On the validity of the method of reduction of dimensionality: area of contact, average interfacial separation and contact stiffness

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It has recently been suggested that many contact mechanics problems between solids can be accurately studied by mapping the problem on an effective one dimensional (1D) elastic foundation model. Using this 1D mapping we calculate the contact area and the average interfacial separation between elastic solids with nominally flat but randomly rough surfaces. We show, by comparison to exact numerical results, that the 1D mapping method fails even qualitatively. We also calculate the normal interfacial stiffness $K$ and compare it with the result of an analytical study. We attribute the failure of the elastic foundation model to the neglect of the long-range elastic coupling between the asperity contact regions.

1 Introduction

The calculation of the stress and displacement field resulting from the contact between elastic solids with rough surfaces is a very complex problem, in part due to the many length scales usually involved, and also because of the long-range elastic coupling between the contact regions. For this reason simplifying approaches are very important. However, most analytical theories, such as the Greenwood-Williamson contact mechanics theory1, or theories based on the elastic foundation model2 (see Fig. 1), neglect the elastic coupling between asperity contact regions. It has recently been shown by exact numerical studies that the neglect of the elastic coupling results in qualitatively wrong contact topography3, and even the relation between the contact force and the area of contact is incorrectly described using this approach4. In Ref. 5, it was also shown that the contact stress-stress correlation function (in wavevector space) $\langle \sigma(q)\sigma(-q) \rangle \sim q^{-\alpha}$, where in the overlap model $\alpha = 2 + H$ (where $H$ is the Hurst exponent), while including the long-range elastic coupling $\alpha = 1 + H$. In a series of papers, Popov and coworkers have proposed that a simple 1D-elastic foundation model can be used to accurately describe the contact between elastic solids5–7. In a recent publication they calculated the normal stiffness between elastic solids with randomly rough but nominally flat surfaces, and argued that the results are in good agreement with exact numerical results8. In this note we will show that in fact the model of Popov et al fails even qualitatively to describe the contact mechanics correctly.

2 Area of real contact and the average interfacial separation

Consider two elastic solids with nominally flat surfaces squeezed together by the nominal pressure $p = F/A_0$. The average interfacial separation is denoted by $\bar{u}$. As $p$ increases, the average interfacial separation $\bar{u}$ monotonically decreases, while the area of real contact $A$ increases10–13. In earlier publications11,14–17 it has been shown that in a large pressure range $A \sim p$ and $\bar{u} \sim \ln p$. This can be understood as follows: As the load increases, existing contact patches grow and new, small contacts are formed. This happens in such a way that the distribution of contact sizes and local pressures remains approximately constant over a wide range of loads13,14. It follows that $A \sim p$ and that the elastic deformation energy (per unit nominal contact area), $U_{el}$, stored at the interface must be proportional to the load or the nominal contact pressure$^{14}$:

$$U_{el} = \mu_0 p,$$

where $\mu_0$ is a length parameter of order the root-mean-square roughness $h_{rms}$. Since the elastic energy is equal to the work done by the external load (assuming hard-wall interactions and no adhesion), it follows that

$$p = -\frac{dU_{el}}{du}.$$

Combining this with (1) gives

$$p = p_0 \exp(-\bar{u}/\mu_0),$$

where $p_0$ is an integration constant. The theory of Persson predicts that $\mu_0 = \alpha h_{rms}$ and $p_0 = \beta E^*$, where $E^*$ is the effective elastic modulus and $\alpha$ (of order unity) and $\beta$ are dimensionless. Both $\alpha$ and $\beta$ only depend on the spectral properties of the surface3,14,16–19.

FIG. 1: In the elastic foundation model the elastic solid is replaced by an array of independent springs.
In the same pressure range where (2) is valid, the area of real contact
\[ \frac{A}{A_0} = \frac{\kappa}{\xi} \frac{p}{E^*}, \]
where \( \xi = \left( \langle \nabla h \rangle^2 \right)^{1/2} \) is the surface rms-slope and \( \kappa \approx 2 \).

Eqs. (2) and (3) are only valid at such high pressures that multi-asperity contact occurs. At very low pressures the solids will only make contact in the vicinity of the highest asperity. In this finite-size pressure region the relation between \( \bar{u} \) and \( p \) will exhibit large fluctuations from one surface realization to another.\[20\]. In the study presented below the finite-size region is too small to be observed on the linear pressure scale used in Fig. 4. In Ref. \[21,22\] we have studied numerically and analytically the relation between the interfacial stiffness and the squeezing pressure in both the finite size pressure region and for higher pressures, and in Sec. 5 we compared the results for the stiffness with the 1D-elastic foundation model of Popov et al.

![Graph](image)

**FIG. 2:** The surface roughness power spectrum as a function of the wavevector (log-log scale) for a self-affine fractal surface with a roll-off.

