Article

Performance Analysis of Identification Codes

Sencer Derebeyoğlu ¹, Christian Deppe ¹ and Roberto Ferrara ¹

¹ Lehr- und Forschungseinheit für Nachrichtentechnik, Technische Universität München, Munich, Germany; christian.deppe@tum.de, roberto.ferrara@tum.de

Abstract: In this paper we analyse the construction of identification codes. Identification codes are based on the question "Is the message I have just received the one I am interested in?" We go over the differences to Shannon’s transmission, where the receiver is interested in not only one, but any message, and show how beneficial identification over transmission is in some special cases. Identification’s advantage is that it allows double exponential growth in capacity at the cost of a new kind of error, which goes to zero in the asymptotic case. We then focus on a special identification code construction using tag codes with two concatenated Reed-Solomon codes. We have a closer look at our implementation of that construction and gain deeper understanding using the results our implementation has yielded. Finally we look towards future works.

Keywords: Identification, Coding, Construction, Reed-Solomon, Simulation

1. Introduction

Shannon’s transmission scheme [1] is the standard model for communication. For some special tasks however, more effective schemes can be used. In 1989 [2], Ahlswede and Dueck proposed a new scheme for communication: identification. The main differences between a transmission scheme and an identification scheme are the following questions: The receiver in the transmission scheme is interested in, "What is the message I just received?", whereas the receiver in the identification scheme is merely interested in, "Is the message I have just received the one I am interested in?" (figs. 1 and 2). In Shannon’s transmission, there is usually one receiver and he is ready to decode any codewords in the codebook he receives into messages. However, there can be multiple receivers in identification and each of them would only be interested in one message of the codebook.

A real life application of identification can be online sales. Since web platforms track user data in their surfing or product viewing behaviors, they can identify whether the user is interested in a

![Figure 1](image1.png)

**Figure 1.** An illustration of a transmission receiver interested in any message.

![Figure 2](image2.png)

**Figure 2.** An illustration of an identification receiver only interested in one message.
certain product type or company. Various categories are possible for data tracking. We can think of each element in these categories as a receiver of an identification scheme. According to the information of the user’s interests gathered with the help of identification, the platform can use optimised advertising when targeting that user [3].

There are advantages and disadvantages to the identification scheme. The scheme allows a negligible amount of overlapping between the decoding sets of codewords. This helps to fit in exponentially more messages [4] in the codebook, but the drawback is that we cannot decode anymore. The additional error introduced by the overlap can still be made arbitrarily close to zero in the asymptotic case, namely for codewords which have large blocklengths.

Tag codes are one way to construct identification codes. They are usually built from error-correction codes. The encoder randomly selects a coloring number and computes its corresponding tag according to a function determined by the identification message. Then the encoder sends this tuple through the channel and each receiver checks if the tag computed by their identification message on the received coloring number matches the received tag. The transmitter and the receivers in the tag code scheme can each be seen as the mapping functions, which map coloring numbers to tags in their own and unique way. The goal is to find mappings unique enough so that the overlapping error is a negligible amount.

In this paper, we focus on a capacity-achieving construction of tag codes via a concatenation of two Reed-Solomon codes. The codewords made with the help of the concatenated Reed-Solomon codes are those mapping functions we have mentioned above. The positions of the symbols in the codewords are the coloring numbers and the tags are nothing but the symbols themselves. Additionally, we thoroughly explain how a tag code is made and how this capacity-achieving scheme works.

For further details on identification we refer to an upcoming survey of identification [5].

2. Preliminaries

Channels are mediums in which the information-carrying codewords are transmitted. Noisy channels disrupt the codewords they are carrying, unless we are using an ideal channel. Error-correction codes (see Subsection 2.5) can be used to allow the receiver to fix the damage a noisy channel does to the codewords. Discrete memoryless channels (DMCs) are channels between discrete values which have a fixed amount of error probability on each codeword symbol sent, where the success or failure in transmitting the previous symbol doesn’t affect the current symbol being transmitted (memoryless). Commonly they are expressed in the tuple \((X, Y, W)\), where \(X\) is the finite set of the input alphabet, \(Y\) is the finite set of the output alphabet and the \(W\) is a stochastic matrix, \(W = \{W(y|x) : x \in X, y \in Y\}\).

The sequences \(x^n = (x_1, x_2, ..., x_n)\) are the inputs to the channel to be transmitted, whereas the sequences \(y^n = (y_1, y_2, ..., y_n)\) are the outputs of the channel. The probability that \(y^n\) is received from the channel as an output, when \(x^n\) was given as an input is:

\[W^n(y^n|x^n) = \prod_{i=1}^{n} W(y_i|x_i).\] (1)

Shannon’s transmission, developed by Claude Elwood Shannon is a system where two communication partners exchange messages via an often noisy channel. The receiver is interested in any message the sender sends, therefore the receiver’s motivation can be efficiently stated with the question: “What is the message?”

To counteract the noise of the channel, the messages are channel-coded into codewords, adding redundancy. Each message is encoded by an encoder before the channel and, after it passes through the channel, the codeword is decoded by a decoder (fig. 3). The decoder is defined by decoding sets,
one for each possible message, which are disjoint\(^1\). If a codeword falls into a decoding set after passing through the channel, the decoder recognises the message associated to the decoding set and outputs that particular message. In this way the message has been decoded and transmitted to the sink, which is the communication partner waiting on the receiving end. Below is a diagram showing the structure of transmission schemes.

2.1. Identification

After Shannon’s transmission theory, there have been some proposals on achieving a better capacity, using schemes that are outside of Shannon’s description. These advanced techniques for communications are sometimes referred to as Post-Shannon schemes. Identification is one of these techniques. From here on, we will be comparing transmission with identification. Identification, making its debut with Ahlswede and Dueck in 1989, differs from Shannon’s transmission in how it carries information between two communication partners. This time the receiver is not interested in all of the messages, but only in one of them, e.g. in message \(j\), chosen by the receiver itself. Compared to transmission, the role of the decoder then becomes that of a verifier which either accepts or rejects the received output as being compatible with the chosen message (fig. 3).

