Global propagation of massive quantum fields in the plane gravitational waves and electromagnetic backgrounds

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Abstract
The behavior of massive quantum fields in the general plane wave spacetime and external, non-plane, electromagnetic waves is studied. The asymptotic conditions, the ‘in’ (‘out’) states and the cross sections are analysed. It is observed that, despite of the singularities encountered, the global form of these states can be obtained: at the singular points the Dirac delta-like behavior emerges and there is a discrete change of phase of the wave function after passing through each singular point. The relations between these phase corrections and local charts are discussed. Some examples of waves of infinite range (including the circularly polarized ones) are presented for which the explicit form of solutions can be obtained. All these results concern both the scalar as well as spin one-half fields; in latter case the change of the spin polarization after the general sandwich wave has passed is studied.

Keywords: plane gravitational waves, exact solutions, electromagnetic backgrounds, quantum fields

1. Introduction

The interaction of quantum fields with the gravitational backgrounds has been studied intensively for various spacetimes for years (let us only mention a few monographs [1–4]). Recently, such investigations concerning the gravitational waves became particularly interesting due to direct observations of the latter from a pair of merging black holes [5] or colliding neutron stars [6]; for the latter case the electromagnetic waves (backgrounds) as well as spin aspects

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seem also to be important. Although the analysis of the quantum phenomena for the general spacetime (in the presence of electromagnetic fields) is very difficult there are some relatively simple configurations of fields which can capture some relevant features of the interaction. For the gravitational waves such models are provided by the plane gravitational waves, which are exact solutions to the Einstein equations. Of course, such models possess physical limitations; however, they can be useful in the analysis of strong or short time processes which can occur in many astrophysical and cosmological phenomena providing there, at least, local or qualitative descriptions.

The analysis of propagation of quantum fields in the plane gravitational wave goes back to the pioneering papers [7, 8], where some basic properties of the scalar fields were established for the plane sandwich waves (e.g. no particles creation and vacuum polarization). Then, these investigations were extended in reference [9] to scattering processes in the fine-tuned (i.e. with special compact supports) plane sandwich waves. Similar results for massive spin one-half fields, but only for linearly polarized sandwiches, were obtained in [10] (see also [11]). Although the plane gravitational waves are relatively simple they exhibit some surprising properties, e.g. non-existence of a spacelike Cauchy hypersurface; when colliding they can produce curvature singularities. Moreover, in spite of the fact that the classical motion in these spacetimes is completely regular the basic wave functions are singular. This fact was related, for compactly supported waves [7, 9], to the focusing behavior of the classical trajectories.

On the other hand, it turns out that a similar situation occurs for some electromagnetic fields in the Minkowski spacetime which belong to the class of the so-called crossed fields (in the terminology of reference [12]). Since these fields include non-plane electromagnetic waves they are interesting in the context of singular optics and ultra intense laser beams (where the plane wave approximation or even paraxial one is not sufficient). The similarity between both the fields is even more evident if one takes into account the double copy conjecture, a part of the colour-kinematic duality which on the quantum level concerns the problem of how scattering amplitudes in gravity can be obtained from those in the gauge theory (see among others [13–18]). In view of this and the above mentioned astrophysical phenomena we find interesting to investigate the behavior of the basic wave functions around the singular points (in consequence, their global form) simultaneously for the gravitational and electromagnet fields (see section 3); of course, both the fields can be considered separately, i.e. gravitational or only electromagnetic fields in the Minkowski spacetime. To make our considerations as general as possible, first, in section 2 we discuss asymptotic conditions which allow to define properly the ‘in’ and ‘out’ states (and, consequently, to analyse the scattering processes, see section 2.3) for arbitrary (not only sandwich) profiles. Moreover, since our aim is to find the global form of states we work in the Brinkmann (B) coordinates; however, to make contact with various results in the literature and the weak field approach in the last part of section 2 we refer to the Baldwin–Jeffery–Rosen (BJR) coordinates which are local because of the above mentioned singular points.

Since the interaction of the scalar fields with plane wave spacetimes was explicitly discussed mainly for the standard (defined by characteristic functions) sandwiches, in section 4 we give some examples of wave pulses of infinite range for which the wave functions and the cross sections can be explicitly written down; these examples include circularly polarized waves (which seem physically more interesting [5, 6, 19, 20]) and in a suitable limit they reduce to impulsive waves [21]. Finally, in section 5 we consider all the above issues for the spin one-half fields; moreover, we discuss the change of the spin polarization after the passage of the general sandwich wave (for standard linearly polarized sandwich waves this fact was discussed,
by means of the Newman–Penrose framework, in reference [11]). Conclusions and possible further directions of investigations are presented in section 6.

2. Asymptotic conditions and scattering

2.1. General discussion

The dynamic of the massive scalar field $\Phi$ in the gravitational $g_{\mu\nu}$ and electromagnetic $A_\mu$ backgrounds is described by the Klein–Gordon (K–G) equation

$$D_\mu D^\mu \Phi - m^2 \Phi = 0,$$

(2.1)

where $D_\mu = \nabla_\mu - i e A_\mu$, and $\nabla$ is the Levi-Civita connection of the metric $g$ (we skip here the term with the scalar curvature since it is zero for the metrics considered below). Equivalently, equation (2.1) can be rewritten in the form

$$\frac{1}{\sqrt{|\text{det}(g)|}} \tilde{D}_\alpha \left( \sqrt{|\text{det}(g)| g^{\alpha\beta} \tilde{D}_\beta \Phi} \right) - m^2 \Phi = 0,$$

(2.2)

with $\tilde{D}_\alpha = \partial_\alpha - i e A_\alpha$.

Let us consider two types of fields: the gravitational one defined by a pp-wave metric

$$g = K(u, x) du^2 + 2 du dv + dx^1 dx^2,$$

(2.3)

where $x$ refers to the $x^1, x^2$ coordinates, while the function $K = K(u, x)$ does not depend on the $v$ coordinate; the electromagnetic background is defined by the following potential

$$A_u = A(u, x), \quad A_v = 0, \quad A_i = 0, \quad i = 1, 2;$$

(2.4)

which also does not depend on $v$. It is well known that for the metric $g$ defined by (2.3) the scalar curvature vanishes and it describes a vacuum solution to the Einstein equations iff $\triangle \perp K \equiv \partial_i \partial_i K = 0$; similarly, one can check that the potential $A_\mu$ describes a vacuum solution of the Maxwell equations (also in the curved spacetime defined by $g$) when $\triangle \perp A = 0$ (note that this condition does not depend on the form of $K$, in particular putting $K = 0$ one obtains an electromagnetic wave in the Minkowski spacetime).

For such fields the coefficients in equation (2.1) do not depend on the $v$ coordinate, thus we look for solutions in the following form

$$\Phi(u, v, x) = e^{i k_v v} \Phi(u, x),$$

(2.5)

where $k_v$ is a non-zero constant related to the conservation law of the momentum in these backgrounds. Substituting (2.5) into equation (2.2) one arrives at the 'time'-dependent ($u$ plays the role of time) Schrödinger equation for the function $\Phi$

$$i \partial_u \Phi = -\frac{1}{2 k_v} \triangle \perp \Phi - \frac{k_v}{2} \left( K + \frac{2 e}{k_v} A \right) \Phi.$$

(2.6)

In order to analyse quantum processes we need the solutions of equation (2.6) as well as an inner product between the full states $\Phi^\dagger$'s; this is, in general, not an easy task.

\footnote{The signature is $(-, +, +, +)$ and the bold symbols refer to the two-dimensional vectors.}
The situation slightly simplifies under the assumption that both $A$ and $K$ are quadratic functions of $x$, i.e.

$$K(u, x) = x^T K(u)x, \quad (2.7)$$

and

$$A(u, x) = x^T A(u)x, \quad (2.8)$$

where $A(u)$ and $H(u)$ are continuous matrix functions (without losing generality assumed to be symmetric ones). Recall also here that the coordinates for which the metric $g$ is of the form (2.3) with (2.7) are called the B ones (see, e.g. [9, 22] and references therein as well as section 3.3).

For such a choice of $A$ and $K$ equation (2.6) reduces to, in general anisotropic and time dependent, quantum linear oscillator which has been intensively studied since the classical papers [23, 24]. However, as we mentioned above, equation (2.6) is an auxiliary one and in order to analyse the processes in such backgrounds one needs an inner product between $\Phi$'s. This is a subtle point since the spacetime $g$ does not possess a Cauchy surface (except the Minkowski case, $K = 0$). However, it turns out [7, 9] that the null surface $u = \text{const}$ can be a good substitute for it; then the inner product can be defined as follows

$$\langle \Phi_1 | \Phi_2 \rangle = -i \int_{\mathbb{R}^3} (\Phi_1 \partial_v \Phi_2^* - \Phi_2 \partial_v \Phi_1^*) dv dx. \quad (2.9)$$

Since $A_v = 0$ the above inner product does not contain electromagnetic potential. Moreover, even for the Minkowski spacetime the inner product (2.9) is related to a null hypersurface; despite this fact it is physically interesting due to its usefulness in the intensive studies of the electromagnetic beams (see [12] and references therein). Finally, let us note that the choice of the gravitational and electromagnetic potentials (2.7) and (2.8) includes many interesting cases, e.g. the exact gravitational waves, $\text{tr}(K) = 0$, and non-plane electromagnetic waves, $\text{tr}(A) = 0$ (although we do not exclude the null fluid solutions\footnote{Sometimes the metric $g$ with an arbitrary $K$ is called the generalized plane gravitational wave.} in our considerations).

From the above we see that in order to analyse the K–G equation (2.1), with the fields defined by (2.7) and (2.8), one should find solutions to equation (2.6). To this end, let us consider the matrix function $P$ which is a solution of the following system of the linear differential equations

$$P = \left( K + \frac{2e}{k_v} A \right) P, \quad (2.10)$$

where dot refers to the derivative w.r.t. to $u$ (let us stress that in the presence of the electromagnetic potential $P$ depends on $k_v$; however, to simplify our notation we skip this subscript). A few remarks are in order. First, one can impose a condition on the solutions of (2.10), namely

$$\dot{p}^T P = P^T \dot{p}. \quad (2.11)$$

Next, following the reasoning presented in reference [9] we conclude that when $\text{tr}(K + \frac{2e}{k_v} A) \leq 0$ and $K + \frac{2e}{k_v} A \neq 0$ (i.e. except the trivial case) there exists at least one point (the so-called singular point) where the determinant of the matrix $P$ vanishes (note that when

\footnote{In many places this assumption can be relaxed to include discontinuous sandwich functions.}
the electromagnetic field is turned on then this singular point depends on \( k_v \). Such a situation appears in the most interesting case when \( g \) is an exact gravitational wave and the electromagnetic field is a non-plane electromagnetic wave; another example of such a situation is provided by \( A = 0 \) and \( g \) satisfying the weak energy condition.

