Solutions of the $T$-system and Baxter equations for supersymmetric spin chains

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Abstract

We propose Wronskian-like determinant formulae for the Baxter $Q$-functions and the eigenvalues of transfer matrices for spin chains related to the quantum affine superalgebra $U_q(\hat{gl}(M|N))$. In contrast to the supersymmetric Bazhanov-Reshetikhin formula (the quantum supersymmetric Jacobi-Trudi formula) proposed in [Z. Tsuboi, J. Phys. A: Math. Gen. 30 (1997) 7975], the size of the matrices of these Wronskian-like formulae is less than or equal to $M+N$. Based on these formulae, we give new expressions of the solutions of the $T$-system (fusion relations for transfer matrices) for supersymmetric spin chains proposed in the abovementioned paper. Baxter equations also follow from the Wronskian-like formulae. They are finite order linear difference equations with respect to the Baxter $Q$-functions. Moreover, the Wronskian-like formulae also explicitly solve the functional relations for Bäcklund flows proposed in [V. Kazakov, A. Sorin, A. Zabrodin, Nucl. Phys. B790 (2008) 345 [arXiv:hep-th/0703147]].

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The \( T \)-system plays important roles in the study of quantum integrable systems (see for example, earlier papers \[1, 2\], which include the \( T \)-system related to \( \widehat{\text{sl}}(M) \)). It is a system of fusion relations for a commuting family of transfer matrices of solvable lattice models. There is a large class of models whose \( R \)-matrices satisfy the graded Yang-Baxter equation \[3, 4\]. In general, their symmetries are described by the quantum affine superalgebras \[5\]. In a series of papers \[6, 7, 8\], we proposed the \( T \)-system for solvable lattice models associated with \( U_q(\widehat{\text{sl}}(M|N)) \) or \( U_q(\widehat{\text{gl}}(M|N)) \). In an appropriate normalization, it has the following form

\[
T^{(a)}_s(xq^{-1})T^{(a)}_s(xq) = T^{(a)}_{s-1}(x)T^{(a)}_{s+1}(x) + T^{(a-1)}_s(x)T^{(a+1)}_s(x)
\]

for \( a \in \{1, 2, \ldots, M-1\} \) or \( s \in \{1, 2, \ldots, N-1\} \) or \( (a, s) = (M, N) \),

\[
T^{(M)}_s(xq^{-1})T^{(M)}_s(xq) = T^{(M)}_{s-1}(x)T^{(M)}_{s+1}(x) \quad \text{for} \quad s \in \mathbb{Z}_{\geq N+1},
\]

\[
T^{(a)}_N(xq^{-1})T^{(a)}_N(xq) = T^{(a-1)}_N(x)T^{(a+1)}_N(x) \quad \text{for} \quad a \in \mathbb{Z}_{\geq M+1},
\]

\[
T^{(M+b)}_N(x) = \varepsilon_b T^{(M)}_{N+b}(x) \quad \text{for} \quad b \in \mathbb{Z}_{\geq 0},
\]

where \( a, s \in \mathbb{Z}_{\geq 1} \) is assumed; the factor \( \varepsilon_b \in \mathbb{C} \) depends on the definition of the transfer matrices (see, \[3,33\] for \( (m, n) = (M, N) \)). \( x \in \mathbb{C} \) is a multiplicative spectral parameter whose origin goes back to the evaluation map from \( U_q(\widehat{\text{sl}}(M|N)) \) to \( U_q(\text{gl}(M|N)) \), or an automorphism of \( U_q(\widehat{\text{gl}}(M|N)) \). \( T^{(a)}_s(x) \) is a transfer matrix (or its eigenvalues) whose auxiliary space (the space where the (super)trace is taken) is an evaluation representation of \( U_q(\widehat{\text{gl}}(M|N)) \) based on a tensor representation of \( U_q(\text{gl}(M|N)) \) labeled by a rectangular Young diagram with a height of \( a \) and a width of \( s \) in the \((M, N)\)-hook (see Figure \[1\]). In particular, \( T^{(1)}_1(x) \) corresponds to the transfer matrix of the Perk-Schultz-type model \[10\]. In contrast to the bosonic algebra \( U_q(\widehat{\text{gl}}(M)) \) case, the index

\[\footnote{A fusion relation corresponding to \[1.1\] for \((M, N) = (2, 1)\) and \( a = s = 1 \) was considered in \[9\].} \]
a can take arbitrary non-negative integer values. For \( N = 0 \), the “duality relation” (1.4) becomes trivial \( T_b^{(M)}(x) = \varepsilon^{-1}_b \), which means that the \( M \)-th antisymmetric tensor representation of \( U_q(gl(M)) \) is trivial; while this is not the case for \( N > 0 \). The functions \( T_0^{(a)}(x), T_b^{(0)}(x) \) appearing in the boundary depend on each model and the normalization. We sometimes chose a normalization where these functions are just 1 (this corresponds to the normalization of the universal \( R \)-matrix). As was pointed out in [6], the above \( T \)-system is a reduction of the so-called Hirota bilinear difference equation [11]. Due to the commutativity of the transfer matrices, the eigenvalues of the transfer matrices obey the same functional relations as the transfer matrices. In this paper, the eigenvalue of the transfer matrix will be called \( T \)-function. These equations (1.1)-(1.4) were transformed into the form of the \( Y \)-system and used [12] to derive the thermodynamic Bethe ansatz equations. We also used [13] these equations (1.1)-(1.4) to derive nonlinear integral equations with a finite number of unknown functions, which are equivalent to the thermodynamic Bethe ansatz equations. We remark that \( T \)-systems closely related to (1.1)-(1.4) also appeared recently in the study of the AdS/CFT correspondence in particle physics [14, 15, 16]. In particular, two copies of a \( gl(2|2) \)-like \( T \)-system coupled\(^2\) each other appeared as the “left wing” and the “right wing” of the AdS/CFT \( T \)-system (in the form of the \( Y \)-system) [16, 17].

The so-called Bazhanov-Reshetikhin formula [2] is a determinant expression of the eigenvalue of the transfer matrix for the fusion [18] model whose auxiliary space is labeled by a general Young diagram for a tensor representation of \( U_q(\hat{sl}(M)) \) (or \( U_q(\hat{gl}(M)) \)). This formula also has a tableaux sum expression. The Bazhanov-Reshetikhin formula allows a supersymmetric generalization for \( U_q(\hat{sl}(M|N)) \) or \( U_q(\hat{gl}(M|N)) \), which may be called “the quantum supersymmetric Jacobi-Trudi formula” or “the supersymmetric Bazhanov-Reshetikhin formula” [6, 7, 8] (cf. (2.35) and (2.36); see also a recent paper [19], and also [20] for \( U_q(B^{(1)}_r) \) case). This formula or its tableaux sum expression for the Young diagram of rectangular shape gives [6, 7, 8] the solution of the \( T \)-system (1.1)-(1.4). The supersymmetric Bazhanov-Reshetikhin formula is a quantum affine superalgebra analogue (or the Yang-Baxterization) of the second Weyl formula for the transfer matrices since this formula reduces to the supersymmetric Jacobi-Trudi formula on the supercharacter of \( gl(M|N) \) [21] if the spectral parameter dependence is dropped (this corresponds to the limit \( x \to 0 \) in our normalization of the spectral parameter \( x \)). Then, it is natural to consider an analogue of the first Weyl formula. The first Weyl formula for the superalgebra \( gl(M|N) \) is often called “Sergeev-Pragacz formula” in mathematical literature [22] (see (A.10)). In addition, the Sergeev-Pragacz formula has a nice Wronskian-like determinant expression [23]. In this paper, we propose Wronskian-like determinant expressions of the \( T \)-functions for \( U_q(\hat{gl}(M|N)) \) (or \( U_q(\hat{sl}(M|N)) \)) for any \( M, N \) (cf. (3.15)-(3.16)). These include formulae in our previous paper [24] on \( U_q(\hat{sl}(2|1)) \) and also similar formulae for

\(^2\)These two \( gl(2|2) \)-like \( T \)-systems decouple in the limit \( L \to \infty \).
the bosonic case $U_q(\hat{gl}(M))$ \[25, 26, 27, 28\] (see also papers from a different approach \[29, 30\]). In particular for the Young diagram of rectangular shape, these formulae give Wronskian-like determinant solutions of the $T$-system \eqref{1.1}-\eqref{1.4} for $U_q(\hat{gl}(M|N))$ (cf. Theorem 3.3). We also remark that these Wronskian-like determinants can be viewed as generalization of the ninth variation of Schur function in terms of the first Weyl formula \[31, 32\].

The so-called Baxter $Q$-operators were introduced by Baxter in his pioneering work on the eight-vertex model \[33\], and have attracted interest from various point of views (see for example, \[34, 26, 27, 29, 28, 24, 30\]). They belong to the same commuting family of operators as the transfer matrices. In this paper, the eigenvalue of the Baxter $Q$-operator will be called the Baxter $Q$-function. Zeros of the Baxter $Q$-function correspond to the roots of the Bethe ansatz equation. We have $2^{M+N}$ kind of the Baxter $Q$-functions. And two of them are “trivial” in the sense that they can be normalized to just 1. Thus we have $2^{M+N} - 2$ kind of non-trivial Baxter $Q$-functions. But they are not independent as they obey the functional relations \eqref{2.17}-\eqref{2.18} (cf. \[35, 27, 37, 38, 36, 29, 39, 40, 41, 24\]). As remarked many times \[42, 7, 37, 38, 29, 39, 41, 24\], there are equivalent, but different, forms of the Bethe ansatz in the model. In our case, there are $(M + N)!$ different Bethe ansatz. They are connected by these functional relations among the Baxter $Q$-functions \eqref{2.17}-\eqref{2.18}. In \[7\], we discussed equivalence of the Bethe ansatz in relation to the Weyl (super) group for any $M, N \in \mathbb{Z}_{\geq 0}$. We also find Wronskian-like determinant expressions of the Baxter $Q$-functions, which solve the functional relations \eqref{2.17}-\eqref{2.18} (cf. Theorem 3.2). These determinant expressions are similar to the ones for the $T$-functions, but labelled by an empty Young diagram. In \[26\], the Baxter $Q$-operators are defined as trace of the universal $R$-matrix over $q$-oscillator representations of the quantum affine algebra $U_q(\hat{sl}(2))$. This construction of the Baxter $Q$-operators was generalized in the subsequent papers \[27, 28, 24\]. In particular, importance of boundary twists or horizontal fields to regularize the trace over the infinite dimensional space was recognized in \[26\] for the first time. Although we do not discuss operator realization of our formulae on the Baxter $Q$-functions, we design our formulae with the construction of the Baxter $Q$-operators in \[26\] in mind.

In \[25\], it was shown for $gl(M)$-related elliptic models that $T$-functions for quantum integrable models can be treated by methods of classical theory of solitons and discrete integrable equations. Moreover a part of discussions in \[25\] was generalized, in the recent papers \[39, 40\], to $gl(M|N)$-related rational models, where $(M + 1)(N + 1)$ kind of Baxter $Q$-functions were treated. In particular, Bäcklund transformations among solutions of the $gl(m|n)$-type $T$-systems ($0 \leq m \leq M, 0 \leq n \leq N$) were proposed in \[39, 40\] (cf. eqs. \eqref{3.58}-\eqref{3.61}), where the problem on $gl(M|N)$ was connected to the one on $gl(0|0)$. The $U_q(\hat{gl}(m|n))$-type $T$-systems ($0 \leq m < M, 0 \leq n < N$; cf. eqs. \eqref{3.50}-\eqref{3.53}) in the intermediate step in the Bäcklund flow have the same form as the original $T$-system \eqref{1.1}-\eqref{1.4} with $(M, N) \to (m, n)$ except for the functions
$T^{(a)}_0(x), T^{(0)}_s(x)$ ($a, s \in \mathbb{Z}_{\geq 0}$) appearing in the boundary. The functions $T^{(a)}_0(x), T^{(0)}_s(x)$ for these intermediate functional relations are proportional to the Baxter $Q$-functions (cf. eqs. (3.37), (3.38)). We find that our Wronskian-like determinant formulae explicitly solve all these functional relations (cf. Theorems 3.3, 3.4).

In the representation theoretical context, the $T$-functions can be interpreted as the so-called $q$-characters [43]. Originally, the $q$-character is defined [43] as a partial trace of the universal $R$-matrix for some representation of the quantum affine algebra. Consequently, the eigenvalue formula of the transfer matrix by the Bethe ansatz has a similar form as the $q$-character if the Baxter $Q$-functions are defined in the normalization of the universal $R$-matrix (this corresponds to case where the vacuum part is formally put to 1). In this sense, the $T$-functions by the analytic Bethe ansatz in [6, 7, 8] are prototypes of the $q$-(super)characters for the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ or $U_q(\widehat{gl}(M|N))$. Thus our new Wronskian-like expression of the $T$-function is yet another form of the $q$-(super)character. In this expression, the Weyl group invariance of the $q$-(super) character becomes manifest.

The size of the Wronskian-like determinant for the $T$-function is less than (for atypical representations) or equal to $M + N$ (for typical representations). On the other hand, the size of the determinant for the supersymmetric Bazhanov-Reshetikhin formula depends on the representation and has no upper bound. In addition, the number of the terms in the tableaux sum expression of the $T$-function is the dimension of the auxiliary space. Thus our new Wronskian-like determinant formulae will be particularly useful to analyze eigenvalues of the transfer matrices for large dimensional representations in the auxiliary space.

In section 2, we reformulate formulae on the analytic Bethe ansatz in our previous papers [6, 7, 8]. In section 3, we propose the Wronskian-like determinant expressions of the $T$-functions and the Baxter $Q$-functions, and give solutions of the $T$-system for $U_q(gl(M|N))$. These are our main results in this paper. We will also mention relation among these Wronskian-like formulae and the formulae in section 2. In section 4, we will briefly comment on how to obtain the $T$-functions for typical representations. The factorization property of the $T$-functions straightforwardly follows from the Wronskian-like determinant formulae. In [8], we showed that the $T$-function for the typical representation can be obtained as a deformation of the tableaux sum expression of the $T$-function. We rederive this result for the typical representation from our new Wronskian-like determinant formula. In section 5, we see that Baxter equations for $U_q(\widehat{gl}(M|N))$ straightforwardly follow from the Wronskian-like determinant formulae. They are finite order difference equations and linear with respect to the Baxter $Q$-functions. Section 6 is devoted to discussions. In Appendix A we briefly mention representations of the superalgebra $gl(M|N)$ and their characters. It will be good to

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3Compare [43] with [43].

4This does not always mean that the quantum space of the model has “zero-spin”. Rather, this means that the quantum space is arbitrary.
see how the $T$-function in this paper reduces to the character formula (in particular, the determinant formula in [23]) of $gl(M|N)$ (or its subalgebras) in the limit $x \to 0$. In Appendix B we will discuss, the normalization of the Baxter $Q$-functions. We will prove our main theorems in Appendix C. In Appendix D we will derive conserved quantities, and obtain determinant formulae of generalized Baxter equations for $U_q(gl(M|N))$. In this paper, we assume that $q$ is not a root of unity.

2 Analytic Bethe ansatz for the Perk-Schultz-type models and their fusion hierarchies revisited

The Perk-Schultz model [10] is a trigonometric vertex model related to the $(M+N)$ dimensional evaluation representation of $U_q(gl(M|N))$. It is just a representative of an infinite set of fusion models. The fusion models related to $U_q(gl(M|N))$ were studies in [45, 46]. In this section, we briefly reformulate formulae for the $T$-functions of the fusion hierarchies of the Perk-Schultz-type models from the analytic Bethe ansatz in our previous papers [6, 7, 8], keeping in mind recent developments [39, 40, 24].

Let us introduce the following sets

\[ \mathcal{J} := \{1, 2, \ldots, M+N\} = \mathcal{B} \sqcup \mathfrak{F}, \]

\[ \mathcal{B} := \{1, 2, \ldots, M\}, \quad \mathfrak{F} := \{M+1, M+2, \ldots, M+N\}, \quad \text{(2.1)} \]

and the grading parameters

\[ p_a = 1 \quad \text{for} \quad a \in \mathcal{B} \quad \text{and} \quad p_a = -1 \quad \text{for} \quad a \in \mathfrak{F}. \quad \text{(2.2)} \]

Consider any one of the permutations of the components of the tuple $(1, 2, \ldots, M+N)$, and write it as a $(M+N)$-tuple $I_{M+N} = (i_1, i_2, \ldots, i_{M+N})$. Of course, $I_{M+N}$ coincides with $\mathcal{J}$ as a set. Let us take the first $a$-components ($0 \leq a \leq M+N$) of the tuple $I_{M+N}$ and write it as an $a$-tuple $I_a = (i_1, i_2, \ldots, i_a)$. We also take the rest of $(M+N-a)$ components of $I_{M+N}$, and write it as $(M+N-a)$-tuple $\mathcal{T}_a = (i_{a+1}, i_{a+2}, \ldots, i_{M+N})$. Similarly, consider any one of the permutations of the components of the tuple $(1, 2, \ldots, M)$ (resp. $(M+1, M+2, \ldots, M+N)$), and write it as a $M$-tuple $B_{M} = (b_1, b_2, \ldots, b_M)$ (resp. $N$-tuple $F_N = (f_1, f_2, \ldots, f_N)$). Let us take the first $m$-components ($0 \leq m \leq M$) of the tuple $B_M$ (resp. the first $n$-components ($0 \leq n \leq N$) of the tuple $F_N$) and write it as a $m$-tuple $B_m = (b_1, b_2, \ldots, b_m)$ (resp. a $n$-tuple $F_n = (f_1, f_2, \ldots, f_n)$). We will use a symbol $B_m \times F_n := (b_1, b_2, \ldots, m; f_1, \ldots, f_n)$. We will denote $I_0, \mathcal{T}_{M+N}, B_0, F_0$ as $\emptyset$. We also use the symbol $I$ to denote any one of the tuples $I_0, I_1, \ldots, I_{M+N}$. In case we need not mind order of the components of these tuples $I_a, B_m, F_n$, we will treat these as just sets, but will use the same symbols. We have $(M+N)!$ different choices of the tuple $I_{M+N}$. There are $\binom{M+N}{a}$ different choices of $I_a$ (as sets) for each $a$, and in total $2^{M+N}$
for all \(a \in \{0, 1, \ldots, M + N\}\). As sets, they form a poset with respect to the inclusion relations. Each \(\{I_a\}_{a=0}^{M+N}\) forms a chain as a family of sets:
\[
\emptyset = I_0 \subset I_1 \subset \cdots \subset I_{M+N}, \quad \text{Card}(I_a \setminus I_{a-1}) = 1, \quad a \in \mathcal{I},
\]
(2.3)
where \(\text{Card}(J)\) is the number of the elements of the set \(J\). We also have an opposite family of sets: \(I_{M+N} = I_0 \supset I_1 \supset \cdots \supset I_{M+N} = \emptyset\).

We will consider functions \(\{Q_{I_a}(x)\}_{a=0}^{M+N}\) of a multiplicative spectral parameter \(x \in \mathbb{C}\) included in \(\{I_a\}_{a=0}^{M+N}\). These are the so-called Baxter \(Q\)-functions, which will be discussed in what follows. We will use a notation
\[
\overline{Q}_{I_a}(x) := Q_{I_a}(x).
\]
(2.4)
We will treat the tuple \(I_a\) just a set for \(Q_{I_a}(x)\) and \(\overline{Q}_{I_a}(x)\), where \(0 \leq a \leq M + N\). For example, \(Q_{(1,2,3)}(x), Q_{(2,3,1)}(x), Q_{(3,1,2)}(x), Q_{(1,3,2)}(x), Q_{(3,2,1)}(x), Q_{(2,1,3)}(x)\) are the same function and denoted as \(Q_{(1,2,3)}(x)\). There are \(2^{M+N}\) Baxter \(Q\)-functions, in total. In this paper, we normalize the Baxter \(Q\)-functions as
\[
Q_J(0) = 1 \quad \text{for any} \quad J \subset \mathcal{I}.
\]
(2.5)

Before we proceed discussions, we must comment on a subset of the \(2^{M+N}\) Baxter \(Q\)-functions. Let us consider the case where the components of the tuple \(I_{M+N}\) are arranged in the following way: for any \(a \in \{0, 1, \ldots, M + N\}\), \(I_a\) has the form \(\{1, 2, \ldots, m\} \sqcup \{M + 1, M + 2, \ldots, M + n\}\) as a set, where \(m = \text{Card}(I_a \cap \mathcal{I})\), \(n = \text{Card}(I_a \cap \overline{\mathcal{I}})\) and \(a = m+n\). There are \(\frac{(M+N)!}{M!N!}\) different choice of such \(I_{M+N}\). And in total, there are \((M+1)(N+1)\)-\(Q\)-functions \(\{Q_{\{1,2,\ldots,m\} \cup \{M+1, M+2, \ldots, M+n\}}(x)\}_{0 \leq m \leq M, 0 \leq n \leq N}\) labeled by such \(\{I_a\}_{a=0}^{M+N}\). These \((M+1)(N+1)\)-\(Q\)-functions (for rational case) essentially correspond to the ones considered in \([39, 40]\).

The eigenvalue formula of the transfer matrix of the Perk-Schultz-type model by the Bethe ansatz has the following form\(^5\)\(^6\): \(\mathcal{F}_{(1)}^{I_{M+N}}(x) = \sum_{a=1}^{M+N} p_a \mathcal{X}_{I_a}(x)\),
(2.6)
where the functions \(\{\mathcal{X}_{I_a}(x)\}_{a=1}^{M+N}\) are defined in the following way. Let us take a \(K\)-tuple \(I = (\gamma_1, \gamma_2, \ldots, \gamma_K)\) whose components are mutually distinct elements of \(\mathcal{I}\),

\(^5\)The Baxter \(Q\)-operators \((A_1(x), A_2(x), A_3(x), \overline{A}_1(x), \overline{A}_2(x), \overline{A}_3(x))\) for \(U_q(\mathfrak{sl}(2|1))\) in \([24]\) corresponds to the functions \((Q_{(2,3)}(x), Q_{(1,3)}(x), Q_{(1,2)}(x), Q_{(1)}(x), Q_{(2)}(x), Q_{(3)}(x))\) for \((M, N) = (2, 1)\) with \(z_1 z_2 z_3^{-1} = 1\) in this paper. \(q\) in \([24]\) corresponds to \(q^{-1}\) here.

\(^6\)Here we are considering the case where the auxiliary space is an \((M + N)\) dimensional evaluation representation of \(U_q(\mathfrak{gl}(M|N))\). The quantum space of the original Perk-Schultz model is also the \((M + N)\) dimensional representation on each site. In our normalization of the Baxter \(Q\)-functions, the vacuum part is included in the Baxter \(Q\)-functions. Thus the formula has the same form even if the model has a more general quantum space \([3, 8, 39, 48]\).
where $1 \leq K \leq M + N$. We also introduce $(K - 1)$-tuples $\hat{I} = (\gamma_1, \gamma_2, \ldots, \gamma_{K-1})$ and $\check{I} = (\gamma_2, \gamma_3, \ldots, \gamma_K)$. Then we define functions labeled by $\hat{I}$ as

$$\mathcal{X}_I(x) := z_{\gamma K} \frac{Q_1(xq^{-\sum_{j=1}^{N} p_j + \frac{M+N}{2}})Q_1(xq^{-\sum_{j=1}^{N} p_j + \frac{M+N}{2}})}{Q_1(xq^{-\sum_{j=1}^{N} p_j + \frac{M+N}{2}})Q_1(xq^{-\sum_{j=1}^{N} p_j + \frac{M+N}{2}})}$$

and

$$\overline{\mathcal{X}}_I(x) := z_{\gamma 1} \frac{\overline{Q}_1(xq^{-\sum_{j=1}^{N} p_1 - \frac{M-N}{2}})\overline{Q}_1(xq^{-\sum_{j=1}^{N} p_1 - \frac{M-N}{2}})}{\overline{Q}_1(xq^{-\sum_{j=1}^{N} p_1 - \frac{M-N}{2}})\overline{Q}_1(xq^{-\sum_{j=1}^{N} p_1 - \frac{M-N}{2}})}.$$  (2.7)

We note that the relation $\overline{\mathcal{X}}_{\hat{I}_{a-1}}(x) = \mathcal{X}_{\check{I}_a}(x)$ holds due the relation $\sum_{j \in \hat{I}_a} p_j + \sum_{j \in \check{I}_a} p_j = M - N$ and (2.11). The parameters $z_a \in \mathbb{C}$ correspond to boundary twists or horizontal fields; namely we are considering a transfer matrix with a twisted boundary condition. The functions $Q_I(x)$ for $I = \emptyset$ or $I = I_{M+N}$ are special. In this paper, we will normalize these as

$$Q_{\emptyset}(x) = 1$$  (2.9)

or

$$\overline{Q}_{\emptyset}(x) = 1.$$  (2.10)

In the normalization of the universal $R$-matrix, the conformal field theoretical models or the $q$-(super)character, we impose both (2.9) and (2.10), at least for the case $M \neq N$ (cf. [26 27 28 24]; see Appendix B). For the Perk-Schultz model, we may use the normalization (2.10) and

$$\overline{Q}_{I_{M+N}}(x) = \prod_{j=1}^{L} \left(1 - \frac{x}{w_j}\right),$$  (2.11)

where $L$ is the number of the lattice site; $w_j$ is an inhomogeneity on the spectral parameter on the $j$-th site of the quantum space (the homogeneous case corresponds to $w_j = 1$). (2.11) corresponds to the so-called quantum Wronskian condition. In this case, $\{Q_{\check{I}_a}(x)\}_{a=1}^{M+N-1}$ are polynomials of $x$ for finite $L$:

$$Q_{\check{I}_a}(x) = \prod_{k=1}^{n_{I_a}} \left(1 - \frac{x}{x_{k}^a}\right),$$  (2.12)

where $\{x_{k}^a\}$ are roots of the Bethe ansatz equation:

$$-1 = \frac{p_{a+1}z_{I_{a+1}}Q_{I_{a+1}}(x_{k}^a q_{p_{a+1}})Q_{I_{a}}(x_{k}^a q_{-2p_{a+1}})Q_{I_{a+1}}(x_{k}^a q_{p_{a+1}})}{p_{a}z_{I_{a}}Q_{I_{a-1}}(x_{k}^a q_{-p_{a}})Q_{I_{a}}(x_{k}^a q_{2p_{a}})Q_{I_{a+1}}(x_{k}^a q_{-p_{a+1}})}$$  (2.13)

for $k \in \{1, 2, \ldots, n_{I_a}\}$ and $a \in \{1, 2, \ldots, M + N - 1\}$.

7Here we treat the tuple $I_a$ just a set for $x_{k}^a$.  

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The condition that the poles of (2.6) from the zeros of \( Q_\mu(x) \) \((1 \leq a \leq M + N - 1)\) chancel each other:

\[
\text{Res}_{x=x_k} \sum_{j \in I_\mu} \frac{m_{\mu+j} - m_{\mu-j}}{x-w_j} (p_n, X_{I_\mu}(x) + p_{n+1} X_{I_{a+1}}(x)) = 0
\]  

(2.14)

produces the Bethe ansatz equation (2.13). One can obtain a Bethe ansatz system, which has one-to-one correspondence with the above mentioned one for the Perk-Schultz model by the normalization (2.9) and

\[
Q_{I_{M+N}}(x) = \prod_{j=1}^L \left(1 - \frac{x}{w_j}\right).
\]  

(2.15)

The rational case of this system corresponds to the one considered in [39, 40]. In the context of the theory of the \( q \)-character [43], (2.6) corresponds to the \( q \)-(super)character \(^8\) for the \( M + N \) dimensional evaluation representation of \( U_q(\mathfrak{gl}(M|N)) \), and \( X_{I_\mu}(x) \) is called the “highest weight monomial”.

Let us write the symmetric group over the components of the tuple \( I \) (or elements of the set \( I \)) as \( S(I) \). We assume that \( \sigma \in S(I) \) acts on \( \mathcal{A} \setminus I \) as \( \sigma(a) = a \). The action on \( \mathcal{F}_{(1)}^{I_{M+N}}(x) \) is defined as \( \sigma[\mathcal{F}_{(1)}^{I_{M+N}}(x)] := \mathcal{F}_{(1)}^{\sigma(I_{M+N})}(x) = \mathcal{F}_{(1)}^{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{M+N})}(x) \). If one interprets the action of \( \sigma \) on \( X_{I_\mu}(x), Q_{I_\mu}(x), z_a \) and \( p_a \) as \( \sigma[X_{I_\mu}(x)] = X_{\sigma(I_\mu)}(x), \sigma[Q_{I_\mu}(x)] = Q_{\sigma(I_\mu)}(x), \sigma(z_a) = z_{\sigma(a)}, \sigma(p_a) = p_{\sigma(a)} \), then one sees that these are compatible with the definitions (2.6) and (2.7). Note that \( \sigma[Q_{I_\mu}(x)] = Q_{I_\mu}(x) \) for any \( \sigma \in S(I) \) such that \( I \subset J \) or \( I \subset J \setminus J \). The direct product of the symmetric groups \( S(\mathfrak{B}) \times S(\mathfrak{F}) \) corresponds to the Weyl group of \( \mathfrak{gl}(M|N) \) if the parameters \( \{z_a\} \) are interpreted as in Appendix A. The \( Q \)-functions \( Q_I(x) \) with the same “level” \( \text{Card}(I) \) are in the same “\( S(\mathfrak{J}) \)-orbit”.

