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The discrete spectrum of Jacobi matrix
related to recurrence relations
with periodic coefficients

Dedicated to Petr Kulish
in connection with the seventieth anniversary.

1 Introduction

The interest in the study of orthogonal polynomials defined by recurrence relations with periodic and asymptotically periodic coefficients increased after the appearance of the article [1] about the properties of the periodic Toda lattices (see, for example, [2]-[13]). In particular, in work [14] recurrence relation with periodic coefficients are investigated. It was shown that such polynomials can be described by the classical Chebyshev polynomials. The aim of this work is to study the discrete spectrum of the Jacobi matrix, connected with polynomials in this class, i.e. polynomials with periodic coefficients in recurrent relations. As an example, we consider:

a) the case when period $N$ of coefficients in recurrence relations equals to three (as a particular case we consider "parametric" Chebyshev polynomials [18]);

b) the elementary $N$-symmetrical Chebyshev polynomials ($N = 3, 4, 5$), that was introduced by authors in studying the "composite model of generalized oscillator" [15].

Let us remind some necessary results from [14]. We denote by $\{ϕ_n(x)\}_{n=0}^{∞}$ the polynomial sequence defined by recurrence relations

$$ϕ_n(x) = (x + a_{n-1})ϕ_{n-1}(x) - b_{n-1}ϕ_{n-2}(x), \quad n ≥ 1, \quad ϕ_0(x) = 1, \quad ϕ_{-1}(x) = 0,$$

where the coefficients are periodic with period $N ≥ 2$:

$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n ≥ 0.$$

We will use the Chebyshev polynomials of the second kind defined by recurrence relations

$$tU_n(t) = U_{n+1}(t) + U_{n-1}(t), \quad n ≥ 0, \quad U_0(t) = 1, U_{-1}(t) = 0.$$

It was proved in [14] that for any $N ≥ 1$ the polynomial $ϕ_{N-1}(x)$ divides the polynomial $ϕ_{2N-1}(x)$

$$ϕ_{2N-1}(x) = ϕ_{N-1}(x)P_N(x),$$

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where the polynomial $P_N(x)$ defined from equality (4). Besides, recurrence relations
\[ \varphi_n(x) = \varphi_{Nn+k}(x) = \varphi_{k+N}(x)U_{m-1}(P_N(x)) - \varphi_k(x)U_{m-2}(P_N(x)). \] (5)
are fulfilled for $n =Nm+k, k = 0; (N - 1), m \geq 2$. The Jacobi matrix $J = \begin{bmatrix} j_{i,k}^{(N)} \end{bmatrix}_{i,k=0}^{\infty}$ related to recurrence relations (1) one can write in following form
\[ j_{i,k}^{(N)} = A\delta_{i+1,k} + B\delta_{i,k} + C\delta_{i-1,k}, \] (6)
where the matrixes
\[ A = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \ldots & 0 & b_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \]
\[ B = \begin{pmatrix} -a_0 & 1 & 0 & \ldots & 0 & 0 \\ b_0 & -a_1 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -a_{N-2} & 1 \\ 0 & 0 & 0 & \ldots & b_{N-1} & -a_{N-1} \end{pmatrix}, \] (7)
have the size $N \times N$. Let $X_\mu = (x_1, x_2, \ldots, x_n, \ldots)^t \in \ell^2$ be an eigenvector for matrix $J$, corresponding to eigenvalue $\mu$:
\[ (J - \mu I)X_\mu = 0. \] (8)
The following necessary and sufficient condition is hold: a solution $\mu$ of the equation (5) is an eigenvalue of matrix $J$ if and only if when
\[ \|X_\mu\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=0}^{\infty} |\varphi_n(\mu)|^2 < \infty \] (9)
In the next paragraph we will obtain the "critical equation" for Jacobi matrix $J$. A solution of the equation (9) is called the "critical value" of Jacobi matrix $J$. The necessary condition for $\mu$ to be an eigenvalue of matrix $J$ is that $\mu$ be a "critical value" of $J$.

2 The critical equation for Jacobi matrix $J$

Let us introduce the notation which was needed in the following:
\[ \sigma_{k,k+m} = \sum_{n=k}^{k+m} \varphi_n^2(\mu), \quad S_1 = \sigma_{N,2N-1}, \quad S_2 = \sigma_{0,N-1}, \quad S = S_1 + S_2; \] (10a)
\[ S_n^N(\mu) = \sigma_{n,n+2N-1}, \quad D_n^N(\mu) = \sum_{k=n}^{n+N-1} \varphi_k(\mu)\varphi_{k+N}(\mu) \] (10b)
\[ \Delta_n^N(\mu) = S_n^N(\mu) - P_n(\mu)D_n^N(\mu), \quad n \geq 0 \] (10c)
It is clear that $S = S_0^N(\mu)$. The following statements are hold:

**Lemma 1.** Let the polynomial system $\{\varphi_n(x)\}_{n=0}^{\infty}$ is defined by recurrence relations (7) and periodic conditions (3). Then for any $n \geq 2N$ the following recurrence relations

$$
\varphi_n(x) = P_N(x)\varphi_{n-N}(x) - \varphi_{n-2N}(x).
$$

(11)

are fulfilled.

