Non-Critical Covariant Superstrings

Pietro Antonio Grassi $^{a,b,c}$ and Yaron Oz $^{d}$

$^a$ CERN, Theory Division, CH-1121 Genevè 23, Switzerland,
$^b$ DISTA, Università del Piemonte Orientale,
Via Bellini, 25/g 15100 Alessandria, Italy,
$^c$ Centro E. Fermi, Compendio Viminale, I-00184, Roma, Italy,
$^d$ Raymond and Beverly Sackler Faculty of Exact Science,
School of Physics and Astronomy, Tel-Aviv University, Ramat-Aviv 69978, Israel

We construct a covariant description of non-critical superstrings in even dimensions. We construct explicitly supersymmetric hybrid type variables in a linear dilaton background, and study an underlying $N = 2$ twisted superconformal algebra structure. We find similarities between non-critical superstrings in $2n + 2$ dimensions and critical superstrings compactified on $CY_{4-n}$ manifolds. We study the spectrum of the non-critical strings, and in particular the Ramond-Ramond massless fields. We use the supersymmetric variables to construct the non-critical superstrings $\sigma$-model action in curved target space backgrounds with coupling to the Ramond-Ramond fields. We consider as an example non-critical type IIA strings on $AdS_2$ background with Ramond-Ramond 2-form flux.

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1 pgrassi@cern.ch
2 yaronoz@post.tau.ac.il
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1. Introduction

The critical dimension for superstrings in flat space is $d = 10$. In dimensions $d < 10$, the Liouville mode is dynamical and should be quantized as well [1,2]. Such strings are called non-critical. The total conformal anomaly vanishes for the non-critical strings due to the Liouville background charge. Our aim is to construct a manifestly supersymmetric and covariant worldsheet description of non-critical superstrings, in particular for curved target space geometries and with coupling to Ramond-Ramond fields.

There are various motivations to study non-critical strings. First, non-critical strings can provide an alternative to string compactifications. Second, non-critical strings can provide a dual description of gauge theories. Examples of backgrounds one wishes to study are

$$ds^2 = d\varphi^2 + a^2(\varphi) d\vec{x}^2,$$

where $\vec{x} = (x_1, \ldots, x_{d-1})$, and with other background fields turned on. String theory on such warped backgrounds is expected to provide a dual description of gauge theories. Depending on the form of the warp factor $a^2(\varphi)$, the gauge theory can be confining, or at a conformal fixed point.

One complication in the study of non-critical superstrings is that unlike the critical case, there is no consistent approximation where supergravity provides a valid effective description. The reason being that the $d$-dimensional supergravity low-energy effective action contains a cosmological constant type term of the form

$$S \sim \int d^d x \sqrt{G} e^{-2\Phi} \left( \frac{d-10}{l_s^2} \right),$$

which vanishes only for $d = 10$. This implies that the low energy approximation $E \ll l_s^{-1}$ is not valid when $d \neq 10$, and the higher order curvature terms of the form $(l_s^2 R)^n$ cannot be discarded. A manifestation of this is that solutions of the $d$-dimensional supergravity equations have typically curvatures of the order of the string scale $l_s^2 R \sim O(1)$ when $d \neq 10$ [4,5,6,7,8].

The second complication is that interesting target space curved geometries include Ramond-Ramond (RR) field fluxes, and we face the need to quantize the strings in such backgrounds. The conventional (Ramond-Neveu-Schwarz) RNS formalism is inadequate, since it does not treat the RR fields on the same footing as the (Neveu-Schwarz) NSNS fields due to the non-polynomial couplings of the RR fields to the spin fields of the CFT.
A framework to study curved geometries that include RR fields fluxes, is a target space covariant formulation of non-critical superstrings. We expect this also to enable us to study D-branes in the non-critical superstrings and their dual configurations in the gauge theories.

We will start by considering the tachyon free non-critical superstrings of [9]. These are \((2n+2)\)-dimensional fermionic strings with \(n = 0, 1, 2, 3\) \((n = 4\) is the ten-dimensional critical superstring). The target space geometry is flat with a linear dilaton field. In the superconformal gauge they are described by \(2n+1\) matter superfields \(X^i, i = 1, \ldots, 2n+1 \equiv D\), and by a Liouville superfield \(\Phi_l\). In components we have \((x^i, \psi^i), (\varphi, \psi_l)\). We will build supersymmetric variables and use them to construct a covariant description of these non-critical superstrings. We will show how the RNS GSO projection is implemented automatically in the covariant formalism, and use this structure to couple the \(\sigma\)-model to curved backgrounds with RR fields. Unlike the Green-Schwarz (GS) \(\sigma\)-model construction, we will work in the framework where \(\kappa\)-symmetry is already fixed. It is also important to stress another fundamental difference compared to the GS formulation which is the field content of the \(\sigma\)-model. Besides the usual bosonic coordinates and the fermionic superspace coordinates \(\theta\)'s, in the present formulation there are also the conjugate momenta to \(\theta\)'s, denoted here by \(p\)'s which provide the linear couplings to the Ramond fields.

The space-time supersymmetry of the \((2n+2)\)-dimensional strings has effectively a supersymmetry structure of \(2n\)-dimensional space-time. In the RNS formalism of a linear dilaton background only half of the supercharges which can be constructed are mutually local with respect to each other [10], and only half of the mutually local supercharges are actual supersymmetries. However, to construct a manifest superspace approach to these string theories we will use a bigger superspace with a double number of fermionic coordinates with respect to the number of supersymmetries manifest. This superspace structure suggests that the non-critical superstrings may have solutions with double the number of supersymmetries of the linear dilaton background. We then follow a similar prescription to that of the hybrid formalism to build the worldsheet \(N = 2\) superconformal generators in a manifestly space-time supersymmetric way. The basic algebraic structure that we will use to compare the RNS variables to the hybrid type variables is a \(\hat{c} = 2\) twisted \(N = 2\) superconformal algebra, where the dimension one current is the BRST current of the non-critical superstring.

We will find a similar algebraic structure for the \((2n+2)\)-dimensional fermionic non-critical superstrings with \(n = 0, 1, 2, 3\) and the compactification independent part of critical
fermionic strings compactified on a $CY_{4-n}$ manifold. More precisely, we will identify an underlying $\hat{c} = n - 2$ twisted $N = 2$ superconformal algebra. Note, however, that the two systems are different. In particular, they differ by the amount of space-time supersymmetry, and the RR fields.

The paper is organized as follows. In the next section we review the structure of the non-critical strings in the RNS formalism. We analyze the BRST symmetry, $N = 2$ superconformal symmetry, the space-time supersymmetry and the spectrum. We construct a $\hat{c} = 2$ twisted $N = 2$ superconformal algebra that will be used as a basic structure to compare with the hybrid formalism. In section 3 we construct the covariant non-critical superstrings using hybrid variables. We compare the twisted $N = 2$ superconformal algebra with the RNS one. We consider in detail the two-dimensional superstrings. We construct the BRST operators and write the spectrum at ghost number one as well as the ground ring generators at ghost number zero in the supersymmetric variables. We show how the RNS GSO projection is implemented in the covariant formalism. We identify an underlying $\hat{c} = -2$ twisted $N = 2$ superconformal algebra, that compares with that of critical fermionic strings compactified on a $CY_4$ manifold. In section 4 we construct the covariant higher dimensional non-critical superstrings. The structure is similar to that of the two-dimensional superstrings. Again, we will enlarge the superspace and compare with critical fermionic strings compactified on $CY$ manifolds. We will comment on some subtleties in the cases $n = 2$ and $n = 3$ associated with the structure of the hybrid formalism. These cases work out much like critical fermionic strings compactified on a $CY_2$ and $CY_1$, respectively. In section 5 we analyze non-critical strings in curved target space backgrounds with coupling to Ramond-Ramond fields. We construct the worldsheet $\sigma$-models and consider the example of non-critical type IIA strings on $AdS_2$ background with Ramond-Ramond 2-form fields flux. Section 6 is devoted to a discussion.

2. Non-critical Superstrings

In this section we will consider fermionic strings propagating on a linear dilaton background of the form (in the string frame)

$$ds^2 = \eta_{ij} dx^i dx^j + dx^2 + d\varphi^2, \quad \Phi = -\frac{Q}{\sqrt{2}} \varphi,$$

where $i, j = 1, \ldots, 2n, n = 0, 1, 2, 3, x \in S^1$ of radius $R = 2/Q (\alpha' = 2), Q = \sqrt{4 - n}$, and $\Phi$ is the dilaton field. Note that $n = 4$ is the critical superstrings case. The analysis will
be performed in the perturbative regime where the string coupling is small: 
\[ g_s \sim e^\Phi \sim e^{-\frac{Q^2}{2}\phi} \ll 1. \]

Since a flat background with constant dilaton field is not a solution of the non-critical string equations, the linear dilaton background will be used by us to make the dictionary between the RNS non-critical strings and the covariant non-critical strings. This dictionary will be used later in order to couple the non-critical strings to curved backgrounds with RR fields.

2.1. RNS variables

We start by discussing the non-critical fermionic strings in the RNS formalism \[3\]. The \((2n + 2)\)-dimensional fermionic strings with \(n = 0, 1, 2, 3\), are described in the superconformal gauge by \(2n + 1\) matter superfields \(X^i, i = 1, \ldots, 2n + 1 \equiv D\), and by a Liouville superfield \(\Phi_l\). In components we have \((x^i, \psi^i), (\varphi, \psi_l)\), where \(\psi^i\) and \(\psi_l\) are Majorana-Weyl fermions. As usual, we have two ghost systems \((\beta, \gamma)\) and \((b, c)\). The central charges are given by \((2n + 1, (2n + 1)/2), (1 + 3Q^2, 1/2), 11\) and \(-26\). The total central charge is given by \(3(2n + 1)/2 + 1/2 + (1 + 3Q^2) - 15 = 3(n + Q^2 - 4)\). It vanishes for

\[ Q(n) = \sqrt{n + Q^2} - n \quad (2.2) \]

For \(n = 4\), we have nine flat coordinates and together with the Liouville field and \(x\), this gives the flat ten dimensional critical superstring. When \(n \neq 4\) we call the resulting systems non-critical superstrings.

The stress energy tensor of the system reads

\[ T = T_m + T_{ghost} \quad (2.3) \]

\[ T_m = \sum_{i=1}^{2n+1} \left( -\frac{1}{2}(\partial x^i)^2 - \frac{1}{2}\psi^i \partial \psi^i \right) + \left( -\frac{1}{2}(\partial \varphi)^2 + \frac{Q(n)}{2}\partial^2 \varphi - \frac{1}{2}\psi_l \partial \psi_l \right) \]

\[ T_{ghost} = -2b \partial c - \partial bc - \frac{3}{2}\beta \partial \gamma - \frac{1}{2}\partial \beta \gamma, \]

The OPE’s conventions that we will be using are

\[ x^i(z)x^j(w) \sim -\eta^{ij} \log(z - w) , \quad \varphi(z) \varphi(w) \sim -\log(z - w), \]

\[ \psi^i \psi^j \sim \eta^{ij} \frac{1}{(z - w)} , \quad \psi_l \psi_l \sim \frac{1}{(z - w)}, \]
\[c(z)b(w) \sim \frac{1}{(z-w)}, \quad \gamma(z)\beta(w) \sim \frac{1}{(z-w)},\]

\[T(z)e^{\rho x}(w) \sim \left(\frac{\rho^2/2 + \partial}{(z-w)^2} \right)e^{\rho x}, \quad T(z)e^{s\varphi}(w) \sim \left(\frac{-s(s-Q(n))/2 + \partial}{(z-w)^2} \right)e^{s\varphi} .\]

We choose an euclidean metric \(\eta^{ij}\) for the bosonic space \(x^i\). However, one of the boson, which we take to be \(x^{2n+1} \equiv x\) is compactified on \(S^1\) of radius \(\frac{2}{Q}\). In this case the system has global \(N=2\) symmetry on the worldsheet, which will be discussed in the next section.

We define \(\Psi = \psi_l + i\psi\) and \(\Psi_I = \psi_i + i\psi_{i+n}\) (with \(I = 1, \ldots, n\)). These are bosonized in the usual way by introducing the bosonic fields \(H\) for \(\partial H = \frac{i}{2} \Psi \Psi^\dagger\) and \(\partial H^I = \frac{i}{2} \Psi^I \Psi^{\dagger I}\) where \(\Psi^\dagger = \psi - i\psi_l\), \(\Psi^{\dagger I} = \psi^i - i\psi^{i+n}\) is complex conjugation in field space. We have

\[H^I(z)H^I(w) \sim - \log(z-w), \quad H(z)H(w) \sim - \log(z-w) .\] (2.4)

We define the spin fields \(\Sigma^\pm = e^{\pm \frac{i}{2} H}\). In addition, we define the other spin fields \(\Sigma^a = e^{\pm \frac{i}{2} H} \ldots \pm \frac{i}{2} H^n\), where the index \(a\) runs over the independent spinor representation of \(SO(2n)\). We have \(H^\dagger = H\) and \(H^{\dagger I} = H^I\).

2.2. BRST symmetry and N=2 superconformal symmetries

The matter and Liouville system has \(N=2\) global superconformal symmetry with central charge \(\hat{c} = 1 + n + Q^2\) generated by

\[G^+ = \frac{1}{2} \sum_{I=1}^n \Psi^\dagger_I \partial(x_I + ix_{I+n}) + \frac{1}{2} \Psi^\dagger \partial(\varphi + ix) - \frac{1}{2} Q(n) \partial \Psi^\dagger ,\] (2.5)

\[G^- = \frac{1}{2} \sum_{I=1}^n \Psi_I \partial(x_I - ix_{I+n}) + \frac{1}{2} \Psi \partial(\varphi - ix) - \frac{1}{2} Q(n) \partial \Psi ,\]

\[ J = \frac{1}{2} \sum_I \Psi^\dagger_I \Psi^{\dagger I} + \frac{1}{2} \Psi \Psi^\dagger + iQ(n) \partial x ,\]

and the energy momentum tensor \(T\) given in (2.3). We have \(T^\dagger = T, (G^\pm)^\dagger = G^\mp, J^\dagger = -J\).

The central charge can be computed by the central term of \(J\)

\[J(z)J(w) \sim \frac{\hat{c}}{(z-w)^2},\] (2.6)

and is \(\hat{c} = n + 1 + Q^2(n) = 5\) for \(Q(n)\) given in (2.2).
To pick an $N = 1$ algebra to gauge inside the $N = 2$ algebra, we consider the combination $G_m = \frac{1}{\sqrt{2}}(G^+ + G^-)$ as the $N = 1$ supersymmetry generator for the matter system. Note that $G^+_m = G_m$. There are other choices for picking the supersymmetry generator. They are parametrized by $SU(2)/U(1)$ ways of choosing the $N = 1$ algebra inside the $N = 2$ algebra of (2.5).

