DUAL PAIRS OF OPERATORS, HARMONIC ANALYSIS OF SINGULAR NON-ATOMIC MEASURES AND KREIN-FELLER DIFFUSION

PALLE E.T. JORGENSEN AND JAMES TIAN

Abstract. We show that a Krein-Feller operator is naturally associated to a fixed measure \( \mu \), assumed positive, \( \sigma \)-finite, and non-atomic. Dual pairs of operators are introduced, carried by the two Hilbert spaces, \( L^2(\mu) \) and \( L^2(\lambda) \), where \( \lambda \) denotes Lebesgue measure. An associated operator pair consists of two specific densely defined (unbounded) operators, each one contained in the adjoint of the other. This then yields a rigorous analysis of the corresponding \( \mu \)-Krein-Feller operator as a closable quadratic form. As an application, for a given measure \( \mu \), including the case of fractal measures, we compute the associated diffusion, semigroup, Dirichlet forms, and \( \mu \)-generalized heat equation.

1. Introduction

Recently there have been several advances to an harmonic analysis of Krein-Feller operators for classes of singular measures. (Intuitively, a Krein-Feller operator is an analogue of a Laplacian for classical domains, as they arise in diffusion problems and in potential theory.)

In fact there are recent papers which cover the theory from the point of fractal analysis, see e.g., [Fre08, Fre03, Iy89, LOSS20, ARCG+20, QS13]; as well as applications to physics and to signal processing, e.g., the papers [AN06, Fle96, Zag87, Iy85, Kas12, Wat98, Fuj87]. Our present approach to Krein-Feller operators is motivated by both of these new trends; but our approach is based on a new duality. It combines a new transformation theory based on the theory of reproducing kernel Hilbert spaces (RKHSs), and a new technology introduced here, based on pairs of unbounded densely defined operators, each one contained in the adjoint of the other.

A word about the terminology “Krein-Feller operator” (details are cited inside the paper): Mark Krein has pioneered a number of powerful Hilbert space-based tools which have found numerous applications, and the present problem is a case in point. Krein’s operator theory (cited below) forms the foundation in our approach to problems for unbounded operators with dense domain in Hilbert space. Sections 2 and 4 below will elaborate on this. William Feller, in the name “Krein-Feller operator” refers to the role of the KF-operator in the study of diffusion. Indeed, W. Feller was one of the pioneers in our understanding of diffusion, diffusion-semigroups, and their analysis. Hence later authors have adopted the name “Krein-Feller operators” for the associated semigroup generators. There are interesting connections to inverse problems, and prediction theory, see [DM76]. A nice presentation of this, and early work of Krein and Feller, is [DM76, chapter 5]. We shall include additional details on this point in Section 3 below.
Starting with a fixed positive non-atomic Borel measure $\mu$ (with support contained in $\mathbb{R}$), then, informally, the associated Krein-Feller operator (denoted $K_F = K_F(\mu)$) is $K_F = \frac{d}{d\mu} \frac{d}{dx}$. The meaning of “$d/d\mu$” will be made precise. If $x > 0$, set $g_\mu(x) = \mu([0,x])$, i.e., the cumulative distribution. For $\psi \in C^1$, we have $\frac{d}{d\mu}(\psi \circ g_\mu) = \psi' \circ g_\mu$. A key step in our consideration is a rigorous study of $K_F$ as an unbounded (symmetric) operator in $L^2(\mu)$.

**Organization:** We begin in Section 2 with the framework for our dual pair analysis. This is presented in the rather general setting of pairs of Hilbert spaces, and associated pairs of densely defined (unbounded) operators. Particular choices of dual pairs of operators are then applied to a rigorous analysis of Krein-Feller operators in Section 3. The framework for these considerations is a fixed measure $\mu$, assumed positive, sigma-finite, and non-atomic. Hence, our starting point is a specified and fixed measure $\mu$ (generally singular, e.g., a Cantor measure). The two Hilbert spaces for the corresponding dual pair of unbounded operators will then be $L^2(\mu)$ and $L^2(\lambda)$, where $\lambda$ denotes Lebesgue measure, or its restriction to a chosen interval. The rest of Section 3 will deal with an analysis of the associated diffusion, Markov process, semigroup, and a corresponding $\mu$-generalized heat equation. Section 6 studies Stieltjes measures $df$ globally. For this purpose, we introduce a Hilbert space $\mathcal{H}_{\text{class}}$ of “sigma functions” as a Hilbert space of certain equivalence classes. Starting with a fixed Stieltjes measure $df$, we then identify its pairwise mutually singular components with corresponding orthogonal “pieces” in the Hilbert space $\mathcal{H}_{\text{class}}$.

In a general framework, these settings correspond to suitably specified Dirichlet forms; the subject of Section 4. An application to iterated function system (IFS) measures is also included in Section 4. Our analysis of specific applications relies on several new tools; one in particular derives from consideration of associated reproducing kernel Hilbert spaces (RKHSs), and Gaussian fields. This is covered in Section 5.

### 2. Dual pairs of operators in Hilbert space

The notion of dual pairs we shall need here is in Definition 2.6 below. But the operators in question will act between two Hilbert spaces to be specified. Hence, we shall first need to recall some properties of unbounded operators with specified dense domains; especially the precise definition (Def 2.1) of the adjoints of such operators. With this accomplished, the dual pair definition (Def 2.6) for a pair of densely defined operators amounts to the assertion that each operator in the pair be contained in the dual of the other (Lemma 2.7). We shall need this in our analysis of classes of Krein-Feller operators introduced in Section 3 below. For a given measure $\mu$ and associated Krein-Feller operator $K_F$, we shall then identify a dual pair which provides a factorization of this Krein-Feller operator $K_F$. Of course, $K_F$ is an unbounded operator, symmetric and semibounded; so our dual pair factorization will present us with a canonical selfadjoint extension, see Lemma 2.7. Background references for this include [CP68, DM72, DS88, Fel54, Kat95, Sch12].

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces. If $\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2$ represents a linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$, we shall denote

$$\text{dom} (T) = \{ \varphi \in \mathcal{H}_1 \mid T\varphi \text{ is well-defined} \},$$

the domain of $T$, and

$$\text{ran} (T) = \{ T\varphi \mid \varphi \in \text{dom} (T) \},$$

the range of $T$. The closure of $\text{ran} (T)$ will be denoted $\overline{\text{ran} (T)}$. 

---

*Palle E.T. Jorgensen and James Tian*
Definition 2.1. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a densely defined operator, and let
\[
\text{dom}(T^*) = \left\{ h_2 \in \mathcal{H}_2 \mid \exists C = C_{h_2} < \infty, \text{ s.t. } |\langle h_2, T \varphi \rangle_2| \leq C \| \varphi \|_1 \right\}
\]
holds for $\forall \varphi \in \text{dom}(T)$.

By Riesz’ theorem, there is a unique $\eta \in \mathcal{H}_1$ for which
\[
\langle \eta, \varphi \rangle_1 = \langle h_2, T \varphi \rangle_2, \quad h_2 \in \text{dom}(T^*), \quad \varphi \in \text{dom}(T),
\]
and the adjoint operator is defined as $T^* h_2 = \eta$. See the diagram below:

![Diagram](image)

Definition 2.2. The graph of $T : \mathcal{H}_1 \to \mathcal{H}_2$ is
\[
G_T := \left\{ \begin{bmatrix} \varphi \\ T \varphi \end{bmatrix} \mid \varphi \in \text{dom}(T) \right\} \subset \mathcal{H}_1 \oplus \mathcal{H}_2,
\]
where $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a Hilbert space under the natural inner product
\[
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} , \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \langle \varphi_1, \psi_1 \rangle_{\mathcal{H}_1} + \langle \varphi_2, \psi_2 \rangle_{\mathcal{H}_2}.
\]

Definition 2.3. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator.

1. $T$ is closed if $G_T$ is closed in $\mathcal{H}_1 \oplus \mathcal{H}_2$.
2. $T$ is closable if $G_T$ is the graph of an operator.
3. If (2) holds, the operator corresponding to $G_T$, denoted $\overline{T}$, is called the closure, i.e.,
\[
\overline{G_T} = G_T.
\]

We shall need the following two results for unbounded operators, see e.g., [DS88, Sch12, Rud91]. To clarify notation, and for the benefit of the reader, we have included them below in the form they are needed.

Theorem 2.4. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a densely defined operator. Then

1. $T^*$ is closed;
2. $T$ is closable $\iff$ dom $(T^*)$ is dense;
3. $T$ is closable $\implies$ $(T^*)^* = T^*$.

Theorem 2.5 (von Neumann, polar decomposition/factorization, [DS88]). Let $\mathcal{H}_i, i = 1, 2$, be two Hilbert spaces, and let $T$ be a closed operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ having dense domain in $\mathcal{H}_1$; then $T^* T$ is selfadjoint in $\mathcal{H}_1$, $T T^*$ is selfadjoint in $\mathcal{H}_2$, both with dense domains.

Moreover, there is a partial isometry $J : \mathcal{H}_1 \to \mathcal{H}_2$ such that
\[
T = J (T^* T)^{\frac{1}{2}} = (T T^*)^{\frac{1}{2}} J
\]
holds on dom $(T)$. (Equation (2.8) is called the polar decomposition of $T$.)

Definition 2.6 (symmetric pair). For $i = 1, 2$, let $\mathcal{H}_i$ be two Hilbert spaces, and suppose $\mathcal{D}_i \subset \mathcal{H}_i$ are given dense subspaces.
We say that a pair of operators \((S, T)\) forms a symmetric pair if \(\text{dom} \ (T) = \mathcal{D}_1\) and \(\text{dom} \ (S) = \mathcal{D}_2\); and moreover,

\[
\langle Tu, v \rangle_{\mathcal{H}_2} = \langle u, Sv \rangle_{\mathcal{H}_1}
\]

holds for \(\forall u \in \mathcal{D}_1, \forall v \in \mathcal{D}_2\). See the diagram below:

\[
\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xleftarrow{S}
\]

**Lemma 2.7** (Dual Pair [JP16]). Let \((S, T)\) be the pair of operators specified in (2.9). Then, we have

\[
T \subset S^*, \quad S \subset T^*
\]

(*containment of graphs.*) Moreover, the two operators \(S^*S\) and \(T^*T\) are selfadjoint.

It is immediate from (2.10) that both \(S\) and \(T\) are closable.

**Definition 2.8.** We say that a symmetric pair is maximal if

\[
\mathcal{T} = S^*, \quad \overline{S} = T^*.
\]

With the starting point, a fixed positive non-atomic Borel measure \(\mu\) with support on an interval, we now show how the operator theoretic framework of dual pairs (Definition 2.6) offers an explicit setting for the study of spectral theory of the associated Krein-Feller operator. In particular, we show in the subsequent sections how key features of our dual pair framework from the discussion above serves to yield explicit new results for the Krein-Feller operator, for example Theorem 3.1, Lemma 3.24, Corollary 3.38.

3. **Krein-Feller operators, and their properties**

For a given measure \(\mu\) we shall offer several tools in our analysis of the associated Krein-Feller operator \(K_F\). One will make use of an appropriate dual pair of operators (see Sec 2, and Theorem 3.1 and Corollary 3.38 below). The other is more direct; it is sketched in the present section. In Theorem 3.22, we present the inverse of \(K_F\) as an explicit integral operator. This will be especially useful in our analysis of the spectrum of diverse selfadjoint extensions of \(K_F\). Background references for this include [AJ12, AJL11, AL08, Hid80, IM74]. For basics on Stieltjes measures, fractals, and transformation rules for measures, readers may wish to consult [BP17, DM72, Hut81, Kol83, Nel67, Roh52, Rud91, SZ09, DJ14].

Perhaps it is appropriate to add a comment on the role of W. Feller, in the name “Krein-Feller operator.” Feller was one of the pioneers in the study of diffusion, diffusion-semigroups, and their analysis. Hence later authors have adopted the name “Krein-Feller operators” for the associated semigroup generators. We shall elaborate this point in the next section. A list of references which covers this viewpoint is long, but it includes the following, [Fel54, FM56, Yos68].

**Terminology convention.** Fixing a measure \(\mu\) as specified, then formally, the notation \(\nabla_\mu\) (see (3.8)) and \(T_\mu\) stand for the same operation, but in the theorem below, we are referring to a specific pair of Hilbert spaces, and the notation \(T_\mu\) is used to stress this point. The conclusion of Theorem 3.1 is that the Krein-Feller operator \(K_F\) then has a symmetric dual-pair realization in the sense of Definition 2.6.

If \(f\) is a function on \(\mathbb{R}\) (or defined on a subinterval), assumed to be locally of bounded variation, then we shall denote by \(df\) the corresponding Stieltjes measure. (Recall \(df\) is defined
first on intervals \((x, y]\) by \(df ((x, y]) := f (y) − f (x)\), and then extended to the Borel \(\sigma\)-algebra \(\mathcal{B}\) by the usual \(\sigma\)-algebra-completion procedure.) If \(\mu\) is a fixed positive Borel measure, we then consider the corresponding Radon-Nikodym derivative, denoted
\[
f^{(\mu)} = \nabla^{(\mu)} f = df/\mu. \tag{3.1}
\]

It is determined by,
\[
f (y) − f (x) = \int_x^y \left( \nabla^{(\mu)} f \right) d\mu; \tag{3.2}
\]
abbreviated \((\nabla^{(\mu)} f) \, d\mu = df\).

