MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV
EXPONENTS FOR $C^r$ SURFACE DIFFEOMORPHISMS WITH LARGE
ENTROPY

DAVID BURGUET

Abstract. We prove a finite smooth version of the entropic continuity of Lyapunov
exponents proved recently by Buzzi, Crovisier and Sarig for $C^\infty$ surface diffeomorphisms [10].
As a consequence we show that any $C^r$, $r > 1$, smooth surface diffeomorphism $f$ with
$h_{top}(f) > \frac{1}{r} \lim\sup_n \frac{1}{n} \log^+ \|df^n\|_{\infty}$ admits a measure of maximal entropy. We also prove
the $C^r$ continuity of the topological entropy at $f$.

1. Statements

We define now some notations to state our main results. Fix a compact Riemannian
surface $(M, \|\cdot\|)$. For $r > 1$ we let $\text{Diff}^r(M)$ be the set of $C^r$ diffeomorphisms of $M$. For
$f \in \text{Diff}^r(M)$ we let $F : \mathbb{PTM} \circlearrowleft$ be the induced map on the projective tangent bundle
$\mathbb{PTM} = T^1M/\pm1$ and we denote by $\phi, \psi : \mathbb{PTM} \to \mathbb{R}$ the continuous observables on $\mathbb{PTM}$
given respectively by $\phi : (x, v) \mapsto \log \|df_x(v)\|$ and $\psi : (x, v) \mapsto \log \|df_x(v)\| - \frac{1}{r} \log^+ \|df_x\|$. 

Date: September 2022.
2010 Mathematics Subject Classification. 37 A35, 37C40, 37 D25.
Theorem (Buzzi-Crovisier-Sarig, Theorem C \[10\]) is phisms with large enough entropy (see Corollary 1).

Let $\lambda^+(x)$ and $\lambda^-(x)$ be the pointwise Lyapunov exponents given by $\lambda^+(x) = \limsup_{n \to +\infty} \frac{1}{n} \log \|d_x f^n\|$ and $\lambda^-(x) = \liminf_{n \to -\infty} \frac{1}{n} \log \|d_x f^n\|$ for any $x \in M$ and $\lambda^+(\mu) = \int \lambda^+(x) \, d\mu(x)$, $\lambda^-(\mu) = \int \lambda^-(x) \, d\mu(x)$, for any $f$-invariant measure $\mu$.

Also we put $\hat{\lambda}^+(f) := \lim_n \frac{1}{n} \log \|d^nf\|$ with $\|d^nf\| = \sup_{x \in M} \|d_x f^n\|$. The function $f \mapsto \hat{\lambda}^+(f)$ is upper semi-continuous in the $C^1$ topology on the set of $C^1$ diffeomorphisms on $M$. For an $f$-invariant measure $\mu$ with $\lambda^+(\mu) > 0 \geq \lambda^-(\mu)$ for $\mu$ a.e. $x$, there are by Oseledets\footnote{We refer to [17] for background on Lyapunov exponents and Pesin theory.} theorem one-dimensional invariant vector spaces $E_+(x)$ and $E_-(x)$, resp. called the unstable and stable Oseledets bundle, such that

$$\forall \mu \text{ a.e. } x \forall v \in E_\pm(x) \setminus \{0\}, \lim_{n \to \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \lambda^\pm(x).$$

Then we let $\hat{\mu}^+$ be the $F$-invariant measure given by the lift of $\mu$ on $\mathbb{P}TM$ with $\hat{\mu}^+(E_+) = 1$. When writing $\hat{\mu}^+$ we assume implicitly that the push-forward measure $\mu$ on $M$ satisfies $\lambda^+(\mu) > 0 \geq \lambda^-(\mu)$ for $\mu$ a.e. $x$.

A sequence of $C^r$, with $r > 1$, surface diffeomorphisms $(f_k)_k$ on $M$ is said to converge $C^r$ weakly to a diffeomorphism $f$, when $f_k$ goes to $f$ in the $C^1$ topology and the sequence $(f_k)_k$ is $C^r$ bounded. In particular $f$ is $C^{r-1}$.

**Theorem** (Buzzi-Crovisier-Sarig, Theorem C \[10\]). Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of $C^r$, with $r > 1$, surface diffeomorphisms converging $C^r$ weakly to a diffeomorphism $f$. Let $(F_k)_{k \in \mathbb{N}}$ and $F$ be the lifts of $(f_k)_{k \in \mathbb{N}}$ and $f$ to $\mathbb{P}TM$. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic $F_k$-invariant measures converging to $\hat{\mu}$.

Then there are $\beta \in [0, 1]$ and $F$-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta \hat{\mu}_1^+$, such that:

$$\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f^{-1})}{r - 1}.$$

In particular when $f (= f_k$ for all $k$) is $C^\infty$ and $h(\nu_k)$ goes to the topological entropy of $f$, then $\beta$ is equal to $1$ and therefore $\lambda^+(\nu_k)$ goes to $\lambda^+(\mu)$:

**Corollary** (Entropic continuity of Lyapunov exponents \[10\]). Let $f$ be a $C^\infty$ surface diffeomorphism with $h_{\text{top}}(f) > 0$.

Then if $(\nu_k)_k$ is a sequence of ergodic measures converging to $\mu$ with $\lim_k h(\nu_k) = h_{\text{top}}(f)$, then

- $h(\mu) = h_{\text{top}}(f)$ \footnote{This follows from the upper semi-continuity of the entropy function $h$ on the set of $f$-invariant probability measures for a $C^\infty$ diffeomorphism $f$ (in any dimension), which was first proved by Newhouse in [19].}
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$. 

We state an improved version of Buzzi-Crovisier-Sarig Theorem, which allows to prove the same entropy continuity of Lyapunov exponents for $C^r$, $1 < r < +\infty$, surface diffeomorphisms with large enough entropy (see Corollary \[1\]).
Main Theorem. Let \((f_k)_{k \in \mathbb{N}}\) be a sequence of \(C^r\) surface diffeomorphisms converging \(C^r\) weakly to a diffeomorphism \(f\). Let \((F_k)_{k \in \mathbb{N}}\) and \(F\) be the lifts of \((f_k)_{k \in \mathbb{N}}\) and \(f\) to \(\mathbb{PTM}\). Assume there is a sequence \((\tilde{\nu}_k^+)\) of \(\mathcal{F}_k\)-invariant measures converging to \(\tilde{\mu}\).

Then for any \(\alpha > \frac{\lambda^+(f)}{r}\), there are \(\beta = \beta_\alpha \in [0, 1]\) and \(F\)-invariant measures \(\tilde{\mu}_0 = \tilde{\mu}_0,\alpha\) and \(\tilde{\mu}_1^+ = \tilde{\mu}_1^+,\alpha\) with \(\tilde{\mu} = (1-\beta)\tilde{\mu}_0 + \beta\tilde{\mu}_1^+\), such that:

\[
\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha.
\]

In the appendix we explain how the Main Theorem implies Buzzi-Crovisier-Sarig statement. We state now some consequences of the Main Theorem.

Corollary 1 (Existence of maximal measures and entropic continuity of Lyapunov exponents). Let \(f\) be a \(C^r\), with \(r > 1\), surface diffeomorphism satisfying \(h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}\).

Then \(f\) admits a measure of maximal entropy. More precisely, if \((\nu_k)\) is a sequence of ergodic measures converging to \(\mu\) with \(\lim_k h(\nu_k) = h_{\text{top}}(f)\), then

- \(h(\mu) = h_{\text{top}}(f)\),
- \(\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)\).

It was proved in [9] that any \(C^r\) surface diffeomorphism satisfying \(h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}\) admits at most finitely many ergodic measures of maximal entropy. On the other hand, J. Buzzi has built examples of \(C^r\) surface diffeomorphisms for any \(+\infty > r > 1\) with \(h_{\text{top}}(f) > \frac{\lambda^+(f)}{r}\) arbitrarily close to \(1/r\) without a measure of maximal entropy [7]. It is expected that for any \(r > 1\) there are \(C^r\) surface diffeomorphisms satisfying \(h_{\text{top}}(f) = \frac{\lambda^+(f)}{r} > 0\) without measure of maximal entropy or with infinitely many such ergodic measures, but these questions are still open. Such results were already known for interval maps [3, 6, 8].

Proof. We consider the constant sequence of diffeomorphisms equal to \(f\). By taking a subsequence, we can assume that \((\tilde{\nu}_k^+)\) is converging to a lift \(\tilde{\mu}\) of \(\mu\). By using the notations of the Main Theorem with \(h_{\text{top}}(f) > \alpha > \frac{\lambda^+(f)}{r}\), we have

\[
h_{\text{top}}(f) = \lim_{k \to +\infty} h(\nu_k),
\]

\[
\leq \beta h(\mu_1) + (1-\beta)\alpha,
\]

\[
\leq \beta h_{\text{top}}(f) + (1-\beta)\alpha,
\]

\[
(1-\beta)h_{\text{top}}(f) \leq (1-\beta)\alpha.
\]

But \(h_{\text{top}}(f) > \alpha\), therefore \(\beta = 1\), i.e. \(\tilde{\mu}_1^+ = \tilde{\mu}\) and \(\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)\). Moreover \(h_{\text{top}}(f) = \lim_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha = h(\mu)\). Consequently \(\mu\) is a measure of maximal entropy of \(f\).

\[\square\]

Corollary 2 (Continuity of topological entropy and maximal measures). Let \((f_k)\) be a sequence of \(C^r\), with \(r > 1\), surface diffeomorphisms converging \(C^r\) weakly to a diffeomorphism \(f\) with \(h_{\text{top}}(f) \geq \frac{\lambda^+(f)}{r}\).

Then

\[
h_{\text{top}}(f) = \lim_k h_{\text{top}}(f_k).
\]
Moreover if \( h_{\text{top}}(f) > \frac{\lambda^+(f)}{r} \) and \( \nu_k \) is a maximal measure of \( f_k \) for large \( k \), then any limit measure of \( (\nu_k)_k \) for the weak-* topology is a maximal measure of \( f \).

**Proof.** By Katok’s horseshoes theorem \([15]\), the topological entropy is lower semi-continuous for the \( C^1 \) topology on the set of \( C^r \) surface diffeomorphisms. Therefore it is enough to show the upper semi-continuity.

By the variational principle there is a sequence of probability measures \( (\nu_k)_k \in K, K \subset \mathbb{N} \) with \( K = \infty \), such that:

- \( \nu_k \) is an ergodic \( f_{\beta} \)-invariant measure for each \( k \),
- \( \lim_{k \in K} h(\nu_k) = \limsup_{k \in \mathbb{N}} h_{\text{top}}(f_k) \).

By extracting a subsequence we can assume \( (\hat{\nu}_k^+)_k \) is converging to a \( F \)-invariant measure \( \hat{\mu} \) in the weak-* topology. We can then apply the Main Theorem for any \( \alpha > \frac{\lambda^+(f)}{r} \) to get for some \( f \)-invariant measures \( \mu_1, \mu_0 \) and \( \beta \in [0, 1] \) (depending on \( \alpha \)) with \( \mu = (1 - \beta)\mu_0 + \beta \mu_1 \):

\[
\limsup_k h_{\text{top}}(f_k) = \lim_k h(\nu_k),
\]

\[
\leq \beta h(\mu_1) + (1 - \beta)\alpha,
\]

\[
\leq \beta h_{\text{top}}(f) + (1 - \beta)\alpha,
\]

\[
\leq \max(h_{\text{top}}(f), \alpha).
\]

By letting \( \alpha \) go to \( \frac{\lambda^+(f)}{r} \) we get

\[
\limsup_k h_{\text{top}}(f_k) \leq h_{\text{top}}(f).
\]

If \( h_{\text{top}}(f) > \frac{\lambda^+(f)}{r} \), we can fix \( \alpha \in \left] \frac{\lambda^+(f)}{r}, h_{\text{top}}(f) \right] \) and the inequalities (1.1) may be then rewritten as follows:

\[
\limsup_k h_{\text{top}}(f_k) \leq \beta h(\mu_1) + (1 - \beta)\alpha,
\]

\[
\leq h_{\text{top}}(f).
\]

By the lower semi-continuity of the topological entropy, we have \( h_{\text{top}}(f) \leq \limsup_k h_{\text{top}}(f_k) \) and therefore these inequalities are equalities, which implies \( \beta = 1 \), then \( \mu_1 = \mu \), and \( h(\mu) = h_{\text{top}}(f) \).

