Two-neutrino double-beta decay Fermi transition and two-nucleon interaction

Dušan Štefánik, 1, 2 Fedor Šimkovic, 1, 2, 3 Kazuo Muto, 4 and Amand Faessler 5

1 Comenius University, Mlynská dolina F1, SK-842 48, Slovakia
2 BLTP, JINR, 141980 Dubna, Moscow region, Russia
3 IEAP CTU, 128-00 Prague, Czech Republic
4 Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
5 Institute of Theoretical Physics, University of Tübingen, 72076 Tübingen, Germany

An exactly solvable model for a description of the two-neutrino double beta decay transition of the Fermi type is considered. By using perturbation theory an explicit dependence of the two-neutrino double beta decay matrix element on the like-nucleon pairing, particle-particle and particle-hole proton-neutron interactions by assuming a weak violation of isospin symmetry of Hamiltonian expressed with generators of the SO(5) group. It is found that there is a dominance of double beta decay transition through a single state of the intermediate nucleus. Then, an energy weighted sum rule connecting ΔZ=2 nuclei is presented and discussed. It is suggested that this sum rule can be exploited to study the residual interactions of the nuclear Hamiltonian.

PACS numbers: 21.60.Fw, 21.60.Jz, 23.40.Hc

I. INTRODUCTION

The two-neutrino double-beta decay (2νββ-decay), which involves the emission of two electrons and two antineutrinos

\[
(A, Z) \rightarrow (A, Z+2) + 2e^- + 2\bar{\nu}_e,
\]

has attracted the attention of both experimentalists and theoreticians for a long period and remains of major importance for nuclear physics.

It is a second order process in the weak interaction allowed in the standard model. The 2νββ-decay can be observed, because due to the pairing force even-even nuclei with an even number of protons and neutrons are more stable than the odd-odd nuclei with broken pairs. Thus, the single β-decay transition from the (A,Z) nucleus to neighboring odd-odd nucleus is energetically forbidden.

Till now, the 2νββ-decay has been detected for 11 different nuclei for transition to the ground state and in two cases also to transition to 0+ excited state of the daughter nucleus [1, 2]. This rare process is one of the major sources of background in running and planned experiments looking for a signal of the more fundamental neutrinoless double-beta decay, which occurs if the neutrino is a massive Majorana particle.

The inverse half-life of the 2νββ-decay is free of unknown parameters of particle physics and can be factorized to a good approximation as

\[
(T^2_{1/2})^{-1} = G^{2\nu} g_A^2 |M^{2\nu}_{GT}|^2 - \left(\frac{g_V}{g_A}\right)^2 |M^{2\nu}_{F}|^2,
\]

where \(G^{2\nu}\) is the lepton phase-space factor, \(g_A\) (\(g_V\)) is the axial-vector (vector) coupling constant. The 2νββ-decay is governed by the double Gamow-Teller (GT) and double Fermi (F) matrix elements, which are given by

\[
M_{F,GT}^{2\nu} = \sum_n \frac{\langle f || O_{F,GT} || 1^+_n \rangle \langle 1^+_n || O_{F,GT} || i \rangle}{E_n - (E_i + E_f)/2}
\]

with

\[
O_F = \sum_{k=1}^A \tau^+_k, \quad O_{GT} = \sum_{k=1}^A \tau^+_k \sigma_k.
\]

where \(|i>\) (\(|f>\)) are 0+ ground states of the initial (final) even-even nuclei with energy \(E_i\) (\(E_f\)), and \(|1^+_n>\) \((|0^+_n>\)) are the 1+ (0+) states in the intermediate odd-odd nucleus with energies \(E_n\).

Many attempts have been made in the literature to calculate the 2νββ-decay nuclear matrix elements (NMEs) for nuclei of experimental interest [1, 4, 7–9]. Recent results obtained within the nuclear shell model are in a good agreement with the measured 2νββ-decay half-lives [10]. But, it is achieved by a consideration of significant quenching by a factor \(q=0.4-0.7\) of the Gamow-Teller operator, which is obtained by a normalization of the total theoretical β− strength in the experimental energy window to the measured one.

The quasiparticle random phase approximation (QRPA) has been found to be successful in revealing the suppression mechanism for the 2νββ-decay NMEs [11–13]. However, the predictive power of the QRPA is questionable because of extreme sensitivity of calculated 2νββ-decay matrix elements in the physically acceptable region on the particle-particle strength of nuclear Hamiltonian. In [13] it was shown that if this strength is determined from a QRPA calculation of single β+ decays a reasonable agreement with the measured 2νβ-decay is achieved.

