A FETI-DP type domain decomposition algorithm for three-dimensional incompressible Stokes equations

Xuemin Tu* Jing Li†

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Abstract

The FETI-DP algorithms, proposed by the authors in [SIAM J. Numer. Anal., 51 (2013), pp. 1235–1253] and [Internat. J. Numer. Methods Engrg., 94 (2013), pp. 128–149] for solving incompressible Stokes equations, are extended to three-dimensional problems. A new analysis of the condition number bound for using the Dirichlet preconditioner is given. An advantage of this new analysis is that the numerous coarse level velocity components, required in the previous analysis to enforce the divergence free subdomain boundary velocity conditions, are no longer needed. This greatly reduces the size of the coarse level problem in the algorithm, especially for three-dimensional problems. The coarse level velocity space can be chosen as simple as for solving scalar elliptic problems corresponding to each velocity component. Both Dirichlet and lumped preconditioners are analyzed using a same framework in this new analysis. Their condition number bounds are proved to be independent of the number of subdomains for fixed subdomain problem size. Numerical experiments in both two and three dimensions demonstrate the convergence rate of the algorithms.

Keywords domain decomposition, incompressible Stokes, FETI-DP, BDDC, divergence free
AMS 65F10, 65N30, 65N55

1 Introduction

Mixed finite elements are often used to solve incompressible Stokes and Navier-Stokes equations. Continuous pressures have been used in many mixed finite ele-

*Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045-7594, xtu@math.ku.edu, http://www.math.ku.edu/~xtu/. This author’s work was supported in part by National Science Foundation contract DMS-1115759.
†Department of Mathematical Sciences, Kent State University, Kent, OH 44242, li@math.kent.edu, http://www.math.kent.edu/~li/.
ments, e.g., the well-known Taylor-Hood finite elements \[23\]. However, most domain decomposition methods require that the pressure be discontinuous, when they are used to solve the indefinite linear systems arising from such mixed finite element discretizations; see, e.g., \[4, 5, 6, 8, 9, 14, 15, 17, 19, 20, 24, 25\]. Several domain decomposition algorithms allow to use continuous pressures, e.g., Klawonn and Pavarino \[14\], Goldfeld \[7\], Šístek \emph{et. al.} \[21\], Benhassine and Bendali \[1\], and Kim and Lee \[13\]. But the convergence rate analysis of those approaches cannot be applied to the continuous pressure case due to the indefiniteness of the linear systems; such difficulty can often be removed conveniently when discontinuous pressures are used in the discretization.

Recently, the authors \[16, 26\] proposed and analyzed a FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting method) type domain decomposition algorithm for solving the incompressible Stokes equation in two dimensions. Both discontinuous and continuous pressures can be used in the mixed finite element discretization. In both cases, the indefinite system of linear equations can be reduced to a symmetric positive semi-definite system. Therefore, the preconditioned conjugate gradient method can be applied and a scalable convergence rate of the algorithm has been proved.

The lumped and Dirichlet preconditioners have been studied in \[16\] and \[26\], respectively. For the lumped preconditioner it was shown both experimentally and analytically in \[16\], that the coarse level space can be chosen the same as for solving scalar elliptic problems corresponding to each velocity component to achieve a scalable convergence rate. Similar observations for the lumped preconditioner have also been pointed out earlier by Kim and Lee \[11, 12, 10\], with Park, even though their studies are only for using discontinuous pressures.

For the Dirichlet preconditioner studied in \[26\], a distinctive feature is the application of subdomain discrete harmonic extensions in the preconditioner. In other existing FETI-DP and BDDC (Balancing Domain Decomposition by Constraints) algorithms, cf. \[15, 17\], subdomain discrete Stokes extensions have been used and the coarse level velocity space has to contain sufficient components to enforce divergence free subdomain boundary velocity conditions. Those complicated and numerous coarse level velocity components, especially for three-dimensional problems as discussed in \[17\], are not needed for the implementation of the Dirichlet preconditioner in \[26\]. But they are still required in \[26\] just for the analysis, where subdomain Stokes extensions were used, to obtain a scalable condition number bound.

In this paper, we provide a new analysis for the algorithms in \[16, 26\], which can analyze both lumped and Dirichlet preconditioners in a same framework. It does not use any subdomain Stokes extensions and those additional coarse level velocity components to enforce divergence free subdomain boundary velocity conditions are no longer needed. For both lumped and Dirichlet preconditioners, the coarse level space can be chosen as simple as for solving scalar elliptic problems corresponding to each velocity component. This greatly simplifies the requirements on the coarse level space for the case of Dirichlet preconditioner, especially in three dimensions. This paper is presented in the context of solving three-dimensional problems; the same approach can be applied to
two-dimensional problems as well.

The remainder of this paper is organized as follows. The finite element discretization of the incompressible Stokes equation is introduced in Section 2. A domain decomposition approach is described in Section 3, and the system is reduced to a symmetric positive semi-definite problem in Section 4. A few preliminary results used in the condition number bound estimates are given in Section 5. The lumped and Dirichlet preconditioners are introduced in Section 6, and the condition number bounds of the preconditioned systems are established in Section 7. At the end, numerical results of solving the incompressible Stokes equation in both two and three dimensions are given in Section 8 to demonstrate the convergence rate of the algorithm.

2 Finite element discretization

We consider solving the following incompressible Stokes problem on a bounded, three-dimensional polyhedral domain Ω with a Dirichlet boundary condition,

\begin{align}
\begin{cases}
-\Delta u^* + \nabla p^* = f, & \text{in } \Omega, \\
-\nabla \cdot u^* = 0, & \text{in } \Omega, \\
u^* = u^*_{\partial\Omega}, & \text{on } \partial\Omega,
\end{cases}
\end{align}

where the boundary velocity \( u^*_{\partial\Omega} \) satisfies the compatibility condition \( \int_{\partial\Omega} u^*_{\partial\Omega} \cdot n = 0 \). For simplicity, we assume that \( u^*_{\partial\Omega} = 0 \) without losing any generality.

The weak solution of (1) is given by: find \( u^* \in (H^1_0(\Omega))^3 = \{ v \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \partial\Omega \} \) and \( p^* \in L^2(\Omega) \), such that

\begin{align}
\begin{cases}
a(u^*, v) + b(v, p^*) = (f, v), & \forall v \in (H^1_0(\Omega))^3, \\
b(u^*, q) = 0, & \forall q \in L^2(\Omega),
\end{cases}
\end{align}

where \( a(u^*, v) = \int_\Omega \nabla u^* \cdot \nabla v, \ b(u^*, q) = -\int_\Omega (\nabla \cdot u^*)q, \ (f, v) = \int_\Omega f \cdot v. \) We note that the solution of (2) is not unique, with the pressure \( p^* \) different up to an additive constant.

A mixed finite element is used to solve (2). In this paper we apply a mixed finite element with continuous pressures, e.g., the Taylor-Hood type mixed finite elements. The same algorithm and analysis can be applied to mixed finite elements with discontinuous pressures as well; see [26]. Denote the velocity finite element space by \( W \subset (H^1_0(\Omega))^3 \), and the pressure finite element space by \( Q \subset L^2(\Omega) \). The finite element solution \((u, p) \in W \oplus Q\) of (2) satisfies

\begin{align}
\begin{bmatrix}
A & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
= \begin{bmatrix}
f \\
0
\end{bmatrix},
\end{align}

where \( A, B \), and \( f \) represent respectively the restrictions of \( a(\cdot, \cdot) \), \( b(\cdot, \cdot) \) and \( (f, \cdot) \) to the finite-dimensional spaces \( W \) and \( Q \). We use the same notation in this paper to represent both a finite element function and the vector of its nodal values.
The coefficient matrix in (3) is rank deficient even though $A$ is symmetric positive definite. $\text{Ker}(B^T)$, the kernel of $B^T$, contains all constant pressures in $Q$. $\text{Im}(B)$, the range of $B$, is orthogonal to $\text{Ker}(B^T)$ and consists of all vectors in $Q$ with zero average. For a general right-hand side vector $(f, g)$ in (3), the existence of solution requires that $g \in \text{Im}(B)$, i.e., $g$ has zero average; for the right-hand side given in (3), $g = 0$ and the solution always exists. When the pressure is considered in the quotient space $Q/\text{Ker}(B^T)$, the solution is unique.

In this paper, when $q \in Q/\text{Ker}(B^T)$, we always assume that $q$ has zero average.

