NONCOMMUTATIVE REGULARIZATION
FOR THE PRACTICAL MAN

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ABSTRACT. It has been proposed that the noncommutative geometry of the “fuzzy” 2-sphere provides a nonperturbative regularization of scalar field theories. This generalizes to compact Kähler manifolds where simple field theories are regularized by the geometric quantization of the manifold.

In order to permit actual calculations and the comparison with other regularizations, I describe the perturbation theory of these regularized models and propose an approximation technique for evaluation of the Feynman diagrams. I present example calculations of the simplest diagrams for the $\phi^4$ model on the spaces $S^2$, $S^2 \times S^2$, and $\mathbb{C}P^2$.

This regularization fails for noncompact spaces; I give a brief dimensional analysis argument as to why this is so. I also discuss the relevance of the topology of Feynman diagrams to their ultra-violet and infra-red divergence behavior in this model.

1. INTRODUCTION

It is generally expected that in a full description of quantum gravity the geometry of space-time at small scales will not resemble that of a manifold. The nature of this “quantum geometry” is one of the fundamental issues in the search for a quantum theory of gravity.

One argument for quantum geometry is that quantum field theories give our best descriptions of microscopic physics, yet they must be regularized in order to yield meaningful predictions. The trouble being that quantum field theory suffers from ultra-violet divergences due to physical processes occurring at arbitrarily large momenta, or equivalently, arbitrarily small distances. Some fundamental regularization that fixes these divergences can probably be interpreted as a modification of geometry at extremely short distances, perhaps even such that the concept of arbitrarily small distances is meaningless.

Another argument extends Heisenberg’s classic gedankenexperiment in support of the uncertainty relations. The observation of structures at very small distances requires radiation of very short wavelength and correspondingly large energy. Attempting to observe a sufficiently small structure would thus require such a high concentration of energy that a black hole would be formed and no observation
could be made. If this is so, then distances below about the Plank scale are unobservable — and thus operationally meaningless.

If short distances are meaningless, then perhaps precise locations are as well. This suggests the possibility of uncertainty relations between position and position, analogous to the standard ones between position and momentum; this has been argued on physical grounds (see [7], or [10] for a review) and from string theory (see [19]). An uncertainty relation between, say, x-position and y-position, would mean that the x and y coordinates do not commute. Since coordinates are just functions on space(-time) this suggests that the algebra of functions on space(-time) might not be commutative (see [7]). That is the fundamental idea of noncommutative geometry.

In noncommutative geometry (see [4]), familiar geometric concepts (metric, measure, bundle, etc.) are reformulated in an entirely algebraic way. This allows the generalization of geometry by replacing the algebra of functions on space with a noncommutative algebra. It may be that such a noncommutative generalization of ordinary geometry can describe the true quantum geometry of space-time.

As I will explain, noncommutativity is no guarantor of regularization (see also [9, 2]). In this paper, I will discuss a specific class of noncommutative geometries which do have the requisite regularization property. These models are not physically realistic; they generalize Euclidean (space) rather than Lorentzian (space-time) geometry (a sin shared by lattice models) and they are not gauge theories. However, it is plausible that more realistic models may share some of the characteristics of these ones.

Although regularization in these geometries is quite manifest, a toolkit for coaxing actual predictions from field theory there has been lacking. My aim here is to present an approximation technique for field theory calculations in this regularization. This will show the leading order effects of noncommutativity on quantum field theory.

1.1. Noncommutative geometry. The existing theory of noncommutative geometry (see [4]) is largely inspired by the so-called Gelfand theorem. According to this theorem, there is an exact correspondence between commutative C*-algebras and locally compact topological spaces. For any locally compact topological space, the algebra of continuous functions vanishing at ∞ is a commutative C*-algebra, and any commutative C*-algebra can be realized in this way.

This suggests that C*-algebras in general should be considered as noncommutative algebras of “continuous functions”, and thus that the category of C*-algebras is the category of noncommutative topologies. The next step is to go from noncommutative topology to noncommutative geometry; the most versatile noncommutative version of a Riemannian metric is given by a Dirac operator.

On an ordinary manifold, taking the commutator of the Dirac operator, $D := i\gamma^j \nabla_j$, with a differentiable function gives

$$[D, f]_\pm = i\gamma^j f_j.$$
Taking the norm of this gives \( \| [D, \phi] \| = \| \nabla f \| \); thus, the Dirac operator can detect the maximum slope of a function. From this, a construction for the distance between two points can be obtained (see [4]). This shows that the Dirac operator contains all information of the Riemannian metric. It also knows which functions are differentiable, smooth, Lipschitz, etc.

A \( \mathbb{C}^* \)-algebra with a Dirac operator thus constitutes a noncommutative Riemannian geometry. Using a couple of additional structures, it is possible to characterize noncommutative Riemannian manifolds axiomatically (see [5]). Unfortunately, it is only known how to do this for what amounts to metrics of positive definite signature. This does not allow for the noncommutative generalization of space-time.

Another problem with these noncommutative manifolds is that they do not tend to regularize quantum field theory. A variation of noncommutative geometry that does have this property is matrix geometry (see [18, 11]). In matrix geometry, every structure has only finite degrees of freedom. Unfortunately, there is no axiomatic characterization of matrix geometries, as for noncommutative manifolds.

The problem of characterizing noncommutative space-time is daunting (see [12]). The trouble is that the analytic properties of the Dirac operator which are essential in the positive-definite case are not there in space-time. With everything finite, all analytic considerations evaporate in the case of matrix geometry, suggesting that the problem of noncommutative space-time might be solvable once matrix geometry is better understood.

Most existing applications of noncommutative geometry to physics have concerned Connes-Lott models. In these (with the question of space-time deferred) a simple type of noncommutative manifold provides an interesting interpretation of the standard model of particle physics. In particular, the Higgs field and gauge Bosons are unified.

Here, I am pursuing a different way of applying noncommutative geometry to physics. I am following [18, 11] and exploring the regularization effects of matrix geometry.

1.2. **Summary.** I will begin in Sec. 2 by discussing what it takes to regularize quantum field theory and describing the geometric quantization construction that is the basis of my approach.

In Sec. 3, I discuss how to construct the action functional in this regularization, in slightly greater generality than has previously been discussed explicitly. This is followed by a brief speculation on convergence when the regularization is removed. Perturbation theory in this regularization has not been described in detail before; I present this in Sec. 3.3.

In Sec. 4, I explain (heuristically) how the infra-red and ultra-violet cutoff scales are balanced around the noncommutativity scale. In particular, the ultra-violet cutoff only exists when there is an infra-red cutoff.

In Sec. 5, I describe the Weyl quantization of flat space and the effect of this on quantum field theory, giving a geometric algorithm for the modification. This is
a prelude to the main result of this paper. I present, in Sec. 6, an approximation technique for perturbative calculations in this regularization.

I illustrate exact and approximate calculations with a few examples in Sec. 7, and in Sec. 8, I discuss the effect on noncommutativity on degrees of divergence.

2. Regularization

A Euclidean quantum field theory can be defined by a path integral over the space of classical field configurations. Given an action functional $S[\phi]$, the vacuum expectation value of some functional, $F[\hat{\phi}]$, of the quantum fields is defined by

$$Z \cdot \langle 0 | F[\hat{\phi}] | 0 \rangle := \int_{\Phi} F[\phi] e^{-S[\phi]} D\phi,$$

(2.1)

where the partition function, $Z$, is a normalizing factor such that $\langle 0 | 1 | 0 \rangle = 1$. The celebrated divergences which plague quantum field theory are primarily due to the fact that the space of classical field configurations, $\Phi$, is infinite-dimensional; this leaves the functional integral measure $D\phi$ formal and awkwardly ill-defined.

In the usual treatment of quantum field theory, perturbative Feynman rules are derived from the formal path integral. Unfortunately, these Feynman rules typically lead to infinite results. In order to get meaningful results from computations, the Feynman rules are usually regularized ad hoc. This is quite effective for perturbative calculations since the details are independent of the choice of regularization.

However, reality is not a perturbation. Physical phenomena such as quark confinement are not reflected in strictly perturbative theories. A complete description of reality will presumably involve a nonperturbative regularization. In a Euclidean quantum field theory, a nonperturbative regularization is implemented at the level of the path-integral rather than perturbation theory. One approach to regularization is to replace the original space of field configurations with some finite-dimensional approximation; this essentially guarantees a finite theory.

If $\phi$ is a single scalar field on a compact manifold, $M$, then the space of field configurations is the algebra of smooth functions, $C^\infty(M)$. Where algebra goes, other structures will surely follow; for this reason, and simplicity, I shall largely restrict attention to a scalar field. To regularize, we would like to approximate $C^\infty(M)$ by a finite dimensional algebra. The standard approach is to use the algebra of functions on some finite set of points — a lattice — which approximates $M$.

Unfortunately, a lattice is symmetry’s mortal enemy. If the space $M$ possesses a nontrivial group of isometries, it would be desireable to preserve these as symmetries in the regularized theory; but for instance, in a lattice approximation to $S^2$, the best possible approximation to the SO(3)-symmetry is the 60-element icosahedral group.

2.1. Geometric quantization. We can maintain much greater symmetry with noncommutative approximating algebras. Geometric quantization provides a method of constructing noncommutative approximations.
As the name suggests, geometric quantization was originally intended as a systematic mathematical procedure for constructing quantum mechanics from classical mechanics. Geometric quantization applies to a symplectic manifold (originally, phase space) with some additional structure (a “polarization”). The terminology of quantization is unfortunate here, since I am concerned with quantum field theory. Insofar as geometric quantization goes here, the manifold is not to be interpreted as phase space, the Hilbert spaces are not to be interpreted as spaces of quantum-mechanical states, and the algebras do not consist of observables. In this paper, “quantization” and “quantum” have nothing to do with each other.

Here I shall use geometric quantization with a “complex polarization”. For a compact Kähler manifold, $M$, this generates a sequence of finite-dimensional matrix algebras $A_N$ which approximate the algebra $C^\infty(M)$ in a sense that I shall explain below.

