Inverse problem for recovery of temporal component of source term for multi-term time fractional parabolic equation with nonlocal boundary datum

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Abstract

Inverse problem for multi-term fractional parabolic equation in two dimensional space, involving \(m+1\) Caputo fractional derivatives in time, is investigated. Presence of nonlocal boundary conditions leads to a non-self-adjoint spectral problem. A bi-orthogonal system of functions is used to construct the solution that involves double infinite series. Properties of multinomial Mittag-Leffler function and eigenfunctions are used to prove the classical nature of the solution under certain regularity conditions on the given datum.

Keywords: Multi-term time fractional differential equation, Smarskii-Ionkin boundary datum, Bi-orthogonal system of functions, Fourier’s method, Multinomial Mittag-Leffler function

1. Introduction

This research article is devoted to the study of inverse problem of extracting the temporal component of the source term \(a(t)\) for the following multi-term time fractional parabolic differential system

\[
\left((^\alpha\!C D_{0+}^\alpha + \sum_{i=1}^{m} \psi_i ^\alpha\!C D_{0+}^\alpha_i)u(x,y,t) + \Delta^2 u(x,y,t) = a(t)f(x,y,t), \quad (x,y,t) \in \Pi, \right.
\]

alongside the homogeneous boundary conditions

\[
u_x(0,y,t) = 0 = u_{xxx}(0,y,t), \quad u_y(x,0,t) = 0 = u_y(x,1,t),
\]

\[
u_{yyy}(x,0,t) = 0 = u_{yyy}(x,1,t),
\]

nonlocal boundary conditions

\[
u(0,y,t) = u(1,y,t), \quad u_{xx}(0,y,t) = u_{xx}(1,y,t),
\]

and initial condition

\[
u(x,y,0) = \phi(x,y), \quad (x,y) \in \Omega : (0,1) \times (0,1),
\]

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where \( \Pi := \Omega \times (0, T) \), \( \{D_{0+}^\alpha \}_{\alpha \in \mathbb{R}^+} \) and \( \{D_{0+}^\alpha \}_{\alpha \in \mathbb{R}^+} \) stands for left sided Caputo fractional derivative of order \( \alpha \) and \( \alpha \), respectively such that \( 0 < \alpha_m < \cdots < \alpha_1 < \alpha \leq 1 \), \( m \in \mathbb{Z}^+ \), \( \psi_i \geq 0 \) and \( \Delta^2 := \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} \).

The solution of the inverse problem is usually recovered via an additional information of the system. By additional information, we mean some extra data about the behavior of the system that may be given at intermediate point \( t_i \), on the final time \( t_f \) or on the whole domain \( \Omega \). Such information/data is termed as over-determination/over-specified condition in the context of inverse problems.

For unique solvability of inverse problem (1)-(4), we will consider total energy of the system as over-determination condition, i.e.,

\[
\int_0^1 \int_0^1 u(x, y, t)dx dy = E(t).
\]

History of non integer order integrals and derivatives, termed as fractional calculus (FC), goes back to late seventeenth century. In the last five decades, FC evolved quite rapidly. Most of the books, journals and conferences were held due to its vast applications in many fields of sciences and engineering are summarized nicely in [3, 4]. For detailed evolution of FC, readers are referred [3, 4, 5]. Differential equations involving the non-integer order derivative of unknown function are termed as Fractional differential equations (FDEs). FDEs have been used extensively to model many well known physical phenomena. Just to mention a few, FDEs are used to model different processes in viscoelasticity [6], image processing [7], combustion [8], dynamics and control theory [9], motion of Newtonian fluid [10], earth science dynamics [11], cosmology [12], financial economics [13], biological processes [14].

Much of the work published to date has been concerned with FDEs involving single fractional derivative. For detail, we refer readers to [15, 16, 17, 18, 19] and references cited therein. Recently, many authors (see e.g. [20, 21, 22, 23, 24]) addressed the sub-diffusion processes in which the mean square displacement of the particles is of logarithmic growth. To model such ultra slow diffusive processes, scientists have used distributed order fractional derivatives which is an integral of fractional derivatives with respect to continuously changing orders. Multi-term time-fractional derivative, the special case of distributed order fractional derivative, involves more than one fractional order differential operators. Many real world problems are modelled more appropriately by using multi-term time fractional derivatives.

Let us provide some literature concerning inverse problems for FDEs. Tuan and Nane [25] discussed the inverse problem, of extracting space dependent source term, for time fractional diffusion equation with final temperature over-determination condition. They regularize the solution by using trigonometric method in nonparametric regression. In [26], authors have used revised generalized Tikhonov regularization method to extract the unknown space dependent source term for time fractional diffusion equation. Extraction of space dependent source, by using generalized Fourier’s method, for space time fractional differential equation is considered in [27]. Two inverse problems for fourth-order parabolic fractional differential equation were investigated by Sara et. al in [28]. In [29], uniqueness and a priory estimates of the solution is established by maximum principle for multi-term time fractional diffusion equation.

Jacobi tau method is used for multi-term time space fraction differential equation with Dirichlet boundary conditions was studied in [30]. Analytical and numerical solution for the direct problem for the two dimensional multi-term fractional differential equation with zero Dirichlet boundary conditions were analyzed in [31]. The classical solution of the inverse problem for two dimensional FDE involving a Caputo fractional derivative is determined in [32].

In this paper, we will prove the existence of the classical nature of the solution of the inverse problem given by (1)-(4). By classical solution of the inverse problem, we mean a pair of function \( u(x, y, t) \) and \( a(t) \) such that \( u(x, y, t) \in C(\Pi), \Delta u \in C(\Pi), \{D_{0+}^\alpha \}_{\alpha \in \mathbb{R}^+} \in C(\Pi) \) and \( a(t) \in C(0, T) \).

The rest of the paper is organized as follows: In Section 2, bi-orthogonal system consisting of Riesz bases obtained from eigenfunction of the spectral and conjugate problem is presented. Some key lemmata alongside some basic definitions are given in Section 3. The research findings of the article are demonstrated in Section 4.
2. Bi-orthogonal System

We intend to use Fourier’s method for construction of the solution of the inverse problem. Crust of
the Fourier’s method is to use the eigenfunctions to write the series expansion of the unknown functions.
Set of eigenfunctions can only be used if it is complete, minimal and form the basis in some appropriate
functional space.