The quenching behavior of the 2νββ-decay matrix elements is a puzzle and has attracted the attention of many theoreticians. Recently, it was shown that \(M_{GT}^{2\nu}\) depends strongly on the isovector part of the particle-particle neutron-proton interaction unlike \(M_{GT}^{2\nu}\), which
depends strongly on its isoscalar part \[14\]. The under-
lying symmetries responsible for these suppressions are
assumed to be isospin SU(2) and spin-isospin SU(4) sym-
metries in the cases of double Fermi and double Gamow-
Teller NMEs, respectively \[15\].

The goal of this paper is to discuss the suppression
mechanism of the double Fermi matrix element close
to the point of restoration of isospin symmetry of the
nuclear Hamiltonian in the context of residual nucleon-
nucleon interaction. For the sake of simplicity we con-
sider a schematic Hamiltonian, describing the gross prop-

ties of the beta-decay processes in the simplest case
of monopole Fermi transitions within the SO(5) model
\[10\ \[21\]. In order to find explicit dependence of \(M^2\)
on different parts of the nuclear Hamiltonian the per-
turbation theory is exploited. We note that the SO(5)
model remains a tool for understanding of different nu-
clear physics phenomena even nowadays \[22\ \[24\].

II. SCHEMATIC HAMILTONIAN WITHIN THE
SO(5) MODEL

In the model, protons and neutrons occupy only
a single j-shell. The Hamiltonian includes a single-
particle term, proton-proton and neutron-neutron pair-
ing, and a charge-dependent two-body interaction with
both particle-hole and particle-particle channels as fol-
sows:

\[
H = e_p N_p + e_n N_n - G_p S^\dagger S - G_n S^\dagger_n S_n + 2\chi \beta^- \beta^+-2\kappa P^-P^+,
\]
(5)

where

\[
N_i = \sum_m a^\dagger_{m,t_i} a_{m,t_i}, \quad \beta^- = \sum_m a^\dagger_{m,-\frac{1}{2}} a_{m,\frac{1}{2}}.
\]

\[
S^\dagger_i = \frac{1}{2} \sum_m a^\dagger_{m,t_i} a^\dagger_{m,t_i}, \quad P^- = \sum_m a^\dagger_{m,-\frac{1}{2}} a^\dagger_{m,\frac{1}{2}}.
\]

(6)

with \(i=p, n\) and \(t_{n,p} = \pm 1/2\). \(a^\dagger_{mt}\) (\(a_{mt}\)) is creation (anni-
hilation) operator of single particle state \(|jm, t >\) for pro-
tons and neutrons \((t = \bar{t}, t_n)\) and \(a^\dagger_{mt} = (-1)^{m-t} a^\dagger_{mt}.

We rewrite Hamiltonian (5) with help of operators

\[
A^\dagger (T_z) = \frac{1}{\sqrt{2}} \left[ a^\dagger \otimes a^\dagger \right]_{T_z},
\]

\[
N = N_p + N_n, \quad T_z = \frac{N_n - N_p}{2},
\]

\[
T^- = -\sqrt{2}\Omega \sum_{m} a^\dagger_{m,-\frac{1}{2}} a_{m,\frac{1}{2}}.
\]

(7)

Here, \(A^\dagger (T_z)\) is the nucleon pair creation operator with
angular momentum \(J = 0\), isospin \(T = 1\) and its pro-
jection on z-axis \(T_z\) \((T_z = 0, \pm 1)\). \(N, T_{z}\) and \(T^-\) are
the particle-number operator, the isospin projection and
the isospin lowering operators, respectively. It holds the
identity \(T^2 = (T^-T^++T^+T^-)/2 + T_z^2\). \(\Omega = j+1/2\) de-
notes the semi-degeneracy of the considered single level.
The operators (7) with their Hermitian conju-
gates represent ten generators of the SO(5) group \[23\]. We assume,
the system is in seniority \(s=0\). Then, \([A^\dagger \hat{A}]^0_0\) expressed
with the SO(5) Casimir operator \[23\] is given by

\[
[A^\dagger \hat{A}]^0_0 = \frac{1}{2\sqrt{3}\Omega} [2\Omega + 3 - N/2] N/2 - T(T+1).
\]

(8)

