Darboux transformations of coherent states of the time-dependent singular oscillator

Boris F Samsonov
Physics Department of Tomsk State University, 634050 Tomsk, Russia
Departamento de Física Teórica, Universidad de Valladolid, 47005 Valladolid, Spain
E-mail: samsonov@phys.tsu.ru

Abstract. Darboux transformation of both Barut-Girardello and Perelomov coherent states for the time-dependent singular oscillator is studied. In both cases the measure that realizes the resolution of the identity operator in terms of coherent states is found and corresponding holomorphic representation is constructed. For the particular case of a free particle moving with a fixed value of the angular momentum equal to two it is shown that Barut-Giriardello coherent states are more localized at the initial time moment while the Perelomov coherent states are more stable with respect to time evolution. It is also illustrated that Darboux transformation may keep unchanged this different time behavior.

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Corresponding Author:
B F Samsonov
Physics Department
Tomsk State University
36 Lenin Ave
634050 Tomsk
RUSSIA
Phone: 7 3822 913 019
E-mail: samsonov@phys.tsu.ru
1. Introduction

There exist several definitions of coherent states (CS) [1]-[3]. They lead to the same result for the harmonic oscillator potential and usually to different results for other physical systems. Nevertheless, a careful analysis shows that they possesses common properties which can be taken to define CS for the very general quantum system [3]. It happens that such a definition is very ambiguous. Therefore one can choose between different possible systems the one which has a desirable property. For instance, one can look for a state, the wave function of which can be expressed in terms of known special functions or even in terms of elementary functions only. This simplifies considerably the study of these CS and their use in other applications. In this way one was able to construct different systems of CS for the well-known soliton potentials [4]. It was also shown that different CS may be related between them with a symmetry operator [4] or with the Laplace transform [5].

The most essential progress in studying CS has been achieved for systems possessing symmetries [2]. If it is not possible to associate a symmetry group with a quantum system the problem becomes much more complicated. For instance, after applying a nontrivial supersymmetric (or equivalently Darboux) transformation to a Hamiltonian allowing a symmetry, the symmetry is usually lost or it is transformed to a nonlinear symmetry [6]. This makes impossible to apply group theoretical methods for studying CS. In this respect it was suggested to apply the same transformation to the known CS of the initial quantum system and treat states thus obtained as coherent states for the transformed system [7, 8]. This approach proved to be useful for studying CS for supersymmetrically transformed Harmonic oscillator [8] and time-independent singular oscillator [9] potentials. In particular, using correspondence between classical and quantum systems, which can be realized just by the technique of CS (see e.g. [2]), one was able to give an interpretation of the supersymmetry transformation in terms of classical notions [6]. Here we continue this study at the level of the time-dependent singular oscillator.

The time-dependent singular oscillator Hamiltonian

\[
h_0 = -\partial_x^2 + \omega(t)x^2 + gx^{-2}
\]

plays an essential role in different physical applications between which we would like to mention interesting results in molecular physics [10], in optics [11] and in mathematical physics [12]. Recently a model of a two-ion trap has been proposed (see [13] where a good literature review is also given) based on this Hamiltonian. Exact solvability of the Schrödinger equation with the Hamiltonian (1) may be, in particular, related with the fact that the symmetry operators realize a representation of the \(su(1,1)\) algebra and therefore it can be solved by the method of separation of variables [14]. Moreover, for this algebra two systems of CS are commonly known [13]. The one may be obtained with the help of the \(SU(1,1)\) group translation operator (they are known as Perelomov CS, see e.g. [2]) and the other are eigenstates of the annihilation operator [15] (Barut-Girardelo CS [16]). Here we will study the supersymmetric transformation of either system.

The paper is organized as follows. In the next section to fix the notations we give a review of some properties of the singular oscillator Hamiltonian and its CS. In particular, basing on the Barut-Girardello CS we construct a holomorphic representation of state functions and
operators; for a concrete example of a free particle moving with a fixed value of the angular momentum equal to two we compare the time stability of Barut-Girardello and Perelomov CS. In section 3 we apply the time-dependent Darboux transformation both to Barut-Girardello and to Perelomov CS and study some properties of states thus obtained. Conclusions are drown in the last section.

2. Singular oscillator Hamiltonian

In this section we summarize briefly what is known about this Hamiltonian and its coherent states (see e.g. [6],[9]-[13],[15],[16]) we need further. We would like to stress that although almost everything presented here is already published, but from one side we are adopting here a different approach with respect to the one usually used and from the other side we are bringing together a number of facts spread in numerous literature. In contrast to most of papers using different modifications of the method of quantum invariants [17] for solving the Schrödinger equation we are using the method of separation of variables [14]. The advantage of this approach is that we do not need to be restricted by any quantum mechanical picture. Instead, we consider the Schrödinger equation as a second order parabolic differential equation, which gives us the possibility to get “nonphysical” solutions we need for applying the Darboux algorithm.

