INJECTIVE SUBSETS OF $l_\infty(I)$

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Abstract. We give an explicit characterization of all injective subsets of the model space $l_\infty(I)$ for a general set $I$, in terms of inequalities involving 1-Lipschitz functions. Since the class of all injective metric spaces coincides with the one of all absolute 1-Lipschitz retracts, the present work yields a characterization of all the subsets of $l_\infty(I)$ that are absolute 1-Lipschitz retracts.

1. Introduction

A metric space $(X, d)$ is said to be injective if for every isometric embedding $i: A \to B$ of metric spaces $(A, d_A), (B, d_B)$ and every 1-Lipschitz map $f: A \to X$, there exists a 1-Lipschitz map $f'$ such that $f = f' \circ i$. Basic examples of injective metric spaces include $\mathbb{R}$, the space $l_\infty(I)$ for any set $I$, as well as $\mathbb{R}$-trees. By $l_\infty(I)$ is meant the space of all real-valued bounded functions endowed with the norm $\|f\|_\infty := \sup_{i \in I} |f_i|$ which we denote by $\|f\|$ for notational convenience.

In the context of subsets of $l_\infty(I)$, we need to introduce some pieces of notation. Let moreover $\pi_i: l_\infty(I) \to l_\infty(I \setminus \{i\})$ be the map given by dropping the $i$-th coordinate.

This work proves a metric characterization of the injective subsets of $l_\infty(I)$ in terms of systems of inequalities given by 1-Lipschitz functions. Namely,

1.1. Theorem. A non-empty subset $Q$ of $l_\infty(I)$ is injective if and only if it can be written as

$$Q = \{ x \in l_\infty(I) : (r_i \circ \pi_i) \leq x_i \leq (\bar{r}_i \circ \pi_i) \text{ for all } i \in I \}$$

where $r_i, \bar{r}_i: l_\infty(I \setminus \{i\}) \to \mathbb{R}$ are 1-Lipschitz maps satisfying $r_i \leq \bar{r}_i$, that is $r_i(y) \leq \bar{r}_i(y)$ for all $y \in l_\infty(I \setminus \{i\})$, possibly dropping a subcollection of the collection of all inequalities appearing in (1.1).

The characterization is explicit, namely it provides a concrete expression for each single injective subset of each of the model spaces $l_\infty(I)$. The proof of this characterization is based on the proof by the first author of the characterization in the particular case where $I = \{1, \ldots, n\}$, that is in the particular case where $l_\infty(I)$ corresponds to $\mathbb{R}^n$ endowed with the maximum norm cf. [3]. In the next section, we prove among others the equivalence between hyperconvexity and the absolute 1-Lipschitz retract property.

2. Preliminaries on Absolute 1-Lipschitz Retracts, Hyperconvexity and Isbell’s Injective Hull

Let us start by recalling two characterizations of injective metric spaces. A metric space $(X, d)$ is called an absolute 1-Lipschitz retract (or 1-ALR) if for every
isometric embedding \( i: X \to Y \) into a metric space \( Y \), there exists a 1-Lipschitz retraction of \( Y \) onto \( i(X) \). To show that this property is equivalent to injectivity, assume first that \( X \) is injective. If now \( i: X \to Y \) is an isometric embedding, \( i(X) \) is injective. Thus, the identity map on \( i(X) \) extends to a 1-Lipschitz retraction \( g: Y \to i(X) \) showing that \( X \) is a 1-ALR. Conversely, every metric space \( X \) embeds isometrically into \( l_\infty(X) \) via a Kuratowski embedding \( k_{x_0}: x \mapsto d_x - d_{x_0} \) for an arbitrarily chosen base point \( x_0 \in X \). Hence, if \( X \) is a 1-ALR, then \( k_{x_0}(X) \) is a 1-Lipschitz retract in \( l_\infty(X) \). Therefore, by injectivity of \( l_\infty(X) \) it follows that \( X \) is injective as well.

