Parametric Oscillation with Squeezed Vacuum Reservoirs

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Abstract: Employing the quantum Hamiltonian describing the interaction of two-mode light (signal-idler modes) generated by a nondegenerate parametric oscillator (NDPO) with two uncorrelated squeezed vacuum reservoirs (USVR), we derive the master equation. The corresponding Fokker-Planck equation for the Q-function is then solved employing a propagator method developed by K. Fesseha [J. Math. Phys. 33 2179(1992)]. Making use of this Q-function, we calculate the quadrature fluctuations of the optical system. From these results we infer that the signal-idler modes are in squeezed states. When the NDPO operates below threshold we show that, for a large squeezing parameter, a squeezing amounting to a noise suppression approaching 100% below the vacuum level in the of the quadratures can be achieved.

1 Introduction

Nonclassical effects of light such as squeezing, antibunching and sub-Poissonian statistics have been attracting the attention of several authors in quantum optics over the last decades [1, 2, 3, 4, 5, 6, 7, 8, 9]. A review article on nonclassical states of the first 75 years is found in Ref. [2].

Squeezed states are nonclassical states characterised by a reduction of quantum fluctuations (noise) in one quadrature component below the vacuum level (quantum standard limit), or below that achievable in a coherent state [4, 6] at the expense of increased fluctuations in the other component such that the product of these fluctuations still obeys the uncertainty relation [4, 5].

It was Takahashi [10] who, in 1965, first pointed out that a degenerate parametric amplifier enhances the noise in one quadrature component and attenuates it in the other quadrature. This prediction has been confirmed by several authors for degenerate and nondegenerate parametric amplifiers and oscillators. Operating below threshold, the parametric amplifier is a source of squeezed states. In the initial experiments carried out to observe squeezing, a noise reduction of 4-17% relative to the quantum standard limit has been obtained [11]. In order to increase the gain, the parametric medium may be placed inside an optical cavity where it is coherently pumped and becomes a parametric oscillator [12, 13, 14, 15, 16, 17].

An optical parametric oscillator is a quantum device with a definite threshold for self sustained oscillations. It is one of the most interesting and well characterised optical devices in quantum optics. This simple dissipative quantum system plays an important role in the study of squeezed states. In a parametric oscillator a strong pump photon interacts with a nonlinear-medium (crystal) inside a cavity and is down-converted into two photons of smaller frequencies. In the NDPO we assume that the strong pump photon is down converted into two modes and these modes are referred to as signal and idler modes.

A quantum-mechanical treatment of different optical systems such as the NDPO is essential as they may generate squeezed states with nonclassical properties which have potential applications in quantum optical communications [4] and computation [19], gravitational wave detection [19, 20, 21, 22], interferometry [22, 23, 24], spectroscopical measurements [25] and for the study of fundamental concepts.

For systems with nonclassical features such as the NDPO, for which the Glauber-Sudarshan P-function is highly singular [8, 26, 27], one may use the Q-function. The Q-function is expressible in terms of the Q-function propagator and the initial Q-function. It is possible to determine the Q-function propagator
by directly solving the Fokker-Planck equation. In this paper, we find it convenient to evaluate the Q-function propagator applying the method developed in [28].

The main aim of this paper is to calculate the amount of squeezing that could be generated by the NDPO coupled to two USVR with the help of the Q-function. We show that squeezing of the output may be optimised and reach 100%.

2 The Master Equation

The description of system-reservoir interactions via the master equation is a standard technique in quantum optics [6, 7]. In this section, however, we found it useful to include a non-detailed derivation of the master equation describing the interaction of the signal-idler modes generated by a NDPO coupled to two USVR in order to make the paper more self-contained.

Denoting the density operator of the optical system and the squeezed reservoir modes by \( \hat{\chi}(t) \), the density operator for the system alone is defined by

\[
\hat{\rho}(t) = \text{Tr}_R(\hat{\chi}(t)),
\]

where \( \text{Tr}_R \) indicates that the trace is taken over the reservoir variables only. The density operator \( \hat{\chi}(t) \) evolves in time according to

\[
\frac{d\hat{\chi}(t)}{dt} = \frac{1}{\hbar i} \left[ \hat{H}_{SR}(t), \hat{\chi}(t) \right],
\]

where \( \hat{H}_{SR}(t) \) is the Hamiltonian describing the interaction between the system and the reservoirs. Note that the Hamiltonian describing only the reservoir modes \( \hat{H}_R(t) \) is not involved in the derivation as it cancels out when we apply the cyclic property of the trace. Furthermore, in order to simplify our calculations, the Hamiltonian that describes the interaction of the system with the pump mode \( \hat{H}_S(t) \) will be added at the end of the derivation.

Since initially the system and the reservoirs are uncorrelated, one can write, for the density operator of the system and the reservoirs at the initial time \( t = 0 \), that \( \hat{\chi}(0) = \hat{\rho}(0) \otimes \hat{R} \) [6], where \( \hat{\rho}(0) \) and \( \hat{R} \) are the density operators of the system and the reservoirs at the initial time, respectively. Then in view of this relation, Eq. (1) results in

\[
\frac{d\hat{\chi}(t)}{dt} = \frac{1}{\hbar i} \left[ \hat{H}_{SR}(t), \hat{\rho}(0) \otimes \hat{R} \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \hat{H}_{SR}(t), \left[ \hat{H}_{SR}(t'), \hat{\chi}(t') \right] \right].
\]

Applying the weak coupling approximation which implies that \( \hat{\chi}(t') = \hat{\rho}(t') \otimes \hat{R} \) [6], it follows that

\[
\frac{d\hat{\rho}(t)}{dt} = \frac{1}{\hbar} \text{Tr}_R \left\{ \left[ \hat{H}_{SR}(t), \hat{\rho}(0) \otimes \hat{R} \right] \right\} - \frac{1}{\hbar^2} \int_0^t dt' \text{Tr}_R \left\{ \left[ \hat{H}_{SR}(t), \left[ \hat{H}_{SR}(t'), \hat{\rho}(t') \otimes \hat{R} \right] \right] \right\}.
\]

We consider the system to be a two-mode light with frequencies \( \omega_a \) and \( \omega_b \) in a cavity coupled to two USVR. The interaction between the two-mode light and the squeezed vacuum reservoirs can be calculated, in the interaction picture, by the Hamiltonian

\[
\hat{H}_{SR}(t) = i\hbar \left[ \sum_j \lambda_j (\hat{a}^\dagger \hat{A}_j e^{i(\omega_a - \omega_j)t} - \hat{a} \hat{A}_j^\dagger e^{-i(\omega_a - \omega_j)t}) + \sum_k \lambda_k (\hat{b}^\dagger \hat{B}_k e^{i(\omega_b - \omega_k)t} - \hat{b} \hat{B}_k^\dagger e^{-i(\omega_b - \omega_k)t}) \right],
\]

in which \( \hat{a} (\hat{a}^\dagger) \) and \( \hat{b} (\hat{b}^\dagger) \) are the annihilation (creation) operators for the intracavity modes and \( \hat{A}_j (\hat{A}_j^\dagger) \) and \( \hat{B}_k (\hat{B}_k^\dagger) \) are the annihilation (creation) operators for the reservoir modes with frequencies \( \omega_j \) and \( \omega_k \), respectively. The coefficients \( \lambda_j \) and \( \lambda_k \) are coupling constants describing the interaction between the intracavity modes and the reservoir modes. Applying the cyclic property of trace and the relation \( \text{Tr}_R(\hat{R} \otimes \hat{H}_{SR}(t)) = \langle \hat{H}_{SR}(t) \rangle_R \), and taking into account that, for squeezed vacuum reservoirs [6],

\[
\langle \hat{A}_j \rangle_R = \langle \hat{A}_j^\dagger \rangle_R = \langle \hat{B}_k \rangle_R = \langle \hat{B}_k^\dagger \rangle_R = 0,
\]

one can show

\[
\frac{1}{\hbar} \text{Tr}_R \left\{ \left[ \hat{H}_{SR}(t), \hat{\rho}(0) \otimes \hat{R} \right] \right\} = 0,
\]
and as a result, expression (4) reduces to

