Fermionic corrections to the interference of the electro- and chromomagnetic dipole operators in $\bar{B} \to X_s \gamma$ at $O(\alpha_s^2)$

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Abstract

We calculate the virtual and bremsstrahlung fermionic corrections due to the interference of the electro- and chromomagnetic dipole operators in the inclusive $\bar{B} \to X_s \gamma$ decay at $O(\alpha_s^2)$ and present analytical results for both the total decay rate and the photon energy spectrum.
1 Introduction

The present experimental world average of the branching ratio of $\bar{B} \to X_s \gamma$, which includes measurements by CLEO, BaBar and Belle [1–3], is performed by the Heavy Flavor Averaging Group [4] and, for photon energies $E_\gamma > 1.6$ GeV, is given by

$$\text{Br}(\bar{B} \to X_s \gamma) = (3.52 \pm 0.23 \pm 0.09) \times 10^{-4},$$

where the errors are combined statistical and systematic and due to the extrapolation to the common lower-cut in the photon energy, respectively. Moreover, the total uncertainty, being already below 7%, is expected to reduce down to 5% at the end of the B-factory era.

In order to keep pace with the improving experimental accuracy the theoretical prediction of the $\bar{B} \to X_s \gamma$ branching ratio has to be known at the next-to-next-to-leading order (NNLO) level. A first estimate of the $\bar{B} \to X_s \gamma$ branching ratio at this level of accuracy has been presented in [5]. For $E_\gamma > 1.6$ GeV it reads

$$\text{Br}(\bar{B} \to X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4}. $$

This estimate includes the three-loop dipole operator matching conditions [6], the three-loop mixing of the four-quark operators [7], the four-loop mixing of the dipole operators into the dipole operators [9]. Also the two-loop matrix elements of the electromagnetic dipole operator together with the corresponding bremsstrahlung terms (at $m_c = 0$) [10–13], as well as the three-loop matrix elements of the four-quark operators within the so-called large-$\beta_0$ approximation [14] have been taken into account. Finally, in order to obtain estimates of these matrix elements (together with other still unknown ones) at the physical value of the charm quark mass $m_c$ an interpolation in the latter has been performed in [15].

We should mention here that there are several perturbative and non-perturbative effects that have not been considered when deriving the estimate given in (1.2). Some of them are already available in the literature: the four-loop mixing of $O_1,\ldots,O_6$ into $O_8$ [9]; the bremsstrahlung contributions of the $(O_2, O_5)$, $(O_2, O_7)$ and $(O_7, O_8)$-interference at $O(\alpha_s^2 \beta_0)$ [16]; the exact charm quark mass dependence of the $(O_7, O_7)$-interference at $O(\alpha_s^2)$ [17]; the three-loop virtual corrections due to charm and bottom quark loop insertions into gluon propagators in the $(O_1, O_7)$ and $(O_2, O_7)$-interference [18]; the updated knowledge of the semileptonic normalization factor [19–21]; photon energy cut-off related effects [22–25]; and estimates for the $O(\alpha_s \Lambda_{QCD}/m_b)$ corrections [26]. Other effects are unknown at the moment, like the complete virtual and bremsstrahlung contributions to the $(O_7, O_8)$ and $(O_8, O_8)$-interference at $O(\alpha_s^2)$ (only the contribution of the $(O_7, O_8)$-interference at $O(\alpha_s^2 \beta_0)$ is known [14,16]), and of course the exact $m_c$-dependence of various matrix elements beyond the large-$\beta_0$ approximation, in order to improve (or even remove) the uncertainty due to the interpolation in $m_c$ [15]. The individual contributions listed above are all expected to remain within the uncertainty given in (1.2), nevertheless they should be taken into account in future updates.

In the present paper we repeat the calculation of the $(O_7, O_8)$-interference contribution performed in [14,16] and extend it to include not only the effects of massless quark loops but also those due to massive ones. More precisely, we calculate those $O(\alpha_s^2)$ contributions which can be obtained from the Feynman diagrams contributing to the $(O_7, O_8)$-interference
at \( O(\alpha_s) \) when dressing the gluon propagators with massless up, down and strange quark loops as well as with massive charm and bottom quark loops. We work out the effects of these contributions to the photon energy spectrum \( d\Gamma(b \to X_s^{\text{partonic}})/dE_\gamma \) and to the total decay width \( \Gamma(b \to X_s^{\text{partonic}}) \big|_{E_\gamma > E_0} \), where \( E_0 \) denotes the cut in the photon energy.

