What can we learn from the study of non-perturbative quantum general relativity?

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Abstract

I attempt to answer the question of the title by giving an annotated list of the major results achieved, over the last six years, in the program to construct quantum general relativity using the Ashtekar variables and the loop representation. A summary of the key open problems is also included.
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1 Introduction

Most of what we know about nature at the present time is contained within the realms of either quantum field theory or general relativity. Each of these is a beautiful, powerful and profound theory. However neither can, because of the existence of the other, be said to constitute the basis for a general theory of physics. Thus, while Newtonian physics has been overthrown, it has not been replaced; and it cannot be until we can invent a synthesis of these two great theoretical edifices that can serve as a single foundation for our understanding of all of nature. To do this is the problem of quantum gravity. As such, the problem of quantum gravity is very different from other problems in which we seek to apply a well defined theoretical structure to a new phenomena. It is more open, and more difficult.

Many people who work on this problem complain about its difficulty, and about its distance from experiment. However, I think both complaints are based on a misunderstanding of the nature of the problem. After all, the last time the progress of science required a transformation of this scale in our basic understanding of nature, it took more than 140 years from the publication of Copernicus’s *Revolutionibus* to the publication of Newton’s *Principia*. As far as experiment is concerned, there is a large amount of data about fundamental physics that is presently unexplained, including the undetermined parameters of the standard model of particle physics, the horizon and flatness problem of cosmology and the problem of explaining the formation of structure in the universe. There are good reasons to believe that a quantum theory of gravity will have something important to contribute to the solution of each of these problems.

Many different approaches to this problem have been pursued. This is proper, as there is no way of really knowing from what direction the solution will come. Moreover, as the philosopher of science Paul Feyerabend reminds us, science functions best when the level of consensus remains near the minimum forced on us by the experimental data. In this contribution, I want to discuss only one approach to the problem. This approach has been particularly active during the last six years and it has achieved a certain amount of progress, which I want to summarize here. There are also big unsolved problems, which I will also be mentioning.

The approach I want to describe can be characterized by a list of questions that those who pursue it are seeking to answer. These may be stated as follows:

1) We take it as given that any quantum theory of gravity that has a chance of being a correct description of reality must be non-perturbative. This means that the theory cannot be based on an expansion around a single classical background, in which it is the deviations from the background that are quantized. Instead, all of the geometrical quantities that describe the geometry of spacetime must be treated as quantum operators. This means that very few of the techniques of conventional quantum field theory can be directly applied to it. Can
we invent a new approach to quantum field theory that applies to the cases of field theories defined on differential manifolds without fixed metric structures?

2) Specifically, it is diffeomorphism invariance that expresses the absence of a non-dynamical background geometry in classical general relativity. As such, its implementation in the quantum theory is the central problem of constructing any quantum theory of gravity. We would then like to understand how the requirement of exact diffeomorphism invariance requires us to modify the standard quantum field theory techniques such as regularization, renormalization, operator product expansions, and so forth.

3) We are not trying to construct the quantum theory of gravity. We are trying to construct a theory which is consistent with quantum mechanics and general relativity. For this reason we study quantum general relativity because we have its exact nonperturbative formulation, classically. String theory is, in principle, an attractive program for the unification of physics. However, as long as it lacks a nonperturbative formulation it cannot be the basis of an exploration of the problems of nonperturbative approaches to quantum gravity. We will not be disappointed if, in the end, quantum general relativity is not the theory of nature. But we aim to decide cleanly whether or not there is a consistent mathematical theory, with a sensible physical interpretation, that could be given that name.

4) Certainly it may be true that a complete solution to the problems of quantum gravity and quantum cosmology will require also a solution to the problem of the unification of gravitation with the other fundamental interactions. This could happen through a "microphysics → macrophysics" approach like string theory or a "macrophysics → microphysics" approach, one example of which is proposed in [3]. At the same time, it may also be the case that important aspects of the problem of quantum gravity can be solved independently of what matter fields gravitation is coupled to. This will be the case to the extent that qualitatively new physics emerges from the construction of diffeomorphism invariant quantum field theories. We would like to see if the study of quantum general relativity at the nonpertubative level can lead to the discovery of such phenomena.

5) In any such approach, one of the key questions is what constitutes an observable. This is because, in a diffeomorphism invariant theory, coordinates and clocks have no a-priori meaning; any meaningful observable must express some relationship between physical fields, rather than being defined with respect to a background geometry or an external observer. One aspect of this problem is the problem of time in quantum cosmology\footnote{Good discussion of this problem are in [4, 46]. The author’s point of view about the problem of time in quantum cosmology is described in [5].}. These problems arise already at the classical level, but there they can be solved\footnote{Good discussion of this problem are in [4, 46]. The author’s point of view about the problem of time in quantum cosmology is described in [5].}. We then need to ask: Can we put what we know about the problem of observables and time in classical general relativity together with the technical developments...
in diffeomorphism invariant quantum field theory to learn how to construct operators that correspond to physically meaningful quantities in a quantum theory of gravity and cosmology?

While these are the main questions that have guided us in the work I will be describing below, the fact that we have been able to make any progress is due primarily to two technical developments. These are the Ashtekar formulation of general relativity [6], and the loop representation of quantum field theories [7, 8].

The Ashtekar variables and the loop representation have been the subject of a number of reviews, including three published within the last two years [9, 10, 11]. For this reason, I will not repeat here what the reader can easily find in those reviews or in the original papers. Instead, I will devote myself to describing, in as concise a manner as possible, the basic results that have been, so far, achieved by this program. As my aim is to be brief, I do not give many technical details, except when describing results that have not so far appeared elsewhere.

I should point out that while I have tried to write this paper so it can be read by someone who is not an expert on quantum gravity, I do not include an introduction to the basics of canonical general relativity and the methods of canonical quantization. These can be found in other papers in this proceedings, such as the paper of Kuchar [12], as well as in numerous other places, among them [3, 4, 11].

I begin in the next section by asking how quantum field theory must be modified so that it can make sense in the absence of a background metric. The core of the paper is in the following two sections, where I give, correspondingly, an annotated list of major results and a list of key open problems. In the conclusion I try to provide an answer to the question in the title.

2 Basic ideas of the approach

A good place to begin is with the question of how to construct diffeomorphism invariant quantum field theories. The key problem is how to get rid of the dependence on the background metric that underlie the standard formulations of quantum field theory in Minkowski spacetime. Here are four ways in which Minkowski spacetime quantum field theories depend on the background metric:

A) The Fock spaces of linearized field theories are constructed by associating operators with the solutions of free field theories on a background spacetime.

I want to mention that this is not intended as a review of all of the developments associated with the Ashtekar variables. Particularly, there are several important developments connected with the classical theory that most likely have implications for the quantum theory, such as the Capovilla-Dell-Jacobson [64] formalism and the classical solutions of the diffeomorphism constraints of Newman and Rovelli [65]. However, as my focus is the quantum theory, and as their implications for that have yet to be developed I do not mention them, as well as a number of other very interesting developments, here.
B) The definition of the vacuum and the definition of the creation and annihilation operators depend on the splitting of the solutions of the linearized field equations into positive and negative frequency parts. This splitting is Poincare invariant but, as we know from our experience with quantum field theories in curved spacetime, it is not invariant under any larger group of transformations and depends also on the background metric.

C) The regularization and renormalization procedures necessary to make sense out of operator products and interactions in quantum field theory all depend explicitly on the background metric. The background metric is used in the definition of the cutoff scale, in the separation of terms in the operator product expansion and to measure how fast the cutoffs are removed in taking the limits that define renormalized operator products.

D) The inner products of conventional quantum field theories are defined by the requirement of Poincare invariance with respect to a given flat background metric.

We propose to replace each of these constructions with alternatives that do not depend on any background metric. The way in which we go about doing it defines the approach we are calling non-perturbative quantum general relativity. These alternative constructions are:

A') We construct background independent quantum field theories by constructing new representations of algebras of observables that are unitarily inequivalent to the Fock representations. It is here that the loop representation plays the key role. These representations carry unbroken unitary representations of the diffeomorphism group; this makes possible the exact solutions of the constraints that impose diffeomorphism invariance on the quantum states.

B') We replace the splitting into positive and negative frequency parts by a splitting into self-dual and antiself-dual parts. This latter splitting is defined only in special cases such as that of connection fields in four spacetime dimensions. It has the great virtue that the connection and curvature of any four dimensional spacetime can be split into self-dual and antiself-dual parts; as a result, the self-dual representation, in which states are functions of the self-dual part of the field, can be defined in a nonperturbative quantization. This stands in contrast to a positive frequency representation, which can only be defined with respect to a fixed background metric and a correspondingly restricted time coordinate.

This is one of the main motivations for the use of the Ashtekar variables as the basis for the quantization.

At the linearized level, the self-dual part of a connection (whether or the electromagnetic, Yang-Mills or linearized gravitational field) consists of the positive frequency part of the left handed helicity, plus the negative frequency part of the right handed helicity. The negative frequency part must then be quantized in a kind of "anti- Bargmann" representation. It was not entirely obvious that such a quantization exists, and one of the key results of the program is that self-dual representations exist at the linearized level\[13, 14\].
C') As I will describe below, the process of renormalization, through which the product of local operators is defined to be another local operator, is necessarily dependent on a background metric (or at least on a background volume element [11, 15].) There is then no local background independent renormalization procedure for local operators. At the same time, we have found that there are background independent regularization procedures for certain non-local observables. Their use results in finite and background independent operators [15, 11]. Further, one can make an argument that any operator constructed from functions of local operators through a regularization procedure which is diffeomorphism invariant in the limit that the regulator is removed will necessarily be finite [1, 29].

