Fast cubature of volume potentials over rectangular domains

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\textbf{Abstract}

In the present paper we study high-order cubature formulas for the computation of advection-diffusion potentials over boxes. By using the basis functions introduced in the theory of approximate approximations, the cubature of a potential is reduced to the quadrature of one dimensional integrals. For densities with separated approximation, we derive a tensor product representation of the integral operator which admits efficient cubature procedures in very high dimensions. Numerical tests show that these formulas are accurate and provide approximation of order $O(h^8)$ up to dimension $10^8$.

\textbf{Keywords.} Multi-dimensional convolution; Advection-diffusion potential; Tensor product representation; Higher dimensions

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\section{Introduction}

High-dimensional volume potentials arise in many mathematical models in the field of physics, chemistry, biology, financial mathematics and many others. In recent years, tensor product approximation has been recognized as a successful tool to overcome the "curse of dimensionality" and treat high-dimensional integral operators as described, for example, in \cite{3, 4, 6, 2}.

In the present paper we propose to combine high-order semi-analytic cubature formulas, obtained by using the method of approximate approximations (see \cite{11} and the reference therein), with tensor product approximations.

Cubature formulas based on approximate approximations for volume potentials over $\mathbb{R}^n$ and over bounded domains have been considered in \cite{10} and \cite{9}, respectively (see also \cite{11}). The
cubature of high-dimensional volume potentials over the full space and over half-spaces has been studied in [7] and [8]. Now we consider the volume potential

$$K_{\lambda} f(x) = \int_{[P,Q]} \kappa_{\lambda}(x - y)f(y)dy,$$  (1.1)

with the fundamental solution

$$\kappa_{\lambda}(x) = \frac{1}{(2\pi)^{n/2}} \frac{1}{|x|^{1-n/2}} K_{n/2-1}(\lambda |x|), \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

over rectangular domains $[P, Q] = \prod_{j=1}^n [P_j, Q_j] \subset \mathbb{R}^n$. Here $K_{\nu}$ is the modified Bessel function of the second kind (see [1, 9.6, p.374]).

The function $u = K f$ provides a solution of the modified Helmholtz equation

$$(-\Delta + \lambda^2)u = \begin{cases} f(x) & x \in [P, Q] \\ 0 & otherwise. \end{cases}$$

For $\lambda = 0$, then

$$\kappa_0(x) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x|}, & n = 2, \\ \frac{1}{\Gamma\left(\frac{n}{2} - 1\right)} \frac{1}{4\pi^{n/2}} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

is the fundamental solution of the Laplacian.

The theory of approximate approximations proposes semi-analytic cubature formulas for volume potentials by using quasi-interpolation of the density $f$ by functions for which the integral operator can be taken analytically. Approximate quasi-interpolant has the form

$$M_{h,D} f(x) = D^{-n/2} \sum_{m \in \mathbb{Z}^n} f(h m) \eta \left( \frac{x - hm}{h \sqrt{D}} \right)$$

where $h$ and $D$ are positive parameters and $\eta$ is a smooth and rapidly decaying function which satisfies the moment conditions of order $N$

$$\int_{\mathbb{R}^n} \eta(x) x^\alpha dx = \delta_{0,\alpha}, \quad 0 \leq |\alpha| < N.$$  (1.2)

If $f \in C_0^N(\mathbb{R}^n)$, it is known ([11]) that

$$|f(x) - M_{h,D} f(x)| \leq c(\sqrt{D}h)^N \|\nabla^N f\|_{L^\infty} + \sum_{k=0}^{N-1} \varepsilon_k(\sqrt{D}h)^k |\nabla^k f(x)|$$

with

$$\varepsilon_k \leq \sum_{m \in \mathbb{Z}^n \setminus \{0\}} |\nabla^k \eta(\sqrt{D}m)|, \quad \lim_{D \to \infty} \sum_{m \in \mathbb{Z}^n \setminus \{0\}} |\nabla^k \eta(\sqrt{D}m)| = 0.$$
If we replace $f$ in (1.1) by the quasi-interpolant
\[ D^{-n/2} \sum_{h \cdot m \in [P,Q]} f(hm) \eta \left( \frac{x - hm}{h \sqrt{D}} \right) \] (1.3)
we don’t obtain good approximations because (1.3) approximates $f$ only in a subdomain of $[P,Q]$ with positive distance from the boundary. To avoid this difficulty we extend $f$ with preserved smoothness in a larger domain. Obviously the quasi-interpolant of the continuation $\tilde{f}$ approximates $f$ in $[P,Q]$. Assume that there exists $C > 0$ such that
\[ ||\tilde{f}||_{W^N_\infty} \leq C ||f||_{W^N_\infty([P,Q])}. \]
Since $\eta$ is a smooth and rapidly decaying function, for any error $\epsilon > 0$ one can fix $r > 0$ and the parameter $D > 0$ such that the quasi-interpolant with nodes in a neighborhood of $[P, Q]$
\[ M_{h,D}^r \tilde{f}(x) = D^{-n/2} \sum_{d(hm,[P,Q]) \leq r h \sqrt{D}} \tilde{f}(hm) \eta \left( \frac{x - hm}{h \sqrt{D}} \right) \]
approximates $f$ with
\[ |f(x) - M_{h,D}^r \tilde{f}(x)| = O((\sqrt{D} h)^N + \epsilon) ||f||_{W^N_\infty} \] (1.4)
for all $x \in [P, Q]$.

Then the integral
\[ K_{\lambda,h} \tilde{f}(x) = K_{\lambda}(M_{h,D}^r \tilde{f})(x) = D^{-n/2} \sum_{d(hm,[P,Q]) \leq r h \sqrt{D}} \tilde{f}(hm) \int_{[P,Q]} \kappa_\lambda(x - y) \eta \left( \frac{y - hm}{h \sqrt{D}} \right) dy \]
gives a cubature of (1.1).