The organization of this paper is as follows. In section 2 we present our final results for the total decay width and the photon energy spectrum and describe briefly the calculation of the relevant Feynman diagrams. The numerical impact of the fermionic corrections on \( \text{Br}(\bar{B} \to X_s \gamma) \) is estimated in section 3. Finally, we summarize in section 4.

## 2 Fermionic corrections

Within the low-energy effective theory the partonic \( b \to X_s \gamma \) decay rate can be written as

\[
\Gamma(b \to X_s^{\text{parton}}) \big|_{E_\gamma > E_0} = \frac{G_F^2 \alpha_{\text{em}} \overline{m}_b^2(\mu) m_b^3}{32 \pi^4} |V_{tb} V_{ts}^*|^2 \sum_{i,j} C_{i}^{\text{eff}}(\mu) C_{j}^{\text{eff}}(\mu) G_{ij}(E_0, \mu), \quad (2.1)
\]

where \( m_b \) and \( \overline{m}_b(\mu) \) denote the pole and the running \( \overline{\text{MS}} \) mass of the \( b \) quark, respectively, \( C_{i}^{\text{eff}}(\mu) \) the effective Wilson coefficients at the low-energy scale, and \( E_0 \) the energy cut in the photon spectrum\(^1\).

As already anticipated in the introduction, we will focus on the function \( G_{78}(E_0, \mu) \) corresponding to the interference of the electro- and chromomagnetic dipole operators

\[
O_7 = \frac{e}{16\pi^2} \overline{m}_b(\mu) (\bar{s}s^{\mu\nu} P_R b) F_{\mu\nu} \quad (2.2)
\]

and

\[
O_8 = \frac{g}{16\pi^2} \overline{m}_b(\mu) (\bar{s}s^{\mu\nu} P_R T^a b) C_{\mu\nu}^a, \quad (2.3)
\]

respectively. In NNLO approximation this function can be decomposed as follows,

\[
G_{78}(E_0, \mu) = \frac{\alpha_s(\mu)}{4\pi} C_F Y^{(1)}(z_0, \mu) + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 C_F Y^{(2)}(z_0, \mu) + O(\alpha_s^3), \quad (2.4)
\]

where

\[
Y^{(2)}(z_0, \mu) = C_F Y^{(2,\text{CF})}(z_0, \mu) + C_A Y^{(2,\text{CA})}(z_0, \mu) + T_R N_L Y^{(2,\text{NL})}(z_0, \mu) + T_R N_H Y^{(2,\text{NH})}(z_0, \mu) + T_R N_V Y^{(2,\text{NV})}(z_0, \mu). \quad (2.5)
\]

Here, \( z_0 = 2E_0/m_b \), \( N_L, N_H \) and \( N_V \) denote the number of light \( (m_q = 0) \), heavy \( (m_q = m_b) \), and purely virtual \( (m_q = m_c) \) quark flavors, respectively, \( \alpha_s(\mu) \) is the running coupling constant in the \( \overline{\text{MS}} \) scheme, and \( C_F, C_A \) and \( T_R \) are the color factors with numerical values given by 4/3, 3 and 1/2, respectively. In this paper we present results for the functions \( Y^{(2,i)}(z_0, \mu) \) with \( i = \text{CF}, \text{CA} \) is the subject of another publication [27].

\(^1\)In this paper we assume that the products \( C_{i}^{\text{eff}}(\mu) C_{j}^{\text{eff}}(\mu) \) are real. That is our formulas are not applicable to physics scenarios beyond the standard model which produce complex short distance couplings.
The functions $Y^{(2,i)}$ with $i = \text{NL}, \text{NH}, \text{NV}$ appearing in (2.5) receive contributions from the $b \to s\gamma$, $b \to s\gamma g$, and $b \to s\gamma q\bar{q}$ ($q \in \{u,d,s\}$, $m_q = 0$) transitions. The latter are contained in the $b$ quark selfenergies which arise from those at $O(\alpha_s)$ when dressing the gluon propagators with massless and massive quark loops. Two sample $b$ quark self-energies containing cuts with two, three and four particles in the intermediate state are displayed in figure 1. As far as the diagrams containing massive quarks in the fermion loop are concerned, like, e.g., the one given on the left-hand side of figure 1 we do not have to calculate cuts with four particles in the intermediate state since such cuts would run through the bottom or charm quark loop and (i) $b$ quarks are of course kinematically not allowed to appear in the final state and (ii) events involving charmed hadrons in the final state are not included on the experimental side. On the other hand, for the diagrams containing massless quarks in the fermion loop like, e.g., the one given on the right-hand side of figure 1 the contributions from $q\bar{q}$ production ($q \in \{u,d,s\}$), that is four-particle cuts running through massless quark loops, have to be taken into account.

