Abstract. A network belongs to the monotone separable class if its state variables are homogeneous and monotone functions of the epochs of the arrival process. This framework contains several classical queueing network models, including generalized Jackson networks, max-plus networks, polling systems, multiserver queues, and various classes of stochastic Petri nets. We use comparison relationships between networks of this class with i.i.d. driving sequences and the GI/GI/1/1 queue to obtain the tail asymptotics of the stationary maximal dater under light-tailed assumptions for service times. The exponential rate of decay is given as a function of a logarithmic moment generating function. We exemplify an explicit computation of this rate for the case of queues in tandem under various stochastic assumptions.

1. Introduction

Consider the GI/GI/1 single server queue: we denote $X_n = \sigma_n - \tau_n$ where $\{\sigma_n\}$ and $\{\tau_n\}$ are independent and identically distributed (i.i.d) non-negative random variables, $\sigma_n$ is the amount of service of customer $n$ and $\tau_n$ is the inter-arrival time between customer $n$ and $n + 1$. Assume that $\mathbb{E}[X_1] < 0$, then the supremum of the random walk $S_n = X_1 + \cdots + X_n$ defined by $M := \sup_{n \geq 1} S_n$ is finite almost surely and has the same distribution as the stationary workload of the single server queue. If we assume moreover that $\mathbb{E}[\exp(\epsilon X_1)] < \infty$ for some $\epsilon > 0$, then the following asymptotics is standard:

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(M > x) = -\theta^*, \quad \text{where } \theta^* = \sup \{\theta > 0, \log \mathbb{E}[e^{\theta X}] < 0\}. \quad (1.1)$$

Motivated by queueing applications, this case has been extensively studied in the literature and much finer estimates are available, see the works of Iglehart [10] and Pakes [12]. The main goal of this paper is to derive analogous results to (1.1) for networks.

In the context of a network, we consider the maximal dater $Z$ which is the time to empty the network when stopping further arrivals. Clearly in the single server queue, the maximal dater corresponds to the workload. In the case of queues in tandem, it corresponds to the end to end delay. Our Theorem 2.2 gives the logarithmic tail asymptotics for the maximal dater of a monotone separable network. The main difficulty in our task is the absence of closed form formula for $Z$. The proof of the theorem will proceed by deriving upper and lower bounds for monotone separable networks. This class, which was introduced by Baccelli and Foss in [3], contains several classical queueing network models like generalized Jackson networks, max-plus networks, polling systems and multiserver queues. In this paper, we choose to put a particular

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emphasis on tandem queues that fall in the class of open Jackson networks, and in the class of open (max,plus) systems which both belong to the class of monotone separable networks. It serves as a pedagogical example to apply our main theorem under various stochastic assumptions and it enables us to link our results with existing asymptotics results from queueing literature.

The paper is structured as follows. In Section 2 we give the precise definition of a monotone separable network and its associated maximal dater. We then give the main result of this paper in Section 2.2. The case of queues in tandem is dealt with great details in Section 3. In particular, we show that a kind of phase transition is possible when service times at both station are dependent. We also link our result to the literature. Finally technical proofs are deferred to Section 4.

2. Tail asymptotics for monotone-separable networks

In this paper, we consider open stochastic networks with a single input process \( N \) which is a marked point process with points \( \{T_n\} \) corresponding to exogenous arrival times and marks \( \{ζ_n\} \) which describe the service times and routing decisions.

More precisely a stochastic network is described by the following framework (introduced by Baccelli and Foss [3])

- The network has a single input point process \( N \), with points \( \{T_n\} \); for all \( m \leq n \in \mathbb{Z} \), let \( N_{[m,n]} \) be the \([m,n]\)-restriction of \( N \), namely the point process with points \( \{T_ℓ\}_{m \leq ℓ \leq n} \).
- The network has a.s. finite activity for all finite restrictions of \( N \); for all \( m \leq n \in \mathbb{Z} \), let \( X_{[m,n]}(N) \) be the time of last activity in the network, when this one starts empty and is fed by \( N_{[m,n]} \). We assume that for all finite \( m \) and \( n \) as above, \( X_{[m,n]}(N) \) is finite.

We assume that there exists a set of functions \( \{f_ℓ\}, f_ℓ : \mathbb{R}^ℓ \times K^ℓ \to \mathbb{R} \), such that:

\[
X_{[m,n]}(N) = f_{n-m+1}(\{T_ℓ, ζ_ℓ\}, m \leq ℓ \leq n),
\]

for all \( n, m \) and \( N = \{T_n\} \), where the sequence \( \{ζ_n\} \) is that describing service times and routing decisions.

**Example.** Consider a \( G/G/1/∞ → ./G/1/∞ \) tandem queue. Denote by \( \{σ_n^{(i)}\} \) the sequence of service times in station \( i = 1, 2 \) and \( N = \{T_n\} \) the sequence of arrival times at the first station. With the notation introduced above, we have \( ζ_n = (σ_n^{(1)}, σ_n^{(2)}) \) and the time of last activity is given by,

\[
X_{[m,n]}(N) = \sup_{m \leq k \leq n} \left\{ T_k + \sup_{k \leq i \leq n} \sum_{j=k}^i σ_j^{(1)} + \sum_{j=i}^n σ_j^{(2)} \right\}.
\]

We refer to the Appendix for an explicit derivation of Equation (2.2). \( X_{[m,n]}(N) \) is simply the last departure time from the network, when only customers \( m, m+1, \ldots, n \) enter the network.

