TRAVELING WAVES FOR THE QUARTIC FOCUSING HALF WAVE EQUATION IN ONE SPACE DIMENSION

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Abstract. We consider the quartic focusing Half Wave equation (HW) in one space dimension. We show first that there exist traveling wave solutions with arbitrary small $H^{1/2}(\mathbb{R})$ norm. This fact shows that small data scattering is not possible for (HW) equation and that below the ground state energy there are solutions whose energy travels as a localised packet and which preserve this localisation in time. This behaviour for (HW) is in sharp contrast with classical NLS in any dimension and with fractional NLS with radial data. The second result addressed is the non existence of traveling waves moving at the speed of light. The main ingredients of the proof are commutator estimates and a careful study of spatial decay of traveling waves profile using the harmonic extension to the upper half space.

Aim of this paper is to consider the Half Wave equation in one space dimension, (HW) since now on, with quartic nonlinearity

\[ i \partial_t u = \sqrt{-\Delta} u - u|u|^3, \quad (t, x) \in \mathbb{R} \times \mathbb{R} \]

where $\sqrt{-\Delta}$ stands for the fractional Laplacian, namely $\mathcal{F}(\sqrt{-\Delta} f) = |\xi| \mathcal{F}(f)$. We recall that (HW) enjoy respectively the conservation of the following energy:

\[ \mathcal{E}_{hw}(u) = \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R})}^2 - \frac{1}{5} \|u\|_{L^5(\mathbb{R})}^5, \]

as well as the conservation of the mass $\mathcal{M}(u)$, namely:

\[ \frac{d}{dt} \|u(t, x)\|_{L^2(\mathbb{R})}^2 = 0. \]

Concerning the Cauchy problem associated with (0.1) it has been proved in [12] the existence of global solutions for initial data in $H^1(\mathbb{R})$ and $\sup_{(0, T)} \|u\|_{H^{1/2}(\mathbb{R})} < \infty$, for every $T$. As a consequence of (0.2) and (0.3) global existence is guaranteed for initial data in $H^1(\mathbb{R})$ with small assumptions on $H^{1/2}(\mathbb{R})$.

The aim of the paper is to prove existence/non existence results for a class of solutions whose energy travels as a localised packet and which preserve this localisation in time: the traveling waves. As a byproduct of our existence results and qualitative properties of traveling waves we show that small data scattering cannot occur for one dimensional quartic (HW).

To establish the existence of standing waves solutions $\psi_q(t, x) = e^{it} q(x)$ for (HW)
equation is not difficult to prove, see [6]. The classical strategy introduced by Weinstein is to maximize a suitable functional whose critical points correspond to the standing waves. For (HW) equation one easily verifies that the map

$$u(t, x) \rightarrow \lambda_0^\frac{1}{3} u(\lambda_0 t + t_0, \lambda_0 x + x_0) e^{i\gamma_0}, \quad (\lambda_0, t_0, x_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

yields a group of symmetries. This implies trivially that if $\psi_q(t, x) = e^{itq(x)}$ is the ground state solution to (0.1) with mass $||q||^2_{L^2(\mathbb{R})} = M_0$ and energy $E_{hw}(\psi_q) = E_0$, then $\psi_{q, \lambda_0} = \lambda_0^\frac{1}{3} q(\lambda_0 x) e^{i\lambda_0 t}$ is another ground state solution with mass $||\psi_{q, \lambda_0}||^2_{L^2(\mathbb{R})} = \lambda_0^{-\frac{2}{3}} M_0$ and energy $E_{hw}(\psi_{q, \lambda_0}) = \lambda_0^2 E_0$. The aforementioned scaling computation shows that

$$E_{hw}(\psi_{q, \lambda_0}) M(\psi_{q, \lambda_0})^2 = \text{const.}$$

In particular when the mass of the ground state goes to zero the energy of the ground state increases as the inverse square of the mass. We recall that the term ground state here indicates any positive standing wave solution to (1.1) even, nonnegative that vanish at infinity. Fixing the value of $\lambda_0$ the ground state is unique, see [6]. Notice that the ground state for quartic (HW) does not minimize the energy with a mass constraint (indeed the energy for quartic (HW) is unbounded from below even with a mass constraint).

We shall underline that (HW) equation differs, for instance, from NLS equation and nonlinear Klein-Gordon equation due to the lack of an explicit formula to construct traveling waves from standing waves. Indeed for NLS and NLKG equations thanks to Galilean and Lorentz invariance, the existence of traveling waves solutions can be straightforward obtained simply applying a Galilean or Lorentz boost to the standing waves. This is not the case for HW equation.

In [7], in the context of Boson star equation, i.e. considering a Hartree type nonlinearity, the existence of traveling waves for HW with the ansatz $u(x, t) = e^{it} q_v(x - vt)$ has been investigated. More recently it has been proved in [11] the existence of travelling waves for the cubic one dimensional Half Wave equation with arbitrary small mass, i.e small $L^2$ norm. This result is important because it is a peculiarity of the (HW) equation. As an example of this peculiarity, for classical $L^2$ critical NLS below the minimal mass ground state, all the initial data scatter to the free evolution equation [5]. Let us notice that traveling wave solutions $u(x, t)$ to (0.1) with the ansatz $u(x, t) = e^{it} Q_v(x - vt)$ fulfill the equation

$$(0.5) \sqrt{-\Delta} Q_v + iv \partial_v Q_v + Q_v - |Q_v|^3 Q_v = 0,$$

and therefore the proof of their existence can be obtained maximizing again a suitable Weinstein-like functional, see e.g. [11].

For simplicity we will consider the case $v > 0$, being the negative case identical. The first result is to prove the following theorem concerning the existence and
the asymptotics when \( v \to 1^- \) of traveling waves of the form
\[
\psi = e^{i(1-v)t}Q_v(x-vt),
\]
where \( Q_v \) satisfies
\[
(0.6) \quad \sqrt{-\Delta}Q_v + iv\partial_x Q_v + (1-v)Q_v - |Q_v|^3Q_v = 0.
\]
Notice that the dependence on \( v \) is both on the phase and space shift.

