Global semantic typing for inductive and coinductive computing
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Abstract
Inductive and coinductive types are commonly construed as ontological (Church-style) types, denoting canonical data-sets such as natural numbers, lists, and streams. For various purposes, notably the study of programs in the context of global ("uninterpreted") semantics, it is preferable to think of types as semantical properties (Curry-style).

Intrinsic theories were introduced in the late 1990s to provide a purely logical framework for reasoning about programs and their semantic types [18]. We extend them here to data given by any combination of inductive and coinductive definitions. This approach is of interest because it fits tightly with syntactic, semantic, and proof theoretic fundamentals of formal logic, with potential applications in implicit computational complexity as well as extraction of programs from proofs. We prove a Canonicity Theorem, showing that the global definition of program typing, via the usual (Tarskian) semantics of first-order logic, agrees with their operational semantics in the intended ("canonical") model.

Finally, we show that every intrinsic theory is interpretable in (a conservative extension of) first-order arithmetic. This means that quantification over infinite data objects does not lead, on its own, to proof-theoretic strength beyond that of Peano Arithmetic.

Intrinsic theories are perfectly amenable to formulas-as-types Curry-Howard morphisms, and were used to characterize major computational complexity classes [18, 19, 17, 16]. Their extensions described here have similar potential which has already been applied in [20].

1 Introduction
1.1 A motivation: termination of equational programs
We refer to the well-known dichotomy between the canonical and global interpretations of proofs and programs, often referred to as "interpreted" and "uninterpreted," respectively. The former is exemplified by Peano's Arithmetic, whose canonical model is the standard structure of the natural numbers with basic operations, and by programming languages with primitive types for integers, strings, etc. The global approach is manifest in algebraic theories such as Group Theory, and in pure logic programming. Thus, the axioms of Peano's Arithmetic (PA) are intended to contribute to the delineation of a particular model, whereas the axioms of Group Theory are intended to describe a class of models, a task they perform successfully by definition.

The limitative properties of canonical axiomatization and computing, e.g. the high complexity of program termination in the canonical model, let alone the complexity of semantic truth of first-order formulas of arithmetic, justify
a reconsideration of canonically intended theories, such as PA, as global theories with unintended, “non-standard,” models. Such non-standard models have “non-standard elements,” but the machinery of Tarskian semantics makes no syntactic distinction between intended and non-standard elements, and consequently no explicit distinction between canonical and non-standard models.

A trivial remedy is to enrich the vocabulary with type-identifiers, i.e. Indeed, that is precisely Peano’s original axiomatization of arithmetic [25]: his context is an abstract universe of objects and sets, and the natural numbers form a particular collection \( N \) within that broader universe. The type \( N \) is thus construed semantically, as a collection of pre-existing objects, which happen to satisfy certain properties. This is in perfect agreement with the brand of typing introduced by Haskell Curry [5, 34]: a function \( f \) has type \( \tau \to \sigma \) if it maps objects of type \( \tau \) to objects of type \( \sigma \); \( f \) may well be defined for input values that are not of type \( \tau \).

Semantic types reflect a global perspective, in that they can be considered for any domain of discourse. In contrast, Church’s approach [4] construes types as inherent properties of objects: a function is of type \( \tau \to \sigma \) when its domain consists of the objects of \( \tau \), and its codomain of objects of type \( \sigma \). That is, Church’s types are related to the presence of a canonical model.

1 The difference between semantic and ontological typing disciplines is thus significant in ways that phrases such as “explicit” and “implicit” do not convey.

2 Herbrand’s proposal also called for a constructive proof that a solution exists and is unique. But that additional condition is ill-formed, and cannot be replaced by provability in some intuitionistic theory, since this would imply that the computable total functions form a semi-decidable collection.
\[ N \to N \] in every model of \( P \) (cf. Theorem \( \text{[10]} \) below).

Within the global framework, it makes sense to consider formal theories for proving global typing properties of equational programs. We adopt as programming model equational programs, since these mesh directly with formal reasoning: a program’s equations can be construed as axioms, computations as derivations in equational logic, and types as formulas. Moreover, equational programs are amenable to term-model constructions, which turn out to be a useful meta-mathematical tool. Theories for reasoning directly about equational programs were developed in \( \text{[18]} \), where they were dubbed intrinsic theories. Among other benefits, they support attractive proof-theoretic characterizations of major complexity classes, such as the provable functions of Peano Arithmetic and the primitive recursive functions \( \text{[18, 19]} \).

In this paper we generalize the global semantics approach of \( \text{[18]} \) to data-systems, that is collections of data-types generated by both inductive and coinductive definitions. To do so we shall start by describing a syntactic framework, generalizing term algebras, in which a syntactic representation of the intended data-types is possible. We give an operational semantics for equational programs over data-objects that may be infinite, and prove a Canonicity Theorem, stating that a program over a data-system is correctly typed in the standard structure (e.g. terminating for inductive input and productive for coinductive input) just in case such typing is correct by Tarskian semantics in all reasonable structures. Finally we show that intrinsic theories, that is the obvious first-order theories for proving correct-typing of equational programs are — proof theoretically — no stronger than Peano Arithmetic, in spite of the use of quantification over infinite objects, such as streams.

## 2 Data systems

### 2.1 Symbolic data

A constructor-vocabulary is a finite set \( \mathcal{C} \) of function identifiers, referred to as constructors, each assigned an arity \( \geq 0 \); as usual, constructors of arity 0 are object-identifiers. Given a constructor-vocabulary \( \mathcal{C} \), a hyper-term (over \( \mathcal{C} \)) is an ordered tree of constructors, possibly infinite, where each node with constructor \( c \) of arity \( r \) has exactly \( r \) children. We write \( \mathcal{H}_\mathcal{C} \) for the set of hyper-terms over \( \mathcal{C} \).

The replete \( \mathcal{C} \)-structure is the structure \( \mathcal{H}_\mathcal{C} \) with:

1. \( \mathcal{C} \) as vocabulary (i.e. similarity-type);
2. \( \mathcal{H}_\mathcal{C} \) as universe; and
3. a syntactic interpretation of the constructors: for an \( r \)-ary \( c \in \mathcal{C} \)
   \[ [c](a_1 \ldots a_r) \] is the tree with \( c \) at the root and \( a_1 \ldots a_r \) as immediate sub-trees.

\( ^3 \)We use typewriter font for actual identifiers, boldface for meta-level variables ranging over syntactic objects, and italics for other meta-level variables.
2.2 Inductive data systems

An inductive type, such as the set $\mathbb{N}$ of natural numbers, is defined by its generative closure rules, which for $\mathbb{N}$ are $\mathbb{N}(0)$ and $\forall x \mathbb{N}(x) \rightarrow \mathbb{N}(s(x))$. Similarly, words in $\{0,1\}^*$, construed as terms generated from the constructors $e$, $0$ and $1$, of arities $0,1,1$ respectively, are generated by the three rules $\mathbb{W}(e)$, $\forall x \mathbb{W}(x) \rightarrow \mathbb{W}(0(x))$, and $\forall x \mathbb{W}(x) \rightarrow \mathbb{W}(1(x))$. If $G$ names a type $G$, then the type of binary trees with leaves in $G$ is generated by the rules $G(x) \rightarrow T(x)$, and $T(x) \land T(y) \rightarrow T(p(x,y))$, where $p$ is a binary constructor.

We can similarly consider several types generated jointly (i.e. simultaneously). For example, the following rules generate the $01$-strings with no adjacent $1$'s, by defining jointly the set (denoted by $E$) of such strings that start with $1$, and the set (denoted by $Z$) of those that don't: $Z(e)$, $Z(x) \rightarrow Z(0(x))$, $Z(x) \rightarrow E(1(x))$, and $E(x) \rightarrow Z(0(x))$.

Generally, a definition of inductive types from given types $\vec{G}$ consists of:

1. A sequence $\vec{D} = (D_1 \ldots D_k)$ of unary relation-identifiers, dubbed type identifiers;
2. A set of construction rules, each one of the form $\forall \vec{y} (Q_1(y_1) \land \cdots \land Q_r(y_r)) \rightarrow D_i(c(y_1 \cdots y_r))$ (1) where $c$ is a constructor of arity $r$, and each $Q_\ell$ is one of the type-identifiers in $\vec{G}, \vec{D}$.

These rules delineate the intended meaning of the inductive types $\vec{D}$ from below, as $D_i$ is built up by the construction rules.

Conjuncting the composition rules, we obtain a single rule, consisting of the universal closure of conjunctions of implications into the data-type being defined. The following variant, equivalent to that conjunction in constructive (intuitionistic) first-order logic, will be useful:

$$\psi_1 \lor \cdots \lor \psi_k \rightarrow D_i(x)$$ (2)

where each $\psi_i$, with $x$ a free variable, is of the form

$$\exists y_1 \ldots y_r \ x = c(\vec{y}) \land Q_1(y_1) \land \cdots \land Q_r(y_r)$$ (3)

where $y_1 \ldots y_r$ are distinct variables. We call a formula of the form (3) a constructor-statement (for $x$).

