The Thurston’s program derived from the Langlands global program with singularities

C. Pierre

Institut de Mathématique pure et appliquée
Université de Louvain
Chemin du Cyclotron, 2
B-1348 Louvain-la-Neuve, Belgium
pierre@math.ucl.ac.be

Abstract

The seven geometries $H^3$, $S^3$, $H^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\text{PSL}_2(\mathbb{R})$, Nil and Sol of the Thurston’s geometrization program are proved to originate naturally from singularization morphisms and versal deformations on euclidean 3-manifolds generated in the frame of the Langlands global program.

The Poincare conjecture for a 3-manifold appears as a particular case of this new approach of the Thurston’s program.
1 Introduction

The classification of 3-manifolds is now based upon the Thurston’s geometrization conjecture [Thu5] which states that every three-dimensional manifold has a natural decomposition into building blocks characterized by eight specific geometric structures [Pap].

R. Hamilton [Ham] formalized such approach by introducing the Ricci flow on a Riemannian space and, recently, G. Perelman [Per1], [Per2], developed a new method of surgery of the Ricci flow by extending the flow past the singularities in order to solve the technical difficulties of the Hamilton’s program [Mor], [Sch].

The interesting feature of the Thurston’s geometrization conjecture is that it implies the Poincare conjecture in dimension 3 as a special case.

S. Smale [Sma2] proved the Poincare conjecture in the dimensions superior to four by developing a method for breaking manifolds into handles and M. Freedman using Casson-handles succeeded in proving the four-dimensional case [Free].

But, the three-dimensional case [Sar] still remains difficult, perhaps because it is related to the “real” mathematical world [Sma1].

In this perspective, a new approach of the Thurston’s program is envisaged in this paper in a “natural way”, i.e. without envisaging surgeries, in such a way that:

a) the “unperturbed” Thurston’s program, corresponding to the Euclidean geometry $E^3$ and to the boundary tori $T^2$, proceeds respectively from the global program of Langlands in dimensions three and two [Pie1].

b) the “perturbed” Thurston’s program, associated with the remaining 7 geometries $H^3$, $S^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $PSL_2(\mathbb{R})$, Nil and Sol, results from the singularities and their versal deformations on $E^3$.

As the structure of the space in dimension three [Hem] corresponds to the real world in which we are living, it would be reassuring to know that the unperturbed and perturbed Thurston’s programs are linked to the physical reality. Now, it is the case since it was proved in [Pie3] that the Langlands global program in dimensions two and three with singularities is associated with the generation of the representation spaces of algebraic (bisemi)groups which correspond precisely to the structure of the vacuum space physical fields of elementary particles [Pie4] submitted to strong fluctuations.

In the new context envisaged here, the Thurston’s program [Thu1]–[Thu4] is directly related to algebraic geometry and number theory by means of the Langlands program.
and to quantum field theory by the geometries of 3-manifolds associated with the field structure of elementary particles.

This new approach of the Thurston’s program allows to understand that the origin of the geometries of 3-manifolds [Gor] depends on:

- **the singularization morphisms** which are introduced [Pie2] as the inverse morphisms of desingularizations [Dur].
- **the versal deformations** of these singularities.
- **the types of geometries** characterizing the singularizations and the versal deformations.

More concretely, the eight geometries of 3-manifolds will appear as resulting from the following procedure exempt from surgeries:

1. **we start with the Langlands global program in real dimensions two and three** by generating the representation spaces \( \text{Repsp}(\text{GL}_2(F^T \times F^T_v)) \) and \( \text{Repsp}(\text{GL}_3(F_{\tau} \times F_v)) \) of the algebraic bilinear semigroups \( \text{GL}_2(F^T_{\tau} \times F^T_v) \) and \( \text{GL}_3(F_{\tau} \times F_v) \) respectively over the products \( (F^T_{\tau} \times F^T_v) \) and \( (F_{\tau} \times F_v) \) of sets of toroidal and normal real completions.

According to [Pie1], the representation space \( \text{Repsp}(\text{GL}_2(F^T_{\tau} \times F^T_v)) \) is constituted by a tower of conjugacy classes of \( \text{GL}_2(F^T_{\tau} \times F^T_v) \) composed of products, right by left, \( T^2_R(j) \times T^2_L(j) \) of increasing two-dimensional (semi)tori.

Similarly, the representation space \( \text{Repsp}(\text{GL}_3(F_{\tau} \times F_v)) \) is constituted by a tower of conjugacy classes of \( \text{GL}_3(F_{\tau} \times F_v) \) composed of products, right by left, of **increasing Euclidean semispaces** \( E^3_R(j) \times E^3_L(j) \), each one characterized by the Euclidean geometry \( E^3 \).

This step corresponds to what is called the unperturbed Thurston’s program.

2. **To get the perturbed Thurston’s program, we generate singularities by singularization contracting surjective morphisms** on the Euclidean semispaces \( E^3_R(j) \) and \( E^3_L(j) \).

Afterwards, we consider the versal deformations of these singularities envisaged as extensions of the singularization morphisms [Pie2].

3. **By analysis of the local geometries resulting from these singularization and versal deformation morphisms, we find that:**

   (a) the singularities of corank 3, 2 and 1 are respectively responsible for the generation of the five local geometries \( (H^3) \), \( (H^2 \times \mathbb{R}, \text{PSL}_2(\mathbb{R})) \) and \( (\text{Nil}, \text{Sol}) \) in the neighborhood of these singularities.
(b) the versal deformations of these singularities in codimensions 3, 2 and 1 generate locally respectively the geometries $S^3$, $S^2 \times \mathbb{R}$ and Sol.

(c) the Poincare conjecture for a 3-manifold appears as a particular case of the versal deformation in codimension 3.
2 Origin of the different geometries of 3-manifolds

Singularization morphisms and versal deformations of these involving non-euclidean geometries will be introduced in this chapter.

But, first, the global Langlands program, leading to the unperturbed Thursdon’s program associated with the euclidean geometries, will be recalled.

The developments of this chapter refer to the paper [Pie2].

2.1 Representation spaces of algebraic bilinear semigroups

The Langlands global program is based on the representations of the Weil groups, given by the (representation spaces of) algebraic groups which are in one-to-one correspondence with their cuspidal representations.

So, the representation spaces of algebraic groups have to be taken into account, but, in order to be the most complete and general, bilinear algebraic semigroups will be considered since they cover their linear equivalents [Pie1].

- A bilinear algebraic semigroup $\text{GL}_n(F_R \times F_L)$:
  
  - is a bilinear semigroup whose bielements are submitted to the cross binary operation $\times$ sending products of right and left elements, referring respectively to the lower and upper half spaces, either in diagonal bielements or in cross bielements [Pie1].
  
  - can be decomposed according to:

\[
\text{GL}_n(F_R \times F_L) = T_n^u(F_R) \times T_n(F_L)
\]

where:

* $F_R$ and $F_L$ are right and left finite algebraic symmetric finite extensions of a global number field $k$ of characteristic zero;

* $T_n(F_L)$ (resp. $T_n^u(F_R)$) is a left (resp. right) semigroup of upper (resp. lower) triangular matrices with entries in the semifield $F_L$ (resp. $F_R$).

- The representation (bisemi)space of the algebraic bilinear semigroup of matrices $\text{GL}_n(F_R \times F_L)$ is given by the $\text{GL}_n(F_R \times F_L)$ -bsemimodule $G^{(n)}(F_R \times F_L)$ which is an affine bisemispace $(V_R \otimes_{F_R \times F_L} V_L)$, in such a way that the left (resp. right) affine semispace $V_L$ (resp. $V_R$) is localized in the upper (resp. lower) half space, and is symmetric of $V_R$ (resp. $V_L$).

- The left and right equivalence classes of the real completions of $F_L$ and $F_R$ are respectively the infinite real places noted $v = \{v_1, \ldots, v_j, \ldots, v_t\}$ and $\bar{v} = \{\bar{v}_1, \ldots, \bar{v}_j, \ldots, \bar{v}_t\}$. 

The infinite places associated with the complex completions of $F_L$ and $F_R$ are the sets $w = \{ w_1, \ldots, w_j, \ldots, w_r \}$ and $\overline{w} = \{ \overline{w}_1, \ldots, \overline{w}_j, \ldots, \overline{w}_r \}$.

The real completions are assumed to cover the corresponding complex completions and the infinite real and complex places are characterized by increasing Galois extension degrees as developed in [Pie2].

