APPROXIMATION OF THE TRAJECTORY ATTRACTOR FOR A 3D MODEL OF INCOMPRESSIBLE TWO-PHASE-FLOWS

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Abstract. In this article we study the relations between the long-time dynamics of the 3D Allen-Cahn-LANS-α model and the exact 3D Allen-Cahn-Navier-Stokes system. Following the idea of [26], we prove that bounded set of solutions of the Allen-Cahn-LANS-α model converge to the trajectory attractor $U_0$ of the 3D Allen-Cahn-Navier-Stokes system as time goes to $+\infty$ and $\alpha$ approaches $0^+$. In particular we show that the trajectory attractors $U_\alpha$ of the 3D Allen-Cahn-LANS-α model converges to the trajectory attractor $U_0$ of the 3D Allen-Cahn-Navier-Stokes as $\alpha$ approaches $0^+$. Let us mention that the strong nonlinearity that results from the coupling of the convective Allen-Cahn system and the LANS-α equations makes the analysis of the problem considered in this article more involved.

1. Introduction. In this article we study relations between the long-time dynamics of the 3D Allen-Cahn-LANS-α (AC-LANS-α) model and the exact 3D Allen-Cahn-Navier-Stokes (AC-NS) system. The AC-LANS-α model is derived from the AC-NS system by substituting the Navier-Stokes system by the LANS-α equations. As the LANS-α model is to the Navier-Stokes system, the AC-LANS-α model can be considered as a regularized approximation of the AC-NS system, depending on a small positive parameter $\alpha > 0$, where in some terms, the unknown velocity function $v$ is replaced by a smoother function $u$ which are related by the elliptic system $v = u - \alpha^2 \Delta u$. For $\alpha = 0$, the model reduces to the exact AC-NS system.

Let us recall that because the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D AC-NS system is not proved yet, the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D AC-NS system. This situation is the same for the 3D Navier-Stokes system. Using regular approximation equations to study the classical 3D Navier-Stokes system has become an effective tool both from the numerical and the theoretical point of views. As noted in [26], it was demonstrated analytically and numerically in many works that the LANS-α model gives a good approximation in the study of many problems related to turbulence flows. In particular, it was found that the explicit steady analytical solution of the LANS-α model compare successfully with empirical and numerical experiment

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data for a wide range of Reynolds numbers in turbulent channel and pipe flows [26].
Let us recall that the inviscid 3D LANS-\( \alpha \) equations was first proposed in [20, 19].
As described in [22], the 3D LANS-\( \alpha \) equations are a systems of partial differential
equations for the mean velocity in which a nonlinear dispersive mechanism filters
the small scales. As such, the 3D LANS-\( \alpha \) equations serve as an appropriate model
for turbulent flows and a suitable approximation of the 3D Navier-Stokes equations
as documented in [8, 10, 9, 11]. A successful comparison with data for time-averaged
quantities for a wide range of Reynolds numbers in turbulent channel and pipe flows
was done in [8, 10]. Further studies of the 3D LANS-\( \alpha \) models in the context of
turbulence modeling appear in [13, 21]. Analytical studies of the global existence,
uniqueness and regularity of solutions to the 3D LANS-\( \alpha \) system are performed in
[15] in the case of periodic boundary conditions. Some existence and uniqueness
results are also established in [22, 6, 7, 5]. In [22], the authors prove the global
well-posedness and regularity of the 3D LANS-\( \alpha \) equations in a bounded domain
with a non-slip boundary condition. A non-autonomous LANS-\( \alpha \) model is consid-
ered in [6], where the authors study the asymptotic behavior of the solutions of a
3D LANS-\( \alpha \) with delay forces. They proved the existence of a pullback and forward
attractors for the model. The stochastic version is also studied in [5].

In [23], the author studied the AC-LANS-\( \alpha \) model in a three-dimensional space.
The governing equations consist of the Allen-Cahn equation for the order (phase)
parameter \( \phi \) coupled with the Lagrange averaged Navier-Stokes-\( \alpha \) (LANS-\( \alpha \)) system
for the velocity \( u \). The asymptotic behavior of the solution to the associated initial
and boundary value problem was analyzed. In particular, it was proved in [23] that
the system generates a strongly continuous semigroup on a suitable phase space,
which possesses a global attractor. Then the existence of an exponential attractor
was established, which entails that the global attractor has finite fractal dimension.

Motivated by the above works, we study in this article the relations between
the long-time dynamics of the 3D AC-LANS-\( \alpha \) model and the exact 3D AC-NS
system. We prove that bounded set of solutions of the coupled AC-LANS-\( \alpha \) model
converge to the trajectory attractor \( \mathcal{U}_0 \) of the 3D AC-NS system as time goes to
\( +\infty \) and \( \alpha \) approaches \( 0^+ \). In particular we show that the trajectory attractors
\( \mathcal{U}_\alpha \) of the 3D AC-LANS-\( \alpha \) model converges to the trajectory attractor \( \mathcal{U}_0 \) of the
3D AC-NS as \( \alpha \) approaches \( 0^+ \) in an appropriate topology. In the case of the 3D
Navier-Stokes system, it is proved in [26] that the trajectory attractor of the 3D
LAN-\( \alpha \) converges to the trajectory attractor of the 3D Navier-Stokes system in an
appropriate topology as \( \alpha \rightarrow 0^+ \). Similar results were obtained in [14] for the 3D
magnetohydrodynamic-\( \alpha \) model. The proof of the main result of this article follows
the same steps as in [26] (see also [14]). Let us mention that in the model considered
in this article, the nonlinearity resulting from the coupling of the convective Allen-
Cahn system and the Navier-Stokes equations is stronger than that of the Navier-
Stokes system or the magnetohydrodynamic system considered in [26, 14].

The rest of article is divided as follows. In the next section, we consider the
3D AC-NS system and we construct its trajectory attractor \( \mathcal{U}_0 \). Most of the results
presented in this section are recalled from [18], see also [1, 2, 3, 4, 12]. In Section
3, we consider the 3D AC-LANS-\( \alpha \) system. We recall from [23] some results on the
corresponding initial value problem and for \( 0 < \alpha \leq 1 \), we construct the trajectory
attractor \( \mathcal{U}_\alpha \) for this system. In Section 4, we study the convergence of the solutions
of the 3D AC-LANS-\( \alpha \) system as \( \alpha \) approaches \( 0^+ \). The main result of this article
appear in Section 5, where we prove that the trajectory attractor \( U_{\alpha} \) converges to \( U_0 \) as \( \alpha \) approaches 0\(^+\) in an appropriate topology.

2. Trajectory attractor of the 3D AC-NS system.

2.1. Governing equations. We consider a model of homogeneous incompressible two-phase flow in a three-dimensional domain with periodic boundary condition. More precisely, we assume that the domain occupied by the fluid is \( M = (0, L)^3 \), where \( L > 0 \) is given. Then, we consider the system

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu_1 \Delta u + (u \cdot \nabla) u + \nabla p &= g - K \text{div}(\nabla \phi \otimes \nabla \phi), \\
\text{div} u &= 0, \\
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu &= 0, \\
\mu &= -\nu_2 \Delta \phi + \epsilon f(\phi),
\end{aligned}
\]

in \( M \times (0, +\infty) \).

In (1), the unknown functions are the velocity \( u = (u_1, u_2, u_3) \) of the fluid, its pressure \( p \) and the order (phase) parameter \( \phi \). The quantity \( \mu \) is the variational derivative of the following free energy functional

\[
F(\phi) = \int_M \left( \frac{\nu_2}{2} |\nabla \phi|^2 + F(\phi) \right) ds,
\]

where, e.g., \( F(r) = \int_0^r f(\zeta)d\zeta \). Here, the constants \( \nu_1 > 0 \) and \( K > 0 \) correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively, \( \nu_2, \epsilon > 0 \) are two physical parameters describing the interaction between the two phases. In particular, \( \nu_2 \) is related with the thickness of the interface separating the two fluids. Hereafter, as in [17] we assume that \( \nu_2 \leq \epsilon \).

For the boundary condition, we assume that \( u(x, t) \) and \( \phi(x, t) \) are periodic in \( x = (x_1, x_2, x_3) \) (with period \( 2\pi L \)) with zero mean, that is

\[
\int_M u \, dx = 0, \quad \int_M \phi \, dx = 0.
\]

The initial condition is given by

\[
(u, \phi)(0) = (u_0, \phi_0) \quad \text{in} \quad M.
\]

2.2. Mathematical setting. As in [17, 16, 18] we assume that potential function \( f \) satisfies

\[
\begin{aligned}
\lim_{|r| \rightarrow +\infty} f'(r) > 0, \\
|f'(r)| &\leq c_f (1 + |r|^2), \quad \forall r \in \mathbb{R},
\end{aligned}
\]

where \( c_f \) is some positive constant. It follows from (5) that

\[
|f(r)| \leq c_f (1 + |r|^3), \quad \forall r \in \mathbb{R}.
\]

Note that from (5), we can find \( \gamma > 0 \) such that

\[
\lim_{|r| \rightarrow +\infty} f'(r) > 2\gamma > 0.
\]

Hereafter, if \( X \) is a real Hilbert space with inner product \((\cdot, \cdot)_X\), we will denote the induced norm by \( |\cdot|_X \), while \( X^* \) will indicate its dual. We set

\[
V_1 = \{ u : u \text{ is a vector valued trigonometrical polynomial defined in } M, \\
\text{div} u = 0, \quad \int_M u \, dx = 0 \}.
\]
We denote by $H_1$ and $V_1$ the closure of $V_1$ in $(L^2(M))^3$ and $(H^1(M))^3$ respectively. The scalar product in $H_1$ is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $| \cdot |_{L^2}$. Moreover, the space $V_1$ is endowed with the scalar product

$$(u, v) = \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \| u \| = ((u, u))^{1/2}. $$

We now define the operator $A_0$ by

$$A_0 u = -\mathcal{P} \Delta u, \quad \forall u \in D(A_0) = (H^2(M))^3 \cap H_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $(L^2(M))^3$ onto $H_1$. Then, $A_0$ is a self-adjoint positive unbounded operator in $H_1$ which is associated with the scalar product defined above. Furthermore, $A_0^{-1}$ is a compact linear operator on $H_1$ and $|A_0|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the $H^2-$norm, i.e., there exists a constant $c_1 > 0$, depending only on $M$, such that

$$\| w \|_{H^2} \leq c_1 |A_0 w|_{L^2}, \quad \forall w \in D(A_0),$$

and so $D(A_0)$ is a Hilbert space with the scalar product

$$(v, w)_{D(A_0)} = (A_0 v, A_0 w).$$

Hereafter, we set $\mathcal{H} = V_1$ with the scalar product $(u, v)_{\mathcal{H}} = (\cdot, \cdot)$, and $\mathcal{U} = D(A_0)$, with the scalar product $((u, v))_{\mathcal{U}} = (A_0 u, A_0 v)$. We also denote by $| \cdot |_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{U}}$ the associated norms defined on $\mathcal{H}$ and $\mathcal{U}$ respectively.

