EPIMORPHISMS OF RINGS

ABOLFAZL TARIZADEH

Abstract. The aim of this article is to present more explicitly the basic properties of epimorphisms of rings and their characterizations; especially the interactions between epimorphisms and flat morphisms and also the relationship of epimorphisms with absolutely flat rings are studied and various deep and interesting results are extracted.

1. Introduction

In a given concrete category, in general, epimorphisms (resp. monomorphisms) are not necessarily surjective (resp. injective) maps; in this article, we are primarily concerned which epimorphisms in the category of commutative rings are surjective; for example, as we shall observe this fact at the present article, an epimorphism of rings with source an absolutely flat ring is always surjective. Epimorphisms and their characterizations have been studied extensively for some familiar categories such as groups, Lie algebras, Hopf algebras, von Neumann algebras, compact groups, locally compact groups, etc (cf. [3], [6], [8], [11], [14]); e.g. in the category of commutative $C^*-\text{algebras}$ epimorphisms are exactly surjective maps in fact this statement is equivalent to the Stone-Weierstrass theorem; Reid in his seminal work [14] goes further, in fact he shows that for the category of von Neumann algebras and consequently for the category of $C^*-\text{algebras}$ (not necessarily commutative) epimorphisms are exactly surjective maps and thereby he generalizes, in a certain sense, Stone-Weierstrass theorem. In most of the foregoing categories, epimorphisms are exactly surjective maps though verifying it for some categories are highly non-trivial (cf. [13]); but in the category of rings epimorphisms have kaleidoscopic nature, as we shall observe, in this category the class of epimorphisms are vast than the class of surjective ring maps; characterizing epimorphisms in the category of commutative rings have been also studied by some people (cf. [12], [13]), but finding a reference source on the topic of

\[\text{2013 Mathematics Subject Classification: 13A, 13B, 13C.}\]
\[\text{Key words and phrases: epimorphism, flat morphism, absolutely flat ring and punctual ring.}\]
epimorphisms of rings (into English) in the literature is a hard task; hence we decided to collect at this article the basic properties of epimorphisms in the category of commutative rings and then we establish their relationship with absolutely flat rings. Some parts of the present article are expository and do not due to the author. In preparing this article we have used some references which amongst them the main sources including [7], [12] and [15]. Throughout this article, all rings and algebras are commutative with identity elements and all ring maps transform the unit element to the unit element.

Recall that in a category $\mathcal{C}$, a morphism $f : A \to B$ is said to be an epimorphism (or shortly an epic) when for any two morphisms $g, h : B \to C$ in $\mathcal{C}$, if $g \circ f = h \circ f$ then $g = h$. Monomorphisms also defined in a similar manner. We refer the reader to [2] and [3] to see any categorical concept which appear throughout this article. In the category of rings (commutative or more general rings with identity elements), monomorphisms are exactly the injective ring maps. Because, every injective ring map is a monomorphism. For the converse, let $f : A \to B$ be a monomorphism in the category of rings. Suppose that there exist two elements $a, a' \in A$ so that $f(a) = f(a')$. Take the $\mathbb{Z}$-algebra homomorphisms $g, h : \mathbb{Z}[x] \to A$ given by $g(x) = a$ and $h(x) = a'$. Obviously we have $f \circ g = f \circ h$. Since $f$ is a monomorphism therefore $g = h$; this implies that $a = g(x) = h(x) = a'$. But the class of epimorphisms in the category of rings are vast than the class of surjective ring maps. Obviously every surjective ring map is an epimorphism. But for the reverse, as an example, for given ring $R$ and for each multiplicative subset $S$ of $R$, consider the canonical ring map $\pi : R \to S^{-1}R$ and take two ring maps $f, g : S^{-1}R \to A$ so that $f \circ \pi = g \circ \pi$, by the universal property of the localization, $f = g$ and so $\pi$ is an epimorphism but it is not necessarily surjective; as a specific example, the inclusion ring map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not surjective while it is an epimorphism. Therefore, in the category of rings an epimorphism is not necessarily surjective. Motivated enough by this example, in the next Section we will begin to study epimorphisms in the category of commutative rings more deeply.

We organized this article as follows. In Section 2, we study epimorphisms of rings and their characterizations, moreover we study those epimorphisms of rings which are also flat morphisms; one of the outstanding results of this section is that a flat epimorphism map of rings $R \to S$ is surjective if and only if the induced ring map $R_{\text{red}} \to S_{\text{red}}$ is so, Lemma 2.14.
In Section 3, after proving the basic properties of absolutely flat rings we then study their relationship with epimorphisms of rings; as a striking result of this section we show that any epimorphism of rings with source an absolutely flat ring is surjective, Corollary 3.9.

2. Characterization of epimorphisms of rings

There are some interesting characterizations of epimorphisms of rings which we have collected them in the following result.

**Theorem 2.1.** Let $R$ and $S$ be rings and $\varphi : R \to S$ a ring map. Then the following conditions are equivalent.

(i) The map $\varphi$ is an epimorphism.

(ii) For each $s \in S$, in the $R$-algebra $S \otimes_R S$, $s \otimes 1 = 1 \otimes s$.

(iii) The canonical surjective $R$-algebra homomorphism $p : S \otimes_R S \to S$ is injective (hence, $p$ is an isomorphism).

(iv) The canonical injective ring map $i_1 : S \to S \otimes_R S$ defined by $s \mapsto s \otimes 1$ is surjective.

(iv)$'$ The canonical injective ring map $i_2 : S \to S \otimes_R S$ defined by $s \mapsto 1 \otimes s$ is surjective.

(v) The $R$-module $S \otimes_R S/N$ is zero where $N = \text{Im}(\varphi)$.

(vi) The graded tensor $R$-algebra $T_R(S) = \bigoplus_{p \geq 0} T^p(S)$ is commutative where $T^0(S) = R$, $T^1(S) = S$ and for each $p \geq 2$, $T^p(S) = S \otimes_R \ldots \otimes_R S$ including $p$ factors of $S$.

(vii) For each $R$-algebra $T$ there exists at most one homomorphism of $R$-modules $f : S \to T$ so that $f(1_S) = 1_T$.

(viii) The restriction of scalars functor $\varphi_* : S\text{-mod} \to R\text{-mod}$ is full and faithful.

(ix) There exists a natural isomorphism between the functors $\varphi^* \circ \varphi_*$ and $\text{Id}_{S\text{-mod}}$ where the functor $\varphi^* : R\text{-mod} \to S\text{-mod}$ is defined by $M \mapsto S \otimes_R M$ and each morphism $f : M \to N$ in the category $R\text{-mod}$ is mapped into $\text{Id}_S \otimes f : S \otimes_R M \to S \otimes_R N$ in $S\text{-mod}$.

**Proof.** (i) $\Rightarrow$ (ii) : Since the following diagram is commutative

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow{\varphi} & & \downarrow{i_2} \\
S & \xrightarrow{i_1} & S \otimes_R S
\end{array}
$$
Decompose the canonical projection $s$ in $S$. Hence, $i_1$ is surjective.

(iii) $\Rightarrow$ (iv) : For each $s \in S$, $s \otimes 1 = 1 \otimes s$. Therefore, the element $s \otimes s'$ in $S \otimes_R S$ can be written as $s \otimes s' = (s \otimes 1)(1 \otimes s') = (s \otimes 1)(s' \otimes 1) = ss' \otimes 1 = i_1(ss')$. Hence, $i_1$ is injective.

(iv) $\Rightarrow$ (iv)' : The map $i_1$ is an isomorphism and its inverse is the map $p$. Therefore, $p$ is injective and hence $\text{Ker}(p) = 0$. Thus, for each $s \in S$, $s \otimes 1 = 1 \otimes s$. Now, similarly as above, an element $s \otimes s'$ in $S \otimes_R S$ can be written as $s \otimes s' = (s \otimes 1)(1 \otimes s') = (1 \otimes s)(1 \otimes s') = 1 \otimes ss' = i_2(ss')$. Hence, $i_2$ is surjective.

(iv)' $\Rightarrow$ (iii) : The map $i_2$ is an isomorphism and its inverse is the map $p$. Therefore, $p$ is injective and hence $\text{Ker}(p) = 0$.

