AREFINED CONJECTURE FOR FACTORING ITERATES OF QUADRATIC POLYNOMIALS OVER FINITE FIELDS

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Abstract. Jones and Boston conjectured that the factorization process for iterates of irreducible quadratic polynomials over finite fields is approximated by a Markov model. In this paper, we find unexpected and intricate behavior for some quadratic polynomials, in particular for the ones with tail size one. We also propose a multi-step Markov model that explains these new observations better than the model of Jones and Boston.

1. Introduction

Let \( f \) be an irreducible quadratic polynomial over a finite field \( \mathbb{F}_q \) of odd order \( q \). We are interested in understanding the factorization of iterates of \( f \). This problem was previously studied in [Gomez-Perez et al, 2012], [Gomez-Perez et al, 2011], [Ahmadi et al, 2012], [Ayad and McQuillan, 2000], and [Jones and Boston, 2012]. In [Jones and Boston, 2012], the authors associated a Markov process to \( f \) and conjectured that its limiting distribution explains the shape of the factorization of large iterates of \( f \). In this paper, we give new data that strongly suggest a more complicated model is required in certain cases, and we propose a multi-step Markov model that fits the new data well. Furthermore, we also conjecture that the original Markov model applies except in these certain cases.

The paper is structured as follows. In Section 2, we make some definitions, give preliminary results, and recall background to the problem. In Section 3, we provide some examples with new, unexpected behavior. In Section 4, we propose a multi-step Markov model to describe the factorization of iterates and conjecture that it provides a better explanation for the process. Section 5 supports this model via actual data corresponding to the examples given in

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Section 3. In Section 6, we summarize matters with some further conjectures and list additional computational results we have obtained.

2. Set-up

**Definition 2.1.** Let \( \mathbb{F}_q \) be a finite field of odd order \( q \). Consider a quadratic polynomial \( f(x) \) defined over \( \mathbb{F}_q \). For all \( n \in \mathbb{N} \), we define the \( n \)th iterate of \( f \) to be \( f^n(x) := f(f^{n-1}(x)) \). We make the convention that \( f^0(x) := x \).

For example, suppose \( f(x) = x^2 + 1 \in \mathbb{F}_7[x] \). Then, \( f^2(x) = f(f(x)) = x^4 + 2x^2 + 2 \), \( f^3(x) = f^2(f(x)) = x^8 + 4x^6 + x^4 + x^2 + 5 \), and so on.

**Definition 2.2.** Let \( f(x) = ax^2 + bx + c \in \mathbb{F}_q[x] \) \( (a \neq 0) \) and \( \alpha = \frac{-b}{a} \) be the critical point of \( f \). The critical orbit of \( f \) is the set \( \mathcal{O} := \{ f^{k}(\alpha) \mid k = 1, 2, 3, \ldots \} \) and the number of elements of \( \mathcal{O} \) is the orbit size of \( f \), denoted \( o \).

To illustrate the definition of the critical orbit, we consider the previous example. The critical point of \( f(x) = x^2 + 1 \) is 0 and \( f(0) = 1 \), \( f^2(0) = 2 \), \( f^3(0) = 5 \), \( f^4(0) = 5 \). It follows that \( f^k(0) = 5 \) for all \( k \geq 3 \). Therefore, the critical orbit for \( f(x) = x^2 + 1 \in \mathbb{F}_7[x] \) is \( \{1, 2, 5\} \).

**Definition 2.3.** Let \( f \) be a quadratic polynomial over \( \mathbb{F}_q \) and \( \alpha \) be the critical point of \( f \). We define the tail of \( f \) to be the set

\[
\mathcal{T} := \{ f^{k}(\alpha) \mid k \geq 1, f^{i}(\alpha) \neq f^{k}(\alpha) \forall i \neq k \}.
\]

Similarly, we call the number of elements of \( \mathcal{T} \) the tail size of \( f \) and denote it by \( t \).

