LOW REGULARITY CAUCHY PROBLEM FOR THE FIFTH-ORDER MODIFIED KDV EQUATIONS ON $T$

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Abstract. In this paper, we consider the fifth-order equation in the modified Korteweg-de Vries (modified KdV) hierarchy as following:

$$\begin{cases}
\partial_t u - \partial_x^5 u + 40u\partial_x u \partial_x^2 u + 10u^2\partial_x^4 u + 10(\partial_x u)^3 - 30u^4\partial_x u = 0, & (t, x) \in \mathbb{R} \times T, \\
u(0, x) = u_0(x) \in H^s(T)
\end{cases}$$

We prove the local well-posedness of the fifth-order modified KdV equation for low regularity Sobolev initial data via the energy method. We use some conservation laws of the modified KdV Hamiltonians (or a partial property of complete integrability) to absorb some linear-like resonant terms into the linear propagator. Also, we use the nonlinear transformation, which has the bi-continuity property, to treat the rest of linear-like resonant terms. Besides, it is essential to use the short time $X^{s,b}$ spaces to control the nonlinear terms due to high $\times$ low $\times$ low $\Rightarrow$ high interaction component in the non-resonant nonlinear term. We also use the localized version of the modified energy in order to control the high-low interaction component in the original energy. As a purpose and consequence of this work, we emphasize that under the periodic setting, to study the low regularity well-posedness problem somewhat relies on the theory of complete integrability. This is the first low regularity well-posedness result for the fifth order modified KdV equation under the periodic setting.

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1. Introduction

It is known that the eigenvalues of the time independent Schrödinger operator

$$L(t) = \frac{d^2}{dx^2} - u(t, x)$$

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are not changed when the potential \( u(t, x) \) evolves according to the Korteweg-de Vries (KdV) equation

\[ u_t + u_{xxx} = 6 uu_x. \]

This property was first discovered by Gardner, Greene, Kruskal, and Miura [5]. Afterward, Lax [15] generalized this property and discovered the family of equations (known by the name KdV hierarchy). This observation opened the theory of complete integrable systems of PDEs, and the KdV equation and its hierarchy hold a central example among other integrable systems. Lax also represented those equations by the form of (known as the Lax pair formulation)

\[ \partial_t L = A_n L - LA_n = [A_n; L]. \]

Here \( L \) is defined as above, and \( A_n \) is the differential operator in the form of

\[
A_n = 4^n \partial_x^{2n+1} + \sum_{j=1}^{n} \{ a_{nj} \partial_x^{2j-1} + \partial_x^{2j-1} a_{nj} \}, \quad n = 0, 1, \ldots,
\]

where \( A_0 = \partial_x \) and the coefficient \( a_{nj} = a_{nj}(u) \) be chosen such that the operator \([A_n; L]\) has order zero.

Thereafter, Magri [16] realized that the integrable Hamiltonian systems have an additional structure, so-called the bi-Hamiltonian structure, and then further studies on the theory of complete integrability have been widely progressed by several researchers (see, for instance, [1] and references therein).

On the other hand, the Miura transformation [17]

\[ u = v_x + v^2 \]

converts the equations in the KdV hierarchy into those in the modified KdV hierarchy. Like the equations in the KdV hierarchy, those in the modified KdV hierarchy have the property of the complete integrability. Recently, Choudhuri, Talukdar, and Das [4] derived the Lax representation and constructed the bi-Hamiltonian structure of the equations in the modified KdV hierarchy. The followings are a few equations and their associated Hamiltonians with respect to bi-Hamiltonian structures:

\[
\begin{align*}
\partial_t u - \partial_x u &= 0 \\
\partial_t u - \partial_x^3 u + 6u^2 \partial_x u &= 0 \\
\partial_t u - \partial_x^5 u + 40u \partial_x u \partial_x^2 u + 10u^2 \partial_x^3 u + 10(\partial_x u)^3 - 30u^4 \partial_x u &= 0 \\
&\vdots
\end{align*}
\]

(1.1)

In this paper, we consider the integrable fifth-order equation in the modified KdV hierarchy:

\[
\begin{cases}
\partial_t u - \partial_x^5 u + 40u \partial_x u \partial_x^2 u + 10u^2 \partial_x^3 u + 10(\partial_x u)^3 - 30u^4 \partial_x u = 0, & (t, x) \in \mathbb{R} \times \mathbb{T}, \\
u(0, x) = u_0(x) \in H^s(\mathbb{T}),
\end{cases}
\]

(1.2)

\( \mathbb{T} = [0, 2\pi] \).

Due to the theory of the complete integrability (or inverse spectral method), for any Schwartz initial data, the global solution exists to any equation in the modified KdV hierarchy. Especially, for the integrable KdV equation under periodic boundary condition, Kappeler and Topalov [9] proved the global well-posedness of solution in \( H^s, s \geq -1 \). It is a fine achievement to solve the low regularity well-posedness problem via only the theory of complete integrability. Nevertheless, it is still required the analytic theory of nonlinear dispersive equations to study on the low regularity well-posedness problem for the integrable equations in the hierarchy. In fact, in a number of previous studies on the low regularity well-posedness problem for nonlinear dispersive equations (especially, under the non-periodic setting), the
integrable structures were ignored. The purpose of this work is to partly use the property of the complete integrability in order to solve the low regularity well-posedness problem.

Generalizing coefficients in the nonlinear terms may break the integrable structure. The following equation generalizes (1.2) to non-integrable case:

\[
\begin{align*}
\partial_t u - \partial_x^5 u + a_1u \partial_x u \partial_x^3 u + a_2 u^2 \partial_x^2 u + a_3(\partial_x u)^3 - a_4 u^4 \partial_x u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \\
u(0, x) &= u_0(x) \in H^s(\mathbb{T}),
\end{align*}
\]  

(1.3)

where \(a_i's, i = 1, 2, 3, 4\), are real constants. For studying (1.3), we can rely no longer on the property of the complete integrability.

Meanwhile, one can observe some resonant terms such as

\[
\|u(t)\|_L^2, \quad \|u(t)\|_{H^1}, \quad \|u(t)\|_L^4, \quad \|u(t)\|_{H^1, \partial_x u},
\]

which are appeared in the nonlinear terms of both (1.2) and (1.3). We call those terms the linear-like resonant terms. Since (1.2) enjoys the Hamiltonian conservation laws in (1.1), most of those terms change into

\[
(constant)_1 \cdot \partial_x^3 u + (constant)_2 \cdot \partial_x u,
\]

and hence the linear part of equations can be expressed as

\[
[\partial_x^5 + (constant)_1 \cdot \partial_x^3 + (constant)_2 \cdot \partial_x] u.
\]

That is the reason why we focus on not (1.3) but (1.2), and thus we again emphasize that this work partly relies on the integrability.

The following is the main result in this paper:

**Theorem 1.1.** Let \(s > 2\). For any \(u_0 \in H^s(\mathbb{T})\) specified

\[
\int_T (u_0(x))^2 \, dx = \gamma_1, \quad \int_T (\partial_x u_0(x))^2 + (u_0(x))^4 \, dx = \gamma_2
\]

(1.4)

for some \(\gamma, \gamma_2 \geq 0\), there exists \(T = T(\|u_0\|_{H^s}) > 0\) such that (1.2) has a unique solution on \([-T, T]\) satisfying

\[
u(t, x) \in C([-T, T]; H^s(\mathbb{T})),
\]

\[
\eta(t) \sum_{n \in \mathbb{Z}} e^{i(nx - 2\pi n)} \int_0^T \|u(s)\|_{L^4}^4 \, ds \hat{u}(t, n) \in C([-T, T]; H^s(\mathbb{T})) \cap F^s(T),
\]

(1.5)

where \(\eta\) is any cut-off function in \(C^\infty(\mathbb{R})\) with supp\(\eta \subset [-T, T]\) and the space \(F^s(T)\) will be defined later. Moreover, the flow map \(S_T : H^s \to C([-T, T]; H^s(\mathbb{T}))\) is continuous on the level set in \(H^s\) satisfying (1.3).

Even though there are infinitely many conservation laws associated to (1.2), one of linear-like resonant components (such as \(\|u\|_{L^4}^4, \partial_x u\) term) in (1.2) still remains. However, as we see the result (1.5) in Theorem 1.1, this type of resonant component can be treated by using the appropriate nonlinear transformation, which is defined as in (2.3). This nonlinear transformation has a good property that the transformation is bi-continuous from the ball in \(C([-T, T]; H^s)\) to itself for \(s \geq \frac{1}{2}\). Due to this good property of the nonlinear transformation, only \(L^2\)-conservation law is enough to study (1.2). However, we guess that the higher-order equations in the hierarchy shall depend on a substantial portion of the theory of complete integrability. From this observation, we can get the following corollary as a partial result for the non-integrable equation (1.3):

\[\text{This space also depends on the initial data } u_0 \text{ with specified}
\]

\[
\int_T (u_0(x))^2 \, dx = \gamma_1, \quad \int_T (\partial_x u_0(x))^2 + (u_0(x))^4 \, dx = \gamma_2
\]

for some \(\gamma, \gamma_2 \geq 0\).
Corollary 1.2. Let \( s > 2 \). We assume that \( a_i, i = 1, 2, 3 \) as in (1.3) satisfy
\[
\int_T \left[ a_1 u \partial_x u \partial_x^2 u + a_2 u^2 \partial_x^3 u + a_3 (\partial_x u)^3 \right] u \, dx = 0.
\]
Then, (1.3) is locally well-posed in \( H^s(T) \).  

In order to prove Corollary 1.2, we first need the high regularity well-posedness result for (1.3). To obtain that, we basically follows the Ponce’s idea [18]. Precisely, we first consider the \( \varepsilon \)-parabolic equation. Then for a smooth solution to the parabolic equation, we show the \textit{a-priori} bound for the solution \( u \). Afterward, with \textit{a-priori} bound and bootstrap argument, we use the approximation method of equation to show that the solution of \( \varepsilon \)-parabolic equation converges to the solution of the fifth-order modified KdV equation. We also use Bona-Smith argument to obtain the sufficiently smooth solution of the fifth-order modified KdV equation. Then, we immediately complete the proof by using the similar way in the proof of Theorem 1.1. The main difficulty is to obtain the energy estimate for both the parabolic and fifth-order modified KdV equations. But, we use the modified energy method, which is introduced by Kwon [13] for the fifth-order KdV equation on \( \mathbb{R} \), and the Sobolev embedding to obtain the energy bound of the solution \( u \). We sketch the proof of the well-posedness for high regularity in Appendix A for the convenience of readers.

The fifth-order modified KdV equation under the non-periodic setting has been studied by Kwon [14]. Kwon used the standard \( X^{s, b} \) space (Fourier restrict norm method) and Tao’s \([k, Z]\)-multiplier norm method [19] to prove trilinear estimate, and hence obtained the local well-posedness in \( H^s \) for \( s \geq \frac{3}{4} \) via the contraction mapping principle. In contrast with the non-periodic setting, the trilinear estimate in the \( X^{s, b} \) space \( \| uv \partial_x^3 w \|_{X^{s-b-1}} \leq C \| u \|_{X^{s, b}} \| v \|_{X^{s, b}} \| w \|_{X^{s, b}} \) fails for all \( s \) and \( b \in \mathbb{R} \) under the periodic boundary condition. As a minor result in this paper, we have

**Theorem 1.3.** For any \( s, b \in \mathbb{R} \), the trilinear estimate \[
\| uv \partial_x^3 w \|_{X^{s-b-1}} \leq C \| u \|_{X^{s, b}} \| v \|_{X^{s, b}} \| w \|_{X^{s, b}}
\]
fails.

The counter-example involves in \textit{high} \( \times \) \textit{low} \( \times \) \textit{low} \( \Rightarrow \) \textit{high} interaction component along the non-resonant phenomenon of the following type: \( (P_{\text{low}} u) \cdot (P_{\text{low}} v) \cdot (P_{\text{high}} w_{xxx}) \).

Even though sufficient large resonance appears in this interactions, due to the much more derivatives in the high frequency mode and the lack of dispersive smoothing effect, one cannot control this component in \( X^{s, b} \)-norm. The detailed example will be given in section 3 later. This is remarkable the fifth-order (of course, much higher-order) modified KdV equation is strictly worse behaved in the periodic setting than in the non-periodic setting due to the lack of dispersive smoothing effect.

So far, we observe the principal enemies to study the fifth-order modified KdV equation: \textit{linear-like} resonant terms and the lack of dispersive smoothing effect. As mentioned before, we can defeat the first enemy by using the theory of complete integrability and the nonlinear transformation. From this, the original linear propagator of (1.2) slightly changes, and with this, we use the modified \( X^{s, b} \) structure for a short time interval \( \approx (\text{frequency})^{-2} \) to get over the second difficulty. This type of short time structure was first developed by Ionescu, Kenig and Tataru [8] in the context of KP-I equation.

\[ \int_T u_0^2(x) \, dx = \gamma, \]
for some \( \gamma > 0 \).  

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2Similarly as Theorem 1.1 local well-posedness result depends on the initial data in the level set satisfying
On the other hand, as a by-product of short time structure, we need the energy-type estimates. But, similarly as in [7] and [12] in the context of the fifth-order KdV equation on $\mathbb{R}$, what much more derivatives are in the high frequency mode is the obstacle to closing the energy estimate by using Ionescu-Kenig-Tataru’s method. It turns out that we have to use the modified energy method on each frequency piece. Nevertheless, exact quintic resonant terms still preclude from obtaining the energy estimate. Fortunately, after using symmetries among frequencies and functions, we are able to control all quintic resonant components in the modified energy (see Remark 6.8 and 6.13 in Section 6).

There are another works on the low regularity well-posedness problem of similar nonlinear types of the fifth-order dispersive equations. For instance, the fifth-order KdV equation on $\mathbb{R}$, which has the nonlinearity of the form of $c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u$, was first studied by Ponce [18] as the low regularity well-posedness problem. Since the nonlinearity is the stronger than the advantage from the dispersive smoothing effect, it is required the energy method to prove the local well-posedness. Ponce used the energy method to prove the local well-posedness for Sobolev initial data $u_0 \in H^s$, $s \geq 4$, and afterward, Kwon [13] improved Ponce’s result for $s > \frac{5}{2}$. Kwon developed the modified energy method with the refined Strichartz estimate, Maximal function estimate, and local smoothing estimate. Recently, Guo, Kwon and the author [7], and Kenig and Pilod [12] further improved the local result, independently. The method in both [7] and [12] is the also energy method based on the short time $X^{s,b}$ space while, the key energy estimates were shown by using additional weight and modified energy, respectively.

The paper is organized as follows: In Section 2, we summarize some notations and define function spaces. In Section 3, we prove Theorem 1.3 by giving a counter example. In Section 4, we show the $L^2$ block trilinear estimates which are useful to obtain nonlinear and energy estimates. In Section 5 and 6, we prove the nonlinear estimate and energy estimate, respectively. In Section 7, we give the proof of Theorem 1.1. Finally, we sketch the proof of high regularity result for (1.3) in Appendix A for the convenience of readers.

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2. Preliminaries

For $x, y \in \mathbb{R}_+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leq C y$, and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. We also use $\lesssim_s$ and $\sim_s$ as similarly, where the implicit constants depend on $s$. Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The quantities $a_{\max} \geq a_{\sub} \geq a_{\thd} \geq a_{\min}$ can be conveniently defined to be the maximum, sub-maximum, third-maximum and minimum values of $a_1, a_2, a_3, a_4$ respectively.

For $Z = \mathbb{R}$ or $\mathbb{Z}$, let $\Gamma_k(Z)$ denote $(k-1)$-dimensional hyperplane by

$$\{\tau = (x_1, x_2, \ldots, x_k) \in Z^k : x_1 + x_2 + \cdots + x_k = 0\}.$$

For $f \in \mathcal{S}^\prime(\mathbb{R} \times T)$ we denote by $\tilde{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ with respect to both spatial and time variables,

$$\tilde{f}(\tau, n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_0^{2\pi} e^{-ixn} e^{-it\tau} f(t, x) \, dx \, dt.$$

Moreover, we use $\mathcal{F}_x$ (or $\check{}$) and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variable respectively.
This paper is based on the following observation. We take the Fourier coefficient in the spatial variable of \((1.2)\) to obtain
\[
\partial_t \hat{u}(n) - i n^5 \hat{u}(n) = 30i \sum_{n_1+n_2+n_3+n_4+n_5=n} \hat{u}(n_1)\hat{u}(n_2)\hat{u}(n_3)\hat{u}(n_4)n_5\hat{u}(n_5) \\
+ 10i \sum_{n_1+n_2+n_3=n} \hat{u}(n_1)\hat{u}(n_2)n_3^3\hat{u}(n_3) \\
+ 10i \sum_{n_1+n_2+n_3=n} n_1\hat{u}(n_1)n_2\hat{u}(n_2)n_3\hat{u}(n_3) \\
+ 40i \sum_{n_1+n_2+n_3=n} \hat{u}(n_1)n_2\hat{u}(n_2)n_3^3\hat{u}(n_3)
\]  
\[(2.1)\]

We consider the resonant relation for the cubic term in the right-hand side of \((2.1)\)
\[
H = H(n_1, n_2, n_3)
:= (n_1 + n_2 + n_3)^5 - n_1^5 - n_2^5 - n_3^5 = \frac{5}{2}(n_1 + n_2)(n_1 + n_2)(n_2 + n_3)(n_1^2 + n_2^2 + n_3^2 + n^2).
\]

Then we can observe that the resonant phenomenon appears only when \((n_1 + n_2)(n_1 + n_2)(n_2 + n_3) = 0\). Moreover, we also observe that \[\|u\|_{L^4_x}^4 n\hat{u}(n)\] portion be contained in the resonant term of quintic term when \(n_i + n_j + n_k + n_l = 0\), for \(1 \leq i < j < k < l \leq 5\). Then, by using the conservation laws in \((1.1)\) and gathering resonant terms in right-hand side of \((2.1)\), we can rewrite \((2.1)\) as following:\footnote{By simple calculation \[40u\partial_x u\partial^3_x u + 10u^2\partial^3_x u + 10(\partial_x u)^3 = 100u_x(u^2\partial^3_x u) + 10\partial_x(u_x u\partial_x u) \quad \text{and} \quad 30u^4\partial_x u = 6\partial_x(u^5)\] we can change all nonlinear terms into the divergence form.}
\[
\partial_t \hat{u}(n) - i(n^5 + c_1 n^3 + c_2 n + c_3\|u(t)\|_{L^4_x}^4)\hat{u}(n) = 20i n^3\|\hat{u}(n)\|^2\hat{u}(n)
+ 6in \sum_{N_{5,n}} \hat{u}(n_1)\hat{u}(n_2)\hat{u}(n_3)\hat{u}(n_4)\hat{u}(n_5)
+ 10in \sum_{N_{5,n}} n_1\hat{u}(n_1)\hat{u}(n_2)n_3^2\hat{u}(n_3)
+ 10in \sum_{N_{5,n}} \hat{u}(n_1)\hat{u}(n_2)n_3\hat{u}(n_3),
\]  
\[(2.2)\]

where \(c_1 = 10\|u_0\|_{L^2}^2, c_2 = 10(\|u_0\|_{H^1}^2 + \|u_0\|_{L^4_x}^4), c_3 = 20,\)
\[
N_{3,n} = \{ (n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = n \quad \text{and} \quad (n_1 + n_2)(n_1 + n_2)(n_2 + n_3) \neq 0 \}
\]  
\[(2.3)\]

and
\[
N_{5,n} = \left\{ (n_1, n_2, n_3, n_4, n_5) \in \mathbb{Z}^5 : n_1 + n_2 + n_3 + n_4 + n_5 = n, \quad n_1 + n_2 + n_k + n_l \neq 0, 1 \leq i < j < k < l \leq 5 \right\}
\]  
\[(2.4)\]

We call the first term of the right-hand side of \((2.2)\) the Resonant term and the others Non-resonant term. Due to the term \(c_3\|u(t)\|_{L^4_x}^4\hat{u}(n)\) in the left-hand side of \((2.2)\), \(\partial_x^5 + c_1\partial_x^3 + c_2\partial_x + c_3\|u(t)\|_{L^4_x}^4\partial_x\) does not play a role of the linear operator to \((2.2)\) yet, even we use the partial property of completely integrable system. Hence for our analysis, we define the nonlinear transformation as
\[
\mathcal{N}T(u)(t, x) = v(t, x) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{in(x-20m \int_0^t \|u(s)\|_{L^4_x}^4 ds)} \hat{u}(n, t, n).
\]  
\[(2.5)\]
We from now concentrate on the $v$ instead of $u$. For $v$, we rewrite again (2.2) as
\[ \partial_t \hat{v}(n) - i(n^5 + c_1 n^3 + c_2 n) \hat{v}(n) = -20in^3 |\hat{v}(n)|^2 \hat{v}(n) + 6in \sum_{N_5,n} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n_4) \hat{v}(n_5) + 10in \sum_{N_5,n} \hat{v}(n_1) \hat{v}(n_2) n_2^2 \hat{v}(n_4) + 10in \sum_{N_5,n} \hat{v}(n_1) n_2 \hat{v}(n_2) n_3 \hat{v}(n_3) =: \hat{N}_1(v) + \hat{N}_2(v) + \hat{N}_3(v) + \hat{N}_4(v) \]

$v(0, x) = u(0, x)$. We simply generalize $N_i(v)$ as $N_i(u, v, w)$ for the cubic term, $i = 1, 3, 4$, or $N_2(v_1, v_2, v_3, v_4, v_5)$ for the quintic term.

We introduce that $X^{s,b}$-norm associated to Eq. (2.6) which is given by
\[ \|u\|_{X^{s,b}} = \|\tau - \mu(n)\|^{b} \|\nu\|^{b} \|\mathcal{F}(u)\|_{L_{t}^{2}(\mathbb{R};L_{x}^{2}(\mathbb{R}))} \]
where
\[ \mu(n) = n^5 + c_1 n^3 + c_2 n \] (2.7)
and $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. The $X^{s,b}$ space turns out to be very useful in the study of low-regularity theory for the dispersive equations. The restricted norm method was first implemented in its current form by Bourgain [3] and further developed by Kenig, Ponce and Vega [11] and Tao [19].

Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty]$. For $k \in \mathbb{Z}_+$, we set
\[ I_0 = \{ n \in \mathbb{Z} : |n| \leq 2 \} \quad \text{and} \quad I_k = \{ n \in \mathbb{Z} : 2^{k-1} \leq |n| \leq 2^{k+1} \}, \quad k \geq 1. \]

Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote a smooth bump function supported in $[-2, 2]$ and equal to 1 in $[-1, 1]$ with the following property of regularities:
\[ \partial^j_n \eta_0(n) = O(\eta_0(n)/\langle n \rangle^j), \quad j = 0, 1, 2. \] (2.8)

For $k \in \mathbb{Z}_+$, let
\[ \chi_0(n) = \eta_0(n), \quad \text{and} \quad \chi_k(n) = \eta_0(n/2^k) - \eta_0(n/2^{k-1}), \quad k \geq 1, \] (2.9)
which is supported in $I_k$, and
\[ \chi_{[k_1, k_2]} = \sum_{k = k_1}^{k_2} \chi_k \quad \text{for any } k_1 \leq k_2 \in \mathbb{Z}_+. \]

\{\chi_k\}_{k \in \mathbb{Z}_+} is the inhomogeneous decomposition function sequence to the frequency space. For $k \in \mathbb{Z}_+$ let $P_k$ denote the operators on $L^2(\mathbb{T})$ defined by $P_k v(n) = \chi_k(n) \hat{v}(n)$. For $l \in \mathbb{Z}_+$ let
\[ P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k. \]

For the time-frequency decomposition, we use the cut-off function $\eta_j$, but the same as $\eta_j(\tau - \mu(n)) = \chi_j(\tau - \mu(n))$.

For $k, j \in \mathbb{Z}_+$ let
\[ D_{k,j} = \{(\tau, n) \in \mathbb{R} \times \mathbb{Z} : \tau - \mu(n) \in I_j, n \in I_k\}, \quad D_{k,\leq j} = \cup_{l \leq j} D_{k,l}. \]
For $k \in \mathbb{Z}_+$, we define the $X^{s,\frac{1}{2}}$-type space $X_k$ for frequency localized functions,

$$X_k = \left\{ f \in L^2(\mathbb{R} \times I_k) : f(\tau, n) \text{ is supported in } \mathbb{R} \times I_k \text{ and } \|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2}\|\eta_j(\tau - \mu(n)) \cdot f(\tau, n)\|_{L^2(\mathbb{R} \times I_k)} < \infty \right\}.$$ 

As in [8], at frequency $2^k$ we will use the $X^{s,\frac{1}{2}}$ structure given by the $X_k$-norm, uniformly on the $2^{-2k}$ time scale. For $k \in \mathbb{Z}_+$, we define function spaces

$$F_k = \left\{ f \in L^2(\mathbb{R} \times I_k) : f(\tau, n) \text{ is supported in } \mathbb{R} \times I_k \text{ and } \|f\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[f \cdot \eta_0(2^{2k}(t-t_k))]\|_{X_k} < \infty \right\},$$

$$N_k = \left\{ f \in L^2(\mathbb{R} \times I_k) : f(\tau, n) \text{ is supported in } \mathbb{R} \times I_k \text{ and } \|f\|_{N_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[f \cdot \eta_0(2^{2k}(t-t_k))]\|_{X_k} < \infty \right\}.$$

Since the spaces $F_k$ and $N_k$ are defined on the whole line in time variable, we define then local-in-time versions of the spaces in standard ways. For $T \in (0, 1]$ we define the normed spaces

$$F_k(T) = \left\{ f \in C([-T, T] : L^2) : \|f\|_{F_k(T)} = \inf_{\tilde{f} \in F_k} \|\tilde{f}\|_{F_k} \right\},$$

$$N_k(T) = \left\{ f \in C([-T, T] : L^2) : \|f\|_{N_k(T)} = \inf_{\tilde{f} \in N_k} \|\tilde{f}\|_{N_k} \right\}.$$

We assemble these dyadic spaces in a Littlewood-Paley manner. For $s \geq 0$ and $T \in (0, 1]$, we define function spaces solutions and nonlinear terms:

$$F^s(T) = \left\{ u : \|u\|_{F^s(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk}\|P_k(u)\|_{F_k(T)}^2 < \infty \right\},$$

$$N^s(T) = \left\{ u : \|u\|_{N^s(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk}\|P_k(u)\|_{N_k(T)}^2 < \infty \right\}.$$

We define the dyadic energy space as follows: For $s \geq 0$ and $u \in C([-T, T] : H^\infty)$

$$\|u\|_{E^s(T)}^2 = \|P_0(u(0))\|_{L^2}^2 + \sum_{k \geq 1} \sup_{t_k \in [-T, T]} 2^{2sk}\|P_k(u(t_k))\|_{L^2}^2.$$ 

**Lemma 2.1** (Properties of $X_k$). Let $k, l \in \mathbb{Z}_+$ and $f_k \in X_k$. Then

$$\sum_{j=+1}^{\infty} 2^{j/2}\beta_{k,j} \left\| \eta_j(\tau - \mu(n)) \int_{\mathbb{R}} |f_k(\tau', n)| 2^{-l}(1+2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2(\mathbb{R} \times I_k)} + 2^{l/2} \left\| \eta_{l}(\tau - \mu(n)) \int_{\mathbb{R}} |f_k(\tau', n)| 2^{-l}(1+2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2(\mathbb{R} \times I_k)} \lesssim \|f_k\|_{X_k}. \tag{2.10}$$

In particular, if $t_0 \in \mathbb{R}$ and $\gamma \in \mathcal{S}(\mathbb{R})$, then

$$\|\mathcal{F}[\gamma(2^l(t-t_0)) \cdot f_k]\|_{X_k} \lesssim \|f_k\|_{X_k}. \tag{2.11}$$

Moreover, from the definition of $X_k$-norm,

$$\left\| \int_{\mathbb{R}} |f_k(\tau', n)| d\tau' \right\|_{L^2} \lesssim \|f_k\|_{X_k}.$$

**Proof.** The proof of Lemma 2.1 only depends on the summation over modulations, and there is no difference between the proof in the non-periodic and periodic settings. Hence we omit details and see [7]. □
Remark 2.2. To prove Corollary 1.2, we can also define function spaces $\tilde{X}_k$, $\tilde{F}_k$, $\tilde{N}_k$, $\tilde{F}^s$ and $\tilde{N}^s$ by using

$$\tilde{\mu}(n) = n^5 + c_1 n^3$$

instead of (2.7).

As in [8], for any $k \in \mathbb{Z}_+$ we define the set $S_k$ of $k$-acceptable time multiplication factors

$$S_k = \{ m_k : \mathbb{R} \to \mathbb{R} : \| m_k \|_{S_k} = \sum_{j=0}^{10} 2^{-2j} \| \partial^j m_k \|_{L^\infty} < \infty \}.$$ 

Direct estimates using the definitions and (2.11) show that for any $s \geq 0$ and $T \in (0,1]$

$$\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot F_k(u) \right\|_{\tilde{F}^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k} \right) \| u \|_{\tilde{F}^s(T)};$$

$$\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot F^s_k(u) \right\|_{\tilde{N}^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k} \right) \| u \|_{\tilde{N}^s(T)};$$

$$\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot F^s_k(u) \right\|_{\tilde{E}^s(T)} \lesssim \left( \sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k} \right) \| u \|_{\tilde{E}^s(T)}.$$ 

3. Proof of Theorem 1.3

In this section, we show the Theorem 1.3. The proof basically follows from the section 6 in [11] associated to the modified KdV equation. As mentioned in the introduction, we observe the high $\times$ low $\times$ low $\Rightarrow$ high interaction component in the non-resonance phenomenon, while, Kenig, Ponce, and Vega focused on the high $\times$ high $\times$ high $\Rightarrow$ high interaction component in the resonant term. Actually, our examples of the modified KdV equation can be easily controlled in $X^{s,b}$, because the size of maximum modulation is comparable to the square of high frequency size ($\approx N^2$) and hence this factor exactly eliminates the one derivative in the nonlinear term. In contrast to this, (1.2) has two more derivatives in nonlinear terms, and thus, one cannot control this component in $X^{s,b}$-norm, although the advantage of the non-resonant effect is better than that of modified KdV equation. Now, we give examples satisfying

$$\| u \partial_x^3 w \|_{X^{s,b-1}} \lesssim C \| u \|_{X^{s,b}} \| v \|_{X^{s,b}} \| w \|_{X^{s,b}}. \quad (3.1)$$ 

In the case of our examples, the trilinear estimate does not depend on the regularity $s$. So, it suffices to show (3.1) for any $b \in \mathbb{R}$. Fix $N > 1$. We first consider when $b > \frac{1}{2}$. Let us define the functions

$$f(\tau, n) = a_n \chi_4(\tau - n^5), \quad g(\tau, n) = b_n \chi_4(\tau - n^5), \quad h(\tau, n) = d_n \chi_4(\tau - n^5),$$

where

$$a_n = \begin{cases} 1, & n = -1 \\ 0, & \text{otherwise} \end{cases}, \quad b_n = \begin{cases} 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}, \quad d_n = \begin{cases} 1, & n = N - 1 \\ 0, & \text{otherwise} \end{cases}. $$

We focus on the case that $|\tau - n^5|$ is the maximum modulation. We put

$$\tilde{u}(\tau, n) = f(\tau, n) \quad \tilde{v}(\tau, n) = g(\tau, n) \quad \tilde{w}(\tau, n) = h(\tau, n),$$

then we need to calculate $\mathcal{F}[u \partial_x^3 w](\tau, n)$. Since $\mathcal{F}[u \partial_x^3 w](\tau, n) = (f * g * h)(\tau, n)$, performing the summation and integration with respect to $n_1, \tau_1$ variables gives

$$(f * g)(\tau_2, n_2) = \sum_{n_1} a_{n_1} b_{n_2 - n_1} \int_{\mathbb{R}} \chi_4(\tau_1 - n_1^5) \chi_4(\tau_2 - \tau_1 - (n_2 - n_1)^5) \ d\tau_1$$

$$\cong c \sum_{n_1} a_{n_1} b_{n_2 - n_1} \chi_4(\tau_2 - n_2^5 + 5 n_1 n_2 (n_2 - n_1) (n_1^2 + n_2^2 - n_1 n_2))$$

$$\cong c a_1 \chi_4(\tau_2 - n_2^5 - 30),$$
where
\[ \alpha_n = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases} \]

By performing the summation and integration with respect to \( n_2, \tau_2 \) variables once more, we have
\[
(f * g * h)(\tau, n) = \left((f * g) * h\right)(\tau, n)
\]
\[
= \sum_{n_2} \alpha_{n_2} d_{n-n_2} \int \chi_4^2(\tau_2 - n_2^5 - 30) \chi_4^2(\tau - \tau_2 - (n_2 - n_2^5)^5) \, d\tau_1
\]
\[
\cong c \sum_{n_2} \alpha_{n_2} d_{n-n_2} \chi_2(n_2 - 5) - n_2^5 - 30)
\]
\[
\cong c\beta_n \chi_2(\tau - (n_2 - n_2^5) - 31),
\]
where
\[ \beta_n = \begin{cases} 1, & n = N \\ 0, & \text{otherwise} \end{cases} \]

On the support of \((f * g * h)(\tau, n)\), since we have \(|\tau - n_2^5| \sim N^4\), we finally obtain
\[
\|uv\partial_x^3 w\|_{X^s,b-1} = \|(n_2^5)^{b-1} f[uv\partial_x^3 w](\tau, n)\|_{L^2}\|w\|_{X^{s,b}}
\]
\[
\sim N^4 N^3 N^4(b-1),
\]
while
\[
\|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}} \sim N^s.
\]

This imposes \( b \leq \frac{1}{4} \) to succeed the trilinear estimate and hence, we show \((3.1)\) when \( b > \frac{1}{4} \).