The spectral and the conjugate problem corresponding to (1)-(3) are

\[
\begin{aligned}
\frac{\partial^4 Z}{\partial x^4}(x,y) + \frac{\partial^4 Z}{\partial x^4}(x,y) &= \sigma Z(x,y), \\
\frac{\partial Z}{\partial x}(0,y) &= 0 = \frac{\partial Z}{\partial x}(0,y), \quad \frac{\partial Z}{\partial y}(x,0) = 0 = \frac{\partial Z}{\partial y}(x,1), \\
\frac{\partial^2 Z}{\partial y^2}(x,0) &= 0 = \frac{\partial^2 Z}{\partial y^2}(x,1), \\
Z(0,y) &= Z(1,y), \quad \frac{\partial^2 Z}{\partial y^2}(0,y) = \frac{\partial^2 Z}{\partial y^2}(1,y), \\
\end{aligned}
\]

(6)

and

\[
\begin{aligned}
\frac{\partial^4 W}{\partial x^4}(x,y) + \frac{\partial^4 W}{\partial x^4}(x,y) &= \sigma W(x,y), \\
W(1,y) &= 0 = \frac{\partial^2 W}{\partial x^2}(1,y), \quad \frac{\partial W}{\partial y}(x,0) = 0 = \frac{\partial W}{\partial y}(x,1), \\
\frac{\partial^2 W}{\partial y^2}(x,0) &= 0 = \frac{\partial^2 W}{\partial y^2}(x,1), \\
\frac{\partial W}{\partial x}(0,y) &= \frac{\partial W}{\partial x}(1,y), \quad \frac{\partial^2 W}{\partial x^2}(0,y) = \frac{\partial^2 W}{\partial x^2}(1,y), \\
\end{aligned}
\]

(7)

respectively.

The eigenvalues of the spectral (6) and conjugate problem (7) are

\[
\sigma_{nk} = \mu_k + \lambda_n \quad \text{and} \quad \sigma_{0k} = \mu_k,
\]

where \(\mu_k = (k\pi)^4\) and \(\lambda_n = (2n\pi)^4\), \(k, n \in \mathbb{Z}^+\).

The eigenfunctions of spectral and conjugate problem forms the bi-orthogonal system with the following
one-one correspondence (8)

\[
\begin{aligned}
\{Z_{0k}(x,y), Z_{(2n-1)k}(x,y), Z_{2nk}(x,y)\}, \\
\{W_{0k}(x,y), W_{(2n-1)k}(x,y), W_{2nk}(x,y)\},
\end{aligned}
\]

(8)
where
\[ Z_{0k}(x,y) = \sqrt{2} \cos(k\pi y), \quad Z_{(2n-1)k}(x,y) = \sqrt{2} \cos(2n\pi x) \cos(k\pi y), \]
\[ Z_{2nk}(x,y) = \sqrt{2} x \sin(2n\pi x) \cos(k\pi y), \quad W_{0k}(x,y) = 2\sqrt{2}(1-x) \cos(k\pi y), \]
\[ W_{(2n-1)k}(x,y) = 4\sqrt{2}(1-x) \cos(2n\pi x) \cos(k\pi y), \]
\[ W_{2nk}(x,y) = 4\sqrt{2} \sin(2n\pi x) \cos(k\pi y). \]

By using the fact that \( a^2 + b^2 \geq 2ab, \forall a, b \in \mathbb{R} \), we have
\[ 1/\sigma_{nk} \leq 1/n^2 k^2 \quad \text{and} \quad 1/\mu_k \leq 1/k^2, \quad \forall n, k \in \mathbb{Z}^+. \tag{10} \]

Notice that
\[ |Z_{nk}| \leq \sqrt{2}, \quad |W_{nk}| \leq 4\sqrt{2}, \quad \forall n \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^+. \]

Lemma 2.1. \[ Z_{nk}(x,y) \text{ and } W_{nk}(x,y) \text{ given by \( \mathbf{S} \) and \( \mathbf{M} \) form Riesz bases of } L_2(\Omega). \]

3. Preliminaries

This section constitutes of some basic definitions and terminologies that will be used throughout the article.

Definition 1. \[ 19, 22 \] Let \( h \in L^1_{\text{loc}}[a,b], -\infty < a < z < b < \infty \), then the left sided Riemann-Liouville fractional integral of order \( \xi > 0 \) is defined as
\[ I_{a+}^{\xi} h(z) = \frac{1}{\Gamma(\xi)} \int_{a}^{z} \frac{h(\tau)}{(z-\tau)^{1-\xi}} d\tau, \quad z \in (a,b]. \]

Definition 2. \[ 19, 22 \] Let \( h \in AC[a,b] \), then the left sided Caputo fractional derivative of order \( \xi; 0 < \xi < 1 \) is defined as
\[ ^cD_{a+}^{\xi} h(z) = I_{a+}^{1-\xi} h'(z). \]

Lemma 3.1. \[ 37 \] For \( g, h \in C^1[0,b], \) then
\[ \frac{d}{dz}(g(z) * h(z)) = g(z)h(0) + g(z) * \frac{d}{dz} h(z) = h(z)g(0) + h(z) * \frac{d}{dz} g(z). \]

Definition 3. \[ 38 \] For \( \eta, \xi_j > 0, z_j \in \mathbb{C}; j = 1, 2, ..., n \) multinomial Mittag-Leffler function is defined as
\[ E_{(\xi_1, \xi_2, ..., \xi_n), \eta}(z_1, z_2, ..., z_n) := \sum_{k=0}^{\infty} \sum_{l_1, ... l_n \geq 0} (k; l_1, ..., l_n) \frac{\Pi_{i=1}^{n} \xi_i^{l_i}}{\Gamma(\eta + \sum_{i=1}^{n} \xi_i l_i)}, \]
where \( (k; l_1, ..., l_n) := \frac{k!}{l_1! \times \cdots \times l_n!}. \)
Remark 1. For $z_j = 0, \ j = 2, 3, \ldots, n$ multinomial Mittag-Leffler function be reduced to two parameter Mittag-Leffler function.

$$E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, 0, \ldots, 0) = \sum_{k=0}^{\infty} \frac{z^k_1}{\Gamma(\xi_1 k + \eta)} := E_{\xi_1, \eta}(z_1).$$

Remark 2. By using definition of multinomial Mittag-Leffler function, we have

$$E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, z_2, \ldots, z_n) = E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, z_2, \ldots, z_n).$$

Lemma 3.2. For $0 < \eta < 1$ and $0 < \xi_n < \ldots < \xi_2 < \xi_1 < 1$ be given. Assume that $\xi_1 \pi/2 < \mu < \xi_1 \pi$, $\mu \leq |\arg(z_n)| \leq \pi$ and $z_i < 0, \ i = 1, 2, \ldots, n$. Then, there exists a constant $C_1$ depending only on $\mu, \xi_i; i = 1, 2, \ldots, n$ such that

$$|E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, z_2, \ldots, z_n)| \leq C_1 \frac{1 + |z_n|}{1 + |z_n|} \leq C_1.$$

Proof. From Remark 2, we have

$$E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, z_2, \ldots, z_n) = E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(z_1, z_2, \ldots, z_n).$$

Due to Lemma 3.2 of [39], we can obtain the required result.

