NON-VANISHING OF TAYLOR COEFFICIENTS AND POINCARÉ SERIES

CORMAC O’SULLIVAN AND MORTEN S. RISAGER

ABSTRACT. We prove recursive formulas for the Taylor coefficients of cusp forms, such as Ramanujan’s Delta function, at points in the upper half-plane. This allows us to show the non-vanishing of all Taylor coefficients of Delta at CM points of small discriminant as well as the non-vanishing of certain Poincaré series. At a “generic” point all Taylor coefficients are shown to be non-zero. Some conjectures on the Taylor coefficients of Delta at CM points are stated.

1. INTRODUCTION

1.1. Background. Let \( \Gamma = \text{SL}(2, \mathbb{Z}) \) be the full modular group acting on the upper half plane \( \mathbb{H} \). Then \( \Gamma \) has a cusp at \( \infty \) with stabilizer \( \Gamma_\infty = \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) | n \in \mathbb{Z} \} \). Denote the space of holomorphic cusp forms for \( \Gamma \) of even weight \( k \) by \( S_k(\Gamma) \). Every \( f \in S_k(\Gamma) \) has a \( q \)-expansion of the form

\[
f(z) = \sum_{m=1}^{\infty} c_\infty(f, m)q^m,
\]

for \( q = e^{2\pi i z} \). The famous inhabitant of the one-dimensional space \( S_{12}(\Gamma) \) is

\[
\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^24 = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6),
\]

Ramanujan’s Delta function. Ramanujan discovered many of the remarkable arithmetic properties of the coefficients \( \tau(m) := c_\infty(\Delta, m) \) that bear his name. These properties were later proved, most notably by Ramanujan himself, Mordell and Deligne. In 1947 Lehmer \[16\] asked if the following statement is true:

\[
\tau(m) \neq 0 \quad \text{for every} \quad m \in \mathbb{Z}_{\geq 1}.
\]

We will refer to (1.3) as Lehmer’s conjecture. Lehmer initially verified that \( \tau(m) \neq 0 \) for \( m < 3316799 \). This has been greatly improved over the years and is now verified for \( m < 2 \cdot 10^{19} \) \[2\]. Recently Lehmer’s conjecture has found an interesting interpretation in terms of spherical t-designs: it is equivalent to the shell of norm \( 2m \) of the E8 lattice never being a spherical 8-design for \( m \in \mathbb{Z}_{\geq 1} \). See for example \[1\] and the contained references for this connection.

Associated with each point \( z_0 = \alpha + i\beta \in \mathbb{H} \) there is another natural expansion for \( \Delta \), less well-known than (1.2), that we describe next. Let \( \mathbb{D} \) be the open unit disc, centered at the origin in \( \mathbb{C} \), and set

\[
\sigma_{z_0} = \frac{1}{2i\beta} \left( \begin{smallmatrix} \bar{z}_0 & z_0 \\ -1 & 1 \end{smallmatrix} \right) \in \text{GL}(2, \mathbb{C}).
\]

Then \( \sigma_{z_0}0 = z_0, \sigma_{z_0}\infty = \bar{z}_0 \) and \( z \mapsto \sigma_{z_0}z \) is a biholomorphic map \( \mathbb{D} \to \mathbb{H} \). Let \( f : \mathbb{H} \to \mathbb{C} \). The stroke operator \( \mid \) is defined as usual by

\[
(f|_k \gamma)(z) := \frac{\det(\gamma)^{k/2} f(\gamma z)}{j(\gamma, z)^k} \quad \text{for} \quad \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), \quad j(\gamma, z) := cz + d.
\]

If \( f \) is holomorphic in \( \mathbb{H} \) then \( f|_k \sigma_{z_0}(z) \) is holomorphic in \( \mathbb{D} \) with a Taylor expansion at zero:

\[
f|_k \sigma_{z_0}(z) = \sum_{n=0}^{\infty} c_{z_0}(f, n)z^n.
\]

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Example 1.1. At \( z_0 = i \) and \( z_0 = \omega := e^{2\pi i/3} \), the elliptic fixed points for \( \Gamma \), we have

\[
\frac{\Delta_{12r_i}(z)}{-64\Delta(i)} = 1 - 12\frac{(r_iz)^2}{2!} + 216\frac{(r_iz)^4}{4!} + 10368\frac{(r_iz)^6}{6!} + O(z^8)
\]

\[
\frac{\Delta_{12r_\omega}(z)}{-27\Delta(\omega)} = 1 + 48\frac{(r_\omega z)^3}{3!} + 18432\frac{(r_\omega z)^6}{6!} + 13271040\frac{(r_\omega z)^9}{9!} + O(z^{12})
\]

where \( r_i = -\Gamma(1/4)/(8\sqrt{3}\pi^2) \) and \( r_\omega = -\sqrt{3}\Gamma(1/3)/(16\pi^3) \). See §5.1 and 5.2 for (1.6) and (1.7).

The Taylor expansion (1.5) is a very natural one to use as its radius of convergence is 1 and

\[
f(z) = \frac{(z_0 - z_0)^k/2}{(z - z_0)^k} \sum_{n=0}^{\infty} c_{z_0}(f, n) (\sigma_{z_0}^{-1}z)^n
\]

is valid for all \( z \in \mathbb{H} \). The numbers \( c_{z_0}(f, n) \) in (1.5) have a dual nature. Certainly they are the (normalized) Taylor coefficients of \( f \) at \( z_0 \), and as Taylor coefficients they may be found from the derivatives of \( f \), as in Proposition 3.1. This property will be crucial in §3.5. It is also important to regard the numbers \( c_{z_0}(f, n) \) as Fourier coefficients and obtain them through integration, as in (3). This also highlights their similarities to the Fourier coefficients at cusps, such as \( c_\infty(f, m) \) in (1.1).

Remark 1. We notice that \( c_{z_0}(\Delta, n) = 0 \) unless \( n \) is divisible by 2 in (1.6), and divisible by 3 in (1.7). This vanishing is a basic general phenomenon: If \( f \) is a form of weight \( k \), and \( z_0 \) is a fixed point of order \( N \geq 1 \) for \( \Gamma \) then \( c_{z_0}(f, n) = 0 \) whenever \( n \not\equiv -k/2 \mod N \). See §3 for an explanation. When \( n \not\equiv -k/2 \mod N \) we call the corresponding Fourier coefficients trivial. Note that if \( z_0 \) is not an elliptic fixed point (i.e. if \( N = 1 \)) then there are no trivial Fourier coefficients.

1.2. The main results. We first investigate Fourier expansions at CM points. Let \( D \) be a negative fundamental discriminant, i.e. \( D < 0 \) with \( D \equiv 1 \mod 4 \) and squarefree, or with \( D = 4m \) where \( m \equiv 2, 3 \mod 4 \) and squarefree. We consider CM points of the form

\[
\zeta D := \begin{cases} \sqrt{D}/2 & \text{for } D \text{ even} \\ (1 + \sqrt{D})/2 & \text{for } D \text{ odd}, \end{cases}
\]

such that \( D \) is the discriminant of \( \mathbb{Q}(\sqrt{D}) \), with ring of integers \( \mathcal{O}_{\mathbb{Q}(\sqrt{D})} = \text{span}_{\mathbb{Z}}(1, \zeta D) \).

For \( \zeta D \) a CM point and \( f \in S_k(\Gamma) \) with algebraic Fourier coefficients at \( \infty \), it is known (due to the work of Damerell and others - see for example [6] for references) that there exist non-zero complex numbers \( \kappa \) and \( \lambda \) such that the quotients

\[
c_{\zeta D}(f, n)/(\kappa \lambda^n)
\]

are algebraic for all \( n \). We show, using a technique of Rodriguez Villegas and Zagier, that for \( \Delta(z) \) these algebraic numbers are essentially the constant terms in a recursively defined sequence of polynomials with algebraic integer coefficients:

Theorem 1.2. For every CM point \( \zeta D \) with \( D \) a negative fundamental discriminant, there exists a number field \( K \), constants \( \kappa_\zeta, \lambda_\zeta \in \mathbb{C} \) and \( a(t) \in \mathcal{O}_K[t] \) (all depending on \( \zeta D \) and explicitly given), so that the \( n \)-th Fourier coefficient of \( \Delta(z) \) at \( \zeta D \) is

\[
c_{\zeta D}(\Delta, m) = \kappa_{\zeta D}^{m/3} q_{m, \zeta D}(0) \quad (m \geq 0)
\]

with \( q_{m, \zeta D}(t) \in \mathcal{O}_K[t] \) defined recursively by

\[
q_{0, \zeta D}(t) = 1, \quad q_{1, \zeta D}(t) = a_1(t),
\]

\[
q_{n+1, \zeta D}(t) = (a_1(t) + n a_2(t)) q_{n, \zeta D}(t) + a_3(t) q_{n, \zeta D}'(t) + n(n + 11) a_4(t) q_{n-1, \zeta D}(t) \quad (n \geq 1).
\]
Theorem 1.3 is proved in [5] where \( K, \kappa_j, \lambda_j \) and \( a_i(t) \) for \( i = 1, 2, 3, 4 \) are precisely defined. A set \( \mathcal{D} \) of negative fundamental discriminants we will focus on is
\[
\mathcal{D} := \{-3, -4, -7, -8, -11, -15, -19, -20, -24\}.
\]
Studying the polynomials \( q_{m,j}(t) \) in (1.10) modulo \( l \) we find that, for certain primes \( l \), \( q_{m,j}(0) \) becomes periodic mod \( l \) and all values in a period are non-zero. This in turn allows us to conclude that all the Fourier coefficients \( c_j(\Delta, n) \) are non-zero:

**Theorem 1.3.** For \( D \in \mathcal{D} \), all non-trivial Fourier coefficients of \( \Delta \) at \( \gamma_D \) are non-zero.

**Remark 2.** The nine points we consider in Theorem 1.3 are especially simple, but it is relatively straightforward to test other CM points. We prove in Theorem 6.1 that for every CM point \( \gamma \) and every prime \( l \) the sequence \( q_n(\gamma) \), \( n \geq 0 \) becomes periodic mod \( l \). To prove non-vanishing we require a prime \( l \) such that the whole corresponding period is non-zero mod \( l \). It is tempting to speculate that such a prime always exists and therefore that the non-trivial Fourier coefficients of \( \Delta \) at all CM points are always non-zero.

The sequences \( q_n(\gamma) \) for \( n \geq 0 \) possess many interesting and, as yet, unexplained features. For example, let \( q_0 \) be the coefficient of \((r_1 z)^n/n!\) in (1.6) i.e. \( q_0 = 1, q_1 = 0, q_2 = -12 \), etc. As a special case of more general results in [6] we show the following.

**Theorem 1.4.** For all primes \( l \) with \( 2 < l < 100 \) we have
\[
q_n \to 0 \mod l \iff l \equiv 3 \mod 4.
\]

We next examine the vanishing of \( c_{2n}(f, n) \) for \( f \in S_k(\Gamma) \) and \( z_0 \) a general point in \( \mathbb{H} \). Recall from Remark 1 that for \( z_0 \in \mathbb{H} \) with order \( N \), \( c_{2n}(f, n) \) is non-trivial when \( n \equiv -k/2 \mod N \). (If \( z_0 \) is a cusp then we say all \( c_{2n}(f, n) \) for \( n \geq 1 \) are non-trivial.) To highlight the points we are interested in, define
\[
A_f := \{ z_0 \in \mathbb{H} \cup \{ \text{cusps of } \Gamma \} \mid c_{2n}(f, n) = 0 \text{ for some non-trivial Fourier coefficient} \}.
\]
We would like to know the detailed structure of this set. With this notation, we see for instance that Lehmer’s conjecture may be written succinctly as \( \infty \notin A_{\Delta} \).

**Theorem 1.5.** For \( f \in S_k(\Gamma) \), not identically 0, the set \( A_f \) has measure 0.

Theorem 1.5 asserts that for a generic point all Fourier coefficients are non-vanishing. Since the countable set of cusps has measure 0, it is really a statement about non-vanishing at points in \( \mathbb{H} \). The proof we give in [6] works for general Fuchsian groups \( \Gamma \) that may not have cusps.

**Theorem 1.6.** The set \( A_{\Delta} \) is non-empty. It contains \( z_0 \approx 1.344i \) for which \( c_{2n}(\Delta, 2) = 0 \).

Theorem 1.6 is proved in [7] where we numerically find further elements of \( A_{\Delta} \). Lastly, in [8] we generalize an identity of Petersson, (8.1), and exhibit formulas for the averages
\[
\sum_{f \in \mathcal{F}} c_{2n}(f, m)c_{2n}(f, n) \quad \text{and} \quad \sum_{f \in \mathcal{F}} c_{2n}(f, m)c_{2n}(f, n)
\]
with \( z_0, z'_0 \in \mathbb{H} \) and \( \mathcal{F} \) an orthonormal basis of \( S_k(\Gamma) \).

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2. Applications

In this section we describe two applications of our non-vanishing result, Theorem 1.3

2.1. Non-vanishing of Poincaré series. The problem of proving non-vanishing for Poincaré series is very interesting and has been investigated by numerous authors e.g. [25, 8, 31, 7, 19, 30, 17, 22].