**Theorem 0.1.** For any \( 0 < v < 1 \) there exists \( Q_v \in H^{\frac{1}{2}}(\mathbb{R}) \) such that \( e^{i(1-v)t}Q_v(x-ut) \) solves (0.6) and such that
\[
(0.7) \quad ||Q_v||_{L^2(\mathbb{R})} \sim (1-v)^{\frac{1}{2}}, \quad ||Q_v||_{H^{\frac{1}{2}}(\mathbb{R})} \leq C(1-v)^{\frac{3}{4}}, \quad ||Q_v||_{H^1(\mathbb{R})} \leq C(1-v)^{\frac{3}{4}}.
\]
In particular for \( v \) sufficiently close to 1 the energy of the traveling wave is below the energy of the ground state with the same mass. Moreover, given \( 0 < v_1 < v_2 < 1 \) with
\[
v_1 = 1 - \varepsilon, \quad v_2 = v_1 + \delta, \quad 0 < \delta \ll \varepsilon \ll 1,
\]
we have
\[
(0.8) \quad ||Q_{v_1} - Q_{v_2}||_{L^2(\mathbb{R})} \leq C\delta(1-v_1)^{\frac{1}{4}}.
\]
In particular small data scattering does not occur.

**Remark 0.1.** Notice that the lack of small data scattering should follow, as a matter of fact, as a straightforward consequence of the existence of small traveling waves. Nevertheless the rigorous proof of this implication requires some additional efforts since in our one dimensional context the (HW) equation does not enjoy nice decay estimates.

**Remark 0.2.** The condition \( 0 < v < 1 \) guarantees that the quadratic form associated with (0.6) fulfills for \( u \neq 0 \)
\[
||u||^2_{H^{\frac{1}{2}}(\mathbb{R})} + iv\int_{\mathbb{R}} \bar{u}\partial_x u dx > 0.
\]

We shall underline that our result confirms the peculiarity of (HW) with respect, for instance, to \( L^2 \) supercritical NLS. For NLS, fixed the \( L^2 \) norm of the initial datum, and assuming that the energy of the wave is below the ground state energy, then the long time dynamics is characterized by only two possible alternatives: either scattering to the free equation or blow up in finite time. The classical approach to show this alternative goes back to [10]. For (HW) we show that this situation cannot occur. Moreover we underline that the dynamics of (HW) in one dimension is different from \( L^2 \) supercritical fractional NLS, i.e if one substitute the operator \( \sqrt{-\Delta} \) with \((-\Delta)^{\frac{s}{2}} \) and \( s > \frac{1}{2} \). Indeed for fractional NLS it is still true in dimension \( n \geq 2 \) and radial data the alternative between scattering to the free equation or blow up in finite time for data with energy below the ground state energy, see [4] for blow up and [14] for scattering (only if
$\frac{3}{1} < s < 1$). For a first attempt to describe the dynamics for (HW) equation in high dimension we quote [3]. When $n \geq 2$ and radial data the blow-up in finite time is still an open question.

The second contribution of this paper is to discuss the nonexistence of traveling waves solutions with arbitrary frequency moving at the limit speed $v = 1$.

**Theorem 0.2.** For any $\omega \in \mathbb{R}$ it does not exist a traveling wave solution to (0.1) given by $u(x, t) = e^{i\omega t}Q_1(x - t)$ with $Q_1 \in H^{1\over 2}(\mathbb{R})$.

**Remark 0.3.** We shall notice that the non existence of traveling wave solutions moving at the speed of light is not elementary to prove. As a matter of fact for $v = 1$ the Fourier multiplier $|\xi| - \xi$ in the kinetic term is no more positive but it does not imply with elementary arguments that traveling waves cannot exist. In fact the crucial step in the proof is given by the spatial decay estimate of $Q_1$ (assuming the existence) together with commutator estimates. More precisely our approach is given by the following steps:

1. proving that any traveling wave at the speed of light decays $O\left(\frac{1}{|x|^2}\right)$;
2. thanks to (1) showing that at the speed of light supp $\hat{Q}_v \subset (-\infty, 0]$;
3. noticing that (2) implies that the Fourier multipliers are nothing but classical derivatives;
4. using the equation to conclude that $Q_v = 0$.

The proof of the decay of traveling waves at the speed of light is inspired by the celebrated work of Amick-Toland [11] in the context con Benjamin-Ono equation (see also [9]), while the localization of frequencies in the half space is a consequence of commutator estimates.

It is interesting to underline how the non existence of traveling waves at the speed of light for (HW) is strongly correlated to the existence of traveling waves with arbitrary speed for Szegő equation

$$(0.9) \quad i\partial_t \psi = \Pi_-(|\psi|^3 \psi),$$

where $\Pi_-$ is the Szegő projector onto negative frequencies. Indeed by minimizing a suitable Weinstein functional a traveling wave solution $\psi(t, x) = e^{-it\tilde{q}_v(x - vt)}$ for Szegő equation can be obtained for any $v < 0$, see [13] in the cubic case. These traveling waves can be rescaled in order to solve the equation

$$(0.10) \quad vD\tilde{q}_v + \tilde{q}_v - \Pi_-(|\tilde{q}_v|^3 \tilde{q}_v) = 0.$$ 

On the other hand the traveling waves at the speed of light for (HW) have frequency localization in the negative half space and hence for those solutions the operator $\sqrt{-\Delta + i\partial_x}$ coincides with $-2D$. We underline the interesting fact that for eq. (0.10) and $v = -2$ solutions exist while solutions at the speed of light for (HW) for $v = 1$ does not. The key point in our non existing argument concerns
the decay of traveling wave that implies, thanks to commutator estimates, the
frequency localization. The same argument cannot be applied for Szegő equation.
In [8] it is noticed that while traveling waves moving at the speed of light for cu-
bic Szegő equation decay like \( \frac{1}{x} \), the traveling waves with \( 0 < v < 1 \) for cubic
(HW) decay as \( \frac{1}{x^2} \). Our aforementioned step (1) (Lemma 3.1) proves that at the
speed of light traveling waves for (HW) (independently on the value of exponent
of the nonlinearity) shall decay as in case \( 0 < v < 1 \). As a byproduct we proved
that this decay implies, in fact, that such waves cannot exist.

Remark 0.4. It is immediate to notice that if \( u(x,t) = e^{i\omega t} Q_1(x - t) \) is a traveling
wave solution at the speed of light with \( Q_1 \in H^\frac{1}{2}(\mathbb{R}) \), then \( Q_1 \) fulfills two addi-
tional properties: \( \lim_{|x| \to \infty} |Q_1(x)| = 0 \) and \( Q_1 \) bounded. These two properties
that follows from the fact that \( Q_1 \in H^1(\mathbb{R}) \) and are crucial to prove the non
existence result.

