Model problem for integro-differential Zakai equation with discontinuous observation processes in Hölder spaces

R. Mikulevicius and H. Pragarauskas
University of Southern California, Los Angeles
Institute of Mathematics and Informatics, Vilnius

August 6, 2010

Abstract

The existence and uniqueness of solutions of the Cauchy problem to a stochastic parabolic integro-differential equation is investigated. The equation considered arises in nonlinear filtering problem with a jump signal process and jump observation.

1 Introduction

In a complete probability space \((\Omega, \mathcal{F}, P)\) with a filtration of \(\sigma\)-algebras \(\mathcal{F} = (\mathcal{F}_t)\) satisfying the usual conditions, we consider the linear stochastic integro-differential parabolic equation

\[
\begin{aligned}
    du(t, x) &= (A^{(\alpha)}u(t, x) + f(t, x))dt + \int_U g(t, x, v)q(dt, dv) \\
    u(0, x) &= 0
\end{aligned}
\]

in \(H, \mathbb{R}^d\) (1)

of the order \(\alpha \in (0, 2]\), where \(H = [0, T] \times \mathbb{R}^d\), \(q(dt, dv)\) is a martingale measure on a measurable space \((\{0, \infty\} \times U, \mathcal{B}(\{0, \infty\}) \otimes \mathcal{U})\), \(g\) is an \(\mathcal{F}\)-adapted measurable real-valued function on \(H \times U\), \(f\) is an \(\mathcal{F}\)-adapted measurable real-valued function on \(H\),

\[
A^{(\alpha)}u(t, x) = \int_{\mathbb{R}^d} [u(t, x + y) - u(t, x) - (\nabla u(t, x), y)\chi^{(\alpha)}(y)]m^{(\alpha)}(t, y)\frac{dy}{|y|^{d+\alpha}}
\]

\[
+ (b(t), \nabla u(t, x))1_{\alpha=1} + \frac{1}{2} \sum_{i,j=1}^d B^{ij}(t)\partial_{ij}u(t, x)1_{\alpha=2},
\]

\[
\chi^{(\alpha)}(y) = 1_{\alpha>1} + 1_{|y| \leq 1}\chi^{(1)}(y),
\]

\(m^{(\alpha)}(t, y)\) is a deterministic measurable real-valued function homogeneous in \(y\) of order zero, \(m^{(2)} = 0, \mathbb{R}^d = \mathbb{R}^d \setminus \{0\}\) and \(b(t) = (b^1(t), \ldots, b^d(t)), B(t) = (B^{ij}(t))\)
are deterministic bounded measurable function. It is the model problem for the Zakai equation (see [15]) arising in the nonlinear filtering problem. Assume that the signal process $X_t$ is defined by

$$X_t = X_0 + \int_0^t b(s)1_{\alpha=1}ds + \int_0^t \sqrt{B(s)}dW_1 + \int_0^t \int \chi^{(\alpha)}(y)\tilde{q}(ds, dy) + \int_0^t \int (1 - \chi^{(\alpha)}(y))y\tilde{p}(ds, dy),$$

where $\tilde{p}(ds, dy)$ is a point measure on $[0, \infty) \times \mathbb{R}^d_0$ with a compensator $m^{(\alpha)}(s, y)\frac{dyds}{|y|^{d+\alpha}}$, and $W_t$ is a standard Wiener process in $\mathbb{R}^d$. Assume $X_0$ has a probability density function $u_0(x)$ and the observation $Y_t$ is discontinuous, with jump intensity depending on the signal, such that

$$Y_t = \int_0^t \int_{|y| > 1} yp(ds, dy) + \int_0^t \int_{|y| \leq 1} y\tilde{q}(ds, dy),$$

where $p(ds, dy)$ is a point measure on $[0, \infty) \times \mathbb{R}^d_0$ not having common jumps with $\tilde{p}(ds, dy)$ with a compensator $\rho(X_t, y)\pi(dy)$ and $\tilde{q}(dt, dy) = p(dt, dy) - \pi(dy)dt$. Assume $C_1 \geq \rho(x, y) \geq c_1 > 0$, $\pi(dy)$ is a measure on $\mathbb{R}^d_0$ such that

$$\int |y|^2 \wedge 1\pi(dy) < \infty,$$

and $\int [\rho(x, y) - 1]^2\pi(dy)$ is bounded. Then for every function $\psi$ such that $\mathbb{E}[\psi(X_t)^2] < \infty$, the optimal mean square estimate for $\psi(X_t)$, $t \in [0, T]$, given the past of the observations $\mathcal{F}_t = \sigma(Y_s, s \leq t)$, is of the form

$$\hat{\psi}_t = \mathbb{E}[\psi(X_t)|\mathcal{F}_t] = \frac{\mathbb{E}[\psi(X_t)\zeta_t|\mathcal{F}_t]}{\mathbb{E}[\zeta_t|\mathcal{F}_t]},$$

where $\zeta_t$ is the solution of the linear equation

$$d\zeta_t = \zeta_{t-} \int [\rho(X_{t-}, y) - 1]\tilde{q}(dt, dy)$$

and $d\tilde{P} = \zeta(T)^{-1}dP$. Under some assumptions, one can easily show that if $v(t, x)$ is an $\mathbb{F} = (\mathcal{F}_t^Y)$-adapted unnormalized filtering density function

$$\tilde{E} [\psi(X_t)\zeta_t|\mathcal{F}_t] = \int v(t, x) \psi(x) \, dx,$$ (3)
then it is a solution of the Zakai equation

\[
dv(t, x) = v(t, x) \int [\rho(x, y) - 1] \hat{q}(dt, dy) + \left\{ (b(t), \nabla v(t, x))_{1_\alpha = 1} + \frac{1}{2} \sum_{i, j=1}^d B^{ij}(t) \partial^2 v(t, x)_{1_{\alpha = 2}} + \int_{\mathbb{R}^d} [v(t, x + y) - v(t, x) - (\nabla v(t, x), y) \chi^{(\alpha)}(y)]m^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} \right\},
\]

\[v(0, x) = u_0(x). \quad (4)\]

Since \(Y_t, t \geq 0\), and \(X_t, t \geq 0\), are independent with respect to \(\tilde{P}\), for \(u(t, x) = v(t, x) - u_0(x)\) we have an equation whose model problem is of the type given by (1). Indeed, according to [3], for any infinitely differentiable function \(\psi\) on \(\mathbb{R}^d\) with compact support, the conditional expectation \(\pi(t) = \tilde{E}[\psi(X_t) | \mathcal{F}_t]\) satisfies the equation

\[
d\pi(t) = \int \pi(t) [\rho(\cdot, y) - 1] \hat{q}(dt, dy) + \pi(t) \left\{ (b(t), \nabla v(t, x))_{1_\alpha = 1} + \frac{1}{2} \sum_{i, j=1}^d B^{ij}(t) \partial^2 v(t, x)_{1_{\alpha = 2}} + \int_{\mathbb{R}^d} [\psi(\cdot + y) - \psi - (\nabla \psi, y) \chi^{(\alpha)}(y)]m^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} \right\} dt.
\]

Assuming (3) and integrating by parts, we obtain (4).

The general Cauchy problem for a linear parabolic SPDE of the second order

\[
\begin{cases}
du = (a^{ij} \partial_{ij} u + b^i \partial_i u + cu + f)dt + (\sigma^i \partial_i u + h u + g)dW_t & \text{in } H, \\
u(0, x) = 0 & \text{in } \mathbb{R}^d
\end{cases}
\]

(5)
driven by a Wiener process \(W_t\) has been studied by many authors. When the matrix \((2a^{ij} - \sigma^i \cdot \sigma^j)\) is uniformly non-degenerate there exists a complete theory in Sobolev spaces \(W^{n,2}(\mathbb{R}^d)\) (see [11], [5], [13] and references therein) and in the spaces of Bessel potentials \(H^s_p(\mathbb{R}^d)\) (see [6]). The equation (5) in Hölder classes with \(\sigma = 0\) was considered in [12], [11]. The equation (5) in Hölder classes with \(\sigma = 0\) and integro-differential operator of the order \(\alpha\) in the drift part was studied in [8].

In this paper, following the main steps in [7], [8], we prove the solvability of the Cauchy problem (1). In Section 2, we introduce the notation and state our main result. In Section 3, we prove some auxiliary results concerning moment estimates of discontinuous martingales and the solvability of (1) for smooth input functions. The proof of the main theorem is given in Section 4.
2 Notation and main result

2.1 Notation

The following notation will be used in the paper.

Let \((\Omega, \mathcal{F}, \mathbf{P})\) be a complete probability space with a right-continuous filtration of \(\sigma\)-algebras \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathcal{F}_0\) containing all \(\mathbf{P}\)-null sets of \(\mathcal{F}\). Let \((U, \mathcal{U})\) be a measurable space with a \(\sigma\)-finite non-negative measure \(\Pi(du)\). Throughout the paper we assume that there is an increasing sequence \(U_n \subset U\) such that

\[
U = \cup_n U_n; \quad \Pi(U_n) < \infty, \quad n = 1, 2, \ldots.
\]

We say that a measurable function \(f : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}\) is \(\mathbb{F}\)-progressively measurable if for each \(t \in [0, T]\) the mapping \((s, \omega, x) \to f(s, \omega, x)\) on \([0, t] \times \Omega \times \mathbb{R}^d\) is \(\mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable.

We denote \(\mathcal{P}(\mathbb{F})\) the \(\mathbb{F}\)-predictable \(\sigma\)-algebra on \([0, T] \times \Omega\) and \(\mathcal{O}(\mathbb{F})\) the \(\sigma\)-algebra of \(\mathbb{F}\)-well measurable sets on \([0, T]\). We denote \(\mathcal{R}(\mathbb{F})\) the \(\mathbb{F}\)-progressive \(\sigma\)-algebra on \([0, T] \times \Omega\).

We say that a stochastic process \(X_t, 0 \leq t \leq T\), is cadlag if \(\mathbf{P}\)-a.s. it is right-continuous \((X_{t+} = X_t)\), the left-hand limits \(X_{t-}\) exist for all \(t \in [0, T]\) and \(X_{t-} = X_t\).

