CASCADES OF TORIC LOG DEL PEZZO SURFACES
OF PICARD NUMBER ONE

DONGSEON HWANG

Abstract. We classify toric log del Pezzo surfaces of Picard number one by introducing the notion, cascades. As an application, we show that if such a surface is Kähler-Einstein, then it should admit a special cascade, and it satisfies the equality of the orbifold Bogomolov-Miyaoka-Yau inequality, i.e., \( K^2 = 3e_{\text{orb}} \).

1. Introduction

We work over an algebraically closed field of characteristic zero.

A lot of work has been devoted to classify log del Pezzo surfaces. For example, they are classified up to index 3. See [HW], [AN], [KK], [N], and [FY]. For the Picard number one case, see also [Z] and [KM]. Moreover, toric log del Pezzo surfaces are completely classified up to index 17 ([KKN]). See [GRDB] for the list. Recently, all toric log del Pezzo surfaces with 1 singular point are completely classified in [D], and those with 2 are completely classified in [S] which also contains a partial classification of those with 3 singular points.

In this note, we shall classify toric log del Pezzo surfaces of Picard number one by using the notion of a cascade, which was introduced in [H] for a larger class of rational \( \mathbb{Q} \)-homology projective planes. In fact, even though there exists infinitely many toric log del Pezzo surfaces of Picard number one, one might think that classifying them is not a very difficult task at least in the sense that it is easy to describe their corresponding Fano triangles, see e.g., [HKi, Proof of Proposition 3.10]. But it does not give us any geometric intuition and thus sometimes it is not easy to derive geometric consequences. By describing the classification in terms of cascades, which we will soon define, one can understand the underlying geometry more clear. See Section 4 for the applications. For example, one can easily determine whether a toric log del Pezzo surface of Picard number one with given singularity types exists or not. See Theorem 4.1, Corollary 4.2 and Algorithm 4.7.

Definition 1.1. Let \( S \) be a toric log del Pezzo surface of Picard number one. We say that \( S \) admits a cascade if there exists a diagram as follows:
where for each $k$

1. $\phi_k$ is a toric blow-down,
2. $\pi_k$ is the minimal resolution,
3. $S_k$ is a toric log del Pezzo surface of Picard number one, and
4. $S_0$ is basic. (See Definition 2.5 for the definition.)

In this case, we also say that $S$ admits a cascade to $S_0$, and $S_0$ is the basic surface of $S$.

In the above definition, when $S_k$ is already smooth, by removing the condition of being Picard number one, we can simply take $S'_k := S_k$ and set $\pi_k$ to be the identity morphism. This reminds us the classical construction of smooth del Pezzo surfaces.

The first main result of the present paper is to show the existence of a cascade for every toric log del Pezzo surface of Picard number one.

Theorem 1.2. Every toric log del Pezzo surface of Picard number one admits a cascade.

The proof uses the standard theory of $\mathbb{P}^1$-fibrations. By looking at the dual graph of the torus-invariant divisors, one can immediately read off the information of $\mathbb{P}^1$-fibration structure on the corresponding smooth toric surface. See Notation 2.6 for dual graphs.

Conversely, by inverting the cascade process, one can obtain every toric log del Pezzo surface of Picard number one.

Theorem 1.3. The minimal resolution of every toric log del Pezzo surface of Picard number one that is not basic is obtained from one of the three basic toric surfaces $S(\text{std}_1^n)$, $S(\text{std}_2^n)$ and $S(3A_2)$ by a sequence of toric blowups at the intersection point of a $(-1)$-curve and a torus-invariant curve with self-intersection number at most $-2$. (See Notation 2.6 for the definition of $S(\text{std}_1^n)$, $S(\text{std}_2^n)$ and $S(3A_2)$.)

Since the cascade and its inverse process preserve the number of singular points of $S$, we can describe all toric log del Pezzo surfaces of Picard number one with respect to the given number of singular points.

Theorem 1.4. Let $S$ be a toric log del Pezzo surface of Picard number one. If $S$ is not basic, it admits a cascade to one of the three basic surfaces: $S(3A_2)$, $S(\text{std}_1^n)$, $S(\text{std}_2^n)$. In particular, we have the following.

