An ecological approach to problems of Dark Energy, Dark Matter, MOND and Neutrinos

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Abstract. Modern astronomical data on galaxy and cosmological scales have revealed powerfully the existence of certain dark sectors of fundamental physics, i.e., existence of particles and fields outside the standard models and inaccessible by current experiments. Various approaches are taken to modify/extend the standard models. Generic theories introduce multiple de-coupled fields A, B, C, each responsible for the effects of DM (cold supersymmetric particles), DE (Dark Energy) effect, and MG (Modified Gravity) effect respectively. Some theories use adopt vanilla combinations like AB, BC, or CA, and assume A, B, C belong to decoupled sectors of physics. MOND-like MG and Cold DM are often taken as antagonising frameworks, e.g. in the muddled debate around the Bullet Cluster. Here we argue that these ad hoc divisions of sectors miss important clues from the data. The data actually suggest that the physics of all dark sectors is likely linked together by a self-interacting oscillating field, which governs a chameleon-like dark fluid, appearing as DM, DE and MG in different settings. It is timely to consider an interdisciplinary approach across all semantic boundaries of dark sectors, treating the dark stress as one identity, hence accounts for several “coincidences” naturally.

1. Fewer coincidences and dark sectors of the universe

Dark Matter and Dark Energy are the most fascinating astronomical puzzles presented to modern physics of particles and gravitation. To this date, the terms DM and DE are descriptive terms without a clear and unique underlying physics. Finding a home for DM and DE in the edifice of symmetry-based physics is a challenge. Although analogy is often drawn in the literature about DM, DE and Modified Gravity (MG) effects, the more fundamental links of these three effects have not been systematically shown. First many MG theories are DE in disguise where a special DE field is allowed to be non-uniform and non-minimally coupled to the metric. The modification term enters the Einstein’s equation on the left or right as $G^{ab} - M^{ab} = \frac{8\pi G}{c^4} T^{ab}_{\text{known}}, \rightarrow G^{ab} = \frac{8\pi G}{c^4} \left[T^{ab}_{\text{known}} + N^{ab}\right]$, where $T^{ab}_{\text{known}}$ is due to stress energy of known matter, e.g., neutrinos and baryons, $G^{ab}$ is due to the curvature tensor of the metric, and $M^{ab}$ is due to extra fields coupled to the metric, and can equivalently viewed as the stress energy $N^{ab} = \frac{c^4}{8\pi G} M^{ab}$ coming from New fields in the DE sectors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

Second the vector, spinor or scalar in MG and DE are special kinds of charge-neutral DM fields where DM particles are allowed to self-interact in pairs (e.g., annihilate or be created from vacuum), but are forbidden to interact with fields of baryons. This is because for MG and DE we have $\nabla_b M^{ab} = 0$ exactly for a Lagrangian construction for the DE or the coupling of MG field with the metric. But $\nabla_b N^{ab} \approx 0$ for DM because of a very small rate of exchange of energy...
when DM and known matter collide. So a DM field with a small cross-section to interact with itself and with the baryons is a more general description of MG and DE.

Third MG, DE, DM fields can be described as a dark fluid with a varying equation of state in the framework of General Relativity. The proper treatment involves relativistic hydrodynamics rather than CDM-like N-body simulations because DM particles are not point masses moving on geodesics, they are packets of energy or fluid which propagates according to the equation of motion of the spin field, and the generally non-trivial coupling to itself and to the metric deflects the path of DM particles from geodesics. This deflection appears as a fluid-like pressure in the stress tensor. Depending on the varying sound speed of the fluid and the Compton wavelength, the self-interaction can exhibit as short-range collisions by exchanging massive particles or long-range fifth force by exchanging low-energy particle [14, 15, 16, 17, 18] or both as in context of the lensing and velocity of the Bullet Clusters velocity [19, 20, 21]. The fluid is a mixture of DE, Cold, Warm and Hot DM etc states, and needs not be stable, can make transitions among these as the sound speed changes with position and redshift. The energy of the dark fluid is nearly conserved, and its cosmic abundance of the dark fluid is nearly fixed at decoupling with the radiations. In fact, a plausible candidate for the multi-phased dark fluid is the cosmic mixture of multi-flavored neutrinos and antineutrinos, which can make transition among at least several energy states, some very low in energy, some very high. Already the uncertain low-energy physics of neutrino mass and flavor makes a plausible case for DE in the mass-varying neutrino theory (Mass-Varying Neutrinos, MaVaNs, by [22, 23, 24, 25].

