Optimal Boundary Control of a Nonlinear Reaction Diffusion Equation via Completing the Square and Al’brekht’s Method

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Abstract—The two contributions of this article are as follows. The first is the solution of an infinite dimensional, boundary controlled linear quadratic regulator by the simple and constructive method of completing the square. The second contribution is the extension of Al’brekht’s method to the optimal stabilization of a boundary controlled, nonlinear reaction diffusion system.

Index Terms—Boundary control of an LQR, computational methods, distributed parameter system, optimal control.

I. INTRODUCTION

In 1961, Al’brekht [1] showed how one could compute degree by degree the Taylor polynomials of the optimal cost and optimal feedback of a smooth, nonlinear, infinite horizon, finite dimensional optimal control problem provided the linear part of the dynamics, and the quadratic part of the running cost satisfied the standard linear quadratic regulator (LQR) conditions.

Recently, Krener [16] showed how Al’brekht’s method could be generalized to infinite dimensional problems with distributed control. In this article, we show how Al’brekht’s method can be generalized to infinite dimensional problems with boundary control. In Section II, we present and solve an LQR for the boundary control of a heated rod. We do this in a novel way, by completing the square in infinite dimensions. In Section III, we analyze the closed loop linear dynamics. In Section IV, we show how Al’brekht’s method can be used to stabilize a nonlinear reaction diffusion equation using boundary control.

We are not the first to use Al’brekht’s method on infinite dimensional systems, see the works of Kunisch et al. [2], [3], [18]. The authors in [17] and [21] have had great success stabilizing infinite dimensional systems through boundary control where the nonlinearities are expressed by Volterra integral operators of increasing degrees using backstepping techniques. In our extension of Al’brekht, we assume that the nonlinearities are given by Fredholm integral expressions of increasing degrees.

II. BOUNDARY CONTROL OF A HEATED ROD

We consider a modification of [8, Example 3.5.5]. We have a rod of length one insulated at one end and heated/cooled at the other. The goal is to control the temperature to a constant set point which we conveniently take to be zero.

Let $0 \leq x \leq 1$ be distance along the rod, $z(x,t)$ be the temperature of the rod at $x$ and $t$ and $z^0(x)$ be the initial temperature distribution of the rod at $t = 0$. The goal is to stabilize the temperature to $z = 0$ as $t \to \infty$ using boundary control at $x = 1$.

The rod is modeled by these equations

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) \quad (1)$$
$$z(x,0) = z^0(x) \quad (2)$$
$$\frac{\partial z}{\partial x}(0,t) = 0 \quad (3)$$
$$\frac{\partial z}{\partial x}(1,t) = \beta(u(t) - z(1,t)) \quad (4)$$

for some positive constant $\beta$ where the control is $u(t)$, the temperature applied to the end of the rod.

First, we study the open loop system where $u(t) = 0$ for all $t \geq 0$. We consider the closed linear operator

$$Ah(x) = \frac{d^2 h}{dx^2}(x)$$

where its domain is space of all $h \in L_2[0,1]$ such that $h$ and $\frac{dh}{dx}$ are absolutely continuous and $\frac{dh}{dx}(0) = 0, \frac{dh}{dx}(1) = -\beta h(1)$. Because of the Neumann boundary condition at $x = 0$, the eigenvectors are of form

$$\phi(x) = c \cos \nu x.$$

The Robin boundary condition at $x = 1$ implies that $\nu$ is a root of the equation

$$\nu \sin \nu = \beta \cos \nu \quad (5)$$

or equivalently

$$\nu = \beta \cot \nu. \quad (6)$$

There is one root $\nu_1$ of this equation in each open interval $(i\pi, (i + 1/2)\pi)$ for $i = 0, 1, \ldots$. The $i$th root $\nu_i \to i\pi$ as $i \to \infty$. As $\beta \to 0$ the $i$th root $\nu_i \to i\pi$. If $\beta = 0, \nu_i = i\pi$ and we have an uncontrolled rod with no heat flux at either end. As $\beta \to \infty$ the $i$th root $\nu_i \to (i + 1/2)\pi$. The corresponding eigenvalues are $\lambda_i = -\nu_i^2$. As $\beta \to 0$ the eigenvalues move to right and as
\( \beta \to \infty \) the eigenvalues move to the left. If \( \beta = 1 \), the first five roots are \( \nu_0 = 0.8603, \nu_1 = 3.4256, \nu_2 = 6.4373, \nu_3 = 9.5293, \) and \( \nu_4 = 12.6453. \) So the five least stable eigenvalues are \( \lambda_0 = -\nu_0^2 = -0.7402, \lambda_1 = -\nu_1^2 = -11.7349, \lambda_2 = -\nu_2^2 = -41.4388, \lambda_3 = -\nu_3^2 = -90.8082, \) and \( \lambda_4 = -\nu_4^2 = -159.9033. \)

Because the Laplacian is self adjoint with respect to these boundary conditions, the eigenfunctions are orthogonal. We normalize them

\[
\phi_i(x) = c_i \cos \nu_i x
\]

where

\[
c_i = \text{sign}(\cos \nu_i) \sqrt{\frac{4

\nu_i}{2\nu_i + \sin 2\nu_i}}
\]

to get an orthonormal family satisfying \( \phi_i(1) > 0 \). Because \( \nu_i \in (i\pi, (i + 1/2)\pi) \) it follows that \( \text{sign} c_i = \text{sign}(\cos \nu_i) = (-1)^{i+1} \) and \( 2\nu_i \in (2i\pi, (2i+1)\pi) \) so \( \sin 2\nu_i \) is positive. This implies that \( |c_i| \leq \sqrt{2} \) so

\[
\phi_i(1) = c_i \cos \nu_i \leq \sqrt{2} |\cos \nu_i| \leq \sqrt{2}.
\]

The open loop system is asymptotically stable because all its eigenvalues are in the open left half-plane. Let \( \mathbb{Z}^o \subset L_2[0,1] \) be the closure of the span of \( \phi_i(x) \) for \( i = 0, 1, 2, \ldots \). The superscript \( ^o \) denotes that this is the closure of the domain of the open loop operator \( A \). This operator is densely defined on \( \mathbb{Z}^o \) and generates a strongly continuous semigroup. If \( z^0(x) \in \mathbb{Z}^o \) then

\[
z(x,t) = T^o(t)z^0(x)
\]

\[
= \sum_{i=0}^{\infty} e^{\lambda_i t} \phi_i(x) \int_0^1 \phi_i(x_1) z^0(x_1) \, dx_1.
\]

Again the superscript on \( T^o \) denotes that this is the open loop semigroup.

We seek a feedback control law of the form

\[
u(t) = \int_0^1 K^{[1]}(x)z(x,t) \, dx
\]

to speed up the stabilization. To find \( K^{[1]}(x) \) we solve an LQR problem of minimizing

\[
\int_0^\infty \int_S Q(x_1,x_2)z(x_1,t)z(x_2,t) \, dA \, dt + \int_0^\infty Ru^2(t)
\]

(10)

where \( S \) is the unit square \([0,1] \times [0,1] \), \( dA = dx_1 \, dx_2 \). We require that \( R > 0 \) and \( Q(x_1,x_2) \) is a symmetric function, \( Q(x_1,x_2) = Q(x_2,x_1) \), satisfying

\[
0 \leq \int_S Q(x_1,x_2)\theta(x_1)\theta(x_2) \, dA
\]

for any function \( \theta(x) \). We allow \( Q(x_1,x_2) \) to be a generalized function. For example if \( Q(x_1,x_2) = Q(x_1) \delta(x_1 - x_2) \), where \( Q(x_1) \geq 0 \) for each \( x_1 \) and \( \delta(x_1 - x_2) \) is the Dirac \( \delta \) function then

\[
\int_S Q(x_1,x_2)z(x_1,t)z(x_2,t) \, dA = \int_0^1 \int_S Q(x_1)z^2(x_1,t) \, dx_1.
\]

