DEFORMATION THEORY OF THE CHOW GROUP OF ZERO-CYCLES

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Abstract. We study the deformations of the Chow group of zero-cycles using Bloch’s formula and differential forms. We thereby obtain a new proof of an algebraization theorem for zero-cycles previously obtained using idelic techniques.

1. Introduction

Let $A$ be a henselian discrete valuation ring with uniformising parameter $\pi$ and residue field $k$. Let $X$ be a smooth projective scheme over $\text{Spec}(A)$ of relative dimension $d$. Let $X_n := X \times_A A/(\pi^n)$, i.e. $X_1$ is the special fiber and the $X_n$ are the respective thickenings of $X_1$. Let $K^{M}_{*,X}$ (resp. $K^{M}_{*,X_n}$) be the improved Milnor K-sheaf defined in [14].

One classically studies the formal deformations of Chow groups via Bloch’s formula setting

$$\text{CH}^p(X_n) := H^p(X_1, K^{M}_{p,X_n})$$

Then one uses the commutative diagram

$$
\begin{array}{ccc}
H^p(X_1, K^{M}_{p,X_{n+1}}) & \longrightarrow & H^p(X_1, K^{M}_{p,X_n}) \\
\downarrow & & \downarrow \\
H^p(X_1, \Omega^{p}_{X_{n+1}}) & \longrightarrow & H^p(X_1, \Omega^{p}_{X_n})
\end{array}
\quad \text{induced by the commutative diagram}
\begin{array}{cccc}
0 & \longrightarrow & \Omega^{p-1}_{X_1} & \longrightarrow & K^{M}_{p,X_{n+1}} & \longrightarrow & K^{M}_{p,X_n} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{p-1}_{X_1} \oplus \Omega^{p}_{X_1} & \longrightarrow & \Omega^{p}_{X_{n+1}} & \longrightarrow & \Omega^{p}_{X_n} & \longrightarrow & 0.
\end{array}
$$

To find Hodge-theoretic conditions for the lifting of cycles. For more details on this approach see [9] and [3]. If $p = d$, i.e. in the case of zero-cycles, these conditions are vacuous for dimensional reasons. One can always lift a (thickened) zero-cycle to the next thickening. We may therefore concentrate on studying the algebraization properties of such thickened zero-cycles. We do this through a detailed study of the group $H^d(X_1, \Omega^{d-1}_{X_1})$ which describes the difference between $H^d(X_1, K^{M}_{d,X_{n+1}})$ and $H^d(X_1, K^{M}_{d,X_n})$.

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Throughout this article we assume that either (1) $k$ is of characteristic $0$ and $A = k[[\pi]]$ or that (2) $A$ is the Witt ring $W(k)$ of a perfect field $k$ of $\text{ch}(k) > 2$. In each of these two cases there exists a well-defined exponential map
\[ \exp : \Omega^d_{X_1} \to \mathcal{K}^M_{d,X_n} \]
defined by
\[ x d\log(y_1) \wedge ... \wedge d\log(y_{d-1}) \mapsto \{1 + x^n, y_1, ..., y_{d-1}\} \]
(see [3, Sec. 2] for (1) and [4, Sec. 12] for (2)). In these two cases we therefore get an exact sequence
\[ \Omega^d_{X_1} \to \mathcal{K}^M_{d,X_n} \to \mathcal{K}^M_{d,X_{n-1}} \to 0 \]
which we use to study the restriction map
\[ \text{res}_{X_n} : \text{CH}^d(X) \to H^d(X, \mathcal{K}^M_{d,X_n}) \]
assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$ for the isomorphism on the left. The Gersten conjecture holds in case (1) by [14] and in case (2) with finite coefficients if we assume $\text{ch}(k)$ to be large enough by [18, Prop. 1.1]. Our main theorem is the following:

**Theorem 1.1.** With the above notation, and assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$, the map $\text{res}_{X_n} : \text{CH}^d(X) \to H^d(X_1, \mathcal{K}^M_{d,X_n})$ is surjective. In particular the map
\[ \text{res} : \text{CH}^d(X) \to \text{"lim}_{n} H^d(X_1, \mathcal{K}^M_{d,X_n}) \]
is an epimorphism in pro-$\text{Ab}$.

Theorem 1.1 is also proved in [18] using an idelic method. The approach of this article, using differential forms, is closer to the deformation theory described above. We think that it is of interest as well.

The text is organised as follows: in Section 5 we prove our main theorem for $d = 2$ using, among others, results from Section 2. We then deduce the general case using a Lefschetz theorem which is proved in Section 4. Section 3 is independent and treats the case of $d = 1$. In the last section we list some open questions.

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## 2. Local Cohomology and Some Calculations

In this section we recall some definitions and calculations in local cohomology which we will need later on. A standard reference for the following is [10, Ch. IV].

Let $X$ be a locally noetherian scheme. To any sheaf of abelian groups $\mathcal{F}$ on $X$ we can associate a coniveau complex of sheaves
\[ \mathcal{C}(\mathcal{F}) := \bigoplus_{x \in X^{(0)}} i_{x,*}H^0_x(X, \mathcal{F}) \to \bigoplus_{x \in X^{(1)}} i_{x,*}H^1_x(X, \mathcal{F}) \to ... \]
where $i_x : x \to X$ is the natural inclusion. This complex is also called the Cousin complex of $\mathcal{F}$. 
Definition 2.1. A sheaf $F$ on $X$ is called Cohen-Macaulay, or simply CM, if for every $x \in X$ it holds that $H^i_x(X,F) = 0$ for $i \neq \text{codim}(x)$.

Via the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H^{p+q}_x(X,F) \Rightarrow H^n(X,F)$$

one can easily deduce that the property of being CM for $F$ is equivalent to $\mathcal{C}(F)$ being an acyclic resolution of $F$ (see \[10\] Ch. IV, Prop. 2.6). In that case one can use $\mathcal{C}(F)$ to calculate the cohomology of $F$ modulo $\Omega^1_X$, e.g. $H^*(X,F) \cong H^*(X,\mathcal{C}(F))$. Locally free sheaves are CM (see \[10\] p.239) so in particular the sheaf of differential forms $\Omega^1_X$ and its exterior powers $\Omega^*_X$ are CM if $X$ is a smooth variety over a field.

Lemma 2.2. Let $k$ be a field and $X_1$ be a scheme of dimension 1 over Spec($k$). Let $x \in X_1$ be a regular closed point and $f$ a local parameter at $x$. Then

$$\mathcal{O}_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x} \cong H^1_x(X_1,\mathcal{O}_{X_1}).$$

Proof. We calculate $H^1_x(X_1,\mathcal{O}_{X_1})$ locally as follows: Let $X_{1,x} := \text{Spec}(\mathcal{O}_{X_1,x})$. Applying Motif B of \[10\] p.217 to the triple $(x, X_1, X_1 - x)$, we get a short exact sequence

$$H^0(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) \rightarrow H^0(X_{1,x}, -x, \mathcal{O}_{X_1}|_{X_{1,x}}) \rightarrow H^1_x(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) \rightarrow H^1(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}).$$

Since $H^1(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) = 0$, this gives an isomorphism

$$\mathcal{O}_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x} \cong H^1_x(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}).$$

We now turn to the higher dimensional case. Similar calculations can be found in \[2\] Sec. 5.

