Mathematical simulation of the vibration treatment of parts in a liquefied abrasive working medium

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Abstract

The hydromechanics of the fluidized bed of the processing medium is presented; the conditions for its formation, stability, and the limits of existence are given. The stochasticity of a two-phase hydromechanical process in a fluidized bed has been determined. The boundary conditions on the surface of a spherical processed part and the vibrating machine reservoir surface are considered.

It was found that the velocity of granules during collision with the surface of the part, in addition to the chaotic component, has a normal and tangential component. It is established that the amplitude of oscillations of one of the components of the tangential velocity of the part is twice the amplitude of its normal component. The other component of the tangential velocity of the granule at the surface of the part is ten times less than the other components of the velocity. The equations for the dynamics of a fluidized medium are compiled taking into account the nature of the flow around of the part at low Reynolds numbers. The fields of fluid flow near the part are found, and the mechanism of its flow is established. The dynamics of the movement of abrasive granules in a fluidized medium, under the action of the movement of the part, is presented. The dependences of metal removal during vibration treatment of a rotating and oscillating part in a fluidized medium of abrasive granules are obtained. It is shown that metal removal from the processed part depends on the rotation frequency of the part and vibration exciter. It is noted that the presence of granule velocity components at the surface of the part makes it possible to control the processes of work hardening and micro-cutting.

Keywords Vibration processing · Pseudo-fluidized bed · Processed part · Fluid flow fields · Dynamics of abrasive medium · Metal removal

1 Introduction

In the process of traditional vibration treatment [1–7], the abrasive granules of the working medium moving in the vibration machine reservoir collide with the surfaces of the parts and perform its necessary finishing operation. The granules are moved due to the walls of the oscillating reservoir. The transference of an impulse from the granules in contact with the reservoir walls to the medium granules located near the surface of the parts occurs due to mutual collisions inside the medium. As a result, the mobility of the medium is formed by the oscillations of the reservoir walls. That is, the walls transfer an impulse to the medium granules, sufficient for the necessary surface treatment of the part and, at the same time, form a mobile medium (pseudo-gas) with the necessary conditions for transferring the impulse to the depth of the medium where the processed parts are located.

In the process of transferring the impulse between the granules of an abrasive, the strength of the impulse decreases because of the friction losses during collisions of the granules [8]. This is especially true when using a finely dispersed working medium in the form of a grinding grain and grinding powders.
In the studies [9], the analysis and modeling of the profile of the processed surface was carried out and the removal of metal during the interaction of the abrasive medium granules with the surface of the processed parts was presented. In this case, the granules of the medium are randomly distributed in the general treatment mass, depending on their percentage and the size of the granules of the medium. At the same time, there are no studies related to obtaining the dependence of metal removal during jet-abrasive processing of processed parts. It is not specified how the metal removal from the processed parts depends on the characteristics of their movement in the abrasive medium. In addition, the components of the velocity of movement of the medium granules near the processed surfaces, which allow controlling the processes of work hardening and micro-cutting, are not indicated.

When studying the abrasive finishing processing, the influence of various process parameters, such as the extrusion pressure, the number of cycles, and the viscosity of the medium when changing metal removal and average surface roughness [10], was determined. Such data are of undoubted practical interest in the control of the abrasive finishing processing and its implementation. At the same time, there is no detailed analysis of the interaction of an abrasive granule with a rough surface, which is determined by such factors as physical and mechanical properties of cutting grains, their size, shape, quantity, and location on the medium granules surface; characteristics of the processed material and its physical and mechanical properties; and process parameters depending on the technological mode of abrasive finishing processing.

In work [11], an attempt was made to improve the process of treatment of processed parts in a rotating flow of abrasive by imparting a rotational motion to them. The efficiency of the process was assessed quantitatively by metal removal and qualitatively by the achieved surface roughness. In the experiments, a medium-hard medium saturated with an abrasive was used. However, the effect of hard, highly hard, and extremely hard abrasive media on process efficiency has not been studied. It is also indicated that the velocity of rotation of the processed parts has a significant effect on the achieved surface roughness. However, the values of the velocities of their rotation in an abrasive medium are not indicated.

During multi-energy vibration treatment [12–14], it is possible to reduce the energy loss during impulse transferring by making the processed parts autonomously movable and vibrating. This provides conditions under which the surfaces of these parts have an additional energy action on the working medium granules simultaneously with their processing. However, here, the energy impact on the medium will be insignificant due to the smallness of the part surface in comparison with the surface area of the walls of the vibrating machine reservoir. Moreover, the mass of granules of the abrasive does not create the medium mobility necessary for uniform vibration processing of all surfaces of the parts. Technologically, it is possible to achieve the necessary mobility of granules by pumping fluid through them to form a pseudo-fluidized bed [15]. The movement of fluid through the working medium in the pseudo-fluidized bed becomes turbulent [16]. In turn, the stochastic nature of the fluid motion leads to a stochastic motion of the solid phase — the granules of the medium. Thus, a two-phase mobile medium is formed in which the granules behave like gas molecules. At the same time, the hydrodynamic flow creating a pseudo-fluidized bed is the source of energy necessary for the formation of a mobile medium from abrasive granules.

This paper presents the results of simulating processes accompanying vibration treatment in a pseudo-fluidized medium which is formed by combining vibrational and hydrodynamic effects.

2 The basic provisions of the pseudo-fluidized bed hydrodynamics

2.1 Conditions for the formation, stability, and existence limits of the pseudo-fluidized bed

The movement of a liquid at a low velocity through an immovable working medium at a low flow rate does not affect the porosity of this medium. An increase in the fluid velocity to a certain critical value leads to the mobility of the abrasive granules due to the beginning of a rising trajectory. This process is called pseudo-fluidization, since solid granules in a liquid acquire the properties of a liquid or gas. A further increase in the fluid velocity leads to a uniform expansion of the pseudo-fluidized volume. With a further increase in speed, a moment comes when uniform expansion gives way to the appearance of inhomogeneities [17].

The dependence of the pressure drop in the fluid flow through the mass of solid granules on the velocity in this flow is known (Fig. 1).

With an increase in the flow velocity from zero to a certain critical value \( U_c \), the pressure drop \( \Delta P_n \) initially increases. Upon reaching a critical value of the flow velocity, the hydraulic resistance of the solid granules mass becomes equal to the product of the weight of these granules per unit of their area. A further increase in speed will cause the granules to begin to rebuild in such a way as to reduce flow resistance. In this case, the volume (height) of the granular mass will begin to grow, increasing the porosity of the medium, and the pressure drop will remain constant. A further increase in the fluid velocity will lead to even greater rise in the trajectory of the granules and their free movement in the flow, as a result of which the working medium
becomes pseudo-fluidized. With a further increase in the liquid velocity, the volume of the granule mass grows, and the pressure drop $\Delta P_B$ remains constant. The distribution of granules in the volume remains uniform.

