Single-Deletion Single-Substitution Correcting Codes

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Abstract—Correcting insertions/deletions as well as substitution errors simultaneously plays an important role in DNA-based storage systems as well as in classical communications. This paper deals with the fundamental task of constructing codes that can correct a single insertion or deletion along with a single substitution. A non-asymptotic upper bound on the size of single-deletion single-substitution correcting codes is derived, showing that the redundancy of such a code of length $n$ has to be at least $2 \log n$. The bound is presented both for binary and non-binary codes while an extension to single deletion and multiple substitutions is presented for binary codes. An explicit construction of single-deletion single-substitution correcting codes with at most $6 \log n + 8$ redundancy bits is derived. Note that the best known construction for this problem has to use 3-deletion correcting codes whose best known redundancy is roughly $24 \log n$.

I. INTRODUCTION

Codes correcting insertions/deletions recently attract a lot of attention due to their relevance in DNA-based data storage systems, cf. [1]. In classical communications, insertions/deletions happen during the synchronization of files and symbols of data streams [2] or due to over-sampling and under-sampling at the receiver side [3]. The algebraic concepts correcting insertions and deletions date back to the 1960s when Varshamov and Tenengolts designed a class of binary codes, nowadays called VT codes. These codes were originally designed to correct asymmetric errors in the Z-channel [4], [5] and later proven to be able to correct a single insertion or a single deletion [6]. VT codes are asymptotically optimal length-$n$ single-insertion/deletion correcting codes of redundancy $\log(n+1)$. As a generalization of VT codes, Tenengolts presented $q$-ary single-insertion/deletion correcting codes in [4]. Levenshtein has also proven that for correcting $t$ insertions/deletions, the redundancy is asymptotically at least $t \log n$. In [7], Brakensiek et al. presented binary multiple-insertion/deletion correcting codes with small asymptotic redundancy. For an explicit small number of deletions, their construction however needs redundancy $c \log n$ where $c$ is a large constant. The recent parallel works by Gabrys et al. [8] and Sima et al. [9] have presented constructions to correct two deletions with redundancy $8 \log n + O(\log \log n)$ [8] and $7 \log n + o(\log n)$ [9].

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II. DEFINITIONS AND PRELIMINARIES

This section formally defines the codes and notations that will be used throughout the paper. For two integers $i, j \in \mathbb{N}$ such that $i \leq j$ the set $\{i, i+1, \ldots, j\}$ is denoted by $[i, j]$ and in short $|j|$ if $i = 0$. The alphabet of size $q$ is denoted by $\Sigma_q = \{0,1,\ldots,q-1\}$. A $t$-indel is any combination of $t_D$ deletions and $t_I$ insertions such that $t_D + t_I = t$. Moreover, for two positive integers, $t \leq n$ and $s \leq n - t$, $B_{t,s}^{DS}(x)$ is the set of all words received from $x \in \Sigma_q^n$ after $t$ deletions and at most $s$ substitutions. Note that the order in which the errors occur does not matter and thus we will mostly assume that first the deletions occurred. Finally, $r(x)$ denotes the number of runs in $x \in \Sigma_q^n$. A code $C \subseteq \Sigma_q^n$ is called a $t$-deletion $s$-substitution correcting code if it can correct any combination of at most $t$ deletions and $s$ substitutions. That is, for all $c_1, c_2 \in C$ it holds that $B_{t,s}^{DS}(c_1) \cap B_{t,s}^{DS}(c_2) = \emptyset$. We define similarly $t$-indel $s$-substitution correcting code to be a code that corrects any combination of at most $t$ indels and $s$ substitutions.

The goal of this paper is to study codes correcting indels and substitutions. Similarly to the equivalence between insertion and deletion correcting codes, the following lemma holds.
Lemma 1. A code $C$ is a $t$-indel $s$-substitution correcting code if and only if it is a $t$-deletion $s$-substitution correcting code.

Therefore, the main focus of the paper is on $t$-deletion $s$-substitution correcting codes and specifically for $t = 1$. The size of the largest $q$-ary length-$n$ single-deletion $s$-substitution correcting code is denoted by $DS_{s,q}(n)$.

