Implementing Tidal and Gravitational Wave Energy Losses in Few-body Codes: A Fast and Easy Drag Force Model

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ABSTRACT

We present a drag force model for evolving chaotic few-body interactions with the inclusion of orbital energy losses, such as tidal dissipation and gravitational wave (GW) emission. The main effect from such losses is the formation of two-body captures, that for compact objects result in GW mergers, and for stars lead to either compact binaries, mergers or disruptions. Studying the inclusion of energy loss terms in few-body interactions is therefore likely to be important for modeling and understanding the variety of transients that soon will be observed by current and upcoming surveys. However, including especially tides in few-body codes has been shown to be technically difficult and computationally heavy, which has lead to very few systematic tidal studies. In this paper we derive a drag force term that can be used to model the effects from tidal, as well as other, energy losses in few-body interactions, if the two-body orbit averaged energy loss is known a priori. This drag force model is very fast to evolve, and gives results in agreement with other approaches, including the impulsive and affine tide approximations.

Key words: gravitation – methods: numerical – stars: black holes – stars: kinematics and dynamics

1 INTRODUCTION

Transient events, including gravitational wave (GW) mergers (Abbott et al. 2016b,c,a, 2017a,b,c), stellar mergers (e.g. Tyenda et al. 2011), and stellar tidal disruptions (e.g., Perets et al. 2016, and references therein), are often the product of a two-body or a dynamical few-body system loosing orbital energy through one or more dissipative mechanisms. The most important of such mechanisms include energy dissipation through the emission of GWs (e.g. Peters 1964; Hansen 1972; Turner 1977), and orbital energy losses through tidal excitations (e.g. Press & Teukolsky 1977; Lee & Ostriker 1986) and dissipation (e.g. Ogilvie 2014, and references therein). In the isolated binary problem, these effects will lead to a merger between the two objects within a finite time, and depending on the stellar types the final binary evolution will either be dominated by GWs (e.g. Peters 1964) (if both objects are compact), tides (Ogilvie 2014) (if at least one object is a star), or common envelope evolution (e.g. Paczynski 1976; Iben & Livio 1993; Taam & Sandquist 2000; MacLeod et al. 2018) (if one of the objects evolves to indulge the other).

During chaotic interactions involving three or more objects, the loss or dissipation of orbital energy often results in the formation of eccentric two-body captures (e.g. Kochanek 1992; Samsing et al. 2014; Perets et al. 2016; Samsing et al. 2017b). A capture refers here to a scenario involving a very close approach between two objects with such a small pericenter distance that the energy loss over one orbit is large enough for the two objects to quickly inspiral and detach from the rest of the N-body system. Such captures are well known and studied in the single-single case (e.g. Hansen 1972; Fabian et al. 1975; Clark 1975). Interestingly, recent studies indicate that such captures form at a higher rate during few-body interactions, compared to single-single interactions. For example, it was recently shown by Samsing et al. (2014), that the rate of eccentric binary black hole (BBH) mergers forming through captures mediated by gravitational wave emission likely is dominated by three-body interactions, and not single-single interactions. Similar eccentric mergers can also form through tidal captures in three-body interactions, as shown by Gaburov et al. (2008, 2010); Samsing et al.

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(2017b). The point here is that the majority of such eccentric capture mergers are likely to form in dense stellar systems, compared to say the field, and any observation of such eccentric sources will therefore be an indirect probe of the dynamical channel for BBH and other mergers and the importance of dense stellar environments. Despite their possible importance, energy loss terms are often not included in the N-body equations-of-motion (EOM) (e.g. Fregeau et al. 2004). For this reason how energy losses during strong few-body encounters, including GWs and tides, affect not only the host cluster dynamics, but also the range of relevant observables, is not yet well understood.

Energy dissipation from GW emission is not difficult to include in N-body codes thanks to the development of the post-Newtonian (PN) formalism (e.g. Blanchet 2014), and aspects of such corrections have therefore been studied. For example, using full N-body simulations Kupi et al. (2006) showed how large BHs can form in a GW capture run-away. Similarly, Samsing et al. (2014); Samsing & Ramírez-Ruiz (2017); Samsing (2017); Samsing et al. (2018) performed isolated three-body scatterings which lead them to conclude that the GW captures forming during the interactions are likely to dominate the rate of eccentric BBH mergers forming in globular clusters (GCs) observable by the ‘Laser Interferometer Gravitational-Wave Observatory’ (LIGO). A monte-carlo (MC) approach for studying the evolution of GCs including scatterings up to binary-binary interactions with PN terms was recently presented by Rodriguez et al. (2017), along with a similar study by Samsing et al. (2017a), who both confirmed that GW emission in the N-body EOM is crucial for probing the population of eccentric BBH mergers forming in clusters (Samsing 2017).

