A characterization of weighted local Hardy spaces

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Abstract In this paper, we give a characterization of weighted local Hardy spaces $h^1_\omega(\mathbb{R}^n)$ associated with local weights by using the truncated Reisz transforms, which generalizes the corresponding result of Bui in [1].

1. Introduction

The theory of local Hardy space plays an important role in various fields of analysis and partial differential equations; see [6, 1]. Bui [1] studied the weighted version $h^p_\omega$ of the local Hardy space $h^p$ considered by Goldberg [6], where the weight $\omega$ is assumed to satisfy the condition $(A_\infty)$ of Muckenhoupt. R. Vyacheslav [11] introduced and studied some properties of the weighted local Hardy space $h^p_\omega$ spaces with weights that are locally in $A_p$ but may grow or decrease exponentially. Recently, the author [10] established the weighted atomic decomposition characterizations of weighted local Hardy space $h^p_\omega$ with local weights.

The main purpose of this paper is to give a characterization of weighted local Hardy spaces $h^1_\omega(\mathbb{R}^n)$ associated with local weights by using the truncated Reisz transforms.

Throughout this paper, $C$ denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. Denote by $\mathbb{N}$ the set $\{1, 2, \cdots\}$ and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Statement of the main result

We first introduce weight classes $A^\text{loc}_p$ from [11].

Let $Q$ run through all cubes in $\mathbb{R}^n$ (here and below only cubes with sides parallel to the coordinate axes are considered), and let $|Q|$ denote the volume of $Q$.

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We define the weight class $A_p^\text{loc}(1 < p < \infty)$ to consists of all nonnegative locally integral functions $\omega$ on $\mathbb{R}^n$ for which

$$A_p^\text{loc}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q \omega(x)dx \left( \int_Q \omega^{-p'/p}(x)dx \right)^{p/p'} < \infty, \quad 1/p + 1/p' = 1. \quad (2.1)$$

The function $\omega$ is said to belong to the weight class of $A_1^\text{loc}$ on $\mathbb{R}^n$ for which

$$A_1^\text{loc}(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q \omega(x)dx \left( \sup_{y \in Q} [\omega(y)]^{-1} \right) < \infty. \quad (2.2)$$

**Remark:** For any $C > 0$ we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in (2.1) and (2.2).

In what follows, $Q(x, t)$ denotes the cube centered at $x$ and of the sidelength $t$. Similarly, given $Q = Q(x, t)$ and $\lambda > 0$, we will write $\lambda Q$ for the $\lambda$-dilate cube, which is the cube with the same center $x$ and with sidelength $\lambda t$. Given a Lebesgue measurable set $E$ and a weight $\omega$, let $\omega(E) = \int_E \omega dx$. For any $\omega \in A_\infty^\text{loc}$, $L^p_\omega$ with $p \in (0, \infty)$ denotes the set of all measurable functions $f$ such that

$$\|f\|_{L^p_\omega} \equiv \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{1/p} < \infty,$$

and $L^\infty_\omega = L^\infty$. The space $L^{1, \infty}_\omega$ denotes the set of all measurable function $f$ such that

$$\|f\|_{L^{1, \infty}_\omega} \equiv \sup_{\lambda > 0} \lambda \cdot \omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty.$$

We define the local Hardy-Littlewood maximal operator by

$$M^\text{loc} f(x) = \sup_{x \in Q: |Q| < 1} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

Similar to the classical $A_p$ Muckenhoupt weights, we give some properties for weights $\omega \in A_\infty^\text{loc} := \bigcup_{1 \leq p < \infty} A_p^\text{loc}$.

**Lemma 2.1.** Let $1 \leq p < \infty$, $\omega \in A_p^\text{loc}$, and $Q$ be a unit cube, i.e. $|Q| = 1$. Then there exists a $\tilde{\omega} \in A_\infty$ so that $\tilde{\omega} = \omega$ on $Q$ and

(i) $A_p(\tilde{\omega}) \leq C A_p^\text{loc}(\omega)$.

(ii) if $\omega \in A_p^\text{loc}$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^\text{loc}(\omega)$ for $p > 1$.

(iii) If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^\text{loc} \subset A_{p_2}^\text{loc}$.

(iv) $\omega \in A_p^\text{loc}$ if and only if $\omega^{-\frac{1}{p'-1}} \in A_{p'}^\text{loc}$.