As already remarked many times in various context [12, 7, 37, 38, 29, 39, 41, 24], (2.6) does not depend on the choice of the tuple \( I_{M+N} \) since there are relations among the Baxter \( Q \)-functions (cf. Figure 2). Let us consider a permutation \( \tau \in S(I_{M+N}) = S(\mathfrak{J}) \) such that \( \tau(i_a) = i_{a+1}, \tau(i_{a+1}) = i_a \) and \( \tau(i_b) = i_b \) for \( b \neq a, a + 1 \), for a fixed \( a \in \{1, 2, \ldots, M + N - 1\} \). Note that \( \tau[Q_{I_\mu}(x)] = Q_{I_b}(x) \) if \( b \neq a \). Thus \( \tau[X_{I_b}(x)] = X_{I_b}(x) \) if \( b \neq a, a + 1 \). Let us write \( i_a = i, i_{a+1} = j, I_{a-1} = I \). The condition \( \tau[\mathcal{F}_{(1)}^{I_{M+N}}(x)] = \mathcal{F}_{(1)}^{I_{M+N}}(x) \) is equivalent to

\[
p_iX_{I_a}(x) + p_jX_{I_{a+1}}(x) = p_jX_{\tau(I_a)}(x) + p_iX_{\tau(I_{a+1})}(x).
\]  

(2.16)

This implies the following functional relation (cf. [35, 27, 36, 29, 41, 24]) \(^3\)

\[
(z_i - z_j)Q_I(x)Q_{I\cup\{i,j\}}(x) = z_iQ_{I\cup\{i\}}(xq^{p_i})Q_{I\cup\{j\}}(xq^{-p_i}) - z_jQ_{I\cup\{i\}}(xq^{-p_i})Q_{I\cup\{j\}}(xq^{p_i})
\]  

for \( p_i = p_j \),

(2.17)

\(^3\) Difference between the supercharacter and the character is not essential here as the supercharacter becomes the character by the transformation: \( z_a \rightarrow -z_a \) for \( a \in \mathfrak{F} \). In the theory of the \( q \)-characters, one usually includes the parameters \( \{z_a\} \) into the Baxter \( Q \)-functions and writes the \( q \)-characters in terms of the variables of the form \( Y_{I_\mu}(x) = Q_{I_\mu}(xq)/Q_{I_\mu}(xq^{-1}) \).

\(^8\) Functional relations similar to (2.17) appeared in [35] for rational vertex models related to
In addition, the function $Q_{I_a}(x)$ (2.17) for $I = I_a$ lives on the edge which connects two vertexes for $Q_{I_a}(x)$ and $Q_{I_{a-1}}(x)$. Thus the function $\mathcal{F}_{(1)}(x)$ (2.32) for $\lambda \subset \mu = (1)$, $I = I_a$, $M = N = 2$ lives on a path from $Q_{I_a}(x) = Q_8(x)$ to $Q_{I_a}(x)$; the function $\mathcal{F}_{(1)}(x)$ (cf. (2.33) for $\lambda \subset \mu = (1)$, $I = I_a$, $M = N = 2$) lives on a path from $Q_{I_a}(x)$ to $Q_{I_a}(x) = Q_{(1,2,3,4)}(x)$, where these paths must not contain more than two functions $Q_I(x), Q_J(x)$ of the same level $\text{Card}(I) = \text{Card}(J)$. In particular, they do not depend on the paths.
and also the following (cf. [37, 38, 29, 39, 40, 24]):

\[(z_i - z_j)Q_{I,\{i\}}(x)Q_{I,\{j\}}(x) = z_i Q_I(xq^{-p_i})Q_{I,\{i,j\}}(xq^{p_i}) - z_j Q_I(xq^{p_j})Q_{I,\{i,j\}}(xq^{-p_j})\]

for \(p_i = -p_j\).

(2.18)

In fact, one can easily show the relation (2.16) based on the functional relations (2.17) and (2.18). Functional relations corresponding to (2.18) were discussed in [39, 40] from a point of view of a classical theory of the soliton, and also to (2.17) in [41]. In addition, (2.17) was proved based on decompositions of \(q\)-oscillator representations of \(U_q(\hat{sl}(2))\) [26], \(U_q(\hat{sl}(3))\) [27] and \(U_q(\hat{sl}(2|1))\) [24]. For \(U_q(\hat{gl}(2))\) case, (2.17) reduces to the quantum Wronskian condition. One may also unify the above two relations.

\[z_i Q_{I,\{i\}}(xq^{p_i/2})Q_{I,\{i,j\}}(xq^{p_i/2}) - z_j Q_{I,\{i\}}(xq^{-p_i/2})Q_{I,\{i,j\}}(xq^{p_i/2})\]

\[= z_i Q_I(xq^{-p_i/2})Q_{I,\{i,j\}}(xq^{p_i/2}) - z_j Q_I(xq^{p_i/2})Q_{I,\{i,j\}}(xq^{-p_i/2}).\]

(2.19)

Note that the relations (2.17) and (2.18) can be rewritten as

\[(z_i - z_j)\bar{Q}_I(x)\bar{Q}_{I,\{i,j\}}(x) = z_i \bar{Q}_{I,\{i\}}(xq^{-p_i})\bar{Q}_{I,\{i,j\}}(xq^{p_i}) - z_j \bar{Q}_{I,\{i\}}(xq^{p_i})\bar{Q}_{I,\{i,j\}}(xq^{-p_i})\]

for \(p_i = p_j\).

(2.20)

\[(z_i - z_j)\bar{Q}_{I,\{i\}}(x)\bar{Q}_{I,\{i,j\}}(x) = z_i \bar{Q}_I(xq^{p_i})\bar{Q}_{I,\{i,j\}}(xq^{-p_i}) - z_j \bar{Q}_I(xq^{-p_i})\bar{Q}_{I,\{i,j\}}(xq^{p_i})\]

for \(p_i = -p_j\).

(2.21)

The form of the functional relations (2.17) - (2.18) are invariant under the gauge transformation \(Q_I(x) \to g_1(xq^{\sum k \in I} p_k)g_2(xq^{-\sum k \in I} p_k)Q_I(x)\) for all \(I \subset \mathcal{I}\), where \(g_1(x), g_2(x)\) are arbitrary functions of \(x\). Thus one can always chose the normalization (2.9) based on this freedom. The same discussion can be applied for (2.20) - (2.21) and (2.10). In particular for \(I = \emptyset\) case, (2.18) with the normalization (2.9) reduces to

\[(z_i - z_j)Q_i(x)Q_j(x) = z_i Q_{\{i\}}(xq) - z_j Q_{\{j\}}(xq^{-1}) \quad \text{for} \quad i \in \mathfrak{B}, \quad j \in \mathfrak{F}.\]

(2.22)

\(A_2\); in [39] in relation to the ODE/IM correspondence for \(A_n, B_n, C_n, D_n\). (2.14) appeared in [27] for \(\mathcal{W}_3\) CFT related to \(U_q(\hat{sl}(3))\). As for the superalgebra related models, functional relations similar to (2.17) appeared in [29] for rational non-compact \(sl(2|1)\) spin chains; in [41], in the context of the AdS/CFT correspondence, for rational models related to \(su(M|N)\). In [24], we proposed (2.17) for both CFT and trigonometric vertex models associated with \(U_q(\hat{sl}(2|1))\). We had also reported these functional relations (2.17), (2.18) and some Wronskian-like formulae on \(T^*\) and \(Q\)-operators for \(U_q(\hat{sl}(2|1))\) case, before the preprint version of [24] appeared, on many conferences which include the following two: “Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory”, the university of Melbourne, January, 2007 [http://www.smft2007.ms.unimelb.edu.au/program/LectureSeries.html]; La 79ème Rencontre entre physiciens theoriciens et mathematiciens “Supersymmetry and Integrability”, IRMA Strasbourg, June, 2007 [http://www-irma.u-strasbg.fr/article383.html].
where we used abbreviations $Q_i(x) := Q_{\{i\}}(x)$ and $\overline{Q}_i(x) := \overline{Q}_{\{i\}}(x)$. Thus $Q_{\{i,j\}}(x)$ with (2.10) (resp. $\overline{Q}_{\{i,j\}}(x)$ with (2.10)) can be expressed as a kind of convolution of $Q_i(x)$ and $Q_j(x)$ (resp. $\overline{Q}_i(x)$ and $\overline{Q}_j(x)$): $Q_{\{i,j\}}(x) = \sum_{a=0}^{\infty} a_I^{(k)} x^a (I = \{i\}, \{j\})$, $a_I^{(0)} = 1$, $a_I^{(k)} := a_{(i)}^{(k)}$ and plug this into (2.22). Then we obtain

\[ Q_{\{i,j\}}(x) = \sum_{a=0}^{\infty} \sum_{\beta=0}^{\infty} (\frac{z_i - z_j}{z_i}) a_i^{(\alpha)} a_j^{(\beta)} x^{a+\beta}. \]  

Substitute $x = q^{\sum_{i=0}^{k} I^{(i)}} x^k$ and $x = q^{\sum_{i=0}^{k} I^{(i)}} x^k$ into (2.17) with (2.12), we obtain

\[ (z_i - z_j)Q_I(x_k^{I^{(i)}} q^{-p_i})Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i}) = -z_j Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-2p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}}), \]  

\[ (z_i - z_j)Q_I(x_k^{I^{(i)}} q^{p_i})Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{p_i}) = z_i Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{2p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}}). \]  

Let us divide these two equations by one another:

\[ -1 = \frac{z_j}{z_i} \frac{Q_I(x_k^{I^{(i)}} q^{p_i})Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{2p_i})}{Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i})} \quad \text{for} \quad p_i = p_j. \]  

Substitute $x = x_k^{I^{(i)}}$ into (2.18) with (2.12), we obtain

\[ 1 = \frac{z_j}{z_i} \frac{Q_I(x_k^{I^{(i)}} q^{p_i})Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{2p_i})}{Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i}) Q_{I\cup\{i\}}(x_k^{I^{(i)}} q^{-p_i})} \quad \text{for} \quad p_i = -p_j. \]  

(2.29) and (2.30) for $I = I_{a-1}, i = i_a, j = i_{a+1}$ coincides with the Bethe ansatz equation (2.11).
Let us introduce notations for the partitions and Young diagrams (see [49] for details). A partition is a non-increasing sequence of positive integers \( \mu = (\mu_1, \mu_2, \ldots) \): 
\[ \mu_1 \geq \mu_2 \geq \cdots \geq 0. \]
We often write this in the form \( \mu = (r^{m_r}, (r-1)^{m_{r-1}}, \ldots, 2^{m_2}, 1^{m_1}) \), where \( r = \mu_1 \), and \( m_k = \text{Card}\{j|\mu_j = k\} \) is called the multiplicity of \( k \) in \( \mu \). Two partitions are regarded equivalent if all their non-zero elements of positive multiplicity coincide. Elements of the multiplicity zero should be abbreviated. For example, 
\[(4, 3, 2, 1, 0, 0) = (4, 3, 2, 1, 1) = (5^0, 4, 3, 2, 1^2) = (4, 3, 2, 1^2).\]
We will also express the partitions \( \mu \) as Young diagrams, and will use the same symbol \( \mu \) for the Young diagrams. The Young diagram \( \mu \), corresponding to a partition \( \mu \), has \( \mu_k \) boxes in the \( k \)-th row in the plane (see Figure 3). Each box in the Young diagram is specified by a coordinate \((i, j) \in \mathbb{Z}^2\), where the row index \( i \) increases as one goes downwards, and the column index \( j \) increases as one goes from left to right. The top left corner of \( \mu \) has coordinates \((1, 1)\). The partition \( \mu' = (\mu'_1, \mu'_2, \ldots) \) is called conjugate of \( \mu \), and is defined as \( \mu'_j = \text{Card}\{k|\mu_k \geq j\} \) (see Figure 4). The Young diagram \( \mu' \) is obtained by the transposition of rows and columns of the Young diagram \( \mu \). Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) be two partitions such that \( \mu_i \geq \lambda_i : i = 1, 2, \ldots \) and \( \lambda_{\mu'_i} = \lambda'_{\mu_i} = 0 \). We will express a skew-Young diagram defined by these two partitions as \( \lambda \subset \mu \). This is the domain given by the subtraction \( \mu - \lambda \) (see Figure 5). If \( \lambda \) is an empty set \( \emptyset \), then \( \lambda \subset \mu \) coincides with \( \mu \). Each box on the skew-Young diagram \( \lambda \subset \mu \) is specified by its coordinate on \( \mu \). We will also use the 180 degrees rotation of the skew-Young
Figure 5: The skew Young diagram $\lambda \subset \mu$ with $\lambda = (2, 1)$ and $\mu = (4, 3, 2, 1^2)$.

Figure 6: The 180 degrees rotation of the skew-Young diagram $\lambda \subset \mu$ with $\lambda = (2, 1)$ and $\mu = (4, 3, 2, 1^2)$ in Figure 5 is the skew Young diagram with shape $\tilde{\lambda} \subset \tilde{\mu} = (3^2, 2, 1) \subset (4^3, 3, 2)$. 
diagram $\lambda \subset \mu$, and denote it as $\widetilde{\lambda} \subset \mu$ (see Figure 6):

$$\widetilde{\lambda} \subset \mu := (\mu_1 - \mu_{1'}, \mu_1 - \mu_{1'-1}, \mu_1 - \mu_{1'-2}, \ldots, \mu_1 - \mu_{2}, \mu_{2})$$

$$\subset (\mu_{1} - \lambda_{1}, \mu_{1} - \lambda_{2}, \mu_{1} - \lambda_{3}, \ldots, \mu_{1} - \lambda_{2}, \mu_{2} - \lambda_{1}). \quad (2.31)$$

Note that that $\tilde{\mu} = \mu$ if $\mu$ is a Young diagram of rectangular shape.

Let us take a $K$-tuple $I = (\gamma_1, \gamma_2, \ldots, \gamma_K)$ whose components are mutually distinct elements of $\mathcal{J}$, where $1 \leq K \leq M + N$. We also introduce $k$-tuple $D_k(I) = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $(K - k + 1)$-tuple $D_k(I) = (\gamma_k, \gamma_{k+1}, \ldots, \gamma_{K})$, where $1 \leq k \leq K$. Next, let us define a space of admissible tableaux $\text{Tab}_I(\lambda \subset \mu)$ for the $K$-tuple $I$ on a (skew) Young diagram $\lambda \subset \mu$. We assign an integer $t_{ij}$ in each box $(i, j)$ of the diagram. An admissible tableau $t \in \text{Tab}_I(\lambda \subset \mu)$ is a set of integers $t = \{t_{jk}\}_{(j,k)\in \lambda \subset \mu}$, where all $t_{jk} \in \{1, 2, \ldots, K\}$ satisfy the conditions

(i) $t_{jk} \leq t_{j+1,k}, t_{j,k+1}$

(ii) $t_{jk} < t_{j,k+1}$ if $\gamma_{t_{jk}} \in \mathfrak{F}$ or $\gamma_{t_{j,k+1}} \in \mathfrak{F}$

(iii) $t_{jk} < t_{j+1,k}$ if $\gamma_{t_{jk}} \in \mathfrak{B}$ or $\gamma_{t_{j+1,k}} \in \mathfrak{B}$.

The fusion models are described by the Young diagrams. Here we supposed that the fusion is performed in the auxiliary space of the transfer matrix. In particular, the $T$-functions of the fusion models from the analytic Bethe ansatz can be written as summation over Young tableaux with spectral parameters $[6, 7, 8]$ (see also [2] for $T$-functions of the fusion models from the analytic Bethe ansatz can be written as summation over Young tableaux with spectral parameters $[6, 7, 8]$ (see also [2] for $T$-functions of the fusion models from the analytic Bethe ansatz can be written as summation over Young tableaux with spectral parameters $[6, 7, 8]$ (see also [2] for $T$-functions of the fusion models from the analytic Bethe ansatz can be written as summation over Young tableaux with spectral parameters $[6, 7, 8]$ (see also [2] for $T$-functions of the fusion models from the analytic Bethe ansatz can be written as summation over Young tableaux with spectral parameters $[6, 7, 8]$). Here we introduce slightly generalized versions of such $T$-functions $\mathcal{F}^I_{\lambda \subset \mu}(x)$ and $\mathcal{F}^I_{\lambda \subset \mu}(x)$. We remark that the following discussion on the $T$-functions is essentially independent of the “vacuum part”, and thus quantum space, where the transfer matrices act, is arbitrary. For any skew Young diagram $\lambda \subset \mu$, define functions $\mathcal{F}^I_{\lambda \subset \mu}(x)$ and $\mathcal{F}^I_{\lambda \subset \mu}(x)$:

$$\mathcal{F}^I_{\lambda \subset \mu}(x) = \sum_{t \in \text{Tab}_I(\lambda \subset \mu)} \prod_{(j,k) \in \lambda \subset \mu} p_{\gamma_{t_{jk}}} \mathcal{X}_{D_t(1)}(xq^{\mu_1 - \mu_{1'} + 2j - 2k - \frac{m-n}{2} - \frac{M+N}{2}}), \quad (2.32)$$

$$\mathcal{F}^I_{\lambda \subset \mu}(x) = \sum_{t \in \text{Tab}_I(\lambda \subset \mu)} \prod_{(j,k) \in \lambda \subset \mu} p_{\gamma_{t_{jk}}} \mathcal{X}_{D_t(1)}(xq^{\mu_1 - \mu_{1'} + 2j - 2k + \frac{m-n}{2} + \frac{M+N}{2}}), \quad (2.33)$$

where the summations are taken over all admissible tableaux, and the products are taken over all boxes of the Young diagram $\lambda \subset \mu$; $m := \text{card}(I \cap \mathfrak{B})$, $n := \text{card}(I \cap \mathfrak{F})$.

$\mathcal{F}^1_{\lambda \subset \mu}(x)$ in $\mathcal{F}^{(1,2,3,4,5)}_{\mu}(x)$ is related to the representation of $gl(M|N)$ as $\mathfrak{g}$.  

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$\mathcal{F}^1_{\lambda \subset \mu}(x)$ in $\mathcal{F}^{(1,2,3,4,5)}_{\mu}(x)$ is related to the representation of $gl(M|N)$ as $\mathfrak{g}$.
We also set \( F_0^I(x) = \mathcal{F}_0^I(x) = 1 \), and \( F_\mu^\emptyset(x) = \mathcal{F}_\mu^\emptyset(x) = 0 \) and for the non-empty Young diagram \( \mu \). Note that the admissible tableaux \( \text{Tab}_1(\lambda \subset \mu) \) becomes an empty set if the Young diagram \( \lambda \subset \mu \) contains a rectangular sub-diagram of a height of \( m + 1 \) and a width of \( n + 1 \), and consequently (2.32) and (2.33) vanish for such Young diagram \(^{12}\).

After inspection of (2.7), (2.8), (2.32) and (2.33), one observes the following relation:

\[
\mathcal{F}_{\lambda \subset \mu}^I(x) = \mathcal{F}_{\lambda \subset \mu}^\emptyset(x) \mathcal{Q}_J(xq^\gamma) = \mathcal{Q}_J(q^{-\gamma}) \quad \text{for all } J \subset I,
\]

(2.34)

where \( \gamma \) is any shift of the \( \mathcal{Q} \)-functions in \( \mathcal{F}_{\lambda \subset \mu}^\emptyset(x) \), and \( \emptyset = (\gamma_K, \ldots, \gamma_2, \gamma_1) \) is a reversed tuple of \( \emptyset \).

Let us take an integer \( K \) (\( 0 \leq K \leq M + N \)). The tableau sum formula (2.32) has determinant expressions.

\[
\mathcal{F}_{\lambda \subset \mu}^I(x) = \prod_{1 \leq i, j \leq \mu_1} (\mathcal{F}_{\mu_1}^I(xq^\mu_1 - xq^\mu_1')) \left( xq^{\mu_1 - \mu_1'} + xq^{\mu_1' - \mu_1} \right) = \prod_{1 \leq i, j \leq \mu_1} (\mathcal{F}_{\mu_1}^I(xq^\mu_1 - xq^\mu_1')) \left( xq^{\mu_1 - \mu_1'} + xq^{\mu_1' - \mu_1} \right),
\]

(2.35)

(2.36)

where \( \mathcal{F}_{(1^a)}^I(x) = \mathcal{F}_{(0)}^I(x) = 1 \) and \( \mathcal{F}_{(1^a)}^I(x) = \mathcal{F}_{(a)}^I(x) = 0 \) for \( a < 0 \). We also have the same type of formulae for (2.33). These determinant expressions for \( K = M + N \) correspond to the supersymmetric Bazhanov-Reshetikhin formulæ [6, 7] (see also a recent paper [19]; and for \( U_q(B_r^{(1)}) \) case, see [20]), which are supersymmetric extension of the Bazhanov-Reshetikhin formulæ [2]. The determinants (2.35) and (2.36) for \( U_q(gl(M|N)) \), \( M, N > 0 \) superficially look same as the ones for \( M = 0 \) or \( N = 0 \) case. However, one must note that their properties for \( M, N > 0 \) case are different from the ones for \( M = 0 \) or \( N = 0 \) case since the conditions of the admissible tableaux are different, and that the matrix elements of the determinants are different.

The above combinatorial tableau sum expressions of the \( T \)-functions (2.32) and (2.33) are convenient for computer calculations. Next, we consider other expressions of the \( T \)-functions. One can rewrite the above tableau sum expressions for the Young diagrams of one row or one column in the form of the non-commutative generating series [3, 7] (see also, [25, 39, 40]).

\[
\mathcal{W}_{I_K}(x, X) = (1 - \mathcal{X}_I(x)X)^{-p_{1}}(1 - \mathcal{X}_I(x)X)^{-p_{2}} \cdots (1 - \mathcal{X}_I(x)X)^{-p_{K}}
\]

\[
= \sum_{\alpha = 0}^{\infty} \mathcal{F}_{(\alpha)}^I(xq^{1 - \alpha/2 + M/2})X^\alpha,
\]

(2.37)

\[
\mathcal{W}_{\overline{\mathcal{T}}_K}(x, X)
\]

\[
= (1 - \overline{\mathcal{X}}_{I_K}(x)X)^{-p_{K+1}}(1 - \overline{\mathcal{X}}_{I_{K+1}}(x)X)^{-p_{K+2}} \cdots (1 - \overline{\mathcal{X}}_{M+N-1}(x)X)^{-p_{M+N}}
\]

\[
= (1 - \mathcal{X}_{I_{K+1}}(x)X)^{-p_{K+1}}(1 - \mathcal{X}_{I_{K+2}}(x)X)^{-p_{K+2}} \cdots (1 - \mathcal{X}_{I_{M+N}}(x)X)^{-p_{M+N}}
\]

See [16, 6, 7] for \( I = I_{M+N} \) case.
These generating series (2.37) and (2.39) for $[6, 7]$. In particular, $I$ is a generalization of eq. (E.6) in [6]. On the other hand, "Intermediate" ($0 = 0$), which generate eigenvalue formulae of "real" transfer matrices appeared in the rational case.

To be precise, the ones in [39, 40] were written in terms of functions corresponding to (2.45) for $F$ in [38, 14, 15, 41, 16].

\[
\mathcal{W}_{I_K}(x, X)^{-1} = (1 - \mathcal{X}_{I_K}(x)X^{p_{IK}} \cdots (1 - \mathcal{X}_{I_1}(x)X^{p_{I_1}}(1 - \mathcal{X}_{I_1}(x)X^{p_{I_1}}) = \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \mathcal{F}_{I_K}^{(1)}(xq^{1-\alpha + \frac{\alpha - n}{2} + \frac{M+N}{2}})X^{\alpha}, \tag{2.39}
\]

\[
\mathcal{W}_{T_K}(x, X)^{-1} = (1 - \mathcal{X}_{T_{M+N-1}}(x)X^{p_{M+N}} \cdots (1 - \mathcal{X}_{T_{K+1}}(x)X^{p_{K+2}}(1 - \mathcal{X}_{T_K}(x)X^{p_{K+1}} = \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \mathcal{F}_{T_{K}}^{(1)}(xq^{1-\alpha + \frac{\alpha - n}{2} + \frac{M+N}{2}})X^{\alpha}. \tag{2.40}
\]

They obey the following recurrence relations (cf. [39, 40]):

\[
\mathcal{W}_{I_K}(x, X)(1 - \mathcal{X}_{I_{K+1}}(x)X)^{-p_{K+1}} = \mathcal{W}_{I_{K+1}}(x, X), \tag{2.41}
\]

\[
(1 - \mathcal{X}_{T_K}(x)X)^{-p_{K+1}} \mathcal{W}_{T_{K+1}}(x, X) = \mathcal{W}_{T_{K}}(x, X). \tag{2.42}
\]

These generating series (2.37) and (2.39) for $K = M + N$ (resp. (2.38) and (2.40) for $K = 0$), which generate eigenvalue formulae of "real" transfer matrices appeared in [6, 7]. In particular, $\mathcal{W}_{I_K}(x, X)\mathcal{W}_{T_K}(x, X) = \mathcal{W}_{I_{M+N}}(x, X) = \mathcal{W}_{T_{0}}(x, X)$, corresponds to eq. (3.11) in [7]. In addition, from this relation, one can derive the following relation for $s \in \mathbb{Z}_{\geq 0}$:

\[
\mathcal{F}_{I_{M+N}}^{(s)}(x) = \mathcal{F}_{I_{0}}^{(s)}(x) = \sum_{\alpha=0}^{n} \mathcal{F}_{I_{K}}^{(s)}(xq^{s-\alpha - \frac{\alpha - n}{2} + \frac{M+N}{2}})\mathcal{F}_{I_{K}}^{(s-\alpha)}(xq^{s-\alpha + \frac{\alpha - n}{2} - \frac{M+N}{2}}). \tag{2.43}
\]

This is a generalization of eq. (E.6) in [6]. On the other hand, "Intermediate" ($0 < K < M + N$) generating series (2.37) and (2.39) were considered [3] in [39, 40]. Some "intermediate" generating series were also written in Appendix E in [6]. We remark that the non-commutative generating series of the $T$-functions similar to the ones in [6, 7] were also used recently in the study of the AdS/CFT correspondence in particle physics [38, 14, 15, 41, 16].

We define the action of $\sigma \in S(\mathfrak{I})$ on the $T$-functions as $\sigma[\mathcal{F}_{I_{L\mu}}(x)] = \mathcal{F}_{I_{L\mu}}^{\sigma}(x)$ and $\sigma[\mathcal{F}_{T_{L\mu}}(x)] = \mathcal{F}_{T_{L\mu}}^{\sigma}(x)$. As was pointed out in [39, 40], compatibility of this

\[13\]To be precise, the ones in [39, 40] were written in terms of functions corresponding to (2.43) for the rational case.
type of recurrence relations (2.41)-(2.42), which corresponds to a discrete “zero-curvature condition”, reproduces the functional relations among the Baxter Q-functions (2.17)-(2.21). This corresponds to W_{l+1}(x, X) = W_{\tau(l+1)}(x, X) in the notation of (2.16). Therefore \( \tau[F_{l+1}(x)] = F_{l+1}(x) \) under the functional relations (2.17)-(2.18), and thus \( \tau[F_{\lambda\mu}(x)] = F_{\lambda\mu}(x) \) through (2.36). Similarly, \( \tau[F_{\lambda\mu}(x)] = F_{\lambda\mu}(x) \) under the functional relations (2.20)-(2.21). Thus \( \sigma[F_{\lambda\mu}(x)] = F_{\lambda\mu}(x) \) and \( \sigma[F_{\lambda\mu}(x)] = F_{\lambda\mu}(x) \) for any \( \sigma \in S(I) \) or \( \sigma \in S(3 \setminus I) \). This means that the T-functions \( F_{\lambda\mu}(x) \) and \( F_{\lambda\mu}(x) \) are independent of the order of the components of the tuple \( I \) (the order of the gradings) under the functional relations (2.17)-(2.21). In particular, \( \sigma[F_{\lambda\mu}(x)] = F_{\lambda\mu}(x) \) for \( \sigma \in S(3) \times S(3) \) under (2.17) corresponds to the Weyl group invariance of the T-functions, and for \( \sigma \in S(3)/[S(3) \times S(3)] \) under (2.18) corresponds to the super-Weyl group invariance (cf. [7]). Note that \( S(3)/[S(3) \times S(3)] \) is not a simple permutation on the parameters \( \{z_a\} \); it affects the rule of the admissible tableau \( \text{Tab}_{I_{l+1}}(\lambda \subseteq \mu) \).

