Smooth knot limit sets of the complex hyperbolic plane

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Abstract
It is shown that if a regular knot of class $C^2$ is embedded in the boundary of the complex hyperbolic plane as the limit set of a discrete subgroup of $PU(2, 1)$, then it is either a chain or an $\mathbb{R}$-circle.

Keywords Knots · Chains · Limitset · Complex geometry

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1 Introduction
One of the most beautiful objects in mathematics is the three dimensional unitary sphere

$$S^3 = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}.$$

This set is in fact the ideal boundary of the real hyperbolic space of dimension 4. The isometry group of this space is the group of conformal transformations of $S^3$, denoted by $\text{Conf}(S^3)$, which is identified with the Lorentz group $O(3, 1)$ (see [5, 9]). Also, the space $S^3$ can be considered as the boundary of complex hyperbolic space $H^2_C \subset \mathbb{C}^2$. The isometry group of $H^2_C$ is the group generated by $PU(2, 1)$ and the antiholomorphic involution $\rho_0(Z) = \overline{Z}$, where $Z = (z_1, z_2) \in \mathbb{C}^2$ (see [3]). Both

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groups, Conf($S^3$) and Isom($H^2_C$) act by diffeomorphisms on $S^3$, although the action of Isom($H^2_C$) is not by conformal automorphisms, but by contact morphisms, that means that the elements of Isom($H^2_C$) preserve the standard contact structure of $S^3$ [3]. The discrete subgroups of Conf($S^3$) and Isom($H^2_C$) are considered higher Kleinian groups, for which there are many properties similar to classical Kleinian groups, i.e., discrete subgroups of $\text{PSL}(2, \mathbb{C})$. Nevertheless, there are several differences, which make them interesting objects to study, we recommend the following references about the subject: [1, 3, 5, 10].

If $G$ is a discrete subgroup of Conf($S^3$) or $\text{PU}(2, 1)$, the limit set $L(G)$ is defined as the set of cluster points of the orbit of any point in $S^3$. It is a well-known fact that if $L(G)$ is a tame knot, then it is trivial [6]. In the case $G \subset \text{Conf}(S^3)$, the result is stronger since $L(G)$ is a round circle [5]. In this article we prove the following theorem.

**Theorem 1** Let $G \subset \text{PU}(2, 1)$ be a discrete subgroup acting on the complex hyperbolic space $H^2_C$. If the limit set $L(G) \subset \partial H^2_C$ is the image of a $C^2$ regular knot, $\gamma : S^1 \to \partial H^2_C$, then it is the boundary of a totally geodesic subspace of $H^2_C$, in other words it is either a chain or an $\mathbb{R}$-circle.

It is a well-known fact that the groups $\text{PU}(1, 1)$ and $\text{PO}(2, 1)$ can be embedded in $\text{PU}(2, 1)$ preserving the vertical chain and the canonical $\mathbb{R}$-circle, respectively, see [3]. Also, any chain and any $\mathbb{R}$-circle can be obtained as the limit set of a conjugate subgroup of these embedded groups. Moreover, we have the following straightforward consequence of the Theorem 1:

**Corollary 2** If $G$ is a discrete subgroup of $\text{PU}(2, 1)$ and the limit set $L(G)$ is the image of a class $C^2$ regular knot then $G$ is conjugate to a discrete subgroup of $\text{PU}(1, 1)$ or $\text{PO}(2, 1)$.

In [11], Yue proves a more general result for cocompact lattices of $\text{SO}(n, 1)$ and $\text{SU}(n, 1)$, $n \geq 2$. Roughly speaking, he proves that for any cocompact lattice $G_0$ of $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$ and any injective representation of $G_0$ whose image is denoted by $G$, the Hausdorff dimension of the limit set $L(G)$ is bounded below by the Hausdorff dimension of the limit set $L(G_0)$ with equality if and only if $G$ stabilizes a totally geodesic copy of $H^n_R$ or $H^n_C$, respectively. The Theorems 1.3 and 1.4 of Yue imply the Theorem 1 and the Corollary 2 with the additional hypothesis that the group $G$ acts convex cocompactly. However, this additional hypothesis is not necessary because the condition that the limit set $L(G)$ is of class $C^2$ implies that the Hausdorff dimension of the limit set is 1 and allows to use techniques of classical differential geometry to prove the Theorem 1.

