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Majorana bound states in nanowire-superconductor hybrid systems in periodic magnetic fields

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We study how the shape of a periodic magnetic field affects the presence of Majorana bound states (MBS) in a nanowire-superconductor system. Motivated by the field configurations that can be produced by an array of nanomagnets, we consider spiral fields with an elliptic cross-section and fields with two sinusoidal components. We show that MBS are robust to imperfect helical magnetic fields. In particular, if the amplitude of one component is tuned to the value determined by the superconducting order parameter in the wire, the MBS can exist even if the second component has a much smaller amplitude. We also explore the effect of the chemical potential on the phase diagram. Our analysis is both numerical and analytical, with good agreement between the two methods.

I. INTRODUCTION

Majorana bound states (MBS) have been of great interest for quantum computing over the past two decades due to their non-Abelian statistics and robustness against local perturbations. Different models for creation of MBS have been suggested and studied. One of the models, which has attracted much attention because of its potential experimental feasibility, is a nanowire-superconductor hybrid system. It is constructed from a nanowire with strong spin-orbit interactions in a uniform magnetic field on a superconducting substrate, which induces superconductivity in the nanowire due to the proximity effect. The Hamiltonian for the semiconducting part of this device, i.e. the nanowire with spin orbit interaction and a uniform magnetic field, is related by a unitary transformation to a Hamiltonian for a nanowire with a helical magnetic field and no spin orbit interaction. The presence of MBS in nanowires and carbon nanotubes with helical magnetic fields was studied in Refs. 16 and 32. It was also suggested to create a similar setup with a helical-shaped effective magnetic field via magnetic atoms on top of a superconductor. Non-uniform magnetic fields, created by an array of nanomagnets, can be used to create MBS in a nanowire-superconductor hybrid system. The formation and braiding of MBS via a nanomagnet pattern on a 2D substrate was discussed in Refs. 44 and 45. There are other suggestions for devices with various magnetic field shapes and origins which may host MBS, e.g., Refs. 46–50.

Recent work in Ref. 51 presented detailed modelling of the magnetic field due to an array of nanomagnets acting on a nanowire in a Si heterostructure. As Si is widely used in modern technology, and therefore a material convenient for potential applications, it can be useful for an experimental realization of MBS to study whether certain Si structures can host MBS. Here, we consider a Si nanowire with superconductivity induced by the proximity effect and with nearby nanomagnets that can be made out of Co or SmCo.

In this work, we investigate when a topological superconducting phase in lithographically defined Si nanowires exists. Using parameters that are reasonable for lithographically defined silicon nanowires and magnets (see Sec. II A), we consider 25-nm wide wires with a superconducting gap $\sim 5 \mu$eV and with the magnetic field produced by nanomagnets with strength about 100 mT, see Fig. 1. These conditions are sufficient for a perfect helical magnetic field to produce the topologically non-trivial superconducting phase that supports an MBS with localization length of about 1 $\mu$m. Since an ideal helix is difficult to achieve using micromagnets, we study how different shapes of the magnetic field would affect the presence of an MBS in a Si-based setup. In particular, we consider a spiral magnetic field with an elliptic cross section. For both ideal and non-ideal helical fields, a partial gap opens in the presence of a magnetic field. However in the non-ideal case, a second gap opens that is proportional to the difference between the major and minor axes of the spiral. If the chemical potential is tuned such that it is inside both gaps, the superconductivity and the MBS are both suppressed.

We investigate the phase boundary of the topological superconducting phase as a function of the major and minor axes of the spiral elliptic magnetic field. The boundary between topological and non-topological phases is marked by the vanishing of the superconducting gap around the chemical potential. In the non-topological phase, there are no states below the gap, while in the topological phase two additional states, the MBS localized at the wire edges, develop. The localization length of the MBS decreases quickly as the superconducting gap recovers away from the phase boundary. Thus, we use the existence of two states with eigenenergies below the superconducting gap, together with their localization near edges, as a criterion for the topological phase. We demonstrate that the eigenenergies of the topological state are exponentially suppressed for long enough wires, yielding effective zero-energy modes that are associated with the topological superconducting phase.

MBS develop in the presence of a perfect helical magnetic field when the field magnitude exceeds a threshold value equal to the superconducting order parameter in the wire. A small deformation of the perfect helix is not expected to immediately destroy the MBS. Our analysis demonstrates the robustness of the MBS to relatively strong deformation of the helical field. As we demonstrate below, when one amplitude of the oscillating magnetic field is tuned so that it is about twice the value of the superconducting order parameter in the
The paper is organized as follows. In Sec. II we discuss the experimental constraints on the model parameters for an example system of a nanowire in which the superconductivity is induced by the proximity effect and magnets are patterned lithographically. We then present the Hamiltonian that we analyze in the succeeding sections, II B and II C. We present an analytical derivation of the MBS wave function and the spectrum for a spiral magnetic field with an elliptic cross-section in Sec. III. Sec. IV A presents numerical results for the phase diagrams for the different shapes of the magnetic field. Our conclusions follow in Sec. V.

II. MODEL OF SUPERCONDUCTING NANOWIRE IN PERIODIC MAGNETIC FIELD

A. Estimations of experimental parameters

While the focus of this work is theoretical, it is important to note that the physical regimes are realistic. An example physical system is a silicon nanowire, whose width we estimate below, with superconductivity induced by the proximity effect, either from metals52 or from the superconductivity in a nearby, very highly doped semiconductor region53. Lithographically defined Co and SmCo nanomagnets51,54 deposited nearby give rise to appropriate helical field variations, as shown in Fig. 1.

We now estimate the transverse width of the nanowire and its associated Fermi wavelength. Because the experimental parameters of interest (e.g., the threshold density) are better characterized in two-dimensional (2D) systems than in wires, we will refer to 2D experiments as a starting point. The six-fold valley degeneracy of the conduction band in bulk silicon is lifted by tensile strain or by narrow confinement, leaving just two low-energy valleys to form a quantum device55. The remaining degeneracy is lifted by wavefunction overlap with sharp interfaces, with a valley energy splitting of $\delta_v$. Assuming a parabolic dispersion relation for the two-dimensional electron gas (2DEG), the lower ($l$) and upper ($u$) valley band energies are given by $\varepsilon_l(p) = p^2/2m$ and $\varepsilon_u(p) = p^2/2m + \delta_v$, where $p$ is a two-dimensional quasi-momentum and $m = 1.73 \times 10^{-31}$ kg is the transverse effective electron mass. Here, we choose $\delta_v \lesssim 100$meV as the valley splitting of a typical 2DEG56, although there is some evidence of larger valley splittings in wire geometries, depending on the confinement57.