D') Without a background metric, there is no symmetry principle that can guide the selection of the inner product. We propose to base the selection on an alternative principle: a complete set of real physical observables must be represented by self-adjoint operators [9]. The proposal depends on the construction of a set of physical observables, realized as well defined operators on the space of physical states, for which the classically corresponding observables are known. We can then use the reality conditions satisfied by the corresponding classical observables to posit hermiticity relations for the physical operators. The inner product is then to be determined by the condition that it realize these hermiticity conditions.

Clearly, what I have just stated is a strategy, but it is not a completely defined procedure. Crucial questions such as which observables and how many observables are necessary are left unspecified. In a number of special cases, including Maxwell theory [14] and linearized gravity [14], in 2+1 gravity [17, 18, 9] and other finite dimensional examples [19], this principle can be implemented and leads to the physically correct inner products. However, for reasons that will become clear later, we have not yet been able to test this idea in the case of full quantum gravity.

3 The main results of nonperturbative quantum gravity

I would like now to state, concisely and with a minimum of technicalities, exactly what the main results of this approach are, to date. To frame the discussion, let me begin by recalling the main elements of canonical quantization of general relativity in the Ashtekar formalism, following the method of Dirac [20]. We begin with a phase space, which is taken to be the space of pairs of complex $\SU(2)$ connections, $A^a_i$ and conjugate electric fields $\tilde{E}^{ai}$ on a three manifold $\Sigma$ of fixed topology. For everything that follows it will be crucial that $\tilde{E}^{ai}$ is a vector density field. This is necessary so that the Poisson bracket relation,

$$\{A^a_i(y), \tilde{E}^{ai}(x)\} = \delta^a_y \delta^{ij} \delta^3(x, y)$$

(1)
makes sense, because the delta function must, in the absence of a background metric, be a density.

To quantize the system, we proceed to construct the state space in three steps, which we call kinematic, diffeomorphism invariant and physical. To define the kinematical state space, $V_{\text{kin}}$, which is the starting point for the quantization, one must first pick out of the algebra of functions on the phase space a subalgebra that one wants to have represented exactly by quantum operators. A choice is necessarily involved at this stage; because of the existence of problems of ordering and the regularization of operator products, it is impossible that the whole algebra of functions on the phase space be identically represented as an algebra of operators. We shall call the subalgebra chosen $A$. The one condition it should satisfy is that its elements should coordinatize the phase space of interest.

Once we choose $A$, the kinematical state space is then constructed by finding a linear space $V_{\text{kin}}$ on which there acts an algebra $\hat{A}$, that is isomorphic to $A$, at least up to terms that vanish in the limit that $\hbar \to 0$.

The diffeomorphism invariant and physical state spaces, which will be called $V_{\text{diffeo}}$ and $V_{\text{phys}}$ are then constructed as follows. One constructs operators in $V_{\text{kin}}$ that correspond to the classical diffeomorphism and Hamiltonian constraints. (These will be denoted $\hat{D}(v)$ and $\hat{H}(N)$, respectively, where $v^a$ and $N$ are smearing functions that are, respectively, a vector field and an inverse density field.) $S_{\text{diffeo}}$ is then defined to be the subspace of states in $V_{\text{kin}}$ that are annihilated by $\hat{D}(v)$. $V_{\text{phys}}$ is defined to be the subspace of those states that are annihilated both by $\hat{D}(v)$ and $\hat{H}(N)$.

This process is called Dirac quantization[20]. Unfortunately Dirac, while setting out the procedure in one of his very readable little books, failed to include two details that must arise in any field theoretic application of the method. The first is that, at least in all representations known to this time, the Hamiltonian constraint involves operator products and must be regularized. The second is the question of how to choose the inner product.

The key point is that it is not sufficient to give an inner product at the kinematical level, because, unless one is very lucky, the solutions to the quantum constraint equations will be non-normalizable with respect to the kinematical inner product. (Thus, $V_{\text{diffeo}}$ and $V_{\text{phys}}$ are, in general, subspaces of $V_{\text{kin}}$ as vector spaces, and not as Hilbert spaces.) The physical inner product must be chosen just on the space $V_{\text{phys}}$ (and similarly for the diffeomorphism invariant states).

Let us now begin the quantization along these lines. Conventionally, one chooses to quantize using the algebra (1). This is the starting point of the construction of Fock representations. However, to construct the non-Fock representations we will be interested in, we take a different subalgebra as a starting point. The main idea behind this choice is to implement the $SU(2)$ gauge invariance explicitly by choosing an algebra of gauge invariant functions. To do
this we take the configuration variable is taken to be the holonomies,
\[ T[\gamma] \equiv \frac{1}{2} \text{Tr} \, P \, e^G \int_\gamma A. \]  
(2)

For a conjugate variables we would like to take a family of functions linear in the conjugate momentum. The requirement of gauge invariance then requires us to take functions that are also dependent on loops. To construct them, consider a loop \( \beta \) on which we have a preferred point \( \beta(s) \). We may then define the function,
\[ T^a[\beta](s) \equiv \frac{1}{2} \text{Tr} \, [\tilde{E}^a(\beta(s)) \, P \, e^G \int_s \dot{\beta}^a(t) \, A_a(t)] \]  
(3)

defined by tracing \( \tilde{E} \) at the point \( \beta(s) \) against the holonomy around \( \beta \) starting and ending at \( \beta(s) \). To complete the definition of the conjugate variable, let us consider a rubber band, \( \beta \) defined by one parameter family of loops \( \beta_u(s) \) with \( 0 \geq u \geq 1 \). We may then define
\[ E[\beta] \equiv \int \int duds\epsilon_{abc} \frac{du}{ds} T^c[\beta_u](s) \]  
(4)

The two observables (2) and (4) have a very pretty algebra. Let us define
\[ I[\gamma, \beta] = \int d\gamma(t) \int d^2 \beta^{bc}(s, u) \delta^3(\gamma(t), \beta(s, u)) \epsilon_{abc} \]  
(5)

to be the intersection number of the loop \( \gamma \) with the two dimensional surface \( \beta \). The intersection number is equal to an integer; it counts, with the sign reflecting the orientation, the number of times that the loop intersects the strip. A simple calculation than yields,
\[ \{ T[\gamma], E[\beta] \} = GI[\gamma, \beta] \left[ T[\gamma \circ \beta_u^*] - T[\gamma \circ \beta_u^{-1}] \right]. \]  
(6)

Here, \( \alpha \circ \beta \) refers to the loop made by combining \( \alpha \) and \( \beta \) and \( \beta_u^* \) is the element of the one parameter family that goes through the point of the rubber band where \( \gamma \) intersects it.

This algebra is called the loop-strip algebra, or the loop algebra for short.

It is important to comment on the presence of Newton’s constant, \( G \), in the definitions of the holonomies in these observables. Because the frame field \( \tilde{E}^{ai} \) should be dimensionless, the quantity it is conjugate to, the Ashtekar connection, cannot have the usual dimensions of a connection of inverse length if the Poisson brackets (1) are to hold. Instead, it is \( GA^a_i \) that has dimensions of inverse length. As a result, there is a \( G \) in (6). We will see later that this plays a key role in several results described below.

With this background, I now give an annotated list of the major results so far which have been achieved in the direction of nonperturbative quantum general relativity. The results will be organized according to the three levels of Dirac quantization: kinematical, diffeomorphism invariant and physical.
3.1 Existence of the loop representations at the kinematical level

A loop representation is a quantization of a gauge theory based on the loop algebra given by (6). The loop algebra bears a relationship to the canonical algebra (1) which is somewhat analogous to that the Weyl algebra bears the simple Heisenberg algebra. There are representations of the loop algebra which are also representations of the canonical algebra. Among them is the Fock representation. However, there are also representations of the loop algebra which in which there exist no operator linear in \( A_a^i \), which are inequivalent to the Fock representation. Among these are the representations which are of interest to non-perturbative quantum gravity [22, 21].

A loop representation may be characterized as follows. We introduce a basis of bra states, labeled by loops, such that any ket state \( |\Psi\rangle \) may be written as

\[
\Psi[\gamma] = \langle \gamma | \Psi \rangle \tag{7}
\]

Note that this notation does not assume the existence of an inner product. The elements of the ket space are functions \( \Psi[\gamma] \) which live in some space of functions, the complete specification of which is necessary to define the representation. The bra states, \( \langle \gamma | \) are a linear space dual to the ket space, whose action is defined by (7). For any loop representation these ket states are constrained to satisfy

\[
\sum_i c_i <\gamma_i| = 0 \tag{8}
\]

whenever the holonomies satisfy

\[
\sum_i c_i Tr P e^{\int_{\gamma_i} A} = 0 \tag{9}
\]

for all connections \( A_a^i \). This is the way that the identities satisfied by holonomies-the Mandelstam identities-are imposed on the representation. The loops \( \gamma \) may then be taken to be piecewise differentiable loops. It is convenient also to use a convention in which a loop can refer to a set of loops, if we assume that the trace of the holonomy of a set of loops is taken to be the product of the traces of the holonomies of the individual loops. Indeed, by a loop or a set of loop we really always mean equivalence classes of loops under the relation (8).

