THE SECOND MAIN THEOREM FOR SPHERICALLY
SYMmetric Kähler manifolds

XIANJING DONG & PEICHU HU

Abstract. We investigate the value distribution of holomorphic maps
defined on one class of Kähler manifolds. With the very natural settings,
we establish a Second Main Theorem which is of the similar form as ones
of the classical Second Main Theorem for complex Euclidean spaces and
complex unit balls.

1. Introduction

1.1. Motivation.

Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^m \) or \( \mathbb{B}^m \) and \( a_1, \ldots, a_q \)
be distinct points in \( \overline{\mathbb{C}} \), where \( \mathbb{B}^m \) is the unit ball (with standard Euclidean
metric) in \( \mathbb{C}^m \). We have the familiar notations in Nevanlinna theory (see [20])
such as characteristic function \( T_f(r) \), counting function \( N_f(r, a) \), proximity
function \( m_f(r, a) \) and simple counting function \( N_f(r, a) \), etc., see Noguchi [19] and Ru [21]. We recall the Second Main Theorem in Nevanlinna theory
for \( \mathbb{C}^m \) and \( \mathbb{B}^m \) as follows:

(i) In the case of \( \mathbb{C}^m \): for every \( \delta > 0 \), we have

\[
(q - 2)T_f(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O\left( \log^+ T_f(r) + \delta \log^+ r \right)
\]

holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

(ii) In the case of \( \mathbb{B}^m \): for every \( \delta > 0 \), we have

\[
(q - 2)T_f(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O\left( \log^+ T_f(r) + \log \frac{1}{1 - r} \right)
\]

holds for all \( r \in (0, 1) \) outside a set \( E_\delta \) with \( \int_{E_\delta} (1 - r)^{-1}dr < \infty \).

2010 Mathematics Subject Classification. 32H30, 30D35.
Key words and phrases. Nevanlinna theory; Second Main Theorem; Holomorphic map;
Defect relation; Spherically symmetric manifolds.
Note that \( T_f(r) \) is bounded from below by \( O(\log r) \) as \( r \to \infty \). Hence, the result (i) concludes the Little Picard Theorem asserting that a nonconstant meromorphic function can omit at most two points. This result was extended to the case of complex projective manifolds by Carlson-Griffiths-King \([4, 9]\) under the dimension condition that the dimension of target manifolds is not greater than \( m \). For the \( B^m \)-case, the Little Picard Theorem no longer holds, but one will find from (ii) that \( f \) can omit at most two points if \( T_f(r) \) grows rapidly enough.

In 2010, by utilizing a technique of Brownian motion initialized by Carne \([5]\), Atsuji \([2]\) established a Second Main Theorem of meromorphic functions on a non-positively curved complete Kähler manifold. His theorem extends the classical Nevanlinna theory for \( \mathbb{C}^m \) (see (i) in above), which says that

**Theorem A** (Atsuji, \([2]\)). Let \( M \) be a complete Kähler manifold of non-positive sectional curvature and \( a_1, \ldots, a_q \) be distinct points in \( \overline{\mathbb{C}} \). Let \( f \) be a nonconstant meromorphic function on \( M \). Then for every \( \delta > 0 \),

\[
(q - 2)T_f(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + N(r, \text{Ric}) + O \left( \log^+ T_f(r) + \log^+ (G(r)\Phi(r)) \right)
\]

holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

In Theorem A, the error terms such as \( \log^+(G(r)\Phi(r)) \) and \( N(r, \text{Ric}) \) are involved, where \( G \) is the solution of a certain second order ODE depending on the Green functions for geodesic balls in \( M \) and \( \Phi \) is expressed by \( G \), and the curvature term \( N(r, \text{Ric}) \) is determined by the Ricci curvature of \( M \).

Recently, the first named author \([6]\) investigated Carlson-Griffiths theory \([4]\) for complete Kähler manifolds by using the similar probabilistic method. The author generalized Theorem A by the following

**Theorem B** (Dong, \([6]\)). Let \( M \) be a complete Kähler manifold of non-positive sectional curvature and \( V \) be a complex projective manifold satisfying that \( \dim M \geq \dim V \). Let \( D \in |L| \) be a divisor of simple normal crossing type, where \( L \) is a holomorphic line bundle over \( V \). Fix a Hermitian metric \( \omega \) on \( V \). Let \( f : M \to V \) be a differentiably non-degenerate meromorphic mapping. Then for any \( \delta > 0 \),

\[
T_f(r, L) + T_f(r, K_V) \leq \bar{N}_f(r, D) + O \left( \log^+ T_f(r, \omega) - \kappa(r)r^2 + \delta \log r \right)
\]

holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

In Theorem B, the term \( \kappa(r) \) is the minimal of the pointwise lower bound of the Ricci curvature for geodesic balls in \( M \). Note from the above theorem, if one needs to receive a defect relation in Nevanlinna theory, then \( T_f(r) \) must grow rapidly enough. Both Theorem A and Theorem B cannot conclude (ii),
because these estimate terms are rough. In order to establish a Second Main Theorem with good error terms, some Second Main Theorem with the form like (i) or (ii) is expected. Thus, a natural question is that: for what kind of Kähler manifolds, the Second Main Theorem will be of the similar form as (i) or (ii)? Motivated by that, we give investigations to Nevanlinna theory for a class of Kähler manifolds, i.e., the so-called \textit{spherically symmetric Kähler manifolds} which satisfy the requirements.

1.2. Main results.

Let $M_\sigma$ be a spherically symmetric Kähler manifold of a pole $o$ and radius $R$ (see Section 2.2 for definition). Consider a holomorphic map $f : M_\sigma \to N$ into a complex projective manifold $N$ with $\dim N \leq \dim M_\sigma$. In our settings, we will remove all the restrictions such as completeness and non-positiveness of sectional curvature of a Kähler manifold in Theorems A and B. Without going into the details of notations, we prove the following main result:

\textbf{Theorem I} (=Theorem 3.6). Let $M_\sigma$ be a spherically symmetric Kähler manifold of complex dimension $m$, with a pole $o$ and radius $R$. Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim_C N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differentiably non-degenerate holomorphic map. Then

(a) For $R = \infty$ and every $\delta > 0$,

$$T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{M_\sigma}) \leq \overline{\nabla} f(r, D) + O\left(\log^+ T_f(r, L) + \delta \log^+ \sigma(r)\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

(b) For $R < \infty$ and every $\delta > 0$,

$$T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{M_\sigma}) \leq \overline{\nabla} f(r, D) + O\left(\log^+ T_f(r, L) + \log \frac{1}{R - r}\right)$$

holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} (R - r)^{-1} \, dr < \infty$.

We interpret how results (i) and (ii) can be derived from ours. In the case that $M_\sigma = \mathbb{C}^m$ (with standard Euclidean metric), we have $R = \infty$, $\sigma(r) = r$ and $\mathcal{R}_{M_\sigma} = 0$, where $\mathcal{R}_{M_\sigma}$ denotes the Ricci form of $M_\sigma$. By (a) in the above theorem, it immediately deduces the theorem of Carlson-Griffiths-King (see Corollary 3.7). The Second Main Theorem for the case that $M_\sigma = \mathbb{B}^m$ (with standard Euclidean metric) also follows by noting that $R = 1$, $\sigma(r) = r$ and $\mathcal{R}_{\mathbb{B}^m} = 0$ (see Corollary 3.8).

A manifold is said to be \textit{non-parabolic} if it admits a non-constant positive superharmonic function, and said to be \textit{parabolic} otherwise.

\textbf{Theorem II} (=Theorem 3.12). Let $M_\sigma$ be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension $m$. 

Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim \mathbb{C} N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differentiably non-degenerate holomorphic map. Assume that $M_\sigma$ is parabolic. Then for every $\delta > 0$,

$$T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{M_\sigma}) \leq N_f(r, D) + O\left(\log^+ T_f(r, L) + \delta \log^+ r\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

We consider a defect relation under certain curvature condition. For two holomorphic line bundles $L_1, L_2$ over $N$, set

$$\left[\frac{c_1(L_2)}{c_1(L_1)}\right] = \inf \left\{ t \in \mathbb{R} : \eta_2 < t \eta_1; \exists \eta_1 \in c_1(L_1), \exists \eta_2 \in c_1(L_2) \right\}.$$

Let $\Theta_f(D)$ be the simple defect of $f$ with respect to $D$ defined by

$$\Theta_f(D) = 1 - \limsup_{r \to R} \frac{N_f(r, D)}{T_f(r, L)}.$$

**Theorem III** (=Corollary 3.15). The conditions are assumed as same as in Theorem I. In addition, assume that $M_\sigma$ has non-negative scalar curvature.