*Proof.* From relations (3) and (4) we have $(n = Nm + k)$

$$
P_N(x)\varphi_{n-N}(x) - \varphi_{n-2N}(x) = \varphi_{k+N}(x)P_N(x)U_{m-2}(P_N(x))
$$

$$
- \varphi_k(x)P_N(x)U_{m-3}(P_N(x)) - \varphi_{k+N}(x)U_{m-3}(P_N(x)) + \varphi_k(x)U_{m-4}(P_N(x)) = 
$$

$$
= \varphi_{k+N}(x)U_{m-1}(P_N(x)) + \varphi_k(x)U_{m-4}(P_N(x))
$$

$$
- \varphi_{k+N}(x)U_{m-3}(P_N(x)) + \varphi_k(x)U_{m-4}(P_N(x)) = 
$$

$$
= \varphi_{k+N}(x)U_{m-1}(P_N(x)) - \varphi_k(x)U_{m-2}(P_N(x)) = \varphi_{Nm+k}(x) = \varphi_n(x).
$$

(14)

□

**Lemma 2.** Let the polynomial system $\{\varphi_n(x)\}_{n=0}^{\infty}$ is defined by recurrence relations (7) and periodic conditions (3). Besides, let $\Delta_n = \Delta_n^N(\mu)$ is defined by the equalities (10b) and (10c). Then for all $n \geq 0$ the identity

$$
\Delta_n = \Delta_0
$$

(12)

is fulfilled.

*Proof.* For proof this proposition by induction it is sufficiently to show that for all $n \geq 0$ the equality

$$
\Delta_n = \Delta_{n+1}.
$$

(13)

is fulfilled. Taking into account (10b) and (10c), for proof this relation it is sufficiently to check the equality

$$
\varphi_n^2 + \varphi_{n+1}^2 + \ldots + \varphi_{n+2N-1}^2 - P_N(\varphi_n\varphi_{n+N} + \varphi_{n+1}\varphi_{n+N+1} + \ldots \varphi_{n+N-1}\varphi_{n+2N-1}) = 
$$

$$
\varphi_{n+1}^2 + \ldots + \varphi_{n+2N-1}^2 + \varphi_{n+2N}^2 - P_N(\varphi_{n+1}\varphi_{n+N+1} + \ldots + \varphi_{n+N-1}\varphi_{n+2N-1} + \varphi_{n+N}\varphi_{n+2N})
$$

This equality is equivalent to the following relation

$$
\varphi_n^2 - P_N\varphi_n\varphi_{n+N} = \varphi_{n+2N}^2 - P_N\varphi_{n+N}\varphi_{n+2N}.
$$

(14)

From (11) it is follow that

$$
\varphi_{n+2N}^2 = P_N^2\varphi_{n+N}^2 - 2P_N\varphi_n\varphi_{n+N} + \varphi_n^2.
$$

(15)

Substituting (15) in (14), we see that the equality (14) is fulfilled. Therefore the equality (13) is also hold. □
Theorem 2.1. For $\mu$ to be an eigenvalue of matrix $J$ defined by (6) and (7) it is necessary that $\mu$ be a solution of equation
\[ \Delta^N_0 (\mu) = 0. \] (16)

Proof. Let us denote
\[ \sigma_{k+m} = \sum_{n=k}^{k+m} \varphi_n^2 (\mu). \] (17)
It follows from (10), (11) that
\[ \sigma_{0,2N-1} = S, \]
\[ \sigma_{2N,3N-1} = (P_N \varphi_N - \varphi_0)^2 + \ldots + (P_N \varphi_{2N-1} \varphi_{N-1})^2 = P_N^2 S_1 + S_2 - 2P_N D^N_0 (\mu). \]
From (10c) it follows that
\[ S - P_N D^N_0 = \Delta^N_0 \Rightarrow P_N D^N_0 = S - \Delta^N_0, \]
so we have
\[ \sigma_{0,2N-1} = 2\Delta^N_0 + (P_N^2 - 2) S_1 - S_2. \] (18)
Analogously, using lemma 2, we obtain
\[ \sigma_{2N,3N-1} = 2\Delta^N_0 + (P_N^2 - 2) \sigma_{2N,3N-1} - \sigma_{N,2N-1}. \] (19)
Than we have
\[ \sigma_{kN,(k+1)N-1} = 2\Delta^N_0 + (P_N^2 - 2) \sigma_{(k-1)N,kN-1} - \sigma_{(k-2)N,(k-1)N-1}. \] (20)
Summing these equalities in $k$, we have
\[ \sigma_{0,nN-1} = \sum_{k=0}^{nN-1} \varphi_k^2 = S + 2\Delta^N_0 + (P_N^2 - 2) S_1 - S_2 \]
\[ + \sum_{k=3}^{n} (2\Delta^N_0 + (P_N^2 - 2) \sigma_{(k-1)N,kN-1} - \sigma_{(k-2)N,(k-1)N-1}) = \]
\[ = 2(n-1)\Delta^N_0 + (P_N^2 - 1) S_1 + (P_N^2 - 2) (\sigma_{0,nN-1} - S) - \sigma_{0,nN-1} + S_2 + \sigma_{(n-1)N,nN-1}. \]
From here we obtain the relation
\[ (4 - P_N^2) \sum_{k=0}^{nN-1} \varphi_k^2 = 2(n - 1)\Delta^N_0 + S_1 + (3 - P_N^2) S_2 + \sigma_{(n-1)N,nN-1}. \] (21)
for finding the quantity $\sigma_{0,nN-1} = \sum_{k=0}^{nN-1} \varphi_k^2$. It is clear that if $\sum_{k=0}^{nN-1} \varphi_k^2 (\mu) < \infty$, then
\[ \lim_{n \to \infty} \sigma_{(n-1)N,nN-1} = 0. \] (22)
Then it follows from (21) that the series $\sum_{k=0}^{nN-1} \varphi_k^2 (\mu)$ is convergent if
\[ \Delta^N_0 (\mu) = 0. \]
Remark 1. All eigenvalues of the matrix $J$ must satisfy the "critical equation" \[(15)\]. But it is possible that some critical values of matrix $J$ are not satisfied the necessary and sufficient condition \[(3)\], i.e. the corresponding vector $X_\mu$ are not belonging to $\ell_2$.

