On the asymptotic behavior of the super sonic interstellar gas flow which is made by spiral density wave, propagating in rapidly rotating galaxy disk

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Abstract

Nonlinear solutions for the large-scale flow of interstellar gas in the presence of a spiral gravitational field have received considerable attention because this model allows to explain a lot of observations for galaxy disks. However this investigations were forced to limit by numerable analysis of the problem because of its nonlinearity.

In this paper we wish to carry out the analytical expression which allows us to describe the super sonic nonlinear interstellar gas flow in rapidly rotating galaxy disk which is made by the spiral density wave.

One of characteristic parameters of theory is the amplitude of spiral density wave potential corresponding to separatrix, which separates super sonic flows from flows containing the jump from super sonic flows to subsonic.

We have defined the dependence of perturbing potential value, under which the galaxy shock wave appeared with respect to the parameters characterizing the gas disk (such as sound speed in gas, disk rotation speed, ”spiral design” rotation speed).

In recent years there have been several numerical simulations for nonlinear solutions of large scale flow of interstellar gas in the stellar gravitational field of spiral galaxies (Lin, Yuan & Shu 1969; Roberts 1969; Woodward 1975) by using the wave theory of spiral structure of galaxy. This concept has played a remarkable role in understanding of many processes occurring in the galaxies such as the structure of dust lanes, observed along the inner edges of spiral arms in many galaxies (Fujimoto, 1966; Lynds, 1970), the enhanced synchrotron radiation from spiral arms (Mathewson et
the radio emission of HI at the wavelength $\lambda = 21m$ (Berman & Mishurov 1980), a trigger mechanism for star formation, creating the narrow bands of young highly luminous stars which delineate the spiral arms (Roberts, 1969; Shu et al., 1972; Shu, 1973) and etc. It was shown that the spiral density wave, propagated in the stellar galaxy disk and interpreted as the galaxy spiral arms, is induced large nonlinear perturbations in the axial symmetric gas flow of the galactic disk. Such nonlinear phenomena take place in the gas even though the amplitude of the spiral stellar field is small. This is because the response to a spiral gravitational potential induced by the stellar density wave is roughly proportional to $a^{-2}$, where $a$ is the velocity dispersion, for stars, or sound speed for the gas (Woodward 1975). For the gas $a \sim 8 \, \text{km/s}$, while for the stars $a \sim 40 \, \text{km/s}$. So, if the amplitude of the perturbation field is a few per cent and we can use the liner theory to calculate a disturbance in the stellar disk, the perturbation of the gas disturbance is much stronger and we must use nonlinear theory. Nevertheless, the nonlinearity makes this model more contextual in the sense of physical ideas, which let to explain the morphology peculiarities of building and evolution of galaxies.

In this paper we wish to carry out the analytical expression which allows us to describe the disturbance of the super sonic nonlinear interstellar gas flow in rapidly rotating galaxy disk which is made by the spiral stellar density wave.

The hydrodynamic equations for the steady nonlinear gas flow (self-similar solutions) in the frame rotating with angular velocity $\Omega_p$ are, at position $r$,

$$\nabla \cdot (\rho \mathbf{v}) = 0 \quad (1)$$

$$(\mathbf{v} \cdot \nabla) \mathbf{v} + 2\Omega_p \times \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla (\Phi - \frac{1}{2} \Omega_p^2 r^2) \quad (2)$$

where $\rho$, $\mathbf{v}$ and $P = a^2 \rho$ (we have assumed that the gas is isothermal, i.e. $a$ is isothermal sound speed) are respectively the density, velocity, and pressure of the interstellar gas; $\Phi$ is the gravitational potential of the stars contained an unperturbed axisymmetric part as well as non axisymmetric perturbation induced by spiral wave; $r$ is a distance from the centre of the galaxy disk. In the absence of a stellar density wave the gas flow is supposed to be purely circular, so the gravitational potential $\Phi$ is axisymmetric and can be determined by the condition $\Phi(r) = \frac{1}{2} \Omega^2 r^2$, where $\Omega = \Omega(r)$ is an angular velocity of rotating galaxy disk at position $r$.