(v) If $\omega \in A_p^\text{loc}$ for $1 \leq p < \infty$, then

$$\omega(tQ) \leq \exp(c_\omega t)\omega(Q) \quad (t \geq 1, |Q| = 1).$$
(vi) the local Hardy-Littlewood maximal operator $M^\text{loc}$ is bounded on $L^p_\omega$ if $\omega \in A^\text{loc}_p$ with $p \in (1, \infty)$.

(vii) $M^\text{loc}$ is bounded from $L^1_\omega$ to $L^{1,\infty}_\omega$ if $\omega \in A^\text{loc}_1$.

We remark that Lemma is also true for $|Q| > 1$ with $c$ depending now on the size of $Q$. In addition, it is easy to see that $A_p(M\text{unckenhout weight}) \subset A^\text{loc}_p$ for $p \geq 1$ and $e^{c|x|}, (1 + |x|\ln^\alpha(2 + |x|))\beta \in A^\text{loc}_1$ with $\alpha \geq 0, \beta \in \mathbb{R}$ and $c \in \mathbb{R}$.

Let $\mathcal{N}$ denote the class of $C^\infty$-functions $\varphi$ on $\mathbb{R}^n$, supported on the cube $Q(0,1)$ of center zero and half-side one whose mean value is not equal to zero. For $t > 0$, let $\varphi_t = t^{-2n}\varphi(z/t)$.

Given a distribution $f$, let $\varphi \in \mathcal{N}$, define the smooth maximal function by

$$Mf(z) = \sup_{0 < t < 1} |\varphi_t \ast f(z)|.$$ 

Follows from [10], we introduce the following weighted atoms.

Let $\omega \in A^\text{loc}_1$. A function $a$ on $\mathbb{R}^n$ is said to be a $(1, q)_\omega$-atom for $1 < q \leq \infty$ if

(i) $\text{supp} \ a \subset Q$,

(ii) $\|a\|_{L^q_\omega(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1}$.

(iii) $\int_{\mathbb{R}^n} a(x)dx = 0$ if $|Q| < 1$.

Moreover, we call $a$ a $(1, q)_\omega$ single atom if $\|a\|_{L^q_\omega(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1}$. We introduce weighted local Hardy spaces via smooth maximal functions and weighted local Hardy spaces. Moreover, we study some properties of these spaces.

The weighted local Hardy space is defined by

$$H^1_\omega(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : M(f) \in L^1_\omega(\mathbb{R}^n) \right\}.$$

Moreover, we define $\|f\|_{h^1_\omega(\mathbb{R}^n)} \equiv \|M(f)\|_{L^1_\omega(\mathbb{R}^n)}$. In [10], the author proved that

**Theorem A.** Let $\omega \in A^\text{loc}_1$ and $1 < q \leq \infty$, then for any $f \in h^1_\omega(\mathbb{R}^n)$, there exists numbers $\lambda_0$ and $\{\lambda_k\}_{k \in \mathbb{Z}, i} \subset \mathbb{C}$, $(1, q)_\omega$-atoms $\{a^k_i\}_{k \in \mathbb{Z}, i}$ with radius $r \leq 2$ and single atom $a_0$ such that

$$f = \sum_{k \in \mathbb{Z}} \sum_{i} \lambda_k^i a^k_i + \lambda_0 a_0,$$

where the series converges almost everywhere and in $\mathcal{D}'(\mathbb{R}^n)$, moreover, there exists a positive constant $C$, independent of $f$, such that $\sum_{k \in \mathbb{Z}, i} |\lambda_k^i|^p + |\lambda_0|^p \leq C \|f\|_{h^1_\omega(\mathbb{R}^n)}$.