Let $h$ represent the characteristic diameter of the mixed elements. We assume that the mixed finite element space $W \times Q$, is inf-sup stable in the sense that there exists a positive constant $\beta$, independent of $h$, such that

$$\sup_{w \in W} \frac{\langle q, Bw \rangle^2}{\langle w, Aw \rangle} \geq \beta^2 \langle q, Zq \rangle, \quad \forall q \in Q/\text{Ker}(B^T),$$

(4)

cf. [3, Chapter III, §7]. Here, as always used in this paper, $\langle \cdot, \cdot \rangle$ represents the inner (or semi-inner) product of two vectors. The matrix $Z$ represents the mass matrix defined on the pressure finite element space $Q$, i.e., for any $q \in Q$, $\|q\|_Z^2 = \langle q, Zq \rangle$. It is easy to see, cf. [27, Lemma B.31], that $Z$ is spectrally equivalent to $h^3I$ for three-dimensional problems, i.e., there exist positive constants $c$ and $C$, such that

$$ch^3I \leq Z \leq Ch^3I,$$

(5)

where $I$ represents the identity matrix. Here, as in other places of this paper, $c$ and $C$ represent generic positive constants which are independent of $h$ and the subdomain diameter $H$ (described in the following section).

3 A non-overlapping domain decomposition approach

The domain $\Omega$ is decomposed into $N$ non-overlapping polyhedral subdomains $\Omega_i$, $i = 1, 2, \ldots, N$. Each subdomain is the union of a bounded number of elements, with the diameter of the subdomain in the order of $H$. We use $\Gamma$ to represent the subdomain interface which contains all the subdomain boundary nodes shared by neighboring subdomains; we assume that the subdomain meshes have matching nodes across $\Gamma$. $\Gamma$ is composed of subdomain faces, which are regarded as open subsets of $\Gamma$ shared by two subdomains, subdomain edges, which are regarded as open subsets of $\Gamma$ shared by more than two subdomains, and of the subdomain vertices, which are end points of edges.

The velocity and pressure finite element spaces $W$ and $Q$ are decomposed into

$$W = W_I \bigoplus W_\Gamma, \quad Q = Q_I \bigoplus Q_\Gamma,$$

where $W_I$ and $Q_I$ are direct sums of independent subdomain interior velocity
spaces $W_i^{(i)}$, and interior pressure spaces $Q_i^{(i)}$, respectively, i.e.,

$$W_I = \bigoplus_{i=1}^{N} W_i^{(i)}, \quad Q_I = \bigoplus_{i=1}^{N} Q_i^{(i)}.$$  

$W_\Gamma$ and $Q_\Gamma$ are subdomain interface velocity and pressure spaces, respectively. All functions in $W_\Gamma$ and $Q_\Gamma$ are continuous across $\Gamma$; their degrees of freedom are shared by neighboring subdomains.

To formulate the domain decomposition algorithm, we introduce a partially sub-assembled subdomain interface velocity space $\tilde{W}_\Gamma$,

$$\tilde{W}_\Gamma = W_\Delta \bigoplus W_{\Pi} = \left( \bigoplus_{i=1}^{N} W_i^{(i)}_\Delta \right) \bigoplus W_{\Pi}.$$  

$W_{\Pi}$ is the continuous, coarse level, primal velocity space which is typically spanned by subdomain vertex nodal basis functions, and/or by interface edge/face-cutoff functions with constant nodal values on each edge/face, or with values of positive weights on these edges/faces. The primal, coarse level velocity degrees of freedom are shared by neighboring subdomains. The complimentary space $W_\Delta$ is the direct sum of independent subdomain dual interface velocity spaces $W_i^{(i)}_\Delta$, which correspond to the remaining subdomain interface velocity degrees of freedom and are spanned by basis functions which vanish at the primal degrees of freedom. Thus, an element in $\tilde{W}_\Gamma$ typically has a continuous primal velocity component and a discontinuous dual velocity component.

It is well known that, for domain decomposition algorithms, the coarse space $W_{\Pi}$ should be sufficiently rich to achieve a scalable convergence rate. On the other hand, a large coarse level problem will certainly degrade the parallel performance of the algorithm. Therefore it is important to keep the size of the coarse level problem as small as possible. When the Dirichlet preconditioner was used in the FETI-DP algorithm for solving incompressible Stokes equations [15] and similarly in the BDDC algorithm [17], subdomain discrete Stokes extensions were used and $W_{\Pi}$ has to contain sufficient subdomain interface components such that functions in $W_\Delta$ have zero flux across the subdomain boundaries. Such requirements lead to a large coarse level velocity space, especially for three-dimensional problems, cf. [17].

In [26], a FETI-DP type algorithm is proposed for solving two-dimensional incompressible Stokes problems. A distinctive feature of the Dirichlet preconditioner used in that algorithm is the application of subdomain discrete harmonic extensions, instead of subdomain discrete Stokes extensions. As a result, the divergence free subdomain boundary velocity conditions are not needed in that algorithm. However, the analysis, given in [26] for the Dirichlet preconditioner, still uses subdomain Stokes extensions and requires the same type coarse level velocity space as discussed in [17] to establish a scalable condition number bound estimate. In this paper, a new analysis is offered and it is sufficient for $W_{\Pi}$ to be spanned just by the subdomain vertex nodal basis functions and subdomain
edge-cutoff functions corresponding to each velocity component, as for solving three-dimensional scalar elliptic problems, cf. [27, Section 6.4.2].

The functions $w_{\Delta}$ in $W_{\Delta}$ are in general not continuous across $\Gamma$. To enforce their continuity, we define a Boolean matrix $B_{\Delta}$ of the form

$$B_{\Delta} = \begin{bmatrix} B_{\Delta}^{(1)} & B_{\Delta}^{(2)} & \cdots & B_{\Delta}^{(N)} \end{bmatrix},$$

constructed from $\{0,1,-1\}$. On each row of $B_{\Delta}$, there are only two nonzero entries, 1 and $-1$, corresponding to one velocity degree of freedom shared by two neighboring subdomains, such that for any $w_{\Delta}$ in $W_{\Delta}$, each row of $B_{\Delta}w_{\Delta} = 0$ implies that these two degrees of freedom from the two neighboring subdomains be the same. We note that, in three dimensions, a velocity degree of freedom on a subdomain edge is shared by more than two subdomains, e.g., by four subdomains. In this case, a minimum of three continuity constraints can be applied to enforce the continuity of this velocity degree of freedom among the four subdomains, which corresponds to the use of non-redundant Lagrange multipliers. In this paper, the fully redundant Lagrange multipliers are used, which means, e.g., for a subdomain edge velocity degree of freedom shared by four subdomains, six Lagrange multipliers are used to enforce all the six possible continuity constraints among them, cf. [27, Section 6.3.1].

We denote the range of $B_{\Delta}$ applied on $W_{\Delta}$ by $\Lambda$, the vector space of the Lagrange multipliers. Solving the original fully assembled linear system (3) is then equivalent to: find $(u_I, p_I, u_\Delta, u_\Pi, p_\Gamma, \lambda) \in W_I \oplus Q_I \oplus W_{\Delta} \oplus W_{\Pi} \oplus Q_{\Gamma} \oplus \Lambda$, such that

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} & B_{I\Pi}^T & 0 \\
B_{II} & 0 & B_{I\Delta} & B_{I\Pi} & 0 & 0 \\
A_{I\Delta} & B_{I\Delta}^T & A_{\Delta \Delta} & A_{\Delta \Pi} & B_{\Delta \Pi}^T & B_{\Delta}^T \\
A_{I\Pi} & B_{I\Pi}^T & A_{\Pi \Delta} & A_{\Pi \Pi} & B_{\Pi \Pi}^T & 0 \\
B_{I\Pi} & 0 & B_{\Pi \Delta} & B_{\Pi \Pi} & 0 & 0 \\
0 & 0 & B_{\Delta} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_I \\
p_I \\
u_\Delta \\
u_\Pi \\
p_\Gamma \\
\lambda \end{bmatrix} = \begin{bmatrix} f_I \\
0 \\
f_\Delta \\
f_\Pi \\
0 \\
0 \end{bmatrix},$$

where the sub-blocks in the coefficient matrix represent the restrictions of $A$ and $B$ in (3) to appropriate subspaces. The leading three-by-three block can be made block diagonal with each diagonal block corresponding to one subdomain.

The coefficient matrix in (6) is singular. The trivial null space vectors are those with $\lambda$ in the null space of $B_{\Delta}^T$ and other components zero. Such singularity, due to the rank deficiency of $B_{\Delta}$, needs not to be worried, since the Lagrange multiplier vector $\lambda$ will be confined in $\Lambda$, the range of $B_{\Delta}$. The only meaningful basis vector in the null space of (6) corresponds to the one-dimensional null space of the original incompressible Stokes system (3), and is specified in the following lemmas.

We first need to introduce a positive scaling factor $\delta^I(x)$ for each node $x$ on $\Gamma$. Let $N_x$ be the number of subdomains sharing $x$, and we define $\delta^I(x) = 1/N_x$. Given such scaling factors at the subdomain interface nodes, we can define a
scaled operator $B_{\Delta,D}$. We note that each row of $B_{\Delta}$ has only two nonzero entries, 1 and $-1$, connecting two neighboring subdomains sharing a node $x$ on $\Gamma$. Multiplying each entry by the scaling factor $\delta^\dagger(x)$ gives us $B_{\Delta,D}$. Namely

$$B_{\Delta,D} = \begin{bmatrix} D_{\Delta}B_{\Delta}^{(1)} & \Delta D_{\Delta}B_{\Delta}^{(2)} & \cdots & \Delta D_{\Delta}B_{\Delta}^{(N)} \end{bmatrix},$$

where $D_{\Delta}$ is a diagonal matrix and contains $\delta^\dagger(x)$ on its diagonal. We also see from the definition of $B_{\Delta,D}$ that the scalings on all the Lagrange multipliers related to the same subdomain interface node are the same, from which we have the following lemma.