Since $K_\lambda$ is a bounded mapping between suitable function spaces, the differences $K_{\lambda,h} \tilde{f}(x) - K_\lambda f(x)$ behave like estimate (1.4). Therefore, to construct high order cubature formulas for (1.1), it remains to compute the integrals
\[ \int_{[P,Q]} \kappa_\lambda \left( \frac{x - hm}{h \sqrt{D}} - y \right) \eta(y) dy \]
for nodes with $d(hm,[P,Q]) \leq r h \sqrt{D}$. This is performed by using one-dimensional integral representations. As basis functions we take the tensor products of univariate basis functions
\[ \bar{\eta}(x) = \prod_{j=1}^{2M} \bar{\eta}_{2M}(x_j); \quad \bar{\eta}_{2M}(x_j) = \pi^{-1/2}L_{M-1}^{(1/2)}(x_j^2) e^{-x_j^2} \] (1.5)
which satisfies the moment condition (1.2) of order $N = 2M$ (cf. [11]), where $L_{k}^{(\gamma)}$ are the generalized Laguerre polynomials
\[ L_{k}^{(\gamma)}(y) = \frac{e^y y^{-\gamma}}{k!} \left( \frac{d}{dy} \right)^k \left( e^{-y} y^{k+\gamma} \right), \quad \gamma > -1. \]
Using the representation with a tensor product integrand

\[
\int_{[P,Q]} K_\lambda(x - y)e^{-|y|^2}dy =
\]

\[
\frac{1}{4} \int_0^\infty e^{-\lambda^2t/4} \prod_{j=1}^n \frac{e^{-x_j^2/(1+t)}}{2\sqrt{\pi}} \left( \text{erf} \left( \frac{1 + t}{t} (P_j - x_j) \right) - \text{erf} \left( \frac{1 + t}{t} (Q_j - x_j) \right) \right) dt
\]

(1.6)

we derive a tensor product representation of the integral operator which admits efficient cubature procedures for densities with separated approximation (Section 2). We will consider quasi-interpolants (2.1) on anisotropic grids which use different step size \(h_j > 0, j = 1, ..., n\) along different space dimensions. If \(h_j = \tau h, 0 < \tau \leq 1\) the error of the quasi-interpolant (2.1) is always \(O(h^N)\). In Section 3 we provide numerical tests, showing that these formulas are accurate and provide approximation of order \(O(h^6)\) up to dimension 10^8.

2 Higher order cubature formula based on (1.6)

In this section we describe a high order cubature of \(K_\lambda f\) in the case of rectangular domain in \(\mathbb{R}^n\). Let

\[
[P, Q] = \{x = (x_1, \ldots, x_n) : P_j \leq x_j \leq Q_j, j = 1, \ldots, n\} = \prod_{j=1}^n [P_j, Q_j].
\]

As basis functions we use (1.5).

In order to apply also quasi-interpolants on rectangular grids \((h_1m_1, \ldots, h_nm_n), h_j > 0\), shortly denoted by \(\{hm\}\),

\[
\mathcal{M}_{h,D} f(x) = D^{-n/2} \sum_{m \in \mathbb{Z}^n} \tilde{f}(hm) \prod_{j=1}^n \tilde{\eta}_{2M} \left( \frac{x_j - h_jm_j}{h_j\sqrt{D}} \right), \quad (2.1)
\]

we define the basis function \(\eta(x) = \prod \tilde{\eta}_{2M}(a_jx_j), a_j > 0\), and look for integral representations of the solution of

\[
(-\Delta + \lambda^2) u = \prod_{j=1}^n \chi_{(p_j,q_j)}(x_j) \tilde{\eta}_{2M}(a_jx_j).
\]

(2.2)

Here \(\chi_{(p_j,q_j)}\) is the characteristic function of the interval \((p_j,q_j)\) with \(-\infty \leq p_j < q_j \leq +\infty, j = 1, \ldots, n\).

**Theorem 2.1.** Let \(\text{Re} \lambda^2 \geq 0\) and \(n \geq 3\). The solution of equation (2.2) in \(\mathbb{R}^n\) can be expressed by the one-dimensional integral

\[
u(x) = \frac{1}{4} \int_0^\infty e^{-\lambda^2t/4} \prod_{j=1}^n \left( \Phi_M(a_jx_j, a_j^2t, a_jp_j) - \Phi_M(a_jx_j, a_j^2t, a_jq_j) \right) dt
\]

(2.3)

where the function \(\Phi_M\) is given by

\[
\Phi_M(x, t, p) = \frac{e^{-x^2/(1+t)}}{2\sqrt{\pi}} \left( \text{erf} \left( F(t, x, p) \right) \mathcal{P}_M(t, x) - \frac{e^{-F^2(t,x,p)}}{\sqrt{\pi}} \mathcal{Q}_M(t, x) \right)
\]
with the function

\[ F(t, x, y) = \sqrt{\frac{1 + t}{t}} (y - \frac{x}{1 + t}) , \]

and \( \mathcal{P}_M, \mathcal{Q}_M \) are polynomials in \( x \) of degree \( 2M - 2 \) and \( 2M - 3 \), respectively:

\[
\mathcal{P}_M(t, x) = \sum_{k=0}^{M-1} \frac{1}{(1 + t)^{k+1/2}} F_k^{(-1/2)} \left( \frac{x^2}{1 + t} \right),
\]

\[
\mathcal{Q}_M(t, x, y) = 2 \sum_{k=1}^{M-1} \frac{(-1)^k}{k! 4^k} \sum_{\ell=1}^{2k} \left( \frac{(-1)^\ell}{t^{\ell/2}} \left( H_{2k-\ell}(y) H_{\ell-1} \left( \frac{y - x}{\sqrt{t}} \right) \right. \right.
\]

\[
- \left. \left( \frac{2k}{\ell} \right) H_{2k-\ell} \left( \frac{x}{\sqrt{1 + t}} \right) \left( \frac{H_{\ell-1} \left( F(t, x, y) \right)}{(1 + t)^{k+1/2}} \right) \right).
\]

If \( \text{Re} \lambda^2 > 0 \), then the representation (2.3) is valid for all \( n \geq 1 \).

By \( H_k \) we denote the Hermite polynomials

\[ H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} . \] (2.4)

Proof. The solution of (2.2) can be obtained explicitly by using the parabolic equation

\[ \partial_t w - \Delta w + \lambda^2 w = 0, \quad t \geq 0 , \] (2.5)

with the initial condition

\[ w(x, 0) = \prod_{j=1}^{n} \chi_{(p_j,q_j)}(x_j) \eta_2M(a_jx_j) . \]