2.3 Coinductive deconstruction rules

Inductive construction rules state sufficient reasons for asserting that a (finite) hyper-term has a given type, given the types of its immediate sub-terms. The intended semantics of an inductive type $D$ is thus the smallest set of hyper-terms closed under those rules. Coinductive deconstruction rules state necessary conditions for a term to have a given type, by implying possible combinations
for the types of its immediate sub-terms. The intended semantics is the largest set of hyper-terms satisfying those conditions.

For instance, the type of $\omega$-words over 0/1 is given by the deconstruction rule

$$W_\omega(x) \rightarrow (\exists y \ W_\omega(y) \land x = 0(y)) \lor (\exists y \ W_\omega(y) \land x = 1(y))$$

(4)

Note that this is not quite captured by the implications $W_\omega(0x) \rightarrow W_\omega(x)$ and $W_\omega(1x) \rightarrow W_\omega(x)$, since these do not guarantee that every element of $W_\omega$ is of one of the two forms considered.

Moreover, using a destructor function in stating deconstruction rules fails to differentiate between cases of the argument’s main constructor. For example, in analogy to the inductive definition above of the words with no adjacent 1’s, the $\omega$-words over 0/1 with no adjacent 1’s are delineated jointly by the two deconstruction rules

$$Z(x) \rightarrow (\exists y \ Z(y) \land x = 0(y)) \lor (\exists y \ E(y) \land x = 0(y))$$

and

$$E(x) \rightarrow \exists y \ Z(y) \land x = 1(y)$$

These rules cannot be captured using a destructor, since those do not differentiate between cases for the input’s main constructor.

These observations motivate the following definition.

**Definition 1** A deconstruction definition of coinductive types from given types $\vec{G}$ consists of:

1. A list $\vec{D}$ of type identifiers;
2. For each of the types $D_i$ in $\vec{D}$ a deconstruction rule, of the form

$$D_i(x) \rightarrow \psi_1 \lor \cdots \lor \psi_k$$

(5)

where each $\psi_i$ is a constructor-statement (as in (3) above).

### 2.4 General data-systems

We proceed to define data-systems, in which data-types may be defined by any combination of induction and coinduction. Descriptive and deductive tools for such definitions have been studied extensively, e.g. referring to typed lambda calculi, with operators $\mu$ for smallest fixpoint and $\nu$ for greatest fixpoint. The Common Algebraic Specification Language CASL was used as a unifying standard in the algebraic specification community, and extended to coalgebraic data [24, 25]. Several frameworks combining inductive and coinductive data, such as [24], strive to be comprehensive, including various syntactic distinctions and categories, in contrast to our minimalistic approach.

**Definition 2** A data-system $\mathcal{D}$ over a constructor vocabulary $C$ consists of:
1. A double-list $\vec{D}_1 \ldots \vec{D}_k$ (the order matters) of unary relation-identifiers, dubbed type-identifiers, where each $\vec{D}_i$ is a type-bundle, and designated as either inductive or coinductive.

2. For each inductive bundle $\vec{D}_i$, an inductive definition of $\vec{D}_i$ from the types in $\vec{D}_j$, $j < i$.

3. For each coinductive bundle $\vec{D}_i$ a coinductive definition of $\vec{D}_i$ from $\vec{D}_j$, $j < i$.

**Definition 3** We say that a data-system $\vec{D}_1 \ldots \vec{D}_k$ is $\Sigma_n$ ($\Pi_n$) if $\vec{D}_k$ is inductive (respectively, coinductive), and the list of bundles alternates $n-1$ times between inductive and coinductive bundles. That is, a single bundle is $\Sigma_1$ ($\Pi_1$) if it is inductive (respectively, coinductive); if $\vec{D}_1 \ldots \vec{D}_k$ is $\Sigma_n$ then $\vec{D}_1, \ldots, \vec{D}_k, \vec{D}_{k+1}$ is $\Sigma_n$ if $\vec{D}_{k+1}$ is inductive, and $\Pi_n$ if $\vec{D}_{k+1}$ is coinductive; if $\vec{D}_1 \ldots \vec{D}_k$ is $\Pi_n$ then $\vec{D}_1, \ldots, \vec{D}_k, \vec{D}_{k+1}$ is $\Pi_n$ if $\vec{D}_{k+1}$ is coinductive, and $\Sigma_{n+1}$ if $\vec{D}_{k+1}$ is inductive.

A data system $\mathcal{D} = \vec{D}_1 \ldots \vec{D}_k$ has rank $n$ if it is $\Sigma_n$ or $\Pi_n$. A data-type $\mathcal{D}_{ij}$ of $\mathcal{D}$ has rank $n$ (in $\mathcal{D}$) if the data-system $\vec{D}_1 \ldots \vec{D}_i$ has rank $n$.

2.5 Examples of data-systems

1. Let $\mathcal{C}$ consist of the identifiers 0, 1, e, s, and p, of arities 0,0,0,1, and 2, respectively. Consider the following $\Sigma_3$ data-system, for the double list $((B), (N), (F, S), (L))$ with inductive $B$ and $N$ (booleans and natural numbers), coinductive $F$ and $S$ (streams with alternating boolean and numeral entries starting with booleans (respectively, with natural numbers)), and finally an inductive $L$ for lists of such streams. The defining formulas are, in simplified form,

\[
\begin{align*}
B(0) & \quad B(1) \\
N(0) & \quad \forall y \ N(y) \rightarrow N(s(y)) \\
F(x) & \rightarrow \exists y, z \ (x = p(y, z)) \land B(y) \land S(z) \\
S(x) & \rightarrow \exists y, z \ (x = p(y, z)) \land N(y) \land F(z) \\
L(e) & \quad \forall y, z \ F(y) \land L(z) \rightarrow L(p(y, z)) \\
L(y) & \quad \forall y, z \ S(y) \land L(z) \rightarrow L(p(y, z))
\end{align*}
\]

Note that constructors are reused for different data-types. This is in agreement with our untyped, generic approach, where the data-objects are untyped.
Let the constructors be 0, 1, s, p, and d, of arities 0, 0, 1, 2, and 3 respectively. Consider the \( \Pi_2 \) data system \(((N), (T), (D))\), where \( N \) is intended to denote \( N \), \( T \) the 2-3 trees (finite or infinite) with leaves in \( N \), and \( D \) the infinite binary trees (no leaves) whose internal nodes are decorated by elements in \( T \). The inductive definition of \( N \) is as above; the coinductive definitions of \( T \) and \( D \) are

\[
T(x) \rightarrow N(x) \\
\lor (\exists y_1, y_2 \ x = p(y_1, y_2) \land T(y_1) \land T(y_2)) \\
\lor (\exists y_1, y_2, y_3 \ x = d(y_1, y_2, y_3) \land T(y_1) \land T(y_2) \land T(y_3))
\]

and

\[
D(x) \rightarrow \exists u, y_1, y_2 \ x = d(u, y_1, y_2) \land T(u) \land D(y_1) \land D(y_2)
\]

Note that we construe a “tree of trees” not as a higher-order object, but simply as a tree of constructors, a suitably parsed.

3 Programs over data-systems

3.1 Equational programs

In addition to the set \( C \) of constructors we posit an infinite set \( X \) of variables, and an infinite set \( F \) of function-identifiers, dubbed program-functions, and assigned arities \( \geq 0 \) as well. The sets \( C, X \) and \( F \) are, of course, disjoint. If \( E \) is a set consisting of function-identifiers and (possibly) variables, we write \( \bar{E} \) for the set of terms generated from \( E \) by application: if \( g \in E \) is a function-identifier of arity \( r \), and \( t_1 \ldots t_r \) are terms, then so is \( g(t_1, \ldots, t_r) \). We use informally the parenthesized notation \( g(t_1, \ldots, t_r) \), when convenient.\(^{\neg1}\) We refer to elements of \( C, \ C \cup X \) and \( \bar{C} \cup \bar{X} \cup F \) as data-terms, base-terms, and program-terms, respectively.\(^{\neg2}\) We shall also refer to a variant of data-terms, where the leaves need not have arity 0; we dub these pseudo-terms. We write \( |t| \) for the height of a term (or pseudo-term) \( t \).

We adopt equational programs, in the style of Herbrand-Gödel, as computation model. There are easy inter-translations between equational programs and program-terms such as those of \( \text{FLR}_0 \) \(^{\neg3}\). We prefer however to focus on equational programs because they integrate easily into logical calculi, and are naturally construed as axioms. In fact, codifying equations by terms is a conceptual detour, since the computational behavior of such terms is itself spelled out using equations or rewrite-rules.

A program-equation is an equation of the form \( f(t_1, \ldots, t_k) = q \), where \( f \) is a program-function of arity \( k \geq 0 \), \( t_1 \ldots t_k \) is a list of base-terms with no variable repeating, and \( q \) is a program-term. Two program-equations are compatible if their left-hand sides cannot be unified. A program-body is a finite

\(^{\neg1}\)Note that if \( g \) is of arity 0, it is itself a term, whereas with parentheses we’d have \( g() \).

\(^{\neg2}\)Data-terms are often referred to as values, and base-terms as patterns.

