LIMITING INTERPOLATION SPACES VIA EXTRAPOLATION

SERGEY V. ASTASHKIN, KONSTANTIN V. LYKOV, AND MARIO MILMAN

Abstract. We give a complete characterization of limiting interpolation spaces for the real method of interpolation using extrapolation theory. For this purpose the usual tools (e.g., Boyd indices or the boundedness of Hardy type operators) are not appropriate. Instead, our characterization hinges upon the boundedness of some simple operators (e.g. $f \mapsto f(t^2)/t$, or $f \mapsto f(t^{1/2})$) acting on the underlying lattices that are used to control the $K$- and $J$-functionals. Reiteration formulae, extending Holmstedt’s classical reiteration theorem to limiting spaces, are also proved and characterized in this fashion. The resulting theory gives a unified roof to a large body of literature that, using ad-hoc methods, had covered only special cases of the results obtained here. Applications to Matsaev ideals, Grand Lebesgue spaces, Bourgain-Brezis-Mironescu-Maz’ya-Shaposhnikova limits, as well as a new vector valued extrapolation theorems, are provided.

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1. Foreword

In this section we take up Halmos’ famous maxim “don’t worry if you do not understand the preliminaries” one step further, as we try to briefly explain our perspective to the necessarily technical material that follows.

In this paper we give a characterization of the so-called “limiting interpolation spaces” via extrapolation theory. We also find sharp conditions, under which the reiteration formulae associated with the above spaces hold.

The original motivation comes from the following apparently unrelated results. On the one hand, there is a paper by Gomez-Milman [38] where the authors find a way to extend the classical Lions-Peetre scale of interpolation spaces to limiting values of the parameters. In particular, in this fashion $L\log L$ and Dini type of spaces appear naturally as limiting spaces of real interpolation scales. Moreover, the basic results of the Lions-Peetre theory were extended to this context, including a special “reiteration formula” $1$. The results of [38] were welcomed and sparked a literature devoted to the construction of limiting spaces in order to complete the Lions-Peetre scale of spaces and obtain the corresponding reiteration theorems (cf. [15–17,19–28], as well as the many references therein). On the other hand, not long after that, Jawerth-Milman (cf. [43] and the references therein) developed a far-reaching generalization of Yano’s theorem. In particular, $L\log L$ spaces, can be naturally described by the extrapolation methods of [43]. For example, setting

$$\langle A_0, A_1 \rangle_{0,1} := \{ f \in A_0 + A_1 : \int_0^1 K(t,f;A_0,A_1) \frac{dt}{t} < \infty \},$$

and working on spaces based on a finite measure space, we have

$$L\log L = \langle L^1, L^\infty \rangle_{0,1} = \sum_{p>1} \frac{L^p}{p-1},$$

In extrapolation theory the reiteration formula of [38]

$$\langle L^1, (L^1, L^\infty)_{\theta,p_0} \rangle_{0,1} = \langle L^1, L^\infty \rangle_{0,1}$$

takes the following form: for all $p_0 > 1$, with norm equivalence, we have

$$\sum_{p>1} \frac{L^p}{p-1} = \sum_{1<p<p_0} \frac{L^p}{p-1}.$$ 

$1$In particular, they find an abstract form of an interpolation theorem of Zygmund. In [67], Zygmund had shown that if $T$ is an operator acting on functions defined on a finite measure space, then if $T$ is of weak types $(1,1),(p_0,p_0)$, for some $p_0 > 1$, it follows that $T$ is bounded from $L(\log L)$ to $L^1$. It is easy to give a direct proof or one can proceed by interpolation followed by extrapolation. Indeed, the assumptions of Zygmund’s theorem imply via, the classical Marcinkiewicz interpolation theorem, that $T$ is of strong type $(p,p)$ for $1 < p < p_0$ with $\|T\|_{L^p \to L^p} \leq (p-1)^{-1} (p-1)^{p-1} \approx (p-1)^{-1}$ as $p \to 1$ (cf. [63] Theorem 4.1, page 91). Therefore, the conclusion of Zygmund’s theorem follows by Yano’s extrapolation theorem (cf. [66]).
A similar result holds for the so-called $\Delta$-method of extrapolation. These results are valid not only for $L^p$ spaces but for general interpolation scales, and provide, in particular, an explanation to the results of [35]. Thus, the connection between limiting spaces, reiteration and extrapolation was known early on, at least in some special cases.

However, the vast literature on limiting spaces that has developed afterwards seems to be largely independent from extrapolation theory (cf. the recent survey presented in [61]). The purpose of this paper is to restore the connection between limiting spaces and extrapolation theory in full generality.

The limiting interpolation spaces considered in this paper are of the form $\langle \vec{A} \rangle_K^F$ (resp. $\langle \vec{A} \rangle_J^F$), where $\vec{A}$ is any ordered pair of Banach spaces, and $F$ is a fixed Banach lattice of measurable functions on $[0,1]$, that is a parameter for the corresponding real method used, that controls the usual $K$- (resp. $J$-) functionals. In this context we ask: under what conditions on $F$ can we guarantee that $\langle \vec{A} \rangle_K^F$ (resp. $\langle \vec{A} \rangle_J^F$) can be represented by an extrapolation functor applied to a Lions-Peetre scale $\langle \vec{A}_{\theta,q} \rangle$? To answer this question we use the general class of extrapolation functors introduced in [1] and we impose conditions on $F$. Surprisingly, a complete answer is provided by the boundedness of simple operations acting on the functions of $F$. The basic operators in question are $f \mapsto f(t^2)/t$, or $f \mapsto f(t^{1/2})$, and their variants/or iterations. So, for example the general version of (1.1) is given by Theorem 1: If $f \mapsto f(t^{1/2})$ is bounded on $F$ (resp. if $f \mapsto f(t^2)/t$ is bounded on $F$, cf. Theorem 1) then

$$\langle \vec{A} \rangle^F = \text{Ext}_F \{\vec{A}_{\theta,q}\},$$

where $\text{Ext}_F$ is a suitable extrapolation functor generalizing the $\Sigma$-functor (resp. when $\text{Ext}_F$ is a generalization of the $\Delta$-functor). The corresponding reiteration formula (1.2) takes the following form. If $F$ is an interpolation Banach lattice on $[0,1]$ with respect to the pair $(L^\infty, L^\infty(1/t))$, then the boundedness of $f \mapsto f(t^{1/2})$ on $F$ is equivalent to the following equivalent statement: for every ordered pair $\vec{A} = (A_0, A_1)$ and for every $\theta \in (0,1)$, $1 \leq q \leq \infty$, we have

$$\vec{A}^F_K = \langle A_0, \vec{A}_{\theta,q} \rangle_K^F.$$ (1.3)

The proof of these results requires the use of the deepest parts of real interpolation theory. In particular, the $K$-divisibility theorem of Brudnyi-Kruglyak [13] and some of its consequences or variants: the strong form of the fundamental lemma (cf. [30] and the references therein), the fact that suitable pairs of Banach spaces (e.g., $(L^1, L^\infty)$) are Conv$_{t_0}$-abundant, are combined with the boundedness assumptions we place on the special operators $f \mapsto f(t^2)/t$, or $f \mapsto f(t^{1/2})$ that act on the underlying lattices controlling the $K$ and $J$ functionals.

In our development we uncover applications to Matsaev ideals, generalized versions of Zygmund’s theorem, and using ideas of Pisier we show vector valued versions of Yano’s theorem. We also discuss the connection of extrapolation to limits

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2We should also mention here the important work of Cwikel-Pustylnik [31] on sharp forms of the Marcinkiewicz interpolation theorem. Although we shall not discuss the connection in detail here we should mention that the limiting spaces in [31] can be easily seen to be extrapolation spaces as well.

3As it will be seen later, the particular form of these special operators is connected with the form of the well known Holmstedt reiteration formula (cf. [2]).
of norm inequalities, and show how extrapolation theory leads to slight extensions of limit theorems due to Bourgain-Brezis-Mironescu-Maz’ya-Shaposhnikova (cf. [10], [51], [53], [59], [60]) (cf. Section 9.3).

2. Introduction

Many of the familiar spaces we use in analysis can be described using interpolation methods (cf. [8], [9], [13], [14], [65]). In particular, the real and complex methods of interpolation play a central rôle in the applications of interpolation theory to analysis. For a given pair \( (\theta, q) \) or simply “pairs”, the latter will be our preferred nomenclature.

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Actually, for mutually closed pairs, \( \ref{2.3} \) is essentially an easy consequence of the strong form of the fundamental lemma of interpolation theory (cf. Section 3.3 below). Moreover, applying \( \ref{2.3} \) to the pair \( (A_1, A_0) \), it follows that if \( A_0 \cap A_1 \) is dense in \( A_1 \),

\[
\begin{align*}
\tilde{A}_{1,1}^J &= A_1. \\
\end{align*}
\]

In the remaining cases, we have\(^5\)

\[
\begin{align*}
\tilde{A}_{j,q}^J &= \{0\}, j = 0, 1, 1 < q \leq \infty. \\
\end{align*}
\]

\(^4\)In the literature they are usually referred to as a “compatible couple of Banach spaces” (cf. [9]) or simply “pairs”, the latter will be our preferred nomenclature.

\(^5\)In fact, there is a simple connection between \( \ref{2.3} \) and \( \ref{2.1} \). For example, by [9] see proof of Theorem 3.7.1 part 3, page 54,

\[
\left( \tilde{A}_{0,q}^J \right)^* \subset \tilde{A}_{1,q}^J.
\]

and thus by \( \ref{2.1} \), \( \left( \tilde{A}_{0,q}^J \right)^* = \{0\} \), if \( 1 < q \leq \infty \).
On the other hand, it is possible to modify the Lions-Peetre constructions in such a way that, for the limiting values of the parameter θ, the resulting spaces are non-trivial. Indeed, such modifications have turned out to be useful since a number of spaces of interest in analysis can be identified in this fashion (cf. [38], and Sections 3 and 6 below). Moreover, the resulting “limiting spaces” appear naturally in extrapolation theory [43]. Indeed, one of the main purposes of this paper is to explicitly identify and characterize limiting spaces as extrapolation spaces, and use these representations to prove new qualitative results for limiting spaces, including reiteration theorems.

One of the first modifications of the Lions-Peetre scale was recorded in [38], where the case of “ordered pairs” of spaces was discussed. Let us say that a pair $\vec{A}$ is ordered if $A_1 \subset A_0$. Then, for any element $f \in A_0$ its $K$--functional $K(t, f; \vec{A})$, will be constant for $t > 1$.

This motivated the following definition (cf. [38]):

**Definition 1.** Let $\vec{A}$ be an ordered pair, and let $0 \leq \theta \leq 1$, $0 < q \leq \infty$. We define

$$\langle \vec{A} \rangle_{K, \theta, q} = \{ f \in A_0 : \| f \|_{\langle \vec{A} \rangle_{K, \theta, q}} < \infty \},$$

where

$$\| f \|_{\langle \vec{A} \rangle_{K, \theta, q}} = \begin{cases} \{ \int_0^1 \left[ s^{-\theta} K(s, f; \vec{A}) \right]^q \frac{ds}{s} \}^{1/q}, & q < \infty, \\ \sup_{0 < s < 1} s^{-\theta} K(s, f; \vec{A}), & q = \infty. \end{cases}$$

As shown in [38], this construction gives non-trivial spaces for the limiting value $\theta = 0$ and all $q > 0$. For example, if $\Omega$ is a probability space then $(L^1(\Omega), L^\infty(\Omega))$ is an ordered pair, and we have (cf. [7], [38])

$$\langle L^1(\Omega), L^\infty(\Omega) \rangle_{K, 0, 1} = L(\text{Log}L)(\Omega).$$

Another example is provided by the Dini spaces defined on a smooth domain $\Omega \subset \mathbb{R}^n$ (cf. [38], [44, Theorem 1, and the paragraph that follows it, in page 120]),

$$(\tilde{W}^1_p(\Omega), L^p(\Omega))_{K, 0, 1} = \{ f : \int_0^1 w_{p, f}(s) \frac{ds}{s} < \infty \},$$

where $\tilde{W}^1_p(\Omega)$ is the usual homogeneous Sobolev space (cf. [8]) and $w_{p, f}(s) := \sup_{|h| \leq s} \| (f(\cdot + h) - f(\cdot)) \chi_{\{x \in \Omega : x + h \in \Omega\}} \|_{L^p}$ is the $p$-modulus of continuity of $f$.

Moreover, this modification of the Lions-Peetre scale is consistent in the sense that when the pair $\vec{A}$ is ordered, and the parameters are in the classical range, $0 < \theta < 1$, $0 < q \leq \infty$, we have

$$\langle \vec{A} \rangle_{K, \theta, q} = \vec{A}_{K, \theta, q}.$$  

More precisely, for ordered pairs and $q \geq 1$ we have (cf. Section 3.2, Proposition 1 below or [54, Lemma 3])

$$\| a \|_{\langle \vec{A} \rangle_{K, \theta, q}} \leq \| a \|_{\vec{A}_{K, \theta, q}} \leq \left[ 1 + (1 - \theta)^{1/q} \theta^{-1/q} \right] \| a \|_{\langle \vec{A} \rangle_{K, \theta, q}}.$$  

6In this paper when considering ordered pairs we shall assume, without loss of generality, that the embedding $A_1 \subset A_0$ has norm 1.

7We refer to Section 3.1 for the definition as well as for background information on interpolation and extrapolation theory.
Furthermore, let us note that when the pair is ordered, \( A_0 + A_1 = A_0 \). Therefore, the \( \langle \hat{A} \rangle^K_{\theta,q} \) spaces, \( 0 \leq \theta \leq 1, 0 < q \leq \infty \), can also be defined consistently for arbitrary Banach pairs simply by keeping (2.6), and letting

\[
\langle \hat{A} \rangle^K_{\theta,q} = \{ f \in A_0 + A_1 : \| f \|_{\langle \hat{A} \rangle^K_{\theta,q}} < \infty \}.
\]

The corresponding \( \langle \hat{A} \rangle^J_{\theta,q} \) spaces are defined in an analogous fashion (cf. [42] and Section 3 below), and consist of those elements \( f \in A_0 + A_1 \) that can be represented by integrals of the form \( f = \int_0^1 u(s) \frac{ds}{s} \) in \( A_0 + A_1 \), where \( u : (0,1) \to A_0 \cap A_1 \) with \( \int_0^1 \left[ s^{-\theta} J(s, u(s); \hat{A}) \right]^q \frac{ds}{s} < \infty \); endowed with the corresponding quotient norm. It is known, and easy to see (cf. [43] and Section 3), that for ordered pairs, and \( 0 < \theta < 1, 1 \leq q \leq \infty \), these spaces coincide with the classical \( J \)-spaces \( \hat{A}^J_{\theta,q} \). In fact, we have (cf. Section 3.2, Proposition 2 below)

\[
\| a \|_{\hat{A}^J_{\theta,q}} \leq \| a \|_{\langle \hat{A} \rangle^J_{\theta,q}} \leq \left( 1 + \frac{4}{\log 2} \right) \left( 1 - \theta \right)^{q-1} \| a \|_{\hat{A}^J_{\theta,q}},
\]

where \( 1/q + 1/q' = 1 \).

At this point one can pursue different generalizations, e.g., through the use of more general weights,\(^8\) more general norms, etc. Indeed, the construction of limiting spaces has recently received considerable attention (as a small sample of recent contributions on limiting spaces we mention [15–17, 19–28], as well as the references therein). However, it is worthwhile to emphasize that, in all the papers listed above, only some special classes of limiting spaces were studied, and only fragments of their corresponding theory were considered. In contrast, in this paper we strive to give a complete characterization of the limiting interpolation spaces associated with the real method. A key point of our approach is the clarification of the connection between the class of limiting spaces and extrapolation theory. In particular, we completely describe “limiting spaces” as extrapolation spaces. For this purpose the usual tools (e.g., Boyd indices or the boundedness of Hardy type operators) are not appropriate. Instead, our characterization hinges upon the boundedness properties of some simple operators (e.g., \( f \mapsto f(t^2)/t \), or \( f \mapsto f(t^{1/2}) \)) acting on the underlying lattices that are used to control the \( K \)- and \( J \)-functionals. Reiteration formulae, extending in a meaningful way the classical results of Holmstedt to limiting spaces, are also proved and characterized in this fashion. In short, we gain a new understanding on the limiting spaces for the classical Lions-Peetre scale that goes well beyond mere considerations of numerical parameters, and use these insights to formulate new results that would be more difficult to guess and prove otherwise. Moreover, we believe that the methods developed here can be used to describe and study limiting spaces for other scales and methods of interpolation. Still another consequence of our efforts is that we acquire extrapolation methods to prove inequalities that involve limiting spaces.

\(^8\)Recall that in the classical Lions-Peetre theory the elements in \( J \)-spaces are represented by integrals \( f = \int_0^\infty u(s) \frac{ds}{s} \). In the case of ordered pairs we don’t lose information if we consider only integrals of functions defined on \((0,1)\). See Section 3 below.

\(^9\)In fact, the use of more general weights cannot be avoided if we wish to extend the fundamental results of the classical Lions-Peetre theory to limiting spaces (cf. [2,15] below).
To motivate and illustrate our development in this paper it will be instructive to discuss here how general ideas of extrapolation theory apply in the special case of the $A^K_{0,1}$ spaces and, in the process, allow us to re-formulate some of the fundamental theorems of the Lions-Peetre classical theory to include limiting spaces. We believe that the short review that follows could be useful to the reader, since the relevant tools of the extrapolation theory of [43] do not seem to have been utilized in the literature of limiting spaces.

It seems appropriate to start our discussion about limiting spaces with the very issue of taking limits of norms of real interpolation spaces. Besides the connection with extrapolation theory, that we shall develop in somewhat more detail below (cf. Section 9.2), this topic is of independent interest in other areas of analysis (e.g., in the theory of Sobolev inequalities cf. [10], [46], [59, see page 244], [60], and the references therein). The basic results are very simple to state. For definiteness, below we only display the limits of the Lions-Peetre norms when $\theta \to 0$.

We first normalize the real interpolation spaces $A^K_{0,q}$ by means of multiplying the corresponding norms by

$$c_{\theta,q} = (q\theta(1-\theta))^{\frac{1}{q}},$$

and denote the corresponding normalized spaces by $A^K_{0,q}$. In an analogous fashion we normalize the $A^J_{\theta,q}$ spaces; here the corresponding constants are given by $c_{\theta,q} = (q\theta(1-\theta))^{-1/q', 1/q' + 1/q} = 1$ (cf. (3.3) in Section 3.1 below). Then, we get

$$\lim_{\theta \to 0} \|f\|_{A^K_{0,q}} = \|f\|_{A_0} \quad \text{and} \quad \lim_{\theta \to 0} \|f\|_{A^J_{\theta,q}} = \|f\|_{A_0}.$$  

On the other hand, if $A$ is an ordered pair, then for $f \in A_1$, we have

$$\lim_{\theta \to 0} \|f\|_{A^K_{\theta,q}} = \|f\|_{A_1} \quad \text{and} \quad \lim_{\theta \to 0} \|a\|_{A^K_{\theta,q}} = 2\|a\|_{A_0} \quad \text{(the first indicated limit follows directly from the definitions while the inequality for the second limit is an immediate consequence of (2.9) and (2.10) above).}$$

In Section 9.2 (cf. Theorem 14) we show how these calculations can be combined with the strong form of the fundamental lemma of interpolation theory to prove extrapolation theorems, and in Section 9.4 we show how extrapolation methods provide an extension of (2.12) and (2.13), as well as add another twist to the Bourgain-Brezis-Mironescu-Maz'ya-Shaposhnikova limit theorems for Sobolev norms.

A cornerstone result in the Lions-Peetre theory is the equivalence between the $K$- and $J$-interpolation spaces that takes the form:

$$A^K_{\theta,q} = A^J_{\theta,q}, \quad 0 < \theta < 1, 1 \leq q \leq \infty.$$ 

As a consequence of our previous discussion we see that the equivalence (2.14) does not hold in the limiting cases, and we need to go beyond power weights in

$$t^{-\theta}K(t, f; \bar{A}) \leq \|f\|_{A^K_{\theta,q}} \leq \|f\|_{A_0}^{1-\theta} \|f\|_{A_1}^\theta.$$ 

The corresponding limit in (2.12) follows readily. One can also give a similar proof of the corresponding limit for the $K$-method using [55, Lemma 2, formula (31), page 244].

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10 Actually the results hold for interpolation functors that are of exact type $\theta$ (cf. [54]). For example, by (2.13) applied to $\rho(t) = t^\theta$, the $J$-method $A^J_{\theta,q}$ is exact of order $\theta$, and moreover, we have that for all $t > 0$,

$$t^{-\theta}K(t, f; \bar{A}) \leq \|f\|_{A^J_{\theta,q}} \leq \|f\|_{A_0}^{1-\theta} \|f\|_{A_1}^\theta.$$
order to find the correct results (cf. [43, page 46] and Section 9.1 below). For example, the limiting space $\langle \vec{A} \rangle_{0,1}^K$ can be described as a $J$–space, but in this case the corresponding characterization requires the use of $J$–spaces with logarithmic weights

$$
\langle \vec{A} \rangle_{0,1}^K = \{ f : f = \int_0^1 u(s) \frac{ds}{s} \text{ with } \int_0^1 J(s, u(s); \vec{A}) |\log s| \frac{ds}{s} < \infty \}.
$$

For a far-reaching generalization of this result, and its connections with the sum extrapolation functor of Jawerth-Milman and the strong form of the fundamental lemma of interpolation theory, we refer to [43, (3.9) page 25], and Section 9.1 below. These results are also closely connected with generalizations of the classical interpolation inequalities of the form

$$
\| f \|_{\vec{A}^K_{θ,q}} \leq c(θ, q) \| f \|_{A_{0}}^{1-θ} \| f \|_{A_{1}}^{θ}, 0 < θ < 1, 1 ≤ q ≤ ∞.
$$

For example, for $θ = 0$ we must modify (2.16) as follows (cf. [43, page 58-59] and Example 6 below)

$$
\| f \|_{\langle \vec{A} \rangle_{0,1}^K} \leq c \| f \|_{A_{0}} \log \left( e + \frac{\| f \|_{A_{1}}}{\| f \|_{A_{0}}} \right).
$$

Underlying (2.17) is the limiting equivalence of the $K$– and $J$–methods given by (2.15) and the fact that, for quasi-concave functions $ρ$, we have [43, (5.5.3) page 58]

$$
\inf_{t>0} \left\{ \frac{J(t, f; \vec{A})}{ρ(t)} \right\} = \frac{\| f \|_{A_{0}}}{ρ(\| f \|_{A_{1}})}.
$$

Another key property of the classical Lions-Peetre scale is the reiteration theorem. In particular, let us consider the following reiteration formula (cf. [9]),

$$
(A_0, \vec{A}_{θ,1,q}^K)_{θ,q}^K = \vec{A}_{θθ,1,q}^K, 0 < θ_1 < 1, 0 < θ < 1.
$$

From our previous discussion it follows that (2.19) gives a trivial result if $θ = 0$. On the other hand, if $\vec{A}$ is an ordered pair, we have (cf. [38], [43]) that, for all $θ_1 ∈ (0, 1), 1 ≤ q ≤ ∞$,

$$
(A_0, \vec{A}_{θ_1,q}^K)_{θ,1}^K = \langle \vec{A} \rangle_{θ_1}^K.
$$

Using the modified $K$–spaces, we can formally obtain (2.20) by letting $θ = 0$ in (2.19). However, the reader should also notice that (2.20) exhibits the somewhat surprising fact that the resulting space on the right hand side does not depend on $θ_1$!