1. If $|\text{Sing}(S)| = 0$, then $S \cong \mathbb{P}^2$.
2. If $|\text{Sing}(S)| = 1$, then $S \cong \mathbb{P}(1, 1, n)$ where $n \geq 2$.
3. If $|\text{Sing}(S)| = 2$, then $S \cong \mathbb{P}(1, p, q)$ and it admits a cascade to $S(\text{std}_1^n)$.
4. If $|\text{Sing}(S)| = 3$, then $S$ admits a cascade to either $S(\text{std}_2^n)$ or $S(3A_2)$.

In particular, this reproves the theorem by [D] and [S] for the Picard number one case.
As an application, we consider the orbifold Bogomolov-Miyaoka-Yau inequality. The inequality does not hold in general for Fano manifolds or Fano orbifolds. However, Chan and Leung proposed a Miyaoka-Yau type inequality for Kähler-Einstein toric Fano manifolds.

**Theorem 1.5.** [CL, Theorem 1.2] Let $X$ be a Kähler-Einstein toric Fano manifold of dimension $n$. Then, for any nef class $H$, we have

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

if either $n = 2, 3, 4$, or each facet of the corresponding dual polytope of the Fano polytope of $X$ contains a lattice point in its interior.

It is natural to ask whether the above inequality can be generalized in singular setting.

**Question 1.6.** [HKi, Question 1.8] Let $S$ be a Kähler-Einstein toric log del Pezzo surface. Does the inequality $K^2(S) \leq 3e_{orb}(S)$ holds?

Unfortunately, the answer is negative in general as in [HKi, Example 1.9]. But it holds when the Picard number is one.

**Theorem 1.7.** Let $S$ be a Kähler-Einstein toric log del Pezzo surface of Picard number one. Then, we have the following properties.

1. $K^2_S = 3e_{orb}$.
2. $S$ is either isomorphic to $\mathbb{P}^2$ or $S$ has exactly 3 singular points.
3. If $S$ is not isomorphic to $\mathbb{P}^2$, it admits a cascade to $S(3A_2)$, not to $S(3d_2)$.

We emphasize that the condition of being Kähler-Einstein forces a singular toric log del Pezzo surface of Picard number one to admits a cascade to a particular basic surface, i.e., $S(3A_2)$.

As a final application, we give a simple observation that every finite cyclic group is a Brauer group of a toric log del Pezzo surface of Picard number one. See Theorem 4.10.

2. **Basic toric log del Pezzo surfaces of Picard number one**

Throughout this section, we always denote by $S$ a toric log del Pezzo surface of Picard number one, $f : S' \to S$ be its minimal resolution. Note that if $S$ is singular, i.e., the Picard number of $S'$ is greater than one, the torus-invariant divisors form two sections and two fibers of a suitable $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$. For generalities about $\mathbb{P}^1$-fibrations on rational surfaces, see [M] or [GMM].

**Notation 2.1.** We denote by $[[s_1^2, F_1, s_2^2, F_2]]$ the smooth toric surface $S'$ admitting a $\mathbb{P}^1$-fibration $\Phi : S' \to \mathbb{P}^1$ where $s_1$ and $s_2$ are the two torus-invariant sections of $\Phi$ and; $F_1$ and $F_2$ are the two torus-invariant fibers of $\Phi$.

**Definition 2.2.** Let $F$ be a singular fiber of a $\mathbb{P}^1$-fibration on $S'$.

1. $F$ is said to be of type $I_0$ if its dual graph is of the form $-2 - 1 - 2$.
2. $F$ is said to be of type $I$ if it can be contracted to a fiber of type $I_0$.
3. $F$ is said to be of type $II_0$ if its dual graph is of the form $-1 - 2 - 1 - 2$.
4. $F$ is said to be of type $II$ if it can be contracted to a fiber of type $II_0$.

**Notation 2.3.** Let $\Phi$ be a $\mathbb{P}^1$-fibration.
A smooth fiber is denoted by $F_0$.

A singular fiber of type $I_0$ is denoted by $F_0^0$.

