A LAGRANGIAN PROGRAM DETECTING THE WEIGHTED FERMAT-STEINER-FRECHET MULTITREE FOR A FRECHET $N$–MULTISIMPLEX IN THE $N$–DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In this paper, we introduce the Fermat-Steiner-Frechet problem for a given $\frac{N(N+1)}{2}$–tuple of positive real numbers determining the edge lengths of an $N$–simplex in $\mathbb{R}^N$, in order to study its solution called the "Fermat-Steiner-Frechet multitree," which consist of a union of Fermat-Steiner trees for all derived pairwise incongruent $N$–simplexes in the sense of Blumenthal, Herzog for $N = 3$ and Dekster-Wilker for $N \geq 3$. We obtain a method to determine the Fermat-Steiner Frechet multitree in $\mathbb{R}^N$ based on the theory of Lagrange multipliers, whose equality constraints depend on $N-1$ independent solutions of the inverse weighted Fermat problem for an $N$–simplex in $\mathbb{R}^N$. A fundamental application of the Lagrangian program for the Fermat-Steiner Frechet problem in $\mathbb{R}^N$ is the detection of the Fermat-Steiner tree with global minimum length having $N-1$ equally weighted Fermat-Steiner points among $\frac{N(N+1)}{2}$–tuple of consecutive natural numbers controlled by Dekster-Wilker, Blumenthal-Herzog conditions and enriched with the fundamental evolutionary processes of Nature (Minimum communication networks, minimum mass transfer, maximum volume of incongruent simplexes). Furthermore, we obtain the unique solution of the inverse weighted Fermat problem, referring to the unique set of $(N+1)$ weights, which correspond to the vertices of a Frechet $N$–multisimplex. Additionally, we give a negative answer for an intermediate weighted Fermat-Steiner-Frechet multitree having one node (weighted Fermat point) for $m$ boundary closed polytopes, $(m \geq N+2)$, which is determined by $m$ prescribed rays meeting at a fixed weighted Fermat point, by deriving a linear dependence for the $m$ variable weights (Plasticity of an Intermediate Fermat-Steiner-Frechet multitree for $m$ boundary closed polytopes). By enriching the plasticity of an intermediate Fermat-Steiner-Frechet multitree for $m$ boundary closed polytopes with a two-way mass transport from $k$ vertices to the unique weighted Fermat point and from this point to the $m-k$ vertices and reversely, we derive the equations of "mutation" of intermediate Fermat-Steiner-Frechet multitree for $m$ boundary closed polytopes in $\mathbb{R}^N$. Finally, we apply the unique solution of the inverse weighted Fermat problem for an $N$–simplex in $\mathbb{R}^N$, in order to construct an $\epsilon$ approximation for the weights which correspond to the vertices of a Frechet $N$–multisimplex.

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1. Introduction

The problem considered in the present paper is a result of synthesis of two problems: The weighted Fermat-Steiner problem on the shortest networks for boundary $N$-simplexes in $\mathbb{R}^N$ and Frechet’s problem on seeking the necessary and sufficient conditions for a given $\frac{N(N+1)}{2}$-tuple of positive real numbers determining the edge lengths of an $N$-simplex in $\mathbb{R}^N$ for $N \geq 3$.

The weighted Fermat-Steiner problem for a boundary $N$-simplex is a problem on finding a shortest network of total weighted length connecting $N+1$ non-collinear and non-coplanar points possessing each of them a positive real number (weight) in $\mathbb{R}^N$.

In [39], we find the solution of the weighted Fermat-Steiner problem for a tetrahedron $A_1A_2A_3A_4$ ($N = 3$). The solution is a tree $T$ having two weighted points $A_0$ and $A'_0$ (nodes) inside the tetrahedron. Each node is a weighted Fermat-Steiner (or Torricelli) point. We call $T$ a weighted Steiner tree. $T$ is the union of one line segment between the two weighted Fermat-Steiner points and of 4 line segments joining each weighted Fermat-Steiner point with two fixed neighboring vertices. In [20], Rubinstein, Thomas and Weng solved the unweighted Fermat-Steiner problem for tetrahedra in $\mathbb{R}^3$. The weighted Fermat-Steiner problem for four points in the Euclidean plane is a problem of practical importance first considered by Gauss in a letter to Schumacher, which deal with the construction of a railway network interconnecting four German cities ([12]). The weight of each vertex may be considered as the population of each city. If one of the four weights equals zero and the other three weights are equal, we obtain an equally weighted Fermat-Steiner problem for three points in $\mathbb{R}^2$, we derive the original problem posed by Fermat ([10]). It is worth mentioning that Fermat proved Snell’s law of retraction of light, which requires the computation of the minimum of a function of one variable as a sum of two weighted distances (the third weight is zero), such that the first weight is inversely proportional to the velocity of propagation of light in the upper medium and the second weight is inversely proportional to the velocity of propagation of light in the lower one (see in [30, p. 21]). In [21], Jarnik and Kossler introduced a generalization of the Fermat problem on optimal networks of a finite set of points in $\mathbb{R}^2$. In [5], Courant and Robbins restated and named the problem as the Steiner problem. In [13], Ivanov and Tuzhilin studied properties of weighted minimal binary trees, which is an important generalization of weighted Fermat-Steiner trees in $\mathbb{R}^2$.

The Frechet problem in $\mathbb{R}^3$ states that ([10]): Given a sextuple of positive real numbers $a_{ij} = a_{ji}$, for $i, j = 1, 2, 3, 4$. What are the necessary and sufficient conditions that be the lengths of the tetrahedron $A_1A_2A_3A_4$ in $\mathbb{R}^3$?

In [1, Theorem 2.1], Blumenthal proved that a sextuple of positive real numbers $(a + nd)^{1/2}, n = 0, 1, 2, 3, 4, 5, 0 < 4d \leq a$ form a completely tetrahedral sextuple, which yields thirty incongruent tetrahedra in $\mathbb{R}^3$.

In [11, Theorem 3], Hertog proved a theorem characterizing complete tetrahedral sextuples, which states that: Six positive real numbers $\{a, b, c, d, e, f\}$ satisfying the conditions $a > b > c > d > e > f > 0$ and $a + f$ form a complete tetrahedral sextuple if and only if the Caley-Menger determinant $D(a, b; c, f; e) \geq 0$, (the edges separated by semicolons are pairs of opposite edges of the tetrahedron) that is if and only if the tetrahedron in which the faces are formed by the triples $(a, c, d), (a, e, f), (b, c, e)$ and $(b, d, f)$ is realizable in $\mathbb{R}^3$. 
In \[11\], Remark 4, 5, 6, Hertog gave the following three important applications of his theorem:

i. Six consecutive positive integers \(x+5, x+4, x+3, x+2, x+1, x\) form a complete tetrahedral sextuple if and only if \(x\) is greater than or equal to the positive root \(\rho\) of the equation \(D(x+5, x+4; x+3, x; x+2, x+1) = 0\): \(6.09 < \rho < 6.10\) (Hertog sextuple of consecutive positive integers)

ii. The sextuple of positive real numbers \(\{1, 1-\delta; 1-2\delta, \epsilon; 1-3\delta, 1-4\delta\}\) for \(\delta = \delta(\epsilon) > 0, 0 < \epsilon < 1\), such that \(1-4\delta > \epsilon\) forms a complete tetrahedral sextuple

iii. Blumenthal’s sextuple \(\{a+n\delta\}_{n=0}^{5}\) is a complete tetrahedral sextuple if and only if \(\frac{\rho}{\delta}\) is greater than or equal to a cubic irrationality \(r:\)

\[1.91 < r < 1.92.\]

In \[6\], Dekster and Wilker derived the condition \(\frac{x+5}{x+5} \geq \lambda(3) \equiv \frac{1}{\sqrt{2}}\), which gives that \(x \geq 13\), which leaves six runs from Hertog result sextuple of consecutive integers that works for \(x \geq 7\).

We introduce the Fermat-Steiner-Frechet problem for a given \(N\) \((N+1)\)^2-tuple of positive real numbers determining \(N\)-simplexes in \(\mathbb{R}^N\).

**Problem 1** (The weighted Fermat-Steiner-Frechet problem in \(\mathbb{R}^N\)). *Given a \(2N\)-tuple of positive real numbers (weights), such that \(N-1\) of them are equal and a \(\frac{N(N+1)}{2}\) of positive real numbers determining the edge lengths of \(N\)-simplexes in \(\mathbb{R}^N\), find the shortest networks (weighted Fermat-Steiner trees) of total weighted length for all derived incongruent \(N\)-simplexes in \(\mathbb{R}^N\).*

We call Fermat-Steiner Frechet multitree the solution of the weighted Fermat-Steiner-Frechet problem in \(\mathbb{R}^N\), which is a union of the corresponding weighted Fermat-Steiner trees for all derived incongruent \(N\)-simplexes in \(\mathbb{R}^N\) and Frechet \(N\)-multisimplex the class of incongruent \(N\)-simplexes derived by the same given \(\frac{N(N+1)}{2}\)-tuple of positive real numbers in \(\mathbb{R}^N\).

In \[37\], we solve the weighted Fermat-Frechet problem for \(N = 3\) in \(\mathbb{R}^3\) and we find the condition to locate the corresponding weighted Fermat trees, by applying a generalization of the cosine law in \(\mathbb{R}^3\), which depend only on edge lengths and the Cayley-Menger determinant representing tetrahedral volume w.r.t to edge lengths.

In this paper, we obtain a Lagrangian program to detect a weighted Fermat-Steiner multitree for a boundary \(N\)-Frechet multisimplex in \(\mathbb{R}^N\), which gives:

(i) a characterization of the most natural of natural numbers obtained by a \(\frac{N(N+1)}{2}\)-tuple of consecutive natural numbers

(ii) The plasticity of weighted Fermat-Steiner multitree having one node (intermediate multitrees) for boundary \(m\)-closed polytopes in \(\mathbb{R}^N\)

(iii) The Bessel (random) plasticity of weighted Fermat-Steiner multitree having one node (intermediate multitrees) for boundary \(m\)-closed polytopes in \(\mathbb{R}^N\)

(iv) An \(\epsilon\) approximation method for the weights of a Frechet \(N\)-multisimplex in \(\mathbb{R}^N\).

Our main results are:

1. A Lagrangian program, which detects the weighted Fermat-Steiner-Frechet multitree for a given \(\frac{N(N+1)}{2}\) in \(\mathbb{R}^N\) for \(N \geq 3\)
Theorem 78. Lagrange multiplier rule for the weighted Fermat-Steiner Frechet multitree in $\mathbb{R}^N$. If the admissible point $\tilde{x}_i$ yields a weighted minimum multitree for $1 \leq i \leq \frac{2N(N+1)!}{(N+1)!}$, which correspond to a Frechet $N$–multisimplex derived by a $\frac{N(N+1)}{2}$–tuple of edge lengths determining up to $\frac{2N(N+1)!}{(N+1)!}$ incongruent $N$–simplexes constructed by the Dekster-Wilker domain $DW_{\mathbb{R}^N}(\ell, s)$, then there are numbers $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{(N^2-1)}$, (components of the Lagrangian vector) such that:

$$\frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{ji}} = 0,$$

for $j = 1, 2, \ldots, 2N(N-1)$,

$$\tilde{x}_i = \{a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_{(0,1),4}, \ldots, \beta_{(0,1),N}, \ldots, a_{(0,N-1),1}, a_{(0,N-1),2}, a_{(0,N-1),3}, \beta_{(0,N-1),4}, \ldots, \beta_{(0,N-1),N}, w_{(0,1),1}, \ldots, w_{(0,1),N}, w_{(0,2),1}, \ldots, w_{(0,2),N}, \ldots, w_{(0,N-1),1}, \ldots, w_{(0,N-1),N}\}$$

and

$$\tilde{\lambda}_i = \{\lambda_0, \lambda_1, \ldots, \lambda_{N^2-1}\}$$

is a Lagrangian function.

Theorems 215 are particular cases for $N = 3$ and $N = 4$, respectively. We note that we mention these two particular cases, in order to explain how the Schlafli angle formed by the normals of two planes is eliminated and vanishes from the computation of the Lagrangian function in $\mathbb{R}^3$ and how the Schlafli angle formed by the normals of a plane and a hyperplane cannot be eliminated and is embodied in the computation of the Lagrangian function in $\mathbb{R}^4$. Hence, $N - 3$ Schlafli angles are embodied in the computation of the Lagrangian function in $\mathbb{R}^N$.

2. A characterization of the most natural of natural numbers obtained by a $\frac{N(N+1)}{2}$–tuple of consecutive natural numbers

Theorem 79. The most natural $\frac{N(N+1)}{2}$–tuple of numbers from $\frac{N(N+1)}{2}$ consecutive natural numbers $\{a + \frac{N(N+1)}{2} - 1, \ldots, a + 1, a\}$ for $a \geq a(N)$ is a $\frac{N(N+1)}{2}$–tuple of edge lengths having the maximum volume (maximum $\frac{N(N+1)}{2}$–tuple) among the $\frac{2N(N+1)!}{(N+1)!}$, incongruent $N$–simplexes, which corresponds to a Fermat-Steiner tree of minimum total weighted length (global minimum solution), such that the upper bound for the weight $B_{ST}$ is determined by the rest Fermat-Steiner minimal trees having larger or equal weighted minimal total length.

The function $a_N$ is derived by the Dekster-Wilker function $f(N)$ (Section 8).

We note that for $N = 3$, we use Blumenthal-Herzog sextuples of positive real numbers determining the edge lengths of 30 incongruent tetrahedra (Frechet multitetrahedron), in order to detect the most natural of natural numbers obtained by a consecutive sextuple of natural numbers whose least element $a \geq 7$ (Theorem 8). The corresponding Dekster-Wilker sextuples detects the most natural of natural numbers for $a \geq 13$.

For $N = 4$, we use Dekster-Wilker tentuples of positive real numbers determining the edge lengths of 30.240 incongruent 4-simplexes (Frechet 4–multisimplex), in
order to detect the most natural of natural numbers obtained by a consecutive sextuple of natural numbers whose least element $a \geq 30$ (Theorem 17).

3. A theoretical construction of a weighted Fermat-Steiner Frechet multitree for a Frechet multitetrahedron in $\mathbb{R}^3$, using two variable dihedral angles, which focus on an auxiliary construction of five points that lie on the same circle.

**Theorem 3** There are up to $4! \cdot 30$ weighted Simpson lines defined by the points $T_{12}, T_{34}$, such that the two equally weighted Fermat-Steiner points $O_{12}, O_{34}$ in $[T_{12}, T_{34}]$, which give the position of the weighted Fermat-Steiner-Frechet multltree for 30 incongruent tetrahedra determined by Blumenthal, Herzog and Dekster-Wilker sextuples of edge lengths in $\mathbb{R}^3$.

4. Non-random and random plasticity equations of an intermediate weighted Fermat-Steiner Frechet multitrees.

**A. Theorem 8** [Unique solution of the $4-$ INVWF problem in $\mathbb{R}^4$] The weight $B_i$ is uniquely determined by:

$$B_i = \frac{C}{1 + \frac{\sin \alpha_i, k\ell m}{\sin \alpha_{k\ell m}^i} + \frac{\sin \alpha_j, j\ell l m}{\sin \alpha_{j\ell l m}^j} + \frac{\sin \alpha_k, k\ell m}{\sin \alpha_{k\ell m}^k} + \frac{\sin \alpha_l, l\ell j m}{\sin \alpha_{l\ell j m}^l}},$$

for $i, j, k, l, m = 1, 2, 3, 4, 5$ and $i \neq j \neq k \neq l \neq m$, where $\alpha_{i,k\ell m}$ is the angle formed by the line segment $A_0A_i$ and $A_0, P_i$, where $P_i$ is the trace of the orthogonal projection of $A_i$ to the hyperplane defined by $A_0A_kA_lA_m$.

**Theorem 14** [Solution of the $N-$ INVWF problem in $\mathbb{R}^N$] The weight $B_i$ is uniquely determined by:

$$B_i = \frac{C}{1 + \frac{\sin \alpha_{i,k_1 k_2 \ldots k_{N-1} k_N}^i}{\sin \alpha_{k_1 k_2 \ldots k_{N-1} k_N}^{i,k_1}} + \frac{\sin \alpha_{i,0k_1 k_2 \ldots k_{N-2} k_{N}}^i}{\sin \alpha_{0k_1 k_2 \ldots k_{N-2} k_{N}}^{i,0k_1}} + \ldots + \frac{\sin \alpha_{i,0k_2 \ldots k_{N-1} k_N}^i}{\sin \alpha_{0k_2 \ldots k_{N-1} k_N}^{i,0k_1}}},$$

for $i, k_1, k_2, \ldots, k_N = 1, 2, \ldots, N + 1$ and $k_1 \neq k_2 \neq \ldots \neq k_N$.

The non-uniqueness of the $N-$ INVWF problem in $\mathbb{R}^N$ gives the plasticity equations of non-random plasticity and Bessel(random) plasticity of intermediate weighted Fermat-Steiner Frechet multitrees having one node (weighted Fermat point) for boundary $m-$ closed polytopes in $\mathbb{R}^N$ for $m \geq N + 2$ (Theorems 12 and 20, respectively).

**Theorem 7** An increase to the weight that corresponds to the $(N + 2)th$ ray causes a decrease to the weight that corresponds to the $(N + 1)th$ ray and a variation to the weight that corresponds to the $ith$ ray depends on the difference $(B_i)_{123 \ldots N(N+1)} - (B_i)_{123 \ldots N(N+2)}$, such that the geometric structure of the weighted Fermat-Frechet multitree with respect to a boundary $N-$ multisimplex, remains the same, for $i = 1, 2, \ldots, N$, $1 \leq k \leq \frac{N(N+1)!}{(N+1)!}$.

The Bessel plasticity of multitrees is derived by adding a random weight, which follows a Bessel motion on a new growing branch starting from the weighted Fermat point $A_0$, i.e $A_0A_{N+2}$ (Theorem 25).

**B. Theorem 11** weighted volume equalities in $\mathbb{R}^N$

$$\frac{B_1}{a_1V ol(A_0A_2A_3 \ldots A_{N+1})} = \frac{B_2}{a_2V ol(A_1A_0 \ldots A_{N+1})} = \ldots = \frac{B_{N+1}}{a_{N+1}V ol(A_1A_2 \ldots A_0)} = \frac{\sum_{i=1}^{N+1} \frac{B_i}{a_i}}{V ol(A_1A_2 \ldots A_{N+1})}.$$
The weighted volume equalities are important to compute the weighted Fermat point of an $N$-simplex in $\mathbb{R}^N$ and to construct the Lagrangian function of a weighted Fermat-Steiner multitetree for the Frechet $N$-multisimplex in $\mathbb{R}^{N-1}$, by taking into account a system of variable weighted volume equalities derived by $(N - 1)$ weighted Fermat points located inside each boundary $N$-simplex derived by the Frechet $N$-multisimplex.

C. By adding an optimal mass transport of a two-way communication network along $k$ directions in the plasticity of non-random plasticity of intermediate weighted Fermat-Steiner Frechet multitrees having one weighted Fermat point, for boundary $m$-closed polytopes in $\mathbb{R}^N$ for $m \geq N + 2$, we derive the $(m, k)$ "mutation" of intermediate weighted Fermat-Steiner Frechet multitrees having one weighted Fermat point, for boundary $m$-closed polytopes in $\mathbb{R}^N$ (Theorem 23).

5. Constructive weights for the vertices of an $N$-Frechet multisimplex in $\mathbb{R}^N$.

An $\epsilon$ approximation of the value of the weight $B_{N+1}$, which corresponds to the vertex $A_{N+1}$ of an $N$-simplex in $\mathbb{R}^N$ circumscribed in a $(N-1)$sphere $S^{N-1}$ of radius $r$, and center $O$, is given by Theorem 24 by applying the $N$-INWF problem in $\mathbb{R}^N$. For $\epsilon \to 0$, the limiting floating weighted Fermat tree solution coincides with the absorbing weighted Fermat tree solution of Theorem 1. Therefore, we get an approximation for the weights of a Frechet $N$-multisimplex via an $\epsilon$ approximation of each corner of incongruent $N$-simplexes with a multiweighted Fermat-Frechet multitree in $\mathbb{R}^N$.

In Section 2, we mention some fundamental known results concerning the existence and uniqueness of solution for the weighted Fermat-Steiner problem and the weighted Fermat problem in $\mathbb{R}^N$ and the uniqueness of solution of the inverse weighted Fermat-problem for a boundary tetrahedron and boundary triangles in $\mathbb{R}^3$. In Section 3, we introduce a Lagrangian program, which detects the weighted Fermat-Steiner Frechet multitetree for a Frechet multitetrahedron in $\mathbb{R}^3$. In Section 4, we apply this program to detect a weighted Fermat-Steiner tree with respect to a boundary tetrahedron having the maximum volume whose edge lengths form a Herzog sextuple of six consecutive natural numbers and the global weighted minimum length. In Section 5, we give a theoretical construction of a weighted Fermat-Steiner Frechet multitree for a Frechet multitetrahedron in $\mathbb{R}^3$, using two variable dihedral angles.

In Section 6, we study the uniqueness of solution of the inverse weighted Fermat problem for $N$-simplexes in $\mathbb{R}^N$ and the plasticity solutions of a weighted Fermat-Frechet multitetree with one node (weighted Fermat point) for $m$-boundary closed polytopes in $\mathbb{R}^N$. In Sections 7, 8, we apply the uniqueness of the inverse weighted Fermat problem for $N$-simplexes in $\mathbb{R}^N$ to a Lagrangian program, in order to detect the weighted Fermat-Steiner Frechet multitree for an $N$-Frechet-multisimplex for $N = 4$ and $N > 4$, respectively. In Section 9, we extend the Lagrangian program for intermediate weighted Fermat-Steiner-Frechet multitrees for an $N$-Frechetmultisimplex in $\mathbb{R}^N$ for an $N$-Frechet multisimplex whose $\#$ of nodes is less than $N - 1$. In Section 10, we give the "mutation" equations of an intermediate weighted Fermat-Frechet multitree having one node for $m$-boundary closed polytopes in $\mathbb{R}^N$, which is a generalization of the plasticity solutions given in section 5. In Section 12, we construct the weights of a Frechet $N$-multisimplex in $\mathbb{R}^N$, by introducing a method of $\epsilon$ approximation of variable weighted multitrees.
in \( \mathbb{R}^N \). In Section 13, we introduce the Bessel plasticity equations of an intermediate weighted Fermat-Steiner Frechet multitree having one node for \( m \)—boundary closed polytopes in \( \mathbb{R}^N \), such that one of the weights follows a Bessel motion. In the final section, we conclude with two open questions, which deal with a “partition” of optimal networks in \( \mathbb{R}^N \).

2. Preliminaries: The weighted Fermat-Steiner problem and the inverse weighted Fermat problem for tetrahedra in \( \mathbb{R}^3 \)

In this section, we state the weighted Fermat-Steiner problem for an \( N \)—simplex in \( \mathbb{R}^N \), which is a generalization of the weighted Fermat problem in \( \mathbb{R}^N \) and mention some fundamental results established by Ivanov-Tuzhilin concerning the existence and uniqueness of the Fermat-Steiner tree solutions. We also mention the characterization of solutions with respect to the weighted Fermat problem given by Sturm for the unweighted case and extended by Kupitz-Martini for the weighted case. We continue by giving some recent known results for the weighted Fermat-Steiner problem for tetrahedra in \( \mathbb{R}^3 \) regarding the position of the weighted Fermat-Steiner tree given by the author, which extends previous results given by Rubenstein-Thomas-Weng for the unweighted case. Finally, we state an inverse weighted Fermat problem for \( m \) points in \( \mathbb{R}^N \) and an explicit solution of the inverse weighted Fermat problem for \( N = 3 \) given by Zouzoulas and the author and generalizes the solution of the inverse weighted Fermat problem for \( N = 2 \) given by Gueron and Tessler.

We denote by \( A_{0,1}, A_{0,2}, \ldots , A_{0,N-1}, N-1 \) points inside the \( N \)—simplex \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \), by \( b_i \) a positive real number (weight), which corresponds to each vertex \( A_i \), by \( a_{(0,i),(0,j)} \) the length of the line segment \( A_{0,i} A_{0,j} \) and by \( a_{060} \) the length of the line segment \( A_k A_0 \), for \( i,j = 1,2,\ldots ,N-1, k = 1,2,\ldots ,N+1 \).

**Problem 2** (The weighted Fermat-Steiner problem for \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \)). Find \( A_{0,i} \) in \( \mathbb{R}^N \), for \( i = 1,2,\ldots ,N-1 \) with equal given weight \( B_{ST} \), such that:

\[
f(\{A_{0,i}\}) = \sum_{i=1,j \in \{1,2,\ldots ,N-1\}}^{N+1} b_i a_{i,(0,j)} + b_{ST} \sum_{i=1,j \in \{1,2,\ldots ,N-1\}}^{N-1} a_{(0,i),(0,j)} \rightarrow \min.
\]

(2.1)

In [16] Corollary 1.1, Proposition 1.2, [17] Theorem 1.1, Corollary 1.2, Chapter 3, [18, 20], Ivanov and Tuzhilin studied the weighted Fermat-Steiner problem and many variations starting from an embedding of a Steiner topological graph (one dimensional cell complex) on a manifold \( W \).

The following definitions of an immersed parametric network are given in [17] Definitions p. 56, p. 58: A parametric network \( \Gamma \) of type \( G \) in the manifold \( W \) is an arbitrary mapping \( \phi : G \rightarrow W \). A network \( \Gamma = \{ \phi : G \rightarrow W \} \) is called piecewise smooth if each of its edges is a piecewise curve. A piecewise smooth curve is called immersed if all its edges are nondegenerate. We note that a description of weighted minimum networks in \( \mathbb{R}^N \) is given in [19] Section 3. Local minimal networks of type \( G \) can be realized as weighted minimal networks type \( G \) with constant weight function (positive real number) defined on the edge set of \( G \).

**Lemma 1.** *Weighted Local Minimality criterion,* [17] Corollary 1.2, Chapter 3, [18] Let \( \Gamma \) be an immersed weighted parametric network (Weighted Fermat-Steiner network). The network is local minimal if and only if all the edges of \( \Gamma \) are geodesic.
segments for any mobile vertex (Fermat-Steiner point) \( A_{0,i} \) of \( \Gamma \) the linear combination of weighted unit vectors of the directions of edges of \( \Gamma \) going out of \( A_{0,i} \) (with coefficients equal to the weights of these edges) vanishes.

