OPTIMIZATION OF FOURTH ORDER STURM-LIOUVILLE TYPE DIFFERENTIAL INCLUSIONS WITH INITIAL POINT CONSTRAINTS

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Abstract. The present paper studies a new class of problems of optimal control theory with differential inclusions described by fourth order Sturm-Liouville type differential operators (SLDOs). Then, there arises a rather complicated problem with simultaneous determination of the SLDOs with variable coefficients and a Mayer functional depending of high order derivatives of searched functions. The sufficient conditions, containing both the Euler-Lagrange and Hamiltonian type inclusions and “transversality” conditions are derived. Formulation of the transversality conditions at the endpoints $t = 0$ and $t = 1$ of the considered time interval plays a substantial role in the next investigations without which it is hardly ever possible to get any optimality conditions. The main idea of the proof of optimality conditions of Mayer problem for differential inclusions with fourth order SLDO is the use of locally-adjoint mappings. The method is demonstrated in detail as an example for the semilinear optimal control problem, for which the Weierstrass-Pontryagin maximum principle is obtained.

1. Introduction. In general, optimization of fourth order differential inclusions arise from a wide variety of problems in science, economy, engineering design, industrial optimization, game theory and so on. In each case, their successful resolution requires the specific peculiarities of each problem to be included in the mathematical model. In our presentation, we discuss a special kind of optimization problem with differential inclusions, in which the left hand side of the inclusion is SLDOs. As is pointed out in [21, 22], boundary value problems for second and fourth order differential equations play a very important role in both theory and applications. Along the way, the problems accompanied with the fourth-order discrete and differential inclusions are more complicated due to the fourth-order derivatives and their discrete analogues. In particular, fourth-order linear differential equations [22], subjected to some boundary conditions arise in the mathematical description of some physical systems (for example, the mathematical models of deflection of beams [22]). These beams, which appear in many structures, deflect under their own weight or under the influence of some external forces. For example, if a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam

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undergoes a distortion, and the curve connecting the centroids of all cross sections is called the deflection curve or elastic curve.

Especially, the problem is considerably more complicated by the presence of SLDOs with variable coefficients; the difficulty in the problems with higher order differential inclusions is rather to construct the Euler-Lagrange type higher order adjoint inclusions and the suitable boundary conditions. That is why on the whole in literature only the qualitative properties of higher order differential inclusions are investigated (see [2, 3, 17, 18] and references therein).

The paper [2] gives necessary and sufficient conditions ensuring the existence of a solution to the second order differential inclusion \( x''(t) \in F(x(t), x'(t)), x(0) = x_0, x'(0) = v_0 \) such that \( x(t) \in K \), where \( K \) is a nonempty given subset of \( \mathbb{R}^n \). Furthermore, second order interior tangent sets are introduced and studied to obtain such conditions. The article [3] studies the three-point boundary value problems for second-order perturbed differential inclusions of the form \( u''(t) \in F(t, u(t), u'(t)) + H(t, u(t), u'(t)) \) a.e. on \([0, 1]\). The existence of solutions is proved under nonconvexity condition for the set-valued mapping \( H \). Here it is assumed that \( F : [0, 1] \times E \times E \rightrightarrows E \) (\( E \) is a separable Banach space) is upper semicontinuous on \( E \times E \) and such that, for any fixed \((x, y)\) in \( E \times E \), the set-valued mapping \( F(\cdot, x, y) \) has a measurable graph, \( H : [0, 1] \times E \times E \rightrightarrows E \) is a set-valued mapping with a measurable graph and closed values, lower semicontinuous on \( E \times E \).

Optimization of first order discrete and continuous time processes with lumped and distributed parameters has been expanding in all directions at an astonishing rate during the last few decades (see [1, 4, 5, 6, 7, 8, 11, 12, 13, 23, 24, 25] and their references). The objective of the paper [1] is to briefly summarize some recent results on Differential Inclusions and their optimal control. Then, using vector measures as controls, are presented some new results on the necessary and sufficient conditions of optimality. Further, are considered systems having structural perturbation modeled by operator valued measures. In the work [8], for set-valued maps are introduced the concept of a generalized second-order composed contingent epiderivative. Then, by virtue of the generalized second-order composed contingent epiderivative, are established a unified second-order necessary and sufficient condition of optimality for set-valued optimization problems. In the paper [24] the behavior of the perturbation map is analyzed quantitatively by using the concept of higher-order contingent derivative for the set-valued maps under Henig efficiency. By using the higher-order contingent derivatives and applying a separation theorem for convex sets, some results concerning higher-order sensitivity analysis are established.

The optimization of higher order differential inclusions was first developed by Mahmudov in [14, 15, 16, 17]. Since then this problem has attracted many authors attentions. The paper [15] is mainly concerned with the necessary and sufficient conditions of optimality for the Bolza problem with third order differential inclusions. Applying this approach to problems with geometric constraints, optimality conditions are formulated for arbitrary higher-order discrete inclusions. In the paper [16] a Bolza problem of optimal control theory with a fixed time interval given by convex and nonconvex higher order differential inclusions, is studied. The main goal is to derive sufficient optimality conditions for Cauchy problem of \( s \) th-order differential inclusions with left-hand side \( x^{(s)}(t) \). The sufficient conditions including distinctive transversality condition are proved incorporating the Euler-Lagrange and Hamiltonian type inclusions. Furthermore, the application of these results is demonstrated by solving the problems with third order differential inclusions. The
FOURTH ORDER STURM-LIOUVILLE INCLUSIONS

The present paper is dedicated to one of the most difficult and interesting fields—optimization of the Sturm-Liouville type differential inclusions involving fourth order self-adjoint linear differential operators. Various applications of the Sturm-Liouville Oscillation Theory to differential equations and spectral theory play a significant role in modern mathematics and, thus, the problem is of high interest to the scientific community. To the best of our knowledge, there is no paper which considers optimality conditions for these problems in the literature and we aim to fill this gap. Therefore, the novelty of our formulation of the problem is justified. It should be noted that there are two papers [9, 10] in the literature devoted to the study of some qualitative properties of the Sturm-Liouville type differential inclusions. In the paper [9], the authors investigate the existence of solutions of impulsive boundary value problems for Sturm-Liouville-type differential inclusions which admit nonconvex set-valued mappings on the right-hand side. Two results under weaker conditions are presented. The methods rely on a fixed-point theorem for the contraction of set-valued maps due to Covitz and Nadler and on Schaefers fixed-point theorem combined with lower semi-continuous set-valued operators with decomposable values.

The present paper is ordered in the following manner.

In Section 2 are given the necessary facts and supplementary results from the book of Mahmudov [14]; Hamiltonian function and locally adjoint mapping (LAM) are introduced and the problem (SLP) with initial point constraints for SLDOs governed by time-dependent set-valued mapping $F(\cdot, t)$ are formulated. In Section 3, we present the main results; on the basis of “transversality” conditions at the endpoints $t = 0$ and $t = 1$ are proved sufficient optimality conditions for differential inclusions with SLDOs and with initial point constraints. For the establishment of the Euler-Lagrange and Hamiltonian inclusions are used LAM construction. The case of variable coefficients of SLDO turns out to be more complicated, unless transversality assumptions at the endpoints $t = 0$ and $t = 1$ are imposed. Notice that the proof relies on consideration of a convex case, even though the result remains true for nonconvex problem, too. In Section 4 the problem governed by fourth order polynomial linear differential operators with constant coefficients is considered. Furthermore, practical applications of these results are demonstrated by optimization of some semilinear with respect to the state variable optimal control problems for which the Weierstrass-Pontryagin maximum condition is obtained. Our results allow us to simplify enough the proof of the maximum principle, to obtain a new Euler-Lagrange inclusion for optimal control problems of Mayer type governed by differential inclusions with SLDOs.

2. Preliminaries and problems statements. In this section we recall the key notions of set-valued mappings from the book [14]: let $\mathbb{R}^n$ be a $n$-dimensional Euclidean space, $\langle x, v \rangle$ be an inner product of elements $x, v \in \mathbb{R}^n$, $(x, v)$ be a pair of $x, v$. Let $F: \mathbb{R}^n \Rightarrow \mathbb{R}^n$ be a set-valued mapping from $\mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Therefore $F: \mathbb{R}^n \Rightarrow \mathbb{R}^n$ is a convex set-valued mapping, if its graph $\text{gph} F = \{(x, v) : v \in F(x)\}$ is a convex subset of $\mathbb{R}^{2n}$. A set-valued mapping $F$
is called closed if its $\text{gph} F$ is a closed subset in $\mathbb{R}^{2n}$. The domain of a set-valued mapping $F$ is denoted by $\text{dom} F$ and is defined as $\text{dom} F = \{ x : F(x) \neq \emptyset \}$. A set-valued mapping $F$ is convex-valued if $F(x)$ is a convex set for each $x \in \text{dom} F$.