We denote \(H = [0, T] \times \mathbb{R}^d, \mathcal{N}_0 = \{0, 1, 2, \ldots\}, \mathcal{R}_d^0 = \mathbb{R}^d \setminus \{0\}\). If \(x, y \in \mathbb{R}^d\), we write

\[
(x, y) = \sum_{i=1}^{d} x_i y_i, \quad |x| = \sqrt{(x, x)}.
\]

Let \(L_p(\Omega, \mathcal{F}, \mathbf{P}), p \geq 1\), be the space of random variables \(X\) with finite norm

\[
|X|_p = (\mathbf{E}|X|^p)^{\frac{1}{p}}.
\]

Let \(B_p(H), p \geq 1\), be the space of all \(\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable functions \(u : \Omega \times H \to \mathbb{R}\) such that

\[
\|u\|_p = \sup_{(t, x) \in H} |u(t, x)|_p < \infty.
\]

Similarly, \(B_{r,p}(H \times U), r \geq 1, p \geq 1\), is the space of all \(\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}\)-measurable functions \(g: \Omega \times H \times U \to \mathbb{R}\) such that

\[
\|g\|_{r,p} = \sup_{(t, x, v) \in H} |g(t, x, v)|_{r,p} < \infty,
\]

where

\[
|g(t, x, v)|_{r,p} = \left( \int_U |g(t, x, v)|^r \Pi(du) \right)^{\frac{1}{r}}.
\]

For \(L_p(\Omega, \mathcal{F}, \mathbf{P})\)-valued function \(u\) on \(H\) or \(\mathbb{R}^d\), we denote its partial derivatives in \(x\) in \(L_p(\Omega, \mathcal{F}, \mathbf{P})\)-sense by \(\partial_x u = \partial u/\partial x_i, \partial^2_x u = \partial^2 u/\partial x_i \partial x_j, \text{ etc.} \partial u =\)
\( \nabla u = (\partial_1 u, \ldots, \partial_d u) \) denotes the gradient of \( u \) with respect to \( x \); for a multiindex \( \gamma \in \mathbb{N}_0^d \) we denote
\[
\partial_\gamma^u(t, x) = \frac{\partial^{\gamma_1} u(t, x)}{\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}}.
\]
For \( \alpha \in (0, 2] \), we write
\[
\partial^\alpha u(t, x) = \mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}u(t, \xi)](x),
\]
where \( \mathcal{F} \) denotes the Fourier transform with respect to \( x \in \mathbb{R}^d \) and \( \mathcal{F}^{-1} \) is the inverse Fourier transform, i.e.
\[
\mathcal{F}u(t, \xi) = \int_{\mathbb{R}^d} e^{-i(\xi, x)} u(t, x) dx, \quad \mathcal{F}^{-1}u(t, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x)} u(t, \xi) d\xi.
\]
For \( \alpha \in [0, 2], \beta \in (0, 1) \), \( C_p^{\alpha, \beta}(H) \) is the set of all \( u \in B_p(H) \) with finite norm
\[
\|u\|_{\alpha, \beta, p} = \|u\|_p + \|\partial^\alpha u\|_p + \|\partial^\alpha u\|_{\beta, p},
\]
where
\[
[u]_{\beta, p} = \sup_{t, x \neq y} \frac{|u(t, x) - u(t, y)|}{|x - y|^\beta}.
\]
Similarly, \( C_{r, p}^{\alpha, \beta}(H \times U) \) is the set of all \( g \in B_{r, p}(H \times U) \) with finite norm
\[
\|g\|_{\alpha, \beta, r, p} = \|g\|_{r, p} + \|\partial^\alpha g\|_{r, p} + \|\partial^\alpha g\|_{\beta, r, p},
\]
where
\[
[g]_{\beta, r, p} = \sup_{t, x \neq y} \frac{|g(t, x, \cdot) - g(t, y, \cdot)|}{|x - y|^\beta}.
\]
For \( \alpha = 0 \), we write
\[
\|u\|_{0, \beta, p} = \|u\|_p + [u]_{\beta, p}
\]
and
\[
\|g\|_{0, \beta, r, p} = \|g\|_{r, p} + [g]_{\beta, r, p}.
\]
\( C_p^\infty(H) \) is the set of all \( u \in B_p(H) \) such that \( \text{P-a.s.} \) for all \( t \in [0, T] \) the function \( u(t, x) \) is infinitely differentiable in \( x \) and for every multiindex \( \gamma \in \mathbb{N}_0^d \)
\[
\sup_{(t, x) \in H} \|\partial_\gamma^u(t, x)\|_p < \infty.
\]
Similarly, \( C_{r, p}^\infty(H \times U) \) is the set of all \( g \in B_{r, p}(H \times U) \) such that \( \text{P-a.s.} \) for all \( t \in [0, T], v \in U \) the function \( g(t, x, v) \) is infinitely differentiable in \( x \) and for every multiindex \( \gamma \in \mathbb{N}_0^d \)
\[
\sup_{(t, x) \in H} \|\partial_\gamma^g(t, x, \cdot)\|_{r, p} < \infty.
\]

The counterparts of spaces \( C_p^{\alpha, \beta}(H) \) and \( C_p^\infty(H) \) for nonrandom functions are denoted simply by \( C^{\alpha, \beta}(H) \) and \( C^\infty(H) \). We denote \( C_0^\infty(\mathbb{R}^d) \) the set of all infinitely differentiable functions on \( \mathbb{R}^d \) with compact support.

\( C = C(\ldots, \cdot), c = c(\ldots, \cdot) \) denote constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.
2.2 Main result

Let $\alpha \in (0, 2]$ and $b:\ [0, T] \to \mathbb{R}^d$, $m^{(\alpha)}:\ [0, T] \times \mathbb{R}^d_0 \to \mathbb{R}$ be measurable functions. Throughout the paper we assume that the function $m^{(\alpha)}(t, y)$ is homogeneous in $y$ of order zero, i.e. for all $t \in [0, T]$, $r > 0$, $y \in \mathbb{R}^d_0$

$$m^{(\alpha)}(t, ry) = m^{(\alpha)}(t, y),$$

and

$$\int_{S^{d-1}} w m^{(1)}(t, w) \mu_{d-1}(dw) = 0,$$

where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $\mu_{d-1}$ is the Lebesgue measure on it. Also, $m^{(2)} = 0$.

Let $A^{(\alpha)}$, $\alpha \in (0, 2)$, be the operators defined by (2). In terms of Fourier transform

$$A^{(\alpha)}u(x) = \mathcal{F}^{-1} \left[ \psi^{(\alpha)}(t, \xi) \mathcal{F}u(\xi) \right](x),$$

where

$$\psi^{(\alpha)}(t, \xi) = i(b(t, \xi)1_{\alpha=1} + \frac{1}{2} \sum_{i,j=1}^d B^{ij}(t)\xi_i \xi_j 1_{\alpha=2} - C\int_{S^{d-1}} |(w, \xi)|^\alpha \left[ 1 - i \left( \tan \frac{\alpha \pi}{2} \sgn(w, \xi) 1_{\alpha \neq 1} - \frac{2}{\pi} \sgn(w, \xi) \ln |(w, \xi)| 1_{\alpha=1} \right) \right] m^{(\alpha)}(t, w) \mu_{d-1}(dw)$$

$$= -|\xi|^\alpha M^{(\alpha)}(t, \xi),$$

$C = C(\alpha, d)$ is a positive constant and

$$M^{(\alpha)}(t, \xi) = -i(b(t, \frac{\xi}{|\xi|})1_{\alpha=1} + \frac{1}{2} \sum_{i,j=1}^d B^{ij}(t)\xi_i |\xi|^{-2} 1_{\alpha=2} + C\int_{S^{d-1}} |(w, \frac{\xi}{|\xi|})|^\alpha \left[ 1 - i \left( \tan \frac{\alpha \pi}{2} \sgn \left( w, \frac{\xi}{|\xi|} \right) 1_{\alpha \neq 1} - \frac{2}{\pi} \sgn \left( w, \frac{\xi}{|\xi|} \right) \ln |(w, \frac{\xi}{|\xi|})| 1_{\alpha=1} \right) \right] m^{(\alpha)}(t, w) \mu_{d-1}(dw)$$

is 0-homogeneous function with respect to $\xi$.

Let $p(dt, dv)$ be an $\mathcal{F}$-adapted Poisson point measure on $[0, \infty) \times U$ with a compensator $\Pi(dv)dt$ and

$$q(dt, dv) = p(dt, dv) - \Pi(dv)dt$$

be a martingale measure.

In stochastic Hölder spaces $C_p^{\alpha, \beta}(H)$, we consider the Cauchy problem

$$\begin{cases}
du(t, x) = [(A^{(\alpha)} - \lambda)u(t, x) + f(t, x)] dt + \int_U g(t, x, v)q(dt, dv) & \text{in } H,
\quad \text{in } \mathbb{R}^d,
\end{cases}$$

$$u(0, x) = 0$$

(7)
where $\lambda \geq 0$.

We will need the following assumptions.

**A1.** There is a constant $\mu > 0$ such that for all $t \in [0, T]$

$$\inf_{|\xi|=1} \Re M^{(\alpha)}(t, \xi) \geq \mu;$$

**A2.** The function $m^{(\alpha)}(t,y)$ is differentiable in $y$ up to the order $d_0$ and

$$C^{(\alpha)} = \sup_{0 \leq t \leq T} \left[ \sup_{|\gamma| \leq d_0, |y|=1} |\partial_y^{d_\gamma} m^{(\alpha)}(t, y)| + |b(t)|_{1=1} + |B(t)|_{1=2} \right] < \infty,$$

where $d_0 = \lceil \frac{d}{2} \rceil + 1$ and $\lceil \frac{d}{2} \rceil$ is the integer part of $\frac{d}{2}$.

**Remark 1** Assumption **A1** is satisfied if and only if for all $t \in [0, T], \xi \in \mathbb{R}^d, |\xi|=1$,

$$(B(t)\xi, \xi) \geq \mu, \alpha = 2,$$

$$\int_{S^{d-1}} |(w, \xi)\xi^{d_\alpha} m^{(\alpha)}(t, w)\mu_{d-1}(dw) \geq \mu, \quad \alpha \in (0, 2).$$

The last condition holds with some constant $\mu > 0$ if, for example, there is a Borel set $\Gamma \subseteq S^{d-1}$ such that $\mu_{d-1}(\Gamma) > 0$ and

$$\inf_{t \in [0, T], w \in \Gamma} m^{(\alpha)}(t, w) > 0.$$

**Definition 2** Let $\alpha \in (0, 2], \beta \in (0, 1), p \geq 2, f \in B_p(H), g \in B_{1,p}(H \times U), l = 2, p$. We say that $u \in C^{\alpha,\beta}_p(H)$ is a solution of (7) if for each $(t, x) \in H \text{ P-a.s.}$

$$u(t, x) = \int_0^t \left[ A^{(\alpha)} u(s, x) - \lambda u(s, x) + f(s, x) \right] ds + \int_0^t \int_U g(s, x, v)q(ds, dv).$$

Now we state the main result of this paper.

**Theorem 3** Let $\alpha, \alpha' \in (0, 2], \beta, \beta' \in (0, 1), p \geq 2$ and assumptions **A1**, **A2** be satisfied. Assume that $\alpha(1 - 1/p) + \beta = \alpha' + \beta'$, $f \in C^{\alpha,\beta}_p(H)$ and $g \in C^{\alpha',\beta'}_{r,p}(H \times U), r = 2, p$.

Then there is a unique solution $u \in C^{\alpha,\beta}_p(H)$ to (7). Moreover, there is a constant $C$ depending only on $\alpha, \beta, p, d, \mu, C^{(\alpha)}, T$ such that the following estimates hold:

(i) $\|u\|_{\alpha,\beta,p} \leq C \left( \|f\|_{0,\beta,p} + \sum_{r=2,p} \|g\|_{\alpha',\beta',r,p} \right)$;

(ii) $\|u\|_{0,\beta,p} \leq C \left( T \wedge \frac{1}{\lambda} \right)^{\frac{1}{p}} \left( \|f\|_{0,\beta,p} + \sum_{r=2,p} \|g\|_{0,\beta,r,p} \right)$;
(iii) for $0 \leq t \leq t' \leq T$

\[ \|u(t', \cdot) - u(t, \cdot)\|_{\alpha', \beta'; p} \leq C(t' - t)^{\frac{1}{p}} \left( \|f\|_{0, \beta, p} + \sum_{r=2, p} \|g\|_{\alpha', \beta', r, p} \right). \]

**Remark 4** If $\alpha' = \alpha(1 - 1/p)$, then $\beta' = \beta$. Lemma 11 below indicates that we could assume this without any loss of generality.

If $p = 2$, one can take $\alpha' = \alpha/2$ and $\beta' = \beta$. In this case the estimates of Theorem 3 are similar to the corresponding estimates of Lemma 17 [8] with $p = 2$ for the Zakai equation driven by a Wiener process.

