A UNIFORM VERSION OF THE YAU-TIAN-DONALDSON CORRESPONDENCE FOR EXTREMAL KÄHLER METRICS ON POLARIZED TORIC MANIFOLDS

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Abstract. The aim of this paper is to solve a uniform version of the Yau-Tian-Donaldson conjecture for polarized toric manifolds. Also, we show a combinatorial sufficient condition for uniform relative K-polystability.

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1. Introduction

Let \((X, L)\) be an \(n\)-dimensional polarized algebraic manifold. We denote \(\mathcal{H}(X, L)\) by the set of all Kähler metrics of \(X\) in \(c_1(L)\). A Kähler metric \(\omega \in \mathcal{H}(X, L)\) is called a Calabi’s extremal Kähler metric if it is a critical point of the Calabi functional

\[
\mathcal{H}(X, L) \ni \alpha \mapsto \int_X (s(\alpha) - \overline{s})^2 \alpha^n \frac{n^n}{n!} \in \mathbb{R},
\]

where \(s(\alpha)\) is a scalar curvature of \(\alpha\) and \(\overline{s} = -n(K_X \cdot L^{n-1})/(L^n)\) is its average. Calabi showed in [14] that this condition is equivalent to that \(\text{grad}_{\omega}^{(1,0)}(s(\omega) - \overline{s})\) is a holomorphic vector field on \(X\). Extremal Kähler metrics have been widely studied as canonical Kähler metrics containing constant scalar curvature Kähler metrics and in particular Kähler-Einstein metrics. Especially, the existence problem of extremal Kähler metrics is known as the Yau-Tian-Donaldson conjecture, and is one of the central problem in Kähler geometry.

The Yau-Tian-Donaldson conjecture predicts that the existence of an extremal Kähler metric is equivalent to the stability of \((X, L)\) in some sense of geometric invariant theory. Székelyhidi introduced in [73] the notion of relative K-polystability as a candidate of such a stability condition. Roughly speaking, relative K-polystability requires the positivity of the relative Donaldson-Futaki invariant \(DF_V > 0\) for any nontrivial test object. Here \(V\) is the extremal Kähler vector field of \((X, L)\) ([40]), and test object is a special kind of degeneration of \((X, L)\), which is called a test configuration.

On the other hand, calculations by Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman [4] suggest that relative K-polystability might not be sufficient to ensure the existence of a extremal Kähler metric, and leads to an expectation that we need to strengthen the notion of relative K-polystability. For strengthenings, the following two approaches are known:

(a) Requiring that \(DF_V\) is bounded from below by a multiple of some ‘norm’ of test configurations ([72], [32], [12], [13]).
(b) Testing \(DF_V > 0\) for more general test objects than test configurations ([76], [63]).

In this paper, we will show the equivalence of both approaches for polarized toric manifolds. As a consequence, we will solve a uniform version of the Yau-Tian-Donaldson conjecture for polarized toric manifolds.
An $n$-dimensional polarized toric manifold $(X, L)$ corresponds to an $n$-dimensional integral Delzant polytope $P \subset \mathbb{R}^n$ which is written by an intersection of half spaces
\begin{equation}
P = \{ x \in \mathbb{R}^n \mid \langle \lambda_j, x \rangle + d_j \geq 0 \ (j = 1, \ldots, r) \},
\end{equation}
where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, $r$ is the number of facets of $P$, $\lambda_j \in \mathbb{Z}^n$, $d_j \in \mathbb{Z}$ and each $\lambda_j$ is a primitive vector. Then, any toric Kähler metric representing $c_1(L)$ corresponds to a smooth strictly convex function on $P^0$ which satisfies the Guillemin’s boundary condition (see Section 4.2).

Let $S$ be the set of all strictly convex functions on $P^0$ which satisfy the Guillemin’s boundary condition. Let $C^Q_{PL}$ be the set of all rational piecewise affine functions on $P$. As mentioned above, we can regard $C^Q_{PL}$ as the set of all toric test configurations for $(X, L)$. Also, we define $C_{PL}$ to be the set of all piecewise affine functions on $P$. Let $C_\infty$ be the set of all continuous convex functions on $P$ which is smooth in the interior. Further, let $P^*$ be the union of relative interiors of faces of $P$ up to codimension one, and $C_\ast$ be the set of all continuous convex functions on $P^*$ which are integrable on $\partial P$ with respect to the measure $\sigma$. We fix a point $x_0 \in P$ in the interior, and denote by $\tilde{C}_\ast$ the set of all $f \in C_\ast$ satisfying the normalized condition $\inf_{P^*} f = f(x_0) = 0$. Also, set $\tilde{F} = F \cap \tilde{C}_\ast$ for any $F \subset C_\ast$. Finally, for each $f \in C_\ast$ we denote the $J$-norm by $\| f \|_J$ (see Section 5.4).

Under the notations above, our first main result is the following equivalence.

**Theorem 1.0.1.** Let $F$ be one of the spaces $\mathcal{E}_1$, $C_\ast$, $C_{PL}$, $C^Q_{PL}$, $C_\infty$ or $S$. Then the followings are equivalent.

1. There exists a $\delta > 0$ such that $L_V(f) \geq \delta \| f \|_J$ for any $f \in F$.
2. There exists a $\delta > 0$ such that $L_V(f) \geq \delta \| f \|_J$ for any $f \in \tilde{F}$.
3. $L_V(f) \geq 0$ for any $f \in C_\ast$ and $L_V(f) = 0$ if and only if $f$ is affine.
4. $L_{\mathcal{E}_1}(f) \geq 0$ for any $f \in \mathcal{E}_1$ and $L_{\mathcal{E}_1}(f) = 0$ if and only if $f$ is affine.

Note that the proof of Theorem 1.0.1 does not require the Delzant condition and that this equivalence also holds for polarized toric varieties.

Let $(X, L)$ be a polarized toric manifold associated to an integral Delzant polytope $P \subset \mathbb{R}^n$. We say that $(X, L)$ is uniformly relatively $K$-polystable
if $P$ satisfies the condition $(J)_{\mathcal{L}^{PL}}$. $(X, L)$ is uniformly $K$-polystable if in addition that $V = 0$. By combining Theorem 1.0.1 with the works of [16], [80], [47] and [52], we can prove a uniform version of the Yau-Tian-Donaldson correspondence for polarized toric manifolds.

**Theorem 1.0.2** (The toric Yau-Tian-Donaldson correspondence). Let $(X, L)$ be an $n$-dimensional polarized toric manifold. Let $T$ be a maximal algebraic torus of $\text{Aut}^0(X, L)$, and $S$ be the maximal compact subgroup. Then the followings are equivalent.

1. $(X, L)$ admits an $S$-invariant extremal Kähler metric.
2. $(X, L)$ is uniformly relatively $K$-polystable.
3. The relative $K$-energy of $(X, L)$ is $T$-coercive.

The implications $(1) \Rightarrow (b)_{\mathcal{L}^{PL}}$ and $(b)_{\mathcal{L}^{PL}} \Rightarrow (3)$ are proved in [16] and [80], respectively. The implication $(3) \Rightarrow (1)$ can be obtained by the work of He [47] and its toric reduction by Legendre (see Section 3.2 of [52]). Finally, the equivalence $(J)_{\mathcal{L}^{PL}} \iff (b)_{\mathcal{L}^{PL}}$ is proved in Theorem 1.0.1.

If we consider the case that the extremal Kähler vector field of $(X, L)$ is zero, we obtain the following:

**Corollary 1.0.3.** Let $(X, L), T, S$ be as above. Then the followings are equivalent.

1. $(X, L)$ admits an $S$-invariant Kähler metric of constant scalar curvature.
2. $(X, L)$ is uniformly $K$-polystable.
3. The $K$-energy of $(X, L)$ is $T$-coercive.

We mention the very recent papers [50] and [56], which have some overlaps with the present paper.

As a simple application of Theorem 1.0.2, we obtain the following product theorem for uniform relative $K$-polystability.

**Corollary 1.0.4.** Let $(X_1, L_1)$ and $(X_2, L_2)$ be polarized toric manifolds, and set $X = X_1 \times X_2$ and $L = L_1 \boxtimes L_2$. If $(X_1, L_1)$ and $(X_2, L_2)$ are uniformly relatively $K$-polystable, then so is $(X, L)$.

In fact, if $(X_1, L_1)$ and $(X_2, L_2)$ are uniformly relatively $K$-polystable, then they admit torus invariant extremal Kähler metrics by Theorem 1.0.2. Since the product metric of extremal Kähler metrics is also an extremal Kähler metric, $(X, L)$ is uniformly relatively $K$-polystable by Theorem 1.0.2 again.

Next, we give a combinatorial sufficient condition for uniform relative $K$-polystability. Let $(X, L)$ be a polarized toric manifold associated an integral Delzant polytope $P$ defined by (1.1). We fix a point $x_0 \in P$ in the interior, and put $d_{x_0,j} = \langle \lambda_j, x_0 \rangle + d_j$ for each $j = 1, \ldots, r$. Note that $d_{x_0,j} > 0$ for any $j$. Then the following theorem strengthens [80] Theorem 0.1], where relative $K$-polystability was deduced under the same assumptions.
Theorem 1.0.5. Let $d_{x_0} = \max\{d_{x_0,1}, \ldots, d_{x_0,r}\}$. Suppose that $P$ satisfies either

\begin{equation}
V = 0 \quad \text{and} \quad \overline{s} < \frac{n+1}{d_{x_0}},
\end{equation}

or

\begin{equation}
V \neq 0 \quad \text{and} \quad \overline{s} + \max_P V \leq \frac{n+1}{d_{x_0}}.
\end{equation}

Then $(X, L)$ is uniformly relatively $K$-polystable.

In case that $(X, L) = (X, -K_X)$ is a toric Fano manifold with anticanonical polarization, then $\overline{s} = n$ and $d_j = 1$ for all $j$. Moreover, we can choose $x_0$ to be the origin of $\mathbb{R}^n$. Hence the conditions (1.2) and (1.3) reduce

\begin{equation}
V = 0
\end{equation}

and

\begin{equation}
V \neq 0 \quad \text{and} \quad \max_P V \leq 1,
\end{equation}

respectively. Combining this with the computations in [80] and [66] yields the following:

Corollary 1.0.6. (1) All toric del Pezzo surfaces are uniformly relatively $K$-polystable.

(2) In 3-dimensional case, at least 13 toric Fano manifolds out of 18 are uniformly relatively $K$-polystable.

It is known that all toric del Pezzo surfaces admit an extremal Kähler metric in its anticanonical class by the works of [14] (degree 8 case), [71] and [77] (degree 6 case) and [25] (degree 7 case). Corollary 1.0.6 gives the stability counterpart of this fact.

We finally concern uniform $K$-polystability of toric Fano manifolds. Note that the condition (1.4) is equivalent to that the Futaki invariant of $X$ vanishes ([40]). Also, it is known that the Futaki invariant vanishes if and only if the barycenter of $P$ coincides with the origin of $\mathbb{R}^n$ ([60]). Thus,

Corollary 1.0.7. Let $(X, -K_X)$ be an $n$-dimensional toric Fano manifold with anticanonical polarization associated to an integral Delzant polytope $P$. Then the followings are equivalent:

(1) $(X, -K_X)$ is uniformly $K$-polystable.

(2) $(X, -K_X)$ is $K$-polystable.

(3) $(X, -K_X)$ is $K$-semistable.

(4) The Futaki invariant of $X$ vanishes.

(5) The barycenter of $P$ coincides with the origin of $\mathbb{R}^n$.

(6) $V = 0$. 

2. Background materials

2.1. Notation and conventions. Throughout this paper, we use the following notation and conventions: In this paper, we mean an $n$-dimensional polarized manifold a pair $(X, L)$ of an $n$-dimensional compact complex manifold $X$ and an ample holomorphic line bundle $L$ over $X$.

- $S^1 = \mathbb{R}/\mathbb{Z}, \text{Lie}(S^1) = 2\pi \sqrt{-1}\mathbb{R}, \text{vol}(S^1) = 1$.
- $C = (-1/2)\mathbb{R} \oplus 2\pi \sqrt{-1}\mathbb{R} = (-\sqrt{-1}/4\pi)\text{Lie}(S^1) \oplus \text{Lie}(S^1)$.
- $C^k = C \setminus \{0\} = \exp((-1/2)\mathbb{R} \oplus 2\pi \sqrt{-1}\mathbb{R})$.
- $\omega(\cdot, \cdot) = (1/2\pi)g(J\cdot, \cdot)$ for any Kähler metric $g$. Here $J$ is the integrable almost complex structure of the complex manifold $X$.
- $\Delta = \Delta_d = 2\Delta = 2\int^i\nabla_i = \text{tr}_g\text{Hess}_g$ for any Kähler metric $g$.

Let $T$ be a maximal algebraic torus of $\text{Aut}^0(X)$, with $t = \text{Lie}(T)$.

- $A^1 = \text{Spec} C[t], G_m = \text{Spec} C[t, t^{-1}]$.
- $N = \text{Hom}(G_m, T)$, which is a lattice of rank $\dim C T$.
- $N_k = N \otimes \mathbb{Z} k$ for $k = \mathbb{Q}$ or $\mathbb{R}$.
- $M = \text{Hom}(N, \mathbb{Z}), M_k = M \otimes \mathbb{Z} k$ for $k = \mathbb{Q}$ or $\mathbb{R}$.
- $S = N \otimes \mathbb{Z} S^1$, which is the maximal compact subgroup of $T$, with $s = \text{Lie}(S) = 2\pi \sqrt{-1} N_\mathbb{R}$, which satisfies $t = (-\sqrt{-1}/4\pi)s \oplus s = (-1/2)N_\mathbb{R} \oplus 2\pi \sqrt{-1}N_\mathbb{R}$. Note that $M_\mathbb{R} = s^\vee$.

2.2. Extremal Kähler metrics. In this section, we recall the definition of extremal Kähler metrics. Let $(X, L)$ be an $n$-dimensional polarized manifold, and fix a maximal algebraic torus $T$ of $\text{Aut}^0(X)$. We denote by $\mathcal{H}(X, L)^S$ the set of all $S$-invariant Kähler metrics $\omega$ such that $\omega \in c_1(L)$. By setting $\overline{\sigma} = -n(K_X \cdot L^{n-1})/(L^n)$, we define $\Phi : \mathcal{H}(X, L)^S \to \mathbb{R}$ by

$$
\Phi(\omega) = \int_X (s(\omega) - \overline{\sigma})^2\omega^n = \frac{1}{\text{vol}_\omega(X)} \int_X (s(\omega) - \overline{\sigma})^2\omega^n,
$$

where $\text{vol}_\omega(X) = \int_X \omega^n$ and $\overline{\sigma} = \int_X s(\omega)\omega^n$. Note that $\text{vol}_\omega(X)$ and $\overline{\sigma}$ is independent of choice of $\omega \in \mathcal{H}(X, L)^S$ since $\text{vol}_\omega(X) = (L^n) = \langle c_1(L)^n, [X] \rangle$ and $\overline{\sigma} = -n(K_X \cdot L^{n-1})/(L^n)$. We call $\Phi$ the Calabi functional.

Definition 2.2.1 ([14]). A Kähler metric $\omega \in \mathcal{H}(X, L)^S$ is called an extremal Kähler metric if $\omega$ is a critical point of $\Phi$.

Kähler-Einstein metrics are extremal metrics, and (more generally) constant scalar curvature Kähler metrics in $\mathcal{H}(X, L)^S$ are typical examples of extremal Kähler metrics. The following fact is well-known.

Proposition 2.2.2 ([14] Theorem 2.1). A Kähler metric $\omega \in \mathcal{H}(X, L)^S$ is an extremal Kähler metric if and only if $\text{grad}_g(s(\omega) - \overline{\sigma})$ is a real holomorphic vector field on $X$. 

We fix $\omega \in \mathcal{H}(X,L)^S$. Set
\[
C^\infty(X,\omega,\mathbf{R})_0^S = \{ f \in C^\infty(X,\mathbf{R}) \mid \int_X f \omega^n = 0 \},
\]
\[
\mathfrak{s}(X,\omega)_0 = \{ u \in C^\infty(X,\omega,\mathbf{R})_0^S \mid 4\pi \text{grad}_g u \in \mathfrak{s} \}.
\]
Conversely, for each $W \in \mathfrak{s}$ there exists unique $u^W_\omega \in C^\infty(X,\mathbf{R})_0^S$ such that
\[
i_W \omega = -du^W_\omega, \quad \int_X u^W_\omega \omega^n = 0.
\]
Then we have
\[
-\frac{1}{4\pi}JW_1 + W_2
\]
for some $W_1, W_2 \in \mathfrak{s}$. Then we define $u^W_\omega \in C^\infty(X,\omega,\mathbf{C})_0^S$ by
\[
u^W_\omega = -\frac{1}{4\pi} \sqrt{-1} u^W_1 + u^W_2.
\]

Definition 2.3.1 (\cite{38, 39, 15, 7}). We call the Lie algebra character $F: t \to \mathbf{C}$ defined by
\[
F(W) = \int_X u^W_\omega (s(\omega) - \overline{s}) \omega^n
\]
the Futaki invariant of $(X, L)$.

We remark that (i) the definition of $F$ is independent of choice of $\omega \in \mathcal{H}(X,L)^S$, (ii) if $\mathcal{H}(X,L)^S$ contains a Kähler metric of constant scalar curvature, then $F$ vanishes identically, and (iii) an extremal Kähler metric $\omega \in \mathcal{H}(X,L)^S$ has constant scalar curvature if and only if $F = 0$ (see \cite{38, 39, 15, 7}).

Let $\text{pr}_\omega: C^\infty(X,\omega,\mathbf{R})_0^S \to \mathfrak{s}(X,\omega)_0$ be the $L^2$-orthogonal projection with respect to the volume form $\omega^n$.

Definition 2.3.2 (\cite{10}). Let $\theta_\omega = \text{pr}_\omega (s(\omega) - \overline{s}) \in \mathfrak{s}(X,\omega)_0$. We call the real holomorphic vector field defined by
\[
V := \text{grad}_g \theta_\omega \in N_{\mathbf{R}}
\]
the extremal Kähler vector field of $(X, L)$ with respect to $T$. 
It is known that (i) the definition of \( V \) is independent of choice of \( \omega \in \mathcal{H}(X, L)^S \), (ii) a Kähler metric \( \omega \in \mathcal{H}(X, L)^S \) is extremal if and only if \( s(\omega) - \tau = \theta_\omega \), and (iii) \( \text{grad}_g \theta_\omega \in N_Q \), that is, \( \exp(mV) = \text{id}_X \) for some \( m \in \mathbb{Z}_{>0} \) (see [40] and [64]).

**Remark 2.3.3.** We remark that the definition of \( V \) depends only on the choice of \( T \) and independent of neither the choice of \( K \) nor \( K \)-invariant Kähler metric in \( c_1(L) \). Indeed, by [40] \( V \) is independent of the choice of \( K \)-invariant Kähler metric in \( c_1(L) \) for fixed \( K \). If \( K' \) is another choice, then \( K \) and \( K' \) have a common maximal torus \( S \). Hence there exists \( \tau \in T \) so that \( K' = \tau K \tau^{-1} \) by [55, Proposition A.2]. Finally, by [40, Corollary E] we obtain \( V' = \text{Ad}(\tau)V = V \).

**Definition 2.3.4 ([40]).** We define \( B : t \times t \to \mathbb{C} \) by
\[
B(W, W') = \int_X u_W^W u_{W'}^{W'} \omega^n
\]
and call the Futaki-Mabuchi bilinear form.

It is known that (i) the definition of \( B \) is independent of choice of \( \omega \in \mathcal{H}(X, L)^S \), (ii) \( B \) is a nondegenerate and symmetric bilinear form on \( t \), and (iii) for any \( W \in t \)
\[
B(W, V) = -F(W),
\]

namely, \( F \) is dual to \( -V \) (see [40]). In particular, the Futaki invariant vanishes identically if and only if \( V = 0 \).

### 2.4. Riemannian geometry of the space of Kähler potentials.

In [58], Mabuchi studied a natural Riemannian structure on the space of Kähler potentials. Let \( (X, L) \) be an \( n \)-dimensional polarized manifold and fix \( \omega_0 \in \mathcal{H}(X, L)^S \). By the well-known \( dd^c \)-lemma, \( \mathcal{H}(X, L)^S \) can be identified as the space
\[
\mathcal{H}^S = \mathcal{H}(X, \omega_0)^S := \{ \varphi \in C^\infty(X, \mathbb{R})^S \mid \omega_\varphi := \omega_0 + dd^c\varphi > 0 \}.
\]

If \( \varphi \in \mathcal{H}^S \), the tangent space \( T_\varphi \mathcal{H}^S \) at \( \varphi \) can be identified with \( C^\infty(X, \mathbb{R})^S \).