Once again, (2.20) can be understood using the $\sum$–method of extrapolation of [43, page 48] (cf. Section 3.4 for the definitions). To see this in more detail we recall that the space $\langle \vec{A} \rangle_{0,1}^K$ can be described via the $\sum$–method as follows (cf. [43]): For any $q ∈ [1, ∞]$,

$$
\langle \vec{A} \rangle_{0,1}^K = \sum_{θ}^1 \vec{A}_{θ,q}^K.
$$

11Indeed, variants of (2.17) hold for extrapolation spaces (cf. [43]).
From this point of view, (2.20) simply reflects the following statement: If the pair $\vec{A}$ is **ordered**, then for any $\theta_i \in (0, 1), i = 0, 1$, (cf. [43]),

$$\sum_{\theta < \theta_0} \frac{1}{\theta} \vec{A}^{K}_{\theta, q} = \sum_{\theta < \theta_1} \frac{1}{\theta} \vec{A}^{K}_{\theta, q}.$$ 

Moreover, the characterization (2.21) holds even if the pair $\vec{A}$ is **not** ordered (cf. (2.9) and [43]):

$$\sum_{\theta < \theta_1} 1_{\theta} \vec{A}^{K}_{\theta, q} = \{ f : f \in A_0 + A_1 \text{ s.t. } \int_{0}^{1} K(s, f; \vec{A}) \frac{ds}{s} < \infty \}$$

(2.22)

$$= \langle \vec{A} \rangle^{K}_{0, 1}.$$ 

Furthermore, in either case we have

$$\|f\|_{\sum_{\theta < \theta_1} 1_{\theta} \vec{A}^{K}_{\theta, q}} \asymp \int_{0}^{1} K(s, f; \vec{A}) \frac{ds}{s}.$$ 

**Example 1.** Let $(\Omega, \mu)$ be a measure space. Then,

$$\sum_{\theta} 1_{\theta} (L^1(\Omega), L^\infty(\Omega))^{K}_{\theta, q} = L(\text{Log} L)(\Omega) + L^\infty(\Omega).$$

For details we refer to Section 8, Example 3 below. Note that if $\mu(\Omega) < \infty$, then $L^\infty(\Omega) \subset L(\text{Log} L)(\Omega)$, and we recover (2.7).

We can now explain in more detail the results of this paper. First note that limiting spaces are themselves interpolation spaces so in this paper we shall be more generally concerned with the problem of representing real interpolation spaces as extrapolation spaces, as well as with the validity of suitable versions of the reiteration formula (2.20). Since the latter formula requires a suitable monotonicity condition, in this paper we shall be working mainly with ordered pairs of Banach spaces. Next, instead of considering separately the different available methods of extrapolation we shall formulate our results using two basic general methods of extrapolation introduced in [1] (cf. Section 3.4 below). These extrapolation constructions indeed contain all the known methods of extrapolation (e.g., the $\Sigma$ and $\Delta$ methods, and their corresponding $p$–variants). As it often happens in mathematics the added generality clarifies the issues and leads to a streamlined presentation. In particular, the extrapolation methods of [1] (see also [3] and [4]) utilize functional parameters and this led us to formulate our characterizations in terms of specific properties of the functional parameters involved. This is a particularly felicitous situation for our development since the traditional tools (e.g., Boyd indices or the boundedness of Hardy operators) do not seem appropriate for our goals, instead we formulate our conditions on the behavior of some simple operators acting on the functional parameters.

12Informally the idea is that if $\theta_0 < \theta_1$, say, we can split

$$\sum_{\theta < \theta_1} 1_{\theta} (A_0, A_1)^{K}_{\theta, q} = \sum_{\theta < \theta_0} 1_{\theta} (A_0, A_1)^{K}_{\theta, q} + \sum_{\theta_0 < \theta < \theta_1} 1_{\theta} (A_0, A_1)^{K}_{\theta, q}$$

and then on the right hand side use the fact that we are working with an ordered pair in order to incorporate the second term to the first term.
Let $\vec{A}$ be an ordered pair and let us consider spaces of the form $\langle \vec{A} \rangle^K_F$, where $F$ is a suitable functional parameter for the real method on $[0, 1]$. We ask: under what conditions on $F$ can we represent $\langle \vec{A} \rangle^K_F$ as an extrapolation space? In Theorem 1 below we show that if $F$ has the property that the operation
\[ f(t) \rightarrow f(t^2)/t \]
is bounded on $F$, then the interpolation space $\langle \vec{A} \rangle^K_F$, can be represented as a generalized extrapolation space of the form
\[ \|a\|_{\langle \vec{A} \rangle^K_F} \asymp \|t \cdot \|a\|_{\langle \vec{A} \rangle^K_{\theta(t), q}}\|_F, \quad \text{where } \theta(t) = 1 + \frac{1}{2 \log \frac{t}{e}}. \]
Moreover, Theorem 5 shows that, if $F$ is an interpolation Banach lattice between the spaces $L^\infty$ and $L^\infty(1/t)$, then condition (2.23) is equivalent to the following peculiar reiteration formula
\[ \langle \vec{A}_{\theta, q}, A_1 \rangle^K_F = \langle \vec{A} \rangle^K_F. \]
A similar result also holds for limiting interpolation spaces which are close to the larger “end” of an ordered Banach pair. Namely, the following extension of (2.20) holds
\[ \langle A_0, \vec{A}_{\theta, q} \rangle^K_F = \langle \vec{A} \rangle^K_F \]
if and only if the operator $Rf(t) = f(t^{1/2})$ is bounded on $F$ (see Theorem 6).

If we further assume that $\vec{A}$ is Gagliardo complete then we can show that, more generally, the representation (2.24) holds if we replace the real method by any family $\{I_\theta\}_{\theta \in (0, 1)}$ of exact interpolation functors $I_\theta$ with characteristic functions $t^\theta$ (cf. Theorem 3 below). In particular, it follows that if $F$ satisfies (2.23) then $\langle \vec{A} \rangle^K_F$ can also be represented as an extrapolation space for the scale of complex interpolation spaces $[A_0, A_1]_\theta$, in the sense that
\[ \|a\|_{\langle \vec{A} \rangle^K_F} \asymp \|t \cdot \|a\|_{[A_0, A_1]_\theta(q)}\|_F, \]
where $[\cdot, \cdot]_\theta$ denotes the complex method of interpolation (cf. [8] for background on the complex method of interpolation).

As is well known (cf. [9]), Holmstedt’s formula is an important tool to prove reiteration theorems in the classical Lions-Peetre theory. In our setting the relevant Holmstedt formula takes the following form: Suppose that $F$ satisfies (2.23), then for every pair $(A_0, A_1)$ it holds, with constants independent of $a \in \langle \vec{A} \rangle^K_F + A_1$, $s > 0$ and $q \in [1, \infty]$,
\[ K(s, a; \langle \vec{A} \rangle^K_F, \vec{A}_1) \asymp K(s, t \cdot \|a\|_{\langle \vec{A} \rangle^K_{\theta(t), q}}, F, L^\infty(1/t)), \]
where as above $\theta(t) = 1 + \frac{1}{2 \log(t/e)}$ (cf. Theorem 2 below).

---

13 See (3.4) below for the definition.
14 Examples include (see Remark 13)
\[ \|f\|_F = \int_0^1 |f(s)| \frac{ds}{s}, \quad \|f\| = \sup_{s \in (0, 1)} |f(s)|. \]
Applications to the theory of symmetrically normed operator ideals are given in Section 6. In particular, using an extrapolation description of the limiting Schatten-von Neumann operator spaces (cf. Theorem 10), we obtain a generalization of Mat-

saev’s well-known result on the behavior of the real and the imaginary components of a Volterra operator (cf. Theorem 11). In Section 7 we show how Theorem 1 can be combined with easy to prove rearrangement inequalities to give a streamlined proof of the Fiorenza-Karadzhov description of Grand Lebesgue spaces \( L^{p}, p > 1 \), as extrapolation spaces (see Theorem 12 below). In Section 8 we show how extrapolation theory combined with the formula for the \( K \)-functional for vector valued \( L^{p} \) spaces obtained by Pisier [58] allows us to prove an extrapolation theorem for vector valued spaces that extends Yano’s classical extrapolation theorem. In Section 9 we complete the proofs of some auxiliary results and discuss further results and applications. We have aimed to make the paper accessible for readers who may not be familiar with extrapolation theory. We refer to Section 3 for background information about interpolation and extrapolation theory.

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3. Background and auxiliary results

3.1. Real Interpolation: basic definitions. We assume that the reader is familiar with elementary real interpolation theory, e.g., as presented in [9]. We shall now give a brief summary of relevant notions in order to fix the notation we use in this paper.

Let \( \vec{A} = (A_0, A_1) \) be a Banach pair. The Peetre \( K \)-functional is defined for \( x \in A_0 + A_1, t > 0 \), by

\[
K(t,x; \vec{A}) = \inf \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : x = a_0 + a_1, a_i \in A_i, i = 0, 1 \}.
\]

The corresponding dual construction is the \( J \)-functional defined for \( x \in A_0 \cap A_1, t > 0 \), by

\[
J(t,x; \vec{A}) = \max \{ \|x\|_{A_0}, t \|x\|_{A_1} \}.
\]

Let \( 0 < \theta < 1 \), and \( 1 \leq q \leq \infty \). Let

\[
\Phi_{\theta,q}(f) = \left\{ \begin{array}{ll}
\left\{ \int_0^\infty (s^{-\theta}|f(s)|)^q \frac{ds}{s} \right\}^{1/q} & \text{if } q < \infty \\
\sup_{s > 0} \left\{ s^{-\theta}|f(s)| \right\}^{1/q} & \text{if } q = \infty.
\end{array} \right.
\]

and let \( \Phi_{\theta,q} \) be the function space of measurable functions on \( (0, \infty) \) such that \( \Phi_{\theta,q}(f) < \infty \). The space \( \vec{A}^{K}_{\theta,q} \) consists of the elements \( x \in A_0 + A_1 \), such that \( \|x\|_{\vec{A}^{K}_{\theta,q}} < \infty \), where

\[
\|x\|_{\vec{A}^{K}_{\theta,q}} := \Phi_{\theta,q}(K(s,x; \vec{A})).
\]

The corresponding \( \vec{A}^{J}_{\theta,q} \) spaces consist of all \( x \in A_0 + A_1 \) that can be represented as

\[
(3.1) \quad x = \int_0^\infty u(t) \frac{dt}{t} \text{ in } A_0 + A_1,
\]

for some strongly measurable function \( u(t) \) defined on \( (0, \infty) \) with values in \( A_0 \cap A_1 \), and such that \( \Phi_{\theta,q}(J(s,u(s); \vec{A})) < \infty \). Then we let

\[
\|x\|_{\vec{A}^{J}_{\theta,q}} = \inf \{ \Phi_{\theta,q}(J(s,u(s); \vec{A})) : x = \int_0^\infty u(t) \frac{dt}{t} \}.
\]
For our development in this paper it is convenient to normalize the interpolation norms as follows. We let
\[
\|x\|_{\vec{A}_{\theta,q}^K} := (q\theta(1 - \theta))^\frac{1}{\theta}\|x\|_{\vec{A}_{\theta,q}^\infty},
\]
with the convention that \((q\theta(1 - \theta))^\frac{1}{\theta}\) = 1 when \(q = \infty\). Then \(\vec{A}_{\theta,q}^K\) is equal to the set of elements of \(\vec{A}_{\theta,q}^\infty\) provided with the normalized norm \(\| \cdot \|_{\vec{A}_{\theta,q}^K}\). Likewise, the spaces \(\vec{A}_{\theta,q}^J\) are equal, as sets, to the \(\vec{A}_{\theta,q}^J\) spaces, but are provided with the normalized norms \(\| \cdot \|_{\vec{A}_{\theta,q}^J}\) given by,
\[
\|x\|_{\vec{A}_{\theta,q}^J} := (q\theta(1 - \theta))^{-1/q}\|x\|_{\vec{A}_{\theta,q}^J},
\]
where if \(q = 1\) we set \((q\theta(1 - \theta))^{-1/q}' = 1\).

The spaces \(\langle \vec{A}\rangle_{\theta,q}^K; \theta \in [0,1], q \in [1, \infty]\), introduced in Definition\(\text{II}\) above, consist of the elements \(x\) in \(A_0 + A_1\) with \(\|x\|_{\langle \vec{A}\rangle_{\theta,q}^K} < \infty\), where
\[
\|x\|_{\langle \vec{A}\rangle_{\theta,q}^K} := \Phi_{\theta,q}(K(s,x;\vec{A})\chi((0,1)(s))).
\]
The corresponding \(\langle \vec{A}\rangle_{\theta,q}^J\) spaces consist of all the elements \(x\) of \(A_0 + A_1\) that can be represented by
\[
x = \int_0^1 u(t)\frac{dt}{t},
\]
for some strongly measurable function \(u(t)\) defined on \((0,1)\), with values in \(A_0 \cap A_1\), such that \(\Phi_{\theta,q}(J(s,u(s);\vec{A})\chi((0,1)(s))) < \infty\). We let
\[
\|x\|_{\langle \vec{A}\rangle_{\theta,q}^J} := \inf\{\Phi_{\theta,q}(J(s,u(s);\vec{A})\chi((0,1)(s))) : x = \int_0^1 u(t)\frac{dt}{t}\}.
\]
Furthermore, in analogy with\(3.2\) and\(3.3\) we formally define the spaces \(\langle \vec{A}\rangle_{\theta,q}^K\) (resp. \(\langle \vec{A}\rangle_{\theta,q}^J\)) using the normalizations
\[
\|x\|_{\langle \vec{A}\rangle_{\theta,q}^K} := (q\theta(1 - \theta))^\frac{1}{\theta}\|x\|_{\langle \vec{A}\rangle_{\theta,q}^\infty}, \quad \text{resp.} \quad \|x\|_{\langle \vec{A}\rangle_{\theta,q}^J} := (q\theta(1 - \theta))^{-1/q}\|x\|_{\langle \vec{A}\rangle_{\theta,q}^J}.
\]

More generally,\(mutatis mutandis\), we consider the interpolation spaces \(\vec{A}_F^K\) (resp. \(\vec{A}_F^J\)), obtained by replacing \(\Phi_{\theta,q}\) with a Banach lattice \(F\) of functions on \((0,\infty),\frac{1}{\theta}\) satisfying the continuous embeddings \(L^\infty \cap L^1(1/s) \subset F \subset L^\infty + L^1(1/s)\) (resp. \(L^1 \cap L^1(1/s) \subset F \subset L^1 + L^1(1/s)\)). Here, and in what follows, we use the notation \(\|f\|_{L_p(1/s)} := \|f(s)/s\|_{L_p}, 1 \leq p \leq \infty\). Banach lattices with the above properties will be called\(parameters for the K-method (resp. J-method) of interpolation on (0, \infty)\). More precisely, \(\vec{A}_F^K\) is defined by
\[
\vec{A}_F^K := \{x \in A_0 + A_1 : \|x\|_{\vec{A}_F^\infty} := \|K(s,x;\vec{A})\|_F < \infty\}.
\]
Likewise, we let
\[
\vec{A}_F^J := \{x \in A_0 + A_1 : \|x\|_{\vec{A}_F^J} < \infty\},
\]
\(^{15}\)The normalization constants given by \(\ref{3.2}\) (resp. \(\ref{3.3}\)) are useful when comparing norms and taking limits (cf. \(\ref{3.7}\) above) and moreover make the corresponding interpolation functors \(\vec{A}_{\theta,q}^K\) (resp. \(\vec{A}_{\theta,q}^J\)) exact of exponent \(\theta\) (cf. footnote (10) above).
where
\[ \|x\|_{\tilde{F}_F} := \inf \{ \left\| J(s, u(s); \tilde{A}) \right\|_F : x = \int_0^\infty u(s) \frac{ds}{s}; \] where \( u : (0, \infty) \to A_0 \cap A_1 \) is strongly measurable \}. 

For our purposes in this paper it is important also to consider the modification of the above spaces that one obtains when the parameter space \( F \) is a Banach lattice of functions on \( ((0, 1), \frac{ds}{s}) \) satisfying the continuous embeddings \( L^\infty(1/s)(0, 1) \subset F \subset L^\infty(0, 1) \) (resp. \( L^1(1/s)(0, 1) \subset F \subset L^1(0, 1) \)). We will refer to such Banach lattices as parameters for the \( K \)-method (resp. \( J \)-method) of interpolation on \( (0, 1) \).

The corresponding definitions are formally the same:
\[ \langle \tilde{A} \rangle_F^K := \{ x \in A_0 + A_1 : \|x\|_{\langle \tilde{A} \rangle_F^K} := \left\| K(s, x; \tilde{A}) \right\|_F < \infty \}, \]
and likewise
\[ \langle \tilde{A} \rangle_F^J := \{ x \in A_0 + A_1 : \|x\|_{\langle \tilde{A} \rangle_F^J} < \infty \}, \]
where
\[ \|x\|_{\langle \tilde{A} \rangle_F^J} := \inf_{x=\int_0^1 u(s) 1} \{ \left\| J(s, u(s); \tilde{A}) \right\|_F : u : (0, 1) \to A_0 \cap A_1 \) is strongly measurable \} \}. 

We shall say that \( \tilde{A} \) is an ordered Banach pair (or simply an “ordered pair”) if \( A_1 \subset A_0 \). Moreover, to simplify the discussion we shall always assume that the norm of the embedding \( A_1 \subset A_0 \) is less than or equal to one. An increasing non-negative function \( f \) on \([0, 1]\) is called quasi-concave, if \( f(t)/t \) decreases. In general, given two Banach spaces \( A, B \), we shall write \( A \subset B \) to indicate that the norm of the embedding is bounded by above by \( B \).

3.2. On the equivalence of interpolation norms on ordered pairs. In this section we explicitly compare the interpolation constructions that we introduced in the previous section, on the class of ordered pairs of Banach spaces. Since the results are important for our purposes in this paper, and do not seem to be readily available in the literature, we provide full details, including explicit computation of the constants involved in the norm inequalities.

**Proposition 1.** Let \( \tilde{A} = (A_0, A_1) \) be an ordered pair. Then

(i) for every parameter \( F \) for the \( K \)-method on \((0, \infty)\) we have, with equivalence of norms,
\[ \tilde{A}_F^K = \langle \tilde{A} \rangle_F^K, \]
where \( \tilde{F} \) is the sublattice of \( F \) consisting of all functions \( f \) such that \( \text{supp} f \subset [0, 1] \);

(ii) for all \( 0 < \theta < 1 \), \( 1 \leq q \leq \infty \), we have
\[ \|a\|_{\langle \tilde{A} \rangle_F^K} \leq \|a\|_{\tilde{A}_F^K} \leq \left[ 1 + (1 - \theta)^{1/q} \theta^{-1/q} \right] \|a\|_{\langle \tilde{A} \rangle_F^K}, \]
where we let \( (1 - \theta)^{1/q} \theta^{-1/q} = 1 \) when \( q = \infty \).

**Proof.** (i) It is plain that
\[ \|a\|_{\langle \tilde{A} \rangle_F^K} \leq \|a\|_{\tilde{A}_F^K}. \]
Suppose now that \( a \in \langle \tilde{A} \rangle^K_F \). Since the pair \( \tilde{A} \) is ordered, \( K(s, a; \tilde{A}) = \|a\|_{A_0} \) for \( s \geq 1 \). Then we can write
\[
\|a\|_{\langle \tilde{A} \rangle^K_F} \leq \|K(s, a; \tilde{A})\chi_{[0,1]}(s)\|_F + \|K(s, a; \tilde{A})\chi_{[1,\infty]}(s)\|_F
= \|a\|_{\langle \tilde{A} \rangle^K_F} + \|a\|_{A_0}\|\chi_{[1,\infty]}\|_F.
\]
Moreover, since the function \( K(s, a; \tilde{A}) \) decreases, we have
\[
\|a\|_{A_0} = \frac{K(1, a; \tilde{A})}{1} \frac{\|s\chi_{[0,1]}(s)\|_F}{\|s\chi_{[0,1]}(s)\|_F} \leq \|s\chi_{[0,1]}(s)\|^{-1}_F \|K(s, a; \tilde{A})\|_F.
\]
Collecting estimates we get
\[
(3.5) \quad \|a\|_{\langle \tilde{A} \rangle^K_F} \leq (1 + \|\chi_{[1,\infty]}\|_F \cdot \|s\chi_{[0,1]}(s)\|^{-1}_F) \|a\|_{\langle \tilde{A} \rangle^K_F},
\]
and the desired result follows.

(ii) We apply \((3.3)\) with \( F = \Phi_{\theta,q} \). The desired result now follows computing the \( \Phi_{\theta,q} \)-norms of the functions \( \chi_{[1,\infty]}(s) \) and \( s\chi_{[0,1]}(s) \) and inserting the corresponding results in \((3.5)\). See also [54, Lemma 3].

Likewise, for the \( J \)-method we have,

**Proposition 2.** Let \( \tilde{A} = (A_0, A_1) \) be an ordered pair. Then

(i) for every parameter \( G \) for the \( J \)-method on \((0, \infty)\) we have, with equivalence of norms,
\[
\tilde{A}^J_G = \langle \tilde{A} \rangle^J_G,
\]
where \( \tilde{G} \) is the sublattice of \( G \) consisting of all functions \( f \) such that \( \text{supp} \ f \subset [0, 1] \);

(ii) for all \( 0 < \theta < 1 \), \( 1 \leq q \leq \infty \), we have
\[
\|a\|_{\tilde{A}^J_G} \leq \|a\|_{\langle \tilde{A} \rangle^J_G} \leq (1 + \frac{4}{\log 2} [(1 - \theta)q']^{-1/q'}) \|a\|_{\tilde{A}^J_G},
\]
where \([(1 - \theta)q']^{-1/q'} = 1 \) when \( q = 1 \).

**Proof.** (i) It is obvious that \( \langle \tilde{A} \rangle^J_G \subset \tilde{A}^J_G \) and, furthermore, the norm of the embedding is 1. We now prove the opposite inclusion.

Let \( a \in (A_0, A_1)^J_G \). For any \( \varepsilon > 0 \), we can select a representation \( a = \int_0^\infty u(s) \frac{ds}{s} \) such that
\[
\|J(s, u(s); \tilde{A})\|_G \leq \|a\|_{(A_0, A_1)^J_G} + \varepsilon.
\]
Let us define
\[
\tilde{u}(s) = \begin{cases} u(s) & 0 < s < 1/2 \\ \frac{1}{\log 2} \int_{1/2}^\infty u(s) \frac{ds}{s} & 1/2 \leq s < 1 \end{cases}.
\]
It is easy to see that
\[
a = \int_0^1 \tilde{u}(s) \frac{ds}{s}.
\]
Moreover, \(\tilde{u}(s)\) is an admissible function. Indeed, since the pair \(\tilde{A}\) is ordered, we have \(A_0 \cap A_1 = A_1\), and by Hölder’s inequality,

\[
\left\| \int_{1/2}^{\infty} u(s) \frac{ds}{s} \right\|_{A_1} \leq \int_{1/2}^{\infty} \|u(s)\|_{A_1} \frac{ds}{s}
\]

\[
\leq \int_{1/2}^{\infty} J(s, u(s); \tilde{A}) \frac{ds}{s^2}
\]

\[
\leq \|J(s, u(s); \tilde{A})\|_G \|s^{-1}\chi_{[1/2, \infty]}\|_{G'}
\]

(3.6)

where \(G'\) is the Köthe dual to \(G\) with respect to the bilinear form

\[
(f, g) = \int_0^{\infty} f(s)g(s) \frac{ds}{s}
\]

Since \(G \subset L^1 + L^1(1/s)\), it follows that \(G' \supset L^\infty \cap L^\infty(s)\). Therefore,

\[
\|s^{-1}\chi_{[1/2, \infty]}\|_{G'} < \infty.
\]

In particular, from (3.6) it follows that \(\tilde{u}(s) : (0, 1) \to A_1\), as we wished to show.

