Gauge Theory Techniques in Quantum Cohomology

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Abstract. Quantum cohomology gives a finite dimensional integrable system via the Dubrovin connection. Motivated by Givental's work on mirror symmetry, we use gauge theory techniques and the Frobenius Integrability Theorem to find flat sections for the Dubrovin connection. An explicit calculation is given for projective space.

1. Introduction

The work of Givental [3, 7] and Liu-Lian-Yau [5] on mirror symmetry relates Gromov-Witten invariants of a quintic hypersurface in $\mathbb{P}^3$ to period integrals of Kähler structures on the mirror manifold. Givental's method uses detailed calculations of equivariant GW invariants to produce flat sections of the Dubrovin connection on the tangent bundle to the even cohomology of the hypersurface, which are then related to solutions of the Picard-Fuchs ODE for the periods on the mirror.

Givental’s approach leads to the following general question: given a flat connection on the tangent bundle to a vector space, how can we compute the flat sections? In contrast to Donaldson/Seiberg-Witten/Chern-Simons gauge theories, where gauge directions are quotiented out, the moduli space of flat connections here is trivial. Thus all the information in the Dubrovin connection is contained in the gauge transformation taking the Dubrovin connection to the trivial connection. Since the Dubrovin connection is defined in terms of the quantum product, which in turn encodes GW invariants, this gauge transformation captures all the algebra of the quantum product.

The purpose of this paper is to compute this gauge transformation and the corresponding flat sections from a classical PDE point of view, specifically through a systematic use of the Frobenius Integrability Theorem. It is surprisingly difficult to compute flat sections; even for the cup product, where all quantum corrections are turned off, the Dubrovin connection is rather trivial, but we show in §2 that the flat sections are much more complicated. In §3, we give a general expression for the flat sections, which involves exponentiating an infinite matrix whose entries are matrices. In §4, we reduce the calculation of the flat sections for the quantum

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product on $\mathbb{P}^m$ to exponentials of ordinary matrices, and compare our formula with known solutions.

In this paper, as in [8], we restrict attention to the small quantum product, with some comments at the end on the big quantum product and on coupling to gravity. This last product, with its relation to infinite dimensional integrable systems, is the real case of interest, and the current paper was intended as a warm-up. The unexpected complexity of the calculations in this case indicates the difficulty of proving the higher genus Virasoro conjecture [8].

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2. Constant Products on Vector Spaces

Let $*$ be an associative, commutative product on a complex vector space $\mathcal{H}$. The associated Dubrovin connection on $T\mathcal{H}$ is

$$\nabla_X Y = dX(Y) + \sqrt{-1} Y \ast X$$

with connection one-form $\omega(Y)(X) = \sqrt{-1} Y \ast X$. If $\{T_0, \ldots, T_m\}$ is a basis of $\mathcal{H}$ with

$$T_i \ast T_j = \Gamma^k_{ij} T_k,$$

then $\omega^i_j = -\Gamma^i_{jk} \partial t^k$ (where a typical $\alpha \in \mathcal{H}$ is $\alpha = t_i T^i$). It is fundamental that $\nabla$ is flat because the product is associative and commutative. Note that $\nabla$ stays flat if we replace $\sqrt{-1}$ in (2.1) by $\hbar \in \mathbb{C}$.

The space $\mathfrak{M}$ of flat connections modulo gauge transformations over any manifold $\mathcal{H}$ has formal tangent space $H^1(\mathcal{H}, gl(m, \mathbb{C}))$ at $\nabla$, where dim $\mathcal{H} = m$ and the de Rham cohomology is computed with respect to the exterior derivative coupled to $\nabla$ (see e.g. [1, §7.2]). For our $\mathcal{H}$, the cohomology vanishes by a Poincaré Lemma, and the points of $\mathfrak{M}$ are characterized by the holonomy of the connection. Since $\mathcal{H}$ is contractible, there is no nontrivial holonomy, so $\mathfrak{M}$ consists of one point.

In particular, $\nabla$ is globally gauge equivalent to the trivial connection $d$, when $d_Y X = dX(Y)$. Thus there exists $g: \mathbb{C}^m \to GL(m, \mathbb{C})$ with $g \cdot d = g^{-1} d g = \nabla$, which translates into $g^{-1} d g = \omega$. A basis of the flat sections for $\nabla$ is given by $\{g^{-1} e_i\}_{i=0}^m$, where $e_i = \partial_i$ is the constant section, since $\nabla g^{-1} e_i = (g^{-1} d g) g^{-1} e_i = 0$. This shows

**Lemma 2.1.** Let $g$ be the gauge transformation on $T\mathcal{H}$ such that $g^{-1} d g = \nabla$. Then a basis of the flat sections of $\nabla$ is given by the columns of $g^{-1}$.

Thus finding flat sections is equivalent to solving $g^{-1} d g = \omega$ for $g$. In coordinates, this becomes the system

$$\frac{\partial g^i_j}{\partial t^k} = \sqrt{-1} g^i_s \Gamma^s_{jk},$$

If $A \in GL(m^2, \mathbb{C})$ is considered as a constant gauge transformation, then $g^{-1} d g = \omega$ iff $(A g)^{-1} d (A g) = \omega$. Thus we may assume $g(0, 0, \ldots, 0) = Id$ as an initial condition, which is natural as $g^{-1}$ will then take flat sections $\{e_i\}$ of $d$ at $t \in T_0 \mathcal{H}$ to flat sections of $\nabla$ through $t$.

(2.3) is a classical PDE handled by the Frobenius Integrability Theorem (see e.g. [8, Vol. I, Ch. 6]). We can rewrite (2.3) as

$$\frac{\partial g^i_j}{\partial t^k} = f^i_k(t, g(t)),$$
with \( x, f_\ell(t, x) m \times m \) matrices satisfying
\[
\begin{align*}
f_\ell(t, x)^i_j &= \sqrt{-1}x^i \Gamma_\ell^s \Gamma^s_{ij} \\
&= \sqrt{-1}(x \cdot \Gamma_\ell)_{ij},
\end{align*}
\]
where \( \Gamma_\ell \) is the \( m \times m \) matrix \( (\Gamma_\ell)^i_j = \Gamma^i_j \).

For later purposes, we first treat the case where \( \ast \) is independent of \( t \in \mathcal{H} \). This includes the case of the classical cup product. Here the Frobenius integrability conditions for the system \((2.4)\) are
\[
\Gamma^j_{ik} \Gamma^k_{\ell m} = \Gamma^j_{i \ell k} \Gamma^k_{im}.
\]
(2.5)
Since \( \ast \) is commutative, \( \Gamma^i_{jk} = \Gamma^j_{ki} \), and \((2.5)\) is equivalent to the associativity of \( \ast \). In addition, \((2.5)\) is equivalent to \([\Gamma_i, \Gamma_\ell] = 0\), which implies that the exponentials of the \( \Gamma_i \) commute (see below).