The corresponding result was proved for interval maps in \([5]\) by using a different method. We also refer to \([3]\) for counterexamples of the upper semi-continuity property for interval maps \( f \) with \( h_{\text{top}}(f) < \frac{\lambda^+(f)}{r} \). Finally, in \([7]\), the author built, for any \( r > 1 \), a \( C^r \) surface diffeomorphism \( f \) with \( \limsup_{g \to f} h_{\text{top}}(g) = \frac{\lambda^+(f)}{r} > h_{\text{top}}(f) = 0 \). We recall also that upper semi-continuity of the topological entropy in the \( C^\infty \) topology was established in any dimension by Y. Yomdin in \([19]\).

Newhouse proved that for a \( C^\infty \) system \((M, f)\), the entropy function \( h : \mathcal{M}(M, f) \to \mathbb{R}^+ \) is an upper semi-continuous function on the set \( \mathcal{M}(M, f) \) of \( f \)-invariant probability measure. It follows from our Main Theorem, that the entropy function is upper semi-continuous at ergodic measures with entropy larger than \( \frac{\lambda^+(f)}{r} \) for a \( C^r \), \( r > 1 \), surface diffeomorphism \( f \).

**Corollary 3** (Upper semi-continuity of the entropy function at ergodic measures with large entropy). Let \( f : M \circlearrowleft \) be a \( C^r \), \( r > 1 \), surface diffeomorphism.
Then for any ergodic measure $\mu$ with $h(\mu) \geq \frac{\lambda^+(f)}{r}$, we have

$$\limsup_{\nu \to \mu} h(\nu) \leq h(\mu).$$

**Proof.** By continuity of the ergodic decomposition at ergodic measures and by harmonicity of the entropy function, we have for any ergodic measure $\mu$ (see e.g. Lemma 8.2.13 in [12]):

$$\limsup_{\nu \to \mu} h(\nu) \leq \limsup_{\nu \to \mu} h(\mu).$$

Let $(\nu_k)_{k \in \mathbb{N}}$ be a sequence of ergodic $f$-invariant measures with $\lim_k h(\nu_k) = h(\mu)$. By extracting a subsequence we can assume that the sequence $(\nu_k)$ is converging to some lift $\hat{\mu}$ of $\mu$. Take $\alpha$ with $\alpha > \frac{\lambda^+(f)}{r}$. Then, in the decomposition $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta \hat{\mu}_1^+$ given by the Main Theorem, we have $\mu_1 = \mu_0 = \mu$ by ergodicity of $\mu$. Therefore

$$\lim_k h(\nu_k) \leq \beta h(\mu) + (1 - \beta)\alpha,$$

and

$$\leq \max (h(\mu), \alpha).$$

By letting $\alpha$ go to $\frac{\lambda^+(f)}{r}$ we get

$$\lim_k h(\nu_k) \leq h(\mu).$$

□

2. Main steps of the proof

We follow the strategy of the proof of [10]. We point out below the main differences:

- **Geometric and neutral empirical component.** For $\lambda^+(\nu_k) > \frac{\lambda^+(f)}{r}$ we split the orbit of a $\nu_k$-typical point $x$ into two parts. We consider the empirical measures from $x$ at times lying between to $M$-close consecutive times where the unstable manifold has a “bounded geometry”. We take their limit in $k$, then in $M$. In this way we get an invariant component of $\hat{\mu}$. In [10] the authors consider rather such empirical measures for $\alpha$-hyperbolic times and then take the limit when $\alpha$ go to zero.

- **Entropy computations.** To compute the asymptotic entropy of the $\nu_k$'s, we use the static entropy w.r.t. partitions and its conditional version. Instead the authors in [10] used Katok's like formulas.

- **$C^r$ Reparametrizations.** Finally we use here reparametrization methods from [4] and [2] respectively rather than Yomdin's reparametrizations of the projective action $F$ as done in [10]. This is the principal difference with [10].

2.1. **Empirical measures.** Let $(X, T)$ be an invertible topological system, i.e. $T : X \to X$ is a homeomorphism of a compact metric space. For a fixed Borel measurable subset $G$ of $X$ we let $E(x) = E_G(x)$ be the set of times of visits in $G$ from $x \in X$:

$$E(x) = \{ n \in \mathbb{Z}, T^n x \in G \}.$$ 

When $a < b$ are two consecutive times in $E(x)$, then $[a, b]$ is called a neutral block (by following the terminology of [9]). For all $M \in \mathbb{N}^*$ we let then

$$E^M(x) = \bigcup_{a < b \in E(x), |a - b| \leq M} [a, b].$$
By convention we let $E^\infty(x) = \mathbb{Z}$. For $M \in \mathbb{N}^*$ the complement of $E^M(x)$ is made of disjoint neutral blocks of length larger than $M$. We consider the associated empirical measures:

$$\forall n, \mu_{x,n}^M = \frac{1}{n} \sum_{k \in E^M(x) \cap [0,n[} \delta_{T^k x}.$$ 

We denote by $\chi^M$ the indicator function of $\{x, 0 \in E^M(x)\}$. The following lemma follows straightforwardly from Birkhoff ergodic theorem:

**Lemma 1.** With the above notations, for any $T$-invariant ergodic measure $\nu$, there is a set $G$ of full $\nu$-measure such that the empirical measures $(\mu_{x,n}^M)_{n}$ are converging for any $x \in G$ and any $M \in \mathbb{N}^* \cup \{\infty\}$ to $\chi^M \nu$ in the weak*-topology, when $n$ goes to $+\infty$.

Fix some $T$-invariant ergodic measure $\nu$. We let $\xi^M = \chi^M \nu$ and $\eta^M = \nu - \xi^M$. Moreover we put $\beta_M = \int \chi^M d\nu$, then $\xi^M = \beta_M \cdot \xi^M$ when $\beta_M \neq 0$ and $\eta^M = (1 - \beta_M) \cdot \eta^M$ when $\beta_M \neq 1$ with $\xi^M, \eta^M$ being thus probability measures. Following partially [10], the measures $\xi^M$ and $\eta^M$ are respectively called here the geometric and neutral components of $\nu$. In general these measures are not $T$-invariant, but $d(\xi^M, T_\ast \xi^M) \leq 1/M$ for some standard distance $d$ on the set $\mathcal{M}(X)$ of Borel probability measures on $X$. From the definition one easily checks that $\xi^M \geq \chi^M$ for $M \geq N$. If $\nu(G) = 0$, then for $\nu$-almost every $x$ we have $\mu_{x,n}^M = 0$ for all $n$ and $M$. Assume $G$ has positive $\nu$-measure. Then, when $M$ goes to infinity, the function $\chi^M$ goes to $\chi^\infty = 1$ almost surely with respect to $\nu$, therefore $\xi^M$ goes to $\nu$. However in general this convergence is not uniform in $\nu$. In the following we consider a sequence $(\nu_k)_k$ of ergodic $T$-invariant measures converging to $\mu$. Then, by a diagonal argument, we may assume by extracting a subsequence that $\xi_k^M := \chi^M \nu_k$ is converging for any $M$, when $k$ goes to infinity, to some $\nu^M$, which is a priori distinct from $\chi^M \mu$. We still have $\nu^M \geq \nu^N$ for $M \geq N$, but the limit $\mu_1 = \lim_M \nu^M$ is a $T$-invariant component of $\mu$, which may differ from $\mu$.

The next lemma follows from Lemma 1 and standard arguments of measure theory:

**Lemma 2.** There is a Borel subset $H$ with $\nu(H) > \frac{1}{2}$ such that for any $M \in \mathbb{N}$ and for any continuous function $\varphi : X \to \mathbb{R}$:

$$(2.1) \quad \frac{1}{n} \sum_{k \in E^M(x) \cap [1,n[} \varphi(T^k x) \to \int \varphi \, d\xi^M \text{ uniformly in } x \in H.$$ 

**Proof.** We consider a dense countable family $\mathcal{F} = (\varphi_k)_{k \in \mathbb{N}}$ in the set $C^0(X, \mathbb{R})$ of real continuous functions on $X$ endowed with the supremum norm $\| \cdot \|_\infty$. Let $G$ be as in Lemma 1. Then for all $k, M$, by Egorov’s theorem applied to the pointwise converging sequence $(f_n : G \to \mathbb{R})_n = (x \mapsto \varphi_k \, d\mu_{x,n}^M)_n$, there is a subset $F_k^M$ of $F$ with $\nu(F_k^M) > 1 - \frac{1}{2^{k+M+3}}$ such that $\int \varphi_k \, d\mu_{x,n}^M$ converges to $\int \varphi_k \, d\xi^M$ uniformly in $x \in F_k^M$. Let $H = \bigcap_{k,M} F_k^M$. We have $\nu(H) > \frac{1}{2}$. Then, if $\varphi \in C^0(X, \mathbb{R})$, we may find for any $\epsilon > 0$ a function $\varphi_k \in \mathcal{F}$ with $\| \varphi - \varphi_k \|_\infty < \epsilon$. Let $M \in \mathbb{N}$. Take $N = N_{\epsilon,k}^M$ such that $| \int \varphi_k \, d\mu_{x,n}^M - \int \varphi_k \, d\xi^M | < \epsilon$ for
n > N and for all \(x \in F_k^M\). In particular for all \(x \in H\) we have for \(n > N\)

\[
\left| \int \varphi d\mu_{x,n}^M - \int \varphi d\xi^M \right| \leq \left| \int \varphi_k d\mu_{x,n}^M - \int \varphi d\mu_{x,n}^M \right| + \left| \int \varphi_k d\mu_{x,n}^M - \int \varphi d\xi^M \right| \\
+ \left| \int \varphi_k d\xi^M - \int \varphi d\xi^M \right|, \\
\leq 2 \|\varphi - \varphi_k\|_\infty + \left| \int \varphi_k d\mu_{x,n}^M - \int \varphi d\mu_{x,n}^M \right|, \\
< 3\epsilon.
\]

\[\square\]

### 2.2. Pesin unstable manifolds

We consider a smooth compact riemannian manifold \((M, \|\cdot\|)\). Let \(\exp_x\) be the exponential map at \(x\) and let \(R_{inj}\) be the radius of injectivity of \((M, \|\cdot\|)\). We consider the distance \(d\) on \(M\) induced by the Riemannian structure. Let \(f : M \circlearrowleft\) be a \(C^r, r > 1\), surface diffeomorphism. We denote by \(R\) the set of Lyapunov regular points with \(\lambda^+(x) > 0 > \lambda^-(x)\). For \(x \in M\) we let \(W^u(x)\) denote the unstable manifold at \(x\):

\[
W^u(x) := \left\{ y \in M, \lim_n \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\}.
\]

By Pesin unstable manifold theorem, the set \(W^u(x)\) for \(x \in R\) is a \(C^r\) submanifold tangent to \(E_+(x)\) at \(x\).

For \(x \in R\), we let \(\hat{x}\) be the vector in \(\mathbb{P}TM\) associated to the unstable Oseledets bundle \(E_+(x)\). For \(\delta > 0\) the point \(x\) is called \(\delta\)-hyperbolic with respect to \(\phi\) (resp. \(\psi\)) when we have \(\phi_l(F^{-l}\hat{x}) \geq \delta l\) (resp. \(\psi_l(F^{-l}\hat{x}) \geq \delta l\)) for all \(l > 0\). Note that if \(x\) is \(\delta\)-hyperbolic with respect to \(\psi\) then it is \(\delta\)-hyperbolic with respect to \(\phi\). Let \(H_\delta := \{ \hat{x} \in \mathbb{P}TM, \forall l > 0 \psi_l(F^{-l}\hat{x}) \geq \delta l \}\) be the set of \(\delta\)-hyperbolic points w.r.t. \(\psi\).