We work in $d = 4 - 2\epsilon$ space-time dimensions to regularize ultraviolet, infrared and collinear singularities, and adopt the renormalization prescription from [12, 17]. Most of the renormalization constants necessary to render our results ultraviolet finite can be found there. The only exceptions are those which describe the selfmixing of $O_8$ at one-loop and the mixing of $O_8$ into $O_7$ up to two-loops; they can be extracted from [28]. For some technical details concerning the evaluation of the two-loop integrals involving both the bottom and charm quark mass $m_b$ and $m_c$, respectively, we refer the reader to the end of this section.

In order to obtain a compact presentation of our findings we split the functions $Y^{(2,i)}$ with $i = \text{NL}, \text{NH}, \text{NV}$ into two parts, namely

$$Y^{(2,i)}(z_0, \mu) = Y^{(2,i)}(0, \mu) - \delta Y^{(2,i)}(z_0, \mu),$$

where the first terms give always the contribution to the full inclusive decay rate, and the second ones correct for the fact that in the experiments a lower cut in the photon energy is applied. Performing the same splitting for the function $Y^{(1)}$ appearing in (2.4), our findings

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There are also $b$ quark selfenergies where the photon runs from $O_7$ to the quark loop and which cannot be obtained from those at $O(\alpha_s)$. However, it is easy to show that these diagrams cancel amongst themselves without performing loop and phase-space integrations (see Furry’s theorem).
for the two individual contributions at $O(\alpha_s)$ read

$$Y^{(1)}(0, \mu) = \frac{4}{9} \left(29 - 2\pi^2\right) + \frac{16}{3} L_\mu,$$

$$\delta Y^{(1)}(z_0, \mu) = \frac{2}{9} z_0 \left(z_0^2 + 24\right) - \frac{8}{3} (z_0 - 1) \ln(1 - z_0) - \frac{8}{3} \text{Li}_2(z_0),$$

while those at $O(\alpha_s^2)$ are given by

$$Y^{(2,\text{NL})}(0, \mu) = -\frac{16}{81} \left(328 - 13\pi^2\right) - \frac{64}{27} \left(18 - \pi^2\right) L_\mu - \frac{64}{9} L_\mu^2 - \frac{64}{3} \zeta_3,$$

$$Y^{(2,\text{NH})}(0, \mu) = \frac{8}{81} \left(244 - 27\sqrt{3}\pi - 61\pi^2\right) - \frac{64}{27} \left(18 - \pi^2\right) L_\mu$$

$$- \frac{64}{9} L_\mu^2 - \frac{64}{27} \zeta_3 + 32\sqrt{3} \text{Cl}_2 \left(\frac{\pi}{3}\right),$$

$$Y^{(2,\text{NV})}(0, \mu) = -\frac{16}{81} \left[157 - 279\rho - \pi^2 \left(5 + 9\rho^2 - 42\rho^{3/2}\right)\right] - \frac{64}{27} \left(18 - \pi^2\right) L_\mu$$

$$- \frac{64}{9} L_\mu^2 + \frac{16}{27} \left(22 - \pi^2 + 10\rho\right) \ln \rho + \frac{16}{27} \left(8 + 9\rho^2\right) \ln^2 \rho$$

$$- \frac{16}{27} \ln^3 \rho - \frac{8}{9} (1 - 6\rho^2) \Phi_1(\rho) - \frac{8}{27} (19 - 46\rho) \Phi_2(\rho)$$

$$- \frac{32}{27} (13 + 14\rho) \Phi_3(\rho) - \frac{64}{9} \Phi_4(\rho) - \frac{32}{9} \ln \rho \text{Li}_2(1 - \rho)$$

$$+ \frac{32}{27} \left(5 + 9\rho^2 + 14\rho^{3/2}\right) \text{Li}_2(1 - \rho) - \frac{1792}{27} \rho^{3/2} \text{Li}_2(1 - \sqrt{\rho})$$