Then, $\mathcal{H}$ and $\mathcal{U}$ are two real and separable Hilbert spaces such that $\mathcal{U} \subset \mathcal{H}$, with the injection being compact and dense. We will identify $\mathcal{H}$ with its topological dual $\mathcal{H}^*$, but considering $\mathcal{U}$ as a subspace of $\mathcal{H}^*$, where we identify $v \in \mathcal{U}$ with the element $f_v \in \mathcal{H}^*$ given by

$$f_v(w) = (v, w)_{\mathcal{H}}, \quad \forall w \in \mathcal{H}.$$ We will denote by $\| \cdot \|_{\mathcal{U}^*}$ the norm of $\mathcal{U}^*$, by $(\cdot, \cdot)$, the duality product between $\mathcal{U}^*$ and $\mathcal{U}$.

Now, we define the operator $A$ by

$$(Au, v) = (A_0 u, v) + \alpha^2 (A_0 u, A_0 v), \quad \forall u, v \in D(A_0).$$

Then, we have (see [7, 6])

$$A \in \mathcal{L}(\mathcal{U}, \mathcal{U}^*),$$

$A$ is self-adjoint, there exists $\alpha_1 > 0$ such that $\langle Au, u \rangle \geq \alpha_1 \| u \|^2_{\mathcal{U}}, \quad \forall u \in \mathcal{U}. \tag{11}$

We note that owing to the properties of $A$, we define

$$(u, v)_{\mathcal{A}} = (Au, v), \quad \forall u, v \in \mathcal{U}.$$ It is clear that $((\cdot, \cdot))_{\mathcal{A}}$ is a scalar product in $\mathcal{U}$ whose associated norm is equivalent to the usual norm $\| \cdot \|_{\mathcal{U}}$. From now on, without loss of generality, we simply set

$$(u, v)_{\mathcal{U}} = (Au, v), \quad \forall u, v \in \mathcal{U} \quad \text{and} \quad \| u \|^2_{\mathcal{U}} = (Au, u).$$

It then follows that

$$\lambda_1 \| u \|^2_{\mathcal{H}} \leq \| u \|^2_{\mathcal{U}}, \quad \forall u \in \mathcal{U}, \tag{12}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $A$.

We also note that

$$(u + \alpha^2 A_0 u, u) = \| u \|^2_{L^2} + \alpha^2 \| \nabla u \|^2_{L^2} = \| u \|^2_{\mathcal{H}}, \quad \forall u \in \mathcal{H}, \tag{13}$$
We denote by $H$, $D$ into $D$. Moreover, the space $V$ endowed with the scalar products whose associated norms are

$$\langle u + \alpha^2 A_0 u, A_0 u \rangle = \|A_0 u\|_H^2 + \alpha^2 \|A_0 u\|_D^2$$

We can easily check that the operators $B_0$, $B_1$ (and their associated trilinear forms $b_0$, $b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A_0) \times D(A_0)$ into $H$, $D(A_0) \times D(A_0)$ into $H_2$, and $H_2 \times D(A_1^{3/2})$ into $H_1$, respectively. More precisely, we set

$$B_0(u,v,w) = \int_M [(u \cdot \nabla)v] \cdotwdx = b_0(u,v,w), \ \forall u,v,w \in D(A_0),$$

$$B_1(u,\phi,\rho) = \int_M [(u \cdot \nabla)\phi] \cdot \rho dx = b_1(u,\phi,\rho), \ \forall u \in D(A_0), \ \phi, \rho \in D(A_1),$$

$$R_0(\mu,\phi) = \int_M \mu [\nabla \phi \cdot \omega] dx = b_1(w,\phi,\mu),$$

$$\forall w \in D(A_0), \ \langle \mu, \phi \rangle \in H_2 \times D(A_1^{3/2}).$$

We denote by $H_2$ and $V_2$ the closure of $V_2$ in $L^2(M)$ and $H^1(M)$ respectively. The scalar product in $H_2$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$ and the associated norm by $| \cdot |_{L^2}$. Moreover, the space $V_2$ is endowed with the scalar product

$$((\phi, \rho)) = \sum_{i=1}^{3} (\partial_{x_i} \phi, \partial_{x_i} \rho)_{L^2}, \ |\phi| = ((\phi, \phi))^{1/2}.$$
Using the notations above, we rewrite (1)-(3) as (see [17] for the details)

\[
\begin{cases}
\frac{du}{dt} + \nu_1 A_0 u + B_0(u, u) - \mathcal{K}R_0(\nu_2 A_1 \phi, \phi) = g, & \text{a.e. in } \mathcal{M} \times (0, +\infty), \\
\mu = \nu_2 A_1 \phi + \epsilon f(\phi), & \text{a.e. in } \mathcal{M} \times (0, +\infty), \\
\frac{d\phi}{dt} + \mu + B_1(u, \phi) = 0, & \text{a.e. in } \mathcal{M} \times (0, +\infty).
\end{cases}
\]

\label{24}

\textbf{Remark 1.} In the weak formulation (24), the term } \mu \nabla \phi \text{ is replaced by } \nu_2 A_1 \phi \nabla \phi. \text{ This is justified since } f'(\phi) \nabla \phi \text{ is the gradient } F'(\phi) \text{ and can be incorporated into the pressure gradient, see [17] for details.}

\textbf{Definition 2.1.} Suppose that } (u_0, \phi_0) \in \mathcal{Y}, \ g \in H_1 \text{ and } T > 0. \text{ A pair } (u, \phi) \text{ is called a weak solution to } (24), (4) \text{ on } [0, T] \text{ if it satisfies } (24), (4) \text{ in the distribution in } D(0, T; D(A_0)^* \times V_2^*) \text{ and}

\[(u, \phi) \in L^\infty([0, T]; \mathcal{Y}) \cap L^2([0, T]; \mathcal{V}), \ \text{ such that } (u, \phi)(t) \text{ makes sense in the space } \mathcal{Y} \text{ and in particular the initial condition } (4) \text{ is meaningful.}

\text{Since a weak solution } (u, \phi) \text{ belongs to } L^\infty([0, T]; \mathcal{Y}), \text{ it follows from the well known Lions-Magenes lemma (see [24]) that } (u, \phi) \in C_{w}([0, T]; \mathcal{Y}), \text{ where } C_{w}([0, T]; \mathcal{Y}) \text{ denotes the space of weakly continuous functions from } [0, T] \to \mathcal{Y}. \text{ Therefore for } \text{every } t \geq 0, \text{ the value } (u, \phi)(t) \text{ makes sense in the space } \mathcal{Y} \text{ and in particular the initial condition } (4) \text{ is meaningful.}

\text{The weak formulation of (24), (4) was proposed and studied in [17, 16], and the existence and uniqueness of solution was proved in the two-dimensional case. In the three-dimensional case, the existence trajectory attractor is proved in [18] when the velocity } u \text{ and the order parameter satisfy the Dirichlet's and Neumann boundary conditions respectively. In this section, we adapt the results of [18] in the case when } u \text{ and } \phi \text{ satisfy a periodic boundary condition.}

\textbf{Remark 2.} As in [17, 16, 18], we can check that if } (u, \phi) \text{ satisfies (24) and (25), then } (u, \phi) \in C_{w}(0, T; \mathcal{Y}). \text{ Therefore, the initial condition (4) holds weakly.}

\text{For } (u, \phi) \in \mathcal{Y}, \text{ we define the functional } \mathcal{L}(u, \phi) \text{ by:}

\[\mathcal{L}(u, \phi) = |(u, \phi)|_{2}^2 + 2\epsilon(F(\phi), 1)_{L^2} + C_c,\]

\text{where } C_c > 0 \text{ is a constant large enough to ensure that } \mathcal{L}(u, \phi) \text{ is nonnegative.}

\text{Note that from the property of } F, \text{ we can find } C_c > 0 \text{ large enough such that}

\[2\epsilon(F(\psi), 1)_{L^2} + C_2 > 0 \ \forall \psi \in V_2. \]

\textbf{Theorem 2.2.} Let } g \in H_1 \text{ and } (u_0, \phi_0) \in \mathcal{Y}. \text{ Then for every } T > 0, \text{ there exists a weak solution } (u, \phi) \text{ to } (24) \text{ from the space } L^\infty(0, T; \mathcal{Y}) \cap L^2(0, T; \mathcal{V}) \text{ such that}

\[(u, \phi)(0) = (u_0, \phi_0) \text{ and satisfies the energy inequality}

\[- \int_0^T \mathcal{L}(u, \phi) \Lambda'(s) ds + \rho_1 \int_0^T \mathcal{L}(u, \phi) \Lambda(s) ds

+ \int_0^T \left( \frac{\nu_0}{\mathcal{K}} |u(s)|^2 + 2|\mu(s)|_{L^2}^2 \right) \Lambda(s) ds \leq \int_0^T \left( \frac{2}{\mathcal{K}} (g(u) + C_1) \right) \Lambda(s) ds, \]

\text{for all } \Lambda \in C_0^\infty(0, T; \mathbb{R}_+), \text{ where } C_1 > 0 \text{ and } \rho_1 > 0 \text{ depends on } \mathcal{M} f, \nu_0 \text{ and } \mathcal{K}.

\textbf{Proof.} The proof is given in details in [18] when } u \text{ and } \phi \text{ satisfy respectively the Dirichlet’s and Neumann boundary condition. When } u \text{ and } \phi \text{ satisfy a periodic boundary condition, the proof is similar and we refer the reader to [18] for the details.} \qed
Proposition 1. For any weak solution \((u, \phi)\) of (24), the following inequalities hold for all \(t \geq 0\):

\[
|\langle u, \phi \rangle(t)|_{Y}^{2} \leq Q(|\langle u, \phi \rangle(0)|_{Y}^{2}) e^{-\rho t} + c \left( |g|_{L_{2}}^{2} + C_{1} \right),
\]

\[
\int_{t}^{t+1} (|u(s)|^{2} + |A_{1}\phi(s)|_{L_{2}}^{2})ds \leq Q(|\langle u, \phi \rangle(0)|_{Y}^{2}) e^{-\rho t} + c \left( |g|_{L_{2}}^{2} + C_{1} \right),
\]

where \(Q\) is a monotone non-decreasing function independent of \(t\) and the initial data, \(C_{1} > 0\) depends on the parameters of the problem such as \(M, f, \nu_{0}, \nu_{1}\) and \(K\).

