Radiation from neutral atoms and mirrors following prescribed trajectories

Alex Calogeracos
Department of Physics & Astronomy, University of Sussex, Brighton BN1 9QH, England
Physics Division, Air Force Academy, TG1010 Dekelia Air Force Base, Greece
E-mail: a.calogeracos@yahoo.co.uk

Abstract. Discussions of the Unruh effect are often confused by the unrealistic nature of the trajectory (acceleration ad infinitum) and by the invocation of accelerated observers. We present a formalism that allows the calculation of radiation emission from accelerated atoms and mirrors in time-dependent perturbation theory moving along asymptotically inertial trajectories. The description is unambiguously set in terms pertaining to the laboratory frame. In the limit of constant acceleration of infinite duration one recovers Unruh’s result.

1. Introduction
I shall review some relatively recent work (Barton and Calogeracos 1995, 2005, 2008) on radiation emission by neutral systems such as mirrors and atoms. Consideration of such effects first took place in the seventies in a series of papers seeking deep connections between fundamental quantum physics, thermodynamics, and gravity. The accelerating trajectory of a mirror can provide a useful simulation of a black hole if one wishes to calculate for instance the spectrum of photons created during gravitational collapse. One often requires the (scalar) electromagnetic field to strictly vanish at the position of the mirror thus introducing perfect reflection at all frequencies. The ensuing underlying conformal invariance of the theory allows an elegant mathematical treatment. The absence however of a Hamiltonian formulation renders difficult the comparison with any realistic model which would incorporate partial transmission and dispersion.

A similar distance from experimentally attainable scenarios sometimes marks the work that followed Unruh’s (1976) seminal observation that “an accelerated observer perceives the vacuum as a thermal bath of photons”. If an atom moves through the vacuum (i.e. through the ground state of the quantized Maxwell field) at constant proper acceleration of magnitude α for τ ranging from \(-∞\) to \(+∞\) (τ denotes the proper time) then in its instantaneous rest frame the atom will experience the vacuum as if the atom were at rest, and interacting with a heat bath at temperature \(T_U\) given by

\[ k_B T_U = \frac{\alpha \hbar}{2\pi c}, \]  

(1)

the subscript \(U\) standing for “Unruh”. One consequence of the statement is that an atom accelerated in vacuum should undergo spontaneous excitation, as if by absorbing a heat bath photon; such photons will also affect the rates of downwards transitions. The statement preceding equation (1) tells us very little about what one ought to expect in the case of a realistic...
trajectory i.e. one starting and ending with vanishing acceleration. In this context however we must take note of the work by Marzlin and Audretsch (1998) who calculate the excitation rate of an atom performing uniform circular motion. For a thorough review of previous work on the Unruh effect and for a discussion of various interpretational issues see Crispino, Higuchi and Matsas (2008).

In our work we look at semi-realistic models of mirrors and atoms interacting with the electromagnetic field and following assigned trajectories. In particular our programme is based on the following requirements: (a) the theory should possess an underlying relativistically covariant Lagrangean formulation, (b) the resulting Hamiltonian is used to compute transition amplitudes using nothing more exotic than conventional time-dependent perturbation theory, (iii) asymptotically inertial trajectories (i.e. uniform velocity at \( t = \pm \infty \)) only need be considered.

Although atoms and mirrors seem not to have very much in common and although the velocity regimes will be very different (relativistic speeds for the former and non-relativistic for the latter) it will transpire that the technicalities are similar. For either mirror or atom our programme is divided to three stages. First we construct a Hamiltonian describing the system interacting with the radiation field. Second we construct an effective Hamiltonian describing the system while subject to the constraint that forces it along a prescribed trajectory. It turns out that this step is technically somewhat delicate and it eventually leads to an effective Hamiltonian coinciding with the Routhian. Third we define the process and calculate the spectrum of emitted radiation in the case of a mirror and the rate for an upwards transition in the case of a ground state atom. These steps are presented in sections 2, 3, and 4 for the mirror and in sections 5, 6, and 7 for an atom.

2. A Hamiltonian formulation for a dispersive mirror.

As a preliminary step consider a scalar electromagnetic field interacting with a fixed mirror. A convenient Lagrangean is provided by

\[
L_0 = \int dx \left\{ \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \gamma \phi^2 \delta(x - \xi) \right\}
\]

where the subscript in \( L_0 \) reminds us that the mirror is fixed and \( \xi \) refers to the mirror’s position. The parameter \( \gamma \) controls the dispersivity of the mirror in a way to be discussed presently.

Variation with respect to the field yields the wave equation

\[
-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} - 2\gamma \phi \delta(x - \xi) = 0
\]

Integrating across the singularity at the mirror’s position we obtain

\[
\frac{\partial \phi}{\partial x}(\xi+) - \frac{\partial \phi}{\partial x}(\xi-) = 2\gamma \phi(\xi)
\]

(2)

For left-incident waves (right-incident are treated similarly) of frequency \( \omega \) we write

\[
\psi_{L\omega}(x) = \theta(-x) \left[ e^{i\omega x} + R(\omega)e^{-i\omega x} \right] + \theta(x)T(\omega)e^{i\omega x}
\]

(3)

where \( \theta \) is the Heaviside step-function and \( R, T \) are reflection and transmission coefficients that are determined by substituting (3) in condition (2):

\[
R(\omega) = -\frac{i\gamma}{\omega + i\gamma}, \quad T(\omega) = \frac{\omega}{\omega + i\gamma}.
\]

(4)
Thus $\gamma$ indeed controls the mirror’s transparency in an obvious way.