Including tides is significantly more difficult than gravitational wave emission. This is due not only to our limited understanding of stellar structure and the mechanism(s) via which tides are excited and subsequently dissipated, but also because tidal effects are extremely time consuming to computationally evolve in an N-body code. Some few-body studies have been done using full hydro dynamics (e.g. Gaburov et al. 2010), but doing large systematic studies are not yet possible due to computational limits. Other methods for studying the effects from dynamical tides include the impulsive approximation, where tidal energy and angular momentum losses are included by simply correcting the velocity vectors at pericenter ‘by hand’ every time two of the N objects pass very close to each other (e.g. Baumgardt et al. 2006). A similar approach was also used by Mardling & Aarseth (2001), and does indeed work. But, making such discontinuous corrections to the N-body system often lead to poor performance and complicated decision making. Finally, other approaches include only solving for the evolution of a subset of the tidal modes, which can be done in both linear tidal theory (Mardling 1995a) using the Press and Teukolsky (PT) approach (Press & Teukolsky 1977), and non-linearly using the so-called affine model (e.g. Carter & Luminet 1985; Luminet & Carter 1986); a model we will apply later in this paper. However, such prescriptions are still too computationally expensive for say parameter space studies and derivations of tidal capture cross sections (Samsing et al. 2017b); other strategies are therefore needed.

In this paper we propose to include energy loss effects in few-body codes through a simple drag force term in the equations-of-motion. Many few-body codes have already been optimized to include drag forces, e.g., the 2.5 PN term that accounts for energy dissipation through the emission of GWs is no more than a simple drag force. Our ansatz in this paper is therefore to derive a general drag force that can be used to model any energy loss effects, and again, with tides as the main motivation. The only input our drag force model requires is an estimate for the amount of orbital energy lost if two of the N objects undergo a near parabolic encounter. This has been calculated in several studies for tides (Press & Teukolsky 1977; Lee & Ostriker 1986), and fitting formulae have also been provided to speed up these calculations (e.g. Giersz 1985b; Portegies Zwart & Meinen 1993).

As illustrated in this paper, the use of such fitting formulae together with our proposed drag force model allows one to quickly evolve few-body systems with both energy losses from tides and GW emission. We note here that our model does not give any new insight into the two-body tidal problem, but it will be able to provide insight into how especially tides affect the evolution of chaotic few-body interactions. For that reason, our model has the same limitations as the two-body tidal problem, e.g., we are not able to predict what happens after a tidal capture; do the two objects merge or do they form a stable binary? However, what we are able to accurately probe and resolve the number of tidal captures forming in chaotic few-body interactions. We illustrate this in a few examples, by performing controlled two-body and three-body experiments with different tidal implementations, including our proposed drag force model. In the near future we plan to include this model into the N0CCA (Monte Carlo SimulAtor) code (Hyppki & Giersz 2013a; Giersz et al. 2013), which will allow us to perform systematic studies of how tidal energy losses in chaotic interactions could affect observables and feedback in to the underlying host cluster dynamics. These are key questions that have to be addressed, as new searches for transient phenomena will soon be monitoring the sky, including LSST (LSST Science Collaboration et al. 2009), JWST (Gardner et al. 2006), and WFIRST (Spergel et al. 2013).

The paper is organized as follows. In Section 2 we present our drag force model, and describe how to normalize it for different energy loss mechanisms. We especially discuss how to apply it for describing tidal energy losses, which is the main motivator for this paper. A short step-by-step description of how to implement the model in an N-body code is also given. In Section 2 we numerically evolve a few two-body and three-body scattering experiments involving tidal and compact objects with the inclusion of our proposed drag force model. We especially compare our drag force results with other tidal prescriptions, including the impulsive and affine tidal approximations. Conclusions are given in Section 4.

2 DRAG FORCE MODEL

In this section we describe and derive our proposed tidal drag force model, that in principle can be used to dynamically evolve chaotic few-body systems with the inclusion of any type of energy loss mechanism; however, our main motivation is how tidal effects impact the evolution. In short, our approach is to model orbital energy losses by introducing a drag force that acts against the relative motion between any pair of objects in the few-body system. For deriving the drag force, we assume that the largest energy loss occurs during close pairwise encounters, and that these encounters can be considered as isolated two-body systems during the period where most of the energy is lost (see Figure 1). This is an excellent assumption, as basically all of the relevant energy loss mechanisms depend steeply on the relative distance between the objects, implying that most of the energy loss in few-body systems do indeed take place during close pairwise encounters (e.g. Samsing et al. 2017b). The amount of energy that is lost over a single close passage for an isolated two-body system has been studied extensively in the literature, both for GWs (Peters 1964; Hansen 1972; Turner 1977) and tides