We mention an important theorem of Ivanov-Tuzhilin ([17]), which deals with the uniqueness for networks with boundaries in \( \mathbb{R}^N \). A strong local minimal network \( \Gamma_0 \), if for any point \( x \) of the parametric graph of the reduced network (without degenerate edges) corresponding to \( \Gamma \), there exists a strong local network \( \Gamma_{sloc}^x \), such that any small deformation of the network \( \Gamma_{sloc}^x \) that preserves its boundary does not decrease the length of \( \Gamma_{sloc}^x \) (see in [17] Definitions, p. 91).

**Lemma 2.** Uniqueness theorem of Ivanov-Tuzhilin for networks with boundaries in \( \mathbb{R}^N \). [17, Theorem 3.1 Chapter 2, Proof of Theorem 3.1, Chapter 3] Let \( M \subset \mathbb{R}^N \) be an arbitrary non-empty finite set of points from \( \mathbb{R}^N \); denote by \( G \) an acyclic topological graph and let \( \beta \) be a boundary mapping that maps a subset of vertices of graph \( G \) onto the set \( M \). Assume that among strong minimal networks in class \( \mathcal{M}_G(\beta) \) there exists an embedded network \( \Gamma \) which does not possess mobile vertices \( A_{0,i} \) of degree two. Then all other strong minimal networks in this class coincide with \( \Gamma \) (up to a parameterization); that is \( \Gamma \) is unique.

By substituting \( A_0 \equiv A_{0,i} \), for \( i = 1, 2, \ldots, N-1 \) in (2.1), we derive the weighted Fermat problem \( A_1A_2\ldots A_{N+1} \) in \( \mathbb{R}^N \).

**Problem 3** (The weighted Fermat problem for \( A_1A_2\ldots A_{N+1} \) in \( \mathbb{R}^N \)). Find \( A_0 \) in \( \mathbb{R}^N \), such that:

\[
f(\{A_{0,i}\}) = \sum_{i=1}^{N+1} b_i a_{0i} \rightarrow \min. \tag{2.2}\]

In [29], Sturm gave a complete characterization of the solutions of the unweighted Fermat problem for \( m \) given points in \( \mathbb{R}^N \), and Kupitz and Martini extended these characterization in the weighted case, in [2] Theorem 18.37, pp. 250).

**Theorem 1.** [29,2, Theorem 18.37, pp. 250]

(I) The weighted Fermat point \( A_0 \) with respect exists and is unique.

(II) If for each point \( A_i \in \{A_1, A_2, \ldots, A_m\} \)

\[
\| \sum_{j=1, j \neq i} b_j \bar{u}(A_j, A_i) \| > b_i,
\]

for \( i, j = 1, 2, \ldots, m \), then

(a) the weighted Fermat point \( A_0 \) does not belong to \( \{A_1, A_2, \ldots, A_m\} \),

(b)

\[
\sum_{i=1}^{m} b_i \bar{u}(A_0, A_i) = \vec{0}. \tag{2.3}\]

(Weighted floating Fermat tree solution).

(III) If there is a point \( A_i \in \{A_1, A_2, \ldots, A_m\} \) satisfying

\[
\| \sum_{j=1, j \neq i} b_j \bar{u}(A_j, A_i) \| \leq b_i.
\]
for \( i, j = 1, 2, ..., m \), then the weighted Fermat point \( A_0 \equiv A_i \) (Weighted absorbed Fermat tree solution).

The inverse weighted Fermat problem for \( m \) points (non-collinear and non-coplanar) in \( \mathbb{R}^N \) \((m-1)\text{-INVWF}) states that:

**Problem 4** (The \((m-1)\text{-INVWF} \text{ problem in } \mathbb{R}^N\)). Given a point \( A_0 \not\in \{A_1, A_2, ..., A_m\} \), \( A_0, A_i \in \mathbb{R}^N \), does there exist a unique set of positive weights \( b_{i0} \), such that

\[
b_{10} + b_{20} + ... + b_{m0} = c = \text{const}, \tag{2.4}\]

for which \( A_0 \)

\[
f(A_0) = \sum_{i=1}^{m} b_{i0}a_{i0} \rightarrow \text{min}. \]

The solution of the \(2\text{-INVWF} \text{ problem in } \mathbb{R}^2 \) \((b_4 = b_5 = b_6 = ... = b_m = 0)\) is given in [14].

**Proposition 1. Solution of the 2-INVF problem in } \mathbb{R}^2 [14] \]

\[
b_i = \frac{C}{1 + \frac{\sin \alpha_{i0j}}{\sin \alpha_{jok}} + \frac{\sin \alpha_{j0k}}{\sin \alpha_{jok}}} \tag{2.5}\]

for \( i, j, k = 1, 2, 3 \) and \( i \neq j \neq k \).

The weights \( b_1, b_2, b_3 \) depend on exactly two given angles \( \alpha_{102}, \alpha_{203} \) because \( \alpha_{103} = 2\pi - \alpha_{102} - \alpha_{203} \).

**Problem 5** (The \(3\text{-INVWF} \text{ problem in } \mathbb{R}^3 \)). Given a point \( A_0 \) which belongs to the interior of \( A_1A_2A_3A_4 \) in \( \mathbb{R}^3 \), does there exist a unique set of positive weights \( b_{i0} \), such that:

\[
b_{10} + b_{20} + b_{30} + b_{40} = C > 0, \]

\[
f(A_0) = \sum_{i=1}^{4} b_{i0}a_{i0} \rightarrow \text{min}. \]

The unique solution of the weighted Fermat problem for tetrahedra has been established in [34], by taking into account (2.17)-(2.20) of Lemma 7.

**Lemma 3** (Solution of the inverse weighted Fermat problem for tetrahedra in \( \mathbb{R}^3 \) [34, (3.12),(3.13), p. 120]).

\[
(b_{j0})^2 = \frac{\sin^2 \alpha_{kom} - \cos^2 \alpha_{moj} - \cos^2 \alpha_{koi} + 2 \cos \alpha_{moj} \cos \alpha_{koi} \cos \alpha_{kom}}{\sin^2 \alpha_{kom} - \cos^2 \alpha_{moj} - \cos^2 \alpha_{koi} + 2 \cos \alpha_{moj} \cos \alpha_{koi} \cos \alpha_{kom}} \tag{2.6}\]

for \( i, j, k = 1, 2, 3, 4 \) and \( i \neq j \neq k \neq m \neq i \).

**Lemma 4.** [38, Proposition 1] The ratios \( \frac{b_{j0}}{b_{i0}} \) depend on exactly five given angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203} \) and \( \alpha_{204} \).

The sixth angle \( \alpha_{304} \) is calculated by the following formula:

\[
\cos \alpha_{304} = \frac{1}{4} [4 \cos \alpha_{103} (\cos \alpha_{104} - \cos \alpha_{102} \cos \alpha_{204}) + \\
+ 2 (b + 2 \cos \alpha_{203} (-\cos \alpha_{102} \cos \alpha_{104} + \cos \alpha_{204}))] \csc^2 \alpha_{102} \tag{2.7}\]

where
\[ b = \sqrt{\prod_{i=3}^{4} (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{104}) + \cos(2\alpha_{204}) - 4 \cos \alpha_{102} \cos \alpha_{104} \cos \alpha_{204})} \tag{2.8} \]

for \( i, k, m = 1, 2, 3, 4, \) and \( i \neq k \neq m. \)

Thus, the solution of the 3-INVF problem in \( \mathbb{R}^3 \) depend on exactly five given angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204}. \)

We mention the definitions of a tree topology, the degree of a vertex, a Fermat tree topology, a Fermat-Steiner tree topology, an intermediate Fermat-Steiner tree topology, in order to study the solution of the weighted Fermat-Steiner problem with respect to fixed tree topology for \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N. \)

**Definition 1.** \[13\] A tree topology is a connection matrix specifying which pairs of points from the list \( A_1, A_2, \ldots, A_N, A_{0,1}, A_{0,2}, \ldots, A_{0,N-2} \) have a connecting line segment (edge).

**Definition 2.** \[17, 4\] The degree of a vertex corresponds to the number of connections of the vertex with line segments.

**Definition 3.** \[13\] A Fermat tree topology of a boundary \( N \)-simplex \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \) is a tree topology, such that each boundary vertex \( A_i \) has degree one and the weighted Fermat point \( A_{0,1} \) has degree \( N + 1. \)

**Definition 4.** \[13\] A Fermat-Steiner tree topology of a boundary \( N \)-simplex \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \) is a tree topology, such that each boundary vertex \( A_i \) has degree one and each weighted Fermat point \( A_{0,i} \) has degree three for \( i = 1, 2, \ldots, N - 1. \)

**Definition 5.** \[13\] An intermediate Fermat Steiner tree topology of a boundary \( N \)-simplex \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \) is a tree topology, such that each boundary vertex \( A_i \) has degree one and the number of weighted Fermat points \( A_{0,i} \) is less than \( N - 1 \) having degree less than \( N + 1. \)

**Definition 6.** A tree of weighted minimum length with a Fermat tree topology is called a weighted Fermat-tree.

**Definition 7.** A tree of weighted minimum length with a Fermat tree topology is called a weighted Fermat-Steiner tree.

**Definition 8.** A tree of weighted minimum length with an intermediate Fermat-Steiner tree topology is called an intermediate weighted Fermat-Steiner tree.

By replacing \( b_5 = \ldots = b_{N+1} = 0, A_{0,1} \equiv A_0 \) and \( A_{0,N-1} \equiv A_{0'} \) in Problem 2, we obtain the weighted Fermat-Steiner problem for a boundary tetrahedron \( A_1 A_2 A_3 A_4 \) in \( \mathbb{R}^3. \)

We denote by \( H \) the length of the common perpendicular (line segment) between the two lines defined by \( A_1 A_2, A_4 A_3, \) by \( a_{ij} \) the length of the line segment \( A_i A_j \) and by \( \alpha_{ij,k} \) the angle \( \angle A_i A_j A_k \) at \( A_j, \) for \( i, j, k = 0, 0', 1, 2, 3, 4. \)

The weighted Fermat-Steiner problem for \( A_1 A_2 A_3 A_4 \) in \( \mathbb{R}^3 \) states that \[39\]:

**Problem 6.** \[39\] Problem 5] Find \( A_0(x_0, y_0, z_0) \) and \( A_{0'}(x_{0'}, y_{0'}, z_{0'}) \) with given weights \( b_0 \) in \( A_0 \) and \( b_{0'} \) in \( A_{0'}, \) such that

\[ f(A_0, A_{0'}) = b_1 a_{01} + b_2 a_{02} + b_3 a_{03} + b_4 a_{04} + \frac{b_0 + b_{0'}}{2} a_{00'} \rightarrow \min. \tag{2.9} \]
By substituting $a_{00'} = 0$ in (2.9) for $A_{0'} \rightarrow A_0$, we get the weighted Fermat problem for $A_1 A_2 A_3 A_4$ (34). The solution(s) of the weighted Fermat-Steiner problem is a weighted Fermat-Steiner tree and $A_0$, $A_{0'}$ are named as weighted Fermat-Steiner points. If one of the two weighted Fermat-Steiner points is a fixed boundary vertex of $A_1 A_2 A_3 A_4$, some degenerate cases may occur and the corresponding trees are degenerate Fermat-Steiner trees. The unique solution of the weighted Fermat problem for $A_1 A_2 A_3 A_4$, is a weighted Fermat tree and $A_0$ is named as the weighted Fermat point. If $A_0$ is a fixed boundary vertex of $A_1 A_2 A_3 A_4$, the corresponding tree is a degenerate (absorbing) weighted Fermat tree.

We proceed by giving the definitions of the weighted Fermat-Steiner tree and the weighted Fermat tree for $A_1 A_2 A_3 A_4$ in $\mathbb{R}^3$.

**Definition 9.** A weighted Fermat-Steiner tree $T_S(A_1 A_2; A_3 A_4)$ is a union of the weighted line segments $\{A_1 A_0, A_2 A_0, A_0 A_{0'}, A_3 A_{0'}, A_4 A_{0'}\}$ with corresponding weights $\{b_1, b_2, \frac{b_0 + b_{0'}}{2}, b_3, b_4\}$.

**Definition 10.** A weighted Fermat tree $T_F(A_1 A_2, A_3 A_4)$ is a union of the weighted line segments $\{A_1 A_0, A_2 A_0, A_3 A_0, A_4 A_0\}$ with corresponding weights $\{b_1, b_2, b_3, b_4\}$.

We continue by mentioning two lemmas regarding the necessary and sufficient conditions for the existence of the two non-degenerate weighted Fermat points $A_0$ and $A_{0'}$, (weighted Fermat-Steiner points) and the angular solutions with respect to $A_0$ and $A_{0'}$. 

---

**Figure 1.** A weighted Fermat-Steiner tree for a Frechet tetrahedron in $\mathbb{R}^3$. 

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We set
\[ r_{12} = \frac{a_{12}}{(b_1 + b_2 + \frac{b_0 + b_{0'}}{2})(b_1 + b_2 - \frac{b_0 + b_{0'}}{2})(b_2 + \frac{b_0 + b_{0'}}{2} - b_1)(b_1 + \frac{b_0 + b_{0'}}{2} - b_2)}, \]
\[ r_{34} = \frac{a_{34}}{(b_3 + b_4 + \frac{b_0 + b_{0'}}{2})(b_3 + b_4 - \frac{b_0 + b_{0'}}{2})(b_4 + \frac{b_0 + b_{0'}}{2} - b_3)(b_4 + \frac{b_0 + b_{0'}}{2} - b_3)}, \]
\[ \beta_{12} = \arccos\left(\frac{a_{12}}{2r_{12}}\right), \]
\[ \beta_{34} = \arccos\left(\frac{a_{34}}{2r_{34}}\right). \]

We suppose that \( A_1, A_2 \) lie on the x-axis and satisfy \( x_1 < x_2 \). Let \( C = (x(C), y(C), z(C)) \in \mathbb{R}^3 \).

**Lemma 5.** Existence of a weighted Fermat-Steiner tree in \( \mathbb{R}^3 \).\cite{39} Theorem 2

The following inequalities provide the necessary and sufficient conditions for the existence of the two non-degenerate weighted Fermat points \( A_0 \) and \( A_{0'} \):
\[
\sqrt{y(C)^2 + z(C)^2} \left( \frac{x_1 - x(C)}{|x_1 - x(C)|} \right) > \tan\left( \arccos\left( \frac{(b_0 + b_{0'})^2 - b_1^2 - b_1^2}{2b_1b_2} \right) \right),
\]
\[
(\sqrt{y(C)^2 + z(C)^2} + r_{12}\beta_{12})^2 + \left( \frac{x_1 + x_2}{2} - x(C) \right)^2 > r_{12}^2,
\]
\[
\sqrt{y(C)^2 + z(C)^2} \left( \frac{x_4 - x(C)}{|x_4 - x(C)|} \right) > \tan\left( \arccos\left( \frac{(b_0 + b_{0'})^2 - b_3^2 - b_3^2}{2b_3b_4} \right) \right),
\]
\[
(\sqrt{y(C)^2 + z(C)^2} + r_{34}\sin\beta_{34})^2 + \left( \frac{x_3 + x_4}{2} - x(C) \right)^2 > r_{34}^2.
\]

We set \( b_{ST} \equiv \frac{b_0 + b_{0'}}{2} \).

**Lemma 6.** \cite{39} Theorem 3

The solution of the weighted Steiner problem is a weighted Steiner tree in \( \mathbb{R}^3 \) whose nodes \( A_0 \) and \( A_{0'} \) (weighted Fermat points) are seen by the angles:
\[
\cos \alpha_{102} = \frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1b_2},
\]
\[
\cos \alpha_{012} = \frac{b_2^2 - b_0^2 - b_{ST}^2}{2b_2b_{ST}},
\]
\[
\cos \alpha_{30}^4 = \frac{b_{ST}^2 - b_3^2 - b_4^2}{2b_3b_4},
\]
\[
\cos \alpha_{34}^0 = \frac{b_3^2 - b_0^2 - b_{ST}^2}{2b_3b_{ST}}.
\]

In \cite{24}, Melzak constructed an algorithm of circles to find Steiner tree topologies for an N-convex polygon in \( \mathbb{R}^2 \). Unfortunately, Melzak cannot be extended in \( \mathbb{R}^3 \).

In \cite{20}, Rubinstein, Thomas and Weng succeeded in solving numerically the unweighted Fermat-Steiner problem, by applying a fixed point iteration method to a system of two equations with two variable line segments. In \cite{39} Theorem 4], we extended Rubinstein, Thomas and Weng method to solve the weighted Fermat-Steiner problem for tetrahedra in \( \mathbb{R}^3 \). We note that extended Rubinstein Thomas method takes into account the coordinates \( (x_i, y_i, z_i) \) of each vertex \( A_i \), for \( i = 1, 2, 3, 4 \). Therefore, we need to find a method to use the six edge lengths of the tetrahedron.
and some variable edge lengths, in order to consider the weighted Fermat-Steiner-Frechet problem for Frechet tetrahedra in \( \mathbb{R}^3 \).

We denote by \( D(S) \) the Cayley-Menger determinant:

\[
D(S) = \det \begin{pmatrix}
0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 1 \\
a_{12}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 \\
a_{13}^2 & a_{23}^2 & 0 & a_{34}^2 & 1 \\
a_{14}^2 & a_{24}^2 & a_{34}^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\] (2.15)

We consider the \( 3 - \text{INV W F} \) problem in \( \mathbb{R}^3 \). Let \( A_0 \) be a weighted Fermat point inside \( A_1A_2A_3A_4 \) in \( \mathbb{R}^3 \). In [34], the following relation is proved:

**Lemma 7.** [34, Formula (2.25)] If \( b_{i0} \) is the weight, which corresponds to the vertex \( A_i \):

\[
\| \sum_{j=1, j \neq i}^{4} b_{j0} \tilde{a}_{ij} \| > b_{i0},
\] (2.16)

for \( i, j = 1, 2, 3, 4 \) holds, then

\[
b_{30} = \frac{b_{i0}}{a_{03} \text{Vol}(A_0A_1A_2A_4)} = \frac{b_{40}}{a_{04} \text{Vol}(A_0A_1A_2A_3)},
\] (2.17)

\[
b_{30} = \frac{b_{10}}{a_{03} \text{Vol}(A_0A_1A_2A_4)} = \frac{b_{10}}{a_{01} \text{Vol}(A_0A_2A_3A_4)},
\] (2.18)

\[
b_{30} = \frac{b_{20}}{a_{03} \text{Vol}(A_0A_1A_2A_4)} = \frac{b_{20}}{a_{02} \text{Vol}(A_0A_1A_3A_4)},
\] (2.19)

and

\[
b_{10} = \frac{b_{20}}{a_{01} \text{Vol}(A_0A_2A_3A_4)} = \frac{b_{20}}{a_{02} \text{Vol}(A_0A_1A_3A_4)},
\] (2.20)

where \( C = \sum_{i=1}^{4} \frac{b_{i0}}{\text{Vol}(A_iA_2A_3A_4)} \).

The volumes of \( A_0A_iA_jA_k \) \( \text{Vol}(A_0A_iA_jA_k) \) for \( i, j, k = 1, 2, 3, 4 \), can be computed via the Caley-Menger determinant ([31 pp. 249-255]):

\[
288 \text{Vol}(A_iA_jA_k)^2 = D(\{a_{0i}, a_{0j}, a_{0k}, a_{ij}, a_{ik}, a_{jk}\}).
\]

3. **The weighted Fermat-Steiner-Frechet multitree for a given sextuple of positive real numbers determining the edge lengths of incongruent tetrahedra in \( \mathbb{R}^3 \)**

In this section, we focus on the solution (multitree) of the weighted Fermat-Steiner-Frechet problem \( (P(\text{Fermat-Steiner-Frechet})) \), by inserting some equality constraints derived by two independent solutions for two new variable weighted Fermat problems for the Frechet multitetrahedron derived by incongruent boundary tetrahedra in \( \mathbb{R}^3 \), which correspond to the same sextuple of positive real numbers (edge lengths) and an equality constraint derived by two different expressions of the line segment connecting the two weighted Fermat-Steiner points. The detection of the weighted Fermat-Steiner Frechet multitrees is achieved by applying the Lagrange multiplier rule.
We give a vector proof of the law of cosine law in $\mathbb{R}^3$, which has been introduced in [34], by using addition and inner product of vectors in $\mathbb{R}^3$.

We denote by $\alpha$ the dihedral angle defined by the planes formed by $\triangle A_0 A_1 A_2$ and $\triangle A_1 A_2 A_3$, the dihedral angle $\alpha_{g_i}$ defined by the planes formed by $\triangle A_1 A_2 A_3$ and $\triangle A_1 A_2 A_i$, by $h_{0,12}$ the height of $\triangle A_0 A_1 A_2$ from $A_0$, by $h_{0,12m}$ the distance of $A_0$ from the plane defined by $\triangle A_1 A_2 A_m$, for $i, j, k = 0, 1, 2, 3, 4$ and $m = 3, 4$, by $A_{0,12}$ the trace of the orthogonal projection of $A_0$ to $A_1 A_2$ by $A_{0,123}$, the trace of the orthogonal projection of $A_0$ to the plane defined by $\triangle A_1 A_2 A_3$, by $x_{(0),2}$ the length of the line segment $A_{0,12} A_2$, by $x_{(0),2}$ the length of the line segment $A_{0,123} A_2$.

Lemma 8. Generalized cosine law in $\mathbb{R}^3$, [34]

The line segment $a_{i0}$ depends on $a_{10}, a_{20}$ and $\alpha$ in $\mathbb{R}^3$:

$$a_{i0}^2 = a_{20}^2 + a_{2i}^2 - 2a_{2i}\left[\sqrt{a_{20}^2 - h_{0,12}^2 \cos(\alpha_{12i}) + h_{0,12} \sin(\alpha_{12i}) \cos(\alpha_{gi} - \alpha)}\right], \quad (3.1)$$

or

$$a_{i0}^2 = a_{10}^2 + a_{1i}^2 - 2a_{1i}\left[\sqrt{a_{10}^2 - h_{0,12}^2 \cos(\alpha_{21i}) + h_{0,12} \sin(\alpha_{21i}) \cos(\alpha_{gi} - \alpha)}\right]. \quad (3.2)$$

for $i = 3, 4$.

Proof. First, we start with the elementary observation

$$\bar{a}_{02} = \bar{h}_{0,12} + \bar{x}_{(0),2}.$$
The inner product \( \vec{a}_0 \cdot \vec{a}_0 \) yields the cosine law for \( \triangle A_0 A_1 A_2 \) in \( \mathbb{R}^2 \):

\[
a_{20}^2 = a_{10}^2 + a_{12}^2 - 2a_{10}a_{12} \cos \alpha_{012}.
\]

We consider the following vector equality in \( \mathbb{R}^3 \):

\[
\vec{a}_{03} = \vec{h}_{0,123} + \vec{x}_{(0,123),2} + \vec{a}_{23}.
\]

Taking the inner product \( \vec{a}_{03} \cdot \vec{a}_{03} \) and by setting \( \phi = \angle A_1 A_2 A_0 \), we get:

\[
a_{03}^2 = a_{02}^2 + a_{23}^2 - 2\vec{x}_{(0,123),2} a_{23} \cos(\alpha_{123} - \phi).
\]  \(\text{(3.3)}\)

From \( \triangle A_{0,12} A_2 A_{0,123} \), we get:

\[
\cos \phi = \frac{\sqrt{a_{10}^2 - \vec{h}_{0,12}^2}}{x_{(0,123),2}}
\]  \(\text{(3.4)}\)

\[
\sin \phi = \frac{h_{0,12} \cos \alpha}{x_{(0,123),2}}
\]  \(\text{(3.5)}\)

By substituting (3.4) and (3.5) in (3.3), we obtain (3.6) for \( i = 3 \) and \( \alpha_{g_3} = 0 \).

By changing the index \( 3 \to 4 \), we derive (3.6) for \( i = 4 \) and \( \alpha_{g_4} \neq 0 \).

Taking into account the following vector equality in \( \mathbb{R}^3 \),

\[
\vec{a}_{03} = \vec{h}_{0,123} + \vec{x}_{(0,123),1} + \vec{a}_{13}.
\]

and following the same process, we obtain (3.7) for \( i = 3 \) and \( i = 4 \).

\(\square\)

If we substitute \( \alpha_{g_3} = \alpha_{g_4} = \alpha \), we obtain a generalization of the cosine law in \( \mathbb{R}^2 \):

**Lemma 9** (Generalized Cosine law in \( \mathbb{R}^2 \)). The line segment \( a_{i0} \) depends on \( a_{10}, a_{20} \) in \( \mathbb{R}^2 \):

\[
a_{i0}^2 = a_{i0}^2 + a_{i1}^2 - 2a_{i0}a_{i1}[\sqrt{a_{i0}^2 - h_{0,12}^2 \cos(\alpha_{12i})} + h_{0,12} \sin(\alpha_{12i})],
\]  \(\text{(3.6)}\)

or

\[
a_{i0}^2 = a_{i0}^2 + a_{i1}^2 - 2a_{i0}a_{i1}[\sqrt{a_{i10}^2 - h_{0,12}^2 \cos(\alpha_{21i})} + h_{0,12} \sin(\alpha_{21i})].
\]  \(\text{(3.7)}\)

where

\[
h_{0,12} = \frac{a_{10}a_{20}}{a_{12}} \sqrt{1 - \left(\frac{a_{10}^2 + a_{20}^2 - a_{12}^2}{2a_{10}a_{20}}\right)^2}
\]

for \( i = 3, 4 \).