Identification can be achieved by adding some preprocessing and postprocessing around transmission (fig. 5). These new steps increase the number of possible messages in great measure, while adding a small new kind of error. The idea is that the high increase in the number of messages can be beneficial if we can keep the new kind of error below a certain threshold, and send it to zero in the asymptotic case. This also allows us to first fix the errors in the channel, approximating a noiseless channel, and then code for identification. For this reason, it is enough to study identification for the noiseless channel, which is the case for this work.

The advantage of identification over transmission, given the added complexity, lies in the number of possible messages in the identification scheme. By allowing the decoding sets to collide, the maximum number of messages grows doubly exponentially, while the number of messages in the transmission scheme grows only exponentially. In short, in identification, we allow a small error to happen and we sacrifice decoding for an extreme increase in the number of messages.

\(^1\) Or at least overlap bounded as the blocklength \(n\) increases in the case of soft decoding.
2.1.1. Types of Errors

In identification we have two kinds of errors: missed identification, also known as the error of the first kind, and false alarm, also known as the error of the second kind. The missed identification is a regular transmission error, i.e. the codeword $i$ is not recognized as codeword $i$ on the receiver end. This happens when the receiver is interested in the same message $i$ sent by the sender, and when the message is disrupted in the channel in such a way that the receiver doesn’t identify the received output as message $i$. This error is absent when coding for the noiseless channel and thus it will not appear in this work.

The false alarm will be our main interest in this paper. If the sender sends the message $i$ and the receiver is only interested in a different message $j$, the receiver may still identify the message as message $j$. This is a false identification or a false alarm error. There are two possible contributions to this error probability:

- A transmission error (the same reason as for the missed identification) happens and the channel gives an output of message $j$ although having received the message $i$,
- The scheme uses a randomized identification code (see Subsubsection 2.1.2) and the randomly selected codeword lies in the overlapping section of the decoding sets$^2$ of message $i$ and message $j$.

Since we only look at identification over the noiseless channel, the first contribution is absent.

2.1.2. Types of Identification Codes

A deterministic code is a sequence of encoding codewords and decoders \{\(u_i, D_i\)\}_{i=1,...,N} for each possible message. A randomized code is a sequence of encoding probability distributions and decoders \{\(Q_i, D_i\)\}_{i=1,...,N} for each possible message. In both cases the decoders are subsets of the possible outputs, namely a decoding set $D_i$ is a subset of the $Y^w$ alphabet, where an output codeword is recognised as the

$^2$ More about decoding sets in Subsubsection 2.1.2.
The $i$-th codeword, if it is an element of $D_i$. The larger the decoding sets are, the more tolerance the code has against transmission errors.

As already hinted, we may use deterministic or randomized encoders. $(n, N, \lambda_1, \lambda_2)$ Deterministic identification codes are identification codes with blocklength $n$, i.e. the length of any codeword, $N$ total number of messages and $\lambda_1, \lambda_2$ bounds on the two types of errors. Namely they must satisfy

$$
\mu_1^i = W^n(D_i^c | u_i) \leq \lambda_1, \forall i \tag{2}
$$

$$
\mu_2^{ij} = W^n(D_j | u_i) \leq \lambda_2, \forall i \neq j \tag{3}
$$

where $u_i \in X^n$ represents the $i$-th codeword and $D_i \subset Y^n$ is the decoding set on the receiver side for the $i$-th codeword. $\mu_1$ is the probability of the error of the first kind and $\mu_2$ is the probability of the error of the second kind. Note that the error of the first kind concerns only one codeword, where the error of the second kind happens between two codewords. So in this case, $\lambda_1$ is a bound on the maximum probability of the error of the first kind and $\lambda_2$ is a bound on the maximum probability of the error of the second kind.

$(n, N, \lambda_1, \lambda_2)$ Randomized identification codes are very similar to the deterministic identification codes. The codewords on the sender side are selected randomly, using a conditional probability distribution $Q$ such that:

$$
\mu_1^i = \sum_{x_n \in X^n} Q(x_n | i) W^n(D_i^c | x^n) \leq \lambda_1, \forall i \tag{4}
$$

$$
\mu_2^{ij} = \sum_{x_n \in X^n} Q(x_n | j) W^n(D_j | x^n) \leq \lambda_2, i \neq j \tag{5}
$$

The randomization is necessary to achieve a high number of messages$^3$. The construction we work on in this paper also makes use of a randomized identification code.

**2.2. Tag Codes**

Tag codes, or Coloring codes, are a particular way of constructing identification codes using pre- and post-processing around a transmission scheme. Each identification message, or identity, is assigned a unique mapping function. The elements of the first set are called "coloring numbers" and the elements of the second set are called "colors/tags". The codeword sent in a tag code is nothing but a concatenated pair of codewords, where the first element, the coloring number, is a randomly selected transmission message encoded for the transmission code and the second element is the encoded tag computed by the identity (identification message) mapping function on the coloring number (transmission message). See fig. 6 for an example.

Tag codes are in one-to-one correspondence to error correction codes. Namely, every error correction code can be used as a tag code. For this purpose the error correction code is not used as an input to the noisy channel. Rather, each codeword of the error-correction code is assigned to an identity, so there are as many identities as codewords in the error-correction code. Each codeword is then used as a mapping function by taking as input the coloring number a position in the codeword and giving as the output tag the corresponding symbol of the codeword. This gives pre- and post-processing. The position and the symbol must then be channel coded via a transmission code if the channel is noisy.

---

$^3$ Having local randomness for Shannon’s transmission scheme doesn’t increase capacity [6].
Figure 6. An example of a mapping function $T_i$ which belongs to the identity $i$.

$[N, t, q, 1 - \frac{d}{n}]$ is used to denote the set of tag codes, where $N$ is the number of messages in the tag code, $t$ is the cardinality of the first set, i.e. the number of bits of the coloring numbers, $q$ is the cardinality of the output set, i.e. the number of bits of tag, and $1 - \frac{d}{n}$ is the maximum false alarm probability $\lambda_2$. Where $d$ is the code distance of the tag code, it is seen as an error correction code.