(a) For $R = \infty$, if $T_f(r, L) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then

$$\Theta_f(D) \leq \left[\frac{c_1(K_N^*)}{c_1(L)}\right].$$

(b) For $R < \infty$, if $\log(R - r) = o(T_f(r, L))$ as $r \to R$, then

$$\Theta_f(D) \leq \left[\frac{c_1(K_N^*)}{c_1(L)}\right].$$

2. Spherically symmetric manifolds

2.1. Laplace operators, polar coordinates and Ricci curvatures.

2.1.1. Laplace operators and polar coordinates.

Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The well-known Laplace-Beltrami operator $\Delta_M$ of $\nabla$ is defined by

$$\Delta_M = \sum_{i,j} g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j}),$$

where $\partial_j = \partial/\partial x_j$ and $(g^{ij})$ is the inverse of $(g_{ij})$. When acting on a function, $\Delta_M$ has the implicit formula

$$\Delta_M = \sum_{i,j} \frac{1}{\sqrt{\det(g_{st})}} \partial_i \left( \sqrt{\det(g_{st})} g^{ij} \frac{\partial}{\partial x_j} \right).$$
Fix $o \in M$, one denotes by $B_o(r), S_o(r)$ the geodesic ball and geodesic sphere of radius $r$ with center at $o$ in $M$ respectively, and by $r(x)$ the Riemannian distance function of $x$ from $o$. Set $CUT^*(o) = CUT(o) \cup \{o\}$, where $CUT(o)$ is the cut locus of $o$. For $x \in M \setminus CUT^*(o)$, one can define the polar coordinates $(r, \theta)$ of $x$ with respect to the pole $o$, where $r = r(x)$ is called the polar radius and $\theta \in S^{d-1}$ is called the polar angle which provides the direction $\Gamma_{\theta} \in T_oM$ of the minimal geodesic connecting $o$ with $x$ at $o$, in which $d = \dim M, S^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$ centered at the origin. Now write the metric $g$ of $M$ in the polar coordinate form

$$ds^2 = dr^2 + \sum_{i,j} \tilde{g}_{ij} d\theta_i d\theta_j,$$

where $\theta_j$ are coordinate components of $\theta$, and $\tilde{g}_{ij}$ is the Riemannian metric on $S_o(r) \setminus CUT(o)$. This gives the Riemannian area element on $S_o(r) \setminus CUT(o)$ that

$$dA_r = \sqrt{\det (\tilde{g}_{st})} d\theta_1 \cdots d\theta_{d-1}.$$

If $\theta_j$ are defined almost everywhere on $S^{d-1}$, then we have

$$Area(S_o(r)) = \int_{S^{d-1}} \sqrt{\det (\tilde{g}_{st})} d\theta_1 \cdots d\theta_{d-1}$$

In terms of polar coordinates, the Laplace-Beltrami operator is written as

$$\Delta_M = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \frac{\log \sqrt{\det (\tilde{g}_{st})}}{\partial r} + \Delta_{S_o(r)},$$

where $\Delta_{S_o(r)}$ is the induced Laplace-Beltrami operator on $S_o(r)$.

Now we turn to Hermitian manifolds. Let $(M, h)$ be a Hermitian manifold with Hermitian connection $\tilde{\nabla}$. Note that $M$ can be regarded as a Riemannian manifold with Riemannian metric $g = \Re h$, thus there is also the Levi-Civita connection $\nabla$ on $M$. Now, extend $\nabla$ linearly to $T_cM = TM \otimes \mathbb{C}$. In general, $\nabla \neq \nabla$ since the torsion tensor of $\nabla$ may not vanish for the general Hermitian manifolds. Hence, the Laplace operator $\Delta_M$ of $\nabla$ does not coincide with the Laplace-Beltrami operator $\Delta_M$ of $\nabla$. However, the case for $\nabla = \nabla$ happens when $M$ is a Kähler manifold. Consequently,

$$\Delta_M = \Delta_M = 2 \sum_{i,j} h^{ij} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$$

acting on a function for that $M$ is Kählerian, where $z_j$ are local holomorphic coordinates and $(h^{ij})$ is the inverse of $(h_{ij})$. 
2.1.2. **Ricci curvatures.**

Let \((M, h)\) be an \(m\)-dimensional Hermitian manifold with Kähler form 
\[
\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.
\]

The metric \(h\) induces a Hermitian metric \(\det(h_{ij})\) on the anticanonical bundle \(K^*_M\). The Chern form of \(K^*_M\) associated to this metric is defined by
\[
\mathcal{R}_M := c_1(K^*_M, \det(h_{ij})) = -dd^c \log \det(h_{ij})
\]
which is usually called the Ricci form of \(M\) due to \(\mathcal{R}_M = \text{Ric}(\alpha^m)\), where 
\[
d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial).
\]

Assume that \(h\) is a Kähler metric, then \(\mathcal{R}_M\) can be written as
\[
\mathcal{R}_M = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} R_{ij} dz_i \wedge d\bar{z}_j,
\]
where \(\text{Ric}^c = \sum_{i,j} R_{ij} dz_i \otimes d\bar{z}_j\) is the complex Ricci curvature tensor of \(h\). Regard \(M\) as a Riemannian manifold with Riemannian metric \(g = \Re h\), then there is also a real Ricci curvature tensor written as \(\text{Ric}^R = \sum_{i,j} R_{ij} dx_i \otimes dx_j\) of \(g\). Denote by \(s^c, s^R\) the scalar curvatures of \(h, g\) respectively, i.e.,
\[
s^c = \sum_{i,j} h^{ij} R_{ij}, \quad s^R = \sum_{i,j} g^{ij} R_{ij}.
\]
Then we have
\[
s^R = 2s^c = -\Delta_M \log \det(h_{ij}).
\]

2.2. **Spherically symmetric manifolds.**

Let \((M, g)\) be a Riemannian manifold. We say that \(M\) is a manifold with a pole \(o\) if \(\text{Cut}(o) = \emptyset\). If, in addition, \(M\) is complete or geodesically complete, then \(M\) is diffeomorphic to \(\mathbb{R}^d\), where \(d = \dim M\). A Riemannian manifold with a pole \(o\) is called a *spherically symmetric manifold* if the induced metric \(\tilde{g}_{ij}\) on \(S_o(r)\) is of the form
\[
\sum_{i,j} \tilde{g}_{ij}(r, \theta)d\theta_i d\theta_j = \sigma^2(r) d\theta^2,
\]
where \(d\theta^2 = d\theta_1^2 + \cdots + d\theta_{d-1}^2\) is the standard Euclidean metric on \(S^{d-1}\) and \(\sigma\) is a positive smooth function of \(r\). For convenience, one uses \(M_\sigma\) to denote such manifolds.

Let a smooth positive function \(\sigma\) on \((0, R)\) with \(0 < R \leq \infty\), the necessary and sufficient condition (see [7]) for that such a manifold exists, is that
\[
\sigma(0) = 0, \quad \sigma'(0) = 1.
\]
One calls $R$ the radius of $M_\sigma$ with respect to the pole $o$. Clearly, $R = \infty$ if $M$ is geodesically complete and non-compact. Consider a spherically symmetric manifold $M_\sigma$ of a pole $o$ and radius $R$. For $r < R$, we have
\[ dA_r = \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1}. \]
This means that
\[ \text{Area}(S_o(r)) = \int_{S^{d-1}} \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1} = \omega_{d-1} \sigma^{d-1}(r), \]
\[ \text{Vol}(B_o(r)) = \omega_{d-1} \int_0^r \sigma^{d-1}(t) dt, \]
where $\omega_{d-1}$ is the area of $S^{d-1}$. We also have
\[ \Delta_{M_\sigma} = \frac{\partial^2}{\partial r^2} + (d - 1) \frac{\sigma'}{\sigma} \frac{\partial}{\partial r} + \frac{1}{\sigma^2} \Delta_\theta, \]
where $\Delta_\theta$ is the standard Laplace-Beltrami operator on $S^{d-1}$.

