**Decoherence and Copenhagen Interpretation:**

*A Scenario*

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Abstract

In this paper we give a reasonable explanation (not proof) to the Copenhagen interpretation of Quantum Mechanics from the view point of decoherence theory.

Mathematical physicists with strong mission must prove the Copenhagen interpretation at all costs.

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1 Introduction

When we start studying Quantum Mechanics the most difficult part to understand is the so–called Copenhagen interpretation. Usually beginners skip over this part, which is a wise choice in a certain sense. However, some researchers feel guilty about skipping over this.

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In this paper we try to give a proof to it from the viewpoint of decoherence theory. Namely, we embed it into the theory of decoherence and solve a master equation based on density matrix (not wave function) exactly.

We will perform this by both incorporating the results in [2], [3] and making the idea in [1] clearer. The method is of course not complete, but some researchers may feel relieved. To the best of our knowledge this is the finest method up to the present.

2 Principles of Quantum Mechanics

In order to set the stage and to introduce proper notation, let us start with a system of principles of Quantum Mechanics (QM in the following for simplicity). See for example [4], [5], [6] and [7]. That is,

System of Principles of QM

1. Superposition Principle

If $|a\rangle$ and $|b\rangle$ are physical states then their superposition $\alpha|a\rangle + \beta|b\rangle$ is also a physical state where $\alpha$ and $\beta$ are complex numbers.

2. Schrödinger Equation and Evolution

Time evolution of a physical state proceeds like

$$|\Psi\rangle \rightarrow U(t)|\Psi\rangle$$

where $U(t)$ is the unitary evolution operator ($U^\dagger(t)U(t) = U(t)U^\dagger(t) = 1$ and $U(0) = 1$) determined by a Schrödinger Equation.

3. Copenhagen Interpretation

Let $a$ and $b$ be the eigenvalues of an observable $Q$, and $|a\rangle$ and $|b\rangle$ be the normalized eigenstates corresponding to $a$ and $b$. When a state is a superposition $\alpha|a\rangle + \beta|b\rangle$ and we

\[\text{There are some researchers who are against this terminology, see for example [7]. However, I don't agree with them because the terminology is nowadays very popular in the world}\]
observe the observable $Q$ the state collapses like

$$\alpha|a\rangle + \beta|b\rangle \rightarrow |a\rangle \quad \text{or} \quad \alpha|a\rangle + \beta|b\rangle \rightarrow |b\rangle$$

where their collapsing probabilities are $|\alpha|^2$ and $|\beta|^2$ respectively ($|\alpha|^2 + |\beta|^2 = 1$).

This is called the collapse of the wave function and the probabilistic interpretation.

4. Many Particle State and Tensor Product

A multiparticle state can be constructed by the superposition of the Kronecker products of one particle states, which are called the tensor products. For example,

$$\alpha|a\rangle \otimes |a\rangle + \beta|b\rangle \otimes |b\rangle \equiv \alpha|a, a\rangle + \beta|b, b\rangle$$

is a two particle state.

The target of this paper is to give a proof to the Copenhagen interpretation, so we give a symbolic figure of it for the latter convenience (we take $|0\rangle$ and $|1\rangle$ in place of $|a\rangle$ and $|b\rangle$ in the following).
Here is an important comment. Beginners of QM might think that a quantum state created by an experiment would undergo the unitary time evolution (U) forever.

This is nothing but an illusion because the quantum state is in an environment (a kind of heat bath) and the interaction with it will disturb the quantum state. For example, readers should imagine an oscillator on the desk.

In order to understand QM deeply readers should take decoherence (: interaction with environment) into consideration correctly. For this topic see for example [8].

In this paper we try to prove the Copenhagen interpretation from the view point of decoherence 2. Namely, we consider that measurement is a kind of decoherence forced.

For the purpose we introduce a decoherence time $t_D$, which is not necessarily definite. The quantum coherence of our system will collapse completely when $t > t_D$. Therefore, we

\[ |0\rangle (\text{probability } |\alpha|^2) \]

\[ |1\rangle (\text{probability } |\beta|^2) \]

Figure I: Image of the Copenhagen interpretation

\[ |\alpha\rangle + \beta|1\rangle \]

\[ |0\rangle \]

\[ |\alpha\rangle |0\rangle + \beta|1\rangle \]

Detector

\[ t = 0 \]

\[ t = t_0 \]

\[ \text{time} \]
must finish measuring the system within \( t_D \) \((t_0 \ll t_D)\).

\[\begin{array}{c}
0 & \bullet & t_0 & \bullet & t_D & \bullet & \text{decoherence} & \bullet & \text{time}
\end{array}\]

Figure II : Decoherence time

3 “Proof” of the Copenhagen Interpretation

In this section we try to give a proof to the Copenhagen Interpretation. We perform this by embedding it into decoherence theory. The method developed in the following is based on the paper \[3\].

