In this paper, we present two improved Amati correlations of gamma-ray burst (GRB) data via a powerful statistical tool called copula. After calibrating with the low-redshift GRB data, the improved Amati correlations based on a fiducial $\Lambda$ cold dark matter ($\Lambda$CDM) model with $\Omega_{\text{m}_0} = 0.3$ and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and extrapolating the results to the high-redshift GRB data, we obtain the Hubble diagram of GRB data points. Applying these GRB data to constrain the $\Lambda$CDM model, we find that the improved Amati correlation from copula can give a result well consistent with $\Omega_{\text{m}_0} = 0.3$, while the standard Amati and extended Amati correlations do not. This results suggest that when the improved Amati correlation from copula is used in the low-redshift calibration method, the GRB data can be regarded as a viable cosmological explorer. However, the Bayesian information criterion indicates that the standard Amati correlation remains to be favored mildly since it has the least model parameters. Furthermore, once the simultaneous fitting method rather than the low-redshift calibration one is used, there is no apparent evidence that the improved Amati correlation is better than the standard one. Thus, more work needs to be done in the future in order to compare different Amati correlations.

Unified Astronomy Thesaurus concepts: Gamma-ray bursts (629); Cosmology (343)
describe the evolutionary function of the Amati correlation with redshift, and Wang et al. (2017) selected two redshift-dependent formulas to parameterize the coefficients in the Amati correlation, which is called the extended Amati correlation in this paper (the details can be found in Appendix B). Since the luminosity \( L_{\text{iso}} \), the isotropic energy \( E_{\text{iso}} \), and the collimated-corrected energy \( E_{\gamma} \) in GRBs are cosmology dependent, the Hubble diagram can be obtained from standardized GRB samples.

To use the GRBs as a cosmological probe, we need to calibrate GRB correlations. Let us point out here that the cosmology-dependent calibration method suffers the so-called circularity problem (Ghirlanda et al. 2006; Wang et al. 2015). To avoid this problem, a low-redshift method (Liang et al. 2008; Kodama et al. 2008; Wei & Zhang 2009; Liang et al. 2010) to calibrate the GRB correlations in a cosmology-independent way was proposed, which uses other distance probes such as SN Ia to calibrate the GRBs at low redshifts and then extrapolates the results to high redshifts to constrain the cosmological parameters. For this method, since the intrinsic dispersion of SN Ia data is very much smaller than the GRB intrinsic dispersion, the SN Ia data will dominate over the GRB data in a joint analysis of them on the cosmological constraints and so the resulting cosmological constraints are effectively from SN Ia data. On the other hand, the simultaneous fitting or global fitting (Ghirlanda et al. 2004b; Li et al. 2008) limits the coefficients of the luminosity correlations and the parameters of cosmological models simultaneously from the observational GRB data. In Khadka & Ratra (2020) and Khadka et al. (2021), by simultaneously fitting cosmological and GRB Amati correlation parameters and using a number of different cosmological models, they found that the Amati correlation parameter values are independent of the cosmological model, which seems to mean that there is no circularity problem and that these GRB data sets are standardizable within the error bars. Until now, the GRB data has been used widely to investigate different cosmological models (see recent work in Amati et al. 2019; Demianski et al. 2021; Cao et al. 2021b, 2022; Hu et al. 2021; Luongo & Muccino 2021; Wang et al. 2022).

In this work we aim to improve the Amati correlation, which has been widely used in GRB cosmology. We introduce a powerful statistical tool called copula, which is a specialized tool developed in modern statistics to describe the complicated dependent structures between multivariate random variables. Copula has been widely used in various areas such as mathematical finance and hydrology in the past few decades. In recent years, it has gradually been recognized by the astronomical community as a very useful tool to analyze data. For example, Yuan et al. (2018) successfully determined the luminosity function of the radio cores in active galactic nuclei via copula, which is difficult to do if the traditional method is used. Using copula, Koen (2009) studied the correlation between the GRB peak energy and the associated supernova peak brightness, Benabed et al. (2009) proposed a new approximation for the low multipole likelihood of the CMB temperature, and Jiang et al. (2009) constructed a period-mass function for extrasolar planets. In addition, the copula likelihood function was constructed for the convergence power spectrum from the weak lensing surveys by Sato et al. (2010, 2011). Modeling bivariate astronomical data with copula instead of the conventional Gaussian mixture method was proposed by Koen & Bere (2017). The copula function is also useful in the study of galaxy luminosity functions and the large scale structure fields of matter density (Takeuchi 2010; Takeuchi et al. 2013; Takeuchi & Kono 2020; Scherrer et al. 2010; Qin et al. 2020). Based on the three-dimensional Gaussian copula, we propose, in this work, two improved Amati correlations and compare them with the standard Amati correlation and the extended Amati correlation by using the latest GRB samples (Khadka et al. 2021).

This paper is organized as follows: Section 2 studies the issue of how to construct a three-dimensional probability density function (PDF) through a Gaussian copula. Two improved Amati correlations are obtained in Section 3. In Section 4, a comparison between the improved Amati correlations and the (extended) Amati correlation is made by using both low-redshift calibration and simultaneous fitting methods. The conclusions are given in Section 5.

### 2. Copula

Briefly speaking, copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions (Nelson 2006). Letting \( x, y, \) and \( z \) be three random variables, we use \( F(x), G(y), \) and \( W(z) \) to express their marginal cumulative distribution functions (CDFs), respectively, and \( H(x, y, z) \) as the joint distribution function of the three variables. According to Sklar’s theorem, if \( F, G, \) and \( W \) are continuous, there exists a unique copula \( C \) such that

\[
H(x, y, z; \theta) = C(F(x), G(y), W(z); \theta),
\]

where \( \theta \) denotes the parameters of the copula function \( C \). Letting \( u \equiv F(x), v \equiv G(y), \) and \( q \equiv W(z) \), the joint PDF \( h(x, y, z; \theta) \) can be obtained by

\[
h(x, y, z; \theta) = \frac{\partial^3 H(x, y, z; \theta)}{\partial x \partial y \partial z} = \frac{\partial C(u, v, q; \theta)}{\partial u \partial v \partial q} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \frac{\partial q}{\partial z} = c[u, v, q; \theta]f(x)g(y)w(z),
\]

where \( f(x), g(y), \) and \( w(z) \) are the marginal PDFs of \( x, y \) and \( z \) respectively, and \( c[u, v, q; \theta] \) is the density function of \( C \).

