A NEW LIGHT ON THE FKMM INVARIANT AND ITS CONSEQUENCES

GIUSEPPE DE NITTIS AND KYONORI GOMI

ABSTRACT. “Quaternionic” vector bundles are the objects which describe the topological phases of quantum systems subjected to an odd time-reversal symmetry (class AII). In this work we prove that the FKMM invariant provides the correct fundamental characteristic class for the classification of “Quaternionic” vector bundles in dimension less than, or equal to three (low dimension). The new insight is provided by the interpretation of the FKMM invariant from the viewpoint of the Bredon equivariant cohomology. This fact, along with basic results in equivariant homotopy theory, allows us to achieve the expected result.

MSC 2010: Primary: 14D21; Secondary: 57R22, 55N25, 81Q99.

Keywords: Class AII topological insulators, “Quaternionic” vector bundles, FKMM invariant, Bredon equivariant cohomology.

CONTENTS

1. Introduction ........................................ 2
2. Basic facts about the Bredon cohomology ........... 5
   2.1. Orbit category .................................. 5
   2.2. Equivariant CW complexes ...................... 6
   2.3. Coefficient systems ............................ 7
   2.4. Bredon equivariant cohomology ............... 8
   2.5. Eilenberg-Mac Lane spaces .................... 9
   2.6. Atiyah-Hirzebruch spectral sequence ........... 10
3. The FKMM invariant and Bredon cohomology ......... 11
   3.1. Borel and Bredon equivariant cohomology .... 11
   3.2. The universal FKMM invariant .................. 14
   3.3. A relevant Eilenberg-Mac Lane space ......... 15
   3.4. Bredon cohomology interpretation of the FKMM invariant ... 15
4. Equivariant cohomology of lens space ............... 18
   4.1. Setup and relevant results ..................... 18
   4.2. Restricted “Real” line bundles ................. 19
   4.3. “Quaternionic” structures ..................... 20
   4.4. Physical applications .......................... 22
Appendix A. A short reminder of the equivariant Borel cohomology .... 24
Appendix B. Cohomology of the Eilenberg-Mac Lane space ............ 25

Date: November 16, 2022.
1. **INTRODUCTION**

In its simplest incarnation, a *Topological Quantum System* (TQS) is a continuous matrix-valued map

\[ X \ni x \mapsto H(x) \in \text{Mat}_N(\mathbb{C}) \]  

(1.1)
defined on a \( d \)-dimensional *nice* topological space \( X \). Although a precise definition of TQS requires some more ingredients (see e.g. [DG1, DG2, DG3, DG4, DG5, DG6]), one can certainly state that the most relevant feature of these systems is the nature of the spectrum which is made by \( N \) continuous “energy” bands (taking into account possible degeneracies). It is exactly this peculiar band structure, along with the structure of the related eigenspaces, which may encode topological information. In a nutshell, assume that one can select \( m \) bands that do not cross the other \( N - m \) bands. Then, it is possible to construct a continuous projection-valued map \( X \ni x \mapsto P(x) \in \text{Mat}_N(\mathbb{C}) \) such that \( P(x) \) is the rank \( m \) spectral projection of \( H(x) \) associated to the spectral subspace selected by the \( m \) energy bands at the point \( x \). Due to the classical Serre-Swan construction [Se, Sw] one can associate to \( \lambda \mapsto P(x) \) a unique (up to isomorphisms) rank \( m \) complex vector bundle \( E \rightarrow X \) called the spectral bundle (see e.g. [DG1, Section 2]). The remarkable consequence of this construction is the following principle:

*One can classify the topological phases of a TQS by the elements of the set \( \text{Vec}^m_C(X) \) of isomorphism classes of rank \( m \) complex vector bundles over \( X \).*

Therefore, the problem of the enumeration of the topological phases of a TQS can be converted into the classical problem in topology of the classification of \( \text{Vec}^m_C(X) \). The important result due to F. P. Peterson [Pet] establishes that this classification can be achieved by using the Chern classes which take values in the cohomology groups \( H^{2k}(X, \mathbb{Z}) \). In particular, in *low dimension* the classification is completely specified by the first Chern class \( c_1 \), i.e.

\[ c_1 : \text{Vec}^m_C(X) \xrightarrow{\cong} H^2(X, \mathbb{Z}) , \quad \forall \ m \in \mathbb{N} \text{ if } d \leq 3 . \]  

(1.2)

TQS of type (1.1) are ubiquitous in mathematical physics [BMKNZ, CJ]. They can be used to model systems subjected to *cyclic adiabatic processes* in classical and quantum mechanics [Pan, Ber], or in the description of the *magnetic monopole* [Dir, Yan] and the *Aharonov-Bohm effect* [AB], or in the molecular dynamics in the context of the *Born-Oppenheimer approximation* [Bae, FZ, GR], just to mention some important example.

---

1In this work we will assume that \( X \) is a topological space with the homotopy type of a finite CW complex. The *dimension* \( d \) of \( X \) is, by definition, the maximal dimension of its cells. We will say that \( X \) is low dimensional if \( 0 \leq d \leq 3 \).
A very important example of TQS comes from the Condensed Matter Physics and concerns the dynamics of (independent) electrons in a periodic background (a crystal). In this case the Bloch-Floquet formalism \cite{AM, Kuc} allows us to decompose the Schrödinger operator in a parametric family of operators like in (1.1) labelled by the points of a torus $X = \mathbb{T}^d$ ($d = 1, 2, 3$), called Brillouin zone. In this particular case the classification of the topological phases is completely specified by $H^2(\mathbb{T}^d, \mathbb{Z})$ due to (1.2), and the different topological phases are interpreted as the distinct quantized values of the Hall conductance by means of the celebrated Kubo-Chern formula \cite{TKNN, BES}. The last result is the core of the theoretical explanation of the Quantum Hall Effect which is the prototypical example of topological insulating phases. Nowadays the study of topologically protected phases of topological insulators is a mainstream topic in Condensed Matter Physics (see the reviews \cite{HK} and \cite{AF} for a vast overview on the subject).

The problem of the classification of the topological phases becomes more interesting, and challenging, when the TQS is constrained by certain (pseudo-)symmetries like the time-reversal symmetry (TRS). A system like (1.1) is said to be time-reversal symmetric if there is an involution $\tau : X \to X$ on the base space and an anti-unitary map $\Theta$ such that
\[
\begin{align*}
\Theta H(x) \Theta^* &= H(\tau(x)), & \forall x \in X \\
\Theta^2 &= \epsilon \mathbf{1}_N \\
\epsilon &= \pm 1
\end{align*}
\] (1.3)
where $\mathbf{1}_N$ denotes the $N \times N$ identity matrix. Let us point out that in the definition above a crucial role is played by the pair $(X, \tau)$ which is called involutive space\footref{1}. Its fixed point set will be denoted by $X^\tau := \{ x \in X : \tau(x) = x \}$.

The case $\epsilon = +1$ corresponds to an even TRS. In this case, the spectral bundle $\mathcal{E}$ turns out to be equipped with an additional structure \cite[Section 2]{DG1}, which makes it a “Real” vector bundle in the sense of Atiyah \cite{Ati1}. Therefore, in the presence of an even TRS the classification problem of the topological phases of a TQS is reduced to the study of the set $\text{Vec}^m_{\mathcal{R}}(X, \tau)$ of equivalence classes of rank $m$ “Real” vector bundles over the involutive space $(X, \tau)$. Also in this case, there is a complete classification in low-dimension given by
\[
c_1^{\mathcal{R}} : \text{Vec}^m_{\mathcal{R}}(X, \tau) \to H^2_{\mathcal{Z}_2}(X, \mathbb{Z}(1)),
\] (1.4)
which provides the generalization of (1.2). The cohomology appearing in (1.4) is the equivariant Borel cohomology of the involutive space $(X, \tau)$ with local coefficient system $\mathbb{Z}(1)$ (see Appendix A and references therein for more details). The isomorphism (1.4), called Kahn’s isomorphism, is induced by the “Real” Chern classes $c_1^{\mathcal{R}}$ as defined in \cite{Kah}. Its proof follows from \cite[Proposition 1]{Kah} (see also \cite[Corollary A.5]{Gom}) along with the stable condition for “Real” vector bundles \cite[Theorem 4.25]{DG1}.

\footref{1}In this case we will assume that $(X, \tau)$ has the equivariant homotopy type of a finite $\mathbb{Z}_2$-CW complex in the sense explained in Section 2.2.
The case $\epsilon = -1$ describes an odd TRS. Also in this situation the spectral vector bundle $E$ acquires an additional structure which converts $E$ in a “Quaternionic” vector bundle in the sense of Dupont [Dup]. Therefore, the topological phases of a TQS with an odd TRS are labelled by the set $\text{Vec}^m_{\mathbb{Q}}(X, \tau)$ of equivalence classes of “Quaternionic” vector bundles over $(X, \tau)$. The study of systems with an odd TRS is generally more interesting, and usually harder, than the even case. Historically, the fame of these “fermionic-type” systems is related with the seminal papers [KM, FKM] where the Quantum Spin Hall Effect is interpreted as the manifestation of a non-trivial topology for TQS constrained by an odd TRS.

Due to the relevance of these systems, it would be certainly important to have a formula for the classification of low-dimensional “Quaternionic” vector bundles, which generalizes the classifications (1.2) for complex vector bundles, or the classifications (1.4) for “Real” vector bundles. Such an achievement is precisely the main result of this work. It can be stated as

$$\kappa : \text{Vec}^m_{\mathbb{Q}}(X, \tau) \xrightarrow{\sim} H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)),$$ \hspace{1cm} \forall \ m \in \mathbb{N} \text{ if } d \leq 3 \hspace{1cm} (1.5)$$

and is proved in Theorem 3.7. The cohomology group in the right-hand side of (1.5) is the second Borel equivariant cohomology group of the pair $(X, X^\tau)$ with coefficients in the local system $\mathbb{Z}(1)$ (see Appendix A). The element $\kappa$ is called FKMM invariant and has been studied in [DG2, DG3, DG4]. By virtue of the isomorphism (1.5), and its comparison with (1.2) and (1.4), we can reformulate our main result as follow:

The FKMM-invariant is the fundamental characteristic class for the category of “Quaternionic” vector bundles in the sense that it completely classifies “Quaternionic” vector bundles in low-dimension $(d \leq 3)$.

Let us spend a few words about the history of the isomorphism (1.5). First of all it is worth noting that odd-rank “Quaternionic” vector bundles can be defined only when $X^\tau = \emptyset$, meaning that the $\mathbb{Z}_2$-action induced by $\tau$ on $X$ is free. In such a case the relative cohomology group in (1.5) reduces to the ordinary cohomology group $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1))$ and the result has been proven in [DG4, Theorem 1.2 & Theorem 1.3 (i)]. Henceforth, let us assume $X^\tau \neq \emptyset$, and consequently $m \in 2\mathbb{N}$. Initially, the isomorphism (1.5) has been established in the case of certain special involutive spaces called time-reversal spheres and tori (see Example 2.3) in [DG2]. The generalization to the case of spheres and tori with all possible involutions has been obtained in [DG3]. The isomorphism (1.5) in the case of a general involutive space with the homotopy type of a $\mathbb{Z}_2$-CW complex of dimension $d \leq 2$ has been proven in [DG4, Theorem 1.3 (ii)]. However, in the same claim it is also stated that the map (1.5) in dimension $d = 3$ is only injective. The latter claim was based on the study of a potential counterexample that apparently violates (1.5) in dimension $d = 3$ [DG4, Section 5]. It turns out that one of the key argument for the construction of the counterexample results wrong [DG4, Lemma 5.3], and in fact the isomorphism (1.5) is valid also in this case as checked in Section 4.
On the one hand, the new contribution of this work consists in the correct extension of [DG4, Theorem 1.3 (ii)] to the case \( d = 3 \). However, this is not the only benefit. In fact the technique used in this work to achieve Theorem 3.7, and in turn (1.5), is completely different from that used in [DG4], and sheds a new light on the nature of the FKMM invariant.

In equivariant homotopy theory, the most basic equivariant cohomology theory is the so-called *Bredon equivariant cohomology* [Bre], rather than the Borel equivariant cohomology that appears in the definition of the FKMM invariant. There are, of course, some relationship between these two equivariant cohomology theories. A crucial step in our analysis consists in showing that the Borel equivariant cohomology \( H_2^G(X|X^e,\mathbb{Z}(1)) \) can be indeed interpreted as a Bredon equivariant cohomology (Theorem 3.3). This discovery, along with the specific properties of the Bredon equivariant cohomology, allow us to derive the isomorphism (1.5) along an elegant, and conceptually new path.

**Structure of the paper.** In Section 2, we introduce the basic concepts concerning the Bredon equivariant cohomology. Section 3 is devoted to the study of the FKMM invariant from the viewpoint of the Bredon equivariant cohomology and contains the proofs of Theorem 3.3 and Theorem 3.7. In Section 4, we test our main result, i.e. the isomorphism (1.5), for the 3-dimensional lens space. In particular this gives us a way to amend some errors contained in [DG4]. Appendix A, written for the benefit of the reader, provides a soft introduction to the Borel equivariant cohomology. Finally, Appendix B contains some technical result needed for the proof of Theorem 3.7.

**Acknowledgements.** GD’s research is supported by the grant Fondecyt Regular - 1190204. KG is supported by JSPS KAKENHI Grant Numbers 20K03606.

2. **Basic facts about the Bredon cohomology**

This section is devoted to the introduction of the *Bredon (equivariant) cohomology*. The main reference is Bredon’s original work [Bre], but also the monograph [May] and the paper [Ill] will be useful. Since we will need some concepts of category theory we will refer to the monographs [Fre, MLA] as a useful reference. Let \( G \) be a topological group. By a \( G \)-space \( X \) we mean a topological space \( X \) together with a (left) action of \( G \) on \( X \) by homeomorphisms. The expression *involutive space* will be used as a synonym for \( \mathbb{Z}_2 \)-space. In its full generality the Bredon equivariant cohomology can be defined for \( G \)-spaces with the action of a compact Lie group \( G \). However, for the aims of this work, it will be enough (when necessary) to restrict our interest to the simpler case where \( G \) is a finite group.