2.1. Solutions of the Schrödinger equation

Generators of the $SU(1,1)$ symmetry group, being symmetry operators of the Schrödinger equation with the Hamiltonian $h_0$, in coordinate representation

$$
k_- = 2 \left[ a^2 - \varepsilon^2 g x^{-2} \right] \quad k_+ = 2 \left[ (a^+)^2 - \bar{\varepsilon}^2 g x^{-2} \right], \quad (2)
k_0 = \frac{1}{2} (k_- k_+ - k_+ k_-) \quad (3)
$$

are expressed in terms of the harmonic oscillator creation and annihilation operators

$$a = \varepsilon \partial_x - \frac{i}{2} \dot{\varepsilon} x \quad a^+ = -\bar{\varepsilon} \partial_x + \frac{i}{2} \bar{\varepsilon} x.$$

The dot over a symbol means the derivative with respect to time. Parameters $\varepsilon$ and $\bar{\varepsilon}$ are solutions to the equation of motion for the classical harmonic oscillator

$$\ddot{\varepsilon} + 4\omega^2(t)\varepsilon = 0.$$

In particular, when $\varepsilon$ is complex, $\bar{\varepsilon}$ will denote its complex conjugate and they are such that $\varepsilon \bar{\varepsilon} - \varepsilon \bar{\varepsilon} = \frac{1}{2} i$. Casimir operator $C$ is expressed in terms of the parameter $g$: $C = \frac{1}{2} (k_+ k_- + k_- k_+) - k_0^2 = 3/16 - g/4$. This gives us the relation between $g$ and the representation parameter $k$: $C = k(1 - k)$, $g = 3/4 + 4k(k - 1)$.

Solutions $\psi_n(x,t)$, $n = 0,1,\ldots$ of the Schrödinger equation belonging to the Hilbert space of functions square integrable on semiaxis form a basis of the discrete series irreducible representation $T_k^+(g)$ of the group $SU(1,1)$ at $k = \frac{1}{4} + \frac{1}{4} \sqrt{1 + 4b}$. The vacuum vector $\psi_0(x,t)$ is selected from the set of solutions of the Schrödinger equation by the equations
\[ k_0 \psi_0(x, t) = k \psi_0(x, t), \quad k_+ \psi_0(x, t) = 0. \] All other basis vectors may be obtained by acting with the creation operator \( k_+ \):

\[ \psi_n(x, t) = (-1)^n \sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}} (k_+)^n \psi_0(x, t). \]

The action of the operators (2)-(3) on this basis is given by

\[ k_0 \psi_n(x, t) = (k + n) \psi_n(x, t) \quad k_+ \psi_n(x, t) = -c_n^+ \psi_{n+1}(x, t) \quad (4) \]

\[ c_n^+ = (n + \frac{1}{2} \pm \frac{1}{2})^\frac{1}{2}(n + 2k - \frac{1}{2} \pm \frac{1}{2})^\frac{1}{2}. \]

To solve the Schrödinger equation simultaneously with the eigenvalue equation for \( k_0 \) we are using the method of separation of variables. From coordinates \((u, v)\) we go to curvilinear coordinates \((x, t)\) defined as: \( u = x\gamma^{-1/2}, \quad v = t \) were \( \gamma = \varepsilon\bar{\varepsilon} \). The choice

\[ \psi = e^{\frac{k_0u^2}{\varepsilon^2}} P(v)Q(u) \]  

(5)

guarantees the separation of variables. The eigenvalue equation for \( k_0 \) is reduced to the first order ordinary equation for \( P \) which can readily be integrated to give \( P = \gamma^{-1/4}(\varepsilon/\bar{\varepsilon})^\lambda \). So, replacing the action of \( k_0 \) on a solution by the multiplication on the separation constant \( \lambda \) one obtains from the Schrödinger equation the following second order differential equation:

\[ Q_{yy} - \left[ \frac{1}{4}(y^2 + gy^{-2} - 2\lambda) \right] Q = 0 \quad y = \frac{1}{2}u. \]  

(6)

This equation can be reduced to the equation for the Laguerre polynomials if \( \lambda = n + k, \quad n = 0, 1, \ldots \), which just corresponds to the well-known coordinate representation of the basis \( \psi_n(x, t) \). If \( \lambda = -k - n \) or \( \lambda = k - n - 1, \quad n = 0, 1 \ldots \) it can give rise to Laguerre polynomials also but of course this does not correspond to square integrable solutions of the Schrödinger equation. Finally these solutions have the form

\[ \psi_n = N_n \gamma^{-1/4} \left( \frac{\varepsilon}{\bar{\varepsilon}} \right)^\lambda y^{\alpha + \frac{1}{2}} e^{\pm \frac{1}{2}(\alpha + \gamma + \frac{1}{2})} y^2 L_n^\alpha \left( \mp \frac{1}{2} y^2 \right) \quad y = \frac{1}{2}(\varepsilon\bar{\varepsilon})^{-\frac{1}{2}} x. \]  

(7)

The lower sign and the choice \( \alpha = 2k - 1 \) correspond to the discrete spectrum eigenfunctions. The upper sign permitting \( \alpha \) to be both positive \( \alpha + n < 0 \) and for \( \alpha = 1 - 2k \) they satisfy the equation \( k_0 \psi_n(x, t) = (k - n - 1)\psi_n(x, t) \). The value \( N_n = 2^{3k-\frac{1}{2}} (n!)^\frac{1}{4} \Gamma^{-\frac{1}{4}} (n + 2k) \) guarantees the normalization of the square integrable solutions to the unity. For the non-normalizable solutions \( N_n \) does not play any role.