(iv) $\Rightarrow$ (v) : Apply the right exact functor $S \otimes_R -$ to the canonical homomorphism of $R$–modules $\pi : S \to S/N$ we obtain the surjective homomorphism of $R$–modules $\text{Id}_S \otimes \pi : S \otimes_R S \to S \otimes_R S/N$. On the other hand, $\text{Ker}(\text{Id}_S \otimes \pi) = \langle s \otimes \varphi(r) : s \in S, r \in R \rangle$. Because, the inclusion $\langle s \otimes \varphi(r) : s \in S, r \in R \rangle \subseteq \text{Ker}(\text{Id}_S \otimes \pi)$ is obvious. To prove the reverse inclusion we act as follows. By the universal property of the tensor product, the $R$–bilinear map

$$S \times \frac{S}{N} \to \frac{S \otimes_R S}{(s \otimes \varphi(r) : s \in S, r \in R)}$$

given by $(s, s' + N) \mapsto s \otimes s' + \langle s \otimes \varphi(r) : s \in S, r \in R \rangle$ induces a (unique) homomorphism of $R$–modules

$$\psi : \frac{S \otimes_R S}{\langle s \otimes \varphi(r) : s \in S, r \in R \rangle} \to \frac{S \otimes_R S}{\langle s \otimes \varphi(r) : s \in S, r \in R \rangle}$$

in which $s \otimes (s' + N)$ is mapped into $s \otimes s' + \langle s \otimes \varphi(r) : s \in S, r \in R \rangle$. Decompose the canonical projection

$$\pi' : \frac{S \otimes_R S}{\langle s \otimes \varphi(r) : s \in S, r \in R \rangle} \to \frac{S \otimes_R S}{\langle s \otimes \varphi(r) : s \in S, r \in R \rangle}$$

as $\pi' = \psi \circ \text{Id}_S \otimes \pi$. Take $z \in \text{Ker}(\text{Id}_S \otimes \pi)$, the decomposition $\pi' = \psi \circ \text{Id}_S \otimes \pi$ implies that $z \in \langle s \otimes \varphi(r) : s \in S, r \in R \rangle$ which was desired. According to (iv), $\text{Im}(i_1) = \{s \otimes 1_S : s \in S\} = S \otimes_R S$. Therefore, $\text{Ker}(\text{Id}_S \otimes \pi) = S \otimes_R S$, because $s \otimes 1_S = s \otimes \varphi(1_R) \in \text{Ker}(\text{Id}_S \otimes \pi)$. 
Since \( \text{Id}_S \otimes \pi \) is surjective therefore

\[
S \otimes_R \frac{S}{N} = 0.
\]

(v) \( \Rightarrow \) (iv) : First note that \( \text{Ker}(\text{Id}_S \otimes \pi) = \langle s \otimes \varphi(r) : s \in S, r \in R \rangle = \{ s \otimes 1 : s \in S \} = \text{Im}(i_1) \). By the assumption (v), \( \text{Ker}(\text{Id}_S \otimes \pi) = S \otimes_R S \). This implies that the map \( i_1 \) is surjective.

In fact, we have established the equivalences: (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (iv)' \( \Leftrightarrow \) (v).

(iv)' \( \Rightarrow \) (vi) : Recall that in the graded \( R \)-algebra \( T_R(S) \), for each two homogeneous elements of the form \( s_1 \otimes \ldots \otimes s_p \) and \( s'_1 \otimes \ldots \otimes s'_q \), the multiplication is defined as \( (s_1 \otimes \ldots \otimes s_p)(s'_1 \otimes \ldots \otimes s'_q) = s_1 \otimes \ldots \otimes s_p \otimes s'_1 \otimes \ldots \otimes s'_q \). On the other hand, according to (iii), for each \( s \in S \), we have \( s \otimes 1 = 1 \otimes s \). Therefore, for each two elements \( s, s' \in S \), we have \( s \otimes s' = (s \otimes 1)(1 \otimes s') = (1 \otimes s)(s' \otimes 1) = s' \otimes s \). Therefore, \( (s_1 \otimes \ldots \otimes s_p)(s'_1 \otimes \ldots \otimes s'_q) = (s'_1 \otimes \ldots \otimes s'_q)(s_1 \otimes \ldots \otimes s_p) \). Hence, \( T_R(S) \) is commutative.

(vi) \( \Rightarrow \) (vii) : Let \( f, g : S \rightarrow T \) be two \( R \)-homomorphism in which \( f(1_S) = g(1_S) = 1_T \). By the universal property of the tensor product, the \( R \)-bilinear map \( S \times S \rightarrow T \) given by \( (s, s') \rightarrow f(s)g(s') \) induces a (unique) homomorphism of \( R \)-modules \( \theta : S \otimes_R S \rightarrow T \) in which \( s \otimes s' \) is mapped into \( f(s)g(s') \). Since the graded \( R \)-algebra \( T_R(S) \) is commutative therefore one has \( s.s' = s \otimes s' = s'.s = s' \otimes s \) in \( T_R(S) \). Thus, \( f(s)g(s') = f(s')g(s) \). Now, by taking \( s' = 1_S \), we have \( f(s) = g(s) \) for each \( s \in S \).

(vii) \( \Rightarrow \) (i) : This implication is obvious because each homomorphism of \( R \)-algebras is also a homomorphism of \( R \)-modules.

So far, we have proved that the following conditions are equivalent.

(i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (iv)' \( \Leftrightarrow \) (v) \( \Leftrightarrow \) (vi) \( \Leftrightarrow \) (vii).

(ii) \( \Rightarrow \) (viii) : Recall that the restriction of scalars functor \( \varphi_* : \text{S-mod} \rightarrow \text{R-mod} \) is defined as follows. Each \( S \)-module \( M \) is mapped into \( M \) equipped with the \( R \)-module structure induced via \( \varphi : S \rightarrow R \) (in this case, sometimes the \( R \)-module \( M \) is denoted by \( \varphi M \) in order that not to be confused with the \( S \)-module \( M \)). Moreover, each \( S \)-homomorphism \( f : M \rightarrow N \) is mapped into \( f : M \rightarrow N \) which is obviously an \( R \)-homomorphism. Thus, the functor \( \varphi_* : \text{S-mod} \rightarrow \text{R-mod} \) is always faithful. To prove that the functor \( \varphi_* \) is full we act as follows. Take two \( S \)-modules, say \( M \) and \( N \), and let \( f : M \rightarrow N \) be an \( R \)-homomorphism. To prove the assertion it is enough to show that \( f \) is also an \( S \)-homomorphism. For each \( m \in M \), by the universal property of the tensor product, the \( R \)-bilinear map \( S \times S \rightarrow N \) given by \( (s, s') \rightarrow sf(s'm) \) induces a (unique) homomorphism \( \theta_m : S \otimes_R S \rightarrow N \)
of $R$–modules in which $s \otimes s'$ is mapped into $sf(s'm)$. Now using the hypothesis (ii), we conclude that $f(sm) = sf(m)$ which was desired.

**(viii) ⇒ (ix)**: For each $S$–module $M$, define the $R$–homomorphism $\mu_M : M \to S \otimes_R M$ by $\mu_M(m) = 1 \otimes m$. Then according to (viii), $\mu_M$ is also an $S$–homomorphism (recall that for each two $S$–modules $M$ and $N$, the ring $S$ puts two $S$–module structures on $M \otimes_R N$ which include: $s.(m \otimes n) = sm \otimes n$ and $s*(m \otimes n) = m \otimes sn$, these two structures are not the same in general, i.e. the map $\otimes_R$ is not necessarily $S$–bilinear).

Therefore we have obtained a transformation $\mu_M : \text{Id}_{S\text{-mod}} \to \varphi^* \circ \varphi_*$ which is obviously a natural transformation. It is easy to observe that the natural transformation $\lambda : \varphi^* \circ \varphi_* \to \text{Id}_{S\text{-mod}}$ is the inverse of $\mu$, where for each $S$–module $M$, the homomorphism $\lambda_M : S \otimes_R M \to M$ is defined by $\lambda_M(s \otimes m) = sm$.

**(ix) ⇒ (iii)**: Take $M = S$ and then by (ix), the map $\lambda_S = p : S \otimes_R S \to S$ is an isomorphism. □

The above Theorem has the following important consequences.