### 3 Auxiliary results

#### 3.1 Moment estimates

First we prove some discontinuous martingale moment estimates. Denote $L_{loc}^{2, p}$ the space of all $\mathcal{R}(\mathbb{F}) \otimes \mathcal{U}$-measurable functions $g(t, v) = g(\omega, t, v)$ such that $\mathbb{P}$-a.s.

\[ \int_0^T \int_U |g(t, v)|^p \Pi(dv) dt + \int_0^T \int_U g(t, v)^2 \Pi(dv) dt < \infty. \]

**Lemma 5** Let $p \geq 2, g \in L_{loc}^{2, p}$ and

\[ Q_t = \int_0^t \int_U g(s, v)q(ds, dv), 0 \leq t \leq T. \]

Then there is a constant $C = C(p)$ such that for any $\mathbb{F}$-stopping time $\tau \leq T$,

\[ \mathbb{E}\left[ \sup_{t \leq \tau} |Q_t|^p \right] \leq C \mathbb{E}\left[ \int_0^\tau \int_U |g(s, v)|^p \Pi(dv) ds \right. \]

\[ + \left. \left( \int_0^\tau \int_U g(s, v)^2 \Pi(dv) ds \right)^{p/2} \right] \]

Moreover, if $\sup_{s, v} |g(s, v)| < \infty \mathbb{P}$-a.s., then for each $\varepsilon > 0$ there is a constant $C(\varepsilon, p)$ such that

\[ \mathbb{E}\left[ \sup_{t \leq \tau} |Q_t|^p \right] \leq \varepsilon \mathbb{E}\left[ \sup_{0 \leq s \leq \tau, v} |g(s, v)|^p \right] + \]

\[ + C(\varepsilon, p) \mathbb{E}\left[ \left( \int_0^\tau \int_U g(s, v)^2 \Pi(dv) ds \right)^{p/2} \right]. \]

**Proof.** Let

\[ A_t = \int_0^t \int_U g(s, v)^2 p(ds, dv), \quad L_t = \int_0^t \int_U g(s, v)^2 \Pi(dv) ds, \quad t \geq 0. \]
By the Burkholder–Davis–Gundy inequality, for each \( \mathcal{F} \)-stopping time \( \tau \)
\[
E[\sup_{t \leq \tau} |Q_t|^p] \leq C_p E[A_p^{p/2}].
\]
Denoting \( q = p/2 \geq 1 \), we have
\[
A_q^\tau = \sum_{s \leq \tau} [(A_s - \Delta A_s)^q - A_s^q] = \int_0^\tau \int_U \left[(A_s - g(s, v)^2)^q - A_s^q\right] p(ds, dv),
\]
and
\[
E[A_q^\tau] = E \int_0^\tau \int_U \left[(A_s - g(s, v)^2)^q - A_s^q\right] \Pi(dv)ds.
\]
Since there are two positive constants \( c_q, C_q \) such that for all non-negative numbers \( a, b \)
\[
C_q(b^q + a^{q-1}b) \geq (a + b)^q - a^q \geq c_q(b^q + a^{q-1}b),
\]
we have
\[
C_q E \int_0^\tau \int_U [(g(s, v))^p + A_s^{q-1} g(s, v)^2] \Pi(dv)ds \geq E[A_q^\tau]
\]
\[
\geq c_q E \int_0^\tau \int_U [(g(s, v))^p + A_s^{q-1} g(s, v)^2] \Pi(dv)ds.
\]
Hence,
\[
E[A_q^\tau] \leq C_q \left\{ E \int_0^\tau \int_U |g(s, v)|^p \Pi(dv)ds + E[A_s^{q-1} L_s] \right\}.
\]
According to Young's inequality, for each \( \varepsilon > 0 \) there is a constant \( C_\varepsilon \) such that
\[
A_s^{q-1} L_s \leq \varepsilon A_s^q + C_\varepsilon L_s^2.
\]
Therefore, there is a constant \( C \) such that
\[
E[A_q^\tau] \leq \frac{1}{2} E[A_q^\tau] + CE \left( \int_0^\tau \int_U |g(s, v)|^p \Pi(dv)ds + L_s^2 \right)
\]
and (8) follows.

If \( \sup_{s,v} |g(s, v)| < \infty \) \( P \)-a.s., then by (9)
\[
E[A_q^\tau] \leq CE \left[ \sup_{s,v} |g(s, v)|^{p-2} L_s + L_s^2 \right].
\]
Applying Young's inequality, we obtain (9). The lemma is proved. \( \blacksquare \)

**Remark 6** Under assumptions of Lemma (2) the estimates (11) imply that
\[
E \int_0^\tau \int_U |g(s, v)|^p \Pi(dv)ds \leq C E[A_s^{p/2}] \leq C E[\sup_{t \leq \tau} Q_t^p].
\]
We will need the following estimate of stochastic integrals (cf. [9], Lemma 1).

**Lemma 7** Let \((\mu_s)\) be a measurable family of \(\sigma\)-finite measures on a measurable space \((A, \mathcal{A})\). Let \(g\) be a \(\mathcal{R}(\mathbb{F}) \otimes \mathcal{A} \otimes \mathcal{U}\)-measurable function on \([0, T] \times A \times U\) such that
\[
\int_0^T \int_U \left( \int_A g(s, a, v)\mu_s(da) \right)^2 \Pi(dv)ds < \infty
\]
P-a.s. Then for each \(p \geq 2\) there is a constant \(C\) such that
\[
\left| \sup_{0 \leq t \leq T} \int_0^t \int_U \int_A g(t, a, v)\mu_t(da)q(dt, du) \right|^p \leq C \sum_{l=2,p} \sup_{t,a} |g(t, a, \cdot)|_{l,p} \left( \int_0^T |\mu_t|(A)^l dt \right)^{1/l},
\]
where \(|\mu_t|\) is the variation of \(\mu_t\).

**Proof.** By Lemma[5] we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_U \int_A g(t, a, v)\mu_t(da)q(dt, du) \right|^p \right] \leq C \sum_{l=2,p} \mathbb{E} \left[ \left( \int_0^T \int_U \int_A \left| g(t, a, v)\mu_t(da) \right|^l \Pi(dv)dt \right)^{p/l} \right].
\]
By generalized Minkowsky’s inequality,
\[
\left| \left( \int_0^T \int_U \int_A g(t, a, v)\mu_t(da) \right)^l \Pi(dv)dt \right|^{1/l} \leq \left( \int_0^T \left| \int_A \left| g(t, a, \cdot)\mu_t(da) \right|_{l,p} dt \right|^l \right)^{1/l}
\]
\[
\leq \left( \int_0^T \left( \int_A |g(t, a, \cdot)|_{l,p}|\mu_t|(da) \right)^l dt \right)^{1/l}
\]
\[
\leq \sup_{t,a} |g(t, a, \cdot)|_{l,p} \left( \int_0^T |\mu_t|(A)^l dt \right)^{1/l}.
\]
The lemma is proved. ■

### 3.2 Solution for smooth input functions

First we solve (7) for smooth input functions \(f, g\).

**Lemma 8** Let \(p \geq 2, f \in C_p(H), g \in C_r,p(H \times U), r = 2, p\).
Then there is a unique $u \in C^\infty_p(H)$ solving (7). Moreover, $\mathbf{P}$-a.s. $u(t,x)$ is cadlag in $t$ and smooth in $x$. Also, for each $(t,x)$ $\mathbf{P}$-a.s.

$$u(t,x) = R_\lambda f(t,x) + \tilde{R}_\lambda g(t,x),$$

where

$$R_\lambda f(t,x) = \int_0^t \mathcal{F}_s^{-1}[K^\lambda_{s,t}(\xi)Ff(s,\xi)](x)ds,$$

$$\tilde{R}_\lambda g(t,x) = \int_0^t \int U F^{-1}[K^\lambda_{s,t}(\xi)Fg(s,\xi,\nu)](x)q(ds,d\nu),$$

and

$$K^\lambda_{s,t}(\xi) = \exp \left\{ \int_s^t (\psi(\alpha)(r,\xi) - \lambda)dr \right\}, 0 \leq s \leq t \leq T.$$

**Proof. Existence.** We take a complete probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ with a filtration of $\sigma$-algebras $\mathcal{F}' = (\mathcal{F}'_t)$ satisfying usual conditions and an adapted stable process $Z_t = Z_t(\omega'), \omega' \in \Omega', t \in [0,T]$, on it defined by

$$Z_t = \int_0^t \sqrt{B(s)}dW_s, \alpha = 2,$$

$$Z_t = \int_0^t \int_{\mathbb{R}^d} yq^Z(ds,dy), \alpha \in (1,2),$$

$$Z_t = \int_0^t b(s)ds + \int_0^t \int_{|y| \leq 1} yq^Z(ds,dy) + \int_0^t \int_{|y| > 1} yp^Z(ds,dy), \alpha = 1,$$

$$Z_t = \int_0^t \int_{\mathbb{R}^d} yp^Z(ds,dy), \alpha \in (0,1),$$

where $W_t$ is a standard Wiener process in $\mathbb{R}^d$, $p^Z(ds,dy) = p^Z(\omega',ds,dy)$ is a Poisson point process on $[0,T] \times \mathbb{R}^d$ and

$$q^Z(ds,dy) = p^Z(ds,dy) - m^{(\alpha)}(s)\frac{dy}{|y|^{d+\alpha}}ds$$

is a $(\mathcal{F}', \mathbf{P}')$-martingale measure.

Consider the product of probability spaces

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbf{P} \times \mathbf{P}').$$

We will denote $\tilde{\mathbf{E}}$ the expectation with respect to $\tilde{\mathbf{P}}$. Let $\tilde{\mathcal{F}}$ be the completion of $\mathcal{F}'$ and $\tilde{\mathcal{F}} = (\mathcal{F}_s)$ be the usual augmentation of $\mathcal{F}_s \otimes \mathcal{F}'_s$ (see [2]). Let $\tilde{\mathcal{F}}^q = (\mathcal{F}^q_t)$ be the usual augmentation of $(\mathcal{F} \otimes \mathcal{F}')$ and $\tilde{\mathcal{F}}^q = (\mathcal{F}'_t)$ be the usual augmentation of $(\mathcal{F}' \otimes \mathcal{F})$. Obviously, $q^Z(ds,dy)$ is $(\tilde{\mathcal{F}}^q, \tilde{\mathbf{P}})$- and $(\tilde{\mathcal{F}}, \tilde{\mathbf{P}})$-martingale measure. Note that $q(dt,d\nu)$ is $(\tilde{\mathcal{F}}, \tilde{\mathbf{P}})$- and $(\tilde{\mathcal{F}}^q, \tilde{\mathbf{P}})$-martingale measure as well.
Denote for a measurable function $F$ on $\tilde{\Omega}$

$$\tilde{E}^Z F = \int_{\tilde{\Omega}'} F(\omega, \omega')P'(d\omega').$$

First, we claim that for each $P(\tilde{\mathcal{F}}^q) \otimes U$-measurable function $g(s, v)$ such that

$$\tilde{E} \int_0^T \int_U g(s, v)^2 \Pi(dv)ds < \infty,$$

we have $P$-a.s. for all $t$

$$\tilde{E}^Z \int_0^t \int_U g(s, v)q(ds, dv) = \int_0^t \int_U \tilde{E}^Z [g(s, v)]q(ds, dv).$$  \hfill (14)

It is enough to show that

$$\tilde{E}^Z \int_0^t \int_U 1_{U_m}(v)g(s, v)q(ds, dv) = \int_0^t \int_U 1_{U_m}(v)\tilde{E}^Z [g(s, v)]q(ds, dv)$$  \hfill (15)

for all $m$ (we have $\cup_m U_m = U, \Pi(U_m) < \infty$ for all $m$).