**Lemma 3.** Let \(\nu\) be an ergodic measure with \(\lambda^+(\nu) - \frac{\log^+ \|df\|_\infty}{r} > \delta > 0 > \lambda^-(\nu)\). Then we have

\[
\hat{\nu}^+(H_\delta) > 0.
\]

**Proof.** By applying the Ergodic Maximal Inequality (see e.g. Theorem 1.1 in [1]) to the measure preserving system \((F^{-1}, \hat{\nu}^+)\) with the observable \(\psi^\delta = \delta - \psi \circ F^{-1}\), we get with \(A_\delta = \{ \hat{x} \in \mathbb{P}TM, \exists k \geq 0 \text{ s.t. } \sum_{l=0}^k \psi_l(F^{-l}\hat{x}) > 0\}:

\[
\int_{A_\delta} \psi^\delta \, d\hat{\nu}^+ \geq 0.
\]
Observe that $H_\delta = \mathbb{P} TM \setminus A_\delta$. Therefore

$$
\int_{H_\delta} \psi^\delta \, d\hat{\nu}^+ = \int \psi^\delta \, d\hat{\nu}^+ - \int_{A_\delta} \psi^\delta \, d\hat{\nu}^+, \\
\leq \int \psi^\delta \, d\hat{\nu}^+, \\
\leq \int (\delta - \psi \circ F^{-1}) \, d\hat{\nu}^+, \\
\leq \delta - \lambda^+(\nu) + \frac{1}{r} \int \frac{\log^+ \|d_x f\|}{r} \, d\nu(x), \\
< 0.
$$

In particular we have $\hat{\nu}^+(H_\delta) > 0$. \quad \Box

A point $x \in \mathcal{R}$ is said to have $\kappa$-bounded geometry for $\kappa > 0$ when $\exp^{-1} W^u(x)$ contains the graph of a $\kappa$-admissible map at $x$, which is defined as a 1-Lipschitz map $f : I \to \mathcal{E}_+(x)^\perp \subset T_x M$, with $I$ being an interval of $\mathcal{E}_+(x)$ containing 0 with length $\kappa$. We let $G_\kappa$ be the subset of points in $\mathcal{R}$ with $\kappa$-bounded geometry.

**Lemma 4.** The set $G_\kappa$ is Borel measurable.

**Proof.** For $x \in \mathcal{R}$ we have $W^u(x) = \bigcup_{n \in \mathbb{N}} f^n W^u_{loc}(f^{-n} x)$ with $W^u_{loc}$ being the Pesin unstable local manifold at $x$. The sequence $(f^n W^u_{loc}(f^{-n} x))_n$ is increasing in $n$ for the inclusion. Therefore, if we let $G^n_\kappa$ be the subset of points $x$ in $G_\kappa$, such that $\exp^{-1} f^n W^u_{loc}(f^{-n} x)$ contains the graph of a $\kappa$-admissible map, then we have

$$
G_\kappa = \bigcup_n G^n_\kappa.
$$

There are closed subsets, $(\mathcal{R}_l)_{l \in \mathbb{N}}$, called the Pesin blocks, such that $\mathcal{R} = \bigcup_l \mathcal{R}_l$ and $x \mapsto W^u_{loc}(x)$ is continuous on $\mathcal{R}_l$ for each $l$ (see e.g. [17]). Let $(x_p)_p$ be sequence in $G^n_\kappa \cap \mathcal{R}_l$ which converges to $x \in \mathcal{R}_l$. By extracting a subsequence we can assume that the associated sequence of $\kappa$-admissible maps $f_p$ at $x_p$ is convergent pointwise to a $\kappa$-admissible map at $x$, when $p$ goes to infinity. In particular $G^n_\kappa \cap \mathcal{R}_l$ is a closed set and therefore $G_\kappa = \bigcup_{l,n} (G^n_\kappa \cap \mathcal{R}_l)$ is Borel measurable. \quad \Box

### 2.3. Entropy of conditional measures

We consider an ergodic hyperbolic measure $\nu$, i.e an ergodic measure with $\nu(\mathcal{R}) = 1$. A measurable partition $\zeta$ is **subordinated** to the Pesin unstable local lamination $W^u_{loc}$ of $\nu$ if the atom $\zeta(x)$ of $\zeta$ containing $x$ is a neighborhood of $x$ inside the curve $W^u_{loc}(x)$ and $f^{-1}\zeta > \zeta$. By Rokhlin’s disintegration theorem, there are a measurable set $Z$ of full $\nu$-measure and probability measures $\nu_x$ on $\zeta(x)$ for $x \in Z$, called the **conditional measures** on unstable manifolds, satisfying $\nu = \int \nu_x \, d\nu(x)$. Moreover $\nu_y = \nu_x$ for $x, y \in Z$ in the same atom of $\zeta$. Ledrappier and Strelcyn [13] have proved the existence of such subordinated measurable partitions. We fix such a subordinated partition $\zeta$ with respect to $\nu$. For $x \in M$, $n \in \mathbb{N}$ and $\rho > 0$, we let $B_n(x, \rho)$ be the Bowen ball $B_n(x, \rho) := \bigcap_{0 \leq k < n} f^{-k} B(f^k x, \rho)$ (where $B(f^k x, \rho)$ denotes the ball for $d$ at $f^k x$ with radius $\rho$).
Lemma 5. [14] For all $\iota > 0$, there is $\rho > 0$ and a measurable set $E \subset Z \cap R$ with $\nu(E) > \frac{1}{2}$ such that

$$\forall x \in E, \ \liminf_n \frac{1}{n} \log \nu_x(B_n(x, \rho)) \geq h(\nu) - \iota. \quad (2.2)$$

The natural projection from $\mathbb{P}TM$ to $M$ is denoted by $\pi$. We consider a distance $\hat{d}$ on the projective tangent bundle $\mathbb{P}TM$, such that $\hat{d}(X, Y) \geq d(\pi X, \pi Y)$ for all $X, Y \in \mathbb{P}TM$. We let $\hat{\eta}^M$ and $\hat{\xi}^M$ be the neutral and geometric components of the ergodic $F$-invariant measure $\hat{\nu}^+$ associated to $G = H_\delta \cap \pi^{-1}G_\kappa \subset \mathbb{P}TM$, where the parameters $\delta$ and $\kappa$ will be fixed later on independently of $\nu$. The importance of this choice of $G$ will appear in Proposition 4 to bound from above the entropy of the neutral component. We also consider the projections $\eta^M$ and $\xi^M$ on $M$ of $\hat{\eta}^M$ and $\hat{\xi}^M$ respectively. By Lemma 2 applied to the system $(\mathbb{P}TM, F)$ and to the ergodic measure $\hat{\nu}^+$, there is a Borel subset $\mathcal{H}$ of $\mathbb{P}TM$ with $\hat{\nu}^+(\mathcal{H}) > \frac{1}{2}$ such that for any $M \in N^* \cup \{\infty\}$ and for any continuous function $\varphi : \mathbb{P}TM \to \mathbb{R}$

$$\frac{1}{n} \sum_{k \in \mathcal{E}^M(\hat{x}) \cap [1, n]} \varphi(F^k \hat{x}) \xrightarrow{n} \int \varphi \, d\hat{\xi}^M \text{ uniformly in } \hat{x} \in \mathcal{H}. \quad (2.3)$$

Fix an error term $\iota > 0$ depending on $\nu$ and let $\rho$ and $E$ be as in Lemma 5. Let $F = E \cap \pi(\mathcal{H})$. Note that $\nu(F) > 0$. We fix also $x_\ast \in F$ with $\nu_{x_\ast}(F) > 0$ and we let $\zeta = \nu_{x_\ast}(\cdot)/\nu_{x_\ast}(F)$ be the probability measure induced by $\nu_{x_\ast}$ on $F$. Observe that $\nu_x = \nu_{x_\ast}$ for $\zeta$ a.e. $x$. We let $D$ be the $C^r$ curve given by the Pesin local unstable manifold $W^u_{loc}(x_\ast)$ at $x_\ast$. For a finite measurable partition $P$ and a Borel probability measure $\mu$ we let $H_\mu(P)$ be the static entropy, $H_\mu(P) = -\sum_{A \in P} \mu(A) \log \mu(A)$. Moreover we let $P^n = \bigvee_{k=0}^{n-1} f^{-k} P$ be the $n$-iterated partition, $n \in N$. We also denote by $P^n_x$ the atom of $P^n$ containing the point $x \in M$.

Lemma 6. For any (finite measurable) partition $P$ with diameter less than $\rho$, we have

$$\liminf_n \frac{1}{n} H_\zeta(P^n) \geq h(\nu) - \iota. \quad (2.4)$$

Proof.

$$\liminf_n \frac{1}{n} H_\zeta(P^n) = \liminf_n \frac{1}{n} \int -\frac{1}{n} \log \zeta(P^n_x) \, d\zeta(x), \text{ by the definition of } H_\zeta,$$

$$\geq \liminf_n \int -\frac{1}{n} \log \zeta(P^n_x) \, d\zeta(x), \text{ by Fatou’s Lemma},$$

$$\geq \liminf_n \int -\frac{1}{n} \log \nu_{x_\ast}(P^n_x) \, d\zeta(x), \text{ by the definition of } \zeta,$$

$$\geq \liminf_n \int -\frac{1}{n} \log \nu_x(P^n_x) \, d\zeta(x), \text{ as } \nu_x = \nu_{x_\ast} \text{ for } \zeta \text{ a.e. } x,$$

$$\geq \liminf_n \int -\frac{1}{n} \log \nu_x(B_n(x, \rho)) \, d\zeta(x), \text{ as } \text{diam}(P) < \rho,$$

$$\geq h(\nu) - \iota, \text{ by the choice of } F \subset E \text{ and } (2.2).$$

In the proof of the Main Theorem we will take $\iota = \iota(\nu_k) \to 0$ for the converging sequence of ergodic measures $(\nu_k)_k$. 

\[\square\]
2.4. Entropy splitting of the neutral and the geometric component. In this section we split the entropy contribution of the neutral and geometric components \( \hat{\eta}^M \) and \( \hat{\xi}^M \) of the ergodic F-invariant measure \( \hat{\nu}^+ \) associated to a fixed Borel set \( G \) of \( \mathbb{PTM} \).

Recall that \( E(\hat{x}) \) denotes the set of integers \( k \) with \( F^k\hat{x} \in G \). Fix now \( M \). For each \( n \in \mathbb{N} \) and \( x \in F \) we let \( E_n(x) = E(\hat{x}) \cap [0, n] \) and \( E_n^M(x) = E^M(\hat{x}) \cap [0, n] \). We also let \( E_n^M \) be the partition of \( F \) with atoms \( A_E := \{ x \in D, E_n^M(x) = E \} \) for \( E \subset [0, n] \). Given a partition \( Q \) of \( \mathbb{PTM} \), we also let \( Q^{E_n^M} \) be the partition of \( F := \{ \hat{x}, x \in F \cap D \} \) finer than \( \pi^{-1}E_n^M \) with atoms \( \{ \hat{x} \in F, E_n^M(x) = E \} \) and \( \forall k \in E, F^k\hat{x} \in Q_k \) for \( E \subset [0, n] \) and \( (Q_k)_{k \in E} \subset Q^E \). We let \( \partial Q \) be the boundary of the partition \( Q \), which is the union of the boundaries of its atoms. For a measure \( \eta \) and a subset \( A \) of \( \mathbb{M} \) with \( \eta(A) > 0 \) we denote by \( \eta_A = \frac{n(A\cap \partial Q)}{n(A)} \) the induced probability measure on \( A \). Moreover, for two sets \( A, B \) we let \( A \Delta B \) denote the symmetric difference of \( A \) and \( B \), i.e. \( A \Delta B = (A \setminus B) \cup (B \setminus A) \). Finally, let \( H : [0, 1] \rightarrow \mathbb{R}^+ \) be the map \( t \mapsto -t \log t - (1 - t) \log (1 - t) \). Recall that \( \hat{\zeta}^+ \) is the lift of \( \zeta \) on \( \mathbb{PTM} \) to the unstable Oseledets bundle (with \( \zeta \) as in Subsection 2.3).

