Abstract. The Cox ring of a so-called Mori Dream Space (MDS) is finitely generated and it is graded over the divisor class group. Hence the spectrum of the Cox ring comes with an action of an algebraic torus whose GIT quotient is the variety in question. We present the associated description of this Cox ring as a polyhedral divisor in the sense of [AH]. Via the shape of its polyhedral coefficients, it connects the equivariant structure of the Cox ring with the world of stable loci and stable multiplicities of linear systems.

1. Introduction

Let \( Z \) be a \( \mathbb{Q} \)-factorial projective variety defined over the field of complex numbers such that its divisor class group \( \text{Cl}(Z) \) is a lattice that is a free abelian, finitely generated group. We consider the Cox ring of \( Z \)

\[
\text{Cox}(Z) = \bigoplus_{D \in \text{Cl}(Z)} \Gamma(Z, \mathcal{O}(D))
\]

with multiplicative structure defined by a choice of divisors whose classes form a basis of \( \text{Cl}(Z) \). Our standing assumption in this paper is the finite generation of the \( \mathbb{C} \)-algebra \( \text{Cox}(Z) \). We will call such \( Z \) a Mori Dream Space (or MDS) as it was baptized by Hu and Keel in [HK]. We note that a somewhat more general definition of MDS, without the \( \mathbb{Q} \) factoriality assumption, was developed by Artebani, Hausen and Laface, see [AHL, Thm. 2.3]. However, \( \mathbb{Q} \)-factoriality of \( Z \) is a part of our set up in the present paper.

The \( \text{Cl}(Z) \)-grading of \( \text{Cox}(Z) \) yields an algebraic action of the associated torus \( \text{Hom}_Z(\text{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{	ext{rk}(\text{Cl}(Z))} \) on the affine variety \( \text{Spec}(\text{Cox}(Z)) \). The variety \( Z \) is a GIT quotient of \( \text{Spec}(\text{Cox}(Z)) \) by the action of this torus. More precisely, a choice of an ample divisor on \( Z \) determines an open subset of \( \text{Spec}(\text{Cox}(\langle Z \rangle)) \) such that \( Z \) is a good geometric quotient of this set, see [HK, Prop. 2.9].

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Affine varieties with an algebraic torus action were dealt with by Altmann and Hausen, [AH], who introduced the notion of polyhedral divisors, or p-divisors. Every normal, affine variety $X$ with an algebraic torus action can be described in terms of a polyhedral divisor $D = \sum_i \Delta_i \otimes D_i$ over its Chow quotient $Y$, [AH, Thm. 3.4]. Alternatively, such a p-divisor can be interpreted as a convex, fanwise linear (i.e. piecewise linear and homogeneous, defined on a cone) map from the character lattice $M$ of the torus to $\text{CaDiv}_\mathbb Q(Y)$. See (2.1) for more details. Note that, by abuse of notation, we use the word “Chow quotient” for the normalization of the distinguished component of the inverse limit of the GIT quotients of $X$, cf. [AH, Sect. 6], [Hu].

We apply this formalism to treat the case of $X = \text{Spec}(\text{Cox}(Z))$ for $Z$ as above. Although, in general, the structure of the Chow quotient $Y$ is rather obscure, our main result, Theorem 11, asserts that the associated p-divisor is supported on a finite number of exceptional divisors $D_i$ with polyhedral coefficients $\Delta_i$ described clearly in terms of stabilized multiplicities with respect to these divisors:

$$\Delta_i = \{ C \in \text{Cl}^*(Z)_\mathbb Q \mid C \geq - \text{mult}^\text{st}_{D_i} \} + \text{shift}.$$ 

Thus, polyhedral divisors provide an alternative view of the stabilized base point loci and the asymptotic order of the vanishing of linear series on $Z$, as defined by Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa, [ELMNP].

The composition of the p-divisor associated to $\text{Cox}(Z)$, treated as a fanwise linear map $D: M_\mathbb Q = \text{Cl}_\mathbb Q(Z) \supset \text{Eff}(Z) \to \text{CaDiv}_\mathbb Q(Y)$, with the divisor class map $\text{CaDiv}_\mathbb Q(Y) \to \text{Pic}_\mathbb Q(Y)$ (dividing by $\mathbb Q$-principal divisors), maps the cone of effective divisors on $Z$, denoted by $\text{Eff}(Z)$, to the cone $\text{Nef}(Y)$ of nef (in this case also semiample) divisors on $Y$. In Corollary 12 we show that it is a composition of two other maps $\text{Cl}_\mathbb Q(Z) \supset \text{Eff}(Z) \to \text{Cl}_\mathbb Q(Z) = \text{Pic}_\mathbb Q(Z) \to \text{Pic}_\mathbb Q(Y)$. First, one performs a retraction of $\text{Eff}(Z)$ to the cone of movable divisors $\text{Mov}(Z)$ which is a union of cones $\text{Nef}(Z_i)$ for $Z_i$ being different GIT quotients of $\text{Cox}(Z)$. Second, the chambers $\text{Nef}(Z_i)$ are mapped to faces of $\text{Nef}(Y)$ by pulling the divisors back along the natural morphisms $Y \to Z_i$.

Our starting point, however, is the toric case where both the Chow quotient of $\text{Cox}(Z)$ and the p-divisor can be described explicitly. We discuss this in Section 3 right after the introductory Section 2 where we recall the language of p-divisors. The main toric result, Theorem 7, is obtained by explicit methods. In the subsequent Section 4, we rephrase it by using dual polyhedra and the associated fanwise linear functions. These easy observations lead us to the relation to multiplicities of divisors in base point loci of linear systems forming the core of the proof of Theorem 11. This is contained in Section 5 where we also recall the basic information about MDS.

Finally, in Section 6 we discuss the surface case and provide some further examples. If $Z = S$ with $\dim S = 2$, then the Chow limit $Y$ coincides with $S$. So the p-divisor defines a retraction $\text{Eff}(S)$ to $\text{Nef}(S)$ reflecting the Zariski decomposition on $S$. It is linear on the Zariski chambers, as defined in [BKS]. The coefficients of the p-divisor
on an MDS surface are presented in Theorem 13. For a del Pezzo surface $S$ they look particularly nice:

**Corollary 14.** If $S$ is a del Pezzo with $E_i \subseteq S$ denoting their exceptional curves, then the $p$-divisor encoding $\text{Cox}(S)$ equals $D = \text{id}_{\text{C}(S)} + \sum_i (0E_i + \text{Nef}(S)) \otimes E_i$.

2. The language of $p$-divisors

2.1. **Definition of $p$-divisors.** We start recalling the basic notions of [AH]. Let $T$ be an affine torus over a field of complex numbers $\mathbb{C}$. It gives rise to the mutually dual free abelian groups, or lattices, $M := \text{Hom}_{\text{algGrp}}(T, \mathbb{C}^*)$ and $N := \text{Hom}_{\text{algGrp}}(\mathbb{C}^*, T)$. The pairing of dual lattices (or, also, dual vector spaces) will be denoted by $\langle \cdot, \cdot \rangle$. Via $T = \text{Spec} \mathbb{C}[M] = N \otimes \mathbb{Z} \mathbb{C}^*$, the torus can be recovered from these lattices. Denote by $M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$ and $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ the corresponding vector spaces over $\mathbb{Q}$ (the same notation will be used whenever we extend a lattice to a $\mathbb{Q}$-vector space).

**Definition 1.** If $\sigma \subseteq N_\mathbb{Q}$ is a polyhedral cone, then we denote by $\text{Pol}(N_\mathbb{Q}, \sigma)$ the Grothendieck group of the semigroup

$$\text{Pol}^+(N_\mathbb{Q}, \sigma) := \{ \Delta \subseteq N_\mathbb{Q} \mid \Delta = \sigma + [\text{compact polytope}] \}$$

with respect to Minkowski addition. Via $a \mapsto a + \sigma$, the latter contains $N_\mathbb{Q}$. Moreover, $\text{tail}(\Delta) := \sigma$ is called the tail cone of the elements of $\text{Pol}(N_\mathbb{Q}, \sigma)$.

