Asymptotics of 6j and 10j symbols

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Abstract

It is well known that the building blocks for state sum models of quantum gravity are given by 6j and 10j symbols. In this work we study the asymptotics of these symbols by using their expressions as group integrals. We carefully describe the measure involved in terms of invariant variables and develop new technics in order to study their asymptotics. Using these technics we compute the asymptotics of the various Euclidean and Lorentzian 6j-symbols. Finally we compute the asymptotic expansion of the 10j symbol which is shown to be non-oscillating, in agreement with a recent result of Baez et al. We discuss the physical origin of this behavior and a way to modify the Barrett-Crane model in order to cure this disease.

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I. INTRODUCTION

It is well known that state sum models defined over a triangulation of space-time are a central tool in the construction of discrete transition amplitudes for quantum gravity. It was first understood by Ponzano and Regge [1] that the transition amplitude of 3d Euclidean gravity could be expressed as a state sum model defined over a triangulation of space-time. This state sum model is obtained by summing, over SU(2) representations labelling the edges of the triangulation, a given weight depending on this labelling. This sum over representations is interpreted as a sum over geometry, and the weight is a product of SU(2) $6j$ symbols associated with each colored tetrahedron. This construction can be extended to the case of 4d gravity and 2+1 Lorentzian gravity. It was shown in [2, 3] that the building block in the construction of transition amplitudes for 2+1 Lorentzian gravity are the SL(2, $\mathbb{R}$) $6j$ symbols. In the context of 4 dimensional gravity, Barrett and Crane [4] proposed to use the so-called 10$j$ symbol in order to construct the discretized transition amplitudes.

One of the key arguments in favor of the Ponzano-Regge model as a model for 3d gravity is the fact that the asymptotic behavior of the $6j$ symbol reproduces the discretized Regge action for 3d gravity. This asymptotic has been conjectured in 1968 by Ponzano and Regge but proved only in 1999 by Roberts [5]. In the same way, one of the arguments in favor of the Barrett-Crane amplitude as a building block for quantum gravity was the fact shown by Barrett and Williams [6] that the stationary contribution to the asymptotic of the 10$j$ symbol reproduces also the discretized Regge action of 4d gravity. However, recent numerical simulations and computations by Baez et al. [7, 8] have shown that the asymptotic of the 10$j$ symbol is not dominated by an oscillating contribution.

In this paper we address this issue by using expressions for the $6j$ and 10$j$ symbol as integrals over group elements [9]. This is done by carefully describing the measure involved in terms of gauge invariant variables. We show that the asymptotics separates into a contribution coming from a stationary phase approximation and a contribution coming from degenerates configurations associated with singularities of the integrand. The stationary phase produces an oscillatory Regge behavior, whereas the degenerate contribution is non-oscillating. It should be noted that this general method has been outlined in the recent paper of Baez, Christensen and Egan [8], and even if our approach is independent from their, one can view our paper as giving proofs of the conjectures they made. Also, while we were completing the redaction of this work, we become aware of the very recent article [10], which is leading to the same conclusions as ours but using a different approach.

In section II we express the square of the SU(2) $6j$-symbol as an integral over the space of spherical tetrahedra. We show that the measure of integration is given by the inverse square root of the determinant of the Graham matrix [11] associated with spherical tetrahedra. We show that the integral naturally separates into two parts: one for which the asymptotics is dominated by an oscillating contribution, associated with flat non-degenerate tetrahedra and obtained by a stationary phase approximation; and one for which the asymptotics comes from boundary contributions which label degenerate tetrahedra. We prove that this contribution is given by an integral associated with the Euclidean group, as was conjectured by Baez et al [8].

In section III we express the square of the SL(2, $\mathbb{R}$) $6j$-symbol as an integral over the space of AdS tetrahedra. We also split this integral in two parts, and show that the stationary phase is dominated by an oscillating contribution associated with flat non-degenerate Lorentzian tetrahedra. The phase of this contribution is the Lorentzian Regge action and the
module is the volume of the Lorentzian tetrahedron. In section IV we apply our technics to the $10j$-symbol and show that the leading contribution is a non-oscillating one, dominating the oscillating Regge action term found by Barrett and Williams. Finally in section V we discuss the physical meaning and origin of this non-oscillating behavior for the $10j$-symbol, in the spirit of the statistical mechanics models of 'order by disorder', and we propose a modification of the Barrett-Crane model to avoid this problem and recover an oscillatory Regge action behavior.

II. ASYMPTOTICS OF THE 6J-SYMBOL

A. Integral expression for the square of the 6j-symbol

We are interested in the computation of the asymptotics of the square of the 6j-symbol. The $6j$-symbol is a real number which is associated to the labelling of the edges of a tetrahedra by $SU(2)$ representations. It is obtain by combining four normalized Clebsch-Gordan coefficients along the six edges of a tetrahedra (see [5]). Let us denote by $V_l$ the $SU(2)$ representation of spin $l/2$ and by $\chi_l(g) = \text{tr}V_l(g)$ the character of this representation. Let $I = 0, 1, 2, 3$ be the 4 vertices of a tetrahedra $T$ and $(IJ)$ the edges of $T$. We associate a representation $V_{lIJ}$ to each edge $(IJ)$ of $T$. It is well known [9, 16] that the square of the $6j$-symbol can be expressed as the following Feynman integral over $SU(2)$

$$I(l_{IJ}) = \left\{ \frac{l_{01}}{l_{23}}, \frac{l_{02}}{l_{13}}, \frac{l_{03}}{l_{12}} \right\}^2 = \int \prod_i \frac{dg_i}{G^4} \prod_{I<J} \chi_{l_{IJ}}(g_Jg_I^{-1}),$$

where the normalized Haar measure $dg$ is used. Such an identity is clear since the integral over each group element produces a pair of Clebsch-Gordan coefficient which are then combined into one tetrahedron and its mirror image. It is also clear from this expression that the $I(l_{IJ}) \neq 0$ only if the $|l_{IK} - l_{IL}| \leq l_{IJ} \leq l_{IK} + l_{IL}$ and $l_{IK} + l_{IL} + l_{IJ}$ is an even integer. In this expression, the integral is over four copies of the group, however the integrand is invariant under the transformations

$$g_I \to hkg_I k^{-1},$$

where $h,k$ are $SU(2)$ elements. It is therefore possible to gauge out this symmetry and write the integral purely in term of gauge invariant variables. A natural choice for the gauge invariant variables is given by the 6 angles $\theta_{IJ} \in [0, \pi]$:

$$\cos \theta_{IJ} = \frac{1}{2} \text{tr}V_i(g_Jg_I^{-1}).$$

Note that this gauge symmetry has a geometrical interpretation. The four group elements $g_I$ define 4 points, hence a tetrahedron, in $S^3$. The angles $\theta_{IJ}$ are the spherical lengths of the edges of this tetrahedron, which indeed parameterize its invariant geometry. Notice that the symmetry includes reflexions $g \to -g$. The integral (1) can be written in terms of these angles as follows:

**Theorem 1**

$$I(l_{IJ}) = \frac{2}{\pi^2} \int_{D^4} \prod_I d\theta_{IJ} \prod_{I<J} \frac{\sin((l_{IJ}+1)\theta_{IJ})}{\sqrt{\det[\cos \theta_{IJ}]}},$$

where $D^4$ is the domain of the integration.
where $D_\pi$ is the subset of $[0;\pi]^6$ of angles satisfying the relations:

$$\theta_{IJ} \leq \theta_{IK} + \theta_{JK},$$

$$2\pi \geq \theta_{IJ} + \theta_{IK} + \theta_{JK},$$

for any triple $(I,J,K)$ of distinct elements. Geometrically this domain is the set of all possible spherical tetrahedra.

The denominator of this integral is the square root of the determinant of the $4 \times 4$ Graham matrix $G_{IJ} = \cos\theta_{IJ}$ associated with the corresponding spherical tetrahedron. This determinant is zero if and only if $\theta_{IJ}$ belongs to the boundary of the domain $D_\pi$, which is the set of degenerate tetrahedra having zero volume. Note that if we consider $\theta_{IJ}$ to be the dihedral angles of a spherical tetrahedra, $\partial D_\pi$ is the set of flat tetrahedra (see the remark 3 below).

**Proof:**

**Notations and spherical geometry:** The invariant measure is obtained by a Faddev-Popov procedure. In order to present the details of this procedure, we first need to do a little bit of spherical geometry and introduce a new set of angles denoted $\theta_i, \tilde{\theta}_{ij}, \bar{\theta}_{ij} \in [0,\pi], i = 1,2,3$. This angles are related in the following way to the angle $\theta_{IJ}$ of eq.(3).

$$\theta_i = \theta_{0i},$$

$$\cos\theta_{ij} = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos \tilde{\theta}_{ij},$$

$$\cos \tilde{\theta}_{ij} = \cos \tilde{\theta}_{ik} \cos \tilde{\theta}_{kj} + \sin \tilde{\theta}_{ik} \sin \tilde{\theta}_{kj} \cos \bar{\theta}_{ij},$$

where in the eq.(9), $i, j, k$ is any permutation of $1,2,3$. The geometrical interpretation of these angles is as follows: first $\theta_i$ are the lengths of the edges $0i$, $\tilde{\theta}_{ij}$ is the angle between the edges $(0i)$ and $(0j)$, finally $\bar{\theta}_{ij}$ is the dihedral angle of the edge $(0k)$ opposite to the edge $(ij)$ (see fig.1). In order to see this, let $\sigma_i, i = 1,2,3$ be the Pauli matrices, $\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ij} \sigma_k$ and let us introduce the normalized vectors $\vec{n}_i \in S^2$ such that

$$g_i g_0^{-1} = e^{i \theta_i \vec{n}_i \cdot \vec{\sigma}} = \cos \theta_i + i \sin \theta_i (\vec{n}_i \cdot \vec{\sigma}).$$

Then

$$\cos \theta_{ij} = \frac{1}{2} \text{tr}(g_j g_i^{-1}) = \frac{1}{2} \text{tr} \left( g_j g_0^{-1}(g_i g_0^{-1})^{-1} \right) = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j (\vec{n}_i \cdot \vec{n}_j),$$

so $\cos \tilde{\theta}_{ij} = \vec{n}_i \cdot \vec{n}_j$. Let us introduce the unit normal vector to the face $(0ij)$: $\vec{a}_k = \vec{n}_i \wedge \vec{n}_j / |\vec{n}_i \wedge \vec{n}_j|$, where $ijk$ is a cyclic permutation of $123$. A simple computation shows that

$$\cos \bar{\theta}_{ij} = -\vec{a}_i \cdot \vec{a}_j,$$

which characterizes $\bar{\theta}_{ij}$ as the dihedral angle of the edge $0k$.

**Measure:** Using these geometrical elements, we can compute the invariant measure. In order to get this measure, we first have to fix the symmetry by choosing a gauge. Using the isometry (eq. 2) we can first translate the tetrahedron so that one of its vertex, say $g_0$, is at the identity. This fix one $SU(2)$ invariance, the other corresponds to rotation of the tetrahedron around the identity. We can gauge out part of this invariance by fixing the
direction of one edge say (10). This still let the freedom to rotate the tetrahedron around that edge, we then choose to fix the direction of the plane (120). In term of the variables \( \theta_i, \vec{n}_i \) the Haar measure reads \( dg_i = \frac{2}{\pi} \sin^2 \theta_i d\theta_i d^2 \vec{n}_i \), where \( d^2 \vec{n} \) is the normalized measure on the 2-sphere. So after fixing \( g_0 = 1 \) the measure \( d\mu = dg_0 dg_1 dg_2 dg_3 \) becomes

\[
d\mu = \left( \frac{2}{\pi} \right)^3 \left( \prod_{i=1}^{3} \sin^2 \theta_i d\theta_i \right) d^2 \vec{n}_i.
\]

(13)

The residual gauge invariance is fixed by imposing

\[
\vec{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} \cos \theta_{12} \\ \sin \theta_{12} \\ 0 \end{pmatrix}, \quad \vec{n}_3 = \begin{pmatrix} \cos \theta_{13} \\ \sin \theta_{13} \cos \theta_{23} \\ \sin \theta_{13} \sin \theta_{23} \end{pmatrix}
\]

(14)

In terms of this variables the measure is now

\[
d\mu = \frac{2}{\pi^4} \left( \prod_{i=1}^{3} \sin^2 \theta_i d\theta_i \right) \sin \theta_{12} d\theta_{12} \sin \theta_{13} d\theta_{13} \sin \theta_{23} d\theta_{23}.
\]

(15)