For applying the Darboux algorithm we need nodeless solutions. The zeros of the Laguerre polynomials are well-known [18]. In the negative semiaxis, \( x < 0 \), \( L_n^\alpha(x) \) has only one node provided \( \alpha + n < 0 \). It is clear that since we put \( k = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\beta} \), \( \alpha = 2k - 1 = \frac{1}{2}\sqrt{1 + 4\beta} > 0 \) meaning that in this case any \( \psi_n \) is nodeless and it is such that \( \psi_n^{-1}(\infty) = 0 \) but \( \psi_n^{-1}(0) = \infty \). For a negative \( \alpha \) the result depends on the interval where \( \alpha \) falls. If \(-2m - 1 > \alpha > -2m - 2 \) we have \( n = 0, 2, \ldots, 2m \) and for \( \alpha \geq 2m - 1 \) more values of \( n \) are possible: \( n = 0, 2, \ldots, 2m, 2m + 1, 2m + 2, \ldots \) where \( m = 0, 1, \ldots \). Moreover, for \( \alpha < -3/2 \) the function \( \psi_n^{-1}(x) \) is square integrable and belongs to the domain of definition of \( h_0 \) when it is considered as an operator in the Hilbert space.
2.2. Barut-Girardelo coherent states

The states $\psi_\mu(x,t)$ known as Barut-Girardelo CS are defined as eigenstates of the operator $k_-$

$$k_- \psi_\mu(x,t) = \mu \psi_\mu(x,t) \quad \mu \in \mathbb{C}. \quad (8)$$

To solve this equation simultaneously with the Schrödinger equation we are using the method of separation of variables once again. In coordinates $\{u = x/\varepsilon, v = t\}$ the Schrödinger equation separates if $\psi_\mu = \exp\left(\frac{i}{\varepsilon \dot{\varepsilon}} u^2\right) P(v) Q(u)$. The function $P(v)$ is defined by a first order ordinary equation which is easily integrated to give

$$P = \varepsilon^{-\frac{1}{2}} e^{-2\mu^2 \frac{v}{\varepsilon}}.$$

The Schrödinger equation is reduced to the second order equation for $Q$:

$$Q''(u) - (gu^{-2} + \mu^2)Q(u) = 0 \quad (9)$$

which after the change of the dependent variable $Q = u^\frac{k}{2} I$ becomes the modified Bessel equation. The condition for $\psi_\mu(x,t)$ to belong to the domain of definition of $h_0$ selects us only one solution to this equation: $I = I_{2k-1}(\mu u)$. (We are using the standard notations for the Bessel functions [18].) To calculate the normalization integral we are making use of the tables [19]. So, the normalized solution is

$$\psi_\mu(x,t) = \frac{1}{\varepsilon^{\frac{k}{2}}} I_{2k-1}(\frac{1}{\varepsilon} \mu x) \exp\left(\frac{i}{\varepsilon} \dot{\varepsilon} u^2 - \frac{2\mu^2}{\varepsilon} \right). \quad (10)$$

Expanding the Bessel function in a Taylor series one can get their expansion in terms of Laguerre polynomials and in such a way they are expressed through the basis functions (7) with coefficients depending only on even powers of $\mu$. Therefore, it is more convenient to change the complex variable $\mu$ in favor of $\lambda = 2\mu^2$ which yields

$$\psi_\lambda(x,t) = N_{0\lambda} \sum_{n=0}^{\infty} a_n \lambda^n \psi_n(x,t) \quad a_n = \frac{(-1)^n \sqrt{\Gamma(2k)}}{\sqrt{n!}\Gamma(n+2k)} \quad (11)$$

with

$$N_{0\lambda} := \langle \psi_0 | \psi_\lambda \rangle = (\lambda \lambda')^{\frac{k}{2} - \frac{1}{2}} I_{2k-1}^{-\frac{1}{2}}(2 |\lambda|) \Gamma^{-\frac{1}{2}}(2k). \quad (12)$$

As usual the states $\psi_\lambda(x,t)$ are not orthogonal to each other

$$\langle \psi_\lambda' | \psi_\lambda \rangle = \frac{I_{2k-1}(2 \sqrt{\lambda \lambda'})}{[I_{2k-1}(2 |\lambda|) I_{2k-1}(2 |\lambda'|)]^{\frac{k}{2}}}. \quad \text{(13)}$$

