A PURELY ALGEBRAIC SHORT APPROACH TO THE GENERALIZED JACOBIAN CONJECTURE

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(Dedicated to Tadashi ODA, the Author’s father, who passed away on August 25, 2010.)

Abstract. Our goal is to settle the following faded problem,

The Jacobian Conjecture ($JC_n$): If $f_1, \cdots, f_n$ are elements in a polynomial ring $k[X_1, \cdots, X_n]$ over a field $k$ of characteristic 0 such that the Jacobian $\text{det}(\partial f_i/\partial X_j)$ is a nonzero constant, then $k[f_1, \cdots, f_n] = k[X_1, \cdots, X_n]$.

For this purpose, we generalize it to the following form:

The Generalized Jacobian Conjecture ($GJC$): Let $\varphi : S \to T$ be an unramified homomorphism of Noetherian domains with $T^\times = \varphi(S^\times)$. Assume that $T$ is factorial and that $S$ is a simply connected normal domain. Then $\varphi$ is an isomorphism.

We settle Conjecture ($GJC$), which resolves ($JC_n$) as a corollary.

1. Introduction

Let $k$ be an algebraically closed field, let $A^n_k = \text{Spec}^n(k[X_1, \ldots, X_n])$ be an affine space of dimension $n$ over $k$ and let $f : A^n_k \to A^n_k$ be a morphism of affine spaces over $k$ of dimension $n$. Note here that for a ring $R$, $\text{Spec}(R)$ (resp. $\text{Spec}^n(R)$) denotes the prime spectrum of $R$ (or merely the set of prime ideals of $R$) (resp. the maximal spectrum (or merely the set of the maximal ideals of $R$)). Then $f$ is given by

$A^n_k \ni (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \in A^n_k,$

where $f_i(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$. If $f$ has an inverse morphism, then the Jacobian $J(f) := \text{det}(\partial f_i/\partial X_j)$ is a nonzero constant. This follows from the easy chain rule of differentiations without specifying the characteristic of $k$. The Jacobian Conjecture asserts the converse.

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If $k$ is of characteristic $p > 0$ and $f(X) = X + X^p$, then $df/dX = f'(X) = 1$ but $X$ cannot be expressed as a polynomial in $f$. It follows that the inclusion $k[X + X^p] \hookrightarrow k[X]$ is finite and étale but $f : k[X] \rightarrow k[X]$ is not an isomorphism. This implies that $k[X]$ is not simply connected (i.e., $\text{Spec}(k[X]) = \mathbb{A}^1_k$ is not simply connected, see §2.Definition 2.2) when $\text{char}(k) = p > 0$. Thus we must assume that the characteristic of $k$ is 0.

The algebraic form of The Jacobian Conjecture ($JC_n$) (or the Jacobian Problem ($JC_n$)) is the following:

**The algebraic form ($JC_n$).** If $f_1, \ldots, f_n$ are elements in a polynomial ring $k[X_1, \ldots, X_n]$ over a field $k$ of characteristic 0 such that $\det(\partial f_i/\partial X_j)$ is a nonzero constant, then $k[f_1, \ldots, f_n] = k[X_1, \ldots, X_n]$.

Note that when considering $(JC_n)$, we may assume that $k = \mathbb{C}$ by “Lefschetz-principle” (See [10, (1.1.12)]).

The Jacobian Conjecture ($JC_n$) has been settled affirmatively under a few special assumptions below (See [6]). Let $k$ denote a field of characteristic 0. We may assume that $k$ is algebraically closed. Indeed, we can consider it in the case $k = \mathbb{C}$, the field of complex numbers. So we can use all of the notion of Complex Analytic Geometry. But in this paper, we go forward with the algebraic arguments.

For example, under each of the following assumptions, the Jacobian Conjecture ($JC_n$) has been settled affirmatively (Note that we may assume that $k$ is algebraically closed and of characteristic 0):

**Case(1)** $f : \mathbb{A}^n_k = \text{Spec}(k[X_1, \ldots, X_n]) \rightarrow \text{Spec}(k[f_1, \ldots, f_n]) = \mathbb{A}^n_k$ is injective;
**Case(2)** $k(X_1, \ldots, X_n) = k(f_1, \ldots, f_n)$;
**Case(3)** $k(X_1, \ldots, X_n)$ is a Galois extension of $k(f_1, \ldots, f_n)$;
**Case(4)** $\deg f_i \leq 2$ for all $i$;
**Case(5)** $k[X_1, \ldots, X_n]$ is integral over $k[f_1, \ldots, f_n]$.

A fundamental reference for The Jacobian Conjecture ($JC_n$) is [6] which includes the above Cases.

See also the reference [6] for a brief history of the developments and the state of the art again since it was first formulated and partially proved by Keller in 1939 ([13]), together with a discussion on several false proofs that have actually appeared in print, not to speak of so many other claims of prospective proofs being announced but proofs not seeing the light of the day. The Jacobian Conjecture ($JC_n$), due to the simplicity of its statement, has already faded the reputation of leading to solution with ease, especially because an answer appears to be almost at hand, but nothing has been insight even for $n = 2$. 
The conjecture obviously attracts the attention of one and all. It is no ex-
aggeration to say that almost every makes an attempt at its solution, especially
finding techniques from a lot of branches of mathematics such as algebra (Com-
mutative Ring Theory), algebraic geometry/topology, analysis (real/complex) and
so on, having been in whatever progress (big or small) that is made so far (cf. E.
Formanek, Bass’ Work on The Jacobian Conjecture, Contemporary Mathematics
243 (1999), 37-45).

For more recent arguments about The Jacobian Conjecture, we can refer to [W]
and [K-M].

Throughout this paper, unless otherwise specified, we use the following nota-
tions:

⟨ Basic Notations ⟩
• All fields, rings and algebras are assumed to be commutative with unity.
• A factorial domain $R$ is also called a unique factorization domain,
• A DVR denotes a principal valuation ring (i.e., a discrete rank one valuation
ring)
• $R^\times$ denotes the set of units of $R$,
• $\text{nil}(R)$ denotes the nilradical of $R$, i.e., the set of the nilpotent elements of $R$,
• $K(R)$ denotes the total quotient ring (or the total ring of fractions) of $R$, that is,
letting $S$ denote the set of all non-zerodivisors in $R$, $K(R) := S^{-1}R$,
• $\text{Ht}_1(R)$ denotes the set of all prime ideals of height one in $R$,
• $\text{Spec}(R)$ denotes the affine scheme defined by $R$ (or merely the set of all prime
ideals of $R$),
• Let $A \to B$ be a ring-homomorphism and $p \in \text{Spec}(A)$. Then $B_p$ means $B \otimes_A A_p$.

Our Main Objective is to settle the generalized version as follows:

Conjecture (GJC). Let $\varphi : S \to T$ be an unramified homomorphism of Noetherian
domains with $T^\times = \varphi(S^\times)$. Assume that $T$ is factorial and that $S$ is a simply
connected normal domain. Then $\varphi$ is an isomorphism.

In Section 2, to begin with a theorem, Theorem 2.10, about a Krull domain and
its flat subintersection with the same units group is discussed and in the next place
Conjecture (GJC) is settled as a main result, Theorem 2.15, and consequently the
Jacobian Conjecture ($JC_n$) ($\forall n \in \mathbb{N}$) is resolved.