1. A REMARK ABOUT TRAVELLING WAVES FOR NLS

Consider the classical NLS
\[
(1.1) \quad i\partial_t u = -\Delta u - u|u|^{p-1}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n
\]
We recall that NLS enjoy respectively the conservation of the following energy:
\[
(1.2) \quad \mathcal{E}_{nls}(u) = \frac{1}{2} \|u\|^2_{H^1(\mathbb{R}^n)} - \frac{1}{p+1} \|u\|^{p+1}_{L^{p+1}(\mathbb{R}^n)},
\]
as well as the conservation of the mass,
\[
(1.3) \quad \frac{d}{dt} \|u(t, x)\|^2_{L^2(\mathbb{R}^n)} = 0
\]
and the conservation of the momentum
\[
(1.4) \quad P(u(x, t)) = \left( -i \int_{\mathbb{R}^n} \bar{u} \nabla u dx \right).
\]
It is well known that the standing wave \( \psi_\omega(x, t) = e^{i\omega t} Q_\omega(x) \), where \( Q_\omega \) is the
positive ground state solution that solves
\[
(1.5) \quad -\Delta w + \omega w - |w|^{p-1}w = 0,
\]
is solution to (1.1).

On the other hand, thanks to the Galilean transform we know that given \( \psi(t, x) \) an arbitrary solution to (1.1) and \( v \in \mathbb{R}^n \) then
\[
\psi_v(t, x) = \psi(t, x - vt)e^{i\left(\frac{v}{2} \cdot x - \frac{|v|^2}{4} t\right)}
\]
is solution to (1.1). In the specific case of \( \psi_\omega(x, t) \) this implies that
\[
\psi_{Q,v}(x, t) = Q(x - vt)e^{i\left(\frac{v}{2} \cdot x - \frac{|v|^2}{4} t + \omega t\right)}
\]
is the corresponding travelling wave solution to (1.1) that moves on the line \( x = vt \).
Moreover we notice that if \( \psi = e^{i\omega t}w(x) \) is an arbitrary standing wave, namely \( w \) is solution to (1.5), then the energy of the corresponding boosted solution \( \psi_v(t, x) \) fulfills

\[
E(\psi_v) = E(w) + \frac{|v|^2}{8} \int_{\mathbb{R}^n} |w|^2 dx + \frac{i}{2} \int_{\mathbb{R}^n} \bar{w} (v \cdot \nabla w) dx.
\]

Now we notice that the quantity \( i \int_{\mathbb{R}^n} \bar{w} \nabla wdx \) corresponds to \( -P(w) \) and from the Heisenberg relations we have the following identity

\[
\frac{d}{dt} <u(t), Av(t)> = i <u(t), [H, A]u(t)>
\]

where \([H, A] := HA - AH\) denotes the commutator of \( A \) with \( H := -\Delta - |u|^{p-1} \) and \( u(t) \) is an arbitrary solution to (1.1). This relation implies with an elementary computation that choosing \( A = \frac{1}{2} ddt u(t), xu(t) > = P(u) \).

Therefore, fixed \( v \) and \( w \) being a standing wave for which clearly

\[
\frac{1}{2} \frac{d}{dt} <w(t), xw(t)> = 0,
\]

then we obtain

\[
i \int_{\mathbb{R}^n} \bar{w} \nabla wdx = 0.
\]

This implies immediately that \( E(\psi) < E(\psi_v) \). This simple fact proves that fixed the mass of the wave, say \( ||Q\omega(x)||^2_{L^2} = r \), then the least energy solution among the standing waves \( w \) and the corresponding travelling waves \( \psi_v \) with \( ||w||^2_{L^2(\mathbb{R}^n)} = ||\psi_v||^2_{L^2(\mathbb{R}^n)} = r \) is given by \( Q\omega(x)e^{i\omega t} \) where \( \omega \) is fixed by the value of \( r \).

2. Proof of Theorem 0.1

Lemma 2.1. For every \( f \in L^2(\mathbb{R}) \) there exists one unique solution \( u = Af \in H^1(\mathbb{R}) \) to the following linear problem

\[
\left( \sqrt{-\Delta} u + i(v\partial_x) u \right) + (1 - v)u = f.
\]

Moreover we have the following bounds

\[
\|Af\|_{L^2(\mathbb{R})} \leq \alpha_v \|f\|_{H^{-1}(\mathbb{R})},
\]

\[
\|Af\|_{H^{1/2}(\mathbb{R})} \leq \alpha_v \|f\|_{L^{5/4}(\mathbb{R})}
\]

where \( \alpha_v = O\left(\frac{1}{(1-v)}\right) \) as \( v \to 1 \).
Proof. By using the Fourier transform we have
\[ \hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi| - v\xi + 1 - v}. \]

By Plancharel we get
\[ u \in H^1(\mathbb{R}) \iff (1 + |\xi|)\hat{u}(\xi) \in L^2(\mathbb{R}) \iff \frac{(1 + |\xi|)\hat{f}(\xi)}{|\xi| - v\xi + 1 - v} \in L^2(\mathbb{R}), \]

since
\[ 1 + |\xi| \leq \frac{1}{1 - v}. \]

In order to prove the uniform a-priori bound (2.2) notice that by the computation above we get
\[ \|A_vf\|_{H^1(\mathbb{R})} \leq \alpha_v \|f\|_{L^2(\mathbb{R})} \]
where
\[ \alpha_v = \sup_{\xi} \frac{1 + |\xi|}{|\xi| - v\xi + 1 - v} = \frac{1}{1 - v} \]
and hence (since we are working with an operator with constant coefficients)
\[ \|\langle D\rangle^{-1/2}A_vf\|_{H^1(\mathbb{R})} \leq \alpha_v \|\langle D\rangle^{-1/2}f\|_{L^2(\mathbb{R})} \]
where \( \langle D\rangle^{-1/2} \) is the Fourier multiplier associated with \((1 + |\xi|)^{-1/4}\). In turn it implies
\[ \|A_vf\|_{H^{1/2}(\mathbb{R})} \leq \alpha_v \|f\|_{H^{-1/2}(\mathbb{R})} \]
and hence we conclude since by Sobolev embedding \( L^2(\mathbb{R}) \subset H^{-1/2}(\mathbb{R}) \). Moreover by direct computation we get by (2.3) the bound \( \alpha_v = O(1/(1 - v)) \). \( \square \)

Proof of Theorem 0.1.

