Spectral Analysis of the Non-self-adjoint Mathieu-Hill Operator

O. A. Veliev
Depart. of Math., Dogus University, Acıbadem, Kadiköy, Istanbul, Turkey. e-mail: oveliev@dogus.edu.tr

Abstract

We obtain uniform, with respect to $t \in (\pi, \pi]$, asymptotic formulas for the operators generated in $[0, 1]$ by Mathieu-Hill equation with complex-valued potential and the $t$-periodic boundary conditions. Using these formulas, we prove that there exists a bounded set $S$ which is independent of $t$ such that all the eigenvalues of these operators, lying out of the set $S$, are simple. These results imply the following consequences for the non-self-adjoint Mathieu-Hill operator $H$ generated in $(-\infty, \infty)$ by the same Mathieu-Hill equation: (i) The spectrum of $H$ in a neighborhood of $\infty$ consists of the separated simple analytic arcs. (ii) The distance between the end points of the neighboring arcs of the spectrum of $H$ satisfies an asymptotic formula, that results in the Avron-Simon-Harrell formula for the widths of the instability intervals of the self-adjoint Mathieu-Hill operator. (iii) The number of the spectral singularities in the spectrum of the operator $H$ is finite. Furthermore, we establish the necessary and sufficient conditions for the potential, for which the operator $H$ has no spectral singularity at infinity and $H$ is an asymptotically spectral operator.

Key Words: Hill operator, Spectral singularities, Spectral operator.
AMS Mathematics Subject Classification: 34L05, 34L20.

1 Introduction and Preliminary Facts

Let $L(q)$ be the Hill operator generated in $L_2(-\infty, \infty)$ by the expression

$$-y'' + q(x)y, \quad (1)$$

where $q$ is a complex-valued summable function on $[0, 1]$ and $q(x + 1) = q(x)$. It is well-known that (see [6, 10-12]) the spectrum $S(L(q))$ of the operator $L(q)$ is the union of the spectra $S(L_t(q))$ of the operators $L_t(q)$ for $t \in (-\pi, \pi]$, where $L_t(q)$ is the operator generated in $L_2[0, 1]$ by (1) and the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0). \quad (2)$$

The operators $L_t(q)$ and $L(q)$ are denoted by $H_t$ and $H$ respectively when

$$q(x) = ae^{-i2\pi x} + be^{i2\pi x}, \quad (3)$$

where $a$ and $b$ are complex numbers. In the cases $t = 0$ and $t = \pi$ the operator $H_t$ is investigated, in detail, by Djakov and Mitjagin [3, 4]. In this paper we consider the operators $H$ and $H_t$ for all value of $t \in (-\pi, \pi]$. First we obtain asymptotic formulas,
uniform with respect to $t \in (-\pi, \pi]$, for eigenvalues of the operators $H_t$. Note that, the formula $f(k, t) = O(h(k))$ is said to be uniform with respect to $t$ in a set $I$ if there exists a positive constants $M$ and $N$, independent of $t$, such that $| f(k, t) | < M | h(k) |$ for all $t \in I$ and $| k | \geq N$. Then using these asymptotic formulas, we prove that there exists a positive number $R$, independent of $t$, such that the eigenvalue $\lambda_n(t)$ of $H_t(q)$ for all $t \in (-\pi, \pi]$ lying in $\{ z \in \mathbb{C} : | z | > R \}$ is simple. It implies that the spectrum of $H$ in a neighborhood of $\infty$ consist of the separated simple analytic arcs and the number of the spectral singularities in $S(H)$ is finite. Moreover, we find necessary and sufficient condition on the numbers $a$ and $b$ for which the operator $H$ has no spectral singularity at infinity and $H$ is asymptotically spectral operator.

Note that the spectral singularities of the operator $L(q)$ are the points of its spectrum in neighborhoods of which the projections of $L(q)$ are not uniformly bounded. We say that the operator $H$ has a spectral singularities at infinity if there exists a sequence $\{ \gamma_n \}$ of arcs $\gamma_n \subset S(H)$ such that the norm of the projections $P(\gamma_n)$ of $H$ corresponding to $\gamma_n$ tends to infinity as $\gamma_n$ goes to infinity. The operator $H$ is said to be asymptotically spectral operator if there exists $R > 0$ such that the projections $P(\gamma)$ of $H$ corresponding to the arcs $\gamma$ lying in $S(H) \cap \{ | z | > R \}$ are uniformly bounded.

In [14] we proved that a number $\lambda = \lambda_n(t) \in S(L_t(q)) \subset S(L(q))$ is a spectral singularity of $L(q)$ if and only if the operator $L_t(q)$ has an associated function at the point $\lambda_n(t)$. McGarvey [10-12] proved that $L(q)$ is a spectral operator if and only if the projections of the operator $L(q)$ are uniformly bounded. Recently, Gesztesy and Tkachenko [7,8] proved two versions of a criterion for the Hill operator $L(q)$ with $q \in L_2[0,1]$ to be a spectral operator of scalar type, in sense [5], one analytic and one geometric. The analytic version is stated in term of the solutions of Hill’s equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems. In [16] and in this paper we find conditions on the potential $q$ for which $L(q)$ is an asymptotically spectral operator, since we use the asymptotic methods. Note that if $L(q)$ has no spectral singularity in $\{ | z | \leq R \}$ and is an asymptotically spectral operator, then it is a spectral operator. It seems this may open up a new horizon for solving the long-standing open problem about deciding which Hill operators are spectral operators.

The next we present some preliminary facts, from [15, 16, 4], we need in this paper. We use the following results of [15]:

The eigenvalue $\lambda_n(t)$ and eigenfunction $\Psi_{n,t}(x)$ of the operator $L_t(q)$ for $t \neq 0, \pi$, satisfy the following asymptotic formulas

$$\lambda_n(t) = (2\pi n + t)^2 + O\left(\frac{[n]}{n}\right), \quad \Psi_{n,t}(x) = e^{i(2\pi n + t)x} + O\left(\frac{1}{n}\right). \tag{4}$$

These asymptotic formulas are uniform with respect to $t$ in $[\rho, \pi - \rho]$, where $\rho$ is a sufficiently small fixed number ($\rho \ll 1$). In the other word, there exist positive numbers $N(\rho)$, independent of $t$, such that the eigenvalues $\lambda_n(t)$ for $t \in [\rho, \pi - \rho]$ and $| n | > N(\rho)$ are simple and the terms $O\left(\frac{1}{n}\right), O\left(\frac{[n]}{n}\right)$ in (4) do not depend on $t$.

Thus, the case $t \in [\rho, \pi - \rho]$, where $\rho \ll 1$ is investigated in [15]. In the paper [16] we obtained the uniform asymptotic formulas in the more complicated case $t \in [0, \rho] \cup [\pi - \rho, \pi]$, when the potential $q$ satisfies the following conditions:

1. $q \in W^p_t[0,1], q^{(k)}(0) = q^{(k)}(1), \quad \forall k = 0, 1, ..., s - 1$ with some $s \leq p$.
2. $q_n \sim q, | q_n | > cn^{-s-1}$ and at least one of the following inequalities $\text{Re} q_n q_n' \geq 0$, $\text{Im} q_n q_n' \geq \epsilon | q_n q_n' |$ holds, where $c$ and $\epsilon$ are positive constants and $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$.

It is clear that these results of [16] can not be used for the potential (3). However we use a lot of formulas of [16] that are listed below as Remark 1 and formulas (6)-(21):
Remark 1 There exists a positive integer \( N \) such that disk

\[
U(n, t, \rho) =: \{ \lambda \in \mathbb{C} : \left| \lambda - (2\pi n + t)^2 \right| \leq 15\pi n \rho \}
\]

for \( t \in [0, \rho] \) and \( n > N \) contains two eigenvalues (counting with multiplicities) denoted by \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \). These eigenvalues are continuous function of \( t \) on the interval \([0, \rho]\). In addition to these eigenvalues, the operator \( L_n(q) \) for \( t \in [0, \rho] \) has only \( 2N + 1 \) eigenvalues denoted by \( \lambda_k(t) \) for \( k = 0, \pm 1, \pm 2, \ldots, \pm N \). One can readily see that

\[
|\lambda - (2\pi n - k) + t^2| > |k| |2n - k|, \quad \forall \lambda \in U(n, t, \rho)
\]

for \( k \neq 0, 2n \) and \( t \in [0, \rho] \), where \( n > N \).