For the consistency of our discussion, we assert that the examples appeared in the
papers ([12], [2] and [20]) published by the certain excellent mathematicians, which
would be against our original target Conjecture (GJC), are imperfect or incomplete
counter-examples. We discuss them in detail in another paper (See S.Oda: Some
By the way, the Jacobian Conjecture (JC\(_n\)) is a problem concerning a polynomial ring over a field \(k\) (characteristic 0), so that investigating the structure of automorphisms \(\text{Aut}_k(k[X_1,\ldots,X_n])\) seems to be substantial. Any member of \(\text{Aut}_k(k[X_1,X_2])\) is known to be tame, but for \(n \geq 3\) there exists a wild automorphism of \(k[X_1,\ldots,X_n]\) (which was conjectured by M.Nagata with an explicit example and was settled by Shestakov and Umirbaev(2003)).

In such a sense, to attain a positive solution of (JC\(_n\)) by an abstract argument like this paper may be far from its significance . . . . . . .

Our general references for unexplained technical terms of Commutative Algebra are [14] and [15].

Remark that we often say in this paper that a ring \(A\) is "simply connected" if \(\text{Spec}(A)\) is simply connected, and a ring homomorphism \(f : A \to B\) is "unramified, étale, an open immersion, a closed immersion, · · · · ·" when "so" is its morphism \(^a_f : \text{Spec}(B) \to \text{Spec}(A)\), respectively.

2. The Main Results, Conjectures (GJC) and (JC\(_n\))

In this section, we discuss Conjecture(GJC). To make sure, we begin with the following definitions.

**Definition 2.1** (Unramified, Étale). Let \(f : A \to B\) be a ring-homomorphism of finite type of Noetherian rings. Let \(P \in \text{Spec}(B)\) and put \(P \cap A := f^{-1}(P)\), a prime ideal of \(A\). The homomorphism \(f\) is called unramified at \(P \in \text{Spec}(B)\) if \(PB_P = (P \cap A)B_P\) and \(k(P) := B_P/PB_P\) is a finite separable field-extension of \(k(P \cap A) := A_{P \cap A}/(P \cap A)A_{P \cap A}\). If \(f\) is not unramified at \(P\), we say \(f\) is ramified at \(P\). The set \(R_f := \{P \in \text{Spec}(B) | ^a_f\text{ is ramified at } P \in \text{Spec}(B)\}\) is called the the ramification locus of \(f\), which is a closed subset of \(\text{Spec}(B)\). The homomorphism \(f\) is called étale at \(P\) if \(f\) is unramified and flat at \(P\). The homomorphism \(f\) is called unramified (resp. étale) if \(f\) is unramified (resp. étale) at every \(P \in \text{Spec}(B)\). The
morphism \( f : \text{Spec}(B) \to \text{Spec}(A) \) is called \textit{unramified} (resp. \textit{étale}) if \( f : A \to B \) is unramified (resp. \textit{étale}).

**Definition 2.2** (Scheme-theoretically or Algebraically Simply Connected). A Noetherian ring \( R \) is called \textit{simply connected} if the following condition holds:

Provided any ‘connected’ ring \( A \) (i.e., \( \text{Spec}(A) \) is connected) with a finite \textit{étale} ring-homomorphism \( \varphi : R \to A, \varphi \) is an isomorphism.

**Remark 2.3.** Let \( K \) be a field. It is known that there exists the algebraic closure \( K \) of \( K \) (which is determined uniquely up to \( K \)-isomorphisms). Let \( K_{\text{sep}} \) denote the separable algebraic closure of \( K \) over \( K \). Note that \( K \) and \( K_{\text{sep}} \) are fields. (See [17] for details.) Let \( K \subset L \) be a finite \textit{étale} \( K \)-algebra \( \Longleftrightarrow L \) is a finite separable \( K \)-algebra (cf. [14, (26.9)]) \( \Longleftrightarrow L \) is a finite algebraic separable extension field of \( K \).

So \( K \) is simply connected if and only if \( K = K_{\text{sep}} \) by Definition 2.2, and hence if \( K \) is algebraically closed, then \( K \) is simply connected. In particular, \( \mathbb{Q} \) is not simply connected because \( \mathbb{Q} \subset \mathbb{Q}_{\text{sep}} = \overline{\mathbb{Q}} \). But \( \mathbb{C} \) is simply connected because \( \mathbb{C} \) is algebraically closed.

**Remark 2.4.** Let \( k \) be an algebraically closed field and put \( k[X] := k[X_1, \ldots, X_n] \), a polynomial ring over \( k \).

(i) If \( \text{char}(k) = 0 \), then the polynomial ring \( k[X] \) \((n \geq 1)\) is simply connected by Proposition A.2.

(ii) If \( \text{char}(k) = p > 0 \), then the polynomial ring \( k[X] \) \((n \geq 1)\) is not simply connected. (Indeed, for \( n = 1, k[X_1 + X_1^p] \to k[X_1] \) is a finite \textit{étale} morphism, but is not an isomorphism as mentioned before.)

(iii) An algebraically closed field \( k \) is simply connected (See Remark 2.3). So we see that for a simply connected Noetherian domain \( A \), a polynomial ring \( A[X] \) is no necessarily simply connected (in the case of \( \text{char}(A) = p > 1 \)).

Moreover any finite field \( \mathbb{F}_q \), where \( q = p^n \) for a prime \( p \in \mathbb{N} \), is not simply connected because it is a perfect field.

Now we start on showing our main result.

It is well-known that for a Noetherian ring or a Krull domain \( A \), an element \( a \in A \) is a non-unit if and only if \( a \) is contained in a prime ideal \( p \) of height one.

\[\text{In general, let } X \text{ and } Y \text{ be locally Noetherian schemes and let } \psi : Y \to X \text{ be a morphism locally of finite type. If } \psi \text{ is finite and surjective, then } \psi \text{ (or } Y \text{) is called a (ramified) cover of } X \text{ (cf.} [4,\text{VI(3.8)})]. \text{ If a cover } \psi \text{ is } \text{étale}, \psi \text{ is called an } \text{étale cover of } X. \text{ If every connected } \text{étale cover of } X \text{ is isomorphic to } X, \text{ X is said to be (scheme-theoretically or algebraically) simply connected (cf.} \text{Lemma A.10).}\]

\[\text{Let } k \text{ be a field and } A \text{ a } k\text{-algebra. We say that } A \text{ is separable over } k \text{ (or } A \text{ is a separable } k\text{-algebra) if for every extension field } k' \text{ of } k, \text{ the ring } A \otimes_k k' \text{ is reduced (See [14,p.198].)}\]
Remark 2.5 (cf. [11] or [14, §10-§12]). Here we give some explanations about Krull domains for our usage.

Let $R$ be a Krull domain. Let $\Delta$ be a subset of $\text{Ht}_1(R)$, let $R_\Delta = \bigcap_{P \in \Delta} R_P$, a subintersection of $R$ and let $P \in \text{Ht}_1(R)$. Note first that any $P \in \text{Ht}_1(R)$ (resp. similar for $R_\Delta$) is a maximal divisorial prime ideal of $R$ (resp. $R_\Delta$) (cf. [11,(3.6)-(3.12)] or [14,Ex(12.4)]) and consequently a defining family $\text{Ht}_1(R)$ (resp. $\text{Ht}_1(R_\Delta)$) of a Krull domain $R$ (resp. $R_\Delta$) is \underline{minimal} among defining families of $R$ (resp. $R_\Delta$) (cf. [14,(12.3)], [11,(1.9)]) and that any DVRs $R_P$, $P \in \text{Ht}_1(R)$ (resp. $(R_\Delta)_Q$, $Q \in \text{Ht}_1(R_\Delta)$) are independent DVRs in $K(R)$ (resp. in $K(R_\Delta) = K(R)$), that is, $(R_P)_{P'} = K(R)$ for $P \neq P'$ (resp. similar for $R_\Delta$) in $\text{Ht}_1(R)$ (resp. in $\text{Ht}_1(R_\Delta)$) (cf. [11,(5.1)]), which gives indeed Lemma A 8 ‘The Approximation Theorem for Krull domains’.