Then the natural Riemannian metric on \( T_\varphi \mathcal{H}^S \) is given by
\[
\langle u, v \rangle_{\varphi} := \int_X uv \omega_\varphi^n.
\]

Let \( \{ \varphi^t \} \) be a smooth path in \( \mathcal{H}^S \), and \( \{ u^t \} \) be a smooth vector filed of \( \mathcal{H}^S \) along the path \( \{ \varphi^t \} \), namely, \( u^t \in T_{\varphi^t} \mathcal{H}^S = C^\infty(X, \mathbb{R})^S \) for each \( t \). Then the Levi-Civita connection is given by
\[
\nabla_{\varphi^t} u^t = u^t - \frac{1}{2} \langle d\varphi^t, du^t \rangle_{\varphi^t} g_{\varphi^t}.
\]
This immediately implies that $t \mapsto \varphi^t$ is a geodesic if and only if $\nabla_{\varphi^t} \dot{\varphi}^t = 0$, or equivalently
\[
\varphi^t - \frac{1}{2} |d\varphi^t|_{g_{\varphi^t}}^2 = 0.
\]
(2.1)

As discovered by Semmes [69] and Donaldson [33], the above equation can be understood as a complex Monge-Ampère equation as follows. For each $T \in (0, \infty]$, let $A = \{w \in \mathbb{C} \mid e^{-T/2} \leq |w| \leq 1\}$. Here we propose $A = \{w \in \mathbb{C} \mid 0 < |w| \leq 1\}$ if $T = \infty$. By setting $w = \exp(-t/2 + 2\pi\sqrt{-1}\theta)$, we have $t = -\log |w|^2$ and $e^{-T/2} \leq |w| \leq 1 \iff 0 \leq t \leq T$.

Let $\overline{X} = X \times A$. We regard $\overline{X}$ as a complex manifold with boundary. Let $\pi: \overline{X} \to X$ be the natural projection. For each smooth path $\{\varphi^t\}$ in $\mathcal{H}^S$, we define $\Phi: \overline{X} \to \mathbb{R}$ by
\[
\Phi(x, w) := \varphi - \log |w|^2(x).
\]

**Theorem 2.4.1** (69). We denote the $d$ and $d^c$-operator on $\overline{X}$ by $\overline{d}$ and $\overline{d}^c$, respectively. Then
\[
(\pi^*\omega_0 + \overline{d}\overline{d}^c\Phi)^{n+1} = (n+1) \left( \varphi^t - \frac{1}{2} |d\varphi^t|_{g_{\varphi^t}}^2 \right) \pi^*\omega_{\varphi^t}^n + \frac{\sqrt{-1}}{2\pi} \frac{dw \wedge d\overline{w}}{|w|^2}.
\]
In particular, $\{\varphi^t\}$ is a geodesic if and only if $\Phi$ satisfies
\[
(\pi^*\omega_0 + \overline{d}\overline{d}^c\Phi)^{n+1} = 0.
\]
(2.2)

Unfortunately, the Dirichlet problem of the equation (2.2) does not usually have a smooth solution (see [53]). This means that in general one cannot find a smooth geodesic segment connecting two points in $\mathcal{H}^S_0$. On the other hand, Chen [18] proved that the Dirichlet problem of (2.2) always has a unique solution in a weak sense ([8]). After successive refinement in [11] and [26], it is proved that the weak solution has $C^{1,1}$ regularity.

**2.5. Energy functionals.** In this subsection, we recall functionals on the space of Kähler potentials.

**Definition 2.5.1.** (1) We define the Monge-Ampère energy of $\varphi \in \mathcal{H}^S$ by
\[
E(\varphi) = \frac{1}{n+1} \sum_{i=0}^{n} \int_X \varphi \omega_{\varphi}^i \wedge \omega_0^{n-i}.
\]
(2) We define the $J$-functional of $\varphi \in \mathcal{H}^S$ by
\[
J(\varphi) = \int_X \varphi \omega_0^n - E(\varphi).
\]

**Fact 2.5.2.** (1) $E(\varphi + c) = E(\varphi) + c$ for any $\varphi \in \mathcal{H}^S$ and $c \in \mathbb{R}$.
(2) $J(\varphi) \geq 0$ and $J(\varphi + c) = J(\varphi)$ for any $\varphi \in \mathcal{H}^S$ and $c \in \mathbb{R}$.
(3) For any smooth path \( \{ \varphi^t \} \) in \( \mathcal{H}^S \) we have

\[
\frac{d}{dt} E(\varphi^t) = \int_X \dot{\varphi}^t \omega_{\varphi^t}^n,
\]

\[
\frac{d}{dt} J(\varphi^t) = \int_X \dot{\varphi}^t (\omega_0^n - \omega_{\varphi^t}^n).
\]

Here we mean \( \dot{\varphi}^t \) the derivative of \( \varphi^t \) with respect to \( t \).

By using \( E \), we define

\[
\mathcal{H}^S_0 = \mathcal{H}(X, \omega)_0^S := \{ \varphi \in \mathcal{H}^S \mid E(\varphi) = 0 \}.
\]

Then we have the identification \( \mathcal{H}^S \ni \varphi \mapsto (\varphi - E(\varphi), E(\varphi)) \in \mathcal{H}^S_0 \mathbb{R} \).

**Definition 2.5.3.** We define the functionals on \( \mathcal{H}^S_0 \) as follows:

1. We define the Ricci energy of \( \varphi \in \mathcal{H}^S \) by

\[
R(\varphi) = -\sum_{i=0}^{n-1} \int_X \varphi \text{Ric}(\omega_0) \wedge \omega_{\varphi}^i \wedge \omega_0^{n-i-1}.
\]

2. We define the entropy of \( \varphi \in \mathcal{H}^S \) by

\[
H(\varphi) = \int_X \log \left( \frac{\omega_0^n}{\omega_{\varphi}^n} \right) \omega_{\varphi}^n.
\]

3. We define the K-energy of \( \varphi \in \mathcal{H}^S \) by

\[
M(\varphi) = -\int_0^1 dt \int_X \dot{\varphi}^t (s(\omega_{\varphi^t}) - \bar{\omega}) \omega_{\varphi^t}^n,
\]

where \( \{ \varphi^t \}_{t \in [0,1]} \) is a smooth path in \( \mathcal{H}^S \) so that \( \varphi^0 = 0 \) and \( \varphi^1 = \varphi \).

**Fact 2.5.4.** (1) For any smooth path \( \{ \varphi^t \} \) in \( \mathcal{H}^S \) we have

\[
\frac{d}{dt} R(\varphi^t) = -n \int_X \dot{\varphi}^t \text{Ric}(\varphi_0) \wedge \omega_{\varphi^t}^{n-1}.
\]

(2) \( M(\varphi) = H(\varphi) + R(\varphi) + \bar{\omega} E(\varphi) \) for any \( \varphi \in \mathcal{H}^S_0 \), and in particular the definition of \( M(\varphi) \) is independent of choice of the path \( \{ \varphi^t \}_{t \in [0,1]} \) in \( \varphi \in \mathcal{H}^S_0 \) ([19], [20] Proposition 3.2).

(3) For any smooth path \( \{ \varphi^t \} \) in \( \mathcal{H}^S \) we have

\[
\frac{d}{dt} M(\varphi^t) = -\int_X \dot{\varphi}^t (s(\omega_{\varphi^t}) - \bar{\omega}) \omega_{\varphi^t}^n.
\]

In particular, \( \varphi \in \mathcal{H}^S_0 \) is a critical point of \( M \) if and only if the scalar curvature of \( \omega_{\varphi} \) is constant.

(4) For each \( \varphi \in \mathcal{H}^S_0 \) and \( W \in \mathfrak{s} \), put

\[
\varphi^t = \exp \left( -\frac{t}{4\pi} JW \right)^* \varphi.
\]
Then $\tilde{\varphi}^t = u^W_{\omega^t}$ and

$$\frac{d}{dt} M(\varphi^t) = \int_X u^W_{\omega^t} (s(\omega^t) - \pi) \omega^n_{\omega^t} = F(W).$$

In other words, the K-energy functional is the integration of the Futaki invariant.

(5) For a smooth geodesic $\{\varphi^t\}$ in $H^S_0$ we have

$$\frac{d^2}{dt^2} M(\varphi^t) = \int_X |\tilde{\varphi}_{\omega^t}^t|^2 \omega^n_{\omega^t} \geq 0.$$

Hence $M$ is convex along the geodesic $\{\varphi^t\}$.

### 2.6. Vector field energies and the relative K-energy.

In this subsection, we introduce the vector field energy functionals defined by Mabuchi [61] and the relative K-energy functional defined by [61], [23], [41], and [70] independently.

Let $(X, L)$ be an $n$-dimensional polarized manifold. Fix $\omega_0 \in \mathcal{H}(X, L)^S$, and define

$$H^V(\varphi) = \int_0^1 dt \int_X \tilde{\varphi}^t \theta_{\varphi^t} \omega^n_{\omega^t},$$

for any $\varphi \in \mathcal{H}^S$, where $\{\varphi^t\}_{t \in [0,1]}$ is a smooth path in $\mathcal{H}^S$ so that $\varphi^0 = 0$ and $\varphi^1 = \varphi$.

**Fact 2.6.1** ([61]).

1. For any $\varphi \in \mathcal{H}^S_0$

$$H^V(\varphi) = \frac{1}{n+1} \sum_{i=0}^n \int_X \varphi \theta_{\omega^t} \omega^i_{\omega^t} \wedge \omega^{n-i}_0 - \frac{1}{(n+1)(n+2)} \sum_{i=0}^n \sum_{j=0}^{n-i} \int_X \varphi V \varphi_j \omega^i_{\omega^t} \wedge \omega^{n-j}_0.$$

   In particular, the definition of $H^V(\varphi)$ is independent of choice of the path $\{\varphi^t\}_{t \in [0,1]}$ in $\mathcal{H}^S_0$.

2. For any smooth path $\{\varphi^t\}$ in $\mathcal{H}^S_0$ we have

$$\frac{d}{dt} H^V(\varphi^t) = \int_X \tilde{\varphi}^t \theta_{\varphi^t} \omega^n_{\omega^t}.$$

3. For each $\varphi \in \mathcal{H}^S_0$ and $W \in \mathfrak{s}$, put $\varphi^t = \exp(-tJW/4\pi)^* \varphi$. Then $\varphi^t = u^W_{\omega^t}$ and

$$\frac{d}{dt} H^V(\varphi^t) = \int_X u^W_{\omega^t} \theta_{\omega^t} \omega^n_{\omega^t} = B(W, V).$$

4. For a smooth geodesic $\{\varphi^t\}$ in $\mathcal{H}^S_0$, we have

$$\frac{d^2}{dt^2} H^V(\varphi^t) = \int_X \theta_{\omega^t} \left( \tilde{\varphi}^t - \frac{1}{2} |d\tilde{\varphi}^t|^2_{g^t} \right) \omega^n_{\omega^t} = 0.$$

Hence $H^V$ is affine along the geodesic $\{\varphi^t\}$. 
Definition 2.6.2 ([11], [23], [31], [70]). We define the relative K-energy functional \( M_V : \mathcal{H}^S \to \mathbb{R} \) by

\[
M_V(\varphi) := M(\varphi) + H_V(\varphi) = -\int_0^1 dt \int_X \varphi^t(s(\omega_{\varphi^t}) - \bar{s} - \theta_{\varphi^t})\omega_{\varphi^t}^n.
\]

Fact 2.6.3. (1) For any smooth path \( \{\varphi^t\} \) in \( \mathcal{H}_0^S \) we have

\[
\frac{d}{dt} M_V(\varphi^t) = -\int_X \varphi^t(s(\omega_{\varphi^t}) - \bar{s} - \theta_{\varphi^t})\omega_{\varphi^t}^n.
\]

Hence \( \omega_\varphi \) is an extremal Kähler metric if and only if \( \varphi \in \mathcal{H}_0^S \) is a critical point of \( M_V \).

(2) For each \( \varphi \in \mathcal{H}_0^S \) and \( W \in \mathfrak{s} \), put \( \varphi^t = \exp(-tJW/4\pi)^*\varphi \). Then \( \dot{\varphi}^t = W_W^\varphi \) and

\[
\frac{d}{dt} M_V(\varphi^t) = F(W) + B(W,V) = 0.
\]

(3) For a smooth geodesic \( \{\varphi^t\} \) in \( \mathcal{H}_0^S \) we have

\[
\frac{d^2}{dt^2} M_V(\varphi^t) = \frac{d^2}{dt^2} M(\varphi^t) \geq 0.
\]

Hence \( M_V \) is also convex along the geodesic \( \{\varphi^t\} \).

2.7. \( d_1 \)-distances, reduced \( d_1 \)-distances and the reduced J-functionals.

Let \((X,L)\) be an \( n \)-dimensional polarized manifold. For any smooth path \( \{\varphi^t\}_{t \in [0,1]} \in \mathcal{H}_0^S \), we define the \( L^1 \)-length of \( \{\varphi^t\}_{t \in [0,1]} \) by

\[
\ell_1(\{\varphi^t\}_{t \in [0,1]}) := \int_0^1 dt \int_X |\varphi^t|^n \omega_{\varphi^t}.
\]

Definition 2.7.1. For each \( \varphi_0, \varphi_1 \in \mathcal{H}_0^S \), we define the \( L^1 \)-distance of \( \varphi_0 \) and \( \varphi_1 \) by

\[
d_1(\varphi_0, \varphi_1) := \inf_{\{\varphi^t\}_{t \in [0,1]}} \ell_1(\{\varphi^t\}_{t \in [0,1]}),
\]

where \( \{\varphi^t\}_{t \in [0,1]} \) runs through all smooth paths in \( \mathcal{H}_0^S \) so that \( \varphi^0 = \varphi_0 \) and \( \varphi^1 = \varphi_1 \).

Fact 2.7.2 ([28]). The \( L^1 \)-distance \( d_1 \) is in fact a distance function on \( \mathcal{H}_0^S \). However, the metric space \((\mathcal{H}_0^S, d_1)\) is not complete.

Next we consider the reduced \( L^1 \)-distance. Let us consider the action of \( T \) on \( \mathcal{H}_0^S \). For each \( \tau \in T \) and \( \varphi \in \mathcal{H}_0^S \), define \( \varphi_\tau \in \mathcal{H}_0^S \) by

\[
\tau^* \omega_\varphi = \omega_0 + dd^c \varphi_\tau = \omega_{\varphi_\tau}.
\]

Since

\[
\tau^* \omega_\varphi = \tau^*(\omega_0 + dd^c \varphi) = \omega_0 + dd^c(0_\tau + \tau^* \varphi)
\]

and
\[ E(0_\tau + \tau^* \varphi) = E(0_\tau + \tau^* \varphi) - E(0_\tau) \]
\[ = \frac{1}{n+1} \sum_{i=0}^{n} \int_X (\tau^* \varphi)(\tau^* \omega_{\varphi})^i \wedge (\tau^* \omega_0)^{n-i} \]
\[ = E(\varphi) = 0, \]
we have \( 0_\tau + \tau^* \varphi \in H^S_0 \) and \( \varphi_\tau = 0_\tau + \tau^* \varphi. \)

**Proposition 2.7.3.** The \( T \)-action on \( (H^S_0, d_1) \) is isometric, that is, for each \( \tau \in T \) and \( \varphi_0, \varphi_1 \in H^S_0 \), we have
\[ d_1((\varphi_0)_\tau, (\varphi_1)_\tau) = d_1(\varphi_0, \varphi_1). \]

**Proof.** Let \( \{\varphi^t\}_{t \in [0,1]} \) be a smooth path in \( H^S_0 \) connecting \( \varphi_0 \) and \( \varphi_1 \). Then \( \{\varphi^t_\tau\}_{t \in [0,1]} \) is a smooth path in \( H^S_0 \) connecting \( (\varphi_0)_\tau \) and \( (\varphi_1)_\tau \), and
\[ \dot{\varphi}^t_\tau = \frac{d}{dt}(0_\tau + \tau^* \varphi^t) = \tau^* \dot{\varphi}^t. \]

Hence we have
\[ \ell_1(\{\varphi^t_\tau\}_{t \in [0,1]} = \int_0^1 \int_X |\varphi^t_\tau| \omega_{\varphi^t_\tau}^n \]
\[ = \int_0^1 \int_X |\tau^* \dot{\varphi}^t| \tau^* \omega_{\varphi^t}^n = \ell_1(\{\varphi^t\}_{t \in [0,1]}), \]
as required. \( \square \)

**Definition 2.7.4.** For each \( \varphi_0, \varphi_1 \in H^S_0 \), we define the reduced \( L^1 \)-distance of \( \varphi_0 \) and \( \varphi_1 \) by
\[ d_{1,T}(\varphi_0, \varphi_1) := \inf_{\tau_0, \tau_1 \in T} d_1((\varphi_0)_{\tau_0}, (\varphi_1)_{\tau_1}) = \inf_{\tau \in T} d_1((\varphi_0)_\tau, (\varphi_1)_\tau). \]

Similarly, we introduce the reduced \( J \)-functional.

**Definition 2.7.5.** We define the reduced \( J \)-functional by
\[ J_T(\varphi) := \inf_{\tau \in T} J(\varphi_\tau), \]
for any \( \varphi \in H^S_0 \).

**Proposition 2.7.6.** The \( J \)-functional is equivalent to the \( L^1 \)-distance in the following sense: there exists a \( C > 0 \) such that for any \( \varphi \in H^S_0 \)
\[ \frac{1}{C} J(\varphi) - C \leq d_1(\varphi, 0) \leq C J(\varphi) + C. \]

**Proposition 2.7.7.** The reduced \( J \)-functional is equivalent to the reduced \( L^1 \)-distance in the following sense: there exists a \( C > 0 \) such that for any \( \varphi \in H^S_0 \)
\[ \frac{1}{C} J_T(\varphi) - C \leq d_{1,T}(\varphi, 0) \leq C J_T(\varphi) + C. \]
By using the theory of Chen-Cheng and He [20], [21], [22], [47], and the arguments in [55], we shall show the following theorem.

**Theorem 2.7.8.** For a polarized manifold \((X, L)\), the followings are equivalent.

1. \((X, L)\) admits an \(S\)-invariant extremal Kähler metric.
2. There exists \(\delta, C > 0\) such that
   \[ M_V(\varphi) \geq \delta d_{1,T}(\varphi, 0) - C \]
   for any \(\varphi \in \mathcal{H}_0^S\).
3. There exists \(\delta, C > 0\) such that
   \[ M_V(\varphi) \geq \delta J_T(\varphi) - C \]
   for any \(\varphi \in \mathcal{H}_0^S\).

We say that the relative K-energy is \(T\)-coercive if \(M_V\) satisfies the condition (2) and (or) (3) in Theorem 2.7.8.

**Proof of Theorem 2.7.8.** The equivalence of (2) and (3) is obvious from Proposition 2.7.7. We shall show the equivalence of (1) and (2). For this, we apply Darvas-Rubinstein’s existence/properness principle [30, Theorem 3.4]. In their notation, we consider the following data

\[ R = \mathcal{H}_0^S, \quad d = d_1, \quad F = M_V, \quad G = T. \]

Note that the lower semicontinuity of \(M_V\) is obtained in [47, Corollary 2.2]. For (P1), existence of \(d_1\)-geodesic can be found in [18, Theorem 3] and [28, Theorem 2]. Continuity and convexity of \(M_V\) can be found in [9, Theorem 3.4] and [47, Proposition 2.2]. The property (P2) is obtained by the compactness theorem in [10, Theorem 2.17] and the lower semicontinuity of \(M_V\). The property (P3) is due to [47, Theorem 3.6]. The property (P4) is already showed in Proposition 2.7.3. The property (P6) can be omitted (see Remark 4.8 of [29]). The property (P7) is easily proved.

We need to check the property (P5). Let \(\mathfrak{aut}(X, V) = \{W \in \mathfrak{aut}(X) \mid L_V W = 0\}\). Then \(\mathfrak{aut}(X, V)\) is a Lie subalgebra of \(\mathfrak{aut}(X)\). We denote by \(H := \text{Aut}(X, V)_0\) the connected Lie subgroup of \(\text{Aut}(X)_0\) corresponding to \(\mathfrak{aut}(X, V)\). Let \(\omega_i, i = 1, 2\) be two \(S\)-invariant extremal Kähler metrics in \(\mathcal{H}(X, L)^S\), and

\[ K_i = \text{Isom}(X, \omega_i)_0. \]

By a theorem of Calabi [15], \(\text{Aut}(X, V)_0\) is a reductive and each \(K_i\) is a maximal compact subgroup of \(\text{Aut}(X, V)_0\). Since \(\omega_i\) is \(S\)-invariant, we have \(S \subset K_1 \cap K_2\) and that \(S\) is a maximal compact torus of both \(K_1\) and \(K_2\). Hence by [55, Proposition A.2] \(K_2 = \tau K_1 \tau^{-1}\) for some \(\tau \in T\).

