THE EXACT MASS-GAPS OF THE PRINCIPAL CHIRAL MODELS

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ABSTRACT

An exact expression for the mass-gap, the ratio of the physical particle mass to the \( \Lambda \)-parameter, is found for the principal chiral sigma models associated to all the classical Lie algebras. The calculation is based on a comparison of the free-energy in the presence of a source coupling to a conserved charge of the theory computed in two ways: via the thermodynamic Bethe Ansatz from the exact scattering matrix and directly in perturbation theory. The calculation provides a non-trivial test of the form of the exact scattering matrix.

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1. Introduction

The principal chiral, or $G \times G$, sigma models are two-dimensional quantum field theories that are integrable at the quantum level. The fact that the theories are integrable means that their scattering matrices are factorizable. Such $S$-matrices have been conjectured for all the theories corresponding to the classical Lie algebras [1]. Expressions for the complete $S$-matrices for SU($N$) can be found in [1] and for Sp(2$N$) in [2]. Not all the $S$-matrix elements are known in the case of SO($N$). It is an outstanding problem to prove from first principles that the $S$-matrices actually do describe the lagrangian field theories. This is especially important because a given factorizable $S$-matrix is ambiguous since it may always by multiplied by CDD factors. Connecting the $S$-matrix picture with the lagrangian picture is highly non-trivial since the masses are generated dynamically and the theories are asymptotically free.

In a series of papers such non-trivial tests have been applied to various integrable models: the Gross-Neveu model [3], the O($N$) sigma model [4,5] and the SU($N$) principal chiral sigma model [6,7] (the SU(2) case was also considered in [8,9]), using a technique known as the Thermodynamic Bethe Ansatz (TBA). The central idea is to couple the theory to a particular conserved current and then compute the response of the free-energy for large values of the source in the regime when conventional perturbation theory is valid. The same quantity can then be computed directly from the $S$-matrix using the TBA equations at zero temperature in which the coupling to the source appears as a chemical potential. By comparing the two expressions a non-trivial test of the $S$-matrix is obtained as well as an exact expression for the mass-gap (the ratio of the physical mass to the $\Lambda$-parameter).

In general the solution of the TBA equations at zero temperature coupled to an arbitrary chemical potential would be a formidable problem, even in ultra-violet limit, since the equations are a set of coupled integral equations. However, by a judicious choice of the source the state of the system contains just one particle which undergoes elastic scattering (the particle is the highest weight state of a multiplet). The TBA equations then reduce to a single integral equation which can be solved in the ultra-violet limit using generalized Wiener-Hopf techniques [4,10] (for a summary see the appendix of [3]).

In this paper we extend the results of [6] to the principal chiral models for all the classical Lie algebras and arrive at a universal formula for the exact mass-gap. We also show that by tuning the source we can force the system into inequivalent ground-states which each consist of a single type of particle. The fact that the ground-states are pure for particular values of the source, is presented as a conjecture whose ultimate justification
comes from the agreement with perturbation theory; however, it should be possible to prove this fact directly from the full TBA equations of the models.

The principal chiral models are described by a lagrangian density

\[ \mathcal{L}_0 = -\frac{1}{\lambda^2} \text{Tr} \left( g^{-1} \partial_\mu g \cdot g^{-1} \partial^\mu g \right) , \tag{1.1} \]

where \( g \) is a group valued field. The theory is invariant under a global symmetry corresponding to left and right multiplication by the group \( g \mapsto h_L g h_R^{-1}, h_{L,R} \in G \). \( \lambda \) is a dimensionless coupling constant.