In [16] to obtain the uniform asymptotic formula for eigenvalues \( \lambda_{n,j}(t) \) and corresponding normalized eigenfunctions \( \Psi_{n,j,t}(x) \) for \( t \in [0, \rho] \), we used (5) and the iteration of the formula

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2)(\Psi_{n,j,t}(x), e^{i(2\pi n + t)x}) = (q\Psi_{n,j,t}, e^{i(t \pi n + t)x}),
\]

where \((., .)\) is the inner product in \( L_2[0,1] \). Iterating (6) infinite times we got the following formula:

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2 - A(\lambda_{n,j}(t), t))u_{n,j}(t) = (q_{2n} + B(\lambda_{n,j}(t), t))v_{n,j}(t),
\]

where

\[
A(\lambda, t) = \sum_{k=1}^{\infty} a_k(\lambda, t), \quad B(\lambda, t) = \sum_{k=1}^{\infty} b_k(\lambda, t),
\]

\[
a_k(\lambda, t) = \sum_{n_1, n_2, \ldots, n_k} q_{-n_1-n_2-\ldots-n_k} k \prod_{s=1}^{k} q_{n_s} \left( \lambda - (2\pi (n - n_1 - \ldots - n_s) + t)^2 \right)^{-1},
\]

\[
b_k(\lambda, t) = \sum_{n_1, n_2, \ldots, n_k} q_{2n_1-n_2-\ldots-n_k} k \prod_{s=1}^{k} q_{n_s} \left( \lambda - (2\pi (n + n_1 + \ldots + n_s) + t)^2 \right)^{-1},
\]

The sums in (9), (10) are taken under conditions \( n_s \neq 0 \) and \( n_1 + n_2 + \ldots + n_s \neq 0, 2n \) for \( s = 1, 2, \ldots \). Similarly, we obtained the formula

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2 - A'(\lambda_{n,j}(t), t))v_{n,j}(t) = (q_{-2n} + B'(\lambda_{n,j}(t), t))u_{n,j}(t),
\]

where

\[
A'(\lambda, t) = \sum_{k=1}^{\infty} a_k'(\lambda, t), \quad B'(\lambda, t) = \sum_{k=1}^{\infty} b_k'(\lambda, t),
\]

\[
a_k'(\lambda, t) = \sum_{n_1, n_2, \ldots, n_k} q_{-n_1-n_2-\ldots-n_k} k \prod_{s=1}^{k} q_{n_s} \left( \lambda - (2\pi (n + n_1 + \ldots + n_s) - t)^2 \right)^{-1},
\]

\[
b_k'(\lambda, t) = \sum_{n_1, n_2, \ldots, n_k} q_{-2n-n_1-n_2-\ldots-n_k} k \prod_{s=1}^{k} q_{n_s} \left( \lambda - (2\pi (n + n_1 + \ldots + n_s) - t)^2 \right)^{-1},
\]

where the sums are taken under conditions \( n_s \neq 0 \), \( n_1 + n_2 + \ldots + n_s \neq 0, -2n \) for \( s = 1, 2, \ldots \). Moreover, it is proved [16] that the equalities

\[
a_k(\lambda, t), \quad b_k(\lambda, t), \quad a_k'(\lambda, t), \quad b_k'(\lambda, t) = O \left( \frac{\ln |n|}{n} \right)^k.
\]
hold uniformly for $t \in [0, \rho]$ and $\lambda \in U(n, t, \rho)$ and the following estimations hold:

(See Lemma 2 and Lemma 3 of [16]) There exists a constant $K$, independent of $n$ and $t$, such that the following inequalities are satisfied

$$|A(\lambda, t)| < K^{-1}, \quad |A'(\lambda, t)| < K^{-1}, \quad |B(\lambda, t)| < K^{-1}, \quad |B'(\lambda, t)| < K^{-1}, \quad (16)$$

$$|A(\lambda, t) - A(\mu, t)| < K^{-2} |\lambda - \mu|, \quad |A'(\lambda, t) - A'(\mu, t)| < K^{-2} |\lambda - \mu|, \quad (17)$$

$$|B(\lambda, t) - B(\mu, t)| < K^{-2} |\lambda - \mu|, \quad |B'(\lambda, t) - B'(\mu, t)| < K^{-2} |\lambda - \mu|, \quad (18)$$

$$|C(\lambda, t)| < tk^{-1}, \quad |C(\lambda, t) - C(\mu, t)| < tk^{-2} |\lambda - \mu| \quad (19)$$

for all $n > N$, $t \in [0, \rho]$ and $\lambda, \mu \in U(n, t, \rho)$, where $N$ and $U(n, t, \rho)$ are defined in Remark 1, and $C(\lambda, t) = \frac{1}{2}(A(\lambda, t) - A'(\lambda, t))$.

In this paper we use also the following, uniform with respect to $t \in [0, \rho]$, equalities for the eigenfunction $\Psi_{n,j,t}(x)$ obtained in [16]:

$$\Psi_{n,j,t}(x) = u_{n,j}(t)e^{i(2\pi n+\rho)x} + v_{n,j}(t)e^{-i(2\pi n+\rho)x} + h_{n,j,t}(x), \quad (20)$$

$$(h_{n,j,t}, e^{i(2\pi n+\rho)x}) = 0, \quad \|h_{n,j,t}\| = O\left(\frac{1}{n}\right), \quad \|u_{n,j}(t)\|^2 + \|v_{n,j}(t)\|^2 = 1 + O\left(\frac{1}{n^2}\right). \quad (21)$$

Besides we use the formula (55) of [4] about estimations of $B(\lambda, 0)$ and $B'(\lambda, 0)$ that can be written, in the notations of this paper, as follows:

Let the potential $q$ has the form (5), $\lambda = (2\pi n)^2 + z$, where $|z| < 1$, and

$$p_{n_1, n_2, ..., n_k}(\lambda, 0) = q_{2n-n_1-n_2-...-n_k} \prod_{s=1}^{k} q_{n_s} \left(\lambda - (2\pi(n - n_1 - ... - n_s))^2\right)^{-1} \quad (22)$$

be summands of $b_k(\lambda, t)$ for $t = 0$ (see (10)). Then

$$\sum_{k=2n+3}^{\infty} \sum_{n_1, n_2, ..., n_k} |p_{n_1, n_2, ..., n_k}(\lambda, 0)| = b_{2n-1}(\lambda, 0)O\left(\frac{1}{n^2}\right). \quad (23)$$

At last, note that we consider only the case $t \in [0, \rho]$ due to the following reason. The case $t \in [\rho, \pi - \rho]$, is considered in [15]. The case $t \in [\pi - \rho, \pi]$ is similar to the case $t \in [0, \rho]$. Namely, instead of (5) for $k \neq 0$, $2n$ using the same inequality for $k \neq 0, 2n+1$ and repeating the investigations for $t \in [0, \rho]$, we obtain the same results for $t \in [\pi - \rho, \pi]$. Besides, the eigenvalues of $L_{-1}(q)$ coincides with the eigenvalues of $L_t(q)$. That is why we consider only the case $t \in [0, \rho]$.

In Section 2 we obtain some general results for $L_t(q)$ with locally integrable potential $q$. In Section 3 using the results of Section 2 we obtain asymptotic formulas, uniform with respect to $t \in (-\pi, \pi]$, and prove that all large eigenvalues $\lambda_{n,j}(t)$ of the operators $H_t$ for all $t \in (-\pi, \pi]$ are simple (Theorem 4). These results imply that the spectrum of the operator $H$ in a neighborhood of $\infty$ consist of separated simple analytic arcs and the number of spectral singularities in $S(H)$ is finite (see Theorem 5 and Theorem 6). Besides using the formulas and estimation of Section 3 and the Djakov-Mitjagin [4] estimations (23) we obtain the formulas (see (80) and (81)) for the distance between the end points of the neighboring arcs of the spectrum of $H$ which implies the Avron-Simon-Harrell formula [1, 9] for the widths of the instability intervals of the self-adjoint Mathieu-Hill operator. The main results of the Section 4 is the following: We find the necessary and sufficient conditions on numbers $a$ and $b$ for which the operator $H$ has no spectral singularity at infinity and $H$ is asymptotically spectral operator.
2 Some General Results for \( L_t(q) \) with \( q \in L_1[0,1] \)

First we consider the cases: \( t = 0 \) and \( t = \pi \) which correspond the periodic and antiperiodic boundary conditions (see (2)). These cases for \( q \in L_2[0,1] \) is considered by Djakov and Mitjagin [3,4]. We obtain similar results for the potentials \( q \in L_1[0,1] \) by other methods, namely by methods of our papers [2, 13]. For brevity, we discuss only the periodic problem and denote \( \lambda_{n,j}(t), A(\lambda_{n,j}(t), t), B(\lambda_{n,j}(t), t), A' (\lambda_{n,j}(t), t), B' (\lambda_{n,j}(t), t) \) for \( t = 0 \) (see (7) and (11)) by \( \lambda_{n,j}, A(\lambda_{n,j}), B(\lambda_{n,j}), A' (\lambda_{n,j}), B' (\lambda_{n,j}) \) respectively. The antiperiodic problem is similar to the periodic problem.

