Majorana bound state in the continuum: Coupling between Majorana bound state and quantum dot mediated by continuum

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In this work, we consider a single-level quantum dot (QD) and a Majorana bound state (MBS) placed at the end of a topological superconducting nanowire (TSW). Both are coupled to the continuum and do not have a direct connection between them. We addressed the behavior of MBS leaking phenomena and its consequences into the QD physics in non-interacting and Coulomb blockade regime. By employing Green’s function formalism via the equation of motion procedure, we calculate the physical quantities of interest. Our results show that the leakage of the MBS into the continuum state is achieved and can alter the physics of Coulomb blockade in the system through continuum-mediated coupling between MBS and QD. As a main consequence, we found a robust and non-trivial mechanism to accomplish a bound state in the continuum in the system.

I. INTRODUCTION

The progress of theoretical formulations and experimental techniques in condensed matter physics provides an interesting playground for scientists to investigate at low energy intriguing phenomena, commonly associated to elementary particles that would be possible solely in high energy physics [1]. An example is the possibility to observe Majorana bound states (MBSs) that have similarities with Majorana fermions, predicted to exist as elementary particles [2]. In condensed matter, MBSs are predicted to emerge as collective excitations in p-wave topological superconductors (TSC) [3–5]. The MBSs satisfy non-Abelian statistics and have a great deal of interest towards applications in quantum computation [6–9]. After the theoretical proposal performed by Kitaev [10], in which MBSs would emerge bound to edges of a one-dimensional (1D) TSC [11], several experiments have been carried out that verified the physical realization of Kitaev model, finding signatures of their presence through anomalies in physical quantities measurements [12–18]. However, from these early physical realizations to practical use, a long way still has to be paved. For example, a full understanding of the transport properties of MBS-based systems are mandatory to use them as electronic devices [19].

Owing to the great flexibility to control their electronic properties, the quantum dots (QDs) have proven to be a convenient platform to study MBS in condensed matter [20–25]. In a first attempt along these ideas, an attracting system was proposed by Liu and Baranger [26]. In their proposal, a QD was coupled simultaneously to two normal metallic contacts and to the end of a topological quantum wire holding a MBS. From a theoretical point of view—and with potential practical application—an interesting feature noted by these authors was the half-integer conductance between the normal contact across the QD. Later on, it was showed that this characteristic is obtained regardless the QD energy level, describing it as a MBS leaking phenomena [27]. In fact this leaking phenomena was observed experimentally [17].

In the setup discussed above, the Majorana mode leaked into the QD because they were directly coupled to each other. The reader may ask what would happen if the Majorana mode were not directly coupled to the dot. In particular, if a continuum of states mediated the coupling between the MBS and the QD. Our results show that no matter how strong the MBS is coupled to the continuum, its bound state character remains unchanged [28]. As such, as far as the MBS plus the contacts concern, this problem can be viewed as a bound state in the continuum (BIC) akin to the prediction by von Neumann and Wigner in a generic framework of engineered potential [29] and later investigated in many fermion systems (see Ref. [30] and references therein). However, this bound state does not represent a full fermion, as in the traditional case, it rather corresponds to an MBS or “half fermion”; as it is commonly referred. Hence we refer to this state as Majorana bound state in the continuum (MBIC). Recently, BICs has gained considerable attention as it has been observed in photonic systems. Motivated by the interference phenomena taking place in electronic systems in analogy with the photonic counterpart, the presence of BICs promoted by MBS has been investigated [31, 32]. Related to this problem, interplay between MBSs and BICs have been proposed as a useful tool to perform applications in quantum computing, allowing, for instance, to read/write information through veil/unveil these states [33–35]. Indeed, MBS provides a quite attractive way to produce BICs as they are topologically protected against local perturbation [36, 37]. As a result, manipulating electronic properties of QDs becomes much suitable as the rest of the system turns out

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to be almost insensitive to applied electric fields.

FIG. 1. Schematic representation of model: A single level QD (gray) and a TSW, hosting MBSs (orange) $\gamma_1$ and $\gamma_2$, coupled with a common metallic lead (green) with a continuum spectrum. The black curve above the topological wire intends to represent the wave function associated to the Majorana bound states (note the peaks at the ends).