Another characterization of injective metric spaces relies upon an intersection property of metric balls. In a metric space \((X, d)\), we use the notation \( B(x, r) := \{ y \in X : d(x, y) \leq r \} \). One says that \((X, d)\) is hyperconvex if every family \( \{ (x_\gamma, r_\gamma) \}_{\gamma \in \Gamma} \) in \( X \times \mathbb{R} \) satisfying \( r_\beta + r_\gamma \geq d(x_\beta, x_\gamma) \) for all pairs of indices \( \beta, \gamma \in \Gamma \), has the property that \( \bigcap_{\gamma \in \Gamma} B(x_\gamma, r_\gamma) \neq \emptyset \). (As a matter of convention, the intersection equals \( X \) if \( \Gamma = \emptyset \), so that hyperconvex spaces are non-empty by definition.) To see that injectivity implies hyperconvexity, let \( \{ (x_\gamma, r_\gamma) \}_{\gamma \in \Gamma} \subset X \times \mathbb{R} \) be a family with \( r_\beta + r_\gamma \geq d(x_\beta, x_\gamma) \) for all \( \beta, \gamma \in \Gamma \). Let \( A := \{ (x_\gamma)_{\gamma \in \Gamma} \} \) be endowed with the metric \( d_A \) induced by \((X, d)\). Set \( B := A \cup \{ b \} \) where \( d_B(x_\gamma, b) := r_\gamma \). By our assumptions, \( d_B \) defines a metric on \( B \). By injectivity of \( X \), there is a 1-Lipschitz map \( j: B \to X \) such that the inclusions \( j: A \to B \) and \( i: A \to B \) satisfy \( j \circ i = j \). It follows that \( j(b) \in \bigcap_{\gamma \in \Gamma} B(x_\gamma, r_\gamma) \), which in turn shows that \( X \) is hyperconvex. For the proof of the converse, note that if \( f: A \to X \) is 1-Lipschitz, \( i: A \to B \) is an isometric embedding, and \( b \in B \setminus i(A) \), then \( d_B(i(a), b) + d_B(i(a'), b) \geq d_A(a, a') \geq d(f(a), f(a')) \) for all \( a, a' \in A \). Hence, if \( X \) is hyperconvex, then \( S := \bigcap_{a \in A} B(f(a), d_B(i(a), b)) \) is non-empty, and one obtains a 1-Lipschitz map \( f_b: i(A) \cup \{ b \} \to X \) by setting \( f_b(i(a)) := f(a) \) on \( A \) and taking \( f_b(b) \) to be any point in \( S \). Using Zorn’s lemma one can prove the existence of a map \( f \) with the desired properties, from which one deduces that \( X \) is injective. A direct consequence of this characterization is that the intersection of any family of closed balls in an injective metric space is itself injective if and only if it is non-empty.

Isbell proved that every metric space \((X, d)\) possesses an injective hull \((i, Y)\) (later, when considering Isbell’s injective hull \((e, E(X))\) we usually write \( E(X) \) for simplicity) which is unique up to isometry and minimal among injective spaces containing an isometric copy of \( X \). This means that \( i: X \to Y \) is an isometric embedding with the following property: whenever there is a metric space \( Z \) and a 1-Lipschitz map \( h: Y \to Z \) so that \( h \circ i \) is an isometric embedding, it follows that \( h \) is an isometric embedding as well.

We now need to give a short outline of some elements of Isbell’s construction for later use. For a more comprehensive introduction to injective spaces and the construction of \( E(X) \), see for instance [9]. Given a metric space \((X, d)\), we denote by \( \mathbb{R}^X \) the vector space of real-valued functions defined on \( X \) and we set

\[
\Delta(X) := \{ f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X \}. \tag{2.1}
\]

For \( f, g \in \mathbb{R}^X \), the inequality \( g \leq f \) means that \( g(x) \leq f(x) \) for all \( x \). A function \( f \in \Delta(X) \) is called extremal if there is no \( g \leq f \) in \( \Delta(X) \) different from \( f \). One can show that the collection \( E(X) \) of all extremal functions in \( \Delta(X) \) is equivalently
given by
\[ E(X) = \{ f \in \mathbb{R}^X : f(x) = \sup_{y \in X} \langle d(x, y) - f(y) \rangle \text{ for all } x \in X \}. \tag{2.2} \]
Thus, \( f \in E(X) \) if and only if \( f \in \Delta(X) \) and for every \( x \in X \) and \( \varepsilon > 0 \), there is an \( y \in X \) so that
\[ f(x) + f(y) \leq d(x, y) + \varepsilon. \tag{2.3} \]
Applying the equation defining the members of \( E(X) \) twice, we obtain that for \( f \in E(X) \) and for all \( x, x' \in X \), one has
\[ f(x) - d(x, x') = \sup_{y \in X} (d(x, y) - d(x, x') - f(y)) \leq f(x'). \]
This implies that extremal functions are 1-Lipschitz. One can also prove that the map \((f, g) \mapsto \sup_{x \in X} |f(x) - g(x)| \) endows \( E(X) \) with a metric. It follows from Lemma 3.2. (ii) that every function in \( \Delta(X) \) is non-negative. It is easy to see that for each \( x \in X \), the function \( d_x \) given by the assignment \( y \mapsto d(x, y) \) is extremal, and the functions of this form are the only extremal functions with zeros. One can show that \((e, E(X))\) is in fact an injective hull of \( X \). Thus, to every metric space \((X, d)\) corresponds Isbell’s injective hull \((e, E(X))\) where \( E(X) \) isometrically embeds into \( l_\infty(X) \) via a Kuratowski embedding.