III. Bounds

The method used to compute a non-asymptotic upper bound for the cardinality of any single-deletion $s$-substitution code is described in [11] and [12]. For clarity of the results, the principal concepts of this method are briefly reviewed. The main idea is to construct a hypergraph $H_s(X, E)$ out of the channel graph with vertices $X = \Sigma^n_q = \{x_1, \ldots, x_m\}$ and hyperedges $E_s = \{E_1, \ldots, E_\ell\} = \{B_{1,s}^D(x) : x \in \Sigma^n_q\}$. The objective is to find the smallest size of a transversal $T \subseteq X$ in $H_s$, i.e. $T$ intersects all hyperedges in $H_s$. Let $I$ be the $m \times \ell$ incidence matrix of $H$ where $I(i, j) = 1$ if $x_j \in E_i$. A transversal $w \in \Sigma^n_q$ satisfies that $I^T \cdot w \geq 1$. If $w \in (\mathbb{R}^+)^m$, then it is called a fractional transversal. Thus, the objective is to find some $w_y \geq 0$, which needs to fulfill the condition $\sum_{y \in B_{1,s}^D(x)} w_y \geq 1$ for all $x \in \Sigma^n_q$. Consequently, the following expression is an upper bound of the cardinality of a code,

$$|C| \leq \sum_{y \in \Sigma^n_q} w_y.$$

A. Upper Bound on Single-Deletion Single-Substitution Codes

Before determining valid fractional transversals, an important property for any $y \in B_{1,s}^D(x)$ is studied in the following.

Claim 1. For all $x \in \Sigma^n_q$ and $y \in B_{1,s}^D(x)$, it holds that

$$r(x) - (2 + 2s) \leq r(y) \leq r(x) + 2s.$$

Thus, the monotonicity argument $|B_{1,s}^D(x)| \leq |B_{1,s}^D(y)|$ as in the single deletion case of [11] does not necessarily hold and choosing $w_y = \frac{1}{|B_{1,s}^D(y)|}$ does not suffice as a feasible solution.

The following lemma extends the result from [13] and provides the size of $B_{1,1}^D(x)$ for the $q$-ary alphabet.

Lemma 2. For any word $x \in \Sigma^n_q$,

$$|B_{1,1}^D(x)| = \begin{cases} \frac{(n - 1)(q - 1) + 1}{r(x) - [(n - 3)(q - 1) + (q - 2)]} & r(x) \leq 1, \\ \frac{r(y) - [(n - 3)(q - 1) + (q - 2)] + (q + 2)}{r(x) + 2} & r(x) \geq 2. \end{cases}$$

Now, from the results of Lemma 2 and Claim 1, a valid expression of a fractional transversal $w_y$ can be derived.

The following claim will be used in computing the upper bound of the cardinality of the code.

Lemma 3. The following choice of $w_y$ for $y \in \Sigma^{n-1}_q$, $n \geq 3$, is a fractional transversal for $H_1$

$$w_y^1 = \begin{cases} \frac{1}{(n-1)(q-1)+1} & r(y) \leq 3 \\ \frac{1}{(r(y)-2)[(n-3)(q-1)+(q-2)]+(q+2)} & r(y) > 3. \end{cases}$$

Claim 2. For integers $q \geq 2$, $n \geq 5$, and $n \geq q$ it holds that

$$\sum_{k=1}^n \frac{n!}{k!(n-k)!} \frac{1}{(q-1)(n-2)} \leq \frac{q^{n+1}}{(n-5)(n-3)(q-1)} + 5q.$$

Note that the claim is combined from similar statements in [14] Lemma 13, 14. Putting everything together the following upper bound on $DS_{1,q}(n)$ can be presented.

