The String Equation and Solitons

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Lecture 1: Periodic 1- and 2-dimensional Schrodinger Operators
Riemann surfaces, Nonlinear Equations

Introduction. We are going to present here some brief survey of the results of Theory of Solitons (see [1–3]) from the viewpoint of periodic theory including some new results in the theory of 2-dimensional periodic Schrodinger Operators.

A remarkable connection of some very special but highly nontrivial nonlinear (especially one-dimensional) systems with spectral properties of one-dimensional linear Schrodinger Operators was discovered in 1965–68 in the series of works [4–6] for the famous KdV equation

\[ u_t = 6uu_x - u_{xxx} \]

This connection is based on the identification of KdV with Heisenberg type equation for the linear operators ("Lax representation"):

\[ L_t = [A, L], \quad L = -\partial_x^2 + u(x, t), \quad A = -4\partial_x^3 + 3(2u\partial_x + u_x) \]

This equation generates an effective GGKM integration procedure ("Inverse Scattering Transform") for KdV in the class of rapidly decreasing functions \( u(x) \to 0, |x| \to \infty \), using the solution of inverse scattering problem for the one-dimensional Schrodinger Operator. A lot of important results were extracted from this method in the 70-s, using traditional methods in the modern Theoretical and Mathematical Physics like exact multisoliton solutions, asymptotics for \( t \to +\infty \), action-angle type variables for the rapidly decreasing KdV, new classes of systems integrable by the same trick (see [2,1]). Many groups participated in this development, including Zakharov, Shabat, Lamb, Faddeev, Ablowitz, Kaup, Newell, Segur, Henon, Flashka, Manakov, Moser, Calogero and others. They found a lot of new important ODE-s and PDE-s (0+1,1+1 and even 2+1)– systems with Lax-type representation, including some special cases of Einstein equations and Self-Duality equation for the Yang-Mills fields (the last system is 4-dimensional).

Periodic Solitons and Algebraic Geometry.

We discuss here a different part of development of this theory, based on the "Triangle" with following vertices:

1. Algebraic Geometry, associated with Riemann surfaces and their \( \Theta \)-functions (which never has been used in Applied Mathematics and Physics before);
2. Spectral theory of Schrodinger operator on the real line \( x \) with periodic (quasiperiodic) potential;
3. Periodic problems for KdV and higher analogues.

This development started from the work [7] in 1974 and was realized completely (in the case of KdV) by the present author, Dubrovin, Its, Matveev, Lax, McKeen, van Moerbeke in 1974-75 (see full survey of this theory in [8,1,3]).

In the periodic case (I remind that the potential \( u \) is periodic here, not the eigenfunction) the corresponding inverse spectral problem for Schrodinger operators never has been solved before the KdV theory. It was solved only in 1974-5 as a part of KdV theory, using the deep connection of KdV and Schrodinger operator. This solution was based on the new ideology, considering KdV and its higher analogs as a symmetry theory for the Schrodinger operator (for its spectral theory): eigenvalues of Schrodinger operators are the integrals of motion for...
all higher KdV systems. We have an infinite-dimensional commutative symmetry group for
any functional $F_u(x)$, if it depends on the spectrum of the operator $L = -\partial + u$ only. Such
functionals played an important role in the definition and studying of finite-gap potentials
(below). Completely different important example was found later: a very well known Peierls
Free Energy functional in the mean field approximation for some electron-phonon systems,
describing the so-called "charge density waves" in some quasi-one-dimensional media. Its exact
integrability was discovered by the group of physicists in collaboration with experts in the
soliton theory: Belokolos, Dzyaloshinski, Gordyunin, Brozovski, Krichever (see, for example,
in the survey [3]).

An important generalization of finite-gap solutions for some special $2+1$ systems (like KP)
was done by Krichever (see [14]), who extended very far an algebraic part of periodic theory.
Many people worked in this area later. The references may be found in the books [13].