$$+ \frac{64}{9} \text{Li}_3(1 - \rho) + \frac{64}{9} \text{Li}_3 \left(1 - \frac{1}{\rho}\right),$$

$$\delta Y^{(2,\text{NL})}(z_0, \mu) = -\frac{4}{27} \left(7z_0^2 - 17z_0 + 238\right) - \frac{8}{3} \delta Y^{(1)}(z_0, \mu) L_\mu$$

$$+ \frac{8}{27} \left(z_0^3 - 6z_0^2 + 80z_0 - 75 + 6\pi^2\right) \ln(1 - z_0)$$

$$- \frac{16}{3} (z_0 - 1) \ln^2(1 - z_0) - \frac{16}{3} \ln z_0 \ln^2(1 - z_0)$$

$$- \frac{32}{27} (3z_0 - 8) \text{Li}_2(z_0) - \frac{32}{3} \ln(1 - z_0) \text{Li}_2(z_0)$$

$$+ \frac{32}{9} \text{Li}_3(z_0) - \frac{32}{3} \text{Li}_3(1 - z_0) + \frac{32}{3} \zeta_3,$$

$$\delta Y^{(2,\text{NH})}(z_0, \mu) = -\frac{8}{3} \delta Y^{(1)}(z_0, \mu) L_\mu,$$

$$\delta Y^{(2,\text{NV})}(z_0, \mu) = -\frac{4}{3} \delta Y^{(1)}(z_0, \mu) \left(2L_\mu - \ln \rho\right).$$
In writing these equations we introduced the short-hand notations
\[ \rho = \frac{m_e^2}{m_b^2} \quad \text{and} \quad L_\mu = \ln \left( \frac{\mu}{m_b} \right). \] (2.15)

The definitions of the auxiliary functions \( \Phi_n(\rho) \) as well as those of the polylogarithms \( \text{Li}_n(z) \) and the Clausen function \( \text{Cl}_2(z) \) can be found in appendix B. The numerical value of the Clausen function at \( z = \pi/3 \) is approximately given by 1.014942, and \( \zeta_3 \approx 1.202057 \) is equal to Riemann’s theta functions \( \zeta(n) \) at \( n = 3 \). Equations (2.11) and (2.14) hold for \( \rho > 0 \).

Turning now to our findings for the photon energy spectrum, we rewrite the function \( G_{78}(E_0, \mu) \) as an integral over the (rescaled) photon energy,
\[ G_{78}(E_0, \mu) = \int_{z_0}^{1} dz \frac{dG_{78}(z, \mu)}{dz}, \quad z = \frac{2E_\gamma m_b}{m_b}. \] (2.16)

In NNLO approximation the integrand can be written as follows,
\[ \frac{dG_{78}(z, \mu)}{dz} = \frac{\alpha_s(\mu)}{4\pi} C_F \tilde{Y}^{(1)}(z, \mu) + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 C_F \tilde{Y}^{(2)}(z, \mu) + O(\alpha_s^3), \] (2.17)
where, in analogy to (2.5),
\[ \tilde{Y}^{(2)}(z, \mu) = T_R N_L \tilde{Y}^{(2,NL)}(z, \mu) + T_R N_H \tilde{Y}^{(2,NH)}(z, \mu) + T_R N_V \tilde{Y}^{(2,NV)}(z, \mu) + \ldots, \] (2.18)
with the ellipses denoting terms which are proportional to the colorfactors \( C_F \) and \( C_A \). The next-to-leading order (NLO) function \( \tilde{Y}^{(1)}(z, \mu) \) is given by
\[ \tilde{Y}^{(1)}(z, \mu) = Y^{(1)}_{2\text{-cuts}}(0, \mu) \delta(1-z) + \frac{d}{dz} \delta Y^{(1)}(z, \mu), \] (2.19)
where \( \delta Y^{(1)}(z, \mu) \) can be obtained from (2.18) by replacing \( z_0 \) by \( z \), and \( Y^{(1)}_{2\text{-cuts}}(0, \mu) \) can be found in appendix A. The latter function summarizes the contribution of all 2-particle cuts entering the functions \( Y^{(1)}(z_0, \mu) \). The terms proportional to \( N_L, N_H \) and \( N_V \) appearing in (2.18) can be written in complete analogy to (2.19),
\[ \tilde{Y}^{(2,i)}(z, \mu) = Y^{(2,i)}_{2\text{-cuts}}(0, \mu) \delta(1-z) + \frac{d}{dz} \delta Y^{(2,i)}(z, \mu), \] (2.20)
with \( \delta Y^{(2,i)}(z, \mu) \) given in (2.12)–(2.14) and \( Y^{(2,i)}_{2\text{-cuts}}(0, \mu) \) in appendix A. Since the contributions of the 2-particle cuts are by themselves free of infrared and collinear singularities it was not necessary to introduce plus-distributions in (2.20) and (2.19).