Integrating (2.5) in \( t \) we derive

\[ w(x, T) - w(x, 0) - (\Delta - \lambda^2) \int_0^T w(x, t) dt = 0 , \]

hence the solution of (2.2) is expressed as the one-dimensional integral

\[ u(x) = \int_0^\infty w(x, t) dt \]

provided it exists. Obviously, if \( w \) solves (2.5), then \( z = w e^{\lambda^2 t} \) is the solution of the initial value problem for the heat equation

\[ \partial_t z - \Delta z = 0, \quad z(x, 0) = \prod_{j=1}^{n} \chi_{(p_j,q_j)}(x_j) \eta_2M(a_jx_j) , \]
which has, by Poisson’s formula, the solution

\[ z(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_\prod(p_j, q_j) e^{-|x-y|^2/(4t)} \prod_{j=1}^n \eta_{2M}(a_j y_j) \, dy \]

\[ = \prod_{j=1}^n \frac{1}{\pi^{1/2}(4a_j^2 t)^{1/2}} \int_{a_j p_j} e^{-(a_j x_j - y_j)^2/(4a_j^2 t)} \eta_{2M}(y_j) \, dy_j \]

where \( \prod(p_j, q_j) \) is the Cartesian product of the intervals \( (p_j, q_j) \). Denoting

\[ \Phi_M(x, t, p) = \frac{1}{\sqrt{\pi t}} \int_\prod e^{-(x-y)^2/t} \eta_{2M}(y) \, dy \]

we get the one-dimensional integral representation (2.3) of the solution of (2.2), provided this integral exists. Denoting

\[ \varphi_k(x, t, p) = \int_\prod e^{-(x-y)^2/t} \frac{d^{2k}}{dy^{2k}} e^{-y^2} \, dy \]

and using the general representation [11, p.55]

\[ \eta_{2M}(x) = \pi^{-n/2} \sum_{j=0}^{M-1} (-1)^j \frac{1}{j! 4^j} \Delta^j e^{-|x|^2}, \]

we have

\[ \Phi_M(x, t, p) = \frac{1}{\pi \sqrt{t}} \sum_{k=0}^{M-1} (-1)^k \frac{1}{k! 4^k} \varphi_k(x, t, p). \]

From

\[ \varphi_0(x, t, p) = \int_\prod e^{-(x-y)^2/t} e^{-y^2} \, dy = \frac{\sqrt{\pi}}{2} \sqrt{\frac{t}{1+t}} e^{-x^2/(1+t)} \text{erfc} \left( F(t, x, p) \right), \]

for \( k \geq 1 \), integration by parts leads to

\[ \varphi_k(x, t, p) = \frac{\partial^{2k}}{\partial x^{2k}} \varphi_0(x, t, p) - \sum_{\ell=0}^{2k-1} (-1)^\ell \frac{\partial^\ell}{\partial y^\ell} e^{-(x-y)^2/t} \frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}} e^{-y^2} \bigg|_{y=p} \]

and the definition (2.4) gives

\[ \frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}} e^{-y^2} = (-1)^{2k-\ell-1} e^{-y^2} H_{2k-\ell-1}(y), \]

\[ \frac{\partial^\ell}{\partial y^\ell} e^{-(x-y)^2/t} = \frac{(-1)^\ell e^{-(x-y)^2/t}}{t^{\ell/2}} H_{\ell} \left( \frac{y-x}{\sqrt{t}} \right). \]

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In view of
\[
\frac{d^\ell}{dx^\ell} \text{erfc}(x) = \frac{2}{\sqrt{\pi}}(-1)^\ell e^{-x^2} H_{\ell-1}(x), \quad \ell \geq 1,
\]
one gets for \(\ell < 2k\)
\[
\frac{\partial^{2k-\ell}}{\partial x^{2k-\ell}} \text{erfc}(F(t, x, p)) = \frac{(-1)^{2k-\ell}}{(t(1+t))^{k-\ell/2}} \left[ \frac{d^{2k-\ell}}{dz^{2k-\ell}} \text{erfc}(z) \right]_{z=F(t,x,p)} = - \frac{2e^{-F^2(t,x,p)}}{\sqrt{\pi}(t(1+t))^{k-\ell/2}} H_{2k-\ell-1}(F(t, x, p)).
\]

Therefore, since
\[
\frac{d^\ell}{dx^{\ell}} e^{-x^2/(1+t)} = \frac{(-1)^\ell e^{-x^2/(1+t)}}{(1+t)^{\ell/2}} H_\ell\left(\frac{x}{\sqrt{1+t}}\right),
\]
we obtain
\[
\frac{\partial^{2k}}{\partial x^{2k}} \varphi_0(x, t, p) = \frac{\sqrt{\pi} t}{2} \frac{e^{-x^2/(1+t)}}{(1+t)^{k+1/2}} H_{2k}\left(\frac{x}{\sqrt{1+t}}\right) \text{erfc}(F(t, x, p))
\]
\[
- \sqrt{t} e^{-x^2/(1+t)} e^{-F^2(t,x,p)} \frac{2k-1}{(1+t)^{k+1/2}} \sum_{\ell=0}^{2k-1} \left(\frac{2k}{\ell}\right)^{\ell\ell} (-1)^\ell H_\ell\left(\frac{x}{\sqrt{1+t}}\right) H_{2k-\ell-1}(F(t, x, p)).
\]

Thus simple transformations give
\[
\varphi_k(x, t, p) = e^{-x^2/(1+t)} \left( \text{erfc}(F(t, x, p)) H_{2k}\left(\frac{x}{\sqrt{1+t}}\right) \frac{\sqrt{\pi} t}{2(1+t)^{k+1/2}} 
\right.
\]
\[
+ e^{-F^2(t,x,p)} \sum_{\ell=1}^{2k} \frac{(-1)^\ell}{\ell(\ell-1)/2}
\]
\[
\times \left( \left(\frac{2k}{\ell}\right) H_{2k-\ell}\left(\frac{x}{\sqrt{1+t}}\right) H_{\ell-1}(F(t, x, p)) \right)
\]
\[
\left. - H_{\ell-1}\left(\frac{p-x}{\sqrt{t}}\right) H_{2k-\ell-1}(p) \right) \right).
\]

Using the relation \(H_{2k}(x) = (-1)^k 4^k k! L_k^{-2}(x^2)\) we find therefore
\[
\Phi_M(t, x, p) = \frac{e^{-x^2/(1+t)} \text{erfc}(F(t, x, p))}{2\sqrt{\pi}} \sum_{k=0}^{M-1} \frac{1}{(1+t)^{k+1/2}} L_k^{-1/2}(x^2) \left(\frac{x^2}{1+t}\right)
\]
\[
+ \frac{e^{-x^2/(1+t)} e^{-F^2(t,x,p)}}{\pi} \sum_{k=0}^{M-1} \frac{(-1)^k}{k! 4^k} \sum_{\ell=1}^{2k} \frac{(-1)^\ell}{\ell^{\ell/2}}
\]
\[
\times \left( \left(\frac{2k}{\ell}\right) H_{2k-\ell}\left(\frac{x}{\sqrt{1+t}}\right) H_{\ell-1}(F(t, x, p)) \right)
\]
\[
\left. - H_{\ell-1}\left(\frac{p-x}{\sqrt{t}}\right) H_{2k-\ell-1}(p) \right) \right)
\]
\[
= \frac{e^{-x^2/(1+t)}}{2\sqrt{\pi}} \left( \text{erfc}(F(t, x, p)) \mathcal{P}_M(t, x) - \frac{e^{-F^2(t,x,p)}}{\sqrt{\pi}} \mathcal{Q}_M(t, x, p) \right).
\]