2.1. **Orbit category.** Let \( \mathbb{H} \) be a closed subgroup of \( G \) and denote with \( G/\mathbb{H} \) the corresponding (left) coset space. We will write \( [g]_\mathbb{H} \in G/\mathbb{H} \) to mean the element represented by \( g \in G \), namely \( [g]_\mathbb{H} := \{gh \mid h \in \mathbb{H}\} \). Therefore, by definition, \( [g]_\mathbb{H} = [gh]_\mathbb{H} \) for any \( h \in \mathbb{H} \). The operation \( g' \cdot [g]_\mathbb{H} := [g'g]_\mathbb{H} \), with \( g' \in G \), shows that \( G \) acts (on the
left) on $G/H$, and under this action $G/H$ turns out to be an homogeneous space. Let $H$ and $K$ be closed subgroups of $G$. A map $\phi : G/H \to G/K$ is called $G$-equivariant if $\phi(g' \cdot [g]_H) = g' \cdot \phi([g]_H)$ for all $g, g' \in G$. The family of $G$-equivariant maps admits a simple description as proved in [Bre, Section I.3]. In fact any $G$-equivariant map $\phi : G/H \to G/K$ is realized as $\phi([g]_H) = [g\alpha]_K$, where the element $\alpha \in G$ meets $\alpha^{-1}Ha \subseteq K$.

**Definition 2.1.** Let $G$ be a topological group. The orbit category $\text{Orb}_G$ is defined as the category such that the objects are the $G$-space $G/H$, where $H \subseteq G$ are closed subgroups, and the morphisms are $G$-equivariant maps.

Let $X$ be a $G$-space and denote with $O_G(x) := \{g \cdot x \mid g \in G\}$ the orbit of the action through the point $x \in X$. It turns out that $O_G(x) := \{g \cdot x \mid g \in G\}$ is isomorphic, as a topological $G$-space, to a coset space $G/H_x$ where $H_x := \{g \in G \mid g \cdot x = x\}$ is the stabilizer group of the point $x$. Therefore, the orbit category of a group $G$ gives the category of “all kinds” of orbits of $G$.

We will write $X^H \subseteq X$ for the subspace consisting of all the points fixed by the $H$-action, i.e. $X^H := \{x \in X \mid hx = x \ \forall h \in H\}$. It turns out that, if there is $\alpha \in G$ such that $\alpha^{-1}Ha \subseteq K$, then there is a map $X^K \to X^H$ given by $x \mapsto \alpha x$.

**Example 2.2 (Orbit category of $\mathbb{Z}_2$).** In view of its relevance for this work, it will be useful to construct the orbit category of the cyclic group of order two $\mathbb{Z}_2 = \{\pm 1\}$. In this case, there are only two possible subgroups of $\mathbb{Z}_2$, namely $H_0 := \{+1\}$ and $H_1 := \mathbb{Z}_2$. Therefore, the associated coset spaces $Z_i := Z_2/H_i$, with $i = 0, 1$, have a simple description: $Z_0 = \{z_+, z_-\}$ is a two-point space (identifiable with $\mathbb{Z}_2$), while $Z_1 = \{z\}$ is a singleton. These two spaces are the only objects of the orbit category of $\text{Orb}_{\mathbb{Z}_2}$. To complete the description of $\text{Orb}_{\mathbb{Z}_2}$, we need to describe all the possible $\mathbb{Z}_2$-equivariant maps between the objects. In addition to the two identity maps $\text{Id}_i : Z_i \to Z_i$, with $i = 0, 1$, induced by the action of $+1$, there are two more non-trivial maps. The space $Z_0$ admits a second internal map $\varphi : Z_0 \to Z_0$ defined by $\varphi(z_0) := z_0$, which is induced by the action of $-1$. Then, there is the map $\psi : Z_0 \to Z_1$ defined by $\psi(z_{\pm}) := z$, which is induced by the action of $\pm 1$, irrespectively, in view of the fact that $\alpha^{-1}H_0\alpha = H_0$ for every choice $\alpha = \pm 1$. The category $\text{Orb}_{\mathbb{Z}_2}$ can be represented by the following diagram:

$$\text{Id}_0, \varphi \quad \varphi^{-1} \quad Z_0 \xrightarrow{\psi} Z_1 \xleftarrow{\text{Id}_1} .$$

The fact that there are no equivariant maps from $Z_1$ to $Z_0$ is a consequence of the fact that $\alpha^{-1}H_1\alpha = \mathbb{Z}_2$, for every $\alpha = \pm 1$, can never be a subset of $H_0$.

### 2.2. Equivariant CW complexes

Let $G$ be a finite group. A $G$-equivariant CW complex, or $G$-CW complex for short, is a $G$-space which is homotopy equivalent to a CW-complex where the usual $n$-dimensional cells modeled by the $n$-dimensional disk $D^n$ are replaced by $G$-equivariant cells of the form $G/H \times D^n$ for some subgroup $H \subseteq G$. 

For a precise definition of $\mathbb{G}$-CW complex we refer to [AP, Definition 1.1.1] or [Bre, Section I.1] Since the notion of $\mathbb{G}$-CW complex is modeled after the usual definition of CW-complex after replacing the “cells” by “$\mathbb{G}$-cells” it follows that many topological and homological properties of CW-complexes have their “natural” counterparts in the equivariant setting.

**Example 2.3** (Time-reversal spheres and tori). Special examples of $\mathbb{Z}_2$-spaces are provided by the spheres $S^d \subset \mathbb{R}^{d+1}$ (of radius 1) endowed with the time-reversal involution $	au: (x_0, x_1, \ldots, x_d) \mapsto (x_0, -x_1, \ldots, -x_d)$. A similar type of involution (still called $\tau$) can be defined for tori $T^d := S^1 \times \ldots \times S^1$ just acting with the time-reversal involution on each copy of the 1-sphere. The $\mathbb{Z}_2$-spaces $(S^d, \tau)$ and $(T^d, \tau)$ are $\mathbb{Z}_2$-CW complexes, and their structures have been explicitly described in [DG1, Section 4.5].

### 2.3. Coefficient systems. Let $\mathbb{Ab}$ be the category of abelian groups and $\mathbb{G}$ a finite group. Following [Bre, Section I.4], we introduce the following concept:

**Definition 2.4.** A coefficient system for $\mathbb{G}$ is a contravariant functor $\mathcal{M} : \text{Orb}_G \to \mathbb{Ab}$.

If $\mathcal{M}, \mathcal{N} : \text{Orb}_G \to \mathbb{Ab}$ are coefficient systems, a morphism $F : \mathcal{M} \to \mathcal{N}$ is a natural transformation of functors. In this way the collection of coefficient systems for $\mathbb{G}$ form an abelian category denoted with $\text{CoSy}_G$.

Let us provide some basic examples of coefficient systems.

**Example 2.5.** Let $X$ be a (path connected) $\mathbb{G}$-space which admits a fixed point $x_a$. For $n \in \mathbb{N}$, the coefficient system $\pi_n[X]$ is defined as

$$\pi_n[X] : \mathbb{G}/\mathbb{H} \to \pi_n(X^\mathbb{H})$$

for every objects $\mathbb{G}/\mathbb{H} \in \text{Orb}_G$, where $\pi_n(X^\mathbb{H})$ is the $n$-th homotopy group of the space $X^\mathbb{H}$ computed with respect to the fixed point $x_a$ (which is evidently contained in each $X^\mathbb{H}$). Let us notice that $\pi_n(X^\mathbb{H}) \in \mathbb{Ab}$ whenever $n \geq 2$. However, for the case $n = 1$ the condition $\pi_1(X^\mathbb{H}) \in \mathbb{Ab}$ amounts to an assumption on the $\mathbb{G}$-space $X$. If there is a $\mathbb{G}$-equivariant map $\phi : \mathbb{G}/\mathbb{H} \to \mathbb{G}/\mathbb{K}$ realized as $\phi([x]_\mathbb{H}) = [xa]_\mathbb{K}$ by an element $a \in \mathbb{G}$ such that $a^{-1}H \alpha \subseteq \mathbb{K}$, then the map $f_\phi : X^\mathbb{K} \to X^\mathbb{H}$ given by $f_\phi(x) := ax$ induces a map $f_{\phi,*} : \pi_n(X^\mathbb{K}) \to \pi_n(X^\mathbb{H})$ between the homotopy groups. Identifying $f_{\phi,*}$ with the image of $\phi$ under the functor $\pi_n[X]$ one gets that

$$\pi_n[X](\phi) : \pi_n[X](\mathbb{G}/\mathbb{K}) \to \pi_n[X](\mathbb{G}/\mathbb{H})$$

showing that $\pi_n[X]$ is contravariant.

**Example 2.6.** Let $X$ be a $\mathbb{G}$-CW complex. For any non-negative integer $n \in \{0\} \cup \mathbb{N}$, we write $X_n \subset X$ for the $n$-skeleton of $X$. Then, we have a coefficient system $\mathbb{C}_n[X]$ defined by

$$\mathbb{C}_n[X] : \mathbb{G}/\mathbb{H} \to H^C_n(X^\mathbb{H}) := H_n(X^\mathbb{H}_n|X^\mathbb{H}_{n-1}, \mathbb{Z})$$

---

For the definition of abelian category we will refer to [MLa, Chapter VIII] or [Fre].
where on the right hand side there is the $n$-th cellular homology group of $X^H$, which is naturally isomorphic to the relative singular homology (with integer coefficient) of the pair $(X^H_{n-1}, X^H_n)$. The fact that $C_n[X]$ is contravariant can be checked exactly as in Example 2.5. The connecting homomorphism in the exact sequence for the triple $(X^H_n, X^H_{n-1}, X^H_{n-2})$ provides a homomorphism

$$
\partial_n : H_n^{CW}(X^H) \rightarrow H_{n-1}^{CW}(X^H)
$$

which is natural. Therefore it induces a natural transformation $\partial_n : C_n[X] \rightarrow C_{n-1}[X]$ such that $\partial_{n-1} \circ \partial_n = 0$. In other words, $C_n[X]$ gives rise to a contravariant functor from $\text{Orb}_G$ to the category $\text{ChComp}$ of chain complexes.

Example 2.7. Let $h^* = \{h^n\}_{n \in \mathbb{Z}}$ be a $G$-equivariant generalized cohomology theory. By this it is meant that $h^*$ satisfies the straightforward $G$-equivariant generalization of the standard Eilenberg-Steenrod axioms of a generalized cohomology theory [ES, Section I.3]. A more detailed explanation is given in [Bre, Section I.2]. With this, we can define the coefficient system $h^n$, with $n \in \mathbb{Z}$, simply by

$$
h^n : G/H \rightarrow h^n(G/H)
$$

in view of the fact every $h^n$ is a contravariant functor from the category of (pointed) topological spaces to $\text{Ab}$, by definition.

Example 2.8 (Coefficient systems for $\mathbb{Z}_2$). Let us now discuss in some details the structure of the coefficient systems for $\mathbb{Z}_2$ by using the notation introduced in Example 2.2. In this case any coefficient system $\mathcal{M} : \text{Orb}_{\mathbb{Z}_2} \rightarrow \text{Ab}$ is specified by two abelian groups $\mathcal{M}(Z_i)$, with $i = 1, 2$, together with two homomorphisms (in addition to the identity homomorphisms)

$$
\mathcal{M}(\phi_-) : \mathcal{M}(Z_0) \rightarrow \mathcal{M}(Z_0), \quad \mathcal{M}(\psi) : \mathcal{M}(Z_1) \rightarrow \mathcal{M}(Z_0).
$$

Notice that the direction of $\mathcal{M}(\psi)$ is reversed with respect to the direction of $\psi$ since $\mathcal{M}$ must be contravariant. The map $\mathcal{M}(\phi_-)$ endows $\mathcal{M}(Z_0)$ with a $\mathbb{Z}_2$-action, let $\mathcal{M}(Z_0)^{\mathbb{Z}_2} \subseteq \mathcal{M}(Z_0)$ be the subset of invariant points under this action. Observing that $\psi \circ \phi_- = \psi$ one gets that $\mathcal{M}(\phi_-) \circ \mathcal{M}(\psi) = \mathcal{M}(\psi \circ \phi_-) = \mathcal{M}(\psi)$ one obtains that the image of $\mathcal{M}(Z_1)$ under $\mathcal{M}(\psi)$ is made by invariant points. In summary, we showed that a coefficient system $\mathcal{M}$ for $\mathbb{Z}_2$ consists of: (i) an abelian group $\mathcal{M}(Z_1)$; (ii) an abelian group $\mathcal{M}(Z_0)$ endowed with a $\mathbb{Z}_2$-action; (iii) a homomorphism $\mathcal{M}(Z_1) \rightarrow \mathcal{M}(Z_0)^{\mathbb{Z}_2}$. In the following we will use the symbol $\mathcal{M}(Z_1) \rightarrow \mathcal{M}(Z_0)^{\mathbb{Z}_2}$ as a synthetic description of the coefficient system $\mathcal{M}$.