For the Hamiltonian (5) we get

\[
H = \left[ e_n + e_p - \frac{1}{3} \left( 3 + 2\Omega - \frac{N}{2} \right) \left( G_p + G_n/2 \right) \right] \frac{N}{2} + \left[ e_n - e_p - 2\chi (T_z + 1) \right] T_z
\]

\[
+ 2\chi \left( \frac{G_p + G_n}{2} + 2\kappa \right) T(T+1)
\]

\[
+ \frac{\Omega}{\sqrt{2}} \left( \frac{G_p - G_n}{2} \right) [A^\dagger \hat{A}]^0_0 + \sqrt{\frac{2}{3}} \Omega \left( 4\kappa - \frac{G_p + G_n}{2} \right) [A^\dagger \hat{A}]^2_0.
\]

(9)

As a consequence of the presence of the isovector and
isoquadrupole terms in Hamiltonian (9) the isospin is
not conserved in general. It is due to differences be-
tween proton and neutron pairing strengths and an arbitrar-
iness strength of the proton-neutron isovector pairing com-
potent. However, particle number and isospin projection
remains as good quantum numbers.

The \(k^{th}\) eigenstates of the Hamiltonian (9) with quan-
tum numbers \(N, T_z\) can be expressed in terms of a
basis labeled by a chain of irreducible representations
of the SO(5) group (see Appendix A), namely

\[
|k; NT_z\rangle = \sum_{T} c_{NTzT_z}^{(k)} |NTzT_z\rangle.
\]

(10)

A diagonalization of \(H\) requires calculation of
matrix elements \(\langle N, T, T_z|H| N, T, T_z\rangle\) and
(N, T ± 2, Tz) \mid H \mid (N, T, Tz). The corresponding reduced matrix elements are given Appendix [11]. For G_p = G_n and (G_p + G_n)/2 = \kappa the Hamiltonian \([10]\) is diagonal in the basis of states \(|N, T, Tz\rangle\).

III. DOUBLE FERMI MATRIX ELEMENT WITHIN PERTURBATION THEORY

We shall assume a small violation of the isospin symmetry due to isotorser term of nuclear Hamiltonian \([9]\). For the numerical example we consider a large value of \(\kappa\) to simulate the realistic situation corresponding to medium- and heavy-mass nuclei. The parameters chosen are given by

\[
\Omega = 10, \quad N = 20, \quad 1 \leq Tz \leq 5, \\
e_p = 0.3 \text{ MeV}, \quad e_n = 0.1 \text{ MeV}, \quad G = 0.165 \text{ MeV}, \\
G_p = G_n = G, \quad \chi = 0.044 \text{ MeV}, \quad 0.7 \leq 4\kappa/G \leq 1.3. \tag{11}
\]

For 4\kappa/G = 1 the isospin symmetry is restored. In Fig. [11] we present \(0^+\) states with energy \(E_{TTz}\) of different isotopes. This level scheme illustrates the situation for the 2\nu3\beta-decay of \(^{48}\text{Ca}\). The isospin is known to be, to a very good approximation, a valid quantum number in nuclei. The ground states of \(^{48}\text{Ca}\) and \(^{48}\text{Ti}\) can be identified with T=4 \(T_z = 4\) and T=2 \(T_z = 2\), respectively, i.e. they are assigned into different isospin multiplets. As the total isospin projection lowering operator \(T^-\) is not changing the isospin the double Fermi matrix element \(M_{F}^{2\nu}\) is non-zero only to the extent that the Coulomb interaction mixes the high-lying T=4 \(T_z = 2\) analog of the \(^{48}\text{Ca}\) ground state into the T=2 \(T_z = 2\) ground state of \(^{48}\text{Ti}\).

We shall study double Fermi matrix element in the perturbation theory within the discussed model close to a point of a restoration of the isospin symmetry \(4\kappa/G = 1\). The isoscalar and isotensor terms of the Hamiltonian \([9]\) represent the unperturbed and perturbed terms, respectively. We denote perturbed states and their energies with a superscript prime symbol \((|TT'_z\rangle, \ E_{TT'_z})\) unlike the states with a definite isospin \((|TTz\rangle, \ E_{TTz})\). Up to the second order of parameter \((4\kappa - G)\) we find

\[
E_{44}' = 14e_n + 6ep - \frac{110}{3} (G + 2\kappa)
\]

\[
- \frac{\sqrt{2}}{3} \Omega (G - 4\kappa) \langle 44 | \hat{A} | 44 \rangle
\]