The proof of the Frobenius Integrability Theorem leads to a construction of the solution of \((2.3)\). We first solve the \( m \times m \) matrix valued, complex time ODE
\[
\dot{B}(t, 0, \ldots, 0) = f_1((t, 0, \ldots, 0), B(t, 0, \ldots, 0)) = \sqrt{-1}B(t, 0, \ldots, 0) \cdot \Gamma_0, \\
B(0) = \text{Id}.
\]
Thus \( B(t, 0, \ldots, 0) = \exp[\sqrt{-1}t \Gamma_0] \). We then fix \( t^0 \) and solve
\[
\dot{B}(t^0, t, 0, \ldots, 0) = f_1(t^0, t, 0, \ldots, 0, B(t^0, t, 0, \ldots, 0)) = \sqrt{-1}B(t) \cdot \Gamma_1, \\
B(0) = \exp[\sqrt{-1}t^0 \Gamma_0],
\]
so \( B(t^0, t, 0, \ldots, 0) = \exp[\sqrt{-1}t^0 \Gamma_0] \exp[\sqrt{-1}t \Gamma_1] \). Continuing, we get at the last step
\[
g(t^0, t^1, \ldots, t^m) = \prod_{\ell=0}^m \exp[\sqrt{-1}t^\ell \Gamma_\ell] = \exp[\sqrt{-1} \sum_{\ell=0}^m t^\ell \Gamma_\ell]
\]
since the \( \Gamma_\ell \) commute. This shows:

**Proposition 2.2.** Let \( \ast \) be a commutative, associative product on a complex vector space \( \mathcal{H} \). Define matrices \( \Gamma_i, i = 1, \ldots, \dim \mathcal{H} \) by \( T_i \ast T_j = (\Gamma_i)^k_j T_k \) for a basis \( \{T_k\} \) of \( \mathcal{H} \). Then a basis of the flat sections of the associated flat Dubrovin connection on \( T\mathcal{H} \) is
\[
\{\exp[\sqrt{-1}t^\ell \Gamma_\ell]\}_{\ell=0}^m.
\]

For example, if \( \ast \) is the cup product on \( \mathbb{P}^1 \), then the connection one-form on \( T\mathcal{H} = TH^\ast(\mathbb{P}^1, \mathbb{C}) \) is
\[
\omega = \sqrt{-1}\begin{pmatrix} \frac{dt^0}{dt^1} & 0 \\ \frac{dt^0}{dt^1} & 0 \end{pmatrix}, \quad \Gamma_0 = \text{Id}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
and the flat sections are spanned by the columns of
\[
\exp\left[\sqrt{-1}\begin{pmatrix} t^0 & 0 \\ t^1 & t^0 \end{pmatrix}\right] = \begin{pmatrix} e^{-it^0} & 0 \\ -it^1 e^{-it^0} & e^{-it^0} \end{pmatrix}.
\]
3. The Small Quantum Product

In this section, we will find a basis for the flat sections for the Dubrovin connection for the small quantum product. As explained in e.g. [2], this is a deformation of the cup product on the even cohomology $H = H^{2k}(M; \mathbb{C})$ of a symplectic manifold or smooth projective variety $M$, but with deformations only in $H^2(M; \mathbb{C})$ directions. We restrict attention to weakly monotonic symplectic manifolds (e.g. Fano varieties), so that there are no convergence issues for the quantum product.

More precisely, the small quantum product is defined on $i^*T\mathcal{H}$, where $i: H^2 \to \mathcal{H}$ is the inclusion. For $\{T_1, \ldots, T_k\}$ a basis of $H^2$, the small quantum product is defined as in (2.2) by $\Gamma^i_j (t^1, \ldots, t^k) = \sqrt{-1} \psi_{jrt} h^{rt}$, where $(h_{\alpha\beta}) = (T_\alpha \cdot T_\beta)$ is the intersection matrix, $(h^{\alpha\beta})$ its inverse, and

$$
\psi_{jrt} = T_{j \cdot T_r \cdot T_t} + \sum_\beta \exp\left[ \sum_{k=1}^k t^1 \int_{T_\beta} \cdots \int_{T_\beta} I_{\beta}(T_{j_1}, T_{r_1}, T_{t_1}) \right]
$$

(3.1)

$[$2, §2.1$]$. In (3.1), $I_{\beta}(T_j, T_r, T_t)$ is the three-pointed Gromov-Witten invariant, and $\beta$ ranges over $H_2(M, \mathbb{Z}) \setminus \{0\}$.

Since the product now depends on $(t^1, \ldots, t^k) \in H^2$, the flatness of $\nabla$ is more complicated. More precisely, we extend (2.1) to a family of connections

$$
(\nabla_h)_Y X = dx(Y) + hY \ast X,
$$

$h \in \mathbb{C}$, with curvature $h dw + h^2 \omega \wedge \omega$, $\omega_j^i = \Gamma^i_{j\ell} dt^\ell$. Then $\omega \wedge \omega = 0$ because $\ast$ is commutative and associative, and $d\omega = 0$ because the small quantum product is potential (i.e. $\omega$ is exact). For convenience we will set $h = \sqrt{-1}$ in this section.

To find a basis of flat sections, we apply the Frobenius Integrability Theorem method. We must first solve the ODE for $t^1$:

$$
\frac{\partial g_j^i}{\partial t} = \sqrt{-1} g_j^i [T_j \cdot T_1 \cdot T_1] + \sum_{\beta \in H^2 \setminus \{0\}} \exp[t \int_{T_\beta} I_{\beta}(T_j, T_1, T_1)] h^{rt}.
$$

This is of the form

(3.3)

$$
\dot{B}(t) = B(t) \left[ A + \sum_{i=1}^n e^{a_i t} C_i \right]
$$

with $A_j^\prime = \sqrt{-1} (T_j \cdot T_1 \cdot T_1) h^{rt}$, $a_i = \int_{T_\beta} T_1$, and $(C_i)_j^\prime = I_{\beta i}(T_j, T_1, T_1) h^{rt}$. Here $\{\beta_1, \ldots, \beta_n\}$ is the set of classes in $H_2$ such that $I_{\beta i}(T_j, T_1, T_1) \neq 0$ for some $j, \ell$. We may assume that $a_i \in \mathbb{Z}$.