We say that a network described as above is monotone-separable if the functions \( f_n \) are such that the following properties hold for all input point process \( N \):

1. **Causality:** for all \( m \leq n \),

\[
X_{[m,n]}(N) \geq T_n;
\]
2.1. Stability and stationary maximal daters. In this section, we introduce stochastic assumptions ensuring the stability of the network. More general results can be found in Baccelli and Foss \cite{Baccelli2009} and we refer to it for the statements given in this section without proof.

By definition, for \( m \leq n \), the \([m, n]\) maximal dater is

\[
Z_{[m,n]}(N) := X_{[m,n]}(N) - T_n.
\]

Note that \( Z_{[m,n]}(N) \) is a function of \( \{\zeta_{m,\ell}\}_{m \leq \ell \leq n} \) and \( \{\tau_{m}\}_{m \leq \ell \leq n} \), where \( \tau_n = T_{n+1} - T_n \). In particular, \( Z_n := Z_{[n,n]}(N) \) is not a function of \( N \) (which makes the notation consistent).

**Lemma 2.1.** \textbf{Internal monotonicity of \( X \) and \( Z \)}

Under the above conditions, the variables \( X_{[m,n]} \) and \( Z_{[m,n]} \) satisfy the internal monotonicity property: for all \( N, m \leq n \),

\[
X_{[m-1,n]}(N) \geq X_{[m,n]}(N),
\]

\[
Z_{[m-1,n]}(N) \geq Z_{[m,n]}(N).
\]

In particular, the sequence \( \{Z_{[-n,0]}(N)\} \) is non-decreasing in \( n \). We define the stationary maximal dater as

\[
Z := Z_{(-\infty,0]}(N) = \lim_{n \to \infty} Z_{[-n,0]}(N) \leq \infty.
\]

**Example.** In the case of the tandem queues, the stationary maximal dater is given by:

\[
Z = \sup_{\rho \leq q \leq 0} \left\{ \sum_{k=p}^{q} \sigma_k^{(1)} + \sum_{k=q}^{0} \sigma_k^{(2)} - (T_0 - T_p) \right\}, \tag{2.3}
\]

and \( Z \) is the stationary end to end delay of the network.

**Lemma 2.2.** \textbf{Subadditive property of \( Z \)}

Under the above conditions, \( \{Z_{[m,n]}(N)\} \) satisfies the following subadditive property: for all \( m \leq \ell < n \), for all \( N \),

\[
Z_{[m,n]}(N) \leq Z_{[m,\ell]}(N) + Z_{[\ell+1,n]}(N).
\]

We assume that the sequence \( \{\tau_n, \zeta_n\}_n \) is a sequence of i.i.d. random variables. The following integrability assumptions are also assumed to hold (recall that \( Z_n = Z_{[n,n]}(N) \) does not depend on \( N \)):

\[
\mathbb{E}[\tau_n] := a < \infty, \quad \mathbb{E}[Z_n] < \infty.
\]
Denote by \( N^0 = \{ T^0_n \} \) the degenerate input process with \( T^0_n = 0 \) for all \( n \). This degenerate point process plays a crucial role for the derivation of the stability condition. The following lemma follows from Lemma 2.2 in which we take as input point process \( N^0 \) (note that the constant \( \gamma \) defined below is denoted \( \gamma(0) \) in [3] to emphasize the fact that the input point process is \( N^0 \)).

**Lemma 2.3.** [3] Under the foregoing stochastic assumption, there exists a non-negative constant \( \gamma \) such that

\[
\lim_{n \to \infty} \frac{Z_{[-n,0]}(N^0)}{n} = \lim_{n \to \infty} \frac{E[Z_{[-n,0]}(N^0)]}{n} = \gamma \text{ a.s.}
\]

The main result on the stability region is the following:

**Theorem 2.1.** [3] Under the foregoing stochastic assumptions, either \( Z = \infty \) a.s. or \( Z < \infty \) a.s.

(a) If \( \gamma < a \), then \( Z < \infty \) a.s.
(b) If \( Z < \infty \) a.s., then \( \gamma \leq a \).

A proof is given in Section 4.1, where we derive an upper bound and a lower bound that will be used for the study of large deviations.

**Example.** In the case of tandem queues, the constant \( \gamma \) is easy to compute. We have

\[
\lim_{n \to \infty} \sup_{-n \leq q \leq 0} \frac{\sum_{k=-n}^{q} \sigma_k^{(1)} + \sum_{k=q}^{0} \sigma_k^{(2)}}{n} = \max \left( \frac{E[\sigma_1^{(1)}]}{E[\sigma_1^{(2)}]} \right).
\]

Hence Theorem 2.1 gives the standard stability condition: \( \max \left( \frac{E[\sigma_1^{(1)}]}{E[\sigma_1^{(2)}]} \right) < E[\tau_1] \).

2.2. Moment generating function and tail asymptotics. In the rest of the paper, we will make the following assumptions:

- Assumption (AA) on the arrival process into the network \( \{ T_n \} \):
  \( \{ T_n \} \) is a renewal process independent of the service time and routing sequences \( \{ \zeta_n \} \).
- Assumption (AZ): the sequence \( \{ \zeta_n \} \) is a sequence of i.i.d. random variables, such that the random variable \( \sigma_0 \) is light-tailed, i.e. for \( \theta \) in a neighborhood of 0,
  \[ E[e^{\theta \sigma_0}] < +\infty. \]
- Stability: \( \gamma < a = E[T_1 - T_0] \) see Theorem 2.1

The subadditive property of \( Z \) directly implies the following property (which is proved in Lemma 4.1): for any monotone separable network that satisfies assumption (AZ), the following limit

\[
\Lambda_Z(\theta) = \lim_{n \to \infty} \frac{1}{n} \log E\left[ e^{\theta Z_{[1,n]}(N^0)} \right],
\]

exists in \( \mathbb{R} \cup \{ +\infty \} \) for all \( \theta \). Note that the subadditive property of \( Z \) is valid regardless of the point process \( N \) (see Lemma 2.2). Like in the study of the stability of the network, it turns out that the right quantity to look at is \( Z_{[m,n]}(N^0) \) where \( N^0 \) is the degenerate input point process with all its point equal to 0. We also define:

\[
\Lambda_T(\theta) = \log E\left[ e^{\theta(T_1 - T_0)} \right].
\]
Theorem 2.2. Under previous assumptions, the tail asymptotics of the stationary maximal dater is given by,

$$\lim_{x \to \infty} \frac{1}{x} \log P(Z > x) = -\theta^* < 0,$$

where $\theta^* = \sup \{\theta > 0, \Lambda_F(-\theta) + \Lambda_Z(\theta) < 0\}$.