We now construct an example when \( b \leq \frac{1}{4} \) and focus on the case that \(|\tau - n_2^5| is too much smaller than the maximum modulation. In this case, we may assume that \(|\tau_1 - n_1^5| is the maximum modulation by symmetry among modulations. Set
\[
a_n = \begin{cases} 1, & n = -N \\ 0, & \text{otherwise} \end{cases} \quad b_n = \begin{cases} 1, & n = 2 \\ 0, & \text{otherwise} \end{cases} \quad d_n = \begin{cases} 1, & n = N - 1 \\ 0, & \text{otherwise} \end{cases}
\]

and
\[
f(\tau, n) = a_n \chi_4^2(\tau - n_2^5), \quad g(\tau, n) = b_n \chi_4^2(\tau - n_5^5), \quad h(\tau, n) = d_n \chi_4^2(\tau - n_2^5).
\]

From the duality, it suffices to consider
\[
\|uv\partial_x^3 w\|_{X^{s,-b},b} \leq C\|u\|_{X^{s,1-b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}},
\]
where
\[
\bar{u}(\tau, n) = f(\tau, n) \quad \bar{v}(\tau, n) = g(\tau, n) \quad \bar{w}(\tau, n) = h(\tau, n).
\]

Similarly as before, we need to calculate \(\mathcal{F}[uv\partial_x^3 w](\tau, n)\). Since \(\mathcal{F}[uv\partial_x^3 w](\tau, n) = (f * g * h)(\tau, n)\), performing the summation and integration with respect to \( n_1, \tau_1 \) variables gives
\[
(f * g)(\tau_2, n_2) = \sum_{n_1} a_{n_1} b_{n_2-n_1} \int \chi_4^2(\tau_1 - n_1^5) \chi_4^2(\tau_2 - \tau_1 - (n_2 - n_1^5)) \, d\tau_1
\]
\[
\cong c \sum_{n_1} a_{n_1} b_{n_2-n_1} \chi_4^2(\tau_2 - n_2^5 + 5n_1n_2(n_2 - n_1)(n_1^2 + n_2^2 - n_1n_2))
\]
\[
\cong c\alpha_n \chi_4^2(\tau_2 - n_2^5 + 10N(N - 2)((N - 1)^2 + 3)),
\]
where
\[ \alpha_n = \begin{cases} 1, & n = 2 - N \\ 0, & otherwise \end{cases} \]

By performing the summation and integration with respect to \( n_2, \tau_2 \) variables once more, we have
\[
(f * g * h)(\tau, n) = \left[(f * g) * h\right](\tau, n)
\]
\[
= \sum_{n_2} \alpha_n d_{n-n_2} \int_{R} \chi_{J_2} (\tau_2 - n_2^5 + 10N(N-2)[(N-1)^2 + 3]) \chi_{J_2} (\tau - \tau_2 - (n-n_2)^5) \, d\tau_2
\]
\[
\cong c \sum_{n_2} \alpha_n d_{n-n_2} \chi_1 (\tau - (n-n_2)^5 - n_2^5 + 10N(N-2)[(N-1)^2 + 3])
\]
\[
\cong c\beta_n \chi_1 (\tau - (n + N-2)^5 + (N-2)^5 + 10N(N-2)[(N-1)^2 + 3]),
\]
where
\[ \beta_n = \begin{cases} 1, & n = 1 \\ 0, & otherwise \end{cases} \]

On the support of \((f * g * h)(\tau, n)\), since we have \(|\tau - n^5| \sim N^4\), we finally obtain
\[
\|uv\partial_x^2 w\|_{X^{s,b}} = \|\langle n \rangle^{-s} (\tau - n^5)^{-b} F[uv\partial_x^2 w](\tau, n)\|_{L^2_x} \sim N^3 N^{-4b},
\]
while
\[
\|u\|_{X^{s,1-b}} \|v\|_{X^{s,b}} \|w\|_{X^{s,b}} \sim N^{-s} N^s \sim 1.
\]
This imposes \( b \geq \frac{3}{4} \) and hence, we show \([3.1]\) when \( b \leq \frac{1}{4} \), which complete the proof of Theorem 1.3.

4. \( L^2 \)-block estimates

In this section, we will give \( L^2 \)-block estimates for trilinear estimates. For \( n_1, n_2, n_3 \in \mathbb{Z} \), let
\[
G(n_1, n_2, n_3) = \mu(n_1) + \mu(n_2) + \mu(n_3) - \mu(n_1 + n_2 + n_3)
\]
be the resonance function, which plays an important role in the trilinear \( X^{s,b} \)-type estimates.

Let \( \zeta_i = \tau_i - \mu(n_i) \). For compactly supported functions \( f_i \in L^2(\mathbb{R} \times T) \), \( i = 1, 2, 3, 4 \), we define
\[
J(f_1, f_2, f_3, f_4) = \sum_{n_4} \int_{R} f_1(\zeta_1, n_1) f_2(\zeta_2, n_2) f_3(\zeta_3, n_3) f_4(\zeta_4 + G(n_1, n_2, n_3), n_4),
\]
where \( \mathbf{N}_{3,n_4} = N_{3,-n_4} \) and \( \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4 + G(n_1, n_2, n_3)) \). From the identities
\[
n_1 + n_2 + n_3 + n_4 = 0
\]
and
\[
\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + G(n_1, n_2, n_3) = 0
\]
on the support of \( J(f_1, f_2, f_3, f_4) \), we see that \( J(f_1, f_2, f_3, f_4) \) vanishes unless
\[
2^{k_{\max}} \sim 2^{k_{\sub}},
\]
\[
2^{j_{\max}} \sim \max(2^{j_{\sub}}, |G|), \quad (4.1)
\]
where \(|n_i| \sim 2^{k_i}\) and \(|\zeta_i| \sim 2^{j_i}, i = 1, 2, 3, 4\). By simple change of variables in the summation and integration, we have
\[
|J(f_1, f_2, f_3, f_4)| = |J(f_2, f_1, f_3, f_4)| = |J(f_3, f_2, f_1, f_4)| = |J(f_4, f_2, f_3, f_1)| = |J(f_1, f_2, f_3, f_4)|, \quad (4.2)
\]
where \( f(\tau, n) = f(-\tau, -n) \).
Lemma 4.1. Let \(k_i, j_i \in \mathbb{Z}_+\), \(i = 1, 2, 3, 4\). Let \(f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{T})\) be nonnegative functions supported in \(I_{k_i} \times I_{j_i}\).

(a) For any \(k_i, j_i \in \mathbb{Z}_+\), \(i = 1, 2, 3, 4\), then we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \lesssim 2^{(j_{\min}+j_{\text{thd}})/2} 2^{k_{\text{thd}}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.3}
\]
(b) Let \(k_{\text{thd}} \leq k_{\max} - 10\).

(b-1) If \(k_i, j_i = (k_{\text{thd}}, j_{\max})\) and \(j_{\sub} \leq 4k_{\max}\), we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \lesssim 2^{(j_1+2j_2+j_3)/2} 2^{-j_{\sub}+j_{\max}} 2^{k_{\text{thd}}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.4}
\]
(b-2) If \(k_i, j_i = (k_{\min}, j_{\max})\) and \(j_{\sub} > 4k_{\max}\), we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \lesssim 2^{(j_1+2j_2+j_3)/2} 2^{-2k_{\max}} 2^{k_{\text{thd}}/2} 2^{-j_{\max}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.5}
\]
(b-3) If \(k_i, j_i \neq (k_{\min}, j_{\max})\) and \(j_{\sub} \leq 4k_{\max}\), we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \lesssim 2^{(j_1+2j_2+j_3+j_4)/2} 2^{-j_{\sub}+j_{\max}} 2^{k_{\text{thd}}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.6}
\]
(b-4) If \(k_i, j_i \neq (k_{\min}, j_{\max})\) and \(j_{\sub} > 4k_{\max}\), we have
\[
J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \lesssim 2^{(j_1+2j_2+j_3+j_4)/2} 2^{-2k_{\max}} 2^{k_{\min}/2} 2^{-j_{\max}/2} \prod_{i=1}^{4} \|f_{k_i, j_i}\|_{L^2}. \tag{4.7}
\]

Proof. For (a), by using the Cauchy-Schwarz inequality with respect to \(\zeta\) and \(n\) variables to \(J(f_1, f_2, f_3, f_4)\), then we can easily obtain (4.3).

For (b), we fix \(\zeta_i, i = 1, 2, 3, 4\). We first consider the summation over frequencies. From (4.2), we may assume that \(j_1 \leq j_2 \leq j_3 \leq j_4\). Since the constraints of modulations
\[
\zeta_i = \tau_i - \mu(n_i) = O(2^i), \quad i = 1, 2, 3,
\]
we need to estimate
\[
\sum_{n_4 \neq n_3 \neq n_2 \neq n_1} f_{k_1, j_1}(n_1) f_{k_2, j_2}(n_2) f_{k_3, j_3}(n_3) f_{k_4, j_4}(n_1 + n_2 + n_3). \tag{4.8}
\]
If \(k_4 \neq k_{\text{thd}}\), let us assume that \(n_1 \leq n_2 \leq n_3 \leq n_4\). Then, by the change of variables \((n'_3 = n_1 + n_2 + n_3)\), (4.8) can be rewritten by
\[
\sum_{n_4 \neq n_3 \neq n_2 \neq n_1} f_{k_1, j_1}(n_1) f_{k_2, j_2}(n_2) f_{k_3, j_3}(n'_3 - n_1 - n_2) f_{k_4, j_4}(n'_3). \tag{4.9}
\]
Since
\[
\partial_{n_2} \left( \mu(n_1) + \mu(n_2) + \mu(n'_3 - n_1 - n_2) \right) = 5n_2^4 - 5(n'_3 - n_1 - n_2) + 3c_1n_2^2 - 3c_1(n'_3 - n_1 - n_2)^2,
\]
\(|n'_3| \sim 2^{k_4}\) and \(k_1 \leq k_2 \leq k_3 - 10\), \(n_2\) is contained in two intervals of length \(O(2^{-4k_4/2^{k_3/2}})\), i.e.

the number of \(n_2 \lesssim 2^{-4k_4/2^{k_3/2}}\).

If \(2^{-4k_4/2^{k_3/2}} \leq 1\), the number of \(n_1\) and \(n_2\) are constants independent on \(k_i\), by the Cauchy-Schwarz inequality with respect to \(n_1, n_2, n'_3\) variables and \(\zeta\) variables, we have (4.6). Otherwise, we also apply the Cauchy-Schwarz inequality to obtain (4.7).
On the other hand, if $k_4 = k_{thd}$, we use similar way as above. Assume that $n_3 \leq n_4 \leq n_2 \leq n_1$ and we change the variable $n_2' = n_1 + n_2 + n_3$, then (5.1) can be rewritten by

$$\sum_{n_1, n_2', n_3} f_{k_1, j_1} (n_1) f_{k_2, j_2} (n_2' - n_1 - n_2) f_{k_3, j_3} (n_3) f_{k_4, j_4} (n_2').$$

Then, since

$$\partial_{n_3} (\mu(n_1) + \mu(n_2' - n_1 - n_3) + \mu(n_3)) = 5n_3^4 - 5(n_2' - n_1 - n_3) + 3c_1 n_3^2 - 3c_1 (n_2' - n_1 - n_3)^2,$$

which implies that $n_3$ is contained in two intervals of length $O(2^{-4k_1} 2^{j_2})$, i.e.

$$\text{the number of } n_3 \lesssim 2^{-4k_1} 2^{j_2},$$

by using the Cauchy-Schwarz inequality with respect to $n_1, n_3, n_2'$ variables and $\zeta$ variables in regular order, we obtain [4.3] and [4.3]. Therefore, we complete the proof of Lemma [4.1].}

As an immediate consequence, we have the following corollary:

**Corollary 4.2.** Let $k_i, j_i \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$. Let $f_{k_i, j_i} \in L^2(\mathbb{R} \times T)$ be nonnegative functions supported in $I_{j_i} \times I_{k_i}$, $i = 1, 2, 3, 4$.

(a) For any $k_i, j_i \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$, then we have

$$\|1_{D_{k_4, j_4}}(n, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \lesssim 2^{(j_{min} + j_{thd})/2} 2^{k_{thd}/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. \quad (4.9)$$

(b-1) If $(k_1, j_1) = (k_{thd}, j_{max})$ and $j_{sub} \leq 4k_{max}$, we have

$$\|1_{D_{k_4, j_4}}(n, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{(j_{sub} + j_{max})/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. \quad (4.10)$$

(b-2) If $(k_1, j_1) = (k_{min}, j_{max})$ and $j_{sub} > 4k_{max}$, we have

$$\|1_{D_{k_4, j_4}}(n, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{k_{thd}/2} 2^{k_{max}/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. \quad (4.11)$$

(b-3) If $(k_1, j_1) \neq (k_{min}, j_{max})$ and $j_{sub} \leq 4k_{max}$, we have

$$\|1_{D_{k_4, j_4}}(n, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{(j_{sub} + j_{max})/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. \quad (4.12)$$

(b-4) If $(k_1, j_1) \neq (k_{min}, j_{max})$ and $j_{sub} > 4k_{max}$, we have

$$\|1_{D_{k_4, j_4}}(n, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3})\|_{L^2} \lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{k_{thd}/2} 2^{k_{max}/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2}. \quad (4.13)$$

5. **Nonlinear estimates**

In this section, we show the trilinear and quintilinear estimates.

**Lemma 5.1 (Resonance estimate).** Let $k \geq 0$. Then, we have

$$\|P_k N_1 (u, v, w)\|_{N_k} \lesssim 2^k \|P_k u\|_{F_k} \|P_k v\|_{F_k} \|P_k w\|_{F_k}. \quad (5.1)$$
Proof. From the definitions of $N_k(u, v, w)$ and $N_k$ norm, the left-hand side of (5.1) is bounded by

$$
sup_{t_k \in \mathbb{R}} \| (\tau - \mu(n) + i2^{2k})^{-1}2^{3k}1_{f_k}(n)F [\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k u] \|
$$

$$\ast \mathcal{F}_{\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k v} \ast \mathcal{F}_{\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k w} \|_{X_k} \tag{5.2}$$

Set $u_k = \mathcal{F}_{\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k u}$, $v_k = \mathcal{F}_{\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k v}$ and $w_k = \mathcal{F}_{\eta_0 \left(2^{2k-2}(t-t_k)\right) P_k w}$. We decompose each of $u_k$, $v_k$ and $w_k$ into modulation dyadic pieces as $u_{k,j_1}(\tau, n) = u_k(\tau, n)\eta_{j_1}(\tau - \mu(n))$, $v_{k,j_2}(\tau, n) = v_k(\tau, n)\eta_{j_2}((\tau - \mu(n))$, and $w_{k,j_3}(\tau, n) = w_k(\tau, n)\eta_{j_3}((\tau - \mu(n))$, respectively, with usual modification like $f_{\leq j}(\tau) = f(\tau)\eta_{\leq j}(\tau - \mu(n))$. Then, from the Cauchy-Schwarz inequality, (5.2) is bounded by

$$2^{3k} \sum_{j_4 \geq 0} \frac{2^{j_4/2}}{\max(2^{j_4}, 2^{2k})} \sum_{j, j_2, j_3 \geq 2} 2^{(j_{\min} + j_{\max})/2} \| u_{k,j_4} \|_{L^2} \| e_k \|_{L^2} \| w_{k,j_3} \|_{L^2} \tag{5.3}$$

Since $j_1, j_2, j_3 \geq 2k$, if $j_4 \leq 2k$, we have $\max(2^{j_4}, 2^{2k})^{-1}2^{(j_{\min} + j_{\max})/2} \leq 2^{j_1 + j_2 + j_3)/2}2^{-3k}$, otherwise, $(\max(2^{j_4}, 2^{2k})^{-1}2^{(j_{\min} + j_{\max})/2} \leq 2^{-j_1 + j_2 + j_3)/2}2^{-3k}$, and hence by performing all summations over $j_1, j_2, j_3$ and $j_4$, we have

$$\begin{align*}
\text{(5.3)} &\leq 2^k \sum_{j, j_2, j_3 \geq 2} \frac{2^{(j_1 + j_2 + j_3)/2} \| u_{k,j_4} \|_{L^2} \| e_k \|_{L^2} \| w_{k,j_3} \|_{L^2}}{\| w_{k,j_3} \|_{L^2}} \\
&\leq 2^k \| u_k \|_{X_k} \| v_k \|_{X_k} \| w_k \|_{X_k},
\end{align*}$$

which implies (5.1). \qed

Next, we show the non-resonance estimates by dividing into several cases. From the support property (4.1), we find from now on, may assume that

$$\max(|\tau - \mu(n)|, |\tau_j - \mu(n_j)|; j = 1, 2, 3) \gtrsim |(n_1 + n_2)(n_1 + n_3)(n_2 + n_3)\| (n_1^2 + n_2^2 + n_3^2 + n^2). \tag{5.4}$$

Lemma 5.2 (High-high-high ⇒ high). Let $k_3 \geq 20$ and $|k_1 - k_4|, |k_2 - k_4|, |k_3 - k_4| \leq 5$. Then, we have

$$\| P_{k_4} N_3(P_{k_4} u, P_{k_4} v, P_{k_4} w) \|_{N_{k_4}} + \| P_{k_4} N_3(P_{k_4} u, P_{k_4} v, P_{k_4} w) \|_{N_{k_4}} \lesssim 2^{3k_3/2} \| P_{k_4} u \|_{F_{k_4}} \| P_{k_4} v \|_{F_{k_4}} \| P_{k_4} w \|_{F_{k_4}}. \tag{5.5}$$

Proof. As in the proof of Lemma 5.1, both terms of the left-hand side of (5.5) is bounded by

$$\sup_{t_k \in \mathbb{R}} \| (\tau - \mu(n) + i2^{2k_3})^{-1}2^{3k_3}1_{f_k}(n_4)F [\eta_0 \left(2^{2k_3-2}(t-t_k)\right) P_k u] \|
$$

$$\ast \mathcal{F}_{\eta_0 \left(2^{2k_3-2}(t-t_k)\right) P_k v} \ast \mathcal{F}_{\eta_0 \left(2^{2k_3-2}(t-t_k)\right) P_k w} \|_{X_{k_4}} \tag{5.6}$$

Set, similarly as in the proof of Lemma 5.1, $f_{k_1} = F [\eta_0 \left(2^{2k_4-2}(t-t_k)\right) P_k u]$, $f_{k_2} = F [\eta_0 \left(2^{2k_4-2}(t-t_k)\right) P_k v]$ and $f_{k_3} = F [\eta_0 \left(2^{2k_4-2}(t-t_k)\right) P_k w]$. Also, we decompose $f_{k_3} \ast f_{k_4}$ into modulation dyadic pieces as $f_{k_1,j_1}(\tau, n) = f_{k_1}(\tau, n)\eta_{j_1}(\tau - \mu(n))$, $j = 1, 2, 3$, with usual modification $f_{\leq j}(\tau, n) = f(\tau, n)\eta_{\leq j}(\tau - \mu(n))$. Then, (5.6) is bounded by

$$2^{3k_3} \sum_{j_4 \geq 0} \frac{2^{j_4/2}}{\max(2^{j_4}, 2^{2k_4})} \sum_{j, j_2, j_3 \geq 2} \| 1_{\mathcal{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3}) \|_{L^2_{\tau_4} L^2_{n_4}} \tag{5.7}$$

and by applying (4.10) to $\| 1_{\mathcal{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3}) \|_{L^2_{\tau_4} L^2_{n_4}}$, then

$$\begin{align*}
\text{(5.7)} &\lesssim 2^{3k_3} \sum_{j_4 \geq 0} \frac{2^{j_4/2}}{\max(2^{j_4}, 2^{2k_4})} \sum_{j, j_2, j_3 \geq 2} 2^{(j_{\min} + j_{\max})/2} \| f_{k_1,j_1} \|_{L^2_{\tau_4} L^2_{n_4}} \| f_{k_2,j_2} \|_{L^2_{\tau_4} L^2_{n_4}} \| f_{k_3,j_3} \|_{L^2_{\tau_4} L^2_{n_4}}.
\end{align*}$$
From \[5.4\], by using \(j_{\max} \geq 3k_3\) and \(j_1, j_2, j_3 \geq 2k_3\), if \(j_4 \leq 2k_4\), then
\[
(\max(2^{j_1}, 2^{k_4}))^{-1/2}g(j_{\min}+j_{\max}/2) \lesssim 2^{-2k_4}2^{(j_1+j_2+j_3)/2}2^{-3k_3}/2.
\]
Otherwise, if \(j_4 = j_{\max} \geq 3k_4\), then
\[
(\max(2^{j_1}, 2^{k_4}))^{-1/2}g(j_{\min}+j_{\max}/2) \lesssim 2^{-j_4}2^{(j_1+j_2+j_3)/2}2^{-k_4},
\]
and if \(j_4 \neq j_{\max}\), then
\[
(\max(2^{j_1}, 2^{k_4}))^{-1/2}g(j_{\min}+j_{\max}/2) \lesssim 2^{-j_4}2^{(j_1+j_2+j_3)/2}2^{-3k_4}/2.
\]
Hence by performing all summations over \(j_1, j_2, j_3\) and \(j_4\), we get
\[
|5.6| \lesssim 2^{3k_3/2} \|P_{k_1} u\|_{F_{k_1}} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}},
\]
which completes the proof of Lemma 5.2 \(\square\)

**Lemma 5.3** (High-high-low \(\Rightarrow\) high). Let \(k_4 \geq 20\), \(|k_2 - k_4|, |k_3 - k_4| \leq 5\) and \(k_1 \leq k_4 - 10\). Then, we have
\[
\|P_{k_4} N_3(P_{k_1} u, P_{k_2} v, P_{k_3} w)\|_{N_{k_4}} + \|P_{k_4} N_4(P_{k_1} u, P_{k_2} v, P_{k_3} w)\|_{N_{k_4}} \lesssim 2^{k_4/2} \|P_{k_1} u\|_{F_{k_1}} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}}.
\]

**Proof.** Since we have \(j_{\max} \geq 5k_4\) from \[5.4\], once we perform the same procedure as in the proof of Lemma 5.2, we can obtain better result than that in Lemma 5.2. Furthermore, from \[2.11\], we obtain
\[
\|f_k\|_{X_{k}} = \|F \left[ \eta_0 \left( \frac{2^{2k_4-2}(t-t_k)}{k_3} \right) \cdot P_{k_1} u \eta_0 \left( \frac{2^{2k_4}(t-t_k)}{k_3} \right) \right] \|_{X_{k}} \lesssim \|P_{k_1} u\|_{F_{k_1}}.
\]
We omit the details. \(\square\)

**Lemma 5.4** (High-high-high \(\Rightarrow\) low). Let \(k_3 \geq 20\), \(|k_1 - k_3|, |k_2 - k_3| \leq 5\) and \(k_4 \leq k_3 - 10\). Then, we have
\[
\|P_{k_4} N_3(P_{k_1} u, P_{k_2} v, P_{k_3} w)\|_{N_{k_4}} + \|P_{k_4} N_4(P_{k_1} u, P_{k_2} v, P_{k_3} w)\|_{N_{k_4}} \lesssim k_3 2^{k_4} 2^{-k_4/2} \|P_{k_1} u\|_{F_{k_1}} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}}.
\]

**Proof.** Since \(k_4 \leq k_3 - 10\), one can observe that the \(N_{k_4}\)-norm is taken on the time intervals of length \(2^{-2k_4}\), while each \(F_{k_i}\)-norm is taken on shorter time intervals of length \(2^{-2k_i}\), \(i = 1, 2, 3\). Thus, we divide the time interval, which is taken in \(N_{k_4}\)-norm, into \(2^{2k_4-2k_4}\) intervals of length \(2^{-2k_4}\) in order to obtain the right-hand side of \[5.7\]. Let \(\gamma : \mathbb{R} \rightarrow [0, 1]\) denote a smooth function supported in \([-1, 1]\) with \(\sum_{m \in \mathbb{Z}} \gamma^3(x - m) \equiv 1\). From the definition of \(N_{k_4}\)-norm, the left-hand side of \[5.7\] is dominated by
\[
\sup_{t_k \in \mathbb{R}} 2^{k_4} 2^{2k_4} \|\tau_4 - \mu(n_4) + i2^{2k_4})^{-1} 1_{k_4} \cdot \sum_{|m| \leq C2^{k_4-2k_4}} \mathcal{F} \left[ \eta_0(2^{2k_4}(t-t_k)) \gamma(2^{2k_4}(t-t_k)-m) P_{k_1} u \right] \quad (5.8)
\]
As similarly in the proof of Lemma 5.2, we have from \[5.10\] that
\[
|5.8| \lesssim 2^{4k_4-k_4} \sum_{k_4 \geq 0} 2^{j_4/2} \max(2^j, 2^{2k_3}) \sum_{j_1, j_2, j_3, j_4 \geq 2k_3} 2^{(j_{\min} + j_{\max})/2} 2^{2(k_3+k_4)/2} \|f_{k_1, j_1} \|_{L^2_{k_1} \ell_{n_1}} \|f_{k_2, j_2} \|_{L^2_{k_2} \ell_{n_2}} \|f_{k_3, j_3} \|_{L^2_{k_3} \ell_{n_3}} \|f_{k_4, j_4} \|_{L^2_{k_4} \ell_{n_4}}.
\]
If \(j_4 < 2k_4\), then since \(j_4 \leq j_1, j_2, j_3\) and \(j_{\max} \geq 5k_3\), we have
\[
(\max(2^{j_1}, 2^{2k_3}))^{-1} 2^{(j_{\min} + j_{\max})/2} \lesssim 2^{-2k_4}2^{(j_1+j_2+j_3)/2}2^{-5k_3}/2^{-k_4}.
\]
If $2k_4 \leq j_4 < 2k_3$, then since we still have $j_4 \leq j_1, j_2, j_3$ and $j_{\max} \geq 5k_3$, and hence we get
\[
\left(\max(2^{j_1}, 2^{2k_4})\right)^{-1} 2^{(j_{\min} + j_{\max})/2} \lesssim 2^{-j_4/2} 2^{(j_1 + j_2 + j_3)/2} 2^{5k_3/2} 2^{-k_3}.
\] (5.9)
Otherwise, we always obtain
\[
\left(\max(2^{j_1}, 2^{2k_4})\right)^{-1} 2^{(j_{\min} + j_{\max})/2} \lesssim 2^{-j_4/2} 2^{(j_1 + j_2 + j_3)/2} 2^{5k_3/2} 2^{-k_3}.
\]
In fact, the worst bound comes from the summation of (5.9) over $j_1, j_2, j_3$ and $j_4$. Indeed,
\[
2^{4k_3/2 - k_4} \sum_{2k_4 \leq j_4 < 2k_3} \frac{2^{j_4/2}}{\max(2^{j_1}, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{(j_{\min} + j_{\max})/2} 2^{(k_3 + k_4)/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2_2 p_n^2}
\]
\[
\lesssim 2^{4k_3/2 - k_4} \sum_{2k_4 \leq j_4 < 2k_3} 2^{j_4/2} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{-j_4/2} 2^{(j_1 + j_2 + j_3)/2} 2^{5k_3/2} 2^{-k_3} 2^{(k_3 + k_4)/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2_2 p_n^2}
\]
\[
\lesssim k_3 2^{k_2 - k_4} \| P_{k_4} u \|_{F_{k_4}} \| P_{k_2} v \|_{F_{k_2}} \| P_{k_3} w \|_{F_{k_3}},
\]
which complete the proof of Lemma 5.4.

Next, we estimate the high-low-low $\Rightarrow$ high non-resonant interaction component in the nonlinear term. As mentioned in Sec. II and III, the trilinear estimate fails in the standard $X^{s,b}$ space due to the much more derivatives in high frequency mode and the lack of dispersive smoothing effect. The following lemma shows that the short time length ($\approx 5 (\text{frequency})^{-2}$) which is taken in $F_k$ or $N_k$ spaces is suitable to estimate the trilinear nonlinearity.

**Lemma 5.5 (High-low-low $\Rightarrow$ high).** Let $k_4 \geq 20$, $|k_3 - k_4| \leq 5$ and $k_1, k_2 \leq k_4 - 10$. Then, we have
\[
\| P_{k_4} N_3 (P_{k_1} u, P_{k_2} v, P_{k_3} w) \|_{N_{k_4}} + \| P_{k_4} N_4 (P_{k_1} u, P_{k_2} v, P_{k_3} w) \|_{N_{k_4}}
\]
\[
\lesssim 2^{k_1 - 2} \| P_{k_1} u \|_{F_{k_1}} \| P_{k_2} v \|_{F_{k_2}} \| P_{k_3} w \|_{F_{k_3}}.
\]

**Proof.** As in the proof of Lemma 5.4 it is enough to consider
\[
2^{4k_4} \sum_{j_4 \geq 0} \frac{2^{j_4/2}}{\max(2^{j_1}, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_4} \| 1_{D_{k_4 - j_4}} (f_{k_1, j_1} \ast f_{k_2, j_2} \ast f_{k_3, j_3}) \|_{L^2_2 E_{k_4}}.
\] (5.10)
By the symmetry of $k_1$ and $k_2$ or the definition of $N_4 (u, v, w)$, we may assume $k_1 \leq k_2$.