We define a twisted $N=2$ superconformal algebra by the generators

$$G'^+ = \gamma G_m + c \left( T_m - \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \gamma \partial \beta - b \partial c \right) - \gamma^2 b + \partial^2 c + \partial (c \xi \eta),$$

(2.7)

$$G'^- = b, \quad J' = cb + \eta \xi, \quad T' = T_m + T_{\text{ghost}}.$$  

The dimension one current $G'^+$ is the BRST current of the superstring. $\xi$ and $\eta$ are defined via the bosonization of the superghosts $\gamma = e^{\phi} \eta, \beta = \partial \xi e^{-\phi}$, as will be discussed shortly, and $J'$ is the ghost current. Notice also that the last two terms in the BRST current do not affect the BRST charge, but only the BRST current in such a way that $G'^+(z)G'^+(w) \rightarrow 0$. The central charge of the twisted $N=2$ superconformal algebra (2.7) can be computed by the central term of $J'$, which gives $\hat{c} = 2$.

Let us bosonize the ghost systems. We define

$$c = e^{\chi}, \quad b = e^{-\chi},$$

(2.8)

with

$$T_{\beta,c} = T_\chi = \frac{1}{2} (\partial \chi)^2 + \frac{3}{2} \partial^2 \chi, \quad c(z)b(w) \sim \frac{1}{z-w} + \partial \chi,$$

(2.9)

$$\chi(z)\chi(w) \sim \log(z-w),$$

$$T_\chi(z)\partial \chi(w) \sim \frac{-3}{(z-w)^3} + \ldots, \quad T_\chi(z)e^{a \chi}(w) \sim \left( \frac{a(a-3)/2}{(z-w)^2} + \frac{\partial}{z-w} \right)e^{a \chi}.$$  

The background charge is $Q_{\beta,c} = 3$ and the total charge is $(1 + 3 \epsilon Q^2) = -26$ with $\epsilon = \pm$ for bosons/fermions.

For the superghosts, we have $\gamma = e^{\phi} \eta, \beta = \partial \xi e^{-\phi}$

$$T_{\beta,\gamma} = T_\phi + T_{\partial \xi} = -\frac{1}{2} (\partial \phi)^2 - \partial^2 \phi - \eta \partial \xi, \quad \gamma(z)\beta(w) \sim \frac{1}{(z-w)} + \partial \phi + \ldots,$$

(2.10)

$$\phi(z)\phi(w) \sim -\log(z-w), \quad \eta(z)\xi(w) \sim \frac{1}{(z-w)},$$

$$T_\phi \partial \phi \sim \frac{2}{(z-w)^3} + \ldots, \quad T_\phi(z)e^{b \phi}(w) \sim \left( \frac{-b(b+2)/2}{(z-w)^2} + \frac{\partial}{z-w} \right)e^{b \phi}.$$  

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The background charge is $Q = -2$ and the total conformal charge is $(1 + 3Q^2) - 2 = 13 - 2 = 11$. We further bosonize the fermions into

$$\eta = e^\kappa, \quad \xi = e^{-\kappa},$$

and

$$T_{\eta, \xi} = T_\kappa = \frac{1}{2}(\partial\kappa)^2 - \frac{1}{2}\partial^2\kappa, \quad \eta(z)\xi(w) \sim \frac{1}{(z - w)} + \partial\kappa + \ldots .$$

$$\kappa(z)\kappa(w) \sim \log(z - w),$$

$$T_\kappa \partial\kappa \sim \frac{1}{(z - w)^3} + \ldots, \quad T_\kappa(z)e^{c_\kappa}(w) \sim \left(\frac{c(c + 1)/2}{(z - w)^2} + \frac{\partial}{(z - w)}\right)e^{c_\kappa}.$$

The background charge is $Q = -1$ and the total charge is $(1 - 3Q^2) = -2$. So, finally we use the ghosts $\gamma = e^{\phi + \kappa}$ and $\beta = \partial\kappa e^{-\phi - \kappa}$.

### 2.3. Supersymmetry

In the following we will discuss the supersymmetry structure in the RNS formalism (see [9] and also [10,11,12,13]), and how it will be realized in terms of the supersymmetric variables. For the $(2n + 2)$-dimensional strings we can construct in the $-\frac{1}{2}$ picture $2n^2$ candidates for supercurrents

$$e^{-\frac{\phi}{2} + \frac{1}{2}(\pm H \pm H^1 \pm \ldots \pm H^n \pm Qx)}.$$

However, only $2^n$ of them are mutually local w.r.t each other and close a supersymmetry algebra. Combining the left and right sectors, one gets an $N = 2$ supersymmetry algebra in $2n$-dimensional space. Type IIA and type IIB strings are distinguished in the way we choose the supersymmetry currents from the left and right sectors. When the target space allows chiral supersymmetry ($n = 1, 3$), type IIA and type IIB have $(1, 1)$ and $(2, 0)$ target space supersymmetry, respectively. In order to work in a covariant formalism we will see that it is convenient to use a bigger superspace with double the amount of supersymmetric coordinates, namely $2^{n+1}$ supersymmetric coordinates from the left sector and $2^{n+1}$ supersymmetric coordinates from the right sector. Such superspace arises naturally when considering the critical superstrings compactified on $CY_{4-n}$ manifolds. It suggests that the non-critical superstrings may have solutions with double the supersymmetry of the linear dilaton background. Let us see how this works in detail.
We start from the simplest model with $D = 1$ ($n = 0$). We have the bosonic fields $(x, \varphi)$. In this case there is only one nilpotent supercharge. We can choose the corresponding supercurrent $q_+(z)$ in the form

$$q_+(z) = e^{-\frac{\varphi}{2} - \frac{iH}{2} - ix}.$$  \hfill (2.14)

$\phi$, $H$ and $x$ are holomorphic parts of scalar fields.

The supercharge $Q_+$ is given by

$$Q_+ = \oint e^{-\frac{1}{2} \phi - \frac{1}{2} H - ix},$$  \hfill (2.15)

with $Q_+^2 = 0$.

One can write another supercurrent in the form

$$q_-(z) = e^{-\frac{\varphi}{2} + \frac{iH}{2} + ix}.$$  \hfill (2.16)

However, it is not local w.r.t. $q_+(z)$. We have $(Q_+)^\dagger = Q_-.$

There is also a supercurrent from the right sector, which we will denote by $\overline{q}$. If we choose, the same supercurrents $(q_+, \overline{q}_+)$ or $(q_-, \overline{q}_-)$ in the left and right sector, we get type IIB with 0 + 0-dimensional $N = 2$ supersymmetry (the two supercharges are nilpotent). If, on the other hand, we choose different supercurrents in the left and right sectors $(q_+, \overline{q}_-)$ or $(q_-, \overline{q}_+)$ we get type IIA with 0 + 0-dimensional $N = 2$ supersymmetry, again with two nilpotent supercharges. The affine current

$$J_R = \frac{2i}{Q(n)} \partial x,$$  \hfill (2.17)

corresponds to the $U(1)_R$ symmetry. $q_+$ and $q_-$ have R-charges $\pm 1$. Note that while the target space is two-dimensional with coordinates $x$ and $\varphi$, the supersymmetry structure is that of two dimensions less. This structure will continue in higher dimensions, namely in $2n + 2$ dimensions we will have $2n$-dimensional supersymmetry algebra for the non-critical superstrings.

In order to construct the covariant hybrid formalism we need to work in a bigger superspace. This can be seen, for instance, by noticing that we have in the RNS formalism four fermionic variables $(\psi, \overline{\psi})$ and $(b, c)$. In the hybrid formalism these four anticommuting fields have to be re-expressed in terms of four anticommuting target space new variables,
namely two target superspace fermionic coordinates and their conjugate momenta, for each of the left and right sectors.

In order to double the superspace, we can add one more supercharge

\[ Q_+ = \oint e^{-\frac{1}{2}\phi - \frac{1}{2}H + ix}, \]  

which has the property that

\[ \{Q_+, Q_+\} = \oint e^{-\phi - iH}, \]  

and therefore it is local w.r.t. \( Q_+ \). Note that the commutation relation between \( Q_+ \) and \( Q_+ \) closes onto the translation generator of the longitudinal mode and Liouville directions (see section 3). The new charge does not impose new constraints on the string spectrum.

Similarly, we introduce

\[ Q_- = \oint e^{-\frac{1}{2}\phi + \frac{1}{2}H - ix}, \]  

which has the same property

\[ \{Q_-, Q_-\} = \oint e^{-\phi + iH}. \]  

After the picture changing operation \( Z \)

\[ Z = \{Q_B, \xi\} = 2\partial\phi b\eta e^{2\phi} + e^{\phi}(G^+ + G^-) + 2b\partial\eta e^{2\phi} + \partial b\eta e^{2\phi} + c\partial\xi, \]  

where \( G^\pm \) are given in (2.5) with \( n = 0 \), is applied, we get on the RHS a translation generator, which involves the Liouville field \( \varphi \) and the space coordinate \( x \) (see next section). Using the two charges \( Q_+, Q_+ \) (or \( Q_-, Q_- \)) we will construct a superspace with two fermionic coordinates \( \theta^+ \) and \( \theta^\pm \) (or \( \theta^- \)) and their conjugate momenta \( p_+, p^\pm \) (or \( p^-, p_- \)). We will follow an hybrid type formalism in order to construct the covariant description of the strings in this superspace.

Consider next \( D = 3 \) (\( n = 1 \)). We have the bosonic fields \( (x_1, x_2, x, \varphi) \). The fermions are \( (\psi_1, \psi_2, \psi, \psi_l) \) which are the fermion super-partners of the coordinates, and the ghost fields \( (b, c, \eta, \xi) \). In the \(-\frac{1}{2}\) picture, we can construct eight candidates for the supercurrents

\[ e^{-\frac{1}{2}\phi + \frac{1}{2}(\pm H^1 \pm H \pm \sqrt{3}x)}. \]  

(2.23)

However, only two of them are mutually local w.r.t. each other and close a supersymmetry algebra. We can choose the supercharges \( Q_{+, \alpha}, \alpha = 1, 2 \) as

\[ Q_{+, 1} = \oint e^{-\frac{1}{2}\phi + \frac{1}{2}(H^1 - H - \sqrt{3}x)}, \quad Q_{+, 2} = \oint e^{-\frac{1}{2}\phi + \frac{1}{2}(H^1 + H + \sqrt{3}x)}. \]  

(2.24)
which satisfy
\[
\{Q_{+,1}, Q_{+,2}\} = \int e^{-\phi + iH^1},
\]  
and zero otherwise. This is the translation operator at the picture $-1$, and after the picture raising operation on the second charge, the anticommutator reads
\[
\{Q_{+,1}, Q_{+,2}\} = \oint \partial y,
\]  
where $y = x_1 + ix_2$.

As before, there is another choice of supersymmetry generators
\[
Q_{-1} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(-H^1 + H + \sqrt{3}x)}, \quad Q_{-2} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(-H^1 - H - \sqrt{3}x)}.
\]  

Adding the right moving sector we have two choices. We can choose the same supercharges on the left and right sectors $(Q_{+,\alpha}, \overline{Q}_{+,\alpha})$ or $(Q_{-\alpha}, \overline{Q}_{-\alpha})$ to have $(2,0)$ supersymmetry for the type IIB string, or we choose different sets as $(Q_{+,\alpha}, \overline{Q}_{-,\alpha})$ or $(Q_{-,\alpha}, \overline{Q}_{+,\alpha})$ to get $(1,1)$ supersymmetry for the type IIA string. We have $(Q_{+,\alpha})^\dagger = Q_{-,\alpha}$.

In addition to the $U(1)$ R-symmetry (2.17) we have the bosonic $SO(2)$ acting as the Lorentz group on two coordinates $(x_1, x_2)$.

Again, in order to construct the covariant hybrid formalism we will need to work in a bigger superspace. We construct the supercharges $Q_{+,\alpha}, \dot{\alpha} = 1, 2$ as
\[
Q_{+,1} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(-H^1 + H + \sqrt{3}x)}, \quad Q_{+,2} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(-H^1 - H - \sqrt{3}x)}.
\]  

We will use four superspace coordinates $\theta_{\alpha}^a$ and $\theta_{\dot{\alpha}}^a$ and their conjugate momenta. Similarly, all the above can be repeated for $Q_{-,\alpha}$ and $Q_{-,\dot{\alpha}}$ (see appendix).

In $D = 5$ ($n = 2$) some subtlety arises. In this case, one can construct four mutually local supercharges $Q_{+,a} = (Q_{+,\alpha}, Q_{+,\alpha'})$ with $\alpha, \alpha' = 1, 2$ from the left sector of the form
\[
Q_{+,a} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(H^1 + H^2) - \frac{i}{2}(H + \sqrt{3}x)}, \quad Q_{+,a'} = \int e^{-\frac{1}{2}\phi + \frac{i}{2}(H^1 - H^2) + \frac{i}{2}(H + \sqrt{3}x)}.
\]  

It can be checked that they close on the translation generators (in the picture $-\frac{1}{2}$) of the four dimensional space. In the same way there are four mutually local supercharges $\overline{Q}_{+,a}$ from the right sector. This gives the $N = 2$ supersymmetry of type IIB strings in $2n = 4$ dimensions. In a similar way, for type IIA strings we use $\overline{Q}_{-,a}$ from the right sector.

In order to construct the covariant hybrid formalism we need to work in a bigger superspace with six dimensional $N = 2$ supersymmetry structure. As before we can add
additional supercharges $Q_{+,a} (a = 1, \ldots, 4)$ which are mutually local w.r.t. $Q_{+,a}$ (see appendix). This, however, does not quiet work as before. The reason is that the original set of anticommuting fields is $(\psi_i, \psi, \bar{\psi}_i)$ with $i = 1, \ldots, 4$ and $b, c$ (plus the fermionization of the superghosts) are not enough to describe eight coordinates plus their conjugate momenta. Even adding the fermionization of the superghost, we have at most ten anticommuting variables to rewrite 16 anticommuting variables.

Comparing to the study of critical superstrings compactified on $CY_2$, the analysis performed in [14] can be repeated. In [14] only half of the supersymmetry is manifest and in a subsequent paper a formalism [15] with manifest $N = 2$ supersymmetry of the target space is constructed by doubling the number of variables and imposing an harmonic constraint. Besides the original $4 + 4$ coordinates, one introduces $4 + 4$ anticommuting new coordinates and momenta which are not obtained from the original fermions and a constraint is implemented at the level of physical states. This will be discussed in section 4. The bosonic symmetry of the non-critical strings is $SO(4) \times U(1)_R$.

The last non-critical string is at $D = 7$. In this case, there are 16 supercharges, which are dependent. Again to fully realize the $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 1)$ supersymmetry, one has to add some auxiliary variables and constrain them by harmonic constraints. It seems that this example can be more easily treated using the pure spinor formulation [16]. the bosonic symmetry of the non-critical strings is $SO(6) \times SU(2)_{R}$. Note that the $R$-symmetry group is now $SU(2)_{R}$ since the scalar $x$ is compactified on a circle of self-dual radius, where there is an enhanced $SU(2)$ affine symmetry [11].

