reduce the problem to another one that is elliptic in nature. Therefore, henceforth we shall look for solutions to problem (11) under the condition that

$$(g_{-1}(\cdot), g_{+1}(\cdot)) \in H^2_{\text{near}, \varepsilon^{-\beta}}(\mathbb{R}) \times H^2_{\text{near}, \varepsilon^{-\beta}}(\mathbb{R}) \subset H^2(\mathbb{R}) \times H^2(\mathbb{R}).$$

4.3. Irrelevant nonlinearities and interaction Lemmas. Plugging $v_{\text{far}}(\cdot) = v_{\text{far}}[\delta, \Omega, v_{\text{near}}(\cdot)]$ into (25a), yields

$$m(\kappa; \mathcal{L}) \tilde{v}_{\text{near}}(\kappa) = \sum_{j=1}^{4} \mathring{p}(\kappa) \circ \mathcal{F} \left[ A^{(j)}[v_{\text{near}} + v_{\text{far}}[\delta, \Omega, v_{\text{near}}], u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] \right](\kappa).$$

The identity Proposition 4.1(i) allows us to rewrite this equation in terms of $z_{\text{near}}^{(\delta)}(\cdot)$; we obtain two equations which, in the blow-up variables $\kappa = \varepsilon^{\beta} \xi$, correspond to

$$m \left( \pm 1 + \varepsilon^{\beta} \xi; \mathcal{L} \right) \tilde{v}_{\text{near}}^{(\pm)}(\varepsilon^{\beta} \xi) = \sum_{j=1}^{4} \mathring{p}(\varepsilon^{\beta} \xi) \circ \mathcal{F} \left[ A^{(j)}[v_{\text{near}} + v_{\text{far}}[\delta, \Omega, v_{\text{near}}], u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] \right](\varepsilon^{\beta} \xi).$$

We shall rewrite this equation in such a way that (i) the linear terms in $v_{\text{near}}(\cdot)$ are written explicitly, and (ii) the non-homogeneous term (that stems from $A^{(4)}[v_{\text{far}}[\delta, \Omega, v_{\text{near}}](\cdot)]$) is highlighted.

The starting point consists of simplifying left hand side of (35); expanding and using the form of $\omega^2 = 1 + \delta \Omega$ as given by (12), it becomes

$$m \left( \pm 1 + \kappa; \mathcal{L} \right) \tilde{v}_{\text{near}}^{(\pm)}(\kappa) = -\omega^2 \varepsilon^{2\beta} \xi^2 \left[ 2 + \omega \varepsilon^{3\beta} \xi \right]^2 \tilde{g}_{\pm 1}(\xi) = -4 \varepsilon^{2\beta} \xi^2 \tilde{g}_{\pm 1}(\xi) + \omega \varepsilon^{3\beta} \xi \tilde{g}_{\pm 1}(\xi),$$

where $A^{(4)}[v_{\text{near}}, \varepsilon] = O \left( \left( \delta \varepsilon^{2\beta} \xi^2 + \varepsilon^{3\beta} \xi^2 + \varepsilon^{4\beta} \xi^2 \right) \tilde{g}_{\pm 1}(\xi) \right) = O \left( \left( \varepsilon^{2\beta} \xi^2 + \varepsilon^{3\beta} \xi^2 + \varepsilon^{4\beta} \xi^2 \right) \tilde{g}_{\pm 1}(\xi) \right)$, the latter equality being also a consequence of (12), namely, $\delta \approx \varepsilon$. Noting that $\nu \mapsto A^{(1)}[v, u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta]$ is linear and that $A^{(4)}[v, u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] = A^{(4)}[u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta]$ for it does not depend on $v(\cdot)$, we rewrite the equation as

$$-4 \varepsilon^{2\beta} \xi^2 \tilde{g}_{\pm 1}(\xi) = A^{(4)}[v_{\text{near}}, u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] + A^{(4)}[u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta],$$

where

$$A^{(1)}[v, u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] = A^{(1)}[u^{(\delta, \omega, \gamma)}_{\text{rolls}}, \delta] = 0.$$
Similarly, one obtains
\[
\|\varepsilon^{4\beta} \xi g_{\pm 1}(\xi)\|_{L^2(\mathbb{R})}^2 \lesssim |\varepsilon|^{8\beta} \int_{|\xi| \leq |\varepsilon|^{-\beta}} |\xi|^{8\beta} |\tilde{g}_{\pm 1}(\xi)|^2 \, d\xi \lesssim |\varepsilon|^{14(\tau + \beta)} \|g_{\pm 1}\|^2_{H^2(\mathbb{R})}.
\]

For the second part of this Proposition, we first need to understand how to control \(g_{\pm 1}(\cdot)\) controls \(v_{\text{near}}(\cdot)\). Later on we study how \(v_{\text{near}}(\cdot)\) and \(v_{\text{far}}(\cdot)\) interact through the nonlinearity.

**Lemma 4.4 (Nonlinear interaction Lemma 1).** Let \(\varepsilon_+ > 0\) be given in Proposition 3.1 and \(\varepsilon \in (0, \varepsilon_+)\). Assume \(v_{\text{near}}(\cdot)\) is of the form \(\mathcal{F}(x)\). Then,
\[
\|v_{\text{near}}\|_{H^1(\mathbb{R})} \lesssim \varepsilon^{\beta(p-\frac{1}{2})} (\|g_{\pm 1}\|_{L^2(\mathbb{R})} + \|g_{\pm 1}\|_{L^2(\mathbb{R})}), \quad p \in \mathbb{N} \setminus \{0\}.
\]

Furthermore, we can use the above inequality to improve the estimate given in Proposition 3.1(i), that is,
\[
\|v_{\text{near}}\|_{H^1(\mathbb{R})} \lesssim \varepsilon^{\beta(p-\frac{1}{2})} (\|g_{\pm 1}\|_{L^2(\mathbb{R})} + \varepsilon^{-2\tau} (\varepsilon^{3-\frac{4}{\beta}} + \varepsilon^{1+\frac{4}{\beta}} + \varepsilon^{2+\frac{2}{\beta}})),
\]
where \(\Lambda(\varepsilon, \tau) = O(\varepsilon^{-2\tau} + \varepsilon^{1+\frac{4}{\beta}-2\tau} + \varepsilon^{3\frac{2}{\beta}-2\tau})\). Furthermore, \(v_{\text{far}}(\cdot)\) scales in \(\varepsilon\) as
\[
\|v_{\text{far}}\|_{H^2(\mathbb{R})} \lesssim \left(\varepsilon^{2-2\tau} + \varepsilon^{2+\frac{2}{\beta} - 2\tau} + \varepsilon^{3\frac{2}{\beta} - 2\tau}\right)
\]

**Remark 4.5.** The main point of this proof is that \((38)\) is not a direct consequence of the inequality \((54)\), which immediately imply that
\[
\|v_{\text{near}}\|_{H^1(\mathbb{R})} \lesssim \|v_{\text{near}}\|^p_{H^1(\mathbb{R})} \lesssim O(\varepsilon^{\frac{2\beta}{\tau}}), \quad p \in \mathbb{N} \setminus \{0\}.
\]
Unfortunately, this bound is not good enough: as we will see later on, in order to obtain a reduced equation we need to derive better bounds.

**Proof.** This is analogous to the proof of [FLTW17 Lemma 6.9]. Using \((33)\),
\[
v_{\text{near}}^p(x) = \varepsilon^{p\beta} (g_{-1}(\varepsilon^\beta x) e^{-ix} + g_1(\varepsilon^\beta x) e^{ix})^p.
\]

In Fourier space, for any \(p \in \mathbb{N} \setminus \{0\}\) the function \(v_{\text{near}}^p(\cdot)\) corresponds to a convolution of band-limited functions, therefore it is also a band limited function (cf. [Bre11 Proposition 4.18]) and we conclude that \(\|v_{\text{near}}^p\|_{L^2} \lesssim \|v_{\text{near}}^p\|_{L^2}^p\). The result follows upon integration, using a change of variables:
\[
\|v_{\text{near}}^p\|_{H^1} \lesssim \|v_{\text{near}}^p\|_{L^2} \lesssim \varepsilon^{p\beta} \left(\|g_{-1}(\varepsilon^\beta \cdot)\|^p_{L^2} + \varepsilon^{p\beta} \left(\|g_{+1}(\varepsilon^\beta \cdot)\|^p_{L^2} \right)\right) \lesssim \varepsilon^{\beta(p-\frac{1}{2})} \|\varepsilon^{p\beta}(\cdot)\|_{L^2} + \varepsilon^{\beta(p-\frac{1}{2})} \|g_{\pm 1}(\cdot)\|_{L^2}.
\]
This proves \((37)\). In order to show \((38)\) and improve the bound in Proposition 3.1(i) we use the previous inequality, Lemma 3.4(ii) and the Sobolev Embedding \(H^4(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\):
\[
\|v_{\text{far}}[\delta, \Omega, v_{\text{near}}]\|_{H^4} = \|\delta(v_{\text{near}}, v_{\text{far}}, \delta, \Omega)\|_{H^4}
\]
\[
\leq CM_1 \left\{ \sum_{j=1}^{3} \varepsilon^{3-j-2\tau} \|v_{\text{near}}^j\|_{L^2} + \sum_{j=1}^{3} \varepsilon^{3-j-2\tau} \|v_{\text{far}}^j\|_{H^4} \right\}
\]
\[
\leq CM_1 \left\{ \sum_{j=1}^{3} \varepsilon^{3-j-2\tau} \|v_{\text{near}}^j\|_{L^2} + \sum_{j=1}^{3} \varepsilon^{3-j-2\tau} \|v_{\text{far}}^j\|_{H^4} \right\}
\]
\[
\leq 3CM_1 (1 + R^2) \left(\varepsilon^{2-2\tau} + \varepsilon^{1+\frac{4}{\beta}-2\tau} + \varepsilon^{3\frac{2}{\beta}-2\tau}\right) \|v_{\text{near}}\|_{H^4}
\]
\[
+ \sum_{j=1}^{3} \left( CM_1 \varepsilon^{3-j-2\tau+\alpha(j-1)} \right) v_{\text{far}}^j \|v_{\text{far}}\|_{H^4} + CM_1 (\varepsilon^{3-\frac{2}{\beta}-2\tau} + \varepsilon^{1+\frac{4}{\beta}-2\tau} + \varepsilon^{2+\frac{2}{\beta}-2\tau}).
\]

Thanks to the parameter conditions in \((27)\), we can absorb the term on the right hand side depending in \(v_{\text{far}}(\cdot)\), then getting
\[
\|v_{\text{far}}[\delta, \Omega, v_{\text{near}}]\|_{H^4} \lesssim \left(\varepsilon^{2-2\tau} + \varepsilon^{1+\frac{4}{\beta}-2\tau} + \varepsilon^{3\frac{2}{\beta}-2\tau}\right) \|v_{\text{near}}\|_{H^4} + \left(\varepsilon^{3-\frac{2}{\beta}-2\tau} + \varepsilon^{1+\frac{4}{\beta}-2\tau} + \varepsilon^{2+\frac{2}{\beta}-2\tau}\right),
\]
and this finishes the proof of \[ \text{(38)}. \] Now we prove \[ \text{(39)}. \]

\[
\| v_{\text{far}}[\delta, \Omega, v_{\text{near}}]\|_{H^4} = \| G[v_{\text{near}}, v_{\text{far}}, \delta, \Omega]\|_{H^4}
\leq CM_1 \left\{ \sum_{j=1}^{3} \epsilon^{3-j-2\tau} \| v_{\text{near}} \|_{L^2} + \sum_{j=1}^{3} \epsilon^{3-j-2\tau} \| v_{\text{far}} \|_{H^4} \right\} + \left( \epsilon^{3-\frac{\beta}{2}-2\tau} + \epsilon^{1+\frac{3\beta}{2}-2\tau} + \epsilon^{2+\frac{\beta}{2}-2\tau} \right)
\leq CM_1 \left\{ \sum_{j=1}^{3} \epsilon^{3-j-2\epsilon^{\beta(j-\frac{1}{2})}} \frac{1}{2} + \sum_{j=1}^{3} \epsilon^{3-j-2\tau} \| v_{\text{far}} \|_{H^4} \right\} + \left( \epsilon^{3-\frac{\beta}{2}-2\tau} + \epsilon^{1+\frac{3\beta}{2}-2\tau} + \epsilon^{2+\frac{\beta}{2}-2\tau} \right).
\]

Now, absorbing the term in \( v_{\text{far}}(\cdot) \) on the right hand side and using \( \text{(37)} \) we obtain

\[
\| v_{\text{far}}[\delta, \Omega, v_{\text{near}}]\|_{H^4} \lesssim \epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau}.
\]

Before we prove the next result, we recall a Lemma in \[ \text{FLTW17}. \]

**Lemma 4.6.** \[ \text{FLTW17} \text{Lemma 6.12} \] For all \( f(\cdot) \in L^2(\mathbb{R}) \) and any fixed \( \xi_0 \in \mathbb{R} \), we have

\[
\| \{ \xi \in \mathcal{E}^{-\tau} B(\xi_0) \} \mathcal{F}[f](\xi_0 + \xi^2) \|_{L^2(\mathbb{R})} \lesssim \frac{1}{\epsilon^{\frac{1}{2}}} \| f \|_{L^2(\mathbb{R})},
\]

followed by a change of variables \( \eta = \xi_0 + \epsilon^b \xi \).

**Lemma 4.7.** (Nonlinear interaction Lemma II). Recall the choice of parameters \( \text{\text{(27)}} \) of Proposition \( \text{3.1} \) that is, \( 0 < \tau < \frac{1}{2} \), and \( \beta \geq 1 \). Let \( f, g \in H^4 \) be given, with \( \| f \|_{H^4} = \mathcal{O}(\epsilon^{\frac{1}{2}}), \| f \|_{H^4} = \mathcal{O}(\epsilon^{\frac{1}{2}}) \), and \( \| g \|_{H^4} = \mathcal{O}(\epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau}) \). Then

(i) \( \max \{ \| f \|_{L^2}^2, \| f \|_{L^2}^2, \| g \|_{L^2}^2, \| g \|_{L^2}^2 \} = \mathcal{O}(\epsilon^{\frac{3\beta}{2}}) \).