2.4. Bredon equivariant cohomology. Let $G$ be a finite group and $X$ a $G$-CW complex. Since the category $\text{CoSy}_G$ of coefficient systems for $G$ is an abelian category, the Hom-set

$$
\text{Hom}_{\text{CoSy}_G}(N, \mathcal{M})
$$

where $N$ is a $G$-CW complex and $\mathcal{M}$ is a coefficient system for $G$, is an abelian group and hence a $G$-equivariant generalized cohomology theory. In particular, it satisfies the straightforward $G$-equivariant generalization of the standard Eilenberg-Steenrod axioms of a generalized cohomology theory [ES, Section I.3].
is an abelian group [Fre, Theorem 2.39] for any pair of coefficient systems $\mathcal{N}$ and $\mathcal{M}$. Now, let us take $\mathcal{N}$ be the coefficient system $C_n(X)$ described in Example 2.6 and define

$$C^n_G(X; \mathcal{M}) := \text{Hom}_{\text{CoSy}_G}(C_n(X), \mathcal{M}).$$

The natural transformation $\delta_n : C_n(X) \Rightarrow C_{n-1}(X)$ induces a homomorphism

$$\delta_n : C^n_G(X; \mathcal{M}) \rightarrow C^{n+1}_G(X; \mathcal{M})$$

satisfying $\delta_{n+1} \circ \delta_n = 0$. This leads to a cochain complex $(C^*_G(X; \mathcal{M}), \delta_*)$.

**Definition 2.9.** Let $G$ be a finite group and $X$ a $G$-CW complex. For any coefficient system $\mathcal{M} \in \text{CoSy}_G$, the $n$-th Bredon $G$-equivariant cohomology of $X$ with coefficients in $\mathcal{M}$ is defined as the $n$-th cohomology of the cochain complex $(C^*_G(X; \mathcal{M}), \delta_*)$, i.e.

$$H^n_G(X, \mathcal{M}) := \text{Ker}(\delta_n)/\text{Im}(\delta_{n-1}).$$

As a matter of fact, for any space $X$ with $G$-action, there is a $G$-CW complex which is $G$-equivariantly weakly homotopy equivalent to $X$ [May]. Using such a $G$-CW complex, the Bredon equivariant cohomology of $X$ is defined. Given an invariant subspace $Y \subseteq X$, it is possible to introduce the relative cohomology $H^n_G(X|Y, \mathcal{M})$. Then the Bredon equivariant cohomology groups with coefficients in $\mathcal{M}$ constitute a $G$-equivariant generalized cohomology theory.

**Example 2.10.** In general, Bredon cohomology is difficult to calculate following the definition. However, in a special case, one can compute it. For example, let $\mathcal{M}$ be the coefficient system such that $\mathcal{M}(G/H) = 0$ for any proper subgroup $H \subset G$. Hence, only $\mathcal{M}(G/G)$ can be non-trivial, and the homomorphisms $\mathcal{M}(\phi)$ are automatically determined. In particular, the $G$-action on $\mathcal{M}(G/G)$ is trivial. In this case, the cochain complex $\text{C}^*_G(X; \mathcal{M})$ is identified with the cellular cochain complex of the fixed point set $X^G$, i.e.

$$\text{C}^*_G(X; \mathcal{M}) \simeq \text{C}^n(X^G; \mathcal{M}(G/G)).$$

It follows that $H^n_G(X, \mathcal{M}) \simeq H^n(X^G, \mathcal{M}(G/G))$. □

### 2.5. Eilenberg-Mac Lane spaces.

The ordinary cohomology $H^n(X, \mathbb{A})$ of a CW-complex $X$ with coefficients in an abelian group $\mathbb{A}$ can be represented as

$$H^n(X, \mathbb{A}) \simeq [X, K(\mathbb{A}, n)],$$

where $K(\mathbb{A}, n)$ is the *Eilenberg-Mac Lane space* of type $(\mathbb{A}, n)$. There exists a parallel representation for the Bredon equivariant cohomology. For simplicity, let $n \geq 1$. Given a coefficient system $\mathcal{M} : \text{Orb}_G \rightarrow \mathbb{A}$, the Eilenberg-Mac Lane space of type $(\mathcal{M}, n)$ is a path connected $G$-space $K(\mathcal{M}, n)$ such that

$$\mathfrak{M}_k(K(\mathcal{M}, n)) = \begin{cases} \mathcal{M} & (k = n) \\ 0 & (k \neq n) \end{cases}.$$
where $\pi_k$ has been described in Example 2.5, and the base point of the homotopy groups is chosen in the set of fixed points of $K(M, n)$ (therefore $K(M, n)^G \neq \emptyset$ is assumed implicitly). It is known [Bre, Section II.6] that there exists a unique $G$-space, up to $G$-equivariant (weak) homotopy equivalence, which provides the identification

$$\mathcal{H}_G^n(X, M) \simeq [X, K(M, n)]_G,$$

(2.1)

where the symbol in the right hand side denotes the set of $G$-equivariant homotopy classes of $G$-equivariant maps $X \to K(M, n)$. By construction, for every $G$-equivariant map $f : X \to K(M, n)$ there exists an element $\chi^n(f) \in \mathcal{H}_G^n(X, M)$ which is the image of the class $[f] \in [X, K(M, n)]_G$ under the isomorphism (2.1). The element $\chi^n(f)$ is called the characteristic class of $f$ according to the definition given in [Bre, Section II.3]. If $\text{id} : K(M, n) \to K(M, n)$ is the identity map, then the corresponding element $\iota := \chi^n(\text{id})$ will be called the universal Bredon class. In view of [Bre, Chapter II, Proposition 3.2] one has that

$$\chi^n(f) = \chi^n(f \circ \text{id}) = f^* \chi^n(\text{id}) = f^* \circ \iota.$$

Therefore, the isomorphism (2.1) is realized by $[f] \mapsto f^* \circ \iota$.

**Remark 2.11.** In the special case $\mathcal{H}_G^n(K(M, n); M) \simeq Z$, the defining property of the universal Bredon class implies that $\iota$ must be a generator identifiable with $+1$ or $-1$. This can be proved by contradiction. Let $\alpha$ be an isomorphism between $\mathcal{H}_G^n(K(M, n); M)$ and $Z$, and assume that $m := \alpha(\iota) \neq \pm 1$. For $t' \in \mathcal{H}_G^n(K(M, n); M)$ such that $\alpha(t') = 1$, there exists a $G$-equivariant map $f : K(M, n) \to K(M, n)$ such that $f^* \circ \iota = t'$. Consequently, one can construct the homomorphism $\alpha \circ f^* \circ \alpha^{-1} : Z \to Z$ which maps $m$ to 1. But this contradicts the fact that $m \neq \pm 1$.

**Example 2.12.** Let $A$ be an abelian group, and $\underline{A}$ the constant coefficient system such that $\underline{A}(G/H) = A$ for all $G/H \in \mathcal{O}r_G$, and $\underline{A}(\phi)$ is the identity map for any morphism $\phi$ in $\mathcal{O}r_G$. Let $K(A, n)$ be the usual Eilenberg-Mac Lane space of type $(A, n)$, namely a path connected space such that $\pi_k(K(A, n)) = A$ if $k = n$, and $\pi_k(K(A, n)) = 0$ otherwise. Let $G$ act on $K(A, n)$ trivially. Then $K(A, n)^H = K(A, n)$ for any subgroup $H \subseteq G$. Hence, the $G$-space $K(A, n)$ realizes $K(\underline{A}, n)$. It follows that

$$\mathcal{H}_G^n(X, \underline{A}) \simeq [X, K(\underline{A}, n)]_G \simeq [X/G, K(\underline{A}, n)] \simeq H^n(X/G, \underline{A})$$

for any $G$-space $X$. In other words, the Bredon equivariant cohomology of a $G$-space $X$ with coefficients in $\underline{A}$ agrees with the ordinary cohomology of the quotient space $X/G$ with coefficients in $A$.

**2.6. Atiyah-Hirzebruch spectral sequence.** Let $X$ be a finite CW complex with base point $*$, and $h^* = \{h^n\}_{n \in \mathbb{Z}}$ a generalized cohomology theory satisfying the Eilenberg-Steenrod axioms [ES, Section I.3]. Then the Atiyah-Hirzebruch spectral sequence is a spectral sequence such that its $E_2$-term is

$$E_2^{p,q} = H^p(X, h^q(*))$$
and converges to a graded quotient of $h^*(X)$ [Spa, Chapter 9] or [DK, Chapter 9]. The construction of the Atiyah-Hirzebruch spectral sequence can be generalized to $G$-CW complexes, and one gets:

**Theorem 2.13 ([Bre, Section IV.4]).** Let $\mathbb{G}$ be a finite group, $X$ a finite $\mathbb{G}$-CW complex, and $h^*$ a $\mathbb{G}$-equivariant generalized cohomology theory. Then there is an Atiyah-Hirzebruch spectral sequence such that its $E_2$-term is

$$E_2^{p,q} = \mathcal{H}^p_G(X, h^q),$$

where the coefficient system $h^q$ has been described in Example 2.7, and which converges to a graded quotient of $h^*(X)$.

**Example 2.14.** As a trivial application, let us take a generalized cohomology theory $h^*$, and define an equivariant generalized cohomology theory $h^*_G$ by $h^n_G(X) := h^n(X/\mathbb{G})$ through the quotient $X/\mathbb{G}$ of the $G$-CW complex $X$. The coefficient system $h^*_G$ turns out to be the constant coefficient system $h^n(\{\ast\})$. Then the $E_2$-term of the spectral sequence is

$$E_2^{p,q} = \mathcal{H}^p_G(X, h^q) = \mathcal{H}^p(X, h^n(\{\ast\})) = H^p(X/\mathbb{G}, h^q(\{\ast\})),
$$

and reduces to the usual Atiyah-Hirzebruch spectral sequence of the orbit $X/\mathbb{G}$. 

### 3. The FKMM Invariant and Bredon Cohomology

This section is devoted to the study of the FKMM invariant from the viewpoint of the Bredon equivariant cohomology. This will provide the framework for the proof of the main Theorem 3.7.

#### 3.1. Borel and Bredon equivariant cohomology

The first necessary step is to relate the Borel equivariant cohomology with the Bredon equivariant cohomology. In order to help the reader, a short review about the Borel equivariant cohomology is presented in Appendix A. In the following we will focus on the case of a $\mathbb{Z}_2$-space $X$ with involution $\tau$. The set of the fixed point will be denoted with $X^\tau$. A $\mathbb{Z}_2$-CW pair $(X, Y)$ is given by a $\mathbb{Z}_2$-CW complex $X$ a $\mathbb{Z}_2$-CW subcomplex $Y$ and a sub-complex inclusion $Y \hookrightarrow X$. In particular one has that $(X, X^\tau)$ is a $\mathbb{Z}_2$-CW pair. Given a $\mathbb{Z}_2$-CW pair $(X, Y)$, we will denote with $H^*_G(X|Y, \mathbb{Z})$ the equivariant Borel cohomology of $X$ relative to $Y$ with local coefficients in $\mathbb{Z}$. By convention we will fix $H^n_{\mathbb{Z}_2}(X|Y, \mathbb{Z}) = 0$ for every $n < 0$.

**Lemma 3.1.** For any $\mathbb{Z}_2$-CW pair $(X, Y)$ and $n \in \mathbb{Z}$, we use the Borel equivariant cohomology to define

$$h^n(X, Y) := H^p_{\mathbb{Z}_2}(X|Y \cup X^\tau, \mathbb{Z}(j)), \quad j = 0, 1.$$ 

Then $h^*_0 := \{h^0_n\}_{n \in \mathbb{Z}}$ and $h^*_1 := \{h^1_n\}_{n \in \mathbb{Z}}$ are $\mathbb{Z}_2$-equivariant generalized cohomologies.

**Proof.** The Borel equivariant cohomology is an equivariant generalized cohomology [AP, Theorem 1.2.6], meaning that it satisfies the equivariant version of the Eilenberg-Steenrod
axioms of a generalized cohomology theory. Thus, for any $\mathbb{Z}_2$-CW complex $X$ and its subcomplexes $Y, Y' \subset X$, one has the exact sequence for the triad $(X, Y, Y')$

\[ \cdots \to H^*_{\mathbb{Z}_2}(X|Y \cup Y', \mathbb{Z}) \to H^*_{\mathbb{Z}_2}(X|Y, \mathbb{Z}) \to \cdots \]

where the isomorphism

\[ H^*_{\mathbb{Z}_2}(Y \cup Y'|Y', \mathbb{Z}) \cong H^*_{\mathbb{Z}_2}(Y|Y', \mathbb{Z}) \]

has been used in third place. Setting $Y' = X^r$ and $Z = \mathbb{Z}(j)$ with $j = 0$ or $j = 1$, one gets the exact sequence

\[ \cdots \to h^n_i(X, Y) \to h^n_i(X, \emptyset) \to h^n_i(Y, \emptyset) \to h^{n+1}_i(X, Y) \to \cdots \]

which shows that $h^n_i$ is subject to the exactness axiom. The other axioms for the equivariant generalized cohomology theory for $h^n_i$ follow from those of $H^*_{\mathbb{Z}_2}$. 

The group $\mathbb{Z}_2$ can act on $\mathbb{Z}$ in two ways. There is the trivial action induced by $-1 \in \mathbb{Z}_2$ on $n \in \mathbb{Z}$ by $-1 \cdot n \mapsto n$, and there is the flip action given by $-1 : n \mapsto -n$. We will write $\tilde{\mathbb{Z}}$ for the group $\mathbb{Z}$ endowed with the flip action and we will use the symbol $\mathbb{Z}$ for the case of the trivial action. Using the notation introduced in Example 2.8 we will introduce three coefficient systems for $\mathbb{Z}_2$. The first one, denoted with $0 \sim Z$, consists in identifying both the orbits $Z_0$ and $Z_1$ with the trivial group 0. In the second coefficient system $0 \sim \tilde{\mathbb{Z}}$ the orbit $Z_0$ is again identified with 0 while the orbit $Z_1$ is identified with $\tilde{\mathbb{Z}}$ (trivial action). The last coefficient system $0 \sim \mathbb{Z}$ is obtained as the previous one, but now $Z_1$ is identified with $\tilde{\mathbb{Z}}$ (flip action).