Furthermore, we have

\[
\|J(s, u(s); \tilde{A})\chi_{[0, 1]}\|_G \leq (\|a\|_{\tilde{A}'} + \varepsilon) + \frac{1}{\log 2} \left\| J \left(s, \int_{1/2}^{\infty} u(r) \frac{dr}{r}; \tilde{A}\right) \chi_{[1/2, 1]}(s) \right\|_G.
\]

Since \(J(s, \cdot; \tilde{A})\) is a norm for each \(s > 0\), and the pair \(\tilde{A}\) is ordered, from (3.6) we get

\[
\left\| \frac{J(s, \int_{1/2}^{\infty} u(r) \frac{dr}{r}; \tilde{A})}{\chi_{[1/2, 1]}} \right\|_G \leq \left\| \int_{1/2}^{\infty} J(s, u(r); \tilde{A}) \frac{dr}{r} \cdot \chi_{[1/2, 1]}(s) \right\|_G
\]

\[
\leq \left\| \int_{1/2}^{\infty} \|u(r)\|_{A_1} \frac{dr}{r} \cdot \chi_{[1/2, 1]}(s) \right\|_G
\]

\[
\leq \int_{1/2}^{\infty} \|u(r)\|_{A_1} \frac{dr}{r} \cdot \|\chi_{[1/2, 1]}\|_G
\]

\[
\leq (\|a\|_{\tilde{A}'} + \varepsilon) \cdot \|s^{-1}\chi_{[1/2, \infty]}\|_{G'} \cdot \|\chi_{[1/2, 1]}\|_G.
\]

Combining inequalities and letting \(\varepsilon \to 0\), we obtain

(3.7) \[\|a\|_{\tilde{A}'} \leq \left(1 + \frac{1}{\log 2} \cdot \|s^{-1}\chi_{[1/2, \infty]}\|_{G'} \cdot \|\chi_{[1/2, 1]}\|_G\right) \cdot \|a\|_{\tilde{A}'}.
\]

This concludes the proof of (i) since we obviously have \(\|\chi_{[1/2, 1]}\|_G < \infty\).

(ii) We apply (3.7) with \(G = \Phi_{\theta, q}\). We need to estimate the norms \(\|\chi_{[1/2, 1]}\|_{\Phi_{\theta, q}}\) and \(\|s^{-1}\chi_{[1/2, \infty]}\|_{(\Phi_{\theta, q})'}\) for this purpose note that,

\[
\|g\|_{(\Phi_{\theta, q})'} = \left(\int_0^{\infty} (s^q g(s))^q \frac{ds}{s}\right)^{1/q'},
\]

where \(1/q + 1/q' = 1\) (with the natural modification if \(q' = \infty\)). Consequently,

\[
\|s^{-1}\chi_{[1/2, \infty]}\|_{(\Phi_{\theta, q})'} = \left(\int_{1/2}^{\infty} s^{(\theta-1)q'} \frac{ds}{s}\right)^{1/q'} \leq 2 ((1 - \theta)q')^{-1/q'}.
\]
Furthermore, it can be easily verified that
\[
\| \chi_{[1/2,1]} \|_{*\theta,q} = \left( \int_{1/2}^{1} s^{-\theta q} \frac{ds}{s} \right)^{1/q} \left( \frac{2^{\theta q} - 1}{\theta q} \right)^{1/q} \leq 2.
\]

The desired result follows upon inserting this information in (3.7). \qed

**Remark 1.** We generally use Banach lattices on \([0,1]\) as parameters for the \(K\)- and \(J\)-methods of interpolation. However, we should like to observe that from Proposition 1 it follows that, for every ordered pair \(\vec{A}\) and all \(\theta \in [1/2,1)\), we have
\[
\| a \|_{(\vec{A})_{\theta,q}^{K}} \leq \| a \|_{\vec{A}_{\theta,q}^{K}} \leq 2 \| a \|_{(\vec{A})_{\theta,q}^{K}}.
\]
Similarly, by Proposition 2,
\[
\| a \|_{\vec{A}_{\theta,q}^{J}} \leq \| a \|_{(\vec{A})_{\theta,q}^{J}} \leq \left( 1 + \frac{8}{\log 2} \right) \| a \|_{\vec{A}_{\theta,q}^{J}},
\]
for all \(\theta \in (0,1/2]\). Therefore, for ordered pairs \(\vec{A} = (A_0, A_1), A_1 \subset A_0\), the extrapolation descriptions of limiting interpolation spaces obtained in Theorems 1, 2 and 4 below will still hold if we replace the scale \(\{ (\vec{A})_{\theta,q}^{K} \}_{\theta \in [1/2,1)}\) (resp. \(\{ (\vec{A})_{\theta,q}^{J} \}_{\theta \in (0,1/2]}\)) with the classical scale \(\{ A_{\theta,q}^{K} \}_{\theta \in [1/2,1)}\) (resp. \(\{ A_{\theta,q}^{J} \}_{\theta \in (0,1/2]}\)).

3.3. **Strong Fundamental Lemma and Peetre’s limit theorem.** Although we will formally recall the strong form of the fundamental lemma later (cf. (4.19) below), it will be instructive to present now a proof of Peetre’s limit formula (cf. (2.3) above), both because Peetre’s result is useful for our purposes in this paper and furthermore, because the method of proof illustrates, in a simple context, an argument that appears several times in our development in this paper (cf. also Sections 9.1 and 9.2 below). More precisely, we now provide a simple approach (compare with [57, Lemma 1.1]) to obtain the non-trivial part of (2.3) for Gagliardo complete pairs.

Let \(\vec{A}\) be a Gagliardo complete pair with \(A_0 \cap A_1\) dense in \(A_0\). Let \(f \in A_0\). By the strong form of the fundamental lemma we can find a decomposition such that
\[
f = \int_{0}^{\infty} u(s) \frac{ds}{s},
\]
such that for every \(t > 0\), it holds
\[
\int_{0}^{t} J(s, u(s), \vec{A}) \frac{ds}{s} \leq \gamma K(t, f, \vec{A}),
\]
where \(\gamma\) is a universal constant. But then, taking supremum over all \(t > 0\), it follows that
\[
\| f \|_{A_{0,1}^{J}} \leq \gamma \| f \|_{A_{0}},
\]
as we wished to show.
3.4. Extrapolation methods. In this section we continue the exposition started in the Introduction of the paper and provide a brief background survey on the extrapolation methods we shall use in this paper. For more information and results we refer to the recent survey \[5\] and the references therein.

It will be useful to start by pointing out the common root between interpolation and extrapolation. In interpolation we work with functors $F$, that assign to each pair $\tilde{A}$ an interpolation space $F(\tilde{A})$. On the other hand, extrapolation functors $\mathcal{E}$ are defined on some set of families $\{A_\theta\}_{\theta \in (0,1)}$ of compatible Banach spaces (in the sense that there exist two Banach spaces $A_0$ and $A_1$, such that for each $\theta \in (0,1)$, we have with continuous inclusions $A_1 \subset A_\theta \subset A_0$) and assign to each such family an extrapolation space $\mathcal{E}(\{A_\theta\}_{\theta \in (0,1)})$ with the following interpolation property. If $T$ is an operator such that $T : A_\theta \to B_\theta$, for each $\theta \in (0,1)$, then $T$ can be extended to $T : \mathcal{E}(\{A_\theta\}_{\theta \in (0,1)}) \to \mathcal{E}(\{B_\theta\}_{\theta \in (0,1)})$. The simplest, and at the same time more important extrapolation functors, are $\Sigma$ and $\Delta$ methods which we now describe (cf. \[43\]).

Suppose that the norms $M_0(\theta)$ of the inclusions $A_\theta \subset A_0$ satisfy the condition: $\sup_{\theta \in (0,1)} M_0(\theta) < \infty$. Then we can form the space $\Sigma_{\{A_\theta\}_{\theta \in (0,1)}}$ of all the elements $x \in A_0$ that can be represented by $x = \Sigma a_\theta$, $a_\theta \in A_\theta$, with $\Sigma \|a_\theta\|_{A_\theta} < \infty$. We endow $\Sigma_{\{A_\theta\}_{\theta \in (0,1)}}$ with the corresponding quotient norm. Likewise, if the norms $M_1(\theta)$ of the embeddings $A_1 \subset A_\theta$ are uniformly bounded, we form the space $\Delta_{\{A_\theta\}_{\theta \in (0,1)}}$ of all elements $x \in \cap_{\theta \in (0,1)} A_\theta$, such that $\|x\|_{\Delta_{\{A_\theta\}_{\theta \in (0,1)}}} := \sup_{\theta \in (0,1)} \|x\|_{A_\theta} < \infty$.

The theory of extrapolation originated from the classical results of Yano \[66\]. One of the objectives of modern extrapolation theory is to characterize family of inequalities and, in particular, reverse the interpolation process and find the best possible end point inequalities. In this sense modern extrapolation is a qualitative evolution from the initial extrapolation theorems of Yano. Indeed, in the classical extrapolation theorems of Yano type, e.g., from the assumption that $T : L^p(0,1) \to L^p(0,1)$, with $\|T\|_{L^p(0,1) \to L^p(0,1)} \leq \frac{c}{p-1}$, for all $p > 1$, we can conclude that $T : LLog L(0,1) \to L^1(0,1)$ but we have the drawback that, in general, $T : LLog L(0,1) \to L^1(0,1) \Rightarrow T : L^p(0,1) \to L^p(0,1)$ (unless we have more structural assumptions on the underlying measure space and the operator $T$ (cf. \[62\])).

On the other hand, by \[43\], the assumptions of Yano’s theorem above are equivalent to the inequality

$$K(t, tf; L^1, L^\infty) \leq c \int_0^t K(s, f; L^1, L^\infty) \frac{ds}{s},$$

or informally, denoting $x^{**}(t) := \frac{1}{t} \int_0^t x^*(s) \, ds$, we have

$$\|T\|_{L^p(0,1) \to L^p(0,1)} = \frac{c}{p-1} \text{ for all } p > 1 \Leftrightarrow (Tf)^{**}(t) \leq \frac{c}{t} \int_0^t f^{**}(s) \, ds.$$

The $\Sigma$- and $\Delta$-methods are the natural prototypes of two general families of extrapolation functors that were introduced in \[1\] and then were studied in \[2\]-\[5\]. Let $F$ be a Banach function lattice on the interval $[0,1]$ (with respect to the usual Lebesgue measure). A given family $\{A_\theta\}_{\theta \in (0,1)}$ of compatible Banach spaces, we define the Banach space $F(\{A_\theta\}_{\theta \in (0,1)})$, consisting of all $a \in \cap_{\theta \in (0,1)} A_\theta$ such that the function $\theta \in (0,1) \mapsto \|a\|_{A_\theta}$ belongs to $F$, endowed with the norm $\|a\| := \|a\|_{A_0}$. In particular, if $F = L^\infty[0,1]$, we arrive at the definition of
the \( \Delta \)-functor. In analogous way one can define a family of extrapolation functors generalizing the \( \Sigma \)-functor (see the definition of the \( \vec{A}_{\chi,\eta,G} \) spaces in Section 4 below).

4. A NEW CHARACTERIZATION OF LIMITING INTERPOLATION SPACES

We begin with the following key result of this paper.

**Theorem 1.** Let \( \vec{A} = (A_0, A_1) \) be a Banach pair, and let \( F \) be a parameter for the \( K \)-method on \([0, 1]\). Suppose that the operator \( T f(t) := f(t^2)/t \) is bounded on \( F \). Then there exist absolute constants such that for all \( a \in \langle \vec{A} \rangle^F \), \( q \in [1, \infty] \), it holds

\[
\|a\|_{\langle \vec{A} \rangle^F} \geq t \cdot \|a\|_{\langle \vec{A} \rangle^F},
\]

where \( \theta(t) = 1 + \frac{1}{2 \log(t/e)} \).

**Proof.** From the inequality

\[
\min(1, t/s)K(s, a; \vec{A}) \leq K(t, a; \vec{A})
\]

we get

\[
\|a\|_{\langle \vec{A} \rangle^F} \geq (\theta(1 - \theta)q) \frac{q}{2} K(s, a; \vec{A}) \left( \int_0^1 (t^{-\theta} \min(1, t/s))^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\geq (\theta(1 - \theta)q) \frac{q}{2} K(s, a; \vec{A}) \left( \frac{s^{-\theta q}}{(1 - \theta)q} \right)^{\frac{1}{q}}
\]

\[
= \theta^\frac{q}{2} s^{-\theta} K(s, a; \vec{A}).
\]

Hence, if \( 1 \leq q \leq r < \infty \),

\[
\|a\|_{\langle \vec{A} \rangle^F} \geq (\theta(1 - \theta)q) \frac{q}{2} \left( \int_0^1 (t^{-\theta} K(t, a; \vec{A}))^q (t^{-\theta} K(t, a; \vec{A}))^q dt \right)^{\frac{1}{q}}
\]

\[
\leq (r/q)^{\frac{1}{q}} \|a\|_{\langle \vec{A} \rangle^F} \cdot \theta^{1 - \frac{1}{q}} \cdot \|a\|_{\langle \vec{A} \rangle^F} = (r/q)^{\frac{1}{q}} \theta^{1 - \frac{1}{q}} \|a\|_{\langle \vec{A} \rangle^F}.
\]

On the other hand, from the definition of \( \theta(t) \) it follows readily that for \( 0 < t \leq 1 \), we have \( 1/2 \leq \theta(t) < 1 \). Combining this observation with the preceding inequality, we obtain that for all \( 0 < t \leq 1 \),

\[
\|a\|_{\langle \vec{A} \rangle^F} \leq \|a\|_{\langle \vec{A} \rangle^F} \leq 4 \|a\|_{\langle \vec{A} \rangle^F}, \quad 1 \leq q \leq r \leq \infty.
\]

Moreover, on account of the fact that

\[
\log t^{1-\theta(t)} = -\frac{\log t}{2 \log(t/e)} \geq -\frac{1}{2}, \quad 0 < t \leq 1,
\]

it follows that

\[
t^{1-\theta(t)} \geq e^{-1/2}, \quad 0 < t \leq 1.
\]
Therefore, for all \(0 < t \leq 1\),

\[
t\|a\|_{\tilde{A}^{K_{t},q}} \geq \frac{t}{2} \|a\|_{\tilde{A}^{K_{t},\infty}} = \frac{t}{2} \sup_{0 < s \leq 1} \left( s^{-\theta(t)} K(s, a; \tilde{A}) \right)
\]

(4.4)

\[
\geq \frac{1}{2} t^{1-\theta(t)} K(t, a; \tilde{A}) = \frac{1}{2\sqrt{e}} K(t, a; \tilde{A}).
\]

Consequently,

\[
\left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}} \right\|_{F} \geq \frac{1}{2\sqrt{e}} \|a\|_{\tilde{A}^{K_{t},q}}.
\]

To prove the converse inequality let us write,

(4.5) \[
\left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}} \right\|_{F} \leq \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(0,1/e)} \right\|_{F} + \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(1/e,1)} \right\|_{F}.
\]

We will now show that the second term on the right hand side of (4.5) can be absorbed into the first one. Indeed, the definition of \(\theta(t)\) implies that for \(1/e < t \leq 1\), we have \(\theta(t) \leq 3/4\); consequently

\[
\left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(1/e,1)} \right\|_{F} \leq C_{1} \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(1/e,1)} \right\|_{L^{\infty}(1/t)} \leq C_{1} \|a\|_{\tilde{A}^{K_{t},q}, \chi(1/e,1)}
\]

\[
\leq \frac{4}{3} C_{1} \|a\|_{\tilde{A}^{K_{t},q}}
\]

\[
\leq \frac{4}{3} C_{1} e \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(1/e,1)} \right\|_{L^{\infty}}
\]

\[
\leq \frac{4}{3} C_{1} C_{2} \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(0,1/e)} \right\|_{F},
\]

where \(C_{1}\) and \(C_{2}\) are the constants of the embeddings \(L^{\infty}(1/t) \subset F\) and \(F \subset L^{\infty}\), respectively. Inserting this estimate in (4.5) we find

(4.6) \[
\left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}} \right\|_{F} \leq \left(1 + \frac{4}{3} C_{1} C_{2} e\right) \left\| t \cdot \|a\|_{\tilde{A}^{K_{t},q}, \chi(0,1/e)} \right\|_{F}.
\]

We now estimate the right-hand side of (4.6). From (4.2) it follows that for every \(1/2 \leq \theta < 1\), and \(1 \leq q \leq \infty\),

(4.7) \[
\|a\|_{\tilde{A}^{K_{t},q}} \leq 4 \|a\|_{\tilde{A}^{K_{t},1}} \leq 4(1 - \theta) \int_{0}^{1} s^{-\theta} K(s, a; \tilde{A}) \frac{ds}{s}.
\]

Furthermore, for all \(0 < t \leq 1/e\), we have

\[
\log t^{1-\theta(t)} = -\frac{\log t}{2 \log(t/e)} \leq -\frac{1}{4}.
\]

Hence,

\[
t^{1-\theta(t)} \leq e^{-1/4}.
\]
The last inequality, combined with the fact that $K(s, a; \vec{A})/s$ is a decreasing function, yields that for all $0 < t \leq 1/e$,

$$
\begin{align*}
&\quad t(1 - \theta(t)) \int_0^1 s^{-\theta(t)} K(s, a; \vec{A}) \frac{ds}{s} = t(1 - \theta(t)) \int_0^1 s^{-\theta(t)} K(s, a; \vec{A}) \frac{ds}{s} \\
&\quad + t(1 - \theta(t)) \sum_{k=1}^{\infty} \int_{t^{2k}}^{t^{2k-1}} s^{-\theta(t)} K(s, a; \vec{A}) \frac{ds}{s} \\
&\quad \leq K(t, a; \vec{A}) + \sum_{k=1}^{\infty} \frac{K(t^{2k}, a; \vec{A})}{t^{2k}} t \cdot t((1 - \theta(t))^{2k-1} \\
&\quad \leq K(t, a; \vec{A}) + \sum_{k=1}^{\infty} e^{-2^{k-3}} \frac{K(t^{2k}, a; \vec{A})}{t^{2k}} t \\
&\quad \leq \sum_{k=0}^{\infty} e^{1-2^{k-3}} T^k \left( K(t, a; \vec{A}) \right).
\end{align*}
$$

From (4.7), and the fact that $T$ is bounded on $F$, we obtain

$$
\left\| t \cdot \left\| a \right\|_{(A)_{\theta(1), q}} \chi(0, 1/e) \right\|_F \leq 4 \left\| \sum_{k=0}^{\infty} e^{1-2^{k-3}} T^k \left( K(t, a; \vec{A}) \right) \right\|_F \\
\leq 4 \sum_{k=0}^{\infty} e^{1-2^{k-3}} \left\| T^k \left( K(t, a; \vec{A}) \right) \right\|_F \\
\leq 4 \sum_{k=0}^{\infty} e^{1-2^{k-3}} \left\| T \right\|_{F \rightarrow F}^k \left\| K(t, a; \vec{A}) \right\|_F \\
\leq C \left\| a \right\|_{(A)^F}.
$$

Finally, combining the last inequality with (4.3) we obtain the desired result. □

**Remark 2.** An inspection of the proof of Theorem [4] shows that the constants of equivalence (4.1) depend on the norm of the operator $T$ acting on $F$, and on the norm of the embeddings $L^\infty(1/t) \subset F \subset L^\infty$. The latter dependence can be eliminated if $F$ is an interpolation space with respect to the pair $(L^\infty, L^\infty(1/t))$. Indeed, arguing in the same way as in the proof of Theorem [4] we obtain

$$
\left\| t \cdot \left\| a \right\|_{(A)_{\theta(1), q}} \chi(0, 1/e) \right\|_F \leq C \left\| a \right\|_{(A)^F}.
$$
where \( \theta_1(t) = 1 + \frac{1}{2 \log t} \), \( 0 < t < 1/e \). Therefore, if \( C' \) denotes the norm of the dilation operator \( \sigma_e f(t) := f(t/e) \) on \( F \), we see that
\[
\left\| t \cdot \| a \|_{(\mathcal{A}^K_{\theta,q})} \right\|_F = e \left\| \sigma_e(t \cdot \| a \|_{(\mathcal{A}^K_{\theta,q})}) \chi[0,1/e) \right\|_F \\
\leq eC' \left\| t \cdot \| a \|_{(\mathcal{A}^K_{\theta,q})} \chi[0,1/e) \right\|_F \\
\leq eCC' \| a \|_{(\mathcal{A}^K_{\theta,q})},
\]
and our claim follows.

In particular, we note that the constants of equivalence in (4.1) are independent of \( q \). It is now easy to see that
\[(4.8) \quad \| a \|_{(\mathcal{A}^K_{\theta,q})} \simeq \left\| t \cdot \| a \|_{(\mathcal{A}^K_{\theta,q})} \right\|_F
\]
for each continuous function \( q(t) : (0,1) \to [1,\infty] \).

**Remark 3.** We remind the reader that, according to Remark 7 when dealing with ordered pairs \( \mathcal{A} = (A_0, A_1) \), we can replace the scale \( (\mathcal{A}^K_{\theta,q}) \) in Theorem 7 with the scale of the normalized Lions-Peetre spaces \( (\mathcal{A}^K_{\theta,q}) \). Similarly, in (4.8) we can replace the scale \( (\mathcal{A}^K_{\theta,q}) \) with the scale \( (\mathcal{A}^K_{\theta,q}) \). As an example, consider the ordered pair \( \mathcal{A} = (L^1[0,1], L^\infty[0,1]) \). Observe that if \( \theta \geq 1/2 \), then setting \( q = 1/(1-\theta) \) we have \( q \geq 2 \), and one can easily verify that
\[
\frac{1}{\sqrt{2}} \leq (q\theta(1-\theta))^{1/q} \leq 1.
\]
Therefore, combining
\[
\| a \|_{(\mathcal{A}^K_{\theta,q})} \simeq \left( \int_0^1 \left( \frac{1}{t} \int_0^t a^*(s) \, ds \right)^q \, dt \right)^{1/q},
\]
(see the formula for the K-functional of the couple \( (L^1, L^\infty) \) in [2, Theorem 5.2.1]) and the fact that the norm of the Hardy operator \( a(s) \mapsto 1/t \int_0^t a^*(s) \, ds \) on \( L_q[0,1] \) is equal to \( q/(q-1) \), we obtain
\[
\frac{1}{\sqrt{2}} \| a \|_{L_q[0,1]} \leq \| a \|_{(\mathcal{A}^K_{\theta,q})} \leq 2 \| a \|_{L_q[0,1]}.
\]
Consequently, for a Banach lattice \( F \) satisfying the conditions of Theorem 7 we obtain
\[
\| a \|_{(L^1[0,1], L^\infty[0,1])} \simeq \left\| t \cdot \| a \|_{L^\infty[0,1]} \right\|_F, \quad \text{where } q(t) = 2 \log(e/t).
\]

On the other hand, for non-ordered pairs \( \mathcal{A} \) it follows readily from the definitions that for \( \theta \geq 1/2 \) we can write
\[
(\mathcal{A}^K_{\theta,q}) = (A_0 + A_1, A_1)^{K_{\theta,q}},
\]
with the norm equivalence independent of \( \theta \). Indeed, it is easy to see that
\[
K(t,a; A_0 + A_1, A_1) = \begin{cases} 
K(t,a; A_0, A_1) & \text{for } 0 < t \leq 1, \\
K(1,a; A_0, A_1) & \text{for } t \geq 1.
\end{cases}
\]
Hence, for $\theta \geq 1/2$ we have,
\[
\|a\|_{\langle \vec{A}, K \rangle_{\theta, q}} \leq \|a\|_{(A_0 + A_1, K \theta, q)} \leq (1 + (1 - \theta)^{1/2}) \|a\|_{\langle \vec{A}, K \theta \rangle_{\theta, q}} \\
\leq (1 + (1 - \theta)^{1/2}) \|a\|_{\langle \vec{A}, K \theta \rangle_{\theta, q}} \leq 2 \|a\|_{\langle \vec{A}, K \theta \rangle_{\theta, q}}.
\]

In particular, let us consider the (unordered) pair $\vec{A} = (L^1(0, \infty), L^\infty(0, \infty))$. Then, for every Banach lattice $F$ that satisfies the conditions of Theorem 1 we have the equivalence
\[
\|a\|_{\langle \vec{A}, K \rangle_{\theta, q}} = \|a\|_{(L^1, L^\infty)_{\theta, q}} \propto \|t \cdot a\|_{L^{s(t)} + L^\infty} \|_F, \quad \text{where } q(t) = 2 \log(e/t), \ t \in [0, 1].
\]

For example, if $F = L^\infty(1/t)(0, 1)$ we get the trivial equality
\[
L^\infty(0, \infty) = \Delta_{q \geq 2}(L^0(0, \infty) + L^\infty(0, \infty)).
\]