This approach is based on the special finite-dimensional families of exact solutions, whose
$x$-dependence is specified by the Commutativity Relation of 2 different linear OD operators
$[C, B] = 0$. For $1+1$ (or $x, t$)-systems the corresponding operator $C$ is necessarily equal to the
operator $L$ in the Lax pair for our system. The operator $B = \sum_i c_i A_i$ is some linear combination
of corresponding $A$-operators for the so-called "Higher KdV" systems, associated with the same
1-dimensional Schrodinger Operator in the case of KdV. Commutativity Relation is equivalent
to the family of Completely Integrable Hamiltonian OD systems in the variable $x$, admitting
some very useful "Lax-type representations"

\[ \Lambda(\lambda)_x = [\Lambda, Q] \]

for $2 \times 2$-matrices, depending on the additional parameter $\lambda$. Riemann surface $\Gamma$ appears
as a polynomial equation

\[ \det[\Lambda(\lambda) - \mu] = 0 \]

whose coefficients are the integrals in $x$.

**Example:** In the simplest case of stationary waves for KdV we have $C = L, B = A$, a
Riemann surface $\Gamma$ with genus 1, extracted from the matrices:

\[ \Lambda = \begin{pmatrix} -u_x & u + 2\lambda \mu & 2u + 4\lambda \\ -4\lambda^2 + 2\lambda u - u_{xx} + 2u^2 & u_x \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} \quad (0.1) \]

Generic finite-gap solutions $u(x)$ are periodic (or quasiperiodic) in $x$. They can be written
by the formula

\[ u = -[2\log\Theta(Ux + Wt + U_0)]_{xx} + \text{Const} \]

with Riemann $\Theta$-function, vectors $U, W$ and constant $C$ determined by the Riemann surface
$\Gamma$; these potentials have a remarkable Spectral Property: Corresponding Schrodinger Operator
$-\partial_x^2 + u(x) = L$ has only a finite number of gaps in the Spectrum on the line, whose endpoints
are exactly the branching points of Riemann surface above.

The Spectral Problem of Bloch

\[ L\Psi = \lambda\Psi, \Psi(x + T) = e^{ipT}\Psi(x) \]

is completely solvable in $\Theta$-functions.
This class of functions generates the "Finite-Gap Solutions" of KdV [8]. This Family is dense in the class of all smooth periodic functions -see [1].

For the 2+1-dimensional KP–system the corresponding Lax-Zakh arov-Shabat operators are

\[ L = \sigma \partial_y - \partial_x^2 + u(x, y, t), \sigma^2 = \pm 1, A = -4\partial_x^3 + 3(2u\partial_x + u_x) + w \]

There is no relation here between the form of linear OD operators \([C, B] = 0\), describing special solutions, associated with Riemann surfaces, and Lax operators \(L, A\). All family of "Krichever Solutions" for KP is much more broad than finite–gap families in the case of 1+1 systems, They have more or less the same analytical form as above, but the class of parameters (compact Riemann surfaces with marked point) is unrestricted. This class of solutions was used later several times for the different goals: for the solution of the classical problems of the theory of \(\Theta\)–functions (like new approach to the Riemann–Schottki Problem, started in [11] and finished by Shiota), for some applications in the Conformal 2-d Field Theory and in the theory of bosonic strings on the base of new beautiful algebraic [12] and functional [13] interpretations. An extension of this class, associated with such modern aspects of Algebraic Geometry as holomorphic vector bundles over algebraic curves and their deformations, was constructed in [14].

**Solitons and strings:** In particular, in the works [13] we realized the following program. As everybody know, in the late 60-ies and early 70-ies a large group of physicists (Veneziano, Virasoro, Alessandrini, Mandelstam and many others) developed the very beautiful theory of the bosonic quantum strings. They used standard operator quantization, decomposing fields in the Fourier series and replacing c-numbers by operators with standard canonical commutators. This program was effectively realized for the "zero-loop" or "tree-like diagrams" only (i.e. for the processes, described by the Riemann surfaces of the zero genus). The so-called Virasoro algebra and its representations played an important role in these constructions. This program stopped because nobody was able to quantize fields in such a way for the multiloop case (i.e. for the Riemann surfaces of nonzero genus). In early 80-ies Polyakov solved the problem of quantization of bosonic strings using a functional ("path") integral. No objects like Virasoro algebra appear in this approach. In the works [13] we constructed a right analog of the Fourier-Laurent series on the Riemann surfaces, using the analytical constructions of the Soliton Theory. After that, an operator quantization of strings was done very easily. Some beautiful analogs of the Virasoro algebra appeared here. This area was not active in the last 3 years, so we shall not discuss it here.