This article is organized as follows: In Sect. 2, we outline the required preliminaries about complex hyperbolic geometry. In Sect. 3, the notion of $\mathbb{R}$-circle is introduced together with basic notions of contact geometry. In Sect. 4, the notions of chain and tangent chain to a smooth curve in a point are introduced. In Proposition 3, we show that tangent chains to curves are mapped by elements of $\text{PU}(2, 1)$ to tangent chains to the corresponding mapped curves. In Proposition 4, it
is shown that a knot is Legendrian at a point if and only if the tangent chain at the point is degenerate. In Sect. 5, we prove the Theorem 1, the idea is to prove, using the dynamics of a loxodromic element, that if the limit set of a discrete subgroup of $\text{PU}(2,1)$ is a knot then it is either a chain or a Legendrian knot. Finally, if the limit set is a Legendrian knot, with help of a suitable Heisenberg coordinate system, we take the vertical projection to the horizontal plane and prove that the projection is an infinite $\mathbb{R}$-circle.

2 Preliminaries

2.1 Projective geometry

We denote by $F$ the field $\mathbb{C}$ or $\mathbb{R}$, the projective space $\mathbb{P}^2_F$ is defined as

$$\mathbb{P}^2_F := (F^3 \setminus \{0\})/F^*,$$

where $F^* = F \setminus \{0\}$ acts on $F^3 \setminus \{0\}$ by the usual scalar multiplication. The space $\mathbb{P}^2_F$ is called the complex projective plane when $F = \mathbb{C}$, and it is called the real projective plane when $F = \mathbb{R}$. Let $[\cdot] : F^3 \setminus \{0\} \to \mathbb{P}^2_F$ be the quotient map, if $v = (x, y, z) \in F^3 \setminus \{0\}$ then we write $[v] = [x : y : z]$. A set of the form

$$\ell = \{[x : y : z] \in \mathbb{P}^2_F : Ax + By + Cz = 0\}$$

for some $A, B, C \in F$ not all zero is called an $F$-line (or line when there is no confusion). We notice that any two distinct points $p, q \in \mathbb{P}^2_F$ define a unique $F$-line containing $p$ and $q$. Consider the action of $F^* = F \setminus \{0\}$ on $\text{GL}(3, F)$ given by the usual scalar multiplication, then

$$\text{PGL}(3, F) = \text{GL}(3, F)/F^*$$

is a Lie group whose elements are called projective transformations. Let $[\cdot] : \text{GL}(3, F) \to \text{PGL}(3, F)$ be the quotient map. If $g \in \text{PGL}(3, F)$ and $g \in \text{GL}(3, F)$, we say that $g$ is a lift of $g$ whenever $[g] = g$. One can show that $\text{PGL}(3, F)$ is a Lie group that acts transitively, effectively and by diffeomorphisms on $\mathbb{P}^2_F$ by $[g][v]) = [gv]$, where $v \in F^3 \setminus \{0\}$ and $g \in \text{GL}(3, F)$ [3]. Notice that any element $g \in \text{PGL}(3, F)$ maps $F$-lines to $F$-lines. The real projective plane $\mathbb{P}^2_R$ can be embedded in a natural way in the complex projective plane $\mathbb{P}^2_C$, in the following way:

$$\mathbb{P}^2_R \hookrightarrow \mathbb{P}^2_C$$

$$[x : y : z] \mapsto [x : y : z].$$

In what follows, $\mathbb{P}^2_R$ denotes the image of this embedding.

2.2 Complex hyperbolic geometry

Let $\mathbb{C}^{2,1}$ denote $\mathbb{C}^3$ equipped with the Hermitian form
\[ \langle z, w \rangle_1 = z_1 \overline{w}_1 + z_2 \overline{w}_2 - z_3 \overline{w}_3, \]

where \( z = (z_1, z_2, z_3) \), \( w = (w_1, w_2, w_3) \). Denote by

\[
V_- = \{ z \in \mathbb{C}^{2,1} : \langle z, z \rangle_1 < 0 \},
\]

\[
V_0 = \{ z \in \mathbb{C}^{2,1} \setminus \{ 0 \} : \langle z, z \rangle_1 = 0 \},
\]

