The Role of Strict Positivity in Quantum Dynamics

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Motivated by quantum thermodynamics we investigate the notion of strict positivity, that is, linear maps which map positive definite states to something positive definite again. We show that strict positivity is decided by the action on any full-rank state, and that the image of not-strictly positive channels—up to something unitary—lives inside a lower-dimensional block. This implies that such channels have maximal distance from the identity channel. We use this to conclude that Markovian dynamics are strictly positive and investigate connections between strict positivity and other notions of divisibility.

A fundamental aspect of resource theories is finding conditions which characterize state-transfers via “allowed” operations. In quantum thermodynamics, for example, one usually asks whether a state can be generated from an initial state via a Gibbs-preserving quantum channel, that is, a CPTP (completely positive and trace-preserving) map which sustains the Gibbs-state of the system. [5, 12, 18, 26, 32] $^1$

As Gibbs states are of the form $e^{-\beta H}/\text{tr}(e^{-\beta H})$ for some system’s Hamiltonian $H$ and some inverse temperature $\beta > 0$, these states in particular are of full rank (equivalently: invertible)—actually every full-rank state $D$ is the Gibbs state of some system, simply choose $\beta = 1$ and $H = \ln(D^{-1})$. Therefore gaining a better understanding of the set of all channels with a given full-rank fixed point and its geometry, properties, etc. would greatly benefit the aforementioned state-conversion problem.

As a notable special case if the initial and final state commute with the Hamiltonian then one is in the classical realm and the problem reduces to $d$-vector majorization, and Gibbs-preserving channels become $d$-stochastic matrices [5, Appendix E]. In this case the state-conversion problem reduces to $n$ vector-1-norm inequalities (where $n$ is the dimension of the system) and the vectors $d$-majorized by some initial vector form a convex polytope with at most $n!$ extreme points. In addition, these extreme points can be easily computed analytically [14].

Every PTP (positive trace-preserving) map acting on a finite-dimensional system—and thus every quantum channel—maps the set of states into itself which by, e.g., Brouwer’s fixed-point theorem [6] guarantees the existence of a fixed point state. The question thus becomes how to guarantee that this state is of full rank.

Channels with a full-rank fixed point, sometimes called faithful [1], are characterized by not containing a decaying subspace (under the asymptotic projection). Also conserved quantities of such channels commute with all Kraus operators up to a phase [1, Prop. 1]. Topics related to faithful channels and fixed-point analysis of CPTP maps are (mean-)ergodic channels [7, 25], irreducible channels [13, 28], zero-error [4], [19, Ch. 4] and relaxation properties of discrete-time [11] and continuous-time [16, 29, 30] Markovian systems.

It turns out that Gibbs-preserving (i.e. faithful) channels belong to the larger class of strictly positive channels, that is, channels which map every full-rank state to something of full rank again (Prop. 1.2 below). In this work we study strictly positive maps, what distinguishes them from positive (but not strictly positive) ones, and which channels other than the faithful ones fall into this class. By doing so we hope to gain some further insight into faithful channels, or their complement, and the general structure of the set of quantum channels.
Section 1 features the definition of strict positivity, its properties and how it “interacts” with complete positivity. In Section 2 we study channels which are not strictly positive and prove that their image has to live inside a lower-dimensional block. Finally the interplay between strict positivity, dynamical systems and divisibility is explored in Section 3, before coming to conclusions and outlook.

1 Strict Positivity

As for notions and notations, the standard basis vectors of $\mathbb{C}^n$ will be denoted by $e_j$ or $|e_j\rangle$. Also recall that a matrix $X \in \mathbb{C}^{n \times n}$ is positive semi-definite, denoted by $X \geq 0$, if $\langle \psi, X \psi \rangle \geq 0$ for all $\psi \in \mathbb{C}^n$ and is positive definite, denoted by $X > 0$, if $\langle \psi, X \psi \rangle > 0$ for all non-zero $\psi \in \mathbb{C}^n$. One readily verifies that $X > 0$ if and only if $X \geq 0$ and $X$ is of full rank (equivalently: invertible).

Now positivity for linear maps can be adjusted to positive definiteness as follows (cf. [3, Ch. 2.2]):

**Definition 1.1.** A linear map $T : \mathbb{C}^{n \times n} \to \mathbb{C}^{k \times k}$ where, here and henceforth, $k, n \in \mathbb{N}$ are arbitrary, is called strictly positive (SP) if $T(X) > 0$ whenever $X > 0$. Moreover $T$ is called completely strictly positive (CSP) if $T \otimes \text{id}_m$ is strictly positive for all $m \in \mathbb{N}$.