We remark that the terms proportional to \( N_L \), that is the functions \( Y^{(2,NL)}(0, \mu) \) and \( \delta Y^{(2,NL)}(z_0, \mu) \), are already known in the literature [14,16] and we completely agree with the results given there. The functions \( Y^{(2,i)}(z_0, \mu) \) with \( i = NH, NV \) are however new.

In the remainder of this section we will summarize the technical details of the calculation. However, we refrain from repeating the algebraic reduction procedure of the 2-, 3- and 4-particle cuts of the three-loop \( b \) quark selfenergies to a set of so-called master integrals as

\[ ^3 \text{I would like to thank Christoph Greub for checking equation (A.5), which contains the contribution of the 2-particle cuts being proportional to } N_H, \text{ numerically.} \]
well as from discussing appropriate parametrizations of the phase-space integrals here since this has already been done in great detail in [12, 13]. Instead, we will briefly describe how we solved the non-trivial two-loop integrals involving the two mass scales $m_b$ and $m_c$. First, we introduced Feynman parameters in the standard way and performed the loop-integrations. Subsequently, we applied the Mellin-Barnes technique [29, 30] based on the relation

$$\frac{1}{(x+y)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_C \frac{ds}{2\pi i} \frac{x^s}{y^{\lambda+s}} \Gamma(-s)\Gamma(\lambda+s),$$

where the integration contour $C$ runs from $-i\infty$ to $+i\infty$ such that it separates the poles generated by the two $\Gamma$ functions. In this way all powers of a sum of several terms could be replaced by one- or two-fold Mellin-Barnes integrals which made the integration over the Feynman parameters trivial. Finally, we closed the integration contours $C$ sideward by a half-circle with infinite radius and summed up the enclosed residues. In our case all infinite sums involving the mass ratio $m_c/m_b$ could be reduced to the inverse binomial sums given in [31], and we obtained solutions for all two-loop integrals which are valid for arbitrary values of $m_c/m_b$. We checked our analytical results for the master integrals for several values of $m_c/m_b$ by numerically integrating over the Feynman parameter representations.

Two other checks of our calculation are provided by taking the limits $m_c \to 0$ and $m_c \to m_b$ of the contributions of the 2-particle cuts proportional to $N_V$,

$$\lim_{\rho \to 0} Y_{2\text{-cuts}}^{(2,NV)}(0, \mu) = Y_{2\text{-cuts}}^{(2,NL)}(0, \mu),$$

$$\lim_{\rho \to 1} Y_{2\text{-cuts}}^{(2,NV)}(0, \mu) = Y_{2\text{-cuts}}^{(2,NH)}(0, \mu),$$

which reproduce our results proportional to $N_L$ and $N_H$. Also the limit $\rho \to 1$ of the complete expression $Y^{(2,i)}(z_0, \mu)$ for $i = NV$ reduces to that for $i = NH$. We note, however, that it is not possible to take the limit $\rho \to 0$ of the complete expression for $i = NV$ since we excluded the contributions with massive $c\bar{c}$-pairs in the final state, and hence some $\ln \rho$ terms being present in $Y^{(2,NV)}(z_0, \mu)$ parametrize infrared and collinear singularities. The last check concerns the asymptotic behavior for $m_c \gg m_b$. In this limit our result for the complete expression with $i = NV$ reduces to

$$Y^{(2,NV)}(z_0, \mu) = \frac{364}{81} - \left[ \frac{224}{27} + \frac{8}{3} Y^{(1)}(z_0, \mu) \right] \ln \left( \frac{\mu}{m_c} \right) + \frac{64}{9} \ln^2 \left( \frac{\mu}{m_c} \right) + O \left( \frac{1}{\rho} \right),$$

which is in agreement with the asymptotic form found in [6] (see equation (5.10) of that reference).