Lemma 2.3. Let $k$ be a field and $X_1$ be a scheme of dimension $d > 1$ over Spec($k$). Let $x \in X_1$ be a regular closed point and $f_1, \ldots, f_d \in \mathfrak{m}_x$ a local parameter system at $x$. Then $H^d_x(X,\Omega^{d-1}_X)$ is generated by elements of the form

$$\frac{df_1 \wedge \ldots \wedge df_i \wedge \ldots \wedge df_d}{f_1^{n_1} \cdots f_d^{n_d}}$$

modulo $\frac{df_1 \wedge \ldots \wedge df_i \wedge \ldots \wedge df_d}{f_1^{n_1} \cdots f_d^{n_d}}$ over $\mathcal{O}_{X_1,x}$.

Proof. Let $U$ be an affine neighbourhood of $x \in X_1$. Let $\mathcal{V} = \{ V_i := U - V(f_i) \}$ be a covering of $U - x$. Then the Čech complex

$$0 \rightarrow \prod_{i \neq j} \Omega^d_{X_1}(V_i) \rightarrow \prod_{i \neq j} \Omega^d_{X_1}(V_i \cap V_j) \rightarrow \Omega^d_{X_1}(V_1 \cap \ldots \cap V_d)$$

gives an isomorphism

$$\text{coker}(\prod \Omega^d_{X_1}(V_1 \cap \ldots \cap \hat{V}_i \cap \ldots \cap V_d)) \rightarrow \Omega^d_{X_1}(V_1 \cap \ldots \cap V_d) \cong \Gamma(U, R^{d-1}j_*(\Omega^{d-1}|_{U-x})).$$
where $j$ is the inclusion $X_1 - x \hookrightarrow X_1$. By Motif B of [10, p.220] and since $d \geq 2$ there is an isomorphism

$$R^{d-1}j_*\left(\Omega^{d-1}|_{X_1-x}\right) \cong H_x^d(X_1, \Omega^{d-1}_{X_1}).$$

In other words, $\Gamma(U, \mathcal{H}_x^d(X, \Omega^{d-1}_{X_1}))$ is generated by elements of the form $\frac{d f_1 \wedge \ldots \wedge d f_d}{f_1 \ldots f_d}$ modulo $\frac{d f_1 \wedge \ldots \wedge d f_d}{f_1 \ldots f_d}$ over $\mathcal{O}(U)$. Passing to the limit, we get the desired result. 

In case (1) of the introduction, i.e. for $k$ a field of characteristic $0$, $S_n = \text{Spec}k[t]/(t^n)$, $S = \text{Spec}[t]$ and $X$ smooth, separated and of finite type over $S$, there exists a short exact sequence

$$0 \to \Omega_{X_1}^{-1} \to K_{r,X_1}^M \to K_{r,X_1}^M \to 0$$

(see [3, Prop. 2.3]). In particular, $K_{r,X_1}^M$ is CM for all $n \geq 1$ (see [3, Prop. 3.5]). We now show analogous statements for case (2).

**Proposition 2.4.** Let $k$ be a perfect field with $\text{ch}(k) = p > 2$ and let $X$ be a smooth scheme over $A := W(k)$. Then there is an exact sequence

$$0 \to \Omega_{X_1}^{-1}/B_{n-1}\Omega_{X_1}^{-1} \to K_{r,X_1}^M \to K_{r,X_1}^M \to 0.$$  

**Proof.** Let $R_n$ be an essentially smooth local ring over $A/\pi^n$. We define a filtration $U^iK_r^M(R_n) \subset K_r^M(R_n)$ by

$$U^iK_r^M(R_n) := \langle 1 + \pi^i x, x_2, \ldots, x_r \mid x \in R_n, x_2, \ldots, x_r \in R_n^* \rangle.$$  

The $U^i$ fit into the following exact sequences:

$$0 \to U^nK_r^M(R_{n+1}) \to K_r^M(R_{n+1}) \to K_r^M(R_n) \to 0.$$  

By [4, Proof of Prop. 12.3, Step 3] there is an isomorphism

$$\frac{\Omega_{R_1}^{-1}/B_{i-1}\Omega_{R_1}^{-1}}{\Omega_{R_1}^{-1}/B_{i-1}\Omega_{R_1}^{-1}} \cong gr^iK_r^M(R_n) \cong U^iK_r^M(R_n)/U^{i+1}K_r^M(R_n)$$

and since $U^{n+1}(K_r^M(R_{n+1})) = 0$, this implies that $U^n(K_r^M(R_{n+1})) \cong \Omega_{R_1}^{-1}/B_{n-1}\Omega_{R_1}^{-1}$ and therefore the exact sequence

$$0 \to \Omega_{R_1}^{-1}/B_{n-1}\Omega_{R_1}^{-1} \to K_r^M(R_{n+1}) \to K_r^M(R_n) \to 0.$$  

**Corollary 2.5.** Let $X$ be as in Proposition [2,4]. Then the sheaf $K_{r,X_1}^M$ is CM.

**Proof.** Applying the derived functor $H_x^i(X_1, -)$ to (2.1), we get the exact sequence

$$H_x^i(X_1, \Omega_{R_1}^{-1}/B_{n-1}\Omega_{R_1}^{-1}) \to H_x^i(X_1, K_{r,X_1}^M) \to H_x^i(X_1, K_{r,X_1}^M).$$

By [12, Cor. 3.9, p. 572], the sheaf $\Omega_{X_1}^{-1}/B_{n-1}\Omega_{X_1}^{-1}$ is locally free and therefore CM. The sheaf $K_{r,X_1}^M$ is CM by [14] and [15]. The result follows inductively. 

□
3. The relative dimension 1 case

Before we state the main proposition of this section, we quickly review the theory of pro-objects. Standard references are [1] and [9].

Let $C$ be a category. The category of pro-objects $\text{pro-}C$ in $C$ is defined as follows: A pro-object is a contravariant functor $X : I^\circ \to C$, from a filtered index category $I$ to $C$, i.e. an inverse system of objects $X_i$ in $C$. We denote $X$ also by $\lim_i X_i$ or $(X_i)_i$. The morphisms between two objects $X = \lim_i X_i$ and $Y = \lim_j Y_j$ in $\text{pro-}C$ are given by

$$\text{Hom}(X, Y) = \lim_{\leftarrow j} (\lim_{\rightarrow i} \text{Hom}(X_i, Y_j)).$$

There is a natural fully faithful embedding of $C$ into $\text{pro-}C$ which associates to an object $C \in C$ the constant diagram $C$. This functor has a right adjoint $\text{pro-}C \to C$, $\lim_i X_i \mapsto \lim_{\leftarrow i} X_i$. If $C$ has finite direct (inverse) limits, then the functor $\text{Hom}(I^\circ, C) \to \text{pro-}C$ commutes with finite direct (inverse) limits. In particular if $C$ has finite direct and inverse limits, then the above functor is exact (see [1, p.163]).

