RELAXATION IN N-BODY SIMULATIONS OF DISK GALAXIES

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ABSTRACT

I use N-body simulations with two mass species of particles to demonstrate that disk galaxy simulations are subject to collisional relaxation at a higher rate than is widely assumed. Relaxation affects the vertical thickness of the disk most strongly, and drives the velocity ellipsoid to a moderately flattened shape similar to that observed for disk stars in the solar neighborhood. The velocity ellipsoid in simulations with small numbers of particles quickly approaches this shape, but shot noise also dominates the in-plane behavior. Simulations with higher, but reachable, numbers of particles relax slowly enough to be considered collisionless, allowing the in-plane dispersions to rise due to spiral activity without heating the vertical motions. Relaxation may have affected many previously published simulations of the formation and evolution of galaxy disks.

Key words: galaxies: kinematics and dynamics – methods: numerical

Online-only material: color figures

1. INTRODUCTION

Simulations over several decades have contributed a great deal to our understanding of the complex dynamical behavior of stellar systems. The numbers of particles employed in a typical work has risen steadily with the available computational power, but even today we are unable to employ as many particles as there are stars in a galaxy, at least on a routine basis. Thus particles in simulations are typically fewer in number and more massive than stars in a galaxy, and therefore the density distribution suffers from a higher level of shot noise than expected in the stellar component of the system being mimicked.

Chandrasekhar (1941) estimated the relaxation time, or the time required for deflections by other stars to cause a star to lose all memory of its original trajectory, to be

$$\tau_{\text{relax}} \approx \frac{v^3}{8\pi G^2 \mu^2 n \ln \Lambda}$$

(1)

where $\mu$ is the mass of a star that moves at the typical speed $v$ and $n$ is the number of stars per unit volume. The ratio $\Lambda \approx R_e/\theta_{\text{min}}$, with $R_e$ being the half-mass radius of the galaxy, and $b_{\text{min}} \approx 2G\mu/v^2$, the impact parameter below which the approximation that deflections are small fails badly. The Coulomb logarithm, $\ln \Lambda$, indicates that every decade of increase of the impact parameter makes an equal contribution to the relaxation rate, because the number of stars rises with distance to compensate for the diminishing influence of each individual encounter. In simulations, one generally replaces $b_{\text{min}}$ with the gravity softening length, $\epsilon$, but distant encounters are unaffected by softening and fully contribute to relaxation. The reason for softening the interparticle force is mostly to avoid the need for shorter time steps during close approaches between particles (Binney & Tremaine 2008, hereafter BT08, p. 123).

Following BT08, we set $v^2 \approx GN\mu/R_e$, $n \approx 3N/(4\pi R_e^3)$, and define a dynamical time $\tau_{\text{dyn}} = R_e/v$. With these substitutions, Equation (1) becomes

$$\tau_{\text{relax}} \approx \frac{N}{6 \ln N} \tau_{\text{dyn}}$$

(2)

indicating that star–star encounters are utterly negligible for galaxies. The effect of a halo of particles whose masses are much smaller than those of stars is to further lengthen the relaxation time, since it increases $v$ without contributing to the deflections. Equation (2) also suggests that simulations with $N \geq 10^5$ should be safely collisionless, especially as the $\ln N$ term in the denominator is an overestimate for the values of $\epsilon$ usually employed.

While scattering in disks is more rapid (see Section 2), relaxation caused by star–star encounters remains much too slow to determine the random speeds of stars in the solar neighborhood. The local distribution of peculiar velocities (e.g., Nordström et al. 2004; Holmberg et al. 2009) has a flattened tri-axial shape. The dispersion in the direction of the Galactic center, $\sigma_R$, is the largest component, with that in the orbital direction, $\sigma_\phi$, being about 70% as great as expected from epicycle theory (BT08, p. 170), while the dispersion, $\sigma_z$, normal to the Galactic plane is some 60% of $\sigma_R$, and is the smallest.

