THREE-DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATIONS WITH MARKOV SWITCHING

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ABSTRACT. A finite-state Markov chain is introduced in the noise terms of the three-dimensional stochastic Navier-Stokes equations in order to allow for transitions between two types of multiplicative noises. We call such systems as stochastic Navier-Stokes equations with Markov switching. To solve such a system, a family of regularized stochastic systems is introduced. For each such regularized system, the existence of a unique strong solution (in the sense of stochastic analysis) is established by the method of martingale problems and pathwise uniqueness. The regularization is removed in the limit by obtaining a weakly convergent sequence from the family of regularized solutions, and identifying the limit as a solution of the three-dimensional stochastic Navier-Stokes equation with Markov switching.

1. INTRODUCTION

Let \( G \) be an open bounded domain in \( \mathbb{R}^3 \) with a smooth boundary. Let the three-dimensional vector-valued function \( u(x, t) \) and the real-valued function \( p(x, t) \) denote the velocity and pressure of the fluid at each \( x \in G \) and time \( t \in [0, T] \). The motion of viscous incompressible flow on \( G \) with no slip at the boundary is described by the Navier-Stokes system:

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u - \nabla p &= f(t) \quad \text{in} \quad G \times [0, T], \\
\nabla \cdot u &= 0 \quad \text{in} \quad G \times [0, T], \\
u u(x, t) &= 0 \quad \text{on} \quad \partial G \times [0, T], \\
\end{align*}
\]

where \( \nu > 0 \) denotes the viscosity coefficient, and the function \( f(t) \) is an external body force. The equation (1.1) can be written in the abstract evolution form on a suitable space as follows:

\[
\begin{align*}
du(t) + [\nu Au(t) + B(u(t))]dt &= f(t)dt, \\
\end{align*}
\]

where \( A \) is the Stokes operator and \( B \) is the nonlinear inertial operator introduced in Section 2.

A random body force, in the form of a multiplicative noise driven by a Wiener process \( W(t) \), is added to the model (see, e.g., [3]) so that one obtains

\[
du(t) + [\nu Au(t) + B(u(t))]dt = f(t)dt + \sigma(t, u(t))dW(t).
\]

Originally, it was Kolmogorov who suggested the introduction of white noise on the right side of equation (1.2) in order to investigate the existence of invariant measures.

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where the operator $B$. From then on, several works on stochastic Navier-Stokes equations appeared with an additive or more generally, a multiplicative noise driven by a Wiener process.

In addition, if the noise is allowed to be “discontinuous,” then a term driven by a compensated Poisson random measure $\tilde{N}_1(dz, dt)$ (which is independent of $W(t)$) is added so that the equation becomes

$$\begin{align*}
(1.3) \quad & du(t) + [\nu Au(t) + B(u(t))] dt \\
& = f(t) dt + \sigma(t, u(t)) dW(t) + \int_Z G(t, u(t-), z) \tilde{N}_1(dz, dt),
\end{align*}$$

where $\tilde{N}_1(dz, dt) := N_1(dz, dt) - \nu_1(dz) dt$ and $\nu_1(dz) dt$ is the intensity measure of $N_1(dz, dt)$. The rationale for the presence of a discontinuous noise (driven by a Poisson random measure) in equation (1.3) is given in Birnir [3]. In short, discontinuities arise from the prevalence of point vorticities in fluid flows in turbulent regime.

Stochastic Navier-Stokes systems have been studied by a number of authors at various levels of generality. Spurred by the works of Bensoussan and Temam [2], and Viot [27], there was an active growth in the area with notable contributions by Flandoli and Gatarek [9], Flandoli and Maslowski [10], Debussche and Da Prato [6], Menaldi and Sritharan [17], Mattingly [16], Röckner and Zhang [21] and Sritharan and Sundar [23], to name a few. The references in the articles by Flandoli as well as Albeverio [1] would provide a more complete list of research work on stochastic Navier-Stokes equations.

A novelty of this paper consists in the introduction of a right continuous Markov chain $\{\tau(t) : t \in \mathbb{R}^+\}$ in order to allow for transitions in the type of random forces that perturb the Navier-Stokes equation. The equation under study appears as

$$\begin{align*}
(1.4) \quad & du(t) + [\nu Au(t) + B(u(t))] dt \\
& = f(t) dt + \sigma(t, u(t), \tau(t)) dW(t) + \int_Z G(t, u(t-), \tau(t-), z) \tilde{N}_1(dz, dt)
\end{align*}$$

with initial condition $u(0) = u_0$ in a specified space. The Markov chain is assumed to be independent of the Wiener process and the Poisson random measure, and it brings transitions between smooth (e.g., laminar) and turbulent flows into stochastic Navier-Stokes equations. We shall call such equations as stochastic Navier-Stokes equations with Markov switching.

The objective of this article is to construct a weak solution (in the sense of stochastic analysis and partial differential equations) to equation (1.4), and it is achieved by the following steps. First, we regularize the nonlinear term in (1.1) and solve the regularized equation: for each $\epsilon > 0$,

$$\begin{align*}
(1.5) \quad & \partial_t u - \nu \Delta u + ((k_\epsilon u) \cdot \nabla) u - \nabla p = f(t),
\end{align*}$$

the operator $k_\epsilon$ is a mollification operator (see equation (2.1) below). The abstract evolution form of equation (1.5) is

$$\begin{align*}
(1.6) \quad & du(t) + [\nu Au(t) + B_{k_\epsilon}(u(t))] dt = f(t) dt,
\end{align*}$$

where the operator $B_{k_\epsilon}$ will be introduced in Section 2. Thus, its stochastic analog with Markov switching is given by

$$\begin{align*}
(1.7) \quad & du(t) + [\nu Au(t) + B_{k_\epsilon}(u(t))] dt \\
& = f(t) dt + \sigma(t, u(t), \tau(t)) dW(t) + \int_Z G(t, u(t-), \tau(t-), z) \tilde{N}_1(dz, dt)
\end{align*}$$
with initial condition \( u(0) = u_0 \) in a specified space. Our first objective is to show that equation (1.7) admits a unique strong solution (in the sense of stochastic analysis) under suitable growth and Lipschitz conditions on the noise coefficients (listed later as Hypotheses \( H \)):

**Theorem 1.1.** Assume that \( \mathbb{E}|u(0)|^3 < \infty \) and \( f \in L^3(0, T; V') \). Then under Hypotheses \( H \), there exists a unique strong solution to the stochastic system (1.7) for each fixed \( \epsilon > 0 \).

Let \( u^\epsilon \) denote the solution to the regularized equation (1.7). The next step is to show that there exists a sequence from the family \( \{u^\epsilon\}_{\epsilon > 0} \) which converges weakly to a limit as \( \epsilon \to 0 \). Let the limit be denoted by \( u \). This sets up the stage to identify the limit, \( u \), as a solution of the stochastic Navier-Stokes equation with Markov switching (1.4) without regularization. The idea of using a regularization for the nonlinear term in the Navier-Stokes system goes back to Leray (see, e.g., [15, 19]). In its stochastic context, we have

**Theorem 1.2.** Assume that \( \mathbb{E}|u(0)|^3 < \infty \) and \( f \in L^3(0, T; V') \). Then under Hypotheses \( H \), there exists a weak solution \( u \) to the three-dimensional stochastic Navier-Stokes equation with Markov switching (1.4).

Ergodic behavior of the solution of the regularized stochastic Navier-Stokes system with Markov switching and its limit (along a sequence) is being prepared by us as a separate article. Coupling of the stochastic Navier-Stokes system and the Markov switching to reflect onset of turbulence is currently under study.

The present article is organized as follows. The background results and the functional analytic setup for the Navier-Stokes system are introduced in Section 2. A priori estimates appear in Section 3. Section 4 is devoted to the proof of Theorem 1.1. In section 5, the proof of Theorem 1.2 is presented.

2. PRELIMINARIES AND FUNCTIONAL ANALYTIC SETUP

2.1. Basic Results on Convolution. First, we recall some properties on convolution in order to explain regularization. The interested reader may consult, e.g., [7, App. C.5.] for more details. If \( U \subset \mathbb{R}^3 \) is open and \( \epsilon > 0 \), we write \( U_\epsilon := \{ x \in U : \text{dist}(x, \partial U) > \epsilon \} \). Define the function \( \eta \in C^\infty(\mathbb{R}^3) \) by

\[
\eta(x) := \begin{cases} 
C \exp \left( \frac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]

where the constant \( C > 0 \) is selected so that \( \int_{\mathbb{R}^3} \eta dx = 1 \). For each \( \epsilon > 0 \), set \( \eta_\epsilon(x) := \frac{1}{\epsilon^3} \eta \left( \frac{x}{\epsilon} \right) \). We call \( \eta \) the standard mollifier; the function \( \eta_\epsilon \) is a smooth function on \( \mathbb{R}^3 \) with support in \( B(0, \epsilon) \) and satisfy \( \int_{\mathbb{R}^3} \eta_\epsilon dx = 1 \). If \( f : U \to \mathbb{R} \) is locally integrable, define the mollification operator by

\[
k_\epsilon f := \eta_\epsilon * f \quad \text{in} \quad U_\epsilon,
\]

i.e., \( k_\epsilon f = \int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta(y)f(x-y)dy \) for \( x \in U_\epsilon \). The next lemma collects some properties of the mollification operator. The interested reader may consult, e.g., [7, Thm. 7 in App. C.5.] or [19, Lem. 6.3] for details.

**Lemma 2.1.** The mollification operator enjoys the following properties:

1. \( k_\epsilon f \in C^\infty(U_\epsilon) \).
2. If \( 1 \leq p < \infty \) and \( f \in L^p_{\text{loc}}(U) \), then \( k_\epsilon f \to f \) in \( L^p_{\text{loc}}(U) \).
3. If \( 1 \leq p < \infty \) and \( f \in L^p_{\text{loc}}(U) \), then \( \| k_\epsilon f \|_{L^p_{\text{loc}}(U)} \leq \| f \|_{L^p_{\text{loc}}(U)} \).
2.2. Function Space and Operators. Let $\mathcal{D}(G)$ be the space of $C^\infty$-functions with compact support contained in $G$ and $\mathcal{V} := \{u \in \mathcal{D}(G) : \nabla \cdot u = 0\}$. Let $H$ and $V$ be the completion of $\mathcal{V}$ in $L^2(G)$ and $W_0^{1,2}(G)$, respectively. Then it can be shown that (see, e.g., [26, Sec. 1.4, Ch. I])

\[
H = \{u \in L^2(G) : \nabla \cdot u = 0, \, u \cdot n|_{\partial G} = 0\}, \\
V = \{u \in W_0^{1,2}(G) : \nabla \cdot u = 0\},
\]

and we denote the $H$-norm ($V$-norm, resp.) by $| \cdot |$ ($\| \cdot \|$) resp.) and the inner product on $H$ (on $V$, resp.) by $(\cdot, \cdot)$ ($(\cdot, \cdot)$, resp.). The duality pairing between $V'$ and $V$ is denoted by $\langle \cdot, \cdot \rangle_V$, or simply by $(\cdot, \cdot)$ when there is no ambiguity. In addition, we have the following inclusion between the spaces: $V \hookrightarrow H \hookrightarrow V'$, and both of the inclusions $V \hookrightarrow H$ and $H \hookrightarrow V'$ are dense, compact embeddings (see, e.g., [25, Lem. 1.5.1 and 1.5.2, Ch. II]).

Let $A : V \to V'$ be the Stokes operator. The set $\{e_i\}_{i=1}^\infty$ is reserved for the orthonormal basis in $H$ (orthogonal in $V$) that consists of the eigenvector of the Stokes operator $A$ (see, e.g., [5, Thm. IV. 5.5]), and $H_n := \text{span}\{e_i\}_{i=1}^n$, $\Pi_n$ is the orthogonal projection of $H$ on $H_n$. In addition, for all $u \in \mathcal{D}(A)$, one has (see, e.g., [8, Sec. 6, Ch. II])

\[
(Au, u)_V = \|u\|.
\]

Define $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by

\[
b(u, v, w) := \sum_{i,j=1}^3 \int_G \frac{\partial v_j}{\partial x_i} w_j dx.
\]

Then $b$ is a trilinear form which induces a bilinear form $B(u, v)$ by $b(u, v, w) = (B(u, v), w)_V$. In addition, $b$ enjoys the following properties (see, e.g., [26, Lem. 1.3, Sec. 1, Ch. II]):

\[
(2.3) \quad b(u, v, v) = 0, \\
(2.4) \quad b(u, v, w) = -b(u, w, v).
\]

For each $\epsilon > 0$, define $b(\epsilon \cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by

\[
b(\epsilon u, v, w) := \sum_{i,j=1}^3 \int_G (\eta_\epsilon * u)_i \frac{\partial v_j}{\partial x_i} w_j dx,
\]

which induces a bilinear form $B_\epsilon(u, v)$ by $b(\epsilon u, v, w) = (B_\epsilon(u, v), w)_V$. The regularization raises the regularity of the first component in $b$, therefore, one may employ the (generalized) Hölder inequality and the Young convolution inequality to deduce

\[
|b(\epsilon u, v, w)| \leq \|\eta_\epsilon * u\|_6 \|\nabla v\|_2 \|w\|_3 \leq C_\epsilon' \|u\|_3 \|\nabla v\|_2 \|w\|_3,
\]

where $C_\epsilon' = \|\eta_\epsilon\|_{\frac{6}{5}}$. This together with Sobolev embedding and interpolation inequalities further implies

\[
|b(\epsilon u, v, w)| \leq C_\epsilon \|u\|^{\frac{2}{3}} \|v\|^{\frac{4}{3}} \|w\|^\frac{2}{3}.
\]

In particular, when $u = w$, we have

\[
|b(\epsilon u, v, u)| \leq C_\epsilon \|u\| \cdot |u| \cdot \|v\|.
\]

As shall be seen later, we first work with a fixed $\epsilon$. Therefore, we shall assume that $C_\epsilon = 1$ for the sake of simplicity.
2.3. Noise Terms.

(i) Let $Q \in \mathcal{L}(H)$ be a nonnegative, symmetric, trace-class operator. Define $H_0 := Q^{\frac{1}{2}}(H)$ with the inner product given by $(u, v)_0 := (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)_H$ for $u, v \in H_0$, where $Q^{-\frac{1}{2}}$ is the inverse of $Q$. Then it follows from [20, Prop. C.0.3 (i)] that $(H_0, (\cdot, \cdot)_0)$ is again a separable Hilbert space. Let $L_2(H_0, H)$ denote the separable Hilbert space of the Hilbert-Schmidt operators from $H_0$ to $H$. Then it can be shown that (see, e.g., [13, Sec. 3, Ch II.])

Let $T > 0$ be a fixed real number and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space. Let $W$ be an $H$-valued Wiener process with covariance $Q$.

Let $\sigma : [0, T] \times \Omega \rightarrow L_2(H_0, H)$ be jointly measurable and adapted. If we have $\mathbb{E}\int_0^T \|\sigma(s)\|_{L_2(H_0,H)}^2 ds < \infty$, then for $t \in [0, T]$, the stochastic integral $\int_0^t \sigma(s) dW(s)$ is well-defined and is an $H$-valued continuous square integrable martingale.

