Critical behavior of the XY model on uncorrelated and correlated random networks

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\textbf{Abstract.} We numerically study the critical behavior of the XY model on the Erdős–Rényi random graph and a growing random network model, representing the uncorrelated and the correlated random networks, respectively. We also checked the dependence of the critical behavior on the choice of order parameters: the ordinary unweighted and the degree-weighted magnetization. On the Erdős–Rényi random network, the critical behavior of the XY model is found to be of the second order with the estimated exponents consistent with the standard mean-field theory for both order parameters. On the growing random network, on the contrary, we found that the critical behavior is not of the standard mean-field type. Rather, it exhibits behavior reminiscent of that in the infinite-order phase transition for both order parameters, such as the lack of discontinuity in specific heat and the non-divergent susceptibility at the critical point, as observed in the percolation and the Potts models on some growing network models.
1. Introduction

After the complex networks [1]–[5] were introduced, studies of the phase transition and the critical phenomena have been performed actively (for a review, see [6]). Especially, degree–degree correlations, either imposed intrinsically or by the growth process, have been shown to have a significant effect on the critical behavior of the complex networks [7]–[12]. Callaway et al [7] studied the percolation on the growing networks and the static random networks and concluded that the growing random networks are fundamentally different from the static random networks, and the growing networks are correlated, while the static networks are uncorrelated. Bauer et al [10] found analytically that the ferromagnetic Ising model on a highly inhomogeneous network created by a growth process exhibits the Berezinskii–Kosterlitz–Thouless-like transition, while that on an uncorrelated network shows the standard mean-field behavior. Noh [12] suggested that the unusual transitions on the correlated networks would be caused by the assortative degree–degree correlation by studying percolation transition.

Studies on the relations between the critical behavior and the degree–degree correlation for the XY model on the complex networks have also been performed [13, 14]. The critical behavior of the XY model on the uncorrelated scale-free networks [13] is in agreement with the ordinary mean-field calculations [15], while on the correlated scale-free networks this is not found to be the case [14].

In this paper, we study the critical behavior of the XY model on the uncorrelated and correlated random networks, by using the Erdős–Rényi (ER) random graph [16, 17] and the randomly growing network [7] as the representative network models in each class, respectively. The XY model has infinite numbers of ground states and takes a longer relaxation time, so we used the histogram reweighting method [18]–[20] for numerical efficiency.

2. Background

2.1. XY model

The Hamiltonian of the XY model is given by

\[ H = -J \sum_{(j,k)} \cos(\phi_j - \phi_k), \]  

(1)
where the sum goes over all the nearest-neighbor pairs and \( \phi \) is the phase variable taking values in \((-\pi, \pi]\). We set the coupling constant to be \( J = 1 \).

2.2. Order parameters

The (ordinary) magnetization \( m_1 \) is defined as

\[
M_1 = \langle m_1 \rangle = \frac{1}{N} \left| \sum_j e^{i\phi_j} \right|,
\]

where \( N \) is the system size, the number of vertices in networks and \( i = \sqrt{-1} \). Usually, the magnetization is used as an order parameter in the Monte Carlo simulations. For the spin models on heterogeneous networks, however, a different form of order parameter \( m_2 \) has been suggested, which is as follows \([6, 21]\):

\[
M_2 = \langle m_2 \rangle = \frac{1}{\kappa N} \left| \sum_j k_j e^{i\phi_j} \right|,
\]

where \( k_j \) is the degree of the \( j \)th node and \( \kappa \) is the mean degree of the network.

In this paper, we use both \( m_1 \) and \( m_2 \) as the order parameters to compare the corresponding critical behavior.

2.3. Random network models

We compare the critical behavior of the XY model on the uncorrelated random networks and the correlated random networks. We use the ER random graph as the representative uncorrelated random network, which is constructed by making \( L \) edges randomly in a pre-existing set of \( N \) vertices \([16, 17]\). Here, no new vertices are ever added so the network is of static type. For the correlated random networks, we used the random growing network model slightly modified from that of Callaway et al \([7]\), in which at each step a new vertex is added to the network and \( l \) links are made between randomly chosen pairs of vertices. In both cases, we set the mean degree of the network \( \langle k \rangle = 8 \), by choosing \( L = 4N \) in the former case and \( l = 4 \) in the latter case.