Fourth the self-interaction cross-section must be small and a running function. Many theories introducing a zero or a fixed cross-section for self-interaction are excluded by data on either small scales (e.g., Cold DM) or large scales (e.g., Collisional DM). Deviations from the ΛCDM model must be small and gradual to preserve its success on Large Scale. Previous multi-field models ignore the regularity of DM in galaxies and introduce more degrees of freedoms than justified by data. From the perspective of measuring weak lensing, dynamics or SNe distances, astronomers detect only the bending of the space-time by the stress energy of all dark fields summed together, yet we see a cosmic “coincidence” over 10^{30} order in length scale: everywhere a characteristic pressure exists \( T_{11} \sim T_{22} \sim T_{33} \sim \Lambda c^4/G \sim (\Delta m^2_\nu)^2 c^5/h^3 \). This pressure adapts to the environment satisfying a Tully-Fisher-Milgrom-McGaugh (TFMM) relation[26, 27, 28], where the cosmological constant \( \Lambda \) is related to Milgrom’s \( a_0 \sim \Lambda^{1/2} \sim \frac{\sigma_{DM}}{\sigma_{Mbaryon}} \). We expect this small pressure \( T_{11} \sim (100 \text{ km/s})^2 \times 10^{9} M_\odot/kpc^3 \) can prevent the formation of Cold DM cusps in galaxies. A possible home for MG and DE in fundamental physics might well be based on low-energy self-interactions of DM, especially what give neutrino mass difference \( \Delta m^2_\nu \sim (0.03eV)^2 \).

For each galaxy, especially each spiral galaxy, the radial distribution of DM mass \( M_{DM}(R) \) and baryons \( M_{baryon}(R) \) could have enjoyed a large scatter as typically seen in CDM simulations depending on formation history, but instead the two strictly follow the very tight rule of Tully-Fisher-McGaugh globally and the formula of Milgrom [26, 27, 28] at every observable radius without significant scatter. These rules hold tightly over many orders of magnitude scatter in the gas and stellar surface brightness and a wide range of formation history, including tidal dwarfs[33].

On bigger scale the universe is in a state of very delicate balance, with an extremely fine-tuned ratio of the stress energy coming from the DM and DE, neutrinos and baryons. These effects are all within about one order of magnitude of each other at the present instead of some 100 orders of magnitude apart if they come from truly independent physics. The way to reduce fine-tuning is to insist symmetry or restrict degrees of freedom. E.g., these come from a common field coupling to all sectors with a cross-section comparable to that of neutrino self-coupling.

In short we argue for a theory-independent approach to DM, DE and MG. We propose to parametrize all these effects and the effects of neutrinos in terms of a Chameleon-like “Dark Fluid” (CDF) with an approximately conserved stress energy tensor \( T_{\alpha\beta} \) plus GR. It is a more
Figure 1. An ecological view of the Cosmos Most-wanted Pizza (CMP), where different dark components (DE, Cold Dark Matter, Neutrino Hot Dark Matter) need not be separately conserved. They might be phases of one Dark Fluid. An analogy is drawn with the distribution of \(H_2O\) molecule on earth in phases of water-ice-vapor; any theory with a built-in conservation of mass of ice would fail since ice melts. A pure \(\Lambda\)CDM approach would fail to capture the possibility of non-conservation of (Cold) Dark Matter, and while an dynamical/ecological approach allows for the possibility of neutrino, CDM and DE to be phase change into each other, and explain the coincidence of scales in these three forms of non-baryonic energies in the universe.
systematic approach to DM interactions, since it combines the strength of previous methods using purely MOND modified gravity, or Warm DM, or neutrino or Fifth-Force. A natural way to extend ΛCDM is to build in the TFMM relation as the local minimum of the action of the self-interacting DM fields. If the DM-DE coupling injects an uneven tiny pressure of order $\Delta c^2/G \sim 10^9 M_\odot/kpc^3 \times (100 km/s)^2$ to the random motion pressure of Dark Matter in dwarf galaxies, this would suffice to remove the undesired DM cusps in dwarf galaxies.

The general inspirations of the models are illustrated by Figure. 1, where we seek to address the coincidence problems of DE, CDM/MOND and HDM (neutrinos) by one mechanism. In the following §2 and §3 we illustrate two possible approaches to unify the treatment of Dark Matter and Modified Gravity. §4 illustrates a mathematical equivalence of these two treatments by a metric redefinition.

2. Neutrino approach
What gives mass to active neutrinos is an open question, and it is not even clear whether the mass is constant or not. Interaction with Higgs-like field in other sectors or perhaps self-interactions in the neutrino sector (perhaps with sterile neutrinos) give an effective mass $m(z, x)$ to the neutrinos observed (chirally left-handed). This argues that the active neutrinos form a self-collisional fluid with free-streaming reduced by the pressure. Since the exact mechanism of interactions are unknown, one could assume most generally that the effective mass could depend on environment.

The original MaV aNs corresponds to a neutrino fluid of pressure $P = -f(n)$ so that the neutrinos can be modeled as a nearly polytropic fluid with an imaginary sound speed in an expanding universe. More generally the neutrino pressure $P$ is likely some function of a scalar field and its temporal and spatial gradient, $P = nE_0 \beta^2$, where $\beta = \beta(n, \dot{n}/n, |\nabla \Phi|)$ depends on the density $n$ and its rate of change and spatial gradient $(n^{-1} \nabla n)$, and $E_0$ is a constant energy scale. Unlike MaV aNs, here we assume that certain species of neutrinos have zero mass in the absence of perturbations, and the effects of mass appear only in collapsed systems, like galaxies and the solar system. We assume that the cosmic fluid of neutrinos has a pressure $P$ due to interaction of a current vector $nu^\alpha$ of neutrinos and a current vector $\bar{n}\bar{u}^\alpha$ of anti-neutrinos. These two generally do not follow the same equations of motion due to their opposite spin-gravity coupling. It is possible for these two currents to interact (creating a pressure) in a way reminiscent of how positive and negative charged particles interact in a plasma to keep average charge-neutrality on scales larger than the spacing of particles. One expects $\bar{n} \sim n$, and a coupling like $\bar{u}^\alpha u^\beta R_{\alpha\beta}$ eventually reduces to terms like $\nabla_\alpha \bar{u}^\beta \nabla_\beta u^\alpha$, which in galaxies reduces to terms like $|\nabla \Phi|^2$.

