Multiplicity of positive solutions for a critical quasilinear Neumann problem

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Abstract. We establish the multiplicity of positive solutions to a quasilinear Neumann problem in expanding balls and hemispheres with critical exponent in the boundary condition.

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1. Introduction. We consider the following problem

\[
\begin{cases}
\Delta_p u := \text{div}(\|\nabla u\|^{p-2} \nabla u) = \|u\|^{q-2} u & \text{in } B_R, \\
\|\nabla u\|^{p-2} \langle \nabla u; n \rangle = \|u\|^{q-2} u & \text{on } S_R, \\
u > 0 & \text{in } B_R,
\end{cases}
\]

where \( B_R \) and \( S_R \) are the ball and the sphere with radius \( R \), respectively, in \( \mathbb{R}^n \). Here \( 1 < p < n \) and \( q = p^{**} = \frac{(n-1)p}{(n-p)} \) is the critical exponent for the trace embedding.

We establish the multiplicity effect for weak solutions to (1). Namely we prove that the number of positive rotationally non-equivalent solutions is unbounded as \( R \to \infty \).

The effect of multiplicity was discovered by Coffman [6] who considered the Dirichlet problem

\[
\begin{cases}
-\Delta_p u = \|u\|^{q-2} u & \text{in } \Omega_R, \\
u = 0 & \text{on } \partial \Omega_R, \\
u > 0 & \text{in } \Omega_R,
\end{cases}
\]
where $\Omega_R$ is the annulus $B_R \setminus B_{R-1} \subset \mathbb{R}^n$ for $n = 2$ and $p = 2$. The problems (1) and (2) were studied later by many authors for subcritical $q$ (see, e.g., [4,9,11,12,18]). In [21] the multiplicity result was obtained for the Neumann problem

$$\begin{cases}
-\Delta u + \lambda u = |u|^{p^*-2}u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{in } \Omega,
\end{cases}$$

where $\Omega$ satisfies some symmetry conditions and $p^*$ is the critical exponent for the Sobolev trace embedding.

One can easily show that after suitable rescaling, solutions of (1) are solutions to the following problem:

$$\begin{cases}
\Delta_p u = \lambda |u|^{p-2}u & \text{in } B, \\
|\nabla u|^{p-2} \langle \nabla u; n \rangle = |u|^{q-2}u & \text{on } S, \\
u > 0 & \text{in } B,
\end{cases}$$

(3)

where $B = B_1, S = S_1$, and $\lambda(R) = R^p$ as $R \to \infty$.

We look for distinct solutions of the problem (3) by minimizing the functional

$$I^\lambda[u] := \frac{\|\nabla u\|_{L^p(B)}^p + \lambda \|u\|_{L^p(B)}^p}{\|u\|_{L_q(S)}^p}$$

(4)
on different subsets of $W^1_p(B)$.

In order to construct solutions to problem (3), let us introduce the following notation:

**Definition 1.** Let $A \subset S$ and $\varkappa > 0$. We denote by $A^\varkappa$ the $\varkappa$-neighborhood of a set $A$, i.e.

$$A^\varkappa = \{z \in S | \text{dist}(z, A) \leq \varkappa\}.$$  

The following definition was introduced in [5].

**Definition 2.** Let $G$ be a closed subgroup of $O(n)$. We call set $A \subset S$ a locally minimal orbital set under the action of $G$ if $A$ is invariant under the action of $G$ and satisfies the following conditions:

- for any $x \in A$ the orbit $Gx$ is a discrete set and $m(A) := |Gx|$ is independent of $x$.
- there exists $\varkappa > 0$ such that for any $y \in A^{\varkappa} \setminus A$ and $x \in A$, we have $|Gx| < |Gy|$.

We denote as $m(G)$ the number of elements in the minimal orbit of $G$, and $K(n, p)$ stands for the best Sobolev trace constant in half-space defined as

$$K(n, p) = \inf_{v \in C_0^\infty(\mathbb{R}^n_+)} \frac{\|\nabla v\|_{L^p(\mathbb{R}^n_+)}^p}{\|v(\cdot, 0)\|_{L_q(\mathbb{R}^{n-1})}^p}.$$
The value of $K(n,p)$ is calculated explicitly in [10] for $p = 2$ and in [14] for arbitrary $p$.