By this way, we get a left (resp. right) tower

$F_v = \{ F_{v_1}, \ldots, F_{v_{j,m_j}}, \ldots, F_{v_t} \}$ (resp. $F_\overline{v} = \{ F_{\overline{v}_1}, \ldots, F_{\overline{v}_{j,m_j}}, \ldots, F_{\overline{v}_t} \}$)

of packets of equivalent real completions covering the left (resp. right) tower

$F_w = \{ F_{w_1}, \ldots, F_{w_{j,m_j}}, \ldots, F_{w_t} \} \equiv r$ (resp. $F_\overline{w} = \{ F_{\overline{w}_1}, \ldots, F_{\overline{w}_{j,m_j}}, \ldots, F_{\overline{w}_t} \}$)

of corresponding complex completions.

**•** Let $G^{(n)}(F_\overline{v} \times F_v)$ be the bilinear algebraic semigroup with entries in the product, right by left, $F_\overline{v} \times F_v$ of towers of packets of equivalent real completions.

$G^{(n)}(F_\overline{v} \times F_v)$ is then composed of conjugacy class representatives $G^{(n)}(F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}})$, $1 \leq j \leq t \leq \infty$, having multiplicities $1 \leq m_j \leq m^{(j)}$ and being $\text{GL}_n(F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}})$-subbisemimodules $\subset G^{(n)}(F_\overline{v} \times F_v)$.

Similarly, the complex bilinear algebraic semigroup $G^{(n)}(F_\overline{w} \times F_w)$ with entries in the product, right by left, $F_\overline{w} \times F_w$ of complex completions, is composed of a tower of increasing conjugacy class representatives $G^{(n)}(F_{\overline{w}_{j,m_j}} \times F_{w_{j,m_j}})$, $1 \leq j \leq r \leq \infty$, $t = r$, being $\text{GL}_n(F_{\overline{w}_{j,m_j}} \times F_{w_{j,m_j}})$-subbisemimodules covered by the corresponding packets $G^{(n)}(F_{\overline{v}_{j,m_j}} \times F_{v_{j,m_j}})$ of real conjugacy class representatives.

### 2.2 Proposition: Double tower of increasing 3D-euclidean subspaces

Let $G_L^{(3)}(F_v) = \text{Repsp}(\text{GL}_3(F_v)) \equiv \text{Repsp}(T_3(F_v))$ (resp. $G_R^{(3)}(F_\overline{v}) = \text{Repsp}(\text{GL}_3(F_\overline{v})) \equiv \text{Repsp}(T_3(F_\overline{v}))$) denote the representation space of the left (resp. right) linear algebraic semigroup of real dimension 3 with entries in $F_v$ (resp. $F_\overline{v}$) such that:

- $G_L^{(3)}(F_v) \subset G^{(3)}(F_\overline{v} \times F_v)$;
- $G_R^{(3)}(F_\overline{v}) \subset G^{(3)}(F_\overline{v} \times F_v)$.

Then, $G_L^{(3)}(F_v)$ (resp. $G_R^{(3)}(F_\overline{v})$) is composed of a left (resp. right) tower

$G_L^{(3)}(F_{v_{j,m_j}}, m_j) \equiv j, m_j \equiv t$ (resp. $G_R^{(3)}(F_{\overline{v}_{j,m_j}}, m_j) \equiv j, m_j \equiv t$), $1 \leq j \leq t \leq \infty$, of increasing conjugacy class representatives which are three-dimensional euclidean subspaces localized in the upper (resp. lower) half 3D-space.
Thus we have that:

\[ G^{(3)}_L = E^3_L = \{ E^3_L(j, m_j) \} \quad \text{(resp.} \quad G^{(3)}_R = E^3_R = \{ E^3_R(j, m_j) \}) \]

where:

- \( E^3_L \) (resp. \( E^3_R \)) is the 3D-half upper (resp. lower) euclidean space;
- \( E^3_L(j, m_j) \) (resp. \( E^3_R(j, m_j) \)) is a 3D-half upper (resp. lower) euclidean subspace characterized by a rank \( r_{E^3} \approx (j \cdot N)^3 \).

Proof. 1. The conjugacy class representatives \( G^{(3)}_L(F_{v, j, m_j}) \) (resp. \( G^{(3)}_R(F_{v, j, m_j}) \)) are 3D-half upper (resp. lower) euclidean subspaces \( E^3_L(j, m_j) \) (resp. \( E^3_R(j, m_j) \)) because they are constructed from flat real completions \( F_{v, j, m_j} \) (resp. \( F_{\overline{v}, j, m_j} \)) submitted to the operator “flat” morphism [Pie1] (of a fibre bundle)

\[
T_3(F_{v, j, m_j}) : \quad F_{v, j, m_j} \longrightarrow G^{(3)}_L(F_{v, j, m_j})
\]

(resp. \( T_3(F_{\overline{v}, j, m_j}) : \quad F_{\overline{v}, j, m_j} \longrightarrow G^{(3)}_R(F_{\overline{v}, j, m_j}) \))

sending the completion \( F_{v, j, m_j} \) (resp. \( F_{\overline{v}, j, m_j} \)) into the left (resp. right) GL_3(F_{v, j, m_j})-subsemimodule \( G^{(3)}_L(F_{v, j, m_j}) \) (resp. GL_3(F_{\overline{v}, j, m_j})-subsemimodule \( G^{(3)}_R(F_{\overline{v}, j, m_j}) \)) which is euclidean because there is no deviation to euclidicity generated by the injective morphism \( T_3(F_{v, j, m_j}) \) (resp. \( T_3(F_{\overline{v}, j, m_j}) \)) as developed in [Pie2].

2. \( E^3_L(j, m_j) \) and \( E^3_R(j, m_j) \) are characterized by a rank \( r_{E^3} = (j \cdot N)^3 \) because they are built from the completions \( F_{v_j} \) and \( F_{\overline{v}_j} \) having a Galois extension degree

\[ [F_{v_j} : k] = [F_{\overline{v}_j} : k] = j \cdot N \]

where \( j \) is a global residue degree and \( N \) is the degree of an irreducible completion [Pie1].

\[ \blacksquare \]

2.3 Proposition: Double tower of increasing 2D-tori

Let \( F_v^T \) (resp. \( F_{\overline{v}}^T \)) be the set of packets of real completions compactified toroidally.

Let \( G^{(2)}_L(F_v^T) = \text{Repsp}(T_2(F_v^T)) \) (resp. \( G^{(2)}_R(F_{\overline{v}}^T) = \text{Repsp}(T_2(F_{\overline{v}}^T)) \)) denote the representation space of the left (resp. right) linear algebraic semigroup of real dimension 2 with entries in \( F_v^T \) (resp. \( F_{\overline{v}}^T \)).

Then, \( G^{(2)}_L(F_v^T) \) (resp. \( G^{(2)}_R(F_{\overline{v}}^T) \)) \( \subset G^{(2)}(F_v^T \times F_v^T) \) is composed of a left (resp. right) tower of increasing conjugacy class representatives which are two-dimensional (semi)tori \( T^2_L(j) \) (resp. \( T^2_R(j) \)), \( 1 \leq j \leq t \leq \infty \), localized in the upper (resp. lower) half 3D-space.
2.4 Three-dimensional and two-dimensional (semi)sheaves of differentiable functions

- Let $\phi_{GL}^{(3)}(x_{(3)})$ (resp. $\phi_{GR}^{(3)}(x_{(3)})$) denote the set $\{\phi_{G_{j,m}L}^{(3)}(x_{(3)})\}_{j,m,j}$ (resp. $\{\phi_{G_{j,m}R}^{(3)}(x_{(3)})\}_{j,m,j}$) of smooth differentiable functions on the set $\{E_{L}^{3}(j,m)\}$ (resp. $\{E_{R}^{3}(j,m)\}$) of increasing conjugacy class representatives of $G_{L}^{(3)}(F_{v})$ (resp. $G_{R}^{(3)}(F_{v})$).

- Similarly, let $\phi_{GL}^{(2)}(x_{(2)})$ (resp. $\phi_{GR}^{(2)}(x_{(2)})$) denote the set $\{\phi_{G_{j,m}L}^{(2)}(x_{(2)})\}_{j,m,j}$ (resp. $\{\phi_{G_{j,m}R}^{(2)}(x_{(2)})\}_{j,m,j}$) of smooth differentiable functions on the set $\{T_{L}^{2}(j)\}_{j}$ (resp. $\{T_{R}^{2}(j)\}_{j}$) of increasing two-dimensional (semi)tori of $G_{L}^{(2)}(F_{v})$ (resp. $G_{R}^{(2)}(F_{v})$). This set $\phi_{GL}^{(2)}(x_{(2)})$ (resp. $\phi_{GR}^{(2)}(x_{(2)})$) of smooth differentiable functions is the set of sections of a semisheaf of rings $\theta_{GL}^{(2)}$ (resp. $\theta_{GR}^{(2)}$) on the left (resp. right) algebraic semigroup $G_{L}^{(2)}(F_{v})$ (resp. $G_{R}^{(2)}(F_{v})$).

