A vanishing theorem for log canonical pairs

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A VANISHING THEOREM FOR LOG CANONICAL PAIRS

By TOMMASO DE FERNEX and LAWRENCE EIN

Abstract. Using inversion of adjunction, we deduce from Nadel’s theorem a vanishing property for ideals sheaves on projective varieties, a special case of which recovers a result due to Bertram–Ein–Lazarsfeld. This enables us to generalize to a large class of projective schemes certain bounds on Castelnuovo–Mumford regularity previously obtained by Bertram–Ein–Lazarsfeld in the smooth case and by Chardin–Ulrich for locally complete intersection varieties with rational singularities. Our results are tested on several examples.

1. Introduction. The following vanishing property and its implications to questions related to Castelnuovo–Mumford regularity are the main goal of the paper.

THEOREM 1.1. Let $X$ be a locally complete intersection projective variety with rational singularities, and let $V \subset X$ be a pure-dimensional proper subscheme with no embedded components. Suppose that $V$ is scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

for some divisors $H_i \in |L^{d_i}|$, where $L$ is a globally generated line bundle on $X$ and $d_1 \geq \cdots \geq d_t$. If the pair $(X, eV)$ is log canonical where $e = \text{codim}_X V$, and $A$ is a nef and big line bundle on $X$, then

$$H^i(X, \omega_X \otimes L^k \otimes A \otimes I_V) = 0 \quad \text{for } i > 0, \; k \geq d_1 + \cdots + d_e.$$ 

In the special case when both $X$ and $V$ are assumed to be smooth this result was proven in [BEL] using vanishing theorems on certain residual intersections on the blow up of $X$ along $V$.

The proof of Theorem 1.1 follows a novelle approach, based on a result on inversion of adjunction from [EM1]. The key idea is to produce a suitable $\mathbb{Q}$-scheme $Z$ on $X$ whose multiplier ideal $\mathcal{J}(X, Z)$ coincides with $I_V$ in a neighborhood of $V$. We do this by taking $Z = (1 - \delta)B + \delta eV$, where $B$ is the base scheme of $|L^{d_1 + \cdots + d_e} \otimes I_V^c|$ and $0 < \delta \ll 1$. We reduce in this way to a setting...
in which one can directly apply Nadel’s vanishing theorem on $X$, without having to blow up. Our argument, once restricted to the case of smooth varieties, gives in particular a very simple proof of the main result in [BEL]. We remark that there are other $\mathbb{Q}$-schemes $Z$ such that $\mathcal{J}(X, Z)$ coincides with $\mathcal{I}_V$ near $V$ (e.g., $Z = eV$), but the vanishing properties one can deduce from them are in general not as strong (cf. Remark 4.6).

Vanishing theorems of the above kind are motivated by questions concerning Castelnuovo–Mumford regularity. We recall that a proper closed subscheme $V$ of $\mathbb{P}^n$ is said to be $m$-regular (in the sense of Castelnuovo–Mumford) if

$$H^i(X, \mathcal{I}_V(m - i)) = 0 \quad \text{for } i > 0. \quad (1)$$

The regularity of $V$ is the minimum value of $m$ for which (1) holds, and is denoted by $\text{reg}(V)$. The regularity of a scheme is related to the complexity of the associated ideal sheaf (and its syzygies) and, apart from a few special cases, is in general difficult to compute. For more on the notion of regularity, we refer to [La2, Section 1.8].

When $V$ is a smooth projective variety, bounds on regularity were determined in terms of the degrees of $V$ or of its defining equations. It is expected that $\text{reg}(V) \leq \deg V - e + 1$ for any non-degenerate variety $V \subset \mathbb{P}^n$ of codimension $e$. The curve case, originally studied by Castelnuovo [Ca], was completely settled (also for possibly singular curves) in [GLP], and the bound has been established for smooth surfaces in [Pi], [La1] and smooth threefolds in [Ra]; see also [Kw].

The results of [BEL] showed that in general potentially stronger bounds can be determined in terms of the degrees of a set of equations defining $V$, rather than the degree of $V$. It is convenient to introduce the notation

$$d(V) := \min\{d_1 + \cdots + d_e \mid V \text{ is cut out by forms of degrees } d_1 \geq \cdots \geq d_e\},$$

where $e = \text{codim}_{\mathbb{P}^n}(V)$. Then the result in [BEL] says that if $V \subset \mathbb{P}^n$ is a smooth variety of codimension $e$, then

$$\text{reg}(V) \leq d(V) - e + 1,$$

and equality holds if and only if $V$ is a complete intersection in $\mathbb{P}^n$.

Here we address the same question allowing $V$ to a singular variety or, more generally, a singular scheme in $\mathbb{P}^n$. It was proven in [CU] using liaison theory that the above bound also holds in the case $V$ is a locally complete intersection variety with rational singularities. By contrast, the regularity is in general sensitive to the singularities, and there are in fact examples of subschemes whose regularity grows doubly exponentially in the generating degrees (cf. [BS]).
In light of Theorem 1.1, it becomes natural to impose conditions within the context of singularities of pairs. We first discuss the case in which $V$ is a variety, where we can impose conditions directly on $V$.

If $V$ is a locally complete intersection variety, then we simply assume that $V$ has log canonical singularities. Since locally complete intersection varieties have canonical (and hence log canonical) singularities, this particular case of our result includes, essentially, the main result in [CU] (cf. Remarks 5.2 and 5.4).