The S-matrices that have been conjectured to describe the scattering of the states of the model describe \( r = \text{rank}(G) \) particles which are associated to the fundamental representations of \( G \). The masses of the particles, except for those associated to the spinors of \( \text{SO}(N) \), can be described by the universal formula [1]

\[ m_a = m \frac{\sin(\pi a / g)}{\sin(\pi / g)} , \quad a = 1, 2, \ldots , \tag{1.2} \]

where \( g \) is the dual Coxeter number of the Lie algebra associated to the group: for \( A_r, B_r, C_r \) and \( D_r \) it is \( r + 1, 2r - 1, 2(r + 1) \) and \( 2(r - 1) \), respectively. The masses of the spinors of \( B_r \) and \( D_r \) are

\[ B_r : \quad m_r = \frac{m}{2 \sin(\pi / g)} , \quad D_r : \quad m_{r-1} = m_r = \frac{m}{2 \sin(\pi / g)} . \tag{1.3} \]

The particle with mass \( m_a \) transforms in the following representation of \( G \times G \) [1]:

\[ A_r : \quad W_a = V_a \otimes V_a , \quad a = 1, 2, \ldots , r , \]
\[ B_r : \quad W_a = \sum_{k=0}^{a-2k \geq 0} V_{a-2k} \otimes \sum_{j=0}^{a-2j \geq 0} V_{a-2j} , \quad a = 1, 2, \ldots , r - 1 , \]
\[ W_r = V_r \otimes V_r , \tag{1.4} \]
\[ C_r : \quad W_a = V_a \otimes V_a , \quad a = 1, 2, \ldots , r , \]
\[ D_r : \quad W_a = \sum_{k=0}^{a-2k \geq 0} V_{a-2k} \otimes \sum_{j=0}^{a-2j \geq 0} V_{a-2j} , \quad a = 1, 2, \ldots , r - 2 , \]
\[ W_{r-1} = V_{r-1} \otimes V_{r-1} , \quad W_r = V_r \otimes V_r , \]

where \( V_a \) is the \( a \)-th fundamental representation of \( G \) with the standard labelling of the Dynkin diagram [1]. Notice that although the particles are associated to the fundamental representations they are sometimes reducible in the case of \( \text{SO}(N) \).

Fortunately, we shall not require the expression for the complete S-matrices but only those elements for the particles of the highest weight in each multiplet (so with quantum
numbers $|\omega_a, \omega_a\rangle$ where the $\omega_a$’s are the fundamental weights). The $S$-matrix amongst these states is purely elastic and their expressions can be extracted from [1]:

$$S_{ab}(\theta) = \exp\left\{i\pi \delta_{ab} + 2i \int_0^\infty \frac{dx}{x} \sin(\theta x) [R_{ab}(x) - \delta_{ab}]\right\},$$

where $\theta$ is the rapidity difference of the incoming particles and the kernel $R_{ab}(\theta)$ has the following form for all the particles except the spinors:

$$A_r: R_{ab}(x) = \frac{2 \sinh\left(\frac{\min(a,b)}{r+1} \pi x\right) \sinh\left(\frac{r+1 - \max(a,b)}{r+1} \pi x\right)}{\sinh\left(\frac{1}{r+1} \pi x\right)},$$

$$B_r, C_r, D_r: R_{ab}(x) = \frac{2 \sinh\left(\frac{\min(a,b)}{g} \pi x\right) \cosh\left(\frac{g - 2 \max(a,b)}{2g} \pi x\right)}{\cosh\left(\frac{1}{2g} \pi x\right)}.$$

The $S$-matrix elements involving the spinors can also be deduced from the formulas of [1] but we shall not require them.

In the following two section we calculate the free-energy in the presence of a source coupling to the conserved charge of the $G \times G$ symmetry in two ways: from the lagrangian using perturbation theory and from the $S$-matrix using the thermodynamic Bethe Ansatz.