2.4. Spectrum

The BRST cohomology of the RNS non-critical strings has not been fully computed for every $n$. It has been computed for the case $D = 1$ $(n = 0)$ in [17,18,19].

Two-dimensional strings: $D = 1$, $n = 0$

The BRST cohomology consists of states at ghost numbers zero, one and two. At ghost number one there are two types of vertex operators. In the NS sector we have in the $-1$ picture

$$ T_k = e^{-\phi + ikx + p_l \varphi} . \quad (2.30) $$

Locality with respect to the space-time supercharges $Q_+$ and $Q_-$ projects on half integer values of the momentum in the $x$-direction

$$ x : k \in Z + \frac{1}{2} . \quad (2.31) $$
The introduction of a second supercharge does not change the constraint on the spectrum.

In the Ramond sector we have in the $-\frac{1}{2}$ picture the vertex operators

\[ V_k = e^{-\phi + \frac{i}{2} \epsilon H + i k x + p_l \phi} , \]

where $\epsilon = \pm 1$. Locality with respect to the space-time supercharges $Q_+$ and $Q_-$ \( (2.13) \) implies $k \in Z + \frac{1}{2}$ for $\epsilon = 1$ and $k \in Z$ for $\epsilon = -1$. An interesting operator that we will consider later is the supercurrent

\[ V_{k=1,\epsilon=-1} = e^{-\phi - \frac{i}{2} H + i x} . \]

The Liouville dressing is determined by requiring conformal invariance of the integrated vertex operators. Thus, the coefficient $p_l$ has to be a solution to the equation

\[ \frac{k^2}{2} - \frac{1}{2} p_l (p_l - 2) = \frac{1}{2} . \]

This equation can be solved by $p_l = 1 \pm k$. Furthermore the locality constraint requires $p_l \leq \frac{Q}{2} = 1$. Being in the BRST cohomology imposes an additional constraint in the Ramond sector $|k| = -\epsilon k$ \( [13] \).

Note that

\[ (T_k)^\dagger = T_{-k}, \quad (V_{k,\epsilon=\pm1})^\dagger = V_{-k,\epsilon=\mp1} . \]

At ghost number zero there are spin zero BRST invariant operators that generate a commutative, associative ring

\[ \mathcal{O}(z)\mathcal{O}'(0) \sim \mathcal{O}''(0) + \{Q_B, \ldots \} , \]

called the ground ring, where $Q_B$ is the worldsheet $N = 1$ BRST operator.

The main objects in the construction of the ring are the R-sector operators

\[ x(z) = \left( e^{-\frac{H}{2}} e^{-\frac{i}{2} \phi} - \frac{1}{\sqrt{2}} e^{\frac{H}{2} \partial \xi} e^{-\frac{\phi}{2} c} \right) e^{\frac{1}{2} z - \frac{1}{2} \phi} , \]

\[ y(z) = \left( e^{\frac{H}{2}} e^{-\frac{i}{2} \phi} - \frac{1}{\sqrt{2}} e^{-\frac{H}{2} \partial \xi} e^{-\frac{\phi}{2} c} \right) e^{-\frac{1}{2} z - \frac{1}{2} \phi} , \]

and the NS-sector operators

\[ u = x^2, \quad v = y^2, \quad w = xy . \]
Locality with respect to the space-time supercharges \( Q_+ \) and \( Q_{\dot{+}} \) implies the projection \( x \to -x \). Thus, the basic invariant elements are \( x^2 \) and \( y \). We have \( x^\dagger = y \).

The ghost number two operators correspond to spin one currents, that acts as derivations of the ring.

In order to construct the string states we need to combine the left and right moving states in such a way that the momentum along the Liouville direction, which is non-compact, is the same in both sectors. Projecting on the left and the right sectors with the same set of supercharges (2.15) or (2.18) defines the type IIB theory. Projection on the left and the right sectors with different set of supercharges defines the type IIA theory [11].

**Higher dimensions**

As noted above, the complete BRST cohomology has not been computed yet. We consider, for instance at ghost number one, two types of vertex operators similar to (2.30) and (2.32). In the NS sector we have

\[
T_k = e^{-\phi + ikX + p_l \varphi} V(z),
\]

where \( V(z) \) is an \( N = 1 \) primary made of the \( 2n \) superfields \( X^i \). One has

\[
\Delta + \frac{k^2}{2} - \frac{1}{2} p_l (p_l - Q) = \frac{1}{2},
\]

where \( \Delta \) is the dimension of \( V \). If \( V \) has \( U(1) \) charge \( q \) then locality with respect to the supercharges implies that

\[
kQ + q \in 2Z + 1.
\]

When \( Q = 2 \) and \( q = 0 \) we recover (2.31).

In the Ramond sector consider, for instance, the vertex operators

\[
V_k = e^{-\frac{\phi}{2} + \frac{\epsilon}{2} H + \frac{1}{2} \sum_{i=1}^n \epsilon_I H^I + ikX + p_l \varphi},
\]

where \( \epsilon, \epsilon_I = \pm 1 \). Locality with respect to the supercharges implies

\[
kQ \in 2Z - \frac{1}{2} \left( \epsilon + \sum_{I=1}^n \epsilon_I - 1 \right),
\]

and

\[
kQ \in 2Z - \frac{1}{2} \left( \epsilon + \sum_{I=1}^n \epsilon_I - 1 \right).\]
Again for $Q = 2$ we recover the two-dimensional results. These conditions can be solved. As an example consider the case $D = 3$ ($n = 1$). Then we have

$$kQ \in 2Z - \frac{1}{2},$$

(2.45)

when $\epsilon = \epsilon_1 = \pm 1$ and

$$kQ \in 2Z + \frac{1}{2},$$

(2.46)

when $\epsilon = -\epsilon_1 = \pm 1$. Interesting operators that we will consider later are the supercurrents

$$V_{k = \pm \frac{1}{2}, \epsilon_1 = -1, \epsilon = \mp 1} = e^{-\frac{1}{2}\phi - \frac{i}{2}H_1 \mp \frac{i}{2}(H - \sqrt{3})x}.$$

(2.47)

3. Covariant Non-Critical Superstrings

3.1. $D = 1$, $n = 0$

In the following we define superspace variables, which exhibit a similar structure to that of [20], where compactification on a Calabi-Yau 4-fold is discussed. Let us discuss the left-moving sector, and everything should be replicated for the right sector. However, as discussed before, when we combine the left and right sectors, there is a choice corresponding to the type IIA and type IIB GSO projections in the RNS formalism. As already explained before we use two supercharges $Q_+$ and $\dot{Q}_+$ to construct the covariant formalism.

Consider the two supercharges (2.15) and (2.18). In order to define the supersymmetry off-shell, we change pictures and modify $Q_+$ into $ZQ_+$ as

$$Q_+ = \oint \left( b\eta e^{\frac{3}{2}\phi - \frac{i}{2}H - ix} + \frac{1}{2}(\partial \varphi - i\partial x + 2\partial \phi) e^{\frac{1}{2}\phi + \frac{i}{2}H - ix} - e^{\frac{1}{2}\phi - \frac{i}{2}H - ix} \right).$$

(3.1)

The first term is the application of the picture changing operator (PCO) (2.22) on $Q_+$ in the $-\frac{1}{2}$ picture. The additional terms and the $\phi$ dependence is coming from the non-homogeneous term in (2.5). Then $Q_+$ and $\dot{Q}_+$ satisfy

$$\{Q_+, \dot{Q}_+\} = \oint \partial(\varphi - ix + 2\phi),$$

(3.2)

where we notice that the translation operator contains in addition to $x$, both the Liouville field $\varphi$ and the superghost $\phi$.

---

3 A way to check that this gives the correct answer is to apply the PCO to $\dot{Q}_+$ and on the r.h.s. of the (2.19) to check that they give the same answer.
We construct superspace variables as the dimension zero combinations
\[ \theta^+ = c \xi e^{-\frac{3}{2} \phi + \frac{i}{4} H + ix}, \quad \theta^+ = e^{\frac{i}{2} \phi + \frac{3}{4} H - ix}. \] (3.3)

The variables \( \theta^+ \) and \( \theta^+ \) have regular OPE, and
\[ q_+(z)\theta^+(w) \sim \frac{1}{(z-w)}, \quad q_+(z)\theta^+(w) \sim \frac{1}{(z-w)}. \] (3.4)

The conjugate momenta to \( \theta^+ \) and \( \theta^+ \) are the dimension one objects
\[ p_+ = b \eta e^{\frac{3}{2} \phi - \frac{i}{2} H - ix}, \quad p_+ = e^{-\frac{i}{2} \phi - \frac{3}{4} H + ix} \] (3.5)

and
\[ p_+(z) \theta^+(w) \sim \frac{1}{(z-w)}, \quad p_+(z) \theta^+(w) \sim \frac{1}{(z-w)}. \] (3.6)

Notice that we defined the conjugate momenta with the a different sign for the \( x \) part, which does not change the conformal spin. With this choice \( p_+ \) and \( p_+ \) have regular OPE.

Let us discuss the hermiticity properties and the periodicity of \( x \). As we discussed before in the RNS formalism, \( Q_+^\dagger = Q_- \) and \( Q_+^\dagger = Q_- \). However, in order to have manifest space-time supersymmetry, we had to apply the picture changing operation. With this, the definition of the hermiticity conditions gets more complicated [21]. Let us define now the compatible hermiticity conditions.

We define the hermiticity conditions by
\[ (x_m)^\dagger = e^R x_m e^{-R}, \quad (\varphi)^\dagger = e^R \varphi e^{-R}, \quad (\psi_m)^\dagger = e^R \psi_m e^{-R}, \] (3.7)
\[ (e^\phi)^\dagger = e^{2R} e^{\phi + \Delta \phi} e^{-R}, \quad (e^\chi)^\dagger = e^{R} e^{\chi + \Delta \chi} e^{-R}, \quad (e^\kappa)^\dagger = e^{2R} e^{\kappa + \Delta \kappa} e^{-R}, \]
where \( R \) is given by (see appendix (7.2))
\[ R = \oint \left[ (G^+ + G^-) e^{\chi - \phi - \kappa} + \frac{1}{2} \partial \phi e^{2(\chi - \phi - \kappa)} \right], \] (3.8)
and \( G^\pm \) are the supersymmetry generators for the matter system given in (2.4). The shifts \( \Delta \) in the ghost fields are given by
\[ \Delta \phi = 2 \chi - 2 \kappa - 4 \phi, \quad \Delta \chi = -2 \kappa - 2 \phi, \quad \Delta \kappa = -2 \chi + 2 \phi. \] (3.9)

The importance of these shifts will become apparent shortly. Note that in addition one applies in the hermiticity definition ordinary complex conjugation.
With this definition of hermiticity we can check that $Q_+^\dagger = Q_-$. This can be seen by observing that the supersymmetry charge $Q_+$ can be rewritten

$$Q_+ = e^R \left( \oint b \eta e^{\frac{3}{2} \phi - \frac{i}{2} H - i x} \right) e^{-R},$$

and that with the definitions (3.7) and (3.9)

$$Q_-^\dagger = \oint b \eta e^{\frac{3}{2} \phi - \frac{i}{2} H - i x}.$$  \hspace{1cm} (3.10)

Also,

$$\theta_+^\dagger = \theta_-, \quad p_+^\dagger = p_-.$$  \hspace{1cm} (3.12)

Note that since $R^\dagger = R$, we have $(O^\dagger)^\dagger = O$, as required.

The hermitian conjugation does not commute with the stress energy-momentum tensors. This implies that the conformal weights of some fields and their hermitian conjugates are different. In addition, we observe that there is a combination of the ghost fields $\phi, \chi, \kappa$, namely $\phi + \kappa - \chi$, which is invariant under the shifts (3.9). This combination appears in the definition of $R$.

The same hermiticity conditions will be used in higher dimensions. The only difference is that in the definition of $R$ (3.8) we will need the appropriate $G^\pm$.

The fermionic fields $\theta_+, \theta_+^\dagger, p_+, p_+^\dagger$ have singular OPE’s with the field $x$. A way to solve this problem is to redefine the variable $x$ such that there are no singular OPE’s by performing a similarity transformation on the operators $q_+$ and $q_+^\dagger$ and the translation generator. Also the energy-momentum tensor $T$ is modified. The similarity transformation is given by

$$U = exp \oint (ix + iH - \varphi - \phi) \partial \kappa.$$ \hspace{1cm} (3.13)

The combination $(ix + iH - \varphi - \phi)$ has several nice properties: it has no singularities with itself, it has no singularities with the Grassman variables $\theta^+, \theta^\dagger, p_+, p_+^\dagger$ (but it has singularities with the supercurrents), and it shifts the translation operator in the r.h.s. of (3.2) by $2i \oint \partial \kappa$. It is therefore convenient to introduce the new coordinate $x'$ defined by

$$x' = x + 2i(\phi + \kappa),$$ \hspace{1cm} (3.14)

such that (3.2) becomes

$$\{Q_+, Q_+\} = \oint \partial(\varphi - ix').$$ \hspace{1cm} (3.15)
It can be easily verified that

$$x'(z)x'(w) = -\ln(z - w), \quad (3.16)$$

since the contributions of $\phi$ and $\kappa$ cancels. The primary field $e^{a x'}$, where $a$ is complex, transforms under the hermiticity condition as follows

$$\left( e^{a x'} \right)^\dagger = e^R (e^{\overline{a} x'}) e^{-R}, \quad (3.17)$$

where we use $\phi + \kappa + \Delta(\phi + \kappa) = -(\phi + \kappa)$. So, the new variable $x'$ is (modulo the similarity transformation $e^R$) hermitian.

We stress again that as in the RNS description where the hermitian conjugation relates the model to its conjugate (with the GSO projection being performed by the charges $Q_-$ and $Q_-^\dagger$), the same holds here. In particular, the supersymmetry algebra (3.15) is mapped into the conjugated relation

$$\{Q_-, Q_-^\dagger\} = \oint \partial(\varphi + i x'). \quad (3.18)$$

Obviously, the spectrum of the theory and its conjugate are equivalent as can be easily observed by mapping the observables using the hermitian conjugation.

As we stressed before, in order to have space-time supersymmetry the field $x$ is taken to be periodic of period $Q^{-1}(n)$. One may worry whether the field redefinition or the hermitian conjugation interferes with periodicity. However, this is not the case since the field redefinition is obtained by acting with $U$ given in (3.13), which is invariant under constant translations of $x$. The periodicity of $x'$ (which is the same as of $x$) is also not affected by the hermitian conjugation. In the RNS framework, the periodicity of $x$ is needed in order to ensure that the interacting theory is space-time supersymmetric. The worldsheet operator that changes the radius $\int \partial x \overline{\partial x}$ is not $N = 2$ invariant, and when switched on it breaks the worldsheet $N = 2$ superconformal algebra leading to the breaking of space-time supersymmetry. Similarly, one can argue the same using the supersymmetric variables.

