Geometry of Dynamical Systems

A. A. Kocharyan

Department of Mathematics, Monash University, Clayton Victoria 3168, Australia

Abstract. We propose a geometrical approach to the investigation of Hamiltonian systems on (Pseudo) Riemannian manifolds. A new geometrical criterion of instability and chaos is proposed. This approach is more generic than well known reduction to the geodesic flow. It is applicable to various astrophysical and cosmological problems.

1 Introduction

Our main aim is to study global behaviour of dynamical systems on (Pseudo) Riemannian manifolds using their local, geometric properties. The guiding principle comes from the well known example of dynamical system: geodesic flow.

If geodesic flow on Riemannian manifold is given, then to work out principal features of the flow, so-called sectional curvature should be investigated. And if it turns out that sectional curvature is negative definite, then one deduces that geodesic flow is Anosov with hyperbolic geodesics.

As is well known, any Hamiltonian system (with the Hamiltonian of the form kinetic plus potential) is reducible to a geodesic flow. Therefore, above mentioned procedure may be used for the new system to get some characteristic features of the original system. In our opinion it is not a proper way to study dynamical systems. First of all, it does not work for all Hamiltonian systems (there must not be turning points i.e. kinetic energy must be positive), then in order to reduce to a geodesic flow one has to change the metric, Levi-Civita connection, and time parameter. And since sectional curvature essentially depends on the above mentioned values, one must make inverse transforms, in order to obtain characteristic parameters for the original system, which is not a trivial problem for typical systems.

Here, in particular, we are looking for a local, geometrical object (it should be the sectional curvature in the case of the geodesic flow) for Hamiltonian systems which should characterise instability, hyperbolicity of dynamical systems.

We adopt the following approach to the problem, which may be considered as an alternative to the approach mentioned above. Instead of changing metric, Levi-Civita connection (covariant derivative), and time, we change only the derivative. We define a new covariant derivative $D_u$ for any flow, which is determined uniquely and for simple cases, coincides with the well known derivatives ($\nabla_u$-covariant derivative, $F_u$-Fermi derivative). By means of the derivative we introduce a coordinate independent equation for the flow invariant separation vector (perpendicular to the velocity). For the geodesic flow it is nothing but the Jacobi equation. As a by-product we represent any invariant vector field as a sum of vectors, which are perpendicular and parallel to the velocity respectively.

\footnote{Published in “Proceedings of the Fourth Monash General Relativity Workshop” Eds A.Lun, L.Brewin, E.Chow (Department of Mathematics, Monash University, 1993), p.38}
Then we introduce a geometrical object $\Omega_u$ negativity of which means hyperbolicity of the corresponding flow. This method does work even for systems with turning points and moreover, which is crucial for the Einstein dynamics, the geometrical approach applies for Pseudo-Riemannian manifolds as well.

## 2 Geometry of Dynamical Systems

### 2.1 Dynamical Systems

Let $S$ be a smooth vector field on the tangent bundle $\tau : TM \to M$ of a Riemannian manifold endowed with a Riemannian metric $g$, its associated Levi-Civita connection $K$ with covariant metric $\nabla$ (cf. [1])

$$S : TM \to TTM : u \mapsto (u, -\partial V) \in Hor(u) \oplus Ver(u),$$

where $V : M \to \mathbb{R}$ and $\partial V$ is the gradient vector of $V$, i.e. vector field such that

$$\langle dV, X \rangle = g(\partial V, X), \quad \forall X \in TM,$$

$Hor(u)$ and $Ver(u)$ are horizontal and vertical subspaces of $T_u TM$ respectively. We denote by $u(t) = f^t u_0$ an integral curve of the vector field $S$ passing through the initial point $u_0 \in TM$ i.e.

$$\dot{u} = S(u),$$

and $u(0) = u_0$. The group of diffeomorphisms $\{f^t\}, t \in \mathbb{R}$

$$f^t : TM \to TM : u_0 \mapsto u(t),$$

is called a dynamical system or flow of the vector field $S$ on $TM$. The curve $u(t) = f^t u_0$ is called the flow line or integral curve starting at $u_0$, $c = \tau \circ u$ is a trajectory.

**Lemma 2.1** If $u(t) = f^t u_0$ is an integral curve of the vector field $S$ starting at $u_0$, then $c(t) = \tau \circ u(t)$ is a solution of the following equation:

$$\begin{cases}
\dot{c} = u, \\
\nabla_u u = -\partial V,
\end{cases}$$

(1)

determined by $c(0) = \tau \circ u_0$ and $\dot{c}(0) = u_0$.