2.5 Proposition: Langlands global correspondence – 2D real case

Let $\sigma_{j}^{(2)}(W_{F_{\overline{\nu}_{j}}} \times W_{F_{v_{j}}}) = G^{(2)}(F_{\overline{\nu}_{j}} \times F_{v_{j}})$ denote the 2-dimensional representation subspace of the product, right by left, $W_{F_{\overline{\nu}_{j}}} \times W_{F_{v_{j}}}$ of the Weil subgroups restricted to $F_{\overline{\nu}_{j}}$ and $F_{v_{j}}$.

Let $\phi_{GL}^{(2)}(T_{L}^{2}(j))$ (resp. $\phi_{GR}^{(2)}(T_{R}^{2}(j))$), denoting the smooth differentiable left (resp. right) function on the (semi)torus $T_{L}^{2}(j)$ (resp. $T_{R}^{2}(j)$), be the cuspidal representation $\Pi_{j}(GL_{2}(F_{v_{j}}))$ (resp. $\Pi_{j}(GL_{2}(F_{\overline{\nu}_{j}}))$) of the $j$-th conjugacy class representative of the algebraic semigroup $GL_{2}(F_{v})$ (resp. $GL_{2}(F_{\overline{\nu}})$).
Then, there exists a Langlands global correspondence

\[ T_L : \sigma^{(2)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) \longrightarrow \Pi(\text{GL}_2(F_{\wp}^{ab} \times F_{\wp}^{ab})) \]

between the sum \( \sigma^{(2)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) \) of the 2-dimensional conjugacy class representatives of the product, right by left, of the Weil subgroups given by the algebraic bilinear semigroup \( G^{(2)}(F_{\wp}^{ab} \times F_{\wp}^{ab}) \) and its cuspidal representation given by \( \Pi(\text{GL}_2(F_{\wp}^{ab} \times F_{\wp}^{ab})) \), where \( F_{\wp} = \bigoplus_{j,m_j} F_{\wp,j,m_j} \).

Proof. This is rather immediate if we refer to the preprint [Pie1]. Indeed, the product, right by left, \( T_R^2(j) \times T_L^2(j) \) of the (semi)tori \( T_R^2(j) \) and \( T_L^2(j) \) results from a bijective toroidal compactification of the conjugacy class \( G^{(2)}(F_{\tau_j} \times F_{\tau_j}) \) of \( G^{(2)}(F_{\tau} \times F_{\tau}) \).

So, we have that: \( T_L^2(j) \times T_R^2(j) = G^{(2)}(F_{\tau}^{j} \times F_{\tau}^{j}) \).

And, the smooth differentiable bifunction (i.e. the product of a right function by its left equivalent) \( \phi_G^{(2)}(T_R(j)) \otimes \phi_G^{(2)}(T_L(j)) \) on \( (T_R(j) \times T_L(j)) \) constitutes the cuspidal representation \( \Pi_j(\text{GL}_2(F_{\tau_j} \times F_{\tau_j})) \) of \( \text{GL}_2(F_{\tau_j} \times F_{\tau_j}) \) or \( G^{(2)}(F_{\tau_j} \times F_{\tau_j}) \).

So, the sum \( \sigma^{(2)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) \) of the 2-dimensional conjugacy class representatives of the Weil subgroups given by:

\[ \sigma^{(2)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) = \bigoplus_j (G^{(2)}(F_{\tau_j} \times F_{\tau_j})) \]

is in one-to-one correspondence with the searched cuspidal representation \( \Pi(\text{GL}_2(F_{\wp}^{ab} \times F_{\wp}^{ab})) \) since:

\[ \sigma^{(2)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) \simeq \bigoplus_j (\Pi_j(\text{GL}_2(F_{\tau_j} \times F_{\tau_j}))) \]

\[ = \Pi(\text{GL}_2(F_{\wp}^{ab} \times F_{\wp}^{ab})). \]

It then appears that to the set of products, right by left, of the (semi)tori \( T_R^2(j) \) and \( T_L^2(j) \), \( 1 \leq j \leq t \leq \infty \), referring to the toroidal representation space of the two-dimensional bilinear algebraic semigroup \( G^{(2)}(F_{\tau} \times F_{\tau}) \), corresponds a cuspidal representation \( \Pi(\text{GL}_2(F_{\wp}^{ab} \times F_{\wp}^{ab})) \), i.e. a Langlands global correspondence.

In the three-dimensional case, as we are dealing with the set of products, right by left, of euclidean (semi)spaces \( E_R^3(j,m_j) \) and \( E_L^3(j,m_j) \), referring to the representation subspaces of the 3D-bilinear algebraic semigroup \( G^{(3)}(F_{\tau} \times F_{\tau}) \), a holomorphic representation of \( G^{(3)}(F_{\tau} \times F_{\tau}) \) can be built according to [Pie1] but not a cuspidal representation. Nevertheless, if a toroidal compactification of these subspaces \( E_R^3(j,m_j) \) and \( E_L^3(j,m_j) \), giving rise to the corresponding 3D-tori \( T_R^3(j,m_j) \) and \( T_L^3(j,m_j) \), is envisaged, then a cuspidal representation of \( \text{GL}_3(F_{\tau} \times F_{\tau}) \) can be obtained, leading to a 3D-Langlands global correspondence as it was developed for the two-dimensional case in Proposition 2.5.
2.6 Introducing singularizations

The generation of singularities, called singularizations, will now be recalled; they consist of collapses of normal crossings divisors into the singular loci and correspond to contracting surjective morphisms being inverse of those of resolutions of singularities [Dej], [Hir], [Zar].

These singularizations will be envisaged on the smooth differentiable functions $\phi^{(3)}_L(E^3_L(j, m_j))$ (resp. $\phi^{(3)}_R(E^3_R(j, m_j))$) and $\phi^{(2)}_L(T^2_L(j))$ (resp. $\phi^{(2)}_R(T^2_R(j))$) respectively on 3D-euclidean (semi)spaces and on 2D-(semi)tori. To facilitate the notations, they will be written indistinctly $\phi_L$ (resp. $\phi_R$) until the section 2.10.

A normal crossings divisor will be assumed to be a function on one or a set of real irreducible completions of rank $N$ [Pie2].

2.7 Proposition: Contracting surjective morphism of singularization

Let $D_L$ (resp. $D_R$) be a normal crossings divisor of the regular function $\overline{\phi}_L$ (resp. $\overline{\phi}_R$) given by:

$$\overline{\phi}_L = \phi_L \cup D_L \quad (\text{resp.} \quad \overline{\phi}_R = \phi_L \cup D_R).$$

The singularization of $\overline{\phi}_L$ (resp. $\overline{\phi}_R$) into the singular locus $\Sigma_L$ (resp. $\Sigma_R$) results from the contracting surjective morphism:

$$\overline{\rho}_L : \overline{\phi}_L \longrightarrow \phi_L^* \quad (\text{resp.} \quad \overline{\rho}_R : \overline{\phi}_R \longrightarrow \phi_R^*)$$

in such a way that:

a) $\Sigma_L \subset \phi_L^*$ (resp. $\Sigma_R \subset \phi_R^*$) be the union of the homotopic image of $D_L \subset \overline{\phi}_L$ (resp. $D_R \subset \overline{\phi}_R$) and of a possible closed singular sublocus $\Sigma_L^S \subset \Sigma_L$ (resp. $\Sigma_R^S \subset \Sigma_R$) of $\phi_L^*$ (resp. $\phi_R^*$):

$$\Sigma_L = \overline{\rho}_L(D_L) \cup \Sigma_L^S \quad (\text{resp.} \quad \Sigma_R = \overline{\rho}_R(D_R) \cup \Sigma_R^S);$$

b) $\overline{\rho}_L$ (resp. $\overline{\rho}_R$) restricted to:

$$\overline{\rho}_L^S : \phi_L \setminus \overline{\rho}_L^{-1}(\Sigma_L^S) \longrightarrow \phi_L^* \setminus \Sigma_L \quad (\text{resp.} \quad \overline{\rho}_R^S : \phi_R \setminus \overline{\rho}_R^{-1}(\Sigma_R^S) \longrightarrow \phi_R^* \setminus \Sigma_R)$$

be an isomorphism.

Proof. The singular sublocus $\Sigma_L^S \subset \phi_L^*$ (resp. $\Sigma_R^S \subset \phi_R^*$) results from singularizations anterior to that of $\overline{\rho}_L(D_L)$ (resp. $\overline{\rho}_R(D_R)$) and then becomes the singular locus of a possible future blowup.
• Let $\overline{\rho}_L^S : D_L \to R \setminus \Sigma_L^S$ (resp. $\overline{p}_R^S : D_R \to R \setminus \Sigma_R^S$) be the singularization morphism restricted to the singular locus $\Sigma_L \setminus \Sigma_L^S$ (resp. $\Sigma_R \setminus \Sigma_R^S$).

