Robustness of noise-present Bell’s inequality violation by entangled state

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Abstract

The robustness of Bell’s inequality (in CHSH form) violation by entangled state in the simultaneous presence of colored and white noise in the system is considered. A twophoton polarization state is modeled by twoparameter density matrix. Setting parameter values one can vary the relative fraction of pure entangled Bell’s state as well as white and colored noise fractions. Bell’s operator-parameter dependence analysis is made. Computational results are compared with experimental data [9] and with results computed using a one-parameter density matrix [8], which one can get as a special case of the model considered in this work.

1 Introduction

After the famous work by Einstein, Podolsky, Rosen (EPR) [1], where they expressed the concept of incompleteness of quantum description of physical reality, there were numerous attempts to build more complete theories, which would not violate the causality principle in the classical meaning.

In 1964 Bell [2], accepting EPR as the working hypothesis, formalized it as a deterministic world idea in terms of local hidden variable model (LHVM) based on the following principles: 1) measurement results are determined by properties the particles carry prior to, and independent of the measurement (“realism”); 2) results obtained at one location are independent of any actions performed at spacelike separation (“locality”); 3) the setting of local apparatus are independent of the hidden variables which determine the local results (“free will”) [3].

Bell showed that the above assumptions impose some constraints on statistical correlations in experiments involving bipartite systems. Such constraints were formulated in the form of the nowadays well-known Bell’s inequalities. Further Bell showed that the corresponding correlations, which one can obtain by quantum mechanical rules, violate these inequalities for some quantum mechanical states called entangled. In this way entanglement...
is that feature of the quantum formalism that gives specific purely quantum correlations that
can’t be simulated within any classical model. Later Bell’s inequalities were reformulated in
the form suitable for experimental verification or refutation.

In 1982 Aspect’s group (Alain Aspect et. al. [4]) performed a verification experiment
for possible violation of Bell’s inequalities in Clauser-Horne-Shimony-Holt (CHSH) form [5],
where a correlation measurement of twophoton polarization states was provided. Measurement
results correspond well with the quantum mechanics predictions. Experimental data
give Bell’s inequality violation by five standard deviations. Numerous later experiments
showed that their results are in agreement with the quantum mechanical description of nature.

Thus, specific quantum correlations obtained the status of reality and entangled states,
which provide such correlations, became an object of intensive research. It turned out that
entanglement can play in essence the role of a new resource in such scientific areas as quantum
cryptography, quantum teleportation, quantum communication and quantum computation.
This became a great stimulus for researching the methods of creation, accumulation, dis-
tributing and broadcasting of this resource.

2 Noise-present entanglement detection

One of the most important questions in the considered topic concerns methods of identifying
the presence of entanglement in one or another realistic quantum mechanical state. So far
as entangled states violate Bell’s inequalities, thus, the violation of Bell’s inequalities can
be the basic tool to detect entanglement. In realistic applications pure entangled states
become mixed states due to different types of noise. Thus a question about robustness of
Bell’s inequalities violation against the noise arises. In other words, one wants to know,
under what proportion of an entangled state and noise in a realistic mixed state can be
uncovered the presence of entanglement. The most reliable source of two-party entangle-
ment are polarization-entangled photons created by the parametric down-conversion process
(PDC) [6].

The entangled singlet twophoton state from the PDC process can be described as a
spherically symmetric function, which is one of the known Bell’s states:

\[ |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \],

where \(|0\rangle\) and \(|1\rangle\) are two mutually orthogonal photon polarization states. The density
matrix for the photon pair state in the presence of white noise is the following:

\[ \hat{\rho}_W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\hat{I}, \]

where \(\hat{I}\) is the \(4 \times 4\) identity matrix. These states are called Werner states [7].

