Axiomatic systems and topological semantics for intuitionistic temporal logic

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Abstract

We propose four axiomatic systems for intuitionistic linear temporal logic and show that each of these systems is sound for a class of structures based either on Kripke frames or on dynamic topological systems. Our topological semantics features a new interpretation for the ‘henceforth’ modality that is a natural intuitionistic variant of the classical one. Using the soundness results, we show that the four logics obtained from the axiomatic systems are distinct. Finally, we show that when the language is restricted to the ‘henceforth’-free fragment, the set of valid formulas for the relational and topological semantics coincide.

1 Introduction

Intuitionistic logic enjoys a myriad of interpretations based on computation, information or topology, making it a natural framework to reason about dynamic processes in which these phenomena play a crucial role. In the areas of non-monotonic reasoning, knowledge representation (KR), and artificial intelligence, intuitionistic and intermediate logics have played an important role within the successful answer set programming (ASP) ⁰³ paradigm for practical KR. Great part of its success is due to the impressive advances in implementation of efficient solvers ²⁸, ¹⁸ and its use in a wide range of domains such as computational biology ²⁰, spatial reasoning ³⁸, or configuration ¹⁹.
Central to this paradigm is equilibrium logic [35], which characterises the ASP semantics in terms of the intermediate logic of here and there [21] plus a minimisation criterion. Such a definition has led to several extensions of modal ASP [8, 14] that are supported by intuitionistic modal logics like temporal here and there [4] and are crucial when characterising the theorem of strong equivalence [30, 13, 7].

There are also several potential applications for intuitionistic temporal logics that are unrelated to ASP. Davies [9] has suggested an extension of the Curry-Howard isomorphism [12] to partially evaluated programs by adding a next-time operator \( \circ \). Maier [31] observed that an intuitionistic temporal logic with infinitary operators including a henceforth operator \( \Box \) could be used for reasoning about safety and liveness conditions in possibly-terminating reactive systems. Fernández-Duque [15] has suggested that a logic with ‘eventually’ \( \diamond \) can be used to provide a decidable framework in which to reason about topological dynamics. It is thus surprising that the computational and proof-theoretic properties of these logics are far from being well-understood.

State-of-the-art. There have, however, been some notable efforts in this direction. Kojima and Igarashi [23] endowed Davies’ logic with Kripke semantics and provided a complete deductive system. Bounded-time versions of logics with henceforth were later studied by Kamide and Wansing [22]. Both use semantics based on Simpson’s bi-relational models for intuitionistic modal logic [36]. Since then, Balbiani and the authors have shown that temporal here-and-there is decidable and enjoys a natural axiomatization [4]. They have identified two natural, semantically-defined intuitionistic temporal logics, ITL\( ^e \) and ITLP, studied bisimulations for these logics [3], and shown ITL\( ^e \) to be decidable [5]. However, the decision procedure does not provide a natural axiomatization, and moreover the decidability of ITLP remains open, despite the latter logic being attractive due to it validating the familiar Fischer Servi axioms [17].

Topological semantics for intuitionistic modal and tense logics have also been studied by Davoren et al. [11, 10], and Kremer suggested an intuitionistic variant of LTL [26] similar to dynamic topological logic (DTL) [2, 27]. DTL is a tri-modal system which gained interest due to its potential applications to automated theorem proving for topological dynamics, but was later shown to be undecidable [24]. On the other hand, the decidability of Kremer’s intuitionistic temporal logic remains open, but Fernández-Duque has shown that a logic with ‘eventually’ \( \diamond \) instead of \( \Box \) is decidable [15]. Both intuitionistic temporal logics can be seen as sublogics of DTL via the Gödel-Tarski translation [37].

Our contribution. The above decidability results for intuitionistic temporal logics are based on semantical methods. The primary goal of this paper is to lay the groundwork for an axiomatic treatment of intuitionistic linear temporal logics. We will introduce a ‘minimal’ intuitionistic temporal logic, ITLP\( ^i \), defined by adding standard axioms of LTL to intuitionistic propositional logic. We also consider additional Fischer Servi axioms and a ‘constant domain’ axiom
\(\Box(p \lor q) \rightarrow \Box p \lor \Diamond q\). Combining these, we obtain four intuitionistic temporal logics. As we will see, each of these logics is sound for a class of structures; the two logics with the constant domain axiom are sound for the class of dynamic posets, and the Fischer Servi axioms correspond to backwards-confluence of the transition function.

The constant domain axiom is not derivable from the others, and to show this, we will consider topological semantics for intuitionistic temporal logic. As our axioms involve both \(\Diamond\) and \(\Box\), we would like to be able to interpret both tenses. Kremer observed that his semantics for \(\Box\) do not satisfy some key validities of \(\text{LTL}\), namely \(\Box p \rightarrow \Diamond \Box p\), \(\Box \Diamond p \rightarrow \Diamond \Box p\), and \(\Box p \rightarrow \Box \Box p\). This makes a proof-theoretic treatment of Kremer’s logic difficult, as \(\Box \varphi \rightarrow \Diamond \Box \varphi\) is one of the defining properties of \(\Box\) and it is hard to tell what weaker principle could replace it.

To avoid this issue, we propose an alternative interpretation for \(\Box\). Our approach is natural from an algebraic perspective, as we define the interpretation of \(\Box \varphi\) via a greatest fixed point in the Heyting algebra of open sets. On the other hand, this fixed point is not definable in the classical language and hence we no longer obtain a sub-logic of \(\text{DTL}\). We will show that dynamic topological systems provide semantics for the logics without the constant domain axiom, from which we conclude the independence of the latter. Moreover, we show that the Fischer Servi axioms are valid for the class of open dynamical topological systems.

The constant domain axiom shows that the \(\{\Diamond, \Box\}\)-logic of dynamic posets is different from that of dynamic topological systems. We show via an alternative axiom that the \(\{\Diamond, \Box\}\)-logics are also different. On the other hand, our main technical contribution is a proof that the \(\{\Diamond, \Box\}\)-logics coincide, for which we use quasimodels, introduced in the context of intuitionistic temporal logics by Fernández-Duque [15]. This suggests that a completeness proof as in [16] could be adapted to give a complete deductive calculus for the \(\{\Diamond, \Box\}\)-logic over both the class of dynamic posets and the class of dynamic topological systems.

**Layout.** Section 2 introduces the syntax and the four axiomatic systems we propose for intuitionistic temporal logic. Section 3 reviews dynamic topological systems, which are used in Section 4 to provide semantics for our formal language. Section 5 shows that each of the four logics is sound for a class of dynamical systems. These soundness results are used in Section 6 to show that the four logics are pairwise distinct. Section 7 reviews non-deterministic quasimodels, which are used in Section 8 to show that the topological and the Kripke \(\{\Diamond, \Box\}\)-logics coincide. Finally, Section 9 lists some open questions.

## 2 Syntax and axiomatics

In this section we will introduce four natural intuitionistic temporal logics. All of the axioms have appeared either in the intuitionistic logic, the temporal logic,
or the intuitionistic modal logic literature. They will be based on the language of linear temporal logic, as defined next.

Fix a countably infinite set \( P \) of ‘propositional variables’. The language \( \mathcal{L} \) of intuitionistic (linear) temporal logic \( \text{ITL} \) is given by the grammar

\[
\perp \mid p \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \rightarrow \psi \mid \circ \varphi \mid \Diamond \varphi \mid \Box \varphi,
\]

where \( p \in P \). As usual, we use \( \neg \varphi \) as a shorthand for \( \varphi \rightarrow \perp \) and \( \varphi \leftrightarrow \psi \) as a shorthand for \( (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \). We read \( \circ \) as ‘next’, \( \Diamond \) as ‘eventually’, and \( \Box \) as ‘henceforth’. Given any formula \( \varphi \), we denote the set of subformulas of \( \varphi \) by \( \text{sub}(\varphi) \) and its length by \( |\varphi| \). The language \( \mathcal{L}_0 \) is defined as the sublanguage of \( \mathcal{L} \) without the modality \( \Box \). Similarly, \( \mathcal{L}_\Box \) is the language without \( \Diamond \).

We begin by establishing our basic axiomatization. It is obtained by adapting the standard axioms and inference rules of \( \text{LTL} \) \[29\], as well as their dual versions, to propositional intuitionistic logic \[32\]. The logic \( \text{ITL}^0 \) is the least set of \( \mathcal{L} \)-formulas closed under the following rules and axioms.

(i) All intuitionistic tautologies.

(ii) \( \neg \circ \perp \)

(iii) \( \circ (\varphi \land \psi) \leftrightarrow (\circ \varphi \land \circ \psi) \)

(iv) \( \circ (\varphi \lor \psi) \leftrightarrow (\circ \varphi \lor \circ \psi) \)

(v) \( \circ (\varphi \rightarrow \psi) \rightarrow (\circ \varphi \rightarrow \circ \psi) \)

(vi) \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

(vii) \( \Diamond (\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi) \)

(viii) \( \Diamond (\varphi \lor \psi) \rightarrow (\Diamond \varphi \lor \Diamond \psi) \)

(ix) \( \Box \varphi \rightarrow \varphi \land \circ \Box \varphi \)

(x) \( \varphi \lor \circ \Diamond \varphi \rightarrow \Diamond \varphi \)

(xi) from \( \varphi \rightarrow \circ \varphi \) infer \( \varphi \rightarrow \Box \varphi \)

(xii) from \( \circ \varphi \rightarrow \varphi \) infer \( \Diamond \varphi \rightarrow \varphi \)

(xiii) from \( \varphi \) and \( \varphi \rightarrow \psi \) infer \( \psi \)

(xiv) from \( \varphi \) infer \( \circ \varphi \).

