ON BANACH-MAZUR DISTANCE BETWEEN PLANAR CONVEX BODIES

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Abstract. Upper estimates of the diameter and the radius of the family of planar convex bodies with respect to the Banach-Mazur distance are obtained. Namely, it is shown that the diameter does not exceed $\frac{19 - \sqrt{73}}{4} \approx 2.614$, which improves the previously known bound of 3, and that the radius does not exceed $\frac{117}{70} \approx 1.671$.

1. Introduction

Let $C^n$ be the family of all convex bodies (convex compact sets with non-empty interior) in Euclidean space $\mathbb{R}^n$, and $M^n$ be the subfamily of all centrally symmetric convex bodies (representing unit balls in $n$-dimensional real Banach spaces). For $\mathcal{A}, \mathcal{B} \in C^n$, the Banach-Mazur distance between $\mathcal{A}$ and $\mathcal{B}$ is

$$d(\mathcal{A}, \mathcal{B}) = \inf_{T, h_\lambda}\{\lambda : T(\mathcal{A}) \subset \mathcal{B} \subset h_\lambda(T(\mathcal{A}))\},$$

where $T$ is an affine transform and $h_\lambda$ is a homothety with ratio $\lambda > 0$. For two families $\mathcal{A}, \mathcal{B} \subset C^n$, we extend the notation by setting $d(\mathcal{A}, \mathcal{B}) := \sup\{d(\mathcal{A}, B) : \mathcal{A} \in \mathcal{A}, \mathcal{B} \in \mathcal{B}\}$, and also let $d(\mathcal{A}, \mathcal{B}) := d(\{\mathcal{A}\}, \mathcal{B})$ for $\mathcal{A} \in C^n$. For $\mathcal{A} \subset C^n$, the Banach-Mazur diameter and radius of $\mathcal{A}$ are defined as $\text{diam}(\mathcal{A}) := d(\mathcal{A}, \mathcal{A})$ and $\text{rad}(\mathcal{A}) := \inf\{d(\mathcal{A}, \mathcal{A}) : \mathcal{A} \in \mathcal{A}\}$, respectively.

Perhaps one of the most well-known results implying estimates on the Banach-Mazur distance is the theorem by John [J] characterizing the ellipsoid of largest volume inscribed into a body from $M^n$ or from $C^n$. As a consequence, if $B^n$ is the unit ball in $\mathbb{R}^n$, then $d(B^n, M^n) = \sqrt{n}$ and $d(B^n, C^n) = n$, where examples of bodies with largest distance are a cube and a simplex, respectively. Hence, $\text{diam}(M^n) \leq n$, which is asymptotically sharp due to the bound $\text{diam}(M^n) \geq Cn$ established by Gluskin [G] (here and below $C$ denotes positive absolute constants). For the non-symmetric case, John’s theorem implies $\text{diam}(C^n) \leq n^2$, but this can be improved significantly as Rudelson [R] established $\text{diam}(C^n) \leq Cn^{4/3}(\ln(n + 1))^{9}$. It is still
an open question to find the asymptotic behavior of diam($\mathcal{C}^n$). Regarding the radii of $\mathcal{M}^n$ and of $\mathcal{C}^n$, John’s theorem implies $\text{rad}(\mathcal{M}^n) \leq \sqrt{n}$ and $\text{rad}(\mathcal{C}^n) \leq n$. While $\text{rad}(\mathcal{M}^n) \geq C\sqrt{n}$ due to Gluskin’s result, we do not seem to know much about lower bounds on $\text{rad}(\mathcal{C}^n)$ except for $\text{rad}(\mathcal{C}^n) \geq \sqrt{n}$ trivially obtained by considering a ball and a simplex. A generalization of John’s theorem obtained in [GLMP] implies $d(\mathcal{M}^n, \mathcal{C}^n) = n$, so one can take any centrally symmetric body as a “center” to show that $\text{rad}(\mathcal{C}^n) \leq n$.

