Uniquesolvabilityofacrackproblem withSignorini-type andTrescafriction conditions in a linearized elastodynamic body

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Weconsiderdynamicmotionofalinearizedelastic body withacrack subject to a modified contact law, which we call theSignorini contact condition of dynamic type, and to theTrescafriction condition. Whereas the modifiedcontactlaw involves bothdisplacement andvelocity, it formally includes the usual non-penetrationcondition as a special case. We prove that thereexists a unique strong solution to this model. It is remarkable that not only existence but also uniqueness is obtained and that no viscosity term that serves as a parabolic regularization is added in our model.

Thisarticle is part of the themeissue ‘Non-smooth variational problems and applications’.

1..Introduction

Analysis of crack motion is one of the most important topics in fracture mechanics and it has also attracted much attention in material science or in seismology (e.g. [1–3]). However, at least from the mathematical point of view, it is far from being understood because of the highly nonlinear and singular behaviour of cracks. Even if we put aside problems regarding crack propagation, which are difficult even at the stage of modelling and will not be addressed in this paper, there still remain many mathematical difficulties as explained below.
In the static case, one of the basic models is the linearized elasticity with interfacial conditions representing the non-penetration contact law (also known as the Signorini condition) and the Coulomb friction law on the crack (see [4,5]). In principle, the former condition implies that normal stress acts on the crack only when its both sides are in contact, and the latter means that slip velocity across it occurs only when tangential stress reaches a threshold (see also remark 2.1 below). We leave two side remarks regarding this model: optimal regularity of weak solutions for the Signorini problem is obtained by Andersson [6] (see also [7]), and non-monotone friction laws that lead to hemi-variational inequalities can also be employed in place of the Coulomb law (see [8]).

The dynamical version of the above model, however, becomes much more difficult and no mathematical results seem to have been obtained. For related problems, in which some conditions mentioned above are modified or simplified, there are several known studies.

First, for the wave equation with the Signorini condition, unique solvability is established for the halfspace in [9]. Existence of a weak solution for general domains is proved by Kim [10], but uniqueness remains open. Generalization of these results to the linearized elasticity equations is also unsolved. If the Kelvin–Voigt viscoelastic model, in which a term serving as parabolic regularization is added to the linearized elasticity, is considered instead, then existence of a weak solution is obtained, e.g. in [11,12] and that of a strong solution is shown by Petrov & Schatzman [13]. If the contact law is furthermore modified in such a way that the Signorini condition is imposed on velocity rather than on displacement, then uniqueness of a weak solution is shown as well (see [11], Section 4.4.2).

Second, dynamic friction problems also exhibit a difficulty. In case of the Tresca friction law, where the threshold parameter of the tangential traction is a given function $g$, under the assumption that $g$ does not depend on the time variable unique solvability of the linearized elasticity equations (without contact conditions) is established in [14]. This result was extended to the time-dependent $g$ in our previous paper [15]. If the Coulomb friction law which is considered to be more realistic but is more complex is employed, in ([11], Chapter 5), existence of a solution to the Kelvin–Voigt viscoelastic model combined with the Signorini condition in velocity is proved. In the context of crack problems, a weak solution of the Kelvin–Voigt viscoelastic model with the Signorini condition in velocity is proved. In the context of crack problems, a weak solution of the Kelvin–Voigt viscoelastic model with the Signorini condition in displacement and with the non-local (approximated) Coulomb friction law is constructed in [12,16].

Namely, when the contact condition is imposed on displacement and is combined with some friction law, only existence of a solution is established in the presence of viscosity terms. In view of such a situation, one would like to mathematically explore a dynamic elasticity model with contact and friction having the following properties:

(i) classical linear elasticity is exploited without viscosity;
(ii) not only existence but also uniqueness of a solution is ensured;
(iii) contact law is formulated in terms of displacement, which is considered to be more realistic.

In this paper, we propose to impose a contact condition to linear combination of normal displacement and normal velocity on the interface with some constant coefficient $\delta > 0$; see (2.2a) below. Since $\delta = 0$ and $\delta = \infty$ correspond to the contact conditions in displacement and in velocity, respectively, it can be regarded as an intermediate between them. We call (2.2a) the Signorini contact condition of dynamic type (hereinafter, referred to as SCD condition). With the SCD and Tresca friction conditions, we prove unique existence of a strong solution for the linearized elastodynamic equations, thus having properties (i) and (ii). Moreover, property (iii) is also approached by our model because $\delta > 0$ can be fixed to an arbitrarily small value (however it is not possible to make exactly $\delta = 0$).

An expository interpretation of our result may be that making the Signorini contact condition in displacement ‘dynamic a bit’ (recall that boundary conditions having quantities with time derivative are called dynamic) leads to some stabilization effect to the system. We expect that
this fact has some connection with Baumgarte-like stabilization techniques known in numerical simulations of non-smooth mechanics (see [17]), which is to be investigated in the future. The present result will also be of basic interest when we make an attempt to more involved crack problems, e.g. propagation and singular behaviour of crack tips.

This paper is organized as follows. In §2, we introduce notation and the precise mathematical setting to be studied. In §3, a variational inequality formulation as well as the definition of a strong solution is introduced, and we present the main theorem. Section 4 is devoted to its proof based on regularization of a variational inequality and Galerkin’s method. The strategy basically follows our previous study [15]; nevertheless, the analysis, in particular a priori estimates and a uniqueness proof, becomes more intricate to deal with the contact condition.

2. Preliminaries

(a) Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial \Omega$ consisting of two parts $\Gamma_D \neq \emptyset$ and $\Gamma_N$ that are mutually disjoint. Let $\Gamma$ be a two-dimensional closed smooth interface that separates $\Omega$ into two subdomains $\Omega_{\pm}$, that is,

$$\Omega = \Omega_+ \cup \Omega_- \cup \Gamma, \quad \Gamma = \overline{\Omega}_+ \cap \overline{\Omega}_-.$$ 

We assume that $\partial \Omega_{\pm}$ satisfy the Lipschitz condition and that $\partial \Omega_{\pm} \cap \Gamma_D \neq \emptyset$. A crack is supposed to be represented by an open subset $\Gamma_c$ of $\Gamma$ such that $\overline{\Gamma}_c \subset \Gamma \setminus \partial \Gamma$ (namely, $\Gamma_c \Subset \Gamma$); we refer to $\Omega_c := \Omega \setminus \Gamma_c$ as the domain with a crack. The unit normal vector associated with $\partial \Omega$ is denoted by $\nu_{\partial \Omega}$, and the unit normal vector on $\Gamma$ pointing from $\Omega_-$ to $\Omega_+$ is denoted by $\nu$. The geometric situation explained so far is schematically summarized in figure 1.

We mainly deal with functions defined in $\Omega_c$ in this paper. For such a function $u$, we let $u^{\pm} := u|_{\Omega_{\pm}}$ be its restrictions to subdomains $\Omega_{\pm}$. If $u^{\pm}$ are smooth enough, we define the jump discontinuity of $u$ across $\Gamma$ by

$$[u] := u^+|_{\Gamma} - u^-|_{\Gamma},$$

and that of $\nabla u$ by $[\nabla u] := (\nabla u^+)|_{\Gamma} - (\nabla u^-)|_{\Gamma}$.

For function spaces, we employ the usual Lebesgue spaces $L^p(\Omega_c)$ ($1 \leq p \leq \infty$) and the Sobolev space $H^1(\Omega_c)$, which have the characterization

$$L^p(\Omega_c) = L^p(\Omega_+) \times L^p(\Omega_-)$$

and

$$H^1(\Omega_c) = (u^+, u^-) \in H^1(\Omega_+) \times H^1(\Omega_-) : [u] = 0 \text{ on } \Gamma \setminus \Gamma_c.$$ 

Accordingly, their norms are given by $\|u\|_{L^p(\Omega_c)} := (\|u^+\|_{L^p(\Omega_+)}^p + \|u^-\|_{L^p(\Omega_-)}^p)^{1/p}$ and $\|u\|_{H^1(\Omega_c)} := (\|u^+\|^2_{H^1(\Omega_+)} + \|u^-\|^2_{H^1(\Omega_-)})^{1/2}$. Note in particular that if $u \in H^1(\Omega_c)$ then $[u] \in H_{00}^{1/2}(\Gamma_c)$, which is the Lions–Magenes space (see [18]).