By Berman-Berndtsson’s uniqueness theorem of extremal Kähler metrics, we can find \(f \in H\) satisfying \(\omega_1 = f^* \omega_2\) (see the proof of [9, Theorem 4.16]). Then we have \(K_1 = f K_2 f^{-1} = (f \tau) K_1 (f \tau)^{-1}\) and hence \(f \tau \in N_H(K_1)\), where \(N_H(K_1)\) is the normalizer of \(K_1\) in \(H\). Furthermore, by
Proposition A.1] we have $f \tau \in C(H)_0 K_1$, where $C(H)_0$ is the identity component of the center of $H$. Hence there exists $\tau' \in C(H)_0 \subset T$ and $k_1 \in K_1$ such that $f \tau = \tau' k_1$. By setting $\tau_1 = \tau' \tau^{-1}$, we have $\tau_1 \in T$, $f = k_1 \tau_1$ and

$$\omega_1 = f^* \omega_2 = (k_1 \tau_1)^* \omega_2 = \tau_1^* k_1^* \omega_2 = \tau_1^* \omega_2.$$ 

□

3. Test configurations and relative K-stability

3.1. Test configurations. Let $(X, L)$ be an $n$-dimensional polarized manifold.

Definition 3.1.1. A test configuration for $(X, L)$ of exponent $r$ consists of

1. a scheme $\mathcal{X}$ with a $\mathbb{G}_m$-action,
2. a $\mathbb{G}_m$-equivariant flat and proper morphism of schemes $\pi: \mathcal{X} \to \mathbb{A}^1$,
3. where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication,
4. a $\mathbb{G}_m$-linearized $\pi$-very ample line bundle $\mathcal{L}$ over $\mathcal{X}$, and
5. an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L')$.

Here $\mathcal{X}_t$ is the fiber of $\pi$ over $t \in \mathbb{A}^1$, and $\mathcal{L}_t = \mathcal{L}|_{\mathcal{X}_t}$. A test configuration $(\mathcal{X}, \mathcal{L})$ is called product if $\mathcal{X}$ is isomorphic to the fiber product of $X$ and $\mathbb{A}^1$ over $\mathbb{A}^1$, and trivial if in addition $\mathbb{G}_m$ acts only on the second factor. A test configuration $(\mathcal{X}, \mathcal{L})$ is called normal if $\mathcal{X}$ is a normal variety. For an algebraic subgroup $G$ of $\text{Aut}(X, L)$, a $G$-equivariant test configuration is a test configuration $(\mathcal{X}, \mathcal{L})$ with a lifted $G$-action on $(\mathcal{X}, \mathcal{L})$ which preserves each fiber, commutes with the fiber $\mathbb{G}_m$-action of $(\mathcal{X}, \mathcal{L})$, and coincides with $G$-action when acting on $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L')$.

For our purpose, it is convenient to compactify test configurations. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for $(X, L)$ of exponent $r$. The $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$ and the isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L')$ induces an equivariant trivialization $(\mathcal{X}, \mathcal{L})|_{\mathcal{G}_m} \cong (X \times \mathbb{G}_m, p_1^* L')$, where $p_1: X \times \mathbb{G}_m \to X$ is the projection to the first factor, and $\mathbb{G}_m$ acts only on the second factor. Then we can compactify $(\mathcal{X}, \mathcal{L})$ by gluing it with the product $(X \times (\mathbb{P}^1 \setminus \{0\}), p_1^* L')$ along their respective open sets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{A}^1 \setminus \{0\})$. The resulting $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ with $\overline{\pi}: \overline{\mathcal{X}} \to \mathbb{P}^1$ is a $\mathbb{G}_m$-equivariant flat family over $\mathbb{P}^1$ with fibers isomorphic to $(X, L')$ except over 0. We call $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ the canonical compactification of the test configuration $(\mathcal{X}, \mathcal{L})$. Note that the action of $\mathbb{G}_m$ on the infinite fiber $(\overline{\mathcal{X}}_{\infty}, \overline{\mathcal{L}}_{\infty})$ is trivial.

Example 3.1.2. Since $L$ is ample, there exists an embeddings

$$\Phi_m: X \to \mathbb{P}(H^0(X, L^m))$$

for any sufficiently divisible $m \in \mathbb{Z}_{>0}$. By replacing $m$ if necessary, we obtain a faithful representation

$$\theta_m: T \to \text{GL}(H^0(X, L^m))$$
satisfying

$$\theta_m(\tau) \circ \Phi_m = \Phi_m \circ \tau$$

for any \( \tau \in T \). Then the infinitesimal representation \( \theta_m^* : t \to \mathfrak{gl}(H^0(X, L_m^m)) \) is given by

$$-\frac{1}{4\pi} \theta_m^*(JW) = \nabla_W + m\omega_W \mathrm{id}_{H^0(X, L_m^m)}.$$

Let \( C_m(T) \) be the centralizer of a subset \( \theta_m(T) \) in \( \text{GL}(H^0(X, L_m^m)) \). For any morphism \( \lambda : G_m \to C_m(T) \), we define \( \mathcal{X} \) as the Zariski closure of the set

$$\bigcup_{\tau \in G_m} \lambda^\vee(\Phi_m(X)) \times \{ \tau \}$$

in \( \mathbf{P}(H^0(X, L_m^m))^\vee \times \mathbf{A}^1 \), and set \( \mathcal{L} = \mathcal{O}_\mathcal{X}(1) \). Then \( (\mathcal{X}, \mathcal{L}) \) is a test configuration for \( (X, L) \) of exponent \( m \). The central fiber \( \mathcal{X}_0 \) is obtained as the flat limit of the image of \( \Phi_m(X) \) under \( \lambda^\vee \) as \( \tau \to 0 \).

**Example 3.1.3.** The set of test configurations for \( (X, L) \) admits two natural operations: translations and scalings. \( (X, L) \) be a test configuration for \( (X, L) \) of exponent \( r \).

(1) Let \( c \in \mathbf{Q} \). A translation of \( (\mathcal{X}, \mathcal{L}) \) is defined to be \( (\mathcal{X}, \mathcal{L} + c\mathcal{X}_0) := (\mathcal{X}, \mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(c\mathcal{X}_0)) \).

(2) Let \( d \in \mathbf{Z}_{>0} \). A scaling of \( (\mathcal{X}, \mathcal{L}) \) is defined to be the normalization of the base change of \( (\mathcal{X}, \mathcal{L}) \) by \( \tau \mapsto \tau^d \). We denote such a test configuration by \( (\mathcal{X}_d, \mathcal{L}_d) \).

Associated to a test configuration, the Donaldson-Futaki invariant is defined as follows. Let \( (\mathcal{X}, \mathcal{L}) \) be a test configuration for \( (X, L) \) of exponent \( r \).

By the flatness of \( \pi : \mathcal{X} \to \mathbf{A}^1 \) we have

$$N_{rm} := \dim C H^0(\mathcal{X}_0, \mathcal{L}_0^m) = \dim C H^0(X, L^{rm})$$

for sufficiently large \( m \in \mathbf{Z}_{>0} \). Moreover, the asymptotic Riemann-Roch theorem tells us that

$$N_m = a_0m^n + a_1m^{n-1} + O(m^{n-2}),$$

where

$$a_0 = \frac{(L^n)}{n!}, \quad a_1 = -\frac{1}{2} \frac{(K_X \cdot L^{n-1})}{(n-1)!}.$$

Note that

$$\xi = -n \frac{(K_X \cdot L^{n-1})}{(L^n)} = -2 \frac{a_1}{a_0}.$$

Since \( 0 \in \mathbf{A}^1 \) is fixed by the \( G_m \)-action, the \( G_m \)-action on \( (\mathcal{X}, \mathcal{L}) \) induces a coaction \( \lambda_m : H^0(\mathcal{X}_0, \mathcal{L}_0^m) \to H^0(\mathcal{X}_0, \mathcal{L}_0^m) \otimes C[t, t^{-1}] \). Let

$$H^0(\mathcal{X}_0, \mathcal{L}_0^m) = \bigoplus_{i=1}^{N_{rm}} H^0(\mathcal{X}_0, \mathcal{L}_0^m)_{\lambda_i^{rm}}$$
be the weight decomposition. Here $H^0(X_0, L^m_0) = \{ s \in H^0(X_0, L^m_0) \mid \lambda_m(s) = s \otimes \tau^\lambda \}$ is the $\lambda$-weight space for this action. By [12, Theorem 3.1], we have

$$w_m := \sum_{i=1}^{N_m} \lambda_i^{(m)} = b_0 m^{n+1} + b_1 m^n + O(m^{n-1})$$

for sufficiently large $m$ divisible by $r$. We then consider the coefficients

$$\frac{w_m}{mn} = \frac{F_0(X, L) + F_1(X, L)m^{-1} + O(m^{-2})}{n}$$

$$F_0(X, L) = \frac{b_0}{a_0}, \quad F_1(X, L) = \frac{a_0 b_1 - a_1 b_0}{a_0^2}.$$

**Definition 3.1.4.** We call $DF(X, L) := -2F_1(X, L) = -2 \frac{a_0 b_1 - a_1 b_0}{a_0^2}$ the Donaldson-Futaki invariant of a test configuration $(X, L)$.

When the test configuration $(X, L)$ is normal, we can describe the Donaldson-Futaki invariant via intersection numbers ([78, Proposition 17], [57, Proposition 6], [12, Proposition 3.12]). Indeed, by using the canonical compactification and asymptotic Riemann-Roch theorem, we have

$$w_{rn} = \chi(X, L^{rm}) = \chi(X, L^{rn})$$

$$= \frac{1}{r^{n+1}(n+1)!} (rm)^{n+1} - \frac{(K_{\overline{X}} \cdot \overline{L}^n)}{2r^n n!} (rn)^n + O(m^{n-1})$$

$$- \frac{(L^n)}{n!} (rn)^n + O(m^{n-1})$$

$$= \frac{(\overline{L}^{n+1})}{r^{n+1}(n+1)!} (rn)^{n+1} - \frac{1}{2r^n n!} ((K_{\overline{X}} \cdot \overline{L}^n) + 2r^n (L^n)) (rn)^n$$

$$+ O(m^{n-1})$$

for any sufficiently large $m \in \mathbb{Z}_{>0}$ (see the proof of [12, Proposition 3.12]). By setting $K_{X/P^1} := K_{\overline{X}} - \overline{\pi}^* K_{P^1}$, we obtain

$$2r^n (L^n) = 2((L^n)^n) = 2(\overline{L}^n)^n = 2(\overline{X}_1 \cdot \overline{L}_1^n)$$

$$= (\overline{\pi}^* (2[1]) \cdot \overline{L}^n) = -(\overline{\pi}^* K_{P^1} \cdot \overline{L}^n)$$

and

$$b_0 = \frac{(\overline{L}^{n+1})}{r(n+1)!}, \quad b_1 = -\frac{1}{2r^n n!} (K_{\overline{X}/P^1} \cdot \overline{L}^n).$$

Therefore, we obtain the intersection number formula

$$DF(X, L) = \frac{1}{r(n+1)} (\overline{L}^{n+1}) + \frac{1}{(L^n)} (K_{\overline{X}/P^1} \cdot \overline{L}^n).$$
3.2. Duistermaat-Heckman measures. Let \((\mathcal{X}, \mathcal{L})\) be a test configuration for \((X, L)\) of exponent \(r\). Then the central fiber \((\mathcal{X}_0, \mathcal{L}_0)\) is a polarized \(\mathbb{G}_m\)-scheme. For sufficiently divisible \(m \in \mathbb{Z}_{>0}\), we can consider the induced \(\mathbb{G}_m\)-action on \(H^0(\mathcal{X}_0, \mathcal{L}_0^m)\). Then there is a weak limit of the measure
\[
DH(\mathcal{X}, \mathcal{L}) = \lim_{m \to \infty} \frac{1}{N_{rm}} \sum_{\lambda \in \mathbb{Z}} \dim H^0(\mathcal{X}_0, \mathcal{L}_0^m) \delta_{\lambda/rm}
\]
on \(\mathbb{R}\), where \(\delta_{\lambda/rm}\) is the Dirac delta measure concentrated at \(\lambda/rm\) (see \([12]\)). This is called the Duistermaat-Heckman measure of the test configuration \((\mathcal{X}, \mathcal{L})\).

3.3. Non-Archimedean functionals. In this subsection we introduce non-Archimedean functionals. Let \((X, L)\) be an \(n\)-dimensional polarized manifold and \((\mathcal{X}, \mathcal{L})\) a test configuration for \((X, L)\) of exponent \(r\).

**Definition 3.3.1** ([12]). (1) We define the non-Archimedean Monge-Ampère energy of \(\varphi\) by
\[
E^{NA}(\mathcal{X}, \mathcal{L}) := \frac{\mathcal{L}^{n+1}}{(n+1)(L^n)} = \int_{\mathbb{R}} \lambda dDH(\mathcal{X}, \mathcal{L}).
\]

(2) We define the non-Archimedean \(J\)-functional of \(\varphi\) by
\[
J^{NA}(\mathcal{X}, \mathcal{L}) := \sup \text{supp}(\text{supp}(DH(\mathcal{X}, \mathcal{L}))) - E^{NA}(\mathcal{X}, \mathcal{L}).
\]

When \(\mathcal{X}\) dominates the product \(X \times \mathbb{P}^1\), namely, there exists \(\rho: \mathcal{X} \to X \times \mathbb{P}^1\) which is equivariant with respect to the trivial action on the target, \(J^{NA}\) can be written as
\[
J^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{(L^n)(\rho^* p_1^* L^n)} - \frac{\mathcal{L}^{n+1}}{r(n+1)(L^n)}.\]

Let
\[
K^{log}_{\mathcal{X}} := K_{\mathcal{X}} + \mathcal{X}_0,_{\text{red}} + \mathcal{X}_\infty,_{\text{red}} = K_{\mathcal{X}} + \mathcal{X}_0,_{\text{red}} + \mathcal{X}_\infty,
\]
\[
K^{log}_{\mathbb{P}^1} := K_{\mathbb{P}^1} + [0] + [\infty],
\]
\[
K^{log}_{\mathcal{X}/\mathbb{P}^1} := K^{log}_{\mathcal{X}} - \pi^* K^{log}_{\mathbb{P}^1} = K_{\mathcal{X}/\mathbb{P}^1} + (\mathcal{X}_0,_{\text{red}} - \mathcal{X}_0).
\]

**Definition 3.3.2** ([12]). Let \((\mathcal{X}, \mathcal{L})\) be a test configuration for \((X, L)\).

(1) The non-Archimedean Ricci energy is defined by
\[
R^{NA}(\mathcal{X}, \mathcal{L}) := \frac{1}{(L^n)} (\rho^* K^{log}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^n).
\]

(2) The non-Archimedean entropy is defined by
\[
H^{NA}(\mathcal{X}, \mathcal{L}) := \frac{1}{(L^n)} (K^{log}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^n) - \frac{1}{(L^n)} (\rho^* K^{log}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^n).\]
(3) The non-Archimedean K-energy is defined by
\[ M^{NA}(\varphi) := H^{NA}(\mathcal{X}, \mathcal{L}) + R^{NA}(\mathcal{X}, \mathcal{L}) + \bar{\eta}E^{NA}(\mathcal{X}, \mathcal{L}) \]
\[ = \bar{\eta} \frac{(\mathcal{L}^{n+1})}{(n+1)(L^n)} + \frac{1}{L^n}(K_{\mathcal{X}/\mathcal{P}^1} \cdot \mathcal{L}^n). \]

The non-Archimedean K-energy is closely related to the Donaldson-Futaki invariant. In fact, by comparing (3.1) we have
\[ M^{NA}(\mathcal{X}, \mathcal{L}) = DF(\mathcal{X}, \mathcal{L}) + \frac{1}{L^n}(K_{\mathcal{X}/\mathcal{P}^1} \cdot \mathcal{L}^n) \]
\[ = DF(\mathcal{X}, \mathcal{L}) + \frac{1}{L^n}((\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \cdot \mathcal{L}^n). \]

Hence \( M^{NA}(\mathcal{X}, \mathcal{L}) \leq DF(\mathcal{X}, \mathcal{L}) \) for any test configuration \((\mathcal{X}, \mathcal{L})\) and equality holds if and only if \( \mathcal{X} \) is regular in codimension one and \( \mathcal{X}_0 \) is generically reduced ([12] Proposition 7.15, [13] Proposition 2.8). An advantage of \( M^{NA} \) is the following homogeneity with respect to base change, which fails for \( DF \) when the central fiber is non-reduced.

**Proposition 3.3.3** ([12] Proposition 7.14]). Let \((\mathcal{X}, \mathcal{L})\) be a test configuration for \((X, L)\). For each \( d \in \mathbb{Z}_{>0} \), let \((\mathcal{X}_d, \mathcal{L}_d)\) be the test configuration obtained by the base change \( z \mapsto z^d \). Then \( M^{NA}(\mathcal{X}_d, \mathcal{L}_d) = dM^{NA}(\mathcal{X}, \mathcal{L}) \).

On the other hand, Mumford’s semistable reduction theorem ([51] p.53, pp.100–101]. See [5] Theorem 3.8 and [57] Lemma 5] for equivariant version) gives us the following proposition.

**Proposition 3.3.4** ([12] Proposition 7.16]). For each test configuration \((\mathcal{X}, \mathcal{L})\) for \((X, L)\), there exists \( d_0 \in \mathbb{Z}_{>0} \) such that \( DF(\mathcal{X}_d, \mathcal{L}_d) = M^{NA}(\mathcal{X}_d, \mathcal{L}_d) = dM^{NA}(\mathcal{X}, \mathcal{L}) \) for all \( d \in \mathbb{Z}_{>0} \) divisible by \( d_0 \).

Now we consider the \( T \)-action on \((X, L)\). Let \((\mathcal{X}, \mathcal{L})\) be a \( T \)-equivariant test configuration for \((X, L)\). Let \( V \) be the extremal Kähler vector field for \( T \), and \( T_V \) the one-dimensional algebraic torus generated by \( V \). Then \( T_V \subset T \) and induces a coaction
\[ \theta_m : H^0(\mathcal{X}_0, \mathcal{L}_0^m) \to H^0(\mathcal{X}_0, \mathcal{L}_0^m) \otimes \mathbb{C}[t, t^{-1}] \]
for each \( m \in \mathbb{Z}_{>0} \). We denote by \((\theta_1^{(rm)}, \ldots, \theta_{N_{rm}}^{(rm)})\) the weight of \( \theta_m^\vee \). Let \( \Theta_{rm} \) be the generator of \( \theta_m \) and \( \Theta_{rm}^\circ \) the its traceless part.

**Definition 3.3.5.** Let \((\mathcal{X}, \mathcal{L})\) a \( T \)-equivariant test configuration for \((X, L)\).

1. \( H^{NA}_V(\mathcal{X}, \mathcal{L}) := \lim_{m \to \infty} \frac{1}{(rm)^2N_{rm}} \text{tr}(\Lambda_{rm}^0 \Theta_{rm}^\circ). \)
2. We define the non-Archimedean relative K-energy of \( \varphi \) by
\[ M^{NA}_V(\mathcal{X}, \mathcal{L}) := M^{NA}(\mathcal{X}, \mathcal{L}) + H^{NA}_V(\mathcal{X}, \mathcal{L}). \]
3. We define the relative Donaldson-Futaki invariant of \( \varphi \) by
\[ DF_V(\mathcal{X}, \mathcal{L}) := DF(\mathcal{X}, \mathcal{L}) + H^{NA}_V(\mathcal{X}, \mathcal{L}). \]
The following slope formula relates the Archimedean and non-Archimedean functionals. Let \((X, L)\) be a \(T\)-equivariant test configuration for \((X, L)\) and \(\lambda: G_\lambda \to \text{Aut}^0(X, L)\) the \(G_\lambda\)-action. We denote by \(h_{FS}\) the pull back of the Fubini-Study metric on \(PH^0(\mathcal{X}_0, \mathcal{L}_0^m)^\vee\) by the composition
\[
X \hookrightarrow PH^0(\mathcal{X}_0, \mathcal{L}_0^m)^\vee \times A^1 \to PH^0(\mathcal{X}_0, \mathcal{L}_0^m)^\vee,
\]
and \(h_{FS}|_\tau\) the restriction of \(h_{FS}\) to \((\mathcal{X}_\tau, \mathcal{L}_\tau)\) for any \(\tau \in G_\lambda\). By setting
\[
h_t := \lambda(e^{-t/2})^* h_{FS}|_{e^{-t/2}}, \quad \omega_0 := c_1(L, h_0)
\]
for each \(t \in [0, \infty)\), we have \(\varphi^t \in \mathcal{H}(X, \omega_0)^S\) by
\[
c_1(L, h_t) - \omega_0 = dd^c \varphi^t.
\]

**Theorem 3.3.6** ([13, Theorem 3.6], [79, Theorem 12]). Let \((X, L)\) and \(\{\varphi^t\}_{t \in [0, \infty)}\) be as above, and \(F\) be one of the following functionals: \(E, J, M, H_V, \text{ and } M_V\). Then
\[
\lim_{t \to \infty} \frac{F(\varphi^t)}{t} = F^{NA}(X, L).
\]

### 3.4. Filtrations

To introduce the non-Archimedean version of the reduced \(J\)-functional, it is convenient to use the notion of the *filtration* of the section ring
\[
R = R(X, L) := \bigoplus_{m=0}^{\infty} R_m, \quad R_m = H^0(X, L^m)
\]
of a polarized manifold \((X, L)\). Following [12], we give a brief review of filtrations.