Having taken \( f(x) = x^2 + c \), the critical orbit of \( f(x) \in \mathbb{F}_q[x] \) becomes \( \{c, c^2 + c, (c^2 + c)^2 + c, \ldots\} \).

**Definition 2.4.** Noting that \( f^n(c) \) is the \((n+1)\)th element of the critical orbit of \( f(x) = x^2 + c \), we define the difference polynomial \( p_{a,b}(c) \) to be

\[
\frac{f^a(c) - f^b(c)}{\text{LCM}(f^{b+1}(c) - f^b(c), f^{a-1}(c) - f^{b-1}(c), f^{a-2}(c) - f^{b-2}(c), \ldots, f^{a-b}(c) - f^0(c))}
\]

if \( a \neq b + 1 \) and

\[
\frac{c \cdot \text{LCM}(f^{b+1}(c) - f^b(c), f^{a-1}(c) - f^{b-1}(c), f^{a-2}(c) - f^{b-2}(c), \ldots, f^{a-b}(c) - f^0(c))}{f^a(c) - f^b(c)}
\]

if \( a = b + 1 \).

**Remark 2.1.** \( x^2 + c_0 \in \mathbb{F}_q[x] \) is a quadratic polynomial with orbit size \( o \) and tail size \( t \) if \( c_0 \) is a root of \( p_{o,t} \) in \( \mathbb{F}_q \). The polynomials in the denominators rescue us from an earlier repetition that would lead to the correct tail size but smaller orbit size or vice versa.
To illustrate this, suppose we want a quadratic polynomial $x^2 + c \in \mathbb{F}_q[x]$ of orbit size 3 and tail size 1. This immediately yields $f^3(c) - f^1(c) = c^3 + 4c^2 + 6c^6 + 6c^5 + 5c^4 + 2c^3 = 0$. This is not sufficient, however, because, for instance, if we set $c = -2$ in this equation, it holds but the critical orbit is only $\{-2, 2\}$.

In fact, the above octic factors as $c^3(c + 1)^2(c + 2)(c^2 + 1)$. All the factors other than $c^2 + 1$ (= $p_{3,1}(c)$) lead to degenerate cases.

**Definition 2.5.** Let $f(x) \in \mathbb{F}_q[x]$ be an irreducible quadratic polynomial with critical orbit $\mathcal{O}$ and $g(x) \in \mathbb{F}_q[x]$. We define the type of $g(x)$ at $\beta$ to be $s$ if $g(\beta)$ is a square in $\mathbb{F}_q$ and $n$ if it is not a square. The type of $g$ is a string of length $|\mathcal{O}|$ whose $k$th entry is the type of $g(x)$ at the $k$th entry of $\mathcal{O}$. The $k$th entry is also called the $k$th digit.

For instance, given $x^2 + 1 \in \mathbb{F}_7[x]$, consider $g(x) = x^2 + 2x + 2$. Then, $g(1) = 5, g(2) = 3, g(5) = 2$, which implies that the type of $g$ is $nns$.

**Definition 2.6.** Given an irreducible quadratic polynomial $f(x) \in \mathbb{F}_q[x]$ and a polynomial $g(x) \in \mathbb{F}_q[x]$, we call the factors of $g(f(x))$ the children of $g$. Also, for any natural number $m$, the factors of $g(f^m(x))$ are called the $m$–step descendants of $g$.

**Definition 2.7.** Let $f(x) \in \mathbb{F}_q[x]$ be a quadratic polynomial and $\gamma$ the unique critical point of $f$. We say $\gamma$ is periodic if there exists an $i \in \mathbb{N}$ s.t. $f^i(\gamma) = \gamma$.

Next we quote a lemma which is one of the building blocks of our paper.