These notions compare to usual positivity (P) and complete positivity (CP) as follows:

- Positive
- Completely positive
- Strictly positive
- Completely strictly positive

Figure 1: Relation between P, CP, SP and CSP. By definition CSP implies SP. A simple continuity-type argument shows that SP ⇒ P and CSP ⇒ CP. Moreover SP and CP are incomparable: the transposition map is obviously SP but not CP and, on the other hand, the trace projection $X \mapsto \text{tr}(X)|\psi\rangle \langle \psi|$ for some pure state $\psi$ is a quantum channel (hence CP) but evidently not SP, unless the dimension equals one.

Bhatia [3] observed that “a positive linear map $\Phi$ is strictly positive if and only if $\Phi(1) > 0$” so strict positivity can be easily checked. This, however, turns out to be a mere corollary of the following stronger result: the image of positive linear maps admit a “universal kernel” completely characterized by any positive definite matrix.

**Proposition 1.2.** Let $T : \mathbb{C}^{n \times n} \to \mathbb{C}^{k \times k}$ linear and positive be given. For all $X,Y,A \in \mathbb{C}^{n \times n}$, $X,Y > 0$ one has

$$\ker(T(X)) = \ker(T(Y)) \subseteq \ker(T(A)). \quad (1)$$

In particular, the following are equivalent:

(i) $T$ is strictly positive.

(ii) $T(1) > 0$

(iii) There exists $X > 0$ such that $T(X) > 0$.

**Proof.** Let $X \in \mathbb{C}^{n \times n}$ and a vector $\psi$ be given such that $X > 0$ and $T(X)\psi = 0$. Then for all $Y \in \mathbb{C}^{n \times n}$ positive semi-definite one finds $\lambda \in \mathbb{R}$ such that $\lambda X - Y \geq 0$. But then $T(\lambda X - Y) \geq 0$ by positivity of $T$ so linearity shows

$$0 \leq \langle \psi, T(\lambda X - Y)\psi \rangle = -\langle \psi, T(Y)\psi \rangle \leq 0.$$ 

Now $\|T(Y)\psi\|^2 = \langle \psi, T(Y)\psi \rangle = 0$ is equivalent to $T(Y)\psi = 0$ which shows $\ker(T(X)) \subseteq \ker(T(Y))$. The case of a general $A \in \mathbb{C}^{n \times n}$ follows from the fact that every matrix can be written as a linear combination of four positive semi-definite matrices [23, Coro. 4.2.4], together with linearity of $T$. Finally if $Y > 0$ then we can interchange the roles of $X,Y$ in the above argument to obtain $\ker(T(X)) = \ker(T(Y))$.

For the second statement—while (i) ⇒ (ii) ⇒ (iii) is obvious—for (iii) ⇒ (i) note that if $T(X) > 0$ for some $X > 0$, meaning $\ker(T(X)) = \{0\}$, then the same holds for all positive definite matrices by (1).

Prop. 1.2 shows that every faithful channel is strictly positive, whereas the converse does not hold (Example A.1).

**Remark 1.** (i) While strict positivity tells us that one cannot leave the relative interior of all states (the invertible states), the boundary (the non-invertible states) can be either mapped onto the boundary or into the interior. The former is achieved, e.g., by every

$^y$ E.g., choose $\lambda = \frac{y}{x}$ where $y$ is the largest eigenvalue of $Y$ and $x > 0$ is the smallest eigenvalue of $X$. *
unitary channel while the latter can be done via a trace projection onto some positive definite state.

(ii) In the generic case the inclusion in (1) is not an equality. Even “worse” there is no general statement to make about the rank regarding the image of strictly positive maps (see Example A.2).

So far we looked at SP and CP separately so let us study their interplay via the Kraus representation next: recall that a linear map $T : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{k \times k}$ is completely positive if and only if there exist $\{K_i\}_i \subset \mathbb{C}^{n \times k}$ such that $T(A) = \sum_i K_i^* A K_i$ for all $A \in \mathbb{C}^{n \times n}$ [10]. Unsurprisingly the "universal kernel" property from the previous proposition appears in this representation, as well.

**Lemma 1.3.** Let $T \in \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{k \times k}$ be linear and completely positive. Then the following are equivalent.

(i) $T$ is strictly positive.

(ii) For all sets of Kraus operators $\{K_i\}_{i \in I}$ of $T$ one has $\bigcap_{i \in I} \ker(K_i) = \{0\}$.