**Remark 4.** It will be useful to present a detailed reformulation of Theorem 1 for interpolation spaces constructed using the $K$-method and parameter spaces defined on $[1, \infty)$. Suppose that $G$ is a Banach lattice of functions on $[1, \infty)$ such that $L^\infty[1, \infty) \subset G \subset L^\infty(1/t)[1, \infty)$, and let $\|b\|_{\langle \vec{B}, K \rangle_G} := \|K(t, b, \vec{B})\chi_{(1, \infty)}\|_G$. Then, if the operator $S f(t) := f(t^2)$ is bounded on $G$, it follows that for each $q \in [1, \infty]$,
\[
\|b\|_{\langle \vec{B}, K \rangle_G} \approx \left\{ \int_1^\infty \left( s^{-\eta(t)} K(s, b, \vec{B}) \right)^q \frac{ds}{s} \right\}^{1/q},
\]
where $\eta(t) = \frac{1}{2 \log(et)}$, $t \geq 1$, and
\[
\text{with the natural modification if } q = \infty.
\]

Indeed, since $K(t, b, \vec{B}) = tK(1/t, b, \vec{B})$, $t > 0$, we readily see that
\[
\langle \vec{B}, K \rangle_G = (B_1, B_0)^K_F,
\]
where $F$ is the Banach lattice on $[0, 1]$ normed by $\|f\|_F := \|tf(1/t)\|_G$. It is plain that the operator $S$ is bounded on $G$ if and only if the operator $T f(t) := f(t^2)/t$ is bounded on $F$. Therefore, combining with Theorem 1 we have
\[
\|b\|_{\langle \vec{B}, K \rangle_G} \approx \left\{ \int_1^\infty \left( s^{-\eta(t)} K(s, b, \vec{B}) \right)^q \frac{ds}{s} \right\}^{1/q},
\]
where $\theta(t) = 1 + \frac{1}{2 \log(t/e)}$. On the other hand, it is easy to see that (see also Theorem 3.4.1(a)) $\langle B_1, B_0 \rangle_{\theta, q}^{K} = \langle B_0, B_1 \rangle_{1-\theta, q}^{K}$ for all $0 < \theta < 1$ and $1 \leq q \leq \infty$, isometrically. Therefore, if we insert this information back in (4.10) and observe that $1 - \theta(1/t) = \eta(t)$, we obtain
\[
\|b\|_{\langle \vec{B}, K \rangle_G} \approx \left\{ \int_1^\infty \left( s^{-\eta(t)} K(s, b, \vec{B}) \right)^q \frac{ds}{s} \right\}^{1/q},
\]
as desired. Moreover, proceeding as in Remark 3 we see that, if $B_0 \subset B_1$, we can replace the scale $\{\langle B, K \rangle_{\theta, q}^{K}\}$ in (4.10) by the scale $\{\vec{B}^{K}_{\theta, q}\}$; the same considerations apply when dealing with scales where $q(t) : [1, \infty) \rightarrow [1, \infty]$ is an arbitrary continuous function.
Remark 5. The proof of Theorem 7 is based on two-sided pointwise estimates, as a consequence, in the context of Remark 4, we can replace the lattice $G$ of functions defined on $[1, \infty)$ by the canonical Banach lattice of sequences $G_d$ modelled on $G$. Since this remark will be useful later in Section 6, we now develop this point in detail. We let $G_d$ be the sequence space defined by

$$
\|\{\xi_n\}_{n=1}^\infty\|_{G_d} = \left\| \sum_{n=1}^\infty \xi_n X_{n, n+1}(t) \right\|_G.
$$

$G_d$ is a Banach sequence lattice, and a number of properties of $G_d$ can be derived from corresponding properties of $G$. In particular, we see that $\ell^\infty(N) \subset G_d \subset \ell^\infty(1/n)(N)$. Moreover, in this context, the discrete version of the operator $S$ can be defined by $S_df(n) := f(n^2)$, and we readily see that if $S$ is bounded on $G$ we have that $S_d$ is bounded on $G_d$. Consequently, for every Banach pair $\vec{B}$ we have

$$
\|\{K(n, f; \vec{B})\}_{n=1}^\infty\|_{G_d} \gtrsim \left\| \{\|f\|(B_0, B_1)^{\eta(n)}\}_{n=1}^\infty \right\|_{G_d},
$$

where $\eta(n) = \frac{1}{2 \log(\log n)}$ and $q(n) : N \to [1, \infty]$ is an arbitrary function.

Our next result shows that, if we impose more conditions on the parameter space $F$, we can obtain a converse to Theorem 1.

We shall say that a Banach pair $\vec{A}$ is $Conv_0$-abundant on $[0, 1]$, if there is a constant $C > 0$ such that for every concave increasing function $f$ on $[0, 1]$ such that $\lim f(t) = 0$, one can find $a \in A_0 + A_1$ that satisfies the inequality

$$
C^{-1}f(t) \leq K(t, a; \vec{A}) \leq Cf(t), \quad 0 \leq t \leq 1.
$$

Example 2. The pair $(L^1[0, 1], L^\infty[0, 1])$ is $Conv_0$-abundant on $[0, 1]$ (cf. [13]).

Theorem 2. Let $F$ be an interpolation Banach lattice on $[0, 1]$ with respect to the pair $(L^\infty, L^\infty(1/t))$. Then, the following conditions are equivalent:

(a) the operator $Tf(t) = f(t^2)/t$ is bounded on $F$;

(b) for every Banach pair $\vec{A} = (A_0, A_1)$ and for each $q \in [1, \infty]$, the equivalence \(1.1\) holds for the space $\langle \vec{A} \rangle^K_F$;

(c) for every Banach pair $\vec{A} = (A_0, A_1)$, we have, with constants independent of $a \in \langle \vec{A} \rangle^K_F + \vec{A}$, $s > 0$ and $q \in [1, \infty]$,

$$
K(s, t; \vec{A}) \asymp K(s, t \cdot \|a\|_{\langle \vec{A} \rangle^K_{\theta(t)\eta}}; F, L^\infty(1/t)),
$$

where $\theta(t) = 1 + \frac{1}{2 \log(\log t / \log s)}$, $t \in (0, 1)$;

(d) there exist a Banach pair $\vec{A}$ that is $Conv_0$-abundant on $[0, 1]$ and $q \in [1, \infty]$, such that equivalence \(1.1\) holds for the space $\langle \vec{A} \rangle^K_F$.

Proof. Since the implication (a) $\Rightarrow$ (b) was proved in Theorem 1 and the implications (c) $\Rightarrow$ (d) and (b) $\Rightarrow$ (d) are trivial (for example, to prove (c) $\Rightarrow$ (d) simply apply (c) to the pair $\vec{A} = (L^1(0, 1), L^\infty(0, 1))$ and $s = 1$), it therefore only remains to prove that (a) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c). It is easy to verify that the norm of the operator $T$ on the space $L^\infty(1/t)$ equals one. Moreover, since $T$ is bounded on $F$, we see that for each fixed $s > 0$, $K(s, T_f; F, L^\infty(1/t)) = \max\{1, \|T\|_{F \to F}\}K(s, f; F, L^\infty(1/t))$. In other words, if for each $s > 0$, we denote by $\Sigma_s$ the space $L^\infty(1/t) + F$ endowed with the norm $K(s, \cdot; F, L^\infty(1/t))$, then $T : \Sigma_s \to \Sigma_s$ is bounded, and the norm of $T$ on $\Sigma_s$.
does not exceed \( \max\{1, \|T\|_{F \to F}\} \). Consequently, if we apply Theorem \(^1\) using the functional parameter \( \Sigma_s \), then, for any pair \( \tilde{A} \), we have (with absolute constants independent of \( a \) and \( s \); cf. Remark \(^2\))

\[
K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) = \|a\|_{(\tilde{A})_{\Sigma_s}^K} \lesssim t \cdot \|a\|_{(\tilde{A})_{\Theta(t,a)}^K} \|_{\Sigma_s} \\
\lesssim K(s, t \cdot \|a\|_{(\tilde{A})_{\Theta(t,a)}^K}; F, L^\infty(1/t)).
\]

Therefore, it suffices to check that for all \( s > 0 \) we have,

\[(4.11) \quad K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) \lesssim K(s, a; (\tilde{A})_{\tilde{F}, \tilde{A}_1}^K).
\]

Let us consider an arbitrary representation \( a = a_0 + a_1, a_0 \in (\tilde{A})_{\tilde{F}}^K, a_1 \in \tilde{A}_1 \). Then,

\[
K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) \leq K(s, \left(K(\cdot, a_0; \tilde{A}) + K(\cdot, a_1; \tilde{A})\right); F, L^\infty(1/t)) \\
\leq \|K(\cdot, a_0; \tilde{A})\|_F + s \|K(\cdot, a_1; \tilde{A})\|_{L^\infty(1/t)} \\
\leq \|K(\cdot, a_0; \tilde{A})\|_F + s \sup_{0 < t \leq 1} \frac{K(t, a_1; \tilde{A})}{t} \\
= \|a_0\|_{(\tilde{A})_{\tilde{F}}^K} + s \|a_1\|_{\tilde{A}_1}.
\]

Consequently, taking the infimum over all admissible representations, we obtain

\[
K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) \leq K(s, a; (\tilde{A})_{\tilde{F}, \tilde{A}_1}^K).
\]

We now prove the reverse inequality. By the definition of the \( K \)-functional, we can select a decomposition of \( K(t, a; \tilde{A}) \) such that

\[(4.12) \quad K(t, a; \tilde{A}) = f_0(t) + f_1(t), \text{ with } f_0 \in F \text{ and } f_1 \in L^\infty(1/t)
\]

and

\[(4.13) \quad K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) \geq \frac{1}{2} \left(\|f_0\|_F + s \|f_1\|_{L^\infty(1/t)}\right).
\]

Consider the mapping \( f \mapsto \tilde{f} \), where

\[(4.14) \quad \tilde{f}(t) := \sup_{0 < s \leq 1} \min\{1, t/s\} |f(s)|, \quad 0 < t \leq 1.
\]

It can be readily verified that \( \tilde{f} \) is quasi-concave, and \( \tilde{f}(t) \geq |f(t)|, 0 < t \leq 1 \). Moreover, since the mapping \( f \mapsto \tilde{f} \) is bounded on both, \( L^\infty \) and \( L^\infty(1/t) \), we find, by interpolation, that \( f \mapsto \tilde{f} \) is bounded on \( F \). Hence, it follows from \((4.13)\) that

\[(4.15) \quad K(s, K(\cdot, a; \tilde{A}); F, L^\infty(1/t)) \geq c \left(\|f_0\|_F + s \|f_1\|_{L^\infty(1/t)}\right).
\]

On the other hand, by \((4.12)\), we have

\[
K(t, a; \tilde{A}) \leq f_0(t) + \tilde{f}_1(t), \quad 0 < t \leq 1.
\]

Hence, on account of the \( K \)-divisibility property (we extend the functions \( \tilde{f}_i, i = 0, 1 \), to the half-line \([0, \infty)\) setting \( \tilde{f}_i(t) = f_i(t)K(\cdot, a; \tilde{A}) \) for \( t > 1 \), we can find \( a_0 \in A_0 \) and \( a_1 \in A_1 \) such that \( a = a_0 + a_1 \) and

\[
K(t, a; \tilde{A}) \leq D\tilde{f}_i(t) \quad \text{for all } 0 < t \leq 1 \text{ and } i = 0, 1,
\]

respectively.
where $D$ is a universal constant. Since $F$ and $L^\infty(1/t)$ are lattices, we get
\[
\|K(\cdot, a_0; \vec A)\|_F \leq D \|f_0\|_F \quad \text{and} \quad \|K(\cdot, a_1; \vec A)\|_{L^\infty(1/t)} \leq D \|f_1\|_{L^\infty(1/t)}.
\]
Combining with (4.15), we obtain
\[
K(s, K(\cdot, a; \vec A); F, L^\infty(1/t)) \geq c \left( \|K(\cdot, a_0; \vec A)\|_F + s \|K(\cdot, a_1; \vec A)\|_{L^\infty(1/t)} \right)
\]
\[
= c \left( \|a_0\|_{\vec A^K} + s \|a_1\|_{\vec A^K} \right)
\]
\[
\geq cK(s, a; \vec A^K, \vec A_1),
\]
concluding the proof of (4.11).

(d) ⇒ (a). Let $\vec A$ be a Conv_{00}-abundant on $[0, 1]$ Banach pair, and let $1 \leq q \leq \infty$ be such that the space $(\vec A)^{K_1}$ satisfies (4.1). The argument in (4.1) above shows that
\[
t \cdot \|a\|_{(\vec A)^{K_1}, q} \geq \frac{1}{2\epsilon t} K(t^2, a; \vec A).
\]
Applying the $F$-norm, and using the hypothesis, we obtain
\[
\|t^{-1} K(t^2, a; \vec A)\|_F \leq 2 \epsilon \|t \cdot \|a\|_{(\vec A)^{K_1}, q}\|_F \leq C \|K(t, a; \vec A)\|_F.
\]
Consequently, since $\vec A$ is a Conv_{00}-abundant pair on $[0, 1]$, and $T$ is a monotone linear operator, it follows that for every concave increasing function $f \in F$ such that $\lim_{t \to 0} f(t) = 0$, we have
\[
(4.16) \quad \|Tf\|_F \leq C \|f\|_F.
\]
We now rule out the possibility that there could exist a concave increasing function $f_0$, such that $\lim_{t \to 0} f_0(t) > 0$ and $f_0 \in F$. For, if this were the case, then it would follow that $F = L^\infty$. But if $F = L^\infty$ we can show that the equivalence (4.1) fails for every Conv_{00}-abundant Gagliardo complete pair $\vec A$. Indeed, we first observe that
\[
\|a\|_{(\vec A)^{K_1}, \infty} = \sup_{0 < t \leq 1} K(t, a; \vec A) = \|a\|_{A_0 + A_1},
\]
whence $(\vec A)^{K_1}_{L^\infty} = A_0 + A_1$. Suppose now that, to the contrary, (4.1) holds. Then applying successively (4.2), (2.8), and the fact that $\theta(t) \geq 1/2$ for all $0 < t \leq 1$, yields
\[
\|a\|_{A_0 + A_1} \approx \left\| t \cdot \|a\|_{(\vec A)^{K_1}, q} \right\|_{L^\infty}
\]
\[
\geq \frac{1}{2} \|t \cdot \|a\|_{(\vec A)^{K_1}, q}\|_{L^\infty}
\]
\[
\geq \frac{1}{2} \lim_{t \to 1} \inf \|a\|_{(\vec A)^{K_1}, \infty}
\]
\[
\geq \frac{1}{2} \|a\|_{(\vec A)^{K_1}_{L^\infty}}.
\]
This implies that $(\vec A)^{K_1}_{L^\infty} = A_0 + A_1$ and, consequently, for all $a \in A_0 + A_1$ there exists $C_a$ such that $K(t, a; \vec A) \leq C_a \sqrt{t}$. This obviously contradicts the assumption that the pair $\vec A$ is Conv_{00}-abundant. As a result it follows that $F \neq L^\infty$ and hence (4.16) is fulfilled for all concave increasing functions $f \in F$. Moreover, since
every quasi-concave function \( f \) is equivalent to its least concave majorant \( f \), the inequality (4.16) can be easily extended to the set of all quasi-concave functions.

Finally, recall that the mapping \( f \mapsto \tilde{f} \) (see (4.14)) is bounded on \( F \), \( \tilde{f}(t) \geq |f(t)| \) if \( 0 < t \leq 1 \), and \( \tilde{f} \) is a quasi-concave function for each \( f \) (cf. [13]). Since the operator \( T \) is monotone, the properties of \( \tilde{f} \) permit us to extend the inequality (4.16) to the whole lattice \( F \), and therefore complete the proof. □

Theorem 1 can be strengthened when a given ordered pair \((A_0, A_1)\) is Gagliardo complete, i.e., when the smaller space \( A_1 \) is Gagliardo complete with respect to the larger space \( A_0 \).

Theorem 3. Let \( \tilde{A} = (A_0, A_1) \) be a Gagliardo complete ordered pair. Let \( F \) be a parameter for the K-method on \([0,1]\), and, moreover, suppose that the operator \( T f(t) = f(t^2)/t \) is bounded on \( F \). Then, for every family \( \{I_{\theta}\}_{\theta \in (0,1)} \) of exact interpolation functors of exponent \( \theta \),

we have,

\[
\|a\|_{(\tilde{A})^K_{\theta,1}} \lesssim \|t \cdot \|a\|_{I_{\theta}(\tilde{A})}\|_P',
\]

where \( \theta(t) = 1 + \frac{1}{2 \log(t/e)} \), \( 0 < t \leq 1 \), and the constants of this equivalence are universal.

Proof. It is well-known that (cf. [43, p. 11, (2.10)]),

\[
(4.17) \quad \tilde{A}^I_{\theta,1} \triangleleft I_{\theta}(\tilde{A}) \subset \tilde{A}^K_{\theta,\infty}.
\]

Moreover, we also have \( \tilde{A}^K_{\theta,\infty} = (\tilde{A})^K_{\theta,\infty} \) and \( \tilde{A}^I_{\theta,1} = \tilde{A}^I_{\theta,1} \) isometrically and, using (2.8), we see that for \( 1/2 \leq \theta < 1 \) the embedding \( (\tilde{A})^K_{\theta,1} \hookrightarrow \tilde{A}^I_{\theta,1} \) holds. Therefore, the desired result would follow from Theorem 1 (see also Remark 2) and embeddings (4.17) if we could show that, for some constant \( C > 0 \), independent of \( \theta \), \( \theta \in (0,1) \), it holds that

\[
(4.18) \quad \tilde{A}^K_{\theta,1} \subset \tilde{A}^I_{\theta,1},
\]

or equivalently,

\[
(4.19) \quad \theta(1 - \theta) \tilde{A}^K_{\theta,1} \subset \tilde{A}^I_{\theta,1}.
\]

The embedding (4.18) is known (cf. [43, page 34 line 5], a more recent proof is given in [18, Theorem 1]). For the sake of completeness we prove (4.18) using the argument implicit in [43]. By hypothesis, \( \tilde{A} \) is a mutually closed pair and, moreover, \( A_1 \) is dense in \( \tilde{A}^I_{\theta,1} \). Therefore, by the strong form of the fundamental lemma (cf. [30]), any element \( a \in \tilde{A}^I_{\theta,1} \) can be represented by \( a = \int_0^\infty u(s) \frac{ds}{s} \), with

\[
(4.19) \quad \int_0^\infty \min(1, t/s) J(s, u(s); \tilde{A}) \frac{ds}{s} \leq \gamma K(t, a; \tilde{A}), \quad t > 0,
\]

\[16\] in fact we have \( f \leq f \leq 2f \) (see e.g., [47, Ch. II, §1, Corollary after Theorem 1.1]).

\[17\] i.e. for each \( \theta \), the characteristic function of \( I_{\theta} \) is \( t^\theta \).
where $\gamma$ is a universal constant independent of $a$. Consequently,

$$
\theta(1 - \theta)\|a\|_{\widetilde{A}_{\theta}^p} = \theta(1 - \theta) \int_0^\infty t^{-\theta} K(t, a; \widetilde{A}) \frac{dt}{t}
$$

\[\geq \theta(1 - \theta)\gamma^{-1} \int_0^\infty t^{-\theta} \int_0^\infty \min(1, t/s) J(s, u(s); \widetilde{A}) \frac{ds}{s} \frac{dt}{t} \]

\[= \theta(1 - \theta)\gamma^{-1} \int_0^\infty \left( \int_0^\infty t^{-\theta} \min(1, t/s) \frac{dt}{t} \right) J(s, u(s); \widetilde{A}) \frac{ds}{s} \]

\[= \theta(1 - \theta)\gamma^{-1} \int_0^\infty \left( \int_0^s t^{1-\theta} \frac{dt}{s} + \int_s^\infty t^{-\theta} \frac{dt}{t} \right) J(s, u(s); \widetilde{A}) \frac{ds}{s} \]

\[= \theta(1 - \theta)\gamma^{-1} \int_s^\infty \frac{s^{-\theta}}{\theta(1 - \theta)} J(s, u(s); \widetilde{A}) \frac{ds}{s} \]

\[\geq \gamma^{-1} \int_0^\infty s^{-\theta} J(s, u(s); \widetilde{A}) \frac{ds}{s} \geq \gamma^{-1}\|a\|_{\widetilde{A}_{\theta}^p}.
\]

Therefore we have shown that (4.18) holds, and the proof is complete. \hfill \Box

We conclude this section with an extrapolation description of the limiting spaces associated with the $J$-method.

Let $\xi(t) := \frac{1}{2 \log(e/t)}$, $0 < t \leq 1$, and $1 \leq q \leq \infty$. Furthermore, suppose that $\widetilde{A}$ is an ordered pair and $G$ is a parameter for the $J$-method on $(0, 1]$, i.e., a Banach lattice on $((0, 1], \frac{dt}{t})$ such that $L^1(1/t) \subset G \subset L^1$. Denote by $\widehat{A}_{\xi, q, G}^l$ the space of all $a \in A_0$ that admit a representation

$$
a = \int_0^1 u(t) \frac{dt}{t},
$$

where $u(t) : (0, 1] \to A_1$ is a strongly measurable function such that $\|u(t)\|_{\widetilde{A}_{\xi(t), q}} \in G$. We provide $\widehat{A}_{\xi, q, G}^l$ with the quotient norm

$$
\|a\|_{\widehat{A}_{\xi, q, G}^l} := \inf \left\| \|u(t)\|_{\widetilde{A}_{\xi(t), q}} \right\|_G,
$$

where the infimum is taken over all $u(t)$ satisfying (4.20).

Let $G'$ be the Köthe dual lattice to $G$ with respect to the bilinear form

$$(f, g) := \int_0^1 f(t)g(t) \frac{dt}{t^2}.
$$

**Theorem 4.** Let $G$ be a separable parameter for the $J$-method on $(0, 1]$, and let the operator $Rf(t) := f(\sqrt{t})$ be bounded on $G$. Suppose that $\widetilde{A} = (A_0, A_1)$ is an ordered pair such that $A_1$ is dense in $A_0$ and $A_0^*\ominus$ is dense in the space $\langle A_1, A_0^*\ominus \rangle_{G'}$. Then, for each $q \in [1, \infty]$, we have, with equivalence of norms,

$$
\langle \widehat{A}_{\xi}^l \rangle_G = \langle \widetilde{A}_{\xi, q, G}^l \rangle_G.
$$
In the proof of this theorem we shall make use of the following auxiliary result. Let \( \eta > 0 \) and let \( H_{\eta,\infty} \) be the Banach lattice of measurable functions on \([0, 1]\) such that

\[
\|f\|_{H_{\eta,\infty}} := \sup_{0 < s \leq 1} (s^{-\eta}|f(s)|) < \infty.
\]

**Lemma 1.** Suppose that \( G \) is a parameter for the \( J \)-method on \([0, 1]\) such that the operator \( Rf(t) = f(\sqrt{t}) \) is bounded on \( G \). Then, for each \( \eta > 0 \) the embedding \( H_{\eta,\infty} \subset G \) holds.