For convenience, we introduce the following notation

$$E_{(m_1 \xi_1, m_2 \xi_2, \ldots, m_n \xi_n), \eta}(z) := z^{-\eta} E_{(\xi_1, \xi_2, \ldots, \xi_n), \eta}(-m_1 z^{\xi_1}, -m_2 z^{\xi_2}, -m_3 z^{\xi_3}, \ldots, -m_n z^{\xi_n}).$$

Lemma 3.3. For $\eta, \xi_j > 0, m_j > 0; \ j \in \mathbb{Z}^+$, we have

$$\int_0^z E_{(m_1 \xi_1, m_2 \xi_2, \ldots, m_n \xi_n), \eta}(\tau) \ d\tau = E_{(m_1 \xi_1, m_2 \xi_2, \ldots, m_n \xi_n), \eta+1}(z).$$

Proof. By using definition of $E_{(m_1 \xi_1, m_2 \xi_2, \ldots, m_n \xi_n), \eta+1}$ and term by term integration, required result can be obtained.

Remark 3. For $m_j = 0, \ j = 2, \ldots, n$, in Lemma 3.3 we get

$$\int_0^z E_{(m_1 \xi_1, 0, 0, \ldots, 0), \eta}(\tau) \ d\tau = \int_0^z \tau^{\eta-1} E_{\xi_n, \eta}(-m_n \tau) \ d\tau = z^n E_{\xi_1, \eta+1}(-m_1 z^{\xi_1}).$$

Above relation was mentioned in [31].
Lemma 3.4. For \( h \in C^1[0,b] \) and \( \xi_i > 0, \psi_i > 0, \) for \( i = 1,2,\ldots,n, \) we have

\[
|h(z) \ast e_{(m_1,\xi_1-\xi_2),\ldots,m_1,\xi_1)}(z)| \leq \frac{C_1}{m_1} \|h\|_1,
\]

where \( \|h\|_1 = \max_{a < z < b} |h(z)| \) and "\( \ast \)" represents the Laplace convolution.

Proof. By using the fact that \( h \in C^1[a,b] \) alongside Lemma 3.3 and Lemma 3.2 we can obtain required result.

Remark 4. In [37], upper bound for the convolution of the Mittag-Leffler function with any continuously differentiable function is established, which is given by

\[
|g(z) \ast z^{\xi_1} E_{\xi_1,\xi_1+1}(-m_n z^{\xi_1})| \leq \frac{C_1}{m_1} \|g(z)\|. \tag{11}
\]

Equation (11) can also be established by setting \( m_i = 0, i = 1,\ldots,n-1 \) where \( n \in N \) in Lemma.

Some formulae for Laplace transform are mentioned here as we intend to use Laplace transform to solve system of multi-term time fractional differential equations.

\[
\begin{align*}
\mathcal{L}(D_0^\eta h) + \sum_{i=1}^m \psi_i \mathcal{L}(D_0^\eta h)h(z) &= (s^\eta + \sum_{i=1}^m \psi_i s^\eta)\mathcal{L}(h(z)) - (s^{\eta-1} + \sum_{i=1}^m \psi_i s^{\eta-1})h(0), \\
\mathcal{L}(e_{(m_1,\xi_1),\ldots,m_n,\xi_1)}(z) &= \frac{s^{-\eta}}{1 + \sum_{i=2}^n m_i s^{-\xi_i}}.
\end{align*}
\]

4. Inverse Source Problem

In this section, first we are going to present some relevant lemmata that would be advantageous in ultimately proving the main result i.e., the classical nature of the solution. Certain regularity conditions are imposed on given datum to prove the well-posedness of the inverse problem (11)-(13).

Lemma 4.1. For \( h \in C_{x,y,t}^{2,1,0}(\Pi) \) such that \( h(0,y,t) = h(1,y,t), \) then we have

\[
\begin{align*}
|h_{0k}(t)| &\leq \frac{2}{k \pi} \left| \frac{\partial h}{\partial y} \sqrt{2} \sin(k \pi y) \right|, \\
|h_{(2n-1)k}(t)| &\leq \frac{4}{(2n \pi)(k \pi)} \left| \frac{\partial^2 h}{\partial x \partial y} \sqrt{2} \sin(2n \pi) \sin(k \pi y) \right|, \\
|h_{2nk}(t)| &\leq \frac{4}{(2n \pi)(k \pi)} \left| \frac{\partial^2 h}{\partial x \partial y} \sqrt{2} \cos(2n \pi) \sin(k \pi y) \right|, \\
|h_{(2n-1)k}(t)| &\leq \frac{4}{(2n \pi)^2} \left( \left| \frac{\partial h(1,y)}{\partial x} + \frac{\partial h(0,y)}{\partial x} \sqrt{2} \cos(k \pi y) \right| + \left| \frac{\partial^2 h}{\partial x^2} \sqrt{2} \cos(2n \pi) \cos(k \pi y) \right| \right),
\end{align*}
\]
Proof. We are going to prove the fourth inequality, other inequalities can be obtained in a similar way.

By definition

\[ |h_{(2n-1)k}| = |\langle h(x,y), W_{(2n-1)k} \rangle| \leq 4\sqrt{2} \int_0^1 \int_0^1 h(x,y) \cos(2n\pi x) \cos(k\pi y) dx dy, \]

where we have used the fact that \(|1-x| < 1\) in \(\Omega\).

Integration by parts leads to

\[ |h_{(2n-1)k}| \leq \frac{4\sqrt{2}}{2n\pi} \int_0^1 \int_0^1 h(x,y) \sin(2n\pi x) \cos(k\pi y) dx dy. \]

Required inequality can be obtained by performing integration by parts with respect to \(x\), once more.

Lemma 4.2. For \(g \in C^{5,5}_{x,y}(\Omega)\), \(i = 0, 1, 2, 3\) and \(j = 1, 3\) such that

\[ \frac{\partial^i g}{\partial x^i}(0,y) = \frac{\partial^i g}{\partial x^i}(1,y), \quad \frac{\partial^j g}{\partial x^j}(x,0) = 0 = \frac{\partial^j g}{\partial x^j}(x,1), \]

