An existence criterion for Hall subgroups of finite groups

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Abstract. We obtain an existence criterion for Hall subgroups of finite groups in terms of a composition series. As a corollary we provide a solution to Problem 5.65 from the Kourovka notebook.

1 Introduction

Let \( p \) be a set of primes. We denote by \( \{p\} \) the set of all primes not in \( p \), by \( p(n) \) the set of prime divisors of a positive integer \( n \), for a finite group \( G \) we denote \( p(|G|) \) by \( p(G) \).

A positive integer \( n \) with \( p(n) \subseteq \{p\} \) is called a \( p \)-number, a group \( G \) with \( p(G) \subseteq \{p\} \) is called a \( p \)-group. A subgroup \( H \) of \( G \) is called a \( p \)-Hall subgroup, if \( p(H) \subseteq \{p\} \) and \( p(|G:H|) \subseteq \{p\} \). According to [3] we say that \( G \) satisfies \( E_p \) (or briefly \( G \) is an \( E_p \)-group).

A group satisfying \( E_p \) (resp. \( C_p \), \( D_p \)) is also called an \( E_p \)-group (resp. a \( C_p \)-group, a \( D_p \)-group). Let \( A, B, H \) be subgroups of \( G \) such that \( B \leq A \) and \( H \leq G \). Then \( N_H(A/B) = N_H(A) \cap N_H(B) \) is the normalizer of \( A/B \) in \( H \). If \( x \in N_H(A/B) \), then \( x \) induces an automorphism of \( A/B \) by \( Bx^{-1}ax \). Thus there exists a homomorphism \( N_H(A/B) \rightarrow Aut(A/B) \). The image of \( N_H(A/B) \) under this homomorphism is denoted by \( Aut_H(A/B) \) and is called a group of induced automorphisms of \( A/B \), while the kernel of this homomorphism is denoted by \( C_H(A/B) \) and is called the centralizer of \( A/B \) in \( H \).

All results of the paper depend on the classification of finite simple groups. We can avoid its use with the following definition. A finite group is said to be a \( K \)-group, if all of its composition factors are known simple groups, i.e., are either cyclic of prime order, or alternating, or of Lie type, or sporadic. If we replace the term ‘finite group’ by the term ‘finite \( K \)-group’, then all results remain valid without the use of the classification of finite simple groups.

In [6, Theorem 7.7] we proved the following theorem.

Theorem 1. Let \( A \) be a normal subgroup of a finite group \( G \). Then \( G \in D_p \) if and only if \( A \in D_p \) and \( G/A \in D_p \).
By using this theorem we obtain that a finite group $G$ satisfies $D_\pi$ if and only if every composition factor satisfies $D_\pi$. In [5] an arithmetic description of finite simple $D_\pi$-groups was obtained; thus there exists a precise arithmetic criterion for determining whether a finite group $G$ satisfies $D_\pi$. Our aim is to find a similar criterion for $E_\pi$.

The following easy proposition is well known (see [3, Lemma 1]).

**Proposition 2.** Let $A$ be a normal subgroup of $G$. If $H$ is a $\pi$-Hall subgroup of $G$, then $H \cap A$ is a $\pi$-Hall subgroup of $A$, and $HA/A$ is a $\pi$-Hall subgroup of $G/A$.

It follows from this that a normal subgroup and a factor group of an $E_\pi$-group satisfy $E_\pi$. But an extension of an $E_\pi$-group by an $E_\pi$-group may fail to satisfy $E_\pi$ (see Example 14). Thus a criterion for a finite group to satisfy $E_\pi$ will be different in nature to a criterion for $D_\pi$.

In [1, Theorem 3.5] Gross proved the following theorem.

**Theorem 3.** Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a composition series of a finite group $G$ which is a refinement of a chief series of $G$. Then the following are equivalent:

(a) $H \in E_\pi$ for all subgroups $H$ such that $H^{(\infty)}$ is subnormal in $G$, where $H^{(\infty)}$ is the intersection of all members of the derived series of $H$;

(b) $\text{Aut}_G(A/A^*) \in E_\pi$ for all $A \in \mathcal{A}(G)$;

(c) $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ for all $i$ with $1 \leq i \leq n$;

(d) $\text{Aut}_G(H/K) \in E_\pi$ for all composition factors $H/K$ of $G$.