3 Numerical impact

In the left frame of figure 2 we show the dependence of the functions $Y_{2\text{-cuts}}^{(2,i)}(0, m_b)$ on the mass ratio $\rho = m_c^2/m_b^2$ for $i = NL$, NH, NV. As can be seen, the function for $i = NH$ (blue dashed line) differs from that for $i = NL$ (red solid line) by a factor of about $-0.5$. On the other hand, at the physical value $\rho \approx (0.262)^2$, the function for $i = NV$ (black dotted curve) has a smaller value of about 15% compared to that for $i = NL$. 

Figure 2: $Y_{2\text{-cuts}}^{(2,i)}(0, m_b)$ (left) and $Y_{2\text{-cuts}}^{(2,i)}(z_0, m_b)$ (right) as a function of $\rho = m_c^2/m_b^2$ for $i = \text{NL}$ (red solid line), $i = \text{NH}$ (blue dashed line) and $i = \text{NV}$ (black dotted curve). The vertical lines indicate the physical value $\rho \approx (0.262)^2$.

The right frame of figure 2 displays the functions $Y_{2\text{-cuts}}^{(2,i)}(z_0, m_b)$ for $i = \text{NL}, \text{NH}, \text{NV}$ as functions of the mass ratio $\rho$. They differ from the ones shown in the left frame by adding the contributions from the 3- and 4-particle cuts, with the latter depending on the (rescaled) photon energy cutoff $z_0$. The numerical value we chose in this illustration is given by $z_0 = 0.68$ and corresponds to $E_0 = 1.6$ GeV. As seen, the main effect of the bremsstrahlung corrections is to shift the function for $i = \text{NL}$ (red solid line) and $i = \text{NV}$ (black dotted curve) down by a factor of about 1.7 and 1.3, respectively. Also the aforementioned logarithmic singularity for $i = \text{NV}$ can be observed for $\rho \to 0$. There is no shift for $i = \text{NH}$ (blue dashed line) since we set $\mu = m_b$ in our illustration (see (2.13)).

We remark that other values for the renormalization scale $\mu$ than $m_b$ lead merely to a shift of the three curves plotted in the left frame of figure 2 by the same amount up or down. The reason for this is that the three quantities given in (A.4)–(A.6) have exactly the same $\mu$-dependence. The same comment is also true for the three curves shown in the right frame of figure 2 as can be seen from (2.9)–(2.14). However, it is clear that a variation of $\mu$ will change the relative importance with which each individual contribution enters the function $G_{78}(E_0, \mu)$. For example, the choice $\mu = 1.2 m_b$ leads to $Y_{2\text{-cuts}}^{(2,NH)}(z_0, \mu) \approx 0$, and hence the fermionic corrections will be dominated by the two functions with $i = \text{NL}$ and $i = \text{NV}$. On the other hand, for smaller values of $\mu$ than $m_b$ the situation can be reversed.

Next, we compare the fermionic corrections at NNLO with the NLO result. Using the numerical values $\alpha_s(m_b) = 0.22, N_L = 3, N_H = 1$ and $N_V = 1$ (for the other input parameters we use the same values as before), we find

$$G_{78}(E_0, m_b) = 0.086 - 0.009 + \ldots = 0.077 + \ldots$$

(3.1)

where the two numbers after the first equality sign correspond to the $O(\alpha_s)$ and the fermionic $O(\alpha_s^2)$ contributions given in (2.4), and the ellipses denote the still unknown $O(\alpha_s^2)$ terms proportional to $C_F$ and $C_A$ as well as higher order corrections. Thus, at $\mu = m_b$, the effect of the NNLO fermionic corrections is to lower the NLO value of $G_{78}(E_0, m_b)$ by around 10%. For $\mu = 2.5$ and 7.5 GeV, the $O(\alpha_s)$ term in $G_{78}(E_0, \mu)$ changes to 0.009 and 0.124, respectively, and the $O(\alpha_s^2)$ correction shifts these values by around 3% and -11%, respectively.