**Lemma 2.2** If $\nabla_c \dot{c} = -\partial V$, then $\dot{c}(t) = f^t \dot{c}(0)$.

We are interested in the behaviour of the nearby integral curves of the integral curve $u(t) = f^t u_0$. To deal with that problem, next we introduce an equation for the invariant vector field along the curve $c(t)$.

**Lemma 2.3** Invariant vector fields $Z(t)$ along the integral curve $u(t) = f^t u_0$, i.e.

$$Z(t) = Tf^t Z_0, \quad Z_0 \in T_{u_0} TM$$
are in correspondence with the solutions of the following equation:
\[ \nabla_u^2 z + \Re_u(z) - H_V(z) = 0 , \tag{2} \]
where \( \Re_u(z) = \text{Riem}(z, u)u, H_V(z) = -\nabla_z \partial V. \) The correspondence is given by
\[ z(t) \leftrightarrow Z(t) = (z(t), \nabla_u z(t)) \in \text{Hor}(u(t)) \oplus \text{Ver}(u(t)) . \]

Note that one might consider Eq. (2) as an equation obtained from Eq. (1) merely by covariant differentiation with respect to \( z. \)

Thus we arrive at the following system of variational equations
\[
\begin{cases}
\dot{c} = u , \\
\nabla_u u = -\partial V , \\
\nabla_u^2 z + \Re_u(z) - H_V(z) = 0 .
\end{cases}
\tag{3}
\]

It is worth mentioning that the vector \( z \) describes the motion of points (but not integral curves) of the nearby integral curves of \( u(t). \) However, in most cases it is important to know the behaviour of nearby integral curves (which means \( z \) is orthogonal to velocity \( u \)) with the same integrals.

If \( V = 0 \) then the flow corresponding to \( S \) is called a geodesic flow and Eq. (3) reads as
\[
\begin{cases}
\dot{c} = u , \\
\nabla_u u = 0 , \\
\nabla_u^2 z + \Re_u(z) = 0 ,
\end{cases}
\]
the last equation is the Jacobi equation.

We say that the Jacobi field \( z \) (solution of the Jacobi equation) is orthogonal to \( u \) if the following conditions hold \( g(u, z) = 0 \) and \( z(E) = g(u, \nabla_u z) = 0 \) along integral curve. One can represent any Jacobi field as follows:
\[ z = n + [n(E)t + g(u, z)]u , \tag{4} \]
where \( n \) is orthogonal to \( u, \) i.e. \( g(u, n) = g(u, \nabla_u n) = 0 . \) One may prove (see [2]) that in the case of the geodesic flow \( n \) is orthogonal to \( u \) for all \( t \) if it is so at \( t = 0. \) So if \( z(0) = n(0) \) and \( n(E) = 0 \) then \( z(t) = n(t) \) for arbitrary \( t. \) It is not true for a general flow.

It is an important property of the geodesic flow, because it means that the equation for \( n \) is the same as for \( z \) and one should only carefully choose the initial conditions.

As is well known (see [3]) by changing the time parameter \( t, \) metric \( g, \) and linear connection \( K \) (covariant derivative \( \nabla \)) one can reduce a Hamiltonian system to a geodesic one.

**Theorem 2.1** On \( TM|_E = \{ u \in TM \mid g(u, u) = 2(E - V(\tau \circ u)) \neq 0 \} , \) there exist

**G1.** \( s = \varphi(t) \) parameter such that \( \dot{\varphi} = \sqrt{2}|E - V| , \)

**G2.** Metric \( \hat{g} = |E - V|g , \)
Linear Levi-Civita connection $\hat{K}$ with a covariant derivative

\[
\hat{\nabla}_X Y = \nabla_X Y + X(\phi)Y + Y(\phi)X - g(X, Y)\partial \phi ,
\]

where $\phi = \frac{1}{2} \ln |E - V|$, such that instead of Eq. (3) one has

\[
\begin{cases}
\gamma' = v, \\
\nabla_v v = 0, \\
\nabla^n_2 v + \hat{R}(v) = 0 ,
\end{cases}
\]

where $\gamma' = \frac{d\gamma}{ds}$, $\hat{R}$ is the Riemannian tensor of the linear connection $\hat{K}$,

\[
\frac{1}{2}g(u, u) = E - V \iff \hat{g}(v, v) = \text{sgn}(E - V) .
\]

So any Hamiltonian system can be reduced to a geodesic one. But this procedure has several demerits:

1) time parameter $t$ changes to affine parameter $s$, to recover $t$ one has to find function $\varphi$;

2) metric $g$ changes to $\hat{g}$ it changes Levi-Civita linear connection and the measure of length of vectors.

