Four-dimensional Integral Model of Dry Friction on the Example of Wheel Movement

M S Salimov¹, G R Saypulaev², I V Merkuriev³

¹Department of Robotics, mechatronics, dynamics and strength of machines
National Research University “Moscow Power Engineering Institute”
Moscow, Russia
²Department of Robotics, mechatronics, dynamics and strength of machines
National Research University “Moscow Power Engineering Institute”
Moscow, Russia
³Department of Robotics, mechatronics, dynamics and strength of machines
National Research University “Moscow Power Engineering Institute”
Moscow, Russia

E-mail: imaxsalimov@gmail.com, saypulaevgr@mail.ru, merkuryeviv@ya.ru

Abstract. A four-dimensional model of dry friction in the interaction of a solid wheel and a horizontal rough surface is investigated. It is assumed that there is no separation between the wheel and the horizontal surface. The movement of the body occurs in conditions of combined dynamics, when in addition to the sliding movement, the body participates in spinning and rolling. The equation of motion of the wheel is compiled using the Appel equation. The resulting model of sliding, spinning, and rolling friction is given for the case where the contact area is a circle. The cumbersome integral expressions were replaced by fractional-linear Pade approximations. Pade approximations accurately describe the behavior of the components of the friction model. A mathematical model is proposed that describes the simultaneous sliding, spinning and rolling of a solid wheel. The dependences of the parallel and perpendicular components of the friction force and the torque of the spinning friction were plotted with respect to the parameter that characterizes the movement of the wheel. Comparisons of the integral friction model and the model based on Pade approximations are presented. The results of the comparison showed a qualitative correspondence of the models. After obtaining the equation of motion, the simulation of motion at a constant control torque of the wheel is carried out. The graphs allow you to match the logical behavior of the wheel movement.

1. Introduction
The increased interest in solving the problem of studying the movement of the wheel in recent years is primarily due to the possibility of widely using the results obtained in the field of robotics. The dynamics of the wheel are not much different from the dynamics of a cart or car. However, research on the dynamic behavior of the wheel has become more important in recent years. The relevance is explained by the development of vehicle control systems, the increasing demand for pedestrian vehicles, such as scooters, skateboards and subsequent analogues. [1], [2]. The movement of mobile robots is due to the movement of the wheel. Thanks to the simple and efficient control of the
movement of the robot on wheels, such mobile robots can be used in a wide class of tasks. For example, the creation of one-wheeled robots for the exploration of outer space and planets. For industrial purposes, for conducting research in severe conditions, for example, in an aquatic environment or during underground work [3], [4].

Thus, research in the field of robotics is very relevant. Consequently, there are new problems that require theoretical research, for example, the study of models of friction between the body and the surface in the conditions of combined dynamics [5]. Since the movement of the mobile robot occurs in different directions, it is necessary to take into account the longitudinal movement and rotation. And in the case of the wheel, there is also rolling, which makes the task more difficult. A similar study was made in [6], the authors considered the dynamics of a cylinder on a vibrating base, but without using a combined friction model.

Some scientists have developed models of friction, which take into account the relationship between the sliding and spinning speeds [7]. A detailed description of this relationship was made in [8]. The resulting integral expressions for describing the main moment and the vector of friction forces are complex. To simplify the solutions for the obtained dependencies, the authors in [9] constructed fractional-linear Pade approximations.

Pade approximations not only simplify analytical expressions, but also help to explain the actions of combined dry friction for linear and angular velocities [10]. Thus, based on the Pade approximations, it became possible to create new models of friction [11], which later began to be classified for better interpretation [12].

The model of sliding, spinning and rolling friction, which is proposed in this article, allows us to take into account the dynamic relationship of those components that determine the force interaction of the wheel and the support surface.

2. Problem statement, basic notation

A wheel with a mass $m$, and radius $R$ are considered. A fixed coordinate system is introduced $O_1x_1y_1z_1$, as well as a movable coordinate system $CXYZ$ with the start in the center of the wheel. The point of contact of the wheel and the horizontal plane $K$ and the center of mass $C$ lie on the same axis $CZ$.

