A thin power law fluid underside a horizontal flat plane

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Abstract. The dynamics of a thin Power-law fluid underside of a horizontal flat plane is investigated. Using lubrication approximation, we derive governing equation for height of the thin film. The gravity and the surface tension are the main forces that act to the fluid. The equations solved numerically. Based on simulations, results show that the characteristic of the fluid significantly influences the dynamics of the thin film.

1. Introduction
Thin layer fluids are common in nature and are part of technological developments. In nature, they appear among others as membranes, lavas, and rain running down a window. In technology, they occur as paints layers, foams, adhesives, lubricants, and so on. Depends on situation, it may occur on an incline plate, vertical wall, circular substrate, or between two moving objects. Since more than half of practical fluids are non-Newtonian, thus studies of the thin film for generalized fluids are necessary.

In this paper, we investigate the dynamic of thin layer underside of a horizontal flat plane. The problem can be observed in some applications, for example paints on ceiling, and a coating of a silicone oil on the underside of a horizontal flat. In case the thickness of the thin film becomes too large, the paints layer may unstable, the fluid may opaque, or the screen may invisible [1].

For non-Newtonian fluid, where the shear stress is not proportional to the rate of shear, the shear stress–rate of shear relations is often modelled as power-law equation. Its mathematical simplicity and its wide application have been preferred in the present problem. Investigations about dynamics of power-law thin liquid film have received considerable attention. Perazzo [2] studied the slow flow of a power-law liquid on an inclined, theoretically. For one-dimensional case, they found solutions for travelling wave. Chu, et al [3] derived a 1-D modified Reynolds equation for a thin power-law fluid bounded by two adsorption layers. They showed that the pressure distribution is significantly affected by flow index. Myers [4] developed an approximate solution method for flow of a power law fluid over a flat plane. This new method was validated by similarity solution for the Newtonian fluid case. Hu and Kieweg [5] also observed spreading and speed of power-law fluids on an incline under the gravity force. They found that surface tension affects both the spreading shape and speed of the fluid. Chu, et al [6] studied numerically effects of plane roughness on the pressure and the film thickness based on power law fluid model under point contact elastohydrodynamic lubrication conditions. They showed that plane roughness and flow index are significant parameters that affect the pressure and film thickness. Thete [7] analysed dynamic a thin free films for power law fluid under the influences of surface tension pressure and van der waals force. Investigations for thin film fluid on underside a plane, to the best of our knowledge, are still limited. Oron [1] constructed mathematical model and reviewed the stability of
the problem in case for Newtonian fluid. In this study, we use the power law fluid as generalized for the Newtonian fluids.

2. Methods
In this section, a mathematical model for dynamic of thin Power law fluid underside of a horizontal flat plane is constructed. We consider a fluid lying on the underside of a horizontal plane where the gravity is the main force that acts to the fluid. The coordinate $x$ lies in the plane, and the coordinate $z$ is perpendicular to the plane in the negative direction, as shown in Figure 1.

![Figure 1. Sketch of the thin fluid.](image)

Let $z = h(x,t)$ be the height of the thin free surface of the power law fluid. In this model, we assume that the Reynold number of the fluid is considerable high, such that the fluid is in creeping flow. We also assume the characteristic length along the plane $L$ is much wider than the characteristic thickness $H$, i.e. $L \gg H$. Following the work of Perazzo [2] and using the lubrication approximation, governing equations for the generalized Newtonian thin fluids are:

\begin{align*}
-\partial_z p + \partial_x \tau_{xz} &= 0, \quad (1) \\
-\partial_z p + \rho g &= 0, \quad (2) \\
\partial_z u + \partial_x w &= 0, \quad (3)
\end{align*}

where $u$ and $w$ are velocities of the fluid in horizontal and vertical directions, $p$ is pressure, $\rho$ is density, $g$ is gravity, $\tau_{xz}$ is the component of the deviatoric stress tensor, i.e. 

\[
\tau_{xz} = A \left\{ \frac{1}{2} \left[ (\partial_x u)^2 + (\partial_x w)^2 \right] \right\}^{(1-\lambda)/2\lambda} \partial_x u,
\]

where $A$ is consistency index that measures of the consistency of the substance, and $\lambda$ is power-law index which physically represents molecular weight of the fluid. At the free surface ($z = h$), we assume that the pressure is proportional to the curvature of the fluid, i.e. $p = -\sigma \partial_x^2 h$, where $\sigma$ is the surface tension, then the pressure in the thin fluid:

\[
p = \rho gh - \sigma \partial_x^2 h.
\]

