MASSEY PRODUCTS IN COHOMOLOGY OF MOMENT-ANGLE MANIFOLDS CORRESPONDING TO POGORELOV POLYTOPES

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Abstract. In this work we construct nontrivial Massey products in the cohomology of moment-angle manifolds corresponding to polytopes from the Pogorelov class. This class includes the dodecahedron and all fullerenes, i.e., simple 3-polytopes with only 5-gonal and 6-gonal facets. The existence of a nontrivial Massey product implies the nonformality of the space in the sense of rational homotopy theory.

Keywords: Massey products, moment-angle complexes, Pogorelov polytopes, fullerenes.

1. Introduction

In this paper we consider the problem of existence of nontrivial Massey products in the cohomology of moment-angle manifolds corresponding to 3-dimensional simple polytopes $P$. As shown in [1], these manifolds $Z_P$ are 2-connected smooth manifolds of dimension $m + 3$, where $m$ is the number of facets in $P$. First examples of moment-angle manifolds having nontrivial Massey products were found by Baskakov (see [2]). Limonchenko constructed in [3] a family of moment-angle manifolds having nontrivial Massey $n$-products for any $n$.

In dimension 3 the Pogorelov class of simple polytopes is particularly interesting. This class consists of combinatorial 3-dimensional simple polytopes which do not have 3-belts and 4-belts of facets. It is known that the Pogorelov class consists precisely of combinatorial 3-polytopes which admit a right-angled realization in Lobachevsky space $L^3$, and such a realization is unique up to isometry (see [4], [5], [6]). There is a family of hyperbolic 3-manifolds associated with Pogorelov polytopes, known as hyperbolic manifolds of Löbell type (see [7]). Moment-angle manifolds corresponding to Pogorelov polytopes are important for the topological study of hyperbolic manifolds of Löbell type, and also for cohomological rigidity of 6-dimensional (quasi)toric manifolds.

It is known that there are no triple Massey products of 3-dimensional classes in cohomology of moment-angle manifolds corresponding to Pogorelov polytopes (see [6]). For cohomology classes of higher dimension, the existence of Massey products was open. We prove that for any Pogorelov polytope $P$ the corresponding moment-angle manifold $Z_P$ has a nontrivial triple Massey product in the cohomology. This implies that all such manifolds $Z_P$ are non-formal.

Our construction of nontrivial Massey products is based on the combinatorial description of the cohomology of moment-angle complexes and certain combinatorial properties of Pogorelov polytopes. The Pogorelov class contains all fullerenes (simple polytopes with only 5-gonal and 6-gonal facets), in particular, the dodecahedron. In this paper we also consider a particular case of dodecahedron.

The work was supported by the Russian Foundation for Basic Research (grant no. 18-51-50005).
The author is grateful to the advisor Taras Panov for the formulation of the problem and for his permanent attention to this work.

2. Preliminaries

Let $A = \bigoplus_{i \geq 0} A^i$ be a commutative differential graded algebra over $\mathbb{Z}$. Let $\alpha_i \in H^{k_i}(A), i = 1, 2, 3,$ be three cohomology classes such that $\alpha_1 \alpha_2 = 0, \alpha_2 \alpha_3 = 0 \in H(A)$. Choose their representing cocycles $a_i \in A^{k_i}, i = 1, 2, 3$. Since the pairwise cohomology products vanish, there are elements $a_{12} \in A^{k_1+k_2-1}$ and $a_{23} \in A^{k_2+k_3-1}$ such that $da_{12} = a_1 a_2$ and $da_{23} = a_2 a_3$. Then one easily checks that

$$b = (-1)^{k_1+1}a_1 a_{23} + a_{12} a_3$$

is a cocycle in $A^{k_1+k_2+k_3-1}$.

A triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is the set in $H^{k_1+k_2+k_3-1}(A)$ consisting of all elements obtained by this procedure. Since elements $a_{12}$ and $a_{23}$ are defined up to addition of cocycles in $A^{k_1+k_2-1}$ and $A^{k_2+k_3-1}$ respectively, then, more precisely, we have

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \{ b \} + \alpha_1 H^{k_2+k_3-1} + \alpha_3 H^{k_1+k_2-1}.$$  

The subset $\alpha_1 H^{k_2+k_3-1} + \alpha_3 H^{k_1+k_2-1}$ is called the indeterminacy of a Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

A Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is called trivial if $0 \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and nontrivial otherwise.