However, modal intuitionistic logics typically involve additional axioms, due to Fischer Servi \[17\], in order to strengthen the ties with first-order intuitionistic logic. Thus we may also consider logics with the latter; for \( \text{FS}_0 \), recall that \( \circ \) is self-dual.

\[
(\text{FS}_\circ (\varphi, \psi)) \quad (\circ \varphi \rightarrow \circ \psi) \rightarrow \circ (\varphi \rightarrow \psi) ,
\]

\[
(\text{FS}_\Diamond (\varphi, \psi)) \quad (\Diamond \varphi \rightarrow \Box \psi) \rightarrow \Box (\varphi \rightarrow \psi) .
\]

Finally, we consider additional axioms reminiscent of constant domain axioms in first-order intuitionistic logic. As we will see, in the context of intuitionistic temporal logics, these axioms separate Kripke semantics from the more general topological semantics.
(CD(ψ, ψ)) □(ψ ∨ ψ) → □ψ ∨ □ψ,
(BI(ψ, ψ)) □(ψ ∨ ψ) ∧ □(ψ → ψ) → □ψ ∨ ψ.

Here, CD stands for ‘constant domain’ and BI for ‘backward induction’. The axiom BI is meant to be a ♦-free approximation to CD, as witnessed by the following.

**Proposition 1.** ITL⁰ ⊢ CD(p, q) → BI(p, q).

*Proof.* We reason within ITL⁰. Assume that (1) CD(p, q) holds, along with (2) □(cq → q) and (3) □(p ∨ q). From [1] and [3] we obtain □p ∨ □q, which together with [2] and axiom [xii] gives us □p ∨ q, as needed.

With this, we define the following logics:

\[
\text{ITL}^{FS} \equiv \text{ITL}^0 + \text{FS}_o + \text{FS}_\Diamond,
\]

\[
\text{ITL}^{CD} \equiv \text{ITL}^0 + \text{CD},
\]

\[
\text{ITL}^1 \equiv \text{ITL}^{FS} + \text{ITL}^{CD}.
\]

We are also interested in logics over sublanguages of \( \mathcal{L} \). For any logic Λ defined above, let Λ_♦ be the logic obtained by restricting all rules and axioms to \( \mathcal{L}_o \), and let Λ□ be defined by restricting similarly to \( \mathcal{L}_\Box \), except that when CD is an axiom of Λ, we add the axiom BI to Λ□.

## 3 Dynamic topological systems

The four logics defined above are pairwise distinct. We will show this by introducing semantics for each of them. They will be based on dynamic topological systems (or dynamical systems for short), which, as was observed in [15], generalize their Kripke semantics [5].

### 3.1 Topological spaces and continuous functions

Let us recall the definition of a *topological space* [33]:

**Definition 1.** A topological space is a pair \((X, T)\), where \(X\) is a set and \(T\) a family of subsets of \(X\) satisfying

(a) \(\emptyset, X \in T\);

(b) if \(U, V \in T\) then \(U \cap V \in T\), and

(c) if \(O \subseteq T\) then \(\bigcup O \in T\).

The elements of \(T\) are called open sets.
If \( x \in X \), a \textit{neighbourhood} of \( x \) is an open set \( U \subseteq X \) such that \( x \in U \). Given a set \( A \subseteq X \), its \textit{interior}, denoted \( A^\circ \), is the largest open set contained in \( A \). It is defined formally by

\[
A^\circ = \bigcup \{ U \in \mathcal{T} : U \subseteq A \}.
\]

(1)

Dually, we define the \textit{closure} \( \overline{A} \) as \( X \setminus (X \setminus A)^\circ \); this is the smallest closed set containing \( A \).

If \( (X, \mathcal{T}) \) is a topological space, a function \( S : X \to X \) is \textit{continuous} if, whenever \( U \subseteq X \) is open, it follows that \( S^{-1}[U] \) is open. The function \( S \) is \textit{open} if, whenever \( V \subseteq X \) is open, then so is \( S[V] \). An open, continuous function is an \textit{interior map}, and a bijective interior map is a \textit{homeomorphism}.

A dynamical system is then a topological space equipped with a continuous function:

\textbf{Definition 2.} A dynamical (topological) system is a triple \( \mathcal{X} = (X, \mathcal{T}, S) \) such that \( (X, \mathcal{T}) \) is a topological space and \( S : X \to X \) is continuous. We say that \( \mathcal{X} \) is invertible if \( S \) is a homeomorphism, i.e., \( S^{-1} \) is also a continuous function, and open if \( S \) is an interior map.

\section{3.2 Up-set topologies}

Topological spaces generalize posets in the following way. Let \( \mathcal{F} = (W, \leq) \) be a poset; that is, \( W \) is any set and \( \leq \) is a transitive, reflexive, antisymmetric relation on \( W \). To see \( \mathcal{F} \) as a topological space, define \( \uparrow w = \{ v : w \leq v \} \). Then consider the topology \( \mathcal{T}_{\leq} \) on \( W \) given by setting \( U \subseteq W \) to be open if and only if, whenever \( w \in U \), we have \( \uparrow w \subseteq U \). A topology of this form is a \textit{up-set topology} \cite{1}. The interior operator on such a topological space can be computed by

\[
A^\circ = \{ w \in W : \uparrow w \subseteq A \};
\]

(2)
i.e., \( w \) lies on the interior of \( A \) if whenever \( v \geq w \), it follows that \( v \in A \).

Throughout this text we will often identify partial orders with their corresponding topologies, and many times do so tacitly. In particular, a dynamical system generated by a poset is called a \textit{dynamic poset}. It will be useful to characterize the continuous and open functions on posets:

\textbf{Lemma 1.} Consider a poset \( (W, \preceq) \) and a function \( S : W \to W \). Then,

1. \( S \) is continuous with respect to the up-set topology if and only if, whenever \( w \preceq w' \), it follows that \( S(w) \preceq S(w') \), and

2. \( S \) is open with respect to the up-set topology if whenever \( S(w) \preceq v \), there is \( w' \in W \) such that \( w \preceq w' \) and \( S(w') = v \).

These are confluence properties common in multi-modal logics; note that in \cite{5} we referred to maps satisfying the two conditions as \textit{persistent maps}. 

6
4 Semantics

In this section we will see how dynamical systems can be used to provide a natural intuitionistic semantics for the language of linear temporal logic.

4.1 Basic definitions

Formulas are interpreted as open subspaces of a dynamical system. Each propositional variable \( p \) is assigned an open set \( \llbracket p \rrbracket \), and then \( \llbracket \cdot \rrbracket \) is defined recursively for more complex formulas according to the following:

**Definition 3.** Given a dynamical system \( \mathcal{X} = (X, \mathcal{T}, S) \), a valuation on \( \mathcal{X} \) is a function \( \llbracket \cdot \rrbracket : \mathcal{L} \to \mathcal{T} \) such that:

\[
\begin{align*}
\llbracket \top \rrbracket &= \varnothing \\
\llbracket \varphi \land \psi \rrbracket &= \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \\
\llbracket \varphi \lor \psi \rrbracket &= \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket \\
\llbracket \varphi \rightarrow \psi \rrbracket &= \left( (X \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket \right)^\circ \\
\llbracket \varphi^s \rrbracket &= S^{-1} \llbracket \varphi \rrbracket \\
\llbracket \varphi^\Diamond \rrbracket &= \bigcup_{n \geq 0} S^{-n} \llbracket \varphi \rrbracket \\
\llbracket \varphi^\Box \rrbracket &= \bigcup \left\{ U \in \mathcal{T} : S[U] \subseteq \llbracket \varphi \rrbracket \right\}
\end{align*}
\]

A tuple \( \mathcal{M} = (X, \mathcal{T}, S, [\cdot]) \) consisting of a dynamical system with a valuation is a dynamic topological model, and if \( \mathcal{T} \) is generated by a partial order, we will say that \( \mathcal{M} \) is a dynamic poset model.

All of the semantic clauses are standard from either intuitionistic or temporal logic, with the exception of that for \( \Box \varphi \), which we discuss in greater detail below. It is not hard to check by structural induction on \( \varphi \) that \( \llbracket \varphi \rrbracket \) is uniquely defined.

Figure 1: On a dynamic poset the above diagrams can always be completed if \( S \) is continuous or open, respectively. Open, continuous maps on a poset are persistent.
given any assignment of the propositional variables to open sets, and that $[\varphi]$ is always open.

In practice, it is convenient to have a ‘pointwise’ characterization of Definition 3. For a model $M = (X, T, S, [\cdot])$, $x \in X$ and $\varphi \in \mathcal{L}$, we write $M, x \models \varphi$ if $x \in [\varphi]$, and $M \models \varphi$ if $[\varphi] = X$. Then, in view of (1), given formulas $\varphi$ and $\psi$, $M, x \models \varphi \rightarrow \psi$ if and only if there is a neighbourhood $U$ of $x$ such that for all $y \in U$, if $M, y \models \varphi$ then $M, y \models \psi$; note that this is a special case of neighbourhood semantics [32].