The Hamiltonian function and argmaximum set corresponding to a set-valued mapping $F$ are defined by the following relations

$$
H_F(x, v^*) = \sup_v \langle v, v^* \rangle : v \in F(x),
$$

$$
F_{\text{Arg}}(x, v^*) = F_{A}(x, v^*) = \{ v \in F(x) : \langle v, v^* \rangle = H_F(x, v^*) \}
$$

respectively. For a convex $F$ we put $H_F(x, v^*) = -\infty$ if $F(x) = \emptyset$. In other terms, $H_F(x, v^*)$ is the support function to the set $F(x)$, evaluated at $v^*$. As usual, $\text{int} M$ denotes the interior of the set $M \subset \mathbb{R}^{2n}$ and $\text{ri} M$ denotes the relative interior of a set $M$, i.e. the set of interior points of $M$ with respect to its affine hull $\text{Aff} M$. A convex cone $K_M(z_0)$, $z_0 = (x^0, v^0)$, is called a cone of tangent directions at a point $z_0 \in M$ to the set $M$ if from $\bar{z} = (\bar{x}, \bar{v}) \in K_M(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set $M$ at a point $z_0 \in M$, i.e., there exists such function $q : R^1 \rightarrow \mathbb{R}^{2n}$ that $z_0 + \alpha \bar{z} + q(\alpha) \in M$ for sufficiently small $\alpha > 0$ and $\alpha^{-1} q(\alpha) \rightarrow 0$, as $\alpha \downarrow 0$.

For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the set-valued mapping defined by $F^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$
F^*(v^*; (x, v)) := \{ x^* : (x^*, -v^*) \in K^*_{\text{gph} F}(x, v) \};
$$

$$
K_{\text{gph} F}(x, v) = \text{cone}[\text{gph} F - (x, v)], \forall (x, v^*) \in \text{gph} F
$$

is called the LAM to $F$ at a point $(x, v) \in \text{gph} F$, where $K^* = \{ z^* : \langle \bar{z}, z^* \rangle \geq 0, \forall \bar{z} \in K \}$ denotes the dual cone to the cone $K$, as usual. Below we will define another LAM definition to a set-valued mapping $F$ by using the Hamiltonian function, associated to $F$. Thus, the LAM to “nonconvex” mapping $F$ is defined as follows

$$
F^*(v^*; (x, v)) := \{ x^* : H_F(x^*, v^*) - H_F(x, v^*) \geq \langle x^*, x^1 - x \rangle, \forall x^1 \in \mathbb{R}^n \},
$$

$$(x, v) \in \text{gph} F, v \in F_{A}(x, v^*).$$

Clearly, for the convex mapping $F$ the Hamiltonian function $H_F(\cdot, \cdot, v^*)$ is concave and the latter definition of LAM coincides with the previous definition of LAM [14, p.62]. Note that prior to the LAM the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [20], (however, for the smooth convex maps the two notions are equivalent). In the most interesting settings for the theory and applications, coderivatives are nonconvex-valued and hence are not tangentially /derivatively generated. This is the case of the first coderivative for general finite dimensional set-valued mappings for the purpose of applications to optimal control.

In Section 3 our goal is to give Euler-Lagrange and Hamiltonian optimality conditions for the following general Mayer problem governed by ordinary differential inclusions with fourth order SLDos and with initial point constraints

$$
\text{minimize } \varphi(x(1), x'(1), x''(1), x'''(1)), 
$$

(SLP) \hspace{1cm} \begin{align*}
\left( p(t)x''(t) \right)'' - \left( s(t)x'(t) \right)' & \in F(x(t), t), \text{ a.e. } t \in [0, 1], \\
x(0) & \in Q_0, \quad x'(0) \in Q_1, \quad x''(0) \in Q_2, \quad x'''(0) \in Q_3,
\end{align*}
$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a time dependent set-valued mapping, $\varphi$ is a continuous function; $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^1, p$ and $s$ are second and first order continuously differentiable scalar functions, correspondingly, $p(\cdot) : [0, 1] \rightarrow (0, \infty), s(\cdot) : [0, 1] \rightarrow \mathbb{R}$.
Indeed, we rewrite this operator in the form defined in three different forms which can be converted (theoretically at least) easily.

**Remark 1.** It can be shown, that the fourth order Sturm-Liouville operator \( Lx = (px'')'' - (sx')' \) is a self-adjoint linear differential operator. Indeed, we rewrite this operator in the form

\[
Lx = px^{(4)} + 2p'x'' + (p'' - s)x' = pD^4x + 2p'D^3x + (p'' - s)D^2x - s'Dx,
\]

where \( D^k(k = 1, 2, 3, 4) \) is the operator of \( k \)-th order derivatives. Then this property can be proven by using the formal adjoint definition:

\[
L^*x^* = (-1)^4D^4(px^*) + (-1)^3D^3(2p'x^*) + (-1)^2D^2[(p'' - s)x^*] \\
+ (-1)^1D(-s'x^*) = D^4(px^*) - D^3(2p'x^*) + D^2[(p'' - s)x^*] - D(-s'x^*).
\]

By using Leibniz’s formula for higher order derivatives of a product of two functions (see, for example [19, p. 125]) it can easily be checked that

\[
L^*x^* = \left( px^{(4)} + 4p'x'' + 6p''x' + 4p'''x + p^{(4)}x^* \right) - 2\left( px''' + 3p''x'' \right) \\
+ 3p'''x + p^{(4)}x^* + \left[ (p^{(4)} - s'')x^* + 2(p''' - s')x' + (p'' - s)x'' \right] \\
+ s''x + s'x^* = pD^4x^* + 2p'D^3x^* + (p'' - s)D^2x^* - s'Dx^*.
\]

**Remark 2.** Recall that a cost functional in optimal control theory can be in general defined in three different forms which can be converted (theoretically at least) easily from one to another. For example, every Bolza problem with cost functional \( (g(\cdot, t) : \mathbb{R}^{4n} \to \mathbb{R}) \)

\[
J[x(t)] = \int_0^1 g(x(t), x'(t), x''(t), x'''(t), t) dt + \varphi(x(1), x'(1), x''(1), x'''(1))
\]

can be converted to a Mayer problem through the following two steps: an auxiliary state \( x_{n+1} \) is added to \( 2 \) with the equation \( x'_{n+1} = g(x, x', x'', x''', t) \), thereby, modifying the state inclusion \( 2 \) to \( Lx \in F(x(t), x'_{n+1} = g(x, x', x'', x''', t) \). The cost functional can now be rewritten as \( J[x(1)] = \varphi(x(1), x'(1), x''(1), x'''(1)) + x_{n+1} \) with \( x_{n+1}(0) = 0 \). In general, using different forms of cost functionals, one can consider different control problems.

**Remark 3.** We note that the method of discrete-approximation of (SLP) with fourth order differential inclusions involving SLDOs has been very effective in the investigation of optimality conditions, where the basic idea was to study the fourth-order discrete-approximation problem:

\[
\minimize \varphi\left(x(1 - 3\delta), \Delta x(1 - 3\delta), \Delta^2 x(1 - 3\delta), \Delta^3 x(1 - 3\delta)\right),
\]

\[
p(t)\Delta^4x(t) + 2\Delta p(t)\Delta^3x(t) + (\Delta^2 p(t) - s(t))\Delta^2x(t) - \Delta s(t)\Delta x(t) \in F(x(t), t),
\]

\[
t = 0, \ldots, 1 - 4\delta; \ x(0) \in Q_0; \ \Delta x(0) \in Q_1; \ \Delta^2 x(0) \in Q_2; \ \Delta^3 x(0) \in Q_3.
\]
Here \( m \)-th order difference operator is defined as follows:

\[
\Delta^m x(t) = \frac{1}{\delta^m} \sum_{k=0}^{m} (-1)^k C_m^k x(t + (m - k)\delta),
\]

\[
C_m^k = \frac{m!}{k!(m-k)!}, \quad m = 1, 2, 3, 4, \quad t = 0, \ldots, 1 - \delta.
\]

It appears that by passing to the limit in necessary and sufficient conditions of optimality for discrete-approximate problem as \( \delta \to 0 \) (at least formally), we can establish the sufficient optimality conditions for continuous problem (SLP). But in the presented paper to avoid a long calculations connected with the discretization method, establishment of optimality and transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \) for the discrete-approximate problem are omitted. In the next section we will derive sufficient optimality conditions for problem (SLP).