Let $\mathcal{K}$ be the collection of all bounded functions $h$ on $\tilde{\Omega} \times U$ of the form

$$h(s, v) = \sum_{k=1}^n \bar{g}_k(s, v)1_{A_k}, n \geq 1,$$

where $A_k \in \mathcal{F}'$, $\bar{g}_k$ are bounded real-valued $\mathcal{P}(\mathcal{F}) \otimes U$-measurable functions on $\Omega \times U$. Since $\mathcal{F}' \subseteq \tilde{\mathcal{F}}^+$, we have $\tilde{P}$-a.s. for all $t$ and $m$

$$\int_0^t \int_U 1_{U_m}(v)h(s, v)q(ds, dv) = \sum_{k=1}^n \int_U 1_{A_k} \int_0^t 1_{U_m}(v)\bar{g}_k(s, v)q(ds, dv)$$

and

$$\tilde{E}^Z \int_0^t \int_U 1_{U_m}(v)h(s, v)q(ds, dv) = \sum_{k=1}^n \mathcal{P}'(A_k) \int_0^t \int_U 1_{U_m}(v)\bar{g}_k(s, v)q(ds, dv)$$

$$= \int_0^t \int_U \sum_{k=1}^n 1_{U_m}(v)\mathcal{P}'(A_k)\bar{g}_k(s, v)q(ds, dv)$$

$$= \int_0^t \int_U \tilde{E}^Z [h(s, v)]1_{U_m}(v)q(ds, dv).$$

Therefore (15) holds by the monotone class theorem (see Theorem I-21 in [2]), and (14) follows as well.

Applying Lemma 15 (see Appendix) in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with the filtration $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)$ for

$$\int_0^t \int_U e^{\lambda s}g(s, x - Z_s, v)q(ds, dz), 0 \leq t \leq T,$$
we find that there is a \( \mathcal{P}(\tilde{\mathbb{F}}) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable real-valued function \( w(t,x) \) such that \( \mathbb{P} \)-a.s. \( w(t,x) \) is c.d.l.g in \( t \) and smooth in \( x \). Moreover, for each \( x \in \mathbb{R}^d, \gamma \in \mathbb{N}_0^d, \mathbb{P} \)-a.s. for all \( t \)

\[
\partial_t w(t,x) = \int_0^t e^{\lambda s} \partial_x^2 f(s, x - Z_s) ds + \int_0^t \int_U e^{\lambda s} \partial_x^2 g(s, x - Z_s, v) q(ds, dv).
\]

Also, for each \( \gamma \in \mathbb{N}_0^d, R > 0, \)

\[
\mathbb{E} \left[ \sup_{t \leq T, \|x\| \leq R} \left| \partial_t^\gamma w(t,x) \right|^p \right] + \mathbb{E} \left[ \sup_{t \leq T} \left| \partial_t^\gamma w(t,x) \right|^p \right] < \infty.
\]

Let \( (\Delta_k^n)_{k \geq 1} \) be a sequence of measurable partitions of \( \mathbb{R}^d \) such that every \( \Delta_k^n \) has a diameter smaller than \( 1/n \) and let \( z_k^n \in \Delta_k^n \). We fix \( t, x \) and define \( Z_t^n = z_k^n \) if \( Z_t \in \Delta_k^n, k \geq 1 \).

Let \( \tilde{g}(s,x,v) \) be \( \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U} \)-measurable modification of \( g(s,x,v) \) in Lemma 15. Since \( \tilde{g}(s,x+Z_t-Z_s), \tilde{g}(s,x+Z_t^n-Z_s) \) are \( \mathcal{P}(\mathbb{F}) \otimes \mathcal{F} \subseteq \mathcal{P}(\tilde{\mathbb{F}}) \)-measurable, for each \( t, x, \gamma \in \mathbb{N}_0^d \), \( \mathbb{P} \)-a.s.

\[
\partial_t^\gamma w(t,x+Z_t^n) = \sum_k \partial_t^\gamma w(t,x+z_k^n)1_{\Delta_k^n}(Z_t)
\]

\[
= \int_0^t e^{\lambda s} \partial_x^2 f(s, x + Z_t^n - Z_s) ds + \sum_k 1_{\Delta_k^n}(Z_t) \int_0^t \int_U e^{\lambda s} \partial_x^2 g(s, x + z_k^n - Z_s, v) q(ds, dv) + \int_0^t \int_U e^{\lambda s} \partial_x^2 \tilde{g}(s, x + Z_t^n - Z_s, v) q(ds, dv)
\]

and, by the Minkowsky inequality and Lemma 5

\[
\mathbb{E}|\partial_t^\gamma w(t,x+Z_t^n)|^p \leq C \mathbb{E} \left\{ \left( \int_0^t e^{\lambda s} \partial_x^2 f(s, x + Z_t^n - Z_s) ds \right)^p \right\} + \int_0^t \int_U e^{\lambda s} \partial_x^2 \tilde{g}(s, x + Z_t^n - Z_s, v) q(ds, dv) \right|^p \right\} \leq C \mathbb{E} \left\{ \int_0^t e^{\lambda s} \partial_x^2 \tilde{g}(s, x + Z_t^n - Z_s) ds \right\} \leq C \sup_{s,y} \left\{ \mathbb{E} \left| \partial_x^2 f(s,y) \right|^p + \sum_{l=2,p} \mathbb{E} \left( \int_U \left| \partial_x^2 g(s,y,v) \right|^l \Pi(du) ds \right)^{p/l} \right\},
\]
where the constant $C$ does not depend on $n$.

Therefore, passing to the limit in (16) as $n \to \infty$ (the stochastic integral is regarded as an integral of $\mathcal{P}(\mathcal{F}_t)$ function), we obtain that for each $(t, x)$, $\gamma \in \mathbb{N}_0^d$, $\mathbb{P}$-a.s.

$$
\partial_2^\gamma w(t, x + Z_t) = \int_0^t e^{\lambda s} \partial_2^\gamma f(s, x + Z_t - Z_s) ds
+ \int_0^t \int_U e^{\lambda s} \partial_2^\gamma \tilde{g}(s, x + Z_t - Z_s, \upsilon) q(ds, d\upsilon).
$$

By (17) and the Fatou lemma, we have for all $\gamma \in \mathbb{N}_0^d$

$$
\sup_{t, x} \mathbb{E} |\partial_2^\gamma w(t, x + Z_t)|^p \leq Ce^{p\lambda T} \sup_{s, y} \mathbb{E} |\partial_2^\gamma f(s, y)|^p
+ \sum_{l=2}^p \mathbb{E} \left( \int_U |\partial_2^\gamma g(s, y, \upsilon)|^l \Pi(d\upsilon) \right)^{p/l}.
$$

Therefore, for each for each $R > 0$, $\gamma \in \mathbb{N}_0^d$,

$$
\sup_t \mathbb{E} \int_{|x| < R} |\partial_2^\gamma w(t, x + Z_t)|^p dx < \infty,
$$

and, by the Sobolev embedding theorem,

$$
\sup_t \mathbb{E} \left[ \sup_{|x| \leq R} |\partial_2^\gamma w(t, x + Z_t)|^p \right] < \infty.
$$

Therefore for each $t$, $\mathbb{P}$-a.s. the function $\tilde{u}(t, x) = \mathbb{E}_Z^Z w(t, x + Z_t)$ is smooth in $x$ and according to (14), for each $(t, x)$, $\gamma \in \mathbb{N}_0^d$, $\mathbb{P}$-a.s.

$$
\partial_2^\gamma \mathbb{E}_Z^Z w(t, x + Z_t) = \mathbb{E}_Z^Z \partial_2^\gamma w(t, x + Z_t)
= \int_0^t e^{\lambda s} \mathbb{E}_Z^Z \partial_2^\gamma f(s, x + Z_t - Z_s-) ds
+ \int_0^t \int_U e^{\lambda s} \mathbb{E}_Z^Z [\partial_2^\gamma g(s, x + Z_t - Z_s-, \upsilon)] q(ds, d\upsilon).
$$

In addition, by Lemma 15 and (20), for each $R > 0$, $\gamma \in \mathbb{N}_0^d$,

$$
\sup_{t, x} \mathbb{E} |\tilde{u}(t, x)|^p + \sup_t \mathbb{E} \sup_{|x| < R} |\partial_2^\gamma \tilde{u}(t, x)|^p < \infty.
$$
On the other hand, by the Itô-Wentzell formula (see e.g. [10]), for each $x$ $\tilde{P}$-a.s. for all $\gamma \in \mathbb{N}_0^d, t \in [0, T],$

$$e^{-\lambda t} \partial_x^2 w(t, x) + Z_t =$$

$$= \int_0^t \partial_x^2 f(s, x)ds + \int_0^t \int_U \partial_x^2 g(s, x, \nu)q(ds, d\nu)$$

$$+ \int_0^t e^{-\lambda s} (\nabla \partial_x^2 w(s, x + Z_s), b(s))1_{\alpha = 1}ds$$

$$+ \int_0^t \int_0^1 e^{-\lambda s}\|\partial_x^2 w(s, x + Z_s - y) - \partial_x^2 w(s, x + Z_s)\|_1 |y| \leq 1 p^Z(ds, dy)$$

$$+ \int_0^t \int_0^1 e^{-\lambda s} 1_{\alpha > 1} (\nabla \partial_x^2 w(s, x + Z_s - y), y)q^Z(ds, dy)$$

$$- \int_0^t \lambda e^{-\lambda s} \partial_x^2 w(s, x + Z_s)ds.$$ 

By (14) and (21), for each $(t, x)$ we have $P$-a.s. for all $\gamma \in \mathbb{N}_0^d,$

$$\partial_x^2 u(t, x) = e^{-\lambda t} \partial_x^2 \bar{w}(t, x) = \int_0^t \partial_x^2 f(s, x)ds + \int_0^t \int_U \partial_x^2 g(s, x, \nu)q(ds, d\nu)$$

$$+ \int_0^t \left(\nabla \partial_x^2 u(s, x), b(s)1_{\alpha = 1} - \int_{|y| > 1} 1_{\alpha > 1} ym^{(\alpha)}(s, y) \frac{dy}{|y|^{d+\alpha}}\right)ds$$

$$- \int_0^t \lambda \partial_x^2 u(s, x)ds + \int_0^t \int |\partial_x^2 u(s, x + y) - \partial_x^2 u(s, x)\|_1 |y| \leq 1 m^{(\alpha)}(x, y)\frac{dyds}{|y|^{d+\alpha}}.$$ 

According to Lemma [15] in Appendix, the right-hand side of this equation has $P$-a.s. cadlag in $t$ and smooth in $x$ modification.

Finally, note that (20) implies (11).

**Uniqueness.** Assume $f = 0, g = 0$ and $u \in C^\infty(H)$ is a deterministic solution of (7). We fix $(t_0, x)$ and show that $u(t_0, x) = 0.$ Let $y_t, t \in [0, t_0]$ be a stable process defined on some probability space by the same formulas (12) as the process $Z$ with $p^Z$ and $q^Z$ replaced by $p^Y$ and $q^Y,$ where

$$q^Y(ds, dy) = p^Y(ds, dy) - m^{(\alpha)}(t_0 - s, y) \frac{dy}{|y|^{d+\alpha}}ds$$

is a martingale measure, $p^Y(ds, dy)$ is a Poisson point process on $[0, t_0] \times \mathbb{R}_0^d.$ By Itô formula,

$$-u(t_0, x) = e^{-\lambda t_0}u(0, x + Y_{t_0}) - u(t_0, x) =$$

$$= \int_0^{t_0} e^{-\lambda t} \left(\frac{\partial u}{\partial t} + A^{(\alpha)} u - \lambda u\right)(t_0 - t, x + Y_t)dt = 0.$$
The lemma is proved. ■

The uniqueness for (7) holds in $C_{p}^{\alpha,\beta}(H)$ as well.

