On a microscopic representation of spacetime

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We start from a noncompact Lie algebra isomorphic to the Dirac algebra and relate this Lie algebra in a brief review to low energy hadron physics described by the compact group SU(4). This step permits an overall physical identification of the operator actions. Then we discuss the geometrical origin of this noncompact Lie algebra and 'reduce' the geometry in order to introduce in each of these steps coordinate definitions which can be related to an algebraic representation in terms of the spontaneous symmetry breakdown along the Lie algebra chain $\text{su}^*(4) \rightarrow \text{usp}(4) \rightarrow \text{su}(2) \times \text{u}(1)$. Standard techniques of Lie algebra decomposition(s) as well as the (physical) operator identification give rise to interesting physical aspects and lead to a rank-1 Riemannian space which provides an analytic representation and leads to a 5-dimensional hyperbolic space $H_5$ with SO(5,1) isometries. The action of the (compact) symplectic group decomposes this (globally) hyperbolic space into $H_2 \oplus H_3$ with SO(2,1) and SO(3,1) isometries, respectively, which we relate to electromagnetic (dynamically broken SU(2) isospin) and Lorentz transformations. Last not least, we attribute this symmetry pattern to the algebraic representation of a projective geometry over the division algebra $\mathbb{H}$ and subsequent coordinate restrictions.

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I. INTRODUCTION

This paper is intended as a conceptual paper focussing on the background of a spontaneously broken symmetry pattern, thus summarizing some of our previous results ([4], [5], [6]). This research originated in effective descriptions of hadronic interactions [3], however,
instead of approaches like 'chiral perturbation theory' or QCD we’ve chosen another way of realizing (and breaking) hadronic symmetries thus avoiding wellknown deficits of the standard approaches. However, in order to avoid political and theological discussions on these subjects in what one has to believe, we go back to a very simple ansatz by using nothing but the standard (15-dimensional) Dirac algebra which is common to all 'quantum' approaches. And because the Dirac algebra is the very foundation of calculations in quantum field theory (QFT) and gauge theories, an identification of the operators of this 15-dimensional algebra is of considerable physical interest to (quantum) gauge theories, too.

In the subsequent sections, we try to develop the idea and the concept stepwise while we report on some work which originated in hadron physics and which became more and more interesting as we illuminated the underlying geometry. So in addition to some recently published calculations [6] it is noteworthy to present the geometrical and conceptual background which we understand as underlying the usual Dirac and QFT description. It is interesting to see that (in agreement with Klein’s 'Erlanger Programm’) we can use (Lie) group and algebra theory to obtain a finite as well as an infinitesimal (algebraic) description of this geometry, and the affine and differential geometry used nowadays turns out to be a subsidiary concept of this scheme. Moreover, it seems that Weyl’s separation of 'ältere' from 'infinitesimale Geometrie’ (see e.g. [15]) has lead over the decades to a tilt in favour of pure technical issues in terms of affine concepts and gauge theories whereas some of the underlying superior geometrical concepts fell into oblivion (especially those going beyond just the first approach by a tangential space).

II. SU(4) VERSUS SU∗(4)

So in order to find an ’entry point' to the subsequent discussion we summarize our (physically motivated) assumption(s) which serve as ansatz - the identification of the number space we are going to use and an assumption about relativistic symmetry and its breaking.

As such, we can observe in chiral hadron theories ([3], [2]) that with higher energies, the symmetry breaking becomes larger and that the symmetry scheme becomes worse and worse. If we cover chiral SU(2)×SU(2) symmetry by the larger compact group SU(4), we see that SU(4) has some interesting properties when compared to the spectrum [3], and in describing axial symmetry properties and charges, however, the SU(4) multiplets are not
realized as (Wigner-Weyl) supermultiplets in the spectrum. Nevertheless, counting multiplet members and spin-isospin quantum numbers in the spectrum, the SU(4) dimensions can be identified if we group observable (distributed) multiplets together [3]. Moreover, it is known from other observations e.g. in nuclear physics that SU(4) can serve as a good low energy symmetry, although it is broken for higher energies.

So instead of writing the (nonlinear) chiral transformation in terms of (complexified) Pauli matrices (which is a certain representation but mathematically and physically misleading) we choose a description in terms of real quaternions right from the beginning. Yes, at a first glance this seems to be artificial, however, we benefit twice from the fact that the (real) quaternions constitute a division algebra: mathematically in that we have well-defined inverses throughout our calculations, and we can use a well-defined and consistent division when working with the real quaternions, thus avoiding some magic (commuting) ‘i’s which otherwise occur at various places within the calculations. This obvious benefit allows to treat nonlinear chiral transformations completely instead of calculating with certain awkward ‘expansions’ and/or weakly justified ‘power series’. The much larger advantage, however, is an identification of usual chiral transformations as subsets of quaternionic Möbius transformations which of course suggests to study a projective quaternionic geometry and start from scratch with geometry. Physically we benefit in that this identification is much closer to the physical measurement process and much more evident in that we compare a certain measure (respectively, an observable) with a known unit measure at various distances, i.e. we need unique inverses to compare with the unit by dividing the unit, and we need the full tool set of a projective geometry, too, to transport and scale the observations appropriately.