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}_R \left\{ [\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\rho}(t') \otimes \hat{R}]] \right\}. \tag{5}$$

Applying the Markov approximation, in which $\hat{\rho}(t')$ is replaced by $\hat{\rho}(t)$, and using the cyclic property of the trace, the above equation can be expressed as

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{1}{\hbar^2} \int_0^t dt' \left[ (\hat{H}_{SR}(t)\hat{H}_{SR}(t'))R\hat{\rho}(t) - (\hat{H}_{SR}(t')\hat{H}_{SR}(t))R\hat{\rho}(t) - \hat{\rho}(t)(\hat{H}_{SR}(t)\hat{H}_{SR}(t'))R + \hat{\rho}(t)(\hat{H}_{SR}(t')\hat{H}_{SR}(t))R \right]. \tag{6}$$

We note again that for squeezed vacuum reservoirs [6]

$$\langle \hat{A}_j \hat{A}_l \rangle_R = -M_A \delta_{j,2j_a-j}, \tag{7a}$$
$$\langle \hat{A}_j^\dagger \hat{A}_l \rangle_R = N_A \delta_{j,l}, \tag{7b}$$
$$\langle \hat{A}_j \hat{A}_l^\dagger \rangle_R = (N_A + 1) \delta_{j,l}, \tag{7c}$$

where $\delta_{j,l}$ is the Kronecker delta symbol and

$$\langle \hat{A}_j \hat{B}_m \rangle_R = \langle \hat{A}_j^\dagger \hat{B}_m^\dagger \rangle_R = \langle \hat{A}_j \hat{B}_m^\dagger \rangle_R = 0. \tag{8}$$

This equation is a consequence of the fact that the two squeezed vacuum reservoirs are uncorrelated. The parameters $N_A, N_B, M_A$ and $M_B$ describe the effects of squeezing of the reservoir modes. Actually, the parameters $N$ and $M$ represent the mean photon number and the phase property of the reservoirs, respectively, and are related as $|M|^2 = N(N + 1)$. Furthermore, introducing the density of states $g(\omega)$, where

$$\sum_j \lambda_j \lambda_{2j_a-j} \int_0^\infty d\omega g(\omega) \lambda(\omega) \lambda(2\omega_a - \omega),$$

and setting $t - t' = \tau$, one can easily show that

$$\int_0^t dt' e^{\pm i(\omega_a - \omega)(t-t')} = \int_0^\tau d\tau e^{\pm i(\omega_a - \omega)\tau}. \tag{9}$$

Since the exponential is a rapidly decaying function of time, the upper limit of integration can be extended to infinity. Making use of the approximate relation

$$\int_0^\infty e^{\pm i(\omega_a - \omega)\tau} = \pi \delta(\omega_a - \omega), \tag{10}$$

and applying the property of the Dirac delta function to the integrals of Eq. (6), we get

$$\pi g(\omega_a) \lambda^2(\omega_a) = \frac{\gamma_A}{2}. \tag{11}$$

where $\gamma_A = 2\pi g(\omega_a) \lambda^2(\omega_a)$ is the cavity damping constant for mode $A$. Similarly one can also show that the cavity damping constant for mode $B$ is given by $\gamma_B = 2\pi g(\omega_b) \lambda^2(\omega_b)$.

In view of Eqs. (7-11), after evaluating lengthy but straightforward consecutive integrations, Eq. (6) takes the form

$$\frac{d\hat{\rho}(t)}{dt} = \frac{\gamma_A}{2} (N_A + 1) \left[ 2\hat{a}\hat{\rho}(t)\hat{a}^\dagger - \hat{a}\hat{\rho}(t) - \hat{\rho}(t)\hat{a}\hat{a}^\dagger \right] + \frac{\gamma_A M_A}{2} \left[ 2\hat{a}\hat{\rho}(t)\hat{a}^\dagger + 2\hat{a}\hat{\rho}(t)\hat{a} - \hat{a}\hat{a}^\dagger \hat{\rho}(t) - \hat{\rho}(t)\hat{a}\hat{a}^\dagger \right]$$
$$+ \frac{\gamma_B}{2} (N_B + 1) \left[ 2\hat{b}\hat{\rho}(t)\hat{b}^\dagger - \hat{b}\hat{\rho}(t) - \hat{\rho}(t)\hat{b}\hat{b}^\dagger \right] + \frac{\gamma_B M_B}{2} \left[ 2\hat{b}\hat{\rho}(t)\hat{b}^\dagger + 2\hat{b}\hat{\rho}(t)\hat{b} - \hat{b}\hat{b}^\dagger \hat{\rho}(t) - \hat{\rho}(t)\hat{b}\hat{b}^\dagger \right]. \tag{12}$$
In the cavity we consider two-modes of light known as the signal and idler modes produced by the NDPO. The cavity has one single-port mirror in which light can enter but can not leave. In this system we assume that a strong pump light of frequency $\omega_p$ interacts with a nonlinear-medium (crystal) inside the cavity and gives rise to a two-mode squeezed light (the signal-idler modes) with frequencies $\omega_a$, and $\omega_b$, such that $\omega_0 = \omega_a + \omega_b$.

With the pump mode treated classically (the amplitude of the pump mode is assumed to be real and constant), the interaction of the system is described, in the interaction picture, by the Hamiltonian

$$\hat{H}_I = \hbar \kappa \gamma_0 (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger), \quad (13)$$

where $\kappa$ is the coupling constant and $\gamma_0$ is the amplitude of the pump mode. Hence the master equation for the NDPO coupled to USVR, in view of Eqs. (12) and (13), takes the form

$$\frac{d\hat{\rho}(t)}{dt} = -\kappa \gamma_0 \left[ \hat{a} \hat{b} \hat{\rho}(t) - \hat{\rho}(t) \hat{a} \hat{b} + \hat{\rho}(t) \hat{a} \hat{b}^\dagger - \hat{a} \hat{b}^\dagger \hat{\rho}(t) \right]$$

$$+ \frac{\gamma_A}{2} (N_A + 1) \left[ 2 \hat{a} \hat{\rho}(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}(t) - \hat{\rho}(t) \hat{a}^\dagger + \hat{a} \hat{\rho}(t) \right]$$

$$+ \frac{\gamma_A N_A}{2} \left[ 2 \hat{a}^\dagger \hat{\rho}(t) \hat{a} + 2 \hat{a} \hat{\rho}(t) \hat{a} - \hat{a} \hat{a} \hat{\rho}(t) - \hat{\rho}(t) \hat{a}^\dagger - \hat{a} \hat{\rho}(t) \right]$$

$$+ \frac{\gamma_B}{2} (N_B + 1) \left[ 2 \hat{b} \hat{\rho}(t) \hat{b}^\dagger - \hat{b}^\dagger \hat{b} \hat{\rho}(t) - \hat{\rho}(t) \hat{b}^\dagger + \hat{b} \hat{\rho}(t) \right]$$

$$+ \frac{\gamma_B N_B}{2} \left[ 2 \hat{b}^\dagger \hat{\rho}(t) \hat{b} - \hat{b} \hat{\rho}(t) \hat{b}^\dagger + \hat{b} \hat{\rho}(t) \hat{b}^\dagger - \hat{b} \hat{b} \hat{\rho}(t) - \hat{\rho}(t) \hat{b} \right] \quad (14)$$

This equation is the basis of our analysis and describes the interactions inside the cavity as well the interaction of the signal-idler modes produced by the NDPO and the squeezed vacuum reservoirs via the partially transmitting mirror. This master equation is consistent to that given in Ref. [7] except that the expression there is for a single mode in a cavity coupled to a single mode vacuum reservoir.