The definition of $\text{Aut}_G(A/B)$ is given above; the symbol $\mathcal{A}(G)$ denotes the set of all atoms of $G$. Recall that a subgroup $A$ of $G$ is called an atom, if $A$ is subnormal in $G$, $A = A'$, and $A$ has exactly one maximal normal subgroup. If in statement (a) we take $H = G$, then we obtain that $G \in E_\pi$. Thus the condition $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ for all $i$ with $1 \leq i \leq n$ implies that $G \in E_\pi$. The main goal of the present work is to prove that the converse is true. The following result is our main theorem.

**Theorem 4.** Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a composition series of a finite group $G$. If, for some $i$, $\text{Aut}_G(G_i/G_{i-1}) \notin E_\pi$, then $G \notin E_\pi$.

The recently completed classification of $\pi$-Hall subgroups in finite simple groups (see [7]) is the main technical tool for the proof of the theorem. Combining Theorems 3 and 4 we obtain a criterion for a finite group to satisfy $E_\pi$.

**Corollary 5.** Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a composition series of a finite group $G$ which is a refinement of a chief series of $G$. Then the following are equivalent:

(a) $H \in E_\pi$ for all subgroups $H$ such that $H^{(\infty)}$ is subnormal in $G$;

(b) $\text{Aut}_G(A/A^*) \in E_\pi$ for all $A \in \mathcal{A}(G)$;
(c) $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ for all $i$ with $1 \leq i \leq n$;
(d) $\text{Aut}_G(H/K) \in E_\pi$ for all composition factors $H/K$ of $G$;
(e) $G \in E_\pi$.

If $G_i/G_{i-1}$ is cyclic, then $\text{Aut}_G(G_i/G_{i-1}) \leq \text{Aut}(G_i/G_{i-1})$ is also cyclic, hence $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$. Thus we need to check that $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ only for non-abelian composition factors. Moreover, let

$$1 = G_0 < G_1 < G_2 < \cdots < G_k = G$$

be a refinement of a chief series

$$1 = G_0 = G_{i_0} < G_{i_1} < \cdots < G_{i_m} = G_k = G.$$

Then $G_{i_{j+1}}/G_{i_j} = T_1 \times \cdots \times T_s$ is a direct product of isomorphic simple groups. Assume that $G_{i_{j+1}}/G_{i_j}$ is non-abelian; thus $T_1 \simeq T_2 \simeq \cdots \simeq T_s$ are non-abelian finite simple groups. We may choose the numbering of $T_1, \ldots, T_s$ so that $G_{i_{j+1}}$ is the complete preimage of $T_1$ in $G$, $G_{i_{j+2}}$ is the complete preimage of $T_1 \times T_2$ in $G$, etc. Since for every $l \leq s$ we have $G_{i_l} \leq C(G_{i_{j+1}}/G_{i_{j+1}-1})$, we have by [11, Lemma 1.2] that $\text{Aut}_G(G_{i_{j+1}}/G_{i_{j+1}-1}) \simeq \text{Aut}_{G_{i_j}}((G_{i_{j+1}}/G_{i_j})/(G_{i_{j+1}-1}/G_{i_j}))$. Assume that $j = 0$, so that $T_1, \ldots, T_s$ are subgroups of $G$. Since $T_1 \times \cdots \times T_s$ is a minimal normal subgroup of $G$, then $G$ acts transitively by conjugation on the set $\{T_1, \ldots, T_s\}$ and $N_G(T_1), \ldots, N_G(T_s)$ are all conjugate in $G$. By [8, Theorem 3.3.10] we obtain that

$$N_G((T_1 \times \cdots \times T_l)/(T_1 \times \cdots \times T_{l-1})) \leq N_G(T_l)$$

and

$$C_G((T_1 \times \cdots \times T_l)/(T_1 \times \cdots \times T_{l-1})) = C_G(T_l) \cap N_G((T_1 \times \cdots \times T_l)/(T_1 \times \cdots \times T_{l-1}))$$