(ii) Let $(Z, B(Z))$ be a measurable space, $M$ be the collection of all nonnegative integer-valued measures on $(Z, B(Z))$, and $B(M)$ be the smallest $\sigma$-field on $M$ with respect to which all $\eta \rightarrow \eta(B)$ are measurable, where $\eta \in M$, $\eta(B) \in \mathbb{Z}^+ \cup \{\infty\}$, and $B \in B(Z)$. Let $N : \Omega \rightarrow M$ be a Poisson random measure with intensity measure $\nu$.

For a Poisson random measure $N(dz, ds)$, $\tilde{N}(dz, ds) := N(dz, ds) - \nu(dz)ds$ defines its compensation. Then it can be shown that (see, e.g., [13, Sec. 3, Ch II.]) $\tilde{N}(dz, ds)$ is a square integrable martingale, and for predictable $f$ such that

$$
\mathbb{E} \int_0^{t+} \int_Z |f(\cdot, z, s)| \nu(dz)ds < \infty, \text{ then }
\int_0^{t+} \int_Z f(\cdot, z, s) \tilde{N}(dz, ds)
= \int_0^{t+} \int_Z f(\cdot, z, s) N(dz, ds) - \int_0^t \int_Z f(\cdot, z, s) \nu(dz)ds
$$

is a well-defined $\mathcal{F}_t$-martingale.

(iii) Let $m \in \mathbb{N}$. Let $\{\tau(t) : t \in \mathbb{R}^+\}$ be a right continuous Markov chain with generator $\Gamma = (\gamma_{ij})_{m \times m}$ taking values in $S := \{1, 2, 3, \ldots, m\}$ such that

$$
\mathcal{R}_t(i, j) = \mathcal{R}(\tau(t + h) = j | \tau(t) = i) = \left\{
\begin{array}{ll}
\gamma_{ij}h + o(h) & \text{if } i \neq j, \\
1 + \gamma_{ii}h + o(h) & \text{if } i = j,
\end{array}
\right. \text{ and } \gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}.
$$

In addition, $\tau(t)$ admits the following stochastic integral representation (see, e.g., [24, Sec. 2.1, Ch. 2]): Let $\Delta_{ij}$ be consecutive, left closed, right open intervals
of the real line each having length \( \gamma_{ij} \) such that
\[
\Delta_{12} = [0, \gamma_{12}), \quad \Delta_{13} = [\gamma_{12}, \gamma_{12} + \gamma_{13}), \ldots
\]
\[
\Delta_{1m} = \left[ \sum_{j=2}^{m-1} \gamma_{1j}, \sum_{j=2}^{m} \gamma_{1j} \right), \ldots
\]
\[
\Delta_{2m} = \left[ \sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1, j \neq 2}^{m-1} \gamma_{2j}, \sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1, j \neq 2}^{m} \gamma_{2j} \right)
\]
and so on. Define a function \( h : S \times \mathbb{R} \rightarrow \mathbb{R} \) by
\[
h(i, y) = \begin{cases} 
  j - i & \text{if } y \in \Delta_{ij}, \\
  0 & \text{otherwise.}
\end{cases}
\]
(2.9)

Then
\[
d\tau(t) = \int_{\mathbb{R}} h(\tau(t-), y) N_2(dt, dy),
\]
with initial condition \( \tau(0) = \tau_0 \), where \( N_2(dt, dy) \) is a Poisson random measure with intensity measure \( dt \times \mathcal{L}(dy) \), in which \( \mathcal{L} \) is the Lebesgue measure on \( \mathbb{R} \).

We assume that such a Markov chain, Wiener process, and the Poisson random measure are independent.

2.4. Hypotheses and Stochastic System. The noise coefficients \( \sigma : [0, T] \times H \times S \rightarrow L^2(H_0, H) \) and \( G : [0, T] \times H \times S \times Z \rightarrow H \) are assumed to satisfy the following Hypotheses H:

**H1.** For all \( t \in [0, T] \) and all \( i \in S \), there exists a constant \( K > 0 \) such that
\[
\|\sigma(t, u, i)\|_{L^q} \leq K(1 + |u|^p)
\]
for \( p \) equal to 2, 3 (growth condition on \( \sigma \)).

**H2.** For all \( t \in [0, T] \), there exists a constant \( L > 0 \) such that for all \( u, v \in H \) and \( i, j \in S \)
\[
\|\sigma(t, u, i) - \sigma(t, v, i)\|_{L^q}^2 \leq L(|u - v|^2)
\]
(Lipschitz condition on \( \sigma \)).

**H3.** For all \( t \in [0, T] \) and all \( i \in S \), there exist a constant \( K > 0 \) such that
\[
\int_Z |G(t, u, i, z)|^p \nu(dz) \leq K(1 + |u|^p)
\]
for all \( p = 1, 2, \) and 3 (growth condition on \( G \)).

**H4.** For all \( t \in [0, T] \), there exists a constant \( L > 0 \) such that for all \( u, v \in H \) and \( i, j \in S \),
\[
\int_Z |G(t, u, i, z) - G(t, v, i, z)|^2 \nu(dz) \leq L(|u - v|^2)
\]
(Lipschitz condition on \( G \)).

The transformation from equation (1.1) to (1.2) is sketched as follows (the transformation from (1.5) to (1.6) can be achieved in a similar manner). Invoking the Helmholtz decomposition, one decomposes the space \( L^2(G) \) into the direct sum of \( H \) and its orthogonal complement, namely, \( L^2(G) = H \oplus H^\perp \). Moreover, by applying the Leray projection to each term of (1.1), one may write (1.1) as (1.2). The interested reader is referred to, e.g., [14, 25] for more details.
Let $F : [0, T] \times V \times S \to \mathbb{R}^+$ be a continuous function with its Fréchet derivatives $F_t$, $F_v$, and $F_w$ are bounded and continuous. Define the operator

\begin{equation}
\mathcal{L}F(t, v, i) := F_t(t, v, i) + \langle -\nu Av - B(v) + f(t), F_v(t, v, i) \rangle_V \\
+ \sum_{j=1}^{m} \gamma_j F(t, v, j) + \frac{1}{2} \text{tr} \left( F_{vw}(t, v, i) \sigma(t, v, i) \sigma^*(t, v, i) \right)
\end{equation}

(2.11)

\begin{align*}
\int_Z \left( F(t, v + G(t, v, i, z), i) - F(t, v, i) \\
- \left( F_v(t, v, i), G(t, v, i, z) \right)_H \right) \nu_1(dz).
\end{align*}

Then we have the following change of variables formula due to Itô (see, e.g., [24, Lem. 3 in Sec. 2.1, Ch. 2]):

\begin{align*}
F(t, u(t), r(t)) &= F(0, u(0), r(0)) + \int_0^t \mathcal{L}F(s, u(s), r(s)) ds \\
&\quad + \int_0^t \langle F_x(s, u(s), r(s)), \sigma(s, u(s), r(s)) dW(s) \rangle \\
&\quad + \int_0^t \int_Z \left( F(s, u(s) -) + G(s, u(s) -), r(s) -) \right) \tilde{N}_1(dz, ds) \\
&\quad - F(s, u(s) -), r(s) -) \right) \tilde{N}_1(dz, ds) \\
&\quad + \int_0^t \int_{\mathbb{R}} \left( F(s, u(s) -) + h(r(s) -), y \right) \\
&\quad - F(s, u(s) -), r(s) -) \right) \tilde{N}_2(ds, dy),
\end{align*}

where $\tilde{N}_1(dz, ds)$ is the compensated Poisson random measure introduced earlier; $\tilde{N}_2(ds, dy) := N_2(ds, dy) - \mathcal{E}(dy)ds$ where $N_2(ds, dy)$ and $\mathcal{E}(dy)ds$ are defined in (2.10); the function $h(s, y)$ is defined as in (2.9). In particular, if $F(t, u(t), i) = |u(t)|^2$, then $\sum_{j=1}^{m} \gamma_j |u(t)|^2 = 0$. We therefore obtain the following energy equality:

\begin{equation}
|u(t)|^2 = |u(0)|^2 + 2 \int_0^t \langle -\nu Au(s) - B(u(s)) + f(s), u(s) \rangle_V ds \\
+ \int_0^t \|\sigma(s, u(s), r(s))\|_L^2 ds + 2 \int_0^t \langle u(s), \sigma(s, u(s), r(s)) dW(s) \rangle \\
+ \int_0^t \int_Z \left( |u(s) -) + G(s, u(s) -), r(s) -) |^2 - |u(s) -) |^2 \right) \tilde{N}_1(dz, ds) \\
+ \int_0^t \int_{\mathbb{R}} \left( |u(s) + G(s, u(s), r(s), z) |^2 - |u(s) |^2 \\
- 2 \left( u(s), G(s, u(s), r(s), z) \right)_H \right) \nu_1(dz) ds.
\end{equation}

(2.12)
2.5. Path Space and its Topology. Denoted by \( \{\tau_i\}_{i=1}^4 \) the topologies
\[
\begin{align*}
\tau_1 &= J\text{-topology} \quad \text{on} \quad \mathcal{D}([0, T]; V'), \\
\tau_2 &= \text{weak topology} \quad \text{on} \quad L^2(0, T; V), \\
\tau_3 &= \text{weak-star topology} \quad \text{on} \quad L^\infty(0, T; H), \\
\tau_4 &= \text{strong topology} \quad \text{on} \quad L^2(0, T; H),
\end{align*}
\]
and \( \Omega_i \) the spaces
\[
\begin{align*}
\Omega_1 &= \mathcal{D}([0, T]; V'), \\
\Omega_2 &= L^2(0, T; V), \\
\Omega_3 &= L^\infty(0, T; H), \\
\Omega_4 &= L^2(0, T; H).
\end{align*}
\]
Then \( \{\{(\Omega_i, \tau_i)\}_{i=1}^4 \) are all Lusin spaces (a topological space that is homeomorphic to a Borel set of a Polish space).

**Definition 1.** Define the space \( \Omega^* \) by \( \Omega^* = \bigcap_{i=1}^4 \Omega_i \). Let \( \tau \) be the supremum of the topologies\(^1\) induced on \( \Omega^* \) by all \( \tau_i \). Then it follows from a result of Metivier\(^2\) that\(^3\)
\[
\begin{align*}
(1) \quad (\Omega^*, \tau) &\text{ is a Lusin space.} \\
(2) \quad \text{Let } \{\mu_k\}_{k\in\mathbb{N}} \text{ be a sequence of Borel probability laws on } \Omega^* \text{ (on the Borel } \sigma\text{-algebra } \mathcal{B}(\tau)) \text{ such that their images } \{\mu_k^i\}_{k\in\mathbb{N}} \text{ on } (\Omega_i, \mathcal{B}(\tau_i)) \text{ are tight for } \tau_i \text{ for all } i. \text{ Then } \{\mu_k\}_{k\in\mathbb{N}} \text{ is tight for } \tau. \\
(3) \quad \text{Let } (\Omega, \mathcal{F}, \mathcal{P}) \text{ be a (complete) probability space on which the following are defined:} \\
\quad (1) \quad W = \{W(t) : 0 \leq t \leq T\}, \text{ an } H\text{-valued } Q\text{-Wiener process.} \\
\quad (2) \quad N = \{N(z, t) : 0 \leq t \leq T \land z \in Z\}, \text{ the Poisson random measure.} \\
\quad (3) \quad \tau = \{\tau(t) : 0 \leq t \leq T\}, \text{ the Markov chain.} \\
\quad (4) \quad \xi, \text{ an } H\text{-valued random variable.}
\end{align*}
\]
Assume that \( \xi, W, N, \) and \( \tau \) are mutually independent. For each \( t \), define the \( \sigma\)-field \( \mathcal{F}_t \) to be \( \sigma(\xi, \tau(t), W(s), N(z, s) : z \in Z, 0 \leq s \leq t) \cup \{\text{all } \mathcal{P}\text{-null sets in } \mathcal{F}\} \). Then it is clear that \( (\mathcal{F}_t) \) satisfies the usual conditions, and both \( W(t) \) and \( N(z, t) \) are \( \mathcal{F}_t \)-adapted processes.

Denoting by \( \mathcal{J} \) the \( J\text{-topology} \) in the space \( \mathcal{D}([0, T]; S) \), the path space of solutions of (1.4) is given by
\[
\Omega_1 := \Omega^* \times \mathcal{D}([0, T]; S), \\
\tau_1 := \tau \times \mathcal{J}.
\]

**Definition 2 (Weak solution).** Suppose that, on some probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \), there exists an increasing family \( \{\mathcal{G}_t\} \) of sub \( \sigma\)-field of \( \mathcal{F} \), an \( H \times S\)-valued random vector \( (\xi, r) \) with given distribution \( \mu \), and \( \mathcal{G}_t \)-adapted processes \( W(t), \tilde{N}(z, t), \) \( u(t), \) and \( \tau(t) \) such that
\[
\begin{align*}
(1) \quad & (W(t), \mathcal{G}_t, \mathcal{P}) \text{ is an } H\text{-valued } Q\text{-Wiener martingale.} \\
(2) \quad & \tilde{N}(z, t) \text{ is an square integrable martingale with respect to } (\mathcal{G}_t). \\
(3) \quad & W(t), \tilde{N}(z, t), \text{ and } (\xi, r) \text{ are mutually independent.} \\
(4) \quad & \text{For all } v \in V, \\
\mathcal{P}\left\{ \omega : \int_0^T \langle v A u(\omega, s) + B(u(\omega, s)), \rho \rangle_V ds < \infty \right\} = 1.
\end{align*}
\]

\(^1\)The coarsest topology that is finer than each \( \tau_i \). See, e.g., [11] Sec. 5.2
\(^2\)Note that all the natural inclusion \( \Omega_i \hookrightarrow \Omega_4, i = 2, 3, 4, \) are continuous.
Then the family $(\Omega, \mathcal{F}, (G_t), \mathcal{P}, \xi, r, \{W(t)\}, \{N(z, t)\}, \{u(t)\}, \{\tau(t)\})$ is called a weak solution of the stochastic Navier-Stokes equation (1.4).

Definition 3 (Pathwise uniqueness). A weak solution of the stochastic Navier-Stokes equation (1.4) is said to be pathwise unique if, for any two weak solutions give by

$$
(\Omega, \mathcal{F}, (G_t), \mathcal{P}, \xi^i, r^i, \{W(t)\}, \{N(z, t)\}, \{u^i(t)\}, \{\tau^i(t)\})
$$

for $i = 1, 2$, the following holds:

$$
\mathcal{P}\left\{ (u^1(t), \tau^1(t)) = (u^2(t), \tau^2(t)) \quad \forall t \geq 0 \right\} = 1.
$$

Remark. The definition of weak solutions to equation (1.7) follow from Definition 2 with $B_k$ in the place of $B$.

We collect some lemmata here for the benefit of the reader.

Lemma 2.2. Let $f \in C(\Omega)$ and $\sup_n \mathbb{E}^n[|f|^{1+\delta}] \leq C$ for some $\delta > 0$. Let $\{P_n\}$ be a sequence of probability measures on $\Omega$ with $P_n \Rightarrow P$, as $n \to \infty$. Then we have $\mathbb{E}^P_n(|f|) \to \mathbb{E}^P(|f|)$.