3. Sufficient optimality conditions for differential inclusions with fourth order SLDOs. First of all we associate with the problem (SLP) the following so-called the fourth-order Euler-Lagrange differential inclusion with SLDO and the transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \) (it is important to note a subtlety in our definition of transversality conditions for the problem (SLP):

(i) \( L^* x^*(t) \in F^*(x^*(t); \hat{x}(t), L\hat{x}(t), t), \) a.e. \( t \in [0, 1] \), where \( L^* x^*(t) = p(t)D^4 x^*(t) \)

\[
+ 2p'(t)D^3 x^*(t) + [p''(t) - s(t)]D^2 x^*(t) - s'(t)Dx^*(t) \text{ is the adjoint SLDO of the primal operator } L.
\]

(ii) \( p(0)x^{(n''}_0) + p'(0)x^{(n')_0} - s(0)x^{(n')_0} \in K^*_Q(0); \)

\[
- p(0)x^{(n')_0} + s(0)x^{(n')_0} \in K^*_{Q_1}(\hat{x}(0));
\]

\[
p(0)x^{(n)_0} - p'(0)x^{(n)_0} \in K^*_{Q_2}(\hat{x}''(0)); \quad -p(0)x^{(n)_0} \in K^*_{Q_3}(\hat{x}'''(0)).
\]

(iii) \( p(1)x^{(n''}_1) + p'(1)x^{(n')_1} - s(1)x^{(n')_1} - p(1)x^{(n')_1} + s(1)x^{(n')_1}, \)

\[
p(1)x^{(n')_1} - p'(1)x^{(n')_1} - p(1)x^{(n')_1} \in \partial \varphi(\hat{x}(1), \hat{x}'(1), \hat{x}''(1), \hat{x}'''(1)).
\]

Later on we assume that \( x^{(k)}(\cdot)(k = 0, 1, 2, 3), t \in [0, 1] \), are absolutely continuous functions and \( x^{(k)}(\cdot) \in L^1(0, 1) \).

At last we formulate the condition ensuring that the \( \text{LAMF}^* \) is nonempty at a given point:

(iv) \( L\hat{x}(t) \in F_A(x^*(t); \hat{x}(t), x^*(t), t), \) a.e. \( t \in [0, 1] \), or, equivalently,

\[
\langle L\hat{x}(t), x^*(t) \rangle = H_F(\hat{x}(t), x^*(t)), L\hat{x}(t) \in F(\hat{x}(t), t).
\]

With this tool we are now ready for the main result, which gives sufficient optimality conditions for differential inclusions with SLDOs.

**Theorem 3.1.** Let \( \varphi : \mathbb{R}^{4n} \to \mathbb{R}^1 \) be a continuous and convex function and \( F(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n \) be a convex time dependent set-valued mapping. Moreover let \( Q_k, k = 0, 1, 2, 3 \) be convex sets. Then for optimality of the trajectory \( \hat{x}(t) \) in the problem (SLP) with differential inclusions and SLDOs it is sufficient that there exists an absolutely continuous function \( x^*(t), t \in [0, 1] \) satisfying a.e. the Euler-Lagrange differential inclusion with SLDOs (i), (iv) and transversality conditions (ii), (iii) at the endpoints \( t = 0 \) and \( t = 1 \).
Proof. For notational simplicity of description let us denote
\[ p_4(t) = p(t), \quad p_3(t) = 2p'(t), \quad p_2(t) = p''(t) - s(t), \quad p_1(t) = -s'(t), \quad t \in [0,1] \quad (4) \]

In view of 4 it is not hard to see that by using Theorem 2.1 [14, p.62] in term of Hamiltonian function from the condition (i) we obtain the following important relation
\[ H_F(x(t), x^*(t), t) - H_F(\tilde{x}(t), x^*(t), t) \leq \langle L^*x^*(t), x(t) - \tilde{x}(t) \rangle \]
or its useful reformulation
\[ H_F(x(t), x^*(t), t) - H_F(\tilde{x}(t), x^*(t), t) \leq \left\langle \sum_{k=1}^{4} (-1)^k \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle. \quad (5) \]

In turn by using definition of the Hamiltonian function, 5 can be converted to the inequality
\[ 0 \geq \left\langle Lx(t) - L\tilde{x}(t), x^*(t) \right\rangle - \left\langle L^*x^*(t), x(t) - \tilde{x}(t) \right\rangle \]
or
\[ 0 \geq \left\langle \sum_{k=1}^{4} p_k(t) \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, x^*(t) \right\rangle - \left\langle \sum_{k=1}^{4} (-1)^k \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle. \quad (6) \]

Integrating 6 over the interval [0, 1] we have
\[ 0 \geq \int_{0}^{1} \left[ \left\langle \sum_{k=1}^{4} p_k(t) \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, x^*(t) \right\rangle - \left\langle \sum_{k=1}^{4} (-1)^k \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle \right] dt. \quad (7) \]

Let us denote
\[ G = \sum_{k=1}^{4} \left\langle \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, p_k(t)x^*(t) \right\rangle - \sum_{k=1}^{4} \left\langle (-1)^k \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle. \]

In what follows our approach lies in reducing \( G \) in a relationship consisting of four sums from \( k(k = 1, 2, 3, 4) \) to four of suitable derivatives of scalar products; thus, after some transformations we can deduce an important representation for a first term of \( G \) as follows
\[
\begin{align*}
\sum_{k=1}^{4} \left\langle \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, p_k(t)x^*(t) \right\rangle &= \sum_{k=1}^{4} \left[ \frac{d}{dt} \left(x^{(k-1)}(t) - \tilde{x}^{(k-1)}(t), p_k(t)x^*(t) \right) \right] \\
&- \sum_{k=2}^{4} \left[ \frac{d}{dt} \left(x^{(k-2)}(t) - \tilde{x}^{(k-2)}(t), \frac{d(p_k(t)x^*(t))}{dt} \right) \right] \\
&+ \sum_{k=3}^{4} \left[ \frac{d}{dt} \left(x^{(k-3)}(t) - \tilde{x}^{(k-3)}(t), \frac{d^2(p_k(t)x^*(t))}{dt^2} \right) \right] \\
&- \frac{d}{dt} \left(x(t) - \tilde{x}(t), \frac{d^3(p_k(t)x^*(t))}{dt^3} \right) + \sum_{k=1}^{4} \left[ \left(x(t) - \tilde{x}(t), (-1)^k \frac{d^k(p_k(t)x^*(t))}{dt^k} \right) \right].
\end{align*}
\quad (8)
\]
Then in view of (8) in the definition of $G$ we have an efficient formula:

$$G = \sum_{k=1}^{4} \left[ \frac{d}{dt} \langle x^{(k-1)}(t) - \tilde{x}^{(k-1)}(t), p_k(t)x^*(t) \rangle \right]$$

$$- \sum_{k=2}^{4} \left[ \frac{d}{dt} \langle x^{(k-2)}(t) - \tilde{x}^{(k-2)}(t), \frac{d(p_k(t)x^*(t))}{dt} \rangle \right]$$

$$+ \sum_{k=3}^{4} \left[ \frac{d}{dt} \langle x^{(k-3)}(t) - \tilde{x}^{(k-3)}(t), \frac{d^2(p_k(t)x^*(t))}{dt^2} \rangle \right]$$

$$- \frac{d}{dt} \langle x(t) - \tilde{x}(t), \frac{d^3(p_4(t)x^*(t))}{dt^3} \rangle.$$

