Monotone Complexity of Spanning Tree Polynomial Re-visited

Arkadev Chattopadhyay (TIFR, Mumbai), Rajit Datta (Goldman-Sachs), Utsab Ghosal (CMI), Partha Mukhopadhyay (CMI)

March 31, 2023
Summary

1. Basic Model of Computation

2. Strongly Exponential Lower Bound Against Monotone Circuits

3. $\epsilon$-Sensitive Monotone Lower Bound

4. Summary and Open Problems
Basic Model of Computation
Arithmetic Circuits

- Arithmetic circuits are a model for computing polynomials.
- Size of the circuit is the number of nodes.
- Monotone Circuits: Only non-negative scalars are allowed on edges. They naturally compute monotone polynomials.

\[ f(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + 5x_3 + 5x_4)(x_2 + x_3) \]
Arithmetic Circuits

- Arithmetic circuits are a model for computing polynomials.
- Size of the circuit is the number of nodes.
- Monotone Circuits: Only non-negative scalars are allowed on edges. They naturally compute monotone polynomials.

\[ f(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + 5x_3 + 5x_4)(x_2 + x_3) \]
Arithmetic Circuits

Arithmetic circuits are a model for computing polynomials.

- **Size of the circuit is the number of nodes.**

**Monotone Circuits:** Only non-negative scalars are allowed on edges. They naturally compute monotone polynomials.

\[
f(x_1, x_2, x_3, x_4) = (2x_1 + 3x_2 + 5x_3 + 5x_4)(x_2 + x_3)
\]
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \in [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

Monotone circuits are universal for monotone polynomials.

\[ f = \sum \alpha_m m, \quad \alpha_m \geq 0 \]
Monotone Computation

Important monotone polynomials:

- \[ S_{n,k} = \sum_{S \subseteq [n]} \prod_{i \in S} x_i \]

- \[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

Monotone circuits are universal for monotone polynomials.

\[ f = \sum \alpha_m m , \alpha_m \geq 0 \]
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{\sigma(i)} \]

Monotone circuits are universal for monotone polynomials.

\[ f = \sum \alpha_m m, \quad \alpha_m \geq 0 \]
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

Monotone circuits are universal for monotone polynomials.

\[ f = \sum \alpha_m m, \quad \alpha_m \geq 0 \]
Important monotone polynomials:

- \( S_{n,k} = \sum_{S \in [n], |S|=k} \prod_{i \in S} x_i \)

- \( \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \)

Monotone circuits are universal for monotone polynomials.

\[
f = \sum \alpha_m \cdot m , \quad \alpha_m \geq 0
\]
Monotone Computation

Important monotone polynomials:

- $S_{n,k} = \sum_{S \in [n], |S| = k} \prod_{i \in S} x_i$

- $\text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}$

Monotone circuits are universal for monotone polynomials.

$$f = \sum \alpha_m m , \quad \alpha_m \geq 0$$
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \in [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

\[ S_{n,k} \text{ has an efficient monotone circuit.} \]
Important monotone polynomials:

\[ S_{n,k} = \sum_{S \in [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

\( S_{n,k} \) has an efficient monotone circuit.
Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \]

\[ \text{Perm}_{n \times n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i, \sigma(i)} \]

\( S_{n,k} \) has an efficient monotone circuit.
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n]} \prod_{i \in S} x_i \quad |S| = k \]

- Number of variables in \( \text{Perm}_{n \times n} \) is \( n^2 \).
- Ckt\(^+\) – size(\( \text{Perm}_{n \times n} \)) \( \geq 2^{\Omega(n)} \).
- The known u.b. for \( \text{Perm}_{n \times n} \) is \( 2^{O(n \log n)} \).
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n]} \prod_{i \in S} x_i \]

Number of variables in \( \text{Perm}_{n \times n} \) is \( n^2 \).

\[ \text{Ckt}^+ - \text{size}(\text{Perm}_{n \times n}) \geq 2^{\Omega(n)}. \]

The known u.b. for \( \text{Perm}_{n \times n} \) is \( 2^{O(n \log n)} \).
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \in [n]} \prod_{i \in S} x_i \]

\( \forall S \subseteq [n], |S| = k \)

Number of variables in \( \text{Perm}_{n \times n} \) is \( n^2 \).

\( \text{Ckt}^+ - \text{size}(\text{Perm}_{n \times n}) \geq 2^{\Omega(n)} \).

The known u.b. for \( \text{Perm}_{n \times n} \) is \( 2^{O(n \log n)} \).
Monotone Computation

Important monotone polynomials:

\[ S_{n,k} = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \]

- Number of variables in \( \text{Perm}_{n \times n} \) is \( n^2 \).

