Rank \( t \mathcal{H} \)-primes in quantum matrices.

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Abstract

Let \( K \) be a (commutative) field and consider a nonzero element \( q \) in \( K \) which is not a root of unity. In \([5]\), Goodearl and Lenagan have shown that the number of \( \mathcal{H} \)-primes in \( R = O_q(\mathcal{M}_n(K)) \) which contain all \((t+1) \times (t+1)\) quantum minors but not all \( t \times t \) quantum minors is a perfect square. The aim of this paper is to make precise their result: we prove that this number is equal to \( (t!)^2 S(n+1, t+1)^2 \), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \). This result was conjectured by Goodearl, Lenagan and McCammond. The proof involves some closed formulas for the poly-Bernoulli numbers that were established in \([10]\) and \([1]\).

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1 Introduction.

Fix a (commutative) field \( K \) and an integer \( n \) greater than or equal to 2, and choose an element \( q \) in \( K^* := K \setminus \{0\} \) which is not a root of unity. Denote by \( R = O_q(\mathcal{M}_n(K)) \) the quantization of the ring of regular functions on \( n \times n \) matrices with entries in \( K \) and by \( (Y_{i,\alpha})_{(i, \alpha) \in [1,n]^2} \) the matrix of its canonical generators. The bialgebra structure of \( R \) gives us an action of the group \( \mathcal{H} := (\mathbb{C}^*)^{2n} \) on \( R \) by \( K \)-automorphisms (See \([5]\)) via:

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_{\alpha} Y_{i,\alpha} \quad ((i, \alpha) \in [1,n]^2).
\]

In \([9]\), Goodearl and Letzter have shown that \( R \) has only finitely many \( \mathcal{H} \)-invariant prime ideals (See \([9]\), 5.7. (i)) and that, in order to calculate the prime and primitive spectra of \( R \), it is enough to determine the \( \mathcal{H} \)-invariant prime ideals of \( R \) (See \([9]\), Theorem 6.6). Next, using the theory of deleting derivations, Cauchon has found a formula for the exact number of \( \mathcal{H} \)-invariant prime ideals in \( R \) (See \([4]\), Proposition 3.3.2). In this paper, we investigate these ideals.

In \([12]\) (See also \([13]\)), we have proved, assuming that \( \mathcal{K} = \mathbb{C} \) (the field of complex numbers) and \( q \) is transcendental over \( \mathbb{Q} \), that the \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(\mathbb{C})) \) are generated by quantum minors, as conjectured by Goodearl and Lenagan (See \([5]\) and \([6]\)). Next, using this result together with Cauchon’s description for the set of \( \mathcal{H} \)-invariant prime ideals of \( O_q(\mathcal{M}_n(\mathbb{C})) \) (See \([4]\), Théorème 3.2.1), we have constructed an algorithm which provides an explicit generating set of quantum minors for each \( \mathcal{H} \)-invariant prime ideal in \( O_q(\mathcal{M}_n(\mathbb{C})) \) (See \([11]\) or \([13]\)).
On the other hand, Goodearl and Lenagan have shown (in the general case where \( q \in \mathbb{K}^* \) is not a root of unity) that, in order to obtain descriptions of all the \( \mathcal{H} \)-invariant prime ideals of \( R \), we just need to determine the \( \mathcal{H} \)-invariant prime ideals of certain "localized step-triangular factors" of \( R \), namely the algebras

\[
R^+_\tau := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle} \left[ \sum_{r_t \geq 1} \cdots \sum_{r_1 \geq 1} \right] \]

and

\[
R^-_\tau := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > c_t \text{ or } i < r_{\alpha} \rangle} \left[ \sum_{r_t \geq 1} \cdots \sum_{r_1 \geq 1} \right],
\]

where \( t \in [0, n] \) and where \( r = (r_1, \ldots, r_t) \) and \( c = (c_1, \ldots, c_t) \) are strictly increasing sequences of integers in the range \( 1, \ldots, n \) (See [3], Theorem 3.5). Using this result, Goodearl and Lenagan have computed the \( \mathcal{H} \)-invariant prime ideals of \( O_q(\mathcal{M}_2(\mathbb{K})) \) (See [3] and \( O_q(\mathcal{M}_3(\mathbb{K})) \) (See [3]).

The aims of this paper are to provide a description for the set \( \mathcal{H} \)-\text{Spec}(\( R^+_\tau \)) of \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \) and to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) (\( t \in [0, n] \)), that is, those \( \mathcal{H} \)-invariant prime ideals of \( R \) which contain all \( (t+1) \times (t+1) \) quantum minors but not all \( t \times t \) quantum minors. In [3], the authors have shown that the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) is a perfect square. More precisely, they have established (See [3], 3.6) that, for any \( t \in [0, n] \):

\[
| \mathcal{H} \text{-Spec}^t(R) | = \left( \sum_{r=(r_t, \ldots, r_1)} \right)^2
\]

where \( \mathcal{H} \text{-Spec}^t(R) \) denotes the set of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) and \( \mathcal{H} \text{-Spec}(R^+_\tau) \) denotes the set of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \). The above relation (1) opens a potential route to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \): if we can compute the number of \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \), then we will be able to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \).

So, to compute the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \), the first step is to study the \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \). Since this algebra is induced from \( R \) by factor and localization, we first construct (See Section 2), by using the deleting derivations theory (See [3]), \( \mathcal{H} \)-invariant prime ideals of \( R \) that provide, after factor and localization, \( 2^{r_t-r_1} \cdots t^r_{t-r_{t-1}}(t+1)^{n-r_{t-1}} \) \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \) (See Section 3.2). Next, by using (1), we are able to show that the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) is greater than or equal to \((t!)^2 S(n+1, t+1)^2\), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \) (See Proposition 3.4). Finally, after observing that the number of \( \mathcal{H} \)-invariant prime ideals of \( R \) is equal to the poly-Bernoulli number \( B_n^{(-n)} \) (See Proposition 2.7), we use a closed formula for the poly-Bernoulli number \( B_n^{(-n)} \) (See [1], Theorem 2) in order to prove our main result: the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) is actually equal to \((t!)^2 S(n+1, t+1)^2\). This result was conjectured by Goodearl, Lenagan and McCammond. As a corollary, we obtain a description for the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_\tau \) (See Section 3.3).

2 \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(\mathbb{K})) \).