For the planar case $n = 2$, the Banach-Mazur distance between a square and a regular hexagon is $\frac{3}{2}$. It was shown by Stromquist [S] that $\text{rad}(\mathcal{M}^2) = \sqrt{\frac{3}{2}}$ and a “central” body was explicitly constructed, so, consequently, $\text{diam}(\mathcal{M}^2) = \frac{3}{2}$. Similarly to the asymptotic situation, the non-symmetric planar case appears to be a more challenging question. The bound $\text{diam}(\mathcal{C}^2) \leq 4$ implied by the John’s theorem was improved to $\text{diam}(\mathcal{C}^2) \leq 3$ by Lassak in [L]. For the lower bound in the general case, the Banach-Mazur distance between a regular pentagon and a triangle is $1 + \sqrt{\frac{5}{2}} \approx 2.118$, see [FMR+] and [L2], so, in summary, $1 + \sqrt{\frac{5}{2}} \leq \text{diam}(\mathcal{C}^2) \leq 3$. It is believed that $\text{diam}(\mathcal{C}^2) = 1 + \sqrt{\frac{5}{2}}$. We improve the upper bound by proving the following:

**Theorem 1.1.** $\text{diam}(\mathcal{C}^2) \leq \frac{19 - \sqrt{73}}{4} < 2.614$.

The main geometric argument used in [L] to show that $\text{diam}(\mathcal{C}^2) \leq 3$ is due to Besicovitch [B]. Namely, it asserts that any $\mathcal{A} \in \mathcal{C}^2$ has an inscribed affine-regular hexagon, in other words, there exists an affine transform $T$ such that the boundary of $T(\mathcal{A})$ contains the vertices of a regular hexagon $\mathcal{H}_1$. By convexity, this implies $\mathcal{H}_1 \subset T(\mathcal{A}) \subset \mathcal{H}_2$, where $\mathcal{H}_2$ is certain regular hexagon which depends only on $\mathcal{H}_1$. Our key auxiliary result is an improvement of the inclusions $\mathcal{H}_1 \subset T(\mathcal{A}) \subset \mathcal{H}_2$.

Now let us turn our attention to the estimates of Banach-Mazur radius of planar convex bodies. Summarizing already mentioned results, it is known that $1.455 \approx \sqrt{1 + \frac{\sqrt{5}}{2}} \leq \text{rad}(\mathcal{C}^2) \leq 2$. We obtain the following improvement of the upper bound:

**Theorem 1.2.** $\text{rad}(\mathcal{C}^2) \leq d(\mathcal{A}, \mathcal{C}^2) \leq \frac{117}{109} < 1.672$, where $\mathcal{A}$ is the 7-gon with the vertices $(0, \frac{2}{3})$, $(\pm1, 1)$, $(\pm2, 2)$ and $(\pm1, 3)$.

We state and prove our key auxiliary result Lemma 2.2 in Section 2, which also includes some important technical computations. The upper bounds on the Banach-Mazur diameter and radius of the planar convex bodies are proved in Sections 3 and 4 respectively.
2. An auxiliary result and some computations

**Definition 2.1.** For any \( a \in [0, 1] \), we define \( \mathcal{L}_a \) to be the convex hull of \((0, \frac{2a}{1+a})\), \((\pm 1, 1)\), \((\pm 2, 2)\) and \((\pm 1, 3)\). If \( a \in [0, \frac{1}{2}] \), we define \( \mathcal{U}_a \) to be the convex hull of \((\pm a, a)\), \((\pm (2+a), 2-a)\), \((\pm (3-a), 3-a)\), \((\pm (3-2a), 3)\) and \((\pm a, 4-a)\). If \( a \in [\frac{1}{2}, 1] \), we let \( \mathcal{U}_a \) be the convex hull of \((\pm a, a)\), \((\pm (3-2a), 1)\), \((\pm (3-a), 1+a)\), \((\pm (3-a), 3-a)\), \((\pm (3-2a), 3)\) and \((\pm a, 4-a)\).

Our key auxiliary result is the following lemma.

**Lemma 2.2.** For any \( \mathcal{A} \in C^2 \) there exist \( a \in [0, 1] \) and an affine transform \( T \) such that \( \mathcal{L}_a \subset T(\mathcal{A}) \subset \mathcal{U}_a \).