Note also that the current

$$J_{x'} = \partial x' = \partial x + 2i(\partial \phi + \partial \kappa) \quad (3.19)$$
has no singularity with $\theta^+$ and $\theta^\dagger$. The combination (3.14) will appear again in the higher dimensional cases in the form

$$x' = x + iQ(n)(\phi + \kappa).$$  \hspace{1cm} (3.20)

In the next step we rewrite the ghost fields in terms of new chiral bosons $\omega$ and $\rho$ by imposing the following two equations

$$b = p_+ e^{\omega - \rho}, \quad -\gamma^2 b = p_+ e^{\omega + \rho}. \hspace{1cm} (3.21)$$

The conformal spins of the combinations $e^{\omega - \rho}$ and $e^{\omega + \rho}$ are 1 and 0, respectively. These conditions lead to the following equations

$$\kappa + \frac{3}{2} \phi - \frac{i}{2} H - i x + \omega - \rho = 0,$$

$$-2(\phi + \kappa) + \chi - \frac{1}{2} \phi - \frac{i}{2} H + i x + \omega + \rho = 0.$$  \hspace{1cm} (3.22)

From this we get the solution for $\omega$ and $\rho$

$$\omega = \frac{1}{2} (\phi + \kappa - \chi + iH), \quad \rho = 2\phi + \frac{1}{2} (3\kappa - \chi) - ix = -\frac{1}{2} \kappa - \frac{1}{2} \chi - ix'.$$  \hspace{1cm} (3.23)

The hermiticity properties of these new fields can be deduced from the transformation laws of the original fields (3.7)

$$(e^\rho)^\dagger = e^R e^{\rho + \Delta \rho} e^{-R}, \quad (e^\omega)^\dagger = e^R e^\omega e^{-R}, \hspace{1cm} (3.24)$$

where $\Delta \rho = -2\rho$. Under the hermitian conjugation defined in (3.9) the chiral bosons $\omega$ and $\rho$ are mapped into the corresponding chiral boson of the conjugate theory. This can be deduced by constructing the conjugate theory and the corresponding ghosts, from the set of charges: $Q_-$ and $Q_-.$

It is easy to check using the above definitions that the OPE’s of these new fields are

$$\rho(z)\rho(w) \sim -\frac{1}{2} \log(z - w), \quad \omega(z)\omega(w) \sim \frac{1}{2} \log(z - w), \quad \rho(z)\omega(w) \sim 0.$$  \hspace{1cm} (3.25)

The stress energy tensor reads

$$T_{\omega, \rho} = (\partial \omega)^2 - \partial^2 \omega - (\partial \rho)^2 - \partial^2 \rho. \hspace{1cm} (3.26)$$
Therefore, the conformal spin of $e^\omega$ is $3/4$, $e^\rho$ is $-3/4$ and $e^{-\rho}$ is $1/4$ which is consistent with $e^{\omega+\rho}$ which has conformal spin $0$ and $e^{\omega-\rho}$ with conformal spin $1$.

Notice that the combination

\[ J' = -2\partial \rho - 2iJ_x, \]

reads

\[ J' = -2\partial \left(2\phi + \frac{3}{2}\kappa - \frac{1}{2}x - ix\right) - 2i\partial \left(x + 2i\phi + 2i\kappa\right) = \partial \kappa + \partial \chi = cb + \eta \xi, \]

as in (2.7) where we used the definitions (3.23).

Using these new variables, we write the energy momentum tensor as

\[ T' = -p_+ \partial \theta^+ - p_+ \partial \theta^+ + (\partial \omega)^2 - \partial^2 \omega - (\partial \rho)^2 - \partial^2 \rho \]

\[ - \frac{1}{2}(\partial x')^2 - i\partial^2 x' - \frac{1}{2}(\partial \varphi)^2 + \partial^2 \varphi. \]

To summarize: we replaced the four bosonic variables $(x, \varphi, \beta, \gamma)$ and four fermionic variables $(\psi, \psi_l, b, c)$ in the RNS formulation by four bosonic variables $(x', \varphi, \omega, \rho)$ and four fermionic variables $(p_+, \theta^+, p_+, \theta^+)$. Let us now compute the total central charge to check the consistency of the above manipulations. We have the following contributions

\[ (-2)p_+ \theta^+ + (-2)p_+ \theta^+ + (1 - 6)\omega + (1 + 6)\rho + (1 - 12)x' + (1 + 12)\varphi = 0. \]

Now we have to look for an $N = 2$ superconformal algebra written in terms of these new variables. The generators $T', J'$ are the generators that appear in the twisted $N = 2$ superconformal algebra (2.7) written in the new variables. The other $N = 2$ generators are given by the supersymmetry charge

\[ G' = p_+ e^{\omega-\rho}, \]

which has conformal spin $2$ and ghost number $-1$ w.r.t. $J'$, and the BRST charge $G'^+$ (see appendix)

\[ G'^+ = (p_+ + \theta^+(\partial \varphi - i\partial x)) e^{\rho+\omega} + p_+ \left[(\partial \varphi + i\partial x') - 2(\theta^+ p_+ + \partial \omega + \partial \rho)\right] e^{\rho-\omega} \]

The OPE of $T$ with $e^{\alpha \omega}$ and $e^{\beta \rho}$ are $T_{\omega, \rho} e^{\alpha \omega} = \left(\frac{\alpha(\alpha/2+1)/2}{(z-w)^2} + \frac{\partial}{(z-w)}\right)e^{\alpha \omega}$ and $T_{\omega, \rho} e^{\beta \rho} = \left(-\beta(\beta/2+1)/2 \right) \frac{(z-w)^2}{(z-w)^2} + \frac{\partial}{(z-w)}\right)e^{\beta \rho}$. |
\[ + \theta^+ e^\rho \omega T' + \partial (\theta^+ e^\rho \omega J') + \partial^2 (\theta^+ e^\rho \omega) . \]

There is another set of supersymmetric invariant variables that will be useful when considering curved target spaces with RR background fields. It is given by

\[ \Pi_{++} = \partial (\varphi - ix'), \quad \Pi^{++} = \partial (\varphi + ix') - 2\theta^+ \partial \theta^+, \]

\[ d_+ = p_+ , \quad d_+ = p_+ + \theta^+ \Pi_{++} , \quad (3.32) \]

which satisfy the algebra

\[ d_+(z)d_+(w) \sim \frac{\Pi_{++}}{(z - w)} , \quad \Pi^{++}(z)\Pi^{++}(w) \sim \frac{-2}{(z - w)^2} , \quad (3.33) \]

\[ d_+(z)\Pi_{++}(w) \sim 0 , \quad d_+(z)\Pi_{++}(w) \sim 0 , \quad d_+(z)\Pi^{++}(w) \sim -2\frac{\partial \theta^+}{(z - w)} , \quad d_+(z)\Pi^{++}(w) \sim -2\frac{\partial \theta^+}{(z - w)} . \]

We will use the notation \((++), (+, +), (+, +)\) for the superspace indices

\[ x_{++} = \varphi - ix', x^{++} = \varphi + ix', \theta^+, \theta^+. \quad (3.34) \]

The algebra of fermionic derivatives

\[ D_+ = \partial_{\theta^+} , \quad D_+ = \partial_{\theta^+} + \theta^+ \partial_{x_{++}} , \quad (3.35) \]

is given by

\[ \{D_+, D_+\} = 0 , \quad \{D_+, D_+\} = \oint \Pi_{++} = \partial_{x_{++}} , \quad \{D_+, D_+\} = 0 . \quad (3.36) \]

The supercurrents commute with the covariant derivatives and therefore it is easy to see that they have the form

\[ q_+ = p_+ - \theta^+ \Pi_{++} , \quad q_+ = p_+ . \quad (3.37) \]

The algebra of the supercurrents is very similar to the algebra of covariant derivatives except the sign in front of the translation operator appearing in the OPE \(q_+(z)q_+(w)\). In terms of the new variables, the energy-momentum tensor can be recast as

\[ T' = -d_+ \partial \theta^+ - d_+ \partial \theta^+ - \frac{1}{2} \Pi_{++} \Pi^{++} + \frac{Q}{2} \partial \Pi_{++} \]

\[ +(\partial \omega)^2 - \partial^2 \omega - (\partial \rho)^2 - \partial^2 \rho . \quad (3.38) \]
At this point we are done with the change of variables. However, it is of value to uncover now an algebraic relation between the $2n + 2$ non-critical superstrings and the critical superstrings compactified on $\text{CY}_{4-n}$ manifolds. In the $n = 0$ two-dimensional case we compare to [20]. The form of the energy momentum tensor (3.38) is identical to that given in [20] (see eq. (2.11) there), except for the fact that there is the background charge associated with $x'$ and $\varphi$ and the term $T_{\text{GS}}$ is missing. Let us construct now a twisted $N = 2$ algebra with $\hat{c} = -2$ which will be compared with a similar algebra in [20]. For that we choose as the $N = 2 U(1)$ current the operator $J'$ obtained by adding the contribution of the Liouville

$$J' \rightarrow J' + 2\partial \varphi = -2\partial \rho + 2(\varphi - ix').$$

Now with the modified $U(1)$ current we have

$$J'(z)J'(w) \sim \frac{-2}{(z-w)^2}, \quad T'(z)J'(w) \sim \frac{2}{(z-w)^3}. \quad (3.39)$$

Notice that in computing $T'(z)J'(w)$ the combination $\varphi - ix'$ has no singularities with term $Q\Pi_{++}$ in $T'$. Next we perform a similarity transformation by $T' \rightarrow e^RT'e^{-R}$ and $J' \rightarrow e^RT'e^{-R}$ (not to be confused with $R$ of (3.38)) with

$$R = 2 \oint (\varphi - ix')\partial \rho. \quad (3.40)$$

This removes the terms $Q\Pi_{++}$ and $2\partial(\varphi - ix')$ from $T'$ and from $J'$. Then, the form of $T'$ and $J'$ are

$$T' = -d_+ \partial \theta^+ - d_+ \partial \theta^+ - \frac{1}{2}\Pi_{++}\Pi^{++} + (\partial \omega)^2 - \partial^2 \omega - (\partial \rho)^2 - \partial^2 \rho,$$

$$J' = -2\partial \rho,$$

and the rest of the algebra is given by

$$G^{-'} = d_+ e^{\omega-\rho}, \quad G^{+'} = e^{-\oint \Pi_{+-} e^{-2w}} d_+ e^{\rho-\omega} e^{\oint \Pi_{+-} e^{-2w}}. \quad (3.42)$$

This is exactly the algebra in [20] where the terms $T_{\text{GS}}, G_{\text{GS}}^+, G_{\text{GS}}^-$ and $J_{\text{GS}}$ are set to zero.

Let us comment on the relation between these algebras. In the case studied in [20] the string theory is compactified on a Calabi-Yau fourfold $\text{CY}_4$. The CFT on $\text{CY}_4$ has central charge $\hat{c} = c/3 = 4$. The total charge of string theory is the sum of the CFT on $\text{CY}_4$ and the central charge of the uncompactified part which is $\hat{c} = c/3 = -2$, which is the same
central charge of the algebra (3.41) and (3.42). The sum gives \( \hat{c} = 2 \) which is indeed the central charge of the RNS algebra (2.7).

This structure continues to hold in higher dimensions as we will see. We compare the non-critical strings for \( D = 2n + 1, n = 0, 1, 2, 3 \) with the string theory compactified on \( CY_{4-n} \). The uncompactified sector has central charge \( \hat{c}_{un} = 2 - (4 - n) = n - 2 \). So, the central term of the appropriate \( J' \) in the algebra for the non-critical string in \( D = 1 + 2n \) should be \( (n - 2)/(z - w)^2 \). In the case \( n = 0 \) we have exactly \(-2/(z - w)^2\) as in (3.39).

We point out that in the new formulation the \( N = 2 \) superconformal algebra plays a fundamental role. Indeed it is used to characterize the physical (supersymmetric) states of the theory. In the original formulation the spectrum is characterized by the BRST cohomology in the small Hilbert space (without the zero mode of \( \xi \)). The BRST cohomology can be also computed in the large Hilbert space (with the zero mode of \( \xi \)) by selecting a finite number of pictures. In the new framework (which is built on the large Hilbert space) the BRST condition on vertex operators is replaced by the condition of being chiral primary w.r.t. the \( N = 2 \) superconformal algebra. The selection of a finite number of pictures is obtained by selecting a finite number of power of \( e^\rho \). In [21], the equivalence of the two descriptions is exploited.

### 3.2. The two-dimensional spectrum in the new variables

Let us compute the spectrum in the new variables. It is convenient to find the inverse map between the original variables and the new supersymmetric variables. Bosonizing the fermions \( \theta^+, \dot{\theta}^+, p_+ \) and \( p_+ \) by

\[
\theta^+ = e^\alpha, p_+ = e^{-\alpha}, \dot{\theta}^+ = e^\beta, p_+ = e^{-\beta},
\]

we have the relations

\[
iH = \frac{1}{2}(\alpha + \beta + 2\omega), \quad \phi = \frac{1}{2}(-3\alpha + \beta + 2\omega - 8\rho - 4ix'), \quad (3.44)
\]

\[
\chi = \alpha - \omega + \rho, \quad \kappa = -\alpha + \omega - 3\rho - 2ix'.
\]

In the NS sector, see (2.30), we have

\[
T_k = e^{\left(\alpha(-\frac{1}{2}+k)+\beta(\frac{1}{2}-k)+\omega+(-2+k)(2\rho+ix')+p_i\varphi\right)}, \quad k \in Z + \frac{1}{2},
\]

(3.45)
and in the R sector, see (3.40) for \( \epsilon = \pm \), we have

\[
V_{k, \epsilon = +1} = e^{ \left( \beta \left( \frac{1}{2} - k \right) + \alpha \left( -\frac{1}{2} + k \right) + \omega + (1+k)(2\rho+ix^\prime)+p_l\varphi \right) } \quad k \in \mathbb{Z} + \frac{1}{2},
\]

\[
V_{k, \epsilon = -1} = e^{ \left( \alpha (-1+k) - \beta k + (1+k)(2\rho+ix^\prime)+p_l\varphi \right) } \quad k \in \mathbb{Z}.
\]

We notice that in both cases, the powers of \( \omega \) and \( \rho \) have only integer values. The value of \( p_l \) is fixed by the conformal dimension of the integrated vertex operator. Note that \( e^{k\alpha} \) and \( e^{k\beta} \) have conformal weight \( k(k-1)/2 \), and \( e^{kx^\prime} \) has conformal weight \( k(k+2)/2 \). One can easily check that \( T_k \) and \( V_k \) have the correct dimensions.

From equations (3.45) we immediately see that, in the case of NS vertex operators, the momentum \( k \) must be half integer in order to rewrite it in terms of the new variables, and for the R sector we have that for \( \epsilon = +1 \) \( k \) must be half integer, while for \( \epsilon = -1 \), the momentum should be an integer. This is the way that the locality with respect to the space-time supercharges in the RNS formalism is realized in the hybrid variables.