for each $l \leq s$; hence $\text{Aut}_G(G_{i_0+1}/G_{i_0}) \leq \text{Aut}_G(G_{i_0+1}/G_{i_0})$. Similarly, in the general case $\text{Aut}_G(G_{i_{j+1}}/G_{i_{j+1}-1}) \leq \text{Aut}_G(G_{i_{j+1}}/G_{i_j})$. Now [1, Corollary 3.3] implies that if $\text{Aut}_G(G_{i_{j+1}}/G_{i_j}) \in E_\pi$, then $\text{Aut}_G(G_{i_{j+1}}/G_{i_{j+1}-1}) \in E_\pi$ for each $l \leq s$. Thus we obtain the following

**Corollary 6.** Let $1 = G_0 < G_1 < G_2 < \cdots < G_k = G$ be a refinement of a chief series $1 = G_0 = G_{i_0} < G_{i_1} < \cdots < G_{i_m} = G_k = G$. Then $G \in E_\pi$ if and only if for every non-abelian $G_{i_{j+1}}/G_{i_j}$ we have $\text{Aut}_G(G_{i_{j+1}}/G_{i_j}) \in E_\pi$.

So we have the following problem: describe the almost simple $E_\pi$-groups. Recall that a finite group $G$ is called almost simple if the generalized Fitting subgroup $F^*(G)$ of $G$ is simple, i.e., if $S \leq G \leq \text{Aut}(S)$ for a non-abelian finite simple group
S. If an arithmetic description of finite almost simple $E_\pi$-groups could be obtained, then Corollary 6 would give an exhaustive arithmetic criterion for determining whether a finite group $G$ satisfies $E_\pi$.

At the end of the paper we prove some corollaries to the main result. Corollary 7 is the main result of [12], but the solution presented in [12] is known to contain a gap\textsuperscript{1}, which at the time of writing has yet to be filled.

**Corollary 7.** A finite subdirect product of $E_\pi$-groups satisfies $E_\pi$.

Combining Proposition 2 and Corollary 7 one can obtain an affirmative answer to [13, Problem 5.65] and [9, Problem 18].

**Corollary 8.** For every set of primes $\pi$ the class of all $E_\pi$-groups is a formation.

In view of Proposition 2 a homomorphic image of a $\pi$-Hall subgroup is a $\pi$-Hall subgroup (in the homomorphic image of the $E_\pi$-group). The following corollary shows that all $\pi$-Hall subgroups of a homomorphic image can be obtained in this way.

**Corollary 9.** Every $\pi$-Hall of a homomorphic image of an $E_\pi$-group $G$ is the image of a $\pi$-Hall subgroup of $G$.

In contrast, there are examples showing that in general a normal subgroup of an $E_\pi$-group may possess $\pi$-Hall subgroups that do not lie in any $\pi$-Hall subgroup of the whole group (cf. Lemma 13 and Example 15).

The main technical tool in the proof of Theorem 4 is the following lemma, which is of independent interest.

**Lemma 10.** Let $S$ be a non-abelian finite simple $E_\pi$-group and suppose that $S < G \leq \text{Aut}(S)$ and $G \not\in E_\pi$. Then $2, 3 \in \pi \cap \pi(S)$ and there exists an element $x$ of $G$ such that $\langle x, S \rangle \not\in E_\pi$ and the order of $x$ is either a power of 2 or a power of 3.

To prove this lemma we shall use a theorem on the number of conjugacy classes of $\pi$-Hall subgroups in finite simple groups ([7, Theorem 1.1]).

## 2 Notation and preliminary results

The term ‘group’ always means ‘finite group’. By $\pi$ we always denote a set of primes. The expressions $H \leq G$ and $H \triangleleft G$ mean respectively that $H$ is a subgroup and a normal subgroup of $G$. The symbol $K_\pi(G)$ denotes the set of conjugacy classes of $\pi$-Hall subgroups of $G$ and $k_\pi(G) = |K_\pi(G)|$. The following lemma follows from the

\textsuperscript{1}In the proof of Lemma 3 the equality $H \cap \text{Aut}(P_i) = \text{Aut}_H(P_i)$ in line 16 of p. 220 is incorrect, so the statement $\text{Aut}_H(P_i) \in E_\pi$ is not proven.
Schur–Zassenhaus Theorem (see [3, Theorems D6 and D7] or [10, Chapter 5, Theorem 3.7], for example).