The uniform distribution of granules in the pseudo-fluidized bed is maintained up to a certain limiting value of the flow velocity $U_i$, at which the liquid begins to break through the mass of granules, forming jets, and bubbles. The distribution of granules in the volume becomes inhomogeneous, and the hydraulic resistance decreases [18].

A diagram of the vibrating machine reservoir equipped with a fluid pumping system is presented (Fig. 2). The height of the pseudo-fluidized bed of abrasive granules is associated with the densities of water $\rho_w$ and the material of granules $\rho_g$. Here $h$ is the coordinate of the height of the medium layer at which the parameters of the pseudo-fluidized bed are determined.

The relationship between the layer porosity $\varepsilon$ and the pressure drop $\Delta P_B$ caused by its hydraulic resistance is expressed by the following relations (1), (3) and (4):

$$\Delta P_B = (\rho_g - \rho_w) \cdot (1 - \varepsilon) H g$$  \hspace{1cm} (1)

where $g$ is the gravitational constant.

From relation (1), we can write a formula that determines the pressure drop $\Delta P_{B_i}$ corresponding to the beginning of pseudo-fluidization:

$$\Delta P_{B_i} = (\rho_g - \rho_w) \cdot (1 - \varepsilon_i) H_i g$$  \hspace{1cm} (2)

where $H_i$ is the height of the medium before liquefaction, and $\varepsilon_i$ is the porization porosity of the medium before liquefaction.

For large granules, the velocity of the pseudo-fluidization beginning is determined by the relation (3):

$$\frac{\Delta P_{B_i}}{H_i} = 150 \left( \frac{1 - \varepsilon_i}{\varepsilon_i^3} \right)^2 \cdot U_i d \cdot 1.75 \left( \frac{1 - \varepsilon_i^3}{\varepsilon_i^3} \right) \cdot \frac{\rho_w U_i^2}{d}$$  \hspace{1cm} (3)

Taking into account expression (2), Eq. (3) can be written as follows:

$$\frac{\rho_w (\rho_g - \rho_w) g d}{\mu^2} = 150 \frac{U_i d \rho_w}{\mu} \cdot \left( \frac{1 - \varepsilon_i}{\varepsilon_i^3} \right) + 1.75 \left( \frac{U_i d \rho_w}{\mu} \right)^2$$  \hspace{1cm} (4)

If the granules of the medium have a shape close to spherical, then $\varepsilon_i = 0.4$. Substituting in formula (4) the expressions for the Reynolds numbers $Re_i = \frac{U_i d \rho_w}{\mu}$ and Archimedes number $Ar = \frac{\rho_w (\rho_g - \rho_w)}{\mu^2} g d^3$, we get the following:

$$Re_i^2 + 51.4 Re - 0.0366 Ar = 0$$  \hspace{1cm} (5)

Solving Eq. (5), we find the Reynolds number, that is, the fluid flow rate at which fluidization of the working medium begins:

$$Re = 25.7 \left( \sqrt{1 + 5.53 \cdot 10^{-5} Ar} - 1 \right)$$  \hspace{1cm} (6)

As indicated above, when flow rate increases, the volume of the medium grows and thereby the porosity of the granular mass increases. The dependence of the flow rate on the porosity is expressed by the following formula [17]:

$$\frac{U}{U_i} = \varepsilon^n$$  \hspace{1cm} (7)

Here, the value $n$ is determined by the following expressions:

$$n = \left( 4.4 + 18 \frac{d}{D} \right) Re_{10}^{-0.1} \text{ for } 1 < Re_h < 500;$$

$$n = 2.4 \text{ for } Re_h > 500$$  \hspace{1cm} (8)
where $Re_p$ is the Reynolds number for the soaring velocity $U_f$ in a bounded volume. For the pseudo-fluidized bed, the soaring velocity in unlimited space $U_{inf}$ is associated with velocity $U_f$ by the ratio:

$$\log U_f = \log U_{inf} - \frac{d}{D} \quad (9)$$

where $D$ is the diameter or characteristic size of the vibrating machine reservoir.

The relationship between the porosity and the fluid pumping rate is determined by the following relations [19]:

$$Re = \frac{Ar \varepsilon^{0.75}}{18 + 0.6 \sqrt{Ar}^{3/5}}; \quad \varepsilon = \left( \frac{18 Re + 0.36 Re^2}{Ar} \right)^{0.21} \quad (10)$$

When a flow rate reaches a certain value, uniform fluidization becomes non-uniform. The criterion for the transition is the ratio [20]:

$$Re_f = \frac{Ar}{(18 + 0.61 \sqrt{Ar})} \quad (11)$$

### 2.2 The stochasticity of the two-phase hydromechanical process in the pseudo-fluidized bed

Qualitatively, the picture of the hydrodynamic fields in the pseudo-fluidized bed can be represented by a superposition of steady and random motions of the granules of the solid phase. The latter displacements occur due to both collisions between the granules and the action of chaotic fluid flows.

Experimental studies show that the frequency spectrum of the granules oscillations in the pseudo-fluidized bed is continuous and quite extensive [16]. Thus, the description of the motion of the solid-phase granules should be carried out using statistical methods, similar to how this is done when describing the movement of molecules or atoms of a gas. However, the movement of the separate granules of the working medium during vibration processing is determined by the turbulent flow of the liquid element. Because of this, the dependence of the velocity of the solid-phase granules on the velocity of these granules and their direction before collision in the pseudo-fluidized bed is much greater than that for molecules or atoms in a gas [18]. Thus, attempts to use the theoretical apparatus used in the kinetic theory of gases encounter great difficulties.

In this paper, to assess the averaged values of the velocity and the “mean free path” of solid-phase granules, we use the concept of the natural scale. The natural scale of the velocity of the continuity phase flowing in the channels between the granules of the pseudo-fluidized bed and the natural linear scale, that is, the “mean free path,” can be estimated by the following relations [16]:

$$\frac{dP}{\partial Q} = \varepsilon_p \frac{W}{2S} \quad (15)$$

where $S$ is the cross-sectional area of the vibrating machine reservoir.

In the future, we will assume that the working medium granules are similar to molecules or atoms of a gas with an average kinetic temperature determined by the velocity of the liquid $W_p$ and by the mean free path $\overline{r}$, i.e., by the distance at which the velocity changes its value by an order of magnitude. In this case, the velocity of the granule will be determined by both the stochastic effect of the liquid and its motion due to the flow around the processed part by pseudo-fluidized medium.

### 3 Statement of the problem of flow around the processed parts by a pseudo-fluidized working medium

#### 3.1 The general picture of the pseudo-fluidized medium motion

Let us assume that the shape of the processed part is close to spherical. This makes it possible to use analytical solutions for flow around the processed part. Let us consider the scheme of flow around the rotating and oscillating processed part by a pseudo-fluidized medium (Fig. 3).
In Fig. 3, \( A \) is the amplitude of the oscillations caused by the rotation of the part around the eccentric axis, \( R(z) \) is the radius of the circle formed by the boundary of the sphere at a height of \( y \) from the base of the sphere, and \( \Omega \) and \( \omega \) are the angular velocities of rotation of the part around the axes passing through its center and displaced relative to the first by a distance \( A \) correspondingly. \( U \) is the averaged velocity of the liquid in the pseudo-fluidized medium, and \( V_g \) is the chaotic velocity of the granular medium.