\[
V_+ = \{ z \in \mathbb{C}^{2,1} : \langle z, z \rangle_1 > 0 \},
\]

the sets of negative, null and positive vectors, respectively. The projectivization of the set of negative vectors,

\[
[V_-] = \{ [z_1 : z_2 : 1] \in \mathbb{P}_\mathbb{C}^2 : z_-1^2 + z_-2^2 < 1 \},
\]

is a complex 2-dimensional open ball in \( \mathbb{P}_\mathbb{C}^2 \). Moreover, \( [V_-] \) equipped with the quadratic form induced by the Hermitian form \( \langle \cdot, \cdot \rangle_1 \) is a model for the complex hyperbolic space \( \mathbb{H}_\mathbb{C}^2 \). The projectivization of the set of null vectors,

\[
[V_0] = \{ [z_1 : z_2 : 1] \in \mathbb{P}_\mathbb{C}^2 : z_-1^2 + z_-2^2 = 1 \},
\]

is a 3-sphere in \( \mathbb{P}_\mathbb{C}^2 \) and it is the boundary of \( \mathbb{H}_\mathbb{C}^2 \), denoted \( \partial \mathbb{H}_\mathbb{C}^2 \). Finally, the projectivization of the set of positive vectors,

\[
[V_+] = \{ [z_1 : z_2 : z_3] \in \mathbb{P}_\mathbb{C}^2 : z_-1^2 + z_-2^2 - z_-3^2 > 0 \},
\]

is the complement in \( \mathbb{P}_\mathbb{C}^2 \) of the complex 2-dimensional closed ball \( \mathbb{H}_\mathbb{C}^2 = \mathbb{H}_\mathbb{C}^2 \cup \partial \mathbb{H}_\mathbb{C}^2 \).

The group of holomorphic isometries of \( \mathbb{H}_\mathbb{C}^2 \) is \( \text{PU}(2, 1) \), the projectivization in \( \text{PGL}(3, \mathbb{C}) \) of the unitary group, \( U(2, 1) \), with respect to the Hermitian form \( \langle \cdot, \cdot \rangle_1 \):

\[
U(2, 1) = \{ g \in \text{GL}(3, \mathbb{C}) : \langle gz, gw \rangle_1 = \langle z, w \rangle_1 \}.\]

The group \( \text{PU}(2, 1) \) acts transitively in \( \mathbb{H}_\mathbb{C}^2 \) and by diffeomorphisms in the boundary \( \partial \mathbb{H}_\mathbb{C}^2 \cong S^3 \). Another fact we use along this paper is the following: Given any point \( p = [w_1 : w_2 : w_3] \in \partial \mathbb{H}_\mathbb{C}^2 \), there exists a unique complex line, denoted \( \ell_p \), tangent to \( \partial \mathbb{H}_\mathbb{C}^2 \) at \( p \). Moreover, \( \ell_p \) is the set:

\[
\{ [z_1 : z_2 : z_3] \in \mathbb{P}_\mathbb{C}^2 : z_1 \overline{w}_1 + z_2 \overline{w}_2 - z_3 \overline{w}_3 = 0 \}.
\]

If we consider \( \mathbb{C}^3 \) with the Hermitian form

\[
\langle z, w \rangle_2 = z_1 \overline{w}_3 + z_2 \overline{w}_2 + z_3 \overline{w}_1,
\]

where \( z = (z_1, z_2, z_3) \) and \( w = (w_1, w_2, w_3) \), then we have that

\[
\langle Cz, Cw \rangle_1 = \langle z, w \rangle_2,
\]

where \( C \) is the Cayley matrix.
Hence, \( C(V_-), C(V_0), C(V_+ \) are the sets of negative, null and positive vectors for \( \langle \cdot, \cdot \rangle_2 \), respectively. The projectivization of \( C(V_-) \) equipped with the Hermitian form \( \langle \cdot, \cdot \rangle_2 \) is the Siegel model for complex hyperbolic space

\[
\{ [z_1 : z_2 : 1] \in \mathbb{P}_C^2 : 2\Re(z_1) + |z_2|^2 < 0 \}
\]

and its boundary is the set

\[
\{ [z_1 : z_2 : 1] \in \mathbb{P}_C^2 : 2\Re(z_1) + |z_2|^2 = 0 \} \cup \{ [1 : 0 : 0] \}.
\]

Any finite point in this boundary can be written in the form

\[
[ -|\zeta|^2 + iv : \sqrt{2}\zeta : 1 ],
\]

for some \((\zeta, v) \in \mathbb{C} \times \mathbb{R}\). Hence, there is a natural identification of this boundary set with the one point compactification of the Heisenberg space \( \mathcal{H} = \mathbb{C} \times \mathbb{R} \). For more details on complex hyperbolic geometry, see the book [3].