One obvious advantage of Equation (1) is that by using copulas one can model the dependence structure and the marginal distributions separately. All the information on the dependence between the three variables is carried by the copula (see, e.g., Yuan et al. 2018). The next issue is to find a three-dimensional optimal copula function and estimate its parameters to describe the observed data in GRBs. The bivariate copulas are abundant and thus the three-dimensional copula functions are abundant too since most of the two-dimensional copula can be extended easily to the three-dimensional case. Here we use the three-dimensional Gaussian copula function with linear correlation coefficient \( \theta = \{ \rho_1, \rho_2, \rho_3 \} \) to model our data, i.e.,

\[
C(u, v, q; \theta) = \Psi_3[\Psi_1^{-1}(u), \Psi_1^{-1}(v), \Psi_1^{-1}(q); \theta],
\]

where \( \Psi_3 \) and \( \Psi_1 \) are the standard three-dimensional Gaussian CDF and one-dimensional Gaussian CDF, respectively, and \( \Psi_1^{-1} \) denotes the inverse of \( \Psi \). The density function of the
Gaussian copula can be obtained from
\[
c(c, u, v; \theta) = \frac{\partial^3 \Psi_3([\Psi_1^{-1}(u), \Psi_1^{-1}(v), \Psi_1^{-1}(q); \theta])}{\partial u \partial v \partial q} \\
= \frac{1}{\sqrt{\det \Sigma}} \exp \left\{ -\frac{1}{2} [\Psi^{-1}(\Sigma^{-1} - I) \Psi^{-1}] \right\},
\]
where \( \Psi^{-1} = [\Psi^{-1}(u), \Psi^{-1}(v), \Psi^{-1}(q)] \), \( I \) stands for the identity matrix, and \( \Sigma^{-1} \) denotes the inverse of the covariance matrix \( \Sigma \), which reads as
\[
\Sigma = \begin{pmatrix}
1 & \rho_1 & \rho_2 \\
\rho_1 & 1 & \rho_3 \\
\rho_2 & \rho_3 & 1
\end{pmatrix}.
\]

The conditional PDF of \( y \) denotes the probability of variable \( y \) when \( x \) and \( z \) are fixed, which can be expressed as
\[
g_y(y|x, z; \theta) = \frac{h(x, y, z; \theta)}{h_x(z; \rho_2)} = \frac{c(u, v, q; \theta) f(x) g(y) w(z)}{c(u, q; \rho_2) f(x) w(z)} = \frac{c(u, v, q; \theta)}{c(u, q; \rho_2)} g(y),
\]
where \( h_x(z; \rho_2) \) is constructed from a two-dimensional Gaussian copula with the correlation coefficient being \( \rho_2 \). If variable \( y \) obeys a Gaussian distribution with the standard deviation being \( \sigma_y \), then \( g_y \) can be expressed as
\[
g_y(y|x, z; \theta) = \frac{1}{\sqrt{2\pi \sigma^2_{y|x,z}}} \exp \left\{ -\frac{1}{2} S(x, y, z; \theta) \right\},
\]
where
\[
\sigma^2_{y|x,z} = \frac{\sigma^2_y(1 - \rho^2_1 - \rho^2_2 - \rho^2_3 + 2\rho_1\rho_2\rho_3)}{1 - \rho^2_2}.
\]

In Equation (7), \( S(x, y, z; \theta) \) is unknown if the marginal distributions of variables \( x \) and \( z \) are undetermined. Due to the variable \( y \) obeying the Gaussian distribution, the highest probability of \( y \) corresponds to \( S(x, y, z; \theta) = 0 \).

### 3. Improved Amati Correlation

Now, we use the copula function introduced in the above section to investigate the correlation between \( E_p \) and \( E_{iso} \) of GRBs. After using \( x, y, \) and \( z \) to denote \( \log \frac{E_p}{3000keV} \), \( \log \frac{E_{iso}}{1erg} \), and redshift \( z \) of GRBs, respectively, the marginal distribution of \( x, y, \) and \( z \) need to be given in order to determine \( S(x, y, z; \theta) \). We assume that two Gaussian distributions for \( x = \log \frac{E_p}{3000keV} \) and \( y = \log \frac{E_{iso}}{1erg} \) are
\[
f(x; \bar{a}_x, \sigma_x) = \frac{1}{\sqrt{2\pi \sigma_x}} e^{-\frac{(a_x-x)^2}{2\sigma_x^2}}, \quad g(y; \bar{a}_y, \sigma_y) = \frac{1}{\sqrt{2\pi \sigma_y}} e^{-\frac{(a_y-y)^2}{2\sigma_y^2}}.
\]

Here \( \bar{a} \) and \( \sigma \) represent the mean value and the standard deviation of the Gaussian distribution, respectively. And the CDFs of \( x, y \), respectively, have the forms
\[
F(x; \bar{a}_x, \sigma_x) = \int_{-\infty}^{x} f(\tilde{x}; \bar{a}_x, \sigma_x) d\tilde{x},
\]
\[
G(y; \bar{a}_y, \sigma_y) = \int_{-\infty}^{y} g(\tilde{y}; \bar{a}_y, \sigma_y) d\tilde{y}.
\]

We consider two different redshift distributions of GRB data:

1. The first distribution is a special form with the PDF being \( w(z) = ze^{-z} \) (Wang et al. 2017). Thus, the corresponding CDF is
\[
W(z) = 1 - e^{-z}(1 + z).
\]

2. The second CDF of the GRB’s redshift distribution is the empirical distribution (Dekking et al. 2005):
\[
W_N(z) = \begin{cases}
0 & \text{if } z < z_1 \\
n & \text{if } z_i \leq z < z_{i+1} \\
1 & \text{if } z_n \leq z,
\end{cases}
\]

where \( \{z_1, z_2, ..., z_n\} \) is an ordered list of redshifts of GRB data, which satisfies \( z_1 \leq z_2 \leq ... \leq z_n \), and \( n \) denotes the number of data.