3.1 General Theory

We consider an atom flying as in the figure of the preceding section and treat a two level system of the atom in the following, see for example \[9\]. First of all let us prepare some notations from Quantum Optics. Since we treat the two level system of the atom the target space is \(\mathbb{C}^2 = \text{Vect}_{\mathbb{C}}(|0\rangle, |1\rangle)\) with bases

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then Pauli matrices \(\{\sigma_1, \sigma_2, \sigma_3\}\) with the identity \(1_2\)

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

act on the space. For

\[
\sigma_+ \equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
it is easy to see
\[
\sigma_+\sigma_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Here we may assume that the initial state is \( |0\rangle \) at \( t = 0 \) and the intermediate state is \( \alpha|0\rangle + \beta|1\rangle \) for \( 0 < t < t_0 \) and the last state is the one detected at \( t = t_0 \), see the figure in the preceding section once more.

For the initial time \( t = 0 \) we can assume that the Hamiltonian is a diagonal form
\[
H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}
\]
(1)
where \( E_0 \) and \( E_1 \) are the eigenvalues \( (E_0 < E_1 \) for simplicity) of the atom. It is easy to see
\[
H_0|0\rangle = E_0|0\rangle, \quad H_0|1\rangle = E_1|1\rangle.
\]

By subjecting a laser field to the atom (at \( t = 0_+ \)) we can take the Hamiltonian to be
\[
H = U(\alpha, \beta) \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} U(\alpha, \beta)^\dagger
\]
\[
= \begin{pmatrix} |\alpha|^2E_0 + |\beta|^2E_1 & \alpha\bar{\beta}(E_0 - E_1) \\ \bar{\alpha}\beta(E_0 - E_1) & |\beta|^2E_0 + |\alpha|^2E_1 \end{pmatrix}
\]
(2)
for the intermediate time \( 0 < t < t_0 \), where \( U(\alpha, \beta) \) is a special unitary matrix given by
\[
U = U(\alpha, \beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad (|\alpha|^2 + |\beta|^2 = 1).
\]
(3)
In this case, it is easy to see
\[
\alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \bar{\alpha}|0\rangle + \bar{\beta}|1\rangle = \begin{pmatrix} \bar{\beta} \\ \bar{\alpha} \end{pmatrix}
\]
and
\[
H(\alpha|0\rangle + \beta|1\rangle) = E_0(\alpha|0\rangle + \beta|1\rangle), \quad H(-\bar{\beta}|0\rangle + \bar{\alpha}|1\rangle) = E_1(-\bar{\beta}|0\rangle + \bar{\alpha}|1\rangle).
\]
Note that \( H \) and \( H_0 \) are of course hermitian matrices (\( H = H^\dagger, H_0 = H_0^\dagger \)).
Since
\[ (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)\dagger = |\alpha|^2|0\rangle\langle 0| + \alpha\beta|0\rangle\langle 1| + \alpha\bar{\beta}|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| \]
the Copenhagen interpretation may be written as collapsing
\[ (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)\dagger \rightarrow |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|. \]

To treat decoherence in a correct manner we must change models based on from a pure state to a density matrix. The general definition of density matrix \(\rho\) is given by both
\[ \rho = \rho(t) \] as
\[ \rho = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \quad (a = \bar{a}, \ d = \bar{d}, \ a + d = 1). \]

The general form of master equation (10), (11) is well-known to be
\[ \frac{d}{dt}\rho = -i[H, \rho] + D\rho \quad (\Leftrightarrow \hbar = 1 \text{ for simplicity}) \]
where
\[ D\rho = \mu \left( \sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_- \right) + \nu \left( \sigma_+\rho\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho - \frac{1}{2}\rho\sigma_-\sigma_+ \right) \]
and \(\mu, \nu > 0\). Note that \(\mu\) and \(\nu\) are important constants determined later.

We must solve the equation. If we write \(H\) in (2) as
\[ H = \begin{pmatrix} h & k \\ \bar{k} & l \end{pmatrix} \quad (h, \ l \in \mathbb{R}, \ k \in \mathbb{C}) \]
for simplicity, then the master equation above can be rewritten as
\[ \frac{d}{dt} \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix} = \begin{pmatrix} -\mu & ik & -ik & \nu \\ ik & i(l - h) - \frac{\mu + \nu}{2} & 0 & -ik \\ -i\bar{k} & 0 & -i(l - h) - \frac{\mu + \nu}{2} & i\bar{k} \\ \mu & -i\bar{k} & ik & -\nu \end{pmatrix} \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix}. \]

\[ ^3 \text{In standard textbooks of QM decoherence theory is usually not contained, so it may be difficult to} \]
beginners (young students). See for example [12] or [13].
The derivation is left to readers.