By solving (3.6) with respect to \( \alpha \), for \( i = 3 \), we get:

\[
\alpha = \arccos\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{02}a_{23}} - \sqrt{a_{02}^2 - h_{0,12}^2 \cos \alpha_{123}} \right) / h_{0,12} \sin \alpha_{123}.
\]

Therefore, by substituting \( \alpha \) function in (3.6) for \( i = 4 \), we obtain a functional dependence of \( a_{40} \) in terms of lengths of line segments (3.7).
Lemma 10. Proposition 1] The variable length \( a_{40} \) depends on the three variable lengths \( a_{10}, a_{20}, a_{30} \) and the given sextuple of positive real numbers \( S = \{a_{12}, a_{13}, a_{14}, a_{24}, a_{23} \} \) determining the edge lengths of incongruent tetrahedra in \( \mathbb{R}^3 \), by taking into account the following relations:

\[
\cos \alpha_{123} = \frac{a_{12}^2 + a_{23}^2 - a_{13}^2}{2a_{12}a_{23}},
\]

\[
\sin \alpha_{123} = \frac{\sqrt{(a_{12} + a_{23} + a_{13})(a_{23} + a_{13} - a_{12})(a_{12} + a_{13} - a_{23})(a_{12} + a_{23} - a_{13})}}{2a_{12}a_{23}}
\]

\[
h_{0,12} = h_{0,12}(a_{01}, a_{02}; a_{12}) = \frac{a_{01}a_{02}}{a_{12}} \sqrt{1 - \left( \frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01}a_{02}} \right)^2},
\]

\[
\cos \alpha_{124} = \frac{a_{12}^2 + a_{24}^2 - a_{14}^2}{2a_{12}a_{24}},
\]

\[
\sin \alpha_{124} = \frac{\sqrt{(a_{12} + a_{24} + a_{14})(a_{24} + a_{14} - a_{12})(a_{12} + a_{14} - a_{24})(a_{12} + a_{24} - a_{14})}}{2a_{12}a_{24}}
\]

\[
\alpha_{41} = \arccos \left( \frac{\left( \frac{a_{12}^2 + a_{23}^2 - a_{13}^2}{2a_{23}} \right) - \frac{a_{12}^2 - h_{4,12}^2 \cos \alpha_{123}}{h_{4,12} \sin \alpha_{123}}}{h_{4,12} \sin \alpha_{123}} \right)
\]

and

\[
h_{4,12} = h_{4,12}(a_{41}, a_{42}, a_{12}) = \frac{a_{41}a_{42}}{a_{12}} \sqrt{1 - \left( \frac{a_{41}^2 + a_{42}^2 - a_{12}^2}{2a_{41}a_{42}} \right)^2}.
\]

The weighted Fermat-Steiner-Frechet problem for a given sextuple of edge lengths determining tetrahedra in \( \mathbb{R}^3 \), states that:

Problem 7 (The weighted Fermat-Steiner Frechet in \( \mathbb{R}^3 \)). Given a sextuple of weights \( \{b_1, b_2, b_3, b_4, b_{ST}, b_{ST} \} \), and a given sextuple of positive real number (edge lengths) \( \{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \} \), determining a Frechet multitetrahedron \( F(A_1A_2A_3A_4) \), find the position of \( A_0 \) and / or \( A_0' \) with given weights \( b_{ST} \) in \( A_0 \) and \( b_{ST} \) in \( A_0' \), such that

\[
f_0(a_{01}, a_{02}, a_{03}, a_{04}, a_{00}) = b_1a_{01} + b_2a_{02} + b_3a_{03} + b_4a_{04} + b_{ST}a_{00} \rightarrow \text{min.} \quad (3.8)
\]

Definition 11 (A non degenerate weighted Fermat-Steiner tree for \( \{A_1A_2A_3A_4\} \)). A non degenerate weighted Fermat-Steiner tree is the minimum of the weighted Fermat-Steiner trees \( T_5(A_1A_2; A_3A_4) \), \( T_5(A_1A_4; A_2A_3) \), and \( T_5(A_1A_3; A_2A_4) \).

Definition 12 (A degenerate weighted Fermat-Steiner (Gauss) tree for \( A_1A_2; A_3A_4 \)). A degenerate weighted Fermat-Steiner tree or Gauss tree is a weighted minimal tree, such that one of two vertices \( A_0 \) or \( A_0' \) coincides with \( A_1 \) or \( A_2 \) or \( A_3 \) or \( A_4 \), respectively.
We continue by constructing the Lagrangian function
\[ \mathcal{L}(\hat{x}, \hat{\lambda}) = \sum_{i=0}^{7} \lambda_i f_i(\hat{x}). \]
where the point \( \hat{x} = \{x_1, \ldots, x_{12}\} = \{a_{10}, a_{20}, a_{30}, a_{40'}, a_{40}, B_{10}, B_{20}, B_{30}, B_{10'}, B_{20'}, B_{30'}\} \in \mathbb{R}^{12} \)
is inside the parallelepiped \( \Pi(p_1, q_1; \ldots; p_{12}, q_{12}) \), where \( p_i < x_i < q_i \), for \( i = 1, 2, \ldots, 12 \) and the Lagrange multiplication vector is given by:
\[ \hat{\lambda} = \{\lambda_0, \lambda_1, \ldots, \lambda_7\}. \]
We shall deal with the weighted Fermat-Steiner-Frechet problem \((P(\text{Fermat-Steiner-Frechet}))\), by inserting some equality constraints derived by two independent solutions for two new weighted Fermat problems for \( A_1A_2A_3A_4 \) in \( \mathbb{R}^3 \), which give a connection with the initial weighted Fermat-Steiner objective function and an equality constraint derived by two different expressions of \( a_{00'} \).

**Problem 8** (The weighted Fermat-Steiner-Frechet \((P(\text{Fermat-Steiner-Frechet}))\) problem in \( \mathbb{R}^3 \) with equality constraints),

\[
\begin{align*}
\text{min} & \quad f_0(\hat{x}) \\
\text{subject to} & \quad f_i(\hat{x}) = 0, \quad i = 1, \ldots, 7 \\
& \quad f_0(\hat{x}) = b_1a_{01} + b_2a_{02} + b_3a_{03} + b_4a_{04} + b_{ST}a_{00'}, \\
& \quad \text{with equality constraints).}
\end{align*}
\]

\[ f_1(\hat{x}) = \frac{B_{10}}{a_{10} \text{Vol}(A_0A_2A_3A_4)} - \frac{1 - B_{10} - B_{20} - B_{30}}{a_{40}(a_{10}, a_{20}, a_{30}) \text{Vol}(A_0A_1A_2A_3)} \]  \hspace{1cm} (3.10)

\[ f_2(\hat{x}) = \frac{B_{20}}{a_{20} \text{Vol}(A_0A_1A_3A_4)} - \frac{1 - B_{10} - B_{20} - B_{30}}{a_{40}(a_{10}, a_{20}, a_{30}) \text{Vol}(A_0A_1A_2A_3)} \]  \hspace{1cm} (3.11)

\[ f_3(\hat{x}) = \frac{B_{30}}{a_{30} \text{Vol}(A_0A_1A_2A_4)} - \frac{1 - B_{10} - B_{20} - B_{30}}{a_{40}(a_{10}, a_{20}, a_{30}) \text{Vol}(A_0A_1A_2A_3)} \]  \hspace{1cm} (3.12)

\[ f_4(\hat{x}) = \frac{B_{40'}}{a_{40'} \text{Vol}(A_0'A_1A_2A_3)} - \frac{1 - B_{20'} - B_{30'} - B_{40'}}{a_{10'}(a_{20'}, a_{30'}, a_{40'}) \text{Vol}(A_0'A_2A_3A_4)}, \]  \hspace{1cm} (3.13)

\[ f_5(\hat{x}) = \frac{B_{20'}}{a_{20'} \text{Vol}(A_0'A_1A_3A_4)} - \frac{1 - B_{20'} - B_{30'} - B_{40'}}{a_{10'}(a_{20'}, a_{30'}, a_{40'}) \text{Vol}(A_0'A_2A_3A_4)}, \]  \hspace{1cm} (3.14)

\[ f_6(\hat{x}) = \frac{B_{30'}}{a_{30'} \text{Vol}(A_0'A_1A_2A_4)} - \frac{1 - B_{20'} - B_{30'} - B_{40'}}{a_{10'}(a_{20'}, a_{30'}, a_{40'}) \text{Vol}(A_0'A_2A_3A_4)}, \]  \hspace{1cm} (3.15)

\[ f_7(\hat{x}) = a_{00'}(a_{10}, a_{20}, a_{10'}, a_{20'}) - a_{00'}(a_{30}, a_{40}, a_{30}, a_{40}(a_{10}, a_{20}, a_{30})). \]  \hspace{1cm} (3.16)

The next theorem is a direct consequence of the Lagrange multiplier rule given in [30], p. 112 in [3], Theorem 3.1, p. 586, and a particular case of an ordinary convex program involving only equalities in [25], Theorem 28.1]
Theorem 2 (Lagrange multiplier rule for the weighted Fermat-Steiner Frechet multitree in $\mathbb{R}^3$). If the admissible point $\tilde{x}_i$ yields a weighted minimum multitree for $1 \leq i \leq 30$, which correspond to a Frechet multitetrahedron derived by a sextuple of edge lengths determining upto 30 incongruent tetrahedra, then there are numbers $\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{7i}$, such that:

$$\frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{ji}} = 0$$ (3.17)

for $j = 1, 2, \ldots, 12$,

$$\tilde{x}_i = \{(a_{10})_i, (a_{20})_i, (a_{30})_i, (a_{20'})_i, (a_{30'})_i, (a_{40'})_i, (B_{10})_i, (B_{20})_i, (B_{30})_i, (B_{10'})_i, (B_{20'})_i, (B_{30'})_i, \}$$

$$\tilde{\lambda}_i = \{\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{7i}\}.$$

Proof. Taking into account that $\frac{\partial (f_k)_i}{\partial x_{ji}}$ are continuous in each parallelepiped $\Pi$, for $1 \leq i \leq 30, k = 0, 1, 2, \ldots, 7, j = 1, 2, \ldots, 12$ and by applying Lagrange multiplier rule, yields the Lagrangian vector $\tilde{\lambda}_i = \{\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{7i}\}$, such that (3.17) is valid.

We note that the system (3.10)-(3.16), (3.17) contains 19 equations with 19 variables, since we can set one of the Lagrange multipliers $1 ((\lambda_0)_i = 1)$, by the definition of the P(Fermat-Steiner-Frechet) problem.

4. A Lagrange program detecting the most natural sextuples of six consecutive natural numbers

In this section, we apply a Lagrange program to detect weighted Fermat-Frechet multitree for a given sextuple of edge lengths determining 30 incongruent tetrahedra (Frechet multitetrahedron) in $\mathbb{R}^3$, by using Blumenthal, Herzog and Dekster Wilker sextuples. An interesting application of seeking unweighted Fermat-Frechet multitrees with two equally weighted Fermat-Steiner points inside the Frechet multitetrahedron, is to detect the most natural of six consecutive natural numbers (Herzog sextuples) for $N \geq 7$. This result is achieved by seeking an upper bound for these two equal weights, which yield a global weighted Fermat-Steiner tree of minimum length for the boundary tetrahedron having the maximum volume among the 30 incongruent tetrahedra in $\mathbb{R}^3$.

Problem 9 (The Fermat-Steiner-Frechet (P(Fermat-Steiner-Frechet)) problem in $\mathbb{R}^3$ with equality constraints).

$$y_0(\tilde{x}) \to \min,$$

where

$$y_0(\tilde{x}) = a_{01} + a_{02} + a_{03} + a_{04} + b_{ST}a_{00'},$$

and

$$y_i(\tilde{x}) = f_i(\tilde{x}).$$
Proposition 2 (Lagrange multiplier rule for the Fermat-Steiner Frechet multitree in $\mathbb{R}^3$). If the admissible point $\tilde{x}_i$ yields a minimum multitree for $i = 1, 2, \ldots, 30$, which correspond to a Frechet multitetrahedron derived by the Blumenthal, Herzog, or Dekster-Wilker sextuples of edge lengths determining 30 incongruent tetrahedra, then there are numbers $\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{7i}$, such that:

$$
\frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{ji}} = 0
$$

for $j = 1, 2, \ldots, 12$,$\tilde{x}_i = \{(a_{10})_i, (a_{20})_i, (a_{30})_i, (a_{40})_i, (a_{50})_i, (a_{60})_i, (a_{70})_i, \}

(B_{10})_i, (B_{20})_i, (B_{30})_i, (B_{40})_i, (B_{50})_i, (B_{60})_i, (B_{70})_i, \}

\tilde{\lambda}_i = \{\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{7i}\}.$$

Proof. It is a direct consequence of Theorem 2 for Blumenthal, Herzog, Dekster, Wilker sextuples determining thirty incongruent tetrahedra in $\mathbb{R}^3$. □

Remark 1. From Lemmas 2.6, 4, 2.7 yields that the weight $B_{10}$ depends on five given angles $\alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204}$, such that:

$$
\alpha_{102} = \arccos\left(\frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1b_2}\right),
$$

$$
\alpha_{10j} = \arccos\left(\frac{a_{10}^2 + a_{j0}^2 - a_{i0}^2}{2a_{10}a_{j0}}\right).
$$

Therefore, we get:

$$
B_{10} = B_{10}(a_{10}, a_{20}, a_{30}; b_1; b_2; b_{ST}),
$$

for $i = 1, 2, 3, 4$. By following a similar process, we get:

$$
B_{10} = B_{10}(a_{20}, a_{30}, a_{40} b_3; b_4; b_{ST}),
$$

for $i = 1, 2, 3, 4$.

Theorem 3. The most natural sextuple of numbers from six consecutive natural numbers $\{a + 5, a + 4, a + 3, a + 2, a + 1, a, \}$ for $a \geq 7$ is a sextuple of edge lengths having the maximum volume (maximum sextuple) among the 30 incongruent tetrahedra, which corresponds a Fermat-Steiner tree of minimum total weighted length (global minimum solution), such that the upper bound for the weight $B_{ST}$ is determined by the rest Fermat-Steiner minimal trees having larger or equal weighted minimal total length.

Proof. By applying Proposition 2 for the Herzog sextuple of edge lengths $\{a + 5, a + 4, a + 3, a + 2, a + 1, a, \}$ for $a \geq 7$ forming 30 incongruent tetrahedra in $\mathbb{R}^3$, we obtain 90 minimum trees for a given weight $B_{ST}$, $T_S((A_1 A_2; A_3 A_4)_j)$, $T_S((A_1 A_4; A_2 A_3)_j)$ and $T_S((A_1 A_3; A_2 A_4)_j)$ which correspond to each derived tetrahedron $(A_1 A_2; A_3 A_4)_j$, for $j = 1, 2, \ldots, 30$. If $B_{ST}$ yields a global minimum tree of the tetrahedron with edge lengths that belongs to $\{a + 5, a + 4, a + 3, a + 2, a + 1, a, \}$ having maximum volume, then we derive the most natural sextuple of the six consecutive natural numbers $\{a + 5, a + 4, a + 3, a + 2, a + 1, a, \}$ otherwise we consider the variable weight $B_{ST}$, in order to perturb the length of the minimum tree only.
for the maximum sextuple. Hence, we obtain an upper bound for $B_{ST}$, by comparing the length of the minimum tree derived for the maximum sextuple with the rest Fermat-Steiner trees, which correspond to the rest 29 incongruent tetrahedra. □

Example 1. Consider 30 incongruent tetrahedra derived by six consecutive natural numbers having edge lengths \{7, 8, 9, 10, 11, 12\} where $f_0 = a_1 + a_2 + a_3 + a_4$, for $b_1 = b_2 = b_3 = b_4 = 1$, and $a_{00} = 0$. This is the first tetrahedral sextuple of sequential positive integers forming 30 tetrahedra (Blumenthal-Herzog). We take these 30 deformations of a tetrahedron (plasticity of the boundary of a tetrahedron) and we compute the length of each Fermat tree. The radius $R$ corresponds to the circumscribed sphere of a tetrahedron having six edges $a_{12}$, $a_{43}$, $a_{13}$, $a_{23}$, $a_{24}$, $a_{14}$. We may expect that nature chooses the minimum communication among these 30 deformation on a boundary tetrahedron having the maximum volume or on a sphere having maximum volume, but computations do not give such a result for Fermat trees having one node (Fermat point) inside each derived tetrahedron. The minimum of the minimum communication is achieved by the edge lengths \{a_{12}, a_{43}, a_{13}, a_{23}, a_{24}, a_{14}\} = \{12, 7, 11, 10, 8, 9\}, having a Fermat tree of minimum length 22.7838, with respect to the derived boundary tetrahedron without having the maximum volume and the corresponding circumscribed sphere with radius 6.59837 without having the maximum volume.
The maximum volume of the 30 incongruent tetrahedra corresponds to the following edge lengths \( \{a_{12}, a_{43}, a_{13}, a_{23}, a_{24}, a_{14}\} = \{12, 7, 11, 9, 8, 10\} \), which yields a Fermat tree having minimal length 22.9123 > 22.7838.

Hence, we need to add one more node \( A' \), in order to obtain the Fermat-Steiner-Frechet multitree for the thirty incongruent tetrahedra (Frechet multitetrahedron) derived by the consecutive sextuple of natural numbers \( \{12, 11, 10, 9, 8\} \) and to reduce the total length for each component of the Fermat-Steiner-Frechet multitree.

Thus, by applying the Lagrangian program of Theorem 3 for \( \{a_{12}, a_{43}, a_{13}, a_{23}, a_{24}, a_{14}\} = \{12, 7, 11, 9, 8, 10\} \), we may derive an upper bound for the variable weight \( B_{ST} \).

5. A Theoretical construction of a weighted Fermat-Steiner-Frechet multitree for a Frechet multitetrahedron in \( \mathbb{R}^3 \)

In this section, we describe a theoretical construction to locate a weighted Fermat-Steiner-Frechet multitree for Blumenthal-Herzog, Dekster-Wilker sextuples.
determining the edge lengths of Frechet multitetrahedra in $\mathbb{R}^3$, giving all the necessary notations, which are used to develop a system of two equations, which depend on two variable dihedral angles and some given metric Euclidean elements. This system of equations may be solved using fixed point Banach-Peano functional iteration.

We denote by
- $l_1$ a line, which passes through $M_{12}$ and is parallel to the line defined by $A_3, A_4$,
- $l_2$ a line, which passes through $T_{12}$ and is parallel to $l_1$,
- $A'_4, T'_1, H'_3$ the orthogonal projection of $A_4, T_1, H_3$, to $l_1$, respectively,
- $E_{34}$ the trace of the orthogonal projection of $T_{12}$ to $l_1$
- $E_{12}$ the trace of the orthogonal projection of $T_{34}$ to the line defined by $A_1, A_2$.
- $Q$ the intersection point of the line defined by $A_{34}, H_{34}$ with the plane defined by $l_1$ and $A_1$.
- $P$ the intersection point of the line, which passes through $T_{34}$ and is parallel to the line defined by $A_{34}, H_{34}$ with the plane defined by $l_1$ and $A_1$.
- $\delta_{12}$ the dihedral angle defined by $\triangle A_1A_2A_{12}$ and the plane perpendicular to $\bar{a}_{12} \times \bar{a}_{34}$ and by $\delta_{34}$ the dihedral angle defined by the plane $A_3A_4A_{34}$ and the plane perpendicular to $\bar{a}_{12} \times \bar{a}_{34}$.
- $A''_4$ is the trace of the orthogonal projection of $A_{34}$ to the plane, which passes through the line defined by $A_4, A_3$ and is parallel to the plane defined by $l_1, A_1$.
- $A'_3$ is the trace of the orthogonal projection of $A''_{34}$ to the plane defined by $l_1, A_1$.
- $A'_{12}$ is the trace of the orthogonal projection of $A_{12}$ to the plane defined by $l_1, A_1$.

We set
- $\delta_{34} \equiv \angle A''_{34}H_{34}A_{34}, \delta_{12} \equiv \angle A'_{12}H_{12}A_{12},$
- $\omega_{12} \equiv \angle PT_{12}T'_{34}, \omega_{34} \equiv \angle E_{12}T'_{34}T_{12},$
- $\alpha \equiv \angle A_{34}T_{34}A''_{34} = \angle A_{12}T_{12}A'_{12}.$

We assume that $45^\circ < \varphi < 90^\circ$ and $E_{12} \in [M_{12}, A_1], T_{12} \in [A_1, H_{12}]$ and $T_{34} \in [A_4, H_{34}]$ (Fig.3).

**Theorem 4.** $T_{34}, P, T_{12}, E_{12}, E_{34}$ belong to the same circle.

**Proof.** $P$ is the intersection point of the line $l_2$, and the line, which passes through $T_{34}$ and is parallel to the line defined by $H_{34}Q$. Thus, we obtain that $\delta_{34} = \angle T_{34}PT_{34} = \angle H_{34}QH_{34}$ and $T_{12}, P, Q$ are collinear and belong to $l_2$. Therefore, $\angle T_{34}PT_{12} = 90^\circ$ and taking into account that $\angle T_{34}E_{12}T_{12} = \angle T_{12}E_{34}T''_{34} = 90^\circ$, we derive that $T_{34}T_{12}$ is a diameter of a circle, which is seen by $90^\circ$ from $P, E_{12}, E_{34}$. Therefore, $T'_{34}, P, T_{12}, E_{12}, E_{34}$ are concircular. 

**Theorem 5.** The position of the weighted Simpson line defined by $T_{12}, T_{34}$ is given by the following two equations, which depend on $\delta_{12}$ and $\delta_{34}$:

$$\cot \delta_{12} = \frac{M_{34}H_{34} \sin \varphi + h_{34} \cos \delta_{34} \cos \varphi}{H + h_{34} \sin \delta_{34}} \quad (5.1)$$

$$\cot \delta_{34} = \frac{M_{12}H_{12} \sin \varphi + h_{12} \cos \delta_{12} \cos \varphi}{H + h_{12} \sin \delta_{12}} \quad (5.2)$$

**Proof.** From the theoretical construction of the weighted Simpson line defined by $O_{12}O_{34}$, which intersects the two edges $A_4A_3, A_1A_2$ of the tetrahedron $A_1A_2A_3A_4$, we get (Fig.3):
We mention some useful relations derived by this theoretical construction:

\[ \angle T_{34}'E_{12}T_{34} = \delta_{12}, \]

\[ T_{34}'P = H \cot \delta_{34}, \quad (5.3) \]

\[ T_{34}'T_{12} = H \cot \alpha, \quad (5.4) \]

\[ T_{34}'E_{12} = H \cot \delta_{12}, \quad (5.5) \]

\[ \angle E_{12}T_{12}P = \pi - \varphi, \quad (5.6) \]

\[ \angle E_{12}T_{34}'P = \varphi. \quad (5.7) \]

The similarity of \( \triangle A_{12}A_{12}'H_{12} \) and \( \triangle T_{34}T_{34}'E_{12} \), which are perpendicular to the line defined by \( M_{12}A_{1}H_{12}A_{2}E_{12} \) yields:

\[ E_{12}T_{34}' = H \cot \delta_{12} = (M_{12}T_{34}') \sin \varphi. \quad (5.8) \]

The similarity of triangles \( \triangle A_{34}A_{34}'H_{34} \) and \( \triangle T_{34}T_{34}'P \), which are perpendicular to the parallel lines defined by \( M_{34}A_{3}H_{34}A_{3} \) and \( M_{12}A_{1}'H_{34}'A_{3}' \), respectively, yields:
\[ P T'_{34} = H \cot \delta_{34} = (M_{12} T_{12}) \sin \varphi, \quad (5.9) \]

because \( T_{12} P \) is equal and parallel to \( E_{34} T'_{34} \) and \( T_{12} E_{34} \) is equal and parallel to \( P T'_{34} \).

Thus, we get:

\[ M_{34} T_{34} = M_{12} T'_{34} = M_{12} H'_{34} - T'_{34} H'_{34} \quad (5.10) \]

and

\[ T'_{34} H'_{34} = T_{34} H_{34} = \sqrt{-h_{34}^2 \cos^2 \delta_{34} + (h_{34} \sin \delta_{34} \cot \alpha)^2} = h_{34} \sin \delta_{34} \sqrt{\cot^2 \alpha - \cot^2 \delta_{34}} \quad (5.11) \]

By applying Theorem 4, the quintuple of points \( \{E_{34}, T_{12}, E_{12}, P, T'_{34}\} \) belong to the same circle (see Fig. 4).

Therefore, we get:

\[
\frac{\cot \delta_{12}}{\cot \alpha} = \sin(\omega_{12} + \varphi) = \frac{\cot \delta_{34}}{\cot \alpha} \cos \varphi + \frac{\sin \varphi}{\cot \alpha} \sqrt{\cot^2 \alpha - \cot^2 \delta_{34}}
\]

or

\[
\sqrt{\cot^2 \alpha - \cot^2 \delta_{34}} = \frac{\cot \delta_{12} - \cot \delta_{34} \cos \varphi}{\sin \varphi} \quad (5.12)
\]

By substituting (5.11), (5.12), (5.8), (5.10), we obtain
\[
\frac{H \cot \delta_{12}}{\sin \varphi} = M_{12}H'_{34} - h_{34} \sin \delta_{34} \frac{\cot \delta_{12} - \cot \delta_{34} \cos \varphi}{\sin \varphi}
\]
which yields (5.1).

By following a similar process, we get:
\[
M_{12}T_{12} = M_{12}H_{12} - H_{12}T_{12}
\]
(5.13)
where
\[
M_{12}T_{12} = h_{12} \sin \delta_{12} \sqrt{\cot^2 \alpha - \cot^2 \delta_{12}}.
\]
(5.14)

Taking into account that the points \(E_{34}, T_{12}, E_{12}, F, T'_{34}\) belong to the same circle (Fig. 4), we obtain:
\[
\cot \delta_{34} \cot \alpha = \cos(\varphi - \omega_{34}) = \cot \frac{\delta_{12}}{\cot \alpha} \cos \varphi + \frac{\sqrt{\cot^2 \alpha - \cot^2 \delta_{12}}}{\cot \alpha} \sin \varphi
\]
or
\[
\sqrt{\cot^2 \alpha - \cot^2 \delta_{12}} = \frac{\cot \delta_{34} - \cot \delta_{12} \cos \varphi}{\sin \varphi}.
\]
(5.15)

By substituting (5.14), (5.15), (5.9) in (5.13), we get:
\[
\frac{H \cot \delta_{34}}{\sin \varphi} = M_{12}H_{12} + h_{12} \sin \delta_{12} \frac{\cot \delta_{12} \cos \varphi - \cot \delta_{34}}{\sin \varphi},
\]
which yields (5.2).

A fixed point iteration (Banach or Picard-Peano) method may be used, in order to derive the numerical values of \(\delta_{12}, \delta_{34}\). We set \(x = \cot \delta_{12} \) and \(y = \cot \delta_{34} \).