3 The Fokker-Planck Equation

In this section we derive the Fokker-Planck equation for the Q-function. In order to obtain the Fokker-Planck equation for the Q-function corresponding to the master equation (14), one has first to put all terms in normal order. Applying the commutation relations

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \frac{\partial f(\hat{a}, \hat{a}^\dagger)}{\partial \hat{a}}, \quad (15a)$$

$$[\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial f(\hat{a}, \hat{a}^\dagger)}{\partial \hat{a}}, \quad (15b)$$

one can verify that $\hat{a} \hat{\rho} = \hat{\rho} \hat{a} + \frac{\partial \hat{\rho}}{\partial \hat{a}}$, $\hat{a}^\dagger \hat{\rho} = \hat{\rho} \hat{a}^\dagger + \frac{\partial \hat{\rho}}{\partial \hat{a}^\dagger}$, where the density operator $\hat{\rho} = \hat{\rho}(\hat{a}, \hat{a}^\dagger, t)$ is considered to be in normal order. Making use of Eqs. (15), the relation $[\hat{a}, \hat{b}, \hat{c}] = \hat{b}[\hat{a}, \hat{c}] + [\hat{a}, \hat{b}][\hat{c}]$ and the photonic commutation relation $\hat{a} \hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1$, the master equation (14) can be written as

$$\frac{d\hat{\rho}(t)}{dt} = -\kappa \gamma_0 \left[ \hat{a} \hat{b} \hat{\rho}(t) - \hat{\rho}(t) \hat{a} \hat{b} + \hat{\rho}(t) \hat{a} \hat{b}^\dagger - \hat{a} \hat{b}^\dagger \hat{\rho}(t) \right]$$

$$+ \frac{\gamma_A}{2} (N_A + 1) \left[ \hat{a} \hat{\rho}(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{\rho}(t) \hat{a} + \hat{\rho}(t) \hat{a}^\dagger - \hat{a} \hat{\rho}(t) \hat{a} \right]$$

$$+ \frac{\gamma_A}{2} \left[ \hat{a} \hat{\rho}(t) \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \hat{\rho}(t) - \hat{\rho}(t) \hat{a}^\dagger - \hat{a} \hat{\rho}(t) \hat{a} \right]$$

$$+ \frac{\gamma_B}{2} (N_B + 1) \left[ \hat{b} \hat{\rho}(t) \hat{b}^\dagger - \hat{b}^\dagger \hat{b} \hat{\rho}(t) + \hat{\rho}(t) \hat{b}^\dagger - \hat{b} \hat{\rho}(t) \hat{b} \right]$$

$$+ \frac{\gamma_B}{2} \left[ \hat{b} \hat{\rho}(t) \hat{b}^\dagger + \hat{b}^\dagger \hat{b} \hat{\rho}(t) - \hat{\rho}(t) \hat{b}^\dagger - \hat{b} \hat{\rho}(t) \hat{b} \right] \quad (16)$$
In order to transform this equation into a c-number Fokker-Planck equation for the Q-function, one needs to multiply it on the left by $(\alpha, \beta)$ and on the right by $|\alpha, \beta\rangle$, so that

$$\frac{\partial Q}{\partial t} = \left[ \kappa \gamma_0 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \kappa \gamma_0 \left( \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \right] Q,
$$

where

$$Q = Q(\alpha^*, \alpha, \beta^*, \beta, t) = \frac{1}{\pi} \langle \alpha, \beta | \hat{\rho} (\hat{a}^\dagger, \hat{b}^\dagger, \hat{b}, \hat{a}, t) | \alpha, \beta \rangle.$$

Expression (17) is the Fokker-Planck equation for the Q-function for the signal-idler modes produced by the NDPO coupled to two USVR. To obtain the solution of this equation, we introduce the Cartesian coordinates defined by

$$\alpha = x_1 + iy_1, \alpha^* = x_1 - iy_1, \beta = x_2 + iy_2, \beta^* = x_2 - iy_2,$$

and note that

$$x_1 = \frac{1}{2}(\alpha + \alpha^*), y_1 = -i\frac{1}{2}(\alpha - \alpha^*), x_2 = \frac{1}{2}(\beta + \beta^*), y_2 = -i\frac{1}{2}(\beta - \beta^*).$$

One can show that

$$\frac{\partial}{\partial \alpha} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial y_1} \right),$$

$$\frac{\partial}{\partial \beta} = \frac{1}{2} \left( \frac{\partial}{\partial x_2} - i \frac{\partial}{\partial y_2} \right).$$

Thus combining these results and their complex conjugates, one readily obtains

$$\frac{\partial Q}{\partial t} = \left[ \kappa \gamma_0 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) + \kappa \gamma_0 \left( \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial y_1} \right] Q,$$

where $Q = Q(x_1, x_2, y_1, y_2, t)$.

Next, introducing the transformation defined by $x_1 = x + u, x_2 = x - u, y_1 = y + v, y_2 = v - y$, one can verify that

$$x = \frac{1}{2}(x_1 + x_2),$$

$$y = \frac{1}{2}(y_1 + y_2).$$

In view of these relations, it follows that

$$\frac{\partial}{\partial x_1} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right),$$

$$\frac{\partial}{\partial y_1} = \frac{1}{2} \left( \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \right).$$

Making use of Eqs. (22, 23) in Eq. (21) and setting $\gamma_A = \gamma_B = \gamma$, $N_A = N_B = N$ and $M_A = M_B = M$, for convenience, one arrives at

$$\frac{\partial Q}{\partial t} = \left[ \kappa \gamma_0 + \gamma (N - M + 1) \frac{\partial^2}{\partial x^2} + \kappa \gamma_0 + \gamma (N + M + 1) \frac{\partial^2}{\partial y^2} - \kappa \gamma_0 - \gamma (N - M + 1) \frac{\partial^2}{\partial u^2} \right] Q,$$

which is the Fokker-Planck equation for the Q-function where $Q = Q(x, y, u, v, t)$. 

