Force on a moving proton vortex in a superfluid neutron star

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Abstract. A controversy is resolved about the form of the Lorentz-type force on proton vortices moving with respect to the bulk neutron-star matter. The result is important for modeling the evolution of stellar magnetic field.

1. Introduction and formulation of the problem

The magnetic field in neutron stars (NSs) varies in a very wide range from $\sim (10^8 - 10^9)$ G in millisecond pulsars and neutron stars in low-mass X-ray binaries to $\sim 10^{12}$ G in ordinary radio pulsars and up to $\sim 10^{15}$ G in magnetars [1]. It is a challenge for theorists to explain such diverse objects within a unified theoretical model. The problem is significantly complicated by the fact that NS matter can become superfluid/superconducting at stellar temperatures $T \sim (10^8 - 10^{10})$ K. The magnetic field in such matter is confined to proton vortices, also called Abrikosov vortices or proton flux tubes [2]. Therefore, to describe the evolution of the magnetic field in NSs it is necessary to understand in detail the vortex dynamics, a very complicated problem, full of controversies in the literature, which has not been fully solved yet (see [3] for a recent review). Here we would like to focus on one such controversy related to the forces acting on proton vortices in superconducting NSs at zero temperature.

We shall start with the equation, describing the magnetic field evolution in superconducting NSs (see, e.g., [4]),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V}_L \times \mathbf{B}),$$

(1)

where $\mathbf{B}$ is the stellar magnetic field, averaged over the volume containing many vortices; $\mathbf{V}_L$ is the local vortex velocity. This equation simply states, that the magnetic field, confined to proton vortices, is transported with the vortex velocity, $\mathbf{V}_L$. To solve (1) one needs first to define $\mathbf{V}_L$, i.e., to express it through available particle velocities in the system. For example, for the neutron-proton-electron mixture at zero temperature ($T = 0$) and neglecting entrainment effects [5], the relevant velocities are the electron velocity, $\mathbf{u}_e$, and the superfluid proton velocity, $\mathbf{V}_{sp}$. To express $\mathbf{V}_L$ through $\mathbf{u}_e$ and $\mathbf{V}_{sp}$, one should write down a force balance equation for a vortex, which can be customarily presented in the form (see, e.g., [6]):

$$\mathbf{F}_{\text{buoyancy}} + \mathbf{F}_{\text{tension}} + \mathbf{F}_{\text{npe}} = 0.$$  

(2)
In equation (2) $F_{\text{buoyancy}}$ and $F_{\text{tension}}$ are the buoyancy and tension forces, respectively (their actual form is not important for us here; see, e.g., [7] for details); $F_{\text{npe} \rightarrow V}$ is the velocity-dependent force from npe-matter on a vortex. It equals

$$F_{\text{npe} \rightarrow V} = F_M + (D' + \pi h n_p) [e_z \times (u_e - V_L)] - D [e_z \times (e_z \times (u_e - V_L))],$$

where $F_M$ is the Magnus force [6],

$$F_M = -\pi h n_p [e_z \times (V_{sp} - V_L)] = -\pi h n_p [e_z \times (u_e - V_L)];$$

(we make use of the so called ‘screening condition’, $V_{sp} = u_e$, in the second equality [4, 6, 8, 9]).

The first two terms in equation (3) describe the transverse (Lorentz-type) part of the force on a vortex, while the last term ($\propto D$) describes the longitudinal (dissipative) part of the force. In equations above $n_p$ is the proton number density; $e_z$ is the unit vector along the vortex line (see Fig. 1).

At this point we face a controversy in the literature regarding the value of the coefficient $D'$ in equation (3). According to [8, 9] $D' = 0$, hence the Magnus force (4) exactly cancels out the underlined term in equation (3). Jones then interpreted the underlined term as the ‘Lorentz force on the electrons’. Unlike Jones, Ref. [10] postulated (without justification) that $D' = -\pi h n_p$, which vanishes the second term in equation (3). In turn, [6] also assumed that $D' = -\pi h n_p$, arguing that the underlined term is ‘a Lorentz-type force due to the interaction of the local vortex magnetic field and the charged fluids’, i.e., a sum of the (minus) Lorentz forces on the electrons and superconducting protons. So, what is the correct value of $D'$? The answer is very important since, depending on the choice of $D'$, the typical magnetic field evolution timescales, defined by equation (1), can vary by orders of magnitude [7, 9, 11–13]. The present contribution, which is based on the work [14], is devoted to answering this question.