Let us now define the antiderivative matrix \( S \) as \( S = (P^TP)^{-1} \); then the field

\[
\Phi^{\kappa,k}(u, x) = \frac{1}{\sqrt{2|\kappa_i|(2\pi)^3|\det(P(u))|}} \times e^{i\tilde{P}(u)\left[\frac{1}{2}x^TP(x)P^{-1}(u)x - \frac{1}{2}\kappa^TS(0)|k\right]},
\]

(2.12)
depending on the constants \( k_v, \kappa \), satisfies equation (2.6) and leads, by virtue of equation (2.5), to the solution \( \Phi^{\kappa,k} \) of the K–G equation (2.1); at least for all \( u \) where \( P \) is invertible (say, before the singular point). At the singular point we have three not well defined terms in (2.12): \( \det(P) \) in the denominator, \( P^{-1} \) and \( S \) in the exponential factor. However, we will show below that all together combine to a correct, in the distributional sense, behavior of the state (2.12) as \( u \) tends to the singular point; namely, one obtains the one or two-dimensional Dirac delta function (depending on the rank of \( P \) at the singular point). In view of this and the fact that the field defined by the right-hand side of (2.12) is a solution to the K–G equation in all nonsingular intervals, one can try to treat the field \( \Phi^{\kappa,k} \) globally (i.e. for all real \( u \)). Then, however, a problem arises since the antiderivative \( S \) is not uniquely determined (there can appear a constant matrix because of the disconnected domain); thus after passing through the singular point an arbitrary phase factor can appear. In consequence, the behavior of \( \Phi^{\kappa,k} \) around the singular points calls for a careful analysis. In what follows we show that beyond the singular point the dynamic of the states is uniquely determined, though there appears a discrete phase correction (depending on the rank of \( P \) at the singular point only). Such a jump of the phase is similar to the one appearing for the propagator (Green function) passing through a caustic point; for example, it has been extensively studied for the harmonic oscillator [25, 26], as well as encountered in various optical, molecular, nuclear contexts (see [27] and references therein). Such an effect has been also noted for the Green functions [28–30] of the plane gravitational waves for which geodesics exhibit conjugated points (caustics); in consequence, some very interesting results concerning the vacuum polarization and refractive index were obtained [28–32]. It is worth to notice that for the states under considerations the initial conditions, see equation (2.13) below, are different than the ones encountered for the propagator (Green functions); cf equation (3.17) and the further discussion in section 3.2. Moreover, the conjugated points are not generic for the vacuum solutions which are our main interest.

Concluding these preliminaries let us note that the fields \( \Phi^{\kappa,k} \) where \( k_v \in \mathbb{R} \setminus \{0\}, \kappa \in \mathbb{R}^2 \) form, for a fixed \( u \), a complete and orthonormal set of functions with respect to the inner product (2.9) (including the electromagnetic case when \( P \) depends on \( k_v \)). In our convention, the states with \( k_v > 0 \) have negative norms and correspond to antiparticle states (cf (2.21)); moreover, they are orthogonal to the ones with \( k_v < 0 \). When we consider the gravitational backgrounds only (without the electromagnetic fields) then \( P \) does not depend on \( k_v \) and one can replace \( \Phi^{\kappa,k} \) with \( k_v > 0 \) by \( \Phi^{\kappa,k^*} \) with \( k_v < 0 \).

2.2. Asymptotic conditions

In order to analyse the interaction of scalar fields with the gravitational as well as electromagnetic backgrounds (defined by (2.7) and (2.8)) we start with the states which form at minus (plus) null infinity the plane wave solutions of the K–G equation. Since we do not restrict ourselves to the backgrounds which are equal to zero for sufficiently large \(|u|\), i.e. with compact
Applying lemma 1.2 from reference [33] (which generalizes theorem 2 in [34]) we conclude that for the functions\(^4\) concerning the profiles are needed; namely, the functions need to be sufficiently small in a sufficiently small neighborhood). To enforce the conditions (2.14) slightly stronger assumptions concerning the profiles are needed; namely, the functions \(u^2\|K(u)\|\) and \(u^2\|\lambda(A(u))\|\) should belong to the \(L^1(\mathbb{R})\) space\(^5\). In view of this we have two possibilities: either

\[
\lim_{u \to \pm \infty} u^2 K(u) = 0, \quad \lim_{u \to \pm \infty} u^2 \lambda(A(u)) = 0;
\]

(2.15)

or the limits on the left-hand side of (2.15) do not exist. In what follows we will focus on the first case, since the integrable functions for which the above limits do not exist have rather peculiar behavior at infinities, especially if we take into account that they should be related to the gravitational and electromagnetic fields. Moreover, they can lead to the states for which the asymptotic form does not coincide with the one for the Minkowski spacetime\(^6\). These peculiarities can be seen also at the classical level. Namely, it turns out that the geodesic equation (or the Lorentz equation) for the longitudinal coordinate \(v\) in the discussed backgrounds can be once integrated yielding \(\dot{v} = -\frac{1}{2}x^2 - \frac{1}{2}x^2 Kx + c_0\). On the other hand, \(\dot{x}\) tends to a constant while \(x\) is (in general) proportional to \(u\) when \(u\) tends to infinities. In consequence, \(\lim_{u \to \infty} \dot{v}(u) \simeq -\frac{1}{2} \lim_{u \to \infty} u^2 K(u) + \text{const}\) and the longitudinal velocity may be not well defined.

As we have noted above in many physically interesting cases there appear singular points where the determinants of \(P_{in}\) and \(P_{out}\) vanish and consequently the behavior of the states should be carefully analysed. Let us first assume that there is only one point \(u_{in}\) such that \(\det(P_{in}(u_{in})) = 0\) and one point \(u_{out} < u_{in}\) such that \(\det(P_{out}(u_{out})) = 0\) (some criteria enforcing such a situation can be found in [35]); the case of more singular points will be discussed latter (cf section 3.2). In such a case for \(u \in (u_{out}, u_{in})\) there are two orthonormal sets of the states determined by \(\Phi^{L,k}_{in}\) and \(\Phi^{L,l}_{out}\).

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\(^4\) We do not specify the matrix norm, one can take e.g. the Frobenius norm.

\(^5\) For the diagonal profiles this fact follows from theorem 3 [34] (see also the third footnote therein); however, the proof of this theorem can be directly extended to the arbitrary case—in the same way as theorem 2 in [34] is extended by lemma 1.2 in [33].

\(^6\) Namely, instead of equation (2.19) we have \(W = P_{in}(\infty) - \lim_{u \to \infty} \dot{\Phi}^{L}_{out}(u) P_{in}(u)\). Although the right-hand side is well defined, it is difficult to check whether \(W\) is invertible; moreover, even if \(W\) were invertible then the asymptotic form of the state would be different from (2.21).
Before we proceed let us make some useful observations. To this end let us define\(^7\)
\[ W \equiv W(P_{\text{in}}, P_{\text{out}}) = P_{\text{out}}^T \dot{P}_{\text{in}} - \dot{P}_{\text{out}}^T P_{\text{in}}. \] (2.16)
Then \(W = 0\), which implies that \(W\) is a constant matrix and, by virtue of (2.11), the following relation hold for \(u > u_{\text{out}}\)
\[ (S_{\text{out}}(u) - S_{\text{out}}(u_0))W = P_{\text{out}}^{-1}(u)P_{\text{in}}(u) - P_{\text{out}}^{-1}(u_0)P_{\text{in}}(u_0), \] (2.17)
as well as for \(u < u_{\text{in}}\)
\[ (S_{\text{in}}(u) - S_{\text{in}}(u_0))W = -P_{\text{in}}^{-1}(u)P_{\text{out}}(u) + P_{\text{in}}^{-1}(u_0)P_{\text{out}}(u_0), \] (2.18)
where \(u_0\) is such that \(u_{\text{in}} < u_0 < u_{\text{out}}\). Next, let us note that
\[ W = P_{\text{in}}(\infty), \quad W^T = -\dot{P}_{\text{out}}(-\infty). \] (2.19)
Indeed, by virtue of (2.13)–(2.15) and L’Hospital’s rule we have
\[ \lim_{u \to \infty} \dot{P}_{\text{out}}^T(u)P_{\text{in}}(u) = \lim_{u \to \infty} u\dot{P}_{\text{out}}^T(u)P_{\text{in}} = -\lim_{u \to \infty} u^2 \left( K(u) + \frac{2e}{k_0}A(u) \right) P_1 = 0, \] (2.20)
and similarly for the second case in equation (2.19). In what follows we assume that \(W\) is an invertible matrix; this assumption has a physical interpretation that the final momenta are not linearly dependent and the cross section is well defined, see below (in particular, the final momenta cannot vanish after scattering).

Now, we are in the position to analyse the asymptotic behavior of \(\Phi_{\text{in}}^{k, k}\) defined by \(P_{\text{in}}\) near minus infinity. To this end let us note that by virtue of (2.14), (2.18) and (2.19) the antiderivative behaves at minus infinity as \(S_{\text{in}}(u) \simeq uI + C_{\text{in}}\) where \(C_{\text{in}}\) is a constant matrix. However, the matrix \(C_{\text{in}}\) can be eliminated by a suitable choice of \(S\) (since the constant in the antiderivative had not been used yet). In consequence, with such a choice of \(S\) one has the following asymptotic behavior of the ‘in’ states near minus infinity
\[ \Phi_{\text{in}}^{k, k}(u, v, x) \sim e^{-\frac{\sqrt{2}}{2}u \sqrt{x^3 + 4k_3v + 8k_1^2x - 4\sqrt{2}k_0k_1u}} = e^{i(\sqrt{2}x - k_3u)}, \] (2.21)
in our convention, for the Minkowski spacetime, we have the identifications \(u = \frac{1}{\sqrt{2}}(x^3 - x^0)\) and \(k_3 = \frac{\sqrt{2}}{2}(k_3 - k_0)\); moreover, \(k_3 < 0\) iff \(k_0 > 0\). Let us stress that in order to obtain (2.21) not only (2.13) but also the condition (2.15) was used (this coincides with the classical straight-line geodesics analysis). The same analysis can be performed for the states \(\Phi_{\text{out}}^{k, k}\), defined by \(P_{\text{out}}\), at plus infinity.