Let $Y$ be a normal and semiprojective (i.e. $Y \to Y_0$ is projective over an affine $Y_0$) $\mathbb{C}$-variety. By $\text{CaDiv}(Y)$ and $\text{Div}(Y)$ we denote the group of Cartier and Weil divisors on $Y$ with linear equivalence groups by $\text{Pic}(Y)$ and $\text{Cl}(Y)$, respectively. A $\mathbb{Q}$-Cartier divisor on $Y$ is called *semiample* if a multiple of it becomes base point free.

**Definition 2.** An element $D = \sum_i \Delta_i \otimes D_i \in \text{Pol}(N_\mathbb{Q}, \sigma) \otimes \mathbb{Z} \text{CaDiv}(Y)$ with effective divisors $D_i$ and $\Delta_i \in \text{Pol}^+(N_\mathbb{Q}, \sigma)$ is called a *polyhedral divisor* on $(Y, N)$ with tail cone $\sigma$. Moreover, it is called *semiample* if the evaluations $D(u) := \sum_i \min(\Delta_i, u) D_i$ are semiample for $u \in \sigma^\vee \cap M$ and big for $u \in \text{int} \sigma^\vee \cap M$.

Note that the membership $u \in \sigma^\vee := \{ u \in M_\mathbb{Q} \mid \langle \sigma, u \rangle \geq 0 \}$ guarantees that $\min(\Delta_i, u) > -\infty$, and therefore $D$ defines a function $\sigma^\vee \to \text{CaDiv}_\mathbb{Q}(Y)$ which we will denote by the same name. Sometimes, by abuse, we will refer to $D$ as a function defined on the whole lattice $M$ or space $M_\mathbb{Q}$. In such a case, for $u \notin \sigma^\vee$, we have $\min(\Delta_i, u) = -\infty$, and thus, although $-\infty$ as a Cartier divisor coefficient does not make sense, we get as a reasonable conclusion that $\Gamma(Y, \mathcal{O}_Y(D(u))) = 0$.

The common tail cone $\sigma$ of the coefficients $\Delta_i$ will be denoted by $\text{tail}(D)$. Semiample polyhedral divisors will be called *$p$-divisors* for short. Their positivity assumptions imply that $D(u) + D(u') \leq D(u + u')$; hence $\mathcal{O}_Y(D) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(D(u))$ becomes a sheaf of rings, and we can define $X := X(D) := \text{Spec} \Gamma(Y, \mathcal{O}(D))$ over $Y_0$. 


This space does not change if $\mathcal{D}$ is pulled back via a birational modification $Y' \to Y$ or if $\mathcal{D}$ is altered by a polyhedral principal divisor – the latter means an image under $N \otimes \mathbb{Z} \subset \text{CaDiv}(Y)$. P-divisors that differ by (chains of) those operations only are called equivalent. Note that this implies that one can always ask for a smooth $Y$.

**Theorem 3 ([AH], Theorems (3.1), (3.4); Corollary (8.12)).** The map $\mathcal{D} \mapsto X(\mathcal{D})$ yields a bijection between equivalence classes of p-divisors and normal, affine $\mathbb{C}$-varieties with an effective $T$-action.

**Remark.** The $T$-action on $X$ corresponds to the $M$-valued grading of $\Gamma(Y, \mathcal{O}(\mathcal{D}))$. In this context, $\text{tail}(\mathcal{D})$ becomes the cone generated by the weights. Note also that the knowledge of $\mathcal{D} \in \text{Pol}(N_Q, \sigma) \otimes \mathbb{Z} \subset \text{CaDiv}(Y)$. P-divisors that differ by (chains of) those operations only are called equivalent. Note that this implies that one can always ask for a smooth $Y$.

**2.2. Morphisms between p-divisors.** The construction of $X(\mathcal{D})$ is functorial: Up to the above mentioned equivalences of p-divisors, a map $(Y', N', \mathcal{D}') \to (Y, N, \mathcal{D})$ consists of a morphism $\psi : Y' \to Y$ such that the support $\bigcup D_i$ of $\mathcal{D}$ does not contain $\psi(Y')$, and a linear map $F : N' \to N$ with

\[
\sum_i \left( F(\Delta_i) + \text{tail} \mathcal{D} \right) \otimes D_i =: F_*(\mathcal{D}') \subseteq \psi^*(\mathcal{D}) := \sum_i \Delta_i \otimes \psi^*(D_i)
\]

inside $\text{Pol}(N_Q, \text{tail} \mathcal{D}) \otimes \mathbb{Z} \subset \text{CaDiv}(Y')$. The inclusion is understood as a relation between the coefficients of the same divisors. In particular, we ask for $F(\text{tail} \mathcal{D}') \subseteq \text{tail} \mathcal{D}$.

**Theorem 4 ([AH], Corollary (8.14)).** A map $(Y', N', \mathcal{D}') \to (Y, N, \mathcal{D})$ with dominant $\psi : Y' \to Y$ gives rise to an equivariant, dominant map $X(\mathcal{D}') \to X(\mathcal{D})$, and, eventually, this leads to an equivalence of categories.

**2.3. p-divisors encode toric degenerations.** The representation or encoding of a multigraded algebra as a p-divisor has many advantages. First, while one misses direct information about generators and syzygies, one should notice that this construction, however, entails being of finite type. This is based on the fact that only semiample divisors are used to produce the homogeneous parts of the algebra.

However, the main advantage of a p-divisor is that it is possible to read off equivariant and geometric properties of the associated affine $T$-variety $X$. This becomes possible because $X$ is the contraction of $\tilde{X} := \text{Spec}_Y \mathcal{O}(\mathcal{D})$, and this space is a degenerate toric fibration over $Y$. That is, there is a flat map $\tilde{X} \to Y$ with the general fiber being the toric variety $\text{TV}(\text{tail}(\mathcal{D}), N) := \text{Spec} \mathbb{C}[\text{tail}(\mathcal{D}) \cap M]$. Moreover, the divisors $D_i$ and their polyhedral coefficients $\Delta_i$ provide the information about the location and the quality of the degeneration, respectively:

\[
\begin{array}{c}
\text{\tilde{X}} \\
\downarrow \\
Y
\end{array} \quad \begin{array}{c}
\quad X \\
\downarrow \\
Y_0
\end{array}
\]
Special fibers over \( y \in Y \) can be reducible; their components are in a one-to-one correspondence with the vertices of the polyhedron \( \Delta_y := \sum_{D_i \ni y} \Delta_i \).

Thus, also the configuration of \( T \)-orbits and their closures is directly encoded in the presentation of \( X \) as a polyhedral divisor \( D \). The orbits in \( \tilde{X} \) correspond to pairs \((y, F)\) with \( y \in Y \) and faces \( F \leq \Delta_y \). Moreover, as it is known from the toric case, mutual inclusions among orbit closures correspond to opposite inclusions of the corresponding faces. The orbit structure of \( X \) may be obtained from that of \( \tilde{X} \) by keeping track of when certain orbits from \( \tilde{X} \) will be identified in \( X \). This happens in relation to the different contractions of \( Y \) provided by the semiample divisors \( D(u) \).

As an example of how to use this information, see in [Ha09] Hausen’s description of those open subsets \( U \subseteq X \) providing a complete quotient \( U/T \).

3. The toric situation

3.1. Restriction to subtorus actions. If \( T \subseteq (\mathbb{C}^*)^n \) occurs as a subtorus induced by a surjective map \( \deg: \mathbb{Z}^n \to M \) (corresponding to the choice of degrees \( \deg x_i \in M \)), then every affine toric variety \( TV(\delta) \) with \( \delta \subseteq \mathbb{Q}^n \) inherits a \( T \)-action. By [AH §11], the associated p-divisor \( D(\delta) \) can be obtained as follows: Defining \( M_Y := \ker(\deg) \), we have two mutually dual exact sequences

\[
0 \to N \xrightarrow{i} \mathbb{Z}^n \xrightarrow{s} N_Y \to 0
\]

with \( s \) we denote a section of \( \pi \). Then, \( D(\delta) \) lives on the toric variety \( Y := TV(\Sigma) \supseteq N_Y \otimes \mathbb{Z} \mathbb{C}^* =: T_Y \) with \( \Sigma \) denoting the fan in \( N_Y \) being the coarsest common refinement of the image under \( \pi \) of all faces of \( \delta \). As a function, \( D(\delta) \) is given by

\[
D(\delta)(u) = s^*(\deg^{-1}(u) \cap \delta^\vee)
\]

where the right hand side is a polyhedron in \( M_Y \) whose normal fan is refined by \( \Sigma \). Thus, it encodes a semiample, \( T_Y \)-invariant divisor on \( Y \). This implies that

\[
D(\delta) = \sum_{a \in \Sigma(1)} \Delta_a \otimes \overline{\text{orb}(a)} \quad \text{with} \quad \Delta_a = (\pi^{-1}(a) \cap \delta) - s(a) \subseteq N_Q.
\]

Here \( a \in \Sigma(1) \) are primitive lattice elements of rays in \( \Sigma \) and \( \overline{\text{orb}(a)} \) are their associated \( T_Y \)-invariant divisors. The relation between these two representations of \( D \) has been proved in [AH Proposition 8.5] and, in a broader context, in [CM].