Functions and function spaces that are vector- or tensor-valued are written with bold fonts, e.g. $u \in H^1(\Omega_c) = H^1(\Omega_c)^3$, whereas fine fonts mean scalar quantities. We denote the inner products of $L^2(\Omega_c)$ by $(\cdot, \cdot)$, and those of $L^2(\Gamma_N)$, $L^2(\Gamma_c)$ by $(\cdot, \cdot)_{\Gamma_N}$, $(\cdot, \cdot)_{\Gamma_c}$ (the same notation will also be used for vectors and tensors). We also exploit the notation of Bochner spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ for a positive constant $T$ and a Banach space $X$, where $k > 0$ is an integer and $1 \leq p \leq \infty$. Finally, the dual space of $X$ is denoted by $X^*$.

(b) Problem formulation

We assume that $\Omega_c$ is regarded as a reference configuration (or non-deformed state) of an elastic body. The deformation of the body may be described by a displacement field $u: (0,T) \times \Omega_c \to \mathbb{R}^3$. 
If the constitutive law of the material is based on isotropic linear elasticity, the stress tensor is given by
\[ \sigma(u) = (\lambda \operatorname{div} u) \mathbb{I} + 2\mu \mathbb{E}(u), \]  
(2.1)
where \( \lambda, \mu \) are Lamé constants such that \( \mu > 0 \) and \( 3\lambda + 2\mu > 0 \), \( \mathbb{I} \) is the unit tensor, and \( \mathbb{E}(u) = (\nabla u + (\nabla u)^	op)/2 \) means the linearized strain tensor. The dynamic deformation of the body is governed by the hyperbolic system
\[ \rho u'' - \operatorname{div} \sigma(u) = \rho f \quad \text{in} \ (0, T) \times \Omega_c, \]
where \( \rho \) is the density which is a positive constant, the prime stands for the time derivative (i.e. \( u'' = \partial_t^2 u \)), \( f \) is the external body force and \( T > 0 \) stands for a fixed time length. As for the boundary conditions, we consider
\[ u = 0 \quad \text{on} \ (0, T) \times \Gamma_D \]
and
\[ \sigma(u)\nu_{\partial \Omega} = F \quad \text{on} \ (0, T) \times \Gamma_N, \]
where \( F \) is a prescribed traction on \( \Gamma_N \). At \( t = 0 \), the initial displacement and velocity fields are given as
\[ u(0) = u_0, \quad u'(0) = \dot{u}_0 \quad \text{on} \ [0] \times \Omega_c. \]

Before stating the interface conditions on the crack, we introduce the normal and tangential components of the displacement, velocity and traction on \( \Gamma \), restricted from \( \Omega \), by
\[ u^\pm_v = u^\pm \cdot \nu, \quad u^\pm_{\tau} = u^\pm - u^\pm_v \nu, \quad u'^\pm_v = u'^\pm \cdot \nu, \quad u'^\pm_{\tau} = u'^\pm - u'^\pm_v \nu, \]
\[ \sigma^\pm_v = \sigma(u^\pm) \nu \cdot \nu, \quad \sigma^\pm_{\tau} = \sigma(u^\pm) \nu \cdot \nu - \sigma^\pm_v \nu, \]
together with their jumps
\[
\begin{align*}
\llbracket u_v \rrbracket &= u^+_v - u^-_v, \quad \llbracket u_\tau \rrbracket = u^+_\tau - u^-_\tau, \quad \llbracket u'_v \rrbracket = u'^+_v - u'^-_v, \quad \llbracket u'_\tau \rrbracket = u'^+_\tau - u'^-_\tau, \\
\llbracket \sigma_v \rrbracket &= \llbracket \sigma_v(u) \rrbracket = \sigma^+_v - \sigma^-_v, \quad \llbracket \sigma_{\tau} \rrbracket = \llbracket \sigma_{\tau}(u) \rrbracket = \sigma^+_\tau - \sigma^-_{\tau}.
\end{align*}
\]
In this paper, we consider the Signorini contact condition of dynamic type (SCD condition) and Tresca friction condition on the crack \( \Gamma_c \) as follows:
\[
\begin{align*}
\llbracket \sigma_v \rrbracket &= 0, \quad \sigma_v \leq 0, \quad \llbracket u_v + \delta u'_v \rrbracket \geq 0, \quad \sigma_v \llbracket u_v + \delta u'_v \rrbracket = 0 \quad \text{on} \ (0, T) \times \Gamma_c \quad (2.2a) \\
\llbracket \sigma_{\tau} \rrbracket &= 0, \quad |\sigma_{\tau}| \leq g, \quad -\sigma_{\tau} \cdot \llbracket u'_v \rrbracket + g \llbracket u'_v \rrbracket = 0 \quad \text{on} \ (0, T) \times \Gamma_c, \quad (2.2b)
\end{align*}
\]
where \( \delta \in (0, \infty) \) is a constant, \( g = g(t, x) \geq 0 \) is a given function.

Several remarks are in order. First, \( \sigma_v := \sigma^+_v = \sigma^-_v \) and \( \sigma_{\tau} := \sigma^+_\tau = \sigma^-_{\tau} \) are well-defined as single-valued functions on \( \Gamma_c \) because they have no jump by (2.2). Second, if \( \delta = 0 \) in (2.2a) then we formally recover the usual non-penetration condition introduced in [5]. On the other hand, if...
\( \delta = \infty \) then we arrive at the contact condition in terms of velocity given by Eck et al. [11]. To see this we equivalently rewrite (2.2a), with \( \gamma := \delta^{-1} \), as

\[
\sigma_v \leq 0, \quad \| \gamma u_v + u'_v \| \geq 0, \quad \sigma_v \| \gamma u_v + u'_v \| = 0,
\]

and set \( \gamma = 0 \). For simplicity of presentation, we mainly deal with the SCD condition in the form (2.3) with \( \gamma \in [0, \infty) \) rather than (2.2a) in the subsequent analysis.

**Remark 2.1.** (i) The introduction of \( \delta \) in (2.2a) is mainly due to the mathematical reason as explained in the Introduction. From a modelling viewpoint, it can be regarded as a first-order approximation to the case \( \delta = 0 \), i.e. the usual non-penetration condition \( \| u_v \| \geq 0 \). We see that the SCD condition allows for interpenetration of the crack, which is not physically feasible and may be a restriction in applications. However, it remains realistic for a short time interval in the case of no initial slip velocity on the crack (e.g. for the first—and usually strongest—wave of an earthquake as mentioned in [11], Chapter 5).

(ii) If \( g \) in (2.2b) is replaced by \( F|\sigma_v| (F \geq 0 \) is a coefficient), then the resulting condition is known as the *Coulomb friction law*, which is mentioned in the Introduction.

### 3. Variational formulations

#### (a) Variational inequality

As discussed in the previous section, the strong form of the initial boundary value problem considered in this paper is represented as follows:

\[
\rho u'' - \text{div} \, \sigma(u) = \rho f \quad \text{in} \ (0, T) \times \Omega_c,
\]

\[
u = 0 \quad \text{on} \ (0, T) \times \Gamma_D,
\]

\[
\sigma(u)v_\| \in \Omega \quad \text{on} \ (0, T) \times \Gamma_N,
\]

\[
\| \sigma_v \| = 0, \quad \sigma_v \leq 0, \quad \| \gamma u_v + u'_v \| \geq 0, \quad \sigma_v \| \gamma u_v + u'_v \| = 0 \quad \text{on} \ (0, T) \times \Gamma_c,
\]

\[
\| \sigma_\tau \| = 0, \quad | \sigma_\tau | \leq g, \quad \sigma_\tau : \| u'_c \| = g(\| u'_c \|) \quad \text{on} \ (0, T) \times \Gamma_c,
\]

\[
\nu(0) = \nu_0, \quad u'(0) = \dot{u}_0 \quad \text{on} \ [0] \times \Omega_c.
\]

Let us derive a weak formulation to this problem assuming that \( u \) is smooth enough in \([0, T] \times (\Omega \setminus \Gamma_c)\). To this end we introduce the following function spaces and convex cone:

\[
H := L^2(\Omega_c), \quad V := \{ v \in H^1(\Omega_c) : v = 0 \ \text{on} \ \Gamma_D \}, \quad K := \{ v \in V : \| v_v \| \geq 0 \ \text{a.e. on} \ \Gamma_c \}.
\]