The action of the operators that represent the loop variables (2) and (4) are then defined by their action on the bra states:

\[
<\gamma|\hat{T}[\alpha] = <\alpha \cup \gamma | \tag{10}
\]

\( ^3 \)As has been emphasized by Gambini and Trias and their coworkers, the space of these equivalence classes of loops actually forms a group [8]. The basis for their alternative formulation of the loop representation is the representation theory of this group.
\[
<\gamma|\dot{E}|\beta> = \hbar G \left( <\gamma \circ \beta_u^+| - <\gamma \circ \beta_u^-| \right) \tag{11}
\]
In these equations the \( \cup \) represents union in a set of loops while the \( \circ \) represents forming a new loop by a product of two loops. It is not hard to verify that these operators satisfy the algebra (6) (with the \( \hbar \) inserted. It is, in fact, interesting to note that \( l_P^2 = \hbar G \) appears in the algebra even at the kinematical level. This fact (and the fact that it is the Planck area and not the Planck length that appears) plays a crucial role below.

Finally, we may note that the fact that the group is \( SL(2,C) \) (or some real subgroup of it) appears in the form of the right hand side of (6), together with the equivalence relations generated by (8). Connections based on other Lie groups also have loop representations; they differ only in the form of this action and in the equivalence relations generated by the holonomies.

### 3.2 Applications of the loop representations to linearized field theories: Fock representations

An example of the foregoing is the loop representations of abelian connections. The loop representation actually first appeared in a quantization of the free Maxwell field given by Gambini and Trias in 1981 [8]. The loop quantization of free Maxwell theory was also treated later in [16, 13]. In these papers it is shown that one can construct a space of states \( \Psi[\gamma] \) which is isomorphic to the Fock space of free photon states. The whole of free quantum electrodynamics, including the Hamiltonian, creation and annihilation operators and inner product may be written simply in the loop representation.

Indeed, there are several different representations of the Fock space as functions of loops. These correspond to the different ways to write the Fock space as functions of the connection. The two well known connection representations of free field theories are the Schroedinger representation, in which the states are functions of the real connection, \( A_a \) and the positive frequency, or Bargmann representation, in which the states are functions of the positive frequency part of the connection.

In the loop representation that corresponds to the positive frequency connections the ground state is simply written as

\[
\Psi_0[\gamma] = <\gamma|0> = 1 \tag{12}
\]

The state of one photon of momentum \( \vec{p} \) and polarization \( \vec{\epsilon} \) is written,

\[
\Psi_{\vec{p},\vec{\epsilon}}[\gamma] = <\gamma|\vec{p},\vec{\epsilon}> = \oint ds e^{\vec{p}\cdot \gamma'(s)} \gamma'(s) \epsilon_a(s) \equiv F[\vec{p},\vec{\epsilon}] \tag{13}
\]

The \( F[\vec{p},\vec{\epsilon}] \)'s play a dual role in the formalism. First, they provide a useful set of coordinates for the loops modulo the relations (8), with the abelian holonomy

\footnote{Their extension to a set of coordinates on the space of loops modulo the relations that arise from non-abelian holonomies are the basis of a new approach to the loop representation of Gambini and his collaborators [23].}
Second, the multiphoton states are written as polynomials of the \( F[\vec{p}, \vec{\epsilon}] \)'s. The Fock space is then defined to be the space of functions of loops that are analytic functions of the loop coordinates \( F[\vec{p}, \vec{\epsilon}] \).

The Hamiltonian and inner product of quantum Maxwell theory are then written as operators on loop space as follows,

\[
\hat{H} = \sum_{\epsilon} \int d^3p |p| F[\vec{p}, \vec{\epsilon}] \frac{\delta}{F[\vec{p}, \vec{\epsilon}]}
\]

\[
<\Phi|\chi> = \int [dF] \bar{\Phi} \chi e^{-\int \frac{d^3p}{|p|} |F[\vec{p}, \vec{\epsilon}]|^2}
\]

It is interesting to note that in every case where the construction of the loop representation has been completed, it has been found that there exists a transform that connects it to the appropriate connection representation in which the states are functions of the connection. The general form for this transform is

\[
\Psi[\gamma] = \int d\mu[\bar{A}] T[\gamma, A] \psi[A]
\]

where \( d\mu[A] \) is an appropriate measure on the space of connections mod gauge transformations.

### 3.3 Existence of self-dual representations for both connection and loop representations

The key discovery of Ashtekar is that in full general relativity the self-dual part of the connection may be considered as a configuration variable of the theory, i.e. every component at every point commutes with every other one. Because of this, we would like to base the quantization of the full theory on the self-dual representation, in which the observables, (2) and (4), whose algebra we quantize, are taken to be functions of the self-dual part of the connection. If this is to be a successful route to the quantum theory it would be most convenient if the linearized theory can also be quantized in a representation in which the observables are functions of the self-dual part of the linearized connection. This, however, gives rise to the following question: At the linearized level, the self-dual part of the connection is the positive frequency part of the left handed helicity component, plus the negative frequency part of the right handed helicity component. This means that the right handed component must be quantized in a kind of negative frequency, or anti-Bargmann representation. Thus, the first question that must be asked is whether there exist such anti-Bargmann representations, which are diagonal in the annihilation operator rather than in the creation operator. At first sight this seems to be impossible; consider for example the equation that the annihilation operator annihilates the ground state. In such a representation it must read

\[
<\bar{z}|\hat{a}|0> = \bar{z} \psi_0(\bar{z}) = 0.
\]
In fact, such representations exist, but their expression requires distributions rather than holomorphic functions, as in the usual Bargmann representation. Thus, in the sense of distributions, the solution to (17) is

$$\hat{\psi}_{0}(\bar{z}) = \delta(\bar{z}).$$

(18)

Once this obstacle is overcome, it is straightforward to construct the full negative frequency representation, for the harmonic oscillator and for any linear field theory whose configuration variable is a connection. Among these is linearized gravity, which may be quantized in the loop representation.

The existence of the loop representation for linearized general relativity not only serves as a confirmation of the basic program of nonperturbative quantum gravity, it may expected to play a key role in the physical interpretation of the exact theory.

3.4 Non-Fock representations of the loop algebra

The Fock representations are important for the construction of linearized field theories, but they are inappropriate as a starting point for the construction of diffeomorphism invariant theories. This is because they cannot carry unbroken representations of the diffeomorphism group for the simple reason that the diffeomorphisms are broken by the existence of the background metric. Since the diffeomorphisms cannot be represented one cannot use them as a starting point to construct diffeomorphism invariant states.

The key question is then, do there exist representations of the kinematical observable algebra that carry unbroken representations of the spatial diffeomorphism group? I do not know the answer for the case of the canonical algebra (1). But, for the loop algebras of the form of (6) (and their generalizations to other groups) there do exist representations with this property.

These representations are based on the use of the discrete measure on the space of loops. To construct this representation it will be useful to introduce a set of basis states, which we call characteristic states. For a loop $\alpha$ which contains no intersections, we may define the state, such that, for $\gamma$ also nonintersecting,

$$\chi_{\alpha}[\gamma] = <\gamma|\alpha> = 1 \text{ if } \alpha = \gamma \text{ and otherwise vanishes.}$$

(19)

Here, the equality is always meant in the sense of the equivalence relations (8).

There is one more case, which is if $\gamma \neq \alpha$, but contains intersections. In this case $\chi_{\alpha}[\gamma]$ does not necessarily vanish, its actual value is determined by requiring that it be an eigenstate of the operators defined in (25) and (30), below. There are also basis states associated with intersecting loops.

\footnote{recall that by loops I always mean loops modulo the relations (8).}
We will denote these characteristic states abstractly by $|\alpha>$, making use of the standard Dirac formalism in which bras represent elements of function spaces and kets are linear maps from those function spaces to the complex numbers.

Let us then consider the linear space, $\mathcal{V}_{\text{discrete}}$, which consists of states of the form,

$$\Psi[\gamma] = \sum_I c_I \chi_{\alpha I}[\gamma]$$

(20)

where we require that

$$\sum_I |c_I|^2 < \infty$$

(21)

Here the sum is over any countable set of loops in $\Sigma$. It can be easily verified that the formal definitions of the loop operators, (10) and (11), are well defined when acting on the states in $\mathcal{V}_{\text{discrete}}$.

We can impose an inner product on $\mathcal{V}_{\text{discrete}}$ by defining

$$<\alpha| = |\alpha>$$

(22)

Note that this inner product does not realize the kinematical relativity conditions for the $T[\gamma]$ observable$^6$, but it does realize them (at least formally) for functions of $E^a_i$ only. Like any kinematical inner product, it is useful only as a mathematical device.

With any choice of inner product which makes the characteristic states normalizable, the discrete representation is unitarily inequivalent to Fock space. There is a possibility that it may be of use for nonperturbative treatments of gauge theories, because it implements, in the continuum, the quantization of the electric flux. This is a possibility that needs further development. At the present time its use comes from its application to diffeomorphism invariant quantum field theories. This is because an exact, unbroken, unitary representation of the diffeomorphism group can be defined on it as,

$$\hat{U}(\phi)|\gamma> = |\phi^{-1} \circ \gamma>$$

(23)

where $\phi$ is any diffeomorphism. This means that the generator of diffeomorphisms is well defined in this representation,

$$\hat{D}(v)\Psi[\gamma] = \frac{d}{dt} \hat{U}(\phi_t)\Psi[\gamma],$$

(24)

where $\phi_t$ is a one parameter group of diffeomorphisms generated by the vector field $v^a$. $\hat{D}(v)$ may be shown from these definitions to satisfy the algebra of vector fields on $\Sigma$.