3. The Characterization

3.1. Theorem \([2, 4]\). If \( \Theta \) is a totally ordered set and \((H_0)_{\theta \in \Theta}\) is a decreasing family of nonempty bounded hyperconvex spaces, then the intersection is hyperconvex.

Recall that for a metric space \((X, d)\), we write \( B(x, r) := \{ y \in X : d(x, y) \leq r \} \).

A non-empty subset \( A \) of a metric space \((X, d)\) is externally hyperconvex in \( X \) if for any collection \( \{ (x_{\gamma}, r_{\gamma}) \}_{\gamma \in \Gamma} \subset X \times [0, \infty) \) satisfying \( d(x_{\beta}, x_{\gamma}) \leq r_{\beta} + r_{\gamma} \) and \( d(x_{\gamma}, A) \leq r_{\gamma} \) for all \( \beta, \gamma \in \Gamma \), one has \( A \cap \bigcap_{\gamma \in \Gamma} B(x_{\gamma}, r_{\gamma}) \neq \emptyset \).

3.2. Lemma. Let \((X, d)\) be a metric space and \( x_0 \in X \). Then, the following are equivalent

(i) \((X, d)\) is injective,

(ii) \( B(x_0, R) \) is injective for every \( R \in [0, \infty) \).

Proof. It is clear that (i) implies (ii) since \( X \) is hyperconvex. To prove the other implication, let \( \{ (x_{\gamma}, r_{\gamma}) \}_{\gamma \in \Gamma} \subset X \times [0, \infty) \) be such that \( d(x_{\beta}, x_{\gamma}) \leq r_{\beta} + r_{\gamma} \). Fix \( \alpha \in \Gamma \) and for every \( \beta, \gamma \in \Gamma \), define
\[ A_{\gamma} := B(x_{\gamma}, r_{\gamma}) \cap B(x_{\alpha}, r_{\alpha}), \]
\[ R_{\beta\gamma} := \max \{ d(x_{\alpha}, x_{\beta}), d(x_{\alpha}, x_{\gamma}) \} + d(x_{\alpha}, x_0) + r_{\alpha}. \]
One has \( x_{\beta}, x_{\gamma} \in B(x_0, R_{\beta\gamma}) \) and this last set is hyperconvex by assumption, it follows that

(a) \( B(x_{\alpha}, r_{\alpha}) \) is hyperconvex (since \( B(x_{\alpha}, r_{\alpha}) \subset B(x_0, R_{\beta\gamma}) \)),

(b) \( A_{\gamma} \) is externally hyperconvex in \( B(x_{\alpha}, r_{\alpha}) \) for each \( \gamma \in \Gamma \),

(c) \( A_{\beta} \cap A_{\gamma} \neq \emptyset \) for every \( \beta, \gamma \in \Gamma \).

Hence, it follows by \([7]\) Proposition 1.2 that \( \bigcap_{\gamma \in \Gamma} B(x_{\gamma}, r_{\gamma}) = \bigcap_{\gamma \in \Gamma} A_{\gamma} \neq \emptyset. \) □
For the characterization of Theorem 1.1 to hold, we may (as stated there) need to drop an arbitrary number of the inequalities. In order to treat all cases in a uniform way we do the following. Set \( \mathbb{R} := \{ -\infty \} \cup \mathbb{R} \cup \{ \infty \} \) and endow it with the obvious total order (also, for example max\( \{ x, -\infty \} = x \) for all \( x \in \mathbb{R} \)). Then we allow the 1-Lipschitz bounds \( \underline{x}_i, \overline{x}_i \) to take values in \( \mathbb{R} \). And a map \( l_\infty (I \setminus \{ i \}) \to \mathbb{R} \) is called \textit{1-Lipschitz} if it is either constant (allowing the image to be \( \{ -\infty \} \) or \( \{ \infty \} \)) or it is real valued and 1-Lipschitz in the usual sense. Now instead of dropping, say, a lower bound \( \overline{x}_i \), we just set \( \overline{x}_i = -\infty \) and the inequality \( (\overline{x}_i \circ \overline{F}_i)(x) \leq x_i \) means that no condition is imposed. The next proposition proves one of the two implications in Theorem 1.1.

