I. INTRODUCTION

The first experimental achievement of Bose-Einstein condensation (BEC) in ultracold, dilute atomic gases [1] sparked a flurry of activity in the physics of cold atoms, and great strides have since been made in manufacturing and manipulating such condensates. The studies were extended to include the effect of spin-orbit coupling (SOC) when a synthetic SOC was demonstrated experimentally [2]. Due to the Doppler effect, lasers can induce momentum dependent transitions between two pseudospin states, emulating the SOC of spin-1/2 particles. The methods have since been refined, and highly tunable synthetic SOC with different linear combinations of Rashba [3] and Dresselhaus [4] SOC have been achieved experimentally [5–10]. The introduction of SOC to the ultracold gas has many interesting consequences including the lack of Galilean invariance [11] and hence a frame dependent superfluid velocity [12]. This greatly complicates the theoretical treatment of such condensates.

It is also possible to load the atoms onto an optical lattice, since lasers can generate a periodic potential landscape [13]. In that case, the highly tunable experimental setup can be used to simulate numerous condensed matter physics phenomena under completely controlled conditions. Examples where SOC plays an important role are the quantum spin Hall effect and topological insulators [14]. Furthermore, the controllability of atoms trapped in optical lattices means they could find applications in quantum computing [15].

In this paper, we consider a two-dimensional (2D), Rashba SOC, weakly interacting BEC in the presence of a square optical lattice and an external Zeeman field. An important consequence of the SOC is the presence of phases with nonzero condensate momenta, some of which can be viewed as bosonic analogues of Fulde-Ferrell-Larkin-Ovchinnikov states in superconductors [16–18]. The Fulde-Ferrell analogous plane wave (PW) phase with one nonzero condensate momentum was treated in [19] by projection on the lowest single particle band. In this paper we will further explore the Larkin-Ovchinnikov analogous stripe wave (SW) phase with two oppositely directed condensate momenta. This phase has previously been studied in a continuum [20], and was later observed experimentally [21], but its excitation spectrum has not been obtained in the presence of a lattice. The excitation spectra in the two phases are found by the same method used in [19] projecting down on the lowest energy excitations, as well as without any projection. It is found that the excitations in the SW phase have zero sound velocity, unlike the nonzero sound velocity found in [19] for the PW phase. In addition, the method used in [19] is found to be an excellent approximation in the PW phase, while it fails at almost all parameters in the SW phase. The origin of these results will be discussed.

II. BOGOLIUBOV THEORY

We start with a Bose-Hubbard Hamiltonian for a Bose gas with two atomic species akin to that introduced in [22], and include a Rashba SOC discretized to a lattice formulation

\[
H = -\sum_{\alpha} t^\alpha \sum_{\langle i,j \rangle} b_{i}^\alpha d_{j}^\alpha - \sum_{\alpha} \mu^\alpha \sum_{i} b_{i}^\alpha d_{i}^\alpha - i\lambda_R \sum_{\alpha\beta\langle i,n \rangle} \left( b_{i}^{\alpha \dagger} \cdot (\sigma^{\alpha \beta} \cdot \hat{a}_{n}) b_{i+n}^{\beta} - \text{H.c.} \right) \\
+ \frac{1}{2} \sum_{\alpha\beta} U^{\alpha \beta} \sum_{i} b_{i}^{\alpha \dagger} b_{i}^{\beta \dagger} b_{i}^{\beta} b_{i}^{\alpha}.
\]  

(1)
Here, \( b_i^\dagger \) annihilates a boson of species \( \alpha \) at the lattice site \( i \), \( t^\alpha \) is a species dependent nearest neighbor hopping parameter, \( \mu^\alpha \) is a species dependent chemical potential, \( \lambda_R \) is the strength of the Rashba SOC, \( \sigma \) is a vector containing the Pauli matrices, \( a_n \) are the \( d \) primitive vectors of a \( d \)-dimensional Bravais lattice, and \( \hats \) denote unit vectors. The two atomic species are labeled pseudospin up and pseudospin down, while \( \hats \) indicates the Hermitian conjugate of the preceding term. The interactions are assumed to be repulsive and \( U^{\alpha\beta} \) is the interaction parameter for an on-site two-body scattering involving the atomic species \( \alpha \) and \( \beta \).

It is advantageous to consider the system in momentum space, since BEC is associated with the particles’ momentum distribution. The bosonic operators are Fourier transformed using \( b_i^\dagger = (1/\sqrt{N_s}) \sum_k A_k^\alpha e^{-ik \cdot \tau_i} \), where \( N_s \) is the number of lattice sites, \( A_k^\alpha \) is a bosonic operator annihilating a boson of particle species \( \alpha \) with momentum \( k \), and \( \tau_i \) is the position of lattice site \( i \). In momentum space, the Hamiltonian becomes

\[
H = \sum_{k} \sum_{\alpha \beta} \eta_{k} A_k^\dagger A_k^\alpha A_k^\beta + \frac{1}{2N_s} \sum_{kk'pp'} \sum_{\alpha \beta} U^{\alpha\beta} A_k^\dagger A_{k'}^\beta A_p^\dagger A_{p'}^\alpha \delta_{k+k',p+p'},
\]

where

\[
\eta_{k} = \left( \epsilon_k^\alpha - \mu^\alpha \right) s_k
\]

\[
\epsilon_k^\alpha \equiv -2t^\alpha \sum_{n=1}^{d} \cos(\mathbf{k} \cdot \mathbf{a}_n)
\]

and the Rashba SOC term is

\[
s_k \equiv -2\lambda_R \sum_{n=1}^{d} (\hat{a}_n \cdot \hat{y} + i \hat{a}_n \cdot \hat{x}) \sin(\mathbf{k} \cdot \mathbf{a}_n).
\]