Proof. Note that (28) is equivalent to

\[
\frac{d}{dt}L(u, \phi) + \rho L(u, \phi) + \frac{\nu_{0}}{K} ||u||^{2} + 2 ||\mu||^{2}_{L_{2}} \leq \frac{2}{K}(g, u) + C_{1}.
\]

We also recall from [17] that

\[
|\langle u, \phi \rangle|_{Y}^{2} \leq L(u, \phi) \leq Q(|\langle u, \phi \rangle|_{Y}^{2}),
\]

for some monotone non-decreasing function \(Q\) independent of time and on the initial data. Then the proof of (29)_1 follows from (30)-(31) and the Gronwall lemma. For (29)_2, we note that from (6) and (24)_2, we have

\[
\int_{t}^{t+1} |A_{1}\phi(s)|^{2}ds \leq c \int_{t}^{t+1} (|\mu(s)|_{L_{2}}^{2} + (1 + |\phi(s)|_{L^{6}}^{6}))ds \leq Q(|\langle u, \phi \rangle(0)|_{Y}^{2}) e^{-\rho t} + c \left( |g|_{L_{2}}^{2} + C_{1} \right),
\]

and therefore (29)_2 follows from (32), (29)_1 and (30). \(\Box\)

2.3. Construction of the trajectory attractor of the 3D AC-NS system.

We now construct the trajectory space for the 3D AC-NS system (24).

Definition 2.3. The trajectory space \(\mathcal{K}^{+}\) is the set of all Leray-Hopf weak solution \((u, \phi)\) of (24) in the space \(L_{loc}^{2}(\mathbb{R}^{+}; Y) \cap L_{loc}^{\infty}(\mathbb{R}^{+}; Y)\) that satisfies the energy inequality (28).

It follows from Theorem 1 that the trajectory space \(\mathcal{K}^{+}\) is nonempty. More precisely, for any \((u_{0}, \phi_{0}) \in Y\), there exists a trajectory \((u, \phi)\) such that \((u, \phi)(0) = (u_{0}, \phi_{0})\).

Let us define the spaces \(\mathcal{F}_{loc}^{+}, \mathcal{F}_{b}^{+}\) and the topology \(\Theta_{loc}^{+}\). We set

\[
\mathcal{F}_{loc}^{+} = \{ (u, \phi) \in L_{loc}^{2}(\mathbb{R}^{+}; Y) \cap L_{loc}^{\infty}(\mathbb{R}^{+}; Y), \ (\partial_{t}u, \partial_{t}\phi) \in L_{loc}^{2}(\mathbb{R}^{+}; D(A_{0})^{*} \times V_{2}^{*}) \}.
\]

In the space \(\mathcal{F}_{loc}^{+}\), we define the following local weak convergence topology:

By definition, a sequence \(\{(u_{n}, \phi_{n})\} \subset \mathcal{F}_{loc}^{+}\) converges to \((u, \phi) \in \mathcal{F}_{loc}^{+}\) in the topology \(\Theta_{loc}^{+}\) as \(n \to \infty\) if for each \(T > 0\), the following limit relations hold:

\[
(u_{n}, \phi_{n}) \rightharpoonup (u, \phi) \text{ weakly star in } L^{\infty}(0, T; Y),
\]

\[
(u_{n}, \phi_{n}) \rightharpoonup (u, \phi) \text{ weakly in } L^{2}(0, T; Y),
\]

\[
(\partial_{t}u_{n}, \partial_{t}\phi_{n}) \rightharpoonup (\partial_{t}u, \partial_{t}\phi) \text{ weakly in } L^{2}(0, T; D(A_{0})^{*} \times V_{2}^{*}).
\]

Remark 3. The space \(\mathcal{F}_{loc}^{+}\) equipped with the topology \(\Theta_{loc}^{+}\) is a Hausdorff Frechet-Uryshon topological vector space with countable base [26].
We consider the linear subspace $\mathcal{F}_b^+ \subset \mathcal{F}_{loc}^+$ consisting of vector function $(u, \phi) \in \mathcal{F}_{loc}^+$ with finite norm
\[
\|u, \phi\|_{\mathcal{F}_b^+} = \|u, \phi\|_{L^2_0(\mathbb{R}^+, \mathcal{V})} + \|u, \phi\|_{L^\infty(\mathbb{R}^+, \mathcal{Y})} + \|\partial_t u, \partial_t \phi\|_{L^2_0(\mathbb{R}^+, \mathcal{V}^*_b)}.
\]  
where
\[
\|u, \phi\|_{L^2_0(\mathbb{R}^+, \mathcal{V})} = \sup_{t \geq 0} \int_0^{t+1} \left(\|u(s)\|^2 + |A_1 \phi(s)|^2_{L^2}\right)ds,
\]
\[
\|u, \phi\|_{L^\infty(\mathbb{R}^+, \mathcal{Y})} = \text{ess sup}_{t \geq 0} \left(\|u(t)\|^2_{L^2} + \|\phi(t)\|^2\right),
\]
\[
\|\partial_t u, \partial_t \phi\|_{L^2_0(\mathbb{R}^+, \mathcal{V}^*_b)} = \sup_{t \geq 0} \int_0^{t+1} \left(\|\partial_t u(s)\|^2_{D(A^*_0)} + \|\partial_t \phi(s)\|^2_{L^2}\right)ds.
\]

Remark 4. Any ball of $\mathcal{B}_r = \{(u, \phi) \in \mathcal{F}_b^+, \|u, \phi\|_{\mathcal{F}_b^+} \leq r\}$ in the space $\mathcal{F}_b^+$ is compact in the topology $\Theta_{loc}^+$. Moreover, the corresponding topological subspace $\mathcal{B}_r|_{\Theta_{loc}^+}$ is metrisable although the space $\mathcal{F}_{loc}^+|_{\Theta_{loc}^+}$ is not metrisable, [26, 25].

We consider the translation semigroup $\{T(h)\} \equiv \{T(h), h \geq 0\}$ acting on $\Theta_{loc}^+$ by the formula
\[
T(h)(u(t), \phi(t)) = (u(t + h), \phi(t + h)), t \geq 0.
\]
The semigroup $\{T(h)\}$ maps $\mathcal{K}^+$ to itself, that is $T(h) : \mathcal{K}^+ \rightarrow \mathcal{K}^+$ for all $h \geq 0$. We now construct the global attractor for the semigroup $\{T(h)\}$. This attractor will be referred to as the trajectory attractor for the AC-NS system (24). We have the following results.

Proposition 2. Let $g \in H_1$. Then
(i) The space $\mathcal{K}^+ \subset \mathcal{F}_b^+$,
(ii) For any $(u, \phi) \in \mathcal{K}^+$,
\[
\|T(h)(u, \phi)\|_{\mathcal{F}_b^+} \leq Q(\|u, \phi\|_{L^\infty(0, 1; \mathcal{Y})})e^{-\rho_1 h} + R_1^2,
\]
where hereafter $Q$ stands for a monotone, positive non-decreasing function that is independent of time and $(u, \phi)$, and $R_1$ denotes a positive constant depending on $\mathcal{M}, \nu_0, \nu_1, \mathcal{K}$, and $|g|_{L^2}$.

Proposition 3. If $(u, \phi) \in \mathcal{K}^+$, then
\[
\left(\int_0^{t+1} \|\partial_t u, \partial_t \phi\|_{D(A^*_0)}^2 \partial_t \mathcal{V}_2^* ds\right)^{1/2} \leq Q(\|u(0), \phi(0)\|_{\mathcal{K}^+})e^{-\rho_1 t} + R_1^2.
\]

Proof. Note that from (24), we have
\[
\left(\int_0^{t+1} \|\partial_t u\|_{D(A^*_0)}^2 ds\right)^{1/2} \leq \nu_0 \left(\int_0^{t+1} \|A_0 u\|_{D(A^*_0)}^2 ds\right)^{1/2} + |g|_{D(A^*_0)} + \mathcal{K} \left(\int_0^{t+1} \|R_0(\phi, A_1 \phi)\|_{D(A^*_0)}^2 ds\right)^{1/2}.
\]
But from (21) and (29) we derive that
\[
\left(\int_0^{t+1} \|A_0 u(s)\|_{D(A^*_0)}^2 ds\right) \leq Q(\|u(0), \phi(0)\|_{\mathcal{K}^+})e^{-\rho_1 t} + R_1^2,
\]
\[
\left(\int_0^{t+1} \|B_0(u(s), u(s))\|_{D(A^*_0)}^2 ds\right)^{1/2} \leq Q(\|u(0), \phi(0)\|_{\mathcal{K}^+})e^{-\rho_1 t} + R_1^2.
\]
Proposition 7. The proof of the inclusion

From (24)

Therefore (39) follows from (40)-(44).

But from (21), (29) we have

\[ \left( \int_t^{t+1} \| \partial_t \phi(s) \|_{V_x}^2 \, ds \right)^{1/2} \leq \nu_2 \left( \int_t^{t+1} \| A_1 \phi(s) \|_{V_x}^2 \, ds \right)^{1/2} 
+ \left( \int_t^{t+1} \| B_1 (u(s), \phi(s)) \|_{V_x}^2 \, ds \right)^{1/2} + \left( \int_t^{t+1} \| f(\phi(s)) \|_{V_x}^2 \, ds \right)^{1/2}. \]

(43)

Therefore (39) follows from (40)-(44).

Proof of Proposition 6. The proof of (ii) follows from Proposition 2 and Proposition 7. The proof of the inclusion \( K^+ \subset F^+ \) follows from (i) by setting \( h = 0 \).

Proposition 4. The space \( K^+ \) is closed in \( \Theta^+_{loc} \).