Let $B^{(1)}(t)$ be the solution of (3.3) with $B^{(1)}(0) = \text{Id}$. As in §2, we will then solve similar equations for $t^2, \ldots, t^k$ and get

$$
g(t) = B^{(1)}(t^1, 0, \ldots, 0) B^{(2)}(t^2, t^2, 0, \ldots, 0) \cdots B^{(k)}(t^1, \ldots, t^k).
$$

We actually want the columns of $g^{-1}$, so we must invert $B = B^{(i)}$. Since $B^{-1} = -BB^{-1}$, we have

$$
\dot{B}^{-1}(t) = - \left[ A + \sum_{i=1}^n e^{a_i t} C_i \right] B^{-1}(t).
$$

(3.4)

Equations of the form (3.3), (3.4) (which differ only by taking the transpose) have a long history. They arise in the Riemann-Hilbert problem of constructing...
second order ODEs with regular singular points having prescribed monodromy at prescribed points in the complex plane. Classically, (3.4) was treated by a good ansatz leading to complicated recursion formulas for the coefficients. The main technical contribution of this paper is a more modern organization of the recursion relations.

The rest of this section is devoted to solving (3.4) with the assumption \( n = 1 \) and \( a_1 = 1 \). For convenience, we replace \( A, C \) in (3.4) by \(-A, -C\), respectively. The general case is discussed in §5.

Under the substitution \( x = e^t \), (3.4) becomes \( \frac{dB}{dx} = -[x^{-1}A + C]B \). By [4, Mem. I, p. 220], this has a solution, valid on \( 0 < \rho_1 < |x| < \rho_2 < 1 \), as a complicated power series in \( x^k \ln^j x \) (\( k \in \mathbb{Z}, j \in \mathbb{Z}^+ \cup \{0\} \)), with matrix coefficients. The solution is convergent for \( \text{Re}(t) < 0 \). This suggests looking for a solution of (3.4) either of the form \( \sum B_{kj} e^{kt} t^j \) or of the form \( \sum B_n(e^t - 1)^n \). The latter will give a solution near \( t = 0 \); recall that in quantum cohomology solutions are formally constructed in a neighborhood of \( t=\ldots=t_k=0 \). We can also address convergence issues near a fixed value of \( e^t = \alpha \), by looking for a solution of the form \( \sum B_n(e^t - \alpha)^n \). In fact, each substitution gives different information.

**Substitution I:** We plug \( \sum_{n \geq 0} B_n(e^t - 1)^n \) into (3.4), noting that \( B(0) = \text{Id} \) implies \( B_0 = \text{Id} \). We obtain

\[
\sum nB_n(e^t - 1)^{n-1}e^t = \sum AB_n(e^t - 1)^n + \sum CB_n(e^t - 1)^n e^t,
\]

or

\[
\sum nB_n(e^t - 1)^n + \sum nB_n(e^t - 1)^{n-1} = \sum AB_n(e^t - 1)^n + \sum CB_n(e^t - 1)^{n+1} + \sum CB_n(e^t - 1)^n.
\]

This gives the recursion relation

\[
B_{n+1} = \frac{1}{n+1}[(A + C - n)B_n + CB_{n-1}], \quad n \geq 2,
\]

\[
B_0 = \text{Id}, \quad B_1 = A + C.
\]

This two-term relation can be encoded as

\[
\begin{pmatrix}
B_{n+1} \\
B_n
\end{pmatrix}
= \frac{1}{n+1} \begin{pmatrix}
A + C - n & C \\
n + 1 & 0
\end{pmatrix}
\begin{pmatrix}
B_n \\
B_{n-1}
\end{pmatrix}
= \frac{1}{n(n+1)} \begin{pmatrix}
A + C - n & C \\
n + 1 & 0
\end{pmatrix}
\begin{pmatrix}
A + C - (n-1) & C \\
n & 0
\end{pmatrix}
\begin{pmatrix}
B_{n-1} \\
B_{n-2}
\end{pmatrix}
= \ldots = \frac{1}{(n+1)!} \prod_{j=1}^n \begin{pmatrix}
A + C - j & C \\
j + 1 & 0
\end{pmatrix}
\begin{pmatrix}
B_1 \\
B_0
\end{pmatrix}.
\]
with the convention that the \( j \)th matrix is to the right of the \((j + 1)\)st matrix. This can be rewritten as

\[
B_{n+1} = \frac{1}{(n + 1)!} \left[ \prod_{j=1}^{n} \left( \begin{array}{cc} A + C - j & C \\ j + 1 & 0 \end{array} \right) \left( \begin{array}{cc} A + C & C \\ 1 & 0 \end{array} \right) \right]_{(1,1)}
\]

\[
= \frac{1}{(n + 1)!} \left[ \prod_{j=0}^{n} \left( \begin{array}{cc} A + C - j & C \\ j + 1 & 0 \end{array} \right) \right]_{(1,1)}.
\]

As a check, when \( C = 0 \) we must get \( B(t) = e^{tA} \). For fixed \( y \), the Taylor series for \( x^y \) at \( x = 1 \) is

\[
x^y = 1 + \sum_{n \geq 1} \frac{1}{n!} \left[ \prod_{j=0}^{n-1} (y - j) \right] (x - 1)^n,
\]

and so

\[
e^{tA} = \text{Id} + \sum_{n \geq 1} \frac{1}{n!} \left[ \prod_{j=0}^{n-1} (A - j) \right] (e^{t} - 1)^n.
\]

For \( C = 0 \), \( B_n = \frac{1}{n!} \prod_{j=0}^{n-1} (A - j) \), so this case checks. It is also interesting to check the case when \( A, C \in \mathbb{C} \) are 1 \( \times \) 1 matrices. The explicit solution to (3.4) is \( B(t) = e^{tA} e^{(e^{t}-1)C} \), so we get the identity

\[
x^a e^{(x-1)c} = 1 + \sum_{n \geq 1} \frac{1}{n!} \left[ \prod_{j=0}^{n-1} \left( \begin{array}{cc} a + c - j & c \\ j + 1 & 0 \end{array} \right) \right] (x - 1)^n.
\]

Writing \( x^a = ((x - 1) + 1)^a \) and using the binomial expansion, we get

\[
\sum_{k+\ell = n} \binom{a}{k} \frac{e^\ell}{\ell!} = \frac{1}{n!} \left[ \prod_{j=0}^{n-1} \left( \begin{array}{cc} a + c - j & c \\ j + 1 & 0 \end{array} \right) \right]_{(1,1)}.
\]

In summary, Substitution I leads to cute identities.