- \( \text{Ckt}^+ - \text{size}(\text{Perm}_{n \times n}) \geq 2^{\Omega(n)} \).

- The known u.b. for \( \text{Perm}_{n \times n} \) is \( 2^{O(n \log n)} \).
Monotone Computations

Is there a monotone polynomial on $n$ variables that has monotone circuit lower bounds of $2^{\Omega(n)}$, i.e. strongly exponential?

Remark

Perm is not a candidate.
Monotone Computations

Is there a monotone polynomial on \( n \) variables that has monotone circuit lower bounds of \( 2^{\Omega(n)} \), i.e. strongly exponential?

Remark

Perm is not a candidate.
Strongly Exponential Lower Bound Against Monotone Circuits
Known Results

Strongly exp. lower bound

- Gashkov-Sergeev (80’s).
- Raz-Yehudayoff (2009). All polynomials are in VNP.
- Srinivasan (2019) Polynomials in VP ?
- Cavalar-Kumar-Rossman (2020).
- Hrubeš-Yehudayoff (2021)
Known Results

Strongly exp. lower bound

- Gashkov-Sergeev (80’s).
- Raz-Yehudayoff (2009).
- Srinivasan (2019)
- Cavalar-Kumar-Rossman (2020).
- Hrubeš-Yehudayoff (2021)

All polynomials are in $\text{VNP}$. Polynomials in $\text{VP}$?
Strongly Exponential Lower Bound Against Monotone Circuits

Known Results

Strongly exp. lower bound

- Gashkov-Sergeev (80’s).
- Raz-Yehudayoff (2009).
- Srinivasan (2019)
- Cavalar-Kumar-Rossman (2020).
- Hrubeš-Yehudayoff (2021)

All polynomials are in \( \text{VNP} \).

Polynomials in \( \text{VP} \) ?
Known Results

Strongly exp. lower bound

- Gashkov-Sergeev (80’s).
- Raz-Yehudayoff (2009).
- Srinivasan (2019)
- Cavalar-Kumar-Rossman (2020).
- Hrubeš-Yehudayoff (2021)

All polynomials are in VNP.
Polynomials in VP?

Arkadev C., Rajit D., Utsab G., Partha M. () Monotone Complexity of ST Polynomial March 31, 2023
Known Results

- Valiant (1979).
- Jerrum-Snir (1982).

\[ \text{Ckt}^{-\text{+size}}(f_n) \geq 2^{\Omega(\sqrt{n})} \quad \text{for } f_n \in \text{VP}. \]
Known Results

- Valiant (1979).
  \[ \text{Ckt}^{+}\text{size}(f_n) \geq 2^{\Omega(\sqrt{n})}. \]
  \[ f_n \in \text{VP}. \]
- Jerrum-Snir (1982).
Known Results

- Valiant (1979).

\[ \text{Ckt}^+\text{size}(f_n) \geq 2^{\Omega(\sqrt{n})}. \quad f_n \in \text{VP}. \]

- Jerrum-Snir (1982).
Known Results

- Valiant (1979).
  \[ \text{Ckt}^{-+}\text{size}(f_n) \geq 2^{\Omega(\sqrt{n})}. \]

- Jerrum-Snir (1982).
  \[ f_n \in \text{VP}. \]

Any strongly exp. monotone lower bound for VP polynomial?
Known Results

- Valiant (1979).
  \[ \text{Ckt-}^+\text{size}(f_n) \geq 2^{\Omega(\sqrt{n})}. \]

- Jerrum-Snir (1982).

\[ f_n \in \text{VP}. \]

Any strongly exp. monotone lower bound for VP polynomial?

Yes! (Our result)
Our Result

Theorem:

The Spanning tree polynomial defined for a family of constant degree expander graphs on $n$ vertices requires monotone circuits of size $2^{\Omega(n)}$.

Remark

- Number of variables in our polynomial is $\Theta(n)$.
- First strongly exp. monotone l.b for VP.
Our Result

Theorem:

The Spanning tree polynomial defined for a family of constant degree expander graphs on $n$ vertices requires monotone circuits of size $2^{\Omega(n)}$.

Remark

- Number of variables in our polynomial is $\Theta(n)$.
- First strongly exp. monotone l.b for VP.
Our Result

Theorem:

The Spanning tree polynomial defined for a family of constant degree expander graphs on $n$ vertices requires monotone circuits of size $2^{\Omega(n)}$.

Remark

- *Number of variables in our polynomial is $\Theta(n)$.*
- *First strongly exp. monotone l.b for VP.*
What is Spanning Tree Polynomial?