**Theorem 6.** The following two equations
\[
x = \frac{M_{34}H_{34} \sin \varphi \sqrt{1 + y^2} + h_{34} \cos \varphi y}{H \sqrt{1 + y^2} + h_{34}}
\]
(5.16)
\[
y = \frac{M_{12}H_{12} \sin \varphi \sqrt{1 + x^2} + h_{12} \cos \varphi x}{H \sqrt{1 + x^2} + h_{12}}
\]
(5.17)
represent a functional iteration w.r. to \(x\) or \(y\):
\[
x = f(x)
\]
or
\[
y = g(y).
\]

**Proof.** By substituting \(x = \cot \delta_{12}, y = \cot \delta_{34}\) in (5.1) and (5.2), we derive (5.1) and (5.2).

We proceed by computing \(H, h_{12}, h_{34}, M_{12}H_{12}, M_{34}H_{34}\) with respect to the coordinates of \(A_1, A_2, A_3, A_4\) and the weights \(b_1, b_2, b_3, b_4, b_{ST}\). 

The common perpendicular distance $H = M_{12}M_{34}$ is given by:

$$H = \frac{|x_4 - x_1 \ y_4 - y_1 \ z_4 - z_1\ |}{a_{12}a_{34} \sin \varphi}.$$  \hspace{1cm} (5.18)

The angle $\varphi$ is given by:

$$\varphi = \arccos \left( \frac{\vec{a}_{12} \cdot \vec{a}_{43}}{a_{12}a_{43}} \right).$$

The line segments $M_{12}A_1$ and $M_{12}A'_4$ are given by

$$M_{12}A_1 = \left| \begin{array}{ccc}
  x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\
  x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \\
 v_1 & v_2 & v_3
\end{array} \right| a_{12}$$

and

$$M_{12}A'_4 = M_{34}A_4 = \left| \begin{array}{ccc}
  x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\
  x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
 v_1 & v_2 & v_3
\end{array} \right| a_{43}$$

where

$$v_1 = (y_2 - y_1)(z_3 - z_4) - (y_3 - y_4)(z_2 - z_1),$$
$$v_2 = (z_2 - z_1)(x_3 - x_4) - (x_2 - x_1)(z_3 - z_4),$$
and

$$v_3 = (x_2 - x_1)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_1).$$

By taking into account the equations derived by the angular solution of Lemma 6, we get:

$$\cos \alpha_{12} = \frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1b_2},$$
$$\cos \alpha_1 = \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_2b_{ST}},$$
$$\cos \alpha_{34} = \frac{b_{ST}^2 - b_3^2 - b_4^2}{2b_3b_4},$$
$$\cos \alpha_4 = \frac{b_4^2 - b_3^2 - b_{ST}^2}{2b_3b_{ST}}.$$  \hspace{1cm} (5.21)

or

$$\frac{b_1}{\sin \alpha_1} = \frac{b_2}{\sin \alpha_2} = \frac{b_{ST}}{\sin \alpha_{12}}.$$  \hspace{1cm} (5.22)

$$\frac{b_3}{\sin \alpha_3} = \frac{b_4}{\sin \alpha_4} = \frac{b_{ST}}{\sin \alpha_{34}}.$$  \hspace{1cm} (5.23)
\[
\cos \alpha_{12} = \frac{b_{ST}^2 - B_1^2 - b_2^2}{2b_1b_2}
\]
\[
\cos \alpha_1 = \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_2b_{ST}}
\]
\[
\cos \alpha_{34} = \frac{b_{ST}^2 - b_3^2 - b_4^2}{2b_3b_4}
\]
\[
\cos \alpha_4 = \frac{b_4^2 - b_3^2 - b_{ST}^2}{2b_3b_{ST}}
\]
(5.24)

From (5.22) and \(\triangle A_1A_2A_{12}\) we get:
\[
(A_1A_{12}) = a_{12} \frac{b_2}{b_{ST}}
\]
(5.25)

and the height
\[
(A_{12}H_{12}) \equiv h_{12} = A_1A_2 \left( \frac{b_2}{b_{ST}} \right) \sqrt{1 - \left( \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_2b_{ST}} \right)^2},
\]
(5.26)
\[
(A_1H_{12}) = A_1A_2 \frac{b_2}{b_{ST}} - \frac{b_3^2 + b_2^2 + b_{ST}^2}{2b_2b_{ST}}.
\]
(5.27)

By adding (5.19) and (5.27), we obtain:
\[
M_{12}H_{12} = M_{12}A_1 + A_1H_{12}.
\]

From (5.23) and \(\triangle A_3A_4A_{34}\) we get:
\[
(A_4A_{34}) = a_{34} \frac{b_3}{b_{ST}}
\]
(5.28)

and the height
\[
(A_{34}H_{34}) \equiv h_{34} = a_{34} \left( \frac{b_3}{b_{ST}} \right) \sqrt{1 - \left( \frac{b_3^2 - b_4^2 - b_{ST}^2}{2b_3b_{ST}} \right)^2},
\]
(5.29)
\[
(A_4H_{34}) = a_{34} \frac{b_3}{b_{ST}} - \frac{b_4^2 + b_3^2 + b_{ST}^2}{2b_3b_{ST}}.
\]
(5.30)

By adding (5.20) and (5.30), we obtain:
\[
M_{34}H_{34} = M_{34}A_4 + A_4H_{34}.
\]

□

**Theorem 7.** The location of \(O_{12}\) and \(O_{34}\) is given by the computation of the line segments \(A_1O_{12}, A_2O_{12}, A_3O_{34}, A_4O_{34}\), with respect to \(b_1, b_2, b_3, b_{ST}, A_1T_{12}, A_2T_2, A_3T_{34}, A_4T_{34}\):

\[
A_2O_{12} = A_1O_{12} \left( \frac{A_2T_{12}}{A_1T_{12}} \right) \left( \frac{b_2}{b_1} \right),
\]
(5.31)
\[
A_1O_{12} = A_1A_2 \left[ \sqrt{\left( \frac{A_2T_{12}}{A_1T_{12}} \right)^2 \left( \frac{b_2}{b_1} \right)^2 - 2 \left( \frac{A_2T_{12}}{A_1T_{12}} \right) \left( \frac{b_2}{b_1} \right) \cos \alpha_{12} + 1} \right]^{-1},
\]
(5.32)
Theorem 6, we obtain that

\[ \phi = 74 \]

Remark 2. The position of the weighted Fermat-Steiner-Frechet multi tree for 30 incongruent tetrahedra determined by Blumenthal, Herzog and Dekster-Wilker sextuples of edge lengths in \( \mathbb{R}^3 \) may also be derived by a fixed point functional iteration.

Proof. By applying the sine law in \( \triangle A_1 O_{12} T_{12}, \triangle A_2 O_{12} T_{12}, \triangle A_1 O_{12} A_2, \triangle A_2 O_{12} A_2, \), we get (5.31).

By applying the sine law in \( \triangle A_3 O_{34} T_{34}, \triangle A_4 O_{34} T_{34}, \triangle A_3 O_{34} A_4, \triangle A_4 O_{34} A_4, \), we get (5.32), we derive \( A_1 O_{12} \).

By substituting (5.31) and \( \cos \alpha_{12} \) from the cosine law in \( \triangle A_1 O_{12} A_2 \), on the right hand side of (5.32), we derive \( A_1 O_{12} \).

By substituting (5.33) and \( \cos \alpha_{34} \) from the cosine law in \( \triangle A_3 O_{34} A_4 \), on the right hand side of (5.34), we derive \( A_4 O_{34} \).

Example 2. Given

\( b_1 = 0.85, b_2 = 0.88, b_3 = 0.83, b_4 = 1.08, b_{ST} = 1, \)

\( A_1 = (2, 0, 0), A_2 = (6.86, 1.37, 0), A_3 = (0, 6, 5), A_4 = (0, 0, 5) \) we derive that \( \varphi = 74.25^\circ, H = 5, M_{12} = (0, -0.56, 0), M_{34} = (0, -0.56, 5). \) By inserting these data in Theorem 6, we obtain that \( \delta_{12} = 58.58^\circ, \delta_{34} = 57.75^\circ, \alpha = 52.08^\circ. \)

We note that the quintuple of points deduced from this construction \( P = (3.16, 2.61, 0), E_{34} = (0, 0.33, 0), T_{34} = (0, 2.61, 0), T_{12} = (3.16, 0.33, 0), E_{12} = (0.83, -0.33, 0) \) belong to the same circle.

Definition 8. There are up to 4! \cdot 30 weighted Simpson lines defined by \( T_{12}, T_{34}, \) such that \( O_{12}, O_{34} \) in \( [T_{12}, T_{34}], \) which give the position of the weighted Fermat-Steiner-Frechet multtree for 30 incongruent tetrahedra determined by Blumenthal, Herzog and Dekster-Wilker sextuples of edge lengths in \( \mathbb{R}^3 \).

Proof. It is a direct consequence of Theorems 5-7, taking into account 30 incongruent tetrahedra derived by Blumenthal, Herzog, Dekster-Wilker sextuples multiplied by the permutation of the four weights \( \{b_1, b_2, b_3, b_4\} \) 4!, such that the two equal weights \( b_{12} \) and \( b_{34} \) that correspond to the two weighted Fermat-Steiner points have the same constant value \( b_{ST} \). Therefore, we derive up to 4!30 weighted Fermat-Steiner trees, which yield a weighted Fermat-Steiner-Frechet multtree in \( \mathbb{R}^3 \).
method, which computes some variable lengths instead of variable dihedral angles (see in [39]).

6. Plasticity of weighted Fermat-Frechet multitrees for boundary closed polytopes in \( \mathbb{R}^N \)

In this section, we find the unique solution of the 4–INVWF problem in \( \mathbb{R}^4 \), which depends on exactly nine given angles. We continue, by deriving the unique solution w.r to the \( N \)–INVWF problem in \( \mathbb{R}^N \), which depends on exactly \( \frac{N(N+1)}{2} \) and the non-unique solution (dynamic plasticity) of the \( (N+1) \)–INVWF problem in \( \mathbb{R}^N \). The dynamic plasticity of the \( (N+1) \)–INVWF problem in \( \mathbb{R}^N \) leads to the plasticity of weighted Fermat-Frechet multitrees for boundary closed polytopes in \( \mathbb{R}^N \).

**Theorem 9** (Solution of the 4–INVWF problem in \( \mathbb{R}^4 \)). The weight \( B_i \) is uniquely determined by:

\[
B_i = \frac{C}{1 + \left| \frac{\sin \alpha_{i,k} \sin \alpha_{i,l}}{\sin \alpha_{j,m}} \right| + \left| \frac{\sin \alpha_{j,m}}{\sin \alpha_{i,k} \sin \alpha_{i,l}} \right| + \left| \frac{\sin \alpha_{i,k} \sin \alpha_{i,l}}{\sin \alpha_{j,m}} \right|},
\]

for \( i, j, k, l, m = 1, 2, 3, 4, 5 \) and \( i \neq j \neq k \neq l \neq m \).

*Proof.* We consider the following five unit vectors \( \vec{u}(A_0, A_i) \), for \( i = 1, 2, ... , 5 \), which meet at the weighted Fermat point \( A_0 \):

\[
\vec{u}(A_0, A_1) = (1, 0, 0, 0),
\]

\[
\vec{u}(A_0, A_2) = (\cos \alpha_{102}, \sin \alpha_{102}, 0, 0),
\]

\[
\vec{u}(A_0, A_3) = (\cos \alpha_{3102} \cos \omega_{3,102}, \cos \alpha_{3102} \sin \omega_{3,102}, \sin \alpha_{3102}, 0),
\]

\[
\vec{u}(A_0, A_4) = (\cos \alpha_{41023} \cos \omega_{4,1023} \cos \alpha_{41023} \sin \omega_{4,1023}, \cos \alpha_{41023} \sin \omega_{4,1023}, \sin \alpha_{41023}).
\]

\[
\vec{u}(A_0, A_5) = (\cos \alpha_{51023} \cos \omega_{5,1023} \cos \alpha_{51023} \sin \omega_{5,1023}, \cos \alpha_{51023} \sin \omega_{5,1023}, \sin \alpha_{51023}).
\]

Taking into account \( [6.2] - [6.5] \) the inner products \( \vec{u}(A_0, A_i) \cdot \vec{u}(A_0, A_i) \) for \( i = 1, 2, 3 \) yield:

\[
\cos \alpha_{41023} \cos \omega_{4,1023} \cos \omega_{4,1023} \cos \alpha_{41023} = \cos \alpha_{104},
\]

\[
\cos \alpha_{1023} \cos \omega_{4,1023} \sin \omega_{4,1023} = \frac{1}{\sin \alpha_{102}} (\cos \alpha_{204} - \cos \alpha_{102} \cos \alpha_{104}).
\]
\[
\cos a_{4,1023} \sin \omega_{4,1023} = \frac{1}{\sin a_{3,102}} (\cos \alpha_{304} - \cos \alpha_{103} \cos \alpha_{104} - \left( \frac{\cos \alpha_{203} - \cos \alpha_{102} \cos \alpha_{103}}{\sin \alpha_{102}} \right) \left( \frac{\cos \alpha_{204} - \cos \alpha_{102} \cos \alpha_{104}}{\sin \alpha_{102}} \right)).
\]

(6.9)

By squaring both parts of (6.7) and (6.8) and by adding the two derived equations, we eliminate \( z_{4,1023} \):

\[
\cos^2 a_{4,1023} \cos^2 \omega_{4,1023} = \frac{1}{\sin^2 \alpha_{102}} (\cos \alpha_{204} - \cos \alpha_{102} \cos \alpha_{104})^2 + \cos^2 \alpha_{104}.
\]

(6.10)

By squaring both parts of (6.10) and (6.9) and by adding the two derived equations, we eliminate \( \omega_{4,1023} \), which yields that \( \cos^2 a_{4,1023} \) depends on six angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204} \) and \( \alpha_{304} \), because \( a_{3,102} \) depends on \( \alpha_{102}, \alpha_{103}, \alpha_{203} \) (3.10), p. 120):

\[
\cos^2 a_{3,102} = \frac{\cos^2 \alpha_{204} + \cos^2 \alpha_{103} - 2 \cos \alpha_{102} \cos \alpha_{103} \cos \alpha_{204}}{\sin^2 \alpha_{102}}.
\]

By taking into account (6.11)-(6.14) the inner products \( \vec{u}(A_0, A_5) \cdot \vec{u}(A_0, A_i) \) for \( i = 1, 2, 3 \) yield:

\[
\cos a_{5,1023} \cos \omega_{5,1023} \cos z_{5,1023} = \cos \alpha_{105},
\]

(6.11)

\[
\cos a_{5,1023} \cos \omega_{5,1023} \sin z_{5,1023} = \frac{1}{\sin \alpha_{102}} \cos \alpha_{205} - \cos \alpha_{102} \cos \alpha_{105},
\]

(6.12)

\[
\cos a_{5,1023} \sin \omega_{5,1023} = \frac{1}{\sin a_{3,102}} (\cos \alpha_{305} - \cos \alpha_{103} \cos \alpha_{105} - \left( \frac{\cos \alpha_{203} - \cos \alpha_{102} \cos \alpha_{103}}{\sin \alpha_{102}} \right) \left( \frac{\cos \alpha_{205} - \cos \alpha_{102} \cos \alpha_{105}}{\sin \alpha_{102}} \right)).
\]

(6.13)

\[
\cos a_{4,1023} \sin \omega_{4,1023} = \frac{1}{\sin \alpha_{102}} \cos \alpha_{205} - \cos \alpha_{102} \cos \alpha_{105} + \left( \frac{\cos \alpha_{204} - \cos \alpha_{102} \cos \alpha_{104}}{\sin \alpha_{102}} \right) + \cos a_{4,1023} \sin \omega_{4,1023} \cos a_{5,1023} \sin \omega_{5,1023} + \sin \alpha_{4,1023} \sin \alpha_{5,1023} + \sin \alpha_{4,1023} \sin \alpha_{5,1023}.
\]

(6.14)

By squaring both parts of (6.11) and (6.12) and by adding the two derived equations, we eliminate \( z_{5,1023} \):

\[
\cos^2 a_{5,1023} \cos^2 \omega_{5,1023} = \frac{1}{\sin^2 \alpha_{102}} (\cos \alpha_{205} - \cos \alpha_{102} \cos \alpha_{105})^2 + \cos^2 \alpha_{105}.
\]

(6.15)
By squaring both parts of (6.15) and (6.13) and by adding the two derived equations, we eliminate \( \omega_{5,1023} \), which yields that \( \cos^2 a_{5,1023} \) depends on six angles \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204}, \alpha_{304}, \alpha_{305}, \alpha_{306}, \alpha_{307} \). Thus, we get:

\[
a_{5,1023} = a_{5,1023}^{(102, 103, 104, 203, 204, 304, 305)}.
\]

Therefore, (6.14) yields:

\[
\alpha_{i,1023} = \alpha_{i,1023}^{(102, 103, 104, 203, 204, 304, 305)}.
\]

The angle \( \alpha_{i,1023} \) is the angle formed by \( \vec{u}(A_0, A_i) \) and \( \vec{u}(A_0, A_{i,1023}) \), where \( A_{i,1023} \) is the projection of \( A_i \) w.r. to the subspace of \( A_1, A_0, A_2, A_4 \), for \( i = 4, 5 \).

The normal \( \vec{n}_{1023} \) is orthogonal to \( A_1A_0A_2A_3 \).

Thus, we obtain:

\[
\vec{n}_{0,123} \cdot \vec{u}(A_0, A_i) = \sin \alpha_{i,1023},
\]

for \( i = 4, 5 \).

The weighted floating condition (2.3) for \( N = 5 \) yields:

\[
\sum_{i=1}^{5} B_i \vec{u}(A_0, A_i) = \vec{0}.
\]

(6.17)

By taking the inner product of (6.17) with \( \vec{n}_{0,123} \), we derive that:

\[
\sum_{i=1}^{5} B_i \vec{u}(A_0, A_i) \cdot \vec{n}_{0,123} = 0
\]
or

\[
B_4 \sin \alpha_{4,1023} = B_5 \sin \alpha_{5,1023},
\]

where \( A_4 \) belongs to the upper half space defined by \( A_1A_0A_2A_3 \) and \( A_5 \) belongs to the corresponding lower half space.

By working similarly and by exchanging the indices \( \{1, 2, 3, 4\} \) of the vertices \( A_i \) for \( i = 1, ..., 4 \), we obtain:

\[
B_m \sin \alpha_{m,i0jk} = B_n \sin \alpha_{n,i0jk},
\]

(6.19)

for \( m, n, i, j, k = 1, 2, 3, 4, 5 \), \( m \neq n \neq i \neq j \neq k \).

By replacing (6.19) in (2.4) for \( m = 5 \), we derive (6.1). Thus, the weights \( B_i \) depend on nine given angles \( \alpha_{102}, \alpha_{203}, \alpha_{103}, \alpha_{104}, \alpha_{204}, \alpha_{304}, \alpha_{305}, \alpha_{306}, \alpha_{307} \).

\[\Box\]

**Theorem 10** (Solution of the \( N \)-INVWF problem in \( \mathbb{R}^N \)). The weight \( B_i \) is uniquely determined by:

\[
B_i = \frac{C}{1 + \frac{\sin \alpha_{i0k1} \cdots \frac{k_N-1}{k_N}}{\sin \alpha_{i0k1} \cdots \frac{k_N-1}{k_N}} + \frac{\sin \alpha_{i0k2} \cdots \frac{k_{N-1}}{k_N}}{\sin \alpha_{i0k2} \cdots \frac{k_{N-1}}{k_N}} + \cdots + \frac{\sin \alpha_{i0kN-1} \cdots \frac{k_1}{k_N}}{\sin \alpha_{i0kN-1} \cdots \frac{k_1}{k_N}}.}
\]

(6.20)

for \( i, k_1, k_2, ..., k_N = 1, 2, ..., N + 1 \) and \( k_1 \neq k_2 \neq \cdots \neq k_N \).

**Proof.** We consider \( N + 1 \) unit vectors \( \vec{u}(A_0, A_i) \in \mathbb{R}^N \), for \( i = 1, 2, \ldots, N + 1 \), which meet at the weighted Fermat point \( A_0 \) :

\[
\vec{u}(A_0, A_1) = (1, 0, \ldots, 0),
\]

(6.21)
\[ \vec{u}(A_0, A_2) = (\cos \alpha_{102}, \sin \alpha_{102}, 0, \ldots, 0), \]  
(6.22)

\[ \vec{u}(A_0, A_3) = (\cos a_{3,102} \cos \omega_{3,102}, \cos a_{3,102} \sin \omega_{3,102}, \sin a_{3,102}, 0, \ldots, 0), \]  
(6.23)

\[ \vec{u}(A_0, A_4) = (\cos a_{4,1023} \cos \omega_{4,1023} \cos z_{4,1023}, \cos a_{4,1023} \cos \omega_{4,1023} \sin z_{4,1023}, \cos a_{4,1023} \sin \omega_{4,1023}, \sin a_{4,1023}, 0, \ldots, 0), \]  
(6.24)

\[ \vdots \]

\[ \vec{u}(A_0, A_N) = (\cos a_{N,1023} \ldots N-1 \cos \omega^{(1)}(N), 1023 \ldots N-1 \ldots \cos \omega^{(N-2)}(N), 1023 \ldots N-1, \ldots, \sin a_{N,1023} \ldots N-1), \]  
(6.25)

\[ \vec{u}(A_0, A_{N+1}) = (\cos a_{N+1,1023} \ldots N-1 \cos \omega^{(1)}(N+1), 1023 \ldots N-1 \ldots \cos \omega^{(N-2)}(N+1), 1023 \ldots N-1, \ldots, \sin a_{N+1,1023} \ldots N-1). \]  
(6.26)

By following the same process that used in the proof of Theorem 5 and by using induction the angles \( \omega^{(i)}(N), 1023 \ldots N-1, \omega^{(i)}(N+1), 1023 \ldots N-1, \ldots, \omega^{(i)}(N), 1023 \ldots N-1 \), \( \omega_{3,102}, \omega_{4,1023} \) are eliminated.

\[ \cos^2 a_{N,1023} \ldots N-1, \]  
depends on \( \frac{N(N-1)}{2} \) angles \( \alpha_{102}, \alpha_{103}, \alpha_{10(N-2)}, \alpha_{10(N-1)}, \alpha_{203}, \) \( \alpha_{10(N-2)}, \alpha_{10(N-1)}, \alpha_{203}, \ldots, \alpha_{10(N-2)(N+1)}, \alpha_{10(N+1)}, \alpha_{203}, \ldots, \alpha_{10(N-2)(N+1)} \).

The inner product \( \vec{u}(A_0 A_N) \cdot \vec{u}(A_0, A_{N+1}) \) yields:

\[ \alpha_{N+1} = \alpha_{N+1}^{(1)102}, \alpha_{103}, \alpha_{10(N-2)}, \alpha_{10(N-1)}, \alpha_{10(N+1)}, \ldots, \alpha_{10(N-2)(N+1)} \).

The angle \( a_{i,1023} \ldots (N-1) \) is the angle formed by \( \vec{u}(A_0, A_i) \) and \( \vec{u}(A_0, A_{i,1023} \ldots (N-1)) \), where \( A_{i,1023} \ldots (N-1) \) is the projection of \( A_i \) w.r. to the subspace of \( A_1 A_0 A_2 A_4 \ldots A_{N-1} \) for \( i = N, N+1 \).

The normal \( \vec{n}_{1023} \ldots (N-1) \) is orthogonal to \( A_1 A_0 A_2 \ldots A_{N-1} \).

Hence, we get:

\[ \vec{n}_{1023} \ldots (N-1) \cdot \vec{u}(A_0, A_i) = \sin a_{i,1023} \ldots (N-1), \]  
(6.27)

for \( i = N, N+1 \).

The weighted floating condition \( 2.38 \) yields:

\[ \sum_{i=1}^{N+1} B_i \vec{u}(A_0, A_i) = \vec{0}. \]  
(6.28)

By taking the inner product of \( 6.28 \) with \( \vec{n}_{0,123} \ldots (N-1) \), we derive that:

\[ \sum_{i=1}^{N+1} B_i \vec{u}(A_0, A_i) \cdot \vec{n}_{0,123} \ldots (N-1) = \vec{0} \]

or

\[ B_N \sin a_{N,1023} \ldots (N-1) = B_{N+1} \sin a_{N+1,1023} \ldots (N-1). \]  
(6.29)
By working similarly and by exchanging the indices \{1, 2, 3, \ldots, N, N+1\} of the vertices \(A_i\) for \(i = 1, \ldots, N+1\), we obtain:

\[
B_{i_m} \sin a_{i_m, i_1, i_2, \ldots, i_{N-1}} = B_{i_N} \sin a_{i_N, i_1, i_2, \ldots, i_{N-1}},
\]

(6.30)

for \(m, N, i_m, i_N \in \{1, 2, \ldots, N+1\}, m \neq n, i_m \neq i_N\).

By replacing (6.30) in (2.4), we obtain (6.20). Thus, the weights \(B_i\) depend on \(\alpha_{i_{0j}}\) for \(i, j = 1, 2, \ldots, N+1, i \neq j\) and \(\alpha_{N0N+1} = \alpha_{N0N+1}(\alpha_{i_{0j}})\).

We set \(\tilde{a}_i = \frac{A_i}{A^\top}.\) The corner \((A_0, A_1A_2 \ldots A_n)\) with \(N\) edges is determined by \(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N.\) The \(N\)-dimensional polar sine of the corner \((A_0, A_1A_2 \ldots A_N)\) is defined in [8, p. 76]:

\[
polsin_N(A_0, A_1A_2 \ldots A_N) = \frac{|[\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N]|}{|\tilde{a}_1| |\tilde{a}_2| \ldots |\tilde{a}_N|},
\]

where \(|[\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N]|\) is the content of an \(N\) dimensional parallelootope with sides \(|\tilde{a}_1| |\tilde{a}_2| \ldots |\tilde{a}_N|\).