A covariant Lagrangean describing a dispersive moving mirror coupled to the radiation field is given by (Barton and Calogeracos (1995))

$$L = -(M + \gamma \phi^2(\xi, t)) \left(1 - \dot{\xi}^2\right)^{1/2} + \int dx \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2} \left(\frac{\partial \phi}{\partial x}\right)^2 \right\}$$

(5)

where $\xi(t), \dot{\xi}(t)$ are the mirror’s position and momentum respectively. The first term in the Lagrangean (5) coincides with that describing a scalar particle in relativistic mechanics with the particle mass augmented by the extra field contribution $\gamma \phi^2(\xi, t)$ and the second term is the familiar electromagnetic Lagrangean. The jump condition appropriate to a moving mirror (the generalization of (2)) is obtained by variation of $\phi$ in both terms in (5) and taking into account contributions arising from integration by parts of the second term. We quote the Hamiltonian derived from (5)

$$H = \sqrt{p^2 + (M + \gamma \phi^2(\xi, t))^2} + H_{rad}$$

(6)

and the version appropriate for small velocities

$$H_{NR} = M + \gamma \phi^2(\xi, t) + H_{rad} + \frac{p^2}{2M} \left(1 - \gamma \phi^2(\xi, t)/M\right)$$

(7)

where $p$ is the momentum conjugate to the mirror’s position $\xi$ and $H_{rad}$ is the Hamiltonian of a massless scalar field

$$H_{rad} = \int dx \frac{1}{2} \left\{ \Pi^2 + \left(\frac{\partial \phi}{\partial x}\right)^2 \right\}$$

(8)

($\Pi$ is the momentum conjugate to $\phi$).

The Hamiltonian (7) has been used to compute mass and energy shifts of the mirror arising from its interaction with the radiation field. However in this paper we shall discuss only questions that arise when the motion of the mirror is prescribed.

3. The effective Hamiltonian for a mirror following a prescribed trajectory.

In this section we take the mirror to follow a prescribed trajectory thus in the sequel $\xi(t)$ ceases to be an operator and becomes a c-number. The technical complications that we face at this stage must be appreciated. We may trivially rewrite wavefunctions (3) so that they refer to a mirror at a fixed position $\xi$; however the fact that the wavefunctions do depend on $\xi$ in an explicit manner makes the passage to a perturbative scheme highly non-obvious. The first step is to restart with Lagrangean (5) but now construct the Routhian (see for example Landau and Lifshitz (1976), para 41 or Sommerfeld (1952), para 42. We Legendre-transform with respect to the field variables only without touching $\xi$ and $\dot{\xi}$:

$$H^{(1)} = -(M + \gamma \phi^2(\xi, t)) \left(1 - \dot{\xi}^2\right)^{1/2} + H_{rad}.$$  

(9)

The reader may recall that the Routhian $H^{(1)}$ is to be used as a Lagrangean to yield the equations of motion for $\xi$ and $\dot{\xi}$ and as a Hamiltonian to yield the equations of motion for the field variables. The second step is to realize that the present problem may be profitably examined by going over to a frame co-moving with the mirror. The most convenient way to do this in a quantum-mechanical language is via the use of an explicitly time-dependent canonical transformation. It is reminded that if

$$U(t) = \exp (i S(t))$$

(10)
is such a transformation the Hamiltonian (call it $H_{\text{old}}$) transforms to a new Hamiltonian $H_{\text{new}}$ given by

$$H_{\text{new}} = U^{-1}H_{\text{old}}U + \Delta, \quad \Delta = i \frac{\partial U^{-1}}{\partial t} U. \quad (11)$$

If $\partial S/\partial t$ commutes with $S(t)$ then $i \partial U^{-1}/\partial t = (\partial S/\partial t) U$ and (11) assumes the simpler form

$$H_{\text{new}} = U^{-1}H_{\text{old}}U + \Delta, \quad \Delta = \frac{\partial S}{\partial t}. \quad (12)$$

New states $|\bar{\psi}\rangle$ are connected to old states $|\psi\rangle$ via

$$|\bar{\psi}\rangle = U^{-1}(t) |\psi\rangle \quad (13)$$

and their time-evolution is governed by

$$i \frac{\partial}{\partial t} |\bar{\psi}\rangle = H_{\text{new}} |\bar{\psi}\rangle . \quad (14)$$