We consider local minimizers of the functional (4) on sets

$$X_G(A, \beta) = \left\{ u \in W^1_p(B) | u(gx) \equiv u(x) \ \forall g \in G, \right. $$

$$\left. \|u\|_{L_q(S)} = 1, \|u\|_{L_q(A^\omega)}^q \geq 1 - \beta \right\},$$

(5)

where $G$ is some closed subgroup of $O(n)$, $A$ is a locally minimal orbital set, and $\beta$ is some small parameter that we will choose later. We denote $X_G(A, \beta)$ by $X$ if it does not lead to confusion.

The structure of the paper is as follows. In Section 2 we prove some auxiliary lemmas, and in Section 3 we establish main multiplicity result:

Theorem. For any $N > 0$ there is $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ problem (3) has at least $N$ distinct solutions.

We show in Lemma 2 that any sequence of functions $u_\lambda$ from $X_G(A, \beta)$, such that $I_\lambda[u_\lambda]$ is bounded by an appropriate constant $C$, has to concentrate in some sense on a locally minimal orbital set. Next, in Lemma 3 we establish that if the infimum of $I_\lambda$ is strictly lower than $C$ for large $\lambda$, then the minimum is attained. Using a result from [15], we construct a sequence of functions $v_\lambda$ from $X_G(A, \beta)$, such that $I_\lambda[v_\lambda] < C$ for $\lambda$ large enough, which proves the existence of a minimizer. Since we had an additional constraint imposed on functions from $X_G(A, \beta)$, the minimizer of $I_\lambda$ is not necessarily a solution of problem (3), but because it is concentrated on a locally minimal orbital set, the inequality in (5) is strict and does not produce a Lagrange multiplier. After that we construct groups with distinct locally minimal orbital sets. It is easy to see that since corresponding minimizers have to concentrate around locally minimal orbital set, they have to be distinct as well.

2. Auxiliary lemmas. The following fact is well known and will be given here without a proof.

Proposition 1. The functional $I_\lambda[u]$ is Gâteaux differentiable and for any $h \in W^1_p(B)$

$$DI_\lambda[u](h) = \left( \int_B |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx + \lambda \int_B |u|^{p-2} uh \, dx \right) \frac{p}{\|u\|_{L_q(S)}^p}$$

$$- \left( \int_B |\nabla u|^{p} \, dx + \lambda \int_B |u|^{p} \, dx \right) \int_S |u|^{q-2} uh \, dS \frac{p}{\|u\|_{L_q(S)}^{p+q}}.$$

Lemma 1. Let $u_j^\lambda \in W^1_p(B)$ be a bounded Palais-Smale sequence for $I_\lambda$ at the level $c > 0$. Then there is $u_0^\lambda \in W^1_p(B)$ such that up to subsequence $u_j^\lambda \rightharpoonup u_0^\lambda$ and
\[ |\nabla u_\lambda|^p dx \to \mu \geq |\nabla u_0|^p dx + \sum_k \mu_k \delta(x - x_k), \quad (6) \]

\[ |u_\lambda|^q dS \to \nu = |u_0|^q dS + \sum_k \nu_k \delta(x - x_k), \quad (7) \]

where \( \delta(x - x_k) \) are delta measures at points \( x_k \) in \( S \) and \( \mu_k \geq K(n, p)\nu_k^\frac{p}{q} \).

Furthermore, either \( \nu_k = 0 \) or \( \nu_k \geq (c^{-1} \cdot K(n, p))^{\frac{q}{q-p}} \nu(S) \).

**Proof.** Since \( \{u_\lambda\} \) is bounded in \( W_1^p(B) \), the relations (6) and (7) follow by the Lions concentration-compactness principle [13]. Since \( I^\lambda \) is homogeneous, we can assume without loss of generality that \( \|u_\lambda\|_{L_q(S)} = 1 \) and \( \nu(S) = 1 \).

Next we use the argument from [1,7]: let us fix \( x_k \) from (6) and (7). We choose \( \varphi \in C^\infty_c(\mathbb{R}^n) \) such that \( \varphi = 1 \) in \( B(x_k, \varepsilon) \), \( \varphi = 0 \) in \( \mathbb{R}^n \setminus B(x_k, 2\varepsilon) \), \( |\nabla \varphi| \leq \frac{C}{\varepsilon} \).