**Theorem 11** (Weighted volume equalities in \(\mathbb{R}^N\)).

\[
\frac{B_1}{a_1 Vol(A_0A_2A_3 \ldots A_{N+1})} = \frac{B_2}{a_2 Vol(A_1A_0 \ldots A_{N+1})} = \ldots = \frac{B_{N+1}}{a_{N+1} Vol(A_{N+1}A_0 \ldots A_{N+1})} = \frac{\sum_{i=1}^{N+1} B_i}{Vol(A_1A_2 \ldots A_{N+1})}.
\]

(6.31)

**Proof.** By multiplying both members of (6.29) with \(\prod_{i=1}^{N+1} \tilde{a}_i \) \(polsin_{N-1}(A_0, A_1A_2 \ldots A_{N-1})\), we get:

\[
B_N \prod_{i=1}^{N+1} \tilde{a}_i \ powsin_{N-1}(A_0, A_1A_2 \ldots A_{N-1}) \sin a_{N, 1023, \ldots, N-1} =
\]

\[
= B_{N+1} \prod_{i=1}^{N+1} \tilde{a}_i \ powsin_{N-1}(A_0, A_1A_2 \ldots A_{N-1}) \sin a_{N+1, 1023, \ldots, N-1},
\]

(6.32)

The volumes of the \(N\)-dimensional simplexes \(A_0A_1A_2 \ldots A_{N-1}A_N, A_0A_1A_2 \ldots A_{N-1}A_{N+1}\), are given by:

\[
N! Vol(A_0A_1A_2 \ldots A_{N-1}A_N) = \prod_{i=1}^{N+1} \tilde{a}_i \ powsin_{N}(A_0, A_1A_2 \ldots A_N)
\]

(6.33)

\[
N! Vol(A_0A_1A_2 \ldots A_{N-1}A_{N+1}) = \prod_{i=1}^{N+1} \tilde{a}_i \ powsin_{N}(A_0, A_1A_2 \ldots A_{N+1})
\]

(6.34)

By substituting (6.33), (6.34) in (6.32), we obtain:

\[
\frac{B_{N+1}}{a_{N+1} Vol(A_0A_1A_2 \ldots A_{N+1})} = \frac{B_N}{a_N Vol(A_0A_1A_2 \ldots A_{N-1}A_{N+1})}.
\]

(6.35)

By exchanging the indices cyclically \(i \to j\) for \(i, j = 1, 2, \ldots, N, N+1\) and taking into account (6.30), we get (6.31).
For $N = 3$, the following corollary is derived for tetrahedra in $\mathbb{R}^3$:

**Corollary 1.** Weighted volume equalities in $\mathbb{R}^N$, [31]

$$
\frac{B_1}{a_1 \text{Vol}(A_0A_2A_3A_4)} = \frac{B_2}{a_2 \text{Vol}(A_1A_0A_3A_4)} = \frac{B_3}{a_3 \text{Vol}(A_0A_1A_3A_4)} = \frac{B_4}{a_4 \text{Vol}(A_1A_2A_3A_0)} = \frac{\sum_{i=1}^{N} \frac{B_i}{a_i}}{\text{Vol}(A_1A_2A_3A_4)}.
$$

**Definition 13.** We call dynamic plasticity of a variable weighted Fermat tree (weighted network) whose endpoints correspond to a closed polytope in $\mathbb{R}^N$, which is formed by $(N+2)$ weighted line segments meeting at the weighted Fermat point $A_0$, the set of solutions of the $(N+2)$ variable weights with respect to the $(N+1)$-INVF problem in $\mathbb{R}^N$, for a given constant value $c$, which correspond to a family of variable weighted networks that preserve the weighted Fermat point and the boundary of the closed polytope, such that the $(N+1)$ variable weights depend on a variable weight and the value of $c$.

We denote by $(B_i)_{12\ldots N+2}$ the weight which corresponds to the vertex that lies on the ray $A_0A_i$, for $i = 1, 2, \ldots, N + 2$ and the weight $(B_j)_{i_1i_2\ldots i_{N+1}}$ corresponds to the vertex $A_j$ that lies on the ray $A_0A_j$ with respect to the $N$-simplex $A_{i_1}A_{i_2}\ldots A_{i_{N+1}}$, for $i_1, i_2, \ldots, i_{N+1} \in \{1, 2, \ldots, N + 2\}$ and $i_1 \neq i_2 \neq \ldots \neq i_{N+1}$.

**Theorem 12.** The following equations point out the dynamic plasticity of a weighted Fermat tree for $(N+1)$ weighted boundary closed polytopes with respect to the non-negative variable weights $(B_i)_{12\ldots N+2}$ in $\mathbb{R}^N$:

$$
\left(\frac{B_1}{B_{N+1}}\right)_{12\ldots N+2} = \left(\frac{B_1}{B_{N+1}}\right)_{12\ldots N+1}(1 - \left(\frac{B_{N+2}}{B_{N+1}}\right)_{12\ldots (N+2)}\left(\frac{B_{N+1}}{B_{N+2}}\right)_{2\ldots N+2})
$$

(6.37)

$$
\left(\frac{B_2}{B_{N+1}}\right)_{12\ldots N+2} = \left(\frac{B_2}{B_{N+1}}\right)_{12\ldots N+1}(1 - \left(\frac{B_{N+2}}{B_{N+1}}\right)_{12\ldots N+2}\left(\frac{B_{N+1}}{B_{N+2}}\right)_{13\ldots N+2})
$$

(6.38)

$$
\vdots
$$

(6.39)

$$
\left(\frac{B_N}{B_{N+1}}\right)_{12\ldots N+2} = \left(\frac{B_N}{B_{N+1}}\right)_{12\ldots N+1}(1 - \left(\frac{B_{N+2}}{B_{N+1}}\right)_{12\ldots N+2}\left(\frac{B_{N+1}}{B_{N+2}}\right)_{12\ldots (N-1)(N+1)(N+2)}).
$$

Proof. We consider the weighted floating case for $A_0 \notin \{A_1A_2\ldots A_{N+2}\}$:

$$
\sum_{i=1}^{N+2} (B_i)_{12\ldots N+2} \vec{u}(A_0, A_i) = \vec{0}.
$$

(6.40)

The normals $\vec{n}_{0, 23\ldots N}$, $\vec{n}_{0, 13\ldots N}$, and $\vec{n}_{0, 12\ldots (N-1)}$ are orthogonal to the subspaces $A_2A_3\ldots A_N$, $A_1A_3\ldots A_N$, $A_1A_2\ldots A_N$, respectively.
We set

\[ sgn_{i,203...N} = \begin{cases} 
+1, & \text{if } A_i \text{ is upper from the subspace } A_2A_0 \ldots A_N, \\
0, & \text{if } A_i \text{ belongs to the subspace } A_2A_0 \ldots A_N, \\
-1, & \text{if } A_i \text{ is under the subspace } A_2A_0 \ldots A_N, 
\end{cases} \]

for \( i = 1, N + 1, N + 2 \).

The inner product of (6.40) with \( \vec{n}_{0,23...N} \), \( \vec{n}_{0,13...N} \), and \( \vec{n}_{0,12...(N-1)} \) yield

\[ (B_1)_{12...N+2}sgn_{1,203...N} \sin a_{1,203...N} + \\
+(B_{N+1})_{12...N+2}sgn_{N+1,203...N} \sin a_{N+1,203...N} + \\
+(B_{N+2})_{12...N+2}sgn_{N+2,203...N} \sin a_{N+2,203...N} = 0, \]

(6.41)

\[ (B_2)_{12...N+2}sgn_{2,103...N} \sin a_{2,103...N} + \\
+(B_{N+1})_{12...N+2}sgn_{N+1,103...N} \sin a_{N+1,103...N} + \\
+(B_{N+2})_{12...N+2}sgn_{N+2,103...N} \sin a_{N+2,103...N} = 0, \]

(6.42)

\[ \vdots \]

\[ (B_N)_{12...N+2}sgn_{N,102...N-1} \sin a_{N,102...N-1} + \\
+(B_{N+1})_{12...N+2}sgn_{N+1,102...N-1} \sin a_{N+1,102...N-1} + \\
+(B_{N+2})_{12...N+2}sgn_{N+2,102...N-1} \sin a_{N+2,102...N-1} = 0. \]

(6.43)

By substituting \( B_{N+2} = 0 \) in (6.41), (6.42), (6.43), we derive:

\[ \left( \frac{B_1}{B_{N+1}} \right)_{12...N+1} = -\frac{sgn_{N+1,203...N} \sin a_{N+1,203...N}}{sgn_{1,203...N} \sin a_{1,203...N}}, \]

(6.44)

\[ \left( \frac{B_2}{B_{N+1}} \right)_{12...N+1} = -\frac{sgn_{N+1,103...N} \sin a_{N+1,103...N}}{sgn_{2,103...N} \sin a_{2,103...N}}, \]

(6.45)

\[ \left( \frac{B_N}{B_{N+1}} \right)_{12...N+1} = -\frac{sgn_{N,102...N-1} \sin a_{N,102...N-1}}{sgn_{N,102...N-1} \sin a_{N,102...N-1}}. \]

(6.46)

By substituting \( (B_1)_{12...N+2} = 0, (B_2)_{12...N+2} = 0, (B_N)_{12...N+2} = 0 \) in (6.41), (6.42), (6.43), respectively, we derive:

\[ \left( \frac{B_{N+2}}{B_{N+1}} \right)_{2...N+2} = -\frac{sgn_{N+1,203...N} \sin a_{N+1,203...N}}{sgn_{N+2,203...N} \sin a_{N+2,203...N}}, \]

(6.47)

\[ \left( \frac{B_{N+2}}{B_{N+1}} \right)_{13...N+2} = -\frac{sgn_{N+1,103...N} \sin a_{N+1,103...N}}{sgn_{N+2,103...N} \sin a_{N+2,103...N}}, \]

(6.48)

\[ \left( \frac{B_{N+2}}{B_{N+1}} \right)_{12...(N-1)(N+1)(N+2)} = -\frac{sgn_{N+1,102...N-1} \sin a_{N+1,102...N-1}}{sgn_{N+2,102...N-1} \sin a_{N+2,102...N-1}}. \]

(6.49)
By substituting (6.41), (6.43), (6.45), (6.46), (6.47), (6.48), (6.49) in (6.41), (6.42), (6.43), we obtain (6.37), (6.38) and (6.39).

A direct consequence of Theorem 12 is the following corollary, by setting
\[ \sum_{i=1}^{N+2} B_i \equiv \sum_{i=1}^{N+2} (B_i)_{12...N+2}, \]
\[ \sum_{i=1}^{N+2} B_i = (B_{i_1})_{12...N+1} + \ldots + (B_{i_{N+1}})_{i_1...i_{N+1}}, \]
for \( i_1, \ldots, i_{N+1} \in \{1, 2, \ldots, N + 2\} \).

**Corollary 2.** If \( \sum_{i=1}^{N+2} B_i = \sum_{i=1}^{N+2} B_i \), for every \( i_1, \ldots, i_{N+1} \in \{1, 2, \ldots, N + 2\} \), where \( \sum_{i=1}^{N+2} B_i := (B_{N+1})_{12...N+2} + \sum_{i=1}^{N+2} (\frac{b_i}{B_{N+1}})_{12...N+2} \), then
\[ (B_i)_{12...N+2} = a_i (B_{N+2})_{12...N+2} + b_i, \quad i = 1, 2, \ldots, N + 1, \quad (6.50) \]
where
\[ (a_{N+1}, b_{N+1}) = \left( \frac{B_{N+1}}{B_N} \right)_{12...N+1} + \sum_{i=1}^{N+1} \left( \frac{B_{N+1}}{B_N} \right)_{12...N+1}, \]
\[ (B_{N+1})_{12...N+1}, \]
\[ (a_N, b_N) = (a_{N+1} (\frac{B_N}{B_{N+1}}))_{12...N+1} - (\frac{B_N}{B_{N+1}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1}, \]
\[ (B_N)_{12...N+1}, \]
\[ \vdots \]
\[ (a_1, b_1) = (a_{N+1} (\frac{B_1}{B_{N+1}}))_{12...N+1} - (\frac{B_1}{B_{N+1}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1} (\frac{B_{N+1}}{B_{N+2}})_{12...N+1}, \]
\[ (B_1)_{12...N+1}. \]

Suppose that \( A_0 \) is an interior weighted Fermat point for the \( N \)-simplex with respect to the non-negative given weights \( \{B_1(0), B_2(0), \ldots, B_N(0), B_{N+1}(0)\} \) in \( \mathbb{R}^N \). Therefore, the topology of the branches \( A_1, A_0 \) which meet at \( A_0 \) form a unique floating weighted Fermat tree. The unique solution of the \( N \)-INVWF problem for \( A_1, A_2, \ldots, A_{N+1} \) is responsible for the cancellation of the dynamic plasticity of simplices. We assume that after time \( t \) an \( (N + 2) \) branch \( A_0A_{N+2} \) starts to grow from \( A_0 \) and the new branch \( A_0A_{N+2} \) is located inside the cone \( C(\text{ray}(A_0A_1), \text{ray}(A_0A_2), \ldots, \text{ray}(A_0A_{N+1})) \) and \( A_0A_2 \) is located outside \( C(\text{ray}(A_0A_1), \text{ray}(A_0A_2), \ldots, \text{ray}(A_0A_{N+1})) \). We assume that \( A_{N+1} \) is upper from the hyperplane formed by the first, second, \ldots and the \((N-1)\)th ray and \( A_N \) are under this hyperplane.

**Theorem 13.** An increase to the weight that corresponds to the \((N+2)\)th ray causes a decrease to the weight that corresponds to the \((N+1)\)th ray and a variation to the weight that corresponds to the \( i \)th ray depends on the difference \( (B_i)_{123...N(N+1)} - (B_i)_{123...N(N+2)} \), for \( i = 1, 2, \ldots, N \).

**Proof.** First, we will show that \( a_{N+1} < 0 \). Taking into account (6.51), we obtain that
\[ a_{N+1} = \frac{(B_{N+1})_{123\ldots N N+2}}{(B_{N+2})_{123\ldots N N+2}} < 0, \]

because \((B_{N+1})_{123\ldots N+1}, (B_{N+2})_{123\ldots N N+2} > 0\). Thus, we get:

\[
(B_{N+1})_{123\ldots N+2} = -\frac{(B_{N+1})_{123\ldots N+1}}{(B_{N+2})_{123\ldots N N+2}} (B_{N+2})_{123\ldots N+2} + (B_{N+1})_{123\ldots N+1}. \tag{6.51}
\]

By inserting (6.51) into (6.41), (6.42), . . . (6.43), we derive that:

\[
(B_i)_{123\ldots N+2} = \frac{(B_i)_{123\ldots N(N+2)} - (B_i)_{123\ldots N(N+1)}}{(B_{N+2})_{123\ldots N N+2}} (B_{N+2})_{123\ldots N+2} + (B_i)_{123\ldots N+1}, \tag{6.52}
\]

for \(i = 1, 2, \ldots, N\). Therefore, the difference \(\delta B_i\) of the two positive weights \((B_i)_{123\ldots N(N+2)} - (B_i)_{123\ldots(N+1)}\) which corresponds to the \(ith\) branch \(A_0 A_i\) of the boundary simplexes \(A_1 A_2 \ldots A_N A_{N+2}, A_1 A_2 \ldots A_N A_{N+1}\), yields the following two results:

(a) If \(\delta B_i > 0\), the weight \((B_i)_{123\ldots N+2}\) is increased,

(b) If \(\delta B_i < 0\), the weight \((B_i)_{123\ldots N+2}\) is decreased.

\[\square\]

**Proposition 3.** An increase to the weight that corresponds to the \((N+2)th\) ray and a decrease to the weight that corresponds to the \(ith\) ray for \(i = 1, 3, \ldots, N\), causes a decrease to the weight that corresponds to the \((N + 1)th\) ray and an increase to the weights that corresponds to the second ray.

**Proof.** By applying Theorem 13 and taking into account that \(a_1, a_3, \ldots, a_N < 0\), the length of the vectors \(\vec{X}_i = (B_i)_{12\ldots n+2}(\delta t)\vec{u}(A_0, A_i)\) are decreased after time \(\delta t\). We proceed by arranging some vector terms of the weighted floating balancing condition for \(\vec{X}_i\) (Theorem 11):

\[\vec{X}_2 + \vec{X}_{n+1} = - (\sum_{i=1, i\neq 2}^{N} \vec{X}_i + \vec{X}_{N+2}).\]

We observe that after time \(\delta t\) the length of the vector \((\sum_{i=1, i\neq 2}^{N} \vec{X}_i + \vec{X}_{N+2})\) is decreased. Thus, on the right hand side of the above equation a vector of reduced length is composed with a vector of increased length \(\vec{X}_{N+2}\) and on the left hand side a vector of reduced length \(\vec{X}_{N+1}\) is composed with \(\vec{X}_2\) whose length is increased. Therefore, we obtain that \(a_2 > 0\).

\[\square\]

Suppose that at time \(t = 0\), an evolutionary multitree occurs, such that \((N + 1)!</k\)

weighted Fermat trees correspond to \(k\) simplexes, which form a Frechet \(N\)–multisimplex in \(\mathbb{R}^N\), with \((N+1)\) permutation of the weights \(\{B_1(0), B_2(0), \ldots, B_N(0), B_{N+1}(0)\}\) in \(\mathbb{R}^N\) for boundary incongruent \(N\)–simplexes \((A_1)_k(A_2)_k \ldots (A_{N+1})_k\), for \(k \leq \frac{(N(N+1))!}{(N+1)!}\). After time \(t\), a \((N + 2)th\) branch \((A_0)_k(A_{N+2})_k\) starts to grow from the weighted Fermat point \((A_0)_k\) and the new branch \((A_0)_k(A_{N+2})_k\) is located inside the cone \(C(ray((A_0)_k(A_1)_k), ray((A_0)_k(A_3)_k) \ldots, ray((A_0)_k(A_{N+1})_k))\) and \((A_0)_k(A_2)_k\) is located outside \(C(ray(((A_0)_kA_1)_k), ray((A_0)_k(A_3)_k) \ldots, ray((A_0)_k(A_{N+1})_k))\) and let \((A_{N+1})_k\) be upper from the hyperplane formed by the first, second, . . . and the \((N - 1)th\) ray and \((A_N)_k, (A_{N+2})_k\) are under this hyperplane.

**Theorem 14.** An increase to the weight that corresponds to the \((N+2)th\) ray causes a decrease to the weight that corresponds to the \((N + 1)th\) ray and a variation to the
weight that corresponds to the $i$th ray depends on the difference $(B_i)_{123...N(N+1)} - (B_i)_{123...N(N+2)}$, such that the geometric structure of the weighted Fermat-Frechet multitree with respect to a boundary $N$–multisimplex, remains the same, for $i = 1, 2, \ldots N$, $1 \leq k \leq \frac{N(N+1)!}{(N+1)!}$.

**Proof.** By applying Theorem 13 starting from a weighted Fermat-Frechet multitree with respect to a boundary $N$–multisimplex in $\mathbb{R}^N$, we obtain the plasticity of a weighted Fermat-Frechet-multitree by adding the ray $(A_0)_k(A_{N+2})_k$, for $i = 1, 2, \ldots N$, $1 \leq k \leq \frac{N(N+1)!}{(N+1)!}$. \hfill $\square$

7. The weighted Fermat-Steiner-Frechet multitree for a given tentuple of positive real numbers determining the edge lengths of incongruent 4–simplexes in $\mathbb{R}^4$

In this section, we deal with the solution (multitree) of the weighted Fermat-Steiner-Frechet problem (P(Fermat-Steiner-Frechet)) for a given tentuple of positive real numbers determining incongruent 4–simplexes in $\mathbb{R}^4$, by inserting three equality constraints derived by three independent solutions for three variable weighted Fermat problems for the Frechet 4–multisimplex derived by incongruent boundary 4–simplexes in $\mathbb{R}^4$, which correspond to the same tentuple of positive real numbers (edge lengths) and two equality constraints derived by two different expressions of the line segments connecting the three weighted Fermat-Steiner points. The detection of the weighted Fermat-Steiner Frechet multitrees is achieved by applying the Lagrange multiplier rule. By applying a Lagrange program to detect unweighted Fermat-Frechet multitree for a given tentuple of edge lengths determining 30,240 incongruent 4–simplexes (Frechet 4–multisimplex) in $\mathbb{R}^4$, by using Dekster Wilker tenttuples, we derive an interesting result of seeking unweighted Fermat-Frechet multitrees with three equally weighted Fermat-Steiner points inside the Frechet 4–multisimplex. This result may provide an approach to detect the most natural of six consecutive natural numbers (Herzog sextuples) for $N \geq 30$ and it is achieved by seeking an upper bound for these three equal weights, which yield a global weighted Fermat-Steiner tree of minimum length for the boundary tetrahedron having the maximum volume among the 30,240 incongruent 4–simplexes in $\mathbb{R}^4$.

Let $\{A_1, A_2, A_3, A_4, A_5\}$ be the vertices of a 4–simplex in $\mathbb{R}^4$ and $A_0$ be a point inside $A_1A_2A_3A_4A_5$.

We denote by $A_{0,1234}$ the orthogonal projection of $A_0$ to the hyperplane defined by the tetrahedron $A_1A_2A_3A_4$, by $A_{0,123}$ the orthogonal projection of $A_0$ to the plane defined by $\triangle A_1A_2A_3$, by $a_{ij}$ the length of the line segment $A_iA_j$, for $i,j = 1, 2, 3, 4, 5$ by $h_{0,123j}$ the length of $A_0A_{0,123j}$, for $j = 4, 5$, by $h_{0,123}$ the length of $A_0A_{0,123}$, by $h_{0,12}$ the length of $A_0A_{0,12}$.

We set $\beta \equiv \angle A_0A_{0,123}A_{0,1234}$, $\alpha \equiv \angle A_0A_{0,12}A_{0,123}$ $x_i \equiv A_{0,1234}A_i$, $y_i \equiv A_{0,123}A_i$; for $i = 1, 2, 3, 4, 5$ $h_{(0,123), (0,12)} \equiv A_{0,123}A_{0,12}$ $z_j \equiv A_{0,123}A_j$ for $k = 4, 5$, $j = 1, 2, 3$.

**Theorem 15** (Generalized cosine law in $\mathbb{R}^4$). The line segment $a_{40}$, $a_{50}$ depend on $a_{10}, a_{20}, a_{30}$ and $\beta$ in $\mathbb{R}^4$:
The weighted Fermat-Steiner-Frechet multitree.

Proof. From $\triangle A_0A_1A_0A_0, \triangle A_0A_2A_0A_0, \triangle A_0A_3A_0A_0, \triangle A_0A_4A_0A_0, \triangle A_0A_5A_0A_0$, we get, respectively:

\[ a_{40}^2 = h_{0,1234}^2(a_{10}, a_{20}, a_{30}, \beta) + x_{4}^2(a_{10}, a_{20}, a_{30}, \beta) \]  
\[ a_{50}^2 = h_{0,1235}^2(a_{10}, a_{20}, a_{30}, \beta) + y_{5}^2(a_{10}, a_{20}, a_{30}, \beta) \]  

From $\triangle A_0A_0A_0A_0A_1, \triangle A_0A_0A_0A_0A_2$, we obtain a relation for Schlafli’s angle $\beta$:

\[ h_{0,1234} = h_{0,123} \sin \beta \]  
\[ h_{0,123} = h_{0,12} \sin \alpha \]  

where

\[ h_{0,123} = h_{0,12} \sin \alpha \]  

By substituting $A_0 \rightarrow A_{0,1234}$ and the notations $x_i \equiv A_{0,1234}A_i$ in (7.6) from Lemma [5] and taking into account Lemma [10] we derive that:

\[ x_4^2 = x_2^2 + a_{24}^2 - 2a_{24} \sqrt{x_2^2 - h_{0,1234}^2(a_{10}, a_{20}, a_{30}, \alpha_{124})} \cos \alpha_{124} + h_{0,1234}(a_{10}, a_{20}, a_{30}, \alpha_{124}) \sin \alpha_{124} \cos(\alpha_{94} - \alpha') \]  
\[ \sqrt{x_2^2 - h_{0,1234}^2(a_{10}, a_{20}, a_{30}, \alpha_{124})} = \sqrt{a_{20}^2 - h_{0,12}^2} \]  

From $\triangle A_0A_0A_0A_0A_1, \triangle A_0A_0A_0A_0A_2$, we get:

\[ \sqrt{x_2^2 - h_{0,1234}^2(a_{10}, a_{20}, a_{30}, \alpha_{124})} = \sqrt{a_{20}^2 - h_{0,12}^2} \]
where

\[ h_{0,12} = h_{0,12}(a_{01}, a_{02}; a_{12}) = \frac{a_{01}a_{02}}{a_{12}} \sqrt{1 - \left( \frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01}a_{02}} \right)^2}. \]

Hence, (7.8) depends on \( a_{10}, a_{20} \).

From \( \triangle A_0A_{0,12}A_{0,1234} \), we get:

\[ h_{(0,1234),(0,12)} = \sqrt{h_{0,12}^2 - h_{0,123}^2}. \]  \( \text{(7.9)} \)

By substituting (7.8), (7.7), (7.9) in (7.3), we obtain:

\[ a_{40}^2 = a_{20}^2 + a_{24}^2 - 2a_{24}\sqrt{a_{20}^2 - h_{0,12}^2 \cos \alpha_{124} + \sqrt{h_{0,12}^2 - h_{0,123}^2 \sin \alpha_{124} \cos(\alpha_{g4} - \alpha')}} \]  \( \text{(7.10)} \)

By substituting \( A_0 \rightarrow A_{0,1234} \) and the notations \( x_i \equiv A_{0,1234}A_i \) in (3.6) for \( i = 3 \) from Lemma \[ \text{[X]} \] we derive that:

\[ \alpha' = \arccos \left( \frac{x_3^2 + x_2^2 - x_1^2}{2x_2} \right) \]  \( \text{(7.11)} \)

or

\[ \alpha' = \arccos \left( \frac{x_2^2 + x_3^2 - x_1^2}{2x_3} \right) \]  \( \text{(7.12)} \)

By solving (3.6) taken from Lemma \[ \text{[X]} \] with respect to \( \alpha \), we get:

\[ \alpha = \arccos \left( \frac{a_{30}^2 + a_{23}^2 - a_{21}^2}{2a_{23}} \right) \]  \( \text{(7.13)} \)

By replacing (7.10), (7.11) in (7.5), we have:

\[ h_{0,123} = \frac{a_{01}a_{02}}{a_{12}} \sqrt{1 - \left( \frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01}a_{02}} \right)^2} \sin(\arccos \left( \frac{a_{30}^2 + a_{23}^2 - a_{21}^2}{2a_{23}} \right) \) \]

\[ h_{0,1234} = \frac{a_{01}a_{02}}{a_{12}} \sqrt{1 - \left( \frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01}a_{02}} \right)^2} \sin(\arccos \left( \frac{a_{30}^2 + a_{23}^2 - a_{21}^2}{2a_{23}} \right) \) \]

\[ \sin(\arccos \left( \frac{a_{30}^2 + a_{23}^2 - a_{21}^2}{2a_{23}} \right) \) \sin \beta. \]  \( \text{(7.12)} \)

Thus, (7.10), (7.12) yield:

\[ h_{0,123} = h_{0,123}(a_{10}, a_{20}, a_{30}), \]
\[ h_{0,1234} \triangleq h_{0,1234}(a_{10}, a_{20}, a_{30}, \beta) \]

and \( x_i = \sqrt{a_i^2 - h_{0,1234}(a_{10}, a_{20}, a_{30}, \beta)^2} \) for \( i = 2, 3, 4, 5 \).