\(\square\)
The polynomials $P_M(t,x)$ and $Q_M(t,x,p)$ for $M = 1, 2, 3$ are given by

$$P_1(t,x) = \frac{1}{(1+t)^{1/2}}, \quad P_2(t,x) = P_1(t,x) + \frac{1}{2(1+t)^{3/2}} - \frac{x^2}{(1+t)^{5/2}},$$

$$P_3(t,x) = P_2(t,x) + \frac{3}{8(1+t)^{5/2}} - \frac{3x^2}{2(1+t)^{7/2}} + \frac{x^4}{2(1+t)^{9/2}},$$

$$Q_1(t,x,p) = 0, \quad Q_2(t,x,p) = \sqrt{\pi} \left( \frac{x}{1+t} + p \right),$$

$$Q_3(t,x,p) = -\frac{\sqrt{\pi}}{4(1+t)} \left( \frac{2x^3}{(1+t)^3} + \frac{2px^2 - 5x}{(1+t)^2} + \frac{(2p^2 - 5)x - 3p}{1+t} + p(2p^2 - 7) \right).$$

**Remark 2.1.** Since for positive $r$

$$0 < \text{erfc}(r) \leq e^{-r^2} \quad \text{and} \quad 2 - e^{-r^2} < \text{erfc}(-r) < 2$$

from the relation

$$F^2(t,x,p) = p^2 + \frac{(x-p)^2}{t} - \frac{x^2}{1+t}$$

we get

$$|e^{-x^2/(1+t)} \text{erfc}(F(t,x,p))| \leq e^{-p^2} \quad \text{if} \quad p > 0$$

and

$$|e^{-x^2/(1+t)} \text{erfc}(F(t,x,p)) - 2e^{-x^2/(1+t)}| < e^{-p^2} \quad \text{if} \quad p < 0.$$

Thus for sufficiently large $|p|

$$\Phi_M(x,t,p) = \begin{cases} 
\pi^{-1/2} e^{-x^2/(1+t)} P_M(t,x) + O(e^{-p^2}) & \text{if } p < 0, \\
O(e^{-p^2}) & \text{if } p > 0,
\end{cases}$$

and therefore, for sufficiently large $r$ one can use the approximation

$$\Phi_M(x,t,p) - \Phi_M(x,t,q) \approx \begin{cases} 
0, & p,q \geq r \text{ or } p,q \leq -r, \\
\pi^{-1/2} e^{-x^2/(1+t)} P_M(t,x), & p \leq -r \text{ and } q \geq r,
\end{cases}$$

with the error $O(e^{-r^2})$. Similarly, if $q - p \geq 2r$, then

$$\Phi_M(x,t,p) - \Phi_M(x,t,q) \approx \begin{cases} 
\Phi_M(x,t,p), & -r < p < r, \\
\pi^{-1/2} e^{-x^2/(1+t)} P_M(t,x) - \Phi_M(x,t,q), & -r < q < r.
\end{cases}$$

### 3 Implementation and numerical results

We compute the cubature formula

$$K_{\lambda,h} \tilde{f}(x) = D^{-n/2} \sum_{hm \in \Omega_h} \tilde{f}(hm) \int_{[P,Q]} \kappa_\lambda(x-y) \prod_{j=1}^{n} \tilde{\eta}_{2M} \left( \frac{y_j - h_j m_j}{h_j \sqrt{D}} \right) dy$$
where \( \Omega_h = \prod_{j=1}^n(P_j - rh_j \sqrt{D}, Q_j + rh_j \sqrt{D}) \), using the tensor product representation of Theorem 2.1. At the grid points \( \mathbf{hk} = (h_1k_1, \ldots, h_nk_n) \) we obtain

\[
\int_{[P,Q]} \kappa_\lambda(\mathbf{hk} - \mathbf{y}) \prod_{j=1}^n \mathcal{Q}_M \left( \frac{y_j - h_jm_j}{h_j \sqrt{D}} \right) dy = \frac{1}{4} \int_0^\infty e^{-\lambda^2 t/4} \times \prod_{j=1}^n \left( \Phi_M(\frac{k_j - m_j}{\sqrt{D}}, \frac{t}{h_j^2 D}, \frac{P_j - h_jm_j}{h_j \sqrt{D}}) - \Phi_M(\frac{k_j - m_j}{\sqrt{D}}, \frac{t}{h_j^2 D}, \frac{Q_j - h_jm_j}{h_j \sqrt{D}}) \right) dt
\]

and therefore

\[
\kappa_\lambda \bar{f}(\mathbf{x}) = \sum_{\mathbf{hm} \in \Omega_h} \bar{f}(\mathbf{hm}) b^{(M)}_{k,m},
\]

where we introduce the one-dimensional integral

\[
b^{(M)}_{k,m} = \frac{1}{4D^{n/2}} \int_0^\infty e^{-\lambda^2 t/4} \prod_{j=1}^n \left( b^{i}_{k_j,m_j}(P_j) - b^{i}_{k_j,m_j}(Q_j) \right) dt
\]

and use the abbreviation

\[
b^{i}_{k,m}(P) = \pi^{-1/2} e^{-(k-m)^2/(\mathcal{D}(1+t))} \text{erf} \left( \frac{t}{h_j^2 D}, \frac{k-m}{h_j \sqrt{D}} \right) \mathcal{P}_M \left( \frac{t}{h_j^2 D}, \frac{k-m}{h_j \sqrt{D}} \right)
\]

\[-\pi^{-1/2} \exp \left( - F^2(\frac{t}{h_j^2 D}, \frac{k-m}{h_j \sqrt{D}}) \right) Q_\mathcal{M} \left( \frac{t}{h_j^2 D}, \frac{k-m}{h_j \sqrt{D}} \right) \right) / (2\sqrt{\pi}).
\]