(ii) \( \max \{ \| f \|_{L^2}^2, \| g \|_{L^2}^2 \} = \mathcal{O}(\epsilon^{3\beta-1}) \).

In particular, the result holds whenever \( f(\cdot) = v_{\text{near}}(\cdot) \) and \( g(\cdot) = v_{\text{far}}(\cdot) \).

**Proof.** We prove case by case, making repeated use of the Sobolev Embedding \[ \text{[18]}. \] First, we estimate each term in \( \text{(4.7)(i)} \):

\[
\| f \|_{L^2}^2 \lesssim \| f \|_{H^4}^2 \lesssim \epsilon^{\frac{3\beta}{2}} \left( \epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau} \right)
\lesssim \epsilon^{\frac{3\beta}{2}} \left( \epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau} \right),
\]

which is \( \mathcal{O}(\epsilon^{\beta}) \), due to \( \text{(27)} \). Similarly, we have

\[
\| g \|_{L^2}^2 \lesssim \| g \|_{H^4}^2 \lesssim \epsilon^{\beta} \left( \epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau} \right)^2
\lesssim \epsilon^{\frac{5\beta}{2}} \left( \epsilon^{4-4\tau-\beta} + \epsilon^{2-4\tau+\beta} + \epsilon^{3\beta-4\tau} + \epsilon^{6-3\beta-4\tau} \right) = \mathcal{O}(\epsilon^{2\beta}).
\]

Thus, (i) holds. Now we prove (ii).

\[
\| f \|_{L^2}^2 \lesssim \epsilon^{\frac{5\beta}{2}-1} \left( \epsilon^{3-2\tau-\frac{3\beta}{2}} + \epsilon^{2-2\tau+\frac{\beta}{2}} + \epsilon^{1-2\tau+\frac{3\beta}{2}} + \epsilon^{2\beta-2\tau} + \epsilon^{3-\frac{\beta}{2}-2\tau} \right).
\]

Thanks to \( \text{(27)} \), we have \( 3 - \frac{3\beta}{2} - 2\tau = \left( 3 - \frac{1}{2} - \frac{3\beta}{2} \right) + \left( \frac{1}{4} - 2\tau \right) > 0 \) and \( 4 - \frac{5\beta}{2} - 2\tau = \left( 4 - \frac{1}{4} - \frac{5\beta}{2} - 2\tau \right) + \left( \frac{1}{4} - 2\tau \right) > 0 \). Therefore, \( \| f \|_{L^2} = \mathcal{O}(\epsilon^{2\beta-1}) \). Once more, using \( \text{(27)} \), we obtain

\[
\| g \|_{L^2}^2 \lesssim \epsilon^{\frac{5\beta}{2}-1} \left( \epsilon^{5-4\tau-3\beta} + \epsilon^{3-4\tau+\beta} + \epsilon^{1+\frac{3\beta}{2}-4\tau} + \epsilon^{7-\frac{\beta}{2}-4\tau} \right) = \mathcal{O}(\epsilon^{2\beta-1}).
\]
since $5 - 4\tau - 3\xi^2 = (2 - 2\tau - \frac{3\xi^2}{2}) + (3 - 2\tau - \beta) > 0$ and $7 - 7\xi^2 - 4\tau = 2\left(\frac{1}{2} - 2\tau\right) + \left(\frac{13}{2} - 7\xi^2\right) > 0$, and we are done.

These results imply that we can use $v_{\text{near}}(\cdot)$ in the form (32) to approximate the equation (23a). The main result of this section is the following.

Proof. [of Proposition 4.3(ii)] For simplicity, we shall write, $\mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}[\delta, \Omega, v_{\text{near}}]]$ to denote $\mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}[\delta, \Omega, v_{\text{near}}], u_{\text{rolls}}, \varepsilon]$.

To begin with, we apply Lemma 4.6 getting

$$\|\langle \xi \in \varepsilon - \beta \mathcal{B} \rangle \mathcal{F} \mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}]|_{L^2(R^{d+1})} \leq \varepsilon^{-\frac{7}{2}} \|\mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}]|_{L^2(R^{d+1})} =: Q_j.$$ 

In the case $j = 1$ we make use of (39), of Lemma 4.7 and of the linearity of $v_{\text{far}}(\cdot) \mapsto \mathcal{N}(\cdot)[v_{\text{far}}]$ to get

$$Q_1 \lesssim \varepsilon^{-\frac{7}{2}} \|v_{\text{far}}|_{L^2(R^{d+1})} \lesssim \varepsilon^{-\frac{7}{2}} \left(\varepsilon^{2-2\tau + \frac{1}{2}} + \varepsilon^{1-2\tau + \frac{3\xi^2}{2}} + \varepsilon^{2\beta - 2\tau} + \varepsilon^{3 - \frac{7}{2} - 2\tau}\right)$$

$$\lesssim \varepsilon^{2\beta} \left(\varepsilon^{4 - 2\tau - 2\beta} + \varepsilon^{3 - 2\tau - \beta} + \varepsilon^{2 - 2\tau} + \varepsilon^{5 - 3\beta - 2\tau}\right),$$

which is $o(\varepsilon^{2\beta})$, due to the constraints in (27). The $j = 2$ the estimate is a consequence of Lemma 4.7 for

$$Q_2 \lesssim \varepsilon^{-\frac{7}{2}} \|\mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}]|_{L^2(R^{d+1})}$$

$$\lesssim \varepsilon^{1 - \frac{7}{2}} \max\{\|v_{\text{near}}v_{\text{far}}|_{L^2(R^{d+1})}, \|v_{\text{far}}^2|_{L^2(R^{d+1})}, \|v_{\text{near}}^2|_{L^2(R^{d+1})}\}$$

$$= \varepsilon^{1 - \frac{7}{2}} \max\{o(\varepsilon^{2\beta}), \varepsilon^{2\beta}\} \left(\|g^2_{1,1}|_{L^2(R^{d+1})} + \|g^2_{1,1}|_{L^2(R^{d+1})}\right)$$

$$= \max\{o(\varepsilon^{2\beta}), \varepsilon^{2\beta}\} \left(\|g^2_{1,1}|_{L^2(R^{d+1})} + \|g^2_{1,1}|_{L^2(R^{d+1})}\right).$$

Last, Lemma 4.7 also gives the estimate when $j = 3$:

$$Q_3 \lesssim \varepsilon^{-\frac{7}{2}} \|\mathcal{N}(\cdot)[v_{\text{near}} + v_{\text{far}}]|_{L^2(R^{d+1})}$$

$$\lesssim \varepsilon^{-\frac{7}{2}} \max\{\|v_{\text{near}}v_{\text{far}}|_{L^2(R^{d+1})}, \|v_{\text{near}}^2|_{L^2(R^{d+1})}, \|v_{\text{far}}^2|_{L^2(R^{d+1})}\}$$

$$= \varepsilon^{-\frac{7}{2}} \max\{o(\varepsilon^{2\beta}), \varepsilon^{2\beta}\} \left(\|g^2_{1,1}|_{L^2(R^{d+1})} + \|g^2_{1,1}|_{L^2(R^{d+1})}\right)$$

$$= \max\{o(\varepsilon^{2\beta}), \varepsilon^{2\beta}\} \left(\|g^2_{1,1}|_{L^2(R^{d+1})} + \|g^2_{1,1}|_{L^2(R^{d+1})}\right),$$

and this finishes the proof.

The last result of this section concerns the decay of $v_{\text{near}}(\cdot)$ and $v_{\text{far}}(\cdot)$ as $|x| \to +\infty$.

**Proposition 4.8** (Decay of $v_{\text{near}}(\cdot)$ and $v_{\text{far}}(\cdot)$ as $|x| \to +\infty$). For any fixed $\delta > 0$ and $j \in 0, \ldots, 3$, we have that

$$\lim_{|x| \to 0} \partial_x^2 v_{\text{near}}(x) = \lim_{|x| \to 0} \partial_x^2 v_{\text{far}}(x) = 0.$$ 

Proof. The result follows from classical Fourier analysis once we show that $\hat{v}_{\text{near}}(\cdot), \hat{v}_{\text{far}}(\cdot)$ are $L^1(R)$ functions (cf. [SW71] Theorem 1.2]). In the case of $\hat{v}_{\text{near}}(\cdot)$ this is straightforward: since the support of $\hat{v}_{\text{near}}(\cdot)$ is bounded we can use the fact that $L^1_{\text{loc}}(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R})$ to derive the result. The result for $\hat{v}_{\text{far}}(\cdot)$ follows from the fact that $v_{\text{far}}(\cdot) \in H^\delta(\mathbb{R})$ implies $\lim_{|x| \to \infty} v(x) = 0$ (cf. [Bre11] Corollary 8.9)); successive applications of this result to $\partial_x^2 v_{\text{far}}(\cdot) \in H^\delta(\mathbb{R})$, for $j \in 1, 2, 3$ then gives the result.

5. **Simplifications using a lemma of Fefferman, Thorpe and Weinstein, and matched asymptotics**

Before we start, recall that of the blow-up in (Fourier) parameter (33) gives the following structure to $v_{\text{near}}(\cdot)$:

$$v_{\text{near}}(x) = \varepsilon^\beta e^{ix} g^1(\varepsilon^\beta x) + \varepsilon^\beta e^{-ix} g^2(\varepsilon^\beta x).$$

We must solve (25a), which reads after its reformulation as (36),

$$\begin{cases}
-4\varepsilon^2\xi^2 g^{ij}(\xi) - \langle \nabla^{(\cdot)} \rangle_{\text{near}} \mathcal{F} \mathcal{N}(\cdot)[v_{\text{near}}, u_{\text{rolls}}^{(i\cdot)}, \delta] + \mathcal{N}(\cdot)[u_{\text{rolls}}^{(j\cdot)}, \delta] \varepsilon^\beta \xi = -\alpha_1(\cdot) + \alpha_2(\cdot) \\
-4\varepsilon^2\xi^2 g^{ij}(\xi) - \langle \nabla^{(\cdot)} \rangle_{\text{near}} \mathcal{F} \mathcal{N}(\cdot)[v_{\text{near}}, u_{\text{rolls}}^{(i\cdot)}, \delta] + \mathcal{N}(\cdot)[u_{\text{rolls}}^{(j\cdot)}, \delta] \varepsilon^\beta \xi = -\alpha_1(\cdot) + \alpha_2(\cdot),
\end{cases}$$

where, for simplicity, we write $\alpha_1(\cdot)_{\pm}$ and $\alpha_2(\cdot)_{\pm}$ to denote $\alpha_1(\cdot)[v_{\text{near}}|_{\varepsilon}, \delta, \varepsilon]$ and $\alpha_2(\cdot)[v_{\text{near}}|_{\varepsilon}, \delta, \varepsilon]$, respectively. In virtue of (2), one in the above equation we plug $\varepsilon = \varepsilon(\delta, \omega)$.

One of the main results in this section concerns the setting of a fixed value for $\beta$, which shall be chosen according to following reasoning:
Definition 5.1 (Parameter choice using matched asymptotics). Within the parameter range \(27\), \(\beta\) is a valid matching parameter whenever the quantity

\[
\lim_{\varepsilon \downarrow 0} \left( -4\varepsilon^{2\beta} \xi^2 \hat{g}_{\pm 1}(\xi) - \frac{\hat{p}(\pm 1)}{z_{\text{near}}} \ast \mathcal{F} \left[ \mathcal{N}^{(1)}[v_{\text{near}}, \omega_{\text{rolls}}, 1] + \mathcal{N}^{(4)}[v_{\text{rolls}}, 1] \right] (\varepsilon^2 \xi) \right)
\]

exists and is finite for any arbitrary \(g_{-1}(\cdot)\) and \(g_{+1}(\cdot)\) in \(H^2(\mathbb{R})\).

In this section we show that \(\beta = 1\) is the only valid parameter we can use. Furthermore, we show that the limit in Def. 5.1 not only exists, but it is also computable quantity, a fact to which most of this section is devoted to. As a first step, we state some results that will help us to understand the multiple scales nature of the problem \(40\).

5.1. Some auxiliary lemmas on two-scale interactions. Before we move forward we shall develop a few tools. We aim to simplify the equation \(40\), in which the most relevant terms consist of the non-homogeneous and the linear ones. We shall write

\[
I(\xi, k) = \hat{p}(\pm 1) \ast \mathcal{F} \left[ \mathcal{N}^{(1)}[v_{\text{near}}, \omega_{\text{rolls}}, 1] + \mathcal{N}^{(4)}[v_{\text{rolls}}, 1] \right] (\varepsilon^2 \xi) \quad \text{for} \quad j \in \{1, 4\}.
\]

The manipulation of these terms brings about several decompositions and reductions, splitting them in several pieces. These results are essentially given in the following multiscale analysis, which can be seen as a far/near (“slow/fast”) scale interaction estimate.

Lemma 5.2. [FLTW17, Lemma 6.5 and Lemma 6.11] Let \(f(x, \xi)\) and \(g(x)\) denote smooth functions of \((x, \xi) \in \mathbb{R} \times \mathbb{R}\) that are 1-periodic in \(x\). Let \(\Gamma(x, X)\) be defined for \((x, X)\) and such that the next two conditions hold:

\[
\Gamma(x + 1, X) = \Gamma(x, X), \quad \sum_{j=0}^{2} \int_{0}^{1} \| \partial_x^j \Gamma(x, X) \|^2_{L^2(\mathbb{R}^X)} dx < +\infty.
\]

Denote by \(\hat{\Gamma}(x, \omega)\) its Fourier transform with respect to the \(X\) variable. Then,

\[
\theta \frac{2\pi}{\sqrt{4\pi}} \int_{\mathbb{R}} g(x) \Gamma(x, \theta x) \overline{f(x, \theta \xi)} e^{-i\theta \xi x} dx = \sum_{n \in \mathbb{Z}} \int_{0}^{1} e^{2\pi inx} \hat{\Gamma}(x, \frac{2\pi n}{\theta} + \xi) \overline{f(x, \theta \xi)} g(x) dx
\]

Assume further that

\[
C_f := \sup_{0 \leq x \leq 1, |\omega| \leq \theta} |f(x, \omega)| < \infty, \quad D_g := \|g\|_{L^\infty([0,1])} < \infty, \quad \text{and} \quad \left\| \sup_{0 \leq x \leq 1} \hat{\Gamma}(x, \xi) \right\|_{H^1(\mathbb{R}_x)} < \infty.
\]

Define \(\mathcal{F}_n(\xi, \theta) := \int_{0}^{1} e^{2\pi inx} \hat{\Gamma}(x, \frac{2\pi n}{\theta} + \xi) \hat{f}(x, \theta \xi) g(x) dx\). Then, the following bounds hold

\[
\left\| \mathcal{F}_n(\xi, \theta) \right\|_{L^2(\mathbb{R}_x)} \lesssim C_f D_g \theta \left\| \sup_{0 \leq x \leq 1} \hat{\Gamma}(x, \xi) \right\|_{H^1(\mathbb{R}_x)}
\]

Specialized to our context, this result can be adapted in the following manner:

Corollary 5.3. Given \(f(x, \xi)\) and \(g(x)\) and \(\Gamma(x, X)\) as in the previous Lemma and \(\tau\) and \(\beta\) as in and satisfying the constraints \(27\). Then, assuming the properties \(H^1, H^2\) and using the notation in \(43\), the following bounds hold

\[
\left\| \mathcal{F}_n(\xi, \theta) \right\|_{L^2(\mathbb{R}_x)} \lesssim C_f D_g \theta \left\| \sup_{0 \leq x \leq 1} \hat{\Gamma}(x, \xi) \right\|_{H^1(\mathbb{R}_x)}
\]
Before we embark into these calculations, we derive another useful result, whose proof is straightforward.

