CANONICAL SURFACES OF HIGHER DEGREE

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Dedicated to Philippe Ellia on the occasion of his 60th birthday

Abstract. We consider a family of surfaces of general type $S$ with $K_S$ ample, having $K_S^2 = 24, p_g(S) = 6, q(S) = 0$. We prove that for these surfaces the canonical system is base point free and yields an embedding $\Phi_1 : S \to \mathbb{P}^5$.

This result answers a question posed by G. and M. Kapustka [Kap-Kap15].

We discuss some related open problems, concerning also the case $p_g(S) = 5$, where one requires the canonical map to be birational onto its image.

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Introduction

Among the many questions that one may ask about surfaces of general type the following one has not yet been sufficiently considered.

**Question 0.1.** Let $S$ be a smooth surface with ample canonical divisor $K_S$, assume that $p_g(S) = 6$ and that the canonical map $\phi_1 : S \to \mathbb{P}^5$ is a biregular embedding.

Which values of $K_S^2$ can occur, in particular which is the maximal value that $K_S^2$ can reach?

Recall the Castelnuovo inequality, holding if $\phi_1$ is birational (onto its image):

$$K_S^2 \geq 3p_g(S) - 7,$$

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and the Bogomolov-Miyaoka-Yau inequality

\[ K_S^2 \leq 9\chi(S) = 9 + 9p_g(S) - 9q(S). \]

By virtue of these inequalities, under the assumptions of question 0.1 one must have:

\[ 11 \leq K_S^2 \leq 63. \]

In the article [Cat97] methods of homological algebra were used to construct such surfaces with low degree \( K_S^2 \leq 17 \), and also to attempt a classification of them. Recently, M. and G. Kapustka constructed in [Kap-Kap15] such canonical surfaces of degree \( K_S^2 = 18 \), using the method of bilinkage. In a preliminary version of the article they even ventured to ask whether the answer to question 0.1 would be \( K_S^2 \leq 18 \).

Our main result consists in exhibiting such canonically embedded surfaces having degree \( K_S^2 = 24 \), and with \( q(S) = 0 \). The family of surfaces was indeed listed in the article [Cat99], dedicated to applications of the technique of bidouble covers; but it was not a priori clear that their canonical system would be an embedding (this was proven in [Cat84] for bidouble covers satisfying much stronger conditions).

**Theorem 0.1.** Assume that \( S \) is a bidouble cover of the quadric \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \) branched on three curves \( D_1, D_2, D_3 \) of respective bidegrees \( (2, 3), (2, 3)(4, 1) \), which are smooth and intersect transversally. Assume moreover that the 12 intersection points \( D_1 \cap D_2 \) have pairwise different images via the second projection \( p_v : Q \to \mathbb{P}^1 \) (i.e., they have different coordinates \( (v_0 : v_1) \)).

Then \( S \) is a simply connected surface with \( K_S^2 = 24, p_g(S) = 6, q(S) = 0 \), whose canonical map \( \phi_1 : S \to \mathbb{P}^5 \) is a biregular embedding. These surfaces form a (non empty) irreducible algebraic subset of dimension 25 of the moduli space.

We plan to try to describe the equations of the above surfaces \( S \subset \mathbb{P}^5 \); since, by a theorem of Walter [Wal96], each \( S \) is the Pfaffian locus of a twisted antisymmetric map \( \alpha : \mathcal{E}(-t) \to \mathcal{E} \) of a vector bundle \( \mathcal{E} \) on \( \mathbb{P}^5 \), a natural question is to describe the bundle \( \mathcal{E} \).

Concerning question 0.1 we should point out that it is often easier to construct algebraic varieties as parametric images rather than as zero sets of ideals. Note that surfaces with \( p_g(S) = 6, q(S) = 2, K_S^2 = 45 \) (and with \( K_S \) ample since they are ball quotients) were constructed in [BC08]. For these, however, the canonical system has base points.

It would be interesting to see whether there do exist canonically embedded surfaces with \( K_S^2 = 56 \) which are regular surfaces isogenous to a product (see [Cat00]).

\footnote{This follows from theorem 3.8 of [Cat84].}
We finish this introduction pointing out that a similar question is wide open also for $p_g(S) = 5$ (while for $p_g(S) = 4$ some work has been done, see for instance [Cat99]).

**Question 0.2.** Let $S$ be a smooth minimal surface of general type, assume that $p_g(S) = 5$ and that the canonical map $\phi_1 : S \to \mathbb{P}^4$ is birational. What is the maximal value that $K_S^2$ can reach?