This simple reasoning leads to the symmetry group Sl(2,H) which one can embed into complex spaces using appropriate coordinate sets and 2×2 matrix representations (see e.g. [11] or [10]) so that we obtain the isomorphic noncompact symmetry group SU*(4) on complex representation spaces. Simple calculations show that (up to a misplaced and additional commuting ‘i’ in the usual definition) the operators of the Dirac algebra are isomorphic to the Lie algebra generators su*(4). Hence it is self-evident to understand the occurrence of an SU(4) symmetry at low energies as an approximation respectively as the remnant of the original relativistic symmetry SU*(4). Following this reasoning, the ‘difference’ in the Lie algebras su(4) and su*(4) has to be responsible for dynamics whereas the (compact) part common to both su(4) and su*(4) should generate dynamically further observable symme-
tries (as isotropy group). Last not least, we can use the spin-isospin construction of SU(4) with its physical operator identification as well as the state identification (see [3], [2]) in order to check the physical significance of our calculations and results.

The technical machinery to investigate such issues more precisely is well-known from Lie algebra theory, and we find a compact subgroup USp(4) of dimension 10 of both SU(4) and SU*(4) as well as a 5-dimensional operator set p mapped by the exponential onto a coset space exp(p) which we can identify with SU*(4)/USp(4). So we find a spontaneously broken symmetry with a (local) USp(4) symmetry group (see also [5] and [6]). This is an irreducible Riemannian globally symmetric space (type AII, [11]) which we can describe technically in terms of a nonlinear sigma model with appropriate representations (or by differential geometry/ fiber bundles with local USp(4) symmetry or induced representations, resp.). Here, however, we postpone the technical details and focus instead on the geometrical background in terms of 'ältere Geometrie’. Why? First of all, we have to identify and define suitable coordinates and appropriate coordinate (transformation) rules which we can use in a second step to perform calculations, i.e. they serve as an appropriate (algebraic) representation of this geometry. It is only the very last step that we need to use affine coordinates and concepts, which then results in 'local’ coordinate systems with emphasized points (or special (restricted) transformation structures), or which results in certain representation(s) for objects to which we then attribute physical properties or behaviour and where we can apply certain optimization (or action) principles like Lagrangian or (by using 'more suitable’ coordinate definitions) Hamiltonian formalisms to extract (infinitesimal) properties in terms of equations of motion. We will see that we can short-circuit some of these typical standard mechanisms by using geometry to understand the vacuum structure and write down the geodesics to determine the physical behaviour so that infinitesimal equations provide no more information than the finite representation.

III. GEOMETRY

In some preceding publications, we’ve already presented the picture given in figure I which can be seen in analogy to the Riemannian case based on the division algebra C. However, the figure in the given form already anticipates some preceding and important steps which are already incorporated in the given definition of points and axes but which
otherwise would allow for additional degrees of freedom. So at next, we follow the reasoning and the construction process of the projective line and appropriate coordinate systems as discussed in [13] for the (commutative) division algebra(s) of real (and complex) numbers.

A. Projective Geometry and Coordinates

Although at a first glance projective geometry is related to points, lines, connections of points and sections of lines (incidences), in order to establish duality it is possible to use polar relationships [13], i.e. to study conic sections in the ‘plane’ $\mathbb{H}$ by various methods.

Having to include the projection point in the plane at infinity (which maps to $\mathbb{N}$), it is helpful to introduce projective coordinates (or homogeneous coordinates if we fix the fundamental points at 0 and $\infty$) in the projective line by a set $(\rho q_1, \rho q_2)$, $\rho \neq 0 \in \mathbb{H}$. Already here we see that noncommutativity of quaternions gives rise to an additional $\text{Gl}(1,\mathbb{H})$ transformation of the affine coordinates (depending on the definition of division) as $\rho^{-1}$ exists in $\mathbb{H}$. If now in addition to the two fundamental points we define an unit point $1$, $q_1 = q_2$, we have three points which fix the coordinate system and we can use the cross-ratio (‘Doppelverhältnis’) to introduce measures, metric structures and more with respect to each fourth (quaternionic) point, respectively. As an immediate consequence, we can introduce real measures and coordinates by suitable cross-ratios (and geometrical configurations), we can express distances and angles naturally by logarithms of certain cross-ratios, and we end up with metric properties which we can control and whose background we understand geometrically. The projective line offers automatically the typical $4\pi$-behaviour of spinorial reps because the line is closed by one ‘infinite point’. Moreover, the cross-ratio allows straightforward access to invariants of four points under projective transformations of the line or to its value in two projective coordinate systems, respectively, whereas exp and
log reflect in group and algebra theory. The discussion of complex numbers in the plane
in relation to real parameters of \( S^2 \) as well as their occurrence in cross-ratios \( [13] \) can be
(carefully) generalized to \( S^4 \), and we may as well discuss quaternionic and real results of
cross-ratios as we find quaternionic transformation properties reflected in the cross-ratios.