**Substitution II:** For convergence issues, we fix \( t_0 \in \mathbb{C} \), set \( e^{t_0} = \alpha \), and substitute \( \sum B_n (e^t - \alpha)^n \) into (3.4). As above, this leads to the recursion relation

\[
B_{n+1} = \frac{1}{(n + 1)\alpha} [(A + C\alpha - n)B_n + CB_{n-1}],
\]

and so

\[
B_{n+1} = \frac{1}{(n + 1)!\alpha^{n+1}} \left[ \prod_{j=0}^{n} \left( \begin{array}{cc} A + C\alpha - j & C \\ (j + 1)\alpha & 0 \end{array} \right) \right]_{(1,1)}.
\]

Thus

\[
(3.6) \quad B = \text{Id} + \sum_{n \geq 1} \frac{1}{n!} \left[ \prod_{j=0}^{n-1} \left( \begin{array}{cc} A + e^{t_0}C - j & C \\ e^{t_0}(j + 1) & 0 \end{array} \right) \right]_{(1,1)} (e^{t-t_0} - 1)^n.
\]

The \((1,1)\) entry of the matrix coefficient of \((e^{t-t_0} - 1)^n\) consists of \( n \) terms, each of which is a product of \( k \) terms of the form \( A + e^{t_0}C - j \), \( \ell \) occurrences of \( C \), and \( m \) terms of the form \( e^{t_0}(j + 1) \), with \( k + \ell + m \leq n \). Thus the supremum norm of
the (1, 1) entry is bounded above by \( c \cdot n \cdot n! \), where \( c \) is a constant depending on \( t_0 \) and the sup norms of \( A \) and \( C_j \), and below by \( c' \cdot n \cdot n! \) for another constant \( c' \).

The ratio test implies that the right hand side of (3.6) converges if \( |e^{t-t_0} - 1| < 1 \), or equivalently if \( \Re(t-t_0) < 2 \cos \Im(t-t_0) \). In particular, the series converges uniformly on a ball around \( t_0 \) to a solution of (3.4).

**Substitution III:** We substitute \( \sum B_k e^{t_k} t^j \) into (3.4), with \( k \geq 0, j \geq 0, k, j \in \mathbb{Z} \). The omission of terms with \( k < 0 \) is motivated by the fact that it works. From \( B(0) = \text{Id} \), we get \( \sum_k B_{k0} = \text{Id} \), so assume \( B_{k0} = \delta_{k0} \cdot \text{Id} \). The substitution gives

\[
(3.7) \quad \sum kB_{kj} e^{t_k} t^j + \sum (j + 1)B_{kj+1} e^{t_k} t^j = \sum AB_{kj} e^{t_k} t^j + \sum CB_{k-1,j} e^{t_k} t^j,
\]

so

\[
(3.8) \quad B_{k,j+1} = \frac{1}{j + 1} [(A - k)B_{kj} + CB_{k-1,j}].
\]

For \( j > 0 \), this encodes as

\[
\begin{pmatrix}
B_{kj} \\
B_{k-1,j} \\
\vdots \\
B_{1j} \\
B_{0j}
\end{pmatrix}
= \frac{1}{j} \begin{pmatrix}
A - k & C \\
A - k + 1 & C \\
\vdots & \ddots & \ddots \\
A - 1 & C & A
\end{pmatrix}
\begin{pmatrix}
B_{k,j-1} \\
B_{k-1,j-1} \\
\vdots \\
B_{1,j-1} \\
B_{0,j-1}
\end{pmatrix}
\]

\[
= \cdots = \frac{1}{j!} \begin{pmatrix}
A - k & C \\
A - k + 1 & C \\
\vdots & \ddots & \ddots \\
A - 1 & C & A
\end{pmatrix}^j
\begin{pmatrix}
B_{k0} \\
B_{k-1,0} \\
\vdots \\
B_{10} \\
B_{00}
\end{pmatrix}
\]

Since \( B_{k0} = \delta_{k0} \cdot \text{Id} \), we get for \( k \geq 0, j > 0, \)

\[
(3.9) \quad B_{kj} = \frac{1}{j!} \begin{pmatrix}
A - k & C \\
A - k + 1 & C \\
\vdots & \ddots & \ddots \\
A - 1 & C & A
\end{pmatrix}^j
\begin{pmatrix}
(1, k+1)
\end{pmatrix}
\]
Note that the assumption $B_{kj} = 0$ for $k < 0$ is consistent with (3.8). The theory in 4 gives uniform convergence of $\sum B_{kj} e^{kt} t^j$ to a solution for $-\frac{1}{\epsilon} < \text{Re}(t) < -\epsilon$ for any $\epsilon > 0$, and so it must coincide with the solution given in Substitution II.

This again leads to identities.

(3.8) has an infinite dimensional interpretation. Let $H = L^2(S^1, \mathbb{C}^m)$ denote the Hilbert space of $L^2 \mathbb{C}^m$-valued functions on $S^1$ with Fourier expansions containing only $e^{in\theta}$ with $n \geq 0$. Let $D : H \rightarrow H$ be the first order operator

\[
D = -\sqrt{-1} \frac{d}{d\theta} + A + e^{\sqrt{-1}n\theta} C.
\]

(The range of $D$ is $H$ plus the span of $e^{-\sqrt{-1}n\theta}$, so we actually compose $D$ with the projection $L^2 \rightarrow H$.) Let $e_{(j)}^{\sqrt{-1}n\theta}$ denote $(0, \ldots, 0, e^{\sqrt{-1}n\theta}, 0, \ldots, 0)$, where $e^{\sqrt{-1}n\theta}$ occurs in the $j^{th}$ slot. Then $De_{(j)}^{\sqrt{-1}n\theta} = ne_{(j)}^{\sqrt{-1}n\theta} + A_{jk} e_{(k)}^{\sqrt{-1}n\theta} + C_{jk} e_{(k)}^{\sqrt{-1}(n+1)\theta}$, so in the basis $\{e_{(j)}^{\sqrt{-1}n\theta}\}_{n \geq 0, j=1, \ldots, m}$, $D$ has matrix

\[
\begin{pmatrix}
A & C & 0 & 0 \\
 A + 1 & C & \ddots & 0 \\
 A + 2 & C & \ddots & \ddots \\
 & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

which we also denote by $D$.

We compute

\[
B = \sum_{k,j \geq 0} \frac{1}{j!} \left[ \begin{pmatrix}
A - k & C & 0 & 0
\\
 A - k + 1 & C & \ddots & 0 \\
 & \ddots & \ddots & \ddots
\\
 & & \ddots & \ddots & \ddots \\
 & & & \ddots & \ddots & \ddots \\
\end{pmatrix}^{(1,k+1)} \right] \cdot e^{kt} t^j
\]

\[
= \sum_{k \geq 0} \exp \left[ \begin{pmatrix}
A - k & C \\
 A - k + 1 & C \\
 & \ddots & \ddots \\
 & & \ddots & \ddots \\
 & & & \ddots & \ddots \\
\end{pmatrix}^{(1,k+1)} \right] e^{kt}
\]

\[
= \sum_{k \geq 0} \exp \left[ \begin{pmatrix}
A - k & C \\
 A - k + 1 & C \\
 & \ddots & \ddots \\
 & & \ddots & \ddots \\
 & & & \ddots & \ddots \\
\end{pmatrix}^{(1,k+1)} \right] e^{kt}
\]
It is easy to check that the $(1, k+1)$ entry of this last matrix, denoted $\exp[tDk]$, equals the $(1, k + 1)$ entry of $\exp[tD]$ by comparing entries for $D^k$ and $D^j$. This gives:

**Proposition 3.1.** The solution of $\dot{B}(t) = (A + e^{tC})B(t), B(0) = \text{Id}$ is given by

$$B = \sum_{k \geq 0} \{\exp(tD)\}_{(1,k+1)}.$$  

(3.11)

with $D$ given by (3.10).