A singular fiber of type $I$ is denoted by $F_1$.

A singular fiber of type $II_0$ is denoted by $F_0^2$.

A singular fiber of type $II$ is denoted by $F_2$.

The below lemma immediately follows from the standard theory of smooth projective rational surfaces.

**Lemma 2.4.** Let $S$ be a toric log del Pezzo surface of Picard number one and $S'$ be its minimal resolution. Denote by $n$ the number of torus-invariant curves on $S'$ and by $N$ the sum of all self-intersection numbers of the torus-invariant curves. Then, we have

$$N = 12 - 3n.$$ 

The following notion is essential in the description of the cascades.

**Definition 2.5.** $S$ is said to be basic if $D^2 \geq -2$ for every torus-invariant curve $D$ on $S'$ intersecting $C$ where $C$ is any $(-1)$-curve on $S'$.

For later use, we introduce the following five surfaces.

**Notation 2.6.** We define the below five surfaces equipped with a $\mathbb{P}^1$-fibration structure.

1. $S(\mathbb{P}^2) = \mathbb{P}^2$
2. $S(\text{std}_{0}^n) := [-n, F_0, n, F_0]$.
3. $S(\text{std}_{1}^n) := [-n, F_0, n-1, F_0^1]$.
4. $S(\text{std}_{2}^n) := [-n, F_0^1, n-2, F_0^2]$.
5. $S(3A_2) := [-2, F_1^1, -2, F_2^2]$.

Figure 2.6 describes the basic dual graphs, i.e., the dual graphs of the torus-invariant curves on the above five surfaces.

![Figure 1. Basic dual graphs](image_url)

Now we are ready to determine basic toric log del Pezzo surfaces of Picard number one.

**Proposition 2.7.** If $S$ is basic, then its minimal resolution is one of the following five surfaces: $S(\mathbb{P}^2)$, $S(\text{std}_{0}^n)$, $S(\text{std}_{1}^n)$, $S(\text{std}_{2}^n)$, $S(3A_2)$.

**Proof.** Assume that $S$ is not isomorphic to $\mathbb{P}^2$. Then, $S$ is singular and the Picard number of its minimal resolution $S'$ is greater than one. Thus, $S'$ admits a $\mathbb{P}^1$-fibration $\pi: S' \to \mathbb{P}^1$ where the cycle of torus-invariant curves forms two singular fibers and two sections of $\pi$.

If $\pi$ is relatively minimal, then $S'$ is isomorphic to the Hirzebruch surface $F_n = S(\text{std}_{0}^n)$ with $n \neq 1$. In this case, $S$ is isomorphic to $\mathbb{P}(1, 1, n)$. 
From now on, we assume that \( \pi \) is not relatively minimal. In particular, there exists a \((-1)\)-curve on \( S' \). Moreover, since \( S \) is singular, there exists a torus-invariant curve with self-intersection number at most \(-2\).

Note that there exists a \((-1)\)-curve \( E \) meeting one of the exceptional curves of \( f \). Let \( D_1, D_2, \ldots, D_k \) be a chain of torus-invariant curves which contracts to one of the singular points of \( S \) such that \( E \) intersects \( D_1 \). Let \( C \) be the other torus-invariant curve intersecting \( E \). Since \( S \) is basic, \( D_1^2 \geq -2 \) and \( C^2 \geq -2 \). We first consider the case \( D_1^2 = C^2 = -2 \). Since \( D_1 + 2E + C \) induces a \( \mathbb{P}^1 \)-fibration structure on \( S' \) on which it forms a singular fiber, there exists another torus-invariant fiber \( F \). Since \( S \) is basic, it is easy to see that \( F \) is one of the following:

\[
0, -1, -1, -2, -1, -1, -2, -2, -2, -2, -2, -2, \ldots, -2, -2, -1.
\]

In the first and third case, by Lemma \ref{lem:2.4}, we see that the corresponding \( \mathbb{P}^1 \)-fibration structures are \( S(\text{std}_n^1) \) and \( S(\text{std}_n^2) \), respectively, where \( n \geq 2 \). In the second case, one can show that \( S \) is of Picard number 2 or 3, a contradiction. In the final case, since \( S \) is basic and \( \rho(S) = 1 \), the torus-invariant sections have self-intersection number \(-2 \), so \( S \) has only rational double points as singular points. Thus, by Lemma \ref{lem:2.4} one can see that \( F \) should be of type \((II_0)\). Hence, the corresponding surface is \( S(3A_2) \).