In order to express the above vertices in terms of the new variables, we need a dictionary. It is easy to check that

\[
ed^\alpha = \theta^+ , \quad e^{2\alpha} = \partial \theta^+ \theta^+ , \quad e^{3\alpha} = \partial^2 \theta^+ \partial \theta^+ \theta^+ , \ldots ,
\]

\[
ed^{-\alpha} = p_+, \quad e^{-2\alpha} = \partial p_+ p_+ , \quad e^{-3\alpha} = \partial^2 p_+ \partial p_+ p_+ , \ldots .
\]

The order of the expression can be easily determined by observing that \( e^{2\alpha} = \lim_{x \to y} : \partial \theta^+(x) \theta^+(y) : \) \( = \lim_{x \to y} : \partial \alpha \theta^+(x)e^\alpha(y) : \) Using the OPE \( e^\alpha(x)e^\alpha(y) \to (x-y)e^{2\alpha}(y) \), then it is easy to check the above formulas. In addition, one can check the OPE’s between different monomials.

Let us examine the Liouville-independent states. By setting \( p_l = 0 \), we have the state (2.33)

\[
V_{+1, \epsilon = -1} = p_+ .
\]

It has a very simple interpretation using the definition of the supercurrent \( q_\downarrow = p_\downarrow \). The vertex operator \( V_{+1, \epsilon = -1} \) describes the single fermion of the open string theory. It is massless and it does not depend on the coordinate \( x^\prime \). The coupling can be done by adding the deformation

\[
S_\psi = t_R \int d\sigma \psi q_\downarrow ,
\]

(3.49)
where $\psi$ is a constant Grassmann number. Note in comparison that in [20] there are two massless fermions which are compactification independent states.

Combining left and right sectors, we have for constant RR fields of IIB/A string theories the vertex operators

$$V_{RR}^A = q_+ \bar{q}_- , \quad V_{RR}^B = q_+ \bar{q}_+ ,$$

(3.50)

where $\bar{q}_+$ and $\bar{q}_-$ are the right moving charges. Again, the number of independent RR vertex operators is dictated by the level matching, and the BRST invariance. As will be explained in section 5, the coupling of the RR vertex operators to the space-time RR fields strength $F^{\alpha\beta}$ is

$$F^{++} q_+ \bar{q}_- , \quad F^{++} q_+ \bar{q}_+ .$$

(3.51)

Thus we find one RR scalar both for IIA and IIB. In Type IIB this corresponds to a self-dual 1-form field strength in two dimensions. In Type IIA this corresponds to a 2-form field strength, or alternatively, its scalar Hodge dual.

As discussed above, in the RNS description there are ghost number zero dimension zero states that generate the ground ring. Let us write them in the new variables. The ghost number zero dimension zero operators are given by

$$x(z) = \left( e^{-\frac{1}{2} \alpha - \frac{1}{2} \beta} - \frac{\partial \kappa}{\sqrt{2}} e^{\frac{1}{2} \alpha + \frac{1}{2} \beta} \right) e^{-\rho - \frac{1}{2} x' - \frac{1}{2} \varphi} ,$$

(3.52)

$$y(z) = \left( e^{\omega} - \frac{\partial \kappa}{\sqrt{2}} e^{-\omega} \right) e^{-\alpha + \beta - 3 \rho - \frac{3 i}{2} x' - \frac{1}{2} \varphi} .$$

(3.53)

We see that $x(z)$ cannot be written in terms of the variable (3.47), which corresponds to the fact that it has been projected out by the GSO projection in the RNS formalism. $y(z)$ takes the from

$$y(z) = \left( p_+ \theta^+ - \frac{e^{-2 \omega}}{\sqrt{2}} \left[ \partial p_+ \theta^+ + p_+ \theta^+ \left( \partial \omega - 3 \partial \rho - 2 i \partial x' \right) \right] \right) e^{\omega - 3 \rho - \frac{3 i}{2} x' - \frac{1}{2} \varphi} .$$

(3.54)

Notice that compared with [13] the surviving state is $y$ and not $x$ because we project here with respect to the $q_+$ supercurrent.

The operators $u = x^2, v = y^2$, which are not projected out by the GSO projection in the RNS formalism [13] take the form $\theta^+, \theta^+, p_+$ and $p_+

$$u(z) = - p_+ \partial p_+ \theta^+ \partial \theta^+ e^{\omega - 3 \rho - \frac{3 i}{2} x' - \frac{3 i}{2} x'} + \ldots ,$$

(3.55)

$$v(z) = p_+ p_+ e^{-2 \rho - \varphi - i x'} + \ldots ,$$

while $w = xy$ which is projected out cannot be written in the supersymmetric variables. One can now repeat the analysis of [13] in a manifestly space-time supersymmetric manner.
4. Higher Dimensions

4.1. \( D = 3, n = 1 \)

In this case the bosonic fields are \( x_1, x_2, x, \varphi \). It is convenient to introduce the coordinates \( y = x_1 + ix_2, \overline{y} = x_1 - ix_2 \). As discussed in section 2.3, we have two possible target space supersymmetries. If we choose \( (Q_{+}\alpha, \overline{Q}_{+}\alpha), \alpha = 1, 2 \), as the supercharges we have the type IIB superstrings with \( (2,0) \) supersymmetry. If we choose \( (Q_{+}\alpha, \overline{Q}_{-}\alpha) \) as the supercharges we get type IIA superstrings with \( (1,1) \) supersymmetry.

Consider first the type IIB superstrings. As before, we construct a bigger superspace. We add the supercharges \( (Q_{+}\alpha, \overline{Q}_{+}\alpha) \) defined in (2.28). The superspace coordinates and their conjugate momenta are

\[
\theta^\alpha = e^{\frac{3}{2}\phi - \frac{1}{2}H^1 \mp \frac{1}{2}(H + \sqrt{3}x)}, \quad p_{+\alpha} = b\eta e^{\frac{3}{2}\phi - \frac{1}{2}H^1 \mp \frac{1}{2}(H + \sqrt{3}x)},
\]

\[
\theta^\dot{\alpha} = e^{\frac{1}{2}\phi + \frac{1}{2}H^1 \pm \frac{1}{2}(H - \sqrt{3}x)}, \quad p_{+\dot{\alpha}} = e^{-\frac{1}{2}\phi - \frac{1}{2}H^1 \pm \frac{1}{2}(H - \sqrt{3})x}.
\]  

(4.1)

The variables \( \theta^\alpha \) and \( \theta^\dot{\alpha} \) have regular OPE, and

\[
q_{+\alpha}(z)\theta^\beta_{+}(w) \sim \frac{\delta^\beta_\alpha}{(z - w)}, \quad q_{+\dot{\alpha}}(z)\theta^\dot{\beta}_{+}(w) \sim \frac{\delta^\dot{\beta}_{\dot{\alpha}}}{(z - w)}.
\]

(4.2)

Also,

\[
p_{+\alpha}(z)\theta^\beta_{+}(w) \sim \frac{\delta^\beta_\alpha}{(z - w)}, \quad p_{+\dot{\alpha}}(z)\theta^\dot{\beta}_{+}(w) \sim \frac{\delta^\dot{\beta}_{\dot{\alpha}}}{(z - w)}.
\]

(4.3)

We define

\[
x' = x + i\sqrt{3}(\phi + \kappa)
\]

(4.4)

and the current

\[
J_{x'} = \partial x' = \partial x + i\sqrt{3}(\partial \phi + \partial \kappa).
\]

(4.5)

The current \( J_{x'} \) commutes with \( (4.1) \). We have

\[
J'(z)J'(w) \sim \frac{2}{(z - w)^2}.
\]

(4.6)

There is one chiral boson \( \rho \) defined by

\[
\rho = 3\phi - \chi + 2\kappa - i\sqrt{3}x,
\]

(4.7)
and we have
\[ b = \frac{1}{2} p_{+\alpha} p_{+\beta} e^{\alpha\beta} e^{-\rho}, \quad \gamma^2 b = \frac{1}{2} p_{+\dot{\alpha}} p_{+\dot{\beta}} e^{\dot{\alpha}\dot{\beta}} e^{\rho}. \]  
(4.8)

As before we define the current \( J' \)
\[ J' = -\partial \rho - i\sqrt{3} J_x', \]  
(4.9)

which is the ghost current of the BRST symmetry.
\[ J' = -\partial(3\phi - \chi + 2\kappa + i\sqrt{3}x) - i\sqrt{3} J_x = \partial \kappa + \partial \chi = cb + \eta\xi. \]  
(4.10)

The stress tensor takes the form
\[ T' = -p_{+\alpha} \partial \theta^{+\alpha} - p_{+\dot{\alpha}} \partial \theta^{\dot{\alpha}} - \frac{1}{2} \partial y \partial y \]  
(4.11)
\[ -\frac{1}{2}(\partial \rho)^2 - \frac{1}{2} \partial^2 \rho - \frac{1}{2}(\partial x')^2 - i\frac{\sqrt{3}}{2} \partial^2 x' - \frac{1}{2}(\partial \phi)^2 + \frac{\sqrt{3}}{2} \partial^2 \phi. \]

The contribution from the central charge are given by
\[ (-4)_{p_{+\theta}^+} + (-4)_{p_{+\dot{\theta}}^+} + (1)_{\Pi} + (1 - 9)_{x'} + (1 + 9)_{\varphi} + (1 + 3)_{\rho} = 0. \]

Next we define
\[ G'^- = \frac{1}{2} p_{+\alpha} p_{+\beta} e^{\alpha\beta} e^{-\rho}. \]  
(4.12)

\( T', J', G' \) are three of the generators of the twisted \( N = 2 \) superconformal algebra \( (2.7) \). \( G'^+ \) is the BRST current.

Denote
\[ x_{11} = (\varphi - ix'), \quad x_{12} = -y, \quad x_{22} = (\varphi + ix'). \]  
(4.13)

We introduce the supersymmetric invariant variables \( d_{+\alpha}, d_{+\dot{\alpha}} \) and \( \Pi_{\alpha\dot{\alpha}} \):
\[ \Pi_{11} = \partial x_{11} - i(\theta^{+1} \partial \theta^{+1} + \theta^{+1} \partial \theta^{+1}), \quad \Pi_{12} = \partial \eta - i(\theta^{+1} \partial \theta^{+2} + \theta^{+2} \partial \theta^{+1}), \]  
(4.14)
\[ \Pi_{21} = -\partial y - i(\theta^{+2} \partial \theta^{+1} + \theta^{+1} \partial \theta^{+2}), \quad \Pi_{22} = \partial x_{22} - i(\theta^{+2} \partial \theta^{+2} + \theta^{+2} \partial \theta^{+2}), \]
\[ d_{+\alpha} = p_{+\alpha} + i\theta^{+\alpha} \partial x_{\alpha\dot{\alpha}} - \frac{1}{2}(\theta^{+})^2 \epsilon_{\alpha\beta} \partial \theta^{+\beta} + \frac{1}{4} \epsilon_{\alpha\beta\dot{\alpha}\dot{\beta}} \partial (\theta^{+})^2, \]  
(4.15)
\[ d_{+\dot{\alpha}} = p_{+\dot{\alpha}} + i\theta^{+\dot{\alpha}} \partial x_{\alpha\dot{\alpha}} - \frac{1}{2}(\theta^{+})^2 \epsilon_{\dot{\alpha}\dot{\beta}} \partial \theta^{+\beta} + \frac{1}{4} \epsilon_{\dot{\alpha}\dot{\beta}\dot{\alpha}\dot{\beta}} \partial (\theta^{+})^2, \]
in terms of which the energy-momentum tensor can be written as follows
\[ T' = -d_+ \partial \theta^{\dagger \alpha} - d_+ \partial \theta^{\dagger \dot{\alpha}} - \frac{1}{2} \Pi_{\alpha \dot{\alpha}} \Pi^{\alpha \dot{\alpha}} - \frac{1}{2} (\partial \rho)^2 - \frac{1}{2} \partial^2 \left( \rho + \sqrt{3} \Pi_{11} \right). \] (4.16)

It can be checked that by using the above definitions the energy-momentum tensor can be rewritten in terms of free variables. The construction of the supersymmetry currents can be derived easily in a similar way.

The OPE of $d$’s are given by
\[ d_{+1}(z) d_{+1}(w) \sim \frac{\Pi_{11}}{(z-w)} \], \quad \[ d_{+1}(z) d_{+2}(w) \sim \frac{\Pi_{12}}{(z-w)} \], \quad \[ d_{+2}(z) d_{+1}(w) \sim \frac{\Pi_{21}}{(z-w)} \], \quad \[ d_{+2}(z) d_{+2}(w) \sim \frac{\Pi_{22}}{(z-w)} \],

and analogously one can construct the supersymmetry generators. The total supersymmetry for open superstring is $(1, 0)$ and in the case of closed superstring is $(1, 1)$ or $(2, 0)$ for the type IIA or type IIB. This also implies that the spectrum at the massless level is generated for the open string by a two component fermion, and for closed string (in the RR sector) by bilinears of those supersymmetry charges.

Combining left and right sectors, we have for constant RR fields of IIB/A string theories the vertex operators
\[ V_{RR}^A = q_{+, \alpha} \overline{q}_{-\dot{\alpha}} \], \quad \[ V_{RR}^B = q_{+, \dot{\alpha}} \overline{q}_{+\alpha} \], \quad \[ F^{+\dagger -\dot{\alpha} \dot{\alpha}} \sim \overline{q}_{+, \alpha} \overline{q}_{+, \dot{\alpha}} \], \quad \[ F^{++\dot{\alpha} \alpha} \sim q_{+, \alpha} q_{+, \dot{\alpha}} \].

Thus we find four RR degrees of freedom both for IIA and IIB. In Type IIB this corresponds to a 1-form (or its Hodge dual 3-form) field strength in four dimensions. In Type IIA this corresponds to a 0-form (or its Hodge dual 4-form) field strength and a self-dual 2-form field strength in four dimensions.