The Fermi energy $E_F$ should be large enough to allow the nanowire to conduct, where $E_F$ is measured from the bottom of the lower valley in the 2DEG dispersion. Normally the threshold electron density needed for a 2DEG to conduct is smaller than for a wire, since any disorder disrupts the current flow in the one-dimensional case. For a nanowire, we therefore assume an electron density $n_e$ for which the Fermi energy is higher than the maximum value of the disorder potential. The Fermi energy and the electron density are then related by

$$n_e = \frac{2m(2E_F - \delta_v)}{2\pi \hbar^2},$$

assuming spin degeneracy.

To determine the size of the nanowire, we assume a harmonic confinement potential in the transverse direction with a root-mean-square width of the wavefunction, $\sigma_w$, corresponding to an energy level splitting of

$$\hbar \omega_0 = \frac{\hbar^2}{m\sigma_w^2}.$$  

Since the valley degree of freedom represents an unwanted quantum variable, we can suppress the filling of the upper valley band by adjusting the ground-state energy $\hbar \omega_0 / 2$ such that it lies between the highest filled state and the lowest unfilled
To satisfy this constraint, we adopt $\sigma_w = 25$ nm, yielding an upper limit of $n_e = 5.9 \times 10^{10}$ cm$^{-2}$ for the electron density, which as desired is significantly higher than the threshold electron density of a conducting 2DEG, $n_{e,\text{th}} = 2 \times 10^{10}$ cm$^{-2}$, as reported in Ref. 58 for a 100 nm deep Si/SiGe quantum well.

The chemical potential in the wire is counted from the bottom of the one-dimensional conduction channel, such that

$$\mu = E_F - \frac{\hbar^2}{2m\sigma_w^2},$$

with a corresponding Fermi vector of $k_F = \sqrt{2m\mu/\hbar^2}$. Here we choose $\mu \simeq 50$ $\mu$eV, so that the occupation of the higher valley is well suppressed.

Finally, we estimate the values of the magnetic field $B$ and the proximity-induced superconducting gap in the nanowire $\Delta$ that support an MBS. For a perfectly helical magnetic field, the presence of an MBS requires fields with $g\mu_B B > \sqrt{\Delta^2 + \delta \mu^2}$, where the Landé $g$-factor $g \approx 2$ for Si, $\mu_B$ is the Bohr magneton, and $\delta \mu = \mu - \hbar^2 Q^2/2m$ is the detuning of the chemical potential away from the center of the energy gap $(\hbar^2 Q^2/2m)$, caused by a magnetic superlattice with period $\Lambda$ (see Fig. 1) and wavevector $Q = 2\pi/\Lambda$. (Henceforth, we adopt energy units for $B$ by absorbing $g\mu_B$ into its definition.) Because it is difficult to achieve Zeeman splittings in excess of 20 $\mu$eV using nanomagnets, we take $\Delta = 5$ $\mu$eV here. This choice also satisfies the condition $\mu \gg \Delta$, which is necessary for achieving a proximitized superconducting gap in the wire.

B. Magnetic superlattice

In the following two sections, we present the Hamiltonian studied in this work. First, we introduce a periodic magnetic field in the absence of superconductivity. In Sec. II C, we then include the effects of superconductivity.

We consider the Hamiltonian for a single electron in the wire with parabolic energy dispersion in the presence of a magnetic field $\mathbf{B}(r) = \{B_x(r), B_y(r), B_z(r)\}$ that oscillates periodically as a function of the coordinate $r$ along the wire:

$$H_Z = \varepsilon \mathbf{p} + \mathbf{B}(r) \cdot \mathbf{\sigma} - \mu,$$

where $\sigma = \{\sigma_x, \sigma_y, \sigma_z\}$ are Pauli matrices.

The actual magnetic field configuration produced by nanomagnets is complex. Field configurations produced by arrays of bar-shaped nanomagnets as well as electron spectra are calculated in Ref. 51. It was also shown that a special configuration of magnetic fields may improve conditions for the MBS to develop. While the semiconducting part of the device was studied in Ref. 51, we include superconductor pairing term to the Hamiltonian and consider the full setup necessary to obtain MBS. The goal of this paper is to investigate how deviations from the perfect helical magnetic field may affect the topologically non-trivial superconducting phase. For example, the random filed orientations was considered in Ref. 33, and the topological region is different from the one presented in Refs. 11 and 12. As we focus on the field generated by an array of nanomagnets, we use a helical magnetic field with elliptical cross-section, which is very close to the field shape obtained in Ref. 51:

$$B_x = B_{x0} \cos Q r, \quad B_y = B_{y0} \sin Q r, \quad B_z = 0,$$

where $Q = 2\pi/\Lambda$ is the vector of the reciprocal 1D Brave lattice of the magnetic superlattice with period $\Lambda$ and $B_{x0}, B_{y0} \geq 0$. We note, that due to the absence of the spin–orbit interaction in this system, the direction of the wire and magnetic field orientation are completely decoupled. For example, the system properties remain the same regardless of the choice of the magnetic components $x, y$ relative to the wire direction.

For the magnetic field given by Eq. 5, the matrix elements of the magnetic periodic potential are non-zero only for two reciprocal vectors $\pm Q$:

$$W_{\pm Q} = \frac{1}{2} \begin{pmatrix} 0 & B_{x0} \mp B_{y0} \\ B_{x0} \mp B_{y0} & 0 \end{pmatrix}.$$
perlattice has the form
\[
\left( \frac{\hbar^2(k-nQ)^2}{2m} - E(k) \right) c_{k-nQ} + \sum_\pm W_{\pm n} c_{k-(n\pm1)Q} = 0,
\]
where \( n = 0, \pm 1, \pm 2 \ldots \) and only two spinor coefficients \( c_{k-nQ} \) are coupled, which differ from the case of perfect helix, studied in Ref. 31, in the latter case only one off-diagonal element is non-zero. This situation is different from the case of perfect helix, studied in Ref. 31, in the latter case only one off-diagonal element is non-zero and only two spinor coefficients are coupled, which permits to perform the gauge transformation as presented in Ref. 31.