But as is well known local stability/instability does depend on metric and time. And in order to reduce a Hamiltonian system to a geodesic one we have changed the both concepts. Therefore, to make a conclusion about stability/instability of original system one has to make the inverse transform. Which turns out to be a complicated one. To overcome the problem we suggest an alternative method. For this purpose we will introduce a new covariant derivative for dynamical systems and derive desired equation by help of the derivative.

Now we will see how to overcome these difficulties.

### 2.2 Covariant Derivative of Dynamical System

Let us introduce a new covariant derivative along curve $c$. Given a smooth curve $c : R \rightarrow M$, $c \in C^\infty (R, M)$, with the velocity vector $u = \dot{c} = T c \frac{d}{dt}$. We denote the set of all smooth vector fields (functions) along curve $c$ by $\mathcal{X}_c (\mathcal{F}_c)$ respectively. First we define the set of curves $\mathcal{C}$ along which one may define the covariant derivative.

**Definition 2.1** We say that $c \in \mathcal{C}$ if there exists unique smooth $(1,1)$ type tensor field $Q_c$ along $c$ such that

\[
Q_c = \frac{u \otimes u^\flat}{g(u, u)}
\]

for all $u$ such that $g(u, u) \neq 0$.

Clearly, if $g(u, u) \neq 0$ along a curve $c$ then $c \in \mathcal{C}$. Another important subset of $\mathcal{C}$ is established by the following Lemma.
Lemma 2.4 If $c$ is a non-constant solution of the Eq. \((\text{7})\) then $c \in \mathbb{C}$ and

$$Q_c = \begin{cases} \frac{u \otimes u}{\|u\|^2} & \text{if } g(u, u) \neq 0, \\ \frac{\partial V \otimes dV}{\|dV\|^2} & \text{if } g(u, u) = 0. \end{cases}$$

Let $P_c = \delta - Q_c$.

Definition 2.2 Given $c \in \mathbb{C}$ and $X \in \mathcal{X}_c$. We will say that $X \in \mathcal{X}_c^D$ i.e. $X$ is a $D$-differentiable along $c$ if there exists unique smooth vector field $D_u X \in \mathcal{X}_c$ such that if $g(u, u) \neq 0$ then

$$D_u X = P_c \nabla_u X + Q_c \mathcal{L}_u Q_c X.$$ \hspace{1cm} (5)

Obviously, if $g(u, u) \neq 0$ along $c$ then $\mathcal{X}_c^D = \mathcal{X}_c$. If $g(X, u) = fg(u, u)$, where $f \in \mathcal{F}$, then $X \in \mathcal{X}_c^D$. In particular, $u \in \mathcal{X}_c^D$. Besides, if $g(X, u) = 0$ then $X \in \mathcal{X}_c^D$. $D_u : \mathcal{X}_c^D \rightarrow \mathcal{X}_c : X \mapsto D_u X$ is a derivative (see \[2\]).

Some easy-to-check properties of $D_u$:

1. $D_u X = Q_c \mathcal{L}_u X$ if $Q_c X = X$ ($P_c X = 0$).
2. $D_u X = P_c \nabla_u X$ if $P_c X = X$ ($Q_c X = 0$).
3. $D_u = \nabla_u$ if $\nabla_u u = 0$ i.e. $c$ is a geodesic.
4. $D_u = F_u$ if $g(u, u) = \text{const} \neq 0$. $F_u$ is the Fermi derivative.
5. $D_u X = P_c \nabla_u P_c X + \mathcal{L}_u Q_c X$.

The derivative along $c$ can be naturally extended from vector fields to arbitrary tensor fields as follows:

**D1.** $D_u$ is a linear mapping of tensor fields of any type along $c$ to tensor fields of the same type, which commutes with contractions;

**D2.** $D_u(T_1 \otimes T_2) = D_uT_1 \otimes T_2 + T_1 \otimes D_uT_2$;

**D3.** $D_u f = u(f) = \frac{df}{dt}$, for any function $f$.

Theorem 2.2 There exists exactly one derivative determined by \((\text{5})\) with 1-2 and D1-D3 properties.

Definition 2.3 Given $c \in \mathbb{C}$ and $X \in \mathcal{X}_c$. We say that $X$ is a $D$-parallel vector field along $c$ if $D_u X = 0$.

Theorem 2.3 For any $Y \in T_c(0)M$ (if $g(u, u)(0) = 0$ then $Q_c Y = 0$) there exists unique parallel vector field $X$ along $c$ such that $D_u X = 0$ and $X(0) = Y$.  