We can also consider a non-commutative generating series of the T-functions for the general skew-Young diagrams \( \lambda \subseteq \mu \):

\[
\det_{1 \leq i, j \leq \mu_1'}(X_i^{\lambda_{i+j}}) \prod_{j=1}^{\mu_1'} W_{jK}(x_j q^{2j-2+\mu_1-\mu'_{j}+\frac{M-N}{2}}; X_j)
= \sum_{\sigma \in S_{\mu_1'}} \text{sgn}(\sigma) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{\mu_1'}=0}^{\infty} \prod_{j=1}^{\mu_1'} F_{m_j}^{I_K}(x_j q^{-2\lambda_{\sigma(j)}+2\sigma(j)-m_j+\mu_1-\mu'_{j}-1})
\times X_1^{m_1+\lambda_1-\sigma(1)+1} X_2^{m_2+\lambda_2-\sigma(2)+2} \cdots X_{\mu_1'}^{m_{\mu_1'}+\lambda_{\mu_1'}-\sigma(\mu_1')+\mu_1'}
, \tag{2.44}
\]

where \( S_{\mu_1'} \) is the symmetric group of order \( \mu_1' \) and \( \text{sgn}(\sigma) \) is the signature of a permutation \( \sigma \); \( \{X_j\} \) are commutative \( (X_i X_j = X_j X_i) \) shift operators: \( X_j f(x_1, x_2, \ldots, x_j, \ldots) = f(x_1, x_2, \ldots, x_j q^{2j}, \ldots) X_j \) for any function \( f(x_1, x_2, \ldots) \) of \( x_1, x_2, \ldots \). Let us take the coefficient of \( X_1^{m_1} X_2^{m_2} \cdots X_{\mu_1'}^{m_{\mu_1'}} \) in the right-hand side of (2.44), and write it as \( F_{(\lambda_\mu)}^{I_K}(x_1, x_2, \ldots, x_{\mu_1'}) \). We find \( F_{(\lambda_\mu)}^{I_K}(x, x, \ldots, x) = F_{(\lambda_\mu)}^{I_K}(x) \) (cf. (2.36)). Thus \( F_{(\lambda_\mu)}^{I_K}(x_1, x_2, \ldots, x_{\mu_1'}) \) is a refinement of \( F_{(\lambda_\mu)}^{I_K}(x) \). We have also derived a similar formula for \( F_{(\lambda_\mu)}^{I_K}(x) \).

In general, the function (2.32) for \( I = I_K \) (resp. (2.33) for \( I = \overline{I}_K \)) has poles from zeros of the Baxter Q-function \( Q_{I_K}(x) \) (resp. \( \overline{Q}_{I_K}(x) \)). For example, the function \( F_{(1)}^{I_K}(x) \) under (2.20), (2.12) and (2.15) has poles at \( x = x_k^{I_a} q^{\sum_{j \in I_a} p_j \frac{M-N}{2}} \) for \( 1 \leq a \leq K \). And these poles for \( 1 \leq a \leq K - 1 \) chancel each other as in (2.14) under the Bethe

\[\text{The compatibility condition in [39] [40] corresponds to the one for the rational case with } p_a = -p_{a+1}. \] The compatibility condition for the rational case with \( p_a = p_{a+1} \) was discussed in [41].

\[\text{See eq. (3.14) in [50] for “classical” case: } x = 0, \lambda = \emptyset, I_K = (1, 2, \ldots, M + N).\]

\[\text{For } K = M + N \text{ case, the poles for } a = M + N \text{ (poles from the vacuum part) are located at}
\]
ansatz equation $\text{(2.13)}$. But the poles for $a = K$ still survive. To kill\textsuperscript{17} this kind of pole, we introduce the following transformations for any (non-skew) Young diagram $\mu$.

\[
F^K_{\mu}(x) = Q_{IK}(x q^{-\frac{m-a}{2} - \mu_1 + \mu'_1}) F^K_{\mu}(x)
\]  

(2.45)

under the normalization $\text{(2.9)}$ and the functional relations $\text{(2.17)}$-$\text{(2.18)}$, and

\[
\overline{F}^K_{\mu}(x) = \overline{Q}_{IK}(x q^{\frac{m-a}{2} + \mu_1 - \mu'_1}) \overline{F}^K_{\mu}(x)
\]  

(2.46)

under the normalization $\text{(2.10)}$ and the functional relations $\text{(2.20)}$-$\text{(2.21)}$. Here $\bar{\mu}$ is the 180 degrees rotated Young diagram of $\mu$ (see, $\text{(2.31)}$ for $\lambda = 0$).

Let us write these functions for the Young diagram of a rectangular shape $\mu = (s^a)$ as $F^{(a),IK}_s(x) = F^{IK}_s(x) = Q_{IK}(x q^{-\frac{m-a}{2} - s + a}) F^{IK}_s(x)$ and $\overline{F}^{(a),IK}_s(x) = \overline{F}^{IK}_s(x) = \overline{Q}_{IK}(x q^{\frac{m-a}{2} - s - a}) \overline{F}^{IK}_s(x)$. In particular for $a = 0$ or $s = 0$ case, we interpret them\textsuperscript{18} as:

\[
F^{(a),IK}_0(x) = Q_{IK}(x q^{-\frac{m-a}{2} + a}) \quad \text{for } a \in \mathbb{Z}_{\geq 0},
\]

(2.47)

\[
F^{(0),IK}_{s}(x) = Q_{IK}(x q^{\frac{m-a}{2} - s}) \quad \text{for } s \in \mathbb{Z},
\]

(2.48)

\[
\overline{F}^{(a),IK}_0(x) = \overline{Q}_{IK}(x q^{\frac{m-a}{2} - s - a}) \quad \text{for } a \in \mathbb{Z}_{\geq 0},
\]

(2.49)

\[
\overline{F}^{(0),IK}_{s}(x) = \overline{Q}_{IK}(x q^{\frac{m-a}{2} + s}) \quad \text{for } s \in \mathbb{Z}.
\]

(2.50)

For $a, s \in \mathbb{Z}$, $F^{(a),IK}_s(x) = 0$ if $a < 0$, or $a > 0$ and $s < 0$, or $a > m$ and $s > n$; $\overline{F}^{(a),IK}_s(x) = 0$ if $a < 0$, or $a > 0$ and $s < 0$, or $a > m$ and $s > n$.

There are “duality relations” among the functions\textsuperscript{19}.

\[
F^{(m),IK}_{a+n}(x) = \left( \prod_{\gamma \in I_K} p_{\gamma}^{z_{\gamma} p_{\gamma}} \right)^a F^{(a+m),IK}_n(x),
\]

(2.51)

\[
\overline{F}^{(m),IK}_{a+n}(x) = \left( \prod_{\gamma \in I_K} p_{\gamma}^{z_{\gamma} p_{\gamma}} \right)^a \overline{F}^{(a+m),IK}_n(x) \quad \text{for } a \in \mathbb{Z}_{\geq 0}.
\]

(2.52)

\[x = w_k q^{-\sum_{j \in I_{M+N} \setminus p_j - (M+N)} - w_k q^{\frac{M+N}{2}}}.\]

\textsuperscript{17}There are cases where poles from the vacuum part still survive (see \textsuperscript{B.11} in Appendix B). This depends on the normalization of the Baxter Q-functions. In any case, we expect that poles from the roots of the Bethe ansatz equation disappear.

\textsuperscript{18}$\mu_1 - \mu'_1 = s - a$ for $\mu = (s^a)$. And then we formally put $\mu_1 - \mu'_1 = s$ for $a = 0$ and $\mu_1 - \mu'_1 = -a$ for $s = 0$ in $\text{(2.45)}$ and $\text{(2.46)}$. We included non-vanishing terms on $s < 0$ for the case $a = 0$ as in \textsuperscript{39}.

\textsuperscript{19}In the context of the analytic Bethe ansatz, the duality relation for $(m, n) = (M, N)$ appeared in \textsuperscript{6} first. This relation with the factor $\left( \prod_{\gamma \in I_K} p_{\gamma}^{z_{\gamma} p_{\gamma}} \right)^a$ for $(m, n) = (M, N)$ appeared in \textsuperscript{13} (the parameters $\{z_\alpha\}$ were used as chemical potentials). This factor was also written as a superdeterminant in \textsuperscript{10}.
The T-functions of fusion models obey functional relations called the T-system. For $a, s \in \mathbb{Z}_{\geq 1}$, $F_{s}^{(a), I_{K}}(u)$ satisfies the $U_{q}(gl(m|n))$-type T-system:

$$F_{s}^{(a), I_{K}}(xq^{-1})F_{s+1}^{(a), I_{K}}(x) = F_{s-1}^{(a), I_{K}}(x)F_{s+1}^{(a), I_{K}}(x) + F_{s}^{(a), I_{K}}(x)F_{s+1}^{(a+1), I_{K}}(x)$$

for $1 \leq a \leq m - 1$ or $1 \leq s \leq n - 1$ or $(a, s) = (m, n)$,

$$F_{s}^{(m), I_{K}}(xq^{-1})F_{s+1}^{(m), I_{K}}(x) = F_{s-1}^{(m), I_{K}}(x)F_{s+1}^{(m), I_{K}}(x) \quad \text{for} \quad s \in \mathbb{Z}_{\geq n+1},$$

$$F_{n}^{(a), I_{K}}(x)F_{n+1}^{(a), I_{K}}(x) = F_{n}^{(a-1), I_{K}}(x)F_{n+1}^{(a+1), I_{K}}(x) \quad \text{for} \quad a \in \mathbb{Z}_{\geq m+1},$$

and $F_{s}^{(a), \mathcal{T}_{K}}(u)$ satisfies the $U_{q}(gl(m|n))$-type T-system:

$$F_{s}^{(a), \mathcal{T}_{K}}(xq^{-1})F_{s+1}^{(a), \mathcal{T}_{K}}(x) = F_{s-1}^{(a), \mathcal{T}_{K}}(x)F_{s+1}^{(a), \mathcal{T}_{K}}(x) + F_{s}^{(a), \mathcal{T}_{K}}(x)F_{s+1}^{(a+1), \mathcal{T}_{K}}(x)$$

for $1 \leq a \leq m - 1$ or $1 \leq s \leq n - 1$ or $(a, s) = (m, n)$,

$$F_{s}^{(m), \mathcal{T}_{K}}(xq^{-1})F_{s+1}^{(m), \mathcal{T}_{K}}(x) = F_{s-1}^{(m), \mathcal{T}_{K}}(x)F_{s+1}^{(m), \mathcal{T}_{K}}(x) \quad \text{for} \quad s \in \mathbb{Z}_{\geq n+1},$$

$$F_{n}^{(a), \mathcal{T}_{K}}(x)F_{n+1}^{(a), \mathcal{T}_{K}}(x) = F_{n}^{(a-1), \mathcal{T}_{K}}(x)F_{n+1}^{(a+1), \mathcal{T}_{K}}(x) \quad \text{for} \quad a \in \mathbb{Z}_{\geq m+1}. $$

This type of “boundary condition” (2.47), (2.48), (2.49), (2.50) was discussed in [39, 40] (for $N = 0$ case, see [25]). For $K = M + N$ (resp. for $K = 0$), the right hand side of (2.47), (2.48), (resp. (2.49), (2.50)) become model dependent scalar functions (see examples on the inhomogeneous Perk-Schultz models (2.11), (2.15)). And the above T-system corresponds to the original T-system (1.1)–(1.4) for $U_{q}(gl(M|N))$ [6, 7]. These relations (2.53)–(2.58) follow from the determinant formula (2.35) or (2.36) and the Jacobi identity (C.9).

The function $F_{s}^{(a), I_{K}}(x)$ also satisfies the following functional relations, which correspond to the Bäcklund transformations [21, 39, 40].

---

20 This should follow from similar discussions given in [39] for rational case. As we already remarked, the Baxter Q-functions in [39] correspond to the rational case of $(M + 1)(N + 1)$ subset of $2^{M+N}$ Baxter Q-functions. Thus we have to consider actions of the Weyl group $S(k) \times S(\mathfrak{g})$ to get the whole set of the Baxter Q-functions, and also the whole set of $F_{s}^{(a), I_{K}}(x)$ and $F_{s}^{(a), \mathcal{T}_{K}}(x)$.

21 For these relations for $gl(M)$ related elliptic models were proposed in [25]. They were generalized to $gl(M|N)$ related rational models in [39] (without twist parameters case) [40] (with twist parameters case). The normalization of the spectral parameter of the functions in [39, 40] are closer to the following (functions with prime): $F_{s}^{I_{K}}(x) = F_{s}^{I_{K}}(xq^{-\frac{m+n}{2}})$, $F_{s}^{\mathcal{T}_{K}}(x) = F_{s}^{\mathcal{T}_{K}}(xq^{-\frac{m+n}{2}-M-N})$, $F_{s}^{(a), I_{K}}(x) = F_{s}^{(a), I_{K}}(xq^{-\frac{m+n}{2}})$, $F_{s}^{(a), \mathcal{T}_{K}}(x) = F_{s}^{(a), \mathcal{T}_{K}}(xq^{-\frac{m+n}{2}-M-N})$, $Q_{s}^{I_{K}}(x) = Q_{s}^{I_{K}}(xq^{-\sum_{i \in I} p_{i} - M-N})$, $Q_{s}^{\mathcal{T}_{K}}(x) = Q_{s}^{\mathcal{T}_{K}}(xq^{-\sum_{i \in I} p_{i} - M-N})$. In fact, (2.59)–(2.62) have the same type of shift of the spectral parameter as in [39, 40] if they are written in terms of $F_{s}^{(a), I_{K}}(x)$. 

20
The functional relations for \( p_{iK} = 1 \) have the following form:

\[
F_{s}^{(a+1),IK}(x)F_{s}^{(a),IK-1}(xq^{\frac{3}{2}}) - F_{s}^{(a),IK}(xq)F_{s}^{(a+1),IK-1}(xq^{\frac{1}{2}})
= z_{iK}F_{s-1}^{(a+1),IK}(xq)F_{s+1}^{(a),IK-1}(xq^{\frac{1}{2}}), \tag{2.59}
\]

\[
F_{s+1}^{(a),IK}(xq)F_{s}^{(a),IK-1}(xq^{\frac{3}{2}}) - F_{s}^{(a),IK}(x)F_{s+1}^{(a+1),IK-1}(xq^{\frac{3}{2}})
= z_{iK}F_{s+1}^{(a+1),IK}(xq)F_{s}^{(a-1),IK-1}(xq^{\frac{1}{2}}), \tag{2.60}
\]

and the functional relations for \( p_{iK} = -1 \) have the following form:

\[
F_{s}^{(a+1),IK-1}(x)F_{s}^{(a),IK}(xq^{\frac{3}{2}}) - F_{s}^{(a),IK-1}(xq)F_{s}^{(a+1),IK}(xq^{\frac{1}{2}})
= z_{iK}F_{s-1}^{(a+1),IK-1}(xq)F_{s+1}^{(a),IK}(xq^{\frac{1}{2}}), \tag{2.61}
\]

\[
F_{s+1}^{(a),IK-1}(xq)F_{s}^{(a),IK}(xq^{\frac{3}{2}}) - F_{s}^{(a),IK-1}(x)F_{s+1}^{(a+1),IK}(xq^{\frac{3}{2}})
= z_{iK}F_{s+1}^{(a+1),IK-1}(xq)F_{s}^{(a-1),IK}(xq^{\frac{1}{2}}). \tag{2.62}
\]

There are also relations for \( \mathcal{F}_{s}^{(a),IK}(x) \). The functional relations for \( p_{iK} = 1 \) have the following form:

\[
\mathcal{F}_{s}^{(a+1),IK-1}(x)\mathcal{F}_{s}^{(a),IK}(xq^{-\frac{3}{2}}) - \mathcal{F}_{s}^{(a),IK-1}(xq^{-1})\mathcal{F}_{s}^{(a+1),IK}(xq^{-\frac{1}{2}})
= z_{iK}\mathcal{F}_{s-1}^{(a+1),IK-1}(xq^{-1})\mathcal{F}_{s+1}^{(a),IK}(xq^{-\frac{1}{2}}), \tag{2.63}
\]

\[
\mathcal{F}_{s+1}^{(a),IK-1}(xq^{-1})\mathcal{F}_{s}^{(a),IK}(xq^{-\frac{3}{2}}) - \mathcal{F}_{s}^{(a),IK-1}(x)\mathcal{F}_{s+1}^{(a+1),IK}(xq^{-\frac{3}{2}})
= z_{iK}\mathcal{F}_{s+1}^{(a+1),IK-1}(xq^{-1})\mathcal{F}_{s}^{(a-1),IK}(xq^{-\frac{1}{2}}), \tag{2.64}
\]

and the functional relations for \( p_{iK} = -1 \) have the following form:

\[
\mathcal{F}_{s}^{(a+1),IK}(x)\mathcal{F}_{s}^{(a),IK-1}(xq^{-\frac{3}{2}}) - \mathcal{F}_{s}^{(a),IK}(xq^{-1})\mathcal{F}_{s}^{(a+1),IK-1}(xq^{-\frac{1}{2}})
= z_{iK}\mathcal{F}_{s-1}^{(a+1),IK}(xq^{-1})\mathcal{F}_{s+1}^{(a),IK-1}(xq^{-\frac{1}{2}}), \tag{2.65}
\]

\[
\mathcal{F}_{s+1}^{(a),IK}(xq^{-1})\mathcal{F}_{s}^{(a),IK-1}(xq^{-\frac{3}{2}}) - \mathcal{F}_{s}^{(a),IK}(x)\mathcal{F}_{s+1}^{(a+1),IK-1}(xq^{-\frac{3}{2}})
= z_{iK}\mathcal{F}_{s+1}^{(a+1),IK}(xq^{-1})\mathcal{F}_{s}^{(a-1),IK-1}(xq^{-\frac{1}{2}}). \tag{2.66}
\]

The functional relations \((2.63)-(2.66)\) follow from \((2.59)-(2.62)\) through \((2.34)\). The functional relations \((2.59)-(2.62)\) (resp. \((2.63)-(2.66)\)) connect the solution of the \( U_{q}(gl(m|n))\)-type \( T \)-system (resp. \( U_{q}(gl(m|n))\)-type \( T \)-system) and that of the \( U_{q}(gl(m-1|n))\)-type \( T \)-system or the \( U_{q}(gl(m|n-1))\)-type \( T \)-system (resp. the \( U_{q}(gl(m-1|n))\)-type \( T \)-system or the \( U_{q}(gl(m|n+1))\)-type \( T \)-system). Thus an original problem on \( U_{q}(gl(M|N)) \) is connected to the one on \( U_{q}(gl(0|0)) \).
Before closing this section, we would like to mention the T-functions for conjugate representations (cf. Appendix B in [6]; eq. (3.11) in [24]). We conjecture that they are obtained by a manipulation\(^{22}\) \(C\) for the T-functions:

\[
C : z_a \to z_a^{-1}, \quad Q_J(xq^s) \to Q_J(xq^{-s}) \tag{2.67}
\]

for any shift \(s\) of the spectral parameter of the Q-functions in the T-functions and for any \(a \in \mathcal{I}\) and \(J \subset \mathcal{I}\). For example, (2.6) is transformed to

\[
C[F(I^M+NN_N(x))] = \sum_{a=1}^{M+N} p_{ia} C[X(J^M+NN_N(x))], \tag{2.68}
\]

where

\[
C[X(J^M+NN_N(x))] = z_i^{-1} Q_{I(a) - 1}(xq^{\sum j \in I(a) - 1 p_j - 2p_{ia} - M - N}) Q_{I(a)}(xq^{\sum j \in I(a) p_j - M - N}) \quad (2.69)
\]

Note that the functional relations for the Baxter Q-functions (2.17)-(2.23) are invariant under (2.67).

### 3 Wronskian-like formulae for the T- and Q-functions

In this section, we propose Wronskian-like formulae for the T- and Q-functions (cf. (3.15), (3.16)), and claim that they satisfy the functional relations. Theorems 3.2, 3.3, 3.4 are our main results of this paper. We will also mention relation among the Wronskian-like formulae and the formulae in the previous section (cf. (3.67), (3.68)). In Proposition 3.6 we will rewrite the Wronskian-like formulae (3.15) and (3.16) as summations over a direct product of symmetric groups, which suggest T-functions for infinite dimensional representations. There are Wronskian-like formulae for the T-functions for the bosonic case \(N = 0\) [25, 26, 27, 28, 30]. However, our new formulae here are not straightforward generalization of the ones for \(N = 0\). The size of the matrices for the Wronskian-like formulae for the T-functions depends on the representation for \(M, N > 0\) case. The upper bound of the size is \(M + N\). Note that the size is constantly \(M\) for \(N = 0\) case. There are papers related to rational models for \(sl(2|1)\) [29] and trigonometric models for \(U_q(\hat{sl}(2|1))\) [24].

\(^{22}\)The Q-function itself is also a function of \(\{z_a\}\) and \(q\): \(Q_I(x) = Q_I(x, q, z_1, \ldots, z_{M+N})\). But we need not touch these in our normalization of the spectral parameter of the Q-function. Namely, \(C[Q_I(xq^s, q, z_1, \ldots, z_{M+N})] = Q_I(xq^{-s}, q, z_1, \ldots, z_{M+N})\). If one applies \(C\) to (2.67)-(2.68), one will obtain the T-system and the Bäcklund transformations for the conjugate representations. There are cases where T-functions for both fundamental and its conjugate representations are necessary to analyze physically interesting models (see for example, [51]).
We introduce the following infinite matrices

$$\mathcal{A}(x) := \begin{pmatrix} (Z_{k,l})_{1 \leq k \leq M, M+1 \leq l \leq M+N} & (X_{k,l})_{1 \leq k \leq M, l \in \mathbb{Z}} & (W_{k,l})_{k,l \in \mathbb{Z}} \\ (Y_{k,l})_{k \in \mathbb{Z}, M+1 \leq l \leq M+N} & (X_{k,l})_{1 \leq k \leq M, l \in \mathbb{Z}} & (W_{k,l})_{k,l \in \mathbb{Z}} \end{pmatrix},$$ 

(3.1)

$$\mathcal{X}(x) := \begin{pmatrix} (Z_{k,l})_{1 \leq k \leq M, M+1 \leq l \leq M+N} & (X_{k,l})_{1 \leq k \leq M, l \in \mathbb{Z}} & (W_{k,l})_{k,l \in \mathbb{Z}} \\ (Y_{k,l})_{k \in \mathbb{Z}, M+1 \leq l \leq M+N} & (-z_{l})^{k-1}Q_{l}(x^{q^{2k+1}}) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(3.2)

whose matrix elements are given as follows:

$$Z_{k,l} := \frac{Q_{k,l}(x)}{z_{k} - z_{l}}, \quad X_{k,l} := z_{l}^{-1}Q_{k}(xq^{2l-1}), \quad Y_{k,l} := (-z_{l})^{k-1}Q_{l}(x^{q^{2k+1}}),$$

$$W_{k,l} := 0,$$

$$Z_{k,l} := \frac{Q_{k,l}(x)}{z_{k} - z_{l}}, \quad X_{k,l} := z_{l}^{-1}Q_{k}(xq^{2l+1}), \quad Y_{k,l} := (-z_{l})^{k-1}Q_{l}(x^{q^{2k-1}}),$$

$$W_{k,l} := 0.$$

For $B_{m} = (b_{1}, b_{2}, \ldots, b_{m})$, $F_{n} = (f_{1}, f_{2}, \ldots, f_{n})$ and $s_{1}, s_{2}, \ldots, s_{\beta}, r_{1}, r_{2}, \ldots, r_{\alpha} \in \mathbb{Z}$ $(m + \alpha = n + \beta)$, we define minor determinants of $\mathcal{A}(x)$ and $\mathcal{X}(x)$.

$$\Delta_{(b_{1}, b_{2}, \ldots, b_{m}), (r_{1}, r_{2}, \ldots, r_{\alpha})}^{(f_{1}, f_{2}, \ldots, f_{n}), (s_{1}, s_{2}, \ldots, s_{\beta})}(x) := \det \begin{pmatrix} (Z_{b_{k}, f_{l}})_{1 \leq k \leq m, 1 \leq l \leq n} & (X_{b_{k}, s_{l}})_{1 \leq k \leq m, 1 \leq l \leq \beta} \\ (Y_{r_{k}, f_{l}})_{1 \leq k \leq \alpha, 1 \leq l \leq n} & (W_{r_{k}, s_{l}})_{1 \leq k \leq \alpha, 1 \leq l \leq \beta} \end{pmatrix},$$

(3.3)

$$= \det \begin{pmatrix} \left(\frac{Q_{b_{k}, f_{l}}(x)}{z_{b_{k}} - z_{l}}\right)_{1 \leq k \leq m, 1 \leq l \leq n} & \left(z_{l}^{-1}Q_{b_{k}}(xq^{2s_{l}-1})\right)_{1 \leq k \leq m, 1 \leq l \leq \beta} \\ \left((-z_{l})^{k-1}Q_{f_{l}}(xq^{2k+1})\right)_{1 \leq k \leq \alpha, 1 \leq l \leq n} & \left(0_{\alpha \times \beta}\right) \end{pmatrix},$$

(3.4)

$$\Delta_{(b_{1}, b_{2}, \ldots, b_{m}), (r_{1}, r_{2}, \ldots, r_{\alpha})}^{(f_{1}, f_{2}, \ldots, f_{n}), (s_{1}, s_{2}, \ldots, s_{\beta})}(x) := \det \begin{pmatrix} \left(\frac{Q_{b_{k}, f_{l}}(x)}{z_{b_{k}} - z_{l}}\right)_{1 \leq k \leq m, 1 \leq l \leq n} & \left(z_{l}^{-1}Q_{b_{k}}(xq^{2s_{l}-1})\right)_{1 \leq k \leq m, 1 \leq l \leq \beta} \\ \left((-z_{l})^{k-1}Q_{f_{l}}(xq^{2k+1})\right)_{1 \leq k \leq \alpha, 1 \leq l \leq n} & \left(0_{\alpha \times \beta}\right) \end{pmatrix},$$

(3.5)

where $(0_{\alpha \times \beta})$ is a $\alpha \times \beta$ zero matrix. Note that $\{r_{k}, s_{l}\}$ are supposed to be integers, but this determinant are still well defined even if they are complex numbers. And this fact will be used in section 4. For $m, n \in \mathbb{Z}_{\geq 0}$ and Young diagram $\mu$, we introduce a number, called $(m, n)$-index $[\mu]$.

$$\xi_{m, n}(\mu) := \min\{j \in \mathbb{Z}_{\geq 0} | \mu_{j} + m - j \leq n - 1\}.$$ 

(3.6)
In particular, \( 1 \leq \xi_{m,n}(\mu) \leq m+1, \xi_{m,0}(\mu) = m+1 \) and \( \xi_{0,n}(\mu) = 1 \) for \( \mu_{m+1} \leq n \), and \( \xi_{m,n}(\mu) = m+1 \) for \( \mu_{m+1} \leq n \leq \mu_m \). We often abbreviate \( \xi_{m,n}(\mu) \) as \( \xi_{m,n} \).