Again, Castelnuovo’s inequality gives $8 \leq K_S^2$, and Bogomolov-Miyaoka-Yau’s inequality gives $K_S^2 \leq 54$.

The known cases where $\phi_1$ is an embedding are just for $K_S^2 = 8, 9$, where $S$ is a complete intersection of type $(4, 2)$ or $(3, 3)$.

Indeed these are the only cases by virtue of the following (folklore?) theorem which was stated and proven in [Cat97], propositions 6.1 and 6.2, corollary 6.3.

**Theorem 0.2.** Assume that $S$ is the minimal model of a surface of general type with $p_g(S) = 5$, and assume that the canonical map $\phi_1$ embeds $S$ in $\mathbb{P}^4$.

Then $S$ is a complete intersection with $\mathcal{O}_S(K_S) = \mathcal{O}_S(1)$, i.e., $S$ is a complete intersection in $\mathbb{P}^4$ of type $(2, 4)$ or $(3, 3)$.

Moreover, if $\phi_1$ is birational, and $K_S^2 = 8, 9$, then $\phi_1$ yields an embedding of the canonical model of $X$ as a complete intersection in $\mathbb{P}^4$ of type $(2, 4)$ or $(3, 3)$.

Recall that the first main ingredient of proof is the well known Severi’s double point formula ([Sev01], see [Cat79] for the proof of some transversality claims made by Severi).

Write the double point formula in the form stated in [Hart77], Appendix A, 4.1.3. It gives, once we set $d := K_S^2$:

$$12\chi(S) = (17 - d)d.$$ 

Since $\chi(S) \geq 1$, we get $8 \leq d \leq 16$, and viewing the double point formula as an equation among integers, we see that it is only solvable for $d = 8, 9 \Rightarrow \chi(S) = 6 \Rightarrow q(S) = 0$, or for $d = 12, \Rightarrow \chi(S) = 5 \Rightarrow q(S) = 1$.

One sees that the last case cannot occur, since the Albanese map of $S$, $\alpha : S \to A$, has as image an elliptic curve $A$; if one denotes by $g$ the genus of the Albanese fibres, the slope inequalities for fibred surfaces of Horikawa, or Xiao, or Konno ([Hor81], [Xiao87], [Kon93]), give $g = 2$: hence $\phi_1$ cannot be birational.

One could replace in theorem 0.2 the hypothesis that $\phi_1$ is an embedding of $S$ in $\mathbb{P}^4$ by the weaker condition that $\phi_1$ yields an embedding of the canonical model of $X$ via an extension of the Severi double formula to the case of surfaces with rational double points as singularities.

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\[\text{Footnote:}\] for which however we have not yet found a reference.
1. The construction of the family of surfaces

We consider the family of algebraic surfaces $S$, bidouble covers (Galois covers with group $(\mathbb{Z}/2)^2$) of the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$, described in the fourth line of page 106 of [Cat99].

This means that we take three divisors

$$D_j = \{\delta_j = 0\}, \delta_1, \delta_2 \in H^0(\mathcal{O}_Q(2, 3)), \delta_3 \in H^0(\mathcal{O}_Q(4, 1)),$$

we consider divisor classes $L_1 = L_2, L_3$ with

$$\mathcal{O}_Q(L_1) = \mathcal{O}_Q(L_2) = \mathcal{O}_Q(3, 2), \mathcal{O}_Q(L_3) = \mathcal{O}_Q(2, 3)$$

and we define the surface $S$ as

$$\text{Spec}(\mathcal{O}_Q \oplus w_1\mathcal{O}_Q(-L_1) \oplus w_2\mathcal{O}_Q(-L_2) \oplus w_3\mathcal{O}_Q(-L_3)),$$

where the ring structure is given by (here $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$)

$$w_i^2 = \delta_j \delta_k, \ w_i w_j = w_k \delta_k.$$

We refer to [Cat84] and [Cat99] for the basics on bidouble covers, which show that on the surface $S$ the sections $w_i$ can be written as a product of square roots of the sections $\delta_1, \delta_2, \delta_3$:

$$w_i = y_j y_k, \ y_i^2 = \delta_i.$$

As is in [Cat99], page 102, define

$$N := 2K_Q + \sum_{i=1}^{3} D_i = 2K_Q + \sum_{i=1}^{3} L_i,$$

obtaining a description of the canonical ring of $S$, which is a smooth surface with ample canonical divisor if the three curves $D_1, D_2, D_3$ are smooth and intersect transversally. The canonical ring is defined as usual by:

$$\mathcal{R} := \oplus_{m=0}^{\infty} \mathcal{R}_m, \ \mathcal{R}_m := H^0(\mathcal{O}_S(mK_S)),$$

and we have (ibidem)

$$\mathcal{R}_{2m+1} = y_1 y_2 y_3 H^0(\mathcal{O}_Q(K_Q + mN)) \oplus (\oplus_{i=1}^{3} y_i H^0(\mathcal{O}_Q(K_Q + mN + L_i)))$$

$$\mathcal{R}_{2m} = H^0(\mathcal{O}_Q(mN)) \oplus (\oplus_{i=1}^{3} w_i H^0(\mathcal{O}_Q(mN - L_i))).$$

In particular, since in this case $\mathcal{O}_Q(N) = \mathcal{O}_Q(4, 3)$,

$$H^0(\mathcal{O}_S(K_S)) = \oplus_{i=1}^{3} y_i H^0(\mathcal{O}_Q(K_Q + L_i)) = y_1 u_0, y_1 u_1, y_2 u_0, y_2 u_1, y_3 v_0, y_3 v_1 >,$$

where $u_0, u_1$ is a basis of $H^0(\mathcal{O}_Q(1, 0)), v_0, v_1$ is a basis of $H^0(\mathcal{O}_Q(0, 1))$.

In particular, $p_g(S) = 6$; moreover, $q(S) = 0$ since $H^1(\mathcal{O}_Q) = H^1(\mathcal{O}_Q(-L_i)) = 0, \forall i = 1, 2, 3$, as follows easily from the K"unneth formula.

Finally, if $\pi : S \to Q$ is the bidouble cover, then $2K_S = \pi^*(N)$, hence $K_S$ is ample and $K_S^2 = N^2 = 24.$
2. Proof that the canonical map is an embedding

Theorem 2.1. Assume that the three branch curves $D_1, D_2, D_3$ are smooth and intersect transversally, and moreover that the 12 intersection points $D_1 \cap D_2$ have pairwise different images via the second projection $p_v : Q \to \mathbb{P}^1$ (i.e., they have different coordinates $(v_0 : v_1)$).

Then the canonical map $\phi_1 : S \to \mathbb{P}^5$, $\phi_1(x) = (y_1(x)u_0(x) : y_1(x)u_1(x) : y_2(x)u_0(x) : y_2(x)u_1(x) : y_3(x)v_0(x) : y_3(x)v_1(x))$, is a biregular embedding, i.e. we have an isomorphism

$$\phi_1 : S \to \Sigma := \phi_1(S).$$

Proof. Let $R_i \subset S, R_i := \{y_i = 0\}$. The curve $R_i$ maps to $D_i$ with degree 2, and $R_1 \cap R_2 \cap R_3 = \emptyset$ since by our assumption $D_1 \cap D_2 \cap D_3 = \emptyset$.

Claim 1: the canonical system is base-point free.

In fact, for each point $x$ there is $u_i, i \in \{0, 1\}$, such that $u_i(x) \neq 0$, and similarly $v_j, j \in \{0, 1\}$, such that $v_j(x) \neq 0$. Hence $x$ is a base point if $y_1 = y_2 = y_3 = 0$ at $x$, contradicting $R_1 \cap R_2 \cap R_3 = \emptyset$.

Claim 2: $\phi_1$ is a local embedding.

2.1) At the points $x \in R_h \cap R_k$ the two sections $y_h, y_k$ yield local coordinates, hence our assertion.

2.2) For the points $x \in S \setminus R$, where $R = R_1 \cup R_2 \cup R_3$ is the ramification divisor, let us consider the rational map $F_u : \Sigma \to \mathbb{P}^1$, induced by the linear projections

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \to (x_1 : x_2)$$

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \to (x_3 : x_4),$$

respectively the rational map $F_v : \Sigma \to \mathbb{P}^1$, induced by the linear projection

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \to (x_5 : x_6).$$

We have that $(u_0(x) : u_1(x)) = p_u \circ \pi = F_u \circ \phi_1, (v_0(x) : v_1(x)) = p_v \circ \pi = F_v \circ \phi_1$. Moreover, $F_u \circ \phi_1$ is a morphism outside the finite set $R_1 \cap R_2, F_v \circ \phi_1$ is a morphism outside $R_3$.