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If \( Q_c X |_0 = 0 = Q_c Y |_0 \) and \( D_u X = 0 = D_u Y \) then \( g(X, u) = 0, g(Y, u) = 0, \) and \( g(X, Y) = \text{const} \) along curve. Notice that \( D_u u = 0. \)

Now we derive the desired equation for \( n = P_z c \) by terms of the covariant derivative.

**Theorem 2.4** Given \( c \) which satisfies the Eq. (1). Then any solution of the Eq. (2) has the following form (we consider limit-values of all objects at \( g(u, u) = 0 \))

\[
z = n - \left[ \int_0^t \frac{-2n(V) + n(E)}{2(E - V)} dt \right] u,
\]

where \( n = P_z c \) satisfies the following equation

\[
D_u^2 n + \Omega_u(n) = \frac{n(E)}{E - V} P_c \partial V,
\]

\( Q_c n |_{t_0} = Q_c D_u n |_{t_0} = 0, \)

where

\[
\Omega_u(n) = \Re_u(n) + P_c \left[ -H_V(n) + \frac{3n(V)}{2(E - V)} \partial V \right],
\]

\[
E = \frac{1}{2} g(u, u) + V = \text{const} \quad \text{along integral curve},
\]

\( n(E) = z(E) = g(u, \nabla u z) + z(V) = \text{const} \quad \text{along integral curve}.\)

Let us mention that if \( V = 0, \) and \( E = \frac{1}{2} \) along the integral curves (geodesics with \( g(u, u) = 1 \)) then decomposition of \( z \) Eq. (6) coincides with the Jacobi field decomposition Eq. (4). If we consider all integral curves of fixed energy then the following claim holds.

**Corollary 2.1** If \( Q_c z |_{t_0} = 0, z(E) = 0, \) then

\[
z = n - \left[ \int_{t_0}^t \frac{n(V)}{E - V} dt \right] u,
\]

where \( n = P_z c \) satisfies the following equation

\[
D_u^2 n + \Omega_u(n) = 0,
\]

\( Q_c n |_{t_0} = Q_c D_u n |_{t_0} = 0. \)

**Corollary 2.2** If \( g(u, u) \neq 0 \) (there is no turning point) and \( n(E) = 0 \) then

\[
\begin{align*}
\{ & D_u^2 n + \Omega_u(n) = 0, \\
& g(u, u) |_0 = g(u, D_u n) |_0 = 0 \}
\iff
\{ & \nabla^2 n + \Re_\nu(n) = 0, \\
& \tilde{g}(u, n) |_0 = \tilde{g}(u, \nabla u n) |_0 = 0 \}.
\end{align*}
\]

One can derive an equation for the length of \( n \) from Eq. (7). Let \( n = \ell \xi, \) where \( ||n|| = \ell \) and therefore \( g(\xi, \xi) = 1 \) and \( g(\xi, u) = 0. \) Then

\[
\ddot{\ell} + [\omega_u(\xi, \xi) - g(D_u \xi, D_u \xi)] \ell = 0,
\]

\( 6 \)
\[
\omega(u, \xi) = g(\Omega(u), \xi) = K(\xi, u) - h_V(\xi, \xi) + \frac{3|V|^2}{2(E-V)} ,
\]
\[
K(\xi, u) = g(\Re(u), \xi) = \text{Riem}(u, \xi, \xi) ,
\]
\[
h_V(\xi, \xi) = g(H_V(\xi), \xi) .
\]

And
\[
D_u^2 u + 2\lambda D u + (\dot{\lambda} + \lambda^2) u + \Omega(u) = 0 ,
\]
where \(\lambda = \dot{\ell}/\ell\).

It is worth noting that \(\Omega(u)\) and \(\omega(u, \xi)\) are smooth functions with respect to their all arguments \((x, u, \xi)\) along integral curves, even at turning points.

And if \(u(0) = 0\) then
\[
\omega_0(\xi, \xi) = \lim_{t \to 0} \omega(u, \xi) = \langle dV \rangle k(\xi, \xi) ,
\]
where \(k\) is the second fundamental form of the \(V = E\) submanifold in \(M\) and \(\langle dV, \xi \rangle = 0\).

**Theorem 2.5** If \(SM_E = \{ u / T M \mid \frac{1}{2} g(u, u) + V(\tau \circ u) = E \}\) is a compact manifold, \(g(u, u) + \|dV\|^2 \neq 0\) (i.e. there is no singular point), and
\[
\omega(u, \xi) < 0 ,
\]
where \(u \in SM_E\), \(g(\xi, \xi) = 1\), and \(g(u, \xi) = 0\) then the dynamical system \(\ell\) is Anosov.