Using (2), this can be simplified somewhat in the case that $T$ is generated by a partial order $\preccurlyeq$:

**Proposition 2.** If $(X, \triangleleft, S, [\cdot])$ is a dynamic poset model, $x \in X$, and $\varphi, \psi$ are formulas, then $M, x \models \varphi \rightarrow \psi$ if and only if whenever $y \trianglerighteq x$ and $M, y \models \varphi$, it follows that $M, y \models \psi$.

This is the standard relational interpretation of implication, and thus topological semantics are a generalization of the usual Kripke semantics.

4.2 The topological ‘henceforth’

Now let us discuss the topological interpretation of ‘henceforth’, which is the main novelty in our semantics. In classical temporal logic, $[\square \varphi]$ is the largest set contained in $[\varphi]$ which is closed under $S$. In our semantics, $[\square \varphi]$ is the greatest open set which is closed under $S$. From this perspective, our interpretation is the natural intuitionistic variant of the classical one. If $M, x \models \square \varphi$, this fact is witnessed by an open, $S$-invariant neighbourhood of $x$, where $U \subseteq X$ is $S$-invariant if $S[U] \subseteq U$.

**Proposition 3.** If $(X, T, S, [\cdot])$ is a dynamic topological model, $x \in X$, and $\varphi$ is any formula, then $M, x \models \square \varphi$ if and only if there is an $S$-invariant neighbourhood $U$ of $x$ such that for all $y \in U$, $M, y \models \varphi$.

In fact, the open, $S$-invariant sets form a topology; that is, the family of $S$-invariant open sets is closed under finite intersections and arbitrary unions. This topology is coarser than $T$, in the sense that every $S$-invariant open set is (tautologically) open. Thus $\square$ can itself be seen as an interior operator based on a coarsening of $T$, and $[\square \varphi]$ is always an $S$-invariant open set.

**Example 1.** As usual, the real number line is denoted by $\mathbb{R}$ and we assume that it is equipped with the standard topology, where $U \subseteq \mathbb{R}$ is open if and only if it is a union of intervals of the form $(a, b)$. Consider a dynamical system based on $\mathbb{R}$ with $S: \mathbb{R} \to \mathbb{R}$ given by $S(x) = 2x$. We claim that for any model $M$ based on $(\mathbb{R}, S)$ and any formula $\varphi$, $M, 0 \models \square \varphi$ if and only if $M \models \varphi$.

To see this, note that one implication is obvious since $\mathbb{R}$ is open and $S$-invariant, so if $[\varphi] = \mathbb{R}$ it follows that $M, 0 \models \square \varphi$. For the other implication, assume that $M, 0 \models \square \varphi$, so that there is an $S$-invariant, open $U \subseteq [\varphi]$ with $0 \in U$. It follows from $U$ being open that for some $\epsilon > 0$, $(0, \epsilon) \subseteq U$. Now, let $x \in \mathbb{R}$, and let $n$ be large enough so that $2^{-n}x \in U$, and
since $U$ is $S$-invariant, $x = S^n (2^{-n} x) \in U$. Since $x$ was arbitrary, $U = \mathbb{R}$, and it follows that $M \models \varphi$.

On the other hand, suppose that $0 < a < x$ and $(a, \infty) \subseteq \llbracket \varphi \rrbracket$. Then, $(a, \infty)$ is open and $S$-invariant, so it follows that $x \in \llbracket \square \varphi \rrbracket$. Hence in this case we do not require that $\llbracket \varphi \rrbracket = \mathbb{R}$. Similarly, if $x < a < 0$ and $(-\infty, a) \subseteq \llbracket \varphi \rrbracket$, we readily obtain $x \in \llbracket \square \varphi \rrbracket$.

4.3 The relational ‘henceforth’

As was the case for implication, our interpretation for $\square$ becomes familiar when restricted to Kripke semantics.

Lemma 2. Let $M = (W, \preceq, S, \llbracket \cdot \rrbracket)$ be any dynamic poset model, $w \in W$ and $\varphi \in \mathcal{L}$. Then, the following are equivalent:

(a) $M, w \models \square \varphi$;

(b) $w \in \left( \bigcap_{n \in \omega} S^{-n} \llbracket \varphi \rrbracket \right)^\circ$;

(c) for all $n < \omega$, $M, S^n(w) \models \varphi$.

Proof. First we prove that (a) implies (b). Assume that $M, w \models \square \varphi$, so that there is an $S$-invariant neighbourhood $U$ of $w$ with $U \subseteq \llbracket \varphi \rrbracket$. To see that $w \in \left( \bigcap_{n \in \omega} S^{-n} \llbracket \varphi \rrbracket \right)^\circ$, we must show that if $v \succ w$, then $v \in \bigcap_{n \in \omega} S^{-n} \llbracket \varphi \rrbracket$. So fix such a $v$ and $n < \omega$. Since $U$ is $S$-invariant, $S^n(w) \in U$, and since $S^n(v) \succ S^n(w)$ and $U$ is open, $S^n(v) \in U$, as needed. Thus $v \in \bigcap_{n < \omega} S^{-n} \llbracket \varphi \rrbracket$, and since $v \succ w$ was arbitrary, (b) holds.

That (b) implies (c) is immediate from

$$\left( \bigcap_{n < \omega} S^{-n} \llbracket \varphi \rrbracket \right)^\circ \subseteq \bigcap_{n < \omega} S^{-n} \llbracket \varphi \rrbracket,$$

so it remains to show that (c) implies (a). Suppose that for all $n < \omega$, $M, S^n(w) \models \varphi$, and let $U = \bigcup_{n < \omega} \uparrow S^n(w)$. That the set $U$ is open follows from each $\uparrow S^n(w)$ being open and unions of opens being open. If $v \in U$, then $v \succ S^n(w)$ for some $n < \omega$ and hence by upwards persistence, from $M, S^n(w) \models \varphi$ we obtain $M, v \models \varphi$; moreover, $S(v) \succ S^{n+1}(w)$ so $S(v) \in U$. Since $v \in U$ was arbitrary, we conclude that $U$ is $S$-invariant and $U \subseteq \llbracket \varphi \rrbracket$. Thus $U$ witnesses that $M, w \models \square \varphi$.

Remark 1. In fact, Kremer \[26\] uses (b) as the definition of $\llbracket \square \varphi \rrbracket$. However, as we mentioned in the introduction, even our minimal axiomatic system $\mathbf{ITL}^0$ is not sound for such an interpretation over arbitrary dynamical systems.
5 Soundness

Recall that if \( M = (X, T, S, [[]]) \) is any dynamic topological model and \( \varphi \in \mathcal{L} \) is any formula, we write \( M \models \varphi \) if \( [[\varphi]] = X \). Similarly, if \( \mathcal{X} = (X, T, S) \) is a dynamical system, we write \( \mathcal{X} \models \varphi \) if for any valuation \( \mathcal{[} \cdot \mathcal{]} \) on \( \mathcal{X} \), we have that \( (X, [[\cdot]]) \models \varphi \). Finally, if \( \Omega \) is a class of structures, we write \( \Omega \models \varphi \) if for every \( A \in \Omega \), \( A \models \varphi \), in which case we say that \( \varphi \) is valid on \( \Omega \).

For purposes of this discussion, a logic may be any set \( \Lambda \subseteq \mathcal{L} \), and we may write \( \Lambda \vdash \varphi \) instead of \( \varphi \in \Lambda \). Then, \( \Lambda \) is sound for a class of structures \( \Omega \) if, whenever \( \Lambda \vdash \varphi \), it follows that \( \Omega \models \varphi \).

In this section we will show that the four logics we have considered are sound for semantics based on different classes of dynamic topological systems (including dynamic preorders). For this, the following simple observation will be useful.

**Lemma 3.** If \( M = (X, T, S, [[]]) \) is any model and \( \varphi, \psi \in \mathcal{L} \), then \( M \models \varphi \rightarrow \psi \) if and only if \( [[\varphi]] \subseteq [[\psi]] \).

**Proof.** If \( [[\varphi]] \subseteq [[\psi]] \) then \( (X \setminus [[\varphi]]) \cup [[\psi]] = X \), so \( [[\varphi \rightarrow \psi]] = ((X \setminus [[\varphi]]) \cup [[\psi]])^o = X^o = X \). Otherwise, there is \( z \in [[\varphi]] \) such that \( z \notin [[\psi]] \), so that \( z \notin ((X \setminus [[\varphi]]) \cup [[\psi]])^o \), i.e. \( z \notin [[\varphi \rightarrow \psi]] \). \( \square \)

5.1 Soundness of ITL^0

With this in mind, let us now show that our minimal logic is sound for the class of all dynamical systems.

**Theorem 1.** ITL^0 is sound for the class of dynamical systems.

**Proof.** Let \( M = (X, T, S, [[]]) \) be any dynamical topological model; we must check that all the axioms \([\text{i})][\text{x}]\) are valid on \( M \) and all rules \([\text{xi})][\text{xiv}]\) preserve validity. Note that all intuitionistic tautologies are valid due to the soundness for topological semantics \([\text{32}]\). Many of the other axioms can be checked routinely, so we focus only on those axioms involving the continuity of \( S \) or the semantics for \( \Box \).