2.3 The limit set

**Definition 1** If \( G \) is a discrete subgroup of \( \text{PU}(2, 1) \), the *limit set* of \( G \), denoted by \( L(G) \), is defined as the set of cluster points of the \( G \)-orbit of any point in \( \mathbb{H}_C^2 \).

Some useful properties of the limit set are the following:

i. The limit \( L(G) \) does not depend on the choice of the point in \( \mathbb{H}_C^2 \).

ii. The limit set \( L(G) \) is a closed \( G \)-invariant set and it is minimal on the family of closed \( G \)-invariant sets with more than two points.

iii. If \( L(G) \) has more than two points and \( x \in L(G) \) is any point, then the \( G \)-orbit of \( x \) is dense in \( L(G) \).

iv. If \( L(G) \) contains at least two points then there exists a loxodromic element in \( G \). A loxodromic element is an element in \( \text{PU}(2, 1) \) with precisely two fixed points in \( \partial \mathbb{H}_C^2 \), one is *attracting* and the other is *repelling*.

For more details, see [2, 4, 5].

2.4 The Hermitian cross product

If \( z, w \in \mathbb{C}^{2,1} \), then the Hermitian cross product of \( z = (z_1, z_2, z_3) \) and \( w = (w_1, w_2, w_3) \), with respect to the Hermitian form \( \langle \cdot, \cdot \rangle_1 \), is defined as
Analogously, the Hermitian cross product, with respect to the Hermitian form $\langle \cdot, \cdot \rangle_2$ is defined as
\[
\mathbf{z} \times_2 \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\overline{z_1} & \overline{z_2} & -\overline{z_3} \\
\overline{w_1} & \overline{w_2} & -\overline{w_3}
\end{vmatrix}.
\]

When there is no danger of confusion, we will omit the subscripts on the notation of these Hermitian cross products. These products share several similarities to the real cross product in $\mathbb{R}^3$, in particular:

i. $\mathbf{z} \times \mathbf{w} \neq \mathbf{0}$ if and only if $\mathbf{z}$ and $\mathbf{w}$ are linearly independent.

ii. $\mathbf{z} \times \mathbf{w}$ is orthogonal with respect to the corresponding Hermitian form to both $\mathbf{z}$ and $\mathbf{w}$.

iii. It is skew linear and anti-commutative.

iv. For any $\mathbf{g} \in \text{SU}(2, 1)$, $\mathbf{g}(\mathbf{z} \times \mathbf{w}) = \mathbf{g}^{\circ}(\mathbf{z}) \times \mathbf{g}(\mathbf{w})$.

**Definition 2** If $z = [z]$ and $w = [w]$ are two distinct points in $\mathbb{P}^2_\mathbb{C}$ then $\mathbf{z} \times \mathbf{w} \neq \mathbf{0}$ and $\mathbf{z} \times \mathbf{w} = [\mathbf{z} \times \mathbf{w}]$ is a well-defined point in $\mathbb{P}^2_\mathbb{C}$.

### 3 $\mathbb{R}$-circles

The boundaries of totally geodesic subspaces of $\mathbb{H}^2_\mathbb{C}$ which are isometric to the Beltrami model of real hyperbolic geometry are called $\mathbb{R}$-circles.

**Definition 3** The canonical $\mathbb{R}$-circle $R_0$ is defined as $\mathbb{P}^2_\mathbb{R} \cap \partial \mathbb{H}^2_\mathbb{C}$, any other $\mathbb{R}$-circle is the translate of $R_0$ by an element of $\text{PU}(2, 1)$.