2.3. Prime Fields

In this paper we consider only tag codes constructed from Reed-Solomon codes. However, in order to create tag codes, we can use any error-correction codes [7]. Before we begin with them, we need to define the concepts of "groups," "prime fields" and "extension fields."

A group is a set $\mathbb{H}$ with some elements, accompanied by an operation, which we denote by $\ast$ and four axioms. These axioms are the following:

- Closure: if $a$ and $b$ are elements of $\mathbb{H}$, then $a \ast b$ is also an element of $\mathbb{H}$. In other words $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$.
- Associativity: let $a, b$ and $c$ be elements of $\mathbb{H}$. Then $(a \ast b) \ast c = a \ast (b \ast c)$ holds.
- Identity Element: there is an identity element $e$ in every group which reflects the other operand to the result of the operation: $e \ast a = a = a \ast e$, if $a$ is an element of $\mathbb{H}$. ¹
- Invertibility: every element of $\mathbb{H}$ has an inverse element in the same set for which it holds: $a \ast a^{-1} = e$, where $a^{-1}$ is the inverse element of $a \in \mathbb{H}$.

Commutative groups possess all of the axioms a group does, but also an extra axiom:

- Commutativity: the order of the operands doesn’t matter, i.e. $a \ast b = b \ast a$.

¹ This axiom called ‘identity’ has, of course, no correlation to the messages of the identification scheme, which we coincidentally call ‘identities’.
Fields are sets, say $F$, with two operations addition and multiplication, with the following axioms:

- addition forms a commutative group on $F$,
- multiplication forms a commutative group on $F \setminus \{0\}$,
- Distributivity: $a \in F \land b \in F \land c \in F \Rightarrow (a + b) \cdot c = a \cdot c + b \cdot c$.

The most notable examples are the integers modulo $n$, which always form a group, but only form a field when $n$ is prime. Prime fields $F_p$ are fields which consist of all the natural numbers ranging from 0 to $p - 1$, where $p$ is a prime number [8, Chapter 3]. Primitive elements $g$ are any element of a prime field, which can generate all the field elements except zero by its different powers ranging from $g^0$ to $g^{p-2}$.

2.4. Extension Fields

Extension Fields $F_{p^m}$ are based on prime fields, but this time the elements of this special kind of fields are strings made of multiple prime field elements. The parameter $p$ shows which prime field this extension field is based on and $m$ shows the extension degree, i.e. how many elements of the prime field are in a symbol of the extension field [8, Chapter 3].

Since we cannot make operations between the elements as if they were $m$ digit numbers on base $p$ without violating the axioms of fields, we have to come up with a different method to define operations we can use. For that purpose we need the primitive element of the extension field. And to determine the primitive element [8], we need to make use of a primitive polynomial of the extension field.

Primitive polynomials are special functions, which enable us to make operations between members of the extension field they belong to. They are of degree $m$ and they are irreducible polynomials. They are under modulo $p$, since they are defined in the field. Primitive polynomials have one more important feature: primitive elements are their roots. This is the final quality that we can determine if such an irreducible function of degree $m$ is actually a primitive polynomial.

2.5. Error-Correction Codes

Error-correction codes [9, Chapter 1] are used to encode the information, i.e. the bitstream, which is supposed to go through a noisy channel. This encoding happens by adding some redundancy to the original bitstream or message. The added redundancy helps on the decoder end to detect and decode errors.

Linear block codes $[n, k, d]_q$ are error-correction codes and any linear combination of its codewords is also a codeword [9, Chapter 1]. Some are based on prime or extension fields (see Subsections 2.3 and 2.4). The parameter $n$ stands for the blocklength, i.e. the length of a codeword, $k$ for the length of the message, $d$ for the minimum Hamming distance and $q$ for the alphabet size of the linear code. If the message is only in bits, namely when $q = 2$, it is common to omit $q$. By definition there are thus $q^k$ codewords in the codebook of a $[n, k, d]_q$ linear code. The minimum Hamming distance is the Hamming distance of the two closest codewords. In such a code, we can detect $(d - 1)$ errors and decode $\left\lfloor \frac{d - 1}{2} \right\rfloor$ of them. Furthermore, taking the linear dependancy of the codewords into account, we can build a generator matrix $G$ and compute any codeword for any message via matrix multiplication as

$$m \cdot G = c \quad (6)$$

where $m$ is the message, i.e. the original bitstream, $G$ is the generator matrix of the linear code and $c$ is the resulting codeword after the encoding process. $G$ will have full rank and the dimensions $k \times n$ in order to map messages to codewords uniquely.
2.6. Reed-Solomon Code

Reed-Solomon codes [9, Chapter 5] are error-correction codes, developed by Irving S. Reed and Gustave Solomon in the year 1960. They are based on either prime fields or extension fields. They make use of code locators, which are nothing but \( n \) different elements of the field, which the Reed-Solomon code is based on. As in any error-correction code, the Reed-Solomon code is used to channel code a bit string, so that the loss of information due to a noisy channel can be regained by correcting the errors, or at least detecting that some errors have taken place. Reed-Solomon codes also achieve the Singleton Bound

\[ d \leq n - k + 1 \]  

with equality, which means that they are maximum distance separable codes. This is extremely important for tag codes, because greater scaling in the code distance means a smaller false alarm probability in the asymptotic case.

The codewords of Reed-Solomon codes are generated as the evaluation of polynomials on the elements of \( F_q \), also called code locators. To each message corresponds a polynomial, where the symbols of the message are the coefficients of the polynomial. So, if we let \( m = \{ m_0, m_1, m_2, m_3, m_4, ..., m_{k-1} \} \) be the message vector and have \( \beta \in F_q \), then:

\[ u_m(\beta) = \sum_{i=0}^{k-1} m_i \cdot \beta^i \]  

is called the evaluation of \( \beta \). For a Reed-Solomon code of length \( n \leq q \), there are \( n \) such evaluations. One evaluation gives us one symbol of the codeword, whereas \( n \) such evaluations give us the whole codeword.