The the Krein-Feller operator \(K_F\) is defined as
\[
K_F = \frac{d}{d\mu} \frac{d}{dx} = \nabla_{\mu} \frac{d}{dx}. \tag{3.3}
\]

In what follows, we denote by \(J\) the unit interval \([0, 1]\).

**Theorem 3.1** (A symmetric pair for \(\nabla_{\mu}\)). If \(\varphi \in C^\infty_c (J)\) then
\[
- \int \varphi' f \, dx = \int_J \varphi (T_{\mu} f) \, d\mu; \tag{3.4}
\]
so we obtain the dual pair of operators:
\[
\begin{aligned}
\mathcal{L}^2 (\mu) &\xrightarrow{T_{\mu}} L^2 (\mu) \\
\mathcal{L}^2 (\mu) &\xleftarrow{D = - \frac{d}{dx}} L^2 (\mu)
\end{aligned}
\]

Here,
\[
\mathcal{L}^2 (\mu) := \text{dom} (T_{\mu})^{L^2 (\lambda)}, \tag{3.5}
\]
i.e., the \(L^2 (\lambda)\)-closure of \(\text{dom} (T_{\mu})\); see (3.7).

**Proof.** One checks that
\[
\int \varphi \, df = - \int \varphi' f \, dx \tag{3.6}
\]
holds for all \(\varphi \in C^\infty_c (J)\), using integration by parts.

Details: Let \(f\) and \(\varphi\) be as specified, \(\varphi \in C^1_c (J)\), \(f\) locally bounded variation s.t. \(f^{(\mu)} = T_{\mu} f \in L^2_{\text{loc}} (\mu)\) is well defined. For the integral \(\int_J \varphi (T_{\mu} f) \, d\mu\), we therefore get the following approximation via choices of partitions in the interval \(J\): \(x_0 < x_1 < \cdots\):
\[
\int_J \varphi (T_{\mu} f) \, d\mu \simeq \sum_i \int_{x_i}^{x_{i+1}} \varphi (T_{\mu} f) \, d\mu
\]
\[
\simeq \sum_i \varphi (x_i) \int_{x_i}^{x_{i+1}} f^{(\mu)} \, d\mu
\]
\[
= (3.2) \sum_i \varphi (x_i) \frac{df ([x_i, x_{i+1}])}{\text{the Stieltjes measure} \, df}
\]
\[
\simeq \int \varphi \, df = - \int_J \varphi' \, f (x) \, dx.
\]
\(\square\)
3.1. Realization of $T_\mu$ as a skew-symmetric operator with dense domain in $L^2(\mu)$.

Fix a non-atomic measure $\mu$ on $[0,1]$. Let

$$\mathcal{D}_1 := \left\{ f : f(x) = f(0) + \int_0^x f^{(\mu)}(x) \, d\mu, \ f^{(\mu)} \in L^2(\mu), \text{ for all } x \right\}.$$  \hfill (3.7)

Then $\mathcal{D}_1 \subset L^2(\mu) \cap C([0,1])$.

Define

$$\nabla_\mu f = f^{(\mu)}, \ \forall f \in \mathcal{D}_1.$$  \hfill (3.8)

In the lemma below we express eq (3.8) for the operator $\nabla_\mu$ (acting on functions $f$) in terms of associated Stieltjes measures. This point is summarized best in eq (3.11) in Lemma 3.2 below, where $df$ then denotes the Stieltjes measure corresponding to some function $f$. In the sequel we shall reserve the notation $df$ for Stieltjes measure, (not to be confused with notions of differential.) Recall that for the Stieltjes measure $df$ to make sense, the function $f$ must be assumed to be locally of bounded variation.

**Lemma 3.2.** Let $f$ be a function on $\mathbb{R}$, assumed to be locally of bounded variation, so that the Stieltjes measure $df$ is well defined. Let $\mu$ be a positive measure defined on the Borel $\sigma$-algebra $\mathcal{B}$, and assume that $df \ll \mu$,

i.e., that the implication (3.10) below holds:

$$\mu(B) = 0 \implies df(B) = 0.$$  \hfill (3.10)

Let $f^{(\mu)}$ be the corresponding Radon-Nikodym derivative (also denoted $f^{(\mu)} = \nabla_\mu f$), then

$$df = f^{(\mu)} \, d\mu.$$  \hfill (3.11)

**Proof.** The assertion in (3.11) amounts to the identity

$$df(B) = \int_B f^{(\mu)}(x) \, d\mu,$$  \hfill (3.12)

for all $B \in \mathcal{B}$. But since $f$ is locally of bounded variation, (3.12) follows from the corresponding assumption for intervals, i.e., $B = [x, y]$ for all $x < y$; so

$$f(y) - f(x) = \int_x^y f^{(\mu)}(x) \, d\mu.$$  \hfill (3.13)

Condition (3.13) in turn is equivalent to the definition of $f^{(\mu)} = \nabla_\mu f$ given in (3.7) above. \qed

**Remark 3.3.** Let $f$ be a locally bounded variation function, and let $\mu$ be a positive Borel measure. Suppose that the two measures $df$ and $\mu$ are mutually singular; we then set $\nabla_\mu f = 0$. See eq (3.14) below for justification.

**Remark 3.4.** We can decompose the restriction $df \ll \mu$ in the definition of $\nabla_\mu f$ as follows:

(a) Let $f$ and $\mu$ be as stated, and pass to the Jordan-decomposition of the signed measure $df$ (as a Stieltjes measure). Then

$$df = (\nabla_\mu f) \, d\mu + (df)_s$$  \hfill (3.14)

where the term $(df)_s$ in (3.14) is mutually singular w.r.t. $\mu$. 

(b) In section 6, we shall consider a more detailed and global analysis of (3.14) for a given Stieltjes measure $df$. Indeed, when $df$ is given, then the second term on the RHS in (3.14) will typically contain contributions from other measures $\nu$, mutually singular, and each $\nu$ relatively singular w.r.t. $\mu$.

**Lemma 3.5.** For all $f, g \in D_1$, it holds that

$$\nabla_\mu (fg) = f \nabla_\mu g + (\nabla_\mu f) g, \text{ Leibnitz' rule}$$

and

$$(fg)(1) - (fg)(0) = \langle \nabla_\mu f, g \rangle_{L^2(\mu)} + \langle f, \nabla_\mu g \rangle_{L^2(\mu)}. \quad (3.16)$$

**Proof.** In our considerations below we make use of (3.7), and the definition (3.8) for the new “$\mu$-derivative.” And we further make use of basic facts for the corresponding Stieltjes measures; in particular the integration by parts formula for Stieltjes measures.

Details: If $f, g \in D_1$ (see (3.7)), then

$$\int_0^x f \nabla_\mu g \, d\mu = \int_0^x f \, dg$$

$$= fg|_0^x - \int_0^x g \, df$$

$$= fg|_0^x - \int_0^x g \nabla_\mu f \, d\mu.$$

That is,

$$f(1)g(x) - f(0)g(0) = \int_0^x (f \nabla_\mu g + g \nabla_\mu f) \, d\mu,$$

so that

$$\nabla_\mu (fg) = f \nabla_\mu g + (\nabla_\mu f) g,$$

and (3.16) also follows. □

The proof above relies on key facts for Stieltjes integrals which might perhaps not be widely known. For the benefit of readers, we have therefore included the following alternative proof:

**Second proof of Lemma 3.5.** Let $f, g \in D_1$ as above, then

$$f(1)g(1) = \left( f(0) + \int_0^1 \nabla_\mu f \, d\mu \right) \left( g(0) + \int_0^1 \nabla_\mu g \, d\mu \right)$$

$$= f(0)g(0) + f(0) \int_0^1 \nabla_\mu g \, d\mu + g(0) \int_0^1 \nabla_\mu f \, d\mu$$

$$+ \left( \int_0^1 \nabla_\mu f \, d\mu \right) \left( \int_0^1 \nabla_\mu g \, d\mu \right),$$

where

$$\left( \int_0^1 \nabla_\mu f \, d\mu \right) \left( \int_0^1 \nabla_\mu g \, d\mu \right)$$

$$= \int_0^1 \int_0^1 (\nabla_\mu f)(s) (\nabla_\mu g)(t) \mu(ds) \mu(dt)$$

$$= \int_0^1 \left[ \int_0^t (\nabla_\mu f)(s) \mu(ds) + \int_t^1 (\nabla_\mu f)(s) \mu(ds) \right] (\nabla_\mu g)(t) \mu(dt).$$
Thus,
\[
(fg)(1) - (fg)(0) = \int_0^1 f \nabla_\mu g \, d\mu + \int_0^1 g \nabla_\mu f \, d\mu
\]
which is (3.16). □

Remark 3.6.
(a) In a $C^*$-algebraic framework, operators with dense domain and satisfying a general Leibnitz rule of the form (3.15) occur under the name “unbounded derivations.” They arise in a wider applied context, beyond that of fractal analysis, and have been extensively studied. They play an important role in dynamics, see e.g., [BR87].
(b) A non-atomic measure $\mu$ is fixed, and we assume that $\mu$ is supported in the unit interval $[0,1]$. We now turn to the corresponding boundary-value problem for the operator $\nabla_\mu$, see (3.20). Its operator theory will be identified relative to the Hilbert space $L^2(\mu)$. To emphasize choice of Hilbert space, we shall use the terminology $T_{\mu}$ for the operator, and then add subscripts to indicate domains. The theory of von Neumann (see [DS88]) of selfadjoint extensions of symmetric operators will be used. Only, for convenience, we shall use the equivalent formulation in the form where we consider instead skew-adjoint extensions of a fixed (minimal) skew-symmetric operator with dense domain.

Definition 3.7. Set $T_{\mu,0} : L^2(\mu) \to L^2(\mu)$ by
\[
T_{\mu,0} = \nabla_\mu|_{\mathcal{D}_0}
\]
where
\[
\mathcal{D}_0 = C_c(J) \cap \mathcal{D}_1.
\]

Theorem 3.8. The operator $T_{\mu,0}$ from (3.17)–(3.18) is skew-symmetric, densely defined in the complex Hilbert space $L^2(\mu)$. Moreover, $T_{\mu,0}$ has deficiency indices $(1, 1)$, and the corresponding skew-adjoint extensions are specified by
\[
dom(T_{\mu,\alpha}) = \{f \in \mathcal{D}_1 : f(1) = \alpha f(0), \ |\alpha| = 1\}, \tag{3.19}
\]
and
\[
T_{\mu,\alpha} = \nabla_\mu|_{\dom(T_{\mu,\alpha})}.
\]

Proof. From the identity
\[
\langle \nabla_\mu f, g \rangle_{L^2(\mu)} + \langle f, \nabla_\mu g \rangle_{L^2(\mu)} = \langle f g \rangle(1) - \langle f g \rangle(0),
\]
which is valid for all $f, g \in \mathcal{D}_1$, it follows that the right-hand side vanishes if and only if $f, g$ are in $\dom(T_{\mu,\alpha})$, see (3.19). □

For more details on extensions of skew-symmetric operators, we refer to [DS88, Sch12].

We now turn to the unitary one-parameter groups which are generated by the skew-adjoint extension operators above, from Theorem 3.8, eq. (3.20).
Consider the two unitary one parameter groups with the respective skew adjoint generators, and periodic boundary condition \( f(0) = f(1) \).

\[
L^2([0,1], \lambda) \ni \psi \xrightarrow{U_\lambda(t)} \psi([\cdot + t]_F) \in L^2([0,1], \lambda)
\]  \hspace{1cm} (3.21)

where \([\cdot]_F\) denotes the fractional part of a real number.

Let \( \mu \) be a non-atomic Borel measure on \([0,1]\), and let

\[
g(x) = \mu([0,x]) .
\]  \hspace{1cm} (3.22)

\begin{center}
\scalebox{0.7}{
\begin{tikzpicture}
    \node (a) at (0,0) {$0$};
    \node (b) at (1,0) {$1$};
    \node (c) at (2,0) {$2$};
    \node (d) at (3,0) {$\cdots$};
    \node (e) at (4,0) {$n-1$};
    \node (f) at (5,0) {$n$};
    \node (g) at (6,0) {$n+1$};
    \node (h) at (7,0) {$\cdots$};
    \node (i) at (8,0) {$\infty$};

    \draw (a) -- (b) -- (c) -- (d) -- (e) -- (f) -- (g) -- (h) -- (i);

    \node at (9,0) {$\mathbb{R}$};
    \node at (0,-1) {$[\cdot]_F: \mathbb{R} \to [0,1]$};
\end{tikzpicture}
}
\end{center}

\textbf{Figure 3.1.} \([\cdot]_F: \mathbb{R} \to [0,1]\)

We have

\[
U_\mu(t)(\psi \circ g)(\cdot) = \psi([g(\cdot) + t]_F)
\]

\begin{center}
\scalebox{0.7}{
\begin{tikzpicture}
    \node (a) at (0,0) {$\psi$};
    \node (b) at (1,0) {$U_\mu(t)$};
    \node (c) at (2.5,0) {$\psi \circ g$};
    \node (d) at (0.5,1) {$U_\lambda(t)$};
    \node (e) at (1.5,1) {$W_\mu$};
    \node (f) at (2.5,1) {$\psi([\cdot + t]_F)$};
    \node (g) at (4,1) {$\psi([g(\cdot) + t]_F)$};

    \draw [->] (a) to (b);
    \draw [->] (b) to (c);
    \draw [->] (d) to (f);
    \draw [->] (e) to (f);
    \draw [->] (f) to (g);
\end{tikzpicture}
}
\end{center}

(3.23)

and summarized in the diagram below:

\[
\begin{array}{c}
U_\lambda(t) \psi \in L^2(\lambda) \\
\xrightarrow{W_\mu} U_\mu(t) f \in L^2(\mu)
\end{array}
\]