In an important paper, Ida et al. (1993) showed that the shape of the velocity ellipsoid is consistent with scattering by dense mass clumps, such as giant molecular clouds (GMCs). Their predicted flattening depends on the slope of the Galactic rotation curve, but were this locally flat, they predict $\sigma_z \approx 0.6\sigma_R$, as observed. While GMCs readily redirect peculiar motions, they are less efficient at increasing them; GMCs are believed to be insufficiently massive or numerous (Lacey 1991; Hänninen & Flynn 2002) to create the high dispersion of the oldest thin-disk stars.

Disk stars are also believed to be scattered at the Lindblad resonances of transient spirals (Carlberg & Sellwood 1985), which increase only the in-plane dispersions. The vertical dispersion should be unaffected because the rapid vertical oscillation of a star is adiabatically invariant at the slow rate at which it encounters the spiral density variations (Carlberg 1987). Thus spiral scattering is needed to account for the high peculiar speeds of the older thin-disk stars while massive gas clumps redirect the peculiar velocities to maintain the shape of the velocity ellipsoid (Sellwood 2013).

Most simulations of galaxy disks do not include a separate population of heavy particles, and therefore have no agent to redirect the random motions generated by spiral waves into the vertical direction if they were truly collisionless. However, a few authors, notably Quinn et al. (1993) and McMillan & Dehnen...
(2007), have worried that isolated disks thicken in simulations as spiral activity heats the in-plane motions. Here I show that collisional relaxation is the likely origin of their finding, and should be a concern for all simulations of disk galaxies.

2. RELAXATION IN DISKS

Formulae (1) and (2) were derived assuming a mass distribution that is roughly uniform in three dimensions (3D). The relaxation rate in disks differs for at least four reasons, as were mostly pointed out by Rybicki (1972).

First, because disks are rotationally supported, stars pass each other at speeds that are a small fraction of the circular orbit speed, say βν, with β ≃ 0.1. The predominantly slow encounters mean that the typical deflection rate is boosted by the factor β−1, and the time needed for the random impulses to amount to a peculiar speed of βν is shorter by a factor of β3.

Second, Equation (1) results from an integration of mean-square deflections over all impact parameters. The 3D integration volume element becomes an element of area in two dimensions, and therefore the impact parameter enters with one power less. Thus the Coulomb logarithm is replaced by the factor (b−1 min − b−1 max), indicating that scattering is dominated by close encounters. Distant encounters are negligible in disks because the number of stars does not rise rapidly enough with distance to compensate for the diminished force each exerts.

Real galaxy disks are neither razor thin nor spherical. In this case, the spherical dependence applies at ranges up to the typical disk thickness, z0, beyond which the contribution to scattering drops quickly. Thus we should replace Re in the Coulomb logarithm by z0, which slightly reduces the relaxation rate.

Third, the local number density of stars is higher so that N ∼ πR2 z0ρν, which increases the density to be used in Equation (1) by the factor Rz/z0. This factor causes another order of magnitude increase in the relaxation rate.

Combining these three considerations yields a relaxation time in disks that is shorter than in 3D distributions by the factor

\[ \beta^3 \left( \frac{z_0}{R_e} \right) \ln \left( \frac{R_e/b_{\text{min}}}{\ln (z_0/b_{\text{min}})} \right), \]

which is typically a few × 10−4! It should be noted that the cube of β arises from the time needed for scattering to produce a peculiar velocity βν, i.e., equal to that of a typical disk star; the relaxation rate is increased over the 3D rate by only a single factor of β, but together with the higher density, the increase is still ∼100-fold.

The fourth factor in real disks is the existence of GMCs, whose role in determining the shape of velocity ellipsoid was described in the Introduction.

3. SIMULATIONS

Here I report a few simulations to test these theoretical predictions; a fuller study will be presented elsewhere.

Since simulations of all realistic disks heat due to instabilities, simple measurements of the heating rate will not determine the relaxation rate directly. Therefore, following Hohl (1973), I employ separate particle species having different masses, since energy exchange between particles of differing mass is a reliable indicator of relaxation.