We now define a unitary transformation

$$U_{\text{tr}} = \exp (-i \xi (t) P_{\text{rad}}) \quad (15)$$

where $P_{\text{rad}}$ is the momentum operator for the radiation field. We can readily check that

$$U_{\text{tr}}^{-1} \phi (\xi) U_{\text{tr}} = \phi (0).$$

In other words the mirror is now fixed at the origin of the coordinate system (this amounts to the transition to a co-moving frame). The transform $\overline{H}$ of the Routhian (9) is (using the abbreviation $B \equiv d\xi/dt$)

$$\overline{H} = H_{\text{rad}} - (M + \gamma \phi^2 (\xi, t)) \left(1 - B^2\right)^{1/2} - B(t) P_{\text{rad}} \quad (16)$$

and the non-relativistic version is

$$\overline{H}_{NR} = H_{\text{rad}} + \gamma \phi^2 (0) - B(t) P_{\text{rad}} \equiv \overline{H}_0 - B(t) P_{\text{rad}}. \quad (17)$$

Observe that the last term in (16) is due to the $\Delta$ term in (12) (compare exponents in (10) and (15)). The $\overline{H}_0$ part of the effective Hamiltonian is diagonalized by the modes (3). The last term is to be used perturbatively in the velocity $B$; the presence of $P_{\text{rad}}$ entails the creation of pairs of photons under non-trivially varying $B$.

4. The spectrum of the radiation emitted by the mirror.

Hamiltonian (17) was used in Barton and Calogeracos (1995) to calculate the energy spectrum of the emitted radiation and the radiative reaction force. Here we shall discuss the former. We expand the field $\phi$ in terms of left-incident and right-incident modes $\psi^{(\lambda)}$ (see (3)) multiplied by creation and annihilation operators $a^\dagger_\lambda, a_\lambda$ in the standard way and denote by $|\lambda\rangle$ the eigenstates of $\overline{H}_0$. We observe a $|\lambda\rangle$ is not a photon state; rather it is connected to a photon via $U(t)$ of equation (13). The phase carried by the eigenstates of $\overline{H}_0$ will prove crucial in the discussion of the Unruh effect. We assume that assume that at time $t = 0$ we start with the vacuum and write the solution of (14) (with $H_{\text{new}} = \overline{H}_{NR}$) in the form

$$|\bar{\psi}(t)\rangle = |0\rangle + \frac{1}{2} \sum_{\lambda\lambda'} C(\lambda\lambda'; t) e^{-i(\omega+\omega')t} |\lambda\lambda'\rangle \quad (18)$$
Standard time-dependent perturbation theory then yields

\[ C(\lambda\lambda'; t) = i \langle \lambda\lambda' | P_{\text{rad}} | 0 \rangle J_{\omega')(t) \]

where

\[ J_{\omega'}(t) = \int_0^t dt' e^{i(\omega+\omega')t'} B(t'). \]

In the case of an asymptotically inertial trajectory and after the acceleration has stopped the time-dependent part of the phase of \[ |\psi(t)\rangle \] equals just the energy \( \omega \) of the corresponding photon state. Thus the total radiated energy is

\[ E_{\text{rad}}(t) = \frac{1}{2} \sum_{\lambda\lambda'} |\langle \lambda\lambda' | P_{\text{rad}} | 0 \rangle|^2 |J_{\omega'}(t)|^2 (\omega + \omega') e^{-\alpha(\omega+\omega')} \]

where \( \alpha \) is a high-frequency cut-off. The calculation of the matrix element is straightforward and the result is given in terms of the reflection coefficient (4):

\[ E_{\text{rad}}(t) = \frac{1}{\pi} \int_0^\infty \int_0^\infty d\omega d\omega' \frac{\omega\omega'}{(\omega+\omega')^2} |R(\omega)|^2 |J_{\omega'}(t)|^2 e^{-\alpha(\omega+\omega')} \]

5. The Hamiltonian formulation for an atom whose nucleus follows a prescribed trajectory.

5.1. Generalities.

Our model includes a scalar radiation field interacting with a scalar nucleon and with a spin 1/2 electron and resides in three spatial dimensions. It is presented in some detail in Barton and Calogeracos (2005), (2008). I shall not duplicate the papers just cited; rather I will try to emphasize the principles underlying the calculation. The question is stated in very unambiguous terms. We consider an atom that starts at proper time \(-\infty \) is in its ground state \( i \) with no photons present and moves along the z axis at velocity \(-B_0\), then goes through an intermediate acceleration from \( \tau = -\tau_0 \) to time \( \tau = \tau_0 \) when the velocity is given by

\[ B = c \tanh(\alpha\tau/c) \]

and ends up at uniform velocity \(+B_0\) at times \( \tau > \tau_0 \) (\( \tau_0, \alpha \) and \( B_0 \) are connected by the condition that \( B(\tau) \) be continuous at \( \tau = \tau_0 \)). We wish to calculate the rate for the electron to make a transition to an excited state \( f \) with the emission of a photon having wave-vector \( k \). In the limit of \( \tau \to \infty \) the trajectory becomes Unruh’s hyperbolic trajectory and we can compare our answer with his. Observe that the question is phrased wholly in terms of quantities pertaining to inertial observers and that there is no mention of accelerated observers.