One can readily see from (7), (11), (16) and Remark 1 that

\[
\lambda_{n,j}(t) \in d^-(r(n), t) \cup d^+(r(n), t) \subset U(n, t, \rho),
\]

where \( r(n) = \max\{|q_{2n}|, |q_{-2n}|\} + 2Kn^{-1} \) and \( d^\pm(r(n), t) \) is a disk with center \((\pm 2\pi n + t)^2\) and radius \( r(n) \). Indeed if \( \lbrack u_{n,j}(t) \rbrack \geq |v_{n,j}(t)| \), then using (7) (if \( |v_{n,j}(t)| > u_{n,j}(t) \), then using (11) and (16) we get (24). In case \( t = 0 \) the disks \( d^-(r(n), t) \) and \( d^+(r(n), t) \) are the same and is denoted by \( d(r(n)) \).

**Theorem 1** Let \( q \in L_1[0,1] \). If the inequality

\[
|q_{2n} + B(\lambda)| + |q_{-2n} + B'(\lambda)| \neq 0 \quad (25)
\]

holds for \( \lambda \in d(r(n)) \), then the geometric multiplicity of the eigenvalue \( \lambda_{n,j} \) is 1. If (25) holds, then the root functions of \( L_0(q) \) form a Riesz basis if and only if

\[
q_{2n} + B(\lambda) \sim q_{-2n} + B'(\lambda) \quad (26)
\]

for \( \lambda \in d(r(n)) \). If (25) and (26) hold, then the eigenvalue \( \lambda_{n,j} \) of \( L_0(q) \) lying in \( d(r(n)) \) for \( n > N \) is simple.

**Proof.** Suppose that there exist 2 eigenfunctions corresponding to \( \lambda_{n,j} \). Then one can choose the eigenfunction \( \Psi_{n,j,0} \) such that \( u_{n,j}(0) = 0 \). This with (7) and (21) implies that \( q_{2n} + B(\lambda_{n,j}) = 0 \). In the same way we prove that \( q_{-2n} + B'(\lambda_{n,j}) = 0 \). The last two equalities contradict (25).

If (25) and (26) hold, then one can readily see that

\[
q_{-2n} + B'(\lambda_{n,j}) \neq 0, \quad q_{-2n} + B'(\lambda_{n,j}) \neq 0 \quad (27)
\]

These with the formulas (7), (11), (21) imply that

\[
u_{n,j}(t)u_{n,j}(t) \neq 0 \quad (28)
\]

for \( t = 0 \). Indeed if \( u_{n,j}(0) = 0 \) then by (21) \( v_{n,j}(0) \neq 0 \) and by (7) \( q_{2n} + B(\lambda_{n,j}) = 0 \) which contradicts (27). Similarly, if \( v_{n,j}(0) = 0 \) then by (21) and (11) \( q_{-2n} + B'(\lambda_{n,j}) = 0 \) which again contradicts (27). By (27) and (28) the left-hand sides of (7) and (11) for \( t = 0 \) are not zero. Therefore dividing (7) and (11) side by side and using the equality \( A(\lambda_{n,j}) = A' (\lambda_{n,j}) \) noted in [13] (see the proof of Lemma 3), we get

\[
\frac{q_{-2n} + B'(\lambda_{n,j})}{q_{2n} + B(\lambda_{n,j})} = \frac{u_{n,j}^2(0)}{v_{n,j}^2(0)} \quad (29)
\]
Then, by (26) and (21) we have
\[ u_{n,j}(0) \sim v_{n,j}(0) \sim 1. \] (30)
which implies that the set of Jordan chains is finite (see the end of page 118 of [2]). Thus, if (25) and (26) hold, then \( \lambda_{n,j}(0) \) for \( n > N \) and \( j = 1, 2 \) is simple. Moreover, by Theorem 1 of [13] the relation (30) implies that the root functions of \( L_0(q) \) form a Riesz basis.

Now suppose that (25) holds and the root functions of \( L_0(q) \) form a Riesz basis. By Theorem 1 of [13] the set of Jordan chains is finite and (30) holds. On the other hand one of the summand in (25) is not zero. Suppose, without less of generality, that \( q_{2n} + B(\lambda_{n,j}) \neq 0 \). Then, using (7), (11) for \( t = 0 \) and (30) we see that the equality (29) holds (see the proof of (29)). Therefore (30) implies (26). The theorem is proved.

Now we consider the case \( t \in [0, \rho] \).

\textbf{Theorem 2} A number \( \lambda \in U(n, t, \rho) \) is an eigenvalue of \( L_t(q) \) for \( t \in [0, \rho] \) and \( n > N \), where \( U(n, t, \rho) \) and \( N \) are defined Remark 1, if and only if
\[ (\lambda - (2\pi n + t)^2 - A(\lambda, t))(\lambda - (2\pi n - t)^2 - A'(\lambda, t)) = (q_{2n} + B(\lambda, t))(q_{-2n} + B'(\lambda, t)). \] (31)
Moreover \( \lambda \in U(n, t, \rho) \) is a double eigenvalue of \( L_t \) if and only if it is a double root of (31).

\textbf{Proof.} If \( u_{n,j}(t) = 0 \), then by (21) \( v_{n,j}(t) \neq 0 \). Therefore (7) and (11) imply that \( q_{2n} + B(\lambda_{n,j}(t), t) = 0 \) and \( (\lambda_{n,j}(t) - (2\pi n + t)^2 - A'(\lambda_{n,j}(t), t)) = 0 \), that is, right-hand side and left-hand side of (31) vanish when \( \lambda \) is replaced by \( \lambda_{n,j}(t) \). Hence \( \lambda_{n,j}(t) \) satisfies (31). In the same way we prove that if \( v_{n,j}(t) = 0 \) then \( \lambda_{n,j}(t) \) is a root of (31). It remains to consider the case \( u_{n,j}(t)v_{n,j}(t) \neq 0 \). In this case multiplying (7) and (11) side by side and canceling \( u_{n,j}(t)v_{n,j}(t) \) we get an equality obtained from (31) by replacing \( \lambda \) with \( \lambda_{n,j}(t) \). Thus in any case \( \lambda_{n,j}(t) \) is a roots of the equations (31).

Now we prove that the roots of (31) lying in \( U(n, t, \rho) \) are the eigenvalues of \( L_t(q) \). Denote by \( F(\lambda, t, \cdot) \) the difference of left-hand side and right-hand side of (31). Using (16) one can easily verify that the inequality
\[ |F(\lambda, t, \cdot)| < |G(\lambda, t)|, \] (32)
where \( G(\lambda, t) = (\lambda - (2\pi n + t)^2)(\lambda - (2\pi n - t)^2) \), holds for all \( \lambda \) from the boundary of \( U(n, t, \rho) \). Since the function \( (\lambda - (2\pi n + t)^2)(\lambda - (2\pi n - t)^2) \) has two roots in the set \( U(n, t, \rho) \), by the Rouche’s theorem (32) implies that \( F(\lambda, t) \) has two roots in the same set. Thus \( L_t(q) \) has two eigenvalue (counting with multiplicities) lying in \( U(n, t, \rho) \) (see Remark 1) that are the roots of (31). On the other hand, (31) has precisely two roots (counting with multiplicities) in \( U(n, t, \rho) \). Therefore \( \lambda \in U(n, t, \rho) \) is an eigenvalue of \( L_t(q) \) if and only if (31) holds. If \( \lambda \in U(n, t, \rho) \) is a double eigenvalue of \( L_t(q) \) then by Remark 1 \( L_t(q) \) has no other eigenvalue in \( U(n, t, \rho) \) and hence (31) has no other root. This implies that \( \lambda \) is a double root of (31). By the same argument if \( \lambda \) is a double root of (31) then it is double eigenvalue of \( L_t(q) \).

One can readily verify that the equation (31) can be written in the form
\[ (\lambda - (2\pi n + t)^2 - \frac{1}{2}(A + A') + 4\pi nt + \sqrt{D})(\lambda - (2\pi n - t)^2 - \frac{1}{2}(A + A') + 4\pi nt - \sqrt{D}) = 0, \] (33)
where
\[ D(\lambda, t) = (4\pi nt)^2 + q_{2n}q_{-2n} + 8\pi ntC + C^2 + q_{2n}B' + q_{-2n}B + BB'. \] (34)
and, for brevity, we denote \( C(\lambda, t), B(\lambda, t), A(\lambda, t) \) etc. by \( C, B, A \) etc. It is clear that \( \lambda \)
is a root of (33) if and only if it satisfies at least one of the equations

\[ \lambda = (2\pi n + t)^2 + \frac{1}{2} (A(\lambda, t) + A'(\lambda, t)) - 4\pi nt - \sqrt{D(\lambda, t)} \]

(35)

and

\[ \lambda = (2\pi n + t)^2 + \frac{1}{2} (A(\lambda, t) + A'(\lambda, t)) - 4\pi nt + \sqrt{D(\lambda, t)}. \]

(36)

**Theorem 3** (a) A number \( \lambda_0 \in U(n, t, \rho) \) is an eigenvalue of \( L_t(q) \) if and only if it satisfies at least one of (35) and (36). Each of the equations (35) and (36) has a root in \( U(n, t, \rho) \).