Multiplying (3.1a) by \( v - (\gamma u + u') \) with an arbitrary \( v \in K \) and integrating over \( \Omega_c \), we obtain

\[
\rho \left( u''(t), v - (\gamma u(t) + u'(t)) \right) + (\sigma(u(t)), \nabla (v - (\gamma u(t) + u'(t))))
\]

\[
+ (\sigma_v, \| v_v - (\gamma u_v(t) + u'_v(t)) \|)_{\Gamma_c} + (\sigma_\tau(t), \| v_\tau - (\gamma u_\tau(t) + u'_\tau(t)) \|)_{\Gamma_c}
\]

\[
= \rho \left( f(t), v - (\gamma u(t) + u'(t)) \right) + \left( F(t), v - (\gamma u(t) + u'(t)) \right)_{\Gamma_N} \quad \forall t \in (0, T),
\]

where we have used \( \| \sigma_v \| = 0, \| \sigma_\tau \| = 0 \) on \( \Gamma_c \) and the fact that the outer unit normal w.r.t. \( \Omega_\pm \) on \( \Gamma \) is \( \mp v \). By (2.1) we see that

\[
(\sigma(u), \nabla v) = (\sigma(u), \nabla(v)) = \lambda (\text{div} \, u, \text{div} \, v) + 2\mu (\nabla(u), \nabla(v)) := a(u, v) \quad \forall v \in V.
\]

It follows from (3.1d) and (3.1e) that

\[
(\sigma_v(u(t)), \| v_v - (\gamma u_v(t) + u'_v(t)) \|)_{\Gamma_c} \leq 0
\]

and

\[
(\sigma_\tau(u(t)), \| v_\tau - (\gamma u_\tau(t) + u'_\tau(t)) \|)_{\Gamma_c} \leq (g(t), \| v_\tau - (\gamma u_\tau(t)) \| - \| u'_\tau(t) \|)_{\Gamma_c}.
\]
Consequently,
\[
\rho (\mathbf{u}''(t), \mathbf{v} - (\mathbf{y} u(t) + \mathbf{u}'(t))) + a(\mathbf{u}(t), \mathbf{v} - (\mathbf{y} u(t) + \mathbf{u}'(t))) \\
+ (g(t), [\mathbf{v}_t - \mathbf{y} u_t(t)]_t) - [\mathbf{u}_t(t)]_{\Gamma_c} \\
\geq \rho (\mathbf{f}(t), \mathbf{v} - (\mathbf{y} u(t) + \mathbf{u}'(t))) + (\mathbf{F}(t), \mathbf{v} - (\mathbf{y} u(t) + \mathbf{u}'(t)))_{\Gamma_N} \\
\forall \mathbf{v} \in \mathcal{V}, \text{ a.e. } t \in (0, T).
\] (3.4)

This is a variational inequality of hyperbolic type that is equivalent to the strong form (3.1), provided that there is a classical solution, as seen below.

**Proposition 3.1.** Let \( \mathbf{u} \) be smooth enough to satisfy \( \mathbf{u} \in C^2([0, T] \times (\Omega \setminus \Gamma_c)) \). Then \( \mathbf{u} \) solves (3.1) if and only if the following hold:

(i) \( \mathbf{u}(t) \in \mathcal{V} \) for all \( t \in (0, T) \);

(ii) \( \mathbf{u}(0) = \mathbf{u}_0 \) and \( \mathbf{u}'(0) = \mathbf{u}'_0 \);

(iii) \( \mathbf{y} u(t) + \mathbf{u}'(t) \in \mathbf{K} \) for all \( t \in (0, T) \);

(iv) \( \mathbf{u} \) satisfies the hyperbolic variational inequality (3.4).

**Proof.** The proof is essentially similar to ([15, pp. 125–126]). It suffices to show the ‘if’ part. Taking a test function \( \mathbf{v} = \pm \mathbf{w} + \mathbf{y} u(t) + \mathbf{u}'(t) \) with arbitrary \( \mathbf{w} \in \mathcal{V} \) such that \( \|\mathbf{w}\| = 0 \) on \( \Gamma_c \), one can reduce (3.4) to
\[
\rho (\mathbf{u}''(t), \mathbf{w}) + a(\mathbf{u}(t), \mathbf{w}) = \rho (\mathbf{f}(t), \mathbf{w}) + (\mathbf{F}(t), \mathbf{w})_{\Gamma_N},
\]
which implies (3.1a), (3.1c), \( \|\sigma_v\| = 0 \) on \( \Gamma_c \), and \( \|\sigma_t\| = 0 \) on \( \Gamma_c \). Then (3.2) and (3.3) follow from integration by parts (note that each of \( \|\mathbf{v}_t\| \) and \( \|\mathbf{v}_r\| \) can be chosen to an arbitrary smooth function independently).

First we focus on (3.2). Setting \( \|\mathbf{v}_t\| = 0 \) to 0 and \( 2\|\mathbf{y} u_v(t) + \mathbf{u}'_v(t)\| \) gives
\[
(\sigma_v(t), \|\mathbf{y} u_v(t) + \mathbf{u}'_v(t)\|)_{\Gamma_c} = 0.
\]
Therefore, \( (\sigma_v(t), \|\mathbf{v}_t\|)_{\Gamma_c} \leq 0 \) for arbitrary \( \|\mathbf{v}_t\| \geq 0 \), which implies
\[
\sigma_v(t) \leq 0 \quad \text{on } \Gamma_c.
\]
These two relations combined with \( \|\mathbf{y} u_v(t) + \mathbf{u}'_v(t)\| \geq 0 \) on \( \Gamma_c \); deduce the last equality of (3.1d).

Next, in (3.3), setting \( \|\mathbf{v}_r\| = 0 \) to \( \|\mathbf{y} u_r(t)\| \) and \( \|\mathbf{y} u_r(t) + 2\mathbf{u}'_r(t)\| \) gives
\[
(\sigma_r(t), \|\mathbf{u}'_r(t)\|)_{\Gamma_c} = (g(t), \|\mathbf{u}'_r(t)\|)_{\Gamma_c}.
\]
Therefore, \( (\sigma_r(t), \|\mathbf{v}_r\|)_{\Gamma_c} \leq (g(t), \|\mathbf{v}_r\|)_{\Gamma_c} \) for arbitrary \( \|\mathbf{v}_r\| \), which implies \( |\sigma_r(t)| \leq g(t) \) on \( \Gamma_c \). Then the last equality of (3.1e) also follows. This proves that \( \mathbf{u} \) solves (3.1). \( \Box \)

**(b) Main result**

In view of proposition 3.1, let us define a solution of (3.1) based on its variational form.

**Definition 3.2.** Given \( f, F, g, \mathbf{u}_0, \mathbf{u}'_0 \), we say that \( \mathbf{u} \in W^{2,\infty}(0, T; \mathbf{H}) \cap W^{1,\infty}(0, T; \mathcal{V}) \) is a strong solution of (3.1) if \( \mathbf{u} \) satisfies conditions (i)–(iv) in proposition 3.1.

**Remark 3.3.** For second-order hyperbolic problems, one usually considers a weak solution in \( W^{1,\infty}(0, T; \mathbf{L}^2(\Omega_c)) \cap L^{\infty}(0, T; H^1(\Omega_c)) \). However, this class would not be appropriate for dynamic elasticity problems with friction where the trace of velocity explicitly appears on an interface. We also note that in the Kelvin–Voigt viscoelastic case, a natural class of a weak solution becomes \( W^{1,\infty}(0, T; \mathbf{L}^2(\Omega_c)) \cap H^1(0, T; H^1(\Omega_c)) \), avoiding this issue.

Now we are ready to state our main result in this paper.

**Theorem 3.4.** Let \( \gamma \in [0, \infty) \), \( f \in H^1(0, T; H) \), \( F \in H^2(0, T; L^2(\Gamma_N)) \), and let \( g \in H^2(0, T; L^2(\Gamma_c)) \) be non-negative. We assume that \( \mathbf{u}_0 \in \mathcal{V}, \mathbf{u}'_0 \in \mathcal{V} \) and that they satisfy the following compatibility conditions:
Then there exists a unique strong solution of (3.1).