The Routhian \( H^{(1)} \) (the analog of (9)) describing the radiation field coupled to the electron and the nucleon with the nucleon’s trajectory \( \mathbf{R}(t) \) to be prescribed is readily written down:

\[ H^{(1)} = H_{\text{rad}} + (M + Zg\phi(R)) \sqrt{1 - B^2} + H_{\text{electron}} \]

The electromagnetic couplings of the electron and nucleon are coupling \( g \) and \( Zg \) respectively, \( M \) is the nuclear mass, and the electron Hamiltonian \( H_{\text{electron}} \) reads

\[ H_{\text{electron}} = \alpha \cdot \mathbf{p} + \beta m + \beta g\phi(r) \]

Observe the mass-like coupling of the electron to the radiation field.
5.2. The sequence of transformations

We now perform a sequence of unitary transformations, the first of them \( U_{tr} \) being the exact analog of (15):

\[
U_{tr} = \exp (-i \mathbf{R}(t) \cdot (\mathbf{P}_{rad} + \mathbf{p}))
\]  

(22)

where the exponent features the sum of the electron momentum and the radiation field momentum operators. Then \( H^{(1)} \) is transformed via (12) to

\[
H^{(2)}(B) = (M + Z g \phi(0)) \sqrt{1 - B^2} + [H_{rad} - B \cdot \mathbf{P}_{rad}] + [\alpha \cdot \mathbf{p} - B \cdot \mathbf{p} + \beta m + \beta g \phi(r)]
\]  

(23)

The \( B \)-proportional terms in (23) are the exact analogs of the last term in (16).

Our Hamiltonian has to provide the binding potential for the electron. This is supplied in the form of a Lienard-Wiechert potential by the so-called passive source transformation \( U_{ps} \) that eliminates the \( \phi(0) \) term in (23), thus decoupling the nucleon from the field, and also provides the electron-photon interaction that induces the electronic transitions (such transformations are well-known from old fashioned nucleon-meson interactions; see e. g. Wentzel (2005)). We write

\[
U_{ps} = \exp (i S_{ps}), \quad S_{ps} = -i \sum a_k^* a_k
\]  

(24)

where \( a_k, a_k^\dagger \) are the creation and annihilation operators defined by the expansion

\[
\phi(x) = \sum_k \left[ a_k \phi_k - a_k^\dagger \phi_k^*(x) \right]
\]  

(25)

where \( \phi_k \) are plane waves. The c-numbers appearing in (24) are designed so as to eliminate the \( \phi(0) \) and they obviously depend on the velocity.\(^1\) We shall only quote the Hamiltonian \( H^{(3)}(B) \) to which \( U_{ps} \) leads

\[
H^{(3)}(B) = \left[ \Delta M \sqrt{1 - B^2} + \Delta_{ps} \right]_{\text{drop}} + [H_{rad} - B \cdot \mathbf{P}_{rad}] + H^{(3)}_{atom}(B) + H^{(3)}_{int}
\]  

(26)

where

\[
H^{(3)}_{atom}(B) \equiv \alpha \cdot \mathbf{p} - B \cdot \mathbf{p} + \beta m + \beta V_B(r)
\]  

(27)

\[
H^{(3)}_{int} \equiv \beta g \phi(r)
\]  

(28)

The binding potential appearing in \( H^{(3)}_{atom}(B) \) as a result of \( U_{ps} \) is

\[
V_B(r) = -\frac{Z g^2/4\pi}{\sqrt{r_\parallel^2/(1 - B^2(t)) + r_\perp^2}}^{1/2}
\]  

(29)

where the \( ||, \perp \) subscripts refer to direction parallel and perpendicular to \( B \). The \( \Delta M \)-proportional term in (26) is a mass-shift arising from the nucleon-photon interaction and is uninteresting for current purposes. The \( \Delta_{ps} \) term in (26) is of course the second term in (11) as provided by \( U_{ps} \). Recall the remark following (25) that \( U_{ps} \) depends on the velocity \( B \) so \( \Delta_{ps} \) gives rise to acceleration-dependent terms in the Hamiltonian. Because of the presence of creation and annihilation operators in \( S_{ps} \) (24) the term \( \Delta_{ps} \) in fact gives rise to radiation emission from the accelerating nucleon (as opposed to radiation from electronic transitions).