Note and set

\[
\begin{pmatrix}
-\mu & ik & -ik & \nu \\
\ i & (l - h) - \frac{\mu + \nu}{2} & 0 & -ik \\
-ik & 0 & -i(l - h) - \frac{\mu + \nu}{2} & ik \\
\mu & -ik & ik & -\nu
\end{pmatrix}
\]

\[= \begin{pmatrix}
0 & ik & -ik & 0 \\
\ i & (l - h) & 0 & -ik \\
-ik & 0 & -i(l - h) & ik \\
0 & -ik & ik & 0
\end{pmatrix}
\begin{pmatrix}
-\mu & 0 & 0 & \nu \\
0 & -\frac{\mu + \nu}{2} & 0 & 0 \\
0 & 0 & -\frac{\mu + \nu}{2} & 0 \\
\mu & 0 & 0 & -\nu
\end{pmatrix}
\]

\[\equiv \hat{H} + \hat{D}.
\]

The general solution of (7) is given by

\[
\begin{pmatrix}
a(t) \\
b(t) \\
\bar{b}(t) \\
d(t)
\end{pmatrix}
= e^{t(\hat{H} + \hat{D})}
\begin{pmatrix}
a(0) \\
b(0) \\
\bar{b}(0) \\
d(0)
\end{pmatrix}
\]

(8)

However, it is not easy to calculate the term \(e^{t(\hat{H} + \hat{D})}\) exactly, so we use some approximation. In general, the Zassenhaus formula (see for example [14], [15]) is convenient

**Zassenhaus Formula** For operators (or square matrices) \(A\) and \(B\) we have an expansion

\[
e^{t(A+B)} = \ldots e^{-\frac{t^2}{2}([A,B],B) + [[A,B],A]} e^{\frac{t^2}{4}[A,B]} e^{tB} e^{tA}.
\]

The proof is easy. Up to \(O(t^2)\) we obtain
\[
e^{\frac{t^2}{2}[A,B]}e^{tB}e^{tA} = \left( 1 + \frac{t^2}{2}[A,B] \right) \left( 1 + tB + \frac{t^2}{2}B^2 \right) \left( 1 + tA + \frac{t^2}{2}A^2 \right)
\]
\[
= \left( 1 + \frac{t^2}{2}(AB - BA) \right) \left( 1 + t(A + B) + \frac{t^2}{2}(A^2 + 2BA + B^2) \right)
\]
\[
= 1 + t(A + B) + \frac{t^2}{2}(A^2 + 2BA + B^2 + AB - BA)
\]
\[
= 1 + t(A + B) + \frac{t^2}{2}(A^2 + AB + BA + B^2)
\]
\[
= 1 + t(A + B) + \frac{t^2}{2}(A + B)^2
\]
\[
= e^{t(A+B)}.
\]

To check the equation up to \(O(t^3)\) is left to readers, which is a good exercise for undergraduates.

Note that the formula is a bit different from that of [14]. Zassenhaus formula is a kind of converse of the Baker-Campbell-Hausdorff formula

\[
e^Ae^B = e^{A+B+\frac{t^2}{2}[A,B]+\frac{t^4}{4!}([A,B],[B,A])+\cdots},
\]

where \(t = 1\) for simplicity.

### 3.2 Measurement (= Decoherence Forced)

The decoherence time \(t_D\) is in general very short and the measurement must be performed within the time \((0 < t_0 < t_D)\). From this the essential part of \(e^{t(\hat{H}+\hat{D})}\) is

\[
e^{t(\hat{H}+\hat{D})} \approx e^{t\hat{D}}e^{t\hat{H}}
\]

for \(0 < t < t_0\).

To embed the measurement (; decoherence forced) into decoherence theory means that we treat the approximate solution

\[
\begin{pmatrix}
a(t) \\
b(t) \\
\bar{b}(t) \\
d(t)
\end{pmatrix}
\approx
\begin{pmatrix}
a(0) \\
b(0) \\
\bar{b}(0) \\
d(0)
\end{pmatrix} e^{t\hat{D}} e^{t\hat{H}}
\]

\((t \geq 0)\) (10)
instead of treating the full solution (8). See the following figure.

Figure III : Embedding of the measurement into decoherence theory

First, let us calculate \( e^{t \hat{D}} \). For the purpose we set

\[
K = \begin{pmatrix}
-\mu & \nu \\
\mu & -\nu
\end{pmatrix}
\]

and calculate \( e^{tK} \). The eigenvalues of \( K \) are \( \{0, -(\mu + \nu)\} \) and corresponding eigenvectors (not normalized) are

\[
0 \leftrightarrow \begin{pmatrix} \nu \\ \mu \end{pmatrix}, \quad -(\mu + \nu) \leftrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

If we define the matrix

\[
O = \begin{pmatrix}
\nu & 1 \\
\mu & -1
\end{pmatrix}
\implies O^{-1} = \frac{1}{\mu + \nu} \begin{pmatrix} 1 & 1 \\ \mu & -\nu \end{pmatrix}
\]

then it is easy to see

\[
K = O \begin{pmatrix} 0 \\ -(\mu + \nu) \end{pmatrix} O^{-1}
\]

and

\[
e^{tK} = O \begin{pmatrix} 1 \\ e^{-t(\mu + \nu)} \end{pmatrix} O^{-1} = \frac{1}{\mu + \nu} \begin{pmatrix}
\nu + \mu e^{-t(\mu + \nu)} & \nu - \nu e^{-t(\mu + \nu)} \\
\mu - \mu e^{-t(\mu + \nu)} & \mu + \nu e^{-t(\mu + \nu)}
\end{pmatrix}.
\]

Therefore, we have

\[
e^{t \hat{D}} \approx \frac{1}{\mu + \nu} \begin{pmatrix}
\nu & 0 & 0 & \nu \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\mu & 0 & 0 & \mu
\end{pmatrix} \tag{11}
\]
if $t$ is large enough ($t \gg 1/(\mu + \nu)$).