More generally, we consider varieties $V$ that are not necessarily locally complete intersection. We still require that $V$ is normal and $\mathbb{Q}$-Gorenstein. In order to compare the singularities of $V$ with those of the pair $(\mathbb{P}^n, eV)$, we rely on a more general version of inversion of adjunction, proven in [EM2], [Ka]. For any positive integer $r$ such that $rK_V$ is Cartier, there is a subscheme $\Sigma_r \subset V$, only supported on the locus where $V$ is not locally complete intersection, which measures---so to speak---how far $V$ is from being locally complete intersection (see Definitions 3.2 and 3.3 below); one has that $V$ is locally complete intersection if and only if $\Sigma_r = \emptyset$. We therefore assume that the pair $(V, \frac{1}{r} \Sigma_r)$ has log canonical singularities. This condition is independent of the choice of $r$; if $V$ is locally complete intersection, then the condition is equivalent to $V$ being log canonical.

In the same spirit of [BEL], we obtain the following results.

**Corollary 1.2.** Let $V \subset \mathbb{P}^n$ is a normal $\mathbb{Q}$-Gorenstein variety of codimension $e$, and suppose that the pair $(V, \frac{1}{r} \Sigma_r)$ has log canonical singularities, where $\Sigma_r$ is defined as above. Then

$$\text{reg}(V) \leq d(V) - e + 1$$

and equality holds if and only if $V$ is a complete intersection in $\mathbb{P}^n$.

**Corollary 1.3.** With the same assumptions as in Corollary 1.2, if $d(V) \leq n + 1$, then $V$ is projectively normal. Moreover, if $V$ has rational singularities and $d(V) \leq n$, then $V$ is projectively Cohen–Macaulay.

An important feature of our method is that it allows us to consider a much larger class of subschemes of $\mathbb{P}^n$ by lifting our conditions on singularities to the pair $(\mathbb{P}^n, eV)$, as the corresponding notions on $V$ would not make sense in such generality. We obtain in this way the following result (which, by inversion of adjunction, includes Corollary 1.2 as a special case).

**Corollary 1.4.** Let $V \subset \mathbb{P}^n$ be a pure-dimensional subscheme with no embedded components, and assume that the pair $(\mathbb{P}^n, eV)$ is log canonical, where $e = \text{codim}_{\mathbb{P}^n} V$. Then

$$\text{reg}(V) \leq d(V) - e + 1$$

and equality holds if and only if $V$ is a complete intersection in $\mathbb{P}^n$. 
The hypotheses of this result are satisfied, for instance, by examples arising from classical constructions, such as determinantal varieties and generic projections. It is interesting to observe that these constructions in general produce varieties that are not locally complete intersection and, at least in the second case, not normal. Moreover, a construction via residual intersections provides a family of examples whose regularity is precisely one less the one expected for complete intersections (a characterization is given in Proposition 5.3 below). These examples are discussed in the last section of the paper.

A different proof of \([\text{BEL}]\) was given in \([\text{Ber}]\) by applying Nadel vanishing theorem to a certain multiplier ideal on the blow up of \(X\) along \(V\). In fact, the same approach can be extended, without much change, to the case (considered in \([\text{CU}]\)) in which \(V\) is a locally complete intersection variety with rational singularities.

What inversion of adjunction gives us is a way to go beyond these conditions on singularities, as it allows us to avoid altogether to work on the blow up of \(X\) along \(V\). Moreover, once the deep result from inversion of adjunction is granted, the approach followed in this paper gives a quite simple and clean proof which applies, uniformly, to all cases.

The authors are grateful to Rob Lazarsfeld for bringing \([\text{Ber}]\) to their attention.

2. Vanishing theorems for multiplier ideal sheaves. Let \(X\) be a variety with rational singularities, and assume that its canonical class \(K_X\) is Cartier. Let \(\omega_X \cong \mathcal{O}_X(K_X)\) be the dualizing sheaf.

If \(\mathcal{L}\) is a nef and big line bundle on \(X\), then

\[ H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad \text{for} \quad i > 0 \]

by the Kawamata–Viehweg vanishing theorem. Multiplier ideals arise naturally in this context as a way to encode this property from different birational models of \(X\).

Consider a formal linear combination

\[ Z = \sum_{j=1}^{k} c_j Z_j, \]

where \(Z_j \subseteq X\) are proper closed subschemes defined by ideal sheaves \(\mathcal{I}_{Z_j} \subseteq \mathcal{O}_X\), and \(c_j \in \mathbb{R}\). The multiplier ideal sheaf \(\mathcal{J}(X, Z)\) of the pair \((X, Z)\) is then defined by

\[ \mathcal{J}(X, Z) = f_* \mathcal{O}_Y \left( \left[ K_{Y/X} - \sum c_j f^{-1}(Z_j) \right] \right) \subseteq \mathcal{O}_X, \]

where \(f: Y \to X\) is any log resolution of \((X, Z)\), \(K_{Y/X}\) is the relative canonical divisor of \(f\), and \(f^{-1}(Z_j)\) is the subscheme of \(Y\) defined by \(\mathcal{I}_{Z_j} \cdot \mathcal{O}_Y\). We recall that the definition of log resolution requires that \(f\) is a proper birational morphism
from a smooth variety $Y$ such that each $f^{-1}(Z_j)$ and the exceptional locus of $f$ are divisors whose supports form, together, a simple normal crossings divisor.

By applying Kawamata–Viehweg vanishing theorem on log resolutions, one obtains the following vanishing theorem.

**Theorem 2.1.** (Nadel vanishing theorem) With the above notation, suppose that $A_j$ and $M$ are Cartier divisors on $X$ such that $O_X(A_j) \otimes I_{Z_j}$ is globally generated for each $j$ and $M - \sum c_j A_j$ is nef and big. Then

$$H^i(X, \omega_X \otimes O_X(M) \otimes I_{J(X,Z)}) = 0 \quad \text{for } i > 0.$$ 

We refer to [La2] for proofs, more general formulations of these vanishing theorems, and more general settings where multiplier ideal sheaves are defined.

Broadly speaking, the main aim of this paper is to determine sufficient conditions for the vanishing of the higher cohomology groups

$$H^i(X, \omega_X \otimes M \otimes I),$$

where $I \subset O_X$ is an ideal sheaf and $M$ is a line bundle on $X$.