Theorem 4. For $q \leq n$, $n \geq 6$ the following is an upper bound on $DS_{1,q}(n)$

$$DS_{1,q}(n) \leq n^3 \cdot \frac{3 \cdot q^{n-1}}{(n-5)(n-3)(q-1)} + 5q.$$

Proof: Note that the number of words in $\Sigma^{n-1}_q$ with $r$ runs is given by $q(q-1)^{r-1}q^{r-1}(n-2)$. The sum over all words in $\Sigma^{n-1}_q$ using the indicated fractional transversals $w_y^1$ has to be computed. For $r = 1, 2, 3$ define the function $g(q, n) = \Sigma_{r=1}^3 q(q-1)^{r-1}q^{r-1}(n-2)$. The rest is given by

$$\sum_{y \in \Sigma^n_q} q(q-1)^{r-1}q^{r-1}(n-2) \leq \frac{q}{(n-3)(q-1)} \sum_{y \in \Sigma^n_q} q(q-1)^{r-1}(n-2) \frac{1}{r}.$$

For simplicity, first the following analysis is performed.

$$f(q, n) := \sum_{r=4}^{n-1} q(q-1)^{r-1}q^{r-1} \frac{1}{r-2}$$

$$= \sum_{r=2}^{n-3} (q-1)^{r+1} \frac{(n-2)!}{(n-r-3)! r!} \frac{1}{r-1}$$

$$= (n-2)(q-1) \sum_{r=2}^{n-3} (q-1)^r \frac{(n-3)!}{r!} \frac{1}{r-2} - \sum_{r=2}^{n-3} (q-1)^{r+1} \frac{(n-2)!}{(n-r-3)! r!}.$$

In the last expression the right part of the difference can be calculated as follows

$$\sum_{r=2}^{n-3} (q-1)^{r+1} \frac{(n-2)!}{(n-r-3)! r!} = \sum_{r=3}^{n-2} (q-1)^r \frac{(n-2)!}{r!}$$

$$= q^{n-2} - (q-1)^2(n-2)(n-3) - (q-1)(n-2) - 1.$$
Finally, the bound in the theorem results after some basic algebraic steps and the fact that \( q \leq n \).

The last theorem provides the following corollary.

**Corollary 5.** It holds that \( D S_{1,q}(n) \leq \frac{3q^{n-1}}{n^{(q-1)}} \).

**B. Upper Bound on Single-Deletion s-Substitution Codes**

To state a legitimate fractional transversal for the case of \( s \) substitutions, first a lower bound on the cardinality of the ball size \( |B_{1,s}^D(x)| \) has to be derived. In the remaining part of the section only \( \Sigma_2 \) will be considered.

**Claim 3.** For all \( x \in \Sigma_2^n \), it holds that \( |B_{1,s}^D(x)| \geq r(x)(n-1-s) \).

Note that this lower bound is derived based upon an explicit expression of \( |B_{1,s}^D(x)| \) from [13]. Using this result, a fractional transversal for the single-deletion s-substitution case can be formulated.

**Lemma 6.** The following choice of \( w_y \) with \( y \in \Sigma_2^{n-1} \) and \( n \geq 2s + 1 \geq 3 \) is a fractional transversal for \( H_s \):

\[
 w_y^\bullet = \begin{cases} 
 1 & r(y) \leq 2s + 1, \\
 \frac{1}{(r(y) - 2s)(n - s)} & r(y) > 2s + 1.
\end{cases}
\]

As a result of Lemma 6 an upper bound for the cardinality of a single-deletion s-substitution correcting code can be stated.

**Theorem 7.** For \( n \geq 3 \) the following is an upper bound on \( D S_{s,2}(n) \):

\[
 D S_{s,2}(n) \leq \frac{s!}{(n - 2s)!} \left( \frac{n}{2s + 1} \right)^2 \sum_{r=2s+1}^{n-2} \frac{2^{r-1} - 2^{r-2}}{r}.
\]

**Proof:** First, only the words in \( y \in \Sigma_2^{n-1} \) with \( r(y) \leq 2s + 1 \) are considered. Using the inequalities \( \sum_{k=0}^{n} \binom{n}{k} \leq \sum_{k=0}^{n} k^2 \binom{n}{k} \leq (1 + n)^k \) and \( \binom{n-k}{k} \geq \frac{(n-k)!}{k!} \), the sum can be calculated as

\[
 \sum_{i=1}^{2s} \left( \frac{n-2}{r-1} \right) \frac{1}{(n-s)} \sum_{i=1}^{2s} \frac{2^{r-1} - 2^{r-2}}{r} \leq s! (n-1) 2s + 1.
\]