Let $K$ be a simplicial complex on the set $[m] = \{1, \ldots, m\}$. The moment-angle complex (see [1]) corresponding to a simplicial complex $K$ is a topological space defined by

$$Z_K = \bigcup_{I \in K} (\prod_{i \notin I} D^2) \times (\prod_{i \in I} S^1) \subseteq (D^2)^m.$$  

An important class of simplicial complexes $K$ comes from simple polytopes. Recall that an $n$-dimensional polytope $P$ is called simple if exactly $n$ facets meet at each vertex. Denote by $K_P$ the simplicial complex dual to the boundary of a simple polytope $P$. In more detail, if $\{F_1, \ldots, F_m\}$ is the set of faces of codimension $1$ in $P$, then

$$K_P = \{ \{i_1, \ldots, i_k\} \subseteq [m] : F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset \}.$$  

Note that $K_P$ is a triangulation of the $(n-1)$-dimensional sphere.

**Theorem 2.1** ([1] Theorem 6.2.4, Corollary 6.2.5). $Z_K$ is a CW-complex. If $K = K_P$ for a simple $n$-polytope $P$, then $Z_{K_P}$ is a smooth manifold.

Let $I = \{i_1, \ldots, i_k\}$ be a simplex, $i_1 < i_2 < \cdots < i_k$. Denote by $v_I$ the monomials $v_{i_1} \cdots v_{i_k}$ in the polynomial algebra $\mathbb{Z}[v_1, \ldots, v_m]$, and denote by $u_I$ the exterior monomial $u_{i_1} \cdots u_{i_k}$ in the exterior algebra $\Lambda[u_1, \ldots, u_m]$. The face ring of the simplicial complex $K$ on the set $[m]$ is defined as the quotient of the polynomial algebra by the monomial ideal corresponding to non-simplices of $K$:

$$Z[K] = \mathbb{Z}[v_1, \ldots, v_m]/I_K,$$

where $I_K = (v_I : I \notin K)$ is the Stanley-Reisner ideal.

We define the quotient algebra

$$R^*(K) = \Lambda[u_1, \ldots, u_m] \otimes Z[K]/(v_i^2 = u_i v_i = 0, 1 \leq i \leq m).$$
Then $R^* (K)$ is a bigraded differential algebra with an additive basis $\{ u_J v_I \}$, where $I \in K$, $J \subseteq [m]$, $I \cap J = \emptyset$;

\[
\text{bideg } u_i = (-1, 2), \text{ bideg } v_i = (0, 2), \text{ } du_i = v_i, \text{ } dv_i = 0.
\]

It is convenient to consider the $\mathbb{Z} \oplus \mathbb{Z}^m$-grading on $R^* (K)$:

\[
\text{mdeg } u_i = (-1; 2e_i), \text{ mdeg } v_i = (0; 2e_i),
\]

where $e_i$, $i = 1, \ldots, m$, are the elements of the standard basis in $\mathbb{Z}^m$.

The multigrading of the algebra of cellular cochains $C^* (Z_K)$ induced from the standard cell decomposition plays an important role in the proof of the existence of a nontrivial Massey product. Consequently, we have the multigrading of the algebra $H^* (Z_K)$.

**Theorem 2.2** ([3] Lemma 4.5.3]). There is an isomorphism of dg-algebras:

\[
R^* (K) \cong C^* (Z_K),
\]

where $C^* (Z_K)$ is the algebra of cellular cochains with a natural multiplication inducing the standard product in cohomology.

Given $J \subseteq [m]$, define the corresponding full subcomplex $K_J$ of a simplicial complex $K$ as

\[
K_J = \{ I \in K \mid I \subseteq J \}.
\]

For each $K_J$ we consider the simplicial cochain complex $(C^* (K_J), d)$ with the coaugmentation. The group $C^p (K_J)$ is a free abelian group with a basis of cochains $\chi_L$, where $\chi_L$ is the characteristic function of a simplex $L \in K_J$, $|L| = p + 1$.