(b) The following statements are equivalent:

(i) A number \( \lambda_0 \in U(n, t, \rho) \) is a double eigenvalue of \( L_t(q) \).

(ii) \( \lambda_0 \) is a root of both (35) and (36).

(iii) \( \lambda_0 \) is a root of (33) and \( D(\lambda_0, t) = 0 \)

(c) If the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are simple, then one of them, say \( \lambda_{n,1}(t) \), satisfies only (35) and the other satisfies only (36), that is,

\[ \lambda_{n,j}(t) = (2\pi n + t)^2 + \frac{1}{2} (A(\lambda_{n,j}, t) + A'(\lambda_{n,j}, t)) - 4\pi nt + (-1)^j \sqrt{D(\lambda_{n,j}, t)}. \]

(37)

**Proof.** (a) Since (31) is equivalent to (33), by Theorem 2, \( \lambda_0 \in U(n, t, \rho) \) is an eigenvalue of \( L_t(q) \) if and only if it satisfies at least one of the equations (35) and (36).

To prove that (35) and (36) has a root in \( U(n, t, \rho) \), let us consider a family of Hill operators \( L_t(\varepsilon q) \) with potential \( \varepsilon q \), where \( \varepsilon \) is a complex parameter and \( t \) is a fixed number from \( [0, \rho] \). By Remark 1 the operator \( L_t(\varepsilon q) \), when \( \varepsilon \in \{ z \mid z \mid \leq 1 \} \), has two eigenvalues (counting multiplicity) lying in \( U(n, t, \rho) \). The eigenvalues \( (2\pi n + t)^2 \) and \( (2\pi n + t)^2 \) of the operator \( L_t(0) \) are simple and satisfy (35) and (36) respectively, since for \( q = 0 \) we have \( A = A' = 0, D = (4\pi nt)^2 \). It is clear that if \( \lambda \in U(n, t, \rho) \) is a multiple eigenvalue of \( L_t(\varepsilon q) \), then there exists a deleted neighborhood \( V \) of \( \varepsilon_0 \) such that for \( \varepsilon \in V \) the operator \( L_t(\varepsilon) \) has two simple eigenvalues. Therefore there exists a continuous curve \( \gamma \) with end points 0 and 1 such that:

1) \( \gamma \subset \{ z \mid z \mid < 1 \}, 0 \in \gamma, 1 \notin \gamma \).

2) For \( \varepsilon \in \gamma \) the operator \( L_t(\varepsilon q) \) has two simple eigenvalues, denoted by \( \lambda_{n,1}(t, \varepsilon) \) and \( \lambda_{n,2}(t, \varepsilon) \), lying in \( U(n, t, \rho) \) and \( \lambda_{n,1}(t, 0) = (2\pi n + t)^2, \lambda_{n,2}(t, 0) = (2\pi n + t)^2 \). Moreover the functions \( \lambda_{n,1}(t, \varepsilon) \) and \( \lambda_{n,1}(t, \varepsilon) \) are analytic, with respect to \( \varepsilon \), on \( \gamma \) and continuous at \( \varepsilon = 1 \).

3) \( D(\lambda_{n,j}(t), \varepsilon) \neq 0 \) for \( \varepsilon \in \gamma \) and \( j = 1, 2 \).

By 2) \( \lambda_{n,1}(t, 0) \) and \( \lambda_{n,2}(t, 0) \) satisfy (35) and (36) respectively. Let us prove that \( \lambda_{n,j}(t, \varepsilon) \) satisfies (35) for all \( \varepsilon \in \gamma \). Suppose to the contrary that this claim is not true. Then there exists \( \varepsilon \in \gamma \) and the sequences \( p_n \rightarrow \varepsilon \) and \( q_n \rightarrow \varepsilon \), where one of them may be constant sequence, such that \( \lambda_{n,1}(t, p_n) \) and \( \lambda_{n,1}(t, q_n) \) satisfy (35) and (36) respectively. Using the continuity of \( \lambda_{n,1}(t, \varepsilon) \), we conclude that \( \lambda_{n,1}(t, \varepsilon) \) satisfies both (35) and (36). However it is possible only if \( D(\lambda_{n,1}(t, \varepsilon), t) = 0 \) which contradicts 3). Hence \( \lambda_{n,1}(t, \varepsilon) \) satisfies (35) for all \( \varepsilon \in \gamma \). In the same way we prove that \( \lambda_{n,2}(t, \varepsilon) \) satisfies (36) for all \( \varepsilon \in \gamma \). Since 1 is an end point of \( \gamma \) and \( \lambda_{n,j}(t, \varepsilon) \) is continuous at \( \varepsilon = 1 \), we conclude that \( \lambda_{n,1}(t, 1) \) and \( \lambda_{n,2}(t, 1) \) are the roots of the equations (35) and (36). Thus the equations (35) and (36) has the roots in \( U(n, t, \rho) \) for \( t \in (0, \rho) \). Since the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) of \( L_t(q) \) lying in \( U(n, t, \rho) \) are continuous functions of \( t \) on the interval \( [0, \rho] \) (see Remark 1) the equations (35) and (36) for \( t = 0 \) also has the roots \( \lambda_{n,1}(0) \) and \( \lambda_{n,2}(0) \) lying in \( U(n, t, \rho) \).

(b) One can readily see that (ii) and (iii) are equivalent. Let \( \lambda_0 \in U(n, t, \rho) \) be a double eigenvalue of \( L_t(q) \). Then by Theorem 3(a), \( \lambda_0 \) satisfies at least one of the equations (35) and (36). Without loss of generality assume that \( \lambda_0 \) satisfies (35). Again by Theorem 3(a)
On the other hand by (b) (see Remark 1). By (ii) it holds uniformly, with respect to $\lambda$, is a double root of the equation (33) that are equivalent to (31). Therefore from Theorem 1 we conclude that $\lambda_0$ is a double eigenvalue of $L_t(q)$.

(c) Suppose the eigenvalues $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ are simple and hence $\lambda_{n,1}(t) \neq \lambda_{n,2}(t)$ (see Remark 1). By (a), $\lambda_{n,1}(t)$ satisfies at least one of the equations (35) and (36). Suppose that it satisfies (35). By (b) $\lambda_{n,1}(t)$ does not satisfy (36), since it is not a double eigenvalue. On the other hand by (a), (36) has a root, different from $\lambda_{n,1}(t)$, which is an eigenvalue of $L_t(q)$ lying in $U(n, t, \rho)$. Therefore $\lambda_{n,2}(t)$ satisfies (36) and by (b), does not satisfy (35).

## 3 On the Operator $H_t$ for $t \in (−\pi, \pi]$.

In this section we study the operator $H_t$ for $t \in [0, \rho]$. When the potential $q$ has the form (3) then

$$ q_{-1} = a, \ q_1 = b, \ q_n = 0, \ \forall n \neq \pm 1 $$

and hence the formulas (7), (11), (31), (34) have the form

$$ (\lambda_{n,j}(t) - (2\pi n + t)^2 - A(\lambda_{n,j}(t), t))u_{n,j}(t) = B(\lambda_{n,j}(t), t)v_{n,j}(t), \quad (39) $$

$$ (\lambda_{n,j}(t) - (-2\pi n + t)^2 - A'(\lambda_{n,j}(t), t))v_{n,j}(t) = B'(\lambda_{n,j}(t), t)u_{n,j}(t), \quad (40) $$

$$ (\lambda - (2\pi n + t)^2 - A(\lambda, t))(\lambda - (2\pi n - t)^2 - A'(\lambda, t)) = B(\lambda, t)B'(\lambda, t). \quad (41) $$

Moreover, by Theorem 2, $\lambda \in U(n, t, \rho)$ is a double eigenvalue of $L_t(q)$ if and only if it satisfies (41) and the equation

$$ (1 - \frac{d}{d\lambda}A)(\lambda - (2\pi n - t)^2 - A') + (1 - \frac{d}{d\lambda}A')(\lambda - (2\pi n + t)^2 - A) = \frac{d}{d\lambda}(BB') \quad (43) $$

By, (24) and (38) $\lambda_{n,j}(t) \in d^-(2Kn^{-1}, t) \cup d^+(2Kn^{-1}, t) \subset U(n, t, \rho)$. Therefore the formula

$$ \lambda_{n,j}(t) = (2\pi n)^2 + O(n^{-1}) \quad (44) $$

holds uniformly, with respect to $t \in [0, n^{-2}]$, for $j = 1, 2$.