\( S^4 \) can also serve to define an orientation of a path between two point \( p_1 \) and \( p_2 \) of \( S^4 \),
dependent on whether we pass the northpole on a great circle in the order \((p_1, p_2, \infty)\) or \((p_1,
\infty, p_2)\). Last not least, using only projective methods (see \( [13] \), ch. V, §2 and \( [13] \), references)
one can construct a complete coordinate system on the line. Following \( [13] \) yields further
interesting results, however, here we want to apply some of the aspects mentioned above with
respect to our original physical problem by choosing twofold (homogeneous) quaternionic
coordinates \( q_1 \) and \( q_2 \) and discussing \( \text{Sl}(2, \mathbb{H}) \) transformations.

**B. Sphere Rotations**

Related to the different coordinate types mentioned above, we can of course study the
effects of sphere rotations. If we are thus to work with the two projective/homogeneous
quaternionic coordinates \( q_1 \) and \( q_2 \) or the (affine) coordinate \( q \) of the (finite) quaternionic
‘line’ \( \mathbb{H} \), respectively, we can introduce (see e.g. \( [3] \), \( [4] \) or \( [6] \)) a ‘spinor’ \( \psi \) by \((q_1 \ q_2)\) and express
the quaternionic Möbius transformations \( f(q) \) by a mapping onto \( 2 \times 2 \) quaternionic matrices
acting on the two homogeneous coordinates. The associated group is \( \text{Sl}(2, \mathbb{H}) \) but we can
restrict these Möbius transformations of course to subgroups of \( \text{Sl}(2, \mathbb{H}) \), or when represented
on complex spaces, to subgroups of \( \text{SU}^*(4) \). Especially there, we can think of rotations
keeping the ‘norm’ \( \psi^+ \psi \) invariant, which translates to the constraint \( q_1^+ q_1 + q_2^+ q_2 = \text{const} \)
in terms of projective/homogeneous coordinates, and we naturally obtain the ‘Hopf map’
\( q_1^+ q_1 + q_2^+ q_2 = 1 \), i.e. the map \( S^7 \longrightarrow S^4 \). In our geometrical picture of the original geometry,
the sphere rotations correspond to unitary quaternionic transformations (i.e. \( \text{U}(2, \mathbb{H}) \)) of the
homogeneous coordinates which, when expressed in complex coordinates, correspond to the
compact group \( \text{USp}(4) \). So we’ve found a simple geometrical explanation of the maximal
compact subgroup of our original picture, and we find conic sections in the projection ‘plane’.

Moreover, because we can identify ’Dirac spinors’ in this scenario in various contexts\(^2\) it

\(^1\) Internally, we use the shorthand notation 'QPT' (quaternionic projective theory) to denote this framework.
\(^2\) For example, to understand the (massive) Dirac spinor components \( u \) used in \([1]\), we can introduce quater-
is natural to find the Hopf map realized in QFT - this map it is a direct and straightforward consequence of the quaternionic Möbius transformations above and a constraint to bind the two coordinates; because USp(4) is compact, we find a related conserved (unitary) norm in terms of the two homogeneous coordinates. The ‘way back’ to single quaternions is possible due to the very existence of a division in $\mathbb{H}$ since we may represent $q = q_1q_2^{-1}$, $q \in \mathbb{H}$, and moreover, because we still have the freedom of an appropriate $\text{Gl}(1,\mathbb{H})$ coordinate transformation by a real quaternion $\rho \neq 0$ as cited above. Last not least in this context it is important to remember the Cayley-Dickson construction of the division algebras where an additional complexification of the next lower division algebra may be represented by a skew-symmetric $2 \times 2$ matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This construction scheme has important influence not only on the (overall) coordinate definition(s) but also highlights certain matrix transformations in the groups as we will see with the operator\(^3\) $Q_{02}$. USp(4) as a group which respects both an unitary and a symplectic (i.e. mainly an orthogonal) norm benefits in an appropriate representation from a clear separation of conjugation (of the field) and transposition of matrices and spinor reps (see also [11]). Later we’ll discuss $Q_{02}$ as an U(1) generator as well as we need its discrete symmetry properties as presented in [6], especially $Q_{02}^2 = -1$.

C. ‘Ältere Geometrie’

Although we’ve presented above some geometry besides the frameworks used in today’s models and calculations, we do not want to run (affine) coordinate approaches or purely infinitesimal methods down. However, it appears necessary to treat some aspects of ‘äußere Geometrie’ at least on an equal footing as the ‘infinitesimaler Zug’ [15], just in order to understand more background of some current technical frameworks. ‘Ältere Geometrie’ yields appropriate coordinate definitions and coordinate systems which allow to go beyond affine models, it introduces naturally and almost automatically cross-ratios, measures and their logarithmic dependence, metric structures as well as ‘norm conserving’ transformations