**Proof.** The proof is a slight modification of the proof in [FLT17], using the fact that, whenever $|\xi| \leq \theta^{-\beta}$ and $|\theta| < 1$ we have

$$\frac{2\pi n}{\theta} + \xi \gtrsim \frac{|n|}{\theta}, \quad \text{and} \quad 1 + \left|\frac{2\pi n}{\theta} + \xi\right|^2 \gtrsim 1 + |n|^2.$$ 



**Lemma 5.4.** Recall from Lemma 3.3 that

$$u^{(\delta, \omega)}(\cdot) = v^{(\delta, \omega)}(\pi_{\text{near}}(x)), \quad \text{for} \quad v^{(\delta, \omega)}(\cdot) = \cos(x + \gamma) + \epsilon^2 h(x);$$

where $x \mapsto h(x)$ is a $2\pi$-periodic $L^2$ mapping. Consider $g(x)$ and $\Gamma(x, X)$ as in the previous Lemma. Then, for any $k, p \in \mathbb{N}$ we have

$$\int_0^1 e^{ip2\pi x} \left(u^{(\delta, \omega)}(2\pi x) \right)^k \, dx = \int_0^1 e^{ip2\pi x} (\cos(2\pi x + \gamma)) \, dx + O(\epsilon^2).$$

In passing, this simplification will allow for a characterization of the parameter $\beta$ that was introduced in (9) and has being carried out in all our computations since then.

**Observation 5.5.** In the next section the following identities will be used several times:

$$\int_0^1 \cos^4(2\pi z) \, dz = \frac{3}{8}, \quad \int_0^1 \cos^2(2\pi z) \, dz = \frac{1}{2}, \quad \int_0^1 \cos^2(2\pi z) \cos(4\pi z) \, dz = \frac{1}{4}.$$ 

5.2. Simplifying $I^{(4; \pm)}(\xi)$, or “when we finally find that $\beta = 1$”. In what follows, we analyze $I^{(4; \pm)}(\xi)$; the case $I^{(4; -)}(\xi)$ is similar. The dependence of this term in $\epsilon$ is twofold: first due to the scalings $\mathbf{1}(\epsilon^2)$ and $\kappa = \epsilon^2 \xi$; and second, due to the term $v^{(\delta, \omega)}(\cdot) = \epsilon^2 u^{(\delta, \omega)}(\cdot)$. First we break this interaction as

$$I^{(4; \pm)}(\xi) = \mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \int_{\mathbb{R}} \chi(\epsilon^2 x) (\chi(\epsilon^2 x) - 1) u^{(\delta, \omega)}(\cdot)(2\pi x) e^{-i(1 + \epsilon^2 \xi)z} \, dx$$

$$\quad \mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \int_{\mathbb{R}} \chi(2\pi e \beta z) \chi(2\pi e \beta z - 1) u^{(\delta, \omega)}(\cdot)(2\pi x) e^{-i(1 + \epsilon^2 \xi)z} \, dz$$

$$\mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \int_{\mathbb{R}} \chi(2\pi e \beta z) \chi(2\pi e \beta z - 1) u^{(\delta, \omega)}(\cdot)(2\pi x) e^{-i(1 + \epsilon^2 \xi)z} \, dz.$$ 

Once we set $Z = \theta z$, $\Gamma_1(z, Z) = \Gamma_1(Z) = \chi(Z) (\chi^2(Z) - 1)$, $f(z, \xi) = \chi(u^{(\delta, \omega)}(\cdot))(2\pi z)$, $g(z) = e^{-i2\pi z}$, we apply Lemma 5.2 and its Corollary 5.3.

$$S^{(4; +)}(\xi) = \frac{4\pi^2}{\theta} \mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \left[ \Gamma_1(\xi) \int_0^1 g(z) f(z) \, dz + \sum_{|n| \geq 1} \int_0^1 e^{i2\pi n z} \Gamma_1(\xi) \left( \frac{2\pi n}{\theta} + \xi \right) g(z) f(z) \, dz \right]$$

with $\|F_1\|_{L^2(\mathbb{R})} = O(\epsilon^3)$. Now we make use of Lemma 5.4 and of a change of variables to get

$$S^{(4; +)}(\xi) = \frac{4\pi^2}{\theta} \mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \left[ \Gamma_1(\xi) \int_0^1 e^{-2\pi i z} \cos^3(2\pi z + \gamma) \, dz + F_1 \right]$$

$$= \frac{4\pi^2}{\theta} \mathbf{1}_{\xi \in e^{-\gamma}B}(\epsilon^2 \xi) \left[ \Gamma_1(\xi) e^{i\gamma} \int_0^1 \cos^3(2\pi z) \, dz + F_1 \right]$$

$$= \frac{3\pi^2}{2\theta} \mathbf{1}_{\xi \in e^{-\gamma}B}(\xi) e^{i\gamma} \Gamma_1(\xi) + F_1.$$
where \(\|F_1\|_{L^2(\mathbb{R})} = O \left( \frac{1}{\nu^2} + \varepsilon^3 \right)\). Substituting back \(\theta = 2\pi \varepsilon^3\) we get
\[
S_{I}^{(4, \pm)}(\xi) = \frac{3\pi^2 \varepsilon^3}{4\pi \varepsilon^3} R_{\{ |\xi - \delta \cdot n \Omega| < 2\varepsilon \}} \tau \Gamma_{1}(\xi) + F_{1},
\]
in which \(\|F_{1}\|_{L^2(\mathbb{R})} = O (\varepsilon^{1-\beta} + \varepsilon^{3})\). At this point we have enough information to choose a value for \(\beta\); the reasoning lies in the following result:

**Lemma 5.6** (\(\beta\) is necessarily 1). Assume \(\beta\) satisfying \(\left[27\right]\). Then,
\[
\lim_{\varepsilon \to 0} \frac{\|S_{I}^{(4, \pm)}(\xi)\|_{L^2(\mathbb{R})}}{\varepsilon^{2\beta}} = \begin{cases} +\infty, & \text{when } \beta > 1, \\ \text{exists,} & \text{when } \beta = 1. \end{cases}
\]

**Proof.** First, notice that \(\tilde{\Gamma}_{1}(-\cdot) \in L^2(\mathbb{R})\), hence for a sufficiently large \(c > 0\) we have that \(\|I_{cB}(-\cdot)\|_{L^2(\mathbb{R})} > 0\). Thanks to the choice of parameters in \(\left[27\right]\), for a sufficiently small \(\varepsilon > 0\) we have that \(cB \subset \varepsilon^{\tau - \beta} B\). Thanks to the previous computations we have
\[
\frac{\|S_{I}^{(4, \pm)}(\xi)\|_{L^2(\mathbb{R})}}{\varepsilon^{2\beta}} \geq \frac{\varepsilon^3}{\varepsilon^{2\beta}} \|I_{cB}(-\cdot)\|_{L^2(\mathbb{R})},
\]
which proves the result when \(\beta > 1\). In the case \(\beta = 1\), we use the fact that \(I_{\{ \xi \in \varepsilon^{\tau - \beta} B \}}(\xi) \uparrow 1\), invoking the Lebesgue Dominated Convergence Theorem to conclude that \(\frac{S_{I}^{(4, \pm)}(\xi)}{\varepsilon^{2\beta}}\) converges to \(\frac{\pi}{\varepsilon^2} \tilde{\Gamma}_{1}(-\cdot)\) in \(L^2(\mathbb{R})\) as \(\varepsilon \downarrow 0\).

**Observation 5.7** (Matched asymptotics). According to the Definition \(\left[5, 1\right]\) and thanks to the previous result, henceforth we shall take \(\beta = 1\).

For the other terms, using computations in appendix A.1, we write
\[
\mathcal{T}^{(4, \pm)}(\xi) = I_{\{ \xi \in \varepsilon^{\tau - 1} B \}}(\xi) \left\{ \frac{3\pi^2 \varepsilon^3}{4} e^{\pm \xi \tau} \Gamma_{1}(\xi) - 4\pi \varepsilon^2 e^{\pm \xi \tau} \Gamma_{2}(\xi) - \delta^2 \Gamma_{1}^{(\pm)}(\xi) \right\} + .\mathcal{M}^{(4, \pm)}(\xi),
\]
where \(\Gamma_{2}(-\cdot)\) and \(\Gamma_{1}^{(\pm)}(-\cdot)\) have explicit expressions given in A.1. The bounds \(\|.\mathcal{M}^{(4, \pm)}(\xi)\|_{L^2(\mathbb{R})} = O (\varepsilon^{2})\) also hold, with \(\mathcal{M}^{(4, \pm)}(\xi) = I_{\{ \xi \in \varepsilon^{\tau - 1} B \}}(\xi)\). \(\mathcal{M}^{(4, \pm)}(\xi)\).

5.3. The term \(\mathcal{T}^{(1, \pm)}(\xi)\). Calculations in appendix A.2 show that
\[
\mathcal{T}^{(1, \pm)}(\xi) = I_{\{ \xi < 1 \}}(\xi) \mathcal{F} \left[ \delta^2 \mu(x) - 3 \left( \text{u}_{\text{rolls}}(\delta, \omega, \gamma) \right)^2 \right] \text{v}_{\text{hear}}(\xi) \right] (1 + \varepsilon \xi)
\]
gets simplified to
\[
\mathcal{T}^{(1, \pm)}(\xi) = I_{\{ \xi < 1 \}}(\xi) \left\{ -\delta^2 \int_{\mathbb{R}} \mu(X) [g_{\pm 1}(X)] e^{-\xi X} dX + 3\pi \varepsilon^2 g_{\pm 1}(\xi) + \frac{3\pi \varepsilon^2}{2} \tilde{\Gamma}_{1}(\xi) \right\} + .\mathcal{M}^{(1, \pm)}(\xi),
\]
where \(\mathcal{M}^{(1, \pm)}(\xi) = I_{\{ \xi < 1 \}}(\xi)\). \(\mathcal{M}^{(1, \pm)}(\xi)\), with bounds \(\|\mathcal{M}^{(1, \pm)}(\xi)\|_{L^2(\mathbb{R})} = O (\varepsilon^{2})\).

6. Approximation and Solvability of the Reduced Equation: Final Steps

In this section we give a proof of the parametrization of \(g_{-1}(\cdot)\) and \(g_{+1}(\cdot)\) by \((\delta, \Omega, \gamma)\), where \(\omega = \sqrt{1 + \delta \Omega}\). Before we embark in the proof a few auxiliary results are derived. Recall that our main concern is the resolution of \(\left[40\right]\) using \(\left[46\right]\) and \(\left[47\right]\) and dividing it by \(\varepsilon^2 > 0\), we can rewrite it in as equivalent system of reduced equations. Hence, this equation is simply written as
\[
\mathcal{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}] - I_{\{ \epsilon < 1 \}}(\xi) h_{\pm}(\xi) = 0\]  
\[
\mathcal{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}] [-\delta^2 \int_{\mathbb{R}} \mu(X) [g_{\pm 1}(X)] e^{-\xi X} dX + 3\pi \varepsilon^2 g_{\pm 1}(\xi) + \frac{3\pi \varepsilon^2}{2} \tilde{\Gamma}_{1}(\xi) \right] + .\mathcal{M}^{(1, \pm)}(\xi),
\]
where \(\mathcal{M}^{(1, \pm)}(\xi) = I_{\{ \xi < 1 \}}(\xi)\). \(\mathcal{M}^{(1, \pm)}(\xi)\), with bounds \(\|\mathcal{M}^{(1, \pm)}(\xi)\|_{L^2(\mathbb{R})} = O (\varepsilon^{2})\).