**Corollary 2.2.** Any faithfully flat epimorphism is an isomorphism.

**Proof.** First note that any faithfully flat morphism $\varphi : R \to S$ is always injective. Because, the homomorphism $\varphi \otimes 1_S : R \otimes_R S \to S \otimes_R S$ decomposes as $\varphi \otimes 1_S = i_2 \circ \theta$ where $\theta : R \otimes_R S \to S$ is the canonical isomorphism in which $r \otimes s$ is mapped into $\varphi(r)s$. Therefore $\varphi \otimes 1_S$ is injective. But, since $S$ is a faithfully flat $R$–module thus $\varphi : R \to S$ is injective too. To prove surjectivity, by the above Theorem, condition (v), we have $S \otimes_R S/N = 0$ where $N = \text{Im}(\varphi)$. Since $S$ is faithfully flat over $R$, therefore $S/N = 0$. Hence, $\varphi$ is surjective. □

**Corollary 2.3.** Any epimorphism $k \to S$ is an isomorphism where $k$ is a field and $S$ is a nontrivial ring.

**Proof.** Every non-zero vector space over a field is faithfully flat, then apply the above Corollary. □

**Remark 2.4.** Let $\varphi : R \to S$ be a ring map and let $M$ and $N$ be $S$–modules. We can put two $S$–module structures on the $R$–module $M \otimes_R N$ given by $s.(m \otimes n) = sm \otimes n$ and $s*(m \otimes n) = m \otimes sn$ which are, in general, different $S$–module structures. Applying Theorem 3.17, it is then obvious that $\varphi$ is an epimorphism if and only if for any $S$–modules $M$ and $N$, the foregoing $S$–module structures on
Remark 2.5. The category of commutative rings with the identity elements \( \mathcal{C}-\text{Ring} \) has pushouts. Because, let \( \varphi : R \to A \) and \( \psi : R \to B \) be two morphisms in \( \mathcal{C}-\text{Ring} \), the triple \( (A \otimes_R B, i_A, i_B) \) including the \( R \)-algebra \( A \otimes_R B \) together with the morphisms \( i_A : A \to A \otimes_R B \) and \( i_B : B \to A \otimes_R B \) defined respectively by \( a \mapsto a \otimes 1 \) and \( b \mapsto 1 \otimes b \), is the pushout of \( \varphi \) and \( \psi \). In the language of diagrams, the following commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & B \\
\downarrow{\varphi} & & \downarrow{i_B} \\
A & \xrightarrow{i_A} & A \otimes_R B
\end{array}
\]

is the pushout of \( \varphi \) and \( \psi \). Because, let

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & B \\
\downarrow{\varphi} & & \downarrow{g} \\
A & \xrightarrow{f} & T
\end{array}
\]

be an another commutative diagram in \( \mathcal{C}-\text{Ring} \). By the universal property of the tensor product, the \( R \)-bilinear map \( A \times B \to T \) given by \( (a, b) \mapsto f(a)g(b) \) induces a (unique) homomorphism of \( R \)-modules \( \theta : A \otimes_R B \to T \) in which \( a \otimes b \) is mapped into \( f(a)g(b) \). Obviously, \( \theta \) is the unique ring map which satisfies in the conditions \( f = \theta \circ i_A \) and \( g = \theta \circ i_B \).

In each category with pushouts, epimorphisms are stable under base change morphism. More precisely, let \( \mathcal{C} \) be a category with pushouts, and let \( \varphi : R \to A \) be an epimorphism in \( \mathcal{C} \). For each morphism \( \psi : R \to R' \) in \( \mathcal{C} \), let \( P \) together with the morphisms \( \varphi' : R' \to P \) and \( \psi' : A \to P \) be the pushout of \( \varphi \) and \( \psi \) in \( \mathcal{C} \). Consider the following commutative diagram.

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & A \\
\downarrow{\psi} & & \downarrow{\psi'} \\
R' & \xrightarrow{\varphi'} & P
\end{array}
\]

Then it is easy to see that \( \varphi' \) is also an epimorphism in \( \mathcal{C} \).

In particular, if \( \varphi : R \to S \) be an epimorphism in \( \mathcal{C}-\text{Ring} \), then for each morphism \( R \to R' \) in \( \mathcal{C}-\text{Ring} \), the base change morphism \( R' \to R' \otimes_R S \) given by \( r' \mapsto r' \otimes 1 \), is an epimorphism.
Lemma 2.6. A ring map \( \varphi : R \to S \) is an epimorphism if and only if for each maximal ideal \( p \) of \( R \), the induced ring map \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is an epimorphism.

Proof. If \( \varphi \) be an epic then the assertion implies from the Remark 2.5. Conversely, suppose that for each maximal ideal \( p \) of \( R \), the base change morphism \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is an epic. Let \( f, g : S \to T \) be two ring maps so that \( f \circ \varphi = g \circ \varphi \). Then for each maximal ideal \( p \) of \( R \), \( f \otimes 1_p = g \otimes 1_p : S \otimes_R R_p \to T \otimes_R R_p \). Because, we have \( (f \otimes 1_p)(\varphi \otimes 1_p) = (f \circ \varphi) \otimes 1_p = (g \circ \varphi) \otimes 1_p = (g \otimes 1_p)(\varphi \otimes 1_p) \), but by the hypothesis, \( \varphi \otimes 1_p : R_p \to S \otimes_R R_p \) is an epimorphism, and so \( f \otimes 1_p = g \otimes 1_p \). On the other hand, recall that for each \( R \)-module \( M \) the canonical homomorphisms \( M \to M \otimes_R R_p \) where \( p \) is a maximal ideal of \( R \), induce the following injective homomorphism of \( R \)-modules

\[
\omega_M : M \to \prod_{p \in \text{Max}(R)} M \otimes_R R_p
\]

in which \( \omega_M (m) = (m \otimes 1)_{p \in \text{Max}(R)} \). Now, for each \( s \in S \), we have \( \omega_T (f(s) - g(s)) = ((f(s) - g(s)) \otimes 1)_{p \in \text{Max}(R)} = 0 \) where

\[
\omega_T : T \to \prod_{p \in \text{Max}(R)} T \otimes_R R_p.
\]

Since \( \omega_T \) is injective therefore \( f(s) = g(s) \). \( \square \)

There is also another characterization of epimorphisms of rings which has a little algebraic geometry taste.

Proposition 2.7. A ring map \( \varphi : R \to S \) is an epimorphism if and only if the following conditions hold.

(a) The induced map \( \varphi^* : \text{Spec}(S) \to \text{Spec}(R) \) between the corresponding spectra is injective.

(b) For each prime ideal \( q \) of \( S \), the induced map via \( \varphi \) between the corresponding residues \( \kappa(p) \hookrightarrow \kappa(q) \) is an isomorphism where \( p = \varphi^*(q) \).

(c) The kernel of the canonical ring map \( S \otimes_R S \to S \) is a finitely generated and idempotent ideal (idempotency is equivalent to the condition that \( \Omega_R(S) = 0 \)).

Proof. Suppose that \( \varphi : R \to S \) is an epimorphism. To prove (a), it is enough to show that for each prime ideal \( p \) of \( R \), the fiber \( (\varphi^*)^{-1}(p) \) has at most one element. The fiber \( (\varphi^*)^{-1}(p) \) is homeomorphic to
Spec($S \otimes_R \kappa(p)$). On the other hand, by Remark 2.3, the base change ring map $\kappa(p) \to \kappa(p) \otimes_R S$ is an epimorphism. Since $\kappa(p)$ is a field therefore by Corollary 2.3, $\kappa(p) \otimes_R S$ is a field whenever it is non-trivial. Therefore, the fiber $(\varphi^*)^{-1}(p)$ has at most one element. To prove (b), let $q$ be a prime ideal of $S$ laying over $p$, i.e. $\varphi^*(q) = p$. Consider the following commutative diagram of rings

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi = \text{epic}} & S \\
\downarrow & & \downarrow \\
\kappa(p) & \xrightarrow{\text{epic}} & \kappa(q)
\end{array}
\]

since the composition $R \to \kappa(p) \to \kappa(q)$ is an epimorphism hence the induced ring map $\kappa(p) \to \kappa(q)$ is also an epimorphism; in fact it is an isomorphism by Corollary 2.3. The assertion (c) is obvious by Theorem 2.1, condition (iii).