Throughout this paper, we use the following conventions:
• If $I$ is a finite set, $|I|$ denotes its cardinality.
• $\mathbb{K}$ denotes a (commutative) field and we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
• $q \in \mathbb{K}^*$ is not a root of unity.
• $n$ denotes a positive integer with $n \geq 2$.
• $R = O_q(M_n(\mathbb{K}))$ denotes the quantization of the ring of regular functions on $n \times n$ matrices with entries in $\mathbb{K}$; it is the $\mathbb{K}$-algebra generated by the $n \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i, \alpha \leq n$, subject to the following relations:
  If \[
  \begin{pmatrix}
  x & y \\
  z & t
  \end{pmatrix}
\]
is any $2 \times 2$ sub-matrix of $Y := (Y_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$, then
  1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
  2. $tx = xt - (q - q^{-1})yz$.
These relations agree with the relations used in [4], [5], [6], [12] and [11], but they differ from those of [14] and [2] by an interchange of $q$ and $q^{-1}$. It is well known that $R$ can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $Y_{i,\alpha}$ adjoined in lexicographic order. Thus the ring $R$ is a Noetherian domain. We denote by $F$ its skew-field of fractions. Moreover, since $q$ is not a root of unity, it follows from [7, Theorem 3.2] that all prime ideals of $R$ are completely prime.
• It is well known that the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ acts on $R$ by $\mathbb{K}$-algebra automorphisms via:
  \[(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_{\alpha} Y_{i,\alpha} \quad \forall (i, \alpha) \in [1,n]^2.\]
An $\mathcal{H}$-eigenvector $x$ of $R$ is a nonzero element $x \in R$ such that $h(x) \in \mathbb{K}^* x$ for each $h \in \mathcal{H}$. An ideal $I$ of $R$ is said to be $\mathcal{H}$-invariant if $h(I) = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}$-$\text{Spec}(R)$ the set of $\mathcal{H}$-invariant prime ideals of $R$.

The aim of this paragraph is to construct $\mathcal{H}$-invariant prime ideals of $R$ that, after factor and localization, will provide $\mathcal{H}$-invariant prime ideals of $R^+$. (See the introduction for the definition of this algebra). In order to do this, we use the description of the set $\mathcal{H}$-$\text{Spec}(R)$ that Cauchon has obtained by applying the theory of deleting derivations (See [4]).

### 2.1 Standard deleting derivations algorithm and description of $\mathcal{H}$-$\text{Spec}(R)$.

In this section, we provide the background definitions and notations for the standard deleting derivations algorithm (See [4], [12], [11]) and we recall the description of the set $\mathcal{H}$-$\text{Spec}(R)$ that Cauchon has obtained by using this algorithm (See [4]).

**Notations 2.1**

• We denote by $\leq_s$ the lexicographic ordering on $\mathbb{N}^2$. We often call it the standard ordering on $\mathbb{N}^2$. Recall that $(i, \alpha) \leq_s (j, \beta) \iff [i < j]$ or $(i = j$ and $\alpha \leq \beta$).
• We set $E_s = ([1,n]^2 \cup \{(n,n+1)\}) \setminus \{(1,1)\}$.
• Let $(j, \beta) \in E_s$. If $(j, \beta) \neq (n,n+1)$, $(j, \beta)^+$ denotes the smallest element (relatively to $\leq_s$) of the set \{(i, \alpha) \in E_s \mid (j, \beta) <_s (i, \alpha)\}.
In [4], Cauchon has shown that the theory of deleting derivations (See [3]) can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}, \ldots, Y_{n,n}; \sigma_{n,n}, \delta_{n,n}]$ (where the indices are increasing for $\leq s$). The corresponding deleting derivations algorithm is called the standard deleting derivations algorithm. It consists in the construction, for each $r \in E_s$, of the family $(Y_{i,\alpha}^{(r)}(i,\alpha)_{i,\alpha} \in [1,n]^2$ of elements of $F = \text{Fract}(R)$, defined as follows:

1. If $r = (n,n+1)$, then $Y_{i,\alpha}^{(n,n+1)} = Y_{i,\alpha}$ for all $(i, \alpha) \in [1,n]^2$.

2. Assume that $r = (j, \beta) <_s (n,n+1)$ and that the $Y_{i,\alpha}^{(r^+)} ((i, \alpha) \in [1,n]^2)$ are already constructed. Then, it follows from [3, Théorème 3.2.1] that $Y_{i,\alpha}^{(r^+)}$ are already constructed. Then, it follows from [3, Théorème 3.2.1] that $Y_{i,\alpha}^{(r^+)} \neq 0$ and, for all $(i, \alpha) \in [1,n]^2$, we have:

$$Y_{i,\alpha}^{(r)} = \begin{cases} Y_{i,\alpha}^{(r^+)} - Y_{i,\alpha}^{(r^+)} \left(Y_{j,\beta}^{(r^+)}\right)^{-1} Y_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta \\ Y_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

**Notation 2.2**

Let $r \in E_s$. We denote by $R^{(r)}$ the subalgebra of $F = \text{Fract}(R)$ generated by the $Y_{i,\alpha}^{(r)} ((i, \alpha) \in [1,n]^2)$, that is, $R^{(r)} := \mathbb{C}\langle Y_{i,\alpha}^{(r)} \mid (i, \alpha) \in [1,n]^2 \rangle$.

**Notations 2.3**

We set $R := R^{(1,2)}$ and $T_{i,\alpha} := Y_{i,\alpha}^{(1,2)}$ for all $(i, \alpha) \in [1,n]^2$.

Let $(j, \beta) \in E_s$ with $(j, \beta) \neq (n,n+1)$. The theory of deleting derivations allows us to construct embeddings $\varphi_{(j,\beta)} : \text{Spec}(R^{(j,\beta)}) \rightarrow \text{Spec}(R^{(j,\beta^+)})$ (See [3], 4.3). By composition, we obtain an embedding $\varphi : \text{Spec}(R) \rightarrow \text{Spec}(\overline{R})$ which is called the canonical embedding. In [4], Cauchon has described the set $\mathcal{H} \cdot \text{Spec}(R)$ by determining its "canonical image" $\varphi(\mathcal{H} \cdot \text{Spec}(R))$. To do this, he has introduced the following conventions and notations.

**Conventions 2.4**

- Let $v = (l, \gamma) \in [1,n]^2$.

  1. The set $C_v := \{(i, \gamma) \mid 1 \leq i \leq l\} \subset [1,n]^2$ is called the truncated column with extremity $v$.

  2. The set $L_v := \{(l, \alpha) \mid 1 \leq \alpha \leq \gamma\} \subset [1,n]^2$ is called the truncated row with extremity $v$.

- $W$ denotes the set of all the subsets in $[1,n]^2$ which are a union of truncated rows and columns.

**Notation 2.5**

Given $w \in W$, $K_w$ denotes the ideal in $\overline{R}$ generated by the $T_{i,\alpha}$ such that $(i, \alpha) \in w$.

(Recall that $K_w$ is a completely prime ideal in the quantum affine space $\overline{R}$ (See [4], 2.1).)
The following description of the set $\mathcal{H}\text{-Spec}(R)$ was obtained by Cauchon (See [4], Corollaire 3.2.1).

**Proposition 2.6**

1. Given $w \in W$, there exists a (unique) $\mathcal{H}$-invariant (completely) prime ideal $J_w$ in $R$ such that $\varphi(J_w) = K_w$.

2. $\mathcal{H}\text{-Spec}(R) = \{J_w \mid w \in W\}$.