Moreover, for further purposes let us analyse the behavior of the matrix function \(P_{\text{in}}(u)\) near the singular point (the reasoning presented in this paragraph is similar to the one in reference \[30\]; however, we do not restrict ourselves to the caustic points and the initial conditions are different). First, let us consider the degenerate case when \(P_{\text{in}}(u_{\text{in}}) = 0\) i.e. the rank is zero. Then, in a small neighborhood of \(u_{\text{in}}\) we have \(P_{\text{in}}(u) \simeq (u - u_{\text{in}})P_{\text{in}}(u_{\text{in}})\). In consequence, to the lowest order \(\det(P_{\text{in}}(u)) \simeq (u - u_{\text{in}})^2 \det P_{\text{in}}(u_{\text{in}})\). On the other hand
\[ W = P_{\text{out}}^T(u_{\text{in}})\dot{P}_{\text{in}}(u_{\text{in}}), \] (2.22)
\footnote{\(W\) can be defined for arbitrary two solutions of equation (2.10).}
thus due to invertibility of $W$ we find that $\det \hat{P}_{in}(u_{in}) \neq 0$ and $\det P_{in}(u)$ does not change the sign after passing through the point $u_{in}$. The situation is quite different if the rank of the matrix $P_{in}(u_{in})$ is one (for example, this holds for the vacuum linearly polarized waves with non-vanishing profiles; in fact, one component of $P_{in}$ is concave while the second one is convex before $u_{in}$, see equation (2.10), thus only one component of $P_{in}$ can vanish at $u_{in}$). Expanding the function $\det(P_{in}(u))$ we arrive, after some computations, at the first order equality

$$\det(P_{in}(u)) \simeq (u - u_{in}) \text{tr}(\hat{P}_{in}^{-1}(u_{in})P_{in}(u_{in})) \det(\hat{P}_{in}(u_{in})).$$

(2.23)

Thus making non-restrictive assumption$^8$ that $\det(\hat{P}_{in}(u_{in})) \neq 0$ (which holds, for example, in the above mentioned case of linearly polarized waves) we find that the rank of the matrix under the trace is also one. Moreover, by virtue of (2.11) it is a symmetric matrix; in consequence, it has the non-vanishing trace. Concluding, in this case $\det(P_{in}(u))$ changes the sign after passing through the point $u_{in}$. The above results will be useful in what follows.

2.3. The quantum scattering

Let us analyse the scattering of the scalar particle states in the discussed backgrounds (in particular, the quantum cross section). To this end we compute the transition amplitude between ‘in’ and ‘out’ states. We will use the particular, the quantum cross section). To this end we compute the transition amplitude between ‘in’ and ‘out’ states. We will use the general continuous pulses the point $u_0$ cannot be chosen in the region where $P_{in}$ is the identity matrix, cf reference [9]). First, we note that $\langle \Phi_{in}^{k_1} | \Phi_{out}^{l_1} \rangle = 0$ for $k_1 < 0$ and $l_1 > 0$, i.e. ‘in’ vacuum and ‘out’ vacuum can be identified, since the positive-energy ‘in’ modes do not develop negative-energy ‘out’ modes parts; there is no particles creation (this holds for the massive case $m \neq 0$; in the massless case the situation may be different, see [36, 37] and references therein). Next, for $k_1, l_1 < 0$ we find

$$\langle \Phi_{in}^{k_1} | \Phi_{out}^{l_1} \rangle = \delta(k_1 - l_1) \sqrt{\det(P_{in}(u_{in}))} \sqrt{\det(P_{out}(u_{in}))} \int e^{i T M(u_0) y + it^T y} dy,$$

(2.24)

where $z = k - P_{in}^T(u_0)(P_{out}^T(u_0))^{-1}$ and the matrix $M(u)$ defined for $u > u_{out}$ by

$$M(u) = \frac{k}{2}(P_{out}^{-1}(u)P_{in}(u))^T W,$$

(2.25)

is a symmetric one, $M^T = M$. Using the well known formula valid for a nonsingular$^9$ and symmetric two-dimensional matrix

$$\int e^{y^T M y + it^T y} dy = \frac{\pi}{\sqrt{|\text{det}(M)|}} e^{\frac{1}{4} \text{sign}(M) t^T M^{-1} t},$$

(2.26)

where $\text{sign}(M)$ is the signature of $M$ (the number of positive eigenvalues minus the number of negative eigenvalues) one obtains

$^8$ The singular points of $P_{in}$ and $\hat{P}_{in}$ are isolated.

$^9$ For the singular matrix $M$ the right-hand side of equation (2.26) reduces to the one or two dimensional (depending on the rank of $M$) Dirac delta function multiplied by suitable coefficients.
\[ \langle \Phi_{\text{in}}^{k, i} | \Phi_{\text{out}}^{k, i} \rangle (u_0) = \frac{\delta(k_\nu - l_\nu)}{2\pi |k_\nu| \sqrt{|\det(W)|}} e^{i \text{sign}(M(u_0))} \frac{1}{2} \frac{\pi}{k_\nu} \frac{k_\nu}{2} M^{-1}(u_0) u_\nu e^{-\frac{i}{2} \pi T^{-1}(u_0) M^{-1}(u_0) \pi T u_\nu} \chi_{\text{in, out}}(u_0 | k_\nu). \] (2.27)

Equation (2.27) simplifies when \( K \) and \( A \) are even matrix functions and \( u_0 = 0 \). Then

\[ -\frac{i}{4} \pi T M(0)^{-1} z = \frac{-i}{2k_\nu} (k - l)^T W^{-1} (k - l), \] (2.28)

and \( W = 2 P_{\text{in}}^T(0) \tilde{P}_{\text{in}}(0) \); moreover, \( S_{\text{out}}(0) = -S_{\text{in}}(0) \).

It remains to find the signature of \( M(u_0) \). Since the behavior of \( M \) near plus infinity is of the form \( M(u) \sim \frac{1}{u} W^T W \) one obtains (for the particle state, i.e. \( k_\nu < 0 \) \text{sign}(M(u)) = -2 \) for \( u > u_{\text{in}} \). Let us consider two cases. First, the degenerate case, i.e. \( P_{\text{in}}(u_{\text{in}}) = 0 \); then the rank \( M(u_{\text{in}}) \) is zero and expanding it in a neighborhood of \( u_{\text{in}} \), we have \( M(u) \sim \frac{1}{u} (u - u_{\text{in}}) \tilde{P}_{\text{in}}^T(u_{\text{in}}) \tilde{P}_{\text{in}}(u_{\text{in}}) \).

On the other hand taking into account that \( W \) is invertible we get \( \det(\tilde{P}_{\text{in}}(u_{\text{in}})) \neq 0 \); thus both the eigenvalues of \( M(u) \) change the sign after passing through \( u_{\text{in}} \). In consequence, \text{sign}(M(u_{\text{in}})) = 2 \) and an additional factor in (2.27) emerges. Next, let us consider the case when the rank of \( P_{\text{in}}(u_{\text{in}}) \) is one. Then, by virtue of our previous considerations, we can assume that the determinant of \( P_{\text{in}}(u) \), and consequently of \( M(u) \), changes the sign at \( u_{\text{in}} \). In consequence, only one eigenvalue function of \( M(u) \) becomes positive for \( u < u_{\text{in}} \), implying that \text{sign}(M(u_{\text{in}})) = 0 \). For further use let us also note that for \( u > u_{\text{in}} \) the difference \text{sign}(M(u_{\text{in}})) - \text{sign}(M(u)) \) equals 2 (or 4) if the rank of \( P_{\text{in}}(u_{\text{in}}) \) is one (or zero).

In view of the above we can easily find the transition amplitude from an ‘in’ one-particle state with \( (k_\nu, \mathbf{k}) \) to an ‘out’ one-particle state \( (l_\nu, \mathbf{l}) \). In fact, since there is no particle creation both vacua can be identified, and the following relation holds

\[ \langle \text{out}, l_\nu, \mathbf{l} | k_\nu, \mathbf{k}, \text{in} \rangle = \langle \Phi_{\text{in}}^{k, i} | \Phi_{\text{out}}^{k, i} \rangle (u_0), \] (2.29)

thus

\[ |\langle \text{out}, l_\nu, \mathbf{l} | k_\nu, \mathbf{k}, \text{in} \rangle| = \frac{\delta(k_\nu - l_\nu)}{2\pi |k_\nu| \sqrt{|\det(W)|}}. \] (2.30)

Following the standard reasoning (and taking into account the normalization related to infinite range of the \( \nu \) coordinate, see reference [9]) we obtain the quantum cross section

\[ d\sigma = \frac{1}{L |\det(W)|} dl. \] (2.31)

Finally, let us note that, by virtue of equation (2.19), it coincides with the classical differential cross section obtained in [38] (in full agreement with the discussion presented in reference [9]).

3. Global states

3.1. The first approach

Now, let us analyse the behavior of the ‘in’ states at and beyond the singular point. In principle, the behavior of \( \Phi_{\text{in}} \)’s at \( u_{\text{in}} \) can be obtained by carefully taking the limit \( \lim_{u \to u_{\text{in}}} \Phi_{\text{in}}^{k, \mathbf{k}} \).

However, since our aim is to find the global form of the states (i.e. also beyond the singular point) such a reasoning will be considered later and then compared to our global result. To find
the global form of the ‘in’ states we follow two approaches. One is based on the local ‘out’
states (assuming as above that there is one singular point and \( u_{\text{out}} < u_{\text{in}} \); another approach,
based on the evolution operator, is discussed in the next subsection (it works also in the case
of more singular points).

The main idea of the first approach is that the form of the state \( \tilde{\Phi}_{\text{in}}^{k, x} \) (defined for \( u < u_{\text{in}} \))
at and beyond the singular point \( u_{\text{in}} \) can be obtained by means of the ‘out’ states which are
well defined around \( u_{\text{in}} \) (since \( u_{\text{out}} < u_{\text{in}} \), and form also an orthonormal basis. Namely, in this
approach the state \( \tilde{\Phi}_{\text{in}}^{k, x} \) is extended to \( \tilde{\Phi}_{\text{in}}^{k, x} \), for all \( u > u_{\text{in}} \), by the relation

\[
\tilde{\Phi}_{\text{in}}^{k, x}(u, v, x) = \int \Phi_{\text{out}}^{k, x}(u, v, x) \Phi_{\text{in}}^{k, x}(u_{\text{in}}) dl_d dl.
\]

Let us analyse the above solution of the K–G equation for the particle state \( k_\nu \leq 0 \). Substituting
(2.5), (2.12) and (2.27) into (3.1) we are faced with the following integral

\[
\tilde{\Phi}_{\text{in}}^{k, x}(u, v, x) = (\ldots) \int e^{\frac{i}{\hbar} T \left( W^{-1}(P_{\text{out}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T}(S_{\text{out}}(u_{\text{in}}) - S_{\text{out}}(u_{\text{out}})) \right) T \right)} dl_d dl.
\]

where \( (\ldots) \) denotes regular terms skipped in the intermediate steps. Using the identity (2.17),
the definition of \( z \) and making the substitution \( l = P_{\text{out}}^{-1}(u) q \) under integral we arrive at the expression

\[
\tilde{\Phi}_{\text{in}}^{k, x}(u, v, x) = (\ldots) |\det(P_{\text{out}}(u))| e^{\frac{i}{\hbar} k T W^{-1}(P_{\text{out}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T}(S_{\text{out}}(u_{\text{in}}) - S_{\text{out}}(u_{\text{out}})) \right) T} \cdot \int e^{i \left( \frac{i}{\hbar} P_{\text{out}}^{-1}(u_{\text{in}}) T \right)^{-1} k_\nu + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T} P_{\text{out}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T} x \right) q + \frac{i}{2} \Phi_{\text{out}}(u_{\text{in}}) q} dq.
\]

where the matrix \( M(u) = -\frac{i}{\hbar} P_{\text{out}}^{-1}(u) W^{-1} P_{\text{out}}^{-1}(u) \) is a symmetric one.