**Lemma 2.2.** [Jones and Boston, 2012] Suppose that $f \in \mathbb{F}_q[x]$ is quadratic with critical orbit of length $o$ and all iterates separable. Let $g \in \mathbb{F}_q[x]$ be irreducible of even degree. Suppose that $h_1h_2$ is a non-trivial factorization of $g(f(x))$, and let $d_i$ (resp. $e_i$) be the $i$th digit of the type of $h_1$ (resp. $h_2$). Then there is some $k$, $1 \leq k \leq o$, with $d_i = e_k$ and $e_o = d_k$. Moreover, $k = o$ if and only if $\gamma$ is periodic, and in the case $\gamma$ is not periodic, we have $k = t$, where $t$ is the tail size of $f$.

In Jones and Boston, 2012, Jones and Boston tried to explain the distribution of types of factors (weighted by their degree) of iterates of $f$ by a Markov model as follows:

We create a time-homogeneous Markov process $Y_1, Y_2, \ldots$ related to $f$. The state space is the space of types of $f$, namely $\{n, s\}^o$, ordered lexicographically. We define the Markov process by giving its transition matrix $M = (P(Y_m = T_j | Y_{m-1} = T_i))$, where $T_i$ and $T_j$ vary over all types. Note that the entries of each column of $M$ sum to 1. We define $M$ by assuming that all allowable types of children arise with equal probability. To define allowable type, note that $f$ acts on its critical orbit, and thus also on the set of types. Indeed, if $T$ is a type, then $f(T)$ is obtained by shifting each entry one position to the left and using the former $m$th entry as the new final entry, where $m$ is such that $f^{o+1}(\gamma) = f^m(\gamma)$. If $g$ has type $T$ which begins with $n$, then $g$ has only one child, and it will have
type $f(T)$, the only allowable type in this case. If $T$ begins with $s$, then $g$ has two children, whose types have product $f(T)$. Among pairs of types $T_1, T_2$ with $T_1 T_2 = f(T)$, we call allowable those that satisfy the conclusion of Lemma 2.3, namely $d_k = e_o$ and $e_k = d_o$ with $k = o$ if $\gamma$ is periodic, and $k = t$ if $\gamma$ is not periodic, where $t$ is the tail size of $f$.

**Definition 2.8.** We define an $m$-step transition matrix as $M_m = (P(Y_{m+1} = T_j | Y_1 = T_i)$ by assuming that all allowable choices of $m$-step transition arise with equal probability. Here, allowable refers to those that arise for the given $f$, which turns out to be a subtle matter at the heart of this paper.

**Remark 2.3.** Since transitions in a Markov model are independent of each other, the model of Jones and Boston [Jones and Boston, 2012] implies that $M_m = M_1^m$ always holds.

### 3. New Phenomena

Contrary to what Jones and Boston suggested, we discover that the story of these descendants can be quite different in certain cases. More precisely, in these special cases, not every 2-step or 3-step transition permitted by the above model actually occurs. Thus, a Markov model does not apply to all quadratic polynomials. We now illustrate this idea with three kinds of examples:

**Example 3.1.** The first kind has orbit size 3 and tail size 1. As computed earlier, $p_{3,1}(c) = c^2 + 1$, so these are the quadratic polynomials of the form $f(x) = x^2 + i$, where $i$ is a square root of $-1$ in $\mathbb{F}_q$. (Note that to do so, we need $q \equiv 1 \pmod{4}$ and in fact $q \equiv 5 \pmod{8}$ to ensure that $f(x)$ is irreducible). The critical orbit is $\{i, i-1, -i\}$. Using Lemma 2.3, the following 1-step transitions arise:

- $n nn \mapsto n nn$
- $n ms \mapsto n sn$
- $n ns \mapsto n sn$
- $n ss \mapsto s ss$
- $s nn \mapsto s ns/sss$ or $s ns/snn$
- $s ns \mapsto s ns/sss$ or $s ns/snn$
- $s sn \mapsto s nn/nsn$ or $s ns/sns$
- $s ss \mapsto s nn/nsn$ or $s ns/sns$ or $s ss/sss$. 