2.4. Finite-size scaling

The susceptibility \( \chi \) and the reduced fourth-order cumulant \( U \) are defined as follows:

\[
\chi = \frac{N}{T} \left( \langle m^2 \rangle - \langle m \rangle^2 \right),
\]

\[
U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2},
\]

where \( T \) is the temperature and \( m \) is either \( m_1 \) or \( m_2 \). In order to perform the finite-size scaling analysis, the correlation volume exponent \( \nu' \) is more suitable than the correlation length exponent \( \nu \) \([22]\), as the system size \( N \) is the natural scale in complex networks. Therefore, the finite-size scaling form of the order parameter, the susceptibility, and the reduced fourth-order cumulant for networks with \( N \) nodes can be written as

\[
m(N) = N^{-\frac{\nu}{\nu'}} \tilde{m} \left( N^{\frac{\nu'}{\nu}} t \right) : (t < 0),
\]

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\( \chi(N) = N^{\frac{\nu}{\gamma}} \tilde{\chi} \left( N^{\frac{1}{\gamma}} t \right), \quad (7) \)

\( U'(N) = N^{\frac{1}{\gamma}} \tilde{U}' \left( N^{\frac{1}{\gamma}} t \right), \quad (8) \)

where the reduced temperature \( t \) is defined as \( t = (T - T_c)/T_c \), and \( \beta \) and \( \gamma \) are the order parameter and the susceptibility exponents, respectively. \( U'(N) \) is the derivative of the fourth-order cumulant with respect to the temperature. According to the finite-size scaling theory \([23]\), the scaling functions \( \tilde{m}, \tilde{\chi} \) and \( \tilde{U}' \) are universal and \( m(N), \chi(N) \) and \( U'(N) \) are analytical and smooth functions around the transition temperature \( T_c \).

2.5. The histogram reweighting method

Histogram reweighting methods \([18]–[20]\) are applied to equilibrium models to study critical phenomena. Using them, we estimate precise thermodynamic quantities depending on temperatures without performing simulations at different temperatures. If we accumulate the normalized energy histogram \( h_{\beta_0}(E, m) \) from simulations directly at inverse temperature \( \beta_0 \), the probability \( P_{\beta_0}(E, m) \) of the system having energy \( E \) and the order parameter \( m \) at \( h_{\beta_0}(E, m) \) could be written as

\[ P_{\beta_0}(E, m) \equiv h_{\beta_0}(E, m). \quad (9) \]

For the equilibrium system, the density of state \( g(E, m) \) can be expressed as the Boltzmann weight \( e^{-\beta_0 E} \) using equation (9) as follows:

\[ g(E, m) = h_{\beta_0}(E, m)e^{\beta_0 E}Z(\beta_0), \quad (10) \]

where \( Z(\beta_T) \) is the canonical partition function. Using equation (10), we can easily estimate the probability \( P_{\beta_T}(E, m) \) without performing simulation at inverse temperature \( \beta_T \) as follows:

\[ P_{\beta_T}(E, m) = \frac{h_{\beta_0}(E, m)e^{-\Delta \beta_T E}}{\sum_E h_{\beta_0}(E, m)e^{-\Delta \beta_T E}}, \quad (11) \]

where \( \Delta \beta_T = \beta_T - \beta_0 \). The expectation value of thermodynamic quantities \( Q_{\beta_T} \) as a function of \( E \) and \( m \) can be calculated as a continuous function of inverse temperature \( \beta_T \)

\[ \langle Q_{\beta_T}(E, m) \rangle = \sum_{E, m} Q_{\beta_T}(E, m)P_{\beta_T}(E, m). \quad (12) \]

Equations (11) and (12) are referred to as the single-histogram equations \([19]\).

When the simulation time is not sufficient, the entries in a histogram are limited to the left and the right tails of the histogram (histogram wings) generated at \( \beta_0 \). So, the histogram reweighting method does not work for \( \beta_T \) far from \( \beta_0 \) because of the lack of entries in histogram wings.