Then, $\Sigma_L \setminus \Sigma_L^S$ (resp. $\Sigma_R \setminus \Sigma_R^S$) is the contracting homotopic image of $D_L$ (resp. $D_R$) in such a way that $\overline{p}_L \setminus \overline{p}_L^S$ (resp. $\overline{p}_R \setminus \overline{p}_R^S$) be a surjective morphisms.

• The inverse morphism

$$\overline{p}_L^{-1} : \phi_L \to \overline{\phi}_L \quad \text{(resp. } \overline{p}_R^{-1} : \phi_R \to \overline{\phi}_R\text{)}$$

of the singularization $\overline{\rho}_L$ (resp. $\overline{\rho}_R$) corresponds to the blowup of the singular locus $\Sigma_L$ (resp. $\Sigma_R$) of $\phi_L$ (resp. $\phi_R$) since

$$\overline{p}_L^{-1} \setminus (\overline{\rho}_L^{-1})^1 : \Sigma_L \to D_L \quad \text{(resp. } \overline{p}_R^{-1} \setminus (\overline{\rho}_R^{-1})^1 : \Sigma_R \to D_R\text{)}$$

is a projective morphism sending the singular locus $\Sigma_L$ (resp. $\Sigma_R$) into the projective normal crossings divisor $D_L$ (resp. $D_R$).

2.8 Proposition: Sequence of surjective morphisms of singularizations

The singularization $\overline{\rho}_L : \overline{\phi}_L \to \phi_L^*$ (resp. $\overline{\rho}_R : \overline{\phi}_R \to \phi_R^*$) of the smooth function $\overline{\rho}_L$ (resp. $\overline{\phi}_R$) is given by the following sequence of contracting surjective morphisms:

$$\overline{\phi}_L \equiv \phi_L^1 \overline{\rho}_L^1 \to \phi_L^2 \overline{\rho}_L^2 \to \phi_L^3 \overline{\rho}_L^3 \to \ldots$$

$$\text{(resp. } \overline{\phi}_R \equiv \phi_R^1 \overline{\rho}_R^1 \to \phi_R^2 \overline{\rho}_R^2 \to \phi_R^3 \overline{\rho}_R^3 \to \ldots\text{)}$$

where $\overline{\rho}_L^{-1}$ denotes the $(r-1)$-th surjective morphism of singularization of $\overline{\phi}_L$ generating $\phi_L^*(r)$, in such a way that:

a) the singular locus $\Sigma_L \subset \phi_L^*$ (resp. $\Sigma_R \subset \phi_R^*$) is given by:

$$\Sigma_L \equiv \Sigma_L^r = \overline{\rho}_L^1(D_L^0) \cup \overline{\rho}_L^2(D_L^1) \cup \ldots \cup \overline{\rho}_L^r(D_L^{r-1})$$

$$\text{(resp. } \Sigma_R \equiv \Sigma_R^r = \overline{\rho}_R^1(D_R^0) \cup \overline{\rho}_R^2(D_R^1) \cup \ldots \cup \overline{\rho}_R^r(D_R^{r-1})\text{)}$$

where $\Sigma_L^1 = \overline{\rho}_L^1(D_L^0)$ (resp. $\Sigma_R^1 = \overline{\rho}_R^1(D_R^0)$);

b) $\overline{\rho}_L$ (resp. $\overline{\rho}_R$) restricted to:

$$\overline{\rho}_L^{(i)} : \phi_L \to \overline{\rho}_L^{-1}(\Sigma_L) \to \phi_L^* \setminus \Sigma_L$$

$$\text{(resp. } \overline{\rho}_R^{(i)} : \phi_R \to \overline{\rho}_R^{-1}(\Sigma_R) \to \phi_R^* \setminus \Sigma_R\text{)}$$

is an isomorphism;
c) The orders “ℓ” of the singular subloci $\Sigma^{(\ell)}_L$ (resp. $\Sigma^{(\ell)}_R$) form an increasing sequence from left to right, $1 \leq \ell \leq r$.

Proof. This proposition is an evident generalization of proposition 2.7 to a set of successive surjective morphisms of singularization giving rise to a singular locus $\Sigma^{(r)}_L$ (resp. $\Sigma^{(r)}_R$) of order “$r$”.

2.9 Definition: Corank of the singular locus

Let $P(x_L, y_L, z_L)$ (resp. $P(x_R, y_R, z_R)$) be the polynomial characterizing the germ of the singular function $\phi_L^*$ (resp. $\phi_R^*$) and also the singular locus $\Sigma_L$ (resp. $\Sigma_R$).

The number of variables of this polynomial is the corank of the germ $\phi_L^*$ (resp. $\phi_R^*$). This corank is inferior or equal to 3 according to [A-V-G].

If the singular locus $\Sigma_L$ (resp. $\Sigma_R$) is given by a singular point of finite codimension, then the corresponding simple germs of differentiable functions are:

- $A_k$: $P(x) = x^{k+1}$, $k \geq 1$,
- $D_k$: $P(x, y) = x^2y + y^{k-1}$, $k \geq 4$,
- $E_6$: $P(x, y) = x^3 + y^4$,
- $E_7$: $P(x, y) = x^3 + xy^3$,
- $E_8$: $P(x, y) = x^3 + y^5$.

2.10 The Malgrange division theorem for germs of corank 1

The Malgrange division theorem for differentiable functions, being the corner stone of the versal deformation, will now be recalled for germs of functions having a singularity of corank 1 and order $k$.

Let $x_L' = (x_{1L}, x_{2L}, \omega_L)$ (resp. $x_R' = (x_{1R}, x_{2R}, \omega_R)$) be a triple of coordinates in such a way that $x_L'$ (resp. $x_R'$) be localized in the upper (resp. lower) half 3D-space.

A germ $P(\omega_L)$ (resp. $P(\omega_R)$) has a singularity of corank 1 and order $k$ in $\omega_L$ (resp. $\omega_R$) if $P(0, \omega_L) = \omega_L^k e(\omega_L)$ (resp. $P(0, \omega_R) = \omega_R^k e(\omega_R)$) where $e(\omega_L)$ (resp. $e(\omega_R)$) is a differentiable unit verifying $e(0) \neq 0$.

Let $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) be the algebra of polynomials in $\omega_L$ (resp. $\omega_R$) with coefficients $a(x_L)$ (resp. $a(x_R)$) being ideals of functions defined on a domain $D_{BL}$ (resp. $D_{BR}$) included in an open ball centered on $\omega_L$ (resp. $\omega_R$).

$x_L = (x_{1L}, x_{2L})$ (resp. $x_R = (x_{1R}, x_{2R})$) are bituples of coordinates in the upper (resp. lower) half space.
The Malgrange division theorem for a germ $P(\omega_L)$ (resp. $P(\omega_R)$) of corank 1 and order $k$ then corresponds to the versal unfolding of $P(\omega_L)$ (resp. $P(\omega_R)$) and is given by [Mal], [Thom], [Mat]:

$$f_L = P(\omega_L)\, q_L + R_L \quad \text{(resp. } f_R = P(\omega_R)\, q_R + R_R\text{)}$$

where:

- $f_L$ (resp. $f_R$) is a 3D-differentiable function (germ);
- $q_L$ (resp. $q_R$) is a 2D-differentiable function (germ);
- $R_L = \sum_{i=1}^{s} a_i(x_L)\, \omega^i_L \in \theta[\omega_L]$ 
  \hspace{1cm} (resp. $R_R = \sum_{i=1}^{s} a_i(x_R)\, \omega^i_R \in \theta[\omega_R]$)

is a polynomial with degree $s < k$, $s \leq 3$ in the 3D-case.

The division theorem can be easily stated for germs of corank 2 and 3, as developed in [Pie2].

### 2.11 Singularization of semisheaves

We now come back to the notations of section 2.4 where a left (resp. right) semisheaf $\theta^{(3)}_{G_L}$ (resp. $\theta^{(3)}_{G_R}$) of 3D-differentiable functions $\phi_{G_j,m}^{(3)}(E^3_L(j,m))$ (resp. $\phi_{G_j,m}^{(3)}(E^3_R(j,m))$) on upper (resp. lower) 3D-euclidean (semi)spaces was introduced. Similarly, a left (resp. right) semisheaf $\theta^{(2)}_{G_L}$ (resp. $\theta^{(2)}_{G_R}$) of 2D-differentiable functions $\phi_{G_j}^{(2)}(T^2_L(j))$ (resp. $\phi_{G_j}^{(2)}(T^2_R(j))$) on upper (resp. lower) 2D-(semi)tori was envisaged. The two cases will be considered in the following but the developments will only concern here the 3D-case, the 2D-case being treated similarly.