Lemma 2.3. Consider the continuous dense embeddings $V \hookrightarrow H \hookrightarrow V'$ with $V \hookrightarrow H$ and $H \hookrightarrow V'$ being compact. Suppose that a set $B$ in $L^q(0, T; H) \cap \mathcal{D}([0, T]; V')$ is relatively compact in $\mathcal{D}([0, T]; V')$ and bounded in $L^q(0, T; V)$. Then $B$ is relatively compact in $L^q(0, T; H)$.

Proof. The proof is based on the Aubin-Lions Lemma, and we refer the interested reader to [18, Lem. 3, Ch. VI].

Lemma 2.4 (Aldous’ criterion). Let $\{X_n\}_{n=1}^\infty$ be a sequence of processes with paths in the space $\mathcal{D}([0, T]; V')$. Suppose that for each rational numbers $t \in [0, T]$, we have

$$
\lim_{N \to \infty} \limsup_n \mathcal{P}\left( \|X_n(t)\|_{V'} > N \right) = 0.
$$

Then $\{X_n\}_{n=1}^\infty$ is tight in $\mathcal{D}([0, T]; V')$ if the following condition is satisfied:

For every sequence $(T_n, \delta_n)$ where each $T_n$ is a stopping time such that $T_n + \delta_n \leq T$, and $\delta_n > 0, \delta_n \to 0$, we have $\|X_n(T_n + \delta_n) - X_n(T_n)\|_{V'} \to 0$ in probability as $n \to \infty$.

Proof. The interested reader may consult, e.g., [29, p. 353 Thm. 6.8] for the proof of this lemma.

3. A priori estimates

In the section, we establish a priori estimates to the approximation of the regularized equation (1.7). Recalling the definitions of $\{e_i\}_{i=1}^\infty$, $H_n$, and $\Pi_n$ in Section 2, we define $W_n := \Pi_n W$, $\sigma_n := \Pi_n \sigma$, and $G_n := \Pi_n G$. Let $u_0$ be an $H$-valued random variable and $\epsilon > 0$ be fixed throughout this section.
Let $u_n$ be the solution to following equation: for each $v \in H_n$,

$$d(u_n(t), v) = [\left( -\nu A u_n(t) - B_{k_i}(u_n(t)), v \right) dt + \langle f(t), v \rangle_V dt$$

$$+ \langle \sigma_n(t, u_n(t), \pi(t)) dW_n(t), v \rangle_V$$

$$+ \int_Z G_n(t, u_n(t-), \pi(t-), z) N_1(dz, dt), v)$$

with $u_n(0) = \Pi_n u_0$, where $N_1(dz, ds) = N_1(dz, ds) - \nu_1(dz)ds$.

**Proposition 3.1** (A priori estimates). Let $T > 0$ be fixed. Suppose that $\mathbb{E}|u_0|^2 < \infty$ and $f \in L^2(0, T; V')$. Then under Hypotheses $H$, there exist constants $C_1$ and $C_2$ that depend on $\mathbb{E}|u_0|^2, T, N$, and $\nu_1, \nu_2, \nu_3$ such that \( \forall t \in [0, T] \),

$$\mathbb{E}|u_n(t)|^2 + \nu \mathbb{E} \int_0^t \|u_n(s)\|^2 ds \leq C_1,$$  

and

$$\mathbb{E} \sup_{0 \leq t \leq T} |u_n(t)|^2 + \nu \mathbb{E} \int_0^T \|u_n(s)\|^2 ds \leq C_2.$$

Suppose further that $\mathbb{E}|u_0|^3 < \infty$ and $f \in L^3(0, T; V')$. Then under Hypotheses $H$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |u_n(t)|^3 + 2\nu \mathbb{E} \int_0^T \|u_n(s)\||u_n(s)||^2 ds \leq C_3.$$

**Remark.** The constants $C_i$ in (3.2), (3.3), and (3.4) is independent of $\epsilon$.

**Proof.** Let $N > 0$. Define

$$\tau_N := \inf \{ t \in [0, T] : |u_n(t)|^2 + \int_0^t \|u_n(s)\|^2 ds > N \}$$

or $|u_n(t-)|^2 + \int_0^t \|u_n(s)\|^2 ds > N \}$. It follows from the Itô formula that

$$|u_n(t \wedge \tau_N)|^2$$

$$= |u_n(0)|^2 + 2 \int_0^{t \wedge \tau_N} \langle -\nu A u_n(s) - B_{k_i}(u_n(s)) + f(s), u_n(s) \rangle_V ds$$

$$+ \int_0^{t \wedge \tau_N} \|\sigma_n(s, u_n(s), \pi(s))\|^2_{L^2} ds$$

$$+ 2 \int_0^{t \wedge \tau_N} \langle u_n(s), \sigma_n(s, u_n(s), \pi(s)) dW_n(s) \rangle$$

$$+ \int_0^{t \wedge \tau_N} \int_Z \left( |u_n(s-)| + G_n(s, u_n(s-), \pi(s-), z) \right) N_1(dz, ds)$$

$$+ \left( |u_n(s)| + G_n(s, u_n(s), \pi(s), z) \right) \left( |u_n(s)| + 2 \langle u_n(s), G_n(s, u_n(s), \pi(s), z) \rangle_H \right) \nu_1(dz) ds.$$
By (2.2), we have
\[ 2 \int_0^{T \wedge \tau_N} -\nu \langle A u_n(s), u_n(s) \rangle_V \, ds = -2\nu \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds. \]

By (2.3), \( \langle B_n(u_n(s)), u_n(s) \rangle_V = 0 \). For the external force term \( f \), one deduces from the basic Young inequality that
\[ \left| 2 \int_0^{T \wedge \tau_N} \langle f(s), u_n(s) \rangle_V \right| \leq \frac{1}{\nu} \int_0^t \| f(s) \|^2 \, ds + \nu \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds. \]

A simplification of the last term in (3.5) gives
\[ \int_0^{T \wedge \tau_N} \int_Z \| G_n(s, u_n(s), \tau(s), z) \|^2 \nu_1(dz) \, ds. \]

Now, taking expectation on the both side of (3.5), using Hypotheses H1 and H3, and then putting everything together, we obtain
\[ \mathbb{E} \| u_n(t \wedge \tau_N) \|^2 + \nu \mathbb{E} \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds \]
\[ \leq \mathbb{E} \| u_0 \|^2 + \frac{1}{\nu} \mathbb{E} \int_0^{T \wedge \tau_N} \| f(s) \|^2 \, ds + 2K \mathbb{E} \int_0^{T \wedge \tau_N} \| u_n(s \wedge \tau_N) \|^2 \, ds + 2KT. \]

Denoting \( C_T := \mathbb{E} \| u_0 \|^2 + \frac{1}{\nu} \mathbb{E} \int_0^T \| f(s) \|^2 \, ds + 2KT(1 + m^2) \), we utilize the Gronwall inequality to obtain
\[ \mathbb{E} \| u_n(t \wedge \tau_N) \|^2 \leq C_T e^{2KT}. \]

A combination of (3.6) and (3.7) yield
\[ \mathbb{E} \| u_n(t \wedge \tau_N) \|^2 + \nu \mathbb{E} \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds \leq C_T(1 + 2KT e^{2KT}). \]

In particular,
\[ \mathbb{E} \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds \leq \mathbb{E} \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \, ds \leq \frac{1}{\nu} C_T(1 + 2KT e^{2KT}). \]

An application of Davis inequality and basic Young inequality yield
\[ 2\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t \langle u_n(s), \sigma(s, u_n(s), \tau(s)) \rangle_{W_n(s)} \right| \]
\[ \leq 2\sqrt{2} \mathbb{E} \left( \left( \int_0^{T \wedge \tau_N} \| u_n(s) \|^2 \| \sigma(s, u_n(s), \tau(s)) \|^2_{L^2} \right)^{\frac{1}{2}} \right) \]
\[ \leq 2\sqrt{2} e_1 \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^2 + 2\sqrt{2} C_{e_1} \mathbb{E} \int_0^{T \wedge \tau_N} \| \sigma(s, u_n(s), \tau(s)) \|^2_{L^2} \, ds. \]

In a similar manner, we obtain
\[ 2\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} \int_0^t \int_Z \left( u_n(s-), G(s, u_n(s-), \tau(s-), z) \right)_H \hat{N}_1(dz, ds) \]
\[ \leq 2\sqrt{10} \mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)||G(s, u_n(s), \tau(s), z)| \nu_1(dz) \, ds \]
\[ \leq 2\sqrt{10} e_2 \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^2 + 2\sqrt{10} C_{e_2} \mathbb{E} \int_0^{T \wedge \tau_N} \int_Z |G(s, u_n(s), \tau(s), z)|^2 \nu_1(dz) \, ds. \]
Therefore, Itô formula implies

\[
E \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^2 + \nu E \int_0^{T \wedge \tau_N} ||u_n(s)||^2 ds
\]

\[
\leq E|u_0|^2 + \frac{1}{\nu} E \int_0^T ||f(s)||_V^2 ds + 2(\sqrt{2}\epsilon_1 + \sqrt{10}\epsilon_2)E \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^2
\]

\[
+ (2\sqrt{2}C_{\epsilon_1} + 1)E \int_0^{T \wedge \tau_N} ||\sigma(s, u_n(s), \tau(s))||_{L_Q}^2 ds
\]

\[
+ (2\sqrt{10}C_{\epsilon_2} + 1)E \int_0^{T \wedge \tau_N} \int_Z |G(s, u_n(s), \tau(s), z)|^2 \nu(dz) ds.
\]

Take \(\epsilon_1 = \frac{1}{8\sqrt{2}}\) and \(\epsilon_2 = \frac{1}{8\sqrt{10}}\). Then \(C_{\epsilon_1} = 2\sqrt{2}\) and \(C_{\epsilon_2} = 2\sqrt{10}\). One obtains from above and (3.9) that

\[
\frac{1}{2} E \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^2 + \nu E \int_0^{T \wedge \tau_N} ||u_n(s)||^2 ds
\]

\[
\leq E|u_0|^2 + \frac{1}{\nu} E \int_0^T ||f(s)||_V^2 ds + 50KE \int_0^{T \wedge \tau_N} (1 + |u_n(s)|^2) ds
\]

\[
\leq E|u_0|^2 + \frac{1}{\nu} E \int_0^T ||f(s)||_V^2 ds + 50K \int_0^{T \wedge \tau_N} C_T(1 + 2KTe^{2KT}) + 50KT := C_2(T),
\]

(3.10)

which implies that \(\tau_N \wedge T \rightarrow T \) as \(N \rightarrow \infty\). Letting \(N \rightarrow \infty\) in (3.10) and (3.9), we obtain (3.3) and (3.2).

Define

\[
\tau'_N := \inf \{t \in [0, T] : |u_n(t)|^3 + \int_0^t |u_n(s)||u_n(s)||^2 ds > N
\]

or \(|u_n(t^-)|^3 + \int_0^t |u_n(s)||u_n(s)||^2 ds > N\}.

It follows from the Itô formula that (see, e.g., [1] Eq. (16), p. 65)

\[
|u_n(t)|^3 = |u_n(0)|^3
\]

\[
+ 3 \int_0^t |u_n(s)||-\nu A u_n(s) - B_d(u_n(s)) + f(s, u_n(s)) V ds
\]

\[
+ 3 \int_0^t |u_n(s)||u_n(s), \sigma_n(s, u_n(s), \tau(s))dW_n(s)
\]

\[
+ 3 \int_0^t |u_n(s)||\sigma_n(s, u_n(s), \tau(s))|_{L_Q}^2 ds
\]

\[
+ \int_0^t \int_Z \left(|u_n(s^-) + G_n(s, u_n(s^-), \tau(s), z)|^3
\]

\[
- |u_n(s^-)|^3 \right) \tilde{N}_1(dz, ds)
\]

\[
+ \int \left(|u_n(s) + G_n(s, u_n(s), \tau(s), z)|^3 - |u_n(s)|^3
\]

\[
- 3|u_n(s)||u_n(s), G_n(s, u_n(s), \tau(s), z)||_H \nu_1(dz) ds,
\]

(3.11)
where the last integral is over $[0, t] \times Z$. Taking integration up to $t \wedge \tau_N$ and then expectation in (3.11), we obtain

$$
(3.12) \quad \mathbb{E}|u_n(t \wedge \tau_N)|^3 + 3\nu \mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)||u_n(s)||^2 ds
$$

$$
\leq \mathbb{E}|u_0|^3 + 3\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)||f(s)||_{V'}||u_n(s)|| ds
$$

$$
+ 3\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)||\sigma(s, u_n(s), r(s))|^2 ds
$$

$$
+ \mathbb{E} \int_0^{t \wedge \tau_N} \int_Z \left( |u_n(s) + G_n(s, u_n(s), r(s), z)|^3 - |u_n(s)|^3
$$

$$
- 3|u_n(s)|(u_n(s), G_n(s, u_n(s), r(s), z))_H \right) \nu(dz) ds.
$$

An application of triangle inequality and Hypothesis H3 yields

$$
(3.13) \quad \mathbb{E} \int_0^{t \wedge \tau_N} \left( |u_n(s) + G_n(s, u_n(s), r(s), z)|^3 - |u_n(s)|^3
$$

$$
- 3|u_n(s)|(u_n(s), G_n(s, u_n(s), r(s), z))_H \right) \nu(dz) ds
$$

$$
\leq 10K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)|^3 ds + 6K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)|^2 ds
$$

$$
+ 3K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)| ds + KT.
$$

It follows from the basic Young inequality and the property $| \cdot | \leq || \cdot ||$ that

$$
3||f(s)||_{V'}||u_n(s)|| |u_n(s)|| \leq \frac{1}{\nu^2}||f(s)||_{V'}^3 + 2\nu(||u_n(s)||||u_n(s)||)^{\frac{3}{2}}
$$

$$
(3.14) \quad \leq \frac{1}{\nu^2}||f(s)||_{V'}^3 + 2\nu||u_n(s)|| ||u_n(s)||^2.
$$

Using Hypothesis H1, (3.13), and (3.14) in (3.12), one has

$$
(3.15) \quad \mathbb{E}|u_n(t \wedge \tau_N)|^3 + \nu \mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)||u_n(s)||^2 ds
$$

$$
\leq \mathbb{E}|u_0|^3 + \frac{1}{\nu^2} \mathbb{E} \int_0^{t \wedge \tau_N} ||f(s)||_{V'}^3 ds + 6K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)|^2 ds
$$

$$
+ 4K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)| ds + KT + 11K\mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)|^3 ds.
$$

Notice that $\int_0^{t \wedge \tau_N} |u_n(s)| ds \leq \int_0^t |u_n(s)| ds$ since $t \wedge \tau_N \leq t$. Thus, by the Schwarz inequality, the Jensen inequality (for concave functions), the property that $| \cdot | \leq || \cdot ||$, and (3.2), we have

$$
(3.16) \quad \mathbb{E} \int_0^{t \wedge \tau_N} |u_n(s)| ds \leq \mathbb{E} \int_0^t |u_n(s)| ds
$$

$$
\leq \mathbb{E} \left( \int_0^t |u_n(s)|^2 ds \right)^{\frac{3}{2}} \sqrt{T} \leq \sqrt{T} \mathbb{E} \left( \int_0^t ||u_n(s)||^2 ds \right)^{\frac{3}{2}}
$$;
we also have

\begin{equation}
E \int_0^{t \wedge \tau_N} |u_n(s)|^2 ds \leq E \int_0^t \|u_n(s)\|^2 ds \leq C. \tag{3.17}
\end{equation}