Since \( DI^\lambda[u_\lambda] \to 0 \), we obtain

\[ \lim_{j \to \infty} DI^\lambda[u_\lambda](\varphi u_\lambda) = 0. \]

Then

\[ \lim_{j \to \infty} \int_B |\nabla u_\lambda|^p - 2 \nabla u_\lambda \cdot \nabla \varphi_j u_\lambda dx = c \int_S \varphi d\nu - \lambda \int_B |u_0|^p \varphi dx. \quad (8) \]

One can estimate the left hand side as follows:

\[
0 \leq \left| \lim_{j \to \infty} \int_B |\nabla u_\lambda|^p - 2 \nabla u_\lambda \cdot \nabla \varphi_j u_\lambda dx \right| \\
\leq \lim_{j \to \infty} \left( \int_B |\nabla u_\lambda|^p dx \right)^\frac{n-1}{n} \left( \int |\nabla \varphi|^p |u_\lambda|^p dx \right)^\frac{1}{p} \\
\leq C \left( \int_{B(x_k, 2\varepsilon)} |\nabla \varphi|^p |u_0|^p dx \right)^\frac{1}{n} \left( \int_{B(x_k, 2\varepsilon)} |u_\lambda|^\frac{n}{n-p} dx \right)^\frac{n-p}{pn} \\
\leq C \left( \int_{B(x_k, 2\varepsilon)} |u_\lambda|^\frac{np}{n-p} dx \right)^\frac{n-p}{pn} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Taking the limit in (8), we get \( \nu_k = c^{-1} \mu_k \geq c^{-1} K(n, p)\nu_k^\frac{p}{q} \). This means either \( \nu_k \geq (c^{-1} K(n, p))^{\frac{q}{q-p}} \) or \( \nu_k = 0 \). \[\square\]
Lemma 2. Let \( u^\lambda \in X \) be a sequence such that \( I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}} \). Then there is a \( \beta_0 > 0 \) such that for any \( \beta \leq \beta_0 \) there is \( x_0 \in S \) such that we have up to subsequence the following weak convergence in the sense of measures as \( \lambda \to \infty \):

\[
|u^\lambda|^q dS \to \sum_{x_k \in Gx_0} \frac{1}{m(A)} \delta(x - x_k).
\]

Proof. Since \( \|u^\lambda\|_{W^{1,q}_0}^p \leq I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}} \) by the Lions concentration-compactness principle, we get

\[
|\nabla u^\lambda|^p dx \to \mu \geq |\nabla u_0|^p dx + K(n,p) \sum_k \nu_k^\frac{p}{q} \delta(x - x_k),
\]

\[
|u^\lambda|^q dS \to \nu = |u_0|^q dS + \sum_k \nu_k \delta(x - x_k),
\]

where \( \delta(x - x_k) \) are delta measures at some points \( x_k \) in \( S \).

Since \( \lambda \|u^\lambda\|_{L^p_\mu(B)} \) is uniformly bounded, we have \( u^\lambda \to 0 \) in \( L^p_\mu(B) \), so \( u_0 = 0 \). Combining the above with the fact that \( u^\lambda \) are invariant with respect to \( G \) we get:

\[
\lim_{\lambda \to \infty} I^\lambda[u^\lambda] = \mu(B) \geq K(n,p) \sum_k \nu_k^\frac{p}{q} = K(n,p) \sum_j |Gx_j| \left( \frac{\tilde{\nu}_j}{|Gx_j|} \right)^\frac{p}{q} = K(n,p) \sum_j |Gx_j|^{1-\frac{p}{q}} \tilde{\nu}_j^\frac{p}{q}. \quad (10)
\]

Here \( j \) goes over different classes of equivalence of \( x_k \), and \( \tilde{\nu}_j = |Gx_j|\nu_j \) is a total contribution of that class to \( \nu(\partial \Omega) \). The second equality is due to the fact that \( u^\lambda \) are \( G \)-invariant, so for every \( x_k \) there are \( |Gx_k|\delta \)-functions with the same coefficient.

Since \( p < q \), we have \( a^\frac{p}{q} + b^\frac{p}{q} > (a + b)^\frac{p}{q} \), for any \( a > 0, b > 0 \). Recalling that \( A \) is a locally minimal orbital set, we can write

\[
\mu(B) \geq K(n,p)m(A)^{1-\frac{p}{q}} \sum_{j : x_j \in A} \tilde{\nu}_j^\frac{p}{q} + K(n,p) \sum_{i : x_i \notin A} |Gx_i|^{1-\frac{p}{q}} \tilde{\nu}_i^\frac{p}{q} \geq \quad (11)
\]

\[
\geq K(n,p)(m(A)^{1-\frac{p}{q}} \alpha^\frac{p}{q} + m(G)^{1-\frac{p}{q}}(1 - \alpha)^\frac{p}{q}),
\]

where \( 1 - \beta \leq \alpha \leq 1 \) (we recall that \( m(G) \) is the number of elements in the minimal orbit of \( G \)).