Usually in Bell’s inequality violation tests with polarization entangled photons from the
PDC Werner states were used. But experimental evidence and physical arguments show that
the colorless noise model is not good for the description of states received in the PDC process.
A more realistic description is given by an alternative oneparameter noise model, where a singlet state is mixed with the decoherence terms, which are called "colored noise" [8, 9]:

$$\hat{\rho}_C = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{2}(|01\rangle\langle01| + |10\rangle\langle10|),$$

(3)

In the polarization density matrices (2) and (3), varying the parameter $p$ from 0 to 1 one can model different relative proportions for the pure entangled state $|\Psi^-\rangle$ and the noise. Bell’s inequality in the CHSC form:

$$|\beta| \leq 2,$$

(4)

where

$$\beta = -\langle A_0B_0 \rangle - \langle A_0B_1 \rangle - \langle A_1B_0 \rangle + \langle A_1B_1 \rangle$$

(5)

is called the Bell operator.

For maximal Bell’s inequality (4) violation analysis, separately in states with white (2) and colored (3) noise, in Cabello’s work (Adan Cabello at al. [8]) the following onequbit observables were taken:

$$A_0 = \sigma_z$$

(6)

$$A_1 = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x$$

(7)

$$B_0 = \cos(\phi)\sigma_z + \sin(\phi)\sigma_x$$

(8)

$$B_1 = \cos(\phi - \theta)\sigma_z + \sin(\phi - \theta)\sigma_x$$

(9)

The parameters $\theta$ and $\phi$ in (6)-(9) are determining analyzers orientation in experimental devices, $\sigma_x$ and $\sigma_z$ are the usual Pauli matrices. Computations showed, that for the Werner state (2) the maximal value of $\beta$ as $p$-parameter function is the following:

$$\beta_{\text{max}}(p) = 2\sqrt{2}p$$

(10)

and for all values of $p$ the maximal value $\beta$ is obtained by $\theta = \frac{\pi}{4}, \phi = \frac{\pi}{4}$.

Thus, Bell’s inequality (4) is violated only for $p > 1/\sqrt{2} \approx 0.707$. This implies, that in the case, where the entangled state $|\Psi^-\rangle$ is distorted only by white noise, entanglement presence can be detected if noise proportion is less then $\sim 29\%$.

In the colored noise case (3) the maximal value of $\beta$ for different values $p$ is achieved at different values of angles $\theta$ and $\phi$. The most interesting fact is that the state (3) violates the CHSH inequality for all values $0 < p \leq 1$. Thus, Bell’s inequality violation is extremely robust against colored noise.

In 2006 the work by Bovino (Fabio A. Bovino et al. [9]) appeared, discussing the experimental verification of the previously mentioned predictions concerning CHSH inequality robustness against colored noise. A crystal (beta-barium borate) was irradiated by a laser, working in pulsed mode, and in the PDC process photon pairs in polarization-correlated states were created. These states correspond to the following polarization density matrix:

$$\hat{\rho} = p|\Phi^+\rangle\langle\Phi^+| + \frac{1-p}{2}(|00\rangle\langle00| + |11\rangle\langle11|),$$

(11)

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is one of the four entangled Bell’s states. State $|1\rangle$ conforms to ordinary polarization and state $|0\rangle$ conforms to extraordinary ray polarization in the uniaxial crystal.
One-qubit observables in the Bovino’s experiment satisfied the expressions (6)-(9). Computations showed, that the $p$-parameter dependence of the $\beta_{\text{max}}$ value in the state (11) is just the same as in the state (3). The experimental setup made possible the regulation of the colored noise fraction, that is parameter $p$ was varying from zero to almost one. Particular cases in (11) are the pure state $|\Phi^+\rangle$ ($p = 1$) and just noise ($p = 0$).

The experimental values of $\beta_{\text{max}}(p)$ and the theoretical predictions comparison showed, that the polarization state (11) model generally appropriately describes the photon pair state from the PDC process. However for all $0 < p < 1$ experimental values of $\beta_{\text{max}}$ were found to be a little smaller than corresponding theoretical values. Due to that, in concordance with experimental data, the CHSH inequality is violated only for $p \gtrsim 0.2$, but not for all values of $p$, which followed from computations. The reason of such discrepancy can be the presence of some portion of white noise beside colored noise in the realistic polarization state.