**Lemma 3.2.** Let $h^n_i$, with $j = 0, 1$, be the equivariant cohomology theory defined in Lemma 3.1 and $h^n_i$ the associated coefficient systems constructed according to the procedure described in Example 2.7. Then, one has that

\[
\begin{align*}
  h^n_0 &= \begin{cases} 0 \sim \mathbb{Z} & (n = 0) \\
                        0 \sim 0 & (n \neq 0), \end{cases} \\
  h^n_1 &= \begin{cases} 0 \sim \tilde{\mathbb{Z}} & (n = 0) \\
                        0 \sim 0 & (n \neq 0). \end{cases}
\end{align*}
\]

**Proof.** First of all let us evaluate $h^n_0$ on the orbit $Z_1 = \mathbb{Z}_2/\mathbb{Z}_2 = \{z\}$ which is a singleton. From its very definition one gets that

\[ h^n_0(Z_1) = h^n_0(\{z\}, \emptyset) = H^n_{\mathbb{Z}_2}(\{z\} | \{z\}, \mathbb{Z}(j)) = 0 \]

where the last (tautological) equality works independently of $n \in \mathbb{Z}$ and $j = 0, 1$. As the second step let us evaluate $h^n_0$ on the orbit $Z_0 = \mathbb{Z}_2/\{+1\} = \{z_+, z_-\}$ which is a two-point space (identifiable with $\mathbb{Z}_2$) on which $\mathbb{Z}_2$ acts freely. For $j = 0$ one has that

\[ h^n_0(Z_0) = h^n_0(Z_0, \emptyset) = H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(0)) = H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}). \]

To compute the Borel cohomology we need to construct the homotopy quotient of $Z_0$ according to (A.1). However, since $\mathbb{Z}_2$ acts freely on $Z_0$ it follows that the homotopy quotient is homotopy equivalent to the regular quotient $Z_0/\mathbb{Z}_2 \simeq \{\ast\}$ which is a singleton.
Then, starting from the last isomorphism in (3.1) one gets
\[ h^n_\mathbb{Z}(Z_0) \simeq H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}) \simeq H^n(\{\ast\}, \mathbb{Z}) = \left\{ \begin{array}{ll} \mathbb{Z} & (n = 0) \\ 0 & (n \neq 0) \end{array} \right. . \] (3.2)
Moreover, the $\mathbb{Z}_2$-action on $\{\ast\}$ induces the trivial action on $H^0(\{\ast\}, \mathbb{Z}) = \mathbb{Z}$. For $j = 1$ the equivalent of (3.1) reads
\[ h^n_\mathbb{Z}(Z_0) = h^n_\mathbb{Z}(Z_0, \emptyset) = H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1)) . \] (3.3)
By using the exact sequence in [Gom, Proposition 2.3] one obtains
\[ \cdots \rightarrow H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}) \rightarrow H^n(Z_0, \mathbb{Z}) \rightarrow H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1)) \rightarrow H^{n+1}_{\mathbb{Z}_2}(Z_0, \mathbb{Z}) \rightarrow \cdots \]
In view of (3.2) one gets
\[ H^n_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1)) \simeq H^n(Z_0, \mathbb{Z}) = 0 , \quad n \neq 0 \]
and
\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H^0_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1)) \rightarrow 0 \]
where the isomorphism $H^0(Z_0, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ for the two-point space $Z_0$ has been used. This implies that
\[ H^0_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1)) \simeq (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \simeq \mathbb{Z} \]
as an abelian group. To identify the $\mathbb{Z}_2$-action on $H^0_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1))$ let us observe that $\mathbb{Z}_2$ acts on $H^0_{\mathbb{Z}_2}(Z_0, \mathbb{Z}) \simeq \mathbb{Z}$ trivially as discussed after (3.2). On the other hand the induced $\mathbb{Z}_2$-action on $H^0(Z_0, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ is given by $(m, n) \mapsto (n, m)$. As a consequence the induced $\mathbb{Z}_2$-action on $H^0_{\mathbb{Z}_2}(Z_0, \mathbb{Z}(1))$ is the non-trivial one. In summary, we showed that
\[ h^n_\mathbb{Z}(Z_0) \simeq \left\{ \begin{array}{ll} \mathbb{Z} & (n = 0) \\ 0 & (n \neq 0) \end{array} \right. . \] (3.4)
and this concludes the proof. \square

We are now in position to compare the Borel and Bredon equivariant cohomology groups.

**Theorem 3.3.** For any finite $\mathbb{Z}_2$-CW complex $X$ and $n \in \mathbb{Z}$, there are natural isomorphisms of groups
\[ H^n_{\mathbb{Z}_2}(X|X^r, Z(0)) \simeq \mathcal{H}^n_{\mathbb{Z}_2}(X, 0 \longrightarrow \mathbb{Z}) \]
\[ H^n_{\mathbb{Z}_2}(X|X^r, Z(1)) \simeq \mathcal{H}^n_{\mathbb{Z}_2}(X, 0 \longrightarrow \widetilde{\mathbb{Z}}) . \]

**Proof.** The Atiyah-Hirzebruch spectral sequence described in Theorem 2.13 provides
\[ E^{p,q}_2 = \mathcal{H}^p_{\mathbb{Z}_2}(X, h^n_q) \Rightarrow h^n_j(X) , \quad j = 0, 1 . \]
By the description of the coefficient system $h^n_\mathbb{Z}$ given in Lemma 3.2, one deduces that the spectral sequence degenerates at $E_2$, and this yields
\[ H^n_{\mathbb{Z}_2}(X|X^r, Z(j)) = h^n_j(X, \emptyset) \simeq E^{p,0}_\infty \simeq E^{p,0}_2 = \mathcal{H}^n_{\mathbb{Z}_2}(X, h^n_0) . \]
The identification of $h^0$ given in Lemma 3.2 concludes the proof. □

3.2. The universal FKMM invariant. Let us review here some cohomological aspects of the universal FKMM invariant as described in [DG2, Section 6] and [DG3, Section 2.6]. For any positive integer $k \in \mathbb{N}$, let

$$
\mathcal{B}_{2k} := \text{Gr}_{2k}(\mathbb{C}^\infty) = \lim_{n \to \infty} \mathbb{U}(2n)/(\mathbb{U}(2k) \times \mathbb{U}(2n - 2k))
$$

be the Grassmannian (or the classifying space) of the unitary group $\mathbb{U}(2k)$ in dimension $2k$. It is known that $\mathcal{B}_{2k}$ is path-connected, which implies $\pi_0(\mathcal{B}_{2k}) = 0$. Moreover, the homotopy groups of $\mathcal{B}_{2k}$ are related to the homotopy groups of $\mathbb{U}(2k)$ by the formula

$$
\pi_n(\mathcal{B}_{2k}) = \pi_{n-1}(\mathbb{U}(2k)) \quad [DG5, \text{eq. A.3}].
$$

In particular this provides $\pi_1(\mathcal{B}_{2k}) = 0$ and $\pi_2(\mathcal{B}_{2k}) \cong \mathbb{Z}$, showing that $\mathcal{B}_{2k}$ is also simply connected. The cohomology ring of $\mathcal{B}_{2k}$ is

$$
H^\bullet(\mathcal{B}_{2k}, \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_{2k}], \quad c_j \in H^j(\mathcal{B}_{2k}, \mathbb{Z}), \quad (3.5)
$$

which is the ring of polynomials with integer coefficients and $2k$ generators $c_j$ of even degree called universal Chern classes [DG5, Appendix A]. Consider the involution $\rho : \mathbb{U}(2k) \to \mathbb{U}(2k)$ given by $\rho U := QUQ^{-1}$, where $Q \in \mathbb{U}(2k)$ is the matrix

$$
Q := \begin{pmatrix}
0 & -1_{c^k} \\
1_{c^k} & 0
\end{pmatrix}
$$

and $\overline{U}$ denotes the complex conjugate of $U$. We also write $\rho : \mathcal{B}_{2k} \to \mathcal{B}_{2k}$ for the involution induced from the involution on $\mathbb{U}(2k)$. Let $\mathcal{T}_{2k}^\mathbb{Z} \to \mathcal{B}_{2k}$ denote the universal (or tautological) vector bundle of rank $2k$ with total space $\mathcal{T}_{2k}^\mathbb{Z} := \lim_{n \to \infty} \mathbb{T}_{2k}^n$ defined through

$$
\mathcal{T}_{2k}^\mathbb{Z} := \{ (\Sigma, v) \in \text{Gr}_{2k}(\mathbb{C}^n) \times \mathbb{C}^n \mid v \in \Sigma \}.
$$

Since $\mathcal{T}_{2k}^\mathbb{Z}$ serves as the universal “Quaternionic” vector bundle of rank $2k$, it turns out that the $\mathbb{Z}_2$-space $\mathcal{B}_{2k}$ is the classifying space of “Quaternionic” vector bundles of rank $2k$ [DG2, Theorem 2.4]. The universal FKMM invariant is the FKMM invariant of the universal bundle, i.e.

$$
\kappa_{\text{univ}} := \kappa(\mathcal{T}_{2k}^\mathbb{Z}) \in H^2_{\mathbb{Z}_2}(\mathcal{B}_{2k}, \mathbb{B}_{2k, 1}^0, \mathbb{Z}(1)).
$$

For any $\mathbb{Z}_2$-CW complex $X$ and $n \in \mathbb{Z}$, the inclusion $j : (X, \emptyset) \to (X, X^T)$ induces a natural homomorphism

$$
(j^* : H^n_{\mathbb{Z}_2}(X|X^T, \mathbb{Z}(1)) \to H^n_{\mathbb{Z}_2}(X, \mathbb{Z}(1))
$$

which fits into the exact sequence (A.3) of the pair $(X, X^T)$. There is also a natural homomorphism

$$
f : H^n_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) \to H^n(X, \mathbb{Z})
$$

forgetting the $\mathbb{Z}_2$-action, which also fits into an exact sequence [Gom, Proposition 2.3]. If $X$ is path connected, then the Hurewicz homomorphism (in homology) induces a natural
homomorphism

\[ H^n(X,\mathbb{Z}) \rightarrow \text{Hom}(\pi_n(X),\mathbb{Z}). \]

**Proposition 3.4.** Let \( k \in \mathbb{N} \) be a positive integer. Then, there are isomorphisms

\[
H^2_\mathbb{Z}(\mathbb{B}_{2k}|\mathbb{B}_{2k},\mathbb{Z}(1)) \xrightarrow{j^*} H^2_\mathbb{Z}(\mathbb{B}_{2k},\mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{B}_{2k},\mathbb{Z}) \cong \text{Hom}(\pi_2(\mathbb{B}_{2k}),\mathbb{Z}) \cong \mathbb{Z}.
\]

The universal FKMM invariant is related to the (universal) first Chern class \( c_1 \) by the formula

\[
f(j^*(\kappa_{\text{univ}})) = c_1
\]

and provides a basis of \( H^2_\mathbb{Z}(\mathbb{B}_{2k}|\mathbb{B}_{2k},\mathbb{Z}(1)). \)

**Proof.** The isomorphism \( f \) has been proved in [DG3, Proposition A.3], while the isomorphism \( j^* \) has been established in [DG3, Proposition A.5]. The isomorphism \( H^2(\mathbb{B}_{2k},\mathbb{Z}) \cong \mathbb{Z} \) follows from (3.5). In particular \( H^2(\mathbb{B}_{2k},\mathbb{Z}) \) is generated by the (universal) first Chern class \( c_1 \). The fact that \( \kappa_{\text{univ}} \) is proven in [DG3, Proposition 2.15 (3)], along with its relation with the first \( c_1 \). Since \( \pi_1(\mathbb{B}_{2k}) \cong \mathbb{Z} \) one has that \( H_1(\mathbb{B}_{2k}) \cong \mathbb{Z} \) in view of the Hurewicz homomorphism, and in turn \( H^2(\mathbb{B}_{2k},\mathbb{Z}) \cong \text{Hom}(\pi_2(\mathbb{B}_{2k}),\mathbb{Z}) \) as a consequence of the universal coefficient theorem. \( \Box \)

### 3.3. A relevant Eilenberg-Mac Lane space.

This subsection is aimed to show that the Eilenberg-Mac Lane space

\[ \mathcal{K} := \mathcal{K}(0 \xrightarrow{\tau} \mathbb{Z},\mathbb{Z}) \]

has the same cohomological nature as the classifying space \( \mathbb{B}_{2k} \). Let us recall from Section 2.5 that \( \mathcal{K} \) is a path connected space endowed with a \( \mathbb{Z}_2 \)-action denoted with \( \tau \), such that \( \mathcal{K}_\tau \neq \emptyset \) and

\[
H^2_\mathbb{Z}_2(\mathcal{K},0 \xrightarrow{\tau} \mathbb{Z}) \cong [\mathcal{K},\mathcal{K}]_{\mathbb{Z}_2},
\]

for any \( \mathbb{Z}_2 \)-CW complex \( X \).

The relevant cohomology properties of \( \mathcal{K} \) are summarized in the following result which parallels the property of the space \( \mathbb{B}_{2k} \) proved in Proposition 3.4.

**Proposition 3.5.** There are isomorphisms

\[
H^2_\mathbb{Z}_2(\mathcal{K}|\mathcal{K}_\tau,\mathbb{Z}(1)) \xrightarrow{j^*} H^2_\mathbb{Z}_2(\mathcal{K},\mathbb{Z}(1)) \xrightarrow{f} H^2(\mathcal{K},\mathbb{Z}) \cong \text{Hom}(\pi_2(\mathcal{K}),\mathbb{Z}) \cong \mathbb{Z}.
\]

This result is proved in several technical steps mimicking the strategy used in [DG3, Appendix A]. More precisely the proof is contained in Lemmas B.1, B.2, B.3 and B.4.

