Bosonization in Particle Physics

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Abstract. Path integral techniques in collective fields are shown to be a useful analytical tool to reformulate a field theory defined in terms of microscopic quark (gluon) degrees of freedom as an effective theory of collective boson (meson) fields. For illustrations, the path integral bosonization approach is applied to derive a (non)linear $\sigma$ model from a Nambu-Jona-Lasinio (NJL) quark model. The method can be extended to include higher order derivative terms in meson fields or heavy-quark symmetries. It is also approximately applicable to QCD.

1 Introduction

In this lecture, I want to demonstrate the powerfulness of the path integral approach in collective fields for the bosonization of quark models containing (effective) 4-quark interactions [1]. For illustrations, let me consider NJL type of models [2-4] with local quark interactions $\sim G(\bar{q}q)^2$ representing a relativistic version of the superconductor BCS theory [5,6]. These models lead to a gap equation for a dynamical quark mass signalling the dynamical breakdown of chiral symmetry (DBCS). Furthermore, the collective field of Cooper pairs of the superconductor is now replaced by collective meson fields of $(q\bar{q})$-bound states.

To be more explicit, let me consider the generating functional $Z$ of the NJL model defined by a path integral of the exponential of the corresponding action over quark fields $q$ as the underlying microscopic degrees of freedom. Path integral bosonization then means to transform this generating functional into an integral of the exponential of an effective meson action where the new collective integration variables $\sigma, \pi, \rho,...$ denote the observable meson fields,

$$Z = \int D\mu(q) e^{i \int L_{NJL} d^4x} \Rightarrow \int D\mu(\sigma, \pi, \rho,...) e^{i \int L_{Eff} d^4x},$$

with $D\mu(q) = D\bar{q} Dq$, $D\mu(\sigma, \pi, \rho,...) = D\sigma D\pi D\rho...$ being the respective integration measures of fields.

* Invited talk given at the Workshop “Field Theoretical Tools in Polymer and Particle Physics”, University Wuppertal, June 17-19, 1997
The basic ingredient of the path integral bosonization (1) is the use of the Hubbard-Stratonovich transformation [7,8] which replaces the (effective) 4-quark interactions of NJL models by a Yukawa-type coupling of quarks with collective meson fields \( \phi_i = (\sigma, \pi, \rho, \ldots) \). After this the primary path integral over quark fields on the L.H.S. of (1) becomes Gaussian resulting in a quark determinant containing meson fields. Further important steps are:

(a) the use of the loop expansion of the quark determinant in powers of meson fields

(b) the evaluation of the resulting Feynman diagrams in the low-momentum region

(c) the limit of large numbers of colours, \( N_c \to \infty, G N_c \) fixed, \( G \) being the 4-quark coupling constant in order to apply a saddle point approximation to the integration over meson fields.

Quark loop diagrams emitting two \( \phi \)-fields then generate in the low-momentum (two-derivative) approximation kinetic and mass terms of mesons. Finally, quark loops emitting \( n > 2 \) meson fields lead to meson interactions with effective small coupling constants \( g_n \sim O\left(\frac{1}{\sqrt{N_c}}\right)^{n-2} \) allowing for a modified perturbation theory in terms of meson degrees of freedom.

Notice that the NJL quark Lagrangian incorporates the global chiral flavour symmetry of Quantum Chromodynamics (QCD) as well as its explicit and dynamical breaking. The equivalent effective meson theory on the R.H.S. of (1) just reproduces this symmetry breaking pattern at the meson level. As mentioned above, masses and coupling constants of collective mesons are now calculable from quark loop diagrams and expressed by the parameters of the underlying quark theory (including a loop momentum cut-off \( \Lambda \)). The final aim is, of course, bosonization of QCD, i.e. the transformation

\[
Z = \int D\mu (q, G_\mu) e^{i \int L_{QCD} d^4x} \Rightarrow \int D\mu (\sigma, \pi, \rho, \ldots) e^{i \int L_{Eff} d^4x},
\]

(2)

where \( q, G_\mu \) denote quark and gluon fields (ghost fields are not shown explicitly).