If we let \( c = \{ c_0, c_1, c_2, c_3, c_4, ..., c_{n-1} \} \) be the codeword and \( f_j \in F_q \) \( \forall i \), where \( j \) shows us the order of the picked code locator, then the codewords are:

\[ \{ c(m) = \{ \sum_{i=0}^{k-1} m_i \cdot f_0^i, \sum_{i=0}^{k-1} m_i \cdot f_1^i, \sum_{i=0}^{k-1} m_i \cdot f_2^i, ..., \sum_{i=0}^{k-1} m_i \cdot f_{n-1}^i \} \} \]  

which are generated from the message \( m = m_1 \ldots m_{k-1} \) with the \( k \times n \) generator matrix:

\[ G = \begin{pmatrix} f_0^0 & f_1^0 & f_2^0 & f_3^0 & f_4^0 & f_5^0 & \cdots & f_n^0 \\ f_0^1 & f_1^1 & f_2^1 & f_3^1 & f_4^1 & f_5^1 & \cdots & f_n^1 \\ f_0^2 & f_1^2 & f_2^2 & f_3^2 & f_4^2 & f_5^2 & \cdots & f_n^2 \\ f_0^3 & f_1^3 & f_2^3 & f_3^3 & f_4^3 & f_5^3 & \cdots & f_n^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0^{k-1} & f_1^{k-1} & f_2^{k-1} & f_3^{k-1} & f_4^{k-1} & f_5^{k-1} & \cdots & f_n^{k-1} \end{pmatrix} \]  

If we use the Reed-Solomon code as a tag code, then the polynomials \( u_m \) are the tagging functions, \( \{ 0, \ldots, n - 1 \} \) are the coloring numbers, \( F_q \) are the tags and \( (j, u_m(j)) \) is the tuple of the coloring number and tag sent through the transmission code.
2.7. Concatenated Codes

In order to build longer codes out of short codes, multiple codes can be concatenated [10]. Short codes in the concatenated codes are easy to decode, whereas we come closer to the code capacity with longer codes. We distinguish the two codes as inner and outer codes with the usual convention that the outer code’s symbols are encoded by the inner code. In the case of two codes, we have an inner code \([n_1, k_1, d_1]_{q_1}\) and an outer code \([n_2, k_2, d_2]_{q_2}\), satisfying \(q_1 = q_2^{k_2}\), then the concatenated error-correction code is a \([n_c, k_c, d_c]_{q_c} = [n_1n_2, k_1k_2, d_1d_2]_{q_1}\) code. The alphabet of the concatenated code is the alphabet of the inner code \(q_1\). The number of codewords in the concatenated code is the number of codewords in the outer code, \(q_1^{k_1k_2} = q_2^{k_2}\).

The following is an example of concatenated codes where each symbol of the outer code is encoded by the inner code \([10]\), albeit without using block codes (and thus in this case \(k_c = k_1 \cdot k_2\), is not valid). \(C_i = \{0120112, 1201120, 1202102, 2100211, 2101120\}\) with \(n_i = 7, A_i = \{0,1,2\}, |C_i| = 4\) and \(C_o = \{ad, bc, ac, cc, db, ab\}\) with \(n_o = 2, A_o = \{a,b,c,d\}, |C_o| = 6\), the indices \(i\) and \(o\) stand for the inner and the outer code respectively. Then:

\[
C_i \circ C_o = \{01201121201120, 12021021200211, 01201121200211, 21002112100211, 12011201202102, 01201121202102\},
\]

where \(n_{con} = 14, A_{con} = \{0,1\}, |C_i \circ C_o| = 6\). The outer code’s alphabet size and the inner code’s number of codewords in the codebook are the same so that each codeword in the inner code represents one symbol in the alphabet of the outer code. E.g.: 0120112 represents \(a\), 1202102 represents \(b\), 2100211 represents \(c\) and 1201120 represents \(d\). This is clear in the codebook of the concatenated code.

3. Capacity-Achieving Codes for Identification

There is an optimal construction for identification, introduced by Sergio Verdú and Victor K. Wei [11]. It involves the concatenation of two Reed-Solomon codes, where one is based on a prime field and the other is based on an extension field. The construction is defined as:

\[(q, k, \delta)_{RS^2} := (q, k)_{q_1}^{RS} \circ (q^k, q^{k-\delta})_{q_2}^{RS}\]

with \(q\) an increasing prime power and \(q \gg k \gg \delta > 0\).

There are three conditions which a capacity achieving identification code constructed on error-correction codes needs to meet. If we are considering a block code \([M, k, d]_{q}\), the following conditions are necessary in order to achieve the identification capacity [11]:

\[
(q, k, \delta)_{RS^2} := (q, k)_{q_1}^{RS} \circ (q^k, q^{k-\delta})_{q_2}^{RS}
\]
1. The size of the code and thus the size of the identification code must be exponential to the size of the coloring numbers:

$$\lim_{q,k \to \infty} \frac{\log k}{\log M} \to 1; \quad (18)$$

2. The size of the tag in bits must be negligible to the size of the coloring number in bits:

$$\lim_{q,k \to \infty} \frac{\log q}{\log M} \to 0; \quad (19)$$

3. The distance of the code must grow as the blocklength, meaning that the error of the second kind goes to zero:

$$\lim_{q,k \to \infty} \frac{\log d}{\log M} \to 1. \quad (20)$$

Recall that after computing the tag, the tuple must be channel coded. The first two conditions together guarantee that the entire capacity of the channel is used to send the coloring number, and that the number of identities grows doubly exponential in the channel capacity. The last condition guarantees that we can send the false alarm error to zero. The Gilbert-Varshamov bound guarantees the existence of such codes.

