ABSOLUTELY CONTINUOUS FURSTENBERG MEASURES FOR FINITELY-SUPPORTED RANDOM WALKS

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ABSTRACT. In this note, we generalise a Bourgain’s construction of finitely-supported symmetric measures whose Furstenberg measure has a smooth density from the case of \( \text{SL}_2(\mathbb{R}) \) to that of a general simple Lie group. The proof is the same as Bourgain’s, except that the use of Fourier series is replaced by harmonic analysis on a maximal compact subgroup.

1. INTRODUCTION

Let \( \mu \) be a Borel probability measure on a non-compact connected simple Lie group \( G \) with finite centre. Assuming that the support of \( \mu \) generates a Zariski-dense semigroup, by a theorem of Furstenberg and Goldsheï’d-Margulis, there exists a unique measure \( \nu \) which is stationary for the random walk induced by \( \mu \) on the full flag manifold \( \mathcal{F} \) of \( G \), that is \( \mu \ast \nu = \nu \), where \( \mu \ast \nu \) is the image of \( \mu \otimes \nu \) by the action map \( (g, \xi) \mapsto g\xi \). The measure \( \nu \) is called the Furstenberg measure and it controls many properties of the random walk associated to \( \mu \). The study of the geometric properties of \( \nu \) is therefore of interest (see Benoist-Quint [4] for more background).

In this note we focus on the case of finitely-supported \( \mu \). Kaimanovich-Le Prince [13] had conjectured that in this case, the measure \( \nu \) is never absolutely continuous with respect to the Lebesgue measure on \( \mathcal{F} \) and in fact constructed examples with arbitrary small Hausdorff dimension of \( \nu \):

**Theorem 1.** Let \( d \geq 2 \) and \( \Gamma \) be a finitely-generated Zariski-dense subgroup of \( \text{SL}_d(\mathbb{R}) \). Then for every \( \delta > 0 \), there exists a finitely-supported measure \( \mu \) on \( \Gamma \) such that the Furstenberg measure associated to \( \mu \) has Hausdorff dimension less than \( \delta \).

However, Barany-Pollicott-Simon [1] showed this conjecture to be false using the "transversality method" (Pollicott-Simon [19]), initially invented in the context of self-similar sets and which had given striking results for the study of Bernoulli convolutions (see Solomyak [21] and Peres-Solomyak [18]), with which there is a number of analogies to the problem studied here.

The construction of Barany-Pollicott-Simon was not explicit, however, and did not give examples of symmetric measures. Another construction was given by Bourgain [5] in the case of \( \text{SL}_2(\mathbb{R}) \), which gives rather explicit (symmetric) examples, and then Benoist-Quint [4] in the more general case of a connected semi-simple Lie group which is different from \( \text{SL}_2(\mathbb{R}) \). Actually, both proof give more than absolute continuity, as they show that it is possible to have a density of class \( C^r \) for arbitrary large \( r \).
A fundamental part of Bourgain’s proof (and in a different way of Benoist-Quint’s proof) is the use of finitely-supported measure having a restricted spectral gap for the associated Markov operator in $SL_2(\mathbb{R})$, which had been constructed earlier, first in the simpler compact case by the Bourgain-Gamburd method \cite{6,7,2}, and then in the non-compact case for $SL_2(\mathbb{R})$ by Bourgain-Yehudahoff \cite{8}. Since Bourgain’s proof appeared, the restricted spectral gap property was established beyond the $SL_2(\mathbb{R})$ case by Boutonnet-Ioana-Salehi Golsefidy \cite{9}. More precisely, their proof gives the following (see the proof of corollary C in \cite{9}):

**Theoreme 2.** Let $\Gamma < G$ be a topologically dense subgroup. Assume that there exists a basis of $\mathfrak{g}$ with respect to which for every $g \in \Gamma$, the matrix of $\text{Ad} \ g$ has coefficients which are algebraic numbers.

Let $U$ be a neighbourhood of the identity in $G$. Then there exists a symmetric subset $T \subset U \cap \Gamma$ such that the Markov operator $T$ associated to the measure $\mu = \frac{1}{|T|} \sum_{g \in T} \delta_g$ on $L^2(\mathcal{F})$ defined by

$$(Tu)(\xi) := \int u(g\xi) \, d\mu(g)$$

for every $u \in L^2(\mathcal{F})$ and $\xi \in \mathcal{F}$ has a restricted spectral gap, that is, there exists a subspace $V$ of finite dimension in $L^2(\mathcal{F})$ such that for every $u \in V^\perp$, $\|Tu\|_2 \leq \frac{1}{2} \|u\|_2$.