\[^nionic\text{ spinor structures } \chi = (q, q^+)^T \text{ respectively linear combinations } \chi' = (q \pm q^+, q \mp q^+)^T \text{ with only four real dimensions but with well-defined ‘conjugation’ properties under quaternionic conjugation, or we may use an associated } 2 \times 2 \text{ matrix representation acting on this ‘spinor’ space of twofold homogeneous coordinates to exchange the spinor components. However, here we postpone the details of this representation and the related discussion of QFT to an upcoming paper [7].}

\[^3\text{ We use the definitions of } Q_{\alpha\beta} \text{ given in [6].}\]
related to properties of the cross-ratio and to polar relationships. Moreover, we can handle and understand fix points and invariants so that it is possible to study certain conic sections at infinity, i.e. we can treat 'light cones' and 'vacua'. Last not least, we find geodesics, isometries and curvature from geometry, and we can always proceed to certain suitable coordinate definitions to discuss differentiability or analyticity, respectively, because we can realize appropriate tangent structures and linear operators, even in differential representation (but not necessarily!).

In addition to the nonlinear (group based) discussion of sigma models, above we’ve tried to work out the correspondence with the geometrical reduction steps. So the 'geometrical chain' we’ve begun when discussing general transformations of two independent homogeneous coordinates \( q_1 \) and \( q_2 \), which we’ve then restricted via a metric relation/a constraint \( f(q_1, q_2) \) and which could be finally restricted to a single (affine) quaternion \( q \) is reflected in the (matrix) representation of the transformation(s) \( SU^*(4) \rightarrow SU^*(4)/USp(4) \rightarrow USp(4)/Gl(1,H) \). This becomes more apparent if we look at the Lie algebras and remember the fact that \( Gl(1,H) \) is the covering group of \( U(2) \) as well as of \( SU(2) \times U(1) \), i.e. on the Lie algebra level respectively in infinitesimal models for the last reduction step we are discussing here nothing but a (local) representation of a (well-known) \( su(2) \oplus u(1) \) Lie algebra.

IV. FURTHER ASPECTS AND OUTLOOK

To present some physical consequences and results, we discuss some technical aspects.

A. Technical Aspects

As we’ve already defined (see \([6]\) and references therein) an operator basis \( Q_{\alpha\beta} \) in terms of spin⊗isospin operators, we can transform this representation (pointwise) to an alternate basis in terms of an isospin⊗spin representation. To distinguish the two basis systems according to their different physical content, we have introduced the operator set \( Q_{\alpha\beta} := Q_{\beta\alpha} \) for the second representation. Choosing the representation of the Dirac algebra according to \([1]\), we find \( \gamma^0 = iQ_{30}, \gamma^j = -i Q_{2j} \) and \( \gamma^5 = iQ_{10} \), which differs from \( SU^*(4) \) by the additional commutative \(-i\) in the definition of \( \gamma^j \). However, this arbitrary complexification can be traced back in literature to an at that time 'suitable' definition of the Lorentz metric.
i.e. to the relative complexification of the coordinate derivatives as a 'postulate' [3]. Within SU*(4) however, all (relative) phases of the operators are of course fixed, and there is no freedom for arbitrary complexifications, especially not with commuting 'i's. Instead, within SU*(4) the skew-symmetric operator \( Q_{02} \) (resp. \( Q_{20} \)) plays an important rôle with respect to complexification according to the embedding of division algebras by \( 2 \times 2 \) matrix reps and the Caley-Dickson process.

As we’ve already mentioned above, we do not find supermultiplets in the low-energy regime of the spectrum. Instead, the mesons and fermions are pretty well grouped according to their respective spin-content⁴ so in order to understand properties of the spectrum we have to break SU*(4) (resp. SU(4)) by an appropriate mechanism. Because we know of the symplectic symmetries of dynamical systems, it is reasonable to think of a decomposition of the global symmetry with respect to the maximal subgroup USp(4), and we are thus lead directly to a discussion of SU*(4)/USp(4) which is isomorphic to \( H_5 \) (see [6] and references) and has beautiful (differential) properties as globally Riemannian symmetric space. This breakdown can be treated and understood as a spontaneous breakdown of the global SU*(4) symmetry to a local symplectic symmetry which both are acting on the 5-dimensional hyperbolic space \( H_5 \). The consequences of this approach are an USp(4) realization on the 5-dimensional space \( H_5 \) which itself can be either expressed in terms of hypercomplex numbers or embedded into \( \mathbb{R}^6 \) by six (hyperbolic) coordinates which, of course, are not independent, and five Goldstone bosons related to shifts of the vacuum structure of the (local) symplectic symmetry group and it’s 'origin' \( 1 \) (resp. the origin 0 in the Lie algebra).