Let us notice that (0.6) is the Euler-Lagrange equation corresponding to the critical point of the following Weinstein functional
\[ W(u) = \frac{||u||_{L^5(\mathbb{R})}^5}{\left(||u||_{H^{1/2}(\mathbb{R})}^2 + i \int_{\mathbb{R}} \bar{u}(v\partial_x u) \, dx\right)^{\frac{5}{2}}}. \]

Existence of maximizers for the Weinstein functional follows arguing as in Appendix B of [7]. Fixing \( v \) let us call the maximizers \( Q_v \). Moreover arguing as in the proof of Theorem 1.1 of [11] it is easy to show that
\[ iv \int_{\mathbb{R}} \bar{Q}_v \partial_x Q_v \, dx < 0. \]
Now from Gagliardo-Nirenberg inequalities
\[
\|u\|_{L^5(\mathbb{R})}^5 \leq C_0 \|u\|_{H^{1/2}(\mathbb{R})}^3 \|u\|_{L^2(\mathbb{R})}^2
\]
\[
\|u\|_{L^5(\mathbb{R})}^5 \leq C_v \left( \|u\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{u} \partial_x u \, dx \right)^{\frac{2}{v}} \|u\|_{L^2(\mathbb{R})}^2
\]
and choosing \(\psi\) having only positive Fourier components such that
\[
\left( \|\psi\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi} \partial_x \psi \, dx \right) = (1-v)\|\psi\|_{H^{1/2}(\mathbb{R})}^2
\]
we get
\[
(2.4) \quad C_v \geq (1-v)^{-\frac{2}{v}} \frac{\|\psi\|_{L^5(\mathbb{R})}^5}{\left( \|\psi\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi} \partial_x \psi \, dx \right)^{\frac{2}{v}} \|\psi\|_{L^2(\mathbb{R})}^2}.
\]
Upper bound for \(C_v\) can be obtained using the decomposition in positive and negative frequencies
\[
\psi = \psi_+ + \psi_-,
\]
we get
\[
\left( \|\psi_\pm\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi}_\pm \partial_x \psi_\pm \, dx \right) = (1 \mp v)\|\psi_\pm\|_{H^{1/2}(\mathbb{R})}^2
\]
so the standard Gagliardo - Nirenberg inequality
\[
\|\psi\|_{L^5(\mathbb{R})}^5 \leq C \|\psi\|_{H^{1/2}(\mathbb{R})}^3 \|\psi\|_{L^2(\mathbb{R})}^2
\]
and in the case \(\psi = \psi_+\) we find
\[
\|\psi_+\|_{L^5(\mathbb{R})}^5 \leq C(1-v)^{-\frac{2}{v}} ((1-v) |D| \psi_+, \psi_+)^{\frac{2}{v}} \|\psi_+\|_{L^2(\mathbb{R})}^2
\]
\[
= C(1-v)^{-\frac{2}{v}} \left( \|\psi_+\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi}_+ \partial_x \psi_+ \, dx \right)^{3/2} \|\psi_+\|_{L^2(\mathbb{R})}^2.
\]
In a similar way we find
\[
\|\psi_-\|_{L^5(\mathbb{R})}^5 \leq C(1+v)^{-\frac{2}{v}} \left( \|\psi_-\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi}_- \partial_x \psi_- \, dx \right)^{3/2} \|\psi_-\|_{L^2(\mathbb{R})}^2
\]
so we can conclude that
\[
\|\psi\|_{L^5(\mathbb{R})}^5 \leq C \left( \|\psi_+\|_{L^5(\mathbb{R})}^5 + \|\psi_-\|_{L^5(\mathbb{R})}^5 \right) \leq C(1-v)^{-\frac{2}{v}} \left( \|\psi_+\|_{H^{1/2}(\mathbb{R})}^2 + iv \int_{\mathbb{R}} \bar{\psi}_+ \partial_x \psi_+ \, dx \right)^{3/2} \|\psi_+\|_{L^2(\mathbb{R})}^2.
\]
By the fact that

\[ Q \]

So we obtain

\[ (2.7) \]

and therefore a Pohozaev type identity yields to

\[ (2.6) \]

\[ \psi \]

for any

\[ Q \]

\[ C \]

\[ (2.5) \]

To this end we use the Plancherel identity and get

\[ A + B = \langle (|\xi| - v\xi)\hat{\psi}_+,\hat{\psi}_+ \rangle_{L^2(\mathbb{R})} + \langle (|\xi| - v\xi)\hat{\psi}_-,\hat{\psi}_- \rangle_{L^2(\mathbb{R})} = \]

\[ \langle (|\xi| - v\xi)\hat{\psi},\hat{\psi} \rangle_{L^2(\mathbb{R})} = ||\psi||^2_{H^{\frac{1}{2}}(\mathbb{R})} + iv \int_{\mathbb{R}} \hat{\psi} \partial_x \psi dx. \]

So we obtain

\[ ||\psi||^5_{L^5(\mathbb{R})} \leq C(1 - v)^{-\frac{3}{2}} \left( ||\psi||^2_{H^{\frac{1}{2}}(\mathbb{R})} + iv \int_{\mathbb{R}} \hat{\psi} \partial_x \psi dx \right)^\frac{3}{2} ||\psi||^2_{L^2(\mathbb{R})} \]

for any \( \psi \in H^{\frac{1}{2}}(\mathbb{R}) \) and we have

\[ (2.5) \]

\[ C_v \lesssim (1 - v)^{-\frac{3}{2}}. \]

Now taking \( Q_v \) maximizer for \( W \), we can scale \( Q_v \rightarrow aQ_v(bx) \) such that \( Q_v \) solves

\[ (2.6) \]

\[ \sqrt{-\Delta}Q_v + iv\partial_x Q_v + (1 - v)Q_v - |Q_v|^3 Q_v = 0. \]

Notice that \( Q_v \) is now a critical point also of the following functional

\[ E_{\text{boost}}(u) = E_{hw}(u) + \frac{1}{2}(1 - v)||u||^2_{L^2(\mathbb{R})} + \frac{i}{2} \int_{\mathbb{R}} v^2(\partial_x u) dx, \]

and therefore a Pohozaev type identity yields to

\[ (2.7) \]

\[ \frac{1}{2}||Q_v||^2_{H^{1/2}(\mathbb{R})} - \frac{3}{10}||Q_v||^5_{L^5(\mathbb{R})} + \frac{i}{2}v \int_{\mathbb{R}} \hat{Q}_v \partial_x Q_v dx = 0. \]