**Remark 3.5.** Since \( u_0 \in V \) satisfies \(- \text{div} \, \sigma(u_0) \in L^2(\Omega)\), initial tractions \( \sigma(u_0^+) \nu_{3\Omega} \) and \( \sigma(u_0^-) \nu_{\beta \Omega} \) are well-defined in \((H_0^{1/2}(\Gamma_N))^*\) and \((H_0^{1/2}(\Gamma_c))^*\), respectively. The third and fourth conditions above are stronger than just requiring that \( u_0 \) and \( u_0^+ \) satisfy (3.1d) and (3.1e) at \( t = 0 \); however, we are not aware whether they can be weakened.

(c) **Regularized problem**

It is not easy to directly construct a solution of the time-dependent variational inequality (3.4) because it contains non-differentiable relations. To see this, we introduce two convex functions

\[
\psi(x) = \begin{cases} 
+\infty & (x < 0), \\
0 & (x \geq 0),
\end{cases} \quad \varphi(x) = |x| \quad (x \in \mathbb{R}^3),
\]

whose subdifferentials \( \beta := \partial \psi \) and \( \alpha := \partial \varphi \) are maximal monotone graphs given by

\[
\beta(x) = \begin{cases} 
\emptyset & x < 0, \\
(-\infty, 0] & x = 0, \\
0 & x > 0,
\end{cases} \quad \alpha(x) = \begin{cases} 
x/|x| & (x \neq 0), \\
\{y \in \mathbb{R}^3 : |y| \leq 1\} & (x = 0).
\end{cases}
\]

We then observe that the SC and Tresca conditions in (3.1) are concisely expressed as

\[
\sigma_v \in \beta(\|u_v + u_0^+\|), \quad \sigma_\tau \in g\alpha(\|u_0^-\|).
\] (3.5)

To address the difficulty that \( \beta \) and \( \alpha \) are multi-valued functions and non-differentiable, we approximate \( \psi \) and \( \varphi \) by the following functions which are convex and \( W^{3,\infty} \cap C^2 \):

\[
\psi_\epsilon(x) = \frac{1}{2\epsilon} |x|^2, \quad \varphi_\epsilon(x) = \sqrt{|x|^2 + \epsilon^2},
\]

where \( \epsilon > 0 \) is a constant and \([x]_- := \max(-x, 0)\) for \( x \in \mathbb{R} \). Their derivatives \( \beta_\epsilon := \text{d} \psi_\epsilon / \text{d} x \) and \( \alpha_\epsilon := \nabla \varphi_\epsilon \) are given by

\[
\beta_\epsilon(x) = \frac{1}{\epsilon} |x|_-, \quad \alpha_\epsilon(x) = \frac{x}{\sqrt{|x|^2 + \epsilon^2}},
\]

which are monotone and \( W^{2,\infty} \cap C^1 \).

With this preparation we consider the following regularized problem denoted by (VI)\(_\epsilon\): find \( u_\epsilon(t) \in V \) such that \( u_\epsilon(0) = u_0, u_\epsilon'(0) = 0 \) and

\[
\rho(u_\epsilon'(t), v - (\gamma u_v(t) + u_\epsilon'(t))) + a(u_\epsilon(t), v - (\gamma u_v(t) + u_\epsilon'(t))) \\
+ (1, \psi_\epsilon(\|v_v\|) - \psi_\epsilon(\|u_{\epsilon v}(t) + u_\epsilon'(t)\|))_{\Gamma_c} + (g(t), \varphi_\epsilon(\|v_\tau - \gamma u_{\epsilon \tau}(t)\|) - \varphi_\epsilon(\|u_\epsilon'(t)\|))_{\Gamma_c} \\
\geq \rho(f(t), v - (\gamma u_v(t) + u_\epsilon'(t))) + (F(t), v - (\gamma u_v(t) + u_\epsilon'(t)))_{\Gamma_N} \quad \forall v \in V, \; \text{a.e.} \; t \in (0, T).
\] (3.6)

In the proposition below, we find that (VI)\(_\epsilon\) is equivalent to the following variational equality problem denoted by (VE)\(_\epsilon\): find \( u_\epsilon(t) \in V \) such that \( u_\epsilon(0) = u_0, u_\epsilon'(0) = 0 \) and

\[
\rho(u_\epsilon'(t), v) + a(u_\epsilon(t), v) + (\beta_\epsilon(\|u_{\epsilon v}(t) + u_\epsilon'(t)\|), \|v_v\|)_{\Gamma_c} + (g(t)\alpha_\epsilon(\|u_\epsilon'(t)\|), \|v_\tau\|)_{\Gamma_c} \\
= \rho(f(t), v) + (F(t), v)_{\Gamma_N} \quad \forall v \in V, \; \text{a.e.} \; t \in (0, T).
\] (3.7)

**Proposition 3.6.** Let \( u_\epsilon \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; V) \). It solves (VI)\(_\epsilon\) if and only if it solves (VE)\(_\epsilon\).
Proof. Although the proof is standard, we present it for completeness. Let \( \mathbf{u}_\epsilon \) be a solution of (VI). Taking \( \mathbf{v} = \pm h \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \) with arbitrary \( h > 0 \) and \( \mathbf{w} \in \mathbf{V} \), dividing by \( h \), and letting \( h \to 0 \), we deduce (VE) from the relations
\[
\lim_{h \to 0} \psi_\epsilon \left( \frac{\| h \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|}{h} \right) = \beta_\epsilon (\| \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|) \| \mathbf{w} \|
\]
and
\[
\lim_{h \to 0} \frac{\psi_\epsilon (\| h \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|)}{h} = \alpha_\epsilon (\| \mathbf{u}'_\epsilon (t) \|) \cdot \| \mathbf{w} \|.
\]
Conversely, let \( \mathbf{u}_\epsilon \) be a solution of (VE). Note that, since \( \psi_\epsilon \) and \( \varphi_\epsilon \) are convex,
\[
\psi_\epsilon (\| h \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|) = \psi_\epsilon (\| \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|) \geq \beta_\epsilon (\| \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|) \| \mathbf{w} \|
\]
and
\[
\varphi_\epsilon (\| h \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) \|) = \varphi_\epsilon (\| \mathbf{u}'_\epsilon (t) \|) \cdot \| \mathbf{w} \|
\]
for all \( \mathbf{w} \in \mathbf{V} \). Setting this \( \mathbf{w} \) in such a way that \( \mathbf{w} + \gamma \mathbf{u}_\epsilon (t) + \mathbf{u}'_\epsilon (t) = \mathbf{v} \) and using (3.7), we arrive at (3.6).

As a result of proposition 3.6, it suffices to solve an equation problem for obtaining \( \mathbf{u}_\epsilon \).
Furthermore, since it follows from (3.7) that
\[
\sigma_\epsilon (\mathbf{u}_\epsilon) = \beta_\epsilon (\| \gamma \mathbf{u}_\epsilon + \mathbf{u}'_\epsilon \|), \quad \sigma_\tau (\mathbf{u}_\epsilon) = \alpha_\epsilon (\| \mathbf{u}'_\epsilon \|) \quad \text{on} \ (0, T) \times \Gamma_\epsilon,
\]
we expect that \( \mathbf{u}_\epsilon \) should converge to a solution of the original problem (3.1) as \( \epsilon \to 0 \). Justification of this fact, which is actually the idea to prove theorem 3.4, is the task of the next section.

4. Proof of main result

In this section, we establish existence in §4a–d and uniqueness in §4e. Coercivity of \( a(\cdot, \cdot) \) in \( \mathbf{V} \), that is,
\[
a(\mathbf{v}, \mathbf{v}) \geq C \| \mathbf{v} \|^2_{H^1(\Omega_\epsilon)} \quad \forall \mathbf{v} \in \mathbf{V},
\]
which is justified by Korn’s inequality (e.g. [14]), will be frequently used in the proof. Here and in what follows, \( C \) represents a generic constant depending only on the domain \( \Omega_\epsilon \), Lamé constants \( \lambda, \mu \) and density \( \rho \). We will also write \( C(f, g) \), etc. in order to indicate dependency on other quantities.

