Diophantine exponents for systems of linear forms in two variables

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Abstract

We improve on Jarník's inequality between uniform Diophantine exponent $\alpha$ and ordinary Diophantine exponent $\beta$ for a system of $n \geq 2$ real linear forms in two integer variables. Jarník (1949, 1954) proved that $\beta \geq \alpha(\alpha - 1)$. In the present paper we give a better bound in the case $\alpha > 1$. We prove that

$$
\beta \geq \begin{cases} 
\frac{1}{2} \left( \frac{1}{2} \alpha^2 - \alpha + 1 + \sqrt{(\alpha^2 - \alpha + 1)^2 + 4\alpha^2(\alpha - 1)} \right) & \text{if } 1 \leq \alpha \leq 2 \\
\frac{1}{2} \left( \alpha^2 - 1 + \sqrt{(\alpha^2 - 1)^2 + 4\alpha(\alpha - 1)} \right) & \text{if } \alpha \geq 2
\end{cases}
$$

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1. Jarník's theorem. In this paper

$$
\Theta = \begin{pmatrix} 
\theta_1^1 & \cdots & \theta_1^m \\
\cdots & \cdots & \cdots \\
\theta_n^1 & \cdots & \theta_n^m
\end{pmatrix}
$$

stands for a $m \times n$ real matrix and $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$ is an integer vector. In the sequel $|x|^{\text{sup}}$ means the sup-norm of a vector $x$:

$$
|x|^{\text{sup}} = \max_{1 \leq i \leq m} |x_i|.
$$

Consider the function

$$
\psi_{\Theta}^{\text{sup}}(t) = \min_{x \in \mathbb{Z}^n: 0 < |x|^{\text{sup}} \leq t} \max_{1 \leq j \leq n} ||\theta_j^1 x_1 + \cdots + \theta_j^m x_m||.
$$

In this paper we suppose that for every $t \geq 1$ one has

$$
\psi_{\Theta}^{\text{sup}}(t) > 0, \quad \forall t \geq 1.
$$

This is a natural condition on the matrix $\Theta$.

The uniform Diophantine exponent $\alpha(\Theta)$ is defined as follows:

$$
\alpha(\Theta) = \sup \{ \gamma > 0 : \limsup_{t \to +\infty} t^\gamma \psi_{\Theta}^{\text{sup}}(t) < +\infty \},
$$

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From the Minkowski convex body theorem it follows that

$$\alpha(\Theta) \geq \frac{m}{n}. \quad (3)$$

In addition it is a well known fact that in the case \( m = 1 \) one has

$$\alpha(\Theta) \leq 1.$$ 

The ordinary Diophantine exponent \( \beta(\Theta) \) is defined as follows:

$$\beta(\Theta) = \sup \{ \gamma > 0 : \liminf_{t \to +\infty} t^\gamma \psi_{\Theta}^{\sup}(t) < +\infty \}. \quad (4)$$

Obviously

$$\beta(\Theta) \geq \alpha(\Theta). \quad (5)$$

This inequality may be considered as a lower bound for \( \beta(\Theta) \) in terms of \( \alpha(\Theta) \). V. Jarník improved on the trivial bound \((5)\) in several papers. Probably his first paper dealing with this topic is the paper \[6\] published in 'Acta Scientarium Mathematicum Szeged' in 1949. Here we formulate a general result by Jarník from \[7\].

**Theorem A.** (V. Jarník \[7\]) Suppose that \( \Theta \) satisfies \((1)\). Then

(i) if \( m = 1 \) and \( \Theta \) consists of at least two numbers \( \theta_1^1, \theta_1^k \) linearly independent over \( \mathbb{Z} \) together with 1, then

$$\beta(\Theta) \geq \alpha(\Theta) \cdot \frac{\alpha(\Theta)}{1 - \alpha(\Theta)}; \quad (6)$$

(ii) if \( m = 2 \) then

$$\beta(\Theta) \geq \alpha(\Theta) \cdot (\alpha(\Theta) - 1); \quad (7)$$

(iii) in the case \( m \geq 3, n \geq 1 \) under the additional condition \( \alpha(\Theta) \geq (5m^2)^{m-1} \) one has

$$\beta(\Theta) \geq \alpha(\Theta) \cdot (\alpha(\Theta)^{1/(m-1)} - 3). \quad (8)$$

In the cases \( m = 1, n = 2 \) and \( m = 2, n = 2 \) the inequalities of Theorem A are the best possible. In \[8\] M. Laurent proved a general result (so-called 'four exponents theorem') from which he deduced the following theorem as a corollary.

**Theorem B.** (M. Laurent \[8\])

(i) Suppose that \( \beta \geq \frac{\alpha \cdot \frac{\alpha}{1 - \alpha}, \frac{1}{2} \leq \alpha \leq 1 \). Then there exists \( \Theta = (\theta_1^1) \) such that the numbers \( 1, \theta_1^1, \theta_1^2 \) are linearly independent over \( \mathbb{Z} \) and \( \alpha(\Theta) = \alpha, \beta(\Theta) = \beta \).

(ii) Suppose that \( \beta \geq \frac{\alpha \cdot (\alpha - 1), \alpha \geq 2 }{2} \). Then there exists \( \Theta = (\theta_1^1, \theta_1^2) \) such that the numbers \( 1, \theta_1^1, \theta_1^2 \) are linearly independent over \( \mathbb{Z} \) and \( \alpha(\Theta) = \alpha, \beta(\Theta) = \beta \).

In a recent paper \[15\] W.M. Schmidt and L. Summerer developed a new powerful method of analysis of the successive minima of one-parameter families of lattices. This method enabled them to improve the inequalities \((6,8)\) of Theorem A in the cases \( m = 1 \) and \( n = 1 \). As a corollary they obtained the following result.