Conversely, suppose that (a), (b) and (c) hold. To prove the assertion we act as follows. Let $T$ be a reduced ring and let $f, g : S \to T$ be two ring maps so that $f \circ \varphi = g \circ \varphi$. We have $\varphi^* \circ f^* = (f \circ \varphi)^* = (g \circ \varphi)^* = \varphi^* \circ g^*$. Since $\varphi^*$ is injective therefore $f^* = g^*$. For each prime ideal $\mathfrak{p}$ of $T$, set $q = f^*(\mathfrak{p})$, also set $p = \varphi^*(q)$. Denote by

\[
\tilde{\varphi} : \prod_{\mathfrak{p} \in \text{Spec}(T)} \kappa(p) \to \prod_{\mathfrak{p} \in \text{Spec}(T)} \kappa(q)
\]

the induced ring map via $\varphi : R \to S$ which is an isomorphism by the hypothesis (b). Similarly, denote by

\[
\tilde{f}, \tilde{g} : S' = \prod_{\mathfrak{p} \in \text{Spec}(T)} \kappa(q) \to T' = \prod_{\mathfrak{p} \in \text{Spec}(T)} \kappa(\mathfrak{p})
\]

the ring maps induced respectively by $f$ and $g$. From $f \circ \varphi = g \circ \varphi$ we conclude that $\tilde{f} \circ \tilde{\varphi} = \tilde{g} \circ \tilde{\varphi}$. Since $\tilde{\varphi}$ is an isomorphism therefore $\tilde{f} = \tilde{g}$. The following diagram is commutative

\[
\begin{array}{ccc}
S & \xrightarrow{f, g} & T \\
\downarrow{\rho} & & \downarrow{\rho'} \\
S' & \xrightarrow{\tilde{f}, \tilde{g}} & T'
\end{array}
\]

i.e. $\tilde{f} \circ \rho = \rho' \circ f$ and $\tilde{g} \circ \rho = \rho' \circ g$. Thus $\rho' \circ f = \rho' \circ g$. Since $T$ is reduced, therefore $\rho'$ is injective and so $f = g$. Since $i_1 \circ \varphi = i_2 \circ \varphi$ therefore $\eta \circ i_1 \circ \varphi = \eta \circ i_2 \circ \varphi$ where $\eta : S \otimes_R S \to (S \otimes_R S)_{\text{red}}$ is the canonical projection map. By applying what we
have just proved for a morphism with target a reduced ring, we conclude that \( \eta \circ i_1 = \eta \circ i_2 \), therefore for each \( s \in S \), \( s \otimes 1 - 1 \otimes s \) is nilpotent in \( S \otimes_R S \). The ideal \( J \) is generated by the elements of the form \( s \otimes 1 - 1 \otimes s \); on the other hand, by the hypothesis \((c)\), \( J = \text{Ker}(p) \) is finitely generated hence \( J \) is generated by a finite number of nilpotent elements of the form \( s \otimes 1 - 1 \otimes s \), therefore \( J \) is a nilpotent ideal, and since \( J \) is idempotent thus \( J = 0 \). Hence, by the Theorem \[2.1\] condition \((iii)\), \( \varphi \) is an epimorphism. \( \square \)

**Corollary 2.8.** Suppose that the ring map \( \varphi : R \rightarrow S \) is an epimorphism. Then \( \dim(S) \leq \dim(R) \) where \( \dim(R) \) denotes the Krull-dimension of \( R \).

**Proof.** By Proposition \[2.7\] condition \((a)\), the induced map \( \varphi^* : \text{Spec}(S) \rightarrow \text{Spec}(R) \) is injective. Therefore, for every strict chain of prime ideals \( q_0 \subset q_1 \subset \ldots \subset q_d \) of \( S \) of length \( d \), the chain \( \varphi^*(q_0) \subset \varphi^*(q_1) \subset \ldots \subset \varphi^*(q_d) \) of prime ideals of \( R \) is also strict and so it is of length \( d \) too. This implies that \( \dim(S) \leq \dim(R) \). \( \square \)

**Proposition 2.9.** A ring map \( \varphi : R \rightarrow S \) is surjective if and only if it is an epimorphism and \( S/N \) is a finitely generated \( R \)-module where \( N = \text{Im}(\varphi) \).

**Proof.** Suppose that \( \varphi \) is an epimorphism which is not surjective. By Theorem \[3.17\] condition \((v)\), \( S \otimes_R S/N = 0 \), therefore \( S/N \otimes_R S/N = 0 \). Also recall that, if \( M \) be a nonzero finitely generated \( R \)-module, then there exists an increasing sequence of \( R \)-submodules of \( M \) as \( 0 = M_1 \subseteq \ldots \subseteq M_d = M \) so that for each \( 1 \leq i \leq d \), the quotients \( M_i/M_{i-1} \) are isomorphic to \( R/I_i \) for some ideals \( I_i \) of \( R \). In particular, there exists some proper ideal \( I \) of \( R \) and a surjective homomorphism \( M \rightarrow R/I \). Apply this fact, then there exists some proper ideal \( I \) of \( R \) and a surjective homomorphism \( \eta : S/N \rightarrow R/I \). Therefore, the homomorphism \( \eta \otimes \eta : S/N \otimes_R S/N \rightarrow R/I \otimes_R R/I = R/I \) is also surjective. But, since \( S/N \otimes_R S/N = 0 \), thus \( I = R \), a contradiction. \( \square \)

**Proposition 2.10.** Suppose that the ring map \( \varphi : R \rightarrow S \) is a flat epimorphism \((\varphi \) is both flat and an epimorphism). Then the following conditions hold.

(i) For each prime ideal \( q \) of \( S \), the induced local homomorphism \( \varphi_q : \)
$R_p \to S_q$ is an isomorphism where $p = \varphi^*(q)$.

(ii) For each ideal $J$ of $S$, $J = I^c$ where $I = \varphi^{-1}(J)$.

(iii) The induced map $\varphi^* : \text{Spec}(S) \to \text{Spec}(R)$ is an homeomorphism onto its image.

(iv) If $R$ is a noetherian ring then so is $S$.

(v) If $R$ is an Artinian ring then so is $S$.

\textbf{Proof.} (i): Since $\varphi : R \to S$ is flat therefore by [9, Theorem 7.2], $\varphi_q : R_p \to S_q$ is faithfully flat. On the other hand, since the following diagram is commutative

\begin{align*}
R & \xrightarrow{\varphi=\text{epic}} S \\
R_p & \xrightarrow{\varphi_q} S_q
\end{align*}

therefore the composition $R \xrightarrow{\varphi} R_p \xrightarrow{\varphi_q} S_q$ and so the induced ring map $\varphi_q : R_p \to S_q$ is an epimorphism. Now the assertion implies from Corollary 2.2.

(ii): We have $IS \subseteq J$. For the reverse inclusion, apply Theorem 2.1 condition (iii), one has $S/J \cong S/J \otimes_R S$ as $S$–modules. On the other hand, since $S$ is flat over $R$, thus from the exact sequence

\begin{align*}
0 & \longrightarrow R/I \xrightarrow{\varphi} S/J \\
\end{align*}

we obtain the following exact sequence

\begin{align*}
0 & \longrightarrow R/I \otimes_R S \xrightarrow{\varphi \otimes 1_S} S/J \otimes_R S.
\end{align*}

Furthermore, the following diagram is commutative

\begin{align*}
R/I \otimes_R S & \xrightarrow{\varphi \otimes 1_S} S/J \otimes_R S \\
S/IS & \xrightarrow{} S/J
\end{align*}

therefore $S/IS \to S/J$ is injective. Thus, $J \subseteq IS$.

(iii): By (ii), $\varphi^*$ is a closed map onto its image.

(iv): Take an arbitrary ideal $J$ of $S$, the ideal $I = \varphi^{-1}(J) = \langle a_1, ..., a_p \rangle$ is finitely generated since $R$ is a noetherian ring. By (ii), $J = IS = \langle \varphi(a_1), ..., \varphi(a_p) \rangle$, therefore $S$ is a noetherian ring.

(v): By (ii), this is obvious. □
Proposition 2.11. Let \( \varphi : R \to S \) be a ring map. The following conditions are equivalent.

(i) \( \varphi \) is a flat epimorphism.

(ii) For each maximal ideal \( p \) of \( R \), the induced ring map \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is a flat epimorphism.

(iii) For each maximal ideal \( q \) of \( S \), the induced ring map \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is an isomorphism where \( p = \varphi^*(q) \).