- \( G = (V, E), |V| = n \) is bi-directed.
- \( T \) is the set of maps from \([2, \ldots, n]\) to \([n]\) that gives spanning tree rooted at 1.
- \( ST_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i,\theta(i)} \)
- U.b : Via determinantal computation using Matrix Tree Theorem

- \( ST_3 = x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1} \).
- \( x_{1,2} \cdot x_{3,2} \) is not a monomial in \( ST_3 \).
What is Spanning Tree Polynomial?

- \( G = (V, E), |V| = n \) is bi-directed.
- \( T \) is the set of maps from \([2, \ldots, n]\) to \([n]\) that gives spanning tree rooted at 1.
- \( \text{ST}_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i, \theta(i)} \)
- U.b : Via determinantal computation using Matrix Tree Theorem

\[
\text{ST}_3 = \]
\[
x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1}.
\]
\[
x_{1,2} \cdot x_{3,2} \text{ is not a monomial in } \text{ST}_3.
\]
What is Spanning Tree Polynomial?

- \( G = (V, E), |V| = n \) is bi-directed.
- \( T \) is the set of maps from \([2, \ldots, n]\) to \([n]\) that gives spanning tree rooted at 1.
- \( ST_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i,\theta(i)} \)
- U.b : Via determinantal computation using Matrix Tree Theorem

\[
\begin{align*}
ST_3 &= x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1} \\
x_{1,2} \cdot x_{3,2} &\text{ is not a monomial in } ST_3.
\end{align*}
\]
**What is Spanning Tree Polynomial?**

- $G = (V, E), |V| = n$ is bi-directed.
- $T$ is the set of maps from $[2, \ldots, n]$ to $[n]$ that gives spanning tree rooted at 1.
- $ST_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i,\theta(i)}$
- U.b : Via determinantal computation using Matrix Tree Theorem

- $\text{ST}_3 = x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1}$.
- $x_{1,2} \cdot x_{3,2}$ is not a monomial in $\text{ST}_3$. 

![Diagram of a triangle with nodes 1, 2, 3 and edges labeled with terms like $x_{1,2}$, $x_{1,3}$, $x_{2,3}$, $x_{2,1}$, $x_{3,1}$, $x_{3,2}$.]
What is Spanning Tree Polynomial?

- $G = (V, E), |V| = n$ is bi-directed.
- $T$ is the set of maps from $[2, \ldots, n]$ to $[n]$ that gives spanning tree rooted at 1.
- $ST_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i, \theta(i)}$
- U.b: Via determinantal computation using Matrix Tree Theorem

$ST_3 = x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1}$

$x_{1,2} \cdot x_{3,2}$ is not a monomial in $ST_3$. 
What is Spanning Tree Polynomial?

\[ G = (V, E), |V| = n \] is bi-directed.

\[ T \] is the set of maps from \([2, \ldots, n]\) to \([n]\) that gives spanning tree rooted at 1.

\[ \text{ST}_n = \sum_{\theta \in T} \prod_{i=2}^{n} x_{i, \theta(i)} \]

\[ \text{U.b} : \text{Via determinantal computation using Matrix Tree Theorem} \]

\[ \text{ST}_3 = \]
\[ x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1} \cdot x_{1,2} \cdot x_{3,2} \text{ is not a monomial in } \text{ST}_3. \]
What is Spanning Tree Polynomial?

- $G = (V, E), |V| = n$ is bi-directed.
- $T$ is the set of maps from $[2, \ldots, n]$ to $[n]$ that gives spanning tree rooted at 1.
- $\text{ST}_n = \sum_{\theta \in T} \prod_{i=2}^n x_{i, \theta(i)}$
- U.b : Via determinantal computation using Matrix Tree Theorem

$\text{ST}_3 = x_{2,1} \cdot x_{3,1} + x_{2,3} \cdot x_{3,1} + x_{3,2} \cdot x_{2,1}.$

$x_{1,2} \cdot x_{3,2}$ is not a monomial in $\text{ST}_3.$
Set-multilinear Polynomial

\[ \prod_{i=2}^{n} x_{i, \pi(i)} \]

\[ \begin{array}{cccc}
  x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\
  x_{3,1} & x_{3,2} & \cdots & x_{3,n} \\
  x_{4,1} & x_{4,2} & \cdots & x_{4,n} \\
  x_{n,1} & x_{n,2} & \cdots & x_{n,n} \\
\end{array} \]

\( n - 1 \times n \)