It follows that

\[
M_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 \\
0 & 0 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 1/4 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 1 & 0 & 1/4 & 1/4 \\
\end{bmatrix}
\]

The new phenomenon is that the following was observed.

**Observation 3.1.** Let \( q \equiv 5 \pmod{8} \). Let \( f(x) = x^2 + i \in \mathbb{F}_q[x] \), where \( i \) is a square root of \(-1\). Then, the following 2-step transitions never occur:

- \( \text{nsn} \rightarrow \text{nns/snn} \)
- \( \text{nss} \rightarrow \text{nnn/mnn} \)
- \( \text{ssn} \rightarrow \text{sns/sns} \).

In particular, \( M_2 \neq M_1^2 \). That is to say, there is a discrepancy between the proposed Markov model and what actually happens. Since we know which 2-step transitions are forbidden, we explicitly calculate the discrepancy matrix \( A := M_2 - M_1^2 \).

\[
A = \begin{bmatrix}
0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

**Example 3.2.** The second kind has orbit size 4 and tail size 1. We have \( p_{4,1}(c) = c^6 + 2c^5 + 2c^4 + 2c^3 + c^2 + 1 \). Let \( c_0 \) be a root of \( p_{4,1} \) in some \( \mathbb{F}_q \) such that \( f(x) = x^2 + c_0 \) is irreducible. Again applying Lemma 2.3, the following 1-step transitions are valid:

- \( \text{nmm} \rightarrow \text{nmm} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
- \( \text{nms} \rightarrow \text{nms} \)
Observation 3.2. Let $c_0$ be a root of $p_{4,1}$ in $\mathbb{F}_q$ and $f(x) = x^2 + c_0 \in \mathbb{F}_q[x]$ be irreducible. Then the 2-step transitions given below never occur:

$$
\begin{align*}
nssn &\rightarrow nnnn/ssnn \\
nnn &\rightarrow ssns/ssnn \\
nn &\rightarrow nns/nnn \\
nns &\rightarrow nnns/ssnn \\
nss &\rightarrow ssns/ssnn \\
ssn &\rightarrow nnn/nnns \\
ss &\rightarrow nns/nnns \\
nns &\rightarrow sss/nns \\
nn &\rightarrow snns/sns \\
nn &\rightarrow nsns/sns \\
nn &\rightarrow nsss/nsns \\
nn &\rightarrow nssn/ssns \\
nn &\rightarrow nsns/ssnn \\
nn &\rightarrow nsss/ssnn \\
nn &\rightarrow snns/ssss \\
nn &\rightarrow nsns/ssss \\
nn &\rightarrow nsns/ssss \\
nn &\rightarrow nsss/ssss \\
nn &\rightarrow snss/ssss \\
nn &\rightarrow snss/ssss \\
nn &\rightarrow ssns/ssss \\
nn &\rightarrow ssns/ssss.
\end{align*}
$$

It follows that

$$
M_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8/8 & 8/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/8 & 8/8 & 8/8 & 8/8
\end{pmatrix}
$$

Analogously to the first example, however, we observe that once more certain 2-step transitions are forbidden. More precisely, the following is observed:
By the same reasoning as in Example 3.1, we can explicitly calculate the discrepancy matrix $A := M_2 - M_1^2$.

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/8 & -1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{pmatrix}$$

Example 3.3. Lastly, we consider examples with orbit size 3 and tail size 2. In this case, the difference polynomial $p_{3,2}(c) = c^3 + 2c^2 + 2c + 2$. Using Lemma 2.3, the 1-step transitions are as given below:

- $nnn \mapsto nnn$
- $nns \mapsto nss$
- $nsn \mapsto sns$
- $nss \mapsto sss$
- $snn \mapsto nsn/sns$ or $nns/ssn$
- $sns \mapsto nnn/snn$ or $snn/sns$
- $ssn \mapsto nns/nsn$ or $sns/ssn$
- $sss \mapsto nnn/mnn$ or $nss/nsn$ or $snn/snn$ or $sss/ssn$.