then, we have

\begin{itemize}
  \item \(|g_{0k}| \leq \frac{2}{(k\pi)} \left| \left\langle \frac{\partial g}{\partial y}, \sqrt{2} \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{0k}| \leq \frac{2}{(k\pi)^2} \left| \left\langle \frac{\partial^2 g}{\partial y^2}, \sqrt{2} \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{(2n-1)k}| \leq \frac{4}{(2n\pi)(k\pi)} \left| \left\langle \frac{\partial^2 g}{\partial x \partial y}, \sqrt{2} \sin(2n\pi) \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{(2n-1)k}| \leq \frac{4}{(2n\pi)^2} \left\{ \left| \left\langle \frac{\partial g(1,y)}{\partial x} + \frac{\partial g(0,y)}{\partial x}, \sqrt{2} \cos(k\pi y) \right\rangle \right| + \left| \left\langle \frac{\partial^2 g}{\partial x^2}, \sqrt{2} \cos(2n\pi x) \cos(k\pi y) \right\rangle \right| \right\}, \]
  \item \(|g_{(2n-1)k}| \leq \frac{4}{(2n\pi)^3}(k\pi) \left| \left\langle \frac{\partial^3 g}{\partial^3 x \partial y}, \sqrt{2} \sin(2n\pi x) \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{(2n-1)k}| \leq \frac{4}{(2n\pi)^3}(k\pi) \left| \left\langle \frac{\partial^3 g}{\partial^3 y \partial x}, \sqrt{2} \sin(2n\pi x) \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{2nk}| \leq \frac{4}{(2n\pi)(k\pi)} \left| \left\langle \frac{\partial^2 g}{\partial x \partial y}, \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \right\rangle \right|, \]
  \item \(|g_{2nk}| \leq \frac{4}{(2n\pi)^2} \left| \left\langle \frac{\partial^2 g}{\partial x^2}, \sqrt{2} \sin(2n\pi x) \cos(k\pi y) \right\rangle \right|, \]
\end{itemize}
solution of inverse source problem

Under the following regularity conditions on the given data

Theorem 1. Under the following regularity conditions on the given datum \( \phi(x, y) \), \( f(x, y, t) \) and \( E(t) \) the solution of inverse source problem (1)-(5) is classical in nature.

- \( \phi(x, y) \in C_{x,y}^{5,5}(\Omega) \), \( i = 0, 1, 2, 3 \), and \( j = 1, 3 \) such that
  \[
  \frac{\partial^i \phi}{\partial x^i}(0, y) = \frac{\partial^i \phi}{\partial x^i}(1, y), \quad \frac{\partial^j \phi}{\partial x^j}(x, 0) = 0 = \frac{\partial^j \phi}{\partial x^j}(x, 1),
  \]

- \( f(x, y, t) \in C_{x,y,t}^{2,1,0}(\Pi) \) such that \( f(0, y, t) = f(1, y, t) \),

- \( \left( \int_0^1 \int_0^1 f(x, y, t) dx dy \right)^{-1} \leq C_2 \), for some \( C_2 > 0 \),

- \( E(t) \in AC[0, T] \) and \( \int_0^1 \int_0^1 \phi(x, y) dx dy = E(0) \).

Proof. In order to prove the classical nature of the solution of inverse source problem (1)-(5), firstly we will construct the solution of the problem followed by existence, uniqueness and stability results.

Construction of the solution:

We seek the series representation of the solution by making use of bi-orthogonal system \( \mathbf{8}-\mathbf{9} \) to get

\[
  u(x, y, t) = \sum_{k=1}^{\infty} T_{0k}(t)Z_{0k}(x, y) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[ T_{(2n-1)k}(t)Z_{(2n-1)k}(x, y) + T_{2nk}(t)Z_{2nk}(x, y) \right],
\]

where \( T_{nk} \)’s satisfy the following multi-term time fractional differential system

\[
  \left( cD_{0+, t}^\alpha + \sum_{i=1}^{m} \psi_i cD_{0+, t}^{\alpha_i} \right) T_{0k}(t) = -\mu_k T_{0k}(t) + a(t)f_{0k}(t),
\]

\[
  \left( cD_{0+, t}^\alpha + \sum_{i=1}^{m} \psi_i cD_{0+, t}^{\alpha_i} \right) T_{(2n-1)k}(t) = -\sigma_{nk} T_{(2n-1)k}(t) + 4(\lambda_n)^{3/4}T_{2nk}(t) + a(t)f_{(2n-1)k}(t),
\]

\[
  \left( cD_{0+, t}^\alpha + \sum_{i=1}^{m} \psi_i cD_{0+, t}^{\alpha_i} \right) T_{2nk}(t) = -\sigma_{nk} T_{2nk}(t) + a(t)f_{2nk}(t),
\]

where \( f_{nk}(t) = \langle f(x, t), W_{nk}(x, y) \rangle \), \( n \in \mathbb{Z}^+ \cup \{0\} \), \( k \in \mathbb{Z}^+ \).
The solution of the equations (13-15) is obtained by using Laplace transform and is given by

\[ T_{0k}(t) = e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \mu_k \alpha), 1}(t) \phi_{0k} + \sum_{i=1}^{m} \psi_i e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \mu_k \alpha), \alpha+1-\alpha_i}(t) \phi_{0k} + a(t)f_{0k}(t) * e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \mu_k \alpha), \alpha}(t), \quad (16) \]

\[ T_{2nk}(t) = e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), 1}(t) \phi_{2nk} + \sum_{i=1}^{m} \psi_i e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), \alpha+1-\alpha_i}(t) \phi_{2nk} + a(t)f_{2nk}(t) * e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), \alpha}(t), \quad (17) \]

\[ T_{(2n-1)k}(t) = e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), 1}(t) \phi_{(2n-1)k} + a(t)f_{(2n-1)k}(t) * e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), \alpha}(t) + \sum_{i=1}^{m} \psi_i e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), \alpha+1-\alpha_i}(t) \phi_{(2n-1)k} + 4(\lambda_n)^{3/4} T_{2nk}(t) * e_{(\psi_1(\alpha-1), \ldots, \psi_m(\alpha-1), \sigma_{nk} \alpha), \alpha}(t), \quad (18) \]

where \( a(t) \) is still to be determined.

After getting the series expression of \( u(x, y, t) \), we will discuss the existence of the source term \( a(t) \).

Use of the over-determination condition (5), for determination of \( a(t) \),

\[ \int_{0}^{1} \int_{0}^{1} (\partial^6 u - 4 \frac{\partial^4 u}{\partial y^4} + a(t)) f(x, y, t) dx dy = (\partial^6 u + \sum_{i=1}^{m} \psi_i \partial^6 u) E(t). \quad (19) \]

By using (11), we have

\[ \int_{0}^{1} \int_{0}^{1} \left( - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^4 u}{\partial y^4} + a(t) \right) f(x, y, t) dx dy = (\partial^6 u + \sum_{i=1}^{m} \psi_i \partial^6 u) E(t). \quad (20) \]

Consequently, we have following explicit expression of \( a(t) \)

\[ a(t) = \left( \int_{0}^{1} \int_{0}^{1} f(x, y, t) dx dy \right)^{-1} (\partial^6 u + \sum_{i=1}^{m} \psi_i \partial^6 u) E(t). \quad (21) \]

The solution of the inverse source problem is given by (12) and (21) where \( T_{0k}(t), T_{(2n-1)k}(t) \) and \( T_{2nk}(t) \) are given by (16)-(18).