### 2.2 Number of $\mathcal{H}$-invariant prime ideals in $R$.

In [4], Cauchon has used his description of the set $\mathcal{H}\text{-Spec}(R)$ in order to give a formula for the total number $S(n)$ of $\mathcal{H}$-invariant prime ideals of $R$. More precisely, he has established (See [4], Proposition 3.3.2) that:

\[
S(n) = (-1)^{n-1} \sum_{k=1}^{n} (k+1)^n \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} j^n,
\]

that is
\[
S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n \left( \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n \right).
\]

Recall (See [15], p. 34) that \(\frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n\) is equal to the Stirling number of second kind $S(n,k)$ (See, for example, [15] for more details on the Stirling numbers of second kind). Hence, we have:
\[
S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n S(n,k),
\]

that is
\[
S(n) = (-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n,k). \tag{2}
\]

On the other hand, it follows from [10, Theorem 1] that:
\[
(-1)^n \sum_{k=0}^{n} \frac{(-1)^k k!}{(k+1)^{-n}} S(n,k) = B_n^{(-n)},
\]

where $B_n^{(-n)}$ denotes the poly-Bernoulli number associated to $n$ and $-n$ (See [10] for the definition of the poly-Bernoulli numbers). Observing that $S(n,0) = 0$ (See [15]), we get:
\[
(-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^{-n}} S(n,k) = B_n^{(-n)},
\]

and thus, we deduce from (2) that:
Proposition 2.7

\[ |\mathcal{H}\text{-Spec}(R) | = B_n^{(n)} \].

This rewriting of Cauchon’s formula was first obtained by Goodearl and McCammond.

2.3 Vanishing and non-vanishing criteria for the entries of \(q\)-quantum matrices.

Let \( J_w (w \in W) \) be an \( \mathcal{H}\)-invariant prime ideal of \( R \) (See Proposition 2.6). In the next section, we will need to know which indeterminates \( Y_{i,\alpha} \) belong to \( J_w \), that is which \( y_{i,\alpha} := Y_{i,\alpha} + J_w \) are zero. This problem is dealt with in Proposition 2.12 and Proposition 2.16 where we respectively obtain a non-vanishing criterion and a vanishing criterion for the entries of \(q\)-quantum matrices.

For the remainder of this section, \( K \) denotes a \( K\)-algebra which is also a skew-field. Except otherwise stated, all the considered matrices have their entries in \( K \).

Definitions 2.8

Let \( M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2} \) be a \( n \times n \) matrix and let \((j, \beta)\in E_s\).

- We say that \( M \) is a \(q\)-quantum matrix if the following relations hold between the entries of \( M \):
  
  If \( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) is any \(2 \times 2\) sub-matrix of \(M\), then
  
  1. \(yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt.\)
  2. \(tx = xt - (q - q^{-1})yz.\)

- We say that \( M \) is a \((j, \beta)\)-\(q\)-quantum matrix if the following relations hold between the entries of \( M \):
  
  If \( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) is any \(2 \times 2\) sub-matrix of \(M\), then
  
  1. \(yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt.\)
  2. If \( t = x_v \), then \( \begin{cases} v \geq s(j, \beta) \Rightarrow tx = xt \\ v < s(j, \beta) \Rightarrow tx = xt - (q - q^{-1})yz.\end{cases} \)

Conventions 2.9

Let \( M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2} \) be a \(q\)-quantum matrix.

As \( r \) runs over the set \( E_s \), we define matrices \( M^{(r)} = (x_{i,\alpha}^{(r)})_{(i,\alpha)\in[1,n]^2} \) as follows:

1. If \( r = (n, n + 1) \), then the entries of the matrix \( M^{(n, n+1)} \) are defined by \( x_{i,\alpha}^{(n, n+1)} := x_{i,\alpha} \) for all \((i, \alpha)\in[1,n]^2\).

2. Assume that \( r = (j, \beta) \in E_s \setminus \{(n, n + 1)\} \) and that the matrix \( M^{(r^+)} \) is already known.
   The entries \( x_{i,\alpha}^{(r)} \) of the matrix \( M^{(r)} \) are defined as follows:
(a) If \( x_{i,j}^{(r)} = 0 \), then \( x_{i,i}^{(r)} = x_{i,j}^{(r)} \) for all \((i, \alpha) \in [1, n]^2 \).

(b) If \( x_{i,j}^{(r)} \neq 0 \) and \((i, \alpha) \in [1, n]^2 \), then

\[
x_{i,i}^{(r)} = \begin{cases} x_{i,i}^{(r)} - x_{i,j}^{(r)} \left( x_{j,j}^{(r)} \right)^{-1} x_{j,i}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\
0 & \text{otherwise.}
\end{cases}
\]

We say that \( M^{(r)} \) is the matrix obtained from \( M \) by applying the standard deleting derivations algorithm at step \( r \).

3. If \( r = (1, 2) \), we set \( t_{i,i} := x_{i,i}^{(1,2)} \) for all \((i, \alpha) \in [1, n]^2 \).

Observe that the formulas of Conventions 2.9 allow us to express the entries of \( M^{(r)} \) in terms of those of \( M^{(r)} \).

**Proposition 2.10 (Restoration algorithm)**

Let \( M = (x_{i,\alpha}(i, \alpha) \in [1, n]^2 \) be a \( q \)-quantum matrix and let \( r = (j, \beta) \in E_s \) with \( r \neq (n, n + 1) \).

1. If \( x_{j,j}^{(r)} = 0 \), then \( x_{i,i}^{(r)} = x_{i,j}^{(r)} \) for all \((i, \alpha) \in [1, n]^2 \).

2. If \( x_{j,j}^{(r)} \neq 0 \) and \((i, \alpha) \in [1, n]^2 \), then

\[
x_{i,i}^{(r)} = \begin{cases} x_{i,i}^{(r)} + x_{i,j}^{(r)} \left( x_{j,j}^{(r)} \right)^{-1} x_{j,i}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\
0 & \text{otherwise.}
\end{cases}
\]

Note that our definitions of \( q \)-quantum matrix and \((j, \beta)\)-\( q \)-quantum matrix slightly differ from those of 2 (See 2, Dénfinitions III.1.1 and III.1.3). Because of this, we must interchange \( q \) and \( q^{-1} \) whenever carrying over result of 2.

**Lemma 2.11**

Let \((j, \beta) \in E_s \).

If \( M = (x_{i,\alpha}(i, \alpha) \in [1, n]^2 \) is a \( q \)-quantum matrix, then the matrix \( M^{(j, \beta)} \) is \((j, \beta)\)-\( q \)-quantum.

**Proof**: This lemma is proved in the same manner as 2 Proposition III.2.3.1. 

We deduce from the above Lemma 2.11 the following non-vanishing criterion for the entries of a \( q \)-quantum matrix.

**Proposition 2.12**

Let \( M = (x_{i,\alpha}(i, \alpha) \in [1, n]^2 \) be a \( q \)-quantum matrix and let \((i, \alpha) \in [1, n]^2 \).

If \( t_{i,i} \neq 0 \), then \( x_{i,i} \neq 0 \). In other words, if \( x_{i,i} = 0 \), then \( t_{i,i} = 0 \).

**Proof**: Assume that \( x_{i,i} = 0 \). We first prove that \( x_{i,i}^{(j, \beta)} = 0 \) for all \((j, \beta) \in E_s \). To achieve this aim, we proceed by decreasing induction (for \( \leq_s \)) on \((j, \beta) \).