**Proof.** First, observe that for each \( \eta > 0 \),

\[
\text{that } C \, \frac{d}{\eta}, \quad \text{on the other hand, by iteration and the definition of } \eta,
\]

\[
L_{\infty}(t^{-d}) \subset L_{1}(1/t)(\frac{dt}{t}) \subset G.
\]

Consequently, there exists \( C > 0 \) such that

\[
\|f(s^{d/\eta})\|_{G} \leq C \sup_{0 < s \leq 1} \left( s^{-d/\eta} \left| f(s^{d/\eta}) \right| \right) = C \sup_{0 < s \leq 1} \left( s^{-\eta} |f(s)| \right) = C \|f\|_{H_{\eta,\infty}}.
\]

On the other hand, by iteration and the definition of \( d \),

\[
R^{n} \left( f(s^{d/\eta}) \right) = f \left( s^{d/(\eta 2^n)} \right) = f(s).
\]

Therefore, since \( R \) is bounded on \( G \), we obtain

\[
\|f\|_{G} = \left\| R^{n} \left( f(s^{d/\eta}) \right) \right\|_{G} \leq \|R\|^{n} \|f(s^{d/\eta})\|_{G} \leq C \|R\|^{n} \|f\|_{H_{\eta,\infty}},
\]

as we wished to show. \( \square \)

**Proof of Theorem 4.** As a first step we shall prove that

\[
\langle \tilde{A} \rangle_{G}^{J} \subset A_{\xi, q, G}^{J}.
\]

Let \( a \in \langle \tilde{A} \rangle_{G}^{J} \). Pick a strongly measurable function \( u(t) \) supported on \((0, 1]\), with values in the space \( A_{1} \), satisfying (4.20), and such that

\[
\|J(s, u(s); \tilde{A})\|_{G} \leq 2\|a\|_{\langle \tilde{A} \rangle_{G}^{J}}.
\]

Recall that \( \tilde{A} \rightarrow \tilde{A}_{\xi, q}^{J} \) is an exact interpolation functor with characteristic function \( t^{\theta} \), and, moreover, \( 0 < \xi(t) \leq 1/2 \) for all \( 0 < t \leq 1 \). Therefore, applying successively (2.10), (4.17), (9, Theorem 3.2.2) and the inequality \( t^{-\xi(t)} \leq \sqrt{t} \) (see (4.3)), we obtain that, for each \( q \in [1, \infty] \) and \( 0 < t \leq 1 \),

\[
\|u(t)\|_{\langle \tilde{A}_{\xi, q}^{J} \rangle_{t, q}} \leq C \|u(t)\|_{\tilde{A}_{\xi(t), q}^{J}} \leq C \|u(t)\|_{\tilde{A}_{t, q}^{J}},
\]

\[
\leq C_{1} t^{-\xi(t)} J(t, u(t); \tilde{A}) \leq C_{1} \sqrt{t} J(t, u(t); \tilde{A}),
\]

where \( C_{1} > 0 \) does not depend on \( t \) and \( q \). Combining with (4.22), yields

\[
\|a\|_{\tilde{A}_{\xi, q}^{J}} \leq \left\| \|u(t)\|_{\langle \tilde{A}_{\xi, q}^{J} \rangle_{t, q}} \right\|_{G} \leq C_{1} \sqrt{t} \|J(t, u(t); \tilde{A})\|_{G} \leq 2C_{1} \sqrt{t} \|a\|_{\langle \tilde{A}_{\xi, q}^{J} \rangle_{t, q}},
\]

concluding the proof of (4.21).
Next, we prove that $\langle \tilde{\mathcal{A}} \rangle _{G}$ is dense in $\tilde{\mathcal{A}} _{\xi,q,G}$. Let $a \in \tilde{\mathcal{A}} _{\xi,q,G}$. Then we can find a representation \[ (4.20) \], where $u(t) : (0,1) \to A_1$ is a strongly measurable function such that $\| u(t) \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),q}} \in G$, and

\[
\| a \| _{\tilde{\mathcal{A}} _{\xi,q,G}} > \frac{1}{2} \left\| u(t) \right\| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),q}} G.
\]

We shall show in a moment that for every $\delta > 0$

\[
(4.24) \quad a_\delta := \int_\delta^1 u(t) \frac{dt}{t} \in \langle \tilde{\mathcal{A}} \rangle _G.
\]

This given, writing

\[
a - a_\delta = \int_0^1 u(t) \cdot \chi_{(0,\delta)}(t) \frac{dt}{t},
\]

and using the fact that the lattice $G$ is separable, we see that for every $\varepsilon > 0$ we can find $\delta > 0$ such that

\[
\| a - a_\delta \| _{\tilde{\mathcal{A}} _{\xi,q,G}} \leq \left\| \| u(t) \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),q}} \cdot \chi_{(0,\delta)}(t) \right\| _G < \varepsilon.
\]

We now return to the proof of \[ (4.21) \]. First of all, observe that if $\delta \leq t \leq 1$, then $1/2 \geq \xi(t) \geq \eta := (2 \log(e/\delta))^{-1} > 0$. Therefore, for each $a_1 \in A_1$ and $t \in [\delta,1]$ we have

\[
\| a_1 \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),q}} \geq (q \xi(t)(1 - \xi(t)))^{-1/\eta'} \| a_1 \| _{\langle \tilde{\mathcal{A}} \rangle _{\eta,q}} \geq e^{-1/\varepsilon} \| a_1 \| _{\langle \tilde{\mathcal{A}} \rangle _{\eta,q}} .
\]

Moreover, since $A_1 \subset A_0$, then by Proposition \[ 2 \] and \[ 9 \] Theorem 3.4.1 (b) it follows that

\[
\langle \tilde{\mathcal{A}} \rangle _{\eta,q} = \tilde{\mathcal{A}} _{\eta,q} \subset \tilde{\mathcal{A}} _{\eta,\infty} = \langle \tilde{\mathcal{A}} \rangle _{\eta,\infty},
\]

with constants that depend only on $\delta$ and $q$. Hence,

\[
\| a_1 \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),q}} \geq \varepsilon \| a_1 \| _{\langle \tilde{\mathcal{A}} \rangle _{\eta,q}} .
\]

Combining this with \[ (4.23) \], yields

\[
\left\| \left\| u(t) \right\| _{\langle \tilde{\mathcal{A}} \rangle _{\xi(t),\infty}} \cdot \chi_{(\delta,1)}(t) \right\| _G < 2c_{\delta,q}^{-1} \| a \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi,q,G}} .
\]

It follows that, for every $\delta \leq t \leq 1$ there exists a strongly measurable function $v(\cdot,t) : (0,1) \to A_1$ such that

\[
u(t) = \int_0^1 v(s,t) \frac{ds}{s},
\]

and

\[
(4.25) \quad \left\| \sup_{0 < s \leq 1} (s^{-\eta J}(s,v(s,t;\tilde{\mathcal{A}}))) \cdot \chi_{(\delta,1)}(t) \right\| _G < 2c_{\delta,q}^{-1} \| a \| _{\langle \tilde{\mathcal{A}} \rangle _{\xi,q,G}} .
\]

We shall now verify that the integral

\[
\int_0^1 \int_\delta^1 v(s,t) \frac{dt}{t} \frac{ds}{s}
\]
Thus, taking into account (4.26), we see that a \(\langle \cdot \rangle\) concludes the proof that (4.27) \(\langle \cdot \rangle\) can find a strongly measurable function \(u\) (4.28) ordered pair. We shall show that Now, our next aim will be to prove the converse embedding. Furthermore, by Minkowski’s inequality, Lemma [11] and the embedding \(G \subset L_1[0,1]\) (\(\frac{dt}{s}\)), for each \(s \in (0,1]\) we have

\[
\|w(s)\|_{L_1} \leq \int_{\delta}^1 \|v(s,t)\|_{L_1} \frac{dt}{s} \leq \frac{1}{s} \int_{\delta}^1 J(s,v(s,t);\tilde{A}) \frac{dt}{t} \\
\leq \frac{C}{s} \left\| \sup_{0<s \leq 1} (s^{-\gamma}J(s,v(s,t);\tilde{A})) \cdot \chi(s) \right\|_G \\
\leq 2C(s\gamma)^{-1}\|\|A\|_{\tilde{J},q,G} < \infty.
\]

Therefore

\[
a_\delta = \int_{\delta}^1 \int_{\delta}^1 v(s,t) \frac{ds}{s} \frac{dt}{t} = \int_{\delta}^1 \int_{\delta}^1 v(s,t) \frac{ds}{s} \frac{dt}{t},
\]

and we can write

(4.26) \(a_\delta = \int_{\delta}^1 w(s) \frac{ds}{s}, \) where \(w(s) := \int_{\delta}^1 v(s,t) \frac{dt}{t}.
\]

It is easy to see that \(w(s) : (0,1] \rightarrow A_1\). Indeed, using successively Minkowski’s inequality and the embedding \(G \subset L_1[0,1]\) (\(\frac{dt}{s}\)), for each \(s \in (0,1]\) we have

\[
\|w(s)\|_{L_1} \leq \int_{\delta}^1 \|v(s,t)\|_{L_1} \frac{dt}{s} \leq \frac{1}{s} \int_{\delta}^1 J(s,v(s,t);\tilde{A}) \frac{dt}{t} \\
\leq \frac{C}{s} \left\| \sup_{0<s \leq 1} (s^{-\gamma}J(s,v(s,t);\tilde{A})) \cdot \chi(s) \right\|_G \\
\leq 2C(s\gamma)^{-1}\|\|A\|_{\tilde{J},q,G} < \infty.
\]

Furthermore, by Minkowski’s inequality, Lemma [11] and the embedding \(G \subset L_1[0,1]\) (\(\frac{dt}{s}\)), we obtain

\[
\|J(s,w(s);\tilde{A})\|_G \leq \int_{\delta}^1 \|J(s,v(s,t);\tilde{A})\|_G \frac{dt}{t} \\
\leq C' \int_{\delta}^1 \sup_{0<s \leq 1} (s^{-\gamma}J(s,v(s,t);\tilde{A}) \frac{dt}{t} \\
\leq CC' \left\| \sup_{0<s \leq 1} (s^{-\gamma}J(s,v(s,t);\tilde{A}) \cdot \chi(s) \right\|_G \\
\leq 2CC'c_{\delta,q}\|\|A\|_{\tilde{J},q,G} < \infty.
\]

Thus, taking into account (4.26), we see that \(a_\delta \in \langle \tilde{A} \rangle_{\tilde{G}}\), and (4.24) is proved. This concludes the proof that \(\langle \tilde{A} \rangle_{\tilde{G}}\) is dense in \(\tilde{A}_{\xi,q,G}\), and therefore from (4.21) it follows that

(4.27) \((\tilde{A}_{\xi,q,G})^* \subset (\langle \tilde{A} \rangle_{\tilde{G}})^*\).

Now, our next aim will be to prove the converse embedding.

On account of the fact that \(A_1\) is dense in \(A_0\), it follows that \((A_1^*, A_0^*)\) is an ordered pair. We shall show that

(4.28) \((A_1^*, A_0^*)_{\tilde{G}} \subset (\tilde{A}_{\xi,q,G})^*\).

Let \(0 < \theta < 1\) and \(1 < q \leq \infty\), be arbitrary (but fixed). Let \(a \in \langle \tilde{A} \rangle_{\theta,q}\). Then we can find a strongly measurable function \(u(t) : (0,1] \rightarrow A_1\) such that \(a = \int_{0}^{1} u(s) \frac{ds}{s}\),
and, moreover,

\[ (\int_{0}^{1} \left( s^{-\theta} J(s, u(s); \vec{A}) \right)^q \frac{ds}{s} )^{1/q} \leq 2 \|a\|_{(\vec{A})_{\theta,q}}. \]

Let \( b \in A_{\theta}^q \), then, by duality and Hölder’s inequality,

\[ |\langle b, a \rangle| = \left| \int_{0}^{1} u(s) \frac{ds}{s} \right| \leq \int_{0}^{1} |\langle b, u(s) \rangle| \frac{ds}{s} \leq \int_{0}^{1} K(1/s; b; A_{\theta}^q, A_1^q) \cdot J(s, u(s); \vec{A}) \frac{ds}{s} \]

\[ \leq \left( \int_{0}^{1} (s^\theta K(1/s; b; A_{\theta}^q, A_1^q)) \frac{ds}{s} \right)^{1/q} \left( \int_{0}^{1} (s^{-\theta} J(s, u(s); \vec{A})) q \frac{ds}{s} \right)^{1/q} \]

\[ \leq \left( \int_{0}^{1} (s^\theta K(s, b; A_{\theta}^q, A_1^q)) \frac{ds}{s} \right)^{1/q} \left( \int_{0}^{1} (s^{-\theta} J(s, u(s); \vec{A})) q \frac{ds}{s} \right)^{1/q}. \]

Combining with (4.29) yields

\[ |\langle b, a \rangle| \leq 2 \|b\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \|a\|_{(\vec{A})_{\theta,q}}, \]

for \( 0 < \theta < 1 \), \( 1 \leq q \leq \infty \), and for all \( a \in \vec{A}_{\theta,q}^q \) and \( b \in A_{\theta}^q \).

Now, let \( a \in \vec{A}_{\xi,q,G}^q \), pick a strongly measurable function \( u(t) : (0,1) \to A_1 \), such that (4.29) holds, and, moreover,

\[ \left\| u(t) \right\|_{(\vec{A})_{\xi,q,G}} \leq 2 \|a\|_{\vec{A}_{\xi,q,G}}. \]

Observe that \( 0 < \xi(t) = \theta(t) \), \( 0 < t \leq 1 \). Therefore, by (4.30) and (4.31), we see that for each \( b \in A_{\theta}^q \)

\[ |\langle b, a \rangle| \leq \int_{0}^{1} |\langle b, u(t) \rangle| \frac{dt}{t} \leq 2 \|b\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \|u(t)\|_{(\vec{A})_{\xi,q,G}} \frac{dt}{t} \]

\[ \leq 2 \left\| b \right\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \left\| u(t) \right\|_{(\vec{A})_{\xi,q,G}} \left\| a \right\|_{\vec{A}_{\xi,q,G}} \]

One can easily verify that the operator \( Rf(t) = f(\sqrt{t}) \) is bounded on \( G \) if and only if the operator \( T f = f(t^2)/t \) is bounded on \( G' \). Thus, from Theorem 1 applied to the ordered pair \( (A_1^q, A_{\theta}^q) \) we obtain,

\[ \left\| t \cdot \left\| b \right\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \left\| a \right\|_{\vec{A}_{\xi,q,G}} \right\|_{G'} \leq C \|b\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \|a\|_{\vec{A}_{\xi,q,G}}. \]

The preceding inequality combined with (4.32) yields that for all \( b \in A_{\theta}^q \), \( a \in \vec{A}_{\xi,q,G}^q \),

\[ |\langle b, a \rangle| \leq C \|b\|_{(A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q}} \|a\|_{\vec{A}_{\xi,q,G}}. \]

This estimate extends to all \( b \in (A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q} \) on account of the fact that, by assumption, \( A_{\theta}^q \) is dense in \( (A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q} \). Therefore, the proof of (4.29) is complete.

Furthermore, since \( G \) is separable it follows that \( A_1 \) is dense in \( \vec{A}_{G}^q \) and consequently \( (\vec{A}_{G}^q)^* = (A_1^q, A_{\theta}^q)^{\Delta}_{\theta,q} \) (cf. [13]). Thus, from embeddings (4.28) and (4.29) it follows that \( (\vec{A}_{\theta,G}^q)^* = (A_{\theta}^q)^* \). Since \( \vec{A}_{\theta,G}^q \) is embedded into \( \vec{A}_{\xi,q,G}^q \) as a dense
Therefore, applying Theorem 4 to $G$ we can invoke the Hahn-Banach Theorem to conclude that, as we wished to show, $\langle \tilde{A} \rangle^J_{\xi,q,G} = \tilde{A}^J_{\xi,q,G}$. □

**Remark 6.** Let $1 \leq q < \infty$. As it is shown in [19, Theorem 7.6] (generalizing formula (2.15)), for every ordered pair $\tilde{A}$ the modified $K$-interpolation space $\langle \tilde{A} \rangle^J_{0,q}$ (see Definition 1) can be obtained also by using the $J$-method. Namely, the latter space coincides (with equivalence of norms) with the space $\langle \tilde{A} \rangle^J_{0,q,\log}$, generated by the space $L^q_{\log}$ on $(0,1), ds/s$), normed by

$$
\|f\|_{L^q_{\log}} = \left( \int_0^1 (|f(s)| \log(e/s))^q \frac{ds}{s} \right)^{1/q}.
$$

It is easy to see that the operator $Rf(t) = f(\sqrt{t})$ is bounded on the lattice $L^q_{\log}$. Therefore, applying Theorem 4 to $G = L^q_{\log}$ with $1 < q < \infty$, we get an extrapolation description of the spaces $\langle \tilde{A} \rangle^K_{0,q}$.

5. **Reiteration properties of limiting interpolation spaces**

Throughout this section $L^p := L^p[0,1], 1 \leq p \leq \infty$.

**Theorem 5.** Let $F$ be an interpolation Banach lattice between $L^\infty$ and $L^\infty(1/t)$ on $[0,1]$. The following conditions are equivalent:

(i) the operator $Tf(t) = f(t^2)/t$ is bounded on $F$;

(ii) for all ordered pairs $\tilde{A} = (A_0,A_1)$, and for all $0 < \theta < 1$ and $1 \leq q \leq \infty$ we have

$$
\langle \tilde{A}_{\theta,q},A_1 \rangle^K_F = \langle \tilde{A} \rangle^K_F
$$

(with constants depending on $\theta$ and $q$);

(iii) for every ordered pair $\tilde{A} = (A_0,A_1)$ there exist $\theta_1, \theta_2 \in (0,1), \theta_1 \neq \theta_2$, and $1 \leq q_1, q_2 \leq \infty$ such that

$$
\langle \tilde{A}_{\theta_1,q_1},A_1 \rangle^K_F = \langle \tilde{A}_{\theta_2,q_2},A_1 \rangle^K_F;
$$

(iv) there exist $\theta_1, \theta_2 \in (0,1), \theta_1 \neq \theta_2$, and $1 \leq q_1, q_2 \leq \infty$ such that

$$
((L^1,L^\infty)_{\theta_1,q_1},L^\infty)_{\log}^K = ((L^1,L^\infty)_{\theta_2,q_2},L^\infty)_{\log}^K.
$$

**Proof.** (i) ⇒ (ii). Since $A_1 \subset A_0$, it follows that $\tilde{A}_{\theta,q} \subset A_0$ for all $0 < \theta < 1, 1 \leq q \leq \infty$. Consequently,

$$
\langle \tilde{A}_{\theta,q},A_1 \rangle^K_F \subset \langle \tilde{A} \rangle^K_F.
$$

Since $\tilde{A}_{\theta,1} \subset \tilde{A}_{\theta,q}, 1 \leq q \leq \infty$, to prove the converse embedding it will suffice to show that $\langle \tilde{A} \rangle^K_F \subset \langle \tilde{A}_{\theta,1},A_1 \rangle^K_F$.

Given $\theta \in (0,1)$ pick $m \in \mathbb{N}$ such that $2^{-m} \leq 1 - \theta < 2^{1-m}$. Then, for all $0 < t \leq 1$ we have

$$
t^{2m} \leq t^{1/(1-\theta)} < t^{2^{m-1}}.
$$
Using Holmstedt’s formula (see [30] or [9, Corollary 3.6.2(b)]) we obtain

$$K(t, a; \overline{A}_{\theta_1}, A_1) \approx \int_0^{t/(1-\theta)} s^{-\theta} K(s, a; \overline{A}) \frac{ds}{s} = \int_{t/\theta}^{t/(1-\theta)} s^{-\theta} K(s, a; \overline{A}) \frac{ds}{s} + \sum_{n=0}^{\infty} \int_{t/\theta}^{t/\theta+1} s^{-\theta} K(s, a; \overline{A}) \frac{ds}{s}.$$  

(5.1)

We estimate separately each of the terms from the right-hand side of (5.1), assuming that $0 < t \leq 1/2$.

For the first integral we use the concavity of $K(s, a; \overline{A})$ with respect to $s$ [9, Lemma 3.1.1] and the definition of $T$, to obtain

$$\int_{t/\theta}^{t/(1-\theta)} s^{-\theta} K(s, a; \overline{A}) \frac{ds}{s} \leq \frac{1}{1-\theta} K\left(t/\theta, a; \overline{A}\right) \cdot t = \frac{1}{1-\theta} T(t, a; \overline{A}).$$

Similarly, since $(1-\theta)t^2 \geq 1$ for all $n \in \mathbb{N}$ such that $n \geq m$, we have

$$\int_{t/\theta}^{t/\theta+1} s^{-\theta} K(s, a; \overline{A}) \frac{ds}{s} \leq \frac{1}{1-\theta} K\left(t/\theta, a; \overline{A}\right) \cdot t \cdot t(1-\theta)^{2n-1} \leq \frac{1}{1-\theta} T \left(K(t, a; \overline{A})\right) 2^{1-(1-\theta)^{2n}}.$$

Inserting these estimates in (5.1) we find that there exists a constant $C$, that depends only on $\theta$, such that for all $0 < t \leq 1/2$,

$$K(t, a; \overline{A}_{\theta_1}, A_1) \leq C \left(T^m \left(K(t, a; \overline{A})\right) + \sum_{n=m}^{\infty} T^{n+1} \left(K(t, a; \overline{A})\right) 2^{1-(1-\theta)^{2n}} \right).$$

(5.2)

On account of the fact that the $K$--functional is a concave function we have that, for all $t \in (0, 1)$,

$$K(t, a; \overline{A}_{\theta_1}, A_1) \leq 2K(t/2, a; \overline{A}_{\theta_1}, A_1).$$

Consequently, by (5.2)

$$\|K(t, a; \overline{A}_{\theta_1}, A_1)\|_F \leq 2 \left\|K(t/2, a; \overline{A}_{\theta_1}, A_1)\chi_{[0,1]}(t)\right\|_F \leq 2C \left(\|T\|^m + \sum_{n=m+1}^{\infty} \|T\| n^{2^{1-(1-\theta)^{2n-1}}} \right) \|K(t, a; \overline{A})\|_F.$$

Thus, with constants depending on $\theta$ and $q$, we have

$$\langle \overline{A} \rangle_{\theta_1, q}^{K} \subset \langle \overline{A}_{\theta_1}^{K}, A_1 \rangle_{F}^{K} \subset \langle \overline{A}_{\theta_1, q}^{K}, A_1 \rangle_{F}^{K},$$

as we wished to show.

Since the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are immediate, it is only left to prove

(iii) $\Rightarrow$ (i). Without loss of generality we can assume that $\theta_1 < \theta_2$. Choose $\theta_1$ and $\theta_2$ so that $\theta_1 < \theta_1 < \theta_2 < \theta_2$. Then, since $L^\infty \subset L^1$, by [9, Theorem 3.4.1 (c), (d)] we obtain

$$(L^1, L^\infty)_{\theta_2, q_2} \subset (L^1, L^\infty)_{\theta_2, 1/(1-\theta_2)} \subset (L^1, L^\infty)_{\theta_1, 1/(1-\theta_1)} \subset (L^1, L^\infty)_{\theta_1, q_1}.$$
Therefore, setting $q := 1/(1 - \tilde{\theta}_1)$ and $p := 1/(1 - \tilde{\theta}_2)$, we have $1 < q < p < \infty$ and, by (iv),
\begin{equation}
\langle L^p, L^\infty \rangle^K_F = \langle L^q, L^\infty \rangle^K_F.
\end{equation}

We now show that the operator $S_r f(t) := f(t^r)$ is bounded from $L^p$ into $L^q$, if $1 < r < p/q$. Indeed, by Hölder’s inequality, we have
\[
\|S_r f\|_q^q = \int_0^1 |f(s^r)|^q \, ds = \frac{1}{r} \int_0^1 u^{1/r - 1} |f(u)|^q \, du \\
\leq \frac{1}{r} \left( \int_0^1 u^{\frac{(1 - r)p}{q - 1}} \, du \right)^{\frac{q}{q - 1}} \left( \int_0^1 |f(u)|^p \, du \right)^{\frac{q}{p}},
\]
where $\int_0^1 u^{\frac{(1 - r)p}{q - 1}} \, du < \infty$ on account of the fact that $1 < r < p/q$. Moreover, it is plain that for all $r > 1$ the operator $S_r$ is bounded on $L^\infty$. Hence, by interpolation and (5.3),
\[
S_r : \langle L^p, L^\infty \rangle^K_F \to \langle L^p, L^\infty \rangle^K_F = \langle L^q, L^\infty \rangle^K_F,
\]
whenever $1 < r < p/q$. Consequently, if we let $X := \langle L^p, L^\infty \rangle^K_F$, then $S_r$ is bounded on $X$ for $1 < r < p/q$. We now show that this implies that $S_r$ is bounded on $X$ for all $r > 1$. Indeed, if $r \geq p/q > 1$, we can select $n \in \mathbb{N}$ large enough so that $r^{1/n} \in (1, p/q)$. Then $S_{r^{1/n}} : X \to X$ and we conclude since $S_r = (S_{r^{1/n}})^n$.

