Computational explorations of the Thompson group $T$ for the amenability problem of $F$

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Abstract. It is a long standing open problem whether the Thompson group $F$ is an amenable group. In this paper we show that if $A$, $B$, $C$ denote the standard generators of Thompson group $T$ and $D := CBA^{-1}$ then

$$\sqrt{2} + \sqrt{3} < \frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \leq 2 + \sqrt{2}.$$  

Moreover, the upper bound is attained if the Thompson group $F$ is amenable. Here, the norm of an element in the group ring $\mathbb{C}T$ is computed in $B(\ell^2(T))$ via the regular representation of $T$. Using the “cyclic reduced” numbers $\tau((((C + C^2)(D + D^2 + D^3))^n), n \in \mathbb{N}$, and some methods from our previous paper [10] we can obtain precise lower bounds as well as good estimates of the spectral distributions of $\frac{1}{\sqrt{12}}((I + C + C^2)(I + D + D^2 + D^3))^*F_1(I + D + D^2 + D^3)$, where $\tau$ is the tracial state on the group von Neumann algebra $L(T)$. Our extensive numerical computations suggest that

$$\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\| \approx 3.28,$$

and thus that $F$ might be non-amenable. However, we can in no way rule out that $\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\| = 2 + \sqrt{2}.$

1. Introduction

Definition 1.1. The Thompson group $T$ is the group of (cyclic) order preserving homeomorphisms $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ for which:

- $f$ and $f^{-1}$ are piecewise linear with finitely many breakpoints.
- all breakpoints of $f$ and $f^{-1}$ are in $\mathbb{Z}^{1/2}/\mathbb{Z}$.
- all slopes of $f$ are in the set $2\mathbb{Z} := \{2^n | n \in \mathbb{Z}\}$.

$T$ is a countable group. It is generated by the elements $C$, $D$, whose graphs are shown in Fig. 1. Moreover, it has the finite presentation

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Figure 1. The generators C, D of the Thompson group T.

(1) $T = \langle C, D \mid C^3 = D^4 = (CD)^5 = e, \quad [C, D] = [C^2 (CD) D^2, C (D^2 C D) D, C (D^2 D) C] = e \rangle$, where the group commutator is given by the usual definition $[g, h] = ghg^{-1}h^{-1}$. If $A, B, C$ denote the standard generators of $T$, then $D = CBA^{-1}$.

It is a long standing open problem whether the Thompson group $F$ is amenable. The present paper is a continuation of the work started in [10] by the same authors. In that paper, we tested the amenability problem of $F$ by estimating norms of certain elements in the group ring $\mathbb{C}F$ using computers. Thanks to the new algorithms we devised to compute words in $C, D$ in polynomial time (see Section 2) and to some results by Haagerup and Olesen [12], we can now test the amenability problem of $F$ by computing norms of certain elements in $\mathbb{C}T$. Extrapolations of our computational results suggest the same as our previous paper, namely, that $F$ might not be amenable. A recent experimental work by Elder, Rechnitzer and E.J. Janse van Reusburg [7] on the amenability problem of $F$ using statistical methods arrives also to the same conclusion that $F$ might be non-amenable (see also [8]).

As in [10, Section 2], by the norm of $||a||$ of an element $a$ in the group ring of a discrete group $\Gamma$ we mean

$$||a|| = ||\lambda(a)||_{B(\ell^2(\Gamma))},$$

where $\lambda$ is the left regular representation of $\Gamma$. We continue using the standard convention of writing $a \in B(\ell^2(\Gamma))$ instead of $\lambda(a) \in B(\ell^2(\Gamma))$ for any $a \in \mathbb{C}T$. Our starting point is the following theorem (see Section 2 for more details)

**Theorem 1.2.** Let $C, D$ be the generators of $T$, whose graphs are shown in Fig. 1, and let $I$ denote the unit element of $T$. Then

$$\sqrt{2} + \sqrt{3} < \frac{1}{\sqrt{12}}||I + C + C^2)(I + D + D^2 + D^3)|| \leq 2 + \sqrt{2}.$$  

Moreover, the upper bound is attained if the Thompson group $F$ is amenable.

In comparison, one gets for the free product $\mathbb{Z}_3 \ast \mathbb{Z}_4$ on two generators $c \in \mathbb{Z}_3, d \in \mathbb{Z}_4$ that

(2) $$\frac{1}{\sqrt{12}}||e + c + c^2)(e + d + d^2 + d^3)|| = \sqrt{2} + \sqrt{3}.$$
Given a discrete group $\Gamma$, let $L(\Gamma)$ denote its von Neumann algebra. That is, $L(\Gamma)$ is the von Neumann algebra in $B(\ell^2(\Gamma))$ generated by $\Gamma$. Then we can define the normal faithful tracial state $\tau$ on $L(\Gamma)$ by

$$\tau(h) := \langle h \delta_c, \delta_c \rangle, \quad h \in L(\Gamma).$$

(see e.g. [14] Section 6.7]). Moreover, we can express the norm of $h$ in terms of the moments

$$m_n(h^* h) := \tau((h^* h)^n), \quad n \in \mathbb{N}_0,$$

namely, (cf. [10] Section 4)

$$||h|| = ||h^* h||^{1/2} = \lim_{n \to \infty} \tau((h^* h)^n)^{1/2n}.$$

The challenge is to compute all the moments $m_n(h^* h)$, and in practice we can only compute a finite number of them. In this paper, we were able to compute the moments $m_n(h^* h)$, $n = 0, \ldots, 28$ for the element

$$h = \frac{1}{\sqrt{12}} (I + C + C^2)(I + D + D^2 + D^3),$$

using efficient methods, both mathematically and computationally. This procedure can be adapted to elements in a discrete group $\Gamma$ that are expressed similarly. Applying the spectral theorem to the self-adjoint operator

$$\tilde{h} := \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \in M_2(B(\ell^2(T))),$$

(cf. [10] Section 2) one gets that there is a unique probability measure $\mu$ to $\mathbb{R}$. Since $\tau$ is faithful, the support $\text{supp}(\mu) = \sigma(\tilde{h}) \subset [-||h||, ||h||]$. Such measure $\mu$ is invariant under the reflection $t \mapsto -t$ because the odd moments of $\tilde{h}$ are zero, and it satisfies

$$\int_{-||h||}^{||h||} t^{2n} d\mu(t) = m_{2n}(\tilde{h}) = m_n(h^* h), \quad n \in \mathbb{N}_0.$$

Moreover, $||h|| \in \text{supp}(\mu)$ by symmetry of $\mu$. The theory of orthonormal polynomials applied to this measure (cf. [10] Section 4) together with the moments yield an increasing sequence of numbers converging to $||h||$. We can calculate the first 28 of these numbers using our computed moments, and a suitable extrapolation of these numbers ($n = 0, \ldots, 28$) gives

$$3.2026 \leq \frac{1}{\sqrt{12}} ||(I + C + C^2)(I + D + D^2 + D^3)|| \approx 3.28.$$

We found that this sequence of numbers gives actually much better lower bounds than the sequence of “roots” and “ratios” of the moments that also converge to the norm (cf. Section 3). Furthermore, we also estimate the Lebesgue density of $\mu$ with fairly high precision, which shows that the measure $\mu$ is very close to zero on the interval $[3.22, 2 + \sqrt{2}]$. However, we cannot rule out that the measure has very “thin tail” stretching all the way up to $2 + \sqrt{2}$.

In comparison, one gets Eq. (2) when one considers the free product $\mathbb{Z}_3 \ast \mathbb{Z}_4$ on two generators $c \in \mathbb{Z}_3, d \in \mathbb{Z}_4$. The measure $\mu$ based on $c, d$ instead of $C, D$ will be denoted by $\mu_{\text{free}}$, and it can be computed explicitly (See Section 4)

$$\mu_{\text{free}} = \frac{1}{2\pi} \frac{\sqrt{24 - (x^2 - 5)^2}}{x(12 - x^2)} 1_{[-\sqrt{3} - \sqrt{2}, \sqrt{2} - \sqrt{3}) \cup \sqrt{3} - \sqrt{2}, \sqrt{2} + \sqrt{3}}(x) \, dx + \frac{3}{4} \delta_0.$$
In our previous paper, we estimated the norm $3.60613 \leq \|A + B + A^{-1} + B^{-1}\| \approx 3.87$, where $A, B$ are the standard generators of $F$, by using the first 24 moments of $(A + B + A^{-1} + B^{-1})^2$. Elvey-Price [9] succeeded in computing 7 more moments, which pass the test of [10, Theorem B.1] for possible computational errors. Using these 31 moments, the updated estimated norm remains unchanged, while the new lower bound is

$$3.64271 \leq \|A + B + A^{-1} + B^{-1}\| \approx 3.87,$$

with a very likely lower bound of $3.70211$, based on the same list of moments.

In comparison,

$$3.7873 \leq \|C + D + C^{-1} + D^{-1}\| \approx 3.84,$$

which suggests that the actual norm is also much closer to 4, even though it cannot be 4 because the Thompson group $T$ is not amenable [4]. This norm is estimated in the same way as we did with the norm $\|A + B + A^{-1} + B^{-1}\|$ (cf. Appendix B).

2. Action of $C, D$, the cyclic reduced numbers $\zeta_n$, and amenability.

Since the elements of the Thompson group $T$ are piecewise linear functions with breakpoints on the diadics, it makes sense to introduce the set of diadic intervals

$$\mathcal{D} := \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] \mid n \in \mathbb{N}_0, k = 0, \ldots, 2^n - 1 \right\}.$$

Note that the unit interval is the largest diadic interval, and cutting a diadic interval into two equal halves gives two new diadic intervals. Then we write these linear maps as functions $\mathcal{D} \rightarrow \mathcal{D}$. For instance, the element $C$ whose graph is in Figure 1 is the function

$$f_C : \{d_1 := [0, \frac{1}{2}], d_2 := [\frac{1}{2}, \frac{3}{4}], d_3 := [\frac{3}{4}, 1]\} \rightarrow \{r_1 := [0, \frac{1}{2}], r_2 := [\frac{1}{2}, \frac{3}{4}], r_3 := [\frac{3}{4}, 1]\}$$

given by

$$f_C (d_1) = r_3 \quad f_C (d_2) = r_1 \quad f_C (d_3) = r_2.$$

Observe that $\mathcal{D}$ is not closed under union, e.g. the union of the diadic intervals $[\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}] \in \mathcal{D}$ is the interval $[\frac{1}{4}, \frac{3}{4}] \not\in \mathcal{D}$ which is not diadic. This asymmetry is used to write the domain and range as trees. Namely, we represent each diadic interval with a dot, and we join two of these dots with a caret whenever the union is also a diadic interval. The tip of the caret is the joined diadic interval. We do this recursively until we get the unit interval. For instance, in our example, for the domain of $f_C$, we can join $d_2$ and $d_3$ with a caret to obtain the diadic interval $[1/2, 1]$ and then we join this and $d_1$ with a caret to obtain the diadic interval $[0, 1]$. The domain tree of $C$ is illustrated in Figure 2.

The domain tree (resp. range tree) of $x \in T$ will be denoted with $d(x)$ (resp. $r(x)$).