It is relatively easy to see that under our light-tailed assumption the stationary maximal dater $Z$ will be light-tailed (see Corollary 3 in [4]). The main contribution of Theorem 2.2 is to give an explicit way of computing the rate of decay of the tail distribution of $Z$. We refer the interested reader to [11] for more details on the computation of $\Lambda_Z$ in the case of $(\max,\text{plus})$-linear networks. In Section 3, we continue the study of our example and deal with the case of queues in tandem under various stochastic assumptions. This case of study allows us to show a phase transition phenomena and to compare our theorem with results of the literature.

Note that in the context of heavy-tailed asymptotics, the moment generating function is infinite for all $\theta > 0$. There is no general result for the tail asymptotics of the maximal dater of a monotone separable network. However the methodology derived by Baccelli and Foss [4] for subexponential distributions allows to get exact asymptotics for $(\max,\text{plus})$-linear networks [6] and generalized Jackson networks [5].

3. A Case of Study: Queues in tandem

3.1. The impact of dependence. We continue our example and consider a stable $G/G/1/\infty \to G/1/\infty$ tandem queue where $\{\sigma_{n}^{(i)}\}$ is the sequence of service times in station $i = 1, 2$ and $\{\tau_{n}\}$ is the sequence of inter-arrival times at the first station. We assume that the sequences $\{(\sigma_{n}^{(1)}, \sigma_{n}^{(2)})\}$ and $\{\tau_{n}\}$ are sequences of i.i.d. random variables such that $\gamma = \max \left(\mathbb{E}[\sigma_{1}], \mathbb{E}[\sigma_{1}]\right) < \mathbb{E}[\tau_{1}]$.

We consider two cases:

- case 1: the sequences $\{\sigma_{n}^{(1)}\}, \{\sigma_{n}^{(2)}\}; \{\tau_{n}\}$ are independent.
- case 2: the sequences $\{\sigma_{n}^{(1)}\}$ and $\{\tau_{n}\}$ are independent and we have $\sigma_{n}^{(2)} = \sigma_{n}^{(1)}$.

We denote $\Lambda_i(\theta) = \log \mathbb{E}[\exp(\theta \sigma_{i}^{(1)})]$ and $\delta = \sup \{\theta \geq 0, \mathbb{E}_{} \left[ e^{\theta \sigma_{1}^{(1)}} \right] < \infty \}$. A direct application of Theorem 2.2 gives an extension of the results of Ganesh [9]:

Corollary 3.1. The tail asymptotics of the stationary end to end delay for two queues in tandem is given by

$$\lim_{x \to \infty} \frac{1}{x} \log P(Z > x) = -\theta^*,$$

where

- in case 1: $\theta^* = \min(\theta^1, \theta^2)$ with $\theta^i = \sup \{\theta > 0, \Lambda_i(\theta) + \Lambda_T(-\theta) < 0\}$;
- in case 2: $\theta^* = \min(\theta^1, \frac{\delta}{2})$.

In case 1, $\theta^i$ is the rate of exponential decay for the tail distribution of the stationary workload of a single server queue with interarrival $\tau_{n}$ and service time $\sigma_{n}^{(i)}$ and we have $\theta^* = \min(\theta^1, \theta^2)$. It is well-known that the stability of such a network is constraint by the "slowest" component. Here we see that in a large deviations regime, the "bad" behavior of the network is due to
a "bottleneck" component (which is not necessarily the same as the "slowest" component in average). Note that in the particular case where the random variables \( \sigma_1(1), \sigma_2(2), \tau_n \) are exponentially distributed with mean \( 1/\mu_1, 1/\mu_2, a \), we have \( \theta^* = \mu - a^{-1} \), and in this case the "slowest" component in average is also the "bottleneck" component in the large deviations regime.

In the case where the service times are the same at both stations, Corollary 3.1 shows that the tail behavior of the random variable \( \sigma_1(1) \) described by \( \delta \) matters. To simplify and to get a parametric model, assume that the arrival process is Poisson with intensity \( \lambda := a^{-1} \) and the service times are exponentially distributed with mean \( 1/\mu \). Then depending on the intensity of the arrival process \( \lambda \), two situations may occur:

\[
\lambda \leq \mu/2 \Rightarrow \theta^* = \mu/2, \quad \lambda > \mu/2 \Rightarrow \theta^* = \mu - \lambda.
\]

In words, we have

1. if \( \lambda < \mu/2 \), then the tail asymptotics of the end-to-end delay is the same as the total service requirement of a single customer;
2. if \( \lambda > \mu/2 \), then the tail asymptotics of the end-to-end delay is the same as in the independent case.

This shows that the behavior of tandems differs from that of a single server queue. In particular Anantharam \([\text{I}]\) shows that for \( GI/GI/1 \) queues, the build-up of large delays can happen in one of two ways:

- If the service times have exponential tails, then it involves a large number of customers (whose inter-arrival and service times differ from their mean values).
- If the service times do not have exponential tails, then large delays are caused by the arrival of a single customer with large service requirement.