Now we consider the case \( C^2 = -1 \) and assume that there is no \((-1)\)-curve such that its adjacent torus-invariant curves have self-intersection number at most \(-2 \). Then, since \( E + C \) induces a \( \mathbb{P}^1 \)-fibration on which it forms a complete fiber, there exists another torus-invariant fiber \( F \). By assumption, we see that the fiber \( F \) is one of the following:

\[
0, -1, -1, -2, -2, -2, -2, -2, -2, -2, -2, -2, \ldots, -2, -2, -1
\]

One can see that \( \rho(S) > 1 \) in all of the above cases, which is a contradiction.

Finally, we may assume that, for every \((-1)\)-curve \( E \) intersecting an exceptional curve \( D_1 \) of \( f \), the other torus-invariant curve \( C \) intersecting \( E \) have \( C^2 \geq 0 \). Then, by the similar analysis as above, one can see that \( C \) is a section of a \( \mathbb{P}^1 \)-fibration \( \Phi \), \( D_1 \) is part of a fiber of type \( II \) and the other fiber is either of type \( II \) or of the form \( 0 \). In any case, we have \( \rho(S) > 1 \), a contradiction. \( \square \)

**Remark 2.8.** Proposition \ref{prop:2.7} shows that \( D^2 \geq -2 \) can be replaced by \( D^2 = -2 \) in Definition \ref{def:2.5}

Every toric log del Pezzo surface of Picard number one corresponds to a Fano triangle. See \cite{kn} for a general introduction to Fano polytopes. For each basic surface \( S(X) \) in Notation \ref{def:2.6} we denote by \( P(X) \) the corresponding Fano triangle. See Figure \ref{fig:2} for the explicit coordinates for the ray generators of \( P(X) \) that is basic.

| \( P(\mathbb{P}^2) \) | \( \{0, 1, (1, 0), (-1, -1)\} \) |
|-----------------------------|----------------------------------|
| \( P(\text{std}_n^1), n \geq 2 \) | \( \{0, 1, (-1, 0), (1, -n)\} \) |
| \( P(\text{std}_n^2), n \geq 2 \) | \( \{0, 1, (-2, 1), (1, -n)\} \) |
| \( P(3A_2) \) | \( \{-2, 1, (-2, 1), (2, -2n + 1)\} \) |

**Figure 2.** Ray generators for \( P(X) \) that is basic
See Figure 2 for the drawings of reflexive singular basic Fano triangles.

Figure 3. Reflexive singular basic Fano polygons

3. CASCADES OF TORIC LOG DEL PEZZO SURFACES OF PICARD NUMBER ONE

Definition 3.1. Let $S$ be a toric log del Pezzo surface of Picard number one. We say that $S$ admits a one-step cascade if there exists a diagram as follows:

$$
\begin{array}{c}
S' \xrightarrow{\phi} \bar{S}' \\
\pi \downarrow \quad \bar{\pi} \downarrow \\
S \quad \bar{S}
\end{array}
$$

where
1. $\phi$ is a blow-down of a $(-1)$-curve,
2. $\pi$ and $\bar{\pi}$ are minimal resolutions, and
3. $\bar{S}$ is a toric log del Pezzo surface of Picard number one.