A criterion for when a map of pro-systems is an isomorphism is given by the following proposition (see [13, Lem. 2.3]):

**Proposition 3.1.** A level map $A \to B$ in $\text{pro-}C$, i.e. a map between pro-systems with the same index category and maps $A_s \to B_s$ for all $s \in I$, is an isomorphism if and only if for all $s$ there exists a $t \geq s$ and a commutative diagram

$$
\begin{array}{ccc}
A_t & \longrightarrow & B_t \\
\downarrow & & \downarrow \\
A_s & \longrightarrow & B_s.
\end{array}
$$

We now give a proof of Conjecture 6.1 for $d = 1$.

**Theorem 3.2.** Let $k$ be a finite field of characteristic $p > 2$ and $A = W(k)$ the Witt ring of $k$. Let $X$ be a smooth projective scheme of relative dimension 1 over $A$. Then the map

$$\text{res} : \text{CH}^1(X) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \lim H^1(X_1, \mathcal{K}_1^{M_{X_n}}/p^i)$$

is an isomorphism in the category of pro-systems of abelian groups.

**Proof.** We first note that $\text{CH}^1(X) \cong \text{Pic}(X)$ and that $\text{Pic}(X) \cong \lim \text{Pic}(X_n)$ by [8, Thm. 5.1.4]. Furthermore, $H^1(X_1, \mathcal{K}_1^{M_{X_n}}) = H^1(X_1, \mathcal{O}_{X_n}^\times) \cong \text{Pic}(X_n)$. It therefore suffices to show that

$$\lim \text{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \lim \text{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z}$$

is an isomorphism.
Using the $p$-adic logarithm isomorphism $1 + p\mathcal{O}_X \cong p\mathcal{O}_X$, the short exact sequence

$$1 \to (1 + p^i\mathcal{O}_X) \to \mathcal{O}^*_X \to \mathcal{O}^*_X \to 1$$

induces a short exact sequence

$$0 \to H^1(X_1, p^i\mathcal{O}_X) \to H^1(X_1, \mathcal{O}^*_X) \to H^1(X_1, \mathcal{O}^*_X) \to H^2(X_1, p^i\mathcal{O}_X) = 0$$

(the last equality following for dimension reasons). Applying the Functor $\lim$, we get an exact sequence

$$\lim_{n} H^1(X_1, p^i\mathcal{O}_X) \to \lim_{n} \text{Pic}(X_n) \to \text{Pic}(X_j) \to \lim_{n} H^1(X_1, p^i\mathcal{O}_X).$$

Now $\lim_{n} H^1(X_1, p^i\mathcal{O}_X) = 0$ since the inverse system $(H^1(X_1, p^i\mathcal{O}_X))_n$ satisfies Mittag-Leffler being an inverse system of finite dimensional vector spaces. Tensoring with $\mathbb{Z}/p^i\mathbb{Z}$ gives the exact sequence

$$\lim_{n} H^1(X_1, p^i\mathcal{O}_X) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \lim_{n} \text{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \text{Pic}(X_j) \otimes \mathbb{Z}/p^i\mathbb{Z} \to 0.$$

We now apply the exact functor $\left(\lim_{n}\right)_{\otimes}$ to this sequence. By the theorem on formal functions, there is an isomorphism

$$\left(\lim_{n}\right)_{\otimes} H^1(X_1, p^i\mathcal{O}_X) \otimes \mathbb{Z}/p^i\mathbb{Z} \cong \lim_{j} H^1(X, p^i\mathcal{O}_X) \otimes \mathbb{Z}/p^i\mathbb{Z}.$$

Since the image of the inclusion $p^{i+j}\mathcal{O}_X \hookrightarrow p^i\mathcal{O}_X$ vanishes modulo $p^i$, the same holds for the image of the morphism $H^1(X, p^{i+j}\mathcal{O}_X) \to H^1(X, p^i\mathcal{O}_X)$. By Proposition 3.1 this implies that

$$\left(\lim_{n}\right)_{\otimes} H^1(X_1, p^i\mathcal{O}_X) \otimes \mathbb{Z}/p^i\mathbb{Z}$$

is pro-isomorphic to zero and therefore that the theorem holds.

## 4. A Lefschetz theorem

In this section we prove a Kodaira vanishing theorem which implies a Lefschetz theorem allowing us later in Section 5 to reduce our main theorem to relative dimension 2.

To put the following proposition into context, we recall the Kodaira vanishing theorem. A good reference is [3].

**Theorem 4.1.** Let $X$ be a complex projective manifold and $\mathcal{A}$ an ample invertible sheaf. Then

$$H^a(X, \Omega^b_X \otimes \mathcal{A}) = 0$$

for $a + b > \dim X$.

In this section let $X_1$, unless otherwise stated, be a smooth projective scheme over a field $k$. Let $H \subset X_1$ be a hyperplane section and $\mathcal{L}(d) = |dH|$, $d > 0$, be the linear system of hypersurface sections of degree $d$. We say that a hypersurface section $Y_1 \subset X_1$ is of high or sufficiently high degree if $Y_1 \in \mathcal{L}(d)$ with $d$ sufficiently large such that certain higher cohomology groups vanish by Serre vanishing.
**Proposition 4.2.** Let $Y_1$ be a smooth hypersurface of $X_1$ and $d = \dim X_1$. If $Y_1$ is of sufficiently high degree, then

$$H^a(X_1, \Omega^b_{Y_1} \otimes_{O_{X_1}} O_{X_1}(Y_1)) = H^a(Y_1, \Omega^b_{Y_1} \otimes_{O_{Y_1}} O_{X_1}(Y_1)|_{O_{Y_1}}) = 0$$

for $a + b > d - 1$.

**Proof.** Note that the first equality in the statement follows from the projection formula $i_* \Omega^b_{Y_1} \otimes_{O_{X_1}} O_{X_1}(Y_1) = i_* (\Omega^b_{Y_1} \otimes_{O_{Y_1}} i^* O_{X_1}(Y_1))$ for $i$ the inclusion $Y_1 \hookrightarrow X_1$.

We first show that for $\omega_{X_1} = \Omega^d_{X_1}$ and $\omega_{Y_1} = \Omega^{d-1}_{Y_1}$, we have that

$$H^{a>0}(X_1, \omega_{Y_1} \otimes_{O_{X_1}} O_{X_1}(Y_1)) = 0$$

if $Y_1$ is of high degree. By [11, Ch. II, Prop. 8.20] we know that

$$\omega_{Y_1} \cong \omega_{X_1} \otimes O_{Y_1} \otimes O_{X_1}(Y_1).$$

This implies that $\omega_{X_1}|_{Y_1} = \omega_{Y_1}(-Y_1)$ and therefore that the sequences

$$0 \to \omega_{X_1}(Y_1) \to \omega_{X_1}(2Y_1) \to \omega_{Y_1}(Y_1) \to 0$$

and

$$H^a(X_1, \omega_{X_1}(2Y_1)) \to H^a(X_1, \omega_{Y_1}(Y_1)) \to H^{a+1}(X_1, \omega_{X_1}(Y_1))$$

are exact. Since by Serre vanishing $H^a(X_1, \omega_{X_1}(2Y_1)) = H^a(X_1, \omega_{X_1}(Y_1)) = 0$ for $a > 0$ and $Y_1$ of sufficiently high degree, this implies that if $Y_1$ is of sufficiently high degree we also have that $H^{a>0}(X_1, \omega_{Y_1}(Y_1)) = H^{a>0}(Y_1, \omega_{Y_1} \otimes_{O_{X_1}} O_{X_1}(Y_1)|_{O_{Y_1}}) = 0$.