\[
- \frac{2}{3} \Omega^2 (G - 4\kappa) \left\langle 64 | \hat{A}^2 | 44 \right\rangle^2 \left\langle 44 \right\rangle
\]

\[
E_{43}' = 13e_n + 7ep + 16\chi - \frac{110}{3} (G + 2\kappa)
\]

\[
- \frac{\sqrt{2}}{3} \Omega (G - 4\kappa) \langle 43 | \hat{A} | 43 \rangle
\]

\[
- \frac{2}{3} \Omega^2 (G - 4\kappa) \left\langle 63 | \hat{A}^2 | 43 \right\rangle^2 \left\langle 43 \right\rangle
\]

\[
E_{22}' = 12e_n + 8ep - \frac{124}{3} (G + 2\kappa)
\]

\[
- \frac{\sqrt{7}}{3} \Omega (G - 4\kappa) \langle 22 | \hat{A} | 22 \rangle
\]

\[
- \frac{2}{3} \Omega^2 (G - 4\kappa) \left\langle 42 | \hat{A} | 22 \right\rangle^2 \left\langle 22 \right\rangle
\]

\[
E_{42}' = 12e_n + 8ep + 28\chi - \frac{110}{3} (G + 2\kappa)
\]

\[
- \frac{\sqrt{2}}{3} \Omega (G - 4\kappa) \langle 42 | \hat{A} | 42 \rangle
\]

\[
+ \frac{2}{3} \Omega^2 (G - 4\kappa) \left\langle 42 | \hat{A}^2 | 22 \right\rangle^2 \left\langle 22 \right\rangle
\]

\[
- \frac{2}{3} \Omega^2 (G - 4\kappa) \left\langle 62 | \hat{A}^2 | 42 \right\rangle^2 \left\langle 42 \right\rangle
\]

The particular matrix elements of \(\text{SO}(5)\) operators connecting states with a definite isospin and its projection are presented in Appendix [11].

For transition \(|4'4\rangle \rightarrow |2'2\rangle\) the double Fermi matrix element can be written as

\[
M_{F}^{2\nu} = \sum_{T=4,6,8,10}^{10} \frac{(2'2) | T^- | T'3 (T'3) | T^- | 4'4\rangle}{E_{TT3} - (E_{44} - E_{22})/2}. \tag{16}
\]

It contains a sum over the states of the intermediate nucleus \(|T'3\rangle\). However, up to second order of perturbation theory there is only a single contribution through the intermediate state \(|4'3\rangle\). Thus, we have

\[
M_{F}^{2\nu} \simeq \frac{(2'2) | T^- | 4'3\rangle \langle 4'3 | T^- | 4'4\rangle}{E_{33} - (E_{44} - E_{22})/2}. \tag{17}
\]

The involved \(\beta\)-transition amplitudes are given by
\[ \langle 4'3 | T^- | 4'4 \rangle = \langle 43 | T^- | 44 \rangle \left( 1 - \frac{1}{3} \frac{\Omega^2 (4\kappa - G)^2}{(4\chi + 22/3 (G + 2\kappa))^2} \left[ (44 | A^\dagger \tilde{A} | 44)^2 + (43 | A^\dagger \tilde{A} | 63)^2 \right] \right) \]

\[ + \langle 63 | T^- | 64 \rangle \frac{2}{3} \frac{\Omega^2 (4\kappa - G)^2}{(4\chi + 22/3 (G + 2\kappa))^2} \left( 64 | A^\dagger \tilde{A} | 44 \rangle \langle 43 | A^\dagger \tilde{A} | 43 \right) \]

and

\[ \langle 2'2 | T^- | 4'3 \rangle = \langle 22 | T^- | 43 \rangle \left[ \sqrt{\frac{2}{3}} \frac{\Omega (G - 4\kappa)}{(28\chi + 14/3 (G + 2\kappa))^2} \left( \langle 42 | A^\dagger \tilde{A} | 22 \rangle^2 \langle 42 | A^\dagger \tilde{A} | 44 \rangle^2 \langle 42 | A^\dagger \tilde{A} | 22 \rangle^2 \right) \right] \]