The unperturbed axisymmetric solutions of equations (1) and (2) for rapidly rotating thin disk are

$$v_z = 0, \quad v_r = 0, \quad v_\phi = r(\Omega - \Omega_p)$$

where, $v_r, v_\phi, v_z$ are in cylindrical polar coordinates.
Describing the nonlinear gas flow perturbed by the spiral stellar density wave each hydrodynamic variable can be written as a combination of an unperturbed axisymmetric part and a perturbation, denoted by subscripts 0 and 1 respectively (Roberts 1973) then the gravitational potential is \( \Phi = \Phi_0 + \Phi_1 \). According to other authors, beginning from Roberts (1973) it can be suggested that the perturbing potential \( \Phi \) is similar to the spiral. It is then useful to introduce the spiral coordinates \( (\eta, \xi, z) \).

The spiral coordinates is the set of coordinates fixed in the rotating frame that are parallel and perpendicular to the spiral equipotential square so \( \eta \) is constant along the spiral arms while \( \xi \) is constant everywhere on lines orthogonal to the spiral arms and \( z \) is constant everywhere on plane parallel to galaxy disk. This coordinates are related to the plane cylindrical polar coordinates \( (r, \phi) \) by

\[
\eta = -\frac{2}{\tan i} \ln r + 2\phi \\
\xi = -2 \ln r + \frac{2}{\tan i} \phi
\]

where \( i \) is angle of inclination of a spiral arm to the circumferential direction; this angle is related to the radial wavenumber of the spiral wave \( k \) by conditions \( \tan i = -\frac{2}{kr} \), and \( k = -\frac{2\pi}{\lambda} \); \( \lambda \) is the wave length, so we get

\[
d\eta = -kdr + 2d\phi \\
d\xi = -2\frac{dr}{r} - krd\phi
\]

The unperturbed gas velocity components parallel to the \( (\xi, \eta, z) \) directions respectively are

\[
v_{||0} = (\Omega - \Omega_p)r \cos i, \quad v_{\perp0} = (\Omega - \Omega_p)r \sin i = \frac{2}{k}(\Omega - \Omega_p) \cos i,
\]

where \( v_{||}, v_{\perp} \) are the gas velocity components parallel and perpendicular to the spiral arms respectively. It is easy to see that \( \frac{v_{\perp0}}{v_{||0}} \sim \frac{1}{kr} \). Since each hydrodynamic variable can be written as a combination of an unperturbed part and a perturbation we can define as \( v_{||} = v_{||0} + v_{||1}, \quad v_{\perp} = v_{\perp0} + v_{\perp1}, \quad \rho = \rho_0 + \rho_1 \).

According to Roberts (1969) we can assume that the spirals are tightly wound, i.e. \( i \ll 1 \), it corresponds to the short, comparing with scale \( r \), waves, i.e. \( \lambda \ll r \), and allows to use WKB approximation for the spiral wave description. We would like to note that the spirals in this case appear to be ”tightly” wound. Then it can be shown (Nelson & Matsuda 1977), that the gradients for the spiral perturbations satisfy

\[
\frac{\partial}{\partial \xi} \leq \sin^2 i \frac{\partial}{\partial \eta},
\]
and it can be also found that for any function \( \phi(r) \) with scale length \( r \) i.e. \( \frac{d\phi}{dr} \sim 1 \) we get

\[
\frac{\partial}{\partial \eta} \sim \sin i \frac{\partial}{\partial \eta} \log(v_{\parallel 1}, v_{\perp 1}, \rho_1),
\]