Let \( e_m(z) := e^{2\pi imz} \). For \( m \in \mathbb{Z}_{\geq 1} \), the \( m \)-th Poincaré series of weight \( 4 \leq k \in 2\mathbb{Z} \) associated to the cusp at infinity is
\[
P_m(z) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} (e_m|_{k}\gamma)(z) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \frac{e^{2\pi im\gamma z}}{f(\gamma, z)^k}.
\]
These parabolic Poincaré series span the finite dimensional space \( S_k(\Gamma) \). Petersson [23] proved that, in fact, \( P_1, P_2, \ldots, P_{d(k)} \) form a basis for \( S_k(\Gamma) \) with \( d(k) := \dim(S_k(\Gamma)) \). When \( d(k) = 0 \) (which happens precisely when \( k \in \{4, 6, 8, 10, 14\} \)) the Poincaré series \( P_m(z) \) must vanish identically. In the simplest non-trivial case, with \( k = 12 \) and \( d(k) = 1 \), the non-vanishing of every \( P_m(z) \) is equivalent to Lehmer’s conjecture since

\[
(2.2) \quad P_m = \tau(m) \left[ \frac{10!}{(4\pi m)^{11}} \right] \frac{\Delta}{\|\Delta\|^2}
\]

(See e.g. [11 (3.29)]). Equation \((2.2)\) follows from the fact that \( S_{12}(\Gamma) \) is one-dimensional containing \( \Delta \) and the next result, first demonstrated by Petersson in [24]. See also [11, Theorem 3.3] for example, for its proof. Here \( \|\cdot\| \) is the norm corresponding to the Petersson inner product \( \langle \cdot, \cdot \rangle \).

**Proposition 2.1.** For \( f \in S_k(\Gamma) \) and \( n \geq 1 \) we have

\[
\langle f, P_n \rangle = c_{\infty}(f, n) \left[ \frac{(k - 2)!}{(4\pi n)^{k-1}} \right].
\]

We also note that if \( m \ll k^2 - e \) then, using Petersson’s formula (8.1), Rankin [27, Theorem 1] proved that \( P_m \not\equiv 0 \).

A second family of Poincaré series was introduced by Petersson in [24] as Poincaré series of elliptic type. They are associated not with cusps but with points \( z_0 \in \mathbb{H} \). Recall \( \sigma_{z_0} \) from (1.4) with \( \det(\sigma_{z_0}) = 1/(2i\beta) \) and \( \sigma_{z_0}^{-1} = (1 - \frac{z}{z_0}) \) mapping the upper half-plane \( \mathbb{H} \) biholomorphically to the open unit disc \( \mathbb{D} \).

Let \( \mu_m(z) = z^m \). For \( 4 \leq k \in 2\mathbb{Z} \) and \( m \in \mathbb{Z}_{\geq 0} \) define

\[
P_{z_0, m}(z) := \sum_{\gamma \in \Gamma} (\mu_m |_{k} \sigma_{z_0}^{-1} \gamma)(z)
\]

\[
= (2i\beta)^{k/2} \sum_{\gamma \in \Gamma} \frac{(\sigma_{z_0}^{-1} \gamma z)^m}{j(\sigma_{z_0}^{-1} \gamma, z)^k}
\]

Each Poincaré series \( P_{z_0, m} \) is in \( S_k(\Gamma) \) and, for fixed \( z_0 \) with \( m \) varying, they span \( S_k(\Gamma) \). This spanning result is implied by the following analogue of Proposition 2.1 also from [24]. See [10, §4.3] as well.

**Proposition 2.2.** Let \( z_0 \in \mathbb{H} \). Then for \( f \in S_k(\Gamma) \) and \( m \geq 0 \) we have

\[
\langle f, P_{z_0, m} \rangle = c_{z_0}(f, m) \left[ \frac{\pi (k - 2)! m!}{9^{k-3} (m + k - 1)!} \right].
\]

Thus, the series \( P_{z_0, m} \) isolates the \( m \)-th Fourier coefficient at \( z_0 \) in the same manner as \( P_m \) isolates the \( m \)-th Fourier coefficient at the cusp \( \infty \). (Note that \( P_{z_0, m}(z) = 2\Phi_{\mathbb{D}}(z, m, z_0) \) in the notation of [10].)

One of our original motivations for the work in this paper is the following basic question:

\[
(2.4) \quad \text{For which } z_0, m \text{ is } P_{z_0, m} \equiv 0 ?
\]

Clearly there is a lot of cancelation in (2.3) since \( (\sigma_{z_0}^{-1} \gamma z)^m \) circles around \( \mathbb{D} \) as \( \gamma \) runs through \( \Gamma \). We prove the following result about Poincaré series of weight \( k = 12 \) for the modular group:

**Theorem 2.3.** Let \( D \in \mathcal{D}, m \in \mathbb{Z}_{\geq 0} \) and assume \( 2|m \) if \( 3_D = 3_{-4} \) and \( 3|m \) if \( 3_D = 3_{-3} \). Then each Poincaré series \( P_{3_D, m} \in S_{12}(\Gamma) \) is not identically zero.

**Proof.** Proposition 2.2 implies that in \( S_{12}(\Gamma) \),

\[
P_{z, m} = c_z(\Delta, m) \left[ \frac{10! m! \pi}{2^{9}(m + 11)!} \right] \frac{\Delta}{\|\Delta\|^2}.
\]

Therefore \( P_{z, m} \) vanishes identically if and only if \( c_z(\Delta, m) \) is zero. Hence Theorem 2.3 follows from Theorem 1.3 and Remark 2 also applies to the vanishing of \( P_{z, m} \). \( \square \)
It is a consequence of Proposition 2.2 and Remark 1 that \( P_{z_0,m} \) necessarily vanishes identically in \( S_k(\Gamma) \) when \( m \neq -k/2 \mod N \) if \( z_0 \) has order \( N \). We label such a \( P_{z_0,m} \) \textit{trivial}. Petersson proved in [23] that the first \( d(k) \) non-trivial Poincaré series in \( \{P_{z_0,0}, P_{z_0,1}, P_{z_0,2}, \ldots \} \) form a basis for \( S_k(\Gamma) \). We see from (2.5) and Theorem 1.6 that \( P_{z_0,2} \) vanishes non-trivially in \( S_{12}(\Gamma) \) for \( z_0 \approx 1.344i \). We return to question (2.4) in §§7§8.

2.2. \textbf{Non-vanishing of central values of certain \( L \)-functions}. Lastly, we briefly speculate on how this work may be applied to another very interesting question, namely non-vanishing of central critical values of certain character twists of \( L \)-functions.

The relation between non-vanishing of Fourier coefficients at CM points and critical values of certain character twists of \( L \)-functions follows from Waldspurger type formulas (see e.g. [34, 14] and for more recent developments [4, 18, 9] and the references therein.)

Let \( f \in S_k(\Gamma) \). Waldspurger type formulas give, usually through some type of theta correspondence, an equality between the absolute value squared of an adelic twisted toric integral involving \( f \) and (a non-zero constant times) the central critical value of the twisted \( L \)-function of the base change \( \pi_{f,K} \) of \( f \) to \( K = \mathbb{Q}(iD) \), i.e. the value \( L(\pi_{f,K}) \otimes \chi^n, 1/2 \) where \( \chi \) is the basic Grössencharacter of \( K \). In the present case \( \pi_{f,K} \) is the classical Doi-Naganuma lift.

On the other hand, the adelic twisted toric integral can, in certain situations, be related to the future work.

3. \textbf{Fourier coefficients}

In this section we allow \( \Gamma \) to be, more generally, a discrete, finitely generated subgroup of \( \text{SL}(2, \mathbb{R}) \) with \( \text{Vol}(\Gamma \setminus \mathbb{H}) < \infty \). For every \( z_0 \in \mathbb{H} \cup \{ \text{cusps of } \Gamma \} \), each \( f \in S_k(\Gamma) \) has a Fourier expansion that we next describe briefly.

For any \( \Gamma' \subset \Gamma \), denote by \( \overline{\Gamma z_0} \) its image under the map \( \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R})/\pm 1 \). Let \( \Gamma_{z_0} \subset \Gamma \) be the stabilizer of \( z_0 \). Then \( \overline{\Gamma z_0} \) is cyclic and there exists a generator \( \gamma_0 \in \Gamma \) such that \( \overline{\Gamma z_0} = \langle \gamma_0 \rangle \). If \( z_0 \) is a cusp we may pick a scaling matrix \( \sigma_{z_0} \in \text{SL}(2, \mathbb{R}) \) such that

\[
\sigma_{z_0}^{-1} z_0 = \infty, \quad \sigma_{z_0}^{-1} \gamma_0 \sigma_{z_0} z = z + 1.
\]

If \( z_0 \in \mathbb{H} \) we may choose \( \sigma_{z_0} \in \text{Isom}^+(\mathbb{D}, \mathbb{H}) \), given by (1.4) for example, with

\[
\sigma_{z_0}^{-1} z_0 = 0, \quad \sigma_{z_0}^{-1} \gamma_0 \sigma_{z_0} z = \zeta^2 z
\]

and \( \zeta^2 = j(\gamma_0, \overline{z_0})^2 \) necessarily a primitive \( N \)-th root of unity for \( 1 \leq N = |\overline{\Gamma z_0}| < \infty \). See, for example, [32] §1.2 and [10] §§2.1, 4.1 for more details.

For \( z_0 \) a cusp we therefore have the Fourier expansion

\[
f|_k \sigma_{z_0} (x + iy) = \sum_{n=-\infty}^{\infty} a_{z_0}(f, n)e^{2\pi i nx}
\]

with \( a_{z_0}(f, n) \) depending on \( y \). For \( z_0 \in \mathbb{H} \) we consider \( z \in \mathbb{D} \) in polar coordinates \((r, \theta)\), normalized so that moving around \( 0 \) once on a circle corresponds to \( \theta \rightarrow \theta + 1 \). We have the Fourier expansion

\[
f|_k \sigma_{z_0} \left( re^{2\pi i \theta} \right) = \sum_{n=-\infty}^{\infty} a_{z_0}(f, n)e^{2\pi i n \theta} \quad (z_0 \in \mathbb{H})
\]

with

\[
a_{z_0}(f, n) = \int_{0}^{1} f|_k \sigma_{z_0} \left( re^{2\pi i \theta} \right) e^{-2\pi i n \theta} d\theta
\]
depending on \( r < 1 \). Rewrite the \( q \)-expansion (1.1) and the normalized Taylor expansion (1.5) as follows:

\[
(3.4) \quad f|_{k}\sigma_{z_{0}}(x + iy) = \sum_{n=1}^{\infty} \left[ c_{z_{0}}(f, n)e^{-2\pi ny}\right]e^{2\pi inx} \quad \text{for } z_{0} \text{ a cusp},
\]

\[
(3.5) \quad f|_{k}\sigma_{z_{0}}(re^{2\pi i\theta}) = \sum_{n=0}^{\infty} \left[ c_{z_{0}}(f, n)r^{n}\right]e^{2\pi in\theta} \quad \text{for } z_{0} \in \mathbb{H}.
\]

Comparing these with (3.1) and (3.2) yields formulas for each \( a_{z_{0}}(f, n) \). Accordingly,

\[
a_{z_{0}}(f, n) = \begin{cases} 
  c_{z_{0}}(f, n)e^{-2\pi ny} & \text{if } n \geq 1 \text{ and } z_{0} \in \{\text{cusps of } \Gamma\} \\
  c_{z_{0}}(f, n)r^{n} & \text{if } n \geq 0 \text{ and } z_{0} \in \mathbb{H}
\end{cases}
\]

and \( a_{z_{0}}(f, n) = 0 \) otherwise. In light of this, we label \( c_{z_{0}}(f, n) \) the \( n \)-th Fourier coefficient \( f \) at \( z_{0} \) for all \( z_{0} \in \mathbb{H} \cup \{\text{cusps of } \Gamma\} \), as in the introduction.

Since \( f \) is weight \( k \) with respect to \( \Gamma \) and \( \gamma \in \Gamma \) we have

\[
(3.6) \quad \zeta^{k}f|_{k}\sigma_{z_{0}}(\zeta^{2}z) = \frac{f|_{k}\sigma_{z_{0}}(\sigma_{z_{0}}^{-1}\gamma_{0}\sigma_{z_{0}}z)}{j(\sigma_{z_{0}}^{-1}\gamma_{0}\sigma_{z_{0}}, z)^{k}} = f|_{k}\sigma_{z_{0}}(\sigma_{z_{0}}^{-1}\gamma_{0}\sigma_{z_{0}}(z)) = f|_{k}\sigma_{z_{0}}(z).
\]

Using that \( \zeta^{2} = e^{2\pi iy/N} \) where \((m, N) = 1\) and inserting (3.6) into (3.3) we obtain

\[
a_{z_{0}}(f, n) = e^{2\pi i(2mn+ink)/2N}a_{z_{0}}(f, n),
\]

and conclude that unless \((n + k/2) \equiv 0 \mod N\) we must have \( c_{z_{0}}(f, n) = 0 \), as observed before in Remark 4.