$^6$For the reader unfamiliar with the reality conditions, they are the conditions that the three metric and its time derivative both turn out to be real when computed in the Ashtekar formalism. They imply that $A^a_i$ is a complex connection, which, together with its complex conjugate, satisfies a certain polynomial condition. The result is that the loop operators are not real.
3.5 Classification of the loop representations in the real case

I have described two different representations of the loop algebra (6). In one case, in which we require that both both $A_i^a$ and $\tilde{E}^a_i$ are real, the representations of the $SU(2)$ loop algebra have been completely classified. This was done by Ashtekar and Isham [21], who make use of the fact that in this case the loop algebra (6) is a star algebra. The classification can then be done using some of the technology of the representation theory of star algebras developed by Gel’fand and collaborators.

3.6 Application of the loop representation to quantum Yang-Mills theory

As I mentioned above, the loop representation may be applied to non-abelian gauge theories [24]. The loop representation may be developed in the context of the lattice regularization, where the loop states provide a gauge invariant basis for the state space. This formulation has been the starting point for several works in which new numerical approaches to lattice gauge theory, both with and without fermions have been explored. These works involve approximation procedures which are based on the fact that in the loop basis almost all the matrix elements of both the Hamiltonian and inner product are zero. As a result, sparse matrix and cluster techniques may be applied [25, 26]. For example, in $2 + 1$ dimensions extensive numerical calculations have been done for both $SU(2)$ [25] and $SU(3)$ [26], which showed that results for the ground state energy and mass gap (as functions of the coupling constant), obtained previously by Monte Carlo simulations and are reproduced accurately. Furthermore, in $3 + 1$ dimensions numerical work has been, and is being, done for the case of QED with fermions [27].

In addition to these numerical approaches, there have been some very interesting analytical work done on non-abelian gauge theories in the loop representation, by Loll [28], Rovelli [29] and others.

3.7 Nonexistence of local operators in non-perturbative quantum gravity

For the remainder of this section, I will confine myself to the applications of the loop representation to quantum gravity. I begin with several results about observables and the classical limit at the kinematical level. First, of all, it is not trivial to construct quantum observables at the kinematical level because such observables must be invariant under the Yang-Mills gauge transformations, and any such observables that involve the frame fields involve operator products. Thus, regularization is an issue even at the kinematical level.
As the kinematical level is meant to be a stepping stone to the diffeomorphism invariant and physical levels, we will be interested only in regularization procedures that do not introduce extra background dependence into the final definitions of the operators. Any regularization procedure depends on additional structure such as background metrics or coordinate systems as these are needed to specify how the point splitting is done or define the cutoffs. What we must then require is that when finite operators are finally produced as a result of the process they have no dependence on these structures.

We have discovered that this requirement seems to rule out the conventional renormalization procedures of Poincare invariant quantum field theories[15, 11]. Although this was discovered through a painful process in which many possible approaches were tried and discarded, the reason for this can be stated very simply. Local operators are distribution valued and distributions are, in the absence of a background metric, densities of weight one. A renormalization procedure is a procedure by which a product of two local operators is defined to be a third local operator. It is thus a procedure for multiplying two distributions to get a third distribution. However, there is a problem with the density weights, because the product of two distributions should have density weight two, but a local operator will have only density weight one. The result is that any such renormalization procedure must introduce an additional scalar density so that the density weights on the left and right hand sides of the product match. What we found was that in any procedure we tried, such a density always appeared which was a function of the background structures introduced in the regularization and renormalization procedures.

Now, in Minkowski spacetime, or even in quantum field theory in a curved spacetime, there is a preferred density which is given by the determinant of the background metric. In these cases the ambiguity may be reduced to one free renormalization constant. This is the reason for the existence of a free renormalization scale in the conventional renormalizations of operator products.

However, in nonperturbative quantum gravity there is no preferred density and the ambiguity of a scale in the renormalization of a quantum field theory in a classical background becomes an ambiguity up to a density. The result is that it is very difficult to imagine how a renormalization procedure could be constructed for operator products in this context that did not lead to a breaking of diffeomorphism invariance.

This means, in particular, that when using a frame field formalism such as the Ashtekar variables there is no operator to measure the metric at a point. This is because the basic variable is the frame field, $\tilde{E}^{ai}$, which is related to the metric through $\tilde{q}^{ab} = \tilde{E}^{ai} \tilde{E}^{ib}$.
3.8 Existence of finite, background independent non-local operators at the kinematical level

One might think that as a result of the situation I’ve just described it is impossible to define meaningful observables that measure the spatial metric in non-perturbative quantum gravity. Fortunately, this is not the case, because there are non-local observables that are equivalent to the metric in the sense that a complete measurement of them allows the metric to be reconstructed. We have found that it is possible to construct quantum operators that correspond to some of these non-local observables and that these operators are finite and background independent, when constructed by means of the right regularization procedure\[15, 11\]. Thus, as these operators don’t need to be renormalized, they escape the difficulty I described in the previous paragraph.

The idea behind the construction of these operators is very simple: If there is no unambiguous procedure for multiplying two distributions to get a third distribution, we may construct unambiguous procedures that define the square root of the product of two distributions.

I will mention here three examples of such observables\[15, 11\]. First, given any one form $\omega$, we may define the integral of its norm as follows,

$$Q[\omega, \tilde{E}^{ai}] = \int_{\Sigma} \sqrt{\tilde{E}^{ai} \omega_a \tilde{E}^{bi} \omega_b}$$

This observable can be regulated through a modified point splitting procedure. I will not describe it here, the details are given in \[11\]. As may be expected, the hardest part of the construction is taking the operator square root. The result is easiest to express in terms of the bras $<\alpha|$. For the case of a non-intersecting loop, $\alpha$, the bra is, for every $\omega$, an eigenstate of the operator corresponding to $Q[\omega, \tilde{E}^{ai}]$. The action of the operator is given by,

$$<\alpha|\hat{Q}[\omega] = \frac{P_{\text{Planck}}}{2} \int_{\alpha} ds |\dot{\alpha}^a \omega_a(\alpha(s))| <\alpha|.$$  

A second nonlocal operator that can be defined as a quantum operator is the area of any surface. Given a surface $S$, there is a function on the kinematical phase space that is its area, it is given by,

$$A[S, q] = \int_{S} \sqrt{q^{ab} n_a n_b}$$

where $n^a$ is the unit normal of the surface. The problem is how to turn this into an operator when there is no operator for the metric $\tilde{q}^{ab}$? There is a solution, which is the following. Let me represent the surface by a distributional one form, $\pi_{a}^{S}$ which is given by,

$$\pi_{a}^{S}(x) = \int d^2 S^{bc}(\sigma) \text{delta}^3(x, S(\sigma)) \epsilon_{abc},$$
where $\sigma$ are coordinates on the surface and $\epsilon_{abc}$ is the inverse of the Levi-Civita density. I then can consider the expression

$$A(S, \tilde{E}) = Q(\pi, \tilde{E})$$

(29)

It is not difficult to show that this is equal to the area of the surface given by the metric by (27). To show this, we demonstrate that an equivalent expression, when the frame fields are smooth, is given by

$$A(\pi, \tilde{E}) = \lim_{N \to \infty} \sum_{i=1}^{N} \sqrt{A_{\text{approx}}[\mathcal{R}_i]}$$

(30)

where space has been partitioned into $N$ regions $\mathcal{R}_i$ such that in the limit $N \to \infty$ the regions all shrink to points. Here, the observable that is measured on each region is defined by,

$$A^2_{\text{approx}}[\mathcal{R}] = \int_{\mathcal{R}} d^3x \int_{\mathcal{R}} d^3y T^{ab}(x, y) \pi_a(x) \pi_b(y)$$

(31)

Here $T^{ab}(x, y)$ is a loop operator that is quadratic in the frame fields $\tilde{E}^{ai}$. It is constructed in the following way. Pick a background Euclidean metric and use it to define, for every two points, $x$ and $y$, in the spatial manifold $\Sigma$, a circle, $\gamma_{xy}$, such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(\pi) = y$ and such that in the limit that $y$ approaches $x$, the circles shrink to the point $x$. Then, define

$$T^{ab}(x, y) = \frac{1}{2} \text{Tr} \left[ (\mathcal{P} \exp G \int_{y}^{\pi} a d\gamma_{xy}^{a}) \tilde{E}^{a}(x) (\mathcal{P} \exp G \int_{x}^{y} a d\gamma_{xy}^{a}) \tilde{E}^{b}(y) \right].$$

(32)

To show the equivalence between (29) and (30), we start with (29) and regulate it by means of a point splitting procedure by introducing, with respect to the background euclidean coordinate system, a set of test fields $f_\epsilon(x, y)$ by

$$f_\epsilon(x, y) = \frac{1}{\epsilon^3} \theta\left[ \frac{\epsilon}{2} - |x^1 - y^1| \right] \theta\left[ \frac{\epsilon}{2} - |x^2 - y^2| \right] \theta\left[ \frac{\epsilon}{2} - |x^3 - y^3| \right].$$

(33)

In these coordinates

$$\lim_{\epsilon \to 0} f_\epsilon(x, y) = \delta^3(x, y)$$

(34)

We can then write

$$A(\pi, \tilde{E}) = Q(\pi, \tilde{E}) = \lim_{\epsilon \to 0} \int d^3x \int d^3y \int d^3z T^{ab}(y, z) \pi_a(y) \pi_b(z) f_\epsilon(y, x) f_\epsilon(z, x)$$

(35)

When the expression inside the square root is slowly varying in $x$ we can re-express it in the following way. We divide space into regions $\mathcal{R}_i$ which are cubes.
of volume $\epsilon^3$ centered on the points $x_i = (n\epsilon, m\epsilon, p\epsilon)$ for $n, m, p$ integers. We then write,

$$
A(\pi, \tilde{E}) = \lim_{\epsilon \to 0} \sum_i \epsilon^3 \left[ \int d^3 y \int d^3 z T^{ab}(y, z) \pi_a(y) \pi_b(z) f_\epsilon(y, x_i) f_\epsilon(z, x_i) \right]^{\frac{1}{2}}
$$

$$
= \lim_{N \to \infty} \sum_{N=1}^{N} \sqrt{A^2_{\text{approx}}(R_i)}
$$

(36)

If we now plug into these expressions the distributional form (28) it is straightforward to show that

$$
A(\pi, \tilde{E}) = \int_S \sqrt{h}
$$

(37)

where $h$ is the determinant of the metric of the two surface, which is given by $h = q^{ab} n_a n_b$ where $n^a$ is the unit normal of the surface.