To be specific we define $\beta$ to be the sound speed the relativistic neutrino fluid in units of the speed of light $c$, and we adopt a variable sound speed such that

$$\sqrt{\frac{nE_0}{P}} \frac{1}{\beta} = 1 + \sqrt{\frac{nE_0}{P_g}}, \quad P_g = \frac{|\nabla \Phi|^2}{8\pi G}, \quad \frac{A^2}{8\pi G} \equiv nE_0, \quad (1)$$

where $E_0$ is certain fixed low-energy threshold in the neutrino sector respectively, whose microphysics is not specified in detail, but could be thought as certain energy gap in neutrinos; note $\beta = 0$ for the uniform universe with $\nabla \Phi = 0$. The quantity $1/\beta$ carries the meaning of the ratio of gravitational wave speed $c$ vs the sound speed $c\beta$ of the neutrino fluid, or the “refraction index” of the neutrino fluid, which is assumed to be function of the ratio of the gravitational stress

$$P_g = \frac{|\nabla \Phi|^2}{8\pi G} \quad (2)$$
vs a characteristic stress $nE_0 \equiv \frac{nE_0}{\rho_b}$ of the neutrino fluid. One can motivate such a correction by saying quantum effects must be considered to model interactions of the metric with a spin $1/2$ neutrino fluid in a box of deBroglie wavelength $1/n \sim 1/n_0 \sim 5\text{mm}^3$; when the gravitational energy inside this box is just below or above certain threshold $E_0$, one might expect that anti-particles can disappear or appear (reminiscent of the famous Klein paradox), hence can change the energy density of neutrinos and anti-neutrinos, hence the sound speed of the neutrino fluid.

Interpreting in the picture of the dielectric analogy of MOND \cite{30, 31} one would say that the neutrino medium is “polarized” by a variable amount due to a varying gravitational stress. The adopted refraction index is such that the perturbations in the stress of the neutrino fluid propagate as the speed of light (as in vacuum) in strong gravity, but much slower in weak gravity; no propagation or pressure in Minkowski space, where $\nabla \Phi = 0$. In the solar system, where $\nabla \Phi$ is much bigger than $A$, we have $\beta = 1$.

We consider a classical Newtonian theory where one minimizes the action $S = \int dt L$ given by

$$S = \int dt \int dx^3 \mathcal{L}, \quad \mathcal{L} = \rho_b \Phi + P_g - P, \quad P = (nE_0)\beta^2,$$

where the relativistic pressure of neutrino fluid is included by a term $nE_0\beta^2$, and $P_g + \rho_b \Phi$ is the Lagrangian of gravitational field and the baryons of density field $\rho_b$, which couples to gravitational potential $\Phi$. Here we use the non-covariant formulation of gravity as an external field, which is the weak perturbation limit of General Relativity. To be clear, $R$ is the scale factor, so that in the Lagrangian $L$ we are free to add or drop terms like $\text{cst} \int dx^3 R^3 \rho_b$ and $\text{cst} \int dx^3 R^3 n$, which are the total number of baryons or neutrinos respectively; the $\text{cst}$ here plays the role of the chemical potential.

The total Lagrangian now resembles the classical MOND of Bekenstein-Milgrom \cite{32}

$$\mathcal{L} = \frac{A^2}{8\pi G} (y - F(y)) + \rho_b \Phi,$$

where

$$F(y) = \frac{y}{(\sqrt{y} + 1)^2}, \quad y \equiv \frac{\nabla \Phi^2}{A^2}.$$

Apply the Lagrangian equation by varying the total Lagrangian with respect to $\Phi$, we obtain the MOND-like Poisson equation

$$2\nabla [\mu \nabla \Phi] = 8\pi G \rho_b,$$

where

$$\mu = 1 - dF/dy = 1 - (1 + \sqrt{y})^{-3} = 1 - (1 + \frac{\nabla \Phi}{3a_0})^{-3}.$$

It is interesting that a relativistic neutrino fluid with a non-trivial pressure can give the physics of MOND. To match with MOND acceleration, we can set

$$A = 3a_0 = 3.6 \times 10^{-10} \text{m/s}^2.$$

The resulting $\mu$ function here matches that of Zhao (2007), which are shown to be compatible with solar system data and spiral galaxy rotation curves, especially the Tully-Fisher-Milgrom relations. The MOND acceleration scale would corresponds to a mass or energy scale in the neutrino sector

$$E_0 = \frac{3a_0^2}{8\pi G n} \sim 2\text{eV}, n = 200\text{cm}^{-3}.$$

Note that any eV range neutrinos with a normal non-relativistic pressure is insignificant to galaxy potential. The non-linear coupling of the neutrino pressure to the metric through some unspecified quantum effects is the key. In our picture neutrinos are not localized, this can create the effects of a ubiquitous dark matter fluid or modified gravity.
2.1. Relations of neutrinos, Higgs and vector field