In the second lecture we shall discuss a completely different deep connection between solitons and strings.

Let me start now the main subject of this lecture.

**Topologically Trivial Periodic 2-dimensional Schrodinger Operators and Riemann surfaces.**

I have no intention to discuss here all the subjects above. My goal is to explain some less popular ideas, associated with 2-dimensional Schrodinger Operator, in connection with new work of the present author and A.Veselov (in preparation).

It is more or less obvious, that there is no nontrivial Lax equations associated with 2-dimensional Schrodinger operator \(-2L = (\partial + A_c)(\bar{\partial} + B_c) + 2V\). Here \(A_c, B_c\) are the components of vector-potential, \(V\) is a scalar potential, \(\partial = \partial_x - i\partial_y, z = x + iy\)
However, nontrivial integrable nonlinear systems can be obtained from the different equation
\( (L - A - B - \text{triple}) \), which appeared and was investigated since 1976 (see [15,16]). The inverse
spectral problem for double-periodic Schrodinger operators \( L \) is associated with one energy level
only \( (L \Psi = 0) \) in the approach. \( L - A - B \) triple equation has a form:
\[
L_t = [A, L] + BL = LA + (A + B)L
\]
which implies something like Lax representation, corresponding to one energy level:
\[
(L_t - [A, B])\Psi = 0, L\Psi = 0
\]
This representation leads to some beautiful 2-dimensional analogs of KdV, containing KP
as some degenerate limit:
\[
u_t = (\partial^3 + \partial P)u(x, y, t) + C.C., \bar{\partial}u = 3\partial P, u = \bar{u}
\]
and corresponding analogs of "Higher KdV" systems. Nontrivial exact solutions of this
nonlinear systems and periodic Schrodinger operators with zero magnetic field \( -2L = \partial \bar{\partial} + 2V \) and solvable Bloch problem \( L\Psi = \epsilon_0\Psi \) for one energy level were found by the present
author in collaboration with Veselov in 1984 (see [17]). Our Riemann surface \( \Gamma \) in this case is
exactly a Complex Fermi Curve. Our Bloch wave function \( \Psi \) can be expressed through the
so called "Prym" \( \Theta \)–functions, which are more complicated than the standard "Jacobian" \( \Theta \)–
functions in the case of KP above. Generic complex Fermi curve has infinite genus. This theory
was developed by Krichever in 1989–90, who proved that our exactly integrable class (with
Complex Fermi Curve of finite genus) is dense in the class of all double periodic potentials.
Rapidly decreasing class also was investigated by Grinevich, Manakov, R.Novikov and the
present author in 1987–89. It is interesting that for the two-dimensional Schrodinger operator
periodic inverse problem was solved earlier than rapidly decreasing inverse scattering problem
(based on the data, associated with one energy level). There exist simple rational potentials
(found by Grinevich in 1988), for which the scattering amplitude is identically equal to zero for
one energy level.

**Topologically Nontrivial Schrodinger Operators.**

All this class of integrable Schrodinger operators, associated with nonlinear systems, Rie-
mann surfaces and \( \Theta \)–functions, contains only Schrodinger operators with "topologically trivial"
magnetic field: it has a "Chern class" (i.e. magnetic flux through the elementary cell in the
double periodic case) equal to zero.