\(^1\) The c-numbers \( \alpha_k \) are given in BCI. Transformation (24) would have to be implemented in the case of vanishing velocity \( B = 0 \) in order to get rid of the \( \phi(0) \) term and provide the binding of the electron to the static course.
The third transformation $\hat{U}_n(B)$ results as follows. Let $|n; B\rangle$ be the $n$th eigenstate of $H_{\text{atom}}^{(3)}(B)$ and $\varepsilon_n(B)$ the corresponding eigenvalue. According to (27) they satisfy
\[ \{-\gamma_0\varepsilon_n(B) - \gamma_0 B \cdot p + \gamma \cdot p + m + V_B(r)\} |n; B\rangle = 0 \tag{30} \]
Thus the atomic Hamiltonian (27) reads
\[ H_{\text{atom}}^{(3)}(B) = \sum_n |n; B\rangle \varepsilon_n(B) \langle n; B| \tag{31} \]
At zero velocity $|n; 0\rangle$ and $\varepsilon_n \equiv \varepsilon_n(0)$ satisfy
\[ \{-\gamma_0\varepsilon_n + \gamma \cdot p + m + V_0(r)\} |n; 0\rangle = 0 \tag{32} \]
The transformation $\hat{U}_n(B)$ by definition satisfies
\[ |n; B\rangle = \hat{U}_n(B) |n; 0\rangle \]
and is explicitly constructed in BCI where it is also shown that
\[ \varepsilon_n(B) = \varepsilon_n \sqrt{1 - B^2} \tag{33} \]
I restrict myself to a few comments. Transformation $\hat{U}_n(B)$ involves (a) a rescaling of $r_n$ so that the denominator of (29) becomes just $r$, (b) a product of a spinor boost and a gauge transformation so that we get rid of the $\gamma_0 B \cdot p$ in (30), (c) an overall factor that ensures the correct normalization of the wavefunctions
\[ \int d^3r \psi^{(n)}(r, B) \psi^{(q)}(r, B) = \delta_{nq} \]
where
\[ \psi^{(q)}(r, B) \equiv \langle r|n; B\rangle \]
These requirements are met by
\[ \hat{U}_n(B) = (1 - B^2)^{1/4} U_{sc} \hat{U}_{\text{spin}} W_n \tag{34} \]
where the operators appearing above satisfy
\[ U_{sc}^{-1} \{ \gamma_0, r \} U_{sc} = \left\{ r \gamma_0 \gamma_0, r \right\} \tag{35} \]
\[ \hat{U}_{\text{spin}}^{-1} \{ \gamma_0, \gamma_0, r \} \hat{U}_{\text{spin}} = \left\{ \frac{\gamma_0 + B_0}{\sqrt{1 - B^2}}, \frac{\gamma_0}{\sqrt{1 - B^2}}, \frac{B_0 + \gamma_0}{\sqrt{1 - B^2}} \right\} \tag{36} \]
\[ W_n = \exp \left\{ i\varepsilon_n(B) \frac{B \cdot r}{\sqrt{1 - B^2}} \right\} \tag{37} \]
Observe that these transformations carry their own $\Delta$ terms (see (11)) which are acceleration-dependent and in fact describe acceleration stresses within the atom; they will be interesting when if it ever comes to contemplating an actual experiment. Note also that transformation $\hat{U}_n(B)$ is not unitary because the spinor boost (see (b) above) is not.
The final unitary transformation connects the sets of states \( \{|n; B\}\) and \( \{|n; 0\}\) and reads

\[
\mathcal{U}(B) = \sum_n |n; B\rangle \langle n; 0| = \sum_n \hat{U}_n(B) |n; 0\rangle \langle n; 0|
\]

and is unitary by construction:

\[
\mathcal{U}^{-1}(B) = \sum_n |n; 0\rangle \langle n; B| = \sum_n |n; 0\rangle \langle n; 0| \hat{U}_n^\dagger(B) = \mathcal{U}^\dagger(B)
\]

(recall the non-unitarity of \( \hat{U}_n \) and observe the presence of \( \hat{U}_n^\dagger(B) \) in the last equality above.

We can now at last construct the final Hamiltonian by transforming \( H^{(3)}(B) \) of (26):

\[
H^{(4)}(B) = \mathcal{U}^{-1}(B) H^{(3)}(B) \mathcal{U}(B) = \\
[H_{\text{rad}} - B \cdot P_{\text{rad}}] + \sqrt{1 - B^2(t)} \sum_n |n; 0\rangle \varepsilon_n \langle n; 0| \\
+ \sqrt{1 - B^2(t)} \sum_{n,q} |n; 0\rangle \langle n; 0| \exp(-i(\varepsilon_n - \varepsilon_q)B \cdot r) \\
\times \beta \phi (r_g \sqrt{1 - B^2, r_\perp}) |q; 0\rangle \langle q; 0|
\]

We should here pause to record what the end product of this rather long subsection is. We started with the rather forbidding Hamiltonian (23) and ended with (39) where (i) the square bracket is already familiar from the mirror problem, (ii) a diagonal term describing bound electrons, (iii) the third term is the descendant of the interaction (28). The factor \( \sqrt{1 - B^2(t)} \) describes time-dilation. Recall that the Hamiltonians we have been quoting are descendants of the original Routhian (20) hence the eigenvalues are frequencies rather than energies. In the next subsection we shall look closer to the interplay between the frequencies that govern photon and electron states resulting from the first two terms in (39).

5.3. Evolution of photon and electron states under the free Hamiltonian and Doppler shifts. Let \( |k, t\rangle \) be an eigenstate to the first term \( H_{\text{rad}} - B \cdot P_{\text{rad}} \) of (39) corresponding to momentum \( k \). The Schroedinger equation gives the time dependence

\[
|k, t\rangle = \exp \left\{ i \left[ R(t) - R(0) \right] \cdot k \right\} \exp \left\{ -ikt \right\} |k, t = 0\rangle \quad (40)
\]

and in the case of uniform motion

\[
|k, t\rangle = \exp \left\{ -it \left( k - B \cdot k \right) \right\} |k, t = 0\rangle \quad (41)
\]

Similarly the time-evolution of the \( n \)th eigenstate \( |\psi^{(n)}(4)\rangle \) of the second term in (39)

\[
H^{(4)}_{\text{atom}}(B) = \sqrt{1 - B^2(t)} \sum_n |n; 0\rangle \varepsilon_n \langle n; 0| \quad (42)
\]

evolves according to

\[
|\psi^{(n)}(4)(t)\rangle = \exp(-i\varepsilon_n \tau) |n; 0\rangle, \quad \tau \equiv \int^t dt' \sqrt{1 - B^2(t')}
\]