As one may expect, the strategy is very simple, and relies on finding formal linear combinations $Z = \sum c_j Z_j$ of proper closed subschemes of $X$ such that $I$ is realized as the multiplier ideal sheaf

$$I = J(X,Z),$$

so that conditions on $M$ can be determined to ensure to be in the setting of Nadel vanishing theorem. One should notice however that different choices of $Z$ may lead to different conditions on $M$ (cf. Remark 4.6 below).

A simple application of inversion of adjunction, which is the topic of the next section, is the key ingredient that allows us to determine good descriptions of certain ideal sheaves as multiplier ideal sheaves.

### 3. Inversion of adjunction

Let $X$ be a normal variety such that $K_X$ is $\mathbb{Q}$-Cartier (we say that $X$ is $\mathbb{Q}$-Gorenstein), and let $Z = \sum c_j Z_j$ be a formal linear combination of proper closed subschemes of $X$, with $c_j \in \mathbb{R}$.

For any closed subset $W \subseteq X$, we define the minimal log discrepancy

$$\text{mld}(W; X, Z) := \inf_{f(E) \subseteq W} \left\{ \text{ord}_E(K_{Y/X}) + 1 - \sum c_j \text{val}_E(Z_j) \right\},$$

where the infimum is computed by considering all log resolutions $f: Y \to X$ of $(X, Z)$ and all prime divisors $E$ on any such $Y$ with center $f(E)$ contained in $W$. If $W = \{p\}$, where $p \in X$ is a closed point, then we denote this number by $\text{mld}(p; X, Z)$. If $W$ is irreducible and $\eta_W$ is the generic point of $W$, then we
similarly define $\text{mld} (\eta_W; X, Z)$ by taking the infimum over all prime divisors $E$ with center $f(E) = W$. For more on minimal log discrepancies, we refer to [K +].

It is easy to see that minimal log discrepancies can either be $\geq 0$ or $-\infty$. The pair $(X, Z)$ is said to be log canonical (or to have log canonical singularities) if $\text{mld} (X; X, Z) \geq 0$.

We will use the following consequence of inversion of adjunction.

**Proposition 3.1.** Let $X$ be a locally complete intersection normal variety, and let $V \subset X$ be a proper subscheme of codimension $e$, scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

for some divisors $H_i \in |L^{d_1}|$, where $L$ is a globally generated line bundle on $X$ and $d_1 \geq \cdots \geq d_t$. Suppose that both $X$ and $V$ are smooth at the generic point of each irreducible component of maximal dimension of $V$. Then for any point $p \in V$ there are sufficiently general $D_i \in |L^{d_1} \otimes I_V|$ for $i = 1, \ldots, e$, such that

$$\text{mld} (p; X, eV) = \text{mld} (p; X, D_1 + \cdots + D_e).$$

**Proof.** First we observe that $L^{d_1} \otimes I_V$ is globally generated. Since one can compute minimal log discrepancies by using a log resolution where the inverse image of $p$ is a simple normal crossing divisor, it follows in particular that if $D_1 \in |L^{d_1} \otimes I_V|$ is sufficiently general, then

$$\text{mld} (p; X, V + Z) = \text{mld} (p; X, D_1 + Z)$$

for any formal linear combination $Z = \sum c_j Z_j$ of proper subschemes of $X$.

We proceed by induction on $e \geq 1$. The above formula with $Z = 0$ verifies the statement for $e = 1$, so we can assume that $e \geq 2$. We fix general $D_1 \in |L^{d_1} \otimes I_V|$ in the following order: we first choose a general $D_1$, then a general $D_2$ (the generality condition on $D_2$ may depend on the choice of $D_1$), and so forth. Note that

$$V = D_1 \cap \cdots \cap D_e \cap H_{e+1} \cap \cdots \cap H_t.$$

Note that $D_1$ is a locally complete intersection scheme, and it is regular in codimension one, and hence normal, if chosen with sufficient generality. Indeed, since the base locus of $|L^{d_1} \otimes I_V|$ is equal to the support of $V$, the restriction of a general $D_1$ to the open set $X^0 = X \setminus V$ is smooth on the regular locus of $X^0$ and intersects properly the singular locus of $X^0$, and thus it is regular in codimension one. On the other hand, since both $X$ and $V$ are smooth at the generic point of each irreducible component of $V$ of codimension $e$, a general choice of $D_1$ will also be smooth at that point. Recalling that we are assuming that $e \geq 2$, we conclude that a sufficiently general $D_1$ is regular in codimension one.
Since $D_1$ is smooth at the generic point of each irreducible component of $V$ of maximal dimension, we can apply induction by restricting to $D_1$. By (2) with $Z = (e - 1)V$, [EM2, Corollary 8.2] applied twice, and induction, we get

$$\text{mld}(p; X, eV) = \text{mld}(p; X, D_1 + (e - 1)V)$$
$$= \text{mld}(p; D_1, (e - 1)V)$$
$$= \text{mld}(p; D_1, D_2|_{D_1} + \cdots + D_e|_{D_1})$$
$$= \text{mld}(p; X, D_1 + \cdots + D_e).$$

Suppose now that $V$ is a normal $\mathbb{Q}$-Gorenstein variety, and let $r$ be a positive integer such that $rK_V$ is Cartier. The image of the canonical map $(\Omega^\dim V) \otimes r \to \mathcal{O}_V(rK_V)$ is equal to $I_r \cdot \mathcal{O}_V(rK_V)$ for some ideal sheaf $I_r \subseteq \mathcal{O}_V$. Let

$$J_r := (\text{Jac}_r V : I_r) \subseteq \mathcal{O}_V,$$

where $\text{Jac}_V \subseteq \mathcal{O}_V$ is the Jacobian ideal sheaf of $V$. By [EM2, Corollary 9.4], the ideal sheaves $\text{Jac}_r$ and $I_r \cdot J_r$ have the same integral closure. Let $\Sigma_r \subset V$ be the subscheme defined by $J_r$.