It can be checked directly that \( \int d\mu = 1 \). Using relations (9) we can express the measure in terms of \( \theta_i \) and \( \tilde{\theta}_{ij} \)

\[
d\mu = \frac{2}{\pi^4} \left( \prod_{i=1}^{3} \sin^2 \theta_i d\theta_i \right) \prod_{i<j} \sin \tilde{\theta}_{ij} d\tilde{\theta}_{ij} \bigg| \frac{\vec{n}}{(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3} \bigg|.
\]

(16)

This trivially follows from a change of variable and the following fact \( |(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3| = \sin \theta_{12} \sin \theta_{13} \sin \theta_{23} \). If we make the change of variable \( \tilde{\theta}_{ij} \rightarrow \theta_{ij} \), eq.(16) becomes

\[
d\mu = \frac{2}{\pi^4} \prod_{i<j} \sin \theta_{IJ} d\theta_{IJ} \bigg| \frac{\vec{n}}{(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3} \bigg|.
\]

(17)

We can now show that the denominator is indeed

\[
\sqrt{\det[\cos \theta_{IJ}]} = \left( \prod_{i=1}^{3} \sin \theta_i \right) |(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3|
\]

(18)

This is done by considering the 4 \( \times \) 4 determinant \( \det[\cos \theta_{IJ}] \), which can be written as a 3 \( \times \) 3 determinant \( \det(\cos \theta_{ij} - \cos \theta_i \cos \theta_j) \) by appropriate line substraction. Moreover \( \cos \theta_{ij} - \cos \theta_i \cos \theta_j = \sin \theta_i \sin \theta_j \cos \tilde{\theta}_{ij} \) so \( \det[\cos \theta_{IJ}] = \left( \prod_{i=1}^{3} \sin^2 \theta_i \right) \det(\vec{n} \cdot \vec{n}_3) \). If we call \( n_i^o \) the component of \( \vec{n} \) in any given basis we see that \( \det(\vec{n} \cdot \vec{n}_3) = \det(n_i^o)^2 = |(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3|^2 \). Thus we have relation (18). Overall this leads to the measure

\[
d\mu = \frac{2}{\pi^4} \prod_{i<j} \sin \theta_{IJ} d\theta_{IJ} \sqrt{\det[\cos \theta_{IJ}]}
\]

(19)

Domain: The domain of integration \( \mathcal{D}_\pi \) arise naturally since, for instance, the relation (8) imply that

\[
\cos(\theta_i + \theta_j) \leq \cos \theta_{ij} \leq \cos(\theta_i - \theta_j),
\]

(20)
which is equivalent to the condition

$$|\theta_i - \theta_j| \leq \theta_{ij} \leq \pi - |\pi - \theta_i - \theta_j|.$$  \hspace{1cm} (21)

Integrand: Finally, we get the theorem using the character formula

$$\chi_l(g(\theta)) = \frac{\sin(l + 1)\theta}{\sin \theta}.$$ \hspace{1cm} (22)

This completes the proof of the theorem.

![FIG. 1: Dual angles in a spherical tetrahedron. The dual \(\tilde{\theta}_{23}\) of \(\tilde{\theta}_{23}\) in the dashed spherical triangle appears as the dihedral angle of the edge 01.]

Remarks:
1) It is interesting to note, by a direct computation, that in the gauge where \(g_0 = 1\) we have

$$\sqrt{\det[\cos \theta_{IJ}]} = \frac{1}{4}|\tr([g_1, g_2] g_3)|.$$ \hspace{1cm} (23)

Also, from relation (18) it is clear that if \(\det[\cos \theta_{IJ}] = 0\) then \(\theta_i = 0, \pi\) or \(|(\vec{n}_1 \wedge \vec{n}_2) \cdot \vec{n}_3| = 0\) in which case the tetrahedra is degenerate. The reverse proposal is trivial.

2) The integration domain is invariant under the transformation

$$\theta_{IJ} \rightarrow \sigma_I \sigma_J \theta_{IJ} + \frac{1 - \sigma_I \sigma_J}{2} \pi,$$ \hspace{1cm} (24)

with \(\sigma_I = \pm 1\). Under this transformation the Graham matrix becomes \(\sigma_I \sigma_J \cos \theta_{IJ}\), and its determinant is invariant, while the product of sinus produces a factor

$$\prod_{I < J}(\sigma_I \sigma_J)^{l_{IJ}} = \prod_I \sigma_I \sum_{j \neq I} l_{IJ}.$$ \hspace{1cm} (25)
The exponent is always a even integer due to the admissibility condition that the label around a vertex must sum into an even integer. Using this extra discrete symmetry we can restrict furthermore the integration domain to the domain $\tilde{D}_\pi$ which is the subset of $D_\pi$ such that $\theta_i \leq \pi/2$.

3) In this remark we clarify the fact that we will consider the variables $\theta_{IJ}$ both as lengths and dihedral angles of spherical tetrahedra. There is a duality between points and 2-spheres in $S^3$, namely if $g \in SU(2)$ we can define the dual 2-sphere $S_g = \{ x \in SU(2), \tr(xg) = 0 \}$, $S_g$ is the sphere of radius $\pi/2$ centered at $g$. Reciprocally, such a sphere determines 2 points; its center $g$ and its antipodal center $-g$. So, given a set of angles $\theta_{IJ} \in D_\pi$ we can construct -modulo gauge invariance- 4 group elements $g_I \in SU(2)$ such that $\cos \theta_{IJ} = (1/2)\tr(g_IG_J^{-1})$.

We can consider this group elements as being the vertices of a spherical tetrahedron, denoted $T(\theta_{IJ})$, so that $\theta_{IJ}$ is the length of the edge $(IJ)$. We can also consider these group elements as defining 4 spheres forming the 4 faces of a dual tetrahedron denoted $T^*(\theta_{IJ})$. In this case $\theta_{IJ}$ is the dihedral angle of the edge dual to $(IJ)$ of the tetrahedron $T^*(\theta_{IJ})$. But given a tetrahedron $T(\theta_{IJ})$ we can also compute the dihedral angles $\tilde{\theta}_{IJ}$ of this tetrahedron as in 9, and construct the tetrahedron $T(\tilde{\theta}_{IJ})$ : it turns out that we do not get a new tetrahedron since $T(\tilde{\theta}_{IJ}) = T^*(\theta_{IJ})$. It can be checked that the Graham matrix of the dual tetrahedron $T^*(\theta_{IJ}) = T(\tilde{\theta}_{IJ})$ defined by $\tilde{G}_{IJ} = \cos \tilde{\theta}_{IJ}$ is related as follows to the Graham matrix $G_{IJ} = \cos \theta_{IJ}$. Let $\theta_{IJ}$ be in the interior of $D_\pi$ then $G$ is invertible, $(G^{-1})_{II} > 0$ and

$$
\tilde{G}_{IJ} = \frac{(G^{-1})_{IJ}}{\sqrt{(G^{-1})_{II}(G^{-1})_{JJ}}}. \tag{26}
$$

Finally if $\theta_{IJ}$ is in the boundary of $D_\pi$, $T(\theta)$ becomes a degenerate spherical tetrahedron and $T^*(\theta)$ is a flat tetrahedron.

### B. Analysis of the integral

Our aim is now to study the large $N$ asymptotic of the integral expressing the 6$j$-symbol in terms of invariant variables

$$
I(Nl_{IJ}) = \frac{2}{\pi^4} \int_{D_\pi} \prod_{I<J} \sin((Nl_{IJ} + 1)\theta_{IJ}) \det[\cos \theta_{IJ}] \prod_{I<J} d\theta_{IJ} \tag{27}
$$

The form of the integrand – an oscillatory function with an argument proportional to $N$ times a function independent of $N$– suggest that the asymptotic behavior can be studied by stationary phase methods. However, this is not so simple since the integrand is singular on the boundary of the integration domain. We therefore have to make the asymptotic analysis by carefully taking into account this singular behavior. For that purpose, we will split the integration domain into two different parts, which will be analyzed by adapted methods. We split the integration domain $D_\pi$ into

$$
D_\pi = D_{\pi,\epsilon}^< \cup D_{\pi,\epsilon}^> \quad \text{with} \quad D_{\pi,\epsilon}^> = [\epsilon, \pi - \epsilon]^6 \cap D_\pi \quad D_{\pi,\epsilon}^< = D_\pi - D_{\pi,\epsilon}^> \tag{28}
$$

with $\epsilon > 0$ is sufficiently small. Thus the integral $I(l_{IJ})$ splits into $I^>(l_{IJ})$ and $I^<(l_{IJ})$ respectively given by the integrations on $D_{\pi,\epsilon}^>$ and $D_{\pi,\epsilon}^<$. Our main theorem summarizing the results obtained for these asymptotic analysis is the following
Theorem 2 Let $F^*(l_{IJ})$ be the flat tetrahedron which is such that the length of the edge $(IJ)$ is given by $l_{KL}$ (with $I, J, K, L$ all distinct). Let $\Theta_{KL}$ be the angles between outward normals at edge $(IJ)$, and denote by $V^*(l_{IJ})$ its volume. Suppose that $V^*(l) \neq 0$ and that $0 < \epsilon < \min(\Theta_{IJ}, \pi - \Theta_{IJ})$. Then the first terms in the asymptotic expansion of both integrals are

$$I^>(Nl_{IJ}) \sim -\frac{\sin\left(\sum_{I<J}(Nl_{IJ} + 1)\Theta_{IJ}\right)}{3\pi N^2 V^*(l_{IJ})},$$

(29)

and

$$I^<(Nl_{IJ}) \sim \frac{1}{3\pi N^2 V^*(l_{IJ})}.$$  

(30)

Overall leads to the asymptotic for the $6j$-symbol

$$I(Nl_{IJ}) \sim \frac{2}{N^3 3\pi V^*(l_{IJ})} \cos^2 \left(\sum_{I<J} \frac{(Nl_{IJ} + 1)}{2} \Theta_{IJ} + \frac{\pi}{4}\right).$$

(31)

This result is, of course, consistent with the results of Ponzano-Regge and Roberts. The methods are however very different, and to our taste simpler in the sense that they can be extrapolate to the Lorentzian case as we will see in section III.

C. Asymptotic of $I^>(Nl_{IJ})$

Method : The result (29) is proved by the stationary phase method. We recall first the main steps of this method on a one dimensional example. First we will write the integral under a form which is appropriate for the stationary phase, namely something like

$$I_N = \int_a^b dx \, \phi(x)e^{iNf(x)},$$

(32)

with $f$ an analytic function. This is done by proposition 1. The asymptotic expansion of such an integral is given by contributions around the stationary points of the phase, i.e. points in the complex plane such that $f'(x_0) = 0$. In our case such points are given by proposition 2. The contribution of such a point is then obtained by expanding the phase around it at second order and extend the integration to infinity

$$I_N \sim \sum_{x_0} \int_{-\infty}^{+\infty} d(\delta x) \, \phi(x_0)e^{iN(f(x_0) + \frac{1}{2}f''(x_0)(\delta x)^2)}.$$  

(33)

$\phi(x)$ is the part of the integrand independant of $N$ and is evaluated on the stationnary points $x_0$. The constant term can be factored out of the integral and the integration is now a gaussian which can be performed

$$I_N \sim \frac{1}{\sqrt{N}} \sum_{x_0} \phi(x_0)e^{i\pi \text{sgn}(f''(x_0)) \sqrt{\frac{2\pi}{|f''(x_0)|}} e^{iNf(x_0)}}.$$  

(34)

Note that if $x_0$ lies on the boundary of the domain, its contribution has to be half counted. For the n-dimensional gaussian integration (see eq. 35), we need to compute the determinant and the signature of the quadratic form obtained by the expansion. All of this is done in
Proposition 3. Recall that in the $n$ dimensional case, the gaussian integration is performed using the fact that for a real $n \times n$ symmetric invertible matrix $A$ with signature $\sigma(A)$ we have

$$\int_{\mathbb{R}^n} [dX] \exp \left[ i \left( \sum_{i,j} X_i A_{ij} X_j \right) \right] = e^{i\sigma(A)\frac{\pi}{4}} \sqrt{\frac{\pi^n}{|\text{det}A|}}.$$  \hspace{1cm} \hspace{1cm} (35)

**Proposition 1**  The integral $I^>(Nl_{IJ})$ can be written as

$$I^>(Nl_{IJ}) = \frac{2N^2}{(2\pi)^6} \sum_{\{\epsilon_{IJ} = \pm\}} \left( \prod_{I<J} \epsilon_{IJ} \right) I_\epsilon$$  \hspace{1cm} \hspace{1cm} (36)

with

$$I_\epsilon = \int_{\mathbb{R}^4} \int_{D^\prime,\epsilon} e^{i\sum_{I<J} \epsilon_{IJ} \theta_{IJ}} e^{iN(\sum_{I<J} \epsilon_{IJ} l_{IJ} \theta_{IJ} + \sum_{I,J} X_I \cos \theta_{IJ} X_J)} d\theta dX$$  \hspace{1cm} \hspace{1cm} (37)

**Proof :** —

We first split the sinuses into exponentials, which leads to integrals which are still convergent. This is shown by notice that due to the computation of the measure, the integral

$$\int_{D^\prime,\epsilon} [d\theta_{IJ}] \frac{1}{\sqrt{\text{det}[\cos \theta_{IJ}]}}$$  \hspace{1cm} \hspace{1cm} (38)

is just the integral

$$\int_{D^\prime,\epsilon} \frac{dh_1 dh_2 dh_3}{\prod_i \sin \theta_i (\prod_{i<j} \sin \theta_{ij})}$$  \hspace{1cm} \hspace{1cm} (39)

As the domain is a compact one, and exclude the points $\theta_i, \theta_{ij} = 0, \pi$, this integral is convergent. Now we use the equation (35), to rewrite the denominator as arising from such an integration (the signature of the matrix being 4). One finally rescales the $X_I$ by $\sqrt{N}$.