As stated in Section 2, tag codes based on a single Reed-Solomon code are also an option, however they don't fulfill the conditions to achieve capacity. A full size \(n = q\) single Reed-Solomon code would mean a \([q, k, d]_q\) block code which does not allow us to satisfy eq. (19):

$$\lim_{q,k \to \infty} \frac{\log q}{\log M} = \lim_{q,k \to \infty} \frac{\log q}{\log q} \to 1,$$

meaning that the tag is too big compared to the coloring number.

Instead, the double Reed-Solomon code construction \((q, k, \delta)_{RS2}\) is a

$$[q^{k+1}, kq^k - \delta, (q - k + 1)(q^k - q^{k-\delta} + 1)]_q$$

block code. The first condition, eq. (18), can be easily verified. The second condition, eq. (19), is also obvious, because \(q^{k+1}\) will always be bigger and also grow faster than \(q\) itself. In particular, these two conditions are met already, by expanding the extension field into the base field for the outer code only. The crucial condition where the inner code is needed is the third one, eq. (20). Applying the code distance of the double Reed-Solomon code construction to the third condition, we obtain:

$$\lim_{q,k \to \infty} \frac{\log d}{\log M} \to 1 \quad (23)$$

$$\lim_{q,k \to \infty} \frac{\log(q - k + 1)(q^k - q^{k-\delta} + 1)}{\log q^{k+1}} \to 1 \quad (24)$$

$$\lim_{q,k \to \infty} \frac{\log(q^{k+1} + kq^{k-\delta} + 1 + q + q^{k - q^{k-\delta} + 1} - kq^k - k - q^{k-\delta})}{\log q^{k+1}} \to 1. \quad (25)$$

As we can see in Equation 25, if we assume that both \(q\) and \(k\) grow to infinity, then the \(q^{k+1}\) in the numerator will outgrow everything else in the logarithm, so that the other values will lose their importance in the asymptotic case and will become negligible. Hence, we have nothing but \(\log q^{k+1}\) in the denominator, so this fraction will actually go to 1 in the asymptotic case.
Example

We now proceed with an example of how such a concatenated tag code is built. For simplicity and clarity reasons, we will use the smallest possible code with the parameter set \((3, 2, 1)_{RS} \circ (3^2, 3)_{RS}^R\). The concatenated code is an

\[
[q^{k+1}, kq^{k-\delta}, (q - k + 1)(q^{k-\delta} + 1)]_q = [27, 6, 14]_3
\]

error-correction code.

In order to obtain a tag from the tag code, we first need an identity. For example purposes, we pick the 587th identity among the 729 total identities. In symbols of the field \(F_{3^2}\), this is the string

\[
m = 587_{10} = 722_9.
\]

Now, we can use the numbers 7, 2 and 2 as the orders of the field elements. With the knowledge of Section 2, we can order the elements of \(F_{3^2} = \{0, \alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^6, \alpha^7\}\). With this order, we can rewrite our identity as

\[
m = 722_9 = (\alpha^6 \alpha^1 \alpha^1) = (\alpha^6 \alpha \alpha)
\]

where \(\alpha\) is the primitive element of \(F_{3^2}\). Now we are ready to compute the tag function for the identity \(m\). It is time to introduce the generator matrix of the outer code \((3^2, 3)_{RS}^R\) with what we have seen in Section 2:

\[
G_o = \begin{pmatrix}
\alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 \\
0 & \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^6 & \alpha^7 \\
0 & \alpha^0 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha^0 & \alpha^2 & \alpha^4 & \alpha^6
\end{pmatrix}.
\]

Multiplying \(m\) with \(G_o\), we get the outer codeword

\[
c'(m) = m \cdot G_o
\]

\[
= (\alpha^6 \alpha \alpha) \cdot G
\]

\[
= (\alpha^6 \alpha^7 \alpha^3 0 0 \alpha^6 \alpha^4 \alpha^3 \alpha^4) \in F_{3^2}^9.
\]

Before calculating the final codeword, we need two preparation steps. First, we need to expand the symbols of the first codeword by transforming its base field from \(F_{3^2}\) to \(F_3\). The reason is that the next Reed-Solomon code \((3^2, 3)_{RS}^R\) is based on field \(F_3\). We can’t have two operands with different base fields. The way to do this is through representing the symbols back in their string form. If we apply this to the first codeword in our example, the outer codeword will become:

\[
c'(m) = ((22) (21) (12) (00) (00) (22) (20) (12) (20)) \in F_3^{18}.
\]

Now our codeword is ready to be concatenated with the inner code. But since a concatenation means the symbols will be encoded one by one, instead of encoding only the codeword, we need a larger generator matrix than the regular generator matrix of \((3, 2)_{RS}^R\). We need the direct sum of this generator matrix with itself by 9 times, as the second preparation step. The specified block sum looks like this:

\[
G_B = \bigoplus_{\ell=0}^{8} G_i = \bigoplus_{\ell=0}^{8} \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2
\end{pmatrix}
\]
where $G_i$ is the generator matrix of the inner code $(3,2)^{RS}_3$. Now the multiplication

$$c_f = \xi(m) \cdot G_B$$

will yield the final codeword $c_f$. Every symbol of the first codeword $c_1$ will be encoded with $G_i$. Since each symbol now has two trits and $G_i$ transforms two trits into three trits, we get

$$c_f = (210201102000002102210222).$$

At this point we have the mapping function of the $587^{th}$ identity. If, for instance, the corresponding tag of the coloring number 5 in the $587^{th}$ identity is required, we simply calculate this mapping function as done above and look at position 5. The preprocess step is completed as we use the concatenated string

$$(j, T_i(j)) = (5, 1)$$

as encoding of the identity 587, which must now be sent through a channel coding to the receiver.

### 3.1. False Alarm Probability

Recall what false alarm or the error of the second kind is: the sender wants, e.g., to send the identification message 4, while the receiver is checking the message 7. After the decoding phase is complete, the receiver succeeds in verifying the identity 7. It is the case where the randomized encoder sends a transmission message which lies in an overlap area of both identification messages. The receiver has verified it as its identification message. However this is not true.