Let us illustrate some intuition behind the statement of the lemma. The resulting \( a \) in a certain sense measures how far is \( \mathcal{A} \) from an inscribed affine regular hexagon. In particular, if \( a = 1 \), then we have \( T(\mathcal{A}) = \mathcal{L}_1 = \mathcal{U}_1 \), i.e. \( \mathcal{A} \) is an affine regular hexagon. At the other extreme, if \( a = 0 \), then there is a point of \( \mathcal{A} \) (namely, \( T^{-1}(0,0) \)) which is at its “furthest” from the inscribed affine regular hexagon. In this case we have that pretty much “half” of the body is determined since

\[
T(\mathcal{A}) \cap \{(x,y) : y \leq 2\} = \mathcal{L}_0 \cap \{(x,y) : y \leq 2\} = \mathcal{U}_0 \cap \{(x,y) : y \leq 2\} = \{(x,y) : |x| \leq y \leq 2\}.
\]

**Proof of Lemma 2.2.** Using [B], we let \( H_i \in \partial \mathcal{A}, i = 1, \ldots , 6 \), be the vertices of an inscribed affine regular hexagon with the center \( O \), where \( \partial \mathcal{A} \) denotes the boundary of \( \mathcal{A} \). Let \( M_i \) be the midpoint of the segment \( H_iH_{i+1} \) (indices are considered modulo 6), and let \( V_i \) be the point of intersection of the lines \( H_{i-1}H_i \) and \( H_{i+1}H_{i+2} \) (alternatively, \( O\overrightarrow{V_i} = 2O\overrightarrow{M_i} \)). Let \( B_i \) be the point of intersection of the ray \( OM_i \) with \( \partial \mathcal{A} \), \( U_i \) be the point of intersection of the lines \( H_{i+1}B_i \) and \( H_iV_i \), and \( W_i \) be the point of intersection of the lines \( H_iB_i \) and \( H_{i+1}V_i \), see Figure 1. We define \( a_i := |U_iV_i|/|H_iV_i| \) (by symmetry, \( a_i = |W_iV_i|/|H_{i+1}V_i| \)), where \(|XY| \) stands for the Euclidean distance between \( X \) and \( Y \), \( a := \min_i a_i \), and \( j \) be such that \( a_j = a \). Finally, define \( T \) to be the affine transform mapping \( H_{j+1}, \ldots , H_{j+6} \) to \((1,1), (2,2), (1,3), (-1,3), (-2,2), (-1,1)\), respectively. It is straightforward to check that \( T(B_j) = (0, \frac{2a}{1+a}) \).

Since all seven points defining \( \mathcal{L}_a \) belong to \( \partial T(\mathcal{A}) \), we get by convexity that \( \mathcal{L}_a \subset T(A) \).

For every \( i \), let \( U_i^* \) and \( W_i^* \) be the points on the segments \( H_iV_i \) and \( H_{i+1}V_i \) respectively such that \( a = |U_i^*V_i|/|H_iV_i| = |W_i^*V_i|/|H_{i+1}V_i| \). Due to the choice of \( a \), we have \( |U_iV_i| \geq |U_i^*V_i| \) and \( |W_iV_i| \geq |W_i^*V_i| \), so by convexity the segment \( U_i^*W_i^* \) does not contain interior points of \( \mathcal{A} \) (this segment can have common points with \( \partial \mathcal{A} \) only for the “degenerate” cases when \( a = 0 \).
or \( a = 1 \). Therefore, \( \mathcal{A} \) is a subset of the 12-gon with the vertices \( U_i^* \) and \( W_i^* \). One can easily see that \( T(U_i^*) \) and \( T(W_i^*) \) are exactly the points defining \( \mathcal{U}_a \) when \( a \in \left[ \frac{1}{2}, 1 \right] \). Therefore, \( T(\mathcal{A}) \subset \mathcal{U}_a \) if \( a \in \left[ \frac{1}{2}, 1 \right] \). For the remainder of the proof suppose \( a \in [0, \frac{1}{2}) \). We need certain additional considerations in the triangles \( H_{j+1}V_{j+1}H_{j+2} \) and \( H_{j+5}V_{j+5}H_j \). Due to symmetry, let us consider only \( H_{j+1}V_{j+1}H_{j+2} \). Let \( C \) be the point of intersection of the lines \( B_jH_{j+1} \) and \( H_{j+2}V_{j+1} \). Since \( B_j \) and \( H_{j+1} \) belong to \( \partial \mathcal{A} \), by convexity there are no interior points of \( \mathcal{A} \) on the segment \( H_{j+1}C \). Noting that \( T(C) = (2 + a, 2 - a) \), this completes the proof of \( T(\mathcal{A}) \subset \mathcal{U}_a \), see also Figure 2. \[ \square \]