![Figure 1. Wheel movement.](image)

The velocity of the center of mass $v_C$ is decomposed into two projections $v_x$ and $v_y$, which are projected on the $CX$ and $CY$ axes, respectively. A control moment $M$ is applied to the wheel, the angular velocity of the wheel $\omega = \varphi$ relative to the $CY$ axis and the angular velocity $\Omega = \psi$ relative to the $CZ$ axis.

Let $q = (x, y, \psi, \varphi)^T$ be the vector of generalized coordinates of the system, and let $\dot{\pi} = (\dot{v}_x, \dot{v}_y, \dot{\Omega}, \dot{\omega})$ be the vector of pseudo-accelerations. The equation of motion of the wheel was compiled in the form of the Appel equation, which has the form:

$$\frac{dS}{dt} = \Pi,$$

where $S = S(\dot{\pi}, \dot{\pi}, q)$ is energy acceleration of the system, $\Pi = (\Pi_{v_x}, \Pi_{v_y}, \Pi_{\Omega}, \Pi_{\omega})$ is the vector of generalized forces are given by pseudocerastes $\dot{\pi} = (v_x, v_y, \Omega, \omega)$. 

2
The acceleration energy $S$ is:

$$S = \frac{m v_c^2}{2} + \frac{J_{KZ} \hat{\Omega}^2}{2} + \frac{J_{KY} \hat{\omega}^2}{2},$$  \hspace{1cm} (2)

where $v_c$ is the acceleration of the center of mass of the wheel. $J_{KZ}$ and $J_{KY}$ are the moments of inertia of the wheel relative to the axes $CZ$ and $CY$, respectively.

Equation (2) can be rewritten as:

$$S = \frac{m}{2} \left( (\dot{v}_x - \Omega \omega_y)^2 + (\dot{v}_y + \Omega \omega_x)^2 \right) + \frac{J_{KZ} \hat{\Omega}^2}{2} + \frac{J_{KY} \hat{\omega}^2}{2},$$  \hspace{1cm} (3)

After substituting formula (3) into expression (1), we differentiate with respect to pseudo-accelerations $\dot{\pi} = (\dot{v}_x, \dot{v}_y, \dot{\Omega}, \dot{\omega})$:

$$\frac{dS}{d\dot{v}_x} = m(\dot{v}_x - \Omega \omega_y),$$
$$\frac{dS}{d\dot{v}_y} = m(\dot{v}_y + \Omega \omega_x),$$
$$\frac{dS}{d\dot{\Omega}} = J_{KZ} \hat{\Omega},$$
$$\frac{dS}{d\dot{\omega}} = J_{KY} \hat{\omega},$$

Thus, we obtain four equations of motion:

$$\begin{cases} m(\dot{v}_x - \Omega \omega_y) = \Pi_{v_x} \\
 m(\dot{v}_y + \Omega \omega_x) = \Pi_{v_y}, \hspace{1cm} (4) \\
 J_{KZ} \dot{\Omega} = \Pi_{\dot{\Omega}} \\
 J_{KY} \dot{\omega} = \Pi_{\dot{\omega}} \end{cases}$$

Next, we find the generalized forces for the equations of motion. To do this, write down the formula for the power of the active forces:

$$N_\alpha^b = (\bar{M}, \bar{\omega}^b) + (\bar{F}_{\text{trp}}, \bar{\nu}_K^b) + (\bar{M}_{\text{trp}}, \bar{\omega}_K^b),$$  \hspace{1cm} (5)

The "b" index characterizes the possible wheel speeds. The terms $\bar{M}_{\text{trp}}$ and $\bar{F}_{\text{trp}}$ of the equation (5) in projections on the mobile coordinate system $CXZY$:

$$\bar{M}_{\text{trp}} = \begin{bmatrix} M_x \\
 M_y \\
 M_C \end{bmatrix}, \hspace{0.5cm} \bar{F}_{\text{trp}} = \begin{bmatrix} F_x \\
 F_y \end{bmatrix},$$  \hspace{1cm} (6)