**Definition 3.4.1.** A filtration \(\mathcal{F}\) of the graded \(C\)-algebra \(R\) consists of a family of subspaces \(\{F^\lambda R_m\}\) of \(R_m\) for each \(m \in \mathbb{Z}_{\geq 0}\) satisfying:

1. decreasing: \(F^\lambda R_m \subset F^{\lambda'} R_m\) if \(\lambda \geq \lambda'\),
2. left-continuous: \(F^\lambda R_m = \bigcap_{\lambda' < \lambda} F^{\lambda'} R_m\),
3. multiplicative: \(F^\lambda R_m \cdot F^{\lambda'} R_{m'} \subset F^{\lambda + \lambda'} R_{m + m'}\), and
4. linearly bounded: there exist \(e_-, e_+ \in \mathbb{Z}\) such that \(F^{m_+} R_m = R_m\) and \(F^{m_-} R_m = \{0\}\) for all \(m \in \mathbb{Z}_{\geq 0}\).

We say that \(\mathcal{F}\) is a \(\mathbb{Z}\)-filtration if \(F^\lambda R_m = F^{[\lambda]} R_m\) for any \(\lambda \in \mathbb{R}\) and \(m \in \mathbb{Z}_{\geq 0}\). A filtration \(FR\) is \(T\)-equivariant if \(F^\lambda R_m\) is \(T\)-invariant subspace of \(R_m\) for all \(\lambda \in \mathbb{R}\).

**Definition 3.4.2.** Let \(\mathcal{F} = \{F^\lambda R_m\}\) be a \(\mathbb{Z}\)-filtration of \(R\).

1. The Rees algebra of \(\mathcal{F}\) is a \(C[t]\)-algebra defined by
\[
\text{Rees}(\mathcal{F}) := \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda R_m.
\]

We say that \(\mathcal{F}\) is finitely generated if its Rees algebra is a finitely generated graded \(C[t]\)-algebra.
(2) The associated graded algebra of $\mathcal{F}$ is a graded $\mathbf{C}$-algebra defined by

$$\text{Gr}^F\mathcal{R} = \bigoplus_{m=0}^{\infty} \text{Gr}^F_{m}\mathcal{R} := \mathcal{R}/t\mathcal{R} = \bigoplus_{m=0}^{\infty} \mathcal{F}^\lambda R_m / \mathcal{F}^{\lambda+1} R_m.$$ 

Given a finitely generated $\mathbf{Z}$-filtration $\mathcal{F} = \{\mathcal{F}\lambda R_m\}$, one can construct a test configuration for $(X,L)$ as follows. Let $\mathcal{R} := \text{Rees}(\mathcal{F})$ be the Rees algebra of the filtration $FR$. Then $\mathcal{R}$ is a torsion free $C[t]$-algebra. Since $C[t]$ is a principal ideal domain, $\mathcal{R}$ is flat over $C[t]$. Let

$$\mathcal{X} := \text{Proj} \mathcal{R}, \quad \mathcal{L} := \mathcal{O}_{\mathcal{X}}(1),$$

and $\pi: \mathcal{X} \to \mathbf{A}^1$ be the structure morphism. Then $(\mathcal{X},\mathcal{L})$ is a flat family over $\mathbf{A}^1$. Further, for any $a \in \mathbf{A}^1(C)$ we have

$$\mathcal{X}_a = \mathcal{X} \times_{\mathbf{A}^1} \text{Spec}(C[t]/(t-a)C[t]) = \text{Proj}(\mathcal{R}/(t-a)\mathcal{R}) = \begin{cases} \text{Proj} \mathcal{R} & a \neq 0, \\ \text{Proj} \text{Gr}^F\mathcal{R} & a = 0. \end{cases}$$

Hence $(\mathcal{X}_a,\mathcal{L}_a) = (X,L)$ for any $a \in \mathbf{A}^1 \setminus \{0\}$ and $(\mathcal{X},\mathcal{L})$ is a test configuration for $(X,L)$.

**Example 3.4.3** (Test configurations). Let $(\mathcal{X},\mathcal{L})$ be a test configuration for $(X,L)$ of exponent $1$. For each $\lambda \in \mathbf{Z}$, define $\mathcal{F}\lambda R_m$ to be the image of the restriction map

$$H^0(\mathcal{X},\mathcal{L}^m)_{\lambda} \to H^0(\mathcal{X}_1,\mathcal{L}_{1}^m) = H^0(X,L^m).$$

By regarding $H^0(\mathcal{X} \setminus X_0,\mathcal{L}^m)$ as a $C[t,t^{-1}]$ module, $\mathcal{F}\lambda R_m$ can be written as

$$\mathcal{F}\lambda R_m = \{\sigma \in H^0(X,L^m) | t^{-\lambda} \sigma \in H^0(\mathcal{X},\mathcal{L}^m)\},$$

where $\sigma \in H^0(\mathcal{X} \setminus X_0,\mathcal{L}^m)$ is the $G_m$-equivariant section defined by $\sigma \in H^0(X,L^m)$. By setting $\mathcal{F}\lambda R_m = F[\lambda] R_m$ for each $\lambda \in \mathbf{R}$, we obtain a $\mathbf{Z}$-filtration of $R$. We denote the filtration by $\mathcal{F}(\mathcal{X},\mathcal{L})$.

**Example 3.4.4** (Translations and scalings). Let $\mathcal{F} = \{\mathcal{F}\lambda R_m\}$ be a filtration of $R$, and set $\mathcal{R} := \text{Rees}(\mathcal{F}).$

1. Let $c \in \mathbf{R}$. A translation $\mathcal{F}(c) = \{\mathcal{F}(c)\lambda R_m\}$ of $\mathcal{F}$ is defined by

$$\mathcal{F}(c)\lambda R_m := \mathcal{F}\lambda - mc R_m.$$ 

If $c \in \mathbf{Z}$ and $\mathcal{F}$ is a $\mathbf{Z}$-filtration, then $\mathcal{F}(c)$ is also a $\mathbf{Z}$-filtration. Suppose $c \in \mathbf{Z}$, and $\mathcal{F}$ is a finitely generated $\mathbf{Z}$-filtration. Let $(\mathcal{X},\mathcal{L})$ be a test configuration associated to $\mathcal{F}$. Then $\mathcal{X} = \text{Proj} \mathcal{R}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1) = \mathcal{R}(1)$. By setting $\mathcal{I}_{X_0}$ to be the ideal sheaf of $X_0$ in $\mathcal{X}$, we have

$$\mathcal{O}_{\mathcal{X}}(cX_0) = \mathcal{I}_{X_0}^{-c} = t^{-c} \mathcal{R}.$$
Hence we obtain

\[(\mathcal{L} + c\mathcal{X}_0)^m = \mathcal{L}^m \otimes_{\mathcal{O}_X} \mathcal{T}_0^{-mc} = (\mathcal{R}(1)^m \otimes \mathcal{R} t^{-mc}\mathcal{R}) = (t^{-mc}\mathcal{R}(m))\]

and

\[H^0(\mathcal{X}, (\mathcal{L} + c\mathcal{X}_0)^m) = (t^{-mc}\mathcal{R}(m))_0 = t^{-mc}\mathcal{R}_m = t^{-mc} \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda}F^\lambda R_m.

This shows that

\[\bigoplus_{m=0}^{\infty} H^0(\mathcal{X}, (\mathcal{L} + c\mathcal{X}_0)^m) = \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda}F^\lambda R_m = \text{Rees}(\mathcal{F}(c))\]

and that the translation \((\mathcal{X}, \mathcal{L} + c\mathcal{X}_0)\) is the test configuration associated to \(\mathcal{F}(c)\). Note that

\[\mathcal{R}(c) := \text{Rees}(\mathcal{F}(c)) \cong \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda}F^\lambda R_m = \mathcal{R}\]

as a graded ring. Hence we have \(\text{Proj} \mathcal{R}(c) = \text{Proj} \mathcal{R} = \mathcal{X}\).

(2) Let \(d \in \mathbb{Z}_{>0}\). A scaling \(\mathcal{F}_d = \{F^\lambda_d R_m\}\) of \(\mathcal{F}\) is defined by

\[F^\lambda_d R_m := F^{[\lambda/d]} R_m.

If \(\mathcal{F}\) is a \(\mathbb{Z}\)-filtration, then \(\mathcal{F}_d\) is also a \(\mathbb{Z}\)-filtration. Suppose \(\mathcal{F}\) is a finitely generated \(\mathbb{Z}\)-filtration. Let \((\mathcal{X}, \mathcal{L})\) be a test configuration associated to \(\mathcal{F}\). Then the base change \(\mathcal{X}_d\) of \(\mathcal{X}\) by the homomorphism \(\mathbb{C}[t] \ni t \mapsto s^d \in \mathbb{C}[s]\) is given by

\[\mathcal{X}_d = \text{Proj} \mathcal{R} \times_{\text{Spec} \mathbb{C}[t]} \text{Spec} \mathbb{C}[s] = \text{Proj}(\mathcal{R} \otimes \mathbb{C}[t] \mathbb{C}[s]),
\]

\[\mathcal{L}_d = p^*\mathcal{L} = p^*\mathcal{O}_\mathcal{X}(1) = p^*\mathcal{R}(1) = ((\mathcal{R} \otimes \mathbb{C}[t] \mathbb{C}[s])(1)) = \mathcal{O}_\mathcal{X}_d(1),\]
where \( p: X_d \to \text{Proj} \mathcal{R} = X \) is the natural projection. As a \( \mathbb{C}[t] \)-algebra, we have

\[
\mathcal{R} \otimes \mathbb{C}[t] \mathbb{C}[s] = \left( \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda R_m \right) \otimes \mathbb{C}[t] \mathbb{C}[s]
\]

\[
\cong \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in \mathbb{Z}} s^{-\lambda} F^\lambda \mathcal{F}_m
\]

\[
= \text{Rees}(\mathcal{F}_d),
\]

which is also an isomorphism as \( \mathbb{C}[s] \)-algebras. Hence the scaling \((X_d, \mathcal{L}_d)\) is the test configuration associated to \( \mathcal{F}_d \).

### 3.5. Limit measures.

Let \((X, L)\) be a polarized manifold and \( \mathcal{F} \) a filtration of \( R = R(X, L) \). For each \( m \in \mathbb{Z}_{>0} \), we define a Borel probability measure \( \nu_m \) on \( R \) by

\[
\nu_m := -\frac{d}{d\lambda} \left( \frac{1}{N_m} \dim F^m \lambda R_m \right),
\]

where the derivative is taken in the sense of distributions. We call it the normalized weight measure. By the linearly boundedness of \( \mathcal{F} \), \( \nu_m \) has uniformly bounded support. To describe its limit, to each \( \lambda \in \mathbb{R} \), we define a graded subalgebra \( R^{(\lambda)} \) of \( R(X, L) \) by

\[
R^{(\lambda)} := \bigoplus_{m=0}^{\infty} F^m \lambda R_m.
\]

The volume of \( R^{(\lambda)} \) is defined by

\[
\text{vol}(R^{(\lambda)}) := \limsup_{m \to \infty} \frac{1}{N_m} \dim F^m \lambda R_m.
\]

**Theorem 3.5.1** ([12, Theorem 5.3]). The sequence of normalized weight measures \( \{\nu_m\}_{m=0}^{\infty} \) converges weakly to the probability measure

\[
\text{DH}(\mathcal{F}) := -\frac{d}{d\lambda} \text{vol}(R^{(\lambda)}).
\]

We call \( \text{DH}(\mathcal{F}) \) the Duistermaat-Heckman measure of the filtration \( \mathcal{F} \).

**Example 3.5.2** (Test configurations). Let \((X, \mathcal{L})\) be a test configuration for \((X, L)\) of exponent 1 and \( \mathcal{F} \) the filtration of \( R \) as in Example 3.4.4. By [68, Lemma 8.5], the normalized weight measure is given by

\[
\nu_m = -\frac{d}{d\lambda} \left( \frac{1}{N_m} \dim F^m \lambda R_m \right)
\]

\[
= \frac{1}{N_m} \sum_{\lambda \in \mathbb{Z}} \dim H^0(X_0, \mathcal{L}_0^{m}) \delta_{\lambda/m}
\]
for any \( m \in \mathbb{Z}_{\geq 0} \). Hence the limit coincides with the Duistermaat-Heckman measure:

\[
DH(\mathcal{F}) = DH(X, \mathcal{L}).
\]

### 3.6. Reduced non-Archimedean J-functional

Let \((X, L)\) be a polarized manifold, and set \( R = R(X, L) \).

**Definition 3.6.1.** Let \( \mathcal{F} = \{ F^\lambda R_m \} \) be a filtration of \( R \).

1. The non-Archimedean Monge-Ampère energy of \( \mathcal{F} \) is defined by

\[
E^{NA}(\mathcal{F}) := \int_R \lambda dDH(\mathcal{F}).
\]

2. The non-Archimedean J-functional of \( \mathcal{F} \) is defined by

\[
J^{NA}(\mathcal{F}) := \sup_{\text{supp}(DH(\mathcal{F}))} \left( - E^{NA}(\mathcal{F}) \right).
\]

**Definition 3.6.2.** Let \( \mathcal{F} \) be a \( T \)-equivariant filtration of \( R \). We denote the weight decomposition of \( T \)-action on each \( F^\lambda R_m \) by

\[
F^\lambda R_m = \bigoplus_{\alpha \in M} (F^\lambda R_m)_\alpha.
\]

Here \((F^\lambda R_m)_\alpha\) is the \( \alpha \)-weight subspace of \( F^\lambda R_m \). For each \( \xi \in N_R \), we define the \( \xi \)-twist \( \mathcal{F}_\xi = \{ F^\lambda R_m \} \) of \( \mathcal{F} \) by

\[
F^\lambda R_m \equiv \bigoplus_{\alpha \in M} (F^\lambda R_m)_\alpha, \quad (F^\lambda R_m)_\alpha = (F^{\lambda - \langle \alpha, \xi \rangle} R_m)_\alpha
\]

for each \( \lambda \in R \) and \( m \in Z_{\geq 0} \).

Note that the \( \xi \)-twist of a filtration \( \mathcal{F} \) may be an \( R \)-filtration even if \( \mathcal{F} \) is a \( Z \)-filtration.

**Definition 3.6.3.** Let \( \mathcal{F} \) be a \( T \)-equivariant filtration of \( R \). The reduced non-Archimedean J-functional of \( \mathcal{F} \) is defined by

\[
J_T^{NA}(\mathcal{F}) := \inf_{\xi \in N_R} J^{NA}(\mathcal{F}_\xi).
\]

If \((X, L)\) is a test configuration for \((X, L)\), the reduced non-Archimedean J-functional of \((X, L)\) is defined by \( J_T^{NA}(X, L) := J_T^{NA} (\mathcal{F}(X, L)) \).

**Theorem 3.6.4** ([48, Theorem B], [55, Theorem 3.14]). Let \((X, L)\) be a \( T \)-equivariant test configuration for \((X, L)\). Then

\[
\lim_{t \to \infty} \frac{J_T(\varphi^t)}{t} = J_T^{NA}(X, L).
\]

**Definition 3.6.5** (see also [12]). Let \( p \in [1, \infty] \).

1. Let \( \mathcal{F} \) be a filtration of \( R \). The \( L^p \)-norm \( \| \mathcal{F} \|_p \) of \( \mathcal{F} \) is defined as the \( L^p \)-norm of \( \lambda - \overline{\lambda} \) with respect to \( DH(\mathcal{F}) \), where

\[
\overline{\lambda} := \int_R \lambda dDH(\mathcal{F})
\]
is the barycenter of DH(\mathcal{F}). If (\mathcal{X}, \mathcal{L}) is a test configuration for (X, L), the $L^p$-norm of (\mathcal{X}, \mathcal{L}) is defined by $\|\mathcal{X}, \mathcal{L}\|_p := \|\mathcal{F}(\mathcal{X}, \mathcal{L})\|_p$.

(2) Let $\mathcal{F}$ be a $T$-equivariant filtration of $R$. Let $\mathcal{F}$ be a filtration of $R$. The reduced $L^p$-norm $\|\mathcal{F}\|_p$ of $\mathcal{F}$ is defined by $\|\mathcal{F}\|_p := \inf_{\xi \in \mathbb{N}_R} \|\mathcal{F}_\xi\|_p$. If $(X, \mathcal{L})$ is a $T$-equivariant test configuration for $(X, L)$, the reduced $L^p$-norm of $(X, \mathcal{L})$ is defined by $\|X, \mathcal{L}\|_{p,T} := \|\mathcal{F}(X, \mathcal{L})\|_{p,T}$. According to [12, Lemma 7.10], we have the following equivalence between the (reduced) $L^1$-norm and the (reduced) non-Archimedean $J$-functional.

Theorem 3.6.6 (see also [12, Theorem 7.9]). Let $c_n := 2n^2/(n + 1)^{n+1}$.

(1) For each filtration $\mathcal{F}$, we have $c_nJ^NA(\mathcal{F}) \leq \|\mathcal{F}\|_1 \leq 2J^NA(\mathcal{F})$.

(2) For each $T$-equivariant filtration $\mathcal{F}$, we have $c_nJ^NA(\mathcal{F}) \leq \|\mathcal{F}\|_{1,T} \leq 2J^NA_T(\mathcal{F})$.

Corollary 3.6.7. Let $(\mathcal{X}, \mathcal{L})$ be a normal test configuration for $(X, L)$.

(1) ([12, Theorem 6.8 and Theorem 7.9]) The following conditions are equivalent:

(i) $(\mathcal{X}, \mathcal{L})$ is trivial.
(ii) $\|\mathcal{X}, \mathcal{L}\|_p = 0$ for some $p \in [1, \infty]$.
(iii) $J^NA(\mathcal{X}, \mathcal{L}) = 0$.

(2) ([49, Theorem B]) Suppose $(\mathcal{X}, \mathcal{L})$ is $T$-equivariant. Then the following conditions are equivalent:

(i) $(\mathcal{X}, \mathcal{L})$ is product.
(ii) $\|\mathcal{X}, \mathcal{L}\|_{p,T} = 0$ for some $p \in [1, \infty]$.
(iii) $J^NA_T(\mathcal{X}, \mathcal{L}) = 0$.

3.7. Uniform relative K-stability.

Definition 3.7.1. Let $(X, L)$ be a polarized algebraic manifold and $T$ a maximal algebraic torus of $\text{Aut}^0(X)$.

(1) $(X, L)$ is relatively $K$-semistable if $M^N_{TV}(\mathcal{X}, \mathcal{L}) \geq 0$ for any $T$-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$.

(2) $(X, L)$ is relatively $K$-polystable if $(X, L)$ is relatively $K$-semistable and $M^N_{TV}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is product.

(3) $(X, L)$ is relatively $K$-unstable if $(X, L)$ is not relatively $K$-semistable.

(4) $(X, L)$ is uniformly relatively $K$-polystable if there exists a $\delta > 0$ such that $M^N_{TV}(\mathcal{X}, \mathcal{L}) \geq \delta J^NA_T(\mathcal{X}, \mathcal{L})$ for any $T$-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$.

By the slope formula, we have the following
Theorem 3.7.2. A polarized manifold \((X, L)\) is uniformly relatively \(K\)-polystable if the relative \(K\)-energy is \(T\)-coercive.

Proof. Assume that the relative \(K\)-energy is \(T\)-coercive. Then there exists \(\delta, C > 0\) such that

\[ M_V(\varphi) \geq \delta J_T(\varphi) - C \]

for any \(\varphi \in \mathcal{H}_0^S\). Let \(\varphi = (\mathcal{X}, \mathcal{L})\) be a test configuration for \((X, L)\) of exponent \(r\), and \(\{\varphi^t\}_{t \in [0, +\infty)}\) the subgeodesic ray corresponding to \(\varphi\). Then, Theorem 3.3.6 and Theorem 3.6.4 allow us to conclude

\[ M_V^N(\mathcal{X}, \mathcal{L}) \geq \delta J_T^N(\mathcal{X}, \mathcal{L}), \]

as required. \(\square\)

Combining Theorems 2.7.8 and 3.7.2, we further obtain the following corollary.

Corollary 3.7.3. A polarized manifold \((X, L)\) is uniformly relatively \(K\)-polystable if \((X, L)\) admits an \(S\)-invariant extremal \(K\)ähler metric.

4. Convex Preliminaries

4.1. Polarized toric manifolds. In this subsection, we briefly review the definition and construction of polarized toric manifolds. See [42] for details.

A \(2n\)-dimensional symplectic toric manifold is a \(2n\)-dimensional compact connected symplectic manifold \((X, \omega)\) endowed with an effective Hamiltonian action of an \(n\)-dimensional compact torus \(S = (S^1)^n\) and the moment map \(\mu: X \to M^R\). Let \((X, \omega)\) be a \(2n\)-dimensional symplectic toric manifold.