In a next step, by additionally applying the inequality \( \frac{1}{2^{2s}} \leq 2^{-s} \) to the sum for all words with \( 2s + 2 \leq r(y) \leq n - 1 \) can be computed as

\[
 \sum_{r=2s+2}^{n-1} \frac{n-2}{r-1} \frac{1}{(n-s)} \frac{1}{(r-2s)} \leq \frac{n-2}{r-1} \sum_{r=2s+2}^{n-1} \frac{2^{r-1} - 2^{r-2}}{r} = 2^{s+1} \frac{2}{n-1} \frac{n-2}{r-1} \frac{1}{(n-s)} \sum_{r=2s+2}^{n-1} \frac{2^{r-1}}{r}.
\]

Subsequently, the sum of all words with \( r \leq 2s + 1 \) is added to the equation again which results to the expression in the theorem.

The corollary below concludes the previous result.

**Corollary 8.** It holds that \( D S_{s,2}(n) \leq \frac{s!}{(n-2s)!} \left( \frac{n}{2s + 1} \right)^2 \).

Note that unlike the proof of Theorem 4 in the proof of Theorem 7 Claim 3 is not applied. Instead, Claim 3 and a different upper bound is used. For this reason, in Corollary 8 the bound for the value of \( s = 1 \) is not the same as the bound stated in Corollary 5 with \( q = 2 \).

**IV. Properties of Codes**

In this section, several properties of the families of codes studied in the paper are presented. Consider a family of codes which are defined in the following way. A binary code \( C \subseteq \Sigma_2^n \) is called a \( \{ y = (y_1, \ldots, y_n); a, N \} \)-congruent code if it is defined in the following way

\[
 C(y; a, N) = \left\{ c \in \Sigma_2^n \mid \sum_{i=1}^{n} y_i \cdot c_i = a \pmod{N} \right\}.
\]

The lemmas in this section provide several basic properties in case the intersection of the single-deletion single-substitution balls of two codewords in \( C \) is not trivial. For the rest of this section it is assumed that \( x, y \in C(y; a, N) \), where \( B_{1,1}^D(x) \cap B_{1,1}^D(y) \neq \emptyset \) so that there exists \( z \in B_{1,1}^D(x) \cap B_{1,1}^D(y) \). For simplicity of notation, the expression \( x(d, e) \) is defined to be the error-word achieved from \( x \) by deleting the bit in the index \( d \), and substituting the bit in the index \( e \). The variables \( d_x, d_y, e_x, e_y \in [n] \) are indices such that \( z = x(d_x, e_x) = y(d_y, e_y) \). It can be assumed w.l.o.g. that \( d_x < d_y \). In order to shift the values of the substituted bits from binary to \( \pm 1 \), the following notation is used \( \delta_i := 2 \cdot x_i - 1 \).

**Lemma 9.** For \( e_x, e_y \notin [d_x, d_y] \) the following statements hold.

1) For \( i \in [d_x + 1, d_y] \), \( x_i = y_i - 1 \).
2) For \( i \in \{e_x, e_y\}, x_i = 1 - y_i \).
3) For \( i \in [n] \setminus ([d_x, d_y] \cup \{e_x, e_y\}) \), \( x_i = y_i \).
4) \( \sum_{i=d_x}^{d_y} y_i \cdot (x_i - y_i) = x_{d_x} \cdot y_{d_y} + \sum_{i=d_x+1}^{d_y} (y_i - y_{i-1}) \cdot x_i - y_{d_y} \cdot y_{d_y} \).
5) \( \delta_{e_x} \cdot e_y + \delta_{e_y} \cdot y_x + x_{d_x} \cdot y_{d_y} + y_{d_y} \cdot y_{d_y} + \sum_{i=d_x+1}^{d_y} x_i \cdot (y_i - y_{i-1}) = 0 \pmod{N} \).