The inequality above allows us to define the norm of \( \mathbf{V} \) as \( \| \mathbf{v} \| := a(\mathbf{v}, \mathbf{v})^{1/2} \), whereas we use \( \| \mathbf{v} \|_H := \| \mathbf{v} \|_{L^2(\Omega_\epsilon)} \).

(a) Galerkin approximation

We apply Galerkin’s method to solve (3.7). Since \( \mathbf{V} \subset H^1(\Omega_\epsilon) \) is separable, there exist countable members \( \mathbf{w}_1, \mathbf{w}_2, \ldots \in \mathbf{V} \), which are linearly independent, such that \( \bigcup_{m=1}^{\infty} \mathbf{V}_m = \mathbf{V} \) where \( \mathbf{V}_m := \text{span} \{\mathbf{w}_k\}_{k=1}^m \). We may assume that \( \mathbf{u}_0, \mathbf{u}_0 \in \mathbf{V}_m \) for \( m \geq 2 \) (otherwise one can add \( \mathbf{u}_0 \) and \( \mathbf{u}_0 \) to the members \( \{\mathbf{w}_k\}_{k=1}^m \)).

For \( m = 2, 3, \ldots \), the Galerkin approximation problem consists in determining \( c_k(t) (k = 1, \ldots, m) \) such that \( \mathbf{u}_m = \sum_{k=1}^m c_k(t) \mathbf{w}_k \) satisfies
\[
\rho (\mathbf{u}_m''(t), \mathbf{v}) + a (\mathbf{u}_m(t), \mathbf{v}) + \left( \beta_\epsilon (\| \gamma \mathbf{u}_m(t) + \mathbf{u}'_m(t) \|), \| \mathbf{v} \| \right)_{\Gamma_\epsilon} + \left( g(t) \alpha_\epsilon (\| \mathbf{u}'_m(t) \|), \| \mathbf{v} \| \right)_{\Gamma_\epsilon} = \rho (\mathbf{f}(t), \mathbf{v}) + (\mathbf{f}(t), \mathbf{v})_{\Gamma_0}, \quad \forall \mathbf{v} \in \mathbf{V}_m, \forall t \in (0, T),
\]
(4.1)
together with the initial conditions \( \mathbf{u}_m(0) = \mathbf{u}_0, \mathbf{u}_m'(0) = \mathbf{u}_0 \).

This is a finite-dimensional system of ODEs that admits a local-in-time unique solution
\[
c_k \in W^{3, \infty}(0, \bar{T}) \cap C^2([0, \bar{T}]) \quad (k = 1, \ldots, m),
\]
for certain \( 0 < \bar{T} \leq T \) (recall that \( \beta_\epsilon, \alpha_\epsilon \) are \( W^{2, \infty} \cap C^1 \)). Because the a priori estimates below ensure that \( \bar{T} \) can be extended to \( T \), we use \( T \) instead of \( \bar{T} \) from the beginning.
Differentiating (4.1) in $t$ we obtain

$$
\rho(u''_m(t), v) + a(u'_m(t), v) + (\beta'_e (\gamma u_{mv}(t) + u'_{mv}(t)) \|\gamma u_{mv}(t) + u'_{mv}(t)\|_H)_{t'}
+ \left(g'(t) \alpha_e(u'_{mt}(t)), \|v_r\| \right)_{t'} + \left(g(t) \nabla \alpha_e(u'_{mt}(t)) \|u''_{mt}(t)\|_H \|v_r\| \right)_{t'}
= \rho(f'(t), v) + (F'(t), v)_{\Gamma_N} \quad \forall v \in V_m, \forall t \in (0, T).
$$

(4.2)

(b) First a priori estimate

Let us establish an estimate for $u_m \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V)$. For arbitrary $t \in (0, T)$ take $v = \gamma u_m + u'_m \in V_m$ in (4.1) to obtain

$$
\frac{1}{2} \frac{d}{dt}(\rho \|u'_m(t)\|_H^2 + \|u_m(t)\|_V^2) + \gamma \|u_m(t)\|_V^2 + \frac{1}{\epsilon} \|\gamma u_{mv}(t) + u'_{mv}(t)\|_{L^2(\Gamma_v)}^3
+ \rho \gamma (u''_m(t), u_m(t)) \leq \rho(f(t), \gamma u_m(t) + u'_m(t)) + (F(t), \gamma u_m(t) + u'_m(t))_{\Gamma_N}
+ \gamma \left(g(t) \|\nabla \alpha_e(u'_{mt}(t))\|_H \|u''_{mt}(t)\|_H \right)_{t'}
$$

where $\|v\|_-$ means $\|v\|_-$ and we have used $\beta_e(x) = \frac{1}{2} |x|^3 \alpha_e(x) \cdot x \geq 0$. Applying Hölder’s and Young’s inequalities to terms involving $\gamma$ on the right-hand side yields

$$
\frac{1}{2} \frac{d}{dt}(\rho \|u'_m(t)\|_H^2 + \|u_m(t)\|_V^2) + \gamma \|u_m(t)\|_V^2 + \frac{1}{\epsilon} \|\gamma u_{mv}(t) + u'_{mv}(t)\|_{L^2(\Gamma_v)}^3
+ \rho \gamma (u''_m(t), u_m(t)) \leq C \gamma (\|f(t)\|_H^2 + \|F(t)\|_{L^2(\Gamma_N)}^2 + \|g(t)\|_{L^2(\Gamma_v)}^2)
+ \rho(f(t), u'_m(t)) + (F(t), u'_m(t))_{\Gamma_N},
$$

where we have used $|\alpha_e(x)| \leq 1$ and the trace inequality $\|\|v\|\|_{L^2(\Gamma_v)} \leq C \|v\|_V$. Integration of both sides with respect to $t$ gives

$$
\frac{1}{2} \left(\rho \|u'_m(t)\|_H^2 + \|u_m(t)\|_V^2\right) + \gamma \frac{1}{2} \int_0^t \|u_m(s)\|_V^2 \, ds
+ \frac{1}{\epsilon} \int_0^t \|\gamma u_{mv}(s) + u'_{mv}(s)\|_{L^2(\Gamma_v)}^3 \, ds
+ \rho \gamma \int_0^t \|u'_m(s)\|_H^2 \, ds
\leq \frac{1}{2} (\|u_0\|_H^2 + \|u_0\|_V^2)
+ C \gamma (\|f\|_{L^2(0,T;H)}^2 + \|F\|_{L^2(0,T;L^2(\Gamma_N))}^2 + \|g\|_{L^2(0,T;L^2(\Gamma_v))}^2)
+ \frac{\rho}{2} \|f\|_{L^2(0,T;H)}^2 + \frac{\rho}{2} \int_0^t \|u'_m(s)\|_H^2 \, ds + \|F(s), u_m(s)\|_{\Gamma_N}^2 - \int_0^t (F'(s), u_m(s))_{\Gamma_N} \, ds.
$$

In particular,

$$
\rho \|u'_m(t)\|_H^2 + \frac{1}{2} \|u_m(t)\|_V^2 + \rho \gamma \frac{d}{dt}\|u_m(t)\|_H^2 + \frac{2}{\epsilon} \int_0^t \|\gamma u_{mv}(s) + u'_{mv}(s)\|_{L^2(\Gamma_v)}^3 \, ds
\leq C (\gamma + 1) (\|f\|_{L^2(0,T;H)}^2 + \|F\|_{H^1(0,T;L^2(\Gamma_N))}^2 + \|g\|_{L^2(0,T;L^2(\Gamma_v))}^2) + \|u_0\|_H^2 + \|u_0\|_V^2
+ C (\gamma + 1) \int_0^t \left(\rho \|u'_m(s)\|_H^2 + \frac{1}{2} \|u_m(s)\|_V^2\right) \, ds,
$$

where $(F(t), u_m(t))_{\Gamma_N}$ has been bounded by $C \|F\|_{H^1(0,T;L^2(\Gamma_N))}^2 + \frac{1}{4} \|u_m(t)\|_V^2$. Setting $A(t) := \rho \|u'_m(t)\|_H^2 + \frac{1}{2} \|u_m(t)\|_V^2$ and neglecting the last term on the left-hand side (this is just for simplicity
of presentation; if we keep this term, we obtain (4.5) below), we rephrase this estimate as

$$A(t) + \rho \gamma \frac{d}{dt} \|u_m(t)\|^2_H \leq C_1(f, F, g, u_0, \dot{u}_0)(\gamma + 1) + C(\gamma + 1) \int_0^t A(s) \, ds \quad \forall t \in (0, T).$$