A Kähler manifold is simultaneously a Riemannian, symplectic, and complex manifold. These structures are compatible with each other, such that raising one index of the symplectic 2-form, $\omega$, with the Riemannian metric gives the complex structure $J$. I shall assume, for a length $R$ (roughly, the size of $M$), that the integral of $\omega^2/2\pi R^2$ over any closed 2-surface is an integer; this implies the existence of a line bundle (1-dimensional, complex vector bundle), $L$, with a connection $\nabla$ and curvature $R^{-2}\omega$. If $M$ is simply connected, then $L$ is unique.

There is also a unique fiberwise, Hermitian inner product on $L$ compatible with the connection. Given two sections $\psi, \varphi \in \Gamma(M, L)$, their inner product is a function $\overline{\psi}\varphi \in C(M)$. Compatibility with the connection means that for smooth sections, $d(\overline{\psi}\varphi) = \nabla\overline{\psi}\varphi + \overline{\psi}\nabla\varphi$. Using the fiberwise inner product, we can construct a global inner product,

$$\langle \psi | \varphi \rangle := \int_M \overline{\psi}\varphi \, e.$$ (2.2)

It is an elementary property of any Kähler manifold that the Riemannian volume form can also be written in terms of the symplectic form as $e = \omega^n/n!$, where $2n = \dim M$.

A holomorphic section of $L$ is one satisfying the differential equation $J\nabla\psi = i\nabla\psi$, or in index notation $J^i_j \psi_j = i\psi_i$. The space of holomorphic sections $\Gamma_{hol}(M, L)$ is finite dimensional, and the inner product, (2.2), makes it a Hilbert space.

With this notation and structure, I can now present the geometric quantization construction. The tensor power bundle $L^\otimes N$ is much like $L$; it is a line bundle with an inner product, but with curvature $NR^{-2}\omega$. As $N$ increases, the spaces of holomorphic sections are increasingly large, finite-dimensional Hilbert spaces,

$$\mathcal{H}_N := \Gamma_{hol}(M, L^\otimes N).$$

1A 2-index tensor such that $J^i_j J^j_k = -\delta^i_k$. 
The dimension of $\mathcal{H}_N$ grows as a polynomial in $N$ (given by the Riemann-Roch formula, Eq. (4.1)). The algebra $\mathcal{A}_N$ is now defined as

$$\mathcal{A}_N := \text{End} \mathcal{H}_N;$$

that is, the space of $\mathbb{C}$-linear maps from $\mathcal{H}_N$ to itself — in other words, matrices over $\mathcal{H}_N$. The inner product on $\mathcal{H}_N$ gives $\mathcal{A}_N$ an involution (Hermitian adjoint) $\alpha \mapsto \alpha^*$; this makes $\mathcal{A}_N$ a finite-dimensional $C^*$-algebra.

The collection of algebras $\{\mathcal{A}_N\}$ alone knows nothing of $\mathcal{M}$. In order to connect $\mathcal{A}_N$ with $\mathcal{M}$, we will need a structure such as the Toeplitz quantization map $T_N : \mathcal{E}(\mathcal{M}) \rightarrow \mathcal{A}_N$. For any function $f \in \mathcal{E}(\mathcal{M})$, the matrix $T_N(f)$ is defined by giving its action on any $\psi \in \mathcal{H}_N$. Since $\psi$ is a section of $L^{\otimes N}$, the product $\psi f$ is also a (not necessarily holomorphic) section of $L^{\otimes N}$. Using the inner product (2.2) we can project $\psi f$ orthogonally back to $\mathcal{H}_N$ and call this $T_N(f)\psi$. This implicitly defines $T_N$.

If $\mathcal{M}$ were a phase space then $T_N(f)$ would be interpreted as the quantum observable, $\hat{f}$, corresponding to $f$. The Toeplitz maps have the important property of being approximately multiplicative; that is, for any two continuous functions on $\mathcal{M}$,

$$\lim_{N \to \infty} \|T_N(f)T_N(g) - T_N(fg)\| = 0.$$

According to Rieffel [21], quantization should be expressed in terms of a continuous field of $C^*$-algebras. A continuous field $\mathfrak{A}$ of $C^*$-algebras over a compact topological space $\mathfrak{B}$ is essentially a vector bundle whose fibers are $C^*$-algebras (see [5, 16]). The space of continuous sections $\Gamma(\mathfrak{B}, \mathfrak{A})$ is a $C^*$-algebra. For every point $b \in \mathfrak{B}$, the evaluation map from $\Gamma(\mathfrak{B}, \mathfrak{A})$ onto the fiber over $b$ is a $*$-homomorphism (a $C^*$-algebra map). The product of a continuous function on $\mathfrak{B}$ with a continuous section of $\mathfrak{A}$ is again a continuous section of $\mathfrak{A}$. A continuous field of $C^*$-algebras is completely specified by describing its base space, its fibers, and which sections are continuous.

The geometric quantization of $\mathcal{M}$ leads to a natural continuous field of $C^*$-algebras, $\mathfrak{A}$. The fibers of $\mathfrak{A}$ are each of the algebras $\mathcal{A}_N$ and $\mathcal{E}(\mathcal{M})$. This is one algebra for every natural number $N \in \mathbb{N} = \{1, 2, \ldots\}$, plus one extra. The notion is that the sequence of algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ tends toward $\mathcal{E}(\mathcal{M})$, so the natural topology for the base space is the one-point compactification $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, in which $\infty$ is the limit of any increasing sequence. The field $\mathfrak{A}$ is implicitly defined by the requirement that for every $f \in \mathcal{E}(\mathcal{M})$, there exists a section $T(f) \in \Gamma(\hat{\mathbb{N}}, \mathfrak{A})$ whose evaluation at $N$ is $T_N(f)$ and at $\infty$ is $f$.

The existence of this continuous field is the sense in which the algebra $\mathcal{A}_N$ approximates $\mathcal{E}(\mathcal{M})$, but we can do better than this. There is a sense in which $\mathcal{A}_N$ approximates the algebra of smooth functions $C^\infty(\mathcal{M})$. As I discussed in [13], the field $\mathfrak{A}$ has a further structure as a sort of smooth field of $C^*$-algebras.

Obviously, our base space, $\hat{\mathbb{N}}$ is not a manifold. However, there is a reasonable notion of smooth functions on $\hat{\mathbb{N}}$. We can identify $\hat{\mathbb{N}}$ with the homeomorphic set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \subset \mathbb{R}$, and define the smooth functions $C^\infty(\hat{\mathbb{N}})$ as those which are...
restrictions of smooth functions on \( \mathbb{R} \). Equivalently, a smooth function on \( \hat{\mathbb{N}} \) is one which can be approximated to arbitrary order by a power series in \( N^{-1} \).

The smooth structure of \( \mathfrak{A} \) is given by a dense \(*\)-subalgebra \( \Gamma^\infty(\hat{\mathbb{N}}, \mathfrak{A}) \subset \Gamma(\hat{\mathbb{N}}, \mathfrak{A}) \) of “smooth” sections. The product of a smooth function on \( \hat{\mathbb{N}} \) with a smooth section of \( \mathfrak{A} \) is again a smooth section of \( \mathfrak{A} \). The evaluation of any smooth section at \( \infty \) is a smooth function on \( M \). The algebra \( \Gamma^\infty(\hat{\mathbb{N}}, \mathfrak{A}) \) is essentially defined by the condition that any smooth function \( f \in C^\infty(M) \) gives a smooth section \( T(f) \in \Gamma^\infty(\hat{\mathbb{N}}, \mathfrak{A}) \).

2.2. Coadjoint Orbits. Some of the simplest spaces to work with are homogeneous spaces — those such that any point of \( M \) is equivalent to any other point under some isometry. In a homogeneous space with semisimple symmetry group, \( G \), we can do many calculations by simply using group representation theory. The homogeneous Kähler manifolds with semisimple symmetry group are the coadjoint orbits.

A coadjoint orbit of a Lie group, \( G \), is simply a homogeneous space that is naturally embedded in the dual space, \( g^* \), of the Lie algebra, \( g \), of \( G \). The coadjoint orbits of \( G \) are classified by positive weight vectors of \( G \). Weight vectors are naturally embedded in \( g^* \), and corresponding to the weight vector \( \Lambda \) is its \( G \)-orbit \( O_\Lambda \), the set of images of \( \Lambda \) under the action of elements of \( G \).

In the case of \( G = SU(2) \), the space \( g^* = su(2)^* \) is 3-dimensional, and the coadjoint orbits are the concentric 2-spheres around the origin. The weight, \( \Lambda \), is simply a positive number, the radius.

The geometric quantization of a coadjoint orbit is especially simple and respects the action of the symmetry group, \( G \). The Hilbert space, \( \mathcal{H}_N \), constructed in the geometric quantization of \( O_\Lambda \) carries an irreducible representation of \( G \). Namely, the representation with “highest weight” \( N\Lambda \).

In the case of \( SU(2) \), the Hilbert space, \( \mathcal{H}_N \), carries the representation of spin \( \frac{N}{2} \).

The algebras are again defined by \( \mathcal{A}_N := \text{End} \ \mathcal{H}_N \). Again, we need to tie these together into a smooth field, \( \mathfrak{A} \), of \( C^* \)-algebras. In the case of a coadjoint orbit, this can be done by using the Lie algebra structure rather than the Toeplitz quantization maps.

Let’s briefly consider what can be said about \( \mathfrak{A} \), given only the collection of algebras \( \{ \mathcal{A}_N \} \). The restriction of \( \mathfrak{A} \) to \( \mathbb{N} \subset \hat{\mathbb{N}} \) is a rather trivial continuous field; since \( \mathbb{N} \) is discrete, any section is continuous. A section of \( \mathfrak{A} \) over \( \mathbb{N} \) is nothing more than a sequence of matrices, one taken from each \( \mathcal{A}_N \). The bounded sections of \( \mathfrak{A} \) over \( \mathbb{N} \) (norm-bounded sequences) form a \( C^* \)-algebra. So, you see, we already knew the restriction of \( \mathfrak{A} \) to \( \mathbb{N} \).

Since \( \mathcal{H}_N \) carries a \( G \)-representation, there is a linear map \( g \hookrightarrow \text{End} \ \mathcal{H}_N = \mathcal{A}_N \), taking Lie brackets to commutators. Because the representation is irreducible, the image of \( g \) is enough to generate the entire associative algebra \( \mathcal{A}_N \). In fact, \( \mathcal{A}_N \) can be expressed in terms of generators and relations based on this.

