Some comments about A Bayesian criterion for singular models

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It is a well-known fact that the BIC approximation of the marginal likelihood in a given irregular model $\mathcal{M}_k$ fails or may fail. The BIC approximation has the form

$$BIC_k = \log p(Y_n|\hat{\pi}_k, \mathcal{M}_k) - d_k \log n/2$$

where $d_k$ corresponds on the number of parameters to be estimated in model $\mathcal{M}_k$. In irregular models the dimension $d_k$ typically does not provide a good measure of complexity for model $\mathcal{M}_k$, at least in the sense that it does not lead to an approximation of

$$\log m(Y_n|\mathcal{M}_k) = \log \left( \int_{\mathcal{M}_k} p(Y_n|\pi_k, \mathcal{M}_k) dP(\pi_k|k) \right).$$

A way to understand the behaviour of $\log m(Y_n|\mathcal{M}_k)$ is through the effective dimension

$$\tilde{d}_k = -\lim_{n} \frac{\log P(\{KL(p(Y_n|\pi_0, \mathcal{M}_k), p(Y_n|\pi_k, \mathcal{M}_k)) \leq 1/n|k\})}{\log n}$$

when it exists, see for instance the discussions in [Chambaz and Rousseau (2008)] and [Rousseau (2007)].

Watanabe (2009) provided a more precise formula, which is the starting point of the approach of Drton and Plummer:

$$\log m(Y_n|\mathcal{M}_k) = \log p(Y_n|\hat{\pi}_k, \mathcal{M}_k) - \lambda_k(\pi_0) \log n + [m_k(\pi_0) - 1] \log \log n + O_p(1)$$

where $\pi_0$ is the true parameter.

The authors propose a clever algorithm to approximate of the marginal likelihood.

Given the popularity of the BIC criterion for model choice, obtaining a relevant penalized likelihood when the models are singular is an important issue and
we congratulate the authors for it. Indeed a major advantage of the BIC formula is that it is an off-the-shelf criterion which is implemented in many softwares, thus can be used easily by non statisticians.

In the context of singular models, a more refined approach needs to be considered and although the algorithm proposed by the authors remains quite simple, it requires that the functions $\lambda_k(\pi)$ and $m_k(\pi)$ need be known in advance, which so far limitsates the number of problems that can be thus processed. In this regard their equation (3.2) is both puzzling and attractive. Attractive because it invokes nonparametric principles to estimate the underlying distribution; puzzling because why should we engage into deriving an approximation like (3.1) and call for Bayesian principles when (3.1) is at best an approximation. In this case why not just use a true marginal likelihood?

1. Why do we want to use a BIC type formula?

The BIC formula can be viewed from a purely frequentist perspective, as an example of penalized likelihood. The difficulty then stands into choosing the penalty and a common view on these approaches is to choose the smallest possible penalty that still leads to consistency of the model choice procedure, isince it then enjoys better separation rates. In this case a $\log \log n$ penalty is sufficient, as proved in [Gassiat and van Handel (2013)].

Now whether or not this is a desirable property is entirely debatable, and one might advocate that for a given sample size, if the data fits the smallest model (almost) equally well, then this model should be chosen. But unless one is specifying what equally well means, it does not add much to the debate.

This also explains the popularity of the BIC formula (in regular models), since it approximates the marginal likelihood and thus benefits from the Bayesian justification of the measure of fit of a model for a given data set, often qualified of being a Bayesian Ockham’s razor. But then why should we not compute instead the marginal likelihood?

Typical answers to this question that are in favour of BIC-type formula include: (1) BIC is suppossely easier to compute and (2) BIC does not call for a specification of the prior on the parameters within each model. Given that the latter is a difficult task and that the prior can be highly influential in non-regular models, this may sound like a good argument. However, it is only apparently so, since the only justification of BIC is purely asymptotic, namely, in such a regime the difficulties linked to the choice of the prior disappear.

This is even more the case for the sBIC criterion, since it is only valid if the parameter space is compact. Then the impact of the prior becomes less of an issue as non informative priors can typically be used. With all due respect, the solution proposed by the authors, namely to use the posterior mean or the posterior mode to allow for non compact parameter spaces, does not seem to make sense in this regard since they depend on the prior. The same comments apply to the author’s discussion on Prior’s matter for sBIC. Indeed variations
of the sBIC could be obtained by penalizing for bigger models via the prior on the weights, for instance as in Rousseau and Mengersen (2011) or by, considering repulsive priors as in Petralia et al (2012), but then it becomes more meaningful to (again) directly compute the marginal likelihood.

Remains (as an argument in its favour) the relative computational ease of use of sBIC, when compared with the marginal likelihood. This simplification is however achieved at the expense of requiring a deeper knowledge on the behaviour of the models and it therefore loses the off-the-shelf appeal of the BIC formula and the range of applications of the method, at least so far.

Although the dependence of the approximation of $\log m(Y_n|M_k)$ on $M_j$, $j \leq k$ is strange, this does not seem crucial, since marginal likelihoods in themselves bring little information and they are only meaningful when compared to other marginal likelihoods. It becomes much more of an issue in the context of a large number of models.

2. Should we care so much about penalized or marginal likelihoods?

Marginal or penalized likelihoods are exploratory tools in a statistical analysis, as one is trying to define a reasonable model to fit the data. An unpleasant feature of these tools is that they provide numbers which in themselves do not have much meaning and can only be used in comparison with others and without any notion of uncertainty attached to them.

A somewhat richer approach of exploratory analysis is to interrogate the posterior distributions by either varying the priors or by varying the loss functions. The former has been proposed in van Havre et al (2015) in mixture models using the prior tempering algorithm. The latter has been used for instance by Yau and Holmes (2013) for segmentation based on Hidden Markov models. Introducing a decision-analytic perspective in the construction of information criteria sounds to us like a reasonable requirement, especially when accounting for the current surge in studies of such aspects.

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