Proof. We consider a sequence \( (u_n, \phi_n) \in K^+ \) which converges as \( n \to \infty \) in \( \Theta^+_{loc} \) to an element \( (u, \phi) \in F^+_{loc} \). Let us prove that \( (u, \phi) \in K^+ \). By definition of the topology \( \Theta^+_{loc} \), for every \( T > 0 \), we have

\[ (u_n, \phi_n) \to (u, \phi) \text{ weakly star in } L^\infty(0, T; Y) \text{ and weakly in } L^2(0, T; V), \]

and

\[ (\partial_t u_n, \partial_t \phi_n) \to (\partial_t u, \partial_t \phi) \text{ weakly in } L^2(0, T; D(A_0)^* \times V_x^*). \]

In particular, \( (u_n, \phi_n) \) is bounded in \( L^\infty(0, T; Y) \) and in \( L^2(0, T; V) \), \( (\partial_t u_n, \partial_t \phi_n) \) is bounded in \( L^2(0, T; D(A_0)^* \times V_x^*) \). Therefore, \( B_0(u_n, u_n), R_0(\phi_n, A_1 \phi_n) \) are bounded in \( L^{4/3}(0, T; V_1^*) \), \( B_1(u_n, \phi_n) \) is bounded in \( L^{4/3}(0, T; V_2^*) \). Then passing to a subsequence, still denoted \( (u_n, \phi_n) \), we can assume that

\[ B_0(u_n, u_n) \to b_0(x, t) \text{ weakly in } L^{4/3}(0, T; V_1^*), \]

\[ R_0(\phi_n, A_1 \phi_n) \to r_0(x, t) \text{ weakly in } L^{4/3}(0, T; V_1^*), \]

\[ B_1(u_n, \phi_n) \to b_1(x, t) \text{ weakly in } L^{4/3}(0, T; V_2^*), \]

\[ f(\phi_n) \to f_1(x, t) \text{ weakly in } L^{4/3}(0, T; V_2^*), \]

(45)

where \( b_0, r_0 \in L^{4/3}(0, T; V_1^*) \) and \( b_1, f_1 \) are elements of \( L^{4/3}(0, T; V_2^*) \). We also note that \( (u_n, \phi_n) \) satisfies

\[
\begin{cases}
\frac{du_n}{dt} + \nu_1 A_0 u_n + B_0(u_n, u_n) - KR_0(\phi_n, A_1 \phi_n) = g, \\
\frac{d\phi_n}{dt} + \nu_2 A_1 \phi_n + B_1(u_n, \phi_n) + f(\phi_n) = 0.
\end{cases}
\]

(46)
Using (45)-(46), it follows that \((u, \phi)\) satisfies
\[
\begin{cases}
\frac{du}{dt} + \nu_1 A_0 u + b_0(x, t) + r_0(x, t) = g, \\
\frac{d\phi}{dt} - \nu_2 A_1 \phi + b_1(x, t) + f_1(x, t) = 0,
\end{cases}
\tag{47}
\]
in the distribution sense.

From the Aubin compactness theorem, we also have
\[
(u_n, \phi_n) \to (u, \phi) \text{ strongly in } L^2(0, T; \mathcal{Y}).
\tag{48}
\]
Passing to the limit gives
\[
(u_n, \phi_n) \to (u, \phi) \text{ for a.e. } (x, t) \in \mathcal{M} \times (0, T).
\tag{49}
\]
It follows that
\[
\begin{align*}
B_0(u_n, u_n) & \to B_0(u, u) \text{ weakly in } L^{4/3}(0, T; V_1^*), \\
R_0(\phi_n, A_1 \phi_n) & \to R_0(\phi, A_1 \phi) \text{ weakly in } L^{4/3}(0, T; V_1^*), \\
B_1(u, \phi_n) & \to B_1(u, \phi) \text{ weakly in } L^{4/3}(0, T; V_2^*), \\
f(\phi_n) & \to f(\phi) \text{ weakly in } L^{4/3}(0, T; V_2^*).
\end{align*}
\tag{50}
\]
We conclude that \(b_0(x, t) = B_0(u, u), r_0(x, t) = R_0(\phi, A_1 \phi), b_1(x, t) = B_1(u, \phi)\) and \(f_1(x, t) = f(\phi)\), which proves that \((u, \phi)\) is a weak solution to (24). It remains to prove that \((u, \phi)\) satisfies the energy inequality (28). We proceed as in [18]. We recall that \((u_n, \phi_n)\) satisfies the energy inequality
\[
-\int_0^T \mathcal{L}(u_n, \phi_n) \Lambda'(s) \, ds + \rho_1 \int_0^T \mathcal{L}(u_n, \phi_n) \Lambda(s) \, ds \\
+ \int_0^T \left( \frac{\nu_0}{K} \|u_n(s)\|^2 + 2 |\mu_n(s)|^2 \right) \Lambda(s) \, ds \leq \int_0^T \left( \frac{2}{K} (g, u_n) + C_1 \right) \Lambda(s) \, ds,
\tag{51}
\]
for all \(\Lambda \in C_0^\infty(0, T; \mathbb{R}_+)\). From (48) and the Lebesgue dominant convergence theorem, we have
\[
\int_0^T \mathcal{L}(u_n, \phi_n) \Lambda'(s) \, ds \to \int_0^T \mathcal{L}(u, \phi) \Lambda'(s) \, ds.
\tag{52}
\]
Moreover, we have
\[
u_n \Lambda_n^{1/2} \to u \Lambda^{1/2} \text{ weakly in } L^2(0, T; V_1), \\
\mu_n \Lambda_n^{1/2} \to \mu \Lambda^{1/2} \text{ weakly in } L^2(0, T; H_2).
\tag{53}
\]
Hence we deduce that
\[
\int_0^T \|u(s)\|^2 \Lambda(s) \, ds \leq \liminf_{n \to \infty} \int_0^T \|u_n(s)\|^2 \Lambda(s) \, ds,
\int_0^T |\mu(s)|^2 \Lambda(s) \, ds \leq \liminf_{n \to \infty} \int_0^T |\mu_n(s)|^2 \Lambda(s) \, ds
\tag{54}
\]
Therefore we can pass to limit in (51) thanks to (52)-(54) to prove that \((u, \phi)\) satisfies the energy inequality (28). The proof is finished.

We have defined the trajectory space \(\mathcal{K}^+\) of (24) on \(\mathbb{R}_+\). We now extend the definition to \(\mathbb{R}\). The kernel \(\mathcal{K}_0\) of (24) is the set of all weak solution \((u, \phi)(t), t \in \mathbb{R}\) bounded in the space
\[
\mathcal{F}_b = \{(u, \phi) \in L^2_b(\mathbb{R}; V) \cap L^\infty(\mathbb{R}; \mathcal{Y}), (\partial_t u, \partial_t \phi) \in L^2_b(\mathbb{R}; D(A_0)^* \times V_2^*)\}.
\tag{55}
\]

that satisfies the following inequality
\[
- \int_{-\infty}^{+\infty} \mathcal{L}(u, \phi) \Lambda(s) ds + \rho_1 \int_{-\infty}^{+\infty} \left( \rho_1 \mathcal{L}(u, \phi) + \frac{\nu_0}{\text{\textit{K}}} ||u_n(s)||^2 \right) \Lambda(s) ds
+ 2 \int_{-\infty}^{+\infty} |\mu(s)|^2 \Lambda(s) ds \leq \int_{-\infty}^{+\infty} \left( \frac{2}{\text{\textit{K}}} (g, u) + C_1 \right) \Lambda(s) ds,
\]
for all \( \Lambda \in C_0^\infty (\mathbb{R} ; \mathbb{R}_+) \).

The norm in \( \mathcal{F}_b \) is defined in a similar way as the norm in \( \mathcal{F}_b^+ \) replacing \( \mathbb{R}_+ \) by \( \mathbb{R} \).

The same definition also holds for \( \mathcal{F}_{\text{loc}} \) with the topology \( \Theta_{\text{loc}}^- \), where the interval \([0, T]\) is replaced by \([-T, T]\). We denote by \( \Pi_+ \) the restriction operator onto \( \mathbb{R}_+ \). This operator takes a function \( \{\phi(t), t \in \mathbb{R}\} \) to the function \( \{\Pi_+ \phi(t), t \geq 0\} \), where \( \Pi_+ \phi(t) = \phi(t), \forall t \geq 0 \).

Let us now study the translation semigroup \( \{T(h)\} \) acting on the trajectory space \( K^+ \). We first recall some definitions from [26, 25].

**Definition 2.4.** A set \( P \subset K^+ \) is said to be absorbing for the semigroup \( \{T(h)\} \) if for every bounded set \( D \subset K^+ \), there exists \( h_1 = h_1(D) \) such that \( T(h)D \subset P, \forall h \geq h_1 \).

**Definition 2.5.** As set \( P \subset K^+ \) is said to be attracting for the semigroup \( \{T(h)\} \) if any neighborhood \( O(P) \) of the set \( P \) in the topology \( \Theta_{\text{loc}}^+ \) is an absorbing set for \( \{T(h)\} \), i.e., for every bounded set \( D \subset K^+ \) in \( \mathcal{F}_b^+ \), there exists \( h_1 = h_1(D, O) \geq 0 \) such that \( T(h)D \subset O(P), \forall h \geq h_1 \).

**Definition 2.6.** A set \( U \subset K^+ \) is called a trajectory attractor for the semigroup \( \{T(h)\} \) on \( K^+ \) if \( U \) is bounded in \( \mathcal{F}_b^+ \), compact with respect to \( \Theta_{\text{loc}}^+ \), strictly invariant with respect to \( \{T(h)\} \), i.e., \( T(h)U = U, \forall h \geq 0 \), and \( U \) is attracting set for \( \{T(h)\} \).

The existence of the trajectory attractor for the AC-NS system follows from the following theorem.

**Theorem 2.7.** Let \( g \in H_1 \), then the translation semigroup \( \{T(h)\} \) acting on \( K^+ \) has a trajectory attractor \( U_0 \). The set \( U_0 \) is bounded in \( \mathcal{F}_b^+ \) and compact in \( \Theta_{\text{loc}}^+ \).

Moreover
\[
U_0 = \Pi_+ K_0,
\]
the set \( K_0 \) is bounded in \( \mathcal{F}_b \) and compact in \( \Theta_{\text{loc}}^- \).