3.3. \textbf{Proposition.} For every \( i \in I \) let \( \underline{x}_i, \overline{x}_i : l_\infty (I \setminus \{ i \}) \to \mathbb{R} \) be a pair of 1-Lipschitz functions such that \( \underline{x}_i \leq \overline{x}_i \). Define

\[
Q := \{ x \in l_\infty (I) : (\underline{x}_i \circ \overline{F}_i)(x) \leq x_i \leq (\overline{x}_i \circ \overline{F}_i)(x) \text{ for all } i \in I \}
\]

and assume that this set is non-empty. Then \( Q \) is injective.

\textbf{Proof.} Let \( \mathcal{R} \) denote the set \( \{ \underline{x}_i : i \in I \} \cup \{ \overline{x}_i : i \in I \} \). Let \( \pi_i : l_\infty (I) \to \mathbb{R} \) be the \( i \)-th coordinate projection. We divide the proof into three steps.

\textbf{First Step:} We first show the statement in the case \( \mathcal{R} \) is a set of \( \lambda \)-Lipschitz functions for some \( \lambda \in [0, 1) \) and \( F \) is a finite subset of \( I \) such that \( \underline{x}_i = -\infty, \overline{x}_i = \infty \) for all \( i \in I \setminus F \). Thus we only have a finite number of non-trivial inequalities. Assume without loss of generality that \( F = \{ 1, \ldots, N \} \). For \( i \in F \) and any \( x \in l_\infty (I) \), let us define \( g_1 \in \text{Lip}_1(l_\infty (I), l_\infty (I)) \) implicitly through

\[
\pi_j \circ g_i(x) = \begin{cases}
\min \{ \max \{ x_i, (\underline{x}_i \circ \overline{F}_i)(x) \}, (\overline{x}_i \circ \overline{F}_i)(x) \} & \text{if } j = i, \\
x_j & \text{otherwise}.
\end{cases}
\]

Moreover, set \( G_0 := \text{id}_{l_\infty (I)} \), as well as

\[
G_i := g_i \circ \cdots \circ g_1
\]

and

\[
T := G_N = g_N \circ \cdots \circ g_1.
\]

Fix now \( x \in l_\infty (I) \). We show that \( (T^m(x))_{m \in \mathbb{N}} \) converges to a fixed point of \( T \). Let us define the maps \( \{ f_i \}_{i \in F} \subset \text{Lip}_\lambda (l_\infty (I), \mathbb{R}) \) by

\[
f_i : y \mapsto \min \{ \max \{ \alpha_i, (\underline{x}_i \circ \overline{F}_i)(y) \}, (\overline{x}_i \circ \overline{F}_i)(y) \},
\]

where \( \alpha_i := (\pi_i \circ G_i^{-1} \circ T^m)(x) = (\pi_i \circ G_i \circ T^{m-1})(x) \). We further set

\[
\beta_i := \left| \pi_i \left( (G_i \circ T^m)(x) - T^m(x) \right) \right|
\]

for any \( i \in F \) and observe that

\[
\beta_i = \left| \pi_i \left( (G_i \circ T^m)(x) - (G_i \circ T^{m-1})(x) \right) \right|
\]

\[
= \left| \pi_i \left( (G_i \circ T^m)(x) - (g_i \circ G_i \circ T^{m-1})(x) \right) \right|
\]

\[
= \left| (f_i \circ G_i^{-1} \circ T^m)(x) - (f_i \circ G_i \circ T^{m-1})(x) \right|
\]

\[
\leq \lambda \left| \left( G_i^{-1} \circ T^m \right)(x) - \left( G_i \circ T^{m-1} \right)(x) \right|
\]

\[
\leq \lambda \left| T^m(x) - T^{m-1}(x) \right|.
\]
Thus
\[ \|T^{m+1}(x) - T^m(x)\| \leq \max_{i \in F} \beta_i \leq \lambda \|T^m(x) - T^{m-1}(x)\|. \]

It easily follows that \((T^m(x))_{m \in \mathbb{N}}\) is a Cauchy sequence and thus converging to a fixed point \(x^*\) of \(T\). This implies in particular that \(x^* \in Q\). Hence, we can define the 1-Lipschitz retraction \(q : l_\infty(I) \to Q\) to be the pointwise limit of the sequence \((T^m)_{m \in \mathbb{N}}\). It follows that \(Q\) is injective.