Now we are ready to analyse the state \( \tilde{\Phi}_{\text{in}} \). First, let us assume that \( u > u_{\text{out}} \) and \( \nu \neq u_{\text{in}} \). Then
\( M(u) \) is nonsingular and \( \tilde{M} = -\frac{i}{\hbar} P_{\text{in}} M^{-1} P_{\text{in}} \) thus \( \det(M) = -\det(M) \). Using these
facts and the formula (2.26) we find

\[
\tilde{\Phi}_{\text{in}}^{k, x}(u, v, x) = \frac{e^{\frac{i}{\hbar} T \left( \det(M(u)) - \det(M(u)) \right)}}{\sqrt{-2 k_\nu (2\pi)^{T} \left| \det(P_{\text{out}}(u_{\text{in}})) \right|}} \times e^{-\frac{i}{\hbar} T (S_{\text{out}}(u_{\text{in}}) - S_{\text{out}}(u_{\text{out}})) \left( \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T} x \right) \left( \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T} x \right)^{T}} \cdot \exp \left( \frac{i}{\hbar} k_{\nu} T \Phi_{\text{out}}(u_{\text{in}}) \right) \left( P_{\text{out}}^{-1} T(u_{\text{in}}) P_{\text{out}}(u_{\text{in}})^{T} P_{\text{out}}^{-1}(u_{\text{in}}) - P_{\text{out}}^{-1} T(u_{\text{in}}) P_{\text{out}}^{-1}(u_{\text{in}})^{T} x + \frac{i}{2} P_{\text{out}}^{-1}(u_{\text{in}})^{T} P_{\text{out}}^{-1}(u_{\text{in}}) x \right).
\]

It is worth to notice that the above formula does not depend on the choice of \( u_0 \), as one can
expect. Let us consider two cases.

First, assume that \( u \in (u_{\text{out}}, u_{\text{in}}) \). Then \( \det(M(u)) = \det(M(u_{\text{in}})) \); moreover, using (2.18)
we can eliminate \( P_{\text{out}} \) in favor of \( P_{\text{in}} \). As a result one obtains the expected relation
\( \tilde{\Phi}_{\text{in}}(u, v, x) = \Phi_{\text{in}}(u, v, x) \), i.e. \( \tilde{\Phi}_{\text{in}} \) is indeed an extension of the field \( \Phi_{\text{in}} \) defined for \( u < u_{\text{in}} \).

Next, let us analyse the case \( u > u_{\text{in}} \). The last exponential factor in (3.4), corresponding to \( \tilde{S}_{\text{in}} \), is uniquely determined. Moreover, since

\[
S_{\text{in}}(u) = W^{-1}(P_{\text{out}}^{-1}(u_{\text{in}})(P_{\text{in}}^{-1}(u_{\text{in}}) x + \frac{i}{2} P_{\text{in}}^{-1}(u_{\text{in}})^{T} x) \cdot P_{\text{out}}^{-1}(u)(P_{\text{out}}^{-1}(u) x + \frac{i}{2} P_{\text{out}}^{-1}(u) x)) + S_{\text{in}}(u_{\text{in}}),
\]

for \( u < u_{\text{in}} \), we can identify \( \tilde{S}_{\text{in}}(u) = \tilde{S}_{\text{in}}(u) \) also for \( u > u_{\text{in}} \)—despite of the disconnectedness of the domain there is no additional constant matrix in the antiderivative of \( (P_{\text{in}}^{-1} P_{\text{out}})^{-1} \).
In view of this equation (3.4) can be written as follows

\[ \tilde{\Phi}_m^{k,m}(u, v, x) = \frac{e^{i \frac{\pi}{2} \text{sign}(M(u_0)) - \text{sign}(M(u))}}{\sqrt{-4 k^2 \pi |\det(\tilde{P}_m(u_0))|}} \delta^{(2)}(x + \frac{1}{k^2} P_{\text{out}}(u_0)(W^T)^{-1}k), \]  

(3.6)

where \( \Phi_m^{k,m} \) is considered for \( u \neq u_0 \) (with the antiderivative \( S_m(u) \) given by the matrix function occurring on the right-hand side of equation (3.5)). In consequence, there is a discrete change of the phase beyond the singular point related only to the change of the signature of \( M \).

Namely, by virtue of our previous considerations (see the discussion below (2.28)) for \( u > u_m \) the sign(\( M(u_0) \)) - sign(\( M(u) \)) equals 2 (or 4) if the rank of \( P_m(u_m) \) is one (or zero). Thus, for the particle state \( k_c < 0 \), beyond the point \( u_m \) the additional factor \( i \) (or \(-i\), respectively) appears.

Finally, let us consider the behavior of \( \tilde{\Phi}_m^{k,m} \) at \( u_m \). At this point the rank of \( P_m(u_m) \) is zero or one. Let us start with the case \( P_m(u_m) = 0 \). Then the second term under the integral in (3.3) drops out and using (2.22) we arrive, after some computations, at the Dirac delta function

\[ \tilde{\Phi}_m^{k,m}(u_m, v) = \frac{i e^{-i k^2 + i k \cdot v}}{\sqrt{-4 k^2 \pi |\det(\tilde{P}_m(u_m))|}} \delta^{(2)}(x + \frac{1}{k^2} P_{\text{out}}(u_m)(W^T)^{-1}k), \]  

(3.7)

note that (3.7) does not depend on the choice of \( u_0 < u_m \). Now, let us consider the case when the rank of \( P_m(u_m) \) is one. Since the matrix \( P_m W^{-1} P_{\text{out}}^T \) is symmetric there exists an orthogonal matrix \( R \) such that

\[ R^T P_m(u_m) W^{-1} P_{\text{out}}(u_m)^T R = D, \]  

(3.8)

where \( D \) is a diagonal matrix with the (say) second element equal to zero. Then, in the new coordinates \( y = R^T x \), the exponential factor under the integral factorizes: the factor related to the \( y^1 \) coordinate can be computed by means of the one dimensional counterpart of the formula (2.26), the second one, related to \( y^2 \), yields the one-dimensional Dirac delta function \( \delta(y^2 + \frac{1}{k^2} (R^TP_{\text{out}}(u_0)(W^T)^{-1}k)^2) \).

Now, let us compare the form of \( \tilde{\Phi}_m^{k,m} \) at \( u_m \), obtained above, with the following limit

\[ \lim_{u \rightarrow u_m} \tilde{\Phi}_m^{k,m}. \]  

As we noted above \( \tilde{\Phi}_m \)'s contain a few terms related to \( P_m^{-1} \) (which become meaningless at \( u_m \)); moreover, we expect the Dirac delta behavior (see (3.7)); thus the distributional character of that limit should be taken into account. In view of this, we first take the Fourier transform (w.r.t. \( x \)) of the state, then perform the limit and, finally, take the inverse Fourier transform. As previously, we make this procedure for \( k_c < 0 \) and both cases of the rank of \( P_m(u_m) \) separately. Let us start with the rank zero case. Then for \( u < u_m \) and sufficiently close to \( u_m \) one has \( \det(\tilde{P}_m(u)) \neq 0 \) (since \( \det(\tilde{P}_m(u_m)) \neq 0 \) and thus \( \text{sign}(\tilde{P}_m(u)P_m^{-1}(u)) = \text{sign}(\tilde{P}_m(u)P_m^{-1}(u)) = \text{sign}(\tilde{P}_m(u - u_m)1) = 2 \). In consequence, for such \( u \), by virtue of equations (2.5), (2.12) and (2.26), we have

\[ \int e^{-i q \cdot x} \tilde{\Phi}_m^{k,m}(u, v, x) dx = \frac{e^{-\frac{\alpha}{4} + i k \cdot q + \frac{\alpha}{4}} e^{-\frac{\alpha}{4} q^T P_m(u)P_m^{-1}(u)q + \frac{\alpha}{4} k^T P_m^{-1}(u)q}}{\sqrt{-4 k^2 \pi |\det(\tilde{P}_m(u_0))|}} \cdot \delta^{(2)}(q + \frac{1}{k^2} P_{\text{out}}(u_0)(W^T)^{-1}k), \]  

(3.9)
Now, we see that only the last exponential factor in (3.9) is singular. However, by virtue of (2.11), (2.16) and (2.18), the following general identity holds for $u < u_{in}$:

$$S_{in}(u) + P_{in}^{-1}(u)(P_{in}^{-1}(u))^T = S_{in}(u_0) - \hat{P}_{in}^{-1}(u)\hat{P}_{out}(u)(W^T)^{-1} + P_{in}^{-1}(u_0)P_{out}(u_0)(W^T)^{-1};$$  \hspace{1cm} (3.10)

moreover, the right-hand side of equation (3.10) is continuous at $u_{in}$. Using this fact in (3.9) one obtains

$$\lim_{u \to u_{in}} \int e^{-i\mathbf{q}^T \mathbf{x}} \Phi_{in}(k, \mathbf{u}, \mathbf{v}, \mathbf{x}) d\mathbf{x} = \frac{1}{\sqrt{-4k^3\pi |\det(P_{in}(u_{in}))|}} e^{i\mathbf{k}^T \hat{P}_{in}^{-1}(u_{in})\mathbf{q}} \cdot e^{-i\mathbf{k}^T (P_{in}^{-1}(u_{in})P_{out}(u_{in})(W^T)^{-1} + S_{in}(u_0) + P_{in}^{-1}(u_0)P_{out}(u_0)(W^T)^{-1})}.$$

Finally, performing the inverse Fourier transform of the right-hand side of equation (3.11) and taking into account (2.22) we obtain precisely the formula (3.7).