It follows that

$$M_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 \\
0 & 1 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\
0 & 0 & 1 & 1/4 & 0 & 1/4 & 0 & 1/4 \\
0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 \\
0 & 0 & 1 & 0 & 1/4 & 0 & 1/4 & 0
\end{bmatrix}$$

We observe, however, that certain 3-step transitions never arise.
Observation 3.3. Let $c_1$ be a root of $p_{3,2}$ in $\mathbb{F}_q$. Let $f(x) = x^2 + c_1 \in \mathbb{F}_q[x]$ be irreducible. Then the following 3-step transitions do not occur:

\[
nns \mapsto nss \mapsto sss \mapsto nss/nss
\]

\[
nns \mapsto nss \mapsto sss \mapsto snn/snn.
\]

It follows that the discrepancy matrix $A$, which this time is $M_3 - M_1^3$, is:

\[
A = \begin{pmatrix}
0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

4. New Model

The investigations in the previous section show that a Markov model does not always fit the factorization process for iterates of quadratic polynomials. We need a new model to explain the process and we propose the following.

Let $a - 1$ and $b$ be the tail and orbit sizes of an irreducible quadratic polynomial $f$ defined over $\mathbb{F}_q$, respectively. Then the $m$-step transition matrices associated to $f$ satisfy the following recurrence relation:

\[
M_{m+a} = M_{m+a-1}B + M_mA \tag{1}
\]

where $M_{-a+1} = \cdots = M_{-2} = M_{-1} = 0$, $M_0 = I$.

Corollary 4.1. The following hold for the new model:

(i) $B = M_1$.

(ii) $M_i = M_1^i$ for $i = 1, 2, \cdots, a - 1$.

(iii) $A = M_a - M_1^a$.

Proof. (i) Setting $m = -a + 1$ in (1) gives the result.

(ii) We prove this by induction. Assume $M_k = M_1^k$ is true for an integer $1 \leq k < a-1$. Setting $m = k-a+1$ in (1) gives $M_{k+1} = M_k B + M_{k-a+1} A$. Since $M_{k-a+1} = 0$, $M_{k+1} = M_k B = M_k M_1$. By induction, $M_k = M_1^k$, which yields $M_{k+1} = M_1^{k+1}$.
Setting \( m = 0 \) in (1) gives \( M_a = M_{a-1}B + M_0A \). From the previous two parts and the initial conditions, we know \( B = M_1, M_{a-1} = M_1^{a-1}, \) and \( M_0 = I \). Plugging these into (1), the result follows.

**Remark 4.2.** The new model with tail size \( a - 1 \) is called an \( a \)-step Markov model.

**Conjecture 4.3.** The multi-step Markov model given above describes the factorization process for the iterates of an irreducible quadratic polynomial over a finite field of odd order.

Of course, this is only approximate at any finite level, but it leads to predictions as regards the limiting behavior. In particular, the multi-step Markov model predicts that in the limit 100% of the factorization of the iterates will be of type \( nn \cdot \cdots n \) (the unique sink) and also allows us to compute the limiting relative proportions of the other types as follows.

We fix an arbitrary natural number \( m \) and define the vector \( v_i \) to be the vector whose entries are the proportions of all \( 2^b \) types (lexicographically ordered) for the \((m+i)\)th iterate of the polynomial \( f \). Say \( v = (v_1, v_2, \cdots, v_a) \). Then, using (1), the next such \( a \)-tuple will, according to the model, be the vector \((v_2, v_3, \cdots, v_a, Av_1 + Bv_a)\). Denoting the associated \( a2^b \) by \( a2^b \) transition matrix by \( T \), we have

\[
T = \begin{pmatrix}
0 & I & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & I & 0 \\
A & 0 & \cdots & B
\end{pmatrix}.
\]