Since the first distribution of redshift \( z \) of GRBs given in Equation (11) is an assumed special form, we need to evaluate whether the redshift of observed GRB data obeys this form by using the Kolmogorov–Smirnov (K-S) test. The K-S test is based on the distance measure \( D \), which is defined to be
\[
D = \max \{|W_N(z) - W(z)|\},
\]
where \( W_N(z) \) and \( W(z) \) are the empirical and assumed CDFs of the redshift distribution of GRBs, respectively. Apparently, \( D \) represents the maximum deviation between two CDFs. In our K-S test, we set the significance level \( \alpha \) to be \( \alpha = 0.05 \), where \( \alpha = 1 - P(\lambda_N) \) and \( P(\lambda) \) is the Kolmogorov distribution
\[
P(\lambda) = \sum_{i=-\infty}^{+\infty} (-1)^i \exp(-2i^2\lambda^2)
\]

obeying the Kolmogorov stochasticity parameter \( \lambda \). Here \( \lambda \equiv \frac{\sqrt{N}D}{\lambda} \) and \( N \) is the number of data. Then, a critical value \( D_0 = \frac{\lambda}{\sqrt{N}} \approx 0.09 \) can be obtained, which means that if the data satisfies the assumed distribution, the probability that \( D < D_0 \) is about 95%. Conversely, the probability that \( D > D_0 \) is only 5%. If \( D > D_0 \), the assumed distribution will be rejected since it is a rare event. Figure 1 shows the CDFs of the empirical distribution \( W_N(z) \) from the latest 220 GRB data points (Khakda et al. 2021) and the assumed distribution \( W(z) \) used in our discussion. We find that \( D \approx 0.06 \) at \( z \approx 2 \), which is less than \( D_0 \). Thus, the assumed redshift distribution of GRB data passes the K-S test and is acceptable.

Substituting the CDFs given in Equations (10)–(12) into Equations (4) and (7), \( S(x, y, z; \theta) \) can be derived. However, the concrete expression is very complicated, so we do not show it here. From the equation of \( S(x, y, z; \theta) = 0 \), we can obtain
\[
y_{\text{copula}_1} = a + b x + c \text{erfc}^{-1}[2W(z)],
\]
\[
y_{\text{copula}_2} = a + b x + c \text{erfc}^{-1}[2W_N(z)].
\]

Here the subscripts \( \text{copula}_1 \) and \( \text{copula}_2 \) denote the correlation relations from the copula method with the assumed redshift distribution (Equation (11)) and the empirical redshift distribution (Equation (12)), respectively, \( \text{erfc} \) is the...
complementary error function, and coefficients $a$, $b$, and $c$ are defined as

$$a = \bar{a}_y - \frac{(\rho_2 \rho_1 - \rho_1)\bar{a}_x \sigma_y}{(\rho_2^2 - 1)\sigma_x},$$

$$b = \frac{(\rho_2 \rho_1 - \rho_1)\sigma_y}{(\rho_2^2 - 1)\sigma_x},$$

$$c = \frac{\sqrt{2}\sigma_y(\rho_3 - \rho_1 \rho_2)}{\rho_2^2 - 1}. \quad (16)$$

Equation (15) is different from the standard Amati correlation (Equation (A1)) by a redshift-dependent term, and this redshift-dependent correction term is also different from that of the extended Amati correlation (Equation (B2)). We name these luminosity correlations from the Gaussian copula as the improved Amati correlations, and they are the main results of our paper.

4. Hubble Diagrams and GRB Cosmology

4.1. Low-redshift Calibration

To test the viability of the improved Amati correlations given in Equation (15), we use two GRB data samples: one is the latest GRB sample (A220) (Khadka et al. 2021), which contains 220 long GRBs in the redshift range of $z \in [0.03, 8.2]$, and the other is the higher-quality A118 data set (Khadka & Ratra 2020; Wang et al. 2016; Fana Dirirsa et al. 2019) contained in A220 with a redshift range of $z \in [0.34, 8.2]$ since it has a tighter intrinsic scatter. These GRBs are divided into the low-redshift part (79 and 20 GRBs at $z \in [0, 1.4]$ in A220 and A118, respectively) and the high-redshift one (141 and 98 GRBs at $z \in [1.4, 8.2]$ in A220 and A118, respectively). We will use the low-redshift GRB data of two data sets to determine the coefficients in $\gamma_{\text{copula}}$, and then extrapolate these results to the high-redshift data to obtain their luminosity distances, which will then be used to constrain the cosmological model. Since the isotropic equivalent radiated energy $E_{\text{iso}}$ (see Equation (A4)) is dependent on the luminosity distance, a fiducial cosmological model needs to be chosen. Here the spatially flat $\Lambda$CDM with $\Omega_{\text{m0}} = 0.30$ and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is chosen as the fiducial model, where $\Omega_{\text{m0}}$ is the dimensionless present matter density parameter. Then, the allowed regions of $a$, $b$, $c$, and $\sigma_{\text{int}}$ can be obtained by maximizing the D’Agostinis likelihood function

$$L(\sigma_{\text{int}}, a, b, c) \propto \prod_{i=1}^{N} \frac{1}{\sqrt{\sigma_{\text{int}}^4 + \sigma_{y,i}^2 + b^2 \sigma_{x,i}^2}} \times \exp \left[ -\frac{[y_i - \gamma_{\text{copula}}(x_i, z_i; a, b, c)]^2}{2(\sigma_{\text{int}}^4 + \sigma_{y,i}^2 + b^2 \sigma_{x,i}^2)} \right]. \quad (17)$$

Here $N = 79$ or 20, $\sigma_{\text{int}}$ is the intrinsic scatter of GRBs, and $y_i$ denotes the observed log $E_{\text{iso}}$ of the low-redshift GRBs. For comparison, the Amati and extended Amati correlations, which are denoted by $\gamma_{\text{Amati}}$ and $\gamma_{\text{exAmati}}$, respectively, are also investigated. In our analysis, the CosmoMC code is used. The results are summarized in Tables 1 and 2.