In this work, we propose to study the electronic properties of a system composed of a QD and a topological superconducting wire (TSW), both connected to a common metallic contact. The TSW is assumed to be in its topological phase, holding MBSs in its ends. The system is schematically depicted in Fig. 1. Alternatively, this system can be viewed as a QD coupled to an effective continuum exhibiting an MBSs. By employing the Green’s function method the equation of motion techniques, we study the spectral and transport properties of the system. While in the non-interacting regime of the QD, we can access the physical property exactly, in its interacting regime, they are available only under certain approximation. Here we employ the so-called Hubbard I approximation that is known to capture qualitatively well the many-body physics in Coulomb blockade regime [38]. Our results show that no matter how strong the MBS is coupled to the continuum, it features in the QD spectral function as a bound state. This behavior remains unchanged in the strong Coulomb interaction regime of the QD.

This paper is organized as follows: Section II presents the system Hamiltonian and method used to obtain quantities of interest; Section III shows the corresponding results and the related discussion. Finally, our concluding remarks are presented in Section IV.

II. HAMILTONIAN MODEL AND METHOD

For the sake of completeness, the system under study consists of a QD and a MBS located at the end of a TSW, both connected with a common normal metallic lead, as schematically shown in Fig. 1. The Hamiltonian of the system can be written as

$$H = H_c + H_{\text{dot}} + H_{c\text{-dot}} + H_{c\text{-MBS}},$$

where the first three terms of Eq. (1) correspond to the traditional Anderson Hamiltonian describing the QD plus the normal metallic lead and are given by

$$H_c = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma};$$

$$H_{\text{dot}} = \sum_{\sigma} \varepsilon_d d_\sigma^\dagger d_\sigma + Un_\uparrow n_\downarrow;$$

$$H_{c\text{-dot}} = \sum_{k,\sigma} \left( V_k c_{k,\sigma}^\dagger d_\sigma + V_k^* d_\sigma^\dagger c_{k,\sigma} \right);$$

where $c_{k,\sigma}^\dagger$ creates(annihilates) a continuum electron with momentum $k$ and spin $\sigma$; $d_\sigma^\dagger$ does it in the QD with energy level $\varepsilon_d$. Here we have assumed that the Majorana mode is provided by a long TSW that are fully polarized with spin down by an effective magnetic field along the $z$ direction. Hence only electrons with spin down couples to the MBS. Moreover, it is worth of mentioning that we consider a TSW in long-wire limit, then the MBS placed at the opposite end, $\gamma_2$, is strictly equivalent to $\gamma_1$.

We are interested in to study the influence of the MBS onto the physical properties of the QD, mediated by the continuum. To access the relevant physical quantities we employ the Green’s function formalism which allows us to obtain, for instance the spin-resolved local density of states (LDOS) at the QD, $\rho_{d,\sigma}(\varepsilon)$ and transport properties. In terms of the Green’s function (GF), the spin dependent LDOS is given

$$\rho_{d,\sigma}(\varepsilon) = -\frac{1}{\pi} \text{Im} \left[ \langle\langle d_{\sigma}^\dagger d_\sigma \rangle\rangle_\varepsilon \right],$$

where $\langle\langle d_{\sigma}^\dagger d_\sigma \rangle\rangle_\varepsilon$ denotes the spin-resolved retarded GF of the QD in energy domain. In the following, we will address the model either in the non-interacting ($U = 0$) and interacting ($U > 0$) cases. For $U > 0$ it is known that one cannot obtain an exact expression for the Green’s function. However, approximated versions can still be obtained. For instance, the so-called Hubbard I approximation is known to provide a fairly good description of Coulomb blockade phenomena above the Kondo temperature. Such an expression can be derived by using the equation-of-motion technique, as discussed in Appendix A. Within this approximation, the Green’s function acquires the form

$$\langle\langle d_{\sigma}^\dagger d_\sigma \rangle\rangle_\varepsilon = \frac{\varepsilon - \varepsilon_d - U(1 - \langle n_\sigma \rangle)}{(\varepsilon - \varepsilon_d)(\varepsilon - \varepsilon_d - U) - (\varepsilon - \varepsilon_d - U(1 - \langle n_\sigma \rangle))\Sigma_\sigma(\varepsilon)},$$