(iv) For each maximal ideal \( p \) of \( R \), either \( pS = S \) or \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is an isomorphism.

Proof. (i) \( \iff \) (ii): This is obvious by Lemma 3.15 and [1, Proposition 3.10].

(ii) \( \Rightarrow \) (iv): Take a prime ideal \( p \) of \( R \), by the hypothesis the induced ring map \( \varphi \otimes 1_p : R_p = R \otimes_R R_p \to S \otimes_R R_p \) is a flat epimorphism. If \( \varphi \otimes 1_p \) be faithfully flat then by Corollary 2.2, it is an isomorphism; otherwise, by [9, Theorem 7.2, condition (3)], we have \((S \otimes_R R_p) \otimes_{R_p} \kappa(p) = 0\) and so \( S \otimes_R \kappa(p) = 0 \). Since \( S \) is flat over \( R \) via \( \varphi \), from the exact sequence \( 0 \to R/p \to \kappa(p) \) we obtain the following exact sequence \( 0 \to S \otimes_R R/p \to S \otimes_R \kappa(p) \). Thus, \( pS = S \).

(iv) \( \Rightarrow \) (iii): There is nothing to prove.

(iii) \( \Rightarrow \) (i): By [9, Theorem 7.1], \( \varphi : R \to S \) is flat. To prove that \( \varphi \) is an epimorphism, by Theorem 2.1, condition (iv), it is enough to show that for each maximal ideal \( q \) of \( S \), the induced morphism \( (i_1)_q : S_q \to (S \otimes_R S)_q \) is an isomorphism. But the map \((i_1)_q\) is the composition of the following natural isomorphisms.

\[
\begin{align*}
S_q & \sim S_q \otimes_{R_q} R_p \sim S_q \otimes_{R_q} (S \otimes_R R_p) \sim (S \otimes_R S)_q
\end{align*}
\]

where \( p = \varphi^*(q) \). Therefore, \((i_1)_q\) is an isomorphism.

Remark 2.12. Proposition 2.11 holds also if we consider the prime ideals instead of maximal ideals at appropriate places.

We require the following lemma in sequel.

Lemma 2.13. Suppose that a flat ring map \( \varphi : R \to S \) decompose as \( \varphi : R \xrightarrow{\psi} A \xrightarrow{\varphi'} S \) so that \( \varphi' \) is an injective ring map and the
following diagram is commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi'} & S \\
\downarrow{\varphi'} & & \downarrow{i_2} \\
S & \xrightarrow{i_1} & S \otimes_R S
\end{array}
\]

Then \( \varphi' \) is a flat morphism.

**Proof.** The function \( A \times S \to S \) given by \((a, s) \mapsto \varphi'(a)s\) is a \(R\)-bilinear map and so by the universal property of the tensor product there exists a (unique) homomorphism of \(R\)-modules \( \rho : A \otimes_R S \to S \) in which \(a \otimes s\) is mapped into \(\varphi'(a)s\). Obviously, \( \rho \) is a surjective homomorphism of \(R\)-algebras. To prove the assertion, first we shall prove that \( \rho \) is an isomorphism. By the hypotheses, for each \(a \in A\), one has \(\varphi'(a) \otimes_R 1 = 1 \otimes_R \varphi'(a)\) in \(S \otimes_R S\). Hence, the following diagram is commutative

\[
\begin{array}{ccc}
A \otimes_R S & \xrightarrow{\rho} & S \\
\downarrow{\varphi' \otimes 1_S} & & \downarrow{j} \\
S \otimes_R S & \xrightarrow{\eta} & S \otimes_A S
\end{array}
\]

where \(j(s) = 1 \otimes s\) and using the universal property of the tensor product, \(\eta\) is the unique ring map in which maps each \(s \otimes_R s'\) into \(s \otimes_A s'\). The hypotheses imply that the map \(\eta\) is in fact an isomorphism. On the other hand, since \(\varphi'\) is injective and \(S\) is flat over \(R\) therefore \(\varphi' \otimes 1_S\) is also injective. Hence, \(\rho\) is injective too.

Now, suppose that \( 0 \to N \xrightarrow{f} M \) is an exact sequence of \(A\)-modules, the sequence \( 0 \to N \otimes_R S \xrightarrow{f \otimes 1_S} M \otimes_R S \) is exact since \(S\) is flat over \(R\). Then using the isomorphism \(\rho : A \otimes_R S \to S\), we have the natural isomorphisms

\[
\varpi_M : M \otimes_A S \cong M \otimes_A (A \otimes_R S) \cong M \otimes_R S.
\]

Naturalness means that the following diagram is commutative.

\[
\begin{array}{ccc}
N \otimes_A S & \xrightarrow{f \otimes 1_S} & M \otimes_A S \\
\downarrow{\varpi_M} & & \downarrow{\varpi_N} \\
N \otimes_R S & \xrightarrow{f \otimes 1_S} & M \otimes_R S
\end{array}
\]

Therefore, \(f \otimes_A 1_S\) is also injective. Hence, \(S\) is flat over \(A\). \(\square\)
Let $\mathfrak{N}$ being the nil-radical of $R$, the quotient ring $R/\mathfrak{N}$ is a reduced ring, it is usually denoted by $R_{\text{red}}$. For every ring map $\varphi : R \to S$, since $\varphi(\mathfrak{N}R) \subseteq \mathfrak{N}S$, hence $\varphi$ induces the ring map $\varphi_{\text{red}} : R_{\text{red}} \to S_{\text{red}}$. Indeed, the assignment

$$\text{red} : \mathcal{C}\text{-Ring} \to \mathcal{C}\text{-Ring}$$

is a covariant functor. We have the following result.

**Lemma 2.14.** Suppose that $\varphi : R \to S$ is a flat epimorphism. Then $\varphi$ is surjective if and only if the induced map $\varphi_{\text{red}}$ is surjective.

**Proof.** If $\varphi$ is surjective, then $\varphi_{\text{red}}$ is also surjective, because the following diagram is commutative.

$$\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
R_{\text{red}} & \xrightarrow{\varphi_{\text{red}}} & S_{\text{red}}
\end{array}$$

Conversely, suppose that $\varphi_{\text{red}}$ is surjective. Decompose $\varphi$ as

$$\begin{array}{ccc}
R & \xrightarrow{\pi} & R_{\text{Ker(\varphi)}} \\
& & \downarrow \\
& \xrightarrow{\varphi'} & S
\end{array}$$

where $\pi$ is the canonical ring map and $\varphi'$ is the injective ring map induced by $\varphi$. Obviously, $\text{Im}(\varphi) = \text{Im}(\varphi')$, $\varphi'$ is an epimorphism and since $\varphi_{\text{red}} = \varphi'_{\text{red}} \circ \pi_{\text{red}}$ therefore $\varphi'_{\text{red}}$ is surjective. Moreover, by Lemma 2.13 $\varphi'$ is flat. Therefore, without loss of generality, we can assume that $\varphi$ is an injective ring map. In this case, $\varphi_{\text{red}}$ is an isomorphism and so $\varphi^* : \text{Spec}(S) \to \text{Spec}(R)$ is an homeomorphism. Thus for each prime ideal $p$ of $R$, $pS \neq S$. Then, from the exact sequence of $R$–modules

$$0 \longrightarrow R \xrightarrow{\varphi} S \longrightarrow S/R \longrightarrow 0$$

we conclude that for each maximal ideal $p$ of $R$, $S/R \otimes R_p = 0$, because by Proposition 2.11 condition (iv), $\varphi \otimes 1_R : R \otimes R_p \to S \otimes R_p$ is an isomorphism. Therefore, $S/R = 0$ and so $\varphi$ is surjective. □

3. **Epimorphisms and Absolutely Flat Rings**

In this Section after introducing the basic properties of absolutely flat rings we then study their relationship with epimorphisms. A ring $R$ is said to be absolutely flat if each $R$–module is flat.
Proposition 3.1. Let $R$ be a ring. Then the following conditions are equivalent.

(i) $R$ is absolutely flat.
(ii) Every ideal of $R$ is idempotent.
(iii) Every principal ideal of $R$ is idempotent.
(iv) Every finitely generated ideal of $R$ is a direct summand of $R$.