We now consider the exact sequence

$$0 \to \Omega^{b-1}_{Y_1}(-Y_1) \to \Omega^b_{X_1}|_{Y_1} \to \Omega^b_{Y_1} \to 0$$

coming from the conormal exact sequence $0 \to O_{Y_1}(-Y_1) \to \Omega^1_{X_1}|_{Y_1} \to \Omega^1_{Y_1} \to 0$. Tensoring with $O_{Y_1}(2Y_1)$ gives an exact sequence

$$0 \to \Omega^{b-1}_{Y_1}(Y_1) \to \Omega^b_{X_1}(2Y_1)|_{Y_1} \to \Omega^b_{Y_1}(2Y_1) \to 0.$$

This implies that the sequence

$$H^a(X_1, \Omega^b_{Y_1}(2Y_1)) \to H^{a+1}(X_1, \Omega^{b-1}_{Y_1}(Y_1)) \to H^{a+1}(X_1, \Omega^b_{X_1}(2Y_1))$$

is exact. The proposition follows inductively. \qed

We can now deduce the following Lefschetz theorem:

**Proposition 4.3.** Let $Y_1$ be a hypersurface section of $X_1$ and $d = \dim X_1$. Then there is a map

$$\phi : H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^d(X_1, \Omega^{d-1}_{X_1})$$

which is an isomorphism for $d \geq 4$ and surjective for $d = 3$ if $Y_1$ is of high degree.
Proof. Let $i$ denote the inclusion $Y_1 \hookrightarrow X_1$. We define $\phi$ to be the composition
\[ H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^{d-1}(Y_1, R^1i^!\Omega^{d-1}_{X_1}) \cong H^d(Y_1, \Omega^{d-1}_{Y_1}) \to H^d(X_1, \Omega^{d-1}_{X_1}) \]
where the first map is induced by the Gysin map
\[ g : \Omega^{d-2}_{Y_1} \to R^1i^!\Omega^{d-1}_{X_1}, \omega \mapsto \omega \wedge \frac{df_d}{f_d} \]
(see [7, Ch. II, (3.2.13)]) with $f_d$ is the regular parameter defining $Y_1$. Since $H^d(X_1 - Y_1, \Omega^{d-1}_{X_1 - Y_1}) = 0$ for $d \geq 1$, we have that
\[ H^d(Y_1, \Omega^{d-1}_{Y_1}) \cong H^d(X_1, \Omega^{d-1}_{X_1}) \]
for $d \geq 2$. We are therefore reduced to showing that $g$ induces an isomorphism on $H^{d-1}$ for $d - 1 \geq 3$ and a surjection for $d - 1 = 2$. We define a filtration
\[ g(\Omega^{d-2}_{Y_1}) = \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \bigcup_{i \geq 0} \mathcal{F} = R^1i^!\Omega^{d-1}_{X_1}, \]
letting $\mathcal{F}_i$ be the subsheaf of $R^1i^!\Omega^{d-1}_{X_1}$ locally defined by
\[ \langle \omega \wedge \frac{df_d}{f_d} | n_d \geq i \rangle. \]
Here $\omega \in \Omega^{d-2}_{Y_1}$.

Let $gr_iR^1i^!\Omega^{d-1}_{X_1} := \mathcal{F}_{i+1}/\mathcal{F}_i$. Then $gr_iR^1i^!\Omega^{d-1}_{X_1} \cong \Omega^{d-2}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}$ and the short exact sequence
\[ 0 \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to gr_iR^1i^!\Omega^{d-1}_{X_1} \to 0 \]
induces the following exact sequence on cohomology groups:
\begin{align*}
H^{d-2}(Y_1, \Omega^{d-2}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}) & \to H^{d-1}(Y_1, \mathcal{F}_i) \to H^{d-1}(Y_1, \mathcal{F}_{i+1}) \\
& \to H^{d-1}(Y_1, \Omega^{d-2}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}})
\end{align*}
(4.1)

By Proposition 4.2 we have that if $Y_1$ is of high degree, then $H^{d-1}(Y_1, \Omega^{d-2}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}})$ vanishes for $d > 2$ and $H^{d-2}(Y_1, \Omega^{d-2}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}})$ for $d > 3$.

This implies that the maps $H^a(Y_1, \mathcal{F}_i) \to H^a(Y_1, \mathcal{F}_{i+1})$ are isomorphisms for $d \geq 4$ and surjective for $d = 3$ if $Y_1$ is of sufficiently high degree. Since $H^a(Y_1, \lim \mathcal{F}_i) \cong \lim H^a(Y_1, \mathcal{F}_i)$ (see [7, Ch. III, Prop. 2.9]), the same holds for the maps $H^a(Y_1, \mathcal{F}_i = \Omega^{d-2}_{Y_1}) \to H^a(Y_1, R^1i^!\Omega^{d-1}_{X_1})$. In particular, for $d = \dim X_1 = 3$ we get that $H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^d(X_1, \Omega^{d-1}_{X_1})$ is surjective and for $d = \dim X_1 \geq 4$ that $H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^d(X_1, \Omega^{d-1}_{X_1})$ is an isomorphism. 

\[ \square \]

Corollary 4.4. Let $X$ be as in the introduction and $d \geq 3$. Let $Y_1$ a smooth hypersurface section of $X_1$. Let $\alpha \in H^d(X_1, \mathcal{K}^M_{d,X_1})$. If $Y_1$ is of sufficiently high degree and contains the image of $\alpha$ in $\text{CH}_0(X_1)$ under the restriction map $H^d(X_1, \mathcal{K}^M_{d,X_1}) \to H^d(X_1, \mathcal{K}^M_{d,x_1}) \cong \text{CH}_0(X_1)$, then $Y_1$ lifts to a smooth projective subscheme $Y$ of $X$ over $A$ and $\alpha$ is in the image of $H^{d-1}(Y_1, \mathcal{K}^M_{d-1,Y_1}) \to H^d(X_1, \mathcal{K}^M_{d,X_1})$. 


Proof. That $Y_1$ lifts to a smooth projective subscheme $Y$ of $X$ over $A$ if it is of high degree follows from Serre vanishing. We do the $n = 2$ case. The general case follows inductively. Consider the commutative diagram

\[
\begin{array}{ccc}
H^d(X_1, \Omega^{d-1}_{X_1}) & \rightarrow & H^d(X_1, \mathcal{K}^M_{d,X_2}) \\
\downarrow & & \downarrow \\
H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) & \rightarrow & H^{d-1}(Y_1, \mathcal{K}^M_{d-1,Y_2}) \\
\end{array}
\]

induced by the (right-)exact sequence of sheaves

\[
\Omega^{d-1} \rightarrow \mathcal{K}^M_2 \rightarrow \mathcal{K}^M_2 \rightarrow 0
\]
on $X_1$ and $Y_1$. The statement follows from Proposition 4.3 and a simple diagram chase.