The smooth, half-mass Mestel disk (BT08, p. 99) was proven by Toomre (1981) to have no global instabilities, although Sellwood (2012) found subtle instabilities, caused by non-linear effects, that gave rise to spiral activity in large-N models. It seems unlikely that 3D motion would alter the unusual stability properties of this disk, which make it more attractive for this test than other more generic disks that could host global linear instabilities.

3.1. Setting Up the Disk

In order to construct an equilibrium model, I start from the two-integral distribution function (DF) for the razor-thin disk (Zang 1976; Toomre 1977): \[ f_z(E, L_z) \propto L_z^3 e^{-E/\sigma_z^2}, \]
where E is a particle’s specific energy and Lz its specific z-angular momentum. The parameter q determines the radial velocity dispersion σr = V0(1 + q + E)−1/2, with V0 being the circular orbital speed at all radii. The value of Toomre’s local stability parameter for a razor-thin Mestel disk is

\[ Q \equiv \frac{\sigma_r}{\sigma_{\text{R, min}}} = \frac{2^{3/2}}{3.36(1 + q)^{1/2}}, \]

where f is the active mass fraction in the disk; both σr and Q are independent of radius. In order to obtain a disk with β ≃ 0.15, which must also have Q > 1, I adopt f = 0.25, with the remaining mass in a rigid halo.

I apply inner and outer tapers to limit the radial extent of the disk: \[ f_0(E, L_z) = \frac{f_0[1 + (L_z/L_s)^2]^q}{[1 + (L_z/L_o)^q]^{1+q}}, \]
where Ls and Lo are the central angular momentum values of the inner and outer tapers, respectively. I choose L0 = 15Ls, and further restrict the extent of the disk by eliminating all particles whose orbits would take them beyond 20 R, where R = L/Lo is the central radius of the inner taper. With these tapers Rs ≃ 8 R.

I thicken the disk by giving it the Gaussian vertical density profile \[ \rho(R, z) = \Sigma(R) \exp(-z^2/2z_0^2)/(2\pi z_0), \]
where \( \Sigma(R) \) being the vertically integrated surface mass density at radius R. I estimate the equilibrium vertical velocity dispersion at each z-height by integrating the one-dimensional Jeans equation (BT08, Equation (4.271)) in the numerically determined potential. I adopt an unrealistically small value z0 = 0.05 Rs, that is independent of R, in order to reveal the effects of relaxation clearly.

The active mass gives rise to a weaker central attraction in the midplane than that which should arise from a razor-thin, infinite, full-mass Mestel disk. In order to maintain equilibrium, I add a rigid central attraction to the grid-determined forces from the particles in the disk at each step. This unchanging, spherically symmetric, rigid central attraction is pre-tabulated from differencing \( -V_0^2/R \) from the grid-determined attraction of a smooth density created from the thickened, tapered disk.

3.2. Numerical Details

I determine the gravitational field of the particles using the 3D polar grid described in Sellwood & Valluri (1997). The grid has 200 rings, 256 spokes, and 375 vertical planes, and I adopt a cubic spline softening rule that yields the full attraction of a point mass at distances ≥2ϵ. I choose Rs = 5 grid units, the grid planes are 0.02 Rs apart, ϵ = 0.025 Rs, a basic time step of 0.025 R/ν0, and advance particles at radii R > 2 Rs at intervals that double in duration from this radius and again with every factor two increase in R. The duration of the simulations was ~20 full rotations of the disk at Rs, or ~3 Gyr when scaled to the Milky Way.

In order to create two populations of particles with unequal masses, I employ every particle selected from the DF twice, placing the two particles each at separately chosen random
azimuths, and make one nine times more massive than the other. The total masses of both populations are set to yield the desired total disk mass ($f = 0.25$ in Equation (4)) and a combined $Q = 1.5$, which corresponds to $\sigma_R \simeq 0.14 V_0$ and $\sigma_\phi \simeq 0.1 V_0$, i.e., in the ratio expected for a flat rotation curve (BT08). Thus $\sigma_{\text{total}} / V_0 = \beta \simeq 0.17$.