In order to begin with an effective 4-quark interaction as intermediate step, one first has to integrate away the gluon (ghost) fields on the L.H.S. of (2). This can only be done exactly for space-time dimensions \( D = 2 \) in the light-cone gauge, \( G_- = \frac{1}{2} (G_0 - G_1) = 0 \), where all self-interactions of gluon fields vanish. The resulting expression contains an effective nonlocal current x current quark interaction with a known gluon propagator which then can be bosonized by introducing bilocal meson fields \( \phi (x, y) \sim \bar{q}(x) G q(y) \), employing the limit of large numbers of colours \( N_c \) [9]. Clearly, the analogous bosonization of four-dimensional QCD is more complicate. It requires some additional approximations: first a truncation of multilocal quark current interactions retaining only the bilocal two-current term and secondly a modelling of the unknown nonperturbative gluon propagator (Cf. Fig. 1a) (see [1,10-13]). As a final result one derives an effective chiral...
Lagrangian describing the low-energy interactions of observable mesons including nontrivial meson form factors.

\[
\tilde{q} q \qquad q \tilde{q} \\
\text{low energy} \qquad q \tilde{q} = L_{\text{int}}^{\text{NJL}}
\]

**Fig. 1.** a-b) Low-energy approximation of a nonlocal current $\times$ current interaction with nonperturbative gluon propagator (a), by a local Nambu-Jona-Lasinio type interaction (b).

Nevertheless, as anticipated in (1), also simpler local NJL type of models are expected to yield a reasonable low-energy description of the chiral sector of QCD. Indeed, discarding the complicated question of the unknown structure of nonperturbative gluon and quark propagators (related to confinement), one can try to approximate the nonperturbative gluon propagator in the region of low energies by a universal constant $G$ leading to a local NJL type of current $\times$ current interaction (Cf. Fig. 1). (For interesting nonperturbative applications of the above approach the reader is further referred to the nuclear many-body problem [10,14] and the Hubbard model [15,16].)

In conclusion, let us notice that for two space-time dimensions there exists a different realization of the bosonization idea based on an explicit construction of fermionic fields in terms of bosonic fields due to Mandelstam [17]. This then allows one to replace, at the operator level, quark bilinears by bosonic fields, e.g.

\[
\bar{\psi} \gamma_{\mu} \psi \sim -\pi^{-\frac{1}{2}} \varepsilon_{\mu\nu} \partial^\nu \phi, \quad \bar{\psi} \psi \sim M \cos \left(2\sqrt{\pi}\phi\right),
\]

and to prove the equivalence of a given fermionic model (e.g. Thirring model) with a corresponding bosonic model (e.g. Sine-Gordon model). Using in particular Witten’s non-Abelian bosonization rules [18] one has derived in the strong coupling limit (which is contrary to the weak coupling limit, $\text{GN}_c$ fixed, for $N_c \to \infty$, of the above described path integral approach) a low-energy effective action from QCD$_2$ [19]. The lecture of T. Giamarchi at this Workshop [20] discusses just the application of this kind of operator bosonization in Condensed Matter Physics.
In the next section I shall now describe the path integral bosonization of the NJL model along line I closely following the original papers [3,4].

2 NJL Model and $\sigma$ Model

2.1 Linear $\sigma$ Model

Let us consider the following NJL Lagrangian with a global symmetry $[U(2) \times U(2)] \times SU(N_c)$ [3]

$$\mathcal{L}_{NJL} = \bar{q} \left( i \overleftrightarrow{\partial} - m_0 \right) q + \frac{G}{2} \left[ (\bar{q}q)^2 + (\bar{q}i\gamma_5 \tau q)^2 \right],$$