In [6], we have already discussed the necessary algebraic tool sets (see also [11]) and we have given the isometry groups related to the coordinates and coordinate sets of the coset space \( \exp(p) \). These isometry groups (SO(5,1) for \( \exp(p) \) and SO(2,1) and SO(3,1) for the 'generator sets' \( \{iQ_{01}, iQ_{03}\} \) and \( \{Q_{j2}\} \), respectively) can be interpreted as coordinate transformations of coordinate systems introduced within the respective generator sets (which themselves generate hyperbolic (sub-)spaces \( H_2 \) and \( H_3 \)), so it is nothing but the hyperbolic structure (i.e. the negative curvature) of the coset space which furnishes the dynamics with these three symmetry groups. The concept of spontaneous symmetry breakdown, i.e. the

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⁴ Here, we cannot run the discussion to what extent the spin grouping of hadrons in the spectrum and the comparison of standard QFT calculations to experimental data via complex pole analysis are related. For now, we take the grouping in spin multiplets for granted.
concept of a local USp(4) symmetry, then requires to introduce coordinate systems with respect to each new vacuum definition in \( \exp(p) \) so we have to change the respective coordinate representation as well\(^5\). Hence depending on the choice the new vacuum within \( \exp(p) \), we find coordinate transformations either governed only by SO(2,1) or SO(3,1), or we have to consider connections in the 'full' coset space which are described by SO(5,1) covariance respectively by its double covering SU\(^\ast\)(4). So already at this point it is evident that we can think of certain equivalence classes (and 'sub'vacua) in that we understand the various isometries as hyperbolic motions or we can use the gauge (and affine connection) concept as well for these hyperbolic spaces. As a consequence of hyperbolic coordinate representations in \( \mathbb{R}^n \), if we are going to associate the (physical) time with the 0-component in \( H_3 \) (i.e. \( \sim \cosh \)), we naturally obtain strong causality (\( \cosh > 0, \cosh > \sinh \)). Moreover, the (physical) time is not independent from the (physical) space coordinates (i.e. \( \sim \sinh \)) (as we should know from Lorentz 4-space), but instead of formally six real independent coordinates (or four in the case of \( H_3 \)), we have only five (or three, resp.) due to the hyperbolic constraint(s) on these real representation spaces. This, of course, has consequences for time and space within a real (pseudo-)orthogonal coordinate identification as being only a derived concept due to the representation of \( H_3 \) chosen on \( \mathbb{R}^4 \). This carries forward to the differentials \( dx_i \) of these coordinates which are not independent, too. However, such problems can be circumvented by working with the underlying (noncommutative) Lie algebra or an appropriate hypercomplex representation. Nevertheless, we can of course always introduce local/affine coordinate sets as well as appropriate differential operators founding on these coordinates in order to realize at least the linear approximation (the tangent spaces) of the underlying geometry. Differential geometry provides additional tools sets (see [12]) in that due to the rank-1 of \( H_5 \) one can realize the (more general) Laplace-Beltrami operator(s) \( \mathcal{L}_X \) at a point \( p \) of a rank-1 coset \( X = G/K \) in terms of geodesic polar coordinates \((r, \theta_i)\) according to

\[
\mathcal{L}_X = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} \frac{dA}{dr} \frac{\partial}{\partial r} + \mathcal{L}_S = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \sqrt{g} \frac{\partial}{\partial r} + \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \theta_j} \left( g_{ij} \sqrt{g} \frac{\partial}{\partial \theta_i} \right)
\]

\( \mathcal{L}_S \) is the Laplace-Beltrami operator on the sphere \( S \) in \( X \), \( A \) the area of \( S_p(r) \) so that for

\(^5\) In other words, in su\(^\ast\)(4) we may use 10 real parameters to determine the local usp(4) coordinate system whereas we can use 5 more real parameters to connect and compare the respective USp(4) ground states or eliminate superfluous components (see [4], sect. 3 and [5] with respect to the Dirac equation)
the hyperbolic spaces $H_n$ we obtain \[12\]

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + (n - 1) \coth r \frac{\partial}{\partial r} + (\sinh r)^{-2} \mathcal{L}_S.$$  \hspace{1cm} (1)

If we take the view of potential theory and spectral functions for the negatively curved rank-1 space, the potentials expected in the context of $H_2$ and $H_3$ should show the same (radial) behaviour $\sim r^{-1}$ if we take the space $H_5$ as a basis for the dynamics. This behaviour is interesting later after having identified $H_2$ and $H_3$ and the related physics.

With respect to \[6\], sect. 3, we want to emphasize the possibility to decompose the 10-dimensional USp(4) generator set, i.e. the Lie algebra $\mathfrak{usp}(4)$, further and define a CI-type sigma model. As we've set out, it is possible to understand this model in terms of a $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ Lie algebra (whose generated local symmetry group can be covered by $\text{Gl}(1,\mathbb{H})$) and a $Q_{02}$-complexified (noncommutative) vector space of real dimension six. Hence, besides all the differential geometry related to this Hermitean symmetric space and the $Q_{02}$-complexification, we may define a norm on this space and a SU(3) automorphism group to rearrange these three (hyper-)complex coordinates while keeping the norm invariant. However, with this 'internal' SU(3) group, the remaining $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ algebra governing the local symmetry as well as with the isometries discussed above (which are inherent already in the pure SU$^*(4)$ approach, i.e in the Dirac algebra!) one should take care when introducing additional unitary symmetries by hand. At least, it is necessary to think about double counting and/or the correct physical identification of the additional parameters and states.