2. Basic parameters and hierarchy of lengthscales

Schematically, the proton vortex consists of the ‘normal’ core with the radius of the order of the coherence length, $\xi$, surrounded by the more extended region containing the magnetic field (see Fig. 1). The radius of that region is $\sim \lambda$, where $\lambda$ is the London penetration depth. The parameters $\xi$ and $\lambda$ are given by the formulas (e.g., [15])

$$\xi = \frac{\hbar p_{Fp}}{\pi m_p^* \Delta_p} \approx 28 \text{ fm} \left( \frac{n_p}{0.18 n_0} \right)^{1/3} \left( \frac{m_p}{m_p^*} \right)^2 \left( \frac{0.456 \text{ MeV}}{\Delta_p} \right);$$

$$\lambda = \sqrt{\frac{m_p c^2}{4\pi \epsilon_0^2 n_{sp}}} \approx 42.4 \text{ fm} \left( \frac{0.18 n_0}{n_{sp}} \right)^{1/2}.$$  

Here $p_{Fp}$, $n_p$, $m_p$, $m_p^*$, $\epsilon_p$ are the proton Fermi momentum, number density, mass, effective mass, and electric charge, respectively; $n_0 = 0.16 \text{ fm}^{-3}$ is the nuclear matter density; $n_{sp}(T)$

![Figure 1. Scheme of a proton vortex. The vortex magnetic field is directed along the axis $z$. Centre of the vortex corresponds to $x = y = 0$.](image-url)
and $\Delta_p(T)$ are the superfluid proton number density and energy gap, respectively. At $T = 0$ one has $n_{sp} = n_p$ and $\Delta_p \approx k_B T_{cp}/0.567$, where $T_{cp}$ is the proton critical temperature. In particular, $\Delta_p \approx 0.456$ MeV for $T_{cp} = 3 \times 10^9$ K. Following [16], we parametrize the vortex magnetic field as

$$B(r) = e_z \left( \frac{\Phi}{\pi \xi^2} \right) \begin{cases} 1 - (\xi/\lambda) K_1(\xi/\lambda) I_0(r/\lambda), & 0 \leq r < \xi; \\ (\xi/\lambda) I_1(\xi/\lambda) K_0(r/\lambda), & r \geq \xi, \end{cases}$$

(7)

where $\Phi$ is the magnetic flux associated with the vortex line; for a proton vortex $\Phi = \Phi_0 = \pi h c / e_p$.

The important characteristic of magnetized neutron stars is the average distance between neighboring vortices, $d_B = 4.89 \times 10^3$ fm $(10^{12} G / B)^{1/2}$ (e.g., [15]). For not too strong magnetic fields $d_B$ is much larger than $\lambda$. Another important parameter in our problem is the typical electron wavelength (divided by $2\pi$):

$$1/k_{Fe} = 1.05 \text{ fm} (0.18 n_0 / n_e)^{1/3},$$

(8)

where $k_{Fe} = p_{Fe} / h$ with $p_{Fe}$ being the electron Fermi momentum. One sees that it is much smaller than $\xi$, $\lambda$, and $d_B$. This enables us to study the forces acting on an isolated vortex within the quasiclassical approximation and ignoring the presence of other vortices, i.e., the collective effects.