During processing, the liquid flows around the part from bottom to top (see Fig. 3). In this case, the part itself is involved in two rotational movements, accordingly, around one vertical axis passing through the center of the part, and around the second vertical axis, displaced from the first by a distance \( A \).

Thus, in order to identify the collection of factors affecting the vibration processing in the pseudo-fluidized bed, it is necessary to solve the task of flow around the rotating and oscillating part by the granulated fluid flow with the granular medium that rotates and moves randomly.

### 3.2 Boundary conditions on the surface of the spherical processed part and on the surface of the vibrating machine reservoir

As indicated earlier, the boundary conditions are the velocities of motion for the part surface points involved in two rotational movements.

Let us consider the velocity of movement for the circle elements formed by the intersection of a plane perpendicular to the rotation axis at a height of \( y \) and the part rotation axis. To calculate the components for the velocities of movement for any point of the part, a diagram is presented (Fig. 4) with the following notation: \( VR \) — radial velocity of rotation of a circle with a radius \( R(z) \), \( VR = R(z) \cdot \Omega \); \( V_r \) — radial velocity of circular motion for a circle with a radius \( r \), \( V_r = A \omega \); \( \alpha = \Omega \cdot t, \beta = \omega t; R(z) = \sqrt{R_{sp}^2 - (R_{sp} - z)^2} \), where \( R_{sp} \) is the radius of the part.

In practice, the radius of the small circle \( A \) is significantly smaller than the size of the radius of the part. In this regard, we can neglect the change in the radius vector \( \vec{A} \) in the total vector \( \vec{R}(z) = \vec{R} + \vec{A} \) by accepting the following:

\[
|\vec{R}_x(z)| \approx |\vec{R}(z)|
\]

(16)

In this case, the direction of vector \( \vec{R}_x(z) \) will coincide with vector \( \vec{R}(z) \), that is,

\[
\frac{\vec{R}_x(z)}{|\vec{R}_x(z)|} = \frac{\vec{R}(z)}{|\vec{R}(z)|}
\]

(17)

Taking into account the foregoing, we find the components of the total velocity on the part surface, which in Cartesian coordinates have the following form:

\[
V_x = \Omega \cdot R(z) \sin(\Omega \cdot t) + A \omega \sin(\omega t);
\]

(18)

\[
V_y = \Omega \cdot R(z) \cos(\Omega \cdot t) + A \omega \cos(\omega t)
\]

\[
V_z = R(z)(\Omega + \omega) + A \omega \cos((\omega + \Omega) t)
\]

(19)

\[
V = A \omega \sin((\omega + \Omega) t)
\]

(20)
Simplifying the relations (19) and (20) (see Fig. 8), we obtain the following:

\[ V \sum_{\rho \in \mathcal{C}} = R \cdot \Omega \cdot \sin(\theta) + A \omega \cdot \cos((\Omega + \omega) t) \]  

\[ V \sum_{\rho \in \mathcal{C}} = A \omega \cdot \sin((\omega + \Omega) \cdot t) \]  

The perturbation around the stationary flow of a pseudo-fluidized medium created by the part damps in inverse proportion to the square of the distance from the center of the part \( \sim \frac{1}{r^2} \), where \( r \) is the distance from the part center to the calculated point [22, 23]. That is, the action of the reservoir wall on the pseudo-pseudo-fluidized medium flow near the processed part can be neglected if the part radius is significantly smaller than the radius of the vibrating machine reservoir machine. In our case, this condition is satisfied. Thus, it is possible to determine the boundary conditions for solving the task of flowing around a part by a pseudo-fluidized medium as follows:

1. We assume that the part is in a limitless medium.
2. We assume that relations (22) and (21) are valid on the part surface.

It should be noted that the expressions for the radial and tangential components of the velocities of rotation and oscillations contain two components of angular velocity, \( \Omega \) and \( \omega + \Omega \).

### 4 The equations of the pseudo-fluidized medium dynamics

#### 4.1 The nature of flow around the part

Before drawing up the equation of pseudo-fluidized medium motion, it is necessary to determine its main parameters: density and viscosity (Appendix).

For the averaged density \( \rho_{av} \) and effective viscosity \( \mu_{eff} \), the following expressions are valid [20]:

\[ \mu_{eff} = \mu \cdot \frac{(1 + 0.5(1 - \epsilon))}{\epsilon^2} \]  

\[ \rho_{av} = \rho_{H2O} \cdot \epsilon + \rho_{gr} \cdot (1 - \epsilon) \]  

where \( \rho_{H2O} \) and \( \rho_{gr} \) are the densities of the liquid and granule material, respectively.

Knowing the density and viscosity of the pseudo-fluidized bed, we can calculate the Reynolds number, which is characteristic for the processes of flowing around a part by a pseudo-fluidized medium flow, by its rotation and oscillation. We present the dependencies of the kinematic and dynamic viscosity, as well as the dependencies of the average pseudo-fluidized medium density \( \rho_{av} \) and the Reynolds number \( Re \) on the porosity (Figs. 5 and 6).

The effective viscosity at the beginning of fluidization exceeds the viscosity of water by almost an order of magnitude (see Fig. 5). As porosity increases, the dynamic and...
kinematic viscosities tend to the values of those parameters characteristic for pure water.

It is obvious that the averaged medium density at the beginning of the fluidization is more than one and a half times higher than the density of water and approaches $\rho_{H_2O}$ as the porosity increases (see Fig. 6). The Reynolds number for the indicated flow does not exceed 1, and for $\varepsilon = 0.4$, it is much less than unity.

The velocity of any part surface during rotation can be determined from its maximum angular velocity, which is realized in practice at 1200 rpm. Dependencies of the Reynolds numbers for a part rotating $Re_{rot}$ and an oscillating $Re_{osc}$ on porosity when a part radius $= 1$ cm and an oscillation amplitude $= 2$ mm are presented in Fig. 7.

Furthermore (see Figs. 6 and 7), it follows that the Reynolds number, that is, the nature of the flow around the part by the pseudo-fluidized medium, is different for the rectilinear flow around the part, as well as the rotation and oscillation of the part. So, in the first case, the viscosity forces prevail over the inertia forces; in the second case, the inertia forces far exceed the viscosity forces; while in the third case, the inertia forces not only are insignificant, but also exceed the viscosity forces.

To solve the task of determining the pseudo-fluidized medium dynamics in the presence of rotating and oscillating parts, one can use the Navier–Stokes equations with the boundary conditions described above. Thus, we divide the total solution into components, each of which satisfies the Navier–Stokes equations, and the boundary conditions are carried out by the sum of these solutions.