Let us now note that a functional-analytic argument of Benoist-Quint \cite{4} already gives a density in $L^2(\mathcal{F})$ from this last statement: indeed the restricted spectral gap of $T$ implies that the essential spectral radius of $T$ is strictly less than 1 (in fact less than 1/2), which implies that $1 = \dim \ker (T - 1) = \dim \ker (T^* - 1)$, where $T^*$ is the adjoint for the $L^2$ scalar product. But a stationary measure with density in $L^2$ is the same as an eigenvector of $T^*$ for the eigenvalue 1. So the interest of what follows is only to obtain higher regularity. To simplify the proof below, we will make use of the existence of the density in $L^2$.

In this note we focus on generalising Bourgain’s argument which deduces from theorem 2 properties of regularity for the stationary measure associated to $\nu$. Bourgain’s argument for the action of $SL_2(\mathbb{R})$ acting on $\mathbb{P}^1$ relies on the use of Fourier series on $\mathbb{P}^1$ seen as a quotient of the circle. For the general case, we note that a maximal compact subgroup $K$ of $G$ acts transitively on $\mathcal{F}$ and therefore we can similarly use the representation theory of $K$. The argument is then exactly parallel to that of Bourgain. More precisely, below we show the following:

**Proposition 1.** Let $t > 0$ be a real number. Then there exists a neighbourhood $U$ of the identity in $G$ such that the following holds:

Let $\mu$ be a measure with finite support $S$ and which generates a dense subgroup of $G$ such that $S \subset U$. Assume that the operator $T : L^2(\mathcal{F}) \to L^2(\mathcal{F})$ defined by

$$(Tu)(\xi) := \sum_{g \in S} \mu(g)u(g^{-1}\xi)$$

for all $u \in L^2(\mathcal{F})$, satisfies the following condition: there exists a subspace $V$ of finite dimension in $L^2(\mathcal{F})$ such that for every $u \in V^\perp$, $\|Tu\|_2 \leq \frac{1}{2} \|u\|_2$. 
Then the unique $\mu$-stationary measure $\nu$ is absolutely continuous with density in $H^t(F)$.

Combining theorem 2 and proposition 2, we obtain the following result:

**Theorem 3.** Let $t > 0$ be a real number. Let $\Gamma < G$ be a topologically dense subgroup. Assume that there exists a basis of $\mathfrak{g}$ with respect to which for every $g \in \Gamma$, the matrix of $\text{Ad} g$ has coefficients which are algebraic numbers.

Then there exists a symmetric subset $T \subset \Gamma$ such that the random walk associated to the measure $\frac{1}{|T|} \sum_{g \in T} \delta_g$ has a density in $H^t(F)$ (and therefore in $\mathcal{C}^k$ if $t$ is large enough by the Sobolev embedding theorem).

Let us note that the higher the regularity, the closer to the identity the measure is required to be supported. This is somewhat similar in spirit to a result of Erdős-Kahane [17].

It remains an interesting problem to study the regularity of arbitrary finitely-supported measures whose support is close enough to the identity. In this direction, let us mention the results of Hochman-Solomyak [12] on the dimension of the Furstenberg measure.

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2. Preliminaries

**Notations.** Let $G$ be a connected non-compact simple Lie group with finite centre. Let $\mathfrak{g}$ be its Lie algebra. Let $\theta$ be a Cartan involution and write the associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k}$ and $\mathfrak{p}$ the eigenspaces of $\theta$ associated to the eigenvalues 1 and $-1$, respectively.

Let $B : (X, Y) \mapsto \text{tr}(\text{ad} X \text{ad} Y)$ be the Killing form of $\mathfrak{g}$. We consider the scalar product $(X, Y) \mapsto \langle X, Y \rangle := -B(X, \theta Y)$ on $\mathfrak{g}$, which is $\text{Ad}$-invariant, and the associated norm denoted by $\| \cdot \|$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $A = \exp \mathfrak{a}$. Denote by $\Sigma$ the set of restricted roots. We choose a closed Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ and denote by $\Sigma^+$ the set of associated positive roots, that is those which are non-negative on $\mathfrak{a}^+$. Let $\mathfrak{n} := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda := \{ X \in \mathfrak{g} : (\text{ad} H) X = \lambda(H) X \text{ pour tout } H \in \mathfrak{a} \}$ is the root space associated to $\lambda \in \mathfrak{a}^+$ and $N$ the analytic subgroup with Lie algebra $\mathfrak{n}$. Let us also write

$$\rho := \frac{1}{2} \sum_{\lambda \in \Sigma^+} \lambda \in \mathfrak{a}^+$$

for the half-sum of positive roots.