**Theorem C.** (W.M. Schmidt, L. Summerer \[15\])

(i) Suppose that \( m = 1, n \geq 2 \) and the matrix \( \Theta \) consists of numbers \( \theta_1^1, ..., \theta_1^n \) linearly independent over \( \mathbb{Z} \) together with 1. Then

$$\beta(\Theta) \geq \alpha(\Theta) \cdot \frac{\alpha(\Theta) + n - 2}{(n - 1)(1 - \alpha(\Theta))}. \quad (9)$$
(ii) Suppose that \( n = 1, m \geq 2 \) and the matrix \( \Theta \) consists of numbers \( \theta_1^1, \ldots, \theta_1^m \) linearly independent over \( \mathbb{Z} \) together with 1. Then
\[
\beta(\Theta) \geq \alpha(\Theta) \cdot \frac{(m-1)(\alpha(\Theta) - 1)}{1 + (m-2)\alpha(\Theta)}.
\]

The proof of the main result from [15] relies on K. Mahler’s theory of pseudocompound bodies [9] and deals with difficult analysis of special piecewise linear functions. An alternative easy geometric proof was given by O. German and N. Moshchevitin in [5]. The inequalities (9,10) follow from the main result of [15] and transference inequalities by Y. Bugeaud and M. Laurent [1]. Here we should note that the method developed by W.M. Schmidt and L. Summerer in [15] cannot be directly applied to the case \( m > 1, n > 1 \), by some geometric reasons.

One can easily see that in the cases \( n = 2 \) and \( m = 2 \) inequalities (9) and (10) turn into (6) and (7) respectively.

In the case \( m = 1, n = 3 \) the best known inequality is due to N. Moshchevitin.

**Theorem D.** (N. Moshchevitin [12]) Suppose that \( m = 1, n = 3 \) and the collection \( \theta_1^1, \theta_1^2, \theta_1^3 \) consists of numbers which, together with 1, are linearly independent over \( \mathbb{Z} \). Then
\[
\beta(\Theta) \geq \alpha(\Theta) \cdot \left( \frac{\alpha(\Theta)}{1 - \alpha(\Theta)} + \sqrt{\frac{\alpha(\Theta)}{1 - \alpha(\Theta)}} + \frac{4\alpha(\Theta)}{1 - \alpha(\Theta)} \right).
\]

In [10], [11] N. Moshchevitin obtained the bounds in the cases \( m = 3, n = 1 \) and \( m = n = 2 \). We will refer to a result from [11] (Theorem 24) which is the best known up to now in the case \( m = 3, n = 1 \).

**Theorem E.** (N. Moshchevitin [10, 11]) Suppose that \( m = 3, n = 1 \) and the collection \( \theta_1^1, \theta_1^2, \theta_1^3 \) consists of numbers which, together with 1, are linearly independent over \( \mathbb{Z} \). Then
\[
\beta(\Theta) \geq \alpha(\Theta) \cdot \left( \sqrt{\frac{\alpha(\Theta)}{1 - \alpha(\Theta)}} + \frac{1}{\alpha(\Theta)} - \frac{7}{4} + \frac{1}{\alpha(\Theta)} - \frac{1}{2} \right).
\]

To finish this section we would like to formulate a result by V. Jarník from [7] from which he deduces the inequality (7) of Theorem A.

**Theorem F.** (V. Jarník [7]) Suppose that \( n \geq 2 \), and that matrix
\[
\Theta = \begin{pmatrix}
\theta_1^1 & \theta_1^2 \\
\vdots & \vdots \\
\theta_n^1 & \theta_n^2
\end{pmatrix}
\]

satisfy the condition (7). Suppose that a positive function \( \psi(t) \) is such that
\[
\lim_{t \to +\infty} t\psi(t) = 0.
\]
Suppose that
\[
\psi_{\Theta}^{\sup}(t) \leq \psi(t)
\]
for all \( t \) large enough. Then there exist arbitrary large values of \( t \) such that
\[
\psi_{\Theta}^{\sup}(t) \leq \psi\left(\frac{1}{6t\psi(t)}\right).
\]
Some related results are discussed in our recent surveys [11, 13] and in the papers by M. Waldschmidt [16] and O. German [3, 4].

2. The result.

We give few comments on the part (ii) of Jarník’s Theorem A. First of all we note that the inequality (10) is better than the trivial bound (5) in the case \(\alpha(\Theta) > 2\) only. However \(\alpha(\Theta)\) can attain any value from the interval \(\left[\frac{2}{n}, +\infty\right]\). So Theorem A gives nothing for the values of \(\alpha(\Theta)\) in the interval \(\left[\frac{2}{n}, 2\right]\).

As it was mentioned in the previous section, it is possible to improve the inequality (7) in the case \(n \geq 2\). A proof of a certain inequality better than (10) was sketched in [10, 11] (Theorem 22 from [11]). However the inequality from [10, 11] is very weak. Moreover it is better than (7) in the range \(1 < \alpha(\Theta) < \left(\frac{1 + \sqrt{5}}{2}\right)^2\) only.

In the present paper we get an inequality which improves the inequality (7) of Theorem A for all values of \(\alpha(\Theta) > 1\). This inequality is better than that from [10, 11].

Put

\[
G(\alpha) = \begin{cases} 
\frac{1}{2} \left(\alpha^2 - \alpha + 1 + \sqrt{\alpha^2 - \alpha + 1} + 4\alpha^2(\alpha - 1)\right) & \text{if } 1 \leq \alpha \leq 2 \\
\frac{1}{2} \left(\alpha^2 - 1 + \sqrt{\alpha^2 - 1}^2 + 4\alpha(\alpha - 1)\right) & \text{if } \alpha \geq 2
\end{cases}
\]

and define

\[
g(\alpha) = \frac{G(\alpha)}{\alpha}.
\]

Note that for \(\alpha > 1\) the value \(g(\alpha)\) is the largest solution of the equation

\[
\alpha g = \max(\alpha - 1, 1) + \frac{\alpha(\alpha - 1)}{g - \alpha + 1}. \tag{13}
\]

One can see that \(g(1) = 1\) and

\[
g(\alpha) > \max(\alpha - 1, 1)
\]

for \(\alpha > 1\).

Now we formulate the main result of the present paper.

**Theorem 1.** Suppose that \(m = 2\) and \(n \geq 3\). Suppose that among \(n + 2\) two-dimensional vectors

\[
\begin{pmatrix}
\theta_1^1 \\
\theta_2^1
\end{pmatrix}, \ldots, \begin{pmatrix}
\theta_1^n \\
\theta_2^n
\end{pmatrix}, \begin{pmatrix}
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1
\end{pmatrix} \tag{14}
\]

there exist at least four vectors linearly independent over \(\mathbb{Z}\). Suppose that \(\alpha(\Theta) \geq 1\).