4.2 The equations of flow around a processed part with low Reynolds numbers: finding a fluid flow field near a part during the formation of a pseudo-fluidized medium

The Navier–Stokes equation for stationary tasks without taking into account mass forces in vector form is as follows:

$$\left(\nabla \Delta\right) \vec{V} = -\frac{1}{\rho} \operatorname{grad} P + \nu \Delta \vec{V}$$

For $Re \ll 1$, the term on the right-hand side of Eq. (25) can be neglected in comparison with the relations on the right-hand side. Thus, it remains to solve the following equation:

$$\frac{1}{\rho} \operatorname{grad} P = \nu \Delta \vec{V}$$

To solve Eq. (26), we introduce the stream function $\varphi$:

$$V_r = \frac{1}{r^2 \sin \theta} \cdot \frac{\partial \varphi}{\partial \theta}$$

$$V_\theta = -\frac{1}{r \sin \theta} \cdot \frac{\partial \varphi}{\partial r}$$

A diagram explaining the flow around a part by a pseudo-fluidized working medium is shown in Fig. 8. Using the current function $\varphi$, we can write the following equations which determine velocities and pressure [22, 24]:

$$DD\varphi = 0$$

Here $D$ is the Stokes operator in spherical coordinates.
Fig. 7  Dependency of Reynolds numbers for a rotating \( \text{Re}_{\text{rot}} \) and an oscillating \( \text{Re}_{\text{osc}} \) part on porosity \( \varepsilon \) for a part radius = 1 cm and an oscillation amplitude = 2 mm.

![Graph showing dependency of Reynolds numbers](image)

The boundary conditions on the part surface can be written as follows:

\[
V_R = U \cos(\theta); \quad V_\theta = U \sin(\theta) \tag{31}
\]

Moreover, the conditions for infinity are as follows: for \( r \to \infty \), \( V_r \) and \( V_\theta \) → 0.

We will look for a solution in the form:

\[
\varphi = \sin^2(\theta) \cdot F(r) \tag{32}
\]

Substituting expression (32) into the Stokes operator (29), we obtain the following:

\[
D \varphi = \sin^2 \theta \left( \frac{d^2 F(r)}{dr^2} - \frac{1}{r^2} F(r) \right) = \sin^2 \theta \cdot f(r) \tag{33}
\]

Substituting expression (33) into Eq. (28), we obtain the relation for functions \( f(r) \):

\[
\frac{d^2 f}{dr^2} - \frac{1}{r^2} f = 0 \tag{34}
\]

From relation (34), it follows that:

\[
f = A r^2 + \frac{B}{r} \tag{35}
\]

where \( A \) and \( B \) are constants.

Substituting Eqs. (34) and (35) into Eq. (33), we obtain the following:

\[
\frac{d^2 F(r)}{dr^2} - \frac{1}{r^2} F(r) = A r^2 + \frac{B}{r} \tag{36}
\]

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Taking into account the conditions for infinity and boundary conditions, we can write the solution of Eq. (28) in conjunction with expressions (34)–(36) in the following form:

\[
\phi = \frac{1}{4} U \sin^2 \theta \left(3 \frac{R r - R^3}{r^3} \right);
\]

\[
V_r = \frac{1}{2} U \cos \theta \left(3 \frac{R r - R^3}{r^3} \right);
\]

\[
V_\theta = \frac{1}{4} U \sin \theta \left(3 \frac{R r + R^3}{r^3} \right)
\]  (37)

The expression for pressure has the following form [19]:

\[
P = P_0 + \frac{3}{2} R \mu_{\text{eff}} \frac{U \cos \theta}{r^2}
\]  (38)

where \(P_0\) is the pressure for “infinite” distance from the streamlined part.

### 4.3 Flow caused by rotation of a part in a pseudo-fluidized medium

As established in Sect. 3.2, the rotational and vibrational motion of a part leads to the fact that flows arise on the spherical surface of the part with constant circular velocity \(R(\theta) \cdot \Omega\) and variable radial and circular velocities with frequency \(\omega = \Omega\) (Eqs. (21), (22)). The circular velocity on the surface of the processed part (21) consists of constant and variable components. A flow-around pattern with such boundary conditions can be composed of two solutions:

- Determining the dynamics of the pseudo-fluidized bed motion around a rotating part
- Finding the flow field around the part oscillating around its vertical axis

Consider the location of the flow field around a part oscillating around its vertical axis. In this case, the oscillatory movement of the part should be transmitted further from its surface, forming a transverse wave. The direction of oscillations in the wave is perpendicular to their propagation. However, it is known that at Reynolds numbers significantly greater than unity (see Fig. 7), the transverse waves in a liquid decay rapidly [25]. Therefore, the action of the rotational-vibrational motion of the part can be neglected and limited to finding the dynamics of the pseudo-fluidized medium caused only by its rotation.

To determine the dynamics of the pseudo-fluidized bed motion around a rotating part, it is necessary to solve the system of Eq. (25). Since the Reynolds number for the layers of a pseudo-fluidized medium near the rotating part surface is much greater than unity (see Fig. 7), the quadratic terms of the equations on the right-hand side of system (25) must be taken into account.

However, for fluid motion along concentric circles, as in our case, these terms are equal to zero [26].

Thus, the equations describing the dynamics of a pseudo-fluidized medium rotating around its vertical axis of the part in cylindrical coordinates have the following form:

\[
\frac{\partial P}{\partial r} = \mu_{\text{eff}} \left( \Delta V_r - \frac{V_r}{r^2} - \frac{2}{r} \frac{\partial V_r}{\partial \alpha} \right);
\]

\[
\frac{1}{r} \cdot \frac{\partial P}{\partial \alpha} = \mu_{\text{eff}} \left( \Delta V_\alpha - \frac{V_\alpha}{r^2} + \frac{2}{r} \frac{\partial V_r}{\partial \alpha} \right)
\]  (39)

Here \(\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{r} \frac{\partial}{\partial \alpha} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\partial}{\partial \theta}\) is the Laplace operator for spherical coordinates.

According to the accepted conditions (see Fig. 4), the derivative of pressure with respect to angle \(\alpha\) (the task is isotropic with respect to rotation along angle \(\alpha\)) and the radial velocity are equal to zero. In this case, from the second equation of system (39), the relation follows:

\[
\Delta V_\alpha - \frac{V_\alpha}{r^2} = 0
\]  (40)

We solve Eq. (40) in the following form:

\[
V_\alpha = \sin \theta \cdot f(r)
\]  (41)

Evaluating Eq. (40) taking into account relation (41) and equalities \(V_\alpha = \sigma R \sin \theta \cdot \frac{\partial V_\alpha}{\partial \sigma}\), we obtain the equation for \(f(r)\):

\[
\frac{d^2 f}{d r^2} + \frac{2}{r} \frac{d f}{d r} - \frac{2}{r^2} f = 0
\]  (42)

The solution of Eq. (42) under condition \(f(r) \to 0\) with \(r \to \infty\) is function \(f(r) = \frac{\text{const}}{r^2}\). Using the boundary condition \(V_\alpha = (\Omega + \omega) R \sin \theta\) adopted above, we find a solution for the motion of a pseudo-fluidized medium caused by the rotation of a part:

\[
V_\alpha = \frac{(\Omega + \omega) R^3 \sin^2 \theta}{r^2}
\]  (43)

The pressure around a part rotating in a pseudo-fluidized medium is determined by the following relation [26]:

\[
\frac{1}{\rho_{\text{eff}}} \cdot \frac{\partial P}{\partial r} = \frac{V_\alpha^2}{r}
\]  (44)

From the relation (44), the pressure distribution depends on the radius of the granule (or part):

\[
P = P_0 - \frac{1}{4} \rho_{\text{eff}} \left(\frac{(\Omega + \omega) R^3 \sin^2 \theta}{r^2}\right)^2
\]  (45)

where \(P_0\) is pressure at infinity.