In Heisenberg coordinates, the $\mathbb{R}$-circles are infinite or finite according to whether they pass through the point at infinity or not. All the infinite $\mathbb{R}$-circles passing through the origin are horizontal lines $(te^{i\theta}, 0)$, $t \in \mathbb{R}$ and $\theta \in [0, 2\pi]$, any other infinite $\mathbb{R}$-circle can be obtained by Heisenberg translations, thus, the infinite $\mathbb{R}$-circles are simply straight lines. Likewise, finite $\mathbb{R}$-circles are obtained by complex dilations, rotations and Heisenberg translations of the finite $\mathbb{R}$-circle
\[
\left\{ \left( i\sqrt{\cos(2\theta)} e^{i\theta}, -\sin(2\theta) \right) \mid \theta \in [-\pi/4, \pi/4] \cup (3\pi/4, 5\pi/4) \right\}.
\]

More details can be found in [8].
3.1 Contact geometry

Let \( E \subset TM \) be a hyperplane field of real codimension 1 on a smooth manifold \( M \), if there is a 1-form \( \omega : TM \to \mathbb{R} \) such that for any \( x \in M \), \( E_x \) is equal to \( \ker(\omega_x) \), then we call \( \omega \) a contact form provided \( d\omega \) is nondegenerate for all \( x \in M \). Moreover, the hyperplane field \( E \) is called a contact structure. A curve such that each tangent vector is in \( E \) is called Legendrian, we will focus on the three dimensional Heisenberg space with coordinates \((\zeta, v) \in \mathcal{H}\) and the group structure determined by the product

\[
(\zeta_1, v_1) \cdot (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + \Re(\zeta_1 \zeta_2) + v_2),
\]

where \( \eta(\zeta_1, \zeta_2) = \Im(\zeta_1 \zeta_2) \) is the symplectic form determined by the complex structure of \( \mathbb{C} \). In Heisenberg space the form \( \omega = dv - \eta(\zeta, df) \) induces the contact structure, hence if \( \gamma : S^1 \to \mathcal{H} \) is a Legendrian knot with parametrization \( \gamma = (\zeta, v) \), the condition that \( \gamma \) is everywhere in \( \ker \omega \) is equivalent to the equation

\[
\dot{v} - \eta(\zeta, \dot{\zeta}) = 0,
\]

showing that Legendrian knots are determined by their vertical projections \( \gamma \mapsto \zeta \). In particular, if \( v(s) \) is constant, this implies that \( \gamma(s) \) is a straight line.

The \( \mathbb{R} \)-circles are Legendrian knots: First, the canonical \( \mathbb{R} \)-circle in Heisenberg coordinates is \((t, 0), t \in \mathbb{R} \) which is a Legendrian subspace. Since \( \text{PU}(2,1) \) preserves the contact structure, any other \( \mathbb{R} \)-circle is everywhere tangent to the contact structure as well.

4 Chains

Complex geodesics are the totally geodesic subspaces obtained as the intersection of a complex line and \( \mathbb{H}_C^2 \). The boundary at infinity of a complex geodesic is a circle obtained as the intersection of \( \partial \mathbb{H}_C^2 \) and a complex line, these circles are called chains.

If \( p = [v] \in \mathbb{P}_C^2 \setminus \mathbb{H}_C^2 \) then \( v \) is a positive vector, so the orthogonal complement \( \langle v \rangle^\perp \) with respect to the first Hermitian form, is a two-dimensional subspace of \( \mathbb{C}^2 \) and it induces a complex line, \( \ell_p \), called the polar line to \( p \). The chain, \( \mathcal{C}_p \), obtained as the intersection \( \ell_p \cap \partial \mathbb{H}_C^2 \) is the polar chain to \( p \). The points in \( \partial \mathbb{H}_C^2 \) are considered as chains and we call them degenerate chains. Conversely, if \( \ell \) is a complex line transversal to \( \partial \mathbb{H}_C^2 \), then we can write \( \ell = [L \setminus \{0\}] \) where \( L \) is a two dimensional complex vector subspace of \( \mathbb{C}^2 \). Moreover, the orthogonal complement of \( L \), respect to the Hermitian form \( \langle \cdot, \cdot \rangle_1 \), is a one dimensional complex subspace of \( \mathbb{C}^2 \) which induces a point in \( \mathbb{P}_C^2 \setminus \mathbb{H}_C^2 \), this point is called the polar point to the line \( \ell \).