We see that the first behavior is still valid for queues in tandem when the service times are independent at each station or if the intensity of the arrival process is sufficiently large. In contrast, when the service times are the same at both station, we see that a single customer can create large delays in the network even under the assumption of exponential service times (if the intensity of arrivals is sufficiently small). Note that this phenomena is rather simple and results intrinsically from the fact that the network considered is of dimension greater than 2 (i.e. one cannot get such a phenomena with a single server queue).

**Proof.** Recall that we have

\[
Z_{[1,n]}(N^0) = \sup_{1 \leq k \leq n} \sum_{i=1}^{k} \sigma_i(1) + \sum_{i=k}^{n} \sigma_i(2).
\]

In case 1, we have

\[
\log \mathbb{E} \left[ e^{\theta Z_{[1,n]}(N^0)} \right] \leq \log \left( \sum_{k=1}^{n} e^{k \Lambda_1(\theta) + (n-k) \Lambda_2(\theta)} \right) \leq \log n + n \max(\Lambda_1(\theta), \Lambda_2(\theta)).
\]

Hence we have \( \Lambda_Z(\theta) = \max(\Lambda_1(\theta), \Lambda_2(\theta)) \) and the corollary follows.
In case 2, we have $Z_{[1,n]}(N^0) = \sum_{i=1}^{n} \sigma_i^{(1)} + \max_i \sigma_i^{(1)} = \max_i \left(2\sigma_i^{(1)} + \sum_{j \neq i} \sigma_j^{(1)}\right)$, hence we have

$$\log E \left[e^{\theta Z_{[1,n]}(N^0)}\right] \geq \max (n\Lambda_1(\theta), \Lambda_1(2\theta)) \text{ and,}$$

$$\log E \left[e^{\theta Z_{[1,n]}(N^0)}\right] \leq (n-1)\Lambda_1(\theta) + \log n + \Lambda_1(2\theta).$$

It follows that

$$\Lambda_Z(\theta) = \begin{cases} \Lambda_1(\theta) & \theta < \eta/2 \\ \infty & \theta > \eta/2 \end{cases}$$

and the corollary follows.

3.2. Comparison with the literature. In the context of two queues in tandem, if we define

$$Y_n = \sup_{-n \leq q \leq 0} \sum_{k=-n}^{q} \sigma_k^{(1)} + \sum_{k=q}^{0} \sigma_k^{(2)} - (T_0 - T_{-n}),$$

then we have in view of (2.3), $Z = \sup_n Y_n$. The supremum of a stochastic process has been extensively studied in queueing theory but we do not know of any general results that would allow to derive Corollary 3.1. To end this section and to make the connection with the existing literature, we state the following result

**Corollary 3.2.** Consider the system of queues in tandem described above. Under assumptions of Theorem 2.2 and if

1. the sequence $\{Y_n/n\}$ satisfies a large deviation principle (LDP) with a good rate function $I$;
2. there exists $\epsilon > 0$ such that $\Lambda_Z(\theta^* + \epsilon) < \infty$,

where $\theta^*$ is defined as in Theorem 2.2. Then we have

$$\lim_{x \to \infty} \frac{1}{x} \log P(Z > x) = -\theta^* = -\inf_{\alpha > 0} \frac{I(\alpha)}{\alpha}. \quad (3.1)$$

This kind of result has been extensively studied in the queueing literature (we refer to the work of Duffy, Lewis and Sullivan [8]). However, we see that considering the moment generating function instead of the rate function allows us to get a more general result than (3.1) since we do not require the assumption (2) on the tail. Indeed this assumption ensures that the tail asymptotics of $P(Y_n > nc)$ for a single $n$ value cannot dominate those of $P(Z > x)$. In this case, equation (3.1) has a nice interpretation: the natural drift of the process $Y_n$ is $\mu n$, where $\mu < 0$. The quantity $I(\alpha)$ can be seen as the cost for changing the drift of this process to $\alpha > 0$. Now in order to reach level $x$, this drift has to last for a time $x/\alpha$. Hence the total cost for reaching level $x$ with drift $\alpha$ is $xI(\alpha)/\alpha$ and the process naturally choose the drift with the minimal associated cost. As already discussed, this heuristic is valid only if an assumption as (2) holds. Note also that in our framework, we do not assume any LDP to hold for the sequence $\{Y_n/n\}$. In particular, as shown by Corollary 3.1, the computation of the moment generating function $\Lambda_Z$ is much easier than deriving a LDP for $\{Y_n/n\}$. Lastly, we should stress that for general monotone separable networks, the maximal dater $Z$ cannot be expressed as the supremum of a simple stochastic process in which case, the derivation of the tail asymptotics of $Z$ requires new techniques.
Proof. We have only to show that $\theta^* = \inf_{\alpha > 0} \frac{I(\alpha)}{\alpha}$. Thanks to Varadhan’s Integral Lemma (see Theorem 4.3.1 in [7]), we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta Y_n} \right] = \Lambda(\theta) = \sup_x \{ \theta x - I(x) \},$$

for $\theta < \theta^* + \epsilon$, where $\Lambda(\theta) = \Lambda_Z(\theta) + \Lambda_T(-\theta)$. Then, the corollary follows from the following observations for $\theta > 0$,

$$\theta < \inf_{\alpha > 0} \frac{I(\alpha)}{\alpha} \iff \theta \alpha - I(\alpha) < 0, \forall \alpha$$

$$\iff \sup_{\alpha} \{ \theta \alpha - I(\alpha) \} = \Lambda(\theta) < 0.$$

$\square$

4. Proof of the tail asymptotics

4.1. Upper $G/G/1/\infty$ queue and lower bound for the maximal dater. The material of this subsection is not new and may be found in various references (that are given in what follows). For the sake of completeness, we include all the proofs. We derive now upper and lower bounds for the stationary maximal dater $Z$. These bounds allow to prove Theorem 2.1 and will be the main tools for the study of large deviations.