Let us construct now a twisted $N = 2$ algebra with $\hat{c} = -1$ which will be compared with a similar algebra of the uncompactified sector of string theory compactified on $CY_3$. We consider the $U(1)$ charge
\[ J' = -\partial \left( \rho + \sqrt{3} \Pi_{11} \right) \], \quad \[ J'(z) J'(w) \sim \frac{-1}{(z-w)^2} \], \quad \[ T'(z) J'(w) \sim \frac{1}{(z-w)^3} \]. (4.20)

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Next we perform the similarity transformation generated by

\[ R = \sqrt{3} \oint \Pi_{11} \partial \rho, \]

which removes the term \( \frac{1}{2} \sqrt{3} \Pi_{11} \) in \( T' \) and \( J' \), but does not change the central term of \( J' \) and the central charge. Notice that the operator \( R \) will modify also the superderivatives \( d_+, d_\dot{\alpha} \) and the translation generators. However, it will preserve the commutation relation. We denote by an additional hat the new operators \( T', J', d, \ldots \rightarrow \hat{T}, \hat{J}, \hat{d}, \ldots \) Now, the form of the energy-momentum tensor and of \( \hat{J} \) is exactly that of the uncompactified part of \( CY_3 \). In terms on these new variables we can construct the \( N=2 \) superconformal algebra by adding the new generators

\[ \hat{G}^- = \hat{d}_+ \hat{d}_\dot{\beta} e^{\alpha \beta} e^{-\rho}, \quad \hat{G}^+ = \hat{d}_+ \hat{d}_\dot{\beta} e^{\alpha \beta} e^\rho. \]

To check the algebra only the commutation relation and the OPE’s for the chiral boson \( \rho \) are needed. So, the last step is to express the algebra in the original variables by performing the \( R \) similarity transformation back

\[ \mathcal{O}' = e^R \hat{O} e^{-R} \]

where \( \hat{O} = (\hat{T}, \hat{J}, \hat{G}^+, \hat{G}^-) \). The form of the generators \( G^{\pm} \) is established by computing the complete expansion of the similarity transformation. Notice that due to presence of the exponential \( e^{\pm \rho} \) in the definition of \( \hat{G}^\pm \), the new \( G^{\pm} \) is definitely more complicated.

4.2. \( D = 5, n = 2 \)

We have the bosons \((x^1, \ldots, x^4, x, \varphi)\) and the ghosts \( \beta, \gamma \). There are eight fermions obtained by the RNS fermions \( \psi^i \), the super-Liouville partner \( \psi_l \) and the ghosts \( b, c \). We define four spinors \( \theta^\alpha, \theta^{\dot{\alpha}} \) and \( \theta^{\dot{\alpha}}, \theta^\alpha \) and their conjugates

\[ \theta^\alpha = c \xi e^{-\frac{1}{2} \phi \pm \frac{1}{2} (H^1 + H^2) + \frac{1}{4} (H + \sqrt{2} x)}, \quad p_{+, \alpha} = b \eta e^{\frac{1}{2} \phi \mp \frac{1}{4} (H + \sqrt{2} x)}, \quad (4.23) \]

\[ \theta^{\dot{\alpha}} = c \xi e^{-\frac{1}{2} \phi \pm \frac{1}{2} (H^1 - H^2) + \frac{1}{4} (H + \sqrt{2} x)}, \quad p_{+, \dot{\alpha}} = b \eta e^{\frac{1}{2} \phi \mp \frac{1}{4} (H + \sqrt{2} x)}, \]

\[ \theta^\dot{\alpha} = e^{\frac{1}{2} \phi \mp \frac{1}{4} (H^1 + H^2) + \frac{1}{4} (H - \sqrt{2} x)}, \quad p_{+, \dot{\alpha}} = e^{\frac{1}{2} \phi \pm \frac{1}{4} (H^1 + H^2) + \frac{1}{4} (H - \sqrt{2} x)}, \]

\[ \theta^{\dot{\alpha}} = e^{\frac{1}{2} \phi \mp \frac{1}{4} (H^1 - H^2) - \frac{1}{4} (H - \sqrt{2} x)}, \quad p_{+, \dot{\alpha}} = e^{\frac{1}{2} \phi \pm \frac{1}{4} (H^1 - H^2) + \frac{1}{4} (H - \sqrt{2} x)}. \]
However, they are not independent. This is the same situation as in [14]. In order to
display the full supersymmetry, one needs to add some auxiliary variables and to impose
some new constraints [15].

By the counting of bosonic variables, we find that we need chiral bosons $\rho$ and $\chi$, where
\[
\rho = 2\phi + \kappa - i\sqrt{2}x,
\] (4.24)
and the chiral boson $\chi$ coincides with the original chiral boson of the bosonization of
$b,c$-system. We define
\[
x' = x + i\sqrt{2}(\phi + \kappa),
\] (4.25)
and the current $J_{x'}$,
\[
J_{x'} = \partial x + i\sqrt{2}(\partial \phi + \partial \kappa).
\] (4.26)
They combine into
\[
J' = -\partial \rho + \partial \chi - i\sqrt{2}J_x = \partial \chi + \partial \kappa,
\] (4.27)
which is the ghost current of the $N = 2$ superconformal algebra related to the BRST
symmetry. With these definitions, it is easy to check that
\[
\frac{1}{4!}p_{+,a}p_{+,b}p_{+,c}p_{+,d}e^{abcd}e^{2\rho-\chi} = \gamma^2 b.
\] (4.28)

The total conformal charge is obtained by summing the contributions coming from
the bosonic variables $x^i$ and $\varphi$, which give $5 + 7$, the fermionic variables which yield $-8$,
and the chiral bosons whose contribution which is $-4$. Again the contribution of the chiral
bosons to the twisted conformal charge matches the contribution of the chiral boson for
the compactification of critical superstring compactified on $CY_2$.

By twisting the theory with the currents $J = -\partial \rho$, we have that the model can be
compared with the critical superstring on $CY_2$ and therefore similar analysis to the one
performed in [14] can be repeated. In [14] only half of the supersymmetry is manifest and
in a subsequent paper a formalism [15] with manifest $N=2$ supersymmetry of the target
space is constructed by duplicating the number of variables and imposing an harmonic
constraint.

Here, we have to perform a similar construction, in order to have the manifest su-
persymmetry. We duplicate the number of variables (adding 8 fermionic variables $\theta^{a'\alpha}$ and
their conjugate momenta) and pick up only those which are local with respect to each
others. In addition, one has to check that the harmonic constraint is compatible with this choice.

This leads to a superspace, which implements the \( N = 1 \) supersymmetry for \( D=4 \) (for the open superstring) and \( N = (1, 1) \) or \( (2, 0) \) for type IIA/B. Out of the 8 variables \( (\theta^a, \theta^{a'}) \), one selects four fermionic variables of the four dimensional superspace and the algebra of covariant derivatives and supercharges is the usual one.

Note the following interesting aspect: we have mapped the original variables of RNS string theory into the bosonic variables \( x^i, x', \varphi, \rho, \chi \) and into the fermionic variables \( \theta_a, p^a \). For a generic background these variables are entangled in the sigma model. However, if the background has a factorized structure where some of the variables are not mixed, we can have a simplified situation: the variables \( x^i, \theta_a, p^a, \rho \) form a \( N = 2 \) superconformal system by their own with \( \hat{c} = -1 \). The rest of the variables \( x', \varphi, \chi \) form an \( N = 2 \) superconformal algebra with \( \hat{c} = 3 \).\(^5\) Using this framework the analysis of such systems is simplified.

4.3. \( D = 7, n = 3 \)

This is the last interesting example of this class of non-critical superstrings. The transverse space is parametrized by six coordinates \( x^{[ab]} \) (with \( a = 1, \ldots, 4 \)) and the longitudinal space is generated by \( x', \varphi \), where \( x' = x + i(\phi + \kappa) \). The original fermions together with bosonization of the ghost fields cannot be mapped into the 16 \( \theta_i, p^i \) (with \( i = 1, \ldots, 8 \)). In order to have the manifest supersymmetry one has to enlarge the space by addition fermionic variables (such as in \([13]\) and \([22]\)). The present situation is even more complicated than the \( D=5 \) case and we suspect that an analysis using pure spinor might simplify the analysis \([10]\).

5. Non-critical strings in curved target space

In this section we consider non-critical strings on curved target spaces with Ramond-Ramond background fields in the hybrid type formalism.

\(^5\) We thank G. Policastro and T. Dasgupta for discussion on this point.
5.1. Coupling to Ramond-Ramond fields

A simple and local coupling of the worldsheet fields to the RR background is a common fundamental feature of the hybrid formalism and of the pure spinor formalism. Let us briefly review some basic facts and explain the structure of the couplings and the RR vertex operators. In the hybrid formalism, we add to the superspace coordinates \((x^m, \theta^\alpha)\) the conjugate momenta \(p_\alpha\). In order to respect the target space supersymmetry we form the supersymmetric invariant quantities \(\Pi_m, \partial \theta^\alpha, d_\alpha^\alpha\) and \(\Pi^n, \partial \theta^\alpha, \bar{d}_\alpha\), which are worldsheet holomorphic or anti-holomorphic 1-forms. They commute with the supersymmetry charges \(Q_\alpha = \oint \, q_\alpha\), where \(q_\alpha\) are the supersymmetric currents.

At the massless level, the vertex operator is constructed in terms of these fundamental blocks

\[
V = \partial \theta^\alpha \partial \bar{\theta}^\beta A_{\alpha \beta} + \Pi^m \partial \theta^\alpha B_{m \beta} + \partial \theta^\alpha \Pi^n B_{\alpha n} + \Pi^m \Pi^n C_{mn} +
\]

\[
d_\alpha \partial \theta^\beta D^\alpha_{\beta} + \partial \theta^\alpha \bar{d}\bar{\alpha} D_{\alpha}^\beta + d_\alpha \Pi^n E^\alpha_{\beta n} + \Pi^m \bar{d}_\beta E_n^\beta + d_\alpha \bar{d}_\beta F_{\alpha \beta} + \ldots .
\]

where \(A_{\alpha \beta}, \ldots, F_{\alpha \beta}\) are superfields. The ellipsis stand for the additional contributions coming from the ghost fields. The form of these couplings relies on the dimension of the spacetime, the \(R\)-symmetry and upon the Lorentz transformation properties of the ghost fields.

The lowest components of the superfield \(C_{mn}\) are the graviton, the NS-NS 2-form and the dilaton

\[
C_{mn} = g_{mn} + b_{mn} + \eta_{mn} \phi + O(\theta, \bar{\theta}).
\] (5.2)

The lowest component of the superfield \(F_{\alpha \beta}\) is the RR field strength (in spinor indices)

\[
F_{\alpha \beta} = f_{\alpha \beta} + O(\theta, \bar{\theta}).
\] (5.3)

For a complete discussion see [23]. By imposing the equation \(\{Q_B, V\} = \partial U\) where \(Q_B\) is the BRST charge and \(U\) is generic vertex operator with ghost number \((1, 0)\) and conformal spin \((0, 1)\) (as is explained in [23]) one gets superspace relation among the different superfields. In the case of constant RR field strengths (which are solutions of the linearized supergravity equations),

\[
F_{\alpha \beta}(x, \theta, \bar{\theta}) = f_{\alpha \beta},
\] (5.4)

and the expression (5.1) reduces to

\[
V = f_{\alpha \beta} q_\alpha \bar{q}_\beta.
\] (5.5)
This gives a local coupling between the RR field strengths and the supersymmetry currents. For constant RR fields all contributions of the superfields to the vertex operators conspire to give the supercurrents even if the coupling has been written starting with the \( d \)'s. Note that in the same way one obtains the coupling to Ramond fields in the case of open superstrings.

The form of the supersymmetry currents is usually non-linear in the worldsheet fields and, generically, by adding the vertex operators (5.3) it is not easy to compute the contribution of the RR field to the amplitudes exactly. However, all propagators of the sigma model are well-defined and in special cases (constant RR fields) the fields \( p_\alpha \) can be integrated out easily. If we wish to study the sigma models with the addition of these deformations, we have to add also the back-reaction as will be discussed later.

The \((2n + 2)\)-dimensional target space even forms RR field strengths of type IIA are encoded in \( F_{\alpha\beta} \) via

\[
F^\alpha_{\beta} = \delta^\alpha_{\beta} F^{(0)} + \frac{1}{2!} (\gamma^{mn})^\beta_\alpha F_{mn} + \frac{1}{4!} (\gamma^{mnpq})^\beta_\alpha F_{mnpq},
\]

and the odd forms RR field strengths of type IIB as

\[
F_{\alpha\beta} = \gamma^m_{\alpha\beta} F_m + \frac{1}{3!} \gamma^{mnp}_{\alpha\beta} F_{mnp} + \frac{1}{5!} \gamma^{mnpqr}_{\alpha\beta} F_{mnpqr}.
\]

The gamma matrices used in the above equations are the off-diagonal 16 \( \times \) 16 blocks of ten-dimensional Dirac matrices \((\Gamma^0 \Gamma^m)_{\alpha\beta}\). They are real and symmetric and they satisfy the Fierz identities \( \gamma^m_{(\alpha\beta} \gamma^m_{\gamma\delta)} = 0 \). Note that the forms that appear are those whose degree is not higher than the target space dimension. By dimensional reduction one finds all lowest dimensional models with their RR fields couplings.

5.2. Target space effective action

As we have seen, the RR field strengths of non-critical type IIB and type IIA superstrings are odd forms and even forms respectively. In addition, in \( d = 2n + 2 \) dimensions the middle \((n + 1)\)-form is self-dual. This is quiet different than the structure of RR fields of the critical type II superstrings compactified on a \( CY_{4-n} \) manifold.

Let us review the counting of the RR degrees of freedom. In \( d = 2 \) there is one RR degree of freedom. In type IIB it is a self-dual 1-form and in type IIA a scalar (or its Hodge dual 2-form). In \( d = 4 \) there are 4 RR degrees of freedom. In type IIB it a 1-form (or its Hodge dual 3-form) and in type IIA it is a self-dual 2-form. In \( d = 6 \) the there
are 16 RR degrees of freedom. In type IIB these are a 1-form (or its Hodge dual 5-form) and a self-dual 3-form, and in type IIA these are 0-form (or its hodge dual 6-form) and a 2-form (or its hodge dual 4-form). In $d=8$ there are 64 RR degrees of freedom. In type IIB these are a 1-form (or its Hodge dual 7-form) and a 3-form (or its Hodge dual 5-form), and in type IIA these are 0-form (or its Hodge dual 8-form) and a 2-form (or its Hodge dual 6-form) and a self-dual 4-form.

As discussed before, unlike the critical superstrings case, the low energy approximation $E \ll l_s^{-1}$ is not valid. The reason being that the action contains a cosmological constant type term which vanishes only for $d=10$, and the higher order curvature terms $(l_s^2 \mathcal{R})^n$ cannot be discarded. One can still write an action for the massless fields, whose bosonic part takes the form

$$S = \frac{1}{2k_d^2} \int d^d x \sqrt{G} \left( e^{-2\Phi} \left( R + 4(\partial \Phi)^2 + \frac{10 - d}{\alpha'} - \frac{1}{2 \cdot 3!} H^2 \right) - \frac{1}{2 \cdot n!} F^2_{n} \right).$$

(5.8)

However, solutions to the field equations have string scale curvature. For instance, consider curved backgrounds with RR fields, with constant dilaton and vanishing NS-NS field, which will be considered later. Then, the field equations of (5.8) imply that the scalar curvature is

$$l_s^2 \mathcal{R} = d - 10.$$  

(5.9)

One class of such backgrounds of type IIA non-critical strings are $AdS_d$ spaces with a constant dilaton $e^{2\Phi} = \frac{1}{N_c^2}$ and a $d$-form RR field $F_d$

$$l_s^2 F_d^2 = 2(10 - d)d! N_c^2.$$  

(5.10)

We note that even though consistent backgrounds of non-critical strings may be solutions of the field equations of the action (5.8), the analysis of fluctuations is likely to give a wrong spectrum.