It is not difficult to show that that starting from the expressions (30) and (31) we may construct a quantum operator for the area of a surface $S$ in the loop representation. We can show that the expression (30) is equivalent to an expression in which the surface is partitioned into $N$ subsurfaces $S_I, I = 1, ..., N$. We then write

$$
A_S = \lim_{N \to \infty} A^2_{\text{approx}}[S_I].
$$

(38)

where $A^2_{\text{approx}}[S_I]$ denotes an approximate expression for the area of the subsurface, which is defined by

$$
A^2_{\text{approx}}[S_I] = \int_{S_I} d^2 S^{bc} \epsilon_{abc} \int_{S_I} d^2 S'^{c'b'} \epsilon_{a'b'c'} T^{aa'}(x, x')
$$

(39)

This last expression may be written as a quantum operator, by writing an operator for $T^{aa'}(x, x')$. This can be done, but, as the action is a bit complicated, I do not give it here. It may be found in [7, 9, 10, 11]. The result is that the limit (38) may be taken on any loop state, leading to a final expression that is finite and independent of the background structure that went into the definition of the loop operator. The result, for non-intersecting loops $\alpha$ is [11, 12],

$$
< \alpha | \hat{A}[S] = \frac{l_{\text{Planck}}^2}{2} I^+[S, \alpha] < \alpha |.
$$

(40)

Here $I^+[S, \alpha]$ is the positive, unoriented, intersection number, which counts (independent of orientation) the number of intersections of the loop with the surface.

If the loop $\alpha$ has intersections the action of the operator is more complicated, but it is still finite and background independent. Details are given in [11].

The third observable that can be constructed in this way is the volume of any region. It is described in [11]
3.9 Quantization of areas and volumes in the discrete representation

The result (40) says that the bras in the loop representation are, at least for intersecting loops, eigenstates of the operator that measures areas. This does not mean that, in general, there are normalizable states that are eigenstates, this can only be the case if the inner product is defined in such a way that there is a state which is the hermitian conjugate of the bra \( < \alpha | \) which is a normalizable state and if the area operator is hermitian in that inner product.

In general these conditions will not be satisfied, for example, there are no normalizable eigenstates of the area operator in the Fock representation of linearized quantum gravity. But in the context of discrete representations an inner product can be defined by the imposing the condition that the inner product be chosen such that the area operator is hermitian so that its eigenstates comprise an orthonormal basis. For nonintersecting loops this is given by (22), for intersecting loops it is more complicated [11].

With such an inner product, we may say that area is quantized in the discrete representation, because the spectrum of the operator that measures area is discrete. This spectrum consists, first of all, of the eigenvalues \( Nl_{Planck}^2/2 \), for every nonnegative integer \( N \). There is also another discrete sequence of eigenvalues corresponding to eigenstates that are labeled by intersecting loops, these are described in [11].

In the same representation, the volume operator turns out also to have a discrete spectrum. The basic action of the volume operator in this representation turns out to be to annihilate the states \( |\alpha> \) associated with nonintersecting loops \( \alpha \) and to rearrange the routings through the intersections of the intersecting loops. That is, with each intersecting loop, \( \alpha \), one can associate a finite dimensional subspace of the state space which is spanned by the loops which have the same support as \( \alpha \) but differ as to how the loops are routed through the intersection points. The action of the volume operator is then to induce a finite dimensional matrix in each such subspace that rearranges the routings then multiplies by \( l_{Planck}^3 \). Its non-zero eigenvalues are given by \( l_{Planck}^3 \) times the eigenvalues of these finite dimensional matrices.

3.10 The correspondence principle: Existence of states which approximate classical metrics at large scales

We are used to describing the classical limit of quantum theories in situations in which there is a background metric against which to measure distance intervals. It is not a completely trivial problem to understand what it means to take the classical limit in a non-perturbative quantum theory of gravity, in which there is no background metric. To do this we need to first understand two simple points. First, in pure quantum general relativity the classical limit is a limit of large distances. This is because the theory has only one dimensional parameter,
the Planck length, \( l_{\text{Planck}} = \sqrt{\hbar G/c^3} \). This obviously goes to zero as \( \hbar \to 0 \). It is perhaps also significant that it is the Planck area that is proportional to \( \hbar \), this perhaps is the reason why length intervals are not defined in the quantum theory, while areas are both defined and are quantized.

The second thing to be understood is that without a classical metric we don’t know what distance and area means and so we cannot tell which intervals are small or large compared to the Planck scale.

Because of these two points, it is easiest to express the classical limit in a way that may seem backwards, as follows[15, 10, 11]. Given any classical metric \( h_{ab} \), whose curvatures are small compared to the Planck scale, we seek a quantum state \( |\Psi> \) which has the property that it is an eigenstate of the operators \( \hat{Q}[\omega] \) and \( \hat{A}[S] \), and where, for every one form \( \omega \) which is slowly varying with respect to the metric \( h_{ab} \) and every surface which has small extrinsic curvatures, again with respect to \( h_{ab} \) (where, again these area measured with respect to the Planck scale) we have

\[
\hat{Q}[\omega]|\Psi> = (Q(\omega, h) + O(l_{\text{Planck}} | \nabla \omega|))|\Psi>,
\]

and

\[
\hat{A}[S]|\Psi> = \left( A[S, h] + O \left( l_{\text{Planck}}^2 \frac{A[S, h]}{A[S, h]} \right) \right)|\Psi>.
\]

Thus, the eigenvalues are required to give back the corresponding values for the metric \( h_{ab} \) up to terms that are small measured in Planck units. When these conditions are satisfied we say that \( |\Psi> \) is a semiclassical state that approximates the metric \( h_{ab} \).

In the loop representation we call states that have this property weaves, because it can be satisfied by loop states \( |\Delta> \), where the multiloop \( \Delta \) consists of many small loops which are arranged so that, through every surface, \( S \), as described above, approximately one line of a loop pierces \( S \) per half Planck area of the surface, measured in the metric \( h_{ab} \). It is easy to give examples of such states; a particularly simple one, for the case that the metric \( h_{ab} \) is flat, is constructed as follows [15].

We use \( h_{ab} \), to introduce a random distribution of points on \( \Sigma = R^3 \) with density \( n \). This means that in any given volume \( V \) there are \( nV(1 + O(1/\sqrt{nV})) \) points. We center a circle of radius \( a = (1/n)^{1/3} \) at each of these points, with a random orientation. Again, the notion of a random orientation is defined with respect to \( h_{ab} \). We call this whole collection of circles \( \Delta \).

It is now straightforward to show that \( |\Delta> \) is an eigenstate of \( \hat{Q}[\omega] \) and \( \hat{A}[S] \). However the conditions (41) and (42) are only satisfied if the density \( n \) is chosen so that [15],

\[
a = \sqrt{\pi/2} l_p.
\]

(43)
3.11 The necessity of discrete structure at the Planck scale

This last result (43) means that if we require that the state $|\Delta>$ approximate the classical metric $h_{ab}$, when we measure it with operators that average the metric information over scales that are large in Planck units, it is necessary that the state have discrete structure at the Planck scale, where, in both cases, what we mean by the Planck scale is determined by $h_{ab}$. This is a direct consequence of the fact that we were able to construct non-local operators to measure the metric information that are finite and background independent. This result can be generalized by considering families of loops that generalize $\Delta$ by being described by more parameters. In each case it is found that there is one combination of parameters that is fixed to be a certain exact multiple of the Planck scale. The other combinations of parameters with dimensions length are also restricted to be on the order of the Planck scale, so that the requirements on the orders of the errors in (41) and (42) are satisfied[15].

This means that in nonperturbative quantum gravity, at least in the formulation I am describing here, it is possible to have states that are semiclassical on large scales. However, there are no states that are semiclassical on the Planck scale.

This completes my discussion of the kinematical level of the theory. I now turn to results concerning diffeomorphism invariant states.

3.12 Complete solution of the diffeomorphism constraints

We have defined the action of diffeomorphisms on states in the loop representation by (23) and (24). Using these, we define the space of diffeomorphism invariant states, $\mathcal{V}_{\text{diffeo}}$, to be those loop states that satisfy,

$$\Psi[\alpha] = \hat{U}(\phi)\Psi[\alpha] = \Psi[\phi^{-1} \circ \alpha]$$

(44)

for all elements of the connected component$^7$ of the diffeomorphism group of $\Sigma$.