We consider now each of the integrals $I_\epsilon$ and apply the stationary phase method to compute their dominant asymptotic contribution. The following proposition gives the stationary points of the phase for these integrals. They are expressed in terms of the geometrical elements of the tetrahedron $FT^*(l_{IJ})$. In the following we denote $V^*$ its volume, $A_I$ its areas and $\Theta_{IJ}$ the angles between its outward normals.

**Proposition 2**  The stationarity equations of the integral $I_\epsilon$ are given by

$$\sum_J \cos \theta_{IJ} X_J = 0,$$  \hspace{1cm} \hspace{1cm} (40)

$$2X_I X_J \sin \theta_{IJ} = \epsilon_{IJ} l_{IJ}.$$  \hspace{1cm} \hspace{1cm} (41)

These equations admits solutions only when $\epsilon_{IJ} = (-1)^\eta \sigma_I \sigma_J$ with $\sigma_I = \pm 1$ and $\eta = 0, 1$. For each of the 16 admissible choices of $\epsilon_{IJ}$ there are 2 solutions $s = \pm 1$ given by

$$\theta_{IJ} = \sigma_I \sigma_J \Theta_{IJ} + \frac{1 - \sigma_I \sigma_J}{2} \pi, \hspace{1cm} (42)$$

$$X_I = \frac{si^\eta \sigma_I A_I}{\sqrt{3V^*}}. \hspace{1cm} (43)$$
Proof: —

Localisation of the solutions in the complex plane: We begin by looking for general solutions in the complex plane. Indeed in the method of stationary phase, one should look for stationary points in the whole complex plane, as the contours of integration can be analytically deformed in C. We prove that a general solution \((\theta_{IJ}, X_I)\) of the stationarity equations is in fact such that \(\theta_{IJ} \in \mathbb{R}\) and \(X_I \in \mathbb{R}\) or \(i\mathbb{R}\). We start from the following lemma

**Lemma:** Consider \(\alpha_{IJ} \in 4 \times 4\) symmetric matrix such that \(\sum_J \alpha_{IJ} = 0\). Consider the \(5 \times 5\) matrix \(K(\alpha) = K_{IL} = 4(\alpha_{II} \alpha_{JJ} - (\alpha_{IJ})^2), K_{I5} = 1, K_{55} = 0\), then we have

\[
(C_{IJ})^2 = 8(\alpha_{IJ})^2 \det K
\]

where \(C_{IJ}\) denote the \((IJ)\)-cofactor of \(K\).

We apply this lemma with \(\alpha_{IJ} = X_I X_J \cos \theta_{IJ}\) (the hypothesis is satisfied due to the first stationary equation). In that case \(K\) is just the Cayley matrix \(C_{IJ} = l_{2KL}(\text{due to the second stationary equation})\), whose cofactors \(\Delta_{IJ}\) are real. As \(\det C > 0\), eq.45 says that \(\alpha_{IJ} = X_I X_J \cos \theta_{IJ}\) is real. \(X_I X_J \sin \theta_{IJ}\) is also real from the second equation. Summing the square of these two results,

\[
0 < (X_I X_J \sin \theta_{IJ})^2 + (X_I X_J \cos \theta_{IJ})^2 = (X_I X_J)^2
\]

we can conclude that \(X_I X_J\) is real, and thus that \(\theta_{IJ}\) is real. Lets denote \(X_I = e^{\delta_I} |X_I|\). If we look at the phase of the second equation (41) one gets that

\[
e^{i\delta_I + i\delta_J} = \pm \epsilon_{IJ}
\]

This imply that

\[
e^{4i\delta_I} = \left(\frac{\epsilon_{IJ} \epsilon_{IK}}{\epsilon_{JK}}\right)^2 = 1.
\]

Therefore one has \(e^{i\delta_I} = i^\eta \sigma_I\) with \(\sigma_I = \pm, \eta = 0, 1\) and \(\epsilon_{IJ} = (-1)^\eta \sigma_I \sigma_J\). The conclusions of our study of the localization of the solutions are that the only integrals with stationary points are the 16 configurations such that \(\epsilon_{IJ} = (-1)^\eta \sigma_I \sigma_J\) and that the solutions \(\theta_{IJ}\) are real, while \(X_I \in i^\eta \sigma_I \mathbb{R}^+\).

Explicit solutions: Now we compute the explicit form of the solutions. Consider a solution for a admissible configuration \(\epsilon_{IJ} = (-1)^\eta \sigma_I \sigma_J\). The solutions of this configuration and of the configuration all \(\epsilon_{IJ} = +1\) are mapped to each others by

\[
\theta'_{IJ} = \sigma_I \sigma_J \theta_{IJ} + \frac{(1 - \sigma_I \sigma_J) \pi}{2}
X'_{I} = i^\eta \sigma_I X_I
\]

Thus we only need to study the solution for the configuration \(\epsilon_{IJ} = 1\), the solutions for the others configuration will be recovered by this map.

The matrix \(G_{IJ} = \cos \theta_{IJ}\) can be written as arising from the scalar product of four unit vectors \(\vec{n}_I\) of \(\mathbb{R}^4\), i.e \(G_{IJ} = \vec{n}_I \cdot \vec{n}_J\). The first relation (40) imply that \(\det(G) = 0\) so the \(\vec{n}_I\) span a vector space of dimension at most three. It is easy to see that under the hypothesis
of the theorem it is exactly of dimension three. Let denote \( \vec{X}_I = X_I \bar{n}_I \), the second relation imply that \(|\vec{X}_I \wedge \vec{X}_J| = \frac{1}{2} l_{IJ} \). The LHS represent the area of the triangle \((\vec{X}_I, \vec{X}_J)\). If the vector space was of dimension two or less then these area, hence \(l_{IJ}\), would be related by triangular equalities. This is not allowed because the tetrahedron \(FT^*(l_{IJ})\) is supposed to be non-degenerate. Therefore the matrix \(G\) as exactly one null eigenvector.

Thus [11], this matrix \(G\) is the Graham matrix of a flat non-degenerate tetrahedron, whose angles between outward normals are the \(\theta_{IJ}\) and areas are proportional to the unique eigenvector, the \(X_I\) in our case. This tetrahedron is defined up to scale. To fix the ideas, let us consider such a tetrahedron of volume 1, let us call \(\lambda_{IJ}\) its lengths, and \(a_I\) its areas, we thus have \(X_I = \mu a_I\). It is well known (see appendix A) that these geometrical elements satisfy

\[
\sum_J \cos \theta_{IJ} a_J = 0, \tag{50}
\]
\[
a_I a_J \sin \theta_{IJ} = (3/2) \lambda_{IJ}. \tag{51}
\]

By considering this together with the second equation

\[
l_{IJ} = 2 \mu^2 a_I a_J \sin \theta_{IJ} = 3 \mu^2 \lambda_{IJ}, \tag{52}
\]

we see that this tetrahedron is in fact homotethic to the tetrahedron \(FT^*(l_{IJ})\). The \(\theta_{IJ}\) solutions are thus \(\Theta_{IJ}\) the angles between outward normals of \(FT^*(l_{IJ})\). As its geometrical elements satisfy

\[
A_I A_J \sin \Theta_{IJ} = \frac{3}{2} l_{IJ} V^* \tag{53}
\]

we have

\[
X_I = s \frac{1}{\sqrt{3V^*}} A_I, \quad s = \pm 1 \tag{54}
\]

Finally for this configuration \(\epsilon_{IJ} = +1\), there are two solutions. According to the map (49), for each admissible configuration this gives the two solutions given in the proposition.

\[\blacksquare\]

Our task is now to expand the phase around the solutions at the second order, and compute the determinant and the signature of the corresponding quadratic form.

**Proposition 3** For each of the 16 admissible configurations \(\epsilon_{IJ} = (-1)^n \sigma_I \sigma_J\) the expansion of the phase

\[
\sum_{I<J} \epsilon_{IJ} l_{IJ} \theta_{IJ} + \sum_{I,J} X_I \cos \theta_{IJ} X_J \tag{55}
\]

around any of the two \(s = \pm 1\) solutions gives

\[
(-1)^n \sum_{I<J} l_{IJ} \Theta_{IJ} + (-1)^n Q(\delta \theta_{IJ}, \delta X_I) \tag{56}
\]

where \(Q\) is a quadratic form. Its signature is \(\sigma(Q) = 2\) and its determinant is \(\det Q = (\frac{3}{2} V^*)^2\).
Proof: —
Consider first the expansion for $\epsilon_{IJ} = +1$, the constant term is easily seen to be $\sum_{I<J} l_{IJ} \Theta_{IJ}$.

The computation of expansion with fluctuations $(\delta \theta_{IJ}, \delta X_I)$ at the second order leaves us with a quadratic form, which can be expressed as such

$$Q(\delta \beta_{IJ}, \delta X_I) = - \sum_{I<J} \frac{1}{3V^*} A_I A_J \cos \Theta_{IJ} (\delta \beta_{IJ})^2$$

$$+ \sum_I \left[ 2 + \sum_{K \neq I} \left( \frac{A_K}{A_J} \frac{1}{\cos \Theta_{IK}} \right) \right] (\delta X_I)^2$$

$$+ \sum_{I \neq J} \frac{1}{\cos \Theta_{IJ}} \delta X_I \delta X_J$$

(57)

where the $\delta \beta_{IJ}$ are a shift redefinition of the $\delta \theta_{IJ}$.

Consider now the expansion of the phase of an integral $I_\epsilon$ around the corresponding solution. One has just to carefully add the $\sigma_I$ and $\eta$ into the previous results. The constant term becomes

$$\sum_{I<J} \epsilon_{IJ} l_{IJ} (\sigma_I \sigma_J \theta_{IJ} + \frac{(1 - \sigma_I \sigma_J)}{2} \pi) = (-1)^\eta \sum_{I<J} l_{IJ} \theta_{IJ} + \pi (-1)^\eta \sum_{I<J} l_{IJ} \sigma_I \sigma_J \frac{1}{2}$$

(58)

The second term is always 0 mod. $2\pi$, due to the condition that the labels around a face must sum into an even integer. The quadratic form is unchanged, except one has just to redefine $\delta X_I \rightarrow \sigma_I \delta X_I$ to get the same expression.

Our task is now to compute the determinant and the signature of the quadratic form $Q$. This quadratic form is given by a diagonal part (involving the $\delta \beta_{IJ}$), whose determinant is

$$\det Q = \frac{\prod_{I<J} A_I A_J \cos \Theta_{IJ}}{(3V^*)^6}$$

(59)

and a non-diagonal part which is the $4 \times 4$ quadratic form involving only the $\delta X_I$. This non-diagonal part can be computed in terms of the $A_I$ and $\cos \theta_{IJ}$. Using the formula (see the appendix A)

$$A_I A_J \cos \theta_{IJ} = -\frac{1}{16} \Delta_{IJ}$$

(60)

$$A_I^2 = -\frac{1}{16} \Delta_{II}$$

(61)

($\Delta_{IJ}$ denotes the $(I, J)$ cofactor of the Cayley matrix), the total determinant can be rewritten as

$$\det Q = \frac{1}{(3V^*)^6} P(\Delta_{IJ})$$

(62)

with $P$ an homogeneous polynomial of order 6. The $\Delta_{IJ}$ can now be expressed in terms of the $l_{IJ}$, the same can be done for the volume $V^*$ and symbolic computation (Maple) allows to check that

$$P(\Delta_{IJ}) = \frac{(3V^*)^8}{4}$$

(63)
Overall leads to
\[ \text{det} Q = \left( \frac{3}{2} V^* \right)^2 \] (64)

The proposition concerning the signature of the quadratic form is verified only numerically. We have computed numerically the signature of the quadratic form \( Q \) for many randomly chosen tetrahedra. In all cases we obtain that \( \sigma(Q) = 2 \).