Making use of (3.16) and (3.17) in (3.15), we then use the Gronwall inequality to obtain

\begin{equation}
E |u_n(t \wedge \tau_N')|^3 \leq C(E |u_0|^3, E \int_0^T \|f(s)\|_{V_\nu}^3 ds, \nu, K, T). \tag{3.18}
\end{equation}

Utilizing (3.18) on the last term on the right of (3.15), we conclude

\[
E |u_n(t \wedge \tau_N')|^3 + \nu E \int_0^{t \wedge \tau_N} |u_n(s)||u_n(s)|^2 ds \leq C(E |u_0|^3, E \int_0^T \|f(s)\|_{V_\nu}^3 ds, \nu, K, T).
\]

A simplification of the last two terms in (3.11) gives

\[
\begin{align*}
&\int_0^t \int_Z \left( |u_n(s-)+G_n(s,u_n(s-),r(s),z)|^3 - |u_n(s-)|^3 \right) N_1(dz,ds) \\
&+ 3 \int_0^t \int_Z |u_n(s)||u_n(s),G_n(s,u_n(s),r(s),z)|_H \nu_1(dz)ds.
\end{align*}
\]

Plugging the result above into (3.11), we have

\begin{equation}
|u_n(t)|^3 = |u(0)|^3 \\
+ 3 \int_0^t |u_n(s)||-\nu Au_n(s) - B_k(u_n(s)) + f(s), u_n(s), \nu| ds \\
+ 3 \int_0^t |u_n(s)||u_n(s), \sigma_n(s,u_n(s),r(s))dW_n(s)| \\
+ 3 \int_0^t |u_n(s)||\sigma_n(s,u_n(s),r(s))|_{L_Q}^2 ds \\
+ \int \left( |u_n(s-)+G_n(s,u_n(s-),r(s),z)|^3 - |u_n(s-)|^3 \right) N_1 \\
- 3 \int_0^t \int_Z |u_n(s)||u_n(s),G_n(s,u_n(s),r(s),z)|_H \nu_1(dz)ds,
\end{equation}
where the second last integral is over \([0, t] \times Z\) and \(N_1 = N_1(dz, ds)\). Taking supremum over \(T \land \tau'_N\) and then expectation on (3.19), we have

\[
E \sup_{0 \leq t \leq T \land \tau'_N} |u_n(t)|^3 + 3\nu E \int_{0}^{T \land \tau'_N} |u_n(s)||u_n(s)|^2 ds
\]

\[
\leq E|u(0)|^3 + 3\nu E \int_{0}^{T \land \tau'_N} |u_n(s)|(f(s), u_n(s))_V ds
+ E \sup_{0 \leq t \leq T \land \tau'_N} 3 \int_{0}^{t} |u_n(s)| \langle u_n(s), \sigma(s, u_n(s), t(s))dW_s(s) \rangle
+ 3E \int_{0}^{T \land \tau'_N} |u_n(s)||\sigma_n(s, u_n(s), t(s))|^2_{L_Q} ds
+ E \int_{0}^{T \land \tau'_N} \int_{Z} \left(|u_n(s-)+G_n(s, u_n(s-), t(s), z)|^3 - |u_n(s-)|^3 \right)N_1(dz, ds)
\]

By the Davis inequality, we have

\[
3E \sup_{0 \leq t \leq T \land \tau'_N} \int_{0}^{t} |u_n(s)| \langle u_n(s), \sigma(s, u_n(s), t(s))dW_n(s) \rangle
\]

\[
\leq 3\sqrt{2}E \left\{ \left( \int_{0}^{T \land \tau'_N} \|\sigma(s, u_n(s), t(s))(u_n(s)|u_n(s)|^2_{L_Q} \right)^{\frac{1}{2}} \right\}
\]

\[
\leq 3\sqrt{2}E \left\{ \sup_{0 \leq t \leq T \land \tau'_N} |u_n(t)|^2 \left( \int_{0}^{T \land \tau'_N} \|\sigma(s, u_n(s), t(s))\|^2_{L_Q} ds \right)^{\frac{1}{2}} \right\}
\]

invoking the basic Young inequality and Hypothesis H1 and continuing,

\[
\leq 3\sqrt{2}E \left\{ \frac{2}{3} \sup_{0 \leq t \leq T \land \tau'_N} |u_n(t)|^3 + \frac{1}{3}C_\epsilon \left( \int_{0}^{T \land \tau'_N} \|\sigma(s, u_n(s), t(s))\|^2_{L_Q} ds \right)^{\frac{3}{2}} \right\}
\]

keep simplifying, one reaches

\[
\leq 3\sqrt{2}E \left\{ \frac{2}{3} \sup_{0 \leq t \leq T \land \tau'_N} |u_n(t)|^3 + \frac{1}{3}C_\epsilon \sqrt{T} \int_{0}^{T \land \tau'_N} \|\sigma(s, u_n(s), t(s))\|^3_{L_Q} ds \right\}
\]

\[
\leq 2\sqrt{2}\epsilon E \sup_{0 \leq t \leq T \land \tau'_N} |u_n(s)|^3 + \sqrt{2\epsilon}TKC_\epsilon \int_{0}^{T \land \tau'_N} |u_n(s)|^3 ds + \sqrt{2TK^2}C_\epsilon.
\]

In conclusion, one has

\[
3E \sup_{0 \leq t \leq T \land \tau'_N} \int_{0}^{t} |u_n(s)| \langle u_n(s), \sigma(s, u_n(s), t(s))dW_n(s) \rangle
\]

\[
\leq 2\sqrt{2}\epsilon E \sup_{0 \leq t \leq T \land \tau'_N} |u_n(s)|^3 + \sqrt{2\epsilon}TKC_\epsilon \int_{0}^{T \land \tau'_N} |u_n(s)|^3 ds + \sqrt{2TK^2}C_\epsilon.
\]

(3.21)
An application of triangle inequality and expanding the cubic power yields (with the integral on the left side being over $[0, T \wedge \tau'_N]$ and $N_1 = N_1(dz, ds)$.)

$$
\mathbb{E} \int \left( |u_n(s-) + G_n(s-, u_n(s-), v(s-), z)|^3 - |u_n(s-)|^3 \right) N_1
\leq 3 \mathbb{E} \int_0^{T \wedge \tau'_N} \int_Z |u_n(s-)|^2 |G_n(s-, u_n(s-), v(s-), z)| N_1(dz, ds) \\
+ 3 \mathbb{E} \int_0^{T \wedge \tau'_N} \int_Z |u_n(s-)| |G_n(s-, u_n(s-), v(s-), z)|^2 N_1(dz, ds) \\
+ \mathbb{E} \int_0^{T \wedge \tau'_N} \int_Z |G_n(s, u_n(s), v(s), z)|^3 N_1(dz, ds);
$$

invoking Hypothesis $H3$ and continuing,

$$
\leq 7KE \int_0^{T \wedge \tau'_N} |u_n(s)|^3 ds + 3K(1 + m) \mathbb{E} \int_0^{T \wedge \tau'_N} |u_n(s)|^2 ds \\
+ 3KE \int_0^{T \wedge \tau'_N} |u_n(s)| ds + KT.
$$

Employing (3.21) and (3.22) in (3.20) and then using Hypotheses $H1$ and $H3$ and the basic Young inequality, we have

$$
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau'_N} |u_n(t)|^3 + \nu \mathbb{E} \int_0^{T \wedge \tau'_N} |u_n(s)||u_n(s)|^2 ds \\
\leq \mathbb{E}|u(0)|^3 + \frac{1}{\nu^2} \mathbb{E} \int_0^{T \wedge \tau'_N} |f(s)|^3 ds \\
+ 2\sqrt{\tau} \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau'_N} |u_n(s)|^3 + \sqrt{2\tau} K C \mathbb{E} \int_0^{T \wedge \tau'_N} |u_n(s)|^3 ds \\
+ \sqrt{2} KT \frac{1}{2} C, \\
+ 3KE \int_0^{T \wedge \tau'_N} |u_n(s)| ds + 3KE \int_0^{T \wedge \tau_N} |u_n(s)|^3 ds \\
+ 7KE \int_0^{T \wedge \tau'_N} |u_n(s)|^3 ds + 3KE \int_0^{T \wedge \tau_N} |u_n(s)|^2 ds \\
+ 3KE \int_0^{T \wedge \tau'_N} |u_n(s)| ds + KT \\
+ 3KE \int_0^{T \wedge \tau'_N} |u_n(s)|^2 ds + 3KE \int_0^{T \wedge \tau_N} |u_n(s)|^3 ds.
$$
Choose $\epsilon = \frac{1}{3\sqrt{2}}$. Then $C_\epsilon = 4\sqrt{2}$. The inequality above can be simplified as
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^3 + \nu \mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)||u_n(s)|^2 ds
\leq C_1 \mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)|^3 ds + C_2 \mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)|^2 ds
+ 6K\mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)| ds + (8\sqrt{T} + 1)KT,
\]
where $C_1 = (13 + 8\sqrt{T})K$ and $C_2 = 6K$. Using (3.16), (3.17), and (3.18) in above, we conclude, upon a simplification, that
\[
\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_N} |u_n(t)|^3 + 2\nu \mathbb{E} \int_0^{T \wedge \tau_N} |u_n(s)||u_n(s)|^2 ds
\leq C(\mathbb{E}|u(0)|^3, \mathbb{E} \int_0^T ||f(s)||_r^3, ds, \nu, K, T),
\]
which leads $T \wedge \tau_N \to T$ as $N \to \infty$. Therefore, (3.4) is proved. \hfill \Box

4. The Regularized Equation

The proof of Theorem 1.1 consists of two parts: the existence of a weak solution (Theorem 4.12) and the pathwise uniqueness of weak solutions (Theorem 4.13). Once the weak solution is shown to be pathwise unique, then we apply a well-known result of Yamada and Watanabe [30] to deduce Theorem 1.1.

The argument is started by showing the existence of a weak solution; we remind the reader that the parameter $\epsilon$ that appears in $B_k$ is chosen to be greater than 0 and fixed.

4.1. Existence of the solution to the regularized equation. The existence of the weak solution is by studying the martingale problem posed by equation (1.7). Suppose that $\omega^\dagger = (u, \tau)$ is a solution to equation (1.7). Then it is not hard to see from the Itô formula that

\[
M^\omega(t) := F(t, u(t), \tau(t)) - F(0, u(0), \tau(0)) - \int_0^t \mathcal{L}F(s, u(s), \tau(s)) ds
\]
is a $\mu$-martingale, where $\mu := \mathcal{P} \circ \omega^\dagger^{-1}$ is the distribution of $\omega^\dagger$ and $\mathcal{L}$ is the operator introduced (2.11) with $B_k$ in place of $B$.

Recalling (2.13), we have defined $\Omega^\dagger := \Omega^* \times \mathcal{D}([0, T]; S)$. Now let $\omega = (u, i)$ be a generic element in $\Omega^\dagger$. Substituting $(u, \tau)$ by $(u, i)$ in (4.1), we obtain canonical expression:

\[
M^\omega(t) := F(t, u(t), i(t)) - F(0, u(0), i(0)) - \int_0^t \mathcal{L}F(s, u(s), i(s)) ds.
\]

The aim of this subsection is to identify a measure $\mu$ in the path space $\Omega^\dagger$ under which $M(\cdot)$ in (4.2) is a martingale, and this is called the martingale problem posed by the stochastic Navier-Stokes equation with Markov switching (1.7).

Recalling the definition of path space ($\Omega^\dagger, \tau^\dagger$) from (2.13), we let $B$ denote the Borel $\sigma$-field of the topology $\tau^\dagger$. Define $\mathcal{F}_t := \sigma(\omega(s) : 0 \leq s \leq t, \omega \in \Omega^\dagger)$. Recall that $\mathcal{L}$ is the operator introduced in Section 2 we are in a position to introduce the definition of a solution to a martingale problem.
Definition 4. A probability measure $\mu$ on $(\Omega, B)$ is called a solution of the martingale problem with the initial distribution $\mu_0$ and operator $\mathcal{L}$ if the following hold:

1. The time marginal of $\mu$ at $t = 0$ is $\mu_0$, i.e., $\mu|_{t=0} = \mu_0$.
2. The canonical expression $M(t)$ defined in (4.2) is an $\mathcal{F}_t$-martingale.

Let $X_\omega(\omega) = \omega(t)$ for all $\omega \in \Omega$.  Therefore, in terms of the canonical process, the definition becomes:

Definition 5. A process $X = \{X_t\}$ with path in $(\Omega^t, \tau^t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a solution to the martingale problem for the initial distribution $\mu_0$ and operator $\mathcal{L}$ if the following hold:

1. The distribution of $X_0$ is $\mu_0$.
2. For any $F \in \mathcal{D}(\mathcal{L})$, the process (4.2) is a $\mathcal{F}_t^X$-martingale.

There are several equivalent formulations of a solution to a martingale problem (see, e.g., [22]), and we introduce one of them in the following lemma. The interested reader is referred to [12] Prop. 7.1.2 for more details.

Lemma 4.1. The following statements are equivalent.

1. $X$ is a solution to the martingale problem for the operator $\mathcal{L}$.
2. For all $f \in \mathcal{D}(\mathcal{L})$, $0 \leq t_1 < t_2 < \cdots < t_{n+1}$, $h_1, h_2, \ldots, h_n \in C_b$ and $n \geq 1$, we have

$$
\mathbb{E}\left\{ \left( f(X_{t_{n+1}}) - f(X_{t_n}) - \int_{t_n}^{t_{n+1}} \mathcal{L}f(X_s)ds \right) \prod_{j=1}^{n} h_j(X_{t_j}) \right\} = 0.
$$

Let $\phi(t, i)$ be a real-valued bounded smooth function with compact support (in each variables). For $\rho \in \mathcal{D}(A) \subset V$, $0 \leq s \leq t$, and each generic element $\omega = (u, i) \in \Omega^t$, define

$$(4.3) M^\phi(t) - M^\phi(s)
:= \phi((u(t), \rho)_V, i(t)) - \phi((u(s), \rho)_V, i(s))
- \int_s^t \sum_{j=1}^m \gamma_{i(j-1)}(\rho) + \phi(\langle u(r), \rho \rangle_V, j) dr
- \int_s^t \left( \phi'(\langle u(r), \rho \rangle_V, i(r)) \left( -\nu A u(r) - B_k(u(r)) + f(r, \rho)_V \right) dr
- \frac{1}{2} \int_s^t \phi''(\langle u(r), \rho \rangle_V, i(r)) \left( \rho, \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r)) \rho \right)_H dr
- \int_s^t \left( \phi((u(r) + G(r, u(r), i(r), z), \rho)_V, i(r)) - \phi((u(r), \rho)_V, i(r))
- \phi'(u(r), \rho)_V, i(r)) \cdot \langle G(r, u(r), i(r), z), \rho \rangle_V \right) \nu(dz) dr,
$$

where the last integral is over $[s, t] \times Z$.