where $q$ denotes a quark spinor with colour and spinor indices, $\tau_i$'s are the generators of the flavour group $U(2)$, and $G$ is a universal quark coupling constant with dimension mass$^{-2}$. Note that the $\bar{q}q$-combinations in (3) are colour singlets, and we have admitted an explicit chiral symmetry-breaking current quark mass term $-m_0 \bar{q}q$. The integration over the quark fields in the generating functional $Z$ of the NJL model (given by the L.H.S. of (1)) can easily be done after bi-linearizing the four-quark interaction terms with the help of colour-singlet collective meson fields $\sigma, \pi$. To this end, we use the Hubbard-Stratonovich transformation [7,8]

$$\exp \left\{ i \int \frac{G}{2} \left[ (\bar{q}q)^2 + (\bar{q}i\gamma_5 \tau q)^2 \right] d^4 x \right\} = \mathcal{N} \int D\sigma D\pi \exp \left\{ i \int \left[ -\frac{1}{2G} (\sigma^2 + \pi^2) + \bar{q} (\sigma + i\gamma_5 \tau \cdot \pi) q \right] d^4 x \right\},$$

which replaces the 4-quark interaction by a Yukawa coupling with collective fields $\sigma, \pi$, Cf. Fig.2.

![Fig. 2. Graphical representation of the integral identity (4).](image)

Inserting (4) into the L.H.S. of (1) leads to the intermediate result

$$Z = \int D\sigma D\pi \int Dq D\bar{q} e^{i \int \mathcal{L}_{NJL}^{SM} d^4 x}$$

(5)
with the semi-bosonized meson-quark Lagrangian

$$\mathcal{L}_{N}^{qM} = -\frac{1}{2G} \left( (\sigma - m_0)^2 + \pi^2 \right) + \bar{q} \left( i \hat{\partial} - \sigma - i \gamma_5 \tau \cdot \pi \right) q, \quad (6)$$

where we have absorbed the current quark mass in the $\sigma$ field by a shift $\sigma \rightarrow \sigma - m_0$ of the integration variable. The Gaussian integral over the Grassmann variable $q$ in (5) can easily be performed leading to the fermion determinant

$$\det S^{-1} = \exp N_c \text{Tr} \ln S^{-1} = \exp iN_c \int -\text{tr} \left( \ln S^{-1} \right)_{(x,x)} d^4x \quad (7)$$

where

$$S^{-1}(x,y) = \left[ i \hat{\partial}_x - \sigma(x) - i \gamma_5 \tau \cdot \pi(x) \right] \delta^4(x-y) \quad (8)$$

is the inverse quark propagator in the presence of collective fields $\sigma, \pi$ and the trace $\text{tr}$ in (7) runs over Dirac and isospin indices. Note that the factor $N_c$ results from the colour trace. Combining (5)-(7) we obtain

$$Z = \int D\sigma D\pi e^{i \int \mathcal{L}_{\text{Eff.}}(\sigma,\pi) d^4x}, \quad (9)$$

$$\mathcal{L}_{\text{Eff.}}(\sigma,\pi) = -\frac{1}{2G} \left( (\sigma - m_0)^2 + \pi^2 \right) - i N_c \text{tr} \ln \left( i \hat{\partial} - \sigma - i \gamma_5 \tau \cdot \pi \right)_{(x,x)}. \quad (10)$$

Let us analyze (9) in the limit of large $N_c$ with $GN_c$ fixed where one can apply the saddle point approximation. The stationary point $\sigma = \langle \sigma \rangle_0 \equiv m, \pi = 0$ satisfies the condition

$$\left. \frac{\delta \mathcal{L}_{\text{Eff.}}}{\delta \sigma} \right|_{\langle \sigma \rangle_0, \pi = 0} = 0, \quad (11)$$

which takes the form of the well-known Hartree-Fock gap equation determining the dynamical quark mass $m$

$$m = m_0 + 8mGN_c I_1(m), \quad (12)$$

$$I_1(m) = i \int \frac{A}{4 \pi^2} \frac{1}{k^2 - m^2} \quad (13)$$

with $A$ being a momentum cut-off which has to be held finite (see Fig.3).