3.1. Existence of a cascade (=Proof of Theorem 1.2). If $S$ is basic, we are done. Assume that $S$ is not basic. Then, there exists a $(-1)$-curve $E$ that intersects a torus-invariant curve $C$ with $C^2 \leq -3$. Let $D$ be the other torus-invariant curve intersecting $E$. We claim that $D^2 = -2$. By [Z, Lemma 1.4], $D^2 \geq -2$. If $D^2 \geq -1$, then, by contracting $E$ and then contracting all torus-invariant curves with self-intersection number at most $-2$, we get a projective surface of Picard number zero, which is a contradiction. Thus, we have $D^2 = -2$. This can also be derived from [Z, Lemma 4.2]. Now, contracting $E$ induces a one-step cascade.

3.2. Inverting a cascade (=Proof of Theorem 1.3). In the process of each one-step cascade $\phi$, the blowing-up locus of $\phi$, in the notation of Definition 3.1 is exactly the intersection point of two torus-invariant curves, one of them being a $(-1)$-curve and the other one has self-intersection number at most $-2$. Since there are exactly three basic surfaces $S(std^1_n)$, $S(std^2_n)$ and $S(3A_2)$ containing a torus-invariant $(-1)$-curve, the result follows.

3.3. Properties of cascades. To describe the applications of Theorem 1.2 and Theorem 1.3 we introduce the notion of a trace of $S$.

Definition 3.2. The sum of self-intersection numbers of all irreducible components of exceptional curves of $f$ multiplied by $-1$ is called the trace $tr(S)$ of $S$. In other words,

$$
tr(s) = -\sum D_i^2
$$
where the sum runs over all exceptional curves $D_i$ over $f$.

Now, the proof of Theorem 1.3 immediately yields the following.

**Corollary 3.3.** The number of singular points of $S$, and the number $\text{tr}(S) - 3L$ are invariant under a cascade where $L$ denotes the number of exceptional curves of the minimal resolution.

**Proof.** It is enough to consider only a one-step cascade. It is clear that $\text{tr}(S) - 3L$ is invariant under a one-step cascade. This also follows from Lemma 2.4. Note that a one-step cascade does not increase the number of singular points. Assume that a one-step cascade decrease the number of singular points of $S$. Then, there is a chain of torus-invariant curves whose dual graph is of the form

$$-n \circ -1 \circ -2 \circ -m \circ$$

where $m \geq -1$ and $n \leq -3$ since $S$ is not basic. Let $E$ be the $(-1)$-curve in the dual graph intersecting the $(-n)$-curve. By blowing down $E$, we can see that $n = 3$ by [Z, Lemma 4.2], hence we get the following dual graph.

$$-2 \circ -1 \circ -m \circ$$

for some $m \geq -1$ (cf. [Z, Lemma 1.4]). This cannot be possible since the Picard number is one. \qed

**Remark 3.4.** By Corollary 3.3, we can easily compute the trace of toric log del Pezzo surface of Picard number one once we know its basic surface.

| $S$ | $S(\mathbb{P}^2)$ | $S(\text{std}_n^0)$ | $S(\text{std}_n^1)$ | $S(\text{std}_n^2)$ | $S(3A_2)$ |
|-----|------------------|-------------------|-------------------|-------------------|----------|
| tr(S) | $-3$ | $n$ | $3L - 5 + n(\geq 3L - 3)$ | $3L - 7 + n(\geq 3L - 5)$ | $3L - 6$ |

The above table shows that the number of singular points and the trace of $S$ determines uniquely the original surface $S$ and its basic surface and vice versa. See Algorithm 4.7.

4. Applications

We completely classify toric log del Pezzo surfaces of Picard number one and their dual graphs.

**4.1. Classification.**

**Theorem 4.1.** Let $S$ be a toric log del Pezzo surface of Picard number one. Then,

1. Either $S \cong \mathbb{P}(1,1,n)$ with $n \geq 1$, or $S$ admits a cascade to one of the following: $S(\text{std}_n^0)$, $S(\text{std}_n^1)$, and $S(3A_2)$ where $n \geq 2$.

2. Let $T$ be the basic surface of $S$. Then, we have the following.

   (a) If $T = S(\text{std}_n^0)$, then $S' = [-n, F_1, n - 1, F_0]$.

   (b) If $T = S(\text{std}_n^1)$, then $S' = [-n, F_1, n - 2, F_1']$.