**Proof:** For any \( i \in [d_x + 1, d_y] \) the definition of \( z \) leads to the fact that \( z_i = y_i \) and also \( z_i = x_{i+1} \). This proves statement 1, and a similar proof can be shown for statement 3.

For a substituted bit \( i = e_y \), either \( i < d_x \) in which case, \( x_i = y_{i+1} \). As the case of \( i = e_y \) and \( i > d_y \) can be proved in a similar way. This concludes the proof of statement 2.

In order to prove statement 4, the sum is separated in the following manner

\[
 \sum_{i=d_x}^{d_y} y_i \cdot (x_i - y_i) = \sum_{i=d_x}^{d_y} y_i \cdot (x_i - y_{i-1}) + \sum_{i=d_y+1}^{d_y} y_i \cdot y_{i-1}.
\]

Using statement 1, the last element is simplified as follows

\[
 \sum_{i=d_x}^{d_y} y_i \cdot y_{i-1} = \sum_{i=d_y+1}^{d_y} y_i \cdot y_{i-1} = \sum_{i=d_y+1}^{d_y} y_i \cdot x_{i+1}.
\]

Thus, the equality can be rewritten as

\[
 x_{d_x} \cdot y_{d_y} - y_{d_y} \cdot y_{d_y} + \sum_{i=d_x+1}^{d_y} y_i \cdot x_{i+1} - \sum_{i=d_x}^{d_y} y_i \cdot y_i + \sum_{i=d_y+1}^{d_y} (y_i - y_{i-1}) \cdot x_{i+1}.
\]

This concludes the proof of statement 4.

The fact \( x, y \in C(y; a, N) \) implies that \( \sum_{i=1}^{n} y_i \cdot x_i = a \pmod{N} \). From this follows
\[
\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i y_i \equiv 0 \pmod{N}.
\]

Furthermore, Statement 3 implicates that \(\sum_{i=1}^{n-1} y_i \cdot (x_i - y_i) + \sum_{i=d+1}^{n} y_i \cdot (x_i - y_i) = y_c \cdot (x_c - y_c) + y_{\hat{c}} \cdot (x_c + y_c).\) On the other hand, statement 2 leads to the fact that for \(e \in \{e_x, e_y\} : x_e - y_e = x_e - x - 1 = \delta_{xy}.\) Combined together with statement 4 the following equivalence is achieved
\[
\delta_{xy} \cdot y_c + \delta_{xy} \cdot y_{\hat{c}} + x_{d_x} \cdot y_{d_x} + \sum_{i=1}^{d_x} x_i \cdot (y_i - y_i - 1) - y_{d_y} \cdot y_{d_y} = n \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i y_i \equiv 0 \pmod{N}.
\]

This proves statement 5.

**Lemma 10.** The following manners are defined:

1) For \(e \in [d_x, d_y]\) and \(e_y \not\in [d_x, d_y] \delta_{xy} \cdot y_{e-x} + \delta_{xy} \cdot y_{e-y} + x_{d_y} \cdot y_{d_y} + \sum_{i=d_y+1}^{n} x_i \cdot (y_i - y_i - 1) \equiv 0 \pmod{N}

2) For \(e_x, e_y \in [d_x, d_y] \delta_{xy} \cdot y_{e-x} + \delta_{xy} \cdot y_{e-y} + x_{d_y} \cdot y_{d_y} + \sum_{i=d_y+1}^{n} x_i \cdot (y_i - y_i - 1) \equiv 0 \pmod{N}

Define the variable \(e_x\) as 1 if \(e_x \in [d_x, d_y]\) and 0 otherwise. The variable \(e_y\) is defined in a similar manner. An important claim about the property of the code is as follows.

**Claim 4.** For any \(x, y \in \mathbb{Z}_2^n\)

1) \(x(d_x, e_y) = y(d_y, e_y) \iff x(d_x, e_y + e_x) = y(d_y, e_x - e_x).