Several typical models

Let $M_\sigma$ be a spherically symmetric manifold of radius $R$.

(a) If $R = \infty, \sigma(r) = r$, then $M_\sigma \cong \mathbb{R}^d$ (with standard Euclidean metric). The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\theta. \]

(b) If $R = \infty, \sigma(r) = \sinh r$, then $M_\sigma \cong H^d$, where $H^d$ is the $d$-dimensional upper half-space with hyperbolic metric of sectional curvature $-1$ in $\mathbb{R}^d$. The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + (d-1) \cot r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_\theta. \]

(c) If $R = \pi, \sigma(r) = \sin r$, then $M_\sigma \cong S^d$ (endpoint with $r = \pi$ is added to $M_\sigma$), where $S^d$ is the $d$-dimensional unit sphere centered at the origin in $\mathbb{R}^{d+1}$. The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + (d-1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_\theta. \]

2.3. Green functions for spherically symmetric manifolds.

Let $M_\sigma$ be a $d$-dimensional spherically symmetric manifold of a pole $o$ and radius $R$. Establish a polar coordinate system $(o, r, \theta)$ of $M_\sigma$. For $0 < r < R$, we shall compute the harmonic measure $d\pi'_o(x)$ on $S_o(r)$ with respect to $o$. 

as well as the Green function $g_r(o, x)$ of $\Delta_{M_o}/2$ for $B_o(r)$ with pole at $o$ and Dirichlet boundary condition, i.e.,
\[-\frac{1}{2}\Delta_{M_o} g_r(o, x) = \delta_o(x) \text{ for } x \in B_o(r); \quad g_r(o, x) = 0 \text{ for } x \in S_o(r),\]
where $\delta_o$ is the Dirac function.

**Lemma 2.1.** For $0 < r < R$, we have
\[d\pi^r_o(x) = \frac{d\theta_1 \cdots d\theta_{d-1}}{\omega_{d-1}}, \quad g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{r(x)}^r \frac{dt}{\sigma^{d-1}(t)},\]
where $\omega_{d-1}$ is the area of the unit sphere in $\mathbb{R}^d$ with $d \geq 2$.

**Proof.** By the property of spherically symmetric manifolds, the induced area measure
\[dS_r(x) = \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1}\]
on $S_o(r)$ is a rotationally invariant one with respect to $o$. On the other hand, $dS_r(x)/Area(S_o(r))$ is a probability measure on $S_o(r)$. Thus,
\[d\pi^r_o(x) = \frac{dS_r(x)}{Area(S_o(r))} = \frac{d\theta_1 \cdots d\theta_{d-1}}{\omega_{d-1}}.\]
Notice the relation
\[-\frac{1}{2} \frac{\partial g_r(o, x)}{\partial n} = \frac{d\pi^r_o(x)}{dS_r(x)},\]
where $\partial/\partial n$ is the inward normal derivative on $S_o(r)$. Then
\[\frac{\partial g_r(o, x)}{\partial n} = -\frac{2}{\omega_{d-1}\sigma^{d-1}(r)}.\]
On the other hand,
\[-\frac{1}{2}\Delta_{M_o} g_r(o, x) = \delta_o(x)\]
for $x \in B_o(r)$. Combine the above two equations, it is trivial to confirm that
\[g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{r(x)}^r \frac{dt}{\sigma^{d-1}(t)}.\]

\[\square\]

In what follows, we give two examples to compute Green functions.

**Example 1.** $M_o = \mathbb{R}^d$ (with standard Euclidean metric)

Take $o$ as the coordinate origin of $\mathbb{R}^d$. By $\sigma(r) = r$ and $r(x) = \|x\|$, one has
\[g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{\|x\|}^r \frac{dt}{t^{d-1}},\]
which can be computed easily.
Example 2. $M_\sigma = \mathbb{H}$ (Poincaré upper half-plane, i.e., $H^2$ with Poincaré metric)

Take $o = (0, \sqrt{-1})$. Let $\phi : \mathbb{D} \to \mathbb{H}$ be the biholomorphic map as follows

$$\phi(z) = \frac{1 - \sqrt{-1}z}{z - \sqrt{-1}}.$$ 

Note that $\phi^* h$ is the Poincaré metric on $\mathbb{D}$, where $h$ is the Poincaré metric on $\mathbb{H}$. By $\sigma(r) = \sinh r$ and $\omega_1 = 2\pi$, we see that

$$g_r(a, x) = 2 \int_0^r \frac{dt}{t} \frac{\sinh t}{e^t - e^{-t}} = \frac{1}{\pi} \log \frac{(e^r - 1)(e^{r(x)} + 1)}{(e^r + 1)(e^{r(x)} - 1)},$$

where

$$r(x) = \log \frac{1 + |\phi^{-1}(x)|}{1 - |\phi^{-1}(x)|}.$$

3. Holomorphic maps on spherically symmetric Kähler manifolds

3.1. Nevanlinna’s functions and First Main Theorem.

3.1.1. Nevanlinna’s functions.

We extend the notion of Nevanlinna’s functions containing characteristic function, counting function and proximity function to spherically symmetric Kähler manifolds. Let $(M_\sigma, h)$ be a spherically symmetric Kähler manifold of complex dimension $m$, with a pole $o$ and radius $R$. Then the Kähler form of $M$ is written as

$$\alpha = \frac{\sqrt{-1}}{\pi} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.$$ 

Fix a $r_0$ such that $0 < r_0 < R$. For any (1,1)-form $\eta$ on $M_\sigma$, define formally the notation

$$T(r, \eta) = \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{B_o(t)} \eta \wedge \alpha^{m-1}.$$ 

Let $f : M_\sigma \to N$ be a holomorphic map into a complex projective manifold $N$, and $(L, h_L)$ be a positive Hermitian line bundle over $N$. Let $|L|$ be the complete linear system of all effective divisors $D_s$ with $s \in H^0(M, L)$, where $D_s$ denotes the zero divisor of a section $s$. For $r_0 < r < R$, the characteristic function of $f$ with respect to $L$ is defined by

$$T_f(r, L) = T_f(r, c_1(L, h_L))$$
up to a bounded term. Since $M_\sigma$ is Kählerian, then
\[
\Delta_M = 2 \sum_{i,j} h^{ij} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.
\]
Thus,
\[
\Delta_M \log h_L = -4m f^* c_1(L, h) \wedge \alpha^{m-1}.
\]

In terms of Green function, we have
\[
T_f (r, L) = -\frac{1}{4} \int_{B_o (r)} g_r (o, x) \Delta_M \log h_L (x) dV (x)
+ \frac{1}{4} \int_{B_o (r_0)} g_{r_0} (o, x) \Delta_M \log h_L (x) dV (x),
\]
where $dV = \pi^m \alpha^m / m!$ is the Riemannian volume element of $M_\sigma$. Now for a divisor $D \in |L|$, the counting function of $f$ with respect to $D$ is defined by
\[
N_f (r, D) = \int_{r_0}^{r} dt \frac{1}{\sigma_2^{m-1} (t)} \int_{f^* D \cap B_o (t)} \alpha^{m-2}.
\]
Let $s_D$ be the canonical section of $L$, it is of zero divisor $D$. Write $s_D = \tilde{s}_D e$, where $e$ is a local holomorphic frame of $L$. By Poincaré-Lelong formula [4], $N_{f,D} (r)$ has an alternate expression
\[
N_f (r, D) = \int_{r_0}^{r} dt \frac{1}{\sigma_2^{m-1} (t)} \int_{f^* D \cap B_o (t)} \alpha^{m-2}.
\]
In a similar way, we define the simple counting function $\overline{N}_f (r, D)$ for $\text{Supp} f^* D$.