### 3.4. Bredon cohomology interpretation of the The FKMM invariant.

We are now in position to carry out the study of the FKMM invariant from the viewpoint of the Bredon cohomology. Recall that Theorem 3.3 establishes the natural isomorphism

\[
H^2_\mathbb{Z}_2(\mathcal{K}|\mathcal{K}_\tau,\mathbb{Z}(1)) \cong \mathcal{H}^2_\mathbb{Z}_2(X,0 \xrightarrow{\tau} \mathbb{Z}).
\]
for any $\mathbb{Z}_2$-CW complex. We will freely use this isomorphism, and for that we can identify the universal FKMM invariant $\kappa_{\text{univ}}$ with an element in the Bredon cohomology, i.e.

$$\kappa_{\text{univ}} \in \mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{B}_{2k}, \mathbb{Z}) .$$

Let us recall the universal Bredon class

$$\iota \in \mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}) \cong \mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\triangledown, \mathbb{Z}) \cong \mathbb{Z}$$

as defined in Section 2.5. In view of Remark 2.11 it follows that $\iota$ is a generator. By its nature, there exists a $\mathbb{Z}_2$-equivariant map

$$\varphi : \mathcal{B}_{2k} \longrightarrow \mathcal{K}$$

such that $\varphi^*(\iota) = \kappa_{\text{univ}}$.

**Lemma 3.6.** Let $\varphi$ be the $\mathbb{Z}_2$-equivariant map (3.7). Then, for any integer $k \in \mathbb{N}$, it holds true that:

(i) The homomorphism $\varphi_* : \pi_n(\mathcal{B}_{2k}) \rightarrow \pi_n(\mathcal{K})$ is bijective for $n \leq 3$ and surjective for $n = 4$;

(ii) The homomorphism $\varphi_* : \pi_n(\mathcal{B}_{2k}^0) \rightarrow \pi_n(\mathcal{K}^\triangledown)$ is bijective for $n \leq 3$ and surjective for $n = 4$.

**Proof.** (i) The homotopy of the Eilenberg-Mac Lane space $\mathcal{K}$ is given by (B.1). Moreover, it is known that $\pi_n(\mathcal{B}_{2k}) \cong \pi_{n-1}(\mathbb{U}(2k))$ [DG5, eq. A.3]. In summary, one has that

| $\pi_n(\mathcal{B}_{2k})$ | $\pi_n(\mathcal{K})$ |
|-----------------|-----------------|
| $n = 0$  | 0                |
| $n = 1$  | 0                |
| $n = 2$  | $\mathbb{Z}$    |
| $n = 3$  | 0                |
| $n = 4$  | $\mathbb{Z}$    |

Therefore, the claim (i) is evident for $n \neq 2$. For $n = 2$, in view of Propositions 3.4 and 3.5, one has the commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{B}_{2k}|\mathcal{B}_{2k}^0, \mathbb{Z}) & \cong & \text{Hom}(\pi_2(\mathcal{B}_{2k}), \mathbb{Z}) \\
\uparrow & & \uparrow \\
\mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\triangledown, \mathbb{Z}) & \cong & \text{Hom}(\pi_2(\mathcal{K}), \mathbb{Z}) \\
\end{array}
$$

where all vertical maps are induced from the equivariant map $\varphi$. Because $\varphi^*$ relates the generators $\iota$ and $\kappa_{\text{univ}}$, it follows that the left most homomorphism $\varphi^* : \mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\triangledown, \mathbb{Z}) \longrightarrow \mathcal{H}^2_{\mathbb{Z}_2}(\mathcal{B}_{2k}|\mathcal{B}_{2k}^0, \mathbb{Z})$ is indeed an isomorphism. Therefore also the right most homomorphism, which is induced from $\varphi_* : \pi_2(\mathcal{B}_{2k}) \rightarrow \pi_2(\mathcal{K})$, must be an isomorphism. This implies that $\pi_2(\mathcal{B}_{2k}) \cong \pi_2(\mathcal{K})$.

(ii) By design, $\mathcal{K}^\triangledown \neq \emptyset$, and equation (B.2) provides $\pi_n(\mathcal{K}^\triangledown) = 0$ for all $n$. On the other
hand

\[ \mathcal{B}_2^p \simeq \text{Gr}_k(\mathbb{H}^p) = \lim_{n} \text{Sp}(n)/(\text{Sp}(k) \times \text{Sp}(n-k)) \]

where $\mathbb{H}$ denotes the (non-commutative) field of quaternions and $\text{Sp}(k)$ is the compact symplectic group. Therefore, from the the homotopy exact sequence one obtains that $\pi_n(\mathcal{B}_2^p) \simeq \pi_{n-1}(\text{Sp}(k))$. The homotopy groups of $\mathcal{B}_2^p$ and $\mathcal{K}^*$ in low degrees are shown in the following table:

| $\pi_n(\mathcal{B}_2^p)$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|-------------------------|--------|--------|--------|--------|--------|
|                         | 0      | 0      | 0      | 0      | $\mathbb{Z}$ |
| $\pi_n(\mathcal{K}^*)$ | 0      | 0      | 0      | 0      | 0      |

The proof follows directly from the values listed above. □

We are now in position to prove our main result.

**Theorem 3.7.** Let $k \in \mathbb{N}$ be a positive integer. Then, the FKMM invariant

\[ \kappa : \text{Vect}_{\mathbb{Q}}^{2k}(X) \rightarrow H_{2k}^2(X|X^\tau, \mathbb{Z}(1)) \]

is bijective for any $\mathbb{Z}_2$-CW complex $X$ of dimension $d \leq 3$.

**Proof.** For any $\mathbb{Z}_2$-CW complex $X$, the equivariant map (3.7) induces a natural map

\[ \varphi_* : [X, \mathcal{B}_2^k]_{\mathbb{Z}_2} \rightarrow [X, \mathcal{K}]_{\mathbb{Z}_2} . \]

By the Whitehead theorem in equivariant homotopy theory [May, Theorem 3.2] and Lemma 3.6, the above map is bijective if the dimension of $X$ is less than or equal to three. Since $\mathcal{B}_2^k$ classifies “Quaternionic” vector bundles of rank $2k$, we have

\[ [X, \mathcal{B}_2^k]_{\mathbb{Z}_2} \simeq \text{Vect}_{\mathbb{Q}}^{2k}(X) , \]

while $\mathcal{K}$ represents the Bredon cohomology

\[ [X, \mathcal{K}]_{\mathbb{Z}_2} \simeq \mathcal{K}_{\mathbb{Z}_2}(X, 0 \rightarrow \mathbb{Z}) \simeq H_{2k}^2(X|X^\tau, \mathbb{Z}(1)) . \]

It remains to verify that the composition of these bijections

\[ \text{Vect}_{\mathbb{Q}}^{2k}(X) \simeq [X, \mathcal{B}_2^k]_{\mathbb{Z}_2} \xrightarrow{\varphi_*} [X, \mathcal{K}]_{\mathbb{Z}_2} \simeq H_{2k}^2(X|X^\tau, \mathbb{Z}(1)) \]

is the FKMM invariant. Suppose that $\mathcal{E} \rightarrow X$ is a “Quaternionic” vector bundle of rank $2k$. Then there is an equivariant map $f : X \rightarrow \mathcal{B}_2^k$ such that $f^* \mathcal{E} \simeq \mathcal{E}$. The assignment $[\mathcal{E}] \mapsto [f]$ realizes the first bijection. Composing $\varphi$, we get an equivariant map $\varphi \circ f : X \rightarrow \mathcal{K}$. The assignment $[f] \mapsto [\varphi \circ f]$ realizes the second bijection. Finally, by the third bijection, we get a cohomology class $(\varphi \circ f)^* \in H_{2k}^2(X|X^\tau, \mathbb{Z}(1))$. By design, one has

\[ (\varphi \circ f)^* = f^*(\varphi^*)(t) = f^*(\kappa_{\text{univ}}) = f^*(\kappa(\mathcal{E} \simeq)) = \kappa(f^* \mathcal{E}) = \kappa(\mathcal{E}) , \]

where the second-last equality is justified by the naturality of $\kappa$. □
Remark 3.8. By the Whitehead theorem in equivariant homotopy theory, the map \( \varphi_* : [X, \mathcal{B}_{2k}]_{\mathbb{Z}_2} \to [X, \mathcal{K}]_{\mathbb{Z}_2} \) is surjective for any \( \mathbb{Z}_2 \)-CW complex of dimension \( d \leq 4 \). It follows that the FKMM invariant \( \kappa : \text{Vect}^2_{\mathbb{Q}}(X) \to H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \) is also surjective in dimension \( d = 4 \).

4. EQUIVARIANT COHOMOLOGY OF LENS SPACE

The main aim of this part is to clarify the wrong part in [DG4, Section 5] and provide corrected claims.

4.1. Setup and relevant results. The three dimensional sphere can be parametrized as the unit sphere in \( \mathbb{C}^2 \),

\[
S^3 \equiv \left\{ (z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1 \right\} \subset \mathbb{C}^2.
\] (4.1)

This representation allows \( u \in \mathbb{U}(1) \) to act (on the left) on \( S^3 \) through the mapping \((z_0, z_1) \mapsto (uz_0, uz_1)\). This action of \( \mathbb{U}(1) \) on \( S^3 \) is evidently free. The inclusion of \( \mathbb{Z}_p \subset \mathbb{U}(1) \), given by the fact that \( \mathbb{Z}_p \) can be identified with the set of the \( p \)-th roots of the unity, implies that one can define a free action of every cyclic group \( \mathbb{Z}_p \) on \( S^3 \). More precisely we can let \( k \in \mathbb{Z}_p \) act on \( S^3 \) through the rotation

\[
k : (z_0, z_1) \mapsto \left( e^{i2\pi \frac{k}{p}} z_0, e^{i2\pi \frac{k}{p}} z_1 \right).
\]

The quotient space \( L_p := S^3/\mathbb{Z}_p \)

is called the (three-dimensional) lens space (see [BT, Example 18.5] or [Hat, Example 2.43] for more details) and sometime is denoted with the symbol \( L(1; p) \). By combining the facts that \( S^3 \) is simply connected and the \( \mathbb{Z}_p \)-action on \( S^3 \) is free one concludes that \( S^3 \) is the universal cover of \( L_p \).

The parametrization (4.1) allows to equip \( S^3 \subset \mathbb{C}^2 \) with the involution induced by the complex conjugation \((z_0, z_1) \mapsto (\bar{z}_0, \bar{z}_1)\). The computation

\[
e^{i2\pi \frac{k}{p}} = e^{-i2\pi \frac{k}{p}} = \overline{e^{i2\pi \frac{k}{p}}} ,
\]

\( k \in \mathbb{Z}_p \)

shows that \( \mathbb{Z}_p \subset \mathbb{U}(1) \) is preserved by the complex conjugation. Therefore, the involution on \( S^3 \) descends to an involution \( \tau \) on \( L_p \). The involutive space \( (L_p, \tau) \) inherits the structure of a smooth (three-dimensional) manifold with a smooth involution, hence it admits a \( \mathbb{Z}_2 \)-CW-complex structure [May, Theorem 3.6]. Let us point out that it is possible to think of \( L_p \to \mathbb{C}P^1 \) as a “Real” principal \( \mathbb{U}(1) \)-bundle where the “Real” structure on the total space is provided by \( \tau \), and the involution \( \tau' \) on the base space \( \mathbb{C}P^1 \) is still given by the complex conjugation \( \tau' : [z_0, z_1] \mapsto [\bar{z}_0, \bar{z}_1] \).

Henceforth, let us focus now on the even case \( p = 2q > 0 \).
Fixed point set. As proved in [DG4, Lemma 5.1] the fixed point set of $L_{2q}$ has the form
\[ L_{2q}^\tau = S_0 \sqcup S_1 \simeq S^1 \sqcup S^1 \]
where
\[ S_0 := \left\{ \left[ \cos \theta, \sin \theta \right] \in L_{2q} \mid \theta \in \mathbb{R} \right\}, \]
\[ S_1 := \left\{ \left[ e^{-i \frac{2\pi}{q}} \cos \theta, e^{-i \frac{2\pi}{q}} \sin \theta \right] \in L_{2q} \mid \theta \in \mathbb{R} \right\}. \tag{4.2} \]
and $S_0 \simeq S_1 \simeq S^1$. In particular it is the disjoint union of two connected components.

Equivariant Borel cohomology. As proved in [DG4, Section 5.2] (see in particular Tables 2 and 4), one has that
\[ H^1_{Z_2}(L_{2q}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \tag{4.3} \]
and
\[ H^1_{Z_2}(L_{2q}^\tau, \mathbb{Z}(1)) = H^1_{Z_2}(S_0, \mathbb{Z}(1)) \oplus H^1_{Z_2}(S_1, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{4.4} \]
In view of the first isomorphism in (4.2), the generators of these two groups can be represented by equivariant maps. More precisely the generator in (4.3) is represented by the constant map on $L_{2q}$ with value $-1$. Similarly, the two generators in (4.4) are represented by the two constant maps on $S_0 \sqcup S_1$ which take the value $+1$ on one connected component and $-1$ on the other.