Next, we will show that the obtained solution is regular in nature.
Existence of the solution:

We will present the existence result for the solution of the inverse source problem i.e., \( \{ a(t), u(x, y, t) \} \).

Under the given assumptions on \( f(x, y, t) \) and \( E(t), a(t) \in C[0, T] \).

For the existence of the solution \( \{12\} \), we need to show that the series representation of \( u(x, y, t) \), \( \Delta^2 u(x, y, t) \) and \( (\sum_{i=1}^{m} \psi_i D_{0^+}^{\alpha_i} + \sum_{i=1}^{m} \psi_i D_{0^+}^{\alpha_i})u(x, y, t) \) are uniformly convergent.

- **Uniform convergence of series representation of** \( u(x, y, t) \)

In order to establish the uniform convergence of series representation of \( u(x, y, t) \) we will show that the double infinite series

\[
\sum_{k=1}^{\infty} T_{0k}(t), \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} T_{nk}(t)
\]

are uniformly convergent.

From equation \( \{10\} \), we get

\[
|T_{0k}(t)| \leq |e_{(\psi_1, \ldots, \psi_m, \mu_k \alpha, 1)}(t) \phi_{0k}|
+ \sum_{i=1}^{m} |\psi_i| e_{(\psi_1, \ldots, \psi_m, \mu_k \alpha_i, \alpha+1)}(t) \phi_{0k}|
+ |a(t)| f_{0k} \cdot e_{(\psi_1, \ldots, \psi_m, \mu_k \alpha, \alpha)}(t). \tag{22}
\]

Since, \( a(t) \in C[0, T] \) so, \( \exists \) a constant \( C_3 \) such that \( |a(t)| \leq C_3, \forall t \in (0, T] \). By using the fact that \( |a(t)| \leq C_3 \). By using Lemma \( \{3,2\} \) and Lemma \( \{3,4\} \) we get

\[
|T_{0k}(t)| \leq C_1 |\phi_{0k}| \{1 + \sum_{i=1}^{m} |\psi_i| T^{\alpha-\alpha_i} \} + \frac{C_1 C_3}{(k\pi)^4} |f_{0k}(t)|.
\]

By using Lemma \( \{4,2\} \) we have

\[
|\phi_{0k}| \leq \frac{2}{k\pi} \left\{ \left| \frac{\partial \phi}{\partial y} \sqrt{2} \sin(k\pi y) \right| \right\}.
\]

Since, \( ab \leq \frac{1}{2}(a^2 + b^2), \forall a, b \in \mathbb{R} \), we have

\[
|\phi_{0k}| \leq \frac{1}{(k\pi)^2} + \left\{ \left| \frac{\partial \phi}{\partial y} \sqrt{2} \sin(k\pi y) \right| \right\}^2.
\]

From Equation \( \{22\} \), we have

\[
\left| \sum_{k=1}^{\infty} T_{0k}(t) \right| \leq C_1 \{1 + \sum_{i=1}^{m} |\psi_i| T^{\alpha-\alpha_i} \} \sum_{k=1}^{\infty} \left( \frac{1}{(k\pi)^2} + \left| \frac{\partial \phi}{\partial y} \sqrt{2} \sin(k\pi y) \right|^2 \right)
+ \sum_{k=1}^{\infty} \frac{C_1 C_3}{(k\pi)^4} |f_{0k}(t)|.
\]
As \( \{ \sqrt{2} \sin(k\pi y) \}_{k=1}^{\infty} \) forms an orthonormal sequence in \( L_2(0, 1) \), by Bessel Inequality, we have

\[
\left| \sum_{k=1}^{\infty} T_{0k}(t) \right| \leq C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \| \partial \phi / \partial y \|_2^2) + \sum_{k=1}^{\infty} C_1 C_3 \| f \|_3,
\]

where \( \| \phi \|_2 = \sqrt{\langle \phi(x, y), \phi(x, y) \rangle} \) and \( \| f \|_3 = \sqrt{\langle f(x, y, t), f(x, y, t) \rangle} \).

By using Cauchy-Bunyakovsky-Schwarz Inequality and the fact that \( |W_{0k}| \leq 4\sqrt{2} \), we get

\[
\left| \sum_{k=1}^{\infty} T_{0k}(t) \right| \leq C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})(\sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \| \partial \phi / \partial y \|_2^2) + \sum_{k=1}^{\infty} \frac{4 \sqrt{2} C_1 C_3}{(k\pi)^4} \| f \|_3.
\]

Similarly, for \( T_{2nk}(t) \), we have

\[
\left| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} T_{2nk}(t) \right| \leq 2 C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^2} + \| \partial^2 \phi / \partial x \partial y \|_2^2) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2 \sqrt{2} C_1 C_3}{(2n\pi)^2(k\pi)^2} \| f \|_3.
\]

For uniform convergence of series involved in \( u(x, y, t) \) we also need to analyze \( T_{(2n-1)k}(t) \). On same lines as for \( T_{0k}(t) \) and \( T_{2nk}(t) \), \( \{ W_{(2n-1)k} \}_{n=1}^{\infty} \) takes the form

\[
\left| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} T_{(2n-1)k}(t) \right| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[ \frac{4 C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})}{(2n\pi)(k\pi)} \langle \partial^2 \phi / \partial x \partial y, \sqrt{2} \sin(2n\pi x) \sin(k\pi y) \rangle \right.
\]
\[
+ \frac{C_1 C_3}{2(2n\pi)^2(k\pi)^2} |f_{(2n-1)k}| + \frac{8 C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})}{(2n\pi)(k\pi)^2} \langle \partial^2 \phi / \partial x^2, \sqrt{2} \sin(2n\pi x) \cos(k\pi y) \rangle
\]
\[
+ \frac{C_1 C_3}{2(2n\pi)(k\pi)^4} \| f_{2nk} \|_3 \right].
\]

Since, \( \{ \sqrt{2} \sin(2n\pi x) \sin(k\pi y) \}_{n,k=1}^{\infty} \) and \( \{ \sqrt{2} \sin(2n\pi x) \cos(k\pi y) \}_{n,k=1}^{\infty} \) form an orthonormal sequences in \( L_2(\Omega) \), by using the elementary inequality \( 2ab < a^2 + b^2 \), the Bessel and the Cauchy-Bunyakovsky-Schwarz Inequality, we have

\[
\left| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} T_{(2n-1)k}(t) \right| \leq 2 C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^2} + \| \partial^2 \phi / \partial x \partial y \|_2^2) \right.
\]
\[
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2 \sqrt{2} C_1 C_3}{(2n\pi)^2(k\pi)^2} \| f \|_3 + 2 C_1(1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i})(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^4} + \| \partial^2 \phi / \partial x^2 \|_2^2) \right.
\]
\[
+ C_1 C_3 \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^8} + \| f \|_2^2 \right). \quad (25)
\]
Inequalities (23)-(25) along side Weierstrass M test ensure that series representation of \( u(x, y, t) \) is uniformly convergent. Hence, \( u(x, y, t) \) represents a continuous function.