Finally, the last important parameter that should be mentioned here is the electron mean free path, $l$. Generally, it is much larger than $d_B$ and hence than other typical lengthscales discussed above. This means that at distances $r \ll l$ a perturbation of the electron distribution function caused by the vortex can be found from the collisionless kinetic equation for electrons. This property will be used in the next section. Summarizing, there are five relevant lengthscales, $\xi$, $\lambda$, $d_B$, $1/k_{Fe}$, and $l$, in the problem of calculation of the force acting on a vortex, and for typical NS conditions they are related by the inequality $l \gg d_B \gg r_0 \gg \lambda$. The result can be presented in the form similar to that obtained by Sonin [17] in his study of the forces acting on vortices in superfluid helium II:

$$D = \frac{1}{2} \sum_{p_0} \left( -\frac{d n_{p0}}{d \epsilon_p} \right) p_{\perp}^2 v_{\parallel} \sigma_\parallel(p_{\perp}), \quad D' = \frac{1}{2} \sum_{p_0} \left( -\frac{d n_{p0}}{d \epsilon_p} \right) p_{\perp}^2 v_{\perp} \sigma_{\perp}(p_{\perp}).$$

(9)

Here $p_{\perp}$ and $v_{\perp}$ are, respectively, the projections of the electron momentum and velocity on the plane $xy$ (see Fig. 1); $n_{p0}$ and $\epsilon_p$ are the electron Fermi-Dirac distribution function and energy, respectively. Finally, $\sigma_\parallel$ and $\sigma_{\perp}$ are the two cross-sections, which are related to the differential cross-section $\sigma$, describing an electron scattering off the magnetic field of a vortex, by the formulas:

$$\sigma_\parallel = \int_{-\pi}^{\pi} \sigma(\gamma) (1 - \cos \gamma) d\gamma, \quad \sigma_{\perp} = \int_{-\pi}^{\pi} \sigma(\gamma) \sin \gamma d\gamma.$$  

(10)

Because the electron wavelength, $1/k_{Fe}$, is much smaller than the typical lengthscales of the magnetic field variation, $\lambda$, the differential cross-section can be found by solving an elementary
4. Results and comparison with the previous works

Using equations (9) and (11) we can calculate the coefficients $D$ and $D'$:

\[ D = \frac{3\pi}{8} \left( \frac{e_e}{e_p} \right) \left( \frac{\Phi}{\Phi_0} \right) \left( \frac{1}{k_{Fe}} \frac{\lambda}{\xi} \right) G(\lambda/\xi), \]

\[ D' = \frac{e_e}{c} n_e \Phi = \pi \hbar n_e \left( \frac{e_e}{e_p} \right) \left( \frac{\Phi}{\Phi_0} \right), \]

where $n_e = p_{Fe}^3/(3\pi^2\hbar^3)$. One sees that $D > 0$, which means that the (dissipative) longitudinal force on a vortex, $F_{||} \equiv D(\mathbf{u}_e - \mathbf{V}_L)$, acts in the direction of the axis $y$. This is an expected result since the momentum of electrons along the axis $y$ decreases in the course of scattering. In turn, $D' < 0$, i.e., the transverse force on a vortex, $F_{\perp} \equiv D'[\mathbf{e}_z \times (\mathbf{u}_e - \mathbf{V}_L)]$, acts in the direction of the axis $x$. This result is also reasonable, since exactly the same force (Lorentz force) acts on electrons in the opposite direction.