Since \( x_{i,i}^{(n,n+1)} = x_{i,i} \), the case \((j, \beta) = (n, n+1) \) is done. Assume now that \((j, \beta) <_s (n, n+1) \) and \( x_{i,i}^{(j, \beta)} = 0 \). If \( x_{i,i}^{(j, \beta)} = x_{i,i}^{(j, \beta)} \), we obviously have \( x_{i,i}^{(j, \beta)} = 0 \). Next, if \( x_{i,i}^{(j, \beta)} \neq x_{i,i}^{(j, \beta)} \), then
In order to construct, in the next section, as desired. This achieves the induction.

\[
\begin{align*}
&x_{j,\beta}^+(x_{i,\alpha}^++x_{j,\alpha}^-) - x_{i,\alpha}^-x_{j,\beta}^+ = -(q-q^{-1})x_{i,\beta}^+x_{j,\alpha}^-.
\end{align*}
\]

Since \(x_{i,\alpha}^+=0\), we deduce from this equality that, in \(K\), \(x_{i,\beta}^+ x_{j,\alpha}^- = 0\). Thus, \(x_{i,\beta}^+ = 0\) or \(x_{j,\alpha}^- = 0\). On the other hand, since \(i < j\) and \(\alpha < \beta\), we have \(x_{i,\alpha}^+ = x_{i,\beta}^+ - x_{i,\alpha}^- x_{j,\beta}^+ (x_{j,\beta}^+)^{-1} x_{j,\alpha}^-\). Now it follows from the induction hypothesis that \(x_{i,\alpha}^+ = 0\). Hence, we have \(x_{i,\alpha}^+ = -x_{i,\beta}^+ (x_{j,\beta}^+)^{-1} x_{j,\alpha}^-\). Finally, since \(x_{i,\beta}^+ = 0\) or \(x_{j,\alpha}^- = 0\), we get \(x_{i,\alpha}^+ = 0\), as desired. This achieves the induction.

In particular, we have shown that \(x_{i,\alpha}^{(1,2)} = 0\), that is \(t_{i,\alpha} = 0\).

Proposition 2.12 furnishes a non-vanishing criterion for the entries of a \(q\)-quantum matrix. In order to construct, in the next section, \(\mathcal{H}\)-invariant prime ideals of \(R\) that will provide, after factor and localization, \(\mathcal{H}\)-invariant prime ideals of \(R_\mathcal{H}^+ := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle [Y_{r_1,1}, \ldots, Y_{r_t,t}]}
\)

\((\mathbf{r} = (r_1, \ldots, r_t) \text{ with } 1 \leq r_1 < \cdots < r_t \leq n)\), we also need to get a vanishing criterion for the entries \(x_{i,\alpha}^+\), \(\alpha > t\) or \(i < r_{\alpha}\), of a \(q\)-quantum matrix. This is what we do now.

**Notation 2.13**

*If \(t\) denotes an element of \([0, n]\), we set:*

\[
\mathbf{R}_t := \{(r_1, \ldots, r_t) \in \mathbb{N} \mid 1 \leq r_1 < \cdots < r_t \leq n\}.
\]

*(If \(t = 0\), then \(\mathbf{R}_0 = \emptyset\).)*

For the remainder of this section, we fix \(t \in [0, n]\) and \(\mathbf{r} = (r_1, \ldots, r_t) \in \mathbf{R}_t\), and we denote by \(w_{\mathbf{r}}\) the subset of \([1, n]^2\) corresponding to indeterminates \(Y_{i,\alpha}\) that have been set equal to zero in \(R_\mathcal{H}^+\), that is, we set:

\[
w_{\mathbf{r}} := \bigcup_{\alpha \in [1, t]} [1, r_{\alpha} - 1] \times \{\alpha\} \cup [1, n] \times [t + 1, n].
\]

For instance, if \(n = 3\), \(t = 2\) and \(\mathbf{r} = (1, 3)\), we have:

\[
w_{(1,3)} = \begin{array}{cc}
1 & 2 \\
\hline
3 & \text{black}
\end{array}
\]

where the black boxes symbolize the elements of \(w_{(1,3)}\).

Note that \(w_{\mathbf{r}}\) is a union of truncated columns, so that:

**Remark 2.14**

\(w_{\mathbf{r}}\) belongs to \(W\).
Observation 2.15
Let \((i, \alpha) \in w_r\). If \(\beta \in [\alpha, n]\), then \((i, \beta) \in w_r\).

Proof: We distinguish two cases.

• If \((i, \alpha) \in [1, n] \times [t + 1, n]\), then \(\alpha \geq t + 1\). Hence \(\beta \geq \alpha \geq t + 1\) and thus, we have \((i, \beta) \in [1, n] \times [t + 1, n] \subseteq w_r\), as required.

• Assume now that \((i, \alpha) \in \bigcup_{\gamma \in [1, t]} [1, r_{\gamma} - 1] \times \{\gamma\}\), so that we have \(\alpha \leq t\) and \(i \leq r_\alpha - 1\). If \(\beta > t\), we conclude as in the previous case that \((i, \beta) \in w_r\). So we assume that \(\beta \leq t\). Since \(i \leq r_\alpha - 1\) and since \(\alpha \leq \beta \leq t\), we have \(i \leq r_\alpha - 1 \leq r_\beta - 1\). Hence, \((i, \beta) \in [1, r_\beta - 1] \times \{\beta\} \subseteq w_r\), as desired. ■

This observation allows us to prove the following vanishing criterion:

Proposition 2.16
Let \(M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}\) be a \(q\)-quantum matrix.
If \(t_{i,\alpha} = 0\) for all \((i, \alpha) \in w_r\), then \(x_{i,\alpha} = 0\) for all \((i, \alpha) \in w_r\).

Proof: Assume that \(t_{i,\alpha} = 0\) for all \((i, \alpha) \in w_r\). We first prove by induction on \((j, \beta)\) (with respect of \(\leq_s\)) that \(x_{i,\alpha}^{(j,\beta)} = 0\) for all \((i, \alpha) \in w_r\) and \((j, \beta) \in E_s\).

If \((j, \beta) = (1, 2)\), then \(x_{i,\alpha}^{(1,2)} = t_{i,\alpha} = 0\) for all \((i, \alpha) \in w_r\), as required. Assume now that \((j, \beta) <_s (n, n + 1)\) and that \(x_{i,\alpha}^{(j,\beta)} = 0\) for all \((i, \alpha) \in w_r\). Let \((i, \alpha) \in w_r\). If \(x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)}\), the desired result follows from the induction hypothesis. Next, if \(x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)}\), it follows from Proposition 2.10 that \(x_{j,\beta}^{(j,\beta)} \neq 0\), \(i < j\), \(\alpha < \beta\) and \(x_{i,\alpha}^{(j,\beta)} + x_{i,\alpha}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{j,\alpha}^{(j,\beta)} = 0\). Since \((i, \alpha) \in w_r\), we deduce from the induction hypothesis that \(x_{i,\alpha}^{(j,\beta)} = 0\), so that \(x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{j,\alpha}^{(j,\beta)}\). Moreover, since \((i, \alpha) \in w_r\) and \(\alpha < \beta\), it follows from Observation 2.15 that \((i, \beta) \in w_r\). Then, we deduce from the induction hypothesis that \(x_{i,\beta}^{(j,\beta)} = 0\), so that \(x_{i,\alpha}^{(j,\beta)} = x_{i,\beta}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{j,\alpha}^{(j,\beta)} = 0\). This achieves the induction.