Hence $\pi = (F_u \circ \phi_1) \times (F_v \circ \phi_1) : S \setminus R$ is a morphism of maximal rank, so $\phi_1$ is a local embedding outside of $R$.

2.3) Set for convenience $u := p_u \circ \pi, v := p_v \circ \pi$.

At the points of $R_1 \setminus (R_2 \cup R_3)$, then $y_1, u, v$ give local coordinates, so we are done; similarly for the points of $R_2 \setminus (R_1 \cup R_3)$.

At the points of $R_3 \setminus (R_1 \cup R_2)$, we need to show that $y_3$ and $u$ give local coordinates. Here, we make the remark that $D_3$ is a divisor of bidegree $(4, 1)$, hence $p_u : D_3 \to \mathbb{P}^1$ is an isomorphism. Hence $\delta_3, u$ give local coordinates at the points of $D_3$, and we infer that $y_3, u$, give local coordinates at the points of $R_3 \setminus (R_1 \cup R_2)$.

Claim 3: $\phi_1$ is injective.
Assume that \( \phi_1(x) = \phi_1(x') \). We observe preliminarily that this condition implies that, if \( x \in R_i \), then also \( x' \in R_i \), since for instance \( x \in R_1 \Rightarrow x_1(\phi_1(x)) = x_2(\phi_1(x)) = 0 \).

3.1 Assume that \( x, x' \in S \setminus R \).

Then \( \pi(x) = \pi(x') \), and since the elements of Galois group \( G = (\mathbb{Z}/2)^2 = \{ \pm 1 \}^3 / \{ \pm 1 \} \) are determined by \( \epsilon \in \{ \pm 1 \}^3 \), and each of them acts on \( \phi_1(x) = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \) by

\[
(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (\epsilon_1 x_1 : \epsilon_1 x_2 : \epsilon_2 x_3 : \epsilon_2 x_4 : \epsilon_3 x_5 : \epsilon_3 x_6),
\]

the action of \( G \) on \( \phi_1(x) \) has an orbit of cardinality 4, which is precisely the cardinality of \( \pi^{-1}(\pi(x)) \).

Hence \( \phi_1(x) = \phi_1(x') \Rightarrow x = x' \).

3.2 Let \( x, x' \in R_1 \setminus (R_2 \cup R_3) \). Again \( \pi(x) = \pi(x') \), and we see that the orbit of \( \phi_1(x) \) under \( G \) is the set of points

\[
(0 : 0 : \epsilon_2 x_3 : \epsilon_2 x_4 : \epsilon_3 x_5 : \epsilon_3 x_6),
\]

which has cardinality 2; this is precisely the cardinality of \( \pi^{-1}(\pi(x)) \), so again we are done.

The case \( x, x' \in R_2 \setminus (R_1 \cup R_3) \) is completely analogous.

3.3 Let \( x, x' \in R_3 \setminus (R_1 \cup R_2) \). Again, the orbit of \( \phi_1(x) \) under \( G \) has cardinality 2, so we are done if we show that \( \pi(x) = \pi(x') \). On the set \( R_3 \), however, \( \pi \) is not a morphism, so we argue differently.

We use instead that \( u(x) = u(x') \), and that \( p_u : D_3 \to \mathbb{P}^1 \) is an isomorphism to conclude that \( \pi(x) = \pi(x') \).

3.4 If \( x, x' \in R_3 \cap R_2 \) (for \( x, x' \in R_3 \cap R_1 \) the argument is entirely similar), we proceed as follows.

We obtain again \( u(x) = u(x') \), and since \( p_u : D_3 \to \mathbb{P}^1 \) is an isomorphism, we get that \( \pi(x) = \pi(x') \). However, the restriction of \( \pi \) to \( R_3 \cap R_2 \) is bijective, hence \( x = x' \).

3.5 Assume finally that \( x, x' \in R_1 \cap R_2 \) and use again that the restriction of \( \pi \) to \( R_1 \cap R_2 \) \( \pi \) is bijective. It suffices therefore to show that \( \pi(x) = \pi(x') \).

The condition \( \phi_1(x) = \phi_1(x') = (0 : 0 : 0 : 0 : y_3 y_0 : y_3 v_1) \), since \( y_3 \) does not vanish on \( R_1 \cap R_2 \), implies that \( v(x) = v(x') \), so the points \( \pi(x), \pi(x') \in D_1 \cap D_2 \) have the same \( v \) coordinate.