The velocity $\bar{v}_K^b$ of the point $K$ in vector form:

$$\bar{v}_K^b = \begin{bmatrix} \nu_{xK}^b \\
 \nu_{yK}^b \\
 0 \end{bmatrix} + \begin{bmatrix} 0 & -\Omega^b & \omega^b \\
 \Omega^b & 0 & 0 \\
 -\omega^b & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\
 0 \\
 -R \end{bmatrix} = \begin{bmatrix} \nu_{xK}^b - \omega R \\
 \nu_{yK}^b \\
 0 \end{bmatrix},$$  \hspace{1cm} (7)

Then the components of the velocity are equal:

$$v_{Kx} = \nu_{xK} = \nu_x - \omega R$$
$$v_{Ky} = \nu_{yK} = \nu_y$$

Let's rewrite the resulting active power equation by taking out the possible generalized coordinates of the system $\nu_{xK}^b, \nu_{yK}^b, \Omega^b, \omega^b$:

$$N_\alpha^b = F_x \nu_{xK}^b + F_y \nu_{yK}^b + (M + M_y - F_x R) \omega^b + M_C \Omega^b,$$  \hspace{1cm} (8)
Thus, we obtain four generalized forces:

\[
\begin{align*}
\Pi_{u_x} &= F_x, & \Pi_{\Omega} &= M_C, \\
\Pi_{\omega} &= M + M_y - F_x R, & \Pi_{u_y} &= F_y,
\end{align*}
\]

Substituting them into the equations of motion (4), we get

\[
\begin{align*}
m(\ddot{u}_x - \Omega \dot{u}_y) &= F_x \\
m(\ddot{u}_y + \Omega \dot{u}_x) &= F_y \\
I_{zz} \dot{\Omega} &= M_C \\
I_{xy} \dot{\omega} &= M + M_y - F_x R
\end{align*}
\]

Next, we find a combined model of friction.

3. Combined sliding, spinning and rolling model

The proposed associated friction model should take into account the rolling of the body, in addition to sliding and spinning. The distribution of normal stresses for circular contact sites depends on the radius vector with the origin in the center of the contact spot [14]. Such models of friction are applicable for bodies where there is translational and rotational motion. However, experiments [15], [1] suggest the effect of rolling on the symmetry in the contact stress distribution diagram. This effect consists in the curvature of the stress distribution diagram in the direction of the instantaneous rolling speed. This skew is described by a linear function with a single coefficient, which depends on the direction and speed of rolling [16]. A violation in the symmetry leads to the appearance of a component of the friction force perpendicular to the direction of the sliding speed. As a result, the friction model becomes four-dimensional.

The construction of a model of sliding, spinning and rolling friction is carried out for circular contact areas.

![Figure 2. Wheel-surface contact area.](image)

The circular contact area \( G \) has a center at a point \( K \) and an arbitrary point \( M \), which is determined using the polar coordinates \( r \) and \( \xi \); \( dS \) - the elementary contact area; \( \Omega \) - the angular velocity of rotation of the body relative to the axis \( CZ \); \( \omega_r \) - the modulus of the angular rolling speed, \( \nu \) - the linear sliding speed of the point \( K \), the direction of which is characterized by an angle \( \alpha \).

\[
\begin{align*}
\nu_{Kx} &= \nu \cos \alpha \\
\nu_{Ky} &= \nu \sin \alpha
\end{align*}
\]
Coulomb’s law [17] is valid in differential form for a small contact area $dS$. The differential of the main vector $dF$ of the friction force of the relative center of the spot is determined:

$$dF = -f \hat{\sigma}(x, y) \frac{V}{|V_M|} dS,$$

where $f$ is the coefficient of friction, $r(x, y)$ is the radius-vector to the elementary contact area $dS$ relative to the center of the circular contact area. $\hat{\sigma}(x, y)$ - distribution of normal contact stresses.

Elementary moment of friction forces at the point $M$ can be written in vector form:

$$dM = r \times dF = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \times \begin{bmatrix} dF_x \\ dF_y \\ dN \end{bmatrix} = \begin{bmatrix} ydN \\ -xdN \\ x dF_y - y dF_x \end{bmatrix},$$

where $dN = \sigma dS$ – elementary reaction support, $dF_x$ and $dF_y$ - elementary components of the friction force on the vertical and horizontal axis [18], respectively.