In a similar way one can consider the rank one case. However, there appear some technical difficulties (it turns out that equation (2.22) is valid only in one direction); thus we present the relevant changes. Let $\mathbf{v}$ be a null-eigenvector, $\mathbf{v}^T \mathbf{P}_{in}(u_{in}) = 0$, such that $\mathbf{v}^2 = 1$. It is then easy to check that $\mathbf{v} = R\mathbf{e}_2$, where $R$ is the orthogonal matrix defined by equation (3.8) and $\mathbf{e}_2$ is the second vector of the canonical basis. Multiplying equation (3.10) by $\mathbf{v}^T \mathbf{P}_{in}(u)$ one obtains

$$\lim_{u \to u_{in}} \mathbf{v}^T \mathbf{P}_{in}(u) S_{in}(u) = -\mathbf{v}^T (\hat{P}_{in}^{-1}(u_{in}))^T. \hspace{1cm} (3.12)$$

On the other hand, multiplying (2.18) by $\mathbf{v}^T \mathbf{P}_{in}(u)$ one gets

$$\lim_{u \to u_{in}} \mathbf{v}^T \mathbf{P}_{in}(u) S_{in}(u) W^T = -\mathbf{v}^T \mathbf{P}_{out}(u_{in}). \hspace{1cm} (3.13)$$

Thus, for the rank one case we have the following counterpart of the identity (2.22)

$$W^{-1} P_{out}^{-1}(u_{in}) R \mathbf{e}_2 = \hat{P}_{in}^{-1}(u_{in}) R \mathbf{e}_2. \hspace{1cm} (3.14)$$

By virtue of (3.8) and (3.14) we see that the matrix $R^T \mathbf{P}_{in}(u_{in}) \hat{P}_{in}^{-1}(u_{in}) R$ has all entries equal to zero except the first one; let us denote it by $\Theta \in \mathbb{R} \setminus \{0\}$. Using this and equation (2.23) one obtains

$$\lim_{u \to u_{in}} \text{sign}(R^T \hat{P}_{in}(u) \hat{P}_{in}^{-1}(u) R) = -1 + \text{sign}(\Theta).$$

Now, we are ready to compute the Fourier transform of the state $\Phi_{in}^{k, \mathbf{k}}$ (expressed in terms of $\mathbf{y} = R \mathbf{x}$) and then perform the left-sided limit; the result is as follows

$$\lim_{u \to u_{in}} \int e^{-i\mathbf{q}^T \mathbf{y}} \Phi_{in}^{k, \mathbf{k}}(u, \mathbf{v}, \mathbf{y}) d\mathbf{y} = \frac{1}{\sqrt{-4k^3\pi |\det(P_{in}(u_{in}))|}} e^{i\mathbf{k}^T \hat{P}_{in}^{-1}(u_{in})\mathbf{q}} e^{-i\mathbf{k}^T \hat{P}_{in}^{-1}(u_{in}) \mathbf{q}} \cdot e^{-i\mathbf{k}^T (P_{in}^{-1}(u_{in})P_{out}(u_{in})(W^T)^{-1} + S_{in}(u_0) + P_{in}^{-1}(u_0)P_{out}(u_0)(W^T)^{-1})}.$$

Finally, making the inverse Fourier transformation of the right-hand side of equation (3.15) and using (3.14) one recovers the one-dimensional Dirac delta function obtained above for $\Phi_{in}^{k, \mathbf{k}}$ at the singular point $u_{in}$ (note that sign(\Theta k) drops out and there remains $e^{i\mathbf{k}^T \mathbf{y}}$ factor which coincides with $e^{i\mathbf{k}^T \text{sign}(\text{sign}(M(u_{in})))}$, cf equation (3.4)).
Finally, let us compare the above results with the classical picture where the trajectories are continuous functions on the whole real line. To this end, first, let us recall that the transversal part of the geodesic (Lorentz force) equations in the curved spacetime (2.7) with the electromagnetic background (2.8) is of the form $\dot{x} = (K + \frac{2e}{\rho} A) x$. Next, let us note that there is a direct relation between the solutions to the transversal part of the geodesic equations (the classical description) and the solutions to the matrix equation (2.10) (which, in turn, form a basis for the quantum description). In view of this relation $x(u) = -P_{\text{out}}(u)(W^T)^{-1} \frac{\dot{k}_v}{k_v} + P_{\text{in}}(u)x_0$ is the solution of the transversal part of the geodesic (Lorentz force) equations for the discussed backgrounds for which the initial transversal momenta (i.e. at minus infinity) equal $k$ (to see this equation (2.19) can be useful). In consequence, when $P_{\text{in}}(u_{\text{in}}) = 0$ all trajectories with fixed $k$, $k_v$ (but with various initial parameters $x_0$) focus at the point $P_{\text{out}}(u_{\text{in}})(W^T)^{-1} \frac{\dot{k}_v}{k_v}$ which coincides with the Dirac delta behavior of the of $\tilde{\Phi}^{k_v}_{\text{in}}$ at $u_{\text{in}}$, see equation (3.7). The same reasoning can be also applied when the rank of $P_{\text{in}}(u_{\text{in}})$ is one; then the second component $y^2$ of $y(u) = R^T x(u)$ is focused at the second component of the point $\frac{1}{i}(R^T P_{\text{out}}(u_{\text{in}})(W^T)^{-1}k)$ which directly corresponds to the above quantum results. In particular, when the transversal initial momenta are equal to zero then the focusing takes place at the point $x = 0$ or on the line defined by $y^2 = 0$, respectively.

### 3.2. The second approach

In view of the above we find that after passing through the singular point the phase jump develops in the wave functions describing the ‘in’ states. This observation can be confirmed and extended to the case of several singular points using the evolution operator. In fact, the dynamic of the field $\Phi_{\text{in}}$ is directly determined by the dynamics of the field $\Phi_{\text{out}}$; in turn, the evolution of the latter one is governed by that of a time-dependent linear oscillator for various approaches, see, among others, references [39–47]. The propagator for such a system is of the form

$$U(x, u; x_0, u_0) = \frac{k_v}{2\pi i \sqrt{|\det(Q_1)|}} e^{i \frac{2\pi}{i} (\dot{x}^T Q_1^{-1} x - 2x^T \dot{Q}_1^{-1} x + x_0^T \dot{Q}_1^{-1} x_0)},$$

(3.16)

where $Q$’s are the solutions to equation (2.10) satisfying the initial conditions

$$Q_1(u_0) = 0, \quad \dot{Q}_1(u_0) = I, \quad Q_2(u_0) = I, \quad \dot{Q}_2(u_0) = 0.$$  

(3.17)

A few remarks are in order. According to the general theory the singular points of $Q_1$ and $Q_2$ are isolated; as a consequence $Q_1^{-1} Q_2$ is a symmetric matrix (if it exists). Moreover, the above form of the propagator is valid only when $\det Q_1(u) \neq 0$ for $u > u_0$ (i.e. when equation (2.10) is disconjugate, see [48, 49] for more detailed studies concerning (dis)conjugated equations). Then, due to the initial conditions $\det(Q_1(u)) > 0$ for $u > u_0$ and one can skip the modulus. However, as in the case of the ordinary harmonic oscillator, there can exist more singular points (i.e. equation (2.10) is conjugated); and more careful treatment of the evolution operator (3.16) is necessary. This was intensively studied in the context of path integral methods, van Vleck’s determinant, Maslov–Morse’s theory, Niederer’s transformation, see e.g. [26, 40–47]. One explanation of such a situation is based on the observation that the quantum harmonic oscillator can be locally related to the free motion and in order to obtain the global propagator the local transformation have to be appropriately glued—this involves the Maslov–Morse

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10 In fact, by virtue of the initial conditions $(Q_1)^{-1} Q_1^{-1} = \dot{Q}_1 Q_1^{-1} - (Q_1^T)^{-1} Q_1^T$, moreover the right-hand side is a symmetric matrix almost everywhere and $Q$’s are continuous functions.
index (see [41, 42]); more precisely, the propagator given by equation (3.16) has to be corrected by the factor of the form \( e^{\frac{m(u)}{4}} \) (for \( k_i < 0 \)) where \( m(u) \) is the number of times counted with multiplicity such that det\( (Q_1(u)) \) is singular since \( u_0 \). To make contact with our reasoning it is worth to notice that det\( (Q_1(u)) \) vanishes always at \( u_0 \). Thus conjugated points (caustics) are necessary for the phase jump in the propagator to appear; this is in contrast to the matrix \( P_m \) in \( \Phi_m \)’s states, for which the different initial conditions are imposed, see equation (2.13).

Let us now consider the ‘in’ states. First, consider the disconjugate case. Namely, we take \( \bar{\Phi}_m^{k,k}(x_0, u_0) \) where the point \( u_0 \) precedes a singular point, \( u_0 < u_m \) (note that in this approach we do not need ‘out’ field and no relations between \( u_{out} \) and \( u_m \) are required). The evolution of the state is given by the standard formula \( \tilde{\Phi}_m^{k,k}(x, u) = \frac{1}{\sqrt{2\pi}} \int U(x, u, x_0, u_0) \bar{\Phi}_m^{k,k}(x_0, u_0) dx_0 \).

Now, using \( P_m = Q_1 P_m(u_0) + Q_2 P_m(u_0) \), \( Q_1^2 Q_1 - Q_2^2 Q_1 = I, Q_1^2 P_m - Q_2^2 P_m = -P_m(u_0) \), as well as (2.5), (2.12) and (2.26) we obtain, after some straightforward but tedious computations, the following result valid for \( u \geq u_0 \) and \( u \neq u_m \)

\[
\tilde{\Phi}_m^{k,k}(u, v, x) = e^{i \frac{m(u)}{2} + \frac{i k}{2} x_T P_m(u_0) P_m^{-1}(u_0) x + \frac{i k}{2} x T P_m^{-1}(u_0) x} e^{-\frac{i k}{2} x T P_m^{-1}(u_0) x} \sqrt{-2k_c(2\pi)^3 \sqrt{\det(P_m(u))}} \nonumber \]

\[
\tilde{\Phi}_m^{k,k}(u, v, x) = e^{i \frac{m(u)}{2} + \frac{i k}{2} x_T P_m^{-1}(u_0) Q_1(u_0) P_m^{-1}(u_0) x + S_m(u_0) x} \nonumber \]

(3.18)

where \( N \) is a symmetric matrix defined as follows

\[
N(u) = \frac{k}{2} Q_1^{-1}(u) P_m(u) P_m^{-1}(u_0), \quad u > u_0. \quad (3.19)
\]