In particular, we have proved that \(x_{i,\alpha} = x_{i,\alpha}^{(n,n+1)} = 0\) for all \((i, \alpha) \in w_r\). ■

2.4 \(\mathcal{H}\)-invariant prime ideals \(J_w\) with \(w_r \subseteq w\).

As in the previous section, we fix \(t \in [0, n]\) and \(r = (r_1, \ldots, r_t) \in R_t\), and we set:

\[ w_r := \left[ \bigcup_{\alpha \in [1, t]} [1, r_{\alpha} - 1] \times \{\alpha\} \right] \bigcup [1, n] \times [t + 1, n]. \]

Recall (See Proposition 2.6) that, if \(w \in W\), there exists a (unique) \(\mathcal{H}\)-invariant prime ideal of \(R\) associated to \(w\) (See Proposition 2.3) and that the \(J_w\) \((w \in W)\) are exactly the \(\mathcal{H}\)-invariant prime ideals in \(R\). This section is devoted to the \(\mathcal{H}\)-invariant prime ideals \(J_w\) \((w \in W)\) of \(R\) with \(w_r \subseteq w\). More precisely, we want to know which indeterminates \(Y_{i,\alpha}\) belong to these ideals.
Notations 2.17
Let \( w \in W \).

1. Set \( R_w := \frac{R}{J_w} \). It follows from [8, Lemme 5.3.3] that, using the notations of Section 2.1, \( R_w \) and \( \frac{R}{K_w} \) are two Noetherian algebras with no zero-divisors, which have the same skew-field of fractions. We set \( F_w := \text{Fract}(R_w) = \text{Fract}\left(\frac{R}{K_w}\right) \).

2. If \( (i, \alpha) \in \mathbb{I}^2 \), \( y_{i,\alpha} \) denotes the element of \( R_w \) defined by \( y_{i,\alpha} := \gamma_{i,\alpha} + J_w \).

3. We denote by \( M_w \) the matrix, with entries in the \( K_w \)-algebra \( F_w \), defined by:
\[
M_w := (y_{i,\alpha})_{(i,\alpha) \in \mathbb{I}^2}.
\]

Let \( w \in W \). Since \( Y = (Y_{i,\alpha})_{(i,\alpha) \in \mathbb{I}^2} \) is a q-quantum matrix, \( M_w \) is also a q-quantum matrix. Thus, we can apply the standard deleting derivations algorithm to \( M_w \) (See Conventions 2.9 with \( K = F_w \)) and if we still denote \( t_{i,\alpha} := y_{i,\alpha}^{(1,2)} \) for \( (i, \alpha) \in \mathbb{I}^2 \), we get:

Proposition 2.18
\( t_{i,\alpha} = 0 \) if and only if \( (i, \alpha) \in w \).

Proof: By [3, Propositions 5.4.1 and 5.4.2], there exists a \( K_w \)-algebra homomorphism \( f_{(1,2)} : R \to F_w \) such that \( f_{(1,2)}(T_{i,\alpha}) = t_{i,\alpha} \) for \( (i, \alpha) \in \mathbb{I}^2 \). Its kernel is \( K_w \) and its image is the subalgebra of \( F_w \) generated by the \( t_{i,\alpha} \) with \( (i, \alpha) \in \mathbb{I}^2 \). Hence, \( t_{i,\alpha} = 0 \) if and only if \( T_{i,\alpha} \in K_w \), that is, if and only if \( (i, \alpha) \in w \). ■

Consider now an element \( w \) in \( W \) with \( w_r \subseteq w \) and denote by \( J_w \) the (unique) \( H \)-invariant prime ideal of \( R \) associated to \( w \) (See Proposition 2.6). Since \( w_r \subseteq w \), we deduce from Proposition 2.18 that \( t_{i,\alpha} = 0 \) for all \( (i, \alpha) \in w_r \). Hence, we can apply Proposition 2.16 to the q-quantum matrix \( M_w \) and we obtain that \( y_{i,\alpha} = 0 \) for all \( (i, \alpha) \in w_r \), that is, \( Y_{i,\alpha} \in J_w \) for all \( (i, \alpha) \in w_r \).

So we have just established:

Proposition 2.19
Let \( w \in W \) with \( w_r \subseteq w \). If \( (i, \alpha) \in w_r \), then \( Y_{i,\alpha} \) belongs to \( J_w \).

We will now add truncated rows to the "\( w_r \) diagram" in order to obtain \( H \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( H \)-invariant prime ideals of \( R^+_r \). We will see later (See Section 3.4) that the \( H \)-invariant prime ideals of \( R \) obtained by adding truncated rows to the "\( w_r \) diagram" are the only \( H \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( H \)-invariant prime ideals of \( R^+_r \).

Notation 2.20
We set \( \Gamma_r := \{ (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \mid \gamma_k \in [0, l] \text{ if } k \in [r_l + 1, r_{l+1}] \} \). (Here \( r_0 = 0 \) and \( r_{l+1} = n \).)
For instance, if \( n = 3, t = 2 \) and \( r = (1, 3) \), we have:

\[
\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\}.
\]

**Theorem 2.21**

Let \((\gamma_1, \ldots, \gamma_n) \in \Gamma_r\) and set \(w_{r,(\gamma_1, \ldots, \gamma_n)} := w_r \bigcup \bigcup_{k \in [1,n]} \{k \} \times [1, \gamma_k]\).

Then \(w_{r,(\gamma_1, \ldots, \gamma_n)}\) belongs to \( W \) and the \( H \)-invariant prime ideal \(J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) of \( R \) has the following properties:

1. \( Y_{i,\alpha} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) for all \((i, \alpha) \in w_r\).
2. \( Y_{r,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) for all \( k \in [1,t] \).

**Proof:** Since \(w_r\) is a union of truncated columns and since \( \bigcup_{k \in [1,n]} \{k \} \times [1, \gamma_k]\) is a union of truncated rows, \(w_{r,(\gamma_1, \ldots, \gamma_n)}\) is a union of truncated rows and columns, so that \(w_{r,(\gamma_1, \ldots, \gamma_n)} \in W\).

Since \(w_r \subseteq w_{r,(\gamma_1, \ldots, \gamma_n)}\), we deduce from Proposition 2.19 that \( Y_{i,\alpha} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) for all \((i, \alpha) \in w_r\).

Now we want to prove that \( Y_{r,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) for all \( k \in [1,t] \). Assume this is not the case, that is, assume that there exists \( k \in [1,t] \) with \( Y_{r,k} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\). Then, \( y_{r,k} = 0 \) and it follows from Proposition 2.12 that \( y_{r,k}^{(1,2)} = t_{r,k} = 0 \). Thus, we deduce from Proposition 2.18 that \((r,k) \in w_{r,(\gamma_1, \ldots, \gamma_n)}\).

Observe now that, since \( k \leq t \), \((r,k) \notin [1,n] \times [t + 1, n]\). Further, it is obvious that \((r,k) \notin \bigcup_{\alpha \in [1,t]} [1, r_{\alpha} - 1] \times \{\alpha\}\). Hence, \((r,k) \notin w_r\).

All this together shows that \((r,k) \in w_{r,(\gamma_1, \ldots, \gamma_n)} \setminus w_r = \bigcup_{l \in [1,n]} \{l\} \times [1, \gamma_l]\), so that \( k \leq \gamma_{r,k}\).