Using Eq. (7), we can determine that there are energy gaps at the edges of the Brillouin zone with magnitudes \( |B_{z0} \pm B_{q0}| \). For an ideal helical field with \( B_{z0} = B_{q0} \), one branch has a large gap \( B_{z0} + B_{q0} \), while the other branch is gapless. However, when \( B_{z0} \neq B_{q0} \), both gaps are nonzero, and there are no states within the energy window \( |B_{z0} - B_{q0}| \) around \( E = \hbar^2 Q^2 / 8m \).

We solve Eq. (7) for the magnetic field period \( \Lambda = 200 \) nm in Si nanowire. The energy bands are shown in Fig. 2(a) for \( B_{z0} = B_{q0} = 10 \) \( \mu \)eV, in Fig. 2(b) for \( B_{z0} = 2B_{q0} = 10 \) \( \mu \)eV, and in Fig. 2(c) for \( B_{z0} = 10 \) \( \mu \)eV, \( B_{q0} = 0 \). For the last case, the magnetic field is non-chiral, the two spin helicities have identical band structures, and no topologically nontrivial phase is supported.

Using parameters from Subsec. II A, we find that \( \Lambda = 200 \) nm is an acceptable scale for nanofabrication and at the same time allows the lifting of the valley degeneracy of the conducting channel in the wire. Indeed, the magnetic structure would require fabrication of pairs of 50 nm wide nanomagnets with opposite magnetizations, see Fig. 1.

We would like to point out that if the magnetic field shape is not exactly elliptic, higher harmonics will be present. However, as we will consider the regions close to the gap around \( k = Q/2 \), the other harmonics will not change the structure of the phase diagram for MBS.

### C. Hamiltonian for superconducting wire

We now add to the Hamiltonian the superconductivity terms that pair electrons with energies above the chemical potential with holes below the chemical potential. This coupling is conveniently represented by the electron creation and annihilation operators in the Nambu space defined by the vector \( \Psi = \{ \psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger \}^T \). The Hamiltonian of the system can be written as
\[
H = \int \bar{\Psi}^\dagger(r) H \Psi(r) dr ,
\]
where the Hamiltonian matrix in the Nambu space is
\[
H = \begin{pmatrix} H_Z & i \Delta \sigma_y \\ -i \Delta \sigma_y & -H_Z \end{pmatrix} ,
\]
where \( r \) is the position along the wire, \( \Delta \) is the superconducting gap, and the single-electron Hamiltonian \( H_Z \) is given by Eq. (4). This Hamiltonian has eigenvectors that are solutions to the Bogolyubov-De Gennes (BdG) equation
\[
H \Psi = E \Psi.
\]

Below we will investigate the eigenenergies and eigenstates of this Hamiltonian by finding solutions of the BdG equation numerically.

The electron wavefunctions can be rewritten in the Majorana basis instead of the Nambu basis by introducing a unitary transformation of the vector \( \Psi \), where the transformation matrix is given by
\[
\Phi = U_M \Psi , \quad U_M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & -i \\ 0 & i & 0 & -i \end{pmatrix} .
\]

The MBS in this basis is represented by a real function with eigenenergy \( E_0 \to 0 \).

### III. ANALYTICAL CHARACTERIZATION OF THE PHASE DIAGRAM

To provide analytic insight into when a magnetic field that does not have an ideal helical form induces MBS, we generalize the procedure described in Ref. 61 to apply to the case of a field with an elliptical cross section. We consider perfect matching between the Fermi momentum of 1D electrons in the wire and the periodicity of the magnetic superlattice by setting the chemical potential \( \mu = \hbar^2 Q^2 / 8m \). We choose \( B_{z0} > 0 \) and \( B_{q0} > 0 \); this restriction is inessential because different signs of these components correspond to different chiralities of the magnetic field. The continuum and long-length limits examined here are expected to be applicable when the period of the magnetic field oscillations is much less than the length of the wire and when \( B_{z0}, B_{q0}, \Delta \ll \mu \).

We represent the electron wave functions as a superposition of left- and right-movers
\[
\psi_\sigma = R_\sigma e^{iQr/2} + L_\sigma e^{-iQr/2} ,
\]
where \( \sigma = \{ \uparrow, \downarrow \} \), and we linearize the energy dispersion in the kinetic energy term:
\[
H_{\text{kin}} = -i \hbar v_F (R_\uparrow \partial_r R_\uparrow - L_\uparrow \partial_r L_\uparrow + R_\downarrow \partial_r L_\uparrow - L_\downarrow \partial_r R_\uparrow) ,
\]
where the Fermi velocity is \( v_F = \hbar Q / 2m \). The hole part of the Hamiltonian can be obtained from the anticommutation relations of the fermion operators and we do not explicitly show it here. We neglect all fast-oscillating terms, assuming that the localization length of the \( R \) and \( L \) functions is much larger than \( 2\pi/Q \). Later we show that this condition indeed holds for our results. The Zeeman term due to the magnetic field is
\[
H_{\text{mag}} = \frac{1}{2} [(B_{z0} - B_{q0})(R_\uparrow L_\downarrow + L_\downarrow R_\uparrow) + (B_{z0} + B_{q0})(L_\uparrow R_\downarrow + R_\downarrow L_\uparrow)] ,
\]
where \( \Phi \) is the superconducting order parameter.
and the superconductivity term
\[ H_{sc} = \Delta(R_{\uparrow}L_{\uparrow} + L_{\uparrow}R_{\uparrow} - R_{\uparrow}L_{\uparrow} - L_{\uparrow}R_{\uparrow} + H.c.). \] (14)

The Hamiltonian can be decoupled into two non-interacting subspaces \( \chi_- = \{R_{\uparrow}, L_{\uparrow}, R^d_{\uparrow}, L^d_{\uparrow}\}^T \) and \( \chi_+ = \{L_{\uparrow}, R_{\uparrow}, L^d_{\uparrow}, R^d_{\uparrow}\}^T \). In these subspaces, the Hamiltonian has the form
\[ H_{\pm} = \pm i\hbar v_F \delta k \sigma_z - \delta \mu \tau_z + B_{\pm} \sigma_x \tau_z - \Delta \sigma_y \tau_y, \] (15)

where \( B_{\pm} = (B_{x0} \pm B_{y0})/2 \) and \( \delta \mu = \mu - \hbar^2 Q^2/8m \) is the mismatch between the chemical potential and the center of energy gap of electron bands of the magnetic superlattice.