\(^{\neg3}\)We refer to elements of \( C, \ C \cup X \) and \( \bar{C} \cup \bar{X} \cup F \) as data-terms, base-terms, and program-terms, respectively.
set of pairwise-compatible program-equations. A program \((P, f)\) (of arity \(k\)) consists of a program-body \(P\) and a program-function \(f\) (of arity \(k\)) dubbed the program’s principal-function. We identify each program with its program-body when in no danger of confusion. Given a program \(P\), we call the program-terms that use the function-identifiers occurring in \(P\) \(P\)-terms, and write \(Tm(P)\) for the set of \(P\)-terms.

The requirement that program-equations have no repeating variable in the input is essential when the input may be infinite, for else the applicability of such an equation might depend on two inputs being identical, a condition which is not decidable.

Programs of arity 0 can be used to define objects. For example, the singleton program \(T\) consisting of the equation \(f = sss0\) defines 3, in the sense that in every model \(S\) of \(T\) the interpretation of the identifier \(t\) is the same as that of the numeral for 3. We can similarly construct 0-ary programs defining infinite terms, such as the program \(I\) consisting of the single equation \(i = s(i)\). This program does not have any solution in the free algebra of the unary numerals, that is: the free algebra cannot be expanded into the richer vocabulary with \(i\) as a new identifier, so as to satisfy the equation \(I\). But \(I\) is modeled by any structure where \(s\) is interpreted as identity, and \(i\) as any structure element.

### 3.2 Operational semantics of programs

A program \((P, f)\) computes a partial-function \(g : \bar{C} \rightarrow \bar{C}\) when \(g(p) = q\) iff the equation \(f(p) = q\) is derivable from \(P\) in equational logic. Non-trivial replete hyperterm-structures have infinite terms, so the output of a program over \(HC\) must be computed piecemeal from finite information about the input values.

To formally describe computation over infinite data, with a modicum of syntactic machinery, we posit that each program over \(C\) has defining equations for destructors and a discriminator. That is, if the given vocabulary’s \(k\) constructors are \(c_1, \ldots, c_k\), with \(m\) their maximal arity, then the program-functions include the unary identifiers \(\pi_{i,m}(i = 1..m)\) and \(\delta\) (destructors and discriminator), and all programs contain, for each constructor \(c\), of arity \(r\) say, the equations

\[
\begin{align*}
\pi_{i,m}(c(x_1, \ldots, x_r)) &= x_i && (i = 1..r) \\
\pi_{i,m}(c(x_1, \ldots, x_r)) &= c(x_1, \ldots, x_r) && (i = r+1..m) \\
\delta(c_1(t), x_1, \ldots, x_k) &= x_i && (i = 1..k)
\end{align*}
\]

We call a repeated composition of destructors a deep-destructor, and construe it as an address in hyper-terms.

A valuation is a function \(\eta\) from a finite set of variables to \(HC\). If \(\bar{v}\) is a list of \(r\) distinct variables, and \(\bar{t}\) a list of \(r\) hyper-terms, then we write \([\bar{v} \leftarrow \bar{t}]\) for the valuation \(\eta\) defined by \(\eta(v_i) = t_i\).

We posit the presence in \(C\) of at least one nullary constructor \(o\); indeed, adding a nullary constructor to \(C\) does not impact the rest of the discussion.

\[\text{As usual, when a structure is an expansion of another they have the same universe.}\]
For an $r$-ary constructor $c$, we write $c^o$ for the term $c(o, \ldots, o)$. For a deep-destructor $\Pi$ we define

$$\Pi^o(x) = \delta(\Pi(x), c_1^o, \ldots, c_k^o)$$

That is, $\Pi^o(x)$ identifies the constructor of $x$ at address $\Pi$.

**Definition 4** We say that a set $\Gamma$ of equations locally infer an equation $t = q$ between program-terms if, for each deep-destructor $\Pi$, the equation $\Pi^o(t) = \Pi^o(q)$ is derivable in equational logic from $\Gamma$. We write then $\Gamma \vdash t = q$.

The diagram of a valuation $\eta$ is the set $\Delta_\eta$ of equations of the form $\Pi^o(v) = c^o$ where $v$ is in the domain of $\eta$, $\Pi$ a deep-destructor, and $c$ the main constructor of $\Pi(\eta(v))$. That is, $\Delta_\eta$ conveys, node by node, the structure of the hyper-terms $\eta(v)$.

An $r$-ary program $(P, f)$ locally-computes a partial-function $g : H_C^r \rightarrow H_C$ when, for every $\vec{t} \in H_C^r$ and $q \in H_C$, $g(t_1, \ldots, t_r) = q$ iff $P, \Delta_\eta \vdash f(v_1 \ldots v_r) = u$, where $\eta = [\vec{v}, u \leftarrow \vec{t}, q]$.

The notion of local-computability is motivated solely by the presence of infinite data:

**Proposition 5** If $g$ is a partial-function whose domain and co-domain are inductive, then $g$ is locally-computably iff it is computable.

**Proof.** Posit, without loss of generality, that $g$ is unary. Suppose that $g$ is locally-computable by a program $(P, f)$, and $g(t) = q$. i.e. $\Delta_\eta, P \vdash f(v) = u$, where $\eta = [v, u \leftarrow t, q]$. But, for finite hyper-terms $r$, $v = r$ implies $\Delta_{[v \leftarrow t]}$ (in equational logic), so

$$P \vdash f(t) = q$$

Let $P'$ be $P$ augmented with the two equations

$$I(x) = \delta(x, c_1(I(\pi_{1m}(x)), \ldots, I(\pi_{1m}(x))), \ldots, c_k(I(\pi_{1m}(x)), \ldots, I(\pi_{1m}(x))))$$

and

$$f'(x) = I(f(x))$$

So

$$P' \vdash f'(t) = I(q)$$

Note that $I$ computes, over finite hyper-terms, the identity function. (This needs a proof by induction, but $I(r) = r$ is derivable already in equational logic for each fixed finite hyper-term $r$.) So $(P', f')$ computes $g$.

Moreover, if $\Gamma \vdash r = s$, where $r$ and $s$ denote finite hyper-terms, then, by the definition of $\vdash$, $\Gamma \vdash I(r) = I(s)$. So

$$P' \vdash f'(t) = q$$

7Here again we stipulate that $C = \{c_1, \ldots, c_k\}$. 9
I.e. \( g \) is computable.

For the converse, suppose that \((P, f)\) computes \( g \). Let \( P'' \) be defined as \( P' \) above, except that \( f' \) is replaced by \( f'' \) defined by

\[
f''(x) = f(I(x))
\]

Since \( I \) is the identity over finite hyper-terms, \((P'', f'')\) computes \( g \). For \( \eta = \left[ v, u \leftarrow t, q \right] \), we have \( \Delta_\eta, P'' \vdash I(v) = t \), so \( P'', \Delta_\eta \vdash f''(v) = q \). Thus, for every deep-destructor \( \Pi \); \( P'', \Delta_\eta \vdash \Pi^\omega(f''(v)) = \Pi^\omega(q) \). Since \( \Delta_\eta \) includes each equation \( \Pi^\omega(u) = \Pi^\omega(q) \), we conclude that \( P'', \Delta_\eta \vdash \Pi^\omega(f''(v)) = \Pi^\omega(u) \), i.e. \( P'', \Delta_\eta \vdash f''(v) = u \).

\[\Box\]

3.3 Equational vs Turing computation

The equivalence of equational programs over \( \mathbb{N} \) with the \( \mu \)-recursive functions was implicit already in [7], and explicit in [12]. Their equivalence with \( \lambda \)-definability [3, 13] and hence with Turing computability [38] followed readily.

When equational programs are used over infinite data, a match with Turing machines must be based on an adequate representation of infinite data by functions over inductive data. For instance, each infinite 0/1 word \( w \) can be identified with the function \( \hat{w} : \mathbb{N} \rightarrow \mathbb{B} \) defined by \( \hat{w}(k) = \) the \( k \)th constructor of \( w \).

Similarly, infinite binary trees with nodes decorated with 0/1 can be identified with functions from \( \mathbb{W} = \{0, 1\}^* \) to \( \{0, 1\} \). Conversely, a function \( f : \mathbb{N} \rightarrow \mathbb{B} \) can be identified with the \( \omega \)-word \( \hat{f} \) whose \( n \)th entry is \( f(n) \).

It follows that a functional \( g : (\mathbb{N} \rightarrow \mathbb{B}) \rightarrow (\mathbb{N} \rightarrow \mathbb{B}) \) can be identified with the function \( \hat{g} : \mathbb{B}^\omega \rightarrow \mathbb{B}^\omega \), defined by \( \hat{g}(w) = (g(\hat{w}))^\omega \). Conversely, a function \( h : \mathbb{B}^\omega \rightarrow \mathbb{B}^\omega \) can be identified with the functional \( \hat{h} : (\mathbb{N} \rightarrow \mathbb{B}) \rightarrow (\mathbb{N} \rightarrow \mathbb{B}) \) defined by \( \hat{h}(f) = (h(\hat{f}))^\omega \).