**Theorem 2.3** ([3] Theorem 3.2.4]). There is an isomorphism of cochain complexes $(C^* (K_J), d)$ and $(R^* - |J| + 1, 2J (K), d)$:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \xrightarrow{d} & C^0 (K_J) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{p-1} (K_J) & \xrightarrow{d} & \cdots \\
& \xrightarrow{f_0} & 0 & \xrightarrow{f_0} & 0 & \xrightarrow{f_0} & \cdots & \xrightarrow{f_0} & 0 & \xrightarrow{f_0} & \cdots \\
0 & \to & R^* - |J|, 2J & \xrightarrow{d} & R^* - |J| + 1, 2J & \xrightarrow{d} & \cdots & \xrightarrow{d} & R^* - |J| + p, 2J & \xrightarrow{d} & \cdots \\
& \xrightarrow{f_p (\chi_L)} & 0 & \xrightarrow{f_p (\chi_L)} & 0 & \xrightarrow{f_p (\chi_L)} & \cdots & \xrightarrow{f_p (\chi_L)} & 0 & \xrightarrow{f_p (\chi_L)} & \cdots \\
\end{array}
\]

where $f_p (\chi_L) = \varepsilon (L, J) u_{J \setminus L} v_L$, $\varepsilon (L, J) = \pm 1$ is a certain sign.

In this way we have an isomorphism of differential graded algebras

\[
C^* (Z_K) \cong R^* (K) \cong \bigoplus_{p \geq 0, J \subseteq [m]} C^{p-1} (K_J),
\]

(2.1)

and also

\[
H^* (Z_K) \cong H (R^* (K)) \cong \bigoplus_{p \geq 0, J \subseteq [m]} \tilde{H}^{p-1} (K_J).
\]

The product in the direct sum of simplicial cochains is induced from $R^* (K)$ by the isomorphism (2.7).

Since the algebras $H^* (Z_K)$ and $H (R^* (K))$ are multigraded we have:

\[
H^{-p, 2J} (Z_K) \cong H^{-p, 2J} (R^* (K)) \cong \tilde{H}^{-p+|J|-1} (K_J) \subset H^{-p+2|J|} (Z_K).
\]
Theorem 2.4 ([1] Proposition 3.2.10]). The product in \( \bigoplus_{p \geq 0, J \subseteq [m]} C^{p-1}(K_J) \), induced from \( R^*(K) \), coincides up to a sign with the product defined by the maps
\[
\mu: C^{p-1}(K_I) \times C^{q-1}(K_J) \to C^{p+q-1}(K_{I \cup J}),
\]
\[
(\chi_L, \chi_M) \mapsto \begin{cases} 
\chi_{L \cup M}, & \text{if } I \cap J = \emptyset, L \cup M \in K_{I \cup J}, \\
0, & \text{otherwise},
\end{cases}
\] (2.2)
where \( \chi_L \) is the characteristic function of the simplex \( L \).

A simple \( n \)-dimensional polytope \( P \) is called a flag polytope if any set of pairwise intersecting facets \( F_{i_1}, \ldots, F_{i_k}, F_{i_s} \cap F_{i_t} \neq \emptyset, s, t = 1, \ldots, k \), has a nonempty intersection \( F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset \).

Let \( P \) be a simple 3-polytope. Let \( F_{1}, \ldots, F_{m} \) be its facets. Define a \( k \)-belt as a cyclic sequence \( (F_{i_1}, \ldots, F_{i_k}) \) of facets in which only consecutive facets have a nonempty intersection. More precisely: \( F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset \) if and only if \( \{ j_1, \ldots, j_r \} \in \{ \{ 1, 2 \}, \{ 2, 3 \}, \ldots, \{ k-1, k \}, \{ k, 1 \} \} \). Note that a \( k \)-belt corresponds to a chordless cycle in the dual complex \( K_P \), which is a triangulation of 2-sphere.

Proposition 2.5 ([8] Proposition 2.3]). A simple 3-polytope \( P \) is a flag polytope if and only if \( P \neq \Delta^3 \) and \( P \) does not contain 3-belts.

Proposition 2.6 ([8] Proposition 2.5]). A simple 3-polytope \( P \) is a flag polytope if and only if any its facet is surrounded by a \( k \)-belt, where \( k \) is the number of edges of this facet.

We say that a polytope \( P \) belongs to the Pogorelov class \( \mathcal{P} \) (or \( P \) is a Pogorelov polytope) if \( P \) is a simple flag 3-dimensional polytope without 4-belts. In dimension 3 the class of combinatorial polytopes which admit a right-angled realization in Lobachevsky space \( L^3 \) coincides with the Pogorelov class. Proposition 2.5 implies that \( P \in \mathcal{P} \) if and only if \( P \neq \Delta^3 \) and \( P \) is a simple 3-polytope without 3-belts and 4-belts.

Corollary 2.7. A polytope \( P \) from the Pogorelov class does not have 3-gonal and 4-gonal facets.

We have the following characteristic property of Pogorelov polytopes.