5.3. $D=1, n=0$

Using the supersymmetric variables the classical action for IIB in the flat background is given by

$$S_{IIB} = \frac{1}{\alpha'} \int dz d\bar{z} \left( \frac{1}{2} \Pi_+ \Pi^{++} + d_+ \overline{\theta}^+ + d_+ \overline{\theta}^+ + \overline{d}_+ \partial \overline{\theta}^+ + \overline{d}_+ \partial \overline{\theta}^+ \right) + S_B^{\text{flat}},$$

(5.11)

where $S_B$ is the action for the chiral bosons $\omega, \overline{\omega}$ and $\rho, \overline{\rho}$. We also introduced the right-moving sector. As explained above the choice of the right-hand supersymmetry charge
determines if the closed model is IIA or IIB. The classical action for IIA in the flat background takes the form

\[ S_{IIA} = \frac{1}{\alpha'} \int dz d\bar{z} \left( \frac{1}{2} \Pi_+ \Pi_{++} + d_+ \bar{\theta}^+ + d_+ \bar{\theta}^+ + \bar{d}_- \partial \bar{\theta}^- + \bar{d}_- \partial \bar{\theta}^- \right) + S_B^{flat}, \quad (5.12) \]

On flat Riemann surface, there is no coupling with the background charge.

In order to couple the system to the background, we introduce the curved vielbeins \( E^A_M \) where the \( A \) are tangent superspace indices and \( M \) are curved superspace indices. We will use the notation introduced in (3.34) \((++, +, +, +)\) for the tangent superspace indices, and \( Z^M \) for the curved target superspace coordinates. The new supersymmetric variables are given by

\[ \Pi^A = E^A_M \partial Z^M, \quad \Pi^{\dot{A}} = E^{\dot{A}}_M \bar{\partial} Z^M. \quad (5.13) \]

In terms of the vielbeins, one can derive the covariant derivatives \( D^A = (E^{-1})^A_M D^M \) where \( E^{-1} \) is the inverse of the vielbeins and \( D^M \) are the covariant derivatives established in section 3.1. We also introduce the NS-NS 2-form \( B_{AB} \) using the superfield \((G + B)_{AB} = E^M_A E_{BM}\). The action for IIB in curved space can be written as

\[ S_{IIB} = \frac{1}{\alpha'} \int dz d\bar{z} \left( (G + B)_{AB} \Pi^A \Pi^B + d_+ \Pi^+ + d_+ \bar{\Pi}^+ + \bar{d}_- \Pi^- + \bar{d}_- \bar{\Pi}^- \right) + S_B. \quad (5.15) \]

Similarly, the action for type IIA takes the form

\[ S_{IIA} = \frac{1}{\alpha'} \int dz d\bar{z} \left( (G + B)_{AB} \Pi^A \Pi^{\dot{B}} + d_+ \Pi^+ + d_+ \bar{\Pi}^+ + \bar{d}_- \Pi^- + \bar{d}_- \bar{\Pi}^- \right) + S_B. \quad (5.16) \]

\( S_B \) is the action for the chiral bosons \( \rho \) and \( \omega \). Note that there is no worldsheet-covariant formulation for chiral bosons and the action should be supplemented by the chirality condition. In order to write the action \( S_B \) for the chiral bosons we notice that the field \( \rho \) depends on \( x \) (see equation (3.23)) and therefore couples to the \( U(1) \) connection \( A^R_M \) of the R-symmetry \((2.17)\). The action \( S_B \) reads

\[ S_B = S_B^{flat} + \int dz d\bar{z} \left( \bar{A}^R M \partial \rho + \partial Z^M A^R_M \bar{\partial} \rho \right). \quad (5.17) \]
So far we have been discussing the classical sigma model. However, in order for the model to be conformally invariant, one needs to add a Fradkin-Tseytlin term. Following (20) we see that the additional term we should add is of the form

$$S_{FT} = \int d^2 z D_+ \overline{D}_+ (\hat{\Phi} \Sigma_{cc} + h.c.),$$

(5.18)

where $\hat{\Phi}_{cc}$ is a $N = 2$ chiral superfield whose lowest component contains the spacetime dilaton. The superfield $\Sigma_{cc}$ is a chiral superfield and the ordinary worldsheet curvature is given by $D_+ D_- \Sigma_{cc}$ where $D_{\pm}$ are the $\text{N}=2$ worldsheet superderivatives. When the dilaton is constant we get from (5.18) the Euler number of the Riemann surface.

5.4. AdS$_2$

Consider the example of AdS$_2$ background of type IIA non-critical string studied in (24) in the Green-Schwarz formalism. Let $Z, \overline{Z}$ denote the coordinates on AdS$_2$. The dilaton $\Phi$, the metric $G$ and the RR 2-form $F$ take the form

$$e^{2\Phi} = \frac{1}{N_c^2}, \quad G_{ZZ} = -\frac{1}{2(Z - \overline{Z})^2}, \quad F_{ZZ} = \frac{8N_c}{(Z - \overline{Z})^2}.$$  

(5.19)

We denote the curved superspace coordinates by $Z_M = (Z, \overline{Z}, \Theta^+, \overline{\Theta}^-)$. In addition there are two free variables $(\Theta^+, \overline{\Theta}^-)$ needed for the extension of the superspace. The tangent space coordinates are denoted by $z_A = (z, \overline{z}, \theta^+, \overline{\theta}^-)$, and in addition we have $(\theta^+, \overline{\theta}^-)$. For the simplicity of the notation we denote $\Theta = \Theta^+, \overline{\Theta} = \overline{\Theta}^-$ and $\theta = \theta^+, \overline{\theta} = \overline{\theta}^-$. Note that we use the same symbols ($z, \overline{z}$) for the target space tangent coordinates and the worldsheet coordinates. The curved quantities $\Pi^A$ are related to the flat ones by the vielbeins $\Pi^A = E^A_M \partial Z^M$ where

$$E^z_Z = \frac{1}{(Z - \overline{Z} - \Theta \overline{\Theta})}, \quad E^z_\Theta = \frac{\Theta}{(Z - \overline{Z} - \Theta \overline{\Theta})},$$

$$E^\theta_Z = \frac{\Theta - \overline{\Theta}}{(Z - \overline{Z} - \Theta \overline{\Theta})^{3/2}}, \quad E^\theta_\Theta = \frac{1}{(Z - \overline{Z} - \Theta \overline{\Theta})^{1/2}} - \frac{\Theta \overline{\Theta}}{(Z - \overline{Z} - \Theta \overline{\Theta})^{3/2}}.$$  

(5.20)

Together with these vielbeins there are also the conjugated $E^\overline{z}_Z, E^\overline{z}_\overline{\Theta}, E^\overline{\theta}_Z, E^\overline{\theta}_\overline{\Theta}$, we are deduced in the same way as (5.20) from the Maurer-Cartan forms of the $Osp(1,2)$ algebra.

The action takes the form

$$S_{IIA} = \int d^2 z \left( \Pi_z \overline{\Pi}_z + d_+ E^\theta_M \partial Z^M + d_+ \overline{\partial} \theta^+ + d_- E^\overline{\theta}_M \partial \overline{Z}^M + \overline{d}_- \partial \overline{\theta}^- + F^{+-} d_+ \overline{d}_- \right).$$

(5.21)
where $F^{+\dot{-}} = 8N_c$ is the constant RR field strength.

In addition we have the chiral boson action $S_B$. In order to write it we need the $U(1)_R$ connection, which in this case is replaced by the dilatation (there is no $U(1)_R$ part in the $OSp(1,2)$ algebra). It takes the form

$$A = \frac{dZ + \bar{d}Z + \Theta d\bar{\Theta} + \Theta d\Theta}{Z - \bar{Z} - \Theta \bar{\Theta}}.$$  \hspace{1cm} (5.22)

Since the dilaton is constant we get from (5.18) the Euler number of the Riemann surface. Note also that $d_+, d_+, \bar{d}_-, \bar{d}_-$ can be integrated out easily. We will leave the complete analysis for a forthcoming publication.

5.5. $D=3$, $n=1$

Using the supersymmetric variables the classical action for IIB in the flat background is given by

$$S_{IIB} = \frac{1}{\alpha'} \int \! dzd\bar{z} \left( \frac{1}{2} \Pi_{\alpha \dot{\alpha}} \Pi^{\alpha \dot{\alpha}} + d_{+\alpha} \bar{\partial} \theta^{+\alpha} + d_{+\dot{\alpha}} \bar{\partial} \bar{\theta}^{+\dot{\alpha}} + \bar{d}_{+\alpha} \partial \theta^{+\alpha} + \bar{d}_{+\dot{\alpha}} \partial \bar{\theta}^{+\dot{\alpha}} \right) + S^{flat}_B,$$

where $S_B$ is the action for the chiral bosons $\rho, \bar{\rho}$.

The classical action for IIA in the flat background takes the form

$$S_{IIA} = \frac{1}{\alpha'} \int \! dzd\bar{z} \left( \frac{1}{2} \Pi_{\alpha \dot{\alpha}} \Pi^{\alpha \dot{\alpha}} + d_{+\alpha} \bar{\partial} \theta^{+\alpha} + d_{+\dot{\alpha}} \bar{\partial} \bar{\theta}^{+\dot{\alpha}} + \bar{d}_{-\alpha} \partial \theta^{-\alpha} + \bar{d}_{-\dot{\alpha}} \partial \bar{\theta}^{-\dot{\alpha}} \right) + S^{flat}_B.$$  \hspace{1cm} (5.24)

As before, on flat Riemann surface, there is no coupling with the background charge.

In order to couple the system to the background, we introduce the curved vielbeins $E^A_M$ where the $A$ are tangent superspace indices and $M$ are curved superspace indices. We will use the notation $x_{\alpha \dot{\alpha}}$ \hspace{0.5cm} (4.13).

The action for IIB in curved space can be written as

$$S_{IIB} = \frac{1}{\alpha'} \int \! dzd\bar{z} \left( (G + B)_{AB} \Pi^A \Pi^B + d_{+\alpha} \Pi^{+\alpha} + d_{+\dot{\alpha}} \Pi^{+\dot{\alpha}} + \bar{d}_{+\alpha} \Pi^{+\alpha} + \bar{d}_{+\dot{\alpha}} \Pi^{+\dot{\alpha}} \right) + S_B + d_{+\dot{\alpha}} \partial_{+\dot{\alpha}} F^{+\dot{-\alpha\beta}}.$$  \hspace{1cm} (5.25)
Similarly, the action for type IIA takes the form

\[ S_{IIA} = \frac{1}{\alpha'} \int dz d\bar{z} \left( (G + B)_{AB} \Pi^A \Pi^B + d_{+a} \Pi^{+a} + d_{+\dot{a}} \Pi^{+\dot{a}} + d_{-a} \Pi^{-a} + d_{-\dot{a}} \Pi^{-\dot{a}} \right) \]

\[ + d_{+a} \overline{d}_{-\dot{a}} F^{+\dot{a}} + S_B. \]  

(5.26)

Again we have to establish the form of the action \( S_B \) in both cases. In a separate publication [16], we will explore in detail the structure of chiral primary fields and the constraints on the sigma model due to the \( N = 2 \) superconformal symmetry.

The invariance of the action under the superconformal transformations implies the form of the chiral boson couplings. In addition, we should add the FT term \( S_{FT} \) in order to guarantee the conformal invariance also at higher orders in \( \alpha' \). As in the hybrid formalism [25] the form of the FT term is

\[ S_{FT} = \int d^2z D_+ \overline{D}_+ (\Phi_{cc} \Sigma_{cc} + h.c.), \]  

(5.27)

where \( \hat{\Phi} \) is the conformal compensator whose lowest component contains the dilaton field.

5.6. Higher dimensions

In the previous section we established the relation between the original variables and the hybrid formalism variables. In order to have the manifest supersymmetry we use the left-movers \( \Theta^{+a}, \Theta^{+\dot{a}} \) (with \( a, a' = 1, \ldots, 4 \)), the right-movers \( \overline{\Theta}^{+a}, \overline{\Theta}^{+\dot{a}} \) and their conjugate momenta \( \Delta \)'s. The type IIA \( \sigma \)-model takes the form

\[ S_{IIA} = \frac{1}{\alpha'} \int dz d\bar{z} \left( (G + B)_{AB} \Pi^A \Pi^B + d_{+a} \Pi^{+a} + d_{+\dot{a}} \Pi^{+\dot{a}} + d_{-a} \Pi^{-a} + d_{-\dot{a}} \Pi^{-\dot{a}} \right) \]

\[ + d_{+a} \overline{d}_{-\dot{a}} F^{+\dot{a}} + \Delta_{+a} \overline{\Theta}^{+a} + \Delta_{+\dot{a}} \overline{\Theta}^{+\dot{a}} + \Delta_{-a} \overline{\Theta}^{-a} + \Delta_{-\dot{a}} \overline{\Theta}^{-\dot{a}} \right) + S_B + S_{FT}. \]  

(5.28)

Again, the chiral boson action \( S_B \) and the FT term \( S_{FT} \) (5.27) are established using the \( N = 2 \) superconformal invariance. The chiral bosons couple to the worldsheet R-symmetry and to the Lorentz connection accordingly.

In addition, we should also impose an harmonic constraint to remove the doubling variables \( \Theta \)'s and \( \Delta \)'s. The form of these harmonic constraint is discussed in [15].

Note that some simplifications arise when studying the class of curved backgrounds with a constant dilaton and RR flux such as, for instance, \( AdS_{2p} \) or \( AdS_3 \times S^3 \). Since the dilaton is constant the FT term \( S_{FT} \) is simply the Euler number of the Riemann surface. The RR field couples as a constant in the action (5.28) and \( d' \) s and \( \overline{d}' \) s can be integrated out easily. The rest of the action is determined by the supergroup structure.
5.7. The inclusion of open strings

In order to enlarge the possible conformal backgrounds of non-critical strings, open strings can be included. In [6] a Born-Infeld type term corresponding to \( N_f \) branes-antibranes uncharged system has been added

\[
S_{\text{open}} = \frac{-2N_f}{2k^2} \int d^d x \sqrt{G} e^{-\Phi},
\]

which allows for gravity solutions such as \( AdS_5 \times S^1 \). In our framework, such a term is generated by considering worldsheets with boundaries.

The inclusion of open string vertex operator can be done in the same way as for the closed deformations. The only difference is that the vertex operator has to placed on the boundary of the worldsheet. The general form of the massless boundary vertex operator is

\[
V_{\text{open}} = \oint dz (\partial \theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \ldots)\]

(5.30)

where \( A_\alpha, A_m, W^\alpha \) are superfields. The ellipsis stands for the ghost contributions and they depend upon the dimension of the space time. The lowest component of the superfields \( A_m \) is represented by the gluon field and \( W^\alpha = \psi^\alpha + \ldots \) has the gluino as the lowest component. In the D=1 case, the only massless vertex operator which is independent of Liouville field is a constant gluino field and the coupling reduces to

\[
V_{\text{open}} = \oint \psi^\alpha q_\alpha,
\]

(5.31)

where \( q_\alpha \) is the supercurrent.