Note that in the chiral limit, $m_0 \rightarrow 0$, (12) admits two solutions $m = 0$ or $m \neq 0$ in dependence on $GN_c < (GN_c)_{\text{crit}}$. Thus, for large enough couplings we find a nonvanishing quark condensate $\langle \bar{q}q \rangle \sim \text{tr} S(x,x)$ signalling spontaneous breakdown of chiral symmetry.

It is further convenient to perform a shift in the integration variables
In order to obtain from the nonlocal expression (10) a local effective meson Lagrangian we have to apply the following recipes:

(a) loop expansion of the determinant in the fields \( \sigma', \pi \), i.e.

\[
N_c \text{tr} \ln \left\{ \left( i\hat{\partial} - m \right) \left[ 1 - \frac{1}{i\hat{\partial} - m} \left( \frac{\sigma'}{\sqrt{N_c}} + i\gamma_5 \tau \frac{\pi}{\sqrt{N_c}} \right) \right] \right\} =
\]

\[
= \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

finite diagrams.

**Fig. 4.** Loop expansion of the fermion determinant.

Here we have omitted the unimportant constant term \( N_c \text{tr} \ln \left( i\hat{\partial} - m \right) \)
and used the formula \( \ln (1 - x) = - \left[ x + \frac{x^2}{2} + \cdots \right] \). (The tadpole diagram linear in \( \sigma' \) cancels by a corresponding linear term arising from the first term in (10) due to the gap equation (12).)

(b) Low-momentum expansion of loop diagrams corresponding to the so-called gradient expansion of meson fields in configuration space.
(c) Field renormalization,

\[(\sigma', \pi) \rightarrow Z^{\frac{1}{2}} (\sigma', \pi).\]

Discarding the contributions of finite \(n\)-point diagrams with \(n > 4\) in the loop expansion leads to a linear \(\sigma\) model of composite fields (the prime in \(\sigma'\) is now omitted) [3]

\[
\mathcal{L}_{\text{Eff}}(\sigma, \pi) = \frac{1}{2} \sigma \left( -\Box - m_{\sigma}^2 \right) \sigma + \frac{1}{2} \pi \left( -\Box - m_{\pi}^2 \right) \pi
- g_{\sigma \pi \pi} \sigma (\sigma^2 + \pi^2) - g_{4\pi} (\sigma^2 + \pi^2)^2. \tag{14}
\]

Notice that the masses and coupling constants of mesons are fixed by the quark model parameters and the (finite) momentum cut-off \(\Lambda\),

\[
m_{\sigma}^2 = m_0 \frac{g_{\sigma \pi q}^2}{m_G}, \quad m_{\pi}^2 = m_{\pi}^2 + 4m^2, \tag{15}
\]

\[
g_{\pi q q} = g_{\sigma q q} = \left( \frac{Z}{N_c} \right)^{\frac{1}{2}} = \left\{ 4N_c I_2 \right\}^{-\frac{1}{2}} = O \left( \frac{1}{\sqrt{N_c}} \right),
\]

\[
g_{\sigma \pi \pi} = g_{3\sigma} = \frac{m}{(N_c I_2)^{\frac{1}{2}}},
\]

\[
g_{4\pi} = g_{4\sigma} = \frac{1}{8N_c I_2}. \tag{16}
\]

where

\[
I_2 = -i \int_{-i}^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}.
\]

Notice that in the chiral limit \(m_0 \rightarrow 0\), the pion becomes the massless Goldstone boson associated to DBCS. Introducing electroweak interactions at the quark level leads to the additional diagram shown in Fig.5.