In the above, we substitute \(\varepsilon = \varepsilon(\delta, \omega)\)
\[
\mathcal{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}] := \begin{cases} \mathcal{R}^{(\delta, \Omega, -)}[g_{-1}, g_{+1}], & \text{if } \xi > \frac{\varepsilon}{\sqrt{1 + \delta \Omega}}, \\ \mathcal{R}^{(\delta, \Omega, +)}[g_{-1}, g_{+1}], & \text{if } \xi < -\frac{\varepsilon}{\sqrt{1 + \delta \Omega}}. \end{cases}
\]
with \(\mathcal{R}^{(\delta, \Omega, \pm)}[g_{-1}, g_{+1}] = -4\xi^2 g_{\pm 1}(\xi) - I_{\{ \epsilon < 1 \}}(\xi) \left[ 3\pi \tilde{\Gamma}_{1}(\xi) + \frac{3\pi}{2} g_{\pm 1}(\xi) + \mathcal{F} [\mu(\cdot)g_{\pm 1}(\cdot)] \right].\)
Since the operator $\mathcal{R}(\delta, \Omega)[g_{-1}, g_{+1}]$ is a multiplier, we can define the associated operator $\mathcal{R}(\delta, \Omega)[g_{-1}, g_{+1}]$ in physical space as

$$\mathcal{R}(\delta, \Omega)[g_{-1}, g_{+1}](x) := \mathcal{F}^{-1}\left[\mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}](\xi)\right],$$

(49)

an operator with domain $D(\mathcal{R}(\delta, \Omega)[\cdot, \cdot]) = H^2(\mathbb{R}) \times H^2(\mathbb{R})$. On the right hand side are non-homogeneous and nonlinear terms,

$$\mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}] + h_\ast(\xi) = \left(\mathcal{Q}(\delta, \Omega)^{-1}[g_{-1}, g_{+1}] \mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}]\right),$$

with

$$\mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}] = -\partial_\mu^{(\pm)}_1 + \partial_\mu^{(\pm)}_2 + \frac{\mathcal{A}(1; \pm)(\xi) + \mathcal{A}(4; \pm)(\xi)}{\varepsilon^2},$$

and

$$h_\ast(\xi) = \left(\begin{array}{c}
h^{(\pm)}_1(\xi) \\
h^{(\pm)}_2(\xi)
\end{array}\right) = -\ii_{\{\varepsilon^{-1}B\}}(\xi)\left(\begin{array}{c}
3\pi\varepsilon^{-r_1}\Gamma_1(\xi) - 4\pi\varepsilon^{-r_1}\Gamma_2(\xi) - \frac{\varepsilon^2}{2}\Gamma_3^{(\pm)}(\xi)
\end{array}\right).$$

(50)

We remark that $h_\ast(\xi) = \ii_{\{\varepsilon^{-1}B\}}(\xi)h_\ast(\xi)$. Thanks to Proposition 6.3 estimates (46) and (47), we have

$$\|\mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}]\|_{L^2(\mathbb{R})} = \mathcal{O}(1) + \mathcal{O}\left(\|g_{-1}\|_{L^2(\mathbb{R})} + \|g_{+1}\|_{L^2(\mathbb{R})}\right);$$

(51)

(one says that $j(\varepsilon) = o(1)$ whenever $\lim j(\varepsilon) = 0$). After this preamble, we are able to state the one of the main result of this section, which consists of another application of the Lyapunov-Schmidt reduction method; Theorem 6.5 will be obtained as a direct consequence of it.

**Proposition 6.1.** Assume (H1)-(H2) and parameters $\tau$ satisfying the constraints in (27) (fixing $\beta = 1$). Then, there exists a small $\delta_\ast > 0$ and a continuous mapping

$$(0, \delta_\ast) \times (-\delta_\ast, \delta_\ast) \ni (\delta, \Omega) \mapsto \left(g_{-1}^{(\delta, \Omega)}(\cdot), g_{+1}^{(\delta, \Omega)}(\cdot)\right) \in H^2_{\text{near}, \varepsilon^{-1}}(\mathbb{R}) \times H^2_{\text{near}, \varepsilon^{-1}}(\mathbb{R}),$$

which satisfies (48), hence (40). Furthermore, these functions are band-limited, that is

$$g_{\pm 1}^{(\delta, \Omega)}(\cdot) \in H^2(\mathbb{R}),$$

and further, $\text{supp}(g_{\pm 1}^{(\delta, \Omega)}) \subseteq \varepsilon^{-1}B$, where $\varepsilon = \varepsilon(\delta, \Omega)$, as given by (3), and (H3).

**Remark 6.2.** We make a brief digression before we tackle this problem; our discussion is similar to that in Remark 6.3. First, recall that we seek for band limited functions $g_{-1}(\cdot), g_{+1}(\cdot)$ as a solution to this problem. Thus, it is natural to investigate the invertibility of the mapping $(\tilde{g}_{-1}(\cdot), \tilde{g}_{+1}(\cdot)) \mapsto \mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}] : H^2_{\text{near}, \varepsilon^{-1}}(\mathbb{R}) \times H^2_{\text{near}, \varepsilon^{-1}}(\mathbb{R}) \to L^2(\mathbb{R})$. An approach to this problem could go along the following line of reasoning: the parameter choice in (27) implies that $\lim_{\varepsilon \downarrow 0} \ii_{\{\varepsilon^{-1}B\}}(\xi)\uparrow 1$. In this fashion, as we take $\varepsilon \downarrow 0$ and use $\|\mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}]\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^2)$, we formally obtain on the left hand side that

$$\lim_{\varepsilon \downarrow 0} \mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}] \approx \mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}] := \left(\begin{array}{c}
\mathcal{R}(\delta, \Omega)^{-1}[\tilde{g}_{-1}, \tilde{g}_{+1}] \\
\mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}]
\end{array}\right),$$

(52)

where $\approx$ should read as a formal limit

$$\mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}] = -4\ii^2 g_{\pm 1}(\xi) = \frac{3\pi}{2} g_{\pm 1}(\xi) - \mathcal{F}[\mu(\cdot)g_{\pm 1}(\cdot)](\xi).$$

Hence, we could plug this operator in (48), solving instead

$$\mathcal{R}(\delta, \Omega)[\tilde{g}_{-1}, \tilde{g}_{+1}] = \ii_{\{\varepsilon^{-1}B\}}(\xi)h_\ast(\xi) = \mathcal{Q}(\delta, \Omega)[g_{-1}, g_{+1}];$$

(54)

\footnote{It must be highlighted that the operator $\mathcal{R}(\delta, \Omega)[g_{-1}, g_{+1}]$ in fact does not depend on $\Omega$. In spite of the risk of being misleading, we kept the notation in this form for consistency reasons.}
to find a solution \((g_{-1}, g_{+1}) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})\) which, posteriorly, we truncate in Fourier space (for we want \((g_{-1}, g_{+1}) \in H^2_{\text{near,} \varepsilon^{-1}} \times H^2_{\text{near,} \varepsilon^{-1}}\)). The main issue with this argument is that, for some \(f(\cdot) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})\), we might have

\[
\|\{\xi \in e^{-\varepsilon B}\}(\xi) \left(\left(\mathcal{R}^{(0, \Omega)}\right)^{-1} \{f\}(\xi) - \left(\mathcal{R}^{(0, \Omega)}\right)^{-1} \{\|\{\xi \in e^{-\varepsilon B}\}(\cdot) f(\cdot)\}(\xi)\right).\]

A more effective strategy uses \(\mathcal{R}^{(0, \Omega)}[g_{-1}, g_{+1}]\) to approximate the operator \(\mathcal{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}]\) rather than directly dealing with solutions.

We appeal to the formal construction of the operator \((g_{-1}, g_{+1}) \mapsto \mathcal{R}^{(0, \Omega)}[g_{-1}, g_{+1}]\) to rewrite the system \([48]\) in an equivalent form as

\[
\mathcal{R}^{(0, \Omega)}[g_{-1}, g_{+1}] + (\mathcal{R}^{(\delta, \Omega)} - \mathcal{R}^{(0, \Omega)})(g_{-1}, g_{+1}) + \|\{\xi \in e^{-\varepsilon B}\}(\xi) \mathcal{h}_*(\xi)\| = Q(\delta, \Omega, \pm)[g_{-1}, g_{+1}].\]  

(55)

Three ingredients are involved in the resolution of \(\mathcal{R}^{(0, \Omega)}\): (i) proving that the operator \(\mathcal{R}^{(\delta, \Omega)}[g_{-1}, g_{+1}]\) is well approximated by \(\mathcal{R}^{(0, \Omega)}[g_{-1}, g_{+1}]\), and that the latter is invertible; (ii) showing that solutions to problem \([55]\) are indeed band-limited; and finally, (iii) showing that \(h_*(\cdot)\) can be taken with a “small” \(\|\cdot\|_{L^2(\mathbb{R})}\) norm.

**Lemma 6.3** \((\mathcal{R}^{(0, \Omega)}\) is a good, invertible, approximation). \(\mathcal{R}^{(\delta, \Omega)}\) be defined as in \([49]\), while we assume \(\mathcal{R}^{(0, \Omega)}\) do be defined as the limiting operator \([54]\, with \(\mathcal{D}(\mathcal{R}^{(0, \Omega)}) = H^2(\mathbb{R}) \times H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})\).

The following properties hold:

i. There exists a constant \(C > 0\) (independent of \(\varepsilon, \delta, \Omega\)) such that

\[
\left\|(\mathcal{R}^{(\delta, \Omega)} - \mathcal{R}^{(0, \Omega)})(g_{-1}, g_{+1})\right\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim \varepsilon^{1-\tau} \|g_{\pm 1}\|_{H^2(\mathbb{R})}.
\]

ii. Assume that \(g_{\pm 1}(\cdot) \in H^2(\mathbb{R})\). Then

\[
\left\|(\mathcal{R}^{(\delta, \Omega)} - \mathcal{R}^{(0, \Omega)})(g_{-1}, g_{+1})\right\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim \varepsilon^{1-\tau} \|g_{\pm 1}\|_{H^2(\mathbb{R})}.
\]

iii. The following inequality holds:

\[
\left\|(\mathcal{R}^{(\delta, \Omega)} - \mathcal{R}^{(0, \Omega)})(g_{-1}, g_{+1})\right\|_{H^2(\mathbb{R}) \times H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \times H^2(\mathbb{R})} < 1, \quad \varepsilon \in (0, \varepsilon_0).
\]

**Proof.** The inequality in (iii) is a direct consequence of the results in (i) and (ii):

\[
\left\|(\mathcal{R}^{(\delta, \Omega)} - \mathcal{R}^{(0, \Omega)})(g_{-1}, g_{+1})\right\|_{H^2(\mathbb{R}) \times H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \times H^2(\mathbb{R})} \lesssim \varepsilon^{1-\tau} \|g_{\pm 1}\|_{H^2(\mathbb{R})}.
\]

In order to prove (i) we first observe that the operator \(v \mapsto \mathcal{R}^{(0, \Omega)}[v]\) described in \([54]\) converges to the following endstates

\[
v \mapsto \mathcal{R}^{(0, \Omega)}(x = -\infty)[v] = \begin{pmatrix} 4 \partial_x^2 v_1 - (3\pi - 1) v_1 - 3\pi v_2 \\ 4 \partial_x^2 v_2 - (3\pi + 1) v_1 - 3\pi v_2 \end{pmatrix}, \quad \text{as} \ x \rightarrow -\infty;
\]

\[
v \mapsto \mathcal{R}^{(0, \Omega)}(x = +\infty)[v] = \begin{pmatrix} 4 \partial_x^2 v_1 - (3\pi + 1) v_1 - 3\pi v_2 \\ 4 \partial_x^2 v_2 - (3\pi - 1) v_1 - 3\pi v_2 \end{pmatrix}, \quad \text{as} \ x \rightarrow +\infty.
\]

We claim that both these constant coefficient operators are coercive. Indeed, whenever \(f = (f_1, f_2) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})\),

\[
\left\langle \mathcal{R}^{(0, \Omega)}(x = -\infty)[f], f \right\rangle_{L^2(\mathbb{R})} \approx \mathcal{F} \left[ \mathcal{R}^{(0, \Omega)}(x = -\infty)[f] \right]_{L^2(\mathbb{R})} \geq 4 \left( \|\partial_x f_1\|_{L^2(\mathbb{R})}^2 + \|\partial_x f_2\|_{L^2(\mathbb{R})}^2 \right) + (3\pi^2) \left( \|f_1\|_{L^2(\mathbb{R})}^2 + \|f_2\|_{L^2(\mathbb{R})}^2 \right) + 3\pi \frac{\text{Re} \left( \int_{\mathbb{R}} f_1(x) \overline{f_2(x)} dx \right)}{3\pi^2 - \frac{3\pi^2}{2} - 1} \|f(\cdot)\|_{H^2(\mathbb{R}) \times H^2(\mathbb{R})}^2.
\]
and the result follows. The same holds for $R_{(0, \infty)}^{(0)}_{(x=-\infty)}$. Applying the Lax-Milgram Theorem (cf. [LM72 Chapter 2, Section 9]) we assert the invertibility of the asymptotic operators $R_{(x=-\infty)}^{(0)}: H^2(\mathbb{R}) \to L^2(\mathbb{R})$. Hence, arguing as in [RS95 §3] (or similarly, as in [MS18 Proposition 4.3]) we conclude that the operator $R_{(0, \infty)}^{(0)}: H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fredholm operator. In order to prove the invertibility of this operator it suffices to show that its kernel and cokernel are trivial. In both cases these reasoning is the same: a similar calculation shows that the operator and its adjoint are coercive operators, and we conclude that these are trivial spaces. Hence, $\text{Ker}(R_{(0, \infty)}^{(0)})) = \text{coker}(R_{(0, \infty)}^{(0)}) = \{0\}$, and the operator $v \mapsto R_{(0, \infty)}^{(0)}[v]$ is invertible.

With regards to property (ii), we first note that

$$
(R_{(0, \infty)}^{(0)} - R_{(0, \infty)}^{(0)}) \tilde{g}_{\pm 1}(\xi) = \left(1 - \mathbb{1}_{(\varepsilon^{-1} \rightarrow 1)}(\xi)\right) \left[R_{(0, \infty)}^{(0)} \tilde{g}_{\pm 1}(\xi)\right] = \left(-\frac{3\pi}{2} \chi\left(\frac{\xi}{\varepsilon}\right) + 3\pi^2\right) \left[\begin{array}{c}
R_{(0, \infty)}^{(0)} \tilde{g}_{+1}(\xi) \\
R_{(0, \infty)}^{(0)} \tilde{g}_{-1}(\xi)
\end{array}\right]
$$

Now notice that

$$
\left\| \left(1 - \mathbb{1}_{(\varepsilon^{-1} \rightarrow 1)}(\xi)\right) \tilde{g}_{+1}(\xi) \right\|^2_{L^2(\mathbb{R})} \leq \int_{|\xi| > \varepsilon^{-1}} |\tilde{g}_{+1}(\xi)|^2 d\xi \leq \int_{|\xi| > \varepsilon^{-1}} \frac{1}{|\xi|^2} |\tilde{g}_{+1}(\xi)|^2 d\xi \leq \varepsilon^{2(1-\tau)} \|g_{\pm 1}\|^2_{H^2(\mathbb{R})},
$$

which concludes the proof of (ii).

The following result is our first step to validate this approach.