We have already seen that \( c_{z_{0}}(f, m) \) can be expressed as an inner product in Proposition 2.2 and as a polar integral in (3.3). Since it is also a Taylor coefficient it follows, as in [24], (see also for example 20 Prop. 16)], that it can be expressed in terms of derivatives of \( f \):

**Proposition 3.1.** For \( f \in S_{k} \) and \( z_{0} \in \mathbb{H} \),

\[
(3.7) \quad c_{z_{0}}(f, m) = \sum_{r=0}^{m} \binom{m + k - 1}{r + k - 1} \frac{r/2^{r}}{(r!)} f^{(r)}(z_{0}).
\]

The Maass raising operator is

\[
(3.8) \quad \partial_{k}f := \mathcal{D}f - \frac{k}{4\pi y}f \quad \text{for } \mathcal{D}f := \frac{1}{2\pi i} \frac{d}{dz}f(z).
\]

It raises the weight by 2 but does not preserve holomorphy. We follow the convention of writing

\[
\partial^{m} = \partial_{k+2m-2} \circ \cdots \circ \partial_{k+2} \circ \partial_{k}
\]

and similarly for \( \vartheta \) below. (See 8.5 and 8.11 in 8 for the computation of \( \partial^{m}P_{n} \) and \( \partial^{m}P_{m,n} \).)

Use induction to verify that

\[
(3.9) \quad \partial^{m}f(z) = \frac{m!}{(-4\pi y)^{m}} \sum_{r=0}^{m} \binom{m + k - 1}{r} \frac{(2iy)^{r}}{r!} f^{(r)}(z)
\]

and comparing (3.7) with (3.9) yields

\[
(3.10) \quad c_{z}(f, m) = (2\pi i)^{m} \binom{2iy}{m+k/2} \frac{\partial^{m}f(z)}{m!}
\]

so that \( c_{z}(f, m) y^{-k/2-m} \) is a non-holomorphic modular form of weight \( k + 2m \). The identity (3.10) will be the key to finding formulas for the coefficients \( c_{z}(f, m) \) in the next two sections and allows us to prove Theorem 1.5.

**Proof of Theorem 1.5** Consider \( c_{z}(f, m) \) as a function of \( z \). From Proposition 3.1 we see that this function is real analytic. We claim that it does not vanish identically. Once this has been established real analyticity allows us to conclude that its set of zeros has measure zero. Then \( A_{f} \), as the union of countably many sets of measure zero, is again of measure zero and the theorem is proven.
To see that $c_z(f, m)$ is non-vanishing we argue as follows. If we have a cusp (at $\infty$, say) and the $n_0$-th term is the first non-vanishing term in the Fourier expansion of $f$ at $\infty$ then $f^{(r)}/f \to (2\pi i n_0)^{r}$ as $y \to \infty$. From this and Proposition [5,1] it easily follows that $c_z(f, m)$ is not identically zero. If $\Gamma$ does not have any cusps we assume that $c_{z_0}(f, m)$ vanishes identically. Identity (3.10) implies that $c_{z_0}(f, n) \equiv 0$ also for all $n > m$. Thus

$$f(z) = \left(\frac{z_0 - z}{z_0 - z_0}\right)^{k/2} \sum_{n=0}^{m-1} c_{z_0}(f, n) \left(\frac{z - z_0}{z - z_0}\right)^n$$

implying in particular that $f(z)$, as a rational function, has a finite number of zeros in $\mathbb{H}$. But the results [32, Prop. 2.16, Theorem 2.20] imply that $\deg(\text{div}(f)) = k \text{Vol}(\Gamma \backslash \mathbb{H})/(4\pi)$ and hence that the holomorphic $f$ has at least one zero in $\Gamma \backslash \mathbb{H}$. Therefore $f$ has infinitely many zeros in $\mathbb{H}$ and we have contradicted our assumption that $c_{z_0}(f, m)$ vanishes identically. □

4. Recursive formulas for Fourier coefficients at non-cuspidal points

We review the theory of Rodriguez Villegas and Zagier which in certain arithmetic cases allows us to compute the coefficients $c_{z_0}(f, m)$ explicitly. See [33], and [5, p. 50 - 55, p. 88] in particular. To proceed further, we focus exclusively on the group $\Gamma = \text{SL}(2, \mathbb{Z})$ in §4[4,5,6] and [7]. Let

$$E_2(z) := 1 - 24 \sum_{n \in \mathbb{Z}_{\geq 1}} \sigma_1(n) e^{2\pi i n z}$$

be the holomorphic quasimodular Eisenstein series and put $E^*_k(z) = E_2(z) - 3/(\pi y)$ so that $E^*_k$ has weight 2 but is not holomorphic. The weight $k > 2$ holomorphic Eisenstein series is

$$E_k(z) := \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma} (1|k\gamma)(z) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}} \frac{1}{(cz + d)^k}$$

To aid clarity we will often use the alternate notation $Q$ for $E_4$ and $R$ for $E_6$.

A variation of (3.3), defining the Maass raising operator $\partial$, gives the Serre derivative

$$\partial f = \partial_k f := Df - \frac{k}{12} E_2 f$$

mapping $S_k(\Gamma) \to S_{k+2}(\Gamma)$. Our next goal is to show:

**Theorem 4.1.** For $m \geq 0$ the $m$-th Fourier coefficient of $\Delta$ at any point $z \in \mathbb{H}$ is

$$c_z(\Delta, m) = \Delta(z) \left(\frac{\pi i}{6}\right)^m (z - \overline{z})^{m+6} \sum_{r=0}^{m} \frac{1}{r!} \binom{m + 11}{r + 11} (E^*_k)^{m-r} B_r$$

with $B_r \in M_2(r)(\Gamma)$ defined recursively as

$$B_0 = 1, \quad B_1 = 0, \quad B_{n+1} = 12\partial B_n - n(n + 11)QB_{n-1} \quad (n \geq 1).$$

(Computing, we find for example that $B_2 = -12Q, B_3 = 48R, B_4 = 216Q^2, B_5 = -4608QR$ and $B_6 = 1152(9Q^3 + 16R^2)$.)

**Proof.** Rodriguez Villegas and Zagier recursively define a modified Serre derivative that also stays in the space of holomorphic cusp forms:

$$\partial^{[0]} f = f, \quad \partial^{[1]} f = \partial f, \quad \partial^{[n+1]} f = \partial(\partial^{[n]} f) - n(n + k - 1)\frac{E_4}{144}\partial^{[n-1]} f.$$ 

We have $\partial^{[n]} : S_k(\Gamma) \to S_{k+2n}(\Gamma)$. Then an induction argument (or combining equalities (56) and (65) in [5]) yields

$$\partial^m f = \sum_{r=0}^{m} \frac{m!}{r!} \binom{m + k - 1}{r + k - 1} \left(\frac{E^*_k}{12}\right)^{m-r} \partial^{[r]} f.$$
To compute \( \vartheta^{[n]} f \) explicitly we write it as a polynomial in \( Q \) and \( R \). Recall that any element of \( S_{k+2n}(\Gamma) \) may be expressed as a linear combination of terms of the form \( R^a Q^b \) with \( 6a + 4b = k + 2n \). Set

\[
A_n(f) := 12^n \vartheta^{[n]} f
\]

with \( A_n(f) \) a polynomial in \( Q, R \) as described above. The factor \( 12^n \) is included to ensure that we obtain integer coefficients. Then, using results going back to Ramanujan \([5, \text{p. 88}]\), we have

\[
(4.5) \quad B_n = \left( \frac{Q}{R^{3/2}} \right)^a Q^{n/2}.
\]

So, assuming that \( Q \neq 0 \), we may write

\[
(4.6) \quad B_n = Q^{n/2} p_n(RQ^{-3/2}).
\]

Substituting (4.5) into (4.1), we see that \( p_n(t) \in \mathbb{Z}[t] \) satisfies the recursion

\[
(4.7) \quad p_0 = 1, \quad p_1 = 0, \quad p_{n+1} = -2nt p_n + 6(t^2 - 1)p'_n - n(n + 11)p_{n+1} \quad (n \geq 1).
\]

Alternatively, for \( R \neq 0 \) write each term \( Q^a R^b = \left( \frac{Q}{R^{3/2}} \right)^a R^{n/3} \) so that

\[
(4.8) \quad q_0 = 1, \quad q_1 = 0, \quad q_{n+1} = -2nt^2 q_n + 4(t^3 - 1)q'_n - n(n + 11)q_{n+1} \quad (n \geq 1).
\]

5. Fourier developments at CM points

Let \( D \) be a negative fundamental discriminant as before and recall the notation \( z_D \) from (1.8). Define the Chowla-Selberg period

\[
\Omega_D := \frac{1}{\sqrt{2\pi |D|}} \left[ \prod_{j=1}^{[D] - 1} \Gamma(j/|D|) \left( \frac{D}{j} \right)^{\frac{1}{2} - j} \right]^{\frac{1}{2\pi |D|}}
\]

for \( D < -4 \) and also for \( D = -3, -4 \) but with the class number \( h(D) \) replaced by \( 1/3, 1/2 \) respectively. Clearly \( \Omega_D \in \mathbb{R}_{>0} \).
5.1. **Fourier expansion of $\Delta$ at $i$.** We are in the case $D = -4$ with $\mathfrak{z} = i$ and

$$
\Omega_{-4} = \frac{1}{2\sqrt{2\pi}} \left[ \Gamma(1/4) \right] = \frac{\Gamma(1/4)^2}{4\pi^{3/2}} \approx 0.5902.
$$

Using the table on p. 87 of [5]\(^1\)

$$
E_2^*(i) = 0, \quad Q(i) = 12\Omega_{-4}, \quad R(i) = 0, \quad \Delta(i) = \Omega_{-4}^{12}.
$$

Therefore, with Theorem 4.1 and (4.5)

(5.1)

$$
c_i(\Delta, n) = -|D|^3\Delta(i) \left( \frac{-2\pi \Omega_{l}^2}{\sqrt{3}} \right) \frac{p_n(0)}{n!} \quad (n \geq 0)
$$

with $p_n(0) \in \mathbb{Z}$. For example, with $0 \leq n < 12$ the numbers $p_n(0)$ are

$$
1, \quad 0, \quad -12, \quad 0, \quad 216, \quad 0, \quad 10368, \quad 0, \quad -2052864, \quad 0, \quad 47029248, \quad 0.
$$

We are already now in a position to prove part of Theorem 1.3 that $c_i(\Delta, n) \neq 0$ for all even $n$.

**Proposition 5.1.** With $p_n(t)$ defined as above by (4.6) we have

$$
p_{2m}(0) \equiv 1 \mod 5 \quad \text{for } m \text{ even},
$$

$$
p_{2m+1}(0) \equiv 3 \mod 5 \quad \text{for } m \text{ odd}.
$$

*Proof.* Calculating the polynomials $p_n(t)$ modulo $5$ for $0 \leq n < 20$, we find:

(5.2)

$$
1, \quad 0, \quad 3, \quad 3t, \quad 3 + 2t, \quad 3 + 2t^2, \quad 3t + 2t^3, \quad 4t + 4t^3, \quad 4t + 4t^3.
$$

The recursion (4.6) modulo $5$ is

(5.3)

$$
p_{n+1}(t) \equiv -2ntp_n(t) + (t^2 - 1)p_n(t) - n(n+1)p_{n-1}(t)
$$

and we see that it only depends on $n \mod 5$. Computing the next two terms we find $p_{20}(t) \equiv 1$ and $p_{21}(t) \equiv 0$. Thus we must have $p_{n+20}(t) \equiv p_n(t) \mod 5$ and the result follows. \(\square\)

**Remark 3.** A simplification we will need later is possible. From the recursion (5.3) we see that the constant term of $p_{n+1}(t)$ depends on the constant terms of $p_n(t)$ and $p_{n-1}(t)$. It may also depend on the coefficient of $t$ in $p_n(t)$ because of the term $-p'_n(t)$ in (5.3). In turn the coefficient of $t$ may depend on a previous coefficient of $t^2$, and so on. But if we are working mod $5$ the coefficients of $t^5$ and higher powers of $t$ cannot affect the constant term of subsequent $p_n(t)$s because $(d/dt)t^5 \equiv 0 \mod 5$. Thus the terms $4t^5$ and $4t^6$ in (5.2) do not play a role and may be ignored.

Accordingly, we define the ring

(5.4)

$$
\mathcal{R}_l := (\mathbb{Z}/l\mathbb{Z})[t]/(t^l)
$$

of polynomials of degree at most $l - 1$ with coefficients in $\mathbb{Z}/l\mathbb{Z}$. Denote the natural projection $\mathbb{Z}[t] \to \mathcal{R}_l$ by $p \mapsto \overline{p}$. Thus we have proved that if we define a recursion

$$
p_0^* = 1, \quad p_1^* = 0, \quad p_{n+1}^* = -2ntp_n^* + 6(t^2 - 1)(p_n^*)' - n(n+1)p_{n-1}^*
$$

in $\mathcal{R}_l$ then

$$
\overline{p}_n \equiv p_n^* \text{ in } \mathcal{R}_l
$$

and

$$
p_n(0) \equiv \overline{p}_n(0) \equiv p_n^*(0) \mod l.
$$

\(^1\)The table entry $(3 - \sqrt{5})/2$ for $D = -15$ should read $(-3 + \sqrt{5})/2$. 