**Definition 3.2.** With the above notation, we call $\Sigma_r$ the $r$-th lci-defect sub-scheme of $V$.

By taking an embedding of $V$ in a smooth ambient variety $X$, it follows from [EM2, Theorem 8.1] that if $s$ is another positive integer such that $sK_V$ is Cartier, then

$$\text{mld}(W; V, 1/r \Sigma_r) = \text{mld}(W; X, eV) = \text{mld}(W; V, 1/r \Sigma_r)$$

for every closed subset $W \subseteq V$, where $e = \text{codim}_X V$ (see also [Ka]). In particular, the property that $(V, 1/r \Sigma_r)$ be log canonical (in some neighborhood or on the whole $V$) is independent of the choice of $r$. As we will be using this condition on singularities, we fix the following terminology:

**Definition 3.3.** A normal $\mathbb{Q}$-Gorenstein variety $V$ is said to be lci-defectively log canonical, or to have lci-defectively log canonical singularities, if for some (equivalently, for every) positive integer $r$ such that $rK_X$ is Cartier, the pair $(V, 1/r \Sigma_r)$ is log canonical.

**Remark 3.4.** Being lci-defectively log canonical is in general a stronger condition than being log canonical; the two conditions are equivalent if $V$ is a locally complete intersection variety.

One immediately obtains from the proposition the following variant of inversion of adjunction.
Corollary 3.5. Let $X$ be a smooth variety, and let $V \subset X$ be proper subvariety of codimension $e$, scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

for some divisors $H_i \in |L^{\otimes d_i}|$, where $L$ is a line bundle on $X$ and $d_1 \geq \cdots \geq d_t$. Assume that $V$ is a normal variety with $\mathbb{Q}$-Gorenstein singularities, let $r$ be a positive integer such that $rK_V$ is Cartier, and let $\Sigma_r$ be the $r$-th lci-defect subscheme of $V$. Then for every point $p \in V$, there are sufficiently general $D_i \in |L^{\otimes d_i} \otimes I_V|$, for $i = 1, \ldots, e$, such that

$$\mld (p; V, \frac{1}{r}\Sigma_r) = \mld (p; X, D_1 + \cdots + D_e).$$

In particular, $(X, D_1 + \cdots + D_e)$ is log canonical in a neighborhood of $p$ if and only if $V$ is lci-defectively log canonical in a neighborhood of $p$.

Proof. By [EM2, Theorem 8.1] we have

$$\mld (p; V, \frac{1}{r}\Sigma_r) = \mld (p; X, eV),$$

and thus the first assertion follows from Proposition 3.1. The last assertion is a consequence of the general fact that if $Y$ is a normal $\mathbb{Q}$-Gorenstein variety, $Z \subset Y$ is a proper closed subscheme, and $c > 0$, then for every $q \in Y$ the pair $(Y, cZ)$ is log canonical near $q$ if and only if $\mld (q; Y, cZ) \geq 0$.

4. Proof of Theorem 1.1. Let $X$ be a locally complete intersection projective variety with rational singularities, and consider on $X$ a globally generated line bundle $L$ and a nef and big line bundle $A$. Let $V \subset X$ be a pure-dimensional proper subscheme with no embedded points, scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

for some divisors $H_i \in |L^{\otimes d_i}|$, where $d_1 \geq \cdots \geq d_t$. Let $e = \codim_X V$, and suppose that the pair $(X, eV)$ is log canonical.

We consider the base scheme $B \subset X$ of the linear system

$$|L^{\otimes (d_1 + \cdots + d_e)} \otimes I_V|.$$

Given any $p \in V$, we fix sufficiently general divisors $D_i \in |L^{\otimes d_i} \otimes I_V|$ for $i = 1, \ldots, e$. Note that

$$D = D_1 + \cdots + D_e \in |L^{\otimes (d_1 + \cdots + d_e)} \otimes I_V|.$$
The following property is of independent interest.

**Proposition 4.1.** Let $X$ be a locally complete intersection normal variety. Let $V \subset X$ be a closed subscheme, and let $\eta_W$ be the generic point of an irreducible component $W$ of $V$. Suppose that $(X, eV)$ is log canonical at $\eta_W$, where $e$ is the codimension of $W$ in $X$. Then both $X$ and $V$ are smooth at $\eta_W$.

**Proof.** This can easily be seen using inversion of adjunction, as follows. We work locally near $\eta_W$, so that all minimal log discrepancies over $\eta_W$ considered in the proof can be thought as minimal log discrepancies over $W$; this allows us to apply inversion of adjunction where needed.

We also assume that $X$ is a complete intersection subvariety of a smooth variety $M$. Let $c$ be the codimension of $X$ in $M$. If $c = 0$, then $X$ is smooth at $\eta_W$. Suppose otherwise that $c > 0$, and let $H, L \subset M$ be general (sufficiently positive) hyperplane sections vanishing, respectively, along $X$ and $V$. It follows by inversion of adjunction that

$$\text{mld} (\eta_W; M, cH + eL) = \text{mld} (\eta_W; M, cX + eV) = \text{mld} (\eta_W; X, eV) \geq 0$$

On the other hand, since $M$ is smooth and $\text{codim}_M W = c + e$, we have $\text{mld} (\eta_W; M, cH + eL) \leq 0$, with equality holding only if both $H$ and $L$ are smooth at $\eta_W$. We conclude that this is the case, and in particular that $\text{mld} (\eta_W; X, eV) = 0$.

As we can assume that the equation of $H$ belongs to a regular sequence on $M$ locally cutting out $X$ as a complete intersection near $\eta_W$, it follows by induction on $c$ that $X$ is smooth at $\eta_W$.