**Lemma 11.** If every factor of a subnormal series of \( G \) is either a \( \pi \)-group or a \( \pi' \)-group, then \( G \in D_\pi \).

**Lemma 12.** Let \( M, N \) be normal subgroups of a group \( G \) such that \( M \cap N = 1 \), and \( A, B \) subgroups of \( M \) such that \( B \leq A \). Let

\[
\tilde{i} : G \to G/N = \bar{G}
\]

be the natural homomorphism. Then \( N \leq C_G(A/B), \ N_G(A/B) = \bar{N}_G(A/B), \ \bar{C}_G(A/B) = C_{\bar{G}}(\bar{A}/\bar{B}) \), and \( \Aut_G(A/B) \approx \Aut_{\bar{G}}(\bar{A}/\bar{B}) \).

**Proof.** The inclusion \( N \leq C_G(A/B) \) is true since the product of \( M \) and \( N \) is direct. The inclusions \( N_G(A/B) \leq N_{\bar{G}}(\bar{A}/\bar{B}) \) and \( C_G(A/B) \leq C_{\bar{G}}(\bar{A}/\bar{B}) \) are evident. We need to show the reverse inclusions in order to complete the proof.

Let \( x \in N_{\bar{G}}(\bar{A}/\bar{B}) \) for some \( x \in G \). Then \( A^xN = AN \) and \( B^xN = BN \). Since \( M \leq \bar{G} \), then \( A^x \leq M \) and \( B^x \leq M \). Applying the coordinate projection map \( M \times N \to M \) to both parts of the equalities \( A^xN = AN \) and \( B^xN = BN \), we obtain \( A^x = A \) and \( B^x = B \). It follows that \( x \in N_G(A/B) \), and consequently we obtain the inclusion \( N_G(A/B) \supseteq N_{\bar{G}}(\bar{A}/\bar{B}) \).

Let \( x \in C_{\bar{G}}(\bar{A}/\bar{B}) \). Then for every \( a \in A \) there exist \( b \in B, \ g \in N \) such that \( a^x = bg \). Since \( a^x \in A \) and \( A, B \leq M \), the projection \( M \times N \to M \) with kernel \( N \) leaves \( a^x \) stable, and at the same time maps it into \( ba \). Therefore \( a^x = ba \). Thus \( C_G(A/B) \supseteq C_{\bar{G}}(\bar{A}/\bar{B}) \). \( \square \)

The proof of the following lemma can be found in [7, Lemma 2.1(e)], for example.

**Lemma 13.** Let \( A \) be a normal \( E_\pi \)-subgroup of \( G \) such that \( G/A \) is a \( \pi \)-group, and \( M \) a \( \pi \)-Hall subgroup of \( A \). Then there exists a \( \pi \)-Hall subgroup \( H \) of \( G \) satisfying \( H \cap A = M \) if and only if the set \( \{ M^a | a \in A \} \) is closed under conjugation by \( G \).

An extension of an \( E_\pi \)-group by an \( E_\pi \)-group may fail to possess a \( \pi \)-Hall subgroup, as is shown in the following known example.

**Example 14.** Let \( \pi = \{2, 3\} \) and let \( G = \GL_3(2) = \SL_3(2) \); thus \( G \) is a group of order \( 168 = 2^3 \cdot 3 \cdot 7 \). From [4, Theorem 1.2] it follows that \( G \) has exactly two classes of \( \pi \)-Hall subgroups with representatives

\[
\begin{pmatrix}
\GL_2(2) & * \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & * \\
0 & \GL_2(2)
\end{pmatrix}.
\]

The first consists of line stabilizers in the natural representation of \( G \), and the second consists of plane stabilizers. The map \( i : x \in G \mapsto (x^t)^{-1} \), where \( x^t \) denotes the trans-
pose of the matrix $x$, is an automorphism of order 2 of $G$. It interchanges classes of $\pi$-Hall subgroups, hence by Proposition 2 and Lemma 13 the natural extension $\hat{G} = G \rtimes \langle \iota \rangle$ does not possess a $\pi$-Hall subgroup.