Lemma 7. For any finite partition \( P \) with diameter less than \( \rho \) and for any finite partition \( Q \) and any \( m \in \mathbb{N}^* \) with \( \hat{\xi}^M(\partial Q)^m = 0 \) we have

\[
(2.5) \quad h(\nu) \leq \beta_M \frac{1}{m} H_E(M(Q^m)) + \limsup_n \frac{1}{n} H_{\hat{\xi}^+}(\pi^{-1}P^n|Q^{E_n^M}) + H(2/M) + \frac{12 \log \#Q}{M} + \iota.
\]

Before the proof of Lemma 7, we first recall a technical lemma from [2].

Lemma 8 (Lemma 6 in [2]). Let \( (X, T) \) be a topological system. Let \( \mu \) be a Borel probability measure on \( X \) and let \( E \) be a finite subset of \( \mathbb{N} \). For any finite partition \( Q \) of \( X \), we have with \( \mu^E := \frac{1}{|E|} \sum_{k \in E} T^k \mu \) and \( Q^E := \bigvee_{k \in E} T^{-k}Q \):

\[
\frac{1}{|E|} H_{\mu}(Q^E) \leq \frac{1}{m} H_{\mu^E}(Q^m) + 6m \frac{\#(E + 1) \Delta E}{\#E} \log \#Q.
\]

Proof of Lemma 7. As the complement of \( E_n^M(x) \) is the disjoint union of neutral blocks with length larger than \( M \), there are at most \( A_n^M = \sum_{k=0}^{\lfloor 2n/M \rfloor + 1} \binom{n}{k} \) possible values for \( E_n^M(x) \) so that

\[
\frac{1}{n} H_{\hat{\zeta}}(P^n) = \frac{1}{n} H_{\hat{\zeta}}(P^n|E_n^M) + H_{\hat{\zeta}}(E_n^M),
\]

\[
\leq \frac{1}{n} H_{\hat{\zeta}}(P^n|E_n^M) + \log A_n^M,
\]

\[
\liminf_n \frac{1}{n} H_{\hat{\zeta}}(P^n) \leq \limsup_n \frac{1}{n} H_{\hat{\zeta}}(P^n|E_n^M) + H(2/M) \quad \text{by using Stirling's formula.}
\]

Moreover

\[
\frac{1}{n} H_{\hat{\zeta}}(P^n|E_n^M) = \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|\pi^{-1}E_n^M),
\]

\[
\leq \frac{1}{n} H_{\hat{\zeta}^+}(Q^{E_n^M}|\pi^{-1}E_n^M) + \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{E_n^M}).
\]

For \( E \subset [0, n] \) we let \( \hat{\zeta}_{E, n} = \frac{n}{|E|} \int \mu_{X, n}^E d\zeta_{A_E}(x) \), which may be also written as \( \left( \hat{\zeta}_{\pi^{-1}A_E} \right)^E \) by using the notations of Lemma 8. By Lemma 8 applied to the system \( (\mathbb{PTM}, F) \) and the
measures $\mu := \hat{\xi}_{\pi^{-1}A_F}$ for $A_F \in E_n^M$ we have for all $n > m \in \mathbb{N}^*$:

$$H_{\hat{\xi}}^+ \left( Q_{n}^M |_{\pi^{-1} E_n^M} \right) = \sum_{E} \zeta(A_E) H_{\hat{\xi}}^+ (Q_E),$$

$$\leq \sum_{E} \zeta(A_E) \hat{\mu} \left( \frac{1}{m} H_{\hat{\xi}, n}^+ (Q^m) + 6m \hat{\mu} (E + 1) \Delta E \log \hat{\mu} Q \right).$$

Recall again that if $E = E_n^M(x)$ for some $x$ then the complement set of $E$ in $[1, n]$ is made of neutral blocks of length larger than $M$, therefore $\hat{\mu} (E + 1) \Delta E \leq \frac{2M}{n}$. Moreover it follows from $\hat{\xi}^M (\partial Q^m) = 0$ and (2.3), that $\mu^M_{x, n} (A^m)$ for $A^m \in Q^m$ and $\hat{\xi}^M_n (x) / n$ are converging to $\hat{\xi}^M (A^m)$ and $\beta_M$ respectively uniformly in $x \in F$ when $n$ goes to infinity. Then we get by taking the limit in $n$:

$$\limsup_n \frac{1}{n} H_{\hat{\xi}}^+ \left( Q_{n}^M |_{\pi^{-1} E_n^M} \right) \leq \beta_M \frac{1}{m} H_{\hat{\xi}}^M (Q^m) + \frac{12m \log \hat{\mu} Q}{M},$$

$$h(\nu) - \lambda \leq \liminf_n \frac{1}{n} H_{\hat{\xi}} (P^m) \leq \beta_M \frac{1}{m} H_{\hat{\xi}}^M (Q^m) + \limsup_n \frac{1}{n} H_{\hat{\xi}}^+ (\pi^{-1} P^n | Q_{n}^M) + H(2/M) + \frac{12m \log \hat{\mu} Q}{M}.$$ 

\[ \square \]

### 2.5. Bounding the entropy of the neutral component

For a $C^1$ diffeomorphism $f$ on $M$ we put $C(f) := 2A_f H(A_f^{-1}) + \log^+ ||df||_{\infty} + B_r$ with $A_f = \log^+ ||df||_{\infty} + \log^+ ||d(f^{-1})||_{\infty} + 1$ and a universal constant $B_r$ depending only $r$ precised later on. Clearly $f \mapsto C(f)$ is continuous in the $C^1$ topology and $C(f) = \lim_{N \to +\infty} C(f)/p$ whenever $\lambda^+(f) > 0$ (indeed $A_f \overset{p}{\to} +\infty$, therefore $H(A_f^{-1}) \overset{p}{\to} 0$). In particular, if $\lambda^+(f) < \alpha$ and $f_k \overset{p}{\to} f$ in the $C^1$ topology, then there is $p$ with $\lim_{k} C(f)/p < \alpha$.

In this section we consider the empirical measures associated to an ergodic hyperbolic measure $\nu$ with $\lambda^+(\nu) > \frac{1}{r} \log \frac{1}{r} + \delta$, $\delta > 0$. Without loss of generality we can assume $\delta < \frac{r-1}{r} \log 2$. Then by Lemma 3 we have $\hat{\nu}^+(H_{\delta}) > 0$. For $x \in \mathcal{R}$ we let $m_n(x) = \max \{k < n, F^k x \in H_{\delta} \}$. By a standard application of Birkhoff ergodic theorem we have

$$\frac{m_n(x)}{n} \overset{n}{\to} 1 \text{ for } \nu \text{ a.e. } x.$$ 

By taking a smaller subset $F$, we can assume the above convergence of $m_n$ is uniform on $F$ and that $\sup_{x \in F} \min \{ k \leq n, F^k x \in H_{\delta} \} \leq N$ for some positive integer $N$.

We bound the term $\limsup_n \frac{1}{n} H_{\hat{\xi}}^+ (\pi^{-1} P^n | Q_{n}^M)$ in the right hand side of (2.5) Lemma 4 which corresponds to the local entropy contribution plus the entropy in the neutral part.

**Lemma 9.** There is $\kappa > 0$ depending only on $\|d^k f\|_{\infty}$, $2 \leq k \leq r$, $\|\partial^j_g (\exp_{f^{-1}(x)} \circ \exp_{x}) (\cdot)\|_{\infty}$ such that the empirical measures associated to $G := \pi^{-1} G_{\kappa} \cap H_{\delta}$ satisfy the following properties. For all $q, M \in \mathbb{N}^*$, 

Here

$$\|d^k f\|_{\infty} = \sup_{\alpha \in \mathbb{R}, |\alpha| = k} \sup_{x, y} \left\| \partial^j_g (\exp_{f^{-1}(x)} \circ \exp_{x}) (\cdot) \right\|_{\infty}$$
there are \( \epsilon_q > 0 \) depending only on \( \|d^k(f^q)\|_\infty, 2 \leq k \leq r \) and \( \gamma_{q,M}(f) > 0 \) such that for any partition \( Q \) of \( \mathbb{P}T \mathbb{M} \) with diameter less than \( \epsilon_q \), we have:

\[
\limsup_n \frac{1}{n} H_{\xi}(\pi^{-1}P^n|Q_{E_n}^M) \leq (1 - \beta_M)C(f) + \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log^+ \|df^q\|_q}{q} d\xi^M - \int \phi d\xi^M \right) + \gamma_{q,M}(f),
\]

where the error term \( \gamma_{q,M}(f) \) satisfies

\[
\forall K > 0 \limsup_n \sup_{q} \sup_{M} \left( \sup_{f \in \text{Diff}(\mathbb{M})} \left\{ \gamma_{q,M}(f) | \|df\|_\infty \vee \|df^{-1}\|_\infty < K \right\} \right) = 0.
\]

The proof of Lemma 9 appears after the statement of Proposition 4, which is a semi-local Reparametrization Lemma.

**Proposition 4.** There is \( \kappa > 0 \) depending only on \( \|d^k f\|_\infty, 2 \leq k \leq r \), such that the empirical measures associated to \( G := \pi^{-1}G_\kappa \cap H_\delta \) satisfy the following properties. For all \( q, M \in \mathbb{N}^\ast \) there are \( \epsilon_q > 0 \) depending only on \( \|d^k(f^q)\|_\infty, 2 \leq k \leq r \) and \( \gamma_{q,M}(f) > 0 \) satisfying (2.6) such that for any partition \( Q \) with diameter less than \( \epsilon < \epsilon_q \), we have for \( n \) large enough:

Any atom \( F_n \) of the partition \( Q_{E_n}^M \) may be covered by a family \( \Psi_{F_n} \) of \( C^r \) curves \( \psi : [-1, 1] \rightarrow \mathbb{M} \) satisfying \( \|d(f^k \circ \psi)\|_\infty \leq 1 \) for any \( k = 0, \ldots, n - 1 \), such that

\[
\frac{1}{n} \log \# \Psi_{F_n} \leq \left( 1 - \frac{\# E_n^M}{n} \right) C(f) + \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log^+ \|d_x f^q\|_q}{q} \right) \left( \int \phi d\xi^M_t (x) \right) + \gamma_{q,M}(f) + \tau_n,
\]

where \( \lim_n \tau_n = 0, E_n^M = E_n^M(f) \) for \( x \in F_n \), \( \xi^M_{F_n} = \int \mu_{x,n}^M d\xi_{F_n}(x) \) and \( \xi^M_{F_n} = \pi_\ast \xi^M_{F_n} \) its push-forward on \( \mathbb{M} \).

The proof of Proposition 4 is given in the last section. Proposition 4 is very similar to the Reparametrization Lemma in [4]. Here we reparametrize an atom \( F_n \) of \( Q_{E_n}^M \) instead of \( Q^n \) in [4].