According to Lemma 4.1, to show $M^\phi(t)$ is a solution to the martingale problem, it suffices to find a Radon measure $\mu$ such that

$$
\mathbb{E}_\mu\left( \prod_{j=1}^m \psi_j(s_j)(M^\phi(t) - M^\phi(s)) \right) = 0, \ \forall s < s_1 < \cdots < s_m < t,
$$

where $\psi_j \in C_b(\Omega)$ and $\mathcal{F}_s$-measurable.
Define the projection of $M^\phi(t)$ as follows:

\begin{align}
(4.4) \quad M_n^\phi(t) = M_n^\phi(s) \\
&:= \phi(\langle u(t), \rho \rangle_V, i(t)) - \phi(\langle u(s), \rho \rangle_V, i(s)) \\
&\quad - \int_s^t \sum_{j=1}^m \gamma_j(r-\tau) \phi(\langle u(r), \rho \rangle_V, j) dr \\
&\quad - \int_s^t \left( \phi'\left(\langle u(r), \rho \rangle_V, i(r)\right) \cdot \left( -\nu A_n u(r) - B_{k_n}^n (u(r)) + f(r, \rho) \right) \right) dr \\
&\quad - \frac{1}{2} \int_s^t \left( \phi''\left(\langle u(r), \rho \rangle_V, i(r)\right) \cdot \left( \sigma_n(r, u(r), i(r)) \right)_{H}\right) dr \\
&\quad - \int_s^t \left( \phi(\langle u(r) + G_n(r), u(r), i(r), z \rangle, \rho) V, i(r) \right) - \phi(\langle u(r), \rho \rangle_V, i(r)) \\
&\quad \quad - \phi'\left(\langle u(r), \rho \rangle_V, i(r)\right)(G_n(r, u(r), i(r), z), \rho) V, i(r)) V(dz) dr,
\end{align}

where second last integral is over $[s, t]$, the last integral is over $[s, t] \times Z$, $A_n = \Pi_n A$, $B_{k_n}^n = \Pi_n B_{k_n}$, $\sigma_n = \Pi_n \sigma$, $G_n = \Pi_n G$, and $\omega = (u, i) \in \Omega^t$ is a generic element.

Let $(u_n, t)$ be the solution to equation (3.1) and denote by $\mu_n$ the (joint) distribution of $(u_n, t)$. Then it follows from the (finite dimensional) Itô formula that $M_n^\phi(t)$ is a $\mu_n$-martingale, therefore, for all $n$, $E^{\mu_n} \prod_{j=1}^m \psi_j(s_j) M_n^\phi(t) = 0$, for all $s < s_1 < \cdots < s_m < t$ and for $\psi_j \in C_b(\Omega)$ and $F_\mathcal{F}$-measurable. Hence,

$$
\lim_{n \to \infty} E^{\mu_n} \prod_{j=1}^m \psi_j(s_j) M_n^\phi(t) = 0, \ \forall s < s_1 < \cdots < s_m < t,
$$

for $\psi_j \in C_b(\Omega)$ and $F_\mathcal{F}$-measurable.

If we show that

- **M1.** there exists a measure $\mu$ such that $\mu_n$ weakly converges to $\mu$,
- **M2.** $\lim_{n \to \infty} M_n^\phi(t) = M^\phi(t)$, and
- **M3.** $\lim_{n \to \infty} E^{\mu_n} M_n^\phi(t) = E^{\mu} M^\phi(t),$

then it follows that $M^\phi(t)$ is a $\mu$-martingale.

Now we prove M1. Recall that $u_n$ is the solution to (3.1) for each $n$.

**Lemma 4.2.** The sequence $\{u_n\}_{n=1}^\infty$ forms a relative compact set in the Skorohod space $\mathcal{D}([0, T]; V')$.

**Proof.** It is clear that $\{u_n\}$ is a subset of $\mathcal{D}([0, T]; V')$.

Let $N > 0$. By the Markov inequality, the property that $\| \cdot \|_{V'} \leq | \cdot |$, and (3.2), we have

$$
\mathcal{P}\left(\|u_n(t)\|_{V'} > N\right) \leq \frac{1}{N^2} E\|u_n(t)\|^2_{V'} \leq \frac{1}{N^2} E\|u_n(t)\|^2 \leq \frac{C}{N^2}.
$$

Therefore, $\lim_{N \to \infty} \limsup_n \mathcal{P}\left(\|u_n(t)\|_{V'} > N\right) = 0$.

Let $(T_n, \delta_n)$ be a sequence, where $T_n$ is a stopping time with $T_n + \delta_n \leq T$ and $\delta_n > 0$ with $\delta_n \to 0$. For each $\epsilon > 0$, the Chebyshev inequality implies

$$
\mathcal{P}\left(\|u_n(T_n + \delta_n) - u_n(T_n)\|_{V'} > \epsilon\right) \leq \frac{1}{\epsilon^2} E\|u_n(T_n + \delta_n) - u_n(T_n)\|^2_{V'} \leq \frac{1}{\epsilon^2} E\|u_n(T_n + \delta_n) - u_n(T_n)\|^2.
$$
It follows from the Itô formula and the Gronwall inequality that
\[ \mathbb{E}|u_n(T_n + \delta_n) - u_n(T_n)|^2 \leq \left( \frac{1}{\nu} \mathbb{E} \int_0^{\delta_n} |f(s)|^2 V ds + 2K\delta_n \right) e^{2K\delta_n}, \]
which tends to 0, as \( n \to \infty \). Therefore, \( \|u_n(T_n + \delta_n) - u_n(T_n)\|_{V'} \to 0 \) in probability as \( n \to \infty \). By Aldous’ criterion, we conclude that \( \{u_n\} \) is tight in \( \mathcal{D}([0,T]; V') \) and thus relative compact in it.

We can have an even stronger convergence which is proved in the following proposition.

**Proposition 4.3.** The sequence \( \{u_n\}_{n=1}^\infty \) forms a relative compact set in \( L^2(0,T; H) \).

**Proof.** It follows from (3.3) that \( \{u_n\} \) is bounded in \( L^2(0,T; V) \); also, we have that
\[ \mathbb{E} \int_0^T |u_n(t)|^2 dt \leq \mathbb{E} \int_0^T |u_n(t)|^2 dt \leq C, \]
which implies \( \{u_n\} \subset L^2(0,T; H) \cap \mathcal{D}([0,T]; V') \). In addition, by Lemma 4.2 \( \{u_n\} \) is relatively compact in the space \( \mathcal{D}([0,T]; V') \); the proposition follows from Lemma 2.3. \( \square \)

Recalling Definition 1 and \( u_n \) being the solution to (3.1), we deduce from a priori estimates and Banach-Alaoglu theorem that \( \{u_n\} \) is relatively compact in \( (\Omega_2, \tau_2) \) and \( (\Omega_3, \tau_3) \). In addition, Lemma 4.2 and 4.3 imply that \( \{u_n\} \) is compact in \( (\Omega_1, \tau_1) \) and \( (\Omega_4, \tau_4) \), respectively. Therefore, by the Prohorov theorem, the induced distribution \( \{\mu_n^*\} \) is tight on each space \( (\Omega_j, \tau_j) \) for \( j = 1, 2, 3, 4 \). Hence, by (2) in Definition 1 \( \{\mu_n^*\} \) is tight on \( (\Omega^*, \tau) \).

Let \( \mu_n \) be the joint distribution of \( (u_n, \mathfrak{r}) \). Then \( \{\mu_n\}_{n=1}^\infty \) is tight on the space \( (\Omega^1, \tau^1) \), hence, there exists a subsequence \( \{\mu_{n\ell}\}_{\ell=1}^\infty \) and a measure \( \mu \) such that \( \mu_{n\ell} \Rightarrow \mu \).

Next, we consider M2. Recall from Section 2 that \( H_n \) is the span of \( \{e_j\}_{j=1}^n \) and \( \Pi_n \) is a projection operator from \( H \) onto \( H_n \). Denote by \( \{n_\ell\}_{\ell=1}^\infty \) the indices such that \( \mu_{n_{\ell\ell}} \Rightarrow \mu \).

**Lemma 4.4.** For each \( \rho \in D(A), \Pi_{n\ell}\rho \to \rho \text{ in } V, \text{ as } \ell \to \infty. \)

**Proof.** Defining \( f_j := \frac{e_j}{\sqrt{\lambda_j}}, \) one sees \( \|f_j\| = \frac{\|e_j\|^2}{\lambda_j} = \frac{\lambda_j}{\lambda_j} = 1 \). This implies that \( \{f_j\} \) is a complete orthonormal basis in \( V \). Thus,
\[ \rho = \sum_{j=1}^\infty (\langle \rho, f_j \rangle_V f_j) ; \quad \Pi_{n\ell}\rho := \rho_{n\ell} = \sum_{j=1}^{n\ell} (\langle \rho, f_j \rangle_V f_j). \]
As a consequence, \( \|\rho - \Pi_{n\ell}\rho\| = \sum_{j=n\ell+1}^\infty (\langle \rho, f_j \rangle_V f_j) \to 0, \) as \( \ell \) tends to infinity. \( \square \)

**Lemma 4.5.** For each \( \rho \in D(A), \) we have
\[ \lim_{\ell \to \infty} \int_s^t \phi((\langle u(r), \rho \rangle_V, i(r))(-\nu A_{n\ell} u(r), \rho)_V dr = \int_s^t \phi((\langle u(r), \rho \rangle_V, i(r))(-\nu A u(r), \rho)_V dr, \]

**Proof.** A direct computation gives, for almost all \( r \in [t, s], \)
\[ \langle -\nu A_{n\ell} u(r), \rho \rangle_V = -\nu \langle A u(r), \rho_{n\ell} \rangle \to -\nu \langle A u(r), \rho \rangle \]
as \( \ell \to \infty, \) by Lemma 4.4. In addition,
\[ \left| \langle -\nu A_{n\ell} u(r), \rho \rangle_V \right| = \left| -\nu \langle A u(r), \rho_{n\ell} \rangle \right| \leq \nu \|\rho\|_V \|u(r)\|. \]
Notice that \( u \in \Omega^*, \) therefore, \( u \in L^2(0,T; V) \subset L(0,T; V) \). Hence, the lemma follows from the Lebesgue Dominated Convergence Theorem. \( \square \)

\[ \mu_{n\ell}^* := \mathcal{P} \circ u_{n\ell}^{-1}. \]
Lemma 4.6. For each \( \rho \in D(A) \), we have

\[
\lim_{\ell \to \infty} \int_{s}^{t} \phi'(\langle u(r), \rho \rangle V, i(r)) (B_{k}^{*\ell}(u(r)), \rho) V \, dr = \int_{s}^{t} \phi'(\langle u(r), \rho \rangle V, i(r)) (B_{k}(u(r)), \rho) V \, dr
\]

Proof. A similar argument as in Lemma 4.5 shows that

\[
\langle B_{k}^{*\ell}(u(r)), \rho \rangle V \to \langle B_{k}(u(r)), \rho \rangle V
\]
as \( \ell \to \infty \) for almost all \( r \in [s, t] \). In addition

\[
|\langle B_{k}^{*\ell}(u(r)), \rho \rangle V| = |\langle B_{k}(u(r)), \rho_{n} \rangle V| \leq \|u(r)\|\|\rho_{n}\|
\]
by (2.8). Since \( u \in \Omega^{*} \), \( u \in L^{2}(0, T; V) \). Thus,

\[
|\phi'(\langle u(r), \rho \rangle V, i(r)) (B_{k}(u(r)), \rho) V| \leq \|\phi\|_{\infty} \|\rho\|\|u(r)\|^{2},
\]
which is an \( L^{1} \)-function. Therefore, the lemma follows from the Lebesgue Dominated Convergence Theorem. \( \Box \)

Lemma 4.7. For each \( \rho \in D(A) \), as \( \ell \to \infty \),

\[
\int_{s}^{t} \phi''(\langle u(r), \rho \rangle V, i(r)) (\rho, \sigma_{n_{i}}(r, u(r), i(r))Q\sigma_{n_{i}}^{*}(r, u(r), i(r))\rho) H \, dr
\]
converges to

\[
\int_{s}^{t} \phi''(\langle u(r), \rho \rangle V, i(r)) (\rho, \sigma(r, u(r), i(r))Q\sigma^{*}(r, u(r), i(r))\rho) H \, dr.
\]

Proof. Throughout the proof, we write \( \sigma(r) = \sigma(r, u(r), i(r)) \) and \( (\cdot, \cdot) = (\cdot, \cdot)_{H} \). A direct computation shows that \( (\rho, \sigma_{n_{i}}(r)Q\sigma_{n_{i}}^{*}(r)\rho) \) equals to \( (\sigma(r)Q\sigma^{*}(r)\rho_{n_{i}}, \rho_{n_{i}}) \), therefore,

\[
(\rho, \sigma_{n_{i}}(r)Q\sigma_{n_{i}}^{*}(r)\rho) - (\rho, \sigma(r)Q\sigma^{*}(r)\rho) = (\rho_{n_{i}}, \sigma(r)Q\sigma^{*}(r)\rho_{n_{i}}) - (\rho, \sigma(r)Q\sigma^{*}(r)\rho),
\]
which implies

\[
|((\rho, \sigma_{n_{i}}(r)Q\sigma_{n_{i}}^{*}(r)\rho) - (\rho, \sigma(r)Q\sigma^{*}(r)\rho)|
\]
\[
\leq |(\rho_{n_{i}}, \sigma(r)Q\sigma^{*}(r)\rho_{n_{i}}) - \sigma(r)Q\sigma^{*}(r)\rho| + |(\rho_{n_{i}} - \rho, \sigma(r)Q\sigma^{*}(r)\rho)|
\]
\[
\leq |\rho_{n_{i}}|\|\sigma(r)Q\sigma^{*}(r)\rho_{n_{i}} - \rho\| + |\rho_{n_{i}} - \rho|\|\sigma(r)Q\sigma^{*}(r)\rho\|
\]
\[
= 2|\rho|\|\rho_{n_{i}} - \rho\|\|\sigma(r)\|_{L_{Q}} \leq 2|\rho|\cdot \|\rho_{n_{i}} - \rho\|\cdot \|\sigma(r)\|_{L_{Q}}
\]

Thus, by Lemma 4.4, \( (\rho, \sigma_{n_{i}}(r)Q\sigma_{n_{i}}^{*}(r)\rho) \to (\rho, \sigma(r)Q\sigma^{*}(r)\rho) \), as \( \ell \) approaches infinity, for all \( r \in [s, t] \). In addition,

\[
|\phi''(\langle u(r), \rho \rangle V, i(r)) (\rho, \sigma_{n_{i}}(r, u(r), i(r))Q\sigma_{n_{i}}^{*}(r, u(r), i(r))\rho)|
\]
\[
\leq \|\phi''\|_{\infty}\|\rho_{n_{i}}\|\|\sigma(r)\|_{L_{Q}} \leq \|\phi''\|_{\infty} \cdot \|\rho\| \cdot \|\sigma(r, u(r), i(r))\|_{L_{Q}}.
\]

Consider

\[
\int_{0}^{T} \|\sigma(r, u(r), i(r))\|_{L_{Q}} \, dr \leq \sqrt{T} \left( \int_{0}^{T} \|\sigma(r, u(r), i(r))\|_{L_{Q}}^{2} \, dr \right)^{1/2}.
\]

Recall that \( u \in \Omega^{*} \), therefore, \( u \in L^{2}(0, T; H) \); the Hypothesis H1 implies

\[
\int_{0}^{T} \|\sigma(r, u(r), i(r))\|_{L_{Q}}^{2} \, dr \leq \int_{0}^{T} K(1 + |u(r)|^{2} + i^{2}) \, dr < C
\]
for a constant \( C \). Therefore, we conclude that the function

\[
|\phi''(\langle u(r), \rho \rangle V, i(r)) (\rho, \sigma_{n_{i}}(r, u(r), i(r))Q\sigma_{n_{i}}^{*}(r, u(r), i(r))\rho)|
\]
is bounded by an \( L^1 \)-function, hence, the lemma follows from the Lebesgue Dominated Convergence Theorem.

\[ \Box \]

**Lemma 4.8.** For each \( \rho \in \mathcal{D}(\mathcal{A}) \), as \( \ell \to \infty \)

\[
\int_s^t \int_Z \left( \phi (\langle u(r) + G_n(r, u(r), i(r), z, \rho \rangle \cdot V, i(r) \rangle - \phi (\langle u(r), \rho \rangle \cdot V, i(r) \rangle \right)
- \phi' (\langle u(r), \rho \rangle \cdot V, i(r) \rangle) \cdot (G_n(r, u(r), i(r), z, \rho) \cdot V) \right) \nu_1 (dz) dr
\]

converges to

\[
\int_s^t \int_Z \left( \phi (\langle u(r) + G(r, u(r), i(r), z, \rho) \rangle \cdot V, i(r) \rangle - \phi (\langle u(r), \rho \rangle \cdot V, i(r) \rangle \right)
- \phi' (\langle u(r), \rho \rangle \cdot V, i(r) \rangle) \cdot (G(r, u(r), i(r), z, \rho) \cdot V) \right) \nu_1 (dz) dr.
\]

**Proof.** Throughout this proof, we write \( G(r, z) = G(r, u(r), i(r), z) \), for all \( r \in [s,t] \). Therefore, the convergence of the integrand is shown. Next we argue that the integrand is bounded by an \( L^1 \)-function, and thus the lemma follows from the Lebesgue Dominated Convergence Theorem.