4 Solution of the Fokker-Planck Equation

In this section the explicit expression for the Q-function that describes the optical system is derived. In order to solve the differential equation (24) using the propagator method discussed in Ref. [1], one needs to transform the above equation into a Schrödinger-type equation. This can be achieved upon replacing \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \) by \( i\hbar \frac{\partial}{\partial x}, i\hbar \frac{\partial}{\partial y} \) respectively. Hence Eq. (24) can be expressed as

\[
\frac{id}{dt} |Q(t)\rangle = i \left[ -\frac{\lambda_1}{8} p_x^2 - \frac{\lambda_2}{8} p_y^2 + \frac{\lambda_3}{8} p_u^2 + \frac{\lambda_4}{8} p_v^2 + i\frac{\lambda_5}{2} (\hat{p}_x \hat{x} + \hat{p}_y \hat{y}) - i\frac{\lambda_6}{2} (\hat{p}_u \hat{u} + \hat{p}_v \hat{v}) \right] |Q(t)\rangle
\]

where

\[
\lambda_{1,2} = \kappa \gamma_0 + \gamma (N \mp M + 1),
\]
\[
\lambda_{3,4} = \kappa \gamma_0 - \gamma (N \mp M + 1),
\]
\[
\lambda_{5,6} = 2\kappa \gamma_0 \pm \gamma.
\]

A formal solution of Eq. (25) can be put in the form

\[
|Q(t)\rangle = \hat{u}(t) |Q(0)\rangle,
\]

where \( \hat{u}(t) = \exp(-i\hat{H}t/\hbar) \) is a unitary operator and

\[
\hat{H} = -\frac{\lambda_1}{8} p_x^2 - \frac{\lambda_2}{8} p_y^2 + \frac{\lambda_3}{8} p_u^2 + \frac{\lambda_4}{8} p_v^2 - \frac{\lambda_5}{2} (\hat{p}_x \hat{x} + \hat{p}_y \hat{y}) + \frac{\lambda_6}{2} (\hat{p}_u \hat{u} + \hat{p}_v \hat{v})
\]

is a quadratic quantum Hamiltonian. Multiplying (27) by \( \langle x, y, u, v \mid \hat{u}(t) |Q(0)\rangle \), where \( Q(x, y, u, v, t) = \langle x, y, u, v \mid Q(t) \rangle \). Introducing a four-dimensional completeness relation for the position eigenstates \( \hat{I} = \int dx' dy' du' dv' \mid x', y', u', v'\rangle \langle x', y', u', v' \mid \) in expression (29), one can see that

\[
Q(x, y, u, v, t) = \int dx' dy' du' dv' Q(x, y, u, v, t \mid x', y', u', v') Q_{\alpha}(x', y', u', v'),
\]

where

\[
Q_0(x', y', u', v') = \langle x', y', u', v' |Q(0)\rangle
\]

is the initial Q-function and \( Q(x, y, u, v, t \mid x', y', u', v', 0) = \langle x, y, u, v |\hat{u}(t) |x', y', u', v'\rangle \) is the Q-function propagator.

Following Fessela [1], the propagator associated with a quadratic Hamiltonian of the form

\[
\hat{H}(\hat{x}_1, ..., \hat{x}_n, \hat{p}_1, ..., \hat{p}_n, t) = \sum_{i=1}^{n} \left[ a_i \hat{p}_i^2 + b_i(t) \hat{p}_i \hat{x}_i + c_i(t) \hat{x}_i^2 \right]
\]

is expressible as

\[
Q(x_1, ..., x_n, t \mid x'_1, ..., x'_n, 0) = \left[ \frac{i}{2\pi} \right]^n \prod_{j=1}^{n} \sqrt{\frac{\partial^2 S_c}{\partial x_j \partial x_j}} \exp \left[ -\xi \int_{0}^{t} b_j(t') dt' + iS_c \right].
\]

where \( S_c \) is the classical action, \( \xi \) is a parameter related with operator ordering and \( \xi \) is constant different from zero for the Hamiltonian to remain quadratic. Comparing Eqs. (32) and (28), it follows that \( a_1 = a_{x,y,u,v} = -\frac{1}{8} \lambda_{1,2,3,4}, \langle x_1, x_2, x_3, x_4 \rangle = \langle x, y, u, v \rangle, c_x = c_y = c_u = c_v = 0, b_x = b_y = -\frac{\lambda_6}{2}, b_u = b_v = \frac{\lambda_5}{2} \) and the antistandard operator ordering \( \xi = \frac{1}{2} \). Thus the Q-function propagator associated with the Hamiltonian (28) is expressible as

\[
Q(x, y, u, v, t \mid x', y', u', v', 0) = \frac{1}{4\pi^2} \left[ \frac{\partial^2 S_c}{\partial x' \partial x} \frac{\partial^2 S_c}{\partial y' \partial y} \frac{\partial^2 S_c}{\partial u' \partial u} \frac{\partial^2 S_c}{\partial v' \partial v} \right]^\frac{1}{2} e^{iS_c + \frac{(\lambda_5 - \lambda_6) t}{2}}.
\]
In order to obtain the explicit form of this expression, one has first to determine the classical action. To this end, the Hamiltonian function corresponding to the quantum Hamiltonian (28) is given by

\[ H = -\frac{\lambda_1}{8} (p_x)^2 - i \frac{\lambda_2}{8} p_y^2 + i \frac{\lambda_3}{8} p_u^2 + \frac{\lambda_5}{2} (p_x x + p_y y) + \frac{\lambda_6}{2} (p_u u + p_v v). \]  

(35)

With the help of the Lagrangian \( L = \sum_i \dot{x}_i p_i - H \) and the Hamilton equations \( \dot{x}_i = \frac{\partial H}{\partial p_i} \) \((i = x, y, u, v)\) one can readily show that

\[ L = \frac{2i}{\lambda_1} (\dot{x} + \frac{\lambda_5}{2} x)^2 + \frac{2i}{\lambda_2} (\dot{y} + \frac{\lambda_3}{2} y)^2 - \frac{2i}{\lambda_3} (\dot{u} - \frac{\lambda_6}{2} u)^2 - \frac{2i}{\lambda_4} (\dot{v} - \frac{\lambda_6}{2} v)^2. \]  

(36)

Applying the Euler-Lagrange equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0, \]  

(37)

along with Eq. (36), leads to

\[
\begin{align*}
\dot{x} - (\frac{\lambda_5}{2})^2 x &= 0, \\
\dot{y} - (\frac{\lambda_3}{2})^2 y &= 0, \\
\dot{u} - (\frac{\lambda_6}{2})^2 u &= 0, \\
\dot{v} - (\frac{\lambda_6}{2})^2 v &= 0.
\end{align*}
\]

The solutions of these differential equations can be written as

\[
\begin{align*}
x(t) &= a_1 e^{\frac{\lambda_5}{2} t} + a_2 e^{-\frac{\lambda_5}{2} t}, \\
y(t) &= b_1 e^{\frac{\lambda_3}{2} t} + b_2 e^{-\frac{\lambda_3}{2} t}, \\
u(t) &= c_1 e^{\frac{\lambda_6}{2} t} + c_2 e^{-\frac{\lambda_6}{2} t}, \\
v(t) &= d_1 e^{\frac{\lambda_6}{2} t} + d_2 e^{-\frac{\lambda_6}{2} t}.
\end{align*}
\]

(38a)

Now substituting these expressions and their corresponding first order time derivatives into Eq. (36), the Lagrangian takes the form

\[ L = 2i\lambda_5^2 \left( \frac{a_2^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} \right) e^{\lambda_5 t} - 2i\lambda_6^2 \left( \frac{c_2^2}{\lambda_3} + \frac{d_2^2}{\lambda_4} \right) e^{-\lambda_6 t}. \]

On account of the above result, the classical action defined by \( S_c = \int_0^T L(t) dt \) takes the form

\[ S_c = 2i\lambda_5^2 \left( \frac{a_2^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} \right) (e^{\lambda_5 T} - 1) + 2i\lambda_6^2 \left( \frac{c_2^2}{\lambda_3} + \frac{d_2^2}{\lambda_4} \right) (e^{-\lambda_6 T} - 1). \]

(39)

Applying the boundary conditions \( x_i(0) = x'_i \) and \( x_i(T) = x''_i \) in Eq. (38), one can obtain that

\[
\begin{align*}
a_1 &= \frac{x'' e^{\lambda_5 T} - x'}{e^{\lambda_5 T} - 1}, \\
b_1 &= \frac{y'' e^{\lambda_3 T} - y'}{e^{\lambda_3 T} - 1}, \\
c_2 &= \frac{u'' e^{-\lambda_6 T} - u'}{e^{-\lambda_6 T} - 1}, \\
d_2 &= \frac{v'' e^{-\lambda_6 T} - v'}{e^{-\lambda_6 T} - 1}.
\end{align*}
\]