The energies of quasiparticle excitations of \( H_\theta \) with \( \theta = \pm \) for perfect matching of the chemical potential, \( \delta \mu = 0 \), are given by
\[ E_\theta^\pm = \sqrt{\hbar^2 v_F^2 \delta k^2 + [B_{\theta} \pm \Delta]^2}, \] (16)

where \( \delta k = k \pm Q/2 \) denotes momenta counted from Fermi points \( \pm Q/2 \) and we consider only non-negative energies. We note that the two subspaces, \( \theta = \pm \), describe two chiralities of electrons in the spiral magnetic field. Without the proximity effect, \( \Delta = 0 \), Eqs. (II) correspond to two branches with a smaller and larger magnetic gaps at the boundary of the Brillouin zone of the magnetic superlattice, see Subsec. II B and Fig. 2.

We observe that the excitation energy vanishes for \( \delta k = 0 \) when
\[ \text{either } B_+ = \Delta \text{ or } B_- = \Delta. \] (17)

These lines, which define the boundaries between topologically trivial and non-trivial superconducting phases in an infinitely long wire, are shown as solid straight lines in Fig. 3. Our next step is to demonstrate that the internal region in the phase diagram indeed supports the MBS.

Away from the lines defined by Eq. (17), there is no zero-energy eigenstate for real \( \delta k \). However, the purely imaginary values of
\[ \delta k_{\alpha,\theta}^\pm = \pm i \frac{\alpha B_{\theta} + \Delta}{\hbar v_F}, \] (18)

might describe zero-energy solutions that exponentially decrease or grow along the nanowire. The \( \pm \) sign in Eq. (18) defines the solutions that decrease or increase as a function of coordinate \( r \) along the wire and would correspond to two states localized at each of the ends of the nanowire, and \( \alpha = \pm \) identifies the sign choice in Eq. (16). Here we focus on the solution that is localized near \( r = 0 \). In this case, we identify only one pair of \( \delta k_{\alpha,\theta=+}^\pm \) and \( \delta k_{\alpha,\theta=-}^\pm \) that satisfy the boundary condition \( \Psi(r = 0) = 0 \).

The general solution can be written as a linear combination of eight terms
\[ \chi(r) = \sum_{\theta=\pm} \sum_{\alpha=\pm} \sum_{\sigma=\pm} \beta_{\alpha,\theta}^\sigma c_{\alpha,\theta}^\sigma \exp(i r \delta k_{\alpha,\theta}^\sigma), \] (19)

of four linearly independent \( 4 \)-component vectors
\[ e_{\theta,+,+}^\pm = \begin{pmatrix} 1 \\ 1 \\ i \\ -i \end{pmatrix}, \quad e_{\theta,+,+}^\pm = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}, \] (20)

\[ e_{\theta,+,+,+}^\pm = \begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix}, \quad e_{\theta,+,+,+}^\pm = \begin{pmatrix} i \\ -1 \\ 1 \\ -i \end{pmatrix}. \]

The proper solution (19) vanishes at \( r = 0 \), so it must contain a pair of terms formed by one of the vectors of Eq. (20) and multiplied by the exponential functions \( \exp(i r \delta k_{\alpha,\theta}^\sigma) \) with different top index \( \sigma \) in \( \delta k_{\alpha,\theta}^\sigma \). At the same time, each term in the pair must decrease as a function of \( r \), i.e. \( \text{Im}(\delta k_{\alpha,\theta}^\sigma) > 0 \). We find that the first pair, \( e_{\theta,+,+}^\pm = e_{\theta,+,+}^\pm \), satisfies these conditions, provided that
\[ B_{x0} < 2\Delta + B_{y0}, \quad B_{x0} + B_{y0} > 2\Delta. \] (21a)

However, addition requirement to the above inequalities is
\[ B_{y0} < 2\Delta + B_{x0}. \] (21b)

Otherwise, another solution with zero energy develops near \( r = 0 \), formed by the pair \( e_{\theta,+,+}^\pm = e_{\theta,+,+}^\pm \) in Eq. (20). Overall,
a non-degenerate solution of Eq. (15) with eigenenergy $E = 0$ and localized near $r = 0$ can exist within the rectangular region shown by bold solid lines in Fig. 3. Since the energy gap vanishes on these lines, we identify the region inside as the topologically non-trivial superconducting phase that supports the MBS. The outside region is the topologically trivial superconducting phase.

We comment on the localization length of the MBS. The length is determined by $\max\{1/|\delta k_n^{n,0}|\}$. The localization length diverges near the phase boundaries, but then saturates to $\hbar v_F/\Delta$ in the center of the topological superconducting phase at $B_{z0} = B_{y0}$. The predictions yielded by this continuum theory for the dependence of the localization length on the superconducting phase.