Next, let us calculate $e^{i\hat{H}}$. Since we need some properties of tensor product in the following see for example [15]. We can write the equation as

$$\hat{H} = -i \left( H \otimes 1_2 - 1_2 \otimes H^T \right).$$

In fact,

$$\hat{H} = -i \left\{ \begin{pmatrix} h & k \\ \bar{k} & l \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} h & \bar{k} \\ k & l \end{pmatrix} \right\}$$

$$= -i \left\{ \begin{pmatrix} h & 0 & k & 0 \\ 0 & h & 0 & k \\ \bar{k} & 0 & l & 0 \\ 0 & \bar{k} & 0 & l \end{pmatrix} - \begin{pmatrix} h & \bar{k} & 0 & 0 \\ k & l & 0 & 0 \\ 0 & 0 & h & \bar{k} \\ 0 & 0 & k & l \end{pmatrix} \right\}$$

$$= -i \begin{pmatrix} 0 & -\bar{k} & k & 0 \\ -k & -(l - h) & 0 & k \\ \bar{k} & 0 & l - h & -\bar{k} \\ 0 & \bar{k} & -k & 0 \end{pmatrix}.$$

It is well-known that

$$e^{i\hat{H}} = e^{-it(H \otimes 1_2 - 1_2 \otimes H^T)} = e^{-itH \otimes 1_2} e^{it1_2 \otimes H^T} = (e^{-itH} \otimes 1_2) \left(1_2 \otimes e^{itH^T}\right) = e^{-itH} \otimes e^{itH^T}.$$ Since

$$H = U \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} U^\dagger$$

we have

$$e^{-itH} = U \begin{pmatrix} e^{-itE_0} & 0 \\ 0 & e^{-itE_1} \end{pmatrix} U^\dagger$$

11
and
\[
e^{it\hat{H}} = \left\{ U \begin{pmatrix} e^{-itE_0} & 0 \\ 0 & e^{-itE_1} \end{pmatrix} U^\dagger \right\} \otimes \left\{ U \begin{pmatrix} e^{itE_0} & 0 \\ 0 & e^{itE_1} \end{pmatrix} U^\dagger \right\}^T \\
= \left\{ U \begin{pmatrix} e^{-itE_0} & 0 \\ 0 & e^{-itE_1} \end{pmatrix} U^\dagger \right\} \otimes \left\{ (U^\dagger)^T \begin{pmatrix} e^{itE_0} & 0 \\ 0 & e^{itE_1} \end{pmatrix} (U^\dagger)^T \right\}
\]
\[
= U \otimes (U^\dagger)^T \begin{pmatrix} e^{-it(E_1 - E_0)} & 0 \\ 0 & e^{it(E_1 - E_0)} \end{pmatrix} (U \otimes (U^\dagger)^T)^\dagger.
\]

Here we have used well–known formulas on tensor product
\[
(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2, \quad (A_1 \otimes B_1)(A_2 \otimes B_2)(A_3 \otimes B_3) = A_1A_2A_3 \otimes B_1B_2B_3,
\]
\[
(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger, \quad (A \otimes B)^T = A^T \otimes B^T,
\]
see for example [15].

Since
\[
U = U(\alpha, \beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \bar{\alpha} & \beta \end{pmatrix} \quad (|\alpha|^2 + |\beta|^2 = 1)
\]
from (3) we have
\[
U \otimes (U^\dagger)^T =
\begin{pmatrix}
|\alpha|^2 & -\alpha\beta & -\bar{\alpha}\beta & |\beta|^2 \\
\alpha\bar{\beta} & \alpha^2 & -\bar{\beta}^2 & -\alpha\bar{\beta} \\
\bar{\alpha}\beta & -\beta^2 & \bar{\alpha}^2 & -\bar{\alpha}\beta \\
|\beta|^2 & \alpha\beta & \bar{\alpha}\beta & |\alpha|^2
\end{pmatrix}
\]
and, by setting \( J = e^{\mu(E_1 - E_0)} \) for simplicity,

\[
e^{i\tilde{H}} = U \otimes (U^\dagger)^T \begin{pmatrix} 1 & J \\ J^{-1} & 1 \end{pmatrix} (U \otimes (U^\dagger)^T)^\dagger
\]

\[
= \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ * & * & * & * \\ * & * & * & * \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}
\]

where

\[
c_{11} = |\alpha|^4 + (J + J^{-1})|\alpha|^2|\beta|^2 + |\beta|^4,
\]

\[
c_{12} = (|\alpha|^2 - |\alpha|^2 J + |\beta|^2 J^{-1} - |\beta|^2) \bar{\alpha} \beta,
\]

\[
c_{13} = (|\alpha|^2 + |\beta|^2 J - |\alpha|^2 J^{-1} - |\beta|^2) \alpha \bar{\beta},
\]

\[
c_{14} = (2 - J - J^{-1})|\alpha|^2|\beta|^2
\]

and

\[
c_{41} = (2 - J - J^{-1})|\alpha|^2|\beta|^2,
\]

\[
c_{42} = (|\beta|^2 + |\alpha|^2 J - |\beta|^2 J^{-1} - |\alpha|^2) \bar{\alpha} \beta,
\]

\[
c_{43} = (|\beta|^2 - |\beta|^2 J + |\alpha|^2 J^{-1} - |\alpha|^2) \alpha \bar{\beta},
\]

\[
c_{44} = |\beta|^4 + (J + J^{-1})|\alpha|^2|\beta|^2 + |\alpha|^4.
\]

Note that *’s in the matrix are elements not used in the latter. The derivation is left to readers.