**Proof.** This is clear from the above results, see [18].

\[ \square \]

3. The 3D AC-LANS-\( \alpha \) model and its mathematical setting.

3.1. Governing equations. We recall that the domain \( \mathcal{M} \) of the fluid is given by \( \mathcal{M} = (0, L)^3 \), where \( L > 0 \) is given. The state of the system is described by a pair \((u, \phi)\), where \( u = (u_1, u_2, u_3) \) is the velocity field of the fluid and \( \phi \) is the order parameter. The system of equations for \((u, \phi)\) reads:
\[
\begin{align*}
\frac{\partial v}{\partial t} - \nu_0 \Delta v + (u \cdot \nabla) v + \sum_{j=1}^{3} v_j \nabla u_j + \nabla p &= g - \text{\textit{K}} \text{div}(\nabla \phi \otimes \nabla \phi), \\
v &= u - \alpha^2 \Delta u, \\
\text{div} u &= \text{div} v = 0, \\
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi + \mu &= 0, \quad \mu = -\nu_2 \Delta \phi + \epsilon f(\phi), \\
\int_{\mathcal{M}} u(x, t) dx &= 0, \quad \int_{\mathcal{M}} v(x, t) dx = 0.
\end{align*}
\]
Let us first note that (58) is obtained by coupling the well-known Allen-Cahn equations

$$\frac{\partial \phi}{\partial t} = - (\nu_2 \Delta \phi + \epsilon f(\phi))$$  \hspace{1cm} (59)

with the 3D LANS-$\alpha$ system through convection and order parameter.

Note that for $\alpha = 0$, the model (58) reduces to the 3D AC-NS (1). Let us recall from [15] that the positive constant $\alpha$ represents the square of the spacial scale at which the fluid motion is filtered.

For the boundary condition, we assume that $u(x, t)$ and $\phi(x, t)$ are periodic in $x$ (with period $2\pi L$) with zero means.

The initial condition is given by

$$(u, \phi)(0) = (u_0, \phi_0) \text{ in } M.$$ \hspace{1cm} (60)

3.2. Mathematical setting. Following the notations of the AC-NS system (1), we define the bilinear operator $\tilde{B}_0$ by:

$$\tilde{B}_0(u, v) = - \mathcal{P} ((\nabla \times v) \times u), \ \forall u, v \in V_1.$$ \hspace{1cm} (61)

Then we have (see [26])

$$(\tilde{B}_0(u, v), w) = - (\tilde{B}_0(w, v), u), \ \forall u, v, w \in V_1.$$ \hspace{1cm} (62)

We also note that

$$\tilde{B}_0(u, u) = B_0(u, u), \ \forall u \in V_1.$$ \hspace{1cm} (63)

We also have the following properties of the bilinear operator $\tilde{B}_0$, see [26] for details.

The operator $\tilde{B}_0$ maps $V_1 \times V_1$ to $V_1^*$ and the following inequalities hold:

$$|(\tilde{B}_0(u, v), w)| \leq c|u|^{1/4}|v|^{3/4}|w|^{1/4}||v||^{3/4}, \ \forall u, v, w \in V_1,$$

$$|(\tilde{B}_0(u, v), w)| \leq c|u|^{1/2}|v|^{1/2}||v||^{3/4}, \ \forall u, v, w \in V_1,$$

$$\tilde{B}_0(u, v, w) = - (\tilde{B}_0(w, v), u), \ \forall u, v, w \in V_1,$$

$$\tilde{B}_0(u, v, u) = 0, \ \forall u, v \in V_1,$$

$$|(\tilde{B}_0(u, v), w)| \leq c|\nu||v|L^2 |A_0 w||L^2, \ \forall u \in V_1, v \in H_1, w \in D(A_0).$$ \hspace{1cm} (64)

It follows that $\tilde{B}_0$ maps $V_1 \times H_1$ into $D(A_0)^*$ and

$$\|\tilde{B}_0(u, v)\|_{D(A_0)^*} \leq c\|u\|_{L^2} \|v\|_{L^2}, \ \forall u \in V_1, v \in H_1.$$ \hspace{1cm} (65)

Using the notations above, we rewrite (58),(60) as (see [23] for the details)

$$\begin{cases}
\frac{dv}{dt} + \nu_1 A_0 v + \tilde{B}_0(u, v) - \kappa R_0 (\nu_2 A_1 \phi, \phi) = g, \ v = u + \alpha^2 A_0 u,
\mu = \nu_2 A_1 \phi + \epsilon f(\phi),
\frac{d\phi}{dt} + \mu + B_1(u, \phi) = 0.
\end{cases}$$ \hspace{1cm} (66)

Let $(u, \phi) \in L^\infty(0, T; V^*) \cap L^2(0, T; V)$. Then $v = u + \alpha^2 A_0 u \in L^\infty(0, T; V_1^*) \cap L^2(0, T; D(A_0)^*)$ and $A_0 v \in L^2(0, T; D(A_0)^*)$. We also have $\tilde{B}_0(u, v) \in L^2(0, T; D(A_0)^*)$ and $R_0 (A_1 \phi, \phi) \in L^2(0, T; D(A_0)^*)$. Moreover, we can also check that $B_1(u, \phi) \in L^2(0, T; V_2^*)$ and $\mu \in L^2(0, T; V_2^*)$. It follows that the system (66) is meaningful in the distribution space $\mathcal{D}(0, T; D(A_0)^* \times V_2^*)$. We have the following notion of solutions.
Definition 3.1. Suppose that \((u_0, \phi_0) \in V_1 \times V_2\), \(g \in H_1\) and \(T > 0\). A pair \((u, \phi)\) is called a weak solution to (66), (60) on \([0, T]\) if \((u, \phi) \in C([0, T]; V_1 \times V_2) \cap L^2([0, T]; D(A_0) \times (H^2(M) \times H_2))\), \((\partial_t u, \partial_t \phi) \in L^2([0, T]; H_1 \times V^*_2)\), and satisfies (66), (60) in the sense of distribution in \(D'(0, T; D(A_0)^* \times V^*_2)\).

3.3. The Cauchy problem for the 3D AC-LANS-\(\alpha\) model.

Theorem 3.2. Let \(g \in H_1\) and \((u_0, \phi_0) \in V_1 \times V_2\). For every \(T > 0\), the Cauchy problem (66), (60) has a unique weak solution \((u, \phi)\) in the sense of Definition 15. Moreover, \((u, \phi)\) satisfies

\[
\begin{align*}
|u(t)|^2_{L^2} + \alpha^2||u(t)||^2 + ||\phi(t)||^2 \\
\leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1,
\end{align*}
\]

\[
\int_t^{t+1} \left( ||u(s)||^2 + \alpha^2 |A_0 u(t)||^2_{L^2} + ||\mu(s)||^2_{L^2} \right) ds \\
\leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1,
\]

\[
\left( \int_t^{t+1} \left( ||\partial_t u(s)||^2_{D(A_0)^*} + ||\partial_t \phi(s)||^2_{V^*_2} \right) ds \right)^{1/2} \\
\leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1,
\]

where \(Q\) is a monotone nondecreasing function independent of time and the initial data and \(R_1 > 0\) is a constant depending on \(g\) and some physical parameters of the problem.

Proof. The proof of (67)\(_1\) – (67)\(_2\) is given in [23]. For (67)\(_3\), we note that

\[
\left( \int_t^{t+1} ||\partial_t \phi(s)||^2_{V^*_2} ds \right)^{1/2} \leq \left( \int_t^{t+1} (||\mu(s)||^2_{L^2} ds) \right)^{1/2} + \left( \int_t^{t+1} ||B_1(u, \phi)||^2_{V^*_2} ds \right)^{1/2}.
\]

It follows from (66) and (67)\(_1\) – (67)\(_2\) that

\[
\int_t^{t+1} ||\partial_t \phi(s)||^2_{V^*_2} ds \leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1,
\]

\[
\left( \int_t^{t+1} ||B_1(u, \phi)||^2_{V^*_2} ds \right)^{1/2} \leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1.
\]

We also have

\[
\left( \int_t^{t+1} ||\partial_t v(s)||^2_{D(A_0)^*} ds \right)^{1/2} \leq \nu_0 \left( \int_t^{t+1} ||A_0 v||^2_{D(A_0)^*} ds \right)^{1/2} + \left( \int_t^{t+1} ||B_0(u, v)||^2_{D(A_0)^*} ds \right)^{1/2} + K \left( \int_t^{t+1} ||R_0(\phi, A_1 \phi)||^2_{D(A_0)^*} ds \right)^{1/2} + |g|_{L^2}
\]

It follows from (66), (67)\(_1\) – (67)\(_2\) that

\[
\int_t^{t+1} ||A_0 v||^2_{D(A_0)^*} \leq Q(|u(0)|^2_{L^2} + \alpha^2||u(0)||^2 + ||\phi(0)||^2)e^{-\rho_1 t} + R_1.
\]
Proposition 5. Let
\[ \left( \int_t^{t+1} \| \mathcal{B}_0(u, v) \|^2_{D(A_0)} \, ds \right)^{1/2} \leq c \left( \int_t^{t+1} \left( |u|^2 + \alpha^2 |u|^2 \right) \left( |u|^2 + \alpha^2 |A_0 u|^2 \right) \, ds \right)^{1/2} \leq Q(|u(0)|^2_{L^2} + \alpha^2 |u(0)|^2 + \| \phi(0) \|^2) e^{-\rho_1 t} + R_1^2, \] 
(72)

and
\[ \left( \int_t^{t+1} \| R_0(\phi, A_1 \phi) \|^2_{D(A_0)} \, ds \right)^{1/2} \leq c \left( \int_t^{t+1} \| A_1 \phi \|^2_{L^2} \, ds \right)^{1/2} \leq Q(|u(0)|^2_{L^2} + \alpha^2 |u(0)|^2 + \| \phi(0) \|^2) e^{-\rho_1 t} + R_1^2, \] 
(73)

It follows from (68)-(73) that (67)_3 holds true.

3.4. Existence of the trajectory attractors of the 3D AC-LANS-\(\alpha\) model.

We set \( w = (1 + \alpha^2 A_0)^{1/2} u \). Then \((w, \phi)\) satisfies
\[
\begin{cases}
\frac{d w}{dt} + \nu_1 A w + (1 + \alpha^2 A_0)^{-1/2} \mathcal{B}_0((1 + \alpha^2 A_0)^{-1/2} w, (1 + \alpha^2 A_0)^{-1/2} w) \\
- \mathcal{K}(1 + \alpha^2 A_0)^{-1/2} \mathcal{R}_0(\nu_2 A_1 \phi, \phi) = (1 + \alpha^2 A_0)^{-1/2} g, \\
\mu = \nu_2 A_1 \phi + \epsilon f(\phi), \\
\frac{d \phi}{dt} + \mu + B_1 ((1 + \alpha^2 A_0)^{-1/2} w, \phi) = 0.
\end{cases}
\]
(74)

Corollary 1. The solution \((w, \phi)\) of (74), (60) satisfies
\[
\begin{align*}
&|w(t)|^2_{L^2} + \| \phi(t) \|^2 \leq Q_1(|w(0)|^2_{L^2} + \| \phi(0) \|^2) e^{-\rho_1 t} + R_1^2, \\
&\int_t^{t+1} \left( |w(s)|^2 + \| \mu(s) \|^2_{L^2} \right) ds \leq Q_1(|w(w)|^2_{L^2} + \| \phi(0) \|^2) e^{-\rho_1 t} + R_1^2, \\
&\left( \int_t^{t+1} \left( |\partial_t w(s)|^2_{D(A_0)^*} + |\partial_t \phi(s)|^2_{V^*_2} \right) ds \right)^{1/2} \leq Q_1(|w(0)|^2_{L^2} + \| \phi(0) \|^2) e^{-\rho_1 t} + R_1^2,
\end{align*}
\] 
(75)

where \( Q_1 \) is a monotone nondecreasing function independent of time and the initial data and \( R_1 > 0 \) is a constant depending on \( g \) and some physical parameters of the problem.