   (c) If $T = S(3A_2)$, then $S' = [-n, F_1, -m, F_2]$.

where $F'_i$ is a fiber of type $I$ and $F_i$ is a fiber of type $I$ for each $i$.

**Proof.** Since $S(\text{std}_n^0) \cong \mathbb{P}(1,1,n)$, (1) immediately follows from Theorem 1.2 and Theorem 1.3.

We may assume that $S$ is not basic. Then, by taking a finite number of one-step cascade, we can always find three torus-invariant curves whose dual graph is of the
form $\frac{-1}{\circ} - \frac{-2}{\circ}$. Note that they induce a $\mathbb{P}^1$-fibration $\Phi$ on the minimal resolution $S'$ of $S$, on which they form a singular fiber $F$ of type $(I_0)$.

Consider the case $T = S(\text{std}_n^1)$. Since the inverting process only changes the singular fiber $F$ of the $\mathbb{P}^1$-fibration, we see that $G(S) = [\langle -n, F_1, n - 1, 0 \rangle]$ for some integer $n \geq 2$ with the unique singular fiber $F_1$ of type $I$.

Consider the case $T = S(\text{std}_n^2)$. Then, only the two torus-invariant sections of $\Phi$ are invariant under the inverse process among all torus-invariant curves. Thus, we have $G(S) = [\langle -n, F_1, n - 2, F_1' \rangle]$ for some integer $n \geq 2$ where both $F_1$ and $F_1'$ are of type $I$.

Consider the case $T = S(3A_2)$. Since no torus-invariant curve is invariant under the process of cascades in general, the result follows.

$\square$

Now we classify toric log del Pezzo surfaces of Picard number one with given number of singular points.

**Corollary 4.2.** Let $S$ be a toric log del Pezzo surface of Picard number one. Then, we have the following.

1. If $|\text{Sing}(S)| \leq 1$, then $S \cong \mathbb{P}(1, 1, n)$ for some $n \geq 1$.
2. If $|\text{Sing}(S)| = 2$, then $S \cong \mathbb{P}(1, q, (n - 1)q + q_1)$ where $\gcd(q, q_1) = 1$.
3. If $|\text{Sing}(S)| = 3$, then $S$ is obtained by inverting a cascade from $S(3A_2)$ or $S(\text{std}_n^2)$.

In particular, if $|\text{Sing}(S)| \leq 2$, then $S$ is a weighted projective plane.

To prove Corollary 4.2 we recall the Hirzebruch-Jung continued fraction.

**Definition 4.3.** For integers $n_1, n_2, \ldots, n_l$, we set the following notation,

$$[n_1, n_2, \ldots, n_l] := n_1 - \frac{1}{n_2 - \frac{1}{\ddots - \frac{1}{n_l}}}.$$ 

If $n_i \geq 2$ for each $i$, then it is called a *Hirzebruch-Jung continued fraction*.

**Proof of Corollary 4.2.** For (1) and (3), the result follows from Theorem 4.1 and Corollary 3.3. Assume that $|\text{Sing}(S)| = 2$. By Theorem 4.1 and Corollary 3.3, $S$ is obtained by inverting the cascade from $S(\text{std}_n^1)$.

Let $F$ be a singular fiber of a $\mathbb{P}^1$-fibration of the form

$$\begin{array}{c}
\square\ 
f_1
\end{array} - \frac{a}{b} \begin{array}{c}
\square
f_2
\end{array}$$

where $F_1$ is the dual graph corresponding to the Hirzebruch-Jung continued fraction $[n_1, \ldots, n_l]$ and $F_2$ corresponds to $[m_1, \ldots, m_t]$. By Lemma 4.6 below, we let $Q/q = [n, n_1, \ldots, n_l]$ and $\frac{q}{q_1} = [m_1, \ldots, m_t]$ such that $[n_1, \ldots, n_l, 1, m_1, \ldots, m_t] = 0$.