Condition (9) may be fulfilled unless dynamical system has an integral on \(SM_E\). One may average \(\omega(u, \xi)\) by replacing \(\xi \otimes \xi\) with
\[
\xi \otimes \xi = \frac{1}{d-1} \left( g - \frac{u \otimes u}{2(E-V)} \right) ,
\]
to get
\[
\mathcal{B}(u) = \frac{1}{d-1} \left[ \text{Ric}(u, u) - \Delta V + \frac{h_V(u, u)}{g(u, u)} + \frac{3|u \wedge \partial V|^2}{g(u, u)^2} \right] ,
\]
where
\[
|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2 ,
\]
and
\[
\mathcal{B}_0(x) = \lim_{t \to 0} \mathcal{B}(u) = \frac{1}{d-1} \left[ -\Delta V + \frac{h_V(\partial V, \partial V)}{\|dV\|^2} \right] .
\]
One may speculate that a solution of the Eq. \(\mathcal{B}\) in average is greater than a solution of the following “evolution equation” (cf. \(\mathcal{M}\), \(\mathcal{B}\))
\[
\ell(t) + \mathcal{B}(t) \ell(t) = 0 ,
\]
where \(\mathcal{B}(t) = \mathcal{B}(u(t))\). Notice that Eq. \(\mathcal{B}\) is an exact if \(d = 2\).
2.3 Pseudo-Riemannian Manifolds

The impossibility of direct application of the results developed for Riemannian manifolds is evidently connected with the Pseudo-Riemannian signature of the metric, e.g. it is easy to see that two close geodesics can diverge in Pseudo-Riemannian manifold, while the length of the separation vector may remain close to zero or even negative. Therefore not only new criteria are needed here but one should also redefine the stability properties themselves.

Consider a dynamical system on Pseudo-Riemannian manifold $M$ (see [5]). Let $c$ be a trajectory passing through a point $c(0) = m \in M$ and $\{E^0_a\} a = 1, \ldots, d$ basis for $T_m M$. One can propagate $\{E^0_a\}$ along $c$ so that

$$D_a E_a = 0 , \quad E_a(0) = E^0_a , \quad \forall a = 1, \ldots, d ,$$

and get a basis $\{E_a\}$ along the curve $c$ for any $T_{c(t)} M$. We will call it $D$-basis. Any vector $X \in T_{c(t)} M$ can be presented by means of the $D$-basis $\{E_a\}$

$$X(t) = X^a(t) E_a .$$

The expression

$$g_E(X, Y) = \sum_a X^a Y^a$$

defines $E$-metric on $TM$. Length of the vector $X$ with respect to the basis $\{E_a\}$ is defined to be

$$\|X\|_E = (g_E(X, X))^{1/2} = \left(\sum_a (X^a)^2\right)^{1/2} .$$

Let $\{E'_a\}$ be another $D$-basis along the same curve $c$. Then a non-singular matrix $\Phi$ exists, such that

$$E^0_a = \sum_{b'} \Phi_{a}^{b'} E^0_{b'} .$$

So far as both $\{E_a\}$ and $\{E'_a\}$ being $D$-bases are $D$-parallel transported along $c$, this relation must hold for constant $\Phi$. Therefore

$$E_a = \sum_{b'} \Phi_{a}^{b'} E_{b'} .$$

Thus we have

$$X^{b'}(t) = \sum_a \Phi_{a}^{b'} X^a(t) .$$

Since $\Phi$ is non-singular then

$$\sum_{b'} (X^{b'})^2 > 0 ,$$
for if \( \sum_{b'}(X^{b'})^2 = 0 \) we get

\[
\sum_a \Phi_{ab} X^a(t) = 0 ,
\]

it is a contradiction with non-singularity of \( \Phi \). Therefore we have a positive defined matrix

\[
\Psi_{ac}X^a X^c = \sum_{b'} \Phi_{ab} \Phi_{c'b'}X^a X^c = \sum_{b'}(X^{b'})^2 > 0 .
\]

Space of all \( X \) with \( \|X\|_E^2 = 1 \) is a compact space, therefore there are positive constants \( \alpha \) and \( \beta \) (depending only on basis) such that

\[
0 < \alpha \sum_a (X^a)^2 \leq \sum_{b'} (X^{b'})^2 \leq \beta \sum_a (X^a)^2
\]
or

\[
0 < \alpha \|X\|_E^2 \leq \|X\|_{E'}^2 \leq \beta \|X\|_E^2 .
\]

For the vector field \( n \): \( n = n^a E_a \), \( D_u n = n^a E_a \), and

\[
\ell_E(n) = (\|n\|_E^2 + \|D_u n\|_E^2)^{1/2} .
\]

Now we can define stability, hyperbolicity, etc. of integral curves.