2) \(B_{1,1}(x) \cap B_{1,1}(y) = \emptyset \iff B_{1,1}(x) \cap B_{1,1}(y) \neq \emptyset.

3) For some weight vector \(\gamma \in \mathbb{Z}^n, a, N, \) and for any \(x, y \in C(\gamma, a, N)\) there exists such \(b\) so that \(\mathcal{X}, \mathcal{Y} \in C(\gamma; b, N)\)

**V. CONSTRUCTION**

In this section, the main result of the paper is shown. An explicit construction for a single-deletion single-substitution correcting code and its correctness are presented. This construction requires redundancy of at most \(6 \cdot \log(n) + 8\) bits.

**Construction 11.** Define four weight vectors \(\alpha, \beta, \eta, \mathbb{I} \in \mathbb{Z}^n\) in the following way

\[
\alpha = (1, 2, 3, \ldots, n), \beta = (\sum_{i=1}^{n} i, \sum_{i=1}^{n} 2i, \sum_{i=1}^{n} 3i, \ldots, \sum_{i=1}^{n} ni),
\]
\[
\eta = (\sum_{i=1}^{n} i^2, \sum_{i=1}^{n} 2i^2, \sum_{i=1}^{n} 3i^2, \ldots, \sum_{i=1}^{n} ni^2), \mathbb{I} = (1, \ldots, 1).
\]

For fixed integers \(a \in [3n], b \in [3 \cdot n^2], c \in [3 \cdot n^3], d \in [4],\) the code is defined as

\[
C_{a,b,c,d} = C(\alpha; a, 3n+1) \cap C(\beta; b, 3n^2+1) \cap C(\eta; c, 3n^3+1) \cap C(\mathbb{I}; d, 5).
\]

Remember that \(d_x, d_y\) are denoted as the indices of the deleted bits from \(x, y\) respectively. The definition of \(e_x, e_y\) are the indices of the substituted bits from \(x, y\). The definition of \(e_x\) is 1 if \(e_x \in [d_x, d_y]\) and 0 otherwise. The definition of \(e_y\) is similar.

**Theorem 12.** For any indices \(e_x, e_y, d_x, d_y \in [n],\) any two codewords \(x, y \in C_{a,b,c,d}\) fulfill

\[
x(d_x, e_x) \neq y(d_y, e_y).
\]

**Proof:** For simplicity, denote \(\hat{e}_x = e_x - e_x\) and \(\hat{e}_y = e_y + e_y.\) Assume by contradiction that \(x(d_x, e_x) = y(d_y, e_y).\) The following equivalence can be concluded from Lemma 9 (statement 5), Lemma 10 and the fact that \(C_{a,b,c,d} \subseteq C(1; d, 5)\)

\[
\delta_{xy} + \delta_{\hat{e}_x} + x_{d_x} - y_{d_x} + 0 \equiv 0 \pmod{5}
\]

This is equivalent to the following system of equations

\[
x_{d_x} = y_{d_x}, \delta_{xy} = -\delta_{\hat{e}_x}.
\]

Define the set \(S\) as follows.

\[
S := \{d_x + 1 \leq i \leq d_y | x_i = 1\} \subseteq [d_x, d_y].
\]

According to Claim 4 statement 1, it is possible to assume w.l.o.g. that \(e_y < e_y.\) According to Claim 4 statements 2-3, it is also possible to assume w.l.o.g. that \(x_{d_x} = y_{d_x} = 0.\) By substituting these values into Lemma 9 statement 5 and Lemma 10 the following equivalence is achieved.

For any \(k \in \{0, 1, 2\}\)

\[
\delta_{xy} \cdot \sum_{j=1}^{x_k} j + \delta_{\hat{e}_x} \cdot \sum_{j=1}^{y_k} j + \sum_{j \in S} j \equiv 0 \mod 3n^k + 1.
\]

Notice that \(\delta_{xy} \cdot \sum_{j=1}^{x_k} j \leq \sum_{j=1}^{n} j < n^2 = n^k + 1.\) This is also true for \(\delta_{\hat{e}_y} \cdot \sum_{j=1}^{x_k} j^k\) and \(\{|y_k, x_k|\}.\) As a result, the left part of the equivalences is at least \(-3 \cdot n^k + 1\) and at most \(3 \cdot n^k + 1.\) Therefore, the congruences are strict equalities.