Hence, by our assumption, \( \pi(x) = \pi(x') \), exactly as desired.

\[ \square \]

**Proposition 2.2.** The hypotheses of theorem 2.1 define a non empty family of dimension \( 11 + 11 + 9 = 31 \).

**Proof.** \( D_1, D_2 \) vary in a linear system of dimension \( 3 \times 4 - 1 = 11 \), \( D_3 \) varies in a linear system of dimension \( 5 \times 2 - 1 = 9 \).

The condition that the divisors intersect transversally is a consequence of the fact that each of the three linear systems embeds \( Q \) in a projective space.
The final condition amounts to the following: write

\[ \delta_j = u_0^2 A_j(v) + u_0 u_1 B_j(v) + u_1^2 C_j(v) = 0, \quad j = 1, 2. \]

We can view the coefficients of \( \delta_1, \delta_2 \), six degree three homogeneous polynomials in \( v = (v_0 : v_1) \), as giving a map \( \psi \) of \( \mathbb{P}^1 \) inside the \( \mathbb{P}^5 \) with coordinates

\[ (A_1 : B_1 : C_1 : A_2 : B_2 : C_2) \]

parametrizing pairs of homogeneous polynomials of degree 2 in \( u = (u_0 : u_1) \),

\[ P_j = u_0^2 A_j + u_0 u_1 B_j + u_1^2 C_j = 0, \quad j = 1, 2. \]

Let \( \Delta \) be the resultant

\[ \Delta := \text{Res}_u(P_1, P_2) = (A_1 C_2 - A_2 C_1)^2 + (B_2 C_1 - C_2 B_1)(A_1 B_2 - A_2 B_1). \]

The resultant \( \Delta \) defines a hypersurface of degree 4 in \( \mathbb{P}^5 \), which is reduced and irreducible, being a non degenerate quadric in the variables \( B_1, B_2 \).

We can therefore choose the six polynomials in a general way so that the twisted cubic curve \( \psi(\mathbb{P}^1) \) intersects \( \Delta \) transversally in 12 distinct points.

We conclude because the image of \( D_1 \cap D_2 \) via the projection \( p_v \) is given by the 12 zeros of

\[ f(v_0 : v_1) := \text{Res}_u(\delta_1, \delta_2) = \Delta(A_1(v) : B_1(v) : C_1(v) : A_2(v) : B_2(v) : C_2(v)), \]

and by our general choice these are 12 distinct points.

\[ \square \]

**Remark 2.3.** i) Taking into account the group of automorphisms of \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \), we see that the above family gives a locally closed subset of dimension 25 inside the moduli space of surfaces of general type. This dimension is more than the expected dimension \( 10\chi(S) - 2K_S^2 = 70 - 48 = 22 \).

ii) since \( H^0(O_Q(D_i - L_i)) = 0 \) for each \( i = 1, 2, 3 \), there are no natural deformations ([Cat84], [Cat99]); it is not clear that our family yields an irreducible component of the moduli space, since the elementary method of cor. 2.20 and theorem 3.8 of [Cat84] do not apply.

3. **The canonical ring as a module over the symmetric algebra** \( A := \text{Sym}(\mathcal{R}_1) \)

Let us first of all look at the homomorphism \( m_2 : \text{Sym}^2(\mathcal{R}_1) \rightarrow \mathcal{R}_2 \), keeping track of the eigenspace decompositions, and using the notation \( H^0(O_Q(a, b)) =: V(a, b) \). We have then

\[ \mathcal{R}_1 = 0 \oplus y_1 V(1, 0) \oplus y_2 V(1, 0) \oplus y_3 V(0, 1), \]

\[ \mathcal{R}_2 = V(4, 3) \oplus y_2 y_3 V(1, 1) \oplus y_1 y_3 V(1, 1) \oplus y_1 y_2 V(2, 0). \]
Since $V(1, 0) \otimes V(0, 1) \cong V(1, 1)$, and we have a surjection $V(1, 0) \otimes V(1, 0) \to V(2, 0)$, the non trivial character spaces of $R_2$ are in the image of $m_2 : Sym^2(R_1) \to R_2$.

Moreover, the kernel of $y_1V(1, 0) \otimes y_2V(1, 0) \to y_1y_2V(2, 0)$ is 1-dimensional, and provides a quadric $\{q(x) = 0\}$ containing the canonical image $\Sigma$ of $S$. To simplify our notation, we directly assume that $S \subset \mathbb{P}^5$, via the canonical embedding.

Then the quadric that we obtain is: $q(x) := x_2x_3 - x_1x_4$.