Let \( P^{[2]}(x_1,x_2) \) be any symmetric function of \( (x_1,x_2) \). Assume that the control trajectory \( u(t) \) results in \( z(x,t) \rightarrow 0 \) as \( t \rightarrow \infty \). We know that such control trajectories exist because the open loop rod \( u(t) = 0 \) is asymptotically stable. By the fundamental theorem of calculus

\[
0 = \int_S \int_S P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dA
\]

\[
+ \int_0^\infty \int_S \frac{d}{dt} \left( \int_S P^{[2]}(x_1,x_2)z(x_1,t)z(x_2,t) \right) \, dA \, dt
\]

\[
= \int_S P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dA
\]

\[
+ \int_0^\infty \int_S P^{[2]}(x_1,x_2)\frac{\partial^2 z}{\partial x_1^2}(x_1,t)z(x_2,t) \, dA \, dt + \int_0^\infty \int_S P^{[2]}(x_1,x_2)\frac{\partial^2 z}{\partial x_2^2}(x_2,t) \, dA \, dt.
\]

(11)

We assume that \( P^{[2]}(x_1,x_2) \) satisfies Neumann boundary conditions at \( x_i = 0 \)

\[
\frac{\partial P^{[2]}}{\partial x_1}(0,x_2) = 0
\]

(12)

\[
\frac{\partial P^{[2]}}{\partial x_2}(x_1,0) = 0
\]

(13)

and Robin boundary conditions at \( x_i = 1 \)

\[
\frac{\partial P^{[2]}}{\partial x_1}(1,x_2) = -\beta P^{[2]}(1,x_2)
\]

(14)

\[
\frac{\partial P^{[2]}}{\partial x_2}(x_1,1) = -\beta P^{[2]}(x_1,1).
\]

(15)

Because of the symmetry of \( P^{[2]}(x_1,x_2) \), (12) is equivalent to (13) and (14) is equivalent to (15).

When we integrate (11) by parts twice, we get the equation

\[
0 = \int_S \int_S P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dA
\]

\[
+ \int_0^\infty \int_S \nabla^2 P^{[2]}(x_1,x_2)z(x_1,t)z(x_2,t) \, dA \, dt
\]

\[
+ \beta \int_0^\infty \int_0^1 P^{[2]}(1,x_2)u(t)z(x_2,t) \, dx_2 \, dt
\]

\[
+ \beta \int_0^\infty \int_0^1 P^{[2]}(x_1,1)z(x_1,t)u(t) \, dx_1 \, dt
\]

(16)

where \( \nabla^2 \) is the 2-D Laplacian.

We add (16) to the criterion (10) to be minimized to get the equivalent criterion

\[
\int_S \int_S P^{[2]}(x_1,x_2)z^0(x_1)z^0(x_2) \, dA
\]

\[
+ \int_0^\infty \int_S \nabla^2 P^{[2]}(x_1,x_2)z(x_1,t)z(x_2,t) \, dA \, dt
\]

\[
\times z(x_1,t)z(x_2,t) \, dA \, dt
\]
\[
+ \beta \int_0^\infty \int_0^1 P^{[2]}(1, x_2)u(t)z(x_2, t) \, dx_2 \, dt \\
+ \beta \int_0^\infty \int_0^1 P^{[2]}(x_1, 1)z(x_1, t)u(t) \, dx_1 \, dt \\
+ \int_0^\infty Ru^2(t) \, dt.
\]

Suppose this criterion can be written as a double integral with respect to state and a time integral of a perfect square involving the control. If the latter can be made zero by the proper choice of the control then the optimal cost is the double integral with respect to state. So we would like to chose \( K^{[1]}(x) \) so that the time integrand in (17) is a perfect square. In other words, we want (17) to be of the form

\[
0 = \int_S z^0(x_1)P^{[2]}(x_1, x_2)z^0(x_2) \, dA \\
+ \int_0^\infty \int_S R \left( u(t) - K^{[1]}(x_1)z(x_1, t) \right) \\
\times \left( u(t) - K^{[1]}(x_2)z(x_2, t) \right) \, dA \, dt.
\]

Clearly the terms quadratic in \( u(t) \) match so we equate the terms involving \( u(t) \) and \( z(x_1, t) \)

\[
\beta \int_0^\infty \int_0^1 P^{[2]}(x_1, 1)z(x_1, t)u(t) \, dx_1 \, dt \\
- \int_0^\infty \int_0^1 K^{[1]}(x_1)Rz(x_1, t)u(t) \, dx_1 \, dt.
\]

They will match if

\[
K^{[1]}(x_1) = -R^{-1}\beta P^{[2]}(x_1, 1)
\]

By the symmetry of \( P^{[2]}(x_1, x_2) \), \( K^{[1]}(x_2) = -R^{-1}\beta P^{[2]}(1, x_2) \).

Finally, we equate the terms involving \( z(x_1, t) \) and \( z(x_2, t) \)

\[
\int_S \left( \nabla^2 P^{[2]}(x_1, x_2) + Q(x_1, x_2) \right) \quad z(x_1, t)z(x_2, t) \, dA \, dt \\
\times \cdot R \int_0^\infty \int_S K^{[1]}(x_1)K^{[1]}(x_2)z(x_1, t)z(x_2, t) \, dA \, dt.
\]

This yields what we call a Riccati PDE

\[
\nabla^2 P^{[2]}(x_1, x_2) + Q(x_1, x_2) = RK^{[1]}(x_1)K^{[1]}(x_2) \\
= \beta^2 R^{-1}P^{[2]}(x_1, 1)P^{[2]}(1, x_2).
\]

This is to be interpreted in the weak sense, if \( \theta(x) \) is \( C^2 \) on \( 0 \leq x \leq 1 \) then

\[
\iint_S \left( \nabla^2 P^{[2]}(x_1, x_2) + Q(x_1, x_2) \right) \theta(x_1)\theta(x_2) \, dA \\
= \beta^2 \iint_S R^{-1}P^{[2]}(x_1, 1)P^{[2]}(1, x_2)\theta(x_1)\theta(x_2) \, dA.
\]

The boundary conditions (12)-(15) are also to be interpreted in the weak sense

\[
0 = \int_0^1 \frac{\partial P^{[2]}(0, x_2)}{\partial x_1} \theta(x_2) \, dx_2 \\
0 = \int_0^1 \frac{\partial P^{[2]}(1, x_1)}{\partial x_2} \theta(x_1) \, dx_1 \\
0 = \int_0^1 \left( \frac{\partial P^{[2]}(1, x_2)}{\partial x_1} + \beta P^{[2]}(1, x_2) \right) \theta(x_2) \, dx_2 \\
0 = \int_0^1 \left( \frac{\partial P^{[2]}(x_1, 1)}{\partial x_2} + \beta P^{[2]}(x_1, 1) \right) \theta(x_1) \, dx_1.
\]

If we can solve the Riccati PDE subject to these boundary conditions then clearly the optimal cost starting from \( z^0(x) \) is a quadratic functional of the initial state

\[
\iint S P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \, dA
\]

and the optimal feedback is a linear functional of the current state

\[
u(t) = \int_0^1 K^{[1]}(x)z(x, t) \, dx
= -R^{-1}\beta \int_0^1 P^{[2]}(x, 1)z(x, t) \, dx.
\]

We assume that the solution to the Riccati PDE has an expansion in the open loop eigenfunctions (7) of the form

\[
P^{[2]}(x_1, x_2) = \sum_{i_1, i_2=0}^\infty \Pi^{[2]}_{i_1, i_2} \phi_{i_1}(x_1)\phi_{i_2}(x_2).
\]

Clearly, any such expansion satisfies the boundary conditions (12)-(15).