Let \( \{ J_i \} \) be a basis of self-adjoint generators of the complexified Lie algebra, \( g_C \). The \( J_i \)'s can be thought of as sections of \( \mathfrak{A} \) over \( \mathbb{N} \), but they are unbounded sections, because their norms diverge linearly with \( \mathbb{N} \). We can get bounded sections by
dividing by $N$. The operators $N^{-1}J_i$ are noncommutative embedding coordinates for the coadjoint orbit, $O_{\Lambda}$, of radius $\|\Lambda\|$. I would like to work with a coadjoint orbit of radius $R$, so I define

$$X_i := \frac{R}{\|\Lambda\|} N^{-1} J_i.$$  \hspace{1cm} (2.3)

We can construct $\Gamma(N, \mathfrak{a})$ as the C*-subalgebra of bounded sections of $\mathfrak{a}$ over $N$ that is generated by the $X_i$’s. This implicitly defines $\mathfrak{a}$ as a continuous field.

The algebra $\mathcal{A}_N$ can be expressed in terms of the generators $J_i$ or $X_i$ and three types of relations. In the SU(2) case, the $J_i$’s are the three standard angular momentum operators, and $X_i = \frac{1}{2} RN^{-1} J_i$.

The first relations are the commutation relations that define the Lie algebra. In the SU(2)-case, these are $[J_1, J_2] = i J_3$ and cyclic permutations thereof. In terms of the $X_i$’s these relations have a factor of $N^{-1}$ on the right hand side, as $[X_1, X_2] = \frac{i}{2} RN^{-1} X_3$. In the limit of $N \to \infty$, the relations simply become that the $X_i$’s commute.

The second relations are Casimir relations. These enforce that the various Casimir operators have the correct eigenvalues. In the SU(2)-case, there is only one independent Casimir, the quadratic one. The relation is $J^2 = \frac{N}{2} \left( \frac{N}{2} + 1 \right)$, or in terms of the $X_i$’s, $X^2 = R^2(1 + 2N^{-1})$.

The third relations are the Serre relations. These enforce finite-dimensionality. In the SU(2)-case, this can be expressed as $(J_1 + i J_2)^{N+1} = 0$. The Serre relations are equivalent to the requirement that $\mathcal{A}_N$ is a C*-algebra, and become redundant in the $N \to \infty$ limit.

Heuristically at least, in the case of SU(2), as $N \to \infty$, the relations become that the $X_i$’s commute and satisfy $X^2 = R^2$. Clearly, this does generate the algebra of functions on $S^2$ of radius $R$. In general, as $N \to \infty$, the $X_i$’s commute and satisfy polynomial relations which determine the relevant coadjoint orbit.

Only a few facts about the smooth structure of $\mathfrak{a}$ will be needed later. Specifically, the $X_i$’s and $N^{-1}$ are smooth sections, and any smooth section vanishing at $N = \infty$ is a multiple of $N^{-1}$.

3. The Regularized Action

For a single real scalar field, $\phi \in C^\infty(M)$, the general, unregularized action functional is

$$S[\phi] := \int_M \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + V(\phi) \right] \epsilon,$$  \hspace{1cm} (3.1)

where $\epsilon$ is the volume form on $M$, and $V$ is a lower-bounded polynomial self-coupling. Complex conjugation on $C^\infty(M)$ corresponds to the adjoint on $\mathcal{A}_N$, so our approximation to the space of real functions on $M$ will be the subspace $\mathcal{A}^{sa}_N \subset \mathcal{A}_N$ of self-adjoint elements. To construct the regularized theory, we need a regularized action defined on $\mathcal{A}^{sa}_N$ that approximates (3.1).
Let’s be precise about what it means to approximate an action functional in this way. We need a sequence of action functionals, \( S_N : A^s_N \rightarrow \mathbb{R} \), which converge to the unregularized action \( S \). This is nontrivial to define because these are functionals on different spaces. Let \( \phi \in \Gamma^\infty(\hat{N}, \mathfrak{A}) \) be an arbitrary, self-adjoint, smooth section, and denote its evaluations as \( \phi_N \in A^N \); the sequence \( \{\phi_N\} \) can be considered to converge well to the smooth function \( \phi_\infty \in \mathcal{C}^\infty(M, \mathbb{R}) \). My definition for convergence of \( \{S_n\} \) is simply that for any such \( \phi \),

\[
\lim_{N \to \infty} S_N(\phi_N) = S[\phi_\infty].
\]

Several ingredients are needed to construct a regularized action. The simplest is the product. It is the most elementary property of geometric quantization that multiplication in \( \mathcal{C}^\infty(M) \) is approximated by multiplication in \( A^N \).

The normalized trace on \( A^N \) approximates the normalized integral on \( M \). That is, for any \( a \in \Gamma^\infty(\hat{N}, \mathfrak{A}) \),

\[
\tilde{\text{tr}} a_N \equiv \frac{\text{tr} a_N}{\text{tr} 1} = \frac{1}{\text{vol} M} \int_M a_\infty \epsilon + O^{-1}(N).
\]

The unregularized kinetic term can be written in terms of the Laplacian, \( \Delta = -\nabla^2 \), using the elementary identity,

\[
\int_M (\nabla \phi)^2 \epsilon = \int_M \phi \Delta(\phi) \epsilon.
\]

Mimicking this, we can safely write the regularized kinetic term as

\[
\text{vol}(M) \tilde{\text{tr}} \left[ \frac{1}{2} \phi \Delta(\phi) \right],
\]

since in fact, any quadratic functional can be written in this form.

In the coadjoint orbit case, which I will mainly discuss, the Laplacian is simply a multiple of the quadratic Casimir operator. However, the discussion in this section will apply equally to any Kähler manifold for which a suitable approximate Laplacian can be constructed. For this reason, I will write the approximate Laplacian as \( \Delta \) until a more explicit form is required for examples.

The regularized action approximating (3.1) is (using \( \text{tr} 1 = \dim \mathcal{H}_N \))

\[
S_N(\phi) = \frac{\text{vol} M}{\dim \mathcal{H}_N} \tilde{\text{tr}} \left[ \frac{1}{2} \phi \Delta(\phi) + \frac{1}{2} m^2 \phi^2 + V(\phi) \right]. \tag{3.2}
\]

This regularized action was originally formulated by Grosse, Klimčík, and Prešnajder [11] in the special case of \( S^2 \); however, the corresponding perturbation theory has not previously been discussed in detail.

3.1. The commutative limit. Using the action, (3.2) we can now define the regularized theory using a path integral formula,

\[
Z \cdot \langle 0|F(\hat{\phi})|0 \rangle := \int_{A^s_N} F(\phi) e^{-S_N(\phi)} d\phi. \tag{3.3}
\]
This is formally identical to Eq. (2.1), but it differs in that it is not merely a formal expression. The measure $d\phi$ is simply the Lebesgue measure on the finite-dimensional vector space $A_N^{s,a}$. Because $S_N(\phi)$ increases at least quadratically in all directions, Eq. (3.3) is finite for all polynomial functionals $F$.

In the standard lattice regularization, it is necessary to verify that a theory is sufficiently well behaved in the “continuum limit” as the regularization is removed. The limit of removing the regularization in the present case is the commutative limit, $N \to \infty$. At this stage, it is not entirely clear what the correct definition of convergence in the commutative limit should be.

Certainly, renormalization is necessary. That is, the bare mass, $m$, and coupling constants (the coefficients in $V$) must depend on $N$, and the field $\phi$ must also be renormalized by an $N$-dependent factor. Some condition of convergence is then applied to the sequence of renormalized, regularized theories.

A plausible form of the convergence condition is in terms of the one-particle irreducible generating functionals, $\Gamma_N$. These are functions $\Gamma_N : A_N^{s,a} \to \mathbb{R}$, derived from the path-integral. If the generating functionals are renormalized so that $\Gamma_N(0) = 0$, then the condition may be that for any smooth, self-adjoint section $\phi \in \Gamma^\infty(\hat{N}, A)$, with evaluations $\phi_N \in A_N$, the sequence $[\Gamma_N(\phi_N)]$ is convergent.

The derivation of Feynman rules from the path integral can now proceed in essentially the standard, heuristic way (see, e.g., [20]), except that now it is not just a formal calculation.

3.2. Green’s functions. Before we can discuss perturbation theory on a noncommutative space, we must first understand what Green’s functions are from an algebraic perspective. I begin in greater generality than just a scalar field.

In general, the space of classical field configurations, $\Phi$, need not be a vector space. Such is the case for nonlinear $\sigma$-models. However, for a free field theory, $\Phi$ is always a vector space. Since we are going to apply perturbation theory about a free field theory, we must assume that $\Phi$ is a vector space here.

A Green’s function is the vacuum expectation value of a product of fields. An expectation value of a quantum field $\hat{\phi}$ is an element of the space of classical field configurations, $\Phi$, since $\hat{\phi}$ is thought of as a quantum field valued in $\Phi$. An expectation value of a product of two fields is an element of the (real) tensor product $\Phi \otimes \Phi$. Carrying on like this, we see that a $k$-point Green’s function is an element of the real tensor power $\Phi \otimes^k$. The use of real tensor products here is actually not restrictive; for a complex field, $\phi$, a real tensor product will include all possible combinations of $\phi$ and its conjugate field.

Actually, I am overoptimistically allowing a great deal to fall into the ambiguity in a tensor product of infinite dimensional spaces. In the case of a real scalar field, $\Phi = C^\infty(M, \mathbb{R})$, so the tensor product $\Phi \otimes \Phi$ should intuitively be a space of functions of two points on $M$. Indeed, a 2-point Green’s function for a scalar field is a function of two points; however, it has a singularity where the points coincide. This shows that the tensor product needs to be interpreted liberally. Fortunately
this issue is irrelevant to the case at hand. Once $\Phi$ is finite-dimensional, there is no ambiguity in the tensor product.

In discussing divergences, one deals primarily with one-particle-irreducible (1PI) Green’s functions. These can be constructed as derivatives of an $\mathbb{R}$-valued generating functional on $\Phi$. This shows immediately that a $k$-point, 1PI Greens function is a linear map from $\Phi^\otimes k$ to $\mathbb{R}$. One-particle-irreducible Green’s functions thus live in the dual space of the corresponding ordinary Green’s functions.