For the sake of completeness we also prove the following Lefschetz theorem. We will not use it though.

**Proposition 4.5.** Let $Y_1$ be a smooth hypersurface section of $X_1$ and $d = \dim X_1$. Let $i$ denote the inclusion $Y_1 \hookrightarrow X_1$. Then the map

\[
i^* : H^q(X_1, \Omega^p_{X_1}) \rightarrow H^q(Y_1, \Omega^p_{Y_1})
\]
is an isomorphism for $p + q < d - 1$ and injective for $p + q = d - 1$ if $Y_1$ is of high degree.

Proof. We factorise the map $i^* : \Omega^p_{X_1} \rightarrow i_* \Omega^p_{Y_1}$ by

\[
i_* \Omega_{X_1} \rightarrow i_* (\Omega_{X_1})_{|Y_1}
\]
followed by

\[
i_*(\Omega_{X_1})_{|Y_1} \rightarrow i_* \Omega^p_{Y_1}
\]
and show that each of these maps induce isomorphisms, resp. injections, on cohomology in the stated range.

We first consider the exact sequence

\[
0 \rightarrow \Omega^p_{X_1}(-Y_1) \rightarrow \Omega^p_{X_1} \rightarrow \Omega^p_{X_1}|_{Y_1} \rightarrow 0.
\]

This induces the exact sequence

\[
H^q(X_1, \Omega^p_{X_1}(-Y_1)) \rightarrow H^q(X_1, \Omega^p_{X_1}) \rightarrow H^q(Y_1, \Omega^p_{X_1}|_{Y_1}) \rightarrow H^{q+1}(X_1, \Omega^p_{X_1}(-Y_1)).
\]

By Serre duality $H^q(X, \Omega^p_{X_1}(-Y_1)) \cong H^{d-q}(X, \Omega^{d-p}_{X_1}(Y_1))$. This implies that $H^q(X_1, \Omega^p_{X_1}) \rightarrow H^q(Y_1, \Omega^p_{X_1}|_{Y_1})$ is an isomorphism for $p + q < d - 1$ and injective for $p + q = d - 1$ if $Y_1$ is of sufficiently high degree by Serre vanishing.

We now consider the exact sequence

\[
0 \rightarrow \Omega^{p-1}_{Y_1}(-Y_1) \rightarrow \Omega^p_{Y_1}|_{Y_1} \rightarrow \Omega^p_{Y_1} \rightarrow 0
\]
on $Y_1$. This induces the exact sequence

\[
H^q(Y_1, \Omega^{p-1}_{Y_1}(-Y_1)) \rightarrow H^q(Y_1, \Omega^p_{Y_1}|_{Y_1}) \rightarrow H^q(Y_1, \Omega^p_{Y_1}) \rightarrow H^{q+1}(Y_1, \Omega^{p-1}_{Y_1}(-Y_1))
\]
which by Serre duality and Proposition 4.2 implies that $H^q(Y_1, \Omega^p_{X_1}|_{Y_1}) \to H^q(Y, \Omega^p_Y)$ is an isomorphism for $p + q < d + 1$ and injective for $p + q = d + 2$ if $Y_1$.

**Remark 4.6.** Proposition 4.5 is an analogue of the following theorem (see for example [21, Thm. 1.29]): Let $X$ be an $n$-dimensional compact complex variety and let $Y \hookrightarrow X$ be a smooth hypersurface such that the line bundle $\mathcal{O}_X(Y) = (\mathcal{I}_Y)^*$ is ample. Then the restriction

$$j^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

is an isomorphism for $k < n - 1$ and injective for $k = n - 1$.

This theorem may be deduced from Kodaira vanishing using the Hodge decomposition for $Y_C$ and $X_C$. We note that when working over an arbitrary field $k$, one seems to need the extra assumption that $Y$ is of high degree to obtain Lefschetz theorems.

5. **Main theorem**

We return to the situation of the introduction. Let $A$ be a henselian discrete valuation ring with uniformising parameter $\pi$ and residue field $k$. Let $X$ be a smooth projective scheme over $\text{Spec}(A)$ of relative dimension $d$. Let $X_n := X \times_A A/(\pi^n)$, i.e. $X_1$ is the special fiber and the $X_n$ are the respective thickenings of $X_1$. We assume furthermore that either (1) $k$ is of characteristic 0 and $A = k[[\pi]]$ or that (2) $A$ is the Witt ring $W(k)$ of a perfect field $k$ of $\text{ch}(k) > 2$.

Let us first recall how one can lift a regular closed point $x \in X_1$ to a 1-cycle on $X$: Let $\{f_1, ..., f_d\} \subset \mathcal{O}_{X_1, x}$ be a generating set of local parameters and let $\{\tilde{f}_1, ..., \tilde{f}_d\}$ be lifts of these generators to $\mathcal{O}_{X, x}$. The ideal $f_1\mathcal{O}_{X, x} + ... + f_d\mathcal{O}_{X, x}$ defines a subscheme of $\text{Spec}\mathcal{O}_{X, x}$ and its closure in $X$ defines a subscheme $Z$ of $X$. The unique irreducible component of $Z$ containing $x$ is a prime-cycle $C \subset Z_1(X)$ which is flat and finite over $A$. Such liftings are of course not unique.

We also introduce the following notation: Let $X$ be a scheme and $Z$ an effective Cartier divisor on $X$. Let $C$ be a curve in $X$, i.e. an effective 1-cycle on $X$. Let

$$(Z, C)_x := \text{length}_{\mathcal{O}_{X, x}}(\mathcal{O}_{X, x}/I_Z + I_C)$$

be the intersection multiplicity of $Z$ and $C$ at $x$. We say that $Z$ and $C$ intersect transversally at $x$ if $(Z, C)_x = 1$ and if $Z$ and $C$ are regular at $x$. If $Z$ and $C$ intersect transversally everywhere, we denote this by $Z \cap C$.

In this section we show the following proposition:

**Proposition 5.1.** Let $X$ be of relative dimension 2 over $A$. Then, assuming the Gersten conjecture for the Milnor $K$-sheaf $K^M_{2, X_n}$, the map $\text{res}_{X_n} : \text{CH}_1(X) \to H^2(X_1, K^M_{2, X_n})$ is surjective. In particular the map

$$\text{res} : \text{CH}_1(X) \to \text{"lim}_n H^2(X_1, K^M_{2, X_n})$$

is an epimorphism in pro-$\text{Ab}$.
We need some preparation for the proof. From now on we assume that $d = 2$ and in particular that $\dim X_1 = 2$. Consider the (right-)exact sequence

$$\Omega^1_{X_1} \to \mathcal{K}^M_{2,X_2} \to \mathcal{K}^M_{2,X_1} \to 0.$$  

We will lift elements which lie in the kernel of $\text{res} : H^2(X_1, \mathcal{K}^M_{2,X_2}) \to H^2(X_1, \mathcal{K}^M_{2,X_1})$ in a compatible way to $\text{CH}_1(X)$. The kernel of $\text{res}$ is in the image of $H^2(X_1, \Omega^1_{X_1})$. Now since $\Omega^1_{X_1}$ is CM, $H^2(X_1, \Omega^1_{X_1})$ is isomorphic to

$$\text{coker}(\oplus_{x \in X_1} H^1_x(X_1, \Omega^1_{X_1}) \to \oplus_{x \in X_1} H^2_x(X_1, \Omega^1_{X_1})).$$

In order to proceed, we need to study this cokernel and the occurring local cohomology groups a bit further. By Lemma 2.3 we have that $H^2_x(X_1, \Omega^1_{X_1})$ is generated by differential forms of the form

$$\frac{df_1}{f_1^{n_1} f_2^{n_2}} \Omega_{X_1,x} \oplus \frac{df_2}{f_1^{n_1'} f_2^{n_2'}} \Omega_{X_1,x} \mod \frac{df_i}{f_j^{n_j}} \Omega_{X_1,x}$$

for $\{f_1, f_2\}$ a system of local parameters in $\mathcal{O}_{X_1,x}$ and $i, j \in \{1, 2\}$.