Therefore, we derive from (7.1) that \( a_{40} \) depends on \( a_{10}, a_{20}, a_{30} \) and \( \beta \).

By following a similar process for \( i = 5 \), we derive from (7.2) that \( a_{50} \) depend on \( a_{10}, a_{20}, a_{30} \) and \( \beta \).

\[ \square \]

Let \( A_{0,1}, A_{0,2}, A_{0,3} \) three points inside the \( 4 \)-simplex \( A_1 A_2 A_3 A_4 A_5 \) in \( \mathbb{R}^4 \). We denote by \( a_{(0,i),j} \) the length of the line segment \( A_{0,i} A_j \), by \( A_{(0,i),jkl}, A_{(0,i),jklm} \) the orthogonal projections of \( A_{0,i} \), with respect to the plane defined by \( \triangle A_j A_k A_l \) and the hyperplane defined by \( A_j A_k A_l A_m \), respectively, for \( i = 1, 2, 3 \), \( j, k, l, m = 1, 2, 3, 4, 5 \), and we set \( \beta_i \equiv \triangle A_{0,1} A_{0,2} A_{0,3}, \beta_2 \equiv \triangle A_{0,2} A_{0,3} A_{0,4}, \beta_3 \equiv \triangle A_{0,3} A_{0,4} A_{0,5} \), where \( \beta_i \) are the Schaffi angles for \( i = 1, 2, 3 \). The weighted Fermat-Steiner-Frechet problem for a given tentuple of edge lengths determining incongruent \( 4 \)-simplexes in \( \mathbb{R}^4 \), states that:

**Problem 10 (The weighted Fermat-Steiner Frechet in \( \mathbb{R}^4 \)).** Given an octuple of weights \( \{b_1, b_2, b_3, b_4, b_5, b_{ST}, b_{ST}, b_{ST}\} \), and a given tentuple of positive real numbers \( (a_{ij}) \), determining a Frechet \( 4 \)-simplex \( F(A_1 A_2 A_3 A_4 A_5) \), find the position of \( A_{0,1} \) and/or \( A_{0,2} \) and/or \( A_{0,3} \) with given weights \( b_{ST} \) in \( A_{0,1} \), \( b_{ST} \) in \( A_{0,2} \), \( b_{ST} \) in \( A_{0,3} \), such that

\[
\begin{align*}
\sum_{i=0}^{13} \lambda_i f_i(\bar{x}) = 0,
\end{align*}
\]

where the point

\[
\bar{x} = \{x_1, \ldots, x_{24}\} = \\
= \{a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, a_{(0,2),3}, a_{(0,2),4}, a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, a_{(0,3),6}, a_{(0,3),7}, a_{(0,3),8}, a_{(0,3),9}, a_{(0,3),10}, a_{(0,3),11}, a_{(0,3),12}, a_{(0,3),13}, a_{(0,3),14}, a_{(0,3),15}, a_{(0,3),16}, a_{(0,3),17}, a_{(0,3),18}, a_{(0,3),19}, a_{(0,3),20}, a_{(0,3),21}, a_{(0,3),22}, a_{(0,3),23}, a_{(0,3),24}\} \in \mathbb{R}^{24}
\]
is inside the parallelepiped \( \Pi(p_1, q_1; \ldots; p_{24}, q_{24}) \), where \( p_i < x_i < q_i \), for \( i = 1, 2, \ldots, 29 \) and the Lagrange multiplication vector is given by:

\[
\bar{\lambda} = \{\lambda_0, \lambda_1, \ldots, \lambda_{13}\}.
\]

We extend the weighted Fermat-Steiner-Frechet problem \((P(\text{Fermat-Steiner-Frechet}))\) in \( \mathbb{R}^4 \), by inserting 12 equality constraints derived by three independent solutions
Figure 6. A weighted Fermat-Steiner tree for $A_1A_2A_3A_4A_5$ in $\mathbb{R}^4$

for three new weighted Fermat problems for $A_1A_2A_3A_4A_5$ in $\mathbb{R}^4$, which give a connection with the initial weighted Fermat-Steiner objective function and one equality constraint derived by two different expressions of $a_{(0,1),(0,2)}$.

We note that the volume of an $(N-1)$-simplex $A_1A_2\cdots A_N$ in $\mathbb{R}^{N-1}$ is given by the Caley-Menger determinant in terms of edge lengths ($[28,(5.1), p. 125]$):

$$
\text{Vol}(A_1A_2\cdots A_N)^2 = \frac{1}{(-1)^N 2^{N-1}((N-1)!)^2} \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & a_{12}^2 & \cdots & a_{1(N-1)}^2 & a_{1N}^2 \\
1 & a_{21}^2 & 0 & \cdots & a_{2(N-1)}^2 & a_{2N}^2 \\
1 & a_{31}^2 & a_{32}^2 & \cdots & a_{3(N-1)}^2 & 0 \\
\end{vmatrix}.
$$

**Problem 11** (The weighted Fermat-Steiner-Frechet (P(Fermat-Steiner-Frechet)) problem in $\mathbb{R}^4$ with equality constraints),

$$
f_0(\tilde{x}) \rightarrow \text{min},
$$

$$
f_i(\tilde{x}) = 0, \ i = 1, \ldots, 13
$$

$$
f_0(\tilde{x}) = b_1a_{(0,1),1} + b_2a_{(0,1),2} + b_3a_{(0,2),3} + b_4a_{(0,3),4} + b_5a_{(0,3),5} + b_{ST}(a_{(0,1),(0,2)}, a_{(0,1),1}, a_{(0,1),2}, a_{(0,2),3}, a_{(0,2),4}, a_{(0,2),5} + \beta_2) + a_{(0,2),(0,3)}, a_{(0,3),5}, a_{(0,2),4}, a_{(0,2),5}), \ (7.14)
$$
\[ f_1(\tilde{x}) = \frac{w_{(0,1),1}}{a_{(0,1),1} \text{Vol}(A_{0,1}A_2A_3A_4A_5)} - \frac{1 - \sum_{i=1}^{4} w_{(0,1),i}}{a_{(0,1),5}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1) \text{Vol}(A_{0,1}A_2A_3A_4A_5)} \]  

(7.15)

\[ f_2(\tilde{x}) = \frac{w_{(0,1),2}}{a_{(0,1),2} \text{Vol}(A_{1}A_{0,1}A_3A_4A_5)} - \frac{1 - \sum_{i=1}^{4} w_{(0,1),i}}{a_{(0,1),5}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1) \text{Vol}(A_{0,1}A_1A_2A_3A_4)} \]  

(7.16)

\[ f_3(\tilde{x}) = \frac{w_{(0,1),3}}{a_{(0,1),3} \text{Vol}(A_{1}A_2A_{0,1}A_4A_5)} - \frac{1 - \sum_{i=1}^{4} w_{(0,1),i}}{a_{(0,1),5}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1) \text{Vol}(A_{0,1}A_1A_2A_3A_4)} \]  

(7.17)

\[ f_4(\tilde{x}) = \frac{w_{(0,1),4}}{a_{(0,1),4}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1) \text{Vol}(A_{1}A_2A_3A_{0,1}A_5)} - \frac{1 - \sum_{i=1}^{4} w_{(0,1),i}}{a_{(0,1),5}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1) \text{Vol}(A_{0,1}A_1A_2A_3A_4)} \]  

(7.18)

\[ f_5(\tilde{x}) = \frac{w_{(0,2),5}}{a_{(0,2),5} \text{Vol}(A_{1}A_2A_3A_4A_{0,2})} - \frac{1 - \sum_{i=2}^{5} w_{(0,2),i}}{a_{(0,2),1}(a_{(0,2),1}, a_{(0,2),2}, a_{(0,2),3}, \beta_2) \text{Vol}(A_{0,2}A_2A_3A_4A_5)} \]  

(7.19)

\[ f_6(\tilde{x}) = \frac{w_{(0,2),4}}{a_{(0,2),4} \text{Vol}(A_{1}A_2A_3A_4A_{0,2})} - \frac{1 - \sum_{i=2}^{5} w_{(0,2),i}}{a_{(0,2),1}(a_{(0,2),1}, a_{(0,2),2}, a_{(0,2),3}, \beta_2) \text{Vol}(A_{0,2}A_2A_3A_4A_5)} \]  

(7.20)

\[ f_7(\tilde{x}) = \frac{w_{(0,2),5}}{a_{(0,2),5} \text{Vol}(A_{1}A_2A_3A_4A_{0,2})} - \frac{1 - \sum_{i=2}^{5} w_{(0,2),i}}{a_{(0,2),1}(a_{(0,2),1}, a_{(0,2),2}, a_{(0,2),3}, \beta_2) \text{Vol}(A_{0,2}A_2A_3A_4A_5)} \]  

(7.21)

\[ f_8(\tilde{x}) = \frac{w_{(0,2),5}}{a_{(0,2),2}(a_{(0,2),3}, a_{(0,2),4}, a_{(0,2),5}, \beta_2) \text{Vol}(A_{1}A_{0,2}A_3A_4A_{0,2})} - \frac{1 - \sum_{i=2}^{5} w_{(0,2),i}}{a_{(0,2),1}(a_{(0,2),2}, a_{(0,2),3}, a_{(0,2),4}, \beta_2) \text{Vol}(A_{0,2}A_2A_3A_4A_5)} \]  

(7.22)

\[ f_9(\tilde{x}) = \frac{w_{(0,3),5}}{a_{(0,3),5} \text{Vol}(A_{1}A_2A_3A_4A_{0,3})} - \frac{1 - \sum_{i=2}^{5} w_{(0,3),i}}{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_4A_5)} \]  

(7.23)
\[ f_{10}(\bar{x}) = \frac{w_{(0,3),4}}{a_{(0,3),4} \text{Vol}(A_1A_2A_3A_{0,3}A_5)} - \left(1 - \sum_{i=2}^{5} w_{(0,3),i}\right) \frac{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}, \] (7.24)

\[ f_{11}(\bar{x}) = \frac{w_{(0,3),5}}{a_{(0,3),3} \text{Vol}(A_1A_2A_{0,2}A_4A_5)} - \left(1 - \sum_{i=2}^{5} w_{(0,3),i}\right) \frac{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}, \] (7.25)

\[ f_{12}(\bar{x}) = \frac{w_{(0,3),5}}{a_{(0,3),2}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{1}A_0A_3A_4A_0,2)} - \left(1 - \sum_{i=2}^{5} w_{(0,3),i}\right) \frac{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}{a_{(0,3),1}(a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3) \text{Vol}(A_{0,3}A_2A_3A_{0,3}A_5)}, \] (7.26)

\[ f_{13}(\bar{x}) = a_{(0,1),(0,2)}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,2),3}, a_{(0,2),4}, a_{(0,2),5}, b_1; b_2; b_{ST}) - a_{(0,1),(0,2)}(a_{(0,1),3}, a_{(0,2),3}, a_{(0,2),3}, a_{(0,2),4}, a_{(0,2),5}, a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, b_3; b_{ST}), \] (7.27)

We note that:

- \(\{7.14\}\) is the objective function of the weighted Fermat-Steiner problem for \(A_1A_2A_3A_4A_5\) in \(\mathbb{R}^4\) having three weighted Fermat-Steiner points \(A_{0,1}\), such that:

\[ \alpha_{1(0,1)2} = \arccos\left(\frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1 b_2}\right), \]

\[ \alpha_{1(0,1)(0,2)} = \arccos\left(\frac{b_2^2 - b_1^2 - b_{ST}^2}{2b_1 b_{ST}}\right), \]

\[ \alpha_{2(0,1)(0,2)} = \arccos\left(\frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_2 b_{ST}}\right), \]

\(A_{0,1}\) is the weight Fermat-Steiner point with respect to the boundary triangle \(\triangle A_1A_2A_0,2\),

\[ \alpha_{(0,1)(0,2)3} = \arccos\left(\frac{-b_3}{2b_{ST}}\right), \]

\[ \alpha_{(0,1)(0,2)(0,3)} = \arccos\left(\frac{b_3^2 - 2b_{ST}^2}{2b_{ST}}\right), \]

\[ \alpha_{3(0,2)(0,3)} = \arccos\left(\frac{-b_3}{2b_{ST}}\right), \]

\(A_{0,2}\) is the weight Fermat-Steiner point with respect to the boundary triangle \(\triangle A_{0,1}A_3A_{0,3}\),

\[ \alpha_{4(0,3)5} = \arccos\left(\frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1 b_5}\right), \]

\[ \alpha_{4(0,3)(0,2)} = \arccos\left(\frac{b_3^2 - b_4^2 - b_{ST}^2}{2b_4 b_{ST}}\right), \]
\[ \alpha_{5(0,3)(0,2)} = \arccos\left(\frac{b_1^2 - b_5^2 - b_{ST}^2}{2b_5b_{ST}}\right), \]

\(A_{0,3}\) is the weight Fermat-Steiner point with respect to the boundary triangle \( \triangle A_4A_5A_{0,2}\). These weighted angular relations are derived as a special case of Lemmas 11, 12

- (7.13)-(7.18) deal with the solution of the weighted Fermat problem for \(A_1A_2A_3A_4A_5\) with weights \(w_{(0,1),1}, w_{(0,1),2}, w_{(0,1),3}, w_{(0,1),4}\) and \(w_{(0,1),5} = 1 - \sum_{i=1}^{4} w_{(0,1),i}\), which is determined by Theorem 11 for \(N = 4\) and the weighted Fermat point \(A_{0,1}\).
- (7.19)-(7.22) deal with the solution of the weighted Fermat problem for \(A_1A_2A_3A_4A_5\) with weights \(w_{(0,2),2}, w_{(0,2),3}, w_{(0,2),4}, w_{(0,2),5}\) and \(w_{(0,2),1} = 1 - \sum_{i=2}^{5} w_{(0,2),i}\), which is determined by Theorem 11 for \(N = 4\) and the weighted Fermat point \(A_{0,2}\).
- (7.23)-(7.26) deal with the solution of the weighted Fermat problem for \(A_1A_2A_3A_4A_5\) with weights \(w_{(0,3),2}, w_{(0,3),3}, w_{(0,3),4}, w_{(0,3),5}\) and \(w_{(0,3),1} = 1 - \sum_{i=2}^{5} w_{(0,2),i}\), which is determined by Theorem 11 for \(N = 4\) and the weighted Fermat point \(A_{0,3}\).
- (7.27) is a derivation of two expressions of \(a_{(0,1),0,2}\) with respect to the boundary triangles \( \triangle A_1A_2A_{0,2} \triangle A_{0,1}A_{0,3}A_3\), by applying the generalized cosine law in \(\mathbb{R}^2\) given in lemma 4.

**Theorem 16** (Lagrange multiplier rule for the weighted Fermat-Steiner Frechet multistree in \(\mathbb{R}^4\)). If the admissible point \(\tilde{x}_i\) yields a weighted minimum multistree for \(1 \leq i \leq 30.240\), which correspond to a Frechet multifivesimplex derived by a tentuple of edge lengths determining upto 30.240 incongruent 4-simplexes, then there are numbers \(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{13}\), such that:

\[ \frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{ji}} = 0 \quad (7.28) \]

for \(j = 1, 2, \ldots, 24\),

\[ \tilde{x}_i = \{a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1, a_{(0,2),3}, a_{(0,2),4}, a_{(0,5),3}, \beta_2, a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3, \]

\[ w_{(0,1),1}, \ldots, w_{(0,1),4}, w_{(0,2),2}, \ldots, w_{(0,2),5}, w_{(0,3),2}, \ldots, w_{(0,3),5} \}

\[ \tilde{\lambda}_i = \{\lambda_0, \lambda_1, \ldots, \lambda_{13}\}. \]

**Proof.** By taking into account that \(\frac{\partial f_k}{\partial x_{ji}}\) are continuous in each parallelepiped \(\Pi_i\), for \(1 \leq i \leq 30.240, k = 0, 1, 2, \ldots, 13, j = 1, 2, \ldots, 24\) and by applying Lagrange multiplier rule, we obtain the Lagrangian vector \(\tilde{\lambda}_i = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{13}\}\), such that (7.28) occurs.

\[ \square \]

We give the definition of the four dimensional Dekster-Wilker Euclidean domain \(DW_{\mathbb{R}^4}(\ell, s)\) discovered by Dekster-Wilker in [0] and [7]. We denote by \(\ell = \max_{i,j} a_{ij}, s = \min_{i,j} a_{ij}\) of the given tentuple \(a_{ij}\) of positive real numbers.

**Definition 15.** The four dimensional Dekster-Wilker Euclidean domain. [0], [7]

The four dimensional Dekster-Wilker Euclidean domain \(DW_{\mathbb{R}^4}(\ell, s)\) is a closed domain in \(\mathbb{R}^2\) between the ray \(s = \ell\), and the graph of a function \(\lambda_4(\ell) = \ell \sqrt{\frac{\pi}{12}}, \ell \geq 0\), which is less than \(\ell\) for \(\ell \neq 0\),
**Proposition 4** (Lagrange multiplier rule for the Fermat-Steiner Frechet multitree in $\mathbb{R}^4$). If the admissible point $\tilde{x}_i$ yields a minimum multitree for $i = 1, 2, \ldots, 30.240$, which correspond to a Frechet 4—multisimplex derived by the Dekster-Wilker tentuples of edge lengths determining 30.240 incongruent tetrahedra, then there are numbers $\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{13i}$, such that:

$$\frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{ji}} = 0 \quad (7.29)$$

for $j = 1, 2, \ldots, 24$,

$$\tilde{x}_i = \{a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1, a_{(0,2),3}, a_{(0,2),4}, a_{(0,5),3}, \beta_2, a_{(0,3),3}, a_{(0,3),4}, a_{(0,3),5}, \beta_3, w_{(0,1),1}, w_{(0,1),4}, w_{(0,2),2}, w_{(0,2),5}, w_{(0,3),2}, \ldots, w_{(0,3),5}\}$$

$$\tilde{\lambda}_i = \{\lambda_0, \lambda_1, \ldots, \lambda_{13}\}.$$

Proof. It is a direct consequence of Theorem 16 for Dekster-Wilker tentuples $\in DW_{k,l}(l, s)$ determining 30.240 incongruent 4—simplexes in $\mathbb{R}^4$. □

**Remark 3.** Given that:

$$\alpha_{1(0,1)2} = \arccos\left(\frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1b_2}\right),$$

$$\alpha_{1(0,3)5} = \arccos\left(\frac{b_{ST}^2 - b_2^2 - b_3^2}{2b_2b_3}\right),$$

$$\alpha_{(0,1),(0,2),(0,3)} = \arccos\left(\frac{b_3^2 - b_{ST}^2 - b_{ST}^2}{2b_{ST}^2}\right),$$

$$\alpha_{i(0,k)j} = \arccos\left(\frac{a_{(0,k),i}^2 + a_{(0,k),j}^2 - a_{ij}^2}{2a_{(0,k),i}a_{(0,k),j}}\right).$$

for $i, j = 1, 2, 3, 4, 5, k = 1, 2, 3$. Therefore, we get:

$$w_{(0,1),i} = w_{(0,1),i}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_1; b_1); b_2; b_{ST},$$

for $i = 1, 2, 3, 4, 5$. By following a similar process, we get:

$$w_{(0,3),i} = w_{(0,3),i}(a_{(0,3),4}, a_{(0,3),5}, a_{(0,3),3}, \beta_2; b_4); b_5; b_{ST},$$

for $i = 1, 2, 3, 4$.

**Theorem 17.** The most natural tentuple of numbers from ten consecutive natural numbers \{a + 9, a + 8, \ldots, a + 1, a\} for $a \geq 30$ is a tentuple of edge lengths having the maximum volume (maximum tentuple) among the 30.240 incongruent 4—simplexes, which corresponds a Fermat-Steiner tree of minimum total weighted length (global minimum solution), such that the upper bound for the weight $b_{ST}$ is determined by the rest Fermat-Steiner minimal trees having larger or equal weighted minimal total length.

Proof. By applying Proposition 4 for the Dekster-Wilker tentuple of edge lengths \{a + 9, a + 8, \ldots, a + 1, a\} for $a \geq 30$ forming 30.240 incongruent 4—simplexes in $\mathbb{R}^4$, we obtain a class of Fermat-Steiner trees for $b_i = 1$, for $i = 1, 2, 3, 4, 5$ and $b_{ST} = 1$. This class of Fermat-Steiner trees forms the Fermat-Steiner-Frechet multitree in $\mathbb{R}^4$. By selecting the proper tentuple of edge lengths, which yields
the maximum volume of all the 30,240 incongruent 4-simplexes in \( \mathbb{R}^4 \), we consider a variable weighted Fermat-Steiner tree having three equally weighted Fermat Steiner points with weight \( B_{ST} \) and the same boundary weights \( b_i = 1 \). Therefore, by perturbing the length weighted Fermat-Steiner tree structure for the 4-simplex \( (A_1 A_2 A_3 A_4 A_5)_{ST} \) having the maximum volume, we can derive an upper bound for the variable weight \( B_{ST} \), which is calculated by comparing the perturbed length tree structure with the other Fermat-Steiner tree structures, that belong to the unweighted Fermat-Steiner-Frechet multitree in \( \mathbb{R}^4 \). Hence, this particular arrangement of the ten consecutive natural numbers for \( a \geq 30 \) with the upper bound \((B_{ST})_s\) yield the most natural weighted tentuple of natural numbers building a weighted Fermat-Steiner tree with the minimum mass transfer.

\[ \square \]

8. **The weighted Fermat-Steiner-Frechet multitree for a given \( \frac{N(N+1)}{2} \)-tuple of positive real numbers determining the edge lengths of incongruent \( N \)-simplexes in \( \mathbb{R}^N \)**

In this section, we deal with the solution (multitree) of the weighted Fermat-Steiner-Frechet problem \( (P_{(Fermat-Steiner-Frechet)}) \) for a given \( \frac{N(N+1)}{2} \)-tuple of positive real numbers determining incongruent \( N \)-simplexes in \( \mathbb{R}^N \), by inserting \( N(N-1) \) equality constraints derived by \( (N-1) \) independent solutions for \( (N-1) \) variable weighted Fermat problems for the Frechet \( N \)-multisimplex derived by incongruent boundary \( N \)-simplexes in \( \mathbb{R}^N \), which correspond to the same \( \frac{N(N+1)}{2} \)-tuple of positive real numbers (edge lengths) and \( N-2 \) equality constraints derived by two different expressions of each line segments connecting two consecutive weighted Fermat-Steiner points. By applying a Lagrange program, we can detect the weighted Fermat-Frechet multitree for a given \( \frac{N(N+1)}{2} \)-tuple of edge lengths determining incongruent \( N \)-simplexes (Frechet \( N \)-multisimplex) in \( \mathbb{R}^N \), by using the Dekster-Wilker function. By seeking unweighted Fermat-Frechet multitrees with \( (N-1) \) equally weighted Fermat-Steiner points inside the Frechet \( N \)-multisimplex, we can detect the most natural of \( \frac{N(N+1)}{2} \) consecutive natural numbers (Dekster-Wilker \( \frac{N(N+1)}{2} \)-tuple) and it is achieved by seeking an upper bound for these equal weights, which yield a global weighted Fermat-Steiner tree of minimum length for a boundary \( N \)-simplex having the maximum volume among the derived incongruent \( N \)-simplexes in \( \mathbb{R}^N \).

We describe the \( N \)-dimensional Dekster-Wilker Euclidean domain (see in [6],[7]), which gives all incongruent \( N \)-simplexes in \( \mathbb{R}^N \) derived by the same \( \frac{N(N+1)}{2} \)-tuple of positive real numbers \( \{a_{ij}\} \).

Denote by \( \ell = \max_{i,j} a_{ij}, s = \min_{i,j} a_{ij} \) and

\[
\lambda_N(\ell) = \begin{cases} 
\ell \sqrt{1 - \frac{2(N+1)}{N(N+2)}} & \text{for even } N \geq 2, \\
\ell \sqrt{1 - \frac{2}{N+1}} & \text{for odd } N \geq 3
\end{cases}
\]

**Definition 16.** The \( N \)-dimensional Dekster-Wilker Euclidean domain [6],[7] The Dekster-Wilker Euclidean domain \( DW_{\mathbb{R}^N}(\ell, s) \) is a closed domain in \( \mathbb{R}^2 \) between the ray \( s = \ell, \) and the graph of a function \( \lambda_N(\ell), \ell \geq 0, \) which is less than \( \ell \) for \( \ell \neq 0, \)

Let \( A_0, A_1, \ldots, A_{N-1} \) be \( N-1 \) points inside the \( N \)-simplex \( A_1 A_2 \ldots A_N A_{N+1} \) in \( \mathbb{R}^N \). We denote by \( a_{(0,i),j} \) the length of the line segment \( A_{0,i} A_j \), by \( \beta_{(0,i),u} \) the Schafli angle formed by the normals of the subspaces spanned by \( \{A_{0,i} A_1 A_2 \ldots A_u\} \)
and \( \{A_1 A_2 \ldots A_{u-1}\} \) for \( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, N + 1 \), \( k = 1, 2, \ldots, N - 1 \), \( u = 4, 5, \ldots, N \). The weighted Fermat-Steiner-Frechet problem for a given \( \frac{N(N+1)}{2} \)-tuple of edge lengths determining incongruent \( N \)-simplexes in \( \mathbb{R}^N \), states that:

**Problem 12** (The weighted Fermat-Steiner Frechet in \( \mathbb{R}^N \)). Given a \( (2N) \)-tuple of weights \( \{b_1, b_2, \ldots, b_{N+1}, b_{ST}, \ldots, b_{ST}\} \), and a given \( \frac{N(N+1)}{2} \)-tuple of positive real numbers (edge lengths) \( \{a_{ij}\} \), determining a Frechet \( N \)-multisimplex \( F(A_1 A_2 \ldots A_{N+1}) \), find the position of \( A_{0,1} \) and/or \( A_{0,2} \) and/or \( \ldots A_{0,N-1} \) with given weights \( b_{ST} \) in \( A_{0,1}, b_{ST} \) in \( A_{0,2}, \ldots b_{ST} \) in \( A_{0,N-1} \), such that

\[
f_0(T_S) = \sum_{a_{ik,jk} \in T_S} b_k a_{ik,jk} \rightarrow \min. \tag{8.1}
\]

**Definition 17** (A non degenerate weighted Fermat-Steiner tree for \( \{A_1 A_2 \ldots A_{N+1}\} \)). A non degenerate weighted Fermat-Steiner tree \( T_S \) is a tree, which consists of some line segments between the \( N - 1 \) weighted Fermat-Steiner points \( A_{0,j} \), such that each \( A_{0,j} \) has degree (connections) three and of \( N + 1 \) line segments joining each boundary vertex \( A_i \) with some \( A_{0,j} \).