(iii) There exist Kraus operators $\{K_i\}_{i \in I}$ of $T$ with $|I| \leq nk$ such that $\bigcap_{i \in I} \ker(K_i) = \{0\}$.

If one, and hence all, of these conditions hold then $T$ is completely strictly positive.

**Proof.** By Prop. 1.2 strict positivity of $T$ is equivalent to $T(1) > 0$. Observe that, given any set of Kraus operators $\{K_i\}_{i \in I}$ of $T$, one has

$$\langle \psi, T(1) \psi \rangle = \sum_{i \in I} \langle \psi, K_i^* K_i \psi \rangle = \sum_{i \in I} ||K_i \psi||^2$$

so $\langle \psi, T(1) \psi \rangle = 0$ holds if and only if $\psi \in \ker(K_i)$ for all $i \in I$. Combining these two things readily implies the above equivalence.

For the additional statement we have to show that $T \otimes \text{id}_m$ is strictly positive for all $m \in \mathbb{N}$. But again by Proposition 1.2 this is equivalent to det($T(1) \otimes \text{id}_m)(1) \neq 0$ which holds due to

$$\text{det}((T \otimes \text{id}_m)(1_n \otimes 1_m)) = \text{det}(T(1))^m \cdot \text{det}(1_m)^k = \text{det}(T(1))^m \neq 0$$

using that $T$ is strictly positive as well as the determinant formula for the Kronecker product [21, Sec. 4.2].

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Therefore while CP and SP are incomparable, together they are equivalent to CSP. In summary we get the following refinement of Figure 1:

![Inclusion relations between P, CP, SP and CSP](image)

**2 Non-Strictly Positive Channels**

The fact that every positive map “produces” the same kernel on all full-rank states begs the question: what if this kernel is non-zero, that is, what is the footprint of positive maps which are not strictly positive? And how does this kernel manifest?

It turns out that such maps—up to a unitary channel on the target system—map into a sub-block of $\mathbb{C}^{k \times k}$ the size of which is determined by their action on the identity:

**Theorem 2.1.** Let $T : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{k \times k}$ be linear and positive. The following are equivalent.

(i) $T$ is not strictly positive.

(ii) There exists unitary $U \in \mathbb{C}^{k \times k}$ such that

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3While Figure 2 is not entirely accurate—as all the sets involved are convex cones, hence unbounded—it nonetheless gives a good intuition about the inclusion relations as well as underlying topological properties.
\[ \text{im}(\text{Ad}_{U^1} \circ T) \subseteq \mathbb{C}^{(k-m) \times (k-m)} \oplus 0^{n \times m}, \text{ i.e.} \]

\[ U^\dagger T(A)U = \begin{pmatrix} * & 0 \\ 0 & 0_m \end{pmatrix} \tag{2} \]

for all \( A \in \mathbb{C}^{n \times n} \) where \( m = \dim \ker(T(1)) \geq 1. \)

(iii) There exists \( \psi \in \mathbb{C}^n, \langle \psi, \psi \rangle = 1 \) such that 

\[ \langle \psi, T(A)\psi \rangle = 0 \text{ for all } A \in \mathbb{C}^{n \times n}. \]

If, in addition, \( T \) is trace-preserving then \( m < k. \)

**Proof.** (i) \( \Rightarrow \) (ii): Define \( \mathcal{K} := \ker(T(1)) \) and be aware that \( \mathcal{K} = \bigcap_{A \in \mathbb{C}^{n \times n}} T(A) \) by Prop. 1.2. By assumption \( T \) not strictly positive so \( T(1) \) is not invertible and thus \( m = \dim \mathcal{K} \geq 1. \) Now one finds an orthonormal basis of \( \mathcal{K}, \) i.e. \( U \in \mathbb{C}^{k \times k} \)

unitary such that \( \mathcal{K} = \text{span}\{Ue_{k-m+1}, \ldots, Ue_k\}. \)

Then for all indices \( j = k - m + 1, \ldots, k \) and \( A \in \mathbb{C}^{n \times n} \) one finds \( (U^\dagger T(A)U)e_j = 0 — 

because positivity of \( T \) in particular means that \( T \) preserves hermiticity. Hence in the basis induced by \( U, \) the \( j \)-th column as well as the \( j \)-th row are zero which shows (2).

(ii) \( \Rightarrow \) (iii): Choose \( \psi = Ue_n \) so for all \( A \)

\[ \langle \psi, T(A)\psi \rangle = \langle e_n, U^\dagger T(A)Ue_n \rangle \tag{2} \]

because positivity of \( T \) in particular means that \( T \) preserves hermiticity. Hence in the basis induced by \( U, \) the \( j \)-th column as well as the \( j \)-th row are zero which shows (2).