Inserting the above expressions into Eq. (39) and replacing \( (x'', y'', u'', v'', T) \) by \( (x, y, u, v, t) \) yields

\[ S_c = 2i\lambda_5 \left[ \frac{(x' - e^{\frac{\lambda_5}{2}(t)})^2}{\lambda_1 (e^{\lambda_5 t} - 1)} + \frac{(y' - e^{\frac{\lambda_3}{2}(t)})^2}{\lambda_2 (e^{\lambda_3 t} - 1)} \right] + 2i\lambda_6 \left[ \frac{(u' - e^{\frac{\lambda_6}{2}(t)})^2}{\lambda_3 (e^{-\lambda_6 t} - 1)} + \frac{(v' - e^{\frac{\lambda_6}{2}(t)})^2}{\lambda_4 (e^{-\lambda_6 t} - 1)} \right]. \]

(40)

and employing this relation the following results are obtained:

\[
\begin{align*}
\frac{\partial^2 S_c}{\partial x \partial x'} &= -\frac{4i\lambda_5 e^{\frac{\lambda_5}{2} t}}{\lambda_1 (e^{\lambda_5 t} - 1)}, \\
\frac{\partial^2 S_c}{\partial y \partial y'} &= -\frac{4i\lambda_5 e^{\frac{\lambda_3}{2} t}}{\lambda_2 (e^{\lambda_3 t} - 1)}, \\
\frac{\partial^2 S_c}{\partial u \partial u'} &= -\frac{4i\lambda_5 e^{\frac{\lambda_6}{2} t}}{\lambda_3 (e^{-\lambda_6 t} - 1)}, \\
\frac{\partial^2 S_c}{\partial v \partial v'} &= -\frac{4i\lambda_5 e^{\frac{\lambda_6}{2} t}}{\lambda_4 (e^{-\lambda_6 t} - 1)}.
\end{align*}
\]

(41a)
Thus, in view of Eq. (41), the Q-function propagator (34) takes the form

$$Q(x, y, u, v, t|x', y', u', v', 0) = \frac{4\lambda_0\lambda_6}{\pi^2\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}} \frac{e^{(\lambda_5 - \lambda_0)t}}{(e^{\lambda_5 t} - 1)(e^{-\lambda_0 t} - 1)}$$

$$\times \exp\left[- \frac{2\lambda_5}{(e^{\lambda_5 t} - 1)} \left( \frac{x'^2 - 2x e^{\lambda_5 t} x' + x^2 e^{\lambda_5 t}}{\lambda_1} \right. \right.$$

$$\left. + \frac{y'^2 - 2y e^{\lambda_5 t} y' + y^2 e^{\lambda_5 t}}{\lambda_2} \right)$$

$$\left. - \frac{2\lambda_5}{(e^{-\lambda_0 t} - 1)} \left( \frac{u'^2 - 2u e^{-\lambda_0 t} u' + u^2 e^{-\lambda_0 t}}{\lambda_3} \right) \right]$$

$$\left. + \frac{v'^2 - 2v e^{-\lambda_0 t} v' + v^2 e^{-\lambda_0 t}}{\lambda_4} \right\}.$$  \hspace{1cm} (42)

Considering the signal-idler modes produced by the NDPO to be initially in a two-mode vacuum state, the initial Q-function is expressible as

$$Q_0(\alpha', \beta') = \frac{1}{\pi^2} (\alpha'|0, 0\rangle \langle 0, 0| \alpha', \beta') = \exp(-\alpha^* \alpha - \beta^* \beta'),$$

and in terms of the Cartesian variables of expression (19), this equation becomes

$$Q_0(x'_1, x'_2, y'_1, y'_2) = \frac{1}{\pi^2} \exp\left[ - \left( x'^2 + y'^2 + u'^2 + v'^2 \right) \right].$$

Furthermore, in terms of $x', y', u'$ and $v'$, one can write

$$\int dx'_1 dx'_2 dy'_1 dy'_2 Q_0(x'_1, x'_2, y'_1, y'_2) = \int dx' dy' du' dv' Q_0(x', y', u', v'),$$

where

$$Q_0(x', y', u', v') = \frac{|J|}{\pi^2} \exp\left[ - 2(x'^2 + y'^2 + u'^2 + v'^2) \right]$$

and $J$ is the Jacobian of the transformation of $x_1$, $x_2$, $y_1$ and $y_2$ with respect to $x$, $y$ and $v$. Making use of Eq. (19) in the Jacobian, one can show that $|J| = 4$. Hence

$$Q_0(x', y', u', v') = \frac{4}{\pi^2} \exp\left[ - 2(x'^2 + y'^2 + u'^2 + v'^2) \right].$$  \hspace{1cm} (43)

Substituting expression (41) into Eq. (34) and then combining the result with Eq. (43) and finally carrying out the integration in Eq. (30) applying the relation

$$\int_{-\infty}^{\infty} dx' \exp\left[-kx'^2 + dx'\right] = \sqrt{\frac{\pi}{k}} \exp\left[\frac{d^2}{4k}\right], \hspace{1cm} k > 0,$$

the Q-function takes the compact form

$$Q(x, y, u, v, t) = \frac{4}{\pi^2\sqrt{a_1a_2a_3a_4}} \exp\left[ - \frac{2}{a_1} x^2 - \frac{2}{a_2} y^2 - \frac{2}{a_3} u^2 - \frac{2}{a_4} v^2 \right].$$  \hspace{1cm} (44)

where

$$a_{1,2} = \frac{\lambda_{1,2}(e^{\lambda_5 t} - 1) + \lambda_5}{\lambda_5 e^{\lambda_5 t}},$$  \hspace{1cm} (45a)

$$a_{3,4} = \frac{\lambda_{3,4}(e^{-\lambda_0 t} - 1) + \lambda_6}{\lambda_6 e^{-\lambda_0 t}}.$$  \hspace{1cm} (45b)

It can be easily verified that the Jacobian of the inverse transformation is $|J'| = \frac{1}{4}$. One can then write

$$\int dx dy du dv Q(x, y, u, v, t) = \int dx_1 dx_2 dy_1 dy_2 Q'(x_1, x_2, y_1, y_2, t),$$
in which the final expression for \( Q'(x_1, x_2, y_1, y_2, t) \) is obtained from Eq. (44) employing the inverse transformations (22). Upon carrying out further inverse transformations (19), the required final form the Q-function for the signal-idler modes produced by the nondegenerate parametric oscillator (NDPO) coupled to two uncorrelated squeezed vacuum reservoirs takes the form

\[
Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{D}{\pi^2} \exp \left[ - b_1(|\alpha|^2 + |\beta|^2) + b_2(\alpha \beta + \alpha^* \beta^*) + b_3(\alpha^2 + \alpha^* \beta^2) + b_4(\alpha^* + \alpha \beta^*) \right],
\]

(46)

where

\[
D = \frac{1}{\sqrt{a_1 a_2 a_3 a_4}},
\]

(47a)

\[
b_{1,2} = \frac{1}{4} \left[ \frac{1}{\pm a_1} \pm \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right],
\]

(47b)

\[
b_{3,4} = \frac{1}{4} \left[ - \frac{1}{a_1} + \frac{1}{a_2} \pm \frac{1}{a_3} \mp \frac{1}{a_4} \right].
\]

(47c)

This Q-function is useful to calculate the expectation values of antinormally ordered operators and consequently the quadrature variances. It could also be used to calculate the photon number distribution of different optical systems. In this paper, this function is used to calculate the quadrature fluctuations (variances) of the NDPO coupled to two USVR. It can be readily verified that the Q-function (46) is positive and normalised.