Assume that $R \hookrightarrow R_\Delta$ is flat.

We can see in [11] that the following statements (i) \sim (vi) hold:

(i) $R_\Delta$ is a Krull domain with $K(R_\Delta) = K(R)$ (cf. [11,(1.5)]), and $P' \cap R \in \text{Ht}_1(R)$ for any $P' \in \text{Ht}_1(R_\Delta)$ (cf. [11,(6.4)]).

(ii) Any $P \in \text{Ht}_1(R)$ is a divisorial (prime) ideal of $R$. Thus $PR_\Delta$ is divisorial by [11,p.31] (or [11,(3.5)] and [11,(6.5)]) according to the flatness of $R \hookrightarrow R_\Delta$. So we have by [11,(5.5)] (or Remark 2.6),

$$PR_\Delta = \begin{cases} PR_P \cap R_\Delta \neq R_\Delta \quad (P \in \Delta) \\ R_\Delta \quad (P \notin \Delta) \end{cases}$$

Hence $PR_\Delta$ ($P \in \Delta$) is in $\text{Ht}_1(R_\Delta)$.

(iii) $i : R \hookrightarrow R_\Delta$ (i.e., $^\circ i : \text{Spec}(R_\Delta) \rightarrow \text{Spec}(R)$) induces a bijection $\text{Ht}_1(R_\Delta) \rightarrow \Delta$ (cf. [11,(3.15)]), and $R_\Delta = \bigcap_{P' \in \text{Ht}_1(R_\Delta)} (R_\Delta)_{P'}$ by (i) and (ii).

\footnote{Let $R$ be an integral domain which is contained in a field $K$. The integral domain $R$ is said to be \textit{Krull domain} provided there is a family $\{V_i\}_{i \in I}$ of principal valuation rings (i.e., discrete rank one valuation rings), with $V_i \subseteq K$, such that

(i) $R = \bigcap_{i \in I} V_i$.

(ii) Given $0 \neq f \in R$, there is at most a finite number of $i$ in $I$ such that $f$ is not a unit in $V_i$.

Such a family as $\{V_i\}_{i \in I}$ (or $\{p_i \cap R \mid p_i$ is a non-zero prime ideal of $V_i$ ($i \in I$)$\}$) is called a \textit{defining family} of a Krull domain $R$. Note that $\{R_P \mid P \in \text{Ht}_1(R)\}$ (or $\text{Ht}_1(R)$) is indeed a defining family of a Krull domain $R$ (cf. [11,(1.9)]), where $R_P$ is called an \textit{essential valuation over-ring} of $R$ ($\forall P \in \text{Ht}_1(R)$).

\footnote{Let $R$ be an integral domain and $K = K(R)$ its quotient field. We say that an $R$-submodule $I$ of $K$ is a \textit{fractional ideal} of $R$ if $I \neq 0$ and there exists a non-zero element $\alpha \in R$ such that $\alpha I \subseteq R$. If a fractional ideal $I$ satisfies $I = R \cdot x$ ($R$ : $K$ $I$), where $R : K$ $I := \{x \in K \mid xI \subseteq R\}$, we say that $I$ is a \textit{divisorial} fractional ideal of $R$ (If $R$ is a Krull domain, then a (integral) ideal of $R$ is divisorial if and only if it can be expressed as the intersection of a finite number of height one primary ideals (See [14,Ex(12.4)]).}

\footnote{Let $R$ be an integral domain and $K = K(R)$ its quotient field. We say that an $R$-submodule $I$ of $K$ is a \textit{fractional ideal} of $R$ if $I \neq 0$ and there exists a non-zero element $\alpha \in R$ such that $\alpha I \subseteq R$. If a fractional ideal $I$ satisfies $I = R \cdot x$ ($R$ : $K$ $I$), where $R : K$ $I := \{x \in K \mid xI \subseteq R\}$, we say that $I$ is a \textit{divisorial} fractional ideal of $R$ (If $R$ is a Krull domain, then a (integral) ideal of $R$ is divisorial if and only if it can be expressed as the intersection of a finite number of height one primary ideals (See [14,Ex(12.4)]).}
(iv) \( R = R_\Delta \iff \text{Ht}_1(R) = \Delta \). (In fact, \( \text{Ht}_1(R) \supseteq \Delta \) if and only if \( R = \bigcap_{P \in \text{Ht}_1(R)} R_P \subseteq \bigcap_{P \in \Delta} R_P = \bigcap_{P' \in \text{Ht}_1(R_\Delta)} (R_\Delta)_{P'} = R_\Delta (\subseteq K(R)) \) by the minimality of their respective defining families and by (iii).)

(v) For \( P \in \Delta \) and \( n \in \mathbb{Z}_{\geq 0} \), \( P^{(n)} R_\Delta = (PR_\Delta)^{(n)} \), where \( (\cdot)^{(n)} \) denotes the symbolic power \( \text{cf.}[11,\text{p.26}] \). The symbolic power \( P^{(n)} \) of \( P \in \text{Ht}_1(R) \) is a divisorial ideal and any divisorial ideal of \( R \) is expressed as a finite intersection of the symbolic powers of prime ideals in \( Ht_1(R) \) (cf.\([11,\text{(5.7)}] \)).

(vi) \( R^\times = R \setminus \bigcup_{P \in \text{Ht}_1(R)} P \) and \( (R_\Delta)^\times = R_\Delta \setminus \bigcup_{P \in \Delta} PR_\Delta \). Let \( \Delta \) be a prime ideal of \( R_\Delta \), a prime ideal of \( R_\Delta \) \( (\forall P \in \Delta) \).

**Remark 2.6.** Let \( A \) be a Krull domain. Then a fractional ideal \( I \) of \( A \) is divisorial if and only if \( I = \bigcap_{P \in \text{Ht}_1(A)} IA_P \) (See \([11,\text{(5.5)}]\)). In addition, let \( B \) be a Krull domain containing \( A \) such that \( A \hookrightarrow B \) is flat. If \( I \) is a divisorial fractional ideal of \( A \), then \( IB \) is also a divisorial fractional ideal of \( B \) (See \([11,\text{p.31}]\)). Note here that there exists an example of a non-flat subintersection of a Noetherian Krull domain (See \([11,\text{p.32}]\)).

**Lemma 2.7** ([11,(6.5)]). Let \( A \) be an integral domain whose quotient field is \( K \). Let \( B \) be a ring between \( A \) and \( K \). Then \( B \) is flat over \( A \) if and only if \( A_M \leftrightarrow B_M \) for every maximal ideal \( M \) of \( B \).

**Corollary 2.8.** Let \( R \) be a Krull domain, let \( \Delta \subseteq \Delta' \) be subsets of \( \text{Ht}_1(R) \) and put \( R_{\Delta'} := \bigcap_{Q \in \Delta'} R_Q \) and \( R_\Delta := \bigcap_{Q \in \Delta} R_Q \). If \( R \hookrightarrow R_\Delta \) is flat then \( R_{\Delta'} \hookrightarrow R_{\Delta'} \) is also flat.

**Proof:** It is easy to see that \( R_{\Delta'} \subseteq R_\Delta \) and \( K(R_{\Delta'}) = K(R_\Delta) \). Since \( R \hookrightarrow R_\Delta \) is flat, \( R_M \cap R = (R_{\Delta'})_M \cap R_{\Delta'} = (R_{\Delta})_M \) for every maximal ideal \( M \) of \( R_\Delta \) by Lemma 2.7. So using Lemma 2.7 again, \( R_{\Delta'} \hookrightarrow R_\Delta \) is also flat.