In particular,

$$B_{ij} = \sum_{k \geq 0} \langle \exp(tD)(e^{\sqrt{-1}k\theta}), 1(i) \rangle,$$

since $e^{\sqrt{-1}0} = 1$. We can extend $D$ by its matrix representation to act on $L^2_+(S^1, M_{m \times m}(\mathbb{C}))$, and then

$$B = \sum_{k \geq 0} \langle \exp(tD)(e^{\sqrt{-1}k\theta}1)1, 1 \rangle.$$  

(3.12)

As in §2, we can now continue to solve in other $H^2$ directions.

For $\mathbb{P}^m$, $B$ can be computed from (3.11) since $A, C$ are particularly simple. For other spaces, we can make (3.12) more explicit by attempting to diagonalize $D$. Let

$$e_k = (0 \cdots 0 \text{Id} -C C^2/2 -C^3/3! \cdots),$$

with Id in the $k$th slot. The \{e_k\} are linearly independent, and $e_k D = e_k (A + k)$. For

$$E = \begin{pmatrix} \text{Id} & -C & C^2/2 & C^3/3! & \cdots \\ \text{Id} & -C & C^2/2 & \cdots \\ \text{Id} & -C & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \\ \end{pmatrix}$$

we have

$$ED = \begin{pmatrix} A & -CA & \frac{C^2A}{2} & -\frac{C^3A}{3!} & \cdots \\ A + 1 & -C(A + 1) & \frac{C^2A}{2} & \cdots \\ A + 2 & -C(A + 2) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \end{pmatrix}$$

$$= \left( \begin{array}{cccc} A & -AC - [C, A] & A\frac{C^2}{2} + [\frac{C^2}{2}, A] & -A\frac{C^3}{3!} - [\frac{C^3}{3!}, A] & \cdots \\ A + 1 & (A + 1)C - [C, A] & (A + 1)\frac{C^2}{2} + [A, \frac{C^2}{2}] & \cdots \\ & A + 2 & (A + 2)C - [C, A] & \cdots \\ \end{array} \right)$$

$$= GE + T$$
for

\[
\begin{pmatrix}
A & A + 1 \\
A + 2 & \ddots \\
\vdots & \ddots & \ddots
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & -[C, A] & [C^2/2, A] & -[C^3/3, A] & \cdots \\
0 & -[C, A] & [C^2/2, A] & \cdots \\
0 & -[C, A] & \ddots & \ddots
\end{pmatrix}
\]

Since \( E \) is invertible,

\begin{equation}
D = E^{-1}(G + TE^{-1})E,
\end{equation}

so \( TE^{-1} \) measures the obstruction to diagonalizing \( D \) due to the noncommuting of \( A \) and \( C \). Note that

\[
E^{-1} = \begin{pmatrix}
\text{Id} & C & C^2/2 & C^3/3! & \cdots \\
\text{Id} & \text{Id} & C & C^2/2 & \cdots \\
\text{Id} & \text{Id} & \text{Id} & C & \ddots \\
\text{Id} & \text{Id} & \text{Id} & \text{Id} & \ddots
\end{pmatrix}.
\]

By \((3.12), (3.13)\),

\[
B = \sum_{k \geq 0} \left( (E^{-1} e^{t(G+TE^{-1})} E e^{\sqrt{-1}k\theta} \text{Id}) \right)^* \left( E^{-1} e^{t(G+TE^{-1})} E e^{\sqrt{-1}k\theta} \text{Id} \right)
\]

where the final bar denotes complex conjugate. This uses the identity \( \langle Av, w \rangle = \langle A^T w, v \rangle \) for a real endomorphism \( A \) of a complex noncommutative algebra, provided the components of \( v, w \) satisfy \([v_i, \bar{w}_i] = 0\). Since \( E e^{\sqrt{-1}k\theta} \text{Id} \) is the \( k \)th column of \( E \), we get

\[
B = \sum_{k \geq 0} \left( \text{Id} \\
C \\
C^2/2 \\
\vdots
\right) e^{t(G+TE^{-1})} \left( \begin{pmatrix}
(-1)^{k-1} C^{k-1} / (k - 1)! \\
(-1)^{k-2} C^{k-2} / (k - 2)! \\
\vdots \\
\text{Id} \\
0 \\
\vdots
\end{pmatrix}
\right).
\]
Thus

\[ (3.14) \quad \mathcal{B} = \left\langle \begin{pmatrix} \text{Id} & C \\ C^2/2 & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix}, e^{t(G+TE^{-1})} \begin{pmatrix} e^{-C} & \vdots \\ e^{-C} & \vdots \\ \vdots & \vdots \end{pmatrix} \right\rangle. \]

This expression will be more useful than (3.11) if \([A, C^k]\) and hence \(TE^{-1}\) is sparse.

4. Flat sections for \(\mathbb{P}^m\)

In computing the flat sections for \(\mathbb{P}^m\), the only step is the ODE for \(H^2\). Here \(A^i_j\) equals

\[
\left( \int_{\mathbb{P}^n} T_1 \cup T_j \cup T_s \right) h^{si} = \left[ \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \end{pmatrix} \right]^{i} \left[ \begin{pmatrix} 1 & \vdots \\ 0 & \vdots \end{pmatrix} \right]^{j}
\]

and

\[
C = \left( \begin{pmatrix} 1/2 & \cdots \\ \vdots & \vdots \end{pmatrix} \right) = \left( \begin{pmatrix} 1/2 & \cdots \end{pmatrix} \right),
\]

where all empty slots are 0. Recall that we solved (3.4) for \(-A, -C^k\).