**Lemma 1** The null of $B^T_{\Delta}$ is the same as the null of $B^T_{\Delta,D}$; the range of $B_{\Delta}$ is the same as the range of $B_{\Delta,D}$.

The following lemma can be found at [27, Page 175].

**Lemma 2** For any $\lambda \in \Lambda$, $B_{\Delta}B^T_{\Delta,D}\lambda = B_{\Delta,D}B^T_{\Delta}\lambda = \lambda$.

**Lemma 3** Let $1_{p_I} \in Q_I$, $1_{p_F} \in Q_F$ represent vectors with value 1 on each entry. Then

$$\begin{bmatrix} 1_{p_I} \\ B^T_{I \Delta} & B^T_{F \Delta} \end{bmatrix} \begin{bmatrix} 1_{p_I} \\ 1_{p_F} \end{bmatrix} = B^T_{\Delta}\lambda,$$

where

$$\lambda = B_{\Delta,D}[B_{I \Delta}^T & B_{F \Delta}^T] \begin{bmatrix} 1_{p_I} \\ 1_{p_F} \end{bmatrix} \in \Lambda.$$

**Proof:** The left side of (7) contains face integrals of the normal component of the dual subdomain interface velocity finite element basis functions across the subdomain interface. For a face velocity degree of freedom, which is shared by two neighboring subdomains, the face integrals of their normal components on the two neighboring subdomains are negative of each other, since their normal directions are opposite. This pair of opposite values can then be represented by the product of $B^T_{\Delta}$ and a Lagrange multiplier with value equal to the face integral of the corresponding basis function.

Now we consider a subdomain edge velocity degree of freedom, which is shared by more than two subdomains, e.g., by four subdomains $\Omega_i$, $\Omega_j$, $\Omega_k$, and $\Omega_l$. A two-dimensional illustration of such an edge node is shown in Figure 1, where the edge shared by the four subdomains points outward directly. Denote the four faces having this edge in common by $\mathcal{F}_{ij}$, $\mathcal{F}_{jk}$, $\mathcal{F}_{kl}$, $\mathcal{F}_{li}$, where, e.g., $\mathcal{F}_{ij}$ represents the face shared by $\Omega_i$ and $\Omega_j$, while $\Omega_i$ and $\Omega_k$ have no common face. Denote the integration of the normal component of this velocity basis function on these four faces by $I_{ij}$, $I_{jk}$, $I_{kl}$, $I_{li}$, with a chosen normal direction for each face, e.g., upward on $\mathcal{F}_{ij}$ and $\mathcal{F}_{kl}$, to the right on $\mathcal{F}_{jk}$ and $\mathcal{F}_{li}$. Then the entries of the left side vector in (7) corresponding to this edge velocity degree of freedom on the four subdomains $\Omega_i$, $\Omega_j$, $\Omega_k$, and $\Omega_l$, are $I_{ij}+I_{ii}$, $-I_{ij}+I_{jk}$, $-I_{jk}+I_{kl}$, and
$I_{kl} - I_{li}$, respectively. Here two neighboring subdomains sharing a common face have opposite face integral values on that face because their normal directions are opposite of each other. Take $I_{ij}$, $I_{jk}$, $I_{kl}$, $I_{li}$ as the four Lagrange multiplier values as illustrated in Figure 1. Then the four subdomain face integral values $I_{ij} + I_{li}$, $-I_{ij} + I_{jk}$, $-I_{jk} - I_{kl}$, and $I_{kl} - I_{li}$, can be represented as the product of corresponding $B_T$ with a Lagrange multiplier vector containing these four Lagrange multiplier values and zero elsewhere.

The above has just shown that the left side of (7) can be represented by the product of $B_T$ with a Lagrange multiplier vector $\lambda$. If $\lambda$ is not in $\Lambda$, i.e., not in the range of $B_T$, it can always be written as the sum of its components in $\Lambda$ and in the null of $B_T$. Then we just take its component in $\Lambda$ as $\lambda$, which does not change the product $B_T\lambda$. By multiplying $B_T, D$ to both sides of (7) and using Lemma 2, we have (8).

**Lemma 4** The basis vector in the null space of (6), corresponding to the one-dimensional null space of the original incompressible Stokes system (3), is

$$B_T \left[ B_T B_T \right] \left[ \begin{array}{c} 1_{pi} \\ 1_{pc} \end{array} \right].$$

**Proof:** Since the null space of (3) consists of all constant pressures, substituting the vector (9) into (6) gives zero blocks on the right-hand side, except at the third block where

$$f = \left[ B_T B_T \right] \left[ \begin{array}{c} 1_{pi} \\ 1_{pc} \end{array} \right],$$

which also equals zero from (7) and (8) in Lemma 3.
A reduced symmetric positive semi-definite system

The system (6) can be reduced to a Schur complement problem for the variables \((p_G, \lambda)\). Since the leading four-by-four block of the coefficient matrix in (6) is invertible, the variables \((u_I, p_I, u_\Delta, u_{II})\) can be eliminated and we obtain

\[
G \begin{bmatrix} p_G \\ \lambda \end{bmatrix} = g,
\]

where

\[
G = B_C \tilde{A}^{-1} B_C^T, \quad g = B_C \tilde{A}^{-1} \begin{bmatrix} f_I \\ 0 \\ f_\Delta \\ f_{II} \end{bmatrix},
\]

with

\[
\tilde{A} = \begin{bmatrix} A_{II} & B_{II}^T & A_{I\Delta} & A_{I\Pi} \\ B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ A_{I\Delta} & B_{I\Delta}^T & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{I\Pi} & B_{I\Pi}^T & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} B_{II} & 0 & B_{I\Delta} & B_{I\Pi} \\ 0 & 0 & B_\Delta & 0 \end{bmatrix}.
\]

We can see that \(-G\) is the Schur complement of the coefficient matrix of (6) with respect to the last two row blocks, i.e.,

\[
\begin{bmatrix} I & 0 \\ -B_C \tilde{A}^{-1} I \end{bmatrix} \begin{bmatrix} \tilde{A} & B_C^T \\ B_C & 0 \end{bmatrix} \begin{bmatrix} I & -\tilde{A}^{-1} B_C^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & -G \end{bmatrix}.
\]

From the Sylvester law of inertia, namely, the number of positive, negative, and zero eigenvalues of a symmetric matrix is invariant under a change of coordinates, we can see that the number of zero eigenvalues of \(G\) is the same as the number of zero eigenvalues (with multiplicity counted) of the original coefficient matrix of (6), and all other eigenvalues of \(G\) are positive. Therefore \(G\) is symmetric positive semi-definite. The basis vectors of the null space of \(G\) also inherit those from the null space of (6), and the only interesting basis vector is

\[
(14) \quad \begin{bmatrix} 1_{p_G} \\ -B_{\Delta, G} [B_{I\Delta}^T B_{I\Pi}^T] \begin{bmatrix} 1_{f_I} \\ 1_{p_G} \end{bmatrix} \end{bmatrix},
\]

which is derived from Lemma 4. The other null space vectors of \(G\) are all vectors with \(\lambda\) in the null of \(B_\Delta^T\) and \(p_G = 0\). The range of \(G\) contain all vectors orthogonal to those null vectors. Denote \(X = Q_G \bigoplus \Lambda\), where, as defined earlier,
Λ is the range of $B_\Delta$. Then the range of $G$, denoted by $R_G$, is the subspace of $X$ orthogonal to (14), i.e.,

$$R_G = \left\{ \left[ \begin{array}{c} g_{pr} \\ g_\lambda \end{array} \right] \in X \mid g_{pr}^T 1_{pr} - g_\lambda^T \left( B_{\Delta,D} B_{I,\Delta}^T B_{I,\Delta}^T \right) \left[ \begin{array}{c} 1_{pr} \\ 1_{pr} \end{array} \right] = 0 \right\}. $$

The restriction of $G$ to its range $R_G$ is positive definite. The fact that the solution of (6) always exists for any given $(f_I, f_\Delta, f_\Pi)$ on the right-hand side implies that the solution of (11) exists for any $g$ defined by (12). Therefore $g \in R_G$. When the conjugate gradient method (CG) is applied to solve (11) with zero initial guess, all the iterates are in the Krylov subspace generated by $G$ and $g$, which is also a subspace of $R_G$, and where the CG cannot break down. After obtaining $(p_\Gamma, \lambda)$ from solving (11), the other components $(u_I, p_I, u_\Delta, u_\Pi)$ in (6) are obtained by back substitution.