Completely different class of Schrodinger operators with exactly integrable ground level
(which is highly degenerate) was found 15 years ago in rapidly decreasing ( [18]) and periodic
([19]) cases. It corresponds to the nonrelativistic Pauli operator for spin 1/2 in the magnetic
field, orthogonal to the plane, and zero electric potential. The ground energy level is equal to
zero in this case. In particular, for the periodic case [19], this level is isomorphic to the first
Landau level in the constant magnetic field with the same magnetic flux through the elementary
cell. If this flux is an integer, the so called "Magnetic Bloch functions" were found analytically
through the elliptic functions for all this class in [18]:
\[
\Psi(x, y) = e^{\phi} \sigma(z - a_1) \ldots \sigma(z - a_n)e^{az}
\]
\[ \Delta \phi = -H \]

Here \( H \) is a magnetic field, \( a \) is expressed through the constants \( a_1, \ldots, a_n \) and \( H \).

**Cyclic and semicyclic chains of the Laplace transformations. New results of the present author and A.Veselov.**

For the unification of these two theories the present author in collaboration with Veselov used an idea of ”Cyclic Chains” of Laplace transformations. Let me point out that the theory of cyclic chains of Backlund transformations for 1-dimensional Schrodinger operator was developed in the beautiful work of Shabat and Veselov [20] (some first observations were found in [21]). In the early XIX century, Laplace constructed the transformations of second order linear PDE for some goals in geometry. I would like to point out that the ”Laplace transformation” acts on the solutions of the equation \( L_0 \Psi_0 = 0 \) for the two-dimensional Schrodinger operator \( L_0 \)

\[ L = L_0 = -1/2(\partial + B_0)(\partial + A_0) + V_0 \]

by the formula

\[ \Psi_1 = (\partial + A_0)\Psi_0, A_1 = A_0 - (\log V_0)_z, B_1 = B_0, V_1 = V_0 + H_1 \]

Here the magnetic field \( H_0 \) is equal to \( 2H_0 = B_0z - A_0\bar{z} \) and \( H_1, V_1 \) are the magnetic field and scalar potential for the operator \( L_1 \) respectively, such that

\[ L_1 \Psi_1 = 0, H_1 = H_0 + 1/2(\log V_0)_{zz} \]

The requirement that the chain of Laplace transformations is periodic leads to the beautiful elliptic partial differential equations for the magnetic fields and scalar potentials of all operators in the chain. This problem was posed and studied for the first time in the XIX century by Darboux (the operator \( L \) and the corresponding nonlinear systems are always hyperbolic in geometry, but formal calculations are the same). Some useful formal calculations were done by Tsiseika in 1920-s. We applied this stuff to the theory of (elliptic) 2-dimensional Schrodinger operator. Globally nonsingular double periodic solutions of this system in our ”elliptic” case give two-dimensional Schrodinger operators with Complex Fermi Curve of finite genus (Algebro-Geometric operators, as above, in the case of topologically trivial magnetic field.)

Especially beautiful well-known integrable systems appear in the case \( H_0 = H_n = 0 \) for the periods \( n = 3, 4 \). Let \( V_0 = \exp f \). We have:

\[ \Delta f = -2(e^f - e^{-2f}), n = 3, \Delta f = -4\sin f, n = 4 \]

In the topologically nontrivial double periodic case, when the magnetic flux \( [H] \) is nonzero, no cyclic chain is possible. Let be \( [H] > 0 \). Instead of cyclic chains we consider the **Semicyclic Chains** and **Quasicyclic chains**, satisfying one of the two following conditions:

1. Semicyclic chains

\[ H_0 = H_n, V_n = V_0 + n[H_0] \]

It leads to some operators with special algebraic properties: Eigenfunctions of two different energy levels 0 and \( n[H_0] \) are connected by the operator

\[ \Psi_n = (\partial + A_{n-1}) \ldots (\partial + A_0) \Psi_0 \]
For $n = 2$ this condition leads to the equation

$$\Delta f_0 = a - bsh f_0, b > 0,$$

which has a lot of double periodic nonsingular solutions. However, these levels, equal to zero and to $n[H_0]$ can be out of the spectrum, so this connection is formal.