We first derive a lower bound that can also be found in the textbook [2] see proof of Theorem 2.11.3.

Proposition 4.1. We have the following lower bound

$$Z \geq \sup_{n \geq 0} \left( Z_{[-n,0]}(N^0) + T_{-n} - T_0 \right).$$

Proof. For $n$ fixed, let $N^n$ be the point process with point $T^n_j = T_{-n} - T_0$, for all $j$. Then

$$Z_{[-n,0]} = X_{[-n,0]}(N) - T_0 \geq X_{[-n,0]}(N^n)$$

$$= X_{[-n,0]}(N^0) + T_{-n} - T_0 = Z_{[-n,0]}(N^0) + T_{-n} - T_0,$$

where we used external monotonicity in the first inequality and homogeneity between the first and second line. $\square$

Proof. of Theorem 2.1 part (b)

Suppose that $\gamma > a$, then we have

$$\liminf_{n \to \infty} \frac{Z_{[-n,0]}(N)}{n} \geq \gamma - a > 0,$$

which concludes the proof of part (b). $\square$

We assume now that $\gamma < a$. We pick an integer $L \geq 1$ such that

(4.1) \quad $\mathbb{E} \left[ Z_{[-L-1]}(N^0) \right] < La,$

which is possible in view of Lemma 2.3. Without loss of generality, we assume that $T_0 = 0$. Part (a) of Theorem 2.1 follows from the following proposition (that can be found in [4]):
Proposition 4.2. The stationary maximal dater $Z$ is bounded from above by the stationary response time $\hat{R}$ in the $G/G/1/\infty$ queue with service times

$$\hat{s}_n := Z_{[L(n-1)+1,Ln]}(N^0)$$

and inter-arrival times $\hat{\tau}_n := T_{Ln} - T_{L(n-1)}$, where $L$ is the integer defined in (4.1). Since $E[\hat{s}_1] < E[\hat{\tau}_1] = La$, this queue is stable. With the convention $\sum_{0}^{-1} = 0$, we have,

$$Z \leq \hat{s}_0 + \sup_{k \geq 0} \sum_{i=k}^{-1} (\hat{s}_i - \hat{\tau}_{i+1}).$$

Proof. To an input process $N$, we associate the following upper bound process, $N^+ = \{T_n^+\}$, where $T_n^+ = T_{kL}$ if $n = (k-1)L + 1, \ldots, kL$. Note that $T_n^+ \geq T_n$ for all $n$. Then for all $n$, since we assumed $T_0 = 0$, we have thanks to the external monotonicity,

$$(4.2)\quad X_{[-n,0]}(N) = Z_{[-n,0]}(N) \leq X_{[-n,0]}(N^+) = Z_{[-n,0]}(N^+).$$

We show that for all $k \geq 1$,

$$(4.3)\quad Z_{[-kL+1,0]}(N^+) \leq \hat{s}_0 + \sup_{-k+1 \leq i \leq 0} \sum_{j=-i}^{i} (\hat{s}_j - \hat{\tau}_{j+1}).$$

This inequality will follow from the two next lemmas.

Lemma 4.1. Assume $T_0 = 0$. For any $m < n \leq 0$,

$$Z_{[m,0]}(N) \leq Z_{[n,0]}(N) + (Z_{[m,n-1]}(N) - \tau_{n-1})^+.$$

Proof. Assume first that $Z_{[m,n-1]}(N) - \tau_{n-1} \leq 0$, which is exactly $X_{[m,n-1]}(N) \leq T_n$. Then by the separability property, we have

$$Z_{[m,0]}(N) = X_{[m,0]}(N) = X_{[n,0]}(N) = Z_{[n,0]}(N).$$

Assume now that $Z_{[m,n-1]}(N) - \tau_{n-1} > 0$. Let $N' = \{T'_j\}$ be the input process defined as follows

$$\forall j \leq n-1, \quad T'_j = T_j,$$

$$\forall j \geq n, \quad T'_j = T_j + Z_{[m,n-1]}(N) - \tau_{n-1}.$$

Then we have $N' \geq N$ and $X_{[m,n-1]}(N') \leq T'_n$, hence by the external monotonicity, the separability and the homogeneity properties, we have

$$Z_{[m,0]}(N) = X_{[m,0]}(N) \leq X_{[m,0]}(N') = X_{[n,0]}(N) + Z_{[m,n-1]}(N) - \tau_{n-1} = Z_{[n,0]}(N) + Z_{[m,n-1]}(N) - \tau_{n-1}.$$

□

From this lemma we derive directly

Lemma 4.2. Assume $T_0 = 0$. For any $n < 0$,

$$Z_{[n,0]}(N) \leq \sup_{n \leq k \leq 0} \left( \sum_{i=k}^{-1} (Z_i - \tau_{i+1}) \right) + Z_0,$$

with the convention $\sum_{0}^{-1} = 0$. 