Let us consider the functions taking part in (39)-(42) in detail. From (38) we see that the indices in formulas (9) and (10) satisfy the conditions

$$ \{n_1, n_2, ..., n_k\} \subset \{-1, 1\}, \ n_1 + n_2 + ... + n_s \neq 0, 2n \quad (45) $$

and

$$ \{n_1, n_2, ..., n_k, 2n - n_1 - n_2 - ... - n_k\} \subset \{-1, 1\}, \ n_1 + n_2 + ... + n_s \neq 0, 2n \quad (46) $$

for $s = 1, 2, ..., k$ respectively. Hence, by (38) $q_{-n_1-n_2-...-n_k} = 0$ if $k$ is an even number. Therefore, by (9) and (13)

$$ a_{2m}(\lambda, t) = 0, \ a'_{2m}(\lambda, t) = 0, \forall m = 1, 2, ... \quad (47) $$
Since \( a_k(\lambda, t) \) for \( k = 2m - 1 \) are the sum of \( 2^k \) terms of the form

\[
a_k(\lambda, n_1, n_2, ..., n_k, t) = (ab)^m \prod_{s=1, 2, ..., k} (\lambda - (2\pi(n - n_1 - n_2 - ... - n_s) + t)^2)^{-1}
\]

(see (9) and (45)) we have

\[
a_{2m-1}(\lambda, n, t) = (4ab)^m O(n^{-2m+1}). \quad (48)
\]

If \( t \in [0, n^{-2}] \), then one can readily see that

\[
a_1(\lambda, n, t) = \frac{ab}{(2\pi n)^2 + O(n^{-1}) - (2\pi(n-1))^2} + \frac{ab}{(2\pi n)^2 + O(n^{-1}) - (2\pi(n+1))^2}
= \frac{ab}{2\pi(\pi(4n-1))} - \frac{ab}{2\pi(\pi(4n+1))} + O\left(\frac{1}{n^3}\right) = O\left(\frac{1}{n^2}\right).
\]

The same estimations for \( a'_1(\lambda, n, t) \) and \( a'_{2m+1}(\lambda, n, t) \) hold respectively. Thus we have

\[
A(\lambda, n, t) = O(n^{-2}), \quad A'(\lambda, n, t) = O(n^{-2}), \quad \forall t \in [0, n^{-2}]. \quad (49)
\]

Now let us consider the functions \( B(\lambda, t) \) and \( B'(\lambda, t) \) (see (8), (10) and (12), (14)). By (46) if \( k = 2n + 1 \) then \( n_1 = n_2 = ... = n_k = 1 \) and hence

\[
b_{2n-1}(\lambda, t) = b^{2n-1} \prod_{s=1}^{2n-1} (\lambda - (2\pi(n-s) + t)^2)^{-1}. \quad (50)
\]

If \( k < 2n - 1 \) or \( k = 2m \), then by (38), \( q_{2n-1-n_2-...-n_k} = 0 \) and by (10)

\[
b_k(\lambda, t) = 0 \quad (51)
\]

In the same way from (14) we get

\[
b'_{2n-1}(\lambda, t) = a^{2n-1} \prod_{s=1}^{2n-1} (\lambda - (2\pi(n-s) - t)^2)^{-1}, \quad b'_k(\lambda, n, t) = 0 \quad (52)
\]

for \( k < 2n - 1 \) or \( k = 2m \). Now, (15), (51) and (52) imply that the equalities

\[
B(\lambda, t) = O\left(\frac{\ln |n|}{n}^{2n-1}\right), \quad B'(\lambda, t) = O\left(\frac{\ln |n|}{n}^{2n-1}\right) \quad (53)
\]

hold uniformly for \( t \in [0, \rho] \) and \( \lambda \in U(n, t, \rho) \). From (39) and (40) by using (49) and (53) we obtain that the formula

\[
\lambda_{n,j}(t) = (2\pi n)^2 + O(n^{-2}) \quad (54)
\]

hold uniformly, with respect to \( t \in [0, n^{-3}] \), for \( j = 1, 2, \ldots \)

More detail estimations of \( B \) and \( B' \) are given in the following lemma.

**Lemma 1** If \( q \) has the form (3), then the formulas

\[
B(\lambda, t) = \beta_n \left(1 + O(n^{-2})\right), \quad B'(\lambda, t) = \alpha_n \left(1 + O(n^{-2})\right), \quad (55)
\]

\[
\frac{\partial}{\partial \lambda} \left(B'(\lambda, t)B(\lambda, t)\right) \sim \alpha_n \beta_n \left(\frac{\ln n}{n}\right) \quad (56)
\]
hold for
\[ t \in [0, n^{-3}], \quad \lambda = (2\pi n)^2 + O(n^{-2}), \]
where \( \beta_n = b^{2n} ((2\pi)^{2n-1}(2n-1)!)^{-2} \) and \( \alpha_n = a^{2n} ((2\pi)^{2n-1}(2n-1)!)^{-2} \).

**Proof.** By direct calculations we get
\[ b_{2n-1}((2\pi n)^2, 0) = \beta_n, \quad b'_{2n-1}((2\pi n)^2, 0) = \alpha_n. \] (58)

If \( 1 \leq s \leq 2n-1 \) then for any \((\lambda, t)\) satisfying (57) there exists \( \lambda_1 = (2\pi n)^2 + O(n^{-2}) \) and \( \lambda_2 = (2\pi n)^2 + O(n^{-2}) \) such that
\[ |\lambda_1 - (2\pi(n-s))^2| < |\lambda - (2\pi(n-s) + t)^2| < |\lambda_2 - (2\pi(n-s))^2| \] (59)

Therefore from (50) we obtain that
\[ |b_{2n-1}(\lambda_1, 0)| < |b_{2n-1}(\lambda, t)| < |b_{2n-1}(\lambda_2, 0)|. \] (60)

On the other hand, differentiating (50) with respect to \( \lambda \) we see that
\[ \frac{\partial}{\partial \lambda} (b_{2n-1}((2\pi n)^2, 0)) = b_{2n-1}((2\pi n)^2, 0) \sum_{s=1}^{2n-1} \frac{1 + O(n^{-1})}{s(2n-s)} \] (61)

Now taking into account that the last summation is of order \( \ln n \) and using (58) we get
\[ \frac{\partial}{\partial \lambda} b_{2n-1}((2\pi n)^2, 0)) \sim \beta_n \left( \frac{\ln n}{n} \right). \] (62)

Arguing as above one can easily see that the \( m \)-th derivative, where \( m = 2, 3, ..., \) of \( b_{2n-1}(\lambda, 0) \) is \( O(\beta_n) \). Therefore writing the Taylor series of \( b_{2n-1}(\lambda, 0) \) for \( \lambda = (2\pi n)^2 + O(n^{-2}) \) about \( (2\pi n)^2 \) we obtain
\[ b_{2n-1}(\lambda_1, 0) = \beta_n(1 + O(n^{-2})), \forall i = 1, 2, \]

This with (60) implies that
\[ b_{2n-1}(\lambda, t) = \beta_n(1 + O(n^{-2})) \] (63)

for any \((\lambda, t)\) satisfying (57). In the same way we get
\[ \frac{\partial}{\partial \lambda} b'_{2n-1}((2\pi n)^2, 0) \sim \alpha_n \left( \frac{\ln n}{n} \right), \quad b'_{2n-1}(\lambda, t) = \alpha_n(1 + O(n^{-2})). \] (64)

Now let us consider \( b_{2n+1}(\lambda, t) \). By (46) the indices \( n_1, n_2, ..., n_{2n+1} \) taking part in \( b_{2n+1}(\lambda, t) \) are 1 except one, say \( n_{s+1} = -1 \), where \( s = 2, 3, ..., 2n-1 \). Moreover, if \( n_{s+1} = -1, \) then \( n_1 + n_2 + ... + n_{s+1} = n_1 + n_2 + ... + n_{s+1} = s-1 \) and \( n_1 + n_2 + ... + n_{s+2} = n_1 + n_2 + ... + n_s = s \). Therefore by (10) \( b_{2n+1}(\lambda, t) \) for
\[ \lambda = (2\pi n)^2 + O(n^{-1}), \quad t \in [0, n^{-3}] \] (65)

has the form
\[ b_{2n-1}(\lambda, t) \sum_{s=2}^{2n-1} \frac{ab}{(2\pi n)^2 - (2\pi(n-s+1))^2 + O(n^{-1})((2\pi n)^2 - (2\pi(n-s))^2 + O(n^{-1}))}. \]
One can easily see that the last sum is \( O(n^{-2}) \). Thus we have
\[
b_{2n+1}(\lambda, t) = b_{2n-1}(\lambda, t)O(n^{-2}) = \beta_n O(n^{-2})
\] (66)
for any \((\lambda, t)\) satisfying (65).