**Definition 2.1 (Doubletree).** The doubletree of $x \in T$ is the pair of trees $r(x), d(x)$ together with the permutation (a rotation) $\sigma : \text{leaves}(r(x)) \rightarrow \text{leaves}(d(x))$ of its leaves.
2.1. The action of $C$ and $D$ on any element of $T$. We investigate the doubletree composition $C \circ x, x \in T$ in three cases, where the doubletrees of $C$ and $x$ are
Case 1C: (non-degenerate)
Assume that the root of the range tree \( r(x) \) has a left and a right node. The tree starting at the left node is called tree I. Assume that the right node has a left node and a right node. The tree starting at the right-left node is called tree II, and the one starting at the right-right node is called tree III. The trees I, II, III, can be 0-trees, i.e. they can be leaves.

Then the root of the range tree \( r(C \circ x) \) has a left and a right node. The tree starting at the left node is tree II. The right node has a left node and a right node. The tree starting at the right-left node is tree III, and the one starting at the right-right node is tree I. The leaves of tree I, which map to leaves of the domain tree \( d(x) \), stay with the tree, and the same holds for the other two trees II, III. See Figure 4. The rotation permutation starts with \( \sigma(C \circ x)(1) = \sigma(x)(\sigma(C)(1)) \).

Reduction occurs only if tree III and tree I are 0-trees and the leaves \( \sigma_{III}, \sigma_{I} \) on the domain form a caret. Reduction occurs again only if tree II is also a 0-tree; in such case we have the identity. An example of case 1C, is shown in Figure 5.

Case 2C: (degenerate, missing tree II)
Assume that the root of the range tree \( r(x) \) has a left and a right node. Assume that the right node is a 0-tree (a leaf), which maps to another leaf of the domain tree \( d(x) \). Insert a double caret at these two leaves and proceed as in case 1C. An example for this case is shown in Figure 6.

Case 3C: (degenerate, missing trees, I, II, III)
The element \( x \) is the identity. Thus \( C \circ x = C \).

The composition \( D \circ x, x \in T \) is done in 5 cases, where the trees of \( D \) and \( x \) are assumed to be reduced.

Case 1D: (non-degenerate)
Assume that the root of the range tree \( r(x) \) has a left and a right node, which both have also a left node and a right node. The tree starting at the left-left node is called tree I. The one starting at the left-right node is called tree II. The one starting at
the right-left node is called tree III, and the one starting at the right-right node is called tree IV. The trees I, II, III, IV can be 0-trees, i.e. they can be leaves.

Then, the root of the range tree $r(D \circ x)$ has a left and a right node, which both have also left and right nodes. The tree starting at the left-left node is tree II. The tree starting at the left-right node is tree III. The tree starting at the right-left node is tree IV. The tree starting at the right-right node is tree I. The leaves of tree I, which map to leaves of the domain tree $d(x)$, stay with the tree, and the same holds for the other three trees II, III, IV. See Figure 7. The rotation permutation starts with $\sigma(D \circ x)(1) = \sigma(x)(\sigma(D)(1))$.

Reduction occurs only in the following two cases:

(a) II and III are 0-trees and the leaves $\sigma_{II}, \sigma_{III}$ on the domain form a caret.

(b) IV and I are 0-trees and the leaves $\sigma_{IV}, \sigma_{I}$ on the domain form a caret.

Reduction occurs again, only if both (a) and (b) occur; in such case we have the identity. An example of case 1D with reduction (a) is shown in Figure 8.

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**Figure 5.** Composition $C \circ x$, where $x \in T$ satisfies the hypothesis of case 1C (non-degenerate).

**Figure 6.** Composition $C \circ x$, where $x \in T$ satisfies the hypothesis of case 2C (degenerate).
Case 1D: Trees I, II, III, IV appear in the range tree of \( x \in T \). The action of \( D \) simply rotates these trees (and each tree keeps its own leaves).

Figure 7.

Case 2D: (degenerate, missing tree II)
Assume that the root of the range tree \( r(x) \) has a left and a right node. Assume that the left node is a leaf which we call \( v \), while the right node has a left and a right node. The leaf \( v \) maps to a leaf of the domain tree \( d(x) \). Insert a double caret at these two leaves. Then proceed as in case 1D. An example is shown in Figure 9.

Case 3D: (degenerate, missing tree III)
Assume that the root of the range tree \( r(x) \) has a left and a right node. Assume that the right node is a leaf which we call \( v \), while the left node has a left and a right node. The leaf \( v \) maps to a leaf of the domain tree \( d(x) \). Insert a double caret at these two leaves. Then proceed as in case 1D. An example is shown in Figure 10.

Case 4D: (degenerate, missing trees II, III)
Assume that the root of the range tree \( r(x) \) has a left and a right node, such that both trees are 0-trees. Since \( x \) is assumed to be reduced, \( x = D^2 \). Thus \( D \circ x = D^3 = D^{-1} \).

Case 5D: (degenerate, missing trees I, II, III, IV)
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The element $x$ is the identity. Thus $D \circ x = D$.

2.2. The cyclic reduced numbers $\zeta_n$. Let $R : T \to T$, be defined by

$$R(x)(t) = 1 - x(1 - t), \quad x \in T, \; t \in \mathbb{R}/\mathbb{Z}.$$  

Then the graph of $R(x)$ is the graph of $x$ rotated by 180 degrees. The rotation map $R$ is a group isomorphism, and

$$(4) \quad R \circ R = id, \quad C^{-1} = D^2 R(C) D^2, \quad D^{-1} = D^2 R(D) D^2.$$  

The doubltree of $R(x), x \in T$, is obtained by swapping all the nodes of every caret in the domain and range trees of $x$, i.e. left node (resp. right node) becomes right node (resp. left node). An example is shown in Figure 11.

Define

$$a := C + C^2 \quad b := D + D^2 + D^3.$$
and recall that $C^3 = I$, $D^4 = I$. Since $C, D \in L(T)$ are unitaries, $C^* = C^{-1}$ and $D^* = D^{-1}$. Moreover, $a$ and $b$ are self-adjoint, i.e. $a^* = a$ and $b^* = b$. Define the “cyclic reduced” numbers

$$\zeta_n := \tau((ab)^n), \quad n \in \mathbb{N}.$$ 

Define $N_0 := \{D, D^2, D^{-1}\}$. For $n \in \mathbb{N}$ let,

$$N_n := \{(t_0, s_1, t_1, \ldots, s_n, t_n) \mid s_j \in \{C, C^{-1}\}, j = 1, \ldots, n, \quad t_i \in \{D, D^2, D^{-1}\}, i = 0, \ldots, n\},$$

and note that $b(ab)^n = \sum_{(t_0, s_1, t_1, \ldots, s_n, t_n) \in N_n} t_0 s_1 t_1 \cdots s_n t_n$, $n \in \mathbb{N}_0$.

We define the product of a tuple as follows

$$(5) \quad \pi((t_1, t_2, \ldots, t_n)) := t_1 t_2 \cdots t_n, \quad t_1, \ldots, t_n \in \{C, D, C^{-1}, D^2, D^{-1}\},$$

and the inverse of each entry of a tuple as

$$(6) \quad J((t_1, t_2, \ldots, t_n)) := (t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}).$$

For $w = (t_1, \ldots, t_n)$ it holds by Eq. (5)

$$(7) \quad \pi(J(w)) = D^2 R(\pi(w)) D^2.$$ 

We say that the word $\pi(J(w))$ is the “reverse-inverse” of the word $\pi(w)$. For example, if $w = (C, D, C^2)$ then $\pi(J(w)) = C^{-1} D^{-1} C^{-2}$, which corresponds to first reversing the letters of the word $CDC^2$ and then taking the inverse. Define for $n \in \mathbb{N}$,

$$S_n := \{w, J(w') \mid w, w' \in \{C\} \times N_{n-1}, \pi(w) < \pi(J(w)), \pi(J(w')) < \pi(w')\}$$

$$S'_n := \{w, J(w') \mid w, w' \in N_{n-1} \times \{C^{-1}\}, \pi(w) < \pi(J(w)), \pi(J(w')) < \pi(w')\}$$

$$E_n := \{w \in \{C\} \times N_{n-1} \mid \pi(w) = \pi(J(w))\}$$

$$E'_n := \{w \in N_{n-1} \times \{C^{-1}\} \mid \pi(w) = \pi(J(w))\}.$$ 

Here the comparison $\pi(w) < \pi(J(w))$ is done by comparing the lexicographic order of their corresponding serialized (e.g. preorder) reduced doubletrees. The letter $S_n$ stands for smaller, e.g. that the word $\pi(w)$ is smaller than its reverse-inverse.
\( \pi(J(w)) \). Similarly, the letter \( E_n \) stands for equal. Since \( b(ab)^n \) is self-adjoint, \( w \in \{C\} \times N_n \) if and only if \( w^{-1} \in N_n \times \{C^{-1}\} \), where \( w^{-1} \) is the inverse of each entry in the tuple \( w \) and in reverse order. Define for \( n \in \mathbb{N} \),

\[
\zeta^n := \sum_{(x,y) \in S_n \times S_n} \langle x, y \rangle, \quad \zeta^n := \sum_{(x,y) \in E_n \times E_n} \langle x, y \rangle
\]

where \( \langle x, y \rangle := \tau(y^*x) \) is the inner product on \( L(T) \) associated with the trace \( \tau \).

Then

**Lemma 2.2.**

\[ \zeta_n = 2\zeta^n + 4\zeta^n, \quad n \geq 2. \]

**Proof.** For \( n \in \mathbb{N} \), define

\[ X_n := \{(s_1, t_1, \ldots, s_n, t_n) \mid s_i \in \{C, C^{-1}\}, t_i \in \{D, D^2, D^{-1}\}, i = 1, \ldots, n\}. \]

Then

\[ (ab)^n = \sum_{x \in X_n} \pi(x). \]

Since \( (ab)^n = Cb(ab)^n + C^{-1}b(ab)^n \), we can partition \( X_n \) as follows

\[ X_n = \{w, J(w) \mid w \in \{C\} \times N_{n-1}\} = S_n \cup L_n \cup E_n \cup J(E_n), \]

where

\[ L_n := \{w, J(w') \mid w, w' \in \{C\} \times N_{n-1}, \pi(w) > \pi(J(w)), \pi(J(w')) > \pi(w')\}. \]

Thus

\[ (ab)^n = \sum_{x \in S_n} \pi(x) + \sum_{x \in L_n} \pi(x) + 2 \sum_{x \in E_n} \pi(x). \]

Similarly,

\[ (ba)^n = \sum_{x \in S_n} \pi(x) + \sum_{x \in L_n} \pi(x) + 2 \sum_{x \in E_n} \pi(x), \]

where

\[ L'_n := \{w, J(w') \mid w, w' \in N_{n-1} \times \{C^{-1}\}, \pi(w) > \pi(J(w)), \pi(J(w')) > \pi(w')\}. \]

The inner product \( \langle \sum_{S_n} \pi(x), \sum_{L_n} \pi(x) \rangle = 0 \) because if there is a \( w \in S_n, w' \in E_n \), such that \( \pi(w) = \pi(w') \), then by Eq. \( [7] \), \( \pi(J(w)) = \pi(J(w')) \), and thus \( \pi(w) < \pi(J(w)) = \pi(J(w')) = \pi(w') = \pi(w) \), a contradiction.