In the rest of this section, we discuss the implementation of multiplying $G$ by a vector. The main operation is the product of $\tilde{A}^{-1}$ with a vector, cf. (12). We denote

$$A_{rr} = \begin{bmatrix} A_{II} & B_{II}^T \\ B_{II} & 0 & B_{I,\Delta} \\ A_{\Delta I} & B_{I,\Delta}^T & A_{\Delta,\Delta} \end{bmatrix}, \quad A_{r_\Pi} = A_{r_\Pi}^T = \begin{bmatrix} A_{II} & B_{II}^T & A_{II,\Pi} \\ A_{II} & B_{II}^T & A_{II,\Pi} \\ A_{II} & B_{II}^T & A_{II,\Pi} \end{bmatrix},$$

and define the Schur complement

$$S_{II} = A_{II} - A_{r_\Pi} A_{rr}^{-1} A_{r_\Pi},$$

which is symmetric positive definite from the Sylvester law of inertia. $S_{II}$ defines the coarse level problem in the algorithm. The product

$$\begin{bmatrix} A_{II} & B_{II}^T & A_{II,\Delta} & A_{II} \\ B_{II} & 0 & B_{II} \end{bmatrix}^{-1} \begin{bmatrix} f_I \\ 0 \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Delta \end{bmatrix}$$

can then be represented by

$$\begin{bmatrix} A_{rr}^{-1} f_r \\ 0 \end{bmatrix} + \begin{bmatrix} -A_{rr}^{-1} A_{r_\Pi} \\ I_{II} \end{bmatrix} S_{II}^{-1} (f_\Pi - A_{r_\Pi} A_{rr}^{-1} f_r),$$

which requires solving the coarse level problem once and independent subdomain Stokes problems with Neumann type boundary conditions twice.

## 5 Preliminary results

Denote

$$\tilde{W} = W_I \bigoplus \tilde{W}_\Gamma = W_I \bigoplus W_\Delta \bigoplus W_\Pi.$$
For any \( \mathbf{w} \) in \( \tilde{W} \), denote its restriction to subdomain \( \Omega_i \) by \( \mathbf{w}^{(i)} \). A subdomain-wise \( H^1 \)-seminorm can be defined for functions in \( \tilde{W} \) by

\[
|\mathbf{w}|^2_{H^1} = \sum_{i=1}^{N} |\mathbf{w}^{(i)}|^2_{H^1(\Omega_i)}.
\]

We also define

\[
\tilde{V} = W_I \bigoplus Q_I \bigoplus \mathbf{W}_\Delta \bigoplus \mathbf{W}_H,
\]

and its subspace

\[
(17) \quad \tilde{V}_0 = \{ v = (w_I, p_I, w_\Delta, w_H) \in \tilde{V} \mid B_{II}w_I + B_{I\Delta} w_\Delta + B_{III} w_H = 0 \}.
\]

For any \( v = (w_I, p_I, w_\Delta, w_H) \in \tilde{V}_0 \), let \( \mathbf{w} = (w_I, w_\Delta, w_H) \in \tilde{W} \). Then

\[
\langle v, v \rangle_{\tilde{A}} = \begin{bmatrix} w_I \\ w_\Delta \\ w_H \end{bmatrix}^T \begin{bmatrix} A_{II} & A_{I\Delta} & A_{III} \\ A_{I\Delta} & A_{\Delta\Delta} & A_{\Delta II} \\ A_{III} & A_{\Delta II} & A_{III} \end{bmatrix} \begin{bmatrix} w_I \\ w_\Delta \\ w_H \end{bmatrix} = \sum_{i=1}^{N} \begin{bmatrix} w_I^{(i)} \\ w_\Delta^{(i)} \\ w_H^{(i)} \end{bmatrix}^T \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{III}^{(i)} \\ A_{I\Delta}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta II}^{(i)} \\ A_{III}^{(i)} & A_{\Delta II}^{(i)} & A_{III}^{(i)} \end{bmatrix} \begin{bmatrix} w_I^{(i)} \\ w_\Delta^{(i)} \\ w_H^{(i)} \end{bmatrix} = |\mathbf{w}|^2_{H^1(\Omega_i)}
\]

(18)

where the superscript \( ^{(i)} \) is used to represent the restrictions of corresponding vectors and matrices to subdomain \( \Omega_i \). We can see from (18) that for any \( v \in \tilde{V}_0 \), the value \( \langle v, v \rangle_{\tilde{A}} \) is independent of its pressure component \( p_I \). \( \langle \cdot, \cdot \rangle_{\tilde{A}} \) defines a semi-inner product on \( \tilde{V}_0 \); \( \langle v, v \rangle_{\tilde{A}} = 0 \) if and only if the velocity component of \( v \) is constant on \( \Omega \) and is in fact zero due to the zero boundary condition on \( \partial \Omega \), while its pressure component can be arbitrary.

Denote

\[
(19) \quad \tilde{B} = \begin{bmatrix} B_{II} & B_{I\Delta} & B_{III} \\ B_{II} & B_{I\Delta} & B_{III} \end{bmatrix},
\]

cf. (6). The following lemma on the stability of \( \tilde{B} \) can be found at [16, Lemma 5.1].

**Lemma 5** For any \( \mathbf{w} \in \tilde{W} \) and \( q \in Q \), \( \langle \tilde{B} \mathbf{w}, q \rangle \leq |\mathbf{w}|_{H^1} \|q\|_{L^2} \).

The following lemma will also be used and can be found at [8, Lemma 2.3].

**Lemma 6** Let \( (u, p) \in W \bigoplus Q \) satisfy

\[
(20) \quad \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},
\]
where $A$ and $B$ are as in (3), $f \in W$, and $g \in \text{Im}(B) \subset Q$. Let $\beta$ be the inf-sup constant specified in (4). Then

$$\|u\|_A \leq \|f\|_{A^{-1}} + \frac{1}{\beta} \|g\|_{Z^{-1}},$$

where $Z$ is the mass matrix defined in Section 2.

### 6 Jump operators and preconditioners

We first define certain jump operators across the subdomain interface $\Gamma$, which will be used for the analysis of the preconditioners.

Denote the restriction operator from $\tilde{V}$ onto $W_{\Delta}$ by $\tilde{R}_{\Delta}$, i.e., for any $v = (w_I, p_I, w_\Delta, w_\Pi) \in \tilde{V}$, $\tilde{R}_{\Delta}v = w_\Delta$. Define $P_{D,L} : V \to \tilde{V}$, by

$$P_{D,L} = \tilde{R}_{\Delta}^T B_{\Delta,D}^T D B_{\Delta} \tilde{R}_{\Delta}.$$ 

Following this definition, given any $v = (w_I, p_I, w_\Delta, w_\Pi) \in \tilde{V}$, the dual velocity component of $P_{D,L}v$, on any subdomain interface node $x$ in subdomain $\Omega_i$, is given by, cf. [27, Equation (6.70)],

$$\left( \tilde{R}_{\Delta} (P_{D,L}v) \right)^{(i)}(x) = \sum_{j \in N_x} \delta^i(x) \left( w_\Delta^{(i)}(x) - w_\Delta^{(j)}(x) \right),$$

which represents the so-called jump of the dual velocity component $w_\Delta$ across the subdomain interface $\Gamma$. All other components of $P_{D,L}v$ equal zero. We also have

$$\langle P_{D,L}v, P_{D,L}v \rangle_{A_{\Delta}} = \langle \tilde{R}_{\Delta}^T B_{\Delta,D}^T D B_{\Delta} \tilde{R}_{\Delta}v, \tilde{R}_{\Delta}^T B_{\Delta,D}^T D B_{\Delta} \tilde{R}_{\Delta}v \rangle_{A_{\Delta}}.$$

Together with (18), we have the following lemma, which can be found at [18, Section 6.1].

**Lemma 7** There exists a constant $C$ and a function $\Phi_L(H/h)$, such that for all $v \in \tilde{V}_0$, $\langle P_{D,L}v, P_{D,L}v \rangle_{\tilde{A}} \leq C \Phi_L(H/h) \langle v, v \rangle_{\tilde{A}}$. Here, $\Phi_L(H/h) = (H/h)(1 + \log(H/h))$, when the coarse level space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component.

When applying $P_{D,L}$ to a vector, the jump of the dual subdomain interface velocities is extended by zero to the interior of subdomains. To improve the stability of the jump operator, the jump can be extended to the interior of subdomains by subdomain discrete harmonic extension. We define a Schur complement operator $H_\Delta^{(i)} : W_\Delta^{(i)} \to W_\Delta^{(i)}$ by, for any $u_\Delta^{(i)} \in W_\Delta^{(i)}$, 

$$\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix} \begin{bmatrix} u_I^{(i)} \\ u_\Delta^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ H_\Delta^{(i)} u_\Delta^{(i)} \end{bmatrix}. $$


To multiply $H^{(i)}_\Delta$ by a vector $u^{(i)}_\Delta$, a subdomain elliptic problem on $\Omega_i$ with given boundary velocity $u^{(i)}_\Delta$ and $u^{(i)}_{\Delta i} = 0$ needs to be solved. We let $H_\Delta : W_\Delta \rightarrow W_\Delta$ to represent the direct sum of $H^{(i)}_\Delta$, $i = 1, \ldots, N$.