**Lemma 6.4.** (Persistence of band-limited properties under approximation) Any functions $g_{-1}(\cdot), g_{+1}(\cdot) \in H^2(\mathbb{R})$ satisfying (55) are in $H^2_{\text{near}, \varepsilon^{-1} \rightarrow 1}(\mathbb{R})$, that is,

$$g_{\pm 1}(\cdot) \in H^2(\mathbb{R}),$$

and further, $\text{supp}(\tilde{g}_{\pm 1}) \subset \varepsilon^{-1} B$.

**Proof.** It is clear that both systems are equivalent. Hence, one can rewrite system (55) back to the form (48) and multiply it by $1 - \mathbb{1}_{\left(\frac{\xi}{\varepsilon} \rightarrow \infty\right)}(\xi)$, obtaining

$$4 \left(1 - \mathbb{1}_{(\varepsilon^{-1} \rightarrow 1)}(\xi)\right) \xi^2 \mathcal{F} \left[\begin{array}{c}
g_{-1}(\xi) \\
g_{+1}(\xi)
\end{array}\right] = \left(0
\right)$$

which readily implies that $\mathcal{F}[g_{\pm 1}(\cdot)] = \tilde{g}_{\pm 1}(\cdot)$ are both supported at $\varepsilon^{-1} B$.

**Lemma 6.5 (Attaining small norms for $h_\pm(\cdot)$ by choosing $\chi_\pm(\cdot)$.** Recall $h_\pm(\cdot) = (h_{\pm}^{\text{even}}, h_{\pm}^{\text{odd}})$ from (50), where

$$h_{\pm}^{\pm}(\xi) = \frac{3\pi}{4} e^{\mp i\gamma} \Gamma_1(\xi) - 4\pi e^{\pm i\gamma} \Gamma_2(\xi) - \frac{\delta^2}{\varepsilon^2} e^{\mp i\gamma} \Gamma_3(\xi).$$

For any given number $c_0 > 0$, we can choose $\chi(\cdot)$ the latter satisfies the ODE

$$-4\pi \partial^2_x \chi(\cdot) + \frac{3\pi}{4} \left(\chi(\cdot) - \chi^3(\cdot) - c_0 \partial_x \chi(\cdot) = 0, \lim_{x \to -\infty} \chi(x) = 1, \lim_{x \to +\infty} \chi(x) = 0,
$$

for any $0 < c < c_0$. Furthermore, this choice can be done in such a way that following bounds hold,

$$\|h_{\pm}(\cdot)\|_{L^2(\mathbb{R})} \leq c_0.$$

**Proof.** As shown in appendix B, the existence of a solution $\chi(\cdot)$ to the ODE above satisfying the asserted spatial asymptotics in (57) is known. In this manner, choosing a $\chi(\cdot)$ with this quality, we can exploit the explicit formulas for $\Gamma_{1,2}$ (see (52)) to get

$$\frac{3\pi}{4} \Gamma_1(\xi) - 4\pi \Gamma_2(\xi) = \mathcal{F} \left[-4\pi \partial^2_x \chi(\cdot) + \frac{3\pi}{4} \left(\chi(\cdot) - \chi^3(\cdot)\right)\right](\xi) = \mathcal{F} [c \partial_x \chi(\cdot)](\xi).$$

Thus, one can write

$$h_{\pm}(\xi) = \mathbb{1}_{(\varepsilon^{-1} \rightarrow 1)}(\xi) \mathcal{F} [c \partial_x \chi(\cdot)](\xi) - \frac{\delta^2}{\varepsilon^2} \Gamma_3(\xi).$$

We shall estimate both terms on the right hand side. Beginning with the first one: it is shown in appendix B we have that $c \partial_x \chi(\cdot) \in L^2(\mathbb{R})$, and $\|c \partial_x \chi(\cdot)\|_{L^2(\mathbb{R})} \lesssim |c|$. Therefore, applying Plancherel’s identity successively, we obtain

$$\|\mathcal{F} [c \partial_x \chi(\cdot)]\|_{L^2(\mathbb{R})} \lesssim \|c \partial_x \chi(\cdot)\|_{L^2(\mathbb{R})} \lesssim |c| < c_0,$

which can be made as small as necessary by taking $0 < c_0 \ll 1$. With regards to the second term, we begin by invoking (H2) to bound $\frac{\delta^2}{\varepsilon^2} \lesssim 16$. Recall that $\Gamma_{1,2}(\cdot) = \mathbb{1}_{(0, \infty)}(\cdot) \chi(\cdot) e^{-i\gamma} \cos(\xi \pm \gamma)$, we exploit the fact that solutions to the ODE (57) are translation invariant and exponentially decaying to 0 as $x \to +\infty$:
considering solutions to the above ODE of the form \( \chi(· + \tau_x) \), for any given \( c_0 > 0 \) we can choose \( \tau_x^* > 0 \) sufficiently large so that
\[
\frac{1}{2\pi} \| \Gamma_3^{(+)} \|_L^2(\mathbb{R}) = \| \Gamma_3^{(-)} \|_L^2(\mathbb{R}) \leq \int_0^{+\infty} |\chi(x + \tau_x)|^2 dx \leq c_0^2, \quad \text{for all} \quad \tau_x > \tau_x^*;
\]
combining both estimates, the result follows.

**Remark 6.6** (On the use of far/near (spatial) decompositions with numerical analysis purposes). The fact that the bounds in Lemma 6.3 are independent of \( \varepsilon \) is essential in the choice of \( \chi(·) \) in Lemma 6.5. This is what makes the combination of the far/near (spatial) decomposition a technique that can consist of an interesting tool in numerical analysis of bifurcation in extended domains. For other approaches, see MS15 LS17 or 8.5.

We are ready to put things together, combining the previous Lemmas to give a proof Proposition 6.1

**Proof.** [of Proposition 6.1] Due to the equivalence between \( 18 \) and \( 55 \), we can work with the latter. Recall that we can write \( \varepsilon = \varepsilon(\delta, \nu_{\text{near}}(·)) \), as a continuous function of its parameters; throughout the analysis, \( \varepsilon \) should be considered in this form. Moreover, thanks to observations in \( H2 \) and Corollary 3.3 we can replace \( \delta \) for \( \varepsilon \) in the analysis, for they are equivalent. In the following, thanks for property \( H2 \), we shall use \( \Omega \) instead of \( \omega = \omega(\delta, \Omega) = \sqrt{1 + \delta \Omega} \). Therefore, throughout the proof, we shall use \( (\delta, \Omega) \) instead of the parametrization \( (\delta, \Omega) \).

Now we handle the rest as follows: we can use the operator \( \tilde{R}(\delta, \Omega)^{-1} \) obtained in Lemma 6.3 to act on \( 55 \), the equation we wish to find a solution for. Denoting the identity mapping as \( (\tilde{g}_{-1}, \tilde{g}_{+1}) \mapsto \text{Id}[\tilde{g}_{-1}, \tilde{g}_{+1}] \), we rewrite the outcome as
\[
\text{Id}[\tilde{g}_{-1}, \tilde{g}_{+1}] = \mathcal{M}[\tilde{g}_{-1}, \tilde{g}_{+1}, \delta, \Omega],
\]
where
\[
\mathcal{M}[\tilde{g}_{-1}, \tilde{g}_{+1}, \delta, \Omega] := \left( \tilde{R}(\delta, \Omega)^{-1} \left[ \left( \tilde{R}(\delta, \Omega) - \tilde{R}(0, \Omega) \right) [\tilde{g}_{-1}, \tilde{g}_{+1}] - \| (\varepsilon^2 - 1)B(\xi) \hat{H}_1(\xi) + Q(\delta, \Omega, \pm)[\tilde{g}_{-1}, \tilde{g}_{+1}] \right] \right).
\]
Next, we use Lemma 6.3 is several ways: first, Lemma 6.3 implies that \( (\tilde{g}_{-1}, \tilde{g}_{+1}) \mapsto \tilde{R}(0, \Omega)^{-1} \circ \left( \tilde{R}(\delta, \Omega) - \tilde{R}(0, \Omega) \right) [\tilde{g}_{-1}, \tilde{g}_{+1}] \) has a \( O(\varepsilon^2) \) Lipschitz constant; the second term can be made as small as we would like in virtue of both Lemma 6.5 and Lemma 6.3; for the bounds on the inverse mapping \( \tilde{R}(0, \Omega)^{-1} \) are independent of \( (\delta, \Omega) \); for the last term, we also rely on the same bounds, allied to the estimates \( 51 \).

Thus, there exists a \( r_\star > 0 \) sufficiently small such that the set
\[
\{ g_{-1}(·), g_{+1}(·) \in H^2(\mathbb{R}) \| g_{-1}(·) \|_{H^2(\mathbb{R})} + \| g_{-1}(·) \|_{H^2(\mathbb{R})} \leq r_\star \}
\]
is mapped to itself through the mapping \( (g_{-1}, g_{+1}) \mapsto \mathcal{M}[g_{-1}, g_{+1}, \delta, \Omega] \), on which the same mapping is a (uniform) contraction. Thus, another application of the Contraction Mapping Theorem implies that there exists a solution to this problem, parametrized by \( (\delta, \Omega) \), that is, there exists a \( 0 < \delta_\star \ll \frac{1}{\varepsilon} \) for which
\[
(0, \delta_\star) \times (-\delta_\star, \delta_\star) \ni (\delta, \Omega) \mapsto \left( g_{-1}^{(\delta, \Omega)}(·), g_{+1}^{(\delta, \Omega)}(·) \right),
\]
with
\[
\text{Id}[g_{-1}^{(\delta, \Omega)}(·), g_{+1}^{(\delta, \Omega)}(·)] = \mathcal{M}[g_{-1}^{(\delta, \Omega)}(·), g_{+1}^{(\delta, \Omega)}(·), \delta], \quad \text{for all} \quad (\delta, \Omega) \in (0, \delta_\star) \times (-\delta_\star, \delta_\star).
\]
Finally, we invoke Lemma 6.4 to conclude that \( (g_{-1}^{(\delta, \Omega)}(·), g_{+1}^{(\delta, \Omega)}(·) \in H^2_{\text{near}, \varepsilon^2 - 1}(\mathbb{R}) \times H^2_{\text{near}, \varepsilon^2 - 1}(\mathbb{R}) \). Let us now study the regularity of these parametrization; initially, notice that all the bounds obtained in Lemma 6.3 and Lemma 6.5 are uniform in \( 0 < \varepsilon \ll 1 \). Since we have pointwise convergence of these functions in \( \varepsilon \), we can argue using Lebesgue Dominated Convergence Theorem to conclude that
\[
(0, \delta_\star) \times (-\delta_\star, \delta_\star) \ni (\delta, \Omega) \mapsto \mathcal{M}[g_{-1}, g_{+1}, \delta, \Omega] \in H^2(\mathbb{R}) \times H^2(\mathbb{R})
\]
is a continuous mapping for any fixed \( g_{-1}(·), g_{+1}(·) \). Another application of the Contraction Mapping Theorem implies the continuity of the mapping \( 55 \), and this finishes the proof.
7. Wavenumber selection – proof of Theorem 1.15

According to results in the previous section, in combination with and Lemma 7.2, we have

\[(0, \delta, \gamma) \times (-\delta, \delta) \ni (\delta, \Omega) \Rightarrow v_{\text{near}}^{(\delta, \Omega)}(\cdot) \in X_{\text{near}, \xi}^4 \subset H^4(\mathbb{R}), \]

in a continuous fashion. In the equation above and throughout this section, we write \(\varepsilon = \varepsilon(\delta, \Omega)\); we shall also fix \(\gamma \in \mathbb{R}\). Similarly, allying this result to that of Proposition 3.1 gives the continuity of

\[(0, \delta, \gamma) \times (-\delta, \delta) \ni (\delta, \Omega) \Rightarrow v_{\text{far}}^{(\delta, \Omega)}(\cdot) = v_{\text{far}}(\delta, \Omega, v_{\text{near}}^{(\delta, \Omega)}(\cdot)) \in X_{\text{far}, \xi}^4 \subset H^4(\mathbb{R}), \]

Recall from (H2) that we can write \(\omega = \sqrt{1 + \delta^2}\), we shall write any parametrization in \((\delta, \omega)\) as a parametrization in \((\delta, \Omega)\); however, to avoid confusion, we shall write the parametrization in (2) as

\[(\delta, \Omega, \gamma) \Rightarrow v_{\text{rolls}}^{(\delta, \sqrt{1 + \delta^2})}. \]

Thus, we can write the Ansatz (2) as

\[(x, \delta, \Omega) \Rightarrow U(x) = v_{\text{near}}^{(\delta, \Omega)}(x) + v_{\text{far}}^{(\delta, \Omega)}[x] + \chi(\varepsilon x)u_{\text{rolls}}^{(\delta, \sqrt{1 + \delta^2})}(x), \]

which gives a solution to problem (11) with the qualities we were after, as stated in 1.2; see Corollary 7.4 below.

After these clarifications, we are now headed to the last step of our proof, where we show that the solution we have found is indeed a solution in the sense of Definition 1.2. In this case, fixing \(\gamma\), we have a full family of solutions parametrized by two parameters \((\delta, \Omega)\). Now, we unfold all the reductions we performed, and plug \(v(\cdot)\) back into problem (1). In this new context, an interesting phenomenon happens: the conservation laws imposed by the Hamiltonian structure of the problem impose a severe parameter restriction, namely, we obtain a selection mechanism, implying that only one parameter is necessary in the characterization of solutions to problem (11). Moreover, these solutions have the qualities we are concerned with, as stated in Definition 1.2.

Lemma 7.1. (Regularity) Writing \(\varepsilon = \varepsilon(\delta, \Omega)\) and \(v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot)\) as before, the following regularity condition holds

\[x \mapsto U(x) = v(x) + \chi(\varepsilon x)u_{\text{rolls}}^{(\delta, \sqrt{1 + \delta^2})}(x) \in \mathcal{C}^{(\infty)}(\mathbb{R} \setminus \{0\}) \cap \mathcal{C}^{(r)}(\mathbb{R}). \]

Proof. With regards to regularity, the embedding \(H^4(\mathbb{R}) \hookrightarrow \mathcal{C}^{(r)}(\mathbb{R})\) gives part of the result, while smoothness in \(\mathbb{R}\{0\}\) (i.e., away from the quenching front) is proved using either elliptic regularity [LM72 Chapter 2, §3.2] (or using [H93, Corollary 3.1.6]).