(4.3)

If $\gamma = 0$, we find from Gronwall’s inequality that

$$A(t) \leq C_1(f, F, g, u_0, \dot{u}_0) e^{Ct}.$$

Otherwise we further integrate (4.3) with respect to $t$, with $B_1(t) := \int_0^t A(s) \, ds$, to get

$$B_1(t) + \rho \gamma \|u_m(t)\|^2_H \leq C_2(f, F, g, u_0, \dot{u}_0, T)(\gamma + 1) + C(\gamma + 1) \int_0^t B_1(s) \, ds,$$

so that, by Gronwall’s inequality,

$$B_1(t) + \rho \gamma \|u_m(t)\|^2_H \leq C_2(f, F, g, u_0, \dot{u}_0, T)(\gamma + 1) e^{C(\gamma + 1)t}.$$

Since $\rho \gamma (d/dt)\|u_m(t)\|^2_H = 2 \rho \gamma (u'_m(t), u_m(t))$, we find from (4.3) that

$$A(t) \leq C_1(f, F, g, u_0, \dot{u}_0)(\gamma + 1) + C(\gamma + 1)B_1(t) + \frac{\rho}{2} \|u'_m(t)\|^2_H + 2 \rho \gamma^2 \|u_m(t)\|^2_H,$$

which concludes

$$\frac{1}{2} (\rho \|u_m(t)\|^2_H + \|u_m(t)\|^2_V) \leq C_3(f, F, g, u_0, \dot{u}_0, T)(\gamma + 1)^2 e^{C(\gamma + 1)t}.$$  (4.4)

**Remark 4.1.** As we already noted before (4.3), it also holds that, for all $t \in [0, T]$,

$$2 \int_0^t \|\gamma u_{mv}(s) + u''_{mv}(s)\|_{L^3(\Omega)}^3 \, ds \leq C_3(f, F, g, u_0, \dot{u}_0, T)(\gamma + 1)^2 e^{C(\gamma + 1)t}.$$  (4.5)

(c) *Second a priori estimate*

Next let us establish an estimate for $u'_m \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V)$. For arbitrary $t \in (0, T)$ we take $v = \gamma u_m' + u''_m \in V_m$ in (4.2) to obtain

$$\frac{1}{2} \frac{d}{dt} (\rho \|u'_m(t)\|^2_H + \|u'_m(t)\|^2_V) + \gamma \|u'_m(t)\|^2_V + \gamma \rho (u''_m(t), u_m(t))$$

$$\leq \gamma \rho (F', u'_m(t)) + \gamma (F'(t), u'_m(t))_{\Gamma_\gamma} + \rho (F', u''_m(t)) + (F(t), u''_m(t))_{\Gamma_\gamma}$$

$$- \gamma \left( g'(t) \nabla \alpha_e \left( \|u'_m(t)\|, \|u''_m(t)\| \right) \right)_{\Gamma_\gamma} - \gamma \left( g'(t) \alpha_e \left( \|u'_m(t)\|, \|u''_m(t)\| \right) \right)_{\Gamma_\gamma}$$

$$- \left( g'(t) \alpha_e \left( \|u'_m(t)\|, \|u''_m(t)\| \right) \right)_{\Gamma_\gamma},$$

where we have used the fact that $\beta'_e$ and $\nabla \alpha_e$ are non-negative.

Applying Hölder’s and Young’s inequalities to the first three and the sixth terms on the right-hand side, together with $|\alpha_e()| \leq 1$ and the trace inequality $\|\|v\|\|_{L^3(\Gamma_\gamma)} \leq C \|v\|_v$, we have

$$\frac{1}{2} \frac{d}{dt} (\rho \|u''_m(t)\|^2_H + \|u'_m(t)\|^2_V) + \gamma \|u'_m(t)\|^2_V + \gamma \rho (u''_m(t), u_m(t))$$

$$\leq C(\gamma + 1) (\|F'(t)\|_{L^2(\Gamma_\gamma)} + \|g'(t)\|_{L^2(\Gamma_\gamma)} + \|u''_m(t)\|_H)$$

$$+ (F'(t), u''_m(t))_{\Gamma_\gamma} - \gamma \left( g'(t) \frac{d}{dt} \alpha_e \left( \|u'_m(t)\|, \|u''_m(t)\| \right) \right)_{\Gamma_\gamma} - \left( g'(t), \frac{d}{dt} \varphi_e \left( \|u'_m(t)\| \right) \right)_{\Gamma_\gamma}.$$
Integration of both sides with respect to \( t \) yields
\[
\frac{\rho}{2} \|u''_m(t)\|_H^2 + \frac{1}{2} \|u'_m(t)\|_V^2 + \frac{\gamma}{2} \int_0^t \|u'_m(s)\|_V^2 \, ds + \rho \gamma \left( \left( \mathbf{u}'_m(s), \mathbf{u}'_m(s) \right) \right)_0
- \rho \gamma \int_0^t \|u''_m(s)\|_H^2 \, ds
\leq \frac{\rho}{2} \|u''_m(0)\|_H^2 + \frac{1}{2} \|\mathbf{u}_0\|_V^2 + C(\gamma + 1)(\|f\|_{L^2(0, T; H)}^2 + \|F\|_{L^2(0, T; L^2(\Gamma_\gamma))}^2 + \|g\|_{L^2(0, T; L^2(\Gamma_\gamma))}^2)
+ \int_0^t \rho \|u''_m(s)\|_H^2 \, ds + \left( \left( F'(s), u'_m(s) \right) \right) \Gamma_\gamma \, ds
- \gamma \left( \left( g(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \, ds
+ \gamma \int_0^t \left( \left( g(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \, ds
- \left( \left( g(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \right) \| \left( \left( f(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \right) \, ds,
\]
where the eighth term on the right-hand side equals
\[
\gamma \left( \left( g(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \right) \| \left( \left( f(s \alpha_f (\left[ u_{m't}(s) \right]) \mathbf{u}_{m't}(s) \right), \left[ u_{m't}(s) \right] \right) \Gamma_\gamma \right) \, ds.
\]
Hölder’s and Young’s inequalities, combined with the relations
\[
H^1(0, T; L^2(\Gamma_\gamma)) \hookrightarrow C([0, T]; L^2(\Gamma_\gamma)),
\]
\[
\|\mathbf{v}\|_{L^2(\Gamma_\gamma)} \leq C\|\mathbf{v}\|_V
\]
and with \( |\alpha_f(\cdot)| \leq 1 \), \( \mathbf{q}_f(\cdot) = \sqrt{1 - |\cdot|^2 + \epsilon^2} \), lead to
\[
\rho \|u''_m(t)\|_H^2 + \frac{1}{2} \|u'_m(t)\|_V^2 + \gamma \int_0^t \|u'_m(s)\|_V^2 \, ds + \rho \gamma \frac{d}{dt} \|u'_m(t)\|_H^2
\leq C(\gamma + 1)(\|u''_m(0)\|_H^2 + \|f\|_{H^1(0, T; H)}^2 + \|F\|_{H^1(0, T; L^2(\Gamma_\gamma))}^2 + \|g\|_{H^1(0, T; L^2(\Gamma_\gamma))}^2 + \|\mathbf{u}_0\|_V^2 + \epsilon^2)
+ C(\gamma + 1) \int_0^t \left( \rho \|u''_m(s)\|_H^2 + \frac{1}{2} \|u'_m(s)\|_V^2 \right) \, ds + C\gamma^2 \|g\|_{H^1(0, T; L^2(\Gamma_\gamma))}^2 \forall t \in (0, T),
\]
(4.6)
where the last contribution owes to \( \gamma (g(t) \alpha_f (\left[ u_{m't}(t) \right]) \mathbf{u}_{m't}(t)) \Gamma_\gamma \) and \( \gamma (g(t) \alpha_f (\left[ u_{m't}(t) \right]) \mathbf{u}_{m't}(t)) \Gamma_\gamma \).