where $\Sigma_\sigma(\varepsilon)$ represents the self-energy of the $\sigma$ spin component.
in which
\[ \Sigma^\uparrow(\varepsilon) = -i\Gamma \tag{8a} \]
\[ \Sigma^\downarrow(\varepsilon) = -i\frac{\Gamma}{1 - M(\varepsilon)} \tag{8b} \]
are the spin-resolved self-energies of the QD. In the above, \( \langle n_\sigma \rangle \) is the occupation of the QD for a given spin \( \sigma \) and \( \Gamma = (\pi V^2/2D) \Theta(D - |\omega|) \) (with \( D \) being the bandwidth of the metallic contact) represents the energy-independent hybridization parameter between the continuum and QD, which is derived in the wide-band limit (\( D \) much larger than any other energy parameter of the system). Note that \( \Sigma^\downarrow(\varepsilon) \) is modified by the presence of MBS accounted by the function \( M(\varepsilon) \), which is given by (see Appendix A)
\[ M(\varepsilon) = -2i\Lambda x \left[ \varepsilon + \frac{2i\Lambda(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U)}{(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U) + i\Gamma (\varepsilon + \varepsilon_d + U(1 - \langle n_\uparrow \rangle))} \right]^{-1}, \tag{9} \]
where \( \Lambda = \pi\lambda^2/2D \) is the hybridization strength between the MBS and the continuum. Since the GF (7) depends on the occupation, it must be determined self-consistently.

### III. NUMERICAL RESULTS

To show our numerical results let us set the hybridization \( \Gamma \) as the energy unit. In the following, we will analyze the LDOS as a function of the energy for different values of the relevant parameters of the system, e.g., \( \Lambda \), \( \varepsilon_d \) and \( U \). We shall first discuss the result at the non-interacting case, \( U = 0 \) and next; we will address the case of \( U \neq 0 \).

#### A. Non-interacting quantum dot (\( U = 0 \))

For \( U = 0 \), the expression for the GF (7) becomes exact and acquires the form
\[ \langle \langle d_\sigma; d_\sigma^\dagger \rangle \rangle_\varepsilon = \frac{1}{\varepsilon - \varepsilon_d - \Sigma^\downarrow(\varepsilon)} \tag{10} \]
The effect of the MBS in the QD is accounted by the self-energy \( \Sigma^\downarrow \), via
\[ M(\varepsilon, U = 0) = -2i\Lambda \left[ \varepsilon + 2i\Lambda \frac{\varepsilon + \varepsilon_d}{\varepsilon + \varepsilon_d + i\Gamma} \right]^{-1}. \tag{11} \]
Note that since the electron’s spins are decoupled from each other, the spin \( \uparrow \), component is not affected by the MBS. Therefore, we focus only on the electrons with spin \( \downarrow \) in the QD. As usual, the self-energy encompasses the information from the rest of the system by shifting the energy level of the QD by an amount \( \text{Re}[\Sigma^\downarrow(\varepsilon)] \) and broadening the bare level by a quantity \( -\text{Im}[\Sigma^\downarrow(\varepsilon)] \equiv \Gamma \text{eff}(\varepsilon) \).

The latter represents the effective hybridization between the QD and continuum, modified by the MBS. For \( \lambda \to 0, \Gamma \text{eff} \to \Gamma \), that is independent of \( \varepsilon \).

![Figure 2](image2.png)

**FIG. 2.** (a) \( \Gamma \text{eff} \) and (b) Re[\( \Sigma^\downarrow(\varepsilon) \)] as function of the energy for different MBS-lead couplings \( \Lambda \). Here the QD energy level is fixed at \( \varepsilon_d = 0 \). The flat black curve for \( \Lambda = 0 \) reflects the wide band limit assumed in the calculations.

![Figure 3](image3.png)

**FIG. 3.** (a) \( \Gamma \text{eff} \) and (b) Re[\( \Sigma^\downarrow(\varepsilon) \)] as function of the energy for different QD energy levels \( \varepsilon_d \). Here the MBS-continuum coupling is fixed at \( \Lambda/\Gamma = 0.5 \). Note in (a) that \( \Gamma \text{eff} \) vanishes only for \( \varepsilon_d = 0 \).

In Fig. 2 we show Re[\( \Sigma^\downarrow(\varepsilon) \)] and \( \Gamma \text{eff}(\varepsilon) \) as a function of \( \varepsilon \), using \( \varepsilon_d = 0 \) and different values of \( \Lambda \). First of all, it is interesting noting in Fig. 2(a) that \( \Gamma \text{eff}(\varepsilon = 0) = 0 \) for any value of \( \Lambda \neq 0 \). This is somewhat surprising because it results from a destructive quantum interference—involving a “half” fermion—and is very much similar to the case of a conventional fermion in the continuum. This complete antiresonance at \( \varepsilon = 0 \) decouples the electrons with spin \( \downarrow \) of the QD from the continuum. For \( |\varepsilon| \gg \Gamma \) we note that \( \Gamma \text{eff} \) tends to saturate at different
values depending on how big is $\Lambda$. This can be understood analytically. In the limit $\Lambda/\Gamma \gg 1$ and $\varepsilon_d = 0$, Eq. (11) becomes
\[ \Gamma_{\text{eff}}(\varepsilon, \varepsilon_d = 0) = \frac{\Gamma}{2} \frac{\varepsilon^2 + (\Gamma/2)^2}{\varepsilon^2 + (\Gamma/2)^2}, \]
which is independent of $\Lambda$. From this equation it is easy to see that for energies $|\varepsilon| \gg \Gamma$ we obtain $\Gamma_{\text{eff}} = \Gamma/2$.