3.2. The polyhedral coefficients. We will now present a method to describe the coefficients \( \Delta_a \) with inequalities. This observation is as trivial as it is useful.
Lemma 5. In the situation of (3.1), the polyhedral coefficients \( \Delta_a \) are cut out by the inequalities \( \langle \ast, \deg(r) \rangle \geq -\langle s(a), r \rangle \) for \( r \in \delta^\vee \) (or generators of \( \delta^\vee \)).

Proof. \( x \in \Delta_a \Leftrightarrow i(x)+s(a) \in \pi^{-1}(a) \cap \delta \Leftrightarrow \langle i(x)+s(a), r \rangle = \langle x, \deg(r) \rangle + \langle s(a), r \rangle \geq 0 \) for all \( r \in \delta^\vee \).

3.3. Toric Cox rings. Let \( \mathcal{F} \) be a simplicial fan in some lattice \( N_Z \). Identifying again its one-dimensional rays \( \mathcal{F}(1) = \{ a^1, \ldots, a^n \} \) with the first lattice points sitting on them, we assume that \( \mathcal{F}(1) \) generates \( N_Z \). We would like to apply Lemma 4 to understand the Cox ring of the \( \mathbb{Q} \)-factorial toric variety \( Z := \mathbb{TV}(\mathcal{F}) \). As a ring, it is simply \( \text{Cox}(Z) = \mathbb{C}[x_a \mid a \in \mathcal{F}(1)] \); but by setting \( M := \text{Cl}(Z) \), it is then the \( M \)-grading which makes it interesting. The exact sequences from (3.1) become

\[
0 \longrightarrow \text{Cl}(Z)^* \longrightarrow \text{Div}_{eq}^* Z \overset{\pi}{\longrightarrow} N_Z \longrightarrow 0
\]

\[
0 \longleftarrow \text{Cl}(Z) \longleftarrow \text{Div}_{eq} Z \overset{\text{div}}{\longrightarrow} M_Z \longleftarrow 0.
\]

Here we have denoted by \( \text{Div}_{eq} Z \simeq \mathbb{Z}^n \) the group of \( T_Z \)-equivariant divisors; the rays \( a^i \) are the images of the unit vectors \( e^i \). Note that the torus \( T \) acting on \( \text{Cox}(Z) \) is the Picard torus \( T = \text{Hom}(\text{Cl}(Z), \mathbb{C}^*) \). The degree cone of \( \text{Cox}(Z) \) is the cone of effective divisors \( \text{Eff}(Z) \subseteq \text{Cl}_\mathbb{Q}(Z) \). Hence, the tail of the p-divisor \( \mathcal{D}_{\text{Cox}} \) will be the dualized cone \( \text{Eff}(Z)^\vee \subseteq \text{Cl}_\mathbb{Q}(Z)^* \).

According to (3.1), \( \mathcal{D}_{\text{Cox}} \) lives on \( \mathcal{Y} := \mathbb{TV}(\Sigma) \) with \( \Sigma \) being the coarsest fan in \( N_Y = N_Z \) containing all possible cones generated by subsets of \( \mathcal{F}(1) \). In particular, \( \Sigma \) is a subdivision of \( \mathcal{F} \), i.e. \( \Sigma \leq \mathcal{F} \), i.e. there is a proper map \( \psi : \mathcal{Y} \to Z \) that becomes an isomorphism if it is restricted on the tori \( T_Y = T_Z \). In the surface case we have \( \Sigma = \mathcal{F} \); hence \( \mathcal{Y} = Z \) and \( \psi = \text{id} \). Finally, the choice of the section \( s \) will not affect the upcoming result.

3.4. The splitting of \( \mathcal{D}_{\text{Cox}} \). While p-divisors on \( Y \) may be altered by so-called principal p-divisors coming from \( N \otimes \mathbb{Z} \mathbb{C}(Y)^* = \text{Hom}(M, \mathbb{C}(Y)^*) \), this does not mean that \( \mathcal{D} \) is determined by an element of \( \text{Pol}(N_{\mathbb{Q}}, \sigma) \otimes \mathbb{Z} \text{Pic}_\mathbb{Q}(Y) \). However, elements of the group \( N \otimes \mathbb{Z} \text{Pic}_\mathbb{Q}(Y) = \text{Hom}(M, \text{Pic}_\mathbb{Q}(Y)) \) with \( \text{Pic}_\mathbb{Q}(Y) := \text{CaDiv}_\mathbb{Q}(Y) / \text{PDiv}(Y) \neq \text{Pic}(Y) \otimes \mathbb{Z} \mathbb{Q} \) denoting the \( \mathbb{Q} \)-Cartier divisors modulo principal divisors do indeed give a correct description of an equivalent class of a polyhedral divisor. In particular, it makes sense to add those elements to already existing p-divisors.

Definition 6. In the case of (3.3), the pull back map \( M = \text{Cl}(Z) \subseteq \text{Pic}_\mathbb{Q}(Z) \to \text{Pic}_\mathbb{Q}(Y) \) defines an element \( \psi^* \in \text{Hom}(M, \text{Pic}_\mathbb{Q}(Y)) = N \otimes \mathbb{Z} \text{Pic}_\mathbb{Q}(Y) \) giving rise to a splitting \( \mathcal{D}_{\text{Cox}} = \psi^* + \mathcal{D}'_{\text{Cox}} \) with some correction term \( \mathcal{D}'_{\text{Cox}} \).

Remark. Note that although \( \text{Pic}_\mathbb{Q}(Y) \neq \text{Pic}_\mathbb{Q}(Y) \), we nevertheless have a map \( \text{Pic}_\mathbb{Q}(Y) = \text{CaDiv}_\mathbb{Q}(Y) / \text{PDiv}(Y) \to \text{CaDiv}_\mathbb{Q}(Y) / \text{PDiv}_\mathbb{Q}(Y) = \text{Pic}_\mathbb{Q}(Y) \), and therefore \( \mathcal{D}_{\text{Cox}} \) determines a map \( \text{Cl}_\mathbb{Q}(Z) \supset \text{Eff}(Z) \to \text{Nef}(Y) \subseteq \text{Pic}_\mathbb{Q}(Y) \).
3.5. The p-divisor of toric Cox rings. If \( E \subseteq Y = \mathbb{T}(\Sigma) \) and \( P \subseteq Z = \mathbb{T}(\mathcal{F}) \) are toric prime divisors, then there are associated rays \( a(E) \in \Sigma(1) \subseteq N_{YZ} := N_Y \cap N_Z \) and \( a(P) \in \mathcal{F}(1) \subseteq N_{YZ} \), respectively. Remember that we identify a ray with its integral, primitive generator. In particular, each \( a(E) \) sits in a unique minimal cone \( C_E \in \mathcal{F} \); hence there are unique \( \lambda_E(P) \in \mathbb{Q}_{>0} \) such that \( a(E) = \sum_{a(P) \in C_E} \lambda_E(P) a(P) \). (Remember that \( \mathcal{F} \) is a simplicial fan.) Set \( \lambda_E(P) := 0 \) for \( a(P) \notin C_E \).