Using $H^{(i)}_\Delta$, we define the second jump operator $P_{D,D} : \tilde{V} \rightarrow \tilde{V}$, by: for any given $v = (w_I, p_I, \Delta, w_{II}) \in \tilde{V}$, the subdomain interior velocity part of $P_{D,D} v$ on each subdomain $\Omega_i$ is taken as $u^{(i)}_I$ in the solution of (22), with given subdomain boundary velocity $u^{(i)}_\Delta = B^{(i)}_{\Delta,D} \Delta w_\Delta$. Here $B^{(i)}_{\Delta,D}$ represents restriction of $B^{T}_{\Delta,D}$ on subdomain $\Omega_i$ and is a map from $\Lambda$ to $W^{(i)}_\Delta$. The other components of $P_{D,D} v$ are kept zero. Therefore

\[
\langle P_{D,D} v, P_{D,D} v \rangle_{\tilde{V}} = \sum_{i=1}^{N} \left[ \begin{array}{c} u^{(i)}_I \\ u^{(i)}_{\Delta i} \end{array} \right]^T \left[ \begin{array}{cc} A^{(i)}_{II} & A^{(i)}_{\Delta i} \\ A^{(i)}_{\Delta i} & A^{(i)}_{\Delta \Delta} \end{array} \right] \left[ \begin{array}{c} u^{(i)}_I \\ u^{(i)}_{\Delta i} \end{array} \right] = \sum_{i=1}^{N} u^{(i)}_I^T H^{(i)}_\Delta u^{(i)}_I = \sum_{i=1}^{N} w^{(i)}_I B^{(i)}_{\Delta,D} H^{(i)}_\Delta B^{(i)}_{\Delta,D} w_\Delta \]

\[
(23) = \sum_{i=1}^{N} \left[ \begin{array}{c} B^{(i)}_{\Delta,D} w_\Delta \\ 0 \end{array} \right] \left[ \begin{array}{c} B^{(i)}_{\Delta,D} B^{(i)}_{\Delta,D} \end{array} \right] \leq C \Phi_D(H/h) \sum_{i=1}^{N} \left[ \begin{array}{c} w^{(i)}_I \\ w^{(i)}_{II} \end{array} \right] \left[ \begin{array}{c} w^{(i)}_I \\ w^{(i)}_{II} \end{array} \right] = C \Phi_D(H/h) |w|_{H^1(\Omega_i)}^2.
\]

The first inequality in (23) is a well established result, cf., [27, Lemma 6.36]. Since for any $v \in \tilde{V}_0$, $\langle v, v \rangle_{\tilde{V}} = |w|^2_{H^1(\Omega_i)}$, cf. (18), we have the following lemma.

**Lemma 8** There exists a constant $C$ and a function $\Phi_D(H/h)$, such that for all $v \in \tilde{V}_0$, $\langle P_{D,D} v, P_{D,D} v \rangle_{\tilde{V}} \leq C \Phi_D(H/h) \langle v, v \rangle_{\tilde{V}}$. Here $\Phi_D(H/h) = (1 + \log (H/h))^2$, when the coarse level space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component.

To introduce the preconditioners, we write $G$, defined in (12) and (13), in a two-by-two block structure. Denote the first row of $B_C$ by

$\tilde{B}_I = [B_{II} B_{I\Delta} B_{I\Omega}]$,

and note that $\tilde{R}_\Delta$ is the restriction operator from $\tilde{V}$ onto $W_\Delta$. Then $G$ can be written as

\[
(24) G = \begin{bmatrix} G_{pp,pp} & G_{pp,\lambda} \\ G_{\lambda pp} & G_{\lambda \lambda} \end{bmatrix},
\]

where

\[
G_{pp,pp} = \tilde{B}_I \tilde{A}^{-1} \tilde{B}_I^T, \quad G_{pp,\lambda} = \tilde{B}_I \tilde{A}^{-1} \tilde{R}_\Delta \tilde{B}_I^T, \\
G_{\lambda pp} = \tilde{B}_I \tilde{R}_\Delta \tilde{A}^{-1} \tilde{B}_I^T, \quad G_{\lambda \lambda} = B_\Delta \tilde{R}_\Delta \tilde{A}^{-1} \tilde{R}_\Delta \tilde{B}_I^T.
\]
We consider a block diagonal preconditioner for (11). As for two-dimensional problems, the first diagonal block $G_{p_{\text{pr}}}$ of $G$ can be shown spectrally equivalent to $h^3 I_{p_{\text{pr}}}$, where $I_{p_{\text{pr}}}$ is the identity matrix of the same dimension as $G_{p_{\text{pr}}}$; see [16, 26]. Therefore, in the following block diagonal preconditioners, the inverse of $G_{p_{\text{pr}}}$ is approximated by $\alpha h^{-3} I_{p_{\text{pr}}}$. Here $\alpha$ is a given constant. We will show in the next section that $\alpha$ has only a minor effect on the condition number bound of the preconditioned operator and its value is typically taken as 1, cf. Remark 2. We introduce $\alpha$ in the preconditioner just for the convenience in the numerical experiments to demonstrate the convergence rates of the proposed algorithm.

The inverse of the second diagonal block $B_{\Delta} \widetilde{R}_{\Delta} \widetilde{A}^{-1} \widetilde{R}_{\Delta}^T B_{\Delta}^T$, can be approximated by the lumped block
\begin{equation}
M_{\lambda,L}^{-1} = B_{\Delta,D} \widetilde{R}_{\Delta} \widetilde{A} \widetilde{R}_{\Delta}^T B_{\Delta,D}^T.
\end{equation}

This leads to the following lumped preconditioner for solving (11)
\begin{equation}
M_L^{-1} = \begin{bmatrix}
\alpha h^{-3} I_{p_{\text{pr}}}
M_{\lambda,L}^{-1}
\end{bmatrix}.
\end{equation}

Applying subdomain discrete harmonic extensions in the preconditioning step, we have the following Dirichlet preconditioner
\begin{equation}
M_D^{-1} = \begin{bmatrix}
\alpha h^{-3} I_{p_{\text{pr}}}
M_{\lambda,D}^{-1}
\end{bmatrix},
\end{equation}

where
\begin{equation}
M_{\lambda,D}^{-1} = B_{\Delta,D} H_{\Delta} B_{\Delta,D}^T.
\end{equation}

We can see from Lemma 1 that both $M_{\lambda,L}^{-1}$ and $M_{\lambda,D}^{-1}$ are symmetric positive definite when restricted on $\Lambda$. Therefore both the lumped and the Dirichlet preconditioners $M_L^{-1}$ and $M_D^{-1}$ are symmetric positive definite in the range of $G$.

7 Condition number bounds

In the following, we use the same framework to establish the condition number bounds for both lumped and Dirichlet preconditioned operators $M_L^{-1} G$ and $M_D^{-1} G$. Let $M^{-1}$, $M_{\lambda}^{-1}$, $P_D$, and $\Phi$ to represent both $M_L^{-1}$, $M_{\lambda,L}$, $P_{D,L}$, $\Phi_L$, for the lumped preconditioner case, and $M_D^{-1}$, $M_{\lambda,D}$, $P_{D,D}$, $\Phi_D$, for the Dirichlet preconditioner case, respectively, when they apply in the proofs.

When the conjugate gradient method is applied to solving the preconditioned system
\begin{equation}
M^{-1} G x = M^{-1} g,
\end{equation}

with zero initial guess, all iterates belong to the Krylov subspace generated by
the operator $M^{-1}G$ and the vector $M^{-1}g$, which is a subspace of the range of
$M^{-1}G$. We denote the range of $M^{-1}G$ by $R_{M^{-1}G}$ and note that both precon-
tioners are symmetric positive definite in the range of $G$. We have the following
lemma, cf. [26, Lemma 6].

**Lemma 9** The conjugate gradient method applied to solving (29) with zero ini-
tial guess cannot break down.

Proof: We just need to show that for any $0 \neq x \in R_{M^{-1}G}$, $\langle x, Gx \rangle \neq 0$, i.e.,
to show $Gx \neq 0$. Let $0 \neq x = M^{-1}Gy$, for a certain $y \in X$ and $y \neq 0$. Then
$Gx = GM^{-1}Gy$, which cannot be zero since $Gy \neq 0$ and $y^TGM^{-1}Gy \neq 0$.
\[\square\]

The following lemma will be used to provide the upper eigenvalue bound of
the preconditioned operator. It is similar to [16, Lemma 6.4] and [26, Lemmas
8 and 11].

**Lemma 10** There exists a constant $C$, such that for all $v \in \tilde{V}_0$,
\[
\langle M^{-1}B_C v, B_C v \rangle \leq C (\alpha + \Phi(H/h)) \langle \tilde{A} v, v \rangle,
\]
where $\Phi(H/h)$ is defined in Lemmas 7 and 8, respectively.