Then

\[
\beta(\Theta) \geq G(\alpha(\Theta)) = \alpha(\Theta) \cdot g(\alpha(\Theta)). \tag{15}
\]

**Remark 1.** From the conditions of Theorem it follows that for the matrix \(\Theta\) one has (11).

**Remark 2.** The condition concerning linearly independence of vectors (14) cannot be removed in Theorem 1. For example in the case \(m = n = 2\) one may take arbitrary \(\alpha, \beta\) under the conditions \(\beta \geq \alpha(\alpha - 1), \alpha \geq 2\) and consider a matrix

\[
\Theta = \begin{pmatrix}
\theta_1^1 & \theta_2^1 \\
\theta_1^2 & \theta_2^2
\end{pmatrix}
\]

where \(\theta_1^1, \theta_1^2\) come from Theorem B (ii). Then \(\alpha(\Theta) = \alpha, \beta(\Theta) = \beta\) and so (15) may be not true.
Remark 3. Theorem 1 gives a bound which is better than the trivial bound (5) in the case \( \alpha(\Theta) > 1 \) only. In the case \( n = 2 \) we know that \( \alpha(\Theta) \) cannot be less than one (see (3)). However in the case \( n \geq 3 \) we do not know if the trivial bound (5) can be improved upon in the range \( \frac{2}{n} < \alpha(\Theta) < 1 \).

Remark 4. If \( n \geq 3 \) the trivial bound (5) cannot be improved in the case \( \alpha(\Theta) = 1 \), in general. We refer to a result from [11] (Theorem 10 and Corollary to it from [11]). Suppose that \( \xi \in \mathbb{R} \setminus \mathbb{Q} \) has bounded partial quotients in its continued fraction expansion. Consider the matrix

\[
\Theta = \begin{pmatrix}
\theta_1 & \xi_1 \\
\theta_2 & \xi_2 \\
\vdots & \vdots \\
\theta_n & \xi_2
\end{pmatrix}, \quad n \geq 3.
\] (16)

Then for almost all (in the sense of Lebesgue measure) real vectors \((\theta_1, \theta_1, \ldots, \theta_n) \in \mathbb{R}^n\) all but a finite number of the best approximations vectors \(Z_\nu\) (see Sections 3, 4 below) lie in a certain two-dimensional linear subspace of \(\mathbb{R}^n+2\), and so for the matrix (16) one has \(\alpha(\Theta) = 2\). As the partial quotients of \(\xi\) are bounded, one can see that \(\beta(\Theta) = 1\) also. Of course in this example all the elements of the matrix (??matts) can be linearly independent over \(\mathbb{Z}\) together with 1. So this example shows that for \(n \geq 3\) it may happen that
\[
\beta(\Theta) = \alpha(\theta) = 1,
\]
and the trivial bound (5) cannot be improved upon under the general condition of linear independence.

Remark 5. In some very special cases (see the first Remark in Section 6 below) it is possible to improve upon the trivial bound (5) in the case \(\alpha(\Theta) < 1\).

3. Ordinary best approximations.

Recall the definition and the simplest properties of ordinary best approximation vectors. These best approximations were actually used in the original paper [7] as well as in authors papers [10, 11, 12, 13]. As usual the sup-norm was used there to define the sequence of the best approximation vectors.

For an integer vector \(x = (x_1, x_2) \in \mathbb{Z}^2\), put
\[
\zeta^{\sup}(x) = \max_{1 \leq j \leq n} ||\theta_1^j x_1 + \theta_2^j x_2||.
\]

A vector \(x \in \mathbb{Z}^2\) is said to be a best approximation vector if
\[
\zeta^{\sup}(x) = \min_{x'} \zeta^{\sup}(x'),
\]
where the minimum is taken over all \(x' = (x'_1, x'_2) \in \mathbb{Z}^2\) such that \(0 < |x'|^{\sup} \leq |x|^{\sup}\).

Suppose that the matrix \(\Theta\) of the form (12) satisfies the following condition (L.I.) : for any pair \((i, j), 1 \leq i, j \leq n, i \neq j\) the collection
\[
\theta_1^i, \theta_1^j, \theta_2^i, \theta_2^j, 1
\]
consists of numbers linearly independent over \(\mathbb{Z}\). From this condition on the matrix \(\Theta\) we see that all best approximations form the sequence
\[
x^{\sup}_\nu = (x^{\sup}_{\nu,1}, x^{\sup}_{\nu,2}), \quad \nu = 1, 2, 3, \ldots,
\]
in such a way that for the values \(\zeta^{\sup}_\nu = \zeta^{\sup}(x^{\sup}_\nu)\) and \(X^{\sup}_\nu = |x^{\sup}_{\nu,i}|^{\sup}\) form infinite monotone sequences
\[
\zeta^{\sup}_1 > \zeta^{\sup}_2 > \cdots > \zeta^{\sup}_\nu > \zeta^{\sup}_{\nu+1} > \cdots,
\] (17)
\[ X_1^{\text{sup}} < X_2^{\text{sup}} < \ldots < X_n^{\text{sup}} u < X_{\nu+1}^{\text{sup}} < \ldots \]  

For a best approximation vector \( x_{\nu}^{\text{sup}} = (x_{\nu,1}^{\text{sup}}, x_{\nu,2}^{\text{sup}}) \) we consider integers \( y_{\nu,j}, 1 \leq j \leq n \) defined by the equalities

\[ ||\theta_j^1 x_{\nu,1}^{\text{sup}} + \theta_j^2 x_{\nu,2}^{\text{sup}}|| = |\theta_j^1 x_{\nu,1}^{\text{sup}} + \theta_j^2 x_{\nu,2}^{\text{sup}} - y_{\nu,j}| \]

and define the extended best approximation vector

\[ z_{\nu}^{\text{sup}} = (x_{\nu,1}^{\text{sup}}, x_{\nu,2}^{\text{sup}}, y_{\nu,1}^{\text{sup}}, \ldots, y_{\nu,n}^{\text{sup}}) \in \mathbb{Z}^{n+2}. \]

Here we should note that each vector \( z_{\nu}^{\text{sup}} \) is a primitive vector, that is, \( \text{g.c.d.}(x_{\nu,1}^{\text{sup}}, x_{\nu,2}^{\text{sup}}, y_{\nu,1}^{\text{sup}}, \ldots, y_{\nu,n}^{\text{sup}}) = 1. \)

Moreover each couple of consecutive vectors \( z_{\nu}^{\text{sup}}, z_{\nu+1}^{\text{sup}} \) can be extended to a basis of the whole integer lattice \( \mathbb{Z}^{n+2} \). In particular \( z_{\nu}^{\text{sup}} \) and \( z_{\nu+1}^{\text{sup}} \) are linearly independent.