(iii) \( \Rightarrow \) (i): Finding a pure state \( \psi \) such that 

\[ \langle \psi, T(1)\psi \rangle = 0 \text{ means } T(1)\psi = 0 \text{ (because } T(1) \geq 0). \]

In particular \( T(1) \) is not of full rank so \( T \) is not strictly positive.

This establishes the equivalence of (i), (ii) and (iii). If \( T, \) additionally, is trace-preserving then \( T(1) \neq 0 \) so \( m < k. \)

\[ \square \]

In other words lack of strict positivity means that the image of such maps show a “loss of dimension”. Thus, as a direct application, if the co-domain are the \( 2 \times 2 \) matrices then the action of the map is limited to one entry (or it is zero, altogether):

**Corollary 2.2.** Let a qudit-to-qudit channel—i.e. \( T : \mathbb{C}^{n \times n} \to \mathbb{C}^{2 \times 2} \) cptp—be given which is not strictly positive. Then there exists \( \psi \in \mathbb{C}^2, \)

\[ \langle \psi, \psi \rangle = 1 \text{ such that } T(\rho) = \text{tr}(\rho)|\psi\rangle\langle \psi| \text{ for all } \rho \text{ so } T \text{ is the trace-projection onto a pure state.} \]

**Proof.** By Theorem 2.1 there exists \( U \in \mathbb{C}^{2 \times 2} \) unitary such that

\[ U^\dagger T(\rho)U = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \]

for all \( \rho \in \mathbb{C}^{2 \times 2}. \) Because \( T \) is trace-preserving \( * \) has to be of size 1 and thus is equal to \( \text{tr}(\rho). \)

Hence \( T(\rho) = \text{tr}(\rho)|Ue_1\rangle\langle Ue_1| \) for all \( \rho. \) Choosing \( \psi := Ue_1 \) concludes the proof.

\[ \square \]

Of course Corollary 2.2 holds not only for channels but all ptp maps as the proof did not exploit complete positivity. Moreover the requirement of the final system being a qubit system is essential as one can, unsurprisingly, construct cptp maps which are not strictly positive but are not a trace projection (cf. Example A.4 (i)).

**Remark 2.** So far we analyzed channels in the Schrödinger picture, but how does the above phenomenon manifest in the Heisenberg picture where the channels are completely positive and unital, hence SP by definition? Let \( T \) be positive but not strictly positive so, using the identity 

\[ \text{tr}(T(A)B) = \text{tr}(AT^*(B)) \]

which relates a positive linear map and its dual, we get

\[ \text{tr} \left( (\text{Ad}_{U^1} \circ T)(A)B \right) = \text{tr} \left( T(A)UBU^\dagger \right) = \text{tr} \left( AT^*(UBU^\dagger) \right) = \text{tr} \left( AT^* \circ \text{Ad}_{U^1}(B) \right) \]

with \( U \) being the corresponding unitary from Theorem 2.1. This shows that if the (pre-)dual channel \( T \) of a Heisenberg channel \( T^* \) is not strictly positive—so the action of \( \text{Ad}_{U^1} \circ T \) is contained in a \( (k-m) \times (k-m) \) block of \( \mathbb{C}^{k \times k} \)—then \( T^* \) (up to a unitary) is fully determined by a \( (k-m) \times (k-m) \) sub-block of the input \( B. \) To substantiate this we refer to Example A.4 (ii).

### 3 Strict Positivity, Dynamics and Divisibility

So far we learned that lack of strict positivity comes along with a loss of dimension which, when approaching from a more physical point of view, motivates the following question: given a dynamical system, that is, a family \( (T_t)_{t \geq 0} \) of cptp maps which is continuous in \( t \) and satisfies \( T_0 = \text{id}, \) can one determine the exact time when this dimension loss occurs? Do there even exist such processes which lack strict positivity and if so, can one classify these cases?
As the most well-understood and well-structured dynamical systems are those of semigroup structure, meaning they satisfy $T_{t+s} = T_t T_s$ for all $s, t \geq 0$, let us first have a look at these. Then the family $(T_t)_{t \geq 0}$ is a quantum-dynamical semigroup and the corresponding process is termed Markovian, and is of the form $T_t = e^{tL}$ with the generator $L : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ being of specific structure ("cGKL-form") comprised of a Hamiltonian and a dissipative part [17, 24].

**Theorem 3.1.** Let $(T_t)_{t \geq 0}$ be a quantum-dynamical semigroup acting on $\mathbb{C}^{n \times n}$. Then $T_t$ is strictly positive for all $t \geq 0$.