(3.24)

\[
\begin{array}{c}
\psi \in L^2(\lambda) \\
\xrightarrow{W_\mu} f \in L^2(\mu)
\end{array}
\]

\begin{lemma}
Fix \( \mu \) and \( g \), and set \( W_\mu \psi = \psi \circ g \). Then TFAE:
\begin{enumerate}
\item \( W_\mu U_\lambda(t) = U_\mu(t) W_\mu : L^2(\lambda) \to L^2(\mu) \)
\item \( U_\mu(t) = W_\mu U_\lambda(t) W_\mu : L^2(\mu) \to L^2(\mu) \)
\item \( U_\lambda(t) = W_\mu U_\mu(t) W_\mu : L^2(\lambda) \to L^2(\lambda) \)
\end{enumerate}
\end{lemma}

Starting with \( \mu \), specified as before, we then set \( g = g_\mu \), \( g(x) := \mu([0,x]) \), the "cumulative distribution". It follows that then the operator \( W_\mu \), given by \( W_\mu \psi := \psi \circ g \), will be an isometric isomorphism of \( L^2(\lambda) \) onto \( L^2(\mu) \), with adjoint \( W_\mu^* : L^2(\mu) \to L^2(\lambda) \), given by (3.28). We add that a detailed analysis of this operator \( W_\mu \), and its applications, will be undertaken below.
Corollary 3.10 (Time-change). We have

\[(U_\mu (t) f) (x) = W_\mu U_\lambda (t) W_\mu^* f (x) \]
\[= W_\mu f ([g(x) + t]_F), \]  

where

\[W_\mu \psi (\cdot) = \psi \circ g, \]  

and

\[(W_\mu^* f) (y) = \int_{g^{-1}(\{y\})} f \, d\rho_y, \]

and so

\[W_\mu^* f ([g(x) + t]_F) = \int_{g^{-1}(\{y\} + t]_F)} f \, d\rho_{g(x) + t} \cdot \]

In (3.28), \(d\rho_y\) denote conditional measures. Since \(\mu \circ g^{-1} = \lambda\) we have conditional measures \(\{\rho_y\}_{y \in J}\) subject to the partition \(\{g^{-1}(\{y\})\}_{y \in J}\). By the disintegration theorem, applied to \(\mu\), we therefore get the representation:

\[\mu (\cdot) = \int_J \rho_y (\cdot) \, dy. \]  

For details regarding (3.30), we refer readers to the literature, e.g., [BJ18, ch2. pg 13-21] and [Roh49, Roh52, Roh64a, Roh64b].

Remark 3.11 (The middle-third Cantor measure).

(a) For a detailed account of harmonic analyses on fractals, see [Jor18, HJW19b]; as well as the papers cited there.

Below we discuss the special case when \(\mu = \mu_3\) is assumed to be the middle-third Cantor measure. In details, if the pair \(\sigma_0, \sigma_1\) denotes \(\sigma_0 (x) = \frac{x}{3}, \sigma_1 (x) = \frac{x+1}{3}, x \in \mathbb{R}\); then \(\mu_3\) is the corresponding normalized IFS-measure fixed by

\[\mu_3 = \frac{1}{2} (\mu_3 \circ \sigma_0^{-1} + \mu_3 \circ \sigma_1^{-1}), \]  

Figure 3.2. \(g_{\mu_3} (\cdot)\) the middle third Cantor measure as a Stieltjes measure. See Remark 3.11.
supported in the Cantor set. The cumulative distribution function of \( \mu_3 \) is the function \( g(x) := \mu_3([0,x]) \). It is sketched in Figure 3.2.

(b) We show in section 4.1 (see especially Theorem 4.2, eq. (4.1), and Figure 4.1) that the Cantor-measure \( \mu \) solving equation (3.31) arises as a special case of a wider class of self-similar measures, also called IFS-measures. In the general case of IFS-measures, the corresponding equation is (4.1). It is defined from a finite system of endomorphisms \( \sigma_i \) (called function systems), and the corresponding iterated function system (IFS)-measure \( \mu \) will then arise as a Markov chain-average (Figure 4.1) of its corresponding \( \sigma_i \) transforms. In this general case, the solution \( \mu \) can be found, see Theorem 4.2: Every IFS-measure \( \mu \) allows a representation (4.7), defined as a pull-back of an infinite-product measure (4.4).

(c) Note that the function \( g \) defined this way (see (3.22)) will have the following properties: It is monotone (increasing, or more precisely non-decreasing). Moreover, by standard measure theory, it follows that the initial measure \( \mu \) will then agree with the corresponding Stieltjes measure (here denoted \( d g \)), i.e., we have \( \mu = d g \). Hence it is possible to define branches of an inverse to the function \( g \), i.e., \( g^{-1} \) defined a.e., w.r.t. \( \mu \). Intuitively we think of the function \( g \) as a time-change, see (3.26).

(d) In the case when \( \mu \) is the standard middle third Cantor measure, then the corresponding function \( g \) is illustrated in Figure 3.2 (often called “the Devil’s staircase.”) At each iteration in the construction of \( g \), we take it to be constant on the omitted middle-third intervals.

In more details, let \( \sigma_0, \sigma_1 : [0,1] \to [0,1] \) be given by
\[
\sigma_0(x) = \frac{x}{3}, \quad \sigma_1(x) = \frac{x + 2}{3}. \tag{3.32}
\]

Then, for the Cantor set \( C_3 \), we have
\[
\sigma_0(C_3) \sqcup \sigma_1(C_3) = C_3.
\]

Using bit representations, we get
\[
g^{-1} \left( \sum_{n=1}^{\infty} \frac{b_n}{3^n} \right) = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}, \quad b_n \in \{0,1\}.
\]

To see this, recall that every \( x \in C_3 \) has the following representation:
\[
x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad (a_n \in \{0,2\})
\]
\[
= \frac{a_1}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n}
\]
\[
= \frac{a_1}{3} + \frac{1}{3} \left( \frac{a_2}{3} + \frac{a_3}{3^2} + \cdots \right)
\]

(e) Let \( \mu (= \mu_3) \) be the Cantor measure of (3.31) above. We then get the following transformation of pairs of Borel measures on the unit-interval \( J \): \( \mu \circ g^{-1} = \lambda_1 \), where \( \lambda_1 \) is Lebesgue measure on \( J \).
Note, the defining properties of the Cantor measure $\mu$, $\text{supp}(\mu) = C$, and the Lebesgue measure $\lambda$ are as follows:

\[
\begin{align*}
\text{Cantor} & \quad \int \varphi \, d\mu = \frac{1}{2} \left( \int \varphi \circ \sigma_0 \, d\mu + \int \varphi \circ \sigma_1 \, d\mu \right), \forall \varphi \in C; \text{ see (3.31)-(3.32) above;}
\text{Lebesgue} & \quad \int \varphi \, d\lambda = \frac{1}{2} \left( \int \varphi \left( \frac{x}{2} \right) \, d\lambda(x) + \int \varphi \left( \frac{x + 1}{2} \right) \, d\lambda(x) \right), \forall \varphi \in C.
\end{align*}
\]

### 3.2. A symmetric pair of operators for $L^2(\mu)$ and $L^2(\nu)$.

Let $\mu, \nu$ be two non-atomic measures on $J = [0,1]$.

**Lemma 3.12.** For $f \in L^2(\nu) \cap \text{dom} \,(T_\mu)$ and $g \in L^2(\mu) \cap \text{dom} \,(T_\nu)$, we have

\[
(fg)(x) - (fg)(0) = \int_0^x (\nabla_\mu f) \, g \, d\mu + \int_0^x f \, (\nabla_\nu g) \, d\nu.
\]

**Proof.** A direct computation:

\[
\begin{align*}
\int_0^x (\nabla_\mu f) \, g \, d\mu & = \int_0^x g df \\
& = g f \bigg|_0^x - \int_0^x f \, dg \\
& = g f \bigg|_0^x - \int_0^x f \, (\nabla_\nu g) \, d\nu \\
\Downarrow \\
(fg)(x) - (fg)(0) & = \int_0^x (\nabla_\mu f) \, g \, d\mu + \int_0^x f \, (\nabla_\nu g) \, d\nu \\
\Downarrow \\
d(fg) & = (\nabla_\mu f) \, g \, d\mu + f \, (\nabla_\nu g) \, d\nu.
\end{align*}
\]

See also the proof of Lemma 3.5, and that of Theorem 3.1. \hfill \square

**Corollary 3.13.** Given a pair of non-atomic measures $\mu$ and $\nu$ as described; with Lemma 3.12 and the arguments in sect 2, and 3.1, we then arrive at the following dual-pair realization for the associated operators:

\[
\begin{align*}
\text{dom} \,(T_\mu) \cap L^2(\nu) & \quad \xrightarrow{T_\nu} \quad L^2(\nu) \cap \text{dom} \,(T_\nu) \\
\xrightarrow{-T_\nu} \quad L^2(\nu) \cap \text{dom} \,(T_\nu) & \quad \xrightarrow{T_\mu} \quad \text{dom} \,(T_\mu) \cap L^2(\nu)
\end{align*}
\]

(3.33)

\[
T_\mu \subset -T_\nu^*, \quad -T_\nu \subset T_\mu^*.
\]

(3.34)

**Remark 3.14.** Let $K_F = \frac{d}{d\mu} \frac{d}{dx}$ be as before. Note it has a quadratic form representation as follows:

\[
\langle \varphi, K_F \psi \rangle_{L^2(\mu)} = -\langle \varphi', \psi' \rangle_{L^2(\lambda)}
\]

(3.35)

where $\varphi' = \frac{d\varphi}{dx}$, $\psi' = \frac{d\psi}{dx}$. This follows from the dual pair $d/dx$, $d/d\mu$ in Theorem 3.1. Details:

\[
\int \varphi \left( \frac{d}{d\mu} \frac{d}{dx} \psi \right) \, d\mu = -\int \frac{d}{dx} \varphi' \frac{d}{dx} \psi \, d\lambda = -\int \varphi' \psi' \, d\lambda.
\]

(3.36)
Then we get the selfadjoint operator

$$T_\mu T_\mu^* = V (T_\mu^* T_\mu) V^*$$  

(3.37)

where $V$ is a partial isometry, and $T_\mu T_\mu^*$ has dense domain in $L^2(\mu)$. (See Theorem 2.5 and Definition 2.6) So $T_\mu T_\mu^*$ is a selfadjoint operator extension for the quadratic form $QF(K_F)$, where

$$QF(K_F)(\varphi, \psi) = -\int \varphi' \psi' d\lambda.$$  

(3.38)

Similarly, in (3.33) we get

$$QF(K_F) \subset T_\mu T_\mu^*.$$  

(3.39)

The notion of “extension” of a closable positive quadratic form $Q$ is made precise in, for example [Kat95]. It is a quadratic form-version of the analogous extension of Friedrichs (see e.g., [DS88]). If $K$ is such a selfadjoint extension operator for $Q$, then the requirement is that the domain of $Q$ be contained in the domain of the square root of $K$.

**Example 3.15** (see also Figure 3.2, (3.32), and Example 4.4 (2)). Let $\mu = \mu_3$, the middle 1/3 Cantor measure,

$$\mu_3 = \frac{1}{2} (\mu_3 \circ \sigma_0^{-1} + \mu_3 \circ \sigma_1^{-1}),$$  

(3.40)

supported on the Cantor set

$$C_3 = [0, 1] \setminus \bigcup \{\text{middle intervals}\},$$  

(3.41)

so that $\lambda(C_3) = 0$. Let $g_3(x) = \mu_3([0, x])$. Set $K_F^{(3)} = \frac{d}{d\mu_3} \frac{d}{dx}$. Then

$$K_F^{(3)} (\psi \circ g_3) = \frac{d}{d\mu} (\psi' \circ g) \left( \frac{d}{dx} g_3 \right) = 0$$  

(3.42)

since $\frac{d}{dx} g_3 = 0$ in the sense of distribution, i.e., $g_3' = 0$ a.e. $\lambda$.

3.3. The $L^2(\mu)$-boundary value problem. We now turn to a detailed harmonic analysis of the skew-adjoint extension operators introduced in Theorem 3.8. Recall that, when $\alpha$ is fixed (on the complex circle), then the corresponding skew-adjoint extension operator generates a unitary one-parameter group (depending on $\alpha$) of operators $U(t)$ acting in $L^2(\mu)$. The harmonic analysis of this unitary one-parameter group was presented in detail above in Lemma 3.9. The following result offers a complete spectral picture.

**Lemma 3.16.** Set

$$v_x(y) := \mu\left([0, y \wedge x]\right).$$  

(3.43)

Then we have

$$T_\mu v_x = \frac{dv_x}{d\mu} = \chi_{[0, x]}.$$  

(3.44)

Moreover, for any $F \in C^1$, we get

$$T_\mu (F(v_x)) = F'(v_x) \chi_{[0, x]}.$$  

(3.45)

In particular,

$$T_\mu (e^{iv_x}) = ie^{iv_x} \chi_{[0, x]}.$$  

(3.46)
Proof. Note that \( v_x (y) = \int_0^y \chi_{[0,x]} (s) \, d\mu (s) = \mu ([0,y \wedge x]) \), which is (3.44).

Now, if \( F \in C^1 \) then
\[
d (F (v_x)) = F' (v_x) \, dv_x.
\]
That is,
\[
F (v_x (y)) - F (v_x (0)) = \int_0^y F' (v_x (s)) \, ds = \int_0^y F' (v_x) \chi_{[0,x]} \, d\mu,
\]
so that (3.45) holds, and (3.46) follows from this.