“Real” line bundles. From [DG4, Remark 5.2] we know that the Picard group of $L_{2q}$ and the “Real” Picard group of the involutive space $(L_{2q}, \tau)$ coincide and are given by
\[ \text{Pic}_\mathbb{R}(L_{2q}, \tau) \simeq H^2_{Z_2}(L_{2q}, \mathbb{Z}(1)) \simeq \mathbb{Z}_{2q} \simeq H^2_{Z_2}(X, \mathbb{Z}) \simeq \text{Pic}(L_{2q}). \]
Therefore, there are only $2q$ complex line bundles over $L_{2q}$ (up to isomorphisms), and each one of these can be endowed with a unique (up to isomorphisms) “Real” structure (cf. eq. (A.2)). The representatives of these line bundles can be constructed explicitly. For $k \in \mathbb{Z}$, we let $u \in \mathbb{Z}_{2q}$ act on $S^3 \times \mathbb{C}$ by $((z_0, z_1), \lambda) \mapsto ((uz_0, uz_1), \tau^k \lambda)$. Since the action is free on the base space the quotient defines a complex line bundle $L_k \to L_{2q}$ (cf. [Ati2, Proposition 1.6.1]). From the construction it results evident that $L_k = L_{k+2q}$ and $L_0 = L_{2q} \times \mathbb{C}$ is the trivial line bundle. Moreover, $L_1$ provides a basis for $\text{Pic}(L_{2q}) \simeq \mathbb{Z}_{2q}$ in view of the fact that $L_1 \otimes k \simeq L_k$. The “Real” structure on $L_k$ is evidently induced by the complex conjugation $\Theta : [(z_0, z_1), \lambda] \mapsto [\tau(z_0, z_1), \bar{\lambda}] = [(z_0, z_1), \bar{\lambda}]$. The isomorphism class of $L_1$ as a “Real” line bundle provides a generator for $H^2_{Z_2}(L_{2q}, \mathbb{Z}(1))$.

4.2. Restricted “Real” line bundles. In [DG4, Section], the restriction of the “Real” line bundle $L_1 \to L_{2q}$ to the fixed point set $L_{2q}^\tau = S_0 \sqcup S_1$ is studied. In particular in [DG4, Lemma 5.3] it is claimed that the restrictions $L_1|_{S_0}$ and $L_1|_{S_1}$ are trivial. However, this result is wrong. The argument used in the proof of [DG4, Lemma 5.3] is based on the construction of two nowhere vanishing invariant sections $s_j : S_j \to L_1|_{S_j}$, $j = 1, 2,$...
but the constructed sections are actually ill-defined. The correct claim is contained in the following result.

**Lemma 4.1.** Both “Real” line bundles \( \mathcal{L}|_{S_0} \) and \( \mathcal{L}|_{S_1} \) are non-trivial.

**Proof.** Let us start with an equivalent, but more “appropriate”, description of the connected components \( S_0 \) and \( S_1 \) of the fixed point set \( L_2q \) given by

\[
S_0 := \left\{ \left[ \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right] \in L_2q \mid \theta \in \mathbb{R}/2\pi \mathbb{Z} \right\},
\]

\[
S_1 := \left\{ \left[ e^{-i \frac{\pi}{2}} \cos \frac{\theta}{2}, e^{-i \frac{\pi}{2}} \sin \frac{\theta}{2} \right] \in L_2q \mid \theta \in \mathbb{R}/2\pi \mathbb{Z} \right\}.
\]

The comparison with (4.2) is obtained by observing that the action of \( q \in \mathbb{Z}_{2q} \) on \( S^3 \) is obtained by multiplying the component with \( e^{i2\pi \frac{\theta}{q}} = -1 \), i.e. by \((z_0, z_1) \mapsto (-z_0, -z_1)\). Therefore, it is sufficient to parametrize the components \( S_0 \) and \( S_1 \) only with angles contained in the range \([0, \pi]\), or equivalently by using the parameter \( \theta/2 \). In view of (4.5), one can identify \( S_0 \) and \( S_1 \) with \( \mathbb{S}^1 \sim \mathbb{R}/2\pi \mathbb{Z} \). Two nowhere-vanishing sections \( s_j : S_j \rightarrow \mathcal{L}|_{S_j} \), for \( j = 0, 1 \), can be defined by

\[
s_0 \left( \left[ \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right] \right) = \left[ \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right), e^{-i \frac{\theta}{2}} \right],
\]

\[
s_1 \left( \left[ e^{-i \frac{\pi}{2}} \cos \frac{\theta}{2}, e^{-i \frac{\pi}{2}} \sin \frac{\theta}{2} \right] \right) = \left[ \left( e^{-i \frac{\pi}{2}} \cos \frac{\theta}{2}, e^{-i \frac{\pi}{2}} \sin \frac{\theta}{2} \right), e^{i \frac{\pi}{2}} e^{-i \frac{\theta}{2}} \right],
\]

for every \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). A direct computation leads to

\[
\Theta \left( s_0 \left( \left[ \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right] \right) \right) = s_0 \left( \tau \left( \left[ \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right] \right) \right) \cdot e^{i\lambda},
\]

where \( \Theta \) is the “Real” structure on \( \mathcal{L}_1 \) and the notation \( \cdot e^{i\lambda} \) on the right denotes the multiplication by the phase \( e^{i\lambda} \) only on the fiber. A similar result holds also for the section \( s_1 \) as it can be checked by a straightforward computation. As a consequence the two nowhere-vanishing sections \( s_0 \) and \( s_1 \) are not “Real” since \( \Theta \circ s_j \neq s_j \circ \tau \). However, each section \( s_j \) induces an isomorphism between \( \mathcal{L}_1|_{S_j} \rightarrow S_1 \) and the product line bundle \( \mathbb{S}^1 \times \mathbb{C} \rightarrow \mathbb{S}^1 \) (on the circle \( \mathbb{S}^1 \sim \mathbb{R}/2\pi \mathbb{Z} \)) with “Real” structure \((\theta, \lambda) \mapsto (\theta, e^{i\theta} \lambda)\). The latter “Real” line bundle is non-trivial, and this concludes the proof.

**4.3. “Quaternionic” structures.** The different “Quaternionic” structures on \((L_{2q}, \tau)\) have been constructed in [DG4, Proposition 5.7]. As a result one has that classification of equivalence classes of rank \( 2m \) “Quaternionic” vector bundles over \((L_{2q}, \tau)\) is given by

\[
\text{Vec}_{2m}^{\mathbb{Q}}(L_{2q}, \tau) \cong \mathbb{Z}_{2q}, \quad \forall \ m \in \mathbb{N}.
\]
In view of Theorem 3.7, this fact must imply $H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1)) \simeq \mathbb{Z}_{2q}$. In fact this is the correct result, while the claim in [DG4, Proposition 5.4] turns out to be wrong as a consequence of the incorrectness of [DG4, Lemma 5.3].

In order to compute directly $H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1))$, let us make use of the exact sequence (A.3). One gets that

$$
\begin{align*}
\frac{H^1_{Z_2}(L_2q, \mathbb{Z}(1))}{\mathbb{Z}_2} \xrightarrow{i^*} \frac{H^1_{Z_2}(L_2q^\tau, \mathbb{Z}(1))}{\mathbb{Z}_2} \to H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1)) & \to \\
\to \frac{H^2_{Z_2}(L_2q, \mathbb{Z}(1))}{\mathbb{Z}_2} \xrightarrow{i^*} \frac{H^2_{Z_2}(L_2q^\tau, \mathbb{Z}(1))}{\mathbb{Z}_2},
\end{align*}
$$

where the values of the cohomology groups are taken from Tables 2 and 4 in [DG4, Section 5.2]. The homomorphism $i^* : H^1_{Z_2}(L_2q, \mathbb{Z}(1)) \to H^1_{Z_2}(L_2q^\tau, \mathbb{Z}(1))$ induced from the inclusion $i : L_2q^\tau \hookrightarrow L_2q$ coincides with the diagonal map in view of the explicit description of the generators given in Section 4.1. Therefore $i^*$ is injective, and its cokernel is $\mathbb{Z}_2$. Let us identify $\mathcal{L}_1$ with the generator of $H^2_{Z_2}(L_2q, \mathbb{Z}(1))$ and observe that $i^*(\mathcal{L}_1) = \mathcal{L}_1|_{S_0\cup S_1}$ is not trivial by Lemma 4.1. On the other hand one can prove that $i^*(\mathcal{L}_2) = \mathcal{L}_2|_{S_0\cup S_1}$ is the trivial element, where $\mathcal{L}_k \simeq \mathcal{L}_k \otimes \mathcal{L}^\tau$ (see the proof of Proposition 4.6 below). From that one infers that the kernel of the homomorphism $i^* : H^2_{Z_2}(L_2q, \mathbb{Z}(1)) \to H^2_{Z_2}(L_2q^\tau, \mathbb{Z}(1))$ is $\mathbb{Z}_q \subset \mathbb{Z}_{2q}$. Hence the exact sequence above reduces to the short exact sequence

$$
0 \to \mathbb{Z}_2 \to H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1)) \to \mathbb{Z}_q \xrightarrow{i^*} 0. \quad (4.6)
$$

**Proposition 4.2.** It holds true that

$$
H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1)) \simeq \mathbb{Z}_{2q}.
$$

**Proof.** The result follows if one can show that the exact sequence (4.6) is not splitting. Let us start by proving that $\mathcal{L}_2|_{S_0\cup S_1}$ admits a nowhere vanishing “Real” section and therefore is trivial. For that it is sufficient to consider the two sections $\sigma_1 : S_1 \to \mathcal{L}_2|_{S_1}$, with $j = 0, 1$, defined by

$$
\sigma_0 \left( \begin{bmatrix} \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \end{bmatrix} \right) = \left( \begin{bmatrix} \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \end{bmatrix}, 1 \right),
$$

$$
\sigma_1 \left( \begin{bmatrix} e^{-i \frac{\pi}{4}}, \cos \frac{\theta}{2}, e^{-i \frac{\pi}{4}} \sin \frac{\theta}{2} \end{bmatrix} \right) = \left( \begin{bmatrix} e^{-i \frac{\pi}{4}}, \cos \frac{\theta}{2}, e^{-i \frac{\pi}{4}} \sin \frac{\theta}{2} \end{bmatrix}, e^{i \frac{\pi}{4}} \right),
$$

for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The pair $(\mathcal{L}_2, \sigma_0 \sqcup \sigma_1)$ represents an element of $H^2_{Z_2}(L_2q|L_2q^\tau, \mathbb{Z}(1))$ which surjects to a basis of the kernel $\mathbb{Z}_q$ of $i^*$. Therefore, the short exact sequence 4.6 would be split if and only if $(\mathcal{L}_2, \sigma_0 \sqcup \sigma_1)^{\otimes q}$ were trivial. We readily see

$$
(\mathcal{L}_2, \sigma_0 \sqcup \sigma_1)^{\otimes q} \simeq (\mathcal{L}_0, 1 \sqcup (-1))
$$
where $L_0 \cong L_{1}^{\otimes 2q}$ is the trivial line bundle. Since $\sigma_{1}^{\otimes q} = -1$ is not trivial, so is the element above. □

The latter result provides the correct version of [DG4, Proposition 5.4], which is consistent with Theorem 3.7.

4.4. Physical applications. Before concluding, let us suggest some physical situation where the classification of the “Quaternionic” structures over $L_{2q}$ described in Section 4.3 can be of some relevance.

First of all, it is worth mentioning that the lens spaces $L_n$ naturally enter the theory of the Dirac monopole. In fact, it can be shown that $L_n$ is isomorphic to the line bundle $\mathfrak{h}^{\otimes n}$, where $\mathfrak{h}$ is the dual bundle of the tautological line bundle over $\mathbb{CP}^1$, and the magnetic monopole of charge $n \neq 0$ is the curvature of the rotationally invariant $U(1)$ connection over $L_n$ [JS, Appendix A].

The Born-Oppenheimer approximation permits to construct interesting examples of Topological Quantum System (with symmetries) in the sense described in Section 1 [Bae, FZ, GR]. The Born-Oppenheimer approximation can generally be applied when a quantum system is coupled with another comparatively slower system which is treated classically. In quantum mechanics this occurs, for example, in molecular dynamics, where usually the electrons have a fast motion compared to the motion of nuclei. Let us denote with $X$ the classical state space (phase space). For a fixed classical state $x \in X$, one considers an instantaneous operator $H(x)$ acting on the quantum state space $\mathcal{H} \cong \mathbb{C}^M$ which describes the dynamics of the fast degrees of freedom. One immediately recognizes that this framework is summarized by (1.1), which provides the definition of a TQS.

Now, being more specific, let us assume that $X$ is the classical state space of a particle of mass $m$ constrained on the unit sphere $S^2 \subset \mathbb{R}^3$. Therefore, the position of the particle is specified by a vector $q \in \mathbb{R}^3$ such that $|q| = 1$. The momentum of the particle is $p = mv$ where $v$ is the velocity. Since the velocity is tangent to the sphere one gets that $q \cdot p = m(q \cdot v) = 0$. If we denote with $|p|$ the modulus of the momentum and with $\varphi := p/|p|$ its unit vector one obtains that

$$X = \Omega \times [0, +\infty)$$

where

$$\Omega := \{(q, \varphi) \in S^2 \times S^2 \mid q \cdot \varphi = 0\}.$$ Since $[0, +\infty)$ is contractible, $\Omega$ provides the only relevant part for the analysis of topological effects. Following the construction in [Bha, Section III] one can prove that $\Omega \cong L_2$. Let $i : S^3 \to SU(2)$ be the standard identification given by

$$i : (z_0, z_1) \mapsto \left(\begin{array}{cc} z_0 & z_1 \\ -z_1 & z_0 \end{array}\right) = e^{i \varphi} \sigma^3.$$
where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the Pauli matrices, $\theta = 2 \arccos(\text{Re}(z_0))$ and the unit vector $d$ is given by

$$d = \frac{1}{\sin \frac{\theta}{2}} (\text{Im}(z_0), \text{Im}(z_1), \text{Re}(z_1)) .$$

Let $f : SU(2) \to SO(3)$ be the standard double cover given by

$$f : e^{i d \cdot \sigma_3^Q} \mapsto R_d(\theta)$$

where $R_d(\theta) \in SO(3)$ is the matrix that rotates of an angle $\theta$ around the direction $d$. Observe that $f^{-1}(R_d(\theta)) \mapsto \{ \pm e^{i d \cdot \sigma_3^Q} \}$. Finally, for a fixed $(q_0, \varphi_0) \in \Omega$ let

$$g : SO(3) \to \Omega$$

be the map given by

$$g : R_d(\theta) \mapsto R_d(\theta) \cdot (q_0, \varphi_0) := (R_d(\theta)q_0, R_d(\theta)\varphi_0) .$$

One can directly check that the map $\alpha : S^3 \to \Omega$, defined by $\alpha := g \circ f \circ i$ is a double covering map in view of the fact that $f$ is a double cover. In particular, if $(q, \varphi) = R_d(\theta) \cdot (q_0, \varphi_0)$, then

$$\alpha^{-1} : (q, \varphi) \mapsto \{(z_0, z_1), (-z_0, -z_1)\} \in L_2$$

provides the desired identification $\Omega \simeq L_2$. Let us identify now the action of the involution $\tau' := \alpha \circ \tau$ on $\Omega$ induced by the involution $\tau$ on $L_2$. Given the unit vector $d = (d_1, d_2, d_3)$, let $\overline{d} := (-d_1, -d_2, d_3)$. Then a simple check shows that

$$\tau' : R_d(\theta) \cdot (q_0, \varphi_0) \mapsto R_{\overline{d}}(\theta) \cdot (q_0, \varphi_0) .$$

The pair $(\Omega, \tau')$ turns out to be an involutive space equivalent to $(L_2, \tau)$.