Thus,

$$dM_x = ydN,$$
$$dM_y = -xdN,$$
$$dM_z = x dF_y - y dF_x$$

(11)

Next, we find expressions for the velocity of a point, using the Euler formula [13]. The sliding velocity at a point $M$ is decomposed into two components along the axes $CX$ and $CY$:

$$V_M = (v_{kx} - \Omega y, \ v_{ky} + \Omega x, \ 0)^T$$

The velocity modulus $|V_M|$ of an arbitrary point $M$ is expressed as follows using (10):

$$|V_M| = \sqrt{(v_{kx} - \Omega y)^2 + (v_{ky} + \Omega x)^2} = \sqrt{v^2 + \Omega^2(x^2 + y^2) - 2\nu \Omega (y \cos \alpha - x \sin \alpha)}$$

where $v = \sqrt{v_{kx}^2 + v_{ky}^2}$.

At the angular rolling speed, there is a skew in the symmetric diagram of the distribution of contact stresses in a rectangular coordinate system, the axis is directed along the instantaneous sliding speed. The skew is described by the dependence [19]:

$$\hat{\sigma}(x, y) = \sigma(x, y) \left(1 + k_r \frac{x \sin \beta - y \cos \beta}{\sqrt{\omega_x^2 + \omega_y^2}} \right),$$

(12)

$$|k_r| \leq 1, k_r \equiv 0, \omega = 0$$

where is the coefficient $k_r$ whose sign depends on the rolling direction. One of the requirements that is imposed on the value $k_r$ is to limit the action of the center of gravity, whose distribution of contact stresses does not extend beyond the boundaries of the contact spot ($|k_r| \leq R^{-1}$). Area $G = x^2 + y^2 \leq R_r$.

The angle $\beta \in [0, \pi/2]$ relates the values of the projections $\omega_x, \omega_y$ and the instantaneous rolling speed $\omega_r$, in which $\omega_x = 0$ we have:

$$\cos \beta = \omega_x / \omega_r = 0, \sin \beta = \omega_y / \omega_r, \omega_r = \omega_x^2 + \omega_y^2 = \omega_r^2,$$

(13)

After substituting equations (13) into formula (12)
Integrating the expressions for the friction forces and substituting the expressions for the velocity \( V_M \), we get:

\[
F_x = - \int_G f \, \tilde{\sigma}(x,y) \frac{u_k x - \Omega y}{|V_M|} \, dx \, dy,
\]

\[
F_y = - \int_G f \, \tilde{\sigma}(x,y) \frac{u_k y + \Omega x}{|V_M|} \, dx \, dy,
\]

(15,16)

We also make a transition to the projections of the friction force on the axis of the coordinate system \( Kxyz \) from the two components of the friction force \( F_x, F_y \).

\[
F = F_x \cos \alpha + F_y \sin \alpha
\]

\[
F_{\perp} = F_x (-\sin \alpha) + F_y \cos \alpha
\]

After integrating the differentials of the friction forces and its moment along the contact spot for \( \nu = 0, \Omega = 0, \nu \to \infty, \Omega \to \infty \), the dependences of the two components of the friction forces and their three moments relative to the three axes and \( CX, CY, CZ \)

\[
F_{\parallel} = \int_{G} \frac{\tilde{\sigma}(x,y) [(v - \Omega(y \cos \alpha - x \sin \alpha)]}{\sqrt{u^2 + \Omega^2(x^2 + y^2) - 2\nu \Omega(y \cos \alpha - x \sin \alpha)}} \, dx \, dy
\]

\[
F_{\perp} = \int_{G} \frac{\tilde{\sigma}(x,y) [(\Omega(y \sin \alpha - x \cos \alpha)]}{\sqrt{u^2 + \Omega^2(x^2 + y^2) - 2\nu \Omega(y \cos \alpha - x \sin \alpha)}} \, dx \, dy
\]

\[
M_{C} = \int_{G} \frac{\tilde{\sigma}(x,y) [(\Omega(x^2 + y^2) + \nu(x \sin \alpha - y \cos \alpha)]}{\sqrt{u^2 + \Omega^2(x^2 + y^2) - 2\nu \Omega(y \cos \alpha - x \sin \alpha)}} \, dx \, dy
\]

\[
M_x = \int_{G} y \tilde{\sigma}(x,y) \, dS
\]

\[
M_y = - \int_{G} x \tilde{\sigma}(x,y) \, dS
\]