**Proof.** Assume to the contrary that there exists $t > 0$ such that $T_t$ is not strictly positive—note that $T_0 = \text{id}$ is SP so $t$ has to be strictly positive, as well. Writing $T_t = T_{t/2} T_{t/2}$ we get that $T_{t/2}$ cannot be SP either—otherwise $T_t$ is the composition of two SP maps, hence SP itself. Proceeding inductively we obtain a sequence $(T_{t/2^n})_{n \in \mathbb{N}}$ of non-SP maps which converge to $T_0$ by continuity of the semigroup in $t$. But this, as shown in the proof of Lemma A.3 (ii), implies that $T_0 = \text{id}$ is not strictly positive, a contradiction. \hfill \Box

Therefore (Markovian) cooling processes—or any relaxation process the steady state of which is not invertible—have to take an infinite amount of time. This is not too surprising as the dissipation happens exponentially in time.

Picking up the above idea of splitting the (non-SP) channel into smaller pieces finally leads us to the concept of divisibility. After all, just like with (complete) positivity, the strictly positive maps form a semigroup so if a map $T$ which is not SP can be written as $T = T_1 T_2$ then $T_1$ or $T_2$ (or both) are not SP either. To investigate whether this leads to a deeper connection between strict positivity and divisibility recall that a channel is called Markovian if it is an element of a quantum-dynamical semigroup which, arguably, is the strongest notion of divisibility. Hence Theorem 3.1 readily implies:

**Corollary 3.2.** Every Markovian quantum channel is strictly positive.

As for the weaker notions of divisibility, for which we orient ourselves towards [33], we are not as lucky:

- There exist non-divisible channels which are strictly positive (Example A.5).
- There exist divisible channels which are not strictly positive (Example A.6).
- There exist infinitely divisible channels which are not strictly positive: just take the trace projection $T : \rho \mapsto \text{tr}(\rho)\langle \psi | \psi \rangle$ onto a pure state as $T$ is idempotent so $T = T^2 = \ldots = T^n$ for all $n \in \mathbb{N}$.

In-between usual divisibility and infinite divisibility one finds the concept of *infinitesimal divisibility*\(^5\) for which the situation is already more interesting. First one defines $\mathcal{I}$ as the set of channels $T$ with the following property: for all $\varepsilon > 0$ there exist finitely many CPTP maps $\{T_i\}_i$ such that $\|T_i - \text{id}\| \leq \varepsilon$ and $T = \prod_i T_i$. With this a channel is infinitesimal divisible if it belongs to the closure $\overline{\mathcal{I}}$.

Let us focus on the norm condition here: the identity is SP and the latter is open (Lemma A.3) to one finds $\varepsilon' > 0$ such that every linear map $\varepsilon'$-close to $\text{id}$ is strictly positive as well. More precisely one finds:

**Lemma 3.3.** Let $T : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ be linear and positive but not strictly positive. Then

$$\|T - \text{id}\| = \sup_{\|A\|_1 = 1} \|T(A) - A\|_1 \geq 1 \quad (3)$$

with $\|\cdot\|_1 = \text{tr}(\sqrt{\cdot}^\dagger \sqrt{\cdot})$ being the trace norm.

If $T$ additionally is trace-preserving one has $\|T - \text{id}\| = 2$ so the distance between $T$ and the identity channel is maximal.

**Proof.** By Thm. 2.1 one finds $U \in \mathbb{C}^{n \times n}$ unitary with $\operatorname{im} \{\text{Ad}_{U^\dagger} \circ T\} \subseteq \mathbb{C}^{(n-m) \times (n-m)} \oplus 0^{m \times m}$ where $m = \dim \ker(T(\mathbb{I})) \geq 1$. Because $\|U e_n\langle U e_n \|_1 = 1$ we by unitary equivalence of the trace norm compute

$$\|T - \text{id}\| \geq \|T(U e_n\langle e_n | U^\dagger - U| e_n\langle e_n U^\dagger\|_1$$

$$= \|\{\text{Ad}_{U^\dagger} \circ T\}(U e_n\langle e_n | U^\dagger - | e_n\langle e_n\|_1$$