The above example belongs to a more general class of variable mass theory where one minimizes the action \( S = \int dt \int dx^3 \sqrt{-g} \mathcal{L} \), \( \mathcal{L} = \sum_f n_f m_f - M_p^2 R \), where we use the natural units \( \hbar = 1 \) and \( c = 1 \), and \( M_p^2 = \frac{1}{8\pi G} \) is the Planck mass, \( R \) is the Ricci scalar for the metric \( g_{ab} \), and \( n_f = \bar{\Psi} \Psi \) is the proper number density of the fermion of species \( f \) and

\[
m_f \equiv \sqrt{g_{ab} f^a f^b}
\]

is the fermion’s effective mass, which is determined by a vector field \( f^a \) by

\[
f^a \equiv s_f(\theta) m_0 u^a + cf(\theta) \dot{u}^a, \quad \dot{u}^a \equiv u^b \nabla_b u^a,
\]

where \( \theta \) is an auxiliary field, and \( u^a \) is a vector field of unit-norm in a metric \( \tilde{g}_{ab} \), defined by

\[
\tilde{g}_{ab} u^a u^b = 1, \quad \tilde{g}_{ab} \equiv g_{ab} N^{-2}, \quad N \equiv \frac{h}{m_0}
\]

i.e., \( p^a = m_0 u^a \) resembles the four-momentum of a particle of mass \( m_0 \) in the metric \( \tilde{g}_{ab} \), which is related to the metric \( g_{ab} \) by a conformal transformation factor \( N^2 \gg 1 \).

To see the relation to the previous neutrino model, we can set \( N^2 = \text{cst} \) in present day galaxies, and set \( s_f = \text{cst} = O(1) \), and \( c_f = 0 \) for baryons so that baryons have a fixed mass of order \( N m_0 \) and go on geodesics of \( g_{ab} \), and so that \( p^a = N m_0 g_{00}^{-1/2}(1,0,0,0) \) has a fixed norm in galaxies, where \( g_{00} = 1 + 2\Phi \) with \( \Phi \) being the quasi-static gravitational potential. Also set \( s_f = \frac{1}{3N} (1 - \varphi)^3 \ll O(1/N) \) and \( c_f = \varphi \equiv \cos \theta \) for neutrinos so that neutrinos has a varying mass of order \( m_0 \). We can minimize the action against the non-dynamical field \( \theta \) to find

\[
2(c_f^{-1} - 2 + c_f) = (N m_0)^{-2} \dot{u}^a u^b g_{ab} \approx y \equiv \left( \frac{\nabla \Phi}{A} \right)^2, \quad A \equiv m_0 / N
\]

in galaxies. This resembles the MOND-like equation in [5]. So \( c_f \) and the auxiliary field \( \theta \) and the neutrino effective mass \( m_f = m_0 \sqrt{F(y)} \) all track \( y \) or the potential \( \Phi \) with a characteristic scale \( A = m_0 / N \). Minimizing against the metric \( g_{00} \) or the potential \( \Phi \), we find

\[
\nabla \cdot (\mu \nabla \Phi) = 4\pi G \sum_f (n_f m_f), \quad \mu = 1 - \alpha \frac{d\sqrt{F}}{dy}
\]

where \( \alpha \equiv \frac{N^2 m_0}{M_p^2 m_0^2} = \frac{n_f m_f}{A^2 M_p^2} \). These models have properties in between that of classical MOND, MaVaNs and Dark Matter.

In Higgs-doublet models, \( h \) and \( \theta \) are dynamical fields with a Lagrangian made of following kinetic and potential terms

\[
\mathcal{L}_h = \sum_f (s_f h) n_f + g^{ab} \nabla_a h \nabla_b h + h^2 g^{ab} \theta \nabla_b \theta + V(h)
\]

where a potential \( V(h) \sim \text{cst}(h^2 n_f m_0 M_p^2 - m_0^2)^2 \) has a a minimum, can keep \( \alpha = 1 \). The Higgs field \( h \) gives mass to baryons and neutrinos through the term \( s_f(\theta) \), but in our case the symmetry of the Higgs doublet along the phase angle \( 0 \leq \theta < 2\pi \) is broken by the term \( c_f(\theta) \dot{u}^a \) spontaneously except in a Minkowski space where \( \dot{u}^a = 0 \). The scale \( h = N m_0 \) and the auxiliary field \( \theta \) can take the meaning of the norm and the phase angle of a complex vector field \( Z^a = m_0 u^a \exp(i\theta) \) and \( \tilde{Z}^a = m_0 u^a \exp(-i\theta) \) with a (dynamical) norm \( Z^a \tilde{Z}^b g_{ab} = h^2 \), as discussed in dark fluid models [5].
3. The vector approach: a Simple Lagrangian for MOND-like Dark Energy

Here we illustrate how the roles of both DM and DE could be replaced by a vector field in a modified metric theory. This follows from merging two long lines of investigations pursued by Kostelecky, Jacobson, Lim and others on consequences of symmetry-breaking in string theory, and by Milgrom, Bekenstein, Sanders, Skordis and others driven by astronomical needs [1, 7, 39].

