ON THE CLASSIFICATION OF ORIENTED 3-PLANE BUNDLES OVER A 6-COMPLEX

BENJAMIN ANTIEAU AND BEN WILLIAMS

ABSTRACT. In this short note, we complete the description of low-degree characteristic classes of oriented 3-plane bundles over a 6-complex. Our goal is to point out and correct an error in L. M. Woodward’s 1982 paper “The classification of principal $PU_n$-bundles over a 4-complex.”

Key Words. Classification of oriented vector bundles, Postnikov towers.

Mathematics Subject Classification 2010. Primary: 55R10, 55R45.

When $G$ is a compact Lie group, the problem of classifying principal $G$-bundles over a topological space $X$ is one of the central problems of algebraic topology. Attempts to solve this problem in various cases lead to the ideas of Postnikov towers, characteristic classes, and $K$-theory. In algebraic geometry, the analogous problem of understanding vector bundles (principal $GL_n$-bundles) is the heart of an enormous amount of ongoing conjecture and research.

When $G = SO_n$, the problem is to classify oriented $n$-plane bundles on a $k$-complex $X$. A typical approach is to consider $k$-tuples of characteristic classes of oriented $n$-plane bundles and to ask which $k$-tuples of cohomology classes of $X$ occur as the characteristic classes of an oriented $n$-plane bundle. This subject has been studied since the beginnings of algebraic topology; see [4] for an overview and [5] for early work on this problem. Most results are for small $k$, when $n$ is large with respect to $k$. This is thanks to simplifications due to Bott periodicity (see [8]). Another case that has been studied for $k \leq 9$ is when $k = n$. The case when $n < k$ is much, much more difficult. When $n = 2$, the answer is known: an oriented 2-plane bundle is determined by its Euler class in $H^2(X, \mathbb{Z})$, and any class in $H^2(X, \mathbb{Z})$ is the Euler class of a 2-plane bundle. When $n = 3$, previous results allow for a classification when $k \leq 5$ by [4, Theorem 2].

Theorem 1. Let $X$ be a 6-dimensional CW complex. Consider the map

\[(1) \quad [X, BSO_3] \to H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})\]

which sends an oriented 3-plane bundle $\xi$ on $X$ to the pair of cohomology classes $(w_2(\xi), p_1(\xi))$, where $w_2(\xi)$ is the second Stiefel-Whitney class and $p_1(\xi)$ is the first Pontrjagin class. Let $\rho_{4*} : H^4(X, \mathbb{Z}) \to H(X, \mathbb{Z}/4)$ denote reduction modulo 4, and let $P_2 : H^2(X, \mathbb{Z}/2) \to H^4(X, \mathbb{Z}/4)$ be the Pontrjagin square. The image of (1) consists of the set of classes $(x, y)$ satisfying

\[(2) \quad \rho_{4*}(p_1(\xi)) = 3P_2(w_2(\xi)),\]

such that $u(x, y) = 0$, where $u(x, y)$ is a certain function on the set of pairs of cohomology classes satisfying (2) with values in $H^6(X, \mathbb{Z}/2)$, to be defined below. Moreover, there is a 6-dimensional CW complex $X$ and a pair of classes $(x, y)$ satisfying (2) such that $u(x, y) \neq 0$. 

1
Previous work of Woodward [7], using the language of principal $PU_2$-bundles and the exceptional isomorphism $PU_2 \cong SO_3^1$, purported to solve this problem as well. Unfortunately, the part of the main theorem of Woodward dealing with 3-planes on 6-complexes is mistaken, because of the incorrect assumption there that $\pi_3BSO_3 = 0$, which appears on p.521. As shown by Bott [3, Theorem 5], $\pi_3BSO_3 = \mathbb{Z}/2$. We explain how this affects the main theorem of [7], and how to correct the theorem. The correction requires more than simply re-writing Woodward’s proof to take the correct homotopy group into account; we need additional information, which comes from the Postnikov tower of $BSO_3$.

Woodward claimed that the image of the map (1) consists of classes satisfying (2) with no other restrictions when $X$ is a CW complex of dimension at most 6. Only a small portion of this claim is false: when $\dim X = 6$, there are some classes $(x, y) \in H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})$ satisfying $\rho_{4*}(y) = 3p_2(x)$, but which are not the characteristic classes of any $PU_2$-bundle over $X$. The necessary additional condition is as stated in our theorem.