For \( B_m = (b_1, b_2, \ldots, b_m) \), \( F_n = (f_1, f_2, \ldots, f_n) \), \( x \in \mathbb{C} \), and Young diagram \( \mu \), we introduce the following functions.

\[
T_{\mu}^{(b_1, b_2, \ldots, b_m), (f_1, f_2, \ldots, f_n)}(x) := (-1)^{(m+n+1)(\xi_{m,n}(\mu)+1)} \times \Delta_{(f_1, f_2, \ldots, f_n), (s_1, s_2, \ldots, s_{\xi_{m,n}(\mu)-1})} \left( xq^{\frac{3(m-n)}{2}+\mu'_1+\mu'_1} \right), \quad (3.7)
\]

\[
T_{\mu}^{(b_1, b_2, \ldots, b_m), (f_1, f_2, \ldots, f_n)}(x) := (-1)^{(m+n+1)(\xi_{m,n}(\mu)+1)} \times \Delta_{(f_1, f_2, \ldots, f_n), (s_1, s_2, \ldots, s_{\xi_{m,n}(\mu)-1})} \left( xq^{\frac{3(m-n)}{2}+\mu'_1+\mu'_1} \right), \quad (3.8)
\]

where \( s_1 = \mu_{\xi_{m,n}(\mu)-1} + m - n - \xi_{m,n}(\mu) + l + 1, \ r_k = \mu'_{n-m+\xi_{m,n}(\mu)-k} + k - \xi_{m,n}(\mu) + 1 \). By the definition of the \((m,n)\)-index \((3.6)\), we have (cf. Lemma 3.2 in \( [23] \))

\[
\mu_l + m - n - \xi_{m,n}(\mu) + 1 \geq 0 \quad \text{for} \quad 1 \leq l \leq \xi_{m,n}(\mu) - 1, \quad (3.9)
\]

\[
\mu'_k - \xi_{m,n}(\mu) + 1 \geq 0 \quad \text{for} \quad 1 \leq k \leq n - m + \xi_{m,n}(\mu) - 1. \quad (3.10)
\]

Here the increasing sequence of positive integers \((r_1, r_2, \ldots, r_{n-m+\xi_{m,n}(\mu)-1})\) is called the Maya diagram for the Young diagram \((\mu'_1 - \xi_{m,n}(\mu) + 1, \mu'_2 - \xi_{m,n}(\mu) + 1, \ldots, \mu'_{n-m+\xi_{m,n}(\mu)-1} - \xi_{m,n}(\mu) + 1)\); the increasing sequence of positive integers \((s_1, s_2, \ldots, s_{\xi_{m,n}(\mu)-1})\) is the Maya diagram for the Young diagram \((\mu_1 + m - n - \xi_{m,n}(\mu) + 1, \mu_2 + m - n - \xi_{m,n}(\mu) + 1, \ldots, \mu_{\xi_{m,n}(\mu)-1} + m - n - \xi_{m,n}(\mu) + 1)\). Maya diagram is convenient to see relation to the fermion Fock space. Thus the above expressions will be useful when we realize the Baxter Q-operators in terms of the fermion operators. We will use many times the above formula \((3.7)\) for the Young diagram with a rectangular shape \( \mu = (s^n) \). We have to consider the following four cases as the \((m,n)\)-index \((3.6)\) depends on \( m, n, a, s \).

For \( a \leq m - n \), we have \( \xi_{m,n}(s^n) = m - n + 1 \) and

\[
T_{s^n}^{B_m, F_n}(x) = \Delta_{F_n,(1,2,\ldots,m-n,a,m-n+a+1,\ldots,m+n-s)} \left( xq^{-\frac{3(m-n)}{2}+a-s} \right). \quad (3.11)
\]

For \( a - s \leq m - n \leq a \), we have \( \xi_{m,n}(s^n) = a + 1 \) and

\[
T_{s^n}^{B_m, F_n}(x) = (-1)^{(m+n+1)a} \Delta_{F_n,(1,2,\ldots,m-n+a+1,\ldots,m+n-s)} \left( xq^{-\frac{3(m-n)}{2}+a-s} \right). \quad (3.12)
\]

For \( -s \leq m - n \leq a - s \), we have \( \xi_{m,n}(s^n) = m - n + s + 1 \) and

\[
T_{s^n}^{B_m, F_n}(x) = (-1)^{(m+n+1)s} \Delta_{F_n,(1,2,\ldots,m-n+s+1,\ldots,m+n-s)} \left( xq^{-\frac{3(m-n)}{2}+a-s} \right). \quad (3.13)
\]

For \( m - n \leq -s \), we have \( \xi_{m,n}(s^n) = 1 \) and

\[
T_{s^n}^{B_m, F_n}(x) = \Delta_{F_n,(1,2,\ldots,m-n-s,n-m+s+a+1,\ldots,m-n+a)} \left( xq^{-\frac{3(m-n)}{2}+a-s} \right). \quad (3.14)
\]
Now we introduce the most important functions, which correspond to the $T$- and $Q$-functions.

\[
\begin{align*}
T_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(x) &:= \frac{T_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(x)}{T_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(0)}, \quad (3.15) \\
\overline{T}_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(x) &:= \frac{\overline{T}_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(x)}{\overline{T}_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(0)}. \quad (3.16)
\end{align*}
\]

On the right hand side of (3.15) and (3.16), a permutation on the components in the tuples $B_m = (b_1, b_2, \ldots, b_m)$ or $F_n = (f_1, f_2, \ldots, f_n)$ in the numerator and the denominator induces signs, but they cancel each other. Thus the formulae are well defined even if we treat the indexes in the left hand side just a pair of the sets $\{b_1, b_2, \ldots, b_m\}, \{f_1, f_2, \ldots, f_n\}$. In short, they are invariant under the action of $S(B_m) \times S(F_n)$. We suppose that the $T$-functions $T_{\mu}(1, 2, \ldots, M),(M+1, M+2, \ldots, M+N)(x)$ and $\overline{T}_{\mu}(1, 2, \ldots, M),(M+1, M+2, \ldots, M+N)(x)$ correspond to the eigenvalue formulae of the transfer matrices whose auxiliary space are evaluation representations of $U_q(\hat{gl}(M|N))$. As representations of $U_q(\hat{gl}(M|N))$, they are labelled by the Young diagram $\mu$. The Young diagram $\mu$ is related to the Kac-Dynkin label of a representation of $gl(M|N)$ (and thus $U_q(\hat{gl}(M|N))$) as in (A.6). We expect that these two different expressions of the $T$-functions are due to the existence of two kind of evaluation maps from $U_q(\hat{gl}(M|N))$ to $U_q(gl(M|N))$ (see eqs. (4.14)-(4.18) in [27], and also [28]). One can calculate the denominator explicitly (cf. [52]).

\[
\begin{align*}
T_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(0) &= \overline{T}_{\mu}(b_1, b_2, \ldots, b_m)(f_1, f_2, \ldots, f_n)(0) \\
&= (-1)^{(m-n)(m+n-1)} D(b_1, f_1, b_2, f_2, \ldots, b_m, f_n), \quad (3.17) \\
D(b_1, f_1, b_2, f_2, \ldots, b_m, f_n) &:= \prod_{1 \leq i < j \leq m} (z_{b_i} - z_{b_j}) \prod_{1 \leq i < j \leq n} (z_{f_i} - z_{f_j}) \prod_{i=1}^{m} \prod_{j=1}^{n} (z_{b_i} - z_{f_j}). \quad (3.18)
\end{align*}
\]

This is a generalization of the Cauchy identity. This represents a correlation function of the bc-system of the conformal field theory on $\mathbb{P}^1$, and is related to the “boson-fermion correspondence”. (3.18) satisfies the following relations.

\[
\begin{align*}
D(b_1, f_1, b_2, f_2, \ldots, b_m, f_n) &D(b_1, b_2, \ldots, b_m, \alpha, \beta) = (z_{\alpha} - z_{\beta}) D(b_1, f_1, f_2, \ldots, f_n), \quad (3.19) \\
D(b_1, f_1, b_2, f_2, \ldots, b_m, f_n) &D(b_1, b_2, \ldots, b_m, \alpha, \beta) = (z_{\beta} - z_{\alpha}) D(b_1, f_1, f_2, \ldots, f_n), \quad (3.20) \\
D(b_1, f_1, b_2, f_2, \ldots, f_n) &D(b_1, b_2, \ldots, b_m, \alpha, \beta) = (z_{\alpha} - z_{\beta}) D(b_1, f_1, f_2, \ldots, f_n). \quad (3.21)
\end{align*}
\]
The functional relations (2.17)-(2.21) may be viewed as the Yang-Baxterization of (3.19)-(3.21).

For generic \(Z_{k,l}, X_{k,l}, Y_{k,l}, W_{k,l}\) in (3.1), (3.15) is a generalization of the ninth variation of Schur function in terms of the first Weyl formula \([31, 32]\). In fact, \(T_{\mu}^{BM,0}(x)\) corresponds to eq. (1.7) in [32]. It is interesting to reformulate discussions in this paper based on the Gauss decomposition of a sub-matrix of \(A(x)\) along the line in [32].

We will use the following lemma (cf. [46, 6, 7]).

**Lemma 3.1.** If the Young diagram \(\mu\) contains a rectangle with a height of \((m+1)\) and a width of \((n+1)\), the functions (3.15) and (3.16) vanish.

This can be proved by the fact that the \((m, n)\)-index (3.6) becomes larger than or equal to \(m+2\) in this case. We impose the normalization (2.9) (resp. (2.10)) when we use (3.15) (resp. (3.16)) from now on. Now we mention one of the main results in this paper.

**Theorem 3.2.**

\[
Q^{B_{m\downarrow F_n}}(x) = Q_{B_m \times F_n}(x) = T^{B_m,0}(xq^{-\frac{m-n}{2}})
\]

(3.22)

solves the functional relations for the Baxter \(Q\)-functions (2.17)-(2.18) under the relation (2.22) and the normalization (2.9), and

\[
\overline{Q}^{B_{m\downarrow F_n}}(x) = \overline{Q}_{B_m \times F_n}(x) = T^{B_m,0}(xq^{-\frac{m-n}{2}})
\]

(3.23)

solves the functional relations for the Baxter \(Q\)-functions (2.20)-(2.21) under the relation (2.23) and the normalization (2.10).

A proof of this theorem is given in Appendix C.1. Let us write the above formulae explicitly. For \(m \geq n\), we have

\[
\overline{Q}^{\{b_1, b_2, \ldots, b_m, f_1, f_2, \ldots, f_n\}}(x) = \frac{(-1)^{\frac{1}{2}(m-n)(m+n-1)} \prod_{1 \leq k < l \leq m} (z_{b_k} - z_{f_l}) \prod_{1 \leq k \leq m} (z_{b_k} - z_{f_k})}{\prod_{1 \leq k < l \leq n} (z_{f_k} - z_{f_l})}
\]

\[
\times \det \left( \frac{\overline{Q}_{\{b_k, f_l\}}(xq^{m-n})}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m, 1 \leq l \leq n}, \left( z_{b_k}^{-1} \overline{Q}_{b_k}(xq^{m-n-2l+1}) \right)_{1 \leq k \leq m, 1 \leq l \leq n}
\]

(3.24)

\[23\] In the context of the conformal field theory, the author found the identity (3.17) in [53] first. Then he heard from Yasuhiko Yamada that this identity naturally follows from a method, for example, in [54]. One may say that the Baxter \(Q\)-functions or \(Q\)-operators are the Yang-Baxterization of correlation functions of the conformal field theory. There are higher genus analogues of correlation functions of the conformal field theory. They were used to calculate string amplitudes. Whether such correlation functions also allow the Yang-Baxterization as \(Q\)-functions or \(Q\)-operators (or more generally, \(T\)-functions or \(T\)-operators) will be an interesting question.

\[24\] without assuming (3.3)
since $\xi_{m,n}(0) = m - n + 1$. And also for $m \leq n$, we have

$$
\mathfrak{Q}_{\{b_1, b_2, \ldots, b_m, f_1, f_2, \ldots, f_n\}}(x) = \frac{(-1)^\ell (m-n)(m+n-1)}{1 \leq k < l \leq m} \prod_{k=1}^{m} \prod_{l=1}^{n} (z_{b_k} - z_{b_l}) \prod_{1 \leq k < l \leq n} (z_{f_l} - z_{f_k}) \\
\times \det \left( \left( \frac{(Q_{(b_k, f_l)}(x_{q^{m-n}}))}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m, l \leq n} \right) \ (3.25)
$$

since $\xi_{m,n}(0) = 1$. We remark that for $(m, n) = (M, N)$ case, the denominator in the right hand side of (3.15) or (3.16) can be replaced by $T^{BM\cdot FN}_\emptyset(x)$ or $T^{BM\cdot FN}_\emptyset(x)$ if both (2.9) and (2.10) are imposed at the same time. (3.22) and (3.23) for $(m, n) = (M, N)$ are supposed to be model dependent scalar functions (for example, (2.11), (2.15)), which correspond to the quantum Wronskian conditions. One can obtain other relations by applying (2.67) to the determinants (3.22) and (3.23) for $(m, n) = (M, N)$.

We have normalized the Baxter $Q$-functions as in (2.5). This normalization is convenient when we realize the Baxter $Q$-operator as super-trace over a $q$-oscillator representation of $U_q(\hat{gl}(M|N))$ [53] since the final expression of the $Q$-operator does not depend on choice of the vacuum of the Fock space where the super-trace is taken. In this normalization, one can also easy to see that the $Q$-function reduces to the (super)character formula of $gl(M|N)$ (or its subalgebras) in the limit $x \to 0$. Of course, one may adapt different normalizations of the Baxter $Q$-function. Let us consider the following transformation (cf. eq. (65) in [40]):

$$
\Omega_I(x) = a_I x^{S_I} Q_I(x). \quad (3.26)
$$

It will be good to put

$$
S_I = \sum_{i \in I} S_i, \quad q^{p_i S_i} = z_i^{\frac{1}{2}} \quad (3.27)
$$

to simplify the formulae. For $a_I = 1$, (2.17) and (2.18) become

$$
\frac{z_i - z_j}{(z_i z_j)^{\frac{1}{2}}} \Omega_I(x) \Omega_{I \cup \{i, j\}}(x) = \Omega_{I \cup \{i\}}(x q^p) \Omega_{I \cup \{j\}}(x q^{-p}) - \Omega_{I \cup \{i\}}(x q^{-p}) \Omega_{I \cup \{j\}}(x q^p) \quad \text{for} \quad p_i = p_j, \quad (3.28)
$$

$$
\frac{z_i - z_j}{(z_i z_j)^{\frac{1}{2}}} \Omega_{I \cup \{i\}}(x) \Omega_{I \cup \{j\}}(x) = \Omega_I(x q^{-p_i}) \Omega_{I \cup \{j\}}(x q^{p_i}) - \Omega_I(x q^{p_i}) \Omega_{I \cup \{j\}}(x q^{-p_i}) \quad \text{for} \quad p_i = -p_j. \quad (3.29)
$$

If one requires $S_{I_{M+N}} = 0$ for (3.27), one obtains $S_I = -S_I$ and $\prod_{i=1}^{M+N} z_i^{\frac{1}{2}} = 1$ since $S_I + S_I = S_{I_{M+N}}$, where $I = \emptyset \setminus I$. We used (3.28)-(3.29) with $S_{I_{M+N}} = 0$ and $a_I = 1$.

---

25 See, eq. (3.1) in [24], where both (2.9) and (2.10) are imposed.

27
for $U_q(\hat{sl}(2|1))$ case [24]. In the case of the Baxter $Q$-operators, the parameters $S_i$ are certain linear combinations of the Cartan generators of $U_q(\hat{gl}(M|N))$ and external field parameters; and in the case of the Baxter $Q$-functions (eigenvalues), they are conserved quantum numbers (edge occupation numbers). In addition to (3.27), one may require

\[ a_{I \times (i,j)} = \left( \frac{z_i - z_j}{(z_i z_j)^{1/2}} \right)^{p_ip_j} a_{I \times (i) a_{I \times (j)}}. \]  

(3.30)

Thus one obtains:

\[ a_{(i_1,i_2,\ldots,i_a)} = \frac{a_{(i_1)}a_{(i_2)}\cdots a_{(i_a)}}{a_{(0)}} \prod_{1 \leq j < k \leq a} \left( \frac{z_{ij} - z_{ik}}{(z_{ij} z_{ik})^{1/2}} \right)^{p_ip_j}. \]  

(3.31)

In this case, the parameters $\{z_j\}$ in (2.17)-(2.18) disappear. In fact, (2.17) and (2.18) become coefficient free form 27:

\[ \mathcal{Q}_I(x)\mathcal{Q}_{I \times (i,j)}(x) = \mathcal{Q}_{I \times (i)}(xq^{p_i})\mathcal{Q}_{I \times (j)}(xq^{-p_j}) - \mathcal{Q}_{I \times (i)}(xq^{-p_i})\mathcal{Q}_{I \times (j)}(xq^{p_j}) \]

for $p_i = p_j$,  

(3.32)

\[ \mathcal{Q}_{I \times (i)}(x)\mathcal{Q}_{I \times (j)}(x) = \mathcal{Q}_I(xq^{-p_i})\mathcal{Q}_{I \times (i,j)}(xq^{p_j}) - \mathcal{Q}_I(xq^{-p_i})\mathcal{Q}_{I \times (i,j)}(xq^{p_j}) \]

for $p_i = -p_j$.  

(3.33)

Moreover, one can eliminate the parameters $\{z_j\}$ in the $T$-functions by the following transformation (with $a_0 = 1$):

\[ T^{B_m:F_n}_{\mu}(x) = a_{B_m \times F_n}(q^{-m/a} + \mu_1) T^{B_m:F_n}_{\mu}(x). \]  

(3.34)

The discussion on $\overline{Q}_I(x)$ and $T^{B_m:F_n}_{\mu}(x)$ is parallel to the one on $Q_I(x)$ and $T^{B_m:F_n}_{\mu}(x)$.

Let us consider the formulae (3.15) and (3.16) for the rectangular Young diagram $\mu = (s^a)$, and introduce the following symbols:

\[ T^{(a),B_m,F_n}_{s}(x) := \begin{cases} 
T^{B_m,F_n}_{(s^a)}(x) & \text{for } a, s \in \mathbb{Z}_{\geq 1}, \\
T^{B_m,F_n}_{0}(xq^{-s}) & \text{for } a = 0 \text{ and } s \in \mathbb{Z}, \\
T^{B_m,F_n}_{0}(xq^a) & \text{for } s = 0 \text{ and } a \in \mathbb{Z}_{\geq 0}, \\
0 & \text{otherwise},
\end{cases} \]  

(3.35)

26Here we used a notation: $I \times (i,j) = (i_1,i_2,\ldots,i_a,i,j)$ etc. for $I = (i_1,i_2,\ldots,i_a)$.

27The second functional relation in the coefficient free form (3.33) is discussed in [39].

28
and

\[ T_s^{(a),B_m,F_n}(x) := \begin{cases} 
T_{B_m,F_n}^{(s^a)}(x) & \text{for } a, s \in \mathbb{Z}_{\geq 1}, \\
T_{B_m,F_n}^{(a)}(x^s) & \text{for } a = 0 \text{ and } s \in \mathbb{Z}, \\
T_{B_m,F_n}^{(s-a)}(x) & \text{for } s = 0 \text{ and } a \in \mathbb{Z}_{\geq 0}, \\
0 & \text{otherwise.}
\end{cases} \] (3.36)

Due to Lemma \ref{lem:3.1}, \( T_s^{(a),B_m,F_n}(x) = T_s^{(a),B_m,F_n}(x) = 0 \) if \( a \in \mathbb{Z}_{\geq m+1} \) and \( s \in \mathbb{Z}_{\geq n+1} \). Taking note on Theorem \ref{thm:3.2}, one finds:

\[ T_s^{(a),B_m,F_n}(x) = Q_{B_m \cup F_n}(xq^{-\frac{m-n}{2}} - s) \] for \( s \in \mathbb{Z} \), \hspace{1cm} (3.37)

\[ T_s^{(a),B_m,F_n}(x) = Q_{B_m \cup F_n}(xq^{-\frac{m-n}{2}} - a) \] for \( a \in \mathbb{Z}_{\geq 0} \), \hspace{1cm} (3.38)

\[ T_s^{(a),B_m,F_n}(x) = Q_{B_m \cup F_n}(xq^{-\frac{m-n}{2}} + s) \] for \( s \in \mathbb{Z} \). \hspace{1cm} (3.39)

The following formulae follow from the Laplace expansion of the determinants, \((3.37)-(3.40)\) and Lemma \ref{lem:6.1}.

For \( a - s \leq m - n \), we have

\[ T_s^{(a),B_m,F_n}(x) = \sum_{I \subseteq B_m, \text{Card}(I) = a} \frac{\prod_{\gamma \in I} z_\gamma^{s-a+m-n} \prod_{\alpha \in I} \prod_{\beta \in B_m \setminus I} (z_\alpha - z_\beta)}{\prod_{\alpha \in I} \prod_{\beta \in B_m \setminus I} (z_\alpha - z_\beta)} \times Q_{B_m \cup F_n \setminus I}(xq^{-s-\frac{m-n}{2}})Q_I(xq^{s+\frac{m-n}{2}}), \] (3.41)

\[ T_s^{(a),B_m,F_n}(x) = \sum_{I \subseteq B_m, \text{Card}(I) = a} \frac{\prod_{\gamma \in I} z_\gamma^{s-a+m-n} \prod_{\alpha \in I} \prod_{\beta \in B_m \setminus I} (z_\alpha - z_\beta)}{\prod_{\alpha \in I} \prod_{\beta \in B_m \setminus I} (z_\alpha - z_\beta)} \times Q_{B_m \cup F_n \setminus I}(xq^{s+\frac{m-n}{2}})Q_I(xq^{-s-\frac{m-n}{2}}), \] (3.42)

where the summation is taken over all the subset \( I \) of \( B_m \) such that \( \text{Card}(I) = a \). We remark that the right hand side of \((3.41)\) and \((3.42)\) is well defined as a function of \( x \) even if \( s \) is not non-negative integer. Based on this fact, one can derive \( T \)-functions for conjugate representations by considering negative \( s \) (as examples for \( U_q(\hat{sl}(2|1)) \) in \cite{24}). In fact, applying \((2.67)\) to \((3.41)\), we obtain the following relation:

\[ e^a[T_s^{(a),B_m,F_n}(x)] = (-1)^a(n-m+1) \sum_{b \in B_m} \frac{z_b^a}{\prod_{\beta \in F_n} z_b^a} \text{[the right hand side of (3.41)]}_{s \to -s-(m-n)}. \] (3.43)
For $a - s \geq m - n$, we have

$$T_{s(m)}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in F_n} (-z_{\gamma}) \right)^{a-m} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) \sum_{J \subset F_n, \text{Card}(J) = s} \frac{\left( \prod_{\gamma \in J} (-z_{\gamma}) \right)^{a-s-m}}{\prod_{\alpha \in F_n \setminus J} \prod_{\beta \in J} \left( z_{\alpha} - z_{\beta} \right)} \times Q_{B_m \cup F_n \setminus J}(x q^{-\frac{m-n}{2}}) Q_J(x q^{-a+\frac{m-n}{2}}), \quad (3.44)$$

$$T_{s(n)}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in F_n} (-z_{\gamma}) \right)^{a-m} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) \sum_{J \subset F_n, \text{Card}(J) = s} \frac{\left( \prod_{\gamma \in J} (-z_{\gamma}) \right)^{a-s-m}}{\prod_{\alpha \in F_n \setminus J} \prod_{\beta \in J} \left( z_{\alpha} - z_{\beta} \right)} \times Q_{B_m \cup F_n \setminus J}(x q^{-a+\frac{m-n}{2}}) Q_{J}(x q^{-s+\frac{m-n}{2}}), \quad (3.45)$$

where the summation is taken over all the subset $J$ of $F_n$ such that $\text{Card}(J) = s$. We remark that the right hand side of $(3.44)$ and $(3.45)$ is well defined even if $a$ is not non-negative integer. Note that $\overline{Q}_{B_m \cup F_n \setminus J}(x) = Q_J(x)$, $\overline{Q}_{B_m \cup F_n \setminus J}(x) = Q_J(x)$ for $(m, n) = (M, N)$. Thus the relation $T_{s(a),B_m,F_n}(x) = T_{s(a),B_m,F_n}(x)$ holds if both $(2.3)$ and $(2.11)$ are imposed. In particular for $a = m$ or $s = n$ case, the above formulae factorize with respect to the Baxter Q-functions:

$$T_{s(m)}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in B_m} z_{\gamma} \right)^{s-n} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) Q_{B_m}(x q^{-\frac{m-n}{2}}) Q_J(x q^{-a+\frac{m-n}{2}}), \quad (3.46)$$

$$T_{s(n)}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in B_m} z_{\gamma} \right)^{s-n} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) Q_{B_m}(x q^{a+\frac{m-n}{2}}) Q_J(x q^{-s+\frac{m-n}{2}}) \quad (3.47)$$

for $s \geq n$, and

$$T_{n}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in F_n} (-z_{\gamma}) \right)^{a-m} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) Q_{B_m}(x q^{a+\frac{m-n}{2}}) Q_J(x q^{-a+\frac{m-n}{2}}), \quad (3.48)$$

$$T_{n}^{(a,m,F_n)}(x) = \left( \prod_{\gamma \in F_n} (-z_{\gamma}) \right)^{a-m} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} \left( z_{\alpha} - z_{\beta} \right) Q_{B_m}(x q^{-a+\frac{m-n}{2}}) \overline{Q}_{J}(x q^{-a+\frac{m-n}{2}}) \quad (3.49)$$

for $a \geq m$.

In [39], this type of factorization formulae for the T-functions were treated as “boundary conditions” of the Hirota equation. These formulae $(3.46)$-$(3.49)$ are related to the T-functions for typical representations, which will be commented in section 4. Now we mention our main theorem.

\footnote{We expect that each term in the right hand side of $(3.44)$-$(3.45)$ corresponds to a T-function for an infinite dimensional representation of $U_q(\widehat{gl}(M|N))$ (or $U_q(\widehat{sl}(M|N))$). See examples for some special $(M, N)$ in [26, 27, 24].}
Theorem 3.3. For $a, s \in \mathbb{Z}_{\geq 1}$, the determinant formula (3.35) solves the $T$-system

$$T_s^{(a), B_m, F_n}(x q^{-1}) T_s^{(a), B_m, F_n}(x q) = T_{s-1}^{(a), B_m, F_n}(x) T_{s+1}^{(a), B_m, F_n}(x) + T_s^{(a-1), B_m, F_n}(x) T_s^{(a+1), B_m, F_n}(x)$$

(3.50)

for $1 \leq a \leq m-1$ or $1 \leq s \leq n-1$ or $(a, s) = (m, n)$,

$$T_s^{(m), B_m, F_n}(x q^{-1}) T_s^{(m), B_m, F_n}(x q) = T_{s-1}^{(m), B_m, F_n}(x) T_{s+1}^{(m), B_m, F_n}(x)$$

for $s \in \mathbb{Z}_{\geq n+1}$, (3.51)

$$T_n^{(a), B_m, F_n}(x q^{-1}) T_n^{(a), B_m, F_n}(x q) = T_{n-1}^{(a), B_m, F_n}(x) T_{n+1}^{(a), B_m, F_n}(x)$$

for $a \in \mathbb{Z}_{\geq m+1}$, (3.52)

$$T_n^{(m), B_m, F_n}(x) = \left( \prod_{\gamma \in B_m} z_\gamma \right)^b T_n^{(b+m), B_m, F_n}(x)$$

for $b \in \mathbb{Z}_{\geq 0}$, (3.53)

with the boundary conditions (3.37) and (3.38) (and also (3.40) and (3.41)) under the relation (2.22), and the determinant formula (3.36) solves the $T$-system

$$T_s^{(a), B_m, F_n}(x q^{-1}) T_s^{(a), B_m, F_n}(x q) = T_{s-1}^{(a), B_m, F_n}(x) T_{s+1}^{(a), B_m, F_n}(x) + T_s^{(a-1), B_m, F_n}(x) T_s^{(a+1), B_m, F_n}(x)$$

(3.54)

for $1 \leq a \leq m-1$ or $1 \leq s \leq n-1$ or $(a, s) = (m, n)$,

$$T_s^{(m), B_m, F_n}(x q^{-1}) T_s^{(m), B_m, F_n}(x q) = T_{s-1}^{(m), B_m, F_n}(x) T_{s+1}^{(m), B_m, F_n}(x)$$

for $s \in \mathbb{Z}_{\geq n+1}$, (3.55)

$$T_n^{(a), B_m, F_n}(x q^{-1}) T_n^{(a), B_m, F_n}(x q) = T_{n-1}^{(a), B_m, F_n}(x) T_{n+1}^{(a), B_m, F_n}(x)$$

for $a \in \mathbb{Z}_{\geq m+1}$, (3.56)

$$T_n^{(m), B_m, F_n}(x) = \left( \prod_{\gamma \in B_m} z_\gamma \right)^b T_n^{(b+m), B_m, F_n}(x)$$

for $b \in \mathbb{Z}_{\geq 0}$ (3.57)

with the boundary conditions (3.39) and (3.40) (and also (3.41) and (3.42)) under the relation (2.23).

A proof of this theorem is given in Appendix C.2.

Our new determinant formulae also satisfy the Bäcklund transformations (39, 40). The following theorem is also one of our main results.