\( \pi : [2, n] \rightarrow [n] \)
Set-multilinear Polynomial

\[ \prod_{i=2}^{n} x_{i, \pi(i)} \]

\[
\begin{bmatrix}
 x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\
 x_{3,1} & x_{3,2} & \cdots & x_{3,n} \\
 x_{4,1} & x_{4,2} & \cdots & x_{4,n} \\
 x_{n,1} & x_{n,2} & \cdots & x_{n,n}
\end{bmatrix}
\]

\[ n - 1 \times n \]
Set-multilinear Polynomial

\[ \prod_{i=2}^{n} x_{i, \pi(i)} \]

\[ \pi : [2, n] \rightarrow [n] \]

\[ n - 1 \times n \]
Set-multilinear Monotone Structure Theorem

For set-multilinear monotone polynomial \( f \)

if \( C^+(f) = S \) then

\[
 f = \sum_{t=1}^{S+1} \alpha_t \cdot \beta_t
\]

with both \( \alpha_t \) and \( \beta_t \) are monotone

\( \forall t \) and

\[
|I(\alpha_t)|, |I(\beta_t)| \in \left[ \frac{n}{3}, \frac{2n}{3} \right]
\]

Nearly Balanced Partition
Set-multilinear Monotone Structure Theorem

For set-multilinear monotone polynomial $f$

if $C^+(f) = S$ then

$$f = \sum_{t=1}^{S+1} \alpha_t \cdot \beta_t$$

with both $\alpha_t$ and $\beta_t$ are monotone

$\forall$ $t$ and

$$|I(\alpha_t)|, |I(\beta_t)| \in \left[ \frac{n}{3}, \frac{2n}{3} \right] \leftarrow \text{Nearly Balanced Partition}$$
Proof Idea of Result

- The measure is counting spanning tree monomials.
- The non spanning tree monomials are forbidden.

\[
\text{ST}_n = \sum_{t=1}^{S+1} a_t \cdot b_t.
\]

- Using Expander Mixing lemma on a \(d\) regular expander graph, \(\exists C_1\) s.t.
  \[
  |\text{mon}(a_t \cdot b_t)| \leq (C_1 d)^{n-1} \text{ for any } t.
  \]

- Using Matrix Tree theorem \(\exists C_2\) s.t.
  \[
  |\text{mon(ST}_n)| \geq (C_2 d)^{n-1}.
  \]

\[ C_2 > 2C_1 \implies S \geq 2^{\Omega(n)}. \]
Proof Idea of Result

- The measure is counting spanning tree monomials.
- The non spanning tree monomials are forbidden.

- \[ ST_n = \sum_{t=1}^{S+1} a_t \cdot b_t. \]

- Using **Expander Mixing lemma** on a \( d \) regular expander graph, \( \exists C_1 \) s.t.
  \[ |\text{mon}(a_t \cdot b_t)| \leq (C_1d)^{n-1} \text{ for any } t. \]

- Using **Matrix Tree theorem** \( \exists C_2 \) s.t.
  \[ |\text{mon}(ST_n)| \geq (C_2d)^{n-1}. \]

- \( C_2 > 2C_1 \implies S \geq 2^{\Omega(n)}. \)
Proof Idea of Result

- The measure is counting spanning tree monomials.
- The non spanning tree monomials are forbidden.

\[ \text{ST}_n = \sum_{t=1}^{S+1} a_t \cdot b_t. \]

- Using **Expander Mixing lemma** on a \( d \) regular expander graph, \( \exists C_1 \) s.t.
  \[ |\text{mon}(a_t \cdot b_t)| \leq (C_1d)^{n-1} \text{ for any } t. \]

- Using **Matrix Tree theorem** \( \exists C_2 \) s.t.
  \[ |\text{mon}(\text{ST}_n)| \geq (C_2d)^{n-1}. \]

\[ C_2 > 2C_1 \implies S \geq 2^{\Omega(n)}. \]
Proof Idea of Result

- The measure is counting spanning tree monomials.
- The non spanning tree monomials are forbidden.

\[ \text{ST}_n = \sum_{t=1}^{S+1} a_t \cdot b_t. \]

- Using **Expander Mixing lemma** on a \( d \) regular expander graph, \( \exists C_1 \) s.t.
  \[ |\text{mon}(a_t \cdot b_t)| \leq (C_1 d)^{n-1} \text{ for any } t. \]

- Using **Matrix Tree theorem** \( \exists C_2 \) s.t.
  \[ |\text{mon}(\text{ST}_n)| \geq (C_2 d)^{n-1}. \]

\[ C_2 > 2C_1 \implies S \geq 2^{\Omega(n)}. \]
$\epsilon$-Sensitive Monotone Lower Bound
Basic Question

Problem

*Can monotone l.b yield general circuit lower bound?*

Remark

*Boolean world: Slice function (Valiant 1986)*
Basic Question

Problem

Can monotone l.b yield general circuit lower bound?

Remark

Boolean world: Slice function (Valiant 1986)
Problem

Is there a hard polynomial $f$ s.t. for every $\epsilon > 0$, the polynomial $g_\epsilon = E + \epsilon \cdot f$ has large monotone complexity? 

