SYMMETRY BREAKING/RESTORATION
IN A NON-SIMPLY CONNECTED SPACE-TIME

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Abstract

Field theories compactified on non-simply connected spaces, which in general allow
to impose twisted boundary conditions, are found to unexpectedly have a rich phase
structure. One of characteristic features of such theories is the appearance of critical
radii, at which some of symmetries are broken/restored. A phase transition can occur at
the classical level, or can be caused by quantum effects. The spontaneous breakdown of
the translational invariance of compactified spaces is another characteristic feature. As
an illustrative example, the $O(N)$ $\phi^4$ model on $M^3 \otimes S^1$ is studied and the novel phase
structure is revealed.

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The parameter space of field theories compactified on non-simply connected spaces is, in general, wider than that of ordinary field theories on the Minkowski space-time, and is spanned by twist parameters specifying boundary conditions, in addition to parameters appearing in the actions. Physical consequences caused by twisted boundary conditions turn out to be unexpectedly rich and many of them have not been uncovered yet. The purpose of this talk is to report some of interesting properties of such theories overlooked so far.

One of characteristic features of such theories is the appearance of critical radii of compactified spaces, at which some of symmetries are broken/restored. Symmetry breaking patterns are found to be unconventional. A phase transition can occur at the classical level, or can be caused by quantum effects. Radiative corrections would become important when a compactified scale is less than the inverse of a typical mass scale, and then some of broken symmetries could be restored, or conversely some of symmetries could be broken. Another characteristic and probably surprising feature is the spontaneous breakdown of the translational invariance of compactified spaces. Twisted boundary conditions do not allow vacuum expectation values of twisted bosons to be non-vanishing constants. In other words, vacuum expectation values of twisted bosons have to vanish or to be coordinate-dependent if they are non-vanishing. If the minimum of a potential does not lie at the origin, twisted bosons could acquire non-vanishing vacuum expectation values, which should be coordinate-dependent. Then, we have to minimize the total energy, which consists of both the kinetic term and the potential term, to find the vacuum configuration. When non-vanishing vacuum expectation values of twisted bosons are energetically preferable, they should be coordinate-dependent and hence the translational invariance is broken spontaneously. Among other characteristic features, a phenomenologically important observation is that twisted boundary conditions can break supersymmetry spontaneously. This will give a new type of spontaneous supersymmetry breaking mechanisms and it would be of great interest to investigate a possibility to construct realistic supersymmetric models with this supersymmetry breaking mechanism, though this subject will not be treated in this talk.

As an illustrative example, we here concentrate on the $O(N) \phi^4$ model on $M^3 \otimes S^1$. This subject was discussed in the talk given by M. Tachibana at this Conference. The classical analysis of this model has been done in Ref.[3].
The action which consists of $N$ real scalar fields $\phi_i \ (i = 1, \cdots, N)$ is given by

$$S = \int d^3 x \int_0^{2\pi R} dy \left[ \frac{1}{2} \partial_A \phi_i \partial^A \phi_i - \frac{m^2}{2} \phi_i^2 - \frac{\lambda}{8} (\phi_i^2)^2 \right],$$

where $y$ and $R$ denote the coordinate and the radius of $S^1$, respectively. Since $S^1$ is multiply-connected, we can impose a twisted boundary condition on $\phi_i$ such as

$$\phi_i(y + 2\pi R) = U_{ij} \phi_j(y).$$

The matrix $U$ must belong to $O(N)$, otherwise the action would not be single-valued. We shall below consider various boundary conditions and discuss physical consequences.

**1. $U = 1$**

In this case, the fields $\phi_i(y)$ obey the periodic boundary condition. For $m^2 > 0$, the phase structure is trivial: The $O(N)$ symmetry is unbroken in a whole range of $R$. For $m^2 < 0$, $O(N)$ would be broken to $O(N - 1)$. It is well known that the leading correction to the squared mass is proportional to $1/R^2$ for small radius $R$ and that the broken symmetry $O(N - 1)$ can be restored for $R \leq R^* = O(\sqrt{\lambda}/\mu)$ ($\mu^2 \equiv -m^2$), just like the symmetry restoration at high temperature. Thus, we have found no new interesting phenomena with $U = 1$.