For the proximity function of $f$ with respect to $D$, we use the definition
\[
m_f (r, D) = \int_{S_o (r)} \log \frac{1}{\| s_D \circ f (x) \|} d\pi^r_o (x)
= \frac{1}{\omega_{2m-1}} \int_{S_2^{m-1}} \log \frac{1}{\| s_D \circ f (r, \theta) \|} d\theta_1 \cdots d\theta_{2m-1},
\]
where $(r, \theta)$ stands for the polar coordinate of $x$ with respect to the pole $o$, $S^{2m-1}_2$ is the unit sphere in $\mathbb{R}^{2m}$, and $\omega_{2m-1}$ is the area of $S^{2m-1}_2$. The last equality is due to Lemma [2.1]. Denote by $\mathcal{R}_M = -dd^c \log \det (h_{ij})$ the Ricci form of $M_\sigma$, then
\[
T(r, \mathcal{R}_M) = \int_{r_0}^{r} dt \frac{1}{\sigma_2^{m-1} (t)} \int_{B_o (t)} \mathcal{R}_M \wedge \alpha^{m-1}
= \frac{1}{2} \int_{B_o (r)} g_r (o, x) s_{M_\sigma} (x) dV (x) - \frac{1}{2} \int_{B_o (r_0)} g_{r_0} (o, x) s_{M_\sigma} (x) dV (x),
\]
where $s_{M_\sigma}$ is the scalar curvature of $M_\sigma$, see Section 2.1.2.

**Remark.** When $M_\sigma = \mathbb{C}^m$ with standard Euclidean metric, we have $R = \infty$ and $\sigma(r) = r$. Since $d\pi^r_\sigma(z) = d^c \log \|z\|^2 \wedge (d\bar{d} \log \|z\|^2)^{m-1}$, the generalized definition of Nevanlinna’s functions agrees with the classical one, see [19, 21].

### 3.1.2. First Main Theorem

In the classical Nevanlinna theory, Green-Jensen formula [19, 21] deduces the Nevanlinna’s First Main Theorem. For a meromorphic function defined on a complex manifold, we need Dynkin formula (see [1, 14, 15]) which plays the similar role as Green-Jensen formula. Let’s introduce a simple version of Dynkin formula which is viewed as a special case of the original probabilistic version via Brownian motion, see, e.g., [1, 2, 5, 6, 14, 15].

**Dynkin formula.** Let $u$ be a function of $C^2$-class except at most a polar set of singularities on a Riemannian manifold $M$. For $0 < r_0 < r$ or $0 \leq r_0 < r$ with $u(o) \neq \infty$, we have

$$
\int_{S_\varnothing(r)} u(x)d\pi^r_o(x) - \int_{S_\varnothing(r_0)} u(x)d\pi^r_o(x) = \frac{1}{2} \int_{B_\varnothing(r)} g_\varnothing(o,x)\Delta_M u(x)dV(x) - \frac{1}{2} \int_{B_\varnothing(r_0)} g_{r_0}(o,x)\Delta_M u(x)dV(x)
$$

where $B_\varnothing(r), S_\varnothing(r)$ are geodesic ball and geodesic sphere of radius $r$ centered at $o$ respectively, $g_\varnothing(o,x)$ is the Green function of $\Delta_M/2$ for $B_\varnothing(r)$ with pole at $o$ and Dirichlet boundary condition, and $d\pi^r_o$ is the harmonic metric on $S_\varnothing(r)$ with respect to $o$. Here, $\Delta_M u$ should be understood as distributions.

Particularly, when $M = \mathbb{C}^m$, Dynkin formula coincides with Green-Jensen formula (see [19, 21]). According to the definition of Nevanlinna’s functions and Dynkin formula, we can easily obtain the First Main Theorem as follows

$$
F. M. T. \quad T_f(r,L) = m_f(r,D) + N_f(r,D) + O(1).
$$

### 3.2. Logarithmic Derivative Lemma

Let $(M_\sigma, h)$ be a Hermitian model manifold of complex dimension $m$, with a pole $o$ and radius $R$.

**Lemma 3.1 ([23]).** Let $\gamma$ be an integrable function on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $h$ be a nondecreasing function of $C^1$-class on $(0, R)$. Assume that $\lim_{r \to R} h(r) = \infty$ and $h(r_0) > 0$ for some $r_0 \in (0, R)$. Then for every $\delta > 0$

$$
h'(r) \leq h^{1+\delta}(r)\gamma(r)
$$

holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} \gamma(r)dr < \infty$. In particular, when $R = \infty$, we can take $\gamma = 1$. Then for every $\delta > 0$

$$
h'(r) \leq h^{1+\delta}(r)
$$
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

Let \( \Gamma \) be a locally integrable function on \( M_\sigma \). Set

\[
E_\Gamma(r) = \int_{S_0(r)} \Gamma(x) d\pi_0^r(x), \quad T_\Gamma(r) = \int_{t_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{B_0(t)} \Gamma(x) \alpha^m.
\]

We need the following so-called Calculus Lemma

**Lemma 3.2.** Let \( \gamma \) be an integrable function on \((0, R)\) with \( \int_0^R \gamma(r)dr = \infty \). Let \( \Gamma \) be a locally integrable function on \( M_\sigma \). Then for every \( \delta > 0 \)

\[
E_\Gamma(r) \leq \frac{\pi^m}{\omega_{2m-1}m!} \sigma^{2m-1}(r) \gamma(r) T_\Gamma \left( 1 + \delta \right)^2(r)
\]

holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r)dr < \infty \), where \( \omega_{2m-1} \) is the area of the unit sphere \( S_{2m-1} \) in \( \mathbb{R}^{2m} \).

**Proof.** Notice that

\[
\int_{B_0(r)} \Gamma(x) \alpha^m = \frac{m! \omega_{2m-1} \sigma^{2m-1}(r)}{\pi^m} \int_0^r dt \int_{S_0(t)} \Gamma(x) d\pi_0^r(x),
\]

then we have

\[
\frac{d}{dr} \left( \sigma^{2m-1} \frac{dT_\Gamma}{dr} \right) = \frac{m! \omega_{2m-1} \sigma^{2m-1}}{\pi^m} E_\Gamma.
\]

Using Lemma 3.1 twice (first to \( \sigma^{2m-1}T_\Gamma \) and then to \( T_\Gamma \)), then we can prove the lemma. \( \Box \)

Let \( \psi \) be a meromorphic function on \( M_\sigma \). The norm of the gradient of \( \psi \) is defined by

\[
\| \nabla_{M_\sigma} \psi \|^2 = 2 \sum_{i,j} h \overline{\partial} \partial \psi \overline{\partial} \partial \psi / \partial z_i \partial z_j.
\]

Identify \( \psi \) with a meromorphic mapping into \( \mathbb{P}^1(\mathbb{C}) \). The characteristic function of \( \psi \) with respect to the Fubini-Study form \( \omega_{FS} \) on \( \mathbb{P}^1(\mathbb{C}) \) is defined by

\[
T_\psi(r) = \int_{t_0}^{r} dt \int_{B_0(t)} f^* \omega_{FS} \wedge \alpha^{m-1}.
\]

Let \( i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C}) \) be an inclusion, then it induces a (1,1)-form \( i^* \omega_{FS} \) on \( \mathbb{C} \). The *Ahlfords characteristic function* of \( \psi \) is defined by

\[
\hat{T}_\psi(r) = \int_{t_0}^{r} dt \int_{B_0(t)} f^* (i^* \omega_{FS}) \wedge \alpha^{m-1}.
\]

Moreover, we define the *Nevanlinna characteristic function*

\[
T(r, \psi) = m(r, \psi) + N(r, \psi),
\]
where
\[ m(r, \psi) = \int_{S_o(r)} \log^+ |\psi(x)| d\pi_o^*(x), \quad N(r, \psi) = \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{f^* \cap B_o(t)} \alpha^{m-2}. \]

It is trivial to confirm that \( \hat{T}_\psi(r) \leq T_\psi(r) \) and \( T(r, \psi) = \hat{T}_\psi(r) + O(1) \). Thus, we obtain
\[ T(r, \psi) \leq T_\psi(r) + O(1). \]

On \( \mathbb{P}^1(\mathbb{C}) \), take a singular metric
\[ \Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \frac{\sqrt{-1}}{4\pi^2} d\zeta \wedge d\bar{\zeta}. \]

A direct computation gives that
\[ \int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 4m \pi \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m} = \frac{\|\nabla M_{\sigma} \psi\|^2}{|\psi|^2(1 + \log^2 |\psi|)}. \]