Proof. (i) $\Rightarrow$ (ii): Let $I$ be an arbitrary ideal of $R$. $R/I$ is a flat $R$-module since $R$ is absolutely flat. Therefore from the exact sequence $0 \longrightarrow I \longrightarrow R$ we obtain the following exact sequence of $R$-modules.

$0 \longrightarrow I \otimes_R R/I \xrightarrow{i_{I/I^2}} R \otimes_R R/I$. Hence, the composition homomorphism

$I/I^2 \xrightarrow{i} I \otimes_R R/I \xrightarrow{i_{I/I^2}} R \otimes_R R/I \xrightarrow{\sigma} R/I$

given by $a + I^2 \sim a + I$ is both injective and a zero homomorphism. Thus, $I = I^2$.

(ii) $\Rightarrow$ (iii): There is nothing to prove.

(iii) $\Rightarrow$ (iv): First of all we show that every finitely generated ideal of $R$ is a principal ideal and generated by an idempotent element. Let $I = \langle a_1, ..., a_n \rangle$ be a finitely generated ideal of $R$. Since each principal ideal of $R$ is idempotent therefore for each $1 \leq i \leq n$, $Ra_i = Ra_i^2$ thus there exists some $c_i \in R$ so that $a_i = c_ia_i^2$. For each $1 \leq i \leq n$, set $e_i = c_ia_i$, then $e_i$ is an idempotent element of $R$ and one can also observe that $I = \langle e_1, ..., e_n \rangle$. Because, obviously one has $\langle e_1, ..., e_n \rangle \subseteq I$. For the reverse inclusion, for each $1 \leq i \leq n$, $a_i = a_ie_i$ therefore $I \subseteq \langle e_1, ..., e_n \rangle$.

Now by induction on $n$, we show that $I = \langle e_1, ..., e_n \rangle$ is generated by an idempotent element. If $n = 1$ there is nothing to prove. Let $n > 1$, then by the induction hypothesis the ideal $\langle e_1, ..., e_{n-1} \rangle$ is generated by some idempotent element $e \in R$, i.e $\langle e_1, ..., e_{n-1} \rangle = \langle e \rangle$. Therefore, $I = \langle e_1, ..., e_n \rangle = \langle e_1, ..., e_{n-1} \rangle + \langle e_n \rangle = \langle e, e_n \rangle$. But $e + e_n - ee_n$ is an idempotent element of $R$ and also one can observe that $\langle e, e_n \rangle = \langle e + e_n - ee_n \rangle$. Because, obviously $\langle e + e_n - ee_n \rangle \subseteq \langle e, e_n \rangle$. For the reverse inclusion, since $e = e(e + e_n - ee_n)$ and similarly $e_n = e_n(e + e_n - ee_n)$, thus $\langle e, e_n \rangle \subseteq \langle e + e_n - ee_n \rangle$.

Finally, every principal ideal $\langle e \rangle$ of $R$ which is generated by an idempotent element $e \in R$ is a direct summand of $R$. Because, $R = \langle e \rangle + \langle 1 - e \rangle$ and $\langle e \rangle \cap \langle 1 - e \rangle = 0$.

(iv) $\Rightarrow$ (i): Let $M$ be an arbitrary $R$-module. By [9, Theorem 7.7], $M$ is a flat $R$-module if and only if for each finitely generated ideal $I$
of $R$, the natural homomorphism $I \otimes_R M \to M$ given by $a \otimes m \mapsto am$ is injective. Let $I$ be a finitely generated ideal of $R$, by the hypothesis (iv), there exists some ideal $J$ of $R$ in which $R = I \oplus J$. Therefore the following sequence is exact and split.

$$
\begin{array}{cccccc}
0 & \longrightarrow & I & \stackrel{i}{\longrightarrow} & R & \stackrel{p}{\longrightarrow} & J & \longrightarrow & 0
\end{array}
$$

where $i$ is the inclusion map and $p$ is the projection. Since an exact and split sequence is left exact and split by an additive functor, therefore the following sequence is exact and split.

$$
\begin{array}{cccccc}
0 & \longrightarrow & I \otimes_R M & \stackrel{i \otimes 1}{\longrightarrow} & R \otimes_R M & \stackrel{p \otimes 1}{\longrightarrow} & J \otimes_R M & \longrightarrow & 0
\end{array}
$$

thus the composition map $I \otimes_R M \stackrel{i \otimes 1}{\longrightarrow} R \otimes_R M \cong M$ is injective.

The above Proposition have some consequences which we state them at here.

**Corollary 3.2.** If a local ring is absolutely flat, then it is a field.

**Proof.** Let $R$ be a absolutely flat local ring with the maximal ideal $m$. For each element $a \in m$, by the above Proposition, one has $Ra = Ra^2$, so there exists some $c \in R$ in which $a = ca^2$, therefore $a(1 - ca) = 0$. But the element $1 - ca$ is unitary in $R$ therefore $a = 0$.

Recall that for every two ideals $I$ and $J$ of $R$ with $I \subseteq J$ and for each natural number $n \geq 0$, one can easily observe that

$$
\left( \frac{J}{I} \right)^n = \frac{J^n + I}{I}.
$$

**Corollary 3.3.** If $R$ be an absolutely flat ring and let $I$ be an ideal of $R$. Then $R/I$ is absolutely flat.

**Proof.** Let $K = J/I$ be an arbitrary ideal of $R/I$ where $J$ is an ideal of $R$ which contains $I$. Since $R$ is absolutely flat thus $J = J^2$. But

$$
K^2 = \frac{J^2 + I}{I} = \frac{J + I}{I} = \frac{J}{I} = K.
$$
Hence by the above Proposition, $R/I$ is absolutely flat. □

**Corollary 3.4.** Let $\{R_i\}_{i \in I}$ be a family of rings. Then the direct product ring $\prod_{i \in I} R_i$ is absolutely flat if and only if for each $j \in I$, $R_j$ is so.

**Proof.** ($\Leftarrow$): Take an arbitrary element $a = (a_i)_{i \in I}$ of $\prod_{i \in I} R_i$. For each $i \in I$, $R_i$ is absolutely flat and so there exists some $b_i \in R_i$ so that $a_i = a_i^2 b_i$. Take $b = (b_i)_{i \in I}$, then obviously $a = a^2 b$ and so the assertion implies from Proposition 3.1.

($\Rightarrow$): For each $j \in I$, consider the projection ring map $\pi_j : \prod_{i \in I} R_i \to R_j$. Then by Corollary 3.3 $R_j$ is absolutely flat. □

**Corollary 3.5.** Let $R$ be a ring so that for each element $a \in R$ there exists some natural number $n \geq 2$ (depending on $a$) in which $a = a^n$. Then $R$ is absolutely flat. In particular, every Boolean ring is absolutely flat.

**Proof.** For every element $a \in R$, set $I = Ra$. By the hypothesis there exists some $n \geq 2$ so that $a = a^n$, therefore $I = I^n$. Since $n \geq 2$ so $I = I^n \subseteq I^2 \subseteq I$. Hence $I = I^2$ and the assertion implies from the Proposition 3.1. □

**Corollary 3.6.** Let $R$ be a Noetherian ring which is also absolutely flat. Then it is a principal ideal ring. Moreover, if it is a non-trivial ring with the trivial idempotents then it is a field. □

**Proposition 3.7.** Let $R$ be a absolutely flat ring and let $S$ be a multiplicative subset of $R$. Then $S^{-1}R$ is absolutely flat.

**Proof.** Let $M$ be an arbitrary $S^{-1}R$–module and let $0 \longrightarrow N' \overset{f}{\longrightarrow} N$ be an exact sequence of $S^{-1}R$–modules. Since $R$ is absolutely flat then we obtain the following exact sequence $0 \longrightarrow N' \otimes_R M \overset{f \otimes 1}{\longrightarrow} N \otimes_R M$. On the other hand, we have the natural isomorphisms $N \otimes_R M \cong N \otimes_{S^{-1}R} M$ for every $S^{-1}R$–modules $M$ and $N$. Therefore we get the following exact sequence of $S^{-1}R$–modules

$$0 \longrightarrow N' \otimes_{S^{-1}R} M \overset{f \otimes 1}{\longrightarrow} N \otimes_{S^{-1}R} M.$$

□
Corollary 3.8. Let $R$ be a ring. Then $R$ is absolutely flat if and only if for every maximal ideal $p$ of $R$, $R_p$ is absolutely flat.