Proof. Note that
\[ w = (1 + \alpha^2 A_0)^{1/2} u, \quad v = (1 + \alpha^2 A_0) u = (1 + \alpha^2 A_0)^{1/2} w, \] 
(76)
implies that
\[
\begin{align*}
|w|^2_{L^2} &= |u|^2_{L^2} + \alpha^2 |u|^2, \\
|w|^2 &= |u|^2 + \alpha^2 |A_0 u|^2, \\
|\partial_t w|_{D(A_0)^*} &\leq |\partial_t v|_{D(A_0)^*},
\end{align*}
\] 
(77)
and therefore (75) follows from (67),(77).

We consider the Banach space \( \mathcal{F}_b^+ \) defined by
\[
\mathcal{F}_b^+ = \left\{ (w, \phi) \in L^2_0(\mathbb{R}^+; \mathcal{Y}) \cap L^\infty(\mathbb{R}^+; \mathcal{Y}), \quad (\partial_t w, \partial_t \phi) \in L^2_0(\mathbb{R}^+; D(A_0)^* \times V^*_2) \right\}
\] 
(78)

Proposition 5. Let \( g \in H_1 \), for any solution \((u, \phi)\) of (66), the corresponding couple \((w, \phi)\) satisfies
\[
\| T(h)(w, \phi) \|_{\mathcal{F}_b^+} \leq Q_1(|w(0)|^2_{L^2} + \| \phi(0) \|^2) e^{-\rho_1 h} + R_1^2.
\] 
(79)

Proof. It follows from (75).
Definition 3.3. The trajectory space $\mathcal{K}_+^\alpha$ is the union of all couple $(w(t), \phi(t))$, where $(u, \phi)$ is a solution to (66) with an arbitrary initial data $(u_0, \phi_0) \in V_1 \times V_2$.

From Theorem 14, it follows that the trajectory space $\mathcal{K}_+^\alpha \neq \emptyset$. Moreover, Proposition 16 implies that $\mathcal{K}_+^\alpha \subset \mathcal{F}_b^+$, for all $\alpha > 0$.

Remark 6. Let us now construct the trajectory attractor for the 3D AC-LANS-$\alpha$ model (74).

Proof. Similar to that of Proposition 8.

Let $\mathcal{K}_+^\alpha$ be the kernel of the system (66). The set $\mathcal{K}_+^\alpha$ is compact in $\Theta_{loc}^+$. We consider the topology $\Theta_{loc}^+$ on $\mathcal{K}_+^\alpha$. We prove that the space $\mathcal{K}_+^\alpha$ is closed in $\Theta_{loc}^+$.

Proposition 6. The space $\mathcal{K}_+^\alpha$ is closed in $\Theta_{loc}^+$.

Proof. We have $T(t)K^\alpha_+ \subset K^\alpha_+$ for all $t \geq 0$. The set $\mathcal{P}_1 = \{(w, \phi) \in \mathcal{F}_b^+, \|(w, \phi)\|_{\mathcal{F}_b^+} \leq 2R_1\}$ is an absorbing set for $\mathcal{K}_+^\alpha$ (see Proposition 16). The ball $\mathcal{P}_1$ is compact in $\Theta_{loc}^+$ and bounded in $\mathcal{F}_b^+$. Moreover, $\mathcal{P}_1$ does not depend on $\alpha$ since the function $Q_1$ and the constant $R_1$ are independent of $\alpha$. It follows from general result given in [26, 25] that there exists a trajectory attractor $\mathcal{U}_\alpha \subset \mathcal{K}_+^\alpha$ such that $\mathcal{U}_\alpha$ is bounded in $\mathcal{F}_b^+$ and compact in $\Theta_{loc}^+$.

Remark 6. Since $\mathcal{U}_\alpha \subset \mathcal{P}_1$, then the trajectory attractor $\mathcal{U}_\alpha$ are uniformly (w.r.t. $\alpha \in (0, 1]$) bounded in $\mathcal{F}_b^+$, i.e.,

$$\|U_\alpha\|_{\mathcal{F}_b^+} \leq R, \ \forall \alpha \in (0, 1],$$

for some $R > 0$ independent of $\alpha \in (0, 1]$.

4. Convergence of the solutions of the 3D AC-LANS-$\alpha$ model. We formulate and prove the main result of this section concerning the behavior of the solution of the 3D AC-LANS-$\alpha$ model when $\alpha$ approaches 0$^+$.

Theorem 4.1. Let a sequence $(w_n, \phi_n) \subset \mathcal{K}_+^\alpha$ be given such that

1. $(w_n, \phi_n) \in \mathbb{N}$ is bounded in $\mathcal{F}_b^+$,
2. $\alpha_n \to 0^+$ as $n \to \infty$,
3. $(w_n, \phi_n) \to (w, \phi)$ in $\Theta_{loc}^+$ as $n \to \infty$. 

Proof. We consider the topology $\Theta_{loc}^+$ on $\mathcal{K}_+^\alpha$. We prove that the space $\mathcal{K}_+^\alpha$ is closed in $\Theta_{loc}^+$. We also consider the topological space $\Theta_{loc}^+$ introduced in Section 2. Let us recall that $\mathcal{F}_b^+ \subset \Theta_{loc}^+$. We consider the topology $\Theta_{loc}^+$ on $\mathcal{K}_+^\alpha$. We prove that the space $\mathcal{K}_+^\alpha$ is closed in $\Theta_{loc}^+$. 

Remark 6. Since $\mathcal{U}_\alpha \subset \mathcal{P}_1$, then the trajectory attractor $\mathcal{U}_\alpha$ are uniformly (w.r.t. $\alpha \in (0, 1]$) bounded in $\mathcal{F}_b^+$, i.e.,

$$\|U_\alpha\|_{\mathcal{F}_b^+} \leq R, \ \forall \alpha \in (0, 1],$$

for some $R > 0$ independent of $\alpha \in (0, 1]$.
Then \((w, \phi)\) is a weak solution to (66) and satisfies the energy inequality

\[- \int_0^T \mathcal{L}(w, \phi) \Lambda'(s) ds + \int_0^T \left( \rho_1 \mathcal{L}(w, \phi) + \frac{\mu_0}{K} |w(s)|^2 \right) \Lambda(s) ds + 2 \int_0^T |\mu(s)|^2 \Lambda(s) ds \leq \int_0^T \left( \frac{2}{K} (g, w) + C_1 \right) \Lambda(s) ds, \tag{84}\]

for all \(\Lambda \in C^\infty_0(0, T; \mathbb{R}_+)\), that is \((w, \phi) \in \mathcal{K}^+, \) where \(\mathcal{K}^+\) is the trajectory space of the 3D AC-NS system (24).

For the proof, we will need the following result.

**Lemma 4.2.** Let two sequences \((u_n, \phi_n)(t) \in F^+_b\) and \(\{\alpha_n\} \subset (0, 1]\) be given such that \(\alpha_n \to 0^+\) as \(n \to \infty\). We denote by \(w_n = (1 + \alpha_n^2 A_0)^{1/2} u_n\) for \(n \in \mathbb{N}\). We assume that the sequence \((w_n, \phi_n)\) is bounded in \(F^+_b\) and \((w_n, \phi_n) \to (w, \phi)\) in \(\Theta^+_\text{loc}\) as \(n \to \infty\). Then the sequence \((u_n, \phi_n)\) is bounded in \(F^+_b\) and \((u_n, \phi_n) \to (u, \phi)\) in \(\Theta^+_\text{loc}\) as \(n \to \infty\).

**Proof.** The proof is similar to that of [26] (see also [25, 14]). For the reader convenience, we give the details.

Note that

\[
|u_n|_{L^2}^2 \leq |u_n|_{L^2}^2 + \alpha^2 \|u_n\|^2 = |w_n|_{L^2}^2,
\]

\[
\|u_n\|^2 \leq \|u_n\|^2 + \alpha^2 |A_0 u_n|_{L^2}^2 = \|w_n\|^2,
\]

\[
\int_{t_n}^{t_n+1} \|\partial_t u_n\|^2_{D(A_0)} ds \leq \int_{t_n}^{t_n+1} |\partial_t w_n|_{D(A_0)}^2 ds,
\]

which gives

\[
\|(u_n, \phi_n)\|_{F^+_b} \leq \|(w_n, \phi_n)\|_{F^+_b}, \forall n \in \mathbb{N}. \tag{85}\]

It follows that the sequence \(\{(u_n, \phi_n)\}\) is bounded in \(F^+_b\). Since a ball in \(F^+_b\) is weakly compact in \(\Theta^+_\text{loc}\), we can extract a subsequence (still denoted \((u_n, \phi_n)\) that converges to \((u, \phi)\) in \(\Theta^+_\text{loc}\), i.e.,

\[
(u_n, \phi_n) \to (u, \phi) \text{ in } \Theta^+_\text{loc};
\]

\[
(w_n, \phi_n) \to (w, \phi) \text{ in } \Theta^+_\text{loc}. \tag{87}\]

Let us prove that \(w = u\). We proceed as in [26], (see also [14]). We consider an interval \([0, T]\). By assumption we have \(w_n \to w\) in \(L^2(0, T; V_1)\) and \(\partial_t w_n \to \partial_t w\) weakly in \(L^2(0, T; D(A_0)^*)\). Therefore, by the Aubin compactness Theorem, we can derive that \(w_n \to w\) strongly in \(L^2(0, T; H_1)\). Similarly, we can check that \(u_n \to u\) strongly in \(L^2(0, T; H_1)\). We also note that

\[
\|(1 + \alpha^2 A_0)^{-1/2}\|_{L(H_1, H_1)} \leq 1. \tag{88}\]

It follows that

\[
\|(1 + \alpha^2 A_0)^{-1/2}(w_n - w)\|_{L^2(0, T; H_1)} \leq \|w_n - w\|_{L^2(0, T; H_1)} \to 0 \text{ as } n \to \infty. \tag{89}\]

On the other hand, from Lemma 3.2 of [26], we have

\[
\|(1 + \alpha^2 A_0)^{-1/2} w - w\|_{L^2(0, T; H_1)} \to 0 \text{ as } n \to \infty. \tag{90}\]

We have

\[
\|u_n - w\|_{L^2(0, T; H_1)} = \|(1 + \alpha^2 A_0)^{-1/2} w_n - w\|_{L^2(0, T; H_1)} \leq \|(1 + \alpha^2 A_0)^{-1/2} w_n - w\|_{L^2(0, T; H_1)} \tag{91}\]

\[
+ \|(1 + \alpha^2 A_0)^{-1/2} w - w\|_{L^2(0, T; H_1)} \to 0 \text{ as } n \to \infty.
\]
We conclude that \( u_n \to w \) strongly in \( L^2(0,T;H_1) \), i.e., \( w = u \) and the lemma is proved. \( \square \)

**Proof of Theorem 22.** We proceed as in [26]. Note that the model considered in this article has a nonlinearity stronger than the one in the 3D LANS-\( \alpha \) model studied in [26].