It remains to estimate \( \|u''(0)\|_H \). For this purpose we make \( t = 0 \) and take \( \mathbf{v} = u''(0) \in V_m \) in (4.1) to see
\[
(\rho f(0), u''_m(0)) + (F(0), u'_m(0)) \Gamma_\gamma,
\]
and using the compatibility conditions, we deduce
\[
\rho \|u''_m(0)\|_H^2 = (\text{div} \mathbf{u}_0, u''_m(0)),
\]
which implies \( \|u''_m(0)\|_H \leq C(\|\text{div} \mathbf{u}_0\|_H + \|f(0)\|_H) \).
Substituting this into (4.6), we proceed as in the previous subsection assuming \( \epsilon \leq 1 \). If \( \gamma = 0 \), Gronwall’s inequality gives us
\[
\rho \| u_{m}''(t) \|_{H}^{2} + \frac{1}{2} \| u_{m}'(t) \|_{V}^{2} \leq C_{4}(f, F, g, u_{0}, \dot{u}_{0}) e^{Ct}.
\]
If \( \gamma > 0 \), we further integrate (4.6) to have
\[
B_{2}(t) + \rho \gamma \| u_{m}'(t) \|_{H}^{2} \leq C_{5}(f, F, g, u_{0}, \dot{u}_{0}, T)(\gamma + 1)^{2} + C(\gamma + 1) \int_{0}^{t} B_{2}(s) \, ds,
\]
where \( B_{2}(t) := \int_{0}^{t} (\rho \| u_{m}'(t) \|_{H}^{2} + \frac{1}{2} \| u_{m}'(t) \|_{V}^{2}) \, ds \). Applying Gronwall’s inequality above and substituting the resulting estimate into (4.6), in which \( 2\rho\gamma \| u_{m}''(t), u_{m}'(t) \| \) is bounded by \( \frac{\rho}{2} \| u_{m}'(t) \|_{H}^{2} + 2\rho\gamma^{2} \| u_{m}'(t) \|_{V}^{2} \), we conclude
\[
\rho \| u_{m}''(t) \|_{H}^{2} + \| u_{m}'(t) \|_{V}^{2} \leq C_{6}(f, F, g, u_{0}, \dot{u}_{0}, T)(\gamma + 1)^{3} e^{C(\gamma + 1)t}.
\]

(d) Passage to limit

The argument of the passage to the limits \( m \to \infty \) and \( \epsilon \to 0 \) is basically similar to ([15], Section 3.7), the essential difference lying in the verification of the constraint \( \gamma u(t) + u'(t) \in K \). However, for the sake of completeness we present the whole proof.

First let us consider the limit \( m \to \infty \) for fixed \( \epsilon \in (0, 1] \). As a consequence of the \textit{a priori} estimates (4.4) and (4.7), there exist a subsequence of \( \{ u_{m} \} \), denoted by the same symbol, and some \( u_{\epsilon} \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; V) \) such that
\[
u_{m} \to u_{\epsilon} \quad \text{weakly-* in } L^{\infty}(0, T; V),
\]
\[
u_{m}' \to u_{\epsilon}' \quad \text{weakly-* in } L^{\infty}(0, T; V)
\]
and
\[
u_{m}'' \to u_{\epsilon}'' \quad \text{weakly-* in } L^{\infty}(0, T; H),
\]
as \( m \to \infty \). Here, we note the compact embedding \( W^{1,\infty}(0, T; L^{2}(\Omega)) \cap L^{\infty}(0, T; H^{1}(\Omega)) \hookrightarrow C([0, T]; L^{2}(\Omega)) \) (see [19]) and the compactness of the trace operator \( H^{1}(\Omega) \to L^{2}(\Gamma_{c}) \) (e.g. [20]). It then follows that
\[
u_{m} \to u_{\epsilon} \quad \text{and } u_{m}' \to u_{\epsilon}' \quad \text{strongly in } C([0, T]; H),
\]
\[
u_{m}'' \to u_{\epsilon}'' \quad \text{strongly in } C([0, T]; L^{2}(\Gamma_{c})),
\]
as \( m \to \infty \). In particular, the initial conditions \( u_{\epsilon}(0) = u_{0} \) and \( u_{\epsilon}'(0) = \dot{u}_{0} \) hold. By choosing a further subsequence, we may also assume that
\[
u_{m} \to u_{\epsilon} \quad \text{and } u_{m}' \to u_{\epsilon}' \quad \text{a.e. in } (0, T) \times \Gamma_{c}.
\]
For arbitrary \( \eta \in C_{0}^{\infty}(0, T) \) and \( v \in V_{m} (m = 2, 3, \ldots) \), we find from (4.1) that
\[
t_{0}^{T} \eta(t)\left( \rho(\nu_{m}''(t), v) + a(u_{m}(t), v) + (\beta_{\epsilon}(\|\nu_{m}v(t) + \nu_{m}'(t)\|), \|v\|)_{\Gamma_{c}} \right.
\]
\[
+ \left. \begin{array}{c}
\left( g(t)\alpha_{\epsilon}(\|\nu_{m}''(t)\|), \|v\|_{\Gamma_{c}} \right) - \rho(f(t), v) - (F(t), v)_{\Gamma_{c}} \right) \right) \, dt = 0.
\]
Letting \( m \to \infty \), using (4.8), and applying the dominated convergence theorem, we have
\[
t_{0}^{T} \eta(t)\left( \rho(\nu_{\epsilon}''(t), v) + a(u_{\epsilon}(t), v) + (\beta_{\epsilon}(\|\nu_{\epsilon}v(t) + \nu_{\epsilon}'(t)\|), \|v\|)_{\Gamma_{c}} \right.
\]
\[
+ \left. \begin{array}{c}
\left( g(t)\alpha_{\epsilon}(\|\nu_{\epsilon}''(t)\|), \|v\|_{\Gamma_{c}} \right) - \rho(f(t), v) - (F(t), v)_{\Gamma_{c}} \right) \right) \, dt = 0.
\]
Since \( \bigcup_{m=1}^{\infty} V_{m} = V \) and \( \eta \) is arbitrary, we conclude (3.7), that is, \( u_{\epsilon} \) is a solution of (VE)\( \epsilon \) and also of (VI)\( \epsilon \) by virtue of proposition 3.6. Moreover, by making \( m \to \infty \) in (4.4), (4.5) and (4.7), we also
Here, observe that \( \lim \) the same symbol, and some \( \epsilon \) as

\[
\int_0^T \| \gamma u_{\epsilon v}(t) + u'_{\epsilon v}(t) \|_{L^3(G)}^3 \, dt = 0,
\]

which verifies \( \| \gamma u_{\epsilon}(t) + u'_\epsilon(t) \| \geq 0 \) a.e. on \( (0, T) \times \Gamma_c \), that is, \( \gamma u(t) + u'(t) \in K \) for \( t \in (0, T) \).

For arbitrary \( \bar{v} \in L^2(0, T; K) \) we find from (3.6) that

\[
\int_0^T \left( \rho(u''_\epsilon, \bar{v}) - (\gamma u_e + u'_\epsilon) \right) + a(u_e, \bar{v}) \, dt \geq 0,
\]

because \( \psi_e(\| \bar{v}(t) \|) \equiv 0 \) and \( \psi_e(\| \gamma u_{\epsilon v}(t) + u'_{\epsilon v}(t) \|) \equiv 0 \). Consequently,

\[
\int_0^T \left( \rho(u''_\epsilon, \bar{v}) - \gamma u_e \right) + a(u_e, \bar{v}) \, dt \geq 0.
\]

Here, observe that \( \lim_{\epsilon \to 0} \| u''_\epsilon(T) \|_H^2 = \| u'(T) \|_H^2 \) in view of the compact embedding

\[
W^{1,\infty}(0, T; L^2(\Omega_\pm)) \cap L^\infty(0, T; H^1(\Omega_\pm)) \hookrightarrow C([0, T]; L^2(\Omega_\pm)).
\]

We further find that \( \psi_e(\| u''_\epsilon \|) \to \| u''_\epsilon \| \) in \( C([0, T]; L^2(\Gamma_c)) \) as \( \epsilon \to 0 \), and that

\[
\| u(T) \|_V^2 \leq \liminf_{\epsilon \to 0} \| u_e(T) \|_V^2 \quad \text{and} \quad \| u \|_{L^2(0, T; V)}^2 \leq \liminf_{\epsilon \to 0} \| u_e \|_{L^2(0, T; V)}^2.
\]