A. Mean-Field Theory

We assume the temperature is low enough for BEC to occur, such that the condensate is dominant. We will set the temperature to zero in the calculations, and consider quantum fluctuations of the ground state. It is assumed that there are few excitations, and terms in the Hamiltonian involving a product of three or more excitation operators are therefore neglected. \( A_k^\alpha \) is named a condensate operator if \( k \) is any occupied condensate momentum, while \( A_k^\alpha \) is an excitation operator given that \( k \neq k_0 \). The momentum configurations in the interaction terms that include at most two excitation momenta are represented in Table I. The cases 2-5 lead to terms that are linear in excitation operators, and originate from the fact that the momentum conservation may be obeyed by three condensate momenta and one excitation momentum. This possibility, requiring multiple condensate momenta in the system, was first elucidated by Jansson [23] and has to our knowledge not been explored previously. Inserting these momentum configurations yields

\[
H \approx H_0 + H_1 + H_2,
\]

where

\[
H_0 = \sum_{i} \sum_{\alpha \beta} \eta_{k_i} A_{k_i}^\dagger A_{k_i}^\alpha A_{k_i}^\beta
\]

\[
+ \frac{1}{2N_s} \sum_{ij} \sum_{\alpha \beta} U^{\alpha\beta} A_{k_i}^\dagger A_{k_j}^\beta A_{k_0}^\dagger A_{k_0}^\alpha \delta_{k_i+k_j,k_0+k_0},
\]

\[
H_1 = \frac{1}{N_s} \sum_{k} \sum_{ij} \sum_{\alpha \beta} U^{\alpha\beta} \left( A_{k_i}^\dagger A_{k_j}^\dagger A_{k_0}^\dagger A_{k_0}^\alpha \delta_{k+k_0,k_0+k_0} + A_{k_i}^\dagger A_{k_j}^\dagger A_{k_0}^\alpha \right) \delta_{k+k_0,k_0+k_0+k_0},
\]

and

\[
H_2 = \sum_{k} \sum_{ij} \sum_{\alpha \beta} \eta_{k} A_{k_i}^\dagger A_{k_j}^\beta A_{k_0}^\dagger A_{k_0}^\beta
\]

\[
+ \frac{1}{2N_s} \sum_{kk'} \sum_{ij} \sum_{\alpha \beta} U^{\alpha\beta} \left( A_{k_i}^\dagger A_{k_j}^\dagger A_{k_0}^\dagger A_{k_0}^\alpha \delta_{k+k_0,k_0+k_0+k_0} + 2 \left( A_{k_i}^\dagger A_{k_i}^\dagger A_{k_0}^\alpha \right) \delta_{k+k_0+k_0,k_0+k_0,k_0} \right)
\]

The primes indicate that the sums exclude any occupied condensate momenta.

| Case | k | k_0 | k_0 | k_0 | k | k | k | k | k |
|------|---|-----|-----|-----|---|---|---|---|---|
|      | k' | k_0 | k_0 | k_0 | k' | k' | k' | k' | k' |
|      | p | p_0 | p_0 | p_0 | p | p | p | p | p |
|      | p' | p_0 | p_0 | p_0 | p' | p' | p' | p' | p' |

From now on, we specialize to a square 2D lattice and assume that \( t^\dagger = t^\dagger \equiv t, \mu^\dagger \equiv \mu + \Omega, \mu^\dagger = \mu - \Omega, U^\dagger = U^\dagger \equiv U \) and \( U^\dagger = U^\dagger \equiv U \alpha \). We let the chemical potential \( \mu \) control the total number of particles \( N \), while the external Zeeman field \( \Omega \) controls the pseudospin imbalance, i.e. the values of \( N^\alpha \), where \( N^\alpha \) is the total number of particles with pseudospin \( \alpha \). With the lattice constant \( a \) set to 1, we have \( \epsilon_k = -2t(\cos k_x + \cos k_y) \) and \( s_k = -2\lambda_R(\sin k_y + i \sin k_x) \).
To gain some insight into the SOC BEC, we first discuss the single particle problem, i.e. no interactions. The single particle excitation spectrum is given by the eigenvalues of \( \eta_k \) which are \( \lambda^\pm_k = \epsilon_k - \mu \pm \sqrt{\Omega^2 + |s_k|^2} \). We will refer to these as the upper and lower helicity bands. Let \( \Omega \equiv 2\lambda^+_k/t \). For \( \Omega > \Omega_c \), \( \lambda^-_k \) has only one minimum at \( (0, 0) \). For \( \Omega < \Omega_c \), it has four minima at \( k_{01} = (k_0, k_0), k_{02} = (-k_0, k_0), k_{03} = (-k_0, -k_0) \) and \( k_{04} = (k_0, -k_0) \) with \( k_0 = k_{0m} \).

\[
k_{0m} = \arcsin \sqrt{(1 - (\Omega/\Omega_c)^2)/(1 + 2(t/\lambda_R)^2)}.
\]

The same was found in [19] where both \( \Omega > \Omega_c \) and \( \Omega < \Omega_c \) were considered for \( \alpha < 1 \). In this paper we will include \( \alpha \geq 1 \) and focus on \( \Omega < \Omega_c \).