We can now put together the results of the previous propositions to prove the asymptotics of the integrals \( I_\epsilon \), then of the total integral \( I^> (NL_{IJ}) \). Consider an integral \( I_\epsilon \), if the \( \epsilon_{IJ} \) can not be written \( \epsilon_{IJ} = (-1)^n \sigma_I \sigma_J \), the integral has no stationary points and its asymptotics is a \( O(N^{-1/2}) \). If \( \epsilon_{IJ} = (-1)^n \sigma_I \sigma_J \), the oscillatory phase has two stationary points given by proposition 2. Recall that the integrand of \( I_\epsilon \) contains a factor independent of \( N \) (see eq. 37) which has to be evaluated on the stationary points. This evaluation is

\[
\exp \left[ i \sum_{I<J} \epsilon_{IJ} \left( \sigma_I \sigma_J \Theta_{IJ} + (1 - \sigma_I \sigma_J) \frac{\pi}{2} \right) \right] = e^{i(-1)^n \sum_{I<J} \epsilon_{IJ} \left( \prod_{I<J} \epsilon_{IJ} \right)} (65)
\]

Recall that these points are the dihedral angles of a flat tetrahedron and are on the boundary of the domain according to the remark made after theorem 1. Their contributions have to be half-counted. Using the results on the determinant and signature of \( Q \), we obtain that the asymptotics of \( I_\epsilon \) is

\[
I_\epsilon(NL_{IJ}) \sim \frac{\pi^5}{N^5} \left( \prod_{I<J} \epsilon_{IJ} \right) \frac{2i(-1)^n}{3V^*} e^{i(-1)^n \sum_{I<J} (NL_{IJ} + 1) \Theta_{IJ}} (66)
\]

Now there are 8 configurations of the \( \epsilon_{IJ} \) with \( (-1)^n = +1 \) and 8 with \( (-1)^n = -1 \). Summing this according to the expansion (eq. 36), the total contribution is

\[
I^> (NL_{IJ}) \sim -\frac{1}{N^3} \frac{\sin(\sum_{I<J} (NL_{IJ} + 1) \Theta_{IJ})}{3\pi V^*} (67)
\]

which proves the first part (29) of the theorem 2.

**D. Asymptotics of \( I^< (NL_{IJ}) \)**

**Method:** Before going on with the proof of the second part (30) of the theorem let us illustrate the problematic on a one dimensional example. Consider the simple case of a one dimensional integral :

\[
\int_a^b dx \ f(x) e^{iNx} \] (68)

If \( f \) is a regular function on \([a; b]\), this integral has a \( O(N^{-1}) \) asymptotic which has its origin in the contributions of the points \( a, b \) of the lying on the boundary. Now lets consider a function which is regular in \([a; b]\) but posses a singularity on the boundary, for instance

\[
\int_a^b dx \ \frac{f(x)}{(x - a)^\alpha} e^{iNx} (69)
\]
with \( f \) regular and non-vanishing at \( a \), one can show that the asymptotic is an \( O(1/N^{1-\alpha}) \). This suggests that the leading asymptotic behavior of an integral with singular points come from the contributions around the most singular points. We expect that in general the dominant contributions will come from singular points of the integrant. It is easy to show that the zeros of highest order of the denominator \( \det[\cos \theta_{IJ}] \), which will dominate the large \( N \) behavior of the integral \( I^<(Nl_{IJ}) \), is given by the configuration where all \( \theta_{IJ} \) are zero. The proof amounts to write the integral as a one dimensional integral which realize an expansion around this singular configuration. We first express the integral as a one-dimensional oscillatory integral over an auxiliary variable \( S \). This is done in proposition 4. Then we compute the large \( N \) asymptotics of this integral in proposition 5. The result is thus expressed as an integral over Euclidian tetrahedra, involving the inverse of the volume \( V \). This integral is exactly computed due to a remarkable self-duality property given by theorem 3.

First it is useful to note that if \( \epsilon \) is small enough (i.e. \( \epsilon < \pi/3 \)) the domain \( D^<_{\pi,\epsilon} \) splits into eight disconnected components which can be mapped into each other using the discrete symmetry (eq. 24). Using this property we have

\[
I^<(Nl_{IJ}) = -\left(\frac{2}{\pi}\right)^4 \int_{\tilde{D}_\epsilon} \frac{\prod_{I<J} \sin((Nl_{IJ}+1)\theta_{IJ})}{\sqrt{\det[\cos \theta_{IJ}]}} \prod_{I<J} d\theta_{IJ}, \tag{70}
\]

where \( \tilde{D}_\epsilon \) is the subset of \( [0;\epsilon]^6 \) of \( \theta_{IJ} \) satisfying the relations :

\[
\theta_{IJ} \leq \theta_{IK} + \theta_{JK} \tag{71}
\]

for any triple \( (I, J, K) \) of distinct elements. We can now rewrite the integral.

**Proposition 4**

\[
I^<(Nl_{IJ}) = \int_{-\infty}^{\infty} e^{iNS^2} F_l(S) dS \tag{72}
\]

where \( F_l(S) \) is a function of \( S \) depending on the lengths \( l_{IJ} \) and given by

\[
F_l(S) = -\frac{1}{4\pi^4} v.p \int_{\tilde{D}_\epsilon} \sum_{I,J=\pm 1} \delta(S - \sum_{I,J} \epsilon_{IJ} l_{IJ} \theta_{IJ}) \frac{\prod_{I<J} \epsilon_{IJ} e^{i\epsilon_{IJ} \theta_{IJ}}}{S^2 \sqrt{\det[\cos \theta_{IJ}]}} \prod_{I<J} d\theta_{IJ}, \tag{73}
\]

where \( v.p \) means that we take the principal value of the integral i.e \( v.p \int d\theta = \lim_{\alpha \to 0} \int_{\theta > \alpha} d\theta \) and \( \delta(x) \) is the Dirac functional.

The rewriting of the integral is immediate, the key point here is that the function \( F_l(S) \) is well defined since the functionals \( S(\theta_{IJ}) = \sum_{I,J} \epsilon_{IJ} l_{IJ} \theta_{IJ} \) don not possess any stationary points in the domain \( \tilde{D}_\epsilon \) as was shown in the section II C.

The asymptotic expansion of the resulting one-dimensional integral over \( S \) can now be computed.

**Proposition 5**

\[
I^<(Nl_{IJ}) \sim \left(\frac{2}{\pi}\right)^3 \frac{1}{N^3} \int_{D_\infty} du_{IJ} \frac{\prod_{I<J} \sin(l_{IJ}u_{IJ})}{3\pi V(u_{IJ})}, \tag{74}
\]

where \( V(u_{IJ}) \) is the volume of the flat tetrahedron which is such that the length of the edge \( (IJ) \) is given by \( u_{IJ} \). The domain \( D_\infty \) is the set of all Euclidian tetrahedron.
Proof: —

It is well known that in the case of a one dimensional integral the asymptotic is:

\[ I^\langle (Nl_{IJ}) = \int_{-\infty}^{+\infty} dSe^{iNS}S^2F_i(S) \sim \int_{-\infty}^{+\infty} dSe^{iNS}S^2F_i(0). \] (75)

Thus what we need is an evaluation at \( S = 0 \) of the function \( F_i(S) \). This function indeed regular at this point, as it can be seen by changing \( \theta_{IJ} = S u_{IJ} \). We get

\[ F_i(S) = -\frac{1}{4\pi^2} S^3 v.p \int_{\tilde{D}_s} \sum_{i,j=\pm 1} \delta(1 - \sum_{I,J} \epsilon_{IJ} u_{IJ}) \prod_{I<J} \frac{\epsilon_{IJ} e^{iS_{IJ} u_{IJ}}}{\sqrt{[\det \cos(S_{IJ})]} \prod_{I<J} du_{IJ}}. \] (76)

As \( S \to 0 \), the domain becomes the domain of all Euclidian tetrahedra. The denominator can now be expanded as \( S \to 0 \). The proposition follows from this expansion

\[ \sqrt{[\det \cos(S_{IJ})]} \sim_{S \to 0} S^3 V^2 \] (77)

where \( V(u_{IJ}) \) denotes the volume of tetrahedron, whose length \((IJ) \) is \( u_{IJ} \). This lemma can be proved by direct expansion, but if follows from this observation: Consider the determinant \( \Delta = \det \cos \theta_{IJ} \) (with \( \theta_0 = \theta_i \)). By substracting row 4 (resp. line 4) to the three others, one obtains the determinant

\[ \Delta = \det(\cos \theta_{ij} - \cos \theta_i \cos \theta_j) \] (78)

which is rewritten, following relation (eq. 11),

\[ \Delta = \left( \prod_i \sin^2 \theta_i \right) \det(\vec{n}_i \cdot \vec{n}_j) \] (79)

The limit \( \theta_i \to 0 \) of this expression leads to the expression \( \det(\cos \theta_{ij} \vec{n}_i \cdot \vec{n}_j) \) which is the square of the volume of the flat tetrahedron defined by vectors \( \theta_1 \vec{n}_1, \theta_2 \vec{n}_2, \theta_3 \vec{n}_3 \) (see eq.A9). Thus we have the property

\[ \det[\cos \theta_{IJ}] \sim_{\theta \to 0} 6^2 V^2 \] (80)

with \( V \) the volume of the tetrahedron whose lengths are \((\theta_i, \theta_{ij})\).

We have express the asymptotic of the integral \( I^\langle (Nl_{IJ}) \) as an integral (74). This integral can be in fact exactly computed, showing a self-duality property of the inverse of the volume of the tetrahedron.

**Theorem 3** Let \( V(u_{IJ}) \) (resp. \( V^*(l_{IJ}) \)) be the volume of the flat tetrahedron which is such that the length of the edge \((IJ) \) is given by \( u_{IJ} \) (resp. \( l_{KL} \)) The inverse volume satisfy the remarkable self duality property

\[ \left( \frac{2}{\pi} \right)^3 \int_{D_\infty} du_{IJ} \frac{\prod_{I<J} \sin(l_{IJ} u_{IJ})}{3\pi V^*(u_{IJ})} = \frac{1}{3\pi V^*(l_{IJ})}. \] (81)

The proof of this theorem make the use of the following proposition concerning the measures on the spaces of Euclidian tetrahedra.
Proposition 6 Let \( \vec{u}_i \) be 3 vectors of \( \mathbb{R}^3 \), and lets denote \( u_{0i} = |\vec{u}_i|, u_{ij} = |\vec{u}_i - \vec{u}_j| \) then
\[
\prod_i \frac{d^3 \vec{u}_i}{2\pi} = \frac{\prod_{I < J} u_{IJ}}{3\pi V(u_{IJ})} \tag{82}
\]
where \( V(u_{IJ}) \) is the volume of the tetrahedron \( (\vec{u}_1, \vec{u}_2, \vec{u}_3) \) whose edges length are \( u_{IJ} \).

Proof : —

The measure on \( \mathbb{R}^3 \) is given by \( d^3 \vec{u}_i = 4\pi u_i^2 du_i d^2 \vec{n}_i \), where \( u_i = |\vec{u}_i|, \vec{n}_i \in S^2 \) and \( d^2 \vec{n} \) is the normalized measure on the 2-sphere. So the measure \( d\mu = \prod_i \frac{d^3 \vec{u}_i}{2\pi} \) becomes
\[
d\mu = 2^3 \left( \prod_{i=1}^3 u_i^2 du_i \right) d^2 \vec{n}_i. \tag{83}
\]

From there we proceed exactly as in the proof of theorem 1, using the fact that
\[
6V(u_{IJ}) = (\prod_i u_i) \det(\vec{n}_1 \wedge \vec{n}_2 \cdot \vec{n}_3). \tag{84}
\]