We want to show that $S \cong \mathbb{P}(1, q, (n - 1)q + q_1)$. Again, by Lemma 4.6 we see that

$$Q = [n, n_1, \ldots, n_l] = n - \frac{n - q_1}{q} = \frac{(n - 1)q + q_1}{q}.$$ 

Thus, $S$ and $\mathbb{P}(1, q, Q)$ have the same singularity types. This completes the proof since the singularity type uniquely determines the surface when $|\text{Sing}(S)| = 2$. $\square$
Remark 4.4. It is well known that a weighted projective plane is a toric log del Pezzo surface of Picard number one. One can easily construct infinitely many toric surfaces of Picard number one which is not a weighted projective plane by inverting the cascade from $S(3A_2)$. See the construction in the proof of Theorem 4.10.

Remark 4.5. Corollary 4.2 reproves the results in [D] and [S] for the Picard number one case.

Lemma 4.6. Let $[n_1, \ldots, n_l]$ and $[m_1, \ldots, m_t]$ be Hirzebrugh-Jung continued fractions such that $[n_1, \ldots, n_l, 1, m_1, \ldots, m_t] = 0$. If $[m_1, \ldots, m_t] = \frac{q}{q_1}$, then $[n_1, \ldots, n_l] = \frac{q}{q - q_1}$. 

Proof. This lemma is well-known and easy to prove. See [R, Example 1] for the algorithm to compute $[n_1, \ldots, n_l]$ for a given $[m_1, \ldots, m_t]$. □

Algorithm 4.7. By Corollary 4.2 we can determine whether there exists a toric log del Pezzo surface $S$ of Picard number one with given singularity types.

**INPUT:** an $k$-tuple of rational numbers $(\frac{n_1}{m_1}, \ldots, \frac{n_k}{m_k})$ where $k$ denotes the number of singular points of $S$ and each rational number describes the singularity type.

**OUTPUT:** False if there exists no toric log del Pezzo surface of Picard number one having the given singularity type in INPUT. If it exists, we return $S$ if $S$ is basic, or $S$ and its basic surface if otherwise.

**PROCEDURE:** (using notation in Remark 3.4)

1. If the input is empty, i.e., $k = 0$, then $S = \mathbb{P}^2$.
2. If $k = 1$ and $m_1 = 1$, then $S = \mathbb{P}(1, 1, n_1)$.
3. If $k \geq 2$, then reorder the $k$-tuple so that $i \geq j$ if and only if either $n_i > n_j$, or $n_i = n_j$ and $m_i \geq m_j$.
4. If $k = 2$, $m_1 = n_2$ and $\frac{n_1 - m_1}{m_1}$ is a positive integer, then $S \cong \mathbb{P}(1, m_1, n_1)$.
5. If $k = 3$ and $tr = 3L - 6$, then consider the three dual graphs of the singularities corresponding to the triple in INPUT. Form a cycle $G$ by adding one vertex of weight $-1$ between any two of the three dual graphs. Note that there are four possible ways for forming the cycle. If the graph is $G(3A_2)$ after a finite number of ”blowing-down” of the graph, then $S$ is the toric log del Pezzo surface of Picard number one whose dual graph of the torus-invariant divisors is $G$.
6. If $k = 3$ and $tr \geq 3L - 5$, then consider the three dual graphs $G_1, G_2, G_3$ of the singularities corresponding to $\frac{n_1}{m_1}, \frac{n_2}{m_2}, \frac{n_3}{m_3}$. Form a tree $G$ by adding one vertex of weight $-1$ between $G_1$ and $G_2$; and between $G_1$ and $G_3$. If the graph is $G(std^2)$ after a finite number of ”blowing-down” of the graph, then $S$ is the toric log del Pezzo surface of Picard number one whose dual graph of the torus-invariant divisors is $G$.
7. Return False.