We can rewrite the constraint in (12) as

\[-\delta^2 u_{\text{near}}^2(x) = -\delta^2 = \mathcal{H}^{(r)}[u_{\text{near}}^{(\delta, \sqrt{1 + \delta^2})}(\cdot)] = \mathcal{H}^{(r)}[0].\]

As we shall see next, this property is the main ingredient in the selection mechanism that relates \(\delta\) to the wavenumber \(\Omega\) of the rolls \(v_{\text{rolls}}^{(\delta, \Omega)}\). But first we need to expand these functions in powers of \(\delta\) in order to understand how they can be approximated.

Lemma 7.2. Assume (H1)-(H3), and choose parameters \(\tau\) as in (27) of Proposition 3.1. Write the Ansatz as in (62) and (63). Then, the following approximations hold:

\[(i) \quad U^2(x) \bigg|_{x=0} = o(\varepsilon) = o(\delta), \]

\[(ii) \quad \mathcal{H}^{(r)}[u_{\text{near}}^{(\delta, \sqrt{1 + \delta^2})}(\cdot)] = \mathcal{H}^{(r)}[0] = -\delta^2 + \frac{4}{3} \delta^3 \Omega + O(\delta^3). \]

Proof. First we prove (i). Notice that

\[U^2(0) \leq \max \left\{ (\chi(\varepsilon(0))u_{\text{near}}^{(\delta, \sqrt{1 + \delta^2})}(0))^2, v_{\text{near}}^2(0), v_{\text{far}}^2(0) \right\}. \]

We must show that the right hand side in the above equation is \(o(\varepsilon)\) (hence, \(o(\delta)\), thanks to (H2)). From (27), we get \(\chi(0)u_{\text{near}}^{(\delta, \sqrt{1 + \delta^2})}(0) = O(\varepsilon^2)\). The second estimate comes from the fact that \(v_{\text{near}}^p(\cdot)\) is a band limited function for any \(p \in \mathbb{N} \setminus \{0\}\), hence \(v_{\text{near}}^p(\cdot) \in H^s(\mathbb{R})\) for all \(s \geq 1\). In particular, whenever \(p = 2\), we can make use of Remark 1.6 and apply the Sobolev embedding [18] to the band limited function \(v_{\text{near}}^2(\cdot)\), getting

\[v_{\text{near}}^2(0) \leq \|v_{\text{near}}^2(\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \|v_{\text{near}}^2(\cdot)\|_{H^0(\mathbb{R})} \lesssim \|v_{\text{near}}^2(\cdot)\|_{L^2(\mathbb{R})}. \]

An application of (37) of Lemma 1.4 then gives \(v_{\text{near}}^2(0) \lesssim O(\varepsilon^2) = o(\varepsilon) = o(\delta)\). For the last term, we use inequality (39),

\[|v_{\text{far}}(0)| \leq \|v_{\text{far}}(\cdot)\|_{L^\infty(\mathbb{R})} \lesssim \|v_{\text{far}}(\cdot)\|_{H^r(\mathbb{R})} = O\left(\varepsilon^2 - 2r + \frac{2}{3} + \varepsilon^{-2r + \frac{2}{3}} + \varepsilon^{-\frac{2}{r} - 2r} + \varepsilon^{\frac{2}{r} - \frac{2}{3} - 2r}\right). \]
Within the parameter we are considering for $\tau$ and $\beta$, we now combine these estimates with $\delta \approx \epsilon$ to derive the result in (i).

To prove (ii) we expand $H^{(l)}[\nu_{\text{rolls}}(\delta,\sqrt{1+\Omega_{\gamma}})(\cdot)]$ using (2), the fact that $\Omega^2 = 1 + \delta \Omega$ from property [H2], and the equivalence $\delta \approx \epsilon$.

In principle, whenever $\delta = 0$ we have that $\nu_{\text{rolls}}(\delta,\sqrt{1+\Omega_{\gamma}})(\cdot) \equiv 0$, hence $\Omega$ would be allowed to take any value. It turns out that $\Omega$ can be chosen in an unique fashion if we extend it to the value it takes as $\delta \downarrow 0$; our approach is allusive to the technique used in [FLTW17, §6.7].

Lemma 7.3. Assume [H1], [H2], fix $\gamma \in \mathbb{R}$ and choose $\tau$ as in (37) of Proposition 3.1. Recall the parametrization $\epsilon = \epsilon(\delta,\Omega)$, due to (2) and [H2]. Let $\delta > 0$ as in Proposition 6.1. Consider the mapping

$$
(0, \delta_*) \times (-\delta_*, \delta_*) \ni (\delta, \Omega) \mapsto S[\delta, \Omega] := \frac{-\delta^2U^2}{\delta^2} - \delta^2 - H^{(l)}[\nu_{\text{rolls}}(\delta,\sqrt{1+\Omega_{\gamma}})(\cdot)] + H^{(r)}[0],
$$

where $U(x)$ is written as in (62). Then, the following properties hold:

(i) The mapping $S[\delta, \Omega] : (0, \delta_*) \times (-\delta_*, \delta_*) \to \mathbb{R}$ is smooth;

(ii) (Selection mechanism) There exists $\delta_{**}$ satisfying $0 < \delta_{**} < \delta_* < \frac{1}{3}$ and a mapping

$$
\delta \mapsto \Omega^{(\delta)} : (0, \delta_{**}) \to \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}
$$

such that $\delta \mapsto S[\delta, \Omega^{(\delta)}] = 0$ on $(0, \delta_{**})$.

(iii) (Branching) The mapping $\delta \mapsto S[\delta, \Omega^{(\delta)}]$ can be extended continuously in an unique fashion to a mapping

$$
\delta \mapsto \overline{S}[\delta, \Omega^{(\delta)}] = 0,
$$

on $\delta \in [0, \delta_{**})$. Moreover, we must have

$$
\lim_{\delta \downarrow 0} \Omega^{(\delta)} = 0,
$$

and further, we have $\frac{\delta}{4} \leq \epsilon = \epsilon(\delta) \leq 4\delta$.

Proof. We know that $H^4(\mathbb{R})$ is an algebra (cf. [Bre11, Corollary 8.10]), and from the fact that pointwise evaluation is continuous, thanks to [18]. Hence, using (62) and the continuity of the parametrization of $(\delta, \Omega) \mapsto \left(\vartheta_{\Omega_{\gamma}}(\cdot), \vartheta_{\Omega_{\gamma}}(\cdot)\right)$ obtained in Proposition 6.1, the continuity of the mapping

$$(\delta, \Omega) \mapsto U^2_{x=0}$$

is obtained. Since we know from (3) that the parametrization of the rolls is continuous in $H^3_{\text{per}}(\mathbb{R})$, we can use the Sobolev embedding as in [18] to derive continuity in the uniform norm. Thus, assertion (i) follows. Recall from [H2] that $\omega^2 = 1 + \delta \Omega$. Plugging the expansions derived in Lemma 7.2 we obtain

$$
0 = \frac{-\delta^2U^2}{\delta^2} - \delta^2 - H^{(l)}[\nu_{\text{rolls}}(\delta,\sqrt{1+\Omega_{\gamma}})(\cdot)] + H^{(r)}[0] = -U^2_{x=0} + \frac{4}{3} \Omega^3 - \frac{4}{3} \Omega + \mathcal{O}(\delta^2).
$$

which we finally rewrite as

$$
\Omega = -\Omega^3 - \frac{3}{4} U^2_{x=0} + \mathcal{O}(\delta^2) = : \Omega[\Omega, \delta].
$$

(64)

Hence, $\Omega$ can be seen as a fixed point for the mapping $\Omega \mapsto S[\Omega, \delta]$. An application of the Contraction Mapping theorem then shows that $\Omega$ is parametrized by $\delta$; in combination with (i) this implies that the parametrization takes place in a continuous fashion, and this establishes (ii).

Last, (iii) is derived from (64) once we take the limit $\delta \downarrow 0$: we conclude that

$$
\lim_{\delta \downarrow 0} \left\{ \Omega^{(\delta)} + (\Omega^{(\delta)})^3 \right\} = 0.
$$

Since $\Omega^{(\delta)} \in (-\delta_*, \delta_*) \subset \left(-\frac{1}{3}, \frac{1}{3}\right)$ and the mapping $y \mapsto y + y^3$ is a diffeomorphism in a neighborhood of 0, we conclude that $\lim_{\delta \downarrow 0} \Omega^{(\delta)} = \Omega_*$ exists and this limit must satisfy

$$
\Omega_* = -(\Omega_*^3).
$$

Thus, as $\Omega_* \in [-\delta_*, \delta_*] \subset \left[-\frac{1}{3}, \frac{1}{3}\right]$ we must have that $\Omega_* = 0$. Last, since $\Omega \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ we can the parametrization result referred to in (2) to have $\frac{\delta}{4} \leq \epsilon = \epsilon(\delta) \leq 4\delta$ (choosing $\delta_* > 0$ smaller if necessary), and we are done. ■
Corollary 7.4 (Existence of solutions to problem (5)). The mapping
\[ (0, \delta^\ast) \ni \delta \mapsto U(\delta)(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) + \chi(\varepsilon x) u_{\text{rolls}}(\delta, \delta, \gamma)(x) \]
solves problem (5); moreover, \( U(\delta)(\cdot) \) has the properties described in Definition 1.2.

Before putting things together, we go back to the stretching \( x \mapsto z := \frac{x}{\omega} \), which lead us to (63):

Lemma 7.5 (Continuity and loss of continuity of \( u_{\text{near}}(\cdot) \) in the sup norm with respect to parameters \( \delta \)). Let \( \beta = 1 \) and recall the Ansatz \( (62) \) reads as
\[ \delta \mapsto U(\delta)(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) + \chi(\varepsilon x) u_{\text{rolls}}(\delta, \omega, \gamma)(x) \]
Then the mapping \( \delta \mapsto U(\delta, \Omega)(\cdot) \) is continuous in the sup norm, that is,
\[ \left\| U(\delta_1)(\cdot) - U(\delta_2)(\cdot) \right\|_{L^\infty(\mathbb{R})} \lesssim |\delta_1 - \delta_2| \]
Consider \( \omega = \omega(\delta) = \sqrt{1 + \delta \Omega(\delta)} \). Then, undoing the mapping \( x \mapsto \omega z \), it follows that
\[ \delta \mapsto U(\omega(\cdot)) \]
is only continuous in the sup norm if \( \Omega = 0 \), that is, if \( \omega \equiv 1 \).

Proof. According to Proposition 6.1 we already know that the mapping \( (\delta, \Omega) \mapsto v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) \) is continuous in \( H^2(\mathbb{R}) \), which readily implies the result once we apply the Sobolev Embedding \( H^4(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \). Thus, using the parametrization \( \delta \mapsto \Omega(\delta) \) derived in Lemma 7.3 the term to be considered is
\[ \delta \mapsto \chi(\varepsilon x) u_{\text{rolls}}(\delta, \sqrt{1 + \delta \Omega(\delta)}, \gamma)(x) \]
Notice that the function \( u_{\text{rolls}}(\delta, \sqrt{1 + \delta \Omega(\delta)}, \gamma)(x) \notin L^2(\mathbb{R}) \). However, we know from Proposition 3.2 that its \( L^\infty(\mathbb{R}) \) norm scales in \( \varepsilon \). Furthermore, our change of coordinates fix the period of \( u_{\text{rolls}}(\cdot) \). Taking \( \delta_1 \) and \( \delta_2 \) in \( (0, \delta^\ast) \) and writing by \( u_{\text{rolls}}(\cdot) := u_{\text{rolls}}(\delta_1, \omega(\cdot), \gamma)(\cdot) \), for \( j \in \{1, 2\} \), and \( \omega(\delta) = \sqrt{1 + \delta \Omega(\delta)} \), we know that
\[ \left\| u_{\text{rolls}}^{(1)}(x) - u_{\text{rolls}}^{(2)}(x) \right\|_{L^\infty(\mathbb{R})} = \left\| u_{\text{rolls}}^{(1)}(x) - u_{\text{rolls}}^{(2)}(x) \right\|_{L^\infty([0, 2\pi])} \approx |\epsilon(\delta_1, \Omega(\delta_1)) - \epsilon(\delta_2, \Omega(\delta_2))| \]
which readily gives the result. Notice that, if we write the above mapping in the variables \((\delta, z)\), with \( x := \omega z \), then the first equality does not hold unless \( \Omega = 0 \); whenever \( \Omega \neq 0 \) the first equality in (65) does not hold, and we only get
\[ \left\| u_{\text{rolls}}^{(1)}(x) - u_{\text{rolls}}^{(2)}(x) \right\|_{L^\infty(\mathbb{R})} \approx |\epsilon(\delta_1, \Omega(\delta_1)) + \epsilon(\delta_2, \Omega(\delta_2))| \]
in which case continuity does not hold.

After this result, the proof of Theorem 1.5 is readily available.

Proof. ([Theorem 1.5]) The proof unfolds as a successive derivation of equivalent but reduced formulations of the same problem, so we just compile the result in the order it was constructed. From the very beginning we fix \( \gamma \in \mathbb{R} \). Equation (11) sets an equation for the “corrector” \( v(\cdot) \), which gets solved from sections 2 through 6 along this resolution process, the following steps were taken:

(i) In section 2 it was shown that \( v(\cdot) \) could be written as \( v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) \), and using a Lyapunov-Schmidt reduction the equation (11) could be rewritten in an equivalent form as a system of coupled equations (25a, 26a). Then, in Proposition 5.1 of section 7 it was proved that the parametrization
\[ (v_{\text{near}}(\cdot), \delta, \Omega) \mapsto v_{\text{far}}[v_{\text{near}}, \delta, \Omega](\cdot) \]
holds in a continuous fashion.

(ii) Thanks to the previous results, the problem gets reduced to understanding \( (v_{\text{near}}(\cdot), \delta, \Omega) \) only. In section 5 a blow-up in Fourier space is introduced in order to desingularize the limit \( \varepsilon \downarrow 0 \). In passing, we obtain the following characterization of \( v_{\text{near}}(\cdot) \) given in (33).
\[ v_{\text{near}} = \varepsilon^\beta e^{ix} g_{-1}(\varepsilon^\beta x) + \varepsilon^\beta e^{-ix} g_{+1}(\varepsilon^\beta x) \]
and, accordingly, many properties of \( g_{-1}(\cdot) \) and \( g_{+1}(\cdot) \) (and, consequently, of \( v_{\text{near}}(\cdot) \)) are derived.