Proof: Given $v = (w_I, q_I, w_\Delta, w_{\Pi}) \in \tilde{V}_0$, let $g_{pv} = B_I w_I + B_\Delta w_\Delta + B_{\Pi} w_{\Pi}$. From (13), (25)–(28), (21), and (23), we have
\[
\langle M^{-1}B_C v, B_C v \rangle = ah^{-3} \langle g_{pv}, g_{pv} \rangle + \left( B_\Delta \tilde{R}_\Delta v \right)^T M_\lambda^{-1} B_\Delta \tilde{R}_\Delta v
\]
\[
= ah^{-3} \langle g_{pv}, g_{pv} \rangle + \langle P_D v, P_D v \rangle_\tilde{A}
\]
\[
\leq ah^{-3} \langle g_{pv}, g_{pv} \rangle + C\Phi(H/h) \langle v, v \rangle_\tilde{A},
\]
where we used Lemmas 7 and 8 for the last inequality. It is sufficient to bound
the first term of the right-hand side in the above inequality.

Since $v \in \tilde{V}_0$, we have $B_{II} w_I + B_\Delta w_\Delta + B_{III} w_{\Pi} = 0$, cf. (17). Then
\[
\langle g_{pv}, g_{pv} \rangle = \begin{bmatrix} B_{II} w_I + B_\Delta w_\Delta + B_{III} w_{\Pi} \\ B_{II} w_I + B_\Gamma w_\Delta + B_{III} w_{\Pi} \end{bmatrix}^T \begin{bmatrix} B_{II} w_I + B_\Delta w_\Delta + B_{III} w_{\Pi} \\ B_{II} w_I + B_\Gamma w_\Delta + B_{III} w_{\Pi} \end{bmatrix}
\]
\[
= \langle \tilde{B} w, \tilde{B} w \rangle,
\]
where $\tilde{B}$ is defined in (19) and $w = (w_I, w_\Delta, w_{\Pi}) \in \tilde{W}$. From (5) and the
stability of $\tilde{B}$, cf. Lemma 5, we have
\[
(3\alpha h)^{-3} \langle g_{pv}, g_{pv} \rangle = h^{-3} \langle \tilde{B} w, \tilde{B} w \rangle \leq C \langle \tilde{B} w, \tilde{B} w \rangle_{Z^{-1}} = C \max_{q \in Q} \frac{\langle \tilde{B} w, q \rangle^2_{Z^{-1}}}{\langle q, q \rangle_{Z}}
\]
\[
\leq C \max_{q \in Q} \frac{|w|_{H^1}^2 \|q\|_{L^2}^2}{\|q\|^2_{L^2}} = C |w|_{H^1}^2 = C \langle v, v \rangle_\tilde{A},
\]
where for the last equality, we used (18). □

The following lemma will be used to provide the lower eigenvalue bound of the preconditioned operator. In [26, Lemmas 9 and 12], the lower eigenvalue bounds for the lumped and Dirichlet preconditioners were analyzed differently. In the analysis of the Dirichlet preconditioner, subdomain discrete Stokes extensions were used. Such extensions require enforcing the same type divergence free subdomain boundary velocity conditions as discussed in [17], even though they are not necessary for implementing the algorithm in [26]. The new proof given in the next lemma works for both lumped and Dirichlet preconditioners. It does not use the subdomain Stokes extensions and those additional subdomain divergence free boundary conditions are no longer needed. For both type of preconditioners, the coarse level velocity space can be chosen as simple as for solving scalar elliptic problems corresponding to each velocity component.

**Lemma 11** There exists a constant $C$, such that for any nonzero $y = (g_F, g)$ $\in R_G$, there exists $v \in \tilde{V}_0$, which satisfies $B_Cv = y$, $(v, v)_A \neq 0$, and

$$\left\langle Av, v \right\rangle \leq C \max \left\{ 1, \frac{1}{\alpha_1^2} \right\} \left( 1 + \frac{1}{\sigma_1^2} \right) \left( M^{-1}y, y \right).$$

Proof: Given $y = (g_F, g) \in R_G$, take $u^{(l)}_{\Delta} = B_{\Delta,D}g$, $u_{\Pi}^{(l)} = 0$, and $p^{(l)} = 0$. On each subdomain $\Omega_i$, let $u_i^{(l),0}$ be zero for the lumped preconditioner, and be obtained for the Dirichlet preconditioner through the solution of (22) with given subdomain boundary values $u_i^{(l)} = u_i^{(l),i}$. Let $\nu^{(l,i)} = \left( u_i^{(l,i)}, p_i^{(l,i)}, u_{\Delta}^{(l,i)}, u_{\Pi}^{(l,i)} \right)$, the corresponding global vectors $\nu^{(l)} = \left( u_i^{(l)}, p_i^{(l)}, u_{\Delta}^{(l)}, u_{\Pi}^{(l)} \right)$, and $u^{(l)} = \left( u_i^{(l)}, u_{\Delta}^{(l)}, u_{\Pi}^{(l)} \right)$. Then we have

$$B_C\nu^{(l)} = \begin{bmatrix} B_{\Gamma} & 0 & B_{\Gamma,D} & B_{\Gamma,\Pi} \\ 0 & 0 & B_{\Delta} & 0 \end{bmatrix} \begin{bmatrix} u_i^{(l)} \\ p_i^{(l)} \\ u_{\Delta}^{(l)} \\ u_{\Pi}^{(l)} \end{bmatrix} = \begin{bmatrix} B_{\Gamma}u_i^{(l)} + B_{\Gamma,D}u_{\Delta}^{(l)} + B_{\Gamma,\Pi}u_{\Pi}^{(l)} \\ g \end{bmatrix},$$

where we have used Lemma 2. Also

$$|u^{(l)}_{H^1}'|^2 = \begin{bmatrix} u_i^{(l)} \\ u_i^{(l)} \\ u_{\Pi}^{(l)} \end{bmatrix}^T \begin{bmatrix} A_{II} & A_{I\Delta} & A_{I\Pi} \\ A_{I\Delta} & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{I\Pi} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} u_i^{(l)} \\ u_i^{(l)} \\ u_{\Pi}^{(l)} \end{bmatrix} = \begin{cases} |u^{(l)}_{A\Delta}|^2, & \text{for lumped preconditioner,} \\
|u^{(l)}_{A\Pi}|^2, & \text{for Dirichlet preconditioner.} \end{cases}$$

We consider a solution to the following fully assembled system of linear equations of the form (3): find $\left( u_i^{(II)}, p_i^{(II)}, u_{\Gamma}^{(II)}, p_{\Gamma}^{(II)} \right) \in W_I \ominus Q_I \ominus W_G \ominus Q_G$, ...
such that
\[(34)\]
\[
\begin{bmatrix}
A_{II} & B_{II}^T & A_{I} & B_{I}^T \\
B_{II} & 0 & B_{I} & 0 \\
A_{I} & B_{I}^T & A_{II} & B_{II}^T \\
B_{I} & 0 & B_{II} & 0
\end{bmatrix}
\begin{bmatrix}
u^{(II)}_I \\
p^{(II)}_I \\
u^{(II)}_F \\
p^{(II)}_F
\end{bmatrix} =
\begin{bmatrix}
0 \\
-B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta - B_{II}u^{(I)}_\Pi \\
0 \\
g_{pr} - B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta - B_{II}u^{(I)}_\Pi
\end{bmatrix},
\]
where we know that the particularly chosen right-hand side is essentially
\[(35)\]
\[
\begin{bmatrix}
0 \\
-B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta \\
0 \\
g_{pr} - B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta
\end{bmatrix}.
\]
Since \((g_{pr}, g_\lambda) \in R_G\), we have, cf. (15),
\[-B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta \] has zero average, which implies existence of the solution to (34).

Denote \(u^{(II)} = (u^{(II)}_I, u^{(II)}_F)\). Then from Lemma 6 and (5), we have
\[
|u^{(II)}|^2_{H^1} \leq \frac{1}{\beta^2} \left\| \begin{bmatrix}
-B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta - B_{II}u^{(I)}_\Pi \\
g_{pr} - B_{II}u^{(I)}_I - B_I \Delta u^{(I)}_\Delta - B_{II}u^{(I)}_\Pi
\end{bmatrix} \right\|_Z^{-1}^2
\leq \frac{1}{\beta^2} \left\| \begin{bmatrix}
B_{II}u^{(I)}_I + B_I \Delta u^{(I)}_\Delta + B_{II}u^{(I)}_\Pi \\
B_{II}u^{(I)}_I + B_I \Delta u^{(I)}_\Delta + B_{II}u^{(I)}_\Pi
\end{bmatrix} \right\|_Z^{-1}^2 + \frac{1}{\beta^2} \left\| \begin{bmatrix}
0 \\
g_{pr}
\end{bmatrix} \right\|_Z^{-1}^2,
\]
(36) \leq \frac{1}{\beta^2} |u^{(II)}|^2_{H^1} + \frac{C}{\beta^2 \rho^2} \langle g_{pr}, g_{pr} \rangle,
where the bound on the first term is obtained in the same way as in (31).