Because we seek a symmetric weak solution without loss of generality \( \Pi^{[2]}_{i_1, i_2} = \Pi^{[2]}_{i_2, i_1} \). We also assume that \( Q(x_1, x_2) \) has a similar expansion

\[
Q(x_1, x_2) = \sum_{i_1, i_2=0}^\infty Q_{i_1, i_2} \phi_{i_1}(x_1)\phi_{i_2}(x_2).
\]

We plug these into (20) and get an algebraic Riccati equation for the infinite dimensional matrix \( \Pi^{[2]}_{i_1, i_2} \)

\[
(\lambda_{i_1} + \lambda_{i_2})\Pi^{[2]}_{i_1, i_2} + Q_{i_1, i_2}
= R^{-1}\beta^2 \sum_{j_1, j_2=0}^\infty \Pi^{[2]}_{j_1, j_2} \Pi^{[2]}_{i_2, j_1} \phi_{i_1}(1)\phi_{i_2}(1).
\]

Assume \( R = 1 \) and

\[
Q(x_1, x_2) = \delta(x_1 - x_2)
Q_{i_1, i_2} = \delta_{i_1, i_2}
\]

where the first is the Dirac \( \delta \) and the second is the Kronecker \( \delta \). Then we guess that \( \Pi^{[2]}_{i_1, i_2} \) is also diagonal

\[
\Pi^{[2]}_{i_1, i_2} = \delta_{i_1, i_2} \Pi^{[2]}_{i_1, i_1}.
\]
and we get a sequence of quadratic equations for $\Pi_{i,i}$

$$0 = \beta^2 \phi_i^2(1) \left( \Pi_{i,i}^{[2]} \right)^2 - 2\lambda_i \Pi_{i,i}^{[2]} - 1$$

whose roots are

$$\Pi_{i,i}^{[2]} = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 2\beta^2 \phi_i^2(1)}}{\beta^2 \phi_i^2(1)}.$$  

Clearly, we wish to take the positive sign so

$$\Pi_{i,i}^{[2]} = \delta_{i,i} \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 2\beta^2 \phi_i^2(1)}}{\beta^2 \phi_i^2(1)}.$$  

This infinite dimensional diagonal matrix is positive definite because

$$0 < \lambda_i + \sqrt{\lambda_i^2 + 2\beta^2 \phi_i^2(1)}$$

but it is not coercive as $\Pi_{i,i}^{[2]} \to 0$ as $i \to \infty$.

Recall a quadratic form is coercive if it is bounded below by a positive multiple of the squared norm. In finite dimensional LQR, a positive definite optimal cost is a Lyapunov function for the closed loop linear dynamics. In infinite dimensional LQR the optimal cost is usually not coercive so the standard Lyapunov argument fails.

**Lemma:**

$$\frac{1}{2\sqrt{\lambda_i^2 + 2\beta^2 \phi_i^2(1)}} \leq \Pi_{i,i}^{[2]} \leq \frac{1}{2|\lambda_i|}.$$  

**Proof:** For each $i \geq 0$ by the mean value theorem there exist an $s$ between $\lambda_i^2$ and $\lambda_i^2 + 2\beta^2 \phi_i^2(1)$ such that

$$\sqrt{\lambda_i^2 + 2\beta^2 \phi_i^2(1)} - \sqrt{\lambda_i^2} = \frac{1}{2\sqrt{\beta^2 \phi_i^2(1)}}.$$  

The function $\frac{1}{\sqrt{s}}$ is monotonic decreasing between $\lambda_i^2$ and $\lambda_i^2 + 2\beta^2 \phi_i^2(1)$. Its maximum occurs at $\lambda_i^2$ and its minimum occurs at $\lambda_i^2 + 2\beta^2 \phi_i^2(1)$ so the result follows.

QED

**Corollary:** The series $\sum_{i=1}^{\infty} \Pi_{i,i}^{[2]} \phi_i(x_1) \phi_i(x_2)$ converges uniformly to a continuous function $P^{[2]}(x_1, x_2)$.

**Proof:** The eigenfunctions $\phi_i(x)$ are uniformly bounded on $[0,1]$. The proof follows by comparison with the series

$$\sum_{i=0}^{\infty} \Pi_{i,i}^{[2]} \leq \sum_{i=0}^{\infty} \frac{1}{2|\lambda_i|}$$

because $|\lambda_i|$ is going to infinity like $i^2$.

If $i > 0$ then $|\lambda_i| > \pi^2 i^2$ so $\Pi_{i,i}^{[2]} < \frac{1}{2\pi^2 i^2}$. Since $0 \leq \phi_i(1) \leq \sqrt{2}$ it follows that $0 \leq \Pi_{i,i}^{[2]} \phi_i(1) \leq \frac{\sqrt{2}}{2\pi^2 i^2}$

QED

The optimal feedback gain is

$$K^{[1]}(x) = -\beta \sum_{i=0}^{\infty} \Pi_{i,i}^{[2]} \phi_i(1) \phi_i(x).$$  

If $\beta = 1$ then the first three diagonal entries are $\Pi_{0,0}^{[2]} = 0.5608$, $\Pi_{1,1}^{[2]} = 0.0425$, and $\Pi_{2,2}^{[2]} = 0.0121$ and

$$K^{[1]}(x) = 0.4645 \cos(\nu_0 x) - 0.0756 \cos(\nu_1 x) + 0.0233 \cos(\nu_2 x) + \ldots$$

**III. CLOSED LOOP EIGENVALUES AND EIGENVECTORS**

The boundary feedback does not appear in the closed loop dynamics, it is still

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t)$$

but the boundary conditions are changed by the feedback. The boundary condition at $x = 0$ is still the Neumann boundary condition (3) but the Robin boundary condition (4) at $x = 1$ is replaced by

$$\frac{\partial z}{\partial x}(1,t) = \beta \left( \int_0^1 K^{[1]}(x_1) z(x_1,t) \, dx_1 - z(1,t) \right)$$

$$= -\beta \left( \beta \sum_{i=0}^{\infty} \Pi_{i,i}^{[2]} \phi_i(1) \int_0^1 \phi_i(x) z(x,t) \, dx + z(1,t) \right).$$  

Note that this nonstandard boundary condition (28) is linear in $z(x,t)$.

Because of the Neumann BC at $x = 0$ we know that the unit normal closed loop eigenvectors are of the form

$$\psi(x) = c(\rho) \cos \rho x$$  

where the normalizing constant is given by

$$c(\rho) = \text{sign}(\cos \rho) \sqrt{\frac{4\rho}{2\rho + \sin 2\rho}}.$$  

The $\rho$ are chosen so that $g(\rho) = 0$ where

$$g(\rho) = \rho \sin \rho$$

$$= -\beta \sum_{i=0}^{\infty} \Pi_{i,i}^{[2]} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho x \, dx + \cos \rho.$$  

The 1-D Laplacian is not self adjoint under these boundary conditions (3), (28) so there is no reason to expect that the closed loop eigenfunctions are orthogonal.

**Lemma:** There is at least one solution $\rho_i$ to (31) in each interval $(\nu_i, \nu_{i+1})$.

**Proof:** Because $\nu_i$ satisfies (6) and the $\phi_i(x)$ are orthonormal

$$g(\nu_i) = -\beta \Pi_{i,i}^{[2]} \phi_i(1) \int_0^1 \phi_i(x_1) \cos \nu_i x \, dx_1$$

$$= -\beta \frac{\beta^2 \Pi_{i,i}^{[2]} \phi_i(1)}{c_i}.$$  

Now, $\Pi_{i,i}^{[2]} \phi_i(1) > 0$ and the sign $c_i = (-1)^i$ so the result follows from the intermediate value theorem.