According to Remark 2.1, for appropriately chosen \( r > 0 \) we can set within a given accuracy

\[
b^{i}_{k,m}(P) = a^{i}_{k-m} = \pi^{-1/2} e^{-(k-m)^2/(\mathcal{D}(1+t))} \mathcal{P}_M \left( \frac{t}{h_j^2 D}, \frac{k-m}{h_j \sqrt{D}} \right) \quad \text{if } P - h_jm \leq -rh_j \sqrt{D},
\]

\[
b^{i}_{k,m}(P) = 0 \quad \text{if } P - h_jm \geq rh_j \sqrt{D},
\]

which speeds up the computation of (3.2). In particular, we can split (3.1) into

\[
\kappa_{\lambda} \bar{f}(\mathbf{x}) = \sum_{\mathbf{hm} \in \Omega_h} f(\mathbf{hm}) a^{(M)}_{k-m} + \sum_{\mathbf{hm} \in \Omega_h \setminus \Omega_r} \bar{f}(\mathbf{hm}) b^{(M)}_{k,m},
\]

where \( \Omega_r = \prod_{j=1}^n(P_j + rh_j \sqrt{D}, Q_j - rh_j \sqrt{D}) \), and the coefficients in the convolutional sum are given by

\[
a^{(M)}_{k} = \frac{1}{4D^{n/2}} \int_0^\infty e^{-\lambda^2 t/4} \prod_{j=1}^n a^{i}_{k_j} dt
\]

\[
= \frac{1}{4(\pi D)^{n/2}} \int_0^\infty e^{-\lambda^2 t/4} e^{-|k|^2/(\mathcal{D}(1+t))} \prod_{j=1}^n \mathcal{P}_M \left( \frac{t}{h_j^2 D}, \frac{k_j}{h_j \sqrt{D}} \right) dt.
\]
Following [12] the one-dimensional integrals of $a^{(M)}_k$ and $b^{(M)}_{k,m}$ are transformed to integrals over $\mathbb{R}$ with integrands decaying doubly exponentially by making the substitutions

$$t = e^\xi, \quad \xi = \alpha(\sigma + e^\sigma), \quad \sigma = \beta(u - e^{-u}) \quad (3.4)$$

with certain positive constants $\alpha, \beta$, and the computation is based on the classical trapezoidal rule. Then the tensor product structure of the integrands allows the efficient computation of the coefficients $b^{(M)}_{k,m}$ and $a^{(M)}_k$. Moreover, the computation of the convolutional sum is very efficient for integrands, which allow a separated representation, i.e., for given accuracy $\epsilon$ they can be represented as a sum of products of vectors in dimension 1

$$f(h_1m_1, \ldots, h_nm_n) = \sum_{p=1}^R r_p \prod_{j=1}^n f^{(p)}_j(h_jm_j) + O(\epsilon).$$

In [7] we have described this approach to the fast computation of high dimensional volume potentials for compactly supported integrands. To compute the convolutional sum

$$\sum_{hm \in \Omega_{rh}} a^{(M)}_{k-m} f(hm)$$

we get after the substitutions

$$a^{(M)}_k = \frac{1}{4(\pi D)^{n/2}} \int_{-\infty}^{\infty} e^{-\lambda^2 \Phi(u)/4} e^{-|k|^2/(D(1 + \Phi(u)))} \prod_{j=1}^n P_M\left(\frac{\Phi(u)}{h_j^2 D}, \frac{k_j}{\sqrt{D}}\right) \Phi'(u) \, du,$$

where we set

$$\Phi(u) = \exp(\alpha\beta(u - \exp(-u)) + \alpha \exp(\beta(u - \exp(-u))),
\Phi'(u) = \Phi(u)\alpha\beta(1 + e^{-u})(1 + \exp(\beta(u - \exp(-u))))).$$

The quadrature with the trapezoidal rule with step size $\tau$

$$a^{(M)}_k \approx \frac{\tau}{4(\pi D)^{n/2}} \sum_{s=-N_0}^{N_1} e^{-\lambda^2 \Phi(s\tau)/4} e^{-|k|^2/(D(1 + \Phi(s\tau)))} \prod_{j=1}^n P_M\left(\frac{\Phi(s\tau)}{h_j^2 D}, \frac{k_j}{\sqrt{D}}\right) \Phi'(s\tau)$$

provides the approximation via one-dimensional discrete convolutions

$$\sum_{hm \in \Omega_{rh}} a^{(M)}_{k-m} f(hm) \approx \frac{\tau}{4(\pi D)^{n/2}} \sum_{p=1}^R r_p \sum_{s=-N_0}^{N_1} e^{-\lambda^2 \Phi(s\tau)/4} \Phi'(s\tau)$$

$$\times \prod_{j=1}^n \sum_{m_j} e^{-(k_j-m_j)^2/(D(1 + \Phi(s\tau)))} P_M\left(\frac{\Phi(s\tau)}{h_j^2 D}, \frac{k_j-m_j}{\sqrt{D}}\right) f^{(p)}_j(h_jm_j).$$

We provide some numerical tests to the approximation of the potential $K_\lambda f$ over the cube $[-1,1]^n$, $n \geq 3$, with the density

$$f(x) = (-\Delta + \lambda^2) \prod_{j=1}^n u(x_j) = \sum_{p=1}^n \prod_{j=1}^n f^{(p)}_j(x_j), \quad x = (x_1, \ldots, x_n) \in [-1,1]^n; \quad (3.5)$$
\[ f_j^{(p)}(x) = u(x) \quad \text{if} \quad j \neq p; \quad f_j^{(p)}(x) = -u''(x) + \frac{\lambda^2}{n} u(x) \quad \text{if} \quad j = p. \]

Let \( \tilde{f}_j^{(p)} \) be an extension of \( f_j^{(p)} \) outside the interval \([-1, 1]\) with preserved smoothness and

\[
\tilde{f}(x) = \sum_{p=1}^{n} \prod_{j=1}^{n} \tilde{f}_j^{(p)}(x_j), \quad x \in \mathbb{R}^n.
\]

By using Hestenes reflection principle ([5]) we construct an extension of \( f_j^{(p)} \) outside the interval \([-1, 1]\) as

\[
\tilde{f}_j^{(p)}(x) = \begin{cases} 
\sum_{s=1}^{N+1} c_s f_j^{(p)}(-a_s (x + 1) - 1), & x < -1 \\
 f_j^{(p)}(x), & -1 \leq x < 1 \\
\sum_{s=1}^{N+1} c_s f_j^{(p)}(-a_s (x - 1) + 1), & x > 1
\end{cases}
\]

where \( a_1, ..., a_{N+1} \) are different positive constants and the coefficients \( c_N = \{c_1, ..., c_{N+1}\} \) are the unique solution of the \((N + 1) \times (N + 1)\) system of linear equations

\[
\sum_{s=1}^{N+1} c_s (-a_s)^k = 1, \quad k = 0, ..., N.
\]

We provide results for \( \tilde{f}_j^{(p)} = f_j^{(p)} \) and three different Hestenes extensions corresponding to \( a_s = 2^{-s} \) (Extension 1), \( a_s = s^{-1} \) (Extension 2), \( a_s = s \) (Extension 3).

The approximation values are computed by the cubature formula (3.3) for \( h_j = h, j = 1, ..., n \). To have the saturation error comparable with the double precision rounding errors, we have chosen the parameter \( D = 4 \).