There is a natural identification of a chain \( \partial \mathbb{H}_C^2 \cap \ell \) with the polar point to \( \ell \). In fact, there is a bijection, between the space of chains and the complement of the
complex hyperbolic space $\mathbb{P}^2_C \setminus \mathbb{H}_C^2$. We remark that for a degenerate chain \( \{p\} = \partial \mathbb{H}_C^2 \cap \ell \), the corresponding point is \( p \in \partial \mathbb{H}_C^2 \).

### 4.1 Tangent chains

Let $\gamma : \mathbb{S}^1 \to \partial \mathbb{H}_C^2$ be a $C^1$ regular knot which we identify with the curve

$$\gamma : [0, 1] \to \partial \mathbb{H}_C^2,$$

$$t \mapsto \gamma(e^{2\pi it}).$$

If $t_0, t \in [0, 1)$ are distinct then $\gamma(t_0) \neq \gamma(t)$ and the polar chain to the point $\gamma(t_0)\mathbb{H}^2 \gamma(t)$ is the unique chain passing through the points $\gamma(t_0)$ and $\gamma(t)$. We notice that as $t \to t_0$, the chain passing through $\gamma(t_0)$ and $\gamma(t)$ goes to the tangent chain to $\gamma$ at $\gamma(t_0)$. However, this tangent chain cannot be defined as the polar chain to $\gamma(t_0)\mathbb{H}^2 \gamma(t_0)$ because the Hermitian cross product of a vector with itself is equal to the zero vector.

To define the tangent chain to $\gamma$ at $t_0$, we use the following notation: $\mathbf{v}(t) \in \mathbb{C}^{2,1} \setminus \{0\}$ is a vector satisfying $\gamma(t) = [\mathbf{v}(t)]$. Moreover, we can assume that the curve

$$\mathbf{v} : [0, 1] \to \mathbb{C}^{2,1} \setminus \{0\}$$

is of class $C^1$. Now, the chain passing through the points $\gamma(t_0) = [\mathbf{v}(t_0)]$ and $\gamma(t) = [\mathbf{v}(t)]$ is the polar chain to the point $[\mathbf{v}(t_0) \mathbb{H}^2 \mathbf{v}(t)] = [\mathbf{v}(t_0)] \mathbb{H}^2 [\mathbf{v}(t)]$.

Since

$$\lim_{t \to t_0} \mathbf{v}(t_0) \mathbb{H}^2 \left( \frac{\mathbf{v}(t) - \mathbf{v}(t_0)}{t - t_0} \right) = \mathbf{v}(t_0) \mathbb{H}^2 \mathbf{v}'(t_0),$$

we have the following definition (see [7]):

**Definition 4** We define the **tangent chain** to $\gamma$ at $\gamma(t_0)$ as the polar chain to the point $\gamma(t_0) \mathbb{H}^2 \mathbf{v}'(t_0) = [\mathbf{v}(t_0) \mathbb{H}^2 \mathbf{v}'(t_0)] \in \mathbb{P}_C^2 \setminus \mathbb{H}_C^2$.

The tangent chain to $\gamma$ at $t_0$ is denoted by $T_{\gamma}(t_0)$.

**Remark 1** The tangent chain $T_{\gamma}(t_0)$ does not depend of the lifted curve $\mathbf{v} : [0, 1] \to \mathbb{C}^{2,1}$ or parametrization.

In fact, if $\mathbf{w}(t) = \lambda(t)\mathbf{v}(t)$ (where $\lambda(t) \in \mathbb{C} \setminus \{0\}$), then $\mathbf{w}'(t_0) = \lambda'(t_0)\mathbf{v}(t_0) + \lambda(t_0)\mathbf{v}'(t_0)$. Hence, $\mathbf{w}(t_0) \mathbb{H}^2 \mathbf{w}'(t_0) = \lambda(t_0)^2 \mathbf{v}(t_0) \mathbb{H}^2 \mathbf{v}'(t_0)$. Likewise if $\mathbf{w}(t) = \mathbf{v}(f(t))$ for some reparametrization $f : [0, 1] \to [0, 1]$.