The definition (44) of diffeomorphism invariant states may be compared with a similar condition in the metric representation, which says that the quantum states are functions of the three geometry, which are defined to be diffeomorphism equivalence classes of three metrics. The difference is that the diffeomorphism equivalence classes of loops are countable$^8$ and a great deal is known about their classification. If we denote by $\{\alpha\}$ the diffeomorphism equivalence class of the loop $\alpha$ (also known as its knot or link class) the condition (44) means that$^7$

$$\Psi[\alpha] = \Psi[\{\alpha\}],$$

(45)

$^7$There are two ways to treat the large diffeomorphisms: as symmetries or as part of the gauge group. I do not discuss this issue here.

$^8$Assuming certain mild conditions on the finiteness of the components and the intersections.
Because the knot classes are countable the space of diffeomorphism invariant states, $V_{diffeo}$, has a countable basis, which are the characteristic states of the knot classes. These are,

$$\Psi_{\{\alpha\}}[\{\gamma\}] = \delta_{\{\alpha\};\{\gamma\}}. \quad (46)$$

We can make $V_{diffeo}$ a Hilbert space by imposing an inner product. The simplest possibility is one in which these characteristic states are orthogonal, that is if we chose a basis of bra states $< \{\alpha\} |$ such that $\Psi[\{\alpha\}] = < \{\alpha\} | \Psi >$ and we chose the inner product so that $< \{\alpha\} | \Psi_{\{\alpha\}} >$, we have

$$< \{\alpha\} | \Psi_{\{\gamma\}} > = \delta_{\{\alpha\};\{\gamma\}} \quad (47)$$

This is almost certainly not the right inner product for general relativity, because it corresponds to a reality condition in which the connection is real. However, it may be useful as a technical device to bound limits in certain calculations.

It should be mentioned there is a diffeomorphism invariant theory for which (47) is the physical inner product. This is the Husain-Kuchar model\[^{31}\], which is a limit of general relativity in which the speed of light has been taken to infinity\[^{32}\]. In the classical version of this theory, all physical evolution has been frozen, and all solutions are static. Because of this the theory has no Hamiltonian constraint—it has only the gauge and diffeomorphism constraints.

The Husain-Kuchar model is a very interesting model because it is a three plus one dimensional diffeomorphism invariant theory that has an infinite number of physical degrees of freedom. It is solved to the extent that the exact state space and inner product have been constructed. The theory needs to be completed by the construction of a sufficient number of diffeomorphism invariant observables, represented by operators on $V_{diffeo}$, on which the interpretation can be grounded. As there are only the spatial diffeomorphism invariant constraints, this problem is significantly easier than in the case of the full theory. The Husain-Kuchar model is a very useful laboratory to study those problems of the interpretation of diffeomorphism invariant quantum field theory that are not related to the problems of time and time reparametrization invariant observables.

### 3.13 Some finite diffeomorphism invariant operators

In fact, a small number of diffeomorphism invariant observables can be directly written down. I will describe here one of them, which is closely related to the area observable I described in subsection 3.8 above. The idea is to introduce a dynamical field whose configuration can define a surface. The area of that surface will then be a diffeomorphism invariant quantity.\[^{9}\]

\[^{9}\]The idea of using matter fields to define physical and diffeomorphism invariant observables is an old idea, which goes back at least to a paper of DeWitt\[^{33}\]. It has been recently revived\[^{34, 35}\].
One way to do this is to couple an antisymmetric tensor gauge field to gravity. This is a two form, $C_{ab} = -C_{ba}$ subject to a gauge transformation generated by a one form $\Lambda_a$ by,

$$
\delta C_{ab} = d\Lambda_{ab}.
$$

(48)

It’s field strength is a three form which is denoted $W_{abc} = dC_{abc}$. In the Hamiltonian theory its conjugate momenta is given by $\pi^{ab} = -\pi^{ba}$ so that

$$
\{C_{ab}(x), \pi^{cd}(y)\} = \delta^c_a \delta^d_b \delta^3(y, x).
$$

(49)

The gauge transform (48) is then generated by the constraint

$$
G = \partial_c \pi^{cd} = 0
$$

(50)

Any two dimensional surface $S$ defines a distributional configuration of the $\pi^{ab}$ by equation (28), where we now want to understand the field $\pi_a$ in that equation as being the dynamical field dual to $\pi^{bc}$ by $\pi_a = \epsilon_{abc} \pi^{bc}$. These distributional configurations are solutions to the gauge constraint (50) and I thus know of no reason it cannot be considered to be an allowed configuration of the classical field.

We can then interpret equation (29) as the definition of a diffeomorphism invariant observable by reading it as a function of a dynamical $\pi_a$ field and the gravitational field. It has the interpretation that when the $\pi^{ab}$ field defines a surface through a distributional configuration by equation (28), it gives the area of that surface.

This observable can be promoted to an operator if we also quantize the $C_{ab}$ field. This can be done by constructing a surface representation to represent it, completely analogous to the loop representation. To do this we introduce a surface observable,

$$
T[S] = e^{k \int_S C}
$$

(51)

associated to every closed surface $S$. The $k$ is a free constant with dimensions of inverse action. The algebra we will quantize is then the surface algebra

$$
\{T[S], \pi^{bc}(x)\} = k \int d^2 S^{bc}(\sigma) \delta^3(x, S(\sigma)) T[S]
$$

(52)

We can then construct a representation of this algebra in which states are functions of surfaces $\Psi[S]$. We then define the representation by

$$
\hat{T}[S'] \Psi[S] = \Psi[S' \cup S]
$$

(53)

and

$$
\hat{\pi}^{ab}(x) \Psi[S] = \hbar k \int d^2 S^{ab}(\sigma) \delta^3(x, S(\sigma)) \Psi[S].
$$

(54)
To represent the coupled $C_{ab}$-gravity system we take the direct product of this state space with the loop representation for quantum gravity. The states are then functions, $\Psi[\alpha, S]$, of loops and surfaces. We may introduce a set of bra’s, $<\alpha, S|$, labeled by loops and surfaces so that $\Psi[\alpha, S] = <\alpha, S|\Psi>$.

We may then impose the diffeomorphism constraints, suitably extended to the coupled Einstein-$C_{ab}$ system. I will not give the details here, the result is that the diffeomorphism invariant states may be constructed and they are functions of the diffeomorphism equivalence classes of loops and surfaces. Denoting these classes by $\{\alpha, S\}$, the diffeomorphism invariant state space then consists of functions of the form

$$\Psi[\{\alpha, S\}] = <\{\alpha, S\}|\Psi>.$$  \hspace{1cm} (55)

We then want to express the area observable (37) as a diffeomorphism invariant operator and show that it does indeed measure areas. It is straightforward to show that the bras at the kinematical level, $<\alpha, S|$, are, for nonintersecting loops $\alpha$, eigenstates of the operator $\hat{A}$. This operator may be constructed by using the expressions (30) and (31) as the definition of a regularization procedure, in the usual way. As the regularization breaks diffeomorphism invariance, this calculation must be done at the kinematical level. A straightforward calculation shows that

$$<\alpha, S|\hat{A}^2_{\text{approx}}[R] = \left(\frac{\hbar k l^2_{\text{Planck}}}{2}\right)^2 I[\alpha, S \cap R]^2 <\alpha, S|$$ \hspace{1cm} (56)

where $S \cap R$ means the part of the surface that lies inside the region. It then follows from (30) that

$$<\alpha, S|\hat{A} = \frac{\hbar k l^2_{\text{Planck}}}{2} I^+[\alpha, S] <\alpha, S|$$ \hspace{1cm} (57)

This may be compared with (40), we see that the only difference is that now that the surface is dynamical it is specified by the state and not by the operator. In addition, we see that if we want agreement between the units measured by this and the kinematical area operator we must pick $k = 1/\hbar$.

The action of $\hat{A}$ can be lifted to the space of diffeomorphism invariant states, giving us,

$$<\{\alpha, S\}|\hat{A} = \frac{\hbar k l^2_{\text{Planck}}}{2} I^+[\{\alpha, S\}] <\{\alpha, S\}|$$ \hspace{1cm} (58)

We have thus defined a diffeomorphism invariant operator that assigns to the surface an area which is given by $\hbar k l^2_{\text{Planck}}/2$ times the number of intersections of the loop with the surface. Thus, we see that the same techniques that gave us finite and background independent kinematical operators work to give us finite operators acting on diffeomorphism invariant states.

If we use the inner product (47) of the Husain-Kuchar model on the space of diffeomorphism invariant states, suitably extended to include the coupling to
the $C_{ab}$ field \[36\], we see that $\hat{A}$ is a hermitian operator and that its spectrum is quantized. Thus, we see that the technology we have been developing allows us to derive a prediction from a $3 + 1$ dimensional diffeomorphism invariant quantum field theory. This is that the area of any two dimensional surface is quantized in units of the Planck area. While that theory is a model, which corresponds classically only to a limit of full general relativity, this is an encouraging result. Moreover, it does not seem impossible that with the addition of structure corresponding to clocks it will be possible to extend the $\hat{A}$ observable to the full physical case, and that we will find that this prediction stands in full quantum general relativity.

It is known that a few additional diffeomorphism invariant operators can be constructed in this way, in either the case of pure gravity or gravity coupled to matter. Examples of these\[11\] are the volume of the universe and the areas of maximal surfaces (when $\pi^2$ of the spatial manifold is non-trivial.) Of course, there must be an infinite number of diffeomorphism invariant observables, it is still an open problem to show that these techniques allow the construction of an infinite number of such operators.