Remark 2. Apparently for any $N \geq 2$ the polynomial $\varphi_{N-1}(\mu)$ divides the $\Delta^N_0(\mu)$, i.e. the equality
\[
\Delta^N_0(\mu) = \varphi_{N-1}(\mu) Q_N(\mu),
\]
is true. Then the "critical equation" \[(15)\] splits into two equations
\[
\varphi_{N-1}(\mu) = 0, \quad (24)
\]
and
\[
Q_N(\mu) = 0. \quad (25)
\]
Apparently that one can obtain a simple condition that a solution $\mu$ of equation \[(24)\] is an eigenvalue of matrix $J$. But we are not a success to get a sufficient condition that a solution $\mu$ of equation \[(25)\] is an eigenvalue of matrix $J$, which is more simply than the condition \[(22)\].

For illustration we consider a few examples. As the first example we consider the matrix $J$ defined by \[(5), (7)\] for $N = 3$, $b_0 = b_1 = b_2 = 1$ and for any complex $a_0$, $a_1$, $a_2$.

3 The case $N = 3$. The parametric Chebyshev polynomials

1. Let us consider the generalized Chebyshev polynomials system $\{\varphi^{(3)}_n(x)\}_{n=0}^{\infty}$ defined by recurrence relations
\[
x \varphi^{(3)}_n(x) = \varphi^{(3)}_{n+1}(x) + a_n \varphi^{(3)}_n(x) + \varphi^{(3)}_{n-1}(x), \quad \varphi^{(3)}_0(x) = 1, \quad \varphi^{(3)}_{-1}(x) = 0. \quad (26)
\]
The coefficients $a_n$ — complex numbers that are fulfilled the periodicity condition \[(2)\] with $N = 3$. It is follows from \[(26)\] that the first six polynomials are
\[
\begin{align*}
\varphi^{(3)}_0 &= 1, & \varphi^{(3)}_1 &= x - a_0, \\
\varphi^{(3)}_2 &= (x - a_2) \varphi^{(3)}_1 - \varphi^{(3)}_1, & \varphi^{(3)}_3 &= (x - a_0) \varphi^{(3)}_2 - \varphi^{(3)}_2, \\
\varphi^{(3)}_4 &= (x - a_0) \varphi^{(3)}_3 - \varphi^{(3)}_3, & \varphi^{(3)}_5 &= \varphi^{(3)}_2 (\varphi^{(3)}_3 - (x - a_1)),
\end{align*}
\]
and for $n \geq 6$ they can be calculated by formula
\[
\varphi^{(3)}_n(x) = P_3(x) \varphi^{(3)}_{n-3}(x) - \varphi^{(3)}_{n-6}(x). \quad (28)
\]
From \[(21)\] and last relation in \[(27)\] it follows that
\[
P_3(x) = \varphi^{(3)}_3(x) - (x-a_1) = x^3 - (a_0 + a_1 + a_2)x^2 + (a_0 a_2 + a_1 a_2 + a_0 a_1 - 3)x - a_0 a_1 a_2 + (a_0 + a_1 + a_2). \quad (29)
\]
Note that under additional condition $(a_0 + a_1 + a_2) = 0$ the polynomial $P_3(x)$ has more simply form
\[
P_3(x) = x^3 + (a_0 a_2 + a_1 a_2 + a_0 a_1 - 3)x - a_0 a_1 a_2. \quad (30)
\]
Let us find the eigenvalues of the matrix $J^{(3)} = \left[ j_{j,k}^{(3)} \right]_{j,k=0}^{\infty}$ corresponding to the recurrence relations

$$j_{i,k}^{(3)} = B_3 \delta_{i+1,k} + A_3 \delta_{i,k} + B_3 \delta_{i-1,k}, \quad (31)$$

where

$$A_3 = \begin{pmatrix} a_0 & 1 & 0 \\ 1 & a_1 & 1 \\ 0 & 1 & a_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (32)$$

Using the formulas (10a)-(10c), (26) and (27), one can write the left-hand side of the equation (16) in the following form

$$S_0^3(\mu) - D_0^3(\mu) P_3(\mu) = \varphi_2^{(3)}(\mu) Q_3(\mu), \quad (33)$$

where

$$Q_3(\mu) = \left( 1 + \left( \varphi_1^{(3)}(\mu) \right)^2 \right) (\mu - a_1)(\mu - a_2) - 2 + \varphi_2^{(3)}(\mu) \left( 1 - \varphi_1^{(3)}(\mu)(\mu - a_2) \right). \quad (34)$$

In result the equation (16) splits into pair of equations

$$\varphi_2^{(3)}(\mu) = \mu^2 - (a_0 + a_1)\mu + a_0a_1 - 1 = 0, \quad (35)$$

$$Q_3(\mu) = 3\mu^2 - 2(a_0 + a_1 + a_2)\mu + (a_0a_2 + a_1a_2 + a_0a_1 - 3) = 0. \quad (36)$$

The roots $\mu_{1,2}$ of the equations (35) have the following form

$$\mu_{1,2} = \mu^\pm = \frac{1}{2} (a_0 + a_1) \pm \frac{1}{2} \sqrt{4 + (a_1 - a_0)^2}, \quad (37)$$

and the roots of the equations (36) equal to

$$\mu_{3,4} = \nu^\pm = \frac{1}{3} \left( a_0 + a_1 + a_2 \pm \sqrt{a_0^2 + a_1^2 + a_2^2 - a_0a_1 - a_1a_2 - a_0a_2 + 9} \right). \quad (38)$$

In the case $(a_0 + a_1 + a_2) = 0$ the roots $\mu_{3,4}$ are simplified

$$\mu_{3,4} = \pm \sqrt{1 + \frac{a_0^2 + a_1^2 + a_2^2}{6}}. \quad (39)$$

It remains to find those critical values $\mu_k$, $(k = 1, 2, 3, 4)$ which are an eigenvalues of Jacobi matrix $J^{(3)}$.