Define $A^+ := \exp \mathfrak{a}^+$. Then we have the Cartan decomposition $G = KA^+K$: for every $g \in G$, there exists a unique $\kappa(g) \in \mathfrak{a}^+$ such that $g \in K \exp (\kappa(g)) K$. This defines a map $\kappa : G \to \mathfrak{a}$ called the Cartan projection. Similarly, the Iwasawa decomposition $G = KAN$ gives the existence, for every $g \in G$, of a unique $H(g) \in \mathfrak{a}$ such that $g \in K \exp (H(g)) N$. 


Let $M$ be the centraliser of $A$ in $K$. Let $P = MAN$ be the associated standard minimal parabolic subgroup, and $F = G/P$ the associated flag manifold. The group $K$ acts transitively on $F$ and the stabiliser is $M$. We will therefore identify $F$ with $K/M$. The scalar product on $\mathfrak{g}$ induces a Riemannian metric on $K/M$ which is $K$-invariant on the left. Let $m$ be the associated volume measure, which is a Haar measure.

Given $g \in G$ and $\xi = kP \in F$ with $k \in K$, write $\sigma(g, \xi) = H(gk^{-1})$; this defines a cocycle $\sigma: G \times F \to \mathfrak{a}$ called the Iwasawa cocycle. We then have the following lemma [20]:

**Lemma 1.** The Radon-Nikodym derivative of $(g^{-1})_* m$ with respect to $m$ at $\xi \in F$ is $e^{-2\rho(\sigma(g, \xi))}$.

Moreover, there is an inequality between the Iwasawa cocycle and the Cartan projection [3, consequence of corollary 8.20]:

**Lemma 2.** For every $g \in G$ and every $\xi \in F$, we have $\|\sigma(g, \xi)\| \leq \|\kappa(g)\|$.

In the following, given two quantities $A$ and $B$, we write $A \lesssim B$ if there exists a constant $C > 0$ depending only on the group $G$ and the choices made above (the group $K$, the Cartan involution, etc.) such that $A \leq CB$, and $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. If the implied constants depend additionally on other parameters, we will write them in indices, for instance $A \lesssim_s B$ if the implied constant depends on $s$.

**Sobolev spaces on a representation.** In the following, we will need some facts on Sobolev spaces on Lie group representations, as in [11] for instance.

Consider the representation $\pi: G \to U(L^2(F))$ defined by $$(\pi(g)u)(\xi) := u(g^{-1}\xi)e^{-\rho(\sigma(g, \xi))}$$ for every $u \in C^\infty(F)$ and $\xi \in F$. This defines an irreducible unitary representation [14, chapter VII]. Here we use the standard Hilbert space structure on $L^2(F)$ and denote by $\langle \cdot, \cdot \rangle$ the inner product and $\|\cdot\|$ the norm.

By differentiation, we define a representation of the Lie algebra: for every $X \in \mathfrak{g}$ and $v \in C^\infty(F)$

$$\pi(X)v := \frac{d}{dt} \bigg|_{t=0} \pi(e^{tX})v$$

which we can extend to the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_C) \mathfrak{g}_C$, and that we also denote by $\pi$ (see Knapp [14, chapter III]).

We now define Sobolev norms. We will need several equivalent definitions. For definiteness we fix one: let $X_i$ be an orthonormal basis of $\mathfrak{f}$ for the inner product on $\mathfrak{g}$, and similarly let $Y_i$ be an orthonormal basis of $\mathfrak{p}$. Let

$$\Delta = -\sum_i X_i^2 - \sum_i Y_i^2 \in \mathcal{U}(\mathfrak{g}_C),$$

and consider, for any $s \in \mathbb{Z}_{\geq 0}$, the the scalar product $\langle \cdot, \cdot \rangle_{H^s}$ on $C^\infty(F)$ defined by