We need to diagonalize the Hamiltonian (6) in order to obtain the quasiparticle excitation spectrum, and we will consider two methods of obtaining it. One way is to employ the method used in [19] which involves projecting down on the lowest helicity band. The argument for the validity of the helicity projection is that we are considering a BEC at zero temperature, and so, before introducing interactions, almost no helicity quasiparticles should occupy the upper helicity band. The other method will be to treat the system in the original (pseudo)spin basis, which is equivalent to keeping both helicity bands.

Using the spin basis and following the Bogoliubov approach [24, 25], we insert

\[
A^\alpha_{k0i} \to \sqrt{N^\alpha_{0i}} e^{-i\theta^\alpha_{0i}},
\]

where \( N^\alpha_{0i} \equiv \langle A^\dagger_{k0i} A^\alpha_{k0i} \rangle \gg 1 \) is the number of condensate particles with momentum \( k_{0i} \) and pseudospin \( \alpha \). The angle \( \theta^\alpha_{0i} \) is a variational parameter that can be determined by minimizing the free energy [26]. It was found that these angles are important in the phases under consideration in this paper.

We define the helicity operators \( C^+_k \) and \( C^-_k \), which annihilate bosons in the upper and lower helicity bands. These are connected to the spin operators \( A^\alpha_k \) through a unitary matrix containing the eigenvectors of \( \eta_k \). The eigenvector for the lowest helicity band contains the transformation coefficients

\[
u_k = \sqrt{\left(1 + \Omega/\sqrt{\Omega^2 + 4\lambda^+_k(s_k^2k_x + s_k^2k_y)}\right)/2}
\]

and \( v_k = e^{-i\theta_k} \sqrt{1 - \left|\nu_k\right|^2} \) with \( e^{-i\theta_k} \equiv s_k/|s_k| \). The helicity projection involves setting \( C^+_k \equiv C^+_ke^0 \) and \( C^-_k \equiv 0 \). Then we have \( A^+_k = u_k C_k \) and \( A^-_k = v_k C_k \). We transform the Hamiltonian before we use (11), and instead insert \( C_{0i} \to \sqrt{N_{0i}} e^{-i\theta_i} \), where \( N_{0i} \) is the total number of condensate particles with momentum \( k_{0i} \). In the helicity projection, we found that the free energy is independent of the angles \( \theta_i \), and they are therefore set to zero for brevity.

\section{Phases}

Without interactions, most of the helicity quasiparticles should occupy the minima of \( \lambda^-_k \). It is expected that introducing weak interactions will designate certain momenta as the ground state [27], and that a Bogoliubov effect appears such that the condensate momenta become phonon minima of the excitation spectrum, similar to the treatment of the weakly interacting Bose gas [13, 28, 29].

As is often done [19, 27], we will use the operator independent part, \( H_0 \), of the Hamiltonian to determine the possible phases. We are thus assuming that the free energy \( F \approx H_0 \), and the phase with the lowest free energy at a certain set of parameters will be the preferred phase. With a nonzero SOC and \( \Omega < \Omega_c \) the two most interesting phases are the plane and stripe wave phases, named according to the wave patterns they produce in real space, and characterized by

- Plane Wave (PW) Phase: The PW phase involves a single nonzero condensate momentum, chosen to be \( k_{01} \) without loss of generality.
- Stripe Wave (SW) Phase: The SW phase involves condensation at two oppositely directed, nonzero momenta chosen as \( k_{01} \) and \( k_{03} = -k_{01} \).

When \( \Omega = 0 \) the PWSW transition occurs at \( \alpha = 1 \). For \( 0 < \Omega < \Omega_c \), the transition occurs at [19]

\[
\Omega/\Omega_c = \sqrt{(\alpha - 1)/(\alpha + 1 + (\lambda_R/t)^2)}.
\]

This analytic expression was found using the operator independent part of the Hamiltonian after projecting onto the lowest helicity band, and is found to be an adequate approximation. The SW phase is preferred for \( \Omega \) less than the value given above and its excitation spectrum was not treated in [19]. A plot of this transition line is shown in figure 1. See figure 1 in [19] for an \( \Omega - \alpha \) phase diagram based on \( H_0 \).

The obtained PWSW transition at \( \alpha = 1 \) when \( \Omega = 0 \) was also found in [30]. As further elaborated in [10, 27] the wave function in the PW phase gives a uniform density of both pseudospin components, while in the SW phase both components have a periodic, striped density variation with opposite phase. Since this minimizes the overlap of the two components, the SW phase is preferred when the intercomponent interactions are stronger than the intracomponent interactions. Upon introducing a Zeeman field the system obtains a pseudospin imbalance. Hence, minimizing the overlap of the two components becomes less effective, and a higher value of \( \alpha \) is required before the PW phase is energetically favorable.

\section{Generalized Diagonalization Method}

Since the system is bosonic and the Hamiltonian contains terms that individually do not conserve the particle number, diagonalizing the Hamiltonian (6) must be
done by a generalization of the Bogoliubov-Valatin (BV) transformation introduced in [31] with further details in [32]. A unitary transformation is in general not sufficient for a bosonic system as there is no guarantee the quasiparticles will be bosonic. To circumvent this, the BV transformation introduces a matrix $J$ which is quadratic in bosonic operators, written in [19]. The quadratic part of the Hamiltonian can be done by a generalization of the Bogoliubov-Valatin (BV) transformation introduced in [31] with further details in [32]. A unitary transformation is in general not sufficient for a bosonic system as there is no guarantee the quasiparticles will be bosonic. To circumvent this, the BV transformation introduces a matrix $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, where $I$ is the identity matrix. If $M$ is the matrix in a Hamiltonian which is quadratic in bosonic operators, then the BV transformation involves diagonalizing $MJ$. Complex eigenvalues of $MJ$ are interpreted as dynamic instabilities of the system [13, 28].