**Lemma 3.** For that a root $\mu_k$, $(k = 1, 2)$ of the equation (33) is an eigenvalues of the matrix $J^{(3)}$ (31), (32), it is necessary and sufficient that the following inequality

$$|\mu_k - a_0| < 1, \quad k = 1, 2, \quad (40)$$

is fulfilled.
Proof. In fact, we have for $\mu_k$, $(k = 1, 2)$ defined by \( \frac{13}{7} \) the following relation

$$
\sum_{j=0}^{\infty} \left( \varphi_j^{(3)}(\mu_k) \right)^2 = 1 + 2(\mu_k - a_0)^2 \sum_{j=0}^{\infty} (\mu_k - a_0)^{2j}.
$$

The series in the right-hand side of this relation is convergent if and only if the following inequality

$$
|\mu_k - a_0| < 1, \quad k = 1, 2,
$$

is true.

Unfortunately, for $\mu_k$ $(k = 3, 4)$ even in simplest case $(a_0 + a_1 + a_2) = 0$ one cannot find a more simply condition that $\mu_k$ is an eigenvalues of the matrix $J^{(3)}$ than the following condition

$$
\left[ \left( \varphi_{3n}^{(3)}(\mu_k) \right)^2 + \left( \varphi_{3n+1}^{(3)}(\mu_k) \right)^2 + \left( \varphi_{3n+2}^{(3)}(\mu_k) \right)^2 \right] \rightarrow 0
$$
as $n \rightarrow \infty$ (but this is the condition \( \frac{22}{7} \)).

2. We consider the parametric Chebyshev polynomials introduced in \( \frac{6}{7} \) as example to the case when the polynomial $Q_3(\mu)$ has not roots. These polynomials \( \{\Psi_n(x; \alpha)\}_{n=0}^{\infty} \) are defined by recurrent relations with coefficients depending on a parameter $\alpha \in [-1, 1]$. These coefficients are

$$
a_0(\alpha) = \frac{i\sqrt{3}}{2}(\alpha + 1)(3\alpha - 2), \quad a_1(\alpha) = -i\sqrt{3}\alpha, \quad a_3(\alpha) = -\frac{i\sqrt{3}}{2}(\alpha - 1)(3\alpha - 2),
$$

\( a_{n+3}(\alpha) = a_n(\alpha), \quad n \geq 0 \) \hfill (41a)

It is clear, that the following equality

$$
a_0(\alpha) + a_1(\alpha) + a_2(\alpha) = 0,
$$
is true. From \( \frac{28}{7} \), \( \frac{32}{7} \), \( \frac{11}{7} \) it is follow that

$$
\Psi_0(x; \alpha) = 1;
$$

$$
\Psi_1(x; \alpha) = x - \frac{i\sqrt{3}}{2}(\alpha + 1)(3\alpha + 2);
$$

$$
\Psi_2(x; \alpha) = x^2 - \frac{1}{2}(\alpha - 1)(3\alpha + 2)x + \frac{3}{2}\alpha(\alpha + 1)(3\alpha - 2) - 1);
$$

$$
\Psi_3(x; \alpha) = x^3 + (1 - \tilde{\alpha}_1^2)x + i\sqrt{3}\alpha(1 - \tilde{\alpha}_2^2);
$$

$$
\Psi_4(x; \alpha) = (x - a_0(\alpha))\Psi_3(x; \alpha) - \Psi_2(x; \alpha);
$$

\( \Psi_5(x; \alpha) = \Psi_2(x; \alpha)P_3(x; \alpha) \),

where are used notations

$$
\tilde{\alpha}_1^2 = \frac{27}{4}\alpha^2(1 - \alpha^2), \quad \tilde{\alpha}_2^2 = \frac{3}{4}(1 - \alpha^2)(9\alpha^2 - 4),
$$

$$
P_3(x; \alpha) = x^3 - \tilde{\alpha}_1^2x - i\sqrt{3}\alpha\tilde{\alpha}_2^2.
$$

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For $n \geq 6$ one can calculate the polynomials $\Psi_n(x; \alpha)$ by formulas

$$\Psi_n(x; \alpha) = P_3(x; \alpha)\Psi_{n-3}(x; \alpha) - \Psi_{n-6}(x; \alpha).$$

In the paper [18] we obtain the continuous spectrum of the Jacobi matrix $J^{(3)}(\alpha)$, corresponding to the parametric Chebyshev polynomials. The support of the continuous spectrum is represented on the Fig.1, where are used the notations

$$\lambda_k = \lambda(\alpha)e^{\frac{2k\pi}{3}}, \quad \tilde{\lambda}_k = \lambda(\alpha)e^{\frac{(2k+3)\pi}{3}}, \quad k = 0, 1, 2.$$  

(46)

The number $\lambda(\alpha) \geq 0$ is a positive root of the equation

$$\lambda^3 - \tilde{\alpha}^2\lambda - 2 = 0,$$

(47)

where

$$\sqrt{2} \leq \lambda(\alpha) \leq \lambda_{\text{max}},$$

(48)

and

$$\lambda_{\text{max}} = 3\sqrt{1 + \sqrt{1 - \left(\frac{27}{64}\right)^2}} + 3\sqrt{1 - \sqrt{1 - \left(\frac{27}{64}\right)^2}}.$$ 