Similarly, the inner product \( \langle \sum_{S_n} \pi(x), \sum_{L'_n} \pi(x) \rangle = 0 \) because if there is a \( w \in S_n, w' \in L'_n \), such that \( \pi(w) = \pi(w') \), then by Eq. \( [7] \), \( \pi(w) < \pi(J(w)) = \pi(J(w')) = \pi(w') = \pi(w) \), a contradiction. The innerproducts

\[ \langle \sum_{x \in S_n} \pi(x), \sum_{x \in S_n} \pi(x) \rangle = \langle \sum_{x \in L_n} \pi(x), \sum_{x \in L_n} \pi(x) \rangle \]

coincide because \( L_n = J(S_n), L'_n = J(S'_n) \) and because by Eq. \( [7] \)

\[ \pi(x) = \pi(x') \iff \pi(J(x)) = \pi(J(x')). \]
It follows that for \( n \in \mathbb{N} \),
\[
\zeta_{2n} = \langle (ab)^n, (ba)^n \rangle \\
= \langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_n} \pi(x) \rangle + \langle \sum_{x \in L_n} \pi(x), \sum_{x \in L'_n} \pi(x) \rangle + 4\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_n} \pi(x) \rangle \\
= 2\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_n} \pi(x) \rangle + 4\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_n} \pi(x) \rangle \\
= 2\zeta_{2n} + 4\zeta_{2n},
\]
and
\[
\zeta_{2n+1} = \langle (ab)^n, (ba)^{n+1} \rangle \\
= 2\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_{n+1}} \pi(x) \rangle + 4\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_{n+1}} \pi(x) \rangle \\
= 2\zeta_{2n+1} + 4\zeta_{2n+1}. \tag*{\Box}
\]

2.3. Amenability.

**Definition 2.3 (Amenability).** A discrete group \( \Gamma \) is said to be amenable if there exists a finitely additive measure \( \mu : P(\Gamma) \to [0, 1] \) with \( \mu(\Gamma) = 1 \) such that
\[
\mu(xA) = \mu(A), \quad \text{for all } A \in P(\Gamma), \ x \in \Gamma,
\]
where \( P(\Gamma) \) denotes the power set of \( \Gamma \).

Let \( P, Q, R \in L(T) \) be the projections given by
\[
P := \frac{I + S}{2}, \quad Q := \frac{I + C + C^2}{3}, \quad R := \frac{I + D + D^2 + D^3}{4},
\]
where \( S := D^2 \).

**Theorem 2.4.** Let \( P, Q, R \in L(T) \) be the projections in Eq. (10), which are defined in terms of the generators \( C, D \) of Thompson group \( T \). Then
\[
||QR|| \leq 1 + \sqrt{\frac{2}{6}}.
\]
Moreover, the upper bound is attained if the Thompson group \( F \) is amenable.

**Proof.** It is well-known that
\[
T \supset \langle S, C \rangle = \mathbb{Z}_2 \ast \mathbb{Z}_3 \cong PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm I\},
\]
(cf. [13] p.11, Example 1.5.3). By [3] Proposition 2.5.9 the reduced \( C^* \)-algebra of any subgroup \( H \) of a (discrete) group \( G \) is a subalgebra of \( C^*_r(G) \) in a canonical way, namely, one takes the left regular representation of \( G \) and restricts it to \( H \). Hence
\[
C^*_r(\mathbb{Z}_2 \ast \mathbb{Z}_3) = C^*_r(\langle S, C \rangle) \subset C^*_r(T).
\]
By this and Example 4.2 (or [1] Remark 15)) we get
\[
||PQ|| = \frac{1 + \sqrt{2}}{\sqrt{6}}.
\]
By spectral reasons
(11) \( R \leq P \).
Hence \( 0 \leq Q R Q \leq P Q P \), and by the \( C^* \)-identity (i.e. \( \| x^* x \| = \| x \|^2 \) we get
\[
\| Q R \| = \| Q R Q \| \leq \| P Q P \|.
\]
Since the norm of an element is the same as the norm of its adjoint (i.e. \( \| x^* \| = \| x \|) \),
\[
\| Q P Q \|^{1/2} = \| P Q P \|^{1/2} = \| P Q \| = \frac{1 + \sqrt{2}}{\sqrt{6}}.
\]
Thus
(12) \( \| Q R \| \leq \frac{1 + \sqrt{2}}{\sqrt{6}} = 0.97 \ldots \).

We will now prove that \( \| Q R \| \geq \frac{1 + \sqrt{2}}{\sqrt{6}} \) if \( F \) is amenable. Let \( H := \ell^2(\mathbb{Z}[1/2] \cap [0, 1)) \),
and let \( \pi : T \to B(H) \) be the representation given by
\[
\pi(x)\delta_t = \delta_{\pi(x)t}, \quad x \in T, \ t \in \mathbb{Z}[1/2] \cap [0, 1),
\]
where the set \( \{ \delta_t \ | \ t \in \mathbb{Z}[1/2] \cap [0, 1) \} \) is an orthonormal basis for the Hilbert space \( H \). Let \( S_1, S_2 \in B(H) \) be the isometries defined by
\[
S_1\delta_t := \delta_{t/2}, \quad S_2\delta_t := \delta_{(t+1)/2}, \quad t \in \mathbb{Z}[1/2] \cap [0, 1),
\]
and note that they satisfy the relation \( S_1 S_1^* + S_2 S_2^* = I \). Then by [12 Proposition 4.3], \( \pi(CT) \) is contained in the Cuntz algebra \( \mathcal{O}_2 = C^*(S_1, S_2) \). If \( F \) is amenable then by [12 Proposition 4.4], \( \pi \) is weakly contained in the left regular representation of \( T \). This is equivalent to the existence of a \( * \)-homomorphism \( \rho \) from the reduced \( C^* \)-algebra \( C^*_r(T) \) into the \( C^* \)-algebra generated by \( \pi(T) \) such that \( \pi = \rho \circ \lambda \). Hence
\[
\| \pi(x) \|_{B(H)} = \| \rho(\lambda(x)) \|_{B(H)} \leq \| \lambda(x) \|_{C^*_r(T)}, \quad x \in CT.
\]
Thus amenability of \( F \) implies that the norm \( \| Q R \| \) is greater or equal than the norm \( \| \pi(Q R) \| \). Thus, it suffices to show that \( \| \pi(Q R) \| = \frac{1 + \sqrt{2}}{\sqrt{6}} \). We have
\[
\| \pi(Q R) \|^2 = \| \pi(Q R) \|^2 \text{ and } 
\pi(Q R) = \frac{1}{48}(S_1 + S_2)^2(2I + S_1 + S_2)^* (2I + S_1 + S_2)(S_1 + S_2)^2.
\]
Substituting \( x := \frac{1}{\sqrt{2}} (S_1 + S_2) \) gives
\[
\pi(Q R) = \frac{1}{6} x^2(\sqrt{2}I + x)^*(\sqrt{2}I + x)x^*.
\]
Since \( x \) is a proper isometry (i.e. \( x^* x = I \neq x x^* \)),
\[
\| x^2(\sqrt{2}I + x)^*(\sqrt{2}I + x)x^* \| = \| (\sqrt{2}I + x)^*(\sqrt{2}I + x) \|.
\]
Consequently,
\[
\| \pi(Q R) \| = \frac{1}{6} \| (\sqrt{2}I + x)^*(\sqrt{2}I + x) \| = \frac{1}{6} \| \sqrt{2}I + x \|^2.
\]
Now, the spectrum of \( x \) is the entire unit disc because \( x \) is a proper isometry. (One uses the Wold decomposition to show this). Thus the spectral radius of \( \sqrt{2}I + x \) is \( \sqrt{2} + 1 \). Hence the norm is at least this number and by the triangle inequality we get
\[
\| \sqrt{2}I + x \| = 1 + \sqrt{2},
\]
as required.

\[ \square \]

### 3. Two Projections

Let $\Gamma := \langle c, d \rangle$ be a discrete group on two generators such that $c$ is of order $k$ and $d$ of order $\ell$, for some integers $\ell \geq k \geq 2$. Let $L(\Gamma)$ denote its von Neumann algebra, i.e. $L(\Gamma)$ is the von Neumann algebra in $B(\ell^2(\Gamma))$ generated by $\Gamma$. Then $\tau(h) = \langle h\delta_e, \delta_e \rangle$, $h \in L(\Gamma)$ defines a normal faithful tracial state on $L(\Gamma)$ (see e.g. [23, Section 6.7]). Let $P := e/c + \cdots + e^{k-1} \in \mathbb{C} \Gamma \subset \mathbb{C}^r(\Gamma) \subset L(\Gamma) \subset B(\ell^2(\Gamma))$ and $Q := 1 + d + \cdots + d^{\ell-1} \in \mathbb{C} \Gamma \subset \mathbb{C}^r(\Gamma) \subset L(\Gamma) \subset B(\ell^2(\Gamma))$ be fixed projections such that $\|PQ\| < 1$.

Let

$\alpha := \tau(P) = \frac{1}{k}$, $\beta := \tau(Q) = \frac{1}{\ell}$

and notice that $0 < \beta \leq \alpha \leq \frac{1}{2}$.

It follows by P. R. Halmos’ work [11] that the $C^*$-algebra $A := C^*(P, Q, I)$ generated by 2 projections $P, Q$ and the identity $I$ is the direct sum of an abelian $C^*$-algebra and a $C^*$-algebra of type $I_2$ i.e. of the form $C(X) \otimes M_2(\mathbb{C})$ where $X$ equals the spectrum of a certain positive contraction. The abelian part is of dimension at most 4 and its minimal projections are given as

$e_{00} := I - P \vee Q = (I - P) \wedge (I - Q)$

$e_{11} := P \wedge Q$ $\quad e_{10} := P \wedge (I - Q)$ $\quad e_{01} := (I - P) \wedge Q$,

where some, and even all may vanish. The central support for the $I_2$ part of $A$ is denoted $f$ and it is given by

\[ f = I - e_{11} - e_{10} - e_{01} - e_{00}. \]

We define

\[ R := P - e_{11} - e_{10}, \quad S := Q - e_{11} - e_{01} \]

and see that $R$ and $S$ are projections satisfying $R \leq f$, $S \leq f$ because $P e_{11} = e_{11} P = e_{11}$, $P e_{01} = e_{01} P = 0$ and $Q e_{11} = e_{11} Q = e_{11}$, $Q e_{10} = e_{10} Q = 0$, $i = 0, 1$. Moreover,

$R \wedge S = 0$ $\quad R \vee S = f$.