**Case I.** $k_1 \leq k_2 - 10$. In this case, we have from (5.4) that
\[
j_{\max} \geq 4k_4 + k_2.
\] (5.11)
We first rewrite the summation over $j_4$ as follows:
\[
\sum_{j_4 \geq 0} = \sum_{j_4 < 2k_4} + \sum_{2k_4 \leq j_4 < 4k_4 - 5} + \sum_{4k_4 - 5 \leq j_4 < 4k_4 + 5} + \sum_{4k_4 + 5 \leq j_4}
\]
\[
= \sum_{I} + \sum_{II} + \sum_{III} + \sum_{IV}.
\]
For the summation over $I$, if $j_4 = j_{\max}$, then from (4.10) or (4.11), (5.10) is bounded by
\[
2^{4k_4} \sum_{I} 2^{j_4/2} 2^{-2k_4} \sum_{j_1, j_2, j_3 \geq 2k_4} 2^{(j_4 + j_2 + j_3)/2 - (j_{\max} + j_{\sub})/2} 2^{k_2/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2_2 p_n^2}
\]
or
\[
2^{4k_4} \sum_{I} 2^{j_4/2} 2^{-2k_4} \sum_{j_1, j_2, j_3 \geq 2k_4} 2^{(j_4 + j_2 + j_3)/2 - j_{\max} - j_{\sub} - k_4} 2^{k_2/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2_2 p_n^2},
\]
respectively. In fact, one can easily know that the first bound is worse than the second one, so it suffices to consider the first one. By using (5.11) and $j_{\text{sub}} \geq 2k_4$, and performing the summation over $j_1, j_2, j_3$ and $j_4$, we obtain that

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y}$$

or

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_4/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

from (4.12) or (4.13), respectively. In this case, it is also enough to consider the first one and then we can obtain by performing the summation over $j_1, j_2, j_3$ and $j_4$ that

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_4/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

from (4.12) or (4.13), respectively. In this case, it is also enough to consider the first one and then we can obtain by performing the summation over $j_1, j_2, j_3$ and $j_4$ that

For the summation over $I$, we follow the same argument as in the case of summation over $I$, then we have from (4.10) that

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_4/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

since $j_4 \leq j_{\text{sub}}$.

For the summation over $III$, we know $j_4 = j_{\text{max}}$ and hence, similarly as before, we have from (4.12) that

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_4/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

For the last summation, since $j_4, j_{\text{sub}} \geq 4k_4 + k_2$, we obtain from (4.12) that

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_4/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

Case II. $|k_1 - k_2| \leq 5$. In this case, we have from (4.4) that $j_{\text{max}} \geq 4k_4$. As in the proof of Case I, it is enough to consider the case when $(k_1, j_1) \neq (k_{\text{thd}}, j_{\text{max}})$, $2k_4 \leq j_4 < 4k_4$ and $j_{\text{sub}} \leq 4k_4$. Then, by (4.13), we have

$$\sum_{j} 2^{j_4/2 - 2k_4} \sum_{j_1,j_2,j_3 \geq 2k_4} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-\left(j_{\text{max}} + j_{\text{sub}}\right)/2} 2^{k_{\text{min}}/2} \sum_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2_{\tau}L^2_y},$$

Thus, we complete the proof of Lemma 5.5. \qed
Next, we estimate the high-high-low ⇒ low non-resonant interaction component which has similar frequency interaction phenomenon as in the high-low-low ⇒ high non-resonant interaction component in the $L^2$-block estimates. But, similarly as in the proof of Lemma 5.4, we lose the regularity gain from longer time interval which is taken in the resulting frequency.

**Lemma 5.6** (High-high-low ⇒ low). Let $k_3 \geq 20$, $|k_2 - k_3| \leq 5$ and $k_1, k_4 \leq k_3 - 10$. Then, we have

$$\left\| P_{k_1} N_3(P_{k_1} u, P_{k_2} v, P_{k_3} w) \right\|_{N_{k_4}} + \left\| P_{k_4} N_4(P_{k_1} u, P_{k_2} v, P_{k_3} w) \right\|_{N_{k_4}}$$

$$\lesssim k_32^k C(k_1, k_4) \left\| P_{k_1} u \right\|_{F_{k_4}} \left\| P_{k_2} v \right\|_{F_{k_2}} \left\| P_{k_3} w \right\|_{F_{k_3}},$$

where

$$C(k_1, k_4) = \begin{cases} 2^{3k_4/2}2^{k_1/2}, & k_1 \leq k_4 - 10 \\ 2^{-k_4}, & k_4 \leq k_1 - 10 \\ 2^{-k_4}, & |k_1 - k_4| < 10 \end{cases}$$

**Proof.** As in the proof of Lemma 5.4, both terms are bounded by

$$2^{4k_3}2^{-k_2} \sum_{j_4 \geq 0} \max(2^{j_1}, 2^{k_4}) \sum_{j_1, j_2, j_3 \geq 2k_3} \left\| 1_{D_{k_4, j_4} \cdot (f_{k_1, j_1} \ast f_{k_3, j_3} \ast f_{k_5, j_5})} \right\|_{L^2_{k_4}.}

(5.12)

**Case I.** $k_1 \leq k_4 - 10$. In this case we have from (5.4) that $j_{\max} \geq 4k_3 + k_1$.

Similarly as Case I in the proof of Lemma 5.6 we can know that the worst case is when

$$2k_1 \leq j_1 \leq 4k_3 \quad \text{and} \quad j_{\text{sub}} \leq 4k_3,$

and hence, it suffices to perform the summation over (5.13). By using (5.12) and $j_{\max} \geq 4k_3 + k_1$, and performing the summation over $j_1, j_2, j_3$ and $j_4$ with (5.13), we obtain

$$2^{4k_3}2^{-k_2} \sum_{2k_1 \leq j_4 \leq 4k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{j_1 + j_2 + j_3 + j_4} / 2 - (j_{\max} + j_{\text{sub}})/2 \left\| P_{k_1} u \right\|_{F_{k_4}} \left\| P_{k_2} v \right\|_{F_{k_2}} \left\| P_{k_3} w \right\|_{F_{k_3}}.

(5.13)

**Case II.** $k_4 \leq k_1 - 10$. In this case we have from (5.4) that $j_{\max} \geq 4k_3 + k_1$.

Similarly as Case I, we can know that the worst case is when

$$j_1 = j_{\max}, \quad 2k_1 \leq j_1 \leq 4k_3 \quad \text{and} \quad j_{\text{sub}} \leq 4k_3,$

and hence, it suffices to perform the summation over (5.13). By using (5.12) and $j_{\max} \geq 4k_3 + k_1$, and performing the summation over $j_1, j_2, j_3$ and $j_4$ with (5.13), we obtain

$$2^{4k_3}2^{-k_2} \sum_{2k_1 \leq j_4 \leq 4k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{j_1 + j_2 + j_3 + j_4} / 2 - (j_{\max} + j_{\text{sub}})/2 \left\| P_{k_1} u \right\|_{F_{k_4}} \left\| P_{k_2} v \right\|_{F_{k_2}} \left\| P_{k_3} w \right\|_{F_{k_3}}.

(5.14)

**Case III.** $|k_1 - k_4| < 10$. In this case we have from (5.4) that $j_{\max} \geq 4k_3$.

Similarly as Case II in the proof of Lemma 5.4, we can know that the worst case is when

$$(k_1, j_1) \neq (k_{\text{thd}}, j_{\max}), \quad 2k_4 \leq j_4 \leq 4k_3 \quad \text{and} \quad j_{\text{sub}} \leq 4k_3.$$

(5.15)

This case is exactly same as the worst case in Case I, while high modulation effect is weaker. By performing similar procedure, we obtain

$$2^{4k_3}2^{-k_2} \sum_{2k_4 \leq j_4 \leq 4k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{j_1 + j_2 + j_3 + j_4} / 2 - (j_{\max} + j_{\text{sub}})/2 \left\| P_{k_1} u \right\|_{F_{k_4}} \left\| P_{k_2} v \right\|_{F_{k_2}} \left\| P_{k_3} w \right\|_{F_{k_3}}.

(5.16)$$
which completes the proof of Lemma 5.6.

Lemma 5.7 (low-low-low ⇒ low). Let $0 \leq k_1, k_2, k_3, k_4 \leq 200$. Then
\[
\| P_{k_4} N_3(P_{k_1} u, P_{k_2} v, P_{k_3} w) \|_{N_{k_4}} + \| P_{k_4} N_4(P_{k_1} u, P_{k_2} v, P_{k_3} w) \|_{N_{k_4}} \lesssim \| P_{k_1} u \|_{F_{k_1}} \| P_{k_2} v \|_{F_{k_2}} \| P_{k_3} w \|_{F_{k_3}}.
\] (5.16)

Proof. Similarly as in the proof of Lemma 5.2, we can get (5.16).

Now, we concentrate the quintilinear estimate. By the symmetry of $k_i$’s, $i = 1, 2, 3, 4, 5$, we may assume that $k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5$. Since we use the short time $X^{s,b}$ space, we have to consider whether the resulting frequency is the highest or not.

Lemma 5.8. Let $k_5 \geq 20$ and $|k_5 - k_6| \leq 5$. Then, we have
\[
\| P_{k_6} N_2(P_{k_1} v_1, P_{k_2} v_2, P_{k_3} v_3, P_{k_4} v_4, P_{k_5} v_5) \|_{N_{k_6}} \lesssim 2^{(k_1+k_2+k_3+k_4)/2} 2^{-k_5/2} \prod_{i=1}^5 \| P_{k_i} v_i \|_{F_{k_i}}.
\] (5.17)

Proof. We follow the similar arguments as in the proof of above lemmas. Then the right-hand side of (5.17) is bounded by
\[
\sup_{t_k \in \mathbb{R}} \left\| \left( \tau - \mu(n) + i2^{2k_6} \frac{1}{2} \right) \cdot \left[ \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_1} v_1 \right] * \left[ \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_2} v_2 \right] * \left[ \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_3} v_3 \right] * \left[ \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_4} v_4 \right] * \left[ \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_5} v_5 \right] \right\|_{X_{k_6}}.
\] (5.18)

Set, similarly as in the proof of Lemma 5.2, $f_{k_1} = \eta_0(2^{2k_6} - 2(t-t_k)) P_{k_1} v_1$, $i = 1, 2, 3, 4, 5$, and also, we decompose $f_{k_1}$ into modulation dyadic pieces as $f_{k_1,j_1} = f_{k_1,j_1}(\tau,n) = f_{k_1}(\tau,n)\eta_{j_1}(\tau - \mu(n))$, $j_1 = 1, 2, 3, 4, 5$, with usual modification $f_{k_1,j_1}(\tau,n) = f_{k_1}(\tau,n)\eta_{j_1}(\tau - \mu(n))$. Then, (5.18) is bounded by
\[
2^{k_6} \sum_{j_6 \geq 0} \frac{2^{j_6/2}}{\max(2^{j_6}, 2^{k_6})} \sum_{j_1, j_2, j_3, j_4, j_5 \geq 2k_6} \| f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4} * f_{k_5,j_5} \|_{L^2_{t_6} L^2_{x_6}}.
\] (5.19)

We apply the Cauchy-Schwarz inequality to $\| f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} * f_{k_4,j_4} * f_{k_5,j_5} \|_{L^2_{t_6} L^2_{x_6}}$, then
\[
(5.19) \lesssim 2^{k_6} \sum_{j_6 \geq 0} \frac{2^{j_6/2}}{\max(2^{j_6}, 2^{k_6})} \times \sum_{j_1, j_2, j_3, j_4, j_5 \geq 2k_6} 2^{(j_1+j_2+j_3+j_4+j_5)/2} 2^{-j_{\text{max}}+j_{\text{sub}}}/2 \| f_{k_1,j_1} \|_{L^2_{t_6} L^2_{x_6}}.
\]

Then by performing the summation over $j_1, j_2, j_3, j_4, j_5$ and $j_6$ with $j_{\text{max}}+j_{\text{sub}} \geq 2k_6$, we obtain
\[
(5.19) \lesssim 2^{k_6} \sum_{j_1, j_2, j_3, j_4, j_5 \geq 2k_6} 2^{(j_1+j_2+j_3+j_4+j_5)/2} 2^{-j_{\text{max}}+j_{\text{sub}}}/2 \times \sum_{i=1}^5 \| f_{k_1,j_1} \|_{L^2_{t_6} L^2_{x_6}} \lesssim 2^{-k_6} 2^{(k_1+k_2+k_3+k_4)/2} \prod_{i=1}^5 \| P_{k_i} v_i \|_{F_{k_i}}.
\]

Thus, we complete the proof of Lemma 5.8.
Lemma 5.9. Let $k_5 \geq 20$, $|k_4 - k_5| \leq 5$ and $k_6 \leq k_5 - 10$. Then, we have

$$\|P_{k_6} N_2(P_{k_1} v_1, P_{k_3} v_2, P_{k_3} v_3, P_{k_4} v_4, P_{k_5} v_5)\|_{N_{k_6}} \lesssim k_4 2^{(k_1 + k_2 + k_3 + k_4)/2} 2^{k_6} \prod_{i=1}^{5} \|P_{k_i} v_i\|_{F_{k_i}}. \quad (5.20)$$

Proof. Since $k_6 \leq k_5 - 10$, by the same reason as in the proof of Lemma 5.3 we further make a partition of interval which is taken in the $N_{k_6}$-norm. Let $\gamma : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in $[-1, 1]$ with $\sum_{m \in \mathbb{Z}} \gamma^5(x - m) \equiv 1$. Then, from the definition of $N_{k_6}$-norm, the left-hand side of (5.20) is dominated by

$$\sup_{t_k \in \mathbb{R}} \left\| (\tau - \mu(n_6) + i 2^{k_6})^{-1/2} \hat{1}_{t_k} n_6 \right\| \left( \sum_{\left|m\right| \leq C 2^{k_3 - 2k_4}} F \left[ \eta_0 \left( 2^{k_6} (t - t_k) \right) \gamma(2^{k_5} (t - t_k) - m) P_{k_1} v_1 \right] \ast F \left[ \eta_0 \left( 2^{k_6} (t - t_k) \right) \gamma(2^{k_5} (t - t_k) - m) P_{k_2} v_2 \right] \ast F \left[ \eta_0 \left( 2^{k_6} (t - t_k) \right) \gamma(2^{k_5} (t - t_k) - m) P_{k_4} v_4 \right] \ast F \left[ \eta_0 \left( 2^{k_6} (t - t_k) \right) \gamma(2^{k_5} (t - t_k) - m) P_{k_5} v_5 \right] \right\|_{N_{k_6}}. \quad (5.21)$$

Similarly as before, (5.21) is bounded by

$$2^{k_6} 2^{2(k_5 - k_6)} \sum_{j_6 \geq 0} \frac{2^{j_6 / 2}}{\max(2^{j_5}, 2^{k_6})} \left\| \prod_{j_2, j_3, j_4, j_5 \geq 1} 1_{D_{k_6, j_6}} \cdot (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3} * f_{k_4, j_4} * f_{k_5, j_5}) \right\|_{L^2_t X^2_{j_6}}. \quad (5.22)$$

We apply the Cauchy-Schwarz inequality to $\|1_{D_{k_6, j_6}} \cdot (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3} * f_{k_4, j_4} * f_{k_5, j_5})\|_{L^2_t X^2_{j_6}}$, then

$$\lesssim 2^{k_6} 2^{2(k_5 - k_6)} \sum_{j_6 \geq 0} \frac{2^{j_6 / 2}}{\max(2^{j_5}, 2^{k_6})} \left\| \prod_{j_1, j_2, j_3, j_4, j_5, j_6 \geq 2} 2^{(j_1 + j_2 + j_3 + j_4 + j_5 + j_6) / 2} 2^{(j_1 k_1 + j_2 k_2 + j_3 k_3 + j_4 k_4 + j_5 k_5) / 2} \prod_{i=1}^{5} \|f_{k_i, j_i}\|_{L^2_t X^2_{j_i}}. \quad (5.23)$$

Then by performing the summation over $j_1, j_2, j_3, j_4, j_5$ and $j_6$ with $j_{max}, j_{sub} \geq 2k_5$, we obtain

$$\lesssim k_4 2^{k_6} 2^{2(k_5 - k_6)} \sum_{j_1, j_2, j_3, j_4, j_5 \geq 2k_6} 2^{(j_1 + j_2 + j_3 + j_4 + j_5) / 2} 2^{\gamma(2^{k_5} (t - t_k) - m) P_{k_2} v_2} \lesssim k_4 2^{-k_6} 2^{(k_1 k_2 + k_2 k_3 + k_4) / 2} \prod_{i=1}^{5} \|P_{k_i} v_i\|_{F_{k_i}},$$

since the worst term arises in the case when $2k_6 \leq j_6 \leq 2k_5$. Thus, we complete the proof of Lemma 5.9. \hfill \Box

Lemma 5.10. Let $0 \leq k_1, k_2, k_3, k_4, k_5, k_6 \leq 200$. Then, we have

$$\|P_{k_6} N_2(P_{k_1} v_1, P_{k_3} v_2, P_{k_3} v_3, P_{k_4} v_4, P_{k_5} v_5)\|_{N_{k_6}} \lesssim \prod_{i=1}^{5} \|P_{k_i} v_i\|_{F_{k_i}}. \quad (5.23)$$

Proof. Similarly as in the proof of Lemma 5.3, we can get (5.23). \hfill \Box

As a conclusion to this section, we prove the nonlinear estimates for (2.6) by gathering the block estimates obtained above.
Proposition 5.11. (a) If $s > 1$, $T \in (0, 1]$ and $u, v, w, v_i \in F^s(T)$, $i = 1, 2, 3, 4, 5$, then

$$\sum_{i=1,3,4} \|N_i(u, v)|_{N^s(T)} + \|N_2(v_1, v_2, v_3, v_4)|_{N^s(T)} \lesssim \|u\|_{F^s} \|v\|_{F^s} \|w\|_{F^s} + \prod_{i=1}^5 \|v_i\|_{F^s}. \quad (5.24)$$

(b) If $T \in (0, 1]$, $w, v_5 \in F^0(T)$, $i = 1, 2, 3, 4, 5$, then

$$\sum_{i=1,3,4} \|N_i(u, v)|_{N^0(T)} + \|N_2(v_1, v_2, v_3, v_4, v_5)|_{N^0(T)} \lesssim \|u\|_{F^1} \|v\|_{F^1} \|w\|_{F^0} + \prod_{i=1}^4 \|v_i\|_{F^1} \|v_5\|_{F^0}. \quad (5.25)$$

Proof. The proof follows from the dyadic trilinear and quintilinear estimates. See [6] for similar proof. \(\square\)

6. Energy estimates

In this section, we will control $\|v\|_{F^s(T)}$ for (2.4) by $\|v_0\|_{H^s}$ and $\|v\|_{F^s(T)}$. Let us define, for $k \geq 1$, $\psi(n) := n\chi'(n)$ and $\psi_k(n) = \psi(2^{-k}n)$, where $\chi$ is defined in (2.9) and $'$ denote the derivative. Then, we have from the simple observation and the definition of $\chi_k$ that

$$\psi_k(n) = n\chi_k'(n).$$

Remark 6.1. The reason why we define another cut-off function $\psi_k$ is to use the second-order Taylor’s theorem for the commutator estimates (see Lemma 6.7). But, for the other estimates, it does not need to distinguish between $\psi_k$ and $\chi_k$, since both play a role of frequency support in the other estimates.

Recall (2.3) by slightly modifying from the symmetry of $n_1$ and $n_2$. Then we have

$$\partial_t \hat{v}(n) - i(n^5 + c_1 n^3 + c_2 n) \hat{v}(n) = -20in^3 |\hat{v}(n)|^2 \hat{v}(n)$$

$$+ 6in \sum_{N_{5,n}} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n_4) \hat{v}(n_5)$$

$$+ 10in \sum_{N_{4,n}} \hat{v}(n_1) \hat{v}(n_2) n_2^2 \hat{v}(n_3)$$

$$+ 5in \sum_{N_{3,n}} (n_1 + n_2) \hat{v}(n_1) \hat{v}(n_2) n_3 \hat{v}(n_3). \quad (6.1)$$

Denote the last three terms in the right-hand side of (6.1) by $\hat{N}_1(v)(n)$ only in the proof of Proposition 6.7 below. We perform the following procedure for $k \geq 1$,

$$\sum_n \chi_k(n) \times \chi_k(-n) \hat{v}(-n) + \chi_k(n) \times \chi_k(n) \hat{v}(n),$$
where (6.1) means to take the complex conjugate on (6.1). Then we have
\[ \partial_t \| P_k v \|^2_{L^2_x} = -\text{Re} \left[ 12i \sum_{n,\chi_{3,n}} \chi_k(n) n \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n_4) \hat{v}(n_5) \chi_k(n) \hat{v}(n) \right] \]
\[ - \text{Re} \left[ 20i \sum_{n,\chi_{3,n}} \chi_k(n) n \hat{v}(n_1) \hat{v}(n_2) n_2^2 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \right] \]
\[ - \text{Re} \left[ 10i \sum_{n,\chi_{3,n}} \chi_k(n) n(n_1 + n_2) \hat{v}(n_1) \hat{v}(n_2) n_3 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \right] \]
\[ =: E_1 + E_2 + E_3. \]

Unfortunately, we cannot control \( \int_0^T E_2 \) and \( \int_0^T E_3 \) in \( F^s(T) \) space directly, due to much more derivatives in the high frequency mode. To overcome this difficulty, we need to use the modified energy method as in \([13] \) and \([12] \). We, especially, use the localized version of the modified energy in \([12] \), by modifying that adapted to the periodic fifth-order mKdV.

For \( k \geq 1 \), let us define the new localized energy of \( v \) by
\[ E_k(v)(t) = \| P_k v(t) \|^2_{L^2_x} + \text{Re} \left[ \alpha \sum_{n,\chi_{3,n}} \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \right] \]
\[ + \text{Re} \left[ \beta \sum_{n,\chi_{3,n}} \hat{v}(n_1) \hat{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \right], \]
where \( \alpha \) and \( \beta \) are real and will be chosen later. By gathering all localized energies, we finally define the new modified energy by
\[ E_T^s(v) = \| P_0 v(0) \|^2_{L^2_x} + \sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [-T,T]} E_k(v)(t_k). \]

**Remark 6.2.** As mentioned before, this type of the modified energy was first introduced by Kwon \([13] \) and then Kenig-Pilod \([12] \) modified it as the localized version. We also slightly modify it to avoid appearing the resonant interaction component.

The following lemma shows that \( E_T^s(v) \) and \( \| v \|_{E^s(T)} \) are comparable.

**Lemma 6.3.** Let \( s > \frac{1}{2} \). Then, there exists \( 0 < \delta \ll 1 \) such that
\[ \frac{1}{2} \| v \|^2_{E^s(T)} \leq E_T^s(v) \leq \frac{3}{2} \| v \|^2_{E^s(T)}, \]
for all \( v \in E^s(T) \cap C([-T,T]; H^s(T)) \) satisfying \( \| u \|_{L^\infty_T H^s(T)} \leq \delta \).

**Proof.** The proof follows from the Sobolev embedding \( H^s(T) \hookrightarrow L^\infty(T) \), \( s > 1/2 \). See Lemma 5.1 in \([12] \) for the details.

We first prove several lemmas which are useful to estimate the modified energy.

**Lemma 6.4.** Let \( T \in (0,1) \), \( k_1, k_2, k_3, k_4 \in \mathbb{Z}_+ \), and \( v_i \in F_{k_i}(T) \), \( i = 1, 2, 3, 4 \). We further assume \( k_1 \leq k_2 \leq k_3 \leq k_4 \) with \( k_4 \geq 10 \). Then
(a) For $|k_1 - k_4| \leq 5$, we have
\[ \left\| \sum_{n_4 \in \mathbb{N}_{3,n_4}} \int_0^T \hat{\nu}_1(n_1) \hat{\nu}_2(n_2) \hat{\nu}_3(n_3) \hat{\nu}_4(n_4) \, dt \right\| \lesssim 2^{k_i} \prod_{i=1}^4 \| \nu_i \|_{F_{k_i}(T)}. \] (6.4)

(b) For $|k_2 - k_4| \leq 5$ and $k_1 \leq k_4 - 10$, we have
\[ \left\| \sum_{n_4 \in \mathbb{N}_{3,n_4}} \int_0^T \hat{\nu}_1(n_1) \hat{\nu}_2(n_2) \hat{\nu}_3(n_3) \hat{\nu}_4(n_4) \, dt \right\| \lesssim 2^{-k_i} 2^{k_i}/2 \prod_{i=1}^4 \| \nu_i \|_{F_{k_i}(T)}. \] (6.5)

(c) For $|k_3 - k_4| \leq 5$, $k_2 \leq k_4 - 10$ and $|k_1 - k_2| \leq 5$, we have
\[ \left\| \sum_{n_4 \in \mathbb{N}_{3,n_4}} \int_0^T \hat{\nu}_1(n_1) \hat{\nu}_2(n_2) \hat{\nu}_3(n_3) \hat{\nu}_4(n_4) \, dt \right\| \lesssim 2^{-k_i} 2^{k_i}/2 \prod_{i=1}^4 \| \nu_i \|_{F_{k_i}(T)}. \] (6.6)

(d) For $|k_3 - k_4| \leq 5$, $k_2 \leq k_4 - 10$ and $k_1 \leq k_2 - 10$, we have
\[ \left\| \sum_{n_4 \in \mathbb{N}_{3,n_4}} \int_0^T \hat{\nu}_1(n_1) \hat{\nu}_2(n_2) \hat{\nu}_3(n_3) \hat{\nu}_4(n_4) \, dt \right\| \lesssim 2^{-k_i} \prod_{i=1}^4 \| \nu_i \|_{F_{k_i}(T)}. \] (6.7)

Proof. We fix extensions $\tilde{\nu}_i \in F_{k_i}$ so that $\| \tilde{\nu}_i \|_{F_{k_i}} \leq 2 \| \nu_i \|_{F_{k_i}(T)}$, $i = 1, 2, 3, 4$. Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth partition of unity with $\sum_{m \in \mathbb{Z}} \gamma^4(x - m) \equiv 1$, $x \in \mathbb{R}$. Then, we obtain
\[ \left\| \sum_{n_4 \in \mathbb{N}_{3,n_4}} \int_0^T \hat{\nu}_1(n_1) \hat{\nu}_2(n_2) \hat{\nu}_3(n_3) \hat{\nu}_4(n_4) \, dt \right\| \lesssim \sum_{|m| \leq 2^{k_1}} \left\| \int_{\mathbb{R}} \left( \gamma(2^{2k_1} t - m) 1_{[0,T]}(t) \tilde{\nu}_1(n_1) \right) \cdot \left( \gamma(2^{2k_1} t - m) \tilde{\nu}_2(n_2) \right) \right. \]
\[ \left. \cdot \left( \gamma(2^{2k_1} t - m) \tilde{\nu}_3(n_3) \right) \cdot \left( \gamma(2^{2k_1} t - m) \tilde{\nu}_4(n_4) \right) \right\| \] (6.8)

Set
\[ A = \{ m : \gamma(2^{2k_1} t - m) 1_{[0,T]}(t) \text{ non-zero and } \neq \gamma(2^{2k_1} t - m) \}. \]

Then, the summation over $m \lesssim 2^{2k_1}$ in the right-hand side of (6.8) is divided into $A$ and $A^c$. Since $|A| \leq 4$, we can easily handle (see [6] for the details) the right-hand side of (6.8) on $A$ by showing
\[ \sup_{j \in \mathbb{Z}_+} 2^{2j} \| \eta_j(\tau - \mu(n)) \cdot F[1_{[0,1]}(t) \gamma(2^{2k_1} t - m) \tilde{\nu}_1(t)] \|_{\dot{X}_{k_i}} \lesssim \| \gamma(2^{2k_1} t - m) \tilde{\nu}_1(t) \|_{X_{k_i}}. \]

Hence, we only handle the summation on $A^c$ (for $m \in A^c$, $\gamma(2^{2k_1} t - m) 1_{[0,T]}(t) \tilde{\nu}_1(n_1) = \gamma(2^{2k_1} t - m) \tilde{\nu}_1(n_1)$). Let $f_{k_i} = F[\gamma(2^{2k_1} t - m) \tilde{\nu}_1(n_1)]$ and $f_{k_i,j_i} = \eta_j(\tau - \mu(n)) f_{k_i}$, $i = 1, 2, 3, 4$. By Parseval's identity and (2.10), the right-hand side of (6.8) is dominated by
\[ \sup_{m \in A^c} 2^{2k_1} \sum_{j_1,j_2,j_3,j_4 \geq 2^{k_1}} |F(j_{k_1,j_1}, j_{k_2,j_2}, j_{k_3,j_3}, j_{k_4,j_4})|. \] (6.9)
Lemma 6.5. Let

Therefore, we finish the proof of Lemma 6.4. \(\square\)

(a) By the support property (4.1), we know \(j_{\text{max}} \geq 3k_4\). Then, we use (4.3) to obtain that

\[
(6.9) \lesssim 2^{2k_4} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} 2^{(j_{\text{min}}+j_{\text{hd}})/2} 2^{k_4} \prod_{i=1}^{4} \|F_{k_i,j_i}\|_{L^2_f L^2_v}
\]

\[
\lesssim 2^{k_4/2} \|v_1\|_{F_{k_1}(T)} \|v_2\|_{F_{k_2}(T)} \|v_3\|_{F_{k_3}(T)} \|v_4\|_{F_{k_4}(T)}.
\]

(b) We use the same block-estimate (4.3) and argument in (a) with \(j_{\text{max}} \geq 5k_4\), then

\[
(6.9) \lesssim 2^{2k_4} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} \|J(F_{k_1,j_1})\|_{L^2_f L^2_v}^2 \prod_{i=1}^{4} \|F_{k_i,j_i}\|_{L^2_f L^2_v}^2
\]

\[
\lesssim 2^{-k_4} 2^{k_1/2} \|v_1\|_{F_{k_1}(T)} \|v_2\|_{F_{k_2}(T)} \|v_3\|_{F_{k_3}(T)} \|v_4\|_{F_{k_4}(T)}.
\]

(c) Since \(|k_1 - k_2| \leq 5\), the case \((k_1,j_1) = (k_{\text{hd}},j_{\text{max}})\) do not be happened. Moreover, the case \(j_{\text{sub}} \leq 4k_4\) gives worse result in the block estimates than the other cases, let us consider only the case when \(j_{\text{sub}} \leq 4k_3\). Then by using (4.1) and \(j_{\text{max}} \geq 4k_4\), we obtain

\[
(6.9) \lesssim 2^{2k_4} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} 2^{(j_{\text{max}}+j_{\text{sub}})/2} 2^{k_4} \prod_{i=1}^{4} \|F_{k_i,j_i}\|_{L^2_f L^2_v}
\]

\[
\lesssim 2^{-k_4} 2^{k_1/2} \|v_1\|_{F_{k_1}(T)} \|v_2\|_{F_{k_2}(T)} \|v_3\|_{F_{k_3}(T)} \|v_4\|_{F_{k_4}(T)}.
\]

(d) In this case, we observe that \(j_{\text{max}} \geq 4k_4+k_2\). Similarly, the worst bound of \(|J(F_{k_1,j_1}, F_{k_2,j_2}, F_{k_3,j_3}, F_{k_4,j_4})|\) should be appeared when \(j_2 = j_{\text{max}}\) and \(j_{\text{sub}} \leq 4k_4\). Hence we use (4.4) so that

\[
(6.9) \lesssim 2^{2k_4} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} \|J(F_{k_1,j_1})\|_{L^2_f L^2_v} \prod_{i=1}^{4} \|F_{k_i,j_i}\|_{L^2_f L^2_v}
\]

\[
\lesssim 2^{-k_4} \|v_1\|_{F_{k_1}(T)} \|v_2\|_{F_{k_2}(T)} \|v_3\|_{F_{k_3}(T)} \|v_4\|_{F_{k_4}(T)}.
\]

Therefore, we finish the proof of Lemma 6.4. \(\square\)

The next lemma is a kind of commutator estimate which will be helpful to handle bad terms \(\int_0^T E_2\) and \(\int_0^T E_3\) in the original energy.