In fact, the former inequality above results from the following weak convergence:

\[
a(u(T) - u_e(T), w) = \int_0^T \left( a(u'(t) - u'_e(t), \eta(w) + a(u(t) - u_e(t), \eta'(w)) \right) \, dt \to 0 \quad \forall w \in V, \, \epsilon \to 0,
\]

where \( \eta \in C^{\infty}([0, \infty)) \) is chosen so that \( \eta(0) = 0 \) and \( \eta(T) = 1 \). Therefore, making \( \epsilon \to 0 \) in (4.10) deduces

\[
\int_0^T \left( \rho(u'', \bar{v}) - \gamma u \right) + a(u, \bar{v}) + (g, [\bar{v} - \gamma u_e])_{\Gamma_c} - \rho(f, \bar{v} - (\gamma u + u')) \, dt \geq 0.
\]
namely,
\[
\int_0^T \left( \rho(u'', \tilde{v} - (\gamma u + u')) + a(u, \tilde{v} - (\gamma u + u')) + (g, \|\tilde{v}_\tau - \gamma u_t\|)_{\tilde{\Omega}_c} \\
- \rho(f, \tilde{v} - (\gamma u + u')) - (F, \tilde{v} - (\gamma u + u'))_{\tilde{\Omega}_c} \right) dt \geq 0.
\]

This implies the pointwise (in time) variational inequality (3.4) by a technique based on the Lebesgue differentiation theorem (see [14, pp. 57–58]). Thus the existence part of theorem 3.4 has been established.

(e) Uniqueness

Before proceeding to the proof of the uniqueness part of theorem 3.4, we present some preparatory results.

**Lemma 4.2.** There exists a vector function \( N \in H^1(\Omega) \) such that its trace satisfies
\[
N = v \quad \text{on } \Gamma_c, \quad N = 0 \quad \text{on } \partial \Omega.
\]

**Proof.** Let \( \tilde{\Gamma}_c \) be a neighbourhood of \( \Gamma_c \) such that \( \Gamma_c \subseteq \tilde{\Gamma}_c \subseteq \Gamma \). Then there exists \( \tilde{v} \in H^{1/2}(\Gamma) \) such that \( \tilde{v} = v \) on \( \Gamma_c \) and \( \tilde{v} = 0 \) on \( \Gamma \setminus \tilde{\Gamma}_c \). Then one can find some \( N_\pm \in H^1(\Omega_\pm) \) whose trace to \( \partial \Omega_\pm \) equals the zero extension of \( \tilde{v} \) to \( \partial \Omega_\pm \). If we define \( N = N^+ \) in \( \Omega_+ \) and \( N = N^- \) in \( \Omega_- \), this is a desired function. \( \blacksquare \)

Using this lemma we introduce, for \( v \in V \),
\[
\tilde{v} := v - (v \cdot N)N.
\]

Note that \( \|\tilde{v}\|_H \leq C\|v\|_H, \|\tilde{v}\|_V \leq C\|v\|_V \), and that \( \|\tilde{v}_\rho\| = 0, \|\tilde{v}_\tau\| = \|v_\tau\|, \|((v \cdot N)N)_\tau\| = 0 \) on \( \Gamma_c \).

For any solution \( u \) of (3.4), we see that \( \sigma_v(u) \in L^2(0, T; H_0^{1/2}(\Gamma_c)) \) and \( \sigma_v(u) \in L^2(0, T; H_0^{1/2}(\Gamma_c)^*) \) are characterized by
\[
\langle \sigma_v(u(t)), [v_\rho] \rangle_{\Gamma_c} = -\rho(u''(t), v) - a(u(t), v) + \rho(f(t), v) + (F, v)_{\tilde{\Omega}_c} \quad \forall v \in V, \|v\|_V = 0 \quad \text{on } \Gamma_c
\]

and
\[
\langle \sigma_v(u(t)), [v_\tau] \rangle_{\Gamma_c} = -\rho(u''(t), v) - a(u(t), v) + \rho(f(t), v) + (F, v)_{\tilde{\Omega}_c} \quad \forall v \in V, \|v\|_V = 0 \quad \text{on } \Gamma_c,
\]
respectively. The next lemma is essentially a consequence of the monotonicity of \( \beta \) and \( \alpha \) appearing in (3.5).

**Lemma 4.3.** If \( u_1, u_2 \) are two solutions of (3.4), then for a.e. \( t \in (0, T) \)
\[
\langle \sigma_v(u_1(t)) - \sigma_v(u_2(t)), [\gamma u_{1v}(t) + u_{1v}'(t)] - [\gamma u_{2v}(t) + u_{2v}'(t)] \rangle_{\Gamma_c} \geq 0
\]
and
\[
\langle \sigma_v(u_1(t)) - \sigma_v(u_2(t)), [u_{1v}'(t)] - [u_{2v}'(t)] \rangle_{\Gamma_c} \geq 0.
\]

**Proof.** Arguing in the same way as in proposition 3.1, we get
\[
\langle \sigma_v(u_i), [v_\rho] \rangle_{\Gamma_c} \leq 0 \quad \forall v \in K \quad \text{and} \quad \langle \sigma_v(u_i), [\gamma u_{iv} + u_{iv}'] \rangle_{\Gamma_c} = 0,
\]
for \( i = 1, 2 \). The first desired inequality follows from these and \( \gamma u_i + u_i' \in K \).

Again by the same way as in proposition 3.1, we have
\[
\langle \sigma(u_i), [v_\tau] \rangle_{\Gamma_c} \leq (g(t), [v_\tau])_{\Gamma_c} \quad \forall v \in V \quad \text{and} \quad \langle \sigma_v(u_i), [u_{iv}'] \rangle_{\Gamma_c} = (g(t), [u_{iv}'])_{\Gamma_c},
\]
for \( i = 1, 2 \), which lead to the second desired inequality. \( \blacksquare \)
Now we prove the uniqueness of a solution of (3.4). Let $u_1, u_2$ be two solutions of (3.4) and set $w := u_1 - u_2$. Then it follows that

$$\rho(w''(t), v) + a(w(t), v) + \langle \sigma_{\tau}(w(t)), [v, w] \rangle_{L^1} + \langle \sigma_{\tau}(w(t)), [v, w] \rangle_{L^1} = 0 \quad \forall v \in V, \text{ a.e. } t \in (0, T).$$

Taking $v = \gamma w(t) + w'(t)$ and using lemma 4.3, we deduce that

$$\frac{1}{2} \frac{d}{dt} (\rho \|w'(t)\|_H^2 + \|w(t)\|_V^2) + \rho \gamma \langle w''(t), w(t) \rangle + \gamma \|w(t)\|_V^2 \leq -\gamma \langle \sigma_{\tau}(w(t)), [w, w] \rangle_{L^1}$$

which, combined with $(w'(t), w(t) - \bar{w}(t)) = (w'(t) \cdot N, w(t) \cdot N)$, gives

$$\frac{1}{2} \frac{d}{dt} (\rho \|w'(t)\|_H^2 + \|w(t)\|_V^2) + \rho \gamma \|w'(t) \cdot N, w(t) \cdot N \| \leq \gamma a(w(t), \bar{w}(t)) \leq C \gamma \|w(t)\|_V^2.$$

Integrate this with respect to $t$ to obtain (note that $w(0) = w'(0) = 0$)

$$\frac{1}{2} (\rho \|w'(t)\|_H^2 + \|w(t)\|_V^2) + \frac{\rho \gamma}{2} \int_0^t \|w(t) \cdot N\|_H^2 \leq \gamma \int_0^t \langle \rho \|w'(s) \cdot N\|_H^2 + C\|w(s)\|_V^2 \rangle \, ds.$$

Setting $D(t) := \int_0^t (\rho \|w'(s)\|_H^2 + \|w(s)\|_V^2) \, ds$, we find from further integration of this estimate that

$$D(t) + \rho \gamma \|w(t) \cdot N\|_H^2 \leq C \gamma \int_0^t D(s) \, ds.$$

By Gronwall’s inequality, $D(t) \equiv 0$ and hence $w(t) \equiv 0$, which shows the uniqueness.

The proof of theorem 3.4 has been completed.

Data accessibility. No new data were created or analysed in this study.

Authors’ contributions. T.K.: formal analysis, investigation, writing—original draft; H.I.: formal analysis, investigation, writing—original draft.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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