So inside $fAf$, the pair of projections $R, S$ actually fit the description given by Halmos. Thus, there exists an isomorphism $\Phi$ of $fAf$ onto $C(\sigma(RS)) \otimes M_2(\mathbb{C}) = M_2(C(\sigma(RS)))$, such that

$\Phi(f) = I_{C(\sigma(RS))} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\Phi(R) = I_{C(\sigma(RS))} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
and
\[ \Phi(S) = \begin{pmatrix} RSR & (RSR)^{1/2}(f - RSR)^{1/2} \\ (RSR)^{1/2}(f - RSR)^{1/2} & f - RSR \end{pmatrix}. \]

Now,
\[
P - \alpha I = (R - \alpha f) + (1 - \alpha)e_{11} + (1 - \alpha)e_{10} - \alpha e_{01} - \alpha e_{00}
\]
\[
Q - \beta I = (S - \beta f) + (1 - \beta)e_{11} + (1 - \beta)e_{01} - \beta e_{10} - \beta e_{00}
\]
\[
(P - \alpha I)(Q - \beta I) = (R - \alpha f)(S - \beta f) + (1 - \alpha)(1 - \beta)e_{11} - \alpha(1 - \beta)e_{01} - \beta(1 - \alpha)e_{10} + (-\alpha)(-\beta)e_{00}.
\]

Thus since
\[
\tau((P - \alpha I)(Q - \beta I)^n) = \tau((R - \alpha f)(S - \beta f)^n) + \alpha \beta^n \tau(e_{00}) + (-\alpha(1 - \beta))^n \tau(e_{01}) + (-\beta(1 - \alpha))^n \tau(e_{11}).
\]

Similarly, since
\[
PQ = RS + e_{11}
\]
\[
(I - P)(I - Q) = (f - R)(f - S) + e_{00}
\]
\[
P(I - Q) = R(f - S) + e_{10}
\]
\[
(I - P)Q = (f - R)S + e_{01}
\]
we get
\[
\tau((PQ)^n) = \tau((RS)^n) + \tau(e_{11})
\]
\[
\tau\left((I - P)(I - Q)^n\right) = \tau\left((f - R)(f - S)^n\right) + \tau(e_{00})
\]
\[
\tau\left((PQ)^n\right) = \tau\left((R - f - S)^n\right) + \tau(e_{10})
\]
\[
\tau\left((I - P)Q^n\right) = \tau\left((f - R)S^n\right) + \tau(e_{01}).
\]
Since \(|PQ| < 1, ||e_{11}|| = ||e_{11}PQ|| < 1| because the norm is submultiplicative, and thus the minimal projection
\[ e_{11} = 0, \]
and since \(PQP = RSR + e_{11},\) the spectrum
\[ \sigma(RSR) = \sigma(PQP) \subset [0, 1). \]

By the continuous functional calculus for the positive element \(RSR \in fAf\) we can write
\[ \Phi(f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(R) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]
\[ \Phi(S) = \begin{pmatrix} t & \sqrt{t(1 - t)} \\ \sqrt{t(1 - t)} & 1 - t \end{pmatrix}, \quad 0 < t < 1. \]
\[
\begin{align*}
\Phi(RSR) &= \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, & 0 < t < 1. \\
\Phi(R(f-S)R) &= \begin{pmatrix} 1-t & 0 \\ 0 & 0 \end{pmatrix}, & 0 < t < 1. \\
\Phi((f-R)S(f-R)) &= \begin{pmatrix} 0 & 0 \\ 0 & 1-t \end{pmatrix}, & 0 < t < 1. \\
\Phi((f-R)(f-S)(f-R)) &= \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, & 0 < t < 1.
\end{align*}
\]

The restriction of \( \tau \) to \( fAf \) is a trace on this algebra, so by the spectral theorem to the positive element \( RSR \), there must exist a positive measure \( \mu \) on \( \sigma(RSR) \) such that for any \( x \in fAf \) we have
\[
\tau(x) = \int_{t \in \sigma(RSR)} \frac{1}{2} \Phi(x)_{11}(t) + \frac{1}{2} \Phi(x)_{22}(t) \, d\mu(t)
\]
where
\[
\tau_t(x) := \frac{1}{2} \text{Tr}(\Phi(x)(t)),
\]
and \( \frac{1}{2} \text{Tr} \) is the normalized trace of \( M_2(\mathbb{C}) \). Thus
\[
1 = \tau(I) = \int_{0}^{1} \tau_t(f) \, d\mu(t) + \tau(e_{11}) + \tau(e_{10}) + \tau(e_{01}) + \tau(e_{00}),
\]
and since \( \tau((pq)^n) = \tau((pqp)^n) \) for any two projections \( p, q \) and any \( n \in \mathbb{N} \), we get
\[
\tau((PQ)^n) = \int_{0}^{1} \tau_t((RS)^n) \, d\mu(t) + \tau(e_{11})
\]
\[
\tau(((I-P)(I-Q))^n) = \int_{0}^{1} \tau_t(((f-R)(f-S))^n) \, d\mu(t) + \tau(e_{00})
\]
\[
\tau((P(I-Q))^n) = \int_{0}^{1} \tau_t((R(f-S))^n) \, d\mu(t) + \tau(e_{10})
\]
\[
\tau(((I-P)Q)^n) = \int_{0}^{1} \tau_t(((f-R)S)^n) \, d\mu(t) + \tau(e_{01}).
\]

It is not difficult to show that \( \mu(\{0\}) = 0 \). We extend \( \mu \) to a probability measure on \([0,1]\) by putting
\[
\mu(\{0\}) := \mu_{10} + \mu_{01}
\]
\[
\mu(\{1\}) := \mu_{00} + \mu_{11},
\]
where
\[
\mu_{ij} := \tau(e_{ij}), \quad i,j \in \{0,1\}.
\]

**Lemma 3.1.** Define
\[
\nu := \mu - (\mu_{10} - \mu_{01})\delta_0 - (\mu_{00} - \mu_{11})\delta_1.
\]
Then \( \nu \) is a measure on \([0,1]\) with total mass \( \nu([0,1]) = 2\beta \), and
\[
\tau((PQ)^n) = \frac{1}{2} \int_{[0,1]} t^n \, d\nu(t), \quad n \geq 1.
\]
Moreover,
\[\mu_{10} - \mu_{01} = \alpha - \beta \geq 0, \quad \mu_{11} = 0, \quad \mu_{00} = 1 - \alpha - \beta \geq 0, \quad \nu(\{1\}) = 0.\]

Proof. Letting \(n = 1\) in Eq. (16), we get
\[1 = \tau(I) = \mu_{00} + \mu_{01} + \mu_{10} + \mu_{11} + \mu((0, 1)).\]
\[\alpha = \tau(P) = \tau(P(I - Q)) + \tau(PQ) = \mu_{10} + \mu_{11} + \frac{1}{2}\nu((0, 1)).\]
\[\beta = \tau(Q) = \tau((I - P)Q) + \tau(PQ) = \mu_{01} + \mu_{11} + \frac{1}{2}\nu((0, 1)).\]
\[1 - \alpha = \tau(I - P) = \mu_{00} + \mu_{01} + \frac{1}{2}\nu((0, 1)).\]
\[1 - \beta = \tau(I - Q) = \mu_{00} + \mu_{10} + \frac{1}{2}\nu((0, 1)).\]

Combining these equations we get
\[0 \leq \alpha - \beta = \tau(P) - \tau(Q) = \mu_{10} - \mu_{01}\]
\[0 \leq 1 - \alpha - \beta = \tau(I - P) - \tau(Q) = \mu_{00} - \mu_{11}.\]

Since \(\|PQ\| < 1\), \(\mu_{11} = 0\). Thus
\[\mu_{00} = 1 - \alpha - \beta.\]

We can write \(\mu_{ij}\), \(i = 0, 1\) as follows
\[\mu_{01} = \frac{1}{2}\mu(\{0\}) - \frac{\mu_{10} - \mu_{01}}{2}\]
\[\mu_{10} = \frac{1}{2}\mu(\{0\}) + \frac{\mu_{10} - \mu_{01}}{2}\]
\[\mu_{00} = \frac{1}{2}\mu(\{1\}) + \frac{\mu_{00} - \mu_{11}}{2}\]
\[\mu_{11} = \frac{1}{2}\mu(\{1\}) - \frac{\mu_{00} - \mu_{11}}{2},\]
and the measure \(\nu\) on \([0, 1]\) defined in the lemma satisfies
\[\nu(\{0\}) = \mu(\{0\}) - (\mu_{10} - \mu_{01})\]
\[\nu(\{1\}) = \mu(\{1\}) - (\mu_{00} - \mu_{11})\]
\[\mu_{01} = \frac{\nu(\{0\}) + (\mu_{10} - \mu_{01})}{2} - \frac{\mu_{10} - \mu_{01}}{2} = \frac{1}{2}\nu(\{0\})\]
\[\mu_{10} = \frac{1}{2}\nu(\{0\}) + (\mu_{10} - \mu_{01})\]
\[\mu_{00} = \frac{\nu(\{1\}) + (\mu_{00} - \mu_{11})}{2} + \frac{\mu_{00} - \mu_{11}}{2} = \frac{1}{2}\nu(\{1\}) + (\mu_{00} - \mu_{11})\]
\[\mu_{11} = \frac{1}{2}\nu(\{1\}).\]

Moreover, the total mass of \(\nu\) is
\[\nu([0, 1]) = \nu(\{0\}) + \nu(\{1\}) + \mu((0, 1))\]
\[= 1 - (\mu_{10} - \mu_{01} + \mu_{00} - \mu_{11}) = 1 - (\alpha - \beta + 1 - \alpha - \beta)\]
\[= 2\beta,\]
and
\[
\int_{[0,1]} \frac{1}{2} t^n d\nu(t) = \frac{1}{2} \int_{(0,1)} t^n d\mu(t) + \frac{1}{2} \nu(\{1\}) = \tau((PQ)^n), \quad n \in \mathbb{N}.
\]

By the following proposition and Lemma 3.1 we can state the “cyclic numbers”
\[
\tau((P - \alpha I)(Q - \beta I))^n), \quad n \in \mathbb{N},
\]
in terms of the moments \(\tau((PQ)^n)\).

**Theorem 3.2.** For \(n \in \mathbb{N}\),
\[
\tau((P - \alpha I)(Q - \beta I))^n) = (\alpha \beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + (\alpha(1 - \alpha)\beta(1 - \beta))^n T_n(\frac{\tau -\alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}})d\nu(t),
\]
where \(T_n\) is the Chebyshev polynomial of the first kind.

**Proof.** Let
\[
E := P - \alpha I, \quad F := Q - \beta I, \quad E' := R - \alpha f, \quad F' := S - \beta f.
\]
Then the spectrum \(\sigma(E) = \{-\alpha, 1 - \alpha\}\), \(\sigma(F) = \{-\beta, 1 - \beta\}\), and by Eq. (15) we get
\[
\tau((EF)^n) = ((-\alpha)(-\beta))^n \mu_{00} + (-\alpha(1 - \beta))^n \mu_{01} + (-\beta(1 - \alpha))^n \mu_{10} + ((1 - \alpha)(1 - \beta))^n \mu_{11} + \\
\int_{(0,1)} \tau_t((E'F')^n)d\mu(t)
\]
\[
= (\alpha \beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + \nu(\{0\})\left(\frac{1}{2}(-\beta(1 - \alpha))^n + \frac{1}{2}(-\alpha(1 - \beta))^n\right) +
\]
\[
\frac{1}{2} \nu(\{1\})((1 - \alpha)(1 - \beta))^n + \int_{(0,1)} \tau_t((E'F')^n)d\mu(t)
\]
\[
= (\alpha \beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + \int_{[0,1]} \tau_t((E'F')^n)d\nu(t),
\]
where we used Eq. (17) in the second equality.