Remark. Note that \( \lambda_E(P) > 0 \Leftrightarrow a(P) \in C_E \Leftrightarrow \psi(E) \subseteq P \). In non-toric terms, these coefficients can be expressed as \( \lambda_E(P) = \text{mult}_E(\psi^*P) \). If \( E \) does not get contracted, then we may identify \( E \subseteq Y \) with its divisorial image \( \psi(E) \subseteq Z \), then \( \lambda_E(P) = 1 \) if \( E = P \) and \( \lambda_E(P) = 0 \) if \( E \neq P \). In dimension two, this is always the case (because \( Y = Z \)).

Theorem 7. \( \mathcal{D}'_{\text{Cox}} = \sum_E \Delta_E \otimes E \) with \( E \subseteq Y \) running through the toric prime divisors and \( \Delta_E \subseteq \text{Cl}(Z)_\mathbb{Q} \) being the polyhedron cut out by the inequalities \( \langle \cdot, [P] \rangle \geq -\lambda_E(P) \) for toric prime divisors \( P \). In particular \( \Delta_E \supseteq \text{tail} \mathcal{D}_{\text{Cox}} \).

Proof. In the first exact sequence of (3.3), we add the cosection \( t : \mathbb{Z}^n \to \text{Cl}(Z)^* \) induced from \( s \). Then, the maps satisfy \( it + s \pi = \text{id}_{\mathbb{Z}^n} \), \( ti = \text{id}_{\text{Cl}^*} \), and \( \pi s = \text{id}_{N_{YZ}} \).

\[
0 \longrightarrow \text{Cl}(Z)^* \longrightarrow \mathbb{Z}^n = \text{Div}_{\text{eq}}^* Z \longrightarrow N_{YZ} \longrightarrow 0
\]

Denote by \( \{e(P)\} \subseteq \mathbb{Z}^n = \text{Div}_{\text{eq}}^* Z \) the dual basis with respect to that of the toric prime divisors of \( Z \). In particular, \( \pi(e(P)) = a(P) \in N_{YZ} \).

If \( E \subseteq Y \) is a toric prime divisor (corresponding to the ray \( a(E) \in \Sigma(1) \subseteq N_{YZ} \)), then, by Lemma 4, the true coefficient \( \Delta_{E_{\text{Cox}}} \) is given by the inequalities \( \langle \cdot, [P] \rangle \geq -\langle s(a(E)), P \rangle \) where the latter just means the \( P \)-th entry of \( -s(a(E)) \in \mathbb{Z}^n \). On the other hand, the claimed inequalities for \( \Delta_E \) of \( \mathcal{D}'_{\text{Cox}} \) are \( \langle \cdot, [P] \rangle \geq -\lambda_E(P) = -\langle e(E), P \rangle \). Thus, it remains for us to show that \( b(E) := e(E) - s(a(E)) \) is satisfied in \( \text{Cl}(Z)^* \subseteq \mathbb{Z}^n \) and satisfies \( d := \sum_E b(E) \otimes [E] = \psi^* \in \text{Cl}(Z)^* \otimes \text{Cl}(Y) \).

The first claim follows from \( b(E) = e(E) - s(a(E)) = e(E) - s \pi(e(E)) = it(e(E)) \). Moreover, \( d = \sum_E it(e(E)) \otimes [E] = (it \otimes \text{cl}_Y) \circ (\sum_E e(E) \otimes E \in \text{Div}_{\text{eq}}^* Z \otimes \text{Div}_Y) \), where \( \text{cl} \) denotes the canonical map \( \text{Div} \to \text{Cl} \). On the other hand, since, for a toric prime divisor \( P \subseteq Z \), \( \psi^*P = \sum_E \lambda_E(P) E \), we obtain that \( \psi^* = \sum_{E,P} \lambda_E(P) e(P) \otimes E = \sum_E e(E) \otimes E \), i.e. \( d = (it \otimes \text{cl}_Y) \circ \psi^* \). Restricted, via...
$i$, to $\text{Cl}(Z)^*$, this yields $(ii_\ast \otimes \text{cl}_Y) \circ \psi^* = (i \otimes \text{cl}_Y) \circ \psi^* = (\text{cl}_Z^* \otimes \text{cl}_Y) \circ \psi^* = \psi^*_\text{Cl}$. □

See (6.3) for an example.

4. Duality of polyhedra

4.1. Cones over polyhedra. Dualization of polyhedral cones via $\sigma^\lor := \{x \mid \langle \sigma, x \rangle \geq 0\}$ is a straightforward generalization of the dualization of vector spaces. One has the basic relations $(\sigma^\lor)^\lor = \sigma$ and $(\sigma_1 \cap \sigma_2)^\lor = \sigma_1^\lor + \sigma_2^\lor$. Moreover, via $\tau(\leq \sigma) \mapsto \tau^\lor := \tau^\perp \cap \sigma^\lor(\leq \sigma^\lor)$ it provides a bijection of faces. For the convenience of the reader, we will recall how this theory can be further extended to the set of polyhedra containing the origin.

Let $V$ be a finitely-dimensional $\mathbb{Q}$-vector space and $\Delta \subseteq V$ be a polyhedron containing 0. Then, we define

$$\nabla := \Delta^\lor := \{x \in V^* \mid \langle \Delta, x \rangle \geq -1\}.$$ 

This construction can be understood by the ordinary duality notion of cones. It just requires a definition of the cone $C(\Delta)$ spanned over a polyhedron $\Delta$ located in an affine hyperplane $V \times \{1\} \subset V \times \mathbb{Q}$. Namely, we set

$$C(\Delta) := \mathbb{Q}_{\geq 0} \cdot (\Delta, 1) = \mathbb{Q}_{\geq 0} \cdot (\Delta, 1) \bigcup (\text{tail}(\Delta), 0) \subseteq V \oplus \mathbb{Q}.$$ 

The polyhedron $\Delta$ can be recovered as cross section $\Delta = C(\Delta) \cap (V \times \{1\})$. Then we verify that $C(\nabla) = C(\Delta)^\lor$; hence $\nabla^\lor = (\Delta^\lor)^\lor = \Delta$ and $(\Delta_1 \cap \Delta_2)^\lor = \text{conv}(\Delta_1 \cup \Delta_2)$. Note that $\Delta_1 + \Delta_2 \subseteq 2 \text{conv}(\Delta_1 \cup \Delta_2) \subseteq 2(\Delta_1 + \Delta_2)$ and, in general, $C(\Delta_1 + \Delta_2) \neq C(\text{conv}(\Delta_1 \cup \Delta_2)) = C(\Delta_1) + C(\Delta_2)$.

4.2. Heads and tails. Inside $V$ there are two cones associated to $\Delta$. One is the already mentioned $\text{tail}(\Delta) = C(\Delta) \cap (0, 1]^\perp$; since $0 \in \Delta$, we have $\text{tail}(\Delta) \subseteq \Delta$. The other is $\text{head}(\Delta) := \mathbb{Q}_{\geq 0} \Delta \supseteq \Delta$.

If $\Delta$ was already a polyhedral cone itself, then both cones coincide and are equal to $\Delta$. In general, polyhedral duality interchanges both constructions, i.e. $\text{tail}(\nabla) = \text{head}(\Delta)^\lor$ and $\text{head}(\nabla) = \text{tail}(\Delta)^\lor$. Indeed, $x \in \text{tail}(\nabla) \iff \Delta^\lor + \mathbb{Q}_{\geq 0} x \subseteq \Delta^\lor \iff \langle x, \Delta \rangle \geq 0 \iff \langle x, \mathbb{Q}_{\geq 0} \Delta \rangle \geq 0$. This duality is even more transparent if we note that

$$\text{head}(\Delta) = \bigcup_{t \to -\infty} t \cdot \Delta \quad \text{and} \quad \text{tail}(\Delta) = \bigcap_{t \to 0} t \cdot \Delta$$.
4.3. **Face duality.** Via applying $C$, the nonempty faces $F \leq \Delta$ correspond bijectively to the faces of $C(\Delta)$ not contained in tail($\Delta$) $\leq C(\Delta)$. The inverse map is the intersection with $V \times \{1\}$. Since the dual face (tail $\Delta$)' $\leq C(\Delta)^{\vee} = C(\nabla)$ contains $(0, 1)$, it is not contained in tail $\nabla$, and it corresponds to the minimal face of $\nabla$ that contains 0. Thus, restricting the duality faces($C\Delta$) $\leftrightarrow$ faces($C\nabla$) to those faces with $\not\subseteq$ (tail $\Delta$) and $\not\supseteq$ (tail $\nabla$)' on the left hand side and doing similarly on the right, we obtain an order and dimension reversing bijection

$$\{\text{faces } F \leq \Delta \mid 0 \notin F\} \leftrightarrow \{\text{faces } F' \leq \nabla \mid 0 \notin F'\}.$$ 

The remainings of the bijection faces($C\Delta$) $\leftrightarrow$ faces($C\nabla$) translate into

$$\{\text{faces } F \leq \Delta \mid 0 \in F\} = \text{faces(head } \Delta) \leftrightarrow \text{faces((head } \Delta)^{\vee}) = \text{faces(tail } \nabla)$$

and, analogously, faces(tail $\Delta$) $\leftrightarrow$ {$\nabla$-faces containing 0}.

