We discuss aspects of the non-standard version of F-theory based on the arithmetic of torsion points on elliptic curves. We construct new F-theory vacua in 8-dimensions. They are coming by the projective realizations of F-theory on $K_3$ surfaces admitting double covers onto $P^2$, branched along a plane sextic curve, the so called double sextics. The new vacua are associated with singular $K_3$ surfaces. In this way the stable picture of the heterotic string is mapped at the triple points of the sextic. We argue that this formulation incorporates naturally the $Sp(4, Z)$ invariance that the extrapolating four dimensional vector multiplet sector of all heterotic vacua may possess. In addition, we describe the way that the $4D$ genus two description of (0,2) moduli dependence of $N = 1$ gauge coupling constants may be connected to Riemann surfaces, with natural $Sp(4, Z)$ duality invariance. Here, we recover a novel way to break space-time supersymmetry and fix the moduli parameters in the presence of the Wilson lines. We also consider a novel way to break space-time supersymmetry and fix the moduli parameters in the presence of the Wilson lines. We also consider the heterotic duals to compactifications of F-theory to four dimensions belonging to isomorphic classes of elliptic curves with point cusps of order two. For the latter theories we calculate the $N = 2$ 4$D$ heterotic prepotential corresponding to $\Gamma_o(2)_T \times \Gamma_o(2)_U$ classical perturbative duality group and their conjugate modular theories.
Introduction

Higher dimensional theories like F-theory don’t have any obvious manifestation in twelve dimensions as one should expect. Instead F-theory makes its presence manifest only through its compactifications to lower dimensions that match already known compactifications of the heterotic string (or its equivalent theories IIA, IIB, etc). In turn $\mathcal{N} = 2$ compactifications of the heterotic string, an obvious candidate for F-theory compactification match have a $\mathcal{N} = 2$ vector multiple sector defined by a Kähler potential

$$K = -\log(-i\Omega^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Omega) = -\log(iX^I F_I - iX^I \bar{F}_I), \quad I = 0, \ldots, n \quad (1)$$

where $\Omega = (X^I(M^I), F_I(X^I))$ the holomorphic symplectic period vector, and $X^I$ the special coordinates. Demanding complete gauge fixing, in perturbation theory, of the vector multiplet sector of the four dimensional $\mathcal{N} = 2$ compactifications of the heterotic string requires the knowledge of the holomorphic prepotential $F$. That has been calculated in [1, 2, 3].

The full target space duality transformations act on the space of the symplectic vectors $\Omega$, which includes Wilson lines, in the form

$$K = -\log[(T + \bar{T})(U + \bar{U}) - (B + \bar{C})(\bar{B} + C)] \quad (2)$$

$$M = \begin{pmatrix} T & B \\ -C & U \end{pmatrix}, \quad M \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}). \quad (3)$$

An obvious challenge, for F-theory, that is matching the existing heterotic compactifications, may be directly related to the arithmetic of torsion points on elliptic curves.

In this work, we will give evidence for the presence of new 8-dimensional F-theory realizations [5].

Double Covers

It is known that F-theory compactified on a $\text{K}_3$ surface realized as an elliptic fibration with a section is on the same moduli space, namely

$$O(\Gamma_{2,18})O(2,18)/(O(2) \times O(18)) \quad (4)$$

the perturbative heterotic string on a $T^2$ torus. Schematically,

$$\frac{F}{K_3} \equiv \frac{(E_8 \times E_8)_{\text{het}}}{T^2}. \quad (5)$$
On the right hand side of this equivalence the torus can be represented as an elliptic curve,

$$y^2 = x^3 + ax + b,$$

(6)

that is the double cover of the complex plane $x$. However, on the left hand side of (3) the $K_3$ surface $S_F$ building the compactification of F-theory is represented as an elliptic fibration over a $P^1$ base, namely as the map $\pi : S_F \to P^1$. Comparison of the two sides in (3) reveals the lack of the presence of a double cover on the F-theory side. This constitutes a form of **violation of double cover parity** signalling different treatment of the bases of the F-theory/heterotic duality pairs. By demanding that F-theory respects the double cover treatment of its base we may generate new F-theories. The double cover parity may disappear only when we consider generating surfaces from **double covers onto $P^2$**. Double covers of $K_3$ onto $P^2$, the so-called **double sextics** are branched along a plane sextic curve. In general a curve $X$ which is a double cover of $Y$ branched along a curve $C$, is one to one correspondence with singular $K_3$ surfaces e.g maximal Picard number equal to twenty. Every singular surface is the double cover of a $K_3$ surface. For example, at the limit that the $T^2$ is large the $K_3$ surface degenerates into a variety that is made from the union of two intersecting rational elliptic surfaces $S_1$, $S_2$ intersecting along an elliptic curve $E^*$. 