The trivial character space of $R_2$ is isomorphic to $V(4, 3)$ and contains the image of the subspace

$$W := y_1^2V(2, 0) \oplus y_2^2V(2, 0) \oplus y_3^2V(0, 2) \subset Sym^2(R_1),$$

which maps onto

$$W' := \delta_1V(2, 0) + \delta_2V(2, 0) + \delta_3V(0, 2).$$

**Lemma 3.1.** $W \to W'$ is an isomorphism.

**Proof.** Assume that there is a kernel: then there are bihomogeneous polynomials $P_1, P_2, P_3$ such that $P_1\delta_1 + P_2\delta_2 = P_3\delta_3$.

Since $\delta_3$ does not vanish at the 12 points where $\delta_1 = \delta_2 = 0$, $P_3 = P_3(v)$ should vanish on the projections of these 12 points under $p_v$. But this is a contradiction, since $P_3$ has degree 2, while the 12 projected points are distinct. Hence $P_3 \equiv 0$, and $\delta_1|P_2$, a contradiction again since $P_2$ has bidegree $(2, 0)$.

\[\square\]

**Corollary 3.1.** $S$ is contained in a unique quadric $\{q(x) = 0\}$, and $m_2 : Sym^2(R_1) \to R_2$ has image of dimension $21 - 1 = 20$ and codimension 11.

**Proof.** $R_m$ has dimension $\dim(R_m) = \chi(S) + \frac{1}{2}m(m - 1)K_S^2 = 7 + 12m(m - 1)$, which, for $m = 2$, is equal to 31.

\[\square\]

We shall now choose $z_1, \ldots, z_{11} \in R_2$ which, together with $Im(m_2)$, generate $R_2$. The elements $z_1, \ldots, z_{11}$ induce a basis of the quotient $R_2/Im(m_2)$.

**Theorem 3.2.** Let $A := Sym(R_1)$ be the coordinate ring of $\mathbb{P}^5$, and consider $R$ as an $A$-module. Then 1, $z_1, \ldots, z_{11}$ is a minimal graded system of generators of $R$ as an $A$-module.

**Proof.** In view of the previous observations, it suffices to show that these elements generate $R$.

Observe preliminarily that $V(a, b) \otimes V(c, d) \to V(a + c, b + d)$ is always surjective as soon as $a, b, c, d \geq 0$.

For $R_3$, let us write:

$$R_3 = y_1y_2y_3V(2, 1) \oplus y_1V(5, 3) \oplus y_2V(5, 3) \oplus y_3V(4, 4).$$
The last three eigenspaces are in the image of $V(4, 3) \otimes \mathcal{R}_1 \subset \mathcal{R}_2 \otimes \mathcal{R}_1 \to \mathcal{R}_3$.

Also the first summand is in the image of $y_1 V(1, 0) \otimes y_2 y_3 V(1, 1)$. The same argument works for $\mathcal{R}_{2m+1}$, while for $\mathcal{R}_{2m+2}$ we find surjections

$$H^0(\mathcal{O}_Q(N)) \otimes w_1 H^0(\mathcal{O}_Q(mN - L_i)) \to w_1 H^0(\mathcal{O}_Q((m+1)N - L_i)),$$

and

$$H^0(\mathcal{O}_Q(N)) \otimes H^0(\mathcal{O}_Q(mN)) \to H^0(\mathcal{O}_Q((m+1)N)).$$

Hence the claimed result follows by induction on $m$.

□

Remark 3.3. 1) Since $\text{dim } \mathcal{R}_3 = 79$, $\text{dim } \mathcal{A}_3 = 56$, we see that the 12 minimal generators of the module admit $56 + 6 \times 11 - 79 = 43$ relations in degree 3.

The module $\mathcal{R}$ is Cohen-Macaulay, hence it has a length 3 minimal graded resolution.

2) In general, $\text{dim } \mathcal{R}_m = 7 + 12m(m-1)$,

$$\text{dim } \mathcal{A}_m = \binom{m+5}{m},$$

hence the image of $\mathcal{A}_m \to \mathcal{R}_m$ has dimension less than or equal (because $S$ is contained in a quadric) to $\text{dim } \mathcal{A}_m - \text{dim } \mathcal{A}_{m-2}$.

Hence the Hartshorne-Rao module $\oplus_m H^1(I_S(m))$ is non zero in all degrees $m = 2, 3, 4, 5, 6$. This shows that the bundle $\mathcal{E}$ contructed by Walter using the Horrocks correspondence should be rather interesting.

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