We now show the results of numerical calculations in which the approximations that enable the analytical calculations in Sec. III are not made, obtaining results similar to those of the analytic model, as shown in Fig. 3. We now consider a finite-length wire. We calculate the eigenvalues and eigenstates of a discretized version of Eq. (8a) to determine the energy gap and identify the MBS. We rewrite the Hamiltonian representing the second derivative as a finite difference of the wave function $\Psi_n = \Psi(n\delta r)$ for a set of $n$ points separated by a discretization distance $\delta r$ along the wire; the total number of sites along the wire is $N = L/\delta r$. The full Hamiltonian is given by the $4N \times 4N$ matrix

$$
\hat{H} = \begin{pmatrix}
\hat{T} & \hat{K}_{n-1} & \cdots & \hat{K}_0 & \hat{K}_1 & \cdots \\
\hat{K}_{n-1} & \hat{T} & \hat{K}_{n-1} & \cdots \\
\vdots & \hat{K}_{n-1} & \hat{T} & \hat{K}_{n-1} & \cdots \\
\hat{K}_0 & \hat{K}_1 & \cdots & \hat{T} & \hat{K}_{n-1} & \cdots \\
\hat{K}_1 & \hat{K}_2 & \cdots & \hat{K}_{n-1} & \hat{T} & \\
\vdots & \vdots & \ddots & \hat{K}_{n-1} & \hat{T} & \\
\hat{K}_{n-1} & \cdots & \hat{K}_1 & \hat{K}_0 & \hat{K}_1 & \cdots \\
\end{pmatrix},
$$

(22)

where the diagonal blocks are

$$
\hat{K}_n = \begin{pmatrix}
2\hat{T} - \mu + B_z^n & B_x^n - iB_y^n & 0 & \Delta \\
B_x^n + iB_y^n & 2\hat{T} - \mu - B_z^n & -\Delta & 0 \\
0 & -\Delta & 0 & -2\hat{T} + \mu - B_z^n \\
\Delta & 0 & -B_x^n + iB_y^n & -2\hat{T} + \mu + B_z^n
\end{pmatrix},
$$

(23)

and the off-diagonal blocks are

$$
\hat{T} = \begin{pmatrix}
-\hat{T} & 0 & 0 & 0 \\
0 & -\hat{T} & 0 & 0 \\
0 & 0 & \hat{T} & 0 \\
0 & 0 & 0 & \hat{T}
\end{pmatrix}
$$

(24)

The $\hat{T}$ terms are given by

$$
\hat{T} = \frac{\hbar^2}{2m\delta r^2}
$$

(25)

and originate from the discretized kinetic energy

$$
-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(r)}{\partial r^2} \rightarrow -\hat{T} (\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}).
$$

(26)

The $B^n_{x,y,z}$ terms in Eq. (23) are components of the magnetic field $\mathbf{B}^n = B(n\delta r)$ at site $n$. To be specific, we assume that the wire length $L$ is a multiple of the magnetic period $\Lambda$. We implement the boundary conditions $\Psi(r = 0) = \Psi(r = L) = 0$.

We diagonalize the discretized Hamiltonian and obtain the energy eigenvalues and eigenstates. The MBS, if present in the superconducting nanowire, is a non-degenerate state with energy in the middle of the superconducting gap that is zero in the limit of an infinite length wire and that is spatially localized at the ends of the nanowire. We use these conditions to build phase diagrams for our setup for different amplitudes of the magnetic field components. The eigenstates of the discretized Hamiltonian Eq. (22) are $4N$ vectors, where 4 elements in each of $N$ blocks represent the four components of the electron wavefunction in the Nambu space.

We discuss the precise criteria for how we define zero energy and the gap energy in the numerical calculation for a fi-
nate system and the definition of the localization length in the following subsections of this section.

B. Energy gap and zero-energy excitations

Using the numerical method described above, we analyze the low energy eigenstates of the Hamiltonian Eq. (8a). For the results shown, the discretization length used in the numerical calculations is $\delta r = 20$ nm unless stated otherwise. This value of $\delta r$ satisfies $\delta r \ll \lambda$ and $\delta r \ll \hbar v_F/\Delta$, and we have checked that changing the value of $\delta r$ does not change the numerical results significantly.

To identify the phase transition and the development of the MBS, it is sufficient to focus only on the behavior of the lowest energy eigenstates of the Hamiltonian Eq. (8a). For $\delta r$ checked that changing the value of $\delta r$ does not change the numerical results significantly.

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FIG. 4. Comparison of two lowest-energy excitation energies \( E_0 \) and \( E_1 \) in non-topological and topological phases as a function of magnetic field magnitude. (a): Energies \( E_0 \) and \( E_1 \) versus the magnetic field amplitude \( B_{x,0} = B_{y,0} \) in wires evaluated for different wire lengths, solid lines: \( L = 40 \mu m \) and \( \delta r = 20\mu m \); dashed lines: \( L = 80 \mu m \) and \( \delta r = 40\mu m \). In the topologically nontrivial phase, \( E_0 \) approaches zero for when \( B_{x,0} = B_{y,0} \) is large. The second excitation energy \( E_1 \) reaches its minimum at the phase transition, where the gap in the density of states closes in the infinitely long wire, and then increases as the magnetic field is increased further. (b): Energies of the lowest two excitations as a function of one component of the field, \( B_{x,0} = B \) (solid lines) or \( B_{y,0} = B \) (dashed lines), while the other component is strictly zero, \( B_{y,0} = 0 \) or \( B_{x,0} = 0 \), respectively. When one field component is zero, the magnetic field has no helicity and no topologically protected state can form. All the excitation energies show minima at \( B \approx 2\Delta = 10 \mu eV \) and then all energies increase as \( B \) is increased further. When \( B_{x,0} = 0 \) and the magnetic field is zero at the ends of the wire, \( E_0 = E_1 \), but the energies \( E_0 \) and \( E_1 \) split near \( B_{x,0} = 2\Delta \) when the field at the ends is not zero \( (B_{y,0} = 0 \) case). (c): Energies of the lowest energy excitations as a function of one component of the magnetic field when the other component is fixed at twice the superconducting gap \( \Delta \), either \( B_{x,0} = B \), \( B_{y,0} = 2\Delta \) or \( B_{x,0} = B \), \( B_{y,0} = 2\Delta \). For both cases, the energy of the lowest-energy state monotonically decreases towards zero as \( B \) increases, while all other excitation energies increase as the superconducting gap opens, consistent with a topologically nontrivial phase. For all these plots, the proximity-induced superconducting order parameter \( \Delta = 5 \mu eV \) and the chemical potential \( \mu = h^2Q^2/2m = 49.6 \mu eV \) is matched to the middle of the magnetic superlattice gap. For panels (b) and (c), the wire length is \( L = 40 \mu m \).