Here, we list very important relations among \( \{\alpha\} \) (coming from \( |\alpha|^2 + |\beta|^2 = 1 \))

\[
c_{11} + c_{41} = 1, \quad c_{12} + c_{42} = 0, \quad c_{13} + c_{43} = 0, \quad c_{14} + c_{44} = 1. \quad (13)
\]
Therefore, from (10), (11), (12) and (13) we obtain

\[
\begin{pmatrix}
  a(t) \\
  b(t) \\
  \bar{b}(t) \\
  d(t)
\end{pmatrix}
\approx
\frac{1}{\mu + \nu}
\begin{pmatrix}
  \nu & 0 & 0 & \nu \\
  0 & 0 & 0 & 0 \\
  \mu & 0 & 0 & \mu \\
  \nu & 0 & 0 & \nu \\
  0 & 0 & 0 & 0 \\
  \mu & 0 & 0 & \mu \\
\end{pmatrix}
\begin{pmatrix}
  c_{11} & c_{12} & c_{13} & c_{14} \\
  * & * & * & * \\
  c_{41} & c_{42} & c_{43} & c_{44} \\
\end{pmatrix}
\begin{pmatrix}
  a(0) \\
  b(0) \\
  \bar{b}(0) \\
  d(0)
\end{pmatrix}
\]

\[= \frac{1}{\mu + \nu}
\begin{pmatrix}
  \nu & 0 & 0 & \nu \\
  0 & 0 & 0 & 0 \\
  \mu & 0 & 0 & \mu \\
\end{pmatrix}
\begin{pmatrix}
  a(0) \\
  b(0) \\
  \bar{b}(0) \\
  d(0)
\end{pmatrix}
\]  

(14)

for \( t \gg 1/(\mu + \nu) \).

From (4)

\[
\rho(t) = \begin{pmatrix}
  a(t) & b(t) \\
  \bar{b}(t) & d(t)
\end{pmatrix}
\]

we have

\[
\rho(\infty) = \frac{1}{\mu + \nu}
\begin{pmatrix}
  \nu (a(0) + d(0)) & 0 \\
  0 & \mu (a(0) + d(0))
\end{pmatrix}
\]

The initial density matrix

\[
\rho(0) = |0\rangle\langle 0| = \begin{pmatrix}
  1 & 0 \\
  0 & 0
\end{pmatrix}
\equiv
\begin{pmatrix}
  a(0) & b(0) \\
  \bar{b}(0) & d(0)
\end{pmatrix}
\]

gives

\[
\rho(\infty) = \frac{1}{\mu + \nu}
\begin{pmatrix}
  \nu & 0 \\
  0 & \mu
\end{pmatrix}
= \frac{\nu}{\mu + \nu}|0\rangle\langle 0| + \frac{\mu}{\mu + \nu}|1\rangle\langle 1|.
\]

(15)

Since

\[
\frac{\nu}{\mu + \nu}, \frac{\mu}{\mu + \nu} > 0 \quad \text{and} \quad \frac{\nu}{\mu + \nu} + \frac{\mu}{\mu + \nu} = 1
\]

the structure of probability comes out in a natural way.

Moreover, if we can choose \( \mu \) and \( \nu \) as

\[
\frac{\nu}{\mu + \nu} = |\alpha|^2 \quad \text{and} \quad \frac{\mu}{\mu + \nu} = |\beta|^2 \quad (\Rightarrow |\alpha|^2 + |\beta|^2 = 1)
\]

(16)
from the starting point then we have the final form

\[ \rho(\infty) = |\alpha|^2 |0\> \langle 0| + |\beta|^2 |1\> \langle 1|. \]  

(17)

We can interpret this equation as a mathematical expression of the Copenhagen interpretation: “when a state is superposition \( \alpha |0\> + \beta |1\> \) and we observe the observable \( Q \) the state collapses like \( \alpha |0\> + \beta |1\> \rightarrow |0\> \) (probability \( |\alpha|^2 \)) or \( \alpha |0\> + \beta |1\> \rightarrow |1\> \) (probability \( |\beta|^2 \)).” This finishes the “proof” of the Copenhagen interpretation.

The remaining problem is

**Problem** Why are \( \frac{\nu}{\mu + \nu} = |\alpha|^2 \) and \( \frac{\mu}{\mu + \nu} = |\beta|^2 \) identified when measuring the system?