In Fig. 2(b) we show the real part of the self energy. Note that, by virtue of the wide-band limit, Re$[\Sigma_{\downarrow}(\varepsilon)] = 0$ for $\Lambda = 0$. Moreover, Re$[\Sigma_{\downarrow}(\varepsilon = \varepsilon_d = 0)] = 0$ for any value of $\Lambda$ and becomes finite for $\varepsilon \neq 0$, but restrict to the condition |Re$[\Sigma_{\downarrow}(\varepsilon)]| < \Gamma/2$.

In Fig. 3 we show $\Sigma_{\downarrow}(\varepsilon)$ for fixed $\Lambda$ and different $\varepsilon_d > 0$. In Fig. 3(a) we see that $\Gamma_{\text{eff}}$ vanishes only for $\varepsilon_d = 0$. Moreover, we note that $\Gamma_{\text{eff}}(\varepsilon = 0) \to \Gamma/2$ for large $\varepsilon_d$. This is a remarkable signature of the presence of the Majorana zero mode in the continuum. In the limit $\varepsilon_d \gg \Gamma$, the contribution given by the MBS to $\Gamma_{\text{eff}}$ amounts to $M(\varepsilon; U = 0) = 2i\Lambda[\varepsilon + 2i\Lambda]^{-1}$. With this we obtain
\[ \Gamma_{\text{eff}}(\varepsilon, \varepsilon_d \gg \Gamma) = \Gamma \left( \frac{\varepsilon^2}{\varepsilon^2 + 16\Lambda^2} + \frac{8\Lambda^2}{\varepsilon^2 + 16\Lambda^2} \right). \]

This clearly shows that $\Gamma_{\text{eff}} = \Gamma/2$ as $\varepsilon \to 0$ regardless the value of $\Lambda$. Interestingly, similar to what was observed in Fig. 2(b), in Fig. 3(b) Re$[\Sigma_{\downarrow}(\varepsilon)]$ is also limited as |Re$[\Sigma_{\downarrow}(\varepsilon)]| < \Gamma/2$.

The behavior of the self-energy discussed above have important consequences in the QD LDOS, $\rho_d(\varepsilon)$, calculated from Eq. (6). This quantity is the one that is actually accessible in experiment via transport spectroscopy. Figure 4(a) shows LDOS as a function of $\varepsilon$ and $\Lambda$ for $\varepsilon_d = 0$. For $\Lambda = 0$ (uncoupled MBS) we observe a broad peak placed around $\varepsilon = \varepsilon_d = 0$. Once the coupling of the MBS is turned on ($\Lambda \neq 0$), the amplitude of the LDOS decreases as $\Lambda$ increases, but the height of the peak does not go below 1/2$\pi\Gamma$. Besides, at $\varepsilon = 0$ sharp peak is observed. This sharp peak is a direct consequence of the vanishing effective hybridization function due to the presence of the MBIC. It is better appreciated in Fig. 4(b) where we show $\rho_d$ along the horizontal orange lines of Fig. 4(a). Indeed this behavior can be understood analytically; from Eq. (10), in the limit of strong MBS coupling ($\Lambda \gg \Gamma$), we can write
\[ \pi \rho_d(\varepsilon, \varepsilon_d = 0) = \frac{\Gamma}{2} \left( \frac{1}{\varepsilon^2 + \Gamma^2} \right) + \frac{\pi}{2} \delta(\varepsilon). \]

Clearly, at $\varepsilon_d = 0$ a bound state in the continuum (BIC) is obtained at zero energy whenever $\Lambda \neq 0$. In Fig. 4(c) $\rho_d$ is displayed for fixed $\Lambda \gg \Gamma$ and different values of $\varepsilon_d \neq 0$. Note that the observed BIC feature evolves to a situation with an antiresonance at $\varepsilon = \varepsilon_d$ for $\varepsilon_d \neq 0$. Analytically, for small values of $\varepsilon_d$, as in Fig. 4(c), we can express the LDOS as
\[ \pi \rho_d(\varepsilon) \sim \frac{\Gamma}{2} \frac{(\varepsilon + \varepsilon_d)^2}{(\varepsilon - \varepsilon_d)^2 + \varepsilon^2 \Gamma^2}. \]

From this, we note that indeed there is an anti-resonance at $\varepsilon = \varepsilon_d$. We see, therefore, that tuning $\varepsilon_d$ is relevant to achieve a BIC. At this point, we should emphasize that in this non-interacting scenario BIC seen in the QD LDOS results solely from the leaking of the MBS into the continuum. In the following, we will see that this feature is still present in the interacting regime of the QD.