**Proposition 3** If $\gamma : [0, 1] \to \partial \mathbb{H}_C^2$ is a $C^1$ regular knot and $g \in \text{PU}(2, 1)$ then...
Proof. If $v : [0, 1] \rightarrow \mathbb{C}^{2,1}$ is a $C^1$ lift for $\gamma$ and $g = [g] \in \text{PU}(2, 1)$ then $gv : [0, 1] \rightarrow \mathbb{C}^{2,1}$ is a $C^1$ lift for $g \circ \gamma$ and

$$g \circ \gamma(t) \bar{v}(g \circ \gamma)'(t) = [gv(t) \bar{v}(gv)'(t)]$$

$$= [g(v(t) \bar{v}'(t))]$$

$$= g(\gamma(t) \bar{\gamma}'(t)).$$

Hence, $T_{g \circ \gamma}(t)$ is the polar chain to the point $g(\gamma(t) \bar{\gamma}'(t))$. Since $g(T_{\gamma}(t))$ is the polar chain to the same point, we conclude that $T_{g \circ \gamma}(t) = g(T_{\gamma}(t))$. \hfill \Box

Proposition 4. If $\gamma : [0, 1] \rightarrow \partial \mathbb{H}_\mathbb{C}^2$ is a $C^1$ regular knot then the following are equivalent

i. The curve $\gamma$ is Legendrian at $t$.

ii. The point $\gamma(t) \bar{\gamma}'(t)$ lies in $\partial \mathbb{H}_\mathbb{C}^2$.

iii. The tangent chain $T_{\gamma}(t)$ is degenerate.

Proof. First, if the curve $v : [0, 1] \rightarrow \mathbb{C}^{2,1}$ is a $C^1$ lift for $\gamma$, then the equation

$$\langle v(t) \bar{v}'(t), v(t) \bar{v}'(t) \rangle = \langle v'(t), v(t) \rangle \langle v(t), v'(t) \rangle$$

$$- \langle v'(t), v'(t) \rangle \langle v(t), v(t) \rangle$$

proves the equivalence of the two equations

$$\langle v(t), v'(t) \rangle = 0 \quad \text{and} \quad \langle v(t) \bar{v}'(t), v(t) \bar{v}'(t) \rangle = 0.$$ 

The first equation is equivalent to i) and the second one is equivalent to ii).

Finally, the equivalence of ii) and iii) follows by the definitions of $T_{\gamma}(t)$ and degenerate chain. \hfill \Box

5 Proof of the Theorem

Lemma 5. Let $G \subset \text{PU}(2, 1)$ be a discrete subgroup acting on the complex hyperbolic space $\mathbb{H}_\mathbb{C}^2$. If the limit set $L(G) \subset \partial \mathbb{H}_\mathbb{C}^2$ is the image of a $C^1$ regular knot, $\gamma : [0, 1] \rightarrow \partial \mathbb{H}_\mathbb{C}^2$, then $\gamma$ is Legendrian or a chain.

Proof. First, we claim that if $\gamma$ is Legendrian at $t_0 \in [0, 1]$ then $\gamma$ is Legendrian at every $t \in [0, 1]$. In fact, $\gamma$ is Legendrian at every $t$ such that $\gamma(t)$ lies in the $G$-orbit of the point $\gamma(t_0)$, because the image of $\gamma$ is $G$-invariant and $G \subset \text{PU}(2, 1)$ acts by contact morphisms. For an arbitrary $t \in [0, 1]$, there is a sequence of points $t_n \rightarrow t$ as $n \rightarrow \infty$ such that $\gamma$ is Legendrian at $t_n$ for every $n \in \mathbb{N}$, because the $G$-orbit of $\gamma(t_0)$ is dense in the image of $\gamma$. Hence $\gamma(t_n) \bar{\gamma}'(t_n) \in \partial \mathbb{H}_\mathbb{C}^2$ for every $n \in \mathbb{N}$ and it implies
that $\gamma(t) \Theta \gamma'(t) \in \partial \mathbb{H}^2_C$. By Proposition 4, $\gamma$ is Legendrian at $t$ and we have proved our claim.