It may be difficult to prove the problem without introducing another theory.

A comment is in order. If we use another approximation

\[ e^{t(\hat{H} + \hat{D})} \approx e^{t\hat{D}} e^{t\hat{H}} \implies e^{t(\hat{H} + \hat{D})} \approx e^{t^2 [\hat{H}, \hat{D}]} e^{t\hat{D}} e^{t\hat{H}} \]

we don’t have a “diagonal form” like [15] any more. As a result, we can say that in the framework of decoherence theory the Copenhagen interpretation is nothing but a special approximate phenomenon except for the problem stated above.

### 3.3 Decoherence

Here, we don’t observe the system at \( t_0 \) and solve the equation [8]

\[
\begin{pmatrix}
  a(t) \\
  b(t) \\
  \bar{b}(t) \\
  d(t)
\end{pmatrix} = e^{t(\hat{H} + \hat{D})} \begin{pmatrix}
  a(0) \\
  b(0) \\
  \bar{b}(0) \\
  d(0)
\end{pmatrix}
\]

exactly and take the limit \( t \rightarrow \infty \).

The method is almost equal to that of [2]. However, since to show it is important as composition of the paper, we repeat it within our necessity.
First, we must look for eigenvalues of the matrix \( W \equiv \hat{H} + \hat{D} \)

\[
W = \begin{pmatrix}
-\mu & ik & -ik & \nu \\
-ik & i(l-h) - \frac{\mu+\nu}{2} & 0 & -ik \\
-i\bar{k} & 0 & -i(l-h) - \frac{\mu+\nu}{2} & i\bar{k} \\
\mu & -i\bar{k} & i\bar{k} & -\nu
\end{pmatrix}.
\]

(18)

For the latter convenience we write the transpose of \( W \)

\[
W^T = \begin{pmatrix}
-\mu & ik & -i\bar{k} & \mu \\
i\bar{k} & i(l-h) - \frac{\mu+\nu}{2} & 0 & -i\bar{k} \\
i\bar{k} & 0 & \lambda + i(l-h) + \frac{\mu+\nu}{2} & -i\bar{k} \\
\nu & -i\bar{k} & i\bar{k} & -\nu
\end{pmatrix}.
\]

Since

\[
0 = |\lambda I_4 - W |
\]

\[
\begin{vmatrix}
\lambda + \mu & -i\bar{k} & ik & -\nu \\
-i\bar{k} & \lambda - i(l-h) + \frac{\mu+\nu}{2} & 0 & i\bar{k} \\
i\bar{k} & \lambda + i(l-h) + \frac{\mu+\nu}{2} & -i\bar{k} & 0 \\
-\mu & i\bar{k} & -i\bar{k} & \lambda + \mu + \nu
\end{vmatrix}
\]

\[
= \lambda \begin{vmatrix}
1 & 0 & 0 & 0 \\
-i\bar{k} & \lambda - i(l-h) + \frac{\mu+\nu}{2} & 0 & 2i\bar{k} \\
i\bar{k} & \lambda + i(l-h) + \frac{\mu+\nu}{2} & -2i\bar{k} & 0 \\
-\mu & i\bar{k} & -i\bar{k} & \lambda + \mu + \nu
\end{vmatrix}
\]

\[
= \lambda \left[ \left( \lambda + \frac{\mu + \nu}{2} \right)^2 + (l-h)^2 \right] (\lambda + \mu + \nu) + 2|k|^2(2\lambda + \mu + \nu)
\]

we obtain one trivial root \( \lambda = 0 \) and a cubic equation

\[
\left\{ \left( \lambda + \frac{\mu + \nu}{2} \right)^2 + (l-h)^2 \right\} (\lambda + \mu + \nu) + 2|k|^2(2\lambda + \mu + \nu) = 0.
\]
Let us transform this. By setting

\[ \Lambda = \lambda + \frac{\mu + \nu}{2} \implies \lambda = \Lambda - \frac{\mu + \nu}{2} \]

the cubic equation becomes

\[ \Lambda^3 + \frac{\mu + \nu}{2} \Lambda^2 + \{(l - h)^2 + 4|k|^2\} \Lambda + (l - h)^2 \frac{\mu + \nu}{2} = 0 \]

and some calculation gives

\[ \Lambda^3 + \frac{\mu + \nu}{2} \Lambda^2 + (E_1 - E_0)^2 \Lambda + (E_1 - E_0)^2 (|\alpha|^2 - |\beta|^2)^2 \frac{\mu + \nu}{2} = 0 \] (19)

by (2) and (6).

Here we set

\[ f(\Lambda) = \Lambda^3 + \frac{\mu + \nu}{2} \Lambda^2 + (E_1 - E_0)^2 \Lambda + (E_1 - E_0)^2 (|\alpha|^2 - |\beta|^2)^2 \frac{\mu + \nu}{2} \]

and treat its roots in an abstract way.