### 3.3. Results

Figure 1 shows the evolution of $\sigma_R$ and $\sigma_\phi$, averaged over a broad radial range centered on $R_e$, measured for both the heavy (red) and light (green) particles in three simulations. Each row is from a separate simulation, differing in the number of particles, as indicated in the left-hand panels. Since the simulations start with randomly placed particles, the initial behavior is dominated by swing-amplified shot noise (Toomre & Kalnajs 1991), and I stop the calculations after $\sim 20$ rotations at $R_e$, while spiral activity is ongoing.

The radial velocity dispersion rises due to spiral activity, which has lower initial amplitude as the number of particles rises. The vertical dispersion and rms $z$-thickness (Figure 2) of each population rises rapidly in the smallest simulation, but are almost constant in the largest, demonstrating that the models were in initial equilibrium. However, the tendency for mass segregation, unmistakable in the top row, is still visible in the bottom row.

The different heating rates for the two mass species is clear evidence for relaxation, as is also the rapid rise in the vertical motions of both populations as the number of particles is decreased. Note also, from the top right panel of Figure 2, that the velocity ellipsoid shapes of both populations rapidly become rounder. The larger experiments, on the other hand, manifest a slower energy equipartition rate and also support the theoretical prediction that spirals do not heat vertical motions.

The rate at which $v_z^2$ rises due to encounters, one step before arriving at Equation (1), is

$$\frac{dv_z^2}{dt} = \frac{8\pi G^2 \mu^2 n \ln \Lambda}{\nu}.$$  

In the simulations, $\mu \simeq M_\ell / N_h$ with $N_h$ being the number of heavy particles, $\Lambda = z_0 / \epsilon$, $\nu = \beta V_0$, and $n \sim N_h / (\pi R_e^2 z_0)$. We set $M_\ell = f M_{\text{tot}}$, and $V_0^2 = GM_{\text{tot}} / (R_e)$, and assume that the vertical velocity dispersion, $\sigma_z$, is rising due to relaxation. Thus we expect

$$\frac{d\sigma_z^2}{dt} = \frac{8 \pi^2 V_0^4 \ln(z_0 / \epsilon)}{N_h \beta z_0} \sim \frac{40}{N_h},$$

if $V_0 = 1, f = 0.25, z_0 = R_e / 20 = 2\epsilon$, and $\beta \simeq 0.17$.

I estimate the initial value of $d\sigma_z^2 / dt \sim 1.1 \times 10^{-5}$ from the light particles in the intermediate simulation. This is to be compared with $40 / N_h \sim 2 \times 10^{-4}$, since $N_h = 2 \times 10^5$. Thus the predicted scattering rate is about 20 times that observed. This discrepancy appears to be due to some additional grid smoothing in the simulation. Relaxation is even slower when softening and grid cell sizes are increased, and also when fewer sectoral harmonics, $m$, contribute to the grid-determined forces. In Figures 1 and 2, the force calculation discarded contributions to the forces from $m > m_{\text{max}} = 32$, and the relaxation rate in further simulations drops by about 30% for each factor two reduction in $m_{\text{max}}$. Applying a cutoff in the expansion of the density distribution effectively smooths each particle in the azimuthal direction, weakening the attraction between disk particles and slowing the relaxation rate. Since relaxation is inhibited by the extra smoothing in grid methods, I have deliberately employed a finer grid and more sectoral harmonics than is normally required in order to highlight relaxation.

A dependence on grid smoothing is physically reasonable, but contrasts with the finding of Hernquist & Barnes (1990) that...
the relaxation rate does not depend on the simulation method. Note that they were testing spherical models, where relaxation is dominated by density fluctuations on large scales and is little affected by short-range smoothing, which is the reason they found the rate of relaxation to be the same in every valid numerical method. However, relaxation in disks is dominated by close encounters (Section 2), making the rate much more dependent on the degree of smoothing. Therefore, it seems reasonable to expect a higher relaxation rate for the same $N$ in direct methods, e.g., tree codes, than in grid-based methods.