If isospin symmetry is restored \((4\kappa = G)\) we end up with \(\langle 2'2 | T^- | 4'3 \rangle = \langle 22 | T^- | 43 \rangle = 0\). For the energy denominator in \([17]\) with help of Eqs. \([12], [13]\) and \([14]\) we get

\[ E_{43} - (E_{44} - E_{22})/2 = 16\chi + \frac{7}{3} (G + 2\kappa) \]

\[ + \sqrt{\frac{2}{3}} \frac{\Omega (G - 4\kappa)}{\left( 28\chi + 14/3 (G + 2\kappa))^2 \right) \left( \langle 42 | A^\dagger \tilde{A} | 22 \rangle^2 \langle 42 | A^\dagger \tilde{A} | 44 \rangle^2 \langle 42 | A^\dagger \tilde{A} | 22 \rangle^2 \right) \]

IV. ENERGY WEIGHTED SUM RULE OF \(\Delta Z=2\) NUCLEI

We suggest that a quantity relevant for the \(2\nu\beta\beta\)-decay might be the energy weighted double Fermi (or Gamow-Teller) sum rule associated with \(\Delta Z=2\) nuclei:

\[ S_{F,GT}^{\text{ew}}(i, f) = \sum_n \left( E_n - \frac{E_i + E_f}{2} \right) \langle f | O_{F,GT} | n \rangle \langle n | O_{F,GT} | i \rangle \]

\[ = \frac{1}{2} \langle f | H, O_{F,GT} | i \rangle . \]
Hamiltonian is eliminated unlike it is in the case of energy weighted sum rules related to a single nuclear ground state. We note that the energy weighted double Gamow-Teller sum rule associated with the $2\nu\beta\beta$-decay was discussed within the proton-neutron QRPA in [26, 27].

We analyze the above sum rule for Fermi transitions and Hamiltonian [9] with $G_p = G_n$ within the SO(5) model. By rewriting the Hamiltonian as

$$H = (e_p + e_n)N/2 + (e_p - e_n)T_z + 2\chi T^{−}T^{+}$$

$$− 2GΩ(A^{′}(-1)A(-1) + A^{′}(1)A(1))$$

$$− 4\kappa ΩA^{′}(0)A(0)$$

and exploiting the commutation relations of the SO(5) group [11] we find

$$S_{FW}^{ew}(i, f) = \frac{1}{2} \langle f| [T^{−}, [H, T^{−}]] |i⟩$$

$$= 2Ω(G − 4\kappa) \langle i| [A^{′}A^{′}_2]_2 |f⟩ + 2\chi \langle f| T^{−}T^{−} |i⟩.$$  

(23)

**i)** The case $|i⟩ = |4^{′}4⟩$, $|f⟩ = |2^{′}2⟩$. We have

$$S_{FW}^{ew}(4^{′}4, 2^{′}2)$$

$$= \sum_{T^′}(E_{T^3} - \frac{E_{T^3} + E_{T^3}^{′}}{2}) \langle 2^{′}2|T^{−}|T^{−}⟩ \langle T^{′}3|T^{−}⟩ |4^{′}3⟩$$

$$= 2Ω(G − 4\kappa) (4^{′}4| [A^{′}A^{′}_2]_2 |2^{′}2⟩ + 2\chi (2^{′}2|T^{−}T^{−} |4^{′}3⟩.$$  

(24)

If the first order perturbation theory is applied to any of two expressions for energy weighted sum rule in [25] we find

$$S_{FW}^{ew}(4^{′}4, 2^{′}2) ≃ \left[ 16\chi + \frac{7}{3} (G + 2\kappa) \right] \times$$

$$\sqrt{\frac{2}{3}} Ω(G − 4\kappa) \langle 24| [A^{′}A^{′}_2]_2 |44⟩ \times$$

$$\langle 42|T^{−}|43⟩ \langle 43|T^{−}|44⟩.$$  

(26)

By comparing this expression with Eqs. (18), (19) and (20) we see that only the lowest intermediate state $|4^{′}3⟩$ contributes to the sum rule within the considered approximation. We find again a combination of energies of involved states to be a function of pairing, particle-particle and particle-hole interactions: $E_{43} = (E_{44} + E_{22})/2 ≃ 16\chi + \frac{7}{3} (G + 2\kappa)$.