III. PLANE WAVE PHASE

The PW phase is treated by the helicity projection in [19]. The quadratic part of the Hamiltonian can be written

$$H_2 = \frac{1}{4} \sum_{k \neq k_{01}} C_k^\dagger N_k C_k,$$

(13)

where, with $p = 2k_{01} - k$,

$$C_k = (C_k, C_p, C_k^\dagger, C_p^\dagger)^T,$$

(14)

and

$$N_k = \begin{pmatrix} N_{11}(k) & 0 & 0 & N_{32}(k) \\ 0 & N_{11}(p) & N_{32}(k) & 0 \\ 0 & N_{32}(k) & N_{11}(k) & 0 \\ N_{32}(k) & 0 & 0 & N_{11}(p) \end{pmatrix}.$$

(15)

The matrix elements are

$$N_{11}(k) = \lambda_k - \lambda_{k_{01}} + U_n \left[ 2u_{k_{01}}^2u_{k_{01}}^2 + 2|v_k|^2|v_{k_{01}}|^2 ight. - \left. u_{k_{01}}^4 - |v_{k_{01}}|^4 \right] + U \alpha n \left[ u_k^2 |v_{k_{01}}|^2 \right]$$

$$+ u_{k_{01}}^2 \left( |v_k|^2 - 2|v_{k_{01}}|^2 \right)$$

$$+ 2u_k u_{k_{01}} \text{Re}(v_k v_{k_{01}}^\ast),$$

$$N_{32}(k) = U n \left( u_{k_{01}}^2 u_k v_p + u_{k_{01}}^2 v_k v_p \right)$$

$$+ U \alpha n u_{k_{01}} v_{k_{01}}^\ast \left( u_p v_k + u_p v_k \right),$$

(16)

where $n = N/N_s$. The eigenvalues of $N_k J$ are

$$E_H(k) = \frac{1}{2} \left( N_{11}(k) - N_{11}(p) \right)$$

$$+ \left( \sqrt{(N_{11}(k) + N_{11}(p))^2 - 4|N_{32}(k)|^2} \right),$$

(17)

and its inverse about $k_{01}$. This agrees with the result obtained in [19]. Using this inversion symmetry, it is possible to write the diagonalized Hamiltonian as [33]

$$H_2 = \sum_{k \neq k_{01}} E_H(k) \left( B_k^\dagger B_k + \frac{1}{2} \right).$$

(18)

As discussed in [19] this energy band has a phonon minimum at the condensate momentum, $k_{01}$, and gapped roton minima close to the other minima of the single particle excitation spectrum. This is illustrated in the insets of figure 2. When approaching the PWSW transition line (12) from above, the roton minimum close to $k_{03}$ goes to zero, and eventually becomes negative, indicating an energetic instability [13].

Treating the PW phase in the spin basis requires a numerical solution for the eigenvalues of an $8 \times 8$ matrix. The method follows the same course as the SW phase, to be presented later, and is therefore omitted here. The lowest band, $E_2(k)$, is almost equal to the eigenvalue $E_H(k)$ in the helicity projection at all $k$, while the upper band, $E_1(k)$, is similar to the upper helicity band $\lambda_k^+$. Both bands are shown in figure 2. Using $E_H(k)$ we can find an analytic expression for the anisotropic sound velocity of the excitations close to $k_{01}$ [19]. The numerical sound velocity from the spin basis corresponds to this analytic result, even without any Zeeman field and at weak SOC. Hence, the helicity projection provides a good approximation for the PW phase at all parameters of interest, even though it is expected to be a better approximation at strong SOC and with a Zeeman field $\Omega > \max(U, \alpha U)$ [19]. The latter requirement is intended to reduce interband scatterings between the helicity bands. Apparently, the interband scatterings are not relevant for the speed of sound of the phonon excitations in the PW phase.

Figure 1. The PWSW transition line for $U/t = 0.1$, $n = 1$ and $\lambda_R/t = 1.0, 3.0$ and 10.
expression for $H_0$ is

$$H_0 = (\epsilon_{k_{01}} - \mu)N + \Omega N(1 - 2x) + N\sqrt{x(1 - x)}|s_{k_{01}}| \sum_{i=1,3} \cos(\gamma_{k_{01}} + \Delta \theta_i)$$

$$+ \frac{U N^2}{4N_s} \left(3x^2 + 3(1 - x)^2 + 2\alpha x(1 - x)(2 + \cos(\Delta \theta_1 - \Delta \theta_3))\right),$$

where $\Delta \theta_i \equiv \theta_i^+ - \theta_i^-$. We write $H_2$ as

$$H_2 = \frac{1}{4} \sum_k A_k^\dagger M_k A_k.$$

Introducing $p_{\pm} = k \pm 2k_{01}$ and $q_{\pm} = -k \pm 2k_{01}$, the operator vector is defined by

$$A_k^\dagger = (A_k^\dagger, A_{-k}^\dagger, A_p^\dagger, A_q^\dagger, A_{-p}^\dagger, A_{-q}^\dagger, A_k^\dagger, A_{-k}^\dagger, A_p^\dagger, A_q^\dagger, A_{-p}^\dagger, A_{-q}^\dagger, A_k^\dagger, A_{-k}^\dagger, A_p^\dagger, A_q^\dagger, A_{-p}^\dagger, A_{-q}^\dagger),$$

and $M_k$ is a $24 \times 24$ matrix on the form

$$M_k = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix},$$

The matrix elements are presented in appendix A. They are obtainable from the expression (9), by using commutators and making $-k$ terms explicit, a procedure that produces some additional operator independent terms in the Hamiltonian, relevant for a calculation of the free energy. More details are found in appendix C.