5.2. **Fourier expansion of $\Delta$ at $\omega$.** We have $D = -3$ with $\Delta = (1 + i\sqrt{3})/2 = \omega + 1$ and

$$\Omega_{-3} = \frac{1}{\sqrt{6\pi}} \left[ \frac{\Gamma(1/3)}{\Gamma(2/3)} \right]^{3/2} = \frac{\sqrt{3}\Gamma(1/3)^3}{4\pi^2} \approx 0.6409.$$  

Also

$$E_2^*(\omega) = 0, \quad Q(\omega) = 0, \quad R(\omega) = 24\sqrt{3}\Omega_{-3}^6, \quad \Delta(\omega) = -\Omega_{-3}^{12}.$$  

With Theorem 4.1 and (4.7)

$$c_{\omega}(\Delta, n) = -|D|^3\Delta(\omega)\left(-\pi\Omega_D^3\right)^n q_n(0) \frac{1}{n!} \quad (n \geq 0)$$

with $q_n(0) \in \mathbb{Z}$. For example, with $0 \leq n < 15$ the numbers $q_n(0)$ are

$$1, 0, 0, 48, 0, 0, 18432, 0, 0, 13271040, 0, 0, 1974730752, 0, 0.$$  

**Proposition 5.2.** With $q_n(t)$ defined as above by (4.8), we have

$$q_{3n}(0) \equiv 1 \mod 7 \quad \text{for $m$ even,} \quad q_{3n}(0) \equiv 6 \mod 7 \quad \text{for $m$ odd.}$$

**Proof.** The polynomials $\mathcal{T}_n(t)$ in $\mathcal{R}_7$ for $0 \leq n < 42$ are

$$\begin{align*}
1, & \quad 0, \quad 2t, \quad 6, \quad 6t^2, \quad 5t, \\
1+3t^2, & \quad 5t^2, \quad 2t+5t^4, \quad 6+4t^3, \quad 2t^2, \quad 5t+4t^4, \\
1+6t^3+4t^6, & \quad t^2+2t^5, \quad 2t+2t^4, \quad 6+4t^3+4t^6, \quad 4t^3+4t^8, \quad 5t+5t^4, \\
1+t^3+t^6, & \quad 2t^2+2t^5, \quad 2t+2t^4, \quad 6, \quad 0, \quad 5t, \\
1, & \quad t^2, \quad 2t, \quad 6+6t^3, \quad 2t^2, \quad 5t+2t^4, \\
1+3t^3, & \quad 5t^2, \quad 2t+3t^4, \quad 6+t^3+3t^6, \quad 6t^2+5t^5, \quad 5+5t^4, \\
1+3t^3+3t^6, & \quad 3t^2+3t^6, \quad 2t+2t^4, \quad 6+6t^3+6t^6, \quad 5t^2+5t^5, \quad 5t^4+5t^4.
\end{align*}$$

We find that $\mathcal{T}_{12}(t) \equiv \mathcal{T}_0(t)$ and $\mathcal{T}_{43}(t) \equiv \mathcal{T}_1(t)$ in $\mathcal{R}_7$. Therefore $\mathcal{T}_{n+42}(t) \equiv \mathcal{T}_n(t)$ and the proof is complete.

It follows from Proposition 5.2 that all non-trivial Fourier coefficients of $\Delta$ at $\omega$ are non-zero. The modulus $l = 7$ was the smallest possible to show all $q_{3m}(0) \neq 0 \mod l$, but we could also have used $l = 13, 19$ and 43. Similarly $p_{2m}(0) \neq 0 \mod l$ for $l = 13, 37, 41$ as well as 5 above. Note that the 5 rows in the array (5.2) and the 7 rows in (5.6) are examples of the pattern we see in Proposition 6.9.

5.3. **Fourier expansion of $\Delta$ at other CM points.** Let $\mathfrak{Z} = \mathfrak{Z}_D$ be a CM point of discriminant $D < -4$. We continue with the method outlined by Zagier in [5] p. 88). Using the results and normalization from [5] p. 86 - 87), we may let

$$E_2^*(\mathfrak{z}) = k_1 |D|^{-1/2} \Omega_D^2, \quad Q(\mathfrak{z}) = k_2 \Omega_D^4, \quad R(\mathfrak{z}) = k_3 |D|^{1/2} \Omega_D^6$$

for non-zero $k_1, k_2, k_3$ in some number field $K = K_3$. There exists $k_0 \in \mathbb{Z}_{>0}$ such that

$$k_0 k_1, k_0 k_2, k_0 k_3 |D| \in \mathcal{O}_K.$$  

Choose now $m_1, m_2 \in \mathcal{O}_K$ such that

$$\left( E_2 - \frac{m_1}{m_2} \frac{R}{Q} \right)(\mathfrak{z}) = 0. \quad (5.7)$$

To make the choice definite, set $m_1 = k_0^3 k_1 k_2$ and $m_2 = k_0^3 |D| k_3$, but with any common factors in $\mathbb{Z}_{>1}$ removed. Lastly, define these elements of $\mathcal{O}_K[t]$:

$$\begin{align*}
a_1(t) & := 12k_0^3 k_2 m_1 |D| (t + k_3), \\
a_2(t) & := 2k_0^3 k_2 (m_1 - m_2) |D| (t + k_3), \\
a_3(t) & := 6k_0^3 m_2 (k_2 |D| (t + k_3)^2 - k_2^2), \\
a_4(t) & := -k_0^3 k_2^2 m_2 (m_2 - 6m_1) |D| - k_0^3 k_2^2 m_1 (4m_2 + m_1) |D|^2 (t + k_3)^2.
\end{align*}$$

We are now ready to state
Theorem 5.3. For a CM point \( z = z_{12} \) with \( D < -4 \), define \( K, k, m, a_t \) as above. We have

\[
(5.8) \quad c_3(\Delta, n) = -|D|^3 \Delta(3) \left( \frac{-\pi \Omega_{12}^2}{6k_0^2k_2^2m_2} \right) q_n(0) \frac{n!}{n!} \quad (n \geq 0)
\]

with \( q_n \in \mathcal{O}_K[t] \) satisfying the recurrence

\[
(5.9) \quad q_{0,3}(t) = 1, \quad q_{1,3}(t) = a_1(t), \quad q_{n+1,3}(t) = (a_1(t) + n a_2(t))q_{n,3}(t) + a_3(t)q_{n,3}(t) + n(n + 11)a_4(t)q_{n-1,3}(t) \quad (n \geq 1).
\]

Proof. We let \( \phi := \frac{1}{\pi} \left( E_2 - \frac{m_1 R}{Q} \right) \), which we note is holomorphic except for a pole at the zeros of \( Q \), i.e. at \( \omega \), the third root of unity. Define \( r := D\phi - \phi^2 \). Using (4.4) it is straightforward to verify that in terms of \( E_2, Q \), and \( R \) the function \( r \) is given by

\[
(5.10) \quad r = -\frac{Q}{144m_2^2} \left( m_2(m_2 - 6m_1) + (4m_1m_2 + m_1^2)(R/Q^{3/2})^2 \right).
\]

We next define a derivative \( \partial_{k,\phi} \) from [5, p. 88] by

\[
\partial_{k,\phi} f = \partial_{\phi} f := Df - k\phi f.
\]

It is a further modification of Rodriguez Villegas and Zagier’s (4.2) giving a mapping from the set of meromorphic modular forms of weight \( k \) to the space of meromorphic modular forms of weight \( k + 2 \).

We note that \( \partial_{k,\phi} f = \partial f + V f \) where \( V \) is multiplication by \( \frac{km_1 R}{12m_2 Q} \). We then define a meromorphic modular form \( \varrho^{[n+1]} \phi \) of weight \( k + 2n \) from the recursion

\[
\varrho^{[0]} \phi f = f, \quad \varrho^{[1]} \phi f = \partial_{\phi} f, \quad \varrho^{[n+1]} \phi f = \varrho_{\phi}(\varrho^{[n]} \phi f) + n(n + k - 1)r \varrho^{[n-1]} \phi f.
\]

Lemma 5.4. For \( z \) not a zero of \( Q \) the following holds:

\[
(12m_2)^n \varrho^{[n]} \phi \Delta(z) = \Delta(z)Q^{n/2}(z)p_n(R(z)/Q(z)^{3/2}),
\]

where the polynomials \( p_n(t) \in \mathcal{O}_K[t] \) are given by the recursion

\[
p_0(t) = 1, \quad p_1(t) = 12m_1 t, \quad p_{n+1}(t) = (2(m_1 - m_2)n + 12m_1)tp_n(t) + 6m_2(t^2 - 1)p'_n(t) - n(n + 11) \left( m_2(m_2 - 6m_1) + (4m_1m_2 + m_1^2)t^2 \right)p_{n-1}(t).
\]

Proof. This is a straightforward (if long) induction using that \( \varrho_{\phi} \) is a derivation, (5.10), and the identities

\[
\varrho_{\phi}(\Delta) = (\partial + V)(\Delta) = V\Delta = \frac{m_1}{m_2}Q^{1/2} \frac{R}{Q^{3/2}} \Delta, \\
\varrho_{\phi}(Q) = Q^{3/2} \frac{m_1 - m_2}{3m_2} \frac{R}{Q^{3/2}},
\]

\[
\varrho_{\phi}(R/Q^{3/2}) = \frac{Q^{1/2}}{2} \left( -1 + \left( \frac{R}{Q^{3/2}} \right)^2 \right).
\]

The generalization of (4.3) is then

\[
\partial_{\phi}^m f(z) = \sum_{r=0}^{m} \frac{m!}{r!} \left( m + k - 1 \right) \left( \frac{E_2^*}{12} - \frac{m_1 R}{12m_2 Q} \right)^{m-r} \varrho_{\phi}^{[r]} f(z)
\]

and at \( z = 3 \) we obtain, by the requirement (5.7),

\[
(5.11) \quad (\partial_{\phi}^m f)(3) = (\varrho_{\phi}^{[m]} f)(3).
\]
We have \( R(\delta)/Q(\delta)^{3/2} = \frac{\sqrt{k_2 |D|}}{k_2^3} k_3 \). It turns out to be convenient to change to a different set of polynomials, namely

\[
q_n(t) := \left( t k_0^4 k_2^2 \sqrt{k_2 |D|} \right)^n p_n \left( \frac{\sqrt{k_2 |D|}}{k_2^3} (t + k_3) \right)
\]

so that

\[
q_n(0) = \left( t k_0^4 k_2^2 \sqrt{k_2 |D|} \right)^n p_n(R(\delta)/Q(\delta)^{3/2}).
\]

Since \( p_n(t) \) has degree at most \( n \) and contains only even powers of \( t \) if \( n \) is even and only odd powers if \( n \) is odd, we see (also using the definition of \( k_0 \)) that \( q_n(t) \in \mathcal{O}_K[t] \). (In fact \( k_0^3 \) may be replaced by \( k_3^3 \) in (5.12) and we still have \( q_n(t) \in \mathcal{O}_K[t] \), but it complicates the exposition slightly.)

The recurrence relations of \( p_n \) from Lemma 5.4 translate into relations for \( q_n \) in (5.9). From (3.10), (5.11), Lemma 5.4 and (5.13) we deduce (5.8). \( \square \)

As a consistency check, we may verify that Theorem 5.3 agrees with Proposition 3.1. We obtain

\[
c_\mathfrak{a}(\mathfrak{a}, m) = (2iy)^6 \sum_{n=1}^{\infty} \tau(n)q^n \sum_{r=0}^{m} \frac{(m + 11)(r + 11)}{r!} (-4\pi ny)^r
\]

easily from Proposition 3.1. Then with \( D = -20 \), for example, (5.8) and (5.14) both give numerically

\[
\Delta|_{12\sigma_{3-20}}(z) = -0.0063 + 0.1019z - 0.6803z^2 + 2.3012z^3 - 3.4187z^4 + O(z^5).
\]

See Example 5.10 below.

We may now prove our main result.

**Proof of Theorem 1.2.** Assembling our formulas from this section we see that in the case \( D = -4 \) we have \( \mathfrak{a} = 3D = i \) and

\[
c_\mathfrak{a}(\mathfrak{a}, m) = \kappa_3 \frac{\lambda_3^m}{m!} q_{m,3}(0) \quad (m \geq 0)
\]

for \( \kappa_3 = -|D|^3 \Delta(i), \lambda_3 = -2\pi \Omega_2^D / \sqrt{3} \), by (5.1), and \( q_{m,3}(t) \in \mathcal{O}_K[t] \) with \( K = \mathbb{Q} \) given by the recursion (1.10) for \( a_1 = 0, a_2 = -2t, a_3 = 6(t^2 - 1) \) and \( a_4 = -1 \) using (4.6).

In the case \( D = -3 \) we have \( \mathfrak{a} = 3D = \omega + 1 \) and (5.15) holds for \( \kappa_3 = -|D|^3 \Delta(\omega), \lambda_3 = -\pi \Omega_2^D, \) by (5.5), and \( q_{m,3}(t) \in \mathcal{O}_K[t] \) with \( K = \mathbb{Q} \) given by the recursion (1.10) with \( a_1 = 0, a_2 = -2t^2, a_3 = 4(t^3 - 1) \) and \( a_4 = -t \) using (4.8).