**Lemma 6.5.** Let \(T \in (0,1), k,k_1,k_2 \in \mathbb{Z}_+\) satisfying \(k_1,k_2 \leq k - 10, u_i \in F_{k_i}(T), i = 1,2,\) and \(v \in F^0(T)\). Then, we have

\[
\left| \sum_{n \in \mathbb{N}_3,n} \int_0^T \chi_k(n) n [\chi_{k_1}(n_1) \hat{u}_1(n_1) \chi_{k_2}(n_2) \hat{u}_2(n_2) n_3^2 \hat{v}(n_3)] \chi_k(n) \hat{v}(n) \, dt \right|
\]

\[
+ \frac{1}{2} \sum_{n \in \mathbb{N}_3,n} \int_0^T (n_1 + n_2) \chi_{k_1}(n_1) \hat{u}_1(n_1) \chi_{k_2}(n_2) \hat{u}_2(n_2) \chi_k(n_3) \hat{v}(n_3) \chi_k(n) \hat{v}(n) \, dt
\]

\[
- \sum_{n \in \mathbb{N}_3,n} \int_0^T (n_1 + n_2) \chi_{k_1}(n_1) \hat{u}_1(n_1) \chi_{k_2}(n_2) \hat{u}_2(n_2) \psi_k(n_3) \chi_k(n) \hat{v}(n) \, dt
\]

\[
\lesssim 2^{2k_2} \|P_{k_1} u_1\|_{F_{k_1}(T)} \|P_{k_2} u_2\|_{F_{k_2}(T)} \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F_{k'}(T)}^2.
\]
and
\[
\left| \sum_{n,N_3,n} \int_0^T \chi_k(n)(n_1 + n_2)\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)n_3\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \\
+ \sum_{n,N_3,n} \int_0^T (n_1 + n_2)\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\chi_k(n_3)n_3\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \right| 
\]
(6.11)
\[
\lesssim 2^{2k_2}\|P_{k_1}u_1\|_{F_{k_1}(T)}\|P_{k_2}u_2\|_{F_{k_2}(T)} \sum_{|k-k'|\leq 5} \|P_{k'}v\|_{F_{k'}(T)}^2.
\]

Proof. We first consider (6.10). From \(n_1 + n_2 + n_3 + n_4 = 0\) and the symmetry of \(n_3, n\), we have

\[
\text{LHS of (6.10)} = \left| \sum_{n,N_3,n} \int_0^T \left[ \chi_k(n)n_3^2 - \chi_k(n_3)n_3^2 - (n_1 + n_2)n_3\psi_k(n_3) \right] \\
\times \chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \right|
\]
(\[\text{LHS of (6.10)} \])
\[
= \left| \sum_{n,N_3,n} \int_0^T \left[ \frac{\chi_k(n) - \chi_k(n_3) - (n_1 + n_2)\chi_k'(n_3)}{(n_1 + n_2)^2} \right] \\
\times (n_1 + n_2)^2\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \right|.
\]

Since both \(\chi_k\) and \(\chi_k'\) are even functions, \(-n_3 = n + (n_1 + n_2), |n| \sim |n_3|\) and \(\chi_k''(n) = O(\chi_k(n)/n^2)\) due to (2.8), we know from the Taylor’s theorem that
\[
\left| \frac{\chi_k(n) - \chi_k(n_3) - (n_1 + n_2)\chi_k'(n_3)}{(n_1 + n_2)^2} \right| \lesssim 1.
\]

Hence by the same way as in the proof of Lemma 6.4 (c) and (d), we have from \(2^{j_{\text{max}}} \geq 2^{4k_1}|n_1 + n_2|\) that
\[
\text{LHS of (6.10)} \lesssim 2^{2k_2}\|P_{k_1}u_1\|_{F_{k_1}(T)}\|P_{k_2}u_2\|_{F_{k_2}(T)} \sum_{|k-k'|\leq 5} \|P_{k'}v\|_{F_{k'}(T)}^2,
\]
for \(|k_1 - k_2| \leq 5\), and
\[
\text{LHS of (6.10)} \lesssim 2^{2k_2}\|P_{k_1}u_1\|_{F_{k_1}(T)}\|P_{k_2}u_2\|_{F_{k_2}(T)} \sum_{|k-k'|\leq 5} \|P_{k'}v\|_{F_{k'}(T)}^2,
\]
for \(k_1 \leq k_2 - 10\).

Next, we consider (6.11). Since \(n = -n_3 - (n_1 + n_2)\), we have
\[
\sum_{n,N_3,n} \int_0^T (n_1 + n_2)\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\chi_k(n_3)n_3\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \\
= -\sum_{n,N_3,n} \int_0^T (n_1 + n_2)^2\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\chi_k(n_3)n_3\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt \\
- \sum_{n,N_3,n} \int_0^T (n_1 + n_2)\chi_k(n_1)\hat{u}_1(n_1)\chi_k(n_2)\hat{v}_2(n_2)\chi_k(n_3)n_3^2\hat{v}(n_3)\chi_k(n)\hat{v}(n) \, dt,
\]}
and similarly as before, we have

$$\sum \int_0^T \chi_k(n)[(n_1 + n_2)\chi_k(n_1)\tilde{u}_1(n_1)\chi_k(n_2)\tilde{u}_2(n_2)n_3^2\tilde{v}(n_3)]\chi_k(n)\tilde{v}(n)$$

$$- \sum \int_0^T (n_1 + n_2)\chi_k(n_1)\tilde{u}_1(n_1)\chi_k(n_2)\tilde{u}_2(n_2)\chi_k(n_3)n_3^2\tilde{v}(n_3)\chi_k(n)\tilde{v}(n)$$

$$= \sum \int_0^T \left[ \frac{\chi_k(n) - \chi_k(n_3)}{(n_1 + n_2)} \cdot n_3 \right]$$

$$\times (n_1 + n_2)^2\chi_k(n_1)\tilde{u}_1(n_1)\chi_k(n_2)\tilde{u}_2(n_2)n_3\tilde{v}(n_3)\chi_k(n)\tilde{v}(n) dt,$$

with

$$\left| \frac{\chi_k(n) - \chi_k(n_3)}{(n_1 + n_2)} \cdot n_3 \right| \lesssim 1.$$

Again we use (6.6) and (5.7) so that

$$\text{LHS of (6.11)} \lesssim 2^{2k_2}\|P_{k_1}u_1\|_{F_{k_1}(T)}\|P_{k_2}u_2\|_{F_{k_2}(T)} \sum_{|k-k'|\leq 5} \|P_{k'}v\|^2_{F_{k'}(T)},$$

for both $|k_1 - k_2| \leq 5$ and $k_1 \leq k_2 - 10$ cases.

**Remark 6.6.** By using the same way, we also have

$$\left| \sum \int_0^T \chi_k(n)[(n_1 + n_2)\chi_k(n_1)\tilde{u}_1(n_1)\chi_k(n_2)\tilde{u}_2(n_2)n_3^2\tilde{v}(n_3)]\chi_k(n)n\tilde{v}(n) dt$$

$$- \sum \int_0^T (n_1 + n_2)\chi_k(n_1)\tilde{u}_1(n_1)\chi_k(n_2)\tilde{u}_2(n_2)\chi_k(n_3)n_3^2\tilde{v}(n_3)\chi_k(n)n\tilde{v}(n) dt \right|$$

$$\lesssim 2^{2k_2}\|P_{k_1}u_1\|_{F_{k_1}(T)}\|P_{k_2}u_2\|_{F_{k_2}(T)} \sum_{|k-k'|\leq 5} \|P_{k'}v\|^2_{F_{k'}(T)}.$$ (6.12)

Using above lemmas, we show the energy estimate.

**Proposition 6.7.** Let $s > 2$ and $T \in (0, 1]$. Then, for the solution $v \in C([-T, T]; H^s(\mathbb{T})$ to (6.1), we have

$$E^s_T(v) \lesssim (1 + \|v_0\|^2_{H^s})\|v_0\|^2_{H^s} + (1 + \|v\|^2_{F^{s/2}(T)} + \|v\|^2_{F^{4/2}(T)})\|v\|^2_{F^{s/2}(T)}$$

$$\|v\|^2_{F^{s}(T)}.$$ (6.12)

**Proof.** For any $k \in \mathbb{Z}_+$ and $t \in [-T, T]$, recall the localized modified energy (6.2)

$$E_k(v)(t) = \|P_k v(t)\|^2_{L^2} + \text{Re} \left[ \alpha \sum \tilde{v}(n_1)\tilde{v}(n_2)\nu_k(n_3)\frac{1}{n_3}\tilde{v}(n_3)\chi_k(n)\frac{1}{n}\tilde{v}(n) \right]$$

$$+ \text{Re} \left[ \beta \sum \tilde{v}(n_1)\tilde{v}(n_2)\chi_k(n_3)\frac{1}{n_3}\tilde{v}(n_3)\chi_k(n)\frac{1}{n}\tilde{v}(n) \right],$$

$$=: I(t) + II(t) + III(t)$$
We use the symmetry of \( n_1, n_2, n_3 \) and \( n_1 + n_2 + n_3 + n = 0 \), then the last term in \( E_1 \) can be rewritten as

\[
- \text{Re} \left[ 10i \sum_{n, n_1, n_2, n_3} \chi_k(n) n(n_1 + n_2) \hat{v}(n_1) \hat{v}(n_2) n_3 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \right]
\]

\[
= + \text{Re} \left[ 20i \sum_{n, n_1, n_2, n_3} \chi_k(n) n_1 \hat{v}(n_1) n_2 \hat{v}(n_2) n_3 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \right]
\]

\[
+ \text{Re} \left[ 20i \sum_{n, n_1, n_2, n_3} \chi_k(n) n(n_1 + n_2) \hat{v}(n_1) \hat{v}(n_2) n_3^2 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \right].
\]

We differentiate \( II(t) \) with respect to \( t \), respectively. Then, we have

\[
\frac{d}{dt} II(t) = \text{Re} \left[ \alpha i \sum_{n, n_1, n_2, n_3} (\mu(n_1) + \mu(n_2) + \mu(n_3) + \mu(n)) \right.
\]

\[
\times \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
+ \text{Re} \left[ \alpha \sum_{n, n_1, n_2, n_3} \tilde{N}_1(v)(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \right.
\]

\[
+ \hat{v}(n_1) \tilde{N}_1(v)(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
+ \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
+ \tilde{N}_1(v) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
\left. + \text{Re} \left[ -20 \alpha i \sum_{n, n_1, n_2, n_3} n_1^3 \hat{v}(n_1)^2 \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \right. \right]
\]

\[
+ \hat{v}(n_1) n_2^2 \hat{v}(n_2)^2 \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
+ \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) n_3^2 \hat{v}(n_3)^2 \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n)
\]

\[
+ \hat{v}(n_1) \hat{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) n^2 \hat{v}(n)^2 \hat{v}(n) \right].
\]
We use the following algebraic laws
\[(a + b + c)^3 = 5(a^4b + a^4c + ab^4 + b^4c + ac^4 + bc^4) + 10(a^3b^2 + a^3c^2 + a^2b^3 + b^3c^2 + a^2c^3 + b^2c^3) \]
\[+ 20(ab^2c + ab^2c + abc^3) + 30(a^2b^2c + a^2bc^2 + ab^2c^3) + a^3 + b^3 + c^3 \]
\[(a + b + c)^3 = a^3 + b^3 + c^3 + 3(ab^2 + a^2c + ab^2 + b^2c + ac^2 + bc^2) \]
and the symmetry of \(n_1\) and \(n_2\) so that
\[\frac{d}{dt} II(t) = E_{2,1} + E_{2,2} + E_{2,3} + E_{2,4} =: E_2,\]
where
\[E_{2,1} = \text{Re} \left[ \alpha \sum_{n \in \mathbb{N}_3, n} \left\{ 10n_1n_2^3(n_3 + n) + 5n_1^2n_2^2(n_3 + n) + 30n_1n_2n_3n \right. \right. \]
\[\left. \left. + 10n_2^3n_3n - 5(n_1 + n_2)n_3^2a^2 \right\} \tilde{v}(n_1) \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right], \]
\[E_{2,2} = \text{Re} \left[ c_1 \alpha \sum_{n \in \mathbb{N}_3, n} \left\{ 3n_1n_2(n_3 + n) + 6n_2n_3n \right\} \tilde{v}(n_1) \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right], \]
\[E_{2,3} = \text{Re} \left[ \alpha \sum_{n \in \mathbb{N}_3, n} \tilde{\mathcal{N}}_1(v)(n_1) \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right. \]
\[\left. + \tilde{v}(n_1) \tilde{\mathcal{N}}_1(v)(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right], \]
\[E_{2,4} = \text{Re} \left[ -20 \alpha \sum_{n \in \mathbb{N}_3, n} n_1^2 \left\{ \tilde{v}(n_1)^2 \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right. \right. \]
\[\left. \left. + \tilde{v}(n_1)n_2^2 \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right], \]
and
\[E_{3,1} = \text{Re} \left[ \beta_1 \sum_{n \in \mathbb{N}_3, n} \left\{ 20n_1n_2^3n_3 + 10n_1^2n_2^2n_3 + 30n_1n_2n_3n \right. \right. \]
\[\left. \left. + 10n_2^3n_3n - 5(n_1 + n_2)n_3^2a^2 \right\} \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right], \]
\[E_{3,2} = \text{Re} \left[ c_1 \beta_i \sum_{n \in \mathbb{N}_3, n} \left\{ 6n_1n_2n_3 + 6n_2n_3n \right\} \tilde{v}(n_1) \tilde{v}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\sigma}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right]. \]
\[ E_{3,3} = \text{Re} \left[ \beta \sum_{n, N_3, n} \tilde{N}_1(v)(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \right. \\
+ \tilde{v}(n_1) \tilde{N}_1(v)(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \\
+ \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{N}_1(v)(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \\
+ \left. \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{N}_1(v)(n) \right] \\
\]

and \[ E_{3,4} = \text{Re} \left[ -20\beta i \sum_{n, N_3, n} n_1^3 |\tilde{v}(n_1)|^2 \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \\
+ \tilde{v}(n_1) n_2^2 |\tilde{v}(n_2)|^2 \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \\
+ \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) n_3^2 |\tilde{v}(n_3)|^2 \tilde{v}(n_3) \chi_k(n) \frac{1}{n} \tilde{v}(n) \\
+ \tilde{v}(n_1) \tilde{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3) \chi_k(n) n_2 |\tilde{v}(n)|^2 \tilde{v}(n) \right]. \]

Fix \( t_k \in [0, T] \), by integrating \( \partial_t E_k(v)(t) \) with respect to \( t \) from \( 0 \) to \( t_k \), then we have
\[ E_k(v)(t_k) - E_k(v)(0) \leq \int_0^{t_k} E_1 + E_2 + E_3 \, dt. \]  
(6.13)

We estimate the right-hand side of (6.13) by dividing it into several cases. First, we choose \( \alpha = -4 \) and \( \beta = -2 \) to use Lemma 5.5 then for each \( k \geq 1 \), we have
\[ \left| \int_0^{t_k} E_1 + E_{2,1} + E_{3,1} \, dt \right| \lesssim \sum_{i=1}^8 A_i(k), \]
where
\[ A_1(k) = \sum_{0 \leq k_1, k_2 \leq k-1} \left| \sum_{n, N_3, n} \int_0^{t_k} \chi_k(n) n |\chi_{k_1}(n_1) \tilde{v}(n_1) \chi_{k_2}(n_2) \tilde{v}(n_2) n_3^2 \tilde{v}(n_3)| \chi_k(n) \tilde{v}(n) \, dt \right. \\
+ \left. \frac{1}{2} \sum_{n, N_3, n} \int_0^{t_k} (n_1 + n_2) \chi_{k_1}(n_1) \tilde{v}(n_1) \chi_{k_2}(n_2) \tilde{v}(n_2) \chi_k(n_3) n_3 \tilde{v}(n_3) \chi_k(n) n \tilde{v}(n) \, dt \right|, \]
\[ A_2(k) = \sum_{0 \leq k_1, k_2 \leq k-10} \left| \sum_{n, N_3, n} \int_0^{t_k} \chi_k(n) [(n_1 + n_2) \chi_{k_1}(n_1) \tilde{v}(n_1) \chi_{k_2}(n_2) \tilde{v}(n_2) n_3^2 \tilde{v}(n_3)] \chi_k(n) \tilde{v}(n) \, dt \right. \\
+ \left. \sum_{n, N_3, n} \int_0^{t_k} (n_1 + n_2) \chi_{k_1}(n_1) \tilde{v}(n_1) \chi_{k_2}(n_2) \tilde{v}(n_2) \chi_k(n_3) n_3 \tilde{v}(n_3) \chi_k(n) n \tilde{v}(n) \, dt \right|, \]
\[ A_3(k) = \sum_{\text{max}(k_1, k_2) \geq k-9} \left| \sum_{n, N_3, n} \int_0^{t_k} \chi_{k_1}(n_1) \tilde{v}(n_1) \chi_{k_2}(n_2) \tilde{v}(n_2) \chi_{k_3}(n_3) n_3^2 \tilde{v}(n_3) \chi_k(n) n \tilde{v}(n) \, dt \right|. \]
\[ A_4(k) = \sum_{\max(k_1, k_2) \geq k-9} \left| \sum_{n, n' \in \mathbb{N}} \int_0^t (n_1 + n_2) \chi_{k_2}(n_1) \hat{v}(n_1) \chi_{k_2}(n_2) \hat{v}(n_2) \chi_{k_3}(n_3) n_3^2 \hat{\chi}(n_3) \chi_k(n) \hat{v}(n) \, dt \right|, \]

\[ A_5(k) = \sum_{k_1, k_2, k_3 \geq 0} \left| \sum_{n, n' \in \mathbb{N}} \int_0^t \chi_{k_1}(n_1) n_1 \hat{v}(n_1) \chi_{k_2}(n_2) n_2 \hat{v}(n_2) \chi_{k_3}(n_3) n_3 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \, dt \right|, \]

\[ A_6(k) = \sum_{\max(k_1, k_2) \geq k-9} \left| \sum_{n, n' \in \mathbb{N}} \int_0^t (n_1 + n_2) \chi_{k_2}(n_1) \hat{v}(n_1) \chi_{k_2}(n_2) \hat{v}(n_2) \chi_k(n_3) n_3 \hat{v}(n_3) \chi_k(n) \hat{v}(n) \, dt \right|, \]

\[ A_7(k) = \sum_{k_1, k_2 \geq 0} \left| \sum_{n, n' \in \mathbb{N}} \int_0^t \left\{ 20n_1 n_3 n_3 + 10 n_1^2 n_3^2 + 30n_1 n_2 n_3 n + 10n_2^2 n n \right\} \chi_{k_1}(n_1) \hat{v}(n_1) \chi_{k_2}(n_2) \hat{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \, dt \right|. \]

By using Lemma 6.13 and the Cauchy-Schwarz inequality, we have

\[ A_1(k) + A_2(k) \lesssim \sum_{0 \leq k_1, k_2 \leq k-10} 2^{2k_2} \| P_{k_1} v \|_{F_k'(T)} \| P_{k_2} v \|_{F_k'(T)} \sum_{|k-k'| \leq 3} \| P_{k'} v \|_{F_k'(T)}^2 \lesssim \| v \|_{F^0(T)} \| v \|_{F^2(T)} \sum_{|k-k'| \leq 5} \| P_{k'} v \|_{F_k'(T)}^2. \]

For \( A_3(k) \) and \( A_4(k) \), we divide the summation over \( \max(k_1, k_2) \geq k-9, k_3 \geq 0 \) into

\[ \sum_{k_1, k_2, k_3 \leq k-10} + \sum_{k_1 \leq k-10} \sum_{k_2, k_3 \geq k-9} + \sum_{k_1 \leq k-9} \sum_{k_2, k_3 \geq k-9} + \sum_{k_1, k_2, k_3 \geq k-9}, \]

assuming without loss of generality \( k_1 \leq k_2 \). We restrict \( A_3(k) \) and \( A_4(k) \) to the first summation, by \( 6.13 \) and \( 6.7 \), we have

\[ \sum_{k_1 \leq k-10} 2^{2k_2} \| P_{k_1} v \|_{F_k'(T)} \| P_{k_2} v \|_{F_k'(T)} \sum_{|k-k'| \leq 3} \| P_{k'} v \|_{F_k'(T)}^2 \lesssim \| v \|_{F^0(T)} \| v \|_{F^2(T)} \sum_{|k-k'| \leq 5} \| P_{k'} v \|_{F_k'(T)}^2. \]
For the restriction to the second summation, by using (6.3) and (6.7), we have

\[
\sum_{k_1 \leq k-10} 2^{k_1/2} \|P_{k_1} v\|_{F_{k_1}(T)} \sum_{|k-k'| \leq 5} 2^{2k} \|P_{k'} v\|_{F_{k'}(T)}^3 \\
+ \sum_{k_1 \leq k-10} \|P_{k_1} v\|_{F_{k_1}(T)} \|P_k v\|_{F_k(T)} \sum_{k_2 \geq k+9} 2^{2k_3} \|P_{k_2} v\|_{F_{k_2}(T)} \|P_{k_3} v\|_{F_{k_3}(T)}
\]

\[
\lesssim \|v\|_{F^{2+}(T)} \|v\|_{F^2(T)} \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F_{k'}(T)}^2 + \|v\|_{F^2(T)} \|v\|_{F^{2+}(T)} 2^{-s - \varepsilon k} \|P_k v\|_{F_k(T)},
\]

for \( s \geq 0 \) and \( 0 < \varepsilon \ll 1 \).

For the third summation, we can get better or same bounds compared with the second summation due to two derivatives in the low frequency mode.

For the last restriction, by using (6.3), (6.4), (6.6) and (6.7), we have

\[
\sum_{|k-k'| \leq 5} 2^{7/2} \|P_{k'} v\|_{F_{k'}(T)}^4 + \sum_{k_2 \geq k+9} 2^{k_3/2} \|P_{k'} v\|_{F_{k'}(T)}^3 \|P_k v\|_{F_k(T)}
\]

\[
+ \sum_{|k-k'| \leq 5} 2^{k_3/2} \|P_{k'} v\|_{F_{k'}(T)}^2 \sum_{k_3 \geq k+9} 2^{2k_3} \|P_{k_2} v\|_{F_{k_2}(T)} \|P_{k_3} v\|_{F_{k_3}(T)}
\]

\[
+ \|P_k v\|_{F_k(T)} \sum_{k_1 \geq k+9} \|P_{k_1} v\|_{F_{k_1}(T)} \sum_{k_2 \geq k+9} 2^{k_3} \|P_{k_2} v\|_{F_{k_2}(T)} \|P_{k_3} v\|_{F_{k_3}(T)}
\]

\[
\lesssim \|v\|_{F^{2+}(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F_{k'}(T)}^2 + \|v\|_{F^2(T)} \|v\|_{F^{2+}(T)} 2^{-s - \varepsilon k} \|P_k v\|_{F_k(T)}
\]

\[
+ \|v\|_{F^{2+}(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F_{k'}(T)}^2 + \|v\|_{F^2(T)} \|v\|_{F^{2+}(T)} 2^{-s - \varepsilon k} \|P_k v\|_{F_k(T)},
\]

for \( s \geq 0 \) and \( 0 < \varepsilon \ll 1 \). Hence, we obtain

\[
A_3(k) + A_4(k) \lesssim \|v\|_{F^{2+}(T)} \|v\|_{F^2(T)} \left( \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F_{k'}(T)}^2 + \|v\|_{F^2(T)} 2^{-s - \varepsilon k} \|P_k v\|_{F_k(T)} \right).
\]

For \( A_5(k) \), we may assume that \( k_1 \leq k_2 \leq k_3 \) by the symmetry. We estimate \( A_5(k) \) for \( k = \max(k_1, k_2, k_3, k) \) or not, separately. When \( k = \max(k_1, k_2, k_3, k) \), we divide the summation over \( k_1, k_2, k_3 \) into

\[
\sum_{|k-k'| \leq 5} + \sum_{k_1 \leq k-10} + \sum_{k_2 \leq k-10} + \sum_{k_3 \leq k-10}.
\]
We use (6.4), (6.5), (6.6) and (6.7) to estimate $A_5(k)$ under above summations, respectively, then we obtain
\[
A_5(k) \lesssim 2^{7k/2} \sum_{|k'-k| \leq 5} \|P_{k'}v\|_{F_k'(T)}^2 + \sum_{|k-k'| \leq 5} 2^{3k_1/2}2^{k} \|P_{k_1}v\|_{F_{k_1}(T)} \|P_{k'}v\|_{F_k'(T)}^3 \\
+ \sum_{|k-k'| \leq 5} 2^{5k_2/2} \|P_{k_1}v\|_{F_{k_1}(T)}^2 \|P_{k'}v\|_{F_k'(T)}^2 \\
+ \sum_{|k-k'| \leq 5} 2^{k_1}2^{k_2} \|P_{k_1}v\|_{F_{k_1}(T)} \|P_{k_2}v\|_{F_{k_2}(T)} \|P_{k'}v\|_{F_k'(T)}^2 \\
\lesssim \|v\|_{F_k'(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_k'(T)}^2.
\]

When $k \leq k_3 - 10$, by the support property (4.1), we know $|k_2 - k_3| \leq 5$. Then, also we divide the summation over $k_1, k_2, k_3$ into
\[
\sum_{|k_1-k_2| \leq 5} + \sum_{k_1 \leq k_2-10} + \sum_{k_1 \leq k_2-10} + \sum_{k_1 \leq k_2-10}.
\]

Similarly we use (6.5), (6.6) and (6.7) to estimate $A_5(k)$ under above summations, then we obtain
\[
A_5(k) \lesssim \sum_{|k-k'| \leq 5} 2^{2k_3}2^{k/2} \|P_{k'}v\|_{F_k'(T)}^3 \|P_kv\|_{F_{k_3}(T)} + \sum_{k_3 \geq k+9} 2^{k_3} \|P_{k_3}v\|_{F_{k_3}(T)} \sum_{k_1 \geq k+9} \sum_{|k-k'| \leq 5} 2^{3k/2} \|P_{k'}v\|_{F_k'(T)}^2 \\
+ \sum_{k_3 \geq k+9} 2^{k_3} \|P_{k_3}v\|_{F_{k_3}(T)} \sum_{k_1 \geq k+9} 2^{k_1} \|P_{k_1}v\|_{F_{k_1}(T)}2^{k_2/2} \|P_{k}v\|_{F_{k_1}(T)} \\
+ \sum_{k_3 \geq k+9} 2^{k_3} \|P_{k_3}v\|_{F_{k_3}(T)} \sum_{k_1 \leq k-10} 2^{3k_1/2} \|P_{k_1}v\|_{F_{k_1}(T)} \|P_{k}v\|_{F_k(T)} \\
\lesssim \left(\|v\|_{F_k'(T)}^2 + \|v\|_{F^{1+}(T)} \|v\|_{F_k'(T)} \|v\|_{F_k(T)} \right) \|v\|_{F^{s+}(T)}^2 2^{-s-k} \|P_kv\|_{F_{k}(T)} + \|v\|_{F_k'(T)}^2 \|P_{k'}v\|_{F_{k'}(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_k'(T)}^2,
\]
for $s \geq 0$ and $0 < \varepsilon \ll 1$. Hence, by gathering all bounds, we conclude that
\[
A_5(k) \lesssim \|v\|_{F^{1+}(T)} \|v\|_{F_k'(T)} \|v\|_{F^{s+}(T)} 2^{-s-k} \|P_kv\|_{F_{k}(T)} + \|v\|_{F_k'(T)} \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_{k'}(T)}^2.
\]

For $A_6(k)$, we may assume $k_1 \leq k_2$ without loss of generality. By the support property (4.1), the summation over max$(k_1, k_2) \geq k - 9$ is divided into
\[
\sum_{k_2 \geq k+10} + \sum_{|k_1-k_2| \leq 5} + \sum_{k_1 \leq k-10}.
\]
Hence, we use (6.6), (6.13) and (6.15) to estimate $A_0(k)$ restricted to above summations, respectively, then

$$A_0(k) \lesssim \sum_{k_2 \geq k+10} \|P_{k_2}v\|^2_{F_{k_2}(T)} 2^{5k/2} \|P_{k}v\|^2_{F_{k}(T)} + 2^{7k/2} \sum_{|k-k'| \leq 5} \|P_{k'}v\|^4_{F_{k'}(T)}$$

$$+ \sum_{k_1 \leq k-10} 2^{k_1/2} \|P_{k_1}v\|_{F_{k_1}(T)} 2^{2k} \sum_{|k-k'| \leq 5} \|P_{k'}v\|^3_{F_{k'}(T)}$$

$$\lesssim \left( \|v\|^2_{F_{\frac{k}{4}}(T)} + \|v\|_{F_{\frac{k}{4}}^+(T)} \|v\|_{L^2(T)} \right) \sum_{|k-k'| \leq 5} \|P_{k'}v\|^2_{F_{k'}(T)}.$$

For $A_7(k)$, since $A_7(k)$ has less derivatives in the $2^k$-frequency mode than the other $A_i$, we can also obtain better or same bounds as

$$A_7(k) \lesssim \left( \|v\|^2_{F_{\frac{k}{4}}(T)} + \|v\|_{F_{\frac{k}{4}}^+(T)} \|v\|_{L^2(T)} \right) \sum_{|k-k'| \leq 5} \|P_{k'}v\|^2_{F_{k'}(T)}.$$

For $A_8(k)$, since one derivative is taken on $P_k v$ and we have the symmetry among frequencies $k_1, k_2, k_3, k_4$, and $k_5$, we may assume that $k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5$ and $|k - k_5| \leq 5$. We use only the Cauchy-Schwarz inequality to estimate the quintic term except for the special term (see (6.29) below). Indeed, by similar argument as in the proof of Lemma 6.3, we have

$$\left| \int_{T \times [0, T]} v_1 v_2 v_3 v_4 v_5 v_6 \, dx \, dt \right| \lesssim 2^{2k_0} \sum_{j_i \geq 2k_0} \left| \prod_{i=1}^{6} \mathcal{F}[\gamma(2^{2k_0} t - m)v_i](\tau_i, n_i) \right|$$

$$\lesssim 2^{2k_0} \sum_{j_i \geq 2k_0} 4 \frac{2^{k_0/2}}{2^{(j_{max}+j_{min})/2}} \prod_{i=1}^{6} 2^{k_i/2} \|v_j(\tau_i - \mu(n_i))\|_{L^2(\tau_i, n_i)} \|\mathcal{F}[\gamma(2^{2k_0} t - m)v_i]\|_{L^2(\tau_i, n_i)}$$

$$\lesssim 2^{(k_1+k_2+k_3+k_4)/2} \prod_{i=1}^{6} \|v_i\|_{F_{k_i}(T)},$$

where $v_i = P_{k_i}v \in F_{k_i}(T), i = 1, 2, 3, 4, 5, 6$ and assuming that $k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6$. Moreover, we also have

$$\left| \int_{T \times [0, T]} v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 \, dx \, dt \right| \lesssim 2^{(k_1+k_2+k_3+k_4+k_5+k_6)/2} \prod_{i=1}^{8} \|v_i\|_{F_{k_i}(T)},$$

where $v_i = P_{k_i}v \in F_{k_i}(T), i = 1, 2, 3, 4, 5, 6, 7, 8$ and assuming that $k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6 \leq k_7 \leq k_8$. We will use (6.16) for the septic term later.