### B. Interacting Regime

In this subsection, we study the interacting regime of the QD, $U \neq 0$. We focus on the Coulomb blockade regime, to which the Hubbard approximation is reasonably good. In contrast to the previous subsection, now the LDOS depends on the temperature ($T$), and we as-
sume $T$ larger than the Kondo temperature $T_K$ so that Kondo correlations are thermally suppressed. Again, using the equation of motion procedure in the energy domain, the GF of the QD for this case is given by

$$\langle \langle d_\sigma; d_\sigma^\dagger \rangle \rangle_\varepsilon = \frac{\varepsilon - \varepsilon_d - U(1 - \langle n_\sigma \rangle)}{(\varepsilon - \varepsilon_d)(\varepsilon - \varepsilon_d - U) - |\varepsilon - \varepsilon_d - U(1 - \langle n_\sigma \rangle)| \Sigma_\sigma(\varepsilon)},$$

where $\Sigma_\sigma(\varepsilon)$ has the form of the Eqs. (8a) and (8b), with

$$M(\varepsilon) = -2i\Lambda \times \left[ \varepsilon + \frac{2i\Lambda(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U)}{(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U) + i\Gamma[\varepsilon + \varepsilon_d + U(1 - \langle n_\uparrow \rangle)]} \right]^{-1}.$$  

Here we should emphasize that, as a consequence of the Coulomb interaction, the GF for spin $\sigma$ depends on QD occupation $\langle n_\sigma \rangle$ given by

$$\langle n_\sigma \rangle = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\varepsilon \text{Im} \left[ \langle \langle d_\sigma; d_\sigma^\dagger \rangle \rangle_\varepsilon \right] f(\varepsilon).$$

where $f(\varepsilon)$ is the Fermi’s function. It, therefore, enforce us to perform a self-consistent calculation numerically. To show our numerical result, we set $U = 10\Gamma$ and carry on the numerical calculations at $k_B T = 10^{-2}\Gamma$ which happen to be above $k_B T_K$ for most of the parameters used throughout this paper. In Fig. 5 we show the effect of the MBS in the effective hybridization function [Fig. 5(a)] and the LDOS [Fig. 5(b)] for a fixed $\varepsilon_d = 0$ and various values of $\Lambda$. Figure 5(a) is similar to what displayed in Fig. 2(a) but now, for finite $U$. We observe that $\Gamma_{eff} = \Gamma$ for $\Lambda = 0$ and $\Gamma_{eff} = 0$ for $\varepsilon = \varepsilon_d = 0$, whenever $\Lambda \neq 0$. This behavior is very much similar to the non-interacting case shown in Fig. 2(a). Again, this is a direct consequence of the MBS leaking into the continuum, reaching the physics quantities in the QD. The behavior of the curves of Fig. 5(a) can be obtained analytically from Eq. (17) for large values of $\Lambda$. In fact, for $\Lambda \gg \Gamma, M(\varepsilon)$ it is independent of $\Lambda$. In this limit, setting $\varepsilon_d = 0$, we can write the effective hybridization as

$$\Gamma_{eff}(\varepsilon) = \frac{2\Gamma\varepsilon^2(\varepsilon + U)^2}{4\varepsilon^2(\varepsilon + U)^2 + |\varepsilon + (1 - \langle n_\uparrow \rangle)U|^2\Gamma^2}. $$

This result clearly show that $\Gamma_{eff} = 0$ vanishes at both $\varepsilon = 0$, and $\varepsilon = -U$. Nevertheless, no important consequence in the $P_d, \sigma$ is observed for $\varepsilon_d = -U$ since that energy is far away from the $\varepsilon_d = 0$ Similarly to the non-interacting case, we note also that $\Gamma_{eff} = \Gamma/2$ for all the energies regions such as $|\varepsilon| \gg \Gamma, U$.