**Second Step:** We now prove the statement in case the functions in \(\mathcal{R}\) are only assumed to be 1-Lipschitz but keeping the assumption about the finite subset \(F \subset I\). Moreover, we assume without loss of generality that \(0 \in Q\). By Lemma 3.2, it is enough to show that for any \(R > 0\), the set \(Q \cap B(0, R)\) is injective. For each \(i \in I\), we set
\[
(s_i \circ \pi_i)(x) := \min\{\max\{(s_i \circ \pi_i)(x), -R\}, R\}
\]
\[
(\pi_i \circ \pi_i)(x) := \min\{\max\{(\pi_i \circ \pi_i)(x), -R\}, R\}.
\]

Using \(0 \in Q\), a short calculation yields that \(s_i \leq \pi_i\) implies \(-R \leq s_i \leq \pi_i \leq R\). Set
\[
P := \left\{ x \in B(0, R) : (s_i \circ \pi_i)(x) \leq x_i \leq (\pi_i \circ \pi_i)(x) \text{ for all } i \in F \right\}.
\]

Since the functions \(s_i, \pi_i\) are 1-Lipschitz and using that \(0 \in Q\) again, one has
\[ P = Q \cap B(0, R). \]

We can thus, for \(k \in \mathbb{N}\) and \(i \in F\), set \(\lambda_k := 1 - \frac{1}{k}\) and define \(s_i^k, \pi_i^k\) through
\[
(s_i^k \circ \pi_i^k)(x) = \lambda_k ((s_i \circ \pi_i^k)(x) + R) - R,
\]
\[
(\pi_i^k \circ \pi_i^k)(x) = \lambda_k ((\pi_i \circ \pi_i^k)(x) - R) + R.
\]

Note that
\[ -R \leq s_i^k \leq s_i \leq \pi_i \leq \pi_i^k \leq R. \]

For any \(k \in \mathbb{N}\), we now set
\[
Q_k := \left\{ x \in B(0, R) : (s_i^k \circ \pi_i^k)(x) \leq x_i \leq (\pi_i^k \circ \pi_i^k)(x) \text{ for all } i \in F \right\}.
\]

The functions in \(\mathcal{R}^k := \{s_i^k : i \in F\} \cup \{\pi_i^k : i \in F\}\) are all \(\lambda_k\)-Lipschitz. Hence, we can apply the first step and define the 1-Lipschitz retraction \(q^k : B(0, R) \to Q_k\) to be the pointwise limit of the sequence \((T^m)_{m \in \mathbb{N}}\). Since \(B(0, R)\) is injective, it follows that \(Q_k\) is injective. Finally, since the sequence \((Q_k)_{k \in \mathbb{N}}\) is decreasing for the inclusion and
\[
\bigcap_{k \in \mathbb{N}} Q_k = P = Q \cap B(0, R),
\]
it follows that \(Q \cap B(0, R)\) is injective by Theorem 3.1. So
\[ Q = \{ x \in l_\infty(I) : (s_i \circ \pi_i)(x) \leq x_i \leq (\pi_i \circ \pi_i)(x) \text{ for all } i \in F \}
\]
is injective by Lemma 3.2.

**Third Step:** Let \(\mathcal{F}\) be the family of all finite subsets of \(I\). For every \(F \in \mathcal{F}\), let
\[
Q^F := \{ x \in l_\infty(I) : (s_i \circ \pi_i)(x) \leq x_i \leq (\pi_i \circ \pi_i)(x) \text{ for all } i \in F \}.
\]
As it is shown just above, $Q^F$ is injective. Therefore, for every $R \in [0, \infty)$, the set $A^F := Q^F \cap B(0, R)$ is injective by Lemma 3.2. Let

\[ M := \{ J \subset I : A^{J \cup F} \text{ is injective for all } F \in \mathcal{F} \} \]

be partially ordered by inclusion. By the second step, $\emptyset \in M$. Moreover, if $F \in \mathcal{F}$ and if $(J_\gamma)_{\gamma \in \Gamma}$ is a chain in $M$, we can set $J_F := \bigcup_{\gamma \in \Gamma} J_\gamma$ and we obtain by Theorem 3.1 that $A^{J \cup F} = \bigcap_{\gamma \in \Gamma} A^{J_\gamma \cup F}$ which is injective by Theorem 3.1. We can thus use Zorn’s lemma to deduce the existence of a maximal element $M \in M$. By maximality, it follows that $M = I$, which implies that the set

\[ A^I = Q \cap B(0, R) \]

is injective as well. Again by Lemma 3.2, it follows that $Q$ is injective and this concludes the proof. \qed

3.4. Remark. Let $X$ and $Y$ be metric spaces, and let $i : X \to Y$ be an isometric embedding. As stated in \[3\] (3) in Proposition 3.4, the following are equivalent:

(i) $(i, Y)$ is an injective hull of $X$.
(ii) $(i, Y)$ is a minimal injective extension of $X$, that is, $Y$ is injective and no proper subspace of $Y$ containing $i(X)$ is injective.