However, since \((\gamma_1, \ldots, \gamma_n) \in \Gamma_r\), we have \( \gamma_{r,k} \leq k - 1 \). This is a contradiction and thus we have proved that \( Y_{r,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}\) for all \( k \in [1,t] \).

Let us now give an example for the elements \(w_{r,(\gamma_1, \ldots, \gamma_n)}\) \((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r\) of Theorem 2.21. If \( n = 3, t = 2 \) and \( r = (1, 3) \), we have already noted that

\[
\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\},
\]

so that the elements \(w_{r,(\gamma_1, \ldots, \gamma_n)}\) \((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r\) of Theorem 2.21 are:

\[
\begin{align*}
w_{(1,3),(0,0,0)} &= w_{(1,3)} \quad & w_{(1,3),(0,1,0)} &= \quad & w_{(1,3),(1,1,1)} &= \\
w_{(1,3),(0,0,1)} &= \quad & w_{(1,3),(0,1,1)} &= \quad & w_{(1,3),(1,1,1)} &= 
\end{align*}
\]
(As previously, if \( w \in W \), the black boxes symbolize the elements of \( w \).

3  Number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(\mathbb{K})) \).

In this paragraph, using the previous section, we begin by constructing \( \mathcal{H} \)-invariant prime ideals of the algebra \( R^+_t := \frac{O_q(\mathcal{M}_n(\mathbb{K}))}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle [Y_{r_1,1}, \ldots, Y_{r_t,t}] \) , where \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \) is a strictly increasing sequence of integers in the range \( 1, \ldots, n \). Next, following the route sketched in the introduction, we establish our main result: the number \( |\mathcal{H}-\text{Spec}^t[R]| \) of \( \mathcal{H} \)-invariant prime ideals of \( R = O_q(\mathcal{M}_n(\mathbb{K})) \) which contain all \( (t+1) \times (t+1) \) quantum minors but not all \( t \times t \) quantum minors is equal to \((t!)^2 S(n+1, t+1)^2\), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n \) and \( t+1 \). From this result, we derive a description of the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \).

3.1 \( \mathcal{H} \)-invariant prime ideals in \( R^+_{t,0} \).

Throughout this section, we fix \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in \mathbb{R}_t \), and we define \( w_r \) as in the previous section.

As in [5 2.1], we set \( R^+_{t,0} = \frac{R}{\langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle} \).

Recall (See [5, 2.1]) that \( R^+_{t,0} \) can be written as an iterated Ore extension over \( \mathbb{K} \). Thus, \( R^+_{t,0} \) is a Noetherian domain. Moreover, since \( q \) is not a root of unity, it follows from [7, Theorem 3.2] that all primes of \( R \) are completely prime and thus, since this property survives in factors, all primes in the algebra \( R^+_{t,0} \) are completely prime.

Observe now that, since the indeterminates \( Y_{i,\alpha} \) are \( \mathcal{H} \)-eigenvectors, \( \langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle \) is an \( \mathcal{H} \)-invariant ideal of \( R \). Hence, the action of \( \mathcal{H} \) on \( R \) induces an action of \( \mathcal{H} \) on \( R^+_{t,0} \) by automorphisms. As usually, an \( \mathcal{H} \)-eigenvector \( x \) of \( R^+_{t,0} \) is a nonzero element \( x \in R^+_{t,0} \) such that \( h(x) \in K^* x \) for each \( h \in \mathcal{H} \), and an ideal \( I \) of \( R^+_{t,0} \) is said to be \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \). Further, we denote by \( \mathcal{H}-\text{Spec}(R^+_{t,0}) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_{t,0} \).

Notations 3.1

- We denote by \( \pi^+_{t,0} : R \rightarrow R^+_{t,0} \) the canonical surjective \( \mathbb{K} \)-algebra homomorphism.
- If \( (i, \alpha) \in [1, n]^2 \), \( Y_{i,\alpha} \) denotes the element of \( R^+_{t,0} \) defined by \( Y_{i,\alpha} := \pi^+_{t,0}(Y_{i,\alpha}) \).

Let \( (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \) (See Notation 2.20) and define \( w_r(\gamma_1, \ldots, \gamma_n) \) as in Theorem 2.21. Recall (See Theorem 2.21) that \( w_r(\gamma_1, \ldots, \gamma_n) \) is an element of \( W \) and that the \( \mathcal{H} \)-invariant prime ideal \( J_{w_r(\gamma_1, \ldots, \gamma_n)} \) of \( R \) contains the indeterminates \( Y_{i,\alpha} \) with \( (i, \alpha) \in w_r \), so that \( \langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle \subseteq J_{w_r(\gamma_1, \ldots, \gamma_n)} \). Thus, \( \pi^+_{t,0}(J_{w_r(\gamma_1, \ldots, \gamma_n)}) \) is a (completely) prime ideal of \( R^+_{t,0} \). More precisely, we have:

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Proposition 3.2
\[ J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} := \pi^+_{r,0} \left( J_{w_r,(\gamma_1,\ldots,\gamma_n)} \right) \]
is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which does not contain the \( \overline{Y}_{r,k} \) \( (k \in [1,t]). \)

**Proof:** We already explained that \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is a (completely) prime ideal of \( R^+_{r,0}. \) Moreover, since \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is \( \mathcal{H} \)-invariant, it is easy to check that \( J^+_{w_r,(\gamma_2,\ldots,\gamma_n)} \) is also \( \mathcal{H} \)-invariant. Finally, since \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) does not contain the indeterminates \( Y_{r,k} \) \( (k \in [1,t]). \) (See Theorem 2.21), \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) does not contain the \( \overline{Y}_{r,k} = \pi^+_{r,0}(Y_{r,k}) \) with \( k \in [1,t]. \)

### 3.2 \( \mathcal{H} \)-invariant prime ideals in \( R^+_r \)

As in the previous section, we fix \( t \in [0,n] \) and \( r = (r_1,\ldots,r_t) \in R_t. \) In [5, 2.1], Goodearl and Lenagan have observed that the \( \overline{Y}_{r,k} \) with \( k \in [1,t] \) are regular normal elements in \( R^+_{r,0} \), so that we can form the Ore localization:

\[ R^+_r := R^+_{r,0}S^{-1}_r, \]

where \( S_r \) denotes the multiplicative system of \( R^+_{r,0} \) generated by the \( \overline{Y}_{r,k} \) with \( k \in [1,t]. \)

In the previous section, we have noted that all the primes of \( R^+_{r,0} \) are completely prime. Since this property survives in localization, all the primes of \( R^+_r \) are also completely prime.

Observe now that, since the \( \overline{Y}_{r,k} \) with \( k \in [1,t] \) are \( \mathcal{H} \)-eigenvectors of \( R^+_{r,0} \), the action of \( \mathcal{H} \) on \( R^+_{r,0} \) extends to an action of \( \mathcal{H} \) on \( R^+_r \) by automorphisms. We say that an ideal \( I \) of \( R^+_r \) is \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \) and we denote by \( \mathcal{H}-\text{Spec}(R^+_r) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_r \). Observe now that contraction and extension provide inverse bijections between the set \( \mathcal{H}-\text{Spec}(R^+_r) \) and the set of those \( \mathcal{H} \)-invariant prime ideals of \( R^+_{r,0} \) which are disjoint from \( S_r \).