Now the following assertions in Remark 2.9 are well-known (or easy to see):

**Remark 2.9.** Let \( A \) be a ring, let \( \Delta \) be a subset of \( \text{Spec}(A) \) and let \( I \) be an ideal of \( A \). In general, the implication \( I \subseteq \bigcup_{P \in \Delta} P \Rightarrow I \subseteq P \) \( (\exists P \in \Delta) \) does not necessarily hold. In other words, \( I \nsubseteq P \) \( (\forall P \in \Delta) \) \( \Rightarrow I \nsubseteq \bigcup_{P \in \Delta} P \) does not necessarily hold. Of course, the opposite implication \( I \nsubseteq \bigcup_{P \in \Delta} P \Rightarrow I \nsubseteq P \) \( (\forall P \in \Delta) \) holds trivially.

However the following cases are true.

1. Suppose that \( \Delta \) is a finite set. If \( I \subseteq \bigcup_{P \in \Delta} P \) then \( I \subseteq P \) for some \( P \in \Delta \).
2. Suppose \( I \) is a principal ideal. If \( I \subseteq \bigcup_{P \in \Delta} P \) then \( I \subseteq P \) for some \( P \in \Delta \).

The following theorem is a core result which will lead us to a positive solution for Conjecture (GJC).
Theorem 2.10. Let $R$ be a Krull domain domain and let $\Delta_1$ and $\Delta_2$ be subsets of $\text{Ht}_1(R)$ such that $\Delta_1 \cup \Delta_2 = \text{Ht}_1(R)$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Put $R_i := \bigcap_{Q \in \Delta_i} R_Q$ ($i = 1, 2$), subintersections of $R$. Assume that $\Delta_2$ is a finite set and that $R \hookrightarrow R_1$ is flat. Assume moreover that $R_1$ is factorial and that $R^x = (R_1)^x$. Then the following statements hold:

1. Every $Q \ast \cap R$ ($Q \ast \in \text{Ht}_1(R_1)$) is a principal ideal of $R$.
2. More strongly, $\Delta_2 = \emptyset$ and $R = R_1$.

Proof. Recall first that $\text{Ht}_R$ is flat. Then we have the canonical bijection $\Delta_i$.

Recall first that $\text{Ht}_R$ is flat. Then we have the canonical bijection $\Delta_i$.

Thus we assume that $\Delta_2 = \{Q\}$ and $R_2 = R_{Q \ast}$. Then

$$R = R_1 \cap R_2 = \bigcap_{Q \in \Delta_1} R_Q \cap R_{Q \ast}.$$ 

Note second that $R_2 = R_{Q \ast}$ is a DVR (factorial domain) and $R \hookrightarrow R_{Q \ast} = R_2$ is flat. Then we have the canonical bijection $\Delta_i \hookrightarrow \text{Ht}_1(R_i)$ ($\Delta_i \ni Q \hookrightarrow QR_i \in \text{Ht}_1(R_i)$) (cf. Remark 2.5(iii)). So for $Q \in \text{Ht}_1(R)$, $QR_i$ is either a prime ideal of height one (if $Q \in \Delta_i$) or $R_i$ itself (if $Q \notin \Delta_i$) for each $i = 1, 2$ (cf. Remark 2.5(ii)).

Let $v_Q(\ )$ be the (additive) valuation on $K(R)$ associated to the principal valuation ring $R_Q (\subseteq K(R))$ for $Q \in \text{Ht}_1(R)$. Note here that for any $Q \in \Delta_i$ ($i = 1, 2$), $R_Q = (R_i)QR_i$, and $v_Q(\ ) = v_{QR_i}(\ )$ by Remark 2.5(i),(ii),(iii).

Proof of (1): Put $P_\ast = Q_\ast \cap R$. Since $Q_\ast \in \text{Ht}_1(R_1)$, $P_\ast \in \Delta_1$ by Remark 2.5(i).

Apply “The Approximation Theorem for Krull Domains (Lemma A.8)” to $R$ (or $\text{Ht}_1(R) = \Delta_1 \cup \Delta_2$ with $\Delta_2$ (a finite set)). Then there exists $t \in K(R)$ such that

$$v_{P_\ast}(t) = 1, \ v_{Q_\ast}(t) = 0 \text{ and } v_{Q_\ast}(t) \geq 0 \text{ otherwise } (*).$$

It is easy to see that $t \in R$ and $tR_{Q_\ast} = R_{Q_\ast}$ (i.e., $t \in R \setminus Q_\ast$) by (*). Since $R_1$ is factorial, we have

$$t = (t_1)^{n_1} \cdots (t_s)^{n_s} (\exists s, n_j \geq 1 (1 \leq j \leq s)) \quad (**).$$

with some prime elements $t_j$ ($1 \leq j \leq s$) in $R_1$, where each prime element $t_j$ in $R_1$ is determined up to modulo $(R_1)^x = R^x$. Put $Q_j = t_jR_1 \cap R$ ($1 \leq j \leq s$). Then $Q_jR_1 = t_jR_1$ and $Q_jR_1 \cap R = t_jR_1 \cap R = Q_j$ by Remark 2.5(i),(ii).

Let $\Delta_1 := \{Q \in \text{Ht}_1(R) \mid v_Q(t) > 0\}$. Then $\Delta_1$ is a finite subset of $\Delta_1$ by (*). Note here that $R_Q = (R_i)QR_i$ for each $Q \in \Delta_i$ ($i = 1, 2$). Considering
that $t_j$ is a prime factor of $t$ in $R_1$ and that $(R_1)^x = R^x$, we have a one-to-one correspondences ($\#$):

$$t_j R^x = t_j(R_1)^x \mapsto t_j R_1 = Q_j R_1 (\exists Q_j \in \Delta') \mapsto t_j R_1 \cap R = Q_j R_1 \cap R = Q_j \in \Delta').$$

Note that for each $j$, the value $v_Q(t_j)$ ($Q \in \text{Ht}_1(R)$) remains unaffected by the choice of $t_j$ in (**) and that $v_Q(t) = n_1 v_Q(t_1) + \cdots + n_s v_Q(t_s)$ for every $Q \in \text{Ht}_1(R)$, where $n_j = v_{Q_j R_1}(t) = v_{Q_j}(t) > 0$ ($1 \leq j \leq s$). So we have an irredundant primary decomposition

$$t R = Q_1^{(n_1)} \cap \cdots \cap Q_s^{(n_s)} \cap Q^{(m)}$$

for some $m \in \mathbb{Z}_{\geq 0}$, where $Q_i^{(n_i)}$, $Q^{(m)}$ denote the symbolic power. Since $Q'R_1 = R_1$ and $R \mapsto R_1$ and $R \mapsto R_Q'$ are flat, $t R_1 = Q_1^{(n_1)} \cap \cdots \cap Q_s^{(n_s)} \cap Q^{(m)} = Q_1^{(n_1)} \cap \cdots \cap Q_s^{(n_s)} \cap Q^{(m)}$. Thus $\Delta' = \{Q_1, \ldots, Q_s\}$ by the a one-to-one correspondences ($\#$) above, which is the set of the prime divisors of $t R$. Since $t R Q' = R Q'$, $Q'$ is not a prime divisor of $t R$ and $Q' \notin \Delta'$ (i.e., $m = 0$ so $v_Q(t) = 0$ by (**)).