**Figure 1.** Location of the points \( U_i, W_i, U_i^* \) and \( W_i^* \)

**Figure 2.** Illustration of the inclusion \( T(\mathcal{A}) \subset \mathcal{U}_a \) when \( a \in \left[ 0, \frac{1}{2} \right) \)
For our applications, it will be important to understand how to cover \( U_b \) with a homothetic image of \( L_a \), for certain values of \( a \) and \( b \).

Lemma 2.3. Let \( a \in [0, 1] \), \( b \in [\frac{1}{2}, 1] \) and \( h \) be the homothety with the ratio \( 2-b \) and the center \((0,2)\). Then \( U_b \subset h(L_a) \).

Proof. This is immediate by \( L_1 \subset L_a \) and the fact that \( h(L_1) \) contains all 12 points defining \( U_b \). \(\square\)

Lemma 2.4. For any \( a, b \in [0, \frac{1}{2}] \) there exists \( c \) such that the homothety \( h \) with the ratio \( \lambda(a,b) \) and the center \((0,c)\) satisfies \( U_b \subset h(L_a) \), where

\[
\lambda(a,b) := \begin{cases} 
1 + \frac{(1-b)(1+2a)}{2+a}, & \text{if } b \leq \frac{3a}{2(1+2a)}, \\
\frac{3}{2}, & \text{otherwise}.
\end{cases}
\]

Proof. First let us fix \( c \) satisfying \( \max\{\frac{2a}{1+a}, \frac{2b}{1+b}\} < c < 2 \) and show that \( U_b \subset h(L_a) \) where \( h \) is the homothety with the ratio \( \lambda \) and the center \((0,c)\) provided

\[
\lambda \geq 1 + \max \left\{ \frac{2a(1-b)}{c+ac-2a}, \frac{2b}{c}, \frac{2(1-b)}{4-c} \right\}.
\]

It is convenient to represent \( L_a \) in terms of half-planes, namely,

\[
L_a = \{(x, y) : (1+a)y \geq (1-a)|x| + 2a, y \geq |x|, y \leq 4 - |x|, y \leq 3\}.
\]

We have \( h^{-1}(x, y) = (\frac{1}{1+a}, \frac{1}{a}(y-a) + c) \), so if \((x, y)\) is a point defining \( U_b \), by substitution of \( h^{-1}(x, y) \) into the inequalities of (2.3), one can compute that

\[
\lambda - 1 \geq \max \left\{ \left| \frac{x-a|x|-y-ay+2a}{c+ac-2a} \right|, \frac{|x-y|}{c}, \frac{|x+y-4|}{4-c}, \frac{y-3}{3-c} \right\},
\]

where \( \frac{2a}{1+a} < c < 2 \) was used. Now we substitute the points defining \( U_b \) into (2.4) and simplify the result. For \((x, y) = (b, b)\), since \( a, b \in [0, \frac{1}{2}] \) and \( \frac{2a}{1+a} < c < 2 \), we get

\[
\lambda - 1 \geq \max \left\{ \frac{2a(1-b)}{c+ac-2a}, \frac{2b}{4-c}, \frac{2(1-b)}{3-c} \right\} = \frac{2a(1-b)}{c+ac-2a}.
\]

In the same manner, we substitute \((3-b, 3-b), (3-2b, 3), (b, 4-b)\) and \((2+b, 2-b)\) into (2.4), and get that \( \lambda - 1 \) is at least \( \frac{2(1-b)}{4-c}, \frac{2(1-b)}{3-c} \) and \( \frac{2(b-a)}{c+ac-2a}, \frac{2b}{c} \)}, respectively. Therefore, taking (2.3) into account and noting that \( \frac{2(1-b)}{4-c} > \frac{1-b}{3-c} \) as \( c < 2 \), and that \( \frac{2(b-a)}{c+ac-2a} < \frac{2b}{c} \) as \( \frac{2b}{1+b} < c \), we obtain the desired (2.2).