If $|k_4 - k_5| \leq 5$, by using (6.15), we have

$$\sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4} 2^{(k_1+k_2+k_3)/2} \prod_{j=1}^{5} \|P_{k_j}v\|_{F_{k_j}(T)} \|P_{k_0}v\|_{F_{k_0}(T)}$$

$$\lesssim \|v\|^3_{F_{\frac{k}{4}}(T)} \|v\|_{F_{\frac{k}{4}}^+(T)} \sum_{|k-k'| \leq 5} \|P_{k'}v\|^2_{F_{k'}(T)}.$$
Otherwise, we need to observe the frequency relation carefully. In other words, under the frequency relation with \( k_4 \leq k_5 - 10 \) condition, the one of following cases should be happened (see Section 8 in [3]):

\[
|n_4| \ll |n|^{1/5},
\]
\[
|n_4| \gtrsim |n|^{4/5} \text{ and } |n_3| \sim |n_4|
\]
and

\[
|n_4| \gtrsim |n|^{4/5} \text{ and } |n_3| \ll |n_4|.
\]

For the first case, since

\[
|\mu(n_1) + \mu(n_2) + \mu(n_3) + \mu(n_4) + \mu(n_5) + \mu(n)| \gtrsim |n|^4,
\]
we use \( 2^{-j_{\text{max}}/2} \) instead of \( 2^{-j_{\text{max}}/2} \leq 2^{-k} \) in (6.14), then we obtain

\[
\|v\|_{F^k_{1/2}}^4 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_{k'}^r(T)}^2.
\]

For the second case, since \( |n| \lesssim |n_3|^{\frac{3}{4}}|n_4|^{\frac{5}{4}} \), we use (6.14) so that

\[
\|v\|_{F^{\frac{k}{2}}_{1/2}}^2 \|v\|_{F^{\frac{k}{2}}_{1/2}}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_{k'}^r(T)}^2.
\]

For the last case, since \( n_1 + n_2 + n_3 + n_4 + n_5 + n = 0 \), we have \( |n_5 + n| \sim |n|^{4/5} \), which implies \( |\mu(n_1) + \mu(n_2) + \mu(n_3) + \mu(n_4) + \mu(n_5) + \mu(n)| \gtrsim |n|^{4+\frac{3}{5}} \).

Similarly as the first case, we obtain

\[
A_8(k) \lesssim \|v\|_{F^k_{1/2}}^4 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_{k'}^r(T)}^2.
\]

Together with all bounds of \( A_i(k) \), we obtain

\[
\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0,T]} \left| \int_0^{t_k} E_1 + E_{2,1} + E_{3,1} \, dt \right| \lesssim \left( \|v\|_{F^{k}_{1/2}}^2 + \|v\|_{F^{k}_{1/2}}^4 \right) \|v\|_{F^r_{1/2}}^2.
\] (6.17)

Next, we need to estimate

\[
\left| \int_0^{t_k} E_{2,2} + E_{3,2} \, dt \right|,
\]
but since total number of derivatives in \( E_{2,2} \) and \( E_{3,2} \) is just 1, while that in \( E_1, E_{2,1} \) and \( E_{3,1} \) is 3, we have much better bound as

\[
\left| \int_0^{t_k} E_{2,2} + E_{3,2} \, dt \right| \lesssim \|v\|_{F^{k}_{1/2}}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F_{k'}^r(T)}^2.
\] (6.18)

which implies

\[
\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0,T]} \left| \int_0^{t_k} E_{2,2} + E_{3,2} \, dt \right| \lesssim \|v\|_{F^{k}_{1/2}}^2 \|v\|_{F^r_{1/2}}^2.
\] (6.19)

For

\[
\left| \int_0^{t_k} E_{2,4} + E_{3,4} \, dt \right|,
\]
By the symmetries of \( n_1, n_2 \) and \( n_3, n \), respectively, it suffices to estimate

\[
\sum_{k_1, k_2 \geq 0} \left| \int_0^{t_k} \sum_{n, N, 3, n} \chi_{k_1}(n_1)\hat{\nu}(n_1)\chi_{k_2}(n_2)n_2^2\hat{\nu}(n_2)\psi_k(n_3) \frac{1}{n_3} \hat{\nu}(n_3)\chi_k(n) \frac{1}{n} \hat{\nu}(n) \, dt \right|.
\] (6.20)
and
\[
\sum_{k_1, k_2 \geq 0} \left| \sum_{n_1, n_2, n_3} \chi_{k_1}(n_1) \tilde{\varphi}(n_1) \chi_{k_2}(n_2) \tilde{\varphi}(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{\varphi}(n_3) \chi_k(n) n^2 |\tilde{\varphi}(n)|^2 \tilde{\varphi}(n) \right| dt. \tag{6.21}
\]

For (6.20), we may assume that \( k_1 \leq k_2 \) since there are three derivatives on the \( P_{k_2} v \). Then, by using Lemma 6.4 we obtain that
\[
(6.20) \lesssim \|v\|^2_{F^4(T)} \sum_{|k-k'| \leq 5} 2^{3k/4} \|P_{k'} v\|^4_{F^{k'}(T)}
+ \|v\|^2_{F^4(T)} \sum_{k_1 \geq k + 10} 2^{k_1/2} \|P_{k_1} v\|^2_{F_{k_1}(T)} \sum_{|k-k'| \leq 5} 2^{-3k/2} \|P_{k'} v\|^2_{F^{k'}(T)}
+ \|v\|^2_{F^4(T)} \sum_{k_2 \leq k - 10} 2^{k_2/2} \|P_{k_2} v\|^2_{F_{k_2}(T)} \sum_{|k-k'| \leq 5} 2^{-3k} \|P_{k'} v\|^2_{F^{k'}(T)}
\tag{6.22}
\]

For (6.21), we may also assume that \( k_1 \leq k_2 \) by the symmetry of \( n_1, n_2 \) variables, and then, similarly, we obtain that
\[
(6.21) \lesssim \|v\|^2_{F^4(T)} \sum_{|k-k'| \leq 5} 2^{3k/4} \|P_{k'} v\|^4_{F^{k'}(T)}
+ \|v\|^2_{F^4(T)} \sum_{k_1 \geq k + 10} 2^{-k_1} \|P_{k_1} v\|^2_{F_{k_1}(T)} \sum_{|k-k'| \leq 5} 2^{5k/4} \|P_{k'} v\|^2_{F^{k'}(T)}
+ \|v\|^2_{F^4(T)} \sum_{k_2 \leq k - 10} 2^{k_2/2} \|P_{k_2} v\|^2_{F_{k_2}(T)} \sum_{|k-k'| \leq 5} 2^{-k_2} \|P_{k'} v\|^2_{F^{k'}(T)}
\tag{6.23}
\]

Hence, we obtain
\[
\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0,T]} \left| \int_0^{t_k} E_{1,2,4} + E_{1,3,4} dt \right| \lesssim \|v\|^4_{F^4(T)} \|v\|^2_{F^4(T)}. \tag{6.24}
\]

Lastly, we estimate quintic and septic terms as
\[
\int_0^{t_k} E_{2,3} + E_{3,3} dt. \tag{6.25}
\]
Remark 6.8. Before doing that, we need to check the quintic resonant case in $E_{2,3}$ and $E_{3,3}$. In fact, the only worst terms are of the form of

$$\text{Re}\left[\alpha \sum_{n,N_{3,n}} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n_3) \frac{1}{n_3} \left\{10m_3 \sum_{N_{3,n}} \tilde{v}(n_3,1)\tilde{v}(n_3,2)n^2_{3,3}\tilde{v}(n_3,3)\right\} \chi_k(n)\frac{1}{n}\tilde{v}(n)\right]$$

$$= \text{Re}\left[10\alpha i \sum_{n,N_{3,n}} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n_3)\tilde{v}(n_3,1)\tilde{v}(n_3,2)n^2_{3,3}\tilde{v}(n_3,3)\chi_k(n)\frac{1}{n}\tilde{v}(n)\right],$$

and

$$\text{Re}\left[10\beta i \sum_{n,N_{3,n}} \tilde{v}(n_1)\tilde{v}(n_2)\chi_k(n_3)\tilde{v}(n_3,1)\tilde{v}(n_3,2)n^2_{3,3}\tilde{v}(n_3,3)\chi_k(n)\frac{1}{n}\tilde{v}(n)\right],$$

where $N_{3,n}$ is the same set as $N_{3,n}$ of $n_{3,1}, n_{3,2}$ and $n_{3,3}$ variables. Especially, if $n_{3,3} = -n$ (exact quintic resonant case), we cannot use the maximum modulation effect to attack the derivative in the high frequency mode. But, since $\psi_k$ and $\chi_k$ are real-valued even functions and we have the symmetry on the $n_1 + n_2 + n_{3,1} + n_{3,2} = 0$, we observe that

$$\sum_{\pi \in \Gamma_4(Z)} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n - n_{3,1} - n_{3,2})\tilde{v}(n_{3,1})\tilde{v}(n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2$$

$$= \sum_{\pi \in \Gamma_4(Z)} \tilde{v}(-n_1)\tilde{v}(-n_2)\psi_k(n - n_{3,1} - n_{3,2})\tilde{v}(-n_{3,1})\tilde{v}(-n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2$$

$$= \sum_{\pi \in \Gamma_4(Z)} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n + n_{3,1} + n_{3,2})\tilde{v}(n_{3,1})\tilde{v}(n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2$$

$$= \sum_{\pi \in \Gamma_4(Z)} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n - n_1 - n_2)\tilde{v}(n_{3,1})\tilde{v}(n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2$$

$$= \sum_{\pi \in \Gamma_4(Z)} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n - n_{3,1} - n_{3,2})\tilde{v}(n_{3,1})\tilde{v}(n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2,$$

where $\pi = (n_1, n_2, n_{3,1}, n_{3,2})$. This observation shows

$$10\alpha i \sum_{n, \pi \in \Gamma_4(Z)} \tilde{v}(n_1)\tilde{v}(n_2)\psi_k(n - n_{3,1} - n_{3,2})\tilde{v}(n_{3,1})\tilde{v}(n_{3,2})\chi_k(n)n|\tilde{v}(n)|^2$$

is a purely imaginary number and then the exact quintic resonant interaction component vanishes. Similarly, the quintic resonant term in $E_3$ also vanishes. Hence, we do not need to consider this case any more in the estimation of quintic terms.

We first consider the quinics in $E_{2,3}$. For

$$\sum_{n,N_{3,n}} \tilde{N}_1(v)(n_1)\tilde{v}(n_2)\psi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3)\chi_k(n) \frac{1}{n}\tilde{v}(n) + \tilde{v}(n_1)\tilde{N}_1(v)(n_2)\psi_k(n_3) \frac{1}{n_3} \tilde{v}(n_3)\chi_k(n) \frac{1}{n}\tilde{v}(n),$$

if the frequency support of $n \sim 2^k$ is the widest among the other frequency supports, it suffices to control the following one:

$$\sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq k} 2^{k_4} \left| \sum_{\pi \in \Gamma_4(Z)} \int_0^{t_k} \prod_{i=1}^4 \chi_k(n_i)\tilde{v}(n_i)\chi_k(n_5)\tilde{v}(n_5)\chi_k(n)\tilde{v}(n) \right|. \tag{6.27}$$
From (6.15), we have

\[ 6.24 \lesssim \|v\|_{F^{1/2}_T}^3 \|v\|_{F^{1/2}_T}^4 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F^{1/2}_T}^2. \]

Otherwise, we need to control

\[ \sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \atop |k_5-k_4| \leq 5} 2^{3k_1} 2^{-2k} \sum_{\mathbf{n} \in \mathbb{Z}^5} 4 \prod_{i=1}^4 \chi_k(n_i) \hat{\nu}(n_i) \hat{\nu}(n_3) \hat{\nu}(n_5) \hat{\nu}(n) \]  

(6.28)

but, similarly as before, we have

\[ 6.28 \lesssim \|v\|_{F^{1/2}_T}^3 \|v\|_{F^{1/2}_T}^2 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F^{1/2}_T}^2. \]

For the remainder quintic terms in (6.25), since we do not distinguish between $\psi_k$ and $\chi_k$, it is enough to consider

\[ \sum_{\mathbf{n}, \mathbf{N} \in \mathbb{N}_3} \hat{\nu}(n_1) \hat{\nu}(n_2) \chi_k(n_3) \frac{1}{n_3} \hat{N}_1(v)(n_3) \chi_k(n) \frac{1}{n} \hat{\nu}(n). \]

Even though $\hat{N}_1(v)$ has one total derivative $n$, it is exactly removed by $1/n_3$-factor, and hence we have the following one as the worst term:

\[ \sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4} \sum_{\mathbb{N} \in \mathbb{N}_3} \int_0^T \chi_k(n) \hat{\nu}(n_1) \chi_k(n_2) \hat{\nu}(n_2) \]

\[ \times \chi_k(n_3) \hat{\nu}(n_3) \hat{\nu}(n_4) \chi_k(n_3) \hat{\nu}(n_5) \frac{1}{n} \hat{\nu}(n). \]  

(6.29)

We first focus on the case when $|k-k_3| \leq 5$. If $|k_3-k| \lesssim 5$, since one derivative in the frequency $n_3,3$ can be moved to $n_3,2$ frequency, we, similarly as before, have

\[ 6.24 \lesssim \|v\|_{F^{1/2}_T}^3 \|v\|_{F^{1/2}_T}^3 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F^{1/2}_T}^2. \]

Otherwise, we use the same argument as in the estimation of $A_5(k)$ and then the one of following cases should be happened:

\[ |n_{3,2}| \ll |n|^4/5, \]

\[ |n_{3,2}| \gg |n|^4/5 \text{ and } |n_{3,1}| \sim |n_{3,2}| \]

and

\[ |n_{3,2}| \gg |n|^4/5 \text{ and } |n_{3,1}| \ll |n_{3,2}|. \]

For the first case, since

\[ |\mu(n_1) + \mu(n_2) + \mu(n_3) + \mu(n_4) + \mu(n)| \gg |n|^4, \]

we use $2^{-j_{\max}/2} \lesssim 2^{-2k}$ instead of $2^{-j_{\max}/2} \lesssim 2^{-k}$ in (6.13), then we obtain

\[ 6.24 \lesssim \|v\|_{F^{1/2}_T}^3 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F^{1/2}_T}^2. \]

For the second case, since $|n_{3,2}| \lesssim |n_{3,1}|^{1/2} |n_{3,2}|^{1/2}$, we use (6.14) so that

\[ 6.24 \lesssim \|v\|_{F^{1/2}_T}^3 \|v\|_{F^{1/2}_T}^3 \sum_{|k-k'| \leq 5} \|P_{k'} v\|_{F^{1/2}_T}^2. \]
For the last case, since \( n_1 + n_2 + n_3,1 + n_3,2 + n_3,3 + n = 0 \), we have \(|n_{3,3} + n| \sim |n|^{4/5}\), which implies
\[
\|\mu(n_1) + \mu(n_2) + \mu(n_{3,1}) + \mu(n_{3,2}) + \mu(n_{3,3}) + \mu(n)\| \gtrsim |n|^{4/5}.
\]

Similarly as the first case, we obtain
\[
(6.24) \lesssim \|v\|_{F^{5/2}(T)}^4 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F^{5/2}(T)}^2.
\]

Now, we focus on the case when \( k \leq k_{3,3} - 10 \). From the support property (1.1), we know \(|k_{3,2} - k_{3,3}| \leq 5\), and hence
\[
(6.29) \lesssim \|v\|_{F^{5/2}(T)}^3\|v\|_{F^2(T)}\|v\|_{F^0(T)}\|v\|_{F^0(T)}\|v\|_{F^{3/2}(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F^{5/2}(T)}^2,
\]
for \( s \geq 0 \).

Next, we consider the septic term in (6.25). For the septic term in
\[
\sum_{n,v,N_{3,3},n} \hat{N}_1(v)(n_1)\hat{v}(n_2)\psi_k(n_3)\frac{1}{n_3}\hat{v}(n_3)\chi_k(n)\frac{1}{n}\hat{v}(n) + \hat{v}(n_1)\hat{N}_1(v)(n_2)\psi_k(n_3)\frac{1}{n_3}\hat{v}(n_3)\chi_k(n)\frac{1}{n}\hat{v}(n),
\]
since the quintic term in \( \hat{N}_1(v) \) also has one total derivative, by the symmetry of frequencies, it is enough to control
\[
\sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6} 2^{k_2}2^{-2k} \left| \sum_{\pi \in \Gamma(1)} \int_0^{t_k} \prod_{i=1}^6 \chi_k(n_i)\hat{v}(n_i)\psi_k(n_i)\hat{v}(n_i)\chi_k(n)\hat{v}(n) \right|. \tag{6.30}
\]
We apply (6.10) to (6.30), then we obtain
\[
(6.30) \lesssim \|v\|_{F^{5/2}(T)}^6 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F^{5/2}(T)}^2.
\]
Moreover, for the septic term in
\[
\sum_{n,v,N_{3,3},n} \hat{v}(n_1)\hat{v}(n_2)\psi_k(n_3)\frac{1}{n_3}\hat{N}_1(v)(n_3)\chi_k(n)\frac{1}{n}\hat{v}(n),
\]
since the total derivative of the quintic term in \( \hat{N}_1(v) \) is canceled out by \( 1/n_3 \) factor and there is no difference between \( \psi_k \) and \( \chi_k \) in the septic estimation, it suffices to control
\[
\sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6} 2^{-k} \left| \sum_{\pi \in \Gamma(1)} \int_0^{t_k} \prod_{i=1}^7 \chi_k(n_i)\hat{v}(n_i)\chi_k(n)\hat{v}(n) \right|. \tag{6.31}
\]
By using (6.10), we obtain
\[
(6.31) \lesssim \|v\|_{F^{5/2}(T)}^5\|v\|_{F^0(T)}\|v\|_{F^0(T)}\|v\|_{F^{3/2}(T)}^2 \sum_{|k-k'| \leq 5} \|P_{k'}v\|_{F^{5/2}(T)}^2\|v\|_{F^{5/2}(T)}^2\|v\|_{F^{5/2}(T)}^2\|v\|_{F^{3/2}(T)}^2\|v\|_{F^{3/2}(T)}^2.
\]
Together with all bounds of quintic and septic terms, we conclude that
\[
\sum_{h \geq 1} 2^{2s} \sup_{t \in [0,T]} \left| \int_0^{t_k} E_{2,3} + E_{3,3} \ dt \right| \lesssim (\|v\|_{F^{5/2}(T)}^2\|v\|_{F^{3/2}(T)}^2 + \|v\|_{F^{5/2}(T)}^2\|v\|_{F^{3/2}(T)}^2 + \|v\|_{F^{5/2}(T)}^2\|v\|_{F^{3/2}(T)}^2) \tag{6.32}
\]
and hence, we complete the proof of Proposition 6.7 by recalling the definition of the modified energy (6.3) and gathering (6.11), (6.12), and (6.32).

As a Corollary to Lemma 6.8 and Proposition 6.7, we obtain a priori bound of \( \|v\|_{F^r(T)} \) for a smooth solution \( v \) to the equation (6.1).
Corollary 6.9. Let $s > 2$ and $T \in (0, 1]$. Then, there exists $0 < \delta \ll 1$ such that
\[
\|v\|_{\dot{H}^s(T)}^2 \lesssim \left(1 + \|v_0\|_{\dot{H}^s}^2\right)\|v_0\|_{\dot{H}^s}^2 + \left(1 + \|v\|_{\dot{F}^{s+}_T}^2 + \|v\|_{\dot{F}^{s+}_T}^2\right)\|v\|_{\dot{F}^{s+}_T}^2, \tag{6.33}
\]
for the solution $v \in C([-T, T]; H^\infty(T))$ to (6.1) with $\|v\|_{L^\infty_T H^s_x} \leq \delta$.

In the following, we consider the energy estimate for the difference of two solutions $v_1$ and $v_2$ to the equation in (6.1). Let $w = v_1 - v_2$, then $w$ satisfies
\[
\partial_t \hat{w}(n) - i(n^5 + c_1 n^3 + c_2 n) \hat{w}(n) = \hat{N}_{1,1}(v_1, v_2, w) + \hat{N}_{1,2}(v_1, v_2, w) + \hat{N}_{1,3}(v_1, v_2, w) + \hat{N}_{1,4}(v_1, v_2, w), \tag{6.34}
\]
with $w(0, x) = w_0(x) = v_{1,0}(x) - v_{2,0}(x)$ and where
\[
\hat{N}_{1,1}(v_1, v_2, w) = -20 in^3 (|\hat{v}_1(n)|^2 \hat{w}(n) + \hat{v}_1(n) \hat{v}_2(n) \hat{w}(-n) + |\hat{v}_2(n)|^2 \hat{w}(n)),
\]
\[
\hat{N}_{1,2}(v_1, v_2, w) = 6in \sum_{N_{n}, n} \hat{w}(n_1) \hat{v}_1(n_2) \hat{v}_1(n_3) \hat{v}_1(n_4) \hat{v}_1(n_5)
\]
\[
+ 6in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{w}(n_2) \hat{v}_1(n_3) \hat{v}_1(n_4) \hat{v}_1(n_5)
\]
\[
+ 6in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{w}(n_2) \hat{v}_1(n_3) \hat{v}_1(n_4) \hat{v}_1(n_5)
\]
\[
+ 6in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{v}_2(n_2) \hat{v}_2(n_3) \hat{w}(n_4) \hat{v}_1(n_5)
\]
\[
+ 6in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{v}_2(n_2) \hat{v}_2(n_3) \hat{v}_2(n_4) \hat{w}(n_5),
\]
\[
\hat{N}_{1,3}(v_1, v_2, w) = 10in \sum_{N_{n}, n} \hat{w}(n_1) \hat{v}_1(n_2)n_3^2 \hat{v}_1(n_3)
\]
\[
+ 10in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{w}(n_2)n_3^2 \hat{v}_1(n_3)
\]
\[
+ 10in \sum_{N_{n}, n} \hat{v}_2(n_1) \hat{v}_2(n_2)n_3^2 \hat{w}(n_3)
\]
and
\[
\hat{N}_{1,4}(v_1, v_2, w) = 5in \sum_{N_{n}, n} (n_1 + n_2) \hat{w}(n_1) \hat{v}_1(n_2)n_3 \hat{v}_1(n_3)
\]
\[
+ 5in \sum_{N_{n}, n} (n_1 + n_2) \hat{v}_2(n_1) \hat{w}(n_2)n_3 \hat{v}_1(n_3)
\]
\[
+ 5in \sum_{N_{n}, n} (n_1 + n_2) \hat{v}_2(n_1) \hat{v}_2(n_2)n_3 \hat{w}(n_3).
\]

We denote $\hat{N}_{1,1}(v_1, v_2, w) + \hat{N}_{1,2}(v_1, v_2, w) + \hat{N}_{1,3}(v_1, v_2, w) + \hat{N}_{1,4}(v_1, v_2, w)$ by $\hat{N}_1(v_1, v_2, w)$ only in the proof of Proposition 6.12 below. Similarly as before, for $k \geq 1$, we define the localized modified energy
for the difference of two solutions like

$$E_k(w)(t) = \|P_k w(t)\|_{L^2}^2 + \text{Re} \left[ \bar{\alpha} \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_2(n_1) \bar{\nu}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

(6.35)

and

$$E^*_k(w) = \|P_k w(0)\|_{L^2}^2 + \sum_{k \geq 1} 2^{2k} \sup_{t_k \in [-T, T]} \bar{E}_{1,k}(w)(t_k),$$

where $\bar{\alpha}_1$ and $\bar{\beta}_1$ are real and will be chosen later.

**Remark 6.10.** In fact, in the energy estimates for the difference of two solutions, since the symmetry of functions breaks down, we have to define the modified energy for the difference of two solutions as

$$E_k(w)(t) = \|P_k w(t)\|_{L^2}^2$$

$$+ \text{Re} \left[ \bar{\alpha}_1 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_1(n_1) \bar{\nu}_1(n_2) \psi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

(6.36)

$$+ \text{Re} \left[ \bar{\alpha}_2 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_1(n_1) \bar{\nu}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

$$+ \text{Re} \left[ \bar{\alpha}_3 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_2(n_1) \bar{\nu}_1(n_2) \psi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

$$+ \text{Re} \left[ \bar{\beta}_1 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_1(n_1) \bar{\nu}_1(n_2) \chi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

$$+ \text{Re} \left[ \bar{\beta}_2 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_1(n_1) \bar{\nu}_2(n_2) \chi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right]$$

$$+ \text{Re} \left[ \bar{\beta}_3 \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}_2(n_1) \bar{\nu}_2(n_2) \chi_k(n_3) \frac{1}{n_3} \bar{\omega}(n_3) \chi_k(n) \frac{1}{n} \bar{\omega}(n) \right],$$

where $\bar{\alpha}_j$ and $\bar{\beta}_j$, $j = 1, 2, 3$, are real and will be chosen later.

If one defines the modified energy like (6.35), then one cannot control $\tilde{N}_{1,2}(v_1, v_2, w)$ due to the following term

$$5i \sum_{n \in \mathcal{N}_{3, n}} n_1 \bar{\omega}(n_1) \bar{\nu}_1(n_2) n_3 \bar{v}_1(n_3) + 5i \sum_{n \in \mathcal{N}_{3, n}} n_2 \bar{\nu}_2(n_1) \bar{\omega}(n_2) n_3 \bar{v}_1(n_3),$$

for $|n_2|, |n_3| \ll |n_1| \sim |n|$ or $|n_1|, |n_3| \ll |n_2| \sim |n|$, and one cannot expect the cancellation of the quintic resonant case as in Remark 6.3 either.

On the other hands, in view of (6.4),

$$10i \sum_{n \in \mathcal{N}_{3, n}} \bar{\nu}(n_1) \bar{\nu}(n_2) n_3^2 \bar{v}(n_3)$$
and

\[ 5i \frac{m}{3} \sum_{N_3,n} (n_1 + n_2) n_3 \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \]

can be rewritten as

\[ \frac{10}{3} \frac{m}{3} \sum_{N_3,n} \{ n_1^2 + n_2^2 + n_3^2 \} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \]

and

\[ \frac{5}{3} \frac{m}{3} \sum_{N_3,n} \{ (n_1 + n_2) n_3 + (n_1 + n_3) n_2 + (n_2 + n_3) n_1 \} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3), \]

respectively, by the symmetry of frequencies. Then, by the simple calculation, \( \hat{N}_{1,3}(v_1, v_2, w) \) and \( \hat{N}_{1,4}(v_1, v_2, w) \) can be replaced by

\[
\frac{10}{3} \frac{m}{3} \sum_{N_3,n} \left( \hat{v}_1(n_1) \hat{v}_1(n_2) + \hat{v}_1(n_1) \hat{v}_2(n_2) + \hat{v}_2(n_1) \hat{v}_2(n_2) \right) n_3^2 \hat{w}(n_3) \\
+ \frac{20}{3} \frac{m}{3} \sum_{N_3,n} n_1^2 (\hat{v}_1(n_1) \hat{v}_1(n_2) + \hat{v}_1(n_1) \hat{v}_2(n_2) + \hat{v}_2(n_1) \hat{v}_1(n_2) + \hat{v}_2(n_1) \hat{v}_2(n_2)) \hat{w}(n_3)
\]  \hfill (6.37)

and

\[
\frac{10}{3} \frac{m}{3} \sum_{N_3,n} (n_1 + n_2) \hat{v}_1(n_1) \hat{v}_1(n_2) n_3 \hat{w}(n_3) \\
+ \frac{10}{3} \frac{m}{3} \sum_{N_3,n} (n_1 + n_2) \hat{v}_1(n_1) \hat{v}_2(n_2) n_3 \hat{w}(n_3) \\
+ \frac{10}{3} \frac{m}{3} \sum_{N_3,n} (n_1 + n_2) \hat{v}_2(n_1) \hat{v}_2(n_2) n_3 \hat{w}(n_3) \\
+ 5i \frac{m}{3} \sum_{N_3,n} n_1 \hat{v}_1(n_1) n_2 \hat{v}_1(n_2) \hat{w}(n_3) \\
+ 5i \frac{m}{3} \sum_{N_3,n} n_1 \hat{v}_1(n_1) n_2 \hat{v}_2(n_2) \hat{w}(n_3) \\
+ 5i \frac{m}{3} \sum_{N_3,n} n_1 \hat{v}_1(n_1) n_2 \hat{v}_2(n_2) \hat{w}(n_3)
\]  \hfill (6.38)

respectively. Then, now we can control the first term in \( (6.37) \) and the first three terms in \( (6.38) \) by using the modified energy \( (6.35) \), and also control the rest terms. Moreover, thanks to the formula of the first term in \( (6.37) \), we can get rid of quintic resonant interaction components similarly as in Remark 6.13 (See Remark 6.13 below for more details.) It is the reason why we have to define the modified energy like \( (6.36) \) instead of \( (6.35) \). However, since the exact same argument operates to the each term, we will only consider \( (6.35) \) as the modified energy, and

\[
\hat{N}_{1,3}(v_2, w) = \frac{10}{3} \frac{m}{3} \sum_{N_3,n} \hat{v}_2(n_1) \hat{v}_2(n_2) n_3^2 \hat{w}(n_3) \\
+ \frac{20}{3} \frac{m}{3} \sum_{N_3,n} n_3^2 \hat{v}_2(n_1) \hat{v}_2(n_2) \hat{w}(n_3)
\]  \hfill (6.39)
and
\[ \tilde{N}_{1,4}(v_2, w) = \frac{10}{3} n^2 \sum_{N_{1,n}} (n_1 + n_2) \tilde{v}_2(n_1) \tilde{v}_2(n_2) n_3 \tilde{w}(n_3) \]
(6.40)

as the cubic nonlinear term for the difference of two solutions in order to avoid inconvenience. Also, for the same reason, we use
\[ \tilde{N}_{1,1}(v_2, w) = -20i n |\tilde{v}_2(n)|^2 \tilde{w}(n). \]
(6.41)

Similarly as in Lemma 6.3 we can show that \( \tilde{E}_{1,T}^s(w) \) and \( \|w\|_{E_s(T)} \) are comparable.

**Lemma 6.11.** Let \( s > \frac{1}{2} \). Then, there exists \( 0 < \delta \leq 1 \) such that
\[ \frac{1}{2} \|w\|_{E_s(T)}^2 \leq \tilde{E}_{1,T}^s(w) \leq \frac{3}{2} \|w\|_{E_s(T)}^2, \]
for all \( w \in E_s(T) \cap C([-T, T]; H^s(\mathbb{T})) \) as soon as \( \|v_2\|_{L^\infty_T H^s(\mathbb{T})} \leq \delta \).