Since the basis $\psi_n(x,t)$ is complete in the Hilbert space one can calculate the measure $\rho(\lambda)$ which realizes the resolution of the identity over the coherent states $\psi_\lambda$ [9]

$$\int |\psi_\lambda\rangle \langle \psi_\lambda| d\rho(\lambda) = 1 \quad \rho(\lambda) = \frac{1}{\pi} K_{2k-1}(2|\lambda|) I_{2k-1}(2|\lambda|) d\lambda d\lambda'. \quad (13)$$

All integrals over the variable $\lambda$ are extended to the whole complex plan.
Now one can construct a holomorphic representation of the vectors and operators \([9]\). Any \( \psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x, t) \) from the Hilbert space of square integrable functions defined on the positive semiaxis can be written in coherent state representation \( \psi^c(\lambda) \):

\[
\psi^c(\lambda) := \langle \psi(x, t) \mid \psi(x, t) \rangle = N_{0\lambda} \sum_{n=0}^{\infty} a_n c_n \lambda^n \equiv N_{0\lambda} \psi(\lambda) \quad \lambda \in \mathbb{C} .
\] (14)

The holomorphic function \( \psi(\lambda) = \sum_{n=0}^{\infty} a_n c_n \lambda^n \) can be associated with the usual square integrable function \( \psi(x, t) \) given by its Fourier coefficients \( c_n \) over the basis (7).

Using the complex conjugate form of the resolution of the identity (13) one can define an inner product \( \langle \psi_a(\lambda) \mid \psi_b(\lambda) \rangle \) in the space of the functions \( \psi_{a,b}(\lambda) \) holomorphic in the complex plane

\[
\langle \psi_a(\lambda) \mid \psi_b(\lambda) \rangle = \int \overline{\psi_a(\lambda)} \psi_b(\lambda) d\tilde{\rho}(\lambda) := \langle \psi_a(\lambda) \mid \psi_b(\lambda) \rangle .
\] (15)

This means that the integration in the space of holomorphic functions should be carried out with the measure \( d\tilde{\rho}(\lambda) = |N_{0\lambda}|^2 d\rho(\lambda) \), so that

\[
\langle \psi_a(\lambda) \mid \psi_b(\lambda) \rangle = \int \overline{\psi_a(\lambda)} \psi_b(\lambda) d\tilde{\rho}(\lambda) .
\] (16)

To distinguish this inner product from the one in the space \( L^2(0, \infty) \) we indicate the integration variable inside the brackets. The space of holomorphic functions \( \psi(\lambda) \) such that

\[
\int |\psi(\lambda)|^2 d\tilde{\rho}(\lambda) < \infty
\]
equipped with the inner product (16) becomes a Hilbert space.

The orthonormal basis \( \psi_n(x, t) \) in this representation looks like as follows:

\[
\psi_n(\lambda) = a_n \lambda^n \quad \langle \psi_n(\lambda) \mid \psi_{n'}(\lambda) \rangle = \delta_{nn'} .
\]

From here we find the Dirac-delta function

\[
\delta(\lambda, \lambda') = \sum_{n=0}^{\infty} \psi_n(\lambda) \overline{\psi_n(\lambda')} = \sum_{n=0}^{\infty} a_n^2 (\lambda \lambda')^n = \Gamma(2k)(\lambda \lambda')^{1-2k} I_{2k-1} \left( 2\sqrt{\lambda \lambda'} \right)
\]

which takes off the integration in this space:

\[
\psi(\lambda) = \int \delta(\lambda, \lambda') \psi(\lambda') d\tilde{\rho}(\lambda') .
\]

The CS in the holomorphic representation

\[
\psi_{\lambda'}(\lambda) = (\lambda \lambda')^{\frac{1}{2} - k} I_{2k-1} \left( 2\sqrt{\lambda \lambda'} \right)
\]

and the \( SU(1, 1) \) generators

\[
k_0 = -\lambda \frac{d}{d\lambda} + k \quad k_+ = \lambda \quad k_- = \lambda \frac{d^2}{d\lambda^2} + 2k \frac{d}{d\lambda} .
\] (17)
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are time-independent. The mean values of the operator \( k_0 \) in the coherent state

\[
\langle \psi_\lambda | k_0 | \psi_\lambda \rangle = k + |\lambda| \frac{I_{2k+1}(2|\lambda|)}{I_{2k-1}(2|\lambda|)}
\]  

(18)

and

\[
\langle \psi_\lambda | k_0^2 | \psi_\lambda \rangle = k^2 + |\lambda|^2 + |\lambda| \frac{I_{2k+1}(2|\lambda|)}{I_{2k-1}(2|\lambda|)}
\]  

(19)

may be useful in the following.