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against the orbital motion of the two objects, as further described in Figure 1. In this picture, the two-body system will lose an amount of orbital energy \[ dE = F \times ds \] per differential line element \( ds \) integrated along the orbit. Assuming that the change in orbital angular momentum per orbit is negligible, one can in all relevant cases approximate the total energy loss over one orbit, denoted by \( \Delta E \), by the following integral (see Turner (1977) for a similar procedure applied to GW energy losses),

\[
\Delta E \approx \int_{-\theta_0}^{\theta_0} F \frac{ds}{d\theta} \frac{dt}{d\theta} d\theta,
\]

where \( dt \) is the differential change in time, \( \theta \) is the true anomaly, and \( \theta_0 = \pi \) for a bound orbit and \( \cos^{-1}(-1/e) \) for an unbound orbit (see Figure 1). The term \( ds/d\theta \) is simply the relative velocity between the two interacting objects, denoted by \( v \), which can be written as,

\[
\frac{ds}{d\theta} = \sqrt{2GM \left( \frac{1}{r} - \frac{1}{2a} \right)}^{1/2},
\]

where \( M = m_i + m_j \) is the total mass of the two interacting objects, referred to as \( i \) and \( j \), and \( r \) is their relative distance. The term \( dt/d\theta \) can be derived from Kepler's relation \( r = a(1-e^2)/(1+e \cos \theta) \), from which follows,

\[
\frac{dt}{d\theta} = \frac{a(1-e^2)^{3/2}}{(GM)^{1/2}} \frac{1}{(1+e \cos \theta)^2}.
\]

To proceed we now have to choose a functional form for the drag force, \( F \). As described, the form should both be simple to implement in a few-body code, possibly similar to the 2.5 PN drag force that has been successfully implemented in many recent few-body codes, while ensuring that most of the energy loss happens at pericenter. A first proposed form could be a force that is \( \propto 1/r^4 \), where \( n \) is some power; however, in this case one finds that the integral in Equation (1) does not have an analytical solution for any \( n \), including \( n = 0 \). This ‘problem’ relates to the fact that the circumference of an ellipse cannot be written out in a closed form. This is why ‘elliptical’ integrals always have to be solved numerically. However, if we instead choose a force that is \( \propto 1/r^4 \), then the integral in Equation (1) can be written out in closed form for any \( n \), which allows us to analytically estimate the drag force normalization, or coefficient. This leads to a very fast derivation of the drag force per time step. In this paper we therefore choose to work with the following drag force,

\[
F = -\delta \frac{v}{r^3} \times \frac{v}{v},
\]

where \( \delta \) is a normalization factor that to leading order depends on the orbital parameters for the two-body system and the considered energy loss mechanism. Although this choice of drag force could seem arbitrary, we note that the 2.5 PN drag force is exactly of this type with \( n = 4 \). Therefore, a code that is optimized to run with PN terms, should have no problem in evolving the system with our proposed drag force. Below we illustrate how to estimate the drag force coefficient \( \delta \).

### 2.1 Drag Force Functional Form

We consider two objects on a Kepler orbit, bound or unbound, with an initial semi-major (SMA) \( a \) and eccentricity \( e \). For this system we now consider a drag force with magnitude \( F \) that acts against the orbital motion of the two objects, as further described and illustrated in Figure 1. In this picture, the two-body system will lose an amount of orbital energy \( dE = F \times ds \) per differential line element \( ds \) integrated along the orbit. Assuming that the change in orbital angular momentum per orbit is negligible, one can in all relevant cases approximate the total energy loss over one orbit, denoted by \( \Delta E \), by the following integral (see Turner (1977) for a similar procedure applied to GW energy losses),

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where \( M = m_i + m_j \) is the total mass of the two interacting objects, referred to as \( i \) and \( j \), and \( r \) is their relative distance. The term \( dt/d\theta \) can be derived from Kepler’s relation \( r = a(1-e^2)/(1+e \cos \theta) \), from which follows,

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\frac{dt}{d\theta} = \frac{a(1-e^2)^{3/2}}{(GM)^{1/2}} \frac{1}{(1+e \cos \theta)^2}.
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To proceed we now have to choose a functional form for the drag force, \( F \). As described, the form should both be simple to implement in a few-body code, possibly similar to the 2.5 PN drag force that has been successfully implemented in many recent few-body codes, while ensuring that most of the energy loss happens at pericenter. A first proposed form could be a force that is \( \propto 1/r^4 \), where \( n \) is some power; however, in this case one finds that the integral in Equation (1) does not have an analytical solution for any \( n \), including \( n = 0 \). This ‘problem’ relates to the fact that the circumference of an ellipse cannot be written out in a closed form. This is why ‘elliptical’ integrals always have to be solved numerically. However, if we instead choose a force that is \( \propto 1/r^4 \), then the integral in Equation (1) can be written out in closed form for any \( n \), which allows us to analytically estimate the drag force normalization, or coefficient. This leads to a very fast derivation of the drag force per time step. In this paper we therefore choose to work with the following drag force,

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### 2.2 Drag Force Normalization Coefficient