From the two tables, one can see that in the case of the copula correlation relation the value of intrinsic scatter $\sigma_{\text{int}}$ is
log $L_z$ is the likelihood function, from the 79 low-redshift GRBs in the A118 data set. Here $\sigma_{\text{int}}$ may also be found from the Pearson linear correlation of intrinsic scatter $\sigma_{\text{int}}$ from the A118 GRB data set. The best-fit $\sigma_{\text{int}}$ is the number of data. Here we denote the difference between $\sigma_{\text{int}}$ and $\sigma_{\text{Amati}}$, which means that the quality of the correlation relation is improved slightly when the copula relation is used. A comparison of Tables 1 and 2 reveals that the value of intrinsic scatter $\sigma_{\text{int}}$ from the A118 GRB data set is smaller than the one from the A220 GRB data set. Thus, the 118 data set is a higher-quality one compared to the A220 data set, which agrees with the results obtained in Khadka et al. (2021). Furthermore, Table 1 shows that the coefficient $c$ in $y_{\text{copula}}$ or $y_{\text{copula}}$ and coefficients $\alpha$ and $\beta$ in $y_{\text{Amati}}$ are at most $1 \sigma$ away from 0, which indicates that there is not strong support for nonzero values of these parameters from the A220 data set. These results are consistent with those obtained in (Khadka et al. 2021) where the relation parameters $a$ and $b$ were found to be independent of redshift within the error bars. This character may also be found from the Pearson linear correlation coefficients $\rho_1$, $\rho_2$, and $\rho_3$, which denote the degree of linear correlation between variables $x$, $y$, and $z$. We derive $\rho_1 = 0.771$, $\rho_2 = 0.493$, and $\rho_3 = 0.426$ from the 79 low-redshift GRBs in the A220 data set and $\rho_1 = 0.781$, $\rho_2 = 0.381$, and $\rho_3 = 0.377$ from the 20 low-redshift GRBs in the A118 data set. Since $\rho_2$ and $\rho_3$ have values of about 0.4 and they are clearly smaller than $\rho_1$, the linear correlations between $\log \frac{E_{\gamma}}{300 \text{keV}}$ and $z$ and $\log \frac{E_{\gamma}}{300 \text{keV}}$ and $z$ are weak, and they are weaker than the linear correlation between $\log \frac{E_{\gamma}}{300 \text{keV}}$ and $E_{\text{iso}}$.

To compare the correlation relations from copula function and the (extended) Amati relation, we compute the values of the Akaike information criterion (AIC; Akaike 1974, 1981) and the Bayesian information criterion (BIC; Schwarz 1978), which, respectively, are defined as

$$ AIC = 2p - 2 \ln L, $$

$$ BIC = p \ln N - 2 \ln L, $$

(18)

where $L$ is the likelihood function, $p$ is the number of free parameters in a model, and $N$ is the number of data. Here we also compute the $\Delta \text{AIC}(\Delta \text{BIC})$, which denotes the difference between AIC(BIC) with respect to the reference model (the standard Amati model here). $0 < \Delta \text{AIC}(\Delta \text{BIC}) \leq 2$ indicates difficulty in preferring a given model, $2 < \Delta \text{AIC}(\Delta \text{BIC}) < 6$ means mild evidence against the given model, and $\Delta \text{AIC}(\Delta \text{BIC}) > 6$ suggests strong evidence against the model. The obtained values of $\Delta \text{AIC}$ and $\Delta \text{BIC}$ are summarized in Tables 1 and 2. We find that except for the case of the extended Amati correlation whose $\Delta \text{AIC}$ given by the low-redshift GRBs in the A118 data set is larger than 2, the AIC cannot

| Table 1 | Constraints on Four Correlations from the Low-redshift GRBs of A220 Data Set |
|---------|--------------------------------------------------------------------------------|
| Amati   | Extended Amati | copula$^1$ | copula$^2$ |
| Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL |
| $\sigma_{\text{int}}$ | 0.512(0.045) | $\pm 0.034$ | 0.503(0.046) | $\pm 0.063$ | 0.510(0.045) | $\pm 0.034$ | 0.509(0.045) | $\pm 0.033$ |
| $a$ | 52.710(0.061) | $\pm 0.060$ | 52.587(0.333) | $\pm 0.334$ | 52.847(0.144) | $\pm 0.145$ | 52.812(0.149) | $\pm 0.141$ |
| $b$ | 1.290(0.126) | $\pm 0.123$ | 1.521(0.367) | $\pm 0.301$ | 1.209(0.150) | $\pm 0.148$ | 1.231(0.145) | $\pm 0.156$ |
| $c$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\alpha$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\beta$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