The 24 eigenvalues of $M_k J$ are equally distributed around 0 [31, 32]. 8 eigenvalues are within numerical accuracy 0, while the remaining eigenvalues are doubly degenerate, upon inserting the values of the variational parameters which minimize the free energy. The 2 lowest positive, doubly degenerate eigenvalues have anomalous modes [13], and therefore enter the diagonalized Hamiltonian with a negative sign [32]. By moving the chemical potential controlling the quasiparticles to just below the (negative) minimum of the excitation spectrum, $-E_0$, Bose-Einstein statistics ensure that the majority of the quasiparticles will occupy the minima of the lowest band. Since we prefer to have only positive energy bands, we move the zero of energy by $E_0$. The final diagonalized Hamiltonian reads [33]

$$H_2 = -E_0 N_\alpha + \sum_k \sum_{\sigma=1}^6 \Delta E_\sigma(k) \left(B_{k,\sigma}^\dagger B_{k,\sigma} + \frac{1}{2}\right).$$
Figure 3. The energy bands in the SW phase in the direction $k_x = k_y$. The dashed vertical lines show the positions of $k = \pm k_{01}$. The parameters were set to $U/t = 0.1$, $n = 1$, $\lambda_R/t = 10$, $\alpha = 1.5$ and $\Omega/t \approx 12.57$. This value of $\Omega$ corresponds to 0.9 times the PWSW transition line (12).

where the quantity $N_q \equiv \sum_k \sum_{\sigma=1}^{\delta} (B_{k,\sigma}^\dagger B_{k,\sigma} + 1/2)$ was defined to simplify the expression, and $\Delta$ is used to indicate that the energies have been shifted by $E_0$. The energies are ordered such that $\Delta E_i(k) \geq \Delta E_j(k)$ if $j \geq i$ and are shown in figure 3. Since there are only two degrees of freedom originally, pseudospin up and pseudospin down, the four highest excitation energies will be considered unoccupied. The lowest energy band, $\Delta E_0(k)$, is the most interesting band in the context of BEC and is shown in figure 4. It has its global minima at the condensate momenta, and gapped roton minima at the unoccupied minima of the single particle spectrum. Note the highly unusual feature that, unlike the typical results when introducing interactions, the minima at the condensate momenta show a non-linear behavior. These quadratic minima indicate that the excitations in the SW phase have zero sound velocity. This SW phase excitation spectrum, found numerically in the spin basis, is the main result of this article. In the next subsection we consider the helicity projection, and find that it is a poor approximation in the SW phase. The treatment is however useful, since it provides a way to explain the quadratic behavior found in the SW phase.

B. Helicity Projection

Obtaining the quasiparticle excitation spectrum utilizing the helicity projection, follows the same course as in the spin basis, the difference being the projection onto the lowest helicity band. This reduces the number of components of the basis to 12. The matrix is presented in appendix B.

The excitation energies are, at first glance, similar to the energies $\Delta E_{2i}$, $i = 1, 2, 3$ in the spin basis. We denote them $\Delta E_{2i}^\sigma$, $\sigma = 1, 2, 3$. The lowest band $\Delta E_3^\sigma(k)$ however, has some properties that separates it from the spin basis result $\Delta E_3(k)$. At zero Zeeman field and $\lambda_R/t < \sqrt{6}$, the minima at the condensate momenta show a linear behavior, contrary to the result in the spin basis, but more in accord with the intuition one would have based on a standard single-momentum condensate. However, for $\lambda_R/t > \sqrt{6}$ and $\Omega = 0$ or any $\lambda_R$ with nonzero $\Omega$, a quadratic behavior is found.

The value $\lambda_R/t = \sqrt{6}$ corresponds to $k_0 = \pi/3$. This is the point where $k + 2k_{01}$ at $k = k_{01}$ goes beyond the first Brillouin zone (1BZ). The term $s_k/k_{01}$ involved in the transformation to the helicity basis has discontinuities when $k$ crosses the boundary of the 1BZ. For $\lambda_R/t < \sqrt{6}$ and $\Omega = 0$ certain matrix elements are zero around $k_{01}$ or $-k_{01}$, while they become nonzero when $\lambda_R/t > \sqrt{6}$. This appears to be the root cause for why the linear behavior of the excitation spectrum is replaced by quadratic behavior.

The absolute square of these matrix elements represent the transition rates of the scatterings (c), (d), (e) and (f) in figure 5. There is no reason why the transition rates of these scatterings should be zero, something which
is supported by the fact that they are nonzero in the original spin basis. The conclusion is that the helicity projection fails for weak SOC and zero Zeeman field due to the discontinuities in the transformation to the helicity basis with zero Zeeman field. Furthermore, the helicity projection should be a better approximation at stronger SOC, where it too shows quadratic behavior even without a Zeeman field. However, it is found that when $\lambda_R/t > \sqrt{6}$, the global minima of the excitation spectrum occur at $\pm k_{02}$, instead of at $\pm k_{01}$ as in the spin basis. Since the spin basis is more accurate than the helicity projection, we conclude that the helicity projection fails in the SW phase at almost all parameters, the possible exception being for $\lambda_R/t < \sqrt{6}$ and $\Omega > 0$.