The coefficient \( \delta \) of the drag force introduced in the above Equation (4), can be estimated using Equation (1) assuming that \( \Delta E \) is known a priori for the considered two-body system. After some algebraic manipulations we find from solving equation (1) with our proposed
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drag force from Equation (4) that,
\[
\xi = AE \times \frac{1}{2} \left[ a(1 - e^2)^{n-1/2} \right]^{1/2} (GM)^{1/2} \mathcal{F}(e, n),
\]
where \(\mathcal{F}(e, n)\) is the solution to the following integral
\[
\mathcal{F}(e, n) = \int_{-\theta_0}^{\theta_0} \frac{1 + e \cos \theta - (1 - e^2)/2}{(1 + e \cos \theta)^2 - n} d\theta.
\]
This factor \(\mathcal{F}(e, n)\) can be written out in closed form for any value of \(n\). In this paper we will study the performance for two different values of \(n\), namely for \(n = 4\) and \(n = 10\). For these two cases \(\mathcal{F}\) evaluates to,
\[
\mathcal{F}(e, n = 4, 10) = \begin{cases} 
\frac{2}{\pi} \left( 2 + 7e^2 + e^4 \right), & \text{for } n = 4 \\
\frac{2}{\pi} \left( 128 + 2944e^2 + 10528e^4 \\
+ 8960e^6 + 1715e^8 + 35e^{10} \right), & \text{for } n = 10. 
\end{cases}
\]
where we have assumed that \(\theta_0 = \pi\). In the unbound case for which \(\theta_0 = \cos^{-1}(-1/e)\), the operation \(\cos^{-1}\) has to be performed at each time step, which is computationally heavy. However, for all practical purposes, there is no problem in just using the value of \(\mathcal{F}\) assuming that \(\theta_0 = \pi\), i.e., use relations similar to the one shown in Equation (7), even when the objects are not bound initially. The reason is simply that if the considered energy loss mechanism turns out to be significant, then the two objects in question will immediately get bound to each. For all the numerical simulations shown in this paper, we will therefore assume the bound limit when calculating the factor \(\mathcal{F}\): that is, we will use the relations from Equation (7). Below we will describe how \(AE\) can be estimated for tides, and discuss open problems related to our adopted methodology.

2.3 Orbital Energy Losses

Our introduced drag force given by Equation (4), is only a convenient kernel that leads to a continuous loss of orbital energy. It therefore needs to be normalized at each time step. As seen from Equation (4) and (5), the normalization happens through the parameter \(\xi\), which depends on the expected two-body energy loss over one orbit, \(AE\). From Equation (5), it is clear that the evaluation of \(AE\) is the only computation that potentially could take up a significant amount of CPU time; however, in this section, we point out that in most cases very simple analytical formulae or functional fits exist which can by-pass this expense. One example is the orbital energy loss through GWs, which was shown by Hansen (1972) to simply scale with the two-body pericenter distance, \(r_p\), as \(\propto r_p^{-7/2}\). For tides, which is our main motivator for this paper, the form is more complicated and will therefore be described in greater detail below. We will also comment on potential problems and open questions when describing strong tidal interactions over subsequent passages with this approach.

2.3.1 Tidal Energy Loss

Consider a two-body encounter between a tidal object ‘t’, and a perturber ‘p’. In the linear limit to quadruple order (\(l = 2\)) the energy loss integrated over an unperturbed parabolic orbit can be written as (Press & Teukolsky 1977),
\[
AE = \left( \frac{GM_p}{R_t} \right)^2 \left( \frac{M_p}{M_t} \right)^2 \left( \frac{R_t}{r_p} \right)^6 \times T(\eta),
\]
where \(r_p\) is the initial orbital pericenter distance = \(a(1-e)\), \(T(\eta)\) is a function with no analytic solution that depends on the internal structure of the tidal object in question, and \(\eta\) is a parameter that is defined by,
\[
\eta = \left( \frac{M_t}{M_t + M_p} \right)^{1/2} \left( \frac{r_p}{R_t} \right)^{3/2}.
\]
The above Equation (8) accounts for the energy loss when only one of the objects is a tidal object. If both of the objects are tidal objects, then the total energy loss is simply the sum of the pairwise energy losses (see, e.g., Portegies Zwart & Meinen 1993). Now, the time consuming part in calculating the tidal energy loss \(AE\) from above, is to calculate the value of \(T(\eta)\). To avoid this, fitting functions have been made to approximate \(T(\eta)\) for different stellar polytropes (e.g. Giersz 1985b; Portegies Zwart & Meinen 1993). By the use of such fitting functions, the tidal energy loss over one orbit can therefore quickly be estimated for any combination of objects by the use of Equation (8). For the examples studied in this paper, we use the fitting formulæ presented in Portegies Zwart & Meinen (1993), which allows us to quickly estimate \(AE\) from \(\xi\), and thereby the drag force at each time step. This strategy is on average expected to give reasonable results; however, there are several well known problems related to this description of two-body tidal captures. A few of these problems will be discussed below.