Since for \( u_0 \leq u < u_m \) we have

\[
S_m(u) = P_m^{-1}(u) Q_1(u) P_m^{-1}(u_0)^T + S_m(u_0), \quad (3.20)
\]

we can identify \( \tilde{S}_m(u) = S_m(u) \) also for \( u > u_m \); there is no additional constant matrix. In consequence, for \( u \geq u_0 \) and \( u \neq u_m \) we arrive at the state

\[
\tilde{\Phi}_m^{k,k}(u, v, x) = e^{i \frac{m(u)}{2} + \frac{i k}{2} x_T P_m^{-1}(u_0) Q_1(u_0) P_m^{-1}(u_0) x + S_m(u_0) x} \tilde{\Phi}_m^{k,k}(u, v, x), \quad (3.21)
\]

where \( \tilde{\Phi}_m^{k,k} \), with \( S_m(u) \) given by the right-hand side of equation (3.20), makes sense also for \( u > u_m \) (this agrees with our previous conclusions since it can be shown that for \( u \neq u_m \) the right-hand side of equation (3.20) is equivalent to the right-hand side of equation (3.5)). The matrix \( N(u) \) apart from being symmetric, obeys \( \text{sign}(N(u)) = \text{sign}(N^{-1}(u)) \) for \( u \neq u_m \). Moreover, in the sufficiently small neighborhood of \( u_0 \) the following expansion holds \( N^{-1}(u) \simeq \frac{1}{k_c}(u - u_0)I \); thus \( \text{sign}(N(u)) = -2 \) for \( u \in (u_0, u_m) \) and, consequently, \( \tilde{\Phi}_m^{k,k} \) coincides with \( \Phi_m^{k,k} \) in this interval. If the rank of \( P_m(u_m) \) is one then, as we stated above (see equation (2.23)) det\( (P_m(u)) \) and, consequently, det\( (N(u)) \) change the sign; thus \( \text{sign}(N(u)) = 0 \) for \( u > u_m \). If the rank of \( P_m(u_m) \) is zero then expanding \( N(u) \) around \( u_m \) we find \( N(u) \simeq \frac{1}{k_c}(u - u_m)(Q_1^2 Q_1)^{-1}(u_m) \) and sign\( (N(u)) = 2 \) for \( u > u_m \). In consequence, equation (3.21) reproduces our previous result even though we do not use ‘out’ states. Finally, the Dirac delta behavior of \( \tilde{\Phi}_m^{k,k} \) at \( u_m \) can be also confirmed in this approach by similar reasoning.

Let us now consider the conjugated case. Then it turns out that we can successively repeat the above procedure to obtain the suitable modification of phase factor of \( \Phi_m \). In fact, let \( u_m \)
be one of the points where \( \det(P_m) \) vanishes. Then there exists\(^\text{11}\) a point \( u_0, u_0 < u_{\text{in}} \), such that \( Q_1(u_0) = 0, \dot{Q}_1(u_0) = 1 \) and \( \det(P_m(u_0)) \) and \( \det(Q^*_1(u)) \) do not vanish in an open interval containing \([u_0, u_{\text{in}}] \). Repeating the above procedure one obtains the suitable correction, \((i \text{ or } -1 \text{ depending on the rank of } P_m(u_{\text{in}})) \) after passing through the point \( u_{\text{in}} \). In consequence, for \( k_v < 0 \), the change of the phase at each point \( u_{\text{in}} \), such that \( \det(P_m(u_{\text{in}})) = 0 \), is \( e^{-\frac{k_v}{2} \text{corank}_P(u_{\text{in}})} \) (under our global assumption that \( \det(P_m(u_{\text{in}})) \neq 0 \)). Similarly, for the antiparticle state, \( k_v > 0 \), we have the correction \( e^{\frac{k_v}{2} \text{corank}_P(u_{\text{in}})} \).

### 3.3. Baldwin–Jeffery–Rosen coordinates

In this section we discuss the above results in the context of the BJR coordinates which are more intuitive for various aspects of the interaction of the particle with plane gravitational waves (e.g. linearized Einstein’s equations, the analysis of isometries and integrability of the geodesic equations); in view of this we skip electromagnetic field in this section. Let us recall that the BJR coordinates \((u, \tilde{x}, \tilde{v})\) are related to B coordinates by the formulae

\[
\mathbf{x} = P\mathbf{x}, \quad v = \dot{\tilde{v}} - \frac{1}{4} \dot{\tilde{x}}^T \dot{\mathbf{x}}, \quad R = P^T P,
\]

where \( P \) is a solution to equation (2.10) (satisfying (2.11)) with \( A = 0 \) (then \( P \) does not depend on \( k_v \)), and (3.22) define a coordinate transformation adopted to the gravitational field only). In the BJR coordinates the plane wave metric takes the form

\[
\hat{g} = 2 \, du \, dv + d\tilde{x}^T R \, d\tilde{x}.
\]

In the weak field approach, in transverse traceless gauge, the metric (3.23) (instead of (2.3)) is very often a starting point for considerations. However, according to the weak energy condition (in particular, for vacuum solutions) \( R \) is singular at least at one point and thus the metric (3.23) is only a local one; in order to cover the whole initial manifold one needs at least two BJR charts (metrics). Moreover, the choice of \( R \) (and consequently the localization of singular points) is not fixed \( a \text{ priori} \); the form of the metric is invariant under the transformation related to the choice of \( P \). One of the natural choices seem to be one based on \( P_{\text{in}} \) and \( P_{\text{out}} \); then we obtain the ‘in’ (‘out’ respectively) BJR coordinates and metrics. In this approach the BJR coordinates tend asymptotically, at minus (plus) infinity, to the Minkowski ones; in other words, the metrics \( \hat{g}_{\text{in}} (\hat{g}_{\text{out}}) \) are chosen in such a way that asymptotically they become the Minkowski ones. Moreover, the asymptotically free ‘in’ state \( \Phi_{\text{in}}^{k_v k_v} \) is of the form

\[
\Phi_{\text{in}}^{k_v k_v} (u, \tilde{x}_{\text{in}}, \tilde{v}_{\text{in}}) = \frac{1}{\sqrt{-2k_v(2\pi)^3 \det(R_{\text{in}})}} e^{i \tilde{x}_{\text{in}} k_v - \frac{m^2_{\text{in}}}{2k_v} - \frac{k_v}{2} S_{\text{in}} - \frac{1}{2} \tilde{x}_{\text{in}}^T k_v \tilde{x}_{\text{in}}}, \tag{3.24}
\]

where \( u < u_{\text{in}} \) (\( S_{\text{in}} \) is the antiderivative of \( R_{\text{in}}^{-1} \)). The inner product in the BJR coordinates can be obtained from (2.9) (see [7]).

First, let us consider the case when two maps ‘in’ and ‘out’ are sufficient to cover the initial B manifold (singular points \( u_{\text{in}} \) and \( u_{\text{out}} \) satisfy \( u_{\text{out}} < u_{\text{in}} \)). Above we have seen that there is a phase jump of the state \( \Phi_{\text{in}} \) beyond the singular point \( u_{\text{in}} \). This change of phase is reflected in the ‘out’ region where the state \( \Phi_{\text{in}} \) is expressed in terms of ‘out’ coordinates (on the common domain \([u_{\text{in}}, u_{\text{out}}]\)) it coincides with \( \Phi_{\text{in}} \) expressed in terms of the ‘in’ BJR coordinates.

\(^{11}\)The existence of such a point can be seen from the observation that \( \det(Q^*_1) \neq 0 \) in a neighborhood of \( u_{\text{in}} \), where \( Q^*_1 \) is defined at \( u_{\text{in}} \) and next using (iii’ and iv’) of the theorem 6.3 section 5 in [48].
The situation changes when both ‘in’ and ‘out’ maps are not sufficient to cover the B manifold (for example, when \( u_{\text{in}} = u_{\text{out}} \); see also an example below). Then the change of the phase appears exactly between both the ‘in’ and ‘out’ BJR maps. Of course, one can choose another ‘out’ map, e.g. demanding that its singular point precedes \( u_{\text{in}} \); however, then the metric does not have to be asymptotically a Minkowski one (only flat for vacuum solutions). In the case of more singular points we need more local BJR charts to cover the B manifold and a similar analysis can be performed (see also examples below).

4. Explicit examples

In this section we discuss some examples of gravitational and electromagnetic backgrounds, defined by (2.7) and (2.8), for which explicit solutions of the K–G (and Dirac, see the next section) equation can be found and consequently some aspects of the interaction of quantum fields with gravitational and electromagnetic waves can be analysed more explicitly. In the literature, especially in the context of gravity, the case of (linearly polarized) sandwich profiles was extensively studied. Here we shall analyse some examples of continuous pulses with non-compact support (including circularly polarized ones). In view of the general discussion given in previous sections the crucial role plays the solutions to equation (2.6) describing a \( u \)-dependent, linear oscillator. According to the general theory (see e.g. [50, 51]) this problem can be reduced to a classical \( u \)-dependent linear oscillator. It turns out that for the latter one there are two families of continuous profiles such that the solutions can be explicitly found and, what is more, these solutions can be built from elementary functions [38]. In consequence, the quantum dynamics in the backgrounds defined by these families can be studied in more detail.

Namely, the first family \( g^{(1)} \) of the linearly polarized plane gravitational waves is defined by the profile

\[
K^{(1)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} G^{(1)}(u), \quad G^{(1)}(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(4.1)

where \( \epsilon > 0 \) and \( a \) is an arbitrary number (excluding the trivial Minkowski case and redefining, if necessary, \( x^1 \) and \( x^2 \) one can achieve \( a > 0 \)). Moreover, let us note that taking \( a \sim \epsilon^3 \) one obtains the model of an impulsive gravitational wave with the Dirac delta profile (as \( \epsilon \) tends to zero).

The second family \( g^{(2)} \) provides an example of the circularly polarized plane gravitational waves. It is defined by the following profiles

\[
K^{(2)}(u) = \frac{a}{(u^2 + \epsilon^2)^2} G^{(2)}(u), \quad G^{(2)}(u) = \begin{pmatrix} \cos(\phi(u)) & \sin(\phi(u)) \\ -\sin(\phi(u)) & \cos(\phi(u)) \end{pmatrix},
\]

(4.2)

where

\[
\phi(u) = \frac{2\gamma}{\epsilon} \tan^{-1}(u/\epsilon),
\]

(4.3)

and \( \epsilon, \gamma > 0 \) and \( a \) can be chosen as above. For \( \gamma = 0 \) equation (4.2) reduces to the previous case; however, for physical and mathematical reasons we shall consider both families separately. In fact, the profiles of the first family are even matrix functions; moreover, the quantum dynamics of this family can be also explicitly solved when we change the sign of the first entry in \( G^{(1)} \); this yields a null-fluid metric. Such a situation does not hold, in general, for the second case. One can also consider, in the Minkowski spacetime, the suitable
electromagnetic potentials \( A^{(1)} \) (\( A^{(2)} \), respectively) given by the formulae (4.2) and (4.3). Then the corresponding electromagnetic fields satisfy the vacuum Maxwell equations; even more, one can combine both gravitational and electromagnetic backgrounds, i.e. \( K^{(1)} \) with \( A^{(1)} \) defined by another parameter \( a \), see equation (2.10) (and similarly for the second family). However, to simplify our notations, we restrict ourselves to the gravitational case (after a redefinition of the constant \( a \) the suitable electromagnetic background can be included into considerations, cf equation (2.10)). Moreover, from equations (4.1) and (4.2) we see that the profiles fit into our previous analysis since \( \mathcal{R} \) is explicitly solvable. Namely, for explicit form of the constant \( a \) the suitable electromagnetic background can be included into considerations, cf equation (2.10). Moreover, from equations (4.1) and (4.2) we see that the profiles fit into our previous analysis since \( a^2 \mathcal{R}^{(1)(\mathcal{R})} \) belong to the \( L^1(\mathcal{R}) \) space and satisfy equation (2.15). Finally, it is worth to notice that these two families of the fields are distinguished due to the conformal symmetry. Namely, they exhibit the seven-dimensional conformal symmetry, maximal among all (non-flat) vacuum solutions to the Einstein equations (see \([52, 53]\)); in turn the electromagnetic fields are invariant under the action of a special conformal generator \([54, 55]\).