**Definition 2.4** Integral curve \( u \) is called to be linear stable if \( \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \) such that \( \ell_E(n(0)) < \delta(\varepsilon) \Rightarrow \ell_E(n(t)) < \varepsilon \), for all \( t \geq 0 \), where \( n \) is a solution of the Eq. (7).

If \( u \) is not linear stable it is called linear unstable.

One can readily verify that the definition does not depend on the choice of \( D \)-basis \( \{E_a\} \). Lyapunov characteristic exponents \( \chi \) for an integral curve \( u \) is defined as follows

\[
\chi(u, n) = \limsup_{t \to \infty} \frac{\ln \ell_E(n)}{t} .
\]

Evidently \( \chi \) does not depend on choice of \( \{E_a\} \) either. One can give a definition of hyperbolicity as well (see [4]).

Let \( n = \ell \xi \), where \( \|n\|_E = \ell \) and therefore \( \|\xi\|_E = 1 \), then (cf. Eq. [8])

\[
\ddot{\ell} + \left[ g_E(\Omega_a(\xi), \xi) - g_E(D_u \xi, D_u \xi) \right] \ell = 0 ,
\]

where

\[
g_E(\Omega_a(\xi), \xi) = g_E(\Re_a(\xi), \xi) - g_E(HV(\xi), \xi) + \frac{3g(\xi, \partial V)g_E(\xi, \partial V)}{2(E - V)} .
\]
3 Einstein Dynamics on Superspace

3.1 World, Universe, Superspace

The set of $d$-dimensional Universes will be described as follows (see [6]). We assume, that the Universe is closed (compact and without boundary). By $\mathcal{M}^{d+1}$ we denote the set of all $d+1$-dimensional, smooth (from the class $C^r$, $r > 2$), oriented, compact manifolds ($d > 1$),

$$\mathcal{M}^{d+1} = \{M^{d+1}\} = \{\text{all } d+1\text{-dimensional smooth, oriented, compact manifolds }\}.$$ 

World (i.e. spacetime with material fields) will mean the following triad:

$$(M^{d+1}, g(M), \Phi(M)),$$

where $M^{d+1} \in \mathcal{M}^{d+1}$, and $g(M)$ is a smooth Riemannian metric on $M^{d+1}$, $\Phi(M)$ is a smooth scalar field. Denote the set of worlds by $W^{d+1}$,

$$W^{d+1} = \{w\} = \{(M^{d+1}, g(M), \Phi(M))\} = \{(M_w, g_w, \Phi_w)\}.$$

Let $\mathcal{F}_c(M^{d+1})$ be a set of smooth functions on $M^{d+1}$ without singular critical points (Morse’s function) such that $f \in \mathcal{F}_c(M^{d+1})$ if

$$f : M^{d+1} \to S^1 = [0, 2\pi]/\{0, 2\pi\},$$

$$f \in C^r(M^{d+1}),$$

$$\text{if } \partial M^{d+1} \neq \emptyset \text{ then } f(\partial M^{d+1}) = \partial[f(M^{d+1})],$$

where $\partial N$ is the boundary of the manifold $N$. For every $c \in S^1$, $f \in \mathcal{F}_c(M^{d+1})$ we denote

$$f_c[M^{d+1}] = \{x| x \in M^{d+1}, f(x) = c\},$$

$$Y\{f_c[M^{d+1}]\} = \{x| x \in f_c[M^{d+1}], df|_{f_c[M^{d+1}]}(x) = 0\}.$$

For given $w \in W^{d+1}$, $f \in \mathcal{F}_c(M_w)$, and $c \in S^1$ we have the following triad:

$$u(w, f, c) = (f_c[M_w], g, \phi),$$

where $g$ is the metric induced on $f_c[M_w]$ by $g_w$, $\phi = \Phi_w|_{f_c[M_w]}$. Universes (i.e. space with material fields) are members of the set $\mathcal{U}^d$,

$$\mathcal{U}^d = \bigcup_{w \in W^{d+1}} \bigcup_{f \in \mathcal{F}_c(M_w)} \bigcup_{c \in S^1} u(w, f, c),$$

$$\mathcal{U}^d = \{u\} = \{(T_u, g_u, \phi_u)\}.$$ 

According to Morse’s theory

$$\mathcal{U}^d = \Sigma^d \cup \Omega^d \cup \emptyset,$$

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where $\Sigma^d$ is the set of all $d$-dimensional smooth, oriented, closed manifolds with a smooth Riemannian metric and a smooth scalar field on them. We will consider the empty set $\emptyset$ as a trivial manifold.