$$\langle u, v \rangle_{H^s} := \langle \pi(1 + \Delta)^s u, v \rangle.$$ 

We will write $\|\cdot\|_{H^s}$ for the associated norm. Then we define the space $H^s(F)$ as the closure in $L^2(F)$ of $C^\infty(F)$ for the norm $\|\cdot\|_{H^s}$.
Harmonic analysis on $L^2(\mathcal{F})$. As we have seen, we can identify $\mathcal{F}$ with $K/M$. Actually, denoting by $K_0$ the neutral component of $K$, the compact connected Lie group $K_0$ acts transitively on $\mathcal{F}$, too, with stabiliser $K_0 \cap M$ (see Knapp \cite{Knapp}, lemma 7.33). Therefore we identify $L^2(\mathcal{F})$ as a $K_0$-module to the subspace of right-$(K_0 \cap M)$-invariant elements of $L^2(K_0)$.

Let $\hat{K}_0$ be the unitary dual of $K_0$. The differential operator $\Delta$ can be written as $\Delta = -\mathcal{C} - 2\mathcal{C}_K$, where $\mathcal{C}$ and $\mathcal{C}_K$ are the Casimir operators of $G$ and $K$ respectively (see Knapp \cite{Knapp}, proof of theorem 8.7). Because $\pi$ is an irreducible unitary representation, $\pi(\mathcal{C})$ acts as a constant on $C^\infty(\mathcal{F})$. As the Casimir operator of $K$ acts as an elliptic operator on $\mathcal{F}$, this means that this Sobolev norm coincides with the usual definition of a Sobolev norm on the Riemannian manifold $\mathcal{F}$.

Moreover, for every $\tau \in \hat{K}_0$, the operator $1 + \Delta$ acts on the subspace $L^2(\mathcal{F})_\tau$ of $\tau$-isotypic vectors as a constant $c(\tau)$. Because $\pi(\Delta)$ is self-adjoint and $\langle \pi(\Delta) u, u \rangle \geq 0$ for every $u \in C^\infty(\mathcal{F})$, this constant is a real number and $c(\tau) \geq 1$.

We now define a Littlewood-Paley decomposition. For every non-negative integer $k$, let $L_k$ be the orthogonal sum of the $L^2(\mathcal{F})_\tau$ for all $\tau \in \hat{K}_0$ such that $2^k < c(\tau) < 2^{k+1}$. Then $L^2(\mathcal{F})$ is the Hilbert sum of the $L_k$ $(k \in \mathbb{Z}_{\geq 0})$. For every $k \in \mathbb{Z}_{\geq 0}$, let $P_k$ be the orthogonal projection on $L_k$. Write also $P_{<k} = \sum_{0 \leq j < k} P_j$ and $P_{\geq k} = 1 - P_{<k}$. Then we can give a second, equivalent, definition of the Sobolev norm. Let $s \in \mathbb{Z}_{\geq 0}$. For $u \in L^2(\mathcal{F})$, then

$$\|u\|_{H^s}^2 \sim_s \sum_k 2^{sk} \|P_k u\|_2^2.$$  

Finally, in the course of the proof of lemma \ref{lem:third}, we will need a third definition of Sobolev norms. Let $\mathcal{B}$ be the basis given by the $\{X_i\}$ and the $\{Y_i\}$ as above in the definition of $\Delta$. Given a non-negative integer $s$ and $u \in C^\infty(\mathcal{F})$, we have (see Nelson \cite{Nelson})

$$\|u\|_{H^s}^2 \sim_s \sum_{k=0}^s \sum_{X_1, \ldots, X_k \in \mathcal{B}} \|d\pi(X_1 \ldots X_k) u\|_2^2$$

where for $k = 0$ the sum is reduced to $\|u\|_2^2$.

**Lemma 3.** There exists a constant $c > 0$ such that for every $s \in \mathbb{Z}_{\geq 0}$, and for every $g \in G$ and $u \in C^\infty(\mathcal{F})$, $\|\pi(g) u\|_{H^s} \lesssim_s e^{cs\|\kappa(g)\|^2} \|u\|_{H^s}$.

**Proof.** For $s = 0$, by lemma \ref{lem:second} we have $\|\pi(g) u\|_2 \leq e^{\|\kappa(g)\|^2} \|u\|_2$. For $s = 1$ and $X \in \mathfrak{g}$, we have

$$\|d\pi(X)\pi(g) u\|_2 = \|\pi(g) d\pi((\text{Ad } g^{-1}) X) u\|_2 \leq \|\pi(g)\|_2 \|d\pi((\text{Ad } g^{-1}) X) u\|_2 \lesssim_s e^{cs\|\kappa(g)\|} \|u\|_{H^s}$$

for some constant $c > 0$, which implies the result by equation \ref{eq:third}. Here we used the Cartan decomposition to bound the coefficients of $(\text{Ad } g^{-1}) X$ in the basis $\mathcal{B}$. The general result follows similarly by induction.  