As we noted above the classical motion in gravitational (electromagnetic) fields (4.1) and (4.2) is explicitly solvable. Namely, for \( \mathbf{q}^{(1)} \) the solution of the geodesic equations, in the transverse directions\(^{12} \), for \( a < \epsilon^2 \), is given by

\[
x'(u) = C_1 \sqrt{u^2 + \epsilon^2} \sin(\sqrt{\Lambda_i} \tan^{-1}(u/\epsilon) + C_2), \tag{4.4}
\]

where

\[
\Lambda_i = 1 + (-1)^i \frac{a}{\epsilon^2} \quad i = 1, 2; \tag{4.5}
\]

(for \( a > \epsilon^2 \) the solution \( x'(u) \) does not change but \( x'(u) \) is obtained by replacing the trigonometric functions by their hyperbolic counterparts). In consequence, the matrix \( P_m \) can be explicitly found and the suitable quantum states \( \Phi_m \)'s constructed. In fact, taking \( C_2 = \frac{1}{2} \sqrt{\Lambda_i} \), \( C_1 = \frac{1}{\sqrt{\Lambda_i}} \) and \( C_2 = 0 \) in equation (4.4) we obtain the solution \( x_1(u) \), while taking \( C_1 = 0 \) and \( C_2 = \frac{1}{\sqrt{\Lambda_i}} \), we obtain another solution \( x_2(u) \). Then, \( P_m \) is given by the formula \( P_m(u) = (x_1(u), x_2(u)) \). The solution \( P_{\text{out}} \) can be obtained analogously; alternatively, by virtue of the fact that (4.1) is an even matrix function, one can put \( P_{\text{out}}(u) = P_m(-u) \). For the second family, the solutions are more complicated but they can also be written in terms of elementary functions, see \([38]\).

Of course, we do not have to refer to classical solutions to find some solutions of equation (2.10) in such backgrounds. The general solution can be found directly on the quantum level by reducing the initial problem to the well known one. This alternative approach can be summarized as follows. For the first family we use the Niederer transformation (see e.g. \([42, 56]\)) to reduce equation (2.10) with \( K^{(1)} \) (optionally with \( A^{(1)} \)) to the one describing two separable quantum harmonic oscillators. More precisely, if \( \Phi \) is a solution of (2.6) for the first family, then it is easy to check that

\[
\Psi(\mathbf{x}, \bar{u}) = \frac{\epsilon}{\cos(\bar{u})} e^{\frac{i}{2\epsilon} \tan(\epsilon \bar{u})} \Phi \left( \frac{i \bar{x}}{\cos(\bar{u})}, \epsilon \tan(\bar{u}) \right) \tag{4.6}
\]

is a solution to the Schrödinger equation with potential being the sum of two independent harmonic oscillators with frequencies \( \Lambda_i, i = 1, 2 \), respectively; this observation leads directly

\(^{12}\)These directions are the most important since they can be used to define \( P \)'s, and to analyse deviation equations; the explicit form of the \( \psi \) coordinate can be also found, see \([38]\).
to the solutions of initial equation (2.1). Note that, in general, this transformation is a local one; its extension is related to the analysis of the singular points presented above.

For the second family (4.2) the transformation (4.6) leads, unfortunately, to a $\tilde{u}$-dependent linear oscillator (it contains trigonometric functions of $\tilde{u}$). However, it is easy to check that the global transformation:

$$\Upsilon(y, \tilde{u}) = \Psi(\text{Rot}(\tilde{u})y, \tilde{u}), \quad (4.7)$$

where

$$\text{Rot}(\tilde{u}) = \begin{pmatrix} \cos(\omega \tilde{u}) & -\sin(\omega \tilde{u}) \\ \sin(\omega \tilde{u}) & \cos(\omega \tilde{u}) \end{pmatrix}, \quad \omega = \frac{\gamma}{\epsilon}, \quad (4.8)$$

define the state $\Upsilon$ satisfying the Schrödinger equation with the Hamiltonian $H$ of the form

$$H = \frac{p_y^2}{2k_v \epsilon} + k_v \epsilon \left(1 - a \epsilon^2 \right) (y_1)^2 + \frac{k_v \epsilon}{2} \left(1 + a \epsilon^2 \right) (y_2)^2 - \omega (p_{y_2} y_1 - p_{y_1} y_2). \quad (4.9)$$

The Schrödinger equation (i.e. spectrum, eigenfunctions) of the Hamiltonian of this type has been considered in many papers and in various approaches, see e.g. [57–59]. Here, we use the reasoning presented in [58] and define a unitary operator $V$:

$$V = e^{i\varphi_1 y_1} e^{i\varphi_2 y_2}, \quad (4.10)$$

where $\varphi_1$ and $\gamma$ are constants adjusted in such a way that $V^+ H V$ is separable; these constants can be easily expressed in terms of the know parameters ($k_v, a, \epsilon$) by means of equations (6) and (7) from [58]. Then for the parameters $|\epsilon^2 - \gamma^2| > a$ the transformed Hamiltonian $V^+ H V$ is equal to the sum of two independent Hamiltonians describing harmonic oscillators; they frequencies are as follows

$$\Omega_1^2 = 1 + \omega^2 + \sqrt{4\omega^2 + \Omega^2}, \quad (4.11)$$

$$\Omega_2^2 = 1 + \omega^2 - \sqrt{4\omega^2 + \Omega^2};$$

where

$$\Omega = \frac{a}{\epsilon^2}. \quad (4.12)$$

These results exactly agree with the classical ones (see [38]). Now, the eigenfunctions for $H$ can be built by means of the operator $V$ and the well known Hermite functions. However, since the operator $V$ contains momenta the computations are involved. The final result, see equation (13) in reference [58], is that the eigenfunctions for $H$, given by equation (4.9), contain finite sums of products of two Hermite polynomials with different complex arguments. Now, performing the inverse transformations to (4.6) and (4.7) we can construct explicit solutions of the K–G equation (2.1) in the gravitational field $g^{(2)}$ (and, optionally, including the electromagnetic potential $A^{(2)}$, also in the Minkowski space). Moreover, we have shown above

\footnote{It seems that there is a printing error in the formula for $H_3$ in [58]; namely instead of $(c + 2\alpha a)W$ there should be $(c - 2\alpha a)W$.}
that the quantum cross section coincides with its classical counterpart; the latter one was explicitly found for both families of backgrounds [38] thus we immediately obtain its quantum counterpart.

Finally, let us go back to singularities. For the profile $K^{(1)}$ the zeros of $\det(P_{in})$ are determined by the second diagonal component of $P_{in}$. For $\Lambda_2 < 2$ there is one singular point $u_{in} > 0$ and thus two maps are sufficient to cover the whole manifold (since $K^{(1)}$ is an even function, $P_{out}(u) = P_{in}(-u)$ and $u_{out} = -u_{in} < 0$). For $\Lambda_2 = 2$ (equivalently, $a = 3e^2$) there is again one singular point $u_{in} = 0$ (the second component of $P_{in}$ is given by $-u/\sqrt{u^2 + e^2}$), but this time $u_{out} = u_{in}$; thus, the maps related to $P_{in}$ and $P_{out}$ are not sufficient to cover the B manifold. Of course, one can take instead of $P_{out}$ another $P$ for which the determinant vanishes at some negative point. Then together with $P_{in}$ they cover the B manifold; however, at plus infinity the BJR metric do not coincide with the Minkowski one. For $\Lambda_2 > 2$ the situation gets complicated and the number zeros growths; in consequence, the number of phase corrections and BJR maps growths too. Since for all these points the rank of $P_{in}$ is one the suitable phase correction is always $e^{i\pi}$. Similar situation holds for the second family; then, however, the singular points are given more implicitly, see [38].

5. Fermionic fields

In this section we analyse all the above issues for a massive fermionic fields. To this end let us recall that the coupling of spin one-half fields to the metric $g$ and external electromagnetic potential $A_\mu$ is described by the spinor field $\Psi$ satisfying the Dirac equation

$$\gamma^\mu(\partial_\mu - \Gamma_\mu - i e A_\mu I) + m I \Psi = 0,$$

(5.1)

where $\gamma^\mu \equiv g^{\mu\nu}\gamma_\nu$, $\gamma_\mu \equiv e^{\nu\alpha}\gamma_\nu$, $\Gamma_\mu = -\frac{1}{4}\gamma_\mu e^{\nu\alpha}g^{\rho\sigma}(\partial_\rho e^{\sigma}_\alpha - \Gamma_\rho^\alpha e^{\sigma}_\beta)$,

(5.2)

and the orthonormal tetrad fields are defined by the condition $g_{\mu\nu} = \eta_{\mu\nu} e^{\rho}_\mu e^{\sigma}_\nu$ ($\gamma^\mu_\nu$ are the ordinary gamma matrices in the Minkowski spacetime $x^\mu$ with $\eta_{\mu\nu} = (-, +, +, +)$ i.e. $\gamma^0 = -\gamma_0 = -i\beta$ and $\gamma^i = -i\beta i$).