In Einstein’s theory of gravity, the slightly bent metrics for a galaxy in an uniform expanding background set by the flat FRW cosmology is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -(1 + \frac{2\Phi}{c^2})d(ct)^2 + (1 - \frac{2\Psi}{c^2})a(t)^2dl^2$$  \hspace{1cm} (17)

where $dl^2 = (dx^2 + dy^2 + dz^2)$ is the Euclidian distance in cartesian coordinates. In the collapsed region of galaxies, the metric is quasi-static with the potential $\Phi(t, x, y, z) = \Psi(t, x, y, z)$ due to DM plus baryon, which all follow the geodesics of $g_{\mu\nu}$.

Modified gravity theories are often inspired to preserve the Weak Equivalence Principle, i.e., particles or small objects still go on geodesics of above physical metric independent of their chemical composition. Unlike in Einstein’s theory, the Strong Equivalence Principle and CPT can be violated by, e.g., creating a preferred frame using a vector field by, e.g., a unit time-like vector field $U^\mu$ which is designed to couple only to the metric but not matter directly. It doesn’t violate spatial rotation symmetry since it is time-like. It has a kinetic Lagrangian with linear superposition of quadratic co-variant derivatives $\nabla(c^2U)\nabla(c^2U)$, where $c^2U^\mu$ is constrained to be a time-like four-momentum vector per unit mass by $-g_{\mu\nu}U^\mu U^\nu = 1$. The norm condition means the vector field introduces up to 3 new degrees of freedom; e.g., a perturbation in the FRW metric (Eq.17) has $c^2U^\mu \equiv g_{\mu\nu}c^2U^\nu \approx (c^2 + \Phi, \frac{A_\alpha}{c}, \frac{A_\beta}{c}, \frac{A_\gamma}{c})$, containing a four-vector made of an electric-like potential $\Phi$ and three new magnetic-like potentials. But for spin-0 mode perturbations with a wavenumber vector $k$, we can approximate $U^\mu - (1, 0) \approx (\frac{\Phi}{c^2}, \frac{kU}{c^2})$, which contains just one degree of freedom, i.e., the flow potential $V(t, x, y, z)$. We expect an initial fluctuation of $c|kV| \sim |\Phi| \sim c^2N^{-1} \equiv 10^{-5}c^2$ can be sourced by a standard inflaton; the vector field tracks the spectrum of metric perturbation.

When it comes to writing down a specific Lagrangian of the vector field, simplicity is the guide since GR plus simple $\Lambda$CDM largely works. Let’s start with forming two pressure terms for any four-momentum-like field $A^\mu$ with a positive norm $mc^2 \equiv \sqrt{-g_{\alpha\beta}A^\alpha A^\beta}$ by

$$8\pi G\mathcal{J}(A) \equiv \frac{1}{3} \left( \nabla_\alpha A^\alpha \right)^2, \hspace{1cm} 8\pi G\mathcal{K}(A) \equiv \frac{\nabla_\parallel A^\alpha}{m} \nabla_\parallel A^\alpha \hspace{1cm} (18)$$

where the RHSs are co-variant with dimension of acceleration squared, and $\nabla_\parallel = A^\alpha \nabla_\alpha$ or $\nabla_\alpha$ stands for the co-variant derivative with space-time coordinates along the direction of the vector $A$ or the dummy index $\alpha$ respectively. From these we can generate two simpler pressure terms $K$ and $J$ of the unit vector field $U^\alpha$ by

$$J \equiv \mathcal{J}(U) \sim 0, \hspace{1cm} K \equiv \mathcal{K}(U) \sim \frac{c^2H^2}{8\pi G} \hspace{1cm} \text{in galaxies}$$

$$\sim \frac{c^2H^2}{8\pi G}, \hspace{1cm} \text{in flat universe}$$

where the approximations hold for $U^\alpha$ with negligible spatial components and nearly flat metric (Eq.17). Note the $J$ and $K$ are constructed so that we can control time-like Hubble expansion and space-like galaxy dynamics separately. The $K$-term, with a characteristic pressure scale $\frac{a_0^2}{3c^2} = P_0$ in galaxies, is the key for our model. The $J$-term, meaning critical density, has a characteristic scale $N^2P_0 \sim 10^10P_0$: at the epoch of recombination $z = 1000$ when baryons, neutrinos, and photons contribute $\sim (8, 3, 5) \times 10^9P_0$ respectively to the term $J = \frac{3c^2H^2}{8\pi G}$; so the epochs of equality and recombination nearly coincide.
Now we are ready to construct our total action $S = \int d^4x \sqrt{-g} \mathcal{L}$ in physical coordinates, where the Lagrangian density
\[
\mathcal{L} = \frac{R}{16\pi G} + L_m + L_J + L_K + (U^r U^r + 1)L^m,
\]
where $R$ is the Ricci scalar, $L_m$ is the ordinary matter Lagrangian. For the vector field part, $L^m$ is the Lagrangian multiplier for the unit norm and we propose the new Lagrangian
\[
L_J = \int_0^J dJ\lambda(x)|_{x=\sqrt{\frac{4\pi}{m}}}, \quad L_K = \int_\infty^K dK\lambda(x)|_{x=\sqrt{\frac{4\pi}{m}}}
\]
where the non-negative continuous functions $\lambda(x) = (1 + \frac{\mu}{a})^{-3} - 0$, and $\lambda\to 1 - (1 + \infty/\infty)^{-3} = 1 - \mu_B$, where $\mu_B \equiv 2^{-3} = 1/8$. A more fine-tuned parametrization is given in Zhao (2007), which passes the BBN constraints better.