Let $\Phi$ be a non-negative, radial and $C^\infty$-function on $\mathbb{R}^n$ with compact support $B(0,2)$ and $\Phi \equiv 1$ on $B(0,1)$. Define the truncated Reisz transforms by

$$R_j f(x) = \int_{\mathbb{R}^n} K_j(x-y)f(y)dy, \ K_j(z) = \frac{z_j}{|z|^{n+1}}\Phi(z), \ j = 1, \cdots, n.$$ 

Now let us state the main result of this paper.
Theorem 1. Let $\omega \in \mathcal{A}^{1}_{\omega}$. Then a function $f$ is in $L^1_{\omega}(\mathbb{R}^n)$ if and only if $f \in L^1_{\omega}(\mathbb{R}^n)$ and $R_j f \in L^1_{\omega}(\mathbb{R}^n)$, $j = 1, \cdots, n$. More precisely,

$$
\|f\|_{L^1_{\omega}(\mathbb{R}^n)} \sim \|f\|_{L^1_{\omega}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1_{\omega}(\mathbb{R}^n)}.
$$

We remark that if $\omega \in A_1$, then Theorem 1 has been proved in [1], that is,

Theorem B. Let $\omega \in A_1$. Then a function $f$ is in $L^1_{\omega}(\mathbb{R}^n)$ if and only if $f \in L^1_{\omega}(\mathbb{R}^n)$ and $R_j f \in L^1_{\omega}(\mathbb{R}^n)$, $j = 1, \cdots, n$. More precisely,

$$
\|f\|_{L^1_{\omega}(\mathbb{R}^n)} \sim \|f\|_{L^1_{\omega}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1_{\omega}(\mathbb{R}^n)}.
$$

3. Proof of Theorem 1

Theorem 1 will be deduced by the following lemmas.

Lemma 3.1. Let $\omega \in \mathcal{A}^{1}_{\omega}$. Then

$$
\|f\|_{L^1_{\omega}(\mathbb{R}^n)} \leq C(\|f\|_{L^1_{\omega}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1_{\omega}(\mathbb{R}^n)}). \quad (3.1)
$$

Proof. We will borrow some idea from [9]. Let $Q$ is an unit cube, $\chi_{3Q}^\prime$ is a $C_0^\infty$ nonnegative function supported in $4Q$ and $\chi_{3Q}^\prime = 1$ on $3Q$. By Lemma 2.1, we can set $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $14Q$. Fix $\varphi \in \mathcal{N}$, by Theorem B, we have

$$
\| \sup_{0<t<1} |\varphi_t * f|\|_{L^1_\omega(Q)} = \| \sup_{0<t<1} |\varphi_t * (f\chi_{3Q}^\prime)|\|_{L^1_\omega(\mathbb{R}^n)}
\leq C\|f\chi_{3Q}^\prime\|_{L^1_\omega(\mathbb{R}^n)}
\leq C \left(\|f\chi_{3Q}^\prime\|_{L^1_\omega(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j (f\chi_{3Q}^\prime)\|_{L^1_\omega(\mathbb{R}^n)} \right). \quad (3.2)
$$

On the other hand, by the properties of $\mathcal{A}^{1}_{\omega}$, we obtain

$$
\|R_j (f\chi_{3Q}^\prime) - \chi_{3Q}^\prime R_j (f)\|_{L^1_\omega(\mathbb{R}^n)}
\leq \| \int |R_j (z-y)[\chi_{3Q}^\prime(y) - \chi_{3Q}^\prime(z)]f(y)| \chi_{12I}(y)dy\|_{L^1_\omega(\mathbb{R}^n)}
\leq C \int_{\mathbb{R}^n} \bar{\omega}(z) \int_{\mathbb{R}^n} |R_j (z-y)||z-y||f(y)| \chi_{12I}(y)dydz
\leq C \|f\|_{L^1_{\omega}(14Q)}. \quad (3.3)
$$
Combing (3.2) and (3.3), we obtain
\[
\| \sup_{0 < t < 1} |\varphi_t * f| \|_{L^1_\omega(Q)} \leq C \left( \| f \|_{L^1_\omega(14Q)} + \sum_{j=1}^{n} \| R_j(f) \|_{L^1_\omega(6Q)} \right).
\]

Summing on $Q$, we obtain (3.1).

**Lemma 3.2.** Let $R_j$ be as above, then

(i) $\| R_j f \|_{L^p_\omega(\mathbb{R}^n)} \leq C_{p,\omega} \| f \|_{L^p_\omega(\mathbb{R}^n)}$ for $1 < p < \infty$ and $\omega \in A^p_{\text{loc}}$.

(ii) $\| R_j f \|_{L^1_{\infty}(\mathbb{R}^n)} \leq C \| f \|_{L^1_{\infty}(\mathbb{R}^n)}$ for $\omega \in A^1_{\text{loc}}$.