Theorem 3.4. For $a, s \in \mathbb{Z}_{\geq 0}$, the determinant formula (3.35) satisfies the Bäcklund transformations

$$T_s^{(a+1), B_m, F_n}(x) T_s^{(a), B_m, F_n}(x q^{1/2}) - T_s^{(a), B_m, F_n}(x q^{1/2}) T_s^{(a+1), B_m, F_n}(x q^{1/2}) = z_{bn} T_{s-1}^{(a+1), B_m, F_n}(x q^{1/2}) T_{s+1}^{(a), B_m, F_n}(x q^{1/2}),$$

(3.58)

$$T_{s+1}^{(a), B_m, F_n}(x q) T_{s}^{(a), B_m, F_n}(x q^{1/2}) - T_{s+1}^{(a), B_m, F_n}(x q^{1/2}) T_{s}^{(a), B_m, F_n}(x q^{1/2}) = z_{bn} T_{s-1}^{(a+1), B_m, F_n}(x q) T_{s+1}^{(a), B_m, F_n}(x q^{1/2}),$$

(3.59)
\[ T_{s+1}^{(a+1),B_m,F_{a-1}}(x) T_s^{(a),B_m,F_a}(qx^{\frac{1}{2}}) = T_s^{(a),B_m,F_{a-1}}(x) T_{s+1}^{(a+1),B_m,F_a}(qx^{\frac{1}{2}}) = z_f n T_{s-1}^{(a+1),B_m,F_{a-1}}(x) T_{s+1}^{(a),B_m,F_a}(qx^{\frac{1}{2}}), \]

(3.60)

\[ T_{s+1}^{(a),B_m,F_{a-1}}(x) T_s^{(a),B_m,F_a}(qx^{\frac{1}{2}}) = T_s^{(a),B_m,F_{a-1}}(x) T_{s+1}^{(a),B_m,F_a}(qx^{\frac{1}{2}}) = z_f n T_{s-1}^{(a+1),B_m,F_{a-1}}(x) T_{s+1}^{(a-1),B_m,F_a}(qx^{\frac{1}{2}}), \]

(3.61)

under the relation \[ (2.22), \] and the determinant formula (3.36) satisfies the Bäcklund transformations

\[ T_{s+1}^{(a+1),B_m,F_{a-1}}(x) T_s^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = T_s^{(a+1),B_m,F_{a-1}}(x) T_{s+1}^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = z_{b_m} T_{s-1}^{(a+1),B_m,F_{a-1}}(x) T_{s+1}^{(a-1),B_m,F_a}(xq^{-\frac{1}{2}}), \]

(3.62)

\[ T_{s+1}^{(a),B_m,F_{a-1}}(xq^{-1}) T_s^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = T_s^{(a),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = z_{b_m} T_{s-1}^{(a+1),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a-1),B_m,F_a}(xq^{-\frac{1}{2}}), \]

(3.63)

\[ T_{s+1}^{(a+1),B_m,F_{a-1}}(x) T_s^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = T_s^{(a+1),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = z_f n T_{s-1}^{(a+1),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a-1),B_m,F_a}(xq^{-\frac{1}{2}}), \]

(3.64)

\[ T_{s+1}^{(a),B_m,F_{a-1}}(xq^{-1}) T_s^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = T_s^{(a),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a),B_m,F_a}(xq^{-\frac{1}{2}}) = z_f n T_{s-1}^{(a+1),B_m,F_{a-1}}(xq^{-1}) T_{s+1}^{(a-1),B_m,F_a}(xq^{-\frac{1}{2}}), \]

(3.65)

under the relation \[ (2.23). \]

A proof of this theorem is given in Appendix C.3. Due to Lemma 3.1, some terms of the above functional relations (3.58)-(3.65) vanish for large \( s, a \). As remarked in [39], this type of functional relations contain TQ relations as special cases. Due to (3.38) and (3.40), the functional relations (3.58), (3.60), (3.62) and (3.64) reduce to TQ relations for \( a = 0 \) and \( s \in \mathbb{Z}_{\geq 1} \). Due to (3.37) and (3.39), the functional relations (3.59), (3.61), (3.63) and (3.65) also reduce to TQ relations for \( s = 0 \) and \( a \in \mathbb{Z}_{\geq 1} \). For example, (3.58) reduces to

\[ T_{s}(1,B_m,F_n(x) Q_{B_m,F_n(x)}(qx^{-\frac{m-n}{2}}-s+2) = Q_{B_m,F_n(x)}(qx^{-\frac{m-n}{2}}-s+1), \]

for \( s \in \mathbb{Z}_{\geq 1} \). (3.66)

Considering the action of \( S(\mathfrak{B}) \times S(\mathfrak{F}) \), one sees that the above equation (3.66) essentially corresponds to eq. (4.6) in [39]. These are difference equations on the Baxter Q-functions \( \{Q_I(x)\} \) or \( \{Q_I(x)\} \) for different “levels” \( \text{Card}(I) \). In section 5, we will mention different type of TQ relations (we will call them “Baxter equations”): difference equations on the Baxter Q-functions for the same level.

Let us mention relation among the formulae in the previous section and the ones in this section. Our claim is:
Conjecture 3.5. We conjecture that the following relations on (3.15), (3.16), (2.45) and (2.46) for the (non-skew) Young diagram $\mu$ hold:

$$T_{\mu}^{B_m,F_n}(x) = F_{\mu}^{B_m \times F_n}(x)$$  \hspace{1cm} (3.67)

under the normalization (2.19) and the functional relations (2.17)-(2.18), and

$$\overline{T}_{\mu}^{B_m,F_n}(x) = \overline{F}_{\mu}^{B_m \times F_n}(x)$$  \hspace{1cm} (3.68)

under the normalization (2.20) and the functional relations (2.21)-(2.22).

For an empty Young diagram $\mu = \emptyset$, (3.67) and (3.68) reduce to (3.22) and (3.23), respectively. We have proved the above relations for $\mu = (1)$ based on direct computations similar to the ones for $U_q(\widehat{gl}(M))$ case in section 7 of (56). As for the rectangular Young diagram $\mu = (s^a)$ $(a, s \in \mathbb{Z}_{\geq 1})$, the above relations follow from the fact that $T_{s}^{(a),B_m,F_n}(x)$ and $F_{s}^{(a),B_m \times F_n}(x)$ (resp. $\overline{T}_{s}^{(a),B_m,F_n}(x)$ and $\overline{F}_{s}^{(a),B_m \times F_n}(x)$) satisfy the same the functional relations with the same boundary conditions. As for $U_q(\widehat{gl}(M))$ case, the relation (3.67) follows from Theorem 3.2 in (32). The relations (3.67)-(3.68) are reductions to (3.22) and (3.23). The proof is under investigation.

We have other expressions of the Wronskian-like formulae (3.15), and (3.16). Let us introduce the following functions:

$$t_{\mu}^{(b_1,b_2,\ldots,b_m),(f_1,f_2,\ldots,f_n)}(x) := \prod_{i=1}^{l_{m,n}-1} z_{b_i}^{\mu_i+n-m-i} Q_{b_i}(xq^{2\mu_i-2i+1+\frac{m-n}{2}+\mu_1-\mu})$$

$$\times \prod_{i=1}^{l_{m,n}-1} (-z_{f_i})^{\mu'_i+n-m-i} Q_{f_i}(xq^{2\mu'_i+2i-1+\frac{m-n}{2}+\mu'_1-\mu_1})$$

$$\times \prod_{i = \xi_{m,n}}^{m} \frac{(-z_{f_{m,n+i-\xi_{m,n}}})^{r_{m,n}}}{z_{b_i}^{r_{m,n}}(z_{b_i} - z_{f_{m,n+i-\xi_{m,n}}})} Q_{\{b_i,f_{m,n+i-\xi_{m,n}}\}}(xq^{-2(r_{m,n+m-n}+\frac{m-n}{2}+\mu'_1-\mu_1)})$$

$$\times \prod_{i = l_{m,n+m+1-\xi_{m,n}}}^{n} (-z_{f_i})^{n-i} Q_{f_i}(xq^{2(m-i)-1+\frac{m-n}{2}+\mu'_1-\mu_1})$$  \hspace{1cm} (3.69)

and

$$\overline{t}_{\mu}^{(b_1,b_2,\ldots,b_m),(f_1,f_2,\ldots,f_n)}(x) := \prod_{i=1}^{l_{m,n}-1} z_{b_i}^{\mu_i+n-m-i} \overline{Q}_{b_i}(xq^{-2\mu_i+2i-1-\frac{m-n}{2}-\mu'_1+\mu_1})$$

$$\times \prod_{i=1}^{l_{m,n}-1} (-z_{f_i})^{\mu'_i+n-m-i} \overline{Q}_{f_i}(xq^{2\mu'_i-2i+1-\frac{m-n}{2}-\mu'_1+\mu_1})$$

29But, in our case, we need numerous case study (similar to the ones in Appendix C) depending on the values of $m, n$.  

33
\[
\times \prod_{i=\xi_{m,n}}^{m} \frac{(-z_{f_{m,n+i}})}{z_{b_i} - z_{f_{m,n+i}}} \prod_{i=m,n}^{m,n} (xq^{2(r_{m,n}+m-n)})^{(m,n)} \frac{B_{b_i,f_{m,n+i}}}{B_{b_i,f_{m,n+i}}} (xq^{2(r_{m,n}+m-n)} - \mu^2)_{+1}
\]

where \( l_{m,n} = \mu_{m,n} + 1 \), \( r_{m,n} = n - m + \xi_{m,n} - l_{m,n} \). Then we find

**Proposition 3.6.** The determinant formula (3.15) can be expressed as

\[
\mathcal{F}(\{b_1, b_2, \ldots, b_m\}, \{f_1, f_2, \ldots, f_n\})(x) = \sum_{w \in S(B_m) \times S(F_n)} \text{sgn}(w) w \left[ \mathcal{F}_{\mu}(\{b_1, b_2, \ldots, b_m\}, \{f_1, f_2, \ldots, f_n\})(x) \right]
\]

under the functional relation (2.22); and the determinant formula (3.16) can be expressed as

\[
\mathcal{F}_{\mu}(\{b_1, b_2, \ldots, b_m\}, \{f_1, f_2, \ldots, f_n\})(x) = \sum_{w \in S(B_m) \times S(F_n)/H} \text{sgn}(w) w \left[ \mathcal{F}_{\mu}(\{b_1, b_2, \ldots, b_m\}, \{f_1, f_2, \ldots, f_n\})(x) \right]
\]

under the functional relation (2.23), where \( \text{sgn}(w) \) is the signature of \( w \); \( H \) is a subgroup of \( S(B_m) \times S(F_n) \) whose elements \( w = \sigma_b \times \sigma_f \) are such that \( \sigma_b \) is a permutation of \((b_{\xi_{m,n}}, b_{\xi_{m,n}+1}, \ldots, b_m)\) and \( \sigma_f \) is the same permutation of \((f_{m,n}, f_{m,n+1}, \ldots, f_{m,n+m-\xi_{m,n}})\).

Note that \( \mathcal{F}_{\mu}(B_m,F_n)(x) \) is invariant under the action of \( H \). For \((m,n) = (M,N)\), (3.71) and (3.73) are invariant under the action of the Weyl group \( S(B_M) \times S(F_N) \) of \( gl(M\mid N) \). Proofs of the above formulae are similar to \( \alpha = 0 \) case (see Theorem 3.4 in [23]), where the following relation will be used:

\[
\frac{z_{b}^s Q_{\{b,f\}}(x)}{z_{b} - z_{f}} = \frac{z_{b}^s Q_{\{b,f\}}(xq^{-2r})}{z_{b} - z_{f}} + \sum_{s=0}^{r-1} z_{b}^s z_{f}^{-s-1} Q_{\mu}(xq^{-2r+2s+1}) Q_{\nu}(xq^{-2r+2s+1}),
\]

where \( b \in \mathfrak{B}, f \in \mathfrak{F} \). This relation follows from (2.22). There is a similar relation among \( Q_{\{b,f\}}(x), \mathcal{Q}_b(x), \mathcal{Q}_f(x) \). The finite dimensional modules can be written in terms
of a direct sum of infinite dimensional highest weight modules due to the Bernstein-Gel’fand-Gel’fand (BGG) resolution. In this respect, the function \(3.69\) and \(3.70\) divided by \(3.18\) for \((m, n) = (M, N)\) may be interpreted \(30\) as T-functions for infinite dimensional highest weight representations of \(U_q(gl(M|N))\) \(\text{compare the above formula with eq. (5.21) in [27]}\) for \(U_q(gl(3))\), and also formulae in \([26, 29, 24, 30]\). Of course, this needs further research in view of the fact that the BGG-type resolution of representations of superalgebras is more involved than that of bosonic algebras (cf. \([57]\)).

Let us introduce linear operators, which act on the variables \(\{z_1, z_2, \ldots, z_{M+N}\}\) as

\[
B_a(x) \cdot z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}} = z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}} Q_a(x q^{(2k_a + 1)p_a}),
\]

\[
B_a(x) \cdot z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}} = z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}} \overline{Q}_a(x q^{-(2k_a + 1)p_a})
\]

for \(a \in \{1, 2, \ldots, M + N\}\),

where one should write any term like \(\frac{1}{z_b - z_f}\) as \(\sum_{k=0}^{\infty} \frac{1}{z_b^{-k} z_f^k}\) before the actions of these operators. For example, for \(b \in \mathfrak{B}, f \in \mathfrak{F}\) and \(a \in \mathfrak{I} (a \neq b, f)\), we have

\[
B_a(x)B_b(x)B_f(x) \cdot \frac{1}{z_b - z_f} = \sum_{k=0}^{\infty} B_a(x)B_b(x)B_f(x) \cdot z_b^{-k} z_f^k
\]

\[
= \sum_{k=0}^{\infty} z_b^{-k} z_f^k Q_a(x q^{p_a}) Q_b(x q^{-2k-1}) Q_f(x q^{-2k-1})
\]

\[
= \frac{1}{z_b - z_f} Q_a(x q^{p_a}) Q_{\{b, f\}}(x),
\]

where we used the relation \((2.24)\). Then we find

\[
\prod_{a \in B_m \cup F_n} B_a(x q^{-\delta(m-n) \frac{3}{2} + \mu'_1 - \mu'_1}) \cdot t_{\mu}(b_1, b_2, \ldots, b_m, (f_1, f_2, \ldots, f_n)) (0) = t_{\mu}(b_1, b_2, \ldots, b_m, (f_1, f_2, \ldots, f_n)) (x),
\]

\[
\prod_{a \in B_m \cup F_n} B_a(x q^{-\delta(m-n) \frac{3}{2} - \mu'_1 + \mu'_1}) \cdot \overline{t}_{\mu}(b_1, b_2, \ldots, b_m, (f_1, f_2, \ldots, f_n)) (0) = \overline{t}_{\mu}(b_1, b_2, \ldots, b_m, (f_1, f_2, \ldots, f_n)) (x).
\]

Thus our formulae in this paper are the Yang-Baxterization of the (super)characters of representations of the subalgebras \(gl(m|n)\) of \(gl(M|N)\) by the operators \(\{B_a(x)\}\)

\(30\)A merit of this type of formula is that it contains only one term, and thus is easy to evaluate. The traditional form of the eigenvalue formula by the Bethe ansatz (such as the one in section 2) contains an infinite number of terms if the auxiliary space is an infinite dimensional space, and thus is not always easy to evaluate. Some discussions on T-functions for infinite dimensional representations (in relation to the AdS/CFT correspondence) can be seen, for example, in Appendix B of [55] and section 6.4 of [14].
and \( \{ \mathfrak{B}_a(x) \} \). The \( q \)-characters, which correspond to the \( T \)-functions, appear in the kernel of the Frenkel-Reshetikhin screening operators \([43]\). The transfer matrices for \( Y(\mathfrak{gl}(M|N)) \) were also written in terms of a certain “group derivative” \([19]\) on the (super)characters of representations of \( \mathfrak{gl}(M|N) \). It will be interesting to investigate\(^{31}\) connection among our operators and the operators in \([43, 19]\).

4 T-functions for typical representations

The type 1 quantum superalgebra \( U_q(\mathfrak{gl}(M|N)) \) (or \( U_q(\mathfrak{sl}(M|N)) \)) admits a one-parameter family of finite-dimensional irreducible representations, which correspond to typical representations (see Appendix \( A \)). Thus the evaluation representations of \( U_q(\mathfrak{gl}(M|N)) \) (or \( U_q(\mathfrak{sl}(M|N)) \)) based on the above mentioned representations depend not only on the spectral parameter but also on another continuous parameter. There are \( R \)-matrices for this family of representations \([58]\). In \([8]\), we proposed \( T \)-functions for a wide class of such representations from the analytic Bethe ansatz (see also, \([9, 59, 39]\)). Here we briefly comment on how to obtain the \( T \)-functions for such representations in the auxiliary space from our Wronskian-like formulae.

The determinant formula \([3.16]\) factorizes if \( \xi_{m,n} = m + 1 \) \((\mu_{m+1} \leq n \leq \mu_m)\):

\[
\mathsf{T}^{B_m,F_n}_\mu(x) = \prod_{\alpha \in B_m} \prod_{\beta \in F_n} (z_\alpha - z_\beta) \mathsf{T}^{B_m,0}_\tau(x q^{\frac{\alpha}{2} + \mu_{m+1} - \mu_1}) \mathsf{T}^{0,F_n}_\eta(x q^{\frac{m}{2} - \mu_{m+1} + \mu_1}),
\]

where \( \tau := (\tau_1, \tau_2, \ldots, \tau_m) = (\mu_1 - n, \mu_2 - n, \ldots, \mu_m - n) \) and \( \eta := (\eta_1, \eta_2, \ldots, \eta_{\mu_1}) = (\mu_{m+1}, \mu_{m+2}, \ldots, \mu_{\mu_1}) \). Let us consider a generalization of the above relation. Due to the relations \((C.3)\) and \((C.4)\), the following relations hold for \( c_1, c_2 \in \mathbb{Z}_{\geq 0} \):

\[
\mathsf{T}^{B_m,0}_\tau(x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \mathsf{T}^{B_m,0}_\tau(x q^{-c_1 + (1-\delta_{c_1,0})} \tau_1 - m),
\]

\[
\mathsf{T}^{0,F_n}_\eta(x) = \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2} \mathsf{T}^{0,F_n}_\eta(x q^{-c_2} \eta_1 - n),
\]

where \( \tau_{c_1} := (\tau_1 + c_1, \tau_2 + c_1, \ldots, \tau_m + c_1) \) and \( \eta_{c_2} := (n, \ldots, n, \eta_1, \eta_2, \ldots, \eta_{\mu_1}) \). Let us assume \( m, n \in \mathbb{Z}_{\geq 1} \). Combining \((4.1)\), \((4.2)\) and \((4.3)\), we obtain

\[
\mathsf{T}^{B_m,F_n}_{\mu_1, c_2}(x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} (z_\alpha - z_\beta) \times \mathsf{T}^{B_m,0}_\tau(x q^{-\frac{\alpha}{2} + \mu_{m+1} - \mu_1 - c_1 - c_2}) \mathsf{T}^{0,F_n}_\eta(x q^{\frac{m}{2} - \mu_{m+1} + \mu_1 + c_1 + c_2}),
\]

\(^{31}\)One thing we should do may be to apply the “group derivative” in \([19]\) to the formulae for \( x = 0 \) in this paper.
where \( \mu_{c_1,c_2} := (\mu_1 + c_1, \mu_2 + c_1, \ldots, \mu_m + c_1, n, n, \ldots, n, \mu_{m+1}, \mu_{m+2}, \ldots, \mu_{\mu'} \). Similarly we obtain

\[
\mathcal{T}_{\mu_{c_1,c_2}}^{B_m, F_n} (x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} (z_\alpha - z_\beta)
\times \mathcal{T}_\tau^B (x q^\frac{c_2 \cdot \mu'_{n+1} - c_1 + c_2}{2} + \mu_1 + c_1 + c_2) \mathcal{T}_\eta^F (x q^{-\frac{c_2 \cdot \mu'_m}{2} + \mu_{m+1} - c_1 - c_2}).
\] (4.5)

In particular for \( \mu = (n^m) \) (\( \tau = \eta = \emptyset \)) case, these relations reduce to

\[
\mathcal{T}_{((n+c_1)^m,n^c_2)}^{B_m, F_n} (x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2} \prod_{\alpha \in B_m} \prod_{\beta \in F_n} (z_\alpha - z_\beta)
\times Q_{B_m} (x q^{\frac{m+n + c_1 + c_2}{2}}) Q_{F_n} (x q^{-\frac{m+n}{2} - c_1 - c_2}).
\] (4.6)

There relations (4.6) and (4.7) are generalization of the relations (3.46)-(3.49). The right hand side of the above relations (4.4), (4.5), (4.6) and (4.7) make sense as a function of \( x, c_1 \) and \( c_2 \) even when \( c_1 \) or \( c_2 \) are not integers. And for \( (m,n) = (M,N) \), these should be interpreted as a \( \mathcal{T} \)-functions for the typical representations (see Appendix A).

Next we shall mention relation between (4.4) and our previous results [8]. Combining (4.1) for \( \mu = \tilde{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \) and (4.4), we obtain

\[
\mathcal{T}_{\mu_{c_1,c_2}}^{B_m, F_n} (x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2}
\times \mathcal{T}_\mu^B (x q^{m - c_1 - c_2}) \mathcal{T}_\emptyset^F (x q^{-\frac{m+n}{2} + \mu_{m+1} + c_1 + c_2})
\times \mathcal{T}_\emptyset^{\tilde{\mu}} (x q^{-\frac{m+n}{2} + \mu_{m+1} - c_1 - c_2}) \mathcal{T}_\emptyset^F (x q^{-\frac{m+n}{2} - c_1 - c_2}).
\] (4.8)

where we used the relation \( \mathcal{T}_\emptyset^F (x) = \mathcal{Q}_{F_n} (x q^{-\frac{x}{2}}) \). One can rewrite the above relation via the relations (2.46) and (3.68) as

\[
\mathcal{T}_\mu^{B_m \times F_n} (x) = \left( \prod_{\alpha \in B_m} z_\alpha \right)^{c_1} \left( \prod_{\beta \in F_n} (-z_\beta) \right)^{c_2}
\times \mathcal{Q}_{B_m \cup F_n} (x q^{\frac{c_2 \cdot \mu'_m - c_1 + c_2}{2} + \mu_1 - c_1 - c_2}) \mathcal{Q}_{B_m \cup F_n} (x q^{-\frac{c_2 \cdot \mu'_m}{2} + \mu_1 - c_1 - c_2})
\times \mathcal{Q}_{F_n} (x q^{\frac{3m-n}{2} + \mu_1 - c_1 - c_2}) \mathcal{Q}_{F_n} (x q^{\frac{3m-n}{2} + \mu_1 - c_1 - c_2})
\times \mathcal{Q}_{F_n} (x q^{\frac{3m-n}{2} + \mu_1 - c_1 - c_2}) \mathcal{Q}_{F_n} (x q^{\frac{3m-n}{2} + \mu_1 - c_1 - c_2})
\times \mathcal{T}_\mu (x q^{\frac{c_2 \cdot \mu'_m - c_1 - c_2}{2} + \mu_1 - c_1 - c_2}) \mathcal{T}_\emptyset^F (x q^{-\frac{c_2 \cdot \mu'_m}{2} + \mu_1 - c_1 - c_2}).
\] (4.9)

We find that (4.9) for \( (m,n) = (M,N) \) and \( c_2 = 0 \) essentially corresponds to the \( \mathcal{T} \)-function for the typical representation written in eq. (3.23) in [8].
5 Baxter equations

The Baxter equations are difference equations whose solutions give the Baxter $Q$-functions. The Baxter equations in question are the equations for the $Q$-functions \{$Q_I(x)$\}, \{$\overline{Q}_I(x)$\} of the same “level” Card($I$) = 1. Discussions on the Baxter equations for superalgebra related models can also be seen in [29, 60, 15].

Once Wronskian-type determinant formulae are established, the Baxter equations can be derived easily. In essence, the following Baxter equations are based on a basic theorem on the linear algebra: if two different rows or columns of a matrix coincide, the corresponding determinant vanishes. Thus one can derive the Baxter equations directly from the Wronskian-like formulae (3.15)-(3.16) related to typical representations as follows:

\[
\sum_{a=0}^{m} (-z_k)^{-a} T_{(n+1)^a,n^{m-a}}^B(x q^{-(a-m)\delta_{n,0} - \delta_{a,0}}) Q_k(x q^{-2a + \frac{3n}{2} + \frac{m}{2}}) = 0 \quad \text{for} \quad k \in B_m, \quad (5.1)
\]

\[
\sum_{a=0}^{n} z_k^{-a} T_{m,n}^B(x q^{(a-n)\delta_{m,0} + \delta_{a,0}}) Q_k(x q^{-2a - m + \frac{3n}{2} - \frac{m}{2}}) = 0 \quad \text{for} \quad k \in F_n, \quad (5.2)
\]

and

\[
\sum_{a=0}^{m} (-z_k)^{-a} T_{m,n}^B(x q^{(a-n)\delta_{m,0} - \delta_{a,0}}) \overline{Q}_k(x q^{-2a + \frac{3n}{2} + \frac{m}{2}}) = 0 \quad \text{for} \quad k \in B_m, \quad (5.3)
\]

\[
\sum_{a=0}^{n} z_k^{-a} T_{m,n}^B(x q^{-(a-n)\delta_{m,0} + \delta_{a,0}}) \overline{Q}_k(x q^{-2a + \frac{3n}{2} + \frac{m}{2}}) = 0 \quad \text{for} \quad k \in F_n, \quad (5.4)
\]

where $((n + 1)^a, n^{m-a}) = ((n + 1)^a, n^{m-a})$ at $a = m$, and $(n^m)$ at $a = 0$; $(n^m, a) = (a)$ at $m = 0$; $(n^m) = \emptyset$ at $n = 0$ or $m = 0$. These are finite order linear difference equations on the Baxter $Q$-functions. Note that (5.1) and (5.2) (resp. (5.3) and (5.4)) share the functions $T_{m,n}^B(x)$ and $T_{m,n}^B(x)$ = \{ $\prod_{s \in F_m}$ $z_s$ \} $T_{m,n}^B(x)$ (resp. $T_{m,n}^B(x)$) and $T_{m,n}^B(x)$ = \{ $\prod_{s \in F_n}$ $z_s$ \} $T_{m,n}^B(x)$. The functions $T_{m,n}^B(x)$, $T_{m,n}^B(x)$ and $T_{m,n}^B(x)$ can also be written in terms of $T_{A,m,B,n}^B(x)$ or $T_{(1,1),m,n}^B(x)$ through (2.35), (2.36), (3.67). The equations (5.1)-(5.4) for $(m,n) = (M,N)$ are the Baxter equations for $U_q(gl(M|N))$. These equations for $m < M, n < N$ are in the intermediate steps of the Bäcklund flows. In addition, $N = 0$ and $(m,n) = (M,0)$ case of (5.1) and (5.3) correspond to the Baxter equations for $U_q(\hat{gl}(M))$ (cf. [25, 28]).

One can also factor out overall factors in (5.1)-(5.4) based on the formula (4.1) (and...
the same type of formula for $T^{B_m,F_n}(x)$ as follows:
\[
\sum_{a=0}^{m} (-z_k)^{-a} T^{B_m,0}_{(1^a)} (xq^{-a-\delta_{a,0}}) Q_k(xq^{-2a+m/2}) = 0 \quad \text{for} \quad k \in B_m, \tag{5.5}
\]
\[
\sum_{a=0}^{n} z_k^{-a} T^{0,F_n}_{(a)} (xq^{a+\delta_{a,0}}) Q_k(xq^{-2a+n/2}) = 0 \quad \text{for} \quad k \in F_n, \tag{5.6}
\]
and
\[
\sum_{a=0}^{m} (-z_k)^{-a} T^{B_m,0}_{(1^a)} (xq^{a+\delta_{a,0}}) \overline{Q}_k(xq^{-2a-m/2}) = 0 \quad \text{for} \quad k \in B_m, \tag{5.7}
\]
\[
\sum_{a=0}^{n} z_k^{-a} T^{0,F_n}_{(a)} (xq^{-a-\delta_{a,0}}) \overline{Q}_k(xq^{-2a+n/2}) = 0 \quad \text{for} \quad k \in F_n. \tag{5.8}
\]
These are also kinds of Baxter equations, but in the intermediate steps of the Bäcklund flows except for the case $N = 0$ and $(m, n) = (M, 0)$, or $M = 0$ and $(m, n) = (0, N)$.

There is another type of Baxter equations, which follows from the kernel of the non-commutative generating series (2.37)-(2.40). In contrast to the above equations, this type of Baxter equations has an infinite number of terms. This is because that the non-commutative generating series of the T-functions are infinite order difference operators. This situation is similar to other quantum affine (super)algebras cases such as $U_q(B^{(1)}_{1\ell})$, $U_q(D^{(1)}_{2\ell})$ [20, 61], $U_q(D^{(2)}_{3\ell+1})$, $U_q(D^{(3)}_4)$ [62], $U_q(\widehat{osp}(M|2N))$ [63] where the non-commutative generating series of the T-functions are infinite order difference operators and Wronskian-like formulae for the T-functions have not been established. In this paper, we have overcome this difficulty for $U_q(\widehat{gl}(M|N))$ case. Thus our example on $U_q(\widehat{gl}(M|N))$ will be a good reference to find [55] finite order Baxter equations and Wronskian-like formulae for T-functions for more general quantum affine (super)algebras.