As in (3.11), we have

\[ (4.1) \quad B = \sum_{k,n \geq 0} \frac{t^n}{n!} D_{(1,k+1)}^{n}. \]

Note that the \(k = 0\) summand is

\[
\sum_n \frac{t^n}{n!} (-A)^n = \begin{pmatrix} 1 & -t & t^2/2 & \cdots & (-1)^m t^m / m! \\ 1 & -t & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & t^2/2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -t & \cdots & \cdots & 1 \end{pmatrix},
\]

and that the only other contribution from the sum for \(n = 0\) or \(n = 1\) is \(tC\) when \((n, k) = (1, 1)\). Thus we may assume below that \(n > 1\) and \(k > 0\).
Write \( D = A + C \), where

\[
A = \begin{pmatrix}
-A & -A + 1 \\
-A + 2 & \ddots \\
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & -C & \ddots \\
0 & 0 & \ddots \\
\end{pmatrix}.
\]

Let \( P(a,b) = \{(r_1, \ldots, r_a) : r_i \in \mathbb{Z}^+, \sum r_i = b\} \) be the set of unordered partitions of \( b \) into \( a \) terms. Since \( C^2 = 0 \) and hence \( C^2 = 0 \), we have

\[
D^n = \sum_s \sum_{P(s, n-s)} A^{r_1} C A^{r_2} C \cdots A^{r_s} C
\]

\[
= \sum_s \sum_{P(s, n-s+1)} A^{r_1} C A^{r_2} C \cdots A^{r_s} C + \sum_s \sum_{P(s, n-s)} C A^{r_1} C A^{r_2} C \cdots A^{r_s} C
\]

\[
= \sum_s \sum_{P(s, n-s-1)} C A^{r_1} C A^{r_2} C \cdots A^{r_s} C
\]

\[
= I^{(n)} + \Pi^{(n)} + \Pi^{(n)} + IV^{(n)}.
\]

A fixed partition contributes only one nonzero entry in the first row of \( I^{(n)} \), namely \((-1)^k(-A)^{r_1} C(-A + 1)^{r_2} C \cdots (-A + s - 1)^{r_s} C \) in the \((1, s + 1)\) slot. Thus

\[
I^{(n)}_{1,k+1} = \sum_{P(k, n-k)} (-1)^k(-A)^{r_1} C(-A + 1)^{r_2} C \cdots (-A + k)^{r_k} C.
\]

Since \((-A + c)^r = \sum_{u=0}^{\infty} (-1)^u{r \choose u} A^uc^{r-u} = \sum_{u=0}^{\infty} (-1)^u{r \choose u} \delta_{i,j-u} c^{r-u} \), and since the product of \( n \times n \) matrices \( X^{(1)} C X^{(2)} C \cdots X^{(k)} C \) has only nonzero entries in the first column, with \((f, 1)\) entry \( x_{1n}^{(1)} x_{2n}^{(2)} x_{3n}^{(3)} \cdots x_{kn}^{(k)} \), we get

\[
I^{(n)}_{1,k+1} = \sum_{P(k, n-k)} (-1)^{(k-1)m+k}(-1)^{r_1} \left( \frac{r_2}{m} \right) \left( \frac{r_3}{m} \right) \cdots \left( \frac{r_k}{m} \right) \left( k - 1 \right)^{r_k-m} E_1^{m+1-r_1},
\]

where \( E_j \) is the \((m+1) \times (m+1)\) matrix with one in the \((i, j)\) entry and all other entries zero. Here we set \( \frac{r_j}{m} = 0 \) if \( r < m \). Notice that \( E_1^{m+1} \) does not occur in \( I^{(n)}_{1,k+1} \).

\( IV^{(n)} \) has a nonzero entry in the \((1, k+1)\) slot iff \( A^{r_1} C \cdots A^{r_s} C \) has a nonzero entry in the \((2, k)\) slot. This entry is \((-1)^k C(-A + 1)^{r_1} C \cdots (-A + k-1)^{r_{k-1}} C \), which has a nonzero entry only in the \((1, m+1)\) slot. As above, we get

\[
IV^{(n)}_{1,k+1} = \sum_{P(k-1, n-k)} (-1)^{km+k-m} \left( \frac{r_1}{m} \right) \left( \frac{r_2}{m} \right) 2^{r_2-m} 3^{r_3-m} \cdots \left( \frac{r_{k-1}}{m} \right) (k - 1)^{r_{k-1}-m} E_1^{m+1}.
\]
Letting $r_1 + 1 = j$ run over $\{1, \ldots, m + 1\}$ and then relabeling $(r_2, \ldots, r_k)$ as $(r_1, \ldots, r_{k-1})$ in (4.3), we can combine (4.3), (4.4) to get

\[
I_{(1,k+1)}^{(n)} + IV_{(1,k+1)}^{(n)} = \sum_{j=1}^{m+1} \sum_{P(k-1,n-k-j+1)} (-1)^{(k-1)m+k+j-1} \binom{r_1}{m} 1^{r_1-m} \cdot \binom{r_2}{m} 2^{r_2-m} \cdot \ldots \cdot \binom{r_{k-1}}{m} (k-1)^{r_{k-1}-m} E_{1}^{m+2-j}.
\]

(4.5)

The other terms are handled similarly. We have

\[
\Pi_{(1,k+1)}^{(n)} = \sum_{P(k+1,n-k)} (-1)^{k} (-A)^{r_1} C(-A + 1)^{r_2} C \cdot \ldots \cdot C(-A + k)^{r_k+1}.
\]

The $(i, \ell)$ entry of $X^{1}CX^{(2)} C \cdot \ldots \cdot X^{(k)} CX^{(k+1)}$ is $x_{1n}^{(1)} x_{1n}^{(2)} (3) \ldots x_{1n}^{(k)} x_{1n}^{(k+1)}$. Since $((-A)^{r_1})_{m+1} = (-1)^{r_1} \delta_{m+1-r_1}$, we get

\[
\Pi_{(1,k+1)}^{(n)} = \sum_{P(k+1,n-k)} \sum_{\ell=1}^{m+1} (-1)^{(k-1)m+\ell+k+r_{1}-1} \binom{r_2}{m} 1^{r_{2}-m} \cdot \binom{r_3}{m} 2^{r_{3}-m} \cdot \ldots \cdot \binom{r_{k}}{m} (k-1)^{r_{k}-m} \binom{r_{k+1}}{\ell-1} k^{r_{k+1}-\ell+1} E_{1}^{m-r_{1}+1}.
\]

(4.6)

Since $r_1 > 0$, (4.6) has no entries in the $m + 1$st row. As with IV$^{(n)}$, this missing row appears in III$^{(n)}$. Setting $r_1 + 1 = j$ and shifting indices on the $r_i$ as above, we get

\[
\Pi_{(1,k+1)}^{(n)} + III_{(1,k+1)}^{(n)} = \sum_{j=1}^{m+1} \sum_{P(k,n-k-j+1)} \sum_{\ell=1}^{m+1} (-1)^{(k-1)m+\ell+k+j-2} \binom{r_1}{m} 1^{r_1-m} \cdot \binom{r_2}{m} 2^{r_2-m} \cdot \ldots \binom{r_{k-1}}{m} (k-1)^{r_{k-1}-m} \binom{r_k}{\ell-1} k^{r_k-\ell+1} E_{1}^{m-j+2}.
\]

(4.7)

Finally, finding the flat sections for $\nabla_h$ as in (3.2) requires replacing $A, C$ with $(-\sqrt{-\theta h})^{-1} A, (-\sqrt{-\theta h})^{-1} C$. This multiplies the entry $(-1)^{u} \binom{u}{v} c^{-u}$ in $(-A + c)^{r}$ by $(-\sqrt{-\theta h})^{-u}$. Since there are $k - 1$ terms with binomial coefficient $\binom{r_{i}}{m}$, one term of the form $(-A)^{j-1}$, and $k$ copies of $C$ in each term on the right hand side of (4.5), this term is multiplied by $(-\sqrt{-\theta h})^{-(k-1)m+k+j-1}$. Similarly, each entry in (4.7) and is multiplied by $(-\sqrt{-\theta h})^{-(mk+k+j-1)}$.