To complete the proof we just need to show that
\[
\int_{[0,1]} \tau_t((E'F')^n)d\nu(t) = (\alpha(1 - \alpha)\beta(1 - \beta))^n/2 \int_{[0,1]} T_n(\frac{\tau -\alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}})d\nu(t).
\]

Since
\[
\Phi(E') = \Phi(R - \alpha f) = \begin{pmatrix} 1 - \alpha & 0 \\ 0 & -\alpha \end{pmatrix}
\]
\[
\Phi(F') = \Phi(S - \beta f) = \begin{pmatrix} t - \beta & \sqrt{t(1 - t)} \\ \sqrt{t(1 - t)} & 1 - t - \beta \end{pmatrix}
\]
we get
\[
\Phi(E'F') = \begin{pmatrix} (1 - \alpha)(t - \beta) & (1 - \alpha)\sqrt{t(1 - t)} \\ -\alpha\sqrt{t(1 - t)} & -(1 - t - \beta) \end{pmatrix}.
\]

Since
\[
\text{Tr}(\Phi(E'F')) = (1 - \alpha)(t - \beta) - \alpha(1 - t - \beta) = t - \alpha - \beta + 2\alpha\beta
\]
\[
\det(\Phi(E'F')) = \alpha(1 - \alpha)(\beta(1 - \beta))
\]
the characteristic polynomial \( \lambda^2 + Tr(\Phi(E'F')) + \det(\Phi(E'F')) \) of \( \Phi(E'F') \) is
\[
\lambda^2 - (t - \alpha - \beta + 2\alpha\beta)\lambda + \alpha(1 - \alpha)(1 - \beta).
\]
The eigenvalues of \( \Phi(E'F') \) are
\[
\lambda_1(t) := \frac{t - \alpha - \beta + 2\alpha\beta + \sqrt{(t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1 - \alpha)\beta(1 - \beta)}}{2},
\]
\[
\lambda_2(t) := \frac{t - \alpha - \beta + 2\alpha\beta - \sqrt{(t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1 - \alpha)\beta(1 - \beta)}}{2}.
\]
If \(|t - \alpha - \beta + 2\alpha\beta| \leq 2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}\), then \( \lambda_2 = \overline{\lambda_1} \) and
\[
\lambda_1 = |\lambda_1|e^{i\theta} \quad \lambda_2 = |\lambda_2|e^{-i\theta} \quad |\lambda_1| = |\lambda_2| = \sqrt{\alpha(1 - \alpha)\beta(1 - \beta)},
\]
where
\[
\cos \theta = \frac{\text{Re} \lambda_1}{2|\lambda_1|} = \frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}}.
\]
Let \( t_1, t_2 \) be the roots of \((t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1 - \alpha)\beta(1 - \beta)\) with \( t_1 < t_2 \). Then for \( t_1 < t < t_2 \) it holds
\[
\tau_t((E'F')^n) = \frac{\lambda_1(t)^n + \lambda_2(t)^n}{2} = |\lambda_1|^n e^{in\theta} + e^{-in\theta} = |\alpha(1 - \alpha)\beta(1 - \beta)|^{n/2} \cos(n\theta) = (\alpha(1 - \alpha)\beta(1 - \beta))^{n/2} T_n\left(\frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}}\right),
\]
where \( T_n \) is the Chebyshov polynomials of the first kind. Since both sides of this equality are polynomials which agree on the interval \([t_1, t_2]\), they must be the same polynomials. Hence
\[
\int_{[0,1]} \tau_t((E'F')^n) \, d\nu(t) = \int_{[0,1]} \frac{\lambda_1(t)^n + \lambda_2(t)^n}{2} \, d\nu(t).
\]

\[
= \int_{[0,1]} (\alpha(1 - \alpha)\beta(1 - \beta))^{n/2} T_n\left(\frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}}\right) \, d\nu(t).
\]

\[
\square
\]

4. The free case

Let \( p \in L(\Gamma) \) be a nonzero projection different from the identity. Here projections mean self-adjoint idempotents \((p^2 = p = p^*)\). Let \( \mu \) be the spectral distribution measure of \( p \) with respect to the trace \( \tau \) on \( L(\Gamma) \). Then \( \text{supp}(\mu) = \sigma(p) = \{0, 1\} \) because \( \tau \) is faithful, and
\[
\tau(f(p)) = \int_{\sigma(p)} f \, d\mu = \int_{\sigma(p)} (f(0)1_{\{0\}} + f(1)1_{\{1\}}) \, d\mu = f(0)\mu(\{0\}) + f(1)\mu(\{1\}),
\]
for any \( f \in C(\{0, 1\}) \). In particular, by letting \( f(t) = t \) and \( f(t) = 1 - t \) we get
\[
\mu(\{0\}) = 1 - \tau(p), \quad \mu(\{1\}) = \tau(p).
\]
Moreover, the moments \( m_n(p) = \tau(p^n) = \tau(p) \) for \( n \in \mathbb{N} \).
Theorem 4.1. Let $\Gamma := \mathbb{Z}_k \ast \mathbb{Z}_\ell$ be the free product on two generators $c \in \mathbb{Z}_k$, $d \in \mathbb{Z}_\ell$ for some integers $k, \ell \geq 2$. Let
\[ p := \frac{e + c + \cdots + c^{k-1}}{k}, \quad q := \frac{e + d + \cdots + d^{\ell-1}}{\ell} \in \mathbb{C}\Gamma \]
be two nonzero projections different from the identity. Moreover, assume that $\alpha := \tau(p)$, $\beta := \tau(q)$ satisfy $0 < \beta \leq \alpha < 1$, $\alpha + \beta < 1$ and let
\[ \lambda_1 := \sqrt{\alpha(1-\beta) + \beta(1-\alpha)} \]
\[ \lambda_2 := \sqrt{\alpha(1-\beta) - \beta(1-\alpha)}. \]
Then the spectral distribution measure $\mu_{\text{free}}$ of \( \left( \begin{array}{cc} 0 & (pq)^* \\ pq & 0 \end{array} \right) \in M_2(L(\Gamma)) \) with respect to the trace $\tilde{\tau} := \tau \otimes \tau_2$ on $M_2(L(\Gamma))$ is
\[ \mu_{\text{free}} = \frac{1}{2\pi} \sqrt{(\lambda_1^2 - t^2)(\lambda_2^2 - t^2)} \left. 1_{[-\lambda_1,-\lambda_2]\cup[\lambda_2,\lambda_1]}(t) \right] dt + (1-\beta)\delta_0, \]
where $dt$ is the Lebesgue measure. In particular,
\[ ||pq|| = \sqrt{\alpha(1-\beta) + \beta(1-\alpha)}. \]

Proof. Let $\mu$ (resp. $\nu$) be the spectral distribution measure of $p$ (resp. $q$) with respect to the trace $\tau$ of $L(\Gamma)$. We now compute the S-transform (cf. [17, p.30-32] or [16]):
\[ \psi_\mu(s) := \int_0^\infty \frac{ts}{1-ts} d\mu(s) = \int_0^\infty (ts + t^2s^2 + t^3s^3 + \cdots) d\mu(t) = m_1(p)s + m_2(p)s^2 + m_3(p)s^3 + \cdots \]
\[ = \frac{\alpha s}{1-s}. \]
Similarly,
\[ \psi_\nu(s) = \frac{\beta s}{1-s}. \]
The formal power series $\psi_\mu$ (resp. $\psi_\nu$) has to satisfy the equations $\chi_\mu(\psi_\mu(s)) = s$, $\psi_\mu(\chi_\mu(z)) = z$ and $S_\mu(z) = \frac{z+1}{z} \chi_\mu(z)$ (resp. $\chi_\nu(\psi_\nu(s)) = s$, $\psi_\nu(\chi_\nu(z)) = z$ and $S_\nu(z) = \frac{z+1}{z} \chi_\nu(z)$). Solving for $\chi_\mu, S_\mu, \chi_\nu, S_\nu$, we get
\[ z = \psi_\mu(s) \iff (1-s)z = \alpha s \iff \chi_\mu(z) = s = \frac{z}{\alpha + z}. \]
\[ z = \psi_\nu(s) \implies \chi_\nu(z) = s = \frac{z}{\beta + z}. \]
\[ S_\mu(z) = \frac{z+1}{z} \chi_\mu(z) = \frac{z+1}{\alpha + z} \]
\[ S_\nu(z) = \frac{z+1}{z} \chi_\nu(z) = \frac{z+1}{\beta + z}. \]
By [17, Theorem 3.6.3], the multiplicative free convolution is
\[ S_{\mu \Box \nu}(z) = S_\mu(z)S_\nu(z) = \frac{(z+1)^2}{(\alpha + z)(\beta + z)} \]
Thus
\[ \chi_{\mu \Box \nu}(z) = \frac{z}{z+1} S_{\mu \Box \nu}(z) = \frac{z(z+1)}{(\alpha + z)(\beta + z)}. \]
and
\[ s = \chi_\mu \mathbb{E}_\nu (z) \iff (\alpha + z)(\beta + z)s = z(z + 1) \]
\[ \implies \psi_\mu \mathbb{E}_\nu (s) = z = \frac{1}{2} \frac{1 - (\alpha + \beta)s \pm \sqrt{d(s)}}{s - 1} \]
where \( d(s) \) is the discriminant
\[ d(s) := (\alpha - \beta)^2 s^2 + (4\alpha\beta - 2\alpha - 2\beta)s + 1. \]
Letting \( s = \frac{1}{\lambda} \), the discriminant \( d(s) \) can be written as
\[ d\left(\frac{1}{\lambda}\right) = \frac{(\lambda - \lambda_1^2)(\lambda - \lambda_2^2)}{\lambda^2}, \]
where
\[ \lambda_1 := \sqrt{\alpha(1 - \beta) + \beta(1 - \alpha)} \quad \lambda_2 := \sqrt{\alpha(1 - \beta) - \beta(1 - \alpha)}. \]
Let \( \psi := \psi_\mu \mathbb{E}_\nu \). Then the Cauchy transform \( G(\lambda) \) of \( \mu \boxplus \nu \) is
\[ G(\lambda) = \frac{1}{\lambda} \psi\left(\frac{1}{\lambda}\right) = \frac{1}{2\lambda} \frac{1 - (\alpha + \beta)\frac{1}{\lambda} + \sqrt{d\left(\frac{1}{\lambda}\right)}}{\frac{1}{\lambda} - 1} = \frac{\lambda - \alpha - \beta + \sqrt{(\lambda - \lambda_1^2)(\lambda - \lambda_2^2)}}{2\lambda(1 - \lambda)}. \]
By Stieltjes inversion formula, \( \mu \boxplus \nu \) is given by
\[ (18) \quad \mu \boxplus \nu (x) = \lim_{y \to 0^+} \left(-\frac{1}{\pi} \text{Im} G(x + iy)\right) dx + c_0 \delta_0 \]
\[ = \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - x)(x - \lambda_2^2)}}{2x(1 - x)} 1_{[\lambda_1^2, \lambda_2^2]}(x) dx + c_0 \delta_0. \]
where \( c_0 := 1 - \beta \), is the residue of \( G \) at the simple pole 0, and \( dx \) is the Lebesgue measure. Since \( 0 < \beta \leq \alpha < 1 \) and \( \alpha + \beta < 1 \), the integral
\[ (19) \quad \int_{\lambda_2^2}^{\lambda_1^2} \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - x)(x - \lambda_2^2)}}{2x(1 - x)} dx = \beta, \]
(\text{cf. \cite{15} Example 4.3.3}). By \cite{17} Remarks 3.6.2(iii), we get that \( \mu \boxplus \nu = \mu_{p^2q^2} \), the spectral distribution of \( p^2q^2 \) with respect to the (unique) trace \( \tau \) on \( L(\Gamma) \). Since \( \tau((p^2q^2)^n) = \tau((pq)^n) \) for every \( n \in \mathbb{N}_0 \), \( \mu \boxplus \nu = \mu_{pq} \), the spectral distribution of \( pq \) w.r.t. \( \tau \). Define the self-adjoint operator
\[ S := \begin{pmatrix} 0 & (pq)^* \\ pq & 0 \end{pmatrix} \in M_2(L(\Gamma)), \]
and let \( \mu_{\text{tree}} := \mu_S \) be the spectral distribution of \( S \) with respect to the normal faithful trace \( \tau \otimes \tau_2 \), where \( \tau_2 := \frac{1}{2} T \tau \) is the normalized trace on \( M_2(\mathbb{C}) \). By \cite{10} Eq. (2.6)], \( m_{2\tau}(S) = \tau(((pq)^*pq)^n) = \tau((pq)^n) = \tau((pqp)^n) \). Thus the image measure \( \phi(\mu_S) \) of \( \mu_S \) by the map \( \phi : t \mapsto t^2 \) is \( \phi(\mu_S) = \mu_{pq} = \mu \boxplus \nu \). Substituting \( t^2 = x \) in Eq. (18), we get
\[ \mu_S(t) = \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - t^2)(t^2 - \lambda_2^2)}}{t(1 - t^2)} 1_{[\lambda_2, \lambda_1]}(t) dt + c_0 \delta_0. \]
Symmetrizing,
\[ \mu_S(t) = \frac{1}{2\pi} \frac{\sqrt{(\lambda_1^2 - t^2)(t^2 - \lambda_2^2)}}{t(1 - t^2)} 1_{[-\lambda_1, -\lambda_2] \cup [\lambda_2, \lambda_1]} dt + c_0 \delta_0, \]
and by [10] Eq. (2.7),
\[ ||pq|| = ||S|| = \max(\text{supp}(\mu_S)) = \lambda_1. \]
\[ \square \]