\([\text{v}]\) Suppose that \( x \in [[\circ(\varphi \rightarrow \psi)]] \). Then, \( S(x) \in [[\varphi \rightarrow \psi]] \). Since \( S \) is continuous and \( [[\varphi \rightarrow \psi]] \) is open, \( U = S^{-1}[[\varphi \rightarrow \psi]] \) is a neighbourhood of \( x \). Then, for \( y \in U \), if \( y \in [\circ\varphi] \), it follows that \( S(y) \in [[\varphi]] \cap [[\varphi \rightarrow \psi]] \), so that \( S(y) \in [[\psi]] \) and \( y \in [[\psi]] \). Since \( y \in U \) was arbitrary, \( x \in [\circ\varphi \rightarrow \circ\psi] \), thus \( [\circ(\varphi \rightarrow \psi)] \subseteq [\circ\varphi \rightarrow \circ\psi] \), and by Lemma \([\text{3}]\) (which from now on we will use without mention), \([\text{v}]\) is valid on \( M \).

\([\text{vi}]\) Observe that \( [[\Box(\varphi \rightarrow \psi)]] \) is an \( S \)-invariant open subset of \( [[\varphi \rightarrow \psi]] \). Similarly, \( [[\Box\varphi]] \) is an \( S \)-invariant open subset of \( [[\varphi]] \). Let \( U = [[\Box(\varphi \rightarrow \psi)]] \cap [[\Box\varphi]] \). 

\( U = [[\Box(\varphi \rightarrow \psi)]] \cap [[\Box\varphi]] \).
Since $U$ is open, it suffices to prove that $U \subseteq [\Box \varphi]$. Moreover, $U$ is $S$-
oinvariant, therefore it suffices to prove that $U \subseteq [\psi]$, which is direct because $U \subseteq [\varphi \to \psi] \cap [\varphi]$ and $[\varphi \to \psi] \subseteq (X \setminus [\varphi]) \cup [\psi]$.

Suppose that $x \in [\Box \varphi]$, and let $U \subseteq [\varphi]$ be an $S$-invariant neighbourhood of $x$. Then, $x \in U$, so $x \in [\varphi]$. Moreover, $U$ is also an $S$-invariant neighbourhood of $S(x)$, so $S(x) \in [\Box \varphi]$ and thus $x \in [\Box \Box \varphi]$. We conclude that $x \in [\varphi \land \Box \varphi]$.

If $\varphi \to \Box \varphi$ is valid and $x \in [\varphi]$, then $[\varphi]$ is open (by the intuitionistic semantics) and $S$-invariant, since if $y \in [\varphi]$, from $y \in [\varphi \to \Box \varphi]$ we obtain $S(y) \in [\varphi]$. It follows that $[\varphi]$ is an $S$-invariant neighbourhood of $x$, so $x \in [\Box \varphi]$.

\section{Soundness of stronger logics}

The additional axioms we have considered are valid over specific classes of dynamical systems. Specifically, the constant domain axiom is valid for the class of dynamic posets, while the Fischer Servi axioms are valid for the class of open systems. Let us begin by discussing the former in more detail.

\textbf{Theorem 2.} $\text{ITL}^{\text{CD}}$ and $\text{ITL}_{\Box}^{\text{CD}}$ are sound for the class of dynamic posets.

\textbf{Proof.} Let $\mathcal{M} = (X, <, S, [\cdot])$ be a dynamic poset model; in view of Theorem 1 it only remains to check that CD and BI are valid on $\mathcal{M}$. However, by Proposition 1 BI is a consequence of CD, so we only check the latter.

$(\text{CD}(\varphi, \psi))$ Suppose that $x \in [\Box (\varphi \lor \psi)]$, but $x \notin [\Box \varphi]$. Then, in view of Lemma 2 for some $n \geq 0$, $S^n(x) \notin [\varphi]$. It follows that $S^n(x) \in [\psi]$, so that $x \in [\Box \psi]$.

Note that the relational semantics are used in an essential way, since Lemma 2 is not available in the topological setting. Now let’s turn our attention to the Fischer Servi axioms.

\textbf{Theorem 3.} $\text{ITL}^{\text{FS}}$ is sound for the class of open dynamical systems.

\textbf{Proof.} Let $\mathcal{M} = (X, T, S, [\cdot])$ be a dynamical topological model where $S$ is an interior map. We check that axioms $\text{FS}_\varphi$ and $\text{FS}_{\Box \varphi}$ are valid on $\mathcal{M}$.

$(\text{FS}_\varphi)$ Suppose that $x \in [\Box \varphi \to \Box \psi]$, and let $U \subseteq [\Box \varphi \to \Box \psi]$ be a neighbourhood of $x$. Since $S$ is open, $V = S[U]$ is a neighbourhood of $S(x)$. Let $y \in V \cap [\varphi]$, and choose $z \in U$ so that $y = S(z)$. Then, $z \in U \cap [\Box \varphi]$, so that $z \in [\psi]$, i.e. $y \in [\psi]$. Since $y \in V$ was arbitrary, $S(x) \in [\varphi \to \psi]$, and $x \in [\Box \varphi \to \Box \psi]$.

$(\text{FS}_{\Box \varphi})$ Suppose that $x \in [\Box \varphi \to \Box \psi]$, and let $U \subseteq [\Box \varphi \to \Box \psi]$ be a neighbourhood of $x$. Set $V = \bigcup_{n \in \omega} S^n[U]$; since $S$ is open and unions of opens are open, $V$ is open as well. Moreover, $V$ is clearly $S$-invariant, as if $x \in V$, then $x \in S^n[U]$ for some $n \geq 0$, so that $S(x) \in S^{n+1}[U] \subseteq V$.

We claim that $V \subseteq [\varphi \to \psi]$, from which we obtain a witness that $\mathcal{M}, x = \Box (\varphi \to \psi)$. Suppose that $y \in V \cap [\varphi]$. By the definition of $V$, $y = S^n(z)$ for
some $n < \omega$ and some $z \in U$. Then, $z \in U \cap [\Diamond \varphi]$, so that $z \in [\Box \psi]$. From this we may choose an $S$-invariant neighbourhood $Z \subseteq [\psi]$ of $z$. But $y = S^n(z) \in Z$ so that $y \in [\psi]$, and since $y \in V$ was arbitrary we see that $V \subseteq [\varphi \rightarrow \psi]$, as needed.

As an easy consequence, we mention the following combination of Theorems 2 and 3. Recall that dynamic posets with an interior map are also called persistent.

**Corollary 1.** ITL$^1$ and ITL$^\Box$ are sound for the class of persistent dynamic posets.

6 Independence

In this section we will use our soundness results to show that the four logics we have considered are pairwise distinct.

6.1 Independence of the constant domain axioms

The formulas $\text{CD}(p,q)$ and $\text{BI}(p,q)$ separate Kripke semantics from the general topological semantics.

**Proposition 4.** The formulas $\text{CD}(p,q)$ and $\text{BI}(p,q)$ are not valid over the class of invertible dynamical systems based on $\mathbb{R}$.

*Proof.* Define a model $\mathcal{M}$ on $\mathbb{R}$, with $S(x) = 2x$, $[p] = (-\infty,1)$ and $[q] = (0,\infty)$. Clearly $[p \lor q] = \mathbb{R}$, so that $[\Box (p \lor q)] = \mathbb{R}$ as well.

Let us see that $\mathcal{M},0 \not\models \text{CD}(p,q)$. Since $\mathcal{M},0 \models \Box (p \lor q)$, it suffices to show that $\mathcal{M},0 \not\models \Box p \lor \Diamond q$. It is clear that $\mathcal{M},0 \not\models \Diamond q$ simply because $S^n(0) = 0 \notin [q]$ for all $n$. Meanwhile, by Example 1, $\mathcal{M},0 \models \Box p$ if and only if $[p] = \mathbb{R}$, which is not the case. We conclude that $\mathcal{M},0 \not\models \text{CD}(p,q)$.

To see that $\mathcal{M},0 \not\models \text{BI}(p,q)$ we proceed similarly, where the only new ingredient is observing that $\mathcal{M},0 \models \Box (\Diamond q \rightarrow q)$. But this follows easily from the fact that if $\mathcal{M},x \models \Diamond q$, then $x > 0$ so that $\mathcal{M},x \models q$, hence $[\Diamond q \rightarrow q] = \mathbb{R}$.

**Corollary 2.** ITL$^\Diamond \not\vdash \text{CD}(p,q)$ and ITL$^\Box \not\vdash \text{BI}(p,q)$.

*Proof.* By Theorem 3 ITL$^\Diamond$ is sound for the class of open dynamical systems, but by Proposition 4 CD$(p,q)$ is not valid on this class, hence ITL$^\Box \not\vdash \text{CD}(p,q)$. That ITL$^\Box \not\vdash \text{BI}(p,q)$ is obtained by the same reasoning.

6.2 Independence of the Fischer Servi axioms

The Fischer Servi axioms are also not valid in general, as shown in Boudou et al. (see Figure 2).

**Proposition 5.** FS$_\Box(p,q)$ and FS$_\Diamond(p,q)$ are not valid over the class of dynamic posets.
Figure 2: A dynamic poset model falsifying both Fischer Servi axioms. Propositional variables that are true on a point are displayed; only one point satisfies $p$ and no point satisfies $q$. It can readily be checked that $\text{FS}_\circ(p, q)$ and $\text{FS}_\diamond(p, q)$ fail on the highlighted point on the left. Note that $S$ is continuous but not open, as can easily be seen by comparing to Figure 1.