### 3.14 Connection between finiteness and diffeomorphism invariance of operators

All of the diffeomorphism invariant operators which have so far been constructed are also finite. We may ask whether finiteness is a general property of diffeomorphism invariant operators constructed nonperturbatively. There is a general argument that this is the case; which I would like to sketch here.

The argument begins with the assumption that any diffeomorphism invariant operator that will exist in a nonperturbative quantum theory must be constructed through a regularization procedure. All such procedures which are known require that one introduce both a background metric and a regulator scale. This is necessary, because the scale that the regularization parameter refers to must be described in terms of some metric and, since none other is available, it must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this, the dependence of the regulated operator on the cutoff parameter is related to its dependence on the background metric. This can be formalized into a kind of renormalization group equation \[29\]. When one takes the limit of the regulator parameter going to zero one isolates the nonvanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have a dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, they must be finite.

This point has profound implications for the whole discussion of finiteness and renormalizability of quantum gravity theories. It means that any nonperturbative and diffeomorphism invariant construction of the observables of the
theory must be finite. A particular approach could fail in that there could be no way to construct the diffeomorphism invariant observables as quantum operators. But if it can be done, without breaking diffeomorphism invariance, those operators will be finite.

3.15 Exact physical states of the quantum gravitational field

We come finally to the physical state space, \( \mathcal{V}_{\text{phys}} \), which consists of those states which are solutions to all of the constraints of quantum general relativity, including the Hamiltonian constraint. Although it is logical to put the discussion of these last, this is not the order in which the subject actually developed. The discovery that the Hamiltonian constraint could be exactly solved was made first \([37]\); later the loop representation was invented to solve the diffeomorphism constraint \([37]\). At the time the first solutions to the Hamiltonian constraint were found, it was possible to imagine that the existence of an infinite dimensional space of exact solutions to the Hamiltonian constraint might be some kind of spurious result, having nothing to do with physics. Now, six years later, after the development of the loop representation, and after we have understood how the discreteness of the quantum representation acts, at the kinematical and diffeomorphism invariant level, to allow the existence of finite, background independent operators and discrete structure at the Planck scale, and after we have further understood the role of this discreteness in assuring the existence of the classical limit, that the Hamiltonian constraint can be solved in this way seems much more natural.

In order to define its action in any representation, the Hamiltonian constraint must be regulated. In the literature there are four different proposals for how to carry out this regularization, due to Rovelli and the author \([37]\), Gambini \([23]\), Blencowe \([38]\) and Bruegmann and Pullin \([39]\). These are now understood to be equivalent, at least when acting on a certain class of states \([39]\).

The result of one of these regularization procedures is a sequence of well defined operators \( C^\delta \) in \( \mathcal{V}_{\text{kin}} \), for \( \delta > 0 \). A solution to the quantum Hamiltonian constraint is then taken to be one such that,

\[
\lim_{\delta \to 0} \left( C^\delta \mid \Psi > \right) = 0
\]  

(59)

The physical states are then taken to be those states that are simultaneous solutions to this condition and the diffeomorphism constraint (44).

I would like to make several comments about this condition.

a) There is no necessity that the limit \( \lim_{\delta \to 0} C^\delta \) define an operator in \( \mathcal{V}_{\text{kin}} \). One could construct such an operator by a renormalization procedure in which the operator was multiplied by the appropriate power of \( \delta \) as the limit is taken. But, it is not clear what use this would be. The operator, in any case, must
vanish on the space $V_{phys}$ which we are interested in, and on $V_{kin}$, for the reasons we discussed in section 3.7, it will not be diffeomorphism invariant.

b) The Hamiltonian constraint, even before regularization, is not diffeomorphism invariant; as it is the integral of an arbitrary density with the local function of the fields. Thus, it does not define an operator in $V_{diffeo}$. It would be very interesting to have an expression for the projection of the diffeomorphism constraint into that space. One could do this, for example, by finding an infinite set of functions on the classical phase space that vanish on the same surface as $C$, but which are diffeomorphism invariant, and then represent those as quantum operators.

c) An issue that is often raised is the question of whether the algebra of the constraints, quantum mechanically, has an anomaly which would prevent the existence of simultaneous solutions to all of them. In fact, as I am about to describe, we know of infinitely many simultaneous solutions, so there can be no anomalous terms in the algebra which is proportional to the identity operator in the state space. Furthermore, as we have just remarked, the Hamiltonian constraint does not define a good operator on $V_{kin}$ unless we further break diffeomorphism invariance by the addition of a renormalization procedure, so it is not clear exactly what condition to ask from our quantum operator algebra. However, it would still be interesting to know what the algebra of the regulated operators is like; this problem is presently under study[40].

d) In the condition (59), the limit is taken in the pointwise topology, which means that the limit must vanish when taken over every point of the loop space. As the physically meaningful inner product is constructed on the space of solutions to the constraint, this is sufficient as long as that solution space is large enough. However, it would be surprising if the limit could also not be expressed in terms of a Hilbert structure on $V_{kin}$. I believe that this can be done, but the details have not been worked out.

The result of the regularization procedures is that the Hamiltonian constraint can be given a kind of geometrical interpretation when acting in the loop representation. First of all, acting on states that have support only on loops without intersections, the limit (59) vanishes[37, 23, 39]. Thus, the action of the operator is, in the limit, only sensitive to the behavior of the state at intersecting loops. At an intersection, the action of the operator consists of two parts: first, a rearrangement of the routings through the intersection and second, a loop derivative taken at the intersecting points[7, 23, 39].

At the present time, there are several different sectors of solutions to the physical state space which have been explicitly constructed. First, as I have just mentioned, any state that has support on only nonintersecting loops is a solution. This is an infinite dimensional space; among these is a state corresponding to every invariant of nonintersecting links[4]. Then, there are two different sectors of states which have been constructed which have support on intersecting loops. The first consists of characteristic states, which have support on only a finite number of diffeomorphism equivalence classes of intersecting loops. These
have been constructed for intersections at which two [37], three [41], four and five [42] lines meet. Then, very recently, a new sector of physical states has been discovered by Bruegmann, Gambini and Pullin, which are closely related to the Jones polynomial [43].

At present, the study of exact physical states is ongoing, and there are a number of open problems. It is clear that the full set of solutions is not known, and nothing is known about the relationship between the different sectors of the solution space. Most importantly, it has not been established the extent to which the known types of solutions characterize the general solution to the constraints.

Of course, the construction of the theory is not complete without further elements, particularly the physical observables and the physical inner product. I will discuss the open problems in the next section. For the remainder of this section, I would like to discuss a number of results concerning the application of these nonperturbative methods to models which are simpler than full 3 + 1 dimensional quantum gravity.

3.16 Application of the loop representation to 2+1 gravity

As Witten first pointed out, quantum general relativity in 2 + 1 dimensions is exactly solvable because for each spatial topology one has, after the solutions of the constraints, a finite dimensional phase space [44]. Thus, although the model has only a finite number of degrees of freedom, it provides a good test of many of the ideas and methods that were originally developed in the 3 + 1 dimensional case. The theory can be completely solved in using both the connection representation [44, 9] and the loop representation [17, 9, 18]. The physical operators can be constructed, and they turn out to be closely related to the loop operators (2) and (4). The physical inner product is also easily constructed.

In addition, as Carlip has shown in a very elegant series of papers [45], the difficult problem of time can be resolved completely in this model, along the lines proposed by Rovelli [46].

A case which lies intermediate in difficulty between this case and the full 3 + 1 case is that of 2 + 1 gravity coupled to matter. In particular, with the addition of one scalar field one has a model that can also be interpreted as 3 + 1 gravity with one killing field [47].

Recently Ashtekar and Varadarajan have studied this model, and have found a number of interesting results at the classical level [48]. The most important of these is that it is possible to define a notion of asymptotic flatness, such that the energy is bounded both from above and from below. Using very different methods, Bonacina, Gamba and Martellini have shown that this theory is also perturbatively renormalizable [49]. In addition, there have been two interesting papers in which such systems are treated in the loop representation; which treat
the coupling of Maxwell fields to $2 + 1$ gravity, and $3 + 1$ gravity with one killing field.

3.17 Application of the loop representation and new variables to two killing field reductions

Another very interesting model is general relativity with two killing fields, as this is known to be an integrable system with an infinite number of degrees of freedom. This system has been studied using the new variables and the loop representation, and a number of interesting results were obtained. The key open problem in this area is to represent the generators of the Geroch group as canonical transformations, generated by an infinite dimensional algebra physical observables of the model. Some very interesting partial results in this direction have been obtained by Torre.

3.18 Nonperturbative quantization of the Bianchi models

A last type of model I would like to mention is the class of Bianchi cosmologies. These are finite dimensional model cosmologies, which offer good laboratories for ideas about quantum gravity and quantum cosmology. Most of these have not been solved, in spite of the fact that they have only a few degrees of freedom; these models thus serve as a reminder that in quantum gravity, as in ordinary quantum mechanics, finite dimensional does not imply solvable.

It would be very interesting to be able to solve these models, through approximation methods if not exactly. There are a number of interesting results concerning them which employ Ashtekar’s variables. Among these is the construction of a set of exact states for the Bianchi IX model by Kodama but, as in the full theory, little is known about the physical observables or the inner product in this model.

4 What are the key open questions?

In quantum gravity, it is safe to assume that any important problem is difficult, until the occasion of some progress provides evidence to the contrary. Indeed, in the development of the work I have been describing here, almost every result came out in a surprising way. Actually, once understood correctly, most of the results are not very difficult; the key seems to be to ask the question in precisely the right way.