$\square$
3. Bourgain’s argument

Decay of Fourier coefficients. We now state a slightly more precise form of proposition 2 and prove it:

Proposition 2. There exists a constant $C > 0$ such that the following holds:

Let $\varepsilon > 0$ be small enough. Let $\mu$ be a measure whose support has finite support $S$ and generates a dense subgroup of $G$ such that for every $g \in S$, $\|\kappa(g)\| \leq \varepsilon$. Assume that the operator $T: L^2(\mathcal{F}) \to L^2(\mathcal{F})$ defined by

$$(Tu)(\xi) := \sum_{g \in S} \mu(g) u(g^{-1} \xi)$$

for all $u \in L^2(\mathcal{F})$, satisfies the following condition: there exists a subspace $V$ of finite dimension in $L^2(\mathcal{F})$ such that for every $u \in V^\perp$, $\|Tu\|_2 \leq \frac{1}{2} \|u\|_2$.

Then the unique $\mu$-stationary measure $\nu$ is absolutely continuous with density in $H^t(\mathcal{F})$ for every $t < \frac{C}{\varepsilon}$.

Proof of proposition 2. As we have noted in the introduction, the arguments of Benoist-Quint [4] show that there exists a density in $L^2(\mathcal{F})$ for the stationary measure $\nu$; let us denote it by $g$.

We now reduce to the case where there exists $N$ such that the subspace $V$ is the sum of the $L^k$ for $k < N$. Indeed, we can approximate a finite-dimensional subspace in such a way; this might slightly increase the norm of the operator $T$, but up to replacing it by $T^2$ this is not a problem, as long as $\varepsilon$ is small enough.

Claim 1. There exists a constant $c > 0$ such that for every non-negative integer $s$ which is large enough and every $m \in \mathbb{Z} \geq 0$, we have $\|(T^*)^m\|_{H^s} \lesssim_s c^{sm\varepsilon}$.

Proof. We can reduce to the analogous statement where $\mu$ is a Dirac mass at $g \in G$, in which case for every $u \in L^2(\mathcal{F})$, $Tu = \pi(g) u \cdot \pi(g)1$. The lemma is then a consequence of the bound for the Sobolev norm of a product and lemma 3. □

We now give a bound for the low frequencies. Let $\tau, \sigma \in \hat{K}_0$ and $u \in L(\mathcal{F})_\tau$ et $v \in L(\mathcal{F})_\sigma$. Then for every even non-negative integer $s$, we have:

$$\|\langle (T^m u, v) \rangle \| = \|\langle u, (T^*)^m v \rangle \|$$

$$= \frac{1}{c(\tau)^{s/2}} \| \langle \pi(1 + \Delta)^{s/2} u, (T^*)^m v \rangle \|$$

$$\lesssim_s \frac{1}{c(\tau)^{s/2}} \|u\|_2 \|(T^*)^m v\|_{H^s}$$

$$\leq \frac{1}{c(\tau)^{s/2}} \|(T^*)^m\|_{H^s} \|u\|_2 \|v\|_{H^s}$$

$$\lesssim_s \frac{c(\sigma)^{s/2}}{c(\tau)^{s/2}} \|(T^*)^m\|_{H^s} \|u\|_2 \|v\|_2.$$
number of $\tau \in \hat{K}_0$ such that $2^k \leq c(\tau) < 2^{k+1}$. A classical argument relating $c(\tau)$ to a quadratic expression in the highest weight of $\tau$ (see Warner [22] proof of lemma 4.4.2.3) shows that $N_k \approx 2^{r/2}$, where $r$ is that rank of $K_0$. Therefore

$$|\langle T^m u, v \rangle| \leq N_k^{1/2} \left( \sum_{2^k \leq c(\tau) < 2^{k+1}} \langle T^m u_\tau, v \rangle^2 \right)^{1/2} \leq \| (T^*)^m \|_{H^s} N_k^{1/2} \left( \sum_{2^k \leq c(\tau) < 2^{k+1}} c(\mu)^s \| u_\tau \| \| v \| \right)^{1/2} \leq 2^{rk/4} \| (T^*)^m \|_{H^s} c(\mu)^{s/2} 2^{-sk} \| u \|_2 \| v \|_2$$