Finally, we estimate the effect of the fermionic corrections at $O(\alpha_s^2)$ on the branching
ratio of $\bar{B} \to X_s \gamma$. As seen in figure 2, the contributions with massive charm quark loops ($i = NV$) at the physical value $\rho \approx (0.262)^2$ are of comparable size as those with massless quarks in the loops ($i = NL$). Hence, the charm quark mass effects can with good accuracy be described by a single massless quark entering the large-$\beta_0$ approximation. On the other hand, approximating the contributions due to massive bottom quark loops ($i = NH$) by massless ones is not very accurate (see figure 2). Here, however, one should bear in mind that in the physical application we have three massless and only two massive quarks. That is the leading correction to $\text{Br}(\bar{B} \to X_s \gamma)$ will be given by the sum of the contributions with $i = NL$ and $i = N_V$, where the former is weighted by a factor of three, and the correction due to the contribution with $i = N_H$ will only appear at the subleading level. In fact, the exact result of the fermionic corrections at $\mu = m_b$ can be accurately approximated by setting $N_L = 3.6$ and $N_H = N_V = 0$ in (2.5). Thus, the effect of the massive bottom quark loops at $\mu = m_b$ can be accounted for in the large-$\beta_0$ approximation by reducing the number of massless quark flavors by -0.4. Given that the large-$\beta_0$ corrections of the $(O_7, O_8)$-interference affect the branching ratio of $\bar{B} \to X_s \gamma$ by around 0.7% for $\mu = m_b$, we conclude that this will not be altered drastically when implementing the exact results for the fermionic corrections with massive quarks. We remark here that for other values of the renormalization scale $\mu$ than $m_b$ it happens that the exact result can only be approximated by the massless contribution when using a negative number of massless quark flavors. For example, for $\mu = m_b/2$, the exact result can be approximated by setting $N_L = -1.32$ and $N_H = N_V = 0$ in (2.5). Determining the effect of the new fermionic corrections on $\text{Br}(\bar{B} \to X_s \gamma)$ in this case would require to repeat the interpolation procedure performed in [6]. Since we expect that it will also be 1% at most (when combining large-$\beta_0$ and massive quark loop corrections), that is below the total uncertainty given in (1.2), we postpone this to a forthcoming analysis which will also include other contributions not considered so far.

4 Summary

In this paper we calculated the NNLO fermionic corrections to the total decay rate and the photon energy spectrum induced by the interference of the electro- and chromomagnetic dipole operators. We confirmed the results for the $O(\alpha_s^2 \beta_0)$ terms given in [14, 16] and also presented analytical results for the contributions with massive bottom and charm quark loops. We expect that the combination of both the massless and the massive quark loop contributions affects $\text{Br}(\bar{B} \to X_s \gamma)$ by 1% at most.

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\footnote{Here we should mention that the effect of the large-$\beta_0$ corrections in the $(O_7, O_8)$-interference on the branching ratio of $B \to X_s \gamma$ stays below 1% when varying $\mu$ between 1.25 GeV and 5 GeV.}
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A Two-particle cuts

In this appendix we specify the contributions of the 2-particle cuts entering the functions $Y^{(1)}$ and $Y^{(2,i)}$ with $i = \text{NL}, \text{NH}, \text{NV}$. To this end we write

$$Y^{(1)}(z_0, \mu) = Y^{(1)}_{2\text{-cuts}}(0, \mu) + Y^{(1)}_{3\text{-cuts}}(z_0, \mu)$$

(A.1)

and

$$Y^{(2)}(z_0, \mu) = Y^{(2)}_{2\text{-cuts}}(0, \mu) + Y^{(2)}_{3\text{-cuts}}(z_0, \mu) + Y^{(2)}_{4\text{-cuts}}(z_0, \mu),$$

(A.2)

with $Y^{(1)}_{n\text{-cuts}}$ and $Y^{(2)}_{n\text{-cuts}}$ summing all contributions of the $n$-particle cuts. The contribution of the 2-particle cuts at $O(\alpha_s)$ reads

$$Y^{(1)}_{2\text{-cuts}}(0, \mu) = \frac{2}{9} (33 - 2\pi^2) + \frac{16}{3} L_\mu,$$

(A.3)

and those at $O(\alpha_s^2)$ are given by

$$Y^{(2)}_{\text{NL}}(0, \mu) = -\frac{16}{81} (157 - 8\pi^2) - \frac{16}{27} (47 - 2\pi^2) L_\mu - \frac{64}{9} L_\mu^2 + \frac{64}{9} \zeta_3,$$