For a fixed \( \ell \), writing \( a = \langle u, \rho \rangle \) and \( b = \langle u + G_n(r, z, \rho) \rangle \), we deduce from the mean value theorem that \( \phi (\langle u + G_n(r, z, \rho) \rangle, i) - \phi (\langle u, \rho \rangle, i) = \phi' (c, i) \langle G_n(r, z, \rho) \rangle \), where \( c \in (a, b) \). Therefore,

\[
\left| \phi (\langle u + G_n(r, z, \rho) \rangle, i) - \phi (\langle u, \rho \rangle, i) \right| \leq \| \phi' \|_{\infty} \rho_n \| G(r, z) \|
\]

which implies

\[
\left| \left( \phi (\langle u + G_n(r, z, \rho) \rangle, i) - \phi (\langle u, \rho \rangle, i) \right) - \phi' (\langle u, \rho \rangle, i) \cdot (G_n(r, z, \rho)) \right|
\leq \left| \phi (\langle u + G_n(r, z, \rho) \rangle, i) - \phi (\langle u, \rho \rangle, i) \right| + \left| \phi' (\langle u, \rho \rangle, i) \cdot (G_n(r, z, \rho)) \right|
\leq 2 \| \phi' \|_{\infty} \rho \| G(r, z) \|
\]

By Hypothesis H3, \( \int_Z |G(r, z)| \nu (dz) \) is an \( L^1 \)-function, therefore, the proof is complete.

\[ \Box \]

In light of Lemmata 4.4 to 4.8, M2 has been proved. Moreover, as shown in the proofs of Lemmata 4.4 to 4.8, the expectation and limit is exchangeable, i.e., \( \lim_{n \to \infty} \mathbb{E} M^n_\rho (t) = \mathbb{E} \lim_{n \to \infty} M^n_\rho (t) \).

Lastly, we consider M3. Clearly, if the assumption of Lemma 2.2 is fulfilled, then M3 is obtained. So far, we have a sequence of measures \( \{ \mu_n \}_{n=1}^\infty \) and there exists a measure \( \mu \) such that \( \mu_n \Rightarrow \mu \) as \( \ell \to \infty \). Therefore, it remains to prove that

1. \( M^\rho (t) \) is continuous on \( (\Omega^I, \tau^I) \), and
2. for some \( \delta > 0 \), \( \sup_n \mathbb{E}^\mu \left( |M^\rho (t)|^{1+\delta} \right) < C \), where \( C \) is a constant.

We begin the proof of continuity of \( M^\rho (t) \) with the following auxiliary lemma.

**Lemma 4.9.** Let \( \{ u_n \}_{n=1}^\infty \) and \( u \) be members of \( (\Omega^*, \tau) \) with \( u_n \to u \) as \( n \to \infty \) in \( \tau \)-topology. For almost all \( t \in [0,T] \), \( k = 0, 1, 2 \), and each \( i \in S \), we have

\[
\frac{d^k}{dx^k} \phi (\langle u_n(t), \rho \rangle \cdot V, i) \to \frac{d^k}{dx^k} \phi (\langle u(t), \rho \rangle \cdot V, i),
\]

as \( n \to \infty \).
Proof. Denote \( C(u) := \{ t \in [0, T]; P(u(t) = u(t-)) = 1 \} \). It is known that the complement of \( C(u) \) is at most countable (see, e.g., [4]). Therefore, for almost all \( t \in [0, T] \), one has \( u_n(t) \to u(t) \), as \( n \to \infty \). This further implies that \( \langle u_n(t), \rho \rangle_V \to \langle u(t), \rho \rangle_V \), as \( n \to \infty \) for any \( \rho \in \mathcal{D}(A) \). Therefore, the lemma follows from the smoothness of the function \( \phi \).

\[ \square \]

Lemma 4.10. \( M^\phi(t) \) is continuous in the \((\tau^t)\)-topology.

Proof. It suffices to prove that \( M^\phi(t) \) is continuous in the \( \tau \)-topology since there is no convergence issue in \( r \).

Let \( \{ u_n \} \) and \( u \) be members of \((\Omega^*, \tau)\) with \( u_n \to u \) as \( n \to \infty \) in \( \tau \)-topology. Let \( M^\phi(u_n(t)) \) be the function where \( u_n \) is in place of \( u \) in (4.3). Given \( u_n \to u \). We need to show that \( \lim_{n \to \infty} M^\phi(u_n(t)) = M^\phi(u(t)) \), and we prove it by taking the term-by-term limit.

The first three terms follows from Lemma 4.9 and the Bounded Convergence Theorem.

From now on, we write \( u = u(r), u_n = u_n(r) \), \( i = i(r) \), \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H \), and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V \).

For the \( A \) term, it is not hard to see that \( \langle Au_n, \rho \rangle \) equals to \( \langle u_n, A\rho \rangle \) which converges to \( \langle u, A\rho \rangle \), as \( n \to \infty \), for almost all \( r \in [s, t] \) by Lemma 4.9. Consider \( \|\langle Au_n, \rho \rangle\|^2 \leq \|u_n\|^2\|\rho\|^2_{V'} \), and \( \int_s^t \|u_n\|^2\|\rho\|^2_{V'} dr \leq \|\rho\|^2_{V'} \int_s^t \|u_n\|^2 dr < C \) for all \( n \) and a constant \( C \) since \( u_n \to u \) in \( \tau \)-topology and thus in \( \tau_2 \). This implies that \( \sup_n \int_s^t \|\langle Au_n, \rho \rangle\|^2 dr < \infty \) so that \( \{\langle Au_n, \rho \rangle\} \) is uniformly integrable, and hence

\[
\lim_{n \to \infty} \int_s^t \langle -\nu Au_n, \rho \rangle dr = \int_s^t \langle -\nu Au, \rho \rangle dr.
\]

For the \( B_k \) term, using the definition of \( B_{k, r} \), we have

\[
|\langle B_{k, r}(u_n), \rho \rangle - \langle B_{k, r}(u), \rho \rangle| \leq |b(k, u_n, u_n - u, \rho)| + |b(k, u_n - u, u, \rho)|,
\]

which together with (2.7) further imply

\[
|\langle B_{k, r}(u_n), \rho \rangle - \langle B_{k, r}(u), \rho \rangle| \leq 2\|\rho\|\|u_n\||\|u_n - u\|^{\frac{1}{2}}|u_n - u|^{\frac{1}{2}}.
\]

Therefore, the Schwarz inequality implies (with \( C = 2\|\rho\|\))

\[
\int_0^T |\langle B_{k, r}(u_n), \rho \rangle - \langle B_{k, r}(u), \rho \rangle| dr \leq C\left(\int_0^T \|u_n\|^2 dr\right)^{\frac{1}{2}} \left(\int_0^T \|u_n(r) - u(r)\||u_n - u| dr\right)^{\frac{1}{2}} \leq C\left(\int_0^T \|u_n\|^2 dr\right)^{\frac{1}{2}} \left(\int_0^T \|u_n\||u_n - u| + \int_0^T \|u\||u_n - u| dr\right)^{\frac{1}{2}} \leq C\left(\int_0^T \|u_n\|^2 dr\right)^{\frac{1}{2}} \cdot \left\{ \left(\int_0^T \|u_n\|^2 dr\right)^{\frac{1}{2}} \left(\int_0^T \|u_n - u|^2 dr\right)^{\frac{1}{2}} + \left(\int_0^T \|u\|^2 dr\right)^{\frac{1}{2}} \left(\int_0^T \|u_n - u|^2 dr\right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}.
\]

Since \( u_n \) and \( u \) are members of \( \Omega^* \), the \( L^2(0, T; V) \)-norms are finite. In addition, \( u_n \to u \) in \( \tau \)-topology implies that \( u_n \to u \) in \( \tau_4 \) (the strong topology in \( L^2(0, T; H) \)). Hence, we
conclude that
\[ \lim_{n \to \infty} \int_0^T \left| \langle B_{k_i}(u_n), \rho \rangle - \langle \rho, B_{k_i}(u) \rangle \right| dr = 0, \]
which implies
\[ \lim_{n \to \infty} \int_s^t \langle B_{k_i}(u_n), \rho \rangle dr = \int_s^t \langle B_{k_i}(u), \rho \rangle dr. \]

For the term representing the continuous noise, consider
\[
\left| (\rho, \sigma(r, u_n, i)Q\sigma^*(r, u_n, i)\rho) - (\rho, \sigma(r, u, i)Q\sigma^*(r, u, i)\rho) \right| \\
\leq |\rho| \cdot \left\{ \left| \sigma(r, u_n, i)Q\sigma^*(r, u_n, i)\rho - \sigma(r, u, i)Q\sigma^*(r, u, i)\rho \right| \right\} \\
\leq |\rho|^2 \left\{ \left| \sigma(r, u_n, i)Q\sigma^*(r, u_n, i) - \sigma(r, u, i)Q\sigma^*(r, u, i) \right| \right\}.
\]

Recalling the definition of \( L^Q \)-norm, we see that
\[
\left| \sigma(r, u_n, i)Q\sigma^*(r, u_n, i) - \sigma(r, u, i)Q\sigma^*(r, u, i) \right| = \| \sigma(r, u_n, i) - \sigma(r, u, i) \|_{L^Q}.
\]
Therefore, by Hypothesis \( \text{H2} \), we have
\[
\int_0^T \left| (\rho, \sigma(r, u_n, i)Q\sigma^*(r, u_n, i)\rho) - (\rho, \sigma(r, u, i)Q\sigma^*(r, u, i)\rho) \right|^2 dr \\
\leq |\rho|^4 \int_0^T \| \sigma(r, u_n, i) - \sigma(r, u, i) \|_{L^Q}^2 dr \leq L\|\rho\|^4 \int_0^T |u_n - u|^2 dr,
\]
which approaches 0 as \( n \to \infty \) since \( u_n \to u \) in \( \tau \) means that \( u_n \to u \) in \( \tau_4 \) (the strong topology in \( L^2(0, T; H) \)). Thus, we have
\[
\lim_{n \to \infty} \int_s^t (\rho, \sigma(r, u_n, i)Q\sigma^*(r, u_n, i)\rho) = \int_s^t (\rho, \sigma(r, u, i)Q\sigma^*(r, u, i)\rho).
\]

For the jump noise term, notice that \( G \) is continuous in all of its components, therefore, \( \lim_{n \to \infty} G(r, u_n, i, z) = G(r, u, i, z) \) for almost all \( r \in [s, t] \) and all fixed \( z \). This implies that, by Lemma 4.9,
\[
\phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) = \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle
\]
converges to
\[
\phi(\langle u + G(r, u, i, z), \rho \rangle, i) - \phi(\langle u, \rho \rangle, i) - \phi'(\langle u, \rho \rangle, i) \cdot \langle G(r, u, i, z), \rho \rangle
\]
almost surely in \( [s, t] \times Z \). Writing \( a_n = \langle u_n, \rho \rangle \) and \( b_n = \langle u_n + G(r, u_n, i, z) \rangle \), one infers from the Mean Value Theorem that
\[
\phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) = \phi'(c_n) \langle G(r, u_n, i, z), \rho \rangle,
\]
where \( \phi'(c_n) = \phi'(c_n, i) \) and \( c_n \in (a_n, b_n) \). Therefore,
\[
\phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) - \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle \\
\leq \phi'(c_n) \langle G(r, u_n, i, z), \rho \rangle + \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle \\
= \phi'(c_n) \langle G(r, u_n, i, z), \rho \rangle + \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle \leq 2\|\phi'\|\infty \|\rho\|\|G(r, u_n, i, z)\|,
\]

24
which implies
\[
\int_0^T \int_Z \left| \phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) \right|^2 \nu_1(\text{d}z) \text{d}r \\
- \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle \right| ^2 \nu_1(\text{d}z) \text{d}r \leq 4\|\rho\|^2 \int_0^T \int_Z |G(r, u_n, i, z)|^2 \nu_1(\text{d}z) \text{d}r \leq C \int_0^T \left( 1 + |u_n|^2 \right) \text{d}r,
\]
where \( C = 4K\|\rho\|^2 \) and the last inequality follows from Hypothesis H3. Since \( u_n \to u \) in \( \tau \)-topology, \( u_n \to u \) in \( \tau_a \), which implies that \( \sup_n \int_0^T |u_n|^2 \text{d}r < C \) for a constant \( C \). Therefore,
\[
\sup_n \int_0^T \int_Z \left| \phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) \right|^2 \nu_1(\text{d}z) \text{d}r < C.
\]
Hence, we conclude, as \( n \to \infty \),
\[
\int_0^T \int_Z \left( \phi(\langle u_n + G(r, u_n, i, z), \rho \rangle, i) - \phi(\langle u_n, \rho \rangle, i) \right) \nu_1(\text{d}z) \text{d}r \\
- \phi'(\langle u_n, \rho \rangle, i) \cdot \langle G(r, u_n, i, z), \rho \rangle \nu_1(\text{d}z) \text{d}r
\]
converges to
\[
\int_s^t \int_Z \left( \phi(\langle u + G(r, u, i, z), \rho \rangle, i) - \phi(\langle u(r), \rho \rangle, i) \right) \nu_1(\text{d}z) \text{d}r \\
- \phi'(\langle u, \rho \rangle, i) \cdot \langle G(r, u, i, z), \rho \rangle \nu_1(\text{d}z) \text{d}r.
\]
Herein, the proof is complete. \( \square \)

The following lemma the final piece of the required argument. It is the only place where we require \( \mathbb{E}|u_0|^3 < 0 \) and \( f \in L^3(0, T; V') \).