**Proof of Lemma 9 assuming Proposition 4.** We take \( \kappa > 0 \) and \( \epsilon_q > 0 \) as in Proposition 4. Observe that

\[
H_{\xi}(\pi^{-1}P^n|Q_{E_n}^M) \leq \sum_{F_n \in Q_{E_n}^M} \xi^+(F_n) \log \# \{A^n \in P^n, \pi^{-1}(A^n) \cap \tilde{F} \cap F_n \neq \emptyset\}.
\]

As \( \nu(\partial P) = 0 \), for all \( \gamma > 0 \), there is \( \chi > 0 \) and a continuous function \( \vartheta : \mathbb{M} \rightarrow \mathbb{R}^+ \) equal to 1 on the \( \chi \)-neighborhood \( \partial P \times \partial P \) of \( \partial P \) satisfying \( \int \vartheta d\nu < \gamma \). Then, by applying (2.3) with \( \varphi : \tilde{x} \mapsto \vartheta(x) \) and \( M = \infty \), we have uniformly in \( x \in \mathbb{F} \subset \pi(\mathbb{H}) \):

\[
\limsup_n \frac{1}{n} \# \{0 \leq k < n, f^k x \in \partial P \times \partial P \} \leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \vartheta (f^k x) = \int \vartheta d\nu < \gamma.
\]
Assume that for arbitrarily large $n$ there is $F_n \in Q^{BM}_n$ and $\psi \in \Psi_{F_n}$ with $\sharp \{ A^n \in P^n, \ A^n \cap \psi([-1,0]) \cap F \neq \emptyset \} > ([\chi^{-1}] + 1)P_F^n$. As $\|d(f^k \circ \psi)\|_\infty \leq 1$ for $0 \leq k < n$ we may reparametrize $\psi$ on $F$ by $[\chi^{-1}] + 1$ affine contractions $\theta$ so that the length of $f^k \circ \psi \circ \theta$ is less than $\chi$ for all $0 \leq k < n$ and $(\psi \circ \theta)([-1,0]) \cap F \neq \emptyset$. Then we have $\sharp \{0 \leq k < n, \partial P \cap (f^k \circ \psi \circ \theta)([-1,0]) \neq \emptyset\} > \gamma n$ for some $\theta$. In particular we get $\sharp \{0 \leq k < n, f^k x \in \partial P \} > \gamma n$ for any $x \in \psi \circ \theta([-1,1])$, which contradicts (2.7). Therefore we have

$$\limsup_n \sup_{F_n, \psi \in \Psi_{F_n}} \frac{1}{n} \log \{ A^n \in P^n, \ A^n \cap \psi([-1,0]) \cap F \neq \emptyset \} = 0.$$ 

Together with Proposition 4 and Lemma 2 we get

$$\limsup_n \frac{1}{n} H_{\hat{\xi}}(\pi^{-1}_n P^n|Q^{BM}_n) \leq \limsup_n \sum_{F_n \in Q^{BM}_n} \hat{\xi}^+(F_n) \frac{1}{n} \log \#F_n,$$

$$\leq \limsup_n \sum_{F_n \in Q^{BM}_n} \hat{\xi}^+(F_n) \left( 1 - \frac{2E_n}{n} \right) C(f) +$$

$$+ \limsup_n \sum_{F_n \in Q^{BM}_n} \hat{\xi}^+(F_n) \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log + \|df^q\|}{q} d\xi^F_n - \int \phi d\hat{\xi}^M_n \right),$$

$$\leq (1 - \beta M)C(f) + \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log + \|df^q\|}{q} d\xi^F_n - \int \phi d\hat{\xi}^M_n \right) + \gamma_{q,M}(f).$$

This concludes the proof of Lemma 9.

By combining Lemma 9 and Lemma 7 we get:

**Proposition 5.** Let $\kappa$, $\epsilon_q$ and $\gamma_{q,M}(f)$ as in Proposition 4. Then for any $q, M \in \mathbb{N}^*$ and for any finite partition $Q$ with diameter less than $\epsilon_q$ and with $\hat{\xi}^M(\partial Q^m) = 0$ we have with $\gamma_{q,Q,M}(f) = \gamma_{q,M}(f) + H \left( \frac{2}{M} \right) + \frac{12 \log Q^m}{M}$:

$$h(\nu) \leq \beta M \frac{1}{m} H_{\hat{\xi}}(Q^m) + (1 - \beta M)C(f)$$

$$+ \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log + \|df^q\|}{q} d\xi^F_n - \int \phi d\hat{\xi}^M_n \right),$$

$$+ \gamma_{q,Q,M}(f) + \iota.$$ 

2.6. **Proof of the Main Theorem.** We first reduce the Main Theorem to the following statement.

**Proposition 6.** Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of $C^r$, with $r > 1$, surface diffeomorphisms converging $C^r$ weakly to a diffeomorphism $f$. Assume there is a sequence $(\hat{\nu}^+_k)_k$ of ergodic $F_k$-invariant measures converging to $\hat{\nu}$ with $\lim_k \lambda^+ (\nu_k) > \frac{\log + \|df\|_\infty}{r}$. Then, there are $F$-invariant measures $\hat{\mu}_0$ and $\hat{\mu}^+_1$ with $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta \hat{\mu}^+_1$, $\beta \in [0,1]$, such that:

$$\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta) C(f).$$
Proof of the Main Theorem assuming Proposition 6. Let \((\hat{\nu}^+_k)_k\) be a sequence of ergodic \(F_k\)-invariant measures converging to \(\hat{\mu}\).

As previously mentioned, for any \(\alpha > \lambda^+(f)/r\) there is \(p \in \mathbb{N}^+\) with \(\alpha > C(f_p)/p\). We can also assume \(\frac{\log \|df^p\|_r}{p} = \log \|df^p\|_r < \alpha\). Let \(\hat{\nu}^+_k\) be an ergodic component of \(\hat{\nu}_k\) for \(F_k^p\) and let us denote by \(\nu_k^p\) its push forward on \(\mathcal{M}\). We have \(h_{f_k}^p(\nu_k^p) = ph_{f_k}(\nu_k)\) for all \(k\). By taking a subsequence we can assume that \((\hat{\nu}^+_k)_k\) is converging. Its limit \(\hat{\mu}\) satisfies \(\frac{1}{p} \sum_{0 \leq l < p} F_k^l \hat{\mu}^p = \hat{\mu}\). If \(\lim_k \lambda^+(\nu_k^p) \leq \frac{1}{p} \log \|df^p\|_r \leq \alpha\), then by Ruelle’s inequality we get
\[
\limsup_{k \to +\infty} h_{f_k}(\nu_k) = \limsup_{k \to +\infty} \frac{1}{p} h_{f_k}^p(\nu_k^p),
\]
\[
\leq \lim_{k \to +\infty} \frac{1}{p} \lambda^+(\nu_k^p),
\]
\[
< \alpha.
\]
This proves the Main Theorem with \(\beta = 1\).

We consider then the case \(\lim_k \lambda^+(\nu_k^p) > \frac{1}{p} \log \|df^p\|_r \). By applying Proposition 4 to the \(p\)-power system, we get \(F_k^p\)-invariant measure \(\hat{\mu}_0^p\) and \(\hat{\mu}_1^+\) with \(\hat{\mu} = (1 - \beta)\hat{\mu}_0^p + \beta \hat{\mu}_1^+, \beta \in [0, 1]\), such that we have with \(\mu_0^p = \pi_*\hat{\mu}_0^p\) :
\[
\limsup_{k \to +\infty} h_{f_k}^p(\nu_k^p) \leq \beta h_{f_p}(\mu_0^p) + (1 - \beta)C(f_p).
\]
But \(h_{f_p}(\mu_0^p) = ph_{f}(\mu_1)\) with \(\mu_1 = \frac{1}{p} \sum_{0 \leq l < p} f^k \mu_0^p\). One easily checks that \(\hat{\mu}_1^+ = \frac{1}{p} \sum_{0 \leq l < p} F_k^l \hat{\mu}_1^+\). Then we have :
\[
\limsup_{k \to +\infty} h_{f_k}(\nu_k) = \limsup_{k \to +\infty} \frac{1}{p} h_{f_k}^p(\nu_k^p),
\]
\[
\leq \beta \frac{1}{p} h_{f_p}(\mu_0^p) + (1 - \beta) \frac{C(f_p)}{p},
\]
\[
\leq h_{f}(\mu_1) + (1 - \beta)\alpha.
\]
This concludes the proof of the Main Theorem.

We show now Proposition 6 by using Lemma 9.

Proof of Proposition 6. Without loss of generality we can assume \(\liminf_k h(\nu_k) > 0\). For \(\mu\) a.e. \(x\), we have \(\lambda^-(x) \leq 0\). If not, some ergodic component \(\hat{\mu}\) of \(\mu\) would have two positive Lyapunov exponents and therefore should be the periodic measure at a source \(S\) (see e.g. Proposition 4.4 in [13]). But then for large \(k\) the probability \(\nu_k\) would give positive measure to the basin of attraction of the sink \(S\) for \(f^{-1}\) and therefore \(\nu_k\) would be equal to \(\hat{\mu}\) contradicting \(\liminf_k h(\nu_k) > 0\).

Let \(\delta > 0\) with \(\lim_k \lambda^+(\nu_k) > \frac{1}{p} \log \|df^p\|_r + \delta\). Then take \(\kappa\) as in Lemma 9. We consider the empirical measures associated to \(G = \pi^{-1}G_\kappa \cap H_\delta\). By a diagonal argument, there is a subsequence in \(k\) such that the geometric component \(\hat{\xi}_M^k\) of \(\hat{\nu}_k^+\) is converging to some \(\hat{\xi}_M\) for all \(M \in \mathbb{N}\). Let us also denote by \(\beta_M^\infty\) the limit in \(k\) of \(\beta_M^k\). Then consider a subsequence in \(M\) such that \(\hat{\xi}_M\) is converging to \(\beta\hat{\mu}_1\) with \(\beta = \lim_M \beta_M^\infty\). We also let \((1 - \beta)\hat{\mu}_0 = \hat{\mu} - \beta\hat{\mu}_1\).

In this way, \(\hat{\mu}_0\) and \(\hat{\mu}_1\) are both probability measures.

Lemma 10. The measures \(\hat{\mu}_0\) and \(\hat{\mu}_1\) satisfy the following properties:
• $\hat{\mu}_1$ and $\hat{\mu}_0$ are $F$-invariant,
• $\lambda^+(x) \geq \delta$ for $\mu_1$-a.e. $x$ and $\hat{\mu}_1 = \hat{\mu}_1^+$.

Proof. The neutral blocks in the complement set of $E^M(x)$ have length larger than $M$. Therefore for any continuous function $\varphi : \mathbb{PTM} \to \mathbb{R}$ and for any $k$, we have

$$\left| \int \varphi d\hat{\xi}_k^M - \int \varphi \circ F d\hat{\xi}_k^M \right| \leq \frac{2 \sup_x |\varphi(x)|}{M}.$$ 

Letting $k$, then $M$ go to infinity, we get $\int \varphi d\hat{\mu}_1 = \int \varphi \circ F d\hat{\mu}_1$, i.e. $\hat{\mu}_1$ is $F$-invariant.

We let $K_M$ be the compact subset of $\mathbb{PTM}$ given by $K_M = \{\hat{x} \in \mathbb{PTM}, \exists 1 \leq m \leq M \phi_m(\hat{x}) \geq m\}$. Let $\hat{x} \in G_k$, where $G_k$ is the set where the empirical measures are converging to $\xi_k^M$ (see Lemma 1). Observe that

$$\lim_n \mu_{x,n}^M(K_M) = \hat{\xi}_k^M(K_M) = \hat{\xi}_k^M(\mathbb{PTM}).$$

Indeed for any $k \in E^M(\hat{x})$ there is $1 \leq m \leq M$ with $E^m(F^k\hat{x}) \in G \subset H_\delta$. Moreover, as already mentioned, $\delta$-hyperbolic points w.r.t. $\psi$ are $\delta$-hyperbolic w.r.t. $\phi$. Therefore $\phi_m(F^k\hat{x}) \geq m\delta$. Consequently we have $\lim_n \mu_{x,n}^M(K_M) = \lim_n \mu_{x,n}^M(\mathbb{PTM}) = \hat{\xi}_k^M(\mathbb{PTM})$. The set $K_M$ being compact in $\mathbb{PTM}$, we get $\hat{\xi}_k^M(K_M) \geq \lim_n \mu_{x,n}^M(K_M)$ and (2.8) follows.

Also we have $\hat{\xi}_k^M(K_M) \geq \limsup_k \hat{\xi}_k^M(K_M) = \limsup_k \hat{\xi}_k^M(\mathbb{PTM}) = \beta_{\infty}^M$. Therefore we have $\hat{\mu}_1(\bigcup M K_M) = 1$ as $\hat{\xi}_k^M$ goes increasingly in M to $\beta\hat{\mu}_1$. The $F$-invariant set $\bigcap_{k \in \mathbb{Z}} E^{-k}(\bigcup M K_M)$ has also full $\hat{\mu}_1$-measure and for all $\hat{x} = (x,v)$ in this set we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v)\| \geq \delta$. Consequently the measure $\hat{\mu}_1$ is supported on the unstable bundle $\mathcal{E}_+(x)$ and $\lambda^+(x) \geq \delta$ for $\mu_1$-a.e. $x$. \hfill \Box

Remark 7. In Theorem C of [10], the measure $\beta\hat{\mu}_1^+$ is obtained as the limit when $\delta$ goes to zero of the component associated to the set $G^\delta := \{x, \forall l > 0 \phi_l(\hat{x}) \geq \delta l\} \supset \pi^{-1}G_\delta \cap H_\delta$. Therefore our measure $\beta_{\alpha\hat{\mu}_1^+,\alpha}$ is just a component of their measure $\beta\hat{\mu}_1^+$. 