Then taking into account the form of $G$ in (9) we can compute the integral on the right hand side of (7) as follows:

$$\int_{0}^{1} G dt = \sum_{k=1}^{4} \left[ \int_{0}^{1} d\langle x^{(k-1)}(t) - \tilde{x}^{(k-1)}(t), p_k(t)x^*(t) \rangle \right]$$

$$- \sum_{k=2}^{4} \left[ \int_{0}^{1} d\langle x^{(k-2)}(t) - \tilde{x}^{(k-2)}(t), \frac{d(p_k(t)x^*(t))}{dt} \rangle \right]$$

$$+ \sum_{k=3}^{4} \left[ \int_{0}^{1} d\langle x^{(k-3)}(t) - \tilde{x}^{(k-3)}(t), \frac{d^2(p_k(t)x^*(t))}{dt^2} \rangle \right]$$

$$- \int_{0}^{1} d\langle x(t) - \tilde{x}(t), \frac{d^3(p_4(t)x^*(t))}{dt^3} \rangle.$$

Hence as a result of integration over an interval $[0, 1]$ we deduce that

$$\int_{0}^{1} G dt = \sum_{k=1}^{4} \left[ \langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_k(1)x^*(1) \rangle \right]$$

$$- \langle x^{(k-1)}(0) - \tilde{x}^{(k-1)}(0), p_k(0)x^*(0) \rangle \right]$$

$$- \sum_{k=2}^{4} \left[ \langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_k(1)x^*(1))}{dt} \rangle \right]$$

$$- \langle x^{(k-2)}(0) - \tilde{x}^{(k-2)}(0), \frac{d(p_k(0)x^*(0))}{dt} \rangle \right]$$

$$+ \sum_{k=3}^{4} \left[ \langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_k(1)x^*(1))}{dt^2} \rangle \right]$$

$$- \langle x^{(k-3)}(0) - \tilde{x}^{(k-3)}(0), \frac{d^2(p_k(0)x^*(0))}{dt^2} \rangle \right]$$

$$- \langle x(1) - \tilde{x}(1), \frac{d^3(p_4(1)x^*(1))}{dt^3} \rangle + \langle x(0) - \tilde{x}(0), \frac{d^3(p_4(0)x^*(0))}{dt^3} \rangle.$$

Here by suitable rearrangement and necessary simplification we have

$$\int_{0}^{1} G dt = \sum_{k=1}^{4} \left[ \langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_k(1)x^*(1) \rangle \right]$$

$$- \sum_{k=2}^{4} \left[ \langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_k(1)x^*(1))}{dt} \rangle \right]$$

$$- \sum_{k=1}^{4} \left[ \langle x^{(k-1)}(0) - \tilde{x}^{(k-1)}(0), p_k(0)x^*(0) \rangle \right]$$

$$+ \sum_{k=2}^{4} \left[ \langle x^{(k-2)}(0) - \tilde{x}^{(k-2)}(0), \frac{d(p_k(0)x^*(0))}{dt} \rangle \right]$$

$$+ \sum_{k=3}^{4} \left[ \langle x^{(k-3)}(0) - \tilde{x}^{(k-3)}(0), \frac{d^2(p_k(0)x^*(0))}{dt^2} \rangle \right]$$

$$- \langle x(1) - \tilde{x}(1), \frac{d^3(p_4(1)x^*(1))}{dt^3} \rangle + \langle x(0) - \tilde{x}(0), \frac{d^3(p_4(0)x^*(0))}{dt^3} \rangle,$$
\[ \sum_{k=3}^{4} \left[ \left( x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^3(p_k(1)x^*(1))}{dt^3} \right) + \left( x(0) - \tilde{x}(0), \sum_{k=0}^{4} (-1)^{4-k} D^{4-k}[p_{4-k}(0)x^*(0)] \right) \right. \\
+ \left( x'(0) - \tilde{x}'(0), \sum_{k=0}^{2} (-1)^{3-k} D^{3-k}[p_{4-k}(0)x^*(0)] \right) + \left( x''(0) - \tilde{x}''(0), \sum_{k=0}^{1} (-1)^{2-k} D^{2-k}[p_{4-k}(0)x^*(0)] \right) \\
- \left( x'''(0) - \tilde{x}'''(0), p_{4}(0)x^*(0) \right) \right]. \] (10)

On the other hand, in view of designations 4 the following formulas can be easily verified

\[ \sum_{k=0}^{2} (-1)^{3-k} D^{3-k}[p_{4-k}(t)x^*(t)] = -D^2[p(t)x^*(t)] + 2D[p'(t)x^*(t)] \]

\[ -\left( p^''(t)x^*(t) + 2p'(t)x''(t) + p(t)x'''(t) \right) + 2\left( p^''(t)x^*(t) + p'(t)x''(t) \right) \]

\[ -\left( p^''(t) - s(t) \right) x^*(t) = -p(t)x^*(t) + s(t) x^*(t); \]

\[ \sum_{k=0}^{1} (-1)^{2-k} D^{1-k}[p_{4-k}(t)x^*(t)] = D[p(t)x^*(t)] - 2p'(t)x^*(t) \]

\[ = p(t)x^*(t) - p'(t)x^*(t). \] (11)

In order to make use of the transversality condition (ii) we rewrite it in more relevant form

\[ \langle x(0) - \tilde{x}(0), p(0)x'''(0) + p'(0)x''(0) - s(0)x''(0) \rangle \]

\[ + \langle x'(0) - \tilde{x}'(0), -p(0)x'''(0) + s(0)x''(0) \rangle + \langle x''(0) - \tilde{x}''(0), p(0)x'\rangle \]

\[ - p'(0)x^*(0) \rangle \geq 0. \]

\[ \forall x(0) \in K_{Q_0}(\tilde{x}(0)), \forall x'(0) \in K_{Q_1}(\tilde{x}'(0)), \]

\[ \forall x''(0) \in K_{Q_2}(\tilde{x}''(0)), \forall x'''(0) \in K_{Q_3}(\tilde{x}'''(0)), \]

Thus, taking into account the formulas 11 at point \( t = 0 \) from 7, 10 we have

\[ 0 \geq \sum_{k=1}^{4} \left[ \langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_k(1)x^*(1) \rangle \right] \]

\[ - \sum_{k=2}^{4} \langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_k(1)x^*(1))}{dt} \rangle \]
above can be expressed in a more compact form

\[ +\sum_{k=3}^{4} \left\langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_k(1)x^*(1))}{dt^2} \right\rangle - \left\langle x(1) - \tilde{x}(1), \frac{d^3(p_4(1)x^*(1))}{dt^3} \right\rangle. \]

It is not hard to see that using the derivative operator \( D \) the relation described above can be expressed in a more compact form

\[ 0 \geq \sum_{k=0}^{3} (-1)^{3-k} D^{3-k} [p_{4-k}(1)x^*(1)], x(1) - \tilde{x}(1) \]

\[ \left\langle \sum_{k=0}^{2} (-1)^{2-k} D^{2-k} [p_{4-k}(1)x^*(1)], x'(1) - \tilde{x}'(1) \right\rangle \]

\[ - \left\langle D[p_4(1)x^*(1)] - p_3(1)x^*(1), x'''(1) - \tilde{x}'''(1) \right\rangle + \left\langle p_4(1)x^*(1), x'''(1) - \tilde{x}'''(1) \right\rangle = 0. \]

Furthermore, applying the formulas 11 at a point \( t = 1 \) in this inequality we derive

\[ \left\langle p(1)x'''(1) + p'(1)x''(1) - s(1)x^*(1), x(1) - \tilde{x}(1) \right\rangle \]

\[ \left\langle - p(1)x'''(1) + s(1)x^*(1), x'(1) - \tilde{x}'(1) \right\rangle \]

\[ \left\langle p(1)x''(1) - p'(1)x^*(1), x''(1) - \tilde{x}''(1) \right\rangle \]

\[ - \left\langle p_4(1)x^*(1), x'''(1) - \tilde{x}'''(1) \right\rangle \geq 0. \] (12)

On the other hand, from the transversality condition (iii) for all feasible trajectory \( x(\cdot) \) it follows that

\[ \varphi(x(1), x'(1), x''(1), x'''(1)) - \varphi(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)) \]

\[ \geq \left\langle p(1)x'''(1) + p'(1)x''(1) - s(1)x^*(1), x(1) - \tilde{x}(1) \right\rangle \]

\[ + \left\langle - p(1)x'''(1) + s(1)x^*(1), x'(1) - \tilde{x}'(1) \right\rangle \]

\[ + \left\langle p(1)x''(1) - p'(1)x^*(1), x''(1) - \tilde{x}''(1) \right\rangle \]

\[ - \left\langle p_4(1)x^*(1), x'''(1) - \tilde{x}'''(1) \right\rangle \geq 0. \] (13)