If the image of the knot is not a chain, we proceed by contradiction and we assume that $\gamma$ is not Legendrian, then it is not Legendrian at some $s_0 \in [0, 1]$. It follows that $T_\gamma(s_0)$ is not degenerate by Proposition 4. Since the image of $\gamma$ is not a chain, there exists an arc of $\gamma$ disjoint from $T_\gamma(s_0)$. Moreover, there exists a loxodromic element $g \in G$ such that its fixed points $p, q$ lie in this arc. By Proposition 3, we have that $g^n(T_\gamma(s_0)) = T_{g^n\gamma}(s_0)$. If $p = \gamma(s_p)$ is the attracting fixed point for $g$, then there exists a sequence $(s_n) \subset [0, 1]$ such that $T_{g^n\gamma}(s_0) = T_\gamma(s_n)$ for every $n \in \mathbb{N}$, and $\lim_{n \to \infty} s_n = s_p$. It follows that the diameter of the chain $g^n(T_\gamma(s_0)) = T_\gamma(s_n)$ goes to zero as $n \to \infty$. Hence, $T_\gamma(s_p)$ is degenerate, or equivalently by Proposition 4, $\gamma$ is Legendrian at any $s \in [0, 1]$, which is a contradiction. 

**Lemma 6** Let $I$ be an open interval such that $\gamma : I \to \mathbb{C}$ is an embedded plane curve of class $C^2$ invariant under the homothety $g(z) = \lambda z$, for some $\lambda \in \mathbb{C}$, $|\lambda| < 1$, then $\gamma$ is a straight line segment.

**Proof** Let $k(z)$ be the the curvature of $\gamma$ at a point $z = \gamma(t)$, $k$ is a geometric invariant of $\gamma(t)$, independent of the parametrization chosen. Let $z_0 = \gamma(a)$ be any point of the curve, then the curvature at the $n$-th iterate $z_n = g^n \circ \gamma(a)$ is $k(z_n) = |\lambda|^{-n} k(z_0)$, but $z_n \to 0$ as $n \to \infty$ whereas $k(z_n) \to \infty$ unless $k(z_0) = 0$, since $z_0$ was arbitrary, this means that $\gamma$ has constant curvature $k \equiv 0$, hence it is a straight line segment. 

**Lemma 7** If $\gamma : [0, 1] \to \partial \mathbb{H}^2_C$ is a $C^2$ regular Legendrian knot and there is a loxodromic element $g \in \text{PU}(2, 1)$ such that $\gamma$ is $g$-invariant and contains the attracting and repelling fixed points of $g$, then $\gamma$ is an $\mathbb{R}$-circle.

**Proof** Let $p, q$ be the attracting and repelling fixed points of $g$ respectively. We can choose Heisenberg coordinates for $\partial \mathbb{H}^2_C \setminus \{q\}$ such that $p$ is the origin in $\mathcal{H}$ and $q$ is the point at infinity. Under these coordinates, $g$ has the representation $g(\zeta, v) = (\lambda^{1/2} \zeta, |\lambda| v),$ where $|\lambda| < 1$ since the attracting fixed point is at the origin. Identifying $\gamma$ with its image under Heisenberg coordinates, we can suppose it is a curve $\mathbb{R} \to \mathcal{H}$ joining the origin with the point at infinity, such that $\gamma(0) = 0$. Let $\psi : \mathbb{R} \to \mathbb{C}$ be the vertical projection of $\gamma$, since $\gamma$ is Legendrian, $\psi'(0) \neq 0$, hence by the inverse function theorem, there is an open interval $I$ centered at 0 such that the restriction $\psi|_I$ is embedded in the complex plane at $v = 0$. By the Lemma 6, $\psi|_I$ is a straight line segment, since $\gamma$ is tangent to the contact structure, this means that $\gamma(I)$ is this segment. Also, $\gamma$ is $g$-invariant and connected, hence $\gamma(\mathbb{R}) = \bigcup_{n \in \mathbb{N}} g^{-n} \circ \gamma(I)$ is a horizontal line and therefore it is an $\mathbb{R}$-circle. 

**Proof of the Theorem 1** By the Lemma 5, if $L(G)$ is not a chain, then it is a Legendrian curve in $\partial \mathbb{H}^2_C$. We know that in any complex Kleinian group
there is a loxodromic element $g$, since $L(G)$ is $g$-invariant and contains both the attracting and repelling fixed points of $g$, by the Lemma 7 it is an $\mathbb{R}$-circle.

Declarations

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