When there is no rolling, the distribution of normal contact stresses is symmetric \( \sigma(x,y) = \sigma(r) \), so for convenience, the transition to polar coordinates \( x = r \cos \xi, y = r \sin \xi, r \in [0,1], \xi \in [0,2\pi] \) occurs and we will pre-set \( u = \Omega R_p \). And also immediately apply the formulas for adding and subtracting the arguments of trigonometric functions.
The main moment of the friction forces in a rectangular coordinate system with a center that coincides with the center of the contact spot and with the axes, one of which is directed in the direction of sliding, is equal to

\[ M = \left( M_x, M_y, -M_z \right). \]  

The moment of the friction force \( M_x \) relative to the axis \( CX \) is zero. This is because otherwise the wheel would lose its balance and fall on its side. The resulting integral model accurately describes the relationship of sliding friction, spinning, and rolling, but requires calculations of complex integrals. To avoid calculating such integrals, a model based on Pade approximations is used.

4. Pade approximation for the components of the force and moment of friction

After investigating the behavior of functions of integrals \( F_{||}, F_{\perp}, M \) \((v, u, \beta, \alpha)\) using

\[ \lim_{u \to \infty} F_{\parallel}, \lim_{u \to \infty} F_{\perp}, \lim_{u \to \infty} M \left( \frac{\partial F_{\parallel}}{\partial u} \right)_{u=0}, \left( \frac{\partial F_{\perp}}{\partial u} \right)_{u=0}, \left( \frac{\partial F_{M}}{\partial u} \right)_{u=0}, \]  

for all functions, we will compose a first-order Pade decomposition:

\[ F_{\parallel} = \frac{a_{1}u + a_{2}u}{b_{1}u + b_{2}u} \]
\[ F_{\perp} = \frac{a_{1}u + a_{2}u}{b_{1}u + b_{2}u} \]
\[ M_{z} = \frac{a_{1}u + a_{2}u}{b_{1}u + b_{2}u} \]

where the coefficients are determined from the equalization of the studies (18) and the corresponding polynomials (19).

We write down the resulting formulas of the components of the friction forces and their moments obtained by the Pade decomposition
5. Results

After substituting the obtained parameters (20) into the equation of motion (9), first making the transition according to the formulas

\[
F_{\|} = -f N \frac{\sqrt{u_{Kx}^2 + u_{Ky}^2 + \dot{a}_1 \Omega}}{\sqrt{u_{Kx}^2 + u_{Ky}^2 + \dot{b}_1 \Omega}}
\]

\[
F_{\perp} = -f k_p \pi I_2 \frac{\Omega}{\sqrt{u_{Kx}^2 + u_{Ky}^2 + \Omega}}
\]

\[
M_c = -2f \pi I_2 \frac{\dot{a}_3 \sqrt{u_{Kx}^2 + u_{Ky}^2 + \Omega}}{\dot{b}_3 \sqrt{u_{Kx}^2 + u_{Ky}^2 + \Omega}}
\]

\[
M_y = -\pi k_p I_3 \sin \beta = -\pi k_p I_3 \frac{\omega_y}{\sqrt{\omega_y^2 + \omega_r^2}}
\]

The graphs correspond to the logical behavior of the wheel movement: the angular rolling speed and the parallel dependence of the speed increase linearly, based on the specified parameters. We can conclude that with the help of fractional-linear approximations, Pade managed to replace cumbersome integral expressions. Pade approximations accurately describe the behavior of the components of the friction model.

Analytical expressions of the main vector and the moment of friction forces for circular wheel-surface contact areas are obtained. The proposed mathematical model, which describes the simultaneous sliding, spinning and rolling of the wheel, allows us to study the movement close to the

\[
F_x = F_{\|} \cos \alpha - F_{\perp} \sin \alpha
\]

\[
F_y = F_{\|} \sin \alpha + F_{\perp} \cos \alpha
\]
real one, by taking into account the component forces and moments in the contact area of the body and the surface.