Then from the last two inequalities 12 and 13 for all feasible trajectory \( x(\cdot) \) we have immediately \( \varphi(x(1), x'(1), x''(1), x'''(1)) \geq \varphi(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)) \), that is, \( \tilde{x}(\cdot) \) is optimal trajectory.

\[ \square \]

**Corollary 1.** In particular, suppose that in the problem (SLP) \( Q_k \equiv \mathbb{R}^n, k = 0, 1, 2, 3 \). Then \( K_{Q_k}(\tilde{x}^{(k)}(0)) = \mathbb{R}^n \) and \( K_{Q_k}^\star(\tilde{x}^{(k)}(0)) = \{0\} \). Consequently, the transversality condition (ii) at the initial point \( t = 0 \) is transformed into equalities

\[ p(0)x'''(0) + p'(0)x''(0) - s(0)x^*(0) = 0, -p(0)x'''(0) + s(0)x^*(0) = 0, \]

\[ p(0)x''(0) - p'(0)x^*(0) = 0; p(0)x^*(0) = 0. \]

**Corollary 2.** Let in the problem (SLP) \( Q_k = \{\alpha_k\} \), where \( \alpha_k \) are fixed points; \( \alpha_k \in \mathbb{R}^n, k = 0, 1, 2, 3 \). Then obviously, \( K_{Q_k}(\tilde{x}^{(k)}(0)) = \{0\} \) and \( K_{Q_k}^\star(\tilde{x}^{(k)}(0)) = \mathbb{R}^n \). Therefore, the transversality condition (ii) at the initial point \( p = 0 \) for optimization of the Cauchy problem with differential inclusion 2 is unnecessary.
Corollary 3. Let \( p_k : [0, 1] \rightarrow \mathbb{R}^1, k = 1, 2, 3, 4 \) be arbitrary real-valued functions. Then for optimal control problem of differential inclusions with second order linear differential operator \( Lx = p_4(t)x''(t) + p_3(t)x'''(t) + p_2(t)x''(t) + p_1(t)x'(t) \) having the form
\[
\min_{t \in [0, 1], x(t) \in Q} \phi(x(t), x'(t), x''(t), x'''(t)),
\]
\[
\sum_{k=1}^{4} p_k(t)x^{(k)}(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1],
\]
the conditions (i)-(iii) of Theorem 3.1 consist of the following
\[
(i) \quad \sum_{k=1}^{4} (-1)^k D^k [p_k(t)x^*(t)] \in F^*(x^*(t); (\ddot{x}(t), L\ddot{x}(t), t)) \text{ a.e. } t \in [0, 1]
\]
\[
(ii) \quad \sum_{k=0}^{3} (-1)^{4-k} D^{3-k} [p_{4-k}(0)x^*(0)] \in K^*_{\mathcal{Q}_0}(\dot{x}'(0)),
\]
\[
\sum_{k=0}^{2} (-1)^{3-k} D^{2-k} [p_{4-k}(0)x^*(0)] \in K^*_{\mathcal{Q}_1}(\ddot{x}'(0));
\]
\[
D[p_4(0)x^*(0)] - p_3(0)x^*(0) \in K^*_{\mathcal{Q}_2}(\dddot{x}'(0));
\]
\[
-D[p_4(0)x^*(0)] - p_3(0)x^*(0) \in K^*_{\mathcal{Q}_3}(\ddddot{x}'(0)).
\]
\[
(iii) \quad \left( \sum_{k=0}^{3} (-1)^{4-k} D^{3-k} [p_{4-k}(1)x^*(1)], \sum_{k=0}^{2} (-1)^{3-k} D^{2-k} [p_{4-k}(1)x^*(1)] \right)
\]
\[
D[p_4(1)x^*(1)] - p_3(1)x^*(1), -p_4(1)x^*(1)) \in \partial \phi(\ddot{x}(1), \dddot{x}(1), \ddddot{x}(1)).
\]

Proof. The needed results follows in a straightforward manner from the proof of Theorem 3.1 \qed

Corollary 4. For optimization of differential inclusions with second order SLDO, where \( p(t) \equiv 0, s(t) < 0, t \in [0, 1], \phi : \mathbb{R}^n \rightarrow \mathbb{R}^1 \), the adjoint Euler-Lagrange inclusion and transversality conditions at points \( t = 0 \) and \( t = 1 \) are simplified as follows
\[
(i) \quad -(s(t)x''(t))' \in F^*[x^*(t); (\ddot{x}(t), (-s(t)\dddot{x}(t)))', \text{ a.e. } t \in [0, 1]
\]
\[
(ii) \quad x^*(0) \in K^*_{\mathcal{Q}_0}(\ddot{x}(0)) - x^*(0) \in K^*_{\mathcal{Q}_1}(\dddot{x}(0)),
\]
\[
(iii) \quad s(1) (x^*(1), x^*(1)) \in \partial \phi(\ddot{x}(1), \dddot{x}(1)).
\]
\[
(iv) \quad -(s(t)\dddot{x}'(t))' \in F_A(\ddot{x}(t), x^*(t), t), \text{ a.e. } t \in [0, 1].
\]

Proof. In this case in the conditions of Theorem 3.1 \( p(t) \equiv 0 \) and \( L^*x^*(t) = -(s(t)x''(t))' \). Further, substituting \( p(t) \equiv 0 \) in the transversality condition at \( t = 0 \) we immediately have \( -(s(t)x''(0)) \in K^*_{\mathcal{Q}_0}(\ddot{x}(0)) \) and \( s(0)x^*(0) \in K^*_{\mathcal{Q}_1}(\dddot{x}(0)), \) which imply \( -(s(t) > 0) \) the condition (ii) of corollary. On the other hand, since \( p(t) \equiv 0 \) the transversality condition at a point \( t = 1 \) is \( -(s(1)x''(1), s(1)x^*(1)) \in \partial \phi(\ddot{x}(1), \dddot{x}(1)). \) \qed
Remark 4. Without any doubts, optimization problems with fourth order differential inclusions are the simplest and most valuable:

\[
\begin{align*}
\text{minimize} & \quad \varphi(x(1), x'(1), x''(1), x'''(1)), \\
& \quad x^{(4)}(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1], \\
& \quad x^{(k)}(0) \in Q_k, \ k = 0, 1, 2, 3.
\end{align*}
\]

We can show that the sufficient optimality conditions for such problems are the simple consequence of the Corollary 3. Indeed, substituting \( p_k(t) \equiv 0(k = 1, 2, 3) \) and \( p_4(t) \equiv 1 \) in the conditions of Corollary 3 directly, we derive the sufficient optimality conditions for a second order differential inclusions:

(i) \( x^{(4)}(t) \in F^*(x^*(t); (\bar{x}(t), \dot{x}^{(4)}(t)), t) \), a.e. \( t \in [0, 1] \);

(ii) \( x^{(3)}(0) \in K^*_{Q_0} (\ddot{x}(0)); -x^{(3)}(0) \in K^*_{Q_1} (\dddot{x}(0)) \);

(iii) \( \{x^{(3)}(1), -x^{(3)}(1), x^{(2)}(1), -x^{(2)}(1)\} \in \partial \varphi(\dot{x}(1), \ddot{x}'(1), \dddot{x}''(1), \dddot{x}'''(1)) \).

Under the usual closedness of \( F(\cdot, t) (gphF(\cdot, t) \text{ is closed}) \) the conditions of the Theorem 3.1 can be rewritten in the more symmetrical form.