By the fact that \( Q_v \) solves \((2.6)\) also

\[ \frac{1}{2}||Q_v||^2_{H^{1/2}(\mathbb{R})} + (1 - v)||Q_v||^2_{L^2(\mathbb{R})} - \frac{5}{2}v \int_{\mathbb{R}} \hat{Q}_v \partial_x Q_v dx = 0. \]

we can conclude that

\[ \left( \frac{1}{2}||Q_v||^2_{H^{1/2}(\mathbb{R})} + \frac{i}{2}v \int_{\mathbb{R}} \hat{Q}_v \partial_x Q_v dx \right) = \frac{3}{2}(1 - v)||Q_v||^2_{L^2(\mathbb{R})}, \]

\[ (2.8) \]

\[ ||Q_v||^5_{L^5(\mathbb{R})} = \frac{5}{2}(1 - v)||Q_v||^2_{L^2(\mathbb{R})}. \]

By recalling that

\[ C_v = \frac{||Q_v||^5_{L^5(\mathbb{R})}}{\left( \frac{1}{2}||Q_v||^2_{H^{1/2}(\mathbb{R})} + \frac{i}{2}v \int_{\mathbb{R}} \hat{Q}_v \partial_x Q_v dx \right)^\frac{3}{2} ||Q_v||^2_{L^2(\mathbb{R})}} \]
we get
\[ C_v = \frac{5(1 - v)}{2 \left( \|Q_v\|_{H^1}^2 + iv \int_{\mathbb{R}} \bar{Q}_v \partial_x Q_v \, dx \right)^{\frac{3}{2}}} \]
and
\[ C'_v = \frac{\frac{5}{2} \left( \frac{2}{3} \right)^{\frac{3}{2}}}{(1 - v)^{\frac{3}{2}} \|Q_v\|_{L^2(\mathbb{R})}^3}. \]
Together with (2.5) we conclude that
\[ \lim_{v \to 1} \|Q_v\|_{L^2(\mathbb{R})} = 0, \quad \lim_{v \to 1} \left( \|Q_v\|_{H^1}^2 + iv \int_{\mathbb{R}} \bar{Q}_v \partial_x Q_v \, dx \right) = 0. \]
We choose in Lemma 2.1 the forcing term \( f = |Q_v|^3 \) and hence by looking at the equation solved by \( Q_v \) we get
\[ A_v(|Q_v|^3) = Q_v \]
and hence again by Lemma 2.1 we get
\[ \|Q_v\|_{H^{1/2}(\mathbb{R})} \leq \alpha_v \|Q_v\|_{L^5(\mathbb{R})}^4 \leq C \frac{(1 - v)^{20}}{(1 - v)^{\frac{10}{3}}} = O((1 - v)^{\frac{1}{3}}). \]
From the fact that \( C_v \geq C(1 - v)^{-\frac{3}{2}} \) it follows that
\[ \|Q_v\|_{L^2(\mathbb{R})}^2 = O((1 - v)^{\frac{3}{2}}), \quad \|Q_v\|_{L^5(\mathbb{R})}^5 = O((1 - v)^{\frac{5}{3}}). \]
Since \( \alpha_v = O(\frac{1}{1 - v}) \) then we conclude
\[ \|Q_v\|_{H^{1/2}(\mathbb{R})} \leq \alpha_v \|Q_v\|_{L^5(\mathbb{R})}^4 \leq C \frac{(1 - v)^{20}}{(1 - v)^{\frac{10}{3}}} = O((1 - v)^{\frac{1}{3}}). \]
To estimate \( \|Q_v\|_{H^1(\mathbb{R})} \) we rewrite (0.6) in the form
\[ (2.10) \quad \sqrt{-\Delta} Q_v + iv \partial_x Q_v = -(1 - v)Q_v + |Q_v|^3Q_v, \]
and note that
\[ \left\| \sqrt{-\Delta} Q_v + iv \partial_x Q_v \right\|_{L^2(\mathbb{R})} \geq (1 - v)\|Q_v\|_{H^1(\mathbb{R})}. \]
We can use the Sobolev inequality
\[ \|Q_v\|_{L^8(\mathbb{R})} \leq C \|Q_v\|_{H^{1/2}(\mathbb{R})} = O((1 - v)^{1/3}) \]
estimating the \( L^2 - \) norm of the right side of (2.10) and deducing
\[ \|Q_v\|_{H^1(\mathbb{R})} \leq C(1 - v)^{1/3}. \]
Our final step is to take \( 0 < v_1 < v_2 < 1 \) with
\[ v_1 = 1 - \varepsilon, \quad v_2 = v_1 + \delta, \quad 0 < \delta \ll \varepsilon \ll 1, \]
and prove the inequality (0.8). For the purpose, we set

\[ R = Q_{v_1} - Q_{v_2} \]

so that it is solution to

\[
(2.11)\quad \sqrt{-\Delta} R + (1 - v_1)R + iv_1 \partial_x R = -\delta Q_{v_2} + i\delta \partial_x Q_{v_2} + \left( \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right) R.
\]

By using Lemma 2.1 we rewrite the above equation as

\[ R = A_{v_1} (F(Q_{v_1}, Q_{v_2})) \]

so taking the \( L^2 \)- norm, we can write

\[
\|R\|_{L^2} \leq \|A_{v_1} (\delta Q_{v_2})\|_{L^2} + \|A_{v_1} (\delta \partial_x Q_{v_2})\|_{L^2(\mathbb{R})} +
\]

\[
+ \left\| A_{v_1} \left[ \left( \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right) R \right] \right\|_{L^2(\mathbb{R})}.
\]

and using (2.1) we get the inequalities

\[
\|A_{v_1} (\delta Q_{v_2})\|_{L^2(\mathbb{R})} \leq C\delta \|Q_{v_2}\|_{L^2(\mathbb{R})} \leq C\delta (1 - v_1)^{1/3}
\]

and

\[
\|A_{v_1} (\delta \partial_x Q_{v_2})\|_{L^2(\mathbb{R})} = \|A_{v_1} (\delta Q_{v_2})\|_{H^1(\mathbb{R})} \leq C\delta \|Q_{v_2}\|_{L^2(\mathbb{R})} \leq C\delta (1 - v_1)^{1/3}.
\]

Further, we use the pointwise bound

\[
\left| \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right| \leq C \left( |Q_{v_1}|^3 + |Q_{v_2}|^3 \right).
\]

Using (2.1), we find

\[
\left\| A_{v_1} \left[ \left( \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right) R \right] \right\|_{L^2(\mathbb{R})} \leq
\]

\[
\leq C \left\| \left( \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right) R \right\|_{H^{-1}(\mathbb{R})} \leq
\]

\[
\leq C \left\| \left( \frac{Q_{v_1}|Q_{v_1}|^3 - Q_{v_2}|Q_{v_2}|^3}{Q_{v_1} - Q_{v_2}} \right) R \right\|_{L^1(\mathbb{R})} \leq
\]

\[
\leq C \left( \|Q_{v_1}\|_{L^6(\mathbb{R})}^3 + \|Q_{v_2}\|_{L^6(\mathbb{R})}^3 \right) R_{L^2(\mathbb{R})} \leq
\]

\[
\leq C \left( \|Q_{v_1}\|_{L^6(\mathbb{R})}^3 + \|Q_{v_2}\|_{L^6(\mathbb{R})}^3 \right) \|R\|_{L^2(\mathbb{R})}.
\]