- Hrubeš (2020): if $E = (1 + \sum_{i=1}^{n} x_i)^n$ then strong monotone l.b on $g_\epsilon$ for every sufficiently small $\epsilon > 0 \implies$ general circuit lower bound on $f$.

- $\epsilon \approx 1/2^2$.

- $E = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} \implies$ general set-multilinear circuit l.b against $f$. 
Arithmetic World: Approach of Hrubeš

Problem

Is there a hard polynomial $f$ s.t. for every $\epsilon > 0$, the polynomial $g_\epsilon = E + \epsilon \cdot f$ has large monotone complexity? $E \rightarrow$ Easy for monotone.

- Hrubeš (2020): if $E = (1 + \sum_{i=1}^{n} x_i)^n$ then strong monotone l.b on $g_\epsilon$ for every sufficiently small $\epsilon > 0 \implies$ general circuit lower bound on $f$.

- $\epsilon \approx 1/2^{2^s}$.

- $E = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} \implies$ general set-multilinear circuit l.b against $f$. 
Arithmetic World: Approach of Hrubeš

**Problem**

Is there a **hard** polynomial $f$ s.t. for every $\epsilon > 0$, the polynomial $g_\epsilon = E + \epsilon \cdot f$ has large monotone complexity? $E \rightarrow$ Easy for monotone.

- Hrubeš (2020): if $E = (1 + \sum_{i=1}^{n} x_i)^n$ then strong monotone l.b on $g_\epsilon$ for every sufficiently small $\epsilon > 0 \implies$ general circuit lower bound on $f$.

- $\epsilon \approx 1/2^{2^s}$.

- $E = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} \rightarrow$ general set-multilinear circuit l.b against $f$. 
Arithmetic World: Approach of Hrubeš

Problem

Is there a hard polynomial $f$ s.t. for every $\epsilon > 0$, the polynomial $g_\epsilon = E + \epsilon \cdot f$ has large monotone complexity? $E \rightarrow$ Easy for monotone.

- Hrubeš (2020): if $E = (1 + \sum_{i=1}^{n} x_i)^n$ then strong monotone l.b on $g_\epsilon$ for every sufficiently small $\epsilon > 0 \implies$ general circuit lower bound on $f$.

- $\epsilon \approx 1/2^s$.

- $E = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} \rightarrow$ general set-multilinear circuit l.b against $f$. 
Arithmetic World: Approach of Hrubeš

**Problem**

*Is there a hard polynomial* \( f \) *s.t. for every* \( \epsilon > 0 \), *the polynomial* \( g_\epsilon = E + \epsilon \cdot f \) *has large monotone complexity?* \( E \rightarrow \) *Easy for monotone.*

- Hrubeš (2020): if \( E = (1 + \sum_{i=1}^{n} x_i)^n \) then strong monotone l.b on \( g_\epsilon \) for every sufficiently small \( \epsilon > 0 \) \( \implies \) general circuit lower bound on \( f \).

- \( \epsilon \approx 1/2^{2^s} \).

- \( E = \prod_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} \rightarrow \) general set-multilinear circuit l.b against \( f \).
Some Remark About the Approach

- If $\epsilon = 0$ then $g_\epsilon$ has trivial monotone circuit.

- Monomial support of $\left(1 + \sum_{i=1}^{n} x_i\right)^n + \epsilon \cdot f$ is full. So any support-based l.b technique fails.

- E.g. Our previous technique fails.

- We Use techniques from Communication Complexity.
Some Remark About the Approach

- If $\epsilon = 0$ then $g_{\epsilon}$ has trivial monotone circuit.

- Monomial support of $(1 + \sum_{i=1}^{n} x_i^n + \epsilon \cdot f)$ is full. So any support-based l.b technique fails.

- E.g. Our previous technique fails.

- We Use techniques from Communication Complexity.
Some Remark About the Approach

- If $\epsilon = 0$ then $g_\epsilon$ has trivial monotone circuit.

- Monomial support of $(1 + \sum_{i=1}^{n} x_i)^n + \epsilon \cdot f$ is full. So any support-based l.b technique fails.

- E.g. Our previous technique fails.

- We Use techniques from Communication Complexity.
Some Remark About the Approach

- If $\epsilon = 0$ then $g_\epsilon$ has trivial monotone circuit.

- Monomial support of $(1 + \sum_{i=1}^{n} x_i)^n + \epsilon \cdot f$ is full. So any support-based l.b technique fails.

- E.g. Our previous technique fails.

- We Use techniques from Communication Complexity.
Results on $\epsilon$-Sensitive Monotone Lower Bounds

- **C.D.M (2021):** First $\epsilon$-sensitive monotone l.b against a VNP polynomial family $\{f_n\}$ with $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

- Can we show this for VP polynomial?