As \( E_0 \propto \exp(-L/\xi) \). The behavior of the localization length for helical fields with elliptical cross-section is qualitatively similar to that found for purely helical fields.

Figure 5(a) shows a semilog scale the lowest energy as a function of \( B_{x,0} = B_{y,0} \), corresponding to a perfectly helical field.\(^{84}\) For \( B_{x,0} = B_{y,0} = B_0 > \Delta \) the lowest excitation eigenenergy \( E_0 > 0 \) decreases exponentially with \( B_0 \), as demonstrated in Fig. 5(a). To demonstrate the dependence on wire length, we show the energy versus \( B_0 \) for wires with length \( L = 20 \) (solid line), \( L = 40 \), (dashed line) and \( L = 80 \mu m \) (dashed-dotted line) and compare the result with exponential fit \( \propto \exp(-L/\xi) \), where \( \xi \) is evaluated from Eq. (27) with numerical values of \( E_1 \) \((E_1 \) is presented in Fig. 4(a) for \( L = 40 \) and \( 80\mu m \).\)

Figure 5(b) examines the case of an elliptical helical field: it is a semilog plot of the three lowest excitation energies as a function of one field component (either \( B_{x,0} \) or \( B_{y,0} \)) as the other is held fixed at \( 2\Delta \). As the variable magnetic field component is increased from zero, the lowest energy \( E_0 \) decreases towards zero. The lowest energy \( E_0 \) again decreases exponentially with the wire length \( L \propto \exp(-L/\xi) \) with \( \xi \) evaluated from Eq. (27) with \( E_1 \) shown in Fig. 4(c) by a dashed line for \( B_{x,0} = 2\Delta \) and variable \( B_{y,0} \).

We also calculate the localization length by examining the wavefunction of the lowest energy excitation (the eigenvector with energy eigenvalue \( E_0 \)). To characterize the localization length, we define the following integral expression that effectively evaluates the distance of the ‘center-of-mass’ of the MBS wave function from the wire ends:

\[
\zeta = 2\delta r \left( \sum_{n=0}^{N/2} nP_n + \sum_{n=N/2}^{N} (N-n)P_n \right),
\]

(28)

where \( P_n \) is the probability density for the MBS at site \( n \):

\[
P_n = \Phi_n^\dagger \Phi_n, \quad \Phi_n = \Phi(n\delta r),
\]

(29)

and \( \Phi(r) \) is obtained from the state \( \Psi(r) \) corresponding to the lowest positive eigenenergy of Eq. (9) via transformation (10).

The dependence of the localization length \( \zeta \) on the magnetic field is shown in Fig. 5(c) a perfect helical field as a function of field magnitude and in Fig. 5(d) for the case where one field component is fixed at \( 2\Delta \) and the magnitude of the other component is varied. We calculate the localization length using Eq. (28) for several values of the nanowire length. At \( B_{x,0} = B_{y,0} < \Delta \), evaluated values of the localization length \( \zeta \) is comparable to the length of the wire. At larger fields, \( \zeta \) decreases rapidly and reaches the value \( \zeta \approx 2\Delta/2 \), where \( \xi_{sc} = h^2Q/(2m\Delta) \approx 1.26 \mu m \) the superconducting coherence length that determines \( |\Psi_n|^2 \propto \exp(-2n\delta r/\xi) \). Figs. 5(c) and 5(d) also show that the estimate of the localization length using the dependence of the coherence length on \( E_1 \) agrees well with the estimate of the MBS localization length done using Eq. (28).
D. Dependence of the phase diagram on chemical potential

We now investigate the phase diagram when the chemical potential \( \mu \) is not located in the middle of energy spectrum gap of magnetic superlattice, so that \( \mu = \hbar^2 Q^2/8m + \delta \mu \) with \( \delta \mu \neq 0 \).

It is straightforward to extend the analytic theory developed in Sec. III to the case in which \( \delta \mu \neq 0 \). The excitation energies of Hamiltonian (15) are

\[
E_\theta^\pm = \sqrt{B_0^2 + \Delta^2 + \delta \mu^2 + v_F^2 \delta k^2} \pm 2 D, \tag{30}
\]

with

\[
D = \sqrt{B_0^2(\Delta^2 + \delta \mu^2) + \delta \mu^2 v_F^2 \delta k^2}. \tag{31}
\]

The lowest energy for real \( \delta k \) is achieved for \( \delta k = 0 \) and is given by

\[
E_\theta = |B_\theta - \sqrt{\Delta^2 + \delta \mu^2}|, \tag{32}
\]

and the gap closes for \( B_\theta = \sqrt{\Delta^2 + \delta \mu^2} \). This expression is similar to the condition for the point of the phase transition in perfect helical magnetic field\(^{31,32} \). For fixed \( \Delta \) and \( \delta \mu \), the gap closes when

\[
B_\theta = (B_{x0} + B_{y0})/2 > \sqrt{\Delta^2 + \delta \mu^2}. \tag{33a}
\]

This condition specifies the magnitude of the magnetic field necessary to develop the topologically nontrivial superconducting phase.\(^{11,12,16} \) The second condition for the topological phase is determined by ellipticity of the spiral magnetic
field that limits the relative mismatch between $B_{x0}$ and $B_{y0}$ components. At non-zero $\delta \mu$, the corresponding condition is

$$|B_-| < \sqrt{\Delta^2 + \delta \mu^2},$$

with $B_- = (B_{x0} - B_{y0})/2$. We notice that Eq. (33b) implies that non-zero $\delta \mu$ makes the system more robust to imperfect helical magnetic fields, but at the same time, Eq. (33a) implies that stronger fields are needed to reach the topologically nontrivial phase. The conditions for the existence of the MBS at $\delta \mu = 0$ can be interpreted as requiring that one out of the two electron bands has a gap in the energy interval $\hbar^2Q^2/8m \pm \Delta$ while the other does not. When $\delta \mu \neq 0$, we find that a similar condition applies. If at the Fermi momentum one of the bands is gapped while the other is not, then a topologically nontrivial phase can be supported. The mismatch of the Fermi momentum with the points of 1D Brillouin zone correspond to the superconducting excitation with energy $\hbar^2Q^2/8m \pm \sqrt{\Delta^2 + \delta \mu^2}$, and the MBS exists if these energies cross one and only one band of electrons in the magnetic superlattice, see Fig. 2.