**Lemma 4.11.** Suppose that the Hypotheses H is fulfilled, \( \mathbb{E}|u_0|^3 < \infty \), and \( f \in L^3(0, T; V') \). There exist some \( \delta > 0 \) such that
\[
\sup_{\ell} \mathbb{E}^{\mu_{\nu, \ell}} \left[ |M^\phi|^{1+\delta} \right] \leq C,
\]
where \( C \) is an appropriate constant.

**Proof.** Recalling from (4.3) the definition of \( M^\phi \), we employ inequality \((\sum_{i=1}^n a_i)^p \leq 5^{p-1} \sum_{i=1}^5 a_i^p\) and the Mean Value Theorem (on \( G \)) to deduce that \( |M^\phi(t)|^{1+\delta} \) is less than or equal to
\[
5^\delta |\phi(\langle u(t), \rho \rangle, \nu, i(t))|^{1+\delta} + 5^\delta |\phi(\langle u(s) \rangle, \nu, i(s))|^{1+\delta} \\
+ 5^\delta \|\phi'\|_\infty \int_s^t |\nu A u(r) + B_{k_i}(u(r)) + f(r), \rho \nu|^1 \text{d}r \\
+ 5^\delta \|\phi''\|_\infty \int_s^t |(\rho, \sigma(r, u(r), i(r))Q \sigma^*(r, u(r), i(r)))_H|^1 \text{d}r \\
+ 5^\delta \int_s^t \|\phi'\|_\infty \int_Z |G(r, u(r), i(r), z), \rho \nu(n)\|_V^1 \text{d}r.
\]
where $\phi$ is bounded smooth function.

For the $A$ term,
\[
\int_s^t |\nu(Au(r),\rho)V|^{1+\delta} \, dr \leq \nu^{1+\delta} \int_s^t (|u(r)||A\rho||V|)^{1+\delta} \, dr \leq C_1 \int_s^t |u(r)||V|^{1+\delta} \, dr,
\]
where $C_1 = \nu^{1+\delta}|A\rho||V|$. This implies that
\[
\mathbb{E}^{\mu_t} \int_s^t |\nu(Au(r),\rho)V|^{1+\delta} \, dr \leq C_1 \mathbb{E}^{\mu_t} \int_s^t |u(r)||V|^{1+\delta} \, dr = C_1 \mathbb{E} \int_s^t |u_n(t)||V|^{1+\delta} \, dr,
\]
hence, $\sup_t \mathbb{E}^{\mu_t} \int_s^t |\nu(Au(r),\rho)V|^{1+\delta} \, dr \leq C_A$ if $\delta < 1$.

For the nonlinear term, (2.7) and H"older inequality imply
\[
\mathbb{E}^{\mu_t} \left\{ |\int_s^t \langle B_k(u(r)),\rho \rangle V \rangle^{1+\delta} \right\} \leq \mathbb{E}^{\mu_t} \left\{ \left( \int_s^t |u(r)| |\nu| |V| \, dr \right)^{1+\delta} \right\} \leq \mathbb{E}^{\mu_t} \left\{ \left( \int_s^t |u(r)| \, dr \right) \mathbb{E}^{\mu_t} \left\{ \left( \int_s^t |u(r)|^{1+\delta} \, dr \right)^{\frac{1}{1+\delta}} \right\} \right\}.
\]
where $\frac{1}{p} + \frac{1}{q} = 1$. Choosing $q$ such that $(1+\delta)q = 2$, we have
\[
\mathbb{E}^{\mu_t} \left\{ |\int_s^t \langle B_k(u(r)),\rho \rangle V \rangle^{1+\delta} \right\} \leq \left\{ \left( \int_s^t |u(r)|^{2(1+\delta)} \, dr \right)^{1+\delta} \right\}^{\frac{1}{2}} \left\{ \int_s^t |u(r)|^{1+\delta} \, dr \right\}^{\frac{1}{1+\delta}}.
\]
Taking $\delta = \frac{1}{3}$, we have $2(1+\delta) = 3$. The first expectation on the right of (4.5) will have a uniform bound by (3.1) if we further assume that $\mathbb{E}|u_0|^3 < \infty$. The boundedness of $\mathbb{E} \int_s^t |u_n(t)|^2 \, dr$ is followed from (3.2). Therefore, (4.5) implies that, if $\delta \leq \frac{1}{2}$,
\[
\sup_t \mathbb{E}^{\mu_t} \left\{ |\int_s^t \langle B_k(u(r)),\rho \rangle V \rangle^{1+\delta} \right\} \leq C_B.
\]

For martingale terms, we have
\[
\int_s^t \left| \langle \rho, \sigma(r,u(r),i(r))Q\sigma^t(r,u(r),i(r))\rangle \right|^{1+\delta} \, dr \leq \left| \rho \right|^{2(1+\delta)} \int_s^t \left| \sigma(r,u(r),i(r)) \right|^2 \, dr \leq \left| \rho \right|^{2(1+\delta)} K_{1+\delta} \int_s^t (1 + |u(r)|^2) \, dr \leq \left| \rho \right|^{2(1+\delta)} K_{1+\delta} T^{1+\delta} \left( T + \int_s^t |u(r)|^2 \, dr \right)^{\frac{1}{1+\delta}},
\]
where the second inequality follows from the Hypothesis $H_1$ with $p = 2$, and the last inequality follows from the concavity of the power $\frac{1+\delta}{2}$. Using Hypothesis $H_3$ and inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, we have
\[
\int_s^t \left| \int_Z \langle G(r,u(r),i(r),z),\rho \rangle V \nu(dz) \right|^{1+\delta} \, dr \leq 2^{\delta} \left| \rho \right|^{1+\delta} K_{1+\delta} \left( T + \int_s^t |u(r)|^{1+\delta} \, dr \right).
Therefore, taking expectation and then supremum over \( \ell \) on martingale terms, the above estimates imply

\[
\sup_{\ell} \mathbb{E}^n_t \int_s^t \left| \left( \rho, \sigma(r, u(r), i(r))Q\sigma^*(r, u(r), i(r))\rho \right)_H \right|^{1+\delta} dr \leq C_Q(T)
\]

and

\[
\sup_{\ell} \mathbb{E}^n_t \int_s^t \int_Z |G(r, u(r), i(r), z, \rho)\nu(dz)| dr \leq C_G(T)
\]
since, by (3.2), if \( \delta < 1 \),

\[
\mathbb{E}^\mu_n \int_s^t \|u(r)\|^{1+\delta}_V dr = \mathbb{E} \int_s^t \|u_{n_\ell}(r)\|^{1+\delta}_V dr \\
\leq \mathbb{E} \int_s^t \|u_{n_\ell}(r)\|^2_V dr = \mathbb{E}^\mu_n \int_s^t \|u(r)\|^2_V dr \leq C.
\]

In conclusion, the argument above shows that for \( 0 < \delta \leq \frac{1}{5} \), there is a constant \( C \) such that \( \sup_{\ell} \mathbb{E}^\mu_t [|M^\phi|^{1+\delta}] \leq C \) provided that \( \mathbb{E}|u(0)|^3 \) is finite. Hence, we complete the proof. \( \square \)

As M1, M2, and M3 are shown, the existence theorem follows:

**Theorem 4.12.** Suppose that \( \mathbb{E}|u_0|^3 < \infty \) and \( f \in L^3(0, T; V') \). Then under the Hypotheses H, \( M^\phi(t) \) is a \( \mu \)-martingale, i.e., \( \mu \) is a solution to the martingale problem posed by (1.7).

**4.2. Uniqueness of the solution to the regularized equation.** In this subsection, we prove that the (weak) solution obtained from Theorem 4.12 is pathwise unique.

**Theorem 4.13.** Let \( \mathbb{E}|u_0|^3 < \infty \) and \( f \in L^3(0, T; V') \). Then under Hypotheses H, the solution obtained from Theorem 4.12 is pathwise unique.

**Proof.** Let \( w = u - v \), where \( u, v \) are solutions with same initial data. Let \( F(t, x, i) := e^{-\rho(t)}x \), where \( \rho(t) \) is a function that will be determined later. Then the Itô formula implies

\[
e^{-\rho(t)}|w(t)|^2 + 2\nu \int_s^t e^{-\rho(s)}|w(s)|^2 ds \\
= \int_0^t -\rho'(s)e^{-\rho(s)}|w(s)|^2 ds \\
- \int_0^t e^{-\rho(s)}(B_{k_\phi}(u(s)) - B_{k_\phi}(v(s)), w(s))_V ds \\
+ \int_0^t e^{-\rho(s)}\|\sigma(s, u(s), r(s)) - \sigma(s, v(s), r(s))\|^2_{L_\sigma} ds \\
+ 2\int_0^t e^{-\rho(s)}\langle w(s), [\sigma(s, u(s), r(s)) - \sigma(s, v(s), r(s))]dW(s) \rangle \\
+ 2\int_0^t \int_Z e^{-\rho(s)} \\
\cdot (w(s-), G(s, u(s-), r(s-), z) - G(s, v(s-), r(s-), z))_H \tilde{N}_1 \\
+ \int_0^t e^{-\rho(s)}|G(s, u(s-), r(s-), z) - G(s, v(s-), r(s-), z)|^2 \tilde{N}_1,
\]

where \( \tilde{N}_1 = \tilde{N}_1(dz, ds), N_1 = N_1(ds, dz), \) and the last integral is over \( [s, t] \times Z. \)
For the nonlinear term, it follows from its definition that
\[
\left| (B_k(u(s)) - B_k(v(s)), w(s))_V \right|
\leq \|b(k, u(s), w(s)) - b(k, v(s), w(s))\| \leq \|b\| \|u(s) - v(s)\| \|w(s)\|
\]
where the last equality follows from (2.3). Therefore, by (2.8) and the basic Young inequality, we see that
\[
\left| e^{-\rho(s)} \left( B_k(u(s)) - B_k(v(s)), w(s) \right)_V \right| ds
\leq \int_0^t e^{-\rho(s)} \|w(s)\| \cdot |w(s)| \cdot \|u(s)\| ds
\leq \nu \int_0^t e^{-\rho(s)} \|w(s)\|^2 ds + \frac{1}{4\nu} \int_0^t e^{-\rho(s)} |w(s)|^2 \|u(s)\|^2 ds.
\]
Therefore, choosing \( \rho(t) := \frac{1}{4\nu} \int_0^t \|u(s)\|^2 ds \), we deduce form (4.6) that
\[
(4.7) \quad e^{-\rho(t)} |w(t)|^2 + \nu \int_0^t e^{-\rho(s)} \|w(s)\|^2 ds
\leq \int_0^t e^{-\rho(s)} \|\sigma(s, u(s), \tau(s)) - \sigma(s, v(s), \tau(s))\|^2_{L^2} ds
+ 2 \int_0^t e^{-\rho(s)} \langle w(s), [\sigma(s, u(s), \tau(s)) - \sigma(s, v(s), \tau(s))] \rangle dW(s)
+ 2 \int_0^t \int_Z e^{-\rho(s)} \left( w(s-) \cdot G(s, u(s-), \tau(s-), z) - G(s, v(s-), \tau(s-), z) \right) d\hat{N}_1
+ \int e^{-\rho(s)} \|G(s, u(s-), \tau(s-), z) - G(s, v(s-), \tau(s-), z)\|^2 d\hat{N}_1,
\]
where \( \hat{N}_1 = \hat{N}_1(dz, ds) \), \( N_1 = N_1(dz, ds) \), and the last integral is over \([s, t] \times Z\). Moreover, by the Davis and the basic Young inequalities and Hypotheses H2 and H4, the martingale terms in (4.7) have the following estimates.
\[
\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t 2e^{-\rho(s)} \langle w(s), [\sigma(s, u(s), \tau(s)) - \sigma(s, v(s), \tau(s))] \rangle dW(s)
\leq 2\sqrt{LC_2} \mathbb{E} \left\{ \left( \int_0^T e^{-2\rho(s)} |w(s)|^2 ds \right)^{\frac{1}{2}} \right\}
\leq 2\sqrt{LC_2} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} e^{-\frac{1}{2}\rho(t)} |w(t)| \left( \int_0^T e^{-\rho(s)} |w(s)|^2 ds \right)^{\frac{1}{2}} \right\}
\leq 2\sqrt{LC_2} \left\{ \epsilon \sup_{0 \leq t \leq T} e^{-\rho(t)} |w(t)|^2 + C_\epsilon \int_0^T e^{-\rho(s)} |w(s)|^2 ds \right\}
\]
and
\[
\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \int_Z 2e^{-\rho(s)}
\cdot (w(s), G(s, u(s), r(s), z) - G(s, v(s), r(s), z)) \, \nu_1(dz, ds)
\leq 2\sqrt{L}C_1^2 \mathbb{E} \left\{ \sup_{0 \leq t \leq T} e^{-\frac{\rho(t)}{2}} |w(t)| \left( \int_0^T e^{-\rho(s)} |w(s)|^2 \, ds \right)^{\frac{1}{2}} \right\}
\leq 2\sqrt{L}C_1^2 \mathbb{E} \left\{ \epsilon \sup_{0 \leq t \leq T} e^{-\rho(t)} |w(t)|^2 + C_\epsilon \int_0^T e^{-\rho(s)} |w(s)|^2 \, ds \right\},
\]
where \( \nu_1 = \tilde{\nu}_1(dz, ds) \). As a consequence, taking supremum over \([0, T]\) and then expectation, one obtains from (4.7) the following.
\[
\mathbb{E} \sup_{0 \leq t \leq T} e^{-\rho(t)} |w(t)|^2 + \nu \mathbb{E} \int_0^T |w(s)|^2 \, ds
\leq 2 \mathbb{E} \int_0^T e^{\rho(s)} |w(s)|^2 \, ds + C \epsilon \mathbb{E} \int_0^T e^{-\rho(s)} |w(s)|^2 \, ds,
\]
where \( C = 4\sqrt{L}C_1^2 \). Choosing \( \epsilon \) small enough so that \( C \epsilon < \frac{1}{2} \), one obtains from above that
\[
\mathbb{E} \sup_{0 \leq t \leq T} e^{-\rho(t)} |w(t)|^2 \leq CE \int_0^T e^{-\rho(s)} |w(s)|^2 \, ds \leq CE \int_0^T \sup_{0 \leq r \leq s} e^{-\rho(r)} |w(r)|^2 \, ds,
\]
where \( C \) stands for a generic constant. Furthermore, we employ the Gronwall inequality to obtain \( \mathbb{E} \sup_{0 \leq t \leq T} e^{-\rho(t)} |w(t)|^2 \leq 0 \), which implies the pathwise uniqueness. Hence, we complete the proof. \( \square \)

5. THE NAVIER-STOKES EQUATION WITH MARKOV SWITCHING

In this section, we prove the existence of a weak solution (in the sense of Definition 5) to equation (1.1). Let \( v_0^\epsilon \) be an \( H \)-valued random variable such that \( \mathbb{E} |v_0^\epsilon|^3 < \infty \). Define \( v_0 := k^\epsilon v_0^\epsilon \). Then \( v_0 \) is an \( H \)-valued random variable with \( \mathbb{E} |v_0|^3 = \mathbb{E} |k^\epsilon v_0^\epsilon|^3 \leq \mathbb{E} |v_0^\epsilon|^3 < \infty \). Let \( f \in L^3(0, T; V') \). Then with the given initial data \( v_0 \) and the external forcing \( f \), there exist a unique strong solution \((v^\epsilon(t), r(t))\) to the equation (1.7) for each \( \epsilon > 0 \).