2. Free-energy in perturbation theory

The conserved currents of the left and right symmetry are $J^L_\mu = g^{-1} \partial_\mu g$ and $J^R_\mu = (\partial_\mu g) g^{-1}$. We wish to couple to modify the hamiltonian of the theory by introducing a coupling to the conserved charge of the diagonal action of the symmetry. At the lagrangian level this is described by introducing the “covariant derivative” [6]:

$$D_\mu g = \partial_\mu g - ih\delta_{\mu 0} (Qg + gQ),$$

where $Q$ is a constant element of the Lie algebra. The lagrangian density in the presence of the source is

$$\mathcal{L} = \mathcal{L}_0 - \frac{2hi}{\chi^2} \text{Tr} \left((g^{-1}Q + Qg^{-1})\partial_0 g\right) - \frac{2h^2}{\chi^2} \text{Tr} \left(Q^2 + g^{-1}QgQ\right).$$

The quantity we will calculate is $\delta f(h) = f(h) - f(0)$ where $f(h)$ is the free-energy per unit volume in the presence of the source. We shall perform a perturbative calculation in the running coupling $\lambda(h)$ which in the ultra-violet regime (large $h$) runs to zero and hence is the regime where perturbation theory will be reliable. We shall only perform the
computation to one loop; however this will be sufficient to provide a non-trivial check of
the S-matrix and allow for the evaluation of the exact mass-gap.

An explicit basis for $g$ is provided by

$$
g = \exp \left\{ i \sum_{\alpha} n^{(\alpha)} E_{\alpha} + in \cdot H \right\}, \quad (2.3)$$

where the fields satisfy the reality condition $n^{(\alpha)*} = n^{(-\alpha)}$ and $n^* = n$, and the sum is
over all the roots of algebra. In the above $E_{\alpha}$ is the usual step generator associated to
a root $\alpha$ and $H$ is the generator of the Cartan subalgebra. In what follows we choose a
normalization in which the roots of the simply-laced algebras have length-squared 2 and
the the long roots of $B_r$ and $C_r$ have length-squared 2 and 4, respectively.

Without loss of generality we take $Q$ to be in the Cartan subalgebra so $Q = q \cdot H$, where $q$ is some $r$-dimensional vector. The quadratic part of the (euclidean) lagrangian
density (2.2) is simply

$$
\mathcal{L} = -\frac{4h^2}{\lambda^2} q^2 + \frac{1}{\lambda^2} \sum_{\alpha > 0} \left\{ \partial_\mu n^{(\alpha)} \partial^\mu n^{(-\alpha)} + h^2 (\alpha \cdot q)^2 n^{(\alpha)} n^{(-\alpha)} \right\}, \quad (2.4)
$$

where the sum is over the positive roots and for simplicity we have changed the normalization of some of the $n^{(\alpha)}$’s. Notice that the Cartan subalgebra fields $n$ are completely
decoupled to this order in the loop expansion.

The tree level contribution to $\delta f(h)$ is simply

$$
\delta f(h)_0 = -\frac{4h^2}{\lambda^2} q^2. \quad (2.5)
$$

To evaluate the one-loop contribution we use dimensional regularization. Using standard
methods one finds

$$
\delta f(h)_1 = -\frac{h^2 g}{2\pi \epsilon} q^2 + \frac{h^2}{4\pi} \sum_{\alpha > 0} (\alpha \cdot q)^2 \left\{ 1 - \gamma_E + \ln 4\pi - \ln (h^2 (\alpha \cdot q)^2 / \mu^2) \right\} + \cdots, \quad (2.6)
$$

where $\epsilon = d - 2$, $\mu$ is the usual mass parameter of dimensional regularization and $g$ is the
dual Coxeter number as before. To cancel the divergence in the $\overline{\text{MS}}$-scheme we add to the
lagrangian a counter-term

$$
\delta \mathcal{L} = \frac{h^2 g}{2\pi \epsilon} q^2 + \frac{h^2}{4\pi} q^2 (\gamma_E - \ln 4\pi). \quad (2.7)
$$

The quantity $\delta f(h)$ is renormalization group invariant when $\lambda$ runs with $\mu$. We can use
this freedom to set $\mu = h$. The way that the coupling constant runs with $h$ is determined
from the form of the counter-term. One finds

$$
\frac{h}{\partial h} \lambda^2 = -\frac{g}{8\pi} \lambda^4 - \beta_1 \lambda^6 - \mathcal{O}(\lambda^8), \quad (2.8)
$$
although the second universal coefficient of the beta-function $\beta_1$ is not determined at the one-loop level. The expression for the first coefficient of the beta-function $\beta_0 = g/8\pi$ agrees with [11].