Proof. By Proposition 3.7, the implication $\Rightarrow$ is obvious. For the reverse implication, let $M$ be an arbitrary $R$–module and take the following exact sequence of $R$–modules

$$0 \to N' \xrightarrow{f} N.$$

Let $K$ be the kernel of the morphism $f \otimes 1 : N' \otimes_R M \to N \otimes_R M$. To prove the assertion it is sufficient to show that $K_p = 0$ for each maximal ideal $p$ of $R$. Since $R_p$ is absolutely flat therefore from the exact sequence $0 \to N'_p \to N_p$ we obtain the following exact sequence

$$0 \to N'_p \otimes_{R_p} M_p \to N_p \otimes_{R_p} M_p.$$

This implies that the induced morphism $(f \otimes 1)_p : (N' \otimes_R M)_p \to (N \otimes_R M)_p$ is injective. Therefore $K_p = 0$. □

Corollary 3.9. Let $\varphi : R \to S$ be an epimorphism so that $R$ is absolutely flat. Then $\varphi$ is surjective.

Proof. Decompose $\varphi$ as

$$R \xrightarrow{\pi} R_{\text{Ker}(\varphi)} \xrightarrow{\overline{\varphi}} S,$$

where $\pi$ is the canonical ring map and $\overline{\varphi}$ in the injective ring map induced by $\varphi$. By Corollary 3.3, $R/\text{Ker}(\varphi)$ is an absolutely flat ring. Moreover, $\text{Im}(\varphi) = \text{Im}(\overline{\varphi})$ and yet $\overline{\varphi}$ is an epimorphism. Hence, to prove the assertion, without loss of generality, we can assume that $\varphi$ is an injective ring map. In this case, $\varphi$ is a faithfully flat morphism. Because, suppose that $S \otimes_R M = 0$ for an $R$–module $M$. From the following exact sequence of $R$–modules

$$0 \to R \xrightarrow{\varphi} S \xrightarrow{\pi} S/R \to 0,$$

we obtain the following long exact sequence of $R$–modules

$$\ldots \to \text{Tor}^R_1(S/R, M) \to R \otimes_R M \xrightarrow{\varphi \otimes 1_M} S \otimes_R M \xrightarrow{\pi \otimes 1_M} S/R \otimes_R M \to 0.$$

Since $S/R$ is a flat $R$–module, then by [16 Theorem 7.2], $\text{Tor}^R_1(S/R, M) = 0$. Thus, $M \cong R \otimes_R M = 0$.

Since $\varphi$ is a faithfully flat epimorphism, therefore by Corollary 2.2 it
is an isomorphism. □

**Definition 3.10.** Let \( R \) be a ring and let \( a \in R \). If there exists an element \( b \in R \) so that \( a^2b = a \) and \( ab^2 = b \), then \( b \) is said to be a punctual inverse of \( a \).

**Lemma 3.11.** An element \( b \in R \) is a punctual inverse of \( a \in R \) if and only if \( a \in Ra^2 \). Moreover, the punctual inverse, if it exists, is unique and also there exists some idempotent element \( e \in R \) so that 
\[
(e + a)(e + b) = 1.
\]

**Proof.** Suppose that \( a \in Ra^2 \). Then there exists some \( r \in R \) so that \( a = ra^2 \). Put \( b = ar^2 \), then obviously \( b \) is a punctual inverse of \( a \). Take \( e = 1 - ab \), then the element \( e \) clearly is an idempotent and \((e + a)(e + b) = 1\). Now suppose that \( a \) has another punctual inverse \( c \in R \). Then it is easy to see that \( e = 1 - ac \), hence \( b = c \). □

The punctual inverse of \( a \in R \), if it exists, is usually denoted by \( a^{-1} \).

**Lemma 3.12.** Let \( \varphi : R \to S \) be a ring map. Suppose that the elements \( a, b \in R \) have the punctual inverses, then the punctual inverses of \( \varphi(a) \) and \( ab \) exist and one has \((\varphi(a))^{-1} = \varphi(a^{-1})\) and \((ab)^{-1} = a^{-1}b^{-1}\).

**Proof.** One can verify this without trouble. □

**Proposition 3.13.** Let \( R \) be a ring and let \( S \) be an arbitrary subset of \( R \). Then there exists an \( R \)-algebra \( S^{-1}R \) with the structure morphism \( \eta : R \to S^{-1}R \) so that for each \( s \in S \), the punctual inverse of \( \eta(s) \) in \( S^{-1}R \) exists and the pair \((S^{-1}R, \eta)\) satisfies in the following universal property:

(P) Let \( \varphi : R \to R' \) be a ring map so that for each \( s \in S \) the punctual inverse of \( \varphi(s) \) exists in \( R' \), then there exists a unique ring map \( \psi : S^{-1}R \to R' \) such that \( \varphi = \psi \circ \eta \).
Proof. Consider the polynomial ring $R[x_s : s \in S]$ with distinct variables $x_s$ for each $s \in S$ and then set

$$S^{(-1)} R = \frac{R[x_s : s \in S]}{I}$$

where the ideal $I$ is generated by the elements of the form $sx_s^2 - x_s$ and $s^2x_s - s$ where $s \in S$, let $\eta : R \to S^{(-1)} R$ be the canonical ring map. Obviously, for each $s \in S$, the element $x_s + I$ is the punctual inverse of $\eta(s) = s + I$.

Now, let $\varphi : R \to R'$ be a ring map so that for each $s \in S$, the punctual inverse of $\varphi(s)$ exists in $R'$. The map $\varphi$ induces a unique homomorphism of $R$-algebras $\tilde{\varphi} : R[x_s : s \in S] \to R'$ so that for $s \in S$, $x_s \sim (\varphi(s))^{(-1)}$. It is clear that $\tilde{\varphi}(I) = 0$. Denote by $\psi : S^{(-1)} R \to R'$ the ring map induced by $\tilde{\varphi}$. Obviously, $\psi$ is the unique ring map which satisfies $\varphi = \psi \circ \eta$. Because, suppose that $\psi' : S^{(-1)} R \to R'$ is another such ring map. For each $s \in S$, one has $\psi(x_s + I) = \tilde{\varphi}(x_s) = (\varphi(s))^{(-1)} = (\psi'(\eta(s)))^{(-1)} = \psi'((\eta(s))^{(-1)}) = \psi'(x_s + I)$; therefore $\psi = \psi'$. \Box

We call $S^{(-1)} R$ the punctual ring of $R$ with respect to $S$. Let $S$ be a multiplicative subset of $R$ and then consider the canonical ring map $\pi : R \to S^{-1} R$. By the universal property of the punctuality, there exists a unique ring map $\psi : S^{(-1)} R \to S^{-1} R$ so that $\pi = \psi \circ \eta$.

**Proposition 3.14.** Let $R$ be a ring and let $S$ be an arbitrary subset of $R$. Then

(i) The canonical homomorphism $\eta : R \to S^{(-1)} R$ is an epimorphism.
(ii) The map $\eta^\ast : \text{Spec}(S^{(-1)} R) \to \text{Spec}(R)$ is bijective.
(iii) For each $s \in S$, $(\eta^\ast)^{-1}(V(s))$ is a clopen (both open and closed) subset in $\text{Spec}(S^{(-1)} R)$.
(iv) The punctual ring $S^{(-1)} R$ is nontrivial if and only if $R$ is so.
(v) $\text{Ker}(\eta) \subseteq \mathfrak{N}$ where $\mathfrak{N}$ is the nil-radical of $R$.

Proof. (i) : This implies from the universal property of the punctual ring.

(ii) : Since $\eta$ is an epimorphism then by Proposition 2.7, $\eta^\ast$ is injective. To prove that $\eta^\ast$ is surjective, we act as follows. Take a prime ideal $p$ of $R$ and then consider the canonical ring map $\rho : R \to \kappa(p)$. Since $\kappa(p)$ is a field therefore the image of every element of $R$ under $\rho$ has the punctual inverse in $\kappa(p)$. Hence, by the universal property of the punctuality, there exists a (unique) ring map $\psi : S^{(-1)} R \to \kappa(p)$ so that
$\rho = \psi \circ \eta$. Set $q = \psi^*(\{0\}) \in \text{Spec}(S^{(-1)}R)$, then one has $p = \eta^*(q)$.