For 6-complexes $X$, there is a surjection \([X, BSO_3] \to [X, BSO_3[5]],\) where $BSO_3[5]$ denotes the 5th stage in the Postnikov tower for $BSO_3$. The characteristic classes above are obtained by showing that $BSO_3[4]$ is equivalent to the homotopy fiber of the map $-3p_2 + \rho_{4*}$

\[
K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4) \to K(\mathbb{Z}/4, 4).
\]

Thus, given a 3-plane bundle $\xi$ over $X$, the characteristic classes are given by the composition

\[X \xrightarrow{\xi} BSO_3 \to BSO_3[4] \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4).\]

The relation (2) is expressed in the fact that this map lands in the fiber of (3). We are left with the problem of computing the image of \([X, BSO_2[5]] \to [X, BSO_2[4]].\)

The 5th stage of the Postnikov tower for $BSO_3$ gives a fiber sequence,

\[K(\mathbb{Z}/2, 5) \to BSO_3[5] \to BSO_3[4],\]

which deloops to a map $BSO_3[4] \to K(\mathbb{Z}/2, 6)$ of which $BSO_3[5]$ is the homotopy fiber. Under the correspondence between maps $BSO_3[4] \to K(\mathbb{Z}/2, 6)$ and cohomology classes in $H^6(BSO_3[4], \mathbb{Z}/2)$, the map is classified by a class $u \in H^6(BSO_3[4], \mathbb{Z}/2)$.

**Definition 2.** Given a space $X$ and classes $(x, y) \in H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})$ satisfying (2), one has a uniquely determined map $f : X \to BSO_3[4]$, and hence a cohomology class $f^*(u) \in H^6(X, \mathbb{Z}/2)$. In order for $f$ to lift to a map $X \to BSO_3[5]$, it is necessary for $f^*(u) = 0$. If $\dim X \leq 6$, this is also a sufficient condition. If $f$ is determined by classes $(x, y)$ satisfying (2), write $u(x, y)$ for $f^*(u)$; thus, $u(x, y) \in H^6(X, \mathbb{Z}/2)$.

**Proof of theorem.** Since Woodward identified $BSO_3[4]$ as the fiber of the map $K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4) \to K(\mathbb{Z}/4, 4)$, given by relation (2), the image of the composition

\[\left[ X, BSO_3 \right] \to \left[ X, BSO_3[4] \right] \to H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})\]

consists of pairs of classes satisfying (2). By the theory of Postnikov towers, a map $f : X \to BSO_3[4]$ lifts to $X \to BSO_3[5]$ if and only if $f^*(u) = 0$, which by

\[1\]In fact, it was the classification of principal $PU_2$-bundles which originally sparked our interest in this problem.
our definition, occurs if and only if \( u(x, y) = 0 \). Since on a 6-complex any map \( X \to \text{BSO}_3[5] \) lifts to a map \( X \to \text{BSO}_3 \), this proves the first statement.

To prove the second statement is equivalent to showing that the extension \( K(\mathbb{Z}/2, 5) \to \text{BSO}_3[5] \to \text{BSO}_3[4] \) is non-split. Indeed, if it is non-split, then the 6-skeleton of \( \text{BSO}_3[4] \) together with the composition

\[
\text{sk}_6 (\text{BSO}_3[4]) \to \text{BSO}_3[4] \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)
\]
gives an example.

Recall that \( \text{PU}_2 = \text{SU}_2/\mathbb{Z}/2 \), where \( \mathbb{Z}/2 \) is the center of the special unitary group \( \text{SU}_2 \). The quotient map \( \text{SU}_2 \to \text{PU}_2 \cong \text{SO}_3 \) induces a map on classifying spaces \( \text{BSU}_2 \to \text{BSO}_3 \), which induces an isomorphism on homotopy groups \( \pi_i \) for \( i > 2 \). By the naturality of Postnikov towers, there is thus a map of extensions

\[
\begin{array}{ccc}
K(\mathbb{Z}/2, 5) & \longrightarrow & \text{BSU}_2[5] \\
\| & & \downarrow \\
K(\mathbb{Z}/2, 5) & \longrightarrow & \text{BSO}_3[5] \to \text{BSO}_3[4].
\end{array}
\]

If the class of the extension in \( H^6(K(\mathbb{Z}, 4), \mathbb{Z}/2) \) is non-zero, then by the commutativity of the diagram, the class in \( H^6(\text{BSO}_3[4], \mathbb{Z}) \) is non-zero. It is not hard to show, using the Serre spectral sequence, that \( H^6(K(\mathbb{Z}, 4), \mathbb{Z}/2) = \mathbb{Z}/2 \), generated by a class \( \gamma \). On the other hand, \( H^6(\text{BSU}_2, \mathbb{Z}) = \mathbb{Z}[c_2] \), where the class \( c_2 \) has degree 4. Therefore, \( H^6(\text{BSU}_2, \mathbb{Z}/2) = 0 \). Since \( \text{BSU}_2 \to \text{BSU}_2[5] \) is a 6-equivalence, it follows that \( H^6(\text{BSU}_2[5], \mathbb{Z}/2) = 0 \) as well. If the extension were split, then the pullback of \( \gamma \) to \( \text{BSU}_2[5] \) would be non-zero. Thus the extension is not split. □