Let $j \in \mathbb{Z}_{\geq 0}$. Taking an orthonormal basis of $L_j$ gives:

$$\| P_j T^m u \|_2 \lesssim \| (T^*)^m \|_{H^s} 2^{(r/4-s)k} \| u \|_2.$$  

We will need the following Bernstein-type inequality [10] for the sup-norm $\| \cdot \|_\infty$, which is a simple consequence of the representation theory of $K_0$ and the Cauchy-Schwarz inequality:

**Lemma 4.** Let $\tau \in \hat{K}_0$ and $u \in L^2(K_0)_\tau$. Then $\| u \|_\infty \leq \dim(\tau) \| u \|_2$, where $\dim(\tau)$ denotes the dimension of $\tau$.

Therefore, for every $j \in \mathbb{Z}_{\geq 0}$, we have

$$\| P_j T^m u \|_\infty \lesssim \| (T^*)^m \|_{H^s} 2^{(r/4-s)k} \| u \|_2. \tag{2}$$

We now give Bourgain’s bound, which relies on iterating the restricted spectral gap of $T$ for the high frequencies, and controlling the low frequencies part which appear at each inductive step with equation [2]. Let $k \in \mathbb{Z}_{\geq 0}$ and $u \in L_k$. Then

$$\| T^{m+1} u \|^2_2 \leq \| TP_{< N} T^m u \|^2_2 + \| TP_{\geq N} T^m u \|^2_2 \leq m(\mathcal{F})^{1/2} \| TP_{< N} T^m u \|_\infty + \frac{1}{2} \| P_{\geq N} T^m u \|_2 \leq m(\mathcal{F})^{1/2} \| P_{< N} T^m u \|_\infty + \frac{1}{2} \| T^m u \|_2 \leq C_{s,N} \| (T^*)^m \|_{H^s} 2^{(r/4-s)k} \| u \|_2 + \frac{1}{2} \| T^m u \|_2$$

for some constant $C_{s,N} > 0$.

Iterating, we obtain:

$$\| T^\ell u \|_2 \lesssim_{s,N} \left( \| (T^*)^{\ell-1} \|_{H^s} + \ldots + \| (T^*) \|_{H^s} \right) 2^{(r/4-s)k} \| u \|_2 + 2^{-\ell} \| u \|_2.$$  

Because for $s$ large enough $\| T^m \|_{H^s} \leq 1 \leq C_s e^{c s \varepsilon}$ by the claim, we thus have:

$$\| T^\ell u \|_2 \lesssim_{s,N} (e^{c s \varepsilon} 2^{(r/4-s)k} + 2^{-\ell}) \| u \|_2.$$  

Let $\ell$ be the integer part of

$$\frac{s - r/4}{\log 2 + c s \varepsilon} k.$$
Then
\[
\|T^k u\|_2 \lesssim_{s,N} \exp\left( - \frac{(s-r/4)}{\log 2 + c \varepsilon} k \right) \|u\|_2 .
\]

For \( s \) large enough \( (s-r/4) \frac{\log 2}{\log 2 + c \varepsilon} > \frac{\log 2}{2c \varepsilon} \), which implies that for \( k \) large enough
\[
\|T^k u\|_2 \lesssim_{s,N} 2^{-\frac{k}{2c \varepsilon}} \|u\|_2 ,
\]
and therefore
\[
\left| \int u d\nu \right| = \left| \int T^k u d\nu \right| = \langle T^k u, g \rangle \lesssim_{s,N} 2^{-\frac{k}{2c \varepsilon}} \|u\|_2 .
\]
where we recall that \( g \) is the \( L^2 \) density of \( \nu \). Therefore, for every \( k \in \mathbb{Z}_{\geq 0} \) large enough,
\[
\|P_k g\|_2^2 = \langle P_k g, g \rangle \lesssim_{s,N} 2^{-\frac{k}{2c \varepsilon}} \|P_k g\|_2 .
\]
This implies that \( \|P_k g\|_2 \lesssim_{s,N} 2^{-\frac{k}{2c \varepsilon}} \) for \( k \) large enough and therefore that \( g \in H^t(F) \) for every \( t < \frac{1}{2c \varepsilon} \). \( \square \)

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