2.3. Perelomov coherent states

Perelomov CS \( \psi_z(x, t) \) are obtained by acting on the ground state function with the group translation operator. Therefore their Fourier coefficients over the basis \( \psi_n(x, t) \) are independent on \( t \) and coincide with the ones for the time-independent Hamiltonian

\[
\psi_z(x, t) = N_{0z} \sum_{n=0}^{\infty} a_n z^n \psi_n(x, t) \quad |z| < 1
\]  

(20)

\[
N_{0z} = (1 - |z|^2)^k \quad a_n = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}
\]  

(21)

Using the generating function for the Laguerre polynomials one gets their explicit expression

\[
\psi_z(x, t) = 2^{\frac{1}{2} - 3k} \Gamma^{-\frac{1}{2}}(2k) e^{-2k x^2} 2^{k-\frac{1}{2}} \left( \frac{1 - |\zeta|^2}{(1 - \zeta)^2} \right)^k \exp \left( -\frac{x^2}{16\gamma} \frac{1 + \zeta}{1 - \zeta} + \frac{ix^2 \gamma}{8\gamma} \right)
\]  

(22)

where \( \zeta = \frac{z^\epsilon}{\epsilon} \).

We notice that since the series expansion (20) of the time-dependent CS (22) in terms of the time-dependent basis \( \{ \psi_n(x, t) \} \) is exactly the same as the similar expansion for the time-independent Hamiltonian, all properties of such states known for the stationary case take place for the non-stationary one. In particular, they realize the resolution of the identity operator and, hence, one can map any square integrable on the positive semiaxis function to a function holomorphic in the unit disc getting in such a way a holomorphic representation. We will not go in details for this case since they are very well-known [2].

2.4. Discussion

Since the parameters \( z \) and \( \lambda \) labelling the CS fall into different domains, it is not straightforward to make a comparison between them. One of the possibilities could be to choose these parameters such that the mean value of an operator is the same for both states. As a particular example we take \( \omega = 0 \) (free particle) and \( g = 2 \) (orbital quantum number \( \ell = 1 \)). In this case \( \epsilon = \frac{1}{\sqrt{2}}(t + i) \) and we have chosen real values for \( z \) and \( \lambda \) such that the mean value of \( k_0 \) coincides with its value in the first excited state (7) which is \( k + 1 \). In this case \( z = (2k + 1)^{-1/2} \) and \( \lambda \approx 1.021 \). With these values of \( z \) and \( \lambda \) we plotted the squared modulus
of $\psi_2(x,t)$ (22) and $\psi_3(x,t)$ given by (10) at $\mu = \sqrt{\lambda/2}$ in fig. 1. It is clearly seen from the figure that at the initial time moment the Barut-Girardello CS are much more localized than Perelomov CS but the latter are much more stable in time. At times greater than 2 they are already very spread whereas the Perelomov CS maintain almost the same localization as at the initial time moment. In the next section we shall show that the Darboux transformation keeps the different time behavior of two types of states practically unchanged.

3. Darboux transformation of coherent states

Time-dependent Darboux transformation [20] can create real potential differences only if the Schrödinger equation has at least one solution $u = u(x,t)$ satisfying the reality condition $(\log u/\pi)_{xxx} = 0$. Solutions (5) satisfy this condition for any real function $Q$. So, taking different real solutions of equation (9) one can, in general, obtain a two-parameter family of exactly solvable partners for $h_0$ but only the functions $u = \psi_n(x,t)$ (7) produce potential differences expressed in terms of elementary functions. Between square integrable solutions only $\psi_0(x,t)$ is nodeless. Unfortunately, it produces a new potential of the same kind as $V_0$ only with a different value of $g$ (shape invariance at the time-dependent level). So, to get essentially new potentials we have to keep in (7) only upper sign. For $\alpha > -3/2$ any nodeless function (7) gives rise to a new potential. For simplicity we will consider here only the case $\alpha > 0$ when all square integrable solutions for the transformed Hamiltonian $h_1 = -\partial_x^2 + V_1(x,t)$ can be obtained by acting on corresponding solutions of the initial equation with the transformation operator

$$L = L_1(t)[\partial_x - u_x(x,t)/u(x,t)] \quad L_1(t) = \exp[2 \int dt \operatorname{Im}(\ln u)_{xx}].$$

The potential $V_1(x,t)$ is expressed in terms of the same function $u$:

$$V_1(x,t) = V_0(x,t) + A(x,t) \quad A(x,t) = -[\ln |u(x,t)|^2]_{xx}.$$
Taking one of the function (7) with \( n = m \) (which is supposed to be fixed from now on) one gets the potential difference \( A(x, t) = A_m(x, t) \):

\[
A_m(x, t) = \frac{4k - 1}{x^2} + \frac{1}{8} \left( x L_{m-1}^2(z) + \frac{x L_{m-1}^2(z)}{\gamma L_{m-1}^2(z)} \right)^2 - \frac{8 \gamma^2 L_{m-1}^2(z)}{x^2} - \frac{1}{4\gamma} \tag{24}
\]

\[
z = -\frac{x^2}{8\gamma}.
\]

The function \( L_1(t) \) is determined by (23) up to a constant which we fix to simplify subsequent formulas so that \( L_1 = \sqrt{2\gamma} \) and the explicit expression for the transformation operator (23) is

\[
L = \sqrt{2\gamma} \left[ \partial_x - \frac{x}{8\gamma} - \frac{4k - 1}{2x} \right] = \frac{ix\gamma}{2} - \frac{x L_{m-1}^2(z)}{4\gamma L_{m-1}^2(z)} \tag{25}
\]

Normalized to unity solutions for the Hamiltonian \( h_1 \) are: \( \varphi_n = N_{1n}L\psi_n, N_{1n} = (n+2k+m)^{-\frac{1}{2}} \).