Here in the context of an SU$^*(4)$/USp(4) realization, i.e. with a global SU$^*(4)$ and a local USp(4) symmetry, it is now necessary to identify the vacuum (and its substructure(s)) as well as the transformations and the action of the group on the coset (and thus the geodesics). Note that right from the beginning, we have an overall consistent microscopic (and noncommutative) framework in terms of $Q_{\alpha\beta}$, and we are now trying to identify further physical processes by investigating the 'light cones'/vacua and geodesics. Put in terms of \[8\], if this succeeds we then know the local geometric structure of the manifold in that we know the conformal structure (the action of SU$^*(4)$ and the null cones), the projective structure (geodesics in $H_5$) as well as the affine structure, i.e. around each point of the manifold there exists an infinitesimal affine geometry (determined by 10+5 real parameters).
B. Physical Consequences

So in order to understand more of the underlying physics and the related identification of operators and transformations, we can benefit from our original starting point SU(4) with the thorough operator identification in the low energy-regime of the hadron spectrum, i.e. we know spin, isospin and axial transformations from our basic construction process [3].

Formally, we can act with the ten usp(4) generators on the coset space and see what happens. Hence if we decompose usp(4) into the three operator sets $Q_{02}$, $\{Q_{j0}\}$ and $\{iQ_{j1}, iQ_{k3}\}$ where $Q_{02}$ generates an U(1) symmetry, $\{Q_{j0}\}$ commute with $Q_{02}$ and generate (by themselves) an SU(2) symmetry, and $\{iQ_{j1}, iQ_{k3}\}$ generate the 6-dimensional (type CI) coset space where the sets $\{iQ_{j1}\}$ and $\{iQ_{j3}\}$ are in addition related by a $Q_{02}$ multiplication, i.e. a relative (noncommutative) complexification [6].

If we define the (infinitesimal) action of $Q_{02}$ via $\delta \cdot = [Q_{02}, \cdot ]$, we can act on an element of $\exp p$. In this case we find that $\delta$ acts only on the $H_2$ subspace generated by $\{iQ_{01}, iQ_{03}\}$ whereas $\{Q_{j2}\}$ is invariant. We know already from [3] that $\{iQ_{01}, iQ_{03}\}$ is a Lie triple system (and a totally geodesic submanifold), and that $\{iQ_{01}, Q_{02}, iQ_{03}\}$ generate (noncompact) SO(2,1) isometries of this 2-dimensional hyperbolic space. If we remember that in our original SU(4) scheme the compact generators $\{Q_{01}, Q_{02}, Q_{03}\}$ generate the isospin symmetry (see [3], [2]), then the only and straightforward conclusion is that we have to associate $Q_{02}$ with the electromagnetic/electroweak U(1) symmetry transformation. Hence, in our coset space $H_5$ we identify two degrees of freedom $\{iQ_{01}, iQ_{03}\}$ on which the operator $Q_{02}$ acts by a U(1) rotation so we can try to associate them with two (real) polarization degrees of freedom. On the other hand, we know of the SO(2,1) isometry transformations of this coordinate set which we may equally well express in terms of a SU(1,1) representation. This corresponds once more to the fact that $iQ_{01}$ and $iQ_{03}$ differ by a (noncomutative) $Q_{02}$ multiplication. So we can rewrite the $H_2$-element $a iQ_{01} + b iQ_{03} = iQ_{01}(a + b Q_{02}) = (a Q_{02} + b) iQ_{03} = (a - b Q_{02})iQ_{01} = \ldots$ equally well in terms of a (hyper-) complex coefficient related to only one remaining 'dimension'. This motivates using additional complex representations for masses and charges in terms of either a single complex number or using instead homogeneous complex coordinates by twofold spinor representation like in the Higgs (spinor) description.

However, the main result of this first step is the observation that compact isospin is broken to noncompact SO(2,1) by the spontaneous breakdown of SU*(4)/USp(4) and that
only an U(1) (as generated by $Q_{02}$) survives this breakdown. So there is no conserved isospin but we expect SO(2,1) to govern the spectral distribution instead of SU(2) isospin, especially at higher energies/masses. Nevertheless we have the (local) isometries SO(2,1) and the constraint via $H_5$ and its SO(5,1) isometries to 'rearrange' the coordinates in order 'to heal' and 'absorb' this isospin breaking a little bit\textsuperscript{6}. The fact that $Q_{02}$ commutes with $\{Q_{j2}\}$, i.e. that $H_3$ is invariant under $Q_{02}$, allows to understand $H_3$ as a vacuum ('ground state') with respect to $Q_{02}$ and as bounded by an 'electromagnetic light cone', respectively. So we define $\delta_{em} \cdot = [Q_{02}, \cdot]$.