We can now prove the theorem. This proposition implies that the integral eq. 74 has a form similar to eq. 1
\[
I^<(l_{IJ}) \sim \left( \frac{2}{\pi} \right)^3 \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \prod_i d^3 \vec{u}_i \prod_i \frac{\sin(l_i |\vec{u}_i|)}{|\vec{u}_i|} \prod_{i<j} \frac{\sin(l_{ij} |\vec{u}_i - \vec{u}_j|)}{|\vec{u}_i - \vec{u}_j|}, \tag{85}
\]
where \( G \) is replaced by \( \mathbb{R}^3 \) and the character \( \chi_i(g) \) is replaced by \( K_i(\vec{u}) = \sin(|\vec{u}|)/|\vec{u}| \). This fact was already conjectured in the paper of Baez et al [8] where the kernel \( K_i(\vec{u}) \) has been interpreted in term of spin networks of the Euclidean group. Changing the lengths to their dual \( l_i = L_{jk}, l_{ij} = L_k \) and using the Kirillov formula
\[
\frac{\sin L_i |x|}{|x|} = L \int_{S^2} e^{iL \vec{x} \cdot \vec{n}} d^2 \vec{n} \tag{86}
\]
one obtains
\[
I^<(l_{IJ}) = \frac{1}{\pi^6} (\prod_{I < J} L_{IJ}) \int_{\mathbb{R}^3} \prod_i d^3 \vec{u}_i \int_{S^2} \left( \prod_{I < J} d^2 n_{IJ} \right) \prod_k \exp \left\{ i\vec{u}_k \cdot (L_{ij} \vec{n}_{ij} + L_j \vec{n}_j - L_i \vec{n}_i) \right\} \tag{87}
\]
The integrals over the sphere can be rewritten as integrals over \( \mathbb{R}^3 \):
\[
\int_{S^2} d^2 \vec{n} = \frac{1}{4\pi L^2} \int_{\mathbb{R}^3} d^3 \vec{X} \delta(|\vec{X}|-L), \tag{88}
\]
while the integrals over the \( \vec{u}_i \) can be performed \( \int e^{i\vec{x} \cdot \vec{u}_i} d^3 \vec{u} = (2\pi)^3 \delta^3(\vec{x}), \) leading to others \( \delta \)-functions.
\[
\frac{1}{\pi^6} (2\pi)^9 \int_{(\mathbb{R}^3)^6} \prod_i d^3 X_i \prod_{i < j} d^3 X_{ij} \prod_{i < j} \delta^3 \left( \vec{X}_{ij} - (\vec{X}_i - \vec{X}_j) \right) \prod_i \frac{1}{L_i} \delta(|\vec{X}_i|-L_i) \prod_{i < j} \frac{1}{L_{ij}} \delta(|\vec{X}_{ij}|-L_{ij}) \tag{89}
\]
The integrations on $X_{ij}$ can be performed, due to the $\delta^{(3)}$ functions. It remains

$$I^< (l_{IJ}) = \frac{1}{(2\pi)^3} \int_{(\mathbb{R}^3)^3} d\vec{X}_i \prod_i \frac{1}{L_i} \delta(|\vec{X}_i| - L_i) \prod_{i<j} \frac{1}{L_{ij}} \delta(|\vec{X}_i - \vec{X}_j| - L_{ij})$$

(90)

Changing into variables $X_i = |\vec{X}_i|, X_{ij} = |\vec{X}_i - \vec{X}_j|$ and using again the proposition 6 one gets finally

$$I^< (Nl_{IJ}) \sim \frac{1}{3\pi N^3 V(L_i, L_{ij})} = \frac{1}{3\pi N^3 V^*(l_{IJ})}$$

(91)

**Remark:** The self duality property (81) of the inverse volume shares strong similarity with the self duality property of the square of the quantum $6j$ symbol which has been shown by Barrett [12]. It is therefore tempting to ask whether this property has a natural interpretation in terms of quantum groups where the square root of the inverse volume would be interpreted as a quantum $6j$ symbol.

### III. LORENTZIAN 6J SYMBOL

In [2] it was shown that the partition function and transition amplitudes of Lorentzian 3d gravity can be written as a state sum model. To construct this state sum model one triangulates a 3d manifold, one gives an orientation to the edges of the triangulation and color these edges by unitary representation of SL(2, $\mathbb{R}$). The weight associated with an oriented colored tetrahedron is the corresponding $6j$ symbol of SL(2, $\mathbb{R}$). Among all the possible representations of SL(2, $\mathbb{R}$) only the positive discrete series, the negative discrete series and the principal series play a role. All these representations are unitary and infinite dimensional [13]. We will denote $T_{il^+}$, $l \in \mathbb{N}$, the positive discrete series of weight $l$, $T_{il^-}$, $l \in \mathbb{N}$, the negative discrete series of weight $l$, and $T_{\rho}$, $\rho \in \mathbb{R}^+$, the principal series of weight $\rho$. To simplify the exposition we have adopted a unifying notation, all the representations are denoted $T_\lambda$ where $\lambda = +il^+,-il^-$ or $\rho$ refers respectively to positive discrete series, negative discrete series and principal series. Similarly to the case of SU(2) one can construct the Clebsch-Gordan coefficients, and $6j$-symbols as a recombination of Clebsch-Gordan coefficients (we refer the reader to [14] for a precise definition of these objects). Due to the fact that we can color each oriented edge of the tetrahedron by either $il^+,-il^-$ or $\rho$ there are many different types of $6j$ symbols. However the square of all these $6j$ symbols can be expressed by an integral formula similar to eq.1

$$I(\lambda_{IJ}) = \left\{ \begin{array}{ccc} \lambda_{01} & \lambda_{02} & \lambda_{03} \\ \lambda_{23} & \lambda_{13} & \lambda_{12} \end{array} \right\}^2 = \int_{(SL(2,\mathbb{R}))^4} \prod_{i<j} \chi_{\lambda_{IJ}}(g_j g_i^{-1}) \prod_I dg_I$$

(92)

where $\chi_\lambda(g)$ is the character corresponding to the $\lambda$ representation (see appendix B). In the following we will suppose that the tetrahedra is ordered by the vertices order, i.e. $(IJ)$ is positively oriented if $I < J$. The triples $\lambda_{IJ}, \lambda_{IK}, \lambda_{IL}$ are supposed to satisfy some admissibility conditions described in the appendix B. The measure $\prod_I dg_I$ denotes the, fully gauge fixed, product of Haar measures. It is here necessary to fully gauge fix the symmetry $g_I \to kg_I h$ in order to avoid divergences since the group is non compact. Using techniques similar to the one we used in the previous section we can compute the gauge fixed measure.
Proposition 7

\[
\left[\prod_I dg_I\right] = \left(\prod_{I<J} d\tau_{IJ}\right) \frac{|\sinh(\tau_{IJ})|}{\sqrt{|\det[G_{IJ}(\tau)]|}}
\]  

(93)

where \(\tau_{IJ} = i\theta_{IJ}\) and \(G_{IJ} = \cos\theta_{IJ}\) if \(g_J g_I^{-1}\) is elliptic and conjugated to \(k\delta_{IJ}\) in this case \(d\tau_{IJ} = d\theta_{IJ}\); and \(\tau_{IJ} = |t_{IJ}|, G_{IJ} = \nu_{IJ} \cosh t_{IJ}\) if \(g_J g_I^{-1}\) is hyperbolic and conjugated to \(\nu_{IJ} \alpha_{IJ}\), \(\nu_{IJ} = \pm 1\) see appendix B for the notations.

These gauge fixed variables are interpreted as parametrizing the invariant geometry of an AdS tetrahedron. The technique used to deal with non compact spin network integrals were first introduced in [15]. Similarly to the case of SU(2) one can split the integral expression of the Lorentzian 6j into a sum \(I^>(\lambda_{IJ}) + I^<\lambda_{IJ})\) where \(I^>\) restrict the integration range into a domain where \(|\tau_{IJ}| > \epsilon\) and \(|i\pi - \tau_{IJ}| > \epsilon\). In the following we analyze the asymptotic behavior of \(I^>(N\lambda_{IJ})\) when \(N \to \infty\). In order to illustrate our method, we will treat in detail the case where all \(\lambda_{IJ}\) are real, corresponding to a spacelike Lorentzian tetrahedron, and the case where all \(\lambda_{IJ}\) are imaginary, corresponding to timelike Lorentzian tetrahedron. The last subsection gives a method to solve the stationarity equations in the general case.

A. Spacelike tetrahedra

The case where all representations are continuous is simpler since, first there is no restriction on the set \(\rho_{IJ}\) and second the characters \(\chi_\rho\) are supported only on hyperbolic elements. The integral reads

\[
I^>(\rho_{IJ}) = \sum_{\nu_{IJ}=\pm} \int_{t_{IJ}>\epsilon} \frac{\prod_{I<J} \cos(\rho_{IJ} t_{IJ})}{\sqrt{|\det[\nu_{IJ} \cosh t_{IJ}]|}} \prod_{I<J} dt_{IJ}.
\]

(94)

We have \(\nu_{IJ} = +1\) and \(\nu_{IJ}\) are restricted to the ones actually arising from an AdS tetrahedron. The cosine can be expanded and the denominator expressed as a Gaussian integral, overall this leads to

\[
I^>(N\rho_{IJ}) = N^2 \left\{\frac{1}{2}\right\}^6 \frac{1}{\pi^{2}} \sum_{\epsilon_{IJ}=\pm,\nu_{IJ}=\pm} \int_{t_{IJ}>\epsilon} \int_{\mathbb{R}^4} e^{iNS_{\epsilon_{IJ},\nu_{IJ}} (t_{IJ}, X_I)} \left(\prod_{I<J} dt_{IJ}\right) \left(\prod_{I} dX_I\right),
\]

(95)

where the oscillating phase is

\[
S_{\epsilon_{IJ},\rho_{IJ},\nu_{IJ}}(t_{IJ}, X_I) = \sum_{I<J} \epsilon_{IJ} \rho_{IJ} t_{IJ} + \sum_{I,J} X_I \nu_{IJ} \cosh t_{IJ} X_J.
\]

(96)

The asymptotic behavior of this integral is driven by the stationary points of this action. They satisfy

\[
\sum_{J} \nu_{IJ} \cosh t_{IJ} X_J = 0,
\]

(97)

\[
2X_I X_J \sinh t_{IJ} = -\nu_{IJ} \epsilon_{IJ} \rho_{IJ}.
\]

(98)

The analysis of this system of equation is similar to the one performed in proposition 2. This analysis will be done in subsection III C in the general case of a Lorentzian tetrahedron. The
conclusions are given in terms of the geometrical elements of the tetrahedron $T(\rho)$ given by the spacelike lengths $\rho_{IJ}$. Let us denote $A_I$ its areas, $V^*$ its volume and $(V_{IJ}, T_{IJ})$ its Lorentzian dihedral angles. The possible solutions are given by

$$X_I = \pm i^n \sigma_I \frac{A_I}{\sqrt{3V^*}}$$  \hspace{1cm} (99)$$
$$t_{IJ} = T_{IJ}$$  \hspace{1cm} (100)$$

The expansion of the phase around the stationnary point $T_{IJ}$ gives the constant term

$$S = \sum_{I<J} \epsilon_{IJ} \rho_{IJ} T_{IJ}$$  \hspace{1cm} (101)$$

The quadratic form obtained by expansion around these configurations has a determinant which equals to $-(\frac{2}{3}V^*)^2$. In order to prove the result about the determinant one first write the quadratic form in term of cofactors $\Delta_{IJ}$ of the Cayley matrix (see appendix A). The form we obtain is the same as the one obtained in the SU(2) case except for a global minus sign in front of the non-diagonal part. It was shown there that the determinant of the quadratic form is proportional to the fourth power of determinant of the Cayley matrix. Since this is a polynomial identity it extend when the Cayley matrix comes from a Lorentzian tetrahedron. Numerical simulations suggest that the signature of this quadratic form is $\pm 2$ depending on the number of positive and negative $V_{IJ}$ in the dihedral angles of the spacelike tetrahedron $T(\rho)$. We now have to consider the compatibility equations on the signs to actually determine the possible $\sigma_I, \nu_{IJ}, \epsilon_{IJ}, \eta$. The first stationnarity equation can be written

$$\sum_j A_J \sigma_J \nu_{IJ} \cosh T_{IJ} = \sum_j A_J \sigma_J \nu_{IJ} V_{IJ} G_{IJ} = 0$$  \hspace{1cm} (102)$$

where $G_{IJ} = V_{IJ} \cosh T_{IJ}$ is the Graham matrix of the tetrahedron $T(\rho)$, while the sign of the second equation gives

$$\epsilon_{IJ} = (-1)^n \nu_{IJ} \sigma_I \sigma_J$$  \hspace{1cm} (103)$$

Considering the equation (102), we know (see appendix A) that the $V_{IJ}$ can be written $V_{IJ} = -\alpha_I \alpha_J$. If we consider the factorizable configurations of $\nu_{IJ} = \beta_I \beta_J$, we get for the equation (102)

$$-\alpha_I \beta_I \sum_j A_J \sigma_J \alpha_J \beta_J G_{IJ} = 0$$  \hspace{1cm} (104)$$

The fact that $G_{IJ}$ possess only one null eigenvector given by the $A_J$ amounts to identify $\beta_J = \sigma_J \alpha_J$ and the contributing $\nu_{IJ}$ as $\nu_{IJ} = -\sigma_I \sigma_J V_{IJ}$. Considering now the second compatibility equation (103) gives the possible $\epsilon_{IJ}$ as $\epsilon_{IJ} = (-1)^n V_{IJ}$. Taking into account these contributions, we obtain the following asymptotic term

$$\pm \frac{C}{N^3} \frac{\sin(\sum_{I<J} V_{IJ} \rho_{IJ} T_{IJ})}{3\pi V^*}$$  \hspace{1cm} (105)$$

for $C$ a constant independant of the $\rho_{IJ}$. 