The results obtained allow us to take into account the dynamic connection of the components that determine the force interaction of a solid wheel and a horizontal plane. These results can be used in the field of robotics, for example, the creation of mobile robots that move with the help of wheels, for conducting research in difficult conditions.

6. References

[1] Svendenius J 2003 Tire Models for Use in Braking Applications Department of Automatic Control, Lund Institute of Technology

[2] Kireenkov A, Zhavoronok S, Nushtaev D On tire models accounting for both deformed state and coupled dry friction in a contact spot Computer Research and Modeling Vol 13 1 pp 163–173 (in Russian)

[3] Salimov M, Ramzin N 2019 Movement of the body on a vibrating surface in the case of dry friction Problems of mechanical engineering and automation 4 pp 100–104 (in Russian)

[4] Munitsyn L 2017 Vibrations of a Rigid Body with Cylindrical Surface on a Vibrating Foundation Mech. Solids Vol. 52 6 pp 675–685

[5] Adamov B, Saypulaev G 2020 A study of the dynamics of an omnidirectional platform, taking into account the design of mecanum wheels and multicomponent contact friction 2020 International Conference Nonlinearity, Information and Robotics (NIR) pp 1–6

[6] Salimov M, Ramzin N 2020 Dynamics of a Rigid Body with a Cylindrical Surface on a Vibrating Plane Journal of Machinery Manufacture and Reliability Vol. 49 8 pp 653–658

[7] Kireenkov A, Zhavoronok S 2020 Anisotropic combined dry friction in problems of pneumatics’ dynamics Journal of vibration engineering and technologies Vol. 8 pp 365–372

[8] Zhuravlyov V 1998 On the model of dry friction in the problem of rolling solids Applied Mathematics and Mechanics Vol. 62 5 pp 762–767

[9] Andronov V, Zhuravlev V 2010 Dry friction in problems of mechanics M. Regular and chaotic dynamics

[10] Acary V, Brémond M, Huber O 2018 On solving contact problems with coulomb friction: Formulations and numerical comparisons Transactions of the European Network for Nonsmooth Dynamics pp 375–457

[11] Kireenkov A, Fedotenkov G 2020 Motion of a composite spherical shell on a solid surface taking into account the combined dry friction Mechanics of Composite materials and Structures Vol. 26 3 pp 327-340

[12] Kireenkov A The associated model of sliding and spinning friction Reports of the Academy of Sciences Vol. 441 6 pp 750-757

[13] Markeev A 1999 Theoretical mechanics: A textbook for universities (M) Regular and chaotic dynamics

[14] Kireenkov A, Zhavoronok S 2020 Anisotropic combined dry friction in problems of pneumatics’ dynamics Journal of vibration engineering and technologies Vol. 8 pp 365–372

[15] Alamdarlo M, Hesami S 2021 Statistical analysis of variables affecting tire-pavement friction International Journal of Pavement Research and Technology Vol. 14 pp 378–384

[16] Kireenkov A, Zhavoronok S 2017 On the effect of the deformed state of a tire on the combined wheels rolling, sliding, and spinning with dry friction 9th European nonlinear dynamics conference

[17] Pervozvanskij A 1998 Friction is a familiar but mysterious force Soros Educational journal 2 pp 120–134

[18] Munitsyna M 2013 Dynamics of a body of rotation on a horizontal plane in the framework of an interconnected friction model Mathematical modeling Optimal control Bulletin of the Lobachevsky University of Nizhny Novgorod Vol. 1 3 pp 237-240 (in Russian)
[19] Salimov M S, Merkuriev I V 2021 Three-dimensional integral dry friction model for the motion of a rectangular body *Advanced Engineering Research* Vol 21 1 pp 14-21

**Acknowledgments**

The investigation was carried out within the framework of the project “Development of a prototype of a new autonomous mobile robot for solving problems of monitoring the technical condition of cable equipment” with the support of a grant from NRU “MPEI” for implementation of scientific research programs “Energy”, “Electronics, Radio Engineering and IT”, and “Industry 4.0, Technologies for Industry and Robotics in 2020-2022”.