We have also derived a determinant form of finite order Baxter equations based on conserved quantities in Appendix D.

## 6 Discussions

In this paper, we assume that the deformation parameter $q$ is generic. If $q$ is a root of unity, the Baxter Q-function becomes a periodic function of $x$. Thus, through our new Wronskian-like determinant formulae of the T-functions, one sees that the quantum group truncation occurs in a very different way than for the bosonic algebra case. Namely, the T-function $T^{(a)B_m,F_n}_s(x)$ may vanish for some $s$ or $a$ for $m, n > 0$ case if $U_q(\widehat{osp}(1|2N))$ case is an exception, where the T-function is similar to $U_q(A^{(2)}_{2N})$ case.

\[
^{33}U_q(\widehat{osp}(1|2N)) \text{ case is an exception, where the T-function is similar to } U_q(A^{(2)}_{2N}) \text{ case.}
\]
the twist parameters are also proportional to appropriate roots of unity. Note that
truncation occurs only on the variable $s$ for $n = 0$ case.

We have proposed various formulae for the Baxter $Q$-functions and the $T$-functions.
It is important to realize these as operators. Of course, the same formulae hold true for
both operators and their eigenvalues since the Baxter $Q$-operators and the $T$-operators
belong to the same commuting family of operators. So far, realizations of the Baxter
$Q$-operators based on the $q$-oscillator representations of the quantum affine (super)algebra,
which are relevant to our formulation in this paper, are available for $U_q(\hat{sl}(2))$ [26], $U_q(\hat{sl}(3))$ [27], $U_q(\hat{sl}(M))$ [28], $U_q(\hat{sl}(2|1))$ [24], and the general case
$U_q(\hat{gl}(M|N))$ (or $U_q(\hat{sl}(M|N))$) is under investigation [55]. Although the identities
among $T$ and $Q$-functions in this paper are combinatorial ones and hold true inde-
pendent of the representation theory, it is also important to establish connec-
tion to the universal $R$-matrix and to prove our new formulae from the point of view of the
representation theory.

The supersymmetric Bazhanov-Reshetikhin formulae (eqs. (2.35)-(2.36) for $K = M + N$) [6, 7, 8] played an important role to derive [13] (see also [64]) a system of
nonlinear integral equations (NLIE) for the free energy of integrable spin chains at
finite temperature, which is equivalent to the thermodynamic Bethe ansatz (TBA)
equation [34], from the $T$-system (1.1)-(1.4) [6, 7, 8]. Then a natural question is whether
the Wronskian-like determinant formulae are also useful to derive the NLIE. There is
another type of NLIE [66], which is equivalent to the TBA equation. To generalize this
type of NLIE to models whose underlying algebras have arbitrary rank is not always
an easy problem as one needs considerable trial and errors to find auxiliary functions
with good analytical properties which are needed to derive the NLIE. Although the
$Y$-system itself is not truncated to a finite set of functional relations for generic $q$, an
introduction of appropriate auxiliary functions modifies [67] the $Y$-system to a finite
set of functional relations, which is suited for the NLIE. So far this type of NLIE is
known for at most rank 4 case [68]. We expect that some of our formulae in this paper
are also useful to construct auxiliary functions for this type of NLIE for arbitrary rank.
One can easily transform the $T$-system into the $Y$-system based on a transformation on
the $T$-functions (to the “$Y$-functions”), as in [12]. This type of transformation (applied
for (3.58)-(3.65)) also induces a system of Bäcklund transformations for the $Y$-system.
Thereby Bäcklund transformations for the TBA equations are obtained. Our formulae
for the $T$-functions $T_s^{(a),B_m,F_n}(x)$ and $T_s^{(a),F_m,F_n}(x)$ (or $F_s^{(a),I}(x)$ and $T_s^{(a),I}(x)$ ) will be
convenient to check analyticity of the $Y$-functions for these Bäcklund transformations.

Our new formulae in this paper have potential applicability to the analysis of the
AdS/CFT $T$-system (or the $Y$-system) [10, 17], after some modifications. Note that
many formulae in this paper do not depend on precise function form of the Baxter
$Q$-functions. The modifications should be done so that the Baxter $Q$-functions are
compatible with the Bethe ansatz equation in [69] (for large $L$). We also expect that

\footnote{TBA equations for $gl(M|N)$ related rational models were written in [65].}
this is also the case with the Hubbard model.

To generalize our formulae to mixed representations is also an important problem. So far, the supersymmetric Bazhanov-Reshetikhin type formula for a simplest case was written long time ago (cf. eq. (4.1) in [6]), but more general case is under investigation [55]. We expect that the (super)character formulae for the mixed representations of $gl(M|N)$ also allow the Yang-Baxterization by the Baxter $Q$-functions (or $Q$-operators).

The representation theory of the quantum affine superalgebras is not always well understood even in finite dimensional representations case. In this concern, to generalize our formulae in this paper to other quantum affine superalgebras such as $U_q(\widehat{osp}(M|2N))$, $U_q(D^{(1)}(2,1;\alpha))$ etc. is interesting not only in the study of the integrable system but also in the representation theory in mathematics. Tableaux sum expressions for $T$-functions, some supersymmetric Bazhanov-Reshetikhin-like formulae for fusion models and the $T$-system are available for $U_q(\widehat{osp}(M|2N))$ [63] case, and this will be a starting point to establish [55] Wronskian-like formulae for $T$-functions or $q$-(super)character formulae for the general quantum affine superalgebras.

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Appendix A. Representations of the superalgebra

In this section, we briefly mention representations of the superalgebra $gl(M|N)$ and their characters. There are a lot of literatures on this subject (see for example, [70, 21, 22, 23]).

There are several choices of simple root systems $\{\alpha_1, \alpha_2, \ldots, \alpha_{M+N-1}\}$ depending on the choices of the Borel subalgebras. The simplest system of simple roots is the so-called distinguished one. It leads as follows

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in \{1, 2, \ldots, M - 1\},$$

$$\alpha_M = \epsilon_M - \delta_1$$

(A.1)

[35] “Finite-size Technology in Low Dimensional Quantum System (IV)”, APCTP (Pohang, Korea), June 2008, [http://wimn.ewha.ac.kr/focus2008/]; a meeting of the Physical Society of Japan, September 23, 2008.
\[ \alpha_{j+M} = \delta_j - \delta_{j+1}, \quad j \in \{1, 2, \ldots, N - 1\}, \]

where \( \epsilon_1, \ldots, \epsilon_M, \delta_1, \ldots, \delta_N \) are the basis of the dual space of the Cartan subalgebra with the bilinear form \( \langle \ , \rangle \) such that

\[ \langle \epsilon_i | \epsilon_j \rangle = \delta_{ij}, \quad \langle \epsilon_i | \delta_j \rangle = \langle \delta_i | \epsilon_j \rangle = 0, \quad \langle \delta_i | \delta_j \rangle = -\delta_{ij}. \quad (A.2) \]

\( \{\alpha_i\}_{i \neq M} \) are even roots and \( \alpha_M \) is an odd root with \( \langle \alpha_M | \alpha_M \rangle = 0 \).

Any weight can be expressed in the following form:

\[ \Lambda = \sum_{i=1}^{M} \Lambda_i \epsilon_i + \sum_{j=1}^{N} \Lambda_j \delta_j, \quad \Lambda_i, \Lambda_j \in \mathbb{C}. \quad (A.3) \]

There is a class of irreducible tensor representations of \( gl(M|N) \) whose highest weight \( \Lambda \) is characterized by the Young diagram \( \mu = (\mu_1, \mu_2, \ldots), (\mu_1 \geq \mu_2 \geq \cdots \geq 0, \mu_{M+1} \leq N) \) in the \((M, N)\)-hook (see Figure 1):

\[ \Lambda_i = \mu_i \quad \text{for} \quad 1 \leq i \leq M, \]

\[ \Lambda_j = \eta_j \quad \text{for} \quad 1 \leq j \leq N, \quad (A.4) \]

where \( \eta_j = \max \{\mu'_j - M, 0\} \). There is a set of parameters \([b_1, b_2, \ldots, b_{M+N-1}]\), called the Kac-Dynkin label of \( \Lambda \), defined by

\[ b_j = \begin{cases} \frac{2\langle \Lambda | \alpha_j \rangle}{\langle \alpha_j | \alpha_j \rangle} & \text{for} \quad \langle \alpha_j | \alpha_j \rangle \neq 0, \\ \langle \Lambda | \alpha_j \rangle & \text{for} \quad \langle \alpha_j | \alpha_j \rangle = 0. \end{cases} \quad (A.5) \]

An irreducible representation with the highest weight \( \Lambda \) is finite dimensional if \( b_j \in \mathbb{Z}_{\geq 0} \) for all \( j \neq M \), and \( b_M \in \mathbb{C} \). The Young diagram is related to the Kac-Dynkin label as follows:

\[ b_j = \mu_j - \mu_{j+1} \quad \text{for} \quad 1 \leq j \leq M - 1, \]

\[ b_M = \mu_M + \eta_1, \quad (A.6) \]

\[ b_{j+M} = \eta_j - \eta_{j+1} \quad \text{for} \quad 1 \leq j \leq N - 1. \]

Let \( \Lambda \) be a real dominant weight. An irreducible representation of \( gl(M|N) \) with the highest weight \( \Lambda \) is called atypical if

\[ \langle \Lambda + \rho, \epsilon_i - \delta_j \rangle = 0 \quad \text{for some} \quad (i, j) : \quad 1 \leq i \leq M, \quad 1 \leq j \leq N, \quad (A.7) \]

and typical if there is no such \((i, j)\). Here \( \rho \) is the graded half sum of the positive roots:

\[ \rho = \frac{1}{2} \sum_{i=1}^{M} (M - N - 2i + 1) \epsilon_i + \frac{1}{2} \sum_{j=1}^{N} (M + N - 2j + 1) \delta_j. \quad (A.8) \]
Thus the aforementioned irreducible tensor representation is typical if \( \mu_M \geq N \), and atypical if \( \mu_M < N \).

There is a large class of finite dimensional typical representations, which is not tensor-like. For example, for the above mentioned typical irreducible tensor representations with the highest weight \( \Lambda \), there is a one parameter family of irreducible typical representations with the highest weight

\[
\Lambda(c) = \Lambda + c\omega_1,
\]

where \( \omega_1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_M \). Note that the \( M \)-th Kac-Dynkin label \( b_M = \mu_M + \eta_1 + c \) is not an integer if the parameter \( c \) is not an integer. One may generalize the above representation to the one with the highest weight \( \Lambda(c_1, c_2) = \Lambda + c_1\omega_1 + c_2\omega_2 \), where \( \omega_2 = \delta_1 + \delta_2 + \cdots + \delta_N \). The Kac-Dynkin label of \( \Lambda(c_1, c_2) \) depends only on \( c_1 + c_2 \), and coincides with that of \( \Lambda(c_1 + c_2) \). In this sense, this representation essentially has only one parameter. These parameters \( c_1 \) and \( c_2 \) correspond to the ones in section 4.

There is a \( gl(M|N) \) analog of the first Weyl formula, called the Sergeev-Pragacz formula \[22\]. It is an alternative representation of the supersymmetric Jacobi-Trudi formula, which corresponds to a \( gl(M|N) \) analog of the second Weyl formula. Let us take a non-skew Young diagram \( \mu \). Then the Sergeev-Pragacz formula has the following form \[36\]

\[
S_\mu(\{x_j\}_{j=1}^M|\{y_k\}_{k=1}^N) = \sum_{\sigma \in S_M \times S_N} \text{sgn}(\sigma) \sigma \left[ \prod_{i=1}^{M-1} x_i^{M-i} \prod_{j=1}^{N-1} y_j^{N-j} \prod_{(i,j) \in \mu} (x_i - y_j) \right] \prod_{i<j} (x_i - x_j) \prod_{i<j} (y_i - y_j), \tag{A.10}
\]

where the third product in the numerator is taken over all boxes \((i, j) \in \mu \) of the Young diagram \( \mu \). We assumed that \( x_i = 0 \) if \( i \geq M + 1 \) and \( y_j = 0 \) if \( j \geq N + 1 \). The notation \( S_M \) (resp. \( S_N \)) denotes the symmetric group of order \( M \) (resp. \( N \)). The symbol \( \sigma[\ldots] \) stands for the action of the permutation \( \sigma \), namely the Weyl group of \( gl(M|N) \), on the variables \( x_1, \ldots, x_M, y_1, \ldots, y_N \) in the square brackets. This formula \[A.10\] gives the (super)character of the above mentioned tensor representation labeled by the Young diagram \( \mu \) if the variables are identified with the formal exponentials: \( x_j = e^{\epsilon_j} \) for \( j \in \{1, 2, \ldots, M\} \) and \( y_j = e^{\delta_j} \) for \( j \in \{1, 2, \ldots, N\} \). The Wronskian-like formula \[3.15\] (or \[3.19\]) at \((m, n) = (M, N)\), \((b_1, \ldots, b_M, f_1, \ldots, f_N) = (1, 2, \ldots, M + N)\) and \( x = 0 \) coincides with the Sergeev-Pragacz formula \[A.10\], where the variables \( x_1, \ldots, x_M \) and \( y_1, \ldots, y_N \) are related to the variables \( z_1, \ldots, z_{M+N} \) in the main text as \( x_j = z_j \) for \( j \in \{1, 2, \ldots, M\} \) and \( y_j = z_{j+M} \) for \( j \in \{1, 2, \ldots, N\} \). This coincidence was proved in section 5 in \[23\].

\[36\] If the atypicality condition \[A.7\] holds, this representation will be reducible (but in general indecomposable).

\[37\] We note that the sign of the variables \( y_1, \ldots, y_N \) in \[22\] is opposite to our definition.
The representation theory of the quantum superalgebra $\mathcal{U}_q(\mathfrak{gl}(M|N))$ for generic $q$ is similar (see for example, [71]) to that of $\mathfrak{gl}(M|N)$. The atypicality conditions for both cases are the same. And the (super)characters are also the same.

To make discussions complete, we should mention the representation theory of $\mathcal{U}_q(\hat{\mathfrak{gl}}(M|N))$ (or its Borel subalgebra). This is realized in combination of the evaluation map from (a Borel subalgebra of) $\mathcal{U}_q(\hat{\mathfrak{gl}}(M|N))$ to $\mathcal{U}_q(\mathfrak{gl}(M|N))$ and representations of $\mathcal{U}_q(\mathfrak{gl}(M|N))$. We expect that the functions $\mathcal{T}_{BM\sqcup FN}(x)$ and $\mathcal{T}_{BM\sqcup FN'}(x)$ come from two different evaluation maps (see eqs. (4.14)-(4.18) in [27], and also [28]). $q$-oscillator representations are also needed to describe the Baxter $Q$-operators (cf. [26, 27, 28, 24]). We would like to postpone these issues to a separate publication [55], where the functions in this paper will be realized as operators.

Appendix B. On the normalization of the Baxter $Q$-functions

Let us renormalize the Baxter $Q$-function for any $I \subset \mathcal{I}$ as:

$$Q_I(x) = \frac{Q_I(x)}{\phi_I(x)},$$

so that $Q_I(x)$ is a polynomial of finite degree (or an entire function of $x$):

$$Q_I(x) = \begin{cases} \prod_{j=1}^{i+1} \left(1 - \frac{x}{x_j} \right) & \text{for } I \neq \emptyset, \mathcal{I}, \\ 1 & \text{for } I = \emptyset, \mathcal{I}. \end{cases}$$

(B.2)

Here the function $\phi_I(x)$ (vacuum part), whose precise form will be fixed in what follows, does not contain the roots $x^I_j$ of the Bethe ansatz equation. In our normalization, it satisfies $\phi_I(0) = 1$. Then the functional relations (2.17)-(2.18) are modified to [28]:

$$(z_i - z_j)Q_{I\sqcup\{i,j\}}(x)Q_{I\sqcup\{i\}}(xq^{p_i})Q_{I\sqcup\{j\}}(xq^{-p_j})$$

$$- z_j \Phi_I^{(j,i)}(x)Q_{I\sqcup\{i\}}(xq^{-p_i})Q_{I\sqcup\{j\}}(x q^{p_i})$$

$$\text{for } p_i = p_j, \quad (B.3)$$

$$(z_i - z_j)Q_{I\sqcup\{i\}}(x)Q_{I\sqcup\{j\}}(x) = z_i \Phi_I^{(i,j)}(x)Q_{I\sqcup\{i\}}(xq^{p_i})Q_{I\sqcup\{j\}}(xq^{-p_j})$$

$$- z_j \Phi_I^{(j,i)}(x)^{-1}Q_I(xq^{p_i})Q_{I\sqcup\{i,j\}}(xq^{-p_i})$$

$$\text{for } p_i = -p_j, \quad (B.4)$$

where we set

$$\Phi_I^{(i,j)}(x) = \frac{\phi_I(xq^{p_i/p_j})\phi_I(i,j)(xq^{p_i/p_j})}{\phi_I(xq^{p_i/p_j})\phi_I(i,j)(xq^{-p_i/p_j})}. \quad (B.5)$$

$^{38}$Compare (B.3) with eq. (7.18) in [39].
Taking note on the zeros of the Baxter $Q$-functions (B.2), one obtains Bethe ansatz equations from the functional relations (B.3)+(B.4):

$$
- \frac{\phi_{I_a-1}(x_j^{I_a}q^{p_{ia}})\phi_{I_a}(x_j^{I_a}q^{-2p_{ia}+1})\phi_{I_a+1}(x_j^{I_a}q^{p_{ia}+1})}{\phi_{I_a-1}(x_j^{I_a}q^{-p_{ia}})\phi_{I_a}(x_j^{I_a}q^{2p_{ia}})\phi_{I_a+1}(x_j^{I_a}q^{-p_{ia}+1})}
= \frac{p_{ia+1}z_{ia+1}Q_{I_a-1}(x_j^{I_a}q^{p_{ia}})Q_{I_a}(x_j^{I_a}q^{-2p_{ia}+1})Q_{I_a+1}(x_j^{I_a}q^{p_{ia}+1})}{p_{ia}z_{ia}Q_{I_a-1}(x_j^{I_a}q^{-p_{ia}})Q_{I_a}(x_j^{I_a}q^{2p_{ia}})Q_{I_a+1}(x_j^{I_a}q^{-p_{ia}+1})}

\text{for } j \in \{1, 2, \ldots, n_{I_a}\} \text{ and } a \in \{1, 2, \ldots, M + N - 1\}. \quad (B.6)
$$

Left hand side of the above equation (B.6) should have a standard form of the Bethe ansatz equation (cf. [47, 4, 72, 8, 48, 39])

$$
- \frac{\Phi_a(x_j^{I_a}q^{\sum_{i \in I_a}p_i})}{\Phi_{a+1}(x_j^{I_a}q^{\sum_{i \in I_a}p_i})} = \frac{p_{ia+1}z_{ia+1}Q_{I_a-1}(x_j^{I_a}q^{p_{ia}})Q_{I_a}(x_j^{I_a}q^{-2p_{ia}+1})Q_{I_a+1}(x_j^{I_a}q^{p_{ia}+1})}{p_{ia}z_{ia}Q_{I_a-1}(x_j^{I_a}q^{-p_{ia}})Q_{I_a}(x_j^{I_a}q^{2p_{ia}})Q_{I_a+1}(x_j^{I_a}q^{-p_{ia}+1})}

\text{for } j \in \{1, 2, \ldots, n_{I_a}\} \text{ and } a \in \{1, 2, \ldots, M + N - 1\}, \quad (B.7)
$$

where the function $\Phi_a(x)$ has, for example for the Perk-Schultz model [47, 10], the following form [39]

$$
\Phi_a(x) = \prod_{k=1}^{L} \left(1 - \frac{xq^{-2p_{ia}\delta_{a,1}}}{w_k}\right). \quad (B.8)
$$

Here $L$ is the number of the lattice site and $w_k$ is an inhomogeneity on the spectral parameter. Thus $\phi_{I_a}(x)$ must satisfy the following functional equations.

$$
\frac{\Phi_a(xq^{\sum_{i \in I_a}p_i})}{\Phi_{a+1}(xq^{\sum_{i \in I_a}p_i})} = \frac{\phi_{I_a-1}(xq^{p_{ia}})\phi_{I_a}(xq^{-2p_{ia}+1})\phi_{I_a+1}(xq^{p_{ia}+1})}{\phi_{I_a-1}(xq^{-p_{ia}})\phi_{I_a}(xq^{2p_{ia}})\phi_{I_a+1}(xq^{-p_{ia}+1})}

\text{for } a \in \{1, 2, \ldots, M + N - 1\}. \quad (B.9)
$$

For more general quantum space with the highest weight $\sum_{j=1}^{M} \lambda_{j}^{(k)} \epsilon_{j} + \sum_{j=1}^{N} \lambda_{j+M}^{(k)} \delta_{j}$ for each site ($k = 1, 2, \ldots, L$) of the quantum space, we expect that the function has the following form:

$$
\Phi_a(x) = \prod_{k=1}^{L} \left(1 - \frac{xq^{-2p_{ia}\lambda_{1}^{(k)}}}{w_k}\right). \quad \text{There will be another form of the vacuum part, where the left hand side of (B.7) has the form } - \frac{\Phi_{a+1}(x_j^{I_a}q^{\sum_{i \in I_a}p_i})}{\Phi_a(x_j^{I_a}q^{\sum_{i \in I_a}p_i})}. \quad \text{Here we set } \Phi_a(x) = \prod_{k=1}^{L} \left(1 - \frac{xq^{2p_{ia}\lambda_{1}^{(k)}}}{w_k}\right).
$$

The left hand side of the Bethe ansatz equation for $\Phi_a(x)$ is speculated by applying a method in [20] to a (non-skew) Young diagram, where Drinfeld polynomials can be obtained; while the one for $\Phi_a(x)$ is speculated by applying a method in [20] to 180 degrees rotation of this Young diagram. See a remark in section 7 in [39]. We expect these come from two different evaluation representations of $U_q(\mathfrak{gl}(M|N))$, commented in Appendix A for the quantum space of the models. A precise description of these issue will need detailed Bethe ansatz calculations based on the representation theory of the quantum affine superalgebra, and is beyond the scope of this paper (cf. [48]).
One may solve (B.9) under the condition \( \phi_{I_a}(x) = \phi_{I_{M+N}}(x) = 1 \) and \( \phi_{I_a}(0) = 1 \) for \( 0 \leq a \leq M + N, M \neq N \). This corresponds to impose both (2.9) and (2.10). For example for the Perk-Schultz model, we have the following solution of (B.9).

\[
\phi_{I_a}(x) = \prod_{k=1}^{L} \left( \frac{q^{2(M-N)} \prod_{i \in I_k} \frac{xq^{2(M-N)} \prod_{j \in I_k} \frac{xq^{2(M-N)}}{w_k}}{q^{2(M-N)} \prod_{j \in I_k} \frac{xq^{2(M-N)}}{w_k}} \right)_{\infty} \quad \text{for} \quad a \in \{1, 2, \ldots, N + N - 1\}, \quad (B.10)
\]

where we assumed \( |q^{M-N}| < 1 \) (the case \( |q^{M-N}| > 1 \) will be similar) and introduced a symbol \( (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) \). It is interesting to see that cancellation occurs in \( (B.10) \) for any \( a \) if \( M - N = \pm 1 \). For example, for \((M, N) = (2, 1)\), \( p_1 = p_2 = 1, p_3 = -1 \) case, we have

\[
\begin{align*}
\phi_{(1)}(x) &= \phi_{(2)}(x) = 1, \\
\phi_{(1,2)}(x) &= \frac{1}{f(x)}, \\
\phi_{(1,3)}(x) &= \phi_{(2,3)}(x) = f(x),
\end{align*}
\]

(B.11)

where \( f(x) := \prod_{k=1}^{L} (1 - x/w_k) \). This agrees with the result in [24] (see eqs. (2.87) in [24]). Now the Baxter Q-functions \( Q_{I}(x) \) are meromorphic functions of \( x \), while \( Q_{I}(x) \) are polynomials (or entire functions) of \( x \). The Baxter Q-functions \( Q_{I}(x) \) with both the conditions (2.9) and (2.10) for \( M \neq N \) will be in the normalization of the universal \( R \)-matrix.

Appendix C. Proof of the functional relations

We will use the following lemma in the proof.

**Lemma 6.1.** The following relations hold for the determinant \( [3,4] \), where \( m + \alpha = n + \beta \). If there is \( k \) (\( 1 \leq k \leq \beta \)) such that \( s_k = 0 \), then we have

\[
\Delta_{(f_1, f_2, \ldots, f_n), (s_1, s_2, \ldots, s_\beta)}(xq^2) = (-1)^{\alpha} z_{f_1} z_{f_2} \cdots z_{f_n} \Delta_{(f_1, f_2, \ldots, f_n), (s_1, s_2, \ldots, s_\beta + 1)}(x) \quad (C.1)
\]

under the relation (2.22). If there is \( k \) (\( 1 \leq k \leq \alpha \)) such that \( r_k = 0 \), then we have

\[
\Delta_{(f_1, f_2, \ldots, f_n), (r_1, r_2, \ldots, r_\alpha)}(xq^{-2}) = (-1)^{\alpha} z_{f_1} z_{f_2} \cdots z_{f_n} \Delta_{(f_1, f_2, \ldots, f_n), (r_1, r_2, \ldots, r_\alpha - 1)}(x) \quad (C.2)
\]

under the relation (2.22). One can also derive similar relations for \( [3,5] \) by \( q \rightarrow q^{-1} \).

---

\[40\] In general, one will need the inverse matrix of a \( q \)-deformed Cartan matrix of \( gl(M \mid N) \) to solve (B.9). However, the Cartan matrix of \( gl(M \mid M) \) is degenerate. Then we imposed a condition \( M \neq N \). As for the case \( M = N \), one will have to relax the condition \( \phi_{I_0}(x) = 1 \) or \( \phi_{I_{M+N}}(x) = 1 \).
In particular for $m = \beta = 0$ or $\alpha = n = 0$, the following relations hold for any $c \in \mathbb{C}$:

$$\Delta_{(f_1, f_2, \ldots, f_n), \emptyset}^{\emptyset,(r_1, r_2, \ldots, r_n+c)}(x) = ((-1)^n z_1 z_2 \cdots z_n)^c \Delta_{(f_1, f_2, \ldots, f_n), \emptyset}^{\emptyset,(r_1, r_2, \ldots, r_n)}(x q^{-2c}), \quad (C.3)$$

$$\Delta_{\emptyset,(s_1+c, s_2+c, \ldots, s_m+c)}^{(b_1, b_2, \ldots, b_n), \emptyset}(x) = (z_{b_1} z_{b_2} \cdots z_{b_n})^c \Delta_{\emptyset,(s_1, s_2, \ldots, s_m), \emptyset}^{(b_1, b_2, \ldots, b_n)}(x q^{2c}). \quad (C.4)$$

For any $m \times n$ matrix, we will write a minor determinant whose $j_1, j_2, \ldots, j_\alpha$-th rows and $k_1, k_2, \ldots, k_\beta$-th columns removed from it as $D(j_1, j_2, \ldots, j_\alpha; k_1, k_2, \ldots, k_\beta)$, where $m-\alpha = n-\beta$, $j_1 < j_2 < \cdots < j_\alpha$ and $k_1 < k_2 < \cdots < k_\beta$. We will use the following identities for determinants.

$$D(j_1, j_2) D(j_3, j_4) - D(j_1, j_3) D(j_2, j_4) + D(j_1, j_4) D(j_2, j_3) = 0, \quad (C.5)$$

$$D(k_1, k_2) D(k_3, k_4) - D(k_1, k_3) D(k_2, k_4) + D(k_1, k_4) D(k_2, k_3) = 0, \quad (C.6)$$

$$D(j_1, k_1) D(j_2, k_3) - D(j_1, k_3) D(j_2, k_1) + D(j_1, k_1) D(j_2, k_3) = 0, \quad (C.7)$$

$$D(k_1, j_1) D(k_2, k_3) - D(k_1, j_3) D(k_2, j_1) + D(k_1, j_1) D(k_2, j_3) = 0, \quad (C.8)$$

$$D(j_1, k_1) D(j_2, k_2) - D(j_1, k_2) D(j_2, k_1) + D(j_1, k_1) D(j_2, k_2) = 0. \quad (C.9)$$

(C.5)-(C.8) are specialization of the so-called Plücker identity, and (C.9) is the Jacobi identity. We remark that some Plücker identities on supersymmetric polynomials were also discussed in [73].