It is easy to see that the right-hand side of (11) is a concave function of \( \alpha \). That means that if \( \beta \) is small enough, then the right-hand side is a decreasing function, which achieves its minimum of \( K(n,p)m(A)^{1-\frac{p}{q}} \) at \( \alpha = 1 \).

Since by assumption \( \mu(B) = \lim_{\lambda \to \infty} I^\lambda[u^\lambda] \leq K(n,p)m(A)^{1-\frac{p}{q}} \), we conclude that \( \alpha = 1 \). Recalling that for \( u \in X \|u\|_{L^q(S)} = 1 \), we get (9). \( \square \)

From now on we always assume that \( \lambda \) is fixed and whenever there is a limit, it is taken over \( j \to \infty \) unless specified otherwise.
Lemma 3. The minimum of $I^\lambda$ on $X$ is attained if $\lambda$ is large enough and
$$\inf_{u \in X} I^\lambda [u] < K(n, p)m(A)^{1-\frac{p}{q}}.$$  

Proof. The Ekeland’s variational principle [8] provides the existence of a minimizing sequence $u_j^\lambda \in X$ such that $I'[u_j^\lambda] \to 0$. Since $u_j^\lambda$ is a Palais–Smale sequence at the level $\inf_{u \in X} I^\lambda [u] < K(n, p)m(A)^{1-\frac{p}{q}}$, Lemma 1 gives the estimate on any non-zero $\nu_k$ in (7):
$$\nu_k > m(A)^{-(1-\frac{p}{q})\frac{q}{p-q}} = \frac{1}{m(A)}.$$  

Suppose that there is a $\delta$-function outside of $A$. From (9) follows that for large $\lambda$ almost all of $\nu(S)$ mass is concentrated in a $\varepsilon$-neighbourhood of $A$, and according to (12) there are no $\delta$-functions outside of that neighbourhood.

Let us suppose that there is a $\delta$-function at $x_k \in A^\varepsilon$ with weight $\nu_k$. Since $A$ is a locally minimal orbital set, we know that $|Gx_k| \geq m(A)$. Now from (10) and (12), we derive
$$\lim_{j \to \infty} I^\lambda[u_j^\lambda] \geq K(n, p)|Gx_k| \left(\frac{1}{m(A)}\right)^{\frac{p}{q}} = K(n, p)m(A)^{1-\frac{p}{q}},$$
which is a contradiction.

From that follows that for $u_0^\lambda$ in (9) $\|u_0^\lambda\|_{L_\nu(S)} = \|u_j^\lambda\|_{L_\nu(S)} = 1$. It is well known that weak convergence and convergence of norms imply strong convergence in uniformly convex Banach space (e.g., [3, Proposition 3.32]), and that completes our proof. That way $u_0^\lambda \in X$ and $I^\lambda[u_0^\lambda]$ attains minimal value. \hfill \Box

3. Main results.

Lemma 4. Let $G = H \times O(n-k)$ where $H$ is a finite subgroup of $O(k)$ and $A \subset \mathbb{R}^k$ is a minimal orbital set under the action of $H$.

Then for any fixed $\beta, \lambda$ large enough, and $p \leq \frac{n+1}{2}$, we have
$$\inf_{u \in X} I^\lambda[u^\lambda] < K(n, p)m(A)^{1-\frac{p}{q}}.$$  

Proof. Let $x_0 \in Gx_0$ be a point in $A \times \{0\}$. As was shown in [15] (see also [2]), there is a function $u_R$ in $W^1_p(B_R)$ supported in a small ball around $Rx_0$ and axially symmetric with respect to the axis $Ox_0$, such that $\|u_R\|_{W^1_p(B_R)}^p < K(n, p)\|u\|_{L_q(B_R)}^p$.

Now we construct the function
$$v_R(x) = \sum_{g \in H} u_R(gx).$$

It is easy to see that $v_R$ is $G$-invariant and
$$\frac{\|v_R\|_{W^1_p(B_R)}^p}{\|v_R\|_{L_q(B_R)}^p} = m(A)^{1-\frac{p}{q}} \frac{\|u_R\|_{W^1_p(B_R)}^p}{\|u_R\|_{L_q(B_R)}^p} < K(n, p)m(A)^{1-\frac{p}{q}}.$$  

By rescaling we obtain (14). \hfill \Box
**Theorem 1.** Let $p \leq \frac{n+1}{2}$, and let $G$ be as in Lemma 4. Suppose that $A \subset \mathbb{R}^k$ is some locally minimal orbital set of $H$. Then there is $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ there is a $G$-invariant solution of problem (3) such that it concentrates at $|Gx_0|$ points in the $Gx_0$ for some $x_0 \in A \times \{0\}$, i.e.