Proof. We first note that for $\omega \in A_p$ the inequality (i) is known to be true, see [5]. For $\omega \in A^p_{\text{loc}}$, by Lemma 2.1 (i) for any unit cube $Q$ there is a $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $6Q$. Then
\[
\| R_j f \|_{L^p_\omega(Q)} = \| R_j(\chi_{6Q} f) \|_{L^p_\omega(Q)}
\leq \| R_j(\chi_{6Q} f) \|_{L^p_\omega(Q)}
\leq C \| (\chi_{6Q} f) \|_{L^p_\omega(\mathbb{R}^n)}
\leq C \| f \|_{L^p_\omega(6Q)}.
\]

Summing over all dyadic unit $Q$ gives (i).

For (ii), similar to (i), note that for $\omega \in A_1$ the inequality (ii) is known to be true, see [2]. Since $\omega \in A^p_{\text{loc}}$, by Lemma 2.1 (i) for any unit cube $Q$ there is a $\bar{\omega} \in A_1$ so that $\bar{\omega} = \omega$ on $6Q$. Then for any $\lambda > 0$
\[
\omega(\{x \in Q : |R_j f(x)| > \lambda\}) \leq \omega(\{x \in Q : |R_j(\chi_{6Q} f)(x)| > \lambda\})
= \bar{\omega}(\{x \in Q : |R_j(\chi_{6Q} f)(x)| > \lambda\})
\leq C \lambda^{-1} \| (\chi_{6Q} f) \|_{L^p_\omega(\mathbb{R}^n)}
= C \lambda^{-1} \| f \|_{L^p_\omega(6Q)}.
\]

Summing over all dyadic unit $Q$ gives (ii).

**Lemma 3.3.** Let $\omega \in A^1_{\text{loc}}$. Then
\[
\| R_j f \|_{h^1(\mathbb{R}^n)} \leq C \| f \|_{h^1(\mathbb{R}^n)}.
\] (3.4)
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Proof: We first fix a function $\varphi \in \mathcal{N}$. Let $a(x)$ be a $(1, 2)$ atom in $h^1_\omega(\mathbb{R}^n)$, supported in a cube $Q$ centered at $y_0$ and sidelength $r \leq 2$, or $a(x)$ is a $(1, 2)$ single atom. To prove the (iii), by Theorem A and Theorem 6.2 in [10], it is enough to show that

$$\|\mathcal{M}(R_j a)\|_{L^1_\omega(\mathbb{R}^n)} \leq C,$$

where $C$ is independent of $a$.

If $a$ is a single atom, by $L^2_\omega(\mathbb{R}^n)$ boundedness of $\mathcal{M}$ and $R_j$, then

$$\|\mathcal{M}(R_j a)\|_{L^1_\omega(\mathbb{R}^n)} \leq C\|R_j a\|_{L^2_\omega(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1/2} \leq C.$$

Next we always assume that $a$ is an atom in $h^1_\omega(\mathbb{R}^n)$, supported in a cube $Q$ centered at $y_0$ and sidelength $r \leq 2$.

We first consider the atom $a$ with sidelength $1 \leq r \leq 2$. Then by $L^2_\omega(\mathbb{R}^n)$ of the operators $\mathcal{M}$ and $R_j$ (see Lemma 3.2), we have

$$\|\mathcal{M}(R_j a)\|_{L^1_\omega(\mathbb{R}^n)} = \int_{8Q} \mathcal{M}(R_j a)(x)\omega(y)dy \leq C\omega(8Q)^{1/2}\|a\|_{L^2_\omega(\mathbb{R}^n)} \leq C.$$

If $r < 1$, we write

$$\|\mathcal{M}(R_j a)\|_{L^1_\omega(\mathbb{R}^n)} = \int_{2Q} \mathcal{M}(R_j a)(x)\omega(y)dy + \int_{\mathbb{R}^n \setminus 2Q} \mathcal{M}(R_j a)(x)\omega(y)dy := I + II.$$

For $I$, by $L^2_\omega(\mathbb{R}^n)$ boundedness of the operators $\mathcal{M}$ and $R_j$, we have

$$I \leq \omega(2Q)^{1/2}\|a\|_{L^2_\omega(\mathbb{R}^n)} \leq C.$$

We now estimate $II$. Let $x \notin 2Q$. For $t > 0$ we define the smooth functions

$$R^t_j = \varphi_t * K_j$$

and we observe that they satisfy

$$\sup_{0 < t < 1} |\partial^\beta K^t_j(x)| \leq C|x - y_0|^{-n-|\beta|} \chi_{\{x-y_0\leq 8\omega\}}(x)$$

for all $|\beta| \leq 1$; see their proof in page 507 of [7].