QED

**Lemma:** Suppose $\rho_j$, $j = 0, 1, 2, \ldots$ is a sequence of solutions to (31) such that $\rho_j \in (\nu_j, \nu_{j+1})$. Then $\nu_j - \rho_j \to 0$ as $j \to \infty$.

**Proof:** We will prove this by showing that

$$\sum_{i=0}^{\infty} \Pi_{i,i}^{[2]} \phi_i(1) \int_0^1 \phi_i(x_1) \cos \rho_j x \, dx \to 0$$

as $j \to \infty$ so $g(\rho)$ given by (6) converges to $g_0(\rho) = \rho \sin \rho - \beta \cos \rho$ whose roots are the $\nu_i$. 


Let $k$ be the greatest integer not exceeding $j/2$. We break this sum (32) into three sums

\[
\begin{align*}
&\sum_{i=0}^{j-k-1} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx \\
&+ \sum_{i=j-k}^{j+k} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx \\
&+ \sum_{i=j+k+1}^{\infty} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx.
\end{align*}
\]

The absolute value of the normalizing constants in (8) are less than or equal to $\sqrt{2}$ so

\[
\left| \int_0^1 \phi_i(x_1) \cos \rho_j x \, dx \right| \leq \frac{\sqrt{2}}{2} \left| \frac{\sin(\nu_i + \rho_j) - \sin(\nu_i - \rho_j)}{\nu_i - \rho_j} \right|.
\]

The absolute value of the first sum is bounded above

\[
\begin{align*}
&\sum_{i=0}^{j-k-1} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx \\
&\leq \frac{\sqrt{2}}{2} \sum_{i=0}^{j-k-1} \frac{1}{|\lambda_i|} \left( \frac{1}{\rho_j} + \frac{1}{\rho_j - \nu_i} \right).
\end{align*}
\]

Now, $\nu_i \in (i\pi, (i+1)\pi)$ and $\rho_j \in (j\pi, (j+1)\pi)$ so $\rho_j - \nu_i > k\pi$ so the first sum is bounded above by

\[
\begin{align*}
&\left( \frac{1}{|\lambda_0|} + \sum_{i=1}^{j-k-1} \frac{1}{\pi^2 i^2} \right) \left( \frac{1}{\rho_j} + \frac{1}{\rho_j - \nu_i} \right) \\
&\leq \frac{\sqrt{2}}{2} \frac{3}{2k\pi} \left( \frac{1}{|\lambda_0|} + \frac{1}{\pi^2} \left( 1 + \int_1^{j-k-1} \frac{1}{s^2} \, ds \right) \right) \\
&\leq \frac{\sqrt{2}}{2} \frac{3}{2k\pi} \left( \frac{1}{|\lambda_0|} + \frac{1}{\pi^2} \left( 2 - \frac{1}{j-k-1} \right) \right).
\end{align*}
\]

So the absolute value of this sum goes to zero as $j, k \to \infty$.

The absolute value of the third sum is bounded above

\[
\begin{align*}
&\sum_{i=j+k+1}^{\infty} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx \\
&\leq \frac{\sqrt{2}}{2} \sum_{i=j+k+1}^{\infty} \frac{1}{|\lambda_i|} \left( \frac{1}{\rho_j} + \frac{1}{\nu_i - \rho_j} \right).
\end{align*}
\]

Again $\nu_i \in (i\pi, (i+1)\pi)$ and $\rho_j \in (j\pi, (j+1)\pi)$ so $\nu_i - \rho_j > k\pi$. So the third sum is bounded above by

\[
\frac{\sqrt{2}}{2} \left( \frac{1}{\rho_j} + \frac{1}{k\pi} \right) \sum_{i=j+k+1}^{\infty} \frac{1}{|\lambda_i|}.
\]

and so it goes to zero as $j, k \to \infty$.

Now $\phi_i(x)$ is of norm one and $\cos \rho x$ is of norm $\sqrt{1/2 + \sin \rho/2\rho}$ so by the Cauchy Schwarz inequality

\[
\int_0^1 \phi_i(x) \cos \rho x \, dx \leq \sqrt{1/2 + \sin \rho/2\rho} \leq 1.
\]

The absolute value of the second sum is less than or equal to

\[
\begin{align*}
&\sum_{i=j-k}^{j+k} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \cos \rho_j x \, dx \\
&\leq \frac{\sqrt{2}}{2} \frac{3}{2k\pi} \int_{j+k}^{\infty} \frac{1}{\pi^2 x^2} \, dx \\
&\leq \frac{2}{\pi^2 j^2 - j^2}
\end{align*}
\]

which goes to zero as $j, k \to \infty$.

QED

For the moment we assume $\beta = 1$. If $i > 5$ it follows that $\Pi^{[2]}_{i,i} \leq \frac{1}{2\pi |\mu_i|} \leq \frac{1}{72\pi^2}$. So to find the first few $\mu_i$ we truncate (31) to

\[
\rho \sin \rho = \beta \left( \beta \sum_{i=0}^{5} \Pi^{[2]}_{i,i} \phi_i(1) \int_0^1 \phi_i(x) \psi(x) \, dx + \cos \rho \right)
\]

and solve (33) by Newton’s method starting at $\rho = 1.1\nu_i$ for $i = 0, 1, 2, 3, 4$. The result is $\rho_0 \approx 1.0202$, $\rho_1 \approx 3.4365$, $\rho_2 \approx 6.4391$, $\rho_3 \approx 9.5299$, and $\rho_4 \approx 12.6455$. So the first five closed loop eigenvalues are $\mu_0 \approx -\rho_0^2 = -1.0409$, $\mu_1 \approx -\rho_1^2 = -11.9094$, $\mu_2 \approx -\rho_2^2 = -41.5352$, $\mu_3 \approx -\rho_3^2 = -90.8190$, and $\mu_4 \approx -\rho_4^2 = -159.9095$. Recall the first five open loop eigenvalues are $\lambda_0 = -0.7401$, $\lambda_1 = -11.7347$, $\lambda_2 = -41.4620$, $\lambda_3 = -90.8082$, and $\lambda_4 = -159.9033$. Notice how close $\mu_i$ and $\lambda_i$ are if $i > 0$. The boundary feedback has a significant effect on the least stable open loop eigenvalue but less so on the rest of the open loop eigenvalues because they are already so stable.

We return to a general $\beta$. Let $\mathbb{Z}^e$ denote the closure of the linear subspace of $L_2[0,1]$ spanned by $\psi_0(x), \psi_1(x), \psi_2(x), \ldots$. Since $\phi_i(x) - \psi_i(x) \to 0$ as $i \to \infty$ and the $\phi_i(x)$ are orthonormal it follows that $\psi_0(x), \psi_1(x), \psi_2(x), \ldots$ is a Riesz basis for $\mathbb{Z}^e$. Therefore there exists a biorthogonal sequence $\theta_j(x)$ in $\mathbb{Z}^e$ such that

\[
\int_0^1 \psi_i(x) \theta_j(x) \, dx = \delta_{i,j}.
\]
The \( C^0 \) semigroup on \( \mathbb{Z}^c \) generated by the closed loop dynamics is given
\[
T^c(t)z^0(x) = \sum_{i=0}^{\infty} e^{\mu i t} \psi_i(x) \int_0^1 \theta_i(x) z^0(x) \, dx
\]
for \( z^0 \in \mathbb{Z}^c \).

### IV. Boundary Control of a Nonlinear Reaction Diffusion Equation

To the above system we add a destabilizing nonlinear term to obtain the boundary controlled reaction diffusion system
\[
\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \alpha z^2(x,t) \tag{34}
\]
\[
z(x,0) = z^0(x) \tag{35}
\]
\[
\frac{\partial z}{\partial x}(0,t) = 0 \tag{36}
\]
\[
\frac{\partial z}{\partial x}(1,t) = \beta(u(t) - z(1,t)) \tag{37}
\]
for some positive constants \( \alpha \) and \( \beta \). Vazquez and Krstic used backstepping to stabilize a similar system with \( \alpha = 1 \). They assumed a different boundary condition at \( x = 1 \), namely
\[
\frac{\partial z}{\partial x}(1,t) = u(t).
\]

To find a feedback to stabilize this system we consider the nonlinear quadratic optimal control of minimizing (10) subject to (34)–(37).