Taking variations of the action with respect to the metric and the vector field, we can derive the modified Einstein’s equation (EE) and the dynamical equation for the vector field. The expressions are generally tedious, but the results simplifies in the perturbation and matter-dominated regime that interest us. As anticipated in [40] the $ij$-cross-term of EE yields $\Psi - \Phi = 0$ for all our models, which means incidentally twice as much deflection for light rays as in Newtonian. The tt-equation of Einstein reduces to the simple form
\[
4\pi G\mu = -\nabla^2\Phi - \nabla \cdot \left[ \lambda_n \frac{\nabla\Phi}{a_0} \right] \nabla\Phi, \text{ in galaxies}
\]
and
\[
\frac{8\pi G\bar{\rho}}{3\mu_B} = H^2 - \frac{\Lambda_0}{3\mu_B}, \text{ in matter-dominated FRW}
\]
Here the pressure from the vector field creates new sources for the curvature. The term $\frac{\nabla(a_0 \nabla\Phi)}{4\pi G}$ in the Poisson equation acts as if adding DM for quasi-static galaxies. A cosmological constant in the Hubble equation is created by
\[
\frac{\Lambda_0c^2}{8\pi G} = -\int_0^\infty dP_0 x^2 \approx (3P_0)^2
\]
For binary stars and the solar system, $4\pi G\mu - \nabla^2\Phi \approx 0$ is true because the gravity at distances 0.3AU to 30AU from a Sun-like star is much greater than the maximum vector field gradient strength $a_0$, so $\frac{dL_K}{dK} = 0$; in fact, $|\nabla\Phi| \approx \frac{GM\mu}{r^2} \sim (10^{-9} - 10^{-5})a_0$, and the typical anomalous acceleration is $\sim 10^{-10}a_0$, well-below the current detection limit of $10^{-4}a_0$ (Soreno & Jezter 2006). This might explain why most tests of non-GR effects around binary pulsars, black holes and in the solar system yield negative results; Pluto at 40 AU and the Pioneer satellites at 100 AU might show interesting effects. Extrapolating the analysis of [37], we expect GR-like PPN parameters and gravitational wave speeds in the inner solar system.

Near the edges of galaxies, we recover the non-relativistic theory of Bekenstein & Milgrom [32] with a function
\[
\mu(x) \equiv 1 - \lambda_n(x) \sim \mu_{\text{min}} + x, \text{ if } x = \frac{\nabla\Phi}{a_0} \ll 1.
\]
Note that $\mu(x) \to x$ hence rotation curves are asymptotically flat except for a negligible correction $\mu_{\text{min}} \sim 10^{-15}$. In the intermediate regime $x = 1$ our function with $1 - \lambda_n(x) \sim (0.55 - 0.6)$ for $n = 2 - 5$ respectively. Galaxy rotation curves prefer a relatively sharper transition than $\mu(x) = x/(1 + x) = 0.5$ at $x = 1$ [31] where we can identify $g_B/(g_{DM} + g_B) = \mu(x)$. So our model should fit observed rotation curves.
For the Hubble expansion: the vector field creates cosmological constant-like term \( \frac{\Lambda_0 c^2}{8\pi G} \approx 9P_0 \) below the zero-point of the energy density in the solar system because the zero point of our Lagrangian (Eq.21) is chosen at \( N^2 P_0 \leq K < +\infty \). During matter domination, the contribution of matter \( 8\pi G\rho \) and \( \Lambda_0 \) to the Hubble expansion \( H^2 \) (Eq.(23)) is further scaled-up because the effective Gravitational Constant \( G_{eff} = G/\mu_B = 8G \geq G \). Coming back to the original issue of the 3 : 1 ratio of matter density to our cosmological constant, Eq.(23) predicts that \( \frac{\Lambda_0 c^2}{8\pi G\mu_B} \approx \frac{9P_0}{\mu_B} = \frac{4(1+z)^3 P_0}{2\mu_B} \), which is close to the desired 3 : (1 + z)^3 ratio. Adding neutrinos makes the explanation slightly poorer. So the DE scale is traced back to a separate coincidence of scale, i.e., the present baryon energy density \( \bar{\rho}_b c^2 \sim 4P_0 \), where \( P_0 \) contains a scale \( a_0 \) for the anomalous accelerations on galactic scale. This model predicts that DE is due to a constant of vacuum.