Set
\[ T_\psi(r, \Phi) = \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{B_o(t)} \psi^* \Phi \wedge \alpha^{m-1}. \]

**Lemma 3.3.** We have
\[ T_\psi(r, \Phi) \leq T(r, \psi) + O(1). \]

**Proof.** It yields from Fubini theorem that
\[ T_\psi(r, \Phi) = \int_{\mathbb{P}^1(\mathbb{C})} \Phi(\zeta) \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{\psi^* \zeta \cap B_o(t)} \alpha^{m-2} \]
\[ = \int_{\mathbb{P}^1(\mathbb{C})} N_\psi(r, \zeta) \Phi(\zeta) \]
\[ \leq \int_{\mathbb{P}^1(\mathbb{C})} (T(r, \psi) + O(1)) \Phi \]
\[ = T(r, \psi) + O(1). \]

**Lemma 3.4.** Let \( \gamma \) be an integrable function on \( (0, R) \) with \( \int_0^R \gamma(r) dr = \infty \). Let \( \psi \neq 0 \) be a meromorphic function on \( M_{\sigma} \). Then for every \( \delta > 0 \)
\[ \int_{S_o(r)} \log^+ \frac{\|\nabla M_{\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_o^*(x) \]
\[ \leq (1 + \delta)^2 \log^+ T(r, \psi) + (2 + \delta) \log^+ \gamma(r) + (2m - 1) \delta \log^+ \sigma(r) + O(1) \]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \).
Proof. By Jensen inequality
\[
\int_{S_o(r)} \log^+ \frac{\|\nabla M_p \psi(x)\|^2}{|\psi(x)|^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
\leq \int_{S_o(r)} \log \left(1 + \frac{\|\nabla M_p \psi(x)\|^2}{|\psi(x)|^2 (1 + \log^2 |\psi(x)|)}\right) d\pi_o^r(x)
\leq \log^+ \int_{S_o(r)} \frac{\|\nabla M_p \psi(x)\|^2}{|\psi(x)|^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x) + O(1).
\]

Applying Dykin formula, Lemma 3.2 and (3) to get
\[
\log^+ \int_{S_o(r)} \frac{\|\nabla M_p \psi(x)\|^2}{|\psi(x)|^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
\leq (1 + \delta)^2 \log^+ \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{B(t)} \frac{\|\nabla M_p \psi(x)\|^2}{|\psi(x)|^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
+ (2 + \delta) \log^+ \gamma(r) + (2m - 1) \delta \log^+ \sigma(r) + O(1)
= (1 + \delta)^2 \log^+ T(r, \psi) + (2 + \delta) \log^+ \gamma(r) + (2m - 1) \delta \log^+ \sigma(r) + O(1).
\]

Combining the above, the lemma is proved. □

Define
\[
m \left( r, \frac{\|\nabla M_p \psi\|}{|\psi|} \right) = \int_{S_o(r)} \log^+ \frac{\|\nabla M_p \psi(x)\|}{|\psi(x)|} d\pi_o^r(x).
\]

We have the following Logarithmic Derivative Lemma

**Theorem 3.5.** Let \( \gamma \) be an integrable function on \((0, R)\) with \( \int_0^R \gamma(r) dr = \infty \). Let \( \psi \neq 0 \) be a meromorphic function on \( M_\sigma \). Then for every \( \delta > 0 \)
\[
m \left( r, \frac{\|\nabla M_p \psi\|}{|\psi|} \right) \leq \frac{2 + (1 + \delta)^2}{2} \log^+ T(r, \psi) + \frac{(2 + \delta)}{2} \log^+ \gamma(r)
+ \frac{(2m - 1)\delta}{2} \log^+ \sigma(r) + O(1)
\]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \).
Proof. Notice that
\[
m\left(r, \frac{\|\nabla_M \psi\|}{|\psi|}\right) \leq \frac{1}{2} \int_{S_o(r)} \log^+ \frac{\|\nabla_M \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
+ \frac{1}{2} \int_{S_o(r)} \log^+ (1 + \log^2 |\psi(x)|) d\pi_o^r(x)
\leq \frac{1}{2} \int_{S_o(r)} \log^+ \frac{\|\nabla_M \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
+ \int_{S_o(r)} \log \left(1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|}\right) d\pi_o^r(x).
\]
Lemma 3.3 implies that for every \(\delta > 0\)
\[
\frac{1}{2} \int_{S_o(r)} \log^+ \frac{\|\nabla_M \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
\leq \frac{(1 + \delta)^2}{2} \log^+ T(r, \psi) + \frac{(2 + \delta)}{2} \log^+ \gamma(r) + \frac{(2m - 1)\delta}{2} \log^+ \sigma(r) + O(1)
\]
holds for all \(r \in (0, R)\) outside a set \(E_\delta\) with \(\int_{E_\delta} \gamma(r) dr < \infty\). Using Jensen inequality, it follows that
\[
\int_{S_o(r)} \log \left(1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|}\right) d\pi_o^r(x)
\leq \log \int_{S_o(r)} \left(1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|}\right) d\pi_o^r(x)
\leq \log \left(m(r, \psi) + m(r, \frac{1}{\psi})\right) + O(1)
\leq \log^+ T(r, \psi) + O(1).
\]
Combining the above, we prove the theorem. \(\square\)

3.3. Second Main Theorem.

This subsection aims to prove the following Second Main Theorem

**Theorem 3.6.** Let \(M_\sigma\) be a spherically symmetric Kähler manifold of complex dimension \(m\), with a pole \(o\) and radius \(R\). Let \(L\) be a positive line bundle over a complex projective manifold \(N\) with \(\dim_{\mathbb{C}} N \leq m\), and \(D \in |L|\) be of simple normal crossings. Let \(f : M_\sigma \to N\) be a differentiably non-degenerate holomorphic map. Then for every \(\delta > 0\)
\[
T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{M_\sigma})
\leq N_f(r, D) + O(\log^+ T_f(r, L) + \log^+ \gamma(r) + \delta \log^+ \sigma(r))
\]
holds for all \(r \in (0, R)\) outside a set \(E_\delta\) with \(\int_{E_\delta} \gamma(r) dr < \infty\), where \(\gamma\) is an integrable function on \((0, R)\) such that \(\int_0^R \gamma(r) dr = \infty\). We have two cases:
For \( R = \infty \), we take \( \gamma(r) = 1 \). Then for every \( \delta > 0 \)
\[
T_f(r, L) + T_f(r, K_N) + T(r, R_M \sigma) \\
\leq \mathcal{N}_f(r, D) + O\left( \log^+ T_f(r, L) + \delta \log^+ \sigma(r) \right)
\]
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

For \( R < \infty \), we take \( \gamma(r) = \frac{1}{R-r} \). Then for every \( \delta > 0 \)
\[
T_f(r, L) + T_f(r, K_N) + T(r, R_M \sigma) \\
\leq \mathcal{N}_f(r, D) + O\left( \log^+ T_f(r, L) + \log \frac{1}{R-r} \right)
\]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} (R-r)^{-1} \, dr < \infty \).

We first give several consequences before proving Theorem 3.6.

1. Three classical consequences

(a) \( M_\sigma = \mathbb{C}^m \) (with standard Euclidean metric)

The case implies that \( R = \infty \) and \( \sigma(r) = r \). Then
\[
T_f(r, L) = \int_{r_0}^r \frac{dt}{r^{2m-1}(t)} \int_{B_o(t)} f^* c_1(L, h_L) \wedge \alpha^{m-1},
\]
which coincides with the classical characteristic function. Since \( \mathbb{C}^m \) has sectional curvature 0, then conclusion (a) in Theorem 3.6 derives immediately the classical result of Carlson-Griffiths-King (see [4, 9]) as follows

Corollary 3.7 (Carlson-Griffiths-King). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim \mathbb{C} N \leq m \), and \( D \in |L| \) be of simple normal crossings. Let \( f : \mathbb{C}^m \to N \) be a differentiably non-degenerate holomorphic map. Then for every \( \delta > 0 \)
\[
T_f(r, L) + T_f(r, K_N) \leq \mathcal{N}_f(r, D) + O\left( \log^+ T_f(r, L) + \delta \log^+ \sigma(r) \right)
\]
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

More generalizations of Corollary 3.7 were done by Sakai [24] in terms of Kodaira dimension and by Shiffman [25] in the case of the singular divisor, see also [6, 8, 10, 11, 18, 22, 26, 27].