Now let us estimate \( b_k(\lambda, t) \) for \( k > 2n + 1 \). Since the sums in (10) are taken under conditions (46) we conclude that
\[
1 \leq n_1 + n_2 + \cdots + n_s \leq 2n - 1.
\]
Using this instead of \( 1 \leq s \leq 2n - 1 \) and repeating the proof of (60) we obtain that for any \((\lambda, t)\) satisfying (65) there exists \( \lambda_3 = (2\pi n)^2 + O(n^{-1}) \) and \( \lambda_4 = (2\pi n)^2 + O(n^{-1}) \) such that
\[
|p_{n_1, n_2, \ldots, n_s}(\lambda, 0)| < |p_{n_1, n_2, \ldots, n_s}(\lambda, t)| < |p_{n_1, n_2, \ldots, n_s}(\lambda_4, 0)|, \quad \forall k < 2n - 1,
\]
where \( p_{n_1, n_2, \ldots, n_s}(\lambda, 0) \) is defined in (22). This with (23) and (66) implies that
\[
\sum_{k=2n+1}^{\infty} |b_k(\lambda, t)| = \beta_n O(n^{-2})
\] (67)
for any \((\lambda, t)\) satisfying (65). In the same way we obtain
\[
\sum_{k=2n+1}^{\infty} |b_k'(\lambda, t)| = \alpha_n O(n^{-2})
\] (68)
for the same \((\lambda, t)\). Thus (55) follows from (63), (64), (67) and (68).

Now we prove (51). It follows from (67), (68) and Cauchy inequality that
\[
\frac{\partial}{\partial \lambda} \left( \sum_{k=2n+1}^{\infty} b_k(\lambda, t) \right) = \beta_n O(n^{-1}), \quad \frac{\partial}{\partial \lambda} \left( \sum_{k=2n+1}^{\infty} b_k'(\lambda, t) \right) = \alpha_n O(n^{-1}).
\] (69)

Therefore (56) follows from (62) and (64). \( \blacksquare \)

**Theorem 4** There exists a positive integer \( N \) such that the eigenvalue \( \lambda_{n,j}(t) \) of \( H_t \) for \( n > N, j = 1, 2, t \in (-\pi, \pi] \), where \( N \) is defined in Remark 1, is simple. The eigenvalue \( \lambda_{n,j}(t) \) satisfies (37), (4) and the equation
\[
\lambda_{n,j}(t) = (2\pi n + t)^2 - 2\pi(2n+1)(t-\pi) + \frac{1}{2}(\vec{A} + A) + (-1)^j \sqrt{D(\lambda_{n,j}(t), t)}
\] (70)
for \( t \in [0, \rho], t \in [\rho, \pi - \rho] \) and \( t \in [\pi - \rho, \pi] \) respectively, where \( \vec{A}, A, D \) are defined in the end of the proof; the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are denoted by \( \lambda_n(t) \) and \( \lambda_n(t) \) in formula (4).

**Proof.** Let us consider the case \( t \in [0, \rho] \). Suppose to the contrary that there exists \( n > N \) and \( t \in [0, \rho] \) such that the eigenvalue of \( H_t \) lying in \( U(n, t, \rho) \) is multiple: \( \lambda_{n,1}(t) = \lambda_{n,2}(t) =: \lambda_n(t) \). Then by Theorem 3(b)
\[
D(\lambda_n(t), t) = 0.
\] (71)
This with (42) and (19) implies that
\[(4\pi nt)^2 ((1 + O(n^{-2})) = -B(\lambda_n(t), t)B'(\lambda_n(t), t)\] (72)

Hence, by (53) we have \( t \in [0, n^{-3}] \). Then by (54) \( t \) and \( \lambda = \lambda_n(t) \) satisfy (57). Now, using (55) in (72) we obtain that
\[ (4\pi nt)^2 = -\alpha_n \beta_n \left(1 + O\left(\frac{\ln n}{n}\right)\right) \] (73)

On the other hand by (71) and Theorem 3(a), we have
\[ \lambda_n(t) = (2\pi n + t)^2 + \frac{1}{2} (A(\lambda_n(t) + A'(\lambda_n, t))) - 4\pi nt. \] (74)

Since the double eigenvalue \( \lambda_n(t) \) satisfies (43) putting the right-hand side of (74) into (43) and doing some simplification we obtain
\[ (4\pi nt - \frac{1}{2} C(\lambda_n(t), t)) \frac{d}{d\lambda} C(\lambda_n(t), t) = \frac{d}{d\lambda} (B(\lambda_n(t), t)B'(\lambda_n(t), t)). \] (75)

Now using (19) and (56) we get the estimation
\[ (4\pi nt - \frac{1}{2} C(\lambda_n(t), t)) \frac{d}{d\lambda} C(\lambda_n(t), t)) = O(n^{-1}t^2) = \alpha_n \beta_n O(n^{-3}). \] (76)

for the left-hand side of the equality (75) that contradicts the estimation (56) for the right-hand side of (75). Thus \( \lambda_n(t) \) is a simple eigenvalue for \( n > N, j = 1, 2 \) and \( t \in [0, \rho] \). Hence the proof of Theorem 4 for the case \( t \in [0, \rho] \) follows from Theorem 3(c).

The case \( t \in [\rho, \pi - \rho] \) is considered in [15]. To prove the theorem in the case \( t \in [\pi - \rho, \pi] \) instead of (7), (11) we use
\[ (\lambda_n(t) - (2\pi n + t)^2 - \tilde{A}(\lambda_n(t), t))u_n(t) = (q_{2n+1} + \tilde{B}(\lambda_n(t), t))v_n(t), \] (77)
\[ (\lambda_n(t) - (-2\pi (n + 1) + t)^2 - \tilde{A}(\lambda_n(t), t))v_n(t) = (q_{2n-1} + \tilde{B}'(\lambda_n(t), t))u_n(t), \] (78)
and repeat the proof of the case \( t \in [0, \rho] \), where \( \tilde{A}, \tilde{B}, \tilde{A}', \tilde{B}', D \) are defined as \( A, B, A', B', D \). Note that instead of (5) for \( k \neq 0, 2n \) using the same inequality for \( k \neq 0, 2n + 1 \) and \( t \in [\rho, \pi - \rho] \) from (6) we obtain (77) and (78) instead of (7) and (11) (see [15]). The functions \( \tilde{A}, \tilde{B}, \tilde{A}', \tilde{B}' \) are obtained from \( A, B, A', B' \) by replacing \( a_i, a'_i, b_i, b'_i \) with \( \tilde{a}_k, \tilde{a}'_k, \tilde{b}_k, \tilde{b}'_k \). Here \( \tilde{a}_k, \tilde{a}'_k, \tilde{b}_k, \tilde{b}'_k \) differ from \( a_i, a'_i, b_i, b'_i \) respectively, in the following sense. The sums in \( \tilde{a}_k, \tilde{a}'_k, \tilde{b}_k, \tilde{b}'_k \) are taken from \( a_i, a'_i, b_i, b'_i \) under the conditions \( n_1 + n_2 + ... + n_s \neq 0, \pm (2n + 1) \) instead of the condition \( n_1 + n_2 + ... + n_s \neq 0, \pm 2n \) for \( s = 1, 2, ..., k \). Besides in \( \tilde{b}_k, \tilde{b}'_k \) the multiplicand \( q_{2n-1} \) is replaced by \( q_{2n+1} \).

**Remark 2** From (38), (55) and Theorem 1 one can readily see that the root functions of \( H_0 \) form a Riesz basis if and only if \( |a| = |b| \). The same result can be proved for \( H_0 \) in the same way. These results is obtained by Djakov and Mitjagin [3, 4] by other method.

## 4 Asymptotic Analysis of \( H \)

In this section we investigate the operator \( H \) generated in \( L_2(-\infty, \infty) \) by (1) when the potential \( q \) has the form (3). Since the spectrum \( S(H) \) of \( H \) is the union of the spectra
\begin{equation}
S(H_t) = \{ \lambda_n(t) : n \in \mathbb{Z} \}, \quad S(H) = \cup_{t \in (-\pi, \pi]} S(H_t) = \cup_{n \in \mathbb{Z}} \Gamma_n,
\end{equation}
where \( \Gamma_n =: \{ \lambda_n(t) : t \in [0, \pi] \} \). In this section we use both notation \( \lambda_n(t) \) and \( \lambda_{n,j}(t) \).

**Theorem 5** There exists \( N \) such that for \( |n| > N \) the component \( \Gamma_n \) of the spectrum of the operator \( H \) is separated simple analytic arc with end points \( \lambda_n(0) \) and \( \lambda_n(\pi) \). Moreover the following formulas for the distance between the end points of the neighboring arcs of the spectrum of \( H \) holds

\begin{align}
| \lambda_n(0) - \lambda_{-n}(0) | &= 2 | ab |^n ((2\pi)^{2n-1}(2n-1)!)^{-2}, \\
| \lambda_n(\pi) - \lambda_{-n-1}(\pi) | &= 2 | ab |^{n+\frac{1}{2}} ((2\pi)^{2n}(2n)!)^{-2}.
\end{align}

**Proof.** Theorem 4 immediately imply that the component \( \Gamma_n \) of the spectrum of the operator \( H \) is separated simple analytic arc with end points \( \lambda_n(0) \) and \( \lambda_n(\pi) \). Since \( A(\lambda_{n,j}, 0) = A(\lambda_{n,j}, 0) \) (see the proof of Lemma 3 of \cite{13}) formula (37) in the case \( t = 0 \) has the form

\begin{equation}
\lambda_{n,j}(0) = (2\pi n)^2 + A(\lambda_{n,j}, 0) + (-1)^j \sqrt{D(\lambda_{n,j}, 0)},
\end{equation}

where, by (42) \( D(\lambda_{n,j}, 0) = B(\lambda_{n,j}, 0)B'(\lambda_{n,j}, 0) \). Now using (17), (54) and Lemma 1 we obtain

\begin{equation}
(\lambda_{n,2} - \lambda_{n,1})(1 + O(n^{-2})) = 2\sqrt{\alpha_n \beta_n (1 + O(n^{-2}))}
\end{equation}

which imply (80). Instead of (37) using (70) in the same way we get (81).