(A.4)

$$Y^{(2)}_{\text{NH}}(0, \mu) = \frac{8}{81} \left( 244 - 27\sqrt{3}\pi - 61\pi^2 \right) - \frac{16}{27} (47 - 2\pi^2) L_\mu - \frac{64}{9} L_\mu^2 - \frac{64}{27} \zeta_3 + 32\sqrt{3} \text{Cl}_2 \left( \frac{\pi}{3} \right),$$

(A.5)

$$Y^{(2)}_{\text{NV}}(0, \mu) = -\frac{16}{81} \left[ 157 - 279\rho - \pi^2 (5 + 9\rho^2 - 42\rho^{3/2}) \right] - \frac{16}{27} (47 - 2\pi^2) L_\mu - \frac{64}{9} L_\mu^2 + \frac{8}{27} (19 + 20\rho) \ln \rho + \frac{16}{27} (8 + 9\rho^2) \ln^2 \rho - \frac{16}{27} \ln^3 \rho - \frac{8}{9} (1 - 6\rho^2) \Phi_1(\rho) - \frac{8}{27} (19 - 46\rho) \Phi_2(\rho) - \frac{32}{27} (13 + 14\rho) \Phi_3(\rho) - \frac{64}{9} \Phi_4(\rho) - \frac{32}{9} \ln \rho \text{Li}_2(1 - \rho) + \frac{32}{27} (5 + 9\rho^2 + 14\rho^{3/2}) \text{Li}_2(1 - \rho) - \frac{1792}{27} \rho^{3/2} \text{Li}_2(1 - \sqrt{\rho}) + \frac{64}{9} \text{Li}_3(1 - \rho) + \frac{64}{9} \text{Li}_3 \left( 1 - \frac{1}{\rho} \right).$$

(A.6)

In writing our results at $O(\alpha_s^2)$ we tacitly performed a splitting of $Y^{(2)}_{\text{2-cuts}}(0, \mu)$ into terms being proportional to certain combinations of colorfactors, in complete analogy to what we did in (2.5). We remark that all contributions given above are by themselves free of infrared and collinear singularities, as well as independent of the gauge parameter entering the gluon propagator.
B Auxiliary functions

Here we collect the four auxiliary functions $\Phi_n(\rho)$ introduced in section 2. They are defined as follows,

\begin{align*}
\Phi_1(\rho) &= \theta(1 - 4\rho) \left[ \ln^2 y - \pi^2 \right] - \theta(4\rho - 1) \arccos^2 \left( 1 - \frac{1}{2\rho} \right), \\
\Phi_2(\rho) &= \sqrt{|1 - 4\rho|} \left\{ \theta(1 - 4\rho) \ln y - \theta(4\rho - 1) \arccos \left( 1 - \frac{1}{2\rho} \right) \right\}, \\
\Phi_3(\rho) &= \sqrt{|1 - 4\rho|} \left\{ \theta(1 - 4\rho) \left[ \text{Li}_2 (-y) + \frac{1}{4} \ln^2 y + \frac{\pi^2}{12} \right] \\
&\quad - \theta(4\rho - 1) \text{Cl}_2 \left( 2 \arcsin \left( \frac{1}{2\sqrt{\rho}} \right) \right) \right\}, \\
\Phi_4(\rho) &= \theta(1 - 4\rho) \left[ \text{Li}_3 (-y) + \frac{1}{12} \ln^3 y + \frac{\pi^2}{12} \ln y \right] \\
&\quad + \theta(4\rho - 1) \text{Cl}_3 \left( 2 \arcsin \left( \frac{1}{2\sqrt{\rho}} \right) \right),
\end{align*}

where $\theta(z)$ is Heavyside’s step function,

\begin{equation}
y = \frac{1 - \sqrt{1 - 4\rho}}{1 + \sqrt{1 - 4\rho}},
\end{equation}

and $\rho > 0$. The definitions of the two Clausen functions appearing in the above given equations read [32]

\begin{equation}
\text{Cl}_2(z) = \text{Im} \left[ \text{Li}_2 \left( e^{iz} \right) \right], \quad \text{Cl}_3(z) = \text{Re} \left[ \text{Li}_3 \left( e^{-iz} \right) \right],
\end{equation}

and those of the polylogarithms are given by

\begin{equation}
\text{Li}_2(z) = -\int_0^z dx \frac{\ln(1 - x)}{x}, \quad \text{Li}_3(z) = \int_0^z dx \frac{\text{Li}_2(x)}{x}.
\end{equation}

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