Now note that if $x \notin 2Q$ and and $y \in Q$, then $|x - y_0| \geq 2|y - y_0|$ stays away from $y_0$ and $K_j(x - y)$ is well defined. We have

$$R_j a * \varphi_t(x) = (a * K^t_j)(x) = \int_Q K^t_j(x - y)a(y)dy.$$
Using the cancellation of atoms we deduce
\[
R_j a \ast \varphi_t(x) = \int_Q K_j^t(x - y)a(y)dy \\
\quad = \int_Q \left[ K_j^t(x - y) - K_j^t(x - y_0) \right] a(y)dy \\
\quad = \int_Q \left[ \sum_{|\beta|=1} (\partial^\beta K_j^t(x - y_0 - \theta y - y_0))y^\beta \right] a(y)dy
\]
for some \(0 \leq \theta_y \leq 1\). Using that \(|x - y_0| \geq 2|y - y_0|\) and (3.6) we get
\[
R_j a \ast \varphi_t(x) \leq C|x - y_0|^{-n-1} \chi_{\{|x-y_0| \leq 8n\}}(x) \int_Q |a(y)||y|dy \leq C |x - y_0|^{n+1} \omega(Q)^{-1}\chi_{\{|x-y_0| \leq 8n\}}(x). \tag{3.7}
\]
By (3.7) and using properties of \(A_{loc}^1\), we obtain
\[
II \leq C \int_{2r \leq |x - y_0| \leq 8n} \frac{r^{n+1}}{|x - y_0|^{n+1}} \omega(Q)^{-1}\omega(x) dx \\
\quad \leq C \frac{|Q|}{\omega(Q)} \sum_{k=1}^{k_0} 2^{-k} \frac{\omega(2^k Q)}{|2^k Q|} \leq C,
\]
where \(k_0\) is an integer such that \(8n \leq 2^{k_0} \leq 16n\).

Thus, (3.5) holds. Hence, the proof is complete.

Next, we study weighted \(h^1_\omega(\mathbb{R}^n)\) boundedness for strongly singular integrals.

Given a real number \(\theta > 0\) and a smooth radial cut-off function \(v(x)\) supported in the ball \(\{x \in \mathbb{R}^n : |x| \leq 2\}\), we consider the strongly singular kernel
\[
k(x) = e^{\theta|x|} |x|^n v(x).
\]
Let us denote by \(Tf\) the corresponding strongly singular integral operator:
\[
Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x - y)f(y)dy.
\]
This operator has been studied by several authors, see [8], [12] and [4]. In particular, S. Chanillo [2] established the weighted \(L^p_\omega(\mathbb{R}^n)(\omega \in A_p, 1 < p < \infty)\) and \(H^1_\omega(\mathbb{R}^n)(\omega \in A_1)\) boundedness for strongly singular integrals. The author [10] proved the following results for the strongly singular integrals.

**Theorem C.** Let \(T\) be strongly singular integral operators, then

(i) \(\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L^p(\mathbb{R}^n)}\) for \(1 < p < \infty\) and \(\omega \in A_{loc}^1\).
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(ii) $\|Tf\|_{L^1_{\infty}(\mathbb{R}^n)} \leq C_\omega \|f\|_{L^1_\omega(\mathbb{R}^n)}$ for $\omega \in A_1^{loc}$.

(iii) $\|Tf\|_{L^1_\omega(\mathbb{R}^n)} \leq C_\omega \|f\|_{h_1^1(\mathbb{R}^n)}$ for $\omega \in A_1^{loc}$.

Theorem 1, Lemma 3.3 and (iii) in Theorem C imply immediately that

**Corollary 1.** Let $T$ be strongly singular integral operators, then

$$\|Tf\|_{h_1^1(\mathbb{R}^n)} \leq C_\omega \|f\|_{h_1^1(\mathbb{R}^n)}$$

for $\omega \in A_1^{loc}$.

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