The singularization of the semisheaf $\theta^{(3)}_{G_L}$ (resp. $\theta^{(3)}_{G_R}$) in the sense of proposition 2.8, given by the contracting surjective morphism(s):

$$\overline{\rho}_{G_L} : \theta^{(3)}_{G_L} \longrightarrow \theta^{*}_{G_L} \quad \text{(resp. } \overline{\rho}_{G_R} : \theta^{(3)}_{G_R} \longrightarrow \theta^{*}_{G_R}\text{)}$$

concerns all the sections $\phi_{G_j,m}^{(3)}(E^3_L(j,m)) \subset \theta^{(3)}_{G_L}$ (resp. $\phi_{G_j,m}^{(3)}(E^3_R(j,m)) \subset \theta^{(3)}_{G_R}$) which are affected with germs $P_j(\omega_L)$ (resp. $P_j(\omega_R)$) having degenerate singularities of corank inferior or equal to 3.

### 2.12 Proposition: Versal deformation [G-K]

Let $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) be the algebra of polynomials $R_L$ (resp. $R_R$) of the versal unfolding of the germs $P_j(\omega_L)$ (resp. $P_j(\omega_R)$) of the sections of the semisheaf $\theta^{(3)}_{G_L}$ (resp. $\theta^{(3)}_{G_R}$).
Let \( \theta_{SL} = \{ \theta(\omega_L^1), \ldots, \theta(\omega_L^s) \} \) (resp. \( \theta_{SR} = \{ \theta(\omega_R^1), \ldots, \theta(\omega_R^s) \} \)) denote the family of (semi)sheaves of monomial functions \( \omega_L^i \) (resp. \( \omega_R^i \)) of the polynomials \( R_L \in \theta[\omega_L] \) (resp. \( R_R \in \theta[\omega_R] \)).

Then, the versal deformation of the singular semisheaf \( \theta^*_G^{(3)} \) (resp. \( \theta^*_G^{(3)} \)) is given by the contracting fiber bundle:

\[
D_{SL} : \theta^*_G^{(3)} \times \theta_{SL} \longrightarrow \theta^*_G^{(3)} \quad (\text{resp.} \quad D_{SR} : \theta^*_G^{(3)} \times \theta_{SR} \longrightarrow \theta^*_G^{(3)})
\]

in such a way that \( \theta^{\text{vers}}_{G_L^{(3)}} = \theta^*_G^{(3)} \times \theta_{SL} \) (resp. \( \theta^{\text{vers}}_{G_R^{(3)}} = \theta^*_G^{(3)} \times \theta_{SR} \)), being the versal deformation of \( \theta^*_G^{(3)} \) (resp. \( \theta^*_G^{(3)} \)), is the total space of the fiber bundle \( D_{SL} \) (resp. \( D_{SR} \)) \([G-K], [Mat]\).

Proof. The algebra of polynomials \( \theta[\omega_L] \) (resp. \( \theta[\omega_R] \)) is given by \( \theta[\omega_L] = \theta_{SL} \times \theta_{a_L} \) (resp. \( \theta[\omega_R] = \theta_{SR} \times \theta_{a_R} \)) where \( \theta_{a_L} \) (resp. \( \theta_{a_R} \)) is the sheaf of functions \( a(x_L) \) (resp. \( a(x_R) \)) introduced in section 2.10.

The quotient algebra of polynomials \( \theta[\omega_L] \) (resp. \( \theta[\omega_R] \)) is thus finitely generated: it is the quotient of the algebra \( \mathcal{E}_L \) (resp. \( \mathcal{E}_R \)) of function germs (generally given by integer power series) by the graded ideal \( I_{P_L} \) (resp. \( I_{P_R} \)) of germs \( P(\omega_L) \) (resp. \( P(\omega_R) \)) \([A-V-G]\):

\[
\theta[\omega_L] = \mathcal{E}_L/I_{P_L} \quad (\text{resp.} \quad \theta[\omega_R] = \mathcal{E}_R/I_{P_R}).
\]

The quotient algebra \( \theta[\omega_L] \) (resp. \( \theta[\omega_R] \)) is thus finitely generated: it is composed of the polynomials \( R_L \) (resp. \( R_R \)) which generate vector (semi)spaces of dimension “s” which is the codimension of the versal deformation. So, \( \theta[\omega_L] \) (resp. \( \theta[\omega_R] \)) and, thus, \( \theta_{SL} \) (resp. \( \theta_{SR} \)) define the versal deformation of the singular semisheaf \( \theta^*_G^{(3)} \) (resp. \( \theta^*_G^{(3)} \)).

2.13 Proposition: Sequence of versal subdeformations

The versal unfolding of a germ of differentiable functions is generated by a sequence of contracting morphisms extending the sequence of contracting surjective morphisms of singularization.

Proof. This proposition was proved in [Pie2] for the versal unfolding of the germ \( P(\omega_L) = \omega_L^{k+1} \) (resp. \( P(\omega_R) = \omega_R^{k+1} \)). Indeed, it was shown that a sequence of \((k-1)\) contracting fiber bundles, whose fibers are divisors projected in the neighbourhood of the singular germ, is responsible for a sequence of \((k-1)\) versal subdeformations generating finitely (i.e. term by term) the polynomial \( R_L \) (resp. \( R_R \)) of the quotient algebra \( \theta[\omega_L] \) (resp. \( \theta[\omega_R] \)). The order of the divisors, projected in the neighbourhood of the singular germ, increases in function of the increase of the dimension of the generated vector sub(semi)spaces of the versal unfolding. By this way, the space around the singularity becomes more and more compact in relation with the increase of the (co)dimension of the versal unfolding.
3 Natural generation of the three-dimensional geometries of Thurston

Before proving that the non-euclidean 3D geometries proceed from singularization morphisms and versal deformations of these, the origin of the hyperbolic and spherical geometries will be introduced.

3.1 Left and right actions of the Kleinian group

- The Kleinian group \( G_K \) of \( \mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \} \) is the group of Möbius transformations of \( \mathbb{R}^n \) acting discontinuously somewhere in \( \mathbb{R}^n \).

The action of the Kleinian group \( G_K \) can be extended to \( \mathbb{H}^{n+1} = \mathbb{H}^{n+1} \cup \mathbb{R}^n \) where \( \mathbb{H}^{n+1} = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0 \} \) is the \((n+1)\)-dimensional hyperbolic space: \( G_K \) thus acts as a group of isometries of \( \mathbb{H}^{n+1} \) with the hyperbolic metric.

The orbit space \( M_{G_K} \) of the Kleinian group \( G_K \) is defined by:
\[
M_{G_K} = (\mathbb{H}^{n+1} \setminus L(G_K))/G_K
\]
where \( L(G_K) \) is the limit set of \( G_K \) \([Mil3]\).

This limit set is the closure of the set of fixed points of non-elliptic elements of \( G_K \) \([Abi]\). It is a nowhere dense set whose area measure is zero.

An ordinary set \( \Omega(G_K) \) of the Kleinian group \( G_K \) is given by:
\[
\Omega(G_K) = \mathbb{R}^n \setminus L(G_K).
\]
Recall that a Möbius transformation \( g \) of \( \mathbb{R}^n \) is loxodromic if it is a transformation of the form \( g(x) = \lambda \alpha(x) \) where \( x \in \mathbb{R}^n \), \( \lambda > 1 \) and \( \alpha \in O(n) \) is the orthogonal group of \( \mathbb{R}^n \). \( g \) is hyperbolic if \( \alpha = \text{id.} \), elliptic if \( \lambda = 1 \) and parabolic if \( g \) has the form \( g(x) = \alpha(x) + a \), where \( a \in \mathbb{R}^n \setminus \{0\} \).

- Similarly, left (resp. right) Möbius transformations \( g_L \) (resp. \( g_R \)), \( g_L \equiv g \), acting discontinuously in the upper (resp. lower) half space \( G^{(n)}(F_v) \) (resp. \( G^{(n)}(F_\overline{v}) \)) can be introduced as well as the left (resp. right) action of the Kleinian group \( G_K \) on the upper (resp. lower) \((n+1)\)-dimensional hyperbolic half space \( H^{n+1}_L \equiv H^{n+1} \) (resp. \( H^{n+1}_R \)). The left (resp. right) orbit space \( M_{G_KL} \) (resp. \( M_{G_KR} \)) associated with the left (resp. right) action of the Kleinian group is defined by:
\[
M_{G_KL} = (\mathbb{H}^{n+1}_L \setminus L(G_{K_L}))/G_{K_L} \quad \text{(resp. \( M_{G_KR} = (\mathbb{H}^{n+1}_R \setminus L(G_{K_R}))/G_{K_R} \))}
\]
where \( L(G_{K_L}) \) (resp. \( L(G_{K_R}) \)) is the limit set of the Kleinian group \( G_K \) acting on the upper (resp. lower) half space.