Proceeding as in the argument of Proposition 5.1, we see that \( v^\epsilon(t) \) satisfies
\[
\mathbb{E} \sup_{0 \leq t \leq T} |v^\epsilon(t)|^2 + \nu \mathbb{E} \int_0^T ||v^\epsilon(s)||^2 \, ds \leq C_2.
\]
Making use of the estimate (5.1), we deduce from Lemma 2.4 that the processes \( \{v^\epsilon\}_{\epsilon > 0} \) is tight in \( D([0, T]; V') \).

**Lemma 5.1.** The sequence of processes \( \{v^\epsilon\}_{\epsilon > 0} \) is tight in \( D([0, T]; V') \).

**Proof.** First, it is clear that the paths of the processes \( \{v^\epsilon\}_{\epsilon > 0} \) are in \( D([0, T]; V') \). Let \( N > 0 \). Employing the Markov inequality, the estimate (5.1), and the property \( \| \cdot \|_{V'} < \| \cdot \| \), we obtain, for each rationals \( t \in [0, T] \),
\[
\mathbb{P}(\|v^\epsilon(t)\|_{V'} > N) \leq \frac{1}{N^2} \mathbb{E}\|v^\epsilon(t)\|^2_{V'} \leq \frac{1}{N^2} \mathbb{E} \mathbb{E}\|v^\epsilon(t)\|^2 \leq \frac{C_2}{N^2}.
\]

29
Thus, for each rationals \( t \in [0, T] \),
\[
\lim_{N \to \infty} \limsup_{\epsilon \to 0} \mathcal{P}\left( \|u'(t)\|_{V'} > N \right) = 0
\]
Let \( (T_{\epsilon}, \delta_{\epsilon}) \) be a sequence, where \( T_{\epsilon} \) is a stopping time with \( T_{\epsilon} + \delta_{\epsilon} \leq T \) and \( \delta_{\epsilon} > 0 \) with \( \delta_{\epsilon} \to 0 \) as \( \epsilon \to 0 \). For each \( N > 0 \), the Chebyshev inequality implies
\[
\mathcal{P}\left( \|u'(T_{\epsilon} + \delta_{\epsilon}) - u'(T_{\epsilon})\|_{V'} > N \right) 
\leq \frac{1}{N^2} \mathbb{E}\|u'(T_{\epsilon} + \delta_{\epsilon}) - u'(T_{\epsilon})\|_{V'}^2
\leq \frac{1}{N^2} \mathbb{E}\|u'(T_{\epsilon} + \delta_{\epsilon}) - u'(T_{\epsilon})\|^2.
\]
In addition, the Itô formula and the Gronwall inequality imply that
\[
\mathbb{E}\|u'(T_{\epsilon} + \delta_{\epsilon}) - u'(T_{\epsilon})\|^2 \leq \left( \frac{1}{\nu} \int_0^{\delta_{\epsilon}} \|f(s)\|_{V'}^2 ds + 2K\delta_{\epsilon} \right)e^{2K\delta_{\epsilon}} \to 0
\]
as \( \epsilon \to 0 \). Therefore, \( \|u'(T_{\epsilon} + \delta_{\epsilon}) - u'(T_{\epsilon})\|_{V'} \to 0 \) in probability as \( \epsilon \to 0 \). By Lemma \ref{lem:boundedness} the set \( \{u'\}_{\epsilon > 0} \) is tight in \( D([0, T]; V') \). \( \square \)

The estimate \eqref{eq:estimate} also shows that \( \{u'\}_{\epsilon > 0} \) is bounded in the space \( L^2(\Omega; L^2(0, T; V)) \). Therefore, by an argument analogous to that in Proposition \ref{prop:Proposition 4.3} we have the following proposition.

**Proposition 5.2.** The sequence of processes \( \{u'\}_{\epsilon > 0} \) forms a relative compact set in the space \( L^2(\Omega; L^2(0, T; H)) \).

Denote by \( u \) the limit (along a subsequence) of \( \{u'\}_{\epsilon > 0} \) in \( L^2(\Omega; L^2(0, T; H)) \). The next step is to identify that \( u \) is indeed a solution to the equation \eqref{eq:main_eq} in the sense of Definition \ref{def:solution}.

In order to carry out the arguments similar to those in Section \ref{sec:4.1} we introduce the following functions.

Let \( \phi(t, i) \) be a real-valued, smooth function with compact support (in each variable). For \( \rho \in \mathcal{D}(A) \subseteq V \) with \( \nabla \rho \in (L^\infty(G))^3 \), \( 0 \leq s \leq t \), and each element \( \omega = (u, i) \in \Omega^1 \), define \( M^\phi(t) \)
\[
\phi((u(t), \rho)_V, i(t)) - \int_0^t \sum_{j=1}^m \gamma_{ij}(r-i) \phi((u(r), \rho)_V, j) dr 
- \int_0^t \left( \phi'(u(r), \rho)_V, i(r) \right) (-\nu A u(r) - B(u(r)) + f(r, \rho)_V) dr 
- \frac{1}{2} \int_0^t \phi''((u(r), \rho)_V, i(r)) \left( \rho, \sigma(r, u(r), i(r))Q\sigma^*(r, u(r), i(r)) \right) \mu dr 
- \int_0^t \phi((u(r) + G(r, u(r), i(r), z, \rho)_V, i(r)) - \phi((u(r), \rho)_V, i(r)) 
- \phi'((u(r), \rho)_V, i(r)) \cdot \left( G(r, u(r), i(r), z, \rho)_V \right) \nu_1(dz) dr,
\]
where the second last integral is over \([0, t]\), and the last integral is over \([0, t] \times Z \), and \( M^\phi(t) \) admits a similar expression with \( B_{ki} \) replacing \( B \).

Similar to M2 in Section \ref{sec:4.1} we need to prove that \( M^\phi(t) \) converges to \( M^\phi(t) \) as \( \epsilon \) tends to 0.

**Proposition 5.3.** Let \( M^\phi(t) \) and \( M^\phi(t) \) be as above. Then
\[
\lim_{\epsilon \to 0} M^\phi(t) = M^\phi(t).
\]
Proof. The convergence of the terms other than the nonlinear term $B_k^*$ follows a similar argument as in the proof of assertion $M_2$.

Recall the definitions of $B$ and $B_k^*$ from Section 2. It suffices to show that

$$\lim_{\epsilon \to 0} \int_0^t b(k_\epsilon u(r), u(r), \rho)dr = \int_0^t b(u(r), u(r), \rho)dr.$$ 

Notice that the function $u(r)$ is a generic element in the space $\Omega^*$, therefore, $u(r) \in H$. Then it follows from (2) of Lemma 2.1 that $k_\epsilon u \to u$ as $\epsilon \to 0$, in $H$. It follows from (2.5) and (2.4) that

$$\int_0^t b(k_\epsilon u(r), u(r), \rho)dr = -\sum_{i,j=1}^3 \int_0^t \int_G (\eta_\epsilon \ast u)_i(r) \frac{\partial \rho(j)(x)}{\partial x_i} u_j(r)dxdr.$$

Consider

$$\sum_{i,j=1}^3 \int_0^t \int_G (\eta_\epsilon \ast u)_i(r) \frac{\partial \rho(j)(x)}{\partial x_i} u_j(r)dxdr$$

(5.3)

$$= \sum_{i,j=1}^3 \int_0^t \int_G (\eta_\epsilon \ast u)_i(r) \frac{\partial \rho(j)(x)}{\partial x_i} [u_j(r) - (\eta_\epsilon \ast u)_j(r)]dxdr$$

$$+ \sum_{i,j=1}^3 \int_0^t \int_G (\eta_\epsilon \ast u)_i(r) \frac{\partial \rho(j)(x)}{\partial x_i} (\eta_\epsilon \ast u)_j(r)dxdr.$$

The former term on the right of (5.3) bounded by

$$\|\nabla \rho\|_{L^\infty(G)} \int_0^T |k_\epsilon u(r)||u(r) - k_\epsilon u(r)|dr$$

$$\leq \|\nabla \rho\|_{L^\infty(G)} \left( \int_0^T |k_\epsilon u(r)|^2 dr \right)^2 \left( \int_0^T |u(r) - k_\epsilon u(r)|^2 dr \right)^2 \to 0,$$

as $\epsilon \to 0$ since $k_\epsilon u \to u$. The latter term is bounded by

$$\|\nabla \rho\|_{L^\infty(G)} \int_0^T |k_\epsilon u(r)|^2 dr < C$$

for an appropriate constant $C$, which is independent of $\epsilon$. Therefore, by the Lebesgue Dominated Convergence Theorem, the latter term converges to

$$\sum_{i,j=1}^3 \int_0^t \int_G u_i(r) \frac{\partial \rho_j(x)}{\partial x_i} u_j(r)dxdr.$$

Hence, we conclude that as $\epsilon \to 0$,

$$\int_0^t b(k_\epsilon u(r), u(r), \rho)dr = -\int_0^t b(k_\epsilon u(r), \rho, u(r))dr$$

$$\to -\int_0^t b(u(r), \rho, u(r))dr = \int_0^t b(u(r), u(r), \rho)dr.$$

□

Lemma 5.4. The function $M^\theta(t)$ is continuous in the $(\tau^1)$-topology.

Proof. The proof of this lemma follows from the argument for proving Lemma 4.10. In fact, everything follows along the same lines except for the $B$. The convergence of $B$ follows from an argument analogous to the proof of [26 Lem. 3.2, Ch. III]. □
Denote by $\mu_c$ the distribution of $(u^c(t), r(t))$ and $\mu$ the distribution of $(u(t), r(t))$.

**Lemma 5.5.** Suppose that the Hypotheses $H$ is fulfilled, $\mathbb{E}|u_0|^3 < \infty$, and $f \in L^3(0, T; V')$. There exist some $\delta > 0$ such that

$$
\sup_{\epsilon > 0} \mathbb{E}^\mu_{\epsilon} \left[ |M^\phi|^1_{1+\delta} \right] \leq C,
$$

where $C$ is an appropriate constant.

**Proof.** The proof follows from the same lines as Lemma 4.11 except for the nonlinear term $B$. It follows from [26, Eq. (3.74)] that

$$
\| B(u(r)) \|_{V'} \leq C \| u(r) \|^\frac{1}{2} \| u(r) \|^\frac{3}{2}
$$

for an appropriate constant $C$. Hence,

$$
\mathbb{E}^\mu \left\{ \left| \int_s^t \langle B(u(r)), \rho \rangle_V \right|^{1+\delta} \right\}
\leq \| \rho \|_{V^{1+\delta}} \mathbb{E}^\mu \left\{ \left( \int_s^t \| u(r) \|^\frac{3}{2} \| u(r) \|^\frac{1}{2} dr \right)^{1+\delta} \right\}
\leq \| \rho \|_{V^{1+\delta}} \mathbb{E}^\mu \left\{ \left( \sup_{0 \leq t \leq T} |u(t)|^{2(\frac{1+\delta}{2H})} \right)^{\frac{1}{2}} \left( \mathbb{E}^\mu \left\{ \int_s^t \| u(r) \|^\frac{2(1+\delta)}{H} dr \right\} \right)^{\frac{1}{2}} \right\}
= \| \rho \|_{V^{1+\delta}} \mathbb{E}^\mu \left\{ \sup_{0 \leq t \leq T} |u^c(t)|^{2(\frac{1+\delta}{2H})} \right\} \left( \mathbb{E} \int_s^t \| u^c(r) \|^\frac{2}{1+\delta} dr \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}
\leq \| \rho \|_{V^{1+\delta}} \mathbb{E}^\mu \left\{ \left( \int_s^t \| u^c(r) \|^\frac{2}{1+\delta} dr \right)^{\frac{1}{2}} \right\} \left( \mathbb{E} \int_s^t \| u^c(r) \|^\frac{2}{1+\delta} dr \right)^{\frac{1}{2}}.
$$

Taking $\delta = \frac{1}{1 + H}$, we have $2(\frac{1+\delta}{1+\delta}) = 3$. By (3.4), we have the following estimate

$$
\mathbb{E} \sup_{0 \leq t \leq T} |u^c(t)|^3 \leq C(\mathbb{E}|u_0|^3, \mathbb{E} \int_0^T \| f(r) \|^3_{V'}, \nu, K, T).
$$

Therefore, The first expectation on the right of (5.4) will have a uniform bound by (3.4) if we further assume that $\mathbb{E}|u_0|^3 < \infty$. The boundedness of $\mathbb{E} \int_s^t \| u^c(r) \|^\frac{2}{1+\delta} dr$ follows from (3.2) and (3) of Lemma 2.1. Therefore, (5.4) implies that, if $\delta \leq \frac{1}{1 + H}$,

$$
\sup_{\epsilon > 0} \mathbb{E}^\mu_{\epsilon} \left( \left| \int_0^t \langle B(u(r)), \rho \rangle_V \right|^{1+\delta} dr \right) \leq C_B.
$$

The results of Lemmata 5.4 and 5.5 together with Lemma 2.2 imply the following proposition.

**Proposition 5.6.** Let $u^c$ be the solution (1.7) and $u$ the limit of $\{u^c\}_{\epsilon > 0}$ in $L^2(\Omega; L^2(0, T; H))$. Denote by $\mu_c$ the distribution of $(u^c(t), r(t))$ and $\mu$ the distribution of $(u(t), r(t))$. Then

$$
\lim_{\epsilon \to 0} \mathbb{E}^\mu_{\epsilon} M^\phi(t) = \mathbb{E}^\mu M^\phi(t).
$$

Finally, we are at the stage to prove Theorem 1.2.
Proof of Theorem 1.2. Given $u_0$ satisfying the assumption, we define $u'_0 := k_\epsilon u_0$. Then $u'_0$ is an $H$-valued random variable with $\mathbb{E}|u'_0|^3 = \mathbb{E}|k_\epsilon u_0|^3 \leq \mathbb{E}|u_0|^3 < \infty$.

Given $(u_0, f)$ satisfying the assumption, we consider the pair of initial condition and external forcing $(u'_0, f)$. Then this pair gives a unique strong solution to equation (1.7) by Theorem 1.1 and we denote the sequence of solutions by $\{u^i(t)\}_{i>0}$.

By Proposition 5.2, we see that there is a limit of $u^i$ in the space $L^2(\Omega; L^2(0, T; H))$, and we denote it by $u$. Moreover, we denote by $\mu_t$ the distribution of $(u^i(t), \tau(t))$ and $\mu$ the distribution of $(u(t), \tau(t))$.

Following the same argument as in Section 4.1, it suffices to prove the following:

M1'. There exists a sequence $\{\mu_{\epsilon_n}\}$ that converges weakly to $\mu$ as $\epsilon_n \to 0$.

M2'. $\lim_{n \to \infty} M_{\epsilon_n}^\phi(t) = M^\phi(t)$.

M3'. $\lim_{n \to \infty} \mathbb{E}^{\mu_{\epsilon_n}} M^\phi(t) = \mathbb{E}^{\mu} M^\phi(t)$,

where $M^\phi(t)$ and $M_{\epsilon_n}^\phi(t)$ are defined earlier in this section.

Now, as M1', M2', and M3' are direct consequences of Propositions 5.2, 5.3, and 5.6 respectively, the proof is complete. $\square$

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