We can thereby interpret this multi-step Markov model as a Markov process on a larger number of states, with transition matrix \( T \). The limiting frequencies of the non-absorbing states are given, up to scaling, by the entries of an eigenvector of \( T \) corresponding to its largest eigenvalue less than 1. [Seneta, 1981]

Combining this fact with the following lemma indicates how the limiting proportions can be computed:

**Lemma 4.4.** With the notation as above, let \( e \) be an eigenvector of the transition matrix \( T \) corresponding to eigenvalue \( \lambda \), and \( e_1 \) be its first \( 2^b \) entries. Then \( e = (e_1, \lambda e_1, \lambda^2 e_1, \cdots, \lambda^{k-1} e_1) \)

**Proof.** This is a consequence of Theorem 3.2 in [Dennis, 1976] (or can be easily directly proven).
5. Data

In this section, we provide actual data corresponding to examples 3.1, 3.2 and 3.3. In each case, we use the smallest \( q \) for which the corresponding difference polynomial has a root and that yields an irreducible quadratic. Comparing the limiting proportions predicted by the new model with the data for each example, we will illustrate how well the multi-step Markov model fits.

**Data for Example 3.1**

| Iterate | nns | nsn | nss | snn | sns | ssn |
|---------|-----|-----|-----|-----|-----|-----|
| 20      | 0.0251 | 0.1748 | 0.1163 | 0.0271 | 0.2541 | 0.1143 | 0.2883 |
| 21      | 0.0268 | 0.1661 | 0.1221 | 0.0267 | 0.2635 | 0.1222 | 0.2726 |
| 22      | 0.0300 | 0.1725 | 0.1253 | 0.0271 | 0.2487 | 0.1282 | 0.2681 |
| 23      | 0.0256 | 0.1689 | 0.1223 | 0.0253 | 0.2508 | 0.1226 | 0.2846 |
| 24      | 0.0238 | 0.1686 | 0.1240 | 0.0238 | 0.2542 | 0.1239 | 0.2817 |
| 25      | 0.0276 | 0.1699 | 0.1217 | 0.0272 | 0.2598 | 0.1220 | 0.2748 |
| 26      | 0.0263 | 0.1699 | 0.1276 | 0.0282 | 0.2526 | 0.1256 | 0.2697 |
| 27      | 0.0263 | 0.1677 | 0.1237 | 0.0269 | 0.2502 | 0.1231 | 0.2821 |

Table 1: Relative proportions of types (other than \( nnn \)) for factors of iterates of \( f(x) = x^2 + 2 \in F_5[x] \).

By comparison, if we consider the related block matrix in the previous section, the first part \( e_1 \) of an eigenvector for the eigenvalue \( \lambda \approx 0.9333801995 \) is

\[
\begin{bmatrix}
-1.0000000000 \\
0.026110931 \\
0.170493119 \\
0.123960675 \\
0.026110931 \\
0.254036800 \\
0.123960675 \\
0.275326866 \\
\end{bmatrix}
\]