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4.4 Flow caused by radial oscillations of a part

The circular and oscillatory movement of the part leads to its radial-vibrational action on the pseudo-fluidized medium. We find the flow arising as a result of this action by solving Eq. (25) with an additional term describing the flow with a variable velocity:

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \Delta) \mathbf{V} = -\frac{1}{\rho} \text{grad} P + \nu \Delta \mathbf{V} \tag{46}
\]

The Reynolds number for our case is much greater than unity (Fig. 7). This means that the viscosity forces when finding a solution to Eq. (45) can be neglected in comparison with the inertia forces and with the second term on the right side of this equation.

Before transforming Eq. (46) into a form convenient for calculations, it is necessary to determine the sound velocity in a pseudo-fluidized medium. To do this, we use the relation for the equilibrium sound velocity in two-phase media \(C_{\text{heter}}\) [27, 28]:

\[
C_{\text{heter}} = \sqrt{\frac{1}{((1 - \varepsilon)\rho_{gr} + \varepsilon \rho_{H_2O}) \cdot \left(\frac{(1-\varepsilon)}{\rho_{gr} C_{gr}^2} + \frac{\varepsilon}{\rho_{H_2O} C_{H_2O}^2}\right)}} \tag{47}
\]

where \(\rho_{H_2O}\), \(\rho_{gr}\), \(C_{H_2O}\), and \(C_{gr}\) are the density and the sound velocity in water and in a granule, respectively.

From the graphical dependence (Fig. 9), it can be seen that the sound velocity in a pseudo-fluidized medium varies depending on the porosity in the range of 1800–1500 m/s.

The oscillation velocity of the processed part is described by Eq. (22) and does not exceed 0.2 m/s. This value is much less than the velocity sound in a pseudo-fluidized medium (see Fig. 9). We write the finally simplified Eq. (46) in spherical coordinates:

\[
\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho_{av}} \cdot \frac{\partial P}{\partial r} \tag{48}
\]

Under conditions of a low oscillation velocity and neglecting the viscosity forces, Eq. (48) turns into a wave equation. So, the vibrational motion of a part causes wave motion in a pseudo-fluidized medium [26]. The equation of such a movement has the following form:

\[
\frac{\partial^2 (r P)}{\partial r^2} + \frac{1}{c^2} \cdot \frac{\partial^2 (r P)}{\partial t^2} = 0 \tag{49}
\]

where \(c^2 = \frac{\partial P}{\partial r_{av}}\).

Taking into account the conditions at the boundary of the oscillating part, the solution of Eq. (49) can be represented as follows:

\[
P = f(t - r/c) \frac{1}{r} \tag{50}
\]

where \(f(t - r/c)\) denotes a certain function satisfying the wave Eq. (50).

From relation (48), it follows that the flow velocity created by the pulsating part can be expressed in the following form:

\[
V = -\frac{1}{\rho_{av}} \cdot \frac{\partial}{\partial r} \int_{t_0}^{t} P \, dt \tag{51}
\]

Fig. 9 The dependence of the sound velocity \(C_{\text{heter}}\) in a pseudo-fluidized medium on porosity \(\varepsilon\)
In relation (50), it is assumed that the initial velocity \( V_0 \) is equal to zero at \( t_0 = 0 \). From the relations (50) and (51), it follows that:

\[
V = \frac{P}{\rho_{av} c} + \frac{1}{\rho_{av} R} \int_{t_0}^{t} P \, dt \tag{52}
\]

Based on the relation (52), the granule velocity on the surface of a part with radius \( R \) which creates a spherical wave of type \( P = f(t - r/c)/r \) will be equal to the following:

\[
V(R) = \frac{1}{\rho_{av} R c} \cdot f \left( t - \frac{R}{c} \right) + \frac{1}{\rho_{av} R^2} \cdot \int_{t_0}^{t} f \left( t - \frac{R}{c} \right) \, dt \tag{53}
\]

The spherically symmetric wave described by Eqs. (50) and (53) arises due to monopole radiation, which can be represented as a radially pulsating part with rigid impermeable walls [26]. The movement of the walls of such a part leads to a change in the volume of its environment, which leads to the appearance of spherically symmetric waves.

Thus, to determine function \( f(t - R/c) \) through the boundary condition (53), it is necessary to establish the change in volume caused by the movement of the part spherical wall.

The change in volume \( \Delta W \) caused by the movement of the part can be expressed as follows:

\[
\Delta W = \frac{4}{3} \pi \left( (R + \Delta r)^3 - R^3 \right) \Delta t \approx 4 \pi \left( \frac{\Delta r}{\Delta t} R^2 + \frac{\Delta r}{\Delta t} R \Delta r \right) \tag{54}
\]

We pass from finite differences to infinitesimals. In addition, we assume, without loss of generality, that \( V(R) = A \omega \sin \left( (\omega - \Omega) t \right) \). To simplify, we assume that \( \Delta t = t - t_0 = t \) if \( t_0 = 0 \). From here, we get the following:

\[
\frac{\partial W}{\partial t} = 4 \pi A \omega R^2 \left( 1 - \frac{A}{R} \cos \left( (\omega + \Omega) t \right) \right) \cdot \sin \left( (\omega + \Omega) t \right) \tag{55}
\]

According to the accepted conditions in expression (55), the ratio \( A/R \ll 1 \). Then, we rewrite expression (55) as follows:

\[
\frac{\partial W}{\partial t} \approx 4 \pi A \omega R^2 \cdot \sin \left( (\omega + \Omega) t \right) \tag{56}
\]

On the other hand, the volume change arising due to radial pulsations can be expressed using relation (53):

![Image](image_url)
It follows from dependence (57) that the volume change is proportional to \( f(t - R/c) \). Omitting cumbersome expressions and neglecting small quantities, we obtain the relation connecting \( f(t - R/c) \) and \( \frac{\partial W}{\partial t} \):

\[
\frac{\partial W}{\partial t} = \frac{4\pi}{\rho_{\text{av}}} \int_{t_0}^{t} f(t) \, dt
\]  

\[(58)\]

Taking into account relations (58) and (50–53), it is possible to write down the dependencies for pressure and velocity of a pseudo-fluidized medium resulting from radial pulsations of a processed part as follows:

\[
P(r) \approx \frac{\rho_{\text{av}}}{4\pi r^2} \left( \frac{1}{c} \cdot \frac{\partial^2 W}{\partial t^2} - \frac{1}{c} \cdot \frac{\partial W}{\partial t} \right)
\]

\[
V(r) \approx \frac{1}{4\pi r^2} \int_{t_0}^{t} f(t) \, dt
\]

(59)

5 The metal removal during vibration processing of rotating and oscillating parts in a pseudo-fluidized medium from abrasive granules

Graphically, the dependencies presented earlier make it possible to choose the technological characteristics of vibration processing that have the greatest and proportional influence on the metal removal.