4.2. Kähler-Einstein toric log del Pezzo surfaces of Picard number one.

Proof of Theorem 1.7. Since $\mathbb{P}^2$ is Kähler-Einstein, (2) follows from [HK1 Corollary 3.11].
Let $S$ be a Kähler-Einstein log del Pezzo surface of Picard number one. It is enough to assume that $S$ is singular. Consider the minimal resolution $f : S' \to S$ of $S$. Let $D_1, \ldots, D_k$ be the all irreducible components of the reduced part $D$ of the $f$-exceptional divisor. Then, by [HK] Remark 3.12, $S$ has 3 singular points, each of which has local fundamental group of order $a$. Then, by [HK Section 3 and Lemma 3.6],

$$K_S^2 = tr - 3L + 6 + \frac{9}{a}$$

where $tr = -\sum_{k=1}^L D_k^2$. Since $3e_{\text{orb}} = \frac{9}{a}$, we see that $K_S^2 = 3e_{\text{orb}}$ if and only if $tr = 3L - 6$ if and only if $S$ admits a cascade to $S(3A_2)$. The last equivalence follows from Corollary [4.2] and Lemma [3.4]. Thus, it remains to show that $S$ admits a cascade to $S(3A_2)$. Now the below lemma completes the proof by Corollary [4.2].

Lemma 4.8. Let $S$ be a log del Pezzo surface of Picard number one. If $S$ admits a cascade to $S(\text{std}^2)$, then $S$ is not Kähler-Einstein.

Proof. Let $P$ be the Fano polygon corresponding to $S$. It is enough to show that the barycenter of $P$ is not the origin by [BB Theorem 1.2]. Since $S$ admits a cascade to $S(\text{std}^2)$, $P$ admits a cascade to $P(\text{std}^2)$). Note that the barycenter of $P(\text{std}^2) = \text{conv}\{(1,-1),(1,1),(-3,-1)\}$ is $(-\frac{1}{3}, -\frac{1}{3})$. Since the $y$-coordinate of the barycenter is not increasing during the inverting process of the cascade, the barycenter of $P$ cannot be the origin. □

4.3. Brauer groups. The Brauer group of a toric surface can easily be computed by the following theorem.

Theorem 4.9. [DF] Corollary 2.9] Let $X$ be a toric surface, $\Delta$ be the corresponding complete fan on $\mathbb{R}^2$ and $\Delta(1) = \{\rho_1, \ldots, \rho_n\}$. If $N' = \langle \rho_1 \cap N, \ldots, \rho_n \cap N \rangle$, then $B(X) \cong N/N'$.

Now, as an application of the cascade structure, we show that every finite cyclic group is a Brauer group of a toric log del Pezzo surface of Picard number one.

Theorem 4.10. For each positive integer $n$, there exists a toric log del Pezzo surface $S$ of Picard number one with $Br(S) \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. First, we observe that $Br(\mathbb{P}^2)$ is a trivial group and $Br(\text{std}^2) \cong \mathbb{Z}/2$. For each integer $n \geq 3$, we shall explicitly construct a toric log del Pezzo surface $S$ of Picard number one with $Br(S) \cong \mathbb{Z}/n\mathbb{Z}$ by inverting the cascade from $S(3A_2)$. Let $S_0 = S(3A_2)$ and $f : S'_0 \to S_0$ be its minimal resolution. Choose a chain of two $(-2)$-curves $C_1$ and $C_2$. Let $E_i$ be a $(-1)$-curve intersecting $C_i$ for each $i = 1, 2$. Blow up the intersection point of $C_1$ and $E_1$, and then blow up the intersection point of $C_2$ and $E_2$. Let $S'_1$ be resulting surface and $S_1$ be its anticanonical model. Note that there exists a $(-1)$-curve $E'_1$ intersecting the proper transform $C'_i$ of $C_i$ for $i = 1, 2$. Blow up the intersection point of $C'_1$ and $E'_1$, and then blow up the intersection point of $C'_2$ and $E'_2$. Let $S'_2$ be resulting surface and $S_2$ be its anticanonical model.

One can continue this process. Note that $S_n$ is a toric log del Pezzo surface of Picard number one with 3 singular points of type $2A_{n+2} + [n + 2, n + 2]$. Now it is easy to see that $Br(S_n) \cong \mathbb{Z}/(n + 3)\mathbb{Z}$ by Theorem 4.9. □
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Department of Mathematics, Ajou University, Suwon 16499, Republic Of Korea
Email address: dshwang@ajou.ac.kr