Applying the Sobolev embedding once more, we get

\[
\|Q_{v_1}\|_{L^6(\mathbb{R})}^3 + \|Q_{v_2}\|_{L^6(\mathbb{R})}^3 \leq C \left( \|Q_{v_1}\|_{H^{1/2}(\mathbb{R})}^3 + \|Q_{v_2}\|_{H^{1/2}(\mathbb{R})}^3 \right) \leq C(1 - v_1).
\]
Summing up, the above estimates lead to the inequality
\[ \|R\|_{L^2(\mathbb{R})} \leq C\delta(1 - v_1)^{1/3} + C(1 - v_1)\|R\|_{L^2(\mathbb{R})} \]
so choosing \( 1 - v_1 = \varepsilon \) so small that \( C\varepsilon < 1/2 \), we find
\[ \|R\|_{L^2(\mathbb{R})} \leq C\delta(1 - v_1)^{1/3} \]
and this completes the proof of (0.8).

Now we show that small data scattering does not occur. More precisely we will show that if the wave operators exist they are not continuous. For any 0 < \( v_1 < v_2 < 1 \) let \( \psi_j(t, x) = e^{i(1 - v_j)t}Q_{v_j}(x - v_j t) \) where \( Q_{v_1}, Q_{v_2} \in H^{1/2}(\mathbb{R}) \) are the traveling waves constructed above. Then we show that
\[ ||\psi_1(t) - \psi_2(t)||_{L^2(\mathbb{R})} \sim ||\psi_1(t)||_{L^2(\mathbb{R})}^2 + ||\psi_2(t)||_{L^2(\mathbb{R})}^2. \] (2.12)

We can show that for any \( \varepsilon > 0 \) and any \( f, g \in L^2(\mathbb{R}) \) with \( ||f||_{L^2(\mathbb{R})} = ||g||_{L^2(\mathbb{R})} \) exists \( \tau_0 = \tau(\varepsilon, f, g) \) so that
\[ \left| \int_{\mathbb{R}} f(x)g(x + \tau)dx \right| \leq \varepsilon, \quad \forall \tau \geq \tau_0. \] (2.13)
Indeed, this is a consequence of the fact that the weak limit of \( \{g(\cdot + \tau)\}_{\tau \to \infty} \) has to be zero. The relation (2.12) follows from
\[ \left| e^{i(1-v_1)t} - e^{-i(1-v_2)t} \int_{\mathbb{R}} Q_{v_1}(x - v_1 t)Q_{v_2}(x - v_2 t)dx \right| = \left| \int_{\mathbb{R}} Q_{v_1}(y)Q_{v_2}(y - (v_2 - v_1)t)dy \right| \]
so choosing
\[ f(x) = \frac{Q_{v_1}(x)}{||Q_{v_1}||_{L^2(\mathbb{R})}}, \quad g(x) = \frac{Q_{v_2}(x)}{||Q_{v_2}||_{L^2(\mathbb{R})}} \]
and applying (2.13) we get
\[ ||\psi_1(t) - \psi_2(t)||_{L^2(\mathbb{R})}^2 \leq ||\psi_1(t)||_{L^2(\mathbb{R})}^2 + ||\psi_2(t)||_{L^2(\mathbb{R})}^2 - 2Re(\psi_1(t), \psi_2(t)) \geq ||\psi_1(t)||_{L^2(\mathbb{R})}^2 + ||\psi_2(t)||_{L^2(\mathbb{R})}^2 - \varepsilon ||\psi_1(t)||_{L^2(\mathbb{R})}||\psi_2(t)||_{L^2(\mathbb{R})} \geq \frac{1}{2}||\psi_1(t)||_{L^2(\mathbb{R})}^2 + \frac{1}{2}||\psi_2(t)||_{L^2(\mathbb{R})}^2. \]
and this completes the proof. If the wave operators exist and they are \( L^2 \) bounded, then
\[ \psi_1(0, x) = \psi_1(x) \to W(\psi_1(x)) = \phi_1, \psi_2(0, x) = \psi_2(x) \to W(\psi_2(x)) = \phi_2 \]
satisfy
\[ \lim_{t \to \infty} ||\psi_1(t) - e^{itD}\phi_1||_{L^2(\mathbb{R})} = \lim_{t \to \infty} ||\psi_2(t) - e^{itD}\phi_2||_{L^2(\mathbb{R})} = 0 \]
and since
\[ \| e^{itD} \phi_1 - e^{itD} \phi_2 \|_{L^2(\mathbb{R})} = \| \phi_1 - \phi_2 \|_{L^2(\mathbb{R})}, \]
we would have
\[ (2.14) \quad \| \psi_1(t) - \psi_2(t) \|_{L^2(\mathbb{R})} \leq C \| \phi_1 - \phi_2 \|_{L^2(\mathbb{R})}. \]
From this estimate we easily get a contradiction referring to (2.12) and choosing \( v_2 = v_1 + \delta, \delta \ll 1 - v_1 \). Indeed, using (0.8) we can write
\[ \| Qv_1 - Qv_2 \|_{L^2(\mathbb{R})} \leq C \delta (1 - v_1)^{\frac{1}{3}}, \]
and
\[ \| \psi_1(t) - \psi_2(t) \|_{L^2(\mathbb{R})} \sim \| \psi_1(t) \|_{L^2(\mathbb{R})}^{\frac{2}{3}} + \| \psi_2(t) \|_{L^2(\mathbb{R})}^{\frac{2}{3}} = \| Qv_1 \|_{L^2(\mathbb{R})}^{\frac{2}{3}} + \| Qv_2 \|_{L^2(\mathbb{R})}^{\frac{2}{3}} \geq C (1 - v_1)^{2/3}. \]
Choosing
\[ (1 - v_1)^{2/3} \gg \delta^2 (1 - v_1)^{2/3} \iff \delta \ll 1, \]
we get
\[ \| \psi_1(t) - \psi_2(t) \|_{L^2(\mathbb{R})} \gg \| Qv_1 - Qv_2 \|_{L^2(\mathbb{R})} \]
and this contradicts (2.14).

\[ \square \]