It is well known \([11]\) that this system cannot be globally stabilized but we are only interested in local stabilization around \( z = 0 \). This is a mathematical model of a physical system and the model is not globally valid, there is an absolute zero temperature that the rod cannot go below and at a sufficiently high temperature the rod will melt. So global stabilization is of mathematical but not physical interest.

Let \( P^2(x_1, x_2) \) be the solution of the Riccati PDE (20) and \( K^{[1]}(x) \) be the gain of the optimal linear feedback (26). Let \( P^3(x_1, x_2, x_3) \) be a symmetric function of three variables. By symmetry we mean that the value of the function is invariant under any permutation of the three variables. We further assume that \( P^3(x_1, x_2, x_3) \) weakly satisfies Neumann boundary conditions at \( x_1 = 0 \)
\[
\frac{\partial P^3}{\partial x_1}(0, x_2, x_3) = 0 \tag{38}
\]
and Robin boundary conditions at \( x_1 = 1 \)
\[
\frac{\partial P^3}{\partial x_1}(1, x_2, x_3) = -\beta P^3(1, x_2, x_3). \tag{39}
\]
By symmetry similar boundary conditions hold at \( x_2 = 0 \) and \( x_3 = 0 \).

Assume also that the optimal feedback takes the form
\[
u(t) = \int_0^1 K^{[1]}(x_1)z(x_1,t) \, dx_1 \tag{40}
\]
where the omitted terms are of degrees three and higher in \( z(x,t) \).

Again by the fundamental theorem of calculus if the control trajectory \( u(t) \) takes \( z(x,t) \to 0 \) as \( t \to \infty \) then
\[
0 = \int_S P^2(x_1, x_2)z^0(x_1)z^0(x_2) \, dA + \int_S P^3(x_1, x_2, x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dV
\]
\[
+ \int_0^\infty \int_S \frac{d}{dt} \left( P^2(x_1, x_2)z(x_1,t)z(x_2,t) \right) \, dA \, dt
\]
\[
+ \int_0^\infty \int_S \frac{d}{dt} \left( P^3(x_1, x_2, x_3)z(x_1,t)z(x_2,t)z(x_3,t) \right) \, dV \, dt
\]
\[
\times dV \, dt + O(z(x,t))^4 \tag{42}
\]
where \( C \) denotes the unit cube \([0,1] \times [0,1] \times [0,1]\) and \( dV \) is the volume element \( dV = dx_1 \, dx_2 \, dx_3 \).

Because \( P^2(x_1, x_2) \) is the solution of the Riccati PDE, the terms quadratic in \( z \) in the time integral drop out. But we pick up cubic terms from the boundary when we integrate (42) by parts twice. We use the symmetry of \( P^2(x_1, x_2) \) and \( P^3(x_1, x_2, x_3) \) to simplify them and we obtain
\[
3\beta \int_0^\infty \int_C P^3(1, x_2, x_3)K^{[1]}(x_1)z(x_1,t)z(x_2,t)z(x_3,t) \, dV \, dt
\]
\[
+ 2\beta \int_0^\infty \int_C P^2(1, x_2)K^{[2]}(x_1, x_3)z(x_1,t)z(x_2,t)z(x_3,t) \, dV \, dt.
\]
So we obtain again using symmetry
\[
0 = \int_S P^2(x_1, x_2)z^0(x_1)z^0(x_2) \, dA + \int_0^\infty \int_S P^3(x_1, x_2, x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dV
\]
\[
+ 2\alpha \int_0^\infty \int_S P^2(x_1, x_2)z^0(x_1)z^0(x_2)z^0(x_3) \, dV \, dt
\]
\[
+ \int_0^\infty \int_S \frac{d}{dt} \left( P^2(x_1, x_2)z(x_1,t)z(x_2,t) \right) \, dA \, dt
\]
\[
+ 3\beta \int_0^\infty \int_C P^3(1, x_2, x_3)K^{[1]}(x_1) \times z(x_1,t)z(x_2,t)z(x_3,t) \, dV \, dt
\]
\[
+ 3\beta \int_0^\infty \int_C P^3(1, x_2, x_3)K^{[1]}(x_1) \times z(x_1,t)z(x_2,t)z(x_3,t) \, dV \, dt
\]
\[
+ 2\beta \int_0^\infty \int_C P^3(1, x_2, x_3)K^{[2]}(x_1, x_3) \times z(x_1,t)z(x_2,t)z(x_3,t) \, dV \, dt + O(z(x,t))^4. \tag{43}
\]
If we integrate (43) by parts twice we get the equation
\[
0 = \int_S P^2(x_1, x_2)z^0(x_1)z^0(x_2) \, dA + \int_0^\infty \int_S P^3(x_1, x_2, x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dV
\]
where $\nabla^2$ is now the 3-D Laplacian. Equation (44) is not symmetric in $x_1, x_2, x_3$ but we are looking for a symmetric weak solution. If $P^{[3]}(x_1, x_2, x_3)$ is a weak solution that is not symmetric we can get a symmetric weak solution by averaging over all permutations of $x_1, x_2, x_3$.

We have cancelled the quadratic terms in the criterion by our choice $P^{[2]}(x_1, x_2)$ and $K^{[1]}(x)$ but the quadratic term in the feedback generates a cubic term in the criterion

$$2R \int_0^\infty \int_0^1 \int_0^1 [K^{[1]}(x_1)K^{[2]}(x_2, x_3)] \times z(x_1, t)z(x_2, t)z(x_3, t) \, dV \, dt.$$

Because of (26) this equals

$$-2\beta \int_0^\infty \int_0^1 \int_0^1 C^{[2]}(x_1, x_2, x_3) \times z(x_1, t)z(x_2, t)z(x_3, t) \, dV \, dt$$

and so this cancels out the last term in (44).

So the new criterion to be minimized is $0 = \int_S P^{[2]}(x_1, x_2,z^0(x_1))z^0(x_2) \, dA$

$$+ \int_0^\infty \int_0^1 \int_0^1 \nabla^2 P^{[3]}(x_1, x_2, x_3) \times z^0(x_1)z^0(x_2)z^0(x_3) \, dV \, dt$$

$$+ 3\beta \int_0^\infty \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, x_3)K^{[1]}(x_1) \times z(x_1, t)z(x_2, t)z(x_3, t) \, dV \, dt$$

$$+ 2\alpha \int_0^\infty \int_0^1 \int_0^1 P^{[2]}(x_1, x_2)z^2(x_1, t)z(x_2, t) \, dA \, dt.$$

(45)

We assume that $P^{[3]}(x_1, x_2, x_3)$ is a symmetric weak solution to the symmetric linear elliptic PDE

$$0 = \nabla^2 P^{[3]}(x_1, x_2, x_3)$$

$$+ \beta (P^{[3]}(1, x_2, x_3)K^{[1]}(x_1) + P^{[3]}(x_1, 1, x_3)K^{[1]}(x_2))$$

$$+ \alpha (P^{[2]}(x_1, x_2)\delta(x_1-x_3) + P^{[2]}(x_1, x_2)\delta(x_1-x_3))$$

(46)

subject to the weak boundary conditions (38), (39). Then the optimal cost is

$$\pi(z^0(\cdot)) = \int_S P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \, dA$$

$$+ \int_0^\infty \int_0^1 \int_0^1 \nabla P^{[3]}(x_1, x_2, x_3) \times z^0(x_1)z^0(x_2)z^0(x_3) \, dV$$

$$+ O(z^0(x))^4.$$

(47)