Fig.5 leads to the interaction Lagrangian

\[
\Delta \mathcal{L} = \frac{g}{2} W_{\mu}^i \left( -\frac{m}{g_{\pi q q}} \partial^\mu \pi^i \right), \tag{17}
\]

which yields just the PCAC meson current. Here the ratio \(m/g_{\pi q q}\) has the meaning of the pion decay constant \(F_\pi\). We thus obtain the Goldberger-Treiman relation

\[
F_\pi = \frac{m}{g_{\pi q q}}, \tag{18}
\]

valid at the quark level.

In the following subsection we shall show how one can derive the related nonlinear version of the \(\sigma\) model.
Fig. 5. Quark diagram describing the weak transition $\pi \to W$.

2.2 Nonlinear $\sigma$ model

Nonlinear chiral meson Lagrangians have been introduced a long time ago in the literature [21-23] and are now widely used in low-energy hadron physics [1,24]. Let us demonstrate how they can be derived from a NJL quark model. To this end, it is convenient to use in (6) an exponential parametrization of the meson fields

$$\sigma + i\gamma_5 \tau \cdot \pi = (m + \tilde{\sigma}) e^{-i\gamma_5 \frac{\pi}{\tau_3}},$$

which yields

$$\sigma^2 + \pi^2 = (m + \tilde{\sigma})^2.$$  \hfill (20)

Let us absorb the exponential in (19) by introducing chiralily rotated “constituent” quarks $\chi$,

$$q = e^{i\gamma_5 \frac{\pi}{\tau_3}} \chi; \quad \xi = \frac{\varphi}{F_\pi}$$  \hfill (21)

leading to the Lagrangian$^1$

$$\mathcal{L}_{\text{NJL}}^\chi = -\frac{1}{2G} (m + \tilde{\sigma})^2 + \Delta L_{sb} +$$

$$+ \bar{\chi} \left[ i\gamma_\mu \left( \partial^\mu + e^{-i\gamma_5 \frac{\pi}{\tau_3}} \partial^\mu e^{i\gamma_5 \frac{\pi}{\tau_3}} \right) - m - \tilde{\sigma} \right] \chi,$$  \hfill (22)

with

$$\Delta L_{sb} = \frac{m + \tilde{\sigma}}{16G} m_{tr} \left( e^{-i\gamma_5 \tau \cdot \xi} + \text{h.c.} \right)$$  \hfill (23)

being a symmetry-breaking mass term.

$^1$ There arises also an additional anomalous Wess-Zumino term from the fermion measure of the path integral [4] which is discarded here.
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Note the nonlinear transformation law of the meson field $\xi(x)$ under global chiral transformations $g \in SU(2)_A \times SU(2)_V$ [22,23],

$$ge^{i\gamma_5 \frac{-\xi(x)}{2}} = e^{i\gamma_5 \frac{-\xi'(x)}{2}} h(x),$$

(24)

where

$$h(x) = e^{i\frac{\tau}{2}} u(\xi(x),g) \in SU(2)_V, \text{loc.}$$

is now an element of a local vector group. It is convenient to introduce the Cartan decomposition

$$e^{-i\gamma_5 \frac{\tau}{2} \partial_\mu} e^{i\gamma_5 \frac{\tau}{2}} = i\gamma_5 \frac{\tau}{2} \cdot A_\mu(\xi) + i\frac{\tau}{2} \cdot V_\mu(\xi).$$

(25)

It is then easy to see that the fields transform under the local group SU(2)$_V, \text{loc.}$ as follows ($A_\mu \equiv \frac{\tau}{2} \cdot A_\mu$, etc.) [23]

$$\chi \rightarrow \chi' = h(x) \chi$$

$$V_\mu \rightarrow V'_\mu = h(x)V_\mu h^\dagger(x) - h(x)i\partial_\mu h^\dagger(x)$$

$$A_\mu \rightarrow A'_\mu = h(x)A_\mu h^\dagger(x).$$

(26)

Thus, $V_\mu$ is a gauge field with respect to SU(2)$_V, \text{loc.}$. This allows one to define the following chiral-covariant derivative of the quark field $\chi$

$$D_\mu \chi = (\partial_\mu + iV_\mu) \chi.$$

(27)