(iii) We show in Proposition 4.8 that \( \lim_{|x| \to +\infty} \partial_x^n v(x) = 0 \) whenever \( \alpha \in \{0, 1, 2, 3\} \); this proves that the asymptotic spatial limits in stated in (i) hold.

(iv) In section 5 a pair of equivalent equations (40) on \( g_{-1}(\cdot) \) and \( g_{+1}(\cdot) \) is derived. After slightly adapting a result of [PTLW17] to our purposes, we are able to conclude this section performing the matched asymptotics result that gives \( \beta = 1 \), besides simplification of the equations, that then reduce to the form (48) presented in the beginning of section 6.
(v) In Section 8 we obtain (iii), showing in Lemma 6.5 that \( \chi(\cdot) \) can be chosen in such a way that it satisfies the ODE
\[
-4\pi \partial_x^2 \chi(\cdot) + \frac{3\pi}{4} (\chi^3(\cdot) - \chi(\cdot)) - c \partial_x \chi(x) = 0, \quad \lim_{x \to -\infty} \chi(x) = 1, \quad \lim_{x \to -\infty} \chi(x) = 0.
\]

In passing, this provides nice conditions under which, once more, the Contraction Mapping Theorem can be applied (Proposition 6.1) to show the existence of a mapping
\[
(0, \delta_x) \times (-\delta_x, \delta_x) \ni (\delta, \Omega) \mapsto (g_{\delta}^{(0, \Omega)}(\cdot), g_{\delta}^{(1, \Omega)}(\cdot)) \in H^2_{\text{near}, \epsilon^{-1}}(\mathbb{R}) \times H^2_{\text{near}, \epsilon^{-1}}(\mathbb{R}),
\]
that solves problem (11) for
\[
v(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot), \Omega(\cdot), \delta, \Omega),
\]
where
\[
v_{\text{near}}(x) = \varepsilon e^{-i x} g_{\delta}^{(\Omega)}(\varepsilon x) + \varepsilon e^{i x} g_{\delta}^{(\Omega)}(\varepsilon x).
\]

Hence, the sections summarized above carefully describe the qualities of the corrector \( v(\cdot) \) first introduced in \( \Theta \), which proves (iv). In Section 7 we bring the Hamiltonian structure of the problem to the limelight, which is used in the proof of the Selection mechanism asserted in (ii): more precisely, in Lemma 7.3 another application of a Contraction Mapping Theorem resulted in an implicit description of \( \Omega \) in terms of \( \delta \).

Subsequently, in Corollary 7.4 we obtain a function
\[
\delta \mapsto U^{(\delta)}(\cdot) = v_{\text{near}}(\cdot) + v_{\text{far}}(\cdot) + \chi(\varepsilon x) u_{\text{rolls}}^{(\delta, \sqrt{1 + \delta \Omega} / x)}(x),
\]
with the regularity and properties we were interested at (see definition 1.2). In Lemma 7.1 the regularity of this mapping is studied, and in Lemma 7.5 we showed that this mapping is continuous in the topology of uniform continuity. Now, plugging back \( x = o(\delta^2) \), the latter Corollary also shows considering the mapping
\[
\delta \mapsto \omega^{(\delta)} = \sqrt{1 + \delta \Omega},
\]
then we obtain (13), namely,
\[
\delta \mapsto U^{(\delta)}(\omega^{(\delta)}),
\]
which is a stationary solution to problem (11), which is continuous in the uniform continuity topology if and only if \( \omega^{(\delta)} \equiv 1 \). This concludes the proof of (i), and this establishes Theorem 1.5.

8. Open problems and further comments

This project was highly inspired by the techniques and the perspective in the memoir [FLTW17]. Our results are corroborated by the results in [SW18], which puts the ideas we advocate for in a safe ground for comparison with other mathematical tools. Some of the questions we address below are suggestively related to well established mathematical techniques (8.1-8.3). Others open problems (8.4-8.8) have a pure speculative nature; regardless of their plausibility, they should be read with caution.

8.1. The techniques in [FLTW17]. The results in the memoir [FLTW17] in many aspects seem to be related to the work of Schneider and touches upon interesting issues previously addressed in [NW69]. Other problems (8.1-8.3) have a pure speculative nature; regardless of their plausibility, they should be read with caution.

8.2. Far/near reductions and dynamical systems. As pointed out before, the shape of \( v_{\text{near}}(\cdot) \) in (33) resembles the initial steps in modulation theory that lead to Ginzburg-Landau equation. In general this type of approximations are applied to pattern formation systems close to unstable states [SU17, Part IV]; different approaches to this derivation using multiple scales analysis are also possible, cf. [vH91]. A clear understanding of the relation between the transversality theory, as used in [SW18, Sections 2(e)(f)(g)], and the reduced equation we found in Section 6 as that of the approximate operator \( R_{\text{AV}} [g_{\text{AV}}, g_{\text{AV}}] \), is still to be investigated.
8.3. **On the role of multipliers.** In case of $n$ distinct singularities in the multiplier one can expect to obtain a system of ODEs in $n$ variables. More examples and possibly a more general theory is still necessary to elucidate how the location of the singularities, the stretching of the vicinities around them (imposed by the far/near decomposition) and the subsequent blow-up in Fourier space play a crucial role in the reduced equations obtained in the end. However, many questions are left behind: is it possible to say that the reduced equations are always unique (or, somewhat, equivalent), up to some parametrization? It would also be interesting to see an example of this technique being applied to non-local models where the linearization has an associated multiplier directly obtained by convolution; this scenario is very interesting, because these cases are not directly amenable to ODE techniques as those used in [SW18]. With regards to the existence of traveling waves in non-local models, one can cite for instance the results of [Che97], derived using comparison principles, or the results in [ST17]; with regards to the latter, stationary solutions to a nonlocal (convolution-type) problem are sought and properties of the multiplier are exploited in an interesting fashion, ending in a reduced equation [ST17] §3; in their case, however, no far/field (spectral) decomposition is used.

8.4. **Invasion fronts and the role of $\chi(\cdot)$.** The type of the ODE satisfied by $\chi(\cdot)$ in Lemma 6.5 brings to mind the work of [CE86], in which the equation for the wavefront is found as a fixed point equation, cf. discussion in [CE90] §28. In our view, several analogies can be made, although further investigation is necessary to clarify them. Indeed, in the literature of pattern formation, whenever near/far (spatial) decompositions have been applied, the functions $\chi(\cdot)$ are mostly introduced aiming localization of pattern properties in the far field (cf. [GS15] [GS16] [LS17] [MS15] [MS17] [MS18] [Mon18]), which happens mostly because no effects of bifurcation parameter variations are felt in the far field; the role of the function $\chi(\cdot)$ is essentially that of a partition of unity, whereas under the perspective of [CE86] [CE90] one can say that the function $\chi(\cdot)$ plays a role of an envelope of the modulated (invasion) front. In their case though, the analogy is more evident, for the profile is positive, making a sharp distinction between the invaded part and the wake of this front.

Last, we mention the interesting work [MNT90] (in particular, section III), where multi-dimensional patterns are studied. Roughly speaking, the boundaries of regions filled with rolls with different orientation are investigated using functions $r_1(\cdot)$ and $r_2(\cdot)$ that have a similar role to that of $\chi(\cdot)$; see also §8.6 below.

8.5. **Nanopatterns and numerical aspects.** In case of patterns with characteristic wavelength smaller than computer floating numbers, our result seems to be useful in the computation of possible profiles that would be otherwise undetectable in numerical simulations. The papers [BLBL12] [BLBL15] exploit this question in an interesting fashion, with techniques different from ours.

8.6. **Grain boundaries, defects, and multi-dimensional patterns.** In the study of multidimensional patterns the theory of dynamical systems, in spite of its robustness, seems to not be broad enough to encompass the unsurmountable difficulties of the field. Interesting studies have been done where rigorous numerical analysis has been applied [MS13], harmonic analysis techniques play an important role [JS15] [Jar15] [BLBL12], variational techniques is [Rab94], or more functional-analytic based techniques are exploited [MS18]. Still, many classes of problems remain unsolved, as that of asymmetrical grain boundaries, a case that does not seem to be directly amenable to the spatial dynamics techniques as presented in [HS12] [SW13]; in this scenario the far/near decompositions we presented might be relevant for analytical results. For numerical results which exploit far/near (spatial) decomposition, see [LS17].

8.7. **Directional quenching and wavenumber selection.** In spite of the consistency between our result and that in [SW18] Theorem 1.1], a comparison between them shows that our selection mechanism result is weaker, for we only prove the existence of an implicit parametrization $\Omega = \Omega(\delta)$, while therein the authors show a much stronger result, namely,

$$\Omega(\delta) = \delta \cos(2\gamma) \frac{16}{\gamma^3} + O(\delta^3).$$

The shape of the control parameter $\mu(\cdot)$ was chosen in its simplest form as a parameter of jump type. The sharp discontinuity type was called here *directional quenching*, and should be seen as a contrast to the case of slow decay, physically closer to the process of *annealing*. It is plausible that, in the latter scenario, no wavenumber selection happens. We highlight the interesting discussion in [SW18] §4 about wavenumber selection by ramp discontinuities; for a general overview of pattern selection mechanisms, the discussion in [Nis02] §3.1 is also very illustrative.
8.8. Stability issues. Once we consider roll solutions embedded in a multidimensional space new types of instabilities are seen, as it is the case of Zig-Zag instabilities (cf. [Mie95, §4], [NPL93]; several mathematical and numerical issues related to the formation of these patterns are still not well understood [AGG+18]. Nonlinear stability results have also been investigated, with some interesting recent results in one-dimension: the seminal paper [SS04], the recent work [BNSZ18] and the memoir [DSSS09].

Appendix A. A Few Computations

In this appendix we give a detailed computation of the results presented in sections 5.2 and 5.3. We stress that the results in this section are posterior to the choice \( \beta = 1 \) done in Observation 5.4, therefore, we shall adopt \( \beta = 1 \) from now on.

A.1. Simplifying \( T^{(4; \pm)}(\xi) \): computations. Our goal is to estimate the terms in

\[
T^{(4; \pm)}(\xi) = \frac{p^{(\pm)}_{\text{near}}}{\mathcal{F}} \left( \mathcal{M}^{(4)}[v_{\text{near}}, u^{(\delta, \omega, \gamma)}_{\text{rolls}}] \right)(\xi)
\]

\[
= \frac{p^{(\pm)}_{\text{near}}}{\mathcal{F}} \left( \chi(\chi^2 - 1) \left( u^{(\delta, \omega, \gamma)}_{\text{rolls}} \right)^3 \right)(\xi) + \frac{p^{(\pm)}_{\text{near}}}{\mathcal{F}} \left( (1 + \beta)^2, \chi \right) u^{(\delta, \omega, \gamma)}_{\text{rolls}}(\xi)
\]

\[
+ \frac{p^{(\pm)}_{\text{near}}}{\mathcal{F}} \left( \delta(\mu - 1) \chi u^{(\delta, \omega, \gamma)}_{\text{rolls}} \right)(\xi)
\]

\[
= S^{(4; \pm)}(\xi) + S^{(4; \pm)}_{\text{II}}(\xi) + S^{(4; \pm)}_{\text{III}}(\xi),
\]

The term \( S^{(4; \pm)}_{\text{I}}(\xi) \) has already been studied. For now, we estimate \( S^{(4; \pm)}_{\text{II}}(\xi) \). In order to do so, initially we study the term \([[(1 + \beta)^2, \chi] u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x)\), which can be written, from lowest order terms to higher order terms, as

\[
[(1 + \beta)^2, \chi] u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) = 4\theta_x(\xi x)(\beta_x^3 + \beta_x) u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) + \partial_x^2(\xi x)[6\beta_x^2 + 2] u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) + \ldots
\]

Notice that

\[
4\theta_x(\xi x)(\beta_x^3 + \beta_x) u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) = 4\epsilon\chi'(\xi x)(\beta_x^3 + \beta_x) u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) = 4\epsilon\chi'(\xi x)(\beta_x^3 + \beta_x)(\xi x) \cos(x + \gamma) + O(\epsilon^3).
\]

Since \((\beta_x^3 + \beta_x) \cos(x + \gamma) = 0\), we obtain \(4\theta_x(\xi x)(\beta_x^3 + \beta_x) u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) = O(\epsilon^4)\). The next term in the hierarchy has order \( \epsilon^2 \), coming from \( \beta_x^3(\xi x)[6\beta_x^2 + 2] \cos(x + \gamma) \). Hence

\[
[(1 + \beta)^2, \chi] \cos(x + \gamma) = \partial_x^2(\xi x)[6\beta_x^2 + 2] u^{(\delta, \omega, \gamma)}_{\text{rolls}}(x) + O(\epsilon^4) + \ldots
\]

\[
= \epsilon^3 \chi''(\xi x)[6\beta_x^2 + 2] \cos(x + \gamma) + O(\epsilon^4)
\]

\[
= -4\epsilon^2 \cos(x + \gamma) \chi''(\xi x) + O(\epsilon^4).
\]

Thus, we write \( S^{(4; \pm)}_{\text{II}}(\xi) \) as

\[
S^{(4; \pm)}_{\text{II}}(\xi) = -4\epsilon^3 \mathbb{1}_{\xi \in \epsilon^{1/2}B}(\xi) \mathcal{F} \left[ \chi' \right] (1 + \xi) + O(\epsilon^4)
\]

\[
= -4\epsilon^3 \mathbb{1}_{\xi \in \epsilon^{1/2}B}(\xi) \int_{\mathbb{R}} \left( \cos(x + \gamma) \chi''(\xi x) \right) e^{-i\xi x} e^{-i\xi^2 x^2} dx + O(\epsilon^4)
\]

\[
= -8\pi \epsilon^3 \mathbb{1}_{\xi \in \epsilon^{1/2}B}(\xi) \mathcal{F} \left[ \cos(2\pi z + \gamma) \chi''(\theta z) \right] e^{-i2\pi z} e^{-i\xi 2\pi z} dz + O(\epsilon^4)
\]

\[
= -\frac{3\epsilon}{\pi} \mathbb{1}_{\xi \in \epsilon^{1/2}B}(\xi) \int_{\mathbb{R}} \left( \cos(2\pi z + \gamma) \chi''(\theta z) \right) e^{-i2\pi z} e^{-i\xi z} dz + F_{\text{II}},
\]
where \( \| F_{II} \|_{L^2(\mathbb{R})} = O(\varepsilon^4) \). We now use Lemma 5.2 to rewrite it: set \( Z = \theta z, f(z) = \cos((2\pi \theta + \gamma)), g(z) = e^{-i2\pi z} \) and \( \Gamma_2(z, Z) = \Gamma_2(Z) = \chi'(\theta z) = \chi'(Z) \).