The dependencies that determine the metal removal on all characteristics of the vibration processing are presented in Figs. 10, 11, 12, 13, 14, 15 and 16.
Fig. 12  (a) The dependencies of the total metal removal during vibration processing in a pseudo-fluidized medium on the angle $\theta$ and the oscillation frequency of the exciter: 0 Hz; 5 Hz; 10 Hz; 15 Hz; 20 Hz; 25 Hz; vibration exciter oscillation amplitude 2.0 mm; part rotation frequency 10 Hz; the average radius of the working medium granules 0.5 mm; b–d dependencies of metal removal caused only by radial velocity components (b) of the granule, as well as tangential components along unit vectors $e_\alpha$ (c) and $e_\beta$ (d) on the part surface.
Fig. 13 (a) The dependencies of the total metal removal during vibration processing in a pseudo-fluidized medium on the angle \( \theta \) and the oscillation frequency of the exciter: 0 Hz; 5 Hz; 10 Hz; 15 Hz; 20 Hz; 25 Hz; vibration exciter oscillation amplitude 2.0 mm; part rotation frequency 10 Hz; the average radius of the working medium granules 1.0 mm; b–d dependencies of metal removal caused only by radial velocity components (b) of the granule, as well as tangential components along unit vectors \( e_\theta \) (c) and \( e_\phi \) (d) on the part surface.
The dependencies of the total metal removal during vibration processing in a pseudo-fluidized medium on angle $\theta$ and the oscillation frequency of the vibration exciter: 0 Hz; 5 Hz; 10 Hz; 15 Hz; 20 Hz; 25 Hz; vibration exciter oscillation amplitude 2.0 mm; part rotation frequency 10 Hz; the average radius of the working medium granules 1.5 mm; b–d dependencies of metal removal caused only by radial velocity components (b) of the granule, as well as tangential components along unit vectors $e_{n}$ (c) and $e_{g}$ (d) on the part surface.
Fig. 15 (a) The dependencies of the total metal removal during vibration processing in a pseudo-fluidized medium on angle $\theta$ and the oscillation frequency of the vibration exciter: 0 Hz; 5 Hz; 10 Hz; 15 Hz; 20 Hz; 25 Hz; vibration exciter oscillation amplitude 2.0 mm; part rotation frequency 10 Hz; the average radius of the working medium granules 2.0 mm; b–d dependencies of metal removal caused only by radial velocity components (b) of the granule, as well as tangential components along unit vectors $e_a$ (c) and $e_\phi$ (d) on the part surface.

Fig. 16 The dependencies of the maximum metal removal amount during vibration processing of the part in a pseudo-fluidized medium on the rotation frequency of the vibration exciter for various rotation frequencies of the part: 0 Hz; 5 Hz; 10 Hz; 15 Hz; 20 Hz; 25 Hz.
6 Conclusions

Simulating the processes accompanying vibration treatment in a pseudo-fluidized medium allows us to draw the following conclusions:

1. Due to the fact that the abrasive granules move inside the hydrodynamic flow consisting of abrasive granules and liquid, the velocity of granules upon collision with the part surface, in addition to the chaotic component, has both normal and tangential components and has certain normal component to this surface.

2. The amplitude of the tangent velocity component caused by part rotation exceeds the amplitude of the normal component to this surface.

3. The tangent velocity component of the granule near the processed part surface caused by the flow of liquid creating a pseudo-fluidized medium is much smaller than the other components of this velocity.

4. The metal removal from the processed part surface does not monotonically depend on the rotational speeds of the part and the vibration exciter, which can be explained by interference between movements caused by the part rotating and the vibration exciter (Fig. 16).

5. The presence of deterministic velocity components of the granule near the processed part surface allows the selection of the frequencies of rotation of the part and the vibration exciter to control the methods of vibration treatment in which the processes of hardening or microcutting can occur to a greater extent.

6. The choice of the diameter of the abrasive medium granules in combination with the control of the vibration processing method makes it possible to achieve the desired technological results in an optimal way.

The non-monotonic dependence of the metal removal rate on the rotational frequencies of the part and the vibration exciter revealed during the simulation of the process of vibration treatment makes it possible to optimize the surface treatment of the parts.

7. The established influence on the amount of metal removal, the frequency of rotation of the part around the axis of symmetry, as well as the amplitude and frequency of oscillations caused by the rotation of the part around the eccentric axis makes it possible to adjust these parameters to ensure the conditions necessary to achieve various technological results of vibration processing.

8. The developed mathematical simulation makes it possible to determine the main design parameters of vibrating machines for processing parts in a liquefied abrasive medium, depending on the geometric and physicochemical characteristics of the parts.

Appendix

This Appendix presents the dynamics of abrasive granules in a pseudo-fluidized medium under the action of the processed part movement.

When considering the state of liquid and granules mixture of an abrasive medium exposed to a moving fluid, the velocity of the liquid chaotic movement was determined as a result of its pumping through the abrasive medium granules.

According to the traditional approach to the description of hydromechanics of a pseudo-fluidized medium, we assume that the liquid and solid phases are interpenetrating interacting media occupying the same volume. We will consider the abrasive granules working medium as an association of gas atoms. In this case, the movement of the granules occurs both under the influence of the random action of the liquid and under the influence of the movement caused by the flow around the pseudo-fluidized medium of the processed part.

The action of a moving fluid on abrasive granules will be determined from the following ratio:

$$m_{gr} \frac{\vec{V}_{gr}}{\rho_{gr}} = \frac{\text{sign}(V_{av} - \vec{V}_{gr})}{\C} \frac{\rho_{av}}{\R^2_{gr}} \left( \frac{V_{av} - \vec{V}_{gr}}{2} \right) + 2\pi \R^3_{gr} \text{grad} P$$

where \(C\) is the head resistance coefficient of the granule; \(m_{gr}\) is the mass of the granule; \(V_{gr}\) is granule velocity; and \(V_{H_{2}O}\) is fluid velocity.

The second term on the right-hand side of Eq. (59) is valid under the assumption that the grain radius \(R\) is small compared to the characteristic distance at which the pressure varies significantly.

