THE ALGEBRA OF THE CATALAN MONOID AS AN INCIDENCE ALGEBRA: A SIMPLE PROOF

STUART MARGOLIS AND BENJAMIN STEINBERG

Abstract. We give a direct straightforward proof that there is an isomorphism between the algebra of the Catalan monoid $C_n$, that is, the monoid of all order-preserving, weakly increasing self-maps $f$ of $[n] = \{1, \ldots, n\}$, over any commutative ring with identity and the incidence algebra of a certain poset over the ring.

1. Introduction

The Catalan monoid $C_n$ is the monoid of all order-preserving, weakly increasing self-maps $f$ of $[n] = \{1, \ldots, n\}$. It is well known that the cardinality of $C_n$ is the $n$-th Catalan number. See for example, [5, Ex.6.19(s)]. $C_n$ has appeared in many guises in combinatorics and combinatorial semigroup theory under different names. For example, it has also been called the monoid of non-decreasing parking functions or, simply, the monoid of order-decreasing and order-preserving functions [1, 2, 4].

It was first observed by Hivert and Thiéry [3], via an indirect proof, that $\mathbb{k}C_n$ is isomorphic to the incidence algebra of a certain poset $P_{n-1}$ for any field $\mathbb{k}$; a different indirect proof using the representation theory of $\mathcal{J}$-trivial monoids can be found in the second author’s book [10]. Grensing gave a direct isomorphism from the incidence algebra of $P_{n-1}$ to $\mathbb{k}C_n$ for any base commutative ring with unit $\mathbb{k}$, but her proof is quite involved and long because of the technical recursive construction of a complete set of orthogonal idempotents.

Here we show that there is a straightforward direct isomorphism from $\mathbb{k}C_n$ to the incidence algebra of $P_{n-1}$ over any base commutative ring with unit whose details are trivial to check. The proof is similar to that used by the second author for inverse monoids [9] and Stein for more general monoids [6–8]. The complication in previous approaches is avoided here as we show that the isomorphism is given by a unipotent upper triangular 0/1-matrix.

2. Combinatorics of the Catalan monoid

If $n \geq 0$ is an integer, let $P_n$ be the poset consisting of subsets of $[n]$ ordered by $X \leq Y$ if and only if $|X| = |Y|$ and if $X = \{x_1 < \cdots < x_k\}$

\textit{Date:} June 19, 2018.

\textit{2000 Mathematics Subject Classification.} 20M25, 16G10, 05E99.
and \( Y = \{y_1 < \cdots < y_k\} \), then \( x_i \leq y_i \) for \( i = 1, \ldots, k \). We shall also need the following refinement of this order given by \( X \preceq Y \) if \(|X| < |Y|\) or if \(|X| = |Y|\) and \( X \leq Y \).

There is a well-known bijection between the Catalan monoid \( C_{n+1} \) and the set of ordered pairs \((X,Y)\) of elements of \( P_n \) with \( X \leq Y \) given as follows. If \( X \leq Y \), define \( f_{X,Y} : [n+1] \to [n+1] \) in \( C_{n+1} \) by

\[
f_{X,Y}(i) = \begin{cases} y_1, & \text{if } 1 \leq i \leq x_1 \\ y_j, & \text{if } x_{j-1} < i \leq x_j \\ n+1, & \text{if } i > x_k 
\end{cases}
\]

where we have retained the notation of the previous paragraph for \( X \) and \( Y \). The condition that \( X \leq Y \) guarantees that \( f_{X,Y} \) is order preserving and weakly increasing. Conversely, if \( f \in C_{n+1} \), let \( Y = f([n+1]) \setminus \{n+1\} \) and let \( X = \{i \in f^{-1}(Y) \mid i = \max\{f^{-1}(f(i))\}\} \); so \( X \) consists of the maximum element of each partition block of \( f \) except the block of \( n+1 \). Then it is straightforward to check from \( f \) being order-preserving and weakly increasing that \( X \leq Y \) and that \( f = f_{X,Y} \). See [10, Ch. 17 Sec. 5] for details where we note that a slightly different convention was used.

Let us say that \( S \subseteq [n] \) is a partial cross-section for \( f \in C_{n+1} \) if \( n+1 \notin f(S) \) and \( f\restriction_S \) is injective, i.e., \( |S| = |f(S)| \). We denote by \( \text{PCS}(f) \) the set of partial cross-sections of \( f \). The following proposition is straightforward from the definitions and so we omit the proof.

**Proposition 2.1.** Let \( S \) be a partial cross-section for \( f = f_{X,Y} \) in \( C_{n+1} \).

1. \( S \preceq X \).
2. \( S \preceq f(S) \).
3. \( f(S) \subseteq Y \), hence \( f(S) \preceq Y \).
4. \( X \in \text{PCS}(f) \).

The following lemma is key to our proof.