(49)

Now we consider the discrete spectrum of the matrix $J^{(3)}(\alpha)$. From (39) and (41) we find roots of the equation (37)

$$\mu_{1,2}(\alpha) = \frac{i\sqrt{3}}{4}(\alpha - 1)(3\alpha + 2) \pm \sqrt{1 - \frac{3}{16}f^2(\alpha)},$$

(50)

where

$$f(\alpha) = 3\alpha^2 + 3\alpha - 2,$$

(51)

and

$$\mu_{1,2}(\alpha) - a_1(\alpha) = -\frac{i\sqrt{3}}{4}f^2(\alpha) \pm \sqrt{1 - \frac{3}{16}f^2(\alpha)}.$$ 

(52)
From lemma 3 it follows that \( \mu_k, k = 1, 2 \) is an eigenvalue of the matrix \( J^{(3)}(\alpha) \) if and only if the inequality (10) is true. We introduce notations

\[
\alpha_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4 - 2\sqrt{3}}{3\sqrt{3}}}, \quad \alpha_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4 - 2\sqrt{3}}{3\sqrt{3}}}, \quad \alpha_3 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4 + 2\sqrt{3}}{3\sqrt{3}}},
\]

and made justification test of the inequality (10) on intervals

\([-1, \alpha_1], \quad (\alpha_1, \alpha_2), \quad [\alpha_2, \alpha_3], \quad (\alpha_3, 1]\).

As a result we have

a) The numbers \((\mu_{1,2})(\alpha)\) are not eigenvalues of the matrix \( J^{(3)}(\alpha) \) as \( \alpha \in [-1, \alpha_1] \) or \( \alpha \in [\alpha_2, \alpha_3] \).

b) \( J^{(3)}(\alpha) \) has only one eigenvalue \( \mu_2(\alpha) \) as \( \alpha \in (\alpha_1, \alpha_2) \).

c) \( J^{(3)}(\alpha) \) has only one eigenvalue \( \mu_1(\alpha) \) as \( \alpha \in (\alpha_3, 1] \).

Now we consider the solutions \( \mu_{3,4}(\alpha) \) of the equation (38), that has in this case the following form

\[
Q_3(\mu; \alpha) = 3\mu^2 - \bar{\alpha}_1^2 = 0
\]

These solutions are equal to

\[
\mu_{3,4}(\alpha) = \pm \frac{\bar{\alpha}_1}{\sqrt{3}}.
\]

To prove that \( \mu_{3,4}(\alpha) \) are not eigenvalues of the matrix \( J^{(3)}(\alpha) \) as \( \alpha \in [-1, 1] \) it is sufficient to check that the necessary condition (22) is broken. Using the recurrence relations (11) (and notation (10)) it is easy to show that

\[
S_{3n}^N(\mu) + S_{3(n-1)}^N(\mu) - S(\mu) - S_3^N(\mu) = P_3^2(\mu; \alpha)(\sigma_{3n,3n+2}(\mu) - S_2(\mu)).
\]

Then

\[
\sum_{s=-1}^{1} \sigma_{3(n+s),3(n+s)+2}(\mu) + (1-P_3^2(\mu; \alpha))\sigma_{3n,3n+2}(\mu) = S_2 + S_1(2-P_3^2(\mu; \alpha)) + \sigma_{6,6+2}(\mu) = A_0(\alpha).
\]

The direct calculation shows that for any \( \alpha \in (-1, 0) \cup (0, 1] \) one have \( A_0(\alpha) \neq 0 \). At the same time the left-hand side of equality (56) tends to zero as \( n \to \infty \) if the necessary condition (22) is fulfilled. From this it follows that \( A_0(\alpha) = 0 \). Thus \( \mu_{3,4}(\alpha) \) are not eigenvalues of the Jacobi matrix \( J^{(3)}(\alpha) \) as \( \alpha \in (-1, 0) \cup (0, 1] \). From results of [19] it follows that \( \mu_{3,4}(0) \) and \( \mu_{3,4}(1) \) are not eigenvalues of the Jacobi matrixes \( J^{(3)}(0) \) and \( J^{(3)}(1) \) too. Therefore, \( \mu_{3,4}(\alpha) \) are not eigenvalues of the Jacobi matrix \( J^{(3)}(\alpha) \) as \( \alpha \in [-1, 1] \).

So the Jacobi matrix \( J^{(3)}(\alpha) \) has only one eigenvalue \( \mu_1(\alpha) \) for \( \alpha \in (\alpha_3, 1] \), and only one eigenvalue \( \mu_2(\alpha) \) for \( \alpha \in (\alpha_1, \alpha_2) \).

Further, as another example, we consider the Jacobi matrix for the elementary \( N \)-symmetric Chebyshev polynomials, which belongs to the type of polynomials under consideration. These polynomials appear in studying of "compound model of generalized oscillator" [15], [17]. We consider only cases \( N = 3, 4, 5 \), since it was shown in the paper [15] that such polynomials not exist for \( n \geq 6 \).
Elementary $N$-symmetric Chebyshev polynomials