Case (A) : \(|\alpha| = |\beta|\)

In this case

\[ f(\Lambda) = \Lambda \left\{ \Lambda^2 + \frac{\mu + \nu}{2} \Lambda + (E_1 - E_0)^2 \right\}, \]

so we have solutions

\[ \Lambda_0 = 0, \quad \Lambda_\pm = \frac{1}{2} \left\{ \frac{\mu + \nu}{2} \pm \sqrt{(\frac{\mu + \nu}{2})^2 - 4(E_1 - E_0)^2} \right\}. \]

From these we know

\[ \Lambda_0 = 0, \quad \Lambda_\pm < 0 \]

if \((\frac{\mu + \nu}{2})^2 - 4(E_1 - E_0)^2 \geq 0\) and

\[ \Lambda_0 = 0, \quad \text{Re} \, \Lambda_\pm = -\frac{\mu + \nu}{4} < 0 \]

if \((\frac{\mu + \nu}{2})^2 - 4(E_1 - E_0)^2 < 0\).

Case (B) : \(|\alpha| \neq |\beta|\)
We note that \( f(\Lambda) > 0 \) for \( \Lambda \geq 0 \) because all coefficients are positive. Since
\[
f(0) = (E_1 - E_0)^2 (|\alpha|^2 - |\beta|^2)^2 \frac{\mu + \nu}{2} > 0,
\]
\[
f\left(-\frac{\mu + \nu}{2}\right) = -2(E_1 - E_0)^2 (\mu + \nu) |\alpha|^2 |\beta|^2 < 0
\]
there is (at least) one root \(-\frac{\mu + \nu}{2} < \Lambda_0 < 0\) satisfying \( f(\Lambda_0) = 0 \). By denoting
\[
f(\Lambda) = \Lambda^3 + a\Lambda^2 + b\Lambda + c
\]
for simplicity we have a decomposition
\[
f(\Lambda) = (\Lambda - \Lambda_0)(\Lambda^2 + (\Lambda_0 + a)\Lambda + (\Lambda_0^2 + a\Lambda_0 + b)).
\]
From this we obtain other two roots
\[
\Lambda_\pm = \frac{-\Lambda_0 + a \pm \sqrt{(\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b)}}{2}.
\]
Note that \( \Lambda_0 + a = \Lambda_0 + \frac{\mu + \nu}{2} > 0 \). If \( \Lambda_0^2 + a\Lambda_0 + b < 0 \) then \( \Lambda_+ > 0 \), which is a contradiction. Therefore, \( \Lambda_0^2 + a\Lambda_0 + b > 0 \).

Therefore,
\[
\Lambda_0 < 0, \quad \Lambda_\pm < 0
\]
if \( (\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b) > 0 \) and
\[
\Lambda_0 < 0, \quad \text{Re} \, \Lambda_\pm = -\frac{\Lambda_0 + a}{2} < 0.
\]
if \( (\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b) < 0 \).

As a result, the solutions of the characteristic polynomial of \( W = |\lambda|_4 - W| \) are
\[
\lambda_1 = 0, \quad \lambda_2 = \Lambda_0 - \frac{\mu + \nu}{2}, \quad \lambda_3 = \Lambda_+ - \frac{\mu + \nu}{2}, \quad \lambda_4 = \Lambda_- - \frac{\mu + \nu}{2} \tag{20}
\]
and
\[
\lambda_2 < 0, \quad \lambda_3 < 0, \quad \lambda_4 < 0 \quad \text{or} \quad \lambda_2 < 0, \quad \text{Re} \lambda_3 < 0, \quad \text{Re} \lambda_4 < 0 \tag{21}
\]
under the conditions stated above.
By the same method in [2]: Section 2 we obtain the diagonal form

\[ W = (O^T)^{-1} D_W O^T \quad (\iff W^T = OD_W O^{-1}) \]  

(22)

where \( D_W \) is the diagonal matrix

\[
D_W = \begin{pmatrix}
0 & & & \\
& \lambda_2 & & \\
& & \lambda_3 & \\
& & & \lambda_4 \\
\end{pmatrix}
\]  

(23)

and \( O \) is the matrix consisting of eigenvectors

\[
O = (|0\rangle, |\lambda_2\rangle, |\lambda_3\rangle, |\lambda_4\rangle) = \\
\begin{pmatrix}
1 & x_2 & x_3 & x_4 \\
0 & y_2 & y_3 & y_4 \\
0 & z_2 & z_3 & z_4 \\
1 & w_2 & w_3 & w_4 \\
\end{pmatrix}
\]  

(24)

and

\[
O^{-1} = \frac{1}{|O|} \begin{pmatrix}
\hat{O}_{11} & \hat{O}_{12} & \hat{O}_{13} & \hat{O}_{14} \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{pmatrix}
\]  

(25)

where * denotes cofactors unnecessary in the following.