We now turn to the detailed spectral expansion for the indexed system of skew-adjoint operators \( T_{\mu,\theta} \) discussed in Theorem 3.8.

**Theorem 3.17.** Let \( T_{\mu,\theta} (\alpha = e^{i\theta}) \) be as above. In particular, elements in \( \text{dom} \, (T_{\mu,\theta}) \) satisfy the boundary condition
\[
f (1) = e^{i\theta} f (0), \quad \theta \in \mathbb{R}.
\]
Then \( T_{\mu,\theta} \) has the following spectral representation:
\[
-i T_{\mu,\theta} = \sum_{n \in \mathbb{Z}} \lambda_n |\varphi_n \rangle \langle \varphi_n|
\]
where
\[
\lambda_n = \frac{\theta + 2n\pi}{\mu (J)}, \quad \varphi_n (x) = \frac{1}{\sqrt{\mu (J)}} e^{i\lambda_n \mu ([0,x])}, \quad n \in \mathbb{Z},
\]
and \( \{ \varphi_n \} \) is an ONB in \( L^2 (\mu) \). And the associated unitary one-parameter group \( U (t) = e^{itT_{\mu,\theta}} \) is given by
\[
e^{itT_{\mu,\theta}} = \sum_{n \in \mathbb{Z}} e^{it\lambda_n} |\varphi_n \rangle \langle \varphi_n|.
\]

Proof. With Lemma 3.16 we justify the eigenvalue/eigenfunction assertions in (3.49) in the Theorem. Note that
\[
T_{\mu,\theta} f = i\lambda f \iff f (x) - f (0) = i\lambda \int_0^x f \, d\mu.
\]
It suffices to verify that \( f (x) = e^{i\lambda \mu ([0,x])} \) satisfies (3.51). Indeed,
\[
\int_0^x e^{i\lambda \mu ([0,s])} \, d\mu (s) = \int_0^x e^{i\lambda g (s)} \, dg (s) = \frac{1}{i\lambda} \left( e^{i\lambda g (x)} - 1 \right),
\]
where \( g (x) = \mu ([0,x]) \) as before.

**Lemma 3.18.** The adjoint operator of \( T_{\mu,0} \) is given by \( T_{\mu,0}^* = -\nabla_\mu \), defined on \( \mathcal{D}_1 \).

Proof. For any \( f \in \mathcal{D}_0 \) and \( g \in \mathcal{D}_1 \), one has
\[
\int_0^1 (T_{\mu,0} f) \, g \, d\mu + \int_0^1 f \nabla_\mu g \, d\mu = 0
\]
and so \( T_{\mu,0}^* \subset -\nabla_\mu \big|_{\mathcal{D}_1} \).

Conversely, for all \( g \in \text{dom} \, (T_{\mu,0}^*) \subset L^2 (\mu) \), let \( h = T_{\mu,0}^* g \) then
\[
\int_0^1 (T_{\mu,0} f) \, g \, d\mu = \int_0^1 f h \, d\mu, \quad \forall f \in \mathcal{D}_0.
\]
Set \( H(x) = H(0) + \int_0^x h d\mu \in \mathcal{D}_1 \), then
\[
\int_0^1 (T_{\mu,0} f) g d\mu = -\int_0^1 f \nabla_\mu H d\mu = -\int_0^1 (T_{\mu,0} f) H d\mu, \quad \forall f \in \mathcal{D}_0.
\]
It follows that \( g = -H \) in \( L^2(\mu) \). □

Even though our present focus is on the case when \( \mu \) is assumed singular w.r.t. Lebesgue measure, \( \lambda \), for the sake of illustration, the next two results, Lemma 3.19 and Corollary 3.20, cover the other extreme, i.e., when \( \mu \ll \lambda \) holds.

**Lemma 3.19.** Assume \( \mu \ll dx \), where \( dx = \) the Lebesgue measure on \([0,1]\), and let \( d\mu/dx = M(x) \). Then
\[
\nabla_\mu f(x) = (M^{-1}f')(x).
\]

**Proof.** Indeed, if \( f \in \mathcal{D}_1 \) then
\[
f(x) - f(0) = \int_0^x f'(x) dx = \int_0^x (f'M^{-1})(x) M(x) dx = \int_0^x \nabla_\mu f(x) d\mu(x).
\]
□

**Corollary 3.20.** Let \( M(x) = d\mu/dx \) be as above. The deficiency subspaces of \( T_{\mu,0} \) are determined as follows:
\[
\mathcal{D}_{\pm}(T_{\mu,0}) := \{ g : (g'M^{-1})(x) = \pm g \}
\]
\[
= \text{span} \left\{ \exp \left( \pm \int M(x) dx \right) \right\}.
\]
In particular, this implies that \( T_{\mu,0} \) has deficiency indices \((1,1)\).

**Proof.** One checks that
\[
\nabla_\mu g = \pm g
\]
\[
\Downarrow
\]
\[
g'M^{-1} = \pm g
\]
\[
\Downarrow
\]
\[
(ln g)' = \pm M
\]
and the conclusion follows. □

**Definition 3.21.** Set
\[
\Delta_\mu = \frac{d}{d\mu} \frac{d}{dx} = \nabla_\mu \frac{d}{dx}
\]
defined on
\[
\text{dom}(\Delta_\mu) := \{ f | f' \in \mathcal{D}_1 \}
\]
\[
= \left\{ c + \int_0^x \left( f(0) + \int_0^x \nabla_\mu f d\mu \right) dx \mid \nabla_\mu f \in L^2(\mu) \right\}.
\]
Set
\[
\Delta_{\mu,0} = \Delta_\mu|_{C_c \cap \text{dom}(\Delta_\mu)}.
\]
Then $\Delta_{\mu,0}$ has two particular selfadjoint extensions that correspond to Dirichlet and Neumann boundary conditions.

**Theorem 3.22.** Fix $\mu$. Let $K_F$ denote a corresponding selfadjoint realization of $\Delta_{\mu}$. For every $g \in L^2(\mu)$, set

$$f(t) := \int_0^t \left( \int_0^x g dq \right) dx.$$  \hfill (3.52)

Then

$$(K_F f)(t) = (f')^{\mu} = g,$$  \hfill (3.53)

and the eigenvalue problem may be stated as

$$g'(t) = \lambda \int_0^t g dq.$$  \hfill (3.54)

**Proof.** We have

$$(K_F f)(t) = \lambda f$$

$$g = \lambda f$$

$$g(t) = \lambda \int_0^t \left( \int_0^x g dq \right) dx.$$  

And so $g'(t) = \lambda \int_0^t g dq \mu$. \hfill \Box

**Corollary 3.23.** Assume $\mu \ll dq$, and $dq/dx = M > 0$. Let

$$dW_t^{(\mu)} = M^{-1/2} (x) dB_t,$$

where $B_t$ is standard Brownian motion. Then,

$$u(t,x) := \mathbb{E}_x \left( f(W_t^{(\mu)}) \right)$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2} K_F u.$$  

**Proof.** An application of Ito’s lemma (see e.g., [Hid80]) gives

$$df(W_t) = f'(W_t) M^{-1/2} dB_t + \frac{1}{2} f''(W_t) M^{-1} dt$$

and

$$\frac{d}{dt} \mathbb{E}(f(W_t)) \bigg|_{t=0} = \frac{1}{2} M^{-1} (x) f''(x) = \frac{1}{2} K_F f(x),$$

where $K_F f = M^{-1} f''$, by Lemma 3.19. \hfill \Box
3.4. Krein-Feller diffusion. We now turn to a detailed discussion of diffusion, and its connection to the Krein-Feller operator. As before, our starting point is a fixed measure $\mu$ supported on the real line, and we let $K_F$ be the corresponding Krein-Feller operator. The measure $\mu$ under consideration is assumed to be defined on the Borel-sigma algebra, and further assumed positive, sigma-finite, and non-atomic. The measure $\mu$ may be singular, and the main novelty in our analysis of the diffusion semigroup is for the singular case; including the case of IFS-measures. We first introduce the centered Gaussian process $W(\mu)$ having $\mu$ as its quadratic variation. We then note that Ito’s lemma applies to $W(\mu)$, see (3.94). We further study the Markov semigroup ((3.79) and Lemma 3.29) corresponding to $W(\mu)$. In our two main results Lemma 3.24 and Theorem 3.30 below, we identify the selfadjoint extension of $K_F$ from section 2 as the infinitesimal generator for this diffusion semigroup, Lemma 3.24. In Theorem 3.30 we introduce a time-change in our characterization of the diffusion semigroup.

Below we show that every positive non-atomic Borel measure (see Lemma 3.2) gives rise to a naturally associated dual pair of operators, as per Definition 2.6. The dual pair is made precise in (3.4), and the figure in Theorem 3.1. Fix a measure space $(J, \mathcal{B}, \mu)$ with $J = [0, \infty)$. We introduce the corresponding positive definite kernel:

$$K_\mu (x,y) = \mu ([0, x \wedge y]). \quad (3.55)$$

Note that if $\mu = \lambda = dx$ is the Lebesgue measure then

$$K_\lambda (x,y) = x \wedge y, \quad (3.56)$$

the usual covariance function for Brownian motion.

Starting with $\mu$ (assumed non-atomic) and the RKHS $\mathcal{H}(\mu)$, we arrive at a generalized Brownian motion $W_{x^{(\mu)}}$; i.e., a Gaussian process with

$$\mathbb{E} \left( W_{x^{(\mu)}} \right) = 0, \quad (3.57)$$

and

$$\mathbb{E} \left( W_{x^{(\mu)}} W_{y^{(\mu)}} \right) = K_\mu (x,y), \quad \forall x, y \in [0, \infty). \quad (3.58)$$

A detailed description of $\{W_{x^{(\mu)}}\}$ and its Ito-calculus is contained in many relevant papers, and books; see e.g., [JT20, JT19b, AJL17, AJ15, AJ12, AJL11, IM74, IM63, HOUZ10, JT21]. Here we shall need the following: Let $\varphi \in C^2$, and consider the corresponding diffusion semigroup (depending on $\mu$):

$$u(t,x) := \mathbb{E} \left( \varphi \left( W_{t^{(\mu)}} \right) \mid W_{0^{(\mu)}} = x \right) \quad (3.59)$$

where $\mathbb{E}$ in (3.59) refers to the expectation $\mathbb{E} (\cdot) = \int_\Omega (\cdot) \, d\mathbb{P}$ for the probability space associated to (3.55). Then $\mathbb{E}(\cdot \mid W_{0^{(\mu)}} = x)$ in (3.59) refers to conditioning with all paths $\omega$ s.t. $\omega(0) = x$. Here we also use the representation

$$W_{t^{(\mu)}} (\omega) = \omega(t), \quad \forall \omega \in \Omega \quad (3.60)$$

for the Gaussian process $\{W_{t^{(\mu)}}\}$. Since the Gaussian process $W^{(\mu)}$ has independent increments, it follows that eq. (3.59) defines a semigroup (see e.g., [IM74, Phi61b, Kas12, KS16]), and we shall call it the Markov semigroup. Its properties and its infinitesimal generator will be identified in Lemma 3.24 below, and in the subsequent discussion.

The Gaussian process from (3.58) and (3.60) is often called a generalized Brownian motion, or a Gaussian field. The associated Ito-integral is also used in the proof of Lemma 3.24 below.
We now introduce the Krein-Feller operator

\[ K_F := \frac{\partial^2}{\partial \mu \partial x} \]  

(see above.) In our discussion of (3.59), we shall consider \( K_F \) as acting in the \( x \)-variable.

**Lemma 3.24.** The diffusion (3.59) is generated by the following generalized heat equation:

\[ \frac{\partial u}{\partial t} = \frac{1}{2} K_F u, \quad u(0, x) = \varphi(x) \]  

where \( \varphi \) is a fixed continuous function.

**Remark 3.25.** The special case of (3.62) corresponding to \( \mu = \lambda = \) Lebesgue measure is

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \quad u(0, x) = \varphi(x). \]  

**Proof.** We shall refer to the literature, e.g., [JT20, JT19b, AJL17, AJ15, JT21]. Suffice it to say that the Ito-lemma for the Gaussian process \( \{W_t^{(\mu)}\} \) is a key tool; together with the following fact for the quadratic variation:

\[ (dW_t^{(\mu)})^2 = \mu(dt). \]  

\[ \square \]

Set \( J = [0, 1] \). Fix \( \mu, \sigma \)-finite and non-atomic. Let \( \lambda = dx = \) Lebesgue measure, and consider the following diagram

\[ \mathcal{L}^2(\mu) \xrightarrow{T_{\mu}} L^2(\mu) \]  

with \( D = -d/dx \).

Let \( K_F \) be as in (3.61), then

\[ K_F \subseteq T_{\mu} T_{\mu}^*, \]  

both are operators in \( L^2(\mu) \). But \( T_{\mu} T_{\mu}^* \) is selfadjoint by general theory (see Section 2), and restriction of symmetric is symmetric.