In this approximations it can be neglected derivatives of perturbed quantities in the direction of the "slow" coordinate \( \xi \) along the spiral arms with respect to their derivatives in the direction of the "rapid" coordinate \( \eta \) which is perpendicular to the spiral arms. In addition, it can be said that the typical length in this problem is \( \frac{1}{k} \sim \lambda \), besides that \( \frac{\Delta}{\chi} \ll 1 \) (here \( \Delta \) is a typical thickness of the galaxy disk at position \( r \sim 10kpc \)) than because of equation (1) \( v_z = 0 \) (it is right because in such conditions the last term \( \frac{\partial}{\partial \xi} \) in equation (1) is dominant one) Finally assuming the perturbation potential as

\[
\Phi_1(\eta) = F \frac{\Omega^2 r \cos^2 i}{k} \cos \eta
\]

where \( F \) is the amplitude of the perturbation potential due to the stellar density spiral wave as a fraction of the unperturbed axisymmetric gravitational potential.

It can be introduced scale \( \frac{\chi^2 \cos i}{k} = 11.34 \frac{km}{s} \) for \( v_{\parallel} \) as well as scale for \( v_{\perp} \) \( \chi \cos i = 18.1 \frac{km}{s} \)

than \( v_{\perp 0} \left( \frac{\chi}{\cos i} \right) = 2\Omega - \frac{\Omega p}{\chi} = -\nu = 0.722 \) Here are used the same set of parameters as used by Shu et al (1973): \( \Omega = 24.7 \frac{km}{s kpc} ; \Omega_p = 13.5 \frac{km}{s kpc} \); at position \( r = 10 kpc ; \chi = 2\Omega ((1 + \frac{r \cdot d\Omega}{27 kpc}) \cos^2 i) \frac{1}{2} = 31.0 \frac{km}{s kpc} \).

Eliminating \( \rho_1 \) between equations (1) and (2), according to Shu et al (1973) we can write this equations in dimensionless form

\[
\frac{(-\nu + u)^2 - c}{-\nu + u} \frac{du}{d\eta} = v - f \sin \eta
\]

\[
u + (-\nu + u) \frac{dv}{d\eta} = 0
\]

(3)

where \( u, v \) are the dimensionless velocity variables perpendicular \( v_{\perp} \) and parallel \( v_{\parallel} \) to the spiral arms respectively; \( f = \frac{F \Omega^2}{\chi^2} kr \sim 0.1 - 0.2 \) is the dimensionless amplitude of the stellar density spiral wave; \( c = \frac{a^2}{\chi^2 \cos^2 i} = 0.195 \) is the square of the dimensionless sound speed.

Equations (3) have been solved numerically by other authors mentioned above. Eliminating \( v \) between these equations we can get a equation which is to be solved for \( u \)

\[
\frac{(-\nu + u)^2 - c}{-\nu + u} \frac{d^2 u}{d\eta^2} + \frac{(-\nu + u)^2 + c}{(-\nu + u)^2} \left( \frac{du}{d\eta} \right)^2 + \frac{u}{-\nu + u} = -f \cos \eta
\]

(4)
Beginning with some value of $f$ the solution of equation (4) contains an irregularity is due behavior of the equation solution in neighborhood some point at $u = \nu + \sqrt{c}$. If we introduce a new variable by condition $\tilde{u} = u - \nu - \sqrt{c}$ then we get

$$
\frac{2\sqrt{c} + \tilde{u}}{2(\sqrt{c} + u)} \tilde{u}'' + \left(1 - \frac{2\sqrt{c} + \tilde{u}}{2(\sqrt{c} + u)} \tilde{u}\right) \tilde{u}' - \frac{\nu}{2\sqrt{c}(\sqrt{c} + u)} \tilde{u} = \frac{1}{2}(1 + \frac{\nu}{\sqrt{c}} + f \cos \eta)
$$

In the flow zone, which lies in the neighborhood of this point, gas parameters corresponding to the explosive solution are exposed by sharp changes, and along the spatial coordinate a narrow transitional area appears, in this area gas parameters constantly change: a so called jump of density if in the chosen frame this area is stable, or a shock wave if the transitional area is transferred in space with respect to time. Nevertheless, for the counting of flow parameters in this area we need to take into account the additional non-gas dynamic transfer of impulse and energy, which corresponds to the viscosity and heat of dissipation. The latter may be taken into account if we add in the simplest case "viscous pressure" to the gas-dynamic pressure.