We pursue now the proof of Proposition 6. Let $q,M \in \mathbb{N}^\ast$. Fix a sequence $(t_k)_k$ of positive numbers with $t_k \overset{k}{\rightarrow} 0$. We consider a partition $Q$ satisfying $\text{diam}(Q) < \varepsilon_q$ with $\varepsilon_q$ as in Lemma 3. The sequence $(f_k)_k$ being $C^r$ bounded, one can choose $\varepsilon_q$ independently of $f_k$, $k \in \mathbb{N}$.

By a standard argument of countability we may assume that for all $m \in \mathbb{N}^\ast$ the boundary of $Q^m$ has zero-measure for $\hat{\mu}_1^+$ and all the measures $\hat{\xi}_k^M$, $M \in \mathbb{N}^\ast$ and $k \in \mathbb{N} \cup \{\infty\}$. By applying Proposition 5 to $f_k$ and $\nu_k$ we get:

$$h(\nu_k) \leq \beta_k^M \frac{1}{m} H_{\xi_k^M}(Q^m) + (1 - \beta_k^M)\mathcal{C}(f_k)$$

$$+ \left( \log 2 + \frac{1}{r-1} \right) \left( \int \log^+ \frac{\|df_k\|}{q} d\xi_k^M - \int \phi d\hat{\xi}_k^M \right)$$

$$\leq \gamma_q Q M (f_k) + t_k.$$
By letting $k$, then $M$ go to infinity, we obtain for all $m$:

$$\limsup_k h(\nu_k) \leq \beta \frac{1}{m} H_{\mu_1} (Q^m) + (1 - \beta) C(f)$$

$$+ \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right)$$

$$+ \limsup_k \sup_M H_{\gamma_q, Q, M, f_k}.$$

By letting $m$ go to infinity, we get:

$$\limsup_k h(\nu_k) \leq \beta h(\hat{\mu}_1^+) + (1 - \beta) C(f)$$

$$+ \left( \log 2 + \frac{1}{r - 1} \right) \left( \int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right)$$

$$+ \limsup_k \sup_M \gamma_q, Q, M, f_k.$$

But $h(\hat{\mu}_1^+) = h(\mu_1)$ as the measure preserving systems associated to $\mu_1$ and $\hat{\mu}_1^+$ are isomorphic. Moreover we have $\int \phi d\hat{\mu}_1^+ = \lambda^+(\mu_1) = \lim_q \int \frac{\log^+ \|df^q\|}{q} d\mu_1$. Therefore by letting $q$ go to infinity we finally obtain with the asymptotic property \eqref{eq:asymptotic} of $\gamma_q, M$:

$$\limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta) C(f).$$

This concludes the proof of Proposition 6. \hfill \Box

3. Semi-local Reparametrization Lemma

In this section we prove the semi-local Reparametrization Lemma stated above in Proposition 4.

3.1. Strongly bounded curves. To simplify the exposition (by avoiding irrelevant technical details involving the exponential map) we assume that $M$ is the two-torus $\mathbb{T}^2$ with the usual Riemannian structure inherited from $\mathbb{R}^2$. Borrowing from [2] we first make the following definitions.

A $C^r$ embedded curve $\sigma : [-1, 1] \to M$ is said \textit{bounded} when $\max_{k=2, \ldots, r} \|d^k \sigma\|_{\infty} \leq \|d\sigma\|_{\infty}$. \hfill \Box

\textbf{Lemma 11.} Assume $\sigma$ is a bounded curve. Then for any $x \in \sigma([-1, 1])$, the curve $\sigma$ contains the graph of a $\kappa$-admissible map at $x$ with $\kappa = \frac{\|d\sigma\|_{\infty}}{6}$.

\textit{Proof.} Let $x = \sigma(s)$, $s \in [-1, 1]$. One checks easily (see Lemma 7 in [4] for further details) that for all $t \in [-1, 1]$ the angle $\angle \sigma'(s), \sigma'(t) < \frac{\pi}{6} \leq 1$ and therefore $\int_0^1 \sigma'(t) \cdot \frac{\sigma'(s)}{\|\sigma'(s)\|} dt \geq \frac{\|d\sigma\|_{\infty}}{6}$. Therefore, as $\sigma'(s) \in \mathcal{E}_+(x)$, the image of $\sigma$ contains the graph of an $\frac{\|d\sigma\|_{\infty}}{6}$-admissible map at $x$. \hfill \Box

A $C^r$ bounded curve $\sigma : [-1, 1] \to M$ is said \textit{strongly $\epsilon$-bounded} for $\epsilon > 0$ if $\|d\sigma\|_{\infty} \leq \epsilon$. For $n \in \mathbb{N}^*$ and $\epsilon > 0$ a curve is said \textit{strongly $(n, \epsilon)$-bounded} when $f^k \circ \sigma$ is strongly $\epsilon$-bounded for all $k = 0, \ldots, n - 1$. 

\hfill \Box
We consider a $C^r$ smooth diffeomorphism $g : M \to \mathbb{R}$ with $\mathbb{R} \ni r \geq 2$. For $\hat{x} = (x, v) \in PTM$ with $\pi(\hat{x}) = x$, we let $k_g(x) \geq k'_g(\hat{x})$ be the following integers:

$$k_g(x) := [\log \|d_x g\|],$$

$$k'_g(\hat{x}) := [\log \|d_v g(v)\|] = [\phi_g(\hat{x})].$$

In the next lemma, we reparametrize the image by $g$ of a bounded curve. The proof of this lemma is mostly contained in the proof of the Reparametrization Lemma [2], but we reproduce it for the sake of completeness.

**Lemma 12.** Let $\frac{R_{\epsilon}}{2} > \epsilon > \epsilon_g > 0$ satisfying $\|d^s g^{s}_{2k}\|_{\infty} \leq 3\epsilon \|d_{x} g\|$ for all $s = 1, \cdots, r$ and all $x \in M$, where $g^{s}_{2k} = g \circ \exp_{y}(2\epsilon \cdot 1) = g(x + 2\epsilon \cdot 1) : \{w_{x} \in T_{y} M, \|w_{x}\| \leq 1\} \to M$. We assume $\sigma : [-1, 1] \to M$ is a strongly $\epsilon$-bounded $C^r$ curve and we let $\hat{\sigma} : [-1, 1] \to \mathbb{P} TM$ be the associated induced map.

Then for some universal constant $C_{r} > 0$ depending only on $r$ and for any pair of integers $(k, k')$ there is a family $\Theta$ of affine maps from $[-1, 1]$ to itself satisfying:

- $\hat{\sigma}^{-1} \{ \hat{x} \in \mathbb{P} TM, k_{g}(x) = k$ and $k'_{g}(\hat{x}) = k' \} \subset \bigcup_{\theta \in \Theta} \theta([-1, 1]),$
- $\forall \theta \in \Theta, \text{ the curve } g \circ \sigma \circ \theta \text{ is bounded},$
- $\forall \theta \in \Theta, \| \theta' \| \leq e^{\frac{k'}{k - 1}} / 4,$
- $\sharp \Theta \leq C_{r} e^{\frac{k'}{k - 1}}.$

**Proof.** First step: **Taylor polynomial approximation.** One computes for an affine map $\theta : [-1, 1] \to \mathbb{P} TM$ with contraction rate $b$ precised later and with $y = \sigma(t)$, $k_{g}(y) = k$, $k'_{g}(y) = k'$, $t \in \theta([-1, 1]):$

$$\|d'(g \circ \sigma \circ \theta)\|_{\infty} \leq b'^2 \|d''(g^{y}_{2k} \circ \sigma^{y}_{2k})\|_{\infty},$$

with $\sigma^{y}_{2k} := (2\epsilon)^{-1} \exp_{y} \circ \sigma = 2\epsilon^{-1} (\sigma(\cdot) - y),$

$$\leq b'^2 \|d'' \left( d_{\sigma^{y}_{2k}} g^{y}_{2k} \circ d \sigma^{y}_{2k} \right)\|_{\infty},$$

$$\leq b'^2 2^{r} \max_{s = 0, \cdots, r} \|d^{s} \left( d_{\sigma^{y}_{2k}} g^{y}_{2k} \right)\|_{\infty} \max_{k = 1, \cdots, r} \|d^{k} \sigma^{y}_{2k}\|_{\infty}.$$  

By assumption on $\epsilon$, we have $\|d^{s} g^{s}_{2k}\|_{\infty} \leq 3\epsilon \|d_{x} g\|$ for any $r \geq 1$. Moreover $\max_{k = 1, \cdots, r} \|d^{k} \sigma^{y}_{2k}\|_{\infty} \leq 1$ as $\sigma$ is strongly $\epsilon$-bounded. Therefore by Faà di Bruno’s formula, we get for some constants $C_{r} > 0$ depending only on $r$:

$$\max_{s = 0, \cdots, r - 1} \|d^{s} \left( d_{\sigma^{y}_{2k}} g^{y}_{2k} \right)\|_{\infty} \leq \epsilon C_{r} \|d_{y} g\|,$$

then

$$\|d'(g \circ \sigma \circ \theta)\|_{\infty} \leq \epsilon C_{r} b'^{2} \|d_{y} g\| \max_{k = 1, \cdots, r} \|d^{k} \sigma^{y}_{2k}\|_{\infty},$$

$$\leq C_{r} b'^{2} \|d_{y} g\| \|d \sigma\|_{\infty},$$

$$\leq (C_{r} b'^{2} \|d_{y} g\|) \|d(\sigma \circ \theta)\|_{\infty},$$

$$\leq (C_{r} b'^{2} \epsilon^k) \|d(\sigma \circ \theta)\|_{\infty},$$

because $k(y) = k$, $\leq \epsilon^{k' - 4} \|d(\sigma \circ \theta)\|_{\infty},$ by taking $b = \left( C_{r} \epsilon^{k' - k + 1} \right)^{- \frac{1}{k' - 1}}.$

\footnote{Although these constants may differ at each step, they are all denoted by $C_{r}.$}
Therefore the Taylor polynomial $P$ at 0 of degree $r - 1$ of $d(g \circ \sigma \circ \theta)$ satisfies on $[-1, 1]$: $\|P - d(g \circ \sigma \circ \theta)\|_\infty \leq e^{k' - 4}\|d(\sigma \circ \theta)\|_\infty$.

We may cover $[-1, 1]$ by at most $b^{-1} + 1$ such affine maps $\theta$.

**Second step : Bezout theorem.** Let $a = e^{k'}\|d(\sigma \circ \theta)\|_\infty$. Note that for $s \in [-1, 1]$ with $k(\sigma \circ \theta(s)) = k$ and $k'(\sigma \circ \theta(s)) = k'$ we have $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-2}, ae^{2}]$, therefore $\|P(s)\| \in [ae^{-3}, ae^{3}]$. Moreover if we have now $\|P(s)\| \in [ae^{-3}, ae^{3}]$ for some $s \in [-1, 1]$ we get also $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-4}, ae^{4}]$.

By Bezout theorem the semi-algebraic set $\{s \in [-1, 1], \|P(s)\| \in [e^{-3}a, ae^{3}]\}$ is the disjoint union of closed intervals $(J_i)_{i \in I}$ with $I$ depending only on $r$. Let $\theta_i$ be the composition of $\theta$ with an affine reparametrization from $[-1, 1]$ onto $J_i$.