**Corollary 9** There is at most one solution $u \in C_{p}^{\alpha,\beta}(H)$ of (7).

**Proof.** Let $u \in C_{p}^{\alpha,\beta}(H)$ be a deterministic solution of (7), $\zeta \in C_{0}^{\infty}(\mathbb{R}^{d})$, $\varepsilon > 0$, $\zeta_{\varepsilon}(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$ and

$$u_{\varepsilon}(t, x) = \int u(t, y)\zeta_{\varepsilon}(x - y)dy.$$ 

Taking the convolution of the both sides of (7) with $\zeta_{\varepsilon}$, we find that $u_{\varepsilon} \in C_{\infty}(\mathbb{R}^{d})$ solves (7). Therefore $u_{\varepsilon}(t, x) = 0$ for all $\varepsilon > 0$ and $u(t, x) = 0$. So, the statement follows. ■

Let $Z_{t}$, $t \in [0, T]$, be the random process defined by (12). Notice that the function $K_{s,t}(\xi)$ is the characteristic function of the increment $Z_{t} - Z_{s}$, $0 \leq s \leq t \leq T$. According to assumption A, $\int |K_{s,t}(\xi)|d\xi < \infty$. Therefore the function

$$G_{s,t}(x) = \mathcal{F}^{-1}\{K_{s,t}(\xi)\}(x), \quad 0 \leq s \leq t \leq T,$$

is the probability density of the increment $Z_{t} - Z_{s}$. Hence,

$$G_{s,t}(x) \geq 0, \quad \int G_{s,t}(x)dx = 1. \quad (22)$$

So, if assumption A is satisfied, then for $f \in C_{p}^{\infty}(H)$ and $g \in C_{r,p}^{\infty}(H \times U)$, $r = 2, p$,

$$R_{\lambda}f(t, x) = \int_{0}^{t} [G_{s,t}^{\lambda}(\cdot) * f(s, \cdot)](x)ds,$$

$$\tilde{R}_{\lambda}g(t, x) = \int_{0}^{t} \int [G_{s,t}^{\lambda}(\cdot) * g(s, \cdot, u)](x)q(ds, du), \quad (24)$$

where

$$G_{s,t}^{\lambda}(x) = e^{-\lambda(t-s)}G_{s,t}(x)$$

and $*$ denotes the convolution.

It is easy to derive that $R_{\lambda}f, \tilde{R}_{\lambda}g \in B_{p}(H)$ if $f \in B_{p}(H)$ and $g \in B_{r,p}(H \times U)$, $r = 2, p$, $p \geq 2$. Indeed, by Lemma 7 and (22), (24),

$$|\tilde{R}_{\lambda}g(t, x)|_{p} \leq C \sum_{r=2, p \leq t, y} \sup |g(s, y, \cdot)|_{r,p} \left\{ \int_{0}^{t} \left( \int G_{s,t}^{\lambda}(y)dy \right) ds \right\}^{1/r} \leq C \sum_{r=2, p} \left( t \wedge \frac{1}{\lambda} \right)^{1/r} \|g\|_{r,p}. \quad (25)$$

16
By the Minkowsky inequality and \( [22], [23] \),
\[
|R_\lambda f(t, x)|_p \leq \int_0^t \int G_\lambda^{s,t}(x-y)|f(s, y)|_p dy ds \leq \sup_{s \leq t, y} |f(s, y)|_p \int_0^t e^{-\lambda(t-s)} ds \leq \left(t \wedge \frac{1}{\lambda}\right)\|f\|_p.
\] (26)

### 3.3 Characterization of stochastic Hölder spaces

For a characterization of our function spaces we will use the following construction (see \([1]\)). Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of all real-valued rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^d \). By Lemma 6.1.7 in \([1]\), there exist a function \( \phi \in C_0^\infty(\mathbb{R}^d) \) such that \( \text{supp } \phi = \{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \} \), \( \phi(\xi) > 0 \) if \( 2^{-1} < |\xi| < 2 \) and
\[
\sum_{j=-\infty}^{\infty} \phi(2^{-j} \xi) = 1 \quad \text{if } \xi \neq 0.
\] (27)

Define the functions \( \varphi_k \in \mathcal{S}(\mathbb{R}^d), k = 0, \pm 1, \ldots, \) by
\[
\mathcal{F}\varphi_k(\xi) = \phi(2^{-k} \xi),
\] (28)
and \( \psi \in \mathcal{S}(\mathbb{R}^d) \) by
\[
\mathcal{F}\psi(\xi) = 1 - \sum_{k \geq 1} \mathcal{F}\varphi_k(\xi).
\] (29)

The following results are proved in \([8]\). We simply take \( V = L_p(\Omega, \mathcal{F}, \mathbb{P}) \) or \( V = L_p(\Omega, \mathcal{F}, \mathbb{P}; L_r(U, \mathcal{U}, \Pi)), r > 1, p > 1, \) in Lemma 12 and Corollary 13 of \([8]\).

**Lemma 10** Let \( \alpha \in [0, 2), \beta \in (0, 1), r > 1 \) and \( p > 1 \). Then the following statements hold:

(i) the norm \( \|u\|_{\alpha, \beta; p} \) is equivalent to the norm
\[
\|\psi * u\|_p + \sup_{j \geq 1} 2^{(\alpha + \beta)j} \|\varphi_j * u\|_p;
\]

(ii) the norm \( \|g\|_{\alpha, \beta; r, p} \) is equivalent to the norm
\[
\|\psi * g\|_{r, p} + \sup_{j \geq 1} 2^{(\alpha + \beta)j} \|\varphi_j * g\|_{r, p}.
\]

**Lemma 11** Let \( \alpha \in (0, 2), \beta \in (0, 1), r > 1, p > 1, u \in C_0^{\alpha, \beta}(H), g \in C_0^{\alpha, \beta}(H \times U), \) and
\[
\begin{align*}
 u_n &= \psi * u + \sum_{j=1}^{n} \varphi_j * u, \\
 g_n &= \psi * u + \sum_{j=1}^{n} \varphi_j * g, \quad n \geq 1.
\end{align*}
\]

17
Then \( u_n \in C^\infty_p(H) \), \( g_n \in C^\infty_{r,p}(H \times U) \),
\[
\|u_n\|_{\alpha,\beta;p} \leq 2\|u\|_{\alpha,\beta;p}, \quad \|g_n\|_{\alpha,\beta;r,p} \leq 2\|g\|_{\alpha,\beta;r,p},
\]
and for each \( \beta' \in (0, \beta) \)
\[
\|u_n - u\|_{\alpha,\beta';p} \to 0, \quad \|g_n - g\|_{\alpha,\beta';r,p} \to 0, \quad n \to \infty.
\]

4 Proof of main result

4.1 Estimates of \( R_\lambda f \) and \( \tilde{R}_\lambda g \)

In order to prove Theorem 3, we need some estimates of the functions \( R_\lambda f \) and \( \tilde{R}_\lambda g \).

Let \( \psi, \varphi_j, j \geq 0 \), be the functions defined by (28), (29) and
\( f_0(t, \cdot) = f(t, \cdot) \ast \psi, \quad g_0(t, \cdot) = g(t, \cdot) \ast \psi, \)
\( f_j(t, \cdot) = f(t, \cdot) \ast \varphi_j, \quad g_j(t, \cdot) = g(t, \cdot) \ast \varphi_j, \quad j \geq 1. \)

Obviously,
\[
\psi \ast R_\lambda f(t, \cdot) = R_\lambda f_0(t, \cdot), \quad \psi \ast \tilde{R}_\lambda g(t, \cdot) = \tilde{R}_\lambda g_0(t, \cdot),
\]
\[
\varphi_j \ast R_\lambda f(t, \cdot) = R_\lambda f_j(t, \cdot), \quad \varphi_j \ast \tilde{R}_\lambda g(t, \cdot) = \tilde{R}_\lambda g_j(t, \cdot), \quad j \geq 1.\]

Let us introduce the functions
\[
\tilde{\varphi}_0 = \psi + \varphi_0, \quad \tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j \geq 1.\]

Since
\[
\psi = \psi \ast \tilde{\varphi}_0, \quad \varphi_j = \varphi_j \ast \tilde{\varphi}_j,
\]
we have
\[
R_\lambda f_j(t, x) = \int_0^t \left[ h_{s,t}^{\lambda,j}(\cdot) \ast f_j(s, \cdot) \right](x) ds,
\]
\[
\tilde{R}_\lambda g_j(t, x) = \int_0^t \int_U \left[ h_{s,t}^{\lambda,j}(\cdot) \ast g_j(s, \cdot, v) \right](x) q(ds, dv),
\]
where
\[
h_{s,t}^{\lambda,j}(x) = F^{-1} \left[ K_{s,t}^{\lambda,j}(\cdot)F\varphi_j \right](x), \quad j \geq 0.
\]

According to Remark 10 [8], there is a finite family \( \{\Gamma_j, j = 1, \ldots, N\} \) of open connected sets which covers the unit sphere \( S^{d-1} \) and a family \( \{O_j(\overline{\Pi}), \overline{\Pi} \in \)
where the constants \( \xi/\xi \) of infinitely differentiable orthogonal transforms with the following properties: for each \( w \in S^{d-1}, w = \Gamma_j, j = 1, \ldots, N \), and multiindex \( \gamma \)

\[
(O_j(w, w)) = w_1, \quad |\partial^\gamma O_j| \leq C(\gamma)
\]

and for each \( \xi/\xi \in \Gamma_j, j = 1, \ldots, N \)

\[
\psi^{(\alpha)}(t, \xi) = \int_{S^{d-1}} |w_1|^\alpha \left[ 1 - i \tan \frac{\alpha \pi}{2} \text{sgn } w_1 1_{a \neq 1} - \frac{2}{\pi} \text{sgn } w_1 \ln |w_1| 1_{a=1} \right] m^{(\alpha)}(t, \Omega_j(\frac{\xi}{\xi})) \mu_{d-1}(dw) + i(b(t), \xi) 1_{a=1} + \frac{1}{2} \sum_{i,j=1}^d B^{ij}(t) \xi_i \xi_j 1_{a=2}.
\]

Using these properties and assumptions \( \textbf{A1, A2} \), it is easy to derive the following estimates:

(i) there is a constant \( C = C(\alpha, \mu, C^{(\alpha)}, d) \) such that for \( 0 \leq s \leq t \leq T \) and multiindices \( \gamma, |\gamma| \leq d_0 \)

\[
|\partial^\gamma K^{\lambda}_{s,t}(\xi)| \leq Ce^{-\mu(t-s)(\xi^{\alpha} + \lambda)} \sum_{k \leq |\gamma|} |\xi^{(k-\alpha)}| (t-s)^k;
\]

(ii) for each \( \kappa \in (0, 1) \) there is a constant \( C = C(\alpha, \mu, C^{(\alpha)}, d, \kappa) \) such that for \( 0 \leq s \leq t \leq T \) and multiindices \( \gamma, |\gamma| \leq d_0 \)

\[
|\partial^\gamma [K^{\lambda}_{s,t}(\xi) - K^{\lambda}_{s,t}(\xi)]| \leq Ce^{-\mu(t-s)(\xi^{\alpha} + \lambda)(t' - t)|\xi^{\alpha}|^\kappa} \sum_{k \leq |\gamma|} |\xi^{(k-\alpha)}| (t-s)^k.
\]

**Lemma 12** Let assumptions \( \textbf{A1, A2} \) be satisfied and \( 0 \leq s \leq t \leq t' \leq T \). Then:

(i) for all \( j \geq 1 \)

\[
\int |h_{s,t}^{\lambda, j}(x)| dx \leq Ce^{-c(2^{\alpha} + \lambda)(t-s)} \sum_{k \leq d_0} \left[ 2^{2j}(t-s) \right] k,
\]

\[
\int |h_{s,t}^{\lambda, 0}(x)| dx \leq Ce^{-\lambda(t-s)}
\]

where the constants \( C = C(\alpha, \mu, \kappa, \lambda^{(\alpha)}) \), \( c = c(\alpha, \mu) > 0 \).