Since

\[
\| (w_n, \phi_n) \| \leq C, \quad \forall n \in \mathbb{N},
\]

and \((w_n, \phi_n) \to (w, \phi)\) in \( \Theta^+_{loc} \), we have \( \| (w, \phi) \| \leq C \). We set \( u_n = (1 + \alpha_n^2 A_0)^{-1/2} w_n \). The couple \((u_n, \phi_n)\) is a solution of the original problem (24). The estimates (92), (77) imply that

\[
\begin{align*}
\text{ess sup}_{t \geq 0} \left( |u_n(t)|^2 + \alpha_n^2 \| u_n \|^2 + \| \phi_n \|^2 \right) &\leq C, \\
\sup_{t \geq 0} \int_t^{t+1} \left( \| u_n(s) \|^2 + \alpha_n^2 \| u_n(s) \|^2 + \| \mu_n(s) \|^2 \right) ds &\leq C, \\
\sup_{t \geq 0} \int_t^{t+1} \left( \| \partial_t u_n(s) \|^2 + \| \partial_t \phi_n(s) \|^2 \right) ds &\leq C.
\end{align*}
\]

We now prove that \((w, \phi)\) is a weak solution of the 3D AC-NS system (24) on the interval \([0,T]\). The couple \((w_n, \phi_n)\) satisfies

\[
\begin{aligned}
&\frac{d w_n}{dt} + \nu_1 A_0 w_n + (1 + \alpha_n^2 A_0)^{-1/2} \tilde{B}_0(u_n, v_n) \\
&- \mathcal{K}(1 + \alpha_n^2 A_0)^{-1/2} R_0(\nu_2 A_1 \phi_n, \phi_n) = (1 + \alpha_n^2 A_0)^{-1/2} g, \\
&\mu_n = \nu_2 A_1 \phi_n + \nu f(\phi_n), \\
&\frac{d \phi_n}{dt} + \mu_n + B_1(u_n, \phi_n) = 0,
\end{aligned}
\]

in the sense of distribution. Here \( v_n = (1 + \alpha_n^2 A_0) u_n \). From the assumption (83), we have

\[
\begin{cases}
(w_n, \phi_n) \to (w, \phi) \text{ weakly in } L^2(0,T;\mathbb{V}), \\
(w_n, \phi_n) \to (w, \phi) \text{ weakly in } L^2(0,T;\mathbb{Y}), \\
(\partial_t w_n, \partial_t \phi_n) \to (\partial_t w, \partial_t \phi) \text{ weakly in } L^2(0,T;D(A_0)^* \times V_2^*).
\end{cases}
\]

It follows from (95) that \((A_0 w_n, A_1 \phi_n) \to (A_0 w, A_1 \phi)\) weakly in \( L^2(0,T;\mathbb{V}) \), and hence in the topology of \( \mathcal{D}'(0,T;D(A_0)^* \times V_2^*) \) as well. From Lemma 23, (95) and the Aubin compactness Theorem, we have \((u_n, \phi_n) \to (u, \phi)\) strongly in \( L^2(0,T;\mathbb{Y}) \).

Arguing as in the proof of Proposition 8, we can also check that

\[
\begin{align*}
B_1(u_n, \phi_n) &\to B_1(u, \phi) \text{ weakly in } L^{1/3}(0,T;V_2^*) \text{ and therefore in } \mathcal{D}'(0,T;V_2^*), \\
R_0(A_1 \phi_n, \phi_n) &\to R_0(A_1 \phi, \phi) \text{ weakly in } L^{1/3}(0,T;D(A_0)^*) \\
\text{and therefore in } \mathcal{D}'(0,T;D(A_0)^*).
\end{align*}
\]

From Lemma 3.3 of [26], we deduce that \((1 + \alpha_n^2 A_0)^{-1/2} R_0(A_1 \phi_n, \phi_n) \to R_0(A_1 \phi, \phi)\) weakly in \( L^{4/3}(0,T;D(A_0)^*) \). We also have \((1 + \alpha_n^2 A_0)^{-1/2} g \to g\) strongly in \( L^2(0,T;H_1) \). As in [26, 14], we can also check that

\[
\begin{align*}
(1 + \alpha_n^2 A_0)^{-1/2} \tilde{B}_0(u_n, v_n) &\to \tilde{B}_0(w, w) \text{ weakly in } L^{4/3}(0,T;D(A_0)^*) \\
\text{and therefore in } \mathcal{D}'(0,T;D(A_0)^*),
\end{align*}
\]

(97)
It follows that \((w, \phi)\) satisfies
\[
\begin{aligned}
\begin{cases}
\frac{dw}{dt} + \nu_1 A_0 w + B_0(w, w) - K R_0(\nu_2 A_1 \phi, \phi) = g, \\
\frac{d\phi}{dt} + \nu_2 A_1 \phi + \epsilon f(\phi) + B_1(w, \phi) = 0.
\end{cases}
\end{aligned}
\] (98)

Let us prove that \((w, \phi)\) satisfies the energy inequality (28). Since \((w_n, \phi_n)\) satisfies the energy inequality, we have
\[
\begin{aligned}
&- \int_0^T L(w_n, \phi_n) \Lambda'(s) ds + \int_0^T \left( \rho_1 L(w_n, \phi_n) \frac{\|w_n(s)\|^2}{K} \right) \Lambda(s) ds \\
&+ 2 \int_0^T |\mu_n(s)|^2 \Lambda(s) ds \leq \int_0^T \left( \frac{2}{K} (g, u_n) + C_1 \right) \Lambda(s) ds,
\end{aligned}
\] (99)

for all \(\Lambda \in C_0^\infty(0, T; \mathbb{R}^+)\). We argue as in the proof of (51) to conclude that \((w, \phi)\) satisfies the energy inequality (28).

5. **Convergence of the trajectory attractors.** We consider the AC-NS system
\[
\begin{aligned}
\begin{cases}
\frac{du}{dt} + \nu_1 A_0 u + B_0(u, u) - K R_0(\nu_2 A_1 \phi, \phi) = g, \\
\mu = \nu_2 A_1 \phi + \epsilon f(\phi), \\
\frac{d\phi}{dt} + \mu + B_1(u, \phi) = 0.
\end{cases}
\end{aligned}
\] (100)

Let \(\mathcal{U}_0\) be the trajectory attractor of the AC-NS system (100). We recall that \(\mathcal{U}_0\) is bounded in \(\mathcal{F}_b^+\), compact in \(\Theta_{loc}\) and \(\mathcal{U}_0 \subset \mathcal{K}^+\). We also know that \(\mathcal{U}_0 = \Pi_+ \mathcal{K}_0\), where \(\mathcal{K}_0\) is the kernel of the system (100). Note that \(\mathcal{K}_0\) is the union of all bounded (in the norm of \(\mathcal{F}_b^+\)) complete solutions \((u(t), \phi(t)), t \in \mathbb{R}\), of the AC-NS system (100) that satisfy the energy inequality (28). We denote by
\[
\mathcal{B}_\alpha = \{ (w_\alpha(t), \phi_\alpha(t)) , \ t \geq t, \ 0 < \alpha \leq 1, \}
\] (101)

where \(w_\alpha = (1 + \alpha^2 A_0)^{1/2} u_\alpha(t)\), and \((u_\alpha(t), \phi_\alpha(t))\) is a solution of the system (66).

The norm of \((w_\alpha(t), \phi_\alpha(t))\) are uniformly (w.r.t. \(\alpha, 0 < \alpha \leq 1\)) bounded in \(\mathcal{F}_b^+\), i.e.,
\[
\| (w_\alpha, \phi_\alpha) \|_{\mathcal{F}_b^+} \leq R,
\] (102)

for some constant \(R\) independent of \(\alpha, 0 < \alpha \leq 1\), and
\[
\| (w_\alpha, \phi_\alpha) \|_{\mathcal{F}_b^+} = \| (w_\alpha, \phi_\alpha) \|_{L^2_b(\mathbb{R}^+; \mathcal{Y})} + \| (w_\alpha, \phi_\alpha) \|_{L^\infty(\mathbb{R}^+; \mathcal{Y})} + \| \partial_t w_\alpha, \partial_t \phi_\alpha \|_{L^2_b(\mathbb{R}^+; D(A_0)^* \times V_2^*)}.
\] (103)

Recall that \((w_\alpha, \phi_\alpha)\) satisfies
\[
\begin{aligned}
\begin{cases}
\frac{dw_\alpha}{dt} + \nu_1 A_0 w_\alpha + (1 + \alpha^2 A_0)^{-1/2} B_0(u_\alpha, v_\alpha) \\
- K (1 + \alpha^2 A_0)^{-1/2} R_0(\nu_2 A_1 \phi_\alpha, \phi_\alpha) = (1 + \alpha^2 A_0)^{-1/2} g, \\
\mu_\alpha = \nu_2 A_1 \phi_\alpha + \epsilon f(\phi_\alpha), \\
\frac{d\phi_\alpha}{dt} + \mu_\alpha + B_1(u_\alpha, \phi_\alpha) = 0,
\end{cases}
\end{aligned}
\] (104)

where \(v_\alpha = (1 + \alpha^2 A_0)^{1/2} v_\alpha\) and \(u_\alpha = (1 + \alpha^2 A_0)^{-1/2} u_\alpha\). We also recall that
\[
T(h)(w_\alpha, \phi_\alpha)(t) = (w_\alpha, \phi_\alpha)(t + h).
\] (105)