**ii)** The case $|i⟩ = |4^{′}4⟩$, $|f⟩ = |4^{′}2⟩$. The energy weighted sum rule is given by

$$S_{FW}^{ew}(4^{′}4, 4^{′}2)$$

$$= \sum_{T^′}(E_{T^3} - \frac{E_{T^3} + E_{T^3}^{′}}{2}) \langle 4^{′}2|T^{−}|T^{−}⟩ \langle T^{′}3|T^{−}⟩ |4^{′}4⟩$$

$$= 2Ω(G − 4\kappa) (4^{′}4| [A^{′}A^{′}_2]_2 |4^{′}2⟩ + 2\chi (4^{′}2|T^{−}T^{−} |4^{′}4⟩.$$  

(27)

Within the first order perturbation theory we find

$$S_{FW}^{ew}(4^{′}4, 4^{′}2) ≃ (2\chi +$$

$$\sqrt{1/6Ω(4\kappa − G)} \left[ 2 \langle 43| [A^{′}A^{′}_2]_2 |43⟩$$

$$− \langle 44| [A^{′}A^{′}_2]_2 |44⟩ − \langle 42| [A^{′}A^{′}_2]_2 |42⟩ \right]$$

$$\langle 42|T^{−}|43⟩ \langle 43|T^{−}|44⟩.$$  

(28)

We note that the dominant contribution to $S_{FW}^{ew}(4^{′}4, 4^{′}2)$ comes from the transition through the single intermediate state $|43⟩$ again. For a combination of energies of
involved states we have
\[
E'_{43} - (E'_{44} + E'_{42})/2 = \\
2\chi + \sqrt{1/6\Omega}(4\kappa - G) \left( 2 \langle 43 | A_0^1 | 43 \rangle - \langle 44 | A_0^1 | 44 \rangle - \langle 42 | A_0^1 | 42 \rangle \right).
\]
Thus, the energy weighted sum rule \(S_F^\nu(4', 4')\) implies another useful relation between energies of states and nucleon-nucleon interactions.

In Fig. 3, two different energy weighted sum rules associated with final states \(2' >\) and \(4' >\) are plotted as function of the ratio \(4\kappa/G\) for a considered set of parameters [11]. They exhibit different dependence on \(4\kappa/G\). It is because the final state \(4' >\) belongs \((2' >\) does not belong\) to the same isospin multiplet as the initial nucleus. We see a very good agreement between the exact results and results obtained within the first order perturbation theory, which allows only the lowest intermediate state \(4'3^+\) to contribute to a sum rule. A better agreement would be achieved if the corresponding combination of energies of states would evaluated up to the second order perturbation theory. We note that a contribution from the second lowest intermediate state to the sum rules \(S_F^\nu(4', 2')\) and \(S_F^\nu(4', 4')\) appears only in the third order perturbation theory.

**V. CONCLUSIONS**

An exactly solvable model for the description of the 2\(\nu\beta\beta\)-decay processes of the Fermi type was used to discuss the dependence of the double-beta decay matrix element \(M_F^\nu\) on different components of the residual interaction, namely like-nucleon pairing, particle-particle and particle hole proton-neutron interactions. We note that the model is equivalent to a complete shell-model treatment in a single-j shell for the adopted Hamiltonian. In addition, it reproduces the main features of the results obtained in realistic calculations.

Good isospin forbids the 2\(\nu\beta\beta\)-decay. One needs an isosensor force to mix \(\Delta T = 2\). Naturally, the Coulomb interaction contains such a isosensor force. In our case we break isospin symmetry by hand. The only isosensor violation comes from the difference of the proton-proton \((G_p\) and the neutron-neutron \((G_n\) pairing force compared to the proton-neutron isosensor \(= 1\) pairing force \((\kappa\). By taking the advantage of the perturbation theory up to the second order in the isosensor contribution to the Hamiltonian a dominance of a contribution through a single state of the intermediate nucleus to \(M_F^\nu\) and explicite dependence of \(M_F^\nu\) on different types of nucleon-nucleon interactions was found. The mean-field part of Hamiltonian does not enter explicietly in this decomposition of double Fermi matrix element and is related only to the calculation of unperturbated states of Hamiltonian.

Further, the importance of the energy weighted sum rule associated with \(\Delta Z = 2\) nuclei for fitting different components of residual interaction of the Hamiltonian was pointed out. It goes without saying that a further studies, in particular by considering realistic nuclear Hamiltonian and Gamow-Teller transitions, are of great interest.

**Acknowledgments**

This work is supported in part by the Deutsche Forschungsgemeinschaft within the project "Nuclear matrix elements of Neutrino Physics and Cosmology" FA67/40-1 and by the grant of the Ministry of Education and Science of the Russian Federation (contract 12.74112.0150). F. Š. acknowledges the support by the VEGA Grant agency of the Slovak Republic under the contract No. 1/0876/12 and by the Ministry of Education, Youth and Sports of the Czech Republic under contract LM2011027.