Next, we exploit the properties of the lattice $F$ to prove that the definition of $X$ self improves to
\begin{equation}
X = \langle L^1, L^\infty \rangle^K_F.
\end{equation}
It is immediate that $X = \langle L^p, L^\infty \rangle^K_F \subset \langle L^1, L^\infty \rangle^K_F$. To prove the opposite embedding we use an equivalent expression for $K(t, g; L^p, L^\infty)$ [9] Theorem 5.2.1] as follows
\[
K(t, f(s^{1/p}); L^p, L^\infty) \asymp \left( \int_0^t \left[ f^*(s^{1/p}) \right]^p ds \right)^{1/p} = p^{1/p} \left( \int_0^t [f^*(u)]^p u^{p-1} \, du \right)^{1/p} \\
\leq p^{1/p} \left( \int_0^t f^*(u) \, du \right)^{1/p} \left( \sup_{0 < u \leq t} u f^*(u) \right)^{(p-1)/p} \\
\leq p^{1/p} \left( \int_0^t f^*(u) \, du \right)^{1/p} \left( \int_0^t f^*(u) \, du \right)^{(p-1)/p} \\
\leq p^{1/p} \int_0^t f^*(u) \, du \\
= p^{1/p} K(t, f; L^1, L^\infty).
\]
Consequently, since $S_p$ is bounded on $X$, we obtain
\[
\|f\|_X = \|S_p S_{1/p} (f)\|_X \leq \|S_p\| \|K(t, f(s^{1/p}); L^p, L^\infty)\|_F \\
\leq p^{1/p} \|S_p\| \|K(t, f; L^1, L^\infty)\|_F,
\]
whence
\[
\langle L^1, L^\infty \rangle^K_F \subset X,
\]
and the proof of (5.4) is completed.
Further, from (5.4) and the boundedness of $S_2$ on $X$ we obtain
\[
\left\| \int_0^t S_2 f^*(s) \, ds \right\|_F \leq \|S_2 f\|_X \leq \|S_2\| \|f\|_X \leq C \left\| \int_0^t f^*(s) \, ds \right\|_F.
\]
On the other hand,
\[
\int_0^t S_2 f^*(s) \, ds = \frac{1}{2} \int_0^t u^{-1/2} f^*(u) \, du \geq \frac{1}{2t} \int_0^t f^*(u) \, du,
\]
and therefore
\[
(5.5) \quad \left\| \frac{1}{t} \int_0^t f^*(s) \, ds \right\|_F \leq 2C \left\| \int_0^t f^*(s) \, ds \right\|_F.
\]
We now re-interpret (5.5) as the boundedness of the operator $T$ on a class of concave functions. Indeed, since $K(t, f; L^1, L^\infty) = \int_0^t f^*(s) \, ds$, we have
\[
\frac{1}{t} \int_0^t f^*(s) \, ds = T(K(\cdot, f; L^1, L^\infty))(t).
\]
Furthermore, since the pair $(L^1, L^\infty)$ is Con\textsubscript{0}0-abundant on $[0, 1]$ (cf. Example 2 above), we can rewrite (5.5) as follows: for every concave increasing function $f \in F$ such that $\lim_{t \to 0} f(t) = 0$
\[
(5.6) \quad \|T f\|_F \leq 2C \|f\|_F.
\]
We will show in a moment that (5.6) holds for all concave increasing functions from $F$. Then, the interpolation property of $F$ guarantees (by the same argument we gave in the course of the proof of Theorem 2 above) that $T$ is bounded on all of $F$, concluding the proof. To prove the above claim we argue by contradiction. Suppose that there exists a concave increasing function $f_0 \in F$ such that $\lim_{t \to 0} f_0(t) > 0$. Then clearly $F = L^\infty$. But by Proposition 1 and the first equation from (2.2), we have
\[
((L^1, L^\infty)_{\theta,q}, L^\infty)_{L^\infty} = ((L^1, L^\infty)_{\theta,q}, L^\infty)_{L^\infty}^{K,0,0}
\]
\[
= (L^1, L^\infty)_{\theta,q},
\]
for every $0 < \theta < 1$ and $1 \leq q \leq \infty$. Therefore (iv) fails in this case, which gives a contradiction. So $F \not= L^\infty$ and therefore (5.6) holds for each concave increasing function $f \in F$, as we wished to show. $\square$

Since the pair $(L^1, L^\infty)$ is Con\textsubscript{0}0-abundant (cf. Example 2 above), from Theorems 2 and 5 we obtain

**Corollary 1.** Suppose $F$ is an interpolation Banach lattice of functions with respect to the pair $(L^\infty, L^\infty(1/t))$ on $[0, 1]$. Then, the following conditions are equivalent:

(i) the operator $T f(t) = f(t^2)/t$ is bounded on $F$;

(ii) for every Banach pair $\tilde{A} = (A_0, A_1)$ and every $1 \leq q \leq \infty$ the equivalence (4.1) holds for the space $(\tilde{A})^\infty_F$;

(iii) for every Banach pair $\tilde{A} = (A_0, A_1)$, we have, with constants independent of $a \in (\tilde{A})^\infty_F + \tilde{A}_1$, $s > 0$ and $1 \leq q \leq \infty$,
\[
K(s, a; (\tilde{A})^\infty_F, \tilde{A}_1) \simeq K(s, \|a\|_{(\tilde{A})^\infty_F, q}; F, L^\infty(1/t)),
\]
where $\theta(t) = 1 + \frac{1}{2\log(t/e)}$;
(iv) for every ordered pair \( \vec{A} = (A_0, A_1) \), and for all \( \theta \in (0, 1) \) and \( 1 \leq q \leq \infty \) we have
\[
\langle \vec{A} \rangle^K_F = \langle \vec{A}_{\theta, q}, A_1 \rangle^K_F;
\]
(v) for every ordered pair \( \vec{A} = (A_0, A_1) \), there exist \( \theta_1, \theta_2 \in (0, 1) \), \( \theta_1 \neq \theta_2 \), and \( 1 \leq q_1, q_2 \leq \infty \) such that
\[
\langle \vec{A}_{\theta_1, q_1}, A_1 \rangle^K_F = \langle \vec{A}_{\theta_2, q_2}, A_1 \rangle^K_F;
\]
(vi) there exist \( \theta_1, \theta_2 \in (0, 1) \), \( \theta_1 \neq \theta_2 \), and \( 1 \leq q_1, q_2 \leq \infty \) such that
\[
\langle (L^1, L^\infty)_{\theta_1, q_1}, L^\infty \rangle_F^K = \langle (L^1, L^\infty)_{\theta_2, q_2}, L^\infty \rangle_F^K.
\]

Remark 7. Applying functors \( \langle \cdot, \cdot \rangle^K_F \), with \( F \) satisfying the conditions of Corollary 1 to the pair \((L^1[0, 1], L^\infty[0, 1])\), we obtain the so-called strong extrapolation rearrangement invariant spaces introduced and studied in [3-6, 7]. The latter ones can be characterized via the boundedness of the operator \( S_2f(t) := f(t^2) \) (see the proof of Theorem 1). In particular, in this connection it is instructive to compare Corollary 1 with Theorem 4.3 from [6].

Next, we investigate the reiteration properties of limiting interpolation spaces which are close to the larger "end point" of an ordered pair. To avoid imposing extra conditions needed to use duality (see Theorem 1), we prefer to handle this case using limiting interpolation \( K \)-functors. As was observed in the Introduction, the simplest functor of such a form, \( \vec{A} \mapsto \langle \vec{A} \rangle^K_{0,1} \), was introduced and studied in the paper [38] (see also [19]). The following theorem is a far-reaching generalization and a refinement of these results (cf. Remark 8 below).

Theorem 6. Let \( G \) be an interpolation Banach lattice on \([0, 1]\) with respect to the pair \((L^\infty, L^\infty(1/t))\). The following conditions are equivalent:

(a) the operator \( Rf(t) := f(t^{1/2}) \) is bounded on \( G \);
(b) for every ordered pair \( \vec{A} = (A_0, A_1) \) and for every \( \theta \in (0, 1) \), \( 1 \leq q \leq \infty \), we have
\[
\langle \vec{A} \rangle^K_F = \langle A_0, \vec{A}_{\theta, q} \rangle^K_G;
\]
(c) for every ordered pair \( \vec{A} = (A_0, A_1) \) there exist \( \theta_1, \theta_2 \in (0, 1) \), \( \theta_1 \neq \theta_2 \), \( 1 \leq q_1, q_2 \leq \infty \), such that
\[
\langle A_0, \vec{A}_{\theta_1, q_1} \rangle^K_G = \langle A_0, \vec{A}_{\theta_2, q_2} \rangle^K_G;
\]
(d) there exist \( \theta_1, \theta_2 \in (0, 1) \), \( \theta_1 \neq \theta_2 \), \( 1 \leq q_1, q_2 \leq \infty \), such that
\[
\langle L^1, (L^1, L^\infty)_{\theta_1, q_1} \rangle^K_G = \langle L^1, (L^1, L^\infty)_{\theta_2, q_2} \rangle^K_G.
\]

Proof. The implications (b) \(\Rightarrow\) (c) and (c) \(\Rightarrow\) (d) are straightforward. Therefore we only need to prove the implications (a) \(\Rightarrow\) (b) and (d) \(\Rightarrow\) (a).

(a) \(\Rightarrow\) (b). First, observe that for arbitrary \( p > 1 \), the restriction of the operator \( R_p f(t) := f(t^{1/p}) \) to the set of all increasing functions is bounded on \( G \). Indeed, by iteration, we see that together with \( R \) the operator \( R_{2^m} = R^m \) is bounded on \( G \) for each \( m \in \mathbb{N} \). Given \( p > 1 \) we pick \( m \) such that \( 2^m \geq p \). Let \( g \) be an increasing function \( g \in G \), then on account of the fact that \( t^{1/p} \leq t^{2^{-m}} \) for \( 0 \leq t \leq 1 \), we obtain
\[
\|R_p g\|_G \leq \|R_{2^m} g\|_G \leq \|R_{2^m}\| \|g\|_G.
\]
Let \( f \in A_0 + A_1, \theta \in (0, 1) \), and set \( g = K(t, f; \bar{A}) \). The previous inequality implies that

\[
\|K(t, f; \bar{A})\|_G = \|R_{1/\theta}(K(t^{1/\theta}; f; \bar{A}))\|_G \leq \|R_{1/\theta}\| \|K(t^{1/\theta}; f; \bar{A})\|_G.
\]

On the other hand, by Holmstedt’s formula, we have

\[
K(t, f; A_0, \bar{A}^{K_\theta}_{q, \infty}) \approx t \sup_{|t/t^{1/\theta}| \leq s \leq 1} s^{-\theta} K(s, f; \bar{A}) \geq K(t^{1/\theta}, f; \bar{A}), \text{ for all } 0 < t \leq 1.
\]

It follows that

\[
\|K(t, f; A_0, \bar{A}^{K_\theta}_{q, \infty})\|_G \geq \|K(t, f; A_0, \bar{A}^{K_\theta}_{q, \infty})\|_G \geq c\|K(t^{1/\theta}, f; \bar{A})\|_G, \text{ for all } 1 \leq q \leq \infty.
\]

Thus,

\[
\langle A_0, \bar{A}_{q, q/G}^{K_\theta} \rangle \subset \bar{A}_{G}^{K}.
\]

This gives the desired result since the converse embedding is obvious.

\((d) \Rightarrow (a)\). Proceeding exactly as in the proof of Theorem 5 we see that \((d)\) guarantees the existence of \(1 < q < p < \infty\) such that

\[
(L^1, L^p)^{K}_{\bar{G}} = (L^1, L^q)^{K}_{\bar{G}}.
\]

It will be useful now to consider a family of auxiliary operators defined as follows. For each \(r > 1\), we let \(Q_r\) be the operator defined by \(Q_r f(s) := s^{1/r-1} f(s^{1/r})\), \(0 < s \leq 1\), \(f \in L^1 = L^1[0, 1]\). By an easy change of variables we see that the operators \(Q_r\) are bounded on \(L^1\), for every \(r > 1\). We will now show that there exists \(r_0 > 1\) such that, for all \(1 < r < r_0\), \(Q_r\) is a bounded operator, \(Q_r : L^p \rightarrow L^q\). In fact, we can let \(r_0 := \frac{q(q-1)}{q-p+1} > 1\). Indeed, since for \(1 < r < r_0 := \frac{q(q-1)}{q-p+1}\), we have \(\int_0^1 u^{(r-1)(1-q)p} du < \infty\), Hölder’s inequality yields

\[
\|Q_r f\|_p \leq C \|f\|_p, \text{ with a constant } C \text{ that depends only on } p, q, r. \]

Moreover, since \(Q_r\) is also bounded on \(L^1\), it follows by interpolation and \(5.7\) that \(Q_r\) is bounded on the space \(X := (L^1, L^p)^{K}_{\bar{G}}\), for all \(1 < r < r_0\). Observe that for each \(r > 1\) and \(k \in \mathbb{N}\), we have \(Q_r^k = Q_{r^k}\), whence \(Q_r\) is actually bounded on \(X\) for all \(r > 1\). Furthermore, it is plain that \(X\) is also an interpolation space with respect to the pair \((L^1, L^\infty)\), whence \(X\) is a rearrangement invariant space. As a matter of fact we now show that, more precisely, \(X\) can be described by

\[
X = (L^1, L^\infty)^{K}_{\bar{G}}.
\]

The preceding discussion shows that comparing norms of the spaces from \((5.8)\) we may assume without loss of generality that \(f = f^*\). Let \(p'\) be defined as usual by \(1/p + 1/p' = 1\). Then, by Holmstedt’s formula for the pair \((L^1, L^p)\) (cf. \(9\) Ch. 5.
Section 7, Problem 2, page 124), we have
\[
K(t, Q_{p'} f^*; L^1, L^p) \geq \int_0^{t^{p'}} (Q_{p'} f^*)^*(s) \, ds = \int_0^{t^{p'}} s^{1/p'-1} f^*(s^{1/p'}) \, ds
\]
\[
= p' \int_0^t f^*(u) \, du
\]
\[
= p' K(t, f; L^1, L^\infty).
\]

Thus, for some \( c > 0 \)
\[
\|Q_{p'} f\|_X = \|K(t, Q_{p'} f; L^1, L^p)\|_G \geq cp' \|K(t, f; L^1, L^\infty)\|_G.
\]

On the other hand, on account of the fact that \( Q_{p'} \) is bounded on \( X \), we obtain
\[
\|Q_{p'} f\|_X \leq \|Q_{p'} \| \|f\|_X = \|Q_{p'} \| \|K(t, f; L^1, L^p)\|_G \leq \|Q_{p'} \| \|K(t, f; L^1, L^\infty)\|_G,
\]
and (5.8) follows.

From (5.8), [9, Theorem 5.2.1] and the boundedness of \( Q_2 \) on \( X \) we have
\[
\left\| \int_0^t (Q_2 f^*)^*(s) \, ds \right\|_G \leq C' \|Q_2 f^*\|_X \leq C' \|Q_2 \| \|f\|_X \leq C \left\| \int_0^t f^*(s) \, ds \right\|_G.
\]

Combining this estimate with the equation
\[
\int_0^t (Q_2 f^*)^*(s) \, ds = \int_0^t s^{1/2} f^*(s^{1/2}) \frac{ds}{s} = 2 \int_0^{t^{1/2}} f^*(s) \, ds,
\]
we arrive at the inequality
\[
\left\| \int_0^{t^{1/2}} f^*(s) \, ds \right\|_G \leq C \left\| \int_0^t f^*(s) \, ds \right\|_G.
\]

Equivalently, setting \( g(t) := \int_0^t f^*(s) \, ds \), we have
\[
\|Rg\|_G \leq C \|g\|_G.
\]

Hence, the latter inequality holds for every concave increasing function \( g \in G \) such that \( \lim_{t \to \infty} g(t) = 0 \). Observe that (a) holds if \( G = L^\infty \). Therefore, we can assume that \( R \), restricted to the set of all concave increasing functions from \( G \), is bounded on \( G \). Thus, proceeding in the same way as in the proof of Theorem 2, we can show that the operator \( R \) is bounded on all of \( G \).

**Remark 8.** A straightforward inspection shows that the implication (i) \( \Rightarrow \) (ii) (resp. (a) \( \Rightarrow \) (b)) of Theorem 3 (resp. Theorem 4) holds under the weaker assumptions that \( G \) is a Banach lattice on \([0, 1]\) such that \( G \supset L^\infty \cap L^\infty(1/t) \) and the operator \( Rf(t) = f(t^{1/2}) \) is bounded on \( G \). Clearly, the Banach lattice \( L^1([0, 1], \frac{dt}{t}) \) satisfies the above conditions. Therefore, since \( \langle A \rangle_{K, 0, 1}^{K, 1} = \langle A \rangle_{L^1([0, 1], \frac{dt}{t})}^{K, 1} \), then from the implication (a) \( \Rightarrow \) (b) of Theorem 4 it follows, in particular, the reiteration formula (2.20).

Observe that one of the main motivations for the reiteration theorem obtained in [38] was the following interpolation theorem of Zygmund (cf. [67]) originally stated for the periodic Hilbert transform.
Theorem 7. Let $T$ be a quasilinear operator defined on the space $L^1 := L^1(0,1)$ such that $T$ is of weak type $(1,1)$ and strong type $(p,p)$, for some $p > 1$. Then $T: L\text{Log}L \to L^1$.

Applying Theorem 6 we can extend the latter result to a rather wide class of limiting interpolation spaces close to $L^1$. Recall that $f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)\,ds$, and the space $L^{p,\infty}$, $1 \leq p < \infty$, consists of all measurable functions on $[0,1]$ such that

$$
\|f\|_{L^{p,\infty}} := \sup_{0 < t \leq 1} t^{1/p} f^*(t) < \infty.
$$

Theorem 8. Under the assumptions of Theorem 7, suppose that $\|f\|_{L^{p,\infty}} < \infty$, then $X_G := \{f : \|f\|_{L^{p,\infty}} < \infty\}$.

Remark 9. Note that if $G = L^1([0,1], \frac{dt}{t})$, then $X_G = L^1$ and $\langle L^1, L^\infty \rangle_G^K = \langle L^1, L^\infty \rangle_{0,1} = L\text{Log}L$.

Proof. By interpolation,

$$
T: \langle L^1, L^\infty \rangle_{0,1}^K \to \langle L^1, L^\infty \rangle_{0,1}^K.
$$

Since (cf. [9, Theorem 5.2.1])

$$
L^p = \langle L^1, L^\infty \rangle_{0,1}^{1/p',p},
$$

where $p' = p/(p - 1)$, it follows from Theorem 6 that

$$
\langle L^1, L^p \rangle_G^K = \langle L^1, L^{\infty} \rangle_{0,1}^{1/p',p} \cap \langle L^1, L^\infty \rangle_G
$$

$$
= \langle L^1, L^\infty \rangle_{0,1}^K
$$

$$
= \{f : \|tf^{**}(t)\|_G < \infty\}.
$$

Next we deal with the range space $\langle L^1, L^{p,\infty} \rangle_G^K$. Since $L^{1,\infty}$ is not a Banach space, we cannot apply Theorem 6 to the pair $(L^{1,\infty}, L^{p,\infty})$. Instead, we show directly that

$$
\langle L^{1,\infty}, L^{p,\infty} \rangle_G^K \subset \{f : \|tf^*(t)\|_G < \infty\},
$$

by means of proving that

$$
(5.9) \quad \|f\|_{L^{1,\infty}, L^{p,\infty}} \geq c \|tf^*(t)\|_G,
$$

using the $G$–boundedness of the operator $R_{1/p'}f(t) = f(t^{1/p'})$ restricted to the set of increasing functions. Indeed, writing $L^{p,\infty} = (L^{1,\infty}, L^\infty)_{1/p',\infty}$ (cf. [9, Theorem 5.3.1]) and applying Holmstedt’s formula (cf. [9, Corollary 3.6.2(a)]), for each $0 < t \leq 1$ we have

$$
K(t, f; L^{1,\infty}, L^{p,\infty}) \sim K(t, f; L^{1,\infty}, (L^{1,\infty}, L^\infty)_{1/p',\infty})
$$

$$
\sim t \sup_{t^{1/p'} \leq s \leq 1} s^{-1/p'} K(s, f; L^{1,\infty}, L^\infty)
$$

$$
\geq K(t^{1/p'}, f; L^{1,\infty}, L^\infty).
$$
By the proof of [(a) ⇒ (b)] in Theorem 5 we know that the operator \( R_{1/p} f(t) = f\left(t^{1/p}\right) \) restricted to the set of increasing functions is \( G \)-bounded. Consequently, since \( K(p^q, f; L^{1,\infty}, L^{\infty}) \) is increasing, there exists an absolute constant \( c > 0 \) such that

\[
\|f\|_{(L^{1,\infty}, L^{p,\infty})_G}^K = \|K(t, f; L^{1,\infty}, L^{p,\infty})\|_G \geq c \|K(t, f; L^{1,\infty}, L^{\infty})\|_G.
\]

But it is well known (cf. [64] and the references therein) that

\[
K(t, f; L^{1,\infty}, L^{\infty}) \sup_{s < t} \{sf^*(s)\} \geq tf^*(t).
\]

Hence, (5.9) follows concluding the proof.

\[\square\]

**Remark 10.** As was observed in Introduction, the theory developed in this paper provides a unified roof to a large body of literature devoted to the study of particular examples of limiting interpolation spaces. We now briefly discuss the connections with our work. Let \( \vec{A} = (A_0, A_1) \) be a Banach pair such that \( A_1 \subset A_0 \). Given \( b > 0 \) and \( 1 \leq q \leq \infty \), we define the Banach lattice \( F_{b,q} \) normed by

\[
\|f\|_{F_{b,q}} := \left( \int_0^1 \left( \frac{|f(s)|}{s(1 - \log s)^b} \right)^q \frac{ds}{s} \right)^{1/q} \text{ if } 1 \leq q < \infty,
\]

and

\[
\|f\|_{F_{b,\infty}} := \sup_{0 < s \leq 1} \left( \frac{|f(s)|}{s(1 - \log s)^b} \right)^{q}. \]

The spaces \( (\vec{A})^K_{F_{b,\infty}} \) have been introduced, using a different notation, by Cobos, Fernández-Cabrera, Manzano, and Martínez in [20, see Theorem 2.6]). Later, Cobos, Fernández-Cabrera, Kühn, and Ullrich [19], considered various properties of the family of the spaces \( F_{1,q}, 1 < q \leq \infty \). In the paper [22] one can find a more general construction of limiting interpolation spaces using powers of iterated logarithms. Let \( L_1(s) = \log s \) and \( L_j(s) = \log(L_{j-1}(s)), j > 1 \). For any \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \) we set \( \phi_{\vec{\alpha}}(s) := \prod_{j=1}^m L_j(s)^{\alpha_j}. \) Then, under suitable conditions on \( \vec{\alpha} \), we can define the Banach lattice \( F_{\vec{\alpha},q} \) consisting of all measurable functions \( f \) on \([0, c_m] \) \((c_m \in (0, 1) \text{ depends only on } m)\) such that

\[
\|f\|_{F_{\vec{\alpha},q}} := \left( \int_0^{c_m} \left( \frac{|f(s)|}{\phi_{\vec{\alpha}}(s)} \right)^q \frac{ds}{s} \right)^{1/q} \text{ if } 1 \leq q < \infty,
\]

and

\[
\|f\|_{F_{\vec{\alpha},\infty}} := \sup_{0 < s \leq c_m} \left( \frac{|f(s)|}{\phi_{\vec{\alpha}}(s)} \right)^{q}. \]

In [26], Cobos and Kühn introduced the corresponding spaces \( (\vec{A})^K_{F_{\vec{\alpha},q}} \) and showed their description by means of the J-functional.