Given the remark at the end of the previous subsection the occurrence of the proper time \( \tau \) in (43) is not unexpected. It is of interest to see how the auxiliary state \( |\psi^{(n)}(4)(t)\rangle \) is expressed
in terms of electron eigenstates of the original Hamiltonian (in the language of (13) we wish to construct \( |\psi\rangle \) given \(|\overline{\psi}\rangle\):

\[
|\psi^{(n)}(t)\rangle \equiv U_{tr}(R)U_{ps}(B)|\psi^{(4)}(t)\rangle
\]

We recall that \( U_{ps} \) does not act on electronic degrees of freedom and use (38), (34) and (43):

\[
|\psi^{(n)}(t)\rangle = \exp(-i\varepsilon_n\tau)U_{tr}(R)(1 - B^2)^{1/4}U_{sc}\hat{U}_{spin}W_n|n;0\rangle
\]

The meaning of the above becomes much more transparent if we write the Schrödinger wavefunction (the inner product of the above with \( \langle r| \) and let the above operators act on \( \langle r| \)):

\[
\langle r|\psi^{(n)}(t)\rangle = (1 - B^2)^{1/4}\exp(-i\varepsilon_n\tau)\exp\left(\frac{i\varepsilon_n B (r|| - R||)}{\sqrt{1 - B^2}}\right)
\times \hat{U}_{spin}(B)\psi_n\left(\frac{r|| - R||}{\sqrt{1 - B^2}}, r\perp - R\perp; 0\right)
\]

where

\[
\psi_n (r; 0) \equiv \langle r|n;0\rangle
\]

is the standard electron wavefunction for an atom at rest.

An insight as to the nature of the photon and electron eigenstates of the auxiliary Hamiltonian may be gained as follows. Consider a downwards transition \( q \rightarrow n + \gamma \) \((q,n)\) are electron states) taking place in flight while the nucleon has velocity \( B \). Lowest order time-dependent perturbation provides a \( \delta \)-function that connects frequencies and so it leads to

\[
(\varepsilon_q - \varepsilon_n)\sqrt{1 - B^2} = k - B \cdot k
\]

or

\[
k = \frac{k_0\sqrt{1 - B^2}}{1 - B \cdot k}
\]

which is indeed the standard expression for the Doppler shift for a source of velocity \( B \).

6. Acceleration-induced transitions.

6.1. The amplitude

The treatment in the previous section should have made clear how the phase factors in electron and photon states for a time-dependent \( B \) may well result in the spontaneous (i.e. in the absence of incident photons) excitation of an atom. I shall give an account of how the calculation proceeds (for more details see BCII). At this stage we do not yet commit ourselves to the hyperbolic (Unruh) trajectory. The specific process we are interested in is an atomic up transition from \( i = 1s \) to \( f = 2pm \) (in spectroscopic notation), accompanied by the emission of a single photon \( k \) into the initial vacuum. The transition is induced by the last term in (39) (the rest of the terms form the free Hamiltonian whose eigenstates were discussed in the previous section). Then, in an obvious notation, the standard perturbative Ansatz, starting bare at \( t = -\infty \), i.e. with \( c_{qk}(-\infty) = 0 \), reads

\[
|\psi; t > = |i > |0 > \exp\left\{-i\varepsilon_i\int_0^t dt' \sqrt{1 - B^2(t')}\right\}
\]
\[ + \sum_q \int d^3k \mid q \mid \mathbf{k} > \exp \left\{ -i \int_0^t dt' \left[ k - \mathbf{B}(t') \cdot \mathbf{k} + \varepsilon_q \sqrt{1 - B^2(t')} \right] \right\} c_q(t), \]

where \( \mid 0 \rangle \) is the vacuum (no-photon) state and \( \mid \mathbf{k} \rangle \) the state with one photon having wave-vector \( \mathbf{k} \). This yields the first-order solution

\[ c_{f_k}(\infty) = -i \int_{-\infty}^{\infty} dt F < f \mid H_{\text{int}}(t) \mid i > \exp \left\{ i \int_0^t dt' \left[ \varepsilon_{f_i} \sqrt{1 - B^2(t')} + k - \mathbf{B}(t') \cdot \mathbf{k} \right] \right\}, \tag{49} \]

where the auxiliary function \( F(\tau) \), of the form \( F = \exp(-\lambda |\tau|) \), switches off the interaction adiabatically and covariantly as \( |\tau| \to \pm \infty \). The switching factor is needed in principle to make sense of the outer integrals in (49) over the initial and final uniform velocity stages, but disappears from the end-results upon taking the limit \( \lambda \to 0 \) afterwards.