Finally, for \( D \) a negative fundamental discriminant \( < -4 \), (and hence \( 3D \) not a zero of \( Q \) or \( R \)), (5.15) holds for \( \mathfrak{a} = 3D \) by Theorem 5.3 with \( \kappa_3 = -|D|^3 \Delta(3), \lambda_3 = -\pi \Omega_2^D / (6k_0^4 k_2^2 m_2) \) and the other constants as defined there. \( \square \)

**5.4. Examples.** We demonstrate how Theorem 5.3 may be used to prove non-vanishing of \( c_{3D}(\Delta, n) \), in the same manner as Propositions 5.1 and 5.2 for the seven remaining fundamental discriminants \( D \in \mathcal{D} \). Generalizing (5.4), let

\[
\mathcal{R}_l = \mathcal{R}_{l,3} := (\mathcal{O}_K / l\mathcal{O}_K)[t]/(t^l)
\]

and denote the natural projection from \( \mathcal{O}_K[t] \rightarrow \mathcal{R}_l \) by \( p \mapsto \overline{p} \).

**Example 5.5.** \( D = -7 \): In this case we have \( k_1 = 3, k_2 = 15 \) and \( k_3 = 27 \). Then \( m_1 = 5, m_2 = 21, k_0 = 1 \) and \( K = \mathbb{Q} \). The recursion for the \( q_n \) polynomials becomes \( q_0 = 1, q_1(t) = 60 \cdot 105 \cdot (t + 27), \)

\[
q_{n+1}(t) = 105(-32n + 60)(t + 27)q_n(t)
\]

\[
+ 126 \cdot (105(t + 27)^2 - 154)q_n'(t)
\]

\[
- n(n + 1)105(-19845 \cdot 15^4 + 445 \cdot 105^2 \cdot (t + 27)^2)q_{n-1}(t).
\]
Computing a few values we suspect that \( q_n(0) \not\equiv 0 \mod l \) when \( l = 23 \). Further calculations reveal that
\[
\begin{align*}
\overline{q}_{265}(t) &= 8\overline{q}_{12}(t) = 17 + 20t + 3t^2 + 3t^3 + 16t^4 \\
\overline{q}_{266}(t) &= 8\overline{q}_{13}(t) = 13 + 6t + 7t^2 + 9t^3 + 12t^4 + 19t^5.
\end{align*}
\]

This implies, since \( 265 \equiv 12 \mod l \), that
\[ (5.17) \quad \overline{q}_{n+(265-12)}(t) = 8\overline{q}_n(t) \quad \text{for all} \quad n \geq 12 \]
and, since \( 8 \mod 23 \) is of order 11, that \( \overline{q}_{n+11(265-12)}(t) = \overline{q}_n(t) \) for \( n \geq 12 \). It follows that \( q_{n+2783}(0) \equiv q_n(0) \mod 23 \) for \( n \geq 12 \) and a computation shows that it holds also for \( n = 1, \ldots, 11 \). On inspection we find that \( q_n(0) \not\equiv 0 \mod 23 \) for \( n < 265 \), and by \( (5.17) \) this holds for all \( n \), so we have proved that for every \( n \in \mathbb{Z}_{\geq 0} \)
\[
c_{3-7}(\Delta, n) \neq 0.
\]

Working modulo \( l = 43, 67 \) or 79 gives a similar proof of non-vanishing.

**Example 5.6.** \( D = -8 \): We have \( k_1 = 4, k_2 = 20, k_3 = 28 \), with \( m_1 = 5, m_2 = 14, k_0 = 1, \) and \( K = \mathbb{Q} \). After some experimenting we guess that we should look at \( l = 17 \). We find that
\[
\begin{align*}
\overline{q}_{550}(t) &= 2\overline{q}_{278}(t) = 6 + 11t + 9t^2 \\
\overline{q}_{551}(t) &= 2\overline{q}_{279}(t) = 15 + 3t + 5t^2 + 12t^3
\end{align*}
\]
which implies, since \( 550 \equiv 278 \mod l \), that \( \overline{q}_{n+(550-278)}(t) = 2\overline{q}_n(t) \) for \( n \geq 278 \). And since \( 2 \mod 17 \) is of order 8 we have
\[
\overline{q}_{n+8(550-278)}(t) = \overline{q}_n(t),
\]
for \( n \geq 278 \). It follows that \( q_{n+2786}(0) \equiv q_n(0) \mod 17 \) for \( n \geq 278 \), and by inspection we see that it holds indeed for all \( n > 0 \). By computing \( q_n(0) \mod 17 \) for every \( n < 550 \) and observing that these values are all different from zero we conclude that \( q_n(0) \not\equiv 0 \mod 17 \) for all \( n \geq 0 \). It follows that \( c_{3-8}(\Delta, n) \neq 0 \) for all \( n \).

**Example 5.7.** \( D = -11 \): We have \( k_1 = 8, k_2 = 32, k_3 = 56, m_1 = 32, m_2 = 77, k_0 = 1, \) and \( K = \mathbb{Q} \). Computing mod \( l = 23 \) we obtain \( \overline{q}_{n+253}(t) = 14\overline{q}_n(t) \) for \( n \geq 12 \) which implies, since \( 14 \mod 23 \) is of order 22, that \( q_{n+5566}(0) \equiv q_n(0) \mod 23 \) for \( n \geq 12 \), and as before we may verify that \( q_n(0) \not\equiv 0 \mod 23 \) and \( c_{3-11}(\Delta, n) \neq 0 \) for all \( n \).

**Example 5.8.** \( D = -15 \): We have \( k_1 = 6 + 3\sqrt{5}, k_2 = 15 + 12\sqrt{5}, k_3 = 42 + 63/\sqrt{5} \) with \( m_1 = 13 + 30\sqrt{5}, m_2 = 70 + 21\sqrt{5}, k_0 = 1, \) and \( K = \mathbb{Q}(\sqrt{5}) \). Reducing mod \( 17\mathcal{O}_K \) we find \( \overline{q}_{278}(t) = (13 + 10\sqrt{5})\overline{q}_6(t), \overline{q}_{279}(t) = (15 + 10\sqrt{5})\overline{q}_7(t), \) and \( 278 \equiv 6 \mod 17 \). The number \( (13 + 10\sqrt{5}) \) has order 144 mod \( 17\mathcal{O}_K \) so \( \overline{q}_n(t) = \overline{q}_{n+39168}(t) \) for \( n \geq 6 \). We have \( q_n(0) \not\equiv 0 \mod 17\mathcal{O}_K \) for all \( n < 278 \) so it follows that \( q_n(0) \not\equiv 0 \mod 17\mathcal{O}_K \) for all \( n \geq 0 \). Hence \( c_{3-15}(\Delta, n) \neq 0 \) for all \( n \).

Recall that for \( K = \mathbb{Q}(\sqrt{5}) \) we have \( \mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{5})/2] \). In the above example we are implicitly using the isomorphism
\[
\mathbb{Z}/l\mathbb{Z} \left[ \sqrt{5} \right] \to \mathcal{O}_K/l\mathcal{O}_K \quad \text{given by} \quad a + b\sqrt{5} \mapsto a + b\sqrt{5} + l\mathcal{O}_K \quad (l \text{ odd}).
\]

**Example 5.9.** \( D = -19 \): We have \( k_1 = 8, k_2 = 32, k_3 = 56 \) with \( m_1 = 32, m_2 = 57, k_0 = 1, \) and \( K = \mathbb{Q} \). If we reduce mod 7 we find \( \overline{q}_{21}(t) = 4\overline{q}_0(t) = 4, \overline{q}_{22}(t) = 4\overline{q}_1(t) = 2 + 5t, 21 \equiv 0 \mod l, \) and 4 has order 3, so \( \overline{q}_n(t) = \overline{q}_{n+63}(t) \). Arguing as before, \( c_{3-19}(\Delta, n) \neq 0 \) for all \( n \).

**Example 5.10.** \( D = -20 \): We have \( k_1 = 12 + 4\sqrt{5}, k_2 = 40 + 12\sqrt{5}, k_3 = 72 + 63/\sqrt{5} \) so that \( m_1 = 45 + 19\sqrt{5}, m_2 = 90 + 28\sqrt{5}, k_0 = 1, \) and \( K = \mathbb{Q}(\sqrt{5}) \). Reducing mod \( 7\mathcal{O}_K \) we find \( \overline{q}_{21}(t) = (4 + 6\sqrt{5})\overline{q}_0(t), \overline{q}_{22}(t) = (4 + 6\sqrt{5})\overline{q}_1(t), \) and \( 21 \equiv 0 \mod 7 \). The number \( (4 + 6\sqrt{5}) \) has order 24 mod \( 7\mathcal{O}_K \) so \( \overline{q}_n(t) = \overline{q}_{n+504}(t) \). We have \( q_n(0) \not\equiv 0 \mod 7\mathcal{O}_K \) for all \( n < 21 \) so it follows from the periodicity that \( q_n(0) \not\equiv 0 \mod 7\mathcal{O}_K \) and \( c_{3-20}(\Delta, n) \neq 0 \) for all \( n \).

**Example 5.11.** \( D = -24 \): We have \( k_1 = 12 + 12\sqrt{2}, k_2 = 60 + 24\sqrt{2}, k_3 = 84 + 72\sqrt{2} \) with \( m_1 = 9 + 7\sqrt{2}, m_2 = 14 + 12\sqrt{2}, k_0 = 1, \) and \( K = \mathbb{Q}(\sqrt{2}) \). Reducing mod \( 5\mathcal{O}_K \) yields \( \overline{q}_{n+48}(t) = \overline{q}_n(t) \) for all \( n \geq 0 \). We have \( q_n(0) \not\equiv 0 \mod 5\mathcal{O}_K \) for all \( n < 48 \) and hence \( c_{3-24}(\Delta, n) \neq 0 \) for all \( n \).
Note that in the above examples we have cases with class number \( h(\mathbb{Q}(\zeta_D)) = 2 \) (namely \( D = -15, -20, -24 \)) and class number \( h(\mathbb{Q}(\zeta_D)) = 1 \) (the remaining cases). With Propositions 5.1, 5.2, and Examples 5.5–5.11 we have completed the proof of Theorem 1.3.

6. Arithmetic of Fourier Coefficients at CM Points

6.1. Periodicity. As proved in Theorem 1.2, recall our main formula

\[ c_3(\Delta, m) = \kappa_3 \lambda_m^n q_{m, 3}(0) \quad (m \geq 0) \]

for the \( m \)-th Fourier coefficient of \( \Delta(z) \) at a CM point \( \zeta \in \mathbb{H} \). In this section we examine the integers \( q_{m, 3}(0) \) modulo \( IO_K \) in greater detail. For example, when \( l = 5 \) we have seen in Proposition 5.1 that \( q_{m, 3-4}(0) \mod 5 \) is

\[ (6.1) \quad 1, 0, 3, 0, 1, 0, 3, 0, 1, 0, 3, 0, 1, 0, 3, 0, 1, 0, 3, 0, 1, 0, 3, 0, \cdots \]

for \( m \geq 0 \) with period 4. For \( l = 7 \) we have \( q_{m, 3-4}(0) \mod 7 \) equaling

\[ (6.2) \quad 1, 0, 2, 0, 6, 0, 1, 0, 5, 0, 0, 0, 4, 0, 0, 4, 0, 0, 2, 0, 0, 0, 0, 0, \cdots \]

for \( m \geq 0 \) with all further terms \( \equiv 0 \mod 7 \). Hence \( q_{m, 3-4}(0) \mod 7 \) has period 1 for \( m \geq 21 \). We next prove that (6.1), (6.2) are typical. (In what follows, by period we always understand the least eventual period of a sequence.)

**Theorem 6.1.** Let \( l \) be in \( \mathbb{Z}_{\geq 1} \) and \( \zeta = \zeta_D \) a CM point.

(i) The sequence \( q_{m, 3}(0) \mod IO_K \) becomes periodic.

(ii) If \( q_{m, 3}(0) \mod IO_K \) is periodic from \( m = \alpha \) with period \( \beta \) then \( \alpha + \beta \leq l |O_K/IO_K|^2 l \).

**Proof.** Recall (5.16) and the map \( O_K[t] \to R_l \) given by \( p \mapsto \bar{p} \). Since \( \frac{d}{\theta l} \equiv 0 \mod l \) we see that \( \bar{q}_n(t) \) also satisfies the recursion (1.10) which depends only on \( n \mod l \). See Remark 3. We can therefore conclude that if, for some rational integers \( i_0 < j_0 \),

\[ (6.3) \frac{\bar{\ell}_{i_0}(t) = \bar{\ell}_{j_0}(t), \quad \bar{\ell}_{i_0+1}(t) = \bar{\ell}_{j_0+1}(t), \quad i_0 \equiv j_0 \mod l, \]

then \( \bar{\ell}_{i_0+n}(t) = \bar{\ell}_{j_0+n}(t) \) for all \( n \in \mathbb{Z}_{\geq 0} \). But since

\[ \langle \bar{\ell}_{i_0}, \bar{\ell}_{i_0+1}, \bar{\ell}_{i_0+n} \rangle, \quad \langle \bar{\ell}_{j_0}, \bar{\ell}_{j_0+1}, \bar{\ell}_{j_0+n} \rangle \in R_l^2 \times (\mathbb{Z}/l\mathbb{Z}), \]

the box principle implies that (6.3) is true for some \( 0 \leq i_0 < j_0 \leq |R_l^2 \times (\mathbb{Z}/l\mathbb{Z})| = l |O_K/IO_K|^2 l \).