Let $S$ be, locally near $\eta_W$, a general hyperplane section of $X$ vanishing along $V$ (equivalently, let $S = L \cap X$). Then $\text{mld} (\eta_W; X, eS) = \text{mld} (\eta_W; X, eV) = 0$, which implies that $S$ is smooth at $\eta_W$. Applying inversion of adjunction on $X$, we obtain

$$\text{mld} (\eta_W; S, (e - 1)V) = \text{mld} (\eta_W; X, S + (e - 1)V) = \text{mld} (\eta_W; X, eV) = 0.$$

We conclude by induction on $e$ that $V$ is smooth at $\eta_W$. \qed

**Remark 4.2.** This property is related to a conjecture of Shokurov, which says that if $(X, B)$ is an effective log pair (namely, $X$ is normal and $B$ is an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier) and $\xi \in X$ is a Grothendieck point of codimension $e$, then $\text{mld} (\xi; X, B) \leq e$, and moreover $\text{mld} (\xi; X, B) \leq e - 1$ if $X$ is singular at $\xi$ (cf. [Am]). Similar arguments as in the proof of Proposition 4.1 show that this conjecture is true if $X$ is a locally complete intersection variety. Indeed, it suffices to show that the condition $\text{mld} (\xi; X, B) > e - 1$ implies that $X$ is smooth at $\xi$. Assuming that $X$ is locally complete intersection, one takes, locally near $\xi$, an embedding of $X$ as a complete intersection subvariety of a smooth variety $M$. Then one can reduce the codimension $c$ of the embedding (assuming that $c > 0$) as follows. First notice that $B$ is $\mathbb{Q}$-Cartier, and thus $B = ra$, where $A$ is an effective Cartier divisor and $r \in \mathbb{Q}_+$. We regard $A$ as a subscheme of $X$, and
hence of $M$. Working locally near $\xi$ and taking a general (sufficiently positive) hyperplane section $H \subset M$ vanishing along $X$, inversion of adjunction gives

$$\operatorname{mld} (\xi; M, cH + rA) = \operatorname{mld} (\xi; M, cX + rA) = \operatorname{mld} (\xi; X, rA) > e - 1.$$ 

This implies that $H$ is smooth at $\xi$, and thus we conclude by induction on $c$ that $X$ is smooth too at $\xi$, as required.

We now come back to the setting introduced at the beginning of the section. By Proposition 4.1, we see that both $X$ and $V$ are smooth at the generic point of each irreducible component $V_i$ of $V$. We can therefore apply Proposition 3.1, which implies that $(X, D)$ is log canonical near $p$. Observing that $\mathcal{O}_X(-D) \subseteq \mathcal{I}_B$, we conclude that $(X, B)$ is log canonical near $p$, and therefore in a whole neighborhood of $V$.

Note that

$$\operatorname{val}_E (B) \geq e \cdot \operatorname{val}_E (V)$$

for every prime divisor $E$ over $X$, and this is an equality if $E = E_i$ is the divisor dominating an irreducible component $V_i$ of $V$ that is extracted by the blow up

$$\mu: X' = \text{Bl}_V X \to X$$

of $X$ along $V$. The inequality follows by the inclusion $\mathcal{I}_B \subseteq \mathcal{I}_Y$. For the second assertion, just notice that if $D_1, \ldots, D_e$ are generally chosen as above, then their equations locally cut out $V$ at the generic point of $V_i$, and hence $D$ has multiplicity $e$ at that point.

**Lemma 4.3.** With the above notation, if $0 < \delta \ll 1$, then

$$\mathcal{J}(X, (1 - \delta)B + \delta eV) = \mathcal{I}_{V \cup W},$$

where $W$ is a closed subscheme of $X$ disjoint from $V$.

**Proof.** Observe that if $E$ is a prime divisor on some model $Y$ over $X$, with center contained in $V$, then

$$\operatorname{val}_E ((1 - \delta)B + \delta eV) - \operatorname{ord}_E (K_{Y/X}) \leq \operatorname{val}_E (B) - \operatorname{ord}_E (K_{Y/X}) \leq 1 \leq \operatorname{val}_E (V),$$

and this is a chain of equalities if $E = E_i$ is the divisor dominating an irreducible component $V_i$ that is extracted by $\mu$. On the other hand, if the center of $E$ intersects $V$ but is not contained in it, then

$$\operatorname{val}_E ((1 - \delta)B + \delta eV) - \operatorname{ord}_E (K_{Y/X}) = \operatorname{val}_E ((1 - \delta)B) - \operatorname{ord}_E (K_{Y/X}) < 1,$$
since \((X, (1 - \delta)B)\) is Kawamata log terminal in a neighborhood \(U \subseteq X\) of \(V\), and hence
\[
\lfloor \text{val}_E ((1 - \delta)B + \delta eV) - \text{ord}_E (K_Y/X) \rfloor \leq 0 = \text{val}_E (V).
\]
Since \(V\) is a generically reduced pure-dimensional scheme with no embedded components, its ideal sheaf has primary decomposition
\[
\mathcal{I}_V = \bigcap \mathcal{I}_{V_i},
\]
where \(\mathcal{I}_{V_i}\) is the ideal sheaf of the irreducible component \(V_i\). We have \(\mathcal{I}_{V_i} = \mu_* \mathcal{O}_{X'}( - E_i)\), and hence
\[
\mathcal{J}(X, (1 - \delta)B + \delta eV)|_U = \mathcal{I}_V|_U.
\]
Therefore
\[
\mathcal{J}(X, (1 - \delta)B + \delta eV) = \mathcal{I}_{V\cup W}
\]
for some subscheme \(W\) in the complement of \(U\), and hence disjoint from \(V\). This completes the proof of the lemma. \(\square\)

Applying Nadel’s vanishing theorem, we obtain
\[
H^i(X, \omega_X \otimes L^{\otimes k} \otimes \mathcal{A} \otimes \mathcal{I}_{V\cup W}) = 0 \quad \text{for } i > 0, \ k \geq d_1 + \cdots + d_e.
\]
The deduce the stated vanishing from the following lemma.

