Thermodynamics of SU(3) gauge theory from gradient flow

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(Osaka U.)
Asakawa, Hatsuda, Itou, MK, Suzuki (FlowQCD Collab.),
arXiv:1312.7492[hep-lat]
Noether current / generator of space-time translation
Einstein Equation

$$G_{\mu \nu} + \Lambda g_{\mu \nu} = \kappa T_{\mu \nu}$$

Noether current / generator of space-time translation

Hydrodynamic Eq.

$$\partial_\mu T_{\mu \nu} = 0$$
The definition of $T_{\mu\nu}$ on the lattice is nontrivial...  

... because of the lack of translational symmetry

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} F F$$

cf) Caracciolo+ (1989)
Rough Idea

coarse graining

no translational invariance

translational symmetry is recovered!
lattice regularized gauge theory

YM gradient flow

$T^{R}_{\mu\nu}$
continuum theory (with dimensional regularization)

YM gradient flow

continuum theory (with dimensional regularization)

Luescher, Weiss (2012)
Suzuki (2013)
lattice regularized
gauge theory

YM gradient flow

continuum theory
(with dimensional regularization)

analytic relation
(in perturbation)

YM gradient flow

continuum theory
(with dimensional regularization)

UV finite

Luescher, Weiss (2012)
Suzuki (2013)
What we can measure with $T_{\mu\nu}$:

- bulk thermodynamics (energy density, pressure)
- correlation functions
- viscosity, thermal excitation
- vacuum structure?
- fluctuations, specific heat
- non-Gaussian fluctuations, etc.

Asakawa, Ejiri, MK (2009)

Pink chars: T>0 physics
QCD EoS
(Energy Density, Pressure)

- Rapid increase of $\varepsilon/T^4$ around $T=150-200$ MeV
- Crossover transition
- Rapid but smooth change of medium from hadronic to QGP-like
QCD Thermodynamics

\[ Z(T) = \text{Tr} \left[ e^{-H/T} \right] \]

\[ = \int \mathcal{D}A \exp \left[ - \int_0^{1/T} d\tau \int_V d^3x \mathcal{L}_E \right] \]

Thermodynamic relations

\[ \varepsilon = \frac{T^2}{V} \frac{\partial \ln Z}{\partial T} \]

\[ p = T \frac{\partial \ln Z}{\partial V} \]

How do we take T and V derivatives?
Lattice Spacing Derivative

Changing lattice spacing $a \rightarrow 1/T$ and $V$ change

\[
\frac{\partial \ln Z}{\partial a} \sim \varepsilon - 3p
\]

\[
\frac{\partial \ln Z}{\partial a} = \frac{\partial \beta}{\partial a} \frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial a} \langle S \rangle
\]

$\beta = 2N_c/g^2$

$\left[ \varepsilon - 3p \right]_{\text{thermodyn.}} = \left[ \varepsilon - 3p \right]_T - \left[ \varepsilon - 3p \right]_{\text{vac}}$
Differential Method

- 2 independent “beta functions”
- perturbative result Karsch (1982)
- negative pressure with Karsch coeffs.
- vacuum subtraction

\[
\frac{1}{T} = \frac{a_s N_t}{L} = \frac{1}{T}
\]

\[
L = a_s N_s
\]

\[
\varepsilon \sim \frac{\partial \ln Z}{\partial a_t}
\]

\[
p \sim \frac{\partial \ln Z}{\partial a_s}
\]
Integral Method

$$T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4}$$

$$\frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\varepsilon - 3p}{T^5}$$

- measurements of $\varepsilon - 3p$ for many $T$
- vacuum subtraction for each $T$
- information on beta function

Boyd+ 1996

Graphs showing $\beta - \beta_c$ with $32^3 \times 6$ and $32^3 \times 8$ configurations.
Gradient Flow Method

\[ \langle T_{\mu\nu} \rangle \]
Gradient Flow

$$\partial_t B_\mu(t, x) = D_\nu G_{\mu\nu}$$

$t$: "flow time"

dim: [length$^2$]

steepest descent direction of the action

$$B_\mu(0, x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Luescher, 2010
Gradient Flow

\[ \partial_t B_\mu(t, x) = D_\nu G_{\mu\nu} \]