4.4. **Fanwise linear functions.** A rational (or real) function is called fanwise linear if it is linear on the closed cones of a fan (hence it is continuous on the support of the fan). This is equivalent to being piecewise (affine) linear and homogeneous, that is $f(t \cdot v) = t \cdot f(v)$ for $t \in \mathbb{Q}_{\geq 0}$. For a polyhedron $\Delta \subseteq V$, we define the fanwise linear function $\min(\Delta) : V^* \to \mathbb{Q} \cup \{-\infty\}$ by setting $\min(\Delta)(v) = \min(\Delta, v)$. In particular, $\min(\Delta)^{-1}(\mathbb{Q}) = (\text{tail } \Delta)^{\vee}$. If, additionally, $0 \in \Delta$, then $\min(\Delta) : V^* \to \mathbb{Q}_{\leq 0} \cup \{-\infty\}$ with $\min(\Delta)^{-1}(\mathbb{Q}_{\leq 0}) = \text{head } (\nabla)$. Moreover, $\min(\Delta)^{-1}(0) = \text{tail } (\nabla)$.

**Lemma 8.** If $\Delta$ and $\nabla$ are mutually dual polyhedra containing 0, then

$$\min(\Delta)(v) = -\frac{1}{\max\{t \in \mathbb{Q} \mid t \cdot v \in \nabla\}}.$$ 

Equivalently, the homogeneous, continuous function $\min(\Delta) : \text{head } (\nabla) \to \mathbb{Q}_{\leq 0}$ is characterized by the property that $\min(\Delta) \equiv -1$ on $\partial \nabla \cap \text{int } (\text{head } \nabla)$, where $\partial$ and int denote, respectively, relative boundary and interior of the cone. In particular, $\min(\Delta)$ is equal to $-1$ on all non-zero vertices of $\nabla$.

**Proof.** Let us consider $v \in \partial \nabla \cap \text{int } (\text{head } \nabla)$, then $t \cdot v \notin \nabla$ for every $t > 1$. Moreover, by definition, $\langle v, \Delta \rangle \geq -1$, hence $\min(\Delta)(v) \geq -1$. On the other hand, if $0 > \lambda > -1$ is such that for all $u \in \Delta$ it holds $\langle u, v \rangle \geq \lambda$, then $\langle u, |\lambda|^{-1}v \rangle \geq -1$; hence, by definition of duality of polyhedra, $|\lambda|^{-1}v$ is in $\nabla$ contradicting the assumption. $\square$

Conversely, let $f : \beta \to \mathbb{Q}_{\geq 0}$ be a fanwise linear function defined on a rational, convex polyhedral cone $\beta \subseteq V^*$. We assume that $f$ is also concave, that is $f(v_1 + v_2) \leq f(v_1) + f(v_2)$. Defining

$$\nabla_f := \text{conv } \{ f(v_1)^{-1} \cdot v \mid v \in \beta \} \quad \text{with} \quad 0^{-1} \cdot v := \mathbb{Q}_{\geq 0} \cdot v,$$

we get a polyhedron with head($\nabla_f$) $= \beta$ and tail($\nabla_f$) $= f^{-1}(0)$.

**Lemma 9.** Let $\Delta_f$ be a polyhedron dual to $\nabla_f$ defined above. Then, over the cone $\beta \subseteq V^*$ it holds

$$\min(\Delta_f) = -f.$$
Proof. Let us set $g(v) = \left(\sup\{t \mid tv \in \nabla_f\}\right)^{-1}$. Clearly, both $f$ and $g$ vanish exactly on $\tail(\nabla_f) \subset \sigma$ so we can assume that $v$ is chosen so that both are non-zero. By definition of $\nabla_f$ we have $f(v)^{-1} \cdot v \in \nabla_f$; hence $f(v)^{-1} \leq \sup\{t \mid tv \in \nabla_f\}$ and thus $g(v) \leq f(v)$. Now suppose that $t \cdot v \in \nabla_f$; hence, by definition of $\nabla_f$, for some $v_i \in \sigma$ and positive numbers $a_i$ such that $\sum_i a_i = 1$. Applying the function $f$ to both sides of the equality and using its homogenity and convexity, we get

$$t \cdot v = \sum_i a_i f(v_i)^{-1} \cdot v_i$$

hence $t \cdot f(v) \leq \sum_i a_i f(v_i)^{-1} \cdot f(v_i)$.

Remark. It is possible to weaken the assumption of fanwise linearity to homogeneity of $f$ (see the above proof). Then, $\nabla_f$ and $\Delta_f$ still become well-defined, mutually dual convex bodies – but they lose their polyhedral structure.

4.5. Dualized Cox coefficients. The duality described in (4.1) allows a nicer description of the polyhedral coefficients $\Delta_E \subseteq \Cl(Y)^*$ from Theorem 7. Since they contain the origin, it makes sense to define their duals $\nabla_E := \Delta_E^\vee \subseteq \Cl(Z)$. It follows that

$$\nabla_E = \text{conv}\left\{0, [P]/\lambda_E(P) \mid \psi(E) \subseteq P \subseteq Z\right\} + \sum P \cdot [\psi(E)] \sup_{Q \geq 0} \cdot [P]$$

$$= \text{conv}\left\{[P]/\lambda_E(P) \mid P \subseteq Z\right\} \subseteq \text{head } \nabla_E = \text{Eff}(Z) \ (v/0 := \sum Q \cdot v)$$

with $P$ running through the toric prime divisors of $Z$ and $\lambda_E(P) = \text{mult}_E(\psi^*P)$. Using these polyhedra, we obtain $\mathcal{D}_E = \sum E \nabla_E \otimes E$, and $\mathcal{D}(u)$ contains $E$ with multiplicity

$$\min\langle \Delta_E; u \rangle = -1/\max\{\lambda \in \mathbb{Q} \mid \lambda u \in \nabla_E\} \in \mathbb{Q}_{\leq 0} \cup \{-\infty\}.$$

5. MDS and their Cox $P$-divisor

5.1. Mori dream spaces. Mori dream spaces (MDS) were introduced in [HK]. Recall that $Z$ is a $\mathbb{Q}$-factorial variety with $\text{Cl}(Z)$ being a lattice and $\text{Cox}(Z)$ being finitely generated.

The birational geometry of $Z$ is finite, i.e. $Z$ has finitely many small (i.e. isomorphic in codimension one) $\mathbb{Q}$-factorial modifications $Z_i$ (set $Z_0 := Z$); we will call them SQM models of $Z$. The varieties $Z_i$ are exactly the $\mathbb{Q}$-factorial GIT quotients of $\text{Cox}(Z)$ by the Picard torus arising from linearizations of the trivial bundle depending on the choice of a character of the torus, see [HK]. All models $Z_i$ share the same Cox ring and can be distinguished by pure combinatorics, cf. [Ha08]. In particular, by strict transforms, we can identify $\text{Div}(Z_i)$ and
Cl(Z_i) with Div(Z) and Cl(Z), respectively. The same holds true for the cones Eff(Z_i) = Eff(Z) and Mov(Z_i) = Mov(Z). However, the cones Nef(Z_i) are different, that is int Nef(Z_i) ∩ int Nef(Z_j) = ∅ if Z_i ≠ Z_j, and we have the decomposition Mov(Z) = ∪_i Nef(Z_i), [HK, 1.11(3)]. This chamber decomposition is polyhedral and coincides with that of the stability with respect to the Picard torus, cf. [HK, 2.3] and [DH]. Finally, the maybe most striking feature of Mori dream spaces is that nefness implies semiampleness.