**Theorem 3.26.** Consider a fixed positive non-atomic Borel measure \( \mu \) on \( J = [0, b], b < \infty; \) and define an operator \( A \) on \( L^2(\mu) = L^2(J, \mu) \) as follows:

For \( \varphi \in L^2(\mu) \cap C \), set

\[ (A\varphi)(x) = \int_0^x \left( \int_0^y \varphi(s) \mu(ds) \right) dy, \]  

then

\[ K_F A\varphi = \varphi. \]
Proof. Since \( L^2(\mu) \subset L^1(\mu) \) the function \( y \mapsto \int_0^y \varphi(s) \mu(ds) \) is continuous, and so \( x \mapsto (A\varphi)(x) \) is \( C^1 \) (one time differentiable with \( (A\varphi)' \in C(J) \)). Hence, for the LHS of (3.67) we have
\[
K_F A : \varphi \mapsto \nabla \mu \frac{d}{dx} A\varphi = \nabla \mu \int_0^x \varphi(s) \mu(ds) = \varphi(x),
\]
which is the desired conclusion (3.67). \( \square \)

Corollary 3.27. Let the measure \( \mu \) be specified as above, and set
\[
(A\varphi)(x) = \int_0^x \left( \int_0^y \varphi(s) \mu(ds) \right) dy
\]
(see (3.66)), which defines a compact integral operator in \( L^2(\mu) \).

Then \( A = A_\mu \) is bounded and compact with triangular integral kernel
\[
a(x,s) = \chi_{[0,x]}(s)(x-s).
\]

Proof. Without loss of generality we may work with \( b = 1 \) and real Hilbert space. An easy application of Schwarz to \( L^2(\mu) \) shows that \( A : L^2(\mu) \to L^2(\mu) \) is bounded. We have
\[
\langle A\varphi, \psi \rangle_{L^2(\mu)} = \int_0^1 \int_0^x (x-s) \varphi(s) \psi(x) \mu(ds) \mu(dx).
\]
The asserted symmetry follows from this, or equivalently,
\[
(A\varphi)(x) = \int_0^1 a(x,s) \varphi(s) \mu(ds)
\]
where \( a(x,s) = \chi_{[0,x]}(s)(x-s) \), see Figure 3.3.

To see that the operator \( A \) of (3.66) is compact as an operator in \( L^2(\mu) \), we make use of (3.69) and (3.70) as follows: We recall the fact the compact operators are the norm-closure of finite rank operators. We then create such norm-limits of finite rank operators with the use of the kernel (3.68), and a choice of a filter of partitions \( P \) with disjoint Borel subsets of the support of \( \mu \). For each such partition \( P \), we form rank-one operators from the corresponding indicator functions from pairs of sets \( B, B' \) in \( P \), and we then form the associated span of the rank-one operators \( \langle \chi_B \vert \chi_{B'} \rangle \) by evaluation of (3.68) with sample points chosen from the partition sets. (We use Dirac’s terminology \( \vert \cdot \rangle \langle \cdot \vert \) for rank-one operators.) The Borel sets \( B \), making up partitions, are chosen with \( \mu(B) \) finite, so the corresponding indicator functions are in \( L^2(\mu) \). For a fixed partition, we then form pairs of such indicator functions, and the corresponding rank-one operators. Since \( \mu \) is chosen non-atomic, the partition-limit refinements can be constructed such that the limit of the corresponding numbers \( \mu(B) \) is zero. \( \square \)

**Figure 3.3.** The kernel \( a(x,s) = \chi_{[0,x]}(s)(x-s) = [x-s]_+ = \max(0,x-s) \).
The following result gives a spectral decomposition for the Krein-Feller operator $K_F$.

**Corollary 3.28.** Let $J = [0, 1]$, and $\mu$ be a non-atomic positive Borel measure on $J$. Set $g(x) = \mu([0, x])$.

Consider $K_F = \frac{d}{dx} \frac{d}{dx}$ as a selfadjoint operator in $L^2(\mu)$ with Neumann boundary condition, i.e.,

$$\text{dom}(K_F) = \left\{ \psi(x) = \psi(0) + \int_0^x \left( \int_0^y f \, d\mu \right) \, dx : f \in L^2(\mu), \psi'(0) = \psi'(1) = 0 \right\}.$$

Let $V : L^2(\mu) \to L^2(\mu)$ be the integral operator defined as

$$V f(x) = \int_0^1 H(x, t) f(t) \, dt$$

where

$$H(x, t) = \begin{cases} (g(x) - 1) g(t) & t \leq x, \\ (g(t) - 1) g(x) & x \geq t. \end{cases}$$

Then $V$ is compact and selfadjoint, and we have the following eigenvalue correspondence:

$$c \in \text{sp}(K_F) \iff c^{-1} \in \text{sp}(V).$$

**Proof.** Selfadjointness of $V$ follows from (3.72), and the argument for compactness is the same as that of Corollary 3.27.

Let $\psi(x) = \psi(0) + \int_0^x \left( \int_0^t f \, d\mu \right) \, dt \in \text{dom}(K_F)$, where $f \in L^2(\mu)$, so that $K_F \psi(x) = f(x)$.

Assume that

$$K_F \psi = c \psi,$$

for some constant $c < 0$. Then,

$$K_F \psi = c \psi \Downarrow f(x) = c \int_0^x \left( \int_0^t f \, d\mu \right) \, dt \Downarrow f'(x) = c \int_0^x f \, d\mu.$$

The last line can be written as follows:

Set $g(x) := \mu([0, x])$, then

$$f'(x) = c \int_0^x f \, d\mu = c \int_0^x f \, dg = c \left( f(x) g(x) - \int_0^x g(t) f'(t) \, dt \right) = c \left( f(0) + \int_0^x f'(s) \, ds \right) g(x) - \int_0^x g(t) f'(t) \, dt$$
\[ = cf(0)g(x) + c \int_0^x (g(x) - g(t)) f'(t) \, dt. \]  
(3.74)

Since \( f' = c\psi' \), the boundary condition \( \psi'(1) = 0 \) and (3.74) imply that
\[ cf(0) + c \int_0^1 (1 - g(t)) f'(t) \, dt = 0. \]  
(3.75)

Substitute (3.75) into (3.74), then
\[ f'(x) = c \int_0^x (g(x) - g(t)) f'(t) \, dt + cf(0)g(x) \]
\[ = c \int_0^x (g(x) - 1) g(t) f'(t) \, dt + \int_x^1 g(x)(g(t) - 1) f'(t) \, dt \]
\[ = c \int_1^0 H(x,t) f'(t) \, dt \]
with \( H \) as defined in (3.72), and the assertion (3.73) follows. \( \square \)

3.5. Path-space and Markov transition. It is also of general interest to relate \( K_F \) directly to the generator of the diffusion semigroup. Notation: \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( \Omega \) path space, \( \mathcal{F} \) cylinder \( \sigma \)-algebra, \( \mathbb{P} \) probability measure, \( K_\mu(A \cap B) = \mu(A \cap B) \),
\[ W^{(\mu)}(\omega) = \omega(t), \quad \forall \omega \in \Omega. \]  
(3.76)

\[ \mathbb{E}(\cdot) = \int_{\Omega} \cdots d\mathbb{P} \quad (x \in J) \]  
(3.77)

\[ \mathbb{E}_x(\cdot) = \int_{\Omega_x} \cdots d\mathbb{P}, \quad \omega \in \Omega_x = \{\omega, \omega(0) = x\} = \{\omega, W^0_0 \omega = x\}. \]  
(3.78)

In the discussion below we omit \( \mu \) in \( W^{(\mu)} \) to simplify notation.
\[ (S_t\psi)(x) := \mathbb{E}_x \left( \varphi \circ W^t_0 \right). \]  
(3.79)

We showed that \( K_F \) is the generator of \( S_t \) in (3.79). It is known that \( S_t \) is a semigroup \((t \in \mathbb{R}_+)\) also \( S_0 = I \), so
\[ S_sS_t = S_{s+t}, \quad \forall s, t \in \mathbb{R}_+. \]

For semigroups and generators in the Hilbert space framework, see e.g., [Phi11, CP68, Phi61a], and for diffusion semigroups, we refer to e.g., [Phi61b, Won21, KS16]. Also see [Kni81, EL93, ch 3].

**Lemma 3.29.** \( S_t \) is selfadjoint in \( L^2(\mu) \) \( \forall t \in \mathbb{R}_+ \), so
\[ \int (S_t\varphi)(x) \psi(x) \mu(dx) = \int \varphi(x) (S_t\psi)(x) \mu(dx), \quad \forall \varphi, \psi \in L^2(\mu), \]  
(3.80)

also
\[ \int |S_t\varphi|^2 d\mu \leq \int |\varphi|^2 d\mu. \]  
(3.81)
Proof. The proof of the properties (3.80)–(3.81) is contained in the literature of diffusion semi-groups. But the following proof sketch for (3.80) is new:

Fix $t > 0$ and any $0 \leq s \leq t$, then

$$
\frac{d}{ds} \int (S_{t-s}\varphi)(x)(S_s\psi)(x) \mu(dx) \equiv 0
$$

so $s \to \int (S_{t-s}\varphi)(x)(S_s\psi)(x) \mu(dx)$ is constant value at $s = 0 = \text{value at } s = t$, and (3.80) follows.

Proof of (3.82).

$$
\text{LHS}_{(3.82)} = \frac{d}{ds} \langle S_{t-s}\varphi, S_s\psi \rangle_{L^2(\mu)} = -\langle KFS_{t-s}\varphi, S_s\psi \rangle_{L^2(\mu)} + \langle S_{t-s}\varphi, KFS_s\psi \rangle_{L^2(\mu)} = 0
$$

since $K_F$ is symmetric w.r.t. $L^2(\mu)$. \hfill \Box

3.6. The conditioning $W^{(\mu)}_0 = x$. For our considerations in (3.78) and (3.79) we used the notation $\mathbb{E}_x$ and $\Omega_x$ with reference to conditioning paths $\omega$ which “start” at $x$, so $\omega(0) = x$. The justification is as follows. We have selected the sample space $\Omega$ to be $\prod_{[0,\infty)} \mathbb{R}$ (Cartesian product), and functions $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}$ (infinitely many “paths”). (It is known that the continuous functions will have full measure relative to $(\Omega, \mathcal{C}, \mathbb{P})$ where $\mathcal{C}$ is the usual cylinder $\sigma$-algebra of subsets of $\Omega$.) Here

$$
\Omega_x := \{ \omega \in \Omega : \omega(0) = x \}, \quad \text{and}
$$

$\mathbb{P}$ denotes the probability measure on $(\Omega, \mathcal{C})$, such that

$$
\mathbb{E}(\cdot) = \int_{\Omega} \cdots d\mathbb{P}, \quad \text{and}
$$

$$
\mathbb{E}(W^{(\mu)}_A) = 0, \quad \mathbb{E}(W^{(\mu)}_A W^{(\mu)}_B) = \mu(A \cap B),
$$

for all Borel sets $A, B \in \mathcal{C}$.

From the construction the projection $\pi_0 : \Omega \to \Omega$, $\pi_0(\omega) = \omega(0)$, satisfies

$$
\mathbb{P} \circ \pi_0^{-1} \ll \mu.
$$

Now consider the Radon-Nikodym derivative:

$$
\frac{d\mathbb{P} \circ \pi_0^{-1}}{d\mu} = \mathbb{E}_x,
$$

or equivalently, for all random variables $F$ on $(\Omega, \mathcal{C})$ we have:

$$
\mathbb{E}(F) = \int_{\mathbb{R}} \mathbb{E}_x(F) \mu(dx).
$$

3.7. The $\mu$-heat equation. We assume a fixed non atomic Borel measure $\mu$ supported in an interval $J = [0, \alpha]$ where $\alpha$ may be finite or infinite. We shall denote by $W^{(\mu)}$ the corresponding generalized Brownian motion, i.e., determined by: $W^{(\mu)}$ is Gaussian, real-valued

$$
\mathbb{E}(W^{(\mu)}) = 0, \quad \mathbb{E}(W^{(\mu)}_A W^{(\mu)}_B) = \mu(A \cap B)
$$

for all Borel sets $A, B \subset J$. 
For every continuous function \( \varphi \) on \( \mathbb{R} \), we consider \( \varphi(W^t_A) = \varphi \circ W^t_A \). If \( A = [s, t] \) we make a choice of \( W^t_A \) such that \( W^t_{[s,t]} = W^t_t - W^t_s \), and we set
\[
S^t_A \varphi(x) = \mathbb{E}_x \left( \varphi \circ W^t_A \right), \quad t \in \mathbb{R}^+.
\]
(3.90)

Since, by (3.89), the process \( W^t \) has independent increments, it follows that \( S^t_A \) is a Markov semigroup.

When \( \varphi \) is given, we set
\[
u(t, x) = \left( S^t(x) \right) \varphi(x) = \mathbb{E}_x \left( \varphi(W^t_0) \right), \quad t \in \mathbb{R}^+.
\]
(3.91)

where the conditional expectation \( \mathbb{E}_x \) corresponds to \( W^t_0 = x \). Then the probability space \( \Omega \) consists of continuous \( \omega \), and
\[
W^t_\omega = \omega(t), \quad 0 \leq t \leq \infty.
\]
(3.92)

We then get the boundary condition \( \nu(0, x) = \varphi(x) \) directly from (3.91).

We shall further consider the operator \( d/d\mu \) acting in the \( t \)-variable. For convenience we shall write \( \nabla^t_\mu \).