| Cubic fibration | order F | $K_3$ fibration |
|-----------------|---------|-----------------|
| $II^* 2I_1$     | 1       | $2II^* 2I_2$   |
| $II^* II$       | 1       | $2II^* IV$     |
| $III^* I_2 I_1$ | 2       | $2III^* I_4 I_2$ |
| $III^* III$     | 2       | $2III^* I_6$   |
| $I_4^* 2I_1$    | 2       | $2I_4^* 2I_2$  |
| $I_9 3I_1$      | 3       | $I_{18} I_2 4I_1$, $2I_9 2I_2 2I_1$ |
| $IV^* I_3 I_1$  | 3       | $2IV^* I_6 I_2$ |
| $IV^* IV$       | 3       | $3IV^*$        |
| $I_8 I_2 2I_1$  | 4       | $I_{16} I_4 4I_1$, $I_{16} 3I_2 2I_1$, $I_{16} 3I_2 2I_1$, $2I_8 I_4 I_2 2I_1$, $2I_8 4I_2$ |
| $I_7^* 2I_2$    | 4       | $2I_2^* 2I_4$  |
| $I_7^* I_4 I_1$ | 4       | $2I_1 I_8 I_2$ |
| $2I_5 2I_1$     | 5       | $2I_{10} 4I_1$, $I_{10} 2I_5 I_2 2I_1$, $4I_5 2I_2$ |
| $I_6 I_3 I_2 I_1$ | 6    | $I_{12} I_6 2I_2 2I_1$, $I_{12} I_4 2I_3 2I_1$, $I_{12} I_3 3I_2$, $3I_6 I_4 2I_1$, $3I_6 3I_2$, $2I_6 I_4 2I_3 I_2$ |
| $2I_4 2I_2$     | 8       | $2I_8 4I_2$, $I_8 3I_4 2I_2$, $6I_4$ |
| $4I_3$          | 9       | $2I_6 I_3$     |
All the possible forms of the rational elliptic surface as an elliptic fibration against its double cover and the order of the Mordell-Weyl (MW) group are listed in table 1. The first column gives us the Kodaira fibers appearing in the cubic associated with the corresponding Mordell-Weyl group order of the group of sections given in the second column. The third column is the configuration of allowed Kodaira fibers for the associated $K_3$ fibrations.

The double cover consists of an elliptic fibration $\pi: Y \rightarrow P^1$ and an involution $\sigma$ that respects the fibration. In this case the fibration always have fixed points. The form of the rational elliptic fibration is directly connected to the order of MW that defines a group operation over the rational points $\mathbb{Q}$ of the elliptic curve. Its action restricts the form of the associated Weierstrass fibration and can be written as $E(\mathbb{Q}) \equiv \mathbb{Z}^r \oplus \Phi$, where $r$ is the rank of $E(\mathbb{Q})$ and $\Phi$ the torsion subgroup. So for example for the

- $\Phi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $y^2 = x(x - \beta)(x - \gamma)$, (7)

while for the

- $\Phi \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, $y^2 = x(z + \zeta^2)(x + \lambda^2)$, $\beta = \zeta^2$, $\gamma = \lambda^2$. (8)

Effective String Theories from Double Covers

On the previous section we demanded that the base space of the 8-dimensional F-theory/heterotic duality pair must be treated in terms of double covers. The question that remains now is that if we can find an effective theory that can be formulated in terms of double covers and is defined in lower dimensions, e.g four, that may be defined both in the F-theory and the heterotic side. The case of the heterotic string is considered in this section while the $N = 2$ four dimensional compactification of F-theory counterpart will be treated in a future work.

The hyperelliptic curve for a genus 2g+2 resp. 2 surface is given by

$$y^2 = \prod_{i=1}^{2g+2}(x - e_i) = P_{2g+2}(x, e_i), \ e_i \neq e_j, \text{ for } i \neq j,$$

and represents the double cover of the sphere branched over 2g+2 resp. 2 points. The map from the Jacobian to the complex numbers defines the $\Theta$ functions with the usual build $Sp(2g, 2)$ invariance. So in $g = 2$ the invariance is $Sp(4, \mathbb{Z})$ the required T-duality of the perturbative heterotic string spectrum. In order to define the genus 2 hyperelliptic curve
as the Riemann surface that the \((0,2)\) 4D heterotic string lives we need one more element. The element that we require is that there is a birational correspondence \([4]\) between the projective varieties associated with the graded ring of even projective invariants of binary sextics and with the graded ring of modular forms. In simple terms that means that the projective variety linked with the graded ring of even projective invariants of binary sextics is a compactification of moduli of genus two. Therefore if we denote the six roots \(\phi_I, I = 1, \ldots, 6\) of the sextic \(s_0X^6 + s_1X^5 + \ldots + s_6\) and we denote their difference by \((\phi_i - \phi_j)\) the invariants \(A, B, C, D\) take the following form

\[
A(s) = s_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2
\]

\[
B(s) = s_0^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2
\]

\[
C(s) = s_0^6 \sum_{\text{fifteen}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2
\]

\[
D(s) = s_0^{10} \Pi_{i<j}(jk)^2.
\]

