We introduce and analyze a simpler, practically useful variant of multivariate singular spectrum analysis (mSSA), a known time series method to impute (or de-noise) and forecast a multivariate time series. Towards this, we introduce a spatio-temporal factor model to analyze mSSA. This model includes the usual components used to model dynamics in time series analysis such as trends (low order polynomials), seasonality (finite sum of harmonics) and linear time-invariant systems. We establish that given \( N \) time series and \( T \) observations per time series, the in-sample prediction error for both imputation and forecasting under mSSA scales as \( \frac{1}{\sqrt{N} \min(N,T)} \). This is an improvement over: (i) the \( \frac{1}{\sqrt{T}} \) error scaling of SSA, which is the restriction of mSSA to univariate time series; (ii) the \( \frac{1}{\min(N,T)} \) error scaling for Temporal Regularized Matrix Factorized (TRMF), a matrix factorization based method for time series prediction. That is, mSSA exploits both the ‘temporal’ and ‘spatial’ structure in a multivariate time series. Our experimental results using various benchmark datasets confirm the characteristics of the spatio-temporal factor model and our theoretical findings—our variant of mSSA empirically performs as well or better compared to neural network based time series methods, LSTM and DeepAR.

ACM Reference Format:
Anish Agarwal, Abdullah Alomar, and Devavrat Shah. 2022. On Multivariate Singular Spectrum Analysis and its Variants. In Abstracts of the 2022 ACM SIGMETRICS/IPDPS PERFORMANCE Joint International Conference on Measurement and Modeling of Computer Systems (SIGMETRICS/PERFORMANCE ’22 Abstracts), June 6–10, 2022, Mumbai, India. ACM, New York, NY, USA, 2 pages. https://doi.org/10.1145/3489048.3526952

1 INTRODUCTION
Multivariate time series data is of great interest across many application areas, including cyber-physical systems, finance, retail, healthcare to name a few. The goal across these domains can be summarized as accurate imputation and forecasting of a multivariate time series in the presence of noisy and/or missing data along with providing meaningful uncertainty estimates.

Setup. We consider a discrete time setting with time indexed as \( t \in \mathbb{Z} \). For \( N \in \mathbb{N} \), let \( f_n : \mathbb{Z} \to \mathbb{R} \), \( n \in [N] := \{1, \ldots, N\} \) be the latent time series of interest. For \( t \in [T] \) and \( n \in [N] \), we observe \( X_n(t) \) where for \( \rho \in (0,1] \),

\[
X_n(t) = \begin{cases} 
   f_n(t) + \eta_n(t) & \text{with probability } \rho \\
   \star & \text{with probability } 1 - \rho.
\end{cases}
\]

Here \( \star \) represents a missing observation and \( \eta_n(t) \) represents the per-step noise, which we assume to be an independent (across \( t, n \)) mean-zero random variable. Though \( \eta_n(t) \) is independent, it is worth noting though the underlying time series, \( f_n(\cdot) \), is of course strongly dependent across \( t, n \).

Goal. The objective is two-folds, for \( n \in [N] \): (i) imputation — estimating \( f_n(t) \) for all \( t \in [T] \); (ii) forecasting — learning a model to forecast \( f_n(t) \) for \( t > T \).

2 ALGORITHM
mSSA is a known method to impute and forecast multivariate time series. The main objective of this work is to introduce and theoretically analyze a simpler, practically useful variant of mSSA. Given the simplicity of the algorithm, we start by describing it below.

**Figure 1: Key steps of proposed variant of mSSA.**

There are two parameters: \( L \geq 1 \) and \( k \geq 1 \). For simplicity and without loss of generality, assume that \( T \) is an integer multiple of \( L \). The algorithm consists of the following steps.

**Step 1:** Transform the time series into a matrix. We start by transforming each time series \( X_n(t), t \in [T] n \in [N] \), into an \( L \times T/L \) matrix where the entry of the matrix in row \( i \in [L] \) and column \( j \in [T/L] \) is \( X_n(i+(j-1) \times L) \). This matrix induced by the time series is called the Page matrix. We then create a stacked Page matrix from these \( N \) time series by performing a column wise concatenation of the \( N \) matrices. We denote this matrix as \( SP((X_1, \ldots, X_N), T, L) \), and note that it has \( L \) rows and \( N \times T/L \) columns.

**Step 2:** Singular value thresholding (imputation). We replace missing values (i.e. \( \star \)) in \( SP((X_1, \ldots, X_N), T, L) \) by 0, then, we compute its singular value decomposition \( SP((X_1, \ldots, X_N), T, L) = \sum_{\ell=1}^{\min(L,N \times T/L)} s_\ell u_\ell v_\ell^T \), where \( s_1 \geq s_2 \cdots \geq s_{\min(L,N \times T/L)} \geq 0 \) denote its ordered singular values, and \( u_\ell, v_\ell \in \mathbb{R}^L \) denote its left and right singular vectors, respectively.
As our primary contribution, we provide an answer to this question.

To forecast, we learn a linear model (forecasting).

Note that the collection of latent steps above.

Step 3: Lean a linear model (forecasting). To forecast, we learn a linear model \( \hat{\beta}(x_1, \ldots, x_N, T, L, k) \in \mathbb{R}^{L-1} \), which is the solution to

\[
\text{minimize } \sum_{m=1}^{N} \frac{T_L}{L} \left( y_m - \hat{\beta}^T x_m \right)^2 \quad \text{over } \beta \in \mathbb{R}^{L-1},
\]

where \( y_m \) is the \( m \)th component of \( (1/\hat{\rho})x_1(L), x_2(2xL), \ldots, x_1(T), x_2(L), \ldots, x_2(T), \ldots, x_N(T) \) \( \in \mathbb{R}^{N \times T} \), and \( x_m \in \mathbb{R}^{L-1} \) corresponds to the vector formed by the entries of the first \( L - 1 \) rows in the \( m \)th column of \( \hat{\rho} \). We establish the efficacy of such an extension when the time-varying variance is also modeled through a spatio-temporal factor model. This enables meaningful uncertainty quantification for the estimation of the mean produced by mSSA. Second, we propose a tensor variant of SSA, termed tSSA, which exploits recent developments in tensor estimation. In tSSA, rather than doing a column-wise stacking of the Page Matrices induced by each of the \( N \) time series to form a larger matrix, we instead view each Page matrix as a slice of a \( L \times T/L \times N \) order-three tensor. We characterize the relative performance of tSSA, mSSA, and "vanilla" matrix estimation (ME) with respect to imputation error.

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