The following example shows that a normal subgroup of an $E_\pi$-group may possess $\pi$-Hall subgroups that are not contained in a $\pi$-Hall subgroup of the whole group.

**Example 15.** Let $\pi = \{2, 3\}$. Let $G = \text{GL}_5(2) = \text{SL}_5(2)$; thus $G$ is a group of order $99999360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. Let $\iota : x \in G \mapsto (x^t)^{-1}$ and $\hat{G} = G \rtimes \langle \iota \rangle$ be the natural semidirect product. From [4, Theorem 1.2] it follows that there exist $\pi$-Hall subgroups of $G$, and every such subgroup is a stabilizer of a series of subspaces $V = V_0 < V_1 < V_2 < V_3 = V$, where $V$ is the natural module of $G$, and $\dim V_k/V_{k-1} \in \{1, 2\}$ for $k = 1, 2, 3$. Therefore, there are three conjugacy classes of $\pi$-Hall subgroups of $G$ with representatives

$$H_1 = \begin{pmatrix}
\text{GL}_2(2) & * \\
0 & \text{GL}_2(2)
\end{pmatrix},$$

$$H_2 = \begin{pmatrix}
1 & * \\
\text{GL}_2(2) & * \\
0 & \text{GL}_2(2)
\end{pmatrix}, \text{ and } H_3 = \begin{pmatrix}
\text{GL}_2(2) & * \\
0 & \text{GL}_2(2)
\end{pmatrix}.$$

Note that $N_{\hat{G}}(H_k) = H_k$ for $k = 1, 2, 3$, since $H_k$ is parabolic. The class containing $H_1$ is $\iota$-invariant. So the Frattini argument implies that $\hat{G} = GN_{\hat{G}}(H_1)$, whence $|N_{\hat{G}}(H_1) : N_{\hat{G}}(H_1)| = 2$ and $N_{\hat{G}}(H_1)$ is a $\pi$-Hall subgroup of $\hat{G}$. Moreover $\iota$ interchanges the classes containing $H_2$ and $H_3$. So, as in the previous example, these subgroups are not contained in $\pi$-Hall subgroups of $\hat{G}$.

3 Proof of the main theorem and the corollaries

**Proof of Lemma** 10. Suppose that $S$ is a finite simple $E_\pi$-group and $G$ is chosen so that $S < G \leq \text{Aut}(S)$ and $G \notin E_\pi$. Assume also that for every proper subgroup $M$ of $G$ with $S \leq M$ we have $M \in E_\pi$, i.e., $G$ is a minimal subgroup of $\text{Aut}(S)$ containing $S$ subject to $G \notin E_\pi$.

Notice that $G/S$ is a $\pi$-group. Indeed, by the Schreier conjecture $G/S \leq \text{Aut}(S)/S$ is solvable, hence it satisfies $D_\pi$. Consider a $\pi$-Hall subgroup $M/S$ of $G/S$. If $G/S$ is not a $\pi$-group, then $M$ is a proper subgroup of $G$, hence it satisfies $E_\pi$ in view of the minimality of $G$. So there exists a $\pi$-Hall subgroup $H$ of $M$. Since $[G : M] = [(G/S) : (M/S)]$ and $M/S$ is a $\pi$-Hall subgroup of $G/S$, we obtain that $[G : M]$ is a $\pi'$-number, so $H$ is a $\pi$-Hall subgroup of $G$. This contradicts $G \notin E_\pi$.