Let \( (\gamma_1,\ldots,\gamma_n) \in \Gamma_r \) (See Notation 2.20) and define \( w_{r,(\gamma_1,\ldots,\gamma_n)} \) as in Theorem 2.21. By Proposition 3.2

\[ J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} := \pi^+_{r,0} \left( J_{w_r,(\gamma_1,\ldots,\gamma_n)} \right) \]
is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which does not contain the \( \overline{Y}_{r,k} \) \( (k \in [1,t]). \) Since \( S_r \) is generated by the \( \overline{Y}_{r,k} \) \( (k \in [1,t]), \) \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which is disjoint from \( S_r. \) Thus, we have the following statement:

**Proposition 3.3**
\[ J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}S^{-1}_r \] is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_r. \)

We will prove later (See Section 3.3) that the \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}S^{-1}_r \) \( ((\gamma_1,\ldots,\gamma_n) \in \Gamma_r) \) are exactly the \( \mathcal{H} \)-invariant prime ideals of \( R^+_r. \)

We deduce from the above Proposition 3.3 that:

**Corollary 3.4**
\[ R^+_r \] has at least \( 1^r\cdot 2^{r_2-r_1} \cdots t^{r_t-r_{t-1}}(t+1)^{n-r_t} \) \( \mathcal{H} \)-invariant prime ideals.
Proof: It follows from Proposition 3.3 that $R^+_r$ has at least $|\Gamma_r|$ $H$-invariant prime ideals, and it is obvious that $|\Gamma_r| = 1^{r_1}2^{r_2-r_1}\ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$.

3.3 Number of rank $t$ $H$-invariant prime ideals in $O_q(M_n(\mathbb{K}))$.

For convenience, we recall the following definitions (See [14]):

Definitions 3.5

• Let $m$ be a positive integer and let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,m]^2}$ be a square $q$-quantum matrix. The quantum determinant of $M$ is defined by:

$$
\det_q(M) := \sum_{\sigma\in S_m} (-q)^{l(\sigma)}x_{1,\sigma(1)}\ldots x_{m,\sigma(m)},
$$

where $S_m$ denotes the group of permutations of $[1,m]$ and $l(\sigma)$ denotes the length of the $m$-permutation $\sigma$.

• Let $Y := (Y_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be the $q$-quantum matrix of the canonical generators of $R$. The quantum determinant of a square sub-matrix of $Y$ is called a quantum minor.

We can now define the rank $t$ $H$-invariant prime ideals of $R$, as follows:

Definition 3.6

Let $t \in [0,n]$. An $H$-invariant prime ideal $J$ of $R = O_q(M_n(\mathbb{K}))$ has rank $t$ if $J$ contains all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors.

As in [5, 3.6], we denote by $H$-$\text{Spec}^{[t]}(R)$ the set of rank $t$ $H$-invariant prime ideals of $R$.

Note that there is only one element in $H$-$\text{Spec}^{[0]}(R)$: $\langle Y_{i,\alpha} \mid (i, \alpha) \in [1,n]^2 \rangle$, the augmentation ideal of $R$. Further, Goodearl and Lenagan have observed (See [5, 3.6]) that $|H$-$\text{Spec}^{[1]}(R)| = (2^n - 1)^2$ and $|H$-$\text{Spec}^{[n]}(R)| = (n!)^2$.

Observation 3.7

The sets $H$-$\text{Spec}^{[t]}(R)$ ($t \in [0,n]$) partition the set $H$-$\text{Spec}^{[t]}(R)$.

Proof: Let $P$ be an $H$-invariant prime ideal of $R$. Let $t \in [0,n]$ be maximal such that $P$ does not contain all $t \times t$ quantum minors. Then $P$ clearly belongs to $H$-$\text{Spec}^{[t]}(R)$. Hence, we have proved that $H$-$\text{Spec}(R) = \bigcup_{t\in[0,n]} H$-$\text{Spec}^{[t]}(R)$. Since this union is obviously disjoint, we get

$H$-$\text{Spec}(R) = \bigsqcup_{t\in[0,n]} H$-$\text{Spec}^{[t]}(R)$, as desired.

In [5], the authors have established the following result that will be our starting point to compute the cardinality of $H$-$\text{Spec}^{[t]}(R)$:
Proposition 3.8 (See [5], 3.6)

For all \( t \in [0, n] \), we have \(|\mathcal{H}-\text{Spec}^t(R)| = \left(\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)| \right)^2\).

Before computing \(|\mathcal{H}-\text{Spec}^t(R)|\), we first give a lower bound for \(\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)|\).

Proposition 3.9

For any \( t \in [0, n] \), we have

\[\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)| \geq t!S(n+1, t+1),\]

where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \) (See, for instance, [15] for the definition of \( S(n+1, t+1) \)).

**Proof:** First, we deduce from Corollary 3.4 the following inequality:

\[\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)| \geq \sum_{r \in \mathbb{R}_t} 1^{r_1}2^{r_2-r_1}\ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}. \tag{3}\]

On the other hand, we know (See [15], Exercise 16 p46) that:

\[S(n+1, t+1) = \sum_{a_1+\ldots+a_{t+1}=n+1} 1^{a_1-1}2^{a_2-1}\ldots(t+1)^{a_{t+1}-1}. \tag{4}\]

Observe now that the map \( f : \{(a_1, \ldots, a_{t+1}) \in (\mathbb{N}^*)^{t+1} | a_1+\ldots+a_{t+1} = n+1\} \rightarrow \{(r_1, \ldots, r_t) \in (\mathbb{N}^*)^t | 1 \leq r_1 < \cdots < r_t \leq n\} = \mathbb{R}_t \) defined by \( f(a_1, \ldots, a_{t+1}) = (a_1, a_1+a_2, \ldots, a_1+\ldots+a_t) \) is a bijection and that its inverse \( f^{-1} \) is defined by \( f^{-1}(r_1, \ldots, r_t) = (r_1, r_2-r_1, \ldots, r_t-r_{t-1}, n+1-n-r_t) \) for all \((r_1, \ldots, r_t) \in \mathbb{R}_t\). Thus, by means of the change of variables \((a_1, \ldots, a_{t+1}) = f^{-1}(r_1, \ldots, r_t)\), the above equality (4) is transformed to

\[S(n+1, t+1) = \sum_{1 \leq r_1 < \cdots < r_t \leq n} 1^{r_1-1}2^{r_2-r_1-1}\ldots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t},\]

so that

\[t!S(n+1, t+1) = \sum_{(r_1, \ldots, r_t) \in \mathbb{R}_t} 1^{r_1}2^{r_2-r_1}\ldots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t}.\]

Thus, we deduce from inequality (3) that:

\[\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)| \geq t!S(n+1, t+1),\]

as desired. \( \blacksquare \)

Remark 3.10

The proof of the above Proposition 3.9 shows that, if there exists \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in \mathbb{R}_t \) such that \(|\mathcal{H}-\text{Spec}(R^+_r)| > 1^{r_1}2^{r_2-r_1}\ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}\), then

\[\sum_{r \in \mathbb{R}_t} |\mathcal{H}-\text{Spec}(R^+_r)| > t!S(n+1, t+1).\]
We can now prove our main result which was conjectured by Goodearl, Lenagan and McCammond:

**Theorem 3.11**

If \( t \in [0, n] \), then \( |H-Spec^t(R)| = (t!S(n + 1, t + 1))^2 \).