Theorem 2.8 ([8] Proposition B.2 (b)). A simple 3-polytope \( P \) is a Pogorelov polytope if and only if each pair of its adjacent facets is surrounded by a \( k \)-belt; if the facets have \( k_1 \) and \( k_2 \) edges, then \( k = k_1 + k_2 - 4 \).

3. Massey products and Pogorelov polytopes

For moment-angle complex \( Z_K \), a triple Massey product of minimal dimension is given by:
\[
H^3(Z_K) \otimes H^3(Z_K) \otimes H^3(Z_K) \to H^8(Z_K).
\] (3.1)

These triple Massey products of 3-dimensional cohomology classes were completely described by Denham and Suciu in [9]:
Theorem 3.1 ([9, Theorem 6.1.1]). The following are equivalent:

(1) there exist cohomology classes $\alpha, \beta, \gamma \in H^3(Z_K)$ for which the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined and nontrivial;

(2) the graph $K^1$ (the 1-dimensional skeleton of $K$) contains an induced subgraph isomorphic to one of the five graphs in Figure 1.

Now we consider the problem of existence of nontrivial Massey products in $H^*(Z_P)$ for Pogorelov polytopes $P$. As noted in [6, Proposition 4.8], triple Massey products of 3-dimensional cohomology classes (3.1) are trivial for simple polytopes $P$ without 4-belts, in particular, for Pogorelov polytopes. In this paper we prove the following:

Theorem 3.2. For any Pogorelov polytope $P$, there exists a nontrivial triple Massey product $\langle \alpha, \beta, \gamma \rangle \subset H^{n+4}(Z_P)$ for some $n \geq 5$, where $\alpha \in H^1(Z_P)$, $\beta \in H^{n-2}(Z_P)$, $\gamma \in H^3(Z_P)$. The number $n$ is described in the following lemma. The indeterminacy of this Massey product is $\alpha \cdot H^n(Z_P) + \gamma \cdot H^{n+1}(Z_P)$.

Lemma 3.3. For any Pogorelov polytope $P$, there is a collection of pairwise different facets $\{F_1, \ldots, F_{l+n-1}\}$ for some natural $n \geq 5$ and $l \geq 5$ such that the full subcomplex $K_{\{1,\ldots,l+n-1\}}$ of the complex $K_P$ has the form shown in Figure 2. In other words, there exist a triple of facets $F_1, F_2, F_3$ surrounded by a belt.

Proof. Let $p_k$ be the number of $k$-gonal facets of a simple 3-polytope $P$. By the Euler theorem,

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k-6)p_k.$$
Since $P$ is a Pogorelov polytope, $p_3 = 0$, $p_4 = 0$ and hence $p_5 \geq 12$. In particular, $p_5 \geq 1$.

Choose a 5-gonal facet $F_1$ of $P$. Consider an arbitrary vertex $v$ of $F_1$. Since $P$ is simple, there are exactly 3 facets $F_1, F_2, F_3$ meeting in $v$. Let $F_2$ and $F_3$ be an $l$-gonal facet and an $n$-gonal facet respectively, we denote this by $|F_2| = l, |F_3| = n$.

Also, any two facets of $P$ either do not intersect, or intersect at an edge (they are adjacent facets). So $F_1 \cap F_2 = e_{12}, F_2 \cap F_3 = e_{23}, F_1 \cap F_3 = e_{13}, v \in e_{ij}$. There are two facets intersecting $e_{12} = F_1 \cap F_2$ at a single vertex. One of them is $F_3$, let $F_4$ be the other one. The edge $F_2 \cap F_3$ intersects $F_1$ and another facet $F_5$, and $F_1 \cap F_3$ intersects $F_2$ and $F_6$. Since $P$ is flag, each facet $F \subset P$ is surrounded by a $k$-belt, where $k = |F|$. This implies that the facet $F_1$ is surrounded by a 5-belt and this belt contains $F_2$ and $F_6$. Therefore, the facets $F_2$ and $F_6$ do not intersect.

Similarly we obtain $F_3 \cap F_4 = \emptyset, F_1 \cap F_5 = \emptyset$. Consider the dual complex $K_P$, in which the vertex $i$ corresponds to the facet $F_i$. It follows from the above that the full subcomplex $K_{\{1, \ldots, 6\}}$ of $K_P$ has the form shown in Figure 3.

The pentagonal facet $F_1$ intersects each of the four facets $F_6, F_3, F_2, F_4$ at an edge. The remaining edge of $F_1$ is the intersection of $F_1$ and some other facet $F_7$. The facet $F_1$ is surrounded by a 5-belt and the simplicial complex $K_{\{1,2,3,4,6,7\}}$ is shown in Figure 4.