3. Nonexistence of traveling waves moving at the speed of light

**Lemma 3.1** (Spatial decay for traveling waves for HW). Let \( v = 1 \) and \( Q_1 \in H^\frac{1}{2} (\mathbb{R}) \) be a solution to
\[ (3.1) \quad \sqrt{-\Delta} Q_1 + i \partial_x Q_1 + \omega Q_1 - |Q_1|^3 Q_1 = 0, \]
then
\[ |Q_1(x)| \leq \frac{C}{1 + |x|^2}. \]

**Proof.** It is evident that if \( Q_1 \) is solution to (3.1) then \( Q_1 \in H^1(\mathbb{R}) \) and therefore \( \lim_{|x| \to \infty} |Q_1(x)| = 0 \) and \( Q_1 \) is bounded. The decay for traveling waves moving with speed \( |v| < 1 \) has been proved in [3]. Here we prove that if traveling waves moving at the speed of light exist then they shall fulfill a certain asymptotics. By scaling argument it is clear that \( \omega > 0 \), see e.g. (2.8). For simplicity and by scaling we consider (3.1) in the case \( \omega = 1 \). We recall that an harmonic extension for a smooth function \( f : \mathbb{R} \to \mathbb{C} \) is given by a function \( F : \mathbb{R}^2_+ = \mathbb{R} \times [0, \infty) \to \mathbb{C} \) fulfilling
\[ (3.2) \quad \partial_x^2 F + \partial_y^2 F = 0 \text{ on } \mathbb{R}^2_+ \quad F(x, 0) = f(x). \]
It is elementary to notice (using that \( F \) is harmonic in \( \mathbb{R}^2_+ \)) that
\[ -\partial_y F(x, 0) = \sqrt{-\Delta} f(x). \]
Now, given $Q_1$ solution to \eqref{eq:3.1} with $\omega = 1$ we consider $q_1(x,y)$ its extension to $\mathbb{R}^2_+$ fulfilling
\begin{align}
\partial_x^2 q_1 + \partial_y^2 q_1 &= 0 \text{ on } \mathbb{R}^2_+ \tag{3.3} \\
\partial_y q_1 - i\partial_x q_1 &= q_1 - |q_1|^3 q_1 \text{ on } (x,0) \tag{3.4} \\
\lim_{|x| \to \infty} |q_1(x,0)| &= 0. \tag{3.5}
\end{align}

Our idea it to deduce the decay of $Q_1$ by looking at the decay of the solution $q_1(x,y)$ to the extension problem. The case $v = 0$ it has been studied by Amick-Toland \cite{1} and Kenig-Martel-Robbiano \cite{9}. Consider hence the boundary value problem
\begin{align}
\partial_x^2 w + \partial_y^2 w &= 0 \text{ on } \mathbb{R}^2_+ \tag{3.6} \\
w - \partial_y w + iv\partial_x w &= f \text{ on } (x,0) \tag{3.7} \\
\lim_{|x| \to \infty} |w(x)| &= 0. \tag{3.8}
\end{align}

The crucial point is to show the existence of a function $G(x,y)$ such that
\begin{enumerate}
\item $G$ is harmonic on $\mathbb{R}^2_+$
\item $G - \partial_y G + i\partial_x G = \frac{1}{\pi x + y^2} \equiv b_0 \left( \frac{x}{y} \right)$,
\end{enumerate}
where
\[ b_0(x) = \frac{1}{\pi (1 + x^2)}. \]

Indeed, if $G$ fulfills (1)-(2) then
\begin{equation}
\label{eq:3.9}
w(x,y) = \int_{\mathbb{R}} G(x,t,y)f(t)dt
\end{equation}
is solution to \eqref{eq:3.6}, \eqref{eq:3.7}, \eqref{eq:3.8}.

Concerning \eqref{eq:3.3}, \eqref{eq:3.4}, \eqref{eq:3.5}, by choosing $f = -|q_1|^3 q_1$ we deduce from \eqref{eq:3.9} that $q_v$ fulfills
\begin{equation}
\label{eq:3.10}
q_1(x,y) = \int_{\mathbb{R}} G(x,t,y)|q_1|^3 q_1(t)dt.
\end{equation}

From this last estimate we shall deduce the decay of $q_v$ using the information about the decay of $G$.

In the case $v = 0$ studied in \cite{1} the function $G$ is explicitly given by
\[ G(x,y) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-s}(y+s)}{x^2 + (y+s)^2} ds. \]

Now if we apply inverse Fourier transform in the variable $x$ we get
We look for solution in the form
\[ \hat{G}(\xi, y) = e^{-y|\xi|}C(\xi), \]
where \( C(\xi) \) is defined by the boundary condition
\[ \left( \hat{G} - \partial_y \hat{G} + \xi \hat{G} \right) \bigg|_{y=0} = 1, \]
i.e.
\[ C(\xi) = \frac{1}{1 + |\xi| + \xi}. \]
Hence
\[ G(x, y) = \int_{-\infty}^{\infty} e^{-\gamma|\xi|} e^{-2\pi ix \xi} \frac{d\xi}{(1 + |\xi| + \xi)}. \]
By fundamental theorem of calculus we obtain immediately that
\[ I_1 = \left( \frac{1}{y - 2\pi i x} \right) \int_{-\infty}^{0} \frac{d}{d\xi} \left( e^{\gamma\xi} e^{-2\pi ix \xi} \right) d\xi = \frac{1}{y - 2\pi i x}. \]
By integration by parts we get
\[ I_2 = \left( \frac{1}{y + 2\pi i x} \right) \int_{0}^{\infty} \frac{d}{d\xi} \left( e^{-\gamma\xi} e^{-2\pi ix \xi} \right) \frac{d\xi}{(1 + 2\xi)^2} \]
such that we obtain
\[ G(x, y) = \left( \frac{2y}{y^2 + 4\pi^2 x^2} \right) + \left( \frac{2}{y - 2\pi i x} \right) \int_{0}^{\infty} e^{-\gamma\xi} e^{-2\pi ix \xi} \frac{d\xi}{(1 + 2\xi)^2} \]
that thanks to a second integration by parts can be estimated as
\[ G(x, y) = \left( \frac{2y}{y^2 + 4\pi^2 x^2} \right) + \left( \frac{2}{y + 2\pi i x} \right) \int_{0}^{\infty} e^{-\gamma\xi} e^{-2\pi ix \xi} \frac{d\xi}{(1 + 2\xi)^3}. \]
Eq. (3.14) implies the following decay
\[ |G(x, y)| \leq C \left( \frac{1 + y}{y^2 + 4\pi^2 x^2} \right). \]
Now recalling (3.10)

\[ q_1(x, y) = \int_{\mathbb{R}} G(x - t, y)|q_1|^3q_1(t)dt, \]

and following verbatim the proof at pag. 24 in [2] we obtain

\[ |Q_1(x)| = |q_1(x, 0)| \leq \frac{C}{1 + |x|^2} \]

and hence the desired decay. We give a brief sketch of the argument of [2] for reader’s convenience.