The features observed for $\Gamma_{eff}$ are directly related to the LDOS of the QD, which is shown in Fig. 5(b). For the case with unconnected MBS, $\Lambda = 0$, two peaks are observed, of the same amplitude, localized at energies $\varepsilon = 0$ and $\varepsilon = U$ due to the Coulomb blockade regime in our system. On the other hand, for the cases with $\Lambda \neq 0,$ different modifications are achieved in each of the mentioned peaks. The amplitude of the peak located around $\varepsilon = 0$ decreases as $\Lambda$ increases, while at exactly $\varepsilon = 0$ a very narrow peak, a BIC, arises from the QD effective disconnection ($\Gamma_{eff}(\varepsilon = \varepsilon_d = 0) = 0$), is similar to the one discussed in Sec. IIIA. At this point, it is interesting to note that the peak located at $\varepsilon = U$, becomes narrower increasing its amplitude, although it remains finite since $\Gamma_{eff}$ does not vanishes. Thus, whenever the QD is in Coulomb regime, the leaked MBS into the continuum affects the LDOS substantially, in a similar fashion as in the non-interacting case.

![Fig. 5](image_url)

**FIG. 5.** (a) $\Gamma_{eff}$ and (b) LDOS for spin $\sigma = \downarrow$ as function of the energy for different $\Lambda$. The inset in panel (b) is the LDOS for spin $\sigma = \uparrow$.

![Fig. 6](image_url)

**FIG. 6.** Occupation number (a) $n_\uparrow$ and (b) $n_\downarrow$ of the QD as function of $\varepsilon_d$, for different $\Lambda$ values. The inset in panel (a) shows a zoom in of $n_\uparrow$ around $\varepsilon_d = 0$.

Taking into account the discussion above, the QD local density of the QD for spin down can be written as
\[
\pi \rho_{d\downarrow}(\varepsilon) = \frac{\Gamma}{2} \frac{(\varepsilon + U)^2(\varepsilon - (1 - \langle n_\uparrow \rangle)U)^2}{\varepsilon^6 + (1 - \langle n_\uparrow \rangle)^2U^4\Gamma^2 + \varepsilon^4(\Gamma^2 - 2U^2) + \varepsilon^2U^2(\Gamma^2 - 2(1 - \langle n_\uparrow \rangle)\Gamma^2)} + \frac{\pi}{3} \delta(\varepsilon).
\]

Despite the complexity of the equation above, for the corresponding occupancy, we can extract that the wide peak placed around \(\varepsilon = 0\) asymptotically reach \(\pi \rho_{d\downarrow} \rightarrow 1/2\Gamma\), while the one located at \(\varepsilon = U\) increase up to \(\pi \rho_{d\downarrow} \rightarrow 2/\Gamma\), both in the limit \(\Lambda \gg \Gamma\).

Before closing this section, we show how the presence of the MBIC affects the occupation of the QD. In Fig. 6, we show the spin-resolved occupation number in the QD as a function of \(\varepsilon_d\). From this figure, for \(\Lambda = 0\) (solid black lines) there is spin degeneration in the occupancy, as we expected since the Hamiltonian is spin symmetric for this case. Allowing coupling between the MBS and the continuum, \(\Lambda \neq 0\), the spin symmetry breaks, and deviations are observed. As consequence, in Fig. 6(a) we observe a subtle oscillation of \(n_\uparrow\) around \(\varepsilon_d = 0\) and \(\varepsilon_d = -U\), better seen in the inset for energies near \(\varepsilon_d = 0\). In Fig. 6(b) we show the corresponding curves for \(n_\downarrow\). Here, a more interesting consequence of the MBIC is observed: a subtle oscillation \(n_\downarrow\) deviates from the continuum, \(\Lambda \neq 0\), for both cases, interacting and non-interacting regimes. In the interacting case, the second peak due to Coulomb blockade, placed at \(\varepsilon = \varepsilon_d + U\), is also affected by the MBIC. It becomes narrower and increasing its amplitude as the coupling strength between MBS and continuum increases. Besides, we have performed an analytic treatment of the effective coupling and local density of states in the limit of strong MBS-continuum coupling. Owing to the robustness of the MBS against the applied electric field, MBICs provide an exciting manner to control the QDs electronic properties without changing the energy position of the bound state in the continuum.

**IV. CONCLUSIONS**

We studied a system formed by a QD coupled to the continuum, which is connected to an MBS localized at the end of a TSW. Considering that continuum electrons with a particular spin down couples with the MBS, we found that the leakage of the MBS into the continuum affects the physical properties of the QD greatly. As a consequence of this leakage, the QD becomes effectively decoupled from the rest of the system at energies \(\varepsilon = \varepsilon_d = 0\), for both cases, interacting and non-interacting regimes. In the interacting case, the second peak due to Coulomb blockade, placed at \(\varepsilon = \varepsilon_d + U\), is also affected by the MBIC. It becomes narrower and increasing its amplitude as the coupling strength between MBS and continuum increases. Besides, we have performed an analytic treatment of the effective coupling and local density of states in the limit of strong MBS-continuum coupling. Owing to the robustness of the MBS against the applied electric field, MBICs provide an exciting manner to control the QDs electronic properties without changing the energy position of the bound state in the continuum.