Elementary $N$-symmetric Chebyshev polynomials $\{\varphi_N^n(x)\}_{n=0}^{\infty}$ [15] are defined by recurrence relations

\begin{equation}
x \varphi_n^N(x) = \varphi_{n+1}^N(x) + a_n \varphi_n^N(x) + \varphi_{n-1}^N(x), \quad \varphi_0^N(x) = 1, \quad \varphi_{-1}^N(x) = 0,
\end{equation}

where coefficients $a_n$ for $N = 3, 4, 5$, given by formulas

\begin{align}
a_0^{(3)} &= i\sqrt{3}, \quad a_1^{(3)} = i\sqrt{3}, \quad a_2^{(3)} = 0, \quad a_{n+3}^{(3)} = a_n^{(3)}, \quad n \geq 0; \\
a_0^{(4)} &= 2i, \quad a_1^{(4)} = 0, \quad a_2^{(4)} = -2i, \quad a_3^{(4)} = 0, \quad a_{n+4}^{(4)} = a_n^{(4)}, \quad n \geq 0; \\
a_0^{(5)} &= a_2^{(5)} = a_3^{(5)} = 0, \quad a_1^{(5)} = i\sqrt{5}, \quad a_4^{(5)} = -i\sqrt{5}, \quad a_{n+5}^{(5)} = a_n^{(5)}, \quad n \geq 0.
\end{align}

Using recurrence relations (57), we find first $2N$ ($N = 3, 4, 5$) polynomials

\begin{align}
\varphi_0^{(3)}(x) &= 1, \quad \varphi_1^{(3)}(x) = x - i\sqrt{3}, \quad \varphi_2^{(3)}(x) = x^2 + 2, \\
\varphi_3^{(3)}(x) &= x \varphi_2^{(3)}(x) - \varphi_1^{(3)}(x), \quad \varphi_4^{(3)}(x) = x^3 \varphi_1^{(3)}(x) + 1, \quad \varphi_5^{(3)}(x) = x^3 \varphi_2^{(3)}(x); \\
\varphi_0^{(4)}(x) &= 1, \quad \varphi_1^{(4)}(x) = x - 2i, \quad \varphi_2^{(4)}(x) = x^2 - 2ix - 1, \quad \varphi_3^{(4)}(x) = x^3 + 2x, \\
\varphi_4^{(4)}(x) &= x^4 + x^2 + 2ix + 1, \quad \varphi_5^{(4)}(x) = x^5 - 2ix^4 + 3x - 2i, \\
\varphi_6^{(4)}(x) &= x^6 - 2ix^5 - x^4 + 2x^2 - 4ix - 1, \quad \varphi_7^{(4)}(x) = (x^4 + 2)\varphi_3^{(4)}(x); \\
\varphi_0^{(5)}(x) &= 1, \quad \varphi_1^{(5)}(x) = x, \quad \varphi_2^{(5)}(x) = x^2 - i\sqrt{5}x - 1, \quad \varphi_3^{(5)}(x) = x^3 - i\sqrt{5}x^2 - 2x, \\
\varphi_4^{(5)}(x) &= x^4 - i\sqrt{5}x^3 - 3x^2 + i\sqrt{5}x + 1, \quad \varphi_5^{(5)}(x) = x^5 + x^3 + i\sqrt{5}x^2 - 2x + i\sqrt{5}, \\
\varphi_6^{(5)}(x) &= x^6 + x^2 - 1, \quad \varphi_7^{(5)}(x) = x^7 - i\sqrt{5}x^6 - x^5 + x, \\
\varphi_8^{(5)}(x) &= x^8 - i\sqrt{5}x^7 - 2x^6 + 1, \quad \varphi_9^{(5)}(x) = x^5 \varphi_4^{(5)}(x).
\end{align}

From here and relation (11) we get the following expression for $P_N(x)$

\begin{equation}
P_3(x) = x^3, \quad P_4(x) = x^4 + 2, \quad P_5(x) = x^5.
\end{equation}

In view of (12), the relation (5) takes the following form ($k = 0, 1, 2, m \geq 2$)

\begin{align}
\varphi_{3m+k}^{(3)}(x) &= \varphi_{k+3}^{(3)}(x) U_{m-1}^3(x^3) - \varphi_{k}^{(3)}(x) U_{m-2}^3(x^3); \\
\varphi_{4m+k}^{(4)}(x) &= \varphi_{k+4}^{(4)}(x) U_{m-1}^4(x^4 + 2) - \varphi_{k}^{(4)}(x) U_{m-2}^4(x^4 + 2); \\
\varphi_{5m+k}^{(5)}(x) &= \varphi_{k+5}^{(5)}(x) U_{m-1}^5(x^5) - \varphi_{k}^{(5)}(x) U_{m-2}^5(x^5).
\end{align}
The Jacobi matrix $J^{(N)} = \left[ j_{j,k}^{(N)} \right]_{j,k=0}^{\infty}$, $(N = 3, 4, 5)$ corresponding to the relations (57), has the form

$$J^{(N)} = B_N \delta_{i+1,k} + A_N \delta_{i,k} + B_N^k \delta_{i-1,k}, \quad (64)$$

where

$$A_3 = \begin{pmatrix} \frac{i\sqrt{3}}{3} & 1 & 0 \\ 1 & \frac{-i\sqrt{3}}{3} & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad (65a)$$

$$A_4 = \begin{pmatrix} 2i & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{-2i}{\sqrt{3}} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad (65b)$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{i\sqrt{5}}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{i\sqrt{3}}{3} \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad (65c)$$

Now we turn to evaluation of eigenvalues of the matrix $J^{(N)}$, $(N = 3, 4, 5)$, using the critical equation (16).