Now suppose that $v_{Q'}(t_j) > 0$ for some $j$. Then $v_{Q'}(t_j) \geq 0$ for every $Q \in \text{Ht}_1(R) = \Delta' \cup \{Q'\}$ ($\because t_j \in R_1$), so that $t_j \in R$. Thus $t_j \in Q_j \cap Q'$, which yields that $Q'$ is a prime divisor of $t_j R$ and $t_j R Q' \subseteq Q'R Q'$. Note that $t R Q_j \subseteq t_j R Q_j \subseteq Q_j R Q_j$ by (**). Put $S := R \setminus Q_j \cup Q'$, a multiplicatively closed set of $R$ and consider $S^{-1} R$. Then $S^{-1} R \neq S^{-1} R \subseteq t_j S^{-1} R \subseteq Q_j S^{-1} R \cap Q'S^{-1} R$ because $t R_1 \cap S = Q_\ell R_1 \cap S \supseteq Q_\ell \cap S \neq \emptyset$ for $\forall \ell \neq j$ ($\because Q_\ell \subseteq Q_j \cup Q'$), and hence $Q'S^{-1} R$ is a prime divisor of $t S^{-1} R$. Thus $Q'$ is a prime divisor of $t R$ ($\because Q' \in \text{Ht}_1(R)$), so that $Q' \in \Delta'$, which is a contradiction.

Hence $v_{Q'}(t_j) \leq 0$ for $\forall j$ ($1 \leq j \leq s$). Therefore we obtain

$$\left\{ \begin{array}{ll}
v_{Q_i}(t_j) &= \delta_{ij} \quad \text{(Kronecker's symbol)} \\
v_{Q_i}(t_j) &\leq 0 \quad (1 \leq \forall j \leq s) \end{array} \right. \quad (**).$$

Thus $v_{Q'}(1/t_j) \geq 0$ ($1 \leq \forall j \leq s$), which yields that $1/t_j \in R Q'$ and hence $R Q' \subseteq t_j R Q'$ ($1 \leq \forall j \leq s$) in $K(R)$. It follows from (**) that $(t_1)^{n_1} \cdot \cdots \cdot (t_s)^{n_s} R Q' = t R Q' \subseteq (t_1)^{n_1} \cdot (t_2)^{n_2} \cdot \cdots \cdot (t_s)^{n_s} R Q'$. Therefore $t_1 R Q' \subseteq R Q'$ and hence $t_1 \in R Q'$, which means that $t_1 \in (R Q')^x$. Replacing $t_j$ by $t_j$, we have

$$t_j \in (R Q')^x (1 \leq \forall j \leq s).$$

Since $P_* \in \Delta_1$ by ($*$) and $1 = v_{P_*}(t) = v_{P_* R_1}(t) = n_1 v_{P_* R_1}(t_1) + \cdots + n_s v_{P_* R_1}(t_s)$ by (**), there exists $i$ such that $n_i = 1$ and $v_{P_* R_1}(t_i) = v_{P_* R_1}(t_i) = 1$, say $i = 1$, and then $P_* = Q_1 \in \Delta'$. So we have $t_1 \in (R Q')^x$ and $t_1 R_1 = P_* R_1$, which yields that $v_{Q}(t_1) \geq 0$ for every $Q \in \Delta_1 \cup \{Q'\} = \Delta_1 \cup \Delta_2 = \text{Ht}_1(R)$ and consequently that $t_1 \in R$. Therefore we have

$$t_1 \in R, \quad t_1 R_1 = P_* R_1, \quad \text{and } t_1 \in (R Q')^x.$$
It follows that \( P_\ast = P_\ast(R_1 \cap R_2) \subseteq P_\ast R_1 \cap P_\ast R_2 = t_1 R_1 \cap R_2 = t_1 R_1 \cap R_1 R_Q = t_1(R_1 \cap R_Q) = t_1 R \subseteq R \). Since \( P_\ast \in \text{Ht}_1(R) \), we have \( P_\ast = t_1 R \). Therefore we conclude that \( P_\ast = Q_\ast \cap R \) is a principal ideal of \( R \).

Proof of (2) : We shall show \( \Delta_2 = \emptyset \).

Suppose that \( \Delta_2 \neq \emptyset \) and so that \( \Delta_2 := \{ Q' \} \) (See the first paragraph of Proof).

We divide the proof of (2) into the following two cases.

(2-1) Consider the case that \( Q' \nsubseteq \bigcup_{P \in \Delta_1} P \). Take \( t' \in Q' \) such that \( t' \notin \bigcup_{P \in \Delta_1} P \) (which is the set of the non-units of \( R_1 \)). Since \( t' \) is a unit in \( R_1 \), we have \( t' \in (R_1)^\times = R^\times \), a contradiction.

(2-2) Consider the case that \( Q' \subseteq \bigcup_{P \in \Delta_1} P \) (with \( Q' \not\subseteq P \) (\( \forall P \in \Delta_1 \)) (\( : Q' \notin \Delta_1 \)). Take \( t' \in Q' \) such that \( t'R_Q' = Q'R_Q \). Since \( t' \in \bigcup_{P \in \Delta_1} P \subseteq \bigcup_{P \in \Delta_1} PR_1 \) by Remark \( \ref{rem:principal-ideal} \)(iii), we have \( t' \in Q' \subseteq \bigcup_{P \in \Delta_1} PR_1 \) (which is the set of the non-units in \( R_1 \)), whence \( t' \) is a non-unit in \( R_1 \), that is, \( t' \notin (R_1)^\times = R^\times \). Thus we have an irredundant primary decomposition \( t'R = P_1^{(m_1)} \cap \cdots \cap P_s^{(m_s')} R \) for some \( s' \geq 1 \), \( m_i \geq 1 \) and \( P_i \in \Delta_1 \) \((1 \leq i \leq s') \) with \( P_i \neq P_j \) \((i \neq j) \). It is clear that \( P_i \neq Q' (i = 1, \ldots, s') \). Thus noting that \( R \twoheadrightarrow R_1 \) is a flat subintersection of \( R \) and \( R_1 \) is factorial, we have \( t'R_1 = P_1^{(m_1)} R_1 \cap \cdots \cap P_s^{(m_s')} R_1 \cap Q'R_1 = P_1^{(m_1)} R_1 \cap \cdots \cap P_s^{(m_s')} R_1 \cap Q' \) \((\text{cf. Remark } \ref{rem:principal-ideal} \text{ iii}) \) and (iv)). Thus we may assume that \( t' = a_1^{m_1} \cdots a_s^{m_s'} \) in \( R_1 \). Since each \( P_i \in \Delta_1 \) is a principal ideal \( a_i'R \) of \( R \) generated by a prime element \( a_i' \in R \) by (1), we have \( a_i'R_1 = a_i R_1 \) and hence both \( a_i/a_i' \) and \( a_i'/a_i \) belong to \( R_1 \). Thus \( a_i'/a_i \in (R_1)^\times = R^\times \). So we can assume that \( a_i \in P_i \subseteq R \) and \( a_i R = P_i \) for all \( i \). Since \( t' = a_1^{m_1} \cdots a_s^{m_s'} \) and \( t'R_Q' = a_i^{m_i} R_Q \) for some \( i \), say \( i = 1 \). Since \( Q'R_Q' = t'R_Q' \) and \( a_1 \) is a prime element in \( R \), we have \( m_1 \geq 1 \) and \( t'R_Q' = a_1 R_Q \). So we have \( Q' = Q'R_Q' \cap R = t'R_Q' \cap R = a_1 R_Q' \cap R = a_1 R = P_1 \) because \( a_1 \) is a prime element in \( R \). It follows that \( Q' = P_1 \in \Delta_1 \cap \Delta_2 = \emptyset \), a contradiction.