We also performed numerical investigations of systems with $\delta \mu \neq 0$ for wires of finite length. Figure 6 shows the results for a phase diagram obtained for a wire with length $L = 20 \, \mu$m by showing the energy of the lowest-energy state as a function of the magnitudes of the magnetic field components. We define the regions of field in which the system is in a topologically nontrivial phase and MBS are supported to be those where the lowest-energy state has energy that is much less than that of the superconducting gap. A comparison of the results in Fig. 6 with those in Fig. 3 demonstrate that tuning the chemical potential of the nanowire can play an important role in optimizing the robustness of MBS.

V. CONCLUSIONS

Motivated by the possibility of introducing strong artificial spin-orbit coupling in nanowires fabricated in silicon, we have considered a nanowire-superconductor hybrid structure with a non-uniform magnetic field and studied the conditions for which the superconductivity is topologically nontrivial and MBS appear. We have investigated both analytically and numerically the case of a spiral magnetic field with an elliptical cross-section, which becomes helical for a round cross-section. This spatial dependence is similar to the magnetic field configurations obtained in Ref. 51, which were achieved using nanomagnet arrays compatible with current lithographic techniques. Here, we have shown that this system can support MBS even when the magnitudes of the two components of the helical field are substantially different.

The robustness of topological superconductivity to ellipticity of the helical magnetic field depends strongly on the magnitude of the dominant field component. If the magnitude of this component is optimized, by making its value twice the superconducting pairing energy $\Delta$, then a topological phase appears, even when the other magnetic field component is small. Analytic theory for an infinite length wire with a linearized electronic spectrum provides an excellent guide for interpreting results obtained numerically for a discretized model using finite-length wires.

We have also investigated the localization length of the MBS. The dependence of the energy on wire length, for the lowest energy excitation, is consistent with a simple picture in which the energy is proportional to the overlap of two exponentially localized states at the ends of the wire.

It was shown theoretically that electron interactions can influence significantly the spectrum of the system and even close the gap\textsuperscript{62-64}. However as was shown in Refs. 62 and 63, depending on the parameters, the gap can be reduced, but still open, with a longer MBS localization length. In such a case, the nanowire must be longer than for the non-interacting case, in order to prevent the MBS from interacting with each other. As the helical magnetic field can be unitarily transformed into spin orbit interaction and uniform magnetic field\textsuperscript{31}, we believe that our system will also have such regime. Our understanding is further supported by the results of Ref. 65, where the effects of electron interactions were studied in a system with spatially-varying spin orbit interaction, which has a similar form to our rotating magnetic field.

Our results provide evidence that using lithographically patterned micromagnets is a viable method for creating spatially-dependent magnetic fields that, together with proximity-induced superconductivity, can be used to generate MBS. Because intrinsic spin-orbit coupling is not required,
many materials systems could also be suitable hosts for MBS, in addition to the silicon wires considered here.