A. Discrete spectrum of Jacobi matrix $J^{(3)}$

The Jacobi matrix $J^{(3)}$ is defined by equalities (64) and (65a). From (33), (59), (62), (35), (10a), taking into account that

$$a_0^{(3)} + a_1^{(3)} + a_2^{(3)} = 0, \quad a_0^{(3)} a_1^{(3)} + a_1^{(3)} a_2^{(3)} + a_0^{(3)} a_2^{(3)} = 3,$$

we obtain that left-hand side $\Delta_0^{(3)}(\mu)$ of equation (16) has the following form

$$\Delta_0^{(3)}(\mu) = z_0^{(3)}(\mu) - D_0^{(3)}(\mu) P_3(\mu) = 3\mu^2(\mu^2 + 2).$$

Then the equation (16) for $N = 3$ looks as

$$\mu^2(\mu^2 + 2) = 0.$$

The solutions of this equation are equal to

$$\mu_1 = i\sqrt{2}, \quad \mu_2 = -i\sqrt{2}, \quad \mu_{3,4} = 0. \quad (66)$$

Using the lemma 4, we have

$$|\mu_1 - a_0| = |i\sqrt{2} - i\sqrt{3}| < 1.$$
It means that \( \mu_1 \) is an eigenvalue of the matrix \( J^{(3)} \). Further,

\[ |\mu_2 - a_0| = |-i\sqrt{2} - i\sqrt{3}| > 1, \]

i.e. \( \mu_2 \) is not an eigenvalue of the matrix \( J^{(3)} \). Now we calculate the components of vector \( X_{\mu_3} = X_{\mu_4} \) for \( \mu_{3,4} \). They are equal to

\[ x_1 = 1, \; x_2 = -i\sqrt{3}, \; x_3 = 2, \; x_4 = i\sqrt{3}, \; x_5 = 1, \; x_6 = 0, \; x_{k+6} = -x_k, \; \text{при} \; k \geq 0. \]  

(67)

Taking into account that \( \varphi_{k-1}^{(3)}(\mu) = x_k \), for \( k = 3, 4 \), we have

\[ \left[ \left( \varphi_{3n}^{(3)}(\mu_k) \right)^2 + \left( \varphi_{3n+1}^{(3)}(\mu_k) \right)^2 + \left( \varphi_{3n+2}^{(3)}(\mu_k) \right)^2 \right] \nrightarrow 0, \; \text{при} \; n \rightarrow \infty, \]

i.e. the condition (22) is not realized. Therefore \( X_{\mu_k} \notin \ell^2 \), it means that \( \mu_{3,4} \) are not eigenvalues of the matrix \( J^{(3)} \).

\[ \text{B. Discrete spectrum of Jacobi matrix } J^{(4)} \]

The Jacobi matrix \( J^{(4)} \) is defined by equalities (64) and (65b). Using (33), (60), (62), (36), (10b), we rewrite the equation (16) for \( N = 4 \) in the form

\[ \mu^4(\mu^2 + 2) = 0. \]  

(68)

The solutions of this equation are equal to

\[ \mu_1 = i\sqrt{2}, \; \mu_2 = -i\sqrt{2}, \; \mu_{3,4,5,6} = 0. \]

In the first place we consider zero solutions of this equation. The components of the vector \( X_{\mu_k} \) (for \( k = 3, 4, 5, 6 \)) are equal to

\[ x_1 = 1, \; x_2 = -\frac{i}{2}, \; x_3 = -1, \; x_4 = -\frac{3i}{2}, \; x_{k+4} = x_k, \; \text{при} \; k \geq 0. \]

Then partial sum of the series

\[ \sum_{n=0}^{\infty} \left( \varphi_{n}^{(4)}(\mu_k) \right)^2 = \sum_{n=1}^{\infty} x_k^2 \]

equal to

\[ S_1 = 1, \; s_2 = 1 - \frac{i}{2}, \; S_3 = -\frac{i}{2}, \; S_4 = -2i, \; s_{n+4} = s_n - 2i, \; \text{при} \; n \geq 1. \]

Thus the sequence of partial sums of series \( \sum_{n=1}^{\infty} x_k^2 \) has not limit point as \( n \rightarrow \infty \), i.e. the series is divergent. Hence, \( \mu_{3,4,5,6} \) are not eigenvalues of the matrix \( J^{(4)} \). The components of vector \( X_{\mu_1} = (x_1, x_2, \ldots)^t \), corresponding to critical value \( \mu_1 \), are equal to

\[ x_1 = 1, \; x_2 = i(\sqrt{2} - 2), \; x_3 = (2\sqrt{2} - 3), \; x_4 = 0, \; x_{k+4} = (3 - 2\sqrt{2})x_k, \; \text{при} \; k \geq 1. \]

Because

\[ \|X_{\mu_1}\|^2 = \sqrt{2}, \]

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then critical value \( \mu_1 = i\sqrt{2} \) is eigenvalue of the Jacobi matrix \( J^{(4)} \). The normalized eigenvector \( Y_{\mu_1} = (y_1, y_2, \ldots)^t \) has the following components

\[
y_{4k+1} = \frac{1}{\sqrt{2}} (3 - 2\sqrt{2})^k, \quad y_{4k+2} = \frac{i(\sqrt{2} - 2)}{\sqrt{2}} (3 - 2\sqrt{2})^k, \quad y_{4k+3} = \frac{(2\sqrt{2} - 3)}{\sqrt{2}} (3 - 2\sqrt{2})^k, \quad y_{4k+4} = 0,
\]

with \( k \geq 0 \). Finally, the components of vector \( X_{\mu_2} = (x_1, x_2, \ldots)^t \), corresponding to critical value \( \mu_2 = -i\sqrt{2} \), are equal to

\[
x_1 = 1, \quad x_2 = -(3 + 2\sqrt{2}), \quad x_3 = -i(2 + \sqrt{2}), \quad x_4 = 0, \quad x_{k+4} = (3 + 2\sqrt{2})x_k, \quad \text{при} \quad k \geq 1.
\]

Consequently,

\[
\|X_{\mu_2}\|^2 = \infty
\]
i.e. critical value \( \mu_2 = -i\sqrt{2} \) is not eigenvalue of the Jacobi matrix \( J^{(4)} \).