(b) \( M_\sigma = \mathbb{B}^m \) (unit ball of complex dimension \( m \) with standard Euclidean metric)

The case implies that \( R = 1, \sigma(r) = r \) and \( R_{\mathbb{B}^m} = 0 \). We also have
\[
T_f(r, L) = \int_{r_0}^r \frac{dt}{r^{2m-1}(t)} \int_{B_o(t)} f^* c_1(L, h_L) \wedge \alpha^{m-1},
\]
which agrees with the classical characteristic function. By Theorem 3.6 (b), it yields that
Corollary 3.8. Let $L$ be a positive line bundle over a complex projective
manifold $N$ with $\dim \mathbb{C} N \leq m$, and $D \in |L|$ be of simple normal crossings.
Let $f : \mathbb{B}^m \to N$ be a differentiably non-degenerate holomorphic map, where
$\mathbb{B}^m$ is the unit ball with standard Euclidean metric. Then for every $\delta > 0$
\[ T_f(r, L) + T_f(r, K_N) \leq N_f(r, D) + O \left( \log^+ T_f(r, L) + \log \frac{1}{1-r} \right) \]
holds for all $r \in (0,1)$ outside a set $E_\delta$ with $\int_{E_\delta} (1-r)^{-1}dr < \infty$. 

(c) $M_\sigma = \mathbb{H}$ or $\mathbb{D}$ (Poincaré upper half-plane or Poincaré disc)

$\mathbb{H}$ and $\mathbb{D}$ are two representative models in hyperbolic geometry, marking
many essential differences from Euclidean geometry. It is important to study
the value distribution of a meromorphic function on Poincaré models, which
provides an effective tool to investigate the modular functions
\[ g(\tau) = \sum_{n \geq m} c_n e^{2\pi in\tau} \]
on $\mathbb{H}$, where $g$ is called a modular (resp. cusp) form if $m = 0$ (resp. $m = 1$ ), see, e.g., [10, 11].

In what follows, one establishes a Second Main Theorem of meromorphic
functions on Poincaré models in the sense of hyperbolic metric. When $M_\sigma = \mathbb{H}$ or $\mathbb{D}$ with Poincaré metric, we have $R = \infty$ and $\sigma(r) = \sinh r$. Let $f$ be a
meromorphic function on $\mathbb{H}$ or $\mathbb{D}$. In this situation, we define the Ahlfords characteristic function
\[ T_f(r) = \int_{r_0}^r \frac{dt}{\sinh t} \int_{B_\sigma(t)} d\sigma \log(1 + |f(x)|^2) \]
\[ = \frac{1}{4} \int_{B_\sigma(r)} g_r(o, x) \Delta \log(1 + |f(x)|^2) dV(x) \]
\[ - \frac{1}{4} \int_{B_\sigma(r_0)} g_{r_0}(o, x) \Delta \log(1 + |f(x)|^2) dV(x) \]
where $\Delta$ is the Laplace-Beltrami operator on $\mathbb{H}$ or $\mathbb{D}$. Adopt the spherical
distance $\| \cdot , \cdot \|$ on $\mathbb{P}^1(\mathbb{C})$. By definition, the proximity function is that
\[ m_f(r, a) = \int_{S_\sigma(r)} \log \frac{1}{\| f(x), a \|} d\pi_o'(x) \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\| f(r, \theta), a \|} d\theta \]
for any point $a \in \mathbb{C}$, where $(r, \theta)$ is the polar coordinate of $x$ with respect
to the pole $o$. According to the definition (see Section 2.1.1), the counting
function is defined by
\[ N_f(r, a) = \int_{r_0}^{r} \frac{n_f(t, a)}{\sinh t} dt \]
for a point \( a \in \overline{C} \), where \( n_f(r, a) \) denotes the number of the zeros of \( f - a \) in \( B_o(r) \) counting multiplicities. By Dynkin formula, we have the First Main Theorem
\[ T_f(r) = m_f(r, a) + N_f(r, a) + O(1). \]
For the sake of intuition of \( T_f(r) \), we introduce the Nevanlinna characteristic function
\[ T(r, f) := m(r, f) + N(r, f), \]
where
\[ m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r, \theta)| d\theta, \quad N(r, f) = \int_{r_0}^{r} \frac{n_f(t, \infty)}{\sinh t} dt. \]
Note that
\[ m_f(r, \infty) = m(r, f) + O(1), \quad N_f(r, \infty) = N(r, f), \]
hence
\[ T_f(r) = T(r, f) + O(1). \]
Namely,
\[ T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r, \theta)| d\theta + \int_{r_0}^{r} \frac{n_f(t, \infty)}{\sinh t} dt + O(1). \]

Let \( \omega_{FS} = dd^c \log ||w||^2 \) be the Fubini-Study form on \( \mathbb{P}^1(\mathbb{C}) \), where \( w = [w_0 : w_1] \) is the homogeneous coordinate of \( \mathbb{P}^1(\mathbb{C}) \). Let \( a_1, \cdots, a_q \) be different points in \( \mathbb{P}^1(\mathbb{C}) \), and regard \( f = f_1/f_0 = [f_0 : f_1] \) as a holomorphic map into \( \mathbb{P}^1(\mathbb{C}) \). Theorem 3.6 \((a)\) gives that
\[ (q - 2)T_{f_0FS}(r) + T_{\varphi}(r) \leq \sum_{j=1}^{q} N_{f_0,a_j}(r) + O(\log^+ T_{f_0FS}(r) + \delta \log^+ \sinh r), \]
which implies that
\[ (q - 2)T_f(r) + T_{\varphi}(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O(\log^+ T_f(r) + \delta \log^+ \sinh r). \]
As noted in Sections 2.2 and 2.3,
\[ \text{Area}(S_o(r)) = 2\pi \sinh r, \quad g_r(o, x) = \frac{1}{\pi} \log \frac{(e^r - 1)(e^r(x) + 1)}{(e^r + 1)(e^r(x) - 1)}. \]
Since \( s_H = s_D = -1 \), then a direct computation leads to
\[
T(r, \mathcal{R}) = \frac{1}{2} \int_{B_o(r)} g_r(o, x)s(x)dV(x) - \frac{1}{2} \int_{B_o(r_0)} g_{r_0}(o, x)s(x)dV(x)
\]
\[
= -\frac{1}{2} \int_0^r \log \left( \frac{e^t - 1}{e^t + 1} \right) \cdot \sinh t dt
\]
\[
+ \int_0^{r_0} \log \left( \frac{e^{r_0} - 1}{e^{r_0} + 1} \right) \cdot \sinh t dt
\]
\[
\geq -r + O(1),
\]
which implies that
\[
-T(r, \mathcal{R}) \leq r + O(1).
\]
Moreover,
\[
\log^+ \sinh r \leq r + O(1).
\]
Therefore, we conclude that

**Corollary 3.9.** Let \( f \) be a nonconstant meromorphic function on \( \mathbb{H} \) or \( \mathbb{D} \), where \( \mathbb{H}, \mathbb{D} \) are the Poincaré upper half-plane and Poincaré disc respectively. Let \( a_1, \ldots, a_q \) be distinct points in \( \overline{\mathbb{C}} \). Then for every \( \delta > 0 \)
\[
(q - 2)T_f(r) \leq \sum_{j=1}^q \mathcal{N}_f(r, a_j) + O\left( \log^+ T_f(r) + r \right)
\]
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

**Remark.** In fact, Corollary 3.9 is equivalent to the case where \( m = 1 \) and \( N = \mathbb{P}^1(\mathbb{C}) \) in Corollary 3.8, i.e., the following Second Main Theorem
\[
(q - 2)T_f(r) \leq \sum_{j=1}^q \mathcal{N}_f(r, a_j) + O\left( \log^+ T_f(r) + \log \frac{1}{1 - r} \right).
\]
To see this equivalence, we just need to compare the Second Main Theorem for \( \mathbb{D} \) under the Poincaré metric and Euclidean metric. To avoid confusion, denote by \( r, \tilde{r} \) the geodesic radius under the Poincaré metric and Euclidean metric respectively, by \( r(x), \tilde{r}(x) \) the Riemannian distance functions under the Poincaré metric and Euclidean metric respectively. Similarly, denote by \( g_r(o, x), \tilde{g}_r(o, x) \) as well as \( T_f(r), \tilde{T}_f(\tilde{r}) \) the Green functions and characteristic functions under the metrics.