Because of the coefficient in front of the action, there are powers of (in the case of (3.2)) $C = \frac{\text{vol } M}{\text{dim } \mathcal{H}_N}$ coming from the vertices and propagators. If, instead of setting $\hbar = 1$, we had kept explicit factors of $\hbar$, then the combination $C \hbar$ multiplying the action would be the only appearance of $\hbar$ in the functional integral. This means that the overall power of $C$ for a Feynman diagram is the same as the overall power of $\hbar$ — namely, the number of loops plus 1.

Actually, implicit in most action functionals is an inner product on $\Phi$. This means that $\Phi$ can be more or less naturally identified with a subspace of $\Phi^*$ in general, and with all of $\Phi^*$ when it is finite-dimensional.

This inner product on $\Phi$ tends to suffer an ambiguity of normalization. The most natural inner product of two functions is given by multiplying them and integrating. The most natural inner product on $A_N$ is given by multiplying matrices and taking the trace. Unfortunately, these inner products disagree when we pair 1 with itself. In $C(M)$ that gives $\text{vol } M$; in $A_N$ it gives $\text{dim } \mathcal{H}_N$. We must correct the inner product on $A_N$ by a factor of $C$, for consistency with that on functions.

The natural form of the 1PI Green’s functions is not actually the most useful normalization for comparison with the results of standard perturbation theory. In practice, we deal with ordinary Feynman diagrams in momentum space. The quantities we usually deal with are not the momentum-space Green’s functions themselves, but have an overall, momentum-conserving $\delta$-function divided out.

Consider 2-point Green’s functions in a scalar theory. In momentum space, with the $\delta$-function divided out, these are simply functions of a single momentum. Multiplying two such functions in momentum space corresponds to convolution of Green’s functions. In this normalization, a 2-point Green’s function is a convolution operator; that is a linear map $\Phi \rightarrow \Phi$.

In general, a $k$-point 1PI Green’s function in this normalization is a linear map $\Phi^\otimes (k-1) \rightarrow \Phi$. In the regularized theory, this change simply amounts to dividing our amplitudes by $C$. This simplifies the Feynman rules slightly, so that the overall power of $C$ is now simply the number of loops.

### 3.3. Noncommutative Feynman rules.

I’ll now specialize to the theory given by the action (3.2), although most of the considerations are more general. As usual, the action splits into a free part (the first two terms) and an interaction part (the $V$ term). The free part gives the propagator $(\Delta + m^2)^{-1}$; each monomial of $V(\phi)$ gives a vertex whose valence is the degree of the monomial.

A vertex of valence $r$ essentially represents the trace of a product of $r$ matrices. The term corresponding to a standard Feynman diagram is a sum of terms with
different orderings of the products. It is convenient to represent these subterms diagrammatically. For each vertex, the multiplicands correspond to incoming edges. Because only the cyclic order matters in a trace, we only need to indicate a cyclic order to the edges. This is easily done graphically by drawing the vertex in the plane. The order of multiplication is indicated by the counterclockwise order of the attached edges. The distinct terms are thus labeled by “framings” of the Feynman diagram in the plane.

In ordinary quantum field theory involving a real field, there are symmetry factors to deal with. The contribution of a given diagram is divided by the number of its symmetries. In this case there is an additional type of combinatorial factor present. Since a given ordinary Feynman diagram corresponds to several framed diagrams, these framed diagrams are weighted by coefficients adding up to 1. Determining these coefficients is a matter of enumerating the cyclic orientations of all vertices, and sorting the resultant framings into equivalence classes.

To express the exact Feynman rules, it is convenient to adapt the notation invented by ’t Hooft for discussing the large $N$ limit of $U(N)$-gauge theories (see [22, 3]). In that diagrammar, the gluon propagator is represented by a double line (two directed lines in opposite directions). An outgoing arrow indicates an upper index and an ingoing arrow indicates a lower index. The way lines are connected indicates how indices are contracted. In these diagrams, the two lines of a gluon propagator do not touch. This is appropriate, since they really have nothing to do with each other. The propagator is not just invariant under $U(N)$, but under 2 separate actions of $U(N)$ corresponding to the 2 separate edges.

In the present context, however, the notation needs to be modified. The propagator is not invariant under arbitrary unitary transformations; it is only invariant under the isometries of $M$ (if there are any). The two lines of the propagator are thus no longer independent, and I indicate this by linked double lines as shown in Fig 1.

The upper indices (outgoing arrows) are factors of $\mathcal{H}_N$, while the lower indices (ingoing arrows) are factors of $\mathcal{H}_N^*$. Each factor of $\mathcal{A}_N^{s,a} \subset \mathcal{A}_N \subset \mathcal{H}_N \otimes \mathcal{H}_N^*$ thus gives an upper and a lower index, or an incoming and an outgoing arrow. Figure 2 shows a more complicated doubled diagram. Note that no distinction is made between overcrossings and undercrossings.

A reader might reasonably question that this is truly a regularization of real scalar field theory. After all, the subspace $\mathcal{A}_N^{s,a} \subset \mathcal{A}_N$ is not closed under multiplication, so why should these Feynman rules respect this subspace? The issue is whether the Green’s functions are self-adjoint, in the obvious sense for elements of a tensor power of $\mathcal{A}_N$. In fact, a given (framed) Feynman diagram may not be self-adjoint. However, its adjoint is the mirror-image diagram, which is another

**Figure 1.** Doubled diagram for the propagator.
framing of the same diagram and therefore contributes to the same Green’s function. This makes the Green’s functions themselves self-adjoint, order-by-order in perturbation theory.

4. SCALES

This is a convenient point at which to introduce some further notation. There are three different length scales that are pertinent to quantum field theory on a quantized, compact space, and there are parameters characterizing each of these scales.

I have already introduced the length $R$. This characterizes the overall size of the manifold $M$. It is relevant to quantum field theory as the infra-red cutoff scale; that is, there do not exist modes of wavelength more than about $R$.

The second parameter, $\kappa := \frac{R^2}{N}$, characterizes the scale of noncommutativity. As I have mentioned (Sec. 2.1), geometric quantization was originally intended as a tool for deriving quantum mechanics from classical mechanics, so there is an analogy between some constructions here and in that problem. In this analogy, $\kappa$ corresponds to $\hbar$. Like a classical phase space, the Kähler manifold $M$ has a Poisson bracket. The Poisson bracket of two differentiable functions on $M$ is defined as $\{f, g\} := \pi^{ij} f_i g_j$, where the Poisson bivector, $\pi$, is in turn the inverse of the symplectic form in the sense that $\pi^{ij} \omega_{kj} = \delta^i_k$. The analogy continues with commutation relations,

$$[T_N(f), T_N(g)]_- = -i\kappa T_N(\{f, g\}) + O^2(\kappa),$$

or heuristically,

$$[f, g]_- \approx -i\kappa [f, g].$$

Note that $\kappa$ has the dimensions of an area; this balances the two derivatives occurring in the Poisson bracket.
In quantum mechanics, noncommutativity of observables leads to uncertainty relations. Likewise, noncommutativity here should intuitively lead to uncertainty relations between coordinates (roughly, something like $\Delta x \Delta y \gtrsim \kappa$). This suggests that the best that we can specify a point is to an uncertainty of the order $\kappa^{1/2}$ in all directions. Naively, we might conclude from this that $\mathcal{M}$ is broken up into “cells” of this size and that the noncommutativity effects an ultraviolet cutoff at the mass scale $\kappa^{-1/2}$, but this is not so.

Actually, if we divide $\mathcal{M}$ into cells of size about $\kappa^{1/2}$, then the number of cells will be about $\dim \mathcal{H}_N$. Using the fact that the Todd class, $\text{td } T\mathcal{M}$ is equal to 1 plus higher degree cohomology classes, the Riemann-Roch formula shows,

$$
\dim \mathcal{H}_N = \int_M \text{td } T\mathcal{M} \wedge e^{\omega/2\pi\kappa} = \frac{\text{vol } \mathcal{M}}{(2\pi\kappa)^n} + O^{1-n}(\kappa). \tag{4.1}
$$

This shows that these cells correspond to the degrees of freedom of $\mathcal{H}_N$ rather than of the scalar field.

But what is the ultra-violet cutoff scale really? The number of degrees of freedom associated to a scalar field is $\dim \mathcal{A}_N = (\dim \mathcal{H}_N)^2$, so the volume belonging to each degree of freedom is

$$
\frac{\text{vol } \mathcal{M}}{(\dim \mathcal{H}_N)^2} \approx \frac{(2\pi\kappa)^{2n}}{\text{vol } \mathcal{M}} \sim \left(\frac{\kappa}{\kappa/\mathcal{R}}\right)^{2n}.
$$

The length scale of this cutoff is thus of the order $\kappa/\mathcal{R} = \mathcal{R}/N$, or as a mass (inverse length),

$$
\mathcal{M} := \frac{N}{\mathcal{R}}.
$$

The noncommutativity scale set by $\kappa$ is the geometric mean between the infra-red and ultra-violet scales. If we attempt to remove the infra-red cutoff (decompactify the space) by sending $\mathcal{R} \to \infty$ without $\kappa$ diverging, then we must let $\mathcal{M} \to \infty$. In other words, if the infra-red cutoff is taken away, then the ultra-violet cutoff goes away. This implies that noncommutativity only achieves an ultra-violet cutoff in the presence of an infra-red cutoff! That will be proven in Sec. 5.2.

This is in marked contrast to lattice regularization. For one thing, an unbounded lattice can certainly achieve an ultra-violet cutoff. In a lattice, there is a sharply defined minimum separation between points (the lattice spacing); this is also the scale of the ultra-violet cutoff. On a quantized space, there is a fuzzy minimum observable distance, but this is much larger than the length scale of the ultra-violet cutoff.

5. Flat Space

Just because a computation is well-defined and possible in principle, doesn’t necessarily mean it is easy or practicable. Doing exact calculations on coadjoint orbits requires working with representations of the symmetry group. While this is better than working on a space with no symmetry at all, it is not as easy as working
on flat space. For flat space, calculations in quantum field theory are considerably simplified by the fact that momentum space is a vector space.