18
B. Timelike tetrahedra

In the case where the representations are discrete and equal to \( l_{IJ} \), the integral reads

\[
I^>(iN\epsilon_{IJ}l_{IJ}^{\epsilon/2}) = \int_{D_+} \left( \prod_{I<J} \frac{-\epsilon_{IJ} \sin \theta_{IJ}}{2i \sin \theta_{IJ}} \right) e^{i\sum_{I<J} \epsilon_{IJ}(Nl_{IJ}^{\epsilon/2} - 1)\theta_{IJ}} \frac{1}{\sqrt{|\det[\cos \theta_{IJ}]|}} \prod_{I<J} d\theta_{IJ} + \cdots \tag{106}
\]

where the domain of integration is the set of angles \( \theta_{IJ} \in [\epsilon, \pi - \epsilon] \cup [-\pi + \epsilon, -\epsilon] \) which are such that \( \det[\cos \theta_{IJ}] \) is negative. It corresponds to the set of timelike AdS tetrahedra. The dots refer to the fact there are other terms in the integral coming from the sectors where at least one element \( g_J g_I^{-1} \) is hyperbolic. These terms contain, in the integrand, at least one factor \( \exp[-N|t_{IJ}|] \), such a factor is bounded by \( \exp[-N\epsilon] \) and is exponentially small, therefore they do not contribute to the asymptotic behavior of the integral. Even if the expression eq.106 is suitable for the asymptotic analysis, it is interesting to note that we could restrict the integration to be the set of angles for which \( \theta_{IJ} \in [\epsilon, \pi - \epsilon], \det[\cos \theta_{IJ}] < 0 \) if we use the identity

\[
\int_{-\pi}^{+\pi} P(\theta) \frac{|\sin \theta|}{\sin \theta} \frac{e^{iK\theta}}{2i} d\theta = \int_{0}^{\pi} P(\theta) \sin(K\theta) d\theta. \tag{107}
\]

for \( P \) such that \( P(-\theta) = P(\theta) \), which is the case of the denominator for each of its arguments \( \theta_{IJ} \). The integral now reads

\[
I^>(iN\epsilon_{IJ}l_{IJ}^{\epsilon/2}) = \int_{D_+} \prod_{I<J} \frac{\sin((Nl_{IJ}^{\epsilon/2} - 1)\theta_{IJ})}{\sqrt{|\det[\cos \theta_{IJ}]|}}, \tag{108}
\]

a form which is strikingly similar to the Euclidean integral eq.4.

The analysis of the asymptotic behavior goes along the same line as before and the stationary points are in one to one correspondence with timelike tetrahedra. Modulo permutation of the vertices and change of orientation one can always present such a tetrahedron in a form where all the edges are future timelike vectors and the vertex 3 is in the future of 2 which is in the future of 2... in the future of 0. This amounts to take all \( \lambda_{IJ}, I > J \) to label positive discrete representation. For such a configuration and using again the same technics one expect the asymptotic to be

\[
I^>(iNl_{IJ}^{+\epsilon/2}) \sim \frac{C}{N^3} e^{i\sum_{I<J}(Nl_{IJ}^{+\epsilon/2} - 1)\Theta_{IJ}} \frac{V^*(l)}{V^*(l)} + c.c \tag{109}
\]

where \( \Theta_{IJ} \) are the dihedral angles of the timelike tetrahedron whose edge are \( l_{IJ}^{+\epsilon/2}, V^* \) is its volume. \( C \) is a constant independent of the \( l_{IJ} \) and \( c.c \) stands for complex conjugate.

C. General method for the stationarity equations

In the general case of Lorentzian 6j symbol, the edges are labelled by (discrete or continuous) representations. If the representation is continuous then the character is zero on elliptic elements and if the representation is discrete the character is exponentially small on hyperbolic elements. Thus the only case of interest for the asymptotic behavior is when
the integration is over elliptic elements if the edge is labelled by a discrete representation and over hyperbolic elements if the edge is labelled by a continuous representation. This amounts to integrate over AdS tetrahedra which have spacelike edge when the representation is continuous and timelike edge when the representation is discrete. The asymptotic behavior of this integral is controlled by the solutions of the following stationary equations.

\[ \sum_{J} X_{J} \nu_{IJ} \cosh \tau_{IJ} = 0 \quad (110) \]
\[ 2X_{I}X_{J} \nu_{IJ} \sinh \tau_{IJ} = -\epsilon_{IJ} \lambda_{IJ}, \quad (111) \]

In the elliptic case, \( \lambda_{IJ} \) stands for \( \pm \imath l_{IJ} \), \( \tau_{IJ} \) denotes \( \imath \theta_{IJ} \) and \( \nu_{IJ} = 1 \). In the hyperbolic case \( \lambda_{IJ} = \rho_{IJ} \), \( \tau_{IJ} = t_{IJ} \) and \( \nu_{IJ} = \pm 1 \) for the different sectors of hyperbolic angles. We will express the solutions of these equations in terms of the geometrical elements of the Lorentzian tetrahedra \( \mathcal{T}(\lambda) \) given by the square lengths \( \lambda_{IJ}^2 \). Let us denote by \( \Theta_{IJ} \) and \( (V_{IJ}, T_{IJ}) \) its dihedral angles respectively in the timelike and spacelike case. Let us denote \( A_{I} \) its areas and \( V \) its volume. We will proceed as in the SU(2) case by first identifying where the solutions lie in the complex plane, then compute them explicitly.

Localization: We can repeat the analysis done in the SU(2) case. We apply the same lemma (eq.45) to obtain

\[ (X_{I}X_{J} \cosh \tau_{IJ})^2 = \frac{(\Delta_{IJ})^2}{-8 \text{det} C(\lambda)} \quad (112) \]

with \( C(\lambda) \) the Cayley matrix of the \( \lambda_{IJ} \) and \( \Delta_{IJ} \) its cofactor. This cofactor is real and the determinant of \( C \) is negative. We can conclude that \( X_{I}X_{J} \cosh \tau_{IJ} \) is real. Now the second stationnarity equation tells us that \( X_{I}X_{J} \sinh \tau_{IJ} \) is real in the spacelike case and pure imaginary in the timelike case. Taking the difference of the square of these two equations gives

\[ (X_{I}X_{J})^2 = (X_{I}X_{J} \cosh \tau_{IJ})^2 - (X_{I}X_{J} \sinh \tau_{IJ})^2 = \frac{(\Delta_{IJ})^2}{-8 \text{det} C'(\lambda)} - \frac{\lambda_{IJ}^2}{4} \quad (113) \]

One can conclude that \( X_{I}X_{J} \) is real since the RHS of this equation is always positive. This is clear in the timelike case since the square of \( \lambda_{IJ} = il_{IJ} \) is negative. In the spacelike case, this arise from rewriting it using the relations (A5) and (A8)

\[ \frac{(\Delta_{IJ})^2}{-8 \text{det} C'(\lambda)} - \frac{\lambda_{IJ}^2}{4} = \frac{A_{I}^2}{(3V)^2} \cosh^2 T_{IJ} - \frac{A_{I}^2 A_{J}^2}{(3V)^2} \sinh^2 T_{IJ} = \frac{A_{I}^2 A_{J}^2}{(3V)^2} \quad (114) \]

We thus know that \( X_{I}X_{J} \) is real, \( \cosh \tau_{IJ} \) is real, \( \sinh \tau_{IJ} \) is real in the spacelike case and pure imaginary in the timelike case. This allows to conclude that \( \tau_{IJ} \) is pure imaginary in the timelike case (hence \( \theta_{IJ} \) is real) and \( \tau_{IJ} \) is real modulo \( \imath \pi \) in the spacelike case. However this \( \imath \pi \) ambiguity corresponds to a \( \pm \) ambiguity in the sign of \( \cosh \tau_{IJ} \) and \( \sinh \tau_{IJ} \), and thus leads to the same solution obtained by taking \( \tau_{IJ} \) and changing the sign of \( \nu_{IJ} \). These solutions are already taken into account and we can restrict to the case of \( \tau_{IJ} \) real in the spacelike case. This completes our investigation of the localization of the solutions.

Explicit solutions: We consider first the equation (112) for \( I = J \) and use the equations (A8) and (A7) to rewrite the RHS. We obtain

\[ X_{I}^4 = \frac{A_{I}^4}{(3V)^2} \quad (115) \]

Localization:
Thus it exists $\sigma_I = \pm 1$ and $\eta_I = 0, 1$ such that $X_I = \sigma_I \sqrt{\frac{\eta_I}{3V}}$. The fact that $X_I X_J$ is real for all $IJ$ leads to the fact that all $\eta_I$ are equal and

$$X_I = \sigma_I \frac{A_I}{\sqrt{3V}}$$ (116)

Now we consider the equation (112) for $I \neq J$ and use the solution we found for the $X_I$ to get

$$(\cos \theta_{IJ})^2 = (\cos \Theta_{IJ})^2$$ (117)

$$(\cosh t_{IJ})^2 = (\cosh T_{IJ})^2$$ (118)

We have to solve these equations in $\mathbb{R}$, but since the original integral is symmetric by changing $t \to -t$ and $\theta \to \theta + 2\pi$, we keep only the actually different solutions and we solve these equations for $t \in \mathbb{R}^+$ and $\theta \in [-\pi, \pi]$. This gives the solutions

$$t_{IJ} = T_{IJ}$$
$$\theta_{IJ} = \alpha_{IJ} \left( \beta_{IJ} \Theta_{IJ} + \left(1 - \beta_{IJ}\right) \frac{\pi}{2} \right) \text{ for } \alpha_{IJ}, \beta_{IJ} = \pm 1$$ (120)

Reporting these result in the first stationnarity equation gives

$$\sum_J \sigma_J A_J \left\{ \nu_{IJ} \cosh T_{IJ} \cos \Theta_{IJ} \right\} = 0$$ (121)

We still have to determine the possible $\nu_{IJ}, \sigma_I, \beta_{IJ}$... by considering compatibility equations. In the previous one, one can use the fact that by hypothesis, the Graham matrix of $T(\lambda)$

$$G_{IJ} = \left\{ \begin{array}{c} V_{IJ} \cosh T_{IJ} \\ \cos \Theta_{IJ} \end{array} \right\}$$ (122)

possess only one null eigenvector given by the $A_I$. On the other hand, the sign of the second stationnary equation gives also equations

$$(-1)^n \sigma_I \sigma_J \left\{ \begin{array}{c} \nu_{IJ} \\ \alpha_{IJ} \end{array} \right\} = -\epsilon_{IJ}$$ (123)

which should be satisfy in order for the solution to exists. The bottom line is that the system of equation 110 admits solutions only if $\lambda_{IJ}$ could be interpreted as a set of length of an oriented tetrahedra $T(\lambda_{IJ})$. In that case the on-shell action for this solutions is, up to a sign, proportional to the Regge action of Lorentzian tetrahedra. The determinant of the quadratic form obtained by looking at fluctuation around these configuration is proportional to the volume of the tetrahedron.

IV. 10J SYMBOL

A. Integral expression for the 10j-symbol

To analyze the case of the 10j-symbol along the same lines of the 6j-symbol, we will use higher-dimensional generalizations of propositions used in the previous parts. The analog of the theorem expressing the symbol as an integral over invariant variables is
Theorem 4  The 10j-symbol can be expressed as the following integral

\[
\frac{4}{\pi^6} \int_{\mathcal{D}_s'} \left( \prod_{I<J} \sin((l_{IJ} + 1)\theta_{IJ}) \right) \delta(G(\theta_{IJ})) \left[ \prod_{I<J} d\theta_{IJ}, \right]
\]

(124)

where \( G(\theta) \) denotes the determinant of the 5 × 5 Graham matrix \( G_{IJ} = \cos \theta_{IJ} \) associated with the spherical 4-simplex 01234 and \( \delta \) is the Dirac functional. \( \mathcal{D}_s' \) is the set of all spherical 4-simplices.