From this and the soundness of $\text{ITL}^\text{FS}$ (Theorem 3), we immediately obtain that they are not derivable in $\text{ITL}^0$.

**Corollary 3.** $\text{ITL}^\text{CD} \nvdash \text{FS}_\circ(p, q)$ and $\text{ITL}^\text{CD} \nvdash \text{FS}_\diamond(p, q)$.

The above independence results are sufficient to see that our four logics are distinct.

**Theorem 4.** The logics $\text{ITL}^0$, $\text{ITL}^\text{FS}$, $\text{ITL}^\text{CD}$ and $\text{ITL}^1$ are pairwise distinct, as are $\text{ITL}^\Box_0$, $\text{ITL}^\Box_0$, $\text{ITL}^\Box_0$, $\text{ITL}^\Box_0$ and $\text{ITL}^\Box_1$.

**Proof.** By Corollary 3 and the definition of $\text{ITL}^\text{CD}$, $\text{CD}(p, q) \in \text{ITL}^\text{CD} \setminus \text{ITL}^\text{FS}$; similarly, by Corollary 3 $\text{FS}_\circ(p, q) \in \text{ITL}^\text{FS} \setminus \text{ITL}^\text{CD}$. Thus $\text{ITL}^\text{FS}$ and $\text{ITL}^\text{CD}$ are incomparable, from which we conclude that $\text{ITL}^0$, which is contained in their intersection, is strictly smaller than either of them, while $\text{ITL}^1$, which contains their union, is strictly larger. The argument for the logics over $\mathcal{L}_\Box$ are analogous, except that $\text{CD}$ is replaced with $\text{BI}$.

7 Types and quasimodels

In this section we review non-deterministic quasimodels [15]. Quasimodels will be our fundamental tool for passing from topological to Kripke semantics.

7.1 Two-sided types

Our presentation will differ slightly from that of [15], since it will be convenient for us to use two-sided types, defined as follows.
Definition 4. Let $\Phi^-, \Phi^+ \subseteq \mathcal{L}_0$ be finite sets of formulas. We say that the pair $\Phi = (\Phi^-; \Phi^+)$ is a two-sided type if:

1. $\Phi^- \cap \Phi^+ = \emptyset$,
2. $\bot \notin \Phi^+$,
3. if $\varphi \land \psi \in \Phi^+$, then $\varphi, \psi \in \Phi^+$,
4. if $\varphi \land \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^-$,
5. if $\varphi \lor \psi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\psi \in \Phi^+$,
6. if $\varphi \lor \psi \in \Phi^-$, then $\varphi, \psi \in \Phi^-$,
7. if $\varphi \to \psi \in \Phi^+$, then either $\varphi \in \Phi^-$ or $\psi \in \Phi^+$, and
8. if $\Diamond \varphi \in \Phi^-$ then $\varphi \in \Phi^-$.

The set of finite two-sided types will be denoted $\mathbb{T}$. Whenever $\Xi$ is an expression denoting a two-sided type, we write $\Xi^-$ and $\Xi^+$ to denote its components.

We will consider two partial orders on $\mathbb{T}$. We will write

(a) $\Phi \preceq_T \Psi$ if $\Psi^- \subseteq \Phi^-$ and $\Phi^+ \subseteq \Psi^+$, and
(b) $\Phi \sqsubseteq_T \Psi$ if $\Phi^- = \Psi^-$ and $\Phi^+ \subseteq \Psi^+$.

If $\Phi$ is a two-sided-type and $\Sigma = \Phi^- \cup \Phi^+$, we may say that $\Phi$ is a two-sided $\Sigma$-type. The set of two-sided $\Sigma$-types will be denoted by $\mathbb{T}_{\Sigma}$.

Remark 2. Fernández-Duque [15] uses one-sided $\Sigma$-types, but it is readily checked that a one-sided type $\Phi$ as defined there can be regarded as a two-sided type $\Psi$ by setting $\Psi^+ = \Phi$ and $\Psi^- = \Sigma \setminus \Phi$. Henceforth we will write type instead of two-sided type and explicitly write one-sided type when discussing [15].

7.2 Quasimodels

Quasimodels are similar to models, except that valuations are replaced with a labelling function $\ell$. We first define the more basic notion of labelled frame.

Definition 5. A labelled frame is a triple $\mathcal{F} = (W, \preceq, \ell)$, where $\preceq$ is a partial order on $W$ and $\ell: W \to \mathbb{T}$ is such that

(a) whenever $w \preceq v$ it follows that $\ell(w) \preceq_T \ell(v)$, and
(b) whenever $\varphi \to \psi \in \ell^-(w)$, there is $v \succ w$ such that $\varphi \in \ell^+(v)$ and $\psi \in \ell^-(v)$,

where $(\ell^-(v), \ell^+(v)) \overset{\text{def}}{=} \ell(v)$.

We say that $\mathcal{F}$ satisfies $\varphi \in \mathcal{L}$ if $\varphi \in \ell^+(w)$ for some $w \in W$, and that it falsifies $\varphi$ if $\varphi \in \ell^-(w)$ for some $w \in W$. If $\ell(w) \in \mathbb{T}_{\Sigma}$ for all $w \in W$, we say that $\mathcal{F}$ is a $\Sigma$-labelled frame.
Labelled frames model only the intuitionistic aspect of the logic. For the temporal dimension, let us define a new relation over types.

**Definition 6.** We define a relation $S_T \subseteq T \times T$ by $\Phi S_T \Psi \iff$ for all $\varphi \in L$:

(a) if $\circ \varphi \in \Phi^+$ then $\varphi \in \Psi^+$,
(b) if $\circ \varphi \in \Phi^-$ then $\varphi \in \Psi^-$,
(c) if $\Diamond \varphi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\Diamond \varphi \in \Psi^+$, and
(d) if $\Diamond \varphi \in \Phi^-$, then $\Diamond \varphi \in \Psi^-$. 

Quasimodels are then defined as labelled frames with a suitable binary relation.

**Definition 7.** A quasimodel is a tuple $Q = (W, \preceq, S, \ell)$ where $(W, \preceq, \ell)$ is a labelled frame and $S$ is a binary relation over $W$ that is

- **serial:** for all $w$ there is $v$ such that $w S v$;
- **forward-confluent:** if $w \preceq w'$ and $w S v$, there is $v'$ such that $v \preceq v'$ and $w' S v'$;
- **sensible:** $w S x$ implies $\ell(w) S_T \ell(x)$, and
- **$\omega$-sensible:** whenever $\Diamond \varphi \in \ell^+(w)$, there are $n \geq 0$ and $v$ such that $w S^n v$ and $\varphi \in \ell^+(v)$.

If $(W, \preceq, \ell)$ is a $\Sigma$-labeled frame then $Q$ is a $\Sigma$-quasimodel. If $S$ is a function then $Q$ is deterministic.

The forward confluence condition plays the role of continuity in the non-deterministic setting; indeed, if $S$ is deterministic, then it is easy to see that $S$ is forward-confluent if and only if it is monotone, which as we have discussed, is equivalent to continuity with respect to the up-set topology. In fact, deterministic quasimodels are essentially dynamic posets with a particular valuation, as witnessed by the following version of the ‘truth lemma’:

**Lemma 4.** Let $Q = (W, \preceq, S, \ell)$ be a deterministic quasimodel, and define a valuation $[\ ]^\ell$ on $Q$ by setting $[p]^\ell = \{ w \in W : p \in \ell^+(w) \}$ and extending to all of $L$ recursively. Then, for all formulas $\varphi \in \mathcal{L}_\Diamond$ and for all $w \in W$,

1. if $\varphi \in \ell^+(w)$ then $w \in [\varphi]^\ell$, and
2. if $\varphi \in \ell^-(w)$ then $w \not\in [\varphi]^\ell$.

**Proof.** We proceed by structural induction on $\varphi$. We must consider the following cases.

($\varphi = p$ is an atom) Note that by definition of $[p]^\ell$, if $p \in \ell^+(w)$ then $w \in [p]^\ell$ and if $p \in \ell^-(w)$ then $p \not\in \ell^+(w)$ so $w \not\in [p]^\ell$. 

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Figure 3: If $S$ is forward-confluent, then the above diagram can always be completed.

$(\varphi = \psi \land \theta)$ Assume that $\psi \land \theta \in \ell^+(w)$. By Definition 4 it follows that $\psi \in \ell^+(w)$ and $\theta \in \ell^+(w)$. By induction hypothesis, $w \in [\psi]^\ell$ and $w \in [\theta]^\ell$. Therefore $w \in [\psi \land \theta]^\ell$.

If $\psi \land \theta \in \ell^-(w)$, by definition it follows that either $\psi \in \ell^-(w)$ or $\theta \in \ell^-(w)$. By induction hypotheses we conclude that $w \notin [\psi]^\ell$ or $w \notin [\theta]^\ell$. Therefore $w \notin [\psi \land \theta]^\ell$.

$(\varphi = \psi \lor \theta)$ This case is symmetric, but using the conditions for $\lor$.

$(\varphi = \psi \rightarrow \theta)$ Assume first that $\psi \rightarrow \theta \in \ell^+(w)$. Then for all $y$ such that $w \preceq y$, by condition (a) of Definition 5, $\psi \rightarrow \theta \in \ell^+(y)$. By condition (b) of Definition 4 and by induction hypothesis, $y \notin [\psi]^\ell$ or $y \in [\theta]^\ell$. Therefore, $w \in [\psi \rightarrow \theta]^\ell$.