Using (22),(25) and (27), the inverse propagator of the rotated $\chi$ field takes the form

$$S_{\chi}^{-1} = i\hat{D} - m - \hat{\sigma} - \hat{A}\gamma_5.$$  

(28)

The nonlinear $\hat{\sigma}$ model is now obtained by “freezing” the $\hat{\sigma}$ field, performing the path integral over the $\chi$ field and then using again the loop expansion for the quark determinant $\det S_{\chi}^{-1}$. Choosing a gauge-invariant regularization, the loop diagram with two external $A_\mu$ fields generates a mass-like term for the axial $A_\mu(\xi)$ field contributing to the effective Lagrangian

$$\mathcal{L}_{\text{nlm.}}^{\sigma} = \frac{m^2}{g^2_{\pi qq}} (\text{tr}_F A^2_\mu) + \Delta \mathcal{L}_{ab}.$$  

(29)

Here, the symmetry-breaking term is taken over from (22), and the first constant term has been omitted.

Note that usually generated field strength terms $-\frac{1}{4} A_{\mu\nu}^2$, $-\frac{1}{4} V_{\mu\nu}^2$ vanish for Cartan fields (25). A mass-like term $\sim V_{\mu}^2$ does not appear due to gauge-invariant regularization.
As has been shown in [23], $\mathcal{A}_\mu$ is just the chiral-covariant derivative of the $\xi$ field admitting the expansion

$$\mathcal{A}_\mu^i = D_\mu \xi^i = \partial_\mu \xi^i + O(\xi^3) = \frac{1}{F_\pi} \partial_\mu \varphi^i + O(\varphi^3).$$

(30)

Thus, we obtain the nonlinear $\sigma$ model

$$L_{\text{lin}} = \frac{F_\pi^2}{2} \mathcal{D}_\mu \mathcal{D}^\mu \xi^i + \Delta L_{sb} =$$

$$= \frac{1}{2} \varphi (-\Box - m_\pi^2) \varphi + O(\varphi^3)$$

reproducing Weinberg’s result [21].

### 3 Conclusions and Outlook

In this talk I have shown you that the path integral bosonization approach applied to QCD-motivated NJL models is a powerful tool in order to derive low-energy effective meson Lagrangians corresponding to the (nonperturbative) chiral sector of QCD.

The above considerations have further been extended to calculate higher-order derivative terms in meson fields by applying heat-kernel techniques to the evaluation of the quark determinant [4]. This allows, in particular, to estimate all the low-energy structure constants $L_i$ introduced by Gasser and Leutwyler [24]. Moreover, it is not difficult to include vector and axial-vector currents into the NJL model and to consider the chiral group $U(3) \times U(3)$. In a series of papers [1,3,4,25] it has been shown that the low-energy dynamics of light pseudoscalar, vector and axial-vector mesons is described surprisingly well by effective chiral Lagrangians resulting from the bosonization of QCD-motivated NJL models. These Lagrangians embody the soft-pion theorems, vector dominance, Goldberger-Treiman and KSFR relations and the integrated chiral anomaly.

Finally, we have investigated the path integral bosonization of an extended NJL model including DBCS of light quarks and heavy quark symmetries of heavy quarks [26] (see also [27]). This enables one to derive an effective Lagrangian of pseudoscalar, vector and axial-vector $D$ or $B$ mesons interacting with light $\pi, \rho$ and $a_1$ mesons. Note that the use of the low-momentum expansion in the evaluation of the quark determinant restricts here the applicability of the resulting effective Lagrangian to such (decay) processes where the momentum of the light mesons is relatively small.

Summarizing, I thus hope to have convinced you that the path integral bosonization approach to QCD (in its bilocal or simpler local formulation) is a very interesting nonperturbative analytical method which, being complementary to numerical studies of lattice QCD, is worth to be developed further. In the next lecture, I will consider the related path integral “hadronization” of baryons.
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