2. Quasicyclic chains

Another, more interesting analog of cyclic chains, we obtain from the condition, that for $n = 0$ and for some other value of $n$ the operators $L_0, L_n$ belong to the class [18,19] up to the shift of energy

$$V_0 = H_0, V_n = H_n + n[H_0]$$

For $n = 1$ there exists only a constant solution of this equation, but for $n = 2$ it leads to the elliptic PDE:

$$\Delta g = 4([H_0] - e^g), e^g = V_0$$

This equation has a lot of real nonsingular double periodic solutions on the plane. The corresponding Schrodinger operators $L_n$ have two highly degenerate energy levels:
1. Ground level, equal to zero, with magnetic Bloch eigenfunctions written above (by the results of [19])
2. Second integrable level, equal to $n[H_0]$, with magnetic Bloch eigenfunctions of the form

$$\Psi_n = (\partial + A_{n-1}) \ldots (\partial + A_0) \Psi_0$$

Here $\Psi_0$ is a magnetic Bloch eigenfunction with zero energy level for the operator $L_0$, written by the same formula, but with different magnetic field:

$$H_0 = e^{f_0}, H_2 = 2[H_0] - e^{f_0}, n = 2, [H_0] > 0$$

These levels are isomorphic to the "Landau levels" with numbers 0 and $n$ of the operator $L_0$ with constant homogeneous magnetic field.

In both levels the magnetic Bloch functions can be calculated through the elliptic functions, written above. H. de Vega pointed out that our nonlinear equation for $n = 2$ appeared already in the 70-s as an "Instanton Reduction" for the Landau-Ginzburg equation for the critical value of parameter [separating between superconductors of the first and second kind (see [22])].
Lecture 2: Theory of Solitons and String Equation

In the modern terminology, "String Equation" means exactly the equation

\[ [L, A] = 1 \]

for some linear OD operators (people call it also a "Heisenberg Relation").

This strange terminology appeared in 1989-90 years after the well-known works of Gross, Migdal, Brezin, Kazakov, Douglas, Shanker, David and many others (see [23–26]).

A partition function and Free energy of \( N \times N \)-Matrix Models in Statistical Mechanics in some very special "Double-Scaling Limit", when the size of matrices is going to infinity \( N \to \infty \), have probably a beautiful interpretation in the String theory, which was conjectured by the above-mentioned physicists. These Matrix Models and their "string limit" have a deep connection with the Theory of Solitons, which was the most interesting mathematical discovery of these authors. An analysis of this limiting process was done by Its and others (see [27]). Famous KdV type Systems of the Soliton Theory play here a role of "Renormalization Group", like in Quantum Field Theory. However, completely different classes of special solutions are needed here. It is good to point out, that the Integrability in the sense of Lax-type representation leads to the effective results only for 2 classes: periodic (quasiperiodic) and rapidly decreasing. Sometimes in the theory of Solitons people needed in the self-similar solutions for asymptotic methods and so on. A beautiful idea to study them was invented by Flashka and Newell about 1979 (it was developed by the Japanese school and later used by the Leningrad and Ufa schools for asymptotical studyings)–see [28–32]. However, this approach is complicated; it gives a very few number of the effective results. We have here exactly this case. In particular, the computation of Free energy can be reduced to the Painleve’–1 equation in the simplest nontrivial case

\[ u_{xx} - 3u^2 = x \]

In fact, it is equal in this limit to the special real "Physical solution" on the positive halfline \( x \leq 0 \) with asymptotics

\[ u(x) \sim +\sqrt{(-x)/3} \]

This nonlinear equation is equivalent to the algebraic "String Equation" or "Heisenberg Relation" above \([L, A] = 1\) for the same OD operators which give a Lax pair for the ordinary KdV equation (see Lecture 1). This observation leads to some analog of the Lax representation for this Painleve'–1 equation. Several new approaches were developed for the investigation of this equation on the base of technics of the Theory of Solitons (see [33–37]). I presented in this lecture the ideas of the last joint work of myself with Grinevich ([37]), where a special isomonodromic method for the studying of the physical solution was developed.
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