(10)

Because a sextic can be brought into the general form

\[
X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3)
\]

we can replace each of the three lambda in \(I\) by some theta function of zero argument, namely

\[
\lambda_1 = (\frac{\theta_{1001}\theta_{1000}}{\theta_{0100}\theta_{0000}})^2, \quad \lambda_2 = (\frac{\theta_{1001}\theta_{1000}}{\theta_{0001}\theta_{0000}})^2, \quad \lambda_3 = (\frac{\theta_{1001}\theta_{1000}}{\theta_{0001}\theta_{0000}})^2,
\]

(12)

where

\[
\theta_{g,h_{1}h_{2}}(\tau_1, \tau_2) = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} \frac{d^n}{d \tau_1^n} \theta_{g_1 h_1}(\tau_1) \frac{d^n}{d \tau_2^n} \theta_{g_2 h_2}(\tau_2) \epsilon^{2n}
\]

(13)

theta functions of genus two. It can be proved that \(\lambda_1, \lambda_2, \lambda_3\) can be expanded in terms of even powers of \(\epsilon\) when \(\epsilon\) is small. As a result all the variables in \(I\) are fixed and its roots may be calculated. Moreover the invariants \(A, B, C, D\) are expressed in terms of \(\epsilon\) and \(\theta\), the following relations hold

\[
I_4 = \sum (\theta_m)^8,
\]

\[
I_6 = \sum \sum \pm(\theta_{m_1}\theta_{m_2}\theta_{m_3})^4,
\]

\[
I_{10} = -2^{14} \cdot \chi_{10} = \Pi(\theta_m)^2,
\]

\[
I_{12} = 2^{17}3 \cdot \chi_{12} = \sum(\theta_{m_1}\theta_{m_2}\ldots\theta_{m_5})^4, \quad \text{nonumber}
\]

\[
I_{35} = 2^{39}5^3 \cdot \chi_{35} = (\Pi\theta_m)(\sum_{\text{zygous}} \pm(\theta_{m_1}\theta_{m_2}\theta_{m_3})^{20}).
\]

(14)
It is remarkable that the following relations hold

\[ \frac{D}{A^5} \propto \epsilon^{12}, \frac{(B/A^2)^3}{A^5} \propto j(\tau_1)j(\tau_2)\epsilon^{12}, \frac{((3C - AB)/A^3)^2}{A^5} \propto (j(\tau_1 - j(\tau_2))\epsilon^{12}. \tag{16} \]

or in precise form

\[ I_4 = B, I_6 = \frac{1}{2}(AB - 3C), I_{10} = D, I_{12} = AD, I_{35} = 5^3D^2E. \tag{17} \]

That means that the uniformation parameters of the sextic (11) are fixed genus two elements therefore possessing manifest \(Sp(4, Z)\) invariance. In its non-perturbative form the equation for the sextic \(P_6\) involves the parametric relation (17). Notice that in (17) the fundamental invariant of the full theory are expressed in terms of products of j-invariants. What we have not discuss is the expansion of genus two theta functions in terms of epsilon parameter, that coincide with the Wilson line background fields, represents exactly the fact that the space of projective varieties corresponding to the invariants \(A, B, C, D\) has been blown up such that the jacobian variety of the genus two curve has degenerate to a product of elliptic curves. The blow up process is necessary since the projective variety \(P_6\) does not apriori include the Siegel fundamental domain. The correspondence with the heterotic string comes after identifying \(T = \tau_1, U = \tau_2, \epsilon = \text{Wilson line.}\)

At the points that the discriminant of the projective variety of \(P_6\) degenerates both \(T, U\) and the Wilson line are involved in a non-trivial relation. That means that one or more moduli may be fixed therefore breaking space-time supersymmetry.

At this point we might be tempted to give a description of the \(N = 2\) vector multiplet effective theory in the case that effective heterotic theory comes with target space duality group \(\Gamma_o(2)_T \times \Gamma_o(2)_U\). The one loop perturbative heterotic prepotential can be calculated in this case [5].

**Conclusions**

In the previous sections we introduced a new constraint, that the F-theory/heterotic duality may satisfy in order to correctly reproduce the target space duality transformations of the heterotic string. That involves the treatment of the base in both pairs in terms of double covers.

Particular role in our study is layed by the MW group. Its presence generates new solutions for the stable degeneration limit of the F-theory K3 surface.

At the moment our study is confined on the equivalence of the F-theory/heterotic string at 8-dimensions in terms of double covers. The relevance of our results for the \(N = 2\) 4D theory comes only through the determination of the Riemann surface that is a double cover.
of the sphere. In order for our results to be relevant to the F-theory/heterotic duality in four dimensions we must find the F-theory compactification realized on a four dimensional context. Work is in progress towards this goal.

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