5.2. The Chow limit. Let Y be the Chow quotient of Cox(Z) by the Picard torus, i.e., by abuse of notation, the normalized component of the inverse limit of the models (GIT quotients) Z_i that is birational to the original Z. In particular, we have birational morphisms ψ_i : Y → Z_i.

Note that Y carries two types of exceptional divisors:

(i) An irreducible divisor E ⊆ Y is called of the first kind if it is a component of the exceptional locus of a morphism ψ_i : Y → Z_i. Note that since Z_i is Q-factorial, the exceptional locus of ψ_i is of pure codimension 1. Moreover, since the Z_i are isomorphic outside codimension 2, the set of exceptional divisors is the same for all ψ_i.

(ii) We say that an irreducible divisor E is an exceptional divisor of the second kind if it is a strict transform to Y of a (divisorial) component of an exceptional locus of a birational morphism (divisorial contraction) of a Z_i. In other words, cf. [HK, 1.11(5)], E is a strict transform of a non-movable divisor from Z.

5.3. Stabilized multiplicities. Let ψ : Y → Z be a proper, birational morphism and E ⊆ Y a prime divisor. Then, in the toric case we used in (3.5) and (4.5) the multiplicities λ_E(P) = mult_E(ψ^*P) of a divisor ψ^*P in the general point of E in Z.

In [ELMNP, §2] there is a stable version of these multiplicities. At least for big divisors P, one defines mult^st_E(ψ^*P) either as the E-multiplicity of the stable base locus of P or, by [ELMNP, Lemma 3.3], as

\[ \text{mult}^\text{st}_E(\psi^*[P]) := \inf_{D \in |P|_Q} \text{mult}_E(\psi^*D) \leq \text{mult}_E(\psi^*P). \]

Here D ∈ |P|_Q means that D is an (effective) Q-divisor with mD ∈ |mP| for m ≫ 0. Finally, it follows from [ELMNP, Theorem D] that for a Mori Dream Space Z the stable multiplicity function mult^st_E := mult^st_E ◦ ψ^* can be extended to a concave, fanwise linear function on Eff(Z) ⊆ Cl(Z)_Q. We have the following immediate consequence of Lemma 9.

**Corollary 10.** Let Z be an MDS and ψ : Y → Z the birational morphism from the Chow quotient of Cox(Z). Let E ⊆ Y a prime divisor. Then

\[ \nabla_E := \text{conv} \left\{ \frac{|P|}{\text{mult}_E^\text{st}(\psi^*[P])} \mid [P] \in \text{Eff} Z \right\} \subseteq \text{Cl}(Z)_Q \]
and
\[ \Delta_E := \{ C \in \text{Cl}^*(Z)_{\mathbb{Q}} \mid C \geq -\text{mult}_{E}^{\text{st}} \} \]
are mutually dual polyhedra with \( \min(\Delta_E) = -\text{mult}_{E}^{\text{st}}. \) Moreover, if \( Z \) is toric, then they coincide with those from (4.5).

5.4. The Cox p-divisor of an MDS. Now we are able to present the p-divisor \( \mathcal{D}_{\text{Cox}} \) describing the Cox ring of an MDS. As in Definition 6, we split \( \mathcal{D}_{\text{Cox}} = \psi^* + \mathcal{D}_{\text{Cox}}' \).

**Theorem 11.** The part \( \mathcal{D}_{\text{Cox}}' \) of the p-divisor of the Cox ring of a MDS equals
\[ \mathcal{D}_{\text{Cox}}' = \sum_{E \subset Y} \Delta_E \otimes E, \]
where the coefficients \( \Delta_E \) are defined in Corollary 10 and the sum is formally taken over all divisors \( E \subset Y \). However, if \( E \) is not one of the finitely many exceptional divisors from (5.2) (i) or (ii), then the corresponding coefficient is trivial, i.e. \( \Delta_E = \text{tail}D = \text{Eff}(Z)^* \subset \text{Cl}(Z)_{\mathbb{Q}}^* \), anyway.

**Proof.** We will treat all SQM models \( Z_i \) on equal footing, i.e. we consider \( \mathcal{D}_i := \psi_i^* + \mathcal{D}_i' \) with \( \mathcal{D}_i' := \sum_{E \subset Y} \Delta_E^{i} \otimes E \) and \( \Delta_E^{i} := \{ C \in \text{Cl}^*(Z_i)_{\mathbb{Q}} \mid \langle C, [P] \rangle \geq -\text{mult}_{E}^{\text{st}} \psi_i^*P \}. \) Since the divisors on \( Z_i \) are identified, via the strict transform, with those on \( Z \), we can compare the \( \mathcal{D}_i \) as functions \( \mathcal{D}_i : \text{Div}(Z) = \text{Div}(Z_i) \to \text{CaDiv}_Q(Y) \). Taking, as we did in Corollary 10, the function \( \text{mult}_{E}^{\text{st}} \psi_i^* \) for the fanwise linear map \( f \) in (4.4), we obtain that \( \mathcal{D}_i(D) = \psi_i^*(D) - \sum_{E \subset Y} \text{mult}_{E}^{\text{st}} \psi_i^*(D) \cdot E \) for \( D \in \text{Div}Z \).

We claim that \( \mathcal{D}_i(D) = \mathcal{D}_j(D) \). Indeed, since the multiplicities of \( D \) along divisors \( E \) contained in \( Z \) (isomorphic in codimension 1 to \( Z_i \) and \( Z_j \)) are the same, we conclude that the difference \( \mathcal{D}_i(D) - \mathcal{D}_j(D) \) is supported on divisors contracted by \( \psi \); more precisely we get
\[ \mathcal{D}_i(D) - \mathcal{D}_j(D) = \left( \psi_i^*(D) - \sum_{E \subset \text{Exc}(\psi)} \text{mult}_{E}^{\text{st}} \psi_i^*(D) \cdot E \right) - \left( \psi_j^*(D) - \sum_{E \subset \text{Exc}(\psi)} \text{mult}_{E}^{\text{st}} \psi_j^*(D) \cdot E \right). \]
But \( \psi_i^*(D) - \sum_{E \subset \text{Exc}(\psi)} \text{mult}_{E} \psi_i^*(D) \cdot E \) is the strict transform of the \( \mathbb{Q} \)-Cartier divisor \( D \) from \( Z_i \) to \( Y \) via birational \( \psi_i : Y \to Z_i \); hence, again by isomorphism in codimension 1, it is the same for \( \psi_j : Y \to Z_j \). Thus, passing to the limit from \( \text{mult}_{E} \) to \( \text{mult}_{E}^{\text{st}} \), we get the conclusion of our claim.

Let us recall that, by [HK Prop. 1.11(5)], every big divisor \( D \in \text{Div}Z \), possibly replaced by its multiple, admits a canonical splitting \( D = \text{mov}(D) + \text{fix}(D) \) into the stable movable and fixed part, respectively. Moreover, there is an SQM model \( Z_i \) such that \( \text{mov}(D) \in \text{Nef}(Z_i) \), i.e. \( \text{mov}(D) \) is semiample on \( Z_i \). Thus, the linear system \( | \text{mov}(D) | \) can be assumed base-point-free so that it defines a contraction of \( Z_i \) such that the support of \( \text{fix}(D) \) is in the exceptional locus of the contraction. If \( E_\nu \subset Z_i \) denote divisors contracted by \( | \text{mov}(D) | \), then, by definition, \( \text{fix}(D) =...
\[\sum_{\nu} \text{mult}_{E_{\nu}}^{st}(D) \cdot E_{\nu}.\] We note that we can write \(\text{mult}_{E_{\nu}}^{st}(D) = \text{mult}_{E_{\nu}}(D)\) because \(|\text{mov}(D)|\) is base-point-free and \(D \in |D| = |\text{mov}(D)|\) can be chosen general. Thus,