The matrix element in (49) is

\[ \frac{g}{4\pi^{3/2} k^{1/2}} \sqrt{1 - B^2} < f \mid \beta \exp \left\{ -i \left[ \varepsilon_{f_i} \mathbf{B} \cdot \mathbf{r} + k_\parallel r_\parallel \sqrt{1 - B^2} + \mathbf{k} \cdot \mathbf{r}_\perp \right] \right\} \mid i >. \tag{50} \]

and so in terms of proper time \( \tau \)

\[ c_{f_k}(\infty) = -\frac{ig}{4\pi^{3/2} k^{1/2}} \int_{-\infty}^{\infty} d\tau F(\tau) \exp \left\{ i \int_0^\tau d\tau' \left[ \varepsilon_{f_i} + \frac{k - \mathbf{B}(\tau') \cdot \mathbf{k}}{\sqrt{1 - B^2(\tau')}} \right] \right\} \times < f \mid \beta \exp \left\{ -i \left[ \varepsilon_{f_i} B(\tau) r_\parallel + k_\parallel r_\parallel \sqrt{1 - B^2(\tau)} + \mathbf{k} \cdot \mathbf{r}_\perp \right] \right\} \mid i >. \tag{51} \]

\( \tau \) From here on we consider only the rectilinear trajectory of the type described just before (19). We introduce the scaled variables

\[ h \equiv c \varepsilon_{f_i}/\hbar \alpha, \quad \kappa \equiv k c^2/\alpha. \tag{52} \]

\( h \) ought not to be confused with Planck’s constant. The electron acceleration in the Bohr model is of the order of \( 0.90 \times 10^{23} \text{ m/s}^2 \) whereas laboratory accelerations are unlikely to exceed those of order \( 10^6 \text{ m/s}^2 \) achievable by the ultracentrifuge. Thus we confine our attention to

\[ h \gg 1. \tag{53} \]

The result (76) and a close look at the properties of the modified Bessel function featuring in the latter also entail

\[ \kappa/h \sim \mathcal{O}(1). \tag{54} \]

The atomic matrix element in (17) is given by

\[ < 2p, \pm 1 \mid \ldots \mid 1s > \to \text{Mak}_{\pm}, \quad M = \pm i 128/243. \tag{55} \]

Then

\[ c_{f_k} = \frac{g a M}{4\pi^{3/2}} \cdot \frac{k_\pm}{k^{1/2}} L, \tag{56} \]

where

\[ L = \left\{ \int_{-\infty}^{-\tau_0} + \int_{-\tau_0}^{\tau_0} + \int_{\tau_0}^{\infty} \right\} d\tau \exp(-i I(\tau)) = L_i + L_a + L_f \tag{57} \]

requires the phases

\[ I(\tau) \equiv \int_0^\tau d\tau' \left[ \varepsilon_{f_i} + \frac{k - B(\tau') k_3}{\sqrt{1 - B^2(\tau')}} \right]. \tag{58} \]
It is straightforward to evaluate $I(\tau)$ both in the initial and final stages $|\tau| > \tau_0$, and in the acceleration stage $|\tau| < \tau_0$ defined in (19).

The amplitudes $L_{i,f}$ are easy to evaluate, by virtue of the adiabatic switching factors $F$, and because (58) shows that the phases $L_{i,f}$ are linear in $x$. We define the polar angle $\vartheta$ of $\kappa$ with respect to $\mathbf{B}$:

$$
\kappa = \kappa \cos \vartheta, \quad \kappa_\perp = \kappa \sin \vartheta.
$$

(59)

For nonzero $x_0$ the two amplitudes combine into

$$
L_{i,f} \equiv L_i + L_f = \frac{(-2i/\alpha) \exp \{i\kappa_3 (\cosh x_0 - 1)\}}{[\hbar + \kappa \cosh x_0]^2 - \kappa_3^2 \sinh^2 x_0} 
\times \{ -i(\hbar + \kappa \cosh x_0) \sin(\kappa \sinh x_0 + \hbar x_0) + \kappa_3 \sinh x_0 \cos(\kappa \sinh x_0 + \hbar x_0) \}.
$$

(60)

We turn to the amplitude $L_\alpha$ and write

$$
L_\alpha = \exp \{i [\hbar x_3 - \kappa_3] \} J/\alpha.
$$

(61)

where

$$
J \equiv \int_{u_1}^{u_2} du \exp \{ -i [\hbar u + \kappa_\perp \sinh(u)] \}, \quad u_2 = x_0 + x_3, \quad u_1 = -x_0 + x_3.
$$

(62)

The quantity $x_3$ is defined by

$$
x_3 \equiv \log \left[ \frac{\kappa - \kappa_3}{\kappa_\perp} \right] = \log [\tan(\vartheta/2)]
$$

(63)

As $x_0 \to \infty$ clearly $L_i, L_f$ and therefore $L_{i,f}$ vanish like $\exp(-x_0)$

$$
|L_{i,f}|^2 \to 16 \exp(-2x_0) \frac{\kappa^2 \sin^2 [\kappa \sinh x_0 + \hbar x_0]}{\alpha^2 \kappa_\perp^2} \{ \kappa_3^2 \sinh^2 [\kappa \sinh x_0 + \hbar x_0] + \kappa_3 \cos^2 [\kappa \sinh x_0 + \hbar x_0] \},
$$

(64)

and so we need consider only $L_\alpha$:

$$
\lim_{x_0 \to \infty} L = \lim_{x_0 \to \infty} L_\alpha \equiv L_\infty.
$$

(65)