**Appendix A: The SO(5) algebra and matrix elements**

Following [22] we introduce operators of the SO(5) group, which are expressed with operators [7] as follows:

\[
H_1 = N/2 - \Omega, \quad H_2 = T_z, \\
E_{11} = \sqrt{\Omega}A^1(1), \quad E_{-1-1} = \sqrt{\Omega}A(1), \\
E_{1-1} = -\sqrt{\Omega}A(-1), \quad E_{-11} = -\sqrt{\Omega}A(-1), \\
E_{10} = \sqrt{\Omega}A^1(0), \quad E_{-10} = \sqrt{\Omega}A(0), \\
E_{01} = \frac{1}{2}\sqrt{2T^+}, \quad E_{0-1} = \frac{1}{2}\sqrt{2T^-},
\]

Their commutation relations are [22]

\[
[H_1, H_2] = 0, \quad [H_1, E_{ab}] = aE_{ab}, \\
[H_2, E_{ab}] = bE_{ab}, \\
[E_{ab}, E_{-a-b}] = aH_1 + bH_2
\]

and

\[
[E_{ab}, E_{a'b'}] = \pm E_{a+a'b+b'}, \quad \text{(A1)}
\]

if \(a + a' = 0, \pm 1\) and \(b + b' = 0, \pm 1\). Otherwise, \([E_{ab}, E_{a'b'}] = 0\).

For the present task, states with with seniority \(s = 0\) are considered. Thus, it is sufficient to define them with quantum numbers \(N, T\) and \(T_z\). They are constructed with help of the isospin lowering operator \(T^-\) on the state \(|N, T, T_z = T\rangle\), which is given by [23]

\[
|NTT\rangle = N(a,b)O_7^+ O_0^+ |N = 4\Omega, T = T_z = 0\rangle,
\]

with

\[
O_+ = E_{-11}, \\
O_0 = 2E_{-11}E_{-1-1} + E_{-10}E_{-10} \quad \text{(A2)}
\]
increases the isospin by one unit and $O_1$ reduces the number of particles by four units. $a$ and $b$ are integers:

$$a = T, \quad b = \Omega - \frac{T}{2} - \frac{N - 1}{2}.$$  \hspace{1cm} (A3)

From a construction of the states it follows that a difference in isospin of two states with fixed $N, T_z$ is an even number.

The reduced matrix elements are calculated with help of the Wigner-Eckart theorem in the convention as follows:

$$\langle TT_z'\mid T \rangle = C_{TT_z\,TT_z'}^{T_T} \langle T' \mid T \rangle$$  \hspace{1cm} (A4)

Particular Clebsh-Gordan coefficients of interest are given by [2]

$$C_{TT_{z,20}}^T = \frac{3T_z^2 - T(T + 1)}{(2T - 1)(T + 1)(2T + 3)}$$

We present relevant reduced matrix elements, which agree with those of [20] up to few corrections:

$$\langle T + 2 \mid [A^+ \tilde{A}]^2 \mid T \rangle = -\frac{1}{2\Omega} \sqrt{(T + 2)(T + N/2 + 3)(2\Omega - T - N/2)(T + 1)(N/2 - T)(2\Omega + T - N/2 + 3)}$$  \hspace{1cm} (A5)

$$\langle T \mid [A^+ \tilde{A}]^2 \mid T \rangle = \frac{1}{\sqrt{6C_{TT_{z,20}}^T}} \left[ \langle NTT \mid A^+(1)A(1) \mid NTT \rangle + \langle NTT \mid A^+(-1)A(-1) \mid NTT \rangle 
- 2 \langle NTT \mid A^+(0)A(0) \mid NTT \rangle \right]$$

$$\langle NTT \mid A^+(1)A(1) \mid NTT \rangle = \frac{1}{\Omega} \left[ -\Omega + T + N/2 + \frac{(2\Omega - T - N/2)(T + N/2 + 3)(T + 1)}{2(T + 3)} \right]$$

$$\langle NTT \mid A^+(-1)A(-1) \mid NTT \rangle = \frac{1}{\Omega} \left[ \frac{(2\Omega + T - N/2 + 3)(-T + N/2)(T + 1)}{2(T + 3)} \right]$$  \hspace{1cm} (A6)

$$\langle NTT \mid A^+(0)A(0) \mid NTT \rangle = \frac{1}{\Omega} \left[ -\Omega + N/2 + \frac{(2\Omega - T - N/2)(T + N/2 + 3)\Omega}{(2\Omega + T - N/2 + 1)(-T + N/2 + 2)} \times \langle N + 4TT \mid A^+(0)A(0) \mid N + 4TT \rangle \right]$$  \hspace{1cm} (A7)