**Data for Example 3.2**

| Iterate | nns | nsn | nss | snn | sns | ssn |
|---------|-----|-----|-----|-----|-----|-----|
| 21      | 0.0189 | 0.0932 | 0.0344 | 0.0869 | 0.0283 | 0.1194 | 0.0496 | 0.0124 | 0.1114 | 0.0506 | 0.0230 | 0.1227 | 0.0095 | 0.1152 |
| 22      | 0.0177 | 0.0705 | 0.0483 | 0.1086 | 0.0187 | 0.1039 | 0.0483 | 0.0137 | 0.1021 | 0.0486 | 0.0811 | 0.0210 | 0.1450 | 0.0497 | 0.1228 |
| 23      | 0.0178 | 0.0816 | 0.0414 | 0.0934 | 0.0182 | 0.1135 | 0.0476 | 0.0180 | 0.1305 | 0.0465 | 0.0870 | 0.0171 | 0.1272 | 0.0435 | 0.1166 |
| 24      | 0.0232 | 0.0904 | 0.0493 | 0.1044 | 0.0189 | 0.0992 | 0.0524 | 0.0183 | 0.1116 | 0.0559 | 0.0763 | 0.0169 | 0.1348 | 0.0527 | 0.1057 |
| 25      | 0.0190 | 0.0859 | 0.0466 | 0.1067 | 0.0191 | 0.1135 | 0.0486 | 0.0185 | 0.1254 | 0.0487 | 0.0790 | 0.0199 | 0.1187 | 0.0464 | 0.1114 |
| 26      | 0.0188 | 0.0739 | 0.0486 | 0.1056 | 0.0199 | 0.0209 | 0.0500 | 0.0175 | 0.1247 | 0.0493 | 0.0776 | 0.0194 | 0.1332 | 0.0514 | 0.1115 |
| 27      | 0.0178 | 0.0828 | 0.0497 | 0.0963 | 0.0189 | 0.1107 | 0.0493 | 0.0176 | 0.1266 | 0.0505 | 0.0792 | 0.0186 | 0.1218 | 0.0489 | 0.1115 |
Table 2: Relative proportions of types (other than \(nnn\)) for factors of iterates of \(f(x) = x^2 + 3 \in \mathbb{F}_{11}[x]\).

If we compute the appropriate eigenvector of the related 32 by 32 matrix, its first block \(e_1\) of size 16 is

\[
\begin{bmatrix}
-1.0000000000 \\
0.0186693999 \\
0.0790508066 \\
0.0492462674 \\
0.0991960362 \\
0.0186693999 \\
0.1101985249 \\
0.0492462674 \\
0.0186693999 \\
0.0790508066 \\
0.0186693999 \\
0.1309252656 \\
0.0492462674 \\
0.01101985249 \\
\end{bmatrix}
\]

Data for Example 3.3

| Iterate | nns | nsn | nss | snn | sns | ssn | sss |
|---------|-----|-----|-----|-----|-----|-----|-----|
| 26      | 0.0731 | 0.0728 | 0.1673 | 0.1827 | 0.0718 | 0.0722 | 0.3601 |
| 27      | 0.0760 | 0.0727 | 0.1695 | 0.1863 | 0.0699 | 0.0732 | 0.3523 |
| 28      | 0.0736 | 0.0754 | 0.1798 | 0.1734 | 0.0747 | 0.0729 | 0.3502 |
| 29      | 0.0654 | 0.0761 | 0.1639 | 0.1873 | 0.0772 | 0.0665 | 0.3636 |
| 30      | 0.0747 | 0.0762 | 0.1757 | 0.1876 | 0.0730 | 0.0714 | 0.3414 |
| 31      | 0.0714 | 0.0772 | 0.1735 | 0.1772 | 0.0766 | 0.0707 | 0.3535 |
| 32      | 0.0715 | 0.0713 | 0.1818 | 0.1910 | 0.0703 | 0.0706 | 0.3434 |
| 33      | 0.0716 | 0.0756 | 0.1720 | 0.1738 | 0.0783 | 0.0743 | 0.3544 |
| 34      | 0.0711 | 0.0708 | 0.1859 | 0.1863 | 0.0715 | 0.0718 | 0.3426 |

Table 3: Relative proportions of types (other than \(nnn\)) for factors of iterates of \(f(x) = x^2 + 1 \in \mathbb{F}_7[x]\).

As mentioned before, in [Jones and Boston, 2012], Jones and Boston proposed a Markov process, and they supported this claim by the example \(x^2 + 1\) over \(\mathbb{F}_7\). However, the result given in Observation 3.3 does not follow this claim. To illustrate how the multi-step Markov model fits better, the following table compares the limiting proportions predicted by the Markov model and the multi-step Markov model:
Table 4: Limiting proportions of types (other than nnn) for factors of iterates of \(x^2 + 1 \in \mathbb{F}_7[x]\) predicted by the Markov model and the multi-step Markov model.