It is easy (albeit tedious) to see that a partial function \( h : \mathbb{B}^\omega \rightarrow \mathbb{B}^\omega \) is computable by an equational program iff the functional \( \hat{h} \) is computable by some oracle Turing machine. Dually, a functional \( g : (\mathbb{N} \rightarrow \mathbb{B}) \rightarrow (\mathbb{N} \rightarrow \mathbb{B}) \) is computable by an oracle Turing machine iff the function \( \hat{g} \) is computable by an equational program.

4 Matching Tarskian semantics and operational semantics

4.1 \( \mathcal{D} \)-correct structures

We have focused so far on the canonical setup for data-systems \( \mathcal{D} \), with hyper-terms as objects. We now consider arbitrary structures. We call a structure \( \mathcal{S} \) a \( \mathcal{D} \)-structure if its vocabulary (i.e. signature, symbol set) contains the constructor- and type-identifiers of \( \mathcal{D} \). In a \( \mathcal{D} \)-structure \( \mathcal{S} \) we may have a finite or infinite regression of constructor-eliminations, regardless of the nature of the
structure elements. For example, if a unary \( f \) is among the constructors, and \( f = f_S \) is its interpretation in \( S \), we might have an element \( v_0 \in |S| \) for which there is a \( v_1 \) with \( v_0 = f(v_1) \), and more generally elements \( v_i \in |S| \) (\( i = 0, 1, \ldots \)) where \( v_i = f(v_{i+1}) \). In general \( f \) need not be injective, and so \( v_{i+1} \) need not be uniquely determined by \( v_i \).

We say that a \( D \)-structure \( S \) is \( D \)-correct (or just correct when in no danger of confusion) if

1. \( S \) is separated for \( C \), that is, the interpretations in \( S \) of the constructors are all injective and have pairwise-disjoint codomains.\(^8\) Note that \( HC \) satisfies this property.

2. If \( \vec{D}_i = \langle D_{i1}, \ldots, D_{im} \rangle \) is inductive, then \( \langle [D_{i1}], \ldots, [D_{im}] \rangle \) is the minimal \( m \)-tuple of subsets of \( |S| \) closed under the construction rules for \( \vec{D}_i \), given the sets \( [D_{i1}] \ldots [D_{i-1}] \).\(^9\)

3. Dually, if \( \vec{D}_i \) is coinductive, then \( [D_{i}] \) is the largest vector of subsets of \( HC \) closed under the deconstruction rules for \( \vec{D}_i \), given the sets \( [D_{i1}] \ldots [D_{i-1}] \).

The canonical model \( A \equiv A_D \equiv [D] \) of a data-system \( D \) is the \( D \)-correct expansion of the replete structure \( HC \).

### 4.2 Decomposition in data-correct structures

Let \( S \) be a \( D \)-structure, \( a \) an element of \( |S| \) (the universe of \( S \)). A \( C \)-decomposition of \( a \) is a finitely-branching tree \( T \) of elements of \( |S| \times C \) such that

1. The root of \( T \) is of the form \( \langle a, c \rangle \) with \( c \in C \);
2. if \( \langle b_i, c_i \rangle \) (\( i = 1..r \)) are the children in \( T \) of a node \( \langle b, c \rangle \) of \( T \), then \( b = c_S(b_1, \ldots, b_r) \).

If \( a \) has a \( C \)-decomposition, we say that it is \( C \)-decomposable. Put differently, \( a \) is \( C \)-decomposable if it is in the range of a partial mapping \( \varphi : HC \rightarrow |S| \) that satisfies \( \varphi(c(t_1 \ldots t_r)) = c_S(\varphi(t_1) \ldots \varphi(t_r)) \).

Obviously, an element \( a \in |S| \) may have multiple \( C \)-decompositions, and even uncountably many: it suffices to take the structure with two elements \( a, b \) and two constant functions \( \lambda x.a \) and \( \lambda x.b \).

Recall that a \( D \)-structure \( S \) is separated if the interpretations in \( S \) of the constructors \( c \in C \) are injective and with disjoint codomains.

**Proposition 6** If \( S \) is a separated structure for \( C \), then each element \( a \in |S| \) has at most one decomposition.

\(^8\)For \( \mathbb{N} \) these are Peano’s Third and Fourth Axioms.
\(^9\)We write \( |S| \) for the universe of the structure \( S \).
PROOF. Let \( T, T' \) be decompositions of \( a \in |S| \). We prove by induction on \( n \) that if \( \langle b, c \rangle \) is at address \( \alpha \) of \( T \) of height \( n \), then it is also at address \( \alpha \) of \( T' \). The induction’s basis and step follow outright from the assumption that \( S \) is separated. \( \square \)

If \( T \) is a \( C \)-decomposition of \( a \), let \( \tilde{T} \) be the hyper-term obtained from \( T \) by replacing each node \( \langle b, c \rangle \) by \( c \). We call \( \tilde{T} \) a constructor-decomposition (for short, a decomposition) of \( a \). From the proof of proposition 6 it follows that an element \( a \) of a separated structure has at most one decomposition, which we denote (when it exists) \( a \).

**Proposition 7** Suppose \( S \) is a \( D \)-correct structure. If \( a \in |S| \) has type \( D \) in \( S \), then it has a decomposition, which has type \( D \) in \( A \).

Conversely, if \( t \in H_C \) has type \( D \) in \( A \), then every \( a \in |S| \) which has \( t \) as decomposition, is of type \( D \) in \( S \).

**Proof.** We prove the Proposition by cumulative induction on the rank of \( D \) in \( D \). Suppose the statement holds for types of rank \( < n \). For each type \( D \) of \( D \) define

\[
A(D) = \{ a \in |S| \mid a \text{ has a decomposition, which in } [D]_A \}
\]

and

\[
S(D) = \{ t \in H_C \mid t \text{ is the decomposition of some } a \in [D]_S \}
\]

Suppose \( D \) is in an inductive bundle \( \tilde{D}_i \) of \( D \). The sequence of sets \( \langle A(D_{i,j}) \rangle \) satisfies the inductive closure condition of \( \tilde{D}_i \). To see this, consider a rule of \( D \) for \( \tilde{D}_i \), say (w.l.o.g.)

\[
D(y_1) \land D'(y_2) \land E(y_3) \rightarrow D(c(y_1, y_2, y_3)
\]

where \( D' \) is another type in \( \tilde{D}_i \) and \( E \) is a type of rank \( < n \). We show that

\[
(y_1 \in A(D)) \land (y_2 \in A(D')) \land (y_3 \in [E]_S) \rightarrow c(y_1, y_2, y_3) \in A(D)
\]

The first two premises mean that \( y_1 \) and \( y_2 \) have decompositions \( \tilde{y}_1 \in [D]_A \) and \( \tilde{y}_2 \in [D']_A \), and the third premise implies that \( \tilde{y}_3 \in [E]_S \) by IH, since \( E \) is of rank \( < n \). So the hyper-term \( c(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \) is in \( [D]_A \), since \( A \) is \( D \)-correct. That hyper-term is the decomposition of \( c(y_1, y_2, y_3) \), proving that the latter is in \( A(D) \).

Since \( \langle [D]_{i,j} \rangle \) is the smallest fixpoint of those conditions (given than \( S \) is \( D \)-correct), it follows that \( [D]_S \subseteq A(D) \), i.e. every element of \( |S| \) of type \( D \) in \( S \) has a decomposition, which furthermore is of type \( D \) in \( A \).

For the converse, we observe that the sequence of sets \( \langle S(D_{i,j}) \rangle \) is closed under the inductive closure conditions of the bundle \( \tilde{D}_i \). To see this, consider again a rule

\[
D(y_1) \land D'(y_2) \land E(y_3) \rightarrow D(c(y_1, y_2, y_3)
\]
as above. Assume the premise of

\[(y_1 \in S(D)) \land (y_2 \in S(D')) \land (y_3 \in [E]_A) \rightarrow c(y_1, y_2, y_3) \in S(D)\]

The first two conjuncts mean that \(y_1\) and \(y_2\) are decompositions of some \(a_1 \in [D]_S\) and \(a_2 \in [D']_S\), and the third implies, by IH, that \(y_3\) is the decomposition of some \(a_3 \in [E]_S\). The hyper-term \(c(y_1, y_2, y_3)\) is the decomposition of \(c_S(a_1, a_2, a_3)\), which is in \([D]_S\), since \(S\) is \(D\)-correct. This concludes the case where \(D\) is an inductive type.