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**Appendix A: QD Green’s function**

In this appendix, we show the procedure used to reach an analytic expression for the QD retarded Green’s function in our system. We considered the equation of motion method up to the equations hierarchy that allows describing the Coulomb blockade phenomena in the QD. The system Hamiltonian is given by Eq. (1). Note that it is not symmetrical in spin degree of freedom, since only continuum electrons with spin \(\sigma = \downarrow\) are coupled with the MBS [Eq. (5)]. The general expression for the retarded Green’s function equation of motion in the energy domain is given by

\[
\langle \varepsilon | \langle A; B \rangle \rangle_\varepsilon = \langle \{ A; B \} \rangle + \langle [[A; H]; B] \rangle_\varepsilon, \tag{A1}
\]

where \(A\) and \(B\) are two arbitrary operators, and \(0^+\) an infinitesimal (positive) number. Throughout this section, as in the main text, we display the energy as \(\varepsilon + i0^+ \rightarrow \varepsilon\) for simplicity.

Using Eq. (A1), for spin \(\sigma = \downarrow\) electrons, calculating the corresponding commutators/anticommutators, the first hierarchy of equations are

\[
(\varepsilon - \varepsilon_d)\langle \langle d_{\downarrow}^\dagger; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon = 1 + \sum_k V \langle \langle c_{\downarrow k}; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon + U \langle \langle n_{\downarrow}; d_{\downarrow}; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon \tag{A2}
\]

\[
(\varepsilon - \varepsilon_k)\langle \langle c_{\downarrow k}; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon = V \langle \langle d_{\downarrow}; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon - \lambda \langle \langle \gamma_1; d_{\downarrow}^\dagger \rangle \rangle_\varepsilon, \tag{A3}
\]
where we have suppressed the superscript \( r \) for simplicity. As a consequence of MBS presence, anomalous Green’s function that must be calculated. The next hierarchy of equations is extracted from the last terms in Eqs. (A2) and (A6). They lead to

\[
\begin{align*}
\langle \varepsilon \rangle = & \sum_{k} \left( \langle \varepsilon_{k} \rangle \right) \langle d_{k}^{\dagger} \rangle \langle d_{k} \rangle - \left( \langle \varepsilon_{k} \rangle \right) \langle d_{k}^{\dagger} \rangle \langle d_{k} \rangle,
\end{align*}
\]

where we have considered \( \sum_{k} \langle d_{k}^{\dagger} c_{k,\sigma} \rangle = \sum_{k} \langle c_{k,\sigma} d_{\sigma} \rangle \). Replacing Eq. (A10) into Eq. (A6), we have

\[
\begin{align*}
\langle \varepsilon \rangle &= -\sum_{k} \left( \langle \varepsilon_{k} \rangle \right) \langle d_{k}^{\dagger} \rangle \langle d_{k} \rangle \\
&= -\left( 1 - \frac{U(\langle n_{\uparrow} \rangle)}{\varepsilon + \varepsilon_d + U} \right) \sum_{k} \langle \langle \varepsilon_{k} \rangle \rangle \langle d_{\uparrow} \rangle,
\end{align*}
\]

thus, including this result into Eq. (A5), we obtain

\[
\begin{align*}
\langle \varepsilon \rangle &= -\left( 1 - \frac{U(\langle n_{\uparrow} \rangle)}{\varepsilon + \varepsilon_d + U} \right) \sum_{k} \langle \langle \varepsilon_{k} \rangle \rangle \langle d_{\uparrow} \rangle = \lambda \tilde{g}(\varepsilon) \langle \langle \varepsilon \rangle \rangle, \\
\end{align*}
\]

where we have defined \( \tilde{g}(\varepsilon) = \sum_{k} (\varepsilon + \varepsilon_{k})^{-1} \). Then, the Eq. (A4) is rewritten as

\[
\begin{align*}
\left( \varepsilon - 2\lambda^{2} \tilde{g}(\varepsilon) \left[ 1 - \frac{U(\langle n_{\uparrow} \rangle)}{\varepsilon + \varepsilon_d} \right]^{-1} \right) \langle \langle \varepsilon \rangle \rangle &= -2\lambda \sum_{k} \langle \langle \varepsilon_{k} \rangle \rangle \langle d_{\uparrow} \rangle,
\end{align*}
\]