According to the convexity theorem of Atiyah [6] and Guillemin-Sternberg [44], the image of the moment map \(P := \mu(X) \subset M^R\) is a convex polytope obtained as the convex hull of the image of the fixed points of the \(S\)-action. We call \(P\) the moment polytope. Further, since \((X, \omega)\) is toric, \(P\) satisfies the following special conditions, usually called the Delzant condition:

1. There are exactly \(n\) edges meeting at each vertex.
2. Each edge meeting at the vertex \(v \in P\) is of the form \(v + tu_i, t \geq 0\), where \(u_i \in N\) is primitive.
3. For each vertex, the corresponding \(u_1, \ldots, u_n\) generate the lattice of \(N^R\) over \(Z\).

A convex polytope satisfying the conditions above is called a Delzant polytope. In [31], Delzant showed that there is a one to one correspondence between isomorphism classes of symplectic toric manifolds and Delzant polytopes. More precisely, for each Delzant polytope \(P\), one can construct in a canonical way a symplectic toric manifold in such a way that \(P\) may be identified with the moment polytope. This construction is known as the Delzant construction. Let \(P\) be a Delzant polytope and \((X_P, \omega_P)\) the corresponding symplectic toric manifold with the moment map \(\mu_P: X_P \to M^R\). By the
Delzant construction, there is an $S$-invariant complex structure $J_P$ compatible with $\omega_P$. In particular, $(X_P, \omega_P, J_P)$ is a Kähler manifold. Since the $S$-action preserves the complex structure $J_P$, it can be canonically extended to an action of $T := S^C = (G_m)^n$ preserving $J_P$ ([15, Theorem 4.4]). Then $X_P$ contains $T$ as an open dense orbit, and hence $X_P$ has a structure of a nonsingular toric variety with the fan associated to the polytope $P$ ([54, Lemma 9.2]).

Let $P = \{x \in M_R \mid \langle \lambda_j, x \rangle + d_j \geq 0 \ (j = 1, \ldots, r)\}$

be the facet representation of $P$. Here $\langle \cdot, \cdot \rangle$ is the natural pairing between $N_R$ and $M_R$, $r$ is the number of facets of $P$, $\lambda_j \in N$, $d_j \in R$, and each $\lambda_j$ is primitive. Let $P^0$ denote the interior of $P$. Then $X_P^0 := \mu_P^{-1}(P^0)$ is a dense open subset of $X_P$ where the $S$-action is free. This coincides with the open dense orbit of the $T$-action described above. Also, for any $k$-dimensional face $F$ of $P$, $\mu_P^{-1}(F)$ is a $T$-invariant $k$-dimensional connected complex submanifold of $X_P$. In particular, if $F$ is a facet of $P$ then $\mu_P^{-1}(F)$ is a $T$-invariant prime divisor of $X_P$. For each $j = 1, \ldots, r$, let $F_j$ denote the facet of $P$ defined by

$$F_j = \{x \in P \mid \langle \lambda_j, x \rangle + d_j = 0\},$$

and $D_j := \mu_P^{-1}(F_j)$. By setting $c_j$ the Poincare dual of $D_j$, we have

$$[\omega_P] = \sum_{j=1}^r d_j c_j$$

as a de Rham cohomology class. If the Delzant polytope $P$ is integral, that is, each vertex of $P$ belongs to $Z^n$, then $d_j \in Z \ (j = 1, \ldots, r)$ and hence $[\omega_P] \in H^2(X, Z)$. Hence, in this case there exists a $T$-equivariant holomorphic line bundle $L_P$ over $X_P$ which satisfies $c_1(L_P) = [\omega_P]$. By the Kodaira embedding theorem, $L_P$ is an ample line bundle over $X_P$. We call $(X_P, L_P)$ a polarized toric manifold associated to $P$.

### 4.2. Symplectic potentials

In this subsection, we quickly review differential-geometric aspects of toric Kähler manifolds. For details, see [1] and [2], [42], [43].

Let $P \subset M_R$ be an $n$-dimensional integral Delzant polytope, and $(X_P, L_P)$ the polarized toric manifold corresponding to $P$. We choose an $S$-invariant Kähler metric $\omega \in c_1(L_P)$. Since $X_P^0$ is holomorphically isomorphic to $T$, there is an $S$-invariant smooth function $\phi: T \to R$ so that

$$\omega = dd^c \phi$$

on $X_P^0$ ([43 Theorem 4.3]). Through the identification $X_P^0 \cong (C^\ast)^n \cong R^n \times S$, we can regard $\phi$ as a smooth function defined on $R^n$. 

Remark 4.2.1. Let $\tau = (\tau^1, \ldots, \tau^n)$, $\tau^i = \exp((-1/2)y^i + 2\sqrt{-1}\theta^i)$ ($i = 1, \ldots, n$) denote the coordinate of $T$. Then, for each $i = 1, \ldots, n$ we have
\[
\frac{d\tau^i}{\tau^i} = -\frac{1}{2}dy^i + 2\pi \frac{\sqrt{-1}}{\tau^i}d\theta^i, \quad \frac{d\bar{\tau}^i}{\bar{\tau}^i} = -\frac{1}{2}dy^i - 2\pi \frac{\sqrt{-1}}{\tau^i}d\theta^i
\]
and
\[
\frac{\sqrt{-1}}{2\pi} d\tau^i \wedge d\bar{\tau}^i = dy^i \wedge d\theta^i.
\]
Further, since
\[
\tau^i \frac{\partial}{\partial \tau^i} = -\frac{\partial}{\partial y^i} - \frac{\sqrt{-1}}{4\pi} \frac{\partial}{\partial \theta^i}, \quad \bar{\tau}^i \frac{\partial}{\partial \bar{\tau}^i} = -\frac{\partial}{\partial y^i} + \frac{\sqrt{-1}}{4\pi} \frac{\partial}{\partial \theta^i},
\]
we obtain
\[
dd^c \phi = \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \phi}{\partial \tau^i \partial \bar{\tau}^j} d\tau^i \wedge d\bar{\tau}^j = \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \phi}{\partial y^i \partial y^j} \frac{d\tau^i}{\tau^i} \wedge \frac{d\bar{\tau}^j}{\bar{\tau}^j}
\]
for any $\phi \in C^\infty(T, \mathbb{R})^S$. In particular, $dd^c \phi$ is a Kähler form on $T$ if and only if $\phi$ is a strictly convex function on $\mathbb{R}^n$.

By using the moment map, one can describe $\phi$ in terms of a convex function on $P$, which is called a symplectic potential. Indeed, up to an additive constant, the derivative $\nabla \phi$ coincides with the moment map for the Kähler metric $\omega$. Hence we may assume that $\nabla \phi$ precisely coincides with the moment map by adding a suitable linear function. Since $\phi$ is strictly convex, the moment map $\nabla \phi$ is a diffeomorphism from $\mathbb{R}^n$ onto $P^\circ$. Now we introduce the coordinate
\[
x = (\nabla \phi)(y),
\]
and define the function $u: P^\circ \to \mathbb{R}$ as the Legendre dual of $\phi$, i.e.,
\[
u(x) + \phi(y) = \langle x, y \rangle.
\]
$u$ is called the symplectic potential of the Kähler metric $\omega$.

Example 4.2.2 ([43]). Let $\ell_j(x) := \langle \lambda_j, x \rangle + d_j$ ($j = 1, \ldots, r$) and $\ell_\infty := \sum_{j=1}^r \ell_j$. Then, by setting $\phi_P := \mu_P^*(\ell_\infty - \sum_{j=1}^r d_j \ell_j)$, we have
\[
\omega_P = dd^c \phi_P
\]
on $X_P^\circ$. The symplectic potential of the Kähler metric $\omega_P$ is given by
\[
u_P := \sum_{j=1}^r \ell_j \log \ell_j,
\]
usually called the Guillemin potential.

By the theory of Guillemin [43] and Abreu [1], there is the following correspondence for $S$-invariant Kähler metrics in $c_1(L_P)$ and symplectic potentials. Let $u$ be a strictly convex function on $P^\circ$. The function $u$ is said to satisfy the Guillemin’s boundary condition if it satisfies the following conditions:
\begin{enumerate}
\item $u - u_P \in C^\infty(P)$;
\item For any face $F$ of $P$, $u|_F$ is a strictly convex function on $F^\circ$.
\end{enumerate}

Let $S$ denote the set of all strictly convex functions on $P^\circ$ which satisfy the Guillemin’s boundary condition.

**Theorem 4.2.3** ([1], [3], [35], [43]). Every $S$-invariant Kähler metric in the class $c_1(L_P)$ has a symplectic potential $u$ in $S$. Conversely, every function belongs to $S$ is a symplectic potential for some $S$-invariant Kähler metric in $c_1(L_P)$.

For later use, for each $u \in S$ we denote $\varphi_u \in \mathcal{H}(X_P, \omega_P)^S$ by the Kähler potential corresponding to $u$, which is determined by $\varphi_u|_{X^u_P} = \phi - \phi_P$, where $\phi$ is the Legendre dual of $u$. By using the correspondence in Theorem 4.2.3, the scalar curvature of an $S$-invariant Kähler metric $\omega \in c_1(L_P)$ is given by derivatives of the corresponding symplectic potential $u$ with respect to the symplectic coordinates $(x_1, \ldots, x_n)$.

**Theorem 4.2.4** (Abreu [2]). Let $\omega$ be an $S$-invariant Kähler metric in the class $c_1(L_P)$ and $u$ the symplectic potential of $\omega$. Then the scalar curvature $s(\omega)$ of $\omega$ is given by

$$
\begin{align*}
    s(\omega) &= s(u) := -\sum_{i,j=1}^{n} \frac{\partial^2 u_{ij}}{\partial x_i \partial x_j},
\end{align*}
$$

where $u_{ij}$ is the inverse of the Hessian $(u_{ij})$ of $u$:

$$(u_{ij}) = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right).$$

Furthermore, by using Donaldson’s integration by parts formula ([31]), the average of the scalar curvature $\bar{s}$ is obtained as follows. Let us recall the Borel measure $\sigma$ on $\partial P$ defined by

$$
\sigma(A) = \sum_{j=1}^{r} \frac{|A \cap F_j|_{n-1}}{|\lambda_j|},
$$

where $| \cdot |_{n-1}$ be the $(n-1)$-dimensional Lebesgue measure. This measure arises naturally as the subleading coefficient of the Ehrhart polynomial of an integral polytope

$$
E_P(m) := \#(mP \cap M) = \text{vol}(P)m^n + \frac{\sigma(\partial P)}{2}m^{n-1} + O(m^{n-2}).
$$

Let $C_\infty$ denote the set of all continuous convex functions on $P$ which are smooth in the interior.
Theorem 4.2.5 ([34] Lemma 3.3.5). Let $u \in S$ and $f \in C_\infty$. Then $u^{ij}f_{ij}$ is integrable on $P$ and

$$\int_P u^{ij}f_{ij} \, dx = \int_{\partial P} f \, d\sigma + \int_P f(u^{ij}) \, dx$$

$$= \int_{\partial P} f \, d\sigma - \int_P f(s(u)) \, dx.$$

Corollary 4.2.6. The average of the scalar curvature $\overline{s}$ is given by

$$\overline{s} = \frac{\sigma(\partial P)}{\text{vol}(P)}.$$

4.3. Toric test configurations. In this subsection we explain the construction of toric test configurations. For this, we first recall the algebraic construction of toric varieties following [27, Chapter 7] (see also [81]). Let $P \subset \mathbb{R}^n$ be an $n$-dimensional integral (possibly unbounded) polyhedron. Then $P$ has a unique facet representation

$$P = \{ x \in M_{\mathbb{R}} \mid \langle \lambda_j, x \rangle + d_j \geq 0 \ (j = 1, \ldots, r) \},$$

where $\lambda_j \in N$, $d_j \in \mathbb{Z}$ and $\lambda_j$ is primitive. Also, it can be written as a Minkowski sum

$$P = Q + C,$$

where $Q$ is an integral polytope and $C$ is a strongly convex rational polyhedral cone ([81, Theorem 1.2]). The cone part of $P$ is determined by

$$C = \{ x \in M_{\mathbb{R}} \mid \langle \lambda_j, x \rangle \geq 0 \ (j = 1, \ldots, r) \}.$$

([81, Proposition 1.12]). Following [81], we call $C$ the recession cone of $P$. Let $C(P)$ denote the cone of $P$ defined by

$$C(P) = \{ (x, \rho) \in M_{\mathbb{R}} \times \mathbb{R} \mid \rho \geq 0, \langle \lambda_j, x \rangle + \rho d_j \geq 0 \ (j = 1, \ldots, r) \}.$$

Then it is easy to check that

$$C(P) \cap (M_{\mathbb{R}} \times \{0\}) = C \times \{0\},$$

and $\rho P$ is the slice of $C(P)$ at height $\rho$ for any $\rho > 0$. Let $S_P := C(P) \cap (M \times \mathbb{Z})$. Also, we let $\sigma := C \times \{0\}$ and $S_\sigma := \sigma \cap (M \times \mathbb{Z})$. From Gordan’s lemma, $S_P$ and $S_\sigma$ are affine semigroups. Let $R := \mathbb{C}[S_P]$ be the semigroup ring of $S_P$. Then $R$ is a finitely generated $\mathbb{C}$-algebra. The character associated to $(\alpha, m) \in M \times \mathbb{Z}$ is written $\chi^\alpha s^m$, and $R$ is graded by height, i.e. $\text{deg}(\chi^\alpha s^m) = m$. Consequently, we obtain a graded $\mathbb{C}$-algebra

$$R = \bigoplus_{m=0}^{\infty} R_m, \quad R_0 = \mathbb{C}[S_\sigma].$$

The inclusion $R_0 \hookrightarrow R$ gives a structure of finitely generated $R_0$-algebra on $R$. Let $U_P := \text{Spec} \ R_0$, $X_P := \text{Proj} \ R$, and $\pi_P : X_P \to U_P$ be the structure morphism. Then $U_P$ is an affine toric variety, $X_P$ is a projective toric variety, and $\pi_P$ is a projective toric morphism. For each $j \in \{1, \ldots, r\}$, let $F_j$ denote
the facet of \( P \) defined by \( F_j = \{ x \in P \mid \langle \lambda_j, x \rangle + d_j = 0 \} \). Associated to \( F_j \), we have a \( T \)-invariant prime divisor \( D_j \) of \( X_P \). Then,

\[
D_P := \sum_{j=1}^{r} d_j D_j
\]
is a \( \pi_P \)-ample Cartier divisor of \( X_P \) ([27] Example 7.2.8]). An important fact is that if \( P \) is an integral Delzant polytope then \( (X_P, L_P) \cong (\text{Proj} \, R, \mathcal{O}_{X_P}(D_P)) \).

Now let \( P \subset M_R \) be an integral Delzant polytope, and identify \( (X_P, L_P) \) and \( (X_P, \mathcal{O}_{X_P}(D_P)) \) constructed above. Recall that a function \( f: P \to \mathbb{R} \) is called rational piecewise affine convex if \( f \) is a convex function of the form

(4.2) \[ f = \max \{ \ell_1, \ldots, \ell_m \} \]

with each \( \ell_j \) an affine function having rational coefficients. Given a rational piecewise affine convex function \( f \), we can construct a \( T \)-equivariant test configuration \( (\mathcal{X}_f, \mathcal{L}_f) \) for \( (X_P, L_P) \) as follows. Choose an integer \( L \) so that \( L > \max_P f \), we define a rational convex polyhedron \( \mathcal{P} \subset M_R \times R \) by

\[ \mathcal{P} = \{(x,y) \in M_R \times R \mid x \in P, f(x) - L \leq y \}. \]

By replacing \( f, L, \mathcal{P} \) by \( k f, k L, k \mathcal{P} \) for suitable \( k \in \mathbb{Z}_{>0} \) if necessary, we may assume that each \( \ell_j \) has integral coefficients and \( \mathcal{P} \) is an integral polyhedron. Then the recession cone of \( \mathcal{P} \) is given by \( \mathcal{C} := \{0\} \times R_{\geq 0} \subset M_R \times R \), and the cone of \( \mathcal{P} \) is given by

\[ C(\mathcal{P}) = (\mathcal{C} \times \{0\}) \cup \left( \bigcup_{\rho > 0} \rho \mathcal{P} \times \{\rho\} \right). \]

Let \( S_\mathcal{P} := C(\mathcal{P}) \cap (M \times \mathbb{Z}^2), \sigma := \mathcal{C} \times \{0\}, \) and \( S_\sigma := (\mathcal{C} \times \{0\}) \cap (M \times \mathbb{Z}^2) \). Then \( S_\mathcal{P} \) and \( S_\sigma \) are affine semigroups. Let \( \mathcal{R} := C[S_\sigma] \) be the semigroup ring of \( S_\mathcal{P} \). Then \( \mathcal{R} \) is a finitely generated \( \mathcal{C} \)-algebra. The character associated to \( (\alpha, \lambda, m) \in M \times \mathbb{Z}^2 \) is written \( \chi_\alpha t^\lambda s^m \), and the grading of \( \mathcal{R} \mathcal{P} \) is given by \( \deg(\chi_\alpha t^\lambda s^m) = m \). Consequently, we have the graded \( \mathcal{C} \)-algebra

\[ \mathcal{R} = \bigoplus_{m=0}^{\infty} \mathcal{R}_m, \]

\[ \mathcal{R}_m = \begin{cases} 
\mathcal{C}[S_\sigma] = \mathcal{C}[t] & m = 0, \\
\text{Vect}_{\mathcal{C}} \left\{ \chi_\alpha t^\lambda s^m \left| \begin{array}{l}
\alpha \in mP \cap M, \lambda \in \mathbb{Z}, \\
m(f(\alpha/m) - L) \leq \lambda
\end{array} \right. \right\} & m > 0.
\end{cases} \]

The inclusion \( \mathcal{C}[t] = \mathcal{R}_0 \hookrightarrow \mathcal{R} \) gives a structure of \( \mathcal{C}[t] \)-algebra on \( \mathcal{R} \). Note also that \( \mathcal{R} \) is finitely generated as a \( \mathcal{C}[t] \)-algebra. By setting \( \mathcal{D}_\mathcal{P} \) the divisor of \( \text{Proj} \, \mathcal{R} \) defined above, we have a polarized scheme

\[ (\mathcal{X}_f, \mathcal{L}_f) = (\text{Proj} \, \mathcal{R}, \mathcal{O}_{\text{Proj} \, \mathcal{R}}(\mathcal{D}_\mathcal{P})). \]

Then one can check that \( \mathcal{O}_{\text{Proj} \, \mathcal{R}}(\mathcal{D}_\mathcal{P}) = \mathcal{O}_{\text{Proj} \, \mathcal{R}}(1) \). Let \( \pi_f: \mathcal{X}_f \to \mathbb{A}^1 \) be the structure morphism. Then \( \pi_f \) is a projective morphism and hence proper.
By construction, \( X_f \) is a normal toric variety and \( L_f \) is \( \pi_f \)-ample line bundle over \( X_f \). There is a \( G^{m+1} \)-action on \( R \), which has weight \((\alpha, \lambda) \in M \times Z \) for any \( \chi^\alpha t^\lambda \in Sp \). This induces a \( T_f := T \times G_m \)-action on \( X_f \) over \( A^1 \). We claim \((X_f, L_f)\) is a test configuration for \((X_P, L_P)\). To see this, it is sufficient to show that \( R \) is the Rees algebra of a finitely generated \( Z \)-filtration of \( R \) (see Section 3.4). Let \( F = \{ F^\lambda R_m \} \) be a \( Z \)-filtration of \( R \) defined by

\[
F^\lambda R_0 = \begin{cases} 
C & \lambda \leq 0, \\
\{0\} & \lambda > 0 
\end{cases}
\]

and

\[
F^\lambda R_m = \text{Vect}_C \{ \chi^\alpha s^m | \alpha \in mP \cap M, \lambda \leq m(L - f(\alpha/m)) \}
\]

for any \( \lambda \in R \) and \( m \in Z_{>0} \). Then the Rees algebra of \( F \) is given by

\[
\text{Rees}(F) = \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in Z} t^{-\lambda} F^\lambda R_m \\
= \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in Z} \text{Vect}_C \left\{ \chi^\alpha t^{-\lambda} s^m \left| \begin{array}{l} \alpha \in mP \cap M, \\
\lambda \leq m(L - f(\alpha/m)) \end{array} \right. \right\} \\
= \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \in Z} \text{Vect}_C \left\{ \chi^\alpha t^\lambda s^m \left| \begin{array}{l} \alpha \in mP \cap M, \\
\lambda \leq m(L - f(\alpha/m)) \end{array} \right. \right\} \\
= \bigoplus_{m=0}^{\infty} \mathcal{R}_m = \mathcal{R}.
\]

Furthermore, \( F \) is finitely generated since \( \mathcal{R} \) is a finitely generated \( C[t] \)-algebra. This shows the claim. We call \((X_f, L_f)\) the toric test configuration associated to \( f \). The induced \( G_m \)-coaction \( \mu_m \) on the central fiber \( H^0((X_f)_0, (L_f)_0) = \Gamma_f \mathcal{R} \) is given by

\[
\mu_m(\chi^\alpha s^m) = t^{m(L - f(\alpha/m))} \otimes \chi^\alpha s^m.
\]

Easy computation shows that the canonical compactification of \((X_f, L_f)\) is given as a polarized toric variety \((X_Q, L_Q)\) corresponding to a rational convex polytope

\[
Q = \{(x, y) \in M_R \times R \mid x \in P, f(x) - L \leq y \leq 0\}.
\]

**Example 4.3.1 (Translations and scalings: toric cases).** Let \( f \) be a rational piecewise affine convex function on an integral Delzant polytope \( P \). Let us describe translations and scalings of \((X_f, L_f)\) in terms of the convex function \( f \). For simplicity, we assume \( f \) is integral, i.e. \( f \) is of the form \([1] \) with each \( \ell_j \) an affine function having integral coefficients.