3. Results

We perform the Monte Carlo simulations using the histogram reweighting method for the XY model on the uncorrelated networks and the correlated random networks, where the network size \( N \) is 125, 250, 500, 1000, 2000 and 4000 with the average degree \( \langle k \rangle = 8 \). For the largest
systems $N = 4000$, each run lasts $1 \times 10^5$ Monte Carlo steps, and 240 independent runs are performed. For the smallest systems $N = 125$, each run lasts $1.25 \times 10^4$ Monte Carlo steps, and 12 800 ensembles are averaged.

First we present the results on the uncorrelated ER network. The critical temperature $T_c$ is estimated by the asymptotic limit of the crossing points $T'_c(N)$ of the reduced fourth-order cumulant $U(N)$ for successive system sizes. Figures 1(a) and (b) show $U(N)$ for different system sizes as a function of temperature, which converges to $T'_c = 3.98(3)$ for $m_1$ and $T'_c = 3.97(3)$ for $m_2$. At the critical temperature $T_c$, the correlation volume exponent $\nu'$ can be estimated using equation (8). Figures 1(c) and (d) show the double logarithmic plot of the maximum values of $U'$ as a function of system size $N$, the slope of which is given by $1/\nu'$ with the estimated exponent $\nu' = 1.91(2)$ for $m_1$ and $\nu' = 1.92(4)$ for $m_2$. Using the maximum values of the susceptibility $\chi(N)$, we are able to find $\gamma/\nu'$ (equation (7)). In figures 1(e) and (f), the slope of the straight line gives $\gamma/\nu' = 0.507(6)$ for $m_1$ and $\gamma/\nu' = 0.530(3)$ for $m_2$. From figures 1(g) and (h), $\beta/\nu' = 0.27(1)$ for $m_1$ and $\beta/\nu' = 0.26(3)$ for $m_2$ are estimated by the finite-size scaling of the order parameter. Therefore, we obtain that the critical exponents for the XY model on the ER random network are

$$
\beta = 0.51(2), \quad \gamma = 0.97(3) \quad \text{and} \quad \nu' = 1.91(2)
$$

for $m_1$, and

$$
\beta = 0.51(2), \quad \gamma = 1.02(2) \quad \text{and} \quad \nu' = 1.92(4)
$$

for $m_2$, both of which are in good agreement with the standard mean-field results $\beta_{MF} = 1/2, \gamma_{MF} = 1$ and $\nu'_{MF} = 2$.

Using the mean-field critical exponents, we draw the scaling plots for the order parameter and the susceptibility in figure 2. As shown in figures 2(a)–(d), the scaling plots collapse well with the mean-field critical exponents for the XY model on uncorrelated random networks for both order parameters. On the other hand, such a data collapse is absent for the XY model on the correlated random networks (figures 2(e)–(h)), suggesting that the two cases belong to distinct universality classes.

Such a discordance in the critical behavior on the uncorrelated networks and the correlated networks has already been noted for a number of spin models [7, 8], [10]–[14]. Specifically, the critical behavior of percolation and the Potts models on growing random networks have been shown to exhibit infinite-order transitions, in striking contrast to the ordinary second-order transitions with the mean-field exponents on the uncorrelated networks. Therefore, we speculate on the possibility of such an infinite-order transition in the XY model as well. It is, however, extremely difficult to probe the infinite-order transition quantitatively solely with numerics, so we turn to some qualitative lines of evidence.

In figure 3, we show the specific heat $C$ and the magnetic susceptibility $\chi$ of the XY model on the growing random network. In the infinite-order transition exhibited by the Potts model on the (annealed) growing random network, the specific heat is non-analytic at $T_c$ with essential singularity, yet it is continuous [11], unlike the case of ordinary mean-field type transition with the discontinuity in the specific heat. We observe that the specific heat of the XY model on the growing random network shows no signature of discontinuity as the system size increases, compared to the case of the ER random network, which shows a large change in the specific heat at $T_c$ even on the networks with comparable sizes (figures 3(a) and (b)). The susceptibility for the infinite-order transitions exhibits a discontinuity and yet no divergence at $T_c$, as shown.
Figure 1. Plots of (a)–(b) the reduced fourth-order cumulant $U(N)$ as a function of $T$, (c)–(d) the maximum values of $U'(N)$ as a function of $N$, (e)–(f) the maximum values of the susceptibility $\chi(N)$ as a function of $N$ and (g)–(h) the scaled order parameter $mN^{\beta'/\nu}$ as a function of $T$ for the XY model on an ER random network. The left column shows the plots for $m_1$ and the right column the plots for $m_2$. 
Figure 2. (a)–(d) Scaling plots of (a)–(b) the order parameter $m$ and (c)–(d) the susceptibility $\chi$ of the XY model on the uncorrelated random networks using the mean-field critical exponents, $\beta_{MF} = 1/2$, $\gamma_{MF} = 1$, and $\nu'_{MF} = 2$. (e)–(h) The same plots for the XY model on the random growing network (correlated random network) with the same set of exponents as in (a)–(d) showing the lack of data collapse. The left column is the plots for $m_1$ and the right column is for $m_2$. 