We consider two-dimensional subspace

\[ \mathcal{L} = \{(x_1, x_2, y_1, \ldots, y_n) \in \mathbb{R}^{n+2} : \theta_j^1 x_1 + \theta_j^2 x_2 - y_j = 0, \ 1 \leq j \leq n \}. \]

From (L.I.) condition on the matrix \( \Theta \) it follows that there is no non-zero integer points in \( \mathcal{L} \) and the best approximation vectors \( z_{\nu}^{\text{sup}} \) become more and more close to \( \mathcal{L} \) as \( \nu \) tends to infinity. From Minkowski convex body theorem it follows that

\[ \zeta_{\nu}^{\text{sup}} (X_{\nu+1}^{\text{sup}})^{\frac{2}{n}} \leq 1. \]  

Here we should note that the inequality

\[ \psi_{\Theta}^{\text{sup}}(t) \leq t^{-\alpha} \]

holds for all \( t \) large enough if and only if

\[ \zeta_{\nu}^{\text{sup}} \leq (X_{\nu+1}^{\text{sup}})^{-\alpha} \]

for \( \nu \) large enough.

4. Spherical best approximations.

However consideration of the ordinary best approximations vectors is not very convenient for our purposes. It makes the proofs too cumbersome. To make our proofs easier we need another definition.

In the sequel by dist(\( \mathcal{A}, \mathcal{B} \)) we denote the Euclidean distance between the sets \( \mathcal{A}, \mathcal{B} \subset \mathbb{R}^{n+2} \). We shall consider vectors from \( \mathbb{R}^{n+2} \) of the form

\[ z = (x_1, x_2, y_1, \ldots, y_n). \]

For such a vector by \( Z = Z(z) = \text{dist}(\{z\}, \{0\}) \) we define its Euclidean norm and by \( \zeta(z) = \text{dist}(\{z\}, \mathcal{L}) \) we define the distance from \( z \) to the two dimensional subspace \( \mathcal{L} \) defined in the previous section.

We need a simple geometric observation.

**Lemma 1.** Let \( \pi \) be a two-dimensional linear subspace in \( \mathbb{R}^{n+2} \) such that \( \pi \cap \mathcal{L} = \{0\} \). Suppose that \( \pi \) and \( \mathcal{L} \) are not orthogonal. Then given \( \lambda > 0 \) the set

\[ \mathcal{G}_\lambda = \{ z \in \pi : \text{dist}(\{z\}, \mathcal{L}) = \lambda \} \]
is an ellipse. Moreover for all values of \( \lambda \) all the ellipses \( G_\lambda \) are dilatated form the ellipse \( G_1 \), and hence all their minor axes coinside and all their major axes coinside.

**Remark.** It is clear that in the case \( \dim \pi \cap L = 1 \) the set \( G_\lambda \) consists of two parallel lines. 

Proof of Lemma 1.

We may restrict ourselves on four-dimensional subspace span \((\pi \cup L)\).

Suppose that \( \eta_1, \eta_2 \) are linearly independent vectors from \( L \).

Then the Euclidean distance from \( z \in \pi \) to \( L \) is defined by the formula

\[
\text{dist} (\{z\}, L) = \frac{\text{volume of the parallelepiped spaned by } z, \eta_1, \eta_2}{\text{area of the parallelogram spaned by } \eta_1, \eta_2}.
\]

So \( \text{dist} (\{z\}, L) \) is a quadratic form in \( z \). Being restricted on \( \pi \) it gives a quadratic form in two variables. It is clear that \( G_\lambda \) is a bounded set. So \( G_\lambda \) is an ellipse. Further statements of Lemma 1 are obvious.\( \Box \)

For a two-dimensional linear subspace \( \pi \) the following observation will be of importance. Consider the circle

\[
\mathcal{S} = \{z \in \pi : Z(z) = 1\}.
\]

Let \( \pi \) be not an orthogonal complement to \( L \). Then there exist two orthogonal vectors \( a, b \in \pi \) such that

\[
\min_{z \in \mathcal{S}} \text{dist} (\{z\}, L) = \text{dist} (\{p\}, L), \quad \max_{z \in \mathcal{S}} \text{dist} (\{z\}, L) = \text{dist} (\{q\}, L).
\]

We suppose that the directed angle between vectors \( p \) and \( q \) is equal to +\( \pi/2 \). For \( t \in [0, \pi/2] \) we consider the point \( A(t) \in \mathcal{S} \) obtained by the rotation of the point \( p \in \mathcal{S} \) by the angle \( t \) towards the point \( q \in \mathcal{S} \). We are interested in the function

\[
f(t) = \frac{\text{dist} (\{A(t)\}, L)}{\text{dist} (\{A(t)\}, \text{span } p)} = \frac{\text{dist} (\{A(t)\}, L)}{\sin t}.
\]

**Lemma 2.** In the interval \( 0 < t \leq \pi/2 \) the function \( f(t) \) decreases.

Proof.

We may suppose that both two-dimensional subspaces belong to the same four-dimensional Euclidean subspace \( \mathbb{R}^4 \) with coodrares \( \eta_1, \eta_2, \eta_3, \eta_4 \) and that the subspace \( L \) in these coordinates is determined by the equations

\[
\eta_1 = \eta_2 = 0.
\]

If in these coordinates we have a point \( z = (\eta_1, \eta_2, \eta_3, \eta_4) \), then

\[
\text{dist} (\{z\}, L) = \sqrt{\eta_1^2 + \eta_3^2}.
\]

Let in these coordinates

\[
p = (p_1, p_2, p_3, p_4), \quad q = (q_1, q_2, q_3, q_4).
\]

Then

\[
(f(t))^2 = \frac{(p_1 \cos t + q_1 \sin t)^2 + (p_2 \cos t + q_2 \sin t)^2}{\sin^2 t} = \frac{p_1^2 + p_2^2}{\sin^2 t} + q_1^2 + q_2^2 - p_1^2 - p_2^2
\]

(we should note that the point \( p \) is the closest point to \( L \) and so \( p_1 q_1 + p_2 q_2 = 0 \)). Now it is clear that \( f(t) \) decreases.\( \Box \)

**Remark.** In the case \( \dim \pi \cap L = 1 \) the function \( f(t) \) is a constant as in this case \( p_1^2 + p_2^2 = 0 \). Now we define the sequence of spherical best approximation vectors.
We define $z \in \mathbb{Z}^{n+2}$ to be a spherical best approximation vector if

$$\zeta(z) \leq \zeta(z')$$

for all nonzero integer vectors $z' \in \mathbb{Z}^{n+2}$ with $Z' = \text{dist}(z', \{0\}) \leq Z$.