$\Delta \text{AIC}$ = 121.514, $\Delta \text{BIC}$ = 6.177

| Table 2 | Constraints on Four Correlations from the Low-redshift GRBs of A118 Data Set |
|---------|--------------------------------------------------------------------------------|
| Amati   | Extended Amati | copula$^1$ | copula$^2$ |
| Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL | Best-fit($\sigma$) | 68% CL |
| $\sigma_{\text{int}}$ | 0.429(0.089) | $\pm 0.113$ | 0.438(0.099) | $\pm 0.135$ | 0.418(0.098) | $\pm 0.149$ | 0.430(0.097) | $\pm 0.137$ |
| $a$ | 52.860(0.111) | $\pm 0.105$ | 52.730(0.609) | $\pm 0.649$ | 52.958(0.282) | $\pm 0.281$ | 52.935(0.282) | $\pm 0.265$ |
| $b$ | 0.990(0.205) | $\pm 0.187$ | 1.525(1.158) | $\pm 1.168$ | 0.967(0.231) | $\pm 0.229$ | 0.976(0.235) | $\pm 0.230$ |
| $c$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\alpha$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\beta$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

$\Delta \text{AIC}$ = 23.374, $\Delta \text{BIC}$ = 5.555

Note. The best-fitted values, standard deviations, and the 68% CL of coefficients of $y_{\text{Amati}}$, $y_{\text{exAmati}}$, $y_{\text{copula}}$, and $y_{\text{copula}}$ from the 79 low-redshift ($z < 1.4$) long GRBs in the A220 data set. Here $\Delta \text{AIC}(\Delta \text{BIC})$ denotes the difference between AIC(BIC) and the standard Amati model.

Note. The best-fitted values, standard deviations, and the 68% CL of coefficients of $y_{\text{Amati}}$, $y_{\text{exAmati}}$, $y_{\text{copula}}$, and $y_{\text{copula}}$ from the 20 low-redshift ($z < 1.4$) long GRBs in the A118 data set.
determine the model preferred by the data, while the BIC indicates that the standard Amati correlation remains to be slightly favored since it has the least model parameters.

Extrapolating directly the values of the coefficients in Tables 1 and 2 from the low-redshift GRB data to the high-redshift samples we can construct a Hubble diagram of GRBs. The distance modulus of GRBs and their errors are obtained from Equations (A5) and (A10). As an example, in Figure 2, we show the Hubble diagrams of 220 and 118 long GRBs obtained from three different correlations ($y_{\text{Amati}}$, $y_{\text{exAmati}}$, and $y_{\text{copula}}$), respectively. Here we do not plot the Hubble diagram based on the $y_{\text{copula}}$ correlation since it is very similar to the one of $y_{\text{copula}}$. The values of distance modulus obtained from the copula method are less at low-redshift ($z \lesssim 1$) regions and larger at high-redshift ($z \gtrsim 1$) regions than the ones from the standard Amati correlation because a correction exists as shown in the right-hand side of Equation (15).

To test whether the GRB can be regarded as a viable cosmological indicator, we can constrain the ΛCDM model from the distance modulus of GRBs obtained in the above subsection, and check whether $\Omega_m = 0.3$ is allowed after setting $H_0 = 70 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$, which is used in calibrating the improved Amati correlations from the low-redshift GRBs. The method of minimizing $\chi^2$ is used to constrain $\Omega_m$. We consider two different samples for each data set: the 141 high-redshift GRBs and the total 220 GRBs for A220, and the 98 high-redshift GRBs and the total 118 GRBs for A118. We must emphasize here that since $W_{\text{m}}(8.2) = 1$ when the maximum redshift GRB data ($z = 8.2$) is considered, $\text{erfc}^{-1}[2W_{\text{m}}(8.2)] = -\infty$ in $y_{\text{copula}}$, and thus the data point with $z = 8.2$ will be ignored when the $y_{\text{copula}}$ is used to constrain $\Omega_m$.

The probability density plots of $\Omega_m$ in the ΛCDM model with $H_0 = 70 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$. The left and right panels show the results from the 141 high-redshift GRBs and full-redshift 220 GRBs, respectively. The gray dashed line denotes the $\Omega_m = 0.3$, which is the value of the fiducial model. The probability density plots of $\Omega_m$ for two data sets are shown in Figures 3 and 4, and the best-fitted values with the standard deviation and 68% confidence level (CL) are summarized in Tables 3 and 4, respectively. It is easy to see that for all data sets the results from the standard Amati correlation and the extended Amati correlation deviate apparently from $\Omega_m = 0.3$. The results from the improved Amati correlation based on an empirical distribution of redshift ($y_{\text{copula}}$) are better than the ones from the (extended) Amati correlation although they are $1\sigma$ away from $\Omega_m = 0.3$, while those from $y_{\text{copula}}$ are always consistent with $\Omega_m = 0.3$ at the $1\sigma$ confidence level. Apparently, the results from $y_{\text{copula}}$ are not as good as those from $y_{\text{copula}}$. This is attributed to that the number of GRBs is still inadequate to construct the empirical distribution precisely. Comparing the constraint on $\Omega_m$ from the high-redshift and full-redshift GRB data shown in Tables 3 and 4, we find that the values of $\Omega_m$ from the full-redshift data
Figure 4. Probability density plots of \( \Omega_{m0} \) in the \( \Lambda \text{CDM} \) model with \( H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \). The left and right panels show the results from the 98 high-redshift GRBs and full-redshift 118 GRBs, respectively. The gray dashed line denotes the values of \( \Omega_{m0} \) = 0.3, which is the value of the fiducial model.

### Table 3

Constraints on \( \Omega_{m0} \) from the A220 Data Set

|         | High Redshift | Full Redshift |
|---------|---------------|---------------|
|         | \( \Omega_{m0}(\sigma) \) | 68% CL | \( \chi^2 \) | \( \Omega_{m0}(\sigma) \) | 68% CL | \( \chi^2 \) |
| Amati   | 0.649(0.106)  | 0.114        | 0.096  | 90.535 | 0.589(0.086) | 0.091 | 0.077  | 168.851 |
| Extended Amati | 0.574(0.103) | 0.106        | 0.093  | 86.194 | 0.507(0.083) | 0.083 | 0.047  | 164.098 |
| copula1 | 0.295(0.063)  | 0.067        | 0.057  | 86.500 | 0.296(0.057) | 0.058 | 0.053  | 160.779 |
| copula2 | 0.385(0.077)  | 0.080        | 0.074  | 84.264 | 0.368(0.067) | 0.071 | 0.060  | 159.941 |

**Note.** The best-fitted value of \( \Omega_{m0} \) with the standard deviation \( \sigma \) and the 68% CL. The results are obtained from the A220 data set.