\[
S_{II}^{(4;+)}(\xi) = -\frac{\theta^3}{\pi} \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} \left( \cos(2\pi z + \gamma) \chi''(\theta z) \right) e^{-i2\pi z + \gamma} e^{-i\theta \xi z} dz
\]

\[
= -\frac{2\theta^2}{\pi} \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \sum_{n \in \mathbb{Z}} \int_{0}^{1} e^{2\pi i n z} \Gamma_2(\frac{2\pi n}{\theta} + \xi) \cos(2\pi z + \gamma) e^{-i2\pi z} dz
\]

\[
= -\frac{2\theta^2}{\pi} \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \times \left[ \Gamma_2(\xi) \int_{0}^{1} \cos(2\pi z + \gamma) e^{-i2\pi z} dz + \sum_{|n| \geq 1} \Gamma_2 \left( \frac{2\pi n}{\theta} + \xi \right) \int_{0}^{1} \cos(2\pi z + \gamma)e^{2\pi i n z} e^{-i2\pi z} dz \right]
\]

\[
= -\frac{2\theta^2}{\pi} \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) e^{i\theta \xi} \Gamma_2(\xi) \int_{0}^{1} \cos^2(2\pi z + \gamma) dz + F_{II},
\]

where now \( \| F_{II} \|_{L^2(\mathbb{R})} = O(\theta^3) \); thus, plugging back \( \theta = 2\pi \varepsilon \) gives \( \| F_{II} \|_{L^2(\mathbb{R})} = O(\varepsilon^3) \). Last, we estimate \( S_{III}^{(4;+)}(\xi) \) using the expansion \( u_{rolls}(\delta, \omega, \gamma)(x) = \varepsilon \cos(x) + O(\varepsilon^3) \); we get

\[
S_{III}^{(4;+)}(\xi) = \delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{F} \left[ (\mu(\cdot) - 1)\chi(\cdot)\frac{4}{\theta^2}(\nu_{\xi}(\delta, \omega, \gamma)(\cdot)) \right] \left( 1 + \varepsilon \xi \right)
\]

\[
= \delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{F} \left[ (\mu(\cdot) - 1)\chi(\cdot)\frac{4}{\theta^2}(\nu_{\xi}(\delta, \omega, \gamma)(\cdot)) \right] \left( 1 + \varepsilon \xi \right)
\]

\[
= -\delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{F} \left[ \nu_{\xi}(\delta, \omega, \gamma)(\cdot) \right] \left( 1 + \varepsilon \xi \right)
\]

\[
= \frac{3\theta^3}{4} e^{i\theta \xi} \Gamma_1(\xi) - 4\pi^2 e^{i\theta \xi} \Gamma_2(\xi) - 3\theta^3 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \Gamma_3(\xi) + \mathcal{M}^{(4;+)}(\xi),
\]

where \( \| F_{III} \|_{L^2(\mathbb{R})} = O(\theta^2) \) and \( \Gamma_3(\xi) = \mathbb{1}_{\{0, \infty\}}(\xi) \varepsilon \cos(\varepsilon \xi \pm \gamma) \); the last equality is due to property (17a). Plugging back \( \theta = 2\pi \varepsilon \) and combining the previous estimates, we get

\[
I^{(4;+)}(\xi) = \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \frac{3\theta^3}{4} e^{i\theta \xi} \Gamma_1(\xi) - 4\pi^2 e^{i\theta \xi} \Gamma_2(\xi) - 3\theta^3 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \Gamma_3(\xi) + \mathcal{M}^{(4;+)}(\xi).
\]

Above, we have \( \mathcal{M}^{(4;+)}(\xi) = \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{M}^{(4;+)}(\xi), \) with \( \| \mathcal{M}^{(4;+)} \|_{L^2(\mathbb{R})} = O(\varepsilon^3) \). A similar analysis gives

\[
I^{(4;-)}(\xi) = \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \frac{3\theta^3}{4} e^{i\theta \xi} \Gamma_1(\xi) - 4\pi^2 e^{-i\theta \xi} \Gamma_2(\xi) - 3\theta^3 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \Gamma_3(\xi) + \mathcal{M}^{(4;-)}(\xi),
\]

where \( \mathcal{M}^{(4;-)}(\xi) = \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{M}^{(4;-)}(\xi), \) and \( \| \mathcal{M}^{(4;-)} \|_{L^2(\mathbb{R})} = O(\varepsilon^3) \).

A.2. Simplifying \( I^{(1;\pm)}(\xi) \). We aim to understand the terms

\[
I^{(1;\pm)}(\xi) = \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \mathcal{F} \left[ \delta^2 \mu(x) - 3(\nu_{\xi}(\delta, \omega, \gamma)(\cdot)) \mathcal{F} \left[ \nu_{\xi}(\delta, \omega, \gamma)(\cdot) \right] \right] \left( 1 + \varepsilon \xi \right).
\]

For the moment, let’s focus on \( I^{(1;+)}(\xi) \). Using the relation (33), we can expand this term as

\[
I^{(1;+)}(\xi) = -\delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} \mu(x) g_{+1}(\varepsilon x) e^{-i\xi x} dx
\]

\[
- \delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} \mu(x) g_{-1}(\varepsilon x) e^{-i\xi x} dx
\]

\[
+ 3\pi^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} (\nu_{\xi}(\delta, \omega, \gamma)(\cdot))^2 g_{+1}(\varepsilon x) e^{-i\xi x} dx
\]

\[
+ 3\pi^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} (\nu_{\xi}(\delta, \omega, \gamma)(\cdot))^2 g_{-1}(\varepsilon x) e^{-i\xi x} dx
\]

\[
= S_{II}^{(1;+)}(\xi) + S_{III}^{(1;+)}(\xi) + S_{IV}^{(1;+)}(\xi).
\]

In order to estimate \( S_{II}^{(1;+)}(\xi) \), we do a change variables \( X = \varepsilon x \), and exploit the homogeneity of the parameter \( \mu(\cdot) \), which enables us to write \( \mu \left( \frac{\theta}{\varepsilon} \right) = \mu(X) \). Thus,

\[
S_{II}^{(1;+)}(\xi) = -\delta^2 \mathbb{1}_{\{\xi \in e^{-1}B_1\}}(\xi) \int_{\mathbb{R}} \mu(X) [g_{+1}(X)] e^{-i\xi X} dX.
\]
With regards to the case $S_{II}^{(1,+)}(\xi)$, the analysis is easier. Indeed,

$$S_{II}^{(1,+)}(\xi) = -\delta^2 \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \mu(x) \left[ g_{+1}(\varepsilon x) e^{-i2\varepsilon x} \right] e^{-i\varepsilon_0 x} \, dx$$

$$= -\delta^2 \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \mu(X) \left[ g_{+1}(X) e^{-i(\xi + \varepsilon)X} \right] \, dX$$

$$= -\delta^2 \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \mathcal{F} \left[ \mu(\cdot) g_{+1}(\cdot) \right] \left( \frac{2}{\varepsilon} + \xi \right).$$

As $\frac{2}{\varepsilon} + \varepsilon^{-1}B \subset \{|\xi| \geq \frac{1}{2}\}$, we make use of (2) to get $\|S_{II}^{(1,+)}(\xi)\|_{L^2(\mathbb{R})} \leq \varepsilon^4 \int_{|\xi| \geq \frac{1}{2}} |\mathcal{F} \left[ \mu(\cdot) g_{+1}(\cdot) \right] (\xi)|^2 \, d\xi$, thus obtaining that $\|S_{II}^{(1,+)}(\xi)\|_{L^2(\mathbb{R})} = O(\varepsilon^4)$; see Figure 5.

**Figure 5.** The blow-up of parameter regions as $\theta \downarrow 0$.

Both terms $S_{II}^{(1,+)}(\xi)$ and $S_{IV}^{(1,+)}(\xi)$ can be estimated using formula in Lemma 5.2. Indeed, after change of variables, we set $\theta = 2\pi \varepsilon$ and apply (2), obtaining the equivalent expression

$$S_{II}^{(1,+)}(\xi) = 3\varepsilon^3 \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \int_{\mathbb{R}} \left( u_{\text{rolls}}(\xi) \right)^2 \left[ g_{+1}(\varepsilon x) \right] e^{-i\varepsilon_0 x} \, dx$$

$$= \frac{3\varepsilon^3}{2\pi} \bar{g}_{+1}(\xi) \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \int_{0}^{1} \left( u_{\text{rolls}}(\xi) \right)^2 (2\pi z)^2 \, dz + G_{II}$$

with $\|G_{II}\|_{L^2(\mathbb{R})} = O(\varepsilon^3)$. Likewise,

$$S_{IV}^{(1,+)}(\xi) = 3\varepsilon^3 \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \int_{\mathbb{R}} \left( u_{\text{rolls}}(\xi) \right)^2 \left[ g_{-1}(\varepsilon x) e^{-2ix} \right] e^{-i\varepsilon_0 x} \, dx$$

$$= \frac{3\varepsilon^3}{2\pi} \bar{g}_{-1}(\xi) \mathbb{I}_{[\varepsilon^{-1},B]}(\xi) \int_{0}^{1} \left( u_{\text{rolls}}(\xi) \right)^2 (2\pi z)^2 \, dz + G_{IV},$$

where $\|G_{IV}\|_{L^2(\mathbb{R})} = O(\varepsilon^3)$. And using the expansion of $u_{\text{rolls}}(\xi)$ given in Lemma 3.2 and Lemma 5.4 we get

$$\mathcal{I}^{(1,+)}(\xi) = \mathbb{I}_{[\xi \in \varepsilon^{-1},B]}(\xi) \left\{ -\delta^2 \int_{\mathbb{R}} \mu(X) \left[ g_{+1}(X) \right] e^{-i\varepsilon_0 X} \, dX + 3\pi \varepsilon^2 \bar{g}_{+1}(\xi) + 3\pi \varepsilon^2 \bar{g}_{-1}(\xi) \right\} + \mathcal{M}^{(1,+)}(\xi).$$

where $\mathcal{M}^{(1,+)}(\xi) = \mathbb{I}_{[\xi \in \varepsilon^{-1},B]}(\xi).$ $\mathcal{M}^{(1,+)}(\xi)$ is so that $\|\mathcal{M}^{(1,+)}\|_{L^2(\mathbb{R})} = O(\varepsilon^3)$. A similar analysis gives

$$\mathcal{I}^{(1,-)}(\xi) = \mathbb{I}_{[\xi \in \varepsilon^{-1},B]}(\xi) \left\{ -\delta^2 \int_{\mathbb{R}} \mu(X) \left[ g_{-1}(X) \right] e^{-i\varepsilon_0 X} \, dX + 3\pi \varepsilon^2 \bar{g}_{-1}(\xi) + 3\pi \varepsilon^2 \bar{g}_{+1}(\xi) \right\} + \mathcal{M}^{(1,-)}(\xi),$$

where $\mathcal{M}^{(1,-)}(\xi) = \mathbb{I}_{[\xi \in \varepsilon^{-1},B]}(\xi).$ $\mathcal{M}^{(1,-)}(\xi)$ is so that $\|\mathcal{M}^{(1,-)}\|_{L^2(\mathbb{R})} = O(\varepsilon^3)$.

**APPENDIX B. ON THE EXISTENCE OF TRAVELING WAVES**

**Figure 6.** Phase plane sketch for the ODE $\partial_t^2 \chi + c \partial_x \chi + f(\chi) = 0$, which has a Heteroclinic orbit $\chi(\cdot)$ satisfying $\chi(-\infty) = 1$ and $\chi(+\infty) = 0$.

Following the analysis in [Fif79, §4], whenever $c > 0$ the ODE

$$\partial_t^2 \chi + c \partial_x \chi + f(\chi) = 0,$$
with \( f(0) = f(1) = 0, f'(0) > 0, f'(1) < 0 \) and \( f(x) > 0 \) whenever \( x \in (0, 1) \) admits a heteroclinic orbit \( \chi(\cdot) \) so that
\[
\lim_{x \to -\infty} \chi(x) = 1, \quad \lim_{x \to +\infty} \chi(x) = 0;
\]
see Figure 3. It is also known that this convergence is exponential, that is, there exists constants \( C_* > 0 \) and \( S_* > 0 \) such that
\[
|\chi(x) - 1| \lesssim e^{-C_*|x|}, \quad \text{whenever} \quad x \leq -S_*, \quad \text{and} \quad |\chi(x)| \lesssim e^{-C_*|x|}, \quad \text{whenever} \quad x \geq S_*.
\]
Furthermore, whenever \( G(\cdot) = f'(\cdot) \) is known, it is possible to compute \( \|\chi\|_{L^2(\mathbb{R})}^2 \); indeed,
\[
eq c \int_{\mathbb{R}} |\chi'(s)|^2 \, ds = - \int_{\mathbb{R}} \chi''(c) \left( G'(|\chi|) \right) \, ds = \left( \frac{\chi'(x)^2}{2} + G(\chi(x)) \right)_{x = +\infty} - \left( \frac{\chi'(x)^2}{2} + G(\chi(x)) \right)_{x = -\infty}.
\]
Thus,
\[
\int_{\mathbb{R}} |\chi'(s)|^2 \, ds \leq \left| \frac{G(\chi(+\infty)) - G(\chi(-\infty))}{c} \right| \leq \left| \frac{G(1) - G(0)}{c} \right|.
\]
Moreover,
\[
|c^2 \int_{\mathbb{R}} |\chi'(s)|^2 \, ds| \lesssim |c|.
\]
In the particular case that \( x \mapsto f(x) := x - x^3 \) we have
\[
|c^2 \int_{\mathbb{R}} |\chi'(s)|^2 \, ds| \leq \frac{|c|}{4}.
\]

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