In our model, the effective DM (the dog) follows the baryons (the tail) throughout the universal \((1 + z)^3\) expansion with a ratio set by the parameter \( \mu_B \). To fit the \( \Lambda CDM \)-like expansion exactly, we note the Hubble equation for a flat FRW cosmology with vector field and standard mix of baryons, neutrinos and photons \( \rho_b h^2 \approx 0.02 \), \( \rho_\nu h^2 \approx 0.002 \cdot m_\nu \approx \frac{\rho_{ph} h^2}{0.000025} \sim 1 \) yields at the present epoch

\[
\frac{\Omega_b + \Omega_\nu + \Omega_{ph}}{\mu_B} = 1 - \frac{\Lambda_0}{3\mu_B H_0^2} = \Omega_m^{\Lambda CDM}
\]

The 2nd equality fixes \( \mu_B^{-1} = (8 - 8.4) \) if we adopt \( a_0/c \approx H_0/6 \approx 12\text{km/s/Mpc} \) and \( \Omega_m^{\Lambda CDM} = (0.25 - 0.3) \). The 1st equality would predict an uncertain but very small neutrino mass \( m_\nu \sim 0.3\text{eV} \), consistent with zero. So the role of neutrinos in uniform expansion can be completely replaced by the vector field.

4. A unified framework for Dark Matter, Dark Energy and Modified Gravity

Finally we illustrate the relation between Modified Gravity and Interacting Dark Matter.

4.1. Einsteinian gravity with an interacting Dark Matter field

Let’s consider Einsteinian gravity but with the normal matter (of standard model of particle physics) being coupled to the field of a dark matter particle. Let the dark particle be spin-1, hence it is described by a vector field, hence with 4 degrees of freedom. Unlike the spin-1 photon field, which is a massless gauge-invariant vector field with zero expectation value in vacuum, the dark vector field is given a unit norm, hence it is a massive field and has a non-zero expectation value in vacuum. The vector field is given self-coupling and coupling to normal matter, which break the gauge-invariance.

The gravity sector is now simply described by a metric \( g_{ab} \), with a sign convention \((+,-,-,-)\) and its associated Ricci scalar \( R \), hence the Einstein-Hilbert action plus matter, \( \mathcal{S} = S_g + S_m = \int dx^4 \sqrt{-g} \left[ \frac{R}{16\pi G} + L_m \right] \), where \( L_m = L_A + L_{int} + L_J \) (27)

where matter is consisted of the Lagrangian density \( L_J \) for a pressure-less matter fluid with matter flux vector \( J^a = \rho u^a \), and \( L_A \) for the dark matter vector field \( A^a \), and an interaction term of the two vector fields. Specifically

\[
L_J = \sqrt{g_{ab} J^a J^b - \phi \nabla_a J^a},
\]

where \( \phi \) is a Lagrangian multiplier field for the conservation of matter flux \( J^a = \rho u^a \) of a collisionless dust of density \( \rho \) and four-velocity \( u^a \). It interacts with the vector field \( A^a \) via

\[
L_{int} = C^2 \sqrt{BC} \sqrt{g_{ab} J^a J^b g_{cd} A^c A^d - (1 - B)(g_{ab} J^a A^b)^2 - \sqrt{g_{ab} J^a J^b}}
\]

(29)
where $B$ and $C$ are coupling constants. The vector field contributes via

$$L_A = \frac{m^2}{2} F_{ab} F^{ab} + (1 - g_{ab} A^a A^b) \lambda$$

which consists of a kinetic photon-like term from Faraday tensor $F_{ab}$, and a massive potential in the form of a unit norm constraint for $A^a$ and a Lagrangian multiplier $\lambda$. And to make the argument simpler we assume the mass $m$ and $\lambda \propto m^4$ is very small, $\beta = 16\pi G m^2 \ll 1$, so that we can neglect the dark matter term $L_{\text{DM}}$, keeping only the interaction term $L_{\text{int}}$.

Note that our Lagrangian $L_J$ depends on $g_{ab}$ in a non-linear fashion. However, non-linearity is not a sufficient condition for modified gravity. It can be shown that the stress tensor associated with $L_J$ is $T_{ab} = \frac{\delta(L_J, \sqrt{g})}{\sqrt{g} g^{mn}} = J_a J_b (g^{ab} J_a J_b)^{1/2} = \rho u^a u^b$, as expected for a collisionless dust.

Note that we cannot observe the dark matter $A^a$ field directly, it is observed through its interaction with baryonic dust $J^a$. Here we considered only a species of dust. If we generalize for the interaction/coupling to be the same for all species of the baryonic dust, then we cannot detect the dark matter through differential measurements of the baryons (the strong equivalence principle). However, the dark matter needs not track the baryonic dust exactly, so we don’t expect the baryonic mass center to coincide with its kinematic (gravitational) center (Kesden & Kamionkowski 2006).