Applying Lemma 4.2 to $Z_{[-kL+1,0]}(N^+)$ gives (4.3). We now return to the proof of Proposition 4.2. We have

$$Z = \lim_{k \to \infty} Z_{[-kL+1,0]} \leq \sup_{k \geq 0} Z_{[-kL+1,0]}(N^+) \text{ thanks to (4.2)} \leq \sup_{k \geq 0} \left( \hat{s}_0 + \sup_{-k+1 \leq i \leq 0} \sum_{j=-i}^{-1} (\hat{s}_j - \hat{\tau}_{j+1}) \right) = \hat{R},$$

from Lemma 4.2. \hfill \Box

4.2. Moment generating function.

**Lemma 4.1.** The function $\Lambda_Z(.)$ defined by (2.4) is a proper convex function with $\Lambda_Z(\theta) < \infty$ for all $\theta < \eta$ and $\Lambda_Z(\theta) = \infty$ for all $\theta > \eta$, where $\eta = \sup \{ \theta, E[\exp(\theta Z_0)] < \infty \}$.

**Proof.** Let

$$\Lambda_{Z,n}(\theta) = \log \mathbb{E} \left[ e^{\frac{Z_{[1,n]}(N^0)}{n}} \right].$$

Thanks to the subadditive property of $Z$, we have,

$$Z_{[1,n+m]}(N^0) \leq Z_{[1,n]}(N^0) + Z_{[n+1,n+m]}(N^0),$$

and $Z_{[1,n]}(N^0)$ and $Z_{[n+1,n+m]}(N^0)$ are independent. Hence for $\theta \geq 0$, we have,

$$\Lambda_{Z,n+m}((n+m)\theta) \leq \Lambda_{Z,n}(n\theta) + \Lambda_{Z,m}(m\theta).$$

Hence we can define for any $\theta \geq 0,$

$$\Lambda_Z(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta Z_{[1,n]}(N^0)} \right] = \lim_{n \to \infty} \frac{\Lambda_{Z,n}(n\theta)}{n} = \inf_{n \geq 1} \frac{\Lambda_{Z,n}(n\theta)}{n},$$

as an extended real number. The fact that $\Lambda_Z$ is a proper convex function follows from Lemma 2.3.9 of [7]. The last fact follows from,

$$\Lambda_Z(\theta) \leq \log \mathbb{E} \left[ e^{\theta Z_1} \right] \text{ and, } \log \mathbb{E} \left[ e^{\theta Z_1} \right] \leq \Lambda_{Z,n}(n\theta) \text{ for } \theta \geq 0 \text{ and all } n \geq 1.$$

\hfill \Box

We define

$$\Lambda(\theta) = \Lambda_T(-\theta) + \Lambda_Z(\theta) \text{ and } \Lambda_n(\theta) = \Lambda_T(-\theta) + \Lambda_{Z,n}(\theta).$$

Note that $\Lambda_Z(.)$ and $\Lambda_T(.)$ are proper convex functions, hence $\Lambda(.)$ is a well defined convex function. Recall that $\theta^*$ is defined as follows:

$$\theta^* = \sup \{ \theta > 0, \Lambda(\theta) < 0 \}.$$

The following lemma is used repeatedly in what follows,
Lemma 4.2. Under the foregoing assumptions, we have $\theta^* > 0$ and

\[
\Lambda(\theta) < 0 \quad \text{if} \quad \theta \in (0, \theta^*),
\]
\[
\Lambda(\theta) > 0 \quad \text{if} \quad \theta > \theta^*.
\]

Proof. Let

\[
\theta_n = \sup\{\theta > 0, \Lambda_n(n\theta) < 0\}.
\]

We fix $n$ such that $\mathbb{E}[Z_{[1,n]}(N^0)] \leq na$, which is possible in view of the stability condition.

We first show that $\theta_n > 0$ and $\Lambda_n(n\theta) < 0$ if $\theta \in (0, \theta_n)$,

\[
\Lambda_n(n\theta) > 0 \quad \text{if} \quad \theta > \theta_n
\]

The function $\theta \mapsto \Lambda_n(n\theta)$ is convex, continuous and differentiable on $[0, \eta)$. Hence we have

\[
\Lambda_n(n\delta) = \delta (\mathbb{E}[Z_{[1,n]}(N^0)] - a) + o(\delta),
\]

which is less than zero for sufficiently small $\delta > 0$. Hence, the set over which the supremum in the definition of $\theta_n$ is taken is not empty and $\theta_n > 0$.

We now show that $\theta_n \to \theta^*$ as $n \to \infty$. We have for $\theta \geq 0$

\[
\lim_{n \to \infty} \frac{\Lambda_n(n\theta)}{n} = \inf_{n \geq 1} \frac{\Lambda_n(n\theta)}{n} = \Lambda(\theta).
\]

Hence for $\theta \geq 0$, we have $\frac{\Lambda_n(n\theta)}{n} \geq \Lambda(\theta)$ and

\[
\forall \theta \in (0, \theta_n), \quad \Lambda(\theta) = \frac{\Lambda_n(n\theta)}{n} < 0.
\]

This implies that $\theta^* \geq \theta_n > 0$. If $\theta^* < \infty$, we can choose $\epsilon > 0$ such that $\theta^* - \epsilon > 0$ and then we have $\Lambda_n(n(\theta^* - \epsilon))/n \to \Lambda(\theta^* - \epsilon) < 0$. Hence for sufficiently large $n$, we have $\frac{\Lambda_n(n(\theta^* - \epsilon))}{n} < 0$, hence $\theta^* - \epsilon \leq \theta_n$, and we proved that $\theta_n \to \theta^*$. $\Lambda(.)$ is a convex function and since $\Lambda(0) = 0$, the lemma follows.