**Theorem:**
The Spanning Tree polynomials for complete graph on $n$ vertices require exponential size $\epsilon$-sensitive monotone lower bound in the set-multilinear setting for $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

**Remark**
Our lower bound technique crucially uses the discrepancy measure from Communication Complexity.
Results on $\epsilon$-Sensitive Monotone Lower Bounds

- C.D.M (2021): First $\epsilon$-sensitive monotone l.b against a VNP polynomial family \( \{f_n\} \) with $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

- Can we show this for VP polynomial?

**Theorem:**
The Spanning Tree polynomials for complete graph on $n$ vertices require exponential size $\epsilon$-sensitive monotone lower bound in the set-multilinear setting for $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

**Remark**

*Our lower bound technique crucially uses the discrepancy measure from Communication Complexity.*
Results on $\epsilon$-Sensitive Monotone Lower Bounds

- C.D.M (2021): First $\epsilon$-sensitive monotone l.b against a VNP polynomial family $\{f_n\}$ with $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

- Can we show this for VP polynomial?

**Theorem:**

The **Spanning Tree** polynomials for complete graph on $n$ vertices require exponential size $\epsilon$-sensitive monotone lower bound in the set-multilinear setting for $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

**Remark**

Our lower bound technique crucially uses the discrepancy measure from Communication Complexity.
Results on $\epsilon$-Sensitive Monotone Lower Bounds

- C.D.M (2021): First $\epsilon$-sensitive monotone l.b against a VNP polynomial family $\{f_n\}$ with $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

- Can we show this for VP polynomial?

**Theorem:**
The Spanning Tree polynomials for complete graph on $n$ vertices require exponential size $\epsilon$-sensitive monotone lower bound in the set-multilinear setting for $\epsilon \geq 2^{-\Omega(\sqrt{n})}$.

**Remark**
Our lower bound technique crucially uses the discrepancy measure from Communication Complexity.
Spanning Tree Communication Problem

\[ V_A \cup V_B = V \]

\[ E_A = \{(u \rightarrow v) \mid u \in V_A\} \]

\[ \{ (w \rightarrow v) \mid w \in V_B \} = E_B \]

Goal: \( E_A \cup E_B \) forms spanning tree rooted at 1 or not?
Spanning Tree Communication Problem

\[ V_A \cup V_B = V \]

\[ E_A = \{(u \rightarrow v) \mid u \in V_A\} \]
\[ \{(w \rightarrow v) \mid w \in V_B\} = E_B \]

Goal: \( E_A \cup E_B \) forms spanning tree rooted at 1 or not?
Spanning Tree Communication Problem

\[ V_A \quad V = V_A \cup V_B \quad V_B \]

\[ E_A = \{(u \rightarrow v) \mid u \in V_A\} \quad \{(w \rightarrow v) \mid w \in V_B\} = E_B \]

Goal: \( E_A \cup E_B \) forms spanning tree rooted at 1 or not?
Spanning Tree Communication Problem

\[ V_A \quad V = V_A \cup V_B \quad V_B \]

\[ E_A = \{(u \rightarrow v) \mid u \in V_A \} \quad \{(w \rightarrow v) \mid w \in V_B \} = E_B \]

Goal: \( E_A \cup E_B \) forms spanning tree rooted at 1 or not?
Spanning Tree is Hard Under a Fixed Partition

A gadget reduction from the Inner Product problem to the Spanning Tree problem.

Inner Product: $\text{IP}(x, y) = \sum_{i=1}^{n} x_i \cdot y_i \pmod{2}$ is a well known hard problem.

We show $\text{IP}(x, y) = 1$ iff the gadget graph $G_{x,y}$ has a spanning tree.
A gadget reduction from the Inner Product problem to the Spanning Tree problem.

Inner Product: \[ \text{IP}(x, y) = \sum_{i=1}^{n} x_i \cdot y_i \pmod{2} \] is a well known hard problem.

We show \( \text{IP}(x, y) = 1 \) iff the gadget graph \( G_{x,y} \) has a spanning tree.
Spanning Tree is Hard Under a Fixed Partition

- A gadget reduction from the Inner Product problem to the Spanning Tree problem.

- **Inner Product**: $\text{IP}(x, y) = \sum_{i=1}^{n} x_i \cdot y_i \pmod{2}$ is a well known hard problem.