Firstly, we compare the main error terms, i.e., \( O(r) \) and \( O(-\log(1 - \tilde{r})) \). Take \( o \) as the coordinate origin of \( \mathbb{D} \). By the relation
\[
r(x) = \log \frac{1 + \tilde{r}(x)}{1 - \tilde{r}(x)},
\]
we see that $r$ corresponds to
\[ \log \frac{1 + \tilde{r}}{1 - \tilde{r}} = \log \frac{1}{1 - \tilde{r}} + O(1) \]
due to $\tilde{r} < 1$. Thus, the two error terms are equivalent.

Finally, we compare the characteristic functions, i.e., $T_f(r)$ and $\tilde{T}_{\tilde{f}}(\tilde{r})$. The similar discussions can be applied to the comparisons for counting functions and proximity functions. Under the Euclidean metric, the Green function is written as
\[ \tilde{g}_{\tilde{f}}(o, x) = \frac{1}{\pi} \log \frac{\tilde{r}}{|x|} = \frac{1}{\pi} \log \frac{\tilde{r}}{\tilde{r}(x)}. \]
which corresponds to the Green function $g_r(o, x)$ under the Poincaré metric since (1) and (4). Notice that
\[ \Delta \log(1 + |f|^2) dV = \tilde{\Delta} \log(1 + |\tilde{f}|^2) d\tilde{V}, \]
where $\Delta, \tilde{\Delta}$ denote Laplace-Beltrami operators under the Poincaré metric and Euclidean metric respectively, and $dV, d\tilde{V}$ denote volume elements under the Poincaré metric and Euclidean metric respectively. By the definition of characteristic function, we see that they are a match. Hence, the two Second Main Theorems are actually equivalent under the two metrics.

2. Some other consequences

Let $M$ be a Riemannian manifold with a point $o \in M$. We establish a polar coordinate system $(o, r, \theta)$. For any $x = (r, \theta) \in M$ such that $x \notin Cut^*(o)$, denote by $Ric_o(x)$ the Ricci curvature of $M$ at $x$ in the direction $\partial/\partial r$. Let $\omega = (\partial/\partial r, X)$ be any pair of tangent vectors from $T_x M$, where $X$ is a unit vector orthogonal to $\partial/\partial r$. Indeed, let $K_\omega(x)$ be the sectional curvature of $M$ at $x$ along the 2-section determined by $\omega$. For a $d$-dimensional spherically symmetric manifold $M_\sigma$, a direct computation [3] yields that
\[ Ric_o(x) = -(d - 1) \frac{\sigma''(r)}{\sigma(r)}, \quad K_\omega(x) = -\frac{\sigma''(r)}{\sigma(r)} \]
for all $x = (r, \theta) \in M_\sigma \setminus o$.

Lemma 3.10 (Ichihara, [12, 13]). Let $\psi$ be a smooth positive function on $(0, \infty)$ such that $\psi(0) = 0, \psi'(0) = 1$.

Let $M$ be a $d$-dimensional geodesically complete, non-compact manifold, and $o \in M$. Set
\[ S(r) = \omega_{d-1} \psi^{d-1}(r), \]
where $\omega_{d-1}$ is the area of $S^{d-1}$. Then
(a) If for all \( x = (r, \theta) \notin \text{Cut}^*(o) \)
\[
\text{Ric}_\sigma(x) \geq -(d-1)\frac{\psi''(r)}{\psi(r)}, \quad \int_{1}^{\infty} \frac{dr}{S(r)} = \infty,
\]
them \( \text{M} \) is parabolic.

(b) If for all \( x = (r, \theta) \neq o \) and all \( \omega \)
\[
K_\omega(x) \geq -\frac{\psi''(r)}{\psi(r)}, \quad \int_{1}^{\infty} \frac{dr}{S(r)} < \infty,
\]
then \( \text{M} \) is non-parabolic.

**Corollary 3.11.** Let \( \text{M}_\sigma \) be a \( d \)-dimensional geodesically complete and non-compact spherically symmetric manifold. Then \( \text{M}_\sigma \) is parabolic if and only if
\[
\int_{1}^{\infty} \frac{dr}{\sigma^{d-1}(r)} = \infty.
\]

**Proof.** The conclusion follows immediately from (5) and Lemma 3.10. \( \square \)

By Corollary 3.11, \( T_f(r, L) \to \infty \) as \( r \to \infty \) if \( \text{M}_\sigma \) is parabolic.

**Theorem 3.12.** Let \( \text{M}_\sigma \) be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension \( m \). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim_{\mathbb{C}} N \leq m \), and \( D \in |L| \) be of simple normal crossings. Let \( f : \text{M}_\sigma \to N \) be a differentiably non-degenerate holomorphic map. Assume that \( \text{M}_\sigma \) is parabolic. Then for every \( \delta > 0 \)
\[
T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{\text{M}_\sigma}) \leq \mathcal{N}_f(r, D) + O(\log^+ T_f(r, L) + \delta \log^+ r)
\]
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

**Proof.** The completeness and non-compactness imply that \( \text{M}_\sigma \) have radius \( R = \infty \). By Corollary 3.11 the parabolicity of \( \text{M}_\sigma \) implies that
\[
\int_{1}^{\infty} \frac{dr}{\sigma^{2m-1}(r)} = \infty,
\]
which leads to \( \log^+ \sigma(r) \leq O(\log^+ r) \). Apply Theorem 3.6 (a), we prove the theorem. \( \square \)

**Corollary 3.13.** Let \( \text{M}_\sigma \) be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension \( m \). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim_{\mathbb{C}} N \leq m \),
and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differentiably non-degenerate holomorphic map. Assume that the Ricci curvature of $M_\sigma$ as a Riemannian manifold is non-negative. Then for every $\delta > 0$

$$T_f(r, L) + T_f(r, K_N) \leq N_f(r, D) + O\left(\log^+ T_f(r, L) + \delta \log^+ r\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

**Proof.** The non-negativity of Ricci curvature implies the parabolicity of $M_\sigma$, hence the conclusion holds by using Theorem 3.12. □

**Proof of Theorem 3.6**

Proof. Write $D = \sum_{j=1}^q D_j$ as the union of irreducible components and equip every $L_{D_j}$ with a Hermitian metric $h_j$. Then it induces a natural Hermitian metric $h_L = h_1 \otimes \cdots \otimes h_q$ on $L$, which defines a volume form $\Omega := c_1(L, h_L)$ on $N$. Pick $s_j \in H^0(N, L_{D_j})$ with $D_j = (s_j)$ and $\|s_j\| < 1$. On $N$, one defines a singular volume form $\Phi = \Omega \prod_{j=1}^q \|s_j\|^2$.