We now turn to the corresponding diffusion equation:

**Theorem 3.30.** Let \( \mu, W^t, S^t \), and \( u(t, x) \) be as specified. We then have
\[
\nabla^t_\mu u = \frac{1}{2} \frac{\partial^2}{\partial x^2} u.
\]
(3.93)

**Proof.** Without loss of generality we may assume \( \varphi \in C^2 \). Then by Ito’s lemma, we get
\[
d\varphi \left( W^t \right) = \varphi' \left( W^t \right) dW^t + \frac{1}{2} \varphi'' \left( W^t \right) \mu(\text{d}t).
\]
(3.94)

where \( \varphi'' = (d/dx)^2 \varphi \). For the derivation of (3.94), we refer to the cited papers (e.g., [IM74, KS16, EL93, AJL17]), we also use the familiar quadratic variation formula
\[
QV = \left( dW^t \right)^2 = \mu(\text{d}t).
\]
(3.95)

Note that (3.95) holds since \( \mu \) was assumed non-atomic. Further note that (3.94) refers to Ito-differentials. In general (3.94) is equivalent to the corresponding integral formula version:
\[
\varphi \left( W^t \right) - \varphi \left( W^t_0 \right) = \int_0^t \varphi' \left( W^s \right) dW^s + \frac{1}{2} \int_0^t \varphi'' \left( W^s \right) \mu(\text{d}s).
\]
(3.96)

Now apply the expectation \( \mathbb{E}_x \) to both sides in (3.95) we arrive at
\[
\mathbb{E}_x \left( \varphi \left( W^t \right) \right) - \varphi(x) = \frac{1}{2} \int_0^t \mathbb{E}_x \left( \varphi'' \left( W^s \right) \right) \mu(\text{d}s),
\]
(3.97)
or equivalently:
\[
S^t_A \varphi(x) = \frac{1}{2} \int_0^t \left( \frac{\partial^2}{\partial x^2} u \right)(s, x) \mu(\text{d}s).
\]
(3.98)
From the definition of the operator $T_{(\infty t)}^{(\mu)} = \nabla_{t}^{(\mu)}$ (see Lemma 3.2), we therefore get
\[ \nabla_{t}^{(\mu)} u = \frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} u, \quad (3.99) \]
which is the desired conclusion (3.93) in the Theorem. □

**Proposition 3.31.** Consider the heat equation
\[ \frac{\partial}{\partial t} u(t, x) = K_{F} u(t, x), \quad (t, x) \in \mathbb{R}_{+} \times [0, 1], \quad (3.100) \]
with $K_{F} = \frac{\partial}{\partial \mu} \frac{\partial}{\partial x}$ given a selfadjoint realization in $L^{2}([0, 1], \mu)$. Then the corresponding solution to (3.100) has the following form:
\[ u(t, x) = \sum_{1}^{\infty} e^{-\lambda \lambda_{n}} k_{n}(x), \quad (3.102) \]
where $\lambda_{n}$ are the eigenvalues of $K_{F}$ and $k_{n}$ are the corresponding eigenfunctions.

**Proof.** The argument is based on separation of the two variables $t$ and $x$, and use of spectral data; but now with reference to $K_{F}$ and $\nabla_{\mu}$.

For details about choices of selfadjoint realizations of $K_{F}$, see Corollary 3.28, as well as Remark 3.32 below.

Details as follows: Set
\[ u(t, x) = h(t) k(x). \quad (3.101) \]
Substituting (3.101) into (3.100) leads to
\[ h'(t) k(x) = h(t) \nabla_{x}^{\mu} k'(x), \]
so that
\[ \frac{h'}{h}(t) = \frac{\nabla_{x}^{\mu} k'(x)}{k(x)} = \text{const} = -\lambda. \]
Thus, $h(t) = \text{const} e^{-\lambda \lambda}$, and $\lambda$ is specified by
\[ -\lambda k(x) = \nabla_{x}^{\mu} k'(x). \quad (3.102) \]
Note, the eigenvalue problem (3.102) is equivalent to
\[ h'(t) \int_{0}^{x} k(y) \mu(dy) = h(t) (k'(x) - k'(0)). \quad (3.103) \]

**Remark 3.32.** Solutions to (3.102) depend on choices of boundary conditions, i.e., selfadjoint realizations of the Krein-Feller operator. Two examples are included below:

1. Dirichlet boundary. (For a related discussion, see also Corollary 3.28 above.) Dirichlet conditions: $f(0) = f(1) = 0$. Specifically,
\[ \text{dom} (K_{F}) = \left\{ f \in L^{2}(\mu) : f(x) = \int_{0}^{x} \left( f'(0) + \int_{0}^{s} \varphi(s) \mu(ds) \right) dy, \right. \]
\[ f(1) = 0, \varphi \in L^{2}(\mu) \right\}. \]
In this case, one has

\[(K_F^{-1} \varphi)(x) = \int_0^1 K_{\text{Dirichlet}}(x, s) \varphi(s) \, \mu(ds),\]

where

\[K_{\text{Dirichlet}}(x, s) = \begin{cases} 
(x - 1) s & s \leq x, \\
(x - s - 1) & s \geq x.
\end{cases}\]

(2) \(f(0) = f'(1) = 0\). That is,

\[
\text{dom}(K_F) = \{ f \in L^2(\mu) : f(x) = \int_0^x \left( f'(0) + \int_0^y \varphi(s) \, \mu(ds) \right) dy, \\
f'(1) = 0, \varphi \in L^2(\mu) \}.
\]

Then,

\[(K_F^{-1} \varphi)(x) = -\int_0^1 \min(x, s) \varphi(s) \, \mu(ds).\]

\textbf{Remark 3.33.} Our analysis of \(W(\mu)\) and the associated semigroup is related to what is often referred to as “change of time;” see e.g., [BNS15] and (3.26) in Lemma 3.9.

\textbf{Remark 3.34.} Dym and McKean developed a version of Krein-Feller operators in a context of what they call “strings”, see e.g., [DM72, DM76] and also [Man68]. In principle, there is the following dictionary: string = positive measure \(\mu\) on a finite interval. Reasoning: every positive measure on an interval is a Stieltjes measure by a monotone function, say \(F\). In Dym & McKean, the monotone function \(F\) measures the accumulation of mass as you move forward on the string, and \(\mu = dF\) as a Stieltjes measure. However, Dym & McKean do not seem to distinguish their analysis for the dichotomy: \(\mu\) singular or not. Recall, for the 1/3 Cantor measure, \(\mu = dF\) where \(F\) is the Devil’s staircase function; see Figure 3.2.

The case when \(\mu \ll \text{Lebesgue}\) is covered in many other places, e.g., books and papers by Edward Nelson, e.g., [Nel92, Nel73, Nel67, Nel64].

\textbf{Remark 3.35 (Summary of extension theory for unbounded operators).} There is a theory in the case of unbounded operators in Hilbert space, see e.g., [CP68, Phi61b, Phi11].

Here, we emphasize the correspondence between skew-adjoint operators, and generators of unitary one-parameter groups. In the case when skew-adjoint operators arise as operator extensions, then they are specified by partial isometries. On the other hand, dissipative operators correspond to generators of contraction semigroups; and dissipative extensions are specified by partial contractions.

Generators of unitary one-parameter groups are maximal skew-symmetric extensions. Examples of maximal skew-symmetric extensions that might not be skew-adjoint will be when one of the indices is 0, so the cases (0, \(m\)) or (\(n, 0\)). Generators of contraction semigroups are maximal dissipative extensions (for details, see [DS88].) One can have semigroup generators for the cases (\(n, 0\)). But there are other semigroup generators.

Consider the operator \((d/dx)^2\) in \(L^2([0, 1])\).

- Two particular selfadjoint extensions:
  - Neumann: \(f'(0) = f'(1) = 0\)
  - Dirichlet: \(f(0) = f(1) = 0\)
- Maximal dissipative extension (diffusion semigroups)
Remark 3.36 (Diffusion paths for Brownian motion, and for generalized Brownian motion).

Diffusion paths:

\[ E_x (\varphi(W_t)) = (S_t \varphi)(x) \]

where \( W_t \) is the Brownian motion (and \( W(\mu) \) for the general case).

Two cases:
1. all sample paths \( \omega, \omega(0) = x \)
2. restricted sample paths; e.g., \( \omega(t) \in [0,1] \) or \( \omega(t) \in [-1,1] \).

3.8. Operators and generalized Dirichlet forms.

Lemma 3.37. Let \( \mu \) be a \( \sigma \)-finite positive measure as before, \( (J, \mathcal{B}) \) with Borel \( \sigma \)-algebra \( \mathcal{B} \). Here \( J = [0,1] \) or \( J = [0,\infty) \). Then the following are equivalent:

1. \( f(y) - f(x) = \int_y^x T_\mu f \, d\mu \);
2. \( \int_J \varphi \, df = \int_J \varphi T_\mu f \, d\mu, \ \forall \varphi \in C_c^\infty(J) \);
3. \( df \ll \mu \) and \( \frac{df}{d\mu} = T_\mu f \).

If \( \mathcal{H}(K_\mu) \) is the RKHS of the p.d. kernel \( K_\mu(A, B) := \mu(A \cap B) \), then in (3) we have

\[ \|df\|_{\mathcal{H}(K_\mu)} = \|T_\mu f\|_{L^2(\mu)}. \]  

(3.104)

Proof. In our previous paper (see e.g., [JT17, JT19a]) we studied the RKHS \( \mathcal{H}(K_\mu) \) as a Hilbert space of measures \( \rho \) such that \( \rho \ll \mu \) and \( \frac{d\rho}{d\mu} \in L^2(\mu) \); \( \|\rho\|_{\mathcal{H}(K_\mu)} = \|d\rho/d\mu\|_{L^2(\mu)} \); and so we apply this result to the current setting, with \( \rho = df \) as a Stieltjes measure. We will come back to this point in Section 5.

Details:
(1)⇔(2). We have

\[
\int \varphi df \approx \sum_i \varphi(x_i)(f(x_{i+1}) - f(x_i))
\]

(3.105)

by (1)

\[
= \sum_i \varphi(x_i) \int_{x_i}^{x_{i+1}} T_\mu f d\mu
\]

\[
= \int_J \varphi(T_\mu f) d\mu \quad \text{(standard integral approximation)}.
\]

(2)⇒(3). Rewrite (2) with \(\varphi = \chi_B\), \(B \in \mathcal{B}\), then

\[
\int_B df = \int_B (T_\mu f) d\mu.
\]

(3.106)

Thus \(\mu(B) = 0 \Rightarrow df(B) = 0\), and \(df \ll \mu\) with \(df/d\mu = T_\mu f\) which is (3).

(3)⇒(2) is clear. □

**Corollary 3.38.** From Theorem 3.1 we now get the following dual pair (with dense domains)

\[
T_\mu : L^2(\mu) \longrightarrow L^2(\mu);
\]

and

\[
D = -\frac{d}{dx} : L^2(\mu) \longrightarrow L^2(\mu),
\]

where

\[
L^2(\mu) := \overline{\text{dom}(T_\mu)^{L^2(\lambda)}},
\]

and \(\lambda = d/dx = \) the usual Lebesgue measure restricted to the fixed interval \(J\). Recall, \(\mu\) is assumed supported in \(J\).

We therefore obtain the following two selfadjoint operators:

\[
T_\mu T_\mu^* : L^2(\mu) \longrightarrow L^2(\mu);
\]

(3.107)

and

\[
T_\mu^* T_\mu : L^2(\mu) \longrightarrow L^2(\mu)
\]

(3.108)

with corresponding Dirichlet forms:

\[
\langle \varphi, T_\mu T_\mu^* \varphi \rangle_{L^2(\mu)} = \int_J |\varphi'|^2 d\mu;
\]

(3.109)

and

\[
\langle f, T_\mu^* T_\mu f \rangle_{L^2(\mu)} = \int_J |f(\mu)|^2 dx.
\]

(3.110)

For a given non-atomic measure \(\mu\), we shall refer to the quadratic form (3.109) as the Dirichlet form induced from \(\mu\). It further follows from (3.107) that this Dirichlet form has a selfadjoint, semibounded \((A \geq 0)\) realization, say \(A\), where \(A\) is a selfadjoint extension of our Krein-Feller operator \(K_F\). Here, initially \(K_F\) is considered as a symmetric operator with dense domain in \(L^2(\mu)\). We shall further show that, when the diffusion semigroup is realized in \(L^2(\mu)\), then its infinitesimal generator is \(-A\). It is known from general theory that the Dirichlet form determines the diffusion semigroup; and vice versa. See, e.g., [Kni81, Fel54, FM56, Fuj87], and also Section 3.4.
4. Applications to IFS measures

The present section deals with applications of Krein-Feller operators in the setting of IFS measures. For the operator theoretic framework, see Section 3, especially Corollary 3.38. It is subdivided into two subsections. The first subsection introduces a general class of iterated function system (IFS) measures; while the second specializes to IFS measures with support contained in finite intervals, and with the Krein-Feller operators. Background references for this include [Hut81, Jor18, JP98a, JP98b, Roh49, Roh52, Roh64a, LOSS20, ARCG + 20, QS13, AJL11].

4.1. Iterated function system measures. The theory of iterated function system (IFS) measures is extensive. IFS measures arise in diverse applications, geometric analysis, fractal harmonic analysis, chaotic dynamics and more. Here we wish to cite the following papers of most direct relevance for our current discussion, [JT19a, HJW19b, BJ19, HJW19a, BJ18, JT18, JPT16, JP16, BF21, MP21, WLLZ20, AJL11, Jor18].

Definition 4.1. Let \((X, \mathcal{B}_X)\) be a measurable space, \(N \in \mathbb{N}\), and let \(\{\sigma_i\}_{i=1}^N\) be a system of continuous endomorphisms \(\sigma_i : X \rightarrow X\). Let \(\{p_i\}_{i=1}^N, p_i > 0, \sum_{i=1}^N p_i = 1\) be fixed.