**Proposition 6.12.** Let \( s > 2 \) and \( T \in (0, 1] \). Then, for solutions \( w \in C([-T, T]; H^\infty(\mathbb{T})) \) to \( 6.31 \) and \( v_1, v_2 \in C([-T, T]; H^\infty(\mathbb{T})) \) to \( 6.11 \), we have
\[ \tilde{E}_{1,T}^s(w) \lesssim (1 + \|v_1,0\|^2_{H^s_T} + \|v_1,0\|_{H^s_T} + \|v_2,0\|^2_{H^s_T} + \|v_2,0\|_{H^s_T}) \|w_0\|^2 \]
\[ + \left( \|v_1\|^2_{F^s(T)} + \|v_1\|_{F^s(T)} \|v_2\|_{F^s(T)} + \|v_2\|^2_{F^s(T)} \right) \|w\|^2_{F^s(T)} + \]
\[ + \left( \sum_{i,j,k,l=1,2} \|v_i\|_{F^s(T)} \|v_j\|_{F^s(T)} \|v_k\|_{F^s(T)} \|v_l\|_{F^s(T)} \right) \|w\|^2_{F^s(T)} \]
(6.42)
and
\[ \tilde{E}_{1,T}^s(w) \lesssim (1 + \|v_1,0\|^2_{H^s_T} + \|v_1,0\|_{H^s_T} + \|v_2,0\|^2_{H^s_T} + \|v_2,0\|_{H^s_T}) \|w_0\|^2 \]
\[ + \left( \|v_1\|^2_{F^s(T)} + \|v_1\|_{F^s(T)} \|v_2\|_{F^s(T)} + \|v_2\|^2_{F^s(T)} \right) \|w\|^2_{F^s(T)} + \]
\[ + \left( \sum_{i,j,k,l=1,2} \|v_i\|_{F^s(T)} \|v_j\|_{F^s(T)} \|v_k\|_{F^s(T)} \|v_l\|_{F^s(T)} \right) \|w\|^2_{F^s(T)} \]
\[ + \left( \sum_{i,j,k,l=1,2} \|v_i\|_{F^s(T)} \|v_j\|_{F^s(T)} \|v_k\|_{F^s(T)} \|v_l\|_{F^s(T)} \right) \|w\|^2_{F^s(T)} \]
(6.43)

\footnote{In view of Remark 6.10 \( \text{Lemma } 6.11 \) holds true for all \( w \in E_s(T) \cap C([-T, T]; H^s(\mathbb{T})) \) as soon as \( \|v_1\|_{L^\infty_T H^s(\mathbb{T})} \leq \delta \) and \( \|v_2\|_{L^\infty_T H^s(\mathbb{T})} \leq \delta \).}
Proof. We use similar argument as in the proof of Proposition 6.7. For any \( k \in \mathbb{Z}_+ \) and \( t \in [-T, T] \), we differentiate \( \tilde{E}_k(w) \) with respect to \( t \) and deduce that

\[
\frac{d}{dt} \tilde{E}_k(w) = \frac{d}{dt} \tilde{I}(t) + \frac{d}{dt} \tilde{I}(t) + \frac{d}{dt} \tilde{III}(t),
\]

where

\[
\frac{d}{dt} \tilde{I}(t) = \frac{d}{dt} \|P_k w\|_L^2
\]

\[
= 2i \sum_n \chi^2_k(n) n^3 \tilde{v}_1(-n) \tilde{v}_2(-n) \tilde{w}(n) \tilde{w}(n)
\]

\[
+ 2 \text{Re} \left( \sum_n \chi_k(n) \left( \tilde{N}_{1,2}(v_1, v_2, w) + \tilde{N}_{1,3}(v_1, v_2, w) + \tilde{N}_{1,4}(v_1, v_2, w) \right) \chi_k(n) \tilde{w}(n) \right)
\]

\[
=: \tilde{E}_1,
\]

\[
\frac{d}{dt} \tilde{I}(t) = \tilde{E}_{2,1} + \tilde{E}_{2,2} + \tilde{E}_{2,3} =: \tilde{E}_2,
\]

where

\[
\tilde{E}_{2,1} = \text{Re} \left[ \tilde{\alpha} \sum_{n \in \mathbb{Z}_+, n} \left\{ 10n_1 n_2^3 (n_3 + n) + 5n_1^2 n_2^2 (n_3 + n) + 30n_1 n_2^2 n_3 night.ight.
\]

\[
+ 10n_2 n_3 n - 5(n_1 + n_2) n_2^2 n^2 \left\{ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right\},
\]

\[
\tilde{E}_{2,2} = \text{Re} \left[ c_1 \tilde{\alpha} \sum_{n \in \mathbb{Z}_+, n} \left\{ 3n_1 n_2 (n_3 + n) + 6n_2 n_3 n \right\} \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right],
\]

and

\[
\tilde{E}_{2,3} = \text{Re} \left[ \tilde{\alpha} \sum_{n \in \mathbb{Z}_+, n} \tilde{N}_1(v_2(n_1)) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{N}_1(v_2(n_2)) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{N}_1(v_1, v_2, w)(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{v}_2(n_3) \chi_k(n) \frac{1}{n} \tilde{N}_1(v_1, v_2, w)(n) \right].
\]

Also,

\[
\frac{d}{dt} \tilde{III}(t) = \tilde{E}_{3,1} + \tilde{E}_{3,2} + \tilde{E}_{3,3} =: \tilde{E}_3,
\]

where

\[
\tilde{E}_{3,1} = \text{Re} \left[ \tilde{\beta} \sum_{n \in \mathbb{Z}_+, n} \left\{ 10n_1 n_2^3 (n_3 + n) + 5n_1^2 n_2^2 (n_3 + n) + 30n_1 n_2^2 n_3 n
\right.ight.
\]

\[
+ 10n_2 n_3 n - 5(n_1 + n_2) n_2^2 n^2 \left\{ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right\},
\]

\[
\tilde{E}_{3,2} = \text{Re} \left[ c_1 \tilde{\beta} \sum_{n \in \mathbb{Z}_+, n} \left\{ 3n_1 n_2 (n_3 + n) + 6n_2 n_3 n \right\} \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right],
\]

and

\[
\tilde{E}_{3,3} = \text{Re} \left[ \tilde{\alpha} \sum_{n \in \mathbb{Z}_+, n} \tilde{N}_1(v_2(n_1)) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{N}_1(v_2(n_2)) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{N}_1(v_1, v_2, w)(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n)
\]

\[
+ \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{v}_2(n_3) \chi_k(n) \frac{1}{n} \tilde{N}_1(v_1, v_2, w)(n) \right].
\]
and
\[ \widetilde{E}_{3,3} = \text{Re} \left[ \bar{\beta} \sum_{n \in \mathcal{N}_3, n} \widehat{N}_1(v_2)(n_1) \widehat{v}_2(n_2) \chi_k(n) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right. \\
+ \widehat{v}_2(n_1) \widehat{N}_1(v_2)(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \\
+ \widehat{v}_2(n_1) \widehat{v}_2(n_2) \chi_k(n_3) \frac{1}{n_3} \widehat{N}_1(v_1, v_2, w)(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \\
\left. + \widehat{v}_2(n_1) \widehat{v}_2(n_2) \chi_k(n_3) \frac{1}{n_3} \widehat{v}_2(n_3) \chi_k(n) \frac{1}{n} \widehat{N}_1(v_1, v_2, w)(n) \right]. \]

Similarly as in the proof of Proposition 6.7, we need to control
\[ \left| \int_0^{t_k} \widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3 \ dt \right|. \quad (6.44) \]

We first control the terms
\[ -\text{Re} \left[ \frac{20}{3} i \sum_{n \in \mathcal{N}_3, n} \chi_k(n)n \tilde{v}_2(n_1) \tilde{v}_2(n_2) n_3^2 \tilde{w}(n_3) \chi_k(n) \tilde{w}(n) \right] \]
and
\[ -\text{Re} \left[ \frac{20}{3} i \sum_{n \in \mathcal{N}_3, n} \chi_k(n)n(n_1 + n_2) \tilde{v}_2(n_1) \tilde{v}_2(n_2) n_3 \tilde{w}(n_3) \chi_k(n) \tilde{w}(n) \right] \]
in \( \widetilde{E}_1 \),
\[ \text{Re} \left[ -5 \alpha_i \sum_{n \in \mathcal{N}_3, n} (n_1 + n_2) n_3^2 n_2^2 \tilde{v}_2(n_1) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right] \]
and
\[ \text{Re} \left[ -5 \beta_i \sum_{n \in \mathcal{N}_3, n} (n_1 + n_2) n_3^2 n_2^2 \tilde{v}_2(n_1) \tilde{v}_2(n_2) \chi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \right] \]
in \( \widetilde{E}_2 \) and \( \widetilde{E}_3 \), respectively. In order to use Lemma 6.5, we choose \( \alpha = \frac{4}{3} \) and \( \beta = \frac{4}{3} \) (in fact, we need to choose \( \alpha_j = \frac{4}{3} \) and \( \beta_j = \frac{2}{3} \), \( j = 1, 2, 3 \)). Then it suffices to control the following terms:
\[ \sum_{k_1, k_2 \leq k-10} \left| \sum_{n \in \mathcal{N}_3, n} \int_0^{t_k} \chi_k(n) n[\chi_k(n) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) n_3^2 \tilde{w}(n_3)] \chi_k(n) \tilde{w}(n) \ dt \right. \\
+ \frac{1}{2} \sum_{n \in \mathcal{N}_3, n} \int_0^{t_k} (n_1 + n_2) \chi_k(n) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) n_3 \tilde{w}(n_3) \chi_k(n) n \tilde{w}(n) \ dt \quad (6.45) \\
\left. - \sum_{n \in \mathcal{N}_3, n} \int_0^{t_k} (n_1 + n_2) \chi_k(n) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) \psi_k(n_3) n_3 \tilde{w}(n_3) \chi_k(n) n \tilde{w}(n) \ dt \right|, \]
\[ \sum_{0 \leq k_1, k_2 \leq k-10} \left| \sum_{n \in \mathcal{N}_3, n} \int_0^{t_k} \chi_k(n)[(n_1 + n_2) \chi_k(n) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) n_3 \tilde{w}(n_3)] \chi_k(n) n \tilde{w}(n) \ dt \\
- \sum_{n \in \mathcal{N}_3, n} \int_0^{t_k} (n_1 + n_2) \chi_k(n) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) \chi_k(n_3) n_3 \tilde{w}(n_3) \chi_k(n) n \tilde{w}(n) \ dt \right|. \quad (6.46) \]
\[
\sum_{\max(k_1, k_2) \geq -k + 9} \left| \sum_{n, 3, n} \int_0^{t_k} \chi_{k_1} (n_1) \tilde{\psi}_1(n_1) \tilde{\chi}_{k_2} (n_2) \tilde{\psi}_2(n_2) \chi_{k_3} (n_3) n^2 \tilde{\omega}(n_3) \chi_k^2 (n) n \tilde{\omega}(n) \, dt \right|, \quad (6.47)
\]

\[
\sum_{\max(k_1, k_2) \geq -k + 9} \left| \sum_{n, 3, n} \int_0^{t_k} (n_1 + n_2) \chi_{k_1} (n_1) \tilde{\psi}_1(n_1) \tilde{\chi}_{k_2} (n_2) \tilde{\psi}_2(n_2) \chi_{k_3} (n_3) n \tilde{\omega}(n_3) \chi_k^2 (n) n \tilde{\omega}(n) \, dt \right|, \quad (6.48)
\]

and

\[
\sum_{\max(k_1, k_2) \geq -k + 9} \left| \sum_{n, 3, n} \int_0^{t_k} (n_1 + n_2) n^2 \chi_{k_1} (n_1) \tilde{\psi}_1(n_1) \tilde{\chi}_{k_2} (n_2) \tilde{\psi}_2(n_2) \chi_{k_3} (n_3) n \tilde{\omega}(n_3) \chi_k^2 (n) n \tilde{\omega}(n) \, dt \right|, \quad (6.49)
\]

We apply (6.10) and (6.12) to (6.47) and (6.48), respectively, then we have

\[
0.39 + 0.40 \lesssim \sum_{k_1, k_2 \leq -k + 10} 2^{\max(2k_1, 2k_2)} \| P_{k_1} v_1 \|_{F_{k_1}} (T) \| P_{k_2} v_2 \|_{F_{k_2}} (T) \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}} (T)
\]

\[
\lesssim \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}} (T),
\]

which implies

\[
\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0, T]} \left( \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \| w \|_{F^2 (T)} \right) \lesssim \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \| w \|_{F^2 (T)},
\]

(6.50)

whenever \( s \geq 0 \).

For (6.47) and (6.48), similarly as the estimates of \( A_3 (k) \) and \( A_4 (k) \), we divide the summation over \( k_1, k_2, k_3 \) into

\[
\sum_{k_1, k_3 \leq -10} + \sum_{k_1 \leq -10} + \sum_{k_2 \geq -9} + \sum_{k_1, k_2, k_3 \geq -9}
\]

assuming without loss of generality \( k_1 \leq k_2 \). We restrict (6.47) to the first summation, by (6.6) and (6.7), we have

\[
\sum_{k_1 \leq -10} 2^{2sk} \| P_{k_1} v_1 \|_{F_{k_1}} (T) \| P_{k_2} w \|_{F_{k_3}} (T) \sum_{|k-k'| \leq 5} \| P_{k'} v_2 \|_{F_{k_2}} (T) \| P_{k_2} w \|_{F_{k_3}} (T)
\]

\[
\lesssim \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \| w \|_{F^2 (T)} \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}} (T),
\]

For the restriction to the second summation, by using (6.5) and (6.7), we have

\[
\sum_{k_1 \leq -10} 2^{k_1/2} \| P_{k_1} v_2 \|_{F_{k_1}} (T) \sum_{|k-k'| \leq 5} \sum_{|k-k'| \leq 5} 2^{2sk} \| P_{k_2} v_1 \|_{F_{k_2}} (T) \| P_{k_2} v_2 \|_{F_{k_3}} (T) \| P_{k_2} w \|_{F_{k_3}} (T)
\]

\[
\lesssim \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}} (T) + \| v_2 \|_{F^0 (T)} \| v_2 \|_{F^2 (T)} \| w \|_{F^2 (T)} 2^{-sk-ek} \| P_{k} w \|_{F_{k}} (T),
\]

for \( s \geq 0 \) and \( 0 < \varepsilon \ll 1 \). Due to two derivatives in the low frequency mode, we can obtain better or same bounds from the third summation than the second summation. For the last restriction, by using
We obtain

\[ \sum_{|k-k'| \leq 5} 2^{7k/2} \| P_{k'} v_2 \|_{F_{k'}(T)} \| P_{k'} w \|_{F_{k'}(T)} + \sum_{k_3 \geq k+9 \atop |k_3-k'| \leq 5} 2^{k_3} \| P_{k_3} v_2 \|_{F_{k_3}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \sum_{k_3 \geq k+9 \atop |k_2-k_3| \leq 5} 2^{k_3} \| P_{k_3} v_2 \|_{F_{k_3}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \]

\[ + \sum_{|k-k'| \leq 5} 2^{3k/2} \| P_{k_1} v_2 \|_{F_{k_1}(T)} \| P_{k_1} w \|_{F_{k_1}(T)} \sum_{k_3 \geq k+9 \atop |k_2-k_3| \leq 5} 2^{k_3} \| P_{k_3} v_2 \|_{F_{k_3}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \]

\[ \lesssim \| v_2 \|_{F_{k'}(T)}^2 \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}(T)}^2 \| w \|_{F_{k'}(T)} \| P_{k'} w \|_{F_{k'}(T)} \]

\[ + \| v_2 \|_{F^2(T)} \| w \|_{F_{k'}(T)} \sum_{|k-k'| \leq 5} 2^{-s k} \| P_{k_1} v_2 \|_{F_{k_1}(T)} \| P_{k_1} w \|_{F_{k_1}(T)} \]

\[ + \| v_2 \|_{F^0(T)} \| v_2 \|_{F^2(T)} \| w \|_{F_{k'}(T)} \| P_{k'} w \|_{F_{k'}(T)}, \]

for \( s \geq 0 \) and \( 0 < \varepsilon \ll 1 \).

For (6.38), by using (6.16) and (6.27, 6.38) restricted to the first summation in (6.31) is dominated by

\[ \left( \sum_{k_1 \leq k_3-10} 2^{k_1} \| P_{k_1} v_2 \|_{F_{k_1}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \sum_{|k-k'| \leq 5} 2^{k_3} \| P_{k_3} v_2 \|_{F_{k_3}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \right) \]

\[ \lesssim \| v_2 \|_{F^0(T)} \| w \|_{F^0(T)} \sum_{|k-k'| \leq 5} 2^{2k_2+2k_3} \| P_{k_2} v_2 \|_{F_{k_2}(T)} \| P_{k_2} w \|_{F_{k_2}(T)}, \]

For the restriction to the other summations, we obtain the same result as the estimation of (6.37). Hence, we obtain

\[ \sum_{k_1 \geq k+10} 2^{2k} \sup_{t_k \in [0,T]} \left( (6.41) + (6.38) \right) \lesssim \| v_2 \|_{F^2(T)} \| w \|_{F_{k'}(T)} \| P_{k'} w \|_{F_{k'}(T)} \]

whenever \( s > 2 \) and

\[ \sum_{k_1 \geq k+10} \sup_{t_k \in [0,T]} \left( (6.41) + (6.38) \right) \lesssim \| v_2 \|_{F^0(T)} \| v_2 \|_{F^2(T)} \| w \|_{F^0(T)}, \]

at \( L^2 \)-level.

For (6.43), similarly as (6.14), we obtain

\[ \left( \sum_{k_1 \geq k+10} 2^{k_1/2} \| P_{k_1} v_2 \|_{F_{k_1}(T)} \| P_{k_1} w \|_{F_{k_1}(T)} \sum_{|k-k'| \leq 5} 2^{k_3} \| P_{k_3} v_2 \|_{F_{k_3}(T)} \| P_{k_3} w \|_{F_{k_3}(T)} \right) \]

\[ \lesssim \left( \| v_2 \|_{F_{k'}(T)}^2 + \| v_2 \|_{F_{k'}(T)} \| v_2 \|_{F^2(T)} \right) \sum_{|k-k'| \leq 5} \| P_{k'} w \|_{F_{k'}(T)}^2, \]

which implies

\[ \sum_{k_1 \geq k+10} 2^{2k} \sup_{t_k \in [0,T]} \left( (6.43) \right) \lesssim \left( \| v_2 \|_{F_{k'}(T)}^2 + \| v_2 \|_{F_{k'}(T)} \| v_2 \|_{F^2(T)} \right) \| w \|_{F_{k'}(T)}^2, \]

whenever \( s \geq 0 \).
Now, we focus on the rest terms in $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$. First, we estimate the cubic terms in $\tilde{E}_1$. Since

$$\left| \sum_n \chi_k^2(n) n^3 \tilde{v}_1(-n) \tilde{v}_2(-n) \tilde{w}(n) \tilde{w}(n) \right| \lesssim \|v_1\|_{H^2} \|v_2\|_{H^2} \|P_k w\|_{L^2}$$

and $F^s(T) \mapsto C_T H^s(\mathbb{R})$, the first term in $\tilde{E}_1$ is dominated by

$$\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0,T]} \left| \sum_n \int_0^{t_k} \chi_k^2(n) n^3 \tilde{v}_1(-n) \tilde{v}_2(-n) \tilde{w}(n) \tilde{w}(n) dt \right| \lesssim \|v_1\|_{F^s(T)} \|v_2\|_{F^s(T)} \|w\|_{F^s(T)}^2. \quad (6.55)$$

For the nonresonant interaction components in $\tilde{E}_1$, it suffices from \((6.39)\) and \((6.40)\) to consider

$$\left| \int_0^{t_k} \sum_{n \in \mathbb{Z}, n} \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{w}(n) \tilde{w}(n) \right| \quad (6.56)$$

and

$$\left| \int_0^{t_k} \sum_{n \in \mathbb{Z}, n} \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{w}(n) \tilde{w}(n) \right|. \quad (6.57)$$

For \((6.56)\), since there are two derivatives on the $P_k v_2$ and one derivative on the $P_k w$, it is enough to consider the case when $k = \max(k_1, k_2, k_3, k)$ and $|k_2 - k| \leq 5$. Then, we use Lemma 6.3 to obtain that

$$\sum_{k_1, k_2, k_3 \geq 0} \left| \int_0^{t_k} \sum_{n \in \mathbb{Z}, n} \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{v}_2(n) \chi_k(n) \tilde{w}(n) \tilde{w}(n) \right|$$

$$\lesssim \sum_{|k-k'| \leq 5} 2^{2sk} \|P_{k'} v_2\|_{F^s(T)} \|P_{k'} v_2\|_{F^s(T)} \|P_k w\|_{F^s(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_1} v_2\|_{F_{k_1}(T)} \|P_{k'} v_2\|_{F^s(T)} \|P_k w\|_{F^s(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_2} v_2\|_{F_{k_2}(T)} \|P_{k'} v_2\|_{F^s(T)} \|P_k w\|_{F^s(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_3} v_2\|_{F_{k_3}(T)} \|P_{k'} v_2\|_{F^s(T)} \|P_k w\|_{F^s(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_1} v_2\|_{F_{k_1}(T)} \|P_{k} v_2\|_{F_{k}(T)} \|P_{k} w\|_{F_{k}(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_2} v_2\|_{F_{k_2}(T)} \|P_{k} v_2\|_{F_{k}(T)} \|P_{k} w\|_{F_{k}(T)}$$

$$+ \sum_{|k-k'| \leq 5} 2^{2sk} 2^{k/2} \|P_{k_3} v_2\|_{F_{k_3}(T)} \|P_{k} v_2\|_{F_{k}(T)} \|P_{k} w\|_{F_{k}(T)}$$

$$\lesssim \left( \|v_2\|_{F^s(T)}^2 + \|v_2\|_{F^{s+1}(T)} \|v_2\|_{F^s(T)} \right) \sum_{|k-k'| \leq 5} \|P_k w\|_{F^s(T)}^2$$

$$+ \left( \|v_2\|_{F^{s+1}(T)} \|w\|_{F^{s}(T)} + \|v_2\|_{F^{s+1}(T)} \|v_2\|_{F^{s+2}(T)} \|w\|_{F^s(T)} \right) 2^{-sk-zk} \|P_k w\|_{F^s(T)}.$$
This implies

$$\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0, T]} \sum_{k_1, k_2, k_3 \geq 0} (6.56) \lesssim \left( \|v_2\|_{F^2(T)}^2 + \|v_2\|_{F^2(T)} \|v_2\|_{F^2(T)} \right) \|w\|_{F^2(T)}^2 \tag{6.58}$$

for $s > 2$ and

$$\sum_{k \geq 1} \sup_{t_k \in [0, T]} \sum_{k_1, k_2, k_3 \geq 0} (6.56) \lesssim \left( \|v_2\|_{F^2(T)}^2 + \|v_2\|_{F^2(T)} \|v_2\|_{F^2(T)} \right) \|w\|_{F^2(T)}^2 \tag{6.59}$$

at $L^2$-level.

For (6.57), since derivatives are distributed to several functions, this term is weaker than (6.56) in some sense and hence we omit to estimate it.

For the rest cubic terms in $E_{2,1}, E_{2,2}, E_{3,1}$ and $E_{3,2}$, it is enough to consider

$$\sum_{k_1, k_2 \geq 0} \left| \int_0^{t_k} \sum_{n, N, m, n} \chi_{k_1} (n_1)(n_1) \chi_{k_2} (n_2) \chi_{k_3} (n_3) \chi_{k_4} (n_4) \frac{1}{n} \hat{w}(n) \right| \tag{6.60}$$

and

$$\sum_{k_1, k_2 \geq 0} \left| \int_0^{t_k} \sum_{n, N, m, n} \chi_{k_1} (n_1) \chi_{k_2} (n_2) \chi_{k_3} (n_3) \chi_{k_4} (n_4) \hat{w}(n) \right| \tag{6.61}$$

under the assumption that $k_1 \leq k_2 \leq k_3 \leq k_4$. Moreover, we may consider (6.60) and (6.61) for $k \leq k_1 - 10$ and $k_1 \leq k - 10$, respectively. If $k_2 \leq k - 10$, by using (6.5) and (6.7), we obtain

$$\sum_{k_2 \leq k - 10} 2^{2k_2} 2^{-k} \|P_{k_1} v_2\|_{F_k(T)} \|P_{k_2} v_2\|_{F_k(T)} \|P_k w\|_{F_k(T)}^2$$

$$+ \sum_{k_2 \leq k - 10} 2^{2k_2} 2^{-k} \|P_{k_1} v_2\|_{F_k(T)} \|P_{k_2} v_2\|_{F_k(T)} \|P_k w\|_{F_k(T)}^2 \lesssim \|v_2\|_{F^0(T)}^2 \|v_2\|_{F^2(T)} \|P_k w\|_{F_k(T)}^2 + \|v_2\|_{F^2(T)}^2 \|P_k w\|_{F_k(T)}^2.$$  

If $k_1 \leq k - 10$ and $|k - k_2| \leq 5$, by using (6.5), we have

$$\sum_{k_2 \leq k - 10} 2^{2k_2} 2^{k_2} \|P_{k_1} v_2\|_{F_k(T)} \|P_{k_2} v_2\|_{F_k(T)} \|P_k w\|_{F_k(T)}^2 \lesssim \|v_2\|_{F^2(T)} \|v_2\|_{F^2(T)} \|P_k w\|_{F_k(T)}^2.$$  

\[\text{In fact, since there are two more derivatives in } \tilde{E}_{2,1} \text{ and } \tilde{E}_{3,1} \text{ than } \tilde{E}_{2,2} \text{ and } \tilde{E}_{3,2}, (6.56) \text{ and } (6.61) \text{ dominate all terms in } \tilde{E}_{2,2} \text{ and } \tilde{E}_{3,2}.\]
Otherwise, we use (6.4) and (6.6) to obtain that
\[
\sum_{|k_1-k| \leq 5} 2^{\frac{7}{2}k} \| P_{k_1} v_2 \|_{F_{k_2}(T)} \| P_{k_2} v_2 \|_{F_{k_2}(T)} \| P_k w \|_{F_k(T)}^2 \\
+ \sum_{k \leq k_3 \leq 10 \atop |k_1-k_2| \leq 5} 2^{\frac{5}{2}k} 2^{3k_1} \| P_{k_1} v_2 \|_{F_{k_2}(T)} \| P_{k_2} v_2 \|_{F_{k_2}(T)} \| P_k w \|_{F_k(T)}^2 \\
\lesssim \| v_2 \|_{F_{\tilde{\tau}}(T)}^2 \| P_k w \|_{F_k(T)}^2 + \| v_2 \|_{F_{\tilde{\tau}}(T)} \| v_2 \|_{F_{\tilde{\tau}}(T)} \| P_k w \|_{F_k(T)}^2.
\]
Above results imply that
\[
\sum_{k \geq 1} 2^{2k} \sup_{t_k \in [0,T]} ( (6.60) + (6.61) ) \lesssim \left( \| v_2 \|_{F_{\tilde{\tau}}(T)}^2 + \| v_2 \|_{F_{\tilde{\tau}}(T)} \| v_2 \|_{F_{\tilde{\tau}}(T)} \right) \| w \|_{F_{\tilde{\tau}}(T)}^2, \tag{6.62}
\]
whenever \( s \geq 0 \).

Now, we concentrate quintic and septic terms in (6.4). We first estimate quintic terms in \( E_1 \). Since we can observe the symmetry of functions and one derivative is taken on \( P_k w \), it suffices to consider
\[
\left| \sum_{n,n_1,n_2,n_3} \chi_{k_1}(n_1) \hat{v}_2(n_1) \chi_{k_2}(n_2) \hat{v}_2(n_2) \chi_{k_3}(n_3) \hat{v}_2(n_3) \chi_{k_4}(n_4) \hat{v}_2(n_4) \chi_{k_5}(n_5) \hat{v}(n_5) \chi^2(n) n \hat{w}(n) dt \right|, \tag{6.63}
\]
under the assumption that \( k_1 \leq k_2 \leq k_3 \leq k_4 \) and \( k_5 \leq k \). When \( k_4 \leq k_5 - 10 \), similarly as in the estimation of \( A_3(k) \), the one of the following cases should be handled:
\[
|n_4| \ll |n|^{4/5},
\]
\[
|n_4| \gtrsim |n|^{4/5} \quad \text{and} \quad |n_3| \sim |n_4|
\]
and
\[
|n_4| \gtrsim |n|^{4/5} \quad \text{and} \quad |n_3| \ll |n_4|.
\]
We use \( 2^{-j_{\max}/2} \lesssim 2^{-2k} \) in (6.15), \( |n| \lesssim |n_3|^{5/8} |n_4|^{5/8} \) and \( 2^{-j_{\max}/2} \lesssim 2^{-2j_{\max}/k} \) in (6.15) for each case, respectively, we obtain
\[
\sum_{|k-k_3| \leq 5 \atop k_4 \leq k_5 - 10} (6.63) \lesssim \| v_2 \|_{F_{\tilde{\tau}}(T)}^2 \| v_2 \|_{F_{\tilde{\tau}}(T)} \sum_{|k-k_3| \leq 5} \| P_k w \|_{F_{k_3}(T)}^2
\]
by using (6.15). If \( |k_5 - k_4| \leq 5 \) and \( k_3 \leq k_4 - 10 \), one derivative on \( P_k w \) can be moved to \( P_{k_4} v_2 \), and hence we get from (6.14) that
\[
\sum_{|k-k_3| \leq 5 \atop k_4 \leq k_5 - 10} (6.63) \lesssim \| v_2 \|_{F_{\tilde{\tau}}(T)}^3 \| v_2 \|_{F_{\tilde{\tau}}(T)} \sum_{|k-k_3| \leq 5} \| P_k w \|_{F_{k_3}(T)}^2
\]
Otherwise, similarly as the case when \( k_4 \leq k_5 - 10 \), we obtain
\[
\sum_{|k-k_3| \leq 5 \atop k_4 \leq k_5 - 10} (6.63) \lesssim \| v_2 \|_{F_{\tilde{\tau}}(T)}^2 \| v_2 \|_{F_{\tilde{\tau}}(T)} \| w \|_{F_{\tilde{\tau}}(T)} \sum_{|k-k_3| \leq 5} \| P_k v_2 \|_{F_{k_3}(T)} \| P_k w \|_{F_{k_3}(T)}
\]
at worst. Hence, we conclude that
\[
\sum_{k \geq 1} \sup_{t_\ell \in [0,T]} \sum_{k_1, k_2, k_3, k_4, k_5 \geq 0} 2^{k_1 + k_2 + k_3 + k_4 + k_5} \left| \left| v_{t_\ell} \right| \right|_{L^2(T)}^2 \left| \left| v_{t_\ell} \right| \right|_{L^2(T)}^2 \left| \left| w \right| \right|_{L^2(T)}^2
\]
whenever $s \geq \frac{n}{2}$, and
\[
\sum_{k \geq 1} \sup_{t_\ell \in [0,T]} \sum_{k_1, k_2, k_3, k_4, k_5 \geq 0} 2^{k_1 + k_2 + k_3 + k_4 + k_5} \left| \left| v_{t_\ell} \right| \right|_{L^2(T)}^2 \left| \left| v_{t_\ell} \right| \right|_{L^2(T)}^2 \left| \left| w \right| \right|_{L^2(T)}^2
\]
at $L^2$-level.