**Proof:** First, since the sets \( H-Spec^t(R) \) (\( t \in [0, n] \)) partition \( H-Spec(R) \) (See Observation 3.7), we have:

\[
|H-Spec(R)| = \sum_{t=0}^{n} |H-Spec^t(R)|.
\]

Recall now (See Proposition 2.7) that \( |H-Spec(R)| \) is equal to the poly-Bernoulli number \( B_n^{(-n)} \). Thus, we deduce from the above equality that:

\[
B_n^{(-n)} = \sum_{t=0}^{n} |H-Spec^t(R)|.
\]

Further, by Theorem 2], \( B_n^{(-n)} \) can also be written as follows:

\[
B_n^{(-n)} = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2.
\]

Hence, we have:

\[
\sum_{t=0}^{n} |H-Spec^t(R)| = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2,
\]

that is:

\[
\sum_{t=0}^{n} \left( |H-Spec^t(R)| - (t!S(n + 1, t + 1))^2 \right) = 0. \tag{5}
\]

On the other hand, recall (See 3.6) that \( |H-Spec^t(R)| = \left( \sum_{r \in R_t} |H-Spec(R_t^+)| \right)^2 \).

Thus, since \( \sum_{r \in R_t} |H-Spec(R_t^+)| \geq t!S(n + 1, t + 1) \) (See Proposition 3.9), we have:

\[
|H-Spec^t(R)| \geq (t!S(n + 1, t + 1))^2.
\]

In other words, each of the terms which appears in the sum on the left-hand side of (5) is non-negative. Since this sum is equal to zero, each term of this sum must be zero, that is, for all \( t \in [0, n] \), we have:

\[
|H-Spec^t(R)| = (t!S(n + 1, t + 1))^2. \tag*{\blacksquare}
\]

**Remark 3.12**

The cases \( t = 0 \), \( t = 1 \) and \( t = n \) were already known (See 3.6).

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3.4 Description of the set $\mathcal{H}\text{-}Spec(R^+_t)$.

Throughout this section, we fix $t \in [0, n]$ and $r = (r_1, \ldots, r_t) \in \mathbb{R}_t$. We now use the above Theorem 3.11 to obtain a description of the set $\mathcal{H}\text{-}Spec(R^+_t)$. More precisely, we show that the only $\mathcal{H}\text{-}invariant$ prime ideals of $R^+_t$ are those obtained in Proposition 3.3, that is, in the notations of Section 3.2:

**Theorem 3.13**

\[ \mathcal{H}\text{-}Spec(R^+_t) = \{ J^+_{w_t, (\gamma_1, \ldots, \gamma_n)} S^{-1}_t \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_t \}. \]

**Proof**: We already know (See Proposition 3.3) that \[ \mathcal{H}\text{-}Spec(R^+_t) \supseteq \{ J^+_{w_t, (\gamma_1, \ldots, \gamma_n)} S^{-1}_t \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_t \}. \]

Assume now that \[ \mathcal{H}\text{-}Spec(R^+_t) \nsubseteq \{ J^+_{w_t, (\gamma_1, \ldots, \gamma_n)} S^{-1}_t \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_t \}. \]

Then we have \[ | \mathcal{H}\text{-}Spec(R^+_t) | > | \Gamma_t |. \] Since \[ | \Gamma_t | = 1^{r_1} 2^{r_2 - r_1} \ldots t^{r_t - r_{t-1}} (t + 1)^{n-r_t}, \]

we get \[ | \mathcal{H}\text{-}Spec(R^+_t) | > 1^{r_1} 2^{r_2 - r_1} \ldots t^{r_t - r_{t-1}} (t + 1)^{n-r_t}. \]

Thus, it follows from Remark 3.10 that

\[ \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | > t! S(n + 1, t + 1). \]

Hence we have

\[ \left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | \right)^2 > (t! S(n + 1, t + 1))^2. \]

Recall now (See 3.6) that

\[ | \mathcal{H}\text{-}Spec^{[t]}(R) | = \left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | \right)^2. \]

All this together shows that \[ | \mathcal{H}\text{-}Spec^{[t]}(R) | > (t! S(n + 1, t + 1))^2. \]

However, it follows from Theorem 3.11 that \[ | \mathcal{H}\text{-}Spec^{[t]}(R) | = (t! S(n + 1, t + 1))^2. \]

This is a contradiction and thus we have proved that \[ \mathcal{H}\text{-}Spec(R^+_t) = \{ J^+_{w_t, (\gamma_1, \ldots, \gamma_n)} S^{-1}_t \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_t \}. \]

\[ \blacksquare \]

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References

[1] T. Arakawa and M. Kaneko, *On poly-Bernoulli numbers*, Comment Math. Univ. St. Paul 48 (2) (1999), 159–167.

[2] G. Cauchon, *Quotients premiers de $O_\mathfrak{q}(\mathcal{M}_n(k))$*, J. Algebra 180 (1996), 530–545.

[3] ———, *Effacement des dérivation et spectres premiers d’algèbres quantiques*, J. Algebra. 260 (2003), 476–518.

[4] ———, *Spectre premier de $O_\mathfrak{q}(\mathcal{M}_n(k))$, image canonique et séparation normale*, J. Algebra. 260 (2003), 519–569.

[5] K.R. Goodearl and T.H. Lenagan, *Prime ideals invariant under winding automorphisms in quantum matrices*, Internat. J. Math. 13 (2002), 497–532.

[6] ———, *Winding-invariant prime ideals in quantum $3 \times 3$ matrices*, J. Algebra. 260 (2003), 657–687.

[7] K.R. Goodearl and E.S. Letzter, *Prime factor algebras of the coordinate ring of quantum matrices*, Proc. Amer. Math. Soc. 121 (1994), 1017–1025.

[8] ———, *Prime and primitive spectra of multiparameter quantum affine spaces*, Trends in ring theory (Miskolc, 1996), Canad. Math. Soc. Conf. Proc. Series, vol. 22, 1998, pp. 39–58.

[9] ———, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. 352 (2000), 1381–1403.

[10] M. Kaneko, *Poly-Bernoulli numbers*, J. de Théorie des Nombres de Bordeaux 9 (1997), 221–228.

[11] S. Launois, *Generators for the $\mathcal{H}$-invariant prime ideals in $O_\mathfrak{q}(\mathcal{M}_{m,p}(\mathbb{C}))$*, to appear in Proceedings of the Edinburgh Mathematical Society.

[12] ———, *Les idéaux premiers invariants de $O_\mathfrak{q}(\mathcal{M}_{m,p}(\mathbb{C}))$*, to appear in J. Algebra.

[13] ———, *Idéaux premiers $\mathcal{H}$-invariants de l’algèbre des matrices quantiques*, Thèse de doctorat, Université de Reims, 2003.

[14] B.J. Parshall and J.P. Wang, *Quantum Linear Groups*, Mem. Amer. Math. Soc., 439, 1991.

[15] R.P. Stanley, *Enumerative Combinatorics I*, Cambridge University Press, 1997.