Suppose now that \(D\) is coinductive. Then \(\langle A(D_{ij}) \rangle_j\) satisfies the coinductive closure condition of the bundle \(D_i\). To see this, consider the rule of \(D\) for \(D\), say

\[D(x) \rightarrow \psi_1 \lor \cdots \lor \psi_k\]

where each \(\psi_i\) is a constructor-statement. Assume (w.l.o.g.) that \(k = 1\) and

\[\psi_1 \equiv \exists y_1, y_2, y_3. x = c(y_1, y_2, y_3) \land D(y_1) \land D'(y_2) \land E(y_3)\]

where \(D'\) and \(E\) are as above. We show that

\[a \in A(D) \rightarrow (\exists y_1 \in A(D) \exists y_2 \in A(D') \exists y_3 \in [E]_S. a = c_S(y_1, y_2, y_3))\]

An element \(a \in A(D)\) has \(\bar{a} \in [D]_A\). Since \(A\) is \(D\)-correct, \(\bar{a}\) must be \(c(t_1, t_2, t_3)\) for some \(t_1 \in [D]_A, t_2 \in [D']_A\), and \(t_3 \in [E]_A\). Since \(\bar{a}\) is the decomposition of \(a\), this means that \(a = c_S(b_1, b_2, b_3)\), where \(t_1 = b_1\). So \(b_1 \in A(D), b_2 \in A(D'),\) by the definition of the function \(A\), and \(b_3 \in [E]_S\) by IH, since \(E\) is of rank \(n\).

Since \(S\) is \(D\)-correct, \([D]_S\) is the greatest set closed under the closure conditions for the bundle \(D_i\); it therefore has \(A(D)\) as a subset. That is, every element \(a\) of \(S\) whose decomposition is of type \(D\) in \(A\), is of type \(D\) in \(S\).

For the converse, we similarly prove that \(\langle S(D_{ij}) \rangle_j\) is closed under the coinductive closure conditions of the bundle \(D_i\). Suppose again that the coinductive rule for \(D\) is \((\square)\) above. We show that for every hyper-term \(t\)

\[t \in S(D) \rightarrow (\exists y_1 \in S(D) \exists y_2 \in S(D') \exists y_3 \in [E]_S. a = c_S(y_1, y_2, y_3))\]

Suppose \(t \in S(D)\), i.e. \(t\) is the deconstruction of some \(a \in [D]_S\). Since \(S\) is \(D\)-correct, \(a\) must be \(c(b_1, b_2, b_3)\) for some \(b_1 \in [D]_S, b_2 \in [D']_S\), and \(b_3 \in [E]_S\). So \(t\) must be of the form \(c(t_1, t_2, t_3)\) where \(t_1 \in S(D), t_2 \in S(D')\), by definition of \(S(\cdots)\), and \(t_3 \in [E]_S\) (by IH).

Since \([D]_A\) is the greatest subset of \(H_C\) closed under the rule for \(D\), it follows that it has \(S(D)\) as a subset. That is, if a hyper-term \(t\) is the decomposition of an element of \([D]_S\), then \(t\) is of type \(D\) in \(A\). \(\square\)

**Corollary 8** For any two \(D\)-correct structures \(S\) and \(Q\), if \(a \in |S|\) and \(b \in |Q|\) have the same decomposition, then they have the same types in \(S\) and \(Q\).
4.3 Typing statements

**Definition 9** Given a data-system \( D \) over \( C \), with \( D_1, \ldots, D_r \) and \( E \) among its type-identifiers, we say that a partial function \( g : H_C^r \rightarrow H_C \) is of type \((\times_i D_i) \rightarrow E\) if \( a_i \in \llbracket D_i \rrbracket_A \) implies that \( g(\bar{a}) \) is definable and \( \in \llbracket E \rrbracket_A \).

If \((P, f)\) is a program that computes the partial-function \( g \) above, we say also that \( P \) is of type \((\times_i D_i) \rightarrow E\).

Note that each function, including the constructors, can have multiple types. Also, a program may compute a non-total mapping over \( H_C \), and still be of type \( D \rightarrow E \), i.e. compute a total function from type \( D \) to type \( E \).

When a (total) function \( f : H_C \rightarrow H_C \) fails to be of a type \( D \rightarrow E \) there must be some \( d \in \llbracket D \rrbracket \) for which \( f(d) \not\in \llbracket E \rrbracket \). Thus the value \( f(d) \) can represent divergence with respect to computation over \( \llbracket D \rrbracket \), as for example when \( \llbracket D \rrbracket = \mathbb{N} \) and \( \llbracket E \rrbracket = \mathbb{N} \perp \) with \( f(d) = \perp \). However, to adequately capture the computational behavior of equational programs, multiple representations of divergence might be necessary; see [18] for examples and discussion.

The partiality of computable functions is commonly addressed either by allowing partial structures \([15, 1, 23]\), or by considering semantic domains, with an object \( \perp \) denoting divergence. The approach here is based instead on the “global” behavior of programs in all structures.

4.4 Canonicity for inductive data

Definition 4 provides the computational semantics of a program \((P, f)\). But as a set of equations a program can be construed simply as a first-order formula, namely the conjunction of the universal closure of those equations. As such, a program has its Tarskian semantic, referring to arbitrary structures for the vocabulary in hand, that is the constructors and the program-functions used in \( P \). A model of \( P \) is then just a structure that satisfies each equation in \( P \).

Herbrand proposed to define a (total) function \( g \) as computable just in case there is a program for which \( g \) is the unique solution.\(^{10}\) It is rather easy to show that every computable function is indeed the unique solution of a program. But the converse fails, as illustrated by the following example.\(^{11}\) Let \( G[x] \equiv \exists y. G_0(x, y) \) be undecidable, with \( G_0 \) decidable. Clearly, there is a program \((P, f)\) that conveys equationally the following informal equality:

\[
  f(x, v) = \begin{cases} 
    1 & \text{if } \exists y < v. G_0(x, y) \\
    2 \cdot f(x, v + 1) & \text{otherwise}
  \end{cases}
\]

If, for a given \( x \), \( \exists y. G_0(x, y) \), then \( \lambda v. f(x, v) \) has a unique solution, with \( f(x, 0) > 0 \). Otherwise \( f(x, v) = 0 \) is the unique solution. So if the unique solution of \( P \) were computable, then \( G \) would be decidable.

\(^{10}\) This proposal was made to Gödel in personal communication, and reported in [7]. A modified proposal, incorporating an operational-semantics ingredient, was made in [10].

\(^{11}\) The first counter-example to Herbrand’s proposal is probably due to Kalmar [11]. The example given here is a simplification of an example of Kreisel, quoted in [29].
In fact, Herbrand’s definition yields precisely the hyper-arithmetic functions [29]. But Herbrand was not far off: one only needs to refer collectively to all $D$-correct structures:

**Theorem 10 (Canonicity Theorem for $\mathbb{N}$)** [18] An equational program $(P, f)$ over $\mathbb{N}$ computes a total function iff the formula $\mathbb{N}(x) \rightarrow \mathbb{N}(f(x))$ is true in every $\mathbb{N}$-correct model of $P$.

### 4.5 Canonicity Theorem for Data Systems

We generalize Theorem 10 to all data-systems. Given a (unary) program $(P, f)$ over a data-system $D$, and a valuation $\eta$, we construct a canonical model $M(P, \eta)$ to serve as “test-structure” for the program $P$ and the valuation $\eta$ as input.

We define the equivalence relation $\approx_{P, \eta}$ over hyper-terms to hold between $t$ and $q$ iff $\Delta_{\eta, P}$ locally infer $t = q$, in the sense of Definition 4. When safe, we write $\approx$ for $\approx_{P, \eta}$.

Let $Q(P, \eta)$ be the structure whose universe is the quotient $H_C / \approx$, and where each function-identifier (constructor or program-function) is interpreted as symbolic application: for an $r$-ary identifier $f$, $f_Q$ maps equivalence classes $[(t_i)]_\approx$ to $[f(t)]_\approx$. This symbolic interpretation of the constructors guarantees that the structure is separated for $C$. Let now $M(P, \eta)$ be the $D$-correct expansion of $Q(P, \eta)$, i.e. the expansion of $Q(P, \eta)$ to the full vocabulary of $D$, with type-identifiers, where inductive types are interpreted as the minimal subsets of $H_C$ closed under their closure conditions, and the coinductive types as the maximal subsets closed under their closure conditions.

**Lemma 11** $M(P, \eta)$ is a model of $P$.

**Proof.** If $f(f^{\bar{t}}) = q$ is an equation in $P$, then $f(f^{\bar{t}}) \approx_{P, \eta} q$ is immediate from the definition of $\approx$. Thus

$[f^{\bar{t}}]_\approx = [q]_\approx$

Also, by structural induction on terms one easily prove that

$[t]_{M(P, \eta)} = [t]_\approx$

for each term $t$, since function-identifiers are interpreted in $M(P, \eta)$ symbolically.

We conclude

$[f^{\bar{t}}]_{M(P, \eta)} = [q]_{M(P, \eta)}$

$\Box$

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Theorem 12 (Canonicity Theorem for Data Systems) Let $D$ be a data-system over $C$, and $D, E$ two type-identifiers of $D$. Let $(P, f)$ be an equational program over $C$ computing a function $g : H_C \to H_C$.

The following are equivalent:

1. $g : D \to E$, i.e. $A|A = D(x) \to E(f(x))$.

2. $D(x) \to E(f(x))$ is true in every $D$-correct model of $P$.

The equivalence above generalizes to arities $\neq 1$.