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1 C. Nayak, S. H. Simon, A. Stern, M. Freedman, S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
2 J. Alicea, Rep. Prog. Phys. 75, 076501 (2012).
3 A. Y. Kitaev, Phys. Usp. 44, 131 (2001).
4 L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
5 Y. Tanaka, T. Yokoyama, and N. Nagaosa, Phys. Rev. Lett. 103, 107002 (2009).
6 F. Wilczek, Nature Phys. 5, 614 (2009).
7 A. R. Akhmerov, J. Nilsson, and C. W. J. Beenakker, Phys. Rev. Lett. 102, 216404 (2009).
8 M. Franz, Physics 3, 24 (2010).
9 A. Stern, Nature (London) 464, 187 (2010).
10 J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. 104, 040502 (2010).
11 R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
12 Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
13 J. Alicea, Phys. Rev. B 81, 125318 (2010).
14 K. Flensberg, Phys. Rev. B 82, 180516(R) (2010).
15 M. Duckheim and P. W. Brouwer, Phys. Rev. B 83, 054513 (2011).
16 J. Klinovaja, P. Stano, and D. Loss, Phys. Rev. Lett. 109, 236801 (2012).
17 M. Kjaergaard, K. Wölms, and K. Flensberg, Phys. Rev. B 85, 020503(R) (2012).
18 S. Turcotte, S. Boutin, J. C. Lemyre, I. Garate, and M. Pioro-Ladrière, arXiv:1904.06275 (2019).
19 Y. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
20 M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. 12, 6414 (2012).
21 A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nature Phys. 8, 887 (2012).
22 L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nat. Phys. 8, 795 (2012).
23 A. D. K. Finck, D. J. Van Harlingen, P. K. Mohseni, K. Jung, and X. Li, Phys. Rev. Lett. 110, 126406 (2013).
24 H. O. H. Churchill, V. Fatemi, K. Grove-Rasmussen, M. T. Deng, P. Caroff, H. Q. Xu, and C. M. Marcus, Phys. Rev. B 87, 241401(R) (2013).
25 S. Nadji-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, A. Yazdani, Science 346, 602 (2014).
26 A. P. Higginbotham, S. M. Albrecht, G. Kiršanskas, W. Chang, F. Kuemmeth, P. Krogstrup, T. S. Jespersen, J. Nygård, K. Flensberg, and C. M. Marcus, Nature Phys. 11, 1017 (2015).
27 S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Nature 531, 206 (2016).
28 M. T. Deng, S. Vaitiekėnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, Science 354, 1557 (2016).
29 H. Zhang, Ö. Gül, S. Conesa-Boj, M. P. Nowak, M. Wimmer, K. Zuo, V. Mourik, F. K. de Vries, J. van Veen, M. W. A. de Moor, J. D. S. Bommer, D. J. van Woerkom, D. Car, S. R. Plissard, E. P. A. M. Bakkers, M. Quintero-Pérez, M. C. Cassidy, S. Koelling, S. Goswami, K. Watanabe, T. Taniguchi, and L. P. Kouwenhoven, Nat. Commun. 8, 16025 (2017).
30 H. Zhang, C.-X. Liu, S. Gazibegovic, D. Xu, J. A. Logan, G. Wang, N. van Loo, J. D. S. Bommer, M. W. A. de Moor, D. Car, R. L. M. Op het Veld, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, M. Pendlhart, D. J. Pennachio, B. Shojaei, J. S. Lee, C. J. Palmström, E. P. A. M. Bakkers, S. Das Sarma, and L. P. Kouwenhoven, Nature 556, 74 (2018).
31 B. Braunecker, G. I. Japaridze, J. Klinovaja, and D. Loss, Phys. Rev. B 82, 045127 (2010).
32 R. Egger and K. Flensberg, Phys. Rev. B 85, 235462 (2012).
33 T. P. Choy, J. M. Edge, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 84, 195442 (2011).
34 M. M. Vazifeh and M. Franz, Phys. Rev. Lett. 111, 206802 (2013).
35 B. Braunecker and P. Simon, Phys. Rev. Lett. 111, 147202 (2013).
36 S. Nadji-Perge, I. K. Drozdov, B. A. Bernevig, and A. Yazdani, Phys. Rev. B 88, 020407(R) (2013).
37 F. Pientka, L. I. Glazman, and F. von Oppen, Phys. Rev. B 88, 155420 (2013).
38 J. Klinovaja, P. Stano, A. Yazdani, and D. Loss, Phys. Rev. Lett. 111, 186805 (2013).
39 M. Ruby, F. Pientka, Y. Peng, F. von Oppen, B. W. Heinrich, and K. J. Franke, Phys. Rev. Lett. 115, 197204 (2015).
40 M. Schecter, M. S. Rudner, and K. Flensberg, Phys. Rev. Lett. 114, 247205 (2015).
41 W. Hu, R. T. Scalettar, and R. R. P. Singh, Phys. Rev. B 92, 115133 (2015).
42 B. Braunecker and P. Simon, Phys. Rev. B 92, 241410(R) (2015).
43 R. Pawlak, M. Kisiel, J. Klinovaja, T. Meier, S. Kawai, T. Glatzel, D. Loss, and E. Meyer, npj Quantum Information 2, 16035 (2016).
44 G. L. Fatin, A. Matos-Abiague, B. Scharf, and I. Žutić, Phys. Rev. Lett. 117, 077002 (2016).
45 A. Matos-Abiague, J. Shabani, A. D. Kent, G. L. Fatin, B. Scharf, and I. Žutić, Solid State Commun. 262, 1 (2017).
46 T. Ojanen, Phys. Rev. B 88, 220502(R) (2013).
47 N. Sedlmayr, J. M. Aguilar-Hualde, and C. Bena, Phys. Rev. B 91, 115415 (2015).
48 C.-H. Hsu, P. Stano, J. Klinovaja, and D. Loss, Phys. Rev. B 92, 235435 (2015).
49 S. Boutin, J. C. Lemyre, and I. Garate, Phys. Rev. B 98, 214512
50 M. M. Desjardins, L. C. Contamin, M. R. Delbecq, M. C. Dartiailh, L. E. Bruhat, T. Cubaynes, J. J. Viennot, F. Mallet, S. Rohart, A. Thiaville, A. Cottet, and T. Kontos, Nat. Mater. 18, 1060 (2019).
51 L. N. Maurer, J. K. Gamble, L. Tracy, S. Eley, and T. M. Lu, Phys. Rev. Applied 10, 054071 (2018).
52 P. Kroghstrup, N. L. B. Ziino, W. Chang, S. M. Albrecht, M. H. Madsen, E. Johnson, J. Nygård, C. M. Marcus, T. S. Jespersen, Nat. Mater. 14, 400 (2015).
53 E. Bustarret, C. Marcenat, P. Achatz, J. Kačmarčík, F. Lévy, A. Huxley, L. Ortégá, E. Bourgeois, X. Blase, D. Débarre, and J. Boulmer, Nature 444, 465 (2006).
54 K. R. Sapkota, S. Eley, E. Bussmann, C. T. Harris, L. N. Maurer, and T. M. Lu, AIP Advances 9, 075203 (2019).
55 M. Friesen, S. Chutia, C. Tahan, and S. N. Coppersmith, Phys. Rev. B 75, 115318 (2007).
56 J. K. Gamble, P. Harvey-Collard, N. T. Jacobson, A. D. Baczewski, E. Nielsen, L. Maurer, I. Montaño, M. Rudolph, M. S. Carroll, C. H. Yang, A. Rossi, A. S. Dzurak, and R. P. Muller, Appl. Phys. Lett. 109, 253101 (2016).
57 D. J. Ibberson, L. Bourdet, J. C. Abadillo-Urriel, I. Ahmed, S. Barraud, M. J. Calderón, Y.-M. Niquet, and M. F. Gonzalez-Zalba, Appl. Phys. Lett. 113, 053104 (2018).
58 D. Laroche, S.-H. Huang, E. Nielsen, Y. Chuang, J.-Y. Li, C. W. Liu, and T. M. Lu, AIP Advances 5, 107106 (2015).
59 N. W. Ashcroft and N. D. Mermin, Solid State Physics, (Brooks/Cole, Pacific Grove, CA, 1976).
60 P. G. de Gennes, Superconductivity of Metals and Alloys (W. A. Benjamin, New York, 1966).
61 J. Klinovaja and D. Loss, Phys. Rev. B 86, 085408 (2012).
62 S. Gangadharaiah, B. Braunecker, P. Simon, and D. Loss, Phys. Rev. Lett. 107, 036801 (2011).
63 E. M. Stoudenmire, J. Alicea, O. A. Starykh, and M. P. A. Fisher, Phys. Rev. B 84, 014503 (2011).
64 E. Sela, A. Altland, A. Rosch, Phys. Rev. B 84, 085114 (2011).
65 G. I. Japaridze, H. Johannesson, and M. Malard, Phys. Rev. B 89, 201403(R) (2014).