To avoid the situation when two best approximation vectors with the same value of $Z$ may occur we need to suppose a condition which generalizes the condition (L.I.) from the previous section. However such a condition deal with quadratic relations instead of linear relations. We do not want to suppose additional restrictions on matrix $\Theta$. So we will not define the sequence of the best spherical approximation vectors in a unique way. Analogously to the sequences of the ordinary best approximations $z_{\nu}^{\sup}$ satisfying (17,18) we define the sequence

$$z_{\nu} = (x_{\nu,1}, x_{\nu,2}, y_{\nu,1}, ..., y_{\nu,n}), \quad \nu = 1, 2, 3, ...$$

such that for $\zeta_{\nu} = \zeta(z_{\nu})$ and $Z_{\nu} = Z(z_{\nu})$ one has

$$\zeta_1 > \zeta_2 > ... > \zeta_{\nu} > \zeta_{\nu+1} > ...$$

and

$$Z_1 < Z_2 < ... < Z_{\nu} < Z_{\nu+1} < ...$$

Of course under the conditions of Theorem 1 it may happen that the same values of $\zeta_{\nu}, Z_{\nu}$ are attained on two (or even more) different integer vectors. In such a situation we choose one of the admissable integer vectors in an arbitrary way and define it to be the $\nu$-th best spherical approximation vector. So the sequence (20) may depend on our choice. But the values from the sequences (21,22) do not depend on our choice. After we have chosen the sequence (20) we fix it. Everywhere in the sequel we deal with the fixed sequence of spherical best approximations which was chosen here.

Analogously to the ordinary best approximations, each vector $z_{\nu}$ is a primitive vector, and each couple of consecutive vectors $z_{\nu}, z_{\nu+1}$ can be extended to a basis of the whole integer lattice $\mathbb{Z}^{n+2}$, and in particular $z_{\nu}$ and $z_{\nu+1}$ are linearly independent.

The set

$$\{z \in \mathbb{R}^{n+2} : \zeta(z) < \zeta_{\nu}, Z(z) < Z_{\nu+1}\}$$

has no non-zero integer points inside. So analogously to (19) from Minkowski convex body theorem we have

$$\zeta_{\nu} Z_{\nu+1}^{\frac{2}{n}} \ll_n 1,$$

where the constant in the symbol $\ll_n$ may depend on the dimension $n$.

Put

$$\psi_\Theta(t) = \min_{z \in \mathbb{Z}^{n+2} : 0 < Z \leq t} \text{dist}(\{z\}, \mathcal{L}).$$

As all the norms in Euclidean spaces are equivalent, we see that in the definitions (24) of the exponents $\alpha(\Theta)$ and $\beta(\Theta)$ we may replace the function $\psi_{\Theta}^{\sup}(t)$ by the function $\psi_\Theta(t)$ and the result will be the same. So

$$\psi_\Theta(t) \leq t^{-\alpha}$$

for all $t$ large enough if and only if

$$\zeta_{\nu} \leq Z_{\nu+1}^{-\alpha}.$$  

In particular for any $\alpha < \alpha(\Theta)$ for all $\nu$ large enough one has

$$\zeta_{\nu} < Z_{\nu+1}^{-\alpha}.$$  

5. Successive best approximation vectors in two-dimensional subspace.
It may happen that three or more vectors
\[ z_\nu, z_{\nu+1}, \ldots, z_k \] (25)
lie in a certain two-dimensional linear subspace \( \pi \subset \mathbb{R}^{n+2} \). Then the following statement is valid.

**Lemma 3.** In the case when vectors (25) lie in a certain two-dimensional linear subspace \( \pi \) one has
\[ \zeta_\nu Z_{\nu+1} \leq \frac{12}{\sqrt{52} - 5} \cdot \zeta_{k-1} Z_k. \] (26)

**Corollary.** Suppose that \( \alpha < \alpha(\Theta) \). Then if \( \nu \) is large enough and vectors (25) lie in a certain two-dimensional linear subspace \( \pi \) one has
\[ \zeta_\nu \ll Z_{\nu+1}^{-1} Z_k^{1-\alpha}. \] (27)

**Proof of Lemma 3.**
First of all we consider the case \( \pi \cap \mathcal{L} = \{0\} \).
Consider the collection of ellipses \( \{G_\lambda^\nu\}_{\lambda > 0} \) They have commom major axes. We denote the one-dimensional subspace of major axes by \( \mathcal{P} \). The orthogonal one-dimensional subspace consisting of all common minor axes we denote by \( \mathcal{Q} \).
For every \( l \) from the interval \( \nu \leq l \leq k - 1 \) we define the value of \( \lambda_l \) from the condition
\[ z_l \in G_{\lambda_l}. \]
As
\[ \zeta_\nu > \zeta_{\nu+1} > \ldots > \zeta_{k-1} > \zeta_k \]
we have
\[ \lambda_\nu > \lambda_{\nu+1} > \ldots > \lambda_{k-1} > \lambda_k. \]
For \( \nu \leq l \leq k - 1 \) we put
\[ \xi_l = \text{dist} (\{z_l\}, \mathcal{P}). \]
One can see by the monotonicity argument that
\[ \zeta_\nu > \zeta_{\nu+1} > \ldots > \zeta_{k-1} > \zeta_k. \]
Define \( \Xi_l \) to be the length of a half of the minor axis of the ellipse \( G_{\lambda_l} \). Then
\[ \Xi_l \geq \xi_l. \] (28)
The planar convex set
\[ \mathcal{E}_l = \{z \in \pi : \zeta(z) < \zeta_l, Z(z) < Z_{l+1}\} \subset \pi \]
has no non-zero integer points inside. There are two pairs of independent integer points \( \pm z_l, \pm z_{l+1} \) on its boundary. For the two-dimensional volume of the set \( \mathcal{E}_l \) one has estimates
\[ 2\Xi_l Z_{l+1} \leq \text{vol}_2 \mathcal{E}_l \leq 4\Xi_l Z_{l+1}. \]
Consider the two-dimensional lattice
\[ \Lambda = \mathbb{Z}^{n+2} \cap \pi \]
with the two-dimensional fundamental volume det Λ. Then by Minkowski convex body theorem

$$\frac{\Xi_l Z_{l+1}}{2} \leq \det \Lambda \leq 2\Xi_l Z_{l+1}. $$

So

$$\Xi_\nu Z_{\nu+1} \leq 4\Xi_{k-1} Z_k. $$

(29)