Therefore in any case, \( \Delta_2 \) must be \( \emptyset \) and \( R = R_1 \). 

Here we emphasize the result (2) in Theorem \ref{thm:main-theorem} as follows.

Corollary 2.11. Let \( R \) be a Krull domain domain and let \( \Gamma \) be a finite subset of \( \text{Ht}_1(R) \). Assume that a subintersection \( R_0 := \bigcap_{Q \in \text{Ht}_1(R) \setminus \Gamma} R_Q \) is factorial, that \( (R_0)^\times = R^\times \) and that \( R \twoheadrightarrow R_0 \) is flat. Then \( \Gamma = \emptyset \) and \( R = R_0 \).

Proof. Putting \( \Delta_1 := \text{Ht}_1(R) \setminus \Gamma \) and \( \Delta_2 := \Gamma \), we can apply Theorem \ref{thm:main-theorem}(2) to this case, and we have our conclusion. 

Now we know the following lemma :
Lemma 2.12 ([5,(6.6)]). A flat extension of a Krull domain within its quotient field is a subintersection.

The following proposition gives us a chance of a fundamental approach to Conjecture (GJC).

Proposition 2.13. Let \( i : C \hookrightarrow B \) be Noetherian normal domains such that \( \mathcal{O}_i : \text{Spec}(B) \to \text{Spec}(C) \) is an open immersion. If \( B \) is factorial and that \( C^\times = B^\times \), then \( C = B \).

**Proof.** Note first that \( C \) is a Krull domain because a Noetherian normal domain is completely integrally closed (See [5,(3.13)]).

Since \( i : C \hookrightarrow B \) is flat and \( K(C) = K(B) \), \( B \) is a subintersection \( C^\Gamma = \bigcap_{P \in \text{Ht}_1(C) \setminus \Gamma} C_P \) with a finite subset \( \Gamma \) of \( \text{Ht}_1(C) \) by Lemma 2.12. Therefore it follows from Corollary 2.11 that \( \Gamma = \emptyset \) and \( C = B \).

**Corollary 2.14.** Let \( i : A \hookrightarrow B \) be a quasi-finite homomorphism of Noetherian normal domains such that \( K(B) \) is finite separable algebraic over \( K(A) \). If \( B \) is factorial and \( i \) induces an isomorphism \( A^\times \to B^\times \) of groups, then \( i : A \hookrightarrow B \) is finite.

**Proof.** Let \( C \) be the integral closure of \( A \) in \( K(B) \). Then \( C \) is a Noetherian normal domain by Lemma A.11. \( A \hookrightarrow C \) is finite and \( C \hookrightarrow B \) is an open immersion by Lemma A.11. Since \( A^\times = C^\times = B^\times \), we have \( C = B \) by Proposition 2.13. Therefore we conclude that \( i : A \hookrightarrow C = B \) is finite.

Here is our main result as follows.

**Theorem 2.15 (Conjecture(GJC)).** Let \( \varphi : S \to T \) be an unramified homomorphism of Noetherian domains with \( T^\times = \varphi(S^\times) \). Assume that \( T \) is factorial and that \( S \) is a simply connected domain. Then \( \varphi \) is an isomorphism.

**Proof.** Note first that \( \varphi : S \to T \) is an étale (and hence flat) homomorphism by Lemmas A.12 and A.10 and that \( \varphi \) is injective by Lemma A.3. We can assume that \( \varphi : S \to T \) is the inclusion \( S \hookrightarrow T \). Let \( C \) be the integral closure of \( S \) in \( K(T) \). Then \( S \hookrightarrow C \) is finite and \( C \) is a Noetherian normal domain by Lemma A.11 since \( K(T) \) is a finite separable (algebraic) extension of \( K(S) \) and \( C \to T \) is an open immersion by Lemma A.11 with \( S^\times = C^\times = T^\times \). Thus we have \( C = T \) by

\[ ^\dagger \text{Let } f : A \to B \text{ be a ring-homomorphism of finite type. Then } f \text{ is said to be quasi-finite if, for each } P \in \text{Spec}(B), \ B_P/(P \cap A)B_P \text{ is a finite dimensional vector space over the field } k(P \cap A) := A_{P \cap A}/(P \cap A)A_{P \cap A}. \quad \text{In general, let } X \text{ and } Y \text{ be schemes and } \varphi : X \to Y \text{ a morphism locally of finite type. Then } \varphi \text{ is said to be quasi-finite if, for each point } x \in X, \ \mathcal{O}_x/m_{\varphi(x)} \mathcal{O}_x \text{ is a finite dimensional vector space over the field } k(\varphi(x)). \quad \text{In particular, a finite morphism and an unramified morphism are quasi-finite (cf.[21]).} \]
Corollary 2.14. So $S \hookrightarrow C = T$ is étale and finite, and hence $S = T$ because $S$ is simply connected. $\square$

Corollary 2.16. Let $k$ be a field of characteristic 0 and let $\psi: V \to W$ be an unramified morphism of simply connected (irreducible) $k$-affine varieties whose affine rings $K[V]$ and $K[W]$. If $K[W]$ is normal and $K[V]$ is factorial, then $\psi$ is an isomorphism.

Proof. We may assume that $k$ is an algebraically closed field. By the simple connectivity, we have $K[V]^\times = K[W]^\times = k^\times$ by Proposition A.3. So our conclusion follows from Theorem 2.15. $\square$

On account of Remark A.1, this resolves The Jacobian Conjecture ($JC_n$) as follows:

Corollary 2.17 (The Jacobian Conjecture ($JC_n$)). If $f_1, \ldots, f_n$ are elements in a polynomial ring $k[X_1, \ldots, X_n]$ over a field $k$ of characteristic 0 such that $\text{det}(\partial f_i/\partial X_j)$ is a nonzero constant, then $k[f_1, \ldots, f_n] = k[X_1, \ldots, X_n]$.

Example 2.18 (Remark). In Theorem 2.15 the assumption $T^\times \cap S = S^\times$ seems to be sufficient. However, the following Example implies that it is not the case. It seems that we must really require at least such strong assumptions that $T$ is simply connected or that $S^\times = T^\times$ as a certain mathematician pointed out.

Let $S := \mathbb{C}[x^3 - 3x]$, and let $T := \mathbb{C}[x, 1/(x^2 - 1)]$. Then obviously $\text{Spec}(T) \to \text{Spec}(S)$ is surjective and $T^\times \cap S = S^\times = \mathbb{C}^\times$, but $T$ is not simply connected and $T^\times \not\supset S^\times$. Since $S = \mathbb{C}[x^3 - 3x] \hookrightarrow \mathbb{C}[x] := C$ is finite, indeed $\mathbb{C}[x]$ is the integral closure of $S$ in $K(\mathbb{C}[x])$. Note here that $S, C$ and $T$ are factorial but that $T^\times \neq C^\times = \mathbb{C}^\times$, which means that $T$ is not a simply connected by Proposition A.3. Since $\frac{\partial(x^3 - 3x)}{\partial x} = 3(x - 1)(x + 1)$, $T$ is unramified (indeed, étale) over $S$ (by Lemma A.10).