One may note that $F_{\perp}$ coincides with the Magnus force $F_M$ (see equation 4), which is the transverse force acting on a vortex from superconducting protons and is well-defined for the extreme type-II superconductors (when $\xi \ll \lambda$). To understand this coincidence, assume for a moment that $\xi \ll \lambda$ for our problem and recall that the force $F_{npe \rightarrow V} = F_{||} + F_{\perp}$, calculated by us above, is the total force from the neutron-proton-electron mixture on a vortex. What are the actual mechanism and an actual particle species participating in transferring the momentum to the vortex core we have not discussed. Clearly, this cannot be neutrons, because they in no way interact with a vortex. Also, this cannot be electrons because they, generally, scatter off the magnetic field localized far from the vortex core ($\lambda \gg \xi$). This magnetic field is generated and supported by the superconducting proton currents, consequently, scattered electrons transfer their momentum to superconducting proton component, but not to the vortex. We come to conclusion that in this example only protons are able to transfer the momentum directly to the vortex core. How does it happen? The mechanism of the transverse force appearance is essentially the same as in liquid helium-II [18]. This is because the proton superfluid velocity, generated by the vortex, scales as $\propto 1/r$ at distances $r$ from the vortex centre, such that
\( \xi \ll r \ll \lambda \) [15, 19]. The superfluid velocity near the vortex core in helium-II behaves in exactly the same way and this is known to produce a transverse force on a vortex if a superfluid transport current is applied to the system [18]. This force can be found by considering a momentum, carried by superconducting protons per unit time through the walls of a cylinder of radius \( r \), with the result that it equals \( \mathbf{F}_M \) [18, 19]. Therefore, it is not surprising that \( \mathbf{F}_\perp = \mathbf{F}_M \) if \( \xi \ll \lambda \). But the expression for the force \( \mathbf{F}_\perp \) does not contain \( \xi \) and/or \( \lambda \), so this result should remain unchanged in the more general case of arbitrary ratio between these parameters.

Note that our results for the transverse force \( \mathbf{F}_\perp \) agree with the assumptions about the form of the force made, e.g., in [6, 10] and disagree with the conclusions of [8, 9], where it is argued that \( \mathbf{F}_\perp = 0 \).

Now, let us discuss in some more detail the longitudinal force \( \mathbf{F}_\parallel \) on a vortex and compare it with the results available in the literature. The longitudinal force due to electron scattering off the vortex magnetic field was calculated in [20] within classical mechanics and coincides with our result (12). The force \( \mathbf{F}_\parallel \) was also calculated in [16]. Strictly speaking, [16] considered a bit different problem, namely, the electron scattering off the neutron vortices, which can carry a magnetic field due to the entrainment effect [5]. However, their solution can easily be applied to our problem. [16] used a very different method of derivation of \( \mathbf{F}_\parallel \) and, moreover, worked in the Born approximation. Meanwhile, this approximation is unjustified for relatively large magnetic fluxes, associated with the vortex, \( \Phi \sim \Phi_0 \), for which it can lead to incorrect results. Thus, it is interesting to look whether our force \( \mathbf{F}_\parallel \) differs from that of [16].

Actually, [16] calculated the so called ‘velocity coupling time between the plasma and the core superfluid’, \( \tau_v \). In the limit of \( k_{Fv} \xi \gg 1 \) it is given by

\[
1/\tau_v = 3p_{Fe}c/(2m_p c^2 \alpha \tau_0) \ G(\lambda/\xi),
\]

where \( \alpha = 2p_{Fe} / \hbar \); \( \tau_0^{-1} = \pi N_p \Phi^2 \); and \( N_p \) is defined by equation (25) in [16]. As shown, e.g., in [21], this relaxation time is related to the force on a vortex per unit length by the formula:

\[
\mathbf{F}_{Alpar \parallel} = m_p n_p (\mathbf{u}_e - \mathbf{V}_L)/(n_v \tau_v).
\]

Plugging equation (14) into (15) and comparing the coefficient in front of \( \mathbf{u}_e - \mathbf{V}_L \) with the expression (12), one verifies that, somewhat unexpectedly, \( \mathbf{F}_{Alpar \parallel} = \mathbf{F}_\parallel \), i.e., the longitudinal force calculated by [16] exactly coincides with our result.

Equations (12) and (13) determine the force on a single vortex. However, in astrophysical applications one is usually interested in the force density \( \mathbf{J}_{npe\to V} \) acting on a system of vortices. Assuming that we have a locally rectilinear array of proton vortices with the surface density \( n_v = \mathcal{B}/\Phi \), one can present \( \mathbf{J}_{npe\to V} \) as:

\[
\mathbf{J}_{npe\to V} = -\mathcal{D} [\mathbf{e}_z \times (\mathbf{e}_z \times (\mathbf{u}_e - \mathbf{V}_L))] + \mathcal{D}' [\mathbf{e}_z \times (\mathbf{u}_e - \mathbf{V}_L)],
\]

where \( \mathcal{D} = n_v D \) and \( \mathcal{D}' = n_v D' \). One sees that

\[
\mathcal{D}/\mathcal{D}' = D/D' \approx 3\pi^2/(64k_{Fe} \lambda) \approx 0.46(k_{Fe} \lambda)^{-1}, \quad \lambda \gg \xi.
\]