2.3.2 Open Problems in Describing Tidal Captures

For describing tidal interactions with our drag force model, one needs to know the exact tidal energy loss over each orbit for each object pair. However, even the evolution of a simple isolated two-body tidal capture is currently poorly understood (see, e.g., Stone et al. 2017). Aspects of this problem relate especially to how fast, and in what way(s), the tidal modes that have been excited during the pericenter passage damp. If the modes damp quickly, and the deposited energy is radiated away shortly thereafter, then the energy loss at each pericenter passage will be independent of previous passages. However, if say the tidal modes do not effectively damp between pericenter passages, then the energy loss over a given orbit depends on all previous passages. The reason is that if the tidal modes are not damped, they will couple to the orbital motion through a quadrupole interaction term induced by the tidal bulge (e.g. Kochanek 1992; Samsing et al. 2017b). This implies that the orbital energy stored in tidal oscillations can be put back into the orbit at later times. This can give rise to highly chaotic orbital motions (Kochanek 1992; Mardling 1995a; Samsing et al. 2017b). If the tidal modes damp quickly, but the energy is not radiated away on short time scales, the tidal object in question will likely undergo a momentary expansion, which might lead to a runaway tidal capture followed by a merger. Therefore, the evolution of the capture depends strongly on how the tidal modes damp, and how fast the energy is dissipated.

Another issue relates to how the angular momentum is transferred from the orbit to the stellar object(s), and how it is later dissipated. For example, if a tight binary forms as a result of a capture with an initial pericenter distance \(r_c\), then the binary will end up on a circular orbit with a final SMA = \(2r_c\) if only little angular momentum is dissipated; however, if the angular momentum instead is effectively dissipated, then the final SMA will be smaller, which might lead to unstable mass transfer followed by a merger.

The two-body tidal captures we consider in this work that are formed in resonating few-body systems, are likely to be further affected by secular effects. The reason is that if a tidal capture binary forms during a resonating few-body interaction, then it will
in principle undergo its evolution with at least one bound companion (e.g. Samsing et al. 2017b), which can lead to secular evolution of the binary through the Kozai-Lidov mechanism (e.g. Kozai 1962; Naoz 2016). This implies that even if the two objects in question undergo a tidal capture that first results in a stable binary, they might be driven to merger shortly after by the remaining bound tertiary object(s). This motivates looking into the tidal capture problem in the presence of bound tertiary objects.

Finally, a technical issue, which can be solved but has not been addressed in detail yet, is the amount of energy dissipated over one orbit when the two objects are bound, i.e., not a parabolic orbit as assumed in Press & Teukolsky (1977). In the bound case one expects that the efficiency of the energy dissipation should decrease with decreasing eccentricity $e$, asymptoting to zero for circular orbits $e = 0$. An attempt to modify the PT model to also work for bound orbits was presented in Mardling & Aarseth (2001), which showed that a simple redefinition of $\eta$ can be used. Another paper by Portegies Zwart & Meinen (1993), suggests to simply multiply the PT value by $e$. So, although the problem has not been studied in detail yet, the work discussing this so far indicates that including the effect into our fitting approach is not difficult.

Despite these issues and complications, it was recently illustrated by Samsing et al. (2017b), that for at least resonating three-body interactions, the main effect from including tides is indeed the formation of tidal captures that form during the interactions. This was shown using the affine model with no mode damping, which allows for strong tidal and orbital couplings. In principle, this coupling could lead to resonating and chaotic behaviors that prevent the formation of captures; however, that was not clearly observed in the simulations. We therefore expect that our simple drag force model will predict at least the right number of binary captures. If these binaries will subsequently remain stable or merge is not currently clear, as described above.

In the next section, we describe how to implement our proposed drag force model into an $N$-body code.

### 2.4 $N$-body Implementation

Consider an object pair denoted by $i, j$ in an $N$-body system. In the following we describe step by step how to derive the drag force from Equation (4), exerted by object $j$ on object $i$. This has to be done at each time step. The procedure is as follows:

- Choose a value for the drag force parameter $n$, which determines the steepness of the drag force near pericenter, and from that calculate the analytical expression for $\mathcal{F}$ given by Equation (6).
- From the position and velocity vectors of the two objects, calculate their (osculating) SMA $a$, and eccentricity $e$, assuming they are on an isolated Kepler orbit.
- From the derived $a, e$ estimate how much energy should in theory be lost over such an (isolated two-body) orbit, $\dot{E}$. For tides, this can be done following the prescriptions in Section 2.3.1. Calculate also the value of $\mathcal{F}$ given $e$.
- From $a, e, \dot{E}$, and $\mathcal{F}$, estimate the drag force coefficient $\delta$ by the use of Equation (5).
- Using the value for $\delta$, write out the drag force vector between the two objects, here referred to as $\mathbf{F}_{ij}^\delta$, via the use of Equation (4).
- The final acceleration of object $i$ exerted by object $j$, due to whatever energy loss mechanism that is considered, is now simply given by $\mathbf{a}_i = \mathbf{F}_{ij}^\delta/m_i$.

This procedure is repeated for all objects $j$, and the total drag force acceleration of object $i$ is then simply the corresponding vector sum. This total acceleration is then added to the remaining acceleration terms in the $N$-body code, which of course includes the Newtonian acceleration, and possibly also the PN terms. We note here that all of the quantities that are needed for calculating the drag force acceleration, must also be derived for the 2.5 PN term. The drag force comes therefore at nearly no additional computational cost if the 2.5 PN term is already included.