If $\theta^* = \infty$, we still have $\theta_n \to \infty$ (that will be needed in proof of Lemma 4.4) by the same argument as above with $\theta^* - \epsilon$ replaced by any real number.

\[
4.3. \text{Lower Bound.}
\]

Lemma 4.3. Under previous assumptions, we have

\[
\liminf_{x \to \infty} \frac{1}{x} \log \mathbb{P}(Z > x) \geq -\theta^*.
\]

Proof. We have (see Proposition 4.4)

\[
Z \geq \sup_n \left\{ Z_{[-n,0]}(N^0) + T_{n} - T_0 \right\}.
\]

We denote $Y_n = Z_{[-n,1]}(N^0) + T_{-n} + T_0$, the lemma will follow from the following fact:

\[
\liminf_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\sup Y_n > x) \geq -\theta^*.
\]
Note that we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta Y_n} \right] = \Lambda(\theta).
\]

In particular, we are in the setting of Gärtner-Ellis theorem see Theorem 2.3.6 in [7] which will be the main tool of the proof.

First note that we only need to consider the case \( \theta^* < \infty \). We consider first the case where there exists \( \theta > \theta^* \) such that \( \Lambda(\theta) < \infty \). First note that the function \( \theta \to \Lambda(\theta) \) is convex, hence the left-hand derivatives \( \Lambda'(\theta) \) and the right-hand derivatives \( \Lambda'(\theta+) \) exist for all \( \theta > 0 \). Moreover, we have \( \Lambda'(\theta-) \leq \Lambda'(\theta+) \) and the function \( \theta \to \frac{1}{\theta}\Lambda'(\theta-)+\Lambda'(\theta+) \) is non-decreasing, hence \( \Lambda'(\theta) = \Lambda'(\theta-) = \Lambda'(\theta+) \) except for \( \theta \in \Delta \), where \( \Delta \) is at most countable. Since \( \Lambda(\theta) < \infty \) for \( \theta > \theta^* \), we have \( \Lambda(\theta^*) = 0 \) and \( \Lambda'(\theta^*) > 0 \). To prove this, assume that \( \Lambda'(\theta^*) = 0 \). Take \( \theta < \theta^* \), thanks to Lemma 4.2, we have \( \Lambda(\theta) < 0 \). Choose \( \epsilon > 0 \) such that \( 0 < \Lambda(\theta^*+\epsilon) < \epsilon|\Lambda(\theta)| \). We have

\[
\frac{\Lambda(\theta^*)}{\epsilon} < \frac{-\Lambda(\theta)}{\theta^* - \theta},
\]

which contradicts the convexity of \( \Lambda(\theta) \). Hence, we can find \( t \leq \theta^* + \epsilon \) such that

\[
0 < \Lambda(t), \quad t \notin \Delta.
\]

Note that these conditions imply \( t > \theta^* \) and \( \Lambda'(t) \geq \Lambda'(\theta^*) > 0 \).

Thanks to Gärtner-Ellis theorem (Theorem 2.3.6 in [7]), we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n > n\alpha) \geq -\inf_{x \in \mathcal{F}, x > \alpha} \Lambda^*(x),
\]

where \( \mathcal{F} \) is the set of exposed point of \( \Lambda^* \) and \( \Lambda^*(x) = \sup_{\theta \geq 0}(\theta x - \Lambda(\theta)) \). Note that from the monotonicity of \( \theta x - \Lambda(\theta) \) in \( x \) as \( \theta \) is fixed, we deduce that \( \Lambda^* \) is non-decreasing. Moreover take \( \alpha = \Lambda'(t) \), then \( \Lambda^*(\alpha) = t\alpha - \Lambda(t) \) and \( \alpha \in \mathcal{F} \) by Lemma 2.3.9 of [7].

Given \( x > 0 \), define \( n = [x/\alpha] \). We have

\[
\frac{1}{x} \log \mathbb{P}(Y_n > x) \geq \frac{1}{n\alpha} \log \mathbb{P}(Y_n \geq n\alpha),
\]

taking the limit in \( x \) and \( n \) (while \( \alpha = \Lambda'(t) \) is fixed) gives thanks to (4.8),

\[
\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\sup Y_n > x) \geq -\frac{t\alpha - \Lambda(t)}{\alpha} \geq -t \geq -\theta^* - \epsilon.
\]

We consider now the case where for all \( \theta > \theta^* \), we have \( \Lambda(\theta) = \infty \), i.e. \( \theta^* = \eta \) defined in Lemma 4.1. Take \( K > 0 \) and define \( \tilde{Z}_{[n,m]}^K = Z_{[n,m]}(N^0) \prod_{i=n}^m 1(\xi_i \leq K) \) and \( \tilde{Z}^K = \sup_{n \geq 0} (\tilde{Z}_{[n,m]}^K + T_{-n}) \). By (4.7), we have \( Z \geq \tilde{Z}^K \). It is easy to see that the proof of Lemma 4.1 is still valid (note that the subadditive property carries over to \( \tilde{Z}_{[n,m]}^K \)) and the following limit exists

\[
\tilde{\Lambda}_Z^K(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\theta \tilde{Z}_{[n,m]}^K} \right] = \frac{1}{n} \log \mathbb{E} \left[ e^{\theta \tilde{Z}_{[1,m]}^K} \right].
\]