(iii) : We know that $V(s) = V(\eta(s))$. Moreover, $V(\eta(s)) = D(1 - \eta(s)(\eta(s))^{(-1)})$. Therefore, $(\eta^*)^{-1}(V(s))$ is both open and closed.

(iv) and (v): These are immediate consequences of (ii). $\square$

The punctual ring of $R$ with respect to $S = R$ is usually denoted by $R^{ab}$, i.e. $R^{ab} = R^{(-1)}R$. Indeed, using the universal property of the punctuality, the assignment $ab : \mathbf{C-Ring} \rightarrow \mathbf{C-Ring}$ is a covariant functor.

**Lemma 3.15.** Let $\varphi : R \rightarrow S$ be an epimorphism. Suppose that $S$ is a nontrivial ring with the trivial idempotents and let for each $r \in R$, $\varphi(r)$ has the punctual inverse in $S$. Then $A = \text{Im}(\varphi)$ is an integral domain and $S$ is its field of fractions.

**Proof.** Suppose that $\varphi(r)\varphi(r') = 0$ for some elements $r, r' \in R$. If $\varphi(r) \neq 0$, then it is invertible in $S$. Because, $\varphi(r)(\varphi(r))^{(-1)}$ is a non-zero idempotent element of $S$, since $\varphi(r) = (\varphi(r))^2(\varphi(r))^{(-1)} \neq 0$, thus $\varphi(r)(\varphi(r))^{(-1)} = 1$. Therefore, $\varphi(r') = 0$. Moreover, since $S$ is nontrivial so $A = \text{Im}(\varphi)$ is also nontrivial. Hence, $A$ is an integral domain. Let $K$ be the field of its fractions. Since every non-zero element of $A$ is unitary in $S$ hence by the universal property of the localization, there exists a (unique) ring map $\psi : K \rightarrow S$ so that $i = \psi \circ j$ where $i : A \rightarrow S$ and $j : A \rightarrow K$ are the inclusion ring maps. On the other hand, the map $\varphi$ decomposes as $\varphi = i \circ \varphi'$ where $\varphi' : R \rightarrow A$ is the ring map induced by $\varphi$. Hence, $i$ and so $\psi$ are epimorphisms. By Corollary 2.3, $\psi$ is an isomorphism. $\square$

**Remark 3.16.** Let $g : X \rightarrow Y$ be a continuous map where $X$ is quasi-compact and $Y$ is Hausdorff. Then $g$ is a closed map. Because, since $X$ is quasi-compact therefore each closed subset $F$ of $X$ is also quasi-compact and so $g(F)$ is quasi-compact subset in $Y$ too. But, Hausdorffness of $Y$ implies that $g(F)$ is closed.

**Theorem 3.17.** Let $R$ be a ring and let $\eta : R \rightarrow R^{ab}$ be the canonical ring map. Then

(i) The induced map $\eta^* : \text{Spec}(R^{ab}) \rightarrow \text{Spec}(R)$ is an homeomorphism where $\text{Spec}(R^{ab})$ is equipped with the Zariski topology and $\text{Spec}(R)$ with the constructible topology.
For each prime ideal $q$ of $R^{ab}$, there exists a canonical isomorphism between the localization $(R^{ab})_q$ and the residue field $\kappa(p)$ where $p = \eta^*(q)$.

(iii) The punctual ring $R^{ab}$ is absolutely flat.

Proof. (i) : By Proposition 3.14 (ii), the map $\eta^*$ is bijective, we denote its inverse by $\rho : \text{Spec}(R) \rightarrow \text{Spec}(R^{ab})$. The map $\rho$ is continuous, because for each closed subset $V(J)$ of $\text{Spec}(R^{ab})$ where $J$ is an ideal of $R^{ab}$, we have

$$\rho^{-1}(V(J)) = \psi^*(\text{Spec}(\frac{R^{ab}}{J}))$$

which is a closed subset in $\text{Spec}(R)$ where $\psi : R \xrightarrow{\eta} R^{ab} \xrightarrow{\pi} R^{ab}/J$ is the composition ring map $\psi = \pi \circ \eta$. Moreover, the space $\text{Spec}(R^{ab})$ equipped with the Zariski topology is Hausdorff. Because, choose distinct prime ideals $q$ and $q'$ of $R^{ab}$; then take $p = \eta^*(q)$ and $p' = \eta^*(q')$, since $\eta^*$ is bijective therefore $p$ and $p'$ are distinct prime ideals of $R$. Hence, one can choose some element $a \in p - p'$, and so $q \in V(\eta(a))$ and $q' \in D(\eta(a))$. By Proposition 3.14 (iii), $V(\eta(a)) = (\eta^*)^{-1}(V(a))$ is an open subspace of $\text{Spec}(R^{ab})$ and we win.

Now, using the Remark 3.16 and also the fact that the constructible topology is compact we conclude that $\rho$ is a closed map and so it is an homeomorphism.

(ii) : For each prime ideal $q$ of $R^{ab}$, consider the composition ring map $\varphi : R \xrightarrow{\eta} R^{ab} \xrightarrow{(R^{ab})_q \text{ epic}} (R^{ab})_q$ where $R^{ab} \rightarrow (R^{ab})_q$ is the canonical ring map. The ring map $\varphi$ satisfies all of the hypotheses of Lemma 3.15 and so $(R^{ab})_q$ is a field. Then consider the following commutative diagram

\[
\begin{array}{ccc}
R_p & \xrightarrow{\eta_q=\text{epic}} & (R^{ab})_q \\
\downarrow & & \downarrow \\
\kappa(p) & \xrightarrow{=} & \kappa(q)
\end{array}
\]

where $p = \eta^*(q)$, therefore $\kappa(p) \rightarrow \kappa(q)$ is an epimorphism; in fact by Corollary 2.3, it is an isomorphism.

(iii) : Using Corollary 3.8 then the assertion is an immediate consequence of (ii). □
REFERENCES

[1] Atiyah, M. and Macdonald, G. Introduction to commutative algebra, Addison-Wesley Publishing Company, 1969.
[2] Borceux, F. Handbook of Categorical Algebra 1, Basic category theory, Cambridge University Press 1994.
[3] Borceux, F. Handbook of Categorical Algebra 2, Categories and Structures, Cambridge University Press 1994.
[4] Bourbaki Nicolas. Algèbre commutative, Chap. 1-7, Paris-Hermann, 1961-1965, (Bourbaki, 27, 28, 30 et 31).
[5] Chirvăsită, Alexandru. On epimorphisms and monomorphisms of Hopf algebras, Journal of Algebra, Vol. 323, 2010, p. 1593-1606.
[6] Laursen, Kjeld. Epimorphisms of $C^*$-algebras, North-Holland Mathematics Studies, Vol. 90, 1984, p. 219-232.
[7] Lazard, Daniel. Epimorphismes plats, Séminaire Samuel. Algèbre commutative, tomme 2(1967-1968), exp.$^0$.4, p. 1-12.
[8] Madden, James and Molitor, Andrew. Epimorphisms of frames, Journal of Pure and Applied Algebra, Vol. 70, 1991, p. 129-132.
[9] Matsumura, H. Commutative ring theory, Cambridge university Press, 1989.
[10] Northcott, D.G. Multilinear algebra, Cambridge University Press, 1984.
[11] Nummela, Eric C. On epimorphisms of topological groups, Journal of General Topology and its Applications, Vol. 9, p. 155-167, 1978.
[12] Olivier, Jean-Pierre. Anneaux absolument plats universels et épimorphismes à buts réduits, Séminaire Samuel. Algèbre commutative, tomme 2(1967-1968), exp.$^0$.6, p.1-12.
[13] Ronald, Prather. Epimorphisms of free monoids, Journal of Linear Algebra and its Applications, Vol. 13, 1976, p. 201-205.
[14] Reid, G.A. Epimorphisms and surjectivity, Invent. Math. 9 (1970) p. 295-307.
[15] Roby, Norbert. Diverses caractérisations des épimorphismes, Séminaire Samuel. Algèbre commutative, tomme 2(1967-1968), exp.$^0$.3, p.1-12.
[16] Rotman, J. An introduction to homological algebra, Second edition, Springer 2009.

DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCES, UNIVERSITY OF MARAGHEH, P. O. BOX 55181-83111, MARAGHEH, IRAN.
E-mail address: ebulez1978@gmail.com