Operator \( L \) and its formally adjoint \( L^+ \) factorize the symmetry operator \( k_0: L^+L = k_0 + k + m \). The opposite superposition \( LL^+ \) is a symmetry operator for the transformed Schrödinger equation. If we denote \( p_0 = LL^+ - k - m \) then the functions \( \varphi_n \) are eigenfunctions of \( p_0 \) and the spectrum of \( p_0 \) is identical to the spectrum of \( k_0 \). Using \( k_\pm \) and transformation operators \( L \) and \( L^+ \) one can construct other symmetry operators for the transformed equation: \( p_\pm = Lk_\pm L^+ \). They are ladder operators for the functions \( \varphi_n \):

\[
p_\pm \varphi_n(x, t) = (N_{1n}N_{1(n\pm1)})^{-1} c_{n\pm} \varphi_{n\pm}(x, t).
\]

We notice now that this Hamiltonian gives us an example of the Schrödinger equation, symmetry operators of which do not close a Lie algebra but satisfy a polynomial algebra

\[
[p_0, p_\pm] = \pm p_\pm,
\]

\[
[p_-, p_+] = 2[k(1 - k) + 2p_0(k + m) + 2p_0^2](p_0 + k + m).
\]

Similar algebra was previously obtained for the time-independent singular oscillator in [6].

### 3.1. Transformation of Barut-Girardeau coherent states

To obtain CS for the Hamiltonian \( h_1 \) we act with \( L \) given in (25) on CS \( \psi_\lambda: \varphi_\lambda = N_{1\lambda}L\psi_\lambda \). The factor \( N_{1\lambda} \) being calculated from the formula \( N_{1\lambda}^{-2} = \langle \psi_\lambda | k_0 + k + m | \psi_\lambda \rangle \) and (18) guarantees the normalization of the states \( \varphi_\lambda \) to unity. Their series expansion in terms of the basis \( \{ \varphi_n \} \) can be found by acting with the same operator on the series (11):

\[
\varphi_\lambda = N \sum_{n=0}^{\infty} b_n \lambda^n \varphi_n \quad b_n = a_n(n + 2k + m)^{\frac{3}{2}}(2k + m)^{-\frac{3}{2}} \tag{26}
\]

where \( N = (2k + m)^{\frac{3}{2}}N_{1\lambda}N_{0\lambda} \).

The states \( \varphi_\lambda = \varphi_\lambda(x, t) \) thus obtained may be interpreted as coherent states if they admit the resolution of the identity operator

\[
\int |\varphi_\lambda\rangle \langle \varphi_\lambda| d\tilde{\rho}(\lambda) = 1 \tag{27}
\]
Now we proceed to find the measure $\tilde{\rho}(\lambda)$. We will look for the function $\tilde{\rho}(\lambda)$ depending only on the absolute value of $\lambda$, $|\lambda| = \sqrt{x}$: $d\tilde{\rho} = \frac{1}{2} h(x) dxd\phi$; the function $h(x)$ is to be determined. Therefore, it is convenient to use polar coordinates in the complex plan of the variable $\lambda$, $\lambda = \sqrt{x}\exp(i\phi)$. After being integrated over the variable $\phi$ equation (27) yields

$$1 = \sum_{n=0}^{\infty} \frac{\pi(n + 2k + m)}{n! \Gamma(n + 2k)} \int_0^\infty x^{n+\alpha/2-1/4} I_{\alpha-1/2}^{-1} (2\sqrt{x}) h_0(x) dx |\varphi_n\rangle\langle \varphi_n|.$$  

(28)

From here it follows that if equation

$$\frac{\pi(n + 2k + m)}{\Gamma(n + 1) \Gamma(n + 2k)} \int_0^\infty x^{n+\alpha/2-1/4} I_{\alpha-1/2}^{-1} (2\sqrt{x}) h_0(x) dx = 1$$  

(29)

is satisfied, than (27) will also take place because of the completeness of the system $\{\varphi_n\}$. If now we rewrite (29) as

$$\int_0^\infty x^n \Phi(x) dx = \frac{\Gamma(n + 1) \Gamma(n + 2k)}{(n + 2k + p)}$$  

(30)

where

$$\Phi(x) = |N_0\lambda N_{1\lambda}|^2 h(x)$$  

(31)

we recognize in it a problem of moments on semiaxis (see e.g. [23]). To solve this problem we are using the following integral [24]

$$\int_0^\infty x^n f(x) dx = \frac{\Gamma(n + 1) \Gamma(n + 2k)}{(n + 2k + p)}$$  

(32)

where $f(x) = 2x^{k-\frac{1}{2}}K_{2k-1}(2\sqrt{x})$. It is not difficult to get the expression for $\Phi(x)$ in terms of $f(x)$

$$\Phi(x) = x^{m+2k-1} \int_x^\infty y^{-2k-m} f(y) dy.$$  

(33)

Indeed, first we notice that since $k > \frac{1}{2}$ we have $x\Phi(x) \rightarrow 0$ when $x \rightarrow 0$. Therefore, the integration in (30) by parts under the condition (32) yields just the right hand side of (30) meaning that equations (33) and (31) define the measure $\tilde{\rho}(\lambda)$.