- **Uniform convergence of series representation of \( \Delta^2 u(x, y, t) \)**

Note that

\[
\Delta^2 u(x, y, t) = \sum_{k=1}^{\infty} \mu_k T_{0k}(t) Z_{0k}(x, y) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [\sigma_{nk} T_{(2n-1)k}(t) Z_{(2n-1)k}(x, y) + \sigma_{nk} T_{2nk}(t) Z_{2nk}(x, y)].
\]

Since \( |Z_{nk}| \leq \sqrt{2}, \forall n \in \mathbb{Z}^+ \cup \{0\}, k \in \mathbb{Z}^+ \), so in order to establish the convergence of \( \Delta^2 u(x, y, t) \), we need to prove the convergence of

\[
\sum_{k=1}^{\infty} \mu_k T_{0k}(t), \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} T_{(2n-1)k}(t), \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} T_{2nk}(t).
\]

By using (16) and Lemma 3.4, we have

\[
\left| \sum_{k=1}^{\infty} \mu_k T_{0k}(t) \right| \leq \sum_{k=1}^{\infty} C_1 \mu_k \left( 1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i} \right) |\phi_{0k}| + C_1 C_3 |f_{0k}(t)|.
\]

By using Lemma 4.1, Lemma 4.2 and Bessel Inequality, we get

\[
\sum_{k=1}^{\infty} \mu_k |T_{0k}(t)| \leq C_1 \left( 1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i} \right) \left( \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\| \frac{\partial^5 \phi}{\partial y^5} \right\|_2^2 \right) + C_1 C_3 \left( \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} + \left\| \frac{\partial f}{\partial y} \right\|_2^2 \right).
\]

We will now turn our attention to \( \sigma_{nk} T_{2nk}(t) \), using (17) to have

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} |T_{2nk}(t)| \leq \sum_{k=1}^{\infty} C_1 \left( 1 + \sum_{i=1}^{m} \psi_i T^{\alpha - \alpha_i} \right) \left( (2n\pi)^4 + (k\pi)^4 \right) |\phi_{2nk}| + C_1 C_3 |f_{2nk}(t)|.
\]
due to Lemma 3.1 and 3.2, we have

\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} |T_{2nk}(t)| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{m} C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha-x_i}) \left\{ \left( \left\| \frac{\partial^6 \phi(1, y)}{\partial x^3 \partial y} - \frac{\partial^6 \phi(0, y)}{\partial x^3 \partial y} \right\|_2 \right)^2 + \left\| \frac{\partial^6 \phi}{\partial x^5 \partial y} \right\|_2^2 \right\} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_3 (2n\pi)(k\pi) \left\{ \left( \frac{\partial^6 \phi(1, y)}{\partial x^3 \partial y} - \frac{\partial^6 \phi(0, y)}{\partial x^3 \partial y} \right) \right\} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_3 (2n\pi)(k\pi) \left( f_{xy}, \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \right).

using the inequality $2ab \leq a^2 + b^2$ and the fact that $\{ \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \}_{n,k=1}^{\infty}$ form orthonormal sequences in $L_2(0,1)$ and $L_2(\Omega)$ respectively, together with Bessel’s Inequality, we get

\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} |T_{2nk}(t)| \leq 4C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha-x_i}) \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^2} \right\} + \left\| \frac{\partial^6 \phi(1, y)}{\partial x^3 \partial y} - \frac{\partial^6 \phi(0, y)}{\partial x^3 \partial y} \right\|_2^2 + \left\| \frac{\partial^6 \phi}{\partial x^5 \partial y} \right\|_2^2 \right\} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_3 (2n\pi)(k\pi) \left\{ \left( \frac{\partial^6 \phi(1, y)}{\partial x^3 \partial y} - \frac{\partial^6 \phi(0, y)}{\partial x^3 \partial y} \right) \right\} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_3 (2n\pi)(k\pi) \left( f_{xy}, \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \right).

For the convergence of $\sigma_{nk} T_{(2n-1)k}(t)$, consider (13) together with Lemma 3.2, we have

\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} |T_{(2n-1)k}(t)| \leq 4C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha-x_i}) \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{1}{(2n\pi)(k\pi)} \left( \frac{\partial^6 \phi}{\partial x^3 \partial y} \right) \right) \left( \sqrt{2} \sin(2n\pi x) \sin(k\pi y) \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_3 (2n\pi)(k\pi) \left( \frac{\partial^6 \phi}{\partial x^3 \partial y} \right) \left( \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 16C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha-x_i}) \left( \left( \frac{\partial^6 \phi(1, y)}{\partial x^3 \partial y} - \frac{\partial^6 \phi(0, y)}{\partial x^3 \partial y} \right) \right) \left( \sqrt{2} \sin(k\pi y) \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{\partial^6 \phi}{\partial x^3 \partial y} \right) \left( \sqrt{2} \cos(2n\pi x) \sin(k\pi y) \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 8C_3 (2n\pi)(k\pi)^2 \left( f_{xx}, \sqrt{2} \sin(2n\pi x) \cos(k\pi y) \right).\]
The inequality $2ab \leq a^2 + b^2$ and the fact that $\{\sqrt{2}\sin(2n\pi x)\sin(k\pi y)\}_{n,k=1}^{\infty}$ form orthonormal sequences in $L_2(0,1)$ and $L_2(\Omega)$ respectively, alongside Bessel’s inequality, allow us to write

$$
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} |T_{(2n-1)k}(t)| \leq 2C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha_{i-1}}) \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2}{(2n\pi)^2(k\pi)^2} \right\} + 2C_1 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2n\pi)^2(k\pi)^2} + \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_3^2 \right\}
+ 8C_1 (1 + \sum_{i=1}^{m} \psi_i T^{\alpha_{i-1}}) \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^4(k\pi)^2} \right\} + 2 \left\| \frac{\partial^6 \phi}{\partial x^5 \partial y} \right\|_2^2 \left\| \frac{\partial^6 \phi}{\partial x^5 \partial y} \right\|_2^2 + 2 \left\| \frac{\partial^6 \phi}{\partial x^5 \partial y} \right\|_2^2 \left\| \frac{\partial^2 f}{\partial x^2} \right\|_3^2 \}
+ 4C_1 C_3 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2n\pi)^2(k\pi)^4} + \left\| \frac{\partial^2 f}{\partial x^2} \right\|_3^2 \right\}.
$$

- Uniform convergence of series representation of $(cD_{0+}^\alpha \sum_{i=1}^{m} \psi_i cD_{0+}^\alpha) u(x, y, t)$