Precisely, put $y = x^3 - 3x$. Then $S = \mathbb{C}[y]$, $C = \mathbb{C}[x]$ and $T = \mathbb{C}[x, 1/(x - 1), 1/(x + 1)]$. It is easy to see that $y - 2 = (x + 1)^2(x - 2)$ and $y + 2 = (x - 1)^2(x + 2)$ in $C = \mathbb{C}[x]$. So $(x + 1)C \cap S = (y - 2)S$ and $(x - 1)C \cap S = (y + 2)S$. Since $T = C_{x^2 - 1} = \mathbb{C}[x, x^2 - 1]$, $(x - 2)T = (y - 2)T$ and $(x + 2)T = (y + 2)T$, that is, $y + 2, y - 2 \not\in T^\times$. It is easy to see $(y - b)T \neq T$ for any $b \in \mathbb{C}$, which means that $S \hookrightarrow T$ is faithfully flat and $T^\times \cap S = S^\times = \mathbb{C}^\times$.

Remark 2.19. We see the following result (cf. [10, (4.4.2)] and [3]):

Let $k$ be an algebraically closed field of characteristic 0. Let $f: V \to W$ be an injective morphism between irreducible $k$-affine varieties of the same dimension. If $K[W]$, the coordinate ring of $W$ is factorial then there is equivalence between
(i) $f$ is an isomorphism and (ii) $f^* : K[W] \rightarrow K[V]$ induces an isomorphism $K[W]^\times \rightarrow K[V]^\times$.

This is indeed interesting and is somewhat a generalization of Case(1) in Introduction. But “the factoriality of $K[W]$” and “the injectivity of $f$” seem to be too strong assumptions (for the Generalized Jacobian Conjecture(GJC)) even if $f$ is not necessarily unramified.

3. An Extension of The Jacobian Conjecture(JC$_n$)

In this section, we enlarge a coefficient ring of a polynomial ring and consider the Jacobian Conjecture about it.

**Theorem 3.1.** Let $A$ be an integral domain whose quotient field $K(A)$ is of characteristic 0. Let $f_1, \ldots, f_n$ be elements of a polynomial ring $A[X_1, \ldots, X_n]$ such that

$$f_i = X_i + \text{(higher degree terms)} \quad (1 \leq i \leq n) \quad (*).$$

If $K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n]$, then $A[X_1, \ldots, X_n] = A[f_1, \ldots, f_n]$.

**Proof.** It suffices to prove $X_1, \ldots, X_n \in A[f_1, \ldots, f_n]$.

We introduce a linear order in the set $\{k := (k_1, \ldots, k_n) | k_r \in \mathbb{Z}_{\geq 0} \ (1 \leq r \leq n)\}$ of lattice points in $\mathbb{R}_{\geq 0}^n$ (where $\mathbb{R}$ denotes the field of real numbers) in the following way:

$$k = (k_1, \ldots, k_n) > j = (j_1, \ldots, j_n) \text{ if } k_r > j_r \text{ for the first index } r \text{ with } k_r \neq j_r.$$

(This order is so-called the lexicographic order in $\mathbb{Z}_{\geq 0}^n$).

**Claim.** Let $F(s) := \sum_{j=0}^{s} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n]$ with $c_j \in K(A)$. Then $c_j \in A \ (0 \leq j \leq s)$.

(Proof.) If $s = 0 = (0, \ldots, 0)$, then $F(0) = c_0 \in A$.

Suppose that for $k(< s), c_j \in A \ (0 < j \leq k)$. Then $F(k) \in A[X_1, \ldots, X_n]$ by (*), and $F(s) = F(k) = G := \sum_{j > k} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n]$. Let $k' = (k'_1, \ldots, k'_n)$ be the next member of $k \ (k = (k_1, \ldots, k_n) < (k'_1, \ldots, k'_n) = k')$ with $c_{k'} \neq 0$.

We must show $c_{k'} \in A$. Note that $F(s) = F(k) + G$ with $F(k), G \in A[X_1, \ldots, X_n]$. Developing $F(s) := \sum_{j=0}^{s} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n]$ with respect to $X_1, \ldots, X_n$, though the monomial $X_1^{k'_1} \cdots X_n^{k'_n}$ with some coefficient in $A$ maybe appears in $F(k)$, it appears in only one place of $G$ with a coefficient $c_{k'}$ by the assumption (*). Hence the coefficient of the monomial $X_1^{k'_1} \cdots X_n^{k'_n}$ in $F(s)$ is a form $b + c_{k'}$ with $b \in A$ because $F(k) \in A[X_1, \ldots, X_n]$. Since $F(s) \in A[X_1, \ldots, X_n]$, we have $b + c_{k'} \in A$ and hence $c_{k'} \in A$. Therefore we have proved our Claim by induction.
Next, considering $K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n]$, we have

$$X_1 = \sum c_j f_1^{i_1} \cdots f_n^{i_n}$$

with $c_j \in A$ by Claim above. Consequently, $X_1$ is in $A[f_1, \ldots, f_n]$. Similarly $X_2, \ldots, X_n$ are in $A[f_1, \ldots, f_n]$ and the assertion is proved completely. Therefore $A[f_1, \ldots, f_n] = A[X_1, \ldots, X_n]$.

The Jacobian Conjecture for $n$-variables can be generalized as follows.

**Corollary 3.2** (cf.[10,(1.1.18)]) The Extended Jacobian Conjecture. Let $A$ be an integral domain whose quotient field $K(A)$ is of characteristic 0. Let $f_1, \ldots, f_n$ be elements of a polynomial ring $A[X_1, \ldots, X_n]$ such that the Jacobian $\det(\partial f_i/\partial X_j)$ is in $A^\times$. Then $A[X_1, \ldots, X_n] = A[f_1, \ldots, f_n]$.

**Proof.** We see that $K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n]$ by Corollary 2.17. It suffices to prove $X_1, \ldots, X_n \in A[f_1, \ldots, f_n]$. We may assume that $f_i (1 \leq i \leq n)$ have no constant term. Since $f_i \in A[f_1, \ldots, f_n]$, $f_i = a_{i_1} X_1 + \ldots + a_{i_n} X_n + ($higher degree terms$)$

with $a_{ij} \in A$, where $(a_{ij}) = (\partial f_i/\partial X_j)(0, \ldots, 0)$. The assumption implies that the determinant of the matrix $(a_{ij})$ is a unit in $A$. Let

$$Y_i = a_{i_1} X_1 + \ldots + a_{i_n} X_n \quad (1 \leq i \leq n).$$

Then $A[X_1, \ldots, X_n] = A[Y_1, \ldots, Y_n]$ and $f_i = Y_i + ($higher degree terms$)$. So to prove the assertion, we can assume that without loss of generality

$$f_i = X_i + ($higher degree terms$) \quad (1 \leq i \leq n) \quad (*) .$$

Therefore by Theorem 3.1 we have $A[f_1, \ldots, f_n] = A[X_1, \ldots, X_n]$.

**Example 3.3.** Let $\phi : \mathbb{A}^n \rightarrow \mathbb{A}^n$ be a morphism of affine spaces over $\mathbb{Z}$, the ring of integers. If the Jacobian $J(\phi)$ is equal to either $\pm 1$, then $\phi$ is an isomorphism.

**Appendix A. A Collection of Tools Required in This Paper**

Recall the following well-known results, which are required in this paper. We write down them for convenience.

**Remark A.1** (cf.[10,(1.1.31)]). Let $k$ be an algebraically closed field of characteristic 0 and let $k[X_1, \ldots, X_n]$ denote a polynomial ring and let $f_1, \ldots, f_n \in k[X_1, \ldots, X_n]$. If the Jacobian $\det(\partial f_i/\partial X_i) \in k^\times (= k \setminus \{0\})$, then $k[X_1, \ldots, X_n]$ is
étales over the subring $k[f_1, \ldots, f_n]$. Consequently $f_1, \ldots, f_n$ are algebraically independent over $k$. Moreover, Spec$(k[X_1, \ldots, X_n]) \rightarrow$ Spec$(k[f_1, \ldots, f_n])$ is surjective, which means that $k[f_1, \ldots, f_n] \rightarrow k[X_1, \ldots, X_n]$ is faithfully flat.