It follows that if $X$ is an injective metric space, then $e(X) = E(X)$. In our case it follows that if $Q \subset l_{\infty}(I)$ is injective, then the distance functions $\{d_q\}_{q \in Q}$ are the only extremal functions in $\Delta(Q)$. Therefore, given any point $x \in l_{\infty}(I) \setminus Q$, the function $d_x \in \mathbb{R}^Q$ given by the assignment $q \mapsto \|x - q\|$ verifies $d_x(Q) \in (0, \infty)$, thus $d_x \notin e(Q) = E(Q)$, in other words $d_x \in \Delta(Q)$ is not extremal.

In the context of subsets of $l_{\infty}(I)$, we need to introduce some pieces of notation. Recall that $\pi_i : l_{\infty}(I) \to \mathbb{R}$ is the $i$-th coordinate projection. For $i \in I$, we set

\[ C_i := \{ x \in l_{\infty}(I) : x_i = \|x\| \}. \]

For $S \subset l_{\infty}(I)$, we define $-S := \{ -s : s \in S \}$ and we write $p + S$ for the set $\{ p + s : s \in S \}$, i.e. the translate of $S$ by $p$. Note that the interior of $C_i$ satisfies

\[ \text{Interior}(C_i) := \left\{ x \in l_{\infty}(I) : x_i > \sup_{j \in I \setminus \{i\}} |x_j| \right\}. \]

The next proposition is the last piece needed to prove Theorem 1.1.

3.5. Proposition. If $Q \subset l_{\infty}(I)$ is injective, then $Q$ satisfies the assumptions of Proposition 3.4.

Proof. The injective subsets of $\mathbb{R}$ are exactly the closed intervals and the result clearly holds in this case. Therefore, assume that $|I| \geq 2$. By Remark 3.4 we may assign to every $x \in l_{\infty}(I) \setminus Q$ the positive quantity

\[ \varepsilon(x) := \sup\{ \varepsilon \in \mathbb{R} : \text{ there is } p \in Q \text{ with } \|x - p\| + \|x - q\| \geq \|p - q\| + \varepsilon \text{ for all } q \in Q \} \]

Choosing $q \in Q$ such that $\|x - q\| = d(x, Q)$, one can easily see that $\varepsilon(x) \leq 2d(x, Q)$. For every $x \in l_{\infty}(I) \setminus Q$, let $p_x \in Q$ be such that for every $q \in Q$, one has

\[ \inf_{q \in Q} \left( \|x - p_x\| + \|x - q\| - \|p_x - q\| \right) \geq \frac{\varepsilon(x)}{2}. \]
Next we select a cone $C_x$ for every $x \in l_\infty \setminus Q$. To that end, let $\alpha$ be some positive real parameter. We determine the value of $\alpha$ in the course of the proof. For any $\delta \in (0, \frac{1}{2}\|x - p_x\|)$, one can find $i \in I$ such that

$$|\pi_i(x - p_x)| \geq \|x - p_x\| - \delta > 0.$$  

($\delta$ is the additional parameter needed to generalize from finite to infinite index sets $I$.) Now let $e^i \in l_\infty(I)$ be given by $e^i_j = \delta_{ij}$ i.e. $e^i$ is everywhere equal to zero except at $i$ where it is equal to one. We set

$$C_x := \begin{cases} x - \alpha \varepsilon(x)e^i + C_i & \text{if } \pi_i(x - p_x) > 0, \\ x + \alpha \varepsilon(x)e^i - C_i & \text{if } \pi_i(x - p_x) < 0. \end{cases}$$

Observe that $\alpha \varepsilon(x)e^i \in \text{Interior}(C_x)$ holds for every $x$. Assume that we are in the case $C_x := x - \alpha \varepsilon(x)e^i + C_i$ (the case $C_x := x + \alpha \varepsilon(x)e^i - C_i$ is analogue). Assume that $Q \cap C_x \neq \emptyset$ and pick $q$ in this intersection, one then has

$$w := q + \alpha \varepsilon(x)e^i \in C_x + \alpha \varepsilon(x)e^i = x + C_i.$$  

Therefore $\|w - x\| = \pi_i(w - x)$ and thus

$$\|w - p_x\| \geq |\pi_i(w - p_x)| = |\pi_i(w - x + p_x)| = |\pi_i(w - x)| + |\pi_i(x - p_x)|$$  

$$\geq \|w - x\| + \|x - p_x\| - \delta.$$  

Consequently

$$\|p_x - q\| \geq \|p_x - x\| + \|x - q\| - 2 \alpha \varepsilon(x) - \delta.$$  

Choosing $\alpha < \frac{1}{8}$ and $\delta < \frac{\varepsilon(x)}{4}$ we obtain a contradiction to the definition of $p_x$. Hence, we do so and obtain $Q \cap C_x = \emptyset$ for all $x \in l_\infty(I) \setminus Q$.