By using (13), (15), we have following relation

$$
(cD_{0+}^\alpha \sum_{i=1}^{m} \psi_i cD_{0+}^\alpha) u(x, y, t) = \sum_{k=1}^{\infty} (-\mu_k T_{0k}(t) + a(t) f_{0k}(t)) Z_{0k}(x, y)
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( -\sigma_{nk} T_{(2n-1)k}(t) + 4(\lambda_n)^{3/4} T_{2nk}(t) \right)
+ a(t) f_{(2n-1)k}(t)) Z_{(2n-1)k}(x, y) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( -\sigma_{nk} T_{2nk}(t) + a(t) f_{2nk}(t) \right) Z_{2nk}(x, y).
$$

The series $\sum_{k=1}^{\infty} \mu_k T_{0k}(t), \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} T_{(2n-1)k}(t)$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{nk} T_{2nk}(t)$ are already proved to be uniformly convergent and using the fact that $|a(t)| \in C[0,T]$ and $|Z_{nk}(x, y)| \leq 4\sqrt{2}, n, k \in \mathbb{Z}^+$, it remains to show that $\sum_{k=1}^{\infty} f_{0k}(t), \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{(2n-1)k}(t)$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{2nk}(t)$ are uniformly convergent.
Dirichlet boundary condition ensures the uniqueness of derivative of known function $E$. Hence, in order to establish the uniqueness of the inverse source problem (1)-(5), we need to prove the uniqueness of time dependent source term $a(x,y,t)$. For uniqueness of $u(x,y,t)$, due to Lemma 4.1 and Bessel inequality, we can write

$$
\sum_{k=1}^{\infty} f_{ok}(t) \leq \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} |\langle f_{yy}, W_{ok}(x,y) \rangle| \leq \sum_{k=1}^{\infty} \frac{4\sqrt{2}}{(k\pi)^2} \| f_{yy} \|_2^2.
$$

$$
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{(2n-1)k}(t) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{4}{(2n\pi)(k\pi)} |\langle f_{xy}, \sqrt{2}\sin(2n\pi x) \sin(k\pi y) \rangle| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2 \left( \frac{1}{(2n\pi)^2(k\pi)^2} + |\langle f_{xy}, \sqrt{2}\sin(2n\pi x) \sin(k\pi y) \rangle|^2 \right) \leq 2 \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(2n\pi)^2(k\pi)^2} + \| f_{xy} \|_2^2 \right).
$$

Uniqueness of the solution:

After showing the regularity of the obtained solution, we will show the uniqueness of the solution.

In order to establish the uniqueness of the inverse source problem (1)-(5), we need to prove the uniqueness of time dependent source term $a(t)$ and $u(x,y,t)$.

The expression of $a(t)$, given by (21), involves definite integral of given function $f(x,y,t)$ and fractional derivative of known function $E(t)$. One-one nature of operators (definite integral and fractional derivative) ensures the uniqueness of $a(t)$.

For uniqueness of $u(x,y,t)$, let us consider $u(x,y,t)$ and $v(x,y,t)$ be two solutions, and let $\bar{u}(x,y,t) = u(x,y,t) - v(x,y,t)$. Then $\bar{u}(x,y,t)$ satisfy the equation

$$
\left( c^D_{0+} + \sum_{i=1}^{m} \psi_{i} c^D_{0+} \right) \bar{u}(x,y,t) + \Delta^2 \bar{u}(x,y,t) = 0, \quad (x,y,t) \in \Pi := \Omega \times (0,T). \quad (26)
$$

Dirichlet boundary condition

$$
\bar{u}_x(0,y,t) = 0 = \bar{u}_{xxx}(0,y,t), \quad \bar{u}_y(x,0,t) = 0 = \bar{u}_y(x,1,t),
$$

$$
\bar{u}_{yy}(x,0,t) = 0 = \bar{u}_{yyy}(x,1,t), \quad (27)
$$

15
The solution of the above system is
\[ T(0, y, t) = \bar{u}(1, y, t), \quad \bar{u}_x(0, y, t) = \bar{u}_{xx}(1, y, t), \quad (x, y) \in \Omega : (0, 1) \times (0, 1), \]
and initial condition
\[ \bar{u}(x, y, 0) = 0, \quad (x, y) \in \Omega : (0, 1) \times (0, 1), \]

Consider the functions
\[ T_0(t) = \int_0^1 \int_0^1 \bar{u}(x, y, t)W_0(x, y)dxdy, \]
\[ T_{(2n-1)k}(t) = \int_0^1 \int_0^1 \bar{u}(x, y, t)W_{(2n-1)k}(x, y)dxdy, \]
\[ T_{2nk}(t) = \int_0^1 \int_0^1 \bar{u}(x, y, t)W_{2nk}(x, y)dxdy. \]

Taking the fractional derivative, we get
\[ (^cD_0^\alpha, t + \sum_{i=1}^m \psi_i ^cD_0^\alpha, t)T_0(t) = -\mu_k T_0(t), \]
\[ (^cD_0^\alpha, t + \sum_{i=1}^m \psi_i ^cD_0^\alpha, t)T_{(2n-1)k}(t) = -\sigma_{nk} T_{(2n-1)k}(t) + 4(\lambda_n)^{3/4} T_{2nk}(t), \]
\[ (^cD_0^\alpha, t + \sum_{i=1}^m \psi_i ^cD_0^\alpha, t)T_{2nk}(t) = -\sigma_{nk} T_{2nk}(t). \]

The solution of the above system is
\[ T_0(t) = \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \mu_k, \alpha_1)}(t) \ T_0(0) + \sum_{i=1}^m \psi_i \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \mu_k, \alpha_1), \alpha+1, \alpha_i}(t) \ T_0(0) \]
\[ T_{2nk}(t) = \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \sigma_{nk}, \alpha_1)}(t) \ T_{2nk}(0) + \sum_{i=1}^m \psi_i \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \sigma_{nk}, \alpha_1), \alpha+1, \alpha_i}(t) \ T_{2nk}(0) \]
\[ T_{(2n-1)k}(t) = \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \sigma_{nk}, \alpha_1)}(t) \ T_{(2n-1)k}(0) + 4(\lambda_n)^{3/4} T_{2nk}(t) \ast \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \sigma_{nk}, \alpha_1), \alpha}(t) \]
\[ + \sum_{i=1}^m \psi_i \mathcal{E}_{(\mu_1(\alpha-\alpha_1), \ldots, \mu_m(\alpha-\alpha_m), \sigma_{nk}, \alpha_1), \alpha+1, \alpha_i}(t) \ T_{(2n-1)k}(0). \]

By using the initial condition [29], we have
\[ T_0(t) = 0, \quad T_{(2n-1)k}(t) = 0, \quad T_{2nk}(t) = 0, \quad t \in (0, T). \]
Consequently, the uniqueness of the solution follows from the completeness of the set of function

\[ \{W_{0k}(x,y), W_{(2n-1)k}(x,y), W_{2nk}(x,y)\}, \quad n, k \in \mathbb{Z}^+. \]

**Stability of the solution:**

Now, we will show that the solution depends continuously on given datum, assume that \{u(x,y,t),a(t)\} and \{\tilde{u}(x,y,t), \tilde{a}(t)\} be two solution sets of the inverse source problem (1)-(5), corresponding to given data \{f(x,y,t), \phi(x), E(t)\} and \{\tilde{f}(x,y,t), \tilde{\phi}(x), \tilde{E}(t)\}, respectively.