**Remark 3** Note that formulas (80) and (81) result in the Avron-Simon-Harrell estimation for the widths of the instability intervals of the self-adjoint Mathieu-Hill operator. However, in general, the quantity \( | \lambda_n(0) - \lambda_{-n}(0) | \) being the distance between the end points of the curves \( \Gamma_n \) and \( \Gamma_{-n} \) is not the same as that between the curves \( \Gamma_n \) and \( \Gamma_{-n} \) themselves.

By Theorem 4 any subset \( \gamma =: \{ \lambda_n(t) : t \in [\alpha, \beta] \} \), where \( [\alpha, \beta] \subset [0, \pi] \), of \( \Gamma_n \) for \( |n| > N \) is a regular spectral arc of \( H \) in sense of \( \cite{8} \) (see Definition 2.4 of \cite{8}). For the projections \( P(\gamma) \) corresponding to the arc \( \gamma \) we use the formula

\begin{equation}
\| P(\gamma) \| = \sup_{t \in \delta} | d_n(t) |^{-1}.
\end{equation}

of \([14]\) (see Theorem 2 of \([14]\)), where \( \delta =: \{ t \in (-\pi, \pi] : \lambda_n(t) \in \gamma \} \), \( d_n(t) = (\Psi_{n,t}, \Psi_{n,t}^*) \), \( \Psi_{n,t}(x) \) and \( \Psi_{n,t}^*(x) \) are normalized eigenfunction of \( H_t \) and \( H_t^* \) corresponding to \( \lambda_n(t) \) and \( \lambda_n(t) \). The function \( d_n(t) \) is continuous on \( \delta \) and \( d_n(t) \neq 0 \), since \( \lambda_n(t) \) for \( |n| > N \) is a simple eigenvalue and the system of root functions of \( H_t \) is complete. We also use the following definitions:

**Definition 1** The operator \( H \) is said to be asymptotically spectral operator if there exists \( N \) and \( M \) such that

\begin{equation}
\sup_{\gamma \subset \Gamma_n, |n| > N} \| P(\gamma) \| \leq M,
\end{equation}

where supremum is taken over all arcs \( \gamma \subset \Gamma_n \) for \( |n| > N \).
Definition 2 We say that the operator $H$ has a spectral singularities at infinity if there exists a sequence $\{\gamma_n\}$ of arcs $\gamma_n \subset \Gamma_n$ such that

$$\lim_{n \to \infty} \| P(\gamma_n) \| = \infty. \quad (86)$$

Recall that the spectral singularities of the operator $H$ are the points of $S(H)$ in neighborhoods of which the projections of the operator $H$ are not uniformly bounded. It is well-known that the spectral singularities of $H$ is contained in the set of multiple eigenvalues of $H_1$ (see [8,14]). Therefore the following theorem follows from Theorem 4, definitions 1, 2, and (84):

Theorem 6 (a) For $|n| > N$ the component $\Gamma_n$ of the spectrum does not contain spectral singularities. In other words the number of spectral singularities of $H$ is finite.

(b) The operator $H$ is an asymptotically spectral operator if and only if $H$ has no spectral singularities at infinity.

(c) The operator $H$ has a spectral singularities at infinity if and only if there exists a sequence of pairs $\{(n_k,t_k)\}$ such that

$$\lim_{k \to \infty} d_{n_k}(t_k) = 0 \quad (87)$$

Now using Theorem 6 and the following equalities we prove the main result of the paper. Using (37) for $j = 1, 2$ in (39) and (40) we obtain

$$(-C(\lambda_{n,1}(t), t) - 4\pi nt - \sqrt{D(\lambda_{n,1}(t), t)})u_{n,1}(t) = B(\lambda_{n,1}(t), t)v_{n,1}(t), \quad (88)$$

$$(-C(\lambda_{n,2}(t), t) - 4\pi nt + \sqrt{D(\lambda_{n,2}(t), t)})u_{n,2}(t) = B(\lambda_{n,2}(t), t)v_{n,2}(t), \quad (89)$$

$$(C(\lambda_{n,1}(t), t) + 4\pi nt - \sqrt{D(\lambda_{n,1}(t), t)})v_{n,1}(t) = B'(\lambda_{n,1}(t), t)u_{n,1}(t), \quad (90)$$

$$(C(\lambda_{n,2}(t), t) + 4\pi nt + \sqrt{D(\lambda_{n,2}(t), t)})v_{n,2}(t) = B'(\lambda_{n,2}(t), t)u_{n,2}(t). \quad (91)$$

From (19), (42), (54) and (55) we get the estimations

$$C(\lambda_{n,j}(t), t) + 4\pi nt = 4\pi nt(1 + O(n^{-2})), \forall t \in [0, \rho], \quad (92)$$

$$D(\lambda_{n,j}(t), t) = (4\pi nt(1 + O(n^{-2}))^2 + B(\lambda_{n,j}(t), t)B'(\lambda_{n,j}(t), t), \forall t \in [0, \rho], \quad (93)$$

$$B(\lambda_{n,j}(t), t) = \beta_n \left(1 + O(\frac{\ln n}{n})\right), \quad B'(\lambda_{n,j}(t), t) = \alpha_n (1 + O(n^{-2})), \forall t \in [0, n^{-3}], \quad (94)$$

$$D(\lambda_{n,j}(t), t) = (4\pi nt(1 + O(n^{-2}))^2 + \beta_n \alpha_n (1 + O(n^{-2})), \forall t \in [0, n^{-3}] \quad (95)$$

Besides we use the following. Since the boundary condition (2) is self-adjoint we have $(H_t(q))^* = H_t(\overline{q})$. Therefore all formulas and theorems obtained for $H_t$ if we replace $a$ and $b$ by $\overline{a}$ and $\overline{b}$ respectively. For instance, formula (20) and Theorem 1 holds for the operator $H_t^*$ too, and hence we have

$$\Psi_{n,j,t}(x) = u_{n,j}(t)e^{i(2\pi n^2 + t)x} + v_{n,j}(t)e^{-i(2\pi n^2 + t)x} + h_{n,j,t}(x), \quad \Psi_{n,j,t}(x) = u_{n,j}(t)\overline{v_{n,j}(t)} + v_{n,j}(t)\overline{u_{n,j}(t)} + O(n^{-1}), \forall n > 0 \quad (96)$$

$$(\Psi_{n,1,t}(x), \Psi_{n,1,t}(x)) = u_{n,1}(t)\overline{v_{n,1}(t)} + v_{n,1}(t)\overline{u_{n,1}(t)} + O(n^{-1}) = 0. \quad (97)$$
Lemma 1. Then using (92), (95), (94) and taking into account that $|a| = 1$ and $|b|$ holds uniformly for $t \in [0, \rho]$. Replacing $a^2$ where $a^2$ and $b^2$ respectively, in the same way we obtain that $\Psi^{*}$ from Theorem 7: If $|a| = 1$, then operator $H$ has spectral singularity at infinity if and only if

$$\inf_{q, p \in \mathbb{N}} \{ |q\alpha - (2p - 1)| \} = 0.$$  \hfill (98)

In other words, $H$ is asymptotically spectral operator if and only if (98) fails to hold.

(b) If $\alpha$ is rational number, that is, $\alpha = \frac{m}{q}$, where $m$ and $q$ are irreducible integers, then $H$ has spectral singularity at infinity if and only if $m$ is an odd integer.

If $\alpha$ is irrational number, then $H$ has spectral singularity at infinity if and only if there exists a sequence of pairs $\{(q_k, p_k)\} \subset \mathbb{N}^2$ such that

$$|\alpha - \frac{2p_k - 1}{q_k}| = o\left(\frac{1}{q_k}\right),$$  \hfill (99)

where $2p_k - 1$ and $q_k$ are irreducible integers.

Proof. Suppose without loss of generality that $|a| < |b|$ and put $c = \frac{|a|}{|b|}$. If $t = 0$, then by (29), (38), (54), (55) and (20), (21) we have

$$u_{n,1}(0) = v_{n,1}(0)O(c^n) = O(n^{-1}), \quad \Psi_{n,1,0}(x) = e^{-i2\pi nx} + O(n^{-1}).$$

Replacing $a$ and $b$ by $\bar{a}$ and $\bar{b}$ respectively, in the same way we obtain

$$\Psi_{n,1,0}^{*}(x) = e^{i2\pi nx} + O(n^{-1}).$$

Hence $d_n(0) \to 0$ as $n \to \infty$. Thus the proof of the theorem in the case $|a| \neq |b|$ follows from Theorem 6(c).