By assuming in (4) that $u = f \nu y$ and $\omega^2 = (\nu^2 - c)^{-1}$ we get

$$
y'' + \omega^2 y = \omega^2 \cos \eta + f((2\nu^2 \omega^2 + 1)yy'' + (2\nu^2 \omega^2 - 1)y'^2 + \\
+ \omega^2(y^2 - 2y \cos \eta)) + f^2 \nu^2 \omega^2(-3y^2 y'' - 2yy'^2 + y^2 \cos \eta) + \\
+ f^3 \nu^2 \omega^2(y^3 y'' + y'^2 y'^2)
$$

If we choose the terms in the same powers of $f$ we get the following set of equations which are to be solved successively for $y_1; y_2; y_3; \ldots$

$$
y_0'' + \omega^2 y_0 = \omega^2 \cos \eta
$$

$$
y_1'' + \omega^2 y_1 = (2\nu^2 \omega^2 + 1)y_0 y_0'' + (2\nu^2 \omega^2 - 1)y_0'^2 + \omega^2(y_0^2 - 2y_0 \cos \eta)
$$

$$
y_2'' + \omega^2 y_2 = (2\nu^2 \omega^2 + 1)(y_0 y_1'' + y_1 y_0'') + (2\nu^2 \omega^2 - 1)2y_0'y_1' + \\
+ \omega^2(2y_0 y_1 - 2y_1 \cos \eta) + \nu^2 \omega^2(-3y_0^2 y_0'' - 2y_0 y_0'^2 + y_0^2 \cos \eta)
$$

$$
y_3'' + \omega^2 y_3 = (2\nu^2 \omega^2 + 1)(y_0 y_2'' + y_1 y_1'' + y_2 y_0'') + \\
+ (2\nu^2 \omega^2 - 1)(y_1'^2 + 2y_0'y_2') + \omega^2(y_1^2 + 2y_0 y_2 - 2y_2 \cos \eta) + \\
+ \nu^2 \omega^2(-3(y_0^2 y_1'' + 2y_0 y_1 y_0'') - 2(2y_0 y_0'y_1' + y_1 y_0'^2) + \\
+ 2y_0 y_1 \cos \eta) + \nu^2 \omega^2(y_0^3 y_0'' + y_0^2 y_0'^2)
$$
This set of equations has been used by Shu, Milione & Roberts (1973) for discussing the slightly nonlinear regime and comparing it with the results of the linear theory by Lin et al. (1969). At the same time the authors noted that for such value as realistic gravitational potential, the series which are solutions of this equation set may converge very slowly, if at all. So they used a numerical method of investigation this problem and they got that for the typical value of parameter \( f \), such as \( f = 0.105 \) the behavior of the solution was changed and if \( f \) was being increased above \( f_c \) the solution could not remain smooth everywhere and had to contain a jump. Physically, the jump corresponds to a shock wave which are formed at the transition from supersonic gas flow to subsonic.

The solution \( y(\eta) \) can be written in the form

\[
y(\eta) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} A_{i,k} \cos(k\eta) \tag{5}\n\]

In summation over \( i \) the dominant contribution arising from a first few terms because if \( i \) increases the series coefficient \( A_{i,k} \) decreases very quickly, on the contrary, in summation over \( k \) dominant contribution is arisen from the remote terms of this series because the coefficient \( A_{i,k} \) is changed with respect to \( k \) very slowly. The necessity to take into account the remote terms is so important as the gas flow velocity is more closely to sound speed, this corresponds with that the value of parameter \( f \) is approached to its typical value mentioned above.