The main reason why the discontinuities in the transformation to the helicity basis when $\Omega = 0$ have such a large influence on the SW phase, but apparently no influence on the PW phase, is the presence of two condensate momenta. In the PW phase, the momentum indices of the operators are $k$ and $p = 2k_{01} - k$ only. At the condensate momentum, $p = k_{01}$ and there are no problems with this crossing the boundary of the 1BZ since the condensate momentum is kept in the 1BZ by definition. On the other hand, the presence of two condensate momenta enables more scattering processes such that e.g. $p_+ = k + 2k_{01}$ becomes one of the momentum indices in the operators. Hence, it is possible for the discontinuities of the transformation to the helicity basis to directly influence the excitation spectrum close to the condensate momenta.

The remaining question is why the excitation spectrum in the SW phase shows quadratic behavior close to its minima, contrary to the usual Bogoliubov result when introducing interactions. The simplest explanation is that it is a consequence of the presence of more than one condensate momentum, a situation which has no counterpart in the standard treatments of such interacting condensates. The presence of two condensate momenta is the reason for the large basis, and the number of nonzero matrix elements. Furthermore, removing a certain set of these matrix elements is required to obtain a linear result.

### C. Stability

With $\Omega = 0$ it is found that the SW phase is stable when $\alpha > 1$ [33]. Introducing a Zeeman field, we find that on approaching the PWSW transition line (12) from below, the excitation spectrum in the SW phase becomes complex, indicating a dynamical instability. Like the energetic instability of the PW phase, this occurs very close to the PWSW transition line. Hence, there is a small area close to this line where neither phase is stable. It may be of interest to study which phase the system will enter in this area. The main candidate is the lattice wave (LW) phase involving all four condensate momenta [23, 33], which did not enter the phase diagram when neglecting excitations [19]. This paper will not explore this further.

When $\lambda_R/t = 1.0$ and $\alpha > 2$ we find that energetic instabilities develop for $\Omega$ around approximately half the PWSW transition line and beyond $\sim 0.9$ of the PWSW transition line. A greater set of values for $\Omega$ is affected by these instabilities when $\alpha$ is increased. These energetic instabilities are characterized by a distance between the minima of the excitation spectrum and the condensate momenta considered to be so large that these no longer correspond to the same lattice sites. If the minima of the excitation spectrum are not located at the condensate momenta, then the initial assumption that the system condenses at $\pm k_{01}$ is invalid. Once again, this paper will not explore the system in the region where neither the PW nor the SW phase is stable. We note that upon choosing $\lambda_R/t = 10$, these energetic instabilities disappear inside the region where the SW phase is already dynamically unstable. One can understand this behavior by considering figure 1 showing the PWSW transition line at different SOC strengths. We notice that when the strength of SOC increases, the maximum value of $\Omega/\Omega_c$ found in the SW phase decreases. Hence, a value of $\Omega$ close to the PWSW transition line when $\lambda_R/t = 10$ represents a significantly smaller $\Omega/\Omega_c$ than when $\lambda_R/t = 1.0$.

A calculation of the ground state depletion for a weakly interacting Bose gas can be found in e.g. [28]. The calculation here is completely analogous, except now we must use the numerically constructed BV transformation matrix [31–33]. At zero temperature and for parameters where the SW phase is stable, we find that $(N - N_0)/N$ is lower than 1% when $U/t = 0.1$, confirming the validity of the mean-field theory.

### V. CONCLUSION

We have explored the plane and stripe wave phases of a weakly interacting SOC BEC on a square lattice in the presence of a Zeeman field. It was found that while the helicity projection provides an excellent approximation for the PW phase with only one condensate momentum, it fails to describe the SW phase which has two condensate momenta. While the PW phase has a phonon minimum at its condensate momentum showing a nonzero, anisotropic speed of sound, the minima in the SW phase excitation spectrum show quadratic behavior and hence zero sound velocity. At strong SOC, the phase diagram based on the operator independent part of the Hamiltonian provides a good description of the system. The PW phase develops an energetic instability close to the PWSW transition line, while the SW phase becomes dynamically unstable when approaching the PWSW transition line.
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Appendix A: SW Phase Matrix in Spin Basis

Due to the form of the matrix $M_k$ (22) together with the fact that $M_1$ is hermitian and $M_2$ is symmetric [31, 32] it is enough to specify rows 1, 2, 7 and 8 of $M_1$ and $M_2$. The rest of the matrix may then be filled, and any unspecified elements are 0. With the values of the variational parameters found to minimize the free energy inserted, we have

$$
M_{1,\text{row}1} = (M_{1,1}(k), 0, M_{1,3}, 0, M_{1,13}^*, 0, \\
    s_k, 0, M_{1,9}, 0, -i M_{1,9}^*), \\
M_{1,\text{row}2} = (0, M_{1,1}(k), 0, M_{1,3}, 0, M_{1,13}^*, \\
    0, -s_k, 0, M_{1,9}, 0, -i M_{1,9}^*), \\
M_{1,\text{row}7} = (s_k^*, 0, i M_{1,9}, 0, M_{1,9}^*, 0, \\
    M_{7,7}(k), 0, M_{7,9}, 0, M_{7,9}^*), \\
M_{1,\text{row}8} = (0, -s_k^*, 0, i M_{1,9}, 0, M_{1,9}^*, \\
    0, M_{7,7}(k), 0, M_{7,9}, 0, M_{7,9}^*),
$$