Given this, the key open questions are simply how to construct those elements of a quantum field theory that are, so far, missing. I give here a list of them; more detailed discussions about each of them may be found in the reviews.
4.1 How can we construct the physical observables?

As I mentioned above, the problem of physical observables in quantum gravity is difficult partly because there is already a problem in the classical theory. The problem can be stated this way: any classical observable in general relativity, with cosmological boundary conditions, must be a constant of motion. This is because to be invariant under diffeomorphisms it must commute with the Hamiltonian constraint, but in the cosmological case the Hamiltonian is proportional to a linear combination of constraints. This must be; were there a meaningful nonvanishing Hamiltonian it would be meaningful to ask how fast the universe is evolving, so that evolutions that differed only by the rate at which time progressed would be physically distinct. As there can be no clock outside the universe, this cannot be meaningful.

Thus, the problem of physical observables is closely connected with the notion of time. As such, it is one of those great problems that are both conceptually and mathematically profound.

At the present time, a rather large number of ideas are being studied with an aim towards solving this problem. I list here the ones I am aware of, with references.

i) Coupling the theory to matter, and using this matter to provide a system of clocks with which to make observables meaningful. This is a very old idea [33], recently it has received a lot of attention [34, 32, 36].

ii) Imposing asymptotically flat boundary conditions, which provide an observer and a classical clock at infinity. The problem with such an approach is that spatial infinity is, in a certain sense, too far away, and only a limit number of observables, corresponding essentially to the globally conserved quantities, may be defined there. Still, this is undoubtedly worth doing, and some recent results of Baez are very interesting in this regard [57]. The key open problem with this approach is to show that, with respect to the correct inner product, the quantum Hamiltonian is bounded from below.

iii) Imposing some other kind of boundary conditions, which may allow more observables to be introduced. One such idea, due to Crane, is to define observables on two dimensional surfaces, and use conformal field theory thereby as a kind of measurement theory for quantum cosmology [58]. Another related idea involves choosing boundary conditions so that a Chern-Simon theory is induced on the boundary [11].

iv) Studying certain limits of the theory, where observables can be constructed [32, 59].

v) Finding an approximation scheme, such as a strong coupling expansion or a new form of perturbation theory, that will allow observables to be constructed systematically.

vi) Modifying the interpretative rules of quantum cosmology, so as to make the problem easier to solve [60].

vii) Construct observables that are associated with global properties of the
configuration of the gravitational field. This has led, during the last year, to the construction of the only explicit examples yet discovered of observables of the pure gravitational field\[61\].

ix) Construct a superposition of exact states that corresponds, in the semiclассical sense described above, to Minkowski spacetime. Small perturbations on this state should then correspond to gravitons traveling on Minkowski spacetime, at least for long wavelengths. To show this one can construct a map from a sector of the Fock space of linearized quantum gravity into a subspace the space of exact physical states. This map then can be used to construct an approximate interpretation of the exact states in that subspace. There is some preliminary evidence that this map exists\[66, 67\].

4.2 How can we construct the physical inner product?
The last structure that is necessary to do physics is the inner product. This problem is closely connected to the problem of the observables because, in the absence of a global Poincaré covariance, the inner product must be picked by the requirement that a complete set of real classical observables are represented by self-adjoint operators. Further, since the observables are constants of motion, the problem of determining the inner product is a dynamical problem. As such, this is a problem that will probably have to be solved by some approximation scheme, following such a solution to the problem of the observables.

4.3 Completeness of the physical state space
Where do the exact physical states that have been found fit into the whole space of solutions? This is a problem that is clearly dependent on the inner product and physical observable algebra; what we need in the end is to show that the physical states carry a representation of the physical observable algebra.

4.4 Coupling matter to gravity
The Ashtekar formalism allows coupling to all types of matter, including spin zero and one-half matter, Yang-Mills fields\[62\], supergravity\[63\], and antisymmetric tensor gauge theories\[36\]. It is easy to extend the loop representation to describe coupling to these matter fields at the kinematical and diffeomorphism invariant level. Nothing is known about solutions to the Hamiltonian constraint including matter.

5 Conclusions
The results that I have been describing constitute a collective work in progress, which has been undertaken by a number of people who share a common interest
in the questions I outlined in the introduction. As with any result of a scientific endeavor, from Stonehenge down through the Macintosh computer on which I’m writing this, these results reflect both the knowledge and the aspirations of those who made them. While such a work remains unfinished, it is difficult to judge its ultimate worth. We certainly don’t yet know whether there is a consistent quantum field theory that would go by the name of general relativity, although the steady progress we have been making keeps us confident that it will be possible to cleanly resolve this question. However, this was, and is, not the only goal of this program; it was equally hoped that this work would uncover some general features that would hold for any quantum theory of gravity that could be constructed nonperturbatively. I believe that it is fair to say that a number of such features have emerged, and that as a result of this work we are wiser about how the world will look when we have a satisfactory quantum theory of gravity then we were before. I would like to close by listing several morals that I believe we have learned from the work I’ve described here.

a) To solve the spatial diffeomorphism constraints it is necessary to take a different starting point already at the level of the quantum kinematics than is taken in conventional Minkowski space quantum field theories. To avoid introducing background structures, Fock space must be replaced by representations of the kinematical observable algebra that rely on no background metric and carry unbroken representations of the diffeomorphism group. That is, to get the diffeomorphism invariant physics right, we must make sure that our state space and regularization procedures are background independent already at the kinematical level.

b) At present the only representations known to have these properties are the discrete representations I discussed here. Whether or not there are others is presently an open question, however even without resolving this, these new representations have interesting structures that deserve more investigation. In essence, what they seem to do is to resolve the paradoxes that follow from the uncountable nature of the classical continuum, as each state in these representations has support only on a countable set of loops. It is exactly this structure that makes it possible to solve directly the diffeomorphism constraints, in a way in which the resulting space of diffeomorphism invariant states has a countable basis.

The price we pay for this is that at the kinematical level the state spaces are nonseperable. This would be a serious problem at the level of the physical state space; however it is only a technical inconvenience in our case.

c) It is a further property of this discrete representation that it allows us to construct finite and regularization independent operators to represent non-local functions of the gravitational field. This results in the quantization of the spectra of areas and volumes. This is, moreover, not a spurious result of the kinematics, for we can show by direct construction that the quantization of areas and volumes is maintained at the diffeomorphism invariant level, when they are measured by diffeomorphism invariant operators.
We believe that these results will survive further translation to the physical level. If this is the case they will be the first physical predictions made by a quantum theory of gravity. That is, we propose that any fine enough measurement will reveal that the area of any surface can only lie in a discrete spectrum consisting of integral multiples of $l_{Planck}^2/2$, and certain other values associated with intersections that are described in [11].

d) The necessary dependence of renormalization procedures for local operators on background metrics is, we believe, a general phenomena. As a result, I conjecture that in any quantum theory of gravity there will be no renormalized local operators. Further, all diffeomorphism invariant operators will be finite after an appropriate regularization procedure [11, 29].

e) I believe that another thing we have seen in our construction of a diffeomorphism invariant operator in section 3.13 is quite general. This is that all diffeomorphism invariant operators, and hence all physical operators, will measure topological properties of non-local structures.

f) This means that in the final quantum theory of gravity we will see the continuous geometry of the classical theory emerge from a quantum theory of purely topological structures at the Planck scale. This is a consequence of what we discovered in section 3.10 and 3.11, in which we saw that when the classical limit was formulated carefully, it follows that every state that behaves semiclassically at large scales must be far from the semiclassical limit when probed on Planck scales. So far, in fact, that what is revealed is the discrete structure required by the quantization of the area operator.

g) We believe that it is this behavior, which is apparent already at the kinematical level, and not a pathology of the dynamics of general relativity, that is responsible for the failure of perturbative quantizations of general relativity. That is, the perturbation theory is already wrong at the kinematical level because it is unitarily inequivalent to the correct kinematical state space. Moreover, while at large distances the correct physics can be well approximated by semiclassical states, this approximation becomes worse and worse at shorter and shorter distances.

h) Finally, while the possibility of solving the diffeomorphism constraint exactly is implied only by the existence of the loop representation, which implies only that it is possible to choose a connection as the canonical coordinate of the theory, that it is in exactly the same representation that the Hamiltonian constraint becomes exactly solvable seems the main miracle uncovered so far. (Here, by a miracle I mean something wonderful that happens for a reason we don’t understand.) It seems to be the case that once the diffeomorphisms have been taken care of correctly, the information remaining in the Hamiltonian constraint is very manageable.

i) Although I do not claim to understand completely what is behind this miracle, it is worth pointing out that it, is in fact, exactly the existence of the self-dual connection that makes it possible to write the Hamiltonian constraint as a single term, which in turn makes possible the exact solutions which have
been discovered. It seems, as a result, very possible that self-duality is one of the keys to quantum gravity in the real 3+1 dimensional world. Indeed, self-duality is the key to several very interesting results that have been recently uncovered about the classical theory [64, 65].

Note that only the last two of the nine morals in this list depended on the form of the Hamiltonian constraint, and hence on the conjecture that general relativity is the correct microscopic description of gravitation. The rest depend only on the existence of the loop representation, which needs only that the theory can be expressed in such a way that a connection is the canonical coordinate. Thus, the fact that so many of the key features are present at the kinematic and diffeomorphism invariant levels, before the dynamics has been imposed, makes it, in my opinion, quite likely that whatever dynamics turns out to be right, the description of Planck scale physics in the final theory of quantum gravity will look a great deal like the picture I have been sketching here.

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