We can calculate the maximum probability of this happening. This is the only meaningful error probability that needs to be bound. It has been proven that bad codes exist if we require only the average of the false alarm probability to be small, allowing for an infinite identification rate [12]. The error probability is given by the number of equal $(j, T_i(j))$ pairs between two different codewords, because $j$ is picked at random. Namely we are looking at the complement of the distance between two codewords (the tag functions of two identities).

Consider for example two codewords

$$c_1 = 10121020212000120100$$
$$c_2 = 0202021010021020121.$$

We can see that the positions 4, 8, 14 and 18 match on the two codewords $c_1$ and $c_2$. This means that if the $j$ is picked randomly from these positions, where $c_1$ is the sender’s identity’s mapping function and $c_2$ is the receiver’s identity’s mapping function, a false alarm will happen, as the pair $(j, T_i(j))$ will be the same.

However we have a guarantee on the maximum amount of entries in which this happens by the code distance $d$ of the error-correction code, which is the number of minimum distinct entries. This maximum probability of a false alarm is thus calculated with

$$\lambda_2 \leq \frac{n - d}{n} = 1 - \frac{d}{n}$$

because $j$ is picked at random with the uniform probability $1/n$. 


We should add a final note here: \( \lambda_2 \) is a bound on the error of the second kind only in the absence of an error of the first kind, since an error of the first kind can also invoke an error of the second kind on a different receiver.

4. Comparison of Identification and Transmission

Here we make a comparison between the identification and the transmission schemes at equal blocklengths. Both the single Reed-Solomon code and the double Reed-Solomon code constructions will be compared to their counterparts in the transmission scheme. Since we are coding for the noiseless channel, we will be comparing the amount of bits used in the construction of the \((j, T_i(j))\) to the number of identities achieved by the tag/error-correction code. However, notice that while the tag code will have a certain false alarm probability, the error in the transmission code is completely absent. Therefore part of the comparison is the trade-off between the introduced error and the increase in the number of identities.

4.1. Single Reed-Solomon Code

As we know by this point, the codewords of the Reed-Solomon code represent each identity. This makes us be sure that there are as many identities as the number of possible codewords (see Subsection 2.2). In a \((q, k)_{q}^{RS}\) Reed-Solomon code, there are \(q^k\) codewords and thus that many identities. Therefore the Single Reed-Solomon construction has \(q^k\) identification messages. The blocklength and the field size of a \((q, k)_{q}^{RS}\) Reed-Solomon code are both \(q\), therefore the transmission \((j, T_i(j))\) takes \(q^2\) elements. If we use the \(q^2\) elements to send \(q^2\) messages via noiseless transmission, we would achieve a rate

\[
r_T = \log \frac{q^2}{n} = 2 \log \frac{q}{n},
\]

where \(n\) is now the blocklength of the noiseless channel. In comparison the rate achieved by identification is

\[
r_{ID} = \log \frac{q^k}{n} = k \log \frac{q}{n}
\]

Therefore we have an increase in the rate of

\[
\frac{r_{ID}}{r_T} = \frac{k}{2}
\]

at the cost of introducing, in the worst case, a false alarm error probability of

\[
1 - \frac{d}{q} = 1 - \frac{q - k + 1}{q} = \frac{k - 1}{q}.
\]

As long as \(q\) grows faster than \(k\), in the asymptotic case where they both grow to infinity, we can increase the encoded identification messages and decrease maximum false alarm probability \(\lambda_2\). This shows that for the same blocklength, the single Reed-Solomon code construction has polynomially more messages than its transmission counterpart.
4.2. Double Reed-Solomon Code

We shall begin this comparison analogously to the previous comparison: we compare the number of identification messages with the number of used coloring number/tag pairs. Recall that the double Reed-Solomon code is a $(q, k, \delta)_{RS} \in [q^{k+1}, kq^{k-\delta}, (q - k + 1)(q^k - q^{k-\delta} + 1)]_q$ \hspace{1cm} (41)

There are thus $q^{kq^{k-\delta}}$ identification messages in this construction.

Now let’s check the size of the preprocessed codeword $(j, T_i(j))$. This time there are $q^{k+1}$ possible coloring numbers with again $q$ possible tags, for a total of $q^{k+2}$. The rate achieved by transmission is thus

$r_T = \frac{\log q^{k+2}}{n} = (k + 2) \frac{\log q}{n}.$ \hspace{1cm} (42)

In contrast, the rate achieved by identification is

$r_I = \frac{\log q^{kq^{k-\delta}}}{n} = kq^{(k-\delta)} \frac{\log q}{n}.$ \hspace{1cm} (43)

The resulting increase in the rate is thus

$\frac{r_I}{r_T} = \frac{kq^{k-\delta}}{k + 2} \approx q^{k-\delta} = \exp\left((k - \delta)n \frac{\log q}{n}\right) = \exp\left((k - \delta)n r_T\right)$ \hspace{1cm} (44)

The rate of a double Reed-Solomon code construction is thus exponential to the rate achieved with simple transmission.

The trade-off in the false alarm error introduced to achieve this rate is given by the distance

$d = (q - k + 1)(q^k - q^{k-\delta} + 1) \geq (q - k)(q^k - q^{k-\delta}) \geq q^{k+1} - kq^k - q^{k+1-\delta},$ \hspace{1cm} (45)

where we assumed that $k > 1$. This gives a maximum false alarm probability of

$\lambda_2 \leq 1 - \frac{d}{q^{k+1}} = 1 - \frac{q^{k+1} - kq^k - q^{k+1-\delta}}{q^{k+1}}$ \hspace{1cm} (46)

$= \frac{k}{q} + q^{-\delta} \in O\left(\frac{k}{q}\right).$ \hspace{1cm} (47)

And thus we can still achieve the exponential increase in messages while still sending the error to zero. Notice, however, that the scaling at which the error goes to zero is unchanged and arguably slow compared to the scaling of the amount of identification messages.