A left (resp. right) ordinary set \( \Omega(G_{K_L}) \) (resp. \( \Omega(G_{K_R}) \)) of \( G_{K_L} \) (resp. \( G_{K_R} \)) is given by:
\[
\Omega(G_{K_L}) = G^{(n)}(F_v) \setminus L(G_{K_L}) \quad \text{(resp. \( \Omega(G_{K_R}) = G^{(n)}(F_\overline{v}) \setminus L(G_{K_R}) \))}
\]
where \( G^{(n)}(F_v) \) and \( G^{(n)}(F_\overline{v}) \) are introduced in section 2.1.
3.2 Proposition: Hyperbolic geometry in the neighbourhood of the singular locus

Let $\Sigma_L$ (resp. $\Sigma_R$) be the singular locus of a germ of corank inferior or equal to 3 on the singular function $\phi_L^*$ (resp. $\phi_R^*$).

Let $D_{\Sigma_L}$ (resp. $D_{\Sigma_R}$) denote the neighbourhood of this singular locus.

Thus, we have that:

1. the limit set $L(G_K^L)$ (resp. $L(G_K^R)$) of the Kleinian group $G_K^L$ (resp. $G_K^R$) corresponds to the singular locus $\Sigma_L$ (resp. $\Sigma_R$).

2. The ordinary set(s) $\Omega(G_K^L)$ (resp. $\Omega(G_K^R)$), characterized by a hyperbolic geometry, correspond to the neighbourhood $D_{\Sigma_L}$ (resp. $D_{\Sigma_R}$) of the singular locus $\Sigma_L$ (resp. $\Sigma_R$).

Proof. 1. The limit set $L(G_K^L)$ (resp. $L(G_K^R)$) is a nowhere dense set, and, furthermore, it has a measure equal to zero: thus, it must correspond to the singular locus $\Sigma_L$ (resp. $\Sigma_R$).

2. From section 3.1, it then results that the ordinary set $\Omega(G_K^L)$ (resp. $\Omega(G_K^R)$) of the Kleinian group $G_K^L$ (resp. $G_K^R$) is characterized by a hyperbolic geometry.

It thus corresponds to the neighbourhood $D_{\Sigma_L}$ (resp. $D_{\Sigma_R}$) of the singular locus $\Sigma_L$ (resp. $\Sigma_R$).

On the other hand, it is clear from the sequence of contracting surjective morphisms of singularizations, developed in propositions 2.7 and 2.8, that the neighbourhood of the singular locus must be affected by a hyperbolic geometry.

Finally, it was proved in proposition 2.3.1 of [Pie2] that there is a deviation to Euclidicity in the neighbourhood $D_{\Sigma_L}$ (resp. $D_{\Sigma_R}$) of the singular locus which leads to consider a non-euclidean hyperbolic space of curvature “$-K$” on each stratum of $D_{\Sigma_L}$ (resp. $D_{\Sigma_R}$).

3.3 Proposition: Versal deformation characterized by a spherical geometry

Let $P(\omega_L)$ (resp. $P(\omega_R)$) denote a singular germ of corank inferior or equal to 3 ad codimension $\leq 3$ on the 3D-differentiable function $\phi_{G_{j,m},L}^{(3)}(E^3_L(j,m))$ (resp. $\phi_{G_{j,m},R}^{(3)}(E^3_R(j,m))$): so, $\omega_L = (\omega_1)$ or $\omega_L = (\omega_1,\omega_2)$ or $\omega_L = (\omega_1,\omega_2,\omega_3)$ (resp. $\omega_R = (\omega_1)$ or $\omega_R = (\omega_1,\omega_2)$ or $\omega_R = (\omega_1,\omega_2,\omega_3)$).

Let $f_L = P(\omega_L)q_L + R_L$ (resp. $f_R = P(\omega_R)q_R + R_R$) denote the versal unfolding of the singular germ $P(\omega_L)$ (resp. $P(\omega_R)$) as described in section 2.10.

Then, the stratum $D_{f_L^{(3)}}$ (resp. $D_{f_R^{(3)}}$) of the unfolded function $f_L$ (resp. $f_R$) on $\phi_{G_{j,m},L}^{(3)}(E^3_L(j,m))$ (resp. $\phi_{G_{j,m},R}^{(3)}(E^3_R(j,m))$) is characterized by a spherical
Proof. We are thus concerned with the union of the functions
\( f_L \cup \phi_{G_{j,mjL}}^{(3)}(E^3_L(j,m_j)) \) (resp.
\( f_R \cup \phi_{G_{j,mjR}}^{(3)}(E^3_R(j,m_j)) \)) i.e. with the unfolded function
\( f_L \) (resp. \( f_R \)) on the 3D-dimensional substratum function
\( \phi_{G_{j,mjL}}^{(3)}(E^3_L(j,m_j)) \) (resp. \( \phi_{G_{j,mjR}}^{(3)}(E^3_R(j,m_j)) \)).

According to section 2.10 and proposition 2.12, this is equivalent to consider in the neighbourhood
of the singular germ \( P(\omega_L) \) (resp. \( P(\omega_R) \)), i.e. on the functions
\( a_i(x_L) \subset \phi_{G_{j,mjL}}^{(3)}(E^3_L(j,m_j)) \) (resp. \( a_i(x_R) \subset \phi_{G_{j,mjR}}^{(3)}(E^3_R(j,m_j)) \)), \( a_i(x_L) \in R_L \) (resp. \( a_i(x_R) \in R_R \), \( 1 \leq i \leq 3 \), the projection of the monomial functions \( \omega^i_L \) (resp. \( \omega^i_R \)) of the polynomials \( R_L \) (resp. \( R_R \)) of the quotient algebra of the versal deformation.

Consequently, the stratum \( D_{f_L|\phi_{L}^{(3)}} \) (resp. \( D_{f_R|\phi_{R}^{(3)}} \)) of \( P(\omega_L) \) (resp. \( P(\omega_R) \)) on
\( \phi_{G_{j,mjL}}^{(3)}(E^3_L(j,m_j)) \) (resp. \( \phi_{G_{j,mjR}}^{(3)}(E^3_R(j,m_j)) \)) is overcrowded leading to a deviation of Euclidicity characterized by a positive sectional curvature +\( K > 0 \) and by a spherical geometry as proved in \([Pie2]\). □

### 3.4 From the unperturbed Thurston’s program to the perturbed one

We shall now analyze the possible origin of the eight three-dimensional geometries of the Thurston’s program which can be stated as follows:

“if \( M \) is a (closed) oriented prime 3-manifold, then there is a finite set of disjoint embedded 2-tori \( T^2(j) \) such that each component of the complement in \( M \) of \( \cup T^2(j) \) admits a geometric structure in the sense of admitting a complete metric, the (necessarily complete) universal metric cover of which is one of the eight three-dimensional model geometries \([Gre]\).”

As indicated precedingly, the Thurston’s program will be split here into an unperturbed and into a perturbed one in such a way that the perturbed part of the Thurston’s program results from deformations of the unperturbed Thurston’s program due to singularities.

So, the unperturbed Thurston’s program will be assumed to originate from the global Langlands program in real dimensions three and two in such a way that:

a) the representation space \( \text{Repsp}(GL_3(F_\tau \times F_v)) \) of the algebraic bilinear semigroup \( GL_3(F_\tau \times F_v) \) over the product, right by left, of sets \( F_\tau \) and \( F_v \) of symmetric completions of finite extensions of a global number field \( k \) of characteristic zero generates a double tower of increasing (compact) three-dimensional euclidean subspaces \( E^3_L(j,m_j) \) and
\( E^3_R(j, m_j) \), \( 1 \leq j \leq r \leq \infty \), localized respectively in the upper and lower half 3D-spaces as developed in proposition 2.2;

b) the representation space \( \text{Rep}(\text{GL}_2(F^T_l \times F^T_u)) \) of the two-dimensional algebraic bilinear semigroup \( \text{GL}_2(F^T_l \times F^T_u) \) over the product, right by left, of sets \( F^T_l \) and \( F^T_u \) of symmetric toroidal completions generates a double tower of increasing two-dimensional tori \( T^2_L(j) \) and \( T^2_R(j) \) as indicated in proposition 2.3.

The reason of considering two-dimensional algebraic bilinear semigroups in the unperturbed Thurston’s program is that:

a) the generated tori \( T^2_L(j) \) and \( T^2_R(j) \) are two-dimensional compact manifolds. If they are isotopic to boundary components of 3D-manifolds, then, these are said to be geometrically atoroidal [Thu1], [G-L-T].

b) these tori \( T^2_L(j) \) (resp. \( T^2_R(j) \)) may be glued together pairwise by diffeomorphisms to obtain a closed 3-manifold or a 3-manifold with toral boundary [And].