The expression for the free-energy is then

$$\delta f(h) = -\frac{4h^2}{\lambda^2(h)} q^2 - \frac{h^2}{4\pi} \sum_{\alpha > 0} (q \cdot \alpha)^2 \left[ \ln(q \cdot \alpha)^2 - 1 \right] + \mathcal{O}(\lambda^2),$$

(2.9)

where the explicit $h$ dependence is obtained by expressing the running coupling in terms of the $\Lambda$-parameter by solving (2.8):

$$\frac{1}{\lambda^2(h)} = \beta_0 \ln \frac{h}{\Lambda_{\text{MS}}} + \frac{\beta_1}{\beta_0} \ln \ln \frac{h}{\Lambda_{\text{MS}}} + \mathcal{O}\left(\frac{1}{\ln h / \Lambda_{\text{MS}}}\right),$$

(2.10)

where $\beta_0 = g/8\pi$. Equation (2.9) is the generalization to all the classical Lie algebras of equation (17) of [6] for $A_r$.

For comparing with the expression for the free-energy from the TBA calculation we set $q = \omega_a / (2\omega_a^2)$ (excluding the spinors of $SO(N)$). Writing

$$\delta f(h) = -\frac{h^2}{4} k_a^2 \left[ \ln \frac{h}{\Lambda_{\text{MS}}} + A_a + \frac{\beta_1}{\beta_0} \ln \ln \frac{h}{\Lambda_{\text{MS}}} + \mathcal{O}\left(\frac{1}{\ln h / \Lambda_{\text{MS}}}\right) \right].$$

(2.11)

By explicit computation we find for $A_r$ that

$$k_a^2 = \frac{(r+1)^2}{2\pi a(r+1-a)}, \quad A_a = \ln \left( \frac{r+1}{2a(r+1-a)} \right) - \frac{1}{2},$$

(2.12)

and for the other algebras a universal form applies:

$$k_a^2 = \frac{g}{2\pi a}, \quad A_a = -\ln a - \frac{1}{2} - \frac{d_1-2a}{g} \ln 2,$$

(2.13)

where the quantity $d_1$ is the dimension of the vector representation of the algebra, i.e. $r+1$, $2r+1$, $2r$ and $2r$ for $A_r$, $B_r$, $C_r$ and $D_r$, respectively.

3. Free-energy from the $S$-matrix

In this section we will calculate $\delta f(h)$ in the ultra-violet limit, $h \gg m$ directly from the $S$-matrix. The technique is known as the Thermodynamic Bethe Ansatz (TBA) and in its most general form it allows one to calculate the behaviour of the free-energy of a one-dimensional gas of particles described by a factorizable $S$-matrix on the temperature
and in the presence of a chemical potential. The free-energy is given in terms of a set of functions—in general infinite in number—which satisfy a set of coupled integral equations (the TBA equations).

For our application we working on the plane and hence at zero temperature. The coupling of the theory to the source in (2.2) leads to a particular form for the chemical potential. The one-particle states are labelled by two weight vectors \(|\mu, \nu\rangle\) (as well as the rapidity) and they can be chosen to be eigenstates of the charge \(Q\) with \(Q|\mu, \nu\rangle = q \cdot (\mu + \nu)|\mu, \nu\rangle\). The full TBA equations for the principal chiral models are known \([1]\); however, for particular choices of \(q\), extending the philosophy of \([3-6]\), we conjecture that only one particle contributes to the ground-state and the infinite set of TBA equations reduces to a single equation. The precise formulation of our conjecture is that when \(q = \omega_a/(2\omega_a^2)\) only the unique particle with the highest charge/mass ratio contributes to the ground-state, i.e. the particle \(|\omega_a, \omega_a\rangle\) which is highest weight state of the multiplet \(W_a\). This particle has \(Q\) eigenvalue 1. However, we exclude the the spinor particles of \(SO(N)\) from this conjecture. We shall find that this proposal leads to a result which is perfectly consistent with the perturbative calculation. Notwithstanding this, it should be possible to prove the conjecture directly from the full TBA equations.