### Table 4

Constraints on \( \Omega_{m0} \) from the A118 Data Set

|         | High Redshift | Full Redshift |
|---------|---------------|---------------|
|         | \( \Omega_{m0}(\sigma) \) | 68% CL | \( \chi^2 \) | \( \Omega_{m0}(\sigma) \) | 68% CL | \( \chi^2 \) |
| Amati   | 0.638         | 0.119        | 0.090  | 69.710 | 0.607         | 0.116        | 0.084  | 88.551 |
|         (0.109) |                     |       |      |      |         (0.105) |                     |       |      |
| Extended Amati | 0.579         | 0.118        | 0.123  | 50.431 | 0.540         | 0.123        | 0.108  | 66.905 |
|         (0.133) |                     |       |      |      |         (0.120) |                     |       |      |
| copula1 | 0.390         | 0.105        | 0.086  | 46.979 | 0.381         | 0.109        | 0.081  | 64.609 |
|         (0.101) |                     |       |      |      |         (0.094) |                     |       |      |
| copula2 | 0.466         | 0.130        | 0.097  | 46.563 | 0.445         | 0.110        | 0.090  | 63.794 |
|         (0.117) |                     |       |      |      |         (0.103) |                     |       |      |

**Note.** The best-fitted value of \( \Omega_{m0} \) with the standard deviation \( \sigma \) and the 68% CL. The results are obtained from the A118 data set.

The best-fitted values of \( \Omega_{m0} \) are closer to 0.3 than those from the high-redshift data. This is because the low-redshift data are calibrated with \( \Omega_{m0} = 0.3 \). When changing the GRB data from the high-redshift region to the full-redshift one, the A220 data give the maximum variation of \( \Omega_{m0} \) for the case of the extended Amati correlation, which is about 12%. However, this variation is very small at \( \gamma_{\text{copula}_1} \) and \( \gamma_{\text{copula}_2} \) cases. From Tables 3 and 4 and Figures 3 and 4, one can also see that the values of \( \Omega_{m0} \) from the total A220 data set are much closer to 0.3 than those from the full-redshift A118 data set. This is attributed to the ratio of the number of the calibrated to the uncalibrated GRBs in the A220 data set, which is about 0.56 (79/141), being apparently larger than that in the A118 data set, which is only 0.20 (20/98). In addition, it is easy to find that our constraints on \( \Omega_{m0} \) from the A118 and A220 data samples differ very significantly from those obtained in Khadka et al. (2021) where the A118 and A220 data limit \( \Omega_{m0} > 0.230 \) and >0.455, respectively, at the 95% CL in the \( \Lambda \text{CDM} \) model. This difference is due to utilizing the low-redshift calibration method while the simultaneous fitting was used in Khadka et al. (2021).

### 4.2. Simultaneous Fitting

To clearly show the difference between the results from the GRB low-redshift calibration and the simultaneous fitting, we follow the steps given in Khadka et al. (2021) and Cao et al. (2022) to use the simultaneous fitting method to constrain the cosmological parameter and the coefficients of four correlation relations. After setting \( H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1} \), the constraints on \( \Omega_{m0} \) and the coefficients of four correlation relations can be obtained from the A118 and A220 data samples by using the D’Agostinis likelihood function (Equation (17)).

Figure 5 shows one-dimensional probability density plots and contour plots of \( \Omega_{m0} \) and the coefficients of four correlation relations, and the marginalized mean values with the standard deviation and the 68% CL are summarized in Table 5. We find that the results from the standard Amati correlation are slightly different from those obtained in Cao et al. (2022). This is because the peak energy of the GRB 081121 data point, which
was released in Fana Dirirsa et al. (2019) and used in Cao et al. (2022), is different from the one given in Table 1 of Amati et al. (2009), and it actually corresponds to the distance modulus rather than the peak energy in Table 4 of Wang et al. (2016). Thus, there is an error in the peak energy of GRB 081121 used in Cao et al. (2022). If this error were not corrected, we would obtain the same result as Cao et al. (2022).

In our analysis we have corrected this error. Comparing Table 5 and Tables 1 and 2, we find that the values of $c$ from the simultaneous fitting are smaller and closer to zero than those from the low-redshift calibration. This seems to indicate that the evolutionary character with the redshift of the Amati correlation becomes weaker when more high-redshift data are used. Figure 5 and Table 5 show that when the simultaneous fitting method is used, the results from the improved Amati correlations are similar to those from the standard Amati correlation since the GRB data favor a large value of $\Omega_m0$ and can only give a low bound limit on $\Omega_m0$, although the low bound limits from the improved Amati correlations are clearly smaller than that from the standard Amati correlation. If the extended Amati correlation is considered, there is almost no constraint on $\Omega_m0$ from GRB, which should be attributed to this relation having more coefficients. Thus, once the simultaneous fitting method is used, we cannot find that the improved Amati correlations obtained in this paper are apparently better than the standard and extended Amati correlations. These results are