A Borel measure \(\mu\) on \(X\) is said to be an iterate function system (IFS) measure w.r.t. the data iff (Def) the following identity holds:

\[
\mu = \sum_{i=1}^N p_i \mu \circ \sigma_i^{-1}
\]  

on the Borel \(\sigma\)-algebra \(\mathcal{B}_X\).

We now turn to an explicit realization of the IFS-measure from eq. (4.1).

Theorem 4.2. Let \(X, N, \{\sigma_i\}_{i=1}^N, \{p_i\}_{i=1}^N\) be as above. Consider the infinite product

\[
\Omega_N := \prod_{n=0}^\infty \{1, 2, \ldots, N\},
\]  

and suppose, for all \(\omega \in \Omega_N\), the intersection below is a singleton, i.e.,

\[
\bigcap_{n=1}^\infty \sigma_{i_n} \sigma_{i_{n-1}} \cdots \sigma_{i_1} (X) = \{x(\omega)\};
\]  

then there is an associated IFS measure \(\mu\) constructed from the infinite product

\[
\pi := p \times p \times p \times \cdots \text{ on } \Omega_N.
\]

Because of assumption (4.3), we get a well-defined \(X\)-valued random variable \(W\) (for the probability space \((\Omega_N, \pi)\)), and the IFS-measure \(\mu\) from (4.1) is then the distribution of \(W\). See Figure 4.1.
Proof. With (4.3), we define the random variable \( W(\omega) := x(\omega), \omega \in \Omega_N \). If \((i_1, \ldots, i_n) \in \prod^n \{1, \ldots, N\}\), then the measure \( \pi \) is specified on cylinder sets as follows:

\[
\pi([i_1, \ldots, i_n]) = p_{i_1}p_{i_2}\cdots p_{i_n}. \tag{(4.5)}
\]

The measure \( \pi \) is then defined on \( \Omega_N \) via Kolomogorov’s consistency extension theorem, see [Hid80, Kol83, Kol77]. Let \( W : \Omega_N \rightarrow X \) be the random variable specified by the condition in (4.3), and set

\[
\mu := \pi \circ W^{-1}. \tag{(4.7)}
\]

One checks that \( \mu \) will then satisfy the IFS condition in (4.1). \( \square \)

Corollary 4.3. In this corollary we fix a system \( \{\sigma_i\}_{i=1}^N \) of endomorphisms, and we consider the IFS measure \( \mu \) as it depends on the choice of probability weights \( p = (p_i)_{i=1}^N, \sum p_i = 1 \). Set \( \mu^{(p)} \) the solution to (4.1); (see also (4.6)). Then if \( p \neq q \) (i.e., \( \exists i \) such that \( p_i \neq q_i \)) then the two measures \( \mu^{(p)} \) and \( \mu^{(q)} \) are mutually singular.

Proof. The result is immediate from (4.7) and Kakutani’s dichotomy theorem for infinite product measures, applied to (4.4). By Kakutani [Kak43], the two measures \( \times_N p \) and \( \times_N q \) are mutually singular; and by (4.7), so are the two IFS measures \( \mu^{(p)} \) and \( \mu^{(q)} \). \( \square \)

4.2. IFS measures supported on compact intervals. Here we take

\[
\sigma_i(x) := \lambda_i x + b_i \tag{(4.8)}
\]

where \( 0 < \lambda_i < 1, b_i \in \mathbb{R}, x \in \mathbb{R}, 1 \leq i \leq N \). Fix \( \{p_i\}_{i=1}^N \) as above. Then the corresponding IFS measure \( \mu \) (see (4.4)) will satisfy

\[
\sum_{i=1}^N p_i \int f(\lambda_i x + b_i) \mu(dx) = \int f(x) \mu(dx) \tag{(4.9)}
\]
for all bounded continuous functions \( f \) on \( \mathbb{R} \). One checks that \( \mu \) will then be supported on a compact interval \( J \subset \mathbb{R} \).

**Example 4.4** (Three IFS measures, Lebesgue measure and two Cantor measures). Let \( N = 2 \) and \( \{p_i\} = \{\frac{1}{2}, \frac{1}{2}\} \).

\[
\begin{align*}
(1) & \quad \left\{ \begin{array}{l}
\sigma_1(x) = \frac{x}{2} \\
\sigma_2(x) = \frac{x+1}{2}
\end{array} \right. & J = [0, 1], \, \mu = \text{restricted Lebesgue measure.}

(2) & \quad \left\{ \begin{array}{l}
\sigma_1(x) = \frac{x}{3} \\
\sigma_2(x) = \frac{x+2}{3}
\end{array} \right. & J = [0, 1], \, \mu_3 = \text{middle } \frac{1}{3} \text{ Cantor measure; Hausdorff dim } = \ln 2/\ln 3. \\
(\text{See Figure 3.2})

(3) & \quad \left\{ \begin{array}{l}
\sigma_1(x) = \frac{x}{4} \\
\sigma_2(x) = \frac{x+2}{4}
\end{array} \right. & J = [0, 1], \, \mu_4 = \text{Cantor measure with two omitted intervals; Hausdorff dim } = \frac{1}{2}.
\end{align*}
\]

**Remark 4.5.** There is an important difference between the cases (2) and (3) above. Naturally they have different geometries, different Hausdorff dimension, and they are mutually singular. They are both IFS measures, but the most striking difference is their respective harmonic analysis. For the middle fourth Cantor measure \( \mu_4 \) in (3), the corresponding \( L^2(\mu_4) \) admits an orthogonal Fourier series expansion; while the middle third Cantor measure \( \mu_3 \) in (2) does not. Even more striking is the fact that \( L^2(\mu_3) \) does not admit three orthogonal Fourier exponentials.

For this subject, and related, readers are referred to [JP98b, JP98a, Jor18].

**Corollary 4.6.** Consider a measure \( \mu \) specified as in (4.9) above, so it includes the cases (1)–(3) in Example 4.4.

Then for the Dirichlet form (3.109) in Corollary 3.38 we have

\[
\int_J |\phi'|^2 \, d\mu = \sum_{i=1}^N p_i \lambda_i^2 \int \left| \frac{d}{dx} (\varphi (\lambda_i x + b_i)) \right|^2 \mu(dx).
\]

**Cumulative functions \( g_\mu \) for general IFS measures \( \mu \).** Let \( \mu \) be an IFS measure as in Theorem 4.2, and let

\[
g_\mu(x) := \mu([0, x]).
\]

The scale 3 Cantor measure \( \mu_3 \) with \( g_{\mu_3} \) is discussed in Remark 3.11 and Example 4.4, see also Figure 3.2. The scale 4 Cantor measure \( \mu_4 \) with \( g_{\mu_4} \) from Example 4.4 is shown in Figure 4.2 below. Note these two cases both have equal weights \( p = (p_i)_{i=1}^2 = \{1/2, 1/2\} \).
Consider more general IFS measures on $[0,1]$, i.e., the unique solutions $\mu$ to:

$$\mu = p_1\mu \circ \sigma_1^{-1} + p_2\mu \circ \sigma_2^{-1}.$$ 

Define

$$f_0(x) = x\chi_{[0,1]}(x) + \chi_{(1,\infty)}(x),$$

and

$$f_n(x) := p_1f_{n-1}(\sigma_1^{-1}(x)) + p_2f_{n-1}(\sigma_2^{-1}(x)), \quad n \in \mathbb{N}. \quad (4.11)$$

Then

$$\lim_{n \to \infty} f_n(x) = g_{\mu}(x), \quad \forall x \in [0,1]. \quad (4.12)$$

**Example 4.7.** Let $p = (p_i)_{i=1}^2 = \left\{ \frac{1}{3}, \frac{2}{3} \right\}$. An illustration of $g_{\mu}(\cdot)$ for the two cases below is in Figure 4.3.

1. $\sigma_1(x) = \frac{x}{2}$, $\sigma_2(x) = \frac{x+1}{2}$;
2. $\sigma_1(x) = \frac{x}{3}$, $\sigma_2(x) = \frac{x+2}{3}$.

For case (1), we have

$$\left\{ (a_i) \in \prod_{N} \{0,1\}, \frac{1}{2^m} \leq x \right\} \xrightarrow{\mu} \lim_{m \to \infty} \left( \frac{2}{3} \right)^m \left( \frac{1}{2^{v_1}} + \frac{1}{2^{v_2}} + \cdots + \frac{1}{2^{v_m}} \right) = g_{\mu}(x).$$

**Figure 4.2.** $g_{\mu_4}(\cdot)$ the scale 4 Cantor measure as a Stieltjes measure.
Figure 4.3. \( g_\mu(x) \) with \( p = \{ \frac{1}{3}, \frac{2}{3} \} \).

For the related IFS-measures and their cumulative distributions discussed earlier, see Figure 3.2 (Remark 3.11, scale-3 Cantor), Example 4.4, Figure 4.2 (scale-4 Cantor). For these cases, the fair-coin measure is used. And by contrast, Figure 4.3 illustrates a choice of biased Markov chain-weights.

In particular, it follows from Corollary 4.3 above that the measure \( \mu \) in Example 4.7 (1), see Figure 4.3a, is mutually singular with respect to Lebesgue measure \( \lambda \). They are mutually singular despite the fact that both measures, \( \mu \) and \( \lambda \), on the unit-interval, arise from the same pair of maps, \( \{ \sigma_i \} \) by IFS-recursive iteration.

5. Special case (intervals) vs general measure spaces

Observation: For general measure spaces \((X, \mathcal{B}, \mu)\), in an earlier paper, we established a canonical isometry \( T_\mu \) of the RKHS \( (K_\mu) \) onto \( L^2(\mu) \).

In the special case of \( X = \) an interval, and \( \mathcal{B} = \) the Borel sigma-algebra, \( \mu \) a singular non-atomic measure, we also have an operator \( T_\mu \) and it is a special case of the \( T_\mu \) we introduced in our earlier paper on RKHS theory. Background references for this include [JPT16, Zag87, Nel67, Nel73, Nel92, AJL17, Jor18, HJW19b].

Remark 5.1 (Distinction between first order and second order operators). Our KF-Laplacian (second order) has selfadjoint extensions, for example \( T^*T \). As we study the KF-Laplacian with minimal domain, we study its selfadjoint extensions.

Let \((X, \mathcal{B}, \mu)\) be a \( \sigma \)-finite measure space. We then consider the p.d. kernel on \( \mathcal{B}_{fin} \times \mathcal{B}_{fin} \), defined as

\[
K_\mu(A, B) := \mu(A \cap B), \quad A, B \in \mathcal{B}_{fin}
\]

with \( \mathcal{H}(K_\mu) \) being the associated RKHS.
Theorem 5.2. We have

\[ \mathcal{H} (K_\mu) = \left\{ F : F \text{ } \sigma\text{-finite measure on } (X, \mathcal{B}) \text{ s.t.} \right\} \]

\[ F \ll \mu, T_\mu F := \frac{dF}{d\mu}, \| F \|_{\mathcal{H}(K)} = \| \frac{dF}{d\mu} \|_{L^2(\mu)} \right\}. \]  

(5.2)

Proof. We also included proof details for the conclusions (5.2)-(5.3) when \( K = K_\mu \) is specified as in (5.1).

Recall that

\[ \mu (\cdot \cap A) \in \mathcal{H} (K_\mu). \]  

(5.4)

So if \( F \) is a signed measure on \((X, \mathcal{B})\) and \( F \in \mathcal{H} (K_\mu) \), then we assign the inner product

\[ \langle F, \mu (\cdot \cap A) \rangle_{\mathcal{H}(K_\mu)} = F (A), \]  

(5.5)

using the reproducing property of \( \mathcal{H} (K_\mu) \).

From (5.5), \( F \) is a function on \( \mathcal{B} \), and we showed that from the axioms of the RKHS \( \mathcal{H} (K_\mu) \) that \( F (\cdot) \) will be \( \sigma \)-additive, so a signed measure. Specifically, if \( B = \cup_i B_i, B_i \in \mathcal{B}, B_i \cap B_j = \emptyset \) for \( i \neq j \), one has

\[ F (B) = \sum_i F (B_i). \]  

(5.6)

But we also derive the axioms for \( \mathcal{H} (K_\mu) \) as follows:

\[ F \ll \mu \Rightarrow \frac{dF}{d\mu} = \text{Radon Nikodym derivative} \]  

(5.7)

In particular if \( A \in \mathcal{B}_{fin} \) is fixed, then

\[ \mu (\cdot \cap A) \ll \mu \]  

(5.8)

and

\[ \frac{d\mu (\cdot \cap A)}{d\mu} = \chi_A (\cdot). \]  

(5.9)

To see (5.9), note that

\[ \int_B \chi_A (\cdot) d\mu = \mu (A \cap B). \]

Moreover, for all \( F,G \in \mathcal{H} (K_\mu) \), we have

\[ \langle F,G \rangle_{\mathcal{H}(K_\mu)} = \int_X \frac{dF}{d\mu} \frac{dG}{d\mu} d\mu. \]  

(5.10)

The formula (5.10) offers another way to verify (5.5). Indeed,

\[ \text{LHS}_{(5.5)} \overset{\text{by (5.10)}}{=} \int_X \frac{dF}{d\mu} (\cdot) \frac{d\mu (\cdot \cap A)}{d\mu} d\mu \]

\[ \overset{\text{by (5.9)}}{=} \int_X \frac{dF}{d\mu} (\cdot) \chi_A (\cdot) d\mu \]

\[ = \int_A \frac{dF}{d\mu} d\mu \]

\[ = F (A) = \text{RHS}_{(5.5)}. \]

\[ \square \]
**Conclusion.** Fix \((X, \mathcal{B}, \mu)\). Recall: \(\mathcal{H}(K_\mu)\) RKHS of \(K_\mu\), consisting of signed measures \(F\) s.t. \(F \ll \mu\) and \(dF/d\mu \in L^2(\mu)\).