**2. $U = -1$**

In this case, $\phi_i(y)$ obey the antiperiodic boundary condition. For $m^2 > 0$, nothing happens and the $O(N)$ symmetry remains unbroken in a whole range of $R$, while for $m^2 < 0$, several interesting phenomena occur. For $R > R^* \sim 1/(2\mu)$, the $O(N)$ symmetry is spontaneously broken to $O(N - 2)$ but not $O(N - 1)$! The translational invariance of $S^1$ is also broken spontaneously because of the $y$-dependent vacuum expectation values of $\phi_i(y)$. For $R \leq R^*$, all the broken symmetries are restored. It should be emphasized that the mechanism of this symmetry restoration is different from the previous case of $U = 1$ and that the present symmetry restoration has a classical origin. This may be seen from the fact that $R^*$ is of order $1/\mu$ but not $\sqrt{\lambda}/\mu$. Radiative corrections in this case are less important.

The nontrivial phase structure for $m^2 < 0$ may be understood as follows: We first note that since $\phi_i(y)$ obey the twisted (antiperiodic) boundary condition, a non-vanishing
vacuum expectation value of $\langle \phi_i(y) \rangle$ immediately implies that it is $y$-dependent, otherwise it would not satisfy the boundary condition. The $y$-dependent configuration of $\langle \phi_i(y) \rangle$ will induce the kinetic energy proportional to $1/R^2$. It follows that for large radius $R$, non-vanishing $\langle \phi_i(y) \rangle^2$ is preferable because the origin is not the minimum of the potential for $m^2 < 0$ and because the contribution from the kinetic energy is expected to be small. Therefore, for large radius $R$, the $O(N)$ symmetry and also the translational invariance of $S^1$ are spontaneously broken. Since $\langle \phi_i(y) \rangle$ must obey the antiperiodic boundary condition, non-vanishing $\langle \phi_i(y) \rangle$ cannot be constants and turn out to be of the form $\langle \phi_i(y) \rangle = (v \cos(y/2R), v \sin(y/2R), 0, \cdots, 0)$, where $v = \sqrt{(2\mu^2 - 1/(2R^2))/\lambda}$ at the tree level. Since two of $\langle \phi_i(y) \rangle$’s have non-vanishing expectation values, the $O(N)$ symmetry should be broken to $O(N - 2)$, but not $O(N - 1)$††. On the other hand, for small radius, the contribution from the kinetic energy becomes large, so that the $y$-independent configuration of $\langle \phi_i(y) \rangle$ is preferable and this implies that $\langle \phi_i(y) \rangle$ should vanish.

\begin{equation}
(3) \quad U = \begin{pmatrix}
1 & 0 \\
0 & -1_{N-L}
\end{pmatrix}
\end{equation}

Since the twist matrix $U$ is not proportional to the identity matrix, the boundary condition (4) explicitly breaks $O(N)$ down to $O(L) \times O(N - L)$, which is the subgroup of $O(N)$ commuting with $U$. For $m^2 > 0$, the $O(L) \times O(N - L)$ symmetry is unbroken in a whole range of $R$ if $N > L > (N - 4)/3$, but is broken to $O(L - 1) \times O(N - L)$ for $R < R^* = O(\sqrt{\lambda}/m)$ if $0 < L < (N - 4)/3$, in spite of positive $m^2$. This symmetry breaking for $R < R^*$ comes from the fact that a one-loop self-energy diagram in which a boson obeying the antiperiodic boundary condition propagates gives a negative contribution to the squared mass[7].

For $m^2 < 0$, the $O(L) \times O(N - L)$ symmetry is broken to $O(L - 1) \times O(N - L)$ in a whole range of $R$ if $0 < L < (N - 4)/3$, but is restored for $R \leq R^* = O(\sqrt{\lambda}/\mu)$ if $N > L > (N - 4)/3$. It should be noticed that the translational invariance is not broken in this model because the vacuum expectation values of the untwisted bosons are always $y$-independent and because no twisted bosons acquire non-vanishing vacuum expectation values.†† The exception is the model with $N = 1$. In this case, there is no continuous symmetry and the $O(1)$ model has only a discrete symmetry $Z_2$. The vacuum expectation value of $\phi(y)$ is found to be a kink-like configuration for $R > R^*$.\[3\]
(4) General $U \in O(N)$

We can show that the twisted boundary condition (2) generally breaks $O(N)$ to $O(L_0) \times U(L_1/2) \times \cdots \times U(L_{M-1}/2) \times O(L_M)$ with $L_0 + L_1 + \cdots + L_M = N/2$. A new phenomenon is that in some class of models phase transitions could occur several times when the radius $R$ varies from 0 to $\infty$. The full details of the phase structure will be reported elsewhere.

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