**Lemma 4.4.** Let \(Z_1\) and \(Z_2\) be two disjoint, nonempty, closed subschemes of \(X\), and suppose that for some nef and big line bundle \(\mathcal{M}\) we have
\[
H^i(X, \omega_X \otimes \mathcal{M} \otimes \mathcal{I}_{Z_1 \cup Z_2}) = 0 \quad \text{for } i > 0.
\]
Then for each \(j = 1, 2\) we have
\[
H^i(X, \omega_X \otimes \mathcal{M} \otimes \mathcal{I}_{Z_j}) = 0 \quad \text{for } i > 0.
\]

**Proof.** Since \(Z_1 \cap Z_2 = \emptyset\), we have the exact sequence
\[
0 \to \omega_X \otimes \mathcal{M} \otimes \mathcal{I}_{Z_1 \cup Z_2} \to \omega_X \otimes \mathcal{M} \to (\omega_X \otimes \mathcal{M}|_{Z_1}) \oplus (\omega_X \otimes \mathcal{M}|_{Z_2}) \to 0.
\]
By Kawamata–Viehweg vanishing theorem, the higher cohomology groups of \(\omega_X \otimes \mathcal{M}\) are trivial. Therefore we see by the vanishing \(H^i(X, \omega_X \otimes \mathcal{M} \otimes \mathcal{I}_{Z_1 \cup Z_2}) = 0\) for \(i > 0\) that
\[
H^0(X, \omega_X \otimes \mathcal{M}) \to H^0(Z_1, \omega_X \otimes \mathcal{M}|_{Z_1}) \oplus H^0(Z_2, \omega_X \otimes \mathcal{M}|_{Z_2})
\]
is surjective and

$$H^i(Z_1, \omega_X \otimes \mathcal{M}|_{Z_1}) \oplus H^i(Z_2, \omega_X \otimes \mathcal{M}|_{Z_2}) = 0 \quad \text{for} \ i > 0.$$  

These properties imply the surjectivity of the maps

$$H^0(X, \omega_X \otimes \mathcal{M}) \to H^0(Z_j, \omega_X \otimes \mathcal{M}|_{Z_j})$$

and the vanishings

$$H^i(Z_j, \omega_X \otimes \mathcal{M}|_{Z_j}) = 0 \quad \text{for} \ i > 0.$$  

We deduce from the exact sequences

$$0 \to \omega_X \otimes \mathcal{M} \otimes I_{Z_j} \to \omega_X \otimes \mathcal{M} \to \omega_X \otimes \mathcal{M}|_{Z_j} \to 0$$

that the cohomology groups $H^i(X, \omega_X \otimes \mathcal{M} \otimes I_{Z_j})$ vanish for $i > 0$.  

**Remark 4.5.** By inversion of adjunction (see [EM2, Theorem 8.1]), the theorem applies in particular to the case when $X$ is smooth and $V$ is a $\mathbb{Q}$-Gorenstein subvariety with lci-defectively log canonical singularities. Equivalently, one can use Corollary 3.5 to adapt the above proof to this setting.

**Remark 4.6.** It is easy to see by inversion of adjunction that the ideal sheaf $I_V$ can be realized as the multiplier ideal $J(X, eV)$, and so one gets a very short proof of the theorem in the special case when all the degrees $d_i$ are equal, as the fact that $L^{\otimes d_1} \otimes I_V$ is globally generated implies immediately that

$$H^i(\omega_X \otimes L^{\otimes k} \otimes \mathcal{A} \otimes I_V) = 0 \quad \text{for} \ i > 0, \ k \geq ed_1$$

by Nadel vanishing theorem. However this realization of $I_V$ as a multiplier ideal sheaf does not yield the desired vanishing result when the degrees $d_1, \ldots, d_e$ are not all equal.

**5. Applications to regularity and projective normality.** Given a subscheme $V \subset \mathbb{P}^n$, we have introduced the notation

$$d(V) := \min\{d_1 + \cdots + d_e \mid V \text{ is cut out by forms of degrees } d_1 \geq \cdots \geq d_e\},$$

where $e = \text{codim}_{\mathbb{P}^n}(V)$. As a particular case of Theorem 1.1, we obtain the following vanishing theorem for ideal sheaves on projective spaces:

**Corollary 5.1.** Let $V \subset \mathbb{P}^n$ be a pure-dimensional proper subscheme with no embedded components, and assume that the pair $(\mathbb{P}^n, eV)$ is log canonical, where
e = \text{codim}_{\mathbb{P}^n} V. Then

\[ H^i(\mathbb{P}^n, \mathcal{I}_V(k)) = 0 \quad \text{for } i > 0, \quad k \geq d(V) - n. \]

The first three corollaries stated in the introduction follow from this result by arguments analogous to those in [BEL], that we sketch below.

**Proof of Corollary 1.4.** For short, let \( d = d(V) \). The required vanishing

\[ H^i(\mathbb{P}^n, \mathcal{I}_V(d - e + 1 - i)) = 0 \quad \text{for } i > 0 \]

follows for \( 1 \leq i \leq n - e + 1 \) by Corollary 5.1 and for \( i > n - e + 1 \) by dimensional reasons.

Regarding the second assertion, suppose that \( V \) is non \((d - e)\)-regular. Then necessarily

\[ H^{n-e}(\mathcal{O}_V(d - n - 1)) = H^{n-e+1}(\mathbb{P}^n, \mathcal{I}_V(d - e - (n - e + 1))) \neq 0. \]

It follows by duality that

\[ H^0(\omega_V(-d + n + 1)) = \text{Hom}(\mathcal{O}_V(d - n - 1), \omega_V) \neq 0, \]

where \( \omega_V \) is the dualizing sheaf of \( V \) (cf. [Ha, Proposition III.3.1]).