We use a standard argument to find the quadratic part of the optimal feedback (40). By the principle of optimality the optimal cost starting at $z^0$ at $t = 0$ is for small $\tau \geq 0$

$$\pi(z^0(\cdot)) = \min_{u(\cdot)} \left\{ \pi(z) + \int_0^\tau R_u^2(t) \, dt + \int_0^\tau \int_S Q(x_1, x_2)z(x_1, t)z(x_2, t) \, dA \, dt \right\}.$$  

(48)

Since $\tau \geq 0$ is small

$$\int_0^\tau R_u^2(t) \, dt = \tau R_u^2(0) + O(\tau)^2$$

and

$$\int_0^\tau \int_S Q(x_1, x_2)z(x_1, t)z(x_2, t) \, dA \, dt = \tau \int_S Q(x_1, x_2)z^0(x_1)z^0(x_2) \, dA + O(\tau)^2.$$  

By arguments similar to the above

$$\pi(z(\tau)) - \pi(z(0)) = \tau \frac{\partial \pi(z(\cdot))}{\partial t} \bigg|_{t=0} + O(\tau)^2$$

$$= \tau \left( \int_S \nabla P^{[2]}(x_1, x_2)z^0(x_1)z^0(x_2) \, dA$$

$$+ 2\alpha \int_0^1 \int_0^1 P^{[2]}(1, x_2)u(0)z^0(x_2) \, dx_2$$

$$+ 3\beta \int_0^\infty \int_0^1 \int_0^1 P^{[3]}(x_1, x_2, x_3)z^0(x_1)z^0(x_2)z^0(x_3) \, dV$$

$$+ \alpha \left( P^{[2]}(x_1, x_2)\delta(x_1-x_3) + P^{[2]}(x_1, x_2)\delta(x_1-x_3) \right)$$

$$+ O(z^0(\cdot))^4 + O(\tau)^2.$$  

We plug these approximations into (48) and to find the minimum, we set the derivative with respect to $u(0)$ to zero to obtain

$$-2R_u(0) = 2\beta \int_0^1 P^{[2]}(1, x_2)z^0(x_2) \, dx_2$$

$$+ 3\beta \int_0^\infty \int_0^1 P^{[3]}(x_1, x_2, x_3)z^0(x_1)z^0(x_2) \, dA$$

$$+ O(z^0(x))^4 + O(\tau)^2.$$
So we conclude again that the linear part of the optimal feedback is given by (21). Using the symmetry of $P^{[3]}(x_1, x_2, x_3)$, the kernel of the quadratic part of the optimal feedback is

$$K^{[2]}(x_1, x_2) = -\frac{3}{2}R^{-1}P^{[3]}(x_1, x_2, 1). \quad (49)$$

Because $P^{[3]}(x_1, x_2, x_3)$ is symmetric in its three arguments, $K^{[2]}(x_1, x_2)$ is symmetric in its two arguments.

If it exists and if it is in $L_2(\mathcal{C})$, the symmetric function $P^{[3]}(x_1, x_2, x_3)$ defines a cubic polynomial on $L_2[0, 1]$. The value of the polynomial at $h(x) \in L_2[0, 1]$ is

$$\iint_{\mathcal{C}} P^{[3]}(x_1, x_2, x_3) h(x_1)h(x_2)h(x_3) \, dV.$$  

The first two terms of (46) define a linear operator that maps such a cubic polynomial into the cubic polynomial

$$\iint_{\mathcal{C}} \nabla^2 P^{[3]}(x_1, x_2, x_3) + \beta \left( P^{[3]}(1, x_2, x_3)K^{[1]}(x_1) + P^{[3]}(x_1, 1, x_3)K^{[1]}(x_2) + P^{[3]}(x_1, x_2, 1)K^{[1]}(x_3) \right) h(x_1)h(x_2)h(x_3) \, dV \quad (50)$$

where $\nabla^2$ is now the 3-D Laplacian.

Since the $\phi_i(x)$ are an orthonormal basis for $\mathbb{C}^n$ the monomials $\phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3)$ are orthonormal for the space of all functions $P^{[3]}(x_1, x_2, x_3)$ that are absolutely continuous, have absolutely continuous first partial derivatives and satisfy the Neumann boundary condition at $x_i = 0$ and the Robin boundary condition at $x_i = 1$. So we expand $P^{[3]}(x_1, x_2, x_3)$ in terms of these monomials

$$P^{[3]}(x_1, x_2, x_3) = \sum_{j_1, j_2, j_3=0}^\infty \Pi^{[3]}_{j_1, j_2, j_3} \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3) \quad (51)$$

where $\Pi^{[3]}_{j_1, j_2, j_3}$ is an infinite tensor that is symmetric in its three indices.

This operator (50) takes the cubic monomial

$$\iint_{\mathcal{C}} \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3) h(x_1)h(x_2)h(x_3) \, dV$$

into the cubic monomial

$$(\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3}) \iint_{\mathcal{C}} \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3) \times h(x_1)h(x_2)h(x_3) \, dV$$

$$+ \beta \iint_{\mathcal{C}} \left( \phi_{j_1}(1)\phi_{j_2}(x_2)\phi_{j_3}(x_3)K^{[1]}(x_1) + \phi_{j_1}(x_1)\phi_{j_2}(1)\phi_{j_3}(x_3)K^{[1]}(x_2) + \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(1)K^{[1]}(x_3) \right) \times h(x_1)h(x_2)h(x_3) \, dV.$$

The infinite matrix of this operator relative to the orthonormal basis $\phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3)$ is obtained by computing

$$\begin{align*}
(\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3}) & \iint_{\mathcal{C}} \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(x_3) \\
& \times \phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3) \, dV \\
+ & \beta \iint_{\mathcal{C}} \left( \phi_{j_1}(1)\phi_{j_2}(x_2)\phi_{j_3}(x_3)K^{[1]}(x_1) \\
& + \phi_{j_1}(x_1)\phi_{j_2}(1)\phi_{j_3}(x_3)K^{[1]}(x_2) \\
& + \phi_{j_1}(x_1)\phi_{j_2}(x_2)\phi_{j_3}(1)K^{[1]}(x_3) \right) \\
& \times \phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3) \, dV.
\end{align*}$$

So the infinite matrix of the operator (50) is

$$\Pi^{[3]}_{i_1, i_2, i_3; j_1, j_2, j_3} = \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} (\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3})$$

$$- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \phi_{j_1}(1)\phi_{j_2}(1)\phi_{j_3}(1)K^{[1]}(x_1)$$

$$+ \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \phi_{j_1}(1)\phi_{j_2}(1)\phi_{j_3}(1)K^{[1]}(x_2)$$

$$+ \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \phi_{j_1}(1)\phi_{j_2}(1)\phi_{j_3}(1)K^{[1]}(x_3)$$

and the operator applied to $P^{[3]}(x_1, x_2, x_3)$ given by (51) is

$$\sum_{i_1, i_2, i_3=0}^\infty \sum_{j_1, j_2, j_3=0}^\infty \Omega^{[3]}_{i_1, i_2, i_3; j_1, j_2, j_3} \phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3).$$

Theorem: If $\beta \leq 1$ the infinite matrix $\Omega^{[3]}$ is strictly column diagonally dominant.