Define the set $G_2$ consisting of those facets which intersect $F_2$ and are different from $F_1, \ldots, F_6$:

$$G_2 = \{ F \subset P \mid F \cap F_2 \neq \emptyset, F \neq F_i, i = 1, \ldots, 6 \}.$$ 

It is easy to see that

$$G_2 = \{ F \subset P \mid F \cap F_2 \neq \emptyset, F \cap F_1 = \emptyset, F \cap F_3 = \emptyset \}.$$ 

Since $|F_2| = l$ we have $|G_2| = l - 4$. The facets from $G_2$ are contained in the $l$-belt around $F_2$. This belt corresponds to a chordless cycle in the dual complex $K_P$. So we can enumerate the facets from $G_2$ as follows:

$$G_2 = \{ F_8, \ldots, F_{l+3} \mid F_8 \cap F_4 \neq \emptyset, F_{l+3} \cap F_5 \neq \emptyset \}.$$ 

The full subcomplex $K_{\{1, \ldots, 5, 8, \ldots, l+3\}}$ is shown in Figure 5.
For the facet $F_3$ define the set $G_3$:

$$G_3 = \{ F \subset P \mid F \cap F_3 \neq \emptyset, F \neq F_i, i = 1, \ldots, 6 \}.$$

Since $|G_3| = n - 4$ we also have that

$$G_3 = \{ F \subset P \mid F \cap F_3 \neq \emptyset, F \cap F_1 = \emptyset, F \cap F_2 = \emptyset \}.$$  

The full subcomplex $K_{\{1,2,3,5,6,l+4,\ldots,l+n-1\}}$ is shown in Figure 6.

Note that the complexes shown in Figures 3, 4, 5 and 6 appear as parts of the complex in Figure 2. It remains to show that they are patched together in the right way. That is, we need to show that if $F_i \in G_2$, $F_j \in G_3$ then $F_i \cap F_j = \emptyset, F_5 \cap F_7 = \emptyset, F_i \cap F_7 = \emptyset$; in particular, $F_i, F_j, F_7$ are different. Since $P$ is a Pogorelov polytope, Theorem 2.8 implies that the pair of adjacent facets $F_1$ and $F_2$ is surrounded by an $(l + 1)$-belt ($F_4, F_8, \ldots, F_{l+3}, F_5, F_6, F_7$). That is, the facets from this sequence are pairwise different and only consecutive facets have a nonempty intersections. Then since $G_2 = \{ F_8, \ldots, F_{l+3} \}$, we have $F_i \cap F_7 = = \emptyset$ if $F_i \in G_2$. Similarly considering the pairs of adjacent facets $F_2, F_3$ and $F_3, F_1$ we obtain that $G_2 \cap G_3 = \emptyset, F_7 \cap F_3 = \emptyset$ if $F_j \in G_3$. In particular, $(F_4, F_8, \ldots, F_{l+3}, F_5, F_{l+4}, \ldots, F_{l+n-1}, F_6, F_7)$ is a $(l+n-4)$-belt around the triple of facets $\{F_1, F_2, F_3\}$.  

\[\square\]

**Proof of Theorem 3.2.** In the notation of Figure 2, consider the following three sets of vertices of $K_P$ (see Figure 7):

$$J_1 = \{5, 6, 7\}, \ J_2 = \{2, l + 4, \ldots, l + n - 1\}, \ J_3 = \{3, 4\}.$$
For any $I \in K_J$, denote by $\chi_{I,J}$ the basis $(|I| - 1)$-dimensional simplicial cochain of the complex $K_J$ taking value 1 on the simplex $I$. Now define the following cohomology classes $\alpha, \beta, \gamma$:

$$
\alpha = [\chi_{6,J_1} + \chi_{7,J_1}] \in \tilde{H}^0(K_{J_1}) \subset H^4(\mathbb{Z}_p),
$$

$$
\beta = [\chi_{2,J_2}] \in \tilde{H}^0(K_{J_2}) \subset H^{n-2}(\mathbb{Z}_p),
$$

$$
\gamma = [\chi_{4,J_3}] \in \tilde{H}^0(K_{J_3}) \subset H^3(\mathbb{Z}_p).
$$