In Tables 1, 2 and 3 we report on the absolute error and the approximation rate for the three-dimensional potential \( K_{\lambda} f \), when \( u(x) = \cos^2(\pi x/2) \) (Table 1), \( u(x) = (x^2 - 1)^3 \) (Table 2) and \( u(x) = (x^2 - 1)^2 \) (Table 3), in the case \( \lambda^2 = 1 \) and \( \lambda^2 = 1+i \). We have chosen the parameters \( \alpha = 2, \beta = 2 \) in the transformations (3.4) and \( \tau = 0.005, N_1 = -N_0 = 300 \) in the quadrature formula. The numerical results confirm the \( h^2 -, h^4 - \) and, respectively, \( h^6 - \) convergence of the cubature formulas (3.3) when \( M = 1, 2, 3 \). For extensions 1, 2 and 3 the numerical results are similar with those if using \( \tilde{f}_j^{(p)} = f_j^{(p)} \). In Table 3 we see that the error of the approximate quasi-interpolant of order 6 has reached the saturation bound. This is a feature of the method that approximate quasi-interpolant of order \( N \) reproduces polynomials of degree \( < N \) up to the saturation error.

To check the effectiveness of the method for very high dimension \( n \) we computed the potential over \([-1, 1]^n\) of the density (3.5) with \( u(x) = 1 - \sin(\pi x^2/2) \) (Table 4) and \( u(x) = e^x(1 - x^2)^2 \) (Table 5) in dimension \( n = 10^i, i = 1, ..., 8 \) and different extensions. We have chosen \( a = 6, b = 5, \tau = 0.003, N_0 = -40, N_1 = 200 \). The results show that \( K_{\lambda,h}^{(3)} \) approximates with the predicted approximation rate 6, also for very large \( n \) and the error scales linearly in the space dimension.
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\(\lambda^2 = 1: \)

| \(f(x)\) | \(h^{-1}\) | \(M = 1\) | \(M = 2\) | \(M = 3\) |
|---|---|---|---|---|
| \(10\) | \(0.822E-01\) | \(0.414E-02\) | \(0.135E-03\) |
| \(20\) | \(0.219E-01\) | \(0.272E-03\) | \(3.9267\) |
| \(40\) | \(0.557E-02\) | \(0.172E-04\) | \(3.9821\) |
| \(80\) | \(0.140E-02\) | \(0.108E-05\) | \(3.9955\) |
| \(160\) | \(0.350E-03\) | \(0.675E-07\) | \(3.9989\) |
| \(320\) | \(0.875E-04\) | \(0.422E-08\) | \(3.9997\) |

\[\lambda^2 = 1 + i: \]

| \(f(x)\) | \(h^{-1}\) | \(M = 1\) | \(M = 2\) | \(M = 3\) |
|---|---|---|---|---|
| \(10\) | \(0.821E-01\) | \(0.413E-02\) | \(0.135E-03\) |
| \(20\) | \(0.219E-01\) | \(0.272E-03\) | \(3.9265\) |
| \(40\) | \(0.557E-02\) | \(0.172E-04\) | \(3.9820\) |
| \(80\) | \(0.140E-02\) | \(0.108E-05\) | \(3.9955\) |
| \(160\) | \(0.350E-03\) | \(0.675E-07\) | \(3.9989\) |
| \(320\) | \(0.875E-04\) | \(0.422E-08\) | \(3.9997\) |

\(\lambda^2 = 1 + i:\)

| \(f(x)\) | \(h^{-1}\) | \(M = 1\) | \(M = 2\) | \(M = 3\) |
|---|---|---|---|---|
| \(10\) | \(0.815E-01\) | \(0.410E-02\) | \(0.134E-03\) |
| \(20\) | \(0.217E-01\) | \(0.270E-03\) | \(3.9265\) |
| \(40\) | \(0.557E-02\) | \(0.172E-04\) | \(3.9821\) |
| \(80\) | \(0.140E-02\) | \(0.108E-05\) | \(3.9955\) |
| \(160\) | \(0.350E-03\) | \(0.675E-07\) | \(3.9989\) |
| \(320\) | \(0.875E-04\) | \(0.422E-08\) | \(3.9997\) |

\(\lambda^2 = 1 + i: \)

| \(f(x)\) | \(h^{-1}\) | \(M = 1\) | \(M = 2\) | \(M = 3\) |
|---|---|---|---|---|
| \(10\) | \(0.814E-01\) | \(0.410E-02\) | \(0.134E-03\) |
| \(20\) | \(0.217E-01\) | \(0.270E-03\) | \(3.9265\) |
| \(40\) | \(0.557E-02\) | \(0.172E-04\) | \(3.9821\) |
| \(80\) | \(0.140E-02\) | \(0.108E-05\) | \(3.9955\) |
| \(160\) | \(0.350E-03\) | \(0.675E-07\) | \(3.9989\) |
| \(320\) | \(0.875E-04\) | \(0.422E-08\) | \(3.9997\) |

**Table 1:** Absolute errors and approximation rates for \(K_{\lambda} f(0.3, 0.3, 0)\) using \(K_{\lambda,h}^{(M)} f(0.3, 0.3, 0)\) with the density \(f\) given in (3.5) with \(u(x) = \cos^2(\pi x/2)\) and different extensions, \(M = 1, 2, 3, \lambda^2 = 1\) and \(\lambda^2 = 1 + i\).
\[ \lambda^2 = 1: \]

| \( f(x) \) | \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|---|
| 10 | 0.673E-01 | 0.626E-02 | 0.427E-04 | |
| 20 | 0.159E-01 | 2.0819 | 3.9965 | 0.668E-06 | 5.9997 |
| 40 | 0.439E-01 | 2.0238 | 3.9970 | 0.804E-07 | 6.0000 |
| 80 | 0.973E-03 | 2.0062 | 3.9991 | 0.136E-09 | 6.0000 |
| 160 | 0.243E-03 | 2.0186 | 3.9907 | 0.255E-11 | 6.0000 |
| 320 | 0.607E-04 | 2.0004 | 3.9999 | 0.407E-13 | 6.0000 |

\[ \tilde{f}(x) \]

| \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|
| 10 | 0.637E-01 | 0.634E-02 | 0.427E-04 | |
| 20 | 0.157E-01 | 2.0254 | 3.9999 | 0.804E-07 | 6.0000 |
| 40 | 0.391E-02 | 2.0075 | 3.9970 | 0.104E-07 | 6.0000 |
| 80 | 0.972E-03 | 2.0005 | 3.9991 | 0.156E-09 | 6.0000 |
| 160 | 0.243E-03 | 2.0001 | 3.9998 | 0.255E-11 | 6.0000 |
| 320 | 0.607E-04 | 2.0000 | 3.9999 | 0.407E-13 | 6.0000 |

\[ \lambda^2 = 1 + i: \]