Now let us assume that $\psi \rightarrow \theta \in \ell^-(w)$. By Definition 5 it follows that there exists $v \in W$ such that $w \preceq v$ and $\psi \in \ell^+(v)$ and $\theta \in \ell^-(v)$. By induction hypothesis it follows that $w \in [\psi]^\ell \setminus [\theta]^\ell$, which means that $w \notin [\psi \rightarrow \theta]^\ell$.

$(\varphi = \circ \psi)$ Assume that $\circ \psi \in \ell^+(w)$. Since $S$ is sensible, $\psi \in \ell^+(S(w))$. By induction hypothesis $S(w) \in [\psi]^\ell$. Therefore $w \in [\circ \psi]^\ell$. The case where $\circ \psi \in \ell^-(w)$ is analogous.

$(\varphi = \Diamond \psi)$ If $\Diamond \psi \in \ell^+(w)$, by the fact that $S$ is $\omega$-sensible there exists $v \in W$ such that $w S^n v$ and $\psi \in \ell^+(v)$; since $S$ is deterministic, we must forcibly have $v = S^0(w)$. By induction hypothesis we conclude that $v \in [\psi]^\ell$ and by the satisfaction relation it follows that $w \in [\Diamond \psi]^\ell$.

In case that $\Diamond \psi \in \ell^-(w)$, observe that for all $n$, if $\Diamond \psi \in \ell^-(S^n(w))$ then $\Diamond \psi \in \ell^-(S^{n+1}(w))$; thus by induction, $\Diamond \psi \in \ell^-(S^n(w))$ for all $n < \omega$. In virtue of Definition 11, $\psi \in \ell^-(S^n(w))$ for all $n < \omega$, hence by the induction hypothesis $S^n(w) \notin [\psi]$, from which it follows that $w \notin [\Diamond \psi]$.

In the non-deterministic case quasimodels are not models as they stand, but in [13], it is shown that dynamical systems can be extracted from them.
Theorem 5. A □-free formula ϕ is satisfiable (falsifiable) over the class of dynamic topological systems if and only if it is satisfiable (falsifiable) over the class of finite, sub(ϕ)-quasimodels.

Theorem 5 applies even to formulas in an extended language L_{DY} with the universal modality, but it is shown in \[13\] that there are topologically falsifiable L_{DY}-formulae that are not Kripke falsifiable. As we will see, this is not the case over L_Ω. Note that \[15\] uses quasimodels with one-sided types, but in view of Remark 2, the theorem can easily be modified to obtain quasimodels with two-sided types.

7.3 Restrictions on types

The reason that two-sided types are convenient is that they can easily be restricted to smaller sets of formulas while maintaining the relations between them. To make this precise, if Σ is a set of formulas, first define Ψ↾Σ = (Ψ−, Ψ+ ∩ Σ), and sub(Σ) = \( \bigcup_{\varphi \in \Sigma} \text{sub}(\varphi) \). With this, we have the following:

Lemma 5. Let \( \Phi, \Psi, \Gamma, \Theta \) be types and Σ a set of formulas closed under subformulas. Then,

1. \( \Phi \vdash \Sigma \) is also a type;
2. if \( \Gamma \sqsubseteq_T \Phi \equiv_T \Psi \) or \( \Gamma \equiv_T \Phi \sqsubseteq_T \Psi \) then \( \Gamma \equiv_T \Psi \), and
3. if \( \Gamma \sqsubseteq_T \Phi \); \( S_T \Psi \) and \( \text{sub}(\Gamma^+) \subseteq \Sigma \), then \( \Gamma S_T \Psi \vdash \Sigma \).

Proof. To prove item 1 it is sufficient to check that the conditions of Definition 4 hold. Conditions 1 and 2 of Definition 4 are straightforward. Since \( \Phi^- = (\Phi \vdash \Sigma)^- \), conditions 1 and 2 clearly hold. For condition 3, suppose that \( \varphi \to \psi \in (\Psi \vdash \Gamma)^+ \). Since Σ is closed under subformulas, \( \varphi, \psi \in \Sigma \) and, since \( \Psi \) is a type it follows that either \( \varphi \in \Psi^- \) or \( \psi \in \Psi^+ \). By definition either \( \varphi \in \Psi^- \) or \( \psi \in \Psi^+ \cap \Sigma \). The proofs for conditions 3 and 6 of Definition 4 are similar and left to the reader.

Regarding item 2 of the lemma, on one side, \( \Gamma \sqsubseteq_T \Phi \equiv_T \Psi \) means that \( \Gamma^+ \subseteq \Phi^+ \subseteq \Psi^+ \) and \( \Gamma^- = \Phi^- \supseteq \Psi^- \). Therefore \( \Gamma^+ \subseteq \Psi^+ \) and \( \Psi^- \subseteq \Gamma^- \) so \( \Gamma \equiv_T \Psi \). On the other side \( \Gamma \equiv_T \Phi \equiv_T \Psi \) means by definition that \( \Gamma^+ \subseteq \Psi^+ \subseteq \Psi^+ \) and \( \Gamma^- \supseteq \Phi^- = \Psi^- \). It follows that \( \Gamma^+ \subseteq \Psi^+ \) and \( \Psi^+ \subseteq \Gamma^- \) so \( \Gamma \equiv_T \Psi \).

For item 3 we consider the conditions of Definition 6.

\[ \text{[a]} \] If \( \circ \psi \in \Gamma^+ \), from \( \Gamma \sqsubseteq_T \Phi \; S_T \Psi \) we conclude that \( \circ \psi \in \Phi^+ \) and \( \psi \in \Psi^+ \). Since \( \text{sub}(\Gamma^+) \subseteq \Sigma \) then \( \psi \in \Sigma \). Therefore \( \psi \in \Psi^+ \cap \Sigma \) so \( \psi \in (\Psi \vdash \Sigma)^+ \).

\[ \text{[b]} \] If \( \diamond \psi \in \Gamma^- \), from \( \Gamma \sqsubseteq_T \Phi \; S_T \Psi \) we conclude that \( \diamond \psi \in \Phi^- \) and \( \psi \in \Psi^- \), which by definition means that \( \psi \in (\Psi \vdash \Sigma)^- \).

\[ \text{[c]} \] If \( \dagger \psi \in \Gamma^+ \), since \( \text{sub}(\Gamma^+) \subseteq \Sigma \) then \( \dagger \psi, \psi \in \Sigma \). From \( \Gamma \sqsubseteq_T \Phi \; S_T \Psi \) we conclude that \( \dagger \psi \in \Phi^+ \) and either \( \psi \in \Gamma^+ \) or \( \dagger \psi \in \Psi^+ \). From this it follows that either \( \psi \in \Gamma^+ \) or \( \dagger \psi \in \Psi^+ \cap \Sigma \) (which means that \( \dagger \psi \in (\Psi \vdash \Sigma)^+ \)).
If $\Diamond \psi \in \Gamma^-$, from $\Gamma \subseteq_T \Phi S_T \Psi$ we conclude that $\Diamond \psi \in \Phi^-$, $\psi \in \Phi^-$ (thus $\psi \in \Gamma^-$) and $\Diamond \psi \in \Psi^-$. As a consequence it follows that $\psi \in \Gamma^-$ and $\Diamond \psi \in (\Psi \mid \Sigma)^-$. □

We may also wish to ‘forget’ temporal formulas that have been realized. To make this precise, let $\sup(\varphi)$ denote the set of super-formulas of $\varphi$. Say that a formula $\varphi$ is a temporal formula if it is of the forms $\circ \psi$ or $\Diamond \psi$, and if $\Phi$ is a set of formulas, say that $\varphi \in \Phi$ is maximal in $\Phi$ if it does not have any temporal superformulas in $\Phi$. Then, define $\Phi \setminus \varphi = (\Phi -, \Phi^+ \setminus \sup(\varphi))$.

Lemma 6. Suppose that $\Phi S_T \Psi$.

1. If $\circ \varphi$ is maximal in $\Phi^+$, then $\Phi S_T \Psi \setminus \circ \varphi$.

2. If $\Diamond \varphi$ is maximal in $\Phi^+$ and $\varphi \in \Phi^+$, then $\Phi S_T \Psi \setminus \Diamond \varphi$.

Proof. We prove the first item; the second is analogous. Assuming that $\circ \varphi$ is maximal in $\Phi^+$, let us check that the four conditions of Definition 6 hold.

(a) If $\circ \theta \in \Phi^+$, since $\Phi S_T \Psi$ then $\theta \in \Psi^+$. Moreover, since $\circ \theta \in \Phi^+$ then $\theta \not\in \sup(\circ \varphi)$ by maximality of $\circ \varphi$. Therefore, $\theta \in (\Psi^+ \setminus \sup(\circ \varphi)) = (\Psi \setminus \circ \varphi)^+$.

(b) If $\circ \theta \in \Phi^-$, since $\Phi S_T \Psi$ then $\theta \in \Psi^- = (\Psi \setminus \circ \varphi)^-$.

(c) If $\Diamond \theta \in \Phi^+$, since $\Phi S_T \Psi$ then either $\theta \in \Phi^+$ or $\Diamond \theta \in \Psi^+$. Moreover, since $\circ \varphi$ is maximal in $\Phi^+$ and $\Diamond \theta \in \Phi^+$, it follows that $\Diamond \theta \not\in \sup(\circ \varphi)$. Therefore, $\Diamond \theta \in (\Phi^+ \setminus \sup(\circ \varphi)) = (\Phi \setminus \circ \varphi)^+$.