\[
F \in \mathcal{H}(K_\mu) \quad \xrightarrow{T_\mu} \quad L^2(\mu) \quad \xrightarrow{T_\mu^*} \quad T_\mu F = dF/d\mu \in L^2(\mu)
\]

\[
T_\mu^* T_\mu = I_{\mathcal{H}(K_\mu)}, \quad \text{and} \quad T_\mu T_\mu^* = I_{L^2(\mu)}
\]

\[
(T_\mu^* \psi)(A) = \int_A \psi d\mu, \forall A \in \mathcal{B}
\]

If \(F\) is represented as a signed Stieltjes measure, \(F = df\), then \(\nabla^{(\mu)} f = T_\mu F\).

| RKHS; different settings for \(K_\mu (A, B) = \mu (A \cap B)\) |
|---|
| \((X, \mathcal{B})\) general, \(\mu (A \cap B)\) | special measures on \((X, \mathcal{B})\) |
| \(X \subset \mathbb{R}\), so \([0, 1]\) or \([0, \infty]\) etc. \(A = [0, x]\), \(B = [0, y]\), \(K_\mu (A, B) = \mu ([0, x \wedge y])\) | Special case \(\mu = \lambda = dx = \text{Lebesgue measure}, K_\lambda = x \wedge y\) |

**GENERAL** \(F \in \mathcal{H}(K_\mu)\) with \(\sigma\)-finite measure s.t. \(F \ll \mu\), \(T_\mu F = \frac{dF}{d\mu}\)

\(X \subset \mathbb{R}\), \(F = df\) where \(f\) is a bounded variation function on \(\mathbb{R}\); Stieltjes measure

\(df \ll \mu\), \(T_\mu F = \frac{dF}{d\mu}\)

**Table 5.1**
Stieltjes measures

It follows from Theorem 5.2, that elements $F$ in $\mathcal{H}(K_\mu)$ are signed measures on $(X, \mathcal{B})$ s.t. $F \ll \mu$. Set $T_\mu F := \frac{dF}{d\mu}$, then

$$T_\mu : \mathcal{H}(K_\mu) \cong L^2(\mu)$$

is an isometric isomorphism, i.e.,

$$\|dF/d\mu\|_{L^2(\mu)} = \|F\|_{\mathcal{H}(\mu)}.$$  \hfill (5.12)

In the special case of $(X, \mathcal{B}) = (\mathbb{R}, \mathcal{B})$, or $J$ an interval, e.g., $J = [0,1]$ or $J = [0, \infty]$, the general conclusions (5.11)-(5.12) simplify as follows.

All signed measures on $(\mathbb{R}, \mathcal{B})$ have the form $F = df$ for a bounded variation function $f$ on $(X, \mathcal{B})$, i.e.,

$$\int \varphi dF = \int \varphi df$$

as a Stieltjes measure, where

$$df([x,y]) = f(y) - f(x)$$

for intervals and extend to all $B \in \mathcal{B}$. Moreover the operator $T_\mu f = f^\mu$ from before then agree (5.11)-(5.12), i.e.,

$$T_\mu f = \frac{df}{d\mu}$$

since

$$df \in \mathcal{H}(K_\mu) \iff df \ll \mu,$$

so that the Radon-Nikodym derivative $\frac{df}{d\mu}$ in (5.15) is well defined and (see (5.14))

$$f(y) - f(x) = \int_X (T_\mu f) d\mu,$$
which is the form we used before when $T_\mu f = f^{(\mu)}$. And (5.17)$\implies$

$$df (B) = \int_B f^{(\mu)} d\mu$$

(5.18)

where $df$ is the Stieltjes measure, $B \in \mathcal{B}$, and $\mathcal{B}$ a $\sigma$-algebra.

So we have $T_\mu f := f^{(\mu)}$, and (5.18) is the standard extension of $df$ defined first on intervals $df ([x, y]) = f (y) - f (x)$, and then extended to Borel sets $df (B)$. Note that every $\mu$ is a Stieltjes measure $f (x) := \mu ([0, x])$, so of the form $\mu = df$.

6. A Hilbert space of equivalence classes

In the earlier literature, authors typically only focus their analysis on a fixed positive Borel measure $\mu$. This $\mu$ might be compared to Lebesgue measure $\lambda$. When Stieltjes measures $df$ are considered, it will then be relative to just this one measure $\mu$. So when discussing $\nabla^{(\mu) f} = f^{(\mu)}$, then consideration of the equation $df = f^{(\mu)} d\mu$ is really only picking out one component of $df$.

Recall that the family of Stieltjes measures $df$ account for all Borel measures. And, in general, a Stieltjes measure will contain other non-zero components.

The focus in section 6 is the following: When we apply the Jordan decomposition to a fixed Stieltjes measure $df$, then the part of $df$ that is singular w.r.t. $\lambda$ may contain multiple components, chosen in such a way that each of these components is mutually singular w.r.t. the others. The emphasis below is this: We introduce a Hilbert space $\mathcal{H}_{class}$ of “sigma functions”.

Starting with a Stieltjes measure $df$, we may identify its mutually singular components with orthogonal “pieces” in the Hilbert space $\mathcal{H}_{class}$.

We shall consider pairs $(f, \mu)$ where $f$ is a locally-bounded variation function, and $\mu$ is a positive non-atomic Borel measure. Following [Nel69], one checks that the $\sim$ as specified below will be an equivalence relation on pairs:

$$(f_1, \mu_1) \sim (f_2, \mu_2) \iff \exists \nu \text{ (positive Borel measure)}$$

(6.1)

such that $\mu_i \ll \nu$, and

$$\frac{df_1}{d\mu_1} \sqrt{\frac{d\mu_1}{d\nu}} = \frac{df_2}{d\mu_2} \sqrt{\frac{d\mu_2}{d\nu}} \text{ a.e. } \nu.$$  

(6.2)

Moreover the set of such equivalence classes will form a Hilbert space, with

$$\|\text{class } (f, \mu)\|_{\mathcal{H}_{class}}^2 = \int \left| \frac{df}{d\mu} \right|^2 d\mu.$$  

(6.3)

One further checks that orthogonality in the inner product in $\mathcal{H}_{class}$ happens precisely for classes with measures $\mu_1$ and $\mu_2$ which are mutually singular.

**Theorem 6.1.** If $f$ is given, locally of bounded variation. In addition, assume that the sum in (6.4) is finite, so the Stieltjes measure $df$ is in $\mathcal{H}_{class}$. It follows that $\mathcal{H}_{class}$ induces a Hilbert norm as follows:

$$\|f\|_{\mathcal{H}_{class}}^2 = \sum_\mu \int \left| \frac{df}{d\mu} \right|^2 d\mu.$$  

(6.4)

where the sum on the RHS in (6.4) is over all $\mu$ s.t. $df|_{\text{supp}(\mu)} \ll \mu|_{\text{supp}(\mu)}$, and distinct terms in the sum correspond to mutually singular measures $\mu$. 


Proof. The result is immediate from the discussion above, and [Nel69, ch 6], commutative multiplicity theory. When the function $f$ is fixed, one checks from the definition of the equivalence relation (6.1)–(6.2), and an easy calculation, that each of the individual terms on the RHS in (6.4) in the sum-expression only depends on the equivalence class determined by the measures $\mu$ entering into the summation. (Note that the Hilbert space $\mathcal{H}_{\text{class}}$ of equivalence classes is also called the Hilbert space of sigma-functions.)

Remark 6.2. Nelson’s sigma Hilbert space [Nel69] serves as a tool allowing us to make precise the formal assertion for Stieltjes measures:

$$df = \sum_{\mu} \left(\nabla^{(\mu)} f\right) d\mu$$

(6.5)

where the measures $\mu$ in (6.5) are specified as in (6.4). Hence (6.5) is justified when the function $f$ (in (6.5)) yields a finite sum for the RHS in (6.4).

For the Stieltjes measure $df$, we therefore get the following evaluation formula: For all $B \in \mathcal{B}_1$ (= the Borel $\sigma$-algebra), we have

$$df(B) = \sum_{\mu \in \mathcal{H}_+(df)} \int_{B \cap \text{esssup}(\mu)} \left(\nabla^{(\mu)} f\right) d\mu,$$

(6.6)

where “esssup” refers to essential support.

The following example illustrates that there are choices of functions $f$ for which the sum might be infinite.

Example 6.3. Consider $J = [0, 1]$. Let $\mu_n = \text{Lebesgue measure restricted to } \left[\frac{1}{2^n+1}, \frac{1}{2^n}\right]$, $n \in \mathbb{N}$. Let $f(x) = \sqrt{x}$ over $[0, 1]$, so that $f'(s) = \frac{1}{2\sqrt{x}}$. Then,

$$\|f\|^2_{\mathcal{H}_{\text{class}}} \geq \sum_{n=1}^{\infty} \int \left|\frac{df}{d\mu_n}\right|^2 d\mu_n$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \int_{\frac{1}{2^n+1}}^{\frac{1}{2^n}} \frac{1}{x} dx$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} (\ln 2) = \infty.$$ 

References

[AJ12] Daniel Alpay and Palle E. T. Jorgensen, *Stochastic processes induced by singular operators*, Numer. Funct. Anal. Optim. 33 (2012), no. 7-9, 708–735. MR 2966130

[AJ15] Daniel Alpay and Palle Jorgensen, *Spectral theory for Gaussian processes: reproducing kernels, boundaries, and $L^2$-wavelet generators with fractional scales*, Numer. Funct. Anal. Optim. 36 (2015), no. 10, 1239–1285. MR 3402823

[AJL11] Daniel Alpay, Palle Jorgensen, and David Levanony, *A class of Gaussian processes with fractional spectral measures*, J. Funct. Anal. 261 (2011), no. 2, 507–541. MR 2793121

[AJL17] ———, *On the equivalence of probability spaces*, J. Theoret. Probab. 30 (2017), no. 3, 813–841. MR 3687240

[AL08] Antoine Ayache and Werner Linde, *Approximation of Gaussian random fields: general results and optimal wavelet representation of the Lévy fractional motion*, J. Theoret. Probab. 21 (2008), no. 1, 69–96. MR 2384473
Sergio Albeverio and Leonid Nizhnik, *A Schrödinger operator with a δ′-interaction on a Cantor set and Krein-Feller operators*, Math. Nachr. **279** (2006), no. 5-6, 467–476. MR 2213587

P. Alonso Ruiz, Y. Chen, H. Gu, R. S. Strichartz, and Z. Zhou, *Analysis on hybrid fractals*, Commun. Pure Appl. Anal. **19** (2020), no. 1, 47–84. MR 4025934

Simon Baker and Michael Farmer, *Quantitative recurrence properties for self-conformal sets*, Proc. Amer. Math. Soc. **149** (2021), no. 3, 1127–1138. MR 4211868

Sergey Bezuglyi and Palle E. T. Jorgensen, *Transfer operators, endomorphisms, and measurable partitions*, Lecture Notes in Mathematics, vol. 2217, Springer, Cham, 2018. MR 3793614

P. Alfonso Ruiz, Y. Chen, H. Gu, R. S. Strichartz, and Z. Zhou, *Analysis on hybrid fractals*, Commun. Pure Appl. Anal. **19** (2020), no. 1, 47–84. MR 4025934

Simon Baker and Michael Farmer, *Quantitative recurrence properties for self-conformal sets*, Proc. Amer. Math. Soc. **149** (2021), no. 3, 1127–1138. MR 4211868

Sergey Bezuglyi and Palle E. T. Jorgensen, *Transfer operators, endomorphisms, and measurable partitions*, Lecture Notes in Mathematics, vol. 2217, Springer, Cham, 2018. MR 3793614

Ole E. Barndorff-Nielsen and Albert Shiryaev, *Change of time and change of measure*, second ed., Advanced Series on Statistical Science & Applied Probability, vol. 21, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. MR 1242198

Christopher J. Bishop and Yuval Peres, *Fractals in probability and analysis*, Cambridge Studies in Advanced Mathematics, vol. 162, Cambridge University Press, Cambridge, 2017. MR 3616046

Ola Bratteli and Derek W. Robinson, *Operator algebras and quantum statistical mechanics. 1*, second ed., Texts and Monographs in Physics, Springer-Verlag, New York, 1987, C∗- and W∗- algebras, symmetry groups, decomposition of states. MR 887100

Michael G. Crandall and Ralph S. Phillips, *On the extension problem for dissipative operators*, J. Functional Analysis **2** (1968), 147–176. MR 0231220

Dorin Ervin Dutkay and Palle E. T. Jorgensen, *The role of transfer operators and shifts in the study of fractals: encoding-models, analysis and geometry, commutative and non-commutative*, Geometry and analysis of fractals, Springer Proc. Math. Stat., vol. 88, Springer, Heidelberg, 2014, pp. 65–95. MR 3275999

H. Dym and H. P. McKean, *Fourier series and integrals*, Academic Press, New York-London, 1972, Probability and Mathematical Statistics, No. 14. MR 0442564

Uta Freiberg, *Analytical properties of measure geometric Krein-Feller-operators on the real line*, Linear Algebra Appl. **576** (2019), 51–66. MR 3958137
(Palle E.T. Jorgensen) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, U.S.A.
Email address: palle-jorgensen@uiowa.edu

(James F. Tian) Mathematical Reviews, 416 4th Street Ann Arbor, MI 48103-4816, U.S.A.
Email address: jft@ams.org