1. Let \( c \in Z \), and choose \( L \in Z_{>0} \) so that \( \max\{f, f - c\} < L \) on \( P \). Then the filtration \( F_c \) corresponding to the translation \((X_f, (L_f)_c)\) is given by

\[
F^\lambda_c R_m = F^\lambda - mc R_m.
\]
Claerly we have $F^\lambda_c R_0 = F^\lambda R_0$. If $m > 0$, then we have

$$F^\lambda_c R_m = \text{Vect}_C \{ \chi^\alpha s^m | \alpha \in mP \cap M, \lambda - mc \leq m(L - f(\alpha/m)) \}$$

$$= \text{Vect}_C \{ \chi^\alpha s^m | \alpha \in mP \cap M, \lambda \leq m(L - f(\alpha/m) + c) \}.$$ 

Hence $F^c_c$ coincides with the filtration corresponding to the toric test configuration $(X_{f-c}, L_{f-c})$.

(2) Let $d \in \mathbb{Z}_{>0}$. Then the filtration $F^d_c$ corresponding to the scaling $((X_d f, (L_d f))$ is given by

$$F^\lambda_c R_m = F^\lceil \frac{\lambda}{d} \rceil R_m.$$ 

Claerly we have $F^\lambda_c R_0 = F^\lambda R_0$. If $m > 0$, then we have

$$F^\lambda_c R_m = \text{Vect}_C \{ \chi^\alpha s^m | \alpha \in mP \cap M, \lfloor \frac{\lambda}{d} \rfloor \leq m(L - f(\alpha/m)) \}$$

$$= \text{Vect}_C \{ \chi^\alpha s^m | \alpha \in mP \cap M, \lambda \leq m(dL - df(\alpha/m)) \}.$$ 

Hence $F^d_c$ coincides with the filtration corresponding to the toric test configuration $(X_d f, L_d f)$.

4.4. Energy functionals on polarized toric manifolds. By using Duistermaat-Heckman theorem and Legendre dual functions, we can write down energy functionals in terms of convex functions.

**Lemma 4.4.1.** Let $\{u^t\} \subset S$ be a smooth path, and $\{\phi^s\}$ be their Legendre duals. Then

$$\dot{u}^t(x) = -\dot{\phi}^t(\nabla u^t(x))$$

for any $x \in P^c$.

**Proof.** Since $\phi^s$ is a strictly convex function, $\nabla u^s(x)$ is a unique minimizer of $y \mapsto \phi^s(y) - \langle y, x \rangle$. By setting $y^s = \nabla u^s(x)$, we have

$$0 = \frac{d}{ds} \bigg|_{s=t} (\phi^s(y^s) - \langle y^s, x \rangle) = \frac{d}{ds} \bigg|_{s=t} \nabla u^s(x).$$

Hence we obtain

$$\dot{u}^t(x) = \frac{d}{ds} \bigg|_{s=t} u^s(x)$$

$$= \frac{d}{ds} \bigg|_{s=t} (\langle y^s, x \rangle - \phi^s(y^s))$$

$$= \frac{d}{ds} \bigg|_{s=t} (\langle y^s, x \rangle - \phi^t(y^t)) + \frac{d}{ds} \bigg|_{s=t} (\phi^t(y^t) - \phi^s(y^s))$$

$$= \langle \nabla \phi^t, \dot{y}^t \rangle - \dot{\phi}^t(y^t) - \langle \nabla \phi^t, \dot{y}^t \rangle$$

$$= -\dot{\phi}^t(y^t),$$

as required. \qed
Proposition 4.4.2. Let $\varphi \in \mathcal{H}(X_P, \omega_P)^S$, and $u \in S$ be the symplectic potential of $\omega_\varphi$. Then

$$E(\varphi) = -\int_P (u - u_P) \, dx.$$ 

Proof. Let $\{u^t\}_{t \in [0, 1]} \subset S$ be a smooth path so that $u^0 = u_P$ and $u^1 = u$. Let $\varphi^t := \varphi_{u^t} \in \mathcal{H}(X_P, \omega_P)^S$ for each $t \in [0, 1]$. Note that $\varphi^0 = 0$ and $\varphi^1 = \varphi$. Then we have

$$E(\varphi) = \int^1_0 dt \int_{X_P} \varphi^t \omega^n_{\varphi^t}$$

$$= \int^1_0 dt \int_{X_P} \tilde{\varphi}^t (dd^c \tilde{\varphi}^t)^n$$

$$= -\int^1_0 dt \int_P \dot{u}^t \, dx$$

$$= -\int_P (u - u_P) \, dx.$$

$\square$

We define a subset $S_0$ of $S$ by

$$S_0 := \left\{ u \in S \left| -\int_P (u - u_P) \, dx = 0 \right. \right\}$$

Proposition 4.4.3. Let $\varphi \in \mathcal{H}(X_P, \omega_P)^S$ and $u \in S$ be the symplectic potential of $\omega_\varphi$. 

(1) Let $\phi$ be the Legendre dual of $u$. Then

$$J(\varphi) = \int_{X_P^0} (\phi - \phi_P) \omega^n_P + \int_P (u - u_P) \, dx.$$ 

(2) Let $F(u) := -\int_P \log \det(u_{ij}) \, dx + \frac{1}{\text{vol}(P)} \int_{\partial P} u \, d\sigma - \int_P u \, dx$. Then

$$M(\varphi) = F(u) - F(u_P).$$

Proof. For (1), since $\varphi = \phi - \phi_P$ on $X_P^0$, we have

$$\int_{X_P} \varphi \omega^n_P = \int_{X_P^0} (\phi - \phi_P) \omega^n_P.$$ 

Hence we have

$$J(\varphi) = \int_{X_P} \varphi \omega^n_P - E(\varphi)$$

$$= \int_{X_P^0} (\phi - \phi_P) \omega^n_P + \int_P (u - u_P) \, dx$$

by Proposition 4.4.2.
Next consider (2). Let \( \{ \varphi^t \}_{t \in [0, 1]} \) be a smooth path in \( \mathcal{H}_0^S \) satisfying \( \varphi^0 = 0 \) and \( \varphi^1 = \varphi \), and \( \{ u^t \}_{t \in [0, 1]} \) be the corresponding symplectic potentials. Note that \( u^0 = u_P \) and \( u^1 = u \). Then
\[
\frac{d}{dt} M(\varphi^t) = -\int_{X_P} \dot{\varphi}^t (s(\varphi^t) - \bar{s}) \omega^n_{\varphi^t} = -\int_{X_P} (\dot{u}^t) (-u^t)_{ij} \omega^n_{\varphi^t} dx = -\int_{X_P} (\dot{u}^t)(u^t)_{ij} dx - \bar{s} \int_P \dot{u}^t dx.
\]
By Theorem 4.2.5, we obtain
\[
-\int_{X_P} (\dot{u}^t)(u^t)_{ij} dx = \frac{1}{\operatorname{vol}(P)} \int_{\partial P} \dot{u}^t d\sigma - \int_P (u^t)_{ij} \frac{d}{dt} \log \det((u^t)_{ij}) dx
\]
and hence
\[
\frac{d}{dt} M(\varphi^t) = \frac{1}{\operatorname{vol}(P)} \int_{\partial P} \dot{u}^t d\sigma - \int_P \frac{d}{dt} \log \det((u^t)_{ij}) dx - \bar{s} \int_P \dot{u}^t dx
\]
\[
= \frac{d}{dt} \left( -\int_P \log \det((u^t)_{ij}) dx + \frac{1}{\operatorname{vol}(P)} \int_{\partial P} u^t d\sigma - \bar{s} \int_P u^t dx \right),
\]
as required.

**Proposition 4.4.4.** Let \( \varphi \in \mathcal{H}(X_P, \omega_P)^S \), and \( u \in S \) be the symplectic potential of \( \omega_\varphi \).

1. \( H_V(\varphi) = -\int_P (u - u_P) V dx \).
2. Let \( F_V(u) := F(u) + H_V(\varphi) \). Then \( M_V(\varphi) = F_V(u) - F_V(u_P) \).

**Proof.** For (1), let \( \{ \varphi^t \}_{t \in [0, 1]} \) be a smooth path in \( \mathcal{H}_0^S \) satisfying \( \varphi^0 = 0 \) and \( \varphi^1 = \varphi \), and \( \{ u^t \}_{t \in [0, 1]} \) be the corresponding symplectic potentials. Note that \( u^0 = u_P \) and \( u^1 = u \). Then
\[
\frac{d}{dt} H_V(\varphi^t) = \int_{X_P} \dot{\varphi}^t \theta \omega^n_{\varphi^t} = \int_{X_P} \dot{\varphi}^t (-\left\langle \mu_{\varphi^t}, -\frac{1}{2} \theta \right\rangle) \omega^n_{\varphi^t} = -\int_P u^t \frac{1}{2} \theta dx = \frac{d}{dt} \left( -\int_P u^t \frac{1}{2} \theta dx \right),
\]
as required.

The claim (2) is easily proved from (1) and Proposition 4.4.3. \( \square \)
As shown by Guan in [41], any pair of Kähler potentials in $H^S_0$ can be joined by a unique geodesic segment, which are obtained as a line segment of symplectic potentials. Thus, we have the following:

**Proposition 4.4.5.** Let $\varphi \in H^S_0$, and $u \in S$ be the symplectic potential of $\omega_{\varphi}$. Then

$$d_1(\varphi, 0) = \int_P |u - u_P| \, dx.$$  

**Proof.** Let $u^t = (1 - t)u_P + tu$ ($t \in [0, 1]$), and $\varphi^t := \varphi_{u^t}$. Then $\{\varphi^t\}_{t \in [0, 1]}$ is a unique geodesic segment joining 0 and $\varphi$. Hence we have

$$d_1(\varphi, 0) = \ell_1(\{\varphi^t\}_{t \in [0, 1]}) = \int_0^1 dt \int_X |\dot{\varphi}^t|^\omega^n_{\varphi^t} = \int_0^1 dt \int_P |\dot{u}^t|^dx = \int_P |u - u_P| \, dx.$$  

□

Consider the action of $T$ on $S_0$. Let $u \in S_0$ and $\tau \in T$. Then $(\varphi_u)_\tau \in H^S_0$ can be written as

$$\tau^* \omega_{\varphi_u} = \omega_P + dd^c(\varphi_u)_\tau.$$  

On $X^\circ_P \cong T$, we can further express

$$\tau^* \omega_{\varphi_u} = \tau^* dd^c u^\vee = dd^c(\tau^* u^\vee).$$

If we write $\tau = \exp(-\xi/2)$ for some $\xi \in N^\circ_{\mathbb{R}}$, we have

$$\tau \cdot \exp(-y/2 + 2\pi \sqrt{-1}\theta) = \exp(-(y + \xi)/2 + 2\pi \sqrt{-1}\theta)$$

and

$$\tau^* u^\vee(y) = u^\vee(y + \xi) = \sup_{x \in P^\circ} (\langle x, y + \xi \rangle - u) = (u - \xi)^\vee(y).$$

Therefore we obtain

$$\tau^* \omega_u = dd^c(\tau^* u^\vee) = dd^c(u - \xi) = \omega_{u - \xi}$$

and hence

$$dd^c(\varphi_u)_\tau = \tau^* \omega_u - \omega_P = \omega_{u - \xi} - \omega_P = dd^c \varphi_{u - \xi}$$

on $X^\circ_P$. By replacing the coordinate $(x_1, \ldots, x_n)$ on $M^\circ_{\mathbb{R}}$ so that

$$\int_P x_i \, dx = 0, \quad i = 1, \ldots, n,$$

we obtain $(\varphi_u)_\tau = \varphi_{u - \xi}$. This shows the following
Proposition 4.4.6. Let $\varphi \in H^S_0$, and $u \in S$ be the symplectic potential of $\omega_\varphi$. Then

$$d_{1,T}(\varphi,0) = \inf_{\xi \in N_R^P} \int_P |u - \xi - u_P| \, dx.$$ 

We define the reduced $L^1$-norm by

$$\|u\|_{1,T} := \inf_{\xi \in N_R^P} \int_P |u - \xi - (u_0 - \xi)| \, dx,$$

where

$$(u_0 - \xi) := \int_P (u - \xi) \, dx.$$ 

Then

Corollary 4.4.7. Let $\varphi \in H^S_0$, and $u \in S$ be the symplectic potential of $\omega_\varphi$. Then

$$\|u\|_{1,T} - \|u_P\|_1 \leq d_{1,T}(\varphi,0) \leq \|u\|_{1,T} + \|u_P\|_1.$$ 

Corollary 4.4.8. For a polarized toric manifold $(X_P, L_P)$, the followings are equivalent.

1. The relative K-energy is $T$-coercive.
2. There exists $\delta, C > 0$ such that $F_V(u) \geq \delta \|u\|_{1,T} - C$ for any $u \in S_0$.
3. There exists $\delta, C > 0$ such that $F_V(u) \geq \delta J_T(u) - C$ for any $u \in S_0$.

4.5. Non-Archimedean functionals on polarized toric manifolds. In this subsection, we give convex analytic expression of non-Archimedean functionals for toric test configurations. Let $f \in C^Q_{PL}$. For simplicity we assume $f$ is integral. Choose $L \in \mathbb{Z}_{\geq 0}$ so that $L - f > 0$ on $P$. Then the corresponding $(X_f, L_f)$ is a toric test configuration of exponent 1.

Proposition 4.5.1. The Duistermaat-Heckmann measure of $(X_f, L_f)$ is given by

$$DH(X_f, L_f) = (L - f)_# \frac{dx}{\text{vol}(P)}.$$ 

In particular, $\text{supp}(DH(X_f, L_f)) = [L - \max_P f, L - \min_P f]$.

Proof. Let $\rho$ be a bounded continuous function on $\mathbb{R}$. Then we have

$$\int_{\mathbb{R}} \rho \, dDH(X_f, L_f) = \lim_{m \to \infty} \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} \rho \left( L - f \left( \frac{\alpha}{m} \right) \right)$$

$$= \int_P \rho(L - f) \, dx.$$
Proposition 4.5.2. For each $f \in C^Q_{PL}$ we have

$$E_{NA}(\mathcal{X}_f, \mathcal{L}_f) = L - \int_P f \, dx.$$  

Proof. Let $f \in C^Q_{PL}$. Then we have

$$E_{NA}(\mathcal{X}_f, \mathcal{L}_f) = \int \lambda \, dDH(\mathcal{X}_f, \mathcal{L}_f)$$

$$= \lim_{m \to \infty} \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} \left( L - f \left( \frac{\alpha}{m} \right) \right)$$

$$= \int_P (L - f) \, dx$$

$$= L - \int_P f \, dx,$$

as required.

Proposition 4.5.3. For each $f \in C^Q_{PL}$ we have

$$J_{NA}(\mathcal{X}_f, \mathcal{L}_f) = \|f\|_J := \int_P f \, dx - \min_P f,$$

$$J^T_{NA}(\mathcal{X}_f, \mathcal{L}_f) = \|f\|_{J^T} := \inf_{\xi: \text{affine}} \left( \int_P (f + \xi) \, dx - \min_P (f + \xi) \right).$$

Proof. The formula for $J_{NA}(\mathcal{X}_f, \mathcal{L}_f)$ is a direct consequence of Propositions 4.5.1 and 4.5.2. Let us consider $J^T_{NA}(\mathcal{X}_f, \mathcal{L}_f)$. For the graded ring $R(X_P, L_P)$, $R_m$ can be identified with

$$R_m = H^0(X_P, L_P^m) \cong \text{Vect}_C \{ \chi^{\alpha t^m} \mid \alpha \in mP \cap M \}.$$

The induced $T$-equivariant filtration $\mathcal{F}(\mathcal{X}_f, \mathcal{L}_f)$ is then given by

$$F^\lambda R_m = \text{Vect}_C \{ \chi^{\alpha t^m} \mid \alpha \in mP \cap M, m(f(\alpha/m) - L) \leq \lambda \},$$

$$(F^\lambda R_m)_\alpha = \text{Vect}_C \{ \chi^{\alpha t^m} \mid m(f(\alpha/m) - L) \leq \lambda \}.$$  

Let $\xi \in N_{R^*}$. If necessary, replace $L \in \mathbb{Z}_{>0}$ so that $L - f - \xi > 0$. Then the $\xi$-twist $\mathcal{F}_\xi$ of $\mathcal{F}(\mathcal{X}_f, \mathcal{L}_f)$ is given by

$$(F^\lambda \mathcal{R}_m)_\alpha = (F^{\lambda - (\alpha, \xi)} \mathcal{R}_m)_\alpha$$

$$= \text{Vect}_C \{ \chi^{\alpha t^m} \mid m(f(\alpha/m) - L) \leq \lambda - (\alpha, \xi) \}$$

$$= \text{Vect}_C \{ \chi^{\alpha t^m} \mid m(f(\alpha/m) + \xi(\alpha/m) - L) \leq \lambda \},$$

$$F^\lambda_\xi R_m = \bigoplus_{\alpha \in M} (F^\lambda_\xi R_m)_\alpha.$$
Hence we have
\[ \nu_m = \frac{d}{d\lambda} \frac{1}{E_P(m)} \dim F^{m\lambda}_\xi H^0(X, L^m) \]
\[ = \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} \delta_{L-f(\alpha/m)-\xi(\alpha/m)} \]
and
\[ \int_{\mathbb{R}} \rho \, d\nu = \lim_{m \to \infty} \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} \rho(L - f(\alpha/m) - \xi(\alpha/m)) \]
\[ = \int_P \rho(L - f - \xi) \, dx \]
for any bounded continuous function \( \rho \) on \( \mathbb{R} \). Therefore, for each Borel measurable set \( A \) in \( P \) we have
\[ \nu(A) = \frac{1}{\text{vol}(P)} \text{vol}(P) \{ x \in P \mid L - f - \xi \in A \} \]
and
\[ \int_{\mathbb{R}} \lambda \, d\nu = L - \int_P (f + \xi) \, dx, \]
\[ \text{supp}(\nu) = [L - \max_P (f + \xi), L - \min_P (f + \xi)]. \]
Hence, we obtain
\[ J^{NA}_{TA}(X_f, L_f) = \inf_{\xi: \text{aline}} \left( \int_P (f + \xi) \, dx - \min_P (f + \xi) \right). \]

**Proposition 4.5.4.** For each \( f \in \mathcal{C}_{PL}^Q \) we have
\[ M^{NA}(X_f, L_f) = \frac{1}{\text{vol}(P)} L(f) \]
\[ := \frac{1}{\text{vol}(P)} \left( \int_{\partial P} f \, d\sigma - \pi \int_P f \, dx \right). \]

**Proof.** Let \( f \in \mathcal{C}_{PL}^Q \). First we assume the central fiber of \( (X_f, L_f) \) is generically reduced. Then \( M^{NA}(X_f, L_f) \) coincides with \( DF(X_f, L_f) \). Further, by \([LM]\) and \([SM]\) Lemma 3.3] we have
\[ w_m = \sum_{\alpha \in mP \cap M} m \left( L - f \left( \frac{\alpha}{m} \right) \right) \]
\[ = m \left( m^n \int_P (L - f) \, dx + \frac{m^{n-1}}{2} \int_{\partial P} (L - f) \, d\sigma + O(m^{n-2}) \right) \]
\[ = m^{n+1} \int_P (L - f) \, dx + \frac{m^n}{2} \int_{\partial P} (L - f) \, d\sigma + O(m^{n-1}) \]
for any \( m \in \mathbb{Z}_{>0} \). By noting \( \text{vol}(P) = (L^n)/n! \) and \( \overline{\sigma} = \sigma(\partial P)/\text{vol}(P) \), we obtain

\[
M^{NA}(X_f, L_f) = DF(X_f, L_f)
\]

\[
= \frac{n!}{(L^n)} \left( - \int_{\partial P} (L - f) \, d\sigma + \overline{\sigma} \int_{P} (L - f) \, dx \right)
\]

\[
= \frac{1}{\text{vol}(P)} \left( \int_{\partial P} f \, d\sigma - \overline{\sigma} \int_{P} f \, dx \right).
\]

For general cases, we consider \( df \) for \( d \in \mathbb{Z}_{>0} \). This corresponds to replacing \((X_f, L_f)\) with its base change \((X_{df}, L_{df})\) (see Example 4.3.1 (2)). Then, by Proposition 3.3.3 there exists \( d_0 \in \mathbb{Z}_{>0} \) such that \( DF(X_{df}, L_{df}) = M^{NA}(X_{df}, L_{df}) = dM^{NA}(X_f, L_f) \) for all \( d \in \mathbb{Z}_{>0} \) divisible by \( d_0 \). Further, by Proposition 4.3.3 we have

\[
M^{NA}(X_f, L_f) = \frac{1}{d} M^{NA}(X_{df}, L_{df})
\]

\[
= \frac{1}{d} \left( \frac{1}{\text{vol}(P)} \left( \int_{\partial P} df \, d\sigma - \overline{\sigma} \int_{P} df \, dx \right) \right)
\]

\[
= \frac{1}{\text{vol}(P)} \left( \int_{\partial P} f \, d\sigma - \overline{\sigma} \int_{P} f \, dx \right),
\]

as required. \( \square \)

**Proposition 4.5.5.** For each \( f \in C_P \), we have

\[
H^{NA}_V(X_f, L_f) = -\int_P fV \, dx,
\]

\[
M^{NA}_V(X_f, L_f) := \frac{1}{\text{vol}(P)} \left( \int_{\partial P} f \, d\sigma - \int_{P} (\overline{\sigma} + V) f \, dx \right).
\]

**Proof.** It is enough to show the formula for \( H^{NA}_V \). Since \( \int_P V \, dx = 0 \), we have

\[
H^{NA}_V(X_f, L_f) = \lim_{m \to \infty} \frac{1}{m^2} \left\{ \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} m \left( L - f \left( \frac{\alpha}{m} \right) \right) V(\alpha) \right\}
\]

\[
- \frac{1}{(E_P(m))^2} \sum_{\alpha \in mP \cap M} m \left( L - f \left( \frac{\alpha}{m} \right) \right) \sum_{\alpha \in mP \cap M} V(\alpha)
\]

\[
= \lim_{m \to \infty} \frac{1}{E_P(m)} \sum_{\alpha \in mP \cap M} \left( L - f \left( \frac{\alpha}{m} \right) \right) V(\alpha)
\]

\[
= \frac{1}{\text{vol}(P)} \int_P (L - f) V \, dx
\]

\[
= -\int_P fV \, dx,
\]

as required. \( \square \)
By Proposition 4.5.5, we have the following

**Corollary 4.5.6.** Let $(X_P, L_P)$ be a polarized toric manifold. If $(X_P, L_P)$ is uniformly relatively K-polystable, then there exists $\delta > 0$ such that

\[
L_V(f) \geq \delta \|f\|_J
\]

for any $f \in \mathcal{C}_P^Q$.