The main coefficient \( A_{0,n} \) can be obtained from solution of \( n - 1 \) equation, to do it we must join terms which are proportional to \( \cos(n+1)\eta \), if \( n > 0 \) we get

\[
y_n = \frac{(n+1)^2}{(n+1)^2 - \omega^2} + \frac{\nu^2 \omega^2}{8} \left( S_1(n - 1) - 2S_4(n) + \frac{\nu^2 \omega^2}{8} (S_5(n) + S_6(n)) \right) \cos(n+1)\eta + \text{ the terms contained } \cos((n-1)\eta) \text{ and etc.} \tag{6}\n\]

Where

\[
S_1(n) = \begin{cases} 
\frac{1}{2} a_{\frac{n-1}{2}}^2 + \sum_{k=0}^{\frac{n-1}{2}} a_k a_{n-1-k} & \text{if } n \text{ is odd} \\
\sum_{k=0}^{\frac{n-2}{2}} a_k a_{n-1-k} & \text{if } n \text{ is even}
\end{cases}
\]
$$S_2(n) = \begin{cases} 
\sum_{k=0}^{\frac{n-3}{2}} \left( \frac{n-2k-1}{n+1} \right)^2 a_k a_{n-1-k} & \text{if } n \text{ is odd} \\
\sum_{k=0}^{\frac{n-2}{2}} \left( \frac{n-2k-1}{n+1} \right)^2 a_k a_{n-1-k} & \text{if } n \text{ is even} 
\end{cases}$$

$$S_3(n) = \sum_{k=0}^{n-2} \sum_{m=0}^{n-2-k} \left( \frac{n-1-k-m}{n+1} \right)^2 a_k a_m a_{n-2-k-m}$$

$$S_4(n) = \left( \sum_{k=0}^{n-2} \sum_{m=0}^{n-2-k} \frac{(k+1)(m+1)}{(n+1)^2} a_k a_m a_{n-2-k-m} \right)$$

$$S_5(n) = \sum_{k=0}^{n-3} \sum_{m=0}^{n-3-k} \sum_{l=0}^{n-3-k-m-l} \left( \frac{n-2-k-m-l}{n+1} \right)^2 a_k a_m a_l a_{n-3-k-m-l}$$

$$S_6(n) = \sum_{k=0}^{n-3} \sum_{m=0}^{n-3-k} \sum_{l=0}^{n-3-k-m-l} \frac{(m+1)(n-2-k-m-l)}{(n+1)^2} a_k a_m a_l a_{n-3-k-m-l}$$

where $a_n$ is a coefficient at $\cos((n+1)\eta)$ in the solution $y_n$ which is related to $A_{0,n}$ by relation $A_{0,n+1} = f^n a_n; a_0 = -\omega^2$. Recursion relation for this coefficients can be obtained by using the successive approximations method to solve the equation (6) for $n$ approach. The first approach can be written in form

$$a_n = \omega^2 \left( 1.89 f a_0 (\nu^2 \omega^2 + 1) \right)^n \prod_{m=0}^{n} \frac{(m+1)^2}{(m+1)^2 - \omega^2}$$  \hspace{1cm} (7)$$

For large $n$, taking into account that the product approximates rapidly, it can be substituted by limit $\prod_{m=1}^{\infty} (1 - \frac{\omega^2}{m^2}) = \frac{\sin \omega}{\pi \omega}$ (equality $\omega$ to whole number corresponds to the Lindblad’s resonance). So for series (5) the following expression can be written:

$$y \simeq \frac{\omega^2}{\omega^2 - 1} \cos(\eta) - 0.0342 \frac{\pi \omega^3}{\sin \pi \omega} \sum_{k=1}^{\infty} (1.89 f (\nu^2 \omega^2 + 1) \frac{\omega^2}{\omega^2 - 1})^k \cos (k+1)\eta \hspace{1cm} (8)$$

The numerical coefficients appeared in the last two expressions are the results of averaging the difference between the product of final number of the terms $\prod_{m=0}^{(m+1)^2 - \omega^2}$
and its limit when the number is infinite and the result of the calculation of the sums $S_1(n), S_2(n)$ and etc. The latter can be done while using the method of continued fraction.