Given the vertex operator for the massless sector, we can construct the sigma model in curved space. One starts from IIB case (where the same supercharges are taken in the left- and right-moving sector) and impose the boundary conditions (for the flat case) at the level of superspace variables

\[
(\theta^+ - \bar{\theta}^+)|_{z=\tau} = 0, \quad (\theta^- - \bar{\theta}^-)|_{z=\tau} = 0,
\]

\[
(p^+ - \bar{p}^+)|_{z=\tau} = 0, \quad (p^- - \bar{p}^-)|_{z=\tau} = 0,
\]

(5.32)

(the usual boundary conditions are imposed on bosonic coordinates). At the level of sigma model one has to: \(i\) construct a supersymmetric sigma model by adding the surface terms in order to compensate those supersymmetry variation which vanish because of partial integration, \(ii\) add the vertex operator (5.31) and derive the Dirac-Born-Infeld action as a consequence of the boundary conditions in the given background.
6. Discussion

In this work we constructed a hybrid type formalism in order to study non-critical strings in a manifest space-time supersymmetric way. We started with the linear dilaton background and worked out the precise map of the RNS variables to the supersymmetric variables. We noticed that in order to construct manifest supersymmetric non-critical string theory, we needed to double the superspace coordinates. This suggests that the non-critical strings may have consistent backgrounds with double the number of supersymmetries of the linear dilaton background.

One of the ingredients of the present formalism is the presence of new ghost fields represented by a set of chiral bosons. The relation between the original set of variables and the new GS-like variables determines also the coupling of the ghost fields to the supergravity background in the curved space.

A feature of the present framework is the possibility to couple the sigma-model to curved backgrounds and it provides simple way to couple the RR fields to the worldsheet field. This allows us, in particular, to study the conformal invariance to all orders in $\alpha'$.

We have seen several similarities of the construction with that of Calabi-Yau compactifications of the critical superstrings, though the systems are different.

There are numerous open problems that should be addressed. Let us mention a few of them. (i) The analysis of the superconformal invariance of the sigma model for curved backgrounds. (ii) The analysis of the spectrum for higher dimensional examples extending the results of the lowest dimensional case. (iii) The construction of tree level scattering amplitudes and the higher genus extension. (iv) The relation with pure spinor formulation in ten dimensional superstring and its dimensional reduction. (v) The analysis of specific curved background such as AdS$_{2n}$, AdS$_5 \times$S$^1$ (with open strings) and others.

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7. Appendix

7.1. Supersymmetry currents and charges

For the reader convenience, we add the relevant OPE's for the supersymmetry currents. For \( D = 1, n = 0 \) case, we have the currents

\[
q_+ = e^{-\frac{i}{2} \phi - \frac{i}{4} (H+2x)} , \quad q_- = e^{-\frac{i}{2} \phi + \frac{i}{4} (H+2x)} ,
\]

\[
q_+ = e^{-\frac{i}{2} \phi - \frac{i}{4} (H-2x)} , \quad q_- = e^{-\frac{i}{2} \phi + \frac{i}{4} (H-2x)} .
\]

which are nilpotent and satisfy

\[
q_-(z)q_-(w) \sim \frac{1}{z-w} e^{-\phi+iH} , \quad q_-(z)q_+(w) \sim 0 ,
\]

\[
q_-(z)q_+(w) \sim \frac{1}{(z-w)^{3/2}} e^{-\phi} , \quad q_+(z)q_-(w) \sim \frac{1}{(z-w)^{3/2}} e^{-\phi} ,\]

\[
q_+(z)q_+(w) \sim \frac{1}{z-w} e^{-\phi-iH} , \quad q_+(z)q_-(w) \sim 0 .
\]

As explained in the text, we can use only the set of supercharges (\( \hat{\mathfrak{f}} q_+, \hat{\mathfrak{f}} q_- \)) or (\( \mathfrak{f} q_-, \mathfrak{f} q_- \)) to construct the supersymmetric model.

In the \( D=3, n=1 \) case we have eight combinations.

\[
q_+^\alpha = e^{-\frac{i}{2} \phi + \frac{i}{4} H^1 + \frac{i}{4} (H+\sqrt{3}x)} , \quad q_-^\alpha = e^{-\frac{i}{2} \phi - \frac{i}{4} H^1 \mp \frac{i}{4} (H+\sqrt{3}x)} ,
\]

\[
q_+^{\dot{\alpha}} = e^{-\frac{i}{2} \phi - \frac{i}{4} H^1 \mp \frac{i}{4} (H-\sqrt{3}x)} , \quad q_-^{\dot{\alpha}} = e^{-\frac{i}{2} \phi + \frac{i}{4} H^1 \mp \frac{i}{4} (H-\sqrt{3}x)} .
\]

The relevant OPE’s are

\[
q_+^1(z)q_+^2(w) \sim \frac{1}{(z-w)} e^{-\phi+iH^1} , \quad q_-^1(z)q_-^2(w) \sim \frac{1}{(z-w)} e^{-\phi+iH^1} ,
\]

\[
q_+^1(z)q_-^2(w) \sim \frac{1}{(z-w)} e^{-\phi-iH^1} , \quad q_-^1(z)q_+^2(w) \sim \frac{1}{(z-w)} e^{-\phi-iH^1} ,
\]

\[
q_+^1(z)q_+^1(w) \sim \frac{1}{(z-w)^{1/2}} e^{-\phi+iH^1+iH} , \quad q_-^1(z)q_-^1(w) \sim \frac{1}{(z-w)^{3/2}} e^{-\phi} ,
\]

\[
q_+^1(z)q_-^1(w) \sim \text{reg.} , \quad q_+^1(z)q_-^2(w) \sim (z-w)^{1/2} e^{-\phi+iH^1-i\sqrt{3}x} \sim 0 ,
\]

\[
q_+^1(z)q_+^2(w) \sim (z-w)^{1/2} e^{-\phi-iH-i\sqrt{3}x} \sim 0 , \quad q_-^1(z)q_-^2(w) \sim \frac{1}{(z-w)} e^{-\phi-iH} .
\]
As before, the way to extract two set of charges which have no branching cuts is to consider \((\oint q_+^\alpha, \oint q_+^\alpha')\) or \((\oint q_-^\alpha, \oint q_-^\alpha')\). Such that the charges \(\oint q_+^\alpha\) and \(\oint q_-^\alpha\) represent the supersymmetry algebra in two dimensions and the OPE’s between the two sets give a translation generators in the Liouville direction.

The case \(D=5, n=2\) is described by the following currents

\[
\begin{align*}
q_+^\alpha &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1+H^2)-\frac{i}{2}(H+\sqrt{2}x)} , & q_+^{\alpha'} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1-H^2)+\frac{i}{2}(H+\sqrt{2}x)} , \\
q_-^\alpha &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1+H^2)+\frac{i}{2}(H+\sqrt{2}x)} , & q_-^{\alpha'} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1-H^2)-\frac{i}{2}(H+\sqrt{2}x)} , \\
q_+^{\dot{\alpha}} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1+H^2)-\frac{i}{2}(H-H^2)} , & q_+^{\dot{\alpha}'} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1-H^2)+\frac{i}{2}(H-H^2)} , \\
q_-^{\dot{\alpha}} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1+H^2)+\frac{i}{2}(H-H^2)} , & q_-^{\dot{\alpha}'} &= e^{-\frac{i}{2}\phi + \frac{i}{2}(H^1-H^2)-\frac{i}{2}(H-H^2)} ,
\end{align*}
\]  

In the present case, out of these currents we select the four \((q_+^\alpha, q_-^{\dot{\alpha}})\). They are local w.r.t. each other and they close on the translation generators of the 4-dimensional transverse space. Again, we can extend this set of 4 supercurrents by adding the currents \((q_+^{\dot{\alpha}}, q_-^{\alpha'})\).

7.2. Construction of BRST charge \(D=1, n=0\) case

In order that the BRST charge is nilpotent and has the correct commutation properties with \(G''\), we can rewrite \(G'^+\) of (2.7) in a simpler form. Working in the large Hilbert space, the BRST current can be expressed as follows

\[
G'^+ = - e^{-R} e^{-\chi + 2(\phi + \kappa)} e^R , \quad R = \oint \left[ (G^+ + G^-) e^{\chi - \phi - \kappa} + \frac{1}{2} \partial \phi e^{2(\chi - \phi - \kappa)} \right] .
\]  

where \(G^\pm\) are the supersymmetry generators for the matter system given in (2.5).\(^6\)

Using the above definitions (3.44), it is easy to show that

\[
(G^+ + G^-) = \frac{1}{2} \left( e^{iH} (\partial(\phi - ix) - 2i\partial H) + e^{-iH} (\partial(\phi + ix) + 2i\partial H) \right) ,
\]  

and inserted into the combination of (7.6) we get the three terms in \(R\) expressed in terms of the new variables

\[
G'^+ e^{\chi - \phi - \kappa} = \frac{1}{2} \theta^+ \theta^+ \left( \partial \phi - i\partial x' - 2\partial \omega - 2\partial \rho \right) - i\partial \theta^+ \theta^+ \]  

\(\)Notice that the first term in \(R\) can be written as follows \(G_m \xi e^{-\phi}\). Therefore, we can see that \((G_m \xi e^{-\phi})(z)(G_m \xi e^{-\phi})(w) \to (z-w)^{-1} 15 e^{-2\phi} \xi \partial \xi c \partial c\) which corresponds to the second term in \(R\) up to an overall coefficient.
\[ G^{-e^{\chi-\phi-\kappa}} = \frac{1}{2} e^{-2\omega} \left( \partial \varphi + i \partial x' + 2 \partial \omega + 2 \partial \rho \right) + i e^{-2\omega} \theta^+ p_+ . \]

\[ \partial \phi e^{2(\chi-\phi-\kappa)} = (-3 \partial \theta \dot{\theta} \dot{\theta} + \theta^+ \theta^+) e^{-2\omega} + (2 \partial \omega - 8 \partial \rho - 4 i \partial x') \theta^+ \theta^+ e^{-2\omega} . \]

Finally, from (3.44) we can check that

\[ e^{\chi+2(\phi+\kappa)} = p_+ e^{\rho+\omega}. \]  

(7.9)

from these definition we can deduce the form of the BRST operator by computing the similarity transformation on \( p_+ e^{\rho+\omega} \).
References

[1] A. M. Polyakov, “Quantum Geometry Of Fermionic Strings,” Phys. Lett. B 103, 211 (1981).
[2] A. M. Polyakov, “Quantum Geometry Of Bosonic Strings,” Phys. Lett. B 103, 207 (1981).
[3] A. M. Polyakov, “String theory as a universal language,” Phys. Atom. Nucl. 64, 540 (2001) [Yad. Fiz. 64, 594 (2001 IMPAE,A16,4511-4526.2001)] [arXiv:hep-th/0006132].
[4] S. Kuperstein and J. Sonnenschein, “Non-critical supergravity (d > 1) and holography,” JHEP 0407, 049 (2004) [arXiv:hep-th/0403254], “Non-critical, near extremal AdS(6) background as a holographic laboratory of four dimensional YM theory,” JHEP 0411, 026 (2004) [arXiv:hep-th/0411009].
[5] A. M. Polyakov, “Conformal fixed points of unidentified gauge theories,” Mod. Phys. Lett. A 19, 1649 (2004) [arXiv:hep-th/0405106].
[6] I. R. Klebanov and J. M. Maldacena, “Superconformal gauge theories and non-critical superstrings,” Int. J. Mod. Phys. A 19, 5003 (2004) [arXiv:hep-th/0409133].
[7] M. Alishahiha, A. Ghodsi and A. E. Mosaffa, “On isolated conformal fixed points and noncritical string theory,” JHEP 0501, 017 (2005) [arXiv:hep-th/0411087].
[8] F. Bigazzi, R. Casero, A. L. Cotrone, E. Kiritsis and A. Paredes, “Non-critical holography and four-dimensional CFT’s with fundamentals,” arXiv:hep-th/0505140.
[9] D. Kutasov and N. Seiberg, “Non-critical superstrings”, Phys. Lett. B 251, 67 (1990).
[10] D. Kutasov, “Some properties of (non)critical strings”, [arXiv:hep-th/9110041].
[11] S. Murthy, “Notes on non-critical superstrings in various dimensions,” JHEP 0311, 056 (2003) [arXiv:hep-th/0305197].
[12] N. Seiberg, “Observations on the moduli space of two dimensional string theory,” arXiv:hep-th/0502156.
[13] H. Ita, H. Nieder and Y. Oz, “On type II strings in two dimensions,” arXiv:hep-th/0502187.
[14] N. Berkovits, C. Vafa and E. Witten, “Conformal field theory of AdS background with Ramond-Ramond flux,” JHEP 9903, 018 (1999) [arXiv:hep-th/9902098].
[15] N. Berkovits, “Quantization of the type II superstring in a curved six-dimensional background,” Nucl. Phys. B 565, 333 (2000) [arXiv:hep-th/9908041].
[16] P.A. Grassi and Y. Oz, to appear.
[17] K. Itoh and N. Ohta, “BRST cohomology and physical states in 2-D supergravity coupled to c ≤ 1 matter,” Nucl. Phys. B 377, 113 (1992) [arXiv:hep-th/9110013].
[18] P. Bouwknegt, J. G. McCarthy and K. Pilch, “Ground ring for the 2-D RNS string,” Nucl. Phys. B 377, 541 (1992) [arXiv:hep-th/9112030].
[19] P. Bouwknegt, J. G. McCarthy and K. Pilch, “BRST analysis of physical states for 2-D (super)gravity coupled to (super)conformal matter,” [arXiv:hep-th/9110031].
[20] N. Berkovits, S. Gukov and B. C. Vallilo, “Superstrings in 2D backgrounds with R-R flux and new extremal black holes,” Nucl. Phys. B 614, 195 (2001) [arXiv:hep-th/0107140].

[21] N. Berkovits, “A new description of the superstring,” [arXiv:hep-th/9604123].

[22] N. Berkovits and J. Maldacena, “N = 2 superconformal description of superstring in Ramond-Ramond plane wave backgrounds,” JHEP 0210, 059 (2002) [arXiv:hep-th/0208092].

[23] P. A. Grassi and L. Tamassia, “Vertex operators for closed superstrings,” JHEP 0407, 071 (2004) [arXiv:hep-th/0405072].

[24] H. Verlinde, “Superstrings on AdS(2) and superconformal matrix quantum mechanics,” [arXiv:hep-th/0403024].

[25] N. Berkovits and W. Siegel, “Superspace Effective Actions for 4D Compactifications of Heterotic and Type II Superstrings,” Nucl. Phys. B 462, 213 (1996) [arXiv:hep-th/9510106].