It remains to choose \( c \) to minimize the right hand side of (2.2). The functions \( f_1(c) := \frac{2a(1-b)}{c+ac-2a} \) and \( f_2(c) := \frac{2b}{c} \) are decreasing, while the function \( f_3(c) := \frac{2(1-b)}{4-c} \) is increasing. We can
compute and bound the points of intersection: $f_1 = f_3$ at $c_1 := \frac{6a}{1+2a} \in \left(\frac{2a}{1+a}, 2\right)$ and $f_2 = f_3$ at $c_2 := 4b \in \left(\frac{2b}{1+b}, 2\right)$. Thus, we can set $\lambda(a, b) := 1 + f_3(\max\{c_1, c_2\})$, which leads to (2.1). \hfill \Box

3. Bound on diameter – proof of Theorem 1.1

Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^2$ be arbitrary. By Lemma 2.2, for some $a, b \in [0, 1]$ and affine transforms $T_a$ and $T_b$, we have $\mathcal{L}_a \subset T_\mathcal{A} \subset \mathcal{U}_a$ and $\mathcal{L}_b \subset T_\mathcal{B} \subset \mathcal{U}_b$. Assume $a \geq b$ and fix $\alpha \in (0, \frac{1}{2})$. When $a \geq \alpha$, i.e., when one of the bodies is “close” to an affine-regular hexagon, we will apply a small modification of the construction from [L]. Otherwise, we will use Lemma 2.4.

We claim that $d(\mathcal{A}, \mathcal{B}) \leq 3 - a$. Let $T$ be an affine transform mapping $\mathcal{L}_1$ to the hexagon $\mathcal{H}$ with the vertices $(0, \frac{4}{3})$, $(\pm 1, \frac{5}{3})$, $(\pm 1, \frac{7}{3})$, $(0, \frac{8}{3})$, and so $\mathcal{H}$ is inscribed into $T(T_\mathcal{B})$. Note that the lines defining the sides of $\mathcal{H}$ intersect (apart from the vertices of $\mathcal{H}$) at the vertices of $\mathcal{L}_1$. Therefore, by convexity and Lemma 2.2

$$\mathcal{H} \subset T(T_\mathcal{B}) \subset \mathcal{H}_1 \subset T_\mathcal{A} \subset \mathcal{U}_a,$$

so to prove our claim it suffices to show that the homothety of $\mathcal{H}$ with the center $(0, 2)$ and the ratio $3 - a$ contains $\mathcal{U}_a$, which is straightforward to verify (e.g. the technique of the proof of Lemma 2.4 can be used).

If $a, b \in [0, \alpha]$, then by Lemma 2.2 and Lemma 2.4 there are homotheties $h_1$ and $h_2$ with the ratios $\lambda(a, b)$ and $\lambda(b, a)$ such that $T_\mathcal{B} \subset h_1(T_\mathcal{A})$ and $T_\mathcal{A} \subset h_2(T_\mathcal{B})$. Then $T_\mathcal{A} \subset h_2(T_\mathcal{B}) \subset h_2(h_1(T_\mathcal{A}))$, so $d(\mathcal{A}, \mathcal{B}) \leq \lambda(a, b)\lambda(b, a)$.

We can summarize the preceding paragraphs as

$$(3.1) \quad d(\mathcal{A}, \mathcal{B}) \leq \inf_{\alpha \in \left(0, \frac{1}{2}\right)} \max \left\{ 3 - \alpha, \max_{a, b \in [0, \alpha]} \lambda(a, b)\lambda(b, a) \right\}.$$ 