### 4.2. “Modified” gravity in redefined metric

Alternatively one can redefine the metric

$$\tilde{g}_{ab} = (g_{ab} - (1 - B) A_a A_b) C [31]$$

The nice feature is that the matter action is now simplified to that of a pure matter field $J^a$, with a new Lagrangian density

$$\tilde{L}_J = \left[ \sqrt{\tilde{g}_{ab} J_a J^b} - \phi \nabla_a J^a \right]$$

which is completely decoupled from the vector field $A^a$. However, the new Ricci scalar, formed out of second derivatives of the new metric, differs from the old Ricci scalar by a $K$-term, so that the gravity is “modified” with an effective gravitational constant $\tilde{G}$ given by

$$\frac{\tilde{G}}{G} = \frac{\sqrt{-\tilde{g}}}{C \sqrt{-g}} = \sqrt{BC}. [34]$$

Rewriting the action $S$ in the new metric, we find an action resembling that of a “modified” gravity, specifically

$$S = \tilde{S}_g + \tilde{S}_J = - \int d^4 x \sqrt{-g} \left[ \tilde{R} + \tilde{K} \right] - \int d^4 x \sqrt{-\tilde{g}} \tilde{L}_J,$$ [35]

where

$$\tilde{K} = \left[ K^{ab}_{mn} \nabla_a A^m \nabla_b A^n \right] + (\tilde{g}_{ab} A^a A^b - 1) \lambda,$$ [36]

$$A^a \equiv A^a / \sqrt{BC}$$ [37]

$$K^{ab}_{cd} \equiv (c_1' \tilde{g}_{ab} + c_4' A^a A^b) \delta_{mn} + (c_2' \delta_{m}^a \delta_{n}^b + c_3' \delta_{m}^a \delta_{n}^b)$$ [38]
where $\lambda$ is a Lagrangian multiplier field,

$$
c'_1 = -c'_4 = -\frac{1 - B^{-1}}{2} c'_2 + B \beta, \quad -c'_3 - c'_4 = c'_2 = B - 1. \tag{39}
$$

where we have considered more generally when the mass $m$ of the vector field is not small, the c-parameters are functions of $\beta = 16\pi Gm^2$ as well [36].

The redefined action is that of a special case of the Einstein-Aether modified gravity, where there is no interaction between normal matter and the vector field (called aether). Assuming $A^a$, $J^a$ and $\tilde{g}^{ab}$ as independent freedoms as conventionally done, the Einstein equations are obtained; taking its trace we have

$$
2\tilde{R} - \tilde{g}^{ab}T_{ab}^A = 8\pi \tilde{G} \tilde{T}^J, \quad \tilde{T}^J = \frac{\tilde{g}^{ab}J_aJ_b}{\sqrt{\tilde{g}_{ab}J^aJ^b}}. \tag{40}
$$

where $T^J$ is the trace of the stress tensor of the collisionless dust, and $\tilde{R}$ is the Ricci scalar, $-T_{ab}^A$ is the part of Einstein tensor involving a fairly lengthy expression of second derivatives of the vector field $A^a$ and the metric $\tilde{g}$. For the uniform expansion of the universe in co-moving coordinates $(t, x, y, z)$, the metric is given by

$$
\tilde{g}_{ab}dx^a dx^b = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \tag{41}
$$

the matter current $J^a = (\rho, 0, 0, 0)$ and $A^a = (1, 0, 0, 0)$ with a preferred time-like direction. The Hubble equation is unchanged,

$$
3 \left( \frac{da(t)}{a(t)dt} \right)^2 = 8\pi \frac{G \rho}{\mu_C}, \quad \frac{1}{\mu_C} = \frac{\tilde{G}/G}{1 + (c_1 + 3c_2 + c_3)/2} = \frac{C}{\sqrt{B}}. \tag{42}
$$

except for a correction $\mu_C$ of the effective gravitational constant. For weakly perturbed metric near a static galaxy or the solar system with

$$
\tilde{g}_{ab}dx^a dx^b = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2), \tag{43}
$$

the matter current $J^a = (\rho, 0, 0, 0)$, and the vector field $A^a = (1/\sqrt{\tilde{g}_{ab}}, 0, 0, 0)$ with a preferred time-like direction, and the Poisson equation is changed to

$$
\nabla^2(2\mu_N \Phi) = 8\pi G\rho, \quad \frac{1}{\mu_N} = \frac{\tilde{G}/G}{1 - (c_1 + c_4)/2} = C\sqrt{B}. \tag{44}
$$

with a factor $\mu_N$ correction of the effective gravitational constant.

One can create appropriate amount of dark matter-like effects by selecting the factors $\mu_N$ and $\mu_C$ in solar system, in galaxies and in matter-dominated cosmology. The above arguments can be generalized to the case where $\mu_N$ and $\mu_C$ are scalar fields, which varies with redshift and position to resemble the $V - \Lambda$ model and $F(K)$ models of co-variant MOND [3] and various modified gravities. E.g., the modified source gravity of Carroll et al. can be recovered by models with $\mu_C = \mu_B = C$ being non-dynamical scalar fields tracking the Ricci scalar through an added potential term $V(C)$ in the interaction Lagrangian.

In short, the above example suggests that dark matter and modified gravity can be the two view points of the same phenomena. Our vector field can be viewed as spin-1 dark matter field in Einsteinian gravity and $g_{ab}$ and this dark matter field $A^a$ is co-variantly coupled to the luminous matter current field $J^a$. However, this vector field is not exactly the cold dark matter, and in fact it does not condense in galaxies in our model, but it does decelerates the expansion.
of the universe, and contributes to the critical density. Alternatively, one can view our vector field as a field modifying the gravity sector of metric $\tilde{g}$. In this case, the luminous matter is decoupled from the vector field $A^a$.

In short the coincidences of scales of DM, DE and Neutrinos are intriguing. We advocate that it is theoretically satisfying to find a unified solution to these problems at a fundamental level.

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