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Results of simultaneously fitting the flat $\Lambda$CDM model and four correlations via the A118 (red) and A220 (blue) data sets. The upper left, upper right, lower left, and lower right panels denote the Amati, extended Amati, copula1 and copula2 correlations, respectively.}
\end{figure}
Table 5
Constraints on the Four Correlations using Simultaneous Fitting Method with the A220 and A118 Data Sets, Respectively

| Data Set | Amati Extended Amati | copula1 | copula2 |
|----------|----------------------|---------|---------|
|          | Mean(y) | 68% CL | Mean(y) | 68% CL | Mean(y) | 68% CL | Mean(y) | 68% CL |
| Ω m0    | >0.713   | ...    | ...     | ...    | >0.554  | ...    | >0.550  | ...    |
| σ m0    | 0.459(0.024) | -0.025 | 0.459(0.024) | +0.021 | 0.460(0.024) | +0.022 | 0.461(0.024) | +0.022 |
| a       | 52.630(0.068) | -0.079 | 52.462(0.158) | -0.140 | 52.685(0.118) | -0.130 | 52.692(0.127) | -0.140 |
| A220 b  | 1.286(0.072) | -0.072 | 1.434(0.228) | -0.230 | 1.258(0.078) | -0.074 | 1.262(0.079) | -0.074 |
| c       | ...      | ...    | ...     | ...    | -0.066(0.074) | +0.077 | -0.064(0.075) | -0.076 |
| α       | ...      | ...    | 0.544(0.447) | +0.240 | ...     | ...    | ...     | ...    |
| β       | ...      | ...    | -0.337(0.370) | ...    | ...     | ...    | ...     | ...    |
| Ω m0    | 0.610(0.230) | +0.340 | <0.680  | ...    | >0.470  | ...    | >0.459  | ...    |
| σ m0    | 0.391(0.028) | -0.031 | 0.394(0.028) | +0.024 | 0.392(0.028) | +0.025 | 0.394(0.029) | +0.025 |
| a       | 52.826(0.115) | -0.140 | 52.786(0.228) | -0.210 | 52.846(0.147) | -0.170 | 52.845(0.142) | -0.170 |
| A118 b  | 1.171(0.085) | -0.083 | 1.234(0.422) | -0.420 | 1.183(0.086) | -0.086 | 1.179(0.089) | -0.089 |
| c       | ...      | ...    | ...     | ...    | 0.017(0.097) | +0.100 | 0.010(0.084) | +0.069 |
| α       | ...      | ...    | 0.198(0.520) | +0.587 | ...     | ...    | ...     | ...    |
| β       | ...      | ...    | -0.097(0.629) | -0.623 | ...     | ...    | ...     | ...    |

Note. The marginalized mean values, standard deviations, and the 68% CL of the flat $Λ$CDM model parameter and coefficients of $y_{Amati}$, $y_{ExtendedAmati}$, $y_{copula1}$, and $y_{copula2}$ from the A220 and A118 data set using the simultaneous fitting method.

significantly different from those obtained in the subsection above with the low-redshift calibration method.

5. Conclusions

In this paper, we use the three-dimensional Gaussian copula method to investigate the luminosity correlation of GRB data. By assuming that the logarithms of the special peak energy and the isotropic energy of GRBs satisfy the Gaussian distributions and two different redshift distributions of GRB data (one is the special form given in Equation (11) and the other is an empirical distribution), we obtain two improved Amati correlations of GRB data ($y_{copula1}$ and $y_{copula2}$), which are distinctively different from the standard Amati correlation and the extended Amati correlation. After calibrating the low-redshift GRB data points from A220 and A118 data sets, respectively, these improved Amati correlations based on a fiducial $Λ$CDM model with $Ω_{m0}=0.3$ and $H_0=70$ km s$^{-1}$ Mpc$^{-1}$, and extrapolating the results to the high-redshift GRB data, we obtain the Hubble diagrams of 220 and 118 GRB data points. Applying these GRB data to constrain the $Λ$CDM model, we find that the results from the improved Amati correlations are apparently better than those from the standard Amati and extended Amati correlations, although the BIC favors mildly the standard Amati correlation. The improved Amati correlation based on the special redshift distribution of GRB data gives the best result, which is always consistent with $Ω_{m0}=0.3$ at the 1σ confidence level and is highly consistent with $Ω_{m0}=0.3$ when the A220 data set is used. These results seem to indicate that when the improved Amati correlation with the special redshift distribution ($y_{copula1}$) is used in the low-redshift calibration, the GRB data can be regarded as a viable cosmological explorer. However, the BIC indicates that the standard Amati correlation remains to be slightly favored since it has the least model parameters. Furthermore, once the simultaneous fitting method rather than the low-redshift calibration one is used, we find that the constraints on $Ω_{m0}$ are weak and only the low bound limit on $Ω_{m0}$ can be obtained. Although this low bound limit from the improved Amati correlation is smaller than the one from the standard Amati correlation, there is no apparent evidence that the former is better than the latter. This result is apparently different from the one from the low-redshift calibration method. Therefore, more work needs to be done in the future in order to compare different Amati correlations.

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Appendix A
Amati Correlation

In 2002, Amati et al. (2002) found that in GRB observational data there is a positive correlation between the spectral peak energy $E_p$ and the isotropic equivalent radiated energy $E_{iso}$ (Amati 2006a, 2006b; Amati et al. 2008, 2009), and this correlation has the form

$$y_{Amati} = a + bx, \quad (A1)$$

where

$$y \equiv \log \frac{E_{iso}}{1\text{erg}} \quad x \equiv \log \frac{E_p}{300\text{keV}}, \quad (A2)$$

intercept $a$ and slope $b$ are free coefficients, “log” denotes the logarithm to base 10, and

$$E_P = E_p^{obs}(1 + z), \quad (A3)$$

$$E_{iso} = 4\pi d_L^2(z)S_{bolo}(1 + z)^{-1}. \quad (A4)$$

Here $z$ is the redshift, $E_p^{obs}$ is the observed peak energy of GRB spectrum, $d_L(z)$ is the luminosity distance, and $S_{bolo}$ is the bolometric fluence.
If the coefficients $a$ and $b$ are determined, the luminosity distance of the GRB data point can be obtained from Equations (A1), (A3), and (A4). Then we can obtain the distance modulus of GRB data, which is defined to be

$$
\mu = 5 \log \frac{d_L(z)}{\text{Mpc}} + 25.
$$

(A5)