(ii) for each \( \kappa \in (0, 1) \) there is a constant \( C = C(\kappa, \alpha, \mu, \lambda^{(\alpha)}) \) such that for all \( j \geq 1 \)

\[
\int |h_{s,t}^{\lambda, j}(x) - h_{s,t}^{\lambda, j}(x)| dx \leq C [2^{2j}(t' - t)]^\kappa e^{-c(2^{\alpha} + \lambda)(t-s)} \sum_{k \leq d_0} \left[ 2^{2j}(t-s) \right] k,
\]

where the constant \( c = c(\alpha, \mu) > 0 \).
Proof. By Lemma 16 [8],
\[ \int |h_{s,t}^0(x)| \, dx \leq Ce^{-\lambda(t-s)}. \]

According to (30), (31) and the definition of \( \tilde{\varphi}_j \), for \( \kappa \in (0,1) \), multiindices \( \gamma, |\gamma| \leq d_0 \), and \( j \geq 1 \)
\[
|\partial_\xi^\gamma [K_{s,t}^\lambda(\xi)F\tilde{\varphi}_j(\xi)]| \leq \sum_{\gamma' + \gamma'' = \gamma} |\partial_\xi^{\gamma'} K_{s,t}^\lambda(\xi)| |\partial_\xi^{\gamma''} F\tilde{\varphi}_j(\xi)| \leq C2^{-j|\gamma|} \kappa \sum_{k \leq |\gamma|} [2^{j\alpha}(t-s)]^k \]
and
\[
|\partial_\xi^\gamma [(K_{s,t}^\lambda(\xi) - K_{s,t}^0(\xi))F\tilde{\varphi}_j(\xi)]| \leq \sum_{\gamma' + \gamma'' = \gamma} |\partial_\xi^{\gamma'} (K_{s,t}^\lambda(\xi) - K_{s,t}^0(\xi))| |\partial_\xi^{\gamma''} F\tilde{\varphi}_j(\xi)| \leq C2^{-j|\gamma|} \kappa \sum_{k \leq |\gamma|} [2^{j\alpha}(t-s)]^k. \]

These estimates, together with Parseval’s equality, imply
\[
\int |(ix)^\gamma h_{s,t}^{\lambda,j}(x)|^2 \, dx = \int |\partial_\xi^\gamma [K_{s,t}^\lambda(\xi)F\tilde{\varphi}_j(\xi)]|^2 d\xi \leq C2^j e^{-c(2^{j\alpha} + \lambda)(t-s)} \sum_{k \leq |\gamma|} [2^{j\alpha}(t-s)]^k \]
and
\[
\int |(ix)^\gamma [h_{s,t}^{\lambda,j}(x) - h_{s,t}^{\lambda,j}(x)]|^2 \, dx = \int |\partial_\xi^\gamma [(K_{s,t}^\lambda(\xi) - K_{s,t}^0(\xi))F\tilde{\varphi}_j(\xi)]|^2 d\xi \leq C2^j e^{-c(2^{j\alpha} + \lambda)(t-s)} \sum_{k \leq |\gamma|} [2^{j\alpha}(t-s)]^k. \]

Therefore,
\[
\int |h_{s,t}^{\lambda,j}(x)| \, dx \leq \left( \int \frac{dx}{(1 + |2^j x|)^{2d_0}} \right)^{1/2} \left( \int (1 + |2^j x|)^{2d_0} |h_{s,t}^{\lambda,j}(x)|^2 \, dx \right)^{1/2} \leq Ce^{-c(2^{j\alpha} + \lambda)(t-s)} \sum_{k \leq d_0} [2^{j\alpha}(t-s)]^k \]
and
\[
\int |h_{s,t}^{\lambda,j}(x) - h_{s,t}^{\lambda,j}(x)| \, dx \leq \left( \int \frac{dx}{(1 + |2^j x|)^{2d_0}} \right)^{1/2} \left( \int (1 + |2^j x|)^{2d_0} |h_{s,t}^{\lambda,j}(x) - h_{s,t}^{\lambda,j}(x)|^2 \, dx \right)^{1/2} \leq C[2^{j\alpha}(t' - t)]^\kappa e^{-c(2^{j\alpha} + \lambda)(t-s)} \sum_{k \leq d_0} [2^{j\alpha}(t-s)]^k. \]
The lemma is proved. ■

Lemma 13 Let $\alpha, \alpha' \in (0, 2]$, $\beta, \beta' \in (0, 1)$ and $p \geq 2$. Let assumptions A1, A2 be satisfied, $\alpha(1 - \frac{1}{p}) + \beta = \alpha' + \beta'$ and $g \in C_{r,p}^{\alpha, \beta'}(H \times U)$, $r = 2, p$.

Then there is a constant $C$ depending only on $\alpha, \beta, p, d, \mu, C^{(\alpha)}, T$ such that the following estimates hold:

\((i)\) \quad \|\vec{R}_\lambda g\|_{\alpha, \beta; p} \leq C \sum_{r=2, p} \|g\|_{\alpha', \beta'; r, p};

\((ii)\) \quad \|\vec{R}_\lambda g\|_{0, \beta; p} \leq C \sum_{r=2, p} \left( T \wedge \frac{1}{\lambda} \right)^{1/r} \|g\|_{0, \beta; r, p};

\((iii)\) \quad \|\vec{R}_\lambda g(t', \cdot) - \vec{R}_\lambda g(t, \cdot)\|_{\alpha', \beta'; p} \leq C \sum_{r=2, p} (t'^{1/r}) \|g\|_{\alpha', \beta'; r, p}.

Proof. (i). By Lemma 16 [8],

$$\int |h_{s,t}^{\lambda,0}(x)| dx \leq Ce^{-\lambda(t-s)}.$$  

Hence, for $0 \leq u \leq t$, $r \geq 1$

$$\int_u^t \left( \int |h_{s,t}^{\lambda,0}(x)| dx \right)^r ds \leq C(t-u). \quad (32)$$

Using the estimate of Lemma [12](i), we have for $0 \leq u \leq t$, $r \geq 1, j \geq 1$

$$\int_u^t \left( \int |h_{s,t}^{\lambda,j}(x)| dx \right)^r ds \leq C \int_u^t \left[ e^{-c(2^{j\alpha}+\lambda)(t-s)} \sum_{k \leq d_0} [2^{j\alpha}(t-s)]^k \right]^r ds \leq C 2^{-j\alpha} \int_0^{2^{j\alpha}(t-u)} e^{-crv} \left( \sum_{k \leq d_0} v^k \right)^r dv \leq C 2^{-j\alpha} \min[1, 2^{j\alpha}(t-u)]. \quad (33)$$

These estimates, together with Lemma [7] imply

$$|\vec{R}_\lambda g_j(t, x)|_p \leq C \sum_{r=2, p} \|g_j\|_{r, p} \left\{ \int_0^t \left( \int |h_{s,t}^{\lambda,j}(x)| dx \right)^r ds \right\}^{1/r} \leq C \sum_{r=2, p} 2^{-j\alpha/r} \|g_j\|_{r, p}.$$
By Lemma \ref{lem:10}

\[ \|\tilde{R}\gamma g\|_{\alpha,\beta;p} \leq C \left( \|\tilde{R}\gamma g_0\|_p + \sup_{j \geq 1} 2^{j(\alpha+\beta)} \|\tilde{R}\gamma g_j\|_p \right) \leq C \left( \sum_{r=2,p} \|g_0\|_{r,p} + \sup_{j \geq 1} 2^{j(\alpha'+\beta')} \sum_{r=2,p} \|g_j\|_{r,p} \right) \leq C \sum_{r=2,p} \|g\|_{\alpha',\beta';r,p}. \]

(ii). By Lemma \ref{lem:7} and (22), (24)

\[ \left| \tilde{R}\gamma g(t, x) - \tilde{R}\gamma g(t, x') \right|_p = \left| \int_0^t \int_U \int_{\mathbb{R}^d} G_{s,t}(y) [g(s, x - y, \nu) - g(s, x' - y, \nu)] dy d\nu ds \right|_p \leq C \sum_{r=2,p} \sup_{s \leq t, y} |g(s, x - y, \cdot) - g(s, x' - y, \cdot)|_r \left\{ \int_0^t \left( \int G_{s,t}(y) dy \right)_r ds \right\}^{1/r} \leq C|x - x'|^\beta \sum_{r=2,p} [g]_{\beta;r,p} \left( \int_0^t e^{-\lambda(t-s)} ds \right)^{1/r} \leq C|x - x'|^\beta \sum_{r=2,p} \left( T \wedge \frac{1}{\lambda} \right)^{1/r} [g]_{\beta;r,p}. \]

From this estimate and (25) follows the assertion (ii).

(iii). By Lemma 16 \[8\],

\[ \int |\tilde{h}_{s,t}^\lambda(0)(x) - h_{s,t}^\lambda(0)(x)| dx \leq C(1 + \lambda)e^{-\lambda(t-s)}(t' - t). \]

Hence, for \( r \geq 1 \)

\[ \int_0^t \left( \int |h_{s,t}^\lambda(0)(x) - h_{s,t}^\lambda(0)(x)| dx \right)^r ds \leq C(t' - t)^r. \quad (34) \]

Using the estimate of Lemma \ref{lem:12} (ii) with \( \kappa = 1/r \), we have for \( r > 1 \) and \( j \geq 1 \)

\[ \int_0^t \left( \int |h_{s,t}^\lambda(x) - h_{s,t}^\lambda(x)| dx \right)^r ds \leq C 2^{ja(t' - t)} \int_0^t e^{-cr(2^ja + \lambda)(t-s)} \left( \sum_{k \leq d_0} [2^ja(t-s)]^k \right)^r ds \leq C(t' - t) \int_0^\infty e^{-crv} \left( \sum_{k \leq d_0} v^k \right)^r dv \leq C(t' - t). \quad (35) \]
According to Lemma 7 and estimates (52–55), for $j \geq 0$

$$
|\tilde{R}_\lambda g_j(t', x) - \tilde{R}_\lambda g_j(t, x)|_p \leq 
$$

$$
\leq \left| \int_{t}^{t'} \int_U \left[ h^\lambda_{s,t}^j(\cdot) \ast g_j(s, \cdot, v) \right](x)q(ds, dv) \right|_p + 
$$

$$
+ \left| \int_{t}^{t'} \int_U \left[ h^\lambda_{s,t}^j(\cdot) - h^\lambda_{s,t}^j(\cdot) \ast g_j(s, \cdot, v) \right](x)q(ds, dv) \right|_p \leq 
$$

$$
\leq C \sum_{r=2, p} \|g_j\|_{r,p} \left[ \left( \int_{t}^{t'} \left( \int |h^\lambda_{s,t}^j(x)|dx \right)^r ds \right) \right]^{1/r} \left( \int_{t}^{t'} \left( \int |h^\lambda_{s,t}^j(x) - h^\lambda_{s,t}^j(\cdot) \ast g_j(s, \cdot, v)|dx \right)^r ds \right)^{1/r} \leq C \sum_{r=2, p} (t' - t)^{1/r}\|g_j\|_{r,p}.
$$

Finally, by Lemma 10

$$
\|\tilde{R}_\lambda g(t', \cdot) - \tilde{R}_\lambda g(t, \cdot)\|_{\alpha', \beta', p} \leq C \left( \|\tilde{R}_\lambda g_0(t', \cdot) - \tilde{R}_\lambda g_0(t, \cdot)\|_p + \right. 
$$

$$
+ \sup_{j \geq 1} 2^{j(\alpha' + \beta')} \|\tilde{R}_\lambda g_j(t', \cdot) - \tilde{R}_\lambda g_j(t, \cdot)\|_p \right) \leq 
$$

$$
\leq C \left( \sum_{r=2, p} (t' - t)^{1/r}\|g_0\|_{r,p} + \sup_{j \geq 1} 2^{j(\alpha' + \beta')} \sum_{r=2, p} (t' - t)^{1/r}\|g_j\|_{r,p} \right) \leq 
$$

$$
\leq C \sum_{r=2, p} (t' - t)^{1/r}\|g\|_{\alpha', \beta', r,p}.
$$

The lemma is proved. ■

**Lemma 14** Let $\alpha \in (0, 2]$, $\beta \in (0, 1)$, $p \geq 1$, $f \in C^p_{\alpha, \beta}(H)$, and let assumptions A1, A2 be satisfied.