Locally, any manifold looks like flat space. Topologically, this is the very definition of a manifold. Geometrically, this is the content of Einstein’s principle of equivalence. If we are concerned with issues of small-scale physics (like renormalization) then it would be nice to do calculations in the simplified setting of flat space and not worry about the global structure of our manifold.

The heuristic arguments of Sec. 4 have already indicated that things are not so simple. We cannot ignore global structure, because ultra-violet regularization is dependent upon the global property of compactness. Nevertheless, I will introduce in Sec. 6 an approximation technique which takes advantage of the local resemblance to flat space.

Before I can describe this approximation I must discuss flat space itself. Specifically, I will discuss quantized flat space, which does not have the effect of regularizing quantum field theory, but does modify it in a relevant way.

5.1. Quantized flat space. If we “zoom in” around any point of a symplectic (or even Poisson) manifold, then it will resemble a flat, affine space, with a Poisson bracket determined by a constant Poisson bivector, \( \pi \), as

\[
\{ f, g \} = \pi_{ij} f_{|i} g_{|j}.
\]

Although the symplectic case is what we are really interested in, there is no need to assume that \( \pi \) is nondegenerate in this section.

A tensor product of two functions on the flat space \( \mathbb{R}^n \) is naturally regarded as a function on \( \mathbb{R}^n \times \mathbb{R}^n \). The multiplication map \( m \) is equivalent to the diagonal evaluation map,

\[
m : C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow C^\infty_0(\mathbb{R}^n),
\]

so that \( m(f \otimes g) = fg \). If we regard \( \pi \) as a second order differential operator on \( \mathbb{R}^n \times \mathbb{R}^n \), then the Poisson bracket can be expressed as

\[
\{ f, g \} = m \circ \pi(f \otimes g).
\]

With this notation, the Weyl product corresponding to \( \pi \) is defined as

\[
f \star_\kappa g := m \circ e^{-\frac{i\kappa}{2} \pi}(f \otimes g) = fg - \frac{i\kappa}{2} \{f, g\} - \frac{\kappa^2}{8} \pi_{ij} \pi_{kl} f_{|ik} g_{|jl} + \ldots.
\]

If we treat this as simply a formal power series in \( \kappa \), then \( \star_\kappa \) is an associative product. The space \( \mathcal{C}^\infty(\mathbb{R}^n)[[\kappa]] \) is defined to consist of formal power series in \( \kappa \) whose coefficients are smooth functions on \( \mathbb{R}^n \). The formal deformation quantization algebra \( \mathcal{A}^\kappa(\mathbb{R}^n) \) is \( \mathcal{C}^\infty(\mathbb{R}^n)[[\kappa]] \) with the product \( \star_\kappa \).

Unfortunately, if we insert an arbitrary pair of smooth functions into Eq. (5.1), the series will typically diverge. Fortunately, there is a sufficiently large space of functions for which \( \star_\kappa \) does converge, that we can construct a sensible, concrete quantization of \( \mathbb{R}^n \) from this.
The archetypal functions for which Eq. (5.1) is convergent are plane-wave functions. The Weyl product of two plane-wave functions is simply

\[ e^{-ip \cdot x} \star \kappa e^{-iq \cdot x} = e^{i\frac{\kappa}{2}(p \cdot q)} e^{-i(p+q) \cdot x}, \]

using the shorthand \( \{p, q\} := \pi^{ij}p_ip_j \) since this combination will occur frequently. This notation is justified by the fact that if we think of \( p \) and \( q \) as linear functions on \( \mathbb{R}^n \), then \( \{p, q\} \) really is their Poisson bracket. The "good" subalgebra \( A^G \subset A^\kappa(\mathbb{R}^n) \) consists of those functions whose dependence on \( \kappa \) is entire, and whose Fourier transforms (on \( \mathbb{R}^n \)) are compactly supported.

\( A^G \) is algebraically closed, and consists of convergent power series in \( \kappa \), so we can actually assign \( \kappa \) a concrete value. The mathematically sanctioned way to assign \( \kappa \) a concrete value, \( \kappa_0 \), is to quotient \( A^G \) by the ideal generated by \( \kappa - \kappa_0 \). The quotient algebra \( A^G/(\kappa - \kappa_0) \) can then be completed to a \( C^* \)-algebra. This is the concrete Weyl quantization of \( \mathbb{R}^n \) at \( \kappa_0 \). This is the same algebra that would be obtained by geometric quantization.

If we tried to construct the concrete Weyl algebra directly from \( A^\kappa \), we would have failed because the ideal generated in \( A^\kappa \) by \( \kappa - \kappa_0 \) is all of \( A^\kappa \). The quotient \( A^\kappa/(\kappa - \kappa_0) \) is thus trivial.

5.2. **Field theory on quantized** \( \mathbb{R}^n \). We can construct perturbative quantum field theory on quantized \( \mathbb{R}^n \). The derivation of Feynman rules is formally the same as on a quantized compact space; because of noncommutativity, we still have to distinguish cyclic orderings of edges around vertices. However, there will again be divergent Feynman diagrams which demand regularization; thus justifying the claim that ultra-violet regularization is contingent upon infra-red regularization.

In terms of momentum space, the Feynman rules for vertices are modified by momenta dependent phase factors. These can be understood geometrically. In momentum space, which is dual to the position space \( \mathbb{R}^n \), the bivector \( \pi \) becomes a 2-form. The \( \frac{1}{2}(p, q) \) in Eq. (5.2) is precisely the flux of \(-\pi\) through the triangle formed by \( p \) and \( q \) (see Fig. 3). For a valence \( r \) vertex draw an \( r \)-sided polygon in momentum space such that the difference of the ends of a side is equal to the ingoing momentum of the corresponding propagator line. Momentum conservation requires that the momenta add up to 0, which ensures that the polygon is closed. Decomposing the polygon into triangles shows that the phase associated to the vertex is \( \kappa \) times the flux of \(-\pi\) through the polygon. Note that this polygon is only fixed modulo an overall translation.
In conventional Euclidean real scalar field theory, the amplitudes are real; the same is true here. We must again remember to sum over all framings of a Feynman diagram. The phases lead to cosines of products of momenta.

If the evaluation of a Feynman diagram involves phases depending on internal momenta, then the resulting oscillatory integral may give a finite result where there once was a divergence. However, consistent with the heuristic argument about $M \to \infty$ in Sec. 4, this will not eliminate all divergences.

Planar diagrams remain just as divergent as those for commutative $\mathbb{R}^n$. The following proof is equivalent to that already given by Filk [9]; however, I interpret it geometrically rather than in terms of "special graph-topological properties of cocycles".

Consider a planar Feynman diagram $\Gamma$, such as that shown in Fig. 4. We can construct a dual polygonalization (2-dimensional CW-complex) $\Gamma^*$. This has a vertex for each open space in the planar rendering of $\Gamma$, including both spaces enclosed by internal edges and spaces between external edges. The edges of $\Gamma^*$ are in one-to-one correspondence with the edges of $\Gamma$. The 2-cells (polygons) of $\Gamma^*$ correspond to the vertices of $\Gamma$. Just as for a single vertex, we can embed $\Gamma^*$ in momentum space so that the separation between adjacent vertices of $\Gamma^*$ is the momentum associated to the connecting edge. Equivalently, we can construct a polygon for each vertex of $\Gamma$ (as above) and fit these together. The total phase associated to $\Gamma$ is then given by the total flux of $-\kappa \pi$ through all polygons of $\Gamma^*$. However, $\pi$ is a constant — and thus closed — differential form, so the flux only depends on the shape of the boundary of $\Gamma^*$ in momentum space. Thus, the phase only depends on the momenta of the external lines. Indeed the phase is the same as if the external lines entered a single vertex.
A generalization of this construction to non-planar diagrams will be employed in Sec. 8.

6. The Deformation Approximation

6.1. Deformation Quantization. The deformation quantization of $\mathbb{R}^n$ in Sec. 5.1 is the archetype of a more general construction (see [23] for overview). In the formal deformation quantization of a manifold $M$, the product $\ast_\kappa$ is a formal power series in $\kappa$ whose terms are bidifferential operators on $M$ (as in Eq. (5.1)). Just as for flat space, the algebra $A^\kappa(M)$ is equivalent as a vector space to the space, $C^\infty(M)[[\kappa]]$, of formal power series in $\kappa$ with smooth functions as coefficients.

In general, a deformation quantization can always be constructed to fit $[f, g]_\kappa = -\iota \kappa \{f, g\}$ mod $\kappa^2$, for any Poisson bracket on $M$; see [17].

A deformation quantization can also be derived from a geometric quantization. Geometric quantization can be loosely thought of as making the product of functions on $M$ variable (dependent on $\kappa$); the corresponding deformation quantization is the result of asymptotically expanding the product as a power series in $\kappa$. As I explained in [15], this algebra can be constructed from the smooth field of $C^*$-algebras, $\mathfrak{A}$, given by geometric quantization.

The parameter $\kappa$ is itself a smooth function on $\hat{N}$ and vanishes at the point $N = \infty$. In fact, any smooth section of $\mathfrak{A}$ which vanishes at $\infty$ is a multiple of $\kappa$. The space of smooth sections which vanish to order $j$ at $\infty$ is thus $\kappa^j \Gamma^\infty(\hat{N}, \mathfrak{A})$; this is a 2-sided ideal. In the quotient algebra $\Gamma^\infty(\hat{N}, \mathfrak{A})/\kappa^{j+1}$ (the algebra of “jets” about $\infty$), the variability of the product is preserved to order $\kappa^j$. These quotient algebras naturally form an algebraic inverse system. Taking the algebraic inverse limit gives the deformation quantization algebra corresponding to $\mathfrak{A}$,

$$A^\kappa(M) = \lim_\leftarrow \Gamma^\infty(\hat{N}, \mathfrak{A})/\kappa^j.$$  

6.2. Reconstruction. We have just seen how to construct a deformation quantization from the geometric quantization. In Sec. 5.1, I described the opposite process — reconstructing the geometric quantization from a deformation quantization — in the case of flat space. This involved a “good” subalgebra $A^G \subset A^\kappa(\mathbb{R}^n)$. It may or may not be possible to make an analogous construction in all cases, but it can be done for coadjoint orbits. In this section I will describe a good subalgebra of $A^\kappa(O_\Lambda)$.