Proof: We start by showing the first column where $j_1 = j_2 = j_3 = 0$ is diagonally dominant if $\beta \leq 1$. Using (5) the absolute value of diagonal entry when $i_1 = i_2 = i_3 = 0$ is

$$3\nu_0^2 + 3\beta^2\phi_0^2(1)\Pi^{[2]}_{0, i_1} > 3\nu_0^2 = 3\beta^2\cot^2 \nu_0.$$

The sum of the absolute values of the other entries in the first column where $j_1 = j_2 = j_3 = 0$ is

$$3\beta^2 \sum_{i_1=1}^\infty \phi_0(1)\phi_{i_1}(1)\Pi^{[2]}_{i_1} \leq 6\beta^2 \sum_{i_1=1}^\infty \Pi^{[2]}_{i_1}$$

$$\leq 3\beta^2 \sum_{i_1=1}^\infty \frac{1}{\pi^2 i^2} < 3\beta^2 \left( 1 + \int_1^\infty \frac{1}{s^2} \, ds \right)$$

$$= \frac{6\beta^2}{\pi^2}.$$

Now

$$\frac{6\beta^2}{\pi^2} \leq 3\beta^2 \cot^2 \nu_0.$$
We return to (46) and express its third term in the orthonormal basis of cubic monomials \(\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\). We multiply the third term by \(\phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\) and integrate over the unit cube to get

\[
\Gamma^{[3]}_{i_1,i_2,i_3} = \alpha e_{i_1} e_{i_2} e_{i_3} \frac{1}{4} \Pi^{[2]}_{i_2,i_3} \times \left( \frac{\sin(\nu_{i_1} + \nu_{i_2} + \nu_{i_3})}{\nu_{i_1} + \nu_{i_2} + \nu_{i_3}} + \frac{\sin(\nu_{i_1} + \nu_{i_2} - \nu_{i_3})}{\nu_{i_1} + \nu_{i_2} - \nu_{i_3}} \right)
\]

\[
+ \frac{\sin(\nu_{i_1} - \nu_{i_2} + \nu_{i_3})}{\nu_{i_1} - \nu_{i_2} + \nu_{i_3}} + \frac{\sin(\nu_{i_1} - \nu_{i_2} - \nu_{i_3})}{\nu_{i_1} - \nu_{i_2} - \nu_{i_3}} \right).
\]

Notice this expression is not symmetric in \(i_1, i_2, i_3\) so we symmetrize \(\Gamma^{[3]}_{i_1,i_2,i_3}\) by averaging over all permutations of \(i_1, i_2, i_3\). Then the third term in (46) in terms of the open loop eigenfunctions is

\[
\sum_{i_1,i_2,i_3} \Gamma^{[3]}_{i_1,i_2,i_3} \phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)
\]

and the linear equations that we need to solve for \(\Pi^{[3]}_{i_1,i_2,i_3}\) are

\[
\sum_{j_1,j_2,j_3=0}^{\infty} \Omega^{[3]}_{i_1,i_2,i_3;j_1,j_2,j_3} \Pi^{[3]}_{j_1,j_2,j_3} = -\Gamma^{[3]}_{i_1,i_2,i_3}
\]

for \(i_1, i_2, i_3\) ranging from 0 to \(\infty\).

From (49), the gain of the quadratic part of the optimal feedback is

\[
K^{[2]}(x_1, x_2) = \frac{3}{2} R^{-1} \sum_{i_1,i_2,i_3=0}^{\infty} \Pi^{[3]}_{i_1,i_2,i_3} \phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(1).
\]

Since \(\Pi^{[3]}_{i_1,i_2,i_3}\) is symmetric in \(i_1, i_2, i_3\), it follows that \(K^{[2]}(x_1, x_2)\) is symmetric in \(x_1, x_2\).

If a finite dimensional matrix is row or column strictly diagonally dominant it is invertible. We do not know whether this is true for an infinite matrix. But it does not matter, we are not going to try to solve (52), instead we shall choose a positive integer \(r\) and solve

\[
\sum_{0 \leq j_1+j_2+j_3 \leq r} \Omega^{[3]}_{i_1,i_2,i_3;j_1,j_2,j_3} \Pi^{[3]}_{j_1,j_2,j_3} = -\Gamma^{[3]}_{i_1,i_2,i_3}
\]

for \(i_1, i_2, i_3\) such that \(0 \leq i_1 + i_2 + i_3 \leq r\). This finite set of equations is always solvable if \(\beta \leq 1\) because the leading principle minors of \(\Omega^{[3]}_{i_1,i_2,i_3;j_1,j_2,j_3}\) are strictly diagonally column dominant and hence invertible. The errors that we make in solving (53) rather than (52) are very small because both the open and closed loop eigenvalues go to \(-\infty\) very quickly.

The higher degree terms in the optimal cost and optimal feedback are found in a similar fashion. In particular if we
expand the optimal cost to degree four in $z^0(\cdot)$
\[
\int_S P^2(x_1, x_2) z^0(x_1) z^0(x_2) \, dA \\
+ \int_C P^3(x_1, x_2, x_3) z^0(x_1) z^0(x_2) z^0(x_3) \, dV \\
+ \int_R P^4(x_1, x_2, x_3, x_4) \\
\times z^0(x_1) z^0(x_2) z^0(x_3) z^0(x_4) \, dH
\]
(54)
where $H$ is the hypercube $[0,1] \times [0,1] \times [0,1] \times [0,1]$ and $dH = dx_1 dx_2 dx_3 dx_4$ is the hypervolume element. Again we assume that $P^4(x_1, x_2, x_3, x_4)$ is a symmetric function of four variables which weakly satisfies the Neumann boundary condition at $x_{j_1} = 0$ and the Robin boundary condition at $x_{j_1} = 1$. Equation (54) will be the Taylor polynomial of the optimal cost to degree four if $P^2(x_1, x_2)$ and $P^3(x_1, x_2, x_3)$ are above and $P^4(x_1, x_2, x_3, x_4)$ is a weak symmetric solution of the linear elliptic PDE
\[
0 = \nabla^2 P^4(x_1, x_2, x_3, x_4) \\
+ \beta P^4(x_1, x_2, x_3, x_4)K^1[(x_1) \\
+ \beta P^4(x_1, x_2, x_3, x_4)K^1[(x_2) \\
+ \beta P^4(x_1, x_2, x_3, x_4)K^1[(x_3) \\
+ \alpha P^4(x_1, x_2, x_3)\delta(x_1 - x_4) \\
+ \alpha P^4(x_1, x_2, x_3)\delta(x_2 - x_4) \\
+ \alpha P^4(x_1, x_2, x_3)\delta(x_3 - x_4)
\]
(55)
where $\nabla^2$ is now the 4-D Laplacian.
Then the cubic part of the optimal feedback is
\[
\int_C K^3(x_1, x_2, x_3) z(x_1, t) z(x_2, t) z(x_3, t) \, dV
\]
where
\[
K^3(x_1, x_2, x_3) = -2R^{-1}P^4(x_1, x_2, x_3, 1).
\]
Since $P^4(x_1, x_2, x_3, x_4)$ is symmetric in its four arguments, it follows that $K^3(x_1, x_2, x_3)$ is symmetric in its three arguments.
Again we assume $P^4(x_1, x_2, x_3, x_4)$ has an expansion in terms of the open loop eigenfunctions (7)
\[
P^4(x_1, x_2, x_3, x_4) \\
= \sum_{i_1, i_2, i_3, i_4=0}^\infty \Pi^4_{i_1, i_2, i_3, i_4} \phi_{i_1}(x_1) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \phi_{i_4}(x_4)
\]
(56)
then (55) leads to an infinite set of linear equations for $\Pi^4_{i_1, i_2, i_3, i_4}$
\[
\sum_{j_1, j_2, j_3, j_4=0}^\infty \Omega^4_{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4} \Pi^4_{j_1, j_2, j_3, j_4} = -\Gamma^4_{i_1, i_2, i_3, i_4}
\]
(57)
for $i_1, i_2, i_3, i_4 = 0, \ldots, \infty$.