It is particularly striking how much better the new model fits the data for sss.

6. Conjectures/Speculations

In this last part, we present some conjectures based on the many different computational results we have obtained.

In section 3, we observed that for the irreducible quadratic polynomials with difference polynomials \(p_3,1\) and \(p_4,1\), there are certain missing 2-step transitions. After further investigations with many quadratic polynomials, we conjecture that the same phenomenon happens for every irreducible quadratic polynomial with tail size 1. What would establish that those 2-step transitions are forbidden is the following conjecture.

**Conjecture 6.1.** Let \(f\) be an irreducible quadratic polynomial over \(\mathbb{F}_q\) with tail size \(t = 1\) and orbit size \(o\) and let \(g\) be an even irreducible polynomial over \(\mathbb{F}_q\) whose type begins with ns. Then, the \((o - 1)\)th digit of the type of each irreducible factor of \(g(f(x))\) is s.

**Example 6.1.** Note that the \(o\)th digit is \(-c\) and so the \((o - 1)\)th digit is \(\alpha\) where \(\alpha^2 + c = -c\), i.e. \(\alpha^2 = -2c\). Suppose that \(g(x) = x^4 + ax^2 + b\). Then \(g(x^2 + c)\) factors as \(h(x)h(-x)\) and we must show that \(h(\alpha)\) is a square. If \(h(x) = x^4 + px^3 + qx^2 + rx + s\), then, comparing coefficients on the two sides of (\(*\)), we eliminate \(q, s, a\), leaving that

\[
h(\alpha) = (\alpha^2 + pa/2 + r/p)^2 = (-2c + r/p + pa/2)^2.
\]

**Remark 6.2.** The above conjecture applies to the case \(f(x) = x^2 - 2\), too, which is the simplest with tail size 1. It is, however, vacuous for factors of iterates of \(x^2 - 2\) itself, because, as indicated by Jones and Boston [Jones and Boston, 2012], the factors are entirely of type nn after a finite number of iterates, whatever \(q\) is.

We end by listing other cases investigated, not covered in previous sections:

(i) \(o = 4\).

\[t = 2.\]

\[p_{4,2}(c) = c^3 + c^2 - c + 1.\]

The first example is \(x^2 + 4 \in \mathbb{F}_7[x]\).

This appears to have no missing transitions, i.e. follows a Markov model.

(ii) \(o = 5.\)

\[t = 2.\]
\[ p_{5,2}(c) = c^{12} + 6c^{11} + 14c^{10} + 18c^9 + 18c^8 + 16c^7 + 10c^6 + 6c^5 + 5c^4 + 2c^3 + 1. \]

The first example is \( x^2 + 12 \in F_{17}[x] \).

This appears to have no missing transitions, i.e. follows a Markov model.

(iii) \( o = 4 \).

\[ t = 3. \]

\[ p_{4,3}(c) = c^7 + 4c^6 + 6c^5 + 6c^4 + 4c^2 + 2 + 2. \]

The first example is \( x^2 + 2 \in F_{7}[x] \).

This appears to have no missing transitions, i.e. follows a Markov model.

(iv) \( o = 5 \).

\[ t = 3. \]

\[ p_{5,3}(c) = c^8 + 4c^7 + 6c^6 + 6c^5 + 4c^4 + 1. \]

The first example is \( x^2 + 1 \in F_{11}[x] \).

This appears to have no missing transitions, i.e. follows a Markov model.

The evidence so far suggests that the only cases where a Markov process does not hold are those noted earlier, namely tail size 1 and any orbit size or tail size 2 and orbit size 3.

**Conjecture 6.3.** Let \( f(x) \) be a quadratic irreducible polynomial in \( F_q[x] \) with orbit size \( o \) and tail size \( t \). Then the Markov model fits the factorization process for iterates of \( f \) if and only if \( (o, t) \not\in \{(m, 1)| m \geq 2\} \cup \{(3, 2)\} \).

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