- modify gauge field toward the stationary point of the action
- smoothing similarly to diffusion equation

\[ \partial_t B_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu B_\mu + \cdots \]

- diffusion length \( d \sim \sqrt{8t} \)
- All composite operators at \( t>0 \) are UV finite \( \text{Luescher,Weisz,2011} \)

\[ B_\mu(0, x) = A_\mu(x) \]

\[ G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \]
Operator Relation

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

- an operator at \( t>0 \)
- remormalized operators of original theory
Constructing EMT

\[ \tilde{\mathcal{O}}(t, x) \rightarrow \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- gauge-invariant dimension 4 operators

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^R_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T^R_{\rho\rho}(x) + \mathcal{O}(t) \]

Suzuki coefficients

\[
\begin{align*}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + O(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + O(g^4) \right]
\end{align*}
\]

\[ g = g(1/\sqrt{8t}) \]

\[ s_1 = 0.03296 \ldots \]

\[ s_2 = 0.19783 \ldots \]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^R_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + O(t) \]

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g = g(1/\sqrt{8t})

s_1 = 0.03296 \ldots

s_2 = 0.19783 \ldots

Remormalized EMT

\[ T^R_{\mu\nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{1}{4\alpha_E(t)} \delta_{\mu\nu} E(t, x)_{\text{subt.}} \right] \]
Numerical Simulation on the Lattice
Gauge field has to be sufficiently smeared!

\[ a \ll \sqrt{8t} \]

Perturbative relation has to be applicable!

\[ \sqrt{8t} \ll \Lambda^{-1}, T^{-1} \]
lattice regularized gauge theory

1. generate gauge configurations

2. solve gradient flow for each confs.

3. measure U and E

4. obtain $T_{\mu\nu}$ with Suzuki formula

5. take $a \to 0$, $t \to 0$ limit

continuum theory (with dimensional regularization)

$T^R_{\mu\nu}$

YM gradient flow
Numerical Simulation 1

- SU(3) YM theory
- Wilson gauge action
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10$
- configurations: 100-300

Scale setting:
alpha Collab., NPB538,669(1999)
There exists a wide range of $t$ at which the Suzuki formula is safely used with $N_t=10$. 

Emergent plateau!

$$2a \lessapprox \sqrt{8t} \lessapprox 0.4T^{-1}$$
Emergent plateau!

\[ 2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1} \]

There exists a wide range of \( t \) at which the Suzuki formula is safely used with \( Nt=10 \).
Continuum Limit

\( \pi \)

Statistical error of \( e - 3p \) is significantly smaller than Boyd+1996 which used \( \sim 10000 \) confs.

No integral! Direct measurement of \( e \) and \( p \) at a given \( T \)

no vacuum subtraction for \( e + p \)

\( a \rightarrow 0 \) limit with fixed \( \sqrt{8tT} = 0.4 \)
Comparison with Integral Method
Numerical Simulation 2

- SU(3) YM theory
- Wilson gauge action
- Lattice size: $32^3 \times N_t$
  - $N_t = 6, 8, 10, 32$
  - Configurations: 100-300
  - $\beta = 5.89 - 6.56$
- Lattice size: $64^3 \times N_t$
  - $N_t = 8, 10, \ldots, 16, 18, 64$
  - Configurations: $\sim 2000$
  - $\beta = 6.4 - 7.4$

**On BlueGene/Q @ KEK**

Efficiency of our code:
- Gauge update (HB+OR): $\sim 25\%$
- Gradient flow (RK$^4$): $\sim 40\%$
t Dependence of e+p

\[ N_x = 32, T \approx 1.65T_c \]

\[ \sqrt{8t} = 2a \]

NEW!!

Plateau region extends toward small t!
Summary

\[ T_{\mu \nu}^{R}(x) \]
Summary

**EMT formula from gradient flow**

\[
T_{R \mu \nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu \nu}(t, x) + \frac{1}{4 \alpha_E(t)} \delta_{\mu \nu} E(t, x)_{\text{subt.}} \right]
\]

Our strategy can successfully define the EMT on the lattice in practical simulations.