We define subgroups

$$F_r := \langle \frac{df_1}{f_1^{n_1} f_2^{n_2}} + \frac{df_2}{f_1^{n_1'} f_2^{n_2'}} | n_1 + n_2 - 1 \leq r, n_1' + n_2' - 1 \leq r >$$

of $H^2_x(X_1, \Omega^1_{X_1})$ with respect to a system of local parameters. Sometimes we therefore write $H^2_x(X_1, \Omega^1_{X_1})_{(f_1, f_2)}$ instead of $H^2_x(X_1, \Omega^1_{X_1})$ to indicate with respect to which system of local parameters we are working. Then

$$0 \subset F_1 \subset F_2 \subset \cdots \subset H^2_x(X_1, \Omega^1_{X_1})_{(f_1, f_2)}$$

defines an ascending filtration on $H^2_x(X_1, \Omega^1_{X_1})$. We will call elements of $F_1$ forms with simple poles. The following lemma shows that this definition is in fact independent of the chosen parameter system, meaning that there is a natural isomorphism between $H^2_x(X_1, \Omega^1_{X_1})_{(f_1, f_2)}$ and $H^2_x(X_1, \Omega^1_{X_1})_{(f_1', f_2')}$ for two local parameter systems $\{f_1, f_2\}$ and $\{f_1', f_2'\}$ inducing isomorphisms on the respective filtrations. This isomorphism is given by considering a differential form with respect to the respective defining parameter systems.

**Lemma 5.2.** (1) Let $x \in X_1$ be a closed point. Then subgroups $F_r \subset H^2_x(X_1, \Omega^1_{X_1})$ are independent of the local parameter system we consider them in.

(2) In particular, the subgroup

$$F_1 = \langle \frac{df_1}{f_1^{n_1} f_2^{n_2}} + \frac{df_2}{f_1^{n_1'} f_2^{n_2'}} | \Omega_{X_1,x} \rangle \subset H^2_x(X_1, \Omega^1_{X_1})$$

is independent of the chosen local parameter system of $\mathcal{O}_{X_1,x}$. We therefore denote it by $\Lambda_x$.

(3) The graded pieces

$$F_{r+1}/F_r$$

are independent of the chosen local parameter system of $\mathcal{O}_{X_1,x}$.
Proof. It suffices to show the proposition for two parameter systems \((f_1, f_2)\) and \((f_1, f_2')\) and \(f_2 = f_2' + \beta f_1\). We saw in Section 2 that \(H^2_x(X_1, \Omega_{X_1}^1)\) can be calculated locally as \(\hat{H}^1(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}, \Omega_{X_1}^1)\) with respect to coverings of \(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}\). Now considering \(\frac{df_2}{f_1 f_2'} \in H^1(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}, \Omega_{X_1}^1)\) for the covering \(D(f_1) \cup D(f_2')\) of \(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}\), we can pass to the smaller covering \(D(f_1) \cup D(f_2')\) of \(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}\). With respect to this covering

\[
\frac{df_2}{f_1 f_2'} = \frac{df_2'}{f_1 f_2'} (1 + \beta f_2')^{n_2} + \frac{\beta df_1}{f_1 f_2'} (1 + \beta f_2')^{n_2}
\]

is equivalent to

\[
(\sum_{n=0}^{\infty} (-\beta f_2')^n) df_2' + (\sum_{n=0}^{\infty} (-\beta f_2')^n) \beta df_1
\]

since \(\frac{1}{f_1 f_2' (1 + \beta f_2')^{n_2}}\) converges to \(\frac{(-\beta f_2')^n}{f_1 f_2' (1 + \beta f_2')^{n_2}}\) in \(\mathcal{O}_{X_1, x}[\frac{1}{f_1 f_2'}]/\mathcal{O}_{X_1, x}[\frac{1}{f_1 f_2'}]\), which lies again in \(F_{n_1 + n_2 - 1}\) considering it as an element of \(\hat{H}^1(\text{Spec}\, \mathcal{O}_{X_1, x} \setminus \{x\}, \Omega_{X_1}^1)\) with respect to the covering \(D(f_1) \cup D(f_2')\). This proves (1). (2) and (3) follow immediately. \(\square\)

In order to prove Proposition 5.1 we need to prove key Lemma 5.4. Its proof is inspired by the techniques of [17] from which we cite the following lemma:

**Lemma 5.3.** ([17] Lemma 10.2) Let \(X\) be a noetherian scheme, \(E\) an effective Cartier divisor on \(X\), and \(A\) be an effective Cartier divisor on \(E\). Let \(\mathcal{F}\) be a coherent \(\mathcal{O}_X\)-module such that

\[
H^1(X, \mathcal{F} \otimes \mathcal{O}_X(-E)) = H^1(E, \mathcal{F}|_E \otimes \mathcal{O}_E(-A)) = 0.
\]

Then the restriction \(\text{res}_A : H^0(X, \mathcal{F}) \to H^0(A, \mathcal{F}|_A)\) is surjective.

By Lemma 5.2 we can talk about the pole order of elements of \(H^2_x(X_1, \Omega_{X_1}^1)\) independent of the parameter system chosen. In particular, the following lemma makes sense:

**Key lemma 5.4.** Every element \(\gamma \in H^2_x(X_1, \Omega_{X_1}^1)\) is equivalent to a sum of forms with simple poles in \(H^2_x(X_1, \Omega_{X_1}^1)\).

**Proof.** Without loss of generality we work with \(\gamma = \frac{\partial df_1}{f_1 f_2'} \in H^2_x(X_1, \Omega_{X_1}^1)\). Let \(D_1\) be a regular curve containing \(x\) and \(f'_1\) a local parameter of \(D_1\) at \(x\). We consider \(\frac{\partial df_1}{f_1 f_2'}\) in \(H^2_x(X_1, \Omega_{X_1}^1)(f'_1, f_2')\). By Lemma 5.2 \(\gamma\) is still in \(F_{r+1}\) for \(r + 1 = n_1 + n_2\). We may assume that it is of the form \(\frac{\alpha df_1'}{f_1 f_2'}\).

Let \(H \subset X_1\) be a hyperplane section and for an integer \(d > 0\) let \(\mathcal{L}(d) = |dH|\) be the linear system of hypersurface sections of degree \(d\).