We now state the main result of this article.
Theorem 5.1. (i) The trajectory attractor $\mathcal{U}_\alpha$ of the system (66) converges in the topology $\Theta^+_{loc}$ as $\alpha \to 0^+$ to the trajectory attractor $\mathcal{U}_0$ of the AC-NS system (100):

$$\mathcal{U}_\alpha \to \mathcal{U}_0 \text{ in } \Theta^+_{loc} \text{ as } \alpha \to 0^+. \quad (106)$$

(ii) Let $\mathcal{B}_\alpha = \{(w_\alpha(t), \phi_\alpha(t)), \ t \geq 0\}$, $0 < \alpha \leq 1$, be the bounded set of solutions of (104) that satisfy

$$\|(w_\alpha, \phi_\alpha)\|_{F^+} \leq R, \ \forall \alpha, \ 0 < \alpha \leq 1. \quad (107)$$

Then the set of shifted solutions $\{T(h)\mathcal{B}_\alpha\}$ converges to the trajectory attractors $\mathcal{U}_0$ of the AC-NS system (100) in the topology $\Theta^+_{loc}$ as $h \to +\infty$ and $\alpha \to 0^+$:

$$T(h)\mathcal{B}_\alpha \to \mathcal{U}_0 \text{ in } \Theta^+_{loc} \text{ as } \alpha \to 0^+, h \to +\infty. \quad (108)$$

Proof. We proceed as in [26]. It is enough if we prove (ii), which implies (i) by taking $\mathcal{B}_\alpha = \mathcal{U}_\alpha = T(h)\mathcal{U}_0$, $h \geq 0$. We assume that (108) fails to hold. Then there is a neighborhood $\Theta(\mathcal{U}_0)$ of $\mathcal{U}_0$ in $\Theta^+_{loc}$ and two sequences $\alpha_n \to 0^+$, $h_n \to +\infty$ as $n \to +\infty$ such that $T(h_n)\mathcal{B}_{\alpha_n}$ is not a subset of $\Theta(\mathcal{U}_0)$. Hence, there exist a couple $(w_{\alpha_n}, \phi_{\alpha_n}) \in \mathcal{B}_{\alpha_n}$ such that $X_{\alpha_n} = (w_{\alpha_n}, \phi_{\alpha_n}) \in \mathcal{B}_{\alpha_n}$ and the functions $W_{\alpha_n} = T(h_n)X_{\alpha_n}(t) = (w_{\alpha_n}(t+h_n), \phi_{\alpha_n}(t+h_n))$, $t \geq 0$ do not belong to $\Theta(\mathcal{U}_0)$, i.e.,

$$W_{\alpha_n} \notin \Theta(\mathcal{U}_0). \quad (109)$$

The couple $W_{\alpha_n}(t) = (U_{\alpha_n}(t), \Phi_{\alpha_n}(t))$ is a solution to (104) on the interval $(-h_n, +\infty)$ with $\alpha = \alpha_n$ since $(w_{\alpha_n}(t+h_n), \phi_{\alpha_n}(t+h_n))$ is a solution for $t+h_n \geq 0$ and the system (104) is autonomous. Moreover, it follows from (107) that

$$\sup_{t \geq -h_n} \int_t^{t+1} \left( \|U_{\alpha_n}(s)\|^2 + |A_1\Phi_{\alpha_n}(s)|_{L^2}^2 \right) ds + ess \sup_{t \geq -h_n} \|\Phi_{\alpha_n}(t)\|^2 \leq R. \quad (110)$$

This inequality implies that the sequence $\{(U_{\alpha_n}, \Phi_{\alpha_n}), n \in \mathbb{N}\}$ is weakly compact in the space

$$\Theta_{-T,T} = L^2(-T,T; \mathbb{V}) \cap L^\infty(-T,T; \mathbb{Y}) \cap \{(v, \psi) | (\partial_t v, \partial_t \psi) \in L^2(-T,T; D(A_0)^* \times V_2^*)\}, \quad (111)$$

for every $T > 0$, if we consider $\alpha_n$ with the indices $n$ such that $h_n \geq T$. Therefore, for every fixed $T > 0$, we can choose a subsequence $\{\alpha_{n'}\} \subset \{\alpha_n\}$ such that $W_{\alpha_{n'}} = (U_{\alpha_{n'}}, \Phi_{\alpha_{n'}})$, $n \in \mathbb{N}$ converges in $\Theta_{-T,T}$. Thus using the well known Cantor diagonal procedure, we can construct a couple of function $W(t) = \{(U(t), \Phi(t)), t \in \mathbb{R}\}$ and a subsequence $\{\alpha_{n''}\} \subset \{\alpha_{n'}\}$ such that

$$W_{\alpha_{n''}} = (U_{\alpha_{n''}}, \Phi_{\alpha_{n''}}) \to W = (U, \Phi) \quad (112)$$

weakly in $\Theta_{-T,T}$, as $n'' \to +\infty$ for every $T > 0$. From (110), we obtain that

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} \left( \|U(s)\|^2 + |A_1\Phi(s)|_{L^2}^2 \right) ds + ess \sup_{t \in \mathbb{R}} \|\Phi(t)\|^2 \leq R. \quad (113)$$

In particular, we have

$$W(t) = (U(t), \Phi(t)) \in \mathcal{F}_{b}^+, \quad (114)$$

where

$$\mathcal{F}_{b}^+ = L^2(\mathbb{R}; \mathbb{V}) \cap L^\infty(\mathbb{R}; \mathbb{Y}) \cap \{(v, \psi) | (\partial_t v, \partial_t \psi) \in L^2(\mathbb{R}; D(A_0)^* \times V_2^*)\}. \quad (115)$$
We can now apply Theorem 22, where we assume that all the functions $U_{\alpha''}$, $\Phi_{\alpha''}$ are defined in $[-T, +\infty)$ instead of $[0, +\infty)$. From (110) and (114), we conclude that $W(t) = (U(t), \Phi(t))$ is a weak solution of the 3D AC-NS system for all $t \in \mathbb{R}$ and $W(t) = (U(t), \Phi(t))$ satisfies the energy inequality. Therefore $W(t) = (U(t), \Phi(t)) \in K_0$, where $K_0$ is the kernel of the system (100). Since $\Pi_+K_0 = U_0$ and $W(t) \in K_0$, we have $\Pi_+W \in U_0$.

On the other hand from (110), we have
\[
\Pi_+W_{\alpha''} \rightarrow \Pi_+W \quad \text{in } \Theta^+_0 \quad \text{as } n'' \rightarrow +\infty. \tag{115}
\]
In particular, for $n''$ large enough, we have
\[
\Pi_+W_{\alpha''} \in \Theta(\Pi_+W) \subset \Theta(U_0). \tag{116}
\]
This contradicts (109) and therefore (108) is proved.

6. Convergence to equilibria. In this section, we consider the 3D $\alpha$-regularized system (3.4), (3.10) in the absence of external forces, i.e., $g = 0$. We aim to show that the analysis in [17, Section 5] can be extended to the regularized system for each $\alpha > 0$. In particular, we can show that each trajectory of such a system converges to a single equilibrium provided that $f$ is real analytic.

We first recall some standard implications from Section 3 (see Definition 13 and Theorem 14) and [23].

**Proposition 7.** Let $f \in C^2(\mathbb{R})$ satisfy assumption (2.6) and $\alpha > 0$. Problem (3.4), (3.10) defines a (nonlinear) strongly continuous semigroup $S_\alpha(t) : Z \rightarrow Z$ by setting, for all $t \geq 0$,
\[
S_\alpha(t)(u_0, \phi_0) := (u(t), \phi(t)), \tag{117}
\]
where $(u, \phi)$ is the unique solution to problem (3.4), (3.10). Here and after, $Z$ denotes $V_1 \times H^1(\mathcal{M})$.

It is easy to see that the semigroup $S_\alpha(t)$ has a (strict) global Lyapunov functional defined by the free energy, namely,
\[
\mathcal{L}(u_0, \phi_0) = \frac{1}{2} \left[ v_2 |\nabla \phi_0|^2_{L^2} + v_2 |\phi_0|^2_{L^2} + \frac{1}{\xi} \left( |u_0|^2_{L^2} + \alpha^2 \|u_0\|^2 \right) \right] + \epsilon \int_\mathcal{M} F(\phi_0) dx,
\]
for $(u_0, \phi_0) \in Z$, where $F(r) = \int_0^r f(\zeta) d\zeta$. In particular, we have, for all $t > 0$,
\[
\frac{d}{dt} \mathcal{L}(u(t), \phi(t)) = -\frac{\nu_1}{\xi} \left[ \|u(t)\|^2 + \alpha^2 \|A_0u(t)\|^2_{L^2} \right] - \|\mu(t)\|^2_{L^2}. \quad \tag{118}
\]

As a result of (118) and [23, Propositions 3.6, 3.7] we can immediately infer that $(S_\alpha(t), Z)$ is a gradient system with precompact trajectories (see also [17]).

**Lemma 6.1.** For any $(u_0, \phi_0) \in Z$, the set $\omega(u_0, \phi_0) \subset D(A_0) \times D(A_1)$ is a nonempty compact connected subset of $Z$. Furthermore, we have:
(i) $\omega(u_0, \phi_0)$ is fully invariant for $S_\alpha(t)$;
(ii) $\mathcal{L}$ is constant on $\omega(u_0, \phi_0)$;
(iii) $\text{dist}_Z(S_\alpha(t)(u_0, \phi_0), \omega(u_0, \phi_0)) \rightarrow 0$ as $t \rightarrow +\infty$;
(iv) $\omega(u_0, \phi_0)$ consists only of equilibria satisfying the following stationary problem:
\[
\begin{cases}
  u = 0, & \text{in } \mathcal{M}, \\
  \mu = -v_2 \Delta \psi + \epsilon f(\psi) = 0, & \text{in } \mathcal{M}. 
\end{cases} \tag{119}
\]

The version of the Lojasiewicz-Simon inequality we need is given by the following lemma (see [17, Lemma 5.6]).
Lemma 6.2. Let \((u, \psi) \in Z\) satisfy (119), that is, \((0, \psi)\) is a critical point of \(\mathcal{L}\). Assume that \(f\) is real analytic. There exist constants \(\zeta \in (0, 1/2)\) and \(C_L > 0\), \(\zeta > 0\) depending on \((0, \psi)\) such that, for any \((u, \phi) \in Z\), if
\[
\| (u, \phi) - (0, \