In the sections below, we study the performance of the drag force model in the case of tidal energy losses, by running a few examples, using the implementation procedure described above. For this we use the few-body code presented in Samsing et al. (2017b). Assumptions, limitations, and general results are described in the following.

### 3 COMPARISONS AND EXAMPLES

In the following sections we study and describe the role of tides in two-body and three-body interactions. We especially compare results derived using different tidal prescriptions, including the affine tides model (Luminet & Carter 1986; Samsing et al. 2017b), our drag force model with $n = 4, 10$ (see Section 2.1), and a model from which an analytic solution can be written out in closed form.

#### 3.1 Two-body Interactions with Tides

In this section, we perform a controlled numerical experiment to quantify how the orbital parameters of a fully isolated two-body system, consisting of a tidal object and a perturber, evolve as a function of time when orbital energy losses through tides are included. As described below, we do this using a few different tidal prescriptions, including our proposed tidal drag force model described in Section 2.1. We note that several more detailed studies of the two-body tidal problem have been performed (e.g. Kozhevak & Mardling 1992; Mardling 1995a; Vick & Lai 2018), however, our study is mainly performed to validate our derived drag force model under the imposed assumptions, while also quantifying how our prescription both depends on the free parameter $n$, and compares to other models.

#### 3.1.1 Initial Conditions

We consider the interaction between a tidal object (solar type star modeled as a polytrope with index 3, mass $1M_\odot$, and radius $1R_\odot$) and a compact object ($1.4M_\odot$, point mass, which could represent a neutron star; NS). The two objects are initially located at apocenter on an orbit with initial SMA $a_0 = 0.1$ AU, and pericenter distance $\approx 1.5R_\odot$, which corresponds to an eccentricity of $e \approx 0.93$. This could represent a temporary binary formed during a resonating few-body interaction, often referred to as an intermediate state (IMS) binary (e.g. Samsing et al. 2014). In the following we study the evolution of this binary when tides are taken into account.

#### 3.1.2 Simulations and Results

The orbital evolution of the two-body system described above is shown in Figure 2, and described in the following. The ‘grey’ line (labeled ‘AFT’ (–damping’), where ‘AFT’ is short for ‘affine tides’), shows the system evolved using the affine model with no mode damping included. As described in Luminet & Carter (1986), in the affine model the tidal object in question is modeled as a triaxial polytrope, which is coupled to the orbital motion of the few-body
Two-body Tidal Evolution

![Two-body Tidal Evolution](image)

**Figure 2.** Study of the tidal evolution of an eccentric binary consisting of a solar type star, modeled as a polytrope with index 3, mass $1\,M_\odot$, and radius $1\,R_\odot$, and a compact object with mass $1.4\,M_\odot$. The initial SMA is $a_0 = 0.1\,AU$, and pericenter distance $= 1.5\,R_\odot$. Both figures show results from three simulations with the same initial conditions (ICs), but different tidal prescriptions. As explained in Section 3.1, the grey curves show results using the affine model with no damping, whereas the blue and the orange curves show results from our proposed drag force model with $n = 4$ and $n = 10$, respectively. These three tidal prescriptions are labeled in the figures by ‘AFT(−damping)’, ‘DFT(+damping, $n=4$)’, and ‘DFT(+damping, $n=10$). Top plot: Evolution shown in the orbital plane, with one of the two objects fixed at 0, 0. As seen, all three models lead to an overall decrease in the SMA due to the exchange of orbital energy into tidal excitations. The affine model shows indications of chaotic motion and orbital precession; these chaotic effects are due to the tidal modes coupling to the orbit. Bottom plot: The corresponding evolution of the orbital parameters as a function of time $t$, in units of the initial orbital time $T_0$. The parameters shown are the SMA $a(t)$ (solid line), the pericenter distance $r_p(t)$ (dashed line), and the eccentricity $e(t)$ (dotted line). The dash-dotted line shows the analytical solution given by Equation (11). As seen, both tidal drag force prescriptions ($n = 4, 10$) lead to a smooth decay, which follows the analytic solution very well. The affine model leads instead to chaotic behavior after a few pericenter passages. On average, or if damping is efficient, the affine model is expected to accurately follow the drag force decay. Further descriptions are given in Section 3.1.2.