Therefore \( q_m(0) \mod l \) is periodic from at most \( m = i_0 \) with period dividing \( j_0 - i_0 \). \(\square\)

6.2. A simpler recurrence. Part of the complication of recursion (1.10) is that each polynomial \( q_{n+1,3}(t) \) in the sequence depends on the two before: \( q_{n,3}(t) \) and \( q_{n-1,3}(t) \). Working in \( R_l \), we see next that each element in the subsequence

\[ 1, \bar{\ell}_{1,3}(t), \bar{\ell}_{2,3}(t), \bar{\ell}_{3,3}(t), \cdots \]

depends only on its immediate predecessor.

Let \( X = \bar{\ell}_{nl,3}(t) \in R_l \) for some \( n \). Then \( \bar{\ell}_{nl+1,3}(t) \) is given by (1.10). Since \( nl(nl+1)a_4(t)\bar{\ell}_{nl-1}(t) \equiv 0 \) in \( R_l \) we do not need to know \( \bar{\ell}_{nl-1}(t) \). Continuing inductively, we get

\[ \bar{\ell}_{nl,t,3}(t) \equiv X, \quad \bar{\ell}_{nl+1,t,3}(t) \equiv a_2(t)X + a_3(t)X', \quad \bar{\ell}_{nl+2,t,3}(t) \equiv (a_1(t) + a_2(t)) \left( a_2(t)X + a_3(t)X' \right) + a_3(t) \left( a_2(t)X + a_3(t)X' \right)' + 12a_4(t)X \]

with polynomials \( f_{j,3} \in R_l \) independent of \( X \) and \( n \). Let \( \Psi_{t,3}: R_l \to R_l \) be the linear map

\[ (6.4) \Psi_{t,3}(X) := f_{j,3}^{l,0}(t)X + f_{j,3}^{l,1}(t)a_3(t)X' + \cdots + f_{j,3}^{l,n}(t)a_3(t)^nX^{(l)} \]

We have

\[ (6.5) \bar{\ell}_{i,3} = \Psi_{t,3}(1), \bar{\ell}_{2i,3} = \Psi_{t,3}(\Psi_{t,3}(1)), \cdots, \bar{\ell}_{n,3} = \Psi_{t,3}^n(1) \quad \text{in} \ R_l \]
so that the sequence \( \tau_{ml,3} \) for \( m = 0, 1, 2, \ldots \) satisfies the simple recurrence
\[
\tau_{0,3} = 1, \quad \tau_{(n+1)l,3} = \Psi_{l,3}(\tau_{nl,3}).
\]
The sequence \( \Psi_{l,3}^{(i)}(1) \) must eventually repeat for large enough \( m \) since \( R_l \) is finite. It follows that Theorem 6.1 part (ii) may be improved to:

If \( q_{m,3}(0) \) mod \( lO_K \) is periodic from \( m = \alpha \) with period \( \beta \) then \( \alpha + \beta \leq l|O_K/lO_K|^l \).

6.3. Examples and conjectures. In this part we systematically consider the periodicity of all sequences
\[
\tau_{0,3D}(t), \tau_{1,3D}(t), \tau_{2,3D}(t), \cdots \text{ in } R_l
\]
for \( D \in \mathcal{D} \) and \( l \) prime with \( 2 \leq l < 100 \). For each such pair \((D, l)\) we use the techniques of §6.1 to find the least eventual period of (6.6). Clearly, the period of the corresponding sequence of constant terms, \( \tau_{m,3D}(0) \), will be a divisor. In the simplest case
\[
\tau_{n,3D}(t) = \tau_{n+1,3D}(t) = 0
\]
for some \( n \). Then all subsequent terms in the sequence will vanish, the period is 1, and we have \( q_{m,3D}(0) \rightarrow 0 \) mod \( lO_K \). This always happens for \( l = 2, 3, \) or \( l \) dividing \( |D| \), as in these cases we have \( \tau_{1,3D}(t) = \tau_{2,3D}(t) = 0 \). For primes \( l \) with \( 3 < l < 100 \) and \( l \) not dividing \( |D| \), our computations show that whether \( q_{m,3D}(0) \rightarrow 0 \) mod \( l \) depends on the value of \( l \) modulo \( D \): If \( D \in \{-3, -4, -7, -11, -19\} \) we have
\[
q_{m,3D}(0) \neq 0 \text{ mod } l \quad \Leftrightarrow \quad l \text{ is a quadratic residue mod } D
\]
and if \( D \in \{-8, -15, -20, -24\} \) we have
\[
q_{m,3D}(0) \neq 0 \text{ mod } lO_K \quad \Leftrightarrow \quad \begin{cases} 
  l \equiv 1, 3 \pmod{D} & \text{mod } D = -8 \\
  l \equiv 1, 2, 4, 8 \pmod{D} & \text{mod } D = -15 \\
  l \equiv 1, 3, 7, 9 \pmod{D} & \text{mod } D = -20 \\
  l \equiv 1, 5, 7, 11 \pmod{D} & \text{mod } D = -24.
\end{cases}
\]

Example 6.2. Let \( \mathfrak{f} = 19 \). Modulo \( l = 41 \) we compute that \( \tau_{m,3}(t) = 0 \) in \( R_{41} \) for \( m \geq 1219 \) implying \( q_{m,3}(0) \rightarrow 0 \) mod 41. This agrees with (6.7) since 41 \( \equiv 3 \) is a non-residue mod 19. Modulo \( l = 43 \) we have \( \tau_{m,3}(t) = \tau_{m+43-2,3}(t) \) in \( R_{43} \) for \( m \geq 32 \). We also see that in the first period there exist \( \tau_{m,3}(0) \neq 0 \). Therefore \( q_{m,3}(0) \neq 0 \) mod 43. This also agrees with (6.7) because 43 \( \equiv 5 \) is a quadratic residue mod 19.

On the above evidence, it is natural to conjecture that (6.7) and (6.8) are in fact true for all primes \( l > 3 \). Thus for each \( D \in \mathcal{D} \) we obtain simple criteria for the primes \( l \) for which the sequence \( q_{m,3D}(0) \) mod \( lO_K \) vanishes for large \( m \). Changing the point of view to consider \( D \) modulo \( l \), Gautam Chinta gave a succinct reformulation of (6.7) and (6.8) that we state as a Theorem:

**Theorem 6.3.** Let \( D \in \mathcal{D} \) and \( l \) any prime with \( 3 < l < 100 \) and \( l \) not dividing \( |D| \). Then
\[
q_{m,3D}(0) \neq 0 \text{ mod } lO_K \quad \Leftrightarrow \quad D \text{ is a quadratic residue mod } l.
\]

Proving that (6.7) and (6.8) are equivalent to Theorem 6.3 is an exercise in quadratic reciprocity. Note that (6.7), (6.8) and (6.9) are essentially independent of the normalization (1.9), if \( \lambda_j \) and \( \lambda_j \) are changed by factors in \( O_K \) (so that \( q_{m,3} \) remains in \( O_K/|l| \)) then (6.7), (6.8) and (6.9) will only be affected for finitely many primes \( l \). We may further ask if Theorem 6.3 is also true for all negative fundamental discriminants \( D \).

Next, we look at the maps \( \Psi_{l,3} \) of (6.2) in greater depth. For \( l \) prime, they are strikingly simple. In every case examined, the coefficients \( f_{l,i}^{(1)} \) in (6.4) are identically zero for \( 2 \leq i \leq l - 1 \). Since \( X^{(l)} \equiv 0 \) in \( R_l \) our computations have shown the following.

**Proposition 6.4.** For \( D \in \mathcal{D} \) and all primes \( l < 100 \) there exist \( a, b \in R_l \) such that
\[
\Psi_{l,3}(X) = aX + bX'.
\]
Example 6.5. Let $j = j_{-4}$. Then $a_3(t) = 6(t^2 - 1)$ and we have
\[
\psi_{5,3}(X) = (2t)X + a_3(t)X',
\psi_{7,3}(X) = (5t)X + t^2 \cdot a_3(t)X',
\psi_{11,3}(X) = (7t^3 + 5t)X + t^2 \cdot a_3(t)X',
\psi_{13,3}(X) = (5t^3 + 7t)X + (12t^4 + 5t^2 + 10)a_3(t)X'.
\]

Example 6.6. In the calculations proving Theorem 6.3, the largest period we found for (6.6) was 23439864 when $j = j_{-15}$ and $l = 83$. To see this, we iterate $\psi_{l,3}$ starting at 1 and find
\[
\psi_{l,3}^{83}(1) = \left(11 + 57\sqrt{5}\right)^{83} \psi_{l,3}^{1}(1)
\]
with no two powers $< 83$ being equal up to a factor. The number $11 + 57\sqrt{5}$ has order 3444 mod $O_K$ and so $\psi_{l,3}^{m}(1)$ has period 82 \cdot 3444. It follows that $\overline{q}_{l,82,3444+1,3,0}(t) = \overline{q}_{l,82,3444+1,3,0}(t)$ in $R_l$ for $m \geq 0$. Therefore $\overline{q}_{m,3,0}(t)$ must have period $\beta$ dividing $l \cdot 82 \cdot 3444$. If $l|\beta$ then $\beta/l$ is the period of $\psi_{m,3}(1)$ so $\beta = l \cdot 82 \cdot 344$. If $l \nmid \beta$ then $\beta$ is the period of $\psi_{m,3}(1)$ so $\beta = l \cdot 82 \cdot 344$. Checking this, we find $\beta$ is $l \cdot 82 \cdot 344$. The period of the constant term sequence $\overline{q}_{m,3,0}(0)$ must be a factor and a final check shows it is $82 \cdot 344$.

Thus we notice interesting relationships between the periods of the sequences $\psi_{l,3}^{m}(1)$, $\Psi_{l,3}(t)$ and $\overline{q}_{m,3}(0)$. The first, between $\psi_{l,3}^{m}(1)$ and $\Psi_{l,3}(t)$, can be shown in general. Recall the polynomials $a_i(t)$, depending on $j$, defined in (1.10).

Lemma 6.7. Let $l > 2$ be a prime and $j = j_D$ any CM point. If $\overline{q}_{m,3}(t)$ eventually has period $\beta$, not divisible by $l$, then
\[
\overline{a}_2(t)\overline{q}_{n,3}(t) \to 0 \quad \text{and} \quad \overline{a}_4(t)\overline{q}_{n,3}(t) \to 0 \quad \text{in} \quad R_l.
\]
Proof. Suppose $\overline{q}_{n+\beta,3}(t) = \overline{q}_{n,3}(t)$ for $n \geq \alpha$. Let $n$ be any integer $> \alpha$. Then for every positive $m$ we have
\[
\overline{q}_{n+m\beta+1,3}(t) = \overline{q}_{n+1,3}(t).
\]
Applying the recursion (1.10) to both sides of (6.11) and equating corresponding parts yields
\[
m\beta \left( \overline{a}_2(t)\overline{q}_{n,3}(t) + (2n + 11 + m\beta)\overline{a}_4(t)\overline{q}_{n-1,3}(t) \right) = 0 \quad \text{in} \quad R_l.
\]
Since $\beta$ is coprime to $l$ we may choose $m$ so that $m\beta$ takes any value mod $l$. If this value is non-zero, $m\beta$ is a unit in $R_l$. Therefore
\[
\overline{a}_2(t)\overline{q}_{n,3}(t) + (2n + 11 + v)\overline{a}_4(t)\overline{q}_{n-1,3}(t) = 0 \quad \text{in} \quad R_l
\]
for any non-zero $v$ mod $l$. Using this with $v = 2$ and $v = 1$ and subtracting these two equalities gives $\overline{a}_4(t)\overline{q}_{n-1}(t) = 0$ in $R_l$ when $n > \alpha$. From this it further follows that $\overline{a}_2(t)\overline{q}_{n,3}(t) = 0$ in $R_l$ when $n > \alpha$.

Label the ideal in $O_K/lO_K$ generated by $\overline{a}_2(0)$ and $\overline{a}_4(0)$ as $(\overline{a}_2(0),\overline{a}_4(0))$.

Proposition 6.8. Let $l > 2$ be any prime and $j = j_D$ any CM point. If $(\overline{a}_2(0),\overline{a}_4(0)) = O_K/lO_K$ and $\overline{q}_{m,3}(0) \neq 0$ in $R_l$ then
\[
\text{period}(\overline{q}_{m,3}(t)) = l \cdot \text{period}(\psi_{l,3}^{m}(1)).
\]
Proof. We claim that the period of $\overline{q}_{m,3}(t)$, call it $\beta$, is divisible by $l$. Suppose not. Then it follows from Lemma 6.7 that $\overline{a}_2(t)\overline{q}_{n,3}(t) \to 0$ and $\overline{a}_4(t)\overline{q}_{n,3}(t) \to 0$ in $R_l$. But since $(\overline{a}_2(0),\overline{a}_4(0)) = O_K/lO_K$, there exist $x,y \in O_K/lO_K$ such that $x\overline{a}_2(0) + y\overline{a}_4(0) = 1$ in $O_K/lO_K$. We can conclude that $\overline{q}_{m,3}(0) \to 0$. But this contradicts our assumptions. Therefore $\psi_{l,3}^{m}(1)$ (which equals $\overline{q}_{m,3}(t)$) has period $\beta/l$.