Next we proceed to obtain the expressions for the Q-function for some special cases of interest: For the case when there are no squeezed vacuum reservoirs \((r = 0)\), that is, when the external environment is an ordinary vacuum, the Q-function (46) takes the form

\[
Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{1}{\pi^2 a_1 a_3} \exp \left[ - \frac{1}{2} \left( \frac{a_1 + a_3}{a_1 a_3} \right)(|\alpha|^2 + |\beta|^2) + \frac{1}{2} \left( \frac{a_1 - a_3}{a_1 a_3} \right)(\alpha \beta + \alpha^* \beta^*) \right].
\]

(48)

This is the Q-function for the nondegenerate parametric oscillator coupled to ordinary vacuum. On the other hand, in the absence of damping \((\gamma = 0)\), Eq. (46) reduces to the form

\[
Q(\alpha, \alpha^*, \beta, \beta^*, t) = \frac{\text{sech}\gamma_0 t}{\pi^2} \exp \left[ - |\alpha|^2 - |\beta|^2 - (\text{tanh} \kappa_0 t)(\alpha \beta + \alpha^* \beta^*) \right],
\]

(49)

which is the Q-function for the nondegenerate parametric amplifier.

Next we obtain the Q-function for the single-mode generated by a degenerate parametric oscillator coupled to a single-mode squeezed vacuum reservoir from the Q-function for the NDPO (46). The Q-function for the single-mode can be expressed as

\[
Q(\alpha, \alpha^*, t) = \int d^2 \beta Q(\alpha, \alpha^*, \beta, \beta^*, t),
\]

so that using Eq. (46) and the relation

\[
\int d^2 \alpha \exp \left[ - a'|\alpha|^2 + b' \alpha + c' \alpha^* + A' \alpha^2 + B' \alpha^* \right] = \frac{1}{\sqrt{(a'^2 - 4A'B')}} \exp \left[ \frac{a' b' + A' c' + B' b'^2}{a'^2 - 4A'B'} \right], a' > 0
\]

(50)

the Q-function for the DPO coupled to a single-mode squeezed vacuum reservoir takes the form

\[
Q(\alpha, \alpha^*, t) = \frac{D}{\pi \sqrt{y}} \exp \left[ - a|\alpha|^2 + \frac{A}{2} \left( \alpha^2 + \alpha^* \right) \right],
\]

(51)

where

\[
y = b_1^2 - b_4^2,
\]

(52a)

\[
a = \frac{1}{y} \left[ (b_1 + b_4)(b_1 - b_4) + 2b_2b_3 - b_1(b_2 + b_3)^2 \right],
\]

(52b)

\[
A = \frac{1}{y} \left[ (b_1 + b_4)(b_4(b_1 - b_4) + 2b_2b_3) + b_4(b_2 + b_3)^2 \right].
\]

(52c)
Upon integrating Eqs. (48) and (49) with respect to $\beta$ by employing relation (50), one can also find the Q-function for the DPO in the absence of squeezed vacuum reservoir ($r = 0$) and in the absence of damping ($\gamma = 0$) to be

$$Q(\alpha, \alpha^*, t) = \frac{2}{\pi(a_1 + a_3)} \exp \left[ - \frac{2}{a_1 + a_3} |\alpha|^2 \right]$$

and

$$Q(\alpha, \alpha^*, t) = \frac{\text{sech}^2 \kappa \gamma_0 t}{\pi} \exp \left[ - (\text{sech}^2 \kappa \gamma_0 t)(|\alpha|^2) \right],$$

respectively.

### 5 Quadrature Squeezing

In this section the intracavity quadrature fluctuations for the single-mode generated by the DPO as well as the signal-idler modes produced by the NDPO coupled to the two squeezed vacuum reservoirs using the pertinent Q-functions derived in the previous section are analysed.

Here the first focus is the squeezing properties of the single-mode light. These properties could be described by two Hermitian operators defined as $\hat{a}_1 = \hat{a}^\dagger + \hat{a}$ and $\hat{a}_2 = i(\hat{a}^\dagger - \hat{a})$. These quadrature operators obey the commutation relation $[\hat{a}_1, \hat{a}_2] = 2i$. The variance of these quadrature operators can be put in the form

$$(\Delta \hat{a}_{1,2})^2 = \langle \hat{a}_{1,2}^2 \rangle - \langle \hat{a}_{1,2} \rangle^2$$

We now proceed to calculate the expectation values involved in expression (54). Applying the relation

$$\langle \hat{A}(\hat{a}, \hat{a}^\dagger) \rangle = \int_{-\infty}^{\infty} d^2 \alpha Q(\alpha, \alpha^*, t) A_\alpha(\alpha, \alpha^*),$$

in which $A_\alpha(\alpha, \alpha^*)$ is the c-number equivalent of the operator $\hat{A}(\hat{a}, \hat{a}^\dagger)$ for the antinormal ordering, one arrives at

$$\langle \hat{a} \rangle = \int_{-\infty}^{\infty} d^2 \alpha Q(\alpha, \alpha^*, t) \alpha.$$

Upon using the Q-function (51) for the single mode, the above equation can be expressed as

$$\langle \hat{a} \rangle = \frac{D}{\sqrt{y}} \frac{\partial}{\partial b} \left| \int_{-\infty}^{\infty} d^2 \alpha \frac{\partial}{\partial b} \left[ \exp \left( - a^\alpha + A / 2 \right) \langle \hat{a}^2 \rangle \right] \right|_{b=0},$$

and on the basis of (50) for which $c' = 0$ and $A' = B'$, one can verify that

$$\langle \hat{a} \rangle = \frac{D}{\sqrt{y}} \frac{\partial}{\partial b} \left| \exp \left( \frac{A a^2}{2b^2 - A^2} \right) \right|_{b=0} = 0.$$

In view of this result expression (55) reduces to

$$(\Delta \hat{a}_{1,2})^2 = 1 + 2 \langle \hat{a}^\dagger \hat{a} \rangle \pm |\langle \hat{a}^\dagger \hat{a} \rangle| \pm |\langle \hat{a} \rangle|^2.$$  \hfill (56)

Making use of the fact that the c-number equivalent of $\hat{a}^\dagger \hat{a}$ for the antinormal ordering is $\alpha^* \alpha - 1$ and applying relation (55) in evaluating all the expectation values in Eq. (56), we arrive at

$$(\Delta \hat{a}_{1,2})^2 = \frac{2}{a+1} - 1.$$  \hfill (57)

Finally the quadrature fluctuations of the single-mode at any time $t$, in view of Eqs. (52), (46) and (45), take the form

$$(\Delta \hat{a}_{1,2})^2 = \frac{\lambda_1 \lambda_2 (e^{\lambda_5 t} - 1) + \lambda_5}{\lambda_5 e^{\lambda_5 t}} + \frac{\lambda_3 \lambda_4 (e^{-\lambda_5 t} - 1) + \lambda_6}{\lambda_6 e^{-\lambda_5 t}} - 1.$$
At steady-state \((t \to \infty)\), the variances given above reduce to
\[
(\Delta \hat{a}_{1,2})^2 = \frac{\lambda_{1,2}}{\lambda_5} + \frac{\lambda_{3,4}}{\lambda_6} - 1,
\]
and with the aid of Eq. (26) one can rewrite these expressions as
\[
(\Delta \hat{a}_{1,2})^2 = \frac{2(N \mp M) + 1}{1 - \left(\frac{2\kappa_0}{\gamma}\right)^2}.
\] (58)

Since for squeezed vacuum reservoirs
\[
N = \sinh^2 r, \quad M = \sinh r \cosh r,
\] (59a)
(59b)
where \(r\) is the squeezing parameter taken to be real and positive for convenience, expression (59) takes the form
\[
(\Delta \hat{a}_{1,2})^2 = \frac{e^{\mp 2r}}{1 - \left(\frac{2\kappa_0}{\gamma}\right)^2}.
\] (60)

Using (60) one can show that \((\Delta \hat{a}_1)^2 < 1\), for
\[
r > \frac{1}{2} \ln \left[1 - \left(\frac{2\kappa_0}{\gamma}\right)^2\right]
\] (61)
and \((\Delta \hat{a}_2)^2 > 1\) for all \(r\). This shows that the degenerate parametric oscillator coupled to a squeezed vacuum reservoir is in a squeezed state for the value of \(r\) specified by Eq. (61).