(d) If $\Diamond \theta \in \Phi^-$, since $\Phi S_T \Psi$ is sensible then $\Diamond \theta \in \Psi^- = (\Psi \setminus \circ \varphi)^-$. □

In the next section, we will use Theorem 5 and our results on two-sided types to show that, for $\Box$-free formulas, validity over the class of topological spaces can be reduced to validity over the class of dynamic posets.

8 Conservativity of the $\Box$-free fragment

Our goal for this section is to show that the temporal logics of dynamic posets and of dynamical systems coincide with respect to $\Box$-free formulas:

Theorem 6. A $\Box$-free formula $\varphi$ is satisfiable (falsifiable) over the class of dynamic posets if and only if it is satisfiable (falsifiable) over the class of dynamical systems.

We will show this by ‘unwinding’ a quasimodel to produce a dynamic poset.
8.1 Weak limit models

The unwinding procedure is similar to that in \[15\]. There, the points of the ‘limit model’ obtained from a quasimodel are those infinite paths satisfying all \(\Diamond\)-formulas in their labels. However, in order to obtain a space rather than a topological space, we will instead work with finite paths.

**Definition 8.** If \(Q = (W, \preceq, S, \ell)\) is a quasimodel, we say that a path (on \(Q\)) is a sequence \((w_i)_{i<n}\) \(\subseteq W\) such that \(w_i S w_{i+1}\) for all \(i < n - 1\). We define a typed path (on \(Q\)) to be a sequence \(((w_i, \Phi_i))_{i<n}\) such that

(a) \((w_i)_{i<n}\) is a path,

(b) for all \(i < n\), \(\Phi_i \sqsubseteq T \ell(w_i)\), and

(c) for all \(i < n - 1\), \(\Phi_i S_T \Phi_{i+1}\).

We say that \(((w_i, \Phi_i))_{i<n}\) is proper if \(\text{sub}(\Phi_i + 1) \subseteq \text{sub}(\Phi_{i+1})\) for all \(i < n - 1\), and terminal if \(\Phi_{n-1} = \emptyset\).

Note that we allow \(\Phi_i \sqsubseteq T \ell(w_i)\) and not only \(\Phi_i = \ell(w_i)\). This will allow us to use finite paths, as temporal formulas can be ‘forgotten’ once they have been realized.

**Definition 9.** We define the weak limit model \(\widehat{Q}\) of \(Q\) as follows:

1. Define \(\widehat{W}\) to be the set of terminal typed paths on \(Q\) together with the empty path, which we denote \(\epsilon\).

2. For \(\alpha = ((w_i, \Phi_i))_{i<n}\), \(\beta = ((v_i, \Psi_i))_{i<m}\) \(\in \widehat{W}\), define \(\alpha \preceq \beta\) if \(n \leq m\) and for all \(i < n\), \(w_i \preceq v_i\) and \(\Phi_i \preceq_T \Psi_i\).

3. Define \(\widehat{S}((w_i, \Phi_i))_{i<n}) = ((w_{i+1}, \Phi_{i+1}))_{i<n-1}\); note that \(\widehat{S}(\epsilon) = \epsilon\).

4. If \(n > 0\), define \(\widehat{\ell}(((w_i, \Phi_i))_{i<n}) = \Phi_0\). Then, set \(\widehat{\ell}^-(\epsilon) = \bigcup_{w \in W} \ell^-(w)\) and \(\widehat{\ell}^+(\epsilon) = \emptyset\).

The structure \(\widehat{Q}\) we have just defined is always a deterministic quasimodel. Let us first show that it is deterministic.

**Lemma 7.** If \(Q = (W, \preceq, S, \ell)\) is a quasimodel then \(\left(\widehat{W}, \preceq, \widehat{S}\right)\) is a dynamic poset.

**Proof.** We have to prove the following:

\(\preceq\) is a partial order on \(\widehat{W}\): This follows easily from the fact that \(\preceq\) and \(\preceq_T\) are both partial orders.

\(\widehat{S}\) is a function: This is clear since \(\widehat{S}(\alpha)\) is defined by removing the first element of \(\alpha\) if it exists, otherwise \(\widehat{S}(\alpha) = \alpha\), and thus \(\widehat{S}(\alpha)\) is uniquely defined for all \(\alpha \in \widehat{W}\).
\( \Sigma \) is monotone: If \( ((w_i, \Phi_i))_{i<n} \preceq ((v_i, \Psi_i))_{i<m} \), then \( n \leq m \) and for all \( i < n \), \( w_i \preceq v_i \) and \( \Phi_i \preceq_T \Psi_i \). If \( n > 0 \), then we also have \( n - 1 \leq m - 1 \) and for all \( i < n - 1 \), \( w_{i+1} \preceq v_{i+1} \) and \( \Phi_{i+1} \preceq_T \Psi_{i+1} \), i.e.,

\[
\hat{S}(\alpha) = ((w_i, \Phi_i))_{i<n} \preceq ((v_i, \Psi_i))_{i<n-1} \preceq ((v_{i+1}, \Psi_{i+1}))_{i<m-1} = \hat{S}(\beta),
\]
as needed. If \( n = 0 \) then \( \alpha = \epsilon \), so that \( \hat{S}(\alpha) = \epsilon \) and clearly \( \epsilon \preceq \beta \).  

\[8.2 \] Constructing terminal paths

Next, we must show that \( \hat{Q} \) has ‘enough’ paths. First we show that we can iterate the forward-confluence property.

**Lemma 8.** If \( Q \) is a quasimodel, \( ((w_i, \Phi_i))_{i<n} \) is a typed path in \( Q \), and \( w_0 \preceq v_0 \), then there is a typed path \( ((v_i, \Psi_i))_{i<n} \) such that \( w_i \preceq v_i \) and \( \Phi_i \preceq_T \Psi_i \) for all \( i < n \).

**Proof.** First we find \( v_i \) by induction on \( i \); \( v_0 \) is already given, and once we have found \( v_i \), we use forward confluence to choose \( v_{i+1} \) so that \( v_i \preceq v_{i+1} \) and \( w_{i+1} \preceq v_{i+1} \). Then we set \( \Psi_i = \ell(v_i) \); since \( S \) is sensible, \( \Psi_i \preceq_T \Psi_{i+1} \), and by Lemma 8, \( \Phi_{i+1} \preceq_T \Psi_{i+1} \).

We want to prove that any point can be included in a terminal typed path. For this we will first show that we can work mostly with properly typed paths, thanks to the following.

**Lemma 9.** Let \( Q = (W, \preceq, S, \ell) \) be a quasimodel, \( (w_i)_{i<n} \) be a path on \( W \), and \( \Phi_0 \preceq \ell(w_0) \). Then there exist \( (\Phi_i)_{i<n} \) such that \( ((w_i, \Phi_i))_{i<n} \) is a properly typed path.

**Proof.** For \( i < n - 1 \) define recursively \( \Phi_{i+1} = \ell(w_{i+1}) \upharpoonright \text{sub}(\Phi_i) \); by the assumption that \( S \) is sensible and Lemma 8, \( (\Phi_i, \Phi_{i+1}) \) is sensible for each \( i < n - 1 \). It is easy to see that \( ((w_i, \Phi_i))_{i<n} \) thus defined is proper.

However, the properly typed paths we have constructed need not be terminal. This will typically require extending them to a long-enough path, as we do below.

**Lemma 10.** If \( Q \) is a quasimodel, then any non-empty typed path on \( Q \) can be extended to a terminal path.

**Proof.** Let \( Q = (W, \preceq, S, \ell) \) and \( \alpha = ((w_i, \Phi_i))_{i<m} \) be any typed path on \( Q \). For a type \( \Phi \), define \( \|\Phi\| = |\text{sub}(\Phi^+)\| \). We proceed to prove the claim by induction on \( \|\Phi_m\| \). Consider first the case where \( \Phi_m \) contains no temporal formulas; that is, formulas of the form \( \omega \psi \) or \( \diamond \psi \) for some \( \psi \). In this case, using the seriality of \( S \) choose \( w_m \) such that \( w_{m-1} \preceq w_m \), and define \( \Phi_{m+1} = (\ell^-(w_m); \emptyset) \); it is easy to see that \( ((w_i, \Phi_i))_{i<m} \) is a terminal path.

Otherwise, let \( \varphi \) be a maximal temporal formula of \( \Phi_{m-1}^+ \); i.e., it does not appear as a proper subformula of any other temporal formula in \( \Phi_{m-1}^+ \). We consider two sub-cases.
Assume first that \( \varphi = \circ \psi \). Then, by the seriality of \( S \), we may choose \( w_m \) so that \( w_{m-1} S w_m \). Applying Lemma 9 let \( \Phi_m \) be such that \( ((w_{m-1}, \Phi_{m-1}), (w_m, \Phi_m)) \) is a properly typed path. Setting \( \Phi_m = \Phi_m \setminus \circ \psi \), we see by Lemma 6 that

\[
((w_{m-1}, \Phi_{m-1}), (w_m, \Phi_m))
\]

is a properly typed path, and \( \| \Phi_m \| < \| \Phi_{m-1} \| \), since the left-hand side does not count \( \circ \psi \). Thus we may apply the induction hypothesis to obtain a terminal typed path \( ((w_i, \Phi_i))_{i < n} \) extending \( \alpha \).