Proof: —

We start from the Barrett’s expression for the 10j-symbol [9]

\[
\int_{SU(2)^5} \chi_{l_{IJ}}(g_I g_J^{-1}) \prod_I dg_I
\]

(125)

Then we apply the same method as in the 6j case. By gauge fixing \( g_0 = 1 \) we are left with an integral over \( SU(2)^4 \) with an \( AdG \)-invariance. We have seen in the proof of theorem 1 that if \( g_1, g_2 \) are two groups elements we can express their invariant measure in terms of the angles \( \theta_1, \theta_2, \theta_{12} \).

\[
dg_1 dg_2 = \frac{2}{\pi^2} \sin \theta_1 d\theta_1 \sin \theta_2 d\theta_2 \sin \theta_{12} d\theta_{12}
\]

Moreover if \( g_1 \) is an additional group element its measure is given by

\[
dg_I = \frac{1}{\pi^2} \frac{\sin \theta_I d\theta_I \sin \theta_{1I} d\theta_{1I} \sin \theta_{2I} d\theta_{2I}}{\sqrt{G_{1I}}}
\]

(127)

where \( G_{1I} \) is the determinant of the Graham matrix for the spherical tetrahedron whose vertices are \( 1, g_1, g_2, g_I \). Using these results for \( g_3, g_4 \) allows to prove the following integral expression for the 10j-symbol

\[
\frac{2}{\pi^6} \int_{\mathcal{D}_s} \left[ \prod_I d\theta_{IJ} \right] \frac{\prod_I \sin(l_{IJ} + 1)\theta_{IJ}}{\sqrt{\Lambda_{33} \Lambda_{44}}} \frac{\sin(l_{34} + 1)\theta_{34}}{\sin \theta_{34}}
\]

(128)

where the set \( I \) is the set \( 1 \leq I < J \leq 4, (I, J) \neq (3, 4) \). \( \Lambda_{IJ} \) denotes the cofactor (i.e., determinant of the minor) of the 5 × 5 Graham matrix. The equation (128) follows from eq. (127) since the minor \( \Lambda_{33} \) (resp. \( \Lambda_{44} \)) is the Graham determinant of the 3-simplex 0124 (resp. 0123).

One of the main differences with the case of the 6j is the fact that the angles \( \theta_{IJ} \) are not all independent. It is clear from our parameterization that \( \theta_{34} \) is not needed; it is a function of the 9 others angles. Indeed, from the original integral expression, the 5 integration elements \( g_I \in S^3 \) can be considered as 5 unit vectors in \( \mathbb{R}^4 \). They are not independent and the angles \( \theta_{IJ} \) can therefore be considered as the dihedral angle of an Euclidean 4-simplex. This means that the 5-dimensional Graham matrix \( G_{IJ}(\theta) \) is degenerate, \( G(\theta) = \det [G_{IJ}(\theta)] = 0 \).

In order to get a symmetric expression of the integrant, let \( T(\theta_{IJ}) \) be a spherical 4-simplex whose dihedral angle are \( \theta_{IJ} \), denote by \( G(\theta) \) its Graham determinant, and by \( \Lambda_{IJ} \) its cofactor. It is clear that

\[
\frac{\partial G(\theta)}{\partial \theta_{IJ}} = -2\Lambda_{IJ}(\theta) \sin \theta_{IJ},
\]

(129)
where the factor 2 is due to the symmetry of the Graham matrix. When the 4-simplex is flat (i-e $G=0$) and non degenerate, the Graham matrix is of corank one, which means that the cofactor matrix is of rank 1

$$\Lambda^2_{IJ} = \Lambda_{II} \Lambda_{JJ}. \quad (130)$$

This means that the denominator in the integrand of (128) is $|\partial_{34} G(\theta)|/2$ and the measure is simply

$$\left( \prod_{I<J, (I,J) \neq (3,4)} d\theta_{IJ} \right) \frac{1}{\sin \theta_{34} \sqrt{\Lambda_{33} \Lambda_{44}}} = \left( \prod_{I<J} d\theta_{IJ} \right) 2\delta(G(\theta)). \quad (131)$$

Using this measure in (128) proves the theorem.

B. Asymptotics for the 10j symbol

Following the lines of the method employed for the 6j symbol’s study, we split the integral $I$ into a sum $I = I^< + I^>$, where in $I^>$ the angles $\theta_{IJ}$ are restricted to be in $[\epsilon, \pi - \epsilon]$, $\epsilon > 0$ small enough. The asymptotic of the integral $I^>(Nl_{IJ})$ is governed by stationary phase. Given $l_{IJ}$ we denote by $T^*(l_{IJ})$ the Euclidean 4-simplex such that the area of the face opposite to the edge $(IJ)$ is given by $l_{IJ}$, and denote $\Theta(l)$ the dihedral angles of this tetrahedron. If $l_{IJ}$ are such that $T^*(l_{IJ})$ is not degenerate and $\epsilon < \min(\Theta(l), \pi - \Theta(l))$ then

$$I^>(Nl_{IJ}) \sim \frac{1}{N^{9/2}} \frac{\cos(N(l_{IJ} + 1)\Theta_{IJ})}{V(l)}, \quad (132)$$

where $V(l)$ has to be determined. This property was proven by Barret and Williams [6]. Since the argument of the cosinus is the Regge action, it was a strong evidence that spin foam models constructed from the 10j give amplitude of 4d gravity. Note that if $T^*(l_{IJ})$ is degenerate, then for $\epsilon > 0$, there is no stationary point in the action and $I^>(Nl_{IJ}) = o(1/N^{9/2})$.

However, we have to take into account the asymptotic behavior of $I^<$. First, as for the 6j, we find that the integration domain of $I^<$ is split into 16 disconnected components. All this components are equal and related by the symmetry transformation $\theta_{IJ} \rightarrow \sigma_I \sigma_J \theta_{IJ} + (1 - \sigma_I \sigma_J)\pi/2$. With this symmetry, we can restrict the angles to be in $[0, \epsilon]$. According to our analysis of the 6j the asymptotic behavior of this integral is governed by the singular contributions of the integrand, i-e points where $\partial_{IJ} G(\theta) = 0$. The most singular points are indeed the ones where $\theta_{IJ} = 0$. The behavior of the Graham determinant around this point is given by

$$G(Su) \sim_{S \rightarrow 0} S^8[4!V_{4s}(u)]^2 \quad (133)$$

where $V_{4s}(u)$ is the volume of the Euclidean 4-simplex whose edge lengths are $u_{IJ}$. We can perform an analysis which is almost identical to the one of section II D, therefore we do not repeat it. This analysis shows that the asymptotic of $I^<$ is given by the following contribution

$$I^>(Nl_{IJ}) \sim \frac{16}{N^2} \frac{4}{\pi^6} \int_{\mathcal{D}} \prod_{I<J} \sin(l_{IJ}u_{IJ}) \delta([4!V_{4s}(u)]^2) \left[ \prod_{I<J} du_{IJ} \right] \quad (134)$$
where the integration is on $\mathcal{D}_\infty$, the set of euclidian flat 4-simplices.

By an analysis similar to the one done in the previous section, we can express the measure in term of a fully gauge fixed measure

$$
\frac{4}{\pi^6} \delta([4!V_{4s}(u)]^2) \prod_{I,J} du_{IJ} = \left[ \prod_{I=0}^4 \frac{d^3 \vec{u}_I}{2\pi^2} \right],
$$

(135)

where $u_{IJ} = |\vec{u}_J - \vec{u}_I|$. The asymptotic contribution of $I^<$ is therefore

$$
\frac{16}{N^2} \int_{(\mathbb{R}^3)^4} \prod_I \left[ \frac{d^3 \vec{u}_I}{2\pi^2} \right] \prod_{I<J} \sin(l_{IJ}|\vec{u}_J - \vec{u}_I|) |\vec{u}_J - \vec{u}_I|
$$

(136)

which is the form proposed by Baez et al [8].

V. DISCUSSION

As we mention in the introduction, given a triangulation of a 4-dimensional manifold, we can construct state sum models (usually called Barrett-Crane models) where the amplitude associated with the 4-simplex is given by the $10j$ symbols. The importance of these models lies in the hope that they would give a way to compute transition amplitudes in 4d gravity [4]. Also, it was found that the integral form of the $10j$ [9] lead to the interpretation of these weights as Feynman diagrams [16]. This leads into the striking result that the sum over all triangulation of Barrett-crane amplitude could be interpreted as a sum over Feynman graph of a non local field theory [17].

However this model suffers from two diseases. First, the amplitudes calculated in terms of $10j$ symbols are always positive [7] and second, as we have just seen, their asymptotics is not dominated by the semi-classical Regge action but it is dominated by degenerate configurations. Does this mean that these model are therefore not good candidates for quantum gravity amplitudes? In order to answer this, we have to recall why the Barrett-Crane models are suspected as good candidates for GR amplitudes. The original derivation of the model in [4] did not make any reference to the Einstein-Hilbert action, however there has been work since then linking the Barrett-Crane model with a discretization of the path integral for Euclidean 4d gravity [18, 19]. In these works it was argued that the Barret-Crane model corresponds to a discretization of a constraint $BF$ model introduced by Plebanski [20], and it was shown that the partition function of this model was related to the gravity partition function if one excludes the degenerates metrics from the path integral. However in [19] it was argued that the degenerate sector is likely to dominate to path integral since the number of degenerate configurations is likely to be more important that the number of non degenerate one. A similar fact is well known in statistical mechanics under the name ‘order by disorder’[21]. It is best illustrated in the $XY$ antiferromagnetic model on a Kagome lattice. This model is frustrated and posses a huge degeneracy (the vacua states have a macroscopic entropy). Since all the states have the same energy, only the entropy of the fluctuation around these states could make a difference between them. When the temperature is small, the states which are selected are the soft modes states, (i-e the one around which fluctuation in energy is quartic), whereas the hard modes (the ones for which the fluctuation in energy is quadratic) are suppressed. In Euclidean gravity we are working
with the Einstein-Hilbert action

\[ S = \int e^I \wedge e^J \wedge F^{KL}(A) \epsilon_{IJKL}, \]  

(137)

where \( e^I \) is the frame field \( I \) are SO(4) indices and \( F^{IJ}(A) \) is the curvature of the SO(4) spin connection. If we evaluate the action on solutions of the equation of motion we find that \( S = 0 \). So the on-shell configurations are not distinguished by their action. By a similar argument to the one used in statistical mechanics one expect that the dominant configuration of the path integral are given by the soft modes in the limit where \( \hbar \to 0 \). It is clear that the fluctuations of the action around a configuration where the metric (hence \( e \)) is non degenerate are purely quadratic. The softest modes of \( S \) is given by \( e = 0 \), and one expect that fluctuations around this configuration dominates the path integral in the classical limit if one include degenerate metrics. Note that in 3d the fluctuation around \( e = 0 \) are quadratic, so in this case, we do not expect the degenerate contributions to dominate the non degenerate ones.

This argument suggests that the disease we have just find in the asymptotic behavior of the 10j symbol is in fact a disease of the gravity partition function if one includes degenerate metrics. The cure for this disease is then clear, one has to avoid degenerates contributions. How this can be implemented in the state sum model is also clear. We have seen that given \( \epsilon > 0 \) the integral defining the 10j symbol could be split int \( I^< + I^> \) where \( I^< \) account for the dominant degenerate contributions, whereas \( I^> \) has a nice semi-classical oscillatory behavior. One can therefore propose a model where, instead of taking the 10j symbol as an amplitude for the 4-simplex, one takes the truncated 10j symbol \( I^> \). This cures at once the two diseases mentioned earlier.

One of the key feature of the Barret-Crane weight was the fact that it could be interpreted as a Feynman graph of a group field theory. The modification of the weight we propose still preserve this key property since \( I^> \) can be written as a Feynman integral with the propagator

\[ K_{\epsilon l}(g_1, g_2) = \chi_l(g_1 g_2^{-1}) C_\epsilon(g_1 g_2^{-1}) \]  

(138)

where \( \chi_l \) denotes the character of the representation of weight \( l \) and \( C_\epsilon(e^X) \) is a cut-off function which is 0 if \( |X| < \epsilon \) and 1 if \( |X| > \epsilon \). This let open the possibility of writing the sum over triangulations of state sum model constructed with a truncated 10j symbol as a Feynman graph expansion. The drawback of the truncated model is the presence of the cut-off \( \epsilon \) which still deserve a physical interpretation. We hope to come back to these issues in the near future.