In the current work theoretical analysis for robustness of Bell’s inequality (in CHSH form) violation with simultaneous presence of colored and white noise is performed. The density matrix for the two-photon polarization state in such a generalized model can be expressed in the form:

$$\hat{\rho}_{\text{CW}} = p|\Phi^+\rangle\langle\Phi^+| + \frac{r}{2}(|00\rangle\langle00| + |11\rangle\langle11|) + \frac{1 - (p + r)}{4}\hat{I}. \quad (12)$$

Varying the parameter $p$ in the range from 0 to 1, one can change the pure state $|\Phi^+\rangle$ fraction in (12), and putting $r$ from 0 to $(1 - p)$, with the value of $p$ fixed, one can change relative colored and white noise fractions. For $r = 0$ we have the particular case (2) (colored noise absence), and for $r = 1 - p$ we have (3) (white noise absence).

In the state (12) the quantity $\beta$, which responds to one-qubit observables (6)-(9) is a four-parameter function:

$$\beta_{\text{CW}}(p, r, \theta, \phi) = \cos(\phi)[(2p + r)(\sin^2(\theta) + \cos(\theta)) + r\cos(\theta)] - \sin(\phi)(2p + r)[\cos(\theta) - 1]\sin(\theta). \quad (13)$$

In the colored noise absence ($r = 0$) we have:

$$\beta_{\text{W}}(p, \theta, \phi) = 2p\{\cos(\phi)[\sin^2(\theta) + \cos(\theta)] - \sin(\phi)[\cos(\theta) - 1]\sin(\theta)\}, \quad (14)$$

and in the white noise absence ($r = 1 - p$):

$$\beta_{\text{C}}(p, \theta, \phi) = \cos(\phi)[(1 + p)\sin^2(\theta) + 2\cos(\theta)] - \sin(\phi)(1 + p)[\cos(\theta) - 1]\sin(\theta). \quad (15)$$

For fixed values of the parameters $p$ and $r$ the expression (13) is a function of $\theta$ and $\phi$. Solving the extremum problem for two-variable function, one can find the maximal values $\beta_{\text{CW}}^{\text{max}}(p, r)$, as well as the angles $\theta$ and $\phi$, that provide the maximal $\beta_{\text{CW}}(p, r)$.

In the Fig.1 the shaded surface graphically displays the $\beta_{\text{CW}}^{\text{max}}(p, r)$ as a function of two variables $p$ and $r$. For comparison in the figure the plane $\beta = 2$ is displayed, which is the boundary value of Bell’s inequality. The surface patch above the plane $\beta = 2$ is the CHSH inequality violation area.
In the Fig.2 projections on the \((p, r)\) plane of the traces \(\beta = \text{const}\) with the surface \(\beta_{CW}^{\text{max}}(p, r)\) are represented. From the figure one can see that the straight line \(p + r = 1\) (white noise absence) fully lies in the \(\beta_{\text{max}}^{\text{max}} > 2\) area, which corresponds to the above conclusion, that Bell’s inequality violation is robust against colored noise. For \(r = 0\) (colored noise absence) Bell’s inequality is violated only for \(p > 1/\sqrt{2}\). For any fixed \(p\) (pure entangled state weight factor) the \(\beta_{CW}^{\text{max}}\) decreases with the increasing white noise fraction. Thus, as expected, adding some amount of white noise to the colored one can reach better agreement of theoretically computed \(\beta_{\text{max}}^{\text{max}}\) values with experimental ones. Bell’s inequality violation is unsteady under the increasing white noise fraction for a fixed total amount (white and colored) of noise.
In the Fig.3a) and 3b) the surfaces, which graphically display the angles $\theta(p, r)$ and $\phi(p, r)$, that provide maximal values of $\beta$ as a function of $p$ and $r$ are represented.

In the Fig.4 the curves, which are intersection lines of the vertical planes $r + p = 1$ and $r = 0$ with the surface $\beta_{max}^{CW}(p, r)$ in the case, when in the photon polarization state there is no white noise, are represented. The dashed line illustrates the case of colored noise absence. The boundary case dependencies of $\beta_{max}^{max}$ on $p$ and $r$ coincide with the ones from the work [9].