### 3 Numerical results for \( A(p) \) and \( \bar{u}(p) \)

In the reduction of dimensionality approach a 3D-contact problem is mapped on a 1D-elastic foundation problem. Here we are interested in the contact between two nominally flat but randomly rough surfaces. For frictionless contact, this problem can be mapped on anelastic half space with a randomly rough surface in contact with a rigid substrate with a flat surface. In the contact mechanics theory of Popov et al. the roughness of the 1D-substrate has a power spectrum related to that of the original via the equation:

\[ C_{1D}(q) = \pi q C_{2D}(q). \]

The spring constant of the elastic foundation is related to the effective (or combined) elastic modulus via \( k = a E^* \), where \( a \) is the spacing between the springs. Using standard procedures we have generated randomly rough 1D-surfaces with the power spectra given by (4). As in an earlier study\[22\], the original 2D surface is self affine fractal with the Hurst exponent \( H = 0.7 \) (or fractal dimension \( D_f = 3 - H = 2.3 \)) and with the small and large cut-off wavevectors \( q_0 = 1 \) and \( q_1 = 8192 \). We consider two cases, namely when the substrate surface is fractal-like in the whole interval \( q_0 < q < q_1 \), and when there is a roll-off at \( q_r = 8 \), see Fig. 2.

The red and blue solid lines in Fig. 3 have been calculated following the procedure outlined by Popov et al.\[23,24\]. Each independent spring is compressed into compliance by the 1D rough surface profile where the profile overlaps with the initial relaxed spring positions. In each step we calculated the force \( F \) and area of contact \( A \). The applied force \( F \) was calculated as the sum of forces for all springs in contact:

\[ F = k \sum_{i=1}^{n} \Delta x_i, \]

where \( n \) is the number of springs in contact, \( \Delta x_i \) is the spring compression. After this we have calculated the area of contact \( A \) :

\[ A = \pi \frac{\bar{u}^2}{\mu} \sum_{i=1}^{n} (a n_i)^2, \]

where \( n_i \) is the number of connected regions. In this case all springs in connected regions must be in contact, \( n_i \) is the number of springs in each region, \( a n_i \) are the diameters of these regions. In the case of full contact the area of contact \( A = A_0 \), where

\[ A_0 = \frac{\pi}{4} (aN)^2, \]

where \( N \) is the full number of springs. Then there is only a single connected region, the diameter of which is equal to the length of the system \( aN \). Using (5) and (7) we can also calculate the squeezing pressure \( p = F/A_0 \). The interfacial separation \( \bar{u} \) was calculated using the formula:

\[ \bar{u} = \frac{1}{N} \sum_{i=1}^{N} u_i, \]

where \( u_i \) is the distance between the end of each spring and the substrate surface. For springs in contact \( u_i = 0 \).

Using this simple procedure we have calculated the dependencies shown in Fig. 3 by the solid lines. This dependencies were calculated in range of coordinates of elastic foundation (array of springs) from first contact to full contact.

We will compare the predictions of the theory of Popov et al. with numerical exact results obtained as described in Ref. \[27,28\]. In brief, this method computes the surface displacements using a Fourier-transform technique with a linear surface Green’s function that corresponds to Poisson ratio \( \nu = 1/2 \). The interaction with the rigid surface is treated as a hard-wall repulsion.
by averaging the calculated quantities over 100 realizations of the rough-line topography, with the 1D-power spectrum given by (4). In each realization the elastic foundation has 8192 springs.

4 Contact stiffness
Consider two elastic solids with nominally flat surfaces squeezed together by the nominal pressure \( p = F/A_0 \). From (2) it follows that the contact stiffness

\[
K = -\frac{dp}{d\bar{u}} = \frac{p}{\alpha H_{\text{rms}}}
\]

or

\[
K \frac{h_{\text{rms}}}{E^*} = \frac{1}{\alpha} \frac{p}{E^*}
\]

This equation is only valid at such high pressures that multi-asperity contact occurs. At very low pressures the solids will only make contact in the vicinity of the highest asperity. In this finite-size pressure region the relation between \( K \) and \( p \) will exhibit large fluctuations from one surface realization to another [24].

In Ref. [21] two of us has derived an (approximate) analytical expression for the (ensemble averaged) interfacial stiffness in the finite-size region. The derivation assumes a self-affine fractal surface with the surface roughness power spectrum shown in Fig. 2. The surface is characterized by the Hurst exponent \( H \) and the small and large wavevector cut-off \( q_0 \) and \( q_1 \), as well as a roll-off \( q_r \) (see Fig. 2). For this model the stiffness per unit area in the low pressure, finite size, region is approximately given by (see Ref. [21])

\[
K \approx \left( \frac{E^*}{L^2 q_r} \right)^{H/(1+H)} \left( \frac{p}{h_{\text{rms}}} \right)^{1/(1+H)} \sim p^{1/(1+H)}
\]

where \( L \approx 2\pi/q_0 \) is the linear size of the studied system. Note that we can also write this equation as

\[
K \frac{h_{\text{rms}}}{E^*} \approx \left( \frac{h_{\text{rms}}}{L^2 q_r} \right)^{H/(1+H)} \left( \frac{p}{E^*} \right)^{1/(1+H)}
\]