Now consider the case where \( \varphi = \Diamond \psi \). Since \( S \) is \( \omega \)-sensible, there is a path

\[
w_{m-1} S w_m S \ldots S w_k
\]

so that \( \varphi \in \ell(w_k) \). Using the seriality of \( S \), choose \( w_{k+1} \) so that \( w_k S w_{k+1} \).

By Lemma 9 there are types \( \Phi_i \) for \( m \leq i \leq k \) and a type \( \hat{\Phi}_{k+1} \) such that

\[
((w_{m-1}, \Phi_{m-1}), \ldots, (w_k, \Phi_k), (w_{k+1}, \hat{\Phi}_{k+1}))
\]

is a properly typed path. Then, define \( \Phi_{k+1} = \hat{\Phi}_{k+1} \setminus \Diamond \psi \). Using Lemma 6 we see that \( (\Phi_k, \Phi_{k+1}) \) is sensible; moreover, \( \| \Phi_{k+1} \| < \| \Phi_{m-1} \| \). Hence we can apply the induction hypothesis to obtain a terminal typed path \( ((w_i, \Phi_i))_{i < n} \) extending \( \alpha \).

\[
\square
\]

8.3 From weak limit models to models

With this, we are ready to show that our unwinding is indeed a deterministic quasimodel.

**Lemma 11.** If \( Q \) is a quasimodel, then \( \hat{Q} \) is a deterministic quasimodel.

**Proof.** Let \( Q = (W, \zeta, S, \ell) \). We have already seen in Lemma 7 that \( (\hat{W}, \hat{\zeta}, \hat{S}) \) is a dynamic poset, so it remains to check that \( (\hat{W}, \hat{\zeta}, \hat{\ell}) \) is a labelled frame and \( \hat{S} \) is sensible and \( \omega \)-sensible. Let \( \alpha = ((w_i, \Phi_i))_{i < n} \in \hat{W} \).

First we must check that if \( \alpha \preceq \beta \), then \( \hat{\ell}(\alpha) \preceq_{\hat{\ell}} \hat{\ell}(\beta) \). Consider two cases; if \( n > 0 \), then \( \beta \) is also of the form \( (v_i, \Psi_i)_{i < m} \) with \( m > 0 \) and by definition, \( \hat{\ell}(\alpha) = \Phi_0 \preceq_{\hat{\ell}} \Psi_0 = \hat{\ell}(\beta) \). Otherwise, \( \alpha = \epsilon \), and it is clear from the definition of \( \hat{\ell}(\epsilon) \) that \( \hat{\ell}(\epsilon) \preceq_{\hat{\ell}} \hat{\ell}(\beta) \) regardless of \( \beta \).

Now assume that \( \varphi \rightarrow \psi \in \hat{\ell}^{-}(\alpha) \). If \( n > 0 \), then since \( Q \) is a labeled frame, we can pick \( v_0 \succ v_0 \) with \( \varphi \in \ell^{+}(v_0) \) and \( \psi \in \ell^{-}(v_0) \). Since \( \Phi_0 \sqsubseteq_{\ell} \ell(v_0) \), by Lemma 5, \( \Phi_0 \preceq_{\hat{\ell}} \ell(v_0) \), so that by Lemma 8 there is a typed path \( \beta' = ((v_i, \Psi_i))_{i < n} \) with \( \Psi_0 = \ell(v_0) \) such that \( w_i \preceq v_i \) and \( \Phi_i \preceq_{\Psi_i} \Psi_i \) for all \( i < n \). By Lemma 10 we can extend \( \beta' \) to a terminal path \( \beta \). Then, it is easy to see that \( \alpha \preceq \beta \), \( \varphi \in \hat{\ell}^{-}(\beta) \), and \( \psi \in \hat{\ell}^{-}(\beta) \), as required.

To check that every pair in \( \hat{S} \) is sensible, consider two cases. If \( \hat{S}(\alpha) \neq \epsilon \), then \( \alpha \) has length at least two, but since \( \alpha \) is a typed path,

\[
\hat{\ell}(\alpha) = \Phi_0 \circ S_{\ell} \Phi_1 = \hat{\ell}(\hat{S}(\alpha)).
\]

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Otherwise, \(\hat{S}(\alpha) = \epsilon\); this means that either \(\alpha = \epsilon\) and thus \(\hat{\ell}^+(\alpha) = \emptyset\), or \(\alpha\) has length 1, in which case since \(\alpha\) is terminal, so we also have that \(\hat{\ell}^+(\alpha) = \emptyset\). In either case, there can be no temporal formula in \(\hat{\ell}^+(\alpha)\). Now, if \(\diamond\varphi \in \hat{\ell}^-(\alpha)\), then \(\diamond\varphi \in \ell^-(w)\) for some \(w \in W\), hence \(\varphi \in \ell^-S(v)\) for any \(v \) with \(w S v\) (which exists since \(S\) is serial), and thus \(\varphi \in \hat{\ell}^-(\epsilon)\). Similarly, if \(\Box\varphi \in \hat{\ell}^-(\alpha)\), then by Definition 4.8 \(\varphi \in \ell^-\alpha\), so that \(\varphi \in \hat{\ell}^-(\epsilon)\).

Finally we check that \(\hat{S}\) is \(\omega\)-sensible. Suppose that \(\Box \varphi \in \ell^+(\alpha)\). This means that \(\alpha \neq \epsilon\), so \(\alpha\) is terminal, and hence \(n > 0\) and \(\Box \alpha \not\in \Phi_{n-1}\). But this is only possible if \(\varphi \in \Phi_i\) for some \(i < n - 1\), in which case \(\varphi \in \ell(S^i(\alpha))\).

Let us put all of our work together to prove our main result.

**Proof of Theorem 6.** Suppose that \(\varphi \in L_\Diamond\) is satisfied (falsified) on a dynamical topological model. Then, by Theorem 3 \(\varphi\) is satisfied (falsified) on some point \(w_*\) of a sub(\(\varphi\))-quasimodel \(\hat{Q} = (W, \preceq, S, \ell)\). By Lemma 11 \(\hat{Q}\) is a deterministic quasimodel, and by Lemma 10 \((w_*, \ell(w_*))\) can be extended to a terminal path \(\alpha_* \in \hat{W}\). By Lemma 4 \(\alpha_*\) satisfies (falsifies) \(\varphi\) on the dynamic poset model \((\hat{W}, \hat{\preceq}, \hat{S}, \llbracket \cdot \rrbracket_{\hat{\ell}})\).

9 Concluding Remarks

We have proposed a natural ‘minimalist’ intuitionistic temporal logic, \(\text{ITL}^0\), along with possible extensions including Fischer Servi or constant domain axioms. We have seen that relational semantics validate the constant domain axiom, leading us to consider a wider class of models based on topological spaces, with a novel interpretation for ‘henceforth’ based on invariant neighbourhoods. With this, we have shown that the logics \(\text{ITL}^0, \text{ITL}_{\Box}^0, \text{ITL}_{\Diamond}^0\) and \(\text{ITL}^1\) are sound for the class of all dynamical systems, of all dynamical posets, of all open dynamical systems, and of all persistent dynamical posets, respectively, which we have used in order to prove that the logics are pairwise distinct.

Of course this immediately raises the question of completeness, which we have not addressed. Specifically, the following are left open.

**Question 1.** Are \(\text{ITL}^0\) and \(\text{ITL}_{\Box}^0\) complete for the class of dynamical systems?

**Question 2.** Are \(\text{ITL}_{\Diamond}^0\), \(\text{ITL}_{\Box}^0\) and \(\text{ITL}_{\Diamond}^{\text{CD}}\) complete for the class of dynamic posets?

**Question 3.** Are \(\text{ITL}_{\Diamond}^{\text{FS}}, \text{ITL}_{\Box}^{\text{FS}}\) and \(\text{ITL}_{\Diamond}^{\text{FS}}\) complete for the class of open dynamical systems?

**Question 4.** Are \(\text{ITL}^1\), \(\text{ITL}_{\Diamond}^1\) and \(\text{ITL}_{\Box}^1\) complete for the class of persistent dynamic posets?

Note that by Theorem 4 \(\text{ITL}_{\Diamond}^0\) is complete for the class of dynamical systems if and only if it is complete for the class of dynamic posets, so thanks to the results we have shown here, proving topological completeness would give us...
Kripke completeness for free. It is likely that the techniques employed in \cite{10}, also based on non-deterministic quasimodels, could be adapted to the intuitionistic setting to obtain such a result. The completeness of $\text{ITL}^{FS}$ and $\text{ITL}^1$ is likely to be a more difficult problem, as in these cases it is not even known if the set of valid formulas is computably enumerable.

**Question 5.** Are the sets of formulas of $\mathcal{L}$, $\mathcal{L}_\Diamond$, or $\mathcal{L}_\Box$ valid over the class of all persistent dynamic posets computably enumerable?

**Question 6.** Are the sets of formulas of $\mathcal{L}$, $\mathcal{L}_\Diamond$, or $\mathcal{L}_\Box$ valid over the class of all open dynamical systems computably enumerable?

In both cases a negative answer is possible, since that is the case for their classical counterparts \cite{25}. Nevertheless, the proofs of non-axiomatizability in the classical case do not carry over to the intuitionistic setting in an obvious way, and these remain challenging open problems.

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