Corollary 5. In addition to assumptions of Theorem 3.1 let \( F(\cdot, t) \) be a closed set-valued mapping. Then the conditions (i), (iv) of Theorem 3.1 can be rewritten in terms of Hamiltonian function in the much more suitable form

\[ L^*x^*(t) \in \partial_{x} H_F(\dot{x}(t), x^*(t)); L\ddot{x}(t) \in \partial_{x} H_Q(\dot{x}(t), x^*(t)), \text{ a.e. } t \in [0, 1]. \]

Proof. Indeed, by Theorem 2.1 [14, p.62] the LAM at a given point and argmaximun set are the subdifferentials of the Hamiltonian function on \( x \) and \( v^* \), that is, \( F^*(v^*; (x, v), t) = \partial_x H_F(x, v^*), F_A(x, v^*, t) = \partial_x H_F(x, v^*) \), respectively. Then the assertions of corollary are equivalent with the conditions (i), (iv) of Theorem 3.1.

\[ \Box \]

Theorem 3.2. Let us consider the nonconvex problem (SLP), that is, \( \varphi : \mathbb{R}^{4n} \rightarrow \mathbb{R} \) be nonconvex function and \( F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) be a nonconvex set-valued mapping. Moreover, let \( K_{Q_k}(\ddot{x}^{(k)}(0)), \dddot{x}^{(k)}(0) \in Q_k, k = 0, 1, 2, 3 \) be the cones of tangent directions to \( Q_k \), \( k = 0, 1, 2, 3 \). Then for the optimality of the trajectory \( \tilde{x}(t), t \in [0, 1] \) it is sufficient that there exists an absolutely continuous function \( x^*(t), t \in [0, 1] \) together with the higher order derivatives until three, satisfying the conditions of Theorem 3.1 in the nonconvex case, where the Euler-Lagrange inclusion with SLDO and transversality conditions at point \( t = 1 \) have the forms

(a) \( L^*x^*(t) \in F^*(x^*(t); (\ddot{x}(t), L\ddot{x}(t)), t), \text{ a.e. } t \in [0, 1], \)

(b) \( \varphi(v_0, v_1, v_2, v_3) - \varphi(\ddot{x}(1), \dddot{x}'(1), \dddot{x}''(1), \dddot{x}'''(1)) \)

\[ \geq \langle p(1)x^{(3)}(1) + p'(1)x^{(3)}(1) - s(1)x^{(2)}(1), v_0 - \ddot{x}(1) \rangle \]

\[ + \langle - p(1)x^{(3)}(1) + s(1)x^{(2)}(1), v_1 - \dot{x}'(1) \rangle \]

\[ + \langle p(1)x^{(3)}(1) - p'(1)x^{(3)}(1), v_2 - \dddot{x}'(1) \rangle \]

\[ + \langle - p(1)x^{(3)}(1), v_3 - \dddot{x}''(1) \rangle, \forall v_k \in \mathbb{R}^n, k = 0, 1, 2, 3. \]
Proof. The basic idea in the proof of Theorem 3.1 was to take into account the inequality
\[ H_F(x(t), x^*(t)) - H_F(\tilde{x}(t), x^*(t)) \leq \sum_{k=1}^{4} (-1)^k \frac{d^k}{dt^k}(p_k(t)x^*(t)), x(t) - \tilde{x}(t). \]

Therefore, the furthest proof of theorem is similar to the one for Theorem 3.1. Thus, by the latter inequality the inequality 12 is justified. Moreover, observe that for a nonconvex \( \varphi \) by the conditions (b) for all feasible trajectories \( x(\cdot) \) denoting \( v_k = x^{(k)}(1), k = 0, 1, 2, 3 \) the inequality
\[ \varphi(x(1), x'(1), x''(1), x'''(1)) - \varphi(\tilde{x}(1), \tilde{x}'(1), \tilde{x}''(1), \tilde{x}'''(1)) \]
\[ \geq \left\langle \int_{0}^{1} x''''(1) + p'(1)x''''(1) - s(1)x''(1), x(1) - \tilde{x}(1) \right\rangle \]
\[ + \left\langle -p(1)x''''(1) + s(1)x''(1), x'(1) - \tilde{x}'(1) \right\rangle \]
\[ + \left\langle p(1)x''(1) - p'(1)x''(1), x'''(1) - \tilde{x}''(1) \right\rangle + \left\langle -p(1)x''(1), x'''(1) - \tilde{x}''(1) \right\rangle \]
is satisfied. 

4. Some applications to optimal control problems with PLDOs. In this section we give two applications of our results. The first one is the particular Mayer problem for differential inclusions involving fourth-order polynomial linear differential operator with constant coefficients and the second one concerns optimization of “linear” differential inclusions with SLDOs. Thus, suppose now we have the following optimization problem (for simplicity we consider a convex problem):

\[ \text{minimize } \varphi_0(x(1)), \]
\[ (PDP_C) \quad Lx(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1], \quad Lx = D^4x + p_1D^3x + p_2D^2x + p_3Dx \]
\[ x(0) = \alpha_0, \quad x'(0) = \alpha_1, \quad x''(0) = \alpha_2, \quad x'''(0) = \alpha_3 \]

where \( L \) is the fourth-order polynomial operator; \( p_k, k = 1, 2, 3 \) are some real constants, \( F(\cdot, t): \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is a convex set-valued mapping, \( \varphi_0: \mathbb{R}^n \rightarrow \mathbb{R}^1 \) is a continuous convex function, \( \alpha_j \in \mathbb{R}^n, j = 0, 1, 2, 3 \) are fixed vectors. The problem \( (PDP_C) \) is to find a trajectory \( \tilde{x}(t) \) such that the cost functional \( \varphi_0(x(t)) \) is minimized (it should be noted that along with many others important properties the multiplication operation is commutative for SLOs with constant coefficients).

Recall that the forth-order an adjoint differential operator with constant coefficients is defined as follows:
\[ L^*x^* = D^4x^* - p_1D^3x^* + p_2D^2x^* - p_3Dx^*. \]

Corollary 6. Let \( \varphi_0: \mathbb{R}^n \rightarrow \mathbb{R}^1 \) be a continuous and convex function. Moreover, let \( F(\cdot, t) \) be convex set-valued mapping. Then for the optimality of the trajectory \( \tilde{x}(t) \) in the problem \( (PDP_C) \) it is sufficient that there exists an absolutely continuous function \( x^*(t), t \in [0, 1] \) together with the higher order derivatives until three, satisfying a.e. the Euler-Lagrange differential inclusion
\[ L^*x^*(t) = F^*(x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t); \]
\[ \left\{ L\tilde{x}(t), x^*(t) \right\} = H_F(\tilde{x}(t), x^*(t)), \text{ a.e. } t \in [0, 1] \]
and transversality condition at the endpoint $t = 1$

$$\frac{d^3 x^*(1)}{dt^3} \in \partial \varphi_0(\tilde{x}(1)), \quad \frac{d^k x^*(1)}{dt^k} = 0, \quad k = 0, 1, 2.$$

**Proof.** We conclude this proof by returning to the condition (i), (ii), (iii) of Corollary 3. Clearly, a problem (PDPC) can be reduced to the problem of form (SLP), where

$$\varphi(x(1), x'(1), x''(1), x'''(1)) = \varphi_0(x(1)).$$

It follows that $\partial \varphi(x(1), x'(1), x''(1), x'''(1)) = \partial_x \varphi_0(x(1)) \times \{0\} \times \{0\} \times \{0\}$. On the other hand since $p_4(t) \equiv 1, p_k(t) \equiv p_{4-k}, k = 1, 2, 3$, are nonzero constants, by transversality condition (iii) of Corollary 3 we have

$$\sum_{k=0}^{2} (-1)^{3-k} D^{3-k} \left[ p_{4-k}(1) x^*(1) \right] \in \partial \varphi_0(\tilde{x}(1)),$$

$$\sum_{k=0}^{2} (-1)^{3-k} D^{2-k} \left[ p_{4-k}(1) x^*(1) \right] = 0; \quad D \left[ x^*(1) \right] - p_1 x^*(1) = 0; \quad -x^*(1) = 0.$$

Then by sequentially substitution in the last relations we derive that $D^k [x^*(1)] = 0 (k = 1, 2)$. Consequently, since

$$\sum_{k=0}^{2} (-1)^{4-k} D^{3-k} \left[ p_{4-k}(1) x^*(1) \right] = D^3 \left[ x^*(1) \right] - D^2 \left[ p_1 x^*(1) \right]$$

$$+ D \left[ p_2 x^*(1) \right] - \left( p_1 x^*(1) \right) = D^3 \left[ x^*(1) \right]$$

we have the desired result. \qed

Suppose now that we have so-called linear Mayer problem with SLDOs:

$$\text{minimize } \varphi_0(x(1)), \quad (14)$$

$$L x(t) \in F(x(t), t), \quad \text{a.e. } t \in [0, 1],$$

$$x^{(k)}(0) = \alpha_k, \quad k = 0, 1, 2, 3, \quad F(x, t) = A(t)x + B(t)U \quad (15)$$

where $\varphi_0$ is continuously differentiable convex function, $A(t)$ and $B(t)$ are $n \times n$ and $n \times r$ continuous matrices, respectively, $U$ is a convex closed subset of $\mathbb{R}^r; \alpha_k, k = 0, 1, 2, 3$ are fixed vectors, The problem is to find a controlling parameter $\tilde{w}(t) \in U$ such that the trajectory $\tilde{x}(t)$ corresponding to it minimizes $\varphi_0(x(1))$. In fact, this is optimization of the Cauchy problem for “linear” differential inclusions with SLDO. The controlling parameter $w(\cdot)$ is called admissible if it only takes values in the given control set $U$ which is nonempty, convex, closed set.