Let \( d_1 \geq \cdots \geq d_t \) be the degrees of the equations cutting \( V \) such that \( d(V) = d_1 + \cdots + d_t \), and let \( W \) be a general complete intersection of type \((d_1, \ldots, d_t)\) containing \( V \). Note that \( W \) is pure dimensional with no embedded components. Denoting by \( \mathcal{I}_V \) and \( \mathcal{I}_W \) the saturated ideals of \( V \) and \( W \) in the polynomial coordinate ring of \( \mathbb{P}^n \), let \( R \subseteq \mathbb{P}^n \) be the subscheme defined by the ideal

\[ \mathcal{I}_R = [\mathcal{I}_W : \mathcal{I}_V] \]

(which is automatically saturated). By [Mi, Proposition 4.2.4], \( R \) has no embedded points, is equidimensional (of the same dimension of \( W \)), and is algebraically linked to \( V \) via \( W \). In particular, \( V \) is algebraically linked to \( R \) via \( W \), and thus we have an exact sequence of sheaves

\[ 0 \to \mathcal{I}_W \to \mathcal{I}_R \to \omega_V(n + 1 - d) \to 0 \]

by [Mi, Proposition 4.2.6]. Observing that \( \mathcal{I}_{R/W} \cong \mathcal{I}_{R\cap V/V} \) since \( W \) has no embedded components, we obtain an isomorphism

\[ \omega_V \cong \mathcal{I}_{R\cap V/V}(d - n - 1). \]
Going back to (3), we conclude that $H^0(\mathcal{I}_{R \cap V/V}) \neq 0$. Since $W$ is connected, if $R$ were nonempty, then it would intersect every connected component of $V$, and hence $H^0(\mathcal{I}_{R \cap V/V})$ would be trivial. Therefore $R = \emptyset$.

**Proof of Corollary 1.2.** It follows from Corollary 1.4, since by inversion of adjunction $(\mathbb{P}^n, eV)$ is log canonical (cf. [EM2], [Ka]).

**Remark 5.2.** The bound on regularity obtained in [BEL] also holds allowing $V$ to be singular at a finite number of points. Similarly, in the statement of [CU], $V$ is only assumed to have rational singularities away from a one dimensional set (the hypothesis of $V$ begin locally complete intersection in [CU] is also only needed away from a finite set). At the moment it is unclear to us whether the methods applied in this paper admit a similar extension.

**Proof of Corollary 1.3.** The proof is essentially the same as the one of [BEL, Corollary 2]. Regarding the second part of the statement, in order to get the vanishings

$$H^i(\mathcal{O}_V(k)) = 0 \quad \text{for} \ 0 < i < \dim V \text{ and } k > 0,$$

we take a resolution of singularities $g: V' \to V$. Since $V$ has rational singularities, we have $H^i(V, \mathcal{O}_V(k)) = H^i(V', g^* \mathcal{O}_V(k))$, and thus the required vanishings follow by Serre duality and Kawamata–Viehweg vanishing theorem.

As in Corollary 1.4, let $V \subset \mathbb{P}^n$ be a pure-dimensional subscheme with no embedded components, such that the pair $(\mathbb{P}^n, eV)$ is log canonical, where $e = \text{codim}_{\mathbb{P}^n} V$. Assume that $V$ is Cohen–Macaulay and smooth in codimension 1, and let $d_1 \geq \cdots \geq d_t$ be the degrees of the equations cutting $V$ such that $d(V) = d_1 + \cdots + d_t$. We have seen that the condition $\text{reg}(V) = d(V) - e + 1$ characterizes $V$ being a complete intersection in $\mathbb{P}^n$, and it is natural to ask whether the next case can be classified as well. The arguments in the proof of Corollary 1.4 yield the following characterization.

**PROPOSITION 5.3.** With the above notation, suppose that $V$ is not a complete intersection. Then

$$\text{reg}(V) = d(V) - e$$

if and only either $H^0(\mathcal{I}_{R \cap V/V}(1)) \neq 0$ or $H^1(\mathcal{I}_{R \cap V/V}) \neq 0$, where $R$ is the algebraic linkage of $V$ via a general complete intersection of type $(d_1, \ldots, d_t)$ containing $V$.

**Remark 5.4.** The first of the two conditions considered in Proposition 5.3 is satisfied, for instance, if $V$ is a rational normal curve of degree 3 in $\mathbb{P}^3$ (which in fact it is well known to have regularity 2). A generalization of this is treated in the next example. In particular, there seems to be a shift by 1 in the definition of the notation $\text{reg}(V)$ as adopted in [CU].
Example 5.5. For $1 \leq k \leq n-1$, fix a $k$-dimensional linear subspace $\Lambda \subset \mathbb{P}^n$, and let $V \subset \mathbb{P}^n$ be the residual intersection of $e := n - k$ general quadrics containing $\Lambda$. We claim that $\text{reg}(V) = d(V) - e$.

First note that $V$ is Cohen–Macaulay (see, e.g., [Mi, Corollary 4.2.9]), and it is not difficult to check that it is smooth in codimension one, so in particular $V$ is normal. Notice also that $V$ is not complete intersection, as it is a non-degenerate variety of degree $2^e - 1$ and codimension $e$.

We observe that $(\mathbb{P}^n, eV)$ is log canonical. To see this, we construct a log resolution $f: Y \to \mathbb{P}^n$ as follows. We first take the blow up $g: Y' \to \mathbb{P}^n$ along $\Lambda$. Let $E$ be the exceptional divisor of $g$. For a general choice of the $e$ quadrics vanishing along $\Lambda$, the proper transform $V' \subset Y'$ of $V$ is smooth and intersects transversally $E$. Therefore the blow up of $Y'$ along $V'$ produces the required log resolution of $(\mathbb{P}^n, eV)$, and it is immediate to check that the discrepancies of the two exceptional divisors along this pair are $\geq -1$, so that the pair is log canonical.