\hspace{1cm} (A1)

The matrix elements in $M_1$ are

$$
M_{1,1}(k) = c_k + \frac{U_n}{2} (x + (1 - x)\alpha) + G_{k_0}^\dagger, \\
G_{k_0}^\dagger = 4t \cos k_0 + |s_{k_0}| \sqrt{\frac{1 - x}{x}}, \\
M_{1,7}(k) = c_k + \frac{U_n}{2} (1 - x + x\alpha) + G_{k_0}^\dagger, \\
G_{k_0}^\dagger = 4t \cos k_0 + |s_{k_0}| \sqrt{\frac{x}{1 - x}}, \\
M_{1,3} = \frac{U_n}{4} e^{i(\theta_1^\dagger - \theta_3^\dagger) (2x - (1 - x)\alpha)}, \\
M_{1,9} = \frac{U_n}{4} \sqrt{x(1 - x)} e^{i(\theta_1^\dagger - \theta_3^\dagger)}, \\
M_{7,9} = \frac{U_n}{4} e^{i(\theta_1^\dagger - \theta_3^\dagger) (-2(1 - x) + x\alpha)},
$$

while the elements in $M_2^*$ are

$$
M_{13,2} = U_n e^{i(\theta_1^\dagger + \theta_3^\dagger)}, \\
M_{13,4} = \frac{U_n}{4} x e^{i 2\theta_1^\dagger}, \\
M_{13,6} = \frac{U_n}{4} x e^{i 2\theta_3^\dagger}, \\
M_{13,10} = \frac{U_n}{4} \sqrt{x(1 - x)} e^{i(\theta_1^\dagger + \theta_3^\dagger)}, \\
M_{13,12} = \frac{U_n}{4} \sqrt{x(1 - x)} e^{i(\theta_1^\dagger + \theta_3^\dagger)}, \\
M_{19,8} = U_n (1 - x) e^{i(\theta_1^\dagger + \theta_3^\dagger)}, \\
M_{19,10} = \frac{U_n}{4} (1 - x) e^{i 2\theta_1^\dagger}, \\
M_{19,12} = \frac{U_n}{4} (1 - x) e^{i 2\theta_3^\dagger}.
$$

The angles are left unspecified in the elements where inserting them would not lead to simplifications. We mentioned that setting a certain subset of these matrix elements to zero leads to a linear behavior close to the minima of the excitation spectrum. One choice is $M_{1,3} = M_{7,9} = M_{1,9} = M_{13,2} = M_{13,6} = M_{13,12} = M_{19,8} = M_{19,12} = 0$. These are connected to the scatterings (c), (d), (f) and (g) in figure 5, i.e. mostly the scatterings involving both condensate momenta, supporting the claim that the quadratic behavior in the SW phase is caused by the presence of more than one condensate momentum.

Appendix B: SW Phase Matrix in Helicity Projection

The operator vector is

$$
C_k = (C_k, C_{-k}, C_{p_+}, C_{q_+}, C_{p_-}, C_{q_-}, \\
    C_k^\dagger, C_{-k}^\dagger, C_{p_+}^\dagger, C_{q_+}^\dagger, C_{p_-}^\dagger, C_{q_-}^\dagger)^T.
$$

\hspace{1cm} (B1)
The 12 × 12 matrix $M^H_k$ is of the form (22). It is enough to specify rows 1 and 2 of $M^H_1$ and $M^H_2$.

\[
\begin{align*}
M^H_{1,\text{row1}} &= (M_{11}(k), 0, M_{13}(k), 0, M_{15}(k), 0), \\
M^H_{1,\text{row2}} &= (0, M_{11}(-k), 0, M_{13}(-k), 0, M_{15}(-k)), \\
M^H_{2,\text{row1}} &= (0, M_{72}(k), 0, M_{74}(k), 0, M_{76}(k)), \\
M^H_{2,\text{row2}} &= (M_{72}(k), 0, M_{74}(-k), 0, M_{76}(-k), 0).
\end{align*}
\]

The matrix elements are defined as follows

\[
M_{11}(k) = \lambda_k - \lambda_{\bar{k},1},
\]

\[
\begin{align*}
&- \frac{U_n}{2} (3 u_{k_{01}}^4 + 3|v_{k_{01}}|^4 + 2\alpha v_{k_{01}}^2|v_{k_{01}}|^2) \\
&+ U_n (2u_{k_{01}}^2 u_{k_{01}}^* v_{k_{01}} + 2|v_{k_{01}}|^2|v_{k_{01}}|^2) \\
&+ \alpha v_{k_{01}}^2 |v_{k_{01}}|^2 + \alpha |v_{k_{01}}|^2 u_{k_{01}}^2,
\end{align*}
\]

\[
M_{13}(k) = \frac{U_n}{4} (2u_{k_{01}}^2 u_{k_{01}} v_{k_{01}} + 2|v_{k_{01}}|^2 v_{k_{01}}^* v_{k_{01}}) \\
- \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} + \alpha v_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}},
\]

\[
M_{15}(k) = \frac{U_n}{4} (2u_{k_{01}}^2 u_{k_{01}} v_{k_{01}} - 2|v_{k_{01}}|^2 v_{k_{01}}^* v_{k_{01}}) \\
+ \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} - \alpha v_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}},
\]

\[
M_{72}(k) = \frac{U_n}{2} (u_{k_{01}}^2 + (v_{k_{01}}^* )^2 v_{k_{01}}^* v_{k_{01}}),
\]

\[
M_{74}(k) = \frac{U_n}{4} (u_{k_{01}}^2 u_{k_{01}} v_{k_{01}} + (v_{k_{01}}^* )^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}}),
\]

\[
M_{76}(k) = \frac{U_n}{4} (u_{k_{01}}^2 u_{k_{01}} v_{k_{01}} + (v_{k_{01}}^* )^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}} + \alpha u_{k_{01}}^2 v_{k_{01}}^* v_{k_{01}}).
\]