5 Numerical results for \( K(p) \)
We consider again two cases, when the substrate surface is fractal-like in the whole interval \( q_0 < q < q_1 \), and when there is a roll-off at \( q_r = 8 \), see Fig. 2. In Fig. 4(a) we show the calculated interfacial stiffness using the theory of Persson. We have plotted \( K h_{\text{rms}}/E^* \) as a function of \( p/E^* \) as these dimensionless quantities enter in the theory [see (9) and (10)]. The results in Fig. 4(a) are in excellent agreement with exact numerical simulations for the same system, see Fig. 1 in Ref. [22]. One can distinguish three regions in the stiffness \( K(p) \) relation. For very small pressures the stiffness increases as \( K \sim p^{1/(1+H)} \). This is a finite size effect, which occurs when a single effective Hertz contact region, formed at
the highest substrate asperity, prevails. For higher pressures a region where $K \sim p$ is observed. This region, which becomes wider as the width of the roll-off region increases, results from contact with many asperities, and depends crucially on the long-range elastic coupling between the contact regions. Finally, for very large pressure the interfacial separation approaches zero and the interfacial stiffness increases towards infinite.

Fig. 4(b) shows the results using the contact mechanics theory of Popov et al. The results are obtained by averaging the contact stiffness obtained in 100 realizations of the rough-line topography, with the 1D-power spectrum given by (4). In each realization the elastic foundation has 8192 springs. Fig 4(b) shows that the theory correctly predicts the initial (low pressure) relation $K \sim p^{1/(1+H)}$. This result is expected because the study in Ref. [22] shows that $K \sim p^{1/(1+H)}$ holds even when one neglect the elastic coupling between the asperity contact regions. However, the region where $K$ increases linear with the pressure $p$ is absent in Fig. 4(b). This is also expected because the $K \sim p$ result depends crucially on the elastic coupling between the asperity contact regions, which is not included in the theory of Popov et al. As shown in Fig. 4(a), the linear region is particular large when there is a roll-off in the power spectrum. Most surfaces of engineering interest exhibit a roll-off even larger than for the $q_0/q_0 = 8$ case shown in Fig. 2. Thus in most practical applications, in particular involving elastically soft materials like rubber, one will be in the linear $K \sim p$ region where the approach of Popov et al fails.

We note that whether there is a roll-off or a cut-off at $q_c$ has very little influence on the result. However, this does not imply that the only thing which matters is the range over which the surface is self-affine fractal. The point is that including a roll-off or cut-off at $q_c$ implies roughly that the surface is “periodically” repeated $(q_c/q_0)^2$ times. This implies that there will be many asperities of similar height as the highest asperity. This in turn means that the contact will much quicker (with increasing pressure) come into the multi-asperity contact configuration where the stiffness $K$ depends linearly on the nominal squeezing pressure $p$. This is the origin of why the linear relation between $K$ and $p$ starts at lower pressures, and extends over a larger pressure range, when the surface has a roll-off or cut-off.

The vertical dashed lines in Fig. 4 indicate the pressures where the theory of Popov et al. starts to deviate from the analytical results in Fig. 4(a). Note that these points correspond to the start of the linear $K \sim p$ region in the analytical theory. This is expected as the linear region corresponds to multi-asperity contact, where the elastic coupling between the asperity contact regions, which is neglected in the Popov et al. theory, becomes important. Thus it is the neglect of the long range elasticity, and not the reduction in dimensionality, which is the basic problem with the approach of Popov et al. In fact, a recent study by Scaraggi et al. [29] has shown that if the long-range elastic coupling is included in the analysis, it is possible to make 2D isotropic roughness approximately equivalent to 1D roughness.

6 Summary and conclusion
We have presented a detailed comparison of the theory of Popov et al. with numerical exact results and analytical results for self-affine fractal surfaces with and without a roll-off. The theory of Popov et al. fails qualitatively to describe the $A(p)$ and $\tilde{u}(p)$ relations, and we attribute this to the absence of the elastic coupling between the asperity contact regions in the approach of Popov et al. In fact, a recent study by Scaraggi et al. [29] has shown that if the long-range elastic coupling is included in the analysis, it is possible to make 2D isotropic roughness approximately equivalent to 1D roughness.

We have also presented a detailed comparison of the theory of Popov et al. with analytical results for the nor-
mal contact stiffness. For the case of a roll-off at $q_r = 8$ the Persson theory and the exact numerical results presented in Ref. [22], exhibit a linear $K \sim p$ region extending over 3 decades in pressure, while there is no linear region in the theory of Popov et al. The latter theory predicts $K \sim p^{1/(1+H)}$ in the limit of small pressures, but this result is expected since an effective Hertz single-asperity contact prevails in this case (see Ref. [21]). However, since the elastic coupling between the asperity contact regions is absent in the approach of Popov et al., no linear $K \sim p$ region is obtained.

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