(a) We prove this for $t \in [0, \rho]$. The other cases are similar. If $t > n^{-3}$, then by (92), (93) and (53) the coefficient of $u_{n,1}(t)$ in (88) is essentially greater than the coefficient of $v_{n,1}(t)$. Therefore from (20) and (21) we get

$$\Psi_{n,1,t}(x) = e^{-i(2\pi n + t)x} + O(n^{-1}).$$

In the same way we obtain that $\Psi_{n,1,t}^{*}$ satisfies the same formula and hence the equality

$$(\Psi_{n,j,t}, \Psi_{n,j,t}^{*}) = 1 + O(n^{-1})$$  \hfill (100)

holds uniformly for $t \in [n^{-3}, \rho]$ and $j = 1$. By (4), (100) holds uniformly also for $t \in [\rho, \pi - \rho]$ and $j = 1, 2$.

Now suppose that $t \in [0, n^{-3}]$. Consider the case $nt \geq |\alpha_n|$, where $\alpha_n$ is defined in Lemma 1. Then using (92), (95), (94) and taking into account that $|\alpha_n| = |\beta_n|$ when $|a| \neq |b|$ from (88) we get $|u_{n,1}(t)| < 6 |v_{n,1}(t)|$ and by (21)

$$|u_{n,1}(t)| < \frac{1}{5}, \quad |v_{n,1}(t)| > \frac{4}{5}$$

Similarly $|u_{n,1}^{*}(t)| < \frac{1}{3}$ and $|v_{n,1}^{*}(t)| > \frac{1}{3}$. Therefore by (96) we have

$$|(\Psi_{n,j,t}, \Psi_{n,j,t}^{*})| > \frac{1}{2}$$  \hfill (101)

for $j = 1$. In the same way we prove (100) and (101) for $j = 2$. 

It remains to consider the case $nt < |\alpha_n|$. Using (88), (90) and (91) one can easily see that if $nt \ll |\alpha_n|$ then both of the number

$$-C(\lambda_{n,1},t) - 4\pi nt - \sqrt{D(\lambda_{n,1},t)}, \ C(\lambda_{n,1},t) + 4\pi nt - \sqrt{D(\lambda_{n,1},t)}$$

(102)

if $nt \sim \beta_n$ then at least one of the numbers in (88) is of order $\alpha_n$. Hence (88) and (90) imply that $u_{n,1}(t) \sim v_{n,1}(t)$. In the same way we obtain $u_{n,2}(t) \sim v_{n,2}(t)$. Thus, by (21) we have

$$u_{n,1}(t) \sim v_{n,1}(t) \sim u_{n,2}(t) \sim v_{n,2}(t) \sim 1.$$  

(103)

This with (96) and (97) implies that (87) holds if and only if

$$\lim_{k \to \infty} \frac{v_{n_k,1}(t_k)}{u_{n_k,1}(t_k)} = \lim_{k \to \infty} \frac{v_{n_k,2}(t_k)}{u_{n_k,2}(t_k)}.$$  

(104)

By (88), (89), (92) and (94) the last equality is equivalent to

$$\lim_{k \to \infty} \frac{4\pi n_k t_k + \sqrt{D(\lambda_{n_k,1},t_k)}}{\beta_{n_k}} = \lim_{k \to \infty} \frac{4\pi n_k t_k - \sqrt{D(\lambda_{n_k,1},t_k)}}{\beta_{n_k}}.$$  

(105)

that is possible if and only if

$$\lim_{k \to \infty} \frac{(4\pi n_k t_k)^2 + \beta_{n_k} \alpha_{n_k}}{\beta_{n_k}^2} = 0.$$  

(106)

Thus (87) is equivalent to (106) which holds if and only if both equalities

$$\lim_{k \to \infty} \frac{(4\pi n_k t_k)^2 + \Re(\beta_{n_k} \alpha_{n_k})}{\beta_{n_k}^2} = 0, \ \lim_{k \to \infty} \frac{\Im(\beta_{n_k} \alpha_{n_k})}{\beta_{n_k}^2} = 0.$$  

(107)

hold. By definition of $\beta_n$ and $\alpha_n$ the second equality in (107) holds if and only if

$$\Im((ab)^{2n_k}) = o((ab)^{2n_k}).$$  

(108)

Then $\Re(\beta_{n_k} \alpha_{n_k}) \sim \beta_{n_k}^2$ and there exists $\{t_k\}$ satisfying the first equality in (107) if and only if

$$\lim_{k \to \infty} \sgn(\Re((ab)^{2n_k})) = -1.$$  

(109)

where $\sgn(x)$ is the signum function. It is clear that (98) for even $q$ holds if and only if (108) and (109) hold. Thus (87) under the condition $t_k \in [0,\rho]$ is equivalent to (98) under conditions $q \in 2\mathbb{N}$. In the same way instead of (7), (11), (37) using (77)-(79) we obtain that (87) under the condition $t_k \in [\pi - \rho, \pi]$ is equivalent to (98) under conditions $q \in 2\mathbb{N} + 1$. Thus (87) and (98) are equivalent. Therefore the proof of (a) follows from Theorem 6(c). It is clear that (b) follows from (a). \[\square\]

**Remark 4** Note that if $H$ has no spectral singularity in $\bigcup_{k=1}^{N-1} \Gamma_k$ and is an asymptotically spectral operator, then it is a spectral operator. As it is noted in [4] (see page 539 of [4]) if $|a| \neq |b|$, then it follows from [4] and [8] that $H$ is not a spectral operator. By Theorem 7 the inverse statement is not true. In $|a| = |b|$ then in Theorem 7 we find the necessary and sufficient conditions on $ab$ for the $H$ to be an asymptotically spectral operator, since we use the asymptotic methods. Thus, to find a necessary and sufficient condition on $a$ and $b$ for the $H$ to be a spectral operator we need to consider the dependence of small eigenvalues on $a$ and $b$, using other, say numerical, methods.
References

[1] Avron, J. and B. Simon: , The asymptotics of the gap in the Mathieu equation, Ann. Phys., (134), 76–84 (1981).

[2] N. Dernek, O. A. Veliev, On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operators, Israel Journal of Mathematics 145, 113-123 (2005).

[3] P. Djakov, B. S. Mitjagin, Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials, Doklady Mathematics, Vol.83, No.1, 5-7 (2011).

[4] P. Djakov, B. S. Mitjagin, Convergence of spectral decompositions of Hill operators with trigonometric polynomial potentials, Mathematische Annalen 351, 509-540 (2011).

[5] N. Dunford, J. T. Schwartz, Linear Operators, Part 3, Spectral Operators, Wiley-Interscience, MR 90g:47001c, New York, 1988.

[6] M. S. P. Eastham, The Spectral Theory of Periodic Differential Equations. Edinburgh, Scottish Academic Press 1973.

[7] F. Gesztesy and V. Tkachenko, When is a non-self-adjoint Hill operator a spectral operator of scalar type, C. R. Acad. Sci. Paris, Ser. I, 343, 239-242 (2006).

[8] F. Gesztesy and V. Tkachenko, A criterion for Hill operators to be spectral operators of scalar type, J. Analyse Math. 107, 287–353 (2009).

[9] Harrell, E., On the effect of the boundary conditions on the eigenvalues of ordinary differential equations, Amer. J. Math. (supple), 139–150 (1981).

[10] D. McGarvey, Operators commuting with translations by one. Part I. Representation theorems, J.Math. Anal. Appl. 4, 366–410 (1962).

[11] D. McGarvey, Operators commuting with translations by one. Part II. Differential operators with periodic coefficients in $L_p(-\infty, \infty)$, J. Math. Anal. Appl. 11, 564–596 (1965).

[12] D. McGarvey, Operators commuting with translations by one. Part III. Perturbation results for periodic differential operators, J. Math. Anal. Appl. 12, 187–234 (1965).

[13] A. A. Shkalikov, O. A. Veliev, On the Riesz basis property of the eigen- and associated functions of periodic and antiperiodic Sturm-Liouville problems, Math. Notes, 85(5), 647-660 (2009).

[14] O. A. Veliev, The spectrum and spectral singularities of the differential operators with periodic complex-valued coefficients. Differential Equations, No.8, 1316-1324 (1983).

[15] O. A .Veliev, M. Toppanuk Duman, The spectral expansion for a nonself-adjoint Hill operators with a locally integrable potential, Journal of Math. Analysis and Appl. 265, 76-90 (2002).

[16] O. A. Veliev, Asymptotic Analysis of Non-self-adjoint Hill Operators, arXiv:1107.2552 (2011).