**Theorem 4.1.** The trajectory $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{w}(t)$ is a solution to the problem 14, 15 if there exists an absolutely continuous function $x^*(t)$ together with the higher order derivatives until three, satisfying the following Euler-Lagrange type differential equation with SLDO, the transversality condition at a point $t = 1$ and Weierstrass-Pontryagin maximum principle:

$$L^* x^*(t) = A^*(t)x^*(t), \quad \text{a.e. } t \in [0, 1],$$

$$\frac{d^3 x^*(1)}{dt^3} = \varphi'_{0}(\tilde{x}(t)), \quad \frac{d^k x^*(1)}{dt^k} = 0, \quad k = 0, 1, 2,$$

$$\left\langle B(t)\tilde{w}(t), x^*(t) \right\rangle = \sup_{w \in U} \left\langle B(t)w(t), x^*(t) \right\rangle.$$
Proof. Here we are proceeding on the basis of Theorem 3.1. Clearly, the Hamiltonian is
\[
H_F(x, v^*) = \langle A(t)x, v^* \rangle + \sup_{w \in U} \langle B(t)w, v^* \rangle.
\]

Hence,
\[
F^*(v^*; (x, \hat{v}), t) = \partial_x H_F(x, v^*) = \{A^*(t)v^*\}, \hat{v} \in F_A(x, v^*, t), \hat{v} = A(t)x + B(t)\bar{w},
\]
where the fact that \(\hat{v}\) belongs to \(F_A(x, v^*, t)\) means that \(\langle B(t)\bar{w}, v^* \rangle = \sup_{w \in U} \langle B(t)w, v^* \rangle \neq \emptyset\).

Then by Theorem 3.1 we can write
\[
L^*x^*(t) = A^*(t)x^*(t), \ L\hat{x}(t) \in F_A(\hat{x}(t), x^*(t), t), \ \langle B(t)\bar{w}(t), x^*(t) \rangle = \sup_{w \in U} \langle B(t)w, x^*(t) \rangle.
\]

To finish the proof of theorem it is enough to note that by Corollary 2 the transversality conditions (ii) of Theorem 3.1 is superfluous and by Corollary 6
\[
D^3x^*(1) = \varphi_0(\hat{x}(1)), D^kx^*(1) = 0, k = 0, 1, 2.
\]

\[\left(-x^*(1), x^*(1)\right) \in \partial \varphi(\hat{x}(1), \hat{x}'(1))\].

Remark 5. Suppose that in the definition of linear Mayer problem with SLDO \((Lx = D^2(px'') - D(sx'))\), where \(p(t) \equiv 0, s(t) \equiv -1, \varphi : \mathbb{R}^{2n} \to \mathbb{R}^1\) and \(U \subset \mathbb{R}^r\) is a nonempty closed convex set. Then \(Lx = D^2(px'') - D(sx') \equiv (px'')'' - (sx')' = x''\) and we have a Mayer problem with second order differential inclusion \(x'' \in A(t)x + B(t)U\) in the finite time interval \(t \in [0, 1]\). By Corollary 2 the transversality condition (ii) at the initial point \(t = 0\) for optimization of the Cauchy problem with differential inclusion \(x'' \in A(t)x + B(t)U\) is unnecessary. Obviously, \(L^*x^* = pD^4x^* + 2pD^3x^* + \left[p'' - s\right]D^2x^* - sx'' = x''\) and for such problems along with the Pontryagins maximum principle an adjoint Euler-Lagrange type differential equation and transversality condition at a point \(t = 1\) consist of the following \(\frac{d^2x^*(t)}{dt^2} = A^*(t)x^*(t), (x^*(1), -x^*(1)) \in \partial \varphi(\hat{x}(1), \hat{x}'(1))\), correspondingly.

Example 1. Let us consider the following Mayer problem with second order PLDO
\[
Lx = D^2x = x'';
\]
the infimum \(\varphi(x(1), x'(1))\) subject to \(\frac{d^2x}{dt^2} = u, u \in [-1, 1], x(0) \in Q_0, x'(0) \in Q_1\)
\[
\text{(16)}
\]
where \(\varphi(x(1), x'(1)) = x^2(1) - x'(1)\) and \(Q_0 = \{0\}, Q_1 = \{1\}\).

It is well known that, \(x = u\) is a particular case of Newtoms second law of motion described by equation \(F(t) = ma(t)\), where \(F(t)\) acts on the particle and \(a(t)\) is acceleration (in our case \(F(t) = u(t), x''(t) = a(t), m = 1\)).

It is required to find the optimal control \(\hat{u}(\cdot)\) such that the corresponding arc \(\hat{x}(\cdot)\) minimizes \(\varphi(x(1), x'(1))\). In the present case (see \((PV)\))
\[
F(x, t) = \{u : |u| \leq 1\} \text{ for all } t \in [0, 1], m = 2.
\]

So the Mayer optimal control problem \(\text{(16)}\) with second order PLDO can be written as follows:
\[
\text{minimize } \varphi(x(1), x'(1)) \text{ subject to } Lx \in F(x), t \in [0, 1], x(0) = 0, x'(0) = 1. \text{ (17)}
\]
Obviously, in the adjoint Euler-Lagrange inclusion (i) \(-D(p_1(t)x^*(t)) + D^2(p_2(t)x^*(t)) \in F^*(x^*(t); (\ddot{x}(t), L\dot{\ddot{x}}(t)))\) of Corollary 3, \(p_1(t) = 0\) and \(p_2(t) = 1\), that is, we have the inclusion \(x'' \in F^*(x^*(t); (\ddot{x}(t), \dddot{x}(t)))\). Then by Theorem 2.1 of LAM in the convex case [11, p.62] \(F^*(v^*; (x, v)) = \partial_x H_F(x, v^*)\).

Clearly
\[
H_F(x, v^*) = \max_u \{uv^* : |u| \leq 1\} = |v^*|
\]  
and so
\[
F^*(v^*; (x, v)) = \partial_x H_F(x, v^*) \equiv 0, \quad v \in F_A(x, v^*) = \{-1, +1\}.
\]  

Then taking into account \(Lx^* = \frac{d^2x^*}{dt^2}\), as a result of Theorem 3.1 (see, Corollary 3) we deduce from (19) that
\[
\frac{d^2x^*}{dt^2} = 0, \quad t \in [0, 1],
\]
for which the solution is a linear function of the form \(x^*(t) = C_1t + C_2\), where \(C_1, C_2\) are arbitrary constants. Then (18) implies that \(\ddot{u}(t)x^*(t) = |x^*(t)|\) or
\[
\ddot{u}(t) = \begin{cases} 
sgnx^*(t), & \text{if } x^*(t) \neq 0, \\
\forall u_0 \in [-1, 1], & \text{if } x^*(t) = 0,
\end{cases}
\]

Furthermore, since \(x^*(t), t \in [0, 1]\) as a linear function, does not change sign more than once in the time interval \([0, 1]\), it follows from (20) that every optimal control \(\ddot{u}(t), t \in [0, 1]\) is a piecewise-constant function, having the values \(\pm1\) and having not more than two interval of constancy. Let us return to the transversality conditions of Corollary 3. By Corollary 2 the transversality condition (ii) at the initial point \(t = 0\) for optimization of the Cauchy problem with differential inclusion (16) is superfluous.