$$= \|\star_{n-m} \oplus 0_{m-1} \oplus (-1)\|_1 = \|\star_{n-m} \|_1 + | -1|$$

$$= \|T(U e_n\langle e_n | U^\dagger\|_1 + 1$$

$$= \text{tr}(T(U e_n\langle e_n | U^\dagger)) + 1 \geq 1. \quad (4)$$

\(^5\)As this is known to be equivalent to continuous completely positive evolutions as well as time-dependent Markovian channels [33, Thm. 16] it suffices to only consider channels which are infinitesimal divisible.
In the last step we used that $\|A\|_1 = \text{tr}(A)$ for all $A \geq 0$ as well as positivity of $T$—this proves (3). If $T$ additionally is trace-preserving then (4) is obviously equal to 2. To show that this is also an upper bound recall that $\|T\| = \|\text{id}\| = 1$ because every PTP map is trace norm-contractive [27]. This by the triangle inequality yields

$$2 \leq \|T - \text{id}\| \leq \|T\| + \|\text{id}\| = 2$$

which concludes the proof. \qed

Lemma 3.3 gives a necessary criterion for lack of strict positivity which, however, is not sufficient (cf. Example A.7). Either way this result leads us to the following:

**Proposition 3.4.** Every $T \in I$ is strictly positive. However, there exist channels which are infinitesimally divisible but not strictly positive.

**Proof.** Let $T \in I$ be given so one finds channels $T_1, \ldots, T_m$ such that $\|T_i - \text{id}\| \leq 1$ for all $i = 1, \ldots, m$ and $T = T_1 \cdots T_m$. By Lemma 3.3 every $T_i$ is strictly positive so, because $\text{sp}$ forms a semigroup, $T$ is strictly positive.

For the second statement, the counterexample for the infinitely divisible channels from before $(\rho \mapsto \text{tr}(\rho)|\psi\rangle\langle\psi|)$ stays valid as those channels form a subset of $I$. \qed

While the norm requirement forces $I \subseteq \text{sp}$ taking the closure is what throws us off here: the trace-projection from the counterexample can be approximated arbitrarily well by a Markovian cooling process, i.e. a quantum-dynamical semigroup (which in particular is strictly positive, cf. Example A.8).

### 4 Conclusion and Outlook

Strict positivity is an inherent feature of most quantum processes, in particular the faithful and the Markovian ones as well as those which do not have maximal distance from the identity channel. This is due to the fact that

- the image of any full-rank state under a positive linear map has the same kernel (Proposition 1.2).
- the strictly positive maps form an open, convex subsemigroup of the linear maps and they are dense in the positive maps (Lemma A.3).

• lack of strict positivity comes along with a loss in dimension of the image (Theorem 2.1).

These results may be a step towards gaining a deeper understanding what characterizes faithful channels and, ultimately, solving the quantum thermodynamics problem from the introduction:

*Given two states $\rho, \omega$ and a full-rank state $D$, can one find a finite set of (easy-to-verify) inequalities which characterize the existence of a channel $T$ such that $T(\rho) = \omega$ and $T(D) = D$?*

A possible next step could be to investigate the classical case (i.e. column-stochastic matrices) and what guarantees the existence of a fixed point with strictly positive entries which, admittedly, might be too much of a restriction. Our findings may also be interesting for extension problems of Alberti-Uhlmann type [2, 8, 9, 12, 20, 22]: when deciding whether there exists a quantum channel which maps a set of input states $\{\rho_i\}_i$ to a set of output states $\{\tau_i\}_i$, checking whether span$\{\rho_i\}_i$ contains a full-rank state can restrict the problem to either the strictly positive channels or their complement.

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A  Examples and Further Results

Example A.1. The linear map

\[ T : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2} \]

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \mapsto \begin{pmatrix}
a_{11} + \frac{1}{2}a_{22} & 0 \\
0 & \frac{1}{2}a_{22}
\end{pmatrix}
\]

is obviously CPTP and strictly positive \((T(1)) > 0\) but the only fixed points of \(T\) are of the form \(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\), that is, not of full rank.

Example A.2. Consider the channel

\[ T : \mathbb{C}^{3 \times 3} \to \mathbb{C}^{3 \times 3} \]

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} \mapsto \begin{pmatrix}
a_{22} + a_{33} & 0 & 0 \\
0 & \frac{1}{2}a_{11} & 0 \\
0 & 0 & \frac{1}{2}a_{11}
\end{pmatrix}
\]

In particular this map is strictly positive by Prop. 1.2 (iii) as \(\text{diag}(2,1,1)\) is a fixed point. However

\[
1 = \text{rank}(\langle e_1 | e_1 \rangle) < \text{rank}(T(\langle e_1 | e_1 \rangle)) = 2
\]

\[
2 = \text{rank}(\langle e_2 | e_2 \rangle + \langle e_3 | e_3 \rangle) > \text{rank}(T(\langle e_2 | e_2 \rangle + \langle e_3 | e_3 \rangle)) = 1.
\]

Lemma A.3. Taking \(P, CP, SP\) and \(CSP\) as subsets of the set of all linear maps \(L\) (with the induced subspace topology) the following statements hold.