We consider the following three types of weighted Fermat-Steiner points \( A_{0,i} \):

- Type one, if it is connected with two boundary vertices and one weighted Fermat-Steiner point (see Fig. 7),
- Type two, if it is connected with one boundary vertex and two weighted Fermat-Steiner points (see Fig. 8),
- Type three, if it is connected with three weighted Fermat-Steiner points (see Fig. 10).

**Problem 13** (The weighted Fermat-Steiner-Frechet (P(Fermat-Steiner-Frechet)) problem in \( \mathbb{R}^N \) with equality constraints).

\[
f_0(\tilde{x}) = \sum_{a_{ik,jk} \in T_S} b_k a_{ik,jk}, \quad f_{i,j}(\tilde{x}) = 0, \ i = 1, \ldots, (N - 1), \ j = 1, \ldots, N, \nonumber\]
\[
f_k(\tilde{x}) = 0, \ k = 1, \ldots, N - 2,
\]

where \( N - 1 \) is: \# of neighboring weighted Fermat-Steiner points \( A_{0,i} \) type one, two or three,

\[
f_0(\tilde{x}) = f_0(T_S) = \sum_{a_{ik,jk} \in T_S} b_k a_{ik,jk}, \tag{8.2}
\]

\[
f_{1,1}(\tilde{x}) = \frac{u_{(0,1),1}}{a_{(0,1),1} \text{Vol}(A_{0,1} \ldots A_{N+1})} \nonumber - \frac{1 - \sum_{i=1}^{N} u_{(0,1),i}}{a_{(0,1),N+1}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_{(0,1),1}, \beta_{(0,1),2}, \ldots, \beta_{(0,1),N}) \text{Vol}(A_{0,1} A_1 \ldots A_N)} \tag{8.3}
\]
Figure 7. A weighted Fermat-Steiner point type one for $A_1A_2\ldots A_{N+1}$ in $\mathbb{R}^N$

Figure 8. A weighted Fermat-Steiner point type two for $A_1A_2\ldots A_{N+1}$ in $\mathbb{R}^N$
Figure 9. A weighted Fermat-Steiner point type three for \(A_1A_2 \ldots A_{N+1}\) in \(\mathbb{R}^N\)

\[
f_{1,N}(\bar{x}) = \frac{w_{(0,1),N}}{a_{(0,1),N}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_{(0,1),4}, \beta_{(0,1),5}, \ldots, \beta_{(0,1),N}) \text{Vol}(A_1 \ldots A_0A_{N+1})} - \frac{1 - \sum_{i=1}^{N} w_{(0,1),i}}{a_{(0,1),N+1}(a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, \beta_{(0,1),4}, \beta_{(0,1),5}, \ldots, \beta_{(0,1),N}) \text{Vol}(A_0A_1 \ldots A_N)},
\]

(8.4)

\[
f_{N-1,1}(\bar{x}) = \frac{w_{(0,N-1),1}}{a_{(0,N-1),1} \text{Vol}(A_{0,N-1} \ldots A_{N+1})} - \frac{1 - \sum_{i=1}^{N} w_{(0,N-1),i}}{a_{(0,N-1),N+1} \text{Vol}(A_{0,N-1}A_1 \ldots A_N)},
\]

(8.5)

where

\[
a_{(0,N-1),N+1} = a_{(0,N-1),N+1}(a_{(0,N-1),1}, a_{(0,N-1),2}, a_{(0,N-1),3}, \beta_{(0,N-1),4}, \beta_{(0,N-1),5}, \ldots, \beta_{(0,N-1),N})
\]
\[ f_{N-1,N}(\tilde{x}) = \frac{w_{(0,N-1),N}}{a_{(0,N-1),N} \operatorname{Vol}(A_1 \ldots A_{0,N-1} A_{N+1})} - \frac{1 - \sum_{i=1}^{N} w_{(0,N-1),i}}{a_{(0,N-1),N+1} \operatorname{Vol}(A_{0,N-1}A_1 \ldots A_N)}, \]  

(8.6)

where

\[ a_{(0,N-1),N} = a_{(0,N-1),1} a_{(0,N-1),2} a_{(0,N-1),3} \beta_{(0,N-1),4} \beta_{(0,N-1),5} \cdots \beta_{(0,N-1),N} \]

\[ f_k(\tilde{x}) = a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_j A_{0,k+1} A_j) - a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_{k+2} A_{0,k} A_{k+1}), \]  

(8.7)

or

\[ f_k(\tilde{x}) = a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_j A_{0,k+1} A_j) - a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_{k+2} A_{0,k+1} A_{k+1}), \]  

(8.8)

or

\[ f_k(\tilde{x}) = a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_j A_{0,k+1} A_j) - a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_{k+2} A_{0,k+1} A_{k+1}), \]  

(8.9)

or

\[ f_k(\tilde{x}) = a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_j A_{0,k+1} A_j) - a_{(0,k),(0,k+1)}(\tilde{x}; \triangle A_{k+2} A_{0,k} A_{k+1}), \]  

(8.10)

for \( k = 1, 2, \ldots N - 2. \)

We take into account that:

- \([8.2]\) is the objective function of the weighted Fermat-Steiner problem for \( A_1 A_2 \ldots A_{N+1} \) in \( \mathbb{R}^N \) having \( N - 1 \) weighted Fermat-Steiner points \( A_{0,i} \), such that:

I. Type one weighted Fermat-Steiner point \( A_{0,i} \)

\[ \alpha_{j(0,i)k} = \arccos\left(\frac{b_j^2 - b_i^2 - b_k^2}{2b_j b_k}\right), \]

\[ \alpha_{j(0,i)(0,i+1)} = \arccos\left(\frac{b_j^2 - b_i^2 - b_{ST}^2}{2b_j b_{ST}}\right), \]

\[ \alpha_{k(0,i)(0,i+1)} = \arccos\left(\frac{b_k^2 - b_i^2 - b_{ST}^2}{2b_k b_{ST}}\right), \]

\( A_{0,i} \) is the weight Fermat-Steiner point with respect to the boundary triangle \( \triangle A_j A_k A_{0,i+1} \).

II. Type two weighted Fermat-Steiner point \( A_{0,i} \)

\[ \alpha_{(0,i+1)(0,i)j} = \arccos\left(\frac{-b_j}{2b_{ST}}\right), \]

\[ \alpha_{(0,i+1)(0,i)(0,i+2)} = \arccos\left(\frac{-b_j^2 - b_{ST}^2}{2b_j b_{ST}}\right), \]

\[ \alpha_{(0,i+1)(0,i)j} = \arccos\left(\frac{-b_j}{2b_{ST}}\right), \]
\(A_{0,2}\) is the weight Fermat-Steiner point with respect to the boundary triangle \(\triangle A_{0,i+1}A_{0,i+2}A_j\).

II. Type three weighted Fermat-Steiner point \(A_{0,i}\)

\[a_{(0,i+1)(0,i)(0,i+2)} = a_{(0,i+1)(0,i)(0,i+3)} = 120^\circ,\]

\(A_{0,i}\) is the unweighted Fermat-Steiner point with respect to the boundary triangle \(\triangle A_{0,i+1}A_{0,i+2}A_{i+3}\).

- \([8.3]-[8.4]\) deal with the solution of the weighted Fermat problem for \(A_1 \ldots A_{N+1}\) with weights \(w_{(0,1)}, \ldots, w_{(0,1)},N\) and \(w_{(0,1)},N+1 = 1 - \sum_{i=1}^{N} w_{(0,1)},i\), which is determined by applying Theorem 11 for the unique weighted Fermat point \(A_{0,1}\).

\[\vdots\]

- \([8.7]-[8.10]\) is a derivation of possible expressions of \(a_{(0,1),i},(0,1+1)\) with respect to the boundary \(\triangle A_{1}A_{k+1}A_{j}, \triangle A_{k+2}A_{0,k}A_{j}\), \(\triangle A_{k}A_{0,k+1}A_{j}, \triangle A_{k+2}A_{0,k}A_{k+1}\) by applying the generalized cosine law in \(\mathbb{R}^2\) given in Lemma 9.

- The Schlafli angle \(\beta_{(0,k),u}\) was discovered in [27]. The distance function \(a_{(0,i),j}\) depends

\[a_{(0,i),j} = a_{(0,i),1} a_{(0,i),2} a_{(0,i),3} a_{(0,i),4} a_{(0,i),5} a_{(0,i),j}\]

for \(i = 1, 2, \ldots, N-1, j = N, N+1\).

In \(\mathbb{R}^3\), we derived that \(a_{(0,i),4}\) depend on \(a_{(0,i),1}, a_{(0,i),2}, a_{(0,i),3}\), because \(\beta_{(0,i),4}\) cannot be expressed explicitly as a function with respect to \(a_{(0,i),1}, a_{(0,i),2}, a_{(0,i),3}\).

In \(\mathbb{R}^4\), we derived that \(a_{(0,i),4}\), \(a_{(0,i),5}\), depend on \(a_{(0,i),1}, a_{(0,i),2}, a_{(0,i),3} \beta_{(0,i),4}\), because we derived an implicit function with respect to \(a_{(0,i),1}, a_{(0,i),2}, a_{(0,i),3} \beta_{(0,i),4}\) and \(\beta_{(0,i),4}\) cannot be solved explicitly with respect to \(a_{(0,i),1}, a_{(0,i),2}, a_{(0,i),3}\).

Therefore, Schlafli angles \(\beta_{(0,i),k}\) are embodied in the computation of the distances \(a_{(0,i),N}, a_{(0,i),N+1}\), for \(N \geq 4\) and \(k = 4, 5, \ldots, N\).

**Theorem 18** (Lagrange multiplier rule for the weighted Fermat-Steiner Frechet multitree in \(\mathbb{R}^N\)). If the admissible point \(\tilde{x}_i\) yields a weighted minimum multitree for \(1 \leq i \leq \frac{N(N+N+1)!}{(N+1)!}\), which correspond to a Frechet \(N\)-multisimplex derived by a \(\frac{N(N+N+1)!}{(N+1)!}\)-tuple of edge lengths determining up to \(\frac{N(N+N+1)!}{(N+1)!}\) incongruent \(N\)-simplexes constructed by the Dekster-Wilker domain \(D_{\nu}(\tilde{x}, s)\), then there are numbers \(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{(N^2-1)i}\), such that:

\[
\frac{\partial L_j(\tilde{x}_i, \tilde{\lambda}_j)}{\partial x_{ji}} = 0
\]

for \(j = 1, 2, \ldots, 2N(N-1)\),

\[
\tilde{x}_i = \{a_{(0,1),1}, a_{(0,1),2}, a_{(0,1),3}, a_{(0,1),4}, a_{(0,1),N}, \ldots, a_{(0,N-1),1}, a_{(0,N-1),2}, a_{(0,N-1),3}, a_{(0,N-1),4}, a_{(0,N-1),N}, w_{(0,1),1}, \ldots, w_{(0,1),N}, w_{(0,2),1}, \ldots, w_{(0,2),N}, \ldots, \}
\]
tree structure compared with the other Fermat-Steiner tree structures, that belong to the unweighted Fermat-Steiner-Frechet multitree in $\mathbb{R}^i$

perturbing the length weighted Fermat-Steiner tree structure for the variable weight $B$

The most natural

Proof. By setting $f_{p,q} \equiv f_k$, we consider that $\partial (f_k)_{(x_j)}$ are continuous in each parallelepiped $\Pi_i$, for $1 \leq i \leq 3\frac{2(N+1)!}{(N+1)!}$, $k = 0, 1, 2, \ldots, N^2 - 1$, $j = 1, 2, \ldots, 2N(N - 1)$ and by applying Lagrange multiplier rule, we obtain the Lagrangian vector $\hat{\lambda}_i = \{\lambda_0, \lambda_1, \ldots, \lambda_{N^2 - 1}\}$, such that (8.11) holds.

Denote by

$$a(N) = \begin{cases} \frac{(N+1)!}{N(N+1)!} \sqrt{1 - \frac{2(N+1)!}{N+1}N} & \text{for even } N \geq 2, \\ \frac{(N+1)!}{N(N+1)!} \sqrt{1 - \frac{2(N+1)!}{N+1}N} & \text{for odd } N \geq 3 \end{cases}$$

The function $a(N)$ is obtained by the function $\ell(N)$ of Dekster-Wilker (see in [6 p. 352]) using the inequality

$$\frac{a(N)}{a(N) + \frac{N(N+1)!}{2}} \geq \ell(N).$$

Theorem 19. The most natural $\frac{N(N+1)!}{2}$-tuple of numbers from $\frac{N(N+1)!}{2}$ consecutive natural numbers $\{a + \frac{N(N+1)!}{2}, a + 1, a\}$ for $a \geq a(N)$ is a $\frac{N(N+1)!}{2}$-tuple of edge lengths having the maximum volume (maximum $\frac{N(N+1)!}{2}$-tuple) among the $\frac{N(N+1)!}{2}$ incongruent $N$-simplexes, which corresponds to a Fermat-Steiner tree of minimum total weighted length (global minimum solution), such that the upper bound for the weight $B_{ST}$ is determined by the rest Fermat-Steiner minimal trees having larger or equal weighted minimal total length.

Proof. We will follow the same process that we used for sextuples and tentuples of consecutive natural numbers in $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

By applying Theorem [18] for the Dekster-Wilker $\frac{N(N+1)!}{2}$-tuple of edge lengths $\{a + \frac{N(N+1)!}{2}, a + 1, a\}$ for $a \geq a(N)$ forming $\frac{N(N+1)!}{2}$ incongruent $N$-simplexes in $\mathbb{R}^N$, we obtain a class of Fermat-Steiner trees for $b_i = 1$, for $i = 1, 2, \ldots, N + 1$ and $b_{ST} = 1$, which yields a Fermat-Steiner-Frechet multitree in $\mathbb{R}^N$. By selecting the proper $\frac{N(N+1)!}{2}$-tuple of edge lengths, which yields the maximum volume from all $\frac{N(N+1)!}{2}$ incongruent $N$-simplexes in $\mathbb{R}^N$, we consider a variable weighted Fermat-Steiner tree having three equally weighted Fermat Steiner points with weight $B_{ST}$ and the same boundary weights $b_i = 1$. By perturbing the length weighted Fermat-Steiner tree structure for the $N$-simplex $(A_1 A_2 \ldots A_{N+1})_{\text{Max}}$ having the maximum volume, we can derive an upper bound for the variable weight $B_{ST}$, which is calculated by comparing the perturbed length tree structure compared with the other Fermat-Steiner tree structures, that belong to the unweighted Fermat-Steiner-Frechet multitree in $\mathbb{R}^N$. Hence, this particular arrangement of the $\frac{N(N+1)!}{2}$ consecutive natural numbers for $a \geq a(N)$ with the upper bound $(B_{ST})$, yields the most natural weighted $\frac{N(N+1)!}{2}$-tuple of natural
numbers referring to weighted Fermat-Steiner trees with the minimum mass transfer.

9. **Intermediate weighted Fermat-Steiner-Frechet multitrees for a given \(\frac{N(N+1)}{2}\)-tuple of positive real numbers determining the edge lengths of incongruent \(N\)-simplexes in \(\mathbb{R}^N\)**

In this section, we deal with the solution (multitree) of the intermediate weighted Fermat-Steiner-Frechet problem \((P(I.\text{Fermat-Steiner-Frechet}))\) for a given \(\frac{N(N+1)}{2}\)-tuple of positive real numbers determining incongruent \(N\)-simplexes in \(\mathbb{R}^N\), by inserting \(N(l+s)\) equality constraints derived by \(l+s< N-1\) independent solutions for \((l+s\) variable weighted Fermat problems for the Frechet \(N\)-multisimplex derived by incongruent boundary \(N\)-simplexes in \(\mathbb{R}^N\), which correspond to the same \(\frac{N(N+1)}{2}\)-tuple of positive real numbers (edge lengths) and \(l+s-1\) equality constraints derived by two different expressions of each line segments connecting two consecutive weighted Fermat-Steiner points. By applying a Lagrange program, we can detect intermediate weighted Fermat-Frechet multitrees for a given \(\frac{N(N+1)}{2}\)-tuple of positive real numbers determining the edge lengths of incongruent \(N\)-simplexes in \(\mathbb{R}^N\).

**Definition 18** (A non degenerate intermediate weighted Fermat-Steiner tree for \(\{A_1, A_2, \ldots, A_{N+1}\}\)). A non degenerate weighted Fermat-Steiner tree \(T_{IS}\) is a tree, which consists of some line segments between the weighted Fermat-Steiner points \(A_{0,j}\), and weighted Fermat points \(P_{0,j}\) whose \(\# < N-1\), such that each \(A_{0,j}\) has degree three, each \(P_{0,j}\) has degree more than three and of \(N+1\) line segments joining each boundary vertex \(A_i\) with some \(A_{0,j}\), or \(P_{0,j}\).

**Definition 19.** An intermediate weighted Fermat-Steiner multitree \(MT_{IS}\) is a union of intermediate weighted Fermat-Steiner trees, which correspond to incongruent \(N\)-simplexes derived by the same Dekster-Wilker \(\frac{N(N+1)}{2}\)-tuple of positive real numbers determining edge lengths.

We consider an intermediate weighted Fermat-Steiner multitree in \(\mathbb{R}^N\) having incongruent boundary simplexes, with \(s\) a fixed \# of weighted Fermat-Steiner points and \(l\) a fixed \# of weighted Fermat points in \(\mathbb{R}^N\), such that \(s+l < N-1\).

**Problem 14** (The intermediate weighted Fermat-Steiner-Frechet \((P(I.\text{Fermat-Steiner-Frechet}))\) problem in \(\mathbb{R}^N\) with equality constraints).

\[
\begin{align*}
  f_0(\tilde{x}) &= \sum_{a_{i_k,j_k} \in T_S} b_k a_{i_k,j_k}, \\
  f_{i,j}(\tilde{x}) &= 0, \ i = 1, \ldots, l+s, \ j = 1, \ldots, N, \\
  f_k(\tilde{x}) &= 0, \ k = 1, \ldots, l+s-1,
\end{align*}
\]
where \( s \) is \# of weighted Fermat-Steiner points \( A_{0,i} \) type one, two or three and \( l \) is \# of weighted Fermat-Steiner points \( A_{0,i} \) of degree (connections) more than three.

\[
f_0(\hat{x}) = f_0(T_S) = \sum_{a_{i_k,j_k} \in T_S} b_{k} a_{i_k,j_k}, \tag{9.1}
\]

\[
f_{1,i}(\hat{x}) = \frac{w_{(0,1)},1}{a_{(0,1),1}} \text{Vol}(A_{0,1} \ldots A_{N+1}) - \frac{1 - \sum_{i=1}^{N} w_{(0,1),i}}{a_{(0,1),N+1} \text{Vol}(A_{0,1} \ldots A_{N+1})} \tag{9.2}
\]

\[
f_{i,N}(\hat{x}) = \frac{w_{(0,1),N}}{a_{(0,1),N} a_{(0,1),1} a_{(0,1),2} a_{(0,1),3} \beta_{(0,1),1} \beta_{(0,1),3} \beta_{(0,1),5} \ldots \beta_{(0,1),N}} \text{Vol}(A_{1} \ldots A_{0,1} A_{N+1}) - \frac{1 - \sum_{i=1}^{N} w_{(0,1),i}}{a_{(0,1),N+1} \text{Vol}(A_{0,1} \ldots A_{N+1})}, \tag{9.3}
\]

\[
f_{l+s,1}(\hat{x}) = \frac{w_{(0,l+s)},1}{a_{(0,l+s),1}} \text{Vol}(A_{0,l+s} \ldots A_{N+1}) - \frac{1 - \sum_{i=1}^{N} w_{(0,l+s),i}}{a_{(0,l+s),N+1} \text{Vol}(A_{0,l+s} A_{1} \ldots A_{N})}. \tag{9.4}
\]

where

\[
a_{(0,l+s),N+1} = a_{(0,l+s),N+1} a_{(0,l+s),1} a_{(0,l+s),2} a_{(0,l+s),3} \beta_{(0,l+s),1} \beta_{(0,l+s),3} \beta_{(0,l+s),5} \ldots \beta_{(0,l+s),N}.
\]

\[
f_{l+s,N}(\hat{x}) = \frac{w_{(0,l+s),N}}{a_{(0,l+s),N} a_{(0,l+s),N} \text{Vol}(A_{1} \ldots A_{0,l+s} A_{N+1})} - \frac{1 - \sum_{i=1}^{N} w_{(0,l+s),i}}{a_{(0,l+s),N+1} \text{Vol}(A_{0,l+s} A_{1} \ldots A_{N})}. \tag{9.5}
\]

where

\[
a_{(0,l+s),N} = a_{(0,l+s),N} a_{(0,l+s),1} a_{(0,l+s),2} a_{(0,l+s),3} \beta_{(0,l+s),1} \beta_{(0,l+s),3} \beta_{(0,l+s),5} \ldots \beta_{(0,l+s),N}.
\]

\[
f_k(\hat{x}) = a_{(0,k),(0,k+1)}(\hat{x}; \text{boundarysimplex } 1) - a_{(0,k),(0,k+1)}(\hat{x}; \text{boundarysimplex } 2), \tag{9.6}
\]

for \( k = 1, 2, \ldots l + s - 1. \)
We take into account that:

- (9.1) is the objective function of the intermediate weighted Fermat-Steiner problem for \( A_1, A_2, \ldots, A_{N+1} \) in \( \mathbb{R}^N \) having \( N-1 \) weighted Fermat-Steiner points \( A_{0,i} \), such that:

There are \( l \) weighted Fermat-Steiner points \( A_{0,i} \) of degree three with respect to a boundary triangle and \( s \) weighted Fermat points \( A_{0,i} \) of degree more than three with respect to a boundary \( r \)-simplex in \( \mathbb{R}^N \), for \( r < N \), which can be derived by the local minimality criterion of Lemma \( \text{I} \)

- (9.2)–(9.3) deal with the solution of the weighted Fermat problem for \( A_1, \ldots, A_{N+1} \) with variable weights \( w(0,1), \ldots, w(0,1), N \) and \( w(0,1), N+1 = 1 - \sum_{i=1}^{N} w(0,1), i \), which is determined by applying Theorem \( \text{II} \) for the unique weighted Fermat point \( A_{0,0} \).

- (9.4)–(9.5) deal with the solution of the weighted Fermat problem for \( A_1, \ldots, A_{N+1} \) with variable weights \( w(0,1), s, 1, w(0,1), s, 2, \ldots, w(0,1), s, N, \) and \( w(0,1), s, N+1 = 1 - \sum_{i=1}^{N} w(0,1), s, i \), which is determined by applying Theorem \( \text{II} \) and the unique weighted Fermat point \( A_{0,0} \).

- (9.6) is a derivation of possible expressions of \( a(0,1), (0,1), 2, 1 \) with respect to boundary \( r \)-simplexes for \( 2 \leq r < N \) by applying the generalized cosine law in \( \mathbb{R}^2 \) given in Lemma \( \text{III} \) combined with Theorem \( \text{II} \) for \( A_{1}, A_{2}, \ldots, A_{r+1} \), with \( r + 1 \) fixed given weights taken from the set \( \{b_1, b_2, \ldots, b_{N+1}, b_{ST}\} \).

The following Lagrangian program detects an intermediate weighted Fermat-Steiner-Frechet multicentre in \( \mathbb{R}^N \).

**Theorem 20** (Lagrange multiplier rule for the weighted Fermat-Steiner-Frechet multicentre in \( \mathbb{R}^N \)). If the admissible point \( \tilde{x}_i \) yields an intermediate weighted minimum multicentre for \( 1 \leq i \leq \frac{N(N+1)}{2} \), which correspond to a Frechet \( N \)-multisimplex derived by a \( \frac{N(N+1)}{2} \)-tuple of edge lengths determining up to \( \frac{N(N+1)}{2} \) incongruent \( N \)-simplexes constructed by the Dukster-Wilker domain \( DW_{\mathbb{R}^N}(l, s) \), then there are numbers \( \lambda_0, \lambda_{1,1}, \lambda_{2,2}, \ldots, \lambda_{(l+s-1)(N+1)+1} \), such that:

\[
\frac{\partial L_i(\tilde{x}_i, \tilde{\lambda}_i)}{\partial x_{j,i}} = 0 \quad (9.7)
\]

for \( j = 1, 2, \ldots, 2N(l + s - 1) \),

\[
\tilde{x}_i = \left\{ a(0,1), 1, a(0,1), 2, a(0,1), 3, \beta(0,1), 4, \ldots, \beta(0,1), N, \ldots, a(0,1), s, 1, a(0,1), s, 2, \ldots, a(0,1), s, 3, \beta(0,1), s, 4, \ldots, \beta(0,1), s, N, \ldots, a(0,1), 1, \ldots, w(0,1), N, \right\}
\]

\[
\tilde{\lambda}_i = \left\{ \lambda_0, \lambda_{1,1}, \ldots, \lambda_{(l+s-1)(N+1)+1} \right\}
\]

**Proof.** By setting \( f(p, q) = f_k \) we consider that \( \frac{\partial f_k}{\partial x_{j,i}} \) are continuous in each parallelepiped \( \Pi_i \), for \( 1 \leq i \leq \frac{3N(N+1)}{2} \), \( k = 0, 1, 2, \ldots, N^2 - 1, j = 1, 2, \ldots, 2N(l + s - 1) \) and by applying Lagrange multiplier rule, we obtain the Lagrangian vector \( \tilde{\lambda}_i = \left\{ \lambda_0, \lambda_{1,1}, \lambda_{2,2}, \ldots, \lambda_{(l+s-1)(N+1)+1} \right\} \), such that (9.7) holds. \( \square \)
Example 3. Let \( \{a_{ij}\} \) be a given 15-th tuple of positive real numbers determining incongruent 5-simplexes using Dekster-Wilker conditions in \( \mathbb{R}^5 \) and \( A_1A_2\ldots A_6 \) be a member of the Frechet 5-multisimplex in \( \mathbb{R}^5 \). We consider a fixed tree topology that contains two weighted Fermat-Steiner points \( A_{0,1}, A_{0,2} \) of degree three and one weighted Fermat point of degree four (see Fig. 3).