At Morse’s reconstructions $\Omega^d$ are critical sets with a given metric and scalar fields.

In order to construct the set $U^d$ as a topological space one needs to introduce a topology, a system of open sets. We will do it as follows: first we define the set of “smooth” curves on $U^d$.

A mapping $\lambda : (0, 1) \to U^d$ is called a “smooth” one in $U^d$, if $\exists w \in W^{d+1}$ and $\exists f \in F_c(M_w)$ such that

$$\lambda(w, f)(\tau) = u(w, f, \tau), \quad \tau \in (0, 1) \subset S^1.$$ 

$\mathcal{O}$ is the strongest topology on the set $U^d$, such that every “smooth” curves from the following set

$$\bigcup_{w \in W^{d+1}} \bigcup_{f \in F_c(M_w)} \lambda(w, f)$$

is continuous on $(0, 1)$.

By Superspace we mean an $U^d$ endowed with the topology $\mathcal{O}(U^d, \mathcal{O})$. We denote that space by $U^d$ again.

The space of Universes with given manifold $M$ we denote by $U^d_M$,

$$U^d_M = \{ u \in U^d | T_u = M \}.$$ 

### 3.2 Geometry of Superspace

Let us fix any $M \in \Sigma^d$ and consider $U^d_M$. In this case $U^d_M$ is the space of all smooth Riemannian metrics and smooth scalar fields on $M$. It is known that there exists smooth Banach structure on such spaces.

We will introduce a metric on $U^d_M$ such that kinematical part of the Hamiltonian given by ADM formalism could be expressed by that metric. So we have the following metric on $U^d_M$

$$\mathcal{G}[g, \phi](k, \chi; h, \theta) = \int_M \left[ -\text{tr}(k)\text{tr}(h) + \text{tr}(k \times h) + \chi \theta N^{-1} \right] d\mu(g),$$

where $N$ is a positive function $N : M \to R_+ = \{ x \in R | x > 0 \}$ and

$$\begin{align*}
(g, \phi) &\in U^d_M, \\
(k, \chi), (h, \theta) &\in T(g, \phi)U^d_M, \\
d\mu(g) &= (\det g)^{1/2} dx^1 \wedge \ldots \wedge dx^d, \\
\text{tr}(k) &= g^{ab}k_{ab}, \\
(k \times h)_{ab} &= k_{ac}g^{cd}h_{db}.
\end{align*}$$

The metric has an inverse metric $\mathcal{G}^{-1}$ and

$$\mathcal{G}^{-1}[g, \phi](\pi, p; q, g) = \int_M \left( -\frac{1}{d-1} \text{tr}(\pi')\text{tr}(g') + \text{tr}(\pi' \times g') + p'q' \right) N d\mu(g).$$
where

\[(\pi, p), (q, q) \in T_{(g, \phi)}U^d_M, \quad \pi = \pi' d\mu(g), \ldots, q = q' d\mu(g).\]

By means of this metric one can map \(TU^d_M\) on \(T^*U^d_M\) and vice versa – \(T^*U^d_M\) on \(TU^d_M\). These mappings are of the following form:

\[
G^2 : T^*U^d_M \to TU^d_M,
\]

\[
G^2[g, \phi](k, \chi) = (-tr(k)g^{-1} + k^{-1}, \chi) d\mu(g) = (-tr(k)g^{ab} + k^{ab}, \chi) d\mu(g),
\]

\[
G^3 : \Gamma [g, \phi](\pi, \omega; h, \theta) = 1 \int_M \{ -tr(k)tr(\omega)tr(h)
\]

\[
+3tr(k)tr(\omega \times h) + tr(\omega)tr(k \times h) + tr(h)tr(k \times \omega)
\]

\[
-4tr(k \times \omega \times h) + tr(h)\chi \varphi + tr(\omega)\chi \theta - tr(k)\varphi \theta \} N^{-1}d\mu(g),
\]

and

\[
Riem[g, \phi](\omega, \varphi; k, \chi; l, \sigma; h, \theta)
\]

\[
= \int_M \{(1/4)tr[(\bar{k} \times \bar{\omega} - \bar{\omega} \times \bar{k}) \times (\bar{h} \times \bar{l} - \bar{l} \times \bar{h})]
\]