$$\frac{|u^\lambda|^q}{\|u^\lambda\|_{L_q(S)}} \rightarrow \sum_{k=1} \left|\frac{1}{|G(x_0)|}\delta(x - x_k)\right| \quad \text{as} \quad \lambda \rightarrow \infty.$$

**Proof.** According to Lemmas 4 and 3, there is a minimizer $u \in X$ such that it is concentrated around $m(A)$ points of $A \times \{0\}$. Lemma 2 implies that if $\lambda$ is large enough, the constraint $\|u\|_{L_q(A^s)}^q > 1 - \beta$ is non-active and does not produce a Lagrange multiplier. Since $I^\lambda[u] = I^\lambda[|u|]$, we can assume that $u$ is non-negative. Since $u$ is a local minimizer, we get for $\mu = I^\lambda[u]$ (see Proposition 1):

$$\int_B |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx + \lambda \int_B |u|^{p-2} u h \, dx - \mu \int_S |u|^{q-2} u h \, dS = 0 \quad \forall h \in L_G,$$

where

$$L_G = \{ h \in W^1_p(B) \mid h(gx) = h(x) \ \forall g \in G \}.$$

Due to the principle of symmetric criticality [17], $u$ is a solution to the problem

$$\begin{cases}
\Delta_p u := \lambda |u|^{p-2} u & \text{in } B, \\
|\nabla u|^{p-2} \langle \nabla u, n \rangle = \mu |u|^{q-2} u & \text{on } S.
\end{cases}$$

Since $u \geq 0$ in $B$, we can apply the Harnack inequality (see [19, 20]), and get the positivity of our solution. Since the boundary condition is not homogeneous, it is easy to show that $\mu \frac{1}{p-q} u$ is a solution for problem (3).

**Theorem 2.** For any $N > 0$ there is $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ problem (3) has at least $N$ distinct solutions.

**Proof.** Let us look at the following decomposition of $\mathbb{R}^n$:

$$\mathbb{R}^n = (\mathbb{R}^2)^l \times \mathbb{R}^m.$$ 

Here $l \geq 1, m \geq 0.$ We denote variables in $\mathbb{R}^n$ by $x$, in $\mathbb{R}^2$ by $y$, and in $\mathbb{R}^m$ by $z$. This way,

$$x = (y_1, y_2, \ldots, y_l; z).$$

We introduce the group $G_{k,l} = H_{k,l} \times O(m)$ where $H_{k,l}$ is generated by rotations of every $y_i$ by $\frac{2\pi}{k}$ and by transpositions of $y_i$ and $y_j$ for every $i$ and $j$.

Let $A$ be a globally minimal orbital set for the action of $H_{k,l}$. One can easily check that $A \times \{0\}$ is a locally minimal orbital set for $G_{k,l}$.

Now we show that for $l \geq 1$ and $k > 2$ the minimizers will be non-equivalent. In order to do that, we analyse minimal orbits of $H_{k,l}$. The simple calculation yields that a minimal orbit would be of a point $(y, 0, \ldots, 0) \in \mathbb{R}^{2l}$ where $y \in \mathbb{R}^2$ and it consists of $k \cdot l$ points. Knowing the structure of the minimal orbits, we
can deduce that minimizers would be different for different pairs of \((k,l)\) and \((k',l')\).

\[
\square
\]

Now we consider an analogue of the problem (3) in an \(n\)-dimensional hemisphere.

To prove the multiplicity result, we only need to modify Lemma 4 by using the existence result from [16].

**Lemma 5.** Let \(n \geq 5\) and let \(B\) be an \(n\)-dimensional hemisphere. Let \(G = H \times O(n-k)\) where \(H\) is a finite subgroup of \(O(k)\) such that \(A\) is a minimal orbital set under the action of \(H \times \{0\}\).

Then for any fixed \(\beta,\lambda\) large enough and \(2 < p \leq \frac{n+2}{3}\), we have

\[
\inf_{u \in X} I^\lambda[u^\lambda] < K(n,p)m(A)^{1-\frac{p}{q}}.
\]

Repeating the previous arguments, we get the following theorem:

**Theorem 3.** Let \(n \geq 5\), let \(B\) be an \(n\)-dimensional hemisphere, and let \(2 < p \leq \frac{n+2}{3}\). Then for any \(N > 0\) there is a \(\lambda_0 > 0\) such that for any \(\lambda > \lambda_0\) problem (3) has at least \(N\) rotationally non-equivalent solutions.

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