**Third step : Landau-Kolmogorov inequality.** By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [2]), we have for some constants $C_r \in \mathbb{N}^*$ and for all $1 \leq s \leq r$:

$$\|d^s(g \circ \sigma \circ \theta_i)\|_\infty \leq C_r \left(\|d^r(g \circ \sigma \circ \theta_i)\|_\infty + \|d(g \circ \sigma \circ \theta_i)\|_\infty\right),$$

$$\leq C_r \frac{|J_i|}{2} \left(\|d^r(g \circ \sigma \circ \theta)\|_\infty + \sup_{t \in J_i} \|d(g \circ \sigma \circ \theta)(t)\|\right),$$

$$\leq C_r a \frac{|J_i|}{2}.$$

We cut again each $J_i$ into $1000C_r$ intervals $\tilde{J}_i$ of the same length with

$$\theta(\tilde{J}_i) \cap \sigma^{-1}\{x, k_n(x) = k \text{ and } k'_n(x) = k'\} \neq \emptyset.$$

Let $\tilde{\theta}_i$ be the affine reparametrization from $[-1, 1]$ onto $\theta(\tilde{J}_i)$. We check that $g \circ \sigma \circ \tilde{\theta}_i$ is bounded:

$$\forall s = 2, \ldots, r, \|d^s(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty \leq (1000C_r)^{-2}\|d^s(g \circ \sigma \circ \theta_i)\|_\infty,$$

$$\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} a_n e^{-4},$$

$$\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in \tilde{J}_i} \|d(g \circ \sigma \circ \theta)(s)\|,$$

$$\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in \tilde{J}_i} \|d(g \circ \sigma \circ \theta)(s)\|,$$

$$\leq \frac{1}{6} \|d(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty.$$

This conclude the proof with $\Theta$ being the family of all $\tilde{\theta}_i$’s. \qed

We recall now a useful property of bounded curve (see Lemma 7 in [4] for a proof).

**Lemma 13.** Let $\sigma : [-1, 1] \to M$ be a $C^\infty$ bounded curve and let $B$ be a ball of radius less than $\epsilon$. Then there exists an affine map $\theta : [-1, 1] \cap \sigma^{-1} B$. 

- $\sigma \circ \theta$ is strongly $3\epsilon$-bounded,
- $\theta([-1, 1]) \supset \sigma^{-1} B$. 
3.2. Choice of the parameters $\kappa$ and $\epsilon_q$. For a diffeomorphism $f : M \subset M$ the scale $\epsilon_f$ in Lemma 13 may be chosen such that $\epsilon_f \leq C \epsilon_f \leq \max(1, \|df\|_{\infty})^{-k}$ for any $q \geq k \geq l \geq 1$. We take $\kappa = \frac{\epsilon_f^2}{M}$ and we choose $\epsilon_q < \frac{\epsilon_f^2}{M}$ such that for any $\hat{x}, \hat{y} \in F_n$ which are $\epsilon_q$-close and for any $0 \leq l \leq q$:
\begin{equation}
\begin{aligned}
|k_{f_i}(x) - k_{f_i}(y)| & \leq 1, \\
|k'_{f_i}(\hat{x}) - k'_{f_i}(\hat{y})| & \leq 1.
\end{aligned}
\end{equation}

Without loss of generality we can assume the local unstable curve $D$ (defined in Subsection 2.3) is reparametrized by a $C^r$ strongly $\epsilon_q$-bounded map $\sigma : [-1, 1] \to D$.

Let $F_n$ be an atom of the partition $Q_n$ and let $E_n = E_n(\hat{x})$ for any $\hat{x} \in F_n$. Recall that the diameter of $Q$ is less than $\epsilon_q$. It follows from (3.1) that for any $\hat{x} \in F_n$ we have with $\hat{F}_n = \int f^M_n \, d\hat{F}_n(\hat{x})$:
\begin{equation}
\sum_{i \in E_{n}^{M}} \left| k_{f_i}(f^{l}x) - k'_{f_i}(f^{l}\hat{x}) \right| \leq 10 \epsilon_q E_n + \int \log^+ \|d_g f^q\| \, d\hat{F}_n(y) - \int \phi_q \, d\hat{F}_n.
\end{equation}

Therefore we may fix some $0 \leq c < q$, such that for any $x \in F_n$:
\begin{equation}
\sum_{i \in (c+qN) \cap E_{n}^{M}} \left| k_{f_i}(f^{l}x) - k'_{f_i}(F^{l}\hat{x}) \right| \leq 10 \frac{n}{q} + \frac{1}{q} \left( \int \log^+ \|d_g f^q\| \, d\hat{F}_n(y) - \int \phi_q \, d\hat{F}_n \right),
\end{equation}
\begin{equation}
\leq 10 \frac{n}{q} + 2A_f \frac{q}{M} + \frac{1}{q} \int \log^+ \|d_g f^q\| \, d\hat{F}_n(y) - \int \phi \, d\hat{F}_n.
\end{equation}

3.3. Combinatorial aspects. We put $\partial_l E_n^{M} := \{a \in E_n^{M} \text{ with } a - 1 \notin E_n^{M}\}$. Then we let $A_n := \{0 = a_1 < a_2 < \cdots a_m\}$ be the union of $\partial_l E_n^{M}$, $[0, n] \setminus E_n^{M}$ and $(c + qN) \cap [0, n]$. We also let $b_i = a_{i+1} - a_i$ for $i = 1, \cdots, m - 1$ and $b_m = n - a_m$.

For a sequence $k = (k_t, k'_t) \in A_n$ of integers, a positive integer $m_n$ and a subset $\Sigma$ of $[0, n]$, we let $F_n^k \Sigma_m$ be the subset of points $\hat{x} \in F_n$ satisfying:
\begin{itemize}
  \item $\Sigma = E_n(x) \setminus E_n^{M}(x)$,
  \item $k_{a_i} = k_{f_i}(f^{a_i}x)$ and $k'_{a_i} = k'_{f_i}(F^{a_i}\hat{x})$ for $i = 1, \cdots, m$,
  \item $m_n(x) = m_n$.
\end{itemize}

Lemma 14.
\begin{equation}
\sharp \left\{ (k, \Sigma, \Sigma_n) : F_n^k \Sigma_m \neq \emptyset \right\} \leq ne^{2n A_f H(A_f^{-1}) 3n(1/q + 1/M) \epsilon_n H(1/M)}.
\end{equation}

Proof. First we observe that if $a_i \notin E_n^{M}$ then $b_i = 1$. In particular $\sum_i a_i \notin E_n^{M} k_{a_i} \leq (n - \sharp E_n^{M}) \log^+ \|df\|_{\infty} \leq (n - \sharp E_n^{M})(A_f - 1)$. The number of such sequences $(k_{a_i})_{i, a_i \notin E_n^{M}}$ is therefore bounded above by $(\binom{n}{r_n A_f})$ with $r_n = n - \sharp E_n^{M}$ and its logarithm is dominated by $r_n A_f H(A_f^{-1}) + 1 \leq n A_f H(A_f^{-1}) + 1$. Similarly the number of sequence $(k'_{a_i})_{i, a_i \notin E_n^{M}}$ is less than $n A_f H(A_f^{-1}) + 1$.

Then from the choice of $\epsilon_q$ in (3.1) there are at most three possible values of $k_{a_i}(x)$ for $a_i \in E_n^{M}$ and $x \in F_n$.

Finally as $\sharp E_n \leq n/M$, the number of admissible sets $\Sigma$ is less than $\binom{n}{\lfloor n/M \rfloor}$ and thus its logarithm is bounded above by $n H(1/M) + 1$. Clearly we can also fix the value of $m_n$ up to a factor $n$. 

Existence of maximal measure for $\mathcal{C}^r$ surface diffeos 19
3.4. The induction. We fix $k$, $m_n$ and $\mathcal{E}$ and we reparametrize appropriately the set $F^{k,e,m_n}_n$.

**Lemma 15.** With the above notations there are families $(\Theta_i)_{i \leq m}$ of affine maps from $[-1,1]$ into itself such that :

- $\forall \theta \in \Theta_i \forall j \leq i$ the curve $f^{a_i} \circ \sigma \circ \theta$ is strongly $\epsilon_{f^{b_i}}$-bounded,
- $\delta^{-1}\left(F^{k,e,m_n}_n\right) \subseteq \bigcup_{\theta \in \Theta_i} \theta([-1,1])$,
- $\forall \theta \in \Theta_i \forall j < i , \exists \hat{\theta}_j \in \Theta_j$, $\frac{|\theta_j'|}{|\hat{\theta}_j'|} \leq \ell \leq \prod_{\ell \leq i} e^{\frac{k_{a_i} - k_{a_i} - 1}{\tau - 1}}/4$, 
- $\exists \Theta \leq C \max(1,\|df\|_{\infty})^{2E[1,a]} \prod_{\ell < i} C_{\ell} e^{\frac{k_{a_i} - k_{a_i} - 1}{\tau - 1}}$.

**Proof.** We argue by induction on $i \leq m$. By changing the constant $C$, it is enough to consider $i$ with $a_i > N$. Recall that the integer $N$ was chosen in such a way that for any $x \in F$ there is $0 \leq k \leq N$ with $F^k \hat{x} \in H_k$. We assume the family $\Theta_i$ for $i < m$ already built and we will define $\Theta_{i+1}$. Let $\theta_i \in \Theta_i$. We apply Lemma 12 to the strongly $\epsilon_{f^{b_i}}$-bounded curve $f^{a_i} \circ \sigma \circ \theta_i$ with $g = f^{b_i}$. Let $\Theta$ be the family of affine reparametrizations of $[-1,1]$ satisfying the conclusions of Lemma 12 in particular $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$ is bounded, $|\theta'| \leq e^{\frac{k_{a_i} - k_{a_i} - 1}{\tau - 1}}/4$ for all $\theta \in \Theta$ and $\exists \Theta \leq C_{\ell} e^{\frac{k_{a_i} - k_{a_i} - 1}{\tau - 1}}$. We distinguish three cases:

- $a_{i+1} \in E^{M}_n$. The diameter of $F^{a_{i+1}}_n$ is less than $\epsilon_{\theta} \leq \frac{\epsilon_{f^{b_{i+1}}}}{3}$. By Lemma 13 there is an affine map $\psi : [-1,1] \to \Theta_i$ such that $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \psi$ is strongly $\epsilon_{f^{b_{i+1}}}$-bounded and its image contains the intersection of the bounded curve $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \psi$ with $F^{a_{i+1}}_n$. We let then $\theta_{i+1} = \theta_i \circ \theta_i \circ \psi \in \Theta_{i+1}$.

- $a_{i+1} \in E \setminus E^{M}_n$. Observe that $b_{i+1} = 1$, therefore $\epsilon_{\theta} \leq \epsilon_{f^{b_{i+1}}}$. Then the length of the curve $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$ is less than $3\|df\|_{\infty} \epsilon_{f^{b_{i+1}}}$, thus may be covered by $3\|df\|_{\infty} + 1$ balls of radius less than $\epsilon_{f^{b_{i+1}}}$. We then use Lemma 13 as in the previous case to reparametrize the intersection of this curve with each ball by a strongly $\epsilon_{f^{b_{i+1}}}$-bounded curve. We define in this way the associated parametrizations of $\Theta_{i+1}$.