Now for \(d \gg 0\) there exists an \(F^1 \in \mathcal{L}(d)\) such that

1. \(x \in F^1\),
2. \(F^1 \cap D_1\) at any \(y \in F^1 \cap D_1\).

For \(d'\) sufficiently large relative to \(d\), there exists an \(F^2 \in \mathcal{L}(d')\) such that
(1) $x \in F^2$, 
(2) $F^2 \cap D_1$ at any $y \in F^2 \cap D_1$,  
(3) $F^1 \cap F^2 \cap D_1 - x = \emptyset$.

We choose $F^3, \ldots, F^{n_2}$ analogously. Furthermore, we choose $F^{n_2}$ to be of sufficiently high degree so that 

$$H^1(X_1, \Omega^1_{X_1}((n_1 - 1)D_1 + F^1 + \ldots + F^{n_2})) = H^1(D_1, \Omega^1_{X_1}((n_1 - 1)D_1 + F^1 + \ldots + F^{n_2}) | D_1 \otimes \mathcal{O}_D(-x)) = 0$$

holds by Serre vanishing. By Lemma 5.3, the last condition implies that the restriction map

$$H^0(X_1, \Omega^1_{X_1}((n_1 - 1)D_1 + F^1 + \ldots + F^{n_2})) \xrightarrow{res} H^0(x, \Omega^1_{X_1}((n_1 - 1)D_1 + F^1 + \ldots + F^{n_2}) \otimes k(x))$$

is surjective. Let $y$ be the generic point of $D_1$. By construction, the diagram

$$
\begin{array}{cccc}
H^1_y(X_1, \Omega^1_{X_1}) & \xrightarrow{d_x} & F_{r + 1}/F_r H^2_x(X_1, \Omega^1_{X_1})
\end{array}
$$

is commutative and $\frac{\alpha df_1}{f_1} f_2$ lies in $\Omega^1_{X_1}((n_1 - 1)D_1 + F^1 + \ldots + F^{n_2}) \otimes k(x)$. Notice that the map on the right is well-defined. This implies that there is a $\gamma \in H^1_y(X_1, \Omega^1_{X_1})$ such that $d_x(\gamma) = \frac{\alpha df_1}{f_1} f_2$. Furthermore for any $x' \in |D_1| - x$, the form $d_{x'}(\gamma)$ has at most simple poles in $f_2$ at $x'$, i.e. $d_{x'}(\gamma) \in F_{n_2 + 1}$. Now we apply the same construction to the form $d_{x'}(\gamma)$ which completes the proof. 

**Proof of Proposition 5.4** Let $x \in X$ be a closed point and $X_{1,x}$ be the spectrum of the stalk of $\mathcal{O}_{X_1}$ in $x$. The Čech to derived functor spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

induces an edge map

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}).$$

Since this edge map is functorial in $\mathcal{F}$, we get a commutative diagram

$$
\begin{array}{ccc}
\check{H}^1(X_{1,x} - x, \Omega^1_{X_1}) & \xrightarrow{\cong} & \check{H}^1(X_{1,x} - x, \mathcal{K}^M_{2,X_1})
\end{array}
$$

$$
\begin{array}{ccc}
H^1(X_{1,x} - x, \Omega^1_{X_1}) & \cong H^2_x(X_1, \Omega^1_{X_1}) & \xrightarrow{\cong} \ H^1(X_{1,x} - x, \mathcal{K}^M_{2,X_1}) \cong H^2_x(X_1, \mathcal{K}^M_{2,X_1})
\end{array}
$$

for a closed point $x \in X_1$. We saw in Lemma 2.3 that $\check{H}^1(X_{1,x} - x, \Omega^1_{X_1})$ is generated by elements of the form $\frac{\alpha_1 df_1}{f_1} f_2 + \frac{\alpha_2 df_1}{f_1} f_2$ for a local parameter system $(f_1, f_2) \in \mathcal{O}_{X_1,x}$ and by key Lemma 5.4 we may assume that it has simple poles. The point of the proof is that for forms with simple poles we can write down explicit lifts of these forms to $\mathrm{CH}_1(X)$.

Without loss of generality we consider the form $\alpha df_1/(f_1 f_2)$. We lift $\alpha df_1/(f_1 f_2)$ to

$$V(f_1, f_2 + \pi^{n-1} \alpha) - V(f_1, f_2) \in \mathrm{CH}_1(X)$$
where \( \tilde{f}_1, \tilde{f}_2 \in \mathcal{O}_{X,x} \) are lifts of \( f_1 \) and \( f_2 \). We now show that this lift is mapped to \( \exp(\alpha df_1/(f_1 f_2)) = \{f_1, 1 + \pi^{-1} \alpha/f_2\} \in H^2_z(X_1, K_{2,X}^M) \) by the restriction map. The map

\[
\text{res} : CH_1(X) \to H^2(X, K_{2,X}^M),
\]

induced by the assumption of the Gersten conjecture, sends the cycle \( V(\tilde{f}_1, \tilde{f}_2 + \pi^{-1} \alpha) - V(f_1, f_2) \) to \( \{\tilde{f}_1, 1 + \pi^{-1} \alpha/f_2\} \) in

\[
\check{H}^1(X_{\tilde{f}_1, f_2 + \pi^{-1} \alpha}) - V(f_1, f_2 + \pi^{-1} \alpha), K_{2,X}^M) \oplus \check{H}^1(X_{V(\tilde{f}_1, f_2)} - V(f_1, f_2), K_{2,X}^M).
\]

Finally, the restriction map

\[
H^2(X, K_{2,X}^M) \to H^2(X, K_{2,X}^M)
\]

sends the tuple of Čech-cycles \( \{\tilde{f}_1, 1 + \pi^{-1} \alpha, \tilde{f}_2\} \) to \( \{\tilde{f}_1, 1 + \pi^{-1} \alpha/f_2\} \in \check{H}^1(X_1 - x, K_{2,X}^M) \). In sum this shows in particular that

\[
\ker[H^2(X_1, K_{2,X}^M) \to H^2(X_1, K_{2,X}^M)]
\]

is in the image of \( CH_1(X) \).

The surjectivity of \( CH_1(X) \to H^2(X_n, K_{2,X}^M) \) now follows by induction: For \( n = 1 \) this is just the surjectivity of \( CH_1(X) \to CH_0(X) \). For \( n > 1 \), let \( \alpha \in H^2(X_1, K_{2,X}^M) \). Let \( \alpha_{n-1} \) be the image of \( \alpha \) in \( H^2(X_1, K_{2,X}^M) \). By assumption there is a cycle \( Z \in CH_1(X) \) mapping to \( \alpha_{n-1} \). Denote the image of \( Z \) in \( H^2(X_n, K_{2,X}^M) \) by \( \alpha_Z \). Now \( \alpha - \alpha_Z \) is in the kernel of \( H^2(X_1, K_{2,X}^M) \to H^2(X_1, K_{2,X}^M) \) and by the above construction lifts to an element \( Z' \in CH_1(X) \). Now \( Z' - Z \) maps to \( \alpha \).