\[
\psi^*_i(D) = \psi^*_i(\text{mov}(D)) + \sum_{\nu} \text{mult}_{E_{\nu}}^{st}(D) \cdot \psi^*_i(E_{\nu}) = \psi^*_i(\text{mov}(D)) + \sum_{\nu} \text{mult}_{E_{\nu}}^{st}(D) \cdot (\hat{E}_{\nu} + \sum_{E \subset \text{Exc}(\psi_i)} \text{mult}_{E} \psi^*_i E_{\nu})
\]

with \(\hat{E}_{\nu} \subseteq Y\) denoting the strict transform via \(\psi^*_i\) of \(E_{\nu}\), i.e. being an exceptional divisor of the second kind, and the second summation is restricted to exceptional divisors of the first kind only. In particular, \(\psi^*_i(D) - \psi^*_i(\text{mov}(D))\) is supported exclusively on exceptional divisors (of both kinds). On the other hand, as the pull back of a semiample divisor, \(\psi^*_i(\text{mov}(D))\) does not contain exceptional components at all when \(D\) is general in its linear system. Thus,

\[
\psi^*_i(D) = \psi^*_i(\text{mov}D) + \sum_{E \in Y} \text{mult}_{E}^{st}(\psi^*_i D) \cdot E,
\]

and therefore, if \(\mathcal{D}(D)\) denotes the mutually equal \(\mathcal{D}_i(D)\), we obtain that \(\mathcal{D}(D) = \psi^*_i(\text{mov}(D))\), and \(\mathcal{D}(D)\) inherits the semiampleness from \(\text{mov}(D)\) on \(Z_i\).

Eventually, since \(|\text{mov}(D)| = |D|\) the natural inclusion map \(\iota_i : \Gamma(Y, \mathcal{D}(D)) = \Gamma(Y, \psi^*_i(\text{mov}(D))) \to \Gamma(Y, \psi^*_i(D)) = \Gamma(Z, D)\) becomes an isomorphism. Since both maps \(D \mapsto \mathcal{D}(D)\) and \(D \mapsto \psi^*_i(D) - \sum_{E} \text{mult}_{E}^{st}(\psi^*_i D) \cdot E\) are piecewise linear, this extends to the whole effective cone being the closure of the cone of big divisors, cf. [Laz Theorem 2.2.26]. In particular, \(\mathcal{D}\) is a decent p-divisor with \(\Gamma(Y, \mathcal{D}(D)) \to \Gamma(Z, D)\) being an isomorphism for every \(D \in \text{Eff}(Z) \cap \text{Cl}(Z)\); hence

\[
\bigoplus_{D \in \text{Cl}(Z)} \Gamma(Z, D) = \bigoplus_{D \in \text{Cl}(Z)} \Gamma(Y, \mathcal{D}(D))
\]

gives a presentation of \(\text{Cox}(Z)\) as a p-divisor. \(\square\)

The arguments in the proof of Theorem 11 yield the following observation (cf. the remark following Definition 9).

**Corollary 12.** The fanwise linear map \(\mathcal{D}_{\text{Cox}} : \text{Eff}(Z) \to \text{Nef}(Y)\) associated to p-divisor \(\mathcal{D}_{\text{Cox}}\) is a composition of a fanwise linear retraction \(\text{Eff}(Z) \to \text{Mov}(Z)\) and a fanwise linear map \(\text{Mov}(Z) \to \text{Nef}(Y)\) whose restriction to the cone \(\text{Nef}(Z_i)\), for every SQM model \(Z_i\), coincides with the pull-back map \(\psi^*_i : \text{Nef}(Z_i) \to \text{Nef}(Y)\).

5.5. **Example: Blowing up two points in \(\mathbb{P}^3\).** This is perhaps the simplest three-dimensional example to illustrate Corollary 12. Let \(Z\) be the blow-up of \(\mathbb{P}^3\) in two points, say \(x_1\) and \(x_2\), with exceptional divisors denoted by \(E_1\) and \(E_2\). The strict transform of a general plane, a plane passing through each of these points, and a plane passing through both of them, define divisors whose classes span \(\text{Mov}(Z)\). The rational maps defined by these divisors are onto \(\mathbb{P}^3, \mathbb{P}^2\) and \(\mathbb{P}^1\), respectively. The flop along the strict transform of the line passing through \(x_1\) and \(x_2\) yields another SQM model, let us call it \(Z_1\). The variety \(Y\) results from blowing up this strict transform.
Now the following picture presents sections of cones in spaces of divisor classes. The 3-dimensional cone $\text{Eff}(Z)$ presented on the left hand side gets retracted to $\text{Mov}(Z)$: the regions on which the retraction is linear are denoted by dotted line segments. Next $\text{Mov}(Z) = \text{Nef}(Z) \cup \text{Nef}(Z_1)$ is mapped linearly on each Nef cone to two 3-dimensional faces of the 4-dimensional cone $\text{Nef}(Y)$.

We note that only two of the four faces of the tetrahedron representing the section of the 4-dimensional cone $\text{Nef}(Y)$ are associated to SQM models of $Z$. The other two faces represent contractions of $Y$ to $\mathbb{P}^3$ blown up at one point ($x_1$ or $x_2$) and then along the strict transform of the line passing through $x_1$ and $x_2$. This is equivalent to blowing up the line first and then blowing up the fiber of the exceptional divisor above $x_1$ or $x_2$. In particular, the dotted edge of the tetrahedron represents the contraction of $Y$ to $\mathbb{P}^3$ blown up along the line passing through $x_1$ and $x_2$.

6. **Surfaces**

6.1. **Specializing the general result.** The case of ($\mathbb{Q}$-factorial MD) surfaces $Z = S$ is special for two reasons. First, it does not require the pull back to the Chow quotient, i.e. $Y = Z = S$ with $\psi = \psi_i = \text{id}$, and $\mathcal{D} : \text{Eff}(S) \to \text{Nef}(S)$ simply reflects the Zariski decomposition. Indeed, given any effective divisor $D$ on $S$, we can write it uniquely as the sum $D \equiv P + \sum_i a_i E_i$ where $P \in \text{Nef}(S)$, $E_i$ are exceptional curves (if there are any) such that $(P \cdot E_i) = 0$, and coefficients $a_i = \text{mult}^{\text{st}}_{E_i} D$. Thus $P = D(D)$.

Second, the $\mathbb{Q}$-valued intersection product, denoted simply by a dot, allows one to identify vector spaces $\text{Cl}(S) \otimes^* \mathbb{Q} = \text{Cl}(S) \otimes \mathbb{Q}$ with $\langle C_1, C_2 \rangle = (C_1 \cdot C_2)$. In particular, the polyhedral coefficients $\Delta_E$ will be contained in $\text{Cl}(S) \otimes \mathbb{Q}$ now and have $\text{Nef}(S) = \text{Eff}(S)^\vee$ as their common tail cone. If $S$ is smooth, then we even know that $\text{Cl}(S) = \text{Cl}(S)$. In general, this equation has to be replaced by $\text{Cl}(S) = \{D \in \text{Cl}(S) | \langle D, \text{Cl}(S) \rangle \subseteq \mathbb{Z}\}$. Finally, we recognize the (finitely many) exceptional divisors $E_i \subseteq S$ by their negative self intersection numbers ($E_i^2$).

**Theorem 13.** Let $S$ be an MD-surface with the exceptional divisors $E_i \subset S$. Then, $\mathcal{D}'_{\text{Cox}} = \sum_i \Delta_i \otimes E_i$ with

$$\Delta_i = \{D \in \text{Eff}(S) | (D \cdot E_i) \geq -1 \text{ and } (D \cdot E_j) \geq 0 \text{ for } j \neq i\},$$
and the dual coefficients equal \( \nabla_i = \overline{0}E_i + \sum_{j \neq i} \mathbb{Q}_{\geq 0}[E_j] + \text{Nef}(S) \).

(With \( \overline{0}E_i \) we denote the line segment connecting 0 and \([E_i]\) inside \( \text{Cl}_\mathbb{Q}(S) \).)