Equation (60) is a vector, so we decompose it into components. In spherical coordinates, Eq. (60) will have the following form:

$$
\begin{align*}
\frac{m_{gr}}{dt} & = \text{sign}(V_{av} - V_{gr}) \frac{\rho_{av}}{\R^2_{gr}} \left( V_{av} - V_{gr} \right) + 2\pi \R^3_{gr} \frac{\partial P}{\partial r} ; \\
\frac{m_{gr}}{dt} & = \text{sign}(V_{av} - V_{gr}) \frac{\rho_{av}}{\R^2_{gr}} \left( V_{av} - V_{gr} \right) + 2\pi \frac{\R^3_{gr}}{r \cos \theta} \cdot \frac{\partial P}{\partial \theta} ; \\
\frac{m_{gr}}{dt} & = \text{sign}(V_{av} - V_{gr}) \frac{\rho_{av}}{\R^2_{gr}} \left( V_{av} - V_{gr} \right) + 2\pi \frac{\R^3_{gr}}{r} \cdot \frac{\partial P}{\partial \alpha} .
\end{align*}
$$

\( \text{(61)} \)
where $R_{gr}$ is the radius of the granule. System (61) contains three components of Eq. (60). The first equation describes the projection of the granule velocity $V_{r gr}$ onto the radius of the part. The second and third equations determine the dynamics of the projection of the granule velocities $V_{a gr}$ and $V_{\theta gr}$ onto the unit vectors of the spherical coordinate system $e_a$ and $e_\theta$.

Let us write the well-known expressions in system (60) assuming that the motion of a pseudo-fluidized medium determines the fluid motion in it and using relations (22), (37), (38), (43), (45), (57), and (59) given in the main text of the article:

$$V_{r av} = 0.5 U \cos \theta \left( 3 \frac{R}{r} - \frac{R^3}{r^3} \right) + \frac{R^2}{r^2} A \omega \cos ((\omega + \Omega) t)$$

$$\frac{\partial P}{\partial r} = -3R \rho_{eff} \frac{U \cos \theta}{r^3} + \rho_\omega \left( \frac{(\Omega + \omega) R^3 \sin^2 \theta}{r^5} - \frac{R^2}{r^2} A \omega (\omega + \Omega) \cos ((\omega + \Omega) t) \right)$$

$$V_{a av} = \frac{R(\Omega + \omega) R^2 \sin^2 \theta}{r^2} ; \frac{\partial P}{\partial \alpha} = 0$$

Fig. 17 Dependency of the granule velocity component directed along the part radius on angle $\theta$ and velocity $U$ of pumping fluid through the working medium. The part rotation frequency: (a) 5 Hz; (b) 10 Hz; (c) 15 Hz; (d) 20 Hz. Oscillation frequency and amplitude of vibration exciter: 20 Hz; 1 mm

Fig. 18 Dependency of the granule velocity component directed perpendicular to the part radius along unit vector $e_\theta$ from angle $\theta$ and velocity $U$ of pumping fluid through the working medium. The part rotation frequency: (a) 5 Hz; (b) 10 Hz; (c) 15 Hz; (d) 20 Hz. Oscillation frequency and amplitude of vibration exciter: 20 Hz; 1 mm
\[ V_{\theta, av} = \frac{1}{4} U \sin \theta \left( \frac{3R}{r} + \frac{R^3}{r^3} \right) \quad (65) \]

\[ \frac{\partial P}{\partial \theta} = \left( -\frac{3}{2} R \mu_d U \sin \theta \frac{1}{r^2} + 4 \rho_{av} \frac{(\Omega + \omega)^2 R^6 \sin^2 \theta \cos \theta}{r^4} \right) \quad (66) \]

We make a change of variables and introduce new variables,

\[ AV_{r, gr} = V_{r, gr} - V_{r, av} ; \quad AV_{a, gr} = V_{a, gr} - V_{a, av} ; \quad AV_{\theta, gr} = V_{\theta, gr} - V_{\theta, av} \quad (67) \]

Next, we rewrite Eq. (61) with the new variables:

**Fig. 19** Dependency of the granule velocity component directed perpendicular to the par radius along unit vector \( e_\theta \) from angle \( \theta \) and velocity \( U \) of pumping fluid through the working medium. The part rotation frequency: (a) 5 Hz; (b) 10 Hz; (c) 15 Hz; (d) 20 Hz. Oscillation frequency and amplitude of vibration exciter: 20 Hz; 1 mm

**Fig. 20** The dependence of the abrasive granule velocity \( U_{r, gr} \) caused by stochastic fluid motion through the pseudo-fluidized medium on fluid pumping velocity \( U \)
Fig. 21 Dependencies of the total granule velocity value in the immediate vicinity of the processed part surface on the angle \( \theta \) and velocity \( U \) of pumping fluid through an abrasive medium. Part rotation frequency: (a) 0 Hz; (b) 5 Hz; (c) 10 Hz; (d) 15 Hz; (e) 20 Hz; (f) 25 Hz. Oscillation frequency and amplitude of vibration exciter: 20 Hz; 1 mm

\[
\begin{align*}
\frac{dAV_{gr}}{dt} &= \text{sign}(AV_{gr}) C_{\rho_{av}} \pi R_{gr}^2 AV_{gr}^2 + \\
&+ 2\pi R_{gr}^3 \frac{\partial^2 r}{\partial r^2}; \\
\frac{dAV_{agr}}{dt} &= \text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2 AV_{agr}^2 + \\
&+ 2\pi R_{agr}^3 \frac{\partial^2 r}{\partial r^2}; \\
\frac{dAV_{agr}}{dt} &= \text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2 AV_{agr}^2 + \\
&+ 2\pi R_{agr}^3 \frac{\partial^2 r}{\partial r^2}; \\
\frac{dAV_{rgr}}{dt} &= \text{sign}(AV_{rgr}) C_{\rho_{av}} \pi R_{rgr}^2 AV_{rgr}^2 + \\
&+ 2\pi R_{rgr}^3 \frac{\partial^2 r}{\partial r^2};
\end{align*}
\]

(68)

Equation (68) is inhomogeneous. The complete solution of each of them will consist of solutions for the homogeneous parts of the differential Eq. (68) and particular solutions with an inhomogeneous part.

We write only the homogeneous parts of Eq. (68):

\[
\begin{align*}
\frac{dAV_{rgr}}{dt} &= \text{sign}(AV_{rgr}) C_{\rho_{av}} \pi R_{rgr}^2 AV_{rgr}^2; \\
\frac{dAV_{agr}}{dt} &= \text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2 AV_{agr}^2; \\
\frac{dAV_{agr}}{dt} &= \text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2 AV_{agr}^2;
\end{align*}
\]

(69)

The solutions of relations (69) will have the following form:

\[
\begin{align*}
AV_{rgr} &= -\frac{1}{C_{\rho_{av}}+\text{sign}(AV_{rgr}) C_{\rho_{av}} \pi R_{rgr}^2} r; \\
AV_{agr} &= -\frac{1}{C_{\rho_{av}}+\text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2} r; \\
AV_{agr} &= -\frac{1}{C_{\rho_{av}}+\text{sign}(AV_{agr}) C_{\rho_{av}} \pi R_{agr}^2} r;
\end{align*}
\]

(70)

where \( C_{1rgr}, C_{1agr}, \) and \( C_{1agr} \) are the integration constants. These constants can be determined from the initial conditions.