The equalities can be rewritten as

\[
\sum_{j \in S} j^k = -\delta_{\hat{e}_x} \cdot (-\hat{e}_x \cdot \sum_{j=1}^{x_k} j^k) - \delta_{\hat{e}_x} \cdot \sum_{j \in S} j^k.
\]

Notice that the left hand side is always non-negative. Hence, the sign of the right hand side is non-negative as well. From this follows that \(\delta_{\hat{e}_x} = -1\) and the equation can be transformed to

\[
\sum_{e \in d+1} \sum_{j \in S} j^k = \sum_{e \in S} j^k.
\]

Since the current assumption is that \(d_x < d_y\) and \(\hat{e}_x < e_y,\) there are 6 possible orderings of the 4 indices. A full proof for the cases \(e_x, e_y < d_x, e_x < d_x < e_y < d_y\) will follow, and a guidance for the rest of the cases can be found afterwards.

Assume \(\hat{e}_x, e_y < d_x.\) In this case, two sets are defined as

\[
S_1 := [\hat{e}_x + 1, e_y], S_2 := S.
\]

Notice that for any two indices \(i \in S_1, j \in S_2\) the following holds

\[
i < j.
\]

Hence, \(\mathbb{I}\) can be altered to the following form. For any \(k \in \{0, 1, 2\}\)

\[
\sum_{j \in S_1} j^k = \sum_{j \in S_2} j^k.
\]
For $k = 0$ this equality is $|S_1| = |S_2|$ which means the cardinality of the sets is equal. For $k = 1$ this equality is $\sum_{j \in S_1} j = \sum_{j \in S_2} j$ which means the sum of elements of $S_1, S_2$ is equal as well. However, through equality (2) if the cardinality of the sets is the same then the sum of elements in $S_2$ should be strictly bigger than the sum of elements in $S_1$. This concludes this case.

Assume $\varepsilon_x < d_x < \varepsilon_y < d_y$. In this case, three sets are defined as:

$$S_1 := [\varepsilon_x + 1, d_x], \quad S_2 := [d_x + 1, e_y], \quad S_3 := [e_y + 1, d_y] \cap S.$$ 

Observe that for any three indices $i \in S_1, j \in S_2, \ell \in S_3$ the following holds:

$$i < j < \ell. \quad (3)$$

Now, (1) can be rewritten as follows. For any $k \in \{0, 1, 2\}$

$$\sum_{j \in S_1} j^k + \sum_{j \in S_2} j^k - \sum_{j \in S_3} j^k = 0.$$

This can be written in matrix form. There exist integers $v_1, v_2, v_3$ such that at least one of them is non-zero and the following equality holds

$$A \cdot v := \begin{bmatrix}
\sum_{j \in S_1} j^k & \sum_{j \in S_2} j^k & \sum_{j \in S_3} j^k
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.$$

In this case, $v_1 = v_2 = 1, v_3 = -1$ is such a solution. This equality means $A$ has a non-trivial solution to the homogeneous system of equalities, which also means $\det(A) = 0$.

The determinant of the matrix $A$ can be computed by

$$\left| \begin{array}{ccc}
\sum_{j \in S_1} j & \sum_{j \in S_2} j & \sum_{j \in S_3} j \\
1 & 1 & 1 \\
i & j & k
\end{array} \right| = (j - i) \cdot (k - j) \cdot (k - i).$$

Notice that each element in the sum is a determinant of a Vandermonde matrix. Hence,

$$\left| \begin{array}{ccc}
1 & 1 & 1 \\
1 & j & k \\
1 & j^2 & k^2
\end{array} \right| = (j - i) \cdot (k - j) \cdot (k - i).$$

To summarize, (3) the following contradiction is obtained.

$$0 = \sum_{i \in S_1} \sum_{j \in S_2} \sum_{k \in S_3} \left| \begin{array}{ccc}
1 & 1 & 1 \\
i & j & k \\
i^2 & j^2 & k^2
\end{array} \right| = \sum_{i \in S_1} \sum_{j \in S_2} \sum_{k \in S_3} (j - i) \cdot (k - j) \cdot (k - i) > 0.$$ 

This concludes this case.