For every $i$, the function $\tilde{r}_i$ is defined to be the pointwise infimum over the family of $1$-Lipschitz functions $l_\infty(I \setminus \{i\}) \to \mathbb{R}$ defined by the assignement

$$y \mapsto ||\tilde{\pi}_i(x) - y|| + \pi_i(x) - \alpha \varepsilon(x)$$

where every $x$ such that $C_x = x - \alpha \varepsilon(x)e^i + C_i$ contributes exactly one member. If there is no such $x$, we let $\tilde{r}_i := \infty$. Similarly, $\tilde{r}_i := -\infty$ if there is no $x$ with $C_x = x + \alpha \varepsilon(x)e^i - C_i$ or otherwise the supremum over all functions

$$y \mapsto ||\tilde{\pi}_i(x) - y|| + \pi_i(x) + \alpha \varepsilon(x)$$

for $x$ with $C_x = x + \alpha \varepsilon(x)e^i - C_i$. It is not difficult to deduce from $x \in \text{Interior}(C_x)$ and $Q \cap C_x = \emptyset$ that

$$Q = \{x \in l_\infty(I) : \tilde{\pi}_i(x) \leq x_i \leq \tilde{r}_i(\tilde{\pi}_i(x)) \text{ for } i \in I\}.$$  

It remains to be shown that the inequalities $r_i \leq \tilde{r}_i$ hold. First, note that if $\sum_i(p) > \tilde{r}_i(p)$ at some $p \in l_\infty(I \setminus \{i\})$, then there are points $x, y \in l_\infty(I) \setminus Q$ with $C_x := x - \alpha \varepsilon(x)e^i + C_i$ and $C_y := y + \alpha \varepsilon(y)e^i - C_i$ such that the intersection $\text{Interior}(C_x) \cap \text{Interior}(C_y)$ is non-empty. To show that this can not happen for appropriate choice of $\alpha$, we assume that $\text{Interior}(C_x) \cap \text{Interior}(C_y) \neq \emptyset$ and start by noting that the apex $x - \alpha \varepsilon(x)e^i$ of $C_x$ lies in $\text{Interior}(C_y)$. Therefore $x' := x - \alpha \varepsilon(x)e^i - \alpha \varepsilon(y)e^i$ lies in $\text{Interior}(y - C_i)$ and the same holds for $p_x' := p_x - \alpha \varepsilon(x)e^i - \alpha \varepsilon(y)e^i$ since $p_x \in x - C_i$. 

So we have
\[
\|x - p_x\| + |x - p_y| \leq \|x' - p'_x\| + \|x' - p_y\| + \alpha(\varepsilon(x) + \varepsilon(y))
\]
\[
= \|p'_x - p_y\| + \alpha(\varepsilon(x) + \varepsilon(y))
\]
\[
\leq \|p'_x - p_y\| + 2\alpha(\varepsilon(x) + \varepsilon(y)),
\]
hence by definition of \(p_x\) and \(p_y\) this leads to \(\varepsilon(x) \leq 4\alpha(\varepsilon(x) + \varepsilon(y))\) and \(\varepsilon(y) \leq 4\alpha(\varepsilon(x) + \varepsilon(y))\), respectively. Now take \(\alpha < \frac{1}{4}\). The sum of the last two inequalities involving \(\varepsilon(x)\) and \(\varepsilon(y)\) then yields a contradiction. Therefore, \(\text{Interior}(C_x) \cap \text{Interior}(C_y)\) is empty and this finishes the proof. \(\square\)

4. Examples

We start with a remark.