Let assume now that the operator $H$ acting on the fast degrees of freedom depends in the momentum $p$ only through a term of the type $|p \cdot w| \chi |\varphi \cdot w|$ for a fixed unitary vector $w \in \mathbb{R}^3$. In this case the relevant classical degree of freedom for the momentum is the line detected by $\varphi$, and denoted by $\ell_\varphi$, rather than $\varphi$ itself. Therefore the relevant classical state space becomes

$$\Sigma := \{(q, \ell_\varphi) \in S^2 \times \mathbb{RP}^3 \mid q \cdot \varphi = 0\}$$

which is the space of all tangent lines to the sphere. A generalization of the argument above shows that $\Sigma \simeq L_4$ [Bha, Section III]. The main difference now consists in the fact that the map $g$, defined as in (4.7), becomes a double covering. In fact, given a reference point $(q_0, \ell_{\varphi_0})$ and a generic point $(q, \ell_\varphi) = R_d(\theta) \cdot (q_0, \ell_{\varphi_0})$, it holds true that

$$g^{-1} : (q, \ell_\varphi) \mapsto \{R_d(\theta), R_d(\theta)R_q(\pi)\}$$

in view of the relations $R_q(\pi)q = q$, $R_q(\pi)\varphi = -\varphi$ and $R_q(\pi)\ell_\varphi = \ell_{-\varphi} = \ell_\varphi$. As a consequence, the map $\alpha$ turns out to be a 4-covering as a composition of two double covering and $\alpha^{-1}$ provides the identification $\Sigma \simeq L_4$. 


APPENDIX A. A SHORT REMINDER OF THE EQUIVARIANT BOREL COHOMOLOGY

The proper cohomology theory for the analysis of vector bundle theories in the category of spaces with involution is the equivariant cohomology introduced by A. Borel in [Bor]. This cohomology has been used for the topological classification of “Real” vector bundles [DG1] and plays also a role in the classification of “Quaternionic” vector bundles [DG2, DG3, DG4]. A short self-consistent summary of this cohomology theory can be found in [DG1, Section 5.1] and we refer to [Hsi, Chapter 3] and [AP, Chapter 1] for a more detailed introduction to the subject.

Let us briefly recall the main steps of the Borel construction. The homotopy quotient of an involutive space $(X, \tau)$ is the orbit space

$$X_{\sim \tau} := X \times S^\infty / (\tau \times \theta_\infty).$$

(A.1)

Here $\theta_\infty$ is the antipodal map on the infinite sphere $S^\infty$ (cf. [DG1, Example 4.1]) and $S^0,\infty$ is used as short notation for the pair $(S^\infty, \theta_\infty)$. The product space $X \times S^\infty$ (forgetting for a moment the $\mathbb{Z}_2$-action) has the same homotopy type of $X$ since $S^\infty$ is contractible. Moreover, since $\theta_\infty$ is a free involution, also the composed involution $\tau \times \theta_\infty$ is free, independently of $\tau$. Let $\mathcal{R}$ be any commutative ring (e. g. $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_2, \ldots$). The equivariant cohomology ring of $(X, \tau)$ with coefficients in $\mathcal{R}$ is defined as

$$H^*_\mathcal{Z}_2(X, \mathcal{R}) := H^*(X_{\sim \tau}, \mathcal{R}).$$

More precisely, each equivariant cohomology group $H^j_\mathcal{Z}_2(X, \mathcal{R})$ is given by the singular cohomology group $H^j(X_{\sim \tau}, \mathcal{R})$ of the homotopy quotient $X_{\sim \tau}$ with coefficients in $\mathcal{R}$ and the ring structure is given, as usual, by the cup product. As the coefficients of the usual singular cohomology are generalized to local coefficients (see e. g. [Hat, Section 3.H] or [DK, Section 5]), the coefficients of the Borel equivariant cohomology are also generalized to local coefficients. Given an involutive space $(X, \tau)$ let us consider the homotopy group $\pi_1(X_{\sim \tau})$ and the associated group ring $\mathbb{Z}[\pi_1(X_{\sim \tau})]$. Each module $\mathcal{Z}$ over the group $\mathbb{Z}[\pi_1(X_{\sim \tau})]$ is, by definition, a local system on $X_{\sim \tau}$. Using this local system one defines, as usual, the equivariant cohomology with local coefficients in $\mathcal{Z}$:

$$H^*_\mathcal{Z}_2(X, \mathcal{Z}) := H^*(X_{\sim \tau}, \mathcal{Z}).$$

We are particularly interested in modules $\mathcal{Z}$ whose underlying groups are identifiable with $\mathbb{Z}$. For each involutive space $(X, \tau)$, there always exists a particular family of local systems $\mathcal{Z}(m)$ labelled by $m \in \mathbb{Z}$. Here $\mathcal{Z}(m) \simeq X \times \mathbb{Z}$ denotes the $\mathbb{Z}_2$-equivariant local system on $(X, \tau)$ made equivariant by the $\mathbb{Z}_2$-action $(x, l) \mapsto (\tau(x), (-1)^m l)$. Because the module structure depends only on the parity of $m$, we consider only the $\mathbb{Z}_2$-modules $\mathcal{Z}(0)$ and $\mathcal{Z}(1)$. Since $\mathcal{Z}(0)$ corresponds to the case of the trivial action of $\pi_1(X_{\sim \tau})$ on $\mathbb{Z}$ one has $H^j_\mathcal{Z}_2(X, \mathcal{Z}(0)) \simeq H^j_\mathcal{Z}_2(X, \mathbb{Z})$ [DK, Section 5.2].
We recall the two important group isomorphisms

\[
\begin{align*}
\text{H}^1_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) & \simeq [X, \mathbb{U}(1)]_{\mathbb{Z}_2}, \\
\text{H}^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) & \simeq \text{Vec}_{\mathbb{R}}(X, \tau) \equiv \text{Pic}_{\mathbb{R}}(X, \tau),
\end{align*}
\]

(A.2)

involving the first two equivariant cohomology groups. The first isomorphism [Gom, Proposition A.2] says that the first equivariant cohomology group is isomorphic to the set of \(\mathbb{Z}_2\)-equivariant homotopy classes of \(\mathbb{Z}_2\)-equivariant maps \(\varphi : X \to \mathbb{U}(1)\) where the involution on \(\mathbb{U}(1)\) is induced by the complex conjugation, i.e. \(\varphi(\tau(x)) = \overline{\varphi(x)}\). The second isomorphism is due to B. Kahn [Kah] and expresses the equivalence between the Picard group of “Real” line bundles (in the sense of [Ati1, DG1]) over \((X, \tau)\) and the second equivariant cohomology group of this space.

The fixed point subset \(X^\tau \subset X\) is closed and \(\tau\)-invariant and the inclusion \(\iota : X^\tau \hookrightarrow X\) extends to an inclusion \(\iota : X^\tau, \tau \hookrightarrow X, \tau\) of the respective homotopy quotients. The relative equivariant cohomology can be defined as usual by the identification

\[
\text{H}^*_\mathbb{Z}_2(X|X^\tau, \mathbb{Z}) := \text{H}^*(X, \tau|X^\tau, \mathbb{Z}).
\]

Consequently, one has the related long exact sequence in cohomology

\[
\ldots \rightarrow \text{H}^k_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}) \rightarrow \text{H}^k_{\mathbb{Z}_2}(X, \mathbb{Z}) \overset{r}{\rightarrow} \text{H}^k_{\mathbb{Z}_2}(X^\tau, \mathbb{Z}) \rightarrow \text{H}^{k+1}_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}) \rightarrow \ldots
\]

(A.3)

where the map \(r := \iota^*\) restricts cochains on \(X\) to cochains on \(X^\tau\). The \(k\)-th cokernel of \(r\) is by definition

\[
\text{Coker}^k(X|X^\tau, \mathbb{Z}) := \text{H}^k_{\mathbb{Z}_2}(X, \mathbb{Z}) / r(\text{H}^k_{\mathbb{Z}_2}(X, \mathbb{Z})).
\]

Let us point out that with the same construction one can define relative cohomology theories \(\text{H}^*_\mathbb{Z}_2(X|Y, \mathbb{Z})\) for each \(\tau\)-invariant subset \(Y \subset X^\tau\), or more in general for every \(\mathbb{Z}_2\)-CW pair \((X, Y)\) consisting of a \(\mathbb{Z}_2\)-CW complex \(X\), a \(\mathbb{Z}_2\)-CW subcomplex \(Y\) and a sub-complex inclusion \(Y \hookrightarrow X\) [AP, Remark 1.2.10]. If \(Y = \emptyset\) then \(\text{H}^k_{\mathbb{Z}_2}(X|\emptyset, \mathbb{Z}) \simeq \text{H}^k_{\mathbb{Z}_2}(X, \mathbb{Z})\) by definition, hence it is reasonable to put \(\text{H}^k_{\mathbb{Z}_2}(\emptyset, \mathbb{Z}) = 0\) for consistency with the above long exact sequence. The case \(Y := \{\ast\}\) of a single invariant point is important since it defines the reduced cohomology theory

\[
\tilde{\text{H}}^k_{\mathbb{Z}_2}(X, \mathbb{Z}) := \text{H}^k_{\mathbb{Z}_2}(X|\{\ast\}, \mathbb{Z}).
\]

In this case, the obvious surjectivity of the map \(r\) at each step of the exact sequence (A.3) justifies the isomorphism

\[
\text{H}^k_{\mathbb{Z}_2}(X, \mathbb{Z}) \simeq \tilde{\text{H}}^k_{\mathbb{Z}_2}(X, \mathbb{Z}) \oplus \text{H}^k_{\mathbb{Z}_2}(\{\ast\}, \mathbb{Z})
\]

(A.4)

**APPENDIX B. COHOMOLOGY OF THE EILENBERG-MAC LANE SPACE**

Let us recall the notation

\[
\mathcal{K} := K(0 \rightarrow \mathbb{Z}, 2)
\]
introduced in Section 3.3. The homotopy type of the spaces $\mathcal{K}$ and $\mathcal{K}^\tau$ is described in the following result.

**Lemma B.1.** It holds true that

$$\pi_k(\mathcal{K}) = \begin{cases} \mathbb{Z} & (k = 2) \\ 0 & (k \neq n) \end{cases}$$ \hspace{1cm} \text{(B.1)}$$

and

$$\pi_k(\mathcal{K}^\tau) = 0, \hspace{1cm} \forall \ k \geq 0.$$ \hspace{1cm} \text{(B.2)}$$

As a consequence, forgetting the $\mathbb{Z}_2$-action, $\mathcal{K}$ is homotopy equivalent to the classical Eilenberg-Mac Lane space $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$. Similarly, $\mathcal{K}^\tau$ is homotopy equivalent to a singleton $\{\ast\}$.