Let $\mathfrak{A}$ be the smooth field given by geometric quantization of the coadjoint orbit $O_\Lambda$. The sections defined in Eq. (2.3) are smooth sections, $X_i \in \Gamma^\infty(\hat{N}, \mathfrak{A})$. Since smooth sections form an algebra, any product of $X_i$’s, or linear combination thereof, is also a smooth section. Such sections form a subalgebra I denote...
\[ \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}), \text{the polynomial sections. Since this is a subalgebra of the smooth sections, there is, for each } j, \text{a natural homomorphism} \]
\[ \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \to \Gamma^\infty(\hat{N}, \mathfrak{A})/\kappa^{j+1}. \tag{6.1} \]

The kernel of this is just \( \kappa^{j+1}\Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \), which vanishes as \( j \to \infty \). When the \( j \to \infty \) limit is taken, (6.1) becomes a natural, injective homomorphism,
\[ \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \hookrightarrow \mathcal{A}^\kappa. \]

Thus, \( \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \) is a subalgebra of both \( \Gamma(\hat{N}, \mathfrak{A}) \) and \( \mathcal{A}^\kappa \). In the identification of \( \mathcal{A}^\kappa(\mathcal{O}_\Lambda) \) with \( \mathcal{C}_\infty(\mathcal{O}_\Lambda)[[\kappa]] \) as a vector space, \( \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \) corresponds to \( \mathcal{C}[\mathcal{O}_\Lambda, \kappa] \), the space of polynomials in \( \kappa \) whose coefficients are polynomial functions on \( \mathcal{O}_\Lambda \). This is the good subalgebra I wanted.

In \( \mathcal{A}^\kappa \), \( \kappa - R^2/N \) is invertible, so the ideal it generates is all of \( \mathcal{A}^\kappa \); thus we cannot assign \( \kappa \) a concrete value in \( \mathcal{A}^\kappa \). On the other hand, the functions of \( \kappa \) in \( \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \) are all polynomials; therefore, \( \kappa - R^2/N \) is not invertible and generates a nontrivial ideal in \( \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A}) \). We can meaningfully take the quotient algebra,
\[ \mathcal{A}_N^\kappa := \Gamma_{\text{poly}}(\hat{N}, \mathfrak{A})/(\kappa - R^2/N). \]

In terms of the generators and relations representation of \( \mathcal{A}_N \) in Sec. 2.2, the quotient algebra \( \mathcal{A}_N^\kappa \) is what we get by discarding the Serre relations. There is therefore a natural surjective homomorphism
\[ e : \mathcal{A}_N^\kappa \to \mathcal{A}_N. \tag{6.2} \]

By characterizing the kernel of \( e \), we can effectively reconstruct the geometric quantization from the deformation quantization. As a G-representation, \( \mathcal{A}_N^\kappa \) is indistinguishable from the space of polynomial functions on \( \mathcal{O}_\Lambda \); this is the direct sum of all irreducible G-representations appearing in \( \mathcal{C}(\mathcal{O}_\Lambda) \). In \( \mathcal{A}_N \), this lattice of irreducible representations is cut off; \( \mathcal{A}_N \) is finite-dimensional. The kernel of \( e \) is spanned by those G-representations which occur in \( \mathcal{C}(\mathcal{O}_\Lambda) \), but not in \( \mathcal{A}_N \).

This is the key to doing approximate calculations. Imagine that \( R \) is very large. If we sit at some point of \( \mathcal{O}_\Lambda \), then the region around us appears very close to the flat space \( \mathbb{R}^{2n} \) (where \( 2n = \dim \mathcal{O}_\Lambda \)). We would like to describe the geometric quantization of \( \mathcal{O}_\Lambda \) in this approximation, but the geometric quantization of the non-compact space \( \mathbb{R}^{2n} \) is very different from that of compact \( \mathcal{O}_\Lambda \). Their deformation quantizations, however, are similar, because the deformation quantization product is constructible locally from bidifferential operators. The geometric quantization of \( \mathcal{O}_\Lambda \) can be approximated using the Weyl quantization of \( \mathbb{R}^{2n} \), and an approximation for the cutoff on representations. I refer to this as the deformation approximation.

6.3. **Cutoff shape.** If we are concerned with some field theory on a quantized coadjoint orbit then we would like to exploit the deformation approximation and the relationship with flat space in order to approximate the values of Feynman diagrams for large \( N \). To accomplish this, we must characterize the kernel of the surjective homomorphism \( e \) in (6.2) in terms of momentum space.
Consider the case of $S^2$. As an $\text{SU}(2)$-representation, $\mathcal{C}(S^2)$ contains all the irreducible representations of integer spin. On the other hand, $\mathcal{A}_N$ is the direct sum of representations of integer spin $\leq N$. The kernel of $\varepsilon$ consists of those representations of $\text{SU}(2)$ which are contained in $\mathcal{C}(S^2)$ but not $\mathcal{A}_N$. That means modes whose spin exceeds $N$. The Laplacian on $S^2$ with radius $R$ is $\Delta = R^{-2}J^2$, so the eigenvalue on a spherical harmonic with spin $l$ is $R^{-2}l(l+1)$. It is thus possible to characterize $\ker \varepsilon$ in terms of the Laplacian: $\ker \varepsilon$ is spanned by the eigenfunctions of $\Delta$ with eigenvalue greater than $R^{-2}N(N+1) \approx M^2$.

Now take the flat space approximation. In terms of momentum space, an eigenvalue of $\Delta$ is simply the magnitude-squared of a momentum vector. Modes with momentum greater than $M$ belong to the kernel of $\varepsilon$. We can approximate the algebra $\mathcal{A}_N$ by the Weyl product on $\mathbb{R}^2$, with modes of momentum greater than $M$ set to $0$.

As I have said, geometric quantization can be thought of as a modification of the product. In the Feynman rules, products occur at the vertices. \textit{Ab initio}, one might expect that the Feynman rules for the deformation approximation would differ from the flat space Feynman rules only at the vertices. However, it is actually more convenient to shift some of the modification to the propagators. In the deformation approximation for $S^2$, the product of a sequence of plane-wave functions is equal to their Weyl product, if all the momenta have magnitude $\leq M$, otherwise the product is $0$. In terms of the Feynman rules, this translates into restricting ("cutting off") momentum integration to the region in which all momenta have magnitude $\leq M$.

In general, the cutoff can be more complicated. Momenta are restricted to some region of size $M$. This "shape" of the cutoff depends on the particular coadjoint orbit being considered, although it is significantly limited by symmetry. In our approximation of zooming in around one point of $\mathcal{O}_\Lambda$, there is still a symmetry group of rotations about that point (isotropies). Momentum space carries a representation of the isotropy group, and the cutoff must be invariant under this. This is a difference with lattice regularization; that is much more arbitrary because features of the cutoff are not constrained by symmetry; here, there is no arbitrariness.

As examples, I shall consider the coadjoint orbits of dimension $\leq 4$. There are really only three of these: $S^2$, $S^2 \times S^2$, and $\mathbb{C}P^2$. For $S^2$ we have just seen that the cutoff shape is $D_2^2$, a disc of radius $M$, which is the only convex shape allowed by symmetry anyway. For $S^2 \times S^2$, the cutoff shape derives from that of $S^2$ and is clearly $D_2^2 \times D_2^2$. For $\mathbb{C}P^2$, the cutoff can again be characterized by the Laplacian; it is $D^4$, a ball of radius $M$. Again, this is the only convex shape allowed by symmetry.

6.4. \textbf{Feynman rules.} The Feynman rules in the deformation approximation are a modification of those for quantized flat space. The cutoff is implemented by modifying the propagator. I summarize the Feynman rules here.
For each internal edge carrying momentum $p$, there is a propagator

$$S(p) = \frac{\theta_M(p)}{p^2 + m^2},$$

(6.3)

where $\theta_M$ is a step function, equal to 1 inside the cutoff and 0 outside.

To each independent closed loop in the diagram, there is an integration over the corresponding momentum and a factor of $(2\pi)^{-2n}$, where $2n$ is the dimension of the space.

A diagram may contain vertices of valence $r$ if the potential, $V$, contains a monomial of degree $r$. To each such vertex there is a coupling constant factor coming from the coefficient of the monomial. There is also a phase factor given by the flux of $-\kappa \pi$ through a polygon in momentum space formed by the momenta entering the vertex. As always, the momenta entering a vertex add up to 0.

As with the exact Feynman rules, there are combinatorial factors coming from the symmetries of the diagram, and from the numbers of alternative framings.

### 7. Examples

As examples, I shall consider (in progressively diminishing detail) the evaluation of some one-particle-irreducible Feynman diagrams for the $\phi^4$ model on quantized coadjoint orbits of small dimension.

In this case, the potential is $V(\phi) = \frac{\lambda}{4!} \phi^4$, so all vertices are of valence 4 and carry a factor of $\lambda$.

On a coadjoint orbit, the cutoff Laplacian can be expressed in terms of the appropriately normalized quadratic Casimir operator as $\Delta = R^{-2}J^2$. This has the insurpassable property that $\Delta T_N(f) = T_N(\Delta f)$. The propagator is thus

$$\frac{1}{R^{-2}J^2 + m^2}.$$

#### 7.1. Planar propagator correction

The first diagram to consider is the planar, 1-loop propagator correction, Fig. 5. Ordinarily, this diagram would have a symmetry factor of $\frac{1}{2}$. However, only 2 out of 6 framings are equivalent to this diagram. Therefore, there is an overall combinatorial factor of $\frac{1}{6}$. 

---

**Figure 5. Planar, one-loop propagator correction.**
This is about the only diagram that can easily be evaluated exactly. The doubled diagram is shown in Fig. 6. Notice that the bottom line is detached from the rest of the diagram; this shows that the diagram factorizes as the tensor product of the identity on $\mathcal{H}_N^*$ with some linear map from $\mathcal{H}_N$ to $\mathcal{H}_N$. This must be $G$-invariant, and since $\mathcal{H}_N$ is an irreducible $G$-representation, this map must be proportional to the identity. In other words, the diagram must evaluate to a number.