**C. Discrete spectrum of Jacobi matrix \( J^{(5)} \)**

We consider now the Jacobi matrix \( J^{(5)} \) that is defined by equalities (64) and (65c). Using (33), (61), (62), (36), (10c), we rewrite the equation (16) for \( N = 5 \) in the form

\[
\mu_4 (\mu_4 - i\sqrt{5}\mu_3 - 3\mu^2 + i\sqrt{5}\mu + 1) = 0.
\]

The solutions of this equation are equal to

\[
\mu_{1,2} = \frac{1}{4} \left[ \pm \sqrt{10 - 2\sqrt{5}} + i(1 + \sqrt{5}) \right],
\]

\[
\mu_{3,4} = \frac{1}{4} \left[ \pm \sqrt{10 + 2\sqrt{5}} + i(-1 - \sqrt{5}) \right],
\]

\[
\mu_{5,6,7,8} = 0.
\]

From the same arguments as given above, we see that for \( k = 5; 8 \) the vector \( X_{\mu_k} \notin \ell^2 \), i.e. corresponding critical values \( \mu_k \) are not eigenvalues of the Jacobi matrix \( J^{(5)} \).

For the critical value \( \mu_1 = \frac{1}{4} \left[ \sqrt{10 - 2\sqrt{5}} + i(1 + \sqrt{5}) \right] \) the squared components of vector \( X_{\mu_1} = (x_1, x_2, \ldots)^t \) are equal to

\[
(x_1)^2 = 1, \quad (x_2)^2 = \frac{1 - \sqrt{5}}{4} + \frac{i(1 + \sqrt{5})}{8} \sqrt{10 - 2\sqrt{5}},
\]

\[
(x_3)^2 = \frac{\sqrt{5} - 2}{2} + \frac{i(1 - \sqrt{5})}{8} \sqrt{10 - 2\sqrt{5}}, \quad (x_4)^2 = \frac{\sqrt{5} - 3}{2},
\]

\[
(x_5)^2 = 0, \quad (x_{k+5})^2 = \frac{\sqrt{5} - 3}{2} (x_k)^2, \quad k \geq 1.
\]

Then \( \|X_{\mu_1}\|^2 = 2\sqrt{5} \) and consequently, \( \mu_1 \) is an eigenvalue of the Jacobi matrix \( J^{(5)} \).
Similarly, for critical value \( \mu_2 = \frac{1}{4} \left[ -\sqrt{10} - 2\sqrt{5} + i(1 + \sqrt{5}) \right] \) the squared components of vector \( X_{\mu_2} \) are equal to

\[
(x_1)^2 = 1, \quad (x_2)^2 = \frac{1 - \sqrt{5}}{4} - \frac{i(1 + \sqrt{5})}{8} \sqrt{10 - 2\sqrt{5}},
\]
\[
(x_3)^2 = \frac{\sqrt{5} - 2}{2} + \frac{i(\sqrt{5} - 1)}{8} \sqrt{10 - 2\sqrt{5}}, \quad (x_4)^2 = \frac{\sqrt{5} - 3}{2},
\]
\[
(x_5)^2 = 0, \quad (x_{k+5})^2 = \frac{\sqrt{5} - 3}{2} (x_k)^2, \quad k \geq 1.
\]

Therefore, \( \|X_{\mu_2}\|^2 = 2\sqrt{5} \) and \( \mu_2 \) is an eigenvalue of the Jacobi matrix \( J^{(5)} \).

For critical value \( \mu_3 = \frac{1}{4} \left[ \sqrt{10} + 2\sqrt{5} + i(\sqrt{5} - 1) \right] \) the squared components of vector \( X_{\mu_3} \) are equal to

\[
(x_1)^2 = 1, \quad (x_2)^2 = \frac{1 + \sqrt{5}}{4} + \frac{i(\sqrt{5} - 1)}{8} \sqrt{10 + 2\sqrt{5}},
\]
\[
(x_3)^2 = -\frac{2 + \sqrt{5}}{2} - \frac{i(1 + \sqrt{5})}{8} \sqrt{10 + 2\sqrt{5}}, \quad (x_4)^2 = -\frac{\sqrt{5} + 3}{2},
\]
\[
(x_5)^2 = 0, \quad (x_{k+5})^2 = -\frac{3 + \sqrt{5}}{2} (x_k)^2, \quad k \geq 1.
\]

From these relations follow that \( X_{\mu_3} \notin \ell^2 \), and corresponding critical value \( \mu_3 \) is not eigenvalue of the Jacobi matrix \( J^{(5)} \).

Finally, for critical value \( \mu_4 = \frac{1}{4} \left[ \sqrt{10} + 2\sqrt{5} - i(\sqrt{5} - 1) \right] \) the squared components of vector \( X_{\mu_4} \) are equal to

\[
(x_1)^2 = 1, \quad (x_2)^2 = \frac{1 + \sqrt{5}}{4} - \frac{i(\sqrt{5} - 1)}{8} \sqrt{10 + 2\sqrt{5}},
\]
\[
(x_3)^2 = -\frac{2 + \sqrt{5}}{2} + \frac{i(1 + \sqrt{5})}{8} \sqrt{10 + 2\sqrt{5}}, \quad (x_4)^2 = -\frac{\sqrt{5} + 3}{2},
\]
\[
(x_5)^2 = 0, \quad (x_{k+5})^2 = -\frac{3 + \sqrt{5}}{2} (x_k)^2, \quad k \geq 1.
\]

As above, from these relations it follows that \( X_{\mu_4} \notin \ell^2 \) and corresponding critical value \( \mu_4 \) is not eigenvalue of the Jacobi matrix \( J^{(5)} \).

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