(i) \(P\) and \(CP\) are closed, convex subsets of \(L\).

(ii) \(SP\) is an open, convex subset of \(L\).

(iii) \(SP\) is dense in \(P\) and \(\text{relint}(P) = SP\).

Proof. The convexity statements are obvious so we only prove what remains. (i): Let \((T_m)_{m \in \mathbb{N}}\) be a sequence in \(P\) which converges to \(T \in L\). Then for all \(A \geq 0\) and all \(\psi \in \mathbb{C}^n\)

\[
\langle \psi, T(A)\psi \rangle = \lim_{m \to \infty} \langle \psi, T_m(A)\psi \rangle \geq 0
\]

so \(T(A) \geq 0\), hence \(T\) is in \(P\). The proof for \(CP\) is analogous. (ii): We will show, equivalently, that its complement is closed. Again let \((T_m)_{m \in \mathbb{N}}\) be a sequence in \(L \setminus SP\) which converges to some \(T \in L\). Using Prop. 1.2 we get \(\det(T(1)) = \lim_{m \to \infty} \det(T_m(1)) = 0\) by continuity of the determinant so \(T \in L \setminus SP\) as claimed.

(iii): Density can be shown constructively: if \(T\) is in \(P\) then \(((1 - \frac{1}{m})T + \frac{1}{m} \text{id})_{m \in \mathbb{N}}\) is a sequence in \(SP\) which approximates \(T\). For the relative interior note that because \(P\) is a non-empty convex subset of \(\mathbb{C}^{n^2 \times k^2} \simeq \mathbb{R}^{2n^2k^2}\) one has [31, Thm. 6.4]

\[
\text{relint}(P) = \{T \in P \mid \forall S \in P \exists \lambda > 1 \lambda T + (1 - \lambda)S \in P\}.
\]

“\(\subseteq\)”: Let \(T \in \text{relint}(P)\) so there exists \(\lambda > 1\) such that \(\tilde{T} := \lambda T + (1 - \lambda) \text{id}\) is positive. Defining \(\mu := 1 - \frac{1}{m} \in (0,1)\) we get that \(T = \mu \text{id} + (1 - \mu)\tilde{T}\), hence \(T\) is strictly positive. “\(\supseteq\)”: Let \(T \in SP\) as well as \(S \in P\) be given. We have to find \(\lambda > 1\) such that \(\lambda T + (1 - \lambda)S \in P\). By (ii) \(SP\) is open so \(B_\varepsilon(T) \cap L \subseteq SP\) for some \(\varepsilon > 0\). Defining \(C := \|T\| + \|S\| (> 0\ because T \neq 0\), choosing \(\lambda := 1 + \frac{\varepsilon}{2C}\) does the job:

\[
\|T - (\lambda T + (1 - \lambda)S)\| = |1 - \lambda|\|T - S\| \leq \frac{\varepsilon}{2C}(\|T\| + \|S\|) = \frac{\varepsilon}{2} < \varepsilon
\]

so \(\lambda T + (1 - \lambda)S \in B_\varepsilon(T) \cap L \subseteq SP \subseteq P\).
Example A.4.  (i) The Choi matrix of the linear map

\[ T : \mathbb{C}^{3 \times 3} \to \mathbb{C}^{3 \times 3} \]

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  a_{11} & i \sqrt{2} (a_{12} + a_{13}) \\
  -i \sqrt{2} (a_{21} + a_{31}) & a_{22} + a_{33} \\
  0 & 0 & 0
\end{pmatrix}
\]

has simple eigenvalues 2,1 and the 7-fold eigenvalue 0 so \( T \) is CPTP, not SP and not a trace projection, i.e. not of the form \( A \mapsto \text{tr}(A) \rho \) for any state \( \rho \).

(ii) Via \( \text{tr}(T(A)B) = \text{tr}(AT^*(B)) \) for all \( A,B \in \mathbb{C}^{3 \times 3} \) the dual of \( T \) from (i) turns out to be

\[ T^* : \mathbb{C}^{3 \times 3} \to \mathbb{C}^{3 \times 3} \]

\[
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23} \\
  b_{31} & b_{32} & b_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  b_{11} & -\frac{i}{\sqrt{2}} b_{12} & -\frac{i}{\sqrt{2}} b_{13} \\
  \frac{i}{\sqrt{2}} b_{21} & b_{22} & 0 \\
  \frac{i}{\sqrt{2}} b_{21} & 0 & b_{22}
\end{pmatrix}
\]

Note that

\[ T^*(B) = T^* \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

for all \( B \in \mathbb{C}^{3 \times 3} \) so the action of \( T^* \) is determined by a smaller block of \( B \) already.