Set $\alpha = (-\sqrt{-\theta h})$. Combining (4.4), (4.2), (4.3), (4.7) (and remembering the contribution from $tc$), we obtain:
Theorem 4.1. A basis of the flat sections for the Dubrovin connection $\nabla_h = \nabla_{\sqrt{-1}}$ on $T\mathcal{H}|_{\mathcal{H}^2(p_m)}$, is given by the columns of

$$B(t, h) = A(t, h) + \sum_{n \geq 2, k \geq 1} \frac{n}{n!} \sum_{j=1}^{m+1} \left[ \alpha^{-(k-1)m+k+j-1} \right]$$

$$\cdot \sum_{P(k-1, n-k-j+1)} (-1)^{k-1}m+k+j-1 \left( \prod_{s=1}^{k-1} \left( \frac{r_s}{m} \right) s^{r_s-m} \right) E_1^{m+2-j}$$

$$+ \alpha^{-(mk+k+j-1)}$$

$$\cdot \sum_{P(k,n-k-j+1)} \sum_{\ell=1}^{m+1} \left[ (-1)^{k-1}m+\ell+k+j-2 \left( \prod_{s=1}^{k-1} \left( \frac{r_s}{m} \right) s^{r_s-m} \right) \right]$$

$$\cdot \left( \frac{r_k}{\ell-1} \right) k^{r_k-\ell+1} E_\ell^{m-j+2},$$

with

$$A(t, h) =$$

$$\begin{pmatrix}
1 & \alpha^{-1} t & \alpha^{-2} t^2/2 & \cdots & \alpha^{-m} t^m / m!
0 & 1 & \alpha^{-1} t & \cdots & \vdots
\vdots & \ddots & \ddots & \ddots & \alpha^{-2} t^2 / 2
0 & \ddots & \ddots & \ddots & \alpha^{-1} t
\alpha^{-1} t & 0 & \ldots & 0 & 1
\end{pmatrix}.$$  

Remark 4.2. In non-$H^2$ directions, there are no quantum corrections to the cup product; i.e. the quantum product is $t$-independent in these directions. Specifically, for $T_\ell \neq T_1$, we have $C = 0$ and the corresponding $A$ matrix is

$$A'_j = \left( \int_{g^n} T_\ell \cup T_j \cup T_s \right) h^{si} = \left( \delta^n_{j+s} (\delta^k_s \delta^{n-k}_i) \right)$$

$$= \delta^j_{\ell+j}.$$ 

Using Proposition 2.2, we see that the flat sections for the Dubrovin connection over all of $\mathcal{H}$ are given by

$$\exp \left[ -\sqrt{-1} \sum_{\ell=0, \ell \neq 1} t^\ell \left( \frac{\sqrt{-1}}{h} \right) \Gamma_\ell \right] B(t^1),$$

with $(\Gamma_\ell)^j_i = \delta^j_{\ell+j}$ and $B(t)$ the matrix in (4.8). The lack of significant information in non-$H^2$ directions is a special feature of $\mathcal{H}^m$.

It is interesting to compare (4.8) to the basis of flat sections obtained in [7]. As in [6], for a vector field $f^i T_i$ in $i^* T\mathcal{H}$, the equation (2.3) is equivalent to the system

$$-h^{-1} \partial_t (f^i) = f^{i-1}, \quad i > 0,$$

$$-h^{-1} \partial_t (f^0) = c^i f^m,$$
where \( t = t^1 \) and \( \partial_t \) denotes differentiation by the vector field \( T_1 \). (Our choice of \( \hbar \) equals \(-\hbar^{-1}\) in \([7]\).) Thus \( f^m \) determines the other \( f^i \), and \( f^m \) must satisfy
\[
(-\hbar^{-1}\partial_t)^{m+1} f^m - e^t f^m = 0.
\]
(4.10)

Letting \( H \) be a formal variable (which we think of as the hyperplane class in \( H^2(P_n) \)), it is easy to check that
\[
S = \sum_{d \geq 0} e^{(-\hbar + d)t} \prod_{r=1}^d (H - \hbar^{-1} r)^{m+1} \mod H^{m+1}
\]
formally solves (4.10). (The denominator is one for \( d = 0 \).) For \( S = \sum_{b=0}^{m} \sum_{k=0}^{m} \sum_{j=0}^{m} [B_{kj} e^{t_k t_j}] \), each \( S_b \) must also solve (4.10). For a fixed \( S_b \), a solution of (4.9) is then \( \dot{S}_b(t) = (-\hbar^{-1}\partial_t)^{m-s} (\partial^{m-b}/\partial H^{m-b})|_{H=0} S \).

The uniqueness of flat sections with initial condition gives
\[
M(0,h)B(t,h) = M(t,h).
\]
(4.11)
This produces complicated identities in \( t, h \).

Another approach for \( P_1 \) is in \([9]\).  

5. Remarks on the general case

In the general case of the small quantum product, the analogue of the nontrivial ODE (3.4) is of the form
\[
\dot{B}(t) = B(t)(A + \sum_{\beta} \exp[\sum_{i=1}^p t^i a_{i\beta}] C_{\beta}),
\]
(5.1)
where \( a_{i\beta} = \int_\beta T_i \). We solve this by Substitution III. Namely, we first set
\[
(t^1, \ldots, t^p) = (t, 0, \ldots, 0) \text{ as in } [3,4] \text{ and assume } B(t) = \sum B_{kj} e^{t_k t_j}. \]
(5.2)
\[
\sum k B_{kj} e^{t_k t_j} + \sum (j + 1) B_{k,j+1} e^{t_k t_j} = \sum B_{kj} A e^{t_k t_j} + \sum_{i=1}^n B_{k-a_i C_i} e^{t_k t_j},
\]
for \( a_i = a_{1\beta} \), \( c_i = c_{\beta} \), where \( \{\beta_i\} \) is the set of \( \beta \)'s with \( I_\beta(T_j, T_1, T_\ell) \neq 0 \) for some \( j, \ell \). (5.3) becomes
\[
B_{k,j+1} = \frac{1}{j + 1} [B_{kj} (A - k) + \sum_{i=1}^n B_{k-a_i C_i}]
\]
If $M$ is weakly monotone, $a_i > 0$, and the expression for $j!B_{kj}$ in (the transposed version of) (3.9) becomes

\[
\begin{pmatrix}
A - k \\
\vdots \\
A - k + 1 \\
C_1 \\
\vdots \\
C_1 \\
C_2 \\
\vdots \\
C_2 \\
C_n \\
\vdots \\
C_n \\
\end{pmatrix}
\begin{pmatrix}
A - k + 2 \\
\vdots \\
A - 1 \\
\end{pmatrix}
\]

where $(C_1) = C_1$ if $a_1 = 1$ and 0 otherwise. In all columns, the number of zeros between the $A - k + \ldots$ and $C_1$ is $a_1$, and the number of zeros between $C_i$ and $C_{i+1}$ is $a_i$. Setting

\[
D = -\sqrt{-1} \frac{d}{d\theta} + A + \sum_{i=1}^{n} e^{\sqrt{-1}a_i \theta} C_i,
\]

we get

\[
B(t) = \sum_{k \geq 0} (\exp(tD)\Id, e^{\sqrt{-1}k\theta}\Id).
\]