To find the velocity in the thin fluid, we use lubrication approximation that $\partial_x w \sim O(\varepsilon)$, integrate equation (1) then apply no slip condition at $z = 0$ and no shear stress at $z = h$. Hence we have:

\[
u = \frac{K^\lambda}{\lambda + 1} (-h)^{\lambda+1} (\rho g \partial_x h - \sigma \partial_x^3 h)^{\lambda} \left[ \left( 1 - \frac{z}{h} \right)^{\lambda+1} - 1 \right],
\]

where $\frac{1}{K} = A \left( \frac{1}{2} \right)^{k}. \lambda$ Kinematic condition that ensure conservation of mass on the domain is given by [8]:

\[
h_t + \partial_x \left( \int_0^h u \, dz \right) = 0.
\]
Finally, equation for the dynamic of the free surface thin power-law fluid lying on underside a horizontal flat plane, in dimensionless form read:

$$h_i = \left[ \frac{K^\lambda}{2 + \lambda} \left(-h\right)^{1+\lambda} h(h - S h_{xxx}) \right]_{x_i},$$  \hspace{1cm} (4)

where $S = \frac{\sigma}{\rho g l^2}$. In equation (4) subscript $x$ signifies the derivative with respect to $x$. We apply boundary conditions at ends of domain as follows:

$$h_i(\pm 1, t) = 0,$$
$$h_{xxx}(\pm 1, t) = 0.$$  \hspace{1cm} (5)

3. Results and Discussion

In this section, we show the numerical solution for the dynamic of the thin fluid height. The equation (4) with the boundaries (5) are solved numerically by Finite Different Method. We choose the parameters $K = 1, S = 1$, with different $\lambda$ and some initial conditions that are sinusoidal and parabolic function. The solutions are computed, up to the change almost vanishes, or it tends to the steady solution.

Figure 2 shows the evolution of thin film height for $\lambda = 1$ (Newtonian fluid). One can see that, the dynamics of the thin film would not depend on the initial profile. The height of the thin film tends to constant steady solution. Due to the principle of mass conservation that expressed in the equation (3), then the constant steady solutions are reached at $h(x, t) = 0.5$ and at $h(x, t) = 0.47$, for initial profile sinusoidal and parabolic, respectively. This result is in accordance with the results of Oron et.al. [1] that the surface tension acts as dissipation term which implies no instabilities would arise. In some applications, the constant steady solution is a worth, because it represents a flat contour.

![Figure 2](image-url)  \hspace{1cm} (a)

![Figure 2](image-url)  \hspace{1cm} (b)

Figure 2. The dynamic of the thin film height for $\lambda = 1$, with (a) sinusoidal initial condition, (b) parabolic initial condition.

Figure 3 illustrates the change height of the thin film for power-law fluid with $\lambda = 1.4$. Based on the figure, it can be notice that the evolution would not tend to constant solution but to a shape that depends on the initial profile. In the sinusoidal initial condition case, the gravity causes growth the height of dale. As the consequence of the mass conservation principle, heights of the peaks decrease. Whereas in the parabolic initial condition case, it tends to accumulate into the peak. Form those figures, it can be said that gravity as the body force drives the fluid into its direction. The surface tension that acts as dissipation term shapes up the fluid into its optimal minimizing energy.
Figure 3. The dynamic of the thin film height for $\lambda = 1.4$, with (a) sinusoidal initial condition, parabolic initial condition.

Different with behaviour of fluid with $\lambda = 1$, dynamic of fluid with $\lambda = 1.4$ indicates the process of thickening. Its results are consistent with the results obtained in [9], i.e. fluids with $\lambda > 1$ indicate shear-thickening fluid. Typical examples of these fluids are thick suspension and pastes of kaolin, TiO2, and corn flour in water [9].

4. Conclusion
In this paper we have considered the evolution of the power law thin fluid height that lying on the underside of a horizontal flat plane. We derived the governing equation for height of the thin film under the action of gravity and the surface tension. The equation is solved numerically by the finite difference method. Our numerical results showed that for Newtonian fluid ($\lambda = 1$) the thin film height tends to constant steady solution for any sort of initial condition. Whereas for fluid with flow index $\lambda = 1.4$, the height of the thin film increases, it indicates a thickening.

5. References
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