**Remark 6.13.** From Remark 6.10 and 6.3 we have to check whether quartic resonant interaction components in $\tilde{E}_2$ and $\tilde{E}_3$ really vanish or not. When we use the full modified energy in \[8.3\] and the full nonlinear term in \[8.33\], there should be 9-resonant interaction components in $\tilde{E}_2$ as the worst term as follows:

\[
\sum_{n, \pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_i(n_{3,1})\tilde{v}_i(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2 \quad i = 1, 2,
\]

\[
\sum_{n, \pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_i(n_{3,1})\tilde{v}_i(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2 \quad i = 1, 2,
\]

\[
\sum_{n, \pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_i(n_{3,1})\tilde{v}_i(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2 \quad i = 1, 2
\]

and for $i = 1, j = 2$ or $i = 2, j = 1$,

\[
\sum_{n, \pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_j(n_{3,1})\tilde{v}_j(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2
\]

where $\pi = (n_1, n_2, n_{3,1}, n_{3,2})$. In view of \[6.20\], we can easily know that \[6.60\] and \[6.61\] vanish since those are purely real numbers. Furthermore, since we chose $\tilde{v}_j$ as the same number for all $j = 1, 2, 3$, by combining some of terms in \[6.68\], \[6.69\] and \[6.70\], we can make all terms vanish. For example, we have from the same argument as in \[6.23\] that

\[
\sum_{\pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_i(n_{3,1})\tilde{v}_i(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2
\]

\[
+ \sum_{\pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_1(n_{3,1})\tilde{v}_2(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2
\]

\[
+ \sum_{\pi \in \Gamma_3(\mathbb{Z})} \tilde{v}_i(n_1)\tilde{v}_i(n_2)\chi_k(n - n_{3,1} - n_{3,2})\tilde{v}_i(n_{3,1})\tilde{v}_i(n_{3,2})\chi_k(n)|\tilde{w}(n)|^2
\]

for $i = 1, 2$. Of course, \[6.70\] can vanish by summing $i = 1, j = 2$ term and $i = 2, j = 1$ term. Hence, in the following, we do not need to consider all terms in \[6.60\] – \[6.70\].
For the other quintic resonant interaction components, it suffices to consider
\[
\int_0^{t_k} \sum_{n,m \in \Gamma_3(\mathbb{Z})} \tilde{v}_2(n_1) \tilde{v}_2(n_2) \chi_k(n - n_{3,1} - n_{3,2}) \tilde{v}_2(n_{3,1}) \tilde{w}(n_3,2) \chi_k(n)n \tilde{v}_2(-n) \tilde{w}(n) \, dt .
\] (6.71)

Once we use the similar way as in (6.68) with (6.69), we can obtain
\[
(6.71) \lesssim \sum_{k_1,k_2,k_{3,1},k_{3,2} \geq 0} 2^{(k_m + n + k_{3,1} + k_{3,2})/2} \| P_{k_1} v_2 \|_{\mathcal{F}_1(T)} \| P_{k_2} v_2 \|_{\mathcal{F}_{k_{3,1},1}(T)} \| P_{k_{3,1},2} \|_{\mathcal{F}_{1,k_{3,2},1}(T)} \| P_{k_{3,2}} w \|_{\mathcal{F}_{1,k_{3,2},2}(T)} \times \sum_{|k - k'| \leq 5} 2^k \| P_{k'} v_2 \|_{L^\infty_T L^2_x} \| P_{k'} w \|_{L^\infty_T L^2_x} .
\]

By using the embedding \(F^s(T) \hookrightarrow C_T H^s\) for \(s \geq 0\), we conclude that
\[
\sum_{k \geq 1} 2^{2sk} \sup_{t_k \in [0,T]} (6.71) \lesssim \| v_2 \|_{3,3}^3_{F^{1/2}(T)} \| w \|_{3,3}^3_{F^1(T)} \| v_2 \|_{3,3}^2_{F^{1/2}(T)} \| w \|_{3,3}^2_{F^1(T)},
\] (6.72)
whenever \(s \geq 0\), and
\[
\sum_{k \geq 1} \sup_{t_k \in [0,T]} (6.71) \lesssim \| v_2 \|_{3,3}^3_{F^{1/2}(T)} \| v_2 \|_{3,3}^2_{F^1(T)} \| w \|_{3,3}^2_{F^1(T)}
\] (6.73)
at \(L^2\)-level.

Of course, the same argument holds for the quintic resonant terms in \(\tilde{E}_3\).

For quintic terms in \(\tilde{E}_2\) and \(\tilde{E}_3\), by the symmetries of \(n_1, n_2\) and \(n_3, n\) variables, respectively, it is enough to consider
\[
\sum_{k_1,k_2 \geq 0} \int_0^{t_k} \sum_{n,m,n_3,n} \chi_{k_1}(n_1) \tilde{v}_2(n_1) \chi_{k_2}(n_2) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \, dt .
\] (6.74)
and
\[
\sum_{k_1,k_2 \geq 0} \int_0^{t_k} \sum_{n,m,n_3,n} \chi_{k_1}(n_1) \tilde{v}_2(n_1) \chi_{k_2}(n_2) \tilde{v}_2(n_2) \psi_k(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n_3) \, dt .
\] (6.75)

For \(N(v_2, w)\) in (6.41), we use the same way as (6.22) and (6.23) to estimate (6.74) and (6.75), and then we obtain
\[
\sum_{k \geq 1} 2^{sk} \sup_{t_k \in [0,T]} \sum_{k_1,k_2} (6.74) + (6.75) \lesssim \| v_2 \|_{3,3}^4_{F^{1/2}(T)} \| w \|_{3,3}^2_{F^1(T)},
\] (6.76)
for \(s \geq 0\).

For the rest quintic terms, it suffices to consider
\[
\int_0^{t_k} \sum_{n,m,n_3,n_2} \chi_{k_1}(n_1) \tilde{v}_2(n_1) \chi_{k_2,1}(n_2,1) \tilde{v}_2(n_{2,1}) \chi_{k_{2,2}}(n_{2,2}) \tilde{v}_2(n_{2,2}) \chi_{k_3}(n_3) \frac{1}{n_3} \tilde{w}(n_3) \chi_k(n) \frac{1}{n} \tilde{w}(n) \, dt
\] (6.77)
under the assumption that $k_{2,1} \leq k_{2,2} \leq k_{2,3}$, and

\[
\left| \int_0^{t_k} \sum_{n \in \mathcal{N}_3, n \in \mathcal{N}_3} \chi_k(n_1) \tilde{v}_2(n_1) \chi_k(n_2) \tilde{v}_2(n_2) \chi_k(n_3) \chi_k(n_4) \chi_k(n_5) \chi_k(n_6) \, dt \right|
\]

(6.78)

under the assumption that $k_{3,1} \leq k_{3,2} \leq k_{3,3}$. But, both (6.77) and (6.78) can be controlled by the exact same argument as in the estimation of the quintic term in $\tilde{E}_1$, and hence we conclude that

\[
\sum_{k \geq 1} 2^{2s^2k} \sup_{t_k \in [0,T]} \left( \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.77) + \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.78) \right)
\]

(6.79)

ger \lesssim \left\| v_2 \right\|_{F^{s+1}(T)}^2 \left\| v_2 \right\|_{F^{s+1}(T)} \left\| w \right\|_{F^s(T)}^2 + \left\| v_2 \right\|_{F^{s+1}(T)}^2 \left\| v_2 \right\|_{F^{s+1}(T)} \left\| w \right\|_{F^s(T)}^2,
\]

whenever $s \geq \frac{9}{8}$, and

\[
\sum_{k \geq 1} 2^{2s^2} \sup_{t_k \in [0,T]} \left( \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.77) + \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.78) \right)
\]

(6.80)

ger \lesssim \left\| v_2 \right\|_{F^{s+1}(T)}^2 \left\| v_2 \right\|_{F^{s+1}(T)} \left\| w \right\|_{F^s(T)}^2,
\]
at $L^2$-level.

Finally, we estimate septic terms in $\tilde{E}_2$ and $\tilde{E}_3$. From (6.35) and the symmetry of functions, it is enough to consider

\[
\left| \int_0^{t_k} \sum_{\pi \in \Gamma_8(\mathbb{Z})} n_6 \prod_{j=1}^6 \chi_k(n_j) \tilde{v}_2(n_j) \chi_k(n_j) \frac{1}{n_j} \tilde{w}(n_j) \chi_k(n) \frac{1}{n} \tilde{w}(n) \, dt \right|
\]

(6.81)

and

\[
\left| \int_0^{t_k} \sum_{\pi \in \Gamma_8(\mathbb{Z})} \prod_{j=1}^6 \chi_k(n_j) \tilde{v}_2(n_j) \chi_k(n_j) \frac{1}{n_j} \tilde{w}(n_j) \chi_k(n) \frac{1}{n} \tilde{w}(n) \, dt \right|
\]

(6.82)

under the assumption that $k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq k_6$. Similarly as in the proof of Proposition 6.7, we obtain by using (6.14) that

\[
\sum_{k \geq 1} 2^{2s^2k} \sup_{t_k \in [0,T]} \left( \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.81) + \sum_{k_1, k_2, k_3, k_4, k_5, k_6 \geq 0} (6.82) \right)
\]

(6.83)

ger \lesssim \left\| v_2 \right\|_{F^{s+1}(T)}^5 \left\| v_2 \right\|_{F^{s+1}(T)} \left\| v_2 \right\|_{F^s(T)}^2 + \left\| v_2 \right\|_{F^{s+1}(T)}^2 \left\| v_2 \right\|_{F^{s+1}(T)} \left\| w \right\|_{F^s(T)}^2,
\]

whenever $s \geq 0$.

Therefore, by gathering (6.30), (6.32), (6.33), (6.34), (6.35), (6.36), (6.39), (6.42), (6.43), (6.44), (6.45), (6.46) and (6.83), and by recalling the definition of the modified energy (6.36), we obtain (6.43). Also, by gathering (6.50), (6.53), (6.51), (6.54), (6.55), (6.56), (6.63), (6.64) and (6.83), we get (6.42). □

As a Corollary to Lemma 6.11 and Proposition 6.12 we obtain a priori bound of $\left\| w \right\|_{E^s(T)}$ for the difference of two solutions.
Corollary 6.14. Let $s > 2$ and $T \in (0, 1]$. Then, there exists $0 < \delta \ll 1$ such that
\[
\|w\|_{E^0(T)}^2 \lesssim (1 + \|v_1,0\|_{H_+^5}^2 + \|v_{1,0}\|_{H_+^5} + \|v_{2,0}\|_{H_+^5}) \|w_0\|_{L^2}^2
\]
\[+ \left( \|v_1\|_{F^2(T)}^2 + \|v_1\|_{F^2(T)} \|v_2\|_{F^2(T)} + \|v_2\|_{F^2(T)}^2 \right) \|w\|_{F^0(T)}^2
\]
\[+ \left( \sum_{i,j,k,l=1,2} \|v_i\|_{F^{\frac{1}{2}}(T)} \|v_j\|_{F^{\frac{1}{2}}(T)} \|v_k\|_{F^{\frac{1}{2}}(T)} \|v_l\|_{F^{\frac{1}{2}}(T)} \right) \|w\|_{F^0(T)}^2
\]
\[+ \left( \sum_{i,j,k,l,m,n=1,2} \|v_i\|_{F^{\frac{1}{2}}(T)} \|v_j\|_{F^{\frac{1}{2}}(T)} \|v_k\|_{F^{\frac{1}{2}}(T)} \|v_m\|_{F^{\frac{1}{2}}(T)} \|v_n\|_{F^{\frac{1}{2}}(T)} \right) \|w\|_{F^0(T)}^2.
\]

(6.84)

and
\[
\|w\|_{E^s(T)}^2 \lesssim (1 + \|v_1,0\|_{H_+^5}^2 + \|v_{1,0}\|_{H_+^5} + \|v_{2,0}\|_{H_+^5}) \|w_0\|_{H^s}
\]
\[+ \left( \|v_1\|_{F^s(T)}^2 + \|v_1\|_{F^s(T)} \|v_2\|_{F^s(T)} + \|v_2\|_{F^s(T)}^2 \right) \|w\|_{F^s(T)}^2
\]
\[+ \left( \sum_{i,j=1,2} \|v_i\|_{F^{\frac{1}{2}}(T)} \|v_j\|_{F^{\frac{1}{2}}(T)} \right) \|w\|_{F^0(T)} \|w\|_{F^s(T)}
\]
\[+ \left( \sum_{i,j,k,l=1,2} \|v_i\|_{F^{\frac{1}{2}}(T)} \|v_j\|_{F^{\frac{1}{2}}(T)} \|v_k\|_{F^{\frac{1}{2}}(T)} \|v_l\|_{F^{\frac{1}{2}}(T)} \right) \|w\|_{F^0(T)} \|w\|_{F^s(T)}
\]
\[+ \left( \sum_{i,j,k,l,m,n=1,2} \|v_i\|_{F^{\frac{1}{2}}(T)} \|v_j\|_{F^{\frac{1}{2}}(T)} \|v_k\|_{F^{\frac{1}{2}}(T)} \|v_m\|_{F^{\frac{1}{2}}(T)} \|v_n\|_{F^{\frac{1}{2}}(T)} \right) \|w\|_{F^0(T)}^2.
\]

(6.85)

for solutions $w \in C([-T,T]; H^\infty(T))$ to \[\square\] and $v_1, v_2 \in C([-T,T]; H^\infty(T))$ to \[\square\] satisfying $\|v_1\|_{L^\infty_T H^\frac{1}{2}_x} < \delta$ and $\|v_2\|_{L^\infty_T H^\frac{1}{2}_x} < \delta$.

7. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1. The main ingredients are the multilinear estimates and energy estimates which are shown in section 5 and 6 respectively. The method is the compactness argument which follows basically the idea of Ionescu, Kenig and Tataru.

Proposition 7.1. Let $s \geq 0$, $T \in (0,1]$ and $v \in F^s(T)$, then
\[
\sup_{t \in [-T,T]} \|v(t)\|_{H^s(T)} \lesssim \|v\|_{F^s(T)}.
\]

(7.1)

Proposition 7.2. Let $T \in (0,1]$ and $v, w \in C([-T,T]; H^\infty)$ satisfying
\[
\partial_t \hat{v}(n) + i\mu(n) \hat{v}(n) = \hat{w}(n) \text{ on } (-T,T) \times T.
\]

(7.2)

Then, for any $s \geq 0$, we have
\[
\|v\|_{F^s(T)} \lesssim \|v\|_{E^s(T)} + \|w\|_{N^s(T)}.
\]

Proof. Even though this problem is under the periodic condition, the proofs of Proposition \[\square\] and \[\square\] are exactly same as in \[\square\]. See Appendix A in \[\square\].
7.1. Small data local well-posedness. We first state the local well-posedness result for (1.2) with the small initial data.

**Proposition 7.3.** Let \( s > 2 \) and \( T \in (0,1) \). For any \( u_0 \in H^s(\mathbb{T}) \) with specified

\[
\int_T (u_0(x))^2 \, dx = \gamma_1, \quad \int_T (\partial_x u_0(x))^2 + (u_0(x))^4 \, dx = \gamma_2, \tag{7.3}
\]

for some \( \gamma_1, \gamma_2 \geq 0 \), and \( \|u_0\|_{H^s(\mathbb{T})} \leq \delta_0 \ll 1 \), then (1.2) has a unique solution \( u(t) \) on \([-T,T] \) satisfying

\[
u(t,x) \in C([-T,T]; H^s(\mathbb{T})),
\]

\[
\eta(t) \sum_{n \in \mathbb{Z}} e^{i(nx - 20n)) \|u(s)\|_{L^4} \, ds} \hat{u}(t,n) \in C([-T,T]; H^s(\mathbb{T})) \cap F^s(T),
\]

where \( \eta \) is any cut-off function in \( C(\mathbb{R}) \) with supp \( \eta \subset [-T,T] \).

Moreover, the flow map \( S_T : H^s \to C([-T,T]; H^s(\mathbb{T})) \) is continuous.

**Proof.** In the following proof, we fix \( s > 2 \). From the theory of the complete integrability (or inverse spectral method), we know that there is a smooth solution \( u \) to (1.2) with \( u_0 \in H^\infty(\mathbb{T}) \). For \( v \to 2,0 \), since \( \|v_0\|_{H^s(\mathbb{T})} = \|v_0\|_{H^s(\mathbb{T})} \), we use Proposition 7.3 and Corollary 6.1 in order to obtain that

\[
\left\{ \begin{array}{c}
\|v\|_{F^s(T')} \lesssim \|v\|_{E^s(T)} + \sum_{j=1}^3 \|N_{1,j}(v)\|_{N^s(T')}, \\
\sum_{j=1}^3 \|N_{1,j}(v)\|_{N^s(T')} \lesssim \left( 1 + \|v\|_{F^s(T')} \right) \|v\|_{F^s(T')},
\end{array} \right.
\]

for any \( T' \in [0,T] \). Let \( X(T') = \|v\|_{E^s(T')} + \sum_{j=1}^3 \|N_{1,j}(v)\|_{N^s(T')} \), then we can know that \( X(T') \) is nondecreasing and continuous on \([0,T]\) (see (12)). If \( \delta_0 \) is small enough, then by using the bootstrap argument (see (20)), we can obtain \( X(T') \leq \delta_0 H^s \), and hence

\[
\sup_{t \in [-T,T]} \|v\|_{H^s(\mathbb{T})} \lesssim \|v_0\|_{H^s(\mathbb{T})},
\]

for all \( T' \in [0,T] \). This implies

\[
\text{(7.4)}
\]

by Proposition 7.1.

Fix \( u_0 \in H^s(\mathbb{T}) \) with \( \|u_0\|_{H^s(\mathbb{T})} \leq \delta_0 \ll 1 \), then we can choose a sequence of functions \( \{u_{0,j}\}_{j=1}^{\infty} \subset H^\infty(\mathbb{T}) \) such that \( u_{0,j} \) satisfies (1.2), and \( u_{0,j} \to u_0 \) in \( H^s(\mathbb{T}) \) as \( j \to \infty \). Let \( u_j(t) \in H^s(\mathbb{T}) \) be a solution to (1.2) with the initial data \( u_{0,j} \). Then, we first show the sequence \( \{u_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( C([-T,T]; H^s(\mathbb{T})) \). Let \( \epsilon > 0 \) be given. For \( K \in \mathbb{Z}_+ \), let \( u_{0,K} = P_{\leq K} u_{0,j} \). Then \( u_{0,K}(t,x) \) is a solution to the frequency localized equation as follows:

\[
\partial_t \chi_{\leq K}(n) \hat{v}(n) - i\mu(n) \chi_{\leq K}(n) \hat{v}(n) = -20i \chi_{\leq K}(n)n^3 \hat{v}(n)^2 \hat{v}(n) + 6i \chi_{\leq K}(n)n \sum_{N_5} \hat{v}(n_1) \hat{v}(n_2) \hat{v}(n_3) \hat{v}(n) + 10i \chi_{\leq K}(n)n \sum_{N_3} \hat{v}(n_1) \hat{v}(n_2) n^2 \hat{v}(n_3) + 5i \chi_{\leq K}(n)n \sum_{N_3} (n_1 + n_2) \hat{v}(n_1) \hat{v}(n_2) n^2 \hat{v}(n_3).
\]

Then, by the triangle inequality, we have

\[
\sup_{t \in [-T,T]} \|v_j - v_l\|_{H^s(\mathbb{T})} \leq \sup_{t \in [-T,T]} \|v_j - v_l\|_{H^s(\mathbb{T})} + \sup_{t \in [-T,T]} \|v_{l,K} - v_{l,K}\|_{H^s(\mathbb{T})} + \sup_{t \in [-T,T]} \|v_{l,K} - v_{l,K}\|_{H^s(\mathbb{T})},
\]

\footnote{When we choose \( \delta_0 < \delta \), where \( \delta \) is given by Lemma 6.3, we can use directly Corollary 6.3 instead of Proposition 5.4.}
and it suffices to show that
\[
\sup_{t \in [-T,T]} \|v^K_t - v_l\|_{H^s(T)} < \frac{\epsilon}{3} \tag{7.5}
\]
and
\[
\sup_{t \in [-T,T]} \|v^K_t - v^K_j\|_{H^s(T)} < \frac{\epsilon}{3}. \tag{7.6}
\]
For (7.5), we use (5.24) and (6.33) with (7.4) so that
\[
\sup_{t \in [-T,T]} \|v^K_t - v_l\|_{H^s(T)} \lesssim \|(I - P_{\leq K})v_l\|_{F^s(T)} \leq C_1\|v_{0,l} - v^K_{0,l}\|_{H^s(T)},
\]
for any \(l, K \) and \( C_1 \geq 1 \).

In order to deal with (7.6), from (7.2), (5.25) and (6.84) with (7.4), we have
\[
\|v_1 - v_2\|_{F^s(T)} \lesssim \|v_{1,0} - v_{2,0}\|_{L^2(T)},
\]
and with this, we obtain from (7.2), (5.24) and (6.85) with (7.4) that
\[
\|v_1 - v_2\|_{F^s(T)} \lesssim \|v_{1,0} - v_{2,0}\|_{H^s(T)} + (\|v_{1,0}\|_{H^2(T)} + \|v_{2,0}\|_{H^2(T)})\|v_{1,0} - v_{2,0}\|_{L^2(T)}.
\]
Hence, we conclude that
\[
\sup_{t \in [-T,T]} \|v^K_t - v^K_j\|_{H^s(T)} \lesssim \|v^K_t - v^K_j\|_{F^s(T)}
\]
\[
\lesssim \|v^K_{0,l} - v^K_{0,j}\|_{H^s(T)} + (\|v^K_{0,l}\|_{H^2(T)} + \|v^K_{0,j}\|_{H^2(T)})\|v^K_{0,l} - v^K_{0,j}\|_{L^2(T)}
\]
\[
\lesssim \|v^K_{0,l} - v^K_{0,j}\|_{H^s(T)} + K^s(\|v^K_{0,l}\|_{H^s(T)} + \|v^K_{0,j}\|_{H^s(T)})K^{-s}\|v^K_{0,l} - v^K_{0,j}\|_{H^s(T)}
\]
\[
\leq C_2\|v^K_{0,l} - v^K_{0,j}\|_{H^s(T)},
\]
for any \(j, l, K \) and \( C_2 \geq 1 \).

For constants \( C_1, C_2 \geq 0 \) in (7.7) and (7.8), let \( C = \max(C_1, C_2) \). Since \( v_{0,j} \to v_0 \) in \( H^s(T) \), as \( j \to \infty \), there exists \( N \in \mathbb{Z}_+ \) such that \( j \geq N \) implies
\[
\|v_{0,j} - v_0\|_{H^s(T)} < C^{-1}\frac{\epsilon}{6}. \tag{7.9}
\]
On the other hands, since \( v_0 \in H^s(T) \), there exists \( L \in \mathbb{Z}_+ \) such that \( K \geq L \) implies
\[
\|(I - P_K)v_0\|_{H^s(T)} < C^{-1}\frac{\epsilon}{6}, \tag{7.10}
\]
Therefore, when we choose suitable \( j, l \geq N \) and \( K \geq L \), then by using (7.9) and (7.10), we have
\[
\sup_{t \in [-T,T]} \|v^K_t - v_l\|_{H^s(T)} \leq C_1\|v_{0,l} - v^K_{0,l}\|_{H^s(T)} + \frac{\epsilon}{6}
\]
\[
< C_1\|(I - P_K)v_0\|_{H^s(T)} + \frac{\epsilon}{6}
\]
\[
< \frac{\epsilon}{3}
\]
and
\[
\sup_{t \in [-T,T]} \|v^K_t - v^K_j\|_{H^s(T)} \leq C_2\|v^K_{0,l} - v^K_{0,j}\|_{H^s(T)}
\]
\[
\leq C_2\|v_{0,l} - v_{0,j}\|_{H^s(T)}
\]
\[
\leq C_2\|v_{0,l} - v_0\|_{H^s(T)} + C_2\|v_{0,j} - v_0\|_{H^s(T)} < \frac{\epsilon}{3},
\]
which complete to show (7.3) and (7.4). Hence we obtain the solution as the limit. The uniqueness of the solution and the continuity of the flow map come from a similar argument, so we omit detail.
Now, it remains to show that the local well-posedness of (2.6) implies that of (1.2). In view of the definition of the nonlinear transformation (2.5), it suffices to show the bi-continuity property of the nonlinear transformation in $C([-T,T];H^s(T))$.

**Lemma 7.4.** Let $s \geq \frac{1}{4}$ and $0 < T < \infty$. Then, $\mathcal{N} \mathcal{T}(u)$ defined as in (2.5) is bi-continuous from a ball in $C([-T,T];H^s(T))$ to itself.

**Proof.** We only show the continuity of $\mathcal{N} \mathcal{T}$, since the proof of the continuity of $\mathcal{N} \mathcal{T}$ is similar as and easier than that of $\mathcal{N} \mathcal{T}^{-1}$. Precisely, when $v_k \in C \mathcal{T} \mathcal{H}^s$ converges to $v$ in $C \mathcal{T} \mathcal{H}^s$ as $k \to \infty$, we need to show

$$u_k = \mathcal{N} \mathcal{T}^{-1}(v_k) \to \mathcal{N} \mathcal{T}^{-1}(v) = u \text{ in } C \mathcal{T} \mathcal{H}^s,$$

as $k \to \infty$.

Fix $0 < T < \infty$. We assume that $\|v_k\|_{L^p T \mathcal{H}^s}, \|v\|_{L^p T \mathcal{H}^s} \leq K$, for some $K > 0$. Observe that

$$\hat{u}_k(n) - \hat{u}(n) = e^{ic_n f_0} \|u_k(s)\|_{L^4}^4 \hat{u}_k(n) - e^{ic_n f_0} \|u(s)\|_{L^4}^4 \hat{u}(n)$$

$$= \left[e^{ic_n f_0} \|u_k(s)\|_{L^4}^4 \hat{u}_k(n) - e^{ic_n f_0} \|u(s)\|_{L^4}^4 \hat{u}(n)\right]$$

$$+ e^{ic_n f_0} \|u(s)\|_{L^4}^4 \hat{u}(n)$$

$$= e^{ic_n f_0} \|u(s)\|_{L^4}^4 \left[e^{ic_n f_0} \|u_k(s)\|_{L^4}^4 - \|u(s)\|_{L^4}^4 \right]$$

$$+ e^{ic_n f_0} \|u(s)\|_{L^4}^4 \hat{u}(n)$$

for $n \neq 0$, and

$$\hat{u}_k(0) - \hat{u}(0) = \hat{u}_k(0) - \hat{u}(0).$$

Then, for fixed $s \geq 1/4$ and $t \in [-T,T]$, we have

$$\|u_k(t) - u(t)\|_{H^s}^2 \leq \|\hat{u}_k(0) - \hat{u}(0)\|^2$$

$$\quad + \frac{2^{s+1}}{2} \sum_{|n| \geq 1} |n|^{2s} \left|e^{ic_n f_0} \|u_k(s)\|_{L^4}^4 - \|u(s)\|_{L^4}^4 \right| d |n| \|\hat{u}_k(n)\|^2$$

$$\quad + \frac{2^{s+1}}{2} \sum_{|n| \geq 1} |n|^{2s} \|\hat{u}_k(n) - \hat{u}(n)\|^2.$$  \hfill (7.11)

(7.12)

(7.13)

Let $\varepsilon > 0$ be given. Since $e^{i\theta}$ is continuous at $\theta = 0$, there exists $\delta > 0$ such that

$$|\theta| < \delta \quad \Rightarrow \quad \left|e^{i\theta} - 1\right| < \frac{\varepsilon}{2^{s+1}} \cdot \sqrt{6K},$$

and $\|v\|_{L^p T \mathcal{H}^s} \leq K$ implies that there exists $M > 0$ such that

$$\sum_{|n| > M} |n|^{2s} \|\hat{u}(n)\|^2 < \frac{\varepsilon^2}{2^{s+1} \cdot 24}.$$  \hfill (7.14)

Moreover, since $v_k \to v$ in $C \mathcal{T} \mathcal{H}^s$ as $k \to \infty$, there exist $N_0, N_1 > 0$ such that

$$k \geq N_0 \quad \Rightarrow \quad \|v_k - v\|_{L^p T \mathcal{H}^s} < \frac{\varepsilon}{2^{s+1} \cdot 3}.$$  \hfill (7.15)

and

$$k \geq N_1 \quad \Rightarrow \quad \|v_k - v\|_{L^p T \mathcal{H}^s} < \frac{\delta}{2c_4 MT K^3}.$$  \hfill (7.16)

Let $N := \max(N_0, N_1)$. If $k \geq N$, from (7.16), we can control (7.11) and (7.13) as

$$|\hat{u}_k(0) - \hat{u}(0)|^2 \leq \frac{\varepsilon^2}{3}.$$  \hfill (7.17)

and

$$\|v_k - v\|_{L^p T \mathcal{H}^s} < \frac{\delta}{2c_4 MT K^3}.$$  \hfill (7.18)
and
\[ 2^{s+1} \sum_{|n| \geq 1} |n|^{2s} |\hat{v}_k(n) - \hat{v}(n)|^2 < \frac{\varepsilon^2}{3} \]  
(7.19)

Now, we consider (7.12). Observe from the Plancherel’s theorem that
\[ \|u\|_{L^4}^2 = \|u^2\|_{L^2} = \|\hat{u} \ast \hat{u}\|_{L^1} \]

Then, for \(1 < n \leq M\), from (7.20), (7.21) and (7.17), we have

\[ \|v^2\|_{L^2} = \|v\|_{L^8}^2. \]

Then, by using the triangle inequality and the Sobolev embedding, we have
\[ \left| \int_0^T \|v_k(s)\|_{L^4}^4 - \|v(s)\|_{L^4}^4 \, dt \right| \]

Then, for \(1 \leq |n| \leq M\), if \(k \geq N\), from (7.22) and (7.23), we have
\[ \left| c_3n \int_0^T \|u_k(s)\|_{L^4}^4 - \|u(s)\|_{L^4}^4 \, ds \right| < \delta \]

which implies
\[ \left| e^{ic_3n} f_0^T \|u_k(s)\|_{L^4}^4 - \|u(s)\|_{L^4}^4 \, ds - 1 \right|^2 < \frac{\varepsilon^2}{2^{s+1} \cdot 6K^2}, \]

by using (7.14).

For \(|n| > M\), since
\[ \left| e^{ic_3n} f_0^T \|u_k(s)\|_{L^4}^4 - \|u(s)\|_{L^4}^4 \, ds - 1 \right|^2 \leq 4, \]

This observation is essential to obtain the bi-continuity property of the nonlinear transformation \(N_T(u)\).
by using (7.15), we have
\[ 2^{s+1} \sum_{|n| > M} |n|^{2s} \left| e^{ic_n \int_0^t \| u_k(s) \|_{L^4}^4 - \| u(s) \|_{L^4}^4 \, ds} - 1 \right|^2 |n|^{2s} |\tilde{v}_k(n)|^2 < \frac{\varepsilon^2}{6}. \]

Hence, we have
\[ 2^{s+1} \sum_{|n| \geq 1} |n|^{2s} \left| e^{ic_n \int_0^t \| u_k(s) \|_{L^4}^4 - \| u(s) \|_{L^4}^4 \, ds} - 1 \right|^2 |n|^{2s} |\tilde{v}_k(n)|^2 < \frac{\varepsilon^2}{3}. \] (7.22)

Together with (7.16), (7.19) and (7.22), we obtain
\[ \| u_k - u \|_{H^s} < \varepsilon, \]
which completes the proof of Lemma 7.4.

From Lemma 7.4, we can complete the proof of Proposition 7.3.

7.2. Local well-posedness with arbitrary initial data. Now, let us complete the proof of Theorem 1.1. In order to extend the result of Proposition 7.3 to the local well-posedness for the arbitrary initial data, we can use the scaling argument, since this problem is scaling-subcritical. More precisely, by the scaling symmetry, we know for \( \lambda \geq 1 \) that
\[ u_\lambda(t, x) = \lambda^{-1} u(\lambda^{-5} t, \lambda^{-1} x) \] (7.23)
is also the solution to (1.1) if \( u \) is the solution to (1.2). Since \( u_\lambda \) is the \( 2\pi \lambda \)-periodic function, we need to modify slightly all estimates obtained in previous sections for the small data problem. But, since proofs follow the arguments in Section 2, 4, 5 and 6, so let us point out only different things. We start with introducing some notations adapted to the \( 2\pi \lambda \)-periodic setting.