For the pp-wave metric $g$ defined by (2.4) the tetrad $e^{\rho}_\mu$ can be easily found together with the matrix functions $\gamma^\mu_\nu$. Then one gets the matrices $\Gamma_{\mu}$:

$$\Gamma_u = \frac{1}{8\sqrt{2}} \partial_\xi K^\xi [\gamma^\xi, \gamma_0 + \gamma_3] , \quad \Gamma_v = 0 , \quad \Gamma_k = 0.$$  

(5.3)

Now, introducing new two-components spinorial fields $\Psi_u$ and $\Psi_v$ by an invertible (unitary up to a constant factor) transformation

$$\begin{pmatrix} \Psi_u \\ \Psi_v \end{pmatrix} \equiv \begin{pmatrix} \sigma_3 \\ I \end{pmatrix} \Psi,$$

(5.4)

one finds that the Dirac equation (5.1), in the pp-wave and electromagnetic backgrounds (2.4), is equivalent to the following set of equations

$$i\sqrt{2} \partial_\mu \Psi_u = -e\sqrt{2} A \Psi_u + \frac{iK}{\sqrt{2}} \partial_\xi \Psi_u - (i\sigma_1 \partial_k + m\sigma_3) \Psi_v,$$

(5.5)

$$i\sqrt{2} \partial_\nu \Psi_v = (i\sigma_1 \partial_k + m\sigma_3) \Psi_u.$$

(5.6)
Since the coefficients in the above set of equations do not depend on \( v \), we look for solutions of the form (cf. (2.5))

\[
\Psi_u = e^{ivk} \psi_u, \quad \Psi_v = e^{ivk} \psi_v. \tag{5.7}
\]

Substituting this ansatz into equations (5.5) and (5.6) one concludes that the equation for the spinor field \( \Psi_u \) decouples and its both components satisfy the same Schrödinger equation (cf. (2.6)), namely

\[
\frac{i}{2} \partial_t \Psi_u = \left( \frac{-\Delta}{2} - \frac{k_v}{2} \left( \frac{2e}{k_v} A + K \right) \right) \Psi_u; \tag{5.8}
\]

moreover, the two-components field \( \bar{\Psi}_v \) (and thus \( \psi_v \)) is directly determined by \( \bar{\Psi}_u \)

\[
\bar{\Psi}_v = -\frac{1}{\sqrt{2k_v}} (i \sigma_t \partial_t + m \sigma_3) \Psi_u. \tag{5.9}
\]

Now, as in the spinless case, we restrict ourselves to the backgrounds defined by (2.7) and (2.8); then the solutions of the Dirac equation (5.1) can be directly constructed in terms of the bosonic fields (2.12). Namely, the component \( \Psi_u^{k_1, k_2, w} \) (compare equations (2.6) and (5.8)) takes the form

\[
\Psi_u^{k_1, k_2, w} = \sqrt{2|k_v|} \psi^{k_1, k_2, w}, \tag{5.10}
\]

where \( w \) is a two-dimensional vector. The final form of the fermionic state reads

\[
\Psi^{k_1, k_2, w} = \sqrt{\frac{|k_v|}{2}} \psi^{k_1, k_2, w} \left( \begin{pmatrix} 1 - \frac{m}{\sqrt{2k_v}} & \frac{1}{\sqrt{2k_v}} (\partial_t \sigma_3) \psi \end{pmatrix} w \right) \left( \begin{pmatrix} 1 - \frac{m}{\sqrt{2k_v}} & \frac{1}{\sqrt{2k_v}} (\partial_t \sigma_3) \psi \end{pmatrix} w \right)^{-1}, \tag{5.11}
\]

where \( f = k^T P^{-1} x + \frac{i}{2} x^T PP^{-1} x \) (for the linearly polarized gravitational waves it reduces to one obtained in reference [10] in local BJR coordinates).

Let us introduce the inner product as follows

\[
\langle \Psi_1 | \Psi_2 \rangle = \int \bar{\Psi}_1 \gamma^0 \gamma^\mu \Psi_2 \, dv \, dx. \tag{5.12}
\]

It is worth to notice that \( \Pi = \frac{1}{\sqrt{2}} \gamma^0 \gamma^\mu \) is a projector operator and, consequently, the inner product defined by (5.12) depends on the \( \mu \)-spinorial component only

\[
\langle \Psi_1 | \Psi_2 \rangle = \int \frac{1}{\sqrt{2}} \bar{\Psi}_1^{k_1, k_2} \Psi_2^{k_2, k_1} \, dv \, dx. \tag{5.13}
\]

In view of this the solutions \( \psi^{k_1, k_2} \) (see (5.10)) defined by the vectors \( w_k, k = 1, 2 \), of the form

\[
w_k = 2^{\frac{1}{2}} e_k, \quad k = 1, 2; \tag{5.14}
\]

where \( \{ e_k \}_{k=1}^2 \) is the standard two-dimensional canonical basis, form an orthonormal set of functions with respect to the inner product (5.12)

\[
\langle \psi^{k_1, k_2} | \psi^{l_1, l_2} \rangle = \delta(k_1 - l_1) \delta^{(2)}(k - l) \delta_{k_1, l_1}. \tag{5.15}
\]
Moreover, the states $\Psi^{l,k,k}$ with $k_v \in \mathbb{R} \backslash \{0\}$, $k \in \mathbb{R}^2$ and $k = 1,2$, form a complete set. This can be directly shown by taking into account the projector operator $\Pi$ and the form of $\Psi$’s (equation (5.11)). As we noted above, for the bosonic case and the gravitational backgrounds only, as a complete basis we cannot take the states $\Phi^{l,k}$ and $\Phi^{l,k*}$ with negative $k_v$, this implies a similar situation for the state $\Psi$’s; however, now for the states with positive $k_v$ there are also some sign changes in the spinorial part of (5.11), namely the replacement $m \to -m$ should be made.

In section 4 we noted that the matrices $P$ (and consequently the bosonic states $\Phi$) can be explicitly found for the two gravitational families defined by the profiles (4.1) and (4.2) (optionally with the suitable electromagnetic backgrounds). In view of equation (5.11) the same holds also for the spin one-half particle described by the Dirac equation in these backgrounds. Finally, let us also note that when we switch off the gravitational and electromagnetic fields (and put $P = I$) the states $\Psi^{l,k,k}$ become equivalent to the ones for the Minkowski spacetime known from the standard textbooks.

Now, as in the case of bosonic fields we define, in terms of the matrix $P$ (and consequently the bosonic states $\Phi$) in (respectively ’out’ states) which asymptotically tend to the free ones. Then, using equation (5.13) we can compute the inner product between them and the transition amplitude from an ’in’ one-particle state to an ’out’ one-particle state

$$|\langle \Psi^{l,k,k}_\text{in} | \Psi^{l',l}_\text{out} \rangle| = \frac{\delta(k_v - l_v)}{2\pi|l_v| \sqrt{|\det(W)|}} \delta_{ll'}.$$  

(5.16)

Summing over all the final spin states and then taking averaging over the initial polarization one obtains the final quantum cross section which is of the same form as in the bosonic case, i.e. given by equation (2.31) (extending in this way the results of reference [10] obtained for fine-tuned and linearly polarized plane gravitational sandwiches). In particular, we immediately obtain the explicit form of the cross section for the backgrounds defined by (4.1) or (4.2) (see section 4 and [38]).

Moreover, as for the bosonic field $\Phi^{l,k}_\text{in}$, there is a phase change for the fermionic fields after passing through the singular point. Namely, the $\nu$-component of the field $\Psi^{l,k}_\text{in}$ is proportional to $\Phi^{l,k}_\text{in}$ (cf equation (5.10)); thus it carries the same correction. In view of equation (5.9) this concerns also the $\nu$-component. As a result the phase of the full state $\Psi^{l,k}_\text{in}$ also changes after passing through the singular point.

Finally, let us analyse the change of the spin polarization. It is difficult to define the spin operator for a general curved spacetime; however, in the Minkowskian spacetime in the rest frame it can be easily described by three matrices

$$\bar{\sigma} = \begin{pmatrix} \bar{\sigma}^x & 0 \\ 0 & \bar{\sigma}^z \end{pmatrix}. \quad (5.17)$$

Thus we can try to analyse the change of the spin polarization after the pulse has passed (where the metric is again the Minkowskian one). Such an approach was applied (in the Newman–Penrose framework) in reference [11] to the linearly polarized sandwich waves; here we study this problem, in a slightly different way, for the general plane sandwich waves (for example, one can take the pulses studied in reference [60]). In such a case the Minkowskian region emerges for a sufficiently large parameter $\nu$. We start with the rest particle state, i.e. $k = 0$, $k_v = -\frac{m}{\sqrt{2}}$, at minus null infinity. Then, at a neighborhood of minus infinity, the positive
energy spinor wave functions $\Psi_{m^2,0,w}^{a}$, $a = 1, 2, 3$ defined by the vectors of the form

$$
\begin{align*}
\mathbf{w}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\mathbf{w}_2 &= \begin{pmatrix} 1 \\ -i \end{pmatrix}, \\
\mathbf{w}_3 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 
\end{align*}
$$

are eigenfunctions (with the eigenvalue 1) of the matrix $\Sigma_a$, respectively; i.e. their spins are polarized in the three directions. Now let us analyse the ‘after’ Minkowskian region where the metric is again the Minkowskian one. To this end, in this region, we act with an arbitrary (i.e. defined by arbitrary momenta $\vec{p}$) Lorentz boost in spinor representation on the field $\Psi_{m^2,0,w}^{a}$ and then check, by direct calculations, that the field obtained in this way is not an eigenfunction (with the initial eigenvalue) of the matrix $\Sigma_a$. In consequence there is no rest frame such that the spin polarization is the same as the initial one (the effect of the gravitational wave pulse is that the Dirac particle loses its initial polarization see also [11]). Thus, there is a change of the spin polarization (a kind of the spin memory effect, cf [22]) after passing the general sandwich wave.

6. Conclusions and outlook

In this paper we have discussed some aspects of massive quantum fields in the general plane gravitational wave in the presence of non-plane electromagnetic waves; in particular, the asymptotic conditions, ‘in’ (‘out’) states and the cross sections were obtained. We observed that despite of singularities of these states (corresponding, on the classical level, to the focusing properties) their global form can be established: at the singular points the Dirac delta (distributional) behavior emerges and there is a phase jump of the wave function after passing through each singular point. Moreover, we noted that this phase correction can be described in terms of relations between BJR charts. We also discussed some examples of the waves of infinite range for which the explicit form of the states can be obtained (in particular, the circularly polarized ones). All these results concern both the scalar as well as spin one-half quantum fields; moreover, in the latter case we analysed the change of the spin polarization after passing the general sandwich wave.

Turning to possible further developments let us recall that the Penrose limit [21] of the spacetime yields a plane gravitational wave. In consequence, the propagation of quantum fields in a more realistic spacetime can be approximated in some local region (near the null geodesics) by considering an appropriate plane wave spacetime. Such an approach was successively applied in references [28–32] to the Green functions and then vacuum polarization in QED. In this context, it would be also interesting to describe spacetimes for which the explicitly solvable examples presented in section 4 emerge as the Penrose limit. The other quite natural generalization corresponds to going beyond the plane waves and considering some pp-waves, gravitational wave beams [61] and the linearized theory [62]. It is also worth to notice that the case of massless fields involves also a special analysis due to the possibility of massless particles creation by the gravitational field (in particular, see the recent considerations in [36, 37]); actually, such a possibility can exist even for the massive fields; however, then some additional constraints have to be added [63]. Moreover, the results obtained may be useful in the context of trapping and soft problems in gravity [15, 64, 65]. Finally, following references [66–69] we hope that they can be also useful in the study of the light–matter interaction.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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