The matrix element on the right hand side of Eq. (A7) can be calculated recurrently by keeping in mind that for $N_{\text{max}} = 4\Omega - 2T$ we have

$$\langle N_{\text{max}}TT \mid A^+(0)A(0) \mid N_{\text{max}}TT \rangle = 1 - T/\Omega$$  \hspace{1cm} (A8)

For isospin raising (lowering) operators the Condon Shortley convention is assumed:

$$T^\pm |N, T, T_z\rangle = \sqrt{(T \pm T_z + 1)(T \mp T_z)} |N, T, T_z \pm 1\rangle.$$  \hspace{1cm} (A9)

[1] M. Doi, T. Kotani, and E. Tagasugi, Prog. Theor. Phys. (Supp.) 83, 1 (1985).

[2] W.C. Haxton and G.S. Stephenson Jr., Prog. Part. Nucl. Phys. 32, 1 (1989).
[3] J. Suhonen and O. Civitarese, Phys. Rep. 300, 123 (1998).
[4] A. Faessler and F. Šimkovic, J. Phys. G 24, 2139 (1998).
[5] J.D. Vergados, H. Ejiri, and F. Šimkovic, Rep. Prog. Phys. 75, 106301 (2012).
[6] A.S. Barabash, Phys. Rev. C 81, 035501 (2010).
[7] P. Domin, S. Kovalenko, F. Šimkovic, and S.V. Semenov, Nucl. Phys. A 753, 337 (2005).
[8] S. Singh, R. Chandra, P.K. Rath, P.K. Raina, and J.G. Hirsch, Eur. Phys. J. A 33, 375 (2007).
[9] R. Alvarez-Rodriguez, P. Sarriguren, E. Moya de Guerra, L. Pacearescu, A. Faessler, and F. Šimkovic, Phys. Rev. C 70, 064309 (2004).
[10] E. Caurier, F. Nowacki, and A. Poves, Phys. Lett. B 711, 62 (2012).
[11] P. Vogel and M.R. Zirnbauer, Phys. Rev. Lett. 57, 3148 (1986).
[12] O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. B 149, 11 (1987).
[13] K. Muto, E. Bender, and H.V. Klapdor, Z. Phys. A 334, 177 (1989).
[14] F. Šimkovic, V. Rodin, A. Faessler, and P. Vogel, Phys. Rev. C 87, 045501 (2013).
[15] J. Bernabeu, B. Desplanques, J. Navarro, and S. Noguera, Z. Phys. C 46, 323 (1990).
[16] V.A. Kuz’min and V.G. Soloviev, Nucl. Phys. A 486, 118 (1988).
[17] K. Muto, E. Bender, T. Oda, and H. V. Klapdor-Kleingrothaus, Z. Phys. A 341, 407 (1992).
[18] O. Civitarese and J. Suhonen, J. Phys. G 20, 1441 (1994).
[19] J. G. Hirsch, P.O. Hess, and O. Civitarese, Phys. Lett. B 390, 36 (1997).
[20] J.G. Hirsch, P.O. Hess, and O. Civitarese, Phys. Rev. C 56, 199 (1997).
[21] F. Krmpotić, E.J.V. de Passos, D.S. Delion, J. Dukelsky, and P. Schuck, Nucl. Phys. A 637, 295 (1998).
[22] J. Engel and P. Vogel, Phys. Rev. C 69, 034304 (2004).
[23] F. Šimkovic, A. Faessler, P. Vogel, J. Engel, Phys. Rev. C 77, 045503 (2008).
[24] J. Engel, J. Phys. G 39, 124001 (2012).
[25] J.C. Parikh, Nucl. Phys. 63, 214 (1965).
[26] V.A. Rodin, M.H. Urin, and A. Faessler, Nucl. Phys. A 748, 295 (2005).
[27] V. Rodin and A. Faessler, Phys. Rev. C 84, 014322 (2011).
[28] D.A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii, Kvantovaja teoriya uglovovo momenta, (Izdatelstvo Nauka, Leningrad, 1975).