**Corollary 5.5.** Let \( X \) be as in Proposition 5.1 but of arbitrary relative dimension \( d \) over \( A \). Then, assuming the Gersten conjecture for the Milnor \( K \)-sheaf \( K_{d,X}^M \), the map \( \text{res}_{X_n} : CH_1(X) \to H^d(X_1, K_{d,X}^M) \) is surjective. In particular the map

\[
\text{res} : CH_1(X) \to \lim_n H^d(X_1, K_{d,X}^M)
\]

is an epimorphism in pro-\( Ab \).

**Proof.** This follows immediately from Corollary 4.1 Proposition 5.1 and standard Bertini arguments. \( \square \)
6. Open problems

The proof of Proposition 5.1 can be summed up in the following diagram:

\[
\begin{array}{ccc}
\Lambda_x & \xrightarrow{\text{lift}} & \Lambda_x \\
\downarrow & & \downarrow \\
\text{CH}_1(X) & \xrightarrow{\text{res}} & \text{H}^2(X_1, \Omega^1_{X_1}) \\
\downarrow & & \downarrow \\
\text{H}^2(X, \mathcal{K}^M_{2,X}) & \xrightarrow{\text{res}} & \text{H}^2(X_1, \mathcal{K}^M_{2,X_1}) \\
\downarrow & & \downarrow \\
\text{H}^2(X_1, \mathcal{K}^M_{2,X_1}) & \xrightarrow{\text{res}} & \text{H}^2(X_1, \mathcal{K}^M_{2,X_2})
\end{array}
\]

The question remains if there is a well-defined map \( \Lambda_x \to \text{CH}_1(X)/F_n \) for some filtration

\[ \ldots \subset F_2 \subset F_1 \subset \text{CH}_1(X) \]

making the above diagram commutative. This would make it possible to construct an inverse to the restriction map

\[ \text{res} : \text{CH}_1(X)/F_n \to \text{H}^2(X_1, \mathcal{K}^M_{2,X_n}). \]

Furthermore, if \( \lim_n F_n \otimes \mathbb{Z}/p^n \mathbb{Z} \), then this would imply the following conjecture by Kerz, Esnault and Wittenberg:

**Conjecture 6.1.** [16, Sec. 10] The map

\[ \text{res} : \text{CH}^d(X) \otimes \mathbb{Z}/p^n \mathbb{Z} \to \lim_n \text{H}^d(X_1, \mathcal{K}^M_{d,X_n}/p^n) \]

is an isomorphism in pro-Ab if \( \text{ch} (\text{Quot}(A)) = 0 \) and if \( k \) is perfect of characteristic \( p > 0 \).

Let us consider the case where \( X \) is of relative dimension 1 over \( A \). Let \( F_n \) be the subgroup of \( \text{CH}_1(X) \) generated by all cycles \( Z \) vanishing on \( X_n \), i.e. \( Z|_{X_n} = 0 \). By Lemma 2.2 we know that \( \text{H}^1(X_1, \mathcal{O}_{X_1}) \cong \mathcal{O}_{X_1,x}[1/f]/\mathcal{O}_{X_1,x} \) for a local parameter \( f \in \mathcal{O}_{X_1,x} \). In this case we can define a map

\[ \gamma_x : \mathcal{O}_{X_1,x}[1/f]/\mathcal{O}_{X_1,x} \to \text{CH}_1(X)/F_n \]

by

\[ \alpha = \frac{\alpha_0}{f^m} \rightarrow V(\tilde{f}^m + \alpha_0 \pi^{n-1}) - V(\tilde{f}^m) \]

where \( \tilde{f} \in \mathcal{O}_{X,x} \) is a lifting of \( f \). That \( \gamma_x \) is well-defined can be seen as follows: Let \( \tilde{f}_1, \tilde{f}_2 \in \mathcal{O}_{X,x} \) be liftings of \( f \). Then \( (1 + \frac{\alpha_0}{f_1^m} \pi^{n-1})/(1 + \frac{\alpha_0}{f_2^m} \pi^{n-1}) \equiv (1 + \frac{\alpha_0}{f_1^m} - \frac{\alpha_0}{f_2^m} \pi^{n-1}) = 1 \mod \pi_n \).

Let \( A \) is as in case (2). Then by an argument similar to that in Section 3 where we showed Conjecture [6.1] for \( d = 1 \), we see that \( \lim_n F_n \otimes \mathbb{Z}/p^n \mathbb{Z} = 0 \). Conjecture [6.1] would
therefore hold if \( \ker[\text{res} : \text{CH}_1(X)/F_n \to H^d(X_1, \mathcal{K}_d^{M,X_n})] \) is generated by the filtrations defined above on smooth relative curves. For a similar conjecture see [18, Conj. 5.7].

**Remark 6.2.** Let \( X_K \) be a smooth projective variety over a \( p \)-adic field \( K \). In [12, Sec. 10], a relation between Conjecture 6.1 and a question by Colliot-Thélène is postulated: is there a (non-canonical) isomorphism

\[
\text{CH}_0(X_K) \cong \mathbb{Z} \oplus \mathbb{Z}_p^m \oplus (\text{finite group}) \oplus (\text{divisible group})?
\]

Let us sketch this relationship. Let \( A \) be the ring of integers in \( K \). Let \( X \) be a smooth and projective scheme of relative dimension \( d \) over \( A \). For a smooth projective (over \( A \)) subscheme of codimension one \( Y \subset X \) we expect that the map

\[
\lim_n H^{d-1}(X_1, \mathbb{K}_d^{M,X_n}/p^r) \to \lim_n H^d(X_1, \mathbb{K}_d^{M,X_n}/p^r)
\]

is an isomorphism for \( d \geq 3 \) and surjective for \( d = 2 \) if we choose \( Y \) to be of high degree. For \( n = 1 \), this follows from class field theory and standard Lefschetz theorems for the étale fundamental group. In order to prove this for arbitrary \( n \), the Lefschetz theorem of Section 4: Proposition 4.3 needs to be improved by one degree. This and the injectivity of \( \text{res} \) for arbitrary dimension would imply that the map

\[
\text{CH}_1(Y)/p^r \to \text{CH}_1(X)/p^r
\]

is bijective for \( d \geq 3 \) and surjective for \( d = 2 \). Since \( \text{CH}_1(Y) \to \text{CH}_0(Y_K) \) and \( \text{CH}_1(X) \to \text{CH}_0(X_K) \) are surjective, the same statement would hold for

\[
\text{CH}_0(Y_K)/p^r \to \text{CH}_0(X_K)/p^r.
\]

For a curve \( C_K \) over \( K \) we know that

\[
A_0(C_K) \cong A^m \oplus (\text{finite group})
\]

for some \( m \in \mathbb{N} \) (see [19]). This implies, under the above assumptions, the same result for the \( p \)-completion of \( \text{CH}_1(X_K) \) and therefore a positive answer to the above Question.

The corresponding weak Lefschetz theorem for \( l \) prime to \( p \) saying that the map

\[
\text{CH}_1(Y)/l^r \to \text{CH}_1(X)/l^r
\]

is bijective for \( d \geq 3 \) and surjective for \( d = 2 \) is proved in [20, Cor. 9.6].

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