**Example 4.2** (free case). Let \( \Gamma := \mathbb{Z}_2 \ast \mathbb{Z}_3 \) be the free product with generators \( s \in \mathbb{Z}_2, c \in \mathbb{Z}_3 \). Let \( p = (e + s)/2, q = (e + c + c^2)/3 \in \mathbb{C} \). Then by Theorem 4.1
\[ ||pq|| = \sqrt{\frac{1}{2}(1 - \frac{1}{3})} + \sqrt{\frac{1}{3}(1 - \frac{1}{2})} = 1 + \sqrt{\frac{2}{\sqrt{6}}} = 0.985 \ldots \]

Let \( \Gamma' := \mathbb{Z}_3 \ast \mathbb{Z}_4 \) be the free product with generators \( c \in \mathbb{Z}_3, d \in \mathbb{Z}_4 \). and let \( q = (e + c + c^2)/3, r = (e + d + d^2 + d^3)/4 \in \mathbb{C} \). Then by Theorem 4.1
\[ ||qr|| = \sqrt{\frac{1}{3}(1 - \frac{1}{4})} + \sqrt{\frac{1}{4}(1 - \frac{1}{3})} = \sqrt{\frac{3}{\sqrt{12}}} = 0.908 \ldots \]

**5. RESULTS**

Let \( P, Q, R \in L(T) \) be the projections given by
\[ P := \frac{I + S}{2}, \quad Q := \frac{I + C + C^2}{3}, \quad R := \frac{I + D + D^2 + D^3}{4}, \]
where \( S := D^2 \) and \( C, D \) are the generators of \( T \) shown in Figure 1. Let \( \tau \) be the trace on \( \mathbb{C}T \) coming from the group von Neumann algebra \( L(T) \), as described in [10] Section 2. Using the trace properties of \( \tau \), the even moments for the self-adjoint operator
\[ \tilde{h} := \sqrt{\frac{12}{12}} \begin{pmatrix} 0 & (QR)^* \\ QR & 0 \end{pmatrix} \in M_2(\mathbb{C}_T^\ast(T)) \]
are given by
\[ m_{2n}(\tilde{h}) = \tau((\tilde{h})^{2n}) = 12^n \tau((QRQ)^n) = 12^n \tau((QR)^n), \quad n \in \mathbb{N}, \]
where
\[ h := \sqrt{12}QR = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3). \]

Our goal is to compute as many as possible of the moments
\[ m_n := m_n(h^*h) = m_{2n}(\tilde{h}) = \tau(((I + C + C^2)(I + D + D^2 + D^3))^n), \quad n \in \mathbb{N}, \]
because by Section 4 in [10] we can use them to estimate the norm \( ||h|| \) (which is equal to \( ||\tilde{h}|| || \)):
\[ \lim_{n \to \infty} m_{1/2n} = \lim_{n \to \infty} \sqrt{\frac{12}{12}}\tau((QRQ)^{1/2n}) = \sqrt{\frac{12}{12}}||QR||^{1/2} = \sqrt{\frac{12}{12}}||QR|| = ||h||. \]

Rather than computing the moments \( m_n \) directly, we compute the “cyclic reduced” numbers
\[ \zeta_n := \tau(((C + C^2)(D + D^2 + D^3))^n), \quad n \in \mathbb{N}. \]
These numbers are related by the formula
\[ \zeta_n - \frac{5 + (-2)^n}{12} = 2 \sum_{k=0}^{n} \frac{c_{n,k}}{12^k}m_k, \quad n \in \mathbb{N}_0, \]
where \( m_0 := 1/4, (\zeta_0 := 1), \) and \( c_{n,k} \) is the coefficient of \( t^k \) in the substituted Chebyshev polynomial of the first kind \( 6^n/2T_n(\sqrt{6}(t-\frac{5}{12})) \). To obtain this formula (22), we use Lemma 3.1 and Theorem 3.2 with \( \alpha = \tau(Q) = 1/3, \beta = \tau(R) = 1/4, \) and note that from Lemma 3.1, \( I_{[0,1]} t^k \mathrm{d} \nu = 2m_k/12^k \).

To compute the cyclic reduced numbers \( \zeta_n \), we use the inner product

\[
(x, y) := \tau(y^* x)
\]
on \( L(T) \) associated with the trace \( \tau \). Recall that the trace restricted to the group ring \( \mathbb{C} T \) (considered as a subalgebra of \( C^*_r(T) \subset L(T) \)) satisfies

\[
\tau(\sum_{x \in Y} c_x x) = \begin{cases} 
    c_e & \text{if } e \in Y \\
    0 & \text{else}
\end{cases},
\]

for any finite subset \( Y \subset T \) and any set of complex numbers \( (c_x)_{x \in T} \) indexed by \( T \). In particular, \( \tau(x) = 1 \) if \( x = e \) else it is zero. For instance, expanding \((I + C + C^2)(I + D + D^2 + D^3)\) we see that the identity occurs only once and thus \( m_1 = 1 \). Thus

\[
\langle \sum_{x \in X} c_x x, \sum_{y \in Y} c_y y \rangle = \sum_{x \in X} \sum_{y \in Y} c_x c_y^*,
\]

for any finite subsets \( X, Y \subset T \), and any \( c_x, c_y \in \mathbb{C} \).

Define

\[
a := C + C^2 \quad b := D + D^2 + D^3,
\]

and recall that \( C^3 = I, D^4 = I, \) and that \( a = a^* \) and \( b = b^* \). Then

\[
\zeta_n = \tau((ab)^n), \quad n \in \mathbb{N}.
\]

Using computers, we compute \( \zeta_n \) as follows. We store in one file the terms of the expanded sum \((ab)^n\), (one term per line), and in another file those of \((ba)^n\). Here a term is a word of length \( n \) in the letters \( C, C^{-1}, D, D^{-1}, D^2 \). Composition of the letters in the word is done using the algorithm described in section 2 which yields a reduced doubltree. This doubltree is then converted into a sequence of zeroes and ones and separators and pointers by serializing the range and domain trees using the preorder traversal method. We save this sequence in base 64 in a single line. To save time and space we read the file for \((ab)^n\) and apply \( ab \) (i.e. we multiply each word with \( CD, CD^2, CD^{-1}, C^{-1}D, C^{-1}D^2, \) and \( C^{-1}D^{-1} \)) in order to obtain the file for \((ab)^{n+1}\). To save space we store not only the word but also its frequency. Taking the inverse of each word in the file for \((ab)^n\) gives the file for \((ba)^n\). The inverse of a doubltree \( x \in T \) with range tree \( r(x) \) and domain tree \( d(x) \) is the doubltree with range tree \( d(x) \) and domain tree \( r(x) \) (i.e. it simply swaps the domain and range trees). The inner product

\[
\zeta_{2n} = \langle (ab)^n, (ba)^n \rangle, \quad n \in \mathbb{N}
\]
is the intersection of the sorted files for \((ab)^n\) and \((ba)^n\). We used the GNU sort program to sort the files. The odd “cyclic reduced” numbers \( \zeta_{2n+1} \) are computed similarly

\[
\zeta_{2n+1} = \langle (ab)^n, (ba)^n \rangle = \langle (ab)^n, (ba)^{n+1} \rangle, \quad n \in \mathbb{N}_0.
\]

To reduce the size of the files by about one half, we compute instead the numbers \( \zeta_n, \zeta \), given in Eqs. (8), (9); the idea is the following: Since \((ab)^n = Cb(ab)^n + C^{-1}b(ab)^n\) and \((ba)^n\) is self-adjoint, we can obtain the terms of the expanded sum for \( C^{-1}b(ab)^n \) from those of \( Cb(ab)^n \) by using the “reverse-inverse” map \( \pi \circ J =
Figure 12. Estimating the norm \( \frac{1}{\sqrt{12}} \| (I + C + C^2)(I + D + D^2 + D^3) \| \).

Ad\((D^2)\circ R\) given in Eq. (7). For instance the reverse-inverse of the word \( CDC^2D^2 \) is \( D^2 R(DCDC^2D^2) = C^{-1}D^{-1}C^{-2}D^{-2} \), which corresponds to reversing the order of the letters and taking the inverse. The letter \( s \) (resp. \( e \)) in \( \zeta^s_n \) (resp. \( \zeta^e_n \)) stands for that we are only keeping the words which are smaller than (resp. equal to) their corresponding reverse-inverses. The comparison is done using the lexicographic order of the serialized form of the doubletrees. The relation between these numbers and the reduced cyclic number \( \zeta_n \) is given by Lemma 2.2

\[
\zeta_n = 2\zeta^s_n + 4\zeta^e_n, \quad n \geq 2.
\]

We wrote two programs in C# and Haskell, both using parallel programming, to calculate the numbers \( \zeta^s_n, \zeta^e_n, \zeta_n, m_n \) which can be downloaded at

https://github.com/shaagerup/ThompsonGroupT/
https://github.com/mariars/ThompsonGroupT/

The size of each of the two files for computing \( \zeta^s_{28} \) is about 281 GB and 285 GB. They were run in a desktop computer with 2 TB of SSD hard disk, and on the Abacus 2.0 supercomputer from the DeIC National HPC Centre. The series of numbers \( \zeta^s_n, \zeta^e_n, \zeta_n, m_n \) are shown in Table 1.