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**APPENDIX A: GEOMETRY OF THE EUCLIDEAN AND LORENTZIAN FLAT TETRAHEDRON**

Let \( \vec{e}_I \in \mathbb{R}^3, \ I = 0, 1, 2, 3 \) be the vertices of a non degenerate tetrahedron, we denote by \( \vec{A}_I \) the vectors normal to the face opposite to the vertex \( I \). They are defined as

\[ \vec{A}_I = \frac{1}{2}(\vec{e}_J \wedge_\eta \vec{e}_K + \vec{e}_K \wedge_\eta \vec{e}_L + \vec{e}_L \wedge_\eta \vec{e}_J) \]  

(A1)
where $I, J, K, L$ is an even permutation of $0, 1, 2, 3$, and
\[
(u \wedge v)^a = \eta^{ae} \epsilon_{bce} u^b v^c,
\] (A2)
where $\epsilon_{abc}$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$ and $\eta_{ab} = \eta^{ab}$ is the metric $+++$ in the Euclidean case and $-++$ in the Lorentzian case. In all the following we denote the wedge product simply $\wedge$ and the scalar product with respect to $\eta$ by $u \cdot v$ without referring explicitly to the dependence of the metric. The vectors $\vec{A}_I$ clearly satisfy
\[
\sum_I \vec{A}_I = 0.
\] (A3)

The edge vectors are given by $\vec{l}_{IJ} = \vec{e}_J - \vec{e}_I$, they satisfy with the area vectors a duality relation
\[
\vec{A}_i \cdot \vec{l}_{0j} = 6V \delta_{ij},
\] (A4)
where $V = (\vec{l}_{01} \wedge \vec{l}_{02}) \cdot \vec{l}_{03} / 6$ is the oriented volume of the tetrahedron. Moreover if one uses the identity $(u \wedge v) \wedge w = \eta[(u \cdot w)v - (v \cdot w)u]$, where $\eta$ is the signature of the metric, one gets
\[
\vec{A}_I \wedge \vec{A}_J = \eta \frac{3V}{2} \vec{l}_{KL}
\] (A5)
where $I, J, K, L$ is an even permutation of $0, 1, 2, 3$.

The Cayley matrix associated with the tetrahedron is defined as
\[
C = \begin{pmatrix}
0 & l_{12}^2 & l_{13}^2 & l_{11}^2 & 1 \\
l_{21}^2 & 0 & l_{23}^2 & l_{22}^2 & 1 \\
l_{31}^2 & l_{32}^2 & 0 & l_{33}^2 & 1 \\
l_{1}^2 & l_{2}^2 & l_{3}^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\] (A6)
where $l^2 = \vec{l} \cdot \vec{l}$ and $l_i \equiv l_{0i}$. This matrix encodes a lot about the geometry of the tetrahedron as the following identities show
\[
\det(C) = \eta^2 (6V)^2,
\] (A7)
\[
\vec{A}_I \cdot \vec{A}_J = -\frac{\eta}{16} \Delta_{IJ},
\] (A8)
where $\Delta_{IJ}$ is the cofactor matrix of the Cayley matrix.

The first relation can be proven as follows, we have the identity
\[
\det(\vec{l}_i \cdot \vec{l}_j) = \det(l_i^3 \eta_{ab} \vec{l}_a \vec{l}_b) = \det(l_i^3)^2 \eta = \eta (6V)^2
\] (A9)

Using the relation,
\[
\vec{l}_i \cdot \vec{l}_j = \frac{l_i^2 + l_j^2 - l_{ij}^2}{2}
\] (A10)

The square of the volume is rewritten as
\[
2^3 6^2 V^2 = \begin{vmatrix}
2l_{12}^2 & -l_{12}^2 + l_{13}^2 + l_{12}^2 & -l_{13}^2 + l_{14}^2 + l_{13}^2 & -l_{14}^2 + l_{11}^2 + l_{14}^2 \\
-l_{13}^2 + l_{12}^2 + l_{13}^2 & 2l_{23}^2 & -l_{23}^2 + l_{24}^2 + l_{23}^2 & -l_{24}^2 + l_{21}^2 + l_{24}^2 \\
-l_{14}^2 + l_{13}^2 + l_{14}^2 & -l_{24}^2 + l_{23}^2 + l_{24}^2 & 2l_{34}^2 & -l_{34}^2 + l_{31}^2 + l_{34}^2 \\
-l_{21}^2 + l_{22}^2 + l_{21}^2 & -l_{31}^2 + l_{32}^2 + l_{31}^2 & -l_{32}^2 + l_{33}^2 + l_{32}^2 & 2l_{43}^2
\end{vmatrix}
\] (A11)
This $3 \times 3$ determinant is easily seen to be the Cayley determinant. This follows from subtracting line 4 to lines 1,2 and 3, then row 4 to rows 1,2 and 3 in the Cayley determinant to finally obtains

$$
\begin{vmatrix}
-2l_1^2 & l_1^2 - l_2^2 & l_1^2 - l_3^2 & l_2^2 - l_1^2 & l_2^2 - l_3^2 & l_3^2 - l_2^2 & 0 \\
l_1^2 - l_2^2 & 0 & l_3^2 - l_2^2 & l_1^2 - l_2^2 & l_2^2 - l_3^2 & 0 & l_1^2 - l_3^2 \\
l_1^2 - l_3^2 & l_2^2 - l_3^2 & 0 & l_1^2 - l_3^2 & l_3^2 - l_1^2 & l_2^2 - l_1^2 & 0 \\
l_1^2 - l_2^2 & l_3^2 - l_2^2 & l_2^2 - l_3^2 & 0 & l_1^2 - l_1^2 & l_3^2 - l_2^2 & l_3^2 - l_1^2 \\
l_1^2 - l_3^2 & l_1^2 - l_3^2 & l_2^2 - l_1^2 & l_3^2 - l_2^2 & 0 & l_1^2 - l_2^2 & l_3^2 - l_1^2 \\
l_1^2 - l_2^2 & l_3^2 - l_2^2 & l_1^2 - l_1^2 & l_3^2 - l_1^2 & l_2^2 - l_3^2 & 0 & l_1^2 - l_3^2 \\
l_1^2 - l_2^2 & l_3^2 - l_2^2 & l_1^2 - l_3^2 & l_2^2 - l_1^2 & l_3^2 - l_2^2 & 0 & l_1^2 - l_3^2 \\
l_1^2 - l_3^2 & l_1^2 - l_3^2 & l_2^2 - l_1^2 & l_3^2 - l_2^2 & l_1^2 - l_3^2 & l_2^2 - l_3^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0
\end{vmatrix}
$$

(A12)

which reduces to the $3 \times 3$ determinant.

The relation (A8) is proven by direct computation and the use of the identity

$$(u \wedge v) \cdot (w \wedge x) = \eta[(u \cdot w)(v \cdot x) - (u \cdot x)(v \cdot w)]$$

(A13)

Let's denote $A_I = \sqrt{|\vec{A}_I \cdot \vec{A}_I|}$ the area of the face $I$, lets introduce the normal unit vector $\vec{n}_I$, $\vec{A}_I = A_I \vec{n}_I$ and lets denote $G_{IJ} = \vec{n}_I \cdot \vec{n}_J$ the Graham matrix. The relation (A3) implies that

$$\sum_J A_J G_{IJ} = 0.$$  

(A14)

From relation (A8), we get that for $I, J = 1..4$, the cofactor $\Delta_{IJ}$ are related to the elements of the Graham matrix

$$\Delta_{IJ} = -\eta 16 A_I A_J G_{IJ}$$

(A15)

A consequence of eq.A14 is that

$$\sum_{I=1}^{4} \Delta_{IJ} = 0$$

(A16)

If the tetrahedron is Euclidean $G_{IJ} = \cos \Theta_{IJ}$, and the relation (A5) imply

$$A_I A_J \sin \Theta_{IJ} = \frac{3}{2} |V| l_{KL}.$$  

(A17)

In this case the Cayley determinant is positive.

If the tetrahedron is Lorentzian and time-like (i-e all its edge are time-like $\vec{l}_{IJ} \cdot \vec{l}_{IJ} = -l_{IJ}^2$) then the normal vectors are all space-like and $G_{IJ} = \cos \Theta_{IJ}$ and

$$A_I A_J \sin \Theta_{IJ} = \frac{3}{2} |V| (\pm) l_{KL}.$$  

(A18)

where the sign in the RHS is + (resp. −) if the edge $\vec{l}_{KL}$ is future (resp. past) pointing. In this case the Cayley determinant is negative. Conversely given a matrix $G_{IJ} = \cos \Theta_{IJ}$ which is degenerate (of corank 1) one can associate uniquely to it a pair of Euclidean tetrahedra or a unique Lorentzian tetrahedron.

If the tetrahedron is Lorentzian and space-like (i-e all its edge are space-like $\vec{l}_{IJ} \cdot \vec{l}_{IJ} = \rho_{IJ}^2$) then the normal vectors are all timelike and $G_{IJ} = \nu_{IJ} \cosh T_{IJ}$, $T > 0$, $\nu_{IJ} = \pm 1$ and $\nu_{II} = -1$. $\nu_{IJ}$ is the sign of the scalar product $(\vec{n}_I \cdot \vec{n}_J)$, depending on whether $n_I$ and $n_J$ lie in the same part of the light-cone. The $\nu_{IJ}$ can thus always be written as $\nu_{IJ} = -\alpha_I \alpha_J$ with $\alpha_I = \pm 1$. We call the pair $(\nu_{IJ}, T_{IJ})$ where $T > 0$ $\nu_{IJ} = \pm 1$ the dihedral angle of the edge $(IJ)$. The relation(A5) implies

$$A_I A_J \sinh T_{IJ} = \frac{3}{2} |V| \rho_{KL}.$$  

(A19)
APPENDIX B: SOME FACTS ABOUT SL(2, \mathbb{R})

We denote by $T_{d^+}$, $l \in \mathbb{N}$ the positive discrete series of weight $l$, $T_{d^-}$, $l \in \mathbb{N}$ the negative discrete series of weight $l$, and $T_\rho \rho \in \mathbb{R}^+$ the principal series of weight $\rho$. In this paper we consider only the representations of SL(2, \mathbb{R}) which extend to representations of SO(2, 1), this essentially means that $l \in 2\mathbb{N}$. It is well known that the continuous representation are conjugated to itself, whereas the positive discrete series is conjugated to the negative discrete series. So if one change the orientation of one edge this amounts to change the label $\lambda$ into its complex conjugate $\bar{\lambda}$. Let $(\lambda_1, \lambda_2, \lambda_3)$ be a triple of representations meeting at a vertex of the oriented tetrahedron, modulo a change of orientation we can suppose that all the edges are incoming at the vertex. Such a triplet is always admissible if there is at least one continuous representation. If there is no continuous representation the admissible triplets (for the incoming orientation) are

1. $(l_1^+, l_2^+, l_3^+)$ with $l_3 > l_1 + l_2$ and $l_1 + l_2 + l_3 \in 2\mathbb{Z}$;
2. $(l_1^-, l_2^-, l_3^-)$ with $l_3 > l_1 + l_2$ and $l_1 + l_2 + l_3 \in 2\mathbb{Z}$.

Let $g$ be an element of SL(2, \mathbb{R}). We say that $g$ is elliptic if it is conjugated to the matrix

$$k_\theta = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right).$$

(\text{B1})

$g$ is said to be hyperbolic if it is conjugated to the matrix

$$\pm a_t = \pm \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right).$$

(\text{B2})

Note that $k_\theta$ and $k_{-\theta}$ are not conjugated to each other in SL(2, \mathbb{R}), whereas $a_t$ and $a_{-t}$ are.

The characters of the positive discrete series are given by \[13\]

$$\chi_{d^+}(g) = -\frac{1}{2i} \frac{e^{i(l-1)\theta}}{\sin \theta}$$

if $g$ is conjugated to $k_\theta$ and

$$\chi_{d^+}(g) = \frac{e^{-(l-1)t}}{2|\sinh t|}$$

if $g$ is conjugated to $\pm a_t \; t > 0$. The characters of the negative discrete series are conjugate to the positive discrete series characters

$$\chi_{d^+}(g) = \bar{\chi}_{d^-}(g)$$

(\text{B5})

The characters of the principal series are given by

$$\chi_\rho(g) = 0$$

if $g$ is conjugated to $k_\theta$ and

$$\chi_\rho(g) = \frac{\cos \rho t}{|\sinh t|}$$

if $g$ is conjugated to $\pm a_t \; t > 0$.

Since the representations are infinite dimensional the characters of these representations are not function but distributions. The previous expressions should be consider as a particular representative of the character distribution in terms of a locally integrable function. One
has however the freedom to change the representative function as long as they differ from the previous one on a set of measure 0. It turns out that we effectively need to take an other representative function in order to make sense of the integral (92), since the integral admits singularities around $g_I = g_J$. We propose the following modification. Let's consider $\alpha > 0$ and let's define $C_\alpha(g)$ to be equal to $1/(e^t - e^{-t})$ if $g$ is conjugated to $\pm a_t$, $0 < t < \alpha$ and to be equal to 0 otherwise. For the continuous series $\rho$ let's consider the following distribution

$$\langle \chi'_\rho | f \rangle = \lim_{\alpha \to 0} \int_G (\chi_\rho(g) - C_\alpha(g)) f(g).$$

(B8)

If $f$ is regular at the identity then it is clear that $\langle \chi'_\rho | f \rangle = \langle \chi_\rho | f \rangle$ so $\chi'_\rho$ define the same distribution as $\chi_\rho$. Analogously we define $\chi'_{\pm \ell \pm}$ by subtracting the divergence at 0. And we use these characters in the definition of the integral (92). It is easy to see that with such a representative of the character the integral is convergent.

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