Then there is a constant $C$ depending only on $\alpha, \beta, p, d, \mu, C(\alpha), T$ such that the following estimates hold:

(i)

$$
\|R_\lambda f\|_{\alpha, \beta, p} \leq C\|f\|_{0, \beta, p};
$$

(ii)

$$
\|R_\lambda f\|_{0, \beta, p} \leq C \left( T \wedge \frac{1}{\lambda} \right) \|f\|_{0, \beta, p};
$$

(iii) for $0 \leq t \leq t' \leq T$ and $\nu \in (0, \alpha)$

$$
\|R_\lambda f(t', \cdot) - R_\lambda f(t, \cdot)\|_{\nu, \beta, p} \leq C(t' - t)^{1 - \frac{\nu}{\beta}} \|f\|_{0, \beta, p}.
$$
Proof. (i). Using the Minkowsky inequality and estimates (32), (33) with $r = 1$, we have for $j \geq 0$
\[
|Rf_j(t, x)|_p \leq C\|f_j\|_p \int_0^t \int |h_{s,t}^\lambda(x)| dx ds \leq C \cdot 2^{-j\alpha} \|f\|_p.
\]
By Lemma 10,
\[
\|Rf\|_{\alpha, \beta; p} \leq C\left(\|Rf_0\|_p + \sup_{j \geq 1} 2^{j(\alpha + \beta)} \|Rf_j\|_p\right) \leq C\left(\|f_0\|_p + \sup_{j \geq 1} 2^{j\beta} \|f_j\|_p\right) \leq C\|f\|_{0, \beta; p}.
\]
(ii). According to the Minkowsky inequality and (22), (23), for $(t, x), (t, x') \in H$
\[
|Rf(t, x) - Rf(t, x')|_p = \left|\int_0^t \int G_{s,t}^\lambda(y) [f(s, x - y) - f(s, x' - y)] dy ds\right|_p \leq \sup_{s \leq t, y} |f(s, x - y) - f(s, x' - y)|_p \int_0^t \int G_{s,t}^\lambda(y) dy ds \leq |x - x'|^\beta [f]_{\beta; p} \int_0^t \int G_{s,t}^\lambda(y) dy ds \leq |x - x'|^\beta (T \wedge \frac{1}{\lambda}) [f]_{\beta; p}.
\]
This estimate, together with (24), implies the assertion (ii).
(iii). Using Lemma 12 and Hölder’s inequality, we obtain for $\kappa \in (0, 1)$ and $j \geq 1$
\[
\int_t^{t'} \int |h_{s,t}^\lambda(x)| dx ds \leq C \int_t^{t'} e^{-c(2^{j\alpha} + \lambda)(t-s)} \sum_{k \leq d_0} 2^{j\alpha} (t-s)^k ds \leq C 2^{-j\alpha} \int_0^{2^{j\alpha}(t'-t)} e^{-cs} \left(\sum_{k \leq d_0} s^k\right) ds \leq C 2^{-j\alpha} [2^{j\alpha} (t'-t)]^\kappa \left\{ \int_0^\infty \left[ e^{-cs} \sum_{k \leq d_0} s^k \right]^{1/(1-\kappa)} ds \right\}^{1-\kappa} \leq C 2^{-j\alpha(1-\kappa)} (t'-t)^\kappa
\]
and
\[ \int_0^t \int |h_{s,t}^{\lambda,j}(x) - h_{s,t}^{\lambda,j}(x)| \, dx \, ds \leq \]
\[ \leq C [2^{j\alpha} (t' - t)]^{\kappa} \int_0^t e^{-c(2^{j\alpha} + \lambda)(t - s)} \sum_{k \leq d_0} [2^{j\alpha} (t - s)]^k \, ds \leq \]
\[ \leq C 2^{-j\alpha(1-\kappa)} (t' - t)^\kappa. \]

According to \([32]\) and \([34]\), the same estimates hold in the case \(j = 0\). Therefore, by Minkowsky’s inequality, for \(\kappa \in (0, 1)\) and \(j \geq 0\).

\[ |R_\lambda f_j(t', x) - R_\lambda f_j(t, x)|_p \leq \left| \int_t^{t'} [h_{s,t}^{\lambda,j} \ast f_j(s, \cdot)](x) \, ds \right|_p + \]
\[ + \left| \int_0^t \int [(h_{s,t}^{\lambda,j} - h_{s,t}^{\lambda,j}) \ast f_j(s, \cdot)](x) \, ds \right|_p \leq \]
\[ \leq C \|f_j\|_p \left( \int_t^{t'} \int |h_{s,t}^{\lambda,j}(x)| \, dx \, ds + \int_0^t \int |h_{s,t}^{\lambda,j}(x) - h_{s,t}^{\lambda,j}(x)| \, dx \, ds \right) \leq \]
\[ \leq C 2^{-j\alpha(1-\kappa)} (t' - t)^\kappa \|f_j\|_p. \]

This estimate with \(\kappa = 1 - \nu/\alpha\) and Lemma\([10]\) imply
\[ \|R_\lambda f(t', \cdot) - R_\lambda f(t, \cdot)\|_{\nu, \beta; p} \leq \]
\[ \leq C \left( \|R_\lambda f_0(t', \cdot) - R_\lambda f_0(t, \cdot)\|_p + \sup_{j \geq 1} 2^{j(\nu + \beta)} \|R_\lambda f_j(t', \cdot) - R_\lambda f_j(t, \cdot)\|_p \right) \leq \]
\[ \leq C (t' - t)^{1-\nu/\alpha} \left( \|f_0\|_p + \sup_{j \geq 1} 2^{j(\beta)} \|f_j\|_p \right) \leq \]
\[ \leq C (t' - t)^{1-\nu/\alpha} \|f\|_{0, \beta; p}. \]

The lemma is proved. \(\blacksquare\)

### 4.2 Proof of Theorem \(\[3\]\

Let
\[ u = R_\lambda f + \tilde{R}_\lambda g. \]

According to Lemmas \([13]\) and \([14]\) the function \(u\) belongs to the space \(C^{\alpha, \beta}_p(H)\) and satisfies all the required estimates (we take \(\nu = \alpha' + \beta' - \beta = \alpha(1 - 1/p)\) in Lemma\([13]\)(iii) and notice that, by Lemma\([10]\) the norms \(\|\cdot\|_{\alpha', \beta'; p}\) and \(\|\cdot\|_{\nu, \beta; p}\) are equivalent). By Corollary \([9]\) the equation \((7)\) has at most one solution in the space \(C^{\alpha, \beta}_p(H)\). Hence, it remains to prove that \(u\) is a solution to \((7)\). Let
\[ f_n(t, \cdot) = f(t, \cdot) \ast \psi + \sum_{j=1}^n f(t, \cdot) \ast \varphi_j, \]
\[ g_n(t, \cdot, v) = g(t, \cdot, v) \ast \psi + \sum_{j=1}^n g(t, \cdot, v) \ast \varphi_j, \quad n \geq 1. \]
where the functions $\psi, \varphi_j$, $j \geq 1$, are defined by (28) and (29). By Lemma 8 the function

$$u_n = \tilde{R}_\lambda f_n + \tilde{R}_\lambda g_n$$

is a unique solution in $C^\infty_p(H)$ to the equation (7) with $f, g$ replaced by $f_n, g_n$.

Let $\nu \in (0, \beta')$ be such that

$$\beta \nu = \alpha' + \nu - \alpha \left(1 - \frac{1}{p}\right) > 0.$$ 

Using the estimates of Lemmas 13, 14 and Lemma 11, we get

$$\|u - u_n\|_{\alpha, \beta; p} \leq \|R_\lambda(f - f_n)\|_{\alpha, \beta; p} + \|R_\lambda(g - g_n)\|_{\alpha, \beta; p} \leq C \left(\|f - f_n\|_{0, \beta; p} + \sum_{r=2, p} \|g - g_n\|_{\alpha', \nu; r, p}\right) \to 0$$

as $n \to \infty$.

Let us introduce the function

$$\pi(t, x) = \int_0^t (Au - \lambda u + f)(s, x) ds + \int_0^t \int_U g(s, x, \upsilon) q(ds, d\upsilon).$$

Then $\mathbb{P}$-a.s. for each $(t, x) \in H$

$$(\pi - u_n)(t, x) = \int_0^t [(A - \lambda)(u - u_n) + f - f_n](s, x) ds + \int_0^t \int_U (g - g_n)(s, x, \upsilon) q(ds, d\upsilon).$$

By Minkowsky’s inequality and Lemma 5

$$\|\pi - u_n\|_p \leq C \left(\|(A - \lambda)(u - u_n)\|_p + \|f - f_n\|_p + \sum_{r=2, p} \|g - g_n\|_{\alpha', \nu; r, p}\right).$$

By Lemma 20 [8], for each $v \in C^\alpha_0(H)$, $\kappa \in (0, 1)$,

$$\|Av\|_{0, \kappa; p} = \left\|F^{-1} \{\psi^{(\alpha)}(t, \xi) (1 + |\xi|^\kappa)^{-1} F(v + \partial^\alpha v)\}\right\|_{0, \kappa; p} \leq C \|v + \partial^\alpha v\|_{0, \kappa; p} \leq C \|v\|_{\alpha, \kappa; p}.$$ 

Hence, according to (36),

$$\|A(u - u_n)\|_{0, \beta; p} \leq C \|u - u_n\|_{\alpha, \beta; p} \to 0$$

as $n \to \infty$, and by (37), $\|\pi - u_n\|_p \to 0$ as $n \to \infty$. Thus, for each $(t, x) \in H$, we have $u(t, x) = \pi(t, x)$ $\mathbb{P}$-a.s.

The theorem is proved.
5 Appendix

In the following lemma, we prove the existence of smooth modifications of stochastic integrals.

Lemma 15 Assume \( g \in C_{l,p}^\infty(H \times U), \ l = 2, p \). Then:

a) There is a \( P(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)) \otimes \mathcal{U} \)-measurable function \( \tilde{g}(s,x,v) \) such that \( d\Pi dsd\mathbb{P} \)-a.e.

\[
\partial^\gamma_x g(s,x,v) = \partial^\gamma_x \tilde{g}(s,x,v)
\]

for all \( x \in \mathbb{R}^d, \gamma \in \mathbb{N}_0^d \). In addition, for any \( R > 0 \) and \( \gamma \in \mathbb{N}_0^d, l = 2, p \),

\[
E \int_0^T \int_U \sup_{|x| \leq R} |\partial^\gamma_x \tilde{g}(s,x,v)|^l \Pi(dv) ds < \infty.
\]

(39)

b) There is an \( O(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable real-valued function \( M(t,x) \ \mathbb{P}\text{-a.s.} \) cadlag in \( t \) and smooth in \( x \) and, for each \( x \in \mathbb{R}^d, \gamma \in \mathbb{N}_0^d, \mathbb{P}\text{-a.s.} \)

\[
\partial^\gamma_t M(t,x) = \int_0^t \int_U \partial^\gamma_x g(s,x,v)q(ds,dv) = \int_0^t \int_U \partial^\gamma_x \tilde{g}(s,x,v)q(ds,dv)
\]

for all \( t \in [0,T] \). In addition, for any \( R > 0 \) and \( \gamma \in \mathbb{N}_0^d \)

\[
E[ \sup_{t \leq T, |x| \leq R} |\partial^\gamma_t M(t,x)|^p ] < \infty,
\]

(41)

and

\[
\sup_x E[ \sup_{t \leq T} |\partial^\gamma_t M(t,x)|^p ] < \infty.
\]

(42)