The infinite matrix $\Omega^4$ is
\[
\Omega^4_{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4} \\
= \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} (\lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_{j_4}) \\
- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} (\delta^2_{j_1} (1) \Pi^2_{j_2, j_1} \\
+ \delta^2_{j_2} (1) \Pi^2_{j_2, j_1} + \phi^2_{j_3} (1) \Pi^2_{j_3, j_3} + \phi^2_{j_4} (1) \Pi^2_{j_4, j_4}) \\
- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} \phi_{j_1}(1) \Omega^4_{i_1, i_1, i_2, i_2} \\
- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} \phi_{j_2}(1) \Omega^4_{i_1, i_1, i_2, i_2} \\
- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} \phi_{j_3}(1) \Omega^4_{i_1, i_1, i_2, i_2} \\
- \beta^2 \delta_{i_1, j_1} \delta_{i_2, j_2} \delta_{i_3, j_3} \delta_{i_4, j_4} \phi_{j_4}(1) \Omega^4_{i_1, i_1, i_2, i_2}
\]
and the operator applied to $P^4(x_1, x_2, x_3, x_4)$ given by (56) is
\[
\sum_{i_1, i_2, i_3, i_4=0}^\infty \sum_{j_1, j_2, j_3, j_4=0}^\infty \Omega^4_{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4} \Pi^4_{j_1, j_2, j_3, j_4} \\
\times \phi_{i_1}(x_1) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \phi_{i_4}(x_4).
\]
The right hand side of (57) is given by the symmetric version of
\[
\Gamma^4 = \alpha \int_C K^3(x_1, x_2, x_3) \delta(x_1 - x_4) \\
+ P^3(x_1, x_2, x_3) \delta(x_2 - x_4) \\
+ P^3(x_1, x_2, x_3) \delta(x_3 - x_4) \] dH.

Because the open and closed eigenvalues go to $-\infty$ so quickly we choose an $r > 0$ and truncate (57) to
\[
\sum_{0 \leq \delta_{j_1 + j_2 + j_3 + j_4} \leq r} \Omega^4_{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4} \Pi^4_{j_1, j_2, j_3, j_4} \\
= -\Gamma^4_{i_1, i_2, i_3, i_4}
\]
(58)
for $0 \leq i_1 + i_2 + i_3 + i_4 \leq r$.

By arguments similar to the above we can show that if $\beta \leq 1$ then $\Omega^4$ is also strictly column diagonally dominant.

V. Example

First, we discretize the linear system. Choose a positive integer $n$. Let and $\zeta_k(t) = \zeta(k/n, t)$ for $k = 0, \ldots, n$. The discretization of the differential equation (1) is
\[
\dot{\zeta}_k = \frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{n^2} = n^2(\zeta_{k+1} - 2\zeta_k + \zeta_{k-1})
\]
for $k = 1, \ldots, n - 1$.

We add fictitious points $\zeta_{-1}$ and $\zeta_{n+1}$ to handle the boundary conditions. The Neumann boundary condition (3) at $x = 0$ is discretized by a centered first difference
\[
\frac{\partial \zeta}{\partial x}(0, t) \approx n \frac{\zeta_1 - \zeta_{-1}}{2}.
\]
then
\[
\dot{\zeta}_0 = n^2(\zeta_1 - 2\zeta_0 + \zeta_{-1}) = 2n^2(\zeta_1 - \zeta_0).
\]
The controlled boundary condition (4) at \(x = 1\) is also discretized by a centered first difference
\[
\frac{\zeta_{n+1} - \zeta_{n-1}}{2n} = \beta (u - \zeta_n).
\]
We solve this for \(\zeta_{n+1}\)
\[
\zeta_{n+1} = \zeta_{n-1} + \frac{2\beta}{n} (u - \zeta_n)
\]
and plug this into the differential equation for \(\zeta_n\)
\[
\dot{\zeta}_n = n^2 (\zeta_{n+1} - 2\zeta_n + \zeta_{n-1})
\]
\[
= 2n^2 \left( \zeta_{n-1} - \frac{n + \beta}{n} \zeta_n + \frac{\beta}{n} u \right).
\]
This yields an \(n + 1\) dimensional linear system
\[
\dot{\zeta} = F\zeta + Gu
\]
where
\[
\zeta = \begin{bmatrix} \zeta_0 & \cdots & \zeta_n \end{bmatrix}^T
\]
\[
F = n^2 \begin{bmatrix} -2 & 2 & 1 & & & & & \\ 1 & -2 & 1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & & \\ & & & 2 & -2n+2\beta & & & \end{bmatrix}
\]
and
\[
G = \begin{bmatrix} 0 & 0 & \cdots & 0 & 2n\beta \end{bmatrix}^T.
\]
If \(n = 10\) and \(\beta = 1\) the three least stable poles of \(F\) are
\(-0.7404, -11.6538,\) and \(-40.1566\). Recall that \(\lambda_0 = -0.7402, \lambda_1 = -11.7349,\) and \(\lambda_2 = -45.3075\) so there is substantial agreement between the first three open loop poles of the finite and infinite dimensional systems.

We set \(Q\) to be \(11 \times 11\) matrix
\[
\begin{bmatrix} 0.5 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0.5 \end{bmatrix}
\]
and \(R = 1\) and solved the resulting finite dimensional LQR problem. Its three least stable closed loop poles are \(-1.0396, -11.7270,\) and \(-40.1804\). We found previously that the three least stable closed loop poles of the infinite dimensional system are \(\mu_0 = -1.0409, \mu_1 = -11.8094,\) and \(\mu_2 = -41.4620,\) so there is also substantial agreement between the first three closed loop poles of the finite and infinite dimensional systems.

At the suggestion of Rafael Vazquez, we discretized using the Crank–Nicolson method [7]. Crank–Nicolson is an implicit method that uses an average of forward and backward Euler steps. At each time we took a forward Euler step and then used fixed point iteration to correct for backward Euler. These converged after 5 iterations. The spatial step was \(\Delta x = 1/10\) and the temporal step was \(\Delta t = (\Delta x)^2 = 1/100.\) We used our nonlinear systems toolbox to compute the optimal feedback through cubic terms [13]. It took 1.56 s on a MacBook Pro with a 3.1 GHz Dual-Core Intel Core i5 processor.

The open loop nonlinear system converged slowly when the initial condition was \(\zeta_i(0) = 0.7\) for \(i = 0, \ldots, 10\) but diverged when \(\zeta_i(0) = 0.8\) (see Fig. 1).

The nonlinear system with optimal linear feedback converged when \(\zeta_i(0) = 1\) for \(i = 0, \ldots, 10\) but diverged when \(\zeta_i(0) = 1.1\) (see Fig. 2).

The nonlinear system with optimal linear, quadratic, and cubic feedback converged when \(\zeta_i(0) = 4\) for \(i = 0, \ldots, 10\) but diverged when \(\zeta_i(0) = 4.1\) (see Fig. 3).

In the above simulations we started at \(\zeta^0(x)\) a constant because the least stable eigenfunction of the open loop system with \(\beta = 0\) is a constant function. The optimal linear, quadratic, and cubic feedback was able to stabilize up to \(\zeta^0(x) = 4\) but diverged when \(\zeta^0(x) = 4.1.\) The next less stable eigenfunction is \(\cos \pi x.\) The optimal linear, quadratic, and cubic feedback was able to stabilize up to \(\zeta^0(x) = 4.4 \cos \pi x\) but diverged when \(\zeta^0(x) = 4.5 \cos \pi x.\)
VI. Conclusion

We have solved the LQR problem for the boundary control of an infinite dimensional system by extending the finite dimensional technique of completing the square. We also optimally locally stabilized a nonlinear reaction diffusion system by extending Al’brekht’s method to infinite dimensions. We showed by example that optimal cubic feedback can lead to a much larger basin of stability than optimal linear feedback. We believe that solving an LQR by the completing the square is applicable to other linear infinite dimensional boundary control problems. We are exploring extending it to the wave equation and the beam equation.

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