Consequently, the Eq. (A3) is expressed as

\[
\begin{align*}
\langle \langle \varepsilon \rangle \rangle &= \left[ \lambda \tilde{g}(\varepsilon) \right]^{-1} \sum_{k} \langle \langle \varepsilon_{k} \rangle \rangle \langle d_{\uparrow} \rangle = \tilde{g}(\varepsilon) \langle \langle d_{\uparrow} \rangle \rangle,
\end{align*}
\]

being defined \( g(\varepsilon) = \sum_{k} (\varepsilon - \varepsilon_{k})^{-1} \). On the other hand, after replacing the Eq. (A9) into Eq. (A2) we obtain

\[
\begin{align*}
\langle \langle d_{\uparrow} \rangle \rangle &= 1 + \frac{U(\langle n_{\uparrow} \rangle)}{\varepsilon - \varepsilon_d - U} + \left( 1 + \frac{U(\langle n_{\uparrow} \rangle)}{\varepsilon - \varepsilon_d - U} \right) \sum_{k} \langle \langle \varepsilon_{k} \rangle \rangle \langle d_{\uparrow} \rangle,
\end{align*}
\]

which allow a closed solution for the set of equations. Finally, performing algebraic manipulations we have

\[
\begin{align*}
\langle \langle d_{\uparrow} \rangle \rangle &= \frac{\varepsilon - \varepsilon_d - U(1 - \langle n_{\uparrow} \rangle)}{(\varepsilon - \varepsilon_d)(\varepsilon - \varepsilon_d - U) - (\varepsilon - \varepsilon_d - U(1 - \langle n_{\uparrow} \rangle))V^{2}\tilde{g}(\varepsilon)} \left( 1 - \tilde{M}(\varepsilon) \right),
\end{align*}
\]
where
\[ M(\varepsilon) = 2\lambda^2 g(\varepsilon) \left[ \varepsilon - \frac{2\lambda^2 \hat{g}(\varepsilon)(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U)}{(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U) - V^2 \hat{g}(\varepsilon)(\varepsilon + \varepsilon_d + U(1 - \langle n_+ \rangle))} \right]^{-1}. \] (A17)

At this point, it is interesting to note that the quantities \( V^2 \hat{g}(\varepsilon) \) and \( \lambda^2 \hat{g}(\varepsilon) \), can be treated within the wideband approximation. In this limit, they are energy independent and fulfill electron-hole symmetry, such as
\[ V^2 \hat{g}(\varepsilon) = V^2 \hat{g}(\varepsilon) = -i\Gamma, \] (A18)
\[ \lambda^2 \hat{g}(\varepsilon) = \lambda^2 \hat{g}(\varepsilon) = -i\Lambda. \] (A19)

Then, the QD Green’s function for \( \sigma = \downarrow \) is given by
\[ \langle \langle d_\downarrow, d_\downarrow^\dagger \rangle \rangle_\varepsilon = \frac{\varepsilon - \varepsilon_d - U(1 - \langle n_+ \rangle)}{(\varepsilon - \varepsilon_d)(\varepsilon - \varepsilon_d - U) - (\varepsilon - \varepsilon_d - U(1 - \langle n_+ \rangle))} \Sigma_\downarrow(\varepsilon), \] (A20)
where \( \Sigma_\downarrow(\varepsilon) = -i\Gamma/[1 - M(\varepsilon)] \) and all the MBS contribution is embedded in the function
\[ M(\varepsilon) = -2i\Lambda \left[ \varepsilon + \frac{2i\Lambda(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U)}{(\varepsilon + \varepsilon_d)(\varepsilon + \varepsilon_d + U) + i\Gamma(\varepsilon + \varepsilon_d + U(1 - \langle n_+ \rangle))} \right]^{-1}. \] (A21)

For the component \( \sigma = \uparrow \), we note that up to the hierarchy considered in this paper, there is no MBS explicit contribution in the corresponding Green’s function. Therefore, it can be obtained from Eq. (A20) fixing \( \lambda = \Lambda = M(\varepsilon) = 0 \), then \( \Sigma_\uparrow(\varepsilon) = -i\Gamma \) and
\[ \langle \langle d_\uparrow, d_\uparrow^\dagger \rangle \rangle_\varepsilon = \frac{\varepsilon - \varepsilon_d - U(1 - \langle n_+ \rangle)}{(\varepsilon - \varepsilon_d)(\varepsilon - \varepsilon_d - U) - (\varepsilon - \varepsilon_d - U(1 - \langle n_+ \rangle))} \Sigma_\uparrow(\varepsilon). \] (A22)
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