The expression for the free-energy with \(q = \omega_a/(2\omega_a^2)\) is then given in terms of a quantity \(\epsilon(\theta)\) which satisfies the integral equation:

\[
\epsilon(\theta) - \int_{-B}^{B} d\theta' \phi_a(\theta - \theta')\epsilon(\theta') = m_a \cosh \theta - h. \tag{3.1}
\]

The parameter \(B\) is determined by the boundary condition \(\epsilon(\pm B) = 0\) and the kernel is given by

\[
\phi_a(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \ln S_{aa}(\theta) = \delta(\theta) - \int_0^\infty \frac{dx}{\pi} \cos(x\theta) R_{aa}(x), \tag{3.2}
\]

where \(S_{aa}(\theta)\) is the \(S\)-matrix element of the particle \(|\omega_a, \omega_a\rangle\) with itself (1.5). Once \(\epsilon(\theta)\) is known the expression for the free-energy per unit volume is

\[
\delta f(h) = \frac{m_a}{2\pi} \int_{-B}^{B} d\theta \epsilon(\theta) \cosh \theta. \tag{3.3}
\]

Our problem is to solve the integral equation (3.1). In general it is not possible to find the solution of such an equation in closed form; however, for comparing with the perturbative result we only need to compute the free-energy in the ultra-violet regime \(h \gg m\). In this limit a series solution can be found using generalized Wiener-Hopf techniques \([3,4,10]\). The first problem is to decompose the kernel \(R_{aa}(x)\):

\[
R_{aa}(x) = \frac{1}{G^{(a)}_+(x)G^{(a)}_-(x)}, \tag{3.4}
\]
where $G^{(a)}_{\pm}(x)$ are analytic in the upper/lower half-planes, respectively, and $G^{(a)}_{+}(x) = G^{(a)}_{-}(-x)$. The next step in the solution technique depends upon the form of $G^{(a)}_{+}(x)$. For all the principal chiral models

$$G^{(a)}_{+}(i\xi) = \frac{k'_a}{\sqrt{\xi}} \left\{ 1 - b_a \xi + O(\xi^2) \right\}.$$  

(3.5)

for constants $k'_a$ and $b_a$. So in this respect these models are of similar type to the $O(N)$ sigma model rather than the fermion models. With $G^{(a)}_{+}(i\xi)$ of the form (3.5), [6] gives a formula for the first few terms in the expansion of the free-energy

$$\delta f(h) = -\frac{h^2}{4} k'^2 a \left[ \ln \frac{h}{m_a} + \ln \left( \frac{\sqrt{2\pi k'_a} e^{-b_a}}{G^{(a)}_{+}(i)} \right) - 1 + \frac{1}{2} \ln \ln \frac{h}{m_a} + O \left( \frac{1}{\ln \frac{h}{M_{MS}}} \right) \right].$$  

(3.6)

The explicit expressions for the decompositions are for $A_r$

$$G^{(a)}_{+}(i\xi) = \sqrt{\frac{r + 1}{2\pi a(r + 1 - a)\xi}} \frac{\Gamma \left( 1 + \frac{a\xi}{r + 1} \right) \Gamma \left( 1 + \frac{r + 1 - a}{r + 1} \xi \right)}{\Gamma(1 + \xi)} \times \exp \left\{ -\xi \left( \frac{r + 1 - a}{r + 1} \ln \frac{r + 1 - a}{r + 1} + \frac{a}{r + 1} \ln \frac{a}{r + 1} \right) \right\}. \quad (3.7)$$