Put

$$a = \sqrt{\sqrt{52 - 5} / 3}, \quad b = \sqrt{1 - a^2}. $$

Now we prove the inequalities

$$\xi_l \geq a\Xi_l, \quad \nu \leq l \leq k - 1. $$

(30)

Indeed there are at least two independent points $\mathbf{z}_{l-1}, \mathbf{z}_l$ in $\Lambda$ on the boundary of the set $E_{l-1}$. Thus $\mathbf{z}_{l-1}$ lies on the boundary of the ellipse $G_{\lambda_{l-1}}$ and $\mathbf{z}_l$ lie inside the ellipse $G_{\lambda_{l-1}}$.

Let $H$ be the half of the length of the major axis of $G_{\lambda_{l-1}}$. If the distance from $\mathbf{z}_{l-1}$ to the minor axis of $G_{\lambda_{l-1}}$ is greater than $bH$ then the distance from $\mathbf{z}_l$ to the minor axis of $G_{\lambda_{l-1}}$ is greater than $bH$ also. Easy calculation shows that in this case the point $\mathbf{z}_{l-1} - \mathbf{z}_l$ lies inside $E_{l-1}$. It is not possible. So the distance from $\mathbf{z}_{l-1}$ to the minor axis of $G_{\lambda_{l-1}}$ is not greater than $bH$. Hence the distance from $\mathbf{z}_{l-1}$ to the major axis of $G_{\lambda_{l-1}}$ is not less than $a\Xi_{l-1}$. But this distance is equal to $\xi_{l-1}$.

So we have

$$\xi_{l-1} \geq a\Xi_{l-1}. $$

Inequalities (30) are proved.

Applying the 0-central projection onto $\mathfrak{S}$ and taking into account Lemma 2 we see that

$$\frac{\zeta_{k-1}}{\xi_{k-1}} \geq \frac{\zeta_{\nu}}{\xi_{\nu}}. $$

(31)

From (28, 29, 30, 31) we immediately deduce (26).

The case $\dim \pi \cap \mathcal{L} = 1$ is easier. In this case the set $\mathcal{G}_\lambda$ is a union of two parallel lines and the function $f(t)$ is a constant function. So we may assume that $\xi_l = \Xi_l, \nu \leq l \leq k - 1$, inequalities (29) remain true, and instead of (31) one has $\frac{\zeta_{k-1}}{\xi_{k-1}} = \frac{\zeta_{\nu}}{\xi_{\nu}}$. So (26) follows in this case also. □

**Remark to the proof of Lemma 3.** Similar argument was used not by the author in [10, 11, 12] only, but by some other mathematicians. In particular similar argument was applied by Y. Cheung (see Theorem 1.6 from [2]).

**6. Dimension of subspace of best approximation vectors.**

It may happen that all the best approximation vectors $\mathbf{z}_\nu$ lie in a certain linear subspace of $\mathbb{R}^{n+2}$ of dimension less than $n + 2$. So we consider the value

$$R(\Theta) = \min\{r : \text{there exist a subspace } \mathcal{R} \subset \mathbb{R}^{n+2} \text{ and } \nu_0 \in \mathbb{Z}_+ \text{ s.t. } \forall \nu \geq \nu_j \mathbf{z}_\nu \in \mathcal{R}\}. $$

Let $\mathcal{R} = R(\Theta)$ be the linear subspace from the definition of $R(\Theta)$. We see that the lattice $\mathbb{Z}^{n+2} \cap \mathcal{R}$ is a lattice of dimension $R(\Theta)$. Here we would like to recall the definition of completely rational subspace. A subspace $\pi \subset \mathbb{R}^d$ is defined to be completely rational if the lattice $\mathbb{Z}^d \cap \pi$ has dimension $d$. So $\mathcal{R}$ is a completely rational subspace.

Consider the subspace

$$\mathcal{K} = \mathcal{K}(\Theta) = \mathcal{R} \cap \mathcal{L}. $$

---

2In the sketched proof in [10, 11] only this case was considered.
As there is an infinite sequence of integer points \( z_\nu \in \mathcal{R} \) such that the distance between \( z_\nu \) and \( \mathcal{L} \) tends to zero as \( \nu \) tends to infinity, we see that \( \dim \mathcal{K} > 0 \). So we have only two opportunities. Either \( \dim \mathcal{K} = 1 \), or \( \dim \mathcal{K} = 2 \) and in this case \( \mathcal{K} = \mathcal{L} \subset \mathcal{R} \).

The case \( \dim \mathcal{K} = 1 \) is easy. First of all we note that there is no completely rational subspace \( \pi \subset \mathbb{R}^{n+2} \) such that \( \mathcal{K} \subset \pi \subset \mathcal{R} \) and \( \dim \pi < \dim \mathcal{R} \).

If \( \dim \mathcal{R} = 2 \) then we deal with approximations to a one-dimensional subspace \( \mathcal{K} \) from the two-dimensional rational subspace \( \mathcal{R} \). It this case \( \alpha(\Theta) = 1 \) and there is nothing to prove.

If \( \dim \mathcal{R} > 2 \) then we deal with the approximations to a one-dimensional subspace \( \mathcal{K} \) from the rational subspace \( \mathcal{R} \) of dimension greater than two. In this case

\[
\frac{1}{\dim \mathcal{R} - 1} \leq \alpha(\Theta) \leq 1.
\]

So this case does not considered in Theorem 1.