From (21), we have

\[ a(t) - \tilde{a}(t) = \left( \int_0^1 \int_0^1 f(x,y,t) dxdy \right)^{-1} \left( \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} + \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} \right) E(t) \]

which leads us to the relation

\[ a(t) - \tilde{a}(t) = \frac{\left( \int_0^1 \int_0^1 f(x,y,t) dxdy \right)^{-1} \left( \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} + \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} \right) E(t)}{\int_0^1 \int_0^1 f(x,y,t) dxdy \int_0^1 \int_0^1 \tilde{f}(x,y,t) dxdy} \]

\[ + \frac{\left( \int_0^1 \int_0^1 \tilde{f}(x,y,t) dxdy \right)^{-1} \left( \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} + \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} \right) \tilde{E}(t)}{\int_0^1 \int_0^1 f(x,y,t) dxdy \int_0^1 \int_0^1 \tilde{f}(x,y,t) dxdy}. \]  

(30)

Note that there exist constants \( C_4, C_5, C_6 \) and \( C_7 \) such that

\[ \| \left( \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} \right) E(t) \|_1 \leq C_4, \]

\[ \| \left( \sum_{i=1}^{m} \psi_i^c D_{0,+}^{\alpha_i} \right) (E(t) - \tilde{E}(t)) \|_1 \leq C_5 \| E - \tilde{E} \|_1, \]

\[ \| \int_0^1 \int_0^1 f(x,y,t) dxdy \|_1 \leq C_6, \]

\[ \| \int_0^1 \int_0^1 \tilde{f}(x,y,t) - f(x,y,t) dxdy \|_1 \leq C_7 \| f - \tilde{f} \|_3. \]

Consequently, (30) becomes

\[ \| a - \tilde{a} \|_1 \leq C_2^2 \left( C_4 C_7 \| f - \tilde{f} \|_3 + C_5 C_6 \| E - \tilde{E} \|_1 \right). \]
After establishing the stability result for source term \(a(t)\), we will establish the inequality of \(u(x, y, t)\) that ensures the dependence of \(u(x, y, t)\) on the given datum.

From (12), we have

\[
|u(x, y, t) - \bar{u}(x, y, t)| \leq \sum_{k=1}^{\infty} |T_{0k}(t) - \bar{T}_{0k}(t)| Z_{0k}(x, y) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[ |T_{(2n-1)k}(t) - \bar{T}_{(2n-1)k}(t)| Z_{(2n-1)k}(x, y) + |T_{2nk}(t) - \bar{T}_{2nk}(t)| Z_{2nk}(x, y) \right],
\]

since \(|Z_{nk}(x, y)| \leq \sqrt{2}, \ \forall n \in \mathbb{Z}_0^+, \ k \in \mathbb{Z}^+\), so we have

\[
|u(x, y, t) - \bar{u}(x, y, t)| \leq \sqrt{2} \sum_{k=1}^{\infty} |T_{0k}(t) - \bar{T}_{0k}(t)| + \sqrt{2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[ |T_{(2n-1)k}(t) - \bar{T}_{(2n-1)k}(t)| + |T_{2nk} - \bar{T}_{2nk}(t)| \right],
\]

(31)

By using (23), we have

\[
\left| \sum_{k=1}^{\infty} (T_{0k}(t) - \bar{T}_{0k}(t)) \right| \leq \sum_{k=0}^{\infty} 2\sqrt{2} C_1 \|\phi - \bar{\phi}\|_2 \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_{t^\alpha} \right) + \sum_{k=0}^{\infty} \frac{2\sqrt{2} C_1 C_3 \|f - \bar{f}\|_3}{(k\pi)^4}.
\]

(32)

Inequality (24) is used to get the following

\[
\left| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (T_{2nk}(t) - \bar{T}_{2nk}(t)) \right| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_1 \|\phi - \bar{\phi}\|_2 \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_{t^\alpha} \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_1 C_3 \|f - \bar{f}\|_3 \sqrt{2(n\pi)^2(k\pi)^2},
\]

(33)

Lastly for \(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (T_{(2n-1)k}(t) - \bar{T}_{(2n-1)k}(t))\), we will make use of (24)

\[
\left| \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (T_{(2n-1)k}(t) - \bar{T}_{(2n-1)k}(t)) \right| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_1 \|\phi - \bar{\phi}\|_2}{\sqrt{2(n\pi)^2(k\pi)^2}} \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_{t^\alpha} \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_1 C_3 \|f - \bar{f}\|_3}{\sqrt{2(n\pi)^2(k\pi)^2}}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_1^2 \|\phi - \bar{\phi}\|_2}{2(n\pi)^3(k\pi)^2} \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_{t^\alpha} \right)
\]

\[
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_1^2 C_3 \|f - \bar{f}\|_3}{2(n\pi)^3(k\pi)^4}.
\]

(34)
By using (32)–(34) in (31) takes the form

\[
|u(x, y, t) - \tilde{u}(x, y, t)| \leq \sum_{k=0}^{\infty} \frac{4C_1\|\phi - \tilde{\phi}\|_2}{(k\pi)^4} \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_i \right) + \sum_{k=0}^{\infty} \frac{4C_1C_3\|f - \tilde{f}\|_3}{(k\pi)^4} \\
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2C_1\|\phi - \tilde{\phi}\|_2}{(n\pi)^2(k\pi)^2} \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_i \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2C_1C_3\|f - \tilde{f}\|_3}{(n\pi)^2(k\pi)^2} \\
+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_2^2\|\phi - \tilde{\phi}\|_2}{\sqrt{2}(n\pi)^3(k\pi)^4} \left( \frac{1}{t^\alpha} + \sum_{i=1}^{m} \psi_i \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{C_2^2C_3\|f - \tilde{f}\|_3}{\sqrt{2}(n\pi)^3(k\pi)^4}.
\]

Since, the double series \(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^2k^2}\) and \(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^3k^4}\) alongside the series \(\sum_{k=1}^{\infty} \frac{1}{k^4}\) are convergent, so there exists constants \(C_8\) and \(C_9\) such that

\[
\|u - \tilde{u}\|_3 \leq C_8\|\phi - \tilde{\phi}\|_2 + C_9\|f - \tilde{f}\|_3,
\]

where \(C_8\) and \(C_9\) are constants independent of \(n\) and \(k\).

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