**C.1 Proof of Theorem 3.2**

We will prove Theorem 3.2 for (3.22). The proof for (3.23) is similar to the one for (3.22). Let us introduce a notation:

$$I_{m,n} = \begin{cases} 
(1, 2, \ldots, m-n) & \text{for } m-n > 0, \\
\emptyset & \text{for } m-n \leq 0.
\end{cases} \quad (C.10)$$

That (3.22) satisfies (2.17)-(2.18) for $I = B_m \cup F_n$ is equivalent to:

$$- \Delta_{F_n, I_{m,n}}^{B_m, I_{m,n}}(x q) \Delta_{F_n, I_{m+2,n}}^{B_{m+2}, I_{m,n+2}}(x q^{-1}) = z_{b_{m+1}} \Delta_{F_n, I_{m+1,n}}^{B_m, I_{n+1,m+1}}(x q) \Delta_{F_n, I_{m+1,n}}^{B_{m+2}, I_{m+1,n}}(x q^{-1})$$

$$- z_{b_{m+2}} \Delta_{F_n, I_{m+1,n}}^{B_{m+1}, I_{m,n+1}}(x q^{-1}) \Delta_{F_n, I_{m+1,n}}^{B_m, I_{n+1,m+1}}(x q) \quad (C.11)$$
for \( i = b_{m+1}, j = b_{m+2}; \)

\[
\Delta_{F_{n},I_{m,n}}(xq^{-1})\Delta_{F_{n+2},I_{m,n+2}}(xq) = z_{f_{n+1}} \Delta_{F_{n+1},I_{m+1,m}}(xq^{-1})\Delta_{F_{n}(f_{n+2}),I_{m,n+1}}(xq) \\
- z_{f_{n+2}} \Delta_{F_{n+1},I_{m+1,m}}(xq)\Delta_{F_{n+1},I_{m,n+1}}(xq^{-1}) \quad \text{(C.12)}
\]

for \( i = f_{n+1}, j = f_{n+2}; \)

\[
\Delta_{F_{n},I_{m+1,n}}(xq^{-1})\Delta_{F_{n+1},I_{m+1,m}}(xq) = z_{b_{m+1}} \Delta_{F_{n},I_{n,m}}(xq^{-1})\Delta_{F_{n+1},I_{m+1,m+1}}(xq) \\
- z_{f_{n+1}} \Delta_{F_{n},I_{n,m}}(xq)\Delta_{F_{n+1},I_{m+1,m+1}}(xq^{-1}) \quad \text{(C.13)}
\]

for \( i = b_{m+1}, j = f_{n+1}. \)

We have to consider the following nine cases since the \((m,n)\)-index \((3.6)\) for \( \mu = \emptyset \) depends on \( m,n; \) for \( \text{(C.11)}: \) (i.1) \( m \geq n, \) (i.2) \( m = n-1, \) (i.3) \( m \leq n-2; \) for \( \text{(C.12)}: \) (ii.1) \( m \leq n, \) (ii.2) \( m = n+1, \) (ii.3) \( m \geq n+2; \) for \( \text{(C.13)}: \) (iii.1) \( m \geq n+1, \) (iii.2) \( m = n, \) (iii.3) \( m \leq n-1. \) We find that the proof of the case (ii.1) is similar to the case (i.1); the case (ii.2) is similar to the case (i.2); the case (ii.3) is similar to the case (i.3). So we will treat (i.1), (i.2), (i.3), (iii.1), (iii.2), (iii.3) in what follows.

(i.1) \( \text{(C.11)} \) for the case \( m \geq n \)

In this case \( \text{(C.11)} \) reduces to

\[
- \Delta_{F_{n},(1,2,...,m-n)}(xq)\Delta_{F_{n+2},(1,2,...,m-n+2)}(xq^{-1}) \\
= z_{b_{m+1}} \Delta_{F_{n+1},(1,2,...,m-n+1)}(xq)\Delta_{F_{n+1},(1,2,...,m-n+1)}(xq^{-1}) \\
- z_{b_{m+2}} \Delta_{F_{n+1},(1,2,...,m-n+1)}(xq^{-1})\Delta_{F_{n+1},(1,2,...,m-n+1)}(xq) \quad \text{(C.14)}
\]

Due to \( \text{(C.1)} \), this is identical to the following relation:

\[
- \Delta_{F_{n+1},(1,2,...,m-n)}(x)\Delta_{F_{n+2},(0,1,...,m-n+1)}(x) = \Delta_{F_{n+1},(1,2,...,m-n+1)}(x)\Delta_{F_{n+1},(0,1,...,m-n+1)}(x) \\
- \Delta_{F_{n+1},(0,1,...,m-n)}(x)\Delta_{F_{n+1},(0,1,...,m-n+1)}(x). \quad \text{(C.15)}
\]

This is nothing but the Jacobi identity \( \text{(C.9)}. \)

(i.2) \( \text{(C.11)} \) for the case \( m = n - 1 \)

In this case \( \text{(C.11)} \) reduces to

\[
- \Delta_{F_{m+1},(1)}(xq)\Delta_{F_{m+2},(1)}(xq^{-1}) = z_{b_{m+1}} \Delta_{F_{m+1},(0,1,...,m-n+1)}(xq)\Delta_{F_{m+1},(0,1,...,m-n+1)}(xq) \\
- z_{b_{m+2}} \Delta_{F_{m+1},(0,1,...,m-n+1)}(xq^{-1})\Delta_{F_{m+1},(0,1,...,m-n+1)}(xq). \quad \text{(C.16)}
\]
Let us expand the determinants \(^{41}\) in the left hand side of the above relation:

\[
- \Delta_{F_{m+1},\emptyset}^{B_m}\left(xq\right)\Delta_{F_{m+1},\emptyset}^{B_{m+2},\emptyset}(xq^{-1}) = - \sum_{\alpha=1}^{m+1} (-1)^{m+1+\alpha} Q_{f_\alpha}(x) \Delta_{F_{m+1}\backslash f_\alpha,\emptyset}^{B_m,\emptyset}(xq)
\]

\[
\times \sum_{\beta=1}^{m+2} (-1)^{\beta+m+2} Q_{b_\beta}(x) \Delta_{F_{m+1},\emptyset}^{B_{m+2}\backslash b_\beta,\emptyset}(xq^{-1})
\]

\[
= \sum_{\alpha=1}^{m+1} \sum_{\beta=1}^{m+2} (-1)^{\alpha+\beta} z_{b_\beta} Q_{b_\beta,\emptyset}(x) - z_{f_\alpha} Q_{b_\beta,\emptyset}(xq^{-1}) \Delta_{F_{m+1},\emptyset}^{B_m,\emptyset}(xq) \Delta_{F_{m+1},\emptyset}^{B_{m+2}\backslash b_\beta,\emptyset}(xq^{-1})
\]

\[
= - \sum_{\beta=1}^{m+2} (-1)^{\beta+m+2} z_{b_\beta} \Delta_{F_{m+1},\emptyset}^{B_{m+2}\backslash b_\beta,\emptyset}(xq^{-1}) \left| \begin{array}{c} Q_{(b_\beta,f_\beta)}(x) \\ z_{b_\beta} - z_{f_\alpha} \\ Q_{(b_\beta,f_\beta)}(xq^{-1}) \\ z_{b_\beta} - z_{f_\alpha} \end{array} \right|_{1 \leq l \leq m+1, 1 \leq k \leq m+2} 
\]

\[
+ \sum_{\alpha=1}^{m+1} (-1)^{m+1+\alpha} z_{f_\alpha} \Delta_{F_{m+1}\backslash f_\alpha,\emptyset}^{B_{m+2},\emptyset}(xq) \left| \begin{array}{c} Q_{(b_\beta,f_\beta)}(xq^{-1}) \\ z_{b_\beta} - z_{f_\alpha} \\ Q_{(b_\beta,f_\beta)}(xq^{-1}) \\ z_{b_\beta} - z_{f_\alpha} \end{array} \right|_{1 \leq l \leq m+1, 1 \leq k \leq m+2},
\]

\[\text{(C.17)}\]

where we introduced a notation \(B_{m+2}\backslash b_\beta = (b_1, b_2, \ldots, b_\beta-1, b_\beta+1, \ldots, b_{m+2})\), and used the functional relation \((2.22)\). The determinant \(\cdots\) in the first summand in the right hand side of \((C.17)\) vanishes for \(\beta \neq m+1, m+2\) as it has two identical rows. The determinant in the second summand vanishes as it has two identical columns. Then the right hand side of \((C.17)\) reduces to the right hand side of \((C.16)\).

\[\text{(i.3) (C.11) for the case } m \leq n-2\]

In this case \((C.11)\) reduces to

\[
- \Delta_{F_{n},\emptyset}^{B_m}(xq)\Delta_{F_{n},\emptyset}^{B_{m+2},(1,2,\ldots,n-m-2)}(xq^{-1})
\]

\[
= z_{b_{m+1}} \Delta_{F_{n},\emptyset}^{B_{m+1},(1,2,\ldots,n-m-1)}(xq) \Delta_{F_{n},\emptyset}^{B_{m}(b_{m+2}), (1,2,\ldots,n-m-1)}(xq^{-1})
\]

\[
- z_{b_{m+2}} \Delta_{F_{n},\emptyset}^{B_{m+1},(1,2,\ldots,n-m-1)}(xq^{-1}) \Delta_{F_{n},\emptyset}^{B_{m}(b_{m+2}), (1,2,\ldots,n-m-1)}(xq).
\]

Due to \((C.2)\), this relation can be modified to:

\[
\Delta_{F_{n},\emptyset}^{B_{m}(0,1,\ldots,n-m-1)}(x) \Delta_{F_{n},\emptyset}^{B_{m+2},(1,2,\ldots,n-m-2)}(x)
\]

\[
= \Delta_{F_{n},\emptyset}^{B_{m+1},(0,1,\ldots,n-m-2)}(x) \Delta_{F_{n},\emptyset}^{B_{m}(b_{m+2}), (1,2,\ldots,n-m-1)}(x)
\]

\[
- \Delta_{F_{n},\emptyset}^{B_{m+1},(1,2,\ldots,n-m-1)}(x) \Delta_{F_{n},\emptyset}^{B_{m}(b_{m+2}), (0,1,\ldots,n-m-2)}(x).
\]

\[\text{(C.19)}\]

\[\text{41 In Appendix C we will use the following notation for matrices in the determinant. We use } k \text{ for the row index and } l \text{ for the column index. Thus } (a_{k,l}) \text{ is a matrix whose } k\text{-th row and } l\text{-th column element is } a_{k,l}; (a_k) \text{ is a column vector; } (a_l) \text{ is a row vector.}\]

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This is nothing but the Plücker relation \((C.5)\).

\((iii.1)\) \((C.13)\) for the case \(m \geq n + 1\)

In this case, \((C.13)\) has the form:

\[
\Delta_{\text{F}_n,0}^{m+1,0}(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m,0}(xq)
= z_{m+1} \Delta_{\text{F}_n,0}^{m,0}(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m+1,0}(xq)
- z_{f_{m+1}} \Delta_{\text{F}_n,0}^{m,0}(xq)\Delta_{\text{F}_{m+1},0}^{m+1,0}(xq). \tag{C.20}
\]

Due to \((C.1)\), this relation reduces to:

\[
\Delta_{\text{F}_n,0}^{m+1,0}(0_{1,...,m-n})(xq)\Delta_{\text{F}_{m+1},0}^{m,0}(xq)
= \Delta_{\text{F}_n,0}^{m,0}(0_{1,...,m-n})(xq)\Delta_{\text{F}_{m+1},0}^{m+1,0}(xq)
- \Delta_{\text{F}_n,0}^{m,0}(1_{2,...,m-n})(xq)\Delta_{\text{F}_{m+1},0}^{m+1,0}(0_{1,...,m-n})(xq). \tag{C.21}
\]

This is nothing but the Plücker identity \((C.8)\).

\((iii.2)\) \((C.13)\) for the case \(m = n\)

In this case, \((C.13)\) has the form:

\[
\Delta_{\text{F}_n,1}^{m+1,0}(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m,0}(xq)
= z_{m+1} \Delta_{\text{F}_n,0}^{m,0}(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m+1,0}(xq)
- z_{f_{m+1}} \Delta_{\text{F}_n,0}^{m,0}(xq)\Delta_{\text{F}_{m+1},0}^{m+1,0}(xq). \tag{C.22}
\]

Let us expand the left hand side of \((C.22)\).

\[
\Delta_{\text{F}_n,1}^{m+1,0}(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m,0}(xq)
= \sum_{\alpha=1}^{m+1} (-1)^{\alpha+m+1} Q_{b_{\alpha}}(x) \Delta_{\text{F}_n,0}^{m+1,0}(b_{\alpha},0)(xq^{-1})
\times \sum_{\beta=1}^{m+1} (-1)^{m+1+\beta} Q_{f_{\beta}}(x) \Delta_{\text{F}_{m+1},0}^{m,0}(f_{\beta},0)(xq)
\]

\[
= \sum_{\alpha=1}^{m+1} \sum_{\beta=1}^{m+1} (-1)^{\alpha+\beta} \frac{z_{b_{\alpha}} Q_{b_{\alpha},f_{\beta}}(xq) - z_{f_{\beta}} Q_{b_{\alpha},f_{\beta}}(xq^{-1})}{z_{b_{\alpha}} - z_{f_{\beta}}} \Delta_{\text{F}_n,0}^{m+1,0}(b_{\alpha},0)(xq^{-1})\Delta_{\text{F}_{m+1},0}^{m,0}(f_{\beta},0)(xq)
\]

\[
= \sum_{\alpha=1}^{m+1} (-1)^{\alpha+m+1} z_{b_{\alpha}} \Delta_{\text{F}_n,0}^{m+1,0}(b_{\alpha},0)(xq^{-1}) \begin{vmatrix} Q_{b_{\alpha},f_{l}}(x) \cr z_{b_{\alpha}} - z_{f_{l}} \end{vmatrix}_{1 \leq l \leq m+1, 1 \leq l \leq \alpha} \begin{vmatrix} Q_{b_{\alpha},f_{l}}(xq^{-1}) \cr z_{b_{\alpha}} - z_{f_{l}} \end{vmatrix}_{1 \leq l \leq m+1, 1 \leq l \leq \alpha}
\]

\[
- \sum_{\beta=1}^{m+1} (-1)^{\beta+m+1} z_{f_{\beta}} \Delta_{\text{F}_{m+1},0}^{m+1,0}(f_{\beta},0)(xq) \begin{vmatrix} Q_{b_{k},f_{l}}(x) \cr z_{b_{k}} - z_{f_{l}} \end{vmatrix}_{1 \leq l \leq m+1, 1 \leq l \leq \beta} \begin{vmatrix} Q_{b_{k},f_{l}}(xq^{-1}) \cr z_{b_{k}} - z_{f_{l}} \end{vmatrix}_{1 \leq l \leq m+1, 1 \leq l \leq \beta}, \tag{C.23}
\]

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where we used the functional relation \((2.22)\). The determinant \(|\cdots|\) in the first summand in the right hand side of \((C.23)\) vanishes for \(\alpha \neq m + 1\). The determinant in the second summand vanishes for \(\beta \neq m + 1\). Thus the right hand side of \((C.23)\) coincides with the right hand side of \((C.22)\).

\((iii.3) (C.13)\) for the case \(m \leq n - 1\)
In this case \((C.13)\) has the form:

\[
\Delta_{F_{n,0}} B_{m+1, (1,2,\ldots,n-m-1)} (xq^{-1}) \Delta_{F_{n+1,0}} B_{m, (1,2,\ldots,n-m+1)} (xq) = z_{b_{m+1}} \Delta_{F_{n,0}} B_{m, (1,2,\ldots,n-m)} (xq^{-1}) \Delta_{F_{n+1,0}} B_{m+1, (1,2,\ldots,n-m)} (xq) 
\]

\[- z_{f_{n+1}} \Delta_{F_{n,0}} B_{m, (1,2,\ldots,n-m)} (xq) \Delta_{F_{n+1,0}} B_{m+1, (1,2,\ldots,n-m)} (xq^{-1}). \quad (C.24)\]

Due to \((C.2)\), this reduces to:

\[- \Delta_{F_{n,0}} B_{m+1, (1,2,\ldots,n-m-1)} (x) \Delta_{F_{n+1,0}} B_{m, (1,2,\ldots,n-m)} (x) = \Delta_{F_{n,0}} B_{m, (1,2,\ldots,n-m)} (x) \Delta_{F_{n+1,0}} B_{m+1, (0,1,\ldots,n-m-1)} (x) 
\]

\[- \Delta_{F_{n,0}} B_{m, (0,1,\ldots,n-m-1)} (x) \Delta_{F_{n+1,0}} B_{m+1, (1,2,\ldots,n-m)} (x). \quad (C.25)\]

This is nothing but the Plücker identity \((C.7)\).

C.2 Proof of Theorem 3.3
We will prove Theorem 3.3 for \((3.35)\). The proof for \((3.36)\) is similar to the one for \((3.35)\). For \(a, s \in \mathbb{Z}_{\geq 1}\), we will prove that \((3.35)\) satisfies

\[
T^{(a), B_{m, F_{n}}}(xq^{-1}) T^{(a), B_{m, F_{n}}}(xq) = T^{(a), B_{m, F_{n}}}(x) T^{(a), B_{m, F_{n}}}(x) 
\]

\[+ T^{(a-1), B_{m, F_{n}}}(x) T^{(a+1), B_{m, F_{n}}}(x). \quad (C.26)\]

This is identical to the following functional relation.

\[
T^{B_{m, F_{n}}}(xq^{-1}) T^{B_{m, F_{n}}}(xq) = T^{B_{m, F_{n}}}(x) T^{B_{m, F_{n}}}(x) + T^{B_{m, F_{n}}}(x) T^{B_{m, F_{n}}}(x). \quad (C.27)\]

We have to consider the following seven cases since the \((m, n)\)-index \((3.6)\) depends on \(m, n, a, s:\)

(i) \(a < m - n\),
(ii) \(a = m - n\),
(iii) \(a - s < m - n < a\),
(iv) \(a - s = m - n\),
(v) \(s < m - n < a - s\),
(vi) \(m - n = -s\),
(vii) \(m - n < -s\). The proof for the case

(v) is similar to the case (iii); the case (vi) is similar to the case (ii); the case (vii) is similar to the case (i). So we will treat four cases (i), (ii), (iii), (iv) from now on.

The “duality relations” \((3.53)\) and \((3.57)\) follow from the formulae \((3.46)-(3.49)\). One also have to take into account reduction of the functional relation \((C.27)\) based on Lemma 3.1 for \((3.51)\), \((3.52)\), \((3.55)\) and \((3.56)\), and Theorem 3.2 for the boundary conditions \((3.37),\), \((3.38),\), \((3.39)\) and \((3.40)\).

(i) The case \(a < m - n\)

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Due to (C.1), (C.27) is identical to the following relation.

\[ \Delta_{F_n,(1,2,\ldots,m-n-a,m-n-a+s+1,m-n+s)}(x)\Delta_{F_n,(0,1,\ldots,m-n-a-1,m-n-a+s,\ldots,m-n+s-1)}(x) \]
\[ = \Delta_{F_n,(1,2,\ldots,m-n-a,m-n-a+s+1,m-n+s)}(x)\Delta_{F_n,(0,1,\ldots,m-n-a-1,m-n-a+s+1,m-n+s-1)}(x) \]
\[ + \Delta_{F_n,(1,2,\ldots,m-n-a-1,m-n-a,s+1,m-n+s)}(x)\Delta_{F_n,(0,1,\ldots,m-n-a,m-n+a+1,m-n+s-1)}(x). \]  

(C.28)

This is nothing but the Plücker identity (C.6).

(ii) The case \( a = m - n \)

(C.27) is identical to the following relation.

\[ \Delta_{F_n,(s+1,s+2,\ldots,s+a)}(x)\Delta_{F_n,(s+1,s+2,\ldots,s+a)}(xq^{-2}) \]
\[ = \Delta_{F_n,(s+1,s+1,\ldots,s+a-1)}(x)\Delta_{F_n,(s+2,s+3,\ldots,s+a+1)}(xq^{-2}) \]
\[ + (-1)^{a+1} \Delta_{F_n,(1)}(x)\Delta_{F_n,(s+1,s+1,\ldots,s+a)}(xq^{-2}). \]  

(C.29)

Let us expand the determinants in the second term of the right hand side of the above relation.

\[ \Delta_{F_n,(s+1,s+1,\ldots,s+a)}(x)\Delta_{F_n,(s+1,s+2,\ldots,s+a)}(xq^{-2}) = \sum_{\beta=1}^{n} (-1)^{\beta+m+1} Q_{f_\beta}(xq^{-1}) \]
\[ \times \Delta_{F_n,(s+1,s+1,\ldots,s+a)}(x) \sum_{\alpha=1}^{m} (-1)^{\alpha+n+1} Q_{a_{\alpha}}(xq^{-1})\Delta_{F_n,(s+2,s+3,\ldots,s+a)}(xq^{-2}) \]  

(C.30)

\[ = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} (-1)^{\alpha+\beta+n+m} \frac{z_{a_{\alpha}} Q_{\{a_{\alpha},f_\beta\}}(x) - z_{f_\beta} Q_{\{a_{\alpha},f_\beta\}}(xq^{-2})}{z_{a_{\alpha}} - z_{f_\beta}} \]
\[ \times \Delta_{F_n,(s+1,s+1,\ldots,s+a)}(x)\Delta_{F_n,(s+1,s+2,\ldots,s+a)}(x) \]  

(C.31)

\[ = \sum_{\beta=1}^{n} (-1)^{\beta+m+1} z_{f_\beta} \Delta_{F_n,(s+1,s+1,\ldots,s+a)}(x) \]
\[ \times \left[ \frac{Q_{(b_k,f_1)}(xq^{-2})}{z_{b_k} - z_{f_1}} \right]_{1 \leq k \leq m} \left[ \frac{Q_{(b_k,f_1)}(xq^{-2})}{z_{b_k} - z_{f_1}} \right]_{1 \leq k \leq m} \left[ z_{b_k} Q_{b_k}(xq^{2(s+l)-1}) \right]_{1 \leq k \leq m} \right] \]
\[ + \sum_{\alpha=1}^{m} (-1)^{\alpha+n+1} z_{a_{\alpha}} \Delta_{F_n,(s+2,s+3,\ldots,s+a)}(xq^{-2}) \]
\[ \times \left[ \frac{Q_{(b_k,f_1)}(x)}{z_{b_k} - z_{f_1}} \right]_{1 \leq k \leq m} \left[ z_{b_k} Q_{b_k}(xq^{2(s+l)-3}) \right]_{1 \leq k \leq m} \right] \]
\[ (0)_{1 \times (a+1)}. \]  

(C.32)
where we introduced notations \( B_m \setminus b_\alpha = (b_1, \ldots, b_{\alpha-1}, b_{\alpha+1}, \ldots, b_m) \), \( F_n \setminus f_\beta = (f_1, \ldots, f_{\beta-1}, f_{\beta+1}, \ldots, f_n) \), and used the functional relation \((2.22)\). The determinant \(|\cdots|\) in the first summand in the right hand side of \((C.32)\) vanishes as it has two identical columns. Subtracting the \(\alpha\)-th row from the \((m + 1)\)-st row in the determinant \(|\cdots|\) in the second summand in the right hand side of \((C.32)\), we get:

\[
\left| \begin{array}{c}
(Q_{(b_k,f_1)}(x))_{1 \leq k \leq m, \\ 1 \leq l \leq n} \\
(Q_{(b_{\alpha},f_1)}(x))_{1 \leq \alpha \leq n} \\
(0)_{1 \times (\alpha + 1)}
\end{array} \right| = - \left| \begin{array}{c}
(Q_{(b_k,f_1)}(x))_{1 \leq k \leq m, \\ 1 \leq l \leq n} \\
(Q_{(b_{\alpha},f_1)}(x))_{1 \leq \alpha \leq n} \\
(0)_{1 \times n}
\end{array} \right| \left| \begin{array}{c}
(z^{s+l-2}Q_{b_k}(xq^{2(s+l)-3}))_{1 \leq k \leq m, \\ 1 \leq l \leq a+1} \\
(z^{s+l-2}Q_{b_{\alpha}}(xq^{2(s+l)-3}))_{1 \leq l \leq a+1}
\end{array} \right|
\]

\[
= - \sum_{\gamma=1}^{a+1} (-1)^{m+1+n+\gamma} z_{b_{\alpha}}^{s+\gamma-2} Q_{b_{\alpha}}(xq^{2(s+\gamma)-3}) \Delta_{B_m,0}^{F_n,(s,s+1,\ldots,s+a)\setminus(s+\gamma-1)}(x). \tag{C.33}
\]

Here we expanded the \((m + 1)\)-st row. Thus \((C.32)\) reduces to

\[
- \sum_{\alpha=1}^{m} (-1)^{\alpha+n+1} z_{b_{\alpha}} \Delta_{B_m,0}^{F_n,(s+2,s+3,\ldots,s+a)}(xq^{-2}) \\
\times \sum_{\gamma=1}^{a+1} (-1)^{m+1+n+\gamma} z_{b_{\alpha}}^{s+\gamma-2} Q_{b_{\alpha}}(xq^{2(s+\gamma)-3}) \Delta_{B_m,0}^{F_n,(s,s+1,\ldots,s+a)\setminus(s+\gamma-1)}(x)
\]

\[
= \sum_{\gamma=1}^{a+1} (-1)^{m+1+n+\gamma+1} \Delta_{B_m,0}^{F_n,(s,s+1,\ldots,s+a)\setminus(s+\gamma-1)}(x) \\
\times \left| \begin{array}{c}
(Q_{(b_k,f_1)}(xq^{-2}))_{1 \leq k \leq m, \\ 1 \leq l \leq n} \\
(z^{s+\gamma-1}Q_{b_k}(xq^{2(s+\gamma)-3}))_{1 \leq k \leq m} \\
(z^{s+1}Q_{b_{\alpha}}(xq^{2(s+l)-1}))_{1 \leq l \leq m, \\ 1 \leq k \leq a-1}
\end{array} \right|. \tag{C.34}
\]

The determinant \(|\cdots|\) in the right hand side of the above relation vanishes for \(\gamma \neq 1, a + 1\) as it has two identical columns. Then the right hand side of \((C.34)\) reduces to

\[
(-1)^{a+1} \Delta_{B_m,0}^{F_n,(s+1,s+2,\ldots,s+a)}(x) \Delta_{B_m,0}^{F_n,(s+s+1,\ldots,s+a)}(xq^{-2}) \\
- (-1)^{a+1} \Delta_{B_m,0}^{F_n,(s,s+1,\ldots,s+a-1)}(x) \Delta_{B_m,0}^{F_n,(s+2,s+3,\ldots,s+a+1)}(xq^{-2}). \tag{C.35}
\]

This proves \((C.29)\).

\(iii\) The case \(a - s < m - n < a\)
Due to (C.2), (C.27) is identical to the following relation.

\[
\Delta_{F_n(s-a+m-n+1, s-a+m-n-2, s-a+m-n-3, \ldots, s+m-n)}(x) \Delta_{F_n(s-a+m-n+1, s-a+m-n+2, s-a+m-n-3, \ldots, s+m-n)}(x)
\]

This is nothing but the Jacobi identity (C.9).