To determine this number, close up the upper part of Fig. 6 and divide by $\dim \mathcal{H}_N$ (a circle) to normalize. The factor of $\dim \mathcal{H}_N$ cancels the factor from the Feynman rules. closing up the diagram amounts to taking a trace. The exact evaluation is thus

$$\lambda \frac{6}{\text{vol} M} \text{tr}_{\mathcal{A}_N} \left( R^{-2} J^2 + m^2 \right)^{-1}.$$

Now turn to the deformation approximation. Because the diagram is planar, the phase factor can only depend on external momenta, but because there is only one external momentum, there is no phase factor. Indeed, the diagram is independent of the external momentum — it is simply a number, as in the exact evaluation. In the deformation approximation,

$$\lambda \frac{6}{(2\pi)^{2N}} \int \frac{\theta_M(p) \, d^{2N}p}{p^2 + m^2}.$$

7.1.1. $S^2$. For the exact evaluation, it remains to calculate the trace of the propagator $(R^2 J^2 + m^2)^{-1}$. This is an SU(2)-invariant linear operator on the algebra $\mathcal{A}_N$. As an SU(2) representation, $\mathcal{A}_N$ is the direct sum of the irreducible representations of integer spin $0$ through $N$. The spin $l$ subspace has dimension $2l + 1$, and on that subspace the quadratic Casimir reduces to $J^2 = l(l + 1)$. Using $\text{vol} S^2 = 4\pi R^2$, the exact evaluation is

$$\lambda \frac{6}{6 \times 4\pi R^2} \sum_{l=0}^{N} \frac{2l + 1}{R^{-2l(l + 1)} + m^2}$$

$$= \frac{\lambda}{24\pi} \sum_{l=0}^{N} \frac{2l + 1}{l(l + 1) + m^2 R^2}.$$
However, this is precisely the \((N + 1)\)-part midpoint approximation to the integral

\[
\frac{\lambda}{24\pi} \int_0^{N+1} \frac{2t}{t^2 + m^2 R^2 - \frac{1}{4}} dt = \frac{\lambda}{24\pi} \ln \left[ 1 + \frac{(N + 1)^2}{m^2 R^2 - \frac{1}{4}} \right].
\] (7.2)

Now evaluate Fig. 6 in the deformation approximation. The momentum cutoff is a disc of radius \(M\). This gives

\[
\circled{\lambda} = \frac{\lambda}{6(2\pi)^2} \int_{|p| \leq M} \frac{d^2p}{p^2 + m^2} = \frac{\lambda}{12\pi} \int_0^M \frac{p\, dp}{p^2 + m^2} = \frac{\lambda}{24\pi} \ln \left[ 1 + \frac{M^2}{m^2} \right].
\] (7.3)

In order for the deformation approximation to be valid, we must assume that \(R^{-1} \ll m \ll M\). This means simply that the Compton wavelength (the distance, \(m^{-1}\), determined by the bare mass) should be much smaller than the universe and that \(m\) should be much smaller than the cutoff mass. We need not assume that \(m\) is smaller than the noncommutativity scale. Using the formula for the leading correction to the midpoint approximation, we can find the leading order correction to (7.3); this is

\[
\frac{\lambda}{24\pi} \left( \frac{1}{3m^2 R^2} + \frac{2}{N} \right).
\]

So, in this case, the deformation approximation indeed converges if we take \(R, N \to \infty\).

7.1.2. \(\mathbb{C}P^2\). As an \(SU(3)\)-representation, the algebra \(A_N\) decomposes into a direct sum of irreducible subspaces numbered 0 through \(N\). The 1 subspace has dimension \((l + 1)^3\) and the quadratic Casimir reduces to \(J^2 = l(l + 2)\). For \(\mathbb{C}P^2\) of circumference \(2\pi R\), the volume is \(\text{vol} \mathbb{C}P^2 = 8\pi R^4\).

The exact evaluation of Fig. 6 is,

\[
\circled{\lambda} = \frac{\lambda}{6 \times 8\pi^2 R^4} \sum_{l=1}^{N} \frac{(l + 1)^3}{R^2 l(l + 2) + m^2}
\]

\[= \frac{\lambda}{48\pi^2 R^2} \sum_{j=0}^{N+1} \frac{j^3}{j^2 + m^2 R^2 - 1}.\] (7.4)

As I have said, the momentum cutoff for \(\mathbb{C}P^2\) is a ball of radius \(M\). So, in the deformation approximation,

\[
\circled{\lambda} = \frac{\lambda}{6(2\pi)^4} \int_{|p| \leq M} \frac{d^4p}{p^2 + m^2} = \frac{\lambda}{48\pi^2} \int_0^M \frac{p^3\, dp}{p^2 + m^2}
\]

\[= \frac{\lambda}{96\pi^2} \left[ M^2 - m^2 \ln \left( 1 + \frac{M^2}{m^2} \right) \right].\] (7.5)

The resemblance to Eq. (7.4) is hopefully apparent.
7.1.3. $S^2 \times S^2$. Here, the exact evaluation is the double sum,

$$\lambda \frac{24\pi}{96\pi^2 R^4} \sum_{j,k=0}^{N} \frac{(2j+1)(2k+1)}{R^{-2j(j+1)} + R^{-2k(k+1)} + m^2}.$$ 

The cutoff shape is the more complicated $D^2 \times D^2$. This gives the deformation approximation as,

$$\lambda \frac{6}{6(2\pi)^4} \int_{D^2 \times D^2} \frac{d^4 p}{p^2 + m^2} = \lambda \frac{96\pi}{96\pi^2} \left[ \int_{|p| \leq M} \int_{|q| \leq M} \frac{d^2 p \, d^2 q}{p^2 + q^2 + m^2} = \lambda \frac{24\pi^2}{24\pi^2} \int_0^M \int_0^M \frac{pq \, dp \, dq}{p^2 + q^2 + m^2} \right] = \lambda \frac{96\pi}{24\pi^2} \left[ (2M^2 + m^2) \ln \left( 1 + \frac{M^2}{M^2 + m^2} \right) - m^2 \ln \left( 1 + \frac{M^2}{m^2} \right) \right].$$ 

(7.6)

This demonstrates the effect of the cutoff shape. For $M \gg m$, the results of Eq.’s (7.5) and (7.6) differ by a factor of $2 \ln 2$. The only reason for this difference is the nontrivial cutoff shape for $S^2 \times S^2$.

7.2. Nonplanar propagator correction. The only nonplanar 1-loop propagator correction is shown in Fig. 7. This is equivalent to all 4 out of 6 other framings of the original diagram; so, the combinatorial factor is $\frac{1}{3}$. The doubled diagram is shown in Fig. 8. Observe that Fig. 8 is really just the bare propagator (Fig. 1 on p. 12) with
the lines rearranged. The evaluation of Fig. 8 is $\lambda/3 \text{vol } M$ times the propagator with the $J_{\varepsilon N}$'s exchanged.

Because this is a nonplanar diagram, there is a nontrivial phase factor. This makes the evaluation more interesting. With external momentum $p$, it is (again using the brace notation of Eq. (5.2))

$$
\frac{\lambda}{3(2\pi)^{2n}} \int_{\text{cutoff}} e^{ikp \cdot \mathbf{q}} d^{2n}q = \frac{\lambda}{3(2\pi)^n} \tilde{S}(\kappa Jp)
$$

(7.7a)

$$
= \frac{\lambda}{3(2\pi)^{2n}} \int_{\text{cutoff}} e^{ikp \cdot \mathbf{q}} d^{2n}q = \frac{\lambda}{3(2\pi)^n} \tilde{S}(kp),
$$

(7.7b)

where $\tilde{S}$ is the Fourier transform of the cutoff propagator (6.3) and $J$ is the complex structure ($a \pi/2$-rotation). Equation (7.7a) is independent of any details of the propagator; Eq. (7.7b) used rotational invariance. It seems that rearranging lines in the doubled diagram corresponds to rotating by $J$ and taking a Fourier transform.

7.2.1. $S^2$. Since the propagator is SU(2)-invariant, it can be written as a linear combination of projectors on irreducible representations. The rearranged version is also invariant and can also be so decomposed. Calculating the rearrangement comes down to transforming between these two decompositions. The coefficients for this transformation are the famous 6-$j$ symbols (see [1]).

Since the value of this diagram is an invariant linear operator on $A_N$, it can most conveniently be described by giving its eigenvalues on the irreducible subspaces of $A_N$. The evaluation of Fig. 8, acting on the spin $l$ subspace is

$$
\frac{\lambda}{12\pi} \sum_{j=0}^{N} \left\{ \begin{array}{ccc} j & N/2 & N/2 \\ j + 1 & m^2R^2 & \end{array} \right\} \frac{1}{j(j + 1) + m^2R^2}.
$$

This corresponds to evaluating with external momentum, $p$, of magnitude $|p| \approx R^{-1}l$.

Following Eq. (7.7b), the deformation approximation for Fig. 8 is

$$
\frac{\lambda}{12\pi^2} \int_{|q| \leq M} \frac{e^{ikp \cdot \mathbf{q}} d^{2n}q}{q^2 + m^2} = \frac{\lambda}{6\pi} \int_{0}^{M} q J_0(kq|p|) dq.
$$

If we take $M \to \infty$ while keeping $k$ fixed, then this becomes a hyperbolic Bessel function, $\frac{\lambda}{6\pi} K_0(km|p|)$. This has a logarithmic singularity at $p = 0$, and falls off exponentially for large $p$. Aside from $p = 0$, it is finite. This means that this diagram, unlike the previous, planar diagram, is actually regularized by noncommutativity alone.
7.2.2. $\mathbb{CP}^2$. The deformation approximation gives,

\[
\frac{\lambda}{48\pi^4} \int_{|q| \leq M} \frac{e^{i\kappa \cdot q} d^4 q}{q^2 + m^2} = \frac{\lambda}{12\pi^2 \kappa |p|} \int_0^M \frac{q^2 J_1(\kappa q |p|) dq}{q^2 + m^2}.
\]

If we take $M \rightarrow \infty$, this becomes $\frac{\lambda m}{12\pi^2 \kappa |p|} K_1(\kappa m |p|)$. As in 2 dimensions, this falls off exponentially. The singularity is of the form $|p|^{-2}$, which is rather mild in 4 dimensions.

7.3. **Vertex correction.** There are several distinct framings of the one-loop vertex correction diagram. One of these is shown in Fig. 9. The external momenta are all incoming; the internal momentum $q$ is directed counterclockwise. By momentum conservation, $p_1 + p_2 + p_3 + p_4 = 0$.