At this stage, we can take up the perturbed Thurston’s program which can be summarized in the three main propositions.

### 3.5 Proposition: Local geometries round singular loci on 3-manifolds

Assume that the sections \( \phi^{(3)}_{G^L_{j,m_j}}(E^3_L(j, m_j)) \) (resp. \( \phi^{(3)}_{G^R_{j,m_j}}(E^3_R(j, m_j)) \)) of the semisheaf \( \theta^{(3)}_{G^L_j} \) (resp. \( \theta^{(3)}_{G^R_j} \)) on the euclidean upper (resp. lower) 3-semispaces \( E^3_L(j, m_j) \) (resp. \( E^3_R(j, m_j) \)) are affected by singularization surjective morphisms in such a way that the coranks of their singular germs are inferior to equal of three.

Then, the neighbourhood of the singular loci of these singular germs of corank three, two and one are characterized respectively by the local geometries \( H^3 \), \( (H^2 \times \mathbb{R} \text{ or } \text{SL}_2(\mathbb{R})) \) and (Nil or Sol).

### 3.6 Proposition: Local geometries of versal deformations on 3-manifolds

Assume that the semisheaf \( \theta^{*}_{G^L_{j}} \) (resp. \( \theta^{*}_{G^R_{j}} \)), of which sections on the upper (resp. lower) 3-semisubspaces are affected by degenerate singularities of corank inferior or equal to three, is submitted to versal deformations of codimensions inferior or equal to three.

Then, the neighbourhoods of the unfolded germs in codimensions three, two and one on the sections of \( \theta^{*}_{G^L_{j}} \) (resp. \( \theta^{*}_{G^R_{j}} \)) are characterized respectively by the local geometries \( S^3 \), \( S^2 \times \mathbb{R} \) and Sol.
3.7 Proposition: The Poincare conjecture resulting from the Thurston’s program

Assume that the neighbourhood of an unfolded germ in codimension 3 on a section of $\theta^{(3)}_{G_L}$ or of $\theta^{(3)}_{G_R}$ is a closed simply connected 3-(semi)manifold. Then, it is the 3-“sphere” $S^3$. The elimination of the hypothesis of sphericality leads naturally to the Poincare conjecture.

3.8 Geometric structure on a 3-manifold

- The proofs of propositions 3.5, 3.6 and 3.7 are “clearly” based on the generation of the three-dimensional local geometries depending on singularities of corank 1, 2 and 3 and on their versal unfoldings in codimensions 1, 2 and 3.

These proofs will be developed in the following, but, before approaching this question, the geometric structure on a manifold will be introduced [B-T], [C-M], [Kol].

- The considered manifolds (or, more exactly, semimanifolds since they are localized in the upper or in the lower half space) are connected (differentiable) manifolds of dimension 3 generally without boundary [Sha].

A geometric structure on a manifold $M$ is defined by a locally Riemannian metric given by a positive definite quadratic form.

The isotropy group of the geometric structure at a point $x \in M$ is defined as the group $G_x$ of linear automorphisms of the tangent space $T_x M$ verifying

$$T_x \phi : T_x M \longrightarrow T_x M$$

where $T_x \phi$ are the differentials of the local isometries sending $x$ into itself [Bon].

In this respect, let $G$ be the group acting transitively on $M$ in such a way that the stabilizer $G_x$ of $x \in M$ is compact for the compact open topology.

A complete geometric structure on $M$ defines a complete $(X, G)$-structure on $M$, given by an atlas modelling locally $M$ over $X$, where $X$ is the universal covering of $M$ and where $G$ is the isometry group of $X$ [H-R-S].

A geometry in dimension 3 consists in a pair $(X, G)$ where $X$ is a connected 3-manifold on which the group $G$ acts transitively.

$G$ is generally a Lie group and the consideration of a Lie subgroup $H$ leads to the quotient $G/H$ having dimension 3. $H$ must be isomorphic to a closed subgroup of $O(3)$.

Thurston introduced eight geometries $(X, G)$ for which there is at least one finite volume complete $(X, G)$-structure [Bon].
• For example, the representation space $\text{Repsp}(\text{GL}_3(F_v \times F_v))$ of the algebraic bilinear semi-group $\text{GL}_3(F_v \times F_v)$ is composed of a left and a right (semi)manifold of which charts are respectively three-dimensional euclidean subspaces $E^3_L(j, m_j)$ and $E^3_R(j, m_j)$.

Thus, this left or right (semi)manifold, having by hypothesis a curvature being equal to zero, has a geometric structure $(X, G)$ in such a way that $X$ is isometric to the Euclidean space $E^3$ and $G$ is the isometry group Isom($E^3$) described by the exact sequence:

$$
O \longrightarrow \mathbb{R}^3 \longrightarrow \text{Isom}(E^3) \longrightarrow O(3) \longrightarrow O.
$$

$G$ is then a discrete group of isometries of $E^3$ and is torsion free.

If $G$ is a finite extension of $\mathbb{Z}$, $G$ is infinite cyclic and $E^3/G$ is the interior of a solid torus or a solid Klein bottle [Sco].

• We are now in a position to approach the proofs of propositions 3.5, 3.6 and 3.7.

3.9 Proof of Proposition 3.5: Local geometries round singular loci

We have to prove that the neighbouring chart of the singular locus of a degenerate singular germ of corank three, two and one is characterized respectively by the local geometry $H^3$, $H^2 \times \mathbb{R}$ or $\text{PSL}_2(\mathbb{R})$ and Nil or Sol.

We refer to the excellent paper of P. Scott [Sco] for the description of these geometries.

a) Case of corank 3: the geometry of $H^3$

It appears from proposition 3.2 that the ordinary sets $\Omega(G_K)$ in the neighbourhood of the singular locus of a degenerate singular germ are characterized by a hyperbolic geometry. As the envisaged singularity is of corank 3, the local geometry round the singular locus is the geometry $H^3$ characterized by a negative curvature which is equal to $-1$ if the metric is rescaled [McM], [Mil1], [Mil2].

In fact, the neighbourhood of the singular locus is isometric to the hyperbolic 3-space [Bra]

$$H^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0\}.$$

The group of orientations preserving the isometries of $H^3$ is isomorphic to $\text{PSL}(2, \mathbb{C})$: it is the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ given by maps of the form $z \to \frac{az + b}{cz + d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - cb \neq 0$.

The group of complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{R}^3_+$, extending its natural action on $\mathbb{C} \cup \{\infty\}$.
b) Case of corank 2:

1) The geometry of $H^2 \times \mathbb{R}$:

As we are considering the local geometry round a singular germ of corank 2, the ordinary sets $\Omega(G_K)$ round the singular locus must be characterized by the hyperbolic geometry $H^2$. But, as the charts $M_c$ of the sections of semi-sheaves $\theta^*_L\mathcal{G}(3)$ and $\theta^*_R\mathcal{G}(3)$ are three-dimensional, they must be characterized by a local geometry of type $H^2 \times \mathbb{R}$ leading to the natural action of the group $G = \text{Isom}(H^2) \times \text{Isom}(\mathbb{R})$. Thus, the isometry group of $H^2 \times \mathbb{R}$ is isomorphic to $\text{Isom}(H^2) \times \text{Isom}(\mathbb{R})$, and the local geometry $H^2 \times \mathbb{R}$ is non-isotropic.

If $G$ is the discrete group of isometries of $H^2 \times \mathbb{R}$ having as quotient the chart $M_c$, then the natural foliation of $H^2 \times \mathbb{R}$ by lines gives $M_c$ the structure of a line bundle over some hyperbolic surface in such a way that $M_c$ cannot be closed [Sco], [Zhe].

2) The geometry of $\text{PSL}_2(\mathbb{R})$:

But, there exists also a twisted version $H^2 \times \mathbb{R}$ of the local geometry $H^2 \times \mathbb{R}$ given by $T^1 H^2$ which is the unit tangent bundle of $H^2$, consisting of all tangent vectors of length 1 of $H^2$. Topologically, $H^2 \times \mathbb{R}$ is homeomorphic to $H^2 \times \mathbb{R}$.

The metric of $H^2$ fixes a metric on $T^1 H^2$ by taking into account that the tangent space $T^1 H^2$ at $v \in T^1 H^2$ splits as the direct sum of a line $L_v$ and of a plane $P_v$, where $L_v$ is the tangent line to the fibre $p^{-1}(p(v))$ where:

- $P_v$ consists of all infinitesimal parallel translations of $v$ along geodesics passing through the point $p(v) \in H^2$.
- $p : T^1 H^2 \to H^2$ is the natural projection associating its base point to each $v \in T^1 H^2$.