The conditions of Proposition 6.8 are usually satisfied. For $D \in \mathcal{D}$, $l$ prime with $2 < l < 100$ and $\overline{q}_{m,3}(0) \neq 0$, we have $(\overline{a}_2(0),\overline{a}_4(0)) \neq O_K/lO_K$ only for the three pairs $(D,l)$ with $D = -11,-19,-24$ and $l = 5$.

The next proposition, proved by computation, shows how the periods of $\overline{q}_{m,3}(t)$ and $\overline{q}_{m,3}(0)$ are related.
Proposition 6.9. Let $D \in \mathcal{D}$ and $l$ any prime $< 100$. Suppose $(\pi_2(0), \pi_4(0)) = O_K/lO_K$ and $\tau_{m,\Delta}(t) \not\to 0$. Then

\begin{equation}
\text{period}(\tau_{m,\Delta}(t)) = l \cdot \text{period}(\tau_{m,\Delta}(0)).
\end{equation}

We expect that Propositions 6.4 and 6.9 hold for all negative fundamental discriminants $D$ and all primes $l$.

7. Vanishing of $c_\Delta(\Delta, m)$ and $P_{z,m}$ for $z \in \mathbb{H}$

We have seen, with the proof of Theorem 1.3 in §5, that $A_\Delta$ does not contain CM points $\mathfrak{A}_\Delta$ of small discriminant. To test the inclusion of general points in $\mathbb{H}$ we define

$$\mathcal{E}_m(z) := \sum_{r=0}^{m} \frac{m!}{r!} \left( \frac{m+11}{r+11} \right) (E_2^*)^{m-r} B_r$$

with $B_r$ given by (4.1). From Theorem 4.1 we see that $c_\Delta(\Delta, m) = 0 \iff \mathcal{E}_m(z) = 0$ and from (2.5) this happens precisely when $P_{z,m}$ vanishes identically. The first few $\mathcal{E}_m$ are

\begin{align*}
\mathcal{E}_0 &= 1 \\
\mathcal{E}_1 &= 12 [E_2^*] \\
\mathcal{E}_2 &= 12 [13(E_2^*)^2 - E_4] \\
\mathcal{E}_3 &= 24 [91(E_2^*)^3 - 21E_2^* E_4 + 2E_6] \\
\mathcal{E}_4 &= 72 [455(E_2^*)^4 - 210(E_2^*)^2 E_4 + 40E_2^* E_6 + 3E_4^2].
\end{align*}

Let $\mathfrak{F}$ be the usual fundamental domain for $\Gamma$, as shown twice in Figure 1. For all $m$, we wish to locate the zeros of $\mathcal{E}_m$ in $\mathfrak{F}$.

![Figure 1. The zeros of $\mathcal{E}_6$ and $\mathcal{E}_8$](image)

Lemma 7.1. We have $\lim_{y \to \infty} \mathcal{E}_m(z) = 12^m$, uniformly in $x$.

Proof. Combine (3.10), Theorem 4.1 and the definition of $\mathcal{E}_m$ to see that

\begin{equation}
\mathcal{E}_m = 12^m \partial^m \Delta / \Delta
\end{equation}

so that $\mathcal{E}_m$ is a non-holomorphic modular form of weight $2m$ satisfying the recursion $\mathcal{E}_{m+1} = 12 \partial \mathcal{E}_m + \mathcal{E}_1 \mathcal{E}_m$. Note also that

\begin{equation}
\lim_{y \to \infty} \Delta^{(r)}(z)/\Delta(z) = (2\pi i)^r.
\end{equation}

The lemma now follows from (3.9), (7.1) and (7.2).
Lemma 7.2. For each \( m \), the zeros of \( \mathcal{E}_m \)
(i) form a set of measure 0,
(ii) are contained in a bounded region of \( \mathfrak{F} \),
(iii) are symmetric about the line \( \text{Re}(z) = 0 \).

Proof. As a consequence of Lemma 7.1, \( \mathcal{E}_m \neq 0 \). Therefore, since \( \mathcal{E}_m \) is real analytic, we obtain (i). Part (ii) also follows from Lemma 7.1. The Fourier coefficients at \( \infty \) of \( E_2^*, E_4 \) and \( E_6 \) are real, implying that
\[
\mathcal{E}_m(-\pi) = \overline{\mathcal{E}_m(\pi)}
\]
and hence (iii).

Lemma 7.3. We have \( \mathcal{E}_m(z) \in \mathbb{R} \) on the vertical lines \( \text{Re}(z) = -1/2 \) and \( \text{Re}(z) = 0 \). We have \( e^{im\theta} \mathcal{E}_m(e^{i\theta}) \in \mathbb{R} \) for \( 0 < \theta < \pi \).

Proof. For \( \text{Re}(z) = -1/2 \) and \( \text{Re}(z) = 0 \) we have \( \mathcal{E}_m(z) \in \mathbb{R} \) using (7.3) and that \( \mathcal{E}_m(z+1) = \mathcal{E}_m(z) \). As noted in [26] we have \( e^{ik\theta/2}E_k(e^{i\theta}) \in \mathbb{R} \) for \( 0 < \theta < \pi \) and \( k > 2 \). Employing Hecke’s limiting argument, as in [5] p. 19 for example, we also find that \( e^{i\theta} E^*_2(e^{i\theta}) \in \mathbb{R} \). As \( \mathcal{E}_m \) is a polynomial in \( E_2^*, E_4 \) and \( E_6 \), homogeneous in the weight, we obtain \( e^{im\theta} \mathcal{E}_m(e^{i\theta}) \in \mathbb{R} \) as required.

Thus we may expect zeros for \( \mathcal{E}_m \) along the boundary of the left half of \( \mathfrak{F} \) as the real-valued functions in Lemma 7.3 change sign. At the corners of this boundary we have \( \mathcal{E}_m(i) = 0 \) for \( m \equiv 1 \mod 2 \) and \( \mathcal{E}_m(\omega) = 0 \) for \( m \equiv 1, 2 \mod 3 \) by Remark 1. In what follows we locate numerically the zeros of \( \mathcal{E}_m \) in \( \mathfrak{F} \), using the first 25 terms in the Fourier expansions at \( \infty \) of \( E_2^*, E_4 \) and \( E_6 \) to get approximations to \( \mathcal{E}_m \).

Zeros of \( \mathcal{E}_1 \). We have \( \mathcal{E}_1(z) = 0 \) for \( z = i, \omega \). Numerically there seem to be no further zeros.

Zeros of \( \mathcal{E}_2 \). We have \( \mathcal{E}_2(z) = 0 \) for \( z = \omega \) and the points
\[
1.344i, \quad -1/2 + 1.29i.
\]

Proof of Theorem 7.6 The value 1.344i lies between the CM points \( i = 3_{-4} \) and \( \sqrt{2}i = 3_{-8} \), where we may explicitly compute \( \mathcal{E}_2 \). With the table on p. 87 of [5] we compute \( \mathcal{E}_2(3_{-4}) = -144\Omega_{4}^4 < 0 \) and \( \mathcal{E}_2(3_{-8}) = 72\Omega_{4}^8 > 0 \). Therefore the real-valued \( \mathcal{E}_2(z) \) changes sign between \( i \) and \( \sqrt{2}i \), proving that there is a zero in the vicinity of 1.344i.

Zeros of \( \mathcal{E}_3 \). We have \( \mathcal{E}_3(z) = 0 \) for \( z = i \) and the points
\[
1.666i, \quad -1/2 + 1.642i, \quad -1/2 + 1.55i.
\]

Zeros of \( \mathcal{E}_6 \). The 9 inequivalent zeros of \( \mathcal{E}_6(z) \) are shown on the left of Figure 1. They all have real part 0 or \(-1/2 \) or absolute value 1.

Zeros of \( \mathcal{E}_7 \). This is the smallest \( m \) for which \( \mathcal{E}_m(z) \) appears to have a pair of zeros not on the boundary of the left half of \( \mathfrak{F} \). The pair is \( \pm 0.302 + 1.18i \). The remaining 11 zeros of \( \mathcal{E}_7(z) \) are on the boundary.

Zeros of \( \mathcal{E}_8 \). The 18 zeros of \( \mathcal{E}_8(z) \) are shown on the right of Figure 1. Three pairs do not have real part 0 or \(-1/2 \) or absolute value 1.

8. Analogues of Petersson’s Formula

We return to the case of a general, discrete, finitely generated, finite volume group \( \Gamma \subseteq \text{SL}(2, \mathbb{R}) \), and assume for simplicity that \( \Gamma \) has a cusp of width 1 at \( \infty \). Let \( \mathcal{F} \) be an orthonormal basis for \( S_k(\Gamma) \). Petersson first proved the following formula:

\[
(8.1) \sum_{f \in \mathcal{F}} c_{\infty}(f, n) c_{\infty}(f, m) = \left( \frac{4\pi \sqrt{mn}}{k-2} \right)^{k-1} \left( \delta_{mn} + 2\pi(i^{-k}) \sum_{c > 0} c^{-1} S(m, n; c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right).
\]
Here the sum on the right runs over positive lower right entries of the group \( \Gamma \), \( S(m, n; c) \) is the Kloosterman sum related to \( \Gamma \), and \( J \) is a Bessel function - see for example [13, Prop. 14.5] or [12, Theorem 9.6] for further details. In this section we find explicit expressions for

\[
\sum_{f \in \mathcal{F}} c_{z_0}(f, m)c_{\infty}(f, n) \quad \text{and} \quad \sum_{f \in \mathcal{F}} c_{z_0}(f, m)c_{\infty}(f, n)
\]

with \( z_0, z'_0 \in \mathbb{H} \). Since the non-cuspidal Fourier coefficients \( c_{z_0}(f, m) \) are essentially non-holomorphic weight \( k + 2m \) forms (recall (3.10)), we can relate the two sums in (8.2) to special values of non-holomorphic objects of weight \( k + 2m \) in \( z_0 \) and weight \( k + 2n \) in \( z'_0 \), which we define below.

### 8.1. Averages of products of cuspidal and non-cuspidal Fourier coefficients.

Recall our notation \( \beta = \text{Im}(z_0) \). Consider the Fourier expansion at \( z_0 \) of the Poincaré series associated to \( \infty \) and the Fourier expansion at infinity of the Poincaré series associated to \( z_0 \):

\[
P_{n|k}\sigma_{z_0}(z) = \sum_{l \in \mathbb{Z}_{\geq 0}} c_{z_0}(P_{n, l})z^l,
\]

\[
P_{z_0, m}(z) = \sum_{l \in \mathbb{Z}_{\geq 1}} c_{\infty}(P_{z_0, m, l})q^l.
\]

Computing \( (P_n, P_{z_0, m}) \) with Propositions 2.1 and 2.2 and equating the results we obtain

\[
c_{\infty}(P_{z_0, m}, n) = \frac{\pi(4\pi n)^{k-1}}{2^{k-3}(m+k-1)!} c_{z_0}(P_{n, m}).
\]

Using (3.10) to rewrite the right-hand side gives

\[
c_{\infty}(P_{z_0, m}, n) = \frac{\pi(4\pi n)^{k-1}(-4\pi\beta)^m(2i\beta)^k/2}{2^{k-3}(m+k-1)!} \partial^m P_n(z_0).
\]

To find \( \partial^m P_n(z_0) \) more explicitly set

\[
F_k(z, n, s) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}(\gamma z)^{s-k/2}j(\gamma, z)^{-k}e^{2\pi in\gamma z}.
\]

For \( \text{Re}(s) > 1 \), the series defining \( F_k(z, n, s) \) converges absolutely and uniformly to a non-holomorphic, weight \( k \) Poincaré series. Verify that

\[
\partial_k F_k(z, n, s) = nF_{k+2}(z, n, s+1) - \frac{s+k/2}{4\pi} F_{k+2}(z, n, s)
\]

and by induction

\[
\partial^m F_k(z, n, s) = (-4\pi)^{-m} \sum_{j=0}^m (-4\pi n)^j \frac{m!}{(s+k/2+j)!} F_{k+2}(z, n, s+j).
\]

Since \( F_k(z, n, k/2) = P_n \) we obtain

\[
\partial^m P_n = (-4\pi)^{-m} \sum_{j=0}^m (-4\pi n)^j \frac{(m+k-1)!}{(j+k-1)!} F_{k+2}(z_0, n, k/2+j)
\]

and combining (8.3) with (8.5) proves

**Proposition 8.1.** With the above notation

\[
c_{\infty}(P_{z_0, m}, n) = \frac{2\beta^m+n/2}{n(-2i)^{k/2}} \sum_{j=0}^m \frac{m!}{(j+k-1)!} \frac{(-4\pi n)^{k+j}}{(j+k-1)!} F_{k+2}(z_0, n, k/2+j).
\]

**Theorem 8.2.** For any \( m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{\geq 1} \) we have

\[
\sum_{f \in \mathcal{F}} c_{z_0}(f, m)c_{\infty}(f, n) = \frac{(-2i)^{k/2}\beta^m+n/2}{(k-2)!} \sum_{j=0}^m \frac{(m+k-1)!}{(k-1)!} \frac{(-4\pi n)^{k+j-1}}{j!} F_{k+2}(z_0, n, k/2+j).
\]
Proof. We expand from the basis \( \mathcal{F} \) and use Propositions \([2.1, 2.2]\) to find
\[
\langle P_n, P_{z_0, m} \rangle = \sum_{f \in \mathcal{F}} \langle P_n, f \rangle \langle f, P_{z_0, m} \rangle
\]
\[
= \sum_{f \in \mathcal{F}} c_{z_0}(f, m) \frac{(k - 2)!}{(4\pi!)^{k-1} 2^{k-3}(m + k - 1)!} c_{z_0}(f, m)
\]
On the other hand the left side of \((8.6)\) equals \((k - 2)! (4\pi!)^{1 - k} c_{z_0}(P_{z_0, m}, n)\) and using the expression from Proposition \([8.1]\) completes the proof. \(\square\)