Example A.5. Consider the linear map

\[ T : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2} \]

\[ \rho \mapsto \rho^T + \frac{1}{3} \text{tr}(\rho) \]

which is CPTP and indivisible [33, Ex. 4 & Coro. 10]. But \( T \) is unital hence strictly positive.

Example A.6. The following map is divisible, CPTP but not SP:

\[ S : \mathbb{C}^{3 \times 3} \to \mathbb{C}^{3 \times 3} \]

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  a_{11} & 0 & 0 \\
  0 & a_{22} + a_{33} & 0 \\
  0 & 0 & 0
\end{pmatrix}
\]

To see this note that \( S = T_1 T \) with

\[ T_1 : \mathbb{C}^{3 \times 3} \to \mathbb{C}^{3 \times 3} \]

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  a_{11} & 0 & 0 \\
  0 & a_{22} & 0 \\
  0 & 0 & a_{33}
\end{pmatrix}
\]

being SP (because unital) and \( T \) from Example A.4 not being SP.

Example A.7. Consider the unitary matrix \( \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and the induced channel \( T : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2} \), \( \rho \mapsto \sigma \rho \sigma \). Then

\[ \| T - \text{id} \| \geq \| T(e_1)\langle e_1 | - | e_1 \rangle e_1 \|_1 = \| | e_2 \rangle \langle e_2 | - | e_1 \rangle \langle e_1 | \|_1 = 2 \]

but as a unitary channel, \( T \) is unital hence strictly positive.
Example A.8. Let \( H = \frac{1}{2} \sigma_z = \text{diag}(\frac{1}{2}, -\frac{1}{2}) \), \( V = \sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) \) and

\[
L : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2}
\]

\[
\rho \mapsto -i[H,\rho] - \frac{1}{2}(V^\dagger V\rho + \rho V^\dagger V - 2V\rho V^\dagger)
\]

so

\[
L \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} -\rho_{11} & -(i + \frac{1}{2})\rho_{12} \\ (i - \frac{1}{2})\rho_{21} & \rho_{11} \end{pmatrix}
\]

and thus for all \( t \geq 0 \)

\[
e^{tL}(\rho) = \begin{pmatrix} e^{-t}\rho_{11} & e^{-t(\frac{1}{2}+i)}\rho_{12} \\ e^{-t(\frac{1}{2}-i)}\rho_{21} & (1 - e^{-t})\rho_{11} + \rho_{22} \end{pmatrix}.
\]

Because \( L \) is of GKS-form, \((e^{tL})_{t \geq 0}\) is a quantum-dynamical semigroup \([17, 24]\) which even is relaxing as \( \lim_{t \to \infty} e^{tL}(\rho) = \text{tr}(\rho)|e_2\rangle\langle e_2| =: T(\rho) \). This channel is not strictly positive but is the limit of \((e^{kL})_{k \in \mathbb{N} \subset \mathbb{I}}\), hence \( T \) is infinitesimal divisible.

This example can be generalized to \( n \)-level systems via the spin-\( j \) representation of \( SU(2) \), choosing

\[
H = \sum_{k=1}^{n} \left( \frac{n + 1 - 2k}{2} \right)|e_k\rangle\langle e_k| = \text{diag} \left( \frac{n - 1}{2}, \frac{n - 3}{2}, \ldots, -\frac{n + 3}{2}, -\frac{n + 1}{2} \right)
\]

\[
V = \sum_{k=1}^{n-1} \sqrt{k(n-k)}|e_{k+1}\rangle\langle e_k|.
\]

in (5) one readily verifies that \((e^{tL})_{t \geq 0}\) has steady state \(|e_n\rangle\langle e_n|\) and again is relaxing\(^6\). Thus \( \rho \mapsto \text{tr}(\rho)|e_n\rangle\langle e_n| \) is the limit of \((e^{kL})_{k \in \mathbb{N} \subset \mathbb{I}}\), hence infinitesimal divisible but not strictly positive.

\(^6\) To see this one can show that the diagonal of an arbitrary initial state \( \rho \) converges to \((0, \ldots, 0, 1) \) \([15]\) and because the set of quantum states is closed the limit has to be positive semi-definite, hence the semigroup relaxes into \(|e_n\rangle\langle e_n|\).