If assuming a fiducial cosmological model, the values of coefficients $a$ and $b$ in the Amati relation can be obtained from the observational data by using the following common fitting strategy (D’Agostini 2005):

$$
\mathcal{L}(\alpha, \beta) \propto \prod_i \frac{1}{\sqrt{\sigma_{\text{int}}^2 + \sigma_x^2 + b^2 \sigma_y^2}} \times \exp \left[-\frac{(y_i - a - bx_i)^2}{2(\sigma_{\text{int}}^2 + \sigma_x^2 + b^2 \sigma_y^2)}\right],
$$

where $\sigma_x$ and $\sigma_y$ are the uncertainties of $x$ and $y$, respectively, and $\sigma_{\text{int}}$ is the intrinsic uncertainty of GRB. From the well-known error propagation equation, one can find that $\sigma_x$ and $\sigma_y$ can be derived from Equations (A3) and (A4) and have the expressions

$$
\sigma_x = \frac{1}{\ln 10} \sigma_{\text{Eiso}}, \quad \sigma_y = \frac{1}{\ln 10} \sigma_{E_p}. 
$$

(A7)

with

$$
\sigma_{\text{Eiso}} = 4\pi d_L^2 \sigma_{\text{Sbol}}(1 + z)^{-1}.
$$

(A8)

Here $\sigma_{E_p}$ and $\sigma_{\text{Sbol}}$ are available in observations of GRBs. Therefore, maximizing the likelihood function $\mathcal{L}$ (Equation (A6)), the allowed values of $a$, $b$, and $\sigma_{\text{int}}$ can be obtained. Then the covariance matrix $C_{ij}$ of these fitted parameters can be approximately evaluated from

$$
(C^{-1})_{ij}(\theta_A) = \frac{\partial^2[-\ln \mathcal{L}(\theta_0)]}{\partial \theta_i \partial \theta_j} \bigg|_{\theta_0 = \theta_A},
$$

where $\theta_A = \{\sigma_{\text{int}}, a, b\}$, and $\theta_A$ denote the best-fitted value of $a$, $b$, and $\sigma_{\text{int}}$.

Using the best-fitted values of $a$ and $b$, we can get the luminosity distance of GRBs and the corresponding distance modulus from Equation (A5). By using the error propagation equation, the uncertainty of the distance modulus can be derived from the following equation:

$$
\sigma^2_{\mu} = \frac{5}{2} \left[\frac{\sigma_{\text{Eiso}}}{\text{erg}} \right]^2 + \frac{5}{2} \left[\frac{\sigma_{\text{Sbol}}}{\text{erg}} \right]^2.
$$

(A10)

Here

$$
\sigma^2_{\log E_p} = \sigma^2_a + \left(\frac{E_p}{300 \text{keV}}\right)^2 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \left(\frac{\partial \lambda_{\text{Aman}}(v; \theta_A)}{\partial \theta_i} \frac{\partial \lambda_{\text{Aman}}(v; \theta_A)}{\partial \theta_j} C_{ij} \right)
$$

$$
+ \left(\frac{b \sigma_{E_p}}{\ln 10 \ E_p}\right)^2 + \sigma^2_{\text{int}}.
$$

(A11)

According to the distance modulus of GRBs, the cosmological model can be constrained by minimizing $\chi^2$

$$
\chi^2 = \sum_{i=1}^{N} \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; p)}{\sigma^2_{\mu_i}},
$$

(A12)

where $\mu_{\text{obs}}$ is the distance modulus of GRB and $\mu_{\text{th}}(z_i; p)$ is the theoretical value of the distance modulus in the cosmological model with $p$ representing the model parameters.

### Appendix B

#### Extended Amati Correlation

The extended Amati correlation was proposed in Wang et al. (2017) where the authors used two formulas to parameterize the coefficients $a$ and $b$ in the standard Amati correlation:

$$
\alpha \rightarrow A = a + \alpha \frac{z}{1 + z}, \quad \beta \rightarrow B = b + \beta \frac{z}{1 + z},
$$

(B1)

where $\alpha$ and $\beta$ are two constants. Substituting these parameterized formulas into the Equation (A1), the extended Amati correlation can be obtained:

$$
\gamma_{x_{\text{exAmati}}} = \left( a + \alpha \frac{z}{1 + z} \right) + \left( b + \beta \frac{z}{1 + z} \right) x.
$$

(B2)

Recently, Khadka et al. (2021) used the A22 GRB data set to limit $\alpha$ and $\beta$, and found that the Amati correlation is independent of redshift within the error bars. The coefficient $\theta_{\text{ex}} = \{\sigma_{\text{int}}, a, b, \alpha, \beta\}$ is also estimated from the D’Agostinis likelihood function. The uncertainty of $\log E_p$ in Equation (A10) can be obtained from

$$
\sigma^2_{\log E_p} = \sigma^2_a + \left(\frac{E_p}{300 \text{keV}}\right)^2
$$

$$
+ \left(\frac{\sigma_a}{1 + z}\right)^2 + \left(\frac{\sigma_b}{1 + z}\right)^2 \left(\frac{\log E_p}{300 \text{keV}}\right)^2
$$

$$
+ 2 \sum_{i=1}^{3} \sum_{j=i+1}^{3} \left(\frac{\partial \lambda_{\text{exAmati}}(v; \theta_{\text{ex}})}{\partial \theta_i} \frac{\partial \lambda_{\text{exAmati}}(v; \theta_{\text{ex}})}{\partial \theta_j} C_{ij} \right)
$$

$$
+ \left(\frac{b + \beta \frac{z}{1 + z}}{1 + z} \right) \frac{1}{\ln 10} \left(\frac{E_p}{E_p}\right)^2 + \left(\frac{\sigma_{\text{Eiso}}}{\text{erg}}\right)^2 + \sigma^2_{\text{int}}.
$$

(B3)

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