system through its center-of-mass (COM) and its quadrupole force induced by the tidal excitation (see also Diener et al. 1995). This allows for tidally stored energy to be pumped back into the orbit if the tidal object is excited before the encounter. The ‘blue’ (‘orange’) line shows the system evolved using the drag force model described in Section 2.1 with $n = 4$ ($n = 10$), and $\Delta E$ derived using the PT model assuming the parabolic limit. The ‘dash-dotted’ line shows the analytic solution to the time evolution of the SMA assuming the system loses a constant amount of energy per pericenter passage. This change in energy is set equal to the energy lost during the first passage, referred to here as $\Delta E_1$. To facilitate easy comparisons, we use the parabolic PT model to calculate $\Delta E_1$. The analytic solution was found by assuming the orbit averaged limit, from which the change in orbital energy $\Delta E(t)$ per unit time $dt$ can be written as (e.g. Samsing et al. 2018),

$$\frac{dE}{dt} \approx \frac{\sqrt{2}}{\pi} \frac{\Delta E_1}{GM} \mu^{-3/2} E(t)^{-3/2},$$

where $\mu$ is the reduced mass. This equation can be solved for $E(t) = \frac{G M_1}{2 a(t)}(2\gamma a(t))$, which can then be rewritten in terms of the SMA $a(t)$ of the system as,

$$a(t) \approx \left(\sqrt{2n} - \gamma t\right)^2,$$

where

$$\gamma = \frac{1}{2 \pi} \sqrt{\frac{M_1}{\mu}} \frac{\Delta E_1}{M} \mu^{-3/2}.$$

This solution is expected to describe the orbital evolution relatively accurately in the limit where the energy transferred between the orbit and the tidal object during each pericenter passage are uncorrelated, i.e., when damping and dissipation are efficient. However, it will break down when the system starts to circularize, which from Equation (11) will happen after a time $\approx \sqrt{2n}/\gamma$. This is also known as the inspiral or life time of an eccentric system with energy losses (e.g. Peters 1964).

By comparing these different numerical solutions, we first notice that the two versions of the drag force model ($n = 4, 10$) and the analytic solution from Equation (11) follow each other accurately during the first stages of the tidal inspiral evolution. However, at late times, just before circularization, the $n = 4$ version (blue line) starts to deviate from the other two solutions. The reason is that for lower values of $n$ the energy loss is smeared out over a larger fraction of the orbit leading to a derived pericenter distance that decreases notably over one orbit. This leads to an increase in the energy loss over the orbit, which in turn makes the system evolve faster near the point of circularization. This explains why the $n = 4$ version leads to a slightly faster merger. For higher values of $n$, such as the $n = 10$ version considered here, the loss of energy becomes increasingly more ‘impulsive’, in the sense that it is removed over a smaller fraction of the orbit is more concentrated near pericenter. For $n \to \infty$ the drag force model therefore approaches the impulsive limit. The evolution in that limit is close to the analytic solution given by Equation (11). This explains why the $n = 10$ version follows the analytic prediction better than the $n = 4$ version.

We now move on to the affine model. As seen in the figure, the strong coupling between the excited modes and the orbit lead to highly chaotic motions. After a few passages, where the affine model actually traces the other models recently well, the evolution enters a semi-chaotic phase. This leads the system to be far from undergoing a merger at the point when the drag force model predicts a merger ($t/T_0 \approx 5 - 6$). This is not a generic feature of the affine model, as it also sometimes happens that a merger occurs before the point found from the drag force model. In general, large changes in the outcome of the affine model are found by only marginally changing the initial conditions. This is mainly due to the ‘orbital time’ of the $l = 2$ mode of the star being much shorter than the Keplerian orbital time of the two interacting objects (for a systematic study on this see, e.g., Vick & Lai (2018)). However, despite this semi-chaotic behavior, one still expects the average decrease in SMA to follow the evolution.
found from assuming instantaneous damping and dissipation (i.e., close to our tidal drag force model), where the dispersion around this average should scale with the number of pericenter passages (e.g. Kochanek 1992).

If the chaotic evolution induced by the mode-orbit coupling turns out to play an important role in the formation of tidal capture binaries, one can include this effect by simply adding a harmonic term. This additional term should be added to our presented drag force term, with a period close to that of the \( l = 2 \) mode.

Below, we study the formation of two-body capture binaries assembled during three-body interactions.

### 3.2 Three-body Interactions with Tides

The first systematic study on the chaotic three-body problem with the inclusion of tides was performed by Samsing et al. (2017b) using the affine model for evolving the tides. As described in Samsing et al. (2017b), the main effect from including tides in the chaotic three-body problem is the formation of tidal capture binaries that form during the interaction. As described in Section 2.3.2, whether or not the tidally formed binaries promptly merge or settle onto a semi-stable orbit after reaching the point of circularization is still an open question. Independent of the exact outcome, however, tidal capture binaries do clearly form in chaotic interactions. The question is, how often do such tidal captures form, and are they dynamically or observationally important? As stated in the introduction, no clear studies have to date been performed to properly address these questions. This is mostly because tides are computationally heavy to implement in an \( N \)-body simulation. Our hope is that our drag force model, which is both easy to implement and very fast to evolve, can be used to gain further insight into this.

Below, we consider the evolution of a chaotic three-body interaction, and study how the inclusion of tides affects the outcome.