For the other algebra one finds the universal form

$$G^{(a)}_{+}(i\xi) = \sqrt{\frac{g}{2\pi a\xi}} \frac{\Gamma \left( 1 + \frac{a\xi}{g} \right) \Gamma \left( \frac{1}{2} + \frac{g - 2a}{2g} \xi \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{2}\xi \right)} \times \exp \left\{ -\xi \left( \frac{a}{g} \ln \frac{a}{g} + \frac{g - 2a}{2g} \ln \frac{g - 2a}{2g} - \frac{1}{2} \ln \frac{1}{2} \right) \right\}. \quad (3.8)$$

From these expressions we find that $k'_a$ equals $k_a$ in (2.12) and (2.13) and for $A_r$

$$b_a = \frac{r + 1 - a}{r + 1} \ln \frac{r + 1 - a}{r + 1} + \frac{a}{r + 1} \ln \frac{a}{r + 1}, \quad (3.9)$$

whilst for the other algebras

$$b_a = \frac{a}{g} \ln \frac{a}{g} + \frac{g - 2a}{2g} \ln \frac{g - 2a}{2g} - \frac{1}{2} \ln \frac{1}{2} - \frac{2a}{g} \ln 2. \quad (3.10)$$

Comparing the expression (3.6) with the result of the perturbative calculation (2.11) we see that they are in perfect agreement if $m_a \propto \sin(\pi a/g)$ which is true for all the
particles excluding the spinors of SO($N$) (1.2), and furthermore the expression for the mass-gap has a universal form:

\[
\frac{m}{\Lambda_{\text{MS}}} = \frac{g}{\sqrt{\pi e}} \exp \left\{ \left( \frac{2d_1 + g}{2g} \right) \ln 2 \right\} \sin \left( \frac{\pi}{g} \right),
\]

(3.11)

where $m$ is the mass of the vector particle, which is the lightest particle in the theory (since without loss of generality it is only necessary to consider $B_r$ for $r \geq 3$ and $D_r$ for $r \geq 4$).

In addition the $S$-matrix calculation implies that the universal ratio $\beta_1/\beta_0^2 = 1/2$ in exact agreement with the perturbative calculation of [11] a fact first pointed out for the SU($r + 1$) theories in [7,8]. The expression for the mass-gap (3.11) reduces to that of [6] for $A_r$. The explicit expressions for each group/algebra are

\[
\begin{align*}
\text{SU}(r + 1), A_r : & \quad \frac{m}{\Lambda_{\text{MS}}} = \frac{r + 1}{\sqrt{\pi e}} 2^{3/2} \sin \left( \frac{\pi}{r + 1} \right), \\
\text{SO}(2r + 1), B_r : & \quad \frac{m}{\Lambda_{\text{MS}}} = \frac{2r - 1}{\sqrt{\pi e}} 2^{(6r+1)/(4r-2)} \sin \left( \frac{\pi}{2r - 1} \right), \\
\text{Sp}(2r), C_r : & \quad \frac{m}{\Lambda_{\text{MS}}} = \frac{2r + 2}{\sqrt{\pi e}} 2^{(3r+1)/(2r+2)} \sin \left( \frac{\pi}{2r + 2} \right), \\
\text{SO}(2r), D_r : & \quad \frac{m}{\Lambda_{\text{MS}}} = \frac{2r - 2}{\sqrt{\pi e}} 2^{(3r-1)/(2r-2)} \sin \left( \frac{\pi}{2r - 2} \right).
\end{align*}
\]

(3.12)

The fact that the perturbative result and the $S$-matrix result are consistent provides strong grounds for believing that our conjecture about the structure of the ground-states is correct. As has been pointed out in [6], the fact that the TBA calculation reproduces the universal part of the beta-function $\beta_1/\beta_0^2$ is a highly non-trivial fact. In addition if the $S$-matrix were modified with CDD factors then the thermodynamics would be drastically altered and the perfect agreement with the perturbative result would be destroyed. It would be interesting to compare these results with lattice simulations.

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