In comparison, when one considers the free product \( \mathbb{Z}_3 \ast \mathbb{Z}_4 \) on two generators \( c \in \mathbb{Z}_3, d \in \mathbb{Z}_4 \), the measure \( \mu_{\text{free}} \) in Eq. (3) based on \( c, d \) instead of \( C, D \), is computed explicitly using Theorem 4.1 with \( \alpha = \tau(\frac{c+c^2}{3}) = 1/3, \beta = \tau(\frac{d+d^2+d^3}{4}) = 1/4, \) (and substituting \( x = \sqrt{12t} \)). Hence the moments \( m_n^{\text{free}} \) can be computed explicitly.
with Eq. (3) together with Lemma A.1. One gets \( m^\text{free}_1 = 1 \), and
\[
m^\text{free}_n = 12m^\text{free}_{n-1} - 6 \sum_{j=0}^{\lfloor \frac{n+2}{2} \rfloor} 6^j 5^{-2j+n-2} \binom{2j}{j} \binom{n-2}{j+1}, \quad n \geq 2.
\]

Estimations of the norm \( ||h|| \) are done as follows. By [10, Section 4] and the theory of orthogonal polynomials we can define a sequence of positive numbers
\[
a_n := \left( \frac{D_{n-2}D_n}{D_n^2} \right)^{1/2}, \quad n \in \mathbb{N},
\]
where
\[
D_{-1} := 1, \quad D_n := \det([m_{i+j}(\tilde{h})]_{i,j=0}^n).
\]

| \( n \) | \( \zeta_n^0 \) | \( \zeta_n^1 \) | \( \zeta_n \) | \( m_n(h^*h) \) |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 6 |
| 3 | 0 | 0 | 0 | 42 |
| 4 | 0 | 0 | 0 | 318 |
| 5 | 1 | 0 | 2 | 2528 |
| 6 | 0 | 0 | 0 | 20790 |
| 7 | 0 | 0 | 0 | 175344 |
| 8 | 4 | 0 | 8 | 1508158 |
| 9 | 36 | 0 | 72 | 13177554 |
| 10 | 70 | 3 | 152 | 116636378 |
| 11 | 64 | 1 | 132 | 1043596346 |
| 12 | 524 | 1 | 1052 | 9423929906 |
| 13 | 2228 | 4 | 4472 | 85780131568 |
| 14 | 8160 | 16 | 16384 | 78625290728 |
| 15 | 21617 | 32 | 43362 | 7251162207110 |
| 16 | 81644 | 102 | 163696 | 67241091321510 |
| 17 | 279531 | 697 | 561850 | 626619942680948 |
| 18 | 1006816 | 4990 | 2033592 | 5865627675769158 |
| 19 | 3429416 | 13057 | 6911060 | 55130780282172364 |
| 20 | 12284412 | 35247 | 24709812 | 520110723876289138 |
| 21 | 43215686 | 89274 | 8678468 | 4923701716098043110 |
| 22 | 154863150 | 246102 | 310710708 | 467595409198606581346 |
| 23 | 550890233 | 763137 | 1104833014 | 445382340814268264936 |
| 24 | 1982133410 | 2484953 | 3974206632 | 4253954798148920432622 |
| 25 | 7128125209 | 9275681 | 14293535142 | 40735421620966779279998 |
| 26 | 2579672490 | 34858087 | 51734777328 | 39102223554637841228050 |
| 27 | 93561508424 | 119608865 | 187601452308 | 376199278400540950198026 |
| 28 | 341014479116 | 411320336 | 683674239576 | 36271465945557216051920334 |

Table 1. The series of numbers for \( h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3) \).
For instance

\[ D_4 = \det \begin{pmatrix}
  m_0 & 0 & m_1 & 0 & m_2 \\
  0 & m_1 & 0 & m_2 & 0 \\
  m_1 & 0 & m_2 & 0 & m_3 \\
  0 & m_2 & 0 & m_3 & 0 \\
  m_2 & 0 & m_3 & 0 & m_4
\end{pmatrix}, \]

where \( m_0 := 1/4 \) and \( m_n \) is defined in Eq. (21). Let

\[ M_n = \begin{pmatrix}
  0 & \alpha_1 \\
  \alpha_1 & 0 & \alpha_2 & 0 \\
  & \alpha_2 & \ddots & \ddots \\
  & & \ddots & \ddots & \ddots \\
  & & & \alpha_n & 0
\end{pmatrix}. \]

| \( n \) | \( m_n^{\frac{1}{2}} \) | \( \frac{m_n}{m_{n-1}} \) | \( \alpha_n \) | \( \lambda_{\text{max}}(M_n) \) | \( \alpha_{n-1} + \alpha_n \) |
|---|---|---|---|---|---|
| 1 | 1.0000 | 2.00000 | 2.00000 | 2. | - - - - - |
| 2 | 1.56508 | 2.44949 | 1.41421 | 2.44949 | 3.41421 |
| 3 | 1.86441 | 2.64575 | 1.73205 | 2.71519 | 3.14626 |
| 4 | 2.05496 | 2.75162 | 1.41421 | 2.82843 | 3.14626 |
| 5 | 2.18916 | 2.81952 | 1.77951 | 2.9224 | 3.19373 |
| 6 | 2.28992 | 2.86773 | 1.41111 | 2.97266 | 3.19062 |
| 7 | 2.36899 | 2.90414 | 1.75248 | 3.01653 | 3.16359 |
| 8 | 2.43306 | 2.93277 | 1.42622 | 3.04235 | 3.17870 |
| 9 | 2.48626 | 2.95593 | 1.78930 | 3.06842 | 3.21552 |
| 10 | 2.53129 | 2.97509 | 1.43494 | 3.08578 | 3.22424 |
| 11 | 2.57000 | 2.99123 | 1.75349 | 3.10309 | 3.18844 |
| 12 | 2.60371 | 3.00504 | 1.43767 | 3.11454 | 3.19117 |
| 13 | 2.63339 | 3.01701 | 1.78223 | 3.12665 | 3.21990 |
| 14 | 2.65975 | 3.02753 | 1.45087 | 3.13534 | 3.23310 |
| 15 | 2.68337 | 3.03685 | 1.76147 | 3.14458 | 3.21234 |
| 16 | 2.70467 | 3.04518 | 1.45124 | 3.15121 | 3.21271 |
| 17 | 2.72399 | 3.05270 | 1.77229 | 3.15841 | 3.22535 |
| 18 | 2.74163 | 3.05953 | 1.45749 | 3.16373 | 3.22978 |
| 19 | 2.75780 | 3.06577 | 1.76751 | 3.16956 | 3.22500 |
| 20 | 2.77270 | 3.07150 | 1.46238 | 3.17394 | 3.22989 |
| 21 | 2.78647 | 3.07679 | 1.76938 | 3.1788 | 3.23176 |
| 22 | 2.79926 | 3.08169 | 1.46384 | 3.18249 | 3.23321 |
| 23 | 2.81116 | 3.08625 | 1.76757 | 3.1866 | 3.23141 |
| 24 | 2.82228 | 3.09051 | 1.46904 | 3.18976 | 3.23662 |
| 25 | 2.83269 | 3.09449 | 1.76906 | 3.19332 | 3.23810 |
| 26 | 2.84247 | 3.09824 | 1.47004 | 3.19607 | 3.23909 |
| 27 | 2.85168 | 3.10176 | 1.76808 | 3.19917 | 3.23811 |
| 28 | 2.86036 | 3.10509 | 1.47534 | 3.2016 | 3.24341 |

Table 2. Estimating the norm \( \frac{1}{\sqrt{12}} \| (I + C + C^2)(I + D + D^2 + D^3) \| \).
Then by \cite{10} Proposition 4.4,
\begin{equation}
\liminf_{n \to \infty} (\alpha_{n-1} + \alpha_n) \leq ||h|| \leq \sup_{n \geq 1} (\alpha_{n-1} + \alpha_n).
\end{equation}

Moreover by \cite{10} Proposition 4.1 and Proposition 4.7, the roots \( (m_n^\frac{1}{2})_{n=1}^{\infty}, \) the ratios \( \left( \sqrt{\frac{m_n}{m_{n-1}}} \right)_{n=1}^{\infty}, \) and the norms \( \{||M_n||\}_{n=1}^{\infty} \) are increasing sequences that converge to \( ||h|| \) and satisfy
\[ m_n^\frac{1}{2} \leq \sqrt{\frac{m_n}{m_{n-1}}} \leq ||M_n|| \leq \frac{1}{\sqrt{12}} ||(I + C + C^2)(I + D + D^2 + D^3)||, \quad n \in \mathbb{N}. \]

We list these sequences in Table 2 and plot them in Figure 12 for \( n = 1, \ldots, 28. \) When \( n \geq 16, \) the norm \( ||h|| \geq \sqrt{2} + \sqrt{3}. \) The best lower bound for \( h \) that we can obtain from our data is
\[ \frac{1}{\sqrt{12}} ||(I + C + C^2)(I + D + D^2 + D^3)|| \geq 3.2016, \]
and a very likely lower bound for \( h \) is
\[ \frac{1}{\sqrt{12}} ||(I + C + C^2)(I + D + D^2 + D^3)|| \geq \alpha_{27} + \alpha_{28} = 3.24341, \]
because of Eq (24) and because the sequence \( (\alpha_{n-1} + \alpha_n)_{n=1}^{\infty} \) appear to be monotonically increasing for \( n \geq 19. \) By making a least squares fitting of the numbers \( \lambda_{\max}^2(M_{18}), \ldots, \lambda_{\max}^2(M_{28}) \) to a function of the form \( f(n) = a - b(n - c)^d \) we get
\[ f(n) = 10.79 - \frac{8.952}{(n + 0.2)^{0.841}}. \]

In particular, the extrapolation method suggests that
\begin{equation}
\frac{1}{\sqrt{12}} ||(I + C + C^2)(I + D + D^2 + D^3)|| \approx \sqrt{10.79} = 3.28.
\end{equation}

This and Theorem 2.4 suggest that \( F \) might be non-amenable.

5.1. **Spectral distribution measure.** Recall that \( h = \sqrt{2QR}. \) Since \( \hat{h} \) defined in Eq. (20) is self-adjoint, we get by the spectral theorem that there is a unique probability measure \( \mu_{\hat{h}} \) to \( \mathbb{R} \) with support (cf. \cite{10} Section 2)
\[ \text{supp}(\mu_{\hat{h}}) \subset [-||\hat{h}||, ||\hat{h}||] \subset [-1 + q, 1 + q], \]
where \( ||\hat{h}|| = ||h|| \leq 2 + \sqrt{2} = 1 + q \) by Theorem 2.4. In this subsection, we will estimate the spectral distribution measure \( \mu_{\hat{h}}, \) as it was done in \cite{10} Section 5.

The Hilbert space \( L^2([-q + 1, q + 1], dt) \) can be equipped with the orthonormal basis
\[ \sqrt{n + \frac{1}{2}} P_n \left( \frac{t}{q + 1} \right), \quad n \in \mathbb{N}_0, \]
where \( P_n, n \in \mathbb{N}_0, \) are the Legendre polynomials. The Hilbert space \( L^2([-q + 1, q + 1], \frac{1}{\pi \sqrt{(1+q)^2 - t^2}} dt) \) can be equipped with the orthonormal basis
\[ T_0 \left( \frac{t}{q + 1} \right), \quad \sqrt{2} T_n \left( \frac{t}{q + 1} \right), \quad n \in \mathbb{N}, \]
where \( T_n, n \in \mathbb{N}_0, \) are the Chebyshev polynomials of the first kind.
Figure 13. Estimating the Chebyshev density for $\tilde{\mu}_h$ where $h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3)$.