Proof. We prove the equivalent above by going through

3. For all valuations $\eta$, $M(P, \eta) \models \forall x D(x) \to E(f(x))$.

(1) implies (2): Assume (1), and let $S$ be a $D$-correct model of $P$. Consider an element $a \in [D]_S$. By Proposition 7 $a$ is deconstructable, with for deconstruction-tree some hyper-term $t$. Moreover, since $S$ is $D$-correct, the closure conditions justifying $a \in [D]_S$ also justify $t \in [D]_A$. By (1), this implies that $g(t) \in [E]_A$.

Since $g$ is computed by $P$ we have, for each deep-destructor $\Pi$ that an equation $\Pi(f(v)) = c_text{Ci}$ is derivable in equational logic from $P$ and $\Delta_\eta$, where $c_text{Ci}$ is the main constructor of $\Pi(g(a))$. Since $a$ has the same deconstruction-tree as $t$, all equations $\Delta_\eta$ are true in $S$. Since $S$ is known to be a model of $P$, it follows that $\Pi(f(v)) = c_text{Ci}$ is true in $S$ with $v$ bound to $a$. This being the case for every deep-destructor $\Pi$, it follows that $f_S(a)$ has the same deconstruction-tree as $g(t)$. But $g(t) \in [E]_A$ and $S$ is $D$-correct, so $f_S(a) \in [E]_S$, proving (2).

(2) implies (3): $M(P, \eta)$ is $D$-correct by definition. It is a model of $P$ by Lemma 11. So (3) is a special case of (2).

(3) implies (1): Assume (3). Consider input $a \in [D]_A$, and let $\eta(v) = a$. The class $[v]_\approx$ has then the same decomposition as $a$, and since $M(P, \eta)$ is $D$-correct, it must have type $D$ in $M(P, \eta)$, by Proposition 7. By (3) it follows that

$$f_{M(P, \eta)}([v]_\approx) \in [E]_{M(P, \eta)}$$

But

$$f_{M(P, \eta)}([v]_\approx) = [f(v)]_\approx$$

by definition of $M(P, \eta)$. Since $g$ is computed by $(P, f)$, we have $\Pi(f(v)) = \Pi(g(a))$ for all deep-destructors $\Pi$. So $g(a)$ has the same decomposition as $[f(v)]_\approx$, and therefore is in $[E]_A$. $\square$

5 Intrinsic theories

5.1 Intrinsic theories for inductive data

Intrinsic theories for inductive data-types were introduced in [18]. They support unobstructed reference to partial functions and to non-denoting terms, common
in functional and equational programming. Each intrinsic theory is intended to be a framework for reasoning about the typing properties of programs, including their termination and fairness. In particular, declarative programs are considered as formal theories. This contrasts with two longstanding approaches to reasoning about programs and their termination, namely programs as modal operators \[26, 9\], and programs (and their computation traces) as explicit mathematical objects \[14, 15\].

Let \(D\) be a data-system consisting of a single inductive bundle \(\bar{D}\). The intrinsic theory for \(D\) is a first order theory over the vocabulary of \(D\), whose axioms are

- The closure rules of \(D\).
- Separation axioms for \(C\), stating that the constructors are injective and have pairwise-disjoint codomains. These imply that all data-terms are distinct.
- Inductive delineation (data-elimination, Induction), which mirrors the inductive closure rules. Namely, if a vector \(\bar{\varphi}[x]\) of first order formulas satisfies the construction rules for \(\bar{D}\), then it contains \(\bar{D}\):

\[
\text{Const}[\bar{\varphi}] \rightarrow \land \forall x \ D_i(x) \rightarrow \varphi_i[x]
\]

where \(\text{Const}[\bar{\varphi}]\) is the conjunction of the construction rules for the bundle, with each \(D_i(t)\) replaced by \(\varphi_i[t]\). The formulas \(\bar{\varphi}\) are the induction-formulas of the delineation.

Examples: \(\mathbb{N}\), i.e. \(\mathcal{A}(0, s)\).

The Intrinsic theory for \(\mathbb{N}\) has for vocabulary the constructors \(0\) and \(s\), and a unary relation identifier \(\mathbb{N}\). Aside from Separation, the axioms are the inductive closure properties, which in a natural-deduction format read

- \(\begin{array}{l}
N(0) \\
N(x) \\
N(sx)
\end{array}\)

- and the inductive delineation rule,

\[
\begin{array}{c}
\{\varphi[z]\} \\
N(t) \\
\varphi[0] \\
\varphi[s(z)]
\end{array}
\rightarrow \varphi(t)
\]

Identifying \(W = \{0, 1\}^*\) with the free algebra generated from the nullary constructor \(\varepsilon\) and the unary 0 and 1, the intrinsic theory \(\text{IT}(W)\) has as vocabulary these constructors and a unary type-identifier \(W\). Here we have the
• inductive closure rules:

\[
\begin{array}{c}
W(\varepsilon) \\
W(0(t)) \\
W(1(t))
\end{array}
\]

• and inductive-delineation:

\[
\begin{array}{c}
\{ \varphi[z] \} \\
\{ \varphi[z] \} \\
W(t) \\
\varphi[\varepsilon] \\
\varphi[0(z)] \\
\varphi[1(z)] \\
\varphi(t)
\end{array}
\]

5.2 Provable typing in intrinsic theories

**Definition 13** A unary program \((P, f)\) is provably of type \(D \rightarrow E\) in a theory \(T\) if \(D(x) \rightarrow E(f(x))\) is provable in \(T\) from the universal closure of the equations in \(P\).\(^{12}\)

For example, consider the doubling function \(\text{dbl}\) over \(\mathbb{N}\) defined by the program \(\text{dbl}(0) = 0, \text{dbl}(s(x)) = s(s(\text{dbl}(x)))\). The following is a proof of \(\mathbb{N}(x) \rightarrow \mathbb{N}(\text{dbl}(x))\), using induction on the predicate \(\varphi[z] \equiv \mathbb{N}(\text{dbl}(z))\). The double-bars indicate the omission of trivial steps.

\[
\begin{array}{c}
\mathbb{N}(0) \\
\text{dbl}(s(z)) = s(s(\text{dbl}(z))) \\
\mathbb{N}(\text{dbl}(0)) \\
\mathbb{N}(\text{dbl}(s(z))) \\
\mathbb{N}(\text{dbl}(x))
\end{array}
\]

In fact, we have:

**Theorem 14** \(^{19}\) \(^{18}\).

1. A function \(f\) over \(\mathbb{N}\) has a program provably of type \(\mathbb{N} \rightarrow \mathbb{N}\) in the intrinsic theory \(\text{IT}(\mathbb{N})\) iff it is a provably-recursive function of Peano’s Arithmetic, i.e. a function definable using primitive-recursion in finite types.

2. \(f\) has a program proved to be of type \(\mathbb{N} \rightarrow \mathbb{N}\) using only formulas in which \(\mathbb{N}\) does not occur negatively iff \(f\) is a primitive-recursive function.

Note that this characterization of the provable functions of PA involves no particular choice of base functions (such as additional and multiplication).\(^{12}\) Universal closure is needed, since the logic here is first-order, rather than equational.
5.3 Intrinsic theories for arbitrary data-systems

Let \( \mathcal{D} \) be a data-system. The intrinsic theory for \( \mathcal{D} \), denoted \( \text{IT}(\mathcal{D}) \), is a first order theory over the vocabulary of \( \mathcal{D} \), whose axioms are the Separation axioms, the inductive construction rules and coinductive deconstruction rules of \( \mathcal{D} \), as well as their duals:

- **Inductive delineation (data-elimination, Induction):** If a vector \( \vec{\varphi}[x] \) of first order formulas satisfies the construction rules for an inductive bundle \( \vec{D} \), then it contains \( \vec{D} \):
  \[
  \text{Const}[\vec{\varphi}] \rightarrow \land_i \forall x \ D_i(x) \rightarrow \varphi_i[x]
  \]
  where \( \text{Const}[\vec{\varphi}] \) is the conjunction of the construction rules for the bundle, with each \( D_i(t) \) replaced by \( \varphi_i[t] \).

- **Coinductive delineation (data-introduction, Coinduction):** If a vector \( \vec{\varphi}[x] \) of first order formulas satisfies the deconstruction rule for a coinductive bundle \( \vec{D} \), then it is contained in \( \vec{D} \):
  \[
  \text{Deconst}[\vec{\varphi}] \rightarrow \land_i \forall x \varphi_i[x] \rightarrow D_i(x)
  \]
  where \( \text{Deconst}[\vec{\varphi}] \) is the conjunction of the deconstruction rules for the bundle, with each \( D_i(t) \) replaced by \( \varphi_i[t] \).

A characterization result, analogous to Theorem 14(2), was proved in [20]: A function over a coinductive type is definable using corecurrence iff its productivity is provable using coinduction for formulas in which type-identifiers do not occur negatively. The proof in [20] is for streams, the general result will be proved elsewhere, as well as an analog Theorem 14(1).

6 Proof theoretic strength

6.1 Innocuous function quantification

Our general intrinsic theories refer to infinite basic objects (coinductive data), in contrast to intrinsic theories for inductive data only, as well as traditional arithmetical theories. However, their deductive machinery does not imply the existence of any particular coinductive object, as would be the case, for example, in the presence of some forms of the Axiom of Choice or of a comprehension principle. Coinductive objects can be specified, of course, by nullary programs, but such programs are treated as axioms, i.e. assumptions, and they specify only a finite number of such objects.