The phase associated to this diagram can be split into two factors. The first, $e^{i\kappa p_1, p_2} + e^{i\kappa p_3, p_4}$, is the same as for a bare vertex with the external edges oriented in this way. The second factor is $e^{i\kappa p_1, q}$.

Except for the combinatorial factors, the evaluation of Fig. 9 in the deformation approximation is

\[
\frac{\lambda^2}{(2\pi)^2 n} e^{i\kappa (p_1, p_2) + (p_3, p_4)} \int S(q) S(p_1 + p_2 - q) e^{i\kappa (p_1, q)} d^{2n} q.
\]  

$S(p)$ again denotes the cutoff propagator (6.3). Note that this essentially amounts to a Fourier transform of the product of propagators. In four dimensions, the 1-loop vertex correction is divergent, but if we take $M \rightarrow \infty$ with $\kappa$ fixed then (7.8) remains finite except at $p_1 = 0$.

8. **Divergences**

Let’s consider how these amplitudes diverge when regularization is removed. Actually, since there are both infra-red and ultra-violet cutoffs (controlled by $R$
and $M$), there are many possible ways to remove the cutoffs, one-by-one or simultaneously.

As I have said, what has been described is far from a realistic physical model. Despite this, we can optimistically juxtapose this model with reality and hope that some properties of the model are pertinent to reality. The kind of noncommutativity considered here has not yet been noticed in experiments. This means that the noncommutativity scale set by $\kappa$ is, at best, about the smallest length scale explored by current experiments. On the other hand, $R$ should be something like the size of the universe — a far larger scale, so $N$ must be very large. This shows that the cutoff mass, $M$ should be well beyond the scale of $\kappa$ and thus far beyond the reach of experiments. The approximation relevant to physical predictions of noncommutativity is thus the limit of $M \to \infty$ with $\kappa$ fixed. Note that this means simultaneously taking $R \to \infty$; in other words, we remove the ultra-violet and infra-red cutoffs in unison. In this limit, the quantized compact space becomes, at least formally, quantized flat space. We should thus expect this limit of field theory to be described by field theory on quantized flat space.

As I have claimed, and the above examples have corroborated, nonplanarity of a Feynman diagram tends to decrease its degree of divergence. In the standard regularizations, the ultra-violet divergences are associated with the loops in the Feynman diagram. It appears that the divergences when $M \to \infty$ with $\kappa$ fixed are associated more closely with the loops in the doubled diagram — index loops, in the terminology of ’t Hooft [22]. The number of index loops is always less than or equal to the number of loops in the original diagram.

Although an arbitrary framed Feynman diagram, $\Gamma$, may not fit into the plane, it will always fit onto some oriented surface. Such a surface, $\Sigma$, can be constructed systematically by filling in a 2-cell for each line in the doubled diagram. It will have the topology of a Riemann surface with at least one puncture; each external leg of $\Gamma$ ends at a puncture.

Our first example, Fig. 5, is planar; thus $\Sigma$ is the plane (equivalently, a sphere with 1 puncture). For both Fig’s 7 and 2, $\Sigma$ is a torus with 1 puncture. For Fig. 9, $\Sigma$ is a sphere with 2 punctures.

By construction, an index loop is always contractible in $\Sigma$. We can make this more precise using homology. The group $H_2(\Sigma)$ is trivial because $\Sigma$ is not closed; the group $H_1(\Sigma, \Gamma)$ is trivial because $\Sigma$ is obtained from $\Gamma$ by attaching 2-cells. Inserting these facts into the long exact sequence for the pair $(\Sigma, \Gamma)$, gives the short exact sequence,

$$
0 \to H_2(\Sigma, \Gamma) \to H_1(\Gamma) \to H_1(\Sigma) \to 0.
$$

The group $H_2(\Sigma, \Gamma)$ is generated by the 2-cells in $\Sigma$ that do not touch punctures; these are in one-to-one correspondence with the index loops. The group $H_1(\Gamma)$ classifies the loops of $\Gamma$. The group $H_1(\Sigma)$ classifies incontractible loops in $\Sigma$. These are all free groups, so the sequence splits (unnaturally). The loops of $\Gamma$ can therefore be divided into index loops and incontractible loops of $\Sigma$. 
Consider some loop $\ell$ in $\Gamma$ which is incontractible in $\Sigma$, and examine what happens when we integrate over the momentum, $p$, circulating around $\ell$. If we leave the momenta in $\Gamma$ otherwise fixed then the part of the phase which depends on $p$ will be of the form $e^{i\kappa [q,p]}$, where $q$ is a linear combination of the other momenta in $\Gamma$. In fact, $q$ is the momentum flowing across $\ell$. The other part of the Feynman integrand depending on $p$ is the product of propagators on the edges of $\ell$; each of these has the form $((p + l)^2 + m^2)^{-1}$. Taking the integral over $p$ effectively means taking the Fourier transform of this product of propagators. The result may have an integrable singularity at $q = 0$ (an infra-red divergence because $R \to \infty$), but is otherwise finite, and falls of exponentially. This means that integration over $p$ does not contribute to the ultra-violet divergence; moreover, if $q$ is also an internal momentum, then integration over $q$ will not contribute to the divergence either.

The other important limit is the commutative limit. This is when $R$ is fixed, but $\kappa \to 0$ and $M \to \infty$. In light of the deformation approximation, it appears that the behavior in this limit will be very much like that of the most elementary regularization, a simple momentum cutoff.

9. Generalizations

9.1. Complex scalar field. For a single, complex scalar field, the general $U(1)$-invariant action is

$$S[\phi] := \int_M \left[ (\nabla \phi^*) \cdot (\nabla \phi) + m^2 \phi^* \phi + P(\phi^* \phi) \right] \epsilon,$$

(9.1)

where $P$ is some real polynomial, lower bounded on the positive axis.

The standard perturbation theory for this complex scalar field is not much different from that of the real field. Every edge of a Feynman diagram is now directed, and each vertex must have an equal number of ingoing and outgoing edges.

The regularized action corresponding to this is a simple generalization of the real case. The only new ingredient is the fact that complex conjugation in $C^\infty(M)$ corresponds to the Hermitian adjoint in $A_N$. The regularized version of (9.1) is

$$S_N(\phi) := \frac{\text{vol } M}{\text{dim } H_N} \text{tr} \left[ \phi^* \Delta(\phi) + m^2 \phi^* \phi + P(\phi^* \phi) \right].$$

(9.2)

In every product inside the trace in Eq. (9.2), $\phi$ alternates with $\phi^*$. As a consequence, in the framed diagrams, ingoing and outgoing edges alternate around each vertex. Except for this restriction, the Feynman rules for the complex field are formally the same as for a real field.

This restriction has an interesting interpretation in terms of the doubled diagram. There are really two types of lines in the doubled diagram, those on the left of a directed edge in the original diagram, and those on the right. The restriction is simply that left edges only connect to left edges and right to right.

---

2That is, fix all the momenta in $\Gamma$ and then add $p$ at each edge of $\ell$. 
9.2. **Twisted fields.** So far, I have only discussed topologically trivial scalar fields. A section of a nontrivial vector bundle is not a function, so sections of a vector bundle are not approximated by the algebra $A_N$. As I have argued in [13, 14], the noncommutative generalization of a vector bundle is a (finitely generated, projective) module of an algebra, so the geometric quantization of a vector bundle should be a module of $A_N$. For the matrix algebra $A_N = \text{End} \mathcal{H}_N$, any module is of the form, $V_N = \text{Hom}(F_N^V, \mathcal{H}_N)$, a space of linear maps from some vector space to $\mathcal{H}_N$. I present a general construction in [14]. In the simplest case — a holomorphic vector bundle — $F_N^V := \Gamma_{\text{hol}}(M, \mathbb{C}^{\otimes N} \otimes V^*)$; this generalizes $\mathcal{H}_N := \Gamma_{\text{hol}}(M, \mathbb{C}^{\otimes N})$.

The simplest action for a nontrivial vector field is a trivial generalization of (9.1), although it requires a fiberwise, Hermitian inner product on $V$. The action is simply (9.1) with $\phi \in \Gamma^\infty(M, V)$ and inner products understood between successive $\phi^*$’s and $\phi$’s.

For the regularized fields, the inner product on $V$ leads to an inner product on $F_N^V$, because of this, for $\phi \in V_N$, we can define the product $\phi \phi^*$ to be an element of $A_N$. The regularized action for a nontrivial field is just (9.2) with $\phi \in V_N$.

In the doubled diagrams, the left and right lines are now truly distinct. Left lines correspond to $\mathcal{H}_N$; right lines correspond to $F_N^V$. The complex scalar field really just wanted to be twisted.

There is not much to be said about topologically nontrivial fields in the deformation approximation. From a local perspective, a nontrivial vector bundle is just a vector bundle with a fixed background gauge field. In the limit as $R \to \infty$, this gauge field vanishes.

9.3. **Fermions.** Fermi statistics do not pose any particular obstacle in this regularization scheme. However, the kinetic terms for Fermion action functionals may be more difficult to construct than that for Bosons. Whereas constructing a cutoff Laplacian is elementary on coadjoint orbits, it is less obvious how to deal with the Dirac operator.

Even on $S^2$, there may be trouble. When the spinor bundle on $S^2$ is quantized by my prescription, the left and right chiral subspaces have different dimensions. This means that there does not exist a cutoff Dirac operator which both anti-commutes with the chirality operator ($\gamma_5$) and has no kernel. In fact, the difference of the dimensions of the left an right subspaces grows linearly with $N$; so, if we require the Dirac operator to anti-commute with the chirality operator, then the size of the kernel will diverge as $N \to \infty$.

Whether these problems can be circumvented, or whether they are really problems at all, remains to be seen.

10. **Prospects**

In all this I have largely dwelt on the regularization effects of noncommutativity. However, if there are effects of noncommutativity which are directly detectable by experiment, then these will probably be tree-level effects.
In field theory on quantized flat space, the phase factors from different framings add up to a sum of cosines of products of momenta. This can, for instance, cause the amplitudes for processes with certain combinations of momenta, to vanish. It is plausible that this sort of behavior may go beyond this simple, Euclidean, scalar field model. It may be the experimental signature of noncommutativity. Clearly, further work is required.

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