The limit $x_0 \to \infty$ entails $u_2 \to \infty$ and $u_1 \to -\infty$. In virtue of the increasingly fast oscillation of the integrand, the integral $J$ remains convergent, so that

$$
J_\infty \equiv \lim_{x_0 \to \infty} J = \int_{-\infty}^{\infty} du \exp \{ -i [\hbar u + \kappa_\perp \sinh(u)] \} = 2 \int_0^{\infty} du \{ \cos [\hbar u] \cos [\kappa_\perp \sinh(u)] - \sin [\hbar u] \sin [\kappa_\perp \sinh(u)] \}.
$$

(66)

The relations (Abramowitz & Stegun 1965, eqs 9.6.22 and 9.6.24)

$$
K_\nu(x) = \sec(\nu \pi/2) \int_0^{\infty} ds \cos(x \sinh s) \cosh(\nu s) = \csc(\nu \pi/2) \int_0^{\infty} ds \sin(x \sinh s) \sinh(\nu s), \quad (|\text{Re} \nu| < 1, \ x > 0),
$$

(67)

then lead to

$$
J_\infty = 2 \{ \cos(i\hbar \pi/2) + i \sin(i\hbar \pi/2) \} K_{i\hbar}(\kappa_\perp) = 2 \exp(-\hbar \pi/2) K_{i\hbar}(\kappa_\perp),
$$

(68)

$$
L_\infty = \frac{2}{\alpha} \exp \{i [\hbar x_3 - \kappa_3]\} \exp(-\pi h/2) K_{i\hbar}(\kappa_\perp).
$$

(69)

For transversely emitted photons (69) reduces to

$$
L_\infty(\vartheta = \pi/2) = (2/\alpha) \exp(-\pi h/2) K_{i\hbar}(\kappa).
$$

(70)
6.2. The probability rate.

The transition probability is defined by

$$P_f(\tau) = \int d^3k P_{fk}(\tau), \quad P_{fk}(\tau) = |c_{fk}(\tau)|^2$$

Equations (56), (70) and (71) lead to

$$P_f = g^2 a^2 \frac{|M|^2}{16\pi^3} \int d^3k \frac{k^2}{k} |L|^2,$$  

We restrict ourselves to large durations of the acceleration and accordingly use the limiting form of $L$ (65) which we substitute in expression (72) for the transition probability:

$$P_f(\tau) = g^2 a^2 \frac{|M|^2}{16\pi^3} \alpha \int d^3\kappa \frac{\kappa^2}{\kappa} |L_\infty(\tau)|^2$$

We define the transition rate

$$W = \lim_{\tau \to \infty} \frac{1}{2} \frac{dP_f(\tau)}{d\tau} = \lim_{x_0 \to \infty} \frac{\alpha}{2} \frac{dP_f}{dx_0}$$

(where the factor 1/2 stems from the fact that the duration of the acceleration is $2\tau$). Relation (73) leads to

$$W = g^2 a^2 \frac{|M|^2}{16\pi^3} \alpha \lim_{x_0 \to \infty} \int d^3\kappa \frac{\kappa^2}{\kappa} \text{Re} \left\{ L_\infty^* \frac{dL_\infty}{dx_0} \right\}$$

Eventually using (69)

$$W = g^2 a^2 \frac{|M|^2}{2\pi^2} \alpha \int d\kappa \kappa^3 K_{ih}(\kappa_\perp)$$

We make use of the remarkable integral (Prudnikov et al 1990)

$$\int_0^\infty dx x^2 K_h(x) = \frac{1}{3} \Gamma(2 + i\hbar)^2 = \frac{1}{3} (1 + \hbar^2)^2 \frac{\pi h}{\sinh(\pi h)}.$$  

(77)

to obtain the final expression for the transition rate

$$W = g^2 a^2 \frac{|M|^2}{3\pi} \frac{\alpha}{2} e^{-\pi h} \left( 1 + 1/h^2 \right) e^{2\pi h} - 1$$

(78)

Notice the Planckian factor, with $2\pi h = 2\pi \varepsilon_{fi}/\alpha = \varepsilon_{fi}/k_B T_U$ featuring the Unruh temperature.

The $1/h^2$ in the numerator of (78) is a correction to the Unruh result (see also Marzlin and Audretsch (1998)).

7. Conclusion

We reviewed a relativistic formalism that allowed us to prescribe the trajectory $\mathbf{R}(t)$ of a mirror or of the nucleus of a bound state interacting with the radiation field. We were then able to calculate the emission of the pattern of emitted radiation in the case of a mirror and the transition rate for atomic transitions using time-dependent perturbation theory. The processes were unambiguously described in terms of quantities pertaining to the laboratory frame. In the case of the atom we examined an asymptotically inertial trajectory and we computed the transition rate for an acceleration-induced upwards transition; in other words we started with an atom in its ground state and computed the rate for emission of a photon plus transition of the electron to an excited state. The energy is of course put in by the external agent that accelerates the nucleon. In the limit of constant acceleration of infinite duration the atom behaves as if it were stationary in a heat-bath of photons at the Unruh temperature.
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