- $a_{i+1} \notin E$ and $a_{i+1} \notin E^{M}_n$. We claim that $\|df^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta\| \leq \epsilon_{f}/3$. Take $\hat{x} \in F^{k,e,m_n}_n$ with $x = \pi(\hat{x}) = \sigma \circ \theta_i \circ \theta(s)$. Let $K_x = \max\{k < a_{i+1}, F^k \hat{x} \in H_k\} \geq N$. Observe that $[K_x,a_{i+1}] \cap E^{M}_n = \emptyset$, therefore for $K_x \leq a_i < a_{i+1}$, we have $b_i = 1$, then $a_i = a_{i+1} - 1 - 1 + l$. We argue by contradiction by assuming:

$$(3.2) \quad \|df^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta\| \geq \epsilon_{f}/6 = 6\kappa$$

By Lemma 13, the point $f^{a_{i+1}} \hat{x}$ belongs to $G_{\kappa}$. We will show $F^{a_{i+1}} \hat{x} \in H_\kappa$. Therefore we will get $F^{a_{i+1}} \hat{x} \in G = \pi^{-1}G_{\kappa} \cap H_\kappa$ contradicting $a_{i+1} \notin E$. To prove $F^{a_{i+1}} \hat{x} \in H_\kappa$ it is enough to show $\sum_{j \leq L < a_{i+1}} \psi(F^j \hat{x}) \geq (a_{i+1} - j)\delta$ for any $K_x \leq j < a_{i+1}$ because
\( F^K_x(\hat{x}) \) belongs to \( H_\delta \). For any \( K_x \leq j < a_{i+1} \) we have:

\[
\| d(f^{a_{i+1}} \circ \sigma \circ \theta) \|_\infty \leq 2 \| d_x(f^{a_{i+1}} \circ \sigma \circ \theta \circ \theta) \|, \quad \text{because } f^{a_{i+1}} \circ \sigma \circ \theta \circ \theta \text{ is bounded,}
\]

\[
\leq 2 \| d_x f^{a_{i+1} - j}(\hat{x}) \| \times \| d_x(f^{\sigma \circ \theta_2^j}) \| \times \frac{\| \theta_1^j \|}{\| \theta_2^j \|}, \quad \text{with } a_j = j,
\]

\[
\leq \frac{e_f}{3} \| d_x f^{a_{i+1} - j}(\hat{x}) \| \prod_{j \leq l \leq i} e^{k_{a_j} - k_{a_1} - 1} / r \text{ by induction hypothesis,}
\]

\[
(3.3) \quad \frac{1}{2} \leq \| d_x f^{a_{i+1} - j}(\hat{x}) \| \prod_{j \leq l \leq i} e^{k_{a_j} - k_{a_1} - 1} / r \text{ by assumption [3.2].}
\]

Recall again that for \( j \leq l \leq i \), we have \( b_l = 1 \), thus

\[
| k_{a_l} - \log \| d_x f \| | \leq 1
\]

and

\[
k_{a_l}^l \leq \phi(F^{a_l} \hat{x}).
\]

Therefore we get for any \( K_x \leq j < a_{i+1} \) from (3.3):

\[
2^{a_{i+1} - j} \leq e^{\frac{1}{r-1} \sum_{j \leq l < a_{i+1}} \phi(F^l \hat{x}) - \frac{1}{r-1} \sum_{j \leq l < a_{i+1}} \log^+ \| d_x f \|},
\]

\[
(a_{i+1} - j) \log 2 \leq \frac{r-1}{r} \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \quad \text{by definition of } \psi,
\]

\[
(a_{i+1} - j) \delta \leq \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \quad \text{as } \delta \text{ was chosen less than } \frac{r-1}{r} \log 2.
\]

\[ \square \]

**Lemma 16.**

\[
\sum_{i, \ m_n > a_i \notin E^M_n} \frac{k_{a_i} - k_{a_i}^l}{r - 1} \leq \left( n - \#E^M_n \right) \left( \frac{\log^+ \| df \|_\infty}{r} + \frac{1}{r - 1} \right).
\]

**Proof.** The intersection of \([0, m_n]\) with the complement set of \( E^M_n \) is the disjoint union of integers of the form \([l, m_n]\). In any case \( F^l \hat{x} \) belongs to \( H_\delta \) for such an interval \([i, j]\) for any \( x \in F^k E^{m_n} \). In particular, we have

\[
\sum_{l, a_l \in [i, j[} k_{a_l}^l - k_{a_l} \geq (\delta - 1)(j - i)
\]

Therefore

\[
\sum_{i, \ m_n > a_i \notin E^M_n} \frac{k_{a_i} - k_{a_i}^l}{r - 1} \geq -(n - \#E^M_n),
\]

\[
\sum_{i, \ m_n > a_i \notin E^M_n} \frac{k_{a_i} - k_{a_i}^l}{r - 1} \leq \frac{n - \#E^M_n}{r - 1} + \sum_{i, \ m_n > a_i \notin E^M_n} k_{a_i},
\]

\[
\leq \left( n - \#E^M_n \right) \left( \frac{\log^+ \| df \|_\infty}{r} + \frac{1}{r - 1} \right).
\]
3.5. Conclusion. We let $\Psi_n$ be the family of $C^r$ curves $\sigma \circ \theta$ for $\theta \in \Theta_m = \Theta_m(k, E, m_n)$ with $\Theta_m$ as in Lemma [15] over all admissible parameters $k, E, m_n$. For $\theta \in \Theta_m$ the curve $f^{a_i} \circ \sigma \circ \theta$ is strongly $\epsilon_{f^{a_i}}$-bounded for any $i = 1, \ldots, m$, in particular
\[
\forall i = 1, \ldots, m, \|d(f^{a_i} \circ \sigma \circ \theta)\|_\infty \leq \epsilon_{f^{a_i}} \leq \max(1, \|df\|_\infty)^{-b},
\]
therefore
\[
\forall j = 0, \ldots, n, \|d(f^j \circ \sigma \circ \theta)\|_\infty \leq 1.
\]
By combining the previous estimates, we get moreover:
\[
\sharp \Psi_n \leq \sharp \{ (k, E, m_n), \mathbf{E}^{k, E, m_n} \neq \emptyset \} \times \sup_{k, E, m_n} \sharp \Theta_n(k, E, m_n),
\]
\[
\leq ne^{2(n-\sharp E_n^m)A_f H(A_f)3n(1/q+1/M)\epsilon_n H(1/M)} \sup_{k, E, m_n} \sharp \Theta_n(k, E, m_n), \text{ by Lemma [14]}
\]
\[
\leq ne^{2(n-\sharp E_n^m)A_f H(A_f)3n(1/q+1/M)\epsilon_n H(1/M)} \max(1, \|df\|_\infty)2^{\mathbf{E}} \prod_{j \leq m} C_r e^{\frac{k_{a_j} - k_{a_j}'}{r-1}}, \text{ by Lemma [15]}
\]
Then we decompose the product into four terms :

- $\sum_{i, m_n > a_i \notin E^M_n} \frac{k_{a_i} - k_{a_i}'}{r-1} \leq (n - \sharp E_n^m) \left( \frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right)$ by Lemma [16]
- $\sum_{i, m_n \leq a_i} \frac{k_{a_i} - k_{a_i}'}{r-1} \leq (n - m_n) \frac{A_f}{r-1}$
- $\sum_{i, a_i \in E^M_n \cap (c+qN)} \frac{k_{a_i} - k_{a_i}'}{r-1} \leq 102r + 2A_{q_M} + \frac{1}{r-1} \left( J \frac{\log^+ \|df\|_\infty}{q} d\zeta^M_n(y) - \int \phi d\zeta^M_n \right)$
- $\sum_{i, a_i \in E^M_n \setminus (c+qN)} \frac{k_{a_i} - k_{a_i}'}{r-1} \leq 2A_{q_M}$

By letting
\[
B_r = \frac{1}{r-1} + \log C_r,
\]
\[
\gamma_{q, M}(f) := 2 \left( \frac{1}{q} + \frac{1}{M} \right) \log C_r + H(1/M) + \frac{10 + \log 3}{q} + \frac{4qA_f + \log 3}{M},
\]
\[
\tau_n = \sup_{x \in \mathbf{F}} \left( 1 - \frac{m_n(x)}{n} \right) \frac{A_f}{r-1} + \frac{\log(nC)}{n},
\]
we get with $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+ \|df\|_\infty}{r} + B_r$:
\[
\frac{1}{n} \log \sharp \Psi_n \leq \left( 1 - \frac{\sharp E^M_n}{n} \right) C(f)
\]
\[
+ \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|df\|_\infty}{q} d\zeta^M_n(x) - \int \phi d\zeta^M_n \right)
\]
\[
+ \gamma_{q, M}(f) + \tau_n,
\]
This concludes the proof of Proposition [3]
Appendix

We explain in this appendix how our Main Theorem implies Buzzi-Crovisier-Sarig statement.

Let \((f_k)_k, (\nu^+_k)_k\) and \(\hat{\mu}\) be as in the setting of Theorem \(\bullet\). Then, either \(\lim_k \lambda^+ (\nu_k) = \int \phi \, d\hat{\mu} \leq \frac{\lambda^+(f)}{r}\) and we get by Ruelle inequality, \(\limsup_k h(\nu_k) \leq \frac{\lambda^+(f)}{r}\) or there exists \(\alpha \in \left[\frac{\lambda^+(f)}{r}, \min \left(\int \phi \, d\hat{\mu}, \frac{\lambda^+(f)}{r-1}\right)\right]\). By applying our Main Theorem with respect to \(\alpha\), there is a decomposition \(\hat{\mu} = (1-\beta_\alpha)\hat{\mu}_{0,\alpha} + \beta_\alpha \hat{\mu}^+_{1,\alpha}\) satisfying \(\limsup_k h(\nu_k) \leq \beta_\alpha h(\mu_{1,\alpha}) + (1-\beta_\alpha)\alpha\).

But it follows from the proofs that \(\beta_\alpha \mu_{1,\alpha}\) is a component of \(\beta \mu_1\) with \(\beta\) and \(\mu_1\) being as in Buzzi-Crovisier-Sarig’s statement as they consider empirical measure associated to a larger set \(G\) (see Remark \(7\)). In particular \(\beta_\alpha h(\mu_{1,\alpha}) = \beta h(\mu_1)\), therefore \(\limsup_{k \to +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f)+\lambda^+(f^{-1})}{r-1}\).

In Theorem C \([10]\), the authors also proved \(\int \phi \, d\hat{\mu}_0 = 0\) whenever \(\beta \neq 1\). Therefore we get here \((1-\beta_\alpha) \int \phi \, d\hat{\mu}_{0,\alpha} \geq (1-\beta) \int \phi \, d\hat{\mu}_0 = 0\), then \(\int \phi \, d\hat{\mu}_{0,\alpha} \geq 0\). But maybe we could have \(\int \phi \, d\hat{\mu}_{0,\alpha} > 0\).

References

[1] J. Brown, Ergodic theory and topological dynamics, Pure and applied mathematics, 1976.
[2] Burguet, David, SRB measure for \(C^\infty\) surface diffeomorphisms, [arXiv:2111.06651]
[3] Burguet, David, Existence of measures of maximal entropy for \(C^r\) interval maps, Proc. Amer. Math. Soc. 142 (2014), p. 957-968
[4] Burguet, David, Symbolic extensions in intermediate smoothness on surfaces, Ann. Sci. Éc. Norm. Supér. (4), 45 (2012), no. 2, 337-362
[5] Burguet, David, Jumps of entropy for \(C^r\) interval maps, Fund. Math., 231, (2015), no.3, p.299-317.
[6] Buzzi, Jérôme and Ruette, Sylvie, Large entropy implies existence of a maximal entropy measure for interval maps, Discrete Contin. Dyn. Syst. A, 14, (2006), p.673-688,
[7] Buzzi, Jérôme, \(C^r\) surface diffeomorphisms with no maximal entropy measure, Ergodic Theory Dynam. Systems 34, 2014, p 1770-1793.
[8] Buzzi, Jérôme, Représentation markovienne des applications réelles de l’intervalle, PhD thesis, Université Paris-Sud, Orsay, 1995.
[9] J. Buzzi, S. Crovisier, and O. Sarig, Measures of maximal entropy for surface diffeomorphisms, Ann. of Math. (2) 195 (2022), no. 2, 421-508.
[10] J. Buzzi, S. Crovisier, and O. Sarig, Continuity properties of Lyapunov exponents for surface diffeomorphisms, Invent. Math. 230 (2022), no. 2, 767-849.
[11] J. Buzzi, S. Crovisier, and O. Sarig, In preparation.
[12] T. Downarowicz, Entropy in dynamical systems, New Mathematical Monographs, 18. Cambridge University Press, Cambridge, 2011.
[13] F. Ledrappier and J. M. Strelcyn, A proof of the estimation from below in Pesin’s entropy formula, Ergod. Th. Dynam. Sys. 2: 203-219, 1982.
[14] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin’s entropy formula, Ann. of Math., 122(1985), 505-539.
[15] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publications Mathématiques de l'IHÉS,51(1980),137-174.
[16] S. Newhouse. Continuity properties of entropy, Ann. of Math. (2), 129(2):215–235, 1989.
[17] Y. Pesin and L. Barreira, Lyapunov Exponents and Smooth Ergodic Theory, University Lecture Series, v. 23, AMS, Providence, 2001
[18] Pollicott, Mark, Lectures on ergodic theory and Pesin theory on compact manifolds, London Mathematical Society Lecture Note Series (180) Cambridge University Press (1993).
[19] Yosef Yomdin, *Volume growth and entropy*, Israel J. Math., 57(3), p. 285-300, 1987.

Sorbonne Université, LPSM, 75005 Paris, France

Email address: david.burguet@upmc.fr