In the absence of squeezing, i.e., \(r = 0\), substitution of Eq. (26) into Eq. (57) leads to
\[
(\Delta \hat{a}_1)^2 = (\Delta \hat{a}_2)^2 = \frac{1 - 2\kappa_0 e^{-(\gamma - 2\kappa_0)t}}{1 - \left(\frac{2\kappa_0}{\gamma}\right)^2} \left[1 - e^{-4\kappa_0 t}\right] + \frac{2\kappa_0}{\gamma} \left(1 + e^{-4\kappa_0 t}\right).
\]

At steady-state and when the parametric oscillator is operating below threshold \((\gamma > 2\kappa_0)\), this equation reduces to
\[
(\Delta \hat{a}_1)^2 = (\Delta \hat{a}_2)^2 = \frac{1}{1 - \left(\frac{2\kappa_0}{\gamma}\right)^2}
\]
in which both variances become greater than unity. Hence the single-mode in this case is not in a squeezed state.

In the absence of damping \((\gamma = 0)\), Eq. (57) reduces to
\[
(\Delta \hat{a}_1)^2 = (\Delta \hat{a}_2)^2 = 2\pi + 1,
\]
where \(\pi = \sinh^2 \kappa_0 t\) is the mean photon number for the single-mode. From these variances one can infer that the single-mode in this case is in a chaotic state as expected. Furthermore, in the absence of parametric interaction \((\kappa = 0)\), Eq. (57) could be expressed as
\[
(\Delta \hat{a}_{1,2})^2 = 2\sigma_{1,2} - 1 = 1 - \left[1 - e^{-\gamma t}\right] \left[1 \mp e^{\mp 2r}\right] \leq 1,
\]
and at steady-state these relations reduce to
\[
(\Delta \hat{a}_{1,2})^2 = e^{\mp 2r},
\] (62)
which are the quadrature fluctuations of the squeezed vacuum reservoir \(A\).

Now we proceed to investigate the squeezing properties of the signal-idler modes produced by the NDPO coupled to the two squeezed vacuum reservoirs applying the Q-function (46). The squeezing properties of two-mode light can be described by two quadrature operators defined as
\[
\hat{c}_{1,2} = \frac{1}{\sqrt{2}} (\hat{a}_{1,2} + \hat{b}_{1,2}),
\] (63)
where
\[ \hat{a}_1 = (\hat{a}^\dagger + \hat{a}), \quad \hat{b}_1 = (\hat{b}^\dagger + \hat{b}), \tag{64a} \]
\[ \hat{a}_2 = i(\hat{a}^\dagger - \hat{a}), \quad \hat{b}_2 = i(\hat{b}^\dagger - \hat{b}), \tag{64b} \]

and \(\hat{a} (\hat{b})\) denotes the annihilation operator for the intracavity mode \(a (b)\). The quadrature operators \(\hat{c}_1\) and \(\hat{c}_2\) satisfy the commutation relation \([\hat{c}_1, \hat{c}_2] = 2i\). On account of these expressions, the variances can be expressed as
\[
(\Delta\hat{c}_{1,2})^2 = \langle \hat{c}_{1,2}^2 \rangle - \langle \hat{c}_{1,2} \rangle^2 \]
\[
= \frac{1}{2}\langle \hat{a}_{1,2}^2 \rangle + \frac{1}{2}\langle \hat{b}_{1,2}^2 \rangle + \langle \hat{a}_{1,2}, \hat{b}_{1,2} \rangle, \tag{65} \]
in which
\[ \langle \hat{a}_i, \hat{b}_i \rangle = \langle \hat{a}_i \hat{b}_i \rangle - \langle \hat{a}_i \rangle \langle \hat{b}_i \rangle, \]
and \(i = 1, 2\). In particular, when \(a\) and \(b\) represent the signal and idler modes, respectively, it can be shown that
\[
(\Delta\hat{c}_{1,2})^2 = \frac{1}{2}(\Delta\hat{a}_{1,2})^2 + \frac{1}{2}(\Delta\hat{b}_{1,2})^2 + \langle \hat{a}_{1,2}, \hat{b}_{1,2} \rangle \]
\[
= (\Delta\hat{a}_{1,2})^2 + \langle \hat{a}_{1,2}, \hat{b}_{1,2} \rangle \tag{66} \]
as \((\Delta\hat{a}_{1,2})^2 = (\Delta\hat{b}_{1,2})^2\) and \(\langle \hat{a}_{1,2} \rangle = \langle \hat{b}_{1,2} \rangle = \langle \hat{c}_{1,2} \rangle = 0\). In order to obtain the explicit form of Eq. (66), we proceed as follows. In view of expression (64) and (55), one can express that
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = \int_{-\infty}^{\infty} d^2\alpha d^2\beta (\alpha^* + \alpha)(\beta^* + \beta) Q(\alpha, \alpha^*, \beta^*, \beta, t). \]

Then employing the Q-function (46) the above equation can be further expressed as
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = D \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} (\alpha^* + \alpha) \exp\left[ -b_1\alpha^*\alpha + \frac{b_4}{2}(\alpha^2 + \alpha^*2) \right]
\times \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} (\beta^* + \beta) \exp\left[ -b_1\beta^*\beta + (b_2\alpha + b_3\alpha^*)\beta 
+ (b_2\alpha^* + b_3\alpha)\beta^* + \frac{1}{2}b_4(\beta^2 + \beta^*2) \right]. \]

On setting \(K = b_2\alpha + b_3\alpha^*\),
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = D \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} (\alpha^* + \alpha) \exp\left[ -b_1\alpha^*\alpha + \frac{b_4}{2}(\alpha^2 + \alpha^*2) \right] \left( \frac{\partial}{\partial K} + \frac{\partial}{\partial K^*} \right)
\times \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \exp\left[ -b_1\beta^*\beta + (b_2\alpha + b_3\alpha^*)\beta + (b_2\alpha^* + b_3\alpha)\beta^* + \frac{1}{2}b_4(\beta^2 + \beta^*2) \right], \]
so that performing the integration with respect to \(\beta\) on the basis of relation (50) and carrying out the differentiation we obtain
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = \frac{D}{y^2}(b_1 + b_4)(b_2 + b_3) \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} (\alpha^2 + \alpha^*2 + 2\alpha^*\alpha) \exp\left[ -aa^*\alpha + \frac{A}{2}(\alpha^2 + \alpha^*2) \right]. \]

from which it follows that
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = \frac{D}{y^2}(b_1 + b_4)(b_2 + b_3) \left( 2\frac{\partial}{\partial A} - 2\frac{\partial}{\partial a} \right) \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi} \exp\left[ -aa^*\alpha + \frac{A}{2}(\alpha^2 + \alpha^*2) \right]. \]