Here $A_j^t = (\int_M T_1 \cup T_2 \cup T_3) h^{t,i} = (T_1 \cup T_2, T_3) h^{t,i} = (T_1 \cup T_2)^i$ where $T_1 \cup T_j = (T_1 \cup T_j)^i T_i$. For the next step in the Frobenius Integrability Theorem, we solve

\[
\dot{B}_{(2)}(t) = B_{(2)}(t) \left[ A_{(2)} + \sum_{\beta} e^{\sqrt{-1}I_\beta(T_3, T_2, T_1)h^{t,r}} \right]
\]

\[
= B_{(2)}(t) [A_{(2)} + \sum_{i=1}^{n_2} e^{a_i^{(2)} t} C_i^{(2)}],
\]

\[
B_{(2)}(0) = B(t_0),
\]

with $(C_i^{(2)})_j^{(2)} = \exp[t_0^1 \int \beta \cdot T_1] I_\beta(T_3, T_2, T_1) h^{t,r}$, and \( \{\beta_1, \ldots, \beta_{n_2}\} \) is the set of $\beta \in H_2$ with $I_\beta(T_3, T_2, T_1) \neq 0$ for some $j$, $t$. Set $D^{(2)} = -\sqrt{-1} \frac{d}{d\theta} + A_{(2)} + \sum_{i=1}^{n_2} e^{\sqrt{-1}I_\beta(T_3, T_2, T_1)^t} C_i^{(2)}$ with $(A^{(2)})_j^{(2)} = (T_2 \cup T_j)^t$. Calling $B = B_{(1)}$ and proceeding, we get that a basis of the flat sections are the columns of

\[
g = \prod_{i=1}^{k} \sum_{j \geq 0} (\exp(t^i D_{(i,j)}\Id, e^{\sqrt{-1}j\theta}\Id)
\]
with

\[ D_{(i)} = -\sqrt{-1} \frac{d}{d\theta} + A_{(i)} + \sum_{k=1}^{n_i} e^{\sqrt{-1} \tau_k(i) \theta} C_k^{(i)}, \]

\[ (A^{(i)})_j^r = (T_i \cup T_j)^r, \quad \text{for} \quad T_i \cup T_j = (T_i \cup T_j)^r T_r, \]

\[ (C_k^{(i)})_j^r = \exp \left[ \sum_{\alpha=1}^{t} t^\alpha \int_{\beta_k} T_\alpha \right] I_{\beta_k}(T_j, T_i, T_\ell) h^\ell r, \]

for \( \{ \beta \} \) the set of \( \beta \in H_2 \) with \( I_{\beta_k}(T_j, T_i, T_\ell) \neq 0 \) for some \( i, \ell \).

We can recover all GW invariants from the solution (5.3.). For example, to recover the \( C_1^{(1)} \), set \( t^1 = t, t^2 = \ldots = t^k = 0 \).

\[ Z(t) = \sum_{k \geq 0} \langle \exp(tD_{(1)}) \text{Id}, e^{\sqrt{-1} \theta \text{Id}} \rangle = \sum B_{k,j} e^{kt^j}, \]

we get

\[ C_1^{(1)} = (t^{-1} e^{-t}(Z(t) - \text{Id}))|_{t=0} = (t^{-1} e^{-t}(g(t, 0, \ldots, 0) - \text{Id}))|_{t=0}. \]

We then multiply \( C_1^{(1)} \) by \( h_{ij} \cdot \exp[-t \int_{T_i} T_1] \) to recover \( I_{\beta_1}(T_j, T_i, T_\ell) \). We similarly recover \( C_k^{(1)} \), and then the \( C_k^{(i)} \).

We conclude with some remarks on the big quantum product. As in [2], the big quantum product is an associative, potential product and hence produces a flat connection. To find flat sections, we must now solve the formal ODE

\[ \frac{\partial g^j_i}{\partial t} = \sqrt{-1} g^j_i \left[ T_j \cdot T_1 \cdot T_\ell + \sum_{n} \frac{t^n}{n!} \sum_{\beta} I_{\beta}(T_j, T_1, T_\ell, T_0, \ldots, T_0) \right] h^\ell r \]

at the first step. The sum over \( n, \beta \) is not finite in general. Since there are no exponential terms, we rewrite this as

\[ \dot{B}(t) = B(t)[A + \sum_n t^n C_n] \]

and plug in \( B(t) = \sum_{k \geq 0} B_k t^k \). The recurrence relation is

\[ B_k = \frac{1}{k} \left[ B_{k-1} A + \sum_{n,k} B_{k-n-1} C_n \right] \]
with $B_0 = \text{Id}$, $B_{-j} = 0$ for $j > 0$. This encodes as

\[
\begin{pmatrix}
B_k \\
B_{k-1} \\
\vdots \\
B_2 \\
B_1 \\
\text{Id}
\end{pmatrix} =
\begin{pmatrix}
A & C_1 & C_2 & \cdots & C_{k-1} & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
B_{k-1} \\
B_{k-2} \\
\vdots \\
B_1 \\
\text{Id}
\end{pmatrix}
\]

= \ldots

= \frac{1}{k!} \left( \alpha^{(k)}_k \right)_1.

As before, setting

\[
\alpha =
\begin{pmatrix}
A & C_1 & \cdots & C_{k-1} & C_k & C_{k+1} & \cdots \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]

it is easily checked that $\left( \alpha^{(k)}_{(k)} \right)_1 = (\alpha^k)_1$, and so

\[
B(t) = \sum B_k t^k = \sum \frac{1}{k!} (\alpha^k)_1 t^k = \left( e^t \alpha \right)_1.
\]

We now proceed as before to generate solutions to the big quantum product by diagonalizing $\alpha$ as much as possible. These computations, as well as computations for quantum cohomology coupled to gravity in genus zero will be discussed in future work.

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