\[
+\kappa^2 [tr(\bar{\omega} \times \bar{l})tr(\bar{k} \times \bar{h}) - tr(\bar{h} \times \bar{\omega})tr(\bar{k} \times \bar{l})]
\]

\[
+tr(\bar{\omega} \times \bar{l})\chi \theta - tr(\bar{\omega} \times \bar{h})\chi \sigma + tr(\bar{k} \times \bar{h})\varphi \sigma - tr(\bar{k} \times \bar{l})\varphi \theta \} N^{-1}d\mu(g),
\]

where

\[
\kappa^2 = \frac{d}{16(d-1)}, \quad \bar{\omega} = \omega - \frac{tr(\omega)}{d} g.
\]

Consider now the dynamics with the following Hamiltonian in \(M\)-superspace

\[
\mathcal{H}[g, \phi, \pi, p] = \frac{1}{2} G^{-1}[g, \phi](\pi, p; \pi, p) + V[g, \phi],
\]

where

\[
V[g, \phi] = \int_M \{-R(g) + \frac{1}{2}||d\phi||^2 + F(\phi)\} d\mu(g),
\]

\[
||d\phi||^2 = g^{ab}\phi_a\phi_b.
\]

The dynamics given by this Hamiltonian

\[
\nabla_a u = -\partial V,
\]

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where $\nabla$ is a covariant derivative on the $M$-Superspace, corresponds to Einstein’s equations together with the following constraints equations,

$$\mathcal{H}[g, \phi; \pi, p] = 0,$$

$$\mathcal{P}_b[g, \phi; \pi, p] = -2\pi^a_{b|a} + p\phi|_b = 0.$$ These equations constrain only initial conditions, but not dynamics.

Now to investigate stability/instability of the solutions of the Einstein’s equations the following objects should be used \[7\], (in order to derive $\Omega_u$)

$$\partial V = N\{S^* - c \cdot F g\} N + Hess(N), \{F' + \Delta \phi\} - g(\partial \phi, \partial N)$$

and

$$-H_V(h, \chi) = \nabla(h, \chi) \partial V = \mathcal{P}(h, \chi) + D(h, \chi) + N(h, \chi),$$

where

$$\mathcal{P}(h, \chi) = N^2 \left( \tfrac{1}{2} r(h) + c\{\text{tr}(S \times h) - F'\chi\} g - (\text{tr}(S) + F)h, \right.$$  

$$-\text{tr}(Hess(\phi) \times h) + F''\chi);$$

$$D(h, \chi) = N^2 \left( \tfrac{1}{2}(\Delta h - \alpha \delta h + Hess(\text{tr}(h)) - d\phi \otimes d\text{tr}(h))^*,$$

$$-g(\partial \phi, \delta h + \tfrac{1}{2} d\text{tr}(h)) + \Delta \chi);$$

$$N(h, \chi) = N(-D_g(Hess(N)) h, -g(\partial \chi, \partial N)), \right.$$  

$$c = (2(d - 1))^{-1}, \quad \partial f = g^{ab} f|_{[b},$$

$$T^* = T - c \cdot tr(T) g, \quad S = Ric - \tfrac{1}{2} d\phi \otimes d\phi,$$

$$Hess(f) = -f|_{[a}, \quad \Delta f = \text{tr}(Hess(f)) = -g^{ab} f|_{[a|b},$$

$$\delta h = -h|_{[a|b}, \quad \alpha(X) = \mathcal{L}_X g = X_{a|b} + X_{b|a},$$

$$r(h)_{ab} = -Riem_{[c}a[d}h_{cd} - Riem_{b}c[a|d}h_{cd} + Ric_c^a h_{ab} + Ric_h_{ca}.\right.$$  

In further investigation \[7\] we will employ suggested method to the problem of stability of cosmological solutions.

### 4 Conclusions

Thus, we have proposed a geometrical approach to the investigation of Hamiltonian systems on (Pseudo) Riemannian manifolds. We have derived the equation for the normal to the velocity of the flow invariant vector Eq. \[7\] in terms of the new covariant derivative. We have introduced tensor $\Omega_u$ which characterises stability, instability, hyperbolicity of the integral curves of Hamiltonian systems. This approach is valid for any Hamiltonian system, while well known reduction to the geodesic flow requires some more conditions. Besides, there is no need to change metric and time parameter. This allows obtaining characteristic parameters by means of physical values. As an important application we will consider in details the behaviour of the N-body gravitating systems and cosmological solutions of the Einstein’s equations elsewhere \[7\].
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