Our main theorem guarantees that the converse is also true, that is, a polarized toric manifold $(X_P, L_P)$ satisfying (4.5) for any $f \in \mathcal{C}_P^Q$ is uniformly relatively K-polystable.

5. **Proof of Theorem 1.0.1**

5.1. **Spaces of convex functions.** Let $P \subset M_\mathbb{R}$ be an $n$-dimensional integral Delzant polytope defined by

\[
P = \{x \in M_\mathbb{R} \mid \langle \lambda_j, x \rangle + d_j \geq 0 \ (j = 1, \ldots, r)\},
\]

where $\lambda_j \in \mathbb{N}$, $d_j \in \mathbb{Z}$, $r$ is the number of the facets, and each $\lambda_j$ is primitive. Throughout this section, we assume the origin 0 lies in the interior of $P$. Then $d_j > 0$ for any $j = 1, \ldots, r$. Let $\text{CVX}(P)$ denote the set of all lower semicontinuous convex functions $f : P \to (-\infty, +\infty]$ such that $f \not\equiv +\infty$.

Note that every function in $\text{CVX}(P)$ is bounded from below since $P$ is compact. For each subset $F$ of $\text{CVX}(P)$, set

\[
\tilde{F} := \{f \in F \mid \inf_p f = f(0) = 0\}.
\]

If $F$ is closed under addition of affine functions, then for any function $f \in \tilde{F}$ there exists an affine function $\ell$ such that $f + \ell \in \tilde{F}$ by the supporting hyperplane theorem.

Let $\mathcal{E}_1 := \text{CVX}(P) \cap L^1(P)$. By the toric pluripotential theory [24, Proposition 3.2, Theorem 3.6, Proposition 3.9], $\mathcal{E}_1$ can be identified with the metric completion of the space $(S, d_1)$. For each $j = 1, \ldots, r$, let $F_j$ denote the facet of $P$ defined by

\[
F_j = \{x \in P \mid \langle \lambda_j, x \rangle + d_j = 0\},
\]

and define a subset $P^*$ of $P$ by

\[
P^* = P^o \cup \left( \bigcup_{j=1}^r F_j^o \right).
\]

Here $F_j^o$ denotes the relative interior of $F_j$. Let $\mathcal{C}_*$ be the set of all elements of $\text{CVX}(P)$ which are continuous on $P^*$ and integrable on $\partial P$. Note that the integral of any function in $\mathcal{C}_*$ over the boundary makes sense since the measure $\sigma$ is supported on the facets of $P$. A relation between $\mathcal{C}_*$ and $\mathcal{E}_1$ is given by the following proposition.

**Proposition 5.1.1.** $\mathcal{C}_* = \mathcal{E}_1 \cap L^1(\partial P)$. 
We explain a proof of Proposition 5.1.1 below. The inclusion $C_* \subset E_1 \cap L^1(\partial P)$ is obtained from the following proposition.

**Proposition 5.1.2.** Let $d = \max \{d_1, \ldots, d_r\}$. If $f$ is a nonnegative function in $C_*$, then

$$
\int_P f \, dx \leq \frac{d}{n + 1} \left( \frac{\sigma(\partial P)}{n} f(0) + \int_{\partial P} f \, d\sigma \right).
$$

In particular, for any $f \in \tilde{C}_*$ we have

$$
\int_P f \, dx \leq \frac{d}{n + 1} \int_{\partial P} f \, d\sigma.
$$

**Proof.** For each $j = 1, \ldots, r$, let $C(F_j) = \{ t \zeta \mid t \in [0,1], \zeta \in F_j \}$. Then we have

$$
P = \bigcup_{j=1}^r C(F_j).
$$

Also, by convexity of $f$ we have

$$
\int_{C(F_j)} f \, dx \leq d_j \int_0^1 t^{n-1} \left( (1-t)f(0) + tf \right) d\sigma
$$

$$
= d_j \int_0^1 (1-t)t^{n-1} dt \int_{F_j} f(0) d\sigma + d_j \int_0^1 t^n dt \int_{F_j} f d\sigma
$$

$$
= \frac{d_j}{n+1} \left( \frac{\sigma(F_j)}{n} f(0) + \int_{F_j} f d\sigma \right).
$$

Hence we obtain

$$
\int_P f \, dx \leq \sum_{j=1}^r \int_{C(F_j)} f \, dx
$$

$$
\leq \sum_{j=1}^r \frac{d_j}{n+1} \left( \frac{\sigma(F_j)}{n} f(0) + \int_{F_j} f d\sigma \right)
$$

$$
\leq \frac{d}{n+1} \left( \frac{\sigma(\partial P)}{n} f(0) + \int_{\partial P} f d\sigma \right).
$$

The latter claim is now obvious. □

**Corollary 5.1.3.** Every function in $C_*$ is integrable on $P$. In particular, we have $C_* \subset E_1 \cap L^1(\partial P)$.

**Proof.** Let $f \in C_*$. Then there is an affine function $\ell$ so that $\tilde{f} := f + \ell \in \tilde{C}_*$. By Proposition 5.1.2 we have

$$
\int_P \tilde{f} \, dx \leq \frac{d}{n+1} \int_{\partial P} \tilde{f} \, d\sigma < \infty
$$

and hence $\tilde{f}$ is integrable on $P$. Since $\ell$ is integrable on $P$, $f = \tilde{f} - \ell$ is also integrable on $P$. □
Let us show the converse inclusion. For the purpose, we use the following facts from the Donaldson’s work \[34\]. Let \( \Omega \) be a bounded convex open set of \( \mathbb{R}^n \), and \( f: \Omega \to \mathbb{R} \) be a convex function. For each \( x \in \Omega \), set 
\[
D_x(f) := \sup\{|\lambda| \mid f(y) \geq \langle \lambda, x \rangle + f(y) \ (y \in \Omega)\}.
\]

**Proposition 5.1.4** (\[34, Lemma 5.2.3\]). There is a constant \( \kappa > 0 \) such that if a nonnegative convex function \( f \) on \( \Omega \) is integrable, then 
\[
D_x(f) \leq \kappa d_x^{-(n+1)} \int_\Omega f \, dy
\]
for any \( x \in \Omega \). Here \( d_x \) is the distance from \( x \) to the boundary of \( \Omega \).

**Proposition 5.1.5** (\[34, Lemma 5.2.4\]). For any convex function \( f: \Omega \to \mathbb{R} \) and any two points \( x, y \in P \),
\[
|f(x) - f(y)| \leq \max\{D_x(f), D_y(f)\}|x - y|.
\]

Let \( f \in E_1 \cap L^1(\partial P) \). By adding a suitable affine function, we may assume \( f \in \tilde{E}_1 \). Since \( f \) is integrable over \( P \), \( f < +\infty \) on \( P^o \). Continuity of \( f \) on \( P^o \) is obvious by the convexity. Let \( F \in \{F_1, \ldots, F_r\} \).

**Lemma 5.1.6.** For each \( \zeta \in F \) we have \( f(\zeta) = \lim_{t \to 1} f(t\zeta) \).

*Proof.* Let \( \zeta \in F \) and \( s, t \in [0, 1] \). If \( s < t \), then we have
\[
\frac{f(t\zeta) - f(s\zeta)}{t - s} \geq \frac{f(t\zeta) - f(0)}{t - 0} \geq 0
\]
by the convexity of \( f \). Hence the function \([0, 1] \ni t \mapsto f(t\zeta) \in \mathbb{R}\) is monotonically non-decreasing. Further, since 
\[
f(t\zeta) \leq (1 - t)f(0) + tf(\zeta) = tf(\zeta)
\]
for any \( t \in [0, 1] \), we obtain 
\[
\lim_{t \to 1} f(t\zeta) \leq f(\zeta) \leq \lim_{\xi \to \zeta} f(\xi)
\]
by the lower semicontinuity of \( f \). We claim that \( \lim_{\xi \to \zeta} f(\xi) \leq \lim_{t \to 1} f(t\zeta) \).

Let \( \delta > 0 \) and \( t \in [0, 1] \), and suppose \(|t - 1| < \delta/|\zeta|\). Then \( t\zeta \in B(\zeta, \delta) \) and
\[
\inf_{B(\zeta, \delta) \cap P} f \leq f(t\zeta).
\]
Hence we have
\[
\inf_{B(\zeta, \delta) \cap P} f \leq \lim_{t \to 1} f(t\zeta)
\]
and
\[
\liminf_{\xi \to \zeta} f(\xi) = \sup_{\delta > 0} \inf_{B(\zeta, \delta) \cap P} f \leq \lim_{t \to 1} f(t\zeta),
\]
as required. \( \square \)
For each \( \eta \in (0, 1] \), define a Borel measure \( \sigma_\eta \) on \( \eta(\partial P) \) by

\[
\sigma_\eta(A) := \eta^{n-1}\sigma(\eta^{-1}A).
\]

By setting \( C := \int_{\partial P} f(\xi) \, d\sigma \), we have

\[
\int_{\eta(\partial P)} f(\xi) \, d\sigma_\eta = \eta^{n-1} \int_{\partial P} f(\eta\xi) \, d\sigma \leq \eta^n \int_{\partial P} f(\xi) \, d\sigma = \eta^n C.
\]

Let \( K \) be a compact subset of \( F^0 \). We write \( F \) as

\[
F = \{ x \in P \mid \langle \lambda, x \rangle + d = 0 \}.
\]

Regarding \( f \) as a nonnegative convex function on \( \eta F^0 \), by Proposition 5.1.4 there is a constant \( \kappa > 0 \) such that

\[
D_\zeta(f) \leq |\lambda|\kappa d_\zeta^{-n} \int_{\eta F} f(\xi) \, d\sigma_\eta \leq |\lambda|\kappa d_\zeta^{-n} \eta^n C
\]

for any \( \zeta \in \eta K \). Here \( d_\zeta \) is the distance from \( \zeta \) to the boundary \( \partial(\eta F^0) \).

Also, by setting \( d_{\eta K} \) with the distance between \( \eta K \) and \( \partial(\eta F^0) \), we have

\[
D_\zeta(f) \leq |\lambda|\kappa d_{\eta K}^{-n} \eta^n C \leq |\lambda|\kappa d_{\eta K}^{-n} \eta^n C = |\lambda|\kappa d_K^{-n} C.
\]

Hence, by Proposition 5.1.5 we obtain

\[
|f(\zeta') - f(\zeta'')| \leq \max\{D_\zeta(f), D_{\zeta'}(f)\} |\zeta' - \zeta''| \leq |\lambda|\kappa d_K^{-n} C |\zeta' - \zeta''|
\]

for any \( \zeta', \zeta'' \in \eta K \). Let \( \zeta \in F^0 \), and

\[
K := B(\zeta, \delta|\zeta|) \cap \partial P
\]

for each \( \delta > 0 \). By choosing \( \delta \) to be sufficiently small, we may assume that \( K \subset F^0 \). Set \( C(K) := \{ t\xi \mid t \geq 0, \xi \in K \} \). Then \( C(K) \) is a closed neighborhood of \( \zeta \) in \( M_R \). Let \( x \in C(K) \cap P^0 \), and \( H_x \) be the hyperplane of \( M_R \) which contains \( x \) and is parallel to the hyperplane containing \( F \).

Then \( H_x \) intersects the line \( R\zeta \) at a single point \( \{y\} \). Define \( t(x) \in [0, 1] \) by \( y = t(x)\zeta \). Then the function \( x \mapsto t(x) \) is continuous and \( t(x) \to 1 \) (as \( x \to \zeta \)). Also, since \( x, y \in t(x)K \) we have

\[
|f(x) - f(y)| \leq |\lambda|\kappa d_K^{-n} C |x - y|.
\]

Hence we obtain

\[
|f(\zeta) - f(x)| \leq |f(\zeta) - f(y)| + |f(y) - f(x)|
\]

\[
\leq |f(\zeta) - f(t(x)\zeta)| + |\lambda|\kappa d_K^{-n} C |y - x|
\]

\[
\leq |f(\zeta) - f(t(x)\zeta)| + |\lambda|\kappa d_K^{-n} C |t(x)\zeta - x|
\]

\[
\to 0 \quad \text{(as } x \to \zeta)\]

by Lemma 5.1.6. This shows that \( f \) is continuous at \( \zeta \), and completes the proof of Proposition 5.1.1.
5.2. **Approximations and compactness of convex functions.** We collect various approximation and compactness results for convex functions, which is crucial for our proof of Theorem [1.0.1](#).

**Proposition 5.2.1.** Let \(\{f_i\}_{i=1}^{\infty}\) be a sequence of nonnegative functions in \(C_*\), and \(f \in C_*\). Suppose that \(\{f_i\}_{i=1}^{\infty}\) and \(f\) satisfy the following conditions:

(i) \(\{f_i\}_{i=1}^{\infty}\) converges locally uniformly to \(f\) on \(P^c\).

(ii) \(\sup_{i \in \mathbb{Z}_{>0}} \int_{\partial P} f_i \, d\sigma < \infty\).

Then \(\lim_{i \to \infty} \|f - f_i\|_{L^1(P)} = 0\).

**Proof.** By assumption, there is a constant \(C > 0\) such that

\[
f(0) + \sup_{i \in \mathbb{Z}_{>0}} f_i(0) + \int_{\partial P} f \, d\sigma + \sup_{i \in \mathbb{Z}_{>0}} \int_{\partial P} f_i \, d\sigma \leq C.
\]

Let \(\eta \in (0, 1)\). Then for each \(i \in \mathbb{Z}_{>0}\) we have

\[
\|f - f_i\|_{L^1(P)} = \int_{P \setminus \eta P} |f - f_i| \, dx + \int_{\eta P} |f - f_i| \, dx.
\]

By the convexity of \(f_i\) and \(f\), we have

\[
\int_{P \setminus \eta P} |f - f_i| \, dx \leq \int_{P \setminus \eta P} (f + f_i) \, dx
\]

\[
\leq \sum_{j=1}^{r} d_j \int_{\eta}^{1} (1 - t)^{n-1} \, dt \int_{F_j} (f(0) + f_i(0)) \, d\sigma
\]

\[
+ \sum_{j=1}^{r} d_j \int_{\eta}^{1} t^n \, dt \int_{F_j} (f + f_i) \, d\sigma
\]

\[
\leq 2Cd\sigma(\partial P) \int_{\eta}^{1} (1 - t)^{n-1} \, dt + 2Cd \int_{\eta}^{1} t^n \, dt
\]

\[
\leq 2Cd(\sigma(\partial P) + 1) \int_{\eta}^{1} t^{n-1} \, dt
\]

\[
= 2Cd(\sigma(\partial P) + 1) \frac{1 - \eta^n}{n}.
\]

For each \(\varepsilon > 0\), choose \(\eta \in (0, 1)\) so that

\[
2Cd(\sigma(\partial P) + 1) \frac{1 - \eta^n}{n} < \frac{\varepsilon}{2}.
\]

Since \(\{f_i\}_{i=1}^{\infty}\) converges uniformly to \(f\) on \(\eta P\), there exists \(N \in \mathbb{Z}_{>0}\) such that for any integer \(i \geq N\)

\[
\int_{\eta P} |f - f_i| \, dx < \frac{\varepsilon}{2}.
\]
Hence, for any integer \( i \geq N \) we have
\[
\|f - f_i\|_{L^1(P)} = \int_{P \setminus \eta P} |f - f_i| \, dx + \int_{\eta P} |f - f_i| \, dx < \varepsilon,
\]
as required. \( \square \)

By using Proposition 5.2.1, we obtain the following improvements of [16, Lemma 3.1] and [34, Proposition 5.2.6].

**Proposition 5.2.2** (cf. [16, Lemma 3.1]). Let \( f \in \tilde{C}_* \). Then there exists a sequence \( \{f_i\}_{i=1}^\infty \) in \( C_\infty \cap C^\infty(P) \) such that
(i) \( 0 \leq f_i \) for any \( i \in \mathbb{Z}_{>0} \),
(ii) \( \{f_i\}_{i=1}^\infty \) converges locally uniformly to \( f \) on \( P^c \), and
(iii) \( \int_P |f - f_i| \, dx + \int_{\partial P} |f - f_i| \, d\sigma \to 0 \) (as \( i \to \infty \)).
In particular, we have \( \lim_{i \to \infty} L_V(f_i) = L_V(f) \).

**Proof.** By [16, Lemma 3.1], there exists a sequence \( \{f_i\}_{i=1}^\infty \) in \( C_\infty \) which satisfies (i), (ii), and
\[
(5.2) \quad \lim_{i \to \infty} \int_{\partial P} |f - f_i| \, d\sigma = 0.
\]
Moreover, a careful reading of the proof in [16] shows that we can choose \( \{f_i\}_{i=1}^\infty \) so that \( f_i \in C_\infty \cap C^\infty(P) \) for any \( i \in \mathbb{Z}_{>0} \). Then the sequence \( \{f_i\}_{i=1}^\infty \) satisfies the conditions (i), (ii) of Proposition 5.2.1 and hence it converges to \( f \) in \( L^1 \). \( \square \)

**Proposition 5.2.3** (cf. [34, Proposition 5.2.6]). Let \( \{f_i\}_{i=1}^\infty \) be a sequence in \( \tilde{C}_* \) with
\[
\sup_{i \in \mathbb{Z}_{>0}} \int_{\partial P} f_i \, d\sigma < \infty.
\]
Then there exists a subsequence \( \{f_{i_k}\}_{k=1}^\infty \) of \( \{f_i\}_{i=1}^\infty \) and \( f \in \tilde{C}_* \) such that
(i) \( \{f_{i_k}\}_{k=1}^\infty \) converges locally uniformly to \( f \) on \( P^c \), and
(ii) \( \int_P |f - f_{i_k}| \, dx \to 0 \) (as \( k \to \infty \)),
\[
\int_{\partial P} f \, d\sigma \leq \liminf_{k \to \infty} \int_{\partial P} f_{i_k} \, d\sigma.
\]
In particular, we have \( L_V(f) \leq \liminf_{k \to \infty} L_V(f_{i_k}) \).

**Proof.** By [34, Proposition 5.2.6], there exist a subsequence \( \{f_{i_k}\}_{k=1}^\infty \) of \( \{f_i\}_{i=1}^\infty \) and \( f \in \tilde{C}_* \) which satisfy (i) and
\[
\int_{\partial P} f \, d\sigma \leq \liminf_{k \to \infty} \int_{\partial P} f_{i_k} \, d\sigma.
\]
Then Proposition 5.2.1 shows that the sequence \( \{f_{i_k}\}_{k=1}^\infty \) converges to \( f \) in \( L^1 \). The proposition is proved. \( \square \)
We also use the following approximation result due to Donaldson [34].