The resolution of identity (27) permits us to construct a holomorphic representation for the transformed Hamiltonian $h_1$. The Fourier coefficients $\{c_n\}$ of a function $\varphi(x, t)$ over the basis $\{\varphi_n(x, t)\}$ gives us the same function in CS representation: $\varphi^c(\lambda) = \langle \varphi^c \mid \varphi \rangle = N \varphi(\lambda)$ and the function $\varphi(\lambda) = \sum_{n=0}^{\infty} b_n c_n \lambda^n$ is the holomorphic representative of $\varphi_\lambda(x, t)$. Now one can define a new inner product

$$\langle \varphi_1(\lambda) \mid \varphi_2(\lambda) \rangle = \int \langle \varphi_1 \mid \varphi^c \rangle \langle \varphi^c \mid \varphi_2 \rangle d\tilde{\rho}(\lambda) = \int |\lambda|^2 \varphi_1(\lambda) \varphi_2(\lambda) d\tilde{\rho}(\lambda)$$

in the space of holomorphic functions, which gives us a holomorphic representation of states and operators different from that discussed in section 2.2.
It is not difficult to see that after the Darboux transformation any basis function \( \psi_n(\lambda) \) goes to \( \varphi_n(\lambda) = (n + 2k + m)^{\frac{1}{2}}(2k + m)^{-\frac{1}{2}}\psi_n(\lambda) \). Therefore, if we want for the functions \( \psi_n(\lambda) \) and \( \varphi_n(\lambda) \) to be related by the Darboux transformation, we have to put

\[
\varphi_n(\lambda) = (n + 2k + m)^{-\frac{1}{2}}L(\lambda)\psi_n(\lambda) \quad \psi_n(\lambda) = (n + 2k + m)^{-\frac{1}{2}}L^+(\lambda)\varphi_n(\lambda) .
\]

This gives us the holomorphic representation of the Darboux transformation operators

\[
L(\lambda) = (m + 2k)^{-\frac{1}{2}}[k_0(\lambda) - k - m] \quad L^+(\lambda) = (m + 2k)^{\frac{1}{2}}.
\]

### 3.2. Transformation of Perelomov coherent states

Once again we act with the transformation operator \( L \) given in (25) but now on \( \psi_z(x,t) \) (22) to get the Darboux transformed Perelomov CS, \( \varphi_z(x,t) = N_{1z}L\psi_z(x,t) \). Normalization constant is easily calculated using the equation \( \langle L\psi_z | L\psi_z \rangle = \langle \psi_z | L^+L\psi_z \rangle \) and the factorization property of the transformation operators, \( N_{1z}^{-2} = m + 2k(1 - |z|^2) \). Their Fourier series in terms of the basis \( \{\varphi_n(x,t)\} \) is

\[
\varphi_z(x,t) = N_z \sum_{n=0}^{\infty} b_n z^n \varphi_n(x,t) \quad (34)
\]

\[
N_z = N_{0z}N_{1z}(2k + m)^{\frac{1}{2}} \quad b_n = a_n(n + 2k + m)^{\frac{1}{2}}(2k + m)^{-\frac{1}{2}}.
\]

Here also the series \( (34) \) is exactly the same which has previously been obtained for the time-independent oscillator [6]. We have already found [6] the measure, which realizes the resolution of the identity operator in terms of \( \varphi_z \), constructed new Holomorphic representation for the operators and states, found the Kähler potential and symplectic 2-form meaning that we obtained a classical mechanics, which being quantized à la Berezin gives us back the holomorphic representation of the quantum system. This procedure can be considered as the one giving rise to a classical counterpart of the Darboux transformation valid both for time-dependent and time-independent cases.

### 3.3. Discussion

![Figure 3](image1.png) ![Figure 4](image2.png)

**Figure 3.** Comparison between probability distribution for Barut-Girardello CS before (solid line) and after (dashed line) Darboux transformation.

**Figure 4.** Comparison between probability distribution for Perelomov CS before (solid line) and after (dashed line) Darboux transformation.
We would like to point out that if the Schrödinger equation has a symmetry algebra, this property is usually lost after the Darboux transformation. Nevertheless, if it has ladder operators, the transformed equation may also have them. In such a case it is possible to look for eigenstates of the annihilation operator and call them coherent states. For the case of the time-independent harmonic oscillator potential this approach was realized in [21]. Since ladder operators have now two derivative orders more with respect to the initial ladder operators, the differential equation they satisfy is rather complicated [22] which makes difficult studying such states. Our approach has an advantage that the transformation operator (25) is a simple first order differential operator. So, it is very easy to operate with it. Moreover, in such a way one can get different systems of CS if they are available for the initial Hamiltonian. Usually different systems of CS exhibit different properties [11, 15]. We conjecture that Darboux transformation approximately preserves different behavior of different CS. To support this conjecture we plotted the transformed Barut-Girardelo CS together with the transformed Perelomov CS on Fig. 2. From the first sight the difference between Figs. 1 and 2 is practically invisible. To show it better we plotted Barut-Girardelo CS (solid line) together with their transformed version (dashed line) on Fig. 3. Fig. 4 shows the Perelomov CS before (solid line) and after (dashed line) the Darboux transformation. It is clearly seen from these figures that for both cases the Darboux transformation results mainly in a displacement of the curve while its shape is very little affected.