In the papers [13, 17], the authors have considered constructions, which give limiting interpolation spaces that are close to the larger end point space of an ordered pair, i.e., to the space \( A_0 \). For any \( b \in \mathbb{R}, \) and \( 1 \leq q \leq \infty \), we define the Banach lattice \( G_{b,q} \) normed by

\[
\|f\|_{G_{b,q}} := \left( \int_0^1 \left( \frac{|f(s)|}{(1 - \log s)^b} \right)^q \frac{ds}{s} \right)^{1/q} \text{ if } 1 \leq q < \infty,
\]

and

\[
\|f\|_{G_{b,\infty}} := \sup_{0 < s \leq 1} \left( \frac{|f(s)|}{(1 - \log s)^b} \right)^{q}. \]
and
\[ \|f\|_{G_{b,∞}} := \sup_{0 < s \leq 1} \frac{|f(s)|}{(1 - \log s)^b}. \]

It is easy to see that the operator \( T_f(t) = f(t^2)/t \) (resp. \( R_f(t) = f(\sqrt{t}) \)) is bounded on the lattices \( F_{b,q} \) and \( F_{b,q}' \) (resp. \( R \) is bounded on \( G_{b,q} \)). To illustrate the issues that are involved we shall now verify that \( T \) is bounded on \( F_{b,q} \). Indeed, after a change of variables we have
\[
\|T_f\|_{F_{b,q}} = \left( \int_0^1 \left( \frac{|f(s)|}{s^2(1 - \log s)^b} \right)^q \frac{ds}{s} \right)^{1/q} = 2^{-1/q} \left( \int_0^1 \left( \frac{|f(t)|}{t(1 - \log(\sqrt{t}))^b} \right)^q \frac{dt}{t} \right)^{1/q} \leq 2^{(b-1)/q} \left( \int_0^1 \left( \frac{|f(t)|}{t(1 - \log t)^b} \right)^q \frac{dt}{t} \right)^{1/q} = 2^{(b-1)/q} \|f\|_{F_{b,q}}.
\]

Let us finally observe that some classes of limiting interpolation spaces were also introduced and studied in the case of unordered pairs (see, for instance, [21], [27], and [28]). Nevertheless, usually these spaces can be represented as intersections of the spaces obtained using the constructions described above.

6. Extrapolation description of limiting Schatten-von Neumann operator classes and a generalization of Matsaev’s theorem

We apply some of the results obtained in this paper to the Banach pair \((\ell^1, \ell^\infty) := (\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))\) (For a different approach, based on the direct estimation of the \( \ell^p \)-norms involved, we refer to [49]).

Recall that by Calderón’s theorem (cf. [8]) the pair \((\ell^1, \ell^\infty)\) is \( K \)-monotone; consequently, if \( X := X(\mathbb{N}) \) is an interpolation space with respect to \((\ell^1, \ell^\infty)\) there exists a Banach sequence lattice \( F := F(\mathbb{N}) \) such that, uniformly on \( a = \{a_k\} \in X \),
\[
\|a\|_X \asymp \left\| \left\{ K(n, \{a_k\}; \ell^1, \ell^\infty) \right\}_n \right\|_F.
\]

The corresponding \( K \)-functional is given by
\[
K(n, \{a_k\}; \ell^1, \ell^\infty) = \sum_{j=1}^n a_j^*, \quad n \in \mathbb{N}, \{a_k\} \in \ell^\infty,
\]
where \( \{a_j^*\} \) is the non-increasing rearrangement of the sequence \( \{|a_k|\} \). In other words, \( X \) is an interpolation symmetric sequence space with respect to \((\ell^1, \ell^\infty)\) if and only if there exists Banach sequence lattice \( F \) such that for all \( a = \{a_k\} \in X \)
\[
(6.1) \quad \|a\|_X \asymp \left\| \left\{ \sum_{j=1}^n a_j^* \right\}_n \right\|_F.
\]

Moreover, by Hardy, and reverse Hardy inequalities (cf. [52] Lemma 2.1) or [53] Example 7) it follows that for all \( 1 < p < \infty \)
\[
\|a\|_p \leq \|a\|_{(\ell^1, \ell^\infty)\kappa^{-1}_{\ell^p,p}} \leq \|a\|_{(\ell^\infty, \ell^1)\kappa^{-1}_{\ell^p,p}} \leq \|a\|_p,
\]
and, according to Remarks [13] and [14] for all \( 1 < p \leq 2 \)
\[
\frac{1}{2} \|a\|_{(\ell^\infty, \ell^1)\kappa^{-1}_{\ell^p,p}} \leq \|a\|_{(\ell^1, \ell^\infty)\kappa^{-1}_{\ell^p,p}} = \|a\|_{(\ell^\infty, \ell^1)\kappa^{-1}_{\ell^p,p}} \leq \|a\|_{(\ell^\infty, \ell^1)\kappa^{-1}_{\ell^p,p}}.
\]

Therefore, applying Theorem [1] (see Remark [5]) gives the following result.

\[\text{Since } \ell^1 \subseteq \ell^\infty.\]
Theorem 9. Let $X$ be a symmetric sequence space satisfying (6.1) and let the operator $S(\{a_n\}) := \{a_{n^2}\}$ be bounded on $F$. Then

$$\|a\|_X \cong \|\{\|a\|_{p(n)}\}_n\|_F,$$

where $p(n) = \frac{2\log(en)}{2\log(en) - 1}$, $n \in \mathbb{N}$.

Now, suppose that $\mathcal{H}$ is a separable complex Hilbert space. Recall that the Schatten-von Neumann class $\mathcal{S}^p$ consists of all compact operators $T : \mathcal{H} \to \mathcal{H}$ such that the norm

$$\|T\|_{\mathcal{S}^p} := \left( \sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}} < \infty,$$

where $\{s_j(T)\}_{j=1}^{\infty}$ is the non-increasing sequence of s-numbers of $T$ determined by the Schmidt expansion (cf. [36]). The Schatten-von Neumann classes belong to the larger family of two-sided symmetrically normed ideals. Let $\alpha > 0$ and let $\mathcal{M}^\alpha$ be the dual ideal to the so-called Matsaev class, i.e., the ideal of bounded compact operators in a Hilbert space $\mathcal{H}$, provided with the norm

$$\|T\|_{\mathcal{M}^\alpha} := \sup_{n \in \mathbb{N}} \sum_{j=1}^{n} s_j(T) \log^\alpha(en).$$

Then, it is known that for each $p_0 > 1$ we have

$$\|T\|_{\mathcal{M}^\alpha} \cong \sup_{1 < p < p_0} (p - 1)^\alpha \|T\|_p$$

(see [31, 5.2]). The ideal $\mathcal{M}^\alpha$ is a typical limiting interpolation space with respect to the pair $(\mathcal{S}^1, \mathcal{S}^\infty)$, where by $\mathcal{S}^\infty$ we denote the class of all compact operators on $\mathcal{H}$ with the usual operator norm. Theorem 9 allows to easily get the following general extrapolation description of the limiting interpolation spaces with respect to the pair $(\mathcal{S}^1, \mathcal{S}^\infty)$.

Theorem 10. Let $X$ be a symmetrically normed ideal of compact operators on a Hilbert space $\mathcal{H}$, and let

$$\|T\|_X \cong \left\| \left\{ \sum_{j=1}^{n} s_j(T) \right\}_n \right\|_F,$$

where $F$ is a sequence Banach lattice such that the operator $S(\{a_n\}) := \{a_{n^2}\}$ is bounded on $F$.

Then

$$\|T\|_X \cong \|\{\|T\|_{p(n)}\}_n\|_F,$$

where $p(n) = \frac{2\log(en)}{2\log(en) - 1}$, $n \in \mathbb{N}$, and $\|T\|_{p(n)}$ are the corresponding Schatten-von Neumann norms of $T$.

Proof. Applying Theorem 9 to the equivalent expression of the ideal norm of $T$ in $X$ via the sequence of s-numbers of $T$, we have

$$\|T\|_X \cong \left\| \{\|s_j(T)\|_{p(n)}\}_n \right\|_F = \left\| \{\|T\|_{p(n)}\}_n \right\|_F.$$

$\square$
As an application of Theorem 10 we present a result on the boundedness, in some appropriate symmetrically normed ideals, of the mapping which transfers the imaginary component of a Volterra operator into its real component. Recall that a Volterra operator is a compact operator whose spectrum coincides with the one-point set \( \{0\} \). Denote by \( T_R \) and \( T_J \) the real and the imaginary components of an operator \( T \), respectively, i.e.,

\[
T_R := \frac{1}{2}(T + T^*) \quad \text{and} \quad T_J := \frac{1}{2i}(T - T^*).
\]

By a well-known result of Matsaev (cf. [37, Theorem III.6.2]), if \( T \) is a Volterra operator, then for each \( 1 < p < \infty \), we have that \( T_J \in S^p \) implies that \( T_R \in S^p \).

In addition, by [37, Theorem III.6.3],

\[
\|T_R\|_p \leq \max \left\{ \frac{p}{p-1}, p \right\} \cdot \|T_J\|_p, \quad 1 < p < \infty.
\]

Moreover, if \( T_J \in S^1 \) then \( T_R \in \mathcal{M}^1 \) [37, Theorem III.2.1]. Combining the latter inequality with Theorem 10, we can obtain some information related to the behavior of both components of a Volterra operator in symmetrically normed ideals "close" to the ideal \( S^1 \).

**Theorem 11.** Suppose that a symmetrically normed ideal \( \mathcal{X} \) satisfies the conditions of Theorem 10 and \( T \) is a Volterra operator such that \( T_J \in \mathcal{X} \). Then \( T_R \in \mathcal{X}(\log^{-1}) \), where the ideal \( \mathcal{X}(\log^{-1}) \) consists of all compact operators \( U \) on \( \mathcal{H} \) that satisfy

\[
\|U\|_{\mathcal{X}(\log^{-1})} := \left\| \left\{ \frac{1}{\log(en)} \sum_{j=1}^{n} s_j(U) \right\}_n \right\|_F < \infty.
\]

**Proof.** First, we observe that the boundedness of the operator \( S(\{a_n\}) = \{a_n^2\} \) on \( F \) implies that \( S \) is also bounded on the Banach lattice of all sequences \( \{a_n\}_{n=1}^{\infty} \) such that

\[
\left\| \left\{ \frac{a_n}{\log(en)} \right\}_n \right\|_F < \infty.
\]

Indeed,

\[
\left\| \left\{ \frac{a_n^2}{\log(en)} \right\}_n \right\|_F = 2 \left\| \left\{ \frac{a_n^2}{\log((en)^2)} \right\}_n \right\|_F \leq 2 \left\| \left\{ \frac{a_n^2}{\log(en^2)} \right\}_n \right\|_F
\]

\[
= 2 \left\| S \left( \left\{ \frac{a_n}{\log(en)} \right\}_n \right) \right\|_F \leq C \left\| \left\{ \frac{a_n}{\log(en)} \right\}_n \right\|_F.
\]

Therefore, applying Theorem 10 to the ideals \( \mathcal{X} \) and \( \mathcal{X}(\log^{-1}) \), we obtain

\[
\|T\|_{\mathcal{X}} \asymp \left\| \left\{ \|T\|_{p(n)} \right\}_n \right\|_F,
\]

and

\[
\|T\|_{\mathcal{X}(\log^{-1})} \asymp \left\| \left\{ \frac{1}{\log(en)} \|T\|_{p(n)} \right\}_n \right\|_F.
\]
where, as above, \( p(n) = \frac{2\log(en)}{2\log(en) - 1} \), \( n \in \mathbb{N} \). Finally, from the hypothesis \( T_\mathcal{J} \in \mathcal{X} \) and inequality (6.2) it follows that
\[
\| T_\mathcal{R} \|_{\mathcal{X}(\log^{-1})} \leq C \left\{ \frac{1}{\log(en)} \| T_\mathcal{R} \|_{p(n)} \right\}_n \|_F \\
\leq C \left\{ \frac{1}{\log(en)} \cdot \frac{p(n)}{p(n) - 1} \| T_\mathcal{J} \|_{p(n)} \right\}_n \|_F \\
= 2C \left\{ \| T_\mathcal{J} \|_{p(n)} \right\}_n \|_F \leq C' \| T_\mathcal{J} \|_{\mathcal{X}}.
\]

\[ \square \]

7. Grand Lebesgue spaces via extrapolation

Let \( 1 < p < \infty \). The Grand Lebesgue \( L^p \) space introduced by T. Iwaniec and C. Sbordone \[\text{[41]}\], consists of all measurable functions \( f \) on \([0, 1]\) such that
\[
\| f \|_{L^p} := \sup_{0 < t < 1} (\log(e/t))^{-\frac{1}{p}} \left( \int_t^1 (f^*)(s)^p \, ds \right)^{\frac{1}{p}}.
\]

These spaces have found many applications in analysis, including the study of maximal operators, PDEs, interpolation theory, etc (see \[\text{[34, 42]}\] and the references therein). On the other hand, the expression (7.1) is somewhat difficult to work with. In this context, an important result of Fiorenza-Karadzhov \[\text{[35, Theorem 4.2]}\] gives a concrete description of the Grand Lebesgue spaces \( L^p \). The proof in \[\text{[35, Theorem 4.2]}\] is based on extrapolation methods. In this section we give a simpler proof through the use of Theorem 1 combined with elementary inequalities for rearrangements of functions.

**Theorem 12.** \[\text{[35, Theorem 4.2]}\] Let \( 1 < p < \infty \). Then, with universal constants of equivalence
\[
\| f \|_{L^p} \asymp \sup_{0 < \varepsilon < p - 1} \varepsilon^{\frac{1}{p}} \left\| f \right\|_{L^{p-\varepsilon}} < \infty.
\]

**Proof.** Let \( F \) be the Banach lattice on \([0, 1]\) equipped with the norm
\[
\| f \|_F := \sup_{0 < s \leq \varepsilon} \left\| \frac{f(s)}{s \log^{1/p}(e/s)} \right\|.
\]

Clearly, the operator \( Tf(s) = f(s^2)/s \) is bounded on \( F \). Consequently, for every ordered pair \((A_0, A_1)\), and for each continuous function \( q : (0, 1) \to [1, \infty] \), we have, by Theorem 1 (see also Remark 2),
\[
\| A \|_{\mathcal{X}(t)} \asymp \left\| t \cdot \| A \|_{\mathcal{X}_{(t)}} \right\|_F,
\]

where \( \theta(t) = 1 - \frac{1}{2\log(e/t)} \). Let \( \bar{A} \) be the pair \((L^1, L^p)\) and let
\[
\theta = 1 - \frac{\varepsilon}{(p - 1)(p - \varepsilon)}.
\]

By Holmstedt’s formula (cf. \[\text{[40, Theorem 4.3]}\]) we get
\[
\langle L^1, L^p \rangle_{\theta,p-\varepsilon} = L^{p-\varepsilon},
\]
where the equivalence constants are independent of \( \varepsilon \in (0, \varepsilon_0), \varepsilon_0 = \frac{p(p-1)}{p+1} < p - 1 \). Consider the interpolation space \( \langle L^1, L^p \rangle^K_p \). From (7.2) and (7.3) it follows that, with constants that depend only on \( p \), we have

\[
\|f\|_{\langle L^1, L^p \rangle^K_p} \leq \sup_{0 < t \leq 1} \frac{\|f\|_{\langle L^1, L^p \rangle^K_p (\theta(t), t, f)}}{\log^{1/p}(e/t)} \leq \sup_{0 < t \leq 1} (1 - \theta(t))^{\frac{1}{p}} \|f\|_{L^p(\theta(t))} ^{\frac{1}{p}}
\]

(7.4)

On the other hand, using the quasi-concavity of the \( K \)-functional and Holmstedt's formula [40, Theorem 4.1], we have

\[
\|f\|_{\langle L^1, L^p \rangle^K_p} \leq \sup_{0 < t \leq 1} \frac{K(t, f; L^1, L^p)}{t \log^{1/p}(e/t)} \leq \sup_{0 < t \leq 1/2} \frac{K(t, f; L^1, L^p)}{t \log^{1/p}(e/t)} + \sup_{0 < t \leq 1/2} \left( \frac{1}{t^{p-1}} \int_0^t f^*(s) ds \right)^{\frac{1}{p}}.
\]

We now show that the first term on the right-hand side can be estimated by the second term (see also [33, Lemma 3.1] with another proof). For this purpose we let

\[
A := \sup_{0 < t \leq 1/2} \frac{1}{t \log^{1/p}(e/t)} \int_0^t f^*(s) ds.
\]

Further, we choose \( M > 0 \) so large that \( M > (M + 1)2^{1/p} - 1 \), and pick \( t_0 \in (0, 1/2] \) such that

\[
\frac{1}{t_0 \log^{1/p}(e/t_0)} \int_0^{t_0^{2/p}} f^*(s) ds > \frac{M + 1}{2^{1/p} M} \cdot A.
\]

Then, we have

\[
\int_0^{2^p t_0^{2/p}} f^*(s) ds \leq M \int_0^{2^p t_0^{2/p}} f^*(s) ds.
\]

(7.5)

Indeed, if (7.5) does not hold, then in particular

\[
\int_0^{2^p t_0^{2/p}} f^*(s) ds \geq \frac{M}{M + 1} \int_0^{2^p t_0^{2/p}} f^*(s) ds,
\]

and since \( t_0 \leq 1/2 \) we see that

\[
\frac{1}{t_0^2 \log^{1/p}(e/t_0)} \int_0^{t_0^{2/p}} f^*(s) ds \geq \frac{2^{1-1/p} M}{(M + 1) t_0 \log^{1/p}(e/t_0)} \int_0^{t_0^{2/p}} f^*(s) ds > A,
\]

which is a contradiction.
From (7.5) and Hölder’s inequality we get

\[ A < \frac{C_p}{t_0 \log^{1/p(e/t_0)}} \int_{t_0}^{t_p} f^*(s) \, ds \leq \frac{C_p}{\log^{1/p(e/t_0)}} \left( \int_{t_0}^{t_p} f^*(s)^p \, ds \right)^{1/p} \]

\[ \leq C_p \sup_{0 < t \leq 1/2} \frac{1}{\log^{1/p(e/t)}} \left( \int_{t}^{1} f^*(s)^p \, ds \right)^{1/p}, \]

where \( C_p := 2^{1-1/p}M \). Thus,

\[ \|f\|_{(L^1, L^p)^K} \simeq \sup_{0 < t \leq 1/2} \frac{1}{\log^{1/p(e/t)}} \left( \int_{t}^{1} f^*(s)^p \, ds \right)^{1/p} \]

Combining the last equivalence with (7.4), we arrive at desired result. \(\square\)

Remark 11. It is interesting to note that the approach developed in this paper allows to get a streamlined proof of other results related to Grand and small Lebesgue spaces. In particular, the interpolation description of the spaces \(L^{p, \alpha} \), \(\alpha > 0\), endowed with the norms

\[ \|x\|_{L^{p, \alpha}} := \sup_{0 < \varepsilon < p-1} \varepsilon^{p-1} \|x\|_{p-\varepsilon} \simeq \sup_{0 < \varepsilon < 1} \log^{-\alpha/p(e/t)} \left( \int_{1}^{1} (x^*(s))^\alpha \, ds \right)^{\frac{1}{\alpha}}, \]

which was obtained in [33, Theorem 1.1 and Theorem 3.1], is an immediate consequence of Theorems 1 and 5. Indeed, proceeding as in (7.4), we obtain (see also [33, Lemma 3.2]) that

\[ L^{p, \alpha} = \langle L^1, L^p \rangle^K_F \]

where the lattice \( F := L^\infty (t^{-1} \log^{-\alpha/p(e/t)})(t) \) satisfies the condition (i) of Theorem 6. Then, from Theorem 6 (iii), it follows that for arbitrary \( r \in [1, p) \) it holds (a result first derived in [33, Theorem 3.1])

\[ L^{p, \alpha} = \langle L^r, L^p \rangle^K_F. \]

Furthermore, using the embeddings \( L^1 \supset L^{r, \alpha} \supset L^s, 1 < r < s < p \), we have (cf. [33, Theorem 1.1])

\[ L^{p, \alpha} = \langle L^r \rangle^K_F. \]

\[ \text{(1)} \] See [39] for the original definition of the spaces \( L^{p, \alpha} \) and [33] for the proof of this equivalence. It can be proved also similarly as in the proof of Theorem 12.
8. Lions-Peetre-Pisier formula and vector valued Yano’s theorem

In this section we consider the problem of proving a vector valued version of Yano’s extrapolation theorem. Our development will be based on Pisier’s approach to the following well-known result due to Lions-Peetre (cf. [29], [58] and the references therein). Let \((\Omega, \mu)\) be a measure space and let \(\tilde{A} = (A_0, A_1)\) be a Banach pair, then

\[
(L^1(\Omega, A_0), L^\infty(\Omega, A_1))^\mathbb{K}_{\theta,p_0} = L^{p_0}(\Omega, (A_0, A_1)^\mathbb{K}_{\theta,p_0}), \quad \text{with } 0 < \theta < 1, \frac{1}{p_0} = 1 - \theta.
\]

To simplify the discussion in this section we shall further assume that \(A_0 \cap A_1\) is dense in \(A_0\); then we can write (cf. [8, Proposition 5.1.15, page 303])

\[
K(t, g; \tilde{A}) = \int_0^t k(s, g; \tilde{A}) ds,
\]

where \(k(s, g; \tilde{A})\) is decreasing. As is well known (cf. [53] Example 7) for the scalar case\(^{20}\) from (5.2) and reverse Hardy inequalities we readily get\(^2\) that for \(\theta \in (0, 1), \frac{1}{p_0} = 1 - \theta\), and all \(g \in \tilde{A}^\mathbb{K}_{\theta,p_0},\)

\[
\|f\|_{\tilde{A}^\mathbb{K}_{\theta,p_0}} = c_{\theta,p_0} \left\{ \int_0^\infty \left( s^{-\theta} K(s, f; \tilde{A}) \right)^{p_0} \frac{ds}{s} \right\}^{1/p_0}
\]

\[
\leq \left\{ \int_0^\infty \left( s^{1-\theta} k(s, f; \tilde{A}) \right)^{p_0} \frac{ds}{s} \right\}^{1/p_0},
\]

where the universal constants of equivalence on the right-hand side are independent of \(\theta\). As a consequence,

\[
\|f\|_{L^{p_0}(\Omega, (A_0, A_1)^\mathbb{K}_{\theta,p_0})} \leq \left\{ \int_{\Omega} \left\{ \int_0^\infty \left( s^{1-\theta} k(s, f(w); \tilde{A}) \right)^{p_0} \frac{ds}{s} \right\} d\mu(w) \right\}^{1/p_0}.
\]

Pisier’s method makes it possible to relate \(8.4\) to the \(K\)-functional for the pair \((L^1(\Omega, A_0), L^\infty(\Omega, A_1))\), and crucially for our purposes, allows us to keep track of the constants in the intervening inequalities. Pisier’s formula for the \(K\)-functional for the pair \((L^1(\Omega, A_0), L^\infty(\Omega, A_1))\) is given by

\[
K(t,f;L^1(\Omega, A_0), L^\infty(\Omega, A_1)) = \sup_{\phi \geq 0, \|\phi\|_{L^1} \leq t} \int_{\Omega} K(\phi(w), f(w); \tilde{A}) d\mu(w).
\]

Given \(f \in L^1(\Omega, A_0) + L^\infty(\Omega, A_1)\), we let \(\Psi_f : \Omega \times (0, \infty) \to \mathbb{R}_+\), be defined by \(\Psi_f(w, s) = k(s, f(w); \tilde{A})\), \((w, s) \in \Omega \times (0, \infty)\). Pisier’s method is based on reinterpreting formula \(8.5\) as follows

**Proposition 3.** (Pisier [58])

\[
K(t,f;L^1(\Omega, A_0), L^\infty(\Omega, A_1)) = K(t, \Psi_f; L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty))).
\]

\(^{20}\)Actually letting \(f = k(s, g; \tilde{A})\), it follows that \(f^{**}(t) = \frac{1}{t} \int_0^t k(u, g; \tilde{A}) du\) and one can apply the argument in [53] Example 7] verbatim.