| \( f(x) \) | \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|---|
| 10 | 0.604E-01 | 0.572E-02 | 0.441E-04 | |
| 20 | 0.142E-01 | 2.0834 | 3.9999 | 0.690E-06 | 5.9997 |
| 40 | 0.350E-02 | 2.0242 | 3.9969 | 0.108E-07 | 6.0000 |
| 80 | 0.872E-03 | 2.0062 | 3.9991 | 0.168E-09 | 6.0000 |
| 160 | 0.218E-03 | 2.0016 | 3.9998 | 0.263E-11 | 6.0000 |
| 320 | 0.544E-04 | 2.0001 | 3.9999 | 0.410E-13 | 6.0000 |

\[ \tilde{f}(x) \]

| \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|
| 10 | 0.603E-01 | 0.579E-02 | 0.441E-04 | |
| 20 | 0.140E-01 | 2.0271 | 3.9999 | 0.690E-06 | 5.9997 |
| 40 | 0.349E-02 | 2.0080 | 3.9904 | 0.108E-07 | 6.0000 |
| 80 | 0.871E-03 | 2.0021 | 3.9991 | 0.163E-09 | 6.0000 |
| 160 | 0.218E-03 | 2.0001 | 3.9998 | 0.240E-11 | 6.0000 |
| 320 | 0.544E-04 | 2.0001 | 3.9999 | 0.410E-13 | 6.0000 |

Table 2: Absolute errors and approximation rates for \( K_\lambda f(0.5, 0.5, 0.5) \) using \( K_{\lambda f}^{(M)} (0.5, 0.5, 0.5) \) with the density \( f \) given in (3.5) with \( u(x) = (x^2 - 1)^3 \) and different extensions, \( M = 1, 2, 3, \lambda^2 = 1 \) and \( \lambda^2 = 1 + i. \)
\[ \lambda^2 = 1: \]

| \( f(x) \) | \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|---|
| \( \sim \) | | | | |
| \( \lambda^2 = 1: \) | | | | |
| \( f(x) \) | 10 | 0.935E-01 | 0.166E-02 | 0.222E-15 |
| ext 1 | 20 | 0.241E-01 | 1.956E-03 | 3.9984 |
| & | 40 | 0.607E-02 | 1.988E-05 | 3.9999 |
| & | 80 | 0.152E-02 | 1.997E-06 | 4.0000 |
| & | 160 | 0.380E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.951E-04 | 1.9998 | 1.58E-08 |
| ext 2 | 10 | 0.941E-01 | 0.166E-02 | 0.779E-10 |
| & | 20 | 0.241E-01 | 1.963E-03 | 3.9984 |
| & | 40 | 0.607E-02 | 1.990E-05 | 2.22E-10 |
| & | 80 | 0.152E-02 | 1.997E-06 | 4.0000 |
| & | 160 | 0.380E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.951E-04 | 1.9998 | 1.58E-08 |
| ext 3 | 10 | 0.941E-01 | 0.166E-02 | 0.779E-10 |
| & | 20 | 0.241E-01 | 1.963E-03 | 3.9984 |
| & | 40 | 0.607E-02 | 1.990E-05 | 2.22E-10 |
| & | 80 | 0.152E-02 | 1.997E-06 | 4.0000 |
| & | 160 | 0.380E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.951E-04 | 1.9998 | 1.58E-08 |

\[ \lambda^2 = 1 + i: \]

| \( f(x) \) | \( h^{-1} \) | \( M = 1 \) | \( M = 2 \) | \( M = 3 \) |
|---|---|---|---|---|
| \( \sim \) | | | | |
| \( \lambda^2 = 1 + i: \) | | | | |
| \( f(x) \) | 10 | 0.569E-01 | 0.168E-02 | 0.222E-15 |
| ext 1 | 20 | 0.224E-01 | 1.954E-03 | 3.9984 |
| & | 40 | 0.565E-02 | 1.987E-05 | 3.9999 |
| & | 80 | 0.142E-02 | 1.999E-06 | 4.0000 |
| & | 160 | 0.354E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.886E-04 | 1.9998 | 1.58E-08 |
| ext 2 | 10 | 0.586E-01 | 0.168E-02 | 0.695E-10 |
| & | 20 | 0.224E-01 | 1.961E-03 | 3.9984 |
| & | 40 | 0.565E-02 | 1.989E-05 | 3.9999 |
| & | 80 | 0.142E-02 | 1.999E-06 | 4.0000 |
| & | 160 | 0.354E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.886E-04 | 1.9998 | 1.58E-08 |
| ext 3 | 10 | 0.586E-01 | 0.168E-02 | 0.695E-10 |
| & | 20 | 0.224E-01 | 1.961E-03 | 3.9984 |
| & | 40 | 0.565E-02 | 1.989E-05 | 3.9999 |
| & | 80 | 0.142E-02 | 1.999E-06 | 4.0000 |
| & | 160 | 0.354E-03 | 1.9995 | 2.53E-07 |
| & | 320 | 0.886E-04 | 1.9998 | 1.58E-08 |

Table 3: Absolute errors and approximation rates for \( K_\lambda f(0.4, 0.5, 0) \) using \( K_{\lambda,h}^{(M)} f(0.4, 0.5, 0) \) with the density \( f \) given in (3.5) with \( u(x) = (1 - x^2)^2 \) and different extensions, with \( M = 1, 2, 3 \), \( \lambda^2 = 1 \) and \( \lambda^2 = 1 + i \).
| $h^{-1}$ | $n$ | $f(x)$ | error rate | error rate | error rate | error rate | error rate | error rate |
|---------|-----|--------|------------|------------|------------|------------|------------|------------|
| 10      | 0.33E-03 | 0.459E-02 | 0.487E-01 | 0.70E+00 |
| 20      | 0.60E-05 | 5.8020 | 0.35E+00 | 6.0282 | 0.75E+00 | 6.5491 |
| 40      | 0.97E-07 | 5.9541 | 0.11E+00 | 5.9999 | 0.11E+01 | 6.0070 |
| 80      | 0.15E-08 | 5.9087 | 0.12E-06 | 5.9999 | 0.12E-06 | 6.0000 |
| 160     | 0.24E-10 | 5.9971 | 0.28E-08 | 6.0000 | 0.28E-07 | 5.9999 |
| 320     | 0.37E-12 | 5.9982 | 0.53E-11 | 5.7677 | 0.44E-10 | 6.0005 | 0.44E-09 | 5.9985 |