Next, integrating over \(a\) and carrying out the differentiation, we get
\[
\langle \hat{a}_1 \hat{b}_1 \rangle = \frac{D}{y^2}(b_1 + b_4)(b_2 + b_3) \left( 2A + 2a \right) \frac{1}{\sqrt{a^2 - A^2}}. \]
Making use of expression (47) along with Eq. (52) the above equation reduces to

\[ \langle \hat{a}_1 \hat{b}_1 \rangle = a_1 - a_3. \]

A similar approach leads to

\[ \langle \hat{a}_2 \hat{b}_2 \rangle = -(a_2 - a_4). \]

Now Eq. (66) can be put as

\[ (\Delta \hat{c}_{1,2})^2 = 2a_{1,4} - 1, \]

and at this stage the variances are given by

\[ (\Delta \hat{c}_{1,2})^2 = 2\left\{ \frac{\kappa \gamma_0 \pm \gamma(N \mp M + 1)}{2\kappa \gamma_0 \pm \gamma} \left( 1 - e^{\mp(2\kappa \gamma_0 \pm \gamma)t} \right) + e^{\mp(2\kappa \gamma_0 \pm \gamma)t} \right\} - 1. \]

Finally the quadrature fluctuations of the signal-idler modes at any time \( t \), in view of Eq. (59), take the form

\[ \begin{align*}
(\Delta \hat{c}_1)^2 &= 1 - \left[ \gamma e^{-2r} \right] \left[ 1 - e^{-\left( \gamma \mp 2\kappa \gamma_0 \right)t} \right] < 1, & (67a) \\
(\Delta \hat{c}_2)^2 &= 1 - \left[ \gamma e^{-2r} \right] \left[ 1 - e^{-\left( \gamma \mp 2\kappa \gamma_0 \right)t} \right] > 1. & (67b)
\end{align*} \]

Hence the signal-idler modes generated by the NDPO coupled to the USVR, when operating below threshold \( (\gamma - 2\kappa \gamma_0 > 0) \), are in squeezed states for all values of \( r \).

At steady-state \( (t \to \infty) \), Eq. (67) can be put in the form

\[ \begin{align*}
(\Delta \hat{c}_1)^2 &= \left( \frac{\gamma}{\gamma \mp 2\kappa \gamma_0} \right) e^{-2r} < 1, & (68a) \\
(\Delta \hat{c}_2)^2 &= \left( \frac{\gamma}{\gamma \mp 2\kappa \gamma_0} \right) e^{+2r} > 1. & (68b)
\end{align*} \]

This equation clearly shows the possibility of a very large amount of squeezing (approaching 100%) below the standard quantum limit in the one quadrature at the expense of enhanced fluctuations in the other quadrature, where in this case the standard quantum limit is taken to be \( \sqrt{(\Delta \hat{c}_1)^2 \sqrt{(\Delta \hat{c}_2)^2}} = 1 \).

In addition, at threshold \( (\gamma = 2\kappa \gamma_0) \), one obtains

\[ \begin{align*}
(\Delta \hat{c}_1)^2 &= \frac{1}{2} e^{-2r}, & (69a) \\
(\Delta \hat{c}_2)^2 &\to \infty. & (69b)
\end{align*} \]

In the absence of squeezed vacuum reservoirs \( (r = 0) \), expression (68) becomes

\[ (\Delta \hat{c}_{1,2})^2 = \frac{\gamma \pm \left( 2\kappa \gamma_0 \right) e^{-\left( \gamma \pm 2\kappa \gamma_0 \right)t}}{\gamma \pm 2\kappa \gamma_0} \leq 1. \]  

(70)

This shows that the signal-idler modes produced by the nondegenerate parametric oscillator in the absence of squeezed vacuum reservoirs are also in squeezed states. At steady-state and at threshold, these relations reduce to

\[ \begin{align*}
(\Delta \hat{c}_1)^2 &= \frac{1}{2}, \\
(\Delta \hat{c}_2)^2 &\to \infty. & (71)
\end{align*} \]

In this case one can easily see that there is only a 50% reduction of noise below the vacuum level. By comparing Eqs. (69) and (71) we can conclude that coupling of the NDPO to the squeezed vacuum reservoirs is essential for the generation of a larger amount of squeezing.
In the absence of damping ($\gamma = 0$), expression (67) reduces to
\[ \Delta e_{1,2}^2 = e^{\mp 2\kappa \gamma_0 t} \leq 1, \] (72)
which are the quadrature fluctuations of the signal-idler modes produced by the nondegenerate parametric amplifier. This indicates that the nondegenerate parametric amplifier coupled to ordinary vacuum reservoirs also generates squeezed states.

Finally, when there is no parametric interaction inside the cavity ($\kappa = 0$), Eq. (67) takes the form
\[ (\Delta \hat{c}_{1,2})^2 = 1 - (1 - e^{-\gamma t}) \left[ 1 \mp e^{\mp 2r} \right] \leq 1, \] (73)
which, at steady-state, leads to
\[ (\Delta \hat{c}_{1,2})^2 = e^{\pm 2r}, \] (74)
which are the quadrature fluctuations of the reservoir modes $A$ and $B$. Upon comparing the relations (71) and (74) with (69), one can see that the quadrature variances at steady-state and at threshold are the product of the variances of the NDPO coupled to ordinary vacuum and the variances pertaining to the squeezed vacuum reservoirs. Furthermore, upon comparing expressions (62) and (74) one can observe that at steady-state the variances of a signal mode squeezed vacuum reservoir as well as those of two independent squeezed vacuum reservoirs are the same.

6 Conclusion

We have derived the master equation for the signal-idler modes produced by the nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs and consequently the Fokker-Planck equation. We have solved the pertinent Fokker-Planck equation which is a second order differential equation applying the propagator method [28] and obtained a compact form of the Q-function of the optical system coupled to two independent squeezed vacuum reservoirs. We have also deduced the Q-functions for a NDPO coupled to ordinary vacuum reservoirs, degenerate parametric oscillators coupled to a squeezed vacuum reservoir and an ordinary vacuum reservoir, and for the nondegenerate and degenerate parametric amplifiers from the Q-function for the NDPO coupled to the two USVR.

In general the Q-function can be used to evaluate the expectation values of antinormally ordered operators as well as photon number distributions for the NDPO and other similar optical systems.

We have calculated the nonlinear quantum quadrature fluctuations of the signal-idler modes generated by a nondegenerate parametric oscillator below threshold coupled to two uncorrelated squeezed vacuum reservoirs, using the Q-function. Although it is a well known fact that quantum noise cannot be eliminated, we have shown that the signal-idler modes produced by the optical system are in a two-mode squeezed state at any time $t$. More interestingly, we have shown that at steady-state and below threshold it is possible to generate an optimal squeezing in one of the quadratures below the standard quantum limit at the expense of enhanced fluctuations in the other quadrature so that the Heisenberg uncertainty principle remains valid. Furthermore, calculation of the quadrature fluctuations at threshold clearly shows that it is possible to produce an arbitrarily large squeezing (approaching 100%) in one of the quadratures with an infinitely large noise in the other quadrature. We have also shown that the degenerate parametric oscillator could be in a squeezed state for a squeezing parameter above a certain value when it is coupled to a squeezed vacuum reservoir.

We have shown that the coupling of the optical system to the squeezed vacuum reservoirs is essential in order to get a more suppressed noise in one of the quadratures.

Finally, we have calculated the quadrature fluctuations for the nondegenerate parametric amplifier coupled to ordinary vacuum reservoirs and verified that it also generates squeezed states.

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