The intermediate weighted Fermat-Steiner tree for \( A_1A_2A_3A_4A_5A_6 \) in \( \mathbb{R}^5 \) consists of the line segments

\[
\{ A_{0,1}A_1, A_{0,1}A_2, A_{0,2}A_{0,2}, A_{0,2}A_3, A_{0,2}A_{0,3}, A_{0,3}A_4, A_{0,3}A_5, A_{0,3}A_6 \}.
\]

By applying the Lagrangian program of Theorem 20, we can detect intermediate weighted Fermat-Steiner-Frechet multitrees for incongruent 5-simplexes in \( \mathbb{R}^5 \) derived by a given Dekster-Wilker 15-tuple of edge lengths.

10. "Mutation" of an intermediate weighted Fermat-Steiner-Frechet multitree for \( m \) closed polytopes in \( \mathbb{R}^N \)

In this section, we introduce the "mutation" of an intermediate weighted Fermat Steiner Frechet multitree for \( m \) boundary polytopes, by applying the plasticity solutions of the \( m-\text{INVWF} \) problem for \( A_1A_2\ldots A_{m+1} \) in \( \mathbb{R}^N \), enriched by a two way mass transportation network. The dynamic plasticity solutions of the \( m-\text{INVWF} \) problem in \( \mathbb{R}^N \) combined with conditions derived by optimal mass transport and storage gives the "mutation of an intermediate weighted Fermat-Frechet multitree for boundary \( m \) closed polytopes in \( \mathbb{R}^N \).
We start by deriving the geometric and dynamic plasticity for \( m \) closed polytopes in \( \mathbb{R}^N \) for \( m \geq N + 1 \).

**Definition 20.** We call geometric plasticity of a weighted Fermat tree whose endpoints correspond to a closed polytope in \( \mathbb{R}^N \), which is formed by \( m+1 \) variable line segments meeting at the weighted Fermat point \( A_0 \), for \( m+1 \) given values of the weights, the set of solutions of the \( m+1 \) variable lengths of the line segments, which correspond to a family of weighted networks that preserve the weighted Fermat point \( A_0 \).

We assume that we select \( B_i \) that correspond to each vertex \( A_i \), such that the weighted floating inequalities of Theorem 1 hold:

\[
\| \sum_{j=1, j \neq i}^{m+1} B_j \bar{u}(A_j, A_i) \| > B_i,
\]

for \( i, j = 1, 2, \ldots, m+1 \). Thus, the weighted Fermat point \( A_0 \) of an \( m \) closed polytope \( A_1 A_2 \ldots A_m \) in \( \mathbb{R}^N \) is the unique intersection point of \( m+1 \) line segments \( A_0 A_i \).

**Theorem 21** (Geometric plasticity of \( m \) closed polytopes in \( \mathbb{R}^n \)). *If we select a point \( A'_i \) on the ray defined by \( A_0 A_i \) with corresponding weight \( B_i \), such that*

\[
\| \sum_{j=1, j \neq i}^{m+1} B_j \bar{u}(A'_j, A'_i) \| > B_i,
\]

*for \( i, j = 1, 2, \ldots, m+1 \), then the corresponding weighted Fermat point \( A'_0 \equiv A_0 \).*

**Proof.** The weighted floating inequalities

\[
\| \sum_{j=1, j \neq i}^{m+1} B_j \bar{u}(A_j, A_i) \| > B_i,
\]

\[
\| \sum_{j=1, j \neq i}^{m+1} B_j \bar{u}(A'_j, A'_i) \| > B_i,
\]

yield respectively,

\[
\sum_{i=1}^{m+1} B_i \bar{u}(A_0, A_i) = \bar{0},
\]

\[
\sum_{i=1}^{m+1} B_i \bar{u}(A_0, A'_i) = \bar{0}.
\]

Thus, we derive that \( A'_0 \equiv A_0 \). \( \square \)

We assume that \( m+1 \) line segments \( A_0 A_i \) intersect at \( A_0 \) in \( \mathbb{R}^N \), for \( m \geq N + 1 \). We note that:

(1) for \( n = 2 \), \( m \) angles \( \alpha_{0j} \) determine the dynamic plasticity equations of \( A_1 A_2 \ldots A_{m+1} \) in \( \mathbb{R}^2 \).

(2) for \( n = 3 \), \( m+(m-1) \) angles \( \alpha_{0j} \) determine the dynamic plasticity equations of \( A_1 A_2 \ldots A_{m+1} \) in \( \mathbb{R}^3 \).
(3) for $n > 3$, $\sum_{i=1}^{N-1} (m+1-i)$ angles $\alpha_{ij}$ determine the dynamic plasticity equations of $A_1A_2\ldots A_{m+1}$ in $\mathbb{R}^N$.

We set $\sum_{i=1}^{m+1} B_i = (B_{i_0})_{i=m+1}^{m+1}$, $\sum_{i=1}^{m+1} B_i = (B_{i_0})_{i=m+1}^{m+1} + \ldots + (B_{i_N})_{i=m+1}^{m+1}$, for $i_1, \ldots, i_{m+1} \in \{1,2,\ldots,m+1\}$.

**Lemma 11.** If $\sum_{i=1}^{m+1} B_i = (B_{i_0})_{i=m+1}^{m+1}$, for every $i_1, \ldots, i_{m+1} \in \{1,2,\ldots,m+1\}$, where $\sum_{i=1}^{m+1} B_i = (B_{i_0})_{i=m+1}^{m+1} + \sum_{i=1,i\neq N+1}^{m} (\frac{B_i}{B_{i_0}})_{i=m+1}^{m+1}$, then

$$\sum_{i=1}^{m} a_{i,j} (B_j)_{i=m+1}^{m+1} + b_i, \quad i = 1,2,\ldots,N+1, \quad (10.1)$$

where

$$\begin{align*}
(a_{N+1,j}, b_{N+1}) &= (\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} + \cdots + (\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} - (\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} \\
(B_{N+1})_{i=m+1}^{m+1},
\end{align*}$$

$$\begin{align*}
(a_{N,j}, b_{N}) &= (a_{N+1,j} (B_N)_{i=m+1}^{m+1} - (\frac{B_N}{B_{N+1}})_{i=m+1}^{m+1} (\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} - (\frac{B_N}{B_{N+1}})_{i=m+1}^{m+1}(\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} \\
(B_N)_{i=m+1}^{m+1},
\end{align*}$$

$$\vdots$$

$$\begin{align*}
(a_{1,j}, b_1) &= (a_{N+1,j} (B_1)_{i=m+1}^{m+1} - (\frac{B_1}{B_{N+1}})_{i=m+1}^{m+1} (\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} - (\frac{B_1}{B_{N+1}})_{i=m+1}^{m+1}(\frac{B_{N+1}}{B_j})_{i=m+1}^{m+1} \\
(B_1)_{i=m+1}^{m+1},
\end{align*}$$

for $j = n+2, \ldots, m+1$.

By assuming mass flow continuity and by applying the geometric and dynamic plasticity of $m$ polytopes in $\mathbb{R}^N$, (Theorem [21] Lemma [11]) the corresponding weighted Fermat tree solutions yield some mass transportation networks, in which the weights correspond to an instantaneous collection of images of masses, which satisfy some specific conditions.

**Definition 21.** We call $(m,k)$ "mutation" of an I.Fermat-Steiner tree with respect to boundary am polytope the plasticity solutions of the $m-$INVWF problem for $A_1A_2\ldots A_{m+1}$ in $\mathbb{R}^N$, enriched by a two way mass transportation network, such that $k$ masses (weights) are transferred in $k$ directions from $A_i \rightarrow A_0$, for $i = 1,2,\ldots,k$ (inflow) creating a storage at $A_0$ and $N-k$ masses are transferred in $m-k$ directions from $A_0 \rightarrow A_j$ (outflow), for $j = k+1,\ldots,m+1$ and reversely $m-k$ masses (new weights) are transferred back to $A_0$ along the same $m-k$ directions creating new storage at $A_0$, and $k$ masses (new weights) are transferred back from $A_0$ to $A_i$, $i = 1,2,\ldots,k$.

We denote by $B_i$ a mass flow which is transferred from $A_i$ to $A_0$ for $i = 1,2,\ldots,k$ by $B_0$ a residual weight which remains at $A_0$ and by $B_{k+1},\ldots B_{m+1}$ a mass flow which is transferred from $A_0$ to $A_{k+1},\ldots,A_{m+1}$. 
We denote by \( \tilde{B}_i \) a mass flow which is transferred from \( A_i \) for \( i = 1, 2, \ldots k \) by \( B_0 \) a residual weight which remains at \( A_0 \) and by \( B_{k+1}, \ldots, B_{m+1} \) a mass flow which is transferred from \( A_{k+1}, \ldots A_{m+1} \) to \( A_0 \).

Thus, we derive that:

\[
\sum_{i=1}^{k} B_i = \sum_{k+1}^{m+1} B_i + B_0 \quad (10.2)
\]

and

\[
\sum_{i=1}^{k} \tilde{B}_i + \tilde{B}_0 = \sum_{i=k+1}^{m+1} \tilde{B}_i. \quad (10.3)
\]

By adding (10.2) and (10.3) and by letting \( \bar{B}_0 = B_0 - \tilde{B}_0 \) we get:

\[
\sum_{i=1}^{k} \bar{B}_i = \sum_{i=k+1}^{m+1} \bar{B}_i + \bar{B}_0 \quad (10.4)
\]

such that:

\[
\sum_{i=1}^{m+1} \bar{B}_i = c, \quad (10.5)
\]

where \( c \) is a positive real number.

Thus, we derive the following theorem as a direct consequence of Lemma 11 under the condition for the weights \( \bar{B}_{12} \ldots m+1 \) taken from (10.4), (10.5), which deal with the \((m, k)\) "mutation" of \( m \) closed polytopes in \( \mathbb{R}^N \).

**Theorem 22.** \((m, k)\) "mutation" of an intermediate weighted Fermat-Steiner tree for boundary \( m \) closed polytopes in \( \mathbb{R}^N \)

If \( \sum_{i=1}^{m+1} \bar{B} = \sum_{i=1}^{m+1} B_i \) for every \( i_1, \ldots, i_{n+1} \in \{1, 2, \ldots, m+1\} \), where \( \sum_{i=1}^{m+1} \bar{B} := (\bar{B}_{m+1})_{12} \ldots m+1 (1 + \sum_{i=1, i\neq n+1}^{m+1} \left( \frac{B_i}{B_{N+1}} \right)_{12} \ldots m+1 \), then

\[
(\bar{B}_1)_{12} \ldots m+1 = \sum_{j=n+2}^{m+1} a_{i,j}(\bar{B}_j)_{12} \ldots m+1 + b_i, \quad i = 1, 2, \ldots, N+1, \quad (10.6)
\]

where

\[
(a_{N+1, j}, b_{N+1}) =
\left( \frac{B_{N+1}}{B_{N+1}} \right)_{12} \ldots N+1 (\bar{B}_{N+1})_{12} \ldots j + \cdots + \left( \frac{B_{N+1}}{B_{N+1}} \right)_{12} \ldots N+1 (\bar{B}_{N+1})_{12} \ldots (N-1)(N+1)j - 1
\]

\[
\sum_{i=1}^{n+1} \left( \frac{B_{N+1}}{B_{N+1}} \right)_{12} \ldots N+1
\]

\[
(\bar{B}_{N+1})_{12} \ldots N+1
\]

\[
(a_{N, j}, b_{N}) = (a_{N+1, j} \left( \frac{B_{N}}{B_{N+1}} \right)_{12} \ldots N+1 - \left( \frac{B_{N}}{B_{N+1}} \right)_{12} \ldots N+1 (\bar{B}_{N+1})_{12} \ldots (N-1)(N+1)j, \quad (\bar{B}_{N})_{12} \ldots N+1,
\]
Creating a storage at respect to a given directions from masses (weights) are transferred in \( k \) masses (new weights) are transferred back from

\[
(a_{1,j}, b_1) = (a_{N+1,j}(\frac{\bar{B}_1}{B_{N+1}})_{12...N+1} = (\frac{\bar{B}_1}{B_{N+1}})_{12...N+1}(\frac{\bar{B}_{N+1}}{B_j})_{23...j},
\]

\((\bar{B}_1)_{12...N+1},\)

for \( j = n + 2, \ldots m + 1 \), under the conditions for the weights:

\[
\sum_{i=1}^{k}(\bar{B}_i)_{12...m+1} = \sum_{i=k+1}^{m+1}(\bar{B}_i)_{12...m+1} + (\bar{B}_0)_{12...m+1}
\]

\[
\sum_{i=1}^{m+1}(\bar{B}_i)_{12...m+1} = c.
\]

Suppose that at time \( t = 0 \), an I.Fermat-Steiner-Frechet multitree occurs with respect to a given \( \frac{N(N+1)}{2} \)-tuple of positive real numbers determining the edge lengths of incongruent \( N \)-simplexes generates by the Dekster-Wilker domain in \( \mathbb{R}^N \). Then \( N + 2, \ldots, m + 1 \) rays start to grow from the weighted Fermat-point \((A_0)_i\), which creates a two-way mass transport, in order to obtain an \((m, k)\) mutation of an I.Fermat-Steiner-Frechet multitree having one weighted Fermat point of degree \( m + 1 \).

**Theorem 23** \((m,k)\)’ mutation” of an intermediate weighted Fermat-Steiner Frechet multitree in \( \mathbb{R}^N \). The equations of \((m,k)\)”mutation” of an intermediate weighted Fermat-Steiner Frechet multitree in \( \mathbb{R}^N \) is derived by adding

\((A_0)(A_{N+1}), (A_0)(A_{N+1})\), \ldots, \((A_0)(A_{m+1})\) rays to a given weighted Fermat-Steiner Frechet multitree with respect to a boundary Frechet \( N \)-multisimplex, such that \( k \) masses (weights) are transferred in \( k \) directions from \((A_i)_i \rightarrow (A_0)_i\), for \( i = 1, \ldots k \) (inflow) creating a storage at \((A_0)_i\) and \( N - k \) masses are transferred in \( m - k \) directions from \((A_0)_i \rightarrow (A_j)_i\) (outflow), for \( j = k + 1, \ldots, m + 1 \) and reversely \( m - k \) masses (new weights) are transferred back to \((A_0)_i\) along the same \( m - k \) directions creating new storage at \((A_0)_i\), and \( k \) masses (new weights) are transferred back from \((A_0)_i \rightarrow (A_i)_i\), \( i = 1, 2, \ldots, k \), \( 1 < l \leq \frac{N(N+1)}{(N+1)!}\).

**Proof.** By adding \((A_0)(A_{N+1}), (A_0)(A_{N+1})\), \ldots, \((A_0)(A_{m+1})\) rays to a given weighted Fermat-Steiner Frechet multitree with respect to a boundary Frechet \( N \)-multisimplex in \( \mathbb{R}^N \) and by applying Theorem 22 we derive the \((m, k)\) plasticity equations of an I.Fermat-Frechet multitree for \( m \) boundary closed polytopes in \( \mathbb{R}^N \).

11. **Constructive tree weights for the vertices of a Frechet \( N \)-multisimplex in \( \mathbb{R}^N \)**

In this section, we obtain an \( \varepsilon \) approximation of the value of the weight \( B_{N+1} \), which corresponds to the vertex \( A_{N+1} \) of an \( N \)-simplex in \( \mathbb{R}^N \) circumscribed in a \((N - 1)\)sphere \( S^{N-1} \) of radius \( r \), and center \( O \), by applying the \( N \)-INWF problem in \( \mathbb{R}^N \). For \( \varepsilon \rightarrow 0 \), the limiting floating weighted Fermat tree solution coincides with the absorbing weighted Fermat tree solution of Theorem 1. An application of this method is an approximation for the weights of a Frechet \( N \)-multisimplex via an \( \varepsilon \) approximation of each corner of incongruent \( N \)-simplexes with a multiweighted Fermat-Frechet multitree in \( \mathbb{R}^N \).
Let $A_1A_2 \ldots A_{N+1}$ be an $N$–simplex circumscribed in $S^{N-1}(0,r)$. We consider a point $A_0 \in [A_{N+1},O]$, and we denote by $\epsilon = |A_{N+1}A_0|$.

**Theorem 24.** The weights $B_1(\epsilon), \ldots, B_{N+1}(\epsilon)$ are uniquely determined by

$$B_i(\epsilon) = \frac{C}{1 + \frac{\sin \alpha_{i,0k_1k_2 \ldots k_{N-1}}}{\sin \alpha_{i,N-10k_1k_2 \ldots k_{N-1}}} + \frac{\sin \alpha_{i,0k_2k_3 \ldots k_N}}{\sin \alpha_{i,N-10k_1k_2 \ldots k_N}} + \ldots + \frac{\sin \alpha_{i,0k_{N-2}k_N}}{\sin \alpha_{i,N-10k_1k_2 \ldots k_N}}}$$

(11.1)

for $i, k_1, k_2, \ldots, k_N = 1, 2, \ldots, N+1$ and $k_1 \neq k_2 \neq \ldots \neq k_N$, and the weighted Fermat tree $\{A_1A_0, A_2A_0, \ldots, A_{N+1}A_0\}$ is an $\epsilon$ approximation of the absorbing weighted Fermat tree $\{A_1A_{N+1}, A_2A_{N+1}, \ldots, ANA_{N+1}\}$, such that: $A_0 \equiv A_{N+1}$, with a weighted error estimate for $B_{N+1}$:

$$\sqrt{\frac{\sum_{i=1}^{N} B_i^2(\epsilon) + 2 \sum_{i,j=1,i<j}^{N} B_i(\epsilon)B_j(\epsilon) \cos \alpha_{i(N+1)j} - B_{N+1}(\epsilon)}}.$$  

(11.2)

**Proof.** By applying the cosine law and the sine law in $\triangle A_0A_iA_j$, and $\triangle OA_iA_0$, we derive that $\frac{N(N+1)}{2} - 1$ angles $\alpha_{0ij} = \alpha_{0ij}(\epsilon)$. By substituting these angles in [23, p. 318] we obtain a unique solution of $B_i(\epsilon)$ given by (11.1). By replacing $B_i(\epsilon)$ for $i = 1, 2, \ldots, N$ in the weighted absorbing condition of Theorem 14 we get an error estimate for $B_{N+1}$ (11.2).

\[\square\]

12. Bessel plasticity and "Mutation" of an intermediate weighted Fermat-Steiner-Frechet multitree for $m$ polytopes in $\mathbb{R}^N$

In this section, we will describe the evolutionary structure of an intermediate weighted Fermat-Steiner-Frechet multitree for $m$ boundary polytopes in $\mathbb{R}^N$ with random weights following a Bessel motion.

We start by constructing a Bessel motion in the sense of McKean (23, 15). We consider the $m$–dimensional Brownian motion with sample paths $b(t)$, $(m \geq 2, t \geq 0)$ and generator $\mathcal{G} = \frac{1}{2}(\sum_{i=1}^{m} \frac{d^2}{\partial x_i^2})$. The radial part $r(t) = |b(t)|$ $(t \geq 0)$ is the Bessel motion with generator $\mathcal{G}^+ = \frac{1}{2}(\frac{d^2}{\partial r^2}) + \frac{N-1}{r} \frac{d}{dr}$. The probability of the event $B$ $P(B)$ as a function of the starting point $a = b(0)$ of the Brownian path if $t_1 < t_2$ is given by:

$$P([r(t_2) \leq l([r(s) : s \leq t_1])] =$$

$$= \int_{b-b(t_1)}(2\pi(t_2-t_1))^{-m/2}e^{-\frac{(b-b(t_1))^2}{2r(t_2-t_1)}}db_1 \ldots db_m.$$  

The Bessel motion $[r(t), P]$ is a Markov process, which depends upon $r(t_1)$.

**Lemma 12** (Non-negative solutions of Bessel Motion). [23 p. 318-319] The solution of the singular integral equation

$$r(t) = b(t) + \frac{N-1}{2} \int_{1}^{t} r^{-1} \, ds, \, t \geq 0,$$

(12.1)

is a Bessel motion starting at $r(0) = b(0)$, which takes non-negative real values by neglecting a class of Brownian paths $b$ of Wiener measure 0.
By taking into account the solution of the $N$–INVWF problem for polytopes in $\mathbb{R}^N$, we derive the equations of Bessel plasticity of a weighted Fermat-tree for a boundary polytope in $\mathbb{R}^N$. The Bessel plasticity of a weighted Fermat tree characterizes the combinatorial plasticity (random weights) of non-random polytopes in $\mathbb{R}^N$.

**Theorem 25.** The following equations point out the Bessel plasticity of an intermediate weighted Fermat-Steiner tree with one weighted Fermat point for a boundary $(N+1)$ weighted closed polytope with respect to the non-negative random weights $(B_i)_{i=1}^{N+2}$, which depend on the Bessel motion $r_{N+2}(t) \equiv (B_{N+2})_{i=1}^{N+2}$ in $\mathbb{R}^N$:

\[
\frac{B_1}{B_{N+1}}_{12\ldots N+2} = \left(\frac{B_1}{B_{N+1}}\right)_{12\ldots N+1} \left(1 - \frac{r_{N+2}(t)}{B_{N+1}}\right)_{12\ldots (N+2)} \left(\frac{B_{N+1}}{B_{N+2}}\right)_{2\ldots N+2}
\]

(12.2)

\[
\frac{B_2}{B_{N+1}}_{12\ldots N+2} = \left(\frac{B_2}{B_{N+1}}\right)_{12\ldots N+1} \left(1 - \frac{r_{N+2}(t)}{B_{N+1}}\right)_{12\ldots (N+2)} \left(\frac{B_{N+1}}{B_{N+2}}\right)_{13\ldots N+2}
\]

(12.3)

\[\vdots\]

\[
\frac{B_N}{B_{N+1}}_{12\ldots N+2} = \left(\frac{B_N}{B_{N+1}}\right)_{12\ldots N+1} \left(1 - \frac{r_{N+2}(t)}{B_{N+1}}\right)_{12\ldots (N+2)} \left(\frac{B_{N+1}}{B_{N+2}}\right)_{12\ldots (N-1)(N+1)(N+2)}
\]

(12.4)

such that:

\[
\sum_{i=1}^{n+2} (B_i)_{12\ldots N+2} = c.
\]

**Proof.** By inserting $r_{N+2}(t) \equiv (B_{N+2})_{i=1}^{N+2}$ into (6.37, (6.38), (6.39), we obtain (12.2), (12.3), (12.4). $\square$

Suppose that at time $t = 0$, an I.Fermat-Steiner-Frechet multitree occurs with respect to a given $\frac{N(N+1)}{2}$-tuple of positive real numbers determining the edge lengths of incongruent $N$-simplexes generates by the Dekster-Wilker domain in $\mathbb{R}^N$. Then an $(N+2)$th, ray start to grow from the weighted Fermat-point $(A_0)$. If the weight $(B_{N=2})$ follows a Bessel motion, we derive the Bessel plasticity of a weighted multitree for a boundary $(N+1)$ closed polytope in $\mathbb{R}^N$.

**Theorem 26 (Bessel plasticity of an I.Fermat-Steiner-Frechet multitree for an $N+1$ boundary closed polytope in $\mathbb{R}^N$).** The Bessel plasticity equations of an I.Fermat-Steiner-Frechet multitree for an $N+1$ boundary closed polytope in $\mathbb{R}^N$ are given by the Bessel plasticity equations of a weighted Fermat tree, such the random weight that corresponds to the $(N+2)$th ray follows a Bessel motion.

**Proof.** By applying Theorem 25 for an I.Fermat-Steiner-Frechet multitree for an $N+1$ boundary closed polytope in $\mathbb{R}^N$ which is derived by adding a ray $(A_0A_{N+2})$ with a random weight following a Bessel motion to I.Fermat-Steiner-Frechet multitree for an $N$ boundary Frechet multisimplex in $\mathbb{R}^N$, we get the Bessel plasticity equations of the generated I.Fermat-Steiner-Frechet multitree in $\mathbb{R}^N$. $\square$
Remark 4. We note that the Bessel plasticity of the I.Fermat-Steiner-Frechet multitree for an $N + 1$ boundary closed polytope in $\mathbb{R}^N$ may cause a distortion of the length structure of the initial I.Fermat-Steiner-Frechet multitree for an $N$ boundary Frechet multisimplex in $\mathbb{R}^N$.

13. Open questions

In this final section, we mention two open questions, which deal with the detection of the most natural of natural numbers.

1. How can we detect the most natural of consecutive $\frac{N(N+1)}{2}$ natural numbers, such that an unweighted Fermat-Steiner tree corresponds to a boundary $N-$simplex having the maximum volume with edge lengths this $\frac{N(N+1)}{2}$-tuple of natural numbers for $N \geq 3$ by using unweighted Fermat-Steiner points with weight $b_{ST} = 1$?

2. How can we detect the $\frac{N(N+1)}{2}$-tuples from the topology structure of intermediate unweighted Fermat Steiner Frechet multitrees for consecutive $\frac{N(N+1)}{2}$ natural numbers determining the edge lengths of a Frechet $N-$multisimplex in $\mathbb{R}^N$, which correspond to an intermediate unweighted Fermat-Steiner tree having the $i$th maximal volume?

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