(iv) The case \(a - s = m - n\) (C.27) is identical to the following relation.

\[
\Delta_{F_n(1, 2, \ldots, s)}(x) \Delta_{F_n(1, 2, \ldots, a)}(xq^{-2}) = \sum_{\alpha=1}^{n} (-1)^{\alpha+m+1} Q_{f_\alpha}(xq^{-1}) \Delta_{F_n(1, 2, \ldots, a)}(x)
\]

Let us expand the determinants in the left hand side of the above relation (C.37), and apply (2.22).

\[
\Delta_{F_n(1, 2, \ldots, a)}(x) \Delta_{F_n(1, 2, \ldots, a)}(xq^{-2}) = \sum_{\alpha=1}^{n} (-1)^{\alpha+m+1} Q_{f_\alpha}(xq^{-1}) \Delta_{F_n(1, 2, \ldots, a)}(x)
\]

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\[
\times \left[ \left( \frac{Q_{(b_{k}, f_{l})}(xq^{-2})}{z_{b_{k}} - z_{f_{l}}} \right)_{1 \leq k \leq m, 1 \leq l \leq n} \left( \frac{Q_{(b_{k}, f_{a})}(xq^{-2})}{z_{b_{k}} - z_{f_{a}}} \right)_{1 \leq k \leq m, 1 \leq a \leq n} \left( \frac{z_{b_{k} - 1} Q_{b_{k}}(xq^{2l-3})}{z_{b_{k}} - 2f_{a}} \right)_{1 \leq k \leq m, 2 \leq l \leq a} \right] \\
\left[ (-z_{f_{l}})^{k-1} Q_{f_{l}}(xq^{-2k-1}) \right]_{1 \leq k \leq s, 1 \leq l \leq a} \right]. \\
\]

(C.38)

Subtracting the \( \beta \)-th row from the \((m + 1)\)-st row in the determinant \( \cdots \) in the first summand in the right hand side of (C.38), we obtain:

\[
= \left[ \left( \frac{Q_{(b_{k}, f_{l})}(x)}{z_{b_{k}} - z_{f_{l}}} \right)_{1 \leq k \leq m, 1 \leq l \leq n} \left( z_{b_{k} - 1} Q_{b_{k}}(xq^{2l-1}) \right)_{1 \leq k \leq m, 1 \leq l \leq a} \right] \\
\left[ (-z_{f_{l}})^{k-1} Q_{f_{l}}(xq^{-2k+1}) \right]_{2 \leq k \leq s, 1 \leq l \leq a} \right]. \\
\]

\[
= - \sum_{\gamma = 1}^{a} (-1)^{m+1+n+1} \gamma z_{b_{\gamma}}^{-1} Q_{b_{\gamma}}(xq^{2\gamma-1}) \Delta_{B_{m, (2, 3, \ldots, n)} \setminus F_{n, (1, 2, \ldots, a)}}^{B_{m, (2, 3, \ldots, n)} \setminus F_{n, (1, 2, \ldots, a)}}(x). \\
\]

(C.39)

Subtracting the \( \alpha \)-th column from the \((n + 1)\)-st column in the determinant \( \cdots \) in the second summand in the right hand side of (C.38), we obtain

\[
= \left[ \left( \frac{Q_{(b_{k}, f_{l})}(xq^{-2})}{z_{b_{k}} - z_{f_{l}}} \right)_{1 \leq k \leq m, 1 \leq l \leq n} \left( \frac{Q_{(b_{k}, f_{a})}(xq^{-2})}{z_{b_{k}} - z_{f_{a}}} \right)_{1 \leq k \leq m, 1 \leq a \leq n} \left( z_{b_{k} - 1} Q_{b_{k}}(xq^{2l-3}) \right)_{1 \leq k \leq m, 2 \leq l \leq a} \right] \\
\left[ (-z_{f_{l}})^{k-1} Q_{f_{l}}(xq^{-2k-1}) \right]_{1 \leq k \leq s, 1 \leq l \leq n} \right] \\
\left[ (-z_{f_{a}})^{k-1} Q_{f_{a}}(xq^{-2k-1}) \right]_{1 \leq k \leq s, 1 \leq a \leq n} \right]. \\
\]

\[
= - \sum_{\gamma = 1}^{a} (-1)^{m+1+n+1} (-z_{f_{a}})^{\gamma - 1} Q_{f_{a}}(xq^{-2\gamma-1}) \Delta_{F_{n, (2, 3, \ldots, a)}}^{B_{m, (1, 2, \ldots, n)} \setminus F_{n, (2, 3, \ldots, a)}}(x). \\
\]

(C.40)

Then (C.38) reduces to

\[
- \sum_{\beta = 1}^{m} (-1)^{\beta + n+1} z_{b_{\beta}} \Delta_{F_{n, (2, 3, \ldots, a)}}^{B_{m, (1, 2, \ldots, n)}}(xq^{-2}).
\]

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\[ \times \sum_{\gamma=1}^{a} (-1)^{m+n+\gamma} \gamma^{-1} Q_{b_{\gamma}}(xq^{2\gamma-1}) \Delta_{B_{m},(2,3,\ldots,s)}(x) \]

\[ + \sum_{a=1}^{n} (-1)^{n+m+1} z_{f_{a}} \Delta_{B_{m},(2,3,\ldots,s)}(x) \]

\[ \times \sum_{\gamma=1}^{s} (-1)^{m+n+\gamma+1} (-z_{f_{a}}) \gamma^{-1} Q_{f_{a}}(xq^{-2\gamma-1}) \Delta_{B_{m},(1,2,\ldots,a)}(xq^{-2}) \]

\[ = \sum_{\gamma=1}^{a} (-1)^{m+n+\gamma} \Delta_{B_{m},(2,3,\ldots,s)}(x) \gamma \]

\[ \times \left| \begin{array}{cccc}
\left( Q_{b_{k,f_{l}}}(xq^{-2}) \right)_{1 \leq k \leq m, 
1 \leq l \leq n}, & \left( z_{b_{k}}^{\gamma} Q_{b_{k}}(xq^{2\gamma-1}) \right)_{1 \leq k \leq m}, & \left( z_{b_{k}}^{l-1} Q_{b_{k}}(xq^{2l-3}) \right)_{1 \leq k \leq m, 
2 \leq l \leq a}, \\
\left( (-z_{f_{l}})^{k-1} Q_{f_{l}}(xq^{-2k-1}) \right)_{1 \leq k \leq s, 
1 \leq l \leq n}, & (0)_{s \times 1}, & (0)_{s \times (a-1)}.
\end{array} \right| \]

\[ - \sum_{\gamma=1}^{s} (-1)^{m+n+\gamma+1} \Delta_{B_{m},(1,2,\ldots,a)}(xq^{-2}) \gamma \]

\[ \times \left| \begin{array}{cccc}
\left( Q_{b_{k,f_{l}}}(x) \right)_{1 \leq k \leq m, 
1 \leq l \leq n}, & \left( z_{b_{k}}^{l-1} Q_{b_{k}}(xq^{2l-1}) \right)_{1 \leq k \leq m, 
2 \leq l \leq a}, \\
\left( (-z_{f_{l}})^{k-1} Q_{f_{l}}(xq^{-2k-1}) \right)_{1 \leq k \leq s, 
1 \leq l \leq n}, & (0)_{1 \times a}, & (0)_{(s-1) \times a},
\end{array} \right|. \tag{C.41} \]

The determinant \[ \cdots \] in the first summand in the right hand side of (C.41) vanishes for \( \gamma \neq a \) as it has two identical columns. The determinant \[ \cdots \] in the second summand in the right hand side of (C.41) vanishes for \( \gamma \neq s \). Then the right hand side of (C.41) reduces to

\[ (-1)^{m+n+1} \Delta_{B_{m},(2,3,\ldots,a)}(x) \Delta_{B_{m},(1,2,\ldots,a-1)}(xq^{-2}) \]

\[ - (-1)^{m+n} \Delta_{B_{m},(1,2,\ldots,a-1)}(x) \Delta_{B_{m},(2,3,\ldots,a)}(x). \tag{C.42} \]

This coincides with the right hand side of (C.37) since \( a - s = m - n \).

### C.3 Proof of Theorem 3.4

We will prove Theorem 3.4 for (3.35). The proof for (3.36) is similar to the one for (3.35). For \( a, s \in \mathbb{Z}_{>0} \), we will prove that (3.35) satisfies (3.38). The case \( s = 0 \) is trivial. Then we consider \( s \geq 1 \) case. We have to consider the following six cases since the \( (m,n) \)-index \( \lfloor m/n \rfloor \) depends on \( m, n, a, s \): (i) \( a + 2 \leq m - n \), (ii) \( a + 1 = m - n \), (iii) \( a - s + 2 \leq m - n \leq a \), (iv) \( a - s + 1 = m - n \), (v) \( -s + 1 \leq m - n \leq a - s \), (vi) \( m - n \leq -s \). For \( a = 0 \), the case (v) is void.

The proof of the other functional relations is similar to that of (3.38).
(i) The case \( a + 2 \leq m - n \)

Due to (C.1), (3.58) is identical to the following relation.

\[
\Delta_{F_n(0,1,\ldots,m-n-a-2,m-n-a+s-1,\ldots,m-n-s-1)}(x) \Delta_{F_n(1,2,\ldots,m-n-a-1,m-n-a+s,\ldots,m-n-s-1)}(x)
\]

\[
- \Delta_{F_n(0,1,\ldots,m-n-a-1,m-n-a+s,\ldots,m-n-s-1)}(x) \Delta_{F_n(1,2,\ldots,m-n-a-2,m-n-a+s-1,\ldots,m-n-s-1)}(x)
\]

\[
= \Delta_{F_n(1,2,\ldots,m-n-a-1,m-n-a+s-1,\ldots,m-n-s-1)}(x) \Delta_{F_n(0,1,\ldots,m-n-a-2,m-n-a+s,\ldots,m-n-s-1)}(x).
\]

(C.43)

For \( a = 0 \) case, we must interpret \( (1, 2, \ldots, m-n-a-1, m-n-a+s, \ldots, m-n-s-1) \) as \( (1, 2, \ldots, m-n-1) \) etc. The above relation is nothing but the Plücker identity (C.8).

(ii) The case \( a + 1 = m - n \)

In this case, (3.58) has the form:

\[
\Delta_{F_n(s+1,s+2,\ldots,a+s+1)}(x) \Delta_{F_n(s+1,\ldots,a+s)}(x^2)
\]

\[
- (-1)^{m+n} \Delta_{F_n(s+2,s+3,\ldots,a+s+1)}(x) \Delta_{F_n(s,\ldots,a+s)}(x^2)
\]

\[
= z_{b_m} \Delta_{F_n(s,a+1,\ldots,a+s)}(x^2) \Delta_{F_n(s+2,s+3,\ldots,a+s+1)}(x).
\]

(C.44)

Let us expand the determinants in the second term of the left hand side of (C.44):

\[
\Delta_{F_n(s+1,\ldots,a+s)}(x) \Delta_{F_n(s+1,\ldots,a+s)}(x^2)
\]

\[
= \sum_{a=1}^{m} (-1)^{a+n+1} Q_{ba} \Delta_{F_n(b_a,\ldots)}(x)
\]

\[
\times \sum_{\beta=1}^{n} (-1)^{m+\beta} Q_{f_\beta}(x^2) \Delta_{F_n(f_\beta,\ldots)}(x^2)
\]

\[
= \sum_{a=1}^{m} (-1)^{a+n+1} \sum_{\beta=1}^{n} (-1)^{m+\beta} Q_{ba} Q_{\{b_a,f_\beta\}}(x^2) - z_{f_\beta} Q_{\{b_a,f_\beta\}}(x)
\]

\[
\times z_{b_a} - z_{f_\beta}
\]

\[
\times \Delta_{F_n(s+2,s+3,\ldots,a+s+1)}(x) \Delta_{F_n(s,a+1,\ldots,a+s)}(x^2)
\]

\[
= \sum_{a=1}^{m} (-1)^{a+n+1} z_{b_a} \Delta_{F_n(b_a,\ldots)}(x)
\]

\[
\times \left( \frac{Q_{\{b_a,f_1\}}(x^2)}{z_{b_a} - z_{f_1}} \right)_{1 \leq k \leq m-1, \ 1 \leq \ell \leq \beta} \left( \frac{Q_{\{b_a,f_\beta\}}(x^2)}{z_{b_a} - z_{f_\beta}} \right)_{1 \leq k \leq m-1, \ 1 \leq \ell \leq \beta}
\]

\[
\left( \frac{(s+1-2)Q_{b_k}(x^2(s+1-1))}{z_{b_k} - z_{f_1}} \right)_{1 \leq k \leq m-1, \ 1 \leq \ell \leq \beta}
\]

\[
- \sum_{\beta=1}^{n} (-1)^{m+\beta} z_{f_\beta} \Delta_{F_n(f_\beta,\ldots)}(x^2)
\]

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\[
\times \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m, \ 1 \leq l \leq n} \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m, \ 1 \leq l \leq n}, \quad (C.45)
\]

where we introduced notations \( B_m \setminus b_{\alpha} = (b_1, \ldots, b_{\alpha-1}, b_{\alpha+1}, \ldots, b_m) \), \( F_n \setminus f_\beta = (f_1, \ldots, f_{\beta-1}, f_{\beta+1}, \ldots, f_n) \), and used the functional relation \((2.22)\). The determinant \( \cdots \) in the second summation in the right hand side of \((C.45)\) vanishes. Subtracting the \( \alpha \)-th row \((1 \leq \alpha \leq m-1)\) from the \( m \)-th row in the determinant \( \cdots \) in the first summation in the right hand side of \((C.45)\), we get:

\[
\begin{align*}
(C.45) &= \sum_{\alpha=1}^{m-1} \left( -1 \right)^{\alpha+n+1} z_{b_\alpha} \Delta_{F_n, (s+2, s+3, \ldots, s+a+1)}(x) \\
&\times \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq n} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq a+1} \\
&\times \left( 0 \right)_{1 \times (a+1)} \\
&+ \left( -1 \right)^{m+n+1} z_{b_m} \Delta_{F_{n-1}, \emptyset}(x) \\
&\times \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq n} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq a+1} \\
&\times \left( 0 \right)_{1 \times n} \\
&\times \left( z_{b_m}^* Q_{b_m} \left( x q^2 (s+l+1) \right) \right)_{1 \leq l \leq a+1} \\
&= - \sum_{\alpha=1}^{m-1} \left( -1 \right)^{\alpha+n+1} z_{b_\alpha} \Delta_{F_n, (s+2, s+3, \ldots, s+a+1)}(x) \\
&\times \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq n} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq a+1} \\
&\times \left( 0 \right)_{1 \times n} \\
&\times \left( z_{b_m}^* Q_{b_m} \left( x q^2 (s+l+1) \right) \right)_{1 \leq l \leq a+1} \\
&+ \left( -1 \right)^{m+n+1} z_{b_m} \Delta_{F_{n-1}, \emptyset}(x) \\
&\times \left\{ \Delta_{F_n, (s+1, \ldots, s+a)}(x) - \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq n} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq a+1} \right\} \\
&= - \sum_{\alpha=1}^{m} \left( -1 \right)^{\alpha+n+1} z_{b_\alpha} \Delta_{F_n, (s+2, s+3, \ldots, s+a+1)}(x) \\
&\times \left( \frac{Q_{b_k, x_q}}{z_{b_k} - z_{f_l}} \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq n} \left( z_{b_k}^* Q_{b_k} \left( x q^2 (s+l+1) \right) \right)_{1 \leq k \leq m-1, \ 1 \leq l \leq a+1} \\
&\times \left( 0 \right)_{1 \times n} \\
&\times \left( z_{b_m}^* Q_{b_m} \left( x q^2 (s+l+1) \right) \right)_{1 \leq l \leq a+1} \\
&+ \left( -1 \right)^{m+n+1} z_{b_m} \Delta_{F_{n-1}, \emptyset}(x) \Delta_{F_n, (s+1, \ldots, s+a)}(x q^2).
\end{align*}
\]

\((C.46)\)
Expanding the $m$-th row in the determinant in the right hand side of (C.46), we obtain:

\[-\sum_{a=1}^{m} (-1)^{a+n+1} z_{ba} \Delta_{F_n(s+2,s+3,...,s+a+1)}^{B_{m}\backslash h_n,\emptyset}(x)\]

\[\times \sum_{\gamma=1}^{a+1} (-1)^{m+n+\gamma \sum s^{-\gamma-2}} z_{\alpha b\beta}(xq^{2(s+\gamma)-1}) \Delta_{F_i(s,s+1,...,s+a+1)}^{B_{m-1}\backslash \emptyset}(xq^2)\]

\[+ (-1)^{m+n+1} z_{bn} \Delta_{F_i(s+2,s+3,...,s+a+1)}^{B_{m-1}\backslash \emptyset}(x) \Delta_{F_n(s,s+1,...,s+a)}^{B_{m}\emptyset}(xq^2)\]

\[= -\sum_{\gamma=1}^{a+1} (-1)^{m+n+\gamma} \Delta_{F_i(s+1,...,s+a)}^{B_{m-1}\emptyset}(xq^2)\]

\[\times \left( \frac{Q_{f_1}(x)}{z_{b_1} - z_{f_1}} \right) \sum_{1 \leq k \leq m} \left( z_{b_k}^{s+1} Q_{b_k}(xq^{2(s+1)+1}) \right) \sum_{1 \leq i \leq a} \left( z_{b_i}^{s+1} Q_{b_i}(xq^{2(s+1)+1}) \right)\]

\[\times \frac{1}{z_{b_k} - z_{f_1}} \Delta_{F_i(s+1,...,s+a)}^{B_{m-1}\emptyset}(xq^2)\]

\[-(-1)^{m+n} \Delta_{F_i(s+1,...,s+a)}^{B_{m-1}\emptyset}(xq^2) \Delta_{F_n(s+1,s+2,...,s+a+1)}^{B_{m}\emptyset}(x)\]

\[-(-1)^{m+n} z_{bn} \Delta_{F_i(s+2,s+3,...,s+a+1)}^{B_{m-1}\emptyset}(x) \Delta_{F_n(s,s+1,...,s+a)}^{B_{m}\emptyset}(xq^2).\]  

(C.47)

The determinant $\left| \cdots \right|$ in the right hand side of (C.47) vanished for $\gamma \neq 1$. Then we obtain (C.44).

(iii) The case $a - s + 2 \leq m - n \leq a$

Due to (C.2), (3.58) is identical to the following relation.

\[\Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m}(1,2,...,n-m+1)}(x) \Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m-1}(0,1,...,n-m+1)}(x)\]

\[= \Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m}(1,2,...,n-m+1)}(x) \Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m-1}(0,1,...,n-m+1)}(x)\]

\[= \Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m}(1,2,...,n-m+1)}(x) \Delta_{F_n(m-n-a+s+1,m-n-a+s+2,...,m-n+s)}^{B_{m-1}(0,1,...,n-m+1)}(x).\]  

(C.49)

This is nothing but the Plücker identity (C.7).

(iv) The case $a - s + 1 = m - n$

In this case, (3.58) has the form:

\[\Delta_{F_n(1,2,...,a+1)}^{B_{m}(1,2,...,s)}(x) \Delta_{F_n(1,2,...,a)}^{B_{m-1}(1,2,...,s-1)}(xq^2) = (-1)^{a+s} \Delta_{F_n(1,2,...,a+1)}^{B_{m}(1,2,...,s)}(xq^2)\]

\[+ (-1)^{a+s} z_{bn} \Delta_{F_n(1,2,...,a)}^{B_{m}(2,3,...,s)}(xq^2) \Delta_{F_n(1,2,...,a+1)}^{B_{m-1}(1,2,...,s)}(xq^2).\]  

(C.50)

Let us expand the determinant $\Delta_{F_n(1,2,...,a+1)}^{B_{m}(1,2,...,s)}(x)$ with respect to the $(n+1)$-st column, and the determinant $\Delta_{F_n(1,2,...,a)}^{B_{m-1}(1,2,...,s)}(xq^2)$ with respect to the $m$-th row. After some calculations similar to the ones in the case (ii), we arrive at the right hand side of (C.50).
(v) The case $-s + 1 \leq m - n \leq a - s$

Due to (C.1), (3.58) is identical to the following relation.

$$
\Delta B_{m,n}(m-s+a+3,n-m-s+a+4,...,n-m+a+2,F_{n},(0,1,...,m-n+s-1))
\times \Delta B_{m-1,n}(m-s+a+2,n-m-s+a+3,...,n-m+a),(x)
- \Delta B_{m,n}(m-s+a+2,n-m-s+a+3,...,n-m+a+1,F_{n},(0,1,...,m-n+s-1))
\times \Delta B_{m-1,n}(m-s+a+3,n-m-s+a+4,...,n-m+a+2,F_{n},(0,1,...,m-n+s-1))(x)
= \Delta B_{m-1,n}(m-s+a+2,n-m-s+a+3,...,n-m+a+1,F_{n},(0,1,...,m-n+s-1))
\times \Delta B_{m-1,n}(m-s+a+3,n-m-s+a+4,...,n-m+a+2,F_{n},(0,1,...,m-n+s-1))(x).
$$

(C.51)

This is nothing but the Plücker identity (C.7).

(vi) The case $m - n \leq -s$

Due to (C.2), (3.58) is identical to the following relation.

$$
\Delta B_{m,n}(1,2,...,n-m-s,n-m-s+a+2,n-m-s+a+3,...,n-m+a+1,F_{n},(0))
\times \Delta B_{m-1,n}(0,1,...,n-m-s,n-m-s+a+1,n-m-s+a+2,...,n-m+a),(x)
- \Delta B_{m,n}(1,2,...,n-m-s,n-m-s+a+1,n-m-s+a+2,...,n-m+a),(F_{n},(0))
\times \Delta B_{m-1,n}(0,1,...,n-m-s,n-m-s+a+2,n-m-s+a+3,...,n-m+a+1,F_{n},(0))(x)
+ \Delta B_{m,n}(0,1,...,n-m-s,n-m-s+a+2,n-m-s+a+3,...,n-m+a),(F_{n},(0))
\times \Delta B_{m-1,n}(1,2,...,n-m-s,n-m-s+a+1,n-m-s+a+2,...,n-m+a+1,F_{n},(0))(x) = 0.
$$

(C.52)

This is nothing but the Plücker identity (C.5).

Appendix D. Conserved quantities and Baxter equations

In this section, we will present conserved quantities and generalized Baxter equations in determinant form based on a similar idea in Theorem 5.1 and Lemma 5.1 in [25].

(i) Formulae from (3.41)

Here we assume that the function $T_{a,Bm,F_{n}}^{(a)}(x)$ is defined by the right hand side of (3.41). A condition $a - s \leq m - n$ was supposed in the left hand side of (3.41), but now this condition is relaxed to $s \in \mathbb{C}$ and $a \in \mathbb{Z}_{\geq 0}$. Let us introduce $\left(\begin{array}{c} m \\ a \end{array}\right) + 1$ matrices $T_{s,Bm,F_{n}}^{(a)}(x)$ (resp. $J_{s,Bm,F_{n}}^{(a)}(x)$) whose $(i,j)$ element is given by $T_{s+i+j}^{(a)}(xq^{s+i-j})$, (resp. $T_{s,i-j}^{(a)}(xq^{s+i-j})$) where $i,j \in \{1,2,...,\left(\begin{array}{c} m \\ a \end{array}\right) + 1\}$. For any square matrix $M$, we will write a minor determinant whose $\alpha$-th row and
\( \beta \)-th column removed from \( \mathcal{M} \) as \( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\mathcal{M}) \). Then we obtain conserved quantities as follows.

**Theorem 6.2.** For any \( a \in \mathbb{Z}_{\geq 0} \) and \( \alpha, \beta, \gamma \in \{1, 2, \ldots, (\binom{m}{a} + 1) \} \), the following quantities

\[
\begin{align*}
D \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\mathcal{I}_s^{(a), B_m, F_n}(x)) \\
D \begin{bmatrix} \gamma \\ \beta \end{bmatrix} (\mathcal{I}_s^{(a), B_m, F_n}(x))
\end{align*}
\]

\( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\mathcal{J}_s^{(a), B_m, F_n}(x)) \) does not depend on \( s \).

Let us replace \( r \)-th column of \( \mathcal{I}_s^{(a), B_m, F_n}(x) \) (resp. \( \mathcal{J}_s^{(a), B_m, F_n}(x) \)) with the column vector whose \( i \)-th component is \( (\prod_{\gamma \in I} z_\gamma)^{-1} Q_I(x^{q^{2i + 1} - m}) \) (resp. \( (\prod_{\gamma \in I} z_\gamma)^{-1} Q_{m-1}(x^{q^{2i - m}}) \)) and write it as \( \mathcal{I}_{s; r, I}^{(a), B_m, F_n}(x) \) (resp. \( \mathcal{J}_{s; r, I}^{(a), B_m, F_n}(x) \)), where \( i \in \{1, 2, \ldots, (\binom{m}{a} + 1) \} \), \( I \subset B_m \) and \( \text{Card}(I) = a \). Then we obtain the following lemma.

**Lemma 6.3.** For any \( a \in \mathbb{Z}_{\geq 0} \), \( s \in \mathbb{C} \) and \( r \in \{1, 2, \ldots, (\binom{m}{a} + 1) \} \), the following relations hold.

\[
\begin{align*}
\det(\mathcal{I}_{s; r, I}^{(a), B_m, F_n}(x)) &= \det(\mathcal{J}_{s; r, I}^{(a), B_m, F_n}(x)) = 0, \\
\det(\mathcal{I}_{s; r, I}^{(a), B_m, F_n}(x)) &= \det(\mathcal{J}_{s; r, I}^{(a), B_m, F_n}(x)) = 0.
\end{align*}
\]

**D.3** for \((m, n) = (M, N)\) corresponds to a \( U_q(\mathfrak{gl}(M|N)) \) generalization of the Baxter equation.

**(ii) Formulae from \( 3.44 \)**

Here we assume that the function \( \mathcal{T}_{s}^{(a), B_m, F_n}(x) \) is defined by the right hand side of \( 3.44 \). A condition \( a - s \geq m - n \) was supposed in the left hand side of \( 3.44 \), but now this condition is relaxed to \( a \in \mathbb{C} \) and \( s \in \mathbb{Z}_{\geq 0} \). Let us introduce \( (\binom{n}{s} + 1) \times (\binom{n}{s} + 1) \) matrices \( \mathcal{K}_{s}^{(a), B_m, F_n}(x) \) (resp. \( \mathcal{L}_{s}^{(a), B_m, F_n}(x) \)) whose \((i, j)\) element is given by \( \mathcal{T}_{s; i+j}^{(a), B_m, F_n}(xq^{a-i+j}) \) (resp. \( \mathcal{T}_{s; i-j}^{(a), B_m, F_n}(xq^{a-i+j}) \)) where \( i, j \in \{1, 2, \ldots, (\binom{n}{s} + 1) \} \). Then we obtain conserved quantities as follows.

**Theorem 6.4.** For any \( s \in \mathbb{Z}_{\geq 0} \) and \( \alpha, \beta, \gamma \in \{1, 2, \ldots, (\binom{n}{s} + 1) \} \), the following quantities

\[
\begin{align*}
D \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\mathcal{K}_{s}^{(a), B_m, F_n}(x)) \\
D \begin{bmatrix} \gamma \\ \beta \end{bmatrix} (\mathcal{K}_{s}^{(a), B_m, F_n}(x))
\end{align*}
\]

\( \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\mathcal{L}_{s}^{(a), B_m, F_n}(x)) \) does not depend on \( a \).
Let us replace \( r \)-th column of \( \mathcal{K}^{(a)}_{s,B_m,F_n}(x) \) (resp. \( \mathcal{L}^{(a)}_{s,B_m,F_n}(x) \)) with a column vector whose \( i \)-th component is \( (\prod_{\gamma \in J}(-z_\gamma))^{-i}Q_{\mathcal{J}}(xq^{-2i+m-\gamma}) \) (resp. \( (\prod_{\gamma \in J}(-z_\gamma))^{-i}Q_{\mathcal{J}}(xq^{-2i+m-\gamma}) \)) and write it as \( \mathcal{K}^{(a)}_{s,r,J}(x) \) (resp. \( \mathcal{L}^{(a)}_{s,r,J}(x) \)), where \( i \in \{1, 2, \ldots, \binom{n}{s}+1\} \), \( J \subset F_n \) and \( \text{Card}(J) = s \). Then we obtain the following lemma.

**Lemma 6.5.** For any \( s \in \mathbb{Z}_{\geq 0}, a \in \mathbb{C} \) and \( r \in \{1, 2, \ldots, \binom{n}{s}+1\} \), the following relations hold.

\[
\det(\mathcal{K}^{(a)}_{s,B_m,F_n}(x)) = \det(\mathcal{L}^{(a)}_{s,B_m,F_n}(x)) = 0, \quad \text{(D.5)}
\]
\[
\det(\mathcal{K}^{(a)}_{s,r,J}(x)) = \det(\mathcal{L}^{(a)}_{s,r,J}(x)) = 0. \quad \text{(D.6)}
\]

(D.6) for \( (m, n) = (M, N) \) corresponds to a \( U_q(\hat{gl}(M|N)) \) generalization of the Baxter equation. Theorem 6.4 and Lemma 6.5 become trivial for \( m = 0 \). Thus these are peculiar to the superalgebra case \( U_q(\hat{gl}(M|N)) \) for \( N > 0 \).

We remark that we did not use concrete function form of \( Q_I(x) \) and \( \overline{Q}_I(x) \) in the above theorems and lemmas. When one take into account the condition \( a - s \leq m - n \) in (3.41) or \( a - s \geq m - n \) in (3.44), one have to put some of the matrix elements in \( \mathcal{I}^{(a)}_{s,B_m,F_n} \), \( \mathcal{J}^{(a)}_{s,B_m,F_n} \), \( \mathcal{K}^{(a)}_{s,B_m,F_n}(x) \) and \( \mathcal{L}^{(a)}_{s,B_m,F_n}(x) \) to 0, and needs some restrictions on \( a, s \) in the above theorems and lemmas. We also remark that similar theorems also hold based on the formul\(e \)e (3.42) and (3.45) (they are related to the above theorems by \( Q_I(xq^a) \rightarrow \overline{Q}_I(xq^{-a}) \) for any \( I \subset \mathcal{J} \), where \( a \) is any shift of \( x \) of the \( Q \)-functions).

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