4. Conclusion

We have shown that acting with the Darboux transformation operator on the known CS of the time dependent singular oscillator gives us the states with the similar properties. Thus, the ones obtained from Barut-Girardelo CS may be called Barut-Girardelo-like CS while the others, which are produced using Perelomov CS, may be called Perelomov-like CS. Each system of CS admits a resolution of the identity operator which makes it possible to construct different holomorphic representations. A particular example of a free particle in $p$-state ($\ell = 1$) shows that Barut-Girardelo-like CS are well localized at the initial time moment while Perelomov-like CS are more stable with time evolution. Such a behavior is a reflection of the similar behavior of corresponding states before the transformation. Therefore we hope that the new systems of CS may be useful in similar applications where the known systems have proven to be helpful.

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References

[1] Klauder J R and Skagerstam B S 1985 Coherent states: applications in physics and mathematical physics (Singapore: World Scientific);
Malkin I A and Man’ko V I 1979 Dynamical Symmetries and Coherent States of Quantum Systems (Moscow: Nauka);
Darboux transformations of coherent states of the time-dependent singular oscillator

Nieto M and Simmons L 1979 Phys. Rev. D 20 1321; Beribe-Lavzier Y and Hussin V 1993 J. Phys. A: Math. and Gen. 26 6271

[2] Perelomov A M 1986 Generalized coherent states and their applications (Berlin: Springer)

[3] Gazeau J P Klauder J R 1999 J. Phys. A: Math. and Gen. 32 123

[4] Samsonov B F 1998 JETP 114 130;
Samsonov B F 2000 J. Phys. A: Math. and Gen. 33 591

[5] Brif C, Vourdas A and Mann A 1996 J. Phys. A: Math. and Gen. 29 5873

[6] Samsonov B F 1998 J. Math. Phys. 39 967

[7] Samsonov B F 1996 Phys. Atom. Nucl 59 753

[8] Bagrov V G and Samsonov B F 1996 J. Phys. A: Math. and Gen. 29 1011;
Bagrov V G and Samsonov B F 1996 JETP 109 1105

[9] Bagrov V G and Samsonov B F 1998 Izv. Vuz. Fiz. (Russ. Phys. J.) 41(3) 46

[10] Hartmann A 1972 Theor. Chim. Acta 24 201;
Chumakov S M, Dodonov V V and Man’ko V I 1986 J. Phys. A: Math. and Gen. 19 3229;

[11] Dodonov V V, Man’ko V I, Man’ko O V and Rosa L 1994 phys. Lett A 185 231

[12] Calogero F 1971 J. Math. Phys. 10 2191;
Sutherland B 1971 J. Math. Phys. 12 246

[13] Dodonov V V, Man’ko V I and Rosa L 1998 Phys. Rev. A 57 2851

[14] Miller W (Jr) 1977 Symmetry and separation of variables (Massachusetts: Addison-Wesley)

[15] Agraval G S and Chaturvedi S 1995 J. Phys. A: Math. and Gen. 28 5747

[16] Barut A O and Girardeo L 1971 Commun. Mayh. Phys. 21 41

[17] Lewis H R and Riesenfeld W B 1969 J. Math. Phys. 10 1458;
Dodonov V V, Man’ko V I in 1982 Group theoretical methods in physics I Markov M A, Man’ko V I and Shabad A E (Eds) (Chur: Harward Academic) 591

[18] Erdelyi A 1953 Higher transcendental functions (New York: McGraw-Hill)

[19] Prudnikov A P, Brychlov Yu A and Marichev O I 1983 Integrals and series. Spetial functions (Moscow: Nauka)

[20] Bagrov V G and Samsonov B F 1977 Phys. Part. Nucl. 28 374.

[21] Fernandez C D J, Hussin V and Nieto L M 1994 J. Phys. A: Math. and Gen. 27 3547;
Fernandez C D J, Nieto L M and Rosas-Ortiz O 1995 J. Phys. A: Math. and Gen. 28 2693;
Rosas-Ortiz O J 1996 J. Phys. A: Math. and Gen. 29 3281

[22] Spiridonov V 1995 Phys. Rev. A 52 1999

[23] Akhiezer N I 1961 Classical moments problem (Moscow: Phys.-Math Literature Publishing Hous)

[24] Erdelyi A 1954 Tables of integral transformations V. 1 (New York: McGraw-Hill)