4. DISCUSSION AND CONCLUSIONS

Both the theoretical argument in Section 2 and the numerical results in Section 3 confirm that two-body scattering in disks is much more rapid than that expected in a spheroidal model with the same $N$. This is because disk particles pass each other at speeds much lower than the orbital speed, and because the density of particles in a disk is higher than were the same number spherically distributed. Furthermore, unlike in 3D systems, relaxation is dominated by the particles within a few disk thicknesses, and the cumulative effect of distant encounters is less important.

The experiments also confirm that spirals, which heat the in-plane motions (left panel of Figure 1), do not cause even extremely thin disks to thicken, when relaxation is slow enough (bottom row). With fewer particles, relaxation causes the shape of the velocity ellipsoid of both species to evolve rapidly to extremely thin disks to thicken, when relaxation is slow enough.

Thus any simulation of an isolated stellar disk that thickens probably does so through two-body relaxation. If gas is included, clumps of gas particles may behave as scattering centers, as do the GMCs in real disks, but with a smaller mass ratio to the star particles. Simulations that mimic the full hierarchical evolution will have other sources of vertical heating, such as in-falling dwarf galaxies and sub-halos, etc. However, segregation of star particles of different masses would remain a valid diagnostic of relaxation in all these contexts.

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REFERENCES

Binney, J., & Tremaine, S. 2008, Galactic Dynamics (2nd ed.; Princeton, NJ: Princeton Univ. Press) (BT08)

Bird, J. C., Kazantzidis, S., Weinberg, D. H., et al. 2013, arXiv:1301.0620

Carlberg, R. G. 1987, ApJ, 322, 59

Carlberg, R. G., & Sellwood, J. A. 1985, ApJ, 292, 79

Chandrasekhar, S. 1941, ApJ, 94, 511

D’Onghia, E., Vogelsberger, M., & Hernquist, L. 2013, ApJ, 766, 34

Governato, F., Brook, C., Mayer, L., et al. 2010, Natur, 464, 203

Hanninen, J., & Flynn, C. 2002, MNRAS, 337, 731

Hernquist, L., & Barnes, J. E. 1990, ApJ, 349, 562

Holmberg, J., Nordström, B., & Andersen, J. 2009, A&A, 501, 941

Ida, S., Kokubo, E., & Makino, J. 1993, MNRAS, 263, 875

Lacey, C. G. 1991, in Dynamics of Disc Galaxies, ed. B. Sundelius (Gothenburg: Göteborgs Univ.), 257

McMillan, P. J., & Dehnen, W. 2007, MNRAS, 378, 541

McNeve, I., Famaey, B., Combos, F., et al. 2011, A&A, 527, A147

Moster, B. P., McCaey, A. V., Somerville, R. S., et al. 2010, MNRAS, 403, 1009

Nordström, B., Mayor, M., Andersen, J., et al. 2004, A&A, 418, 989

Quinn, P. J., Hernquist, L., & Fullagar, D. P. 1993, ApJ, 403, 74

Robertson, B., Bullock, J. S., Cox, T. J., et al. 2006, ApJ, 645, 986

Roškar, R., Debattista, V. P., & Loebman, S. R. 2012, arXiv:1211.1982

Rubinski, G. B. 1972, in IAU Colloq. 10, Gravitational N-body Problem, ed. M. Lecar (Dordrecht: Reidel), 22

Scannapieco, C., Wadehpuh, M., Parry, O. H., et al. 2012, MNRAS, 423, 1726

Sellwood, J. A. 2012, ApJ, 751, 44

Sellwood, J. A. 2013, RvMP, submitted

Sellwood, J. A., & Valluri, M. 1997, MNRAS, 287, 124

Solway, M., Sellwood, J. A., & Schönrich, R. 2012, MNRAS, 422, 1363

Toomre, A. 1977, ARA&A, 15, 437

Toomre, A. 1981, in The Structure and Evolution of Normal Galaxies, ed. S. M. Fall & D. Lynden-Bell (Cambridge: Cambridge Univ. Press), 111

Toomre, A., & Kalnajs, A. I. 1991, in Dynamics of Disc Galaxies, ed. B. Sundelius (Gothenburg: Göteborgs Univ.), 341

Zang, T. A. 1976, PhD thesis, MIT