There is a natural identification of $T^1 H^2$ with $\text{PSL}_2(\mathbb{R})$, the orientation preserving the isometry group of $H^2$ on $T^1 H^2$. As the action of this group is transitive and free, the choice of a base point identifies this group with $T^1 H^2$. Every orientation preserving the isometry of $H^2$ is a linear fractional map of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1,$$

which defines a group isomorphism between the orientation preserving the isometry group of $H^2$ and the matrix group $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/(\pm \text{Id})$.

If $G$ is a discrete subgroup of isometries of $\text{PSL}_2(\mathbb{R})$ acting on the chart(s) $M_c$, then the foliation of $\text{PSL}_2(\mathbb{R})$ by vertical lines gives $M_c$ the structure of a line bundle over a non-closed surface $H^2$. 

20
c) Case of corank 1:

1) The geometry of Nil:

As $\text{PSL}_2(\mathbb{R})$ is a line bundle over $H^2$, there exists a geometry Nil which consists in a line bundle over the non-closed Euclidean plane $E^2$: it is a twisted version $E^2 \cong E^1$ characterized on $\mathbb{R}^3$ by the Riemannian metric:

$$ds^2 = dx_1^2 + dx_2^2 + (dx_3 - x_1 dx_2)^2.$$ 

As there is a negative deviation to Euclidicity in the third dimension “$x_3$”, there must exist a singular point of corank 1 in this dimension “$x_3$”. This would correspond to a local geometry $H^1 \times E^2$ round the singularity since $H^1 \subset \mathbb{R} \simeq E^1$.

The charts $M_c$ on the sections of semisheaves $\theta_*^{(3)}$ (resp. $\theta_*^{(3)}$) are characterized by a Nil geometry of which isometry group $G$ is nilpotent and given by $(3 \times 3)$ real upper (resp. lower) unitriangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(resp.} \quad \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \text{)}.$$ 

This leads to the exact sequence [Tho]:

$$O \longrightarrow \mathbb{R} \longrightarrow \text{Nil} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow O$$

where $\mathbb{R}$ consists of the elements of Nil with $a = c = 0$.

2) The geometry of Sol:

There still exists a geometry associated with a singularity of corank 1. It is the Sol geometry characterized by the Riemannian metric

$$ds^2 = e^{+2x_3} \, dx_1^2 + e^{-2x_3} \, dx_2^2 + dx_3^2$$

which is such that the discrete group $G$ of transformations of charts of $\theta_*^{(3)}$ and of $\theta_*^{(3)}$ acts according to:

$$(x_1, x_2, x_3) \longrightarrow (e^{-c} x_1 + a, e^c x_2 + b, x_3 + c)$$

where $a, b, c \in \mathbb{R}$.

This group $G$ is defined as a split extension of $\mathbb{R}^2$ by $\mathbb{R}$ according to the exact sequence:

$$O \longrightarrow \mathbb{R}^2 \longrightarrow \text{Sol} \longrightarrow \mathbb{R} \longrightarrow O$$

in such a way that $t$ in $\mathbb{R}$ acts on $\mathbb{R}^2$ by the map sending $(x_1, x_2)$ to $(e^t x_1, e^{-t} x_2)$: this corresponds to a linear isomorphism of $\mathbb{R}^2$ with determinant one and distinct real eigenvalues. Such a linear map is called a hyperbolic isomorphism of $\mathbb{R}^2$. And, Sol/$G$ is a bundle over a 1-dimensional orbifold with fibre $S^1 \times \mathbb{R}$ and base $\mathbb{R}$. So, the local geometry round the degenerate singularity of corank 1 would be $H^1 \times S^1 \times \mathbb{R}$ since:
• the dimension “$x_2$” undergoes a negative deviation to Euclidicity due to the factor $e^{-2x_3}$ of $dx_2^2$ in $ds^2$; this explains the hyperbolic geometry $H^1 \subset \mathbb{R}$ in $H^1 \times S^1 \times \mathbb{R}$ and the existence of the base $H^1$ of the bundle $\text{Sol}/G$.

• the dimension “$x_1$” undergoes a positive deviation to Euclidicity due to the factor $e^{+2x_3}$ of $dx_1^2$ in $ds^2$.

This is due to the versal unfolding in this dimension, i.e. of codimension one, of the considered degenerate singularity; this explains the spherical geometry $S^1$ in $H^1 \times S^1 \times \mathbb{R}$.

Thus, any 3-dimensional chart $M_c$, endowed with a degenerate singularity of corank 1 and codimension 1, has a geometric structure modelled on $\text{Sol}$ and is characterized by a natural foliation associated with the (sub)geometry $S^1 \times \mathbb{R}$.

3.10 Proof of Proposition 3.6: Local geometries of versal deformations

We have to prove that the neighbouring chart of an unfolded germ in codimension three, two or one is characterized respectively by a local geometry $S^3$, $S^2 \times \mathbb{R}$ or Sol.

a) Case of codimension 3: the geometry of $S^3$:

It appears from proposition 3.3 that the neighbouring chart of an unfolded degenerate singular germ is characterized in the neighbourhood of the singular locus by a spherical geometry. As the codimension of the considered versal unfolding is equal to three, the local geometry round the singular locus must be the geometry of $S^3$ characterized by a positive curvature which is equal to $+1$ if the metric is rescaled [Ber]. In fact, the envisaged neighbouring chart is isometric to the unit sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \sum_{i=1}^{4} x_i^2 = 1\}$$

with the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^4 = E^4$.

The isometry group of $S^3$ is $\text{Isom}(S^3)$ which contains the orthogonal group $O(4)$.

Let $\phi = S^3 \to SO(4)$ be the isometry of $S^3$ sending $x \in S^3$ to $q x q^{-1}$. Then the image of $\phi$ lies in the subgroup of $SO(4)$ fixing 1, which can be identified with $SO(3)$.

Remark that the considered neighbouring chart of the unfolded degenerate singular germ in codimension 3 is not necessarily closed.

b) Case of codimension 2: The geometry of $S^2 \times \mathbb{R}$ [Whi]:

As we are considering the charts or the manifold $M_c$ referring to the versal unfolding in codimension 2 of a degenerate singular germ, (their) its local geometry has to be of type
$S^2$. But, as the envisaged chart(s) is (are) tridimensional, its (their) local geometry is (are) $S^2 \times \mathbb{R}$ of which isometry group is $\text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$.

This manifold (or chart(s)) $M_c$ thus has the structure of a bundle over the 1-dimensional base orbifold $\mathbb{R}$ with fibre $S^2$ resulting locally from a versal deformation in codimension 2.

c) Case of codimension one: The geometry of Sol:

The chart of the versal unfolding in codimension one of a degenerate singular germ must be characterized by the local spherical geometry $S^1$. As the corank of the versal unfolding in codimension one of a singular germ cannot be generally superior to one and as the envisaged chart of manifold is three-dimensional, the local geometry of a 3-chart or a 3-manifold referring to a versal unfolding in codimension one must be $H^1 \times S^1 \times \mathbb{R}$, i.e. the Sol geometry is considered in section 3.9, c), 2).

3.11 Proof of Proposition 3.7: The Poincare conjecture resulting from the Thurston’s program

This proposition asserts that the manifold $M_c$ (or chart) resulting from the versal unfolding in codimension 3 of a degenerate singular germ is the 3-sphere $S^3$ if it is closed and simply connected: this corresponds to a natural strong version of the Poincare conjecture in dimension 3, i.e. when the considered manifold is generated by versal unfolding from a degenerate singular germ.

The Poincare conjecture then appears as originating from the Thurston’s geometrization program in the case of a versal deformation in codimension 3 leading to a manifold having the geometry of $S^3$. On the other hand, from the standard hypothesis of the Poincare conjecture, we know that the manifold $M_c$ must be closed and simply connected, i.e. composed of closed curves. Then, it is clear that this manifold $M_c$ is the 3-sphere $S^3$, which is composed of closed curves (i.e. circles) having the same length: this corresponds to the strong version of the Poincare conjecture. To reduce this strong version of the normal version of the Poincare conjecture, we have to eliminate the hypothesis of sphericality of $M_c$ as resulting from a versal deformation in codimension 3. That is to say, our manifold $M_c$ can be also characterized locally by a hyperbolic and by an euclidean geometry.

Then, $M_c$, being closed, simply connected with all closed curves having the same length, can be deformed continuously by homeomorphism in order to become homeomorphic to $S^3$. This leads to the Poincare conjecture: “Any closed, simply connected 3-manifold of which closed curves have the same length is homeomorphic to $S^3$, ” which differs slightly from the classical Poincare conjecture: “Any closed, simply connected 3-manifold is homeomorphic to $S^3$. ”

23
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26
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