Given \( \delta > 0 \), let \( X(\delta) > 0 \) such that \( |u(x)| < \delta \) if \( |x| > X(\delta) \). Now, let us define, for \( \alpha = 0 \) or \( \alpha = 2 \), the Banach space \( C^\alpha_{\delta} \) defined by continuous functions defined on \( \{ x \in \mathbb{R}, \ |x| \geq X(\delta) \} \) such that

\[ ||f||_{\alpha, \delta} = \sup \{ (1 + |x|^\alpha)|f(x)|, \ |x| \geq X(\delta) \} < +\infty. \]

Now thanks to (3.10) we have

\[ q_1(x, 0) = \int_{|x|<X(\delta)} G(x - t, 0)|q_1|^3q_1(t)dt + \int_{|x| \geq X(\delta)} G(x - t, 0)|q_1|^3q_1(t)dt. \]

Defining the operator \( T_\delta \) defined on \( C^\alpha_{\delta} \) as

\[ T_\delta g(x) = \int_{|x| \geq X(\delta)} G(x - t, 0)|q_1|^3g(t)dt, \]

we have the elementary estimates

\[ |A_\delta(x)| \leq \text{const} \left( \frac{1}{1 + x^2} \right), \ |T_\delta g(x)| \leq \text{const} \cdot \delta^3 \left( \frac{||g||_{\alpha, \delta}}{1 + |x|^\alpha} \right) \]

By the fact that \( q_1(x, 0) \neq 0 \) we can choose \( \delta \) sufficiently small such that \( A_\delta \neq 0 \). Hence, by the contraction mapping principle, the equation \( g = A_\delta + T_\delta g \) has a unique solution in \( C^\alpha_{\delta} \) for \( \delta \) sufficiently small. By the fact that \( q_1(x, 0) \) fulfills (3.10) we get the desired decay.

\[ \square \]

Proof of Theorem 0.2.

By Heisenberg relations we have

\[ \frac{d}{dt} <u, Au> = i <u, [\mathcal{H}, A]u> \]

where \([\mathcal{H}, A] = \mathcal{H}A - A\mathcal{H}\) denotes the commutator and \( \mathcal{H} \) is the time dependent operator \( \sqrt{-\Delta - |u|^3} \).

We recall the definition of the Hilbert transform \( H \) defined

\[ H(f)(x) = \int_P v \left( \frac{1}{x - y} \right) f(y)dy = \int_0^\infty \frac{f(x - y) - f(x + y)}{y} dy, \]
and fulfilling in Fourier variables $\hat{H}f(\xi) = -i\pi \text{sign}(\xi)\hat{f}(\xi)$. We shall use the following commutator relations
\begin{equation}
[AB, C] = A[B, C] + [A, C]B.
\end{equation}
and
\begin{equation}
[x, D_x] = \frac{1}{i}[x, \partial_x] = i,
\end{equation}
\begin{equation}
[x, H](f)(x) = \int (x - y)Pv\left(\frac{1}{x - y}\right)f(y)dy = \int f(y)dy,
\end{equation}
where
\begin{equation}
\partial_x[x, H] = [x, H]\partial_x = 0,
\end{equation}
\begin{equation}
[x, \sqrt{-\Delta}] = \frac{1}{i}[x, \partial_x H] = \frac{1}{\pi}[x, \partial_x]H + \frac{1}{\pi}[x, H]\partial_x = -\frac{H}{\pi}.
\end{equation}

Therefore from Heisenberg relation and using commutator relations we get
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}} x|u(x, t)|^2dx = i < u, [\sqrt{-\Delta}, x]u >
\end{equation}
and we conclude thanks to Cauchy-Schwarz and Plancherel that
\begin{equation}
\left| \frac{d}{dt} \int_{\mathbb{R}} x|u(x, t)|^2dx \right| = \left| < \frac{H(u)}{\pi}, u > \right| \leq \frac{1}{\pi}||u||_{L^2(\mathbb{R})}||H(u)||_{L^2(\mathbb{R})} = ||u||_{L^2(\mathbb{R})}^2.
\end{equation}
By applying (3.18) to $u(x, t) = e^{i\omega t}Q_v(x - vt)$ and thanks to Lemma 3.1 we get
\begin{equation}
\left| \frac{d}{dt} \int_{\mathbb{R}} x|Q_v(x - vt)|^2dx \right| = \left| v \int_{\mathbb{R}} |Q_v(x)|^2dx \right| \leq ||Q_v||_{L^2(\mathbb{R})}^2,
\end{equation}
and hence
\begin{equation}
|v| \leq 1.
\end{equation}
By observing that equality in Cauchy-Schwarz means that $u$ and $\frac{H(u)}{\pi}$ are parallel we deduce that if a travelling wave moves at speed $|v| = 1$, i.e. at the speed of light, therefore supp $\hat{Q}_v \subset (-\infty, 0)$ or supp $\hat{Q}_v \subset [0, \infty)$.

By elementary scaling arguments it is evident that when $v = 1$ then supp $\hat{Q}_v \subset (-\infty, 0]$ and when $v = -1$ then supp $\hat{Q}_v \subset [0, \infty)$. Let us notice that the equation fulfilled by the traveling wave moving at the speed of light
\begin{equation}
\sqrt{-\Delta}Q_1 + i\partial_x Q_1 + Q_1 - |Q_1|^3Q_1 = 0,
\end{equation}
thanks to the fact that supp $\hat{Q}_1 \subset (-\infty, 0]$, can be rewritten as
\begin{equation}
-2\partial_x Q_1 + iQ_1 = i|Q_1|^3Q_1,
\end{equation}
and
\begin{equation}
2\partial_x \bar{Q}_1 + i\bar{Q}_1 = i|Q_1|^3\bar{Q}_1.
\end{equation}
Now thanks to (3.19) and (3.20) we conclude that
\[
\partial_x (|Q_1|^2) = \partial_x (Q_1) \bar{Q}_1 + \partial_x (\bar{Q}_1) Q_1 = 0,
\]
which implies that \( Q_1(x) = c \) and hence \( Q_1 \equiv 0 \).
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