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Figure 3. Plots of (a)–(b) the specific heat $C$ and (c)–(d) the susceptibility $\chi$ of the XY model as a function of the inverse temperature $1/T$ on growing random networks with various system sizes ranging from $N = 250$ to 4000 (open symbols). Also shown in (a)–(b) is the specific heat and in the insets of (c)–(d) the susceptibility on the ER random network with $N = 4000$ (filled triangles), for comparison. The left column shows the plots for $m_1$ and the right column the plots for $m_2$.

by the studies of the percolation on the random growing network [7] and the Potts model on the annealed random growing network [11], whereas on the uncorrelated random network it diverges with the standard mean-field criticality. We observe that for the XY model on the growing random network the susceptibility does not develop the divergence as the system size increases, compared to the case of the ER random network, as shown in figures 3(c) and (d) for both order parameters $m_1$ and $m_2$. There are other distinctive signatures of the infinite-order transitions such as the power-law decaying spin–spin correlations in the entire temperature region below $T_c$ [10, 11], but such a property could not be directly addressed by our numerical ability. Therefore, a firm conclusion would require an analytic understanding, although we have presented some lines of evidence in favor of the infinite-order transition to the ordinary second-order one for the XY model on the growing random network.

It is worth noting that the specific heat curves in figures 3(a) and (b) are almost identical, because they are obtained from the energy that is not influenced by the type of order parameters. Meanwhile, susceptibility curves show difference in magnitude because the susceptibility is directly obtained from the order parameter fluctuations. $\chi$ with $m_2$ is larger than that with $m_1$, because it accounts for the degree fluctuations.
4. Conclusion

We have studied the critical behavior of the XY models on the (uncorrelated) ER random network and on the (correlated) growing random network by means of Monte Carlo simulation with the histogram reweighting method with two different order parameters \( m_1 \) and \( m_2 \). For the XY model on the ER random network, we confirmed that the mean-field predictions for the uncorrelated network hold, in that it shows the second-order transition with the set of critical exponents \( \beta = 0.51(2) \), \( \gamma = 0.97(3) \), \( \nu' = 1.91(2) \) for \( m_1 \) and \( \beta = 0.51(2) \), \( \gamma = 1.02(2) \), \( \nu' = 1.92(4) \) for \( m_2 \), in good agreement with the standard mean-field values for both order parameters, where the Pearson correlation coefficient of the degrees \( r \) [24] is \(-0.00007(1)\) by averaging 400 samples for the ER random network with \( 10^6 \) nodes. Here, \( r \) is defined by

\[
\frac{\langle kk' \rangle_l - \langle (k + k')/2 \rangle_l^2}{\langle (k^2 + k'^2)/2 \rangle_l - \langle (k + k')/2 \rangle_l^2},
\]

where \( \langle \cdot \rangle_l \) denotes the average over all links and \( (k, k') \) denotes the degree of two nodes at either end of links, and \( r \) is positive for assortative, negative for disassortative or 0 for uncorrelated (neutral) networks.

For the XY model on the growing random network, our results show that it is not described by the standard mean-field transitions. Motivated by the previous studies of spin models on random growing networks, we tested the possibility of the infinite-order transition in the XY model on growing random network as well. Our numerical results show signatures in favor of the infinite-order transition also for the XY model on the growing random network. For the growing random network, we obtain \( r = 0.26781(7) \) by averaging 400 samples with \( 10^6 \) nodes. Thus we conjecture that the XY model on the growing random network would also exhibit the infinite-order transition due to the assortative degree–degree correlation.

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