**Proposition 5.2.4** ([34 Proposition 5.2.8]). Let $f$ be a nonnegative function in $C_\ast$. Then there exists a sequence \(\{f_i\}_{i=1}^\infty\) in $C_{PL}$ such that

(i) \(0 \leq f_i \leq f\) for any \(i \in \mathbb{Z}_{>0}\),

(ii) \(\{f_i\}_{i=1}^\infty\) converges locally uniformly to $f$ on $P^\circ$, and

(iii) \(\int_P |f - f_i|\, dx + \int_{\partial P} |f - f_i|\, d\sigma \to 0\) (as $i \to \infty$).

In particular, we have $\lim_{i \to \infty} L_V(f_i) = L_V(f)$.

The following proposition is easily proved from the density of $Q$ in $R$ and convexity.

**Proposition 5.2.5.** Let $f \in C_{PL}$ (possibly irrational). Then there exists a sequence \(\{f_i\}_{i=1}^\infty\) in $C_{QL}$ which converges uniformly to $f$ in $P$.

**Proof.** Choose affine functions $\ell_1, \ldots, \ell_r$ so that

\[ f(x) = \max\{\ell_1(x), \ldots, \ell_m(x)\} \]

for any $x \in P$, and we regard $f$ as a function defined on $R^n$. For any $j \in \{1, \ldots, m\}$, choose a sequence \(\{\ell^{(i)}_j\}_{i=1}^\infty\) of rational affine functions which converges pointwise to $\ell_j$ on $R^n$, and define a rational piecewise affine function $f^{(i)}$ by

\[ f^{(i)}(x) = \max\{\ell^{(i)}_1(x), \ldots, \ell^{(i)}_m(x)\}. \]

Then \(\{f^{(i)}\}_{i=1}^\infty\) converges locally uniformly to $f$ on $R^n$. In particular, \(\{f^{(i)}\}_{i=1}^\infty\) converges uniformly to $f$ on $P$. \(\square\)

### 5.3. The non-Archimedean K-energy

We can define $L(f)$ and $L_V(f)$ for any $f \in E_1$, taking value in $(-\infty, \infty]$. By using these, we can get the following characterization of $C_\ast$.

**Proposition 5.3.1.**

\[ C_\ast = \{f \in E_1 \mid L(f) < \infty\} = \{f \in E_1 \mid L_V(f) < \infty\}. \]

**Proof.** By Proposition 5.1.1, for each $f \in E_1$ we have

\[ f \in C_\ast \iff \int_{\partial P} f\, d\sigma < \infty \iff L(f) < \infty \iff L_V(f) < \infty, \]

as required. \(\square\)

### 5.4. The J-norm

For any convex function $f$ in $E_1$, define

\[ \|f\|_J = \inf_{\xi: \text{affine}} \left( \int_P (f + \xi)\, dx - \inf_P (f + \xi) \right). \]

We call $\|f\|_J$ the $J$-norm of $f$. The followings are the basic properties on the $J$-norm.

**Proposition 5.4.1.** (1) $\|f + \xi\|_J = \|f\|_J$ for any $f \in E_1$ and $\xi \in N_R$. 

(2) \( \| f \|_J \geq 0 \) for any \( f \in \mathcal{E}_1 \), and \( \| f \|_J = 0 \) if and only if \( f \) is affine.

(3) There exists a constant \( C_1 > 0 \) such that
\[
\| f \|_J \leq \| f \|_{L^1(P)} \leq C_1 \| f \|_J
\]
for any \( f \in \tilde{\mathcal{E}}_1 \).

(4) If a sequence \( \{ f_i \}_{i=1}^\infty \) in \( \mathcal{E}_1 \) converges uniformly to \( f \in \mathcal{E}_1 \), then
\[
\lim_{i \to \infty} \| f_i \|_J = \| f \|_J.
\]

Proof. The claim (1) is clear, and (2) is easily deduced from (1) and (3). Let us show the claim (3). The first inequality is obvious. For the latter inequality, it is sufficient to show that there exists a constant \( \kappa > 0 \) such that
\[
\| f \|_J \geq \kappa
\]
for any \( f \in \tilde{\mathcal{E}}_1 \) with \( \| f \|_{L^1(P)} = 1 \). For contradiction, suppose that the conclusion does not hold. Then there is a sequence \( \{ f_i \}_{i=1}^\infty \) in \( \tilde{\mathcal{E}}_1 \) such that
\[
\begin{align*}
(i) & \quad \| f_i \|_{L^1(P)} = 1 \text{ for any } i \in \mathbb{Z}_{>0}, \text{ and} \\
(ii) & \quad \lim_{i \to \infty} \| f_i \|_J = 0.
\end{align*}
\]
By condition (ii), there is a sequence of affine functions \( \{ \ell_i \}_{i=1}^\infty \) such that
\[
\begin{align*}
(iii) & \quad \inf_P (f_i + \ell_i) = 0 \text{ for any } i \in \mathbb{Z}_{>0}, \text{ and} \\
(iv) & \quad \int_P (f_i + \ell_i) \, dx = 0.
\end{align*}
\]
By conditions (i), (ii), (iii), and (iv) and [34 Corollary 5.2.5], we may assume both \( \{ f_i \}_{i=1}^\infty \) and \( \{ f_i + \ell_i \}_{i=1}^\infty \) converge locally uniformly on \( P^o \). Then the sequence \( \{ \ell_i \}_{i=1}^\infty \) converges locally uniformly on \( P^o \). Moreover, since every \( \ell_i \) is affine, the limit \( \ell \) is also affine and the convergence \( \ell_i \to \ell \) is uniform.

Let
\[
f(x) := \lim_{i \to \infty} f_i(x), \quad \ell(x) := \lim_{i \to \infty} \ell_i(x)
\]
for each \( x \in P^o \). By (iv) above, \( f + \ell = 0 \) on \( P^o \) and hence \( f \) is affine on \( P^o \). Moreover, since \( \inf_P f = f(0) = 0 \) we obtain
\[
f = -\ell = 0.
\]
Therefore, we have
\[
1 = \| f_i \|_{L^1(P)} \leq \| f_i + \ell_i \|_{L^1(P)} + \| -\ell_i \|_{L^1(P)} \to 0 \quad \text{(as } i \to \infty),
\]
which is contradiction.

Finally, we show the claim (4). For each \( \varepsilon > 0 \), choose \( N \in \mathbb{Z}_{>0} \) so that
\[
\sup_P |f - f_i| < \frac{\varepsilon}{4\text{vol}(P)}
\]
for any \( i \in \mathbb{Z}_{\geq N} \). Then for any \( i \in \mathbb{Z}_{\geq N} \) and an affine function \( \ell \) we have

\[
\left| \int_P (f + \ell) \, dx - \text{vol}(P) \inf_P (f + \ell) \right| \leq \left| \int_P (f + \ell) \, dx - \int_P (f_i + \ell) \, dx \right| + \text{vol}(P) \left| \inf_P (f + \ell) - \inf_P (f_i + \ell) \right| \\
\leq \left| \int_P (f - f_i) \, dx \right| + \text{vol}(P) \sup_P |f - f_i| \\
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\]

By choosing \( \ell \) so that

\[
\int_P (f + \ell) \, dx - \text{vol}(P) \inf_P (f + \ell) - \|f\|_J < \frac{\varepsilon}{2},
\]

we obtain

\[
\|f_i\|_J \leq \int_P (f_i + \ell) \, dx - \text{vol}(P) \inf_P (f_i + \ell) \\
< \int_P (f + \ell) \, dx - \text{vol}(P) \inf_P (f + \ell) + \frac{\varepsilon}{2} \\
< \|f\|_J + \varepsilon.
\]

Similarly, we have

\[
\|f\|_J < \|f_i\|_J + \varepsilon.
\]

By (5.3) and (5.4), we obtain \( \|f\|_J - \|f_i\|_J < \varepsilon \), as required. \( \Box \)

Similar to the J-norm, the reduced \( L^1 \)-norm of a convex function \( f \in \mathcal{E}_1 \) is defined by

\[
\|f\|_{1,T} := \inf_{\xi: \text{affine}} \int_P |(f + \xi) - \overline{(f + \xi)}| \, dx.
\]

The similar claims as (1), (2), (3) of Proposition 5.4.1 are valid for the \( J \)-norm. In particular, the reduced \( L^1 \)-norm and the \( J \)-norm are equivalent in the following sense.

**Proposition 5.4.2.** There are constants \( C_2, C_3 > 0 \) such that

\[
C_2 \|f\|_{1,T} \leq \|f\|_J \leq C_3 \|f\|_{1,T}
\]

for any \( f \in \mathcal{E}_1 \).

An advantage of the reduced \( L^1 \)-norm is the following \( L^1 \)-continuity.

**Proposition 5.4.3.** If a sequence \( \{f_i\}_{i=1}^\infty \) in \( \mathcal{E}_1 \) converges to \( f \in \mathcal{E}_1 \) in \( L^1 \), then

\[
\lim_{i \to \infty} \|f_i\|_{1,T} = \|f\|_{1,T}.
\]
Proof. For each \( \varepsilon > 0 \), choose \( N \in \mathbb{Z}_{>0} \) so that
\[
\| f - f_i \|_{L^1(P)} < \frac{\varepsilon}{2}
\]
for any \( i \in \mathbb{Z}_{\geq N} \). Then for any \( i \in \mathbb{Z}_{\geq N} \) and an affine function \( \ell \) we have
\[
\| f + \ell \|_{L^1(P)} - \| f_i + \ell \|_{L^1(P)} \leq \| f - f_i \|_{L^1(P)} < \frac{\varepsilon}{2}.
\]
By choosing \( \ell \) so that
\[
\| f + \ell \|_{L^1(P)} - \| f \|_{1,T} < \frac{\varepsilon}{2},
\]
we obtain
\[
(5.5) \quad \| f_i \|_{1,T} \leq \| f_i + \ell \|_{L^1(P)} < \| f + \ell \|_{L^1(P)} + \frac{\varepsilon}{2} < \| f \|_{1,T} + \varepsilon.
\]
Similarly, we have
\[
(5.6) \quad \| f \|_{1,T} < \| f_i \|_{1,T} + \varepsilon.
\]
By (5.5) and (5.6), we obtain \( \| f \|_{1,1} - \| f_i \|_{1,1} < \varepsilon \), as required. \( \square \)

It is not known whether the similar result as Proposition 5.4.3 are valid for the \( J \)-norm or not.

5.5. Proof of Theorem 1.0.1. In this subsection, we give a proof of Theorem 1.0.1. For \((b)_F \Rightarrow (J)_F\), let \( \delta > 0 \) satisfying
\[
L_V(f) \geq \delta \int_{\partial P} f \, d\sigma
\]
for any \( f \in \widetilde{F} \). Let \( f \in F \), and choose an affine function \( \ell \) so that \( f + \ell \in \widetilde{F} \).

Then, by Propositions 5.1.2 and 5.4.1 we have
\[
L_V(f) = L_V(f + \ell) \geq \delta \int_{\partial P} (f + \ell) \, d\sigma
\]
\[
\geq \left( \frac{n+1}{d} \right) \delta \int_{\partial P} (f + \ell) \, dx
\]
\[
\geq \left( \frac{n+1}{2d} \right) \delta \| f + \ell \|_J
\]
\[
= \left( \frac{n+1}{2d} \right) \delta \| f \|_J,
\]
as required.

For \((K)_{C_*} \Rightarrow (b)_F\), first note that
\[
L_V(f) \geq 0
\]
for any \( f \in \widetilde{C}_* \). For contradiction, suppose that \( P \) does not satisfy \((b)_F\). Then there exists a sequence \( \{f_i\}_{i=1}^\infty \) in \( \widetilde{F} \) such that
\[
(\text{a}) \quad \int_{\partial P} f_i \, d\sigma = 1 \quad \text{for any} \quad i \in \mathbb{Z}_{>0}, \quad \text{and}
\]
\[
(\text{b}) \quad \lim_{i \to \infty} L_V(f_i) = 0.
\]
By the condition (a), we may further assume that there exists $f \in \tilde{C}_*$ which satisfies the conditions (i), (ii) of Proposition 5.2.3. Since

$$0 \leq L_V(f) \leq \liminf_{i \to \infty} L_V(f_i) = 0,$$

we have $L_V(f) = 0$ and that $f$ is affine. Since $f \in \tilde{C}_*$, $f$ must be 0, which contradicts the choice of $f$.

The equivalence $(K)_{C_*} \iff (K)_{E_1}$ can be obtained as follows. The implication $(K)_{E_1} \implies (K)_{C_*}$ is clear. Let us assume $(K)_{C_*}$ and $f \in E_1$. If $f \in C_*$, then $L_V(f) \geq 0$ and $L_V(f) = 0$ if and only if $f$ is affine by the assumption $(K)_{C_*}$. If $f \in E_1 \setminus C_*$, then $L_V(f) = +\infty > 0$ by Proposition 5.3.1.

Finally, let us show $(J)_{F} \implies (K)_{C_*}$ in each case.

Case 1: For the case $F = E_1$ or $F = C_*$, it is obvious from Propositions 5.3.1 and 5.4.1.

Case 2: Let $F = C_{PL}$. It is sufficient to show that $L_V(f) \geq 0$ for any $f \in \tilde{C}_*$ and $L_V(f) = 0$ if and only if $f$ is affine. Choose a constant $\delta > 0$ so that

$$L_V(f) \geq \delta \|f\|_J$$

for any $f \in C_{PL}$. Then by Proposition 5.4.2 we have

$$L_V(f) \geq \delta C_2 \|f\|_{1,T}.$$ 

Let $f \in \tilde{C}_*$, and choose a sequence $\{f_i\}_{i=1}^\infty$ in $C_{PL}$ satisfying the conditions (i), (ii), (iii) of Proposition 5.2.4. By conditions (i) and (iii), we have $f_i \in \tilde{C}_{PL}$ for any $i \in \mathbb{Z}_{>0}$ and $\lim_{i \to \infty} L_V(f_i) = L_V(f)$. Since

$$L_V(f_i) \geq \delta C_2 \|f_i\|_{1,T} \geq 0$$

for any $i \in \mathbb{Z}_{>0}$, we have

$$L_V(f) \geq \delta C_2 \|f\|_{1,T} \geq 0$$

by Proposition 5.4.3. Suppose $L_V(f) = 0$. Then $\|f\|_{1,T} = 0$ and hence $f$ is affine.

Case 3: Let $F = C_{PL}^Q$. Then then we can reduce to the Case 2 as follows. Choose a constant $\delta > 0$ so that

$$L_V(f) \geq \delta \|f\|_J$$

for any $f \in C_{PL}^Q$. Let $f \in C_{PL}$. By Proposition 5.2.5, there is a sequence $\{f_i\}_{i=1}^\infty$ in $C_{PL}^Q$ which converges uniformly to $f$ on $P$. Since

$$L_V(f_i) \geq \delta \|f_i\|_J$$

for any $i \in \mathbb{Z}_{>0}$, we have

$$L_V(f) \geq \delta \|f\|_J$$

by Proposition 5.4.4. Hence $P$ satisfies the condition $(J)_{C_{PL}^Q}$. 
Case 4: Let $\mathcal{F} = \mathcal{C}_\infty$. Then, by using Proposition 5.2.2 in place of Proposition 5.2.4, we can also prove $(J)_{\mathcal{C}_\infty} \Rightarrow (K)_{\mathcal{C}_*}$ by the similar argument as Case 2.

Case 5: Finally, let $\mathcal{F} = \mathcal{S}$. As well as Case 2, it is sufficient to show that $L_V(f) \geq 0$ for any $f \in \tilde{\mathcal{C}}_*$ and $L_V(f) = 0$ if and only if $f$ is affine. Choose $\delta > 0$ so that

$$L_V(u) \geq \delta \|u\|_J$$

for any $u \in \mathcal{S}$. Then by Proposition 5.4.2 we have

$$L_V(u) \geq \delta C_2 \|u\|_{1,T}.$$ 

Let $f \in \tilde{\mathcal{C}}_*$, and choose a sequence $\{f_i\}_{i=1}^\infty$ in $\mathcal{C}_\infty \cap C_\infty(P)$ satisfying the conditions (i), (ii), (iii) of Proposition 5.2.2. Fix $u_0 \in \tilde{\mathcal{S}}$ arbitrarily. Then, for each $i \in \mathbb{Z}_{>0}$ and $t \in (0, \infty)$, $u_0 + tf_i \in \mathcal{S}$ and

$$L_V\left(\frac{u_0 + tf_i}{t}\right) \geq \delta C_2 \left\|\frac{u_0 + tf_i}{t}\right\|_{1,T} \geq 0.$$

By taking the limit as $t \to \infty$ and $i \to \infty$, we obtain

(5.7) 

$$L_V(f) \geq \delta C_2 \|f\|_{1,T} \geq 0.$$

Now suppose $L_V(f) = 0$. Then $\|f\|_{1,T} = 0$ and hence $f$ is affine.

**Remark 5.5.1.** Let $\mathcal{B}$ be the set of all bounded lower semicontinuous convex functions defined on the whole of $P$. Then $\mathcal{B}$ is contained in $\mathcal{C}_*$ and consequently $(K)_{\mathcal{C}_*}$ implies $(K)_{\mathcal{B}}$. The latter condition $(K)_{\mathcal{B}}$ can be found in [56].

6. Proof of Theorem 1.0.5

Finally in this section, we prove Theorem 1.0.5. By the implication $(b)_{\mathcal{C}_P} \Rightarrow (J)_{\mathcal{C}_P}$ in Theorem 1.0.1, it is enough to show that $P$ satisfies the condition $(b)_{\mathcal{C}_P}$. First suppose that $P$ satisfies

$$\bar{\sigma} + \max_P V \leq \frac{n + 1}{d}.$$ 

Define a constant $\delta \geq 0$ by

$$\delta = 1 - \frac{d}{n + 1} (\bar{\sigma} + \max_P V).$$
Then, by Proposition 5.1.2 we have

\[ L_V(f) = \int_{\partial P} f \, d\sigma - \int_P (\bar{s} + V) f \, dx \]

\[ \geq \int_{\partial P} f \, d\sigma - (\bar{s} + \max_P V) \int_P f \, dx \]

\[ \geq \int_{\partial P} f \, d\sigma - \frac{d}{n+1}(\bar{s} + \max_P V) \int_P f \, d\sigma \]

\[ = \delta \int_{\partial P} f \, d\sigma \]

for any \( f \in \tilde{C}^{Q}_{P_L} \). Hence, if \( P \) satisfies (1.2) then \( \delta > 0 \) and \( P \) satisfies the condition (b) of Proposition 5.2.3. Now suppose that \( P \) satisfies (1.3). In this case \( \delta \) might be 0, but at least we have

\[ L_V(f) \geq \delta \int_{\partial P} f \, d\sigma \geq 0 \]

for any \( f \in \tilde{C}^{Q}_{P_L} \). For contradiction, suppose that our conclusion does not hold. Then there exists a sequence \( \{f_i\}_{i=1}^{\infty} \) in \( \tilde{C}^{Q}_{P_L} \) such that

\[ \int_{\partial P} f_i \, d\sigma = 1 \]

and

\[ \lim_{i \to \infty} L_V(f_i) = 0. \]

Furthermore, we can find a subsequence \( \{f_{i_k}\}_{k=1}^{\infty} \) of \( \{f_i\}_{i=1}^{\infty} \) and \( f \in \tilde{C}_s \) satisfying the conditions (i), (ii), (iii) of Proposition 5.2.3. Hence we have

\[ L_V(f) \leq \liminf_{k \to \infty} L_V(f_{i_k}) = \lim_{k \to \infty} L_V(f_{i_k}) = 0. \]

Now suppose \( f \not\equiv 0 \) on \( P^* \). Then \( U = \{x \in P^* \mid f(x) > 0\} \) is a nonempty open set of \( P^* \). Since \( V \) is a nonconstant affine function, it holds that \( V < \max_P V \) almost everywhere on \( P \). It follows that

\[ L_V(f) = \int_{\partial P} f \, d\sigma - \int_P (\bar{s} + V) f \, dx \]

\[ > \int_{\partial P} f \, d\sigma - (\bar{s} + \max_P V) \int_U f \, dx \]

\[ \geq \delta \int_{\partial P} f \, d\sigma \]

\[ \geq 0, \]

which contradicts to (6.3). Hence we have \( f \equiv 0 \) on \( P^* \), and

\[ \lim_{k \to \infty} \int_P (\bar{s} + V) f_{i_k} \, dx = 0. \]
On the other hand, by (6.1) and (6.2) we have

$$\lim_{k \to \infty} \int_p (\overline{s} + V) f_{i_k} \, dx = \lim_{k \to \infty} \left( \int_{\partial p} f_{i_k} \, d\sigma - L_V(f_{i_k}) \right) = 1,$$

which contradicts to (6.4).

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