Moreover thanks to the subadditive property of \( Z \), we have \( \mathbb{P}(\tilde{Z}_{[1,m]}^K \leq nK) = 1 \), so that \( \tilde{\Lambda}_Z^K(\theta) \leq \theta K \). Hence by the first part of the proof, we have

\[
\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\tilde{Z}^K > x) \geq -\tilde{\theta}^K,
\]

with \( \tilde{\theta}^K = \sup\{\theta > 0, \tilde{\Lambda}_Z^K(\theta) + \Lambda_T(-\theta) < 0\} \). We now prove that \( \tilde{\theta}^K \to \eta \) as \( K \) tends to infinity which will conclude the proof. Note that for any fixed \( \theta \geq 0 \), the function \( \tilde{\Lambda}_Z^K(\theta) \) is
nondecreasing in $K$ and $\lim_{K \to \infty} \hat{\Lambda}_Z^K(\theta) = \hat{\Lambda}_Z(\theta) \leq \Lambda_Z(\theta)$. This directly implies that $\hat{\theta}^K \geq \eta$. Take $\theta > \eta$, so that $\Lambda_Z(\theta) = \infty$. If $\Lambda_Z(\theta) < \infty$, then for all $K$, we have $\hat{\Lambda}_Z^K(\theta) \leq \Lambda_Z(\theta) < \infty$. But, we have $\hat{\Lambda}_Z^K(\theta) = \inf_n \frac{1}{n} \log \mathbb{E} \left[ e^{\theta Z_{1:n}} \right]$, so that there exists $n$ such that
\[
\mathbb{E} \left[ e^{\theta Z_{1:n}(N_0)}, \max(Z_1, \ldots, Z_n) \leq K \right] \leq e^{n(\hat{\Lambda}_Z^K(\theta)+1)} \leq e^{n(\hat{\Lambda}_Z(\theta)+1)},
\]
but the left-hand side tends to infinity as $K \to \infty$. Hence we proved that for all $\theta > \eta$, we have $\hat{\Lambda}_Z^K(\theta) \to \infty$ as $K \to \infty$. This implies that $\hat{\theta}^K \to \eta$ as $K \to \infty$. □

4.4. Upper bound.

**Lemma 4.4.** Under previous assumptions, we have
\[
\limsup_{x \to \infty} \frac{1}{x} \log \mathbb{P}(Z > x) \leq -\theta^*.
\]

**Proof.** For $L$ sufficiently large, we have with the convention $\sum_0^{-1} = 0$ (see Proposition 4.2),
\[
Z \leq \sup_{n \geq 0} \left( \sum_{i=-n}^{-1} \hat{s}_i(L) - \hat{\tau}_{i+1}(L) \right) + \hat{s}_0(L) =: V(L) + \hat{s}_0(L).
\]

We will show that under previous assumptions, we have
\[
\limsup_{x \to \infty} \frac{1}{x} \log \mathbb{P}(V(L) + \hat{s}_0(L) > x) \leq -\theta_L,
\]
where $\theta_L$ is defined as in (4.4) and the lemma will follow since $\theta_L \to \theta^*$ as $L$ tends to infinity (see Lemma 4.2).

First note that for all $\theta \in (0, \theta_L)$, we have
\[
\max \left\{ \mathbb{E} \left[ e^{\theta \hat{s}_0(L)} \right], \mathbb{E} \left[ e^{\theta V(L)} \right] \right\} < \infty.
\]
Hence for $\theta \in (0, \theta_L)$, we have $\mathbb{E} \left[ e^{\theta (V(L) + \hat{s}_0(L))} \right] = \mathbb{E} \left[ e^{\theta V(L)} \right] \mathbb{E} \left[ e^{\theta \hat{s}_0(L)} \right] \leq A$ for some finite constant $A$. Hence by Chernoff’s inequality,
\[
\mathbb{P}(V(L) + \hat{s}_0(L) \geq x) \leq e^{-\theta x} \mathbb{E} \left[ e^{\theta (V(L) + \hat{s}_0(L))} \right] \leq Ae^{-\theta x}.
\]
Since the above holds for all $0 < \theta < \theta_L$, we get
\[
\limsup_{x \to \infty} \frac{1}{x} \log \mathbb{P}(V(L) + \hat{s}_0(L) \geq x) \leq -\theta_L.
\]

\[\square\]

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Appendix: recursion for queues in tandem

We consider a G/G/1/∞ → .G/1/∞ tandem queue, where \( \{ \sigma_n^{(i)} \} \) denotes the sequence of service times in station \( i = 1, 2 \) and \( N = \{ T_n \} \) is the sequence of arrival times at the first station. For \( m \leq k \leq n \), we denote by \( D_{[m,n]}^{(i)}(k) \) the departure time of customer \( k \) from station \( i = 1, 2 \), when the network starts empty and is fed by \( N_{[m,n]} \). With the notations introduced in Section 2, we have \( X_{[m,n]}(N) = D_{[m,n]}^{(2)}(n) \). We now derive the recursion equations satisfied by the \( D_{[m,n]}^{(i)} \)'s,

\[
\begin{align*}
D_{[m,n]}^{(1)}(m) &= T_m + \sigma_m^{(1)}, \\
D_{[m,n]}^{(2)}(m) &= D_{[m,n]}^{(1)}(m) + \sigma_m^{(2)} = T_m + \sigma_m^{(1)} + \sigma_m^{(2)}, \\
D_{[m,n]}^{(1)}(k) &= \max \left( D_{[m,n]}^{(1)}(k-1), T_k \right) + \sigma_k^{(1)}, \\
D_{[m,n]}^{(2)}(k) &= \max \left( D_{[m,n]}^{(2)}(k-1), D_{[m,n]}^{(1)}(k) \right) + \sigma_k^{(2)},
\end{align*}
\]

for \( m < k \leq n \). From these equations, one can easily check that:

\[
\begin{align*}
D_{[m,n]}^{(1)}(k) &= \sup_{m \leq j \leq k} \left\{ T_j + \sum_{i=j}^{k} \sigma_i^{(1)} \right\}, \\
D_{[m,n]}^{(2)}(k) &= \sup_{m \leq j \leq k} \left\{ T_j + \sup_{j \leq \ell \leq k} \sum_{i=j}^{\ell} \sigma_i^{(1)} + \sum_{i=\ell}^{k} \sigma_i^{(2)} \right\},
\end{align*}
\]

and Equation (2.2) follows.

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