Now assume $a, b \in [0, \alpha]$ and consider several cases in order to estimate $\lambda(a, b)\lambda(b, a)$. If $b > \frac{3a}{2(1+2a)}$ and $a > \frac{3b}{2(1+2b)}$, then $\lambda(a, b)\lambda(b, a) = \frac{9}{4}$. If $b \leq \frac{3a}{2(1+2a)}$ and $a \leq \frac{3b}{2(1+2b)}$, then $\lambda(a, b)\lambda(b, a) = (1 - (1 - b)f(a))(1 + (1 - a)f(b))$, where $f(t) = \frac{1+2t}{2+t}$ which is increasing for $t > -2$. Therefore, $(1 - b)f(a) \leq (1 - b)f\left(\frac{3b}{2(1+2b)}\right) = 2\frac{1+4b-5b^2}{4+11b}$, which, using standard calculus, attains its largest value on $[0, \frac{1}{2}]$ at $b = \frac{3\sqrt{14} - 4}{11}$. Hence, $1 + (1 - b)f(a) \leq \frac{289 - 60\sqrt{14}}{121} \approx 1.52956 < 1.53$. Arguing similarly for $(1 - a)f(b)$, we obtain $\lambda(a, b)\lambda(b, a) < 1.53^2 = 2.3409 < 2.35$. If $b \leq \frac{3a}{2(1+2a)}$ and $a > \frac{3b}{2(1+2b)}$, then $\lambda(a, b)\lambda(b, a) = \frac{3}{2}(1 + (1 - b)\frac{1+2a}{2+a}) \leq \frac{3}{2}(1 + \frac{1+2a}{2+a}) = \frac{9(1+a)}{2(2+a)} \leq \frac{9(1+a)}{2(2+a)}.$
Similarly, if \( b > \frac{3a}{2(1+2a)} \) and \( a \leq \frac{3b}{2(1+2b)} \), then \( \lambda(a, b) \lambda(b, a) \leq \frac{9(1+a)}{2(2+a)} \). In summary, since \( \max\{3-\alpha, 2.35, \frac{9}{4}\} \geq 2.5 \), (3.1) becomes
\[
d(\mathcal{A}, \mathcal{B}) \leq \inf_{\alpha \in (0, \frac{1}{2})} \max \left\{ 3 - \alpha, \frac{9(1+\alpha)}{2(2+\alpha)} \right\},
\]
which is optimal when \( 3 - \alpha = \frac{9(1+\alpha)}{2(2+\alpha)} \), i.e. when \( \alpha = \sqrt{\frac{73}{7} - \frac{7}{4}} \). This implies \( d(\mathcal{A}, \mathcal{B}) \leq \frac{19 - \sqrt{73}}{4} \).

4. Bound on radius – proof of Theorem 1.2

Note that \( \mathcal{A} = L_\frac{1}{4} \), so we need to show for any \( \mathcal{B} \in C^2 \) that \( d(\mathcal{A}, \mathcal{B}) \leq \frac{117}{70} \). Apply Lemma 2.2 to \( \mathcal{B} \), then for some \( b \in [0, 1] \) and an affine transform \( T \) we have \( L_b \subset T(\mathcal{B}) \subset U_b \).

It is rather immediate from Definition 2.1 that if \( h_1 \) is the homothety with the center \((0, 3)\) and the ratio \( \max\{(3 - \frac{2b}{1+b})/(3 - \frac{2}{5}), 1\} \), then \( h_1(L_\frac{1}{4}) \subset L_b \). If \( h_2 \) is a homothety with the ratio \( \lambda > 0 \) such that \( U_b \subset h_2(L_\frac{1}{4}) \), then by
\[
h_1(L_\frac{1}{4}) \subset L_b \subset T(\mathcal{B}) \subset U_b \subset h_2(L_\frac{1}{4})
\]
we get
\[
d(\mathcal{A}, \mathcal{B}) \leq \lambda \max\{(3 - \frac{2b}{1+b})/(3 - \frac{2}{5}), 1\}.
\]
If \( b \in [0, \frac{1}{4}] \), then by Lemma 2.4 we have \( \lambda = \lambda_\frac{1}{4}(b) \leq \lambda_\frac{1}{4}(0) = \frac{5}{3} \), so \( d(\mathcal{A}, \mathcal{B}) \leq \frac{5}{3} \leq \frac{117}{70} \). If \( b \in [\frac{1}{4}, \frac{1}{2}) \), then again by Lemma 2.4 we have \( \lambda = \lambda_\frac{1}{4}(b) = \frac{3}{2} \), and as \( (3 - \frac{2b}{1+b})/(3 - \frac{2}{5}) > \frac{35}{39} \), we obtain \( d(\mathcal{A}, \mathcal{B}) < \frac{117}{70} \). Finally, if \( b \in [\frac{1}{2}, 1] \), we use Lemma 2.3 to take \( \lambda = 2 - b \) and then \( d(\mathcal{A}, \mathcal{B}) \leq \frac{13(2-2b)(1+b)}{5(3-b)} \leq \frac{117}{70} \).

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