\(^2\)cf. (2.11) above for the definition of \(c_{\theta,p_0}\).
Proof. It will be useful to give complete details of Pisier’s proof of (8.6). First, by Fubini’s theorem, we have
\[
(L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty))) = (L^1(\Omega, L^1(0, \infty)), L^\infty(\Omega, L^\infty(0, \infty))).
\]
Therefore we can use (8.5), to write
\[
\theta, \text{norm equivalence independent of formulae for the pair (Fubini's theorem, we have)}
\]
\[
The desired result now follows comparing (8.7) and (8.8).
\]
\[
K(t, \Psi_f; L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty)))
\]
\[
= K(t, \Psi_f; L^1(\Omega, (L^1(0, \infty)), L^\infty(\Omega, (L^\infty(0, \infty)))
\]
\[
= \sup_{\phi \geq 0, \|\phi\|_{L^1} \leq t} \int_0^\infty \int_0^\phi (\psi(w), \psi_s(w); L^1(0, \infty), L^\infty(0, \infty))d\mu(w).
\]
Now, by the known Peetre-Oklander formula for the $K-$functional of the pair $(L^1(0, \infty), L^\infty(0, \infty))$, the fact that in the representation of a $K-$functional given by (8.2), $k(s, g; \tilde{A})$ is decreasing, we can continue with
\[
= \sup_{\phi \geq 0, \|\phi\|_{L^1} \leq t} \int_0^\infty \int_0^\phi \psi(w,s)d\mu(w)
\]
\[
= \sup_{\phi \geq 0, \|\phi\|_{L^1} \leq t} \int_0^\phi \int_0^\phi (\psi(\phi, f(w); \tilde{A})d\mu(w) (by (8.2))
\]
\[
= K(t, f; L^1(\Omega, A_0), L^\infty(\Omega, A_1)). (by (8.5)).
\]
\[
\Box
\]
As a consequence, we obtain the Lions-Peetre-Pisier formula with constants:

**Corollary 2.** $(L^1(\Omega, A_0), L^\infty(\Omega, A_1))^{1\theta} = L^p(\Omega, (A_0, A_1))^{1\theta}_{p_0}$.

**Proof.** Our only contribution here is that we keep track of the constants to be able to introduce the symbol $\blacktriangle$ in the formula. Observe that by the usual interpolation formulae for the pair $(L^1, L^\infty)$ (cf. also (8.3)) and Fubini’s theorem, we have, with norm equivalence independent of $\theta$,
\[
(L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty)))^{1\theta} = L^{p_\theta}(\Omega \times (0, \infty)) = L^{p_\theta}(\Omega, L^{p_\theta}(0, \infty)).
\]

We combine this fact with (8.1) to find that
\[
\|f\|_{(L^1(\Omega, A_0), L^\infty(\Omega, A_1))^{1\theta}}^{1\theta}_{p_0}
\]
\[
= c_{p_\theta} \left\{ \int_0^\infty \left( s^{-\theta} K(s, \Psi_f; L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty))) \right)^{p_0} \frac{ds}{s} \right\}^{1/p_0}
\]
\[
= \|\Psi_f\|_{L^{p_\theta}(\Omega, L^{p_\theta}(0, \infty))}
\]
\[
(8.7)
\]
\[
= \left( \int_\Omega \left\{ \int_0^\infty (s^{1-\theta} \Psi_f(s, w)^{p_0} \frac{ds}{s}) \right\} d\mu(w) \right)^{1/p_0} (\text{since (1 - \theta)p_0 = 1}).
\]

Using the definition of $\Psi_f$ we see that (8.4) states that
\[
(8.8) \quad \|f\|_{L^{p_\theta}(\Omega, (A_0, A_1))^{1\theta}}^{1\theta}_{p_0} \leq \left( \int_\Omega \left\{ \int_0^\infty (s^{1-\theta} \Psi_f(s, w)^{p_0} \frac{ds}{s}) \right\} d\mu(w) \right)^{1/p_0}.
\]

The desired result now follows comparing (8.7) and (8.8). $\Box$
Obviously the same method can be used to find versions of (8.1) that are valid in the limiting cases.

**Corollary 3.** Let $F$ be a lattice on $[0, 1]$. Then

$$
\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_F = \{ f : \Psi_f \in (L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty))) \}.
$$

isometrically, i.e.

$$
\| f \|_{\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_F} = \| \Psi_f \|_{(L^1(\Omega \times (0, \infty)), L^\infty(\Omega \times (0, \infty)))}.
$$

In the next set of results we use Example 1 from Introduction; so it will be useful to provide the details here.

**Example 3.** For any measure space $(\Omega, \mu)$ we have

$$(8.9) \sum \frac{1}{\theta} (L^1(\Omega), L^\infty(\Omega))_\theta^K = (L^1(\Omega), L^\infty(\Omega))_0^K = L(\log L)(\Omega) + L^\infty(\Omega).$$

**Proof.** Since $K(t, f; L^1, L^\infty) = tf^{\ast \ast}(t)$, it follows from (2.22) that

$$
\sum \frac{1}{\theta} (L^1(\Omega), L^\infty(\Omega))_\theta^K = \{ f : \int_0^1 f^{\ast \ast}(s) \, ds \} < \infty.
$$

On the other hand, the $K$–functional for the pair $(L(\log L)(\Omega), L^\infty(\Omega))$ is given by (cf. [7])

$$
K(t, f; L(\log L), L^\infty) \asymp \| f^{\ast \ast}(0, \phi^{-1}(t)) \|_{L(\log L)},
$$

where $\phi^{-1}(t)$ is the inverse of the fundamental function of $L(\log L)$. Without loss of generality we can modify $\phi$ so that $\phi^{-1}(1) = 1$. It follows that

$$
\| f \|_{L(\log L) + L^\infty} = K(1, f; L(\log L), L^\infty)
\asymp \| f^{\ast \ast}(0, 1) \|_{L(\log L)}
\asymp \int_0^1 f^{\ast}(s)(1 + \log \frac{1}{s}) \, ds
\asymp \int_0^1 f^{\ast}(s) \, ds + \int_0^1 f^{\ast \ast}(s) \, ds
\asymp \int_0^1 f^{\ast \ast}(s) \, ds.
$$

□

**Example 4.** Let $w_\alpha(t) = (\log \frac{t}{s})^\alpha$, $\alpha > 0$, and let $F_\alpha = L^1((0, 1), w_\alpha(t) \, dt)$, then

$$
\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_{F_{\alpha - 1}} = \{ f : \Psi_f \in L(\log L)^\alpha(\Omega \times (0, \infty)) + L^\infty(\Omega \times (0, \infty)) \},
$$

and

$$
\| f \|_{\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_{F_{\alpha - 1}}} \asymp \| \Psi_f \|_{L(\log L)^\alpha(\Omega \times (0, \infty)) + L^\infty(\Omega \times (0, \infty))}
= \| k(s, f(w); \tilde{A}) \|_{L(\log L)^\alpha(\Omega \times (0, \infty)) + L^\infty(\Omega \times (0, \infty))}.
$$
Proof. For any measure space Θ, and for all α > 0, a slight generalization of Example 3 yields that
\[
\|g\|_{L(\log L)^{\alpha}(\Theta)} + L^\infty(\Theta) = K(1, g; L(\log L)^{\alpha}(\Theta), L^\infty(\Theta))
\]
(8.10)
By Corollary 2
\[
\int_0^1 K(s, g; L^1(\Theta), L^\infty(\Theta)) \log^{\alpha-1} \frac{1}{s} \, ds
\]
= \|g\|_{L^1(\Theta), L^\infty(\Theta)} \xi_{\alpha-1}.

Let \( f \in \langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_{F_{\alpha-1}} \). Then the desired result obtains letting Θ = \( \Omega \times (0, \infty) \), and \( g = \Psi_f \).

Example 5. Let \( \tilde{A} \) be an ordered pair. Suppose that \( \mu(\Omega) < \infty \). Then
\( \langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle \) is an ordered pair, and since \( k(s, g; \tilde{A}) = 0 \) for \( s > 1 \), using the notation of the previous example we can write
\[
\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_{\tilde{F}_{\alpha-1}} = \{ f : \Psi_f \in L((\log L)^{\alpha}(\Omega \times (0, 1))) + L^\infty(\Omega \times (0, 1)) \}.
\]
Since in this case we have that
\[
L((\log L)^{\alpha}(\Omega \times (0, 1))) + L^\infty(\Omega \times (0, 1)) = L((\log L)^{\alpha}(\Omega \times (0, 1))),
\]
it follows that
\[
\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle^K_{\tilde{F}_{\alpha-1}} = \{ f : \Psi_f \in L((\log L)^{\alpha}(\Omega \times (0, 1))) \}.
\]
Now, we show how the results of this section can be used to extend the classical extrapolation theorems of Yano. A more detailed exposition will be given elsewhere (cf. [10]). To simplify the discussion we work with finite measure spaces. From the point of view of extrapolation theory Yano’s theorem is simply the statement that (there is an analogous formula for the \( \Delta \)-method, which will not be considered here)
\[
\sum_{p>1} \frac{L^p(\Omega)}{(p-1)\alpha} = L((\log L)^{\alpha}(\Omega)).
\]

Theorem 13. Let \( \tilde{A} \) be an ordered pair, and for \( f \in L^1(\Omega, A_0) + L^\infty(\Omega, A_1) \) let \( \Psi_f(s, w) = k(s, f(w); \tilde{A}) \), \( (s, w) \in \Omega \times (0, 1) \). Then,
\[
\sum_{\theta} \frac{L^p(\Omega, (A_0, A_1)_{\theta,p})}{\theta^\alpha} = \{ f : \Psi_f \in L((\log L)^{\alpha}(\Omega \times (0, 1))) \}.
\]
Proof. By Corollary 2
\[
\sum_{\theta} \frac{L^p(\Omega, (A_0, A_1)_{\theta,p})}{\theta^\alpha} = \sum_{\theta} \frac{\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle_{\theta,p}}{\theta^\alpha}
\]
Now by [10]
\[
\sum_{\theta} \frac{\langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle_{\theta,p}}{\theta^\alpha} = \langle L^1(\Omega, A_0), L^\infty(\Omega, A_1) \rangle_{\tilde{F}_{\alpha-1}}
\]
and by Example 5 we can continue with
\[
\{ f : \Psi_f \in L((\log L)^{\alpha}(\Omega \times (0, 1))) \},
\]
as we wished to show.
9. Further applications

9.1. Weights and $K/J$ equivalence. There is a simple mechanism underlying (2.15) that seems worthwhile to discuss in detail. Let $w$ be a weight, $w : (0, \infty) \rightarrow [0, \infty)$ (e.g., $w(s) = \chi_{(0,1)}(s)$), and let $\vec{A}$ be a Gagliardo complete pair of Banach spaces. Define

$$\vec{A}^K_w := \{ f : f \in A_0 + A_1 \text{ s.t. } \| f \|_{\vec{A}^K} < \infty \},$$

where

$$\| f \|_{\vec{A}^K} := \int_0^\infty K(s, f; \vec{A}) w(s) \frac{ds}{s}.$$ 

Likewise, we define

$$\vec{A}^J_w := \{ f = \int_0^\infty u(s) \frac{ds}{s}, \text{ with } u : (0, \infty) \rightarrow A_0 \cap A_1, \| f \|_{\vec{A}^J} < \infty \},$$

where

$$\| f \|_{\vec{A}^J} := \inf_{f = \int_0^\infty u(s) \frac{ds}{s}} \left\{ \int_0^\infty J(s, u(s); \vec{A}) w(s) \frac{ds}{s} \right\}.$$ 

Lemma 2. Let $w$ be a nonnegative locally summable function on $(0, \infty)$ such that $\int_0^\infty w(s) ds = \infty$. For each Gagliardo complete pair $\vec{A}$ such that $A_0 \cap A_1$ is dense in $A_0$ it holds

$$\vec{A}^K_w = \vec{A}^J_w,$$

where

$$\tilde{w}(t) = \int_0^\infty \min\{1, \frac{s}{t}\} w(s) \frac{ds}{s}.$$ 

Proof. Let $f \in \vec{A}^J_w$. We can represent $f = \int_0^\infty u(s) \frac{ds}{s}$ in such a way that

$$\| f \|_{\vec{A}^J} \geq \int_0^\infty J(s, u(s); \vec{A}) \tilde{w}(s) \frac{ds}{s}.$$ 

Using Minkowski’s inequality, and the fact that (cf. [9 Lemma 3.2.1 page 42]), $K(t, a; \vec{A}) \leq \min\{1, \frac{t}{s}\} J(s, a; \vec{A})$, we get

$$K(t, f; \vec{A}) \leq \int_0^\infty \min\{1, \frac{t}{s}\} J(s, u(s); \vec{A}) \frac{ds}{s}.$$ 

Integrating the last inequality and then using successively Fubini’s theorem, and the definition of $\tilde{w}$, we find,

$$\int_0^\infty K(t, f; \vec{A}) \tilde{w}(t) \frac{dt}{t} \leq \int_0^\infty \int_0^\infty \min\{1, \frac{t}{s}\} \int_0^\infty J(s, u(s); \vec{A}) \frac{ds}{s} \tilde{w}(t) \frac{dt}{t}$$

$$= \int_0^\infty J(s, u(s); \vec{A}) \int_0^\infty \min\{1, \frac{t}{s}\} \tilde{w}(t) \frac{dt}{t} \frac{ds}{s}$$

$$= \int_0^\infty J(s, u(s); \vec{A}) \tilde{w}(s) \frac{ds}{s}$$

$$\leq \| f \|_{\vec{A}^J_w}.$$
On the other hand, suppose that \( f \in \tilde{A}^K_w \). Observe that for all \( t > 0 \) we have
\[
\int_0^\infty K(s, f; \tilde{A}) w(s) \frac{ds}{s} \geq \int_0^t K(s, f; \tilde{A}) w(s) \frac{ds}{s} \geq \frac{K(t, f; \tilde{A})}{t} \int_0^t w(s) \, ds.
\]
Hence, since \( \int_0^\infty w(s) \, ds = \infty \) and \( A_0 \cap A_1 \) is dense in \( A_0 \), we deduce that
\[
\lim_{t \to 0} K(t, f; \tilde{A}) = \lim_{t \to \infty} \frac{K(t, f; \tilde{A})}{t} = 0.
\]
Consequently, by the strong form of the fundamental lemma (cf. [30] and the references therein), we can find a representation \( f = \int_0^\infty \tilde{u}(s) \frac{ds}{s} \), such that
\[
\int_0^\infty \min\{1, \frac{t}{s}\} J(s, \tilde{u}(s); \tilde{A}) \frac{ds}{s} \leq \gamma K(t, f; \tilde{A}),
\]
where \( \gamma \) is an absolute constant. Multiplying the last inequality by \( w(t) \) and then integrating, by Fubini’s theorem, we see that
\[
\int_0^\infty J(s, \tilde{u}(s); \tilde{A}) \tilde{w}(s) \frac{ds}{s} \leq \gamma \|f\|_{\tilde{A}^K_w}.
\]
Thus,
\[
\|f\|_{\tilde{A}^\theta \tilde{w}} \leq \gamma \|f\|_{\tilde{A}^\theta \tilde{w}^{\delta_1}},
\]
as we wished to show. \(\square\)

In particular, as have pointed out above, for ordered pairs \( \tilde{A} \) we may restrict the weights to be defined on the interval \((0, 1)\). Then \( \tilde{w}(t) = \int_0^1 \min\{1, \frac{t}{s}\} w(s) \frac{ds}{s} \), \( t \in (0, 1) \), and we have
\[
\langle \tilde{A} \rangle^K_w = \langle \tilde{A} \rangle^\delta_{\tilde{w}}.
\]
For example, if \( w \equiv 1 \), then \( \tilde{w}(t) = 1 + \log \frac{t}{e} \), \( t \in (0, 1) \). More generally, if \( w_\alpha(t) = (\log \frac{t}{e})^\alpha \), \( \alpha > -1 \), then \( \tilde{w}(t) \asymp (\log \frac{t}{e})^{\alpha + 1} \), and with norm equivalence we have
\[
\langle \tilde{A} \rangle^K_{w_\alpha} = \langle \tilde{A} \rangle^\delta_{w_{\alpha + 1}}.
\]
(see Remark 3).

**Example 6.** The following extension of (2.17) holds (cf. [43])
\[
\|f\|_{\tilde{A}^\theta w^{\delta_1}} \leq C \theta (1 - \theta) \|f\|_{\tilde{A}^\theta \tilde{w}}^{\delta_1} \|f\|_{\tilde{A}^\theta \tilde{w}}^{1/\alpha}, \quad f \in A_1.
\]

Inequalities of this type specialize to well-known inequalities in Analysis (cf. [11], [12]).

### 9.2. Limits of interpolation spaces and extrapolation

In this section we show how limit theorems of the form (2.12), (2.13) can be combined with the strong form of the fundamental lemma to prove extrapolation theorems. For more general results see [6].

Firstly, observe that from the strong form of the fundamental lemma precisely as in the proof of embedding (4.18) above (cf. [43]) it follows
\[
\|a\|_{\langle \tilde{A} \rangle^\delta_{\theta^\alpha_1}} \leq C_0 (1 - \theta) \|a\|_{\langle \tilde{A} \rangle^K_{\theta^\alpha_1}}, \quad (9.1)
\]
Theorem 14. (Abstract form of Yano’s theorem). Let \( \tilde{A} \) be an ordered Gagliardo complete pair. Suppose that \( T \) is a bounded linear operator such that

\[
T : \langle \tilde{A} \rangle^J_{\theta,1} \to \langle \tilde{A} \rangle^J_{\theta,1}, \quad \|T\|_{\langle \tilde{A} \rangle^J_{\theta,1} \to \langle \tilde{A} \rangle^J_{\theta,1}} \leq \frac{C}{\theta}, \quad 0 < \theta < 1.
\]

Then,

\[
T : \langle \tilde{A} \rangle^K_{0,1} \to A_0.
\]

**Proof.** Let \( f \in A_1 \). From the assumptions combined with the inequality (9.1), we have that, for all \( \theta \in (0, 1) \),

\[
\|Tf\|_{\langle \tilde{A} \rangle^J_{\theta,1}} \leq CC_0 \theta (1 - \theta) \|f\|_{\langle \tilde{A} \rangle^K_{0,1}}.
\]

Now let \( \theta \to 0 \), and compute the limits using (2.13) to find

\[
\|Tf\|_{A_0} \leq \lim_{\theta \to 0} \|Tf\|_{\langle \tilde{A} \rangle^J_{\theta,1}} \leq CC_0 \lim_{\theta \to 0} \|f\|_{\langle \tilde{A} \rangle^K_{0,1}} \leq 2CC_0 \|f\|_{\langle \tilde{A} \rangle^K_{0,1}}.
\]

Thus, we have obtained the desired inequality under the assumption that \( f \in A_1 \), but this restriction can be eliminated on account of the fact that \( A_1 \) is dense in \( \langle \tilde{A} \rangle^K_{0,1} \) (cf. Lemma 2).

\[\square\]

9.3. Limiting Spaces with “broken logarithms”. In the literature of limiting spaces one finds the so called “spaces with broken logarithms”. In this section we wish to point out how spaces of this type appear in extrapolation theory. We shall be brief and refer to [5] and [43] for details.

Let \( \tilde{A} \) be a Gagliardo complete Banach pair, and let \( M(\theta) \) be a weight defined on \((0, 1)\). One can consider very general classes of weights (cf. [5]) but here we shall only discuss the following example

\[
M(\theta) = \theta^{-\alpha_0} (1 - \theta)^{-\beta_0}, \quad \alpha_0, \beta_0 \geq 0.
\]

It is then shown in [43, Corollary 3.5, page 24, Example 3.15 page 31-32] that if \( T \) is a bounded linear operator, such that \( T : \tilde{A}^J_{\theta,1} \to \tilde{A}^K_{\theta,\infty} \), with norm \( \|T\|_{\tilde{A}^J_{\theta,1} \to \tilde{A}^K_{\theta,\infty}} \leq CM(\theta) \), then

\[
K(t, Tf; \tilde{A}) \leq C' \int_0^\infty \left[ \left( \log^+ \frac{t}{s} \right)^{\alpha_0 - 1} + \left( \log^+ \frac{s}{t} \right)^{\beta_0 - 1} \right] K(s, f; \tilde{A}) \frac{ds}{s},
\]

This is also directly connected with the strong form of the fundamental lemma and with the equivalence [43, Corollary 3.5, page 24]

\[
\sum M(\theta) \tilde{A}^K_{\theta,q} = \sum M(\theta) \tilde{A}^J_{\theta,q},
\]

and where \( \mu \) is the measure representing \( \tau(t) : = \inf_{0 < \theta < 1} \{ M(\theta)t^\theta \} \), that is,

\[
\tau(t) = \inf_{0 < \theta < 1} \{ M(\theta)t^\theta \} = \int_0^\infty \min \{ 1, \frac{t}{r} \} d\mu(r), \quad 0 < t < \infty.
\]

This discussion provides a motivation for the interest in the spaces with “broken powers” and “broken logarithms” defined in the literature.
9.4. Limits, recovery of end points, and extrapolation. One implication of (2.12), (2.13) is that, for the real methods, we can reverse the process of interpolation in the sense that if we know the intermediate norms then we can recover the initial norms by taking suitable limits. On the other hand, the $\Delta-$ and $\Sigma-$extrapolation functors can be also used to recover the $K$- and $J$-functionals, from where we can, once again, recover the initial norms. These methods can be used to complement some classical inequalities in Analysis.

Let $\{ F_\theta \}_{\theta \in (0,1)}$ be a family of interpolation functors, which are exact of exponent $\theta$ (as examples of such families we mention the normalized real interpolation methods $\{ [\cdot, \cdot]_{\theta,p} \}_{\theta \in (0,1)}$, $\{ [\cdot, \cdot]_{\theta,q} \}_{\theta \in (0,1)}$, as well as the complex method $\{ [\cdot, \cdot] \}_{\theta \in (0,1)}$), then (cf. [43, page 15]) for all Gagliardo complete pairs $\tilde{A}$, $t > 0$, $f \in A_0 \cap A_1$, we have, uniformly,

\begin{equation}
\| f \|_{\Delta(t^s F_\theta(\tilde{A}))} \simeq J(t, f ; \tilde{A}),
\end{equation}

where by definition $\| \cdot \|_{t^s F_\theta(\tilde{A})} = t^s \| \cdot \|_{F_\theta(\tilde{A})}$. Therefore,

\[ \lim_{t \to 0} \frac{\| f \|_{\Delta(t^s F_\theta(\tilde{A}))}}{t} = \| f \|_{A_0}, \quad \lim_{t \to \infty} \frac{\| f \|_{\Delta(t^s F_\theta(\tilde{A}))}}{t} \simeq \| f \|_{A_1}. \]

Likewise, we can “recover” the $K-$functional as follows (cf. [43, page 15])

\begin{equation}
\| f \|_{\sum(t^s F_\theta(\tilde{A}))} \simeq K(t, f ; \tilde{A}).
\end{equation}

As a consequence we deduce that

\[ \lim_{t \to \infty} \frac{\| f \|_{\sum(t^s F_\theta(\tilde{A}))}}{t} = \| f \|_{A_0}, \quad \lim_{t \to 0} \frac{\| f \|_{\sum(t^s F_\theta(\tilde{A}))}}{t} \simeq \| f \|_{A_1}. \]

From the previous discussion we see that if the pair $\tilde{A}$ is ordered then, letting $t = 1$ in (9.2) and (9.3), we have

\[ \| f \|_{\Delta(F_\theta(\tilde{A}))} \simeq \| f \|_{A_1}, \quad \| f \|_{\sum(F_\theta(\tilde{A}))} \simeq \| f \|_{A_2}. \]

As an application we show a connection to the celebrated Bourgain-Brezis-Mironescu-Maz’ya-Shaposhnikova Sobolev-Besov inequalities (cf. [10], [46], [59], and the references therein).

Example 7. For more background information concerning this example we refer to [8], [47]. For succinctness for each $(s, r) \in (0, 1) \times (1, \infty)$ we let $W^{s,r}(\mathbb{R}^n) := (L^r(\mathbb{R}^n), W^{1,r}(\mathbb{R}^n))^{K_{s,r}}$. Then, from (9.3) we see that for all $t > 0$,

\[ \| f \|_{\sum(t^s W^{s,r}(\mathbb{R}^n))} \simeq K(t, f ; L^{r}(\mathbb{R}^n), W^{1,r}(\mathbb{R}^n)) \simeq w_{r,f}(t), \]

where $w_{r,f}(t)$ is the $r-$modulus of continuity of $f$ at $t$ (cf. Introduction and Exercise 13 (b), page 431]). As another simple application, let $p, q \in (1, \infty)$ be fixed and suppose that $T$ is a bounded linear operator such that for all $s \in (0, 1)$,

\[ T : W^{s,p}(\mathbb{R}^n) \to W^{s,q}(\mathbb{R}^n), \quad \| T \|_{W^{s,p} \to W^{s,q}} \leq C, \quad \text{independent of } s. \]

Then, from (9.3) (i.e., by extrapolation) we have

\begin{equation}
\| w_{q,Tf}(t) \| \leq C \| w_{p,f}(t) \|,
\end{equation}

where $C$ is an absolute constant.
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