By construction, we have $O_Y(-1) \subseteq I_{\Lambda \cap V}/V$, and thus

$$h^0(I_{\Lambda \cap V}/V(1)) \geq h^0(O_V) > 0.$$ 

Therefore we conclude that $\text{reg}(V) = d(V) - e$ by Proposition 5.3.

We conclude the paper with a discussion of the following classical constructions giving rise to settings that satisfy the assumptions in Corollary 1.4.

Example 5.6. For $s \geq r \geq 1$, let $V \subset \mathbb{P}^{rs-1}$ be defined by the $r$-minors of an $r \times s$ matrix with general linear entries. We can choose coordinates $x_{i,j}$, where $1 \leq i \leq r$ and $1 \leq j \leq s$, so that $\mathbb{P}^{rs-1} = \text{Proj} \mathbb{C}[x_{i,j}]$ and $M = [x_{i,j}]$. For every $k = 1, \ldots, r - 1$, let $V_k \subset \mathbb{P}^{rs-1}$ be the locus where $\text{rk} M \leq k$. Then

$$\text{mult} V_k (V) = r - k \quad \text{and} \quad \text{codim}_{\mathbb{P}^{rs-1}} (V_k) = (s - k)(r - k)$$

for every $k$, so that, in particular, $V = V_{r-1}$ and $\text{Sing} (V) = V_{r-2}$.

We consider the sequence of blowups $g_k: Y_k \to Y_{k-1}$, where $Y_0 = \mathbb{P}^{rs-1}$ and $g_k$ is the blow up of $Y_{k-1}$ along the proper transform of $V_k$. An explicit computation shows that each center of blow up is smooth and that this operation produces a log resolution

$$f: Y = Y_{r-1} \to \mathbb{P}^{rs-1}$$

of $(\mathbb{P}^{rs-1}, V)$. Let $E_1, \ldots, E_{r-1}$ be the exceptional divisors of $f$, labeled so that $f(E_k) = V_k$. Then

$$\text{ord}_{E_k} (K_{Y/\mathbb{P}^{rs-1}}) - e \text{val}_{E_k} (V) = (s - k)(r - k) - 1 - (s - r + 1)(r - k) \geq -1$$
for all $k$, which implies that $(\mathbb{P}^{x-1}, eV)$ is log canonical. Therefore $V$ satisfies the assumptions in Corollary 1.4, and hence, in particular, we recover the well known fact that $\text{reg}(V) \leq d(V) - e + 1$.

**Example 5.7.** Let $V \subset \mathbb{P}^{2e}$ be the image of a smooth $e$-dimensional variety $\tilde{V}$ in $\mathbb{P}^{2e+1}$ under a general projection. Note that $V$ has at most a finite number of double points and is smooth elsewhere. Moreover, if $p$ is a double point of $V$, then the tangent cone of $V$ at $p$ (viewed as a subset of the tangent space $T_p\mathbb{P}^{2e}$) is the union of two linear spaces meeting only at the origin (and hence spanning $T_p\mathbb{P}^{2e}$).

This can be checked as follows. Consider the incidence sets

$I = \{(p, q, o) \in \tilde{V} \times \tilde{V} \times \mathbb{P}^{2e+1} \mid p \neq q, o \in pq\}$,

$I' = \{(p, q, o) \in I \mid \dim(\text{Span}(PT_p\tilde{V}, o) \cap \text{Span}(PT_q\tilde{V}, o)) \geq 2\}$,

where $PT_p\tilde{V} \subset \mathbb{P}^{2e+1}$ denotes the linear subspace of dimension $e$ tangent to $\tilde{V}$ at $p$. Note that $I$ is irreducible and the projection to the last factor $I \to \mathbb{P}^{2e+1}$ is generically finite whenever dominant. On the other hand, $I'$ is a proper closed subset of $I$. Indeed, if $(p, q)$ is general in $\tilde{V} \times \tilde{V}$, then the linear projection $\pi_p: \tilde{V} \setminus \{p\} \to \mathbb{P}^{2e}$ has maximal rank at $q$, which means that $PT_p\tilde{V} \cap PT_q\tilde{V} = \emptyset$, and hence $\text{Span}(PT_p\tilde{V}, o) \cap \text{Span}(PT_q\tilde{V}, o) = pq$ for every $o \in pq$. This implies that the projection to the last factor $I' \to \mathbb{P}^{2e+1}$ is not dominant.

Suppose that $p$ is a double point of $V$. Let $h: Y \to \mathbb{P}^{2e}$ be the blow up of $\mathbb{P}^{2e}$ at $p$, and let $E$ be the exceptional divisor. If $V' \subset Y$ is the proper transform of $V$, then $V' \cap E$ is the union of two disjoint planes, and thus $V'$ is smooth and intersects transversally $E$. Therefore, in order to prove that $(\mathbb{P}^{2e}, eV)$ is log canonical, it suffices to check the discrepancy along $E$. We have

$$\text{ord}_E(K_{Y/\mathbb{P}^{2e}}) - e \text{ val}_E(V) = 2e - 1 - 2e = -1,$$

and thus we can conclude that the pair is log canonical, and hence $V$ satisfies the assumptions in Corollary 1.4, and therefore

$$\text{reg}(V) \leq d(V) - e + 1.$$
divisor. By contrast, if \( e \geq 3 \), then taking a general projection to an hypersurface \( V \subset \mathbb{P}^{e+1} \) produces a pair \((\mathbb{P}^{e+1}, V)\) that in general is not log canonical, as the multiplicities may grow exponentially with respect to \( e \). One can still ask whether there is a function \( \phi(e) \), with \( e < \phi(e) < 2e \) such that a general projection to \( \mathbb{P}^{\phi(e)} \) gives a variety to which Corollary 1.4 applies.

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