Set

$$\xi^m = f^* \Phi \wedge \alpha^{m-n}, \quad \alpha = \sqrt{-1} \pi \sum_{i,j=1}^m h_{ij} dz_i \wedge d\bar{z}_j.$$ 

Note that

$$\alpha^m = m! \det(h_{ij}) \prod_{j=1}^m \sqrt{-1} \pi dz_j \wedge d\bar{z}_j.$$ 

A direct computation leads to

$$dd^c \log \xi \geq f^* c_1(L, h_L) - f^* \text{Ric}(\Omega) + \mathcal{R}_{M_\sigma} - \text{Supp} f^* D$$

in the sense of currents, where $\mathcal{R}_{M_\sigma} = -dd^c \log \det(h_{ij})$. This follows that

$$T(r, dd^c \log \xi) \geq T_f(r, L) + T_f(r, K_N) + T(r, \mathcal{R}_{M_\sigma}) - N_f(r, D) + O(1).$$

Next we give an upper bound of $T(r, dd^c \log \xi)$. The simple normal crossing property of $D$ implies that there exist a finite open covering $\{U_\lambda\}$ of $N$ and finitely many rational functions $w_{\lambda_1}, \ldots, w_{\lambda_n}$ on $N$ such that $w_{\lambda_1}, \ldots, w_{\lambda_n}$ are holomorphic on $U_\lambda$ for each $\lambda$ as well as

$$dw_{\lambda_1} \wedge \cdots \wedge dw_{\lambda_n}(y) \neq 0, \quad \forall y \in U_\lambda,$$

$$D \cap U_\lambda = \{w_{\lambda_1} \cdots w_{\lambda_h} = 0\}, \quad \exists h_\lambda \leq n.$$
Indeed, we can require that \( L_{D_j}|_{U_\lambda} \cong U_\lambda \times \mathbb{C} \) for \( \lambda, j \). On \( U_\lambda \), write
\[
\Phi = \frac{e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \prod_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k},
\]
where \( e_\lambda \) is a positive smooth function on \( U_\lambda \). Set
\[
\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda h_\lambda}|^2} \prod_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \wedge d\bar{w}_{\lambda k},
\]
where \( \{\phi_\lambda\} \) is a partition of unity subordinate to \( \{U_\lambda\} \). Let \( f_{\lambda k} = w_{\lambda k} \circ f \), then on \( f^{-1}(U_\lambda) \)
\[
f^* \Phi_\lambda = \frac{\phi_\lambda \circ f \cdot e_\lambda \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h_\lambda}|^2} \prod_{k=1}^{n} \frac{\sqrt{-1}}{2\pi} df_{\lambda k} \wedge d\bar{f}_{\lambda k}
\]
\[
= \phi_\lambda \circ f \cdot e_\lambda \circ f \sum_{1 \leq i_1 \neq \ldots \neq i_n \leq m} \left| \frac{\partial f_{\lambda 1}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda h_\lambda}}{\partial z_{i_1}} \right|^2 \left| \frac{\partial f_{\lambda(h+1)}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_1}} \right|^2 \left( \frac{\sqrt{-1}}{2\pi} \right)^m dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_n} \wedge d\bar{z}_{i_n}.
\]
Fix any \( x_0 \in M_\sigma \), we may pick a holomorphic coordinate system \( z_1, \ldots, z_m \) near \( x_0 \) and a holomorphic coordinate system \( w_1, \ldots, w_m \) near \( f(x_0) \) so that
\[
\alpha = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^{m} dw_j \wedge d\bar{w}_j, \quad c_1(L, h_L)|_{f(x_0)} = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^{m} dw_j \wedge d\bar{w}_j.
\]
Set
\[
f^* \Phi_\lambda \wedge \alpha^{m-n} = \xi_\lambda \alpha^m.
\]
Then we have \( \xi = \sum_\lambda \xi_\lambda \) and
\[
\xi_\lambda|_{x_0} = \phi_\lambda \circ f \cdot e_\lambda \circ f \sum_{1 \leq i_1 \neq \ldots \neq i_n \leq m} \left| \frac{\partial f_{\lambda 1}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda h_\lambda}}{\partial z_{i_1}} \right|^2 \left| \frac{\partial f_{\lambda(h+1)}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_1}} \right|^2 \left\| \nabla_{M_\sigma} f_{\lambda 1} \right\|^2 \cdots \left\| \nabla_{M_\sigma} f_{\lambda h_\lambda} \right\|^2 \cdots \left\| \nabla_{M_\sigma} f_{\lambda(h+1)} \right\|^2 \cdots \left\| \nabla_{M_\sigma} f_{\lambda n} \right\|^2.
\]
Again, set
\[
(7) \quad f^* c_1(L, h_L) \wedge \alpha^{m-1} = \phi_\alpha^m.
\]
Let \( f_j = w_j \circ f \) for \( 1 \leq j \leq n \), then
\[
f^* c_1(L, h_L) \wedge \alpha^{m-1}|_{x_0} = \frac{(m-1)!}{2} \sum_{j=1}^{m} \left\| \nabla_{M_\sigma} f_j \right\|^2 \alpha^m.
That is,
\[
\vartheta_{x_0} = (m - 1)! \sum_{i=1}^{n} \sum_{j=1}^{m} \left| \frac{\partial f_i}{\partial z_j} \right|^2 = \frac{(m - 1)!}{2} \sum_{j=1}^{n} \| \nabla_{M_{\sigma}} f_j \|^2.
\]

Combine the above, we are led to
\[
\xi_{\lambda} \leq \frac{\phi_{\lambda} \circ f \cdot e_{\lambda} \circ f \cdot (2\varrho)^{n-h_{\lambda}}}{(m - 1)!^{n-h_{\lambda}}} \sum_{1 \leq i_1 \neq \cdots \neq i_n \leq m} \frac{\| \nabla_{M_{\sigma}} f_{\lambda i_1} \|^2 \cdots \| \nabla_{M_{\sigma}} f_{\lambda i_n} \|^2}{|f_{\lambda i_1}|^2 \cdots |f_{\lambda i_n}|^2}
\]
on $f^{-1}(U_{\lambda})$. Note that $\phi_{\lambda} \circ f \cdot e_{\lambda} \circ f$ is bounded on $M_{\sigma}$, then it follows from
\[
\log \xi \leq \sum_{\lambda} \log \xi_{\lambda} + O(1)
\]
on $M_{\sigma}$. By Dynkin formula (see Section 3.1.2)
\[
T(r, dd^c \log \xi) = \frac{1}{2} \int_{S_{\sigma}(r)} \log \xi(x) d\pi_{\sigma}(x) + O(1).
\]
By (8) and (9) with Theorem 3.5
\[
T(r, dd^c \log \xi) \leq O \left( \log^+ \xi + \sum_{k, \lambda} \log^+ \frac{\| \nabla_{M_{\sigma}} f_{\lambda k} \|}{|f_{\lambda k}|} \right) + O(1)
\]
on $M_{\sigma}$. By Dynkin formula (see Section 3.1.2)
\[
T(r, dd^c \log \xi) \leq O \left( \log^+ T_f(r, L) + \log^+ \int_{S_{\sigma}(r)} \varrho(x) d\pi_{\sigma}(x) \right) + O(1).
\]
Lemma 3.2 and (7) imply that for every $\delta > 0$
\[
\log^+ \int_{S_{\sigma}(r)} \varrho(x) d\pi_{\sigma}(x) \leq (1 + \delta)^2 \log^+ T_f(r, L) + (2 + \delta) \log^+ \gamma(r) + (2m - 1) \delta \log^+ \sigma(r)
\]
holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} \gamma(r) dr < \infty$. Thus,
\[
T(r, dd^c \log \xi) \leq O \left( \log^+ T_f(r, L) + \log^+ \gamma(r) + \delta \log^+ \sigma(r) \right) + O(1)
\]
for all $r \in (0, R)$ outside $E_\delta$ with $\int_{E_\delta} \gamma(r) dr < \infty$. Combining (5) with (10), we prove the theorem. □
3.4. Defect relations.

Recall the definition of simple defect $\Theta_f(D)$ in Introduction.

**Theorem 3.14.** Assume the same conditions as in Theorem 3.6.

(a) For $R = \infty$, if $T_f(r, L) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right] - \liminf_{r \to \infty} \frac{T(r, R_{M_\sigma})}{T_f(r, L)}.$$

(b) For $R < \infty$, if $\log \frac{1}{R-r} = o(T_f(r, L))$ as $r \to R$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right] - \liminf_{r \to R} \frac{T(r, R_{M_\sigma})}{T_f(r, L)}.$$

**Proof.** The conclusions follow directly from Theorem 3.6. □

**Corollary 3.15.** Assume the same conditions as in Theorem 3.6. Suppose also that $M_\sigma$ has non-negative scalar curvature.

(a) For $R = \infty$, if $T_f(r, L) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

(b) For $R < \infty$, if $\log(R-r) = o(T_f(r, L))$ as $r \to R$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

**Proof.** The non-negativity of scalar curvature of $M_\sigma$ gives that $T(r, R_{M_\sigma}) \geq 0$, see [2]. □

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SCHOOL OF MATHEMATICS, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, JIANGSU, XUZHOU, 221116, P. R. CHINA

Email address: xjdong05@126.com
Department of Mathematics, Shandong University, Jinan, 250100, Shandong, P. R. China

Email address: pchu@sdu.edu.cn