We show next that, as a consequence, any intrinsic theory is interpretable in a formal theory whose proof theoretic strength is no stronger than that of Peano Arithmetic.

We take as starting point the formalism \( \text{PRA} \) of Primitive Recursive Arithmetic, with function identifiers for all primitive recursive functions, and their
defining equations as axioms. In addition, we have the Separation axioms for \( \mathbb{N} \) (as above), and the schema of Induction for all formulas. It is well known that PRA is interpretable in Peano’s Arithmetic (where only addition and multiplication are given as functions with their defining equations).

Let PRA' be PRA augmented with function variables and quantifiers over them, as well as constants (equivalently, free variables) for functionals (i.e. functions from numeric functions to numeric functions.) The set of terms is built by type-correct explicit definition (i.e. composition and application) from number-, function-, and functional-variables, starting with 0 and identifiers for all primitive-recursive functions. The theory has as axiom schema the Principle of Explicit Definition: for each term \( t[\vec{x}, \vec{f}] \) of the extended language, with number variables \( \vec{x} \) and function variables \( \vec{f} \),

\[
\forall \vec{f} \exists g \forall \vec{x} \ g(\vec{x}) = t[\vec{x}, \vec{f}]
\]

There are no further axioms stipulating the existence of additional functions.

The schema of Induction applies now to all formulas in the extended language.

**Lemma 15** The theory PRA' is conservative over PRA. That is, if a formula in the language of PRA is provable in PRA', then it is provable already in PRA.

Consequently, PRA' is no stronger, proof-theoretically, than PA.

**Proof.** The proof is virtually the same as that in [37, Prop. 1.14, p. 453], that E-HA\(^\omega \) is conservative over HA\(^{\omega} \). The use of classical logic, rather than constructive (intuitionistic) logic, makes here no difference, and PRA' is a sub-theory of (the classical counterpart of) E-HA\(^\omega \).

Let us outline the proof idea. It is routine to define PRA-formulas expressing basic syntactic properties, such as the following ones:

- A formula \( F_1[p] \), stating that \( p \) is the code of a program for a total unary function.
- \( Eq[p,q] \), stating that \( p \) and \( q \) codes programs that compute the same function.
- \( F_2[n] \), stating that \( n \) is the code of a type-2 effective operator (i.e. a type-2 element of HEO [36]). That is, \( n \) is the code of a program for a total unary function \( f \), where if \( Eq[p,q] \) then \( f(p) = f(q) \) and \( f(p) \) is the code of a program for a total unary function.

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13See e.g. [35] for details and related discussions.

14I am grateful to Ulrich Kohlenbach for pointing me to that reference.

15See e.g. [14, 30, 36] for details, related notations, and proofs of the closure of the PR functions and predicates under major operations, such as bounded quantification and minimization.
One defines then a mapping $\varphi \mapsto \varphi^*$ from $\text{PRA}^*$-formulas to $\text{PRA}$-formulas, where the function variables in $\varphi$ are interpreted in $\varphi^*$ as codes of programs for total functions, and functional variables as codes of type-2 extensional operators. One shows then that the axioms of $\text{PRA}^*$ become theorems of $\text{PA}$ under the interpretation; consequently, if $\varphi$ is a theorem of $\text{PRA}^*$, then $\varphi^*$ is a theorem of $\text{PA}$. Since $\varphi^*$ is identical to $\varphi$ for a formula $\varphi$ in the vocabulary of $\text{PA}$, the theorem is proved. \hfill \square

The main result of this section is the following proof theoretic calibration of intrinsic theories.

**Theorem 16** (Arithmetic interpretability) *Every intrinsic theory is interpretable in $\text{PRA}^*$.*

### 6.2 Representing data by numeric functions

We posit for each constructor $c$ a distinct numeric code $c^\#$, and some canonical primitive-recursive coding-scheme $\langle \cdots \rangle$ for sequences of natural numbers. More generally, we assume that basic syntactic operations on finite data-terms, such as application and sub-term extraction, are represented by primitive recursive functions. See e.g. [14, 30] for details, related notations, and proofs of the closure of the primitive-recursive functions and predicates under major operations, such as bounded quantification and minimization.

We say that a hyper-term $t \in H_c$ is *represented* by a function $f : \mathbb{N} \to \mathbb{N}$ if $f$ maps addresses $a = \langle a_0 \cdots a_k \rangle \in \mathbb{N}$ to the code $c^\#$ of the constructor $c$ at address $a$ of $t$, whenever such a constructor exists. (We could insist that $f(a)$ be some flag, say 0, when $t$ has no constructor at address $a$, thereby determining $f$ uniquely from $t$; but this is of no use to us, and implies that it is undecidable whether two functions represent the same finite term.)

For example, the finite term $p(e, 0(e))$ is represented by $f$ provided $f(\langle \rangle) = p^\#$, $f(\langle 0 \rangle) = e^\#$, $f(\langle 1 \rangle) = 0^\#$, and $f(\langle 1, 0 \rangle) = e^\#$. Similarly, the infinite 01-word $(0, 1)^\omega = 0101\cdots$ is represented by $f$ provided $f(\langle 0^{2n} \rangle) = 0^\#$ and $f(\langle 0^{2n+1} \rangle) = 1^\#$.

It follows that an $r$-ary constructor $c$ is represented by the functional $c^\circ$ defined by

\[
\begin{align*}
  c^\circ(f_1, \ldots, f_r)(0) &= c^\# \\
  c^\circ(f_1, \ldots, f_r)((i) * a) &= f_i(a) \quad i = 1..r \\
  c^\circ(f_1, \ldots, f_r)(n) &= 0 \quad (\text{arbitrary}) \text{ for other } n's
\end{align*}
\]

### 6.3 Representing types by formulas of $\text{PRA}^*$

Consider a purely co-inductive data-system. One can state that a hyper-term $t$ is of time $D$ be asserting the existence of a correct type decoration of the nodes of $t$, with the root assigned type $D$. The correctness of the decoration can be expressed by a single numeric $\forall$, but we would need to have an existential function-quantifier to state, in the first place, the existence of the decoration.
We show here that no such function quantification is needed. Referring to Definition 3, we have:

**Theorem 17** $\Sigma_n$ types are defined by $\Sigma^0_n$ formulas, and $\Pi_n$ types are defined by $\Pi^0_n$ formulas.

**Proof.** The proof is by induction on $n \geq 1$. If a type $D$ is $\Sigma_n$, i.e. is in a bundle defined inductively from $\Pi_{n-1}$ types (where we take $\Pi_0$ to be empty), then a hyper-term $t$ has type $D$ iff there is a finite deduction establishing $D(t)$ from typing-statements $E(t')$, with $E$ a type of lower rank, and $t' = \Pi(t)$ for some deepDestructor $\Pi$. By IH the premises $E(t')$ are all defined by $\Pi^0_{n-1}$ formulas $E^0[t']$, and the correctness of the finite type-derivation is clearly a primitive-recursive predicate. Thus $D$ is definable by existential quantification over $\Pi^0_{n-1}$ formulas, i.e. by a $\Sigma^0_n$ formula $D^0$.

Consider now a $\Pi_n$ type $D$. We shall concretize our general argument with the following running example, where $D$ and $E$ are defined by a common coinduction, and $T$ is a (previously defined) $\Sigma_{n-1}$ type:

$$
D(x) \rightarrow \exists y, z \ x = p(y, z) \land D(y) \land E(z)
\lor \exists y, z \ x = p(y, z) \land T(y) \land D(z)
\lor \exists y \ x = f(y) \land E(y)
$$

$$
E(x) \rightarrow \exists y, z \ x = p(y, z) \land E(y) \land D(z)
$$

The decomposition rule for each type has a number of constructor-statements as choices, in our example $D$ has three and $E$ one. Each choice determines a main constructor, and types for the component. The spelling out of $D$ into three options can be represented graphically:

![Diagram]

- $T$, $D$, $E$, $D_f$
- $D_{p}$, $D_{p}$, $D_{f}$
- $D$

We continue an expansion of all typing options for a hyper-term in $D$. That is, we construct a tree $T_D$, where a node of height $h$ consists of a finite tree (of height $\leq h$), with types at the leaves, and a pair of a type and a constructor at internal nodes. Each such node represents a possible partial typing of a hyper-term of type $D$. The children of each such node $N$ are the local expansions of the lowermost-leftmost unexpanded leaf, with a type in the bundle considered. (E.g., in our running example, the leaves with type $T$ are not expandable, and are left alone.) Put differently, the leaves are expanded in a breadth-first order.
(We refrain from expanding all expandable leaves at each step, because the resulting tree, albeit finitely-branching, would have unbounded degree.)

A few nodes of height 3 are given here:

Note that the tree $T_D$ is primitive recursive, i.e. there is a primitive-recursive function that, for every address $\alpha$, gives (a numeric code for) the node at address $\alpha$.