8.2. Averages of products of non-cuspidal Fourier coefficients. To get a similar result for the second sum in \((8.2)\) we put
\[
Q_{m,l}(z) := z^m \left( \frac{\tau}{|z|^2 - 1} \right)^l
\]
and
\[
G_k(z, z_0; m, l) := \sum_{\gamma \in \Gamma} (Q_{m,l}|k\sigma_0^{-1}\gamma)(z)
\]
\[
= (2i\beta)^{k/2} \sum_{\gamma \in \Gamma} (\sigma_0^{-1}\gamma z_0)^{m-l} \left( \frac{|\sigma_0^{-1}\gamma z_0|^2}{|\sigma_0^{-1}\gamma z_0|^2 - 1} \right)^l.
\]
Then \(G_k(z, z_0; m, 0) = P_{z_0, m}(z)\) and for all \(m \geq 0\) and \(0 \leq l < k/2 - 2\) we have (using the methods of \([10, \S\S 4, 5]\)) that \(G_k(z, z_0; m, l)\) converges absolutely and uniformly to a non-holomorphic weight \(k\) modular form. Note also that we may use \(\sigma z_0 = z\) to express \((8.7)\) in the different form:
\[
G_k(z, z_0; m, l) = \frac{(z_0 - z)^{k/2}}{(z - \bar{z})^k} \sum_{\begin{smallmatrix} a \ b \\ c \ d \end{smallmatrix}} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \sigma_0^{-1}\Gamma \sigma_0 \left( \frac{b}{d} \right)^{m-l} \left( \frac{|b|^2}{|b|^2 - 1} \right)^l.
\]

Theorem 8.3. For all \(m, n \in \mathbb{Z}_{\geq 0}\) we have
\[
\langle P_{z_0, m}, P_{z_0', n} \rangle = \sum_{f \in \mathcal{F}} \langle P_{z_0, m}, f \rangle \langle f, P_{z_0', n} \rangle
\]
\[
= \sum_{f \in \mathcal{F}} c_{z_0}(f, m) \frac{\pi(k - 2)! m!}{2^{k-3}(m + k - 1)!} c_{z_0'}(f, n) \frac{\pi(k - 2)! n!}{2^{k-3}(n + k - 1)!}
\]
where the left-hand side is
\[
\frac{\pi(k - 2)! m!}{2^{k-3}(m + k - 1)!} c_{z_0}(P_{z_0, m}, n).
\]
Thus, with \((8.10)\), it remains to compute \(\partial^n P_{z_0, m}(z)|_{z = z_0'}. \) Let \(\tau\) be any element of \(\text{GL}(2, \mathbb{C})\). Then for any differentiable function \(h\) we have \(\frac{d}{dz} h(\tau z) = \text{det} \tau \cdot h'(\tau z) j(\tau, z)^{-2}. \) Also
\[
\frac{d}{dz} j(\tau, z)^{-k} = -\frac{k}{2iy} j(\tau, z)^{-k} + \frac{k}{2iy} j(\tau, z)^{-k-2} j(\tau, z) j(\tau, \bar{z}).
\]
Hence
\[
\partial_k (h|_{k\tau}) = \frac{1}{2\pi i} \left( h'|_{k+2\tau} + \frac{j(\tau, z) j(\tau, \bar{z})}{2iy \text{det} \tau} h|_{k+2\tau} \right).
\]
Now \( j(\tau, z)j(\tau, \overline{z})/(2iy \det \tau) = 1/(\tau z - \tau \overline{z}) \). In the case where \( \tau = \sigma_\gamma - 1 \gamma \) with \( \gamma \in \text{SL}(2, \mathbb{R}) \) we have \( \tau \overline{z} = 1/\overline{\tau z} \) and then it follows from (8.10) that

\[
\partial_k(Q_{l,m}|k\tau) = \frac{1}{2\pi i} \left( (k-l)Q_{l+1,m}|k+2\tau + mQ_{l,m-1}|k+2\tau \right).
\]

Therefore

\[
\partial_k G_k(z, z_0; m, l) = \frac{1}{2\pi i} \left( (k-l)G_{k+2}(z, z_0; m, l+1) + mG_{k+2}(z, z_0; m-1, l) \right)
\]

and by induction

\[
\partial^n G_k(z, z_0; m, l) = \frac{n!}{(2\pi i)^n} \sum_{j=0}^{n} \binom{m}{j} \binom{n-l+k-1}{n-j} G_{k+2n}(z, z_0; m-j, l+n-j).
\]

With \( l = 0 \) we find

\[
\partial^n P_{z_0} = \frac{n!}{(2\pi i)^n} \sum_{j=0}^{n} \binom{m}{j} \binom{n+k-1}{n-j} G_{k+2n}(z, z_0; m-j, n-j)
\]

and the proof is complete. \( \square \)

8.3. A second proof of Proposition [8.1] We give another demonstration of the key Proposition [8.1]. In this argument the Poincaré series \( F_k(z, n, s) \) emerges very naturally.

Lemma 8.4. We have

\[
(\overline{a b}) = \frac{2}{n(2i)^{k/2}} \sum_{j=0}^{m} \beta^{k+j} \binom{m}{j} (-4\pi n)^{k+j} \sum_{\gamma \in \Gamma/\Gamma_\infty \infty} \frac{\sigma_{\gamma}^{-1} \gamma(z+n)}{\gamma(z+n)} m \epsilon^{2\pi i n/c}.
\]

Proof. Write

\[
P_{z_0}(z) = 2(2i\beta)^{k/2} \sum_{\gamma \in \Gamma/\Gamma_\infty} \sum_{n=-\infty}^{\infty} \frac{\sigma_{\gamma}^{-1} \gamma(z+n)}{j(\sigma_{\gamma}^{-1} \gamma, (z+n))}.\]

Using Poisson summation on the inner sum we obtain

\[
P_{z_0}(z) = 2(2i\beta)^{k/2} \sum_{\gamma \in \Gamma/\Gamma_\infty} \sum_{n=-\infty}^{\infty} \epsilon^{2\pi i n/c}.
\]

where, for \( (\overline{a b}) = \sigma_{\gamma}^{-1} \gamma \),

\[
I(\gamma, z, n) = \int_{\mathbb{R}} \left( \frac{a}{c} - \frac{2i\beta}{c(z+t)} \right)^{m} \epsilon^{-2\pi i t d} e^{2\pi i t} dt = \sum_{j=0}^{m} \binom{m}{j} \left( \frac{a}{c} \right)^{m-j} \left( \frac{-2i\beta}{c} \right)^{j} \int_{-\infty}^{\infty} \epsilon^{2\pi i t} e^{2\pi i n/c} e^{2\pi i z/c} dt = \sum_{j=0}^{m} \binom{m}{j} \left( \frac{a}{c} \right)^{m-j} \left( \frac{-2i\beta}{c} \right)^{j} \epsilon^{2\pi i n/c} e^{2\pi i z/c} \int_{-\infty}^{\infty} \epsilon^{2\pi i (c u)k+m-j} du.
\]

We note that \( c \) is never zero when \( \gamma \in \text{SL}(2, \mathbb{R}) \), so division by \( c \) is not a problem in the above calculations. When \( n \leq 0 \) we see, by moving the line of integration upwards, that the integral vanishes. When \( n \) is positive we can move the line of integration downwards and see that the integral equals

\[
-2\pi i \text{Res}_{u=0} \epsilon^{2\pi i n/c} e^{2\pi i z/c} = \frac{(-2\pi i n/c)^{k+j}}{n(k+j-1)!}.
\]

Putting together the different terms we obtain the lemma. \( \square \)
Rewriting the series on the right of (8.12) using the following identities

\[
\sum_{\gamma \in \Gamma/\Gamma_{\infty}} f(a, b, c, d) = \sum_{\gamma \in \Gamma/\Gamma_{\infty}} f(a, b, c, d) \\
\quad \gamma^{-1} \in \Gamma/\Gamma_{\infty} \\
\quad (a \ b \ c \ d)^{-1} = \sigma_{z_0}^{-1}\gamma^{-1} \\
\quad (a \ b \ c \ d)^{-1} = \gamma\sigma_{z_0} \\
\quad (a' \ b' \ c' \ d') \in \Gamma_{\infty} \setminus \Gamma \\
\quad f(c'z_0 + d', -c'z_0 - b', c'z_0 + d', -a'z_0 - b')
\]

we conclude that

\[
\sum_{\gamma \in \Gamma/\Gamma_{\infty}} \frac{a^{m-j}}{e^{\pi in\gamma/c}} e^{2\pi in\gamma/c} = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \frac{j^{m-j}}{j(\gamma, z_0)^{2m} e^{2\pi in\gamma z_0}} \\
\quad = \beta^{m-j} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \text{Im}(\gamma z_0)^{j-m} j(\gamma, z_0)^{-k-2m} e^{-2\pi in\gamma z_0} \\
\quad = \beta^{m-j} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \frac{\text{Im}(\gamma z_0)^{j-m} j(\gamma, z_0)^{-k-2m} e^{-2\pi in\gamma z_0}}{\pi in\gamma z_0} \\
\quad = \beta^{m-j} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \frac{\text{Im}(\gamma z_0)^{j-m} j(\gamma, z_0)^{-k-2m} e^{-2\pi in\gamma z_0}}{\pi in\gamma z_0} \\
\quad = \beta^{m-j} F_{k+2m}(z_0, n, k/2 + j).
\]

Therefore

\[
c_{\infty}(P_{z_0, m, n}) = \frac{2\beta^{k+2+m}}{n(2i)^{k/2}} \sum_{j=0}^{m} \frac{(m-j)}{(k+j-1)!} F_{k+2m}(z_0, n, k/2 + j)
\]

and we have arrived at Proposition 8.1 by another route.

8.4. Examples. We illustrate the results of this section with some particular cases.

Example 8.5. When \( k = 12 \) and \( \Gamma = \text{SL}(2, \mathbb{Z}) \), \( S_k(\Gamma) \) is 1-dimensional and \( \mathcal{F} = \{ \Delta/\|\Delta\| \} \). In this case Theorem 8.2 implies that

\[
\tau(n)c_{z_0}(\Delta, m) = \frac{64}{10!} \|\Delta\|^2 \text{Im}(z_0)^{m+6} \sum_{j=0}^{m} \frac{(m+11)}{(m-j)} \frac{(-4\pi n)^{j+11}}{j!} F_{2m+12}(z_0, n, j + 6).
\]

We may verify (8.13) in the case of \( m = 0 \):

\[
\tau(n)c_{z_0}(\Delta, 0) = \frac{64}{10!} \|\Delta\|^2 \text{Im}(z_0)^{6} (-4\pi n)^{11} F_{12}(z_0, n, 6)
\]

where \( F_{12}(z_0, n, 6) = P_n(z_0) \). With (2.2), equation (8.14) reduces to \( c_{z_0}(\Delta, 0) = (z_0 - \overline{z_0})^6 \Delta(z_0) \) which now follows from (3.7).

Example 8.6. Consider (8.13) for \( z_0 = z_D \) and \( D \in \mathcal{D} \). Divide both sides by the coefficient \( c_{3D}(\Delta, m) \), which is non-zero by Theorem 1.3 (assuming \( 2|m \) if \( j = 3_4 \) or \( 3|m \) if \( j = 3_3 \)), and we find Lehmer’s conjecture is equivalent to showing that for every \( n \) there exists an \( m \) and a \( z_D \) as above with

\[
\sum_{j=0}^{m} \frac{(m+11)}{(m-j)} \frac{(-4\pi n)^{j}}{j!} F_{2m+12}(3D, n, j + 6) \neq 0.
\]
Example 8.7. Specializing Theorem 8.5 to $\Gamma = \text{SL}(2, \mathbb{Z})$, $k = 12$, $m = n$, $z_0 = z'_0 \in \mathbb{H}$, and employing (8.8), we obtain a final vanishing criterion for $P_{z_0,m}$ and $c_{z_0}(\Delta, m)$:

$$P_{z_0,m} \equiv 0 \iff |c_{z_0}(\Delta, m)|^2 = 0 \iff \sum_{j=0}^{m} \binom{m}{j} \binom{m+11}{j} G_{2m+12}(z_0, z_0; j, j) = 0 \iff \sum_{j=0}^{m} \binom{m}{j} \binom{m+11}{j} \sum_{\sigma \in \mathbb{Z}/(\mathbb{Z}+\Gamma)} d^{-2m-12} \left( \frac{\sigma b^2}{|d|^2 - 1} \right)^j = 0.$$

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DEPARTMENT OF MATHEMATICS, THE CUNY GRADUATE CENTER, NEW YORK, NY 10016-4309, U.S.A.
E-mail address: cosullivan@gc.cuny.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK.
E-mail address: risager@math.ku.dk