**Proof.** Using the notation of Example 2.2 one gets $\mathcal{K}^{H_0} = \mathcal{K}$ and $\mathcal{K}^{H_1} = \mathcal{K}^\tau$. Therefore, in view of the definition in Example 2.5 one obtains

$$\pi_k[\mathcal{K}](Z_0) = \pi_k(\mathcal{K}) \, , \, \pi_k[\mathcal{K}](Z_1) = \pi_k(\mathcal{K}^\tau) .$$

Comparing these latter equations with the defining property

$$\pi_k(\mathcal{K}) = \begin{cases} \mathbb{Z} & (k = 2) \\ 0 & (k \neq 2) \end{cases} ,$$

of the Eilenberg-Mac Lane space $\mathcal{K}$, one gets equations (B.1) and (B.2). Now, in view of the uniqueness of the homotopy type of the classical Eilenberg-MacLane spaces [Hat, Proposition 4.30] there is a weak homotopy equivalence between $\mathcal{K}$ and $K(\mathbb{Z}, 2)$, and between $\mathcal{K}^\tau$ and the singleton. Finally, the Whitehead’s Theorem [Hat, Theorem 4.5] ensures that the weak homotopy equivalences above induce respective homotopy equivalences. \[ \square \]

**Lemma B.2.** There are isomorphisms

$$H^n(\mathcal{K}, \mathbb{Z}) \simeq \text{Hom}(\pi_n(\mathcal{K}), \mathbb{Z}) , \hspace{1cm} n = 1, 2 .$$

Moreover, the first integral cohomology groups of $\mathcal{K}$ are

| $n$ | 0  | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|----|
| $H^n(\mathcal{K}, \mathbb{Z})$ | $\mathbb{Z}$ | 0  | $\mathbb{Z}$ | 0  | $\mathbb{Z}$ |

**Proof.** Let us use the homotopy equivalence $\mathcal{K} \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ from Lemma B.1. Since integral cohomology of $\mathbb{C}P^\infty$ is well-known, one has that $H^j(\mathcal{K}, \mathbb{Z}) \simeq \mathbb{Z}$ for $j$ even and $H^j(\mathcal{K}, \mathbb{Z}) \simeq 0$ for $j$ odd. Moreover, the Hurewicz homomorphism $\pi_2(\mathcal{K}) \to H_2(\mathcal{K})$ is an isomorphism. By the universal coefficient theorem, the homomorphism $H^n(\mathcal{K}, \mathbb{Z}) \to \text{Hom}(H_n(\mathcal{K}), \mathbb{Z})$ is an isomorphism for $0 \leq n \leq 2$. \[ \square \]
Lemma B.3. The Borel equivariant cohomology groups in low degrees of the space $\mathcal{K}$ are summarized in the following table:

| $n$ | $0$ | $1$ | $2$ | $3$ |
|-----|-----|-----|-----|-----|
| $H^0_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z})$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $H^1_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z})$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}$ | $0$ |
| $H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1))$ | $0$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ |

In particular, the map that forgets the $\mathbb{Z}_2$-action provides the isomorphism

$$H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)) \cong H^2(\mathcal{K}, \mathbb{Z}) \cong \text{Hom}(\pi_2(\mathcal{K}), \mathbb{Z}) .$$

Proof. The strategy of the proof is very similar to that of [DG3, Lemma A.2] that can be used as a reference for more details. To compute the Borel equivariant cohomology, we use the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(\mathcal{K}, \mathbb{Z})) \Rightarrow H^*_\mathbb{Z}_2(\mathcal{K}, \mathbb{Z}),$$

where the coefficient $H^q(\mathcal{K}, \mathbb{Z})$ in the group cohomology of $\mathbb{Z}_2$ is endowed by the $\mathbb{Z}_2$-action induced from the involution on $\mathcal{K}$. As it has been seen in Lemma B.2 one has

$$H^q(\mathcal{K}, \mathbb{Z}) \cong \text{Hom}(\pi_q(\mathcal{K}), \mathbb{Z}), \quad q = 1, 2 .$$

If one takes the $\mathbb{Z}_2$-action into account, then $H^0(\mathcal{K}, \mathbb{Z}) \cong \mathbb{Z}$ and $H^2(\mathcal{K}, \mathbb{Z}) \cong \mathbb{Z}$ by the very definition of the Eilenberg-Mac Lane space $\mathcal{K}$. Hence the $E_2$-terms can be summarized as follows:

| $q$ | $0$ | $2$ | $1$ | $0$ | $0$ | $0$ |
|-----|-----|-----|-----|-----|-----|-----|
| $E_2^{p,q}$ | $0$ | $\mathbb{Z}_2$ | $0$ | $\mathbb{Z}_2$ | $0$ |

This immediately determines $H^n_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z})$ for $n = 0, 1, 2$. Note that the Eilenberg-Mac Lane space is assumed to have a fixed point $* \in \mathcal{K}^\tau \neq \emptyset$. Then $E_2^{0,0}$ must survive into the direct summand $H^0_{\mathbb{Z}_2}(\{\ast\}, \mathbb{Z})$ in the decomposition

$$H^p_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}) \cong H^p_{\mathbb{Z}_2}(\{\ast\}, \mathbb{Z}) \oplus \tilde{H}^p_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z})$$

by using the reduced cohomology. This shows that

$$\mathbb{Z}_2 \cong E_2^{1,2} = E_3^{1,2} = \ldots = E_{\infty}^{1,2} \cong H^3_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}) .$$

Let us now make use the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(\mathcal{K}, \mathbb{Z}) \otimes \mathbb{Z}) \Rightarrow H^*_\mathbb{Z}_2(\mathcal{K}, \mathbb{Z}(1)) .$$
The $E_2$-terms are summarized as follows:

| $q$   | 0 | 0 | 0 | 0 | 0 |
|------|---|---|---|---|---|
| $q = 3$ | 0 | 0 | 0 | 0 | 0 |
| $q = 2$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ |
| $q = 1$ | 0 | 0 | 0 | 0 | 0 |
| $q = 0$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 |
| $E_2^{p,q}$ | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ |

This also immediately determines $H^n_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1))$ for $n = 0, 1, 2$. By the same argument about a fixed point $\ast \in \mathcal{K}^\tau$ and the reduced cohomology, one also see that

$$\mathbb{Z}_2 \cong E_2^{3,0} = E_3^{3,0} = \ldots = E\infty^{3,0} \cong H^3_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)).$$

From the exact sequence in [Gom, Proposition 2.3]

$$H^1_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}) \to H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)) \to H^2(\mathcal{K}, \mathbb{Z}) \to H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}) \to H^3_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)) \to H^3(\mathcal{K}, \mathbb{Z})$$

one infers that $f$ is an isomorphism. □

Recall that the inclusion $j: (\mathcal{K}, \emptyset) \hookrightarrow (\mathcal{K}, \mathcal{K}^\tau)$ induces a natural homomorphism $j^*$ in the Borel equivariant cohomology.

**Lemma B.4.** The inclusion $j$ induces the isomorphism

$$H^2_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\tau, \mathbb{Z}(1)) \cong H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)).$$

**Proof.** By Lemma B.1 $\mathcal{K}^\tau \neq \emptyset$ is homotopy equivalent to a singleton $\{\ast\}$, on which evidently $\mathbb{Z}_2$ acts trivially. Therefore one has that

$$H^n_{\mathbb{Z}_2}(\mathcal{K}^\tau, \mathbb{Z}(1)) \cong H^n_{\mathbb{Z}_2}(\{\ast\}, \mathbb{Z}(1))$$

and

$$H^n_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\tau, \mathbb{Z}(1)) \cong \tilde{H}^n_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1))$$

where on the right hand side there is the reduced cohomology induced by the inclusion $\{\ast\} \hookrightarrow \mathcal{K}$. The Borel equivariant cohomology of the fixed point is well known [Gom, Proposition 2.4.] and in particular $H^2_{\mathbb{Z}_2}(\{\ast\}, \mathbb{Z}(1)) = 0$. From (A.4) one gets

$$H^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)) \cong \tilde{H}^2_{\mathbb{Z}_2}(\mathcal{K}, \mathbb{Z}(1)) \cong H^2_{\mathbb{Z}_2}(\mathcal{K}|\mathcal{K}^\tau, \mathbb{Z}(1))$$

where the isomorphism is induced by the inclusion. □

**References**

[AB] Aharonov, Y.; Bohm, D.: *Significance of electromagnetic potentials in quantum theory*. Phy. Rev. 115, 485–491, (1959)

[AF] Ando, Y.; Fu, L.: *Topological crystalline insulators and topological superconductors: from concepts to materials*. Annu. Rev. Cond. Matt. Phys. 6, 361–381 (2015)

[AM] Ashcroft, N. W.; Mermin N. D.: *Solid State Physics*. Saunders College Pub., Philadelphia, 1976
[AP] Allday, C.; Puppe, V.: Cohomological Methods in Transformation Groups. Cambridge University Press, Cambridge, 1993

[Ati1] Atiyah, M. F.: K-theory and reality. Quart. J. Math. Oxford Ser. (2) 17, 367-386 (1966)

[Ati2] Atiyah, M. F.: K-theory. W. A. Benjamin, New York, 1967

[Bae] Bauer, M.: Beyond Born-Oppenheimer: Electronic Nonadiabatic Coupling Terms and Conical Intersections. Wiley & Sons, Hoboken, 2006

[Ber] Berry M. V.: Quantal Phase Factors Accompanying Adiabatic Changes. Proc. Roy. Soc. Lond. A. 392, 45-57, (1984)

[BES] Bellissard, J.; van Elst, A.; Schulz-Baldes, H.: The Non-Commutative Geometry of the Quantum Hall Effect. J. Math. Phys. 35, 5373-5451 (1994)

[Bha] Bharath H. M.: Non-Abelian geometric phases carried by the spin fluctuation tensor. J. Math. Phys. 59, 062105 (2018)

[BMKNZ] Böhm, A.; and Mostafazadeh, A.; Koizumi, H.; Niu, Q.; Zwanziger, J.: The Geometric Phase in Quantum Systems. Springer-Verlag, Berlin, 2003

[Bor] Borel, A.: Seminar on transformation groups, with contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies 46, Princeton University Press, Princeton, 1960

[Bre] Bredon, G. E.: Equivariant cohomology theories. Lecture Notes in Mathematics No. 34, Springer-Verlag, Berlin-New York, 1967

[BT] Bott, R.; Tu, L. W.: Differential Forms in Algebraic Topology. Springer-Verlag, Berlin, 1982

[CJ] Chruściński, D.; Jamiołkowski, A.: Geometric Phases in Classical and Quantum Mechanics. Birkhäuser, Basel, 2004

[DG1] De Nittis, G.; Gomi, K.: Classification of “Real” Bloch-bundles: Topological Insulators of type AI. J. Geometry Phys. 86, 303-338 (2014)

[DG2] De Nittis, G.; Gomi, K.: Classification of “Quaternionic” Bloch-bundles: Topological Insulators of type AII. Commun. Math. Phys. 339, 1-55 (2015)

[DG3] De Nittis, G.; Gomi, K.: The cohomological nature of the Fu-Kane–Mele invariant. J. Geometry Phys. 124, 124-164 (2018)

[DG4] De Nittis, G.; Gomi, K.: The FKMM-invariant in low dimension. Lett. Math. Phys. 108, 1225-1277 (2018)

[DG5] De Nittis, G.; Gomi, K.: Chiral vector bundles. Math. Z. 290, 775-830 (2018)

[DG6] De Nittis, G.; Gomi, K.: The Cohomology Invariant for Class DIII Topological Insulators. Ann. Henri Poincaré 23, 755-830 (2022)

[Dir] Dirac, P. A. M.: Quantized singularities in the electromagnetic field. Proc. Roy. Soc. Lond. A. 133, 60-72 (1931)

[DK] Davis, J. F.; Kirk, P.: Lecture Notes in Algebraic Topology. AMS, Providence, 2001

[Dup] Dupont, J. L.: Symplectic Bundles and KR-Theory. Math. Scand. 24, 27-30 (1969)

[ES] Eilenberg, S.; Steenrod, N. E.: Foundations of algebraic topology. Princeton 1952

[FKM] Fu, L.; Kane, C. L.; Mele, E. J.: Topological Insulators in Three Dimensions. Phys. Rev. Lett. 98, 106803 (2007)

[Fre] Freyd, P.: Abelian Categories. An Introduction to the Theory of Functors. Harper & Row, 1964

[FZ] Faure, F.; Zhilinskii, B.: Topological Properties of the Born-Oppenheimer Approximation and Implications for the Exact Spectrum. Lett. Math. Phys. 55, 219-238 (2001)

[Gom] Gomi, K.: A Variant of K-Theory and Topological T-Duality for Real Circle Bundles. Commun. Math. Phys. 334, 923-975 (2015)

[GR] Gat, O.; Robbins, J. M.: Topology of time-invariant energy bands with adiabatic structure. J. Phys. A: Math. Theor. 50, (2017)
[Hat] Hatcher, A.: *Algebraic Topology*. Cambridge University Press, Cambridge, 2002

[HK] Hasan, M. Z.; Kane, C. L.: *Colloquium: Topological insulators*. Rev. Mod. Phys. **82**, 3045-3067 (2010)

[Hsi] Hsiang, W. Y.: *Cohomology Theory of Topological Transformation Groups*. Springer-Verlag, Berlin, 1975

[JS] Jante, R.; Schroers, B. J.: *Dirac operators on the Taub-NUT space, monopoles and SU(2) representations*. J. High Energ. Phys. **2014**, 114 (2014)

[Kah] Kahn, B.: *Construction de classes de Chern équivariantes pour un fibré vectoriel Réel*. Comm. Algebra. **15**, 695-711 (1987)

[KM] Kane, C. L.; Mele, E. J.: *$\mathbb{Z}_2$ Topological Order and the Quantum Spin Hall Effect*. Phys. Rev. Lett. **95**, 146802 (2005)

[Kuc] Kuchment, P.: *Floquet theory for partial differential equations*. Birkhäuser, Boston, 1993

[IlI] Illman, S.: *Equivariant singular homology and cohomology*. Bull. Amer. Math. Soc. **79**, 188-192 (1973)

[May] May, J. P.: *Equivariant homotopy and cohomology theory*. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. CBMS Regional Conference Series in Mathematics **91**, AMS, Providence, 1996

[MLa] Mac Lane, S.: *Categories for the Working Mathematician*. Springer, 1978

[Pan] Pancharatnam S.: *Generalized Theory of Interference, and Its Applications. Part I. Coherent Pencils*. Proc. Indian Acad. Sci. A. **44**, 247-262, (1956)

[Pet] Peterson, F. P.: *Some remarks on Chern classes*. Ann. of Math. **69**, 414-420 (1959)

[Se] Serre, J.-P.: *Faisceaux Algébriques Coherents*. Ann. Math. **61**, 197-278 (1955)

[Spa] Spanier, E. H.: *Algebraic Topology*. McGraw-Hill, New York, 1966

[Sw] Swan, R.-G.: *Vector Bundles and Projective Modules*. Trans. Amer. Math. Soc. **105**, 264?277 (1962)

[TKNN] Thouless, D. J.; Kohmoto, M.; Nightingale, M. P.; den Nijs, M.: *Quantized Hall Conductance in a Two-Dimensional Periodic Potential*. Phys. Rev. Lett. **49**, 405-408 (1982)

[Yan] Yang, C. N.: *Magnetic Monopoles, Fiber Bundles, and Gauge Fields*. In *History of Original Ideas and Basic Discoveries in Particle Physics* pg. 55-65, *Springer, Boston, 1996*

(G. De Nittis) FACULTAD DE MATEMÁTICAS & INSTITUTO DE FÍSICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE.

*Email address*: gidenittis@mat.uc.cl

(K. Gomi) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO, 152-8551, JAPAN.

*Email address*: kgomi@math.titech.ac.jp