We put \( T_\lambda = [0, 2\pi \lambda] \) and \( Z_\lambda := \{ n/\lambda : n \in \mathbb{Z} \} \). For a function \( f \) on \( T_\lambda \), we define
\[ \int_{T_\lambda} f(x) \, dx := \int_0^{2\pi \lambda} f(x) \, dx. \]

For a function \( f \) on \( Z_\lambda \), we define normalized counting measure \( dn \):
\[ \int_{Z_\lambda} f(n) \, dn := \frac{1}{\lambda} \sum_{n \in Z_\lambda} f(n) \]

(7.24)
and \( \ell_2^n(\lambda) \) norm:
\[ \| f \|_{\ell_2^n(\lambda)}^2 := \int_{Z_\lambda} |f(n)|^2 \, dn. \]

We define the Fourier transform of \( f \) with respect to the spatial variable by
\[ \hat{f}(n) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi \lambda} e^{-inx} f(x) \, dx, \quad n \in Z_\lambda, \]
and we have the Fourier inversion formula
\[ f(x) := \frac{1}{\sqrt{2\pi}} \int_{Z_\lambda} e^{inx} \hat{f}(n) \, dn, \quad x \in T_\lambda. \]

Of course, we can naturally define the space-time Fourier transform similarly.

Then the usual properties of the Fourier transform hold:
\[ \| f \|_{L^2_x(T_\lambda)} = \| \hat{f} \|_{\ell_2^n(\lambda)}, \]
\[ \int_0^{2\pi \lambda} f(x) \overline{g(x)} \, dx = \int_{Z_\lambda} \hat{f}(n) \overline{\hat{g}(n)} \, dn, \] (7.25)
Similarly as in Section 2, we take the Fourier coefficient in the spatial variable of (7.27) will denote $H^n$. Together with (7.25) and (7.26), we can define the Sobolev space $N$ and $\mathcal{N}$, and for $m \in \mathbb{Z}_+$,

$$\partial^m f(x) = \int_{\mathbb{T}_\lambda} e^{i\pi n m} \hat{f}(n) \, dn. \quad (7.26)$$

Together with (7.25) and (7.26), we can define the Sobolev space $H^s(\mathbb{T}_\lambda)$ with the norm

$$\|f\|_{H^s(\mathbb{T}_\lambda)} = \|\langle n \rangle^s \hat{f}(n)\|_{L^2(\mathbb{T}_\lambda)}.$$  

Under those observations, we consider (1.2) for a $2\pi \lambda$-periodic solution $u_\lambda$ defined as in (1.28), but we will denote $u_\lambda$ by $u$ (also $v$ and $w$), for simplicity, in this section. Recall the fifth-order modified KdV equation (1.22)

$$\partial_t u - \partial_x^5 u + 40 u \partial_x u \partial_x^3 u + 10 u^2 \partial_x^2 u + 10 (\partial_x u)^3 - 30 u^4 \partial_x u = 0. \quad (7.27)$$

Similarly as in Section 2, we take the Fourier coefficient in the spatial variable of (7.27) to obtain

$$\partial_t \tilde{u}(n) - in^5 \tilde{u}(n) = 10i \int_{\mathbb{T}_\lambda} \tilde{u}(n_1) \tilde{u}(n_2)(n - n_1 - n_2)^2 \tilde{u}(n - n_1 - n_2) \, dn_1 dn_2$$

$$+ 10i \int_{\mathbb{T}_\lambda} \tilde{u}(n_1 n_2) \tilde{u}(n_2)(n - n_1 - n_2) \tilde{u}(n - n_1 - n_2) \, dn_1 dn_2$$

$$+ 6i \int_{\mathbb{T}_\lambda} \tilde{u}(n_1) \tilde{u}(n_2) \tilde{u}(n_3) \tilde{u}(n_4) \tilde{u}(n - n_1 - n_2 - n_3 - n_4) \, dn_1 dn_2 dn_3 dn_4. \quad (7.28)$$

Since $2\pi \lambda$-periodic solution still satisfies all conservation laws, by considering the cubic and the quintic resonant interactions, we can reduce (7.28) to

$$\partial_t \tilde{v}(n) - i(n^5 + c_{1,\lambda} n^3 + c_{2,\lambda} n) \tilde{v}(n)$$

$$= - \frac{20i}{\lambda^2} n^3 \tilde{v}(n)^2 \tilde{v}(n)$$

$$\int_{\mathbb{T}_\lambda} \tilde{v}(n_1) \tilde{v}(n_2)(n - n_1 - n_2)^2 \tilde{v}(n - n_1 - n_2) \, dn_1 dn_2$$

$$+ 10i \int_{\mathbb{T}_\lambda} \tilde{v}(n_1 n_2) \tilde{v}(n_2)(n - n_1 - n_2) \tilde{v}(n - n_1 - n_2) \, dn_1 dn_2$$

$$+ 6i \int_{\mathbb{T}_\lambda} \tilde{v}(n_1) \tilde{v}(n_2) \tilde{v}(n_3) \tilde{v}(n_4) \tilde{v}(n - n_1 - n_2 - n_3 - n_4) \, dn_1 dn_2 dn_3 dn_4,$$

where

$$c_{1,\lambda} = \frac{10}{\lambda} \|u_0\|_{L^4(\mathbb{T}_\lambda)}^2, \quad c_{2,\lambda} = \frac{10}{\lambda} \|u_0\|_{H^1(\mathbb{T}_\lambda)}^2 + \|u_0\|_{L^6(\mathbb{T}_\lambda)}^4,$$

and $N_{3,n,\lambda}$ and $N_{5,n,\lambda}$ are defined similarly as in (2.3) and (2.4), respectively, for $\mathbb{Z}_\lambda$-variables. Moreover, $v$ is also defined similarly as in (2.3) by

$$v(t, x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}_\lambda} e^{i(n x - c_{3,\lambda} n f_0 + \mu_\lambda n^4)} \tilde{v}(t, n) \, dn,$$

where $c_{3,\lambda} = \frac{20}{\lambda}$. Let

$$\mu_\lambda(n) = n^5 + c_{1,\lambda} n^3 + c_{2,\lambda} n.$$
From those observations, we change function spaces $X_k$, $F_k$, $N_k$ and $E_k$ by $X_{k,\lambda}$, $F_{k,\lambda}$, $N_{k,\lambda}$ and $E_{k,\lambda}^2(T)$ with norms
\[
\|f\|_{X_{k,\lambda}} = \sum_{j \geq 0} 2^{j/2} \|\eta_j (\tau - \mu_\lambda(n)) \cdot f(\tau, n)\|_{L^2_2(\lambda)},
\]
\[
\|f\|_{F_{k,\lambda}} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F} \eta_0(2^{2k}(t - t_k)) \cdot f\|_{X_{k,\lambda}},
\]
\[
\|f\|_{N_{k,\lambda}} = \sup_{t_k \in \mathbb{R}} \|(\tau - \mu_{1,\lambda}(n)) + i2^{2k})^{-1} \mathcal{F} \eta_0(2^{2k}(t - t_k)) \cdot f\|_{X_{k,\lambda}}
\]
and
\[
\|u\|^2_{E_{k,\lambda}^2(T)} = \|P_{\leq 0} u(0)\|^2_{L^2_2(\lambda)} + \sum_{k \geq 1} \|P_{k} u(t_k)\|^2_{L^2_2(\lambda)}.
\]

Now, we check the nonlinear estimate
\[
\sum_{i=1,4} \|N_i(u, v, w)\|_{N^2(T)} + \|N_{2}(v_1, v_2, v_3, v_4, v_5)\|_{N^2(T)} \lesssim \lambda \beta \|u\|_{F_1^2(T)} \|v\|_{F_1^2(T)} \|w\|_{F_1^2(T)} + \lambda \beta \prod_{i=1}^5 \|v_i\|_{F_1^2(T)}
\]

and the energy estimate
\[
\|v\|^2_{E_{1,\lambda}^2(T)} \lesssim (1 + \|v\|^2_{H^1(\lambda)}) \|v_0\|^2_{H^2(\lambda)} + \lambda \beta (1 + \|v\|^2_{F_1^2(T)} + \|v\|^2_{F_2^2(T)}) \|v\|^2_{F_1^2(T)} \|v\|^2_{F_2^2(T)},
\]
in Section 5 and 6 respectively.

First, consider the $L^2$-block estimates in Section 4. Define the functional for the trilinear estimate by
\[
J_\lambda(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) = \int_{\mathbb{R}^3} \int_{\Gamma_4(\mathbb{R})} f_{k_1,j_1}(\zeta_1, n_1) f_{k_2,j_2}(\zeta_2, n_2) f_{k_3,j_3}(\zeta_3, n_3) f_{k_4,j_4}(\zeta_4 + G(n_1, n_2, n_3, n_4))
\]

Then, in view of the proof of Lemma 4.1 since we use the normalized counting measure $\mathcal{W}^{2,2}$, we obtain the exact same result as in Lemma 4.4 even for $2\pi\lambda$-periodic functions, while the threshold of restriction $2\pi\lambda 2^{-4k_{\text{max}}} = 1$ is replaced by $2\pi\lambda 2^{-4k_{\text{max}}} = 1$. In fact, since the $L^2$-block estimates still hold independent on $\lambda$, we can use the similar way to obtain nonlinear estimates. The only different thing is to use the fact that
\[
2^{k_{\text{max}}} \gtrsim |(n_1 + n_2)(n_1 + n_3)(n_2 + n_3)(n_1^2 + n_2^2 + n_3^2 + n^2), \quad n_1, n_2n_{3, n} \in \mathbb{Z}_\lambda.
\]

If $|k_{\text{min}} - k_{\text{max}}| \leq 5$, we have
\[
2^{k_{\text{max}}} \gtrsim \lambda^{-2} 2^{3k_{\text{max}}},
\]
and if $k_{\text{th}} \leq k_{\text{max}} - 10$ and $|k_{\text{min}} - k_{\text{th}}| \leq 5$, we obtain
\[
2^{k_{\text{max}}} \gtrsim \lambda^{-1} 2^{4k_{\text{max}}},
\]

To get (7.29), we follow almost same argument as in the nonlinear estimate, while we use the short time advantage $j_{\text{max}} \geq 2k_{\text{max}}$ instead of (7.32) in the proof of Lemma 5.2.

For (7.30), we define the modified energy similarly as in (6.2) and (6.3) by
\[
E_{\lambda,k}(v) = \|P_k v(t)\|^2_{L^2_2(\lambda)} + \text{Re} \left[ \alpha \int_{\mathbb{R}^3} \hat{v}(n_1) \hat{v}(n_2) \hat{v}_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \, dn \, d_2 \, dn \right]
\]
\[
+ \text{Re} \left[ \beta \int_{\mathbb{R}^3} \hat{v}(n_1) \hat{v}(n_2) \chi_k(n_3) \frac{1}{n_3} \hat{v}(n_3) \chi_k(n) \frac{1}{n} \hat{v}(n) \, dn \, d_2 \, dn \right]
\]

10 All properties of $X_{k}$-norm still hold for $X_{k, \lambda}$. 
Let where \( a \) is the a-priori bound for the solution of the \( \epsilon \)-parabolic equation to show that the solution of equation to the parabolic equation converges to the solution of the fifth-order modified KdV equation. But, the high regularity well-posedness result for the fifth-order modified KdV equation. The main difficulty is to obtain the energy estimate for both the parabolic and the fifth-order modified KdV equations. Finally, we use the Bona-Smith argument to obtain the energy estimate for both the parabolic and the fifth-order modified KdV equations. Precisely, we first control the this term for (A.1) without quintic term for avoiding complicated calculations.

From (7.31), we need to change Lemma 6.4 (a) and (c) as follows:

\[
\sum_{n_4,N_3,n_4} \int_0^T \hat{v}_1(n_1)\hat{v}_2(n_2)\hat{v}_3(n_3)\hat{v}_4(n_4) \, dt \lesssim 2^{k_4} \prod_{i=1}^4 \|v_i\|_{F_{k,i}(T)}^4
\]

and

\[
\sum_{n_4,N_3,n_4} \int_0^T \hat{v}_1(n_1)\hat{v}_2(n_2)\hat{v}_3(n_3)\hat{v}_4(n_4) \, dt \lesssim \lambda \hat{v}_1 \lesssim 2^{-k_4} \prod_{i=1}^4 \|v_i\|_{F_{k,i}(T)},
\]

respectively.

For the difference of two solutions, we use the similar argument as above, and then by following the small data well-posedness argument in Section 7.2 with the standard scaling argument, we can complete the proof of Theorem 1.1.

### Appendix A.

In this Appendix, we show the high regularity well-posedness for the non-integrable fifth-order modified KdV equation. We use the energy method with approximation of parabolic equation. Precisely, we first consider the \( \epsilon \)-parabolic equation, and for the smooth solution to the parabolic equation, we show the a-priori bound for the solution. Afterward, with a-priori bound and bootstrap argument, we use the approximation method of equation to show that the solution of \( \epsilon \)-parabolic equation converges to the solution of the fifth-order modified KdV equation. Finally, we use the Bona-Smith argument to obtain the high regularity well-posedness result for the fifth-order modified KdV equation. The main difficulty is to obtain the energy estimate for both the parabolic and the fifth-order modified KdV equations. But, we use the modified energy method, which is introduced by Kwon for the fifth-order KdV equation on \( \mathbb{R} \), and the Sobolev embedding to obtain the energy bound of the solution.

Furthermore, as another purpose of appendix, we emphasize that the generalized fifth-order modified KdV equation is unconditionally locally well-posed for \( s > 7/2 \). As mentioned, we only use the modified energy and Sobolev embedding to solve the local well-posedness problem. In this sense, it is necessary and crucial to construct the modified energy.

We use the symbols \( D^s \) and \( J^s \) as the Fourier multiplier operators defined as

\[
\mathcal{F}_x[D^s f](k) = |n|^s \hat{f}(k), \quad \text{and} \quad \mathcal{F}_x[J^s f](k) = (1 + |n|^2)^{s/2} \hat{f}(k).
\]

We consider the non-integrable fifth-order modified KdV equation\(^\dagger\):

\[
\begin{aligned}
\partial_t u - \partial_x^5 u + c_1 u \partial_x u \partial_x^3 u + c_2 u^2 \partial_x^3 u + c_3 (\partial_x u)^3 &= 0, \quad (t, x) \in \mathbb{R} \times T, \\
 u(0, x) &= u_0(x) \in H^s(T),
\end{aligned}
\]

where \( c_i \)'s be real constants. The following is the main Proposition in Appendix A.

**Proposition A.1.** Let \( s > 7/2 \) and \( u_0 \in H^s(T) \). Then, there is the time \( T = T(\|u_0\|_{H^s}) > 0 \) such that (A.1) is (unconditionally) locally well-posed in \( C([0, T]; H^s) \).

---

\(^\dagger\)Similarly as the nonlinear estimate, we use the short time advantage instead of maximum modulation effect.

\(^\dagger\)As we know, the ordinary fifth-order modified KdV equation has the quintic non-linear term, but, since one can easily control this term for \( s > 3/2 \) by using the Leibnitz rule for fractional derivative, Kato-Ponce commutator estimate and Sobolev embedding, we only consider (A.1) without quintic term for avoiding complicated calculations.
The argument of proof basically follows that in [18] and [13] associated to the fifth-order KdV equations on \( \mathbb{R} \). We prove Proposition A.1 by following the several steps:

**Step I.** This step shows the existence of smooth solution for perturbed equation. We consider the following parabolic problem:

\[
\partial_t u - \partial_x^5 u + c_1 u \partial_x u \partial_x^2 u + c_2 u^2 \partial_x^3 u + c_3 (\partial_x u)^3 = \varepsilon \partial_x^6 u,
\]  

(A.2)

where \( \varepsilon > 0 \). Then, we have

**Lemma A.2.** Let \( \varepsilon > 0 \) be given and \( u_0 \) be in Schwartz class. Then there is \( T_\varepsilon > 0 \) and a unique solution to (A.2) in the class \( S((0, T_\varepsilon) \times \mathbb{T}) \cap C([0, T_\varepsilon]; H^\infty) \).

**Proof.** The proof follows the argument of R. Temam. See [21]. \( \square \)

**Step II.** This step is to show that there is the time \( T \) independent on \( \varepsilon \) such that the solution \( u_\varepsilon \) of (A.2) provided by Lemma A.2 is in the class \( C([0, T]; H^\infty) \) by obtaining a-priori bound of \( u_\varepsilon \) in the \( C([0, T]; H^s) \) norm for \( s > \frac{7}{2} \). From the appropriate energy estimate and standard bootstrap argument, we have the following Lemma:

**Lemma A.3.** Let \( s > \frac{7}{2} \). Then, there exists \( T = T(\|u_0\|_{H^s}) \) such that for any \( \varepsilon > 0 \), the solution \( u_\varepsilon \) to (A.2) provided Lemma A.2 satisfies

\[
u_\varepsilon \in C([0, T]; H^\infty)
\]

and

\[
\sup_{t \in [0, T]} \|u_\varepsilon\|_{H^s} \lesssim \|u_0\|_{H^s}.
\]

**Step III.** This step gives the local well-posedness result for (A.1) for \( s > \frac{7}{2} \). The following is the conclusion in this step, which exactly implies Proposition A.1:

**Lemma A.4.** Let \( s > \frac{7}{2} \). Let \( u_\varepsilon \) be a Schwartz solution to (A.2) with sufficiently small \( H^s \)-norm. Then \( u_\varepsilon \) converges to \( u \) in the class \( C([0, T]; H^s) \) and hence \( u \) is the unique solution to (A.1) in the same class.

Both Ponce [18] and Kwon [13] used the idea of Bona and Smith [2] with energy estimates. The main difficulty in those works is to estimate the energy of solution \( u \). Hence we omit the detailed arguments (such as Bona-Smith argument and bootstrap argument) and finish this section by showing the following energy estimates:

**Lemma A.5.** Let \( s > \frac{7}{2} \) and \( u(t, x) \) be a Schwartz solution to (A.2) with sufficiently small \( H^s \)-norm. Then, there are constants \( C_1 \) and \( C_2 \), we have

\[
\sup_{t \in [0, T]} \|\partial^s u(t)\|_{L^2} \leq C_1 \varepsilon^{C_2} \int_0^t \|u(t')\|_{H^s}^2 \ dt' \|\partial^s u(0)\|_{H^s}.
\]

(A.3)

**Remark A.6.** To complete this section, we, in fact, show the energy estimate for the solution \( u \) to (A.1). But, this exactly follows the proof of Lemma A.5 if one eliminates the \( \varepsilon \)-terms in the proof of Lemma A.5 below.

To obtain (A.3), one needs to control the time increment of \( \|\partial^s u(t)\|_{L^2} \) using itself and other norms with the same size. But, since the nonlinear term of (A.2) has multi-derivatives, the standard energy

\[13\text{For the convenience, we use } u \text{ instead of } u_\varepsilon \text{ as the smooth solution to } \varepsilon\text{-parabolic equation.} \]

\[14\text{Since scaling argument still works on the periodic problem, we may assume the smallness of } \|u\|_{H^s}. \]
method gives\textsuperscript{15}

\[
\frac{d}{dt} \|D^s u\|_{L^2}^2 \lesssim \|\partial_x^3 u\|_{L^\infty}^2 \|D^s u\|_{L^2}^2 + \int uu_x D^s u_x D^s u_x
\] (A.4)

and the last term of the right-hand side of (A.4) is not favorable. Hence we use the modified energy developed by Kwon\textsuperscript{13} as the following:

\[E_s(t) := \|D^s u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + a_s \int u(t)^2 D^{s-2} \partial_x u(t) D^{s-2} \partial_x u(t),\]

where the constant \(a_s\), which eliminates the bad term in the right-hand side of (A.4), will be chosen later.

To prove Lemma A.5, we only need to show that

\[c\|u(t)\|_{H^s}^2 \leq E_s(t) \leq C\|u(t)\|_{H^s}^2,
\]

for some \(c, C > 0\), and

\[
\frac{d}{dt} E_s(t) \lesssim_s \|u(t)\|_{H^s}^2, E_s(t).
\] (A.6)

We begin with mentioning some lemmas, which are useful tools to prove A.6.

**Lemma A.7** (Commutator estimate). Let \(s \geq 1\). Then, we have

\[
\| [D^s; f] g \|_{L^2} \lesssim \| f \|_{L^\infty} \| D^{s-1} g \|_{L^2} + \| D^s f \|_{L^2} \| g \|_{L^\infty},
\]

where \([ \cdot ; \cdot ]\) is standard commutator defined as

\[
[A; B] = A(B) - B(A).
\]

**Lemma A.8** (Kwon, Lemma 2.2 in\textsuperscript{13}). Let \(s > 0\). Then, we have

\[
\left\| D^s (u \partial_x^2 v) - u D^s (\partial_x^2 v) - s \partial_x u D^s (\partial_x^2 v) - \frac{s(s-1)}{2} \partial_x^2 u D^s (\partial_x v) \right\|_{L^2} \\
\lesssim_s \| \partial_x^3 u \|_{L^\infty} \| D^s v \|_{L^2} + \| D^s u \|_{L^2} \| \partial_x^3 v \|_{L^\infty},
\]

and

\[
\left\| D^s (\partial_x u \partial_x^2 v) - \partial_x u D^s (\partial_x^2 v) - s \partial_x^2 u D^s (\partial_x v) \right\|_{L^2} \\
\lesssim_s \| \partial_x^3 u \|_{L^\infty} \| D^s v \|_{L^2} + \| D^s u \|_{L^2} \| \partial_x^3 v \|_{L^\infty},
\]

For the proof of Lemma A.7 and A.8 we refer\textsuperscript{10} and\textsuperscript{13}, respectively.

**Proof of (A.5).** We use the H"older inequality and Sobolev embedding to the third term in \(E_s(t)\) in order that \(E_s(t)\) is bounded by \(|a_s| \| u \|_{H^s}^4\). Then, we have (A.5), when \(|u|_{H^s}^2 \leq \frac{1}{2c|a_s|}\.\]

**Proof of (A.6).** The standard energy method to (A.1) yields

\[
\frac{1}{2} \frac{d}{dt} \|D^s u\|_{L^2}^2 = -\varepsilon \|D^{s+3} u\|_{L^2}^2 \\
- c_1 \int D^s (u \partial_x u \partial_x^2 u) D^s u \\
- c_2 \int D^s (u^2 \partial_x^2 u) D^s u \\
- c_3 \int D^s (\partial_x u \partial_x u \partial_x u) D^s u.
\] (A.7)

\textsuperscript{15}In fact, we have

\[
\frac{d}{dt} \|D^s u\|_{L^2}^2 + 2\varepsilon \|D^{s+3} u\|_{L^2}^2 \lesssim \|\partial_x^3 u\|_{L^\infty}^2 \|D^s u\|_{L^2}^2 + \int uu_x D^s u_x D^s u_x,
\]

but this implies (A.4) for smooth solution \(u\).
We note that
\[ \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u = \partial_x (u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u) - 2(u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u), \] \tag{A.8}

\[ u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u = \frac{1}{2} \partial_x (u^2) \frac{\partial}{\partial x} u. \] \tag{A.9}

Then, (A.7) can be rewritten from (A.8) and (A.9) that
\[
\frac{1}{2} \frac{d}{dt} \| D^s u \|^2_{L^2} = -\varepsilon \| D^{s+3} u \|^2_{L^2} \\
+ \frac{2c_3 - c_1}{2} \int D^s (\partial_x (u^2) \frac{\partial}{\partial x} u) D^s u \\
- c_2 \int D^s (u^2 \frac{\partial}{\partial x} u) D^s u \\
- c_3 \int D^s \partial_x (u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u) D^s u.
\]

In order to use Lemma A.8, we add and subtract some terms, and then
\[
\frac{1}{2} \frac{d}{dt} \| D^s u \|^2_{L^2} = -\varepsilon \| D^{s+3} u \|^2_{L^2} \\
+ \frac{2c_3 - c_1}{2} \int \left[ D^s (\partial_x (u^2) \frac{\partial}{\partial x} u) - \partial_x (u^2) D^s (\frac{\partial}{\partial x} u) - s \frac{\partial}{\partial x} (u^2) D^s (\partial_x u) \right] D^s u \\
- c_2 \int \left[ D^s (u^2 \frac{\partial}{\partial x} u) - u^2 D^s (\frac{\partial}{\partial x} u) - s \partial_x (u^2) D^s (\frac{\partial}{\partial x} u) - \frac{s(s-1)}{2} \frac{\partial}{\partial x} (u^2) D^s (\partial_x u) \right] D^s u \\
- \int \left[ c_3 D^s \partial_x (u \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u) + c_4 \partial_x (u^2) D^s (\frac{\partial}{\partial x} u) + c_5 \frac{\partial}{\partial x} (u^2) D^s (\partial_x u) \right] D^s u \\
= I + II + III + IV.
\]

We use Lemma A.8 product rule for fractional derivative, and Sobolev embedding to estimate terms of II and III, then
\[ |II + III| \lesssim \| u \|_{H^s}^2 \| D^s u \|^2_{L^2}. \]

We also perform the integration by parts to VI, then we obtain
\[ VI = d_1 \int u \frac{\partial}{\partial x} u D^s u D^s u \\
+ d_2 \int \frac{\partial}{\partial x} u \frac{\partial}{\partial x} u D^s u D^s u \\
+ d_3 \int \frac{\partial}{\partial x} (u^2) D^s \frac{\partial}{\partial x} u D^s u. \]
On the other hand, taking the time derivative to the third term in $E_s(t)$ yields

$$\frac{d}{dt} \int u^2 D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$= 2 \int u \partial_x D^{s-2} \partial_x u D^{s-2} \partial_x u + 2 \int (u^2) D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$= -2 \int u \partial_x D^{s-2} \partial_x u D^{s-2} \partial_x u + 2 \int (u^2) D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ 2e \int u \partial_x D^{s-2} \partial_x u D^{s-2} \partial_x u + 2 \int (u^2) D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$- 2c_1 \int u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u + (u^2) D^{s-2} \partial_x(u \partial_x \partial_x \partial_x u) D^{s-2} \partial_x u$$

$$- 2c_2 \int u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u + (u^2) D^{s-2} \partial_x(u \partial_x \partial_x \partial_x u) D^{s-2} \partial_x u$$

$$- 2c_3 \int u (\partial_x \partial_x \partial_x u \partial_x u D^{s-2} \partial_x u + (u^2) D^{s-2} \partial_x(\partial_x \partial_x \partial_x u \partial_x u) D^{s-2} \partial_x u$$

$$= A + B + C.$$

From the following observation

$$\partial_x^5(fg^2) = fxxxxxg^2 + 10fxxxxg^2 + 20fxxxxg^2 + 20fxxxg^2 + 60fxxxg^2 + 20fxxxg^2 + 20fxxxg^2 + 10fxxxg^2 + 10fxxxg^2 + 10fxxxg^2 + 2fgxxxxx,$$

putting $f = u$ and $g = D^{s-2} \partial_x u$ with performing the integration by parts several times yields

$$A = \alpha_1 \int u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_2 \int u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_3 \int u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_4 \int \partial_x u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_5 \int \partial_x u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_6 \int \partial_x u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u$$

$$+ \alpha_7 \int \partial_x^3 u \partial_x \partial_x \partial_x u D^{s-2} \partial_x u D^{s-2} \partial_x u.$$
The rest of distributed terms be absorbed to the other four terms, thus we finally obtain

\[ A = \beta_1 \int u \partial_x u D^{s-2} \partial_x^3 u D^{s-2} \partial_x^1 u + \beta_2 \int \partial_x u \partial_x^3 u D^{s-2} \partial_x^2 u D^{s-2} \partial_x^1 u + \beta_3 \int u \partial_x^3 u D^{s-2} \partial_x^2 u D^{s-2} \partial_x u + \beta_4 \int \partial_x u \partial_x^3 u D^{s-2} \partial_x^2 u D^{s-2} \partial_x u + \beta_5 \int \partial_x^2 u \partial_x^3 u D^{s-2} \partial_x^2 u D^{s-2} \partial_x u. \]

Once we choose \( a_s \) such that \( a_s \cdot \beta_1 + d_3 = 0 \), then

\[ |VI + A| \lesssim \|u\|_{H^s}^2 \|D^s u\|_{L^2}^2. \]

Now we concentrate on the term \( C \). We use the integration by parts, Hölder inequality, Lemma \( \text{A.7} \) and the Sobolev embedding to estimate term \( C \), then we can easily obtain

\[ |C| \lesssim \|u\|_{H^s}^2 \|D^s u\|_{L^2}^2, \]

when \( \|u\|_{H^s} \leq 1 \), for instance

\[
\left| \int u^2 D^{s-2} (u \partial_x u \partial_x^3 u) D^{s-2} \partial_x u \right| \lesssim \int u^2 \|D^{s-2}; u \partial_x u \| \partial_x^3 u \|D^{s-2} \partial_x u \|
\]

\[
+ \int u^2 \partial_x u D^{s-2} \partial_x^3 u D^{s-2} \partial_x u \| \lesssim \|u\|_{L^\infty} \|D^{s-2}; u \partial_x u \| \partial_x^3 u \|D^{s-2} \partial_x u \|_{L^2} \n\]

\[
+ \int \partial_x (u^2 \partial_x u) D^{s-2} \partial_x^3 u D^{s-2} \partial_x u \| + \int u^2 \partial_x u D^{s-2} \partial_x^3 u D^{s-2} \partial_x u \|
\]

\[
\lesssim \|u\|_{H^s}^2 \|D^s u\|_{L^2}^2. \]

We finally consider the term \( B \). From the integration by parts, we have the following observations:

\[
+2\varepsilon \int u \partial_x^3 u D^{s-2} \partial_x u D^{s-2} \partial_x u = -2\varepsilon \int \partial_x^3 u \partial_x^3 u D^{s-1} u D^{s-1} u + 12\varepsilon \int \partial_x^2 u \partial_x^3 u D^s u D^{s-1} u + 8\varepsilon \int \partial_x u \partial_x^3 u D^s u D^s u
\]

\[
-12\varepsilon \int \partial_x^3 u D^{s-1} u D^{s+1} u - 4\varepsilon \int u \partial_x^3 u D^{s+1} u - 4\varepsilon \int u \partial_x^3 u D^{s-1} u D^{s+2} u. \]
and
\[
2\varepsilon \int (u^2) D^{s-2} \partial_x^2 u D^{s-2} \partial_x u = 2\varepsilon \int D^{s+3} u \partial_x^2 [u^2 D^{s-1} u]
+ 4\varepsilon \int \partial_x u \partial_x u D^{s+3} u D^{s-1} u
+ 4\varepsilon \int \partial_x^2 u D^{s+3} u D^{s-1} u
+ 8\varepsilon \int \partial_x u D^{s+3} u D^s u
+ 2\varepsilon \int u^2 D^{s+3} u D^{s+1} u.
\]

From the Sobolev embedding, the last three terms in the first observation and all terms in the second observations are dominated by
\[
38\varepsilon K \|u\|_{H^s}^2 \|D^{s+3} u\|_{L^2}^2,
\]
where the constant $K$ only appears from the Sobolev embedding. Furthermore, the other terms in the first observation can be easily treated by the Sobolev embedding.

Gathering all things yields
\[
\varepsilon \|D^{s+3} u\|_{L^2}^2 + \frac{d}{dt} E_s(t) \lesssim \|u\|_{H^s}^2 E_s(t),
\]
when $\|u\|_{H^s}^2 \leq \frac{1}{38K|a_s|}$, and hence we conclude from the Gronwall’s inequality that
\[
E_s(t) \lesssim e^{\int_0^t \|u(t')\|_{H^s}^2 dt'} E_s(0),
\]
which complete the proof. \hfill \Box

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