5. Implementation and Simulation

The major part of our contribution lies in the implementation of the double Reed-Solomon code construction using Sagemath[13]. Coding the steps as presented so far is an inefficient solution. In particular, producing the generator matrices is costly in memory and a waste of computation. Already at small parameters $(7, 5, 2)_{RS}$ the generator matrices become too large to handle. Even storing only one codeword /mapping function is prohibitive. The only advantage to such a solution would be the instant access to the precomputed tags which is nullified by the impossibility of taking advantage of the doubly exponential growth of the codewords.
Instead, the only way of taking advantage of the doubly exponential growth is to compute each tag on demand, using the Reed-Solomon codes as polynomial evaluations as per the initial definition (see section 2.6), and computing single symbols of each codeword at every transmission of an encoded identity. In the final implementation, a tag is computed as follows:

- Divide the coloring number $j$, ranging from 0 to $q^{k+1} - 1$ by $q$. The quotient $j \div q$ shows us which column of the generator matrix of $(q^k, q^{k-\delta})_{RS}$ we need to use. We don’t need the other columns to calculate the necessary tag. Using only that column, i.e. evaluating the message with that one code locator, we will get one symbol $\tilde{t}$ in the alphabet of size $q^k$. The remainder of the division, $j \mod q$ is used later.
- Expand the symbol $\tilde{t}$ into $k$ symbols of size $q$. The list of these $q$ elements will be called the expanded codeword.
- Find the column - or the code locator - in the generator matrix of $(q,k)_{RS}$ with the index as the remainder of the division in the first step and multiply the expanded codeword with that column scalarly, or simply evaluate it with the picked code locator.

The result of this last evaluation gives us the necessary tag. This method saves us a lot of memory compared to using the generator matrices. The bottleneck of the implementation at this point is listing the elements of the extension field $F_{q^k}$, which renders parameters of the order $(11,8,4)_{RS^2}$ again intractable. Ideally we would want to index the desired field element as a parameter of the field and obtain the desired element. This technique works on prime fields $F_q$, but not on extension fields $F_{q^k}$. On prime fields $F_q(129)$ returns $129 \mod q$, which is correct, however on extension fields $F_{q^k}(129)$ also returns $129 \mod q$, which is not correct for $q \leq 129$.

Further improvements were obtained by changing the way field elements are generated. The "next()" method of the field class does what its name suggests: it gives the next field element. In this way, we can generate the elements by sequentially producing the next element from the zero element. If, for example, we wanted the $10234^{\text{th}}$ element of the field, we would start from the $0^{\text{th}}$ element of the field and using the "next()" method 10234 times to get to the $10234^{\text{th}}$ element of the field. With this method, the computation time of an element remains feasible up to elements of the order $10^8$, with elements larger that $10^9$ taking hours of computation time. A final improvement was made again, changing the generations method. In Section 2, we see that the primitive element of a field can create the field from scratch by its exponents, where 0 and $a^0$ would be the first two elements and $a^0 + 1$ would be the last element, if we are speaking of the field $F_{q^k}$. Generating the elements with this method resulted in several order of magnitude of improvement in the computation time. With this final method, the boundary of the feasible computations lay at parameters $(17,12,6)_{RS^2}$ for which the computation of a single tag took $\sim 1.5$ hours.

At this point the achieved number of identities are far beyond the ones achievable with transmission, however the maximum false alarm ratio $\lambda_2 \approx k/q = 12/17 \approx 60\%$ is too high to be acceptable. In order, to bring the error down, the base field size $q$ must grow much larger than the parameter $k$. In the final simulations we decided to keep fixed $k = 3$ and $\delta = 2$ and simply increase the field size $q$, studying the performance of only $(q,3,2)_{RS^2}$ codes. Recall that the increase in rate is

\[
\frac{r_{ID}}{r_T} = \exp((k-\delta)n_T).
\]  

(48)

By fixing the parameter $k$ we did not achieve identification capacity but we still hit an exponential increase in rate compared to transmission. This also allowed for a vast increase in the range of computable values for $q$, and reduced the false alarm error as rapidly as we could increase $q$. The limiting value in this regime is $q \approx 10^8$ which provides a false alarm error of $\lambda_2 = 10^{-8}$, at the cost of $\approx 2.5$ hours required to calculate one tag. In the next section we present these results in detail.
6. Results

We used 3 for $k$ and 2 for $\delta$ at all times in order for $q$ to grow larger than $k$. To reach channel capacity $k$ would also need to grow, however, making both $q$ and $k$ grow together makes the simulation prohibitive, requiring a much greater RAM in the simulation computer.

Regardless of the algorithm used, the time requirement decreases with the complexity, in turn increasing the error of the second kind $\lambda_2$. If a false alarm error lower than $10^{-5}$ is needed, then with our algorithm this implies a calculation time of at least 1s. And vice versa; requiring the calculation time to be under 1ms, we must allow a smaller number of identities and a false alarm error of at least 10%. These results are displayed in fig. 7. Simply said, complexity reduces the maximum false alarm probability at the cost of calculation time. Note that this calculation time involves everything from building the system with the selected parameters, picking a random identity among all identities and picking a random coloring number to calculate the corresponding tag.

A similar trade-off exists with the number of identification messages. We saw that there is an exponential relationship between the calculation time of one tag and the total number of identities. When we have a tenfold calculation time of one tag, the exponent of the number of identities becomes tenfold.
Figure 8. The relationship between the number of identities and the calculation time of one tag.
For examples, the reader can see Figure 8. These results\(^5\) show us that we can increase the total number of identities exponentially if we can endure a linear increase of the calculation time, depending on the applications.