**Remark.** The situation in the case \( \dim \mathcal{K} = 1, \dim \mathcal{R} > 2 \) is quite similar to the setting which was considered in Theorem A, statement (i). So in this case the inequality \((6)\) is valid. The inequality \((6)\) gives an optimal bound in the case \( \dim \mathcal{R} = 3 \). If \( \dim \mathcal{R} > 3 \) we may apply Theorem C part (i) or Theorem D and obtain an even better lower bound for \( \beta(\Theta) \) in terms of \( \alpha(\Theta) \): for \( R(\Theta) = 4 \) we have the bound \((11)\) from Theorem D, and for \( R(\Theta) > 4 \) from the inequality \((9)\) of Theorem C, part (i) we have

\[
\beta(\Theta) \geq \alpha(\Theta) \cdot \frac{\alpha(\Theta) + \dim \mathcal{R} - 3}{(\dim \mathcal{R} - 2)(1 - \alpha(\Theta))}.
\]

In any case here we have a bound which is better than the trivial bound \((5)\).

Now we consider the case \( \dim \mathcal{K} = 2 \) when \( \mathcal{K} = \mathcal{L} \subset \mathcal{R} \). Here we should not that under the conditions of Theorem 1 among two-dimensional vectors \((14)\) there are at least four vectors linearly independent over \( \mathbb{Z} \). So \( \mathcal{L} \) cannot lie in a completely rational subspace of dimension \( \leq 3 \), and if \( \dim \mathcal{K}(\Theta) = 2 \) then

\[
\dim \mathcal{R}(\Theta) \geq 4.
\]

(For more details see Section 2.1 from \([11]\) and especially formula \((21)\).) In the rest of the paper we suppose that \((32)\) holds.

**Remark.** The author does not know if in the case \( m = n = 2 \) there exists a matrix \( \Theta \) satisfying the condition of Theorem 1 and such that \( \dim \mathcal{R}(\Theta) = 2, \dim \mathcal{K}(\Theta) = 1 \). From Jarník’s result it follows that in the case \( m = n = 2 \) the situation with \( \dim \mathcal{R}(\Theta) = 3, \dim \mathcal{K}(\Theta) = 1 \) never happens (see the discussion in \([11, 13]\)).

**7. Four linearly independent vectors.**

From the condition \((32)\) we see that there exist infinitely many pairs of indices \( \nu < k, \nu \to +\infty \) such that

(a) both triples

\[ z_{\nu-1}, z_\nu, z_{\nu+1}; \quad z_{k-1}, z_k, z_{k+1} \]

consist of linearly independent vectors;

(b) there exists a two-dimensional linear subspace \( \pi \) such that

\[ z_l \in \pi, \quad \nu \leq l \leq k; \quad z_{\nu-1} \notin \pi, \quad z_{k+1} \notin \pi; \]

(c) the vectors

\[ z_{\nu-1}, z_\nu, z_k, z_{k+1} \]

are linearly independent.

If a pair if indices \((\nu, k)\) satisfy (a), (b), (c) we say that \((\nu, k)\) satisfy \((abc)\)-property.
Lemmas 4 and 6 below were actually proved by Jarník in [7]. However they were not stated by him explicitly. So we give a complete proof. Lemma 5 comes from [10] [11].

**Lemma 4.** Suppose that $\alpha(\Theta) > 1$. Then for all $\nu$ large enough one has

$$\Delta_\nu = \begin{vmatrix} x_{\nu,1} & x_{\nu,2} \\ x_{\nu+1,1} & x_{\nu+1,2} \end{vmatrix} \neq 0.$$  

Proof. Suppose that $\Delta_\nu = 0$. Consider the determinants

$$\Delta_{\nu,j} = \begin{vmatrix} x_{\nu,1} & y_{\nu,j} \\ x_{\nu+1,1} & y_{\nu+1,j} \end{vmatrix} = \begin{vmatrix} x_{\nu,1} & y_{\nu,j} - \theta_1^1 x_{\nu,1} - \theta_2^1 x_{\nu,2} \\ x_{\nu+1,1} & y_{\nu+1,j} - \theta_1^2 x_{\nu+1,1} - \theta_2^2 x_{\nu+1,2} \end{vmatrix}, \quad 1 \leq j \leq n.$$  

As $\alpha(\Theta) > 1$ we see that

$$|\Delta_{\nu,j}| \leq 2Z_{\nu+1} \zeta_\nu \to 0, \quad \nu \to \infty.$$  

That is why

$$\Delta_{\nu,j} = 0, \quad 1 \leq j \leq n.$$  

But we have supposed that $\Delta_\nu = 0$ also. This means that the vectors $z_\nu$ and $z_{\nu+1}$ are linearly dependent. This is a contradiction. □

**Lemma 5.** Suppose that $\alpha(\Theta) > 1$ and $1 < \alpha < \alpha(\Theta)$. Suppose that the pair of indices $(\nu, k)$ satisfies (abc)-property and $\nu$ is large enough. Then

$$Z_{k+1} \gg Z_\nu^\alpha Z_k^{\alpha-1}. \quad (33)$$

Proof.

To prove (33) we consider four linearly independent integer vectors $z_{\nu-1}, z_{k-1}, z_k, z_{k+1}$. Consider four-dimensional vectors

$$\begin{pmatrix} x_{\nu-1,1} \\ x_{k-1,1} \\ x_{k,1} \\ x_{k+1,1} \end{pmatrix}, \quad \begin{pmatrix} x_{\nu-2,1} \\ x_{k-2,1} \\ x_{k,2} \\ x_{k+1,2} \end{pmatrix}, \quad \begin{pmatrix} y_{\nu-1,1} \\ y_{k-1,1} \\ y_{k,1} \\ y_{k+1,1} \end{pmatrix}, \ldots, \quad \begin{pmatrix} y_{\nu-1,n} \\ y_{k-1,n} \\ y_{k,n} \\ y_{k+1,n} \end{pmatrix}.$$  

Among these vectors there are four linearly independent ones. From Lemma 4 we know that two vectors

$$\begin{pmatrix} x_{\nu-1,1} \\ x_{k-1,1} \\ x_{k,1} \\ x_{k+1,1} \end{pmatrix}, \quad \begin{pmatrix} x_{\nu-2,1} \\ x_{k-2,1} \\ x_{k,2} \\ x_{k+1,2} \end{pmatrix}$$

are linearly independent. So there exist indices $i \neq j$ such that four vectors

$$\begin{pmatrix} x_{\nu-1,1} \\ x_{k-1,1} \\ x_{k,1} \\ x_{k+1,1} \end{pmatrix}, \quad \begin{pmatrix} x_{\nu-2,1} \\ x_{k-2,1} \\ x_{k,2} \\ x_{k+1,2} \end{pmatrix}, \quad \begin{pmatrix} y_{\nu-1,i} \\ y_{k-1,i} \\ y_{k,i} \\ y_{k+1,i} \end{pmatrix}, \quad \begin{pmatrix} y_{\nu-1,j} \\ y_{k-1,j} \\ y_{k,j} \\ y_{k+1,j} \end{pmatrix}.$$  