When $\Omega = 0$ and $\lambda_R/t < \sqrt{3}$, $M_{13}(k) = 0$ and $M_{76}(k) = 0$ around $k_{01}$, while $M_{15}(k) = 0$ and $M_{74}(k) = 0$ around $-k_{01}$. Such cancellations are considered erroneous upon comparison with the spin basis.

**Appendix C: Free Energy**

In this appendix we will give an overview of the methods involved in calculating the free energy in the SW phase, and hence determining the values of the variational parameters. The use of commutators when setting up the matrix $M_k$ gives a shift $-\sum_k (M_{1,1}(k) + M_{7,7}(k))/2$ of the operator independent part of the Hamiltonian. Employing the BV transformation, we numerically transform $H_1$ to the basis where $H_2$ is diagonal. The terms that are linear in excitation operators may then be removed by completing squares using terms from $H_2$. We shift some operators by complex numbers, which does not alter their interpretation since their commutation relations are conserved. Finally, this procedure leads to a shift of the free energy by a real number. More details can be found in [33].

We consider the free energy at zero temperature, such that $F = \langle H \rangle$. Using that $\langle B_k \sigma B_k \sigma \rangle = 0$ for $k \neq \pm k_{01}$ we may now calculate the free energy numerically at a given set of parameters. To find the minimum of $F$ with respect to a variational parameter, we vary it while keeping the other variational parameters set to the values that minimize $H_0$. The result is that $k_0 = k_{01}$ and $x$ equal to the value that minimizes $H_0$ also minimizes $F$ to a good approximation. Upon choosing $\theta_0$ as a free parameter, the angles that minimize $F$ are

\[
\theta_1 = \theta_1 = \frac{\pi}{4} \quad \text{and} \quad \theta_3 = \theta_3 + \frac{5\pi}{4}.
\]

A more rigorous approach would be to use simulated annealing [34] to find the global minimum of the free energy in terms of the set of variational parameters. This was performed on the SW phase with no Zeeman field in [35], and again the values that minimize $H_0$ were found to minimize $F$.

In general we find that the values of the variational parameters which minimize $F = \langle H \rangle$ can be approximated very well by the values that minimize $H_0$. This can be understood from the order of the terms in the Hamiltonian (6). $H_0$ is of order $N^2/N_\sigma H_1$ of order $\sqrt{N}/N_\sigma$ and $H_2$ of order $N/N_\sigma$. When $n = N/N_\sigma = 1$ and $N_\sigma >> 1$ it is natural that $H_0$ dominates the minimization. In experiments, typical lattice sizes are $N_\sigma \sim 1 - 3 \cdot 10^5$ while $n$ is most often of order unity [13, 36, 37]. We have therefore set $n = 1$ when producing the figures, and have assumed that $\mu$ is set to the value which ensures this.

**Appendix D: Special Momenta and Energetic Instability in SW Phase**

This appendix will briefly mention some subtle points not considered in the article. Firstly, there are some special momenta that require a separate treatment [33]. Considering the SW phase operator vector (21) at $k = 0$ and $\pm 2k_{01}$ several elements become equal, which is not acceptable in the BV transformation. Additionally, at $k = \pm 3k_{01}$ there are elements involving the condensate momenta $\pm k_{01}$ in (21). Such terms should have been excluded from the sum $\sum_{kk'}$ as mentioned after (9).

However, the physical interpretation of these results is problematic. For instance, the special eigenvalues found at $\pm 2k_0$ do not correspond to the eigenvalues of $M_{12} k_{01}$ suggesting the excitation spectrum is discontinuous. For $\alpha$ close to 3 and $\Omega$ close to 0 these special eigenvalues are lower than the minimum of the excitation spectrum, which seems to indicate energetic instabilities. However, we suggest treating this as an artifact of the BV diagonalization, rather than an indication of instability in the SW phase. On physical grounds we expect a continuous excitation spectrum.
For $\alpha = 3$ and no Zeeman field we find that the anomalous modes in the excitation spectrum are zero for all $k$. Hence, the lowest energy band is zero for all $k$, indicating that the phase is unstable. This seems to be caused by the fact that $|M_{1,3}|$ and $|M_{7,9}|$ become equal to $|M_{1,3,4}|$, $|M_{13,6}|$, $|M_{19,10}|$ and $|M_{19,12}|$. This means that the transition rates of the scatterings (c) and (d) become equal to the rates of (e) and (f) in figure 5. These matrix elements are related to scatterings where the $k$-dependent particles have the same spin. A similar behavior is not found for $\Omega > 0$, since then $x \neq 1/2$. At least when ignoring any indications of energetic instability from the special momenta and keeping the SOC strength large, this energetic instability appears to be located at a single point in $\Omega - \alpha$-space and was therefore omitted in the article.

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