On the other hand, by the transversality conditions (iii) of Corollary 3 at a point \(t = 1\) we have
\[
\begin{vmatrix} D(p_2(19x^*(1)) - p_1(1)x^*(1), -p_2(1)x^*(1)) \end{vmatrix} \in \partial \varphi(\hat{x}(1), \hat{\dot{x}}(1)),
\]
where \(p_2(1) = 1\) and \(p_1(1) = 0\) or equivalently, \((x^*(1), -x^*(1)) \in \partial \varphi(\hat{x}(1), \hat{\dot{x}}(1))\). It can be easily checked that \(\varphi(x, y) = x^2 - y\) is a convex function; indeed it is sufficient to show that the Hessian matrix
\[
\varphi''(x, y) = \begin{pmatrix} \varphi''_{xx}(x, y) & \varphi''_{xy}(x, y) \\
\varphi''_{yx}(x, y) & \varphi''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\
0 & 0 \end{pmatrix}
\]
is a positive semidefinite matrix, that is, all eigenvalues of \(\varphi''(x, y)\) are nonnegative. Indeed, denoting this matrix by \(A\) we see that the characteristic equation \(\det(A - \lambda E) = \lambda^2 - 2\lambda = 0\) (\(E\) is a \(2 \times 2\) unique square matrix) have two real nonnegative eigenvalues \(\lambda_1 = 0, \lambda_2 = 2\). Consequently, \(\varphi(x, y)\) is convex and \(\partial \varphi(\hat{x}, \hat{\dot{x}}) = (2x, -1)\). It follows that \(\partial \varphi(\hat{x}(1), \hat{\dot{x}}(1)) = (2\hat{x}(1), -1)\). Comparing this relation with \((x^*(1), -x^*(1)) \in \partial \varphi(\hat{x}(1), \hat{\dot{x}}(1))\) we immediately have \(x^*(1) = 2\hat{x}(1), x^*(1) = 1\). Then from the general solution of the adjoint Euler-Lagrange inclusion (equation) \(x^*(t) = C_1t + C_2\) we have \(1 = x^*(1) = C_1 + C_2, 2\hat{x}(1) = x^*(1) = C_1 (C_1, C_2\) are arbitrary constants) and so \(x^*(t) = 2(t - 1)\hat{x}(1) + 1\), whence \(x^*(t) \neq 0, \quad t \neq 1 - \frac{1}{2\hat{x}(1)}\). Therefore, (20) implies that for optimal control \(\ddot{u}(\cdot)\) there are four possibilities:
\[
\ddot{u}(t) = 1, \quad x^*(t) > 0, \quad t \in [0, 1].
\]
\[
\ddot{u}(t) = -1, \quad x^*(t) < 0, \quad t \in [0, 1].
\]
\[ \ddot{u}(t) = \begin{cases} 1, & \text{if } 0 \leq t < \tau, \\ -1, & \text{if } \tau < t \leq 1, \end{cases} \]

\[ \dot{u}(t) = \begin{cases} -1, & \text{if } 0 \leq t < \tau, \\ 1, & \text{if } \tau < t \leq 1, \end{cases} \]

(observe that \( \tau \) is a point of discontinuity of \( u(\cdot) \) and the values of the control functions \( u(\cdot) \) at a point of discontinuity \( \tau \) are unessential). Moreover, the value of the linear function \( x^*(t) = 2(t - 1)\ddot{x}(1) + 1 \) is positive at a point \( t = 1 \), i.e. \( x^*(1) = 1 \). It follows that either \( x^*(t) > 0 \) for all \( t \in [0,1] \) or \( x^*(t) < 0, 0 \leq t < \tau; \) \( x^*(t) > 0, \tau < t \leq 1 \) for some \( \tau \) in the interval \( 0 < \tau < 1 \). Therefore, since \( \ddot{u}(t) \) is a piecewise-constant function, having not more than two interval of constancy we have either the case 21 or the case 24. In general, using 21-24, by solving the Cauchy problem

\[ \frac{d^2x(t)}{dt^2} = \ddot{u}(t), \quad x(0) = 0, \quad x'(0) = 1 \]

we have a unique solution of the initial value problem 25. Thus for the time interval on which \( u = 1 \) we have \( x'(t) = t + c_1; x(t) = \frac{t^2}{2} + c_1 t + c_2 \) \((c_1, c_2 \text{ are arbitrary constants})\). Taking into account the initial values in 25 we see that

\[ x'(t) = t + 1; \quad x(t) = \frac{t^2}{2} + t. \]

Similarly, we have for the time interval on which \( u = -1 \)

\[ x'(t) = 1 - t; \quad x(t) = -\frac{t^2}{2} + t. \]

Let us denote the parabolas 26 and 27 by \( x_1(t), x_2(t) \), correspondingly. Obviously, in the case 21 \( \ddot{u}(t) = 1, t \in [0,1] \) and from 26 we have \( \ddot{x}_1(1) = \frac{3}{2} + 1 = \frac{5}{2} \) and \( \ddot{x}_2(1) = 2 \). As a consequence, the value of our Mayer problem 16 or 17 would be \( \varphi(\ddot{x}_1(1), \ddot{x}_2(1)) = \ddot{x}_2(1) - \ddot{x}_1(1) = \frac{\varphi(\ddot{x}_1(1), \ddot{x}_2(1))}{2} - 2 = \frac{3}{2} \), if \( \ddot{u}(t) = 1, t \in [0,1] \). On the other hand, in the case 24 the control function \( \ddot{u}(t) \) first is equal to \( -1 \), then equal to \( +1 \) and the trajectory \( \ddot{x}(t) \) consists of two pieces of parabolas \( \ddot{x}_1(t), \ddot{x}_2(t) \), \( (\ddot{x}(t), \ddot{x}(t), \) is continuous and piecewise smooth on the interval \( 0 \leq t \leq 1 \). In this case the solution of the equation 25 on the interval \( 0 \leq t \leq \tau \) is given by 27; at a point \( \tau \) are satisfied \( x_2(\tau) = -\frac{\tau^2}{2} + \tau, x_2'(\tau) = 1 - \tau \). Consider now the initial value problem

\[ \frac{d^2x_2(t)}{dt^2} = 1, \quad x_2(\tau) = -\frac{\tau^2}{2} + \tau, \quad x_2'(\tau) = 1 - \tau, \quad t \in [\tau, 1]. \]

It is clear that \( -\frac{\tau^2}{2} + \tau = x_2(\tau) = \frac{\tau^2}{2} + c_1 \tau + c_2 \) and \( 1 - \tau = x_2'(\tau) = \tau + c_1 \). Then the solution of initial value problem 28 is \( \ddot{x}_2(t) = \frac{\tau^2}{2} + (1 - 2\tau)t + \tau^2 \), whence \( \ddot{x}_2(1) = \ddot{x}(1) = \tau^2 - 2\tau + \frac{3}{2} \). Substituting here the value \( \tau = 1 - \frac{1}{2\ddot{x}(1)} \) we have a cubic equation \( 4\ddot{x}^3(1) - 2\ddot{x}(1) = 0 \) with respect to \( \ddot{x}(1) \). By the familiar Cardano formula the real root of this cubic equation approximately is \( \ddot{x}(1) \approx 0.84781 \). Moreover, \( \dddot{x}_2(t) = t - 2\tau + 1 \) and \( \dddot{x}(1) = \dddot{x}_2(1) = 2 - 2(1 - \frac{1}{2\dddot{x}(1)}) = 0.41025 \) and \( \varphi(\dddot{x}(1), \dddot{x}(1)) = \dddot{x}(1) - \dddot{x}(1) \approx 0.30853 \), where \( \dddot{u}(t) \) is defined as in 24. Thus, the value of our Mayer problem is the minimum of numbers \( 1/4 = 0.25 \) and \( 0.30853 \), that is 0.25.
5. **Conclusion.** In this paper is presented discretization method for solving a new class of problems of optimal control theory, namely Mayer problem with fourth-order Sturm-Liouville type differential operator (SLDOs) which are often used to describe various processes in science and engineering. This approach plays a much more important role in derivation of so-called Euler-Lagrange differential inclusion with fourth-order SLDO and the transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \). But in this paper to avoid a long calculations connected with the discretization method, establishment of optimality and transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \) for the discrete-approximate problem are omitted. Thus, sufficient optimality conditions for such problems are deduced by passing to the limit as \( \delta \to 0 \). There has been a significant development in the study of optimization for differential and difference equations and inclusions in recent years [14, 20, 22]. Finally, it is concluded that the proposed method is reliable for solving the various optimization problems with fourth-order discrete and differential inclusions. Theoretical analysis and practical results show that this method is simple and easy to implement and is efficient for computing optimal solution of the fourth order differential inclusions. At last, an example is given for illustrating the obtained results.

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