In the Fig.5 the values of the angles $\theta$ and $\phi$, that provide maximal values of the Bell operator, are represented. Two solid curves correspond to the case, when in the twophoton...
polarization state (12) white noise is absent \((p + r = 1)\), and two dashed lines correspond to the case, when colored and white noise enter into the expression (12) with the same weight \(r = (1 - p)/2\). Solid curves coincide with the ones plotted in the work [9]. From the figure one can see that the values of the angles \(\theta\) and \(\phi\) for a fixed pure entangled state fraction \((p\) is constant\) depend on the distribution of weighting coefficients of white and colored noise. Thus, the orientation of the analyzers for obtaining maximal values of \(\beta\) depends on the fraction distribution between white and colored noise.

In the Fig.6 the points represent the experimental maximal values of \(\beta\) from the work [9]; the dashed curve displays theoretical predictions for the maximal values of \(\beta\) on the one-parameter colored noise model [8]; the solid curve illustrates theoretical calculations on the two-parameter (generalized) noise model with the white noise fraction being 3.5% of the total noise amount in the system. In the figure we can see that for such a noise proportion experimental data better corresponds to theoretical predictions, i.e. the generalized (two-parameter) noise model is more correct than the one parameter for realistic states description. But in this case too, as one can see in the figure, some experimental points lie above and below the theoretical curve. According to the two-parameter model, this is explained by the fact that by moving from one point to other not only does the total noise amount in the system change, but relative fractions of white and colored noise does too.
Fig. 6. The points represent experimental maximal values of $\beta$ from the work [9]; the dashed curve is the theoretical predictions for the maximal values of $\beta$ in the one-parameter colored noise model [8]; the solid curve shows theoretical calculations in the two-parameter (generalized) noise model with the white noise fraction being 3.5% of the total noise amount in the system.

This kind of interpretation is absolutely logical, because for each measurement experimental setup is tuned up in a new way (particulary, one has to change the analyzers orientation in space). Remaining in the theoretical model, which is considered in this work, and choosing the corresponding parameter $r$ values for each experimental point (for fixed $p$) one can fully conform theoretical computations with the experimental data. Let us recall, that the preselected values of the parameters $p$ and $r$, according to our model, determine the pure entangled state fraction and relative noise fractions. The percentage of white and colored noise fractions, that give coincidence between theoretical values $\beta_{\text{max}}$ and experimental data, is represented in the table. Experimental data was taken from the figure in the work [9].

Table of noise proportions in the system. Correspondence with experimental points in the Fig. 6

| Nr. | $p$   | $1 - p$ | white,% | colored,% | $r$   |
|-----|-------|---------|---------|-----------|-------|
| 1   | 0.02  | 0.98    | 2       | 98        | 0.96  |
| 2   | 0.06  | 0.97    | 3       | 97        | 0.92  |
| 3   | 0.17  | 0.83    | 4       | 96        | 0.80  |
| 4   | 0.24  | 0.76    | 2       | 98        | 0.75  |
| 5   | 0.32  | 0.68    | 2       | 98        | 0.67  |
| 6   | 0.42  | 0.58    | 5       | 95        | 0.55  |
| 7   | 0.52  | 0.48    | 5       | 95        | 0.46  |
| 8   | 0.64  | 0.36    | 7       | 93        | 0.40  |
| 9   | 0.75  | 0.25    | 15      | 85        | 0.21  |
| 10  | 0.85  | 0.15    | 15      | 85        | 0.13  |
3 Conclusions

For adequate modeling of the twophoton polarization state, created in the parametric down-conversion process (PDC type II), one should take into account the presence of colored noise as well as white. While Bell’s inequality violation is extremely robust against the colored noise (Bell’s inequality is violated for all $0 < p \leq 1$), the violation is unsteady under white noise. White noise presence, that is determined by a weighting coefficient of just $0.1$ ($p + r = 0.9$), as one can see in the Fig.2, leads to Bell’s inequality violation only for $p \gtrsim 0.5$. Simultaneously taking into account both colored and white noise gives possibility to conform theoretical computations with experimental data. Taking $p$ and $r$ as adjustable parameters one can determine colored and white noise fractions by comparison of theoretical calculations with experimental data. The best model is the one which explains experimental values of $\beta_{\text{max}}$ as well as the angles $\theta$ and $\phi$, that provide these values.

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