By [10, Eq. (5.3)], the density of $\tilde{\mu}_h$ with respect to the Lebesgue measure can be approximated by

$$\rho'_N(t) = 2N \sum_{n=0}^{2N} \frac{n + \frac{1}{2}}{q + 1} \left( \int_{-(q+1)}^{q+1} P_n\left(\frac{s}{q+1}\right) d\mu_h(s) \right) P_n\left(\frac{t}{1+q}\right)$$

and by

$$\rho_N(t) = 2N \sum_{n=0}^{2N} c_n^2 \left( \int_{-(q+1)}^{q+1} T_n\left(\frac{s}{q+1}\right) d\mu_h(s) \right) T_n\left(\frac{t}{1+q}\right) \frac{1}{\pi \sqrt{(1+q)^2 - t^2}},$$

where $c_0 := 1$ and $c_n := \sqrt{2}, n \in \mathbb{N}$. Since $\mu_h$ is a symmetric measure, all the odd terms in Eq. (26) are zero. Using the moments $m_n$ in Eq. (21) (with $m_0 = 1/4$) and the formula

$$\int_{-(q+1)}^{q+1} t^{2n} d\mu_h = m_{2n}(\tilde{h}) = m_n, \quad n \in \mathbb{N}_0,$$

we compute the “Chebyshev” density $\rho_{28}$ for $0 \leq t \leq 2 + \sqrt{2}$ and plot it in Figure 13 together with the corresponding “free” density. We are more interested in the tail of the measure, however, because it gives us an estimate of the norm $||\tilde{h}|| = ||h||$:

$$||h|| = \max\{||t|| \mid t \in \text{supp}(\mu_h)\},$$

(cf. [10, Section 2]). We plot $\rho_{28}$ in the tail interval $[\sqrt{2} + \sqrt{3}, 2 + \sqrt{2}]$ in Figure 14. This shows that $\mu_h$ has very little mass in $[3.22, 3.414]$, hence $||h||$ can be any number in $[3.22, 3.414]$ including the extrapolated number 3.28 found in Eq. (25).
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Appendix A. Free moments

Lemma A.1. Let

\begin{equation}
m_n^{\text{free}} := \int_{\sqrt{3} - \sqrt{2}}^{\sqrt{3} + \sqrt{2}} \frac{t^{2n} \sqrt{24 - (t^2 - 5)^2}}{\pi t(12 - t^2)} dt, \quad n \in \mathbb{N}.
\end{equation}

Then \( m_1^{\text{free}} = 1 \) and

\[m_{n+1}^{\text{free}} = 12m_n^{\text{free}} - 6 \sum_{j=0}^{\frac{n-1}{2}} C_j \left( \begin{array}{c} n-1 \\ 2j \end{array} \right) 6^j 5^{n-1-2j}, \quad n \in \mathbb{N}\]

where \( C_j := \frac{1}{j+1} \binom{2j}{j} \), \( j \in \mathbb{N}_0 \) are the Catalan numbers.

Proof. Substituting \( x = t^2 \) in Eq.(27), we get

\[m_n^{\text{free}} = \int_{5 + 2\sqrt{6}}^{5 + 2\sqrt{6}} \frac{\sqrt{24 - (x - 5)^2}}{\pi} \frac{1}{12 - x} dx, \quad n \in \mathbb{N}.\]
Then
\[12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = \frac{1}{2\pi} \int_{-\sqrt{5}}^{\sqrt{5}} x^n \sqrt{24 - (x - 5)^2} dx, \quad n \in \mathbb{N}.\]
Substituting \(\frac{x-5}{\sqrt{6}} = y\), we get
\[(28) \quad 12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = \frac{1}{2\pi} \int_{-2}^{2} 6(\sqrt{6}y + 5)^n \sqrt{4 - y^2} dy, \quad n \in \mathbb{N}.\]
Using the binomial formula, we get
\[12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = 6 \sum_{k=0}^{n} \frac{1}{2\pi} \int_{-2}^{2} y^k \sqrt{4 - y^2} dy \left(\frac{n}{k}\right) 6^{k/2} 5^{n-k}, \quad n \in \mathbb{N}.\]
Observing that the integral is zero when \(k\) is odd,
\[12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = 6 \sum_{j=0}^{\lfloor n/2 \rfloor} C_j \left(\frac{n}{2j}\right) 6^{j} 5^{n-2j}, \quad n \in \mathbb{N},\]
where \(C_j := \frac{1}{j+1} \binom{2j}{j}\) is a Catalan number.
Eq. (28), holds also if we let \(n = -1:\)
\[12m_{0}^{\text{free}} - m_{1}^{\text{free}} = \frac{6}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - y^2}}{\sqrt{6}y + 5} dy = 2.\]
It follows that \(m_{1}^{\text{free}} = 1\) because by Eq. (19) (with \(\alpha = 1/3, \beta = 1/4\), \(m_{0}^{\text{free}} = 1/4\)).

\[\square\]

Appendix B. Estimating the norm \(||C + D + C^{-1} + D^{-1}||\)
Here we will describe briefly our estimation of the norm of
\[h := C + D + C^{-1} + D^{-1} \in C T,\]
where \(C, D\) are the generators of \(T\) whose graphs are shown in Figure 1. We use the same procedure as we did in [10]. By [10] Theorem 1.3
\[2\sqrt{3} < ||C + D + C^{-1} + D^{-1}|| < 4.\]
The upper bound is never attained because the Thompson group \(T\) is not amenable.
We compute the first 28 even moments \(m_n(h^* h)\) by first computing the sequence (with \(q = 3\))
\[
\begin{align*}
h_0 &= e \\
h_1 &= h \\
h_2 &= hh_1 - (q+1)h_0 \\
h_{n+1} &= hh_n - qh_{n-1}, \quad n \geq 2.
\end{align*}
\]
Table 3. The series of numbers for \( h = C + C^{-1} + D + D^{-1} \).

| \( n \) | \( ||h_n||^2 \) | \( \eta_n \) | \( \zeta_n \) | \( m_n(h^*h) \) |
|---|---|---|---|---|
| 1 | 4 | 0 | 0 | 4 |
| 2 | 14 | 2 | 2 | 30 |
| 3 | 46 | 6 | 2 | 270 |
| 4 | 182 | 50 | 34 | 2678 |
| 5 | 856 | 360 | 244 | 28418 |
| 6 | 3888 | 1080 | 844 | 317246 |
| 7 | 20536 | 10552 | 6356 | 3681822 |
| 8 | 111366 | 35010 | 1407754 | 44027350 |
| 9 | 642550 | 368762 | 228242 | 538815546 |
| 10 | 3850086 | 10552 | 1407754 | 6713421830 |
| 11 | 2344600 | 8719700 | 1407754 | 84869473770 |
| 12 | 145705728 | 14837859548 | 2382930531465042 |
| 13 | 915645208 | 355133636 | 1410672509722 |
| 14 | 5816241006 | 2288268034 | 182071429751606 |
| 15 | 37273096250 | 63390231984 | 414711515674495370 |
| 16 | 24060840566 | 96703523122 | 31360608130235654 |
| 17 | 1563526262404 | 63390231984 | 414711515674495370 |
| 18 | 1021920908952 | 417463000486 | 5507403086681142854 |
| 19 | 67146704535028 | 2761853001904 | 73415226964469375622 |
| 20 | 44323828667666 | 18347893659506 | 981973882890399349286 |
| 21 | 293993937656674 | 12236106447844838 | 13175045740884220099018 |
| 22 | 19575351631144042 | 81896481405262 | 17726711286150905927594 |
| 23 | 13083502206113204 | 54997636841585104 | 193911297555795301975623294 |
| 24 | 877529231836455728 | 370502892149137828 | 32335124616320091148224950 |
| 25 | 590501992065165584 | 2503367879099490904 | 43821497789410524044150738 |
| 26 | 39857811116595156626 | 1696168753234006854 | 59511906746415918623110822 |
| 27 | 269809546538449104054 | 115227866329705330058 | 80978038411680591548914827558 |
| 28 | 1831388211478414017418 | 78475277424152455900 | 1103891232023903090341589166522 |
Then we compute the sequences
\[
\xi_n = ||h_n||^2 - (q + 1)q^{n-1},
\]
\[
\eta_n = \xi_n - (q - 1)(\xi_{n-1} + \xi_{n-2} + \cdots + \xi_1),
\]
\[
\zeta_n = \eta_n - (q - 1)(\eta_{n-1} + \eta_{n-2} + \cdots + \eta_1)
\]
\[
m_n = \left(\frac{2n}{n}\right)q^n + \sum_{k=1}^{n} \left(\frac{2n}{n-k}\right)(\zeta_k + 1 - q)q^{n-k}, \quad n \in \mathbb{N}.
\]

These series of numbers are listed in Table 3. Next, we compute the increasing sequences of “roots”, “ratios”, “norms” that converge to the norm (cf. Section 5). We list them in Table 4 and plot them in Figure 15. The best lower bound for \(||h||\) that we can obtain from our results is
\[
||C + D + C^{-1} + D^{-1}|| \geq 3.7873.
\]

| \(n\) | \(m_n\) | \(\sqrt{\frac{m_n}{m_{n-1}}}\) | \(\alpha_n\) | \(\lambda_{\max}(M_n)\) | \(\alpha_{n-1} + \alpha_n\) |
|---|---|---|---|---|---|
| 1 | 2.00000 | 2.00000 | 2.00000 | 2. | - - - - |
| 2 | 2.34035 | 2.73861 | 1.87083 | 2.73861 | 3.87083 |
| 3 | 2.54230 | 3.00000 | 1.79284 | 3.0557 | 3.66367 |
| 4 | 2.68211 | 3.14937 | 1.94854 | 3.25861 | 3.74139 |
| 5 | 2.78843 | 3.25755 | 2.04888 | 3.42926 | 3.99743 |
| 6 | 2.87375 | 3.34119 | 1.79284 | 3.51875 | 3.77659 |
| 7 | 2.94445 | 3.40670 | 1.96717 | 3.57859 | 3.69163 |
| 8 | 3.00423 | 3.45804 | 1.78580 | 3.61369 | 3.74972 |
| 9 | 3.05548 | 3.49831 | 1.94497 | 3.63956 | 3.73078 |
| 10 | 3.09992 | 3.53005 | 1.99819 | 3.66407 | 3.94316 |
| 11 | 3.13878 | 3.55259 | 1.75237 | 3.68118 | 3.75056 |
| 12 | 3.17305 | 3.57561 | 2.02259 | 3.69704 | 3.77496 |
| 13 | 3.20348 | 3.59223 | 1.79177 | 3.70881 | 3.81436 |
| 14 | 3.23068 | 3.60604 | 1.95285 | 3.71884 | 3.74462 |
| 15 | 3.25715 | 3.61772 | 1.98142 | 3.72895 | 3.93427 |
| 16 | 3.27727 | 3.62774 | 1.75239 | 3.73636 | 3.73381 |
| 17 | 3.29738 | 3.63648 | 2.03111 | 3.74354 | 3.78351 |
| 18 | 3.31575 | 3.64418 | 1.81639 | 3.74935 | 3.84750 |
| 19 | 3.33261 | 3.65107 | 1.97352 | 3.75482 | 3.78992 |
| 20 | 3.34813 | 3.65727 | 1.93786 | 3.7603 | 3.91138 |
| 21 | 3.36249 | 3.66291 | 1.78748 | 3.76455 | 3.72534 |
| 22 | 3.37581 | 3.66807 | 2.02147 | 3.76876 | 3.80895 |
| 23 | 3.38821 | 3.67283 | 1.82478 | 3.77227 | 3.84625 |
| 24 | 3.39979 | 3.67724 | 2.00115 | 3.77583 | 3.82593 |
| 25 | 3.41062 | 3.68134 | 1.88660 | 3.77924 | 3.88775 |
| 26 | 3.42079 | 3.68518 | 1.83914 | 3.78205 | 3.72575 |
| 27 | 3.43036 | 3.68877 | 2.00637 | 3.7849 | 3.84551 |
| 28 | 3.43939 | 3.69215 | 1.82673 | 3.7873 | 3.83309 |

Table 4. Estimating the norm \(||C + C^{-1} + D + D^{-1}||\).
By making a least squares fitting of the numbers $\lambda_{\text{max}}^2(M_{18}), \ldots, \lambda_{\text{max}}^2(M_{28})$ to a function of the form $f(n) = a - b(n - c)^{-d}$ we get
\[
f(n) = 14.8 - \frac{18.5975}{(n + 0.2)^{1.1097}}.
\]
In particular, the extrapolation method predicts that
\[
||C + D + C^{-1} + D^{-1}|| \approx \sqrt{14.8} = 3.84,
\]
which is closer to 4.
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