A hyper-term $t \in H_C$ is consistent with a node $N$ as above if its constructor-decomposition is consistent with the tree of constructors in $N$, and for every deep-destructor $\Pi$, if $N$ has at address $\Pi$ a type $T$ of lower rank, then $\Pi(t)$ has
type $T$. The consistency of a hyper-term $t$ with a node $N$ is thus definable by a $\Sigma^0_{n-1}$ formula.

A hyper-term has type $D$ iff there is an unbounded (i.e. infinite or terminating) branch of the tree $T$ above, every node of which is consistent with $t$. The existence of such a branch is equivalent, by the Weak König’s Lemma, to the existence, for every $h > 0$, of a node $N$ of height $h$ in $T_D$, which is consistent with $t$. Since consistency of $t$ with $N$ is definable by a $\Sigma^0_{n-1}$ formula, this property is $\Pi^0_n$.

6.4 Interpretation of terms

For each unary constructor $c$ let $\hat{c}$ denote the PR functional that maps functions $f$ coding a hyper-term $t$ to the code $\hat{c}(f)$ of the term-tree obtained by rooting $t$ from the symbol $c$, i.e.

$$\hat{c}(f)(a) = \text{if } a = () \text{ then } c^t \text{ else } f((0) * a)$$

Recall that we posit the presence in the vocabulary of $\mathbb{PRA}^*$ of identifiers for all PR functionals, in particular $\hat{c}$. The definition of $\hat{c}$ for $c$ of arity $\neq 1$ is similar.

Next, we define a mapping $t \mapsto t^0$ from terms of $T$ to terms of $\mathbb{PRA}^*$. We posit that the identifiers of $\mathbb{PRA}^*$ for PR functions and functionals are disjoint from the program identifiers of intrinsic theories.

- For a variable $x$ of $T$ (intended to range over hyper-terms) we let $x^0$ be a fresh function variable of $\mathbb{PRA}^*$ (intended to range over functions coding hyper-terms).
- For a constructor $c$ of arity $r \geq 0$, let $(c(t_1 \ldots t_r))^0 \equiv \hat{c}(t_1^0 \ldots t_r^0)$.
- For a program-function $f$ of arity $r \geq 0$ (i.e. a free variable denoting a function between hyper-terms), we let $(f(t_1 \ldots t_r))^0 \equiv f(t_1^0 \ldots t_r^0)$, where $f^0$ is a fresh functional variable of $\mathbb{PRA}^*$, of arity $r$.

6.5 Interpretation of formulas

Finally, we define a mapping $\varphi \mapsto \varphi^0$ from formulas of $T$ (possibly with program-functions) to formulas of $\mathbb{PRA}^*$. Let $HTrm[g]$ be a PR formula stating that the function $g$ codes a hyper-term.

- $(t = q)^0$ is $t^0 = q^0$.
- $(D(t))^0$ is $D^0[t^0]$, were $D^0$ is the arithmetic formula (possibly with free function and functional variables) that defines $D$ (Theorem 17).
- $(\varphi \land \psi)^0$ is $\varphi^0 \land \psi^0$, and similarly for the other connectives.
- $(\forall x \varphi)^0$ is $\forall x^0 HTrm[x^0] \rightarrow \varphi^0$; $(\exists x \varphi)^0$ is $\exists x^0 HTrm[x^0] \land \varphi^0$. 24
Proposition 18 The mapping $\varphi \mapsto \varphi^o$ is semantically faithful; that is, for each formula $\varphi[\vec{x}, \vec{g}]$ of $T$, with free object variables among $\vec{x}$ and program-variables among $\vec{g}$,

$$A, [\vec{x} \leftarrow \vec{r}, \vec{g} \leftarrow \vec{f}] \models \varphi$$

iff

$$N, [\vec{x}^o \leftarrow \vec{r}^o, \vec{g}^o \leftarrow \vec{f}^o] \models \varphi^o$$

where for a hyper-term $t$ we write $t^o$ for the function coding $t$, and for a function $f$ between hyper-terms we let $f^o$ be the corresponding functional between numeric functions.

In particular, if $\varphi$ is a closed formula of $T$, then $\varphi$ is true in the canonical model $A$ of the data-system iff $\varphi^o$ is true in the standard model of $PRA$.

The proof is straightforward by structural induction on $\varphi$. $\square$

6.6 The Interpretability Theorem

We finally show that the interpretation is proof-theoretically faithful.

Theorem 19 If a closed formula $\varphi$ is provable in the intrinsic theory $T$, then $\varphi^o$ is provable in $PRA^\ast$.

More generally: if a formula $\varphi[\vec{x}]$, with free variables among $\vec{x}$, is provable in $T$, then $HTrm[\vec{x}] \rightarrow \varphi^o[\vec{x}]$ is provable in $PRA^\ast$.

Proof. The proof proceeds by structural induction on derivations.

• Logic. The propositional and quantifier inferences are trivially preserved by the interpretation.

• Separation. The case of the Separation Axioms is immediate by the definition of the interpretation.

• Inductive construction. Consider the construction axiom (2) for an inductive bundle $D_i$,

$$\psi_1 \lor \cdots \lor \psi_k \rightarrow D_i(x)$$

where each $\psi_i$ is of the form

$$\exists y_1 \cdots y_r \ x = c(\vec{y}) \land Q_1(y_1) \land \cdots \land Q_r(y_r)$$

The interpretation of (2) is

$$HTrm[\vec{x}^o] \rightarrow \psi_1^o \lor \cdots \lor \psi_k^o \rightarrow D_i^o[x^o]$$

with

$$\psi_i^o \ of \ the \ form \ \exists y_1^o \cdots y_r^o \in HTrm \ x^o = c^o(\vec{y}) \land Q_1^o(y_1) \land \cdots \land Q_r^o(y_r)$$

25
Recall (from Theorem 17) that $D_0[x^o]$ states the existence of a finite type derivation $\Delta$ of $D[x^o]$ from statements of the form $E[\Pi(x^o)]$ with $E$ of lower rank and $\Pi$ a deep-destroyer. Thus one of the decompositions $\psi_i^o$ must be true for $x^o$, with the correctness of the $Q_j^o[y_j]$ true by induction on the height of $\Delta$.

- **Induction.** Given an inductive bundle $\tilde{D}$, the interpretation of $\tilde{D}$-induction for formulas $\bar{\phi}$ is

\[
D_0^o[x^o] \rightarrow ((\text{Const}^o[\bar{x}^o] \land HTrm[x^o]) \rightarrow \varphi_i^o[x^o]) \quad (7)
\]

Recall that $D_0^o[x^o]$ states the existence of a finite derivation $\Delta$ of $D(x^o)$ from formulas of the form $E[\Pi(x^o)]$, where $E$ is of lower rank than $D$, and $\Pi$ is a deep-destroyer. The conclusion of (7) is straightforward by cumulative (i.e. course-of-value) induction on the height of $\Delta$.

- **Coinductive deconstruction.** A deconstruction axiom $[5]$ for a coinductive bundle $\tilde{D}$ has the interpretation

\[
D_0^o[x^o] \rightarrow \psi_1^o \lor \cdots \lor \psi_k^o
\]

where the formulas $\psi_i$ are as above. Recall that $D_0^o[x^o]$ states the existence of a finite derivation $\Delta$ of $D(x^o)$ from formulas of the form $E[\Pi(x^o)]$, where $E$ is of lower rank than $D$, and $\Pi$ is a deep-destroyer. The conclusion of (7) is straightforward by cumulative (i.e. course-of-value) induction on the height of $\Delta$.

- **Coinduction.** Given a coinductive bundle $\tilde{D}$, the interpretation of $\tilde{D}$-coinduction for formulas $\bar{\phi}$ is

\[
\varphi_i^o[x^o] \rightarrow ((\text{Deconst}^o[\bar{x}^o] \land HTrm[x^o]) \rightarrow D_0^o[x^o]) \quad (8)
\]

The conclusion of (8) is established by showing that the tree $T_D$ (see the proof of Theorem 17) has a node consistent with $x^o$ at any given height $h$. This follows outright from the assumptions of (8) by induction on $h$.

\[\square\]

\[^{16}\text{Note that we do not use here Weak K"onig’s Lemma, as we do not assert the existence of an infinite branch as a consequence.}\]
7 Applications and further developments

Intrinsic theories provide a minimalist framework for reasoning about data and computation. The benefits were already evident when dealing with inductive data only, including a characterization of the provable functions of Peano’s Arithmetic without singling out any functions beyond the constructors, a particularly simple proof of Kreisel’s Theorem that classical arithmetic is \( \Pi^0_2 \)-conservative over intuitionistic arithmetic [18], and a particularly simple characterization of the primitive-recursive functions [19]. The latter application guided a dual characterization of the primitive corecursive functions in terms of intrinsic theories with positive coinduction [20].

Intrinsic theories are also related to type theories, via Curry-Howard morphisms, providing an attractive framework for extraction of computational contents from proofs, using functional interpretations and realizability methods. The natural extension of the framework to coinductive methods, described here, suggests new directions in extracting such methods for coinductive data as well.

Finally, intrinsic theories are naturally amenable to ramification, leading to a transparent Curry-Howard link with ramified recurrence [2] [16] as well as ramified corecurrence [27].

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