The second compact generator set $\{Q_{j0}\}$ of usp(4) generators shows similar behaviour when acting on $H_5$. This time, however, $H_2$ is invariant, i.e. $[Q_{j0}, iQ_{01}] = [Q_{j0}, iQ_{03}] = 0$, so $H_2$ is bounded by a 'dynamic' light cone, but the $H_3$ subspace of $H_5$ is SU(2) rotated according to our original decomposition pattern from the su*(4) Lie algebra, i.e. $[h,p] \subset p$, and as a result we find another (rotated) element of $H_3$, $[Q_{j0}, Q_{k2}] \sim Q_{l2}$. So we state the action of a compact transformation on a noncompact (hyperbolic) space where the generators $\{Q_{k2}\}$ are a Lie triple system and $\{Q_{j0}, Q_{k2}\}$ is isomorphic to the Lorentz algebra. Moreover, the commutator $[Q_{j2}, Q_{k2}] \sim Q_{l0}$, i.e. $[p,p] \subset h$, according to $Q_{02}^2 = -1 \sim Q_{00}$ generates a compact (Wigner) rotation which is evident from the underlying su*(4) algebra.

Accounting for these aspects, it is obvious to associate these symmetry transformations with the Lorentz group and $H_3$ with spacetime. Hence formally (using [9]), we are finished with respect to the title of the paper because we have related a microscopic representation originating from QPT to $H_3$ and Lorentz transformations. So we can choose directly an appropriate (real) representation of $H_3$ on $\mathbb{R}^4$. In pursuing this idea, however, we may identify points in $H_3$ with spacetime events, and we are lead to a hyperbolic geometry of points, geodesic lines, incidences, triangles, etc.) in $H_3$ which map to $\mathbb{R}^4$. This is no new result at all because e.g. in [9] the Lorentz transformations are described as 'motions in $H_3$', in [10] $H_3$ is represented as a coset space SO(3,1)/SO(3), and last not least there is beautiful work [14] discussing some physical patterns originating from hyperbolic geometry of $H_3$. It is also possible to delve into further aspects of points and lines in $H_3$ or projective geometry in $\mathbb{R}^4$ or $\mathbb{R}^5$, however, here we want to emphasize that this SO(3,1) symmetry occurs in the same context as the SO(2,1) symmetry, i.e. both groups act as isometries on the respective

\textsuperscript{6} See also the discussion of isospin breaking in [3] with respect to experimental data.
subspaces $H_2$ and $H_3$ within the coset space $H_5$. The coset space itself, however, occurred because of our procedure of building USp(4) equivalence classes, i.e. we have imposed a certain (quadratic) constraint on the original quaternionic coordinates (see above). So for now we introduce the subscript $\delta_{LG} \cdot = [Q_{j0}, \cdot]$. Regarding Goldstone bosons within the quantum point of view, the photon(s) (and further 'gauge bosons') play(s) the exceptional rôle as we have a common vacuum structure in $H_5$ (also for the boundary of $H_2$ versus $H_3$), and we know that Bremsstrahlung occurs whenever we accelerate charged particles, i.e. when we change the equivalence classes and the 'vacuum definition', respectively. So it is intuitively quite plain (although for now a conjecture), that the photon in classical physics plays a residual-only rôle, however, motivated by geometry. More qualitative and quantitative aspects have to be worked out profoundly by coordinate mappings in $\mathbb{R}^{1,n}$.

Here, the identification of 'classical' coordinates is still open. We parametrize elements in $H_5$ by five real coordinates of the hypercomplex operator set $\{iQ_{01}, iQ_{03}, Q_{j2}\}$ which we map via exp to coordinates of the coset. There, we can introduce six real coordinates which fulfil an hyperbolic constraint with signature $(1,5)$, i.e. by an appropriate involution of the hypercomplex operators it is possible to define a purely real 'norm' with signature $(1,5)$ which corresponds to an appropriate metric. Hence we can understand the hypercomplex description as roots of a real theory (in the sense of Dirac's use of $\gamma$-matrices for SO(3,1)) based on 'vectors' obeying an SO(5,1) symmetry, or if we remember the su*(4) origin of these 'hypercomplex numbers', we can think of a microscopic vs. a 'classical' representation. If we restrict the coset elements to $H_2$ and $H_3$, then according to the su*(4) multiplication table we can realize 3- and 4-dimensional real spaces with signatures $(1,2)$ and $(1,3)$, respectively.

Last not least, we can apply actions of the remaining 6-dimensional usp(4) generator set $\{iQ_{j1}, iQ_{k3}\}$ to $H_5$. This time, we see a transition of elements in $H_2 \rightarrow H_3$ and vice versa from $H_3 \rightarrow H_2$ according to $\delta_{SC} iQ_{01} = -2b_kQ_{k2} \subset H_3$, $\delta_{SC} iQ_{03} = +2a_kQ_{k2} \subset H_3$, $\delta_{SC} Q_{j2} = 2 (b_j iQ_{01} - a_j iQ_{03}) \subset H_2$, if we define $\delta_{SC} \cdot = [a_k iQ_{k1} + b_k iQ_{k3}, \cdot]$. So this transformation couples the subspaces $H_2$ and $H_3$ within $H_5$, and we expect these equations to identify later on charge, mass, $\hbar$ (and combinations thereof) geometrically\(^7\) when comparing

\(^7\) An explicit/overall representation theory with a thorough identification of physical parameters is ongoing but not yet finished. So I invite everybody interested in performing calculations to accelerate this interesting process. However, in analogy to $\vec{v}$ and $c$ related to $H_3$ and Lorentz transformations, it is evident to introduce constants and transformation parameters for $H_2$, $H_5$ and their isometries as well.
to experiment or special relativity.