Split the continuous subdomain interface velocity \(u^{(II)}_\Delta\) into the dual part \(u^{(II)}_\Delta\) and the primal part \(u^{(II)}_\Pi\), and denote \(v^{(II)} = (u^{(II)}_I, p^{(II)}_I, u^{(II)}_\Delta, u^{(II)}_\Pi)\).
Let \( v = v^{(I)} + v^{(II)} \). Then we have from (34) that \( v \in \tilde{V}_0 \), and

\[
B_{CV}^{(II)} = \begin{bmatrix}
B_{\Gamma I} & 0 & B_{\Gamma \Delta} & B_{\Gamma II} \\
0 & 0 & B_{\Delta} & 0
\end{bmatrix}
\begin{bmatrix}
u^{(II)}_I \\
p^{(II)}_I \\
u^{(II)}_\Delta \\
u^{(II)}_{II}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
g_{p_\Gamma} - B_{\Gamma I}u^{(I)}_I - B_{\Gamma \Delta}u^{(I)}_\Delta - B_{\Gamma II}u^{(I)}_{II} \\
0
\end{bmatrix}.
\]

Together with (32), we have \( B_{CV} = y \). From (18) and (36), we have

\[
|v|_A^2 = |u^{(I)} + u^{(II)}|^2_{H^1} \leq |u^{(I)}|^2_{H^1} + |u^{(II)}|^2_{H^1} = \left(1 + \frac{1}{\beta^2}\right)|u^{(I)}|^2_{H^1} + \frac{C}{\beta^2 h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle
\]

\[
= \begin{cases}
\left(1 + \frac{1}{\beta^2}\right)|u^{(I)}_\Delta|^2_{A_{\Delta \Delta}} + \frac{C}{\beta^2 h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle, & \text{for lumped preconditioner}, \\
\left(1 + \frac{1}{\beta^2}\right)|u^{(I)}_\Delta|^2_{H_\Delta} + \frac{C}{\beta^2 h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle, & \text{for Dirichlet preconditioner},
\end{cases}
\]

where we used (33) in the last equality.

On the other hand, we have from (25)–(28)

\[
\langle M^{-1}y, y \rangle = \frac{\alpha}{h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + g_{\Gamma}^T M_{\lambda}^{-1} g_{\lambda}
\]

\[
= \begin{cases}
\frac{\alpha}{h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + g_{\Gamma}^T B_{\Delta, D} A_{\Delta \Delta} B_{\Delta, D}^T g_{\lambda}, & \text{for lumped preconditioner}, \\
\frac{\alpha}{h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + g_{\Gamma}^T B_{\Delta, D} H_{\Delta} B_{\Delta, D}^T g_{\lambda}, & \text{for Dirichlet preconditioner},
\end{cases}
\]

\[
= \begin{cases}
\frac{\alpha}{h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + |u^{(I)}_\Delta|^2_{A_{\Delta \Delta}}, & \text{for lumped preconditioner}, \\
\frac{\alpha}{h^3} \langle g_{p_\Gamma}, g_{p_\Gamma} \rangle + |u^{(I)}_\Delta|^2_{H_\Delta}, & \text{for Dirichlet preconditioner}.
\end{cases}
\]

It is not difficult to see that \( \langle v, v \rangle_A \neq 0 \). Otherwise, all the velocity components of \( v \) would be zero, cf. (18), and then \( B_{CV} \) would be zero, which conflicts with that \( B_{CV} = y \) and \( y \) is nonzero. \( \square \)

The proofs of the following two lemmas can be found at [16, Lemmas 6.6 and 6.3].

**Lemma 12** For any \( v = (w_I, p_I, w_\Delta, w_{II}) \in \tilde{V}_0, B_{CV} \in R_G \).

**Lemma 13** For any \( x \in R_{M^{-1}G} \),

\[
\langle Mx, x \rangle = \max_{y \in R_G, y \neq 0} \frac{\langle y, x \rangle^2}{\langle M^{-1}y, y \rangle}.
\]
The condition number bound of the preconditioned operator $M^{-1}G$ is given in the following theorem.

**Theorem 1** There exist positive constants $c$ and $C$, such that for all $x \in R_{M^{-1}G}$,

$$
\min \left\{ 1, \alpha \right\} \frac{c \beta^2}{1 + \beta^2} \langle Mx, x \rangle \leq \langle Gx, x \rangle \leq C \left( \alpha + \Phi(H/h) \right) \langle Mx, x \rangle.
$$

**Proof:** We only need to prove the above inequalities for any nonzero $x \in R_{M^{-1}G}$. We know from Lemma 9 that

$$
0 \neq \langle Gx, x \rangle = x^T B_C \tilde{A}^{-1} B_C^T x = x^T B_C \tilde{A}^{-1} \tilde{A} \tilde{A}^{-1} B_C^T x = \left\langle \tilde{A}^{-1} B_C^T x, \tilde{A}^{-1} B_C^T x \right\rangle_{\tilde{A}}.
$$

Therefore $\tilde{A}^{-1} B_C^T x \neq 0$. Also note that $\tilde{A}^{-1} B_C^T x \in \tilde{V}_0$ and $\langle \cdot, \cdot \rangle_{\tilde{A}}$ defines a semi-inner product on $\tilde{V}_0$, cf (18), and then we have

$$
\langle Gx, x \rangle = \max_{v \in \tilde{V}_0, \langle v, v \rangle_{\tilde{A}} \neq 0} \frac{\left\langle v, \tilde{A}^{-1} B_C^T x \right\rangle_{\tilde{A}}^2}{\langle v, v \rangle_{\tilde{A}}} = \max_{v \in \tilde{V}_0, \langle v, v \rangle_{\tilde{A}} \neq 0} \frac{\langle B_C v, x \rangle^2}{\langle \tilde{A} v, v \rangle_{\tilde{A}}}.
$$

**Lower bound:** From Lemma 11, we know that for any nonzero $y \in R_G$, there exits $w \in \tilde{V}_0$, such that $B_C w = y$, $\langle w, w \rangle_{\tilde{A}} \neq 0$, $\langle \tilde{A} w, w \rangle \leq \max \left\{ 1, \frac{1}{\alpha} \right\} \frac{c (1 + \beta^2)}{\beta^2} \langle M^{-1}y, y \rangle$.

Then from (37), we have

$$
\langle Gx, x \rangle \geq \frac{\langle B_C w, x \rangle^2}{\langle \tilde{A} w, w \rangle} \geq c \frac{\beta^2}{\max \left\{ 1, \frac{1}{\alpha} \right\} (1 + \beta^2)} \frac{\langle y, x \rangle^2}{\langle M^{-1}y, y \rangle}.
$$

Since $y$ is arbitrary, using Lemma 13, we have

$$
\langle Gx, x \rangle \geq c \frac{\beta^2}{\max \left\{ 1, \frac{1}{\alpha} \right\} (1 + \beta^2)} \max_{y \in R_G, y \neq 0} \frac{\langle y, x \rangle^2}{\langle M^{-1}y, y \rangle} = \min \left\{ 1, \alpha \right\} \frac{c \beta^2}{(1 + \beta^2)} \langle Mx, x \rangle.
$$

**Upper bound:** From (37) and the fact that $\langle Gx, x \rangle \neq 0$, we have

$$
\langle Gx, x \rangle = \max_{v \in \tilde{V}_0, \langle v, v \rangle_{\tilde{A}} \neq 0} \frac{\langle B_C v, x \rangle^2}{\langle \tilde{A} v, v \rangle_{\tilde{A}}} = \max_{v \in \tilde{V}_0, \langle v, v \rangle_{\tilde{A}} \neq 0, B_C v \neq 0} \frac{\langle B_C v, x \rangle^2}{\langle \tilde{A} v, v \rangle_{\tilde{A}}},
$$

where the maximum only needs to be considered among $v$ also satisfying $B_C v \neq 0$. Then using Lemmas 10, 12, and 13, we have

$$
\langle Gx, x \rangle \leq C \left( \alpha + \Phi(H/h) \right) \max_{y \in R_G, y \neq 0} \frac{\langle B_C v, x \rangle^2}{\langle M^{-1}B_C v, B_C v \rangle} \leq C \left( \alpha + \Phi(H/h) \right) \max_{y \in R_G, y \neq 0} \frac{\langle y, x \rangle^2}{\langle M^{-1}y, y \rangle} = C \left( \alpha + \Phi(H/h) \right) \langle Mx, x \rangle.
$$

\[\square\]
Remark 2 We can see from Theorem 1 that, for $\alpha \geq 1$, the condition number bound of $M^{-1}G$ is proportional to $\alpha + \Phi(H, h)$, and we should take smaller $\alpha$ to achieve faster convergence. When $\alpha \leq 1$, the condition number bound is proportional to $1 + \frac{\Phi(H, h)}{\alpha}$ and we should take larger $\alpha$. This explains why the value of $\alpha$ in (26) and (27) is typically taken as 1. We introduce $\alpha$ in the preconditioner just for the convenience to demonstrate the convergence rates of the proposed algorithm in the following section.