Another simulation yielded us the relationship between the total number of identities and the error of the second kind. The thing to notice here, is that when the \(q\) grows ten-fold, the error of the second kind becomes 10% of the previous value. This is no surprise, since we fixed \(k\) to 3 and \(\delta\) to 2:

\(^5\) Note that both in Figure 8 and 9, the vertical axis expresses the common logarithm of number of identities, but not the number of identities themselves. This was the only way we could fit a double exponential growth in a plot.
As we see in Equation 56, for large $q$, the error of the second kind equals, in our example set with $(q, 3, 2)^2_{RS}$, approximately $\frac{1}{q}$, which is why the error of the second kind $\lambda_2$ and $q$ seem inversely proportional. They are indeed inversely proportional. This brings us to the conclusion that lowering the error probability is expensive and the user of the system needs to prioritize between a slow but mostly accurate system, a fast system which has unneglectable errors, or a balanced system. Our simulation results can be seen in Figure 9.

We also ran a different simulation: we fix a randomly chosen identity and one of its coloring numbers (again randomly chosen), then we start to pick different identities at random and compare their tags in
the same position with the tag of our original identity at the fixed position. We repeat this comparison
with 1000 different secondary identities and set the said “false alarm ratio” as the number of recognized
false alarms over 1000 iterations. This simulation can be seen in Figure 10. It is assuring that no false
alarm ratio in the simulation exceeds the theoretical limits.$^6$

One could make two remarks here. First, that the simulated false alarm ratio values are
approximately inversely proportional to $q$. This is no surprise, since the originally-chosen tag is a fixed
number from the set $\{0, 1, 2, \ldots, q - 1\}$. And yet, the secondary identity’s tag at the same position will also
be another number from the same set $\{0, 1, 2, \ldots, q - 1\}$. The broad probability that a position possesses a
certain value is therefore $\frac{1}{q}$, assuming that each possible tag comes up almost the same number of times,
which also seems like the case in this last simulation of ours. So it is no coincidence that we meet the
same number in the same position of other identities with the probability $\frac{1}{q}$.

The second remark, which is the most important aspect here, is that the simulated false alarm ratio
is an average error probability, unlike this paper focusing on the maximum error probability. As the
reader can see, the average error probability can make the user relax, but not reliably. The average error
probability is perhaps lower than the maximum error probability, but it gives you no assurance against
local errors which can happen anywhere and any time, causing a disruption in communication and thus
caus[...]

7. Conclusions

This paper shows that identification can efficiently reach many more messages than transmission at
the cost of a manageable additional error and the inability to decode all messages. We can see from the
results, that making the maximum probability of the error of the second kind $\lambda_2$ smaller also means a
trade-off in the computation time, which contributes to the latency of the scheme. So if one wants smaller
ersors, our paper shows that one needs a more powerful system to calculate the tags under an acceptable
latency.

For future works on identification using tag codes, adding some security element into play is
important to protect the information from being eavesdropped. The benefit of protecting a tag code
is that we only need to protect the tag part of the codeword, because the previous part, e.g. the coloring
number part is created by local randomness, so it carries no useful information at all. Even though the
eavesdropper has all of the mappings of each identity’s mapping functions, it won’t help them. Let’s say
the local randomness decided on the coloring number $j$. Every mapping function has $j$ in the input set,
so it won’t increase the success probability of the eavesdropper.

Furthermore, most of the channel capacity in an identification code is used to produce common
randomness on which to compute the tag. In systems where the common randomness is given as
a resource, or can be synchronized and produced on demand by other means, the resulting uses of
the channel diminish drastically. Then, rather than increasing the size of the messages, an advantage
can be achieved by keeping the number of messages comparable and instead exponentially or even
doubly exponentially reducing the number of channel uses. In such a scenario we can think of
identification as converting the latency of communication into latency of computation, which might
become advantageous in applications where the increased number of messages is not the main interest.

$^6$ Note that in Figure 10, the 3 in the horizontal axis corresponds to a parameter set of $(3, 2, 1)_{R \in \mathbb{R}}$ exceptionally.
Acknowledgements

Christian Deppe and Roberto Ferrara were supported by the Bundesministerium für Bildung und Forschung (BMBF) through the grant 16KIS1005.

References

1. Shannon, C.E. A mathematical theory of communication. *Bell System Technical Journal* 1948, 27, 379–423, 623–656.
2. Ahlswede, R.; Dueck, G. Identification via channels. *IEEE Transactions on Information Theory* 1989, 35, 15–29.
3. Boche, H.; Deppe, C. Secure Identification for Wiretap Channels; Robustness, Super-Additivity and Continuity. *IEEE Transactions on Information Forensics and Security* 2018, 13, 1641–1655.
4. Ahlswede, R.; Dueck, G. Identification in the presence of feedback—a discovery of new capacity formulas. *IEEE Transactions on Information Theory* 1989, 35, 30–36.
5. Ahlswede, R.; Althöfer, I.; Deppe, C.; Tamm, U.; (Eds.). *Identification and Other Probabilistic Models, Rudolf Ahlswede's Lectures on Information Theory 6*, 1st edition, to appear ed.; Springer-Verlag, 2020.
6. Ahlswede, R.; Wolfowitz, J. The structure of capacity functions for compound channels. Probability and Information Theory; Behara, M.; Krickeberg, K.; Wolfowitz, J., Eds.; Springer Berlin Heidelberg: Berlin, Heidelberg, 1969; pp. 12–54.
7. Ahlswede, R.; Zhang, Z. New directions in the theory of identification via channels. *IEEE Transactions on Information Theory* 1995, 41, 1040–1050.
8. Roth, R. *Introduction to Coding Theory*; Cambridge University Press: USA, 2006.
9. Justesen, J.; Hoholdt, T. *A Course in Error-Correcting Codes (EMS Textbooks in Mathematics)*; European Mathematical Society, 2004.
10. Eswaran, K. Identification via Channels and Constant-Weight Codes, 2005.
11. Verdu, S.; Wei, V.K. Explicit construction of optimal constant-weight codes for identification via channels. *IEEE Transactions on Information Theory* 1993, 39, 36–36.
12. Han, T.S.; Verdu, S. New results in the theory of identification via channels. *IEEE Transactions on Information Theory* 1992, 38, 14–25.
13. The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.1)*, 2020. 
https://www.sagemath.org.