4.1. Remark. Any codimension one linear subspace \(V\) of \(l_\infty(I)\) is injective if and only if there is an \(i \in I\) such that
\[
V \subset l_\infty(I) \setminus (\text{Interior}(C_i) \cup \text{Interior}(C_j)).
\]
We first show that if \(V\) is injective, there exists a coordinate \(i\) as in (4.1). Assume that the converse holds, namely that for every \(j \in I\), one can pick
\[
v^j \in V \cap (\text{Interior}(C_j) \cup \text{Interior}(C_j)),
\]
Choose arbitrarily \(p \in l_\infty(I)\) and assume without loss of generality that \(v^j \in V \cap \text{Interior}(C_j)\). Note that for any \(\alpha \in [0, \infty)\), one has \(\alpha v^j \in V \cap \text{Interior}(C_j)\). Let \(A(v^j) := |\pi_j(v^j)|\) and \(B(v^j) := \sup_{t \in I \setminus \{j\}} |\pi_t(v^j)|\). Clearly, there is then an \(\varepsilon \in [0, \infty)\) such that \(A(v^j) \geq (1 + \varepsilon)B(v^j)\). A short calculation shows that putting \(\alpha_j := \frac{2\|p\|}{A(v^j)}\), one obtains \(A(\alpha_j v^j) - B(\alpha_j v^j) \geq 2\|p\|\). By choice of \(\alpha_j\), it then follows that
\[
|\pi_j(p_j - \alpha_j v^j)| = \|p_j - \alpha_j v^j\|
\]
\[
|\pi_j(p_j + \alpha_j v^j)| = \|p_j + \alpha_j v^j\|.
\]
Therefore, one infers that
\[
\{p\} = \bigcap_{j \in I} \left( B(\alpha_j v^j, \|p - \alpha_j v^j\|) \cap B(-\alpha_j v^j, \|p + \alpha_j v^j\|) \right).
\]
Picking \(p \notin V\), it follows that \(V\) is not hyperconvex and thus not injective. This shows that if \(V\) is injective, such a coordinate as in (4.1) exists. Conversely, if such a coordinate \(i\) exists, then \(V\) can be expressed as
\[
V = \left\{ x \in l_\infty(I) : (\underline{\pi_i} \circ \pi_i)(x) \leq x_i \leq (\overline{\pi_i} \circ \pi_i)(x) \right\}
\]
where
\[
(\underline{\pi_i} \circ \pi_i)(x) := \sup_{y \in l_\infty(I) \setminus V} (y_i - \|\pi_i(x) - \pi_i(y)\|)
\]
and
\[
(\overline{\pi_i} \circ \pi_i)(x) := \inf_{y \in l_\infty(I) \setminus V} (y_i + \|\pi_i(x) - \pi_i(y)\|).
\]
It is easy to see by (4.1) that \(V\) is a subset of the right-hand side of (4.2). Now, we prove that the complement of \(V\) is contained in the complement of the right-hand side of (4.2). Consider the map \(\pi_i^V : l_\infty(I) \to V\) which corresponds to the projection onto \(V\) along the \(i\)-th coordinate which is a well-defined map since \(i\) satisfies (4.2).
For any \( y \notin V \), one either has \( \pi_i(y) > \pi_i(\pi_i(y)) \) which implies \( \pi_i(y) > (\pi_i \circ \pi_i)(y) \) or \( \pi_i(y) < \pi_i(\pi_i(y)) \) which implies \( \pi_i(y) < (\pi_i \circ \pi_i)(y) \). This proves the desired inclusion and thus that (4.2) holds. By Proposition 3.3 it follows that \( V \) is injective and this finishes the proof of the equivalence.

4.2. Example. In case \( I := \mathbb{N} \) we consider the set \( V := \ker(\Lambda) \) where \( \Lambda: l_\infty(\mathbb{N}) \to \mathbb{R} \) denotes a real Banach limit, namely \( \Lambda \in (l_\infty(\mathbb{N}))^* \) satisfies the following properties

1. Let \( x := (x_n)_{n \in \mathbb{N}} \) be a sequence with non-negative terms, then \( \Lambda(x) \geq 0 \).
2. If \( S: l_\infty(\mathbb{N}) \to l_\infty(\mathbb{N}) \) denotes the left-shift operator given by the relation \( \pi_n \circ S = \pi_{n+1} \), one has \( \Lambda \circ S = \Lambda \).
3. For every convergent sequence \( x := (x_n)_{n \in \mathbb{N}} \), one has \( \Lambda(x) = \lim_{n \to \infty} x_n \).

One can see that by invariance of the Banach limit under left-shift, \( V \) contains all sequences having only finitely many non-zero entries. It is then easy to see that there is no \( i \in \mathbb{N} \) satisfying (4.1) and thus \( V \) is not injective.

4.3. Example. In case \( I := \mathbb{N} \) we can also consider the set \( V := \ker(\Phi) \) where \( \Phi: l_\infty(\mathbb{N}) \to \mathbb{R} \) denotes an element in \( \alpha(l_1(\mathbb{N})) \) standing for the canonical isometric embedding induced by the one of \( l_1(\mathbb{N}) \) into its double dual. It is then easy to see that (4.1) holds if and only if \( \|\alpha^{-1}(\Phi)\|_1 \leq 2\|\alpha^{-1}(\Phi)\|_\infty \), compare to [8, 9].

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