This operator provides us with novel approaches to measure observables on the lattice!

They are direct, intuitive and less noisy.
Many Future Works

- precision measurement of YM thermodynamics
- EMT correlation functions → measurement of viscosity
- specific heat, non-Gaussian fluctuations, etc.
- scale setting

- taking double limit $a \to 0$, $t \to 0$
- full QCD Makino, Suzuki, 1403.4772

from Gradient flow to Hydrodynamic flow
Two Point Functions

\[ \langle T_{\mu\nu}(x, t)T_{\mu\nu}(0, 0) \rangle \]
EMT Correlator

- Kubo Formula: $T_{12}$ correlator $\leftrightarrow$ shear viscosity

$$\eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3 x \langle T_{12}(x, -i\tau)T_{12}(0, t) \rangle$$

- Hydrodynamics describes long range behavior of $T_{\mu\nu}$

- Energy fluctuation $\leftrightarrow$ specific heat

$$c_V = \frac{\langle \delta E^2 \rangle}{VT^2}$$
EMT Correlator: Noisy...

With naïve EMT operators

\[ \langle T_{12}(\tau)T_{12}(0) \rangle \]

\[ \langle T_{\mu\nu}(\tau)T_{\mu\nu}(0) \rangle \]

Nakamura, Sakai, PRL, 2005

N\text{t}=8

improved action

\( \sim 10^6 \) configurations

N\text{t}=16

standard action

5x10^4 configuration

... no signal
\[
\int d^3x \langle T_{12}(x, \tau) T_{12}(0, 0) \rangle
\]

- Smearing length \(= \sqrt{8t} \)
- \(64^3 \times 16 \)
- \(\beta=7.2 \) (\(T \sim 2.2T_c\))
- 1200 confs

- Converge at \(\tau > \sqrt{8t} \)
- Improvement of the statistics at large \(t\)
Correlation Function

\[ C_{\mu \nu}(\tau) = \int d^3x \langle T_{\mu \nu}(x, \tau) T_{\mu \nu}(0, 0) \rangle \]

- \[64^3 \times 16\] configurations
- \[\beta = 7.2 \ (T \sim 2.2T_c)\]
- \[1200\] confs
- \[t/a^2 = 1.9\]

\[ C_{44}(\tau) : \text{constant} \]
\[ \left( \text{conservation law!} \right) \]
\[ \partial_{\tau} \langle \delta E(\tau) \delta E(0) \rangle = 0 \] (for \( \tau \neq 0 \))

\[ C_{12}(\tau) \]
\[ C_{41}(\tau) \]
\[ \text{negative} \left( i^2 = -1 \right) \]
Energy Fluctuation and Specific Heat

Specific Heat

\[ c_V = \frac{1}{V} \left. \frac{\partial E}{\partial T} \right|_V \]

\[ = \frac{\langle \delta E^2 \rangle}{VT^2} \]

\[ = \frac{\langle \delta E(\tau)\delta E(0) \rangle}{VT^2} \]

\[ \frac{c_V}{T^3} = \frac{\langle \delta E(\beta/2)\delta E(0) \rangle}{VT^5} \]

64³x16
\( \beta = 7.2 \) (T~2.2Tc)
1200 confs
**Energy Fluctuation and Specific Heat**

Specific Heat

\[ c_V = \frac{1}{V} \left. \frac{\partial E}{\partial T} \right|_V \]

\[ = \frac{\langle \delta E^2 \rangle}{VT^2} \]

\[ = \frac{\langle \delta E(\tau)\delta E(0) \rangle}{VT^2} \]

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**Gavai, et al., 2005**

**differential method**

for \( T=2T_c \)

\[ \frac{c_V}{T^3} = \frac{\langle \delta E(\beta/2)\delta E(0) \rangle}{VT^5} \]

\( 64^3 \times 16 \)

\( \beta = 7.2 \ (T \sim 2.2T_c) \)

1200 confs

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**Novel approach to measure** \( c_V \)