**Proof.** This is a reformulation of Theorem 11. \( \Delta_i \) and \( \nabla_i \) are dual with respect to the intersection product. On the other hand, by Lemma 8 the function defined by \( \nabla_i \) is just \( -\text{mult}^\text{st}_{E_i} \). \( \square \)

6.2. **Del Pezzo surfaces.** Let \( S = S_d \) be a smooth del Pezzo surface of degree \( d = K_S^2 \). By definition, \(-K_S\) is ample. Any such \( S_d \) is known to be \( \mathbb{P}^1 \times \mathbb{P}^1 \) (\( d = 8 \)) or a blow-up of \( \mathbb{P}^2 \) at \( r := 9 - d \) general points. It is known that for \( d \leq 7 \) the cone \( \text{Eff}(S) \) is generated by a finite number of (-1)-curves. In fact, any non-movable curve on such \( S \) is a (-1)-curve. In this special case, the polyhedral coefficients from Theorem 13 become especially easy:

**Corollary 14.** If \( S \) is a del Pezzo surface and \( E \) a (-1)-curve, then the only vertices of \( \Delta_E \) are 0 and \([E]\). In particular, the polyhedral coefficients of \( \mathcal{D}_{\text{Cox}} \) are as follows:

\[
\Delta_E = \text{conv} \{0, [E]\} + \text{Nef}(S) = 0[E] + \text{Nef}(S).
\]

**Proof.** If \( D \in \Delta_E \), i.e. if \( D \) is an effective \( \mathbb{Q} \)-divisor with \( (D \cdot E) \geq -1 \) and \( (D \cdot F) \geq -\delta_{EF} \) for (-1)-curves \( F \neq E \), then we have to show that \( D \in 0[E] + \text{Nef}(S) \). If \( D \) was already nef, then we are done. If not, then by rescaling we may assume that \( (D \cdot E) = -1 \), and then we claim that \( D' := D - E \) is nef: First, \( (D' \cdot E) = (D \cdot E) - (E^2) = 0 \). Then, if \( F \) is an arbitrary (-1)-curve different from \( E \), we may write \( D = eE + fF + P \) with \( e, f \geq 0 \) and \( P \) being effective without \( E \) and \( F \) contributions. Thus,

\[
-1 = (D \cdot E) = -e + f(F \cdot E) + (P \cdot E) \geq -e + f(F \cdot E);
\]

hence \( e - 1 \geq f(F \cdot E) \geq 0 \). This implies that \( D' \) is effective and, moreover,

\[
(D' \cdot F) \geq (e - 1)(E \cdot F) - f \geq f(E \cdot F)^2 - f = f((E \cdot F)^2 - 1).
\]

If \( (E \cdot F) \neq 0 \), then we obtain \( (D' \cdot F) \geq 0 \); in the opposite case of \( (E \cdot F) = 0 \), we simply conclude via \( (D' \cdot F) = (D \cdot F) - (E \cdot F) = (D \cdot F) \geq 0 \). \( \square \)

**Remark.** Let \( S = S_d \) be a smooth del Pezzo surface of degree \( d \leq 7 \) which is a blow-up of \( \mathbb{P}^2 \) at \( r := 9 - d \) general points; by \( E_1, \ldots, E_r \subset S \) we denote their preimages. Then \( \text{Cl}(S) = \mathbb{Z}H \oplus (\oplus_{i=1}^r \mathbb{Z}E_i) \), hence id\( \text{Cl}_S = [H] \otimes [H] - \sum_{i=1}^r [E_i] \otimes [E_i] \). In particular,

\[
\mathcal{D}_{\text{Cox}} = ([H] + \text{Nef}(S)) \otimes H + \sum_{i=1}^r ([{-E_i}]0 + \text{Nef}(S)) \otimes E_i + \sum_{E_i \notin \{E_j\}} (0[E] + \text{Nef}(S)) \otimes E.
\]

The above result says that Zariski decomposition on a del Pezzo surface is orthogonal. That is, given any effective divisor \( D \) on \( S \), we can write it uniquely as the sum \( D \equiv P + \sum_i a_i E_i \) where \( P \in \text{Nef}(S) \) and \( E_i \) are (-1)-curves such that \( (P \cdot E_i) = 0 \), and \( a_i = \text{mult}_{E_i} D \) and, moreover, \( (E_i \cdot E_j) = 0 \) if \( i \neq j \). The last of these properties is known and follows from the fact that the birational morphism of a del Pezzo surface associated to \(|mP|, m \gg 0\) contracts disjoint (-1)-curves \( E_i \).
6.3. Example: Blowing up two points in $\mathbb{P}^2$. While the following two examples are just toric, they, nevertheless, illustrate the special shape of $\mathcal{D}_{\text{Cox}}$ for del Pezzo surfaces and indicate the difference to a somewhat more general situation. First, we consider a surface $S_1$ which is an ordinary blowing up of $\mathbb{P}^2$ in two points; second we present a surface $S_2$ which is a $\mathbb{P}^2$ with two infinitesimally near points blown up.

The toric surface $S_1$ is given by the fan $\Sigma_1 = \{(1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1)\}$. The exceptional divisors of the blowing up are $E_1 = \text{orb}(1, 1)$ and $E_2 = \text{orb}(-1, 0)$ together with the strict transform $E_0 = \text{orb}(0, 1)$ of the line connecting the two centers; they are the only $(-1)$-curves in $S_1$.

Let $[H]$ denote the pull back of the line in $\mathbb{P}^2$. Then $[E_0] = [H] - [E_1] - [E_2]$, and the nef cone $\text{Nef}(S_1)$ is formed by the strict transforms $[A] = [H] - [E_1] = [E_0] + [E_2]$, $[B] = [H] - [E_2] = [E_0] + [E_1]$, and by $[H] = [E_0] + [E_1] + [E_2]$ itself. The ample anti-canonical bundle is $[-K] = 3[H] - [E_1] - [E_2] = [A] + [B] + [H]$.

The classes of the $E_i$ form a basis of $\text{Cl}(S_1)$; the associated intersection matrix is

$$
\begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{pmatrix}.
$$

This implies that $\text{id}_{\text{Cl}, S_1} = [H] \otimes [E_0] + [A] \otimes [E_1] + [B] \otimes [E_2]$, and the coefficients of $E_i$ in $\mathcal{D}_{\text{Cox}, S_1}^\prime$ are indeed $\Delta_{E_i} = 0[E_i] + \text{Nef}(S_1)$.

For the second example $S_2$, we obtain the following pictures for the fan and the class group, respectively. Again, $E_1$ is the exceptional curve of the first blow-up, $E_2$ of the second blow-up, and $E_0$ is the strict transform of the line.
Using the basis \{[E_0], [E_1], [E_2]\}, the intersection matrix is as follows:

\[
\begin{pmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{pmatrix}
\]

The pull back of the line is \([H] = [B] = [A] + [E_1]\) with \([A] = [E_2] + [E_0], [B] = [E_0] + [E_1] + 2[E_2]\), and \([C] := 2[E_0] + [E_1] + 2[E_2] = 2[A] + [E_1] = [B] + [E_0]\) generating the nef cone \text{Nef}(S_2). This implies that \(\text{id}_{C_1 S_2} = [A] \otimes [E_1] + [C] \otimes [E_2] + [B] \otimes [E_0]\), and the compact parts of the coefficients of the \(E_i\) in \(D'_{\text{Cox} S_2}\) are \(\Delta_{E_0}^\text{comp} = 0[E_0]\), but

\[
\Delta_{E_1}^\text{comp} = \text{conv}\{0, \frac{1}{2}[E_1], [E_1] + [E_2]\} \quad \text{and} \quad \Delta_{E_2}^\text{comp} = \text{conv}\{0, [E_2], [E_1] + 2[E_2]\}.
\]

The two surfaces are homeomorphic; in fact, there exists a deformation of \(S_2\) to \(S_1\). Thus we can identify respective homology classes and put them in one picture. The cohomology classes \([H], [E_0], [E_2]\) and \([A]\) are the same for both surfaces, the class of the second blow-up we denote by \([E_1]^1\) and \([E_1]^2\), respectively. To make the picture transparent the boundaries of \(\text{Eff}\) cones, as well as their division in Zariski chambers, are denoted by dotted line segments.
This picture describes a typical situation: the effective cone, as the function of a deformation is upper semicontinuous, that is Eff($S_2$) ⋇ Eff($S_1$) while the nef or movable cone is lower semicontinuous, that is Mov($S_2$) ⊂ Mov($S_1$).

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