- We show $\text{IP}(x, y) = 1$ iff the gadget graph $G_{x, y}$ has a spanning tree.
The Measure Discrepancy

\[ R = A \times B \text{ where } A, B \subseteq \{0, 1\}^m \]

**Alice**

**Bob**

\[
\begin{align*}
\text{Disc}(R, \delta) &= \left| \sum_{x \in R \atop F(x) = 0} \delta(x) - \sum_{x \in R \atop F(x) = 1} \delta(x) \right|.
\end{align*}
\]

\[
\text{Disc}(F, \delta) = \max_R \text{Disc}(R, \delta).
\]

\[
\text{Disc}(\mu, \text{IP}(x, y)) \leq 2^{-\Omega(n/2)} \quad \text{[Chor, Goldreich (1988)]} \implies \text{Spanning Tree problem has low discrepancy.}
\]
The Measure Discrepancy

\[ R = A \times B \text{ where } A, B \subseteq \{0, 1\}^m \]

- \( \text{Disc}(R, \delta) = \left| \sum_{x \in R, F(x) = 0} \delta(x) - \sum_{x \in R, F(x) = 1} \delta(x) \right| \).

- \( \text{Disc}(F, \delta) = \max_R \text{Disc}(R, \delta) \).

- \( \text{Disc}(\mu, \text{IP}(x, y)) \leq 2^{-\Omega(n/2)} \) \[\text{Chor, Goldreich (1988)}\] \implies \text{Spanning Tree problem has low discrepancy.}
The Measure Discrepancy

\[ R = A \times B \quad \text{where} \quad A, B \subseteq \{0, 1\}^m \]

\[ \text{Disc}(R, \delta) = \left| \sum_{x \in R, F(x) = 0} \delta(x) - \sum_{x \in R, F(x) = 1} \delta(x) \right| . \]

\[ \text{Disc}(F, \delta) = \max_R \text{Disc}(R, \delta) . \]

\[ \text{Disc}(\mu, \text{IP}(x, y)) \leq 2^{-\Omega(n/2)} \quad [\text{Chor, Goldreich (1988)}] \quad \implies \quad \text{Spanning Tree problem has low discrepancy.} \]
The Measure Discrepancy

\[ R = A \times B \text{ where } A, B \subseteq \{0, 1\}^m \]

- \( \text{Disc}(R, \delta) = \left| \sum_{x \in R, F(x) = 0} \delta(x) - \sum_{x \in R, F(x) = 1} \delta(x) \right| \)

- \( \text{Disc}(F, \delta) = \max_R \text{Disc}(R, \delta) \)

- \( \text{Disc}(\mu, \text{IP}(x, y)) \leq 2^{-\Omega(n/2)} \) [Chor, Goldreich (1988)] \( \implies \) Spanning Tree problem has low discrepancy.
A Subtle Issue

- $\text{ST}_n = \sum_t \alpha_t \cdot \beta_t$.

- Every $\alpha_t \cdot \beta_t$ gives a different rectangle with Alice has $\alpha_t$ and Bob has $\beta_t$.

\[
\text{IP}(X, Y) = \sum_{i=1}^{n} x_i y_i \text{ is not hard under partition } \{(x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}) \cup (x_{n/2+1}, \ldots, x_n, y_{n/2+1}, \ldots, y_n)\}.
\]
A Subtle Issue

- $\text{ST}_n = \sum_t \alpha_t \cdot \beta_t$.

- Every $\alpha_t \cdot \beta_t$ gives a different rectangle with Alice has $\alpha_t$ and Bob has $\beta_t$.

Every Product polynomial may give different partition.

$\text{IP}(X, Y) = \sum_{i=1}^{n} x_i y_i$ is not hard under partition

$\{(x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}) \cup (x_{n/2+1}, \ldots, x_n, y_{n/2+1}, \ldots, y_n)\}$. 
A Subtle Issue

- \( ST_n = \sum_t \alpha_t \cdot \beta_t \).

- Every \( \alpha_t \cdot \beta_t \) gives a different rectangle with Alice has \( \alpha_t \) and Bob has \( \beta_t \).

Every Product polynomial may give different partition.

\[
\text{IP}(X, Y) = \sum_{i=1}^{n} x_i y_i \text{ is not hard under partition}\n\{(x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}) \cup (x_{n/2+1}, \ldots, x_n, y_{n/2+1}, \ldots, y_n)\}.
\]
A Subtle Issue

- $\text{ST}_n = \sum_t \alpha_t \cdot \beta_t$.

- Every $\alpha_t \cdot \beta_t$ gives a different rectangle with Alice has $\alpha_t$ and Bob has $\beta_t$.

Every Product polynomial may give different partition.

$\text{IP}(X, Y) = \sum_{i=1}^{n} x_i y_i$ is not hard under partition
$
\{(x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}) \cup (x_{n/2+1}, \ldots, x_n, y_{n/2+1}, \ldots, y_n)\}.$
Global Measure Via Universal Distribution

- We need a **Universal distribution**, under which for every nearly balanced partition, the discrepancy of Spanning Tree problem remains low.