Generally, \( \mathcal{D} \) and \( \mathcal{D}' \) are much smaller than, respectively, \( \mathcal{D}' \) and \( \mathcal{D}' \) for typical NS conditions, for which \( k_{Fe} \lambda \sim (30 - 50) \). This is also illustrated in Fig. 2, where the coefficients \( \mathcal{D} \) and \( \mathcal{D}' \) are plotted as functions of \( n_v \) for \( \mathcal{B} = 10^{12} \) G. For comparison, we also present the dissipative coefficient \( \mathcal{D}_{Jones} \) used in [8, 9] and, recently, in [13] in their studies of the magnetic field expulsion timescale from the NS cores. (Note, that the latter authors completely ignored the effect of electron scattering by the vortex magnetic field, assuming that \( \mathcal{D} = \mathcal{D}' = 0 \).) The temperature-dependent coefficient \( \mathcal{D}_{Jones} \) enters the expression for the dissipative force, similar to the first term in equation (3). This force arises due to the electron scattering off the unpaired proton quasiparticles localized in the vortex core. The coefficient \( \mathcal{D}_{Jones} \) in Fig. 2 is plotted for three stellar temperatures, \( T = 10^7, 10^8, \) and \( 10^9 \) K. One sees that \( \mathcal{D}_{Jones} \) is always small in comparison to \( \mathcal{D} \). This result is independent of the magnetic field \( \mathcal{B} \), since both these coefficients are proportional to \( \mathcal{B} \). Thus, [9, 13] substantially underestimate the typical timescales of the magnetic field evolution in NSs.
5. Conclusions
We calculated the force acting on a proton vortex from the neutron-proton-electron mixture at vanishing stellar temperature and neglecting, for simplicity, entrainment effects between the superfluid neutrons and superconducting protons. Our main results are summarized as follows:

- For typical NS conditions the electron wavelength is much smaller than all other relevant lengthscales in the problem. This permits us to use a quasiclassical theory in order to determine the electron cross-sections $\sigma_\parallel$ and $\sigma_\perp$, responsible for the appearance of transverse $\mathbf{F}_\perp$ and longitudinal $\mathbf{F}_\parallel$ forces on a vortex.

- The calculated transverse force $\mathbf{F}_\perp$ coincides with the ordinary Magnus force, discussed in the context of superconductors, e.g., by [19]. It also equals to the (minus) Lorentz force acting on electrons in the magnetic field of a vortex. This result proves that the assumptions made, e.g., in [6, 10] about the form of $\mathbf{F}_\perp$ are correct. At the same time, our result disagrees with the conclusion of [8, 9] that $\mathbf{F}_\perp = 0$.

- The longitudinal force on a proton vortex $\mathbf{F}_\parallel$ is, typically, smaller than $\mathbf{F}_\perp$ by a factor of $k_p \lambda \sim (30 - 50)$. It coincides with the force derived in [20] and, after some straightforward adjustment, with the force on a neutron vortex calculated in [16] using a quite different method. The latter coincidence is rather surprising since [16] worked within the Born approximation, which is not justified in our problem.

- [9] and [13] ignored in their analysis electron scattering by the magnetic field of a vortex, thus effectively setting $D = 0$ and $D' = 0$. Instead, they considered a different scattering mechanism, namely scattering of electrons off the proton localized excitations in the vortex core. As is shown in section 4, this mechanism leads to a longitudinal force much smaller than our $\mathbf{F}_\parallel$ (i.e., $D_{\text{Jones}} \ll D$). This means that [9] and [13] substantially underestimate the typical timescale $\tau_B$ for the magnetic field evolution in the NS core.

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