#### 3.2.1 Initial Conditions

We consider a binary consisting of a white dwarf (‘WD’, modeled as a polytrope with index 3/2, mass 0.6\( M_\odot \), and radius 0.0136\( R_\odot \)) and a compact object (‘CO’, with mass 1\( M_\odot \)) with an initial SMA of 0.01 AU, interacting with an incoming identical CO. We perform three scatterings with exactly the same ICs. In the first interaction we do not include any tidal effects, and treat the WD as a solid sphere. In the second interaction, we include tides on the WD using the affine model without damping. In the third interaction, we include tides using our drag force model with \( n = 10 \), assuming \( AE \) from the parabolic PT model using the fitting formulae presented in Portegies Zwart & Meinen (1993). We describe our results below.

#### 3.2.2 Simulations and Results

The orbital evolution of the three binary-single interactions we performed are shown in Figure 3. The top plot shows the results with no tides (labeled ‘No Tides’), the middle plot shows the results assuming affine tides with no damping (labeled ‘AFT(−damping)’), and the bottom plot shows the results with tides modeled using our drag force model with \( n = 10 \) (labeled ‘DFT(+ damping, \( n = 10 \))’).

As seen, when tides are not included, the interaction concludes with a classical exchange outcome in which one of the COs is ejected from the system, leaving behind a WD-CO binary. As seen in the middle and bottom plots, when tides are included the interaction instead ends with the WD undergoing a tidal capture event with
one of the COs as a result of a very close encounter between the two bodies during the interaction. Upon comparing the results from the two tidal models, we see that the tidal capture event happens at almost exactly the same point during the interaction. This serves as a good verification of both tidal implementations. However, there are small differences. These differences are most clearly seen upon considering the zoom boxes shown in each plot. For example, in the affine model, the inspiral is clearly chaotic. This chaotic behavior arises from the tidal modes coupling to the orbit. This is contrast to the evolution from the tidal drag force model, which by construction leads to a smooth inspiral. In all three zoom boxes are shown the COM motion of the WD-CO binary via a ‘green dashed’ line. As should be the case, both of the tidal models and the no tides model all give rise to the same COM motion of the considered WD-CO binary.

4 CONCLUSIONS

Accurate modeling of the dynamical assembly of BBH mergers, as well as stellar mergers and disruptions, requires $N$-body codes that include orbital energy losses in the EOM, such as occur due to tidal excitations and dissipation (e.g. Samsing et al. 2017b), as well as due to GW emission (e.g. Gültekin et al. 2006; Samsing et al. 2014; Samsing 2017; Samsing et al. 2017a; Rodriguez et al. 2017). For example, as recently shown by Samsing (2017), if GW emission is included in the EOM then the estimated rate of eccentric BH mergers forming in globular clusters is ~ 100 times higher than one finds using a standard Newtonian code; a correction that can play a key role in how to observationally distinguish between different BH merger channels. Such PN corrections are relatively easy to implement in $N$-body codes via the use of the PN formalism (Blanchet 2014). This is in contrast to energy losses from dynamical tides, which are much more challenging to include due to both theoretical and numerical limitations, as described in Section 2.3.2. For that reason, only limited work has been done on how dynamical tides affect the dynamics and observables of dense stellar systems. Some of the work that has been done includes full hydro simulations of three-body encounters (Gaburov et al. 2010), $N$-body simulations with ‘impulsive’ tidal corrections (Mardling & Aarseth 2001; Baumgardt et al. 2006), and scatterings using linear (e.g. Mardling 1995a) as well as non-linear (e.g. Samsing et al. 2017b) tidal models. However, all of these prescriptions are either too computationally expensive or too decision heavy for systematic parameter space studies. Something more simple and fast is needed, at least for the few-body codes that currently are used in Monte Carlo studies of dense stellar systems (e.g. Hypki & Giersz 2013b).

To overcome the problems related to the inclusion of especially tidal energy losses, we have described in this paper a method for including these effects in the $N$-body EOM through a simple drag force prescription, given that the amount of energy loss per close encounter is known a priori for every object pair in the system. We point out that pair-wise energy losses have been derived in the literature; for example, for the tidal examples presented in this paper (See Section 3.1 and 3.2) we used the two-body tidal fitting functions given by Portegies Zwart & Meinen (1993). This allows us to compute the magnitude and direction of our proposed drag force almost instantaneously at each time step. This in turn leads to an increase in computational speed by several orders of magnitude compared to previous methods, such as the affine model (e.g. Samsing et al. 2017b), and a much simpler implementation than say the impulsive method used in (Baumgardt et al. 2006).

Our drag force model is particularly well-suited to accurately probe the number of tidal captures forming in chaotic few-body interactions, including binary-single and binary-binary interactions. However, our model is not suitable for modeling secular systems such as Kozai triples; in this case other methods should be used (e.g. Fabrycky & Tremaine 2007; Perets & Naoz 2009; Naoz 2016). In upcoming work we plan to use our new drag force formulation to estimate cross sections for tidal captures forming in chaotic few-body systems, involving different compact and tidal objects. We are further in the process of including the model in the MOCCA code (Hypki & Giersz 2013a; Giersz et al. 2013), which soon will allow us to perform systematic studies on the dynamical and observational consequences of tidal and GW energy losses in dense stellar systems. These studies are essential in preparing for data coming from future and current transient surveys.

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