8 Numerical experiments

We illustrate the convergence rate of the proposed algorithm by solving the incompressible Stokes problem (1) in both two and three dimensions, on $\Omega = [0, 1]^2$ and $\Omega = [0, 1]^3$, respectively. Zero Dirichlet boundary condition is used. The right-hand side $f$ is chosen such that the exact solution is

$$u = \begin{bmatrix} \sin^3(\pi x) \sin^2(\pi y) \cos(\pi y) \\ -\sin^2(\pi x) \sin^3(\pi y) \cos(\pi x) \end{bmatrix}, \quad p = x^2 - y^2,$$

for two dimensions, and for three dimensions

$$u = \begin{bmatrix} \sin^2(\pi x) (\sin(2\pi y) \sin(\pi z) - \sin(\pi y) \sin(2\pi z)) \\ \sin^2(\pi y) (\sin(2\pi z) \sin(\pi x) - \sin(\pi z) \sin(2\pi x)) \\ \sin^2(\pi z) (\sin(2\pi x) \sin(\pi y) - \sin(\pi x) \sin(2\pi y)) \end{bmatrix}, \quad p = xyz - \frac{1}{8}.$$  

The $Q_2$-$Q_1$ Taylor-Hood mixed finite element with continuous pressures is used; its inf-sup stability can be found at [2, 22]. In two dimensions, the velocity space contains piecewise biquadratic functions and the pressure space contains piecewise bilinear functions; in three dimensions, piecewise triquadratic functions for the velocity and piecewise trilinear functions for the pressure.

The preconditioned conjugate gradient method is used to solve (29); the iteration is stopped when the $L^2$-norm of the residual is reduced by a factor of $10^{-6}$.

The following tables list the minimum and maximum eigenvalues of the iteration matrix $M^{-1}G$, and the iteration counts for using both lumped and Dirichlet preconditioners, respectively, for different cases. Here the extreme eigenvalues of $M^{-1}G$ are estimated by using the tridiagonal Lanczos matrix generated in the iteration.

Table 1 shows the performance for solving the two-dimensional problem. The coarse level velocity space in the algorithm is spanned by the subdomain vertex nodal basis functions corresponding to each velocity component. We take $\alpha = 1$ in both the lumped and the Dirichlet preconditioners (26) and (27). We can see from Table 1 that the minimum eigenvalue is independent of the mesh size for both preconditioners. The maximum eigenvalue is independent of the number of subdomains for fixed $H/h$: for fixed number of subdomains, it depends on $H/h$ in the order of $(H/h)(1 + \log (H/h))$ for the lumped preconditioner, and in the order of $(1 + \log (H/h))^2$ for the Dirichlet preconditioner.
Table 1: Performance of solving two-dimensional problem on $[0, 1]^2$, $\alpha = 1$ in (26) and (27).

| $H/h$  | $\#_{\text{sub}}$ | lumped $\lambda_{\min}$ | lumped $\lambda_{\max}$ | lumped iteration | Dirichlet $\lambda_{\min}$ | Dirichlet $\lambda_{\max}$ | Dirichlet iteration |
|--------|------------------|--------------------------|--------------------------|------------------|---------------------------|--------------------------|-------------------|
| 8      | 4 x 4            | 0.3066                   | 32.28                    | 31               | 0.2983                    | 4.40                     | 18                |
| 8 x 8  | 0.3067           | 37.25                    | 46                       | 5.03             | 24                        |
| 16 x 16 | 0.3068           | 38.42                    | 51                       | 25               |
| 24 x 24 | 0.3069           | 38.62                    | 51                       | 25               |
| 32 x 32 | 0.3070           | 38.68                    | 51                       | 25               |

| $\#_{\text{sub}}$ | $H/h$  | lumped $\lambda_{\min}$ | lumped $\lambda_{\max}$ | lumped iteration | Dirichlet $\lambda_{\min}$ | Dirichlet $\lambda_{\max}$ | Dirichlet iteration |
|------------------|-------|--------------------------|--------------------------|------------------|---------------------------|--------------------------|-------------------|
| 8 x 8            | 4     | 0.3024                   | 15.91                    | 34               | 0.2706                    | 4.15                     | 21                |
| 8               | 0.3067 | 37.25                    | 46                       | 5.03             | 24                        |
| 16              | 0.3069 | 85.32                    | 62                       | 6.04             | 25                        |
| 24              | 0.3073 | 137.49                   | 73                       | 6.99             | 26                        |
| 32              | 0.3075 | 192.32                   | 83                       | 7.19             | 27                        |

Tables 2 and 3 are for solving the three-dimensional problem. The coarse level velocity space is spanned by the subdomain vertex nodal basis functions and subdomain edge-cutoff functions corresponding to each velocity component. This coarse space is the same as for solving scalar elliptic problems in [27, Algorithm 6.25] corresponding to each velocity component. In Table 2, $\alpha = 1$; in Table 3, $\alpha = 1/2$.

In Table 2, the minimum eigenvalue is independent of the mesh size for both preconditioners. The maximum eigenvalue is independent of the number of subdomains for fixed $H/h$; for fixed number of subdomains, it depends on $H/h$, but not in the order of $(H/h)(1 + \log(H/h))$ for the lumped preconditioner, nor $(1 + \log(H/h))^2$ for the Dirichlet preconditioner, as $\Phi(H/h)$ does. Moreover, the convergence rate of the algorithm using the Dirichlet preconditioner is only slightly better than using the lumped preconditioner. The reason is that the upper eigenvalue bound in Theorem 1 depends on two terms $\alpha$ and $\Phi(H/h)$, and in this case $\alpha = 1$ dominates when $H/h$ is small. Therefore, even though using the Dirichlet preconditioner can reduce $\Phi(H/h)$ compared with using the lumped preconditioner, this improvement on the upper eigenvalue bound can not show up in Table 2. What shows in Table 2 for $\lambda_{\max}$ is essentially its dependence on $\alpha$. Only for larger $H/h$, e.g., for $H/h = 6$ and $H/h = 8$ in Table 2, the improvement on the upper eigenvalue bound by using the Dirichlet preconditioner becomes visible.

To experiment the case when $\alpha$ is less dominant in the upper eigenvalue bound, we take $\alpha = 1/2$ in Table 3. Consistent with Theorem 1, the lower eigenvalue bounds in Table 3 become half of those in Table 2 and they are also
Table 2: Performance of solving three-dimensional problem on $[0, 1]^3$, $\alpha = 1$ in (26) and (27).

| $H/h$ | $\#_{\text{sub}}$ | $\lambda_{\text{min}}$ | $\lambda_{\text{max}}$ | $\text{iteration}$ | $\lambda_{\text{min}}$ | $\lambda_{\text{max}}$ | $\text{iteration}$ |
|-------|-----------------|----------------|----------------|-----------------|----------------|----------------|----------------|
| $4$   | $3 \times 3 \times 3$ | 0.0776 | 9.13 | 56 | 0.0776 | 8.97 | 56 |
|       | $4 \times 4 \times 4$ | 0.0775 | 9.35 | 54 | 0.0774 | 9.19 | 55 |
|       | $6 \times 6 \times 6$ | 0.0773 | 9.41 | 58 | 0.0773 | 9.23 | 59 |
|       | $8 \times 8 \times 8$ | 0.0773 | 9.51 | 57 | 0.0772 | 9.34 | 61 |

Table 3: Performance of solving three-dimensional problem on $[0, 1]^3$, $\alpha = 1/2$ in (26) and (27).

| $H/h$ | $\#_{\text{sub}}$ | $\lambda_{\text{min}}$ | $\lambda_{\text{max}}$ | $\text{iteration}$ | $\lambda_{\text{min}}$ | $\lambda_{\text{max}}$ | $\text{iteration}$ |
|-------|-----------------|----------------|----------------|-----------------|----------------|----------------|----------------|
| $4$   | $3 \times 3 \times 3$ | 0.0395 | 7.20 | 59 | 0.0395 | 4.89 | 54 |
|       | $4 \times 4 \times 4$ | 0.0394 | 8.15 | 66 | 0.0394 | 5.01 | 53 |
|       | $6 \times 6 \times 6$ | 0.0393 | 8.85 | 70 | 0.0393 | 5.03 | 55 |
|       | $8 \times 8 \times 8$ | 0.0393 | 9.09 | 72 | 0.0393 | 5.09 | 56 |

independent of the mesh size. The upper eigenvalue bounds exhibit the pattern of $\Phi(H/h)$ for both preconditioners. They are independent of the number of subdomains for fixed $H/h$; for fixed number of subdomains, they depend on $H/h$ in the order of $(H/h)(1 + \log(H/h))$ for the lumped preconditioner, and in the order of $(1 + \log(H/h))^2$ for the Dirichlet preconditioner.
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