We have that $G$, acting by conjugation, permutes the elements of $K_\pi(S)$, so we obtain a homomorphism $\varphi : G \rightarrow \text{Sym}(K_\pi(S)) \simeq \text{Sym}_{k_\pi(S)}$. By Lemma 13 we ob-
tain that \( G \notin E_\pi \) if and only if \( G \) does not leave invariant any conjugacy class of \( \pi \)-Hall subgroups of \( S \), i.e., \( G^\phi \) is a subgroup of \( \text{Sym}_{k_\pi(S)} \) acting without fixed points.

If \( 2 \notin \pi \), then by [2, Theorem A], \( S \in C_\pi \), so \( G^\phi \simeq \text{Sym}_1 \) has a fixed point and this case is impossible.

If \( 3 \notin \pi \) then [6, Corollary 5.3] implies that \( S \) always possesses an \( \text{Aut}(S) \)-invariant class of conjugate \( \pi \)-Hall subgroups, so again \( G \) has a fixed point on \( K_\pi(S) \) and this case is impossible.

Suppose that \( 2, 3 \in \pi \). Then [7, Theorem 1.1] implies that either \( k_\pi(S) \in \{1, 2, 3, 4\} \), or \( k_\pi(S) = 9 \). As in the case \( 2 \notin \pi \) we see, that \( k_\pi(S) = 1 \) is impossible. By [7, Lemma 8.2] we obtain that if \( k_\pi(S) = 9 \), then \( S \) possesses an \( \text{Aut}(S) \)-invariant conjugacy class of \( \pi \)-Hall subgroups, hence \( G \) has a fixed point on \( K_\pi(S) \). So we may assume that \( 2 \leq k_\pi(S) \leq 4 \).

Assume that \( k_\pi(S) = 2 \) and \( G \notin E_\pi \). Then \( G^\phi \simeq \text{Sym}_2 \), so there exists an element \( x \in G \) such that \( x^\phi = (1, 2) \). Clearly we may assume that the order of \( x \) is a power of 2. By Lemma 13 we obtain \( \langle x, S \rangle \notin E_\pi \).

Assume that \( k_\pi(S) = 3 \). Since \( G \) has no fixed points on \( K_\pi(G) \), it acts transitively on \( K_\pi(S) \), so there exists an element \( x \in G \) such that \( x^\phi = (1, 2, 3) \). Clearly we may assume that the order of \( x \) is a power of 3. By Lemma 13 we obtain \( \langle x, S \rangle \notin E_\pi \).

Assume finally that \( k_\pi(S) = 4 \). Then \( G^\phi \), as a subgroup of \( \text{Sym}_4 \), acts without fixed points. Now every subgroup of \( \text{Sym}_4 \) that does not fix a point contains an element of type \((i, j)(k, l)\). Consider an element \( x \in G \) in the preimage of such an element \((i, j)(k, l)\) under the homomorphism \( G \to \text{Sym}_4 \). Since \(|(i, j)(k, l)| = 2\), we can choose \( x \) of order a power of 2. Furthermore \( x \) stabilizes no conjugacy class of \( \pi \)-Hall subgroups of \( S \), hence by Proposition 2 and Lemma 13, the group \( \langle x, S \rangle \) has no \( \pi \)-Hall subgroups. □

**Note.** Using the classification of \( \pi \)-Hall subgroups it is possible to show that if \( k_\pi(S) = 3 \), then \( S \) possesses an \( \text{Aut}(S) \)-invariant class of conjugate \( \pi \)-Hall subgroups. So the element \( x \) from Lemma 10 may be chosen to be a 2-element. However this fact needs a rather complicated analysis of simple \( E_\pi \)-groups with three conjugacy classes of \( \pi \)-Hall subgroups and we do not give it here.