Proof. Obviously, for each \( R > 0, \gamma \in \mathbb{N}_0^d, \)

\[
E \int_0^T \int_U \int_{|y| < R} |\partial^\gamma_y g(s,y,v)|^l \Pi(dy)(dv) ds < \infty,
\]

(40)

\( l = 2, p \). Define a sequence of \( P(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)) \otimes \mathcal{U} \)-measurable functions

\[
g_n(t,x,v) = \int_{t_n}^t g(s,x,v)ds, \ n \geq 1,
\]

\[
g^\gamma_n(t,x,v) = \int_{t_n}^t \partial^\gamma_x g(s,x,v)ds, \ \gamma \in \mathbb{N}_0^d, \ n \geq 1,
\]

where \( t_n = (t - 1/n) \vee 0 \). For each \( \gamma \in \mathbb{N}_0^d \) we have \( \mathbb{P}\text{-a.s.} \) for all \( t \geq 0, \varphi \in C_0^\infty(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} g_n(t,y,v) \partial^\gamma_y \varphi(y)dy = \int_{\mathbb{R}^d} n \int_{t_n}^t g(s,y,v)ds \partial^\gamma \varphi(y)dy
\]

\[
= (-1)^{|\gamma|} \int_{\mathbb{R}^d} n \int_{t_n}^t \partial^\gamma g(s,y,v)ds \varphi(y)dy.
\]

(43)
Let $w \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that $w(x) = 0$ if $|x| \geq 1$ and $\int w(x)dx = 1$. We define $P$-a.s. continuous in $t$ and smooth in $x$ functions

\[ g_{n,\varepsilon}(t, x, v) = n \int_{t_n}^t \int_{\mathbb{R}^d} g(s, y, v)w_\varepsilon(x - y)dyds, \]

\[ g_{n,\varepsilon}^\gamma(t, x, v) = n \int_{t_n}^t \int_{\mathbb{R}^d} \partial_\gamma g(s, y, v)w_\varepsilon(x - y)dyds, \]

where $w_\varepsilon(x) = \varepsilon^{-d}w(x/\varepsilon), x \in \mathbb{R}^d, \gamma \in \mathbb{N}_0^d$. According to [43], $P$-a.s. for all $t \geq 0, x \in \mathbb{R}^d, \gamma \in \mathbb{N}_0^d$,

\[ \partial_\gamma^2 g_{n,\varepsilon}(t, x, v) = \int_{\mathbb{R}^d} n \int_{t_n}^t \partial_\gamma^2 g(s, y, v)ds\partial_\gamma^2 w_\varepsilon(x - y)dy = g_{n,\varepsilon}^\gamma(t, x). \]

Therefore for each $R > 0$ and $l = 2, p$,

\[ E \int_0^T \int_{|x|<R} \int_U |\partial_\gamma^2 g_{n,\varepsilon}(t, x, v) - \partial_\gamma^2 g_{n',\varepsilon}(t, x, v)|^l \Pi(dv)dxdt \]

\[ \leq C \int_0^T \int_{|x|<R} |g_{n,\varepsilon}^\gamma(t, x, \cdot) - \partial_\gamma g(t, x, \cdot)|_{l, l}^l dxdt \]

\[ + \int_0^T \int_{|x|<R} |g_{n',\varepsilon}^\gamma(t, x, \cdot) - \partial_\gamma g(t, x, \cdot)|_{l, l}^l dxdt \to 0 \]

as $\varepsilon, \varepsilon' \to 0$ and $n, n' \to \infty$. By the Sobolev embedding theorem, for each $R > 0, \gamma \in \mathbb{N}_0^d, l = 2, p$,

\[ E \int_0^T \int_{U \cap |x|<R} \sup_{|x|<R} |\partial_\gamma^2 g_{n,\varepsilon}(t, x, v) - \partial_\gamma^2 g_{n',\varepsilon}(t, x, v)|^l \Pi(dv)dt \to 0 \]

as $\varepsilon, \varepsilon' \to 0$ and $n, n' \to \infty$. Let $\tilde{g}_n = g_{n,1/n}$ and choose a subsequence $n_k \uparrow \infty$ as $k \to \infty$ such that for $l = 2, p$,

\[ E \int_0^T \int_{U \cap |x|<k,|\gamma|\leq k} \sup_{|x|<k,|\gamma|\leq k} |\partial_\gamma^2 \tilde{g}_{n_k+1}(t, x, v) - \partial_\gamma^2 \tilde{g}_{n_k}(t, x, v)|^l \Pi(dv)dt \leq 2^{-kp}. \]

Then the function

\[ \tilde{g}(t, x, v) = \begin{cases} \lim_k \tilde{g}_{n_k}(t, x) & \text{if } \sum_k \sup_{|x|<k,|\gamma|\leq k} |\partial_\gamma^2 \tilde{g}_{n_k+1} - \partial_\gamma^2 \tilde{g}_{n_k}|(t, x, v) < \infty, \\ 0 & \text{otherwise,} \end{cases} \]

is $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable and satisfies [38] and [39].

b) By Theorem 5.44 in [4], there are $\mathcal{O}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions $\tilde{M}^\gamma(t, x), \gamma \in \mathbb{N}_0^d$ such that for each $x \in \mathbb{R}^d$ $P$-a.s.

\[ \tilde{M}^\gamma(t, x) = \int_0^t \int_U \partial_2^2 \tilde{g}(s, x, v)q(ds, dv) = \int_0^t \int_U \partial_\gamma^2 g(s, x, v)q(ds, dv), t \in [0, T], \]

28
where $\mathcal{O}(\mathbb{F})$ is the $\sigma$-algebra of well measurable subsets of $[0, T] \times \Omega$. By the Burkholder–Davis–Gundy inequality and Lemma 5, for each $R > 0$, $t \in [0, T]$,

$$
E \int_{|x| \leq R} \sup_{t \leq T} |\tilde{M}^\gamma(t, x)|^p dx \leq C \int_{|x| \leq R} \sum_{i=2, p} E \left( \int_0^T \int_U |\partial_x^i g(s, x, v)|^2 \Pi(dv) ds \right)^{p/l} dx
$$

$$
\leq C \sum_{i=2, p} \sup_{t \leq T} \sup_{0 \leq x \leq R} \|\partial_x^i g(s, x, \cdot)\|_{l, p}^p < \infty.
$$

We define $\mathbb{P}$-a.s. cadlag in $t$ and smooth in $x$ functions

$$
\tilde{M}_{\gamma}^z(t, x) = \int_{\mathbb{R}^d} \tilde{M}^\gamma(t, y) w_z(x - y) dx, \gamma \in \mathbb{N}_0^d.
$$

By the stochastic Fubini theorem (see [10]), for every $x \in \mathbb{R}^d$ we have $\mathbb{P}$-a.s. for all $t \in [0, T]$ and $\gamma \in \mathbb{N}_0^d$,

$$
\tilde{M}_{\gamma}^z(t, x) = \int_0^t \int_{\mathbb{R}^d} \tilde{M}^\gamma(s, y) w_z(x - y) dy q(ds, dv) = \int_0^t \int_{\mathbb{R}^d} \partial_x^\gamma g(s, y, v) w_z(x - y) dy q(ds, dv),
$$

$$
\partial_x^\gamma \tilde{M}_{\gamma}^0(t, x) = \int_{\mathbb{R}^d} \tilde{M}^0(t, y) \partial_x^\gamma w_z(x - y) dy = (-1)^{\gamma_l} \int_{\mathbb{R}^d} \tilde{M}^0(t, y) \partial_y^\gamma w_z(x - y) dy
$$

$$
= \int_0^t \int_{\mathbb{R}^d} \partial_x^\gamma g(s, y, v) w_z(x - y) dy q(ds, dv) = \int_0^t \int_{\mathbb{R}^d} \partial_x^\gamma g(s, y, v) w_z(x - y) dy q(ds, dv).
$$

Also, denoting

$$
g^{\gamma, \epsilon}(s, x, v) = \int \partial_y^\gamma g(s, y, v) w_z(x - y) dy
$$

$$
= \int \partial_x^\gamma g(s, y, v) w_z(x - y) dy,
$$

we have, by Lemma 5, for each $R > 0$

$$
E \int_{|x| < R} \sup_{t \leq T} |\partial_x^\gamma \tilde{M}_{\gamma}^0(t, x) - \partial_x^\gamma \tilde{M}_{\gamma}^0(t, x)|^p dx
$$

$$
= \int_{|x| < R} E \sup_{t \leq T} |\partial_x^\gamma \tilde{M}_{\gamma}^0(t, x) - \partial_x^\gamma \tilde{M}_{\gamma}^0(t, x)|^p dx
$$

$$
\leq C E \int_0^T \sum_{i=2, p} \int_{|x| < R} \left( \int_U |g^{\gamma, \epsilon}(s, x, v) - g^{\gamma, \epsilon'}(s, x, v)|^2 \Pi(dv) \right)^{p/l} dx ds \to 0
$$

as $\epsilon, \epsilon' \to 0$. By the Sobolev embedding theorem, for each $R > 0$ and $\gamma \in \mathbb{N}_0^d$,

$$
E \sup_{|x| \leq R, t \leq T} |\partial_x^\gamma \tilde{M}_{\gamma}^0(t, x) - \partial_x^\gamma \tilde{M}_{\gamma}^0(t, x)|^p \to 0
$$

29
as $\varepsilon, \varepsilon' \to 0$. Therefore, there is an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function $M(t, x)$ which is $\mathbb{P}$-a.s. cadlag in $t$ and smooth in $x$, satisfies (40)–(42) hold and for each $R > 0, \gamma \in \mathbb{N}_0^d$
\[
\mathbb{E} \sup_{|x| \leq R, t \leq T} |\partial_x^\gamma \tilde{M}_\varepsilon(t, x) - \partial_x^\gamma M(t, x)|^p \to 0
\]
as $\varepsilon \to 0$. The lemma is proved.

References

[1] Bergh J. and Löfstrom J., Interpolation Spaces. An Introduction, Springer Verlag, 1976.
[2] Dellacherie C. and Meyer P., Probabilities and Potential, North-Holland Mathematics Studies 29, North-Holland Publishing Company, Amsterdam, 1978.
[3] Grigelionis B., Reduced stochastic equations of the nonlinear filtering of random processes, Lithuanian Math. J., 16 (1976) 348–358.
[4] Jacod J., Calcul stochastique et problemes de martingales, Lecture Notes in Mathematics, v. 714, Springer Verlag, Berlin, 1979.
[5] Krylov, N. V. and Rozovskii, B. L., On the Cauchy problem for linear partial differential equations. Math. USSR Izvestija, 11 (1977) 1267–1284.
[6] Krylov, N. V., On $L_p$ theory of stochastic partial differential equations, SIAM J. Math. Anal., 27 (1996) 313–340.
[7] Mikulevicius, R. and Pragarauskas, H., On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces, Lithuanian Math. J., 32 (1992) 238–264.
[8] Mikulevicius, R. and Pragarauskas, H., On Hölder solutions of the integro-differential Zakai equation, Stochastic Processes and their Applications, 119 (2009) 3319–3355.
[9] Mikulevicius, R., On the Cauchy problem for parabolic SPDEs in Hölder classes, Annals of Probability, 28 (2000) 74–108.
[10] Mikulevicius R., Properties of solutions of stochastic differential equations, Lithuanian Math. J., 23 (1983) 367–376.
[11] Pardoux, E., Equations aux dérivées partielles stochastiques non linéaires monotones. Étude de solutions de type Itô: Thèse. Université de Paris Sud, Orsay, 1975.
[12] Rozovskii, B. L., On stochastic partial differential equations, Mat.Sbornik, 96 (1975) 314–341.
[13] Rozovskii, B. L., *Stochastic Evolution Systems*, Kluwer Academic Publishers, Norwell, 1990.

[14] Stein E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[15] Zakai, M., On the optimal filtering of diffusion processes, *Z. Wahrsch.*, 11 (1969) 230–243.