Here, let us go back to the equation (7). If we set

\[
(\hat{\rho} =) \Psi = \begin{pmatrix}
a \\
b \\
\bar{b} \\
d \\
\end{pmatrix}
\]

for simplicity, the equation (7) reads

\[
\frac{d}{dt} \Psi = W \Psi
\]

\footnote{In order to find the eigenvectors of \( W \) it is better to use \( W^T \) rather than \( W \) itself. See [2] in more detail.}
and the general solution is given by (22)

$$\Psi(t) = e^{tW}\Psi(0) = (O^T)^{-1}e^{tDw}O^T\Psi(0).$$

Since we are interested in the final state $\Psi(\infty)$ we must look for the asymptotic limit

$$\lim_{t \to \infty} e^{tDw}.$$ It is easy to see

$$\lim_{t \to \infty} e^{tDw} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left| 0 \right\rangle \langle 0 |, \quad | 0 \rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

by (21) and (23), so we obtain

$$\Psi(\infty) = (O^T)^{-1}|0\rangle \langle 0 |O^T\Psi(0) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & 0 & 0 & \hat{O}_{11} \\ \hat{O}_{12} & 0 & 0 & \hat{O}_{12} \\ \hat{O}_{13} & 0 & 0 & \hat{O}_{13} \\ \hat{O}_{14} & 0 & 0 & \hat{O}_{14} \end{pmatrix} \Psi(0) \quad (26)$$

by (24) and (25).

This equation gives

$$\Psi(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \Psi(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} \\ \hat{O}_{12} \\ \hat{O}_{13} \\ \hat{O}_{14} \end{pmatrix}$$

and it is equivalent to

$$\rho_0(0) = |0\rangle \langle 0 | = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho_0(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{13} & \hat{O}_{14} \end{pmatrix}.$$

Similarly,

$$\Psi(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \Psi(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} \\ \hat{O}_{12} \\ \hat{O}_{13} \\ \hat{O}_{14} \end{pmatrix}$$

20
is equivalent to
\[
\rho_1(0) = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies \rho_1(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{13} & \hat{O}_{14} \end{pmatrix}.
\]
As a result
\[
\rho_0(0) = |0\rangle\langle 0|, \quad \rho_1(0) = |1\rangle\langle 1| \implies \rho_0(\infty) = \rho_1(\infty).
\]
Clearly, the Copenhagen interpretation does not hold (see the equation (17)). We would like to interpret the final density matrix as “classical one”, see [2].

We

4 Concluding Remarks

In this paper we tried to prove the Copenhagen interpretation of Quantum Mechanics. In our understanding measurement is a kind of decoherence forced and our method is performed by embedding it into decoherence theory (which is reasonable at least to the author).

We treated the master equation based on density matrix and introduced a decoherence time $t_D$ (which is in general small). Since measurement must be done within $t_D$ we have only to obtain not the full solution but the approximate one of the master equation.

Our solution gave a proof to the Copenhagen interpretation under some assumption. In order to prove the assumption we must introduce another theory.

Although our method is not complete it will become a starting point to give a complete proof to the Copenhagen interpretation in the near future. Mathematical physicists with strong mission must prove the Copenhagen interpretation at any cost.

References

[1] K. Fujii : “Proof” of the Copenhagen Interpretation, arXiv:1304.1591 [quant-ph].
[2] K. Fujii : Exact Solution of a Master Equation Applied to the Two Level system of an Atom, Int. J. Geom. Methods Mod. Phys, 11 (2014), 1450085 (18 pages), arXiv:1405.2604 [quant-ph].

[3] K. Fujii : Superluminal Group Velocity of Neutrinos : Review, Development and Problems, Int. J. Geom. Methods Mod. Phys, 10 (2013), 1250083 (19 pages), arXiv:1203.6425 [physics].

[4] P. Dirac : The Principles of Quantum Mechanics, Fourth Edition, Oxford University Press, 1958.

[5] H. S. Green : Matrix Mechanics, P. Noordhoff Ltd, Groningen, 1965.

[6] Asher Peres : Quantum Theory : Concepts and Methods, Kluwer Academic Publishers, 1995.

[7] Akio Hosoya : Lectures on Quantum Computation (in Japanese), SGC Library 4, Sainensu-sha Co., Ltd. Publishers (Tokyo), 1999.

[8] W. H. Zurek : Decoherence and the transition from quantum to classical, Physics Today, 44 (1991), 36-44.

[9] W. P. Schleich : Quantum Optics in Phase Space, WILEY–VCH, Berlin, 2001.

[10] G. Lindblad : On the generator of quantum dynamical semigroups, Commun. Math. Phys, 48 (1976), 119.

[11] V. Gorini, A. Kossakowski and E. C. G. Sudarshan : Completely positive dynamical semigroups of N–level systems, J. Math. Phys, 17 (1976), 821.

[12] H. -P. Breuer and F. Petruccione : The theory of open quantum systems, Oxford University Press, New York, 2002.
[13] K. Hornberger: Introduction to Decoherence Theory, in “Theoretical Foundations of Quantum Information”, Lecture Notes in Physics, 768 (2009), 221-276, Springer, Berlin, quant-ph/061211.

[14] C. Zachos: Crib Notes on Campbell-Baker-Hausdorff expansions, unpublished, 1999, see [http://www.hep.anl.gov/czachos/index.html](http://www.hep.anl.gov/czachos/index.html).

[15] K. Fujii and et al: Treasure Box of Mathematical Sciences (in Japanese), Yuseisha, Tokyo, 2010.
I expect that the book will be translated into English.