**Lemma 2.2.** Let \( f, g \in C_{n+1} \). Then \( S \in \text{PCS}(fg) \) if and only if \( S \in \text{PCS}(f) \) and \( g(S) \in \text{PCS}(f) \).

**Proof.** Suppose first that \( S \in \text{PCS}(fg) \). Since \( f(n+1) = n+1 \) and \( n+1 \notin fg(S) \), it follows that \( n+1 \notin g(S) \). Also, \( fg\restriction_S \) is injective and hence \( g\restriction_S \) is injective. Thus \( S \) is a partial cross-section for \( g \). Since \( |fg(S)| = |S| = |g(S)| \) we also have that \( f\restriction_g(S) \) is injective. As \( n+1 \notin fg(S) \), we see that \( g(S) \) is a partial cross-section for \( f \). Conversely, assume that \( S \) is a partial cross-section for \( g \) and \( g(S) \) is a partial cross-section for \( f \). Then \( n+1 \notin fg(S) \) and \( |fg(S)| = |g(S)| = |S| \). Thus \( S \) is a partial cross-section for \( fg \). \( \square \)

### 3. The isomorphism of algebras

Let \( k \) be a commutative ring with unit and let \( n \geq 0 \). Let \( I(P_n, k) \) be the incidence algebra of \( P_n \) over \( k \). It can be viewed as the \( k \)-algebra with basis
all ordered pairs \((X, Y)\) with \(X \leq Y\) in \(P_n\) and where the product is defined on basis elements by

\[
(U, V)(X, Y) = \begin{cases} (X, V), & \text{if } Y = U \\ 0, & \text{else.} \end{cases}
\]

In other words, this is the algebra of the category corresponding to the poset \(P_n\).

We partially order the basis \(C_{n+1}\) of \(kC_{n+1}\) by saying \(f_{U,V}\) comes before \(f_{X,Y}\) if \(U \prec X\) and \(V \preceq Y\) and, similarly, we order the basis of \(I(P_n, k)\) by saying that \((U, V)\) comes before \((X, Y)\) if \(U \preceq X\) and \(V \preceq Y\). We can now prove the main result.

**Theorem 3.1.** There is an isomorphism \(\varphi : kC_{n+1} \rightarrow I(P_n, k)\) of \(k\)-algebras given by

\[
\varphi(f) = \sum_{S \in PCS(f)} (S, f(S))
\]

for \(f \in C_{n+1}\).

**Proof.** It is immediate from Proposition 2.1 that \(\varphi\) is well defined and that the matrix of \(\varphi\) as a homomorphism of free \(k\)-modules with respect to our preferred bases and orderings is unipotent upper triangular. Indeed, \(\varphi(f_{X,Y}) = (X, Y) + a\) where \(a\) is a sum of certain terms \((U, V)\) with \(U \prec X\) and \(V \preceq Y\). It follows that \(\varphi\) is an isomorphism of \(k\)-modules. It remains to show that it is a ring homomorphism. Indeed,

\[
\varphi(f)\varphi(g) = \sum_{T \in PCS(f)} (T, f(T)) \cdot \sum_{S \in PCS(g)} (S, g(S))
\]

\[= \sum_{S \in PCS(g), g(S) \in PCS(f)} (S, f(g(S))) \]

\[= \varphi(fg)\]

where the last equality is by Lemma 2.2. This completes the proof. 

\[\square\]

**References**

[1] O. Ganyushkin and V. Mazorchuk. *Classical finite transformation semigroups, an introduction*. Number 9 in Algebra and Applications. Springer, 2009.

[2] P. Higgins. Combinatorial aspects of semigroups of order-preserving and decreasing functions. In *Semigroups (Luino, 1992)*, pages 103–110. World Scientific, River Edge, NJ, USA, 1993.

[3] F. Hivert and N. Thiéry. The hecke group algebra of a coxeter group and its representation theory. *J. Algebra*, 321(8):2230–2258, 2009.

[4] A. Solomon. Catalan monoids, monoids of local endomorphisms, and their presentations. *Semigroup Forum*, 53:351–368, 1996.

[5] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by G.-C. Rota and appendix 1 by S. Fomin.
[6] I. Stein. The representation theory of the monoid of all partial functions on a set and related monoids as \(e\)-\(i\)-category algebras. *Journal of Algebra*, 450, no. 15:549–569, 2016.

[7] I. Stein. Algebras of ehresmann semigroups and categories. *Semigroup Forum*, 95 issue 3:509–526, 2017.

[8] I. Stein. Erratum to: Algebras of ehresmann semigroups and categories. *Semigroup Forum*, 96 issue 3:603–607, 2018.

[9] B. Steinberg. Möbius functions and semigroup representation theory. *J. Combin. Theory Ser. A*, 113(5):866–881, 2006.

[10] B. Steinberg. *The Representation Theory of Finite Monoids*. Springer, 2016.