- We transfer this discrepancy bound to a lower bound using the following novel correspondence theorem.
We need a Universal distribution, under which for every nearly balanced partition, the discrepancy of Spanning Tree problem remains low.

We transfer this discrepancy bound to a lower bound using the following novel correspondence theorem.
Let $\Delta$ be a Universal distribution and $f$ be a $0-1$ set-multilinear polynomial. If the communication problem $C^f_P$ has discrepancy at most $\gamma$ w.r.t $\Delta$ for every nearly balance partition $P$, then the monotone complexity of $F_{n,m} - \epsilon \cdot f$ is at least $\frac{\epsilon}{3\gamma}$ as long as $\epsilon \geq \frac{6\gamma}{1-3\gamma}$.

We construct an Universal distribution $\Delta$ s.t the discrepancy of Spanning Tree problem w.r.t $\Delta$ for every nearly balance partition is at most $2^{-\Omega(n)}$. 
Theorem

Let $\Delta$ be a Universal distribution and $f$ be a $0–1$ set-multilinear polynomial. If the communication problem $C_f^P$ has discrepancy at most $\gamma$ w.r.t $\Delta$ for every nearly balance partition $P$, then the monotone complexity of $F_{n,m} - \epsilon \cdot f$ is at least $\frac{\epsilon}{3\gamma}$ as long as $\epsilon \geq \frac{6\gamma}{1-3\gamma}$.

We construct an Universal distribution $\Delta$ s.t the discrepancy of Spanning Tree problem w.r.t $\Delta$ for every nearly balance partition is at most $2^{-\Omega(n)}$.
Conclusion and Open Problems

- First strongly exponential separation between Monotone-VP and VP.

- First exponential size $\epsilon$-sensitive lower bound against a VP polynomial.

Open Problems

- Find Polynomial with polynomial size non monotone formulas and strongly exponential monotone circuit complexity.

- Give a strongly exponential size $\epsilon$-sensitive lower bound for a polynomial in VP.

- Give sensitive lower bounds against the following polynomials, $F_{n,n} \pm \epsilon \cdot \text{det}_{n,n}$ and $F_{n,n} \pm \epsilon \cdot \text{Perm}_{n,n}$. 
Conclusion and Open Problems

- First strongly exponential separation between Monotone-VP and VP.
- First exponential size $\epsilon$-sensitive lower bound against a VP polynomial.

Open Problems

- Find Polynomial with polynomial size non monotone formulas and strongly exponential monotone circuit complexity.
- Give a strongly exponential size $\epsilon$-sensitive lower bound for a polynomial in VP.
- Give sensitive lower bounds against the following polynomials, $F_{n,n} \pm \epsilon \cdot \det_{n,n}$ and $F_{n,n} \pm \epsilon \cdot \text{Perm}_{n,n}$. 
Conclusion and Open Problems

- First strongly exponential separation between Monotone-VP and VP.
- First exponential size $\epsilon$-sensitive lower bound against a VP polynomial.

Open Problems

- Find Polynomial with polynomial size non monotone formulas and strongly exponential monotone circuit complexity.
- Give a strongly exponential size $\epsilon$-sensitive lower bound for a polynomial in VP.
- Give sensitive lower bounds against the following polynomials, $F_{n,n} \pm \epsilon \cdot \det_{n,n}$ and $F_{n,n} \pm \epsilon \cdot \Perm_{n,n}$.
Conclusion and Open Problems

- First strongly exponential separation between Monotone-VP and VP.

- First exponential size $\epsilon$-sensitive lower bound against a VP polynomial.

Open Problems

- Find Polynomial with polynomial size non monotone formulas and strongly exponential monotone circuit complexity.

- Give a strongly exponential size $\epsilon$-sensitive lower bound for a polynomial in VP.

- Give sensitive lower bounds against the following polynomials, $F_{n,n} \pm \epsilon \cdot \text{det}_{n,n}$ and $F_{n,n} \pm \epsilon \cdot \text{Perm}_{n,n}$.
Conclusion and Open Problems

- First strongly exponential separation between Monotone-VP and VP.

- First exponential size $\epsilon$-sensitive lower bound against a VP polynomial.

Open Problems

- Find Polynomial with polynomial size non monotone formulas and strongly exponential monotone circuit complexity.

- Give a strongly exponential size $\epsilon$-sensitive lower bound for a polynomial in VP.

- Give sensitive lower bounds against the following polynomials, $F_{n,n} \pm \epsilon \cdot \det_{n,n}$ and $F_{n,n} \pm \epsilon \cdot \text{Perm}_{n,n}$. 
There are more exciting open problems in our paper. We invite you to check the following link
https://arxiv.org/abs/2109.06941

Thank You