We consider the determinant

$$D = \begin{vmatrix} x_{\nu-1,1} & x_{\nu-2,1} & y_{\nu-1,i} & y_{\nu-1,j} \\ x_{k-1,1} & x_{k-2,1} & y_{k-1,i} & y_{k-1,j} \\ x_{k,1} & x_{k,2} & y_{k,i} & y_{k,j} \\ x_{k+1,1} & x_{k+1,2} & y_{k+1,i} & y_{k+1,j} \end{vmatrix} =$$
corresponding to these integer vectors. As $D \neq 0$ we see that

$$1 \leq |D| \leq 24 \zeta_{\nu-1} \zeta_{k-1} Z_k Z_{k+1} \ll Z_{\nu}^{-\alpha} Z_k^{1-\alpha} Z_{k+1}$$

(here we use (24)) and Lemma 5 follows. \hfill \Box

**Lemma 6.** Suppose that $\alpha(\Theta) > 2$. Then for any positive $\varepsilon$ for all $\nu$ large enough if three best approximation vectors $z_{\nu-1}, z_{\nu}, z_{\nu+1}$ are linearly independent then $Z_{\nu+1} \gg Z_{\nu}^{\alpha(\Theta) - \varepsilon - 1}$.

Proof. Consider three-dimensional vectors

$$
\begin{pmatrix}
x_{\nu-1,1} \\
x_{\nu-1,2} \\
x_{\nu-1,1}
\end{pmatrix},
\begin{pmatrix}
x_{\nu-1,2} \\
x_{\nu,1} \\
x_{\nu+1,2}
\end{pmatrix},
\begin{pmatrix}
y_{\nu-1,1} \\
y_{\nu,1} \\
y_{\nu+1,1}
\end{pmatrix}, \ldots, 
\begin{pmatrix}
y_{\nu-1,n} \\
y_{\nu,n} \\
y_{\nu+1,n}
\end{pmatrix}.
\tag{34}
$$

As three vectors $z_{\nu-1}, z_{\nu}, z_{\nu+1}$ are linearly independent we see that among three-dimensional vectors there are three linearly independent vectors. From Lemma 4 we know that three-dimensional vectors

$$
\begin{pmatrix}
x_{\nu-1,1} \\
x_{\nu,1} \\
x_{\nu+1,1}
\end{pmatrix} \text{ and } \begin{pmatrix}
x_{\nu-1,2} \\
x_{\nu,2} \\
x_{\nu+1,2}
\end{pmatrix}
$$

are linearly independent. So there exists $j$ such that

$$\Delta = \begin{vmatrix} x_{\nu-1,1} & x_{\nu-1,2} & y_{\nu-1,j} \\ x_{\nu,1} & x_{\nu,2} & y_{\nu,j} \\ x_{\nu+1,1} & x_{\nu+1,2} & y_{\nu+1,j} \end{vmatrix} = \begin{vmatrix} x_{\nu-1,1} & x_{\nu-1,2} & y_{\nu-1,j} - \theta_j^1 x_{\nu-1,1} - \theta_j^2 x_{\nu-1,2} \\ x_{\nu,1} & x_{\nu,2} & y_{\nu,j} - \theta_j^1 x_{\nu,1} - \theta_j^2 x_{\nu,2} \\ x_{\nu+1,1} & x_{\nu+1,2} & y_{\nu+1,j} - \theta_j^1 x_{\nu+1,1} - \theta_j^2 x_{\nu+1,2} \end{vmatrix} \neq 0.
$$

Now we consider the inequality

$$1 \leq |\Delta| \leq 6 Z_{\nu+1} Z_{\nu} \zeta_{\nu-1}.$$

But for $\nu$ large enough we have $\zeta_{\nu-1} \leq Z_{\nu}^{-\alpha(\Theta) + \varepsilon}$, and Lemma 6 follows. \hfill \Box

8. Proof of Theorem 1.

We take $\alpha < \alpha(\Theta)$ close to $\alpha(\Theta)$. Suppose that $(\nu, k)$ satisfies (abc)-property and $\nu$ is large enough.

If $\alpha(\Theta) > 2$ and $\alpha > 2$ from Lemma 6 we have $Z_{\nu+1} \gg Z_{\nu}^{\alpha-1}$. If $\alpha(\Theta) \leq 2$ we have nothing but trivial bound $Z_{\nu+1} > Z_{\nu}$. So in any case

$$Z_{\nu+1} \gg Z_{\nu}^{\max(\alpha-1, 1)}. \tag{35}$$

Now we prove that either

$$Z_{\nu+1} \gg Z_{\nu}^{g(\alpha)}$$

or

$$Z_{k+1} \gg Z_k^{g(\alpha)}$$

As

$$\zeta_{\nu} < Z_{\nu+1}^{-\alpha}, \quad \zeta_k < Z_{k+1}^{-\alpha}$$

this will be enough to obtain Theorem 1.
Suppose that $g > \alpha - 1$. Either

$$Z_{k+1} \geq Z^g_k,$$

or

$$Z_{k+1} < Z^g_k.$$

The last inequality together with the inequality (33) of Lemma 5 gives

$$Z_k \geq Z^{\frac{\alpha - 1}{g - \alpha + 1}}_k.$$  (36)

Now from inequality (27) of Corollary to Lemma 3 and (35,36) we get

$$\zeta_\nu \leq Z^\nu \max(\alpha - 1,1) - \frac{\alpha(\alpha - 1)}{g - \alpha + 1}.$$  

So

$$\beta(\Theta) \geq \max_{g > \alpha - 1} \min \left( \alpha g, \max(\alpha - 1,1) + \frac{\alpha(\alpha - 1)}{g - \alpha + 1} \right).$$

But $g(\alpha)$ is the solution of (13). So Theorem is proved. □

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