**Proof of Theorem 4.** Assume by contradiction that \( G \) is an \( E_\pi \)-group, with composition series \( 1 = G_0 < G_1 < G_2 < \cdots < G_k = G \), and that \( \text{Aut}_G(G_i/G_{i-1}) \notin E_\pi \) for some \( i \). If \( G_i/G_{i-1} \) is abelian, then \( \text{Aut}(G_i/G_{i-1}) \) is cyclic. Since

\[
\text{Aut}_G(G_i/G_{i-1}) \leq \text{Aut}(G_i/G_{i-1}),
\]
we obtain that \( \text{Aut}_G(G_i/G_{i-1}) \in E_\pi \). Thus \( G_i/G_{i-1} \) is non-abelian. By Lemma 10 we have that \( 2, 3 \in \pi \cap \pi(G) \) and there exists \( x \in \text{Aut}_G(G_i/G_{i-1}) \) such that \( \langle x, G_i/G_{i-1} \rangle \notin E_\pi \) and the order of \( x \) is either a power of 2 or a power of 3. Let \( y \in N_G(G_i/G_{i-1}) \) be a preimage of \( x \); it may be chosen to be a 2-element or a 3-element. By Sylow’s theorem there exists a \( \pi \)-Hall subgroup \( H \) of \( G \) such that
By Proposition 2, $(H \cap G_i)G_{i-1}/G_{i-1}$ is a $\pi$-Hall subgroup of $G_i/G_{i-1}$. Moreover in view of the choice of $y$ we have $y \in N_G((H \cap G_i)G_{i-1}/G_{i-1})$, hence $x$ (as the image of $y$ in $N_G((G_i/G_{i-1})/C_G(G_i/G_{i-1}))$ normalizes a $\pi$-Hall subgroup $(H \cap G_i)G_{i-1}/G_{i-1}$ of $G_i/G_{i-1}$. By Lemma 13 we obtain that $\langle x, G_i/G_{i-1} \rangle \in E_\pi$, a contradiction. 

Now we are able to prove the corollaries.

**Proof of Corollary 7.** It is enough to prove that if $G$ is a finite group with normal subgroups $M$, $N$ such that $M \cap N = 1$ and $G/M$, $G/N \in E_\pi$, then $G \in E_\pi$. Clearly we may assume that both $M$ and $N$ are proper subgroups of $G$. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a composition series of $G$ which is a refinement of a chief series of $G$ through $M$. If $G_i > M$, then the condition $G/M \in E_\pi$ and Theorem 4 imply that $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$. If $G_i \leq M$, then the condition $M \cap N = 1$ and Lemma 12 imply that $\text{Aut}_G(G_i/G_{i-1}) \cong N_{G/N}((G_iN/N)/(G_{i-1}N/N))$. Condition $G/N \in E_\pi$ and Theorem 4 imply that $\text{Aut}_G(G_i/G_{i-1}) \cong (G_iN/N)/(G_{i-1}N/N) \in E_\pi$. Therefore by Corollary 5 we obtain that $G \in E_\pi$. 

**Proof of Corollary 9.** Let $G \in E_\pi$, $A \leq G$ and $M/A$ be a $\pi$-Hall subgroup of $G/A$. Let $1 = G_0 < G_1 < \cdots < G_k = G$ be a composition series of $G$ which is a refinement of a chief series of $G$ through $A$, so $G_n = A$ for some $n$. Let

$$1 = M_n/A < M_{n+1}/A < \cdots < M_m/A = M/A$$

be a composition series which is a refinement of a chief series of $M/A$. Then

$$1 = G_0 < G_1 < \cdots < G_n = A = M_n < M_{n+1} < \cdots < M_m = M$$

is a composition series of $M$ which is a refinement of a chief series of $M$. Since $M/A$ is a $\pi$-group, then, for each $i > n$, $\text{Aut}_M(M_i/M_{i-1})$ satisfies $E_\pi$. For every non-abelian composition factor $G_i/G_{i-1}$ with $i \leq n$ we have that

$$G_i/G_{i-1} \leq \text{Aut}_M(G_i/G_{i-1}) \leq \text{Aut}_G(G_i/G_{i-1}).$$

By Theorem 4, $\text{Aut}_G(G_i/G_{i-1})$ satisfies $E_\pi$ and [1, Corollary 3.3] implies that $\text{Aut}_M(G_i/G_{i-1})$ satisfies $E_\pi$. By Theorem 3, $M$ satisfies $E_\pi$. Hence there exists a $\pi$-Hall subgroup $H$ of $M$ and $H$ is a $\pi$-Hall subgroup of $G$. Since $M/A$ is a $\pi$-group, then $M/A = HA/A$. 

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