With respect to the operator algebra itself we can extend the graphical representation given in [6] by glueing four (multiplicative) triangles together (see figure 2). In identifying further points at the edges of the large triangle, we can construct a tetrahedron where $Q_{02}$ builds not only the U(1) barycenter of the tetrahedron but is also involved in spatial multiplications of the generator sets, i.e. the full tetrahedral symmetry.

FIG. 2: Multiplicative structure of the $\text{su}^*(4)$ generators where identification of the generators leads to a tetrahedron with $iQ_{03}$ on top.

C. Summary and Outlook

We’ve presented an end-to-end identification of operators and symmetries with origin in the low-energy hadron spectrum and spin×isospin SU(4). We’ve related this phenomenological symmetry by the assumption that SU(4) is an approximation of relativistic SU$^*(4)$ in the sense that the USp(4) cosets of SU$^*(4)$ are responsible for the differences between static low-energy and relativistic symmetries, and we’ve thus studied the coset decomposition SU$^*(4)/\text{USp}(4)$ (which differs from SU(4)/USp(4) only by complexification (‘duality’) [10]). We’ve found two kinds of boosts, one related to SO(2,1) and the other one related to SO(3,1) so we find our initial assumption (in that exp($p$) determines the dynamics) as justified. This patterns yields some more exciting results with respect to the local and global hyperbolic structure of the manifold exp($p$), and using Lie triple systems we’ve used an algebraic (and geometrically more interesting) way to determine the isometries as further geometrical properties [6] without using explicit differential operator representations.
Nevertheless, we can always introduce such (local) representations by introducing (affine) coordinates and representing the generators appropriately in terms of differential operators, moreover, we can use as well nonlinear sigma models or the machinery of differential geometry, fibers and differential equations, see e.g. the Laplace-Beltrami operator in eq. (1) or appropriate (differential) representations of \( \text{su}^*(4) \) operators resp. 'equations of motion'.

Anyhow, we act (locally) with a 10-dimensional compact group on a noncompact space (remember \([h, p] \subset p\)) which controls the 'time' development of the system.

However, we've emphasized right from the beginning the background in quaternionic geometry, in that we've represented a noneuclidean (projective) geometry in terms of twofold quaternionic coordinates and related reduction steps. So the very construction of coordinates and the restriction and reduction of the coordinates provides a continuous background for the various symmetry aspects and patterns discussed nowadays in terms of 'separate aspects' or even of 'separate' or 'effective' theories. Here, we've used nothing but a 15-dimensional operator algebra known for decades in slightly modified/complexified form, but we benefit from the enormous power of the division algebra \( \mathbb{H} \). We have not considered additional (unitary or gauge) symmetries but the \( \text{su}^*(4) \) algebra alone was not only able to provide the known symmetries (local and as isometries), but in addition we have seen how the dynamical symmetry breaking breaks isospin \( \text{SU}(2) \) down to an (electromagnetic/electroweak) \( \text{U}(1) \) symmetry in a nontrivial way. Vice versa, strict isospin conservation (or in general the assumption of a conserved \( \text{SU}(n) \) flavour symmetry) contradicts the geometrical pattern presented above. So one should \textit{ab initio} expect identification problems when attaching \( \text{SU}(n) \) symmetries to a relativistic description. With respect to an experimental verification of our approach the interesting energy regime is the \textit{low energy-regime} of the particle spectrum because the symmetry breaking mechanisms as well as the remaining symmetries can be tested in terms of hadron representations and their (electromagnetic) interactions [3].

We hope that the use of the hypercomplex representation \( Q_{\alpha\beta} \) to represent quaternionic Möbius transformations attends to the bell in order to gain more insight into the geometry. Although there is still a lot of work to do, the \( \text{SU}^*(4) \) representation alone has provided an overall and unique identification of well-known unitary symmetries as well as access to their origin(s). Moreover, if we understand the hypercomplex system as a description of quantum theories (which is justified by its isomorphism to the Dirac algebra), the five 'numbers' of the coset basis can be interpreted as roots of Lorentz(-like) vectors obeying (orthogonal) 'norms'
with signature(s) (1, n), moreover they are related to the Dirac equation[4]. Thus QPT yields structural and physical results based on nothing but a well-defined and transparent geometrical foundation. The microscopic structure of spacetime events in \( H_3 \) as given by \( \{Q_2\} \) corresponds to the Pauli representation \( ig_j \sim \sigma_j \) of spacetime events in \( \text{Sl}(2, \mathbb{C}) \) [5], however, that way the noncommuting operators \( Q_2 = Q_{02}Q_{j0} \) are only ‘approximated’. Last not least, an association of quaternions with spacetime events via \( \{Q_2\} \) has some more interesting consequences for gravitational models as well as for (quantum) statistics and for the motivation of (spin) lattices and nets, but that’s beyond the scope of discussion here.

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