| $f(x)$ | $n$ | $h^{-1}$ | $10^3$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|--------|-----|---------|--------|--------|--------|--------|--------|--------|--------|--------|
| 20     | 0.79E-01 | 0.145E+01 | 0.129E+00 | 0.34E+01 |
| 40     | 0.11E-02 | 6.0852 | 0.11E+00 | 6.9443 | 0.75E+00 | 6.5491 |
| 80     | 0.18E-04 | 6.0012 | 0.12E-02 | 6.1364 | 0.18E-01 | 6.0000 |
| 160    | 0.28E-06 | 5.9992 | 0.12E-04 | 5.9999 | 0.28E-07 | 5.9999 |
| 320    | 0.45E-08 | 5.9982 | 0.47E-07 | 5.9999 | 0.51E-06 | 5.8096 | 0.51E-05 | 5.7889 |

| $f(x)$ | $n$ | $h^{-1}$ | $10^3$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|--------|-----|---------|--------|--------|--------|--------|--------|--------|--------|--------|
| 20     | 0.45E-03 | 0.459E-02 | 0.487E-01 | 0.70E+00 |
| 40     | 0.60E-05 | 5.8020 | 0.35E+00 | 6.0282 | 0.75E+00 | 6.5491 |
| 80     | 0.97E-07 | 5.9541 | 0.11E+00 | 5.9999 | 0.11E+01 | 6.0070 |
| 160    | 0.15E-08 | 5.9087 | 0.12E-06 | 5.9999 | 0.12E-06 | 6.0000 |
| 320    | 0.24E-10 | 5.9971 | 0.28E-08 | 6.0000 | 0.28E-07 | 5.9999 |

| $f(x)$ | $n$ | $h^{-1}$ | $10^3$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|--------|-----|---------|--------|--------|--------|--------|--------|--------|--------|--------|
| 20     | 0.35E-03 | 0.459E-02 | 0.487E-01 | 0.70E+00 |
| 40     | 0.60E-05 | 5.8020 | 0.35E+00 | 6.0282 | 0.75E+00 | 6.5491 |
| 80     | 0.97E-07 | 5.9541 | 0.11E+00 | 5.9999 | 0.11E+01 | 6.0070 |
| 160    | 0.15E-08 | 5.9087 | 0.12E-06 | 5.9999 | 0.12E-06 | 6.0000 |
| 320    | 0.24E-10 | 5.9971 | 0.28E-08 | 6.0000 | 0.28E-07 | 5.9999 |

Table 4: Absolute errors and approximation rates for $K_\lambda f(0.5, 0, ..., 0)$ using $K_\lambda^{(3)} f(0.5, 0, ..., 0)$ with the density $f$ given in (3.5) with $u(x) = 1 - \sin(\pi x^2/2)$ and different extensions, $n = 10^i$, $i = 1, ..., 8$, $\lambda^2 = 1$. 
Table 5: Absolute errors and approximation rates for $K_\lambda f(0.4, 0.4, 0, ..., 0)$ using $K_{\lambda, h}^{(3)} f(0.4, 0.4, 0, ..., 0)$ with the density $f$ given in (3.5) with $u(x) = e^x(1 - x^2)^2$ and different extensions, $n = 10^i$, $i = 1, ..., 8$, $\lambda^2 = 1$. 

| $f(x)$ | $n$ | $10$ | $10^2$ | $10^3$ | $10^4$ |
|--------|-----|------|--------|--------|--------|
|        | $h^{-1}$ | error | rate | error | rate | error | rate | error | rate | error | rate |
| $f(x)$ | 10 | 0.699E-03 | 0.596E-02 | 0.595E-01 | 0.759E-00 |
|        | 20 | 0.106E-04 | 6.0400 | 0.902E-04 | 6.0453 | 0.880E-03 | 6.0792 | 0.881E-02 | 6.4288 |
|        | 40 | 0.165E-06 | 6.0100 | 0.140E-05 | 6.0105 | 0.136E-04 | 6.0111 | 0.136E-03 | 6.0162 |
|        | 80 | 0.257E-08 | 6.0026 | 0.218E-07 | 6.0026 | 0.213E-06 | 6.0026 | 0.212E-05 | 6.0027 |
|        | 160 | 0.402E-10 | 6.0005 | 0.341E-09 | 6.0017 | 0.332E-08 | 6.0006 | 0.332E-07 | 6.0005 |
|        | 320 | 0.632E-12 | 5.9990 | 0.491E-11 | 6.1156 | 0.585E-10 | 5.9998 | 0.519E-09 | 5.9973 |
| $f(x)$ | $10^2$ | error | rate | error | rate | error | rate | error | rate | error | rate |
|        | 20 | 0.913E-01 | 0.134E+01 | 0.134E-01 | 0.134E+01 | 0.145E-00 | 0.145E+00 | 0.267E+01 |
|        | 40 | 0.165E-02 | 6.0671 | 0.137E-01 | 6.6101 | 0.145E-00 | 6.0906 | 0.214E-01 | 6.9639 |
|        | 80 | 0.212E-04 | 6.0035 | 0.212E-03 | 6.0113 | 0.212E-02 | 6.0906 | 0.214E-01 | 6.9639 |
|        | 160 | 0.332E-06 | 5.9994 | 0.332E-05 | 5.9966 | 0.333E-04 | 5.9966 | 0.333E-03 | 6.0087 |
|        | 320 | 0.526E-08 | 5.9779 | 0.572E-07 | 5.8504 | 0.632E-06 | 5.7186 | 0.646E-05 | 5.6865 |
| $f(x)$ | $10^3$ | error | rate | error | rate | error | rate | error | rate | error | rate |
|        | 20 | 0.913E-01 | 0.134E+01 | 0.134E-01 | 0.134E+01 | 0.145E-00 | 0.145E+00 | 0.267E+01 |
|        | 40 | 0.165E-02 | 6.0671 | 0.137E-01 | 6.6101 | 0.145E-00 | 6.0906 | 0.214E-01 | 6.9639 |
|        | 80 | 0.212E-04 | 6.0035 | 0.212E-03 | 6.0113 | 0.212E-02 | 6.0906 | 0.214E-01 | 6.9639 |
|        | 160 | 0.332E-06 | 5.9994 | 0.332E-05 | 5.9966 | 0.333E-04 | 5.9966 | 0.333E-03 | 6.0087 |
|        | 320 | 0.526E-08 | 5.9779 | 0.572E-07 | 5.8504 | 0.632E-06 | 5.7186 | 0.646E-05 | 5.6865 |

Table 5: Absolute errors and approximation rates for $K_\lambda f(0.4, 0.4, 0, ..., 0)$ using $K_{\lambda, h}^{(3)} f(0.4, 0.4, 0, ..., 0)$ with the density $f$ given in (3.5) with $u(x) = e^x(1 - x^2)^2$ and different extensions, $n = 10^i$, $i = 1, ..., 8$, $\lambda^2 = 1$. 

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