If we leave in the Eq. (68) only the terms that depend on time (inhomogeneous terms), then we obtain the following relations:

\[
\begin{align*}
\frac{dAV_{rgr}}{dt} &= -m_{gr} AV_{rgr} + 2\pi R_{rgr}^3 \frac{\partial P}{\partial r}; \\
\frac{dAV_{agr}}{dt} &= -m_{gr} AV_{agr} + 2\pi R_{agr}^3 \frac{\partial P}{\partial r}; \\
\frac{dAV_{agr}}{dt} &= -m_{gr} AV_{agr} + 2\pi R_{agr}^3 \frac{\partial P}{\partial r};
\end{align*}
\]

(71)

Omitting the cumbersome calculations, we write down the full expressions for the velocities of the granules movement under the action of the pseudo-fluidized medium flow:

\[
\begin{align*}
V_{rgr} &= -\frac{1}{C_{1rgr} + 0.5 C_{\rho_{av}} \pi R_{rgr}^2} + 0.5 U \cos \theta \left( \frac{3R}{r} - \frac{R_{rgr}^3}{r^3} \right) + \\
&\quad + \frac{R_{rgr}^2}{R^2} \frac{A\omega}{r^2} \sin ((\omega + \Omega) t) + \\
&\quad \frac{3}{2} \frac{\rho_{av}}{\rho_{gr}} \left( \frac{(\Omega + \omega) R_{rgr}^2 \sin^2 \theta}{r^5} \right) - 3 R_{v eff} U \cos \theta \frac{1}{r^3} t - \\
&\quad \frac{R_{rgr}^2}{R^2} \frac{A\omega}{r^2} \sin ((\omega + \Omega) t) + C_{2r}; \\
V_{agr} &= -\frac{1}{C_{1agr} + 0.5 C_{\rho_{av}} \pi R_{agr}^2} + \frac{R(\Omega+\omega)^2 \sin^2 \theta}{r^7} + C_{2a}; \\
V_{agr} &= -\frac{1}{C_{1agr} + 0.5 C_{\rho_{av}} \pi R_{agr}^2} + \frac{1}{4} U \cos \theta \left( \frac{3R}{r} + \frac{R_{agr}^3}{r^3} \right) + \\
&\quad + \frac{3}{2} \frac{\rho_{av}}{\rho_{gr}} \left( 4(\Omega + \omega)^2 R^6 \sin^3 \theta \cot \beta - \frac{3R}{2} R_{v eff} U \cos \theta \frac{1}{r^3} \right) + C_{2a};
\end{align*}
\]

(72)
The integration constants $C_{2r}$, $C_{2a}$, and $C_{2\theta}$ as well as $C_{1gr}$, $C_{1ag}$, and $C_{1ogr}$ are determined from the initial and boundary conditions.

Based on the initial conditions, without loss of generality, equating the constants $C_{2r}$, $C_{2a}$, and $C_{2\theta}$ to zero we find the constants $C_{1gr}$, $C_{1ag}$, and $C_{1ogr}$. Substituting into Eq. (72) $t = 0$, we get the following:

$$
\begin{align*}
C_{1r gr} &= \frac{1}{r} + 0.5U \cos \theta \left( \frac{\pi - \theta}{2} \right), \\
C_{1a gr} &= \frac{1}{R (\Omega + \omega)} R^2 \sin \theta, \\
C_{1\theta gr} &= \frac{4}{U} \sin \theta \left( \frac{\pi + \theta}{2} \right).
\end{align*}
$$

Based on relations (73), we can conclude that all the first terms of Eq. (72) tend to zero with increasing $t$. And since we consider steady flows during vibration processing in a pseudo-fluidized medium, these terms can be ignored later on.

Among relations (72), there remain the constant terms, oscillating, and increasing with time. An infinite increase in the components of the granule velocities is impossible, because if the granules exceed a certain flow rate of the pseudo-fluidized medium, they will invariably begin to slow. This velocity can be determined from Eq. (61) if their left parts are equal to zero (the derivatives of the velocity components of the granule will become lower than the velocity components of the pseudo-fluidized medium). Therefore, the relations for the velocity components of the abrasive granules can be written as follows:

$$
\begin{align*}
V_{r gr last} &= \frac{1}{2} \left( \text{sign}(V_{r} - V_{r gr}) + 1 \right) V_{r gr} + \\
&+ \frac{1}{2} \left( \text{sign}(V_{r gr} - V_{r}) + 1 \right) V_{r}; \\
V_{a gr last} &= \frac{1}{2} \left( \text{sign}(V_{a} - V_{a gr}) + 1 \right) V_{a gr} + \\
&+ \frac{1}{2} \left( \text{sign}(V_{a gr} - V_{a}) + 1 \right) V_{a}; \\
V_{\theta gr last} &= \frac{1}{2} \left( \text{sign}(V_{\theta} - V_{\theta gr}) + 1 \right) V_{\theta gr} + \\
&+ \frac{1}{2} \left( \text{sign}(V_{\theta gr} - V_{\theta}) + 1 \right) V_{\theta}.
\end{align*}
$$

In relation (74), the quantities $V_{r gr last}$, $V_{a gr last}$, and $V_{\theta gr last}$ are the final expressions for the velocity components of the abrasive granules. Full expressions for the above velocity components are not given here because of their bulkiness.

In our case, when vibration processing takes place for a relatively long time, in order to determine the velocity components of the granule before it collides with the processed part surface, it is necessary to solve Eqs. (60) or (61), equating the left side to zero. This is a consequence of the fact that all transient processes in a pseudo-fluidized medium after turning on the vibrating machine will take much less time than the processing time in this medium.

For a complete description of the motion of the granules in a pseudo-fluidized medium, it is necessary to supplement the relations determined by relations (74) by the action of stochastic liquid motion determined by relation (12) given in the main text of the article.

The graphical dependencies (Figs. 17, 18, 19 and 20) are presented by the curves corresponding to the solution of Eq. (61) with the left-hand side equal to zero.

The total abrasive granule velocity is the geometric sum of all velocity components plus the stochastic component. Since the random component of the granule velocity is distributed in all directions equally likely, we can assume that the component is equal to $1/3U_{ag}$.

A graph of the total granule velocity in the immediate vicinity of the processed part surface is presented in Fig. 21.

From the graphs in Figs. 17, 18, 19 and 21, it is obvious that the dependence of the metal removal is not monotonically dependent on the vibration excitation frequency of the vibration exciter (see Fig. 21). The velocity of the liquid creating a pseudo-fluidized medium has an extremely limited effect on metal removal (see Figs. 19 and 21); the tangential velocity component of the granule in the direction of unit vector $e_{a}$ can have a greater effect on the metal removal than the radial velocity component of the granule (see Figs. 17 and 18).

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