In [2], we produce an example of a 6-dimensional smooth affine variety \( X \) and a fixed non-zero class \( x \in H^2(X, \mathbb{Z}/2) \) such that there is no oriented 3-plan \( \xi \) with \( w_2(\xi) = x \). This is despite the fact that there is a pair \( (x, y) \) satisfying (2). Thus, in some sense, Woodward’s statement can fail as badly as possible in some situations.

Now, we prove a corollary, which amounts to determining the class \( u \) in

\[ H^6(\text{BSO}_3[4], \mathbb{Z}/2). \]

By Serre [6, Section 9], the \( \mathbb{Z}/2 \)-cohomology of \( K(\mathbb{Z}/2, 2) \) is a polynomial ring

\[ H^*(K(\mathbb{Z}/2, 2), \mathbb{Z}/2) = \mathbb{Z}/2[u_2, Sq^1 u_2, Sq^2 u_2, \ldots, Sq^k u_2, \ldots], \]

where \( u_2 \) is the fundamental class in degree 2, and \( Sq^i \) denotes the \( i \)-th Steenrod operation. Let \( \text{BSO}_3[4] \to K(\mathbb{Z}/2, 2) \) be denoted by \( p \).

**Corollary 3.** The set \( \{ u, p^* u_3, p^* (\text{Sq}^1 u_2)^2 \} \) forms a basis of the 3-dimensional vector space \( H^6(\text{BSO}_3[4], \mathbb{Z}/2) \).

**Proof.** There is an isomorphism \( H^*(\text{BSO}_3, \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3] \), where \( w_i \) has degree \( i \). The map \( \text{BSO}_3 \to \text{BSO}_3[2] \cong K(\mathbb{Z}/2, 2) \) is a 4-equivalence, so that \( w_2 \) is the pullback of \( u_2 \), and \( w_3 \) is the pullback of \( Sq^1 u_2 \). It follows, in fact, that \( H^*(\text{BSO}_3[n], \mathbb{Z}/2) \) contains the algebra \( \mathbb{Z}/2[w_2, w_3] \) for \( n \geq 2 \). A brief examination of the Serre spectral sequence for the fibration \( K(\mathbb{Z}, 4) \to \text{BSO}_3[4] \to K(\mathbb{Z}/2, 2) \) shows that the dimension of \( H^6(\text{BSO}_3[4], \mathbb{Z}/2) \) is at most 3. The classes \( w_3 \) and \( w_3^2 \) must survive and be distinct, since they do in the cohomology of \( \text{BSO}_3 \). Finally, since we showed in the proof of the theorem that the extension class \( u \) restricts to
the non-zero class in $H^6(K(Z, 4), \mathbb{Z}/2)$, it follows that the asserted classes form a basis for $H^6(BSO_3[4], \mathbb{Z}/2)$, as desired.

\[\square\]

References

[1] B. Antieau and B. Williams, The topological period-index problem for 6-complexes, J. Top., available at http://arxiv.org/abs/1208.4430. To appear.

[2] , Unramified division algebras do not always contain Azumaya maximal orders, Inv. Math. (2013), available at http://dx.doi.org/doi:10.1007/s00222-013-0479-7.

[3] R. Bott, The space of loops on a Lie group, Michigan Math. J. 5 (1958), 35–61.

[4] M. Čadek and J. Vanžura, On the classification of oriented vector bundles over 5-complexes, Czechoslovak Math. J. 43(118) (1993), no. 4, 753–764, available at http://www.math.muni.cz/~cadek/list.html.

[5] A. Dold and H. Whitney, Classification of oriented sphere bundles over a 4-complex, Ann. Math. 69 (1959), 667–677.

[6] J.-P. Serre, Cohomologie modulo 2 des complexes d’Eilenberg-MacLane, Comment. Math. Helv. 27 (1953), 198–232.

[7] L. M. Woodward, The classification of principal $PU_n$-bundles over a 4-complex, J. London Math. Soc. (2) 25 (1982), no. 3, 513–524.

[8] , The classification of orientable vector bundles over CW-complexes of small dimension, Proc. Roy. Soc. Edinburgh Sect. A 92 (1982), no. 3-4, 175–179.