For any of the other orderings, the same proof can be concluded using the following definitions:

1) For $\varepsilon_x, e_y < d_x$, define $S_1 := [\varepsilon_x + 1, e_y], S_2 := S$;

2) For $\varepsilon_x < d_x < \varepsilon_y < d_y$, define $S_1 := [\varepsilon_x + 1, d_x], S_2 := [d_x + 1, e_y], S_3 := [e_y + 1, d_y] \cap S$;

3) For $\varepsilon_x < d_x < d_y < e_y$, define $S_1 := [\varepsilon_x + 1, d_x], S_2 := S, S_3 := [d_y + 1, e_y]$;

4) For $d_x < \varepsilon_x < e_y < d_y$, define $S_1 := S \cap [\varepsilon_x, d_x], S_2 := [\varepsilon_x + 1, e_y], S_3 := [e_y + 1, d_y] \cap S$;

5) For $d_x < \varepsilon_x < d_y < e_y$, define $S_1 := S \cap [d_x, \varepsilon_x], S_2 := [\varepsilon_x + 1, e_y], S_3 := [e_y + 1, d_y]$;

6) For $d_y < \varepsilon_x, e_y$, define $S_1 := S, S_2 := [\varepsilon_x + 1, e_y]$.

This concludes the proof.

In this proof, it is shown that $C_{a,b,c,d}$ guarantees to correct a combination of single-deletion and single-substitution errors. However, a single deletion or a single substitution can be corrected as well due to the $\alpha$ constraint of the code $C(\alpha; a, 3n + 1)$ [6]. Lastly, via the pigeonhole principle the following conclusion about the redundancy is obtained.

**Corollary 13** There exist $a \in [3n + 1], b \in [3n^2 + 1], c \in [3n^2 + 1], d \in [5]$ such that the code $C_{a,b,c,d}$ is a binary single-deletion single-substitution correcting code with at most $6 \cdot \log(n) + 8$ redundancy bits.

VI. CONSTRUCTION FOR NON-BINARY ALPHABETS

In this section, a construction for non-binary codes correcting a single deletion and a single substitution is presented.

For a word $z \in \Sigma_q^n$ we associate its binary signature $z^{01}_1 \in \Sigma_q^n$, where $z^{01}_1 = 1$ if and only if $z_i > z_{i-1}$. The motivation for using the signature vector is to convert the error correction into a binary problem. The following lemma shows the conversion explicitly. We define an adjacent transposition to be the error event in which two adjacent bits switch their positions.

**Lemma 14.** For any word $x \in \Sigma_q^n$, and $y \in \Sigma_q^{n-1}$ the error word achieved by a single-deletion and a single-substitution, $y^{01}$ can be achieved from by $x^{01}$ by one of the following errors:

1) single-deletion;

2) single-deletion and a single-substitution;

3) single-deletion and a single-adjacent-transposition.

For the rest of this section, let $C_2$ be a code correcting either a single deletion and a single substitution or a single deletion and a single adjacent transposition. We are now ready to present the code construction for the non-binary case.

**Construction 15.** For $a \in [2q], b \in [2qn], c \in [2qn^2], d \in [2qn^2 + 1]$, let $C_{a,b,c}$ be the code $C_{a,b,c} \subseteq \Sigma_q^n$ is a single-deletion single-substitution correcting code.

**Theorem 16.** For all $a \in [2q], b \in [2qn], c \in [2qn^2]$ the code $C_{a,b,c} \subseteq \Sigma_q^n$ is a single-deletion single-substitution correcting code with at most $10 \cdot \log(n) + 3 \cdot \log(q) + 11$ redundancy symbols.
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