In fact, put $T = k[X_1, \ldots, X_n]$ and $S = k[f_1, \ldots, f_n]$. We have an exact sequence by [15,(26.H)]:

$$\Omega_k(S) \otimes_S T \rightarrow \Omega_k(T) \rightarrow \Omega_S(T) \rightarrow 0,$$

where

$$v(df_i \otimes 1) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} dX_j \quad (1 \leq i \leq n).$$

So $\det(\partial f_i/\partial X_j) \in k^\times$ implies that $v$ is an isomorphism. Thus $\Omega_S(T) = 0$ and hence $T$ is unramified over $S$ by [4,VI,(3.3)]. So $T$ is étales over $S$ by Lemma A.10 below. Thus $df_1, \ldots, df_n \in \Omega_k(S)$ compounds a free basis of $T \otimes_S \Omega_k(T) = \Omega_k(T)$, which means $K(T)$ is algebraic over $K(S)$ and that $f_1, \ldots, f_n$ are algebraically independent over $k$.

The following proposition is related to the simple-connectivity of affine space $A_k^n$ $(n \in \mathbb{N})$ over a field $k$ of characteristic 0. Its (algebraic) proof is given without the use of the geometric fundamental group $\pi_1(\ )$ after embedding $k$ into $\mathbb{C}$ (the Lefschetz Principle).

**Proposition A.2** ([23]). Let $k$ be an algebraically closed field of characteristic 0. Then a polynomial ring $k[Y_1, \ldots, Y_n]$ over $k$ is simply connected.

**Proposition A.3** ([2,Theorem 2]). Any invertible regular function on a normal, simply connected $\mathbb{C}$-variety is constant.

**Lemma A.4** ([15,(6.D)]). Let $\varphi : A \rightarrow B$ be a homomorphism of rings. Then $\text{a}^{\varphi} : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is dominating (i.e., $\text{a}^{\varphi}($Spec$(B))$ is dense in Spec$(A)$) if and only if $\varphi$ has a kernel $\subseteq \text{nil}(A) := \sqrt{(0)A}$. If, in particular, $A$ is reduced, then $\text{a}^{\varphi}$ is dominating $\Leftrightarrow \text{a}^{\varphi}($Spec$(B))$ is dense in Spec$(A)$ $\Leftrightarrow \varphi$ is injective.

**Lemma A.5** ([14,(9.5)], [15,(6.1)]). Let $A$ be a Noetherian ring and let $B$ be an $A$-algebra of finite type. If $B$ is flat over $A$, then the canonical morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is an open map. (In particular, if $A$ is reduced (eg., normal) in addition, then $A \rightarrow B$ is injective.)

The following is well-known, but we write it down here for convenience.

**Lemma A.6** ([14]). Let $k$ be a field, let $R$ be a $k$-affine domain and let $L$ be a finite algebraic field-extension of $K(R)$. Then the integral closure $R_L$ of $R$ in $L$ is finite over $R$.

Moreover the above lemma can be generalized as follows.
Lemma A.7 ([15,(31.B)]). Let $A$ be a Noetherian normal domain with quotient field $K$, let $L$ be a finite separable algebraic extension field of $K$ and let $A_L$ denote the integral closure of $A$ in $L$. Then $A_L$ is finite over $A$.

Lemma A.8 (The Approximation Theorem for Krull Domains [11,(5.8)]). Let $A$ be a Krull domain. Let $n(P)$ be a given integer for each $P$ in $\text{Ht}_1(A)$ such that $n(P) = 0$ for almost all $P$. For any preassigned set $P_1,\ldots,P_r$ there exists $x \in K$ such that $v_P(x) = n(P_i)$ with $v_P(x) \geq 0$ otherwise, where $v_P(\ )$ denotes the (additive) valuation associated to the DVR $A_P$.

For a Noetherian ring $R$, the definitions of its Normality (resp. its Regularity) is seen in [15,p.116], that is, $R$ is a normal ring (resp. a regular ring) if $R_p$ is a normal domain (resp. a regular local ring) for every $p \in \text{Spec}(R)$.

Lemma A.9 ([14,(23.8)], [15,(17.I)])(Serre’s Criterion on normality)). Let $A$ be a Noetherian ring. Consider the following conditions :

$(R_1) : A_p$ is regular for all $p \in \text{Spec}(A)$ with $\text{ht}(p) \leq 1$ ;

$(S_2) : \text{depth}(A_p) \geq \min(\text{ht}(p), 2)$ for all $p \in \text{Spec}(A)$.

Then $A$ is a normal ring if and only if $A$ satisfies $(R_1)$ and $(S_2)$. ( Note that $(S_2)$ is equivalent to the condition that any prime divisor of $fA$ for any non-zerodivisor $f$ of $A$ is not an embedded prime.)

Lemma A.10 ([SGA,(Exposé I, Cor.9.11)]). Let $S$ be a Noetherian normal domain, let $R$ be an integral domain and let $\phi : S \to R$ be a ring-homomorphism of finite type. If $\phi$ is unramified, then $\phi$ is étale.

Lemma A.11 ([21,p.42](Zariski’s Main Theorem)). Let $A$ be a ring and let $B$ be an $A$-algebra of finite type which is quasi-finite over $A$. Let $\overline{A}$ be the integral closure of $A$ in $B$. Then the canonical morphism $\text{Spec}(B) \to \text{Spec}(\overline{A})$ is an open immersion.

Lemma A.12 (cf.[14,(23.9)]). Let $(A,m)$ and $(B,n)$ be Noetherian local rings and $A \to B$ a local homomorphism. Suppose that $B$ is flat over $A$. Then

(i) if $B$ is normal (or reduced), then so is $A$,

(ii) if both $A$ and the fiber rings of $A \to B$ are normal (or reduced), then so is $B$.

Corollary A.13. Let $A \to B$ be a étale (i.e., flat and unramified) homomorphism of Noetherian rings $A$ and $B$. Then $A$ is a normal ring (resp. a regular ring) if and only if so is $B$.

Proof. Let $M$ be a prime ideal of $B$ and put $m = A \cap M$. Then we have a local homomorphism $A_m \to B_M$. We have only to show that $B_M$ is a Noetherian normal ring if $A_m$ is a Noetherian normal domain.
The “If-Part” follows Lemma A.12(i) immediately.

The “Only if-Part”: Let \( P' \) be a prime ideal of \( B_M \) and put \( p' = P' \cap A_m \). Then the fiber ring at \( p' \) of \( A_m \to B_M \) is \( B_M \otimes_{A_m} k(p') \) with \( k(p') = (A_m)_{p'}/p'(A_m)_{p'} \).

Since \( A_m \to B_M \) is étale, so that \( B_M \otimes_{A_m} k(p') \) is a finite separable algebraic field extension of \( k(p') \) and is a Noetherian normal (resp. regular) ring. Thus by Lemma A.12, \( B_M \) is a Noetherian normal (resp. regular) ring. Therefore \( B \) is a Noetherian normal (resp. regular) ring.

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" 14 For He Himself knows our frame;
     He is mindful that we are (made of) but dust.
 15 As for man, his days are like grass;
     As a flower of the field, so he flourishes.
 16 When the wind has passed over it,
     it is no more,
     and its place acknowledges it no longer.  
"  
——— PSALM 103 (NASB)

" The Long Goodbye ! " — R. Chandler —
" The Last Goodbye ! " — Someone —