We consider $\tilde{H}^i(K_J)$ as subgroups in $H^i(\mathbb{Z}_p)$ through the isomorphism (2.7). Since $\tilde{H}^p(K_J) \cdot \tilde{H}^q(K_I) \subset \tilde{H}^{p+q+1}(K_{J \cup I})$, we obtain that

$$
\alpha \beta \in \tilde{H}^1(K_{J_1 \cup J_2}), \quad \beta \gamma \in \tilde{H}^1(K_{J_2 \cup J_3}).
$$

We have $\tilde{H}^1(K_{J_1 \cup J_2}) = \tilde{H}^1(K_{J_2 \cup J_3}) = 0$ because $K_{J_1 \cup J_2}$ and $K_{J_2 \cup J_3}$ are contractible. Thus $\alpha \beta = \beta \gamma = 0$, and the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined. Next,

$$(\chi_{6,J_1} + \chi_{7,J_1}) \cdot \chi_{2,J_2} = 0, \quad \chi_{2,J_2} \cdot \chi_{4,J_3} = \pm \chi_{(2,4),J_2 \cup J_3} = \pm d(\chi_{4,J_2 \cup J_3}),$$

since the product (2.9) in the algebra $\bigoplus_{p \geq 0, J \subseteq [m]} C^p(\mathbb{Z}_p)$ coincides with the product in $C^\ast(\mathbb{Z}_p)$ up to a sign. Also, we have

$$
(\chi_{6,J_1} + \chi_{7,J_1}) \cdot (\pm \chi_{4,J_2 \cup J_3}) = \pm \chi_{(4,7),J_1 \cup J_2 \cup J_3},
$$

and $\pm [\chi_{(4,7),J_1 \cup J_2 \cup J_3}] \neq 0$ as it is a generator in $H^1(K_{J_1 \cup J_2 \cup J_3}) = \mathbb{Z}$. Then

$$
\langle \alpha, \beta, \gamma \rangle = \pm [\chi_{(4,7),J_1 \cup J_2 \cup J_3}] + \alpha \cdot H^n(\mathbb{Z}_p) + \gamma \cdot H^{n+1}(\mathbb{Z}_p) \subset H^{n+4}(\mathbb{Z}_p).
$$

We need to prove that the Massey product $\langle \alpha, \beta, \gamma \rangle$ is nontrivial. Assume the opposite, $0 \in \langle \alpha, \beta, \gamma \rangle$. Then there exist $\nu \in H^n(\mathbb{Z}_p)$ and $\mu \in H^{n+1}(\mathbb{Z}_p)$ such that

$$
0 = \pm [\chi_{(4,7),J_1 \cup J_2 \cup J_3}] + \alpha \cdot \nu + \gamma \cdot \mu.
$$

Since $\alpha \in \tilde{H}^0(K_{J_1}), \gamma \in \tilde{H}^0(K_{J_3}), [\chi_{(4,7),J_1 \cup J_2 \cup J_3}] \in \tilde{H}^1(K_{J_1 \cup J_2 \cup J_3})$, we can assume that $\nu \in \tilde{H}^0(K_{J_1 \cup J_3}), \mu \in \tilde{H}^0(K_{J_2 \cup J_3})$ using the multigrading. However, $K_{J_1 \cup J_2}$ and $K_{J_2 \cup J_3}$ are contractible, hence, $\mu = 0, \nu = 0$. So $0 = \pm [\chi_{(4,7),J_1 \cup J_2 \cup J_3}]$, a contradiction. \(\square\)
Example 3.4. Let $P$ be a dodecahedron, so $K_P$ is the boundary of an icosahedron.

In this case we have the following sets of vertices of $K_P$ (see Figure 8):

\[ J_1 = \{5, 6, 7\}, \quad J_2 = \{2, 9\}, \quad J_3 = \{3, 4\}. \]

The corresponding cohomology classes are

\[ \alpha = \left[ \chi_{6, J_1} + \chi_{7, J_1} \right] \in \tilde{H}^0(K_{J_1}) \subset H^4(Z_P), \]
\[ \beta = \left[ \chi_{2, J_2} \right] \in \tilde{H}^0(K_{J_2}) \subset H^3(Z_P), \]
\[ \gamma = \left[ \chi_{4, J_3} \right] \in \tilde{H}^0(K_{J_3}) \subset H^3(Z_P). \]

We obtain the following nontrivial Massey product:

\[ \langle \alpha, \beta, \gamma \rangle = \pm \left[ \chi_{\{4, 7\}, J_1 \cup J_2 \cup J_3} \right] \in H^9(Z_P). \]

Note that in the case of dodecahedron we obtain a nontrivial triple Massey product in the lowest possible degree.

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