Trigonometric series in (8) in condition that $|1.89f(\nu^2\omega^2 + \frac{1}{2})\frac{\omega^2}{\omega^2 - 1}| < 1$ converges and, according to Prudnikov, Brychkov & Marichev (1981) can be expressed in elementary functions, so we have:

$$y \simeq \frac{\omega^2}{\omega^2 - 1} \cos(\eta) - 0.0342 \frac{\pi \omega^3}{\sin \pi \omega} f \frac{\cos \eta - \alpha}{1 - 2\alpha \cos \eta + \alpha^2}$$

Here

$$\alpha = 1.89f(\nu^2\omega^2 + \frac{1}{2})\frac{\omega^2}{\omega^2 - 1}$$

Using (9) we will put down at last the expression which consists the main part of speed component perpendicular to wave front, i.e. $u_\perp$, taking into account that (9) to the large extend is adequate to large $k$ from (5), we will change in (9) at least the first member by the exact solution of the equation for the first approximation, so we get:

$$u_\perp = u_{\perp 0} + u_{\perp 1} = -\nu(1 + \frac{f^2}{2}\frac{\omega^2}{\omega^2 - 1})^2 - f \frac{\omega^2}{\omega^2 - 1} \cos \eta$$

$$+ f^2 \frac{\omega^2}{\omega^2 - 1} (\frac{\omega^2}{\omega^2 - 1})^2 \frac{\omega^2}{(\omega^2 - 1)(\omega^2 - 4)} (2\nu^2\omega^2 + \frac{\omega^2}{2} - 1) \cos 2\eta$$

$$+ 0.0342f \frac{\pi \omega^3}{\sin \pi \omega} f \frac{\cos \eta - \alpha}{1 - 2\alpha \cos \eta + \alpha^2}$$

Integral curves (10) of equation (4) for several values of parameter $f$ are given on Figure 1. As we can see from this picture with parameter $f$ increasing, i.e. with the growth of perturbed potential amplitude the gas flow speed, at $\eta = 0$, is decreasing and may reach sound speed in gas. The integral curve, corresponding to this condition, has a sharp jump if $\eta = 0$ and presents a separatrix which separates the solutions corresponding to the gas flow with supersonic speed from the solution, in which there are areas of both supersonic and subsonic gas flow. If one prefers to see a results represented as the velocity plane its pictured at Figure 2.

The amplitude value of perturbed potential $f_c$ which corresponds to this separatrix was defined by Shu et al. numerable. Formula (10) allows to define the dependence $f_c$ respect to the parameters of the rotating gas disk. For this we take in (10) $u_\perp = \sqrt{c}$ if $\eta = 0$, then solving the equation in respect to $f$ we get:

$$f_c \simeq -\frac{1}{2} \frac{\pi \omega}{\pi \omega^3} \frac{\sqrt{c}}{\nu} + 1$$

$$- 0.0342 - 0.0616(1 - 1.89(1 + \frac{\sqrt{c}}{\nu})(\nu^2\omega^2 + \frac{1}{2})) \simeq 0.12$$
Figure 1. The phase portrait of equation. Curve A assumes the amplitude of the perturbation potential such as $f = 0.11$; B assumes $f = 0.10$; C assumes $f = 0.095$; D assumes $f = 0.085$; E assumes $f = 0.055$; The sound speed is marked by broken line.
Figure 2. The supersonic flow in the velocity plane. Curve $A$ assumes the amplitude of the perturbation potential such as $f = 0.11$; $B$ assumes $f = 0.10$; $C$ assumes $f = 0.095$; $D$ assumes $f = 0.085$; $E$ assumes $f = 0.055$; For $f = 0.11$ the flow contains a cusp at the location of the minimum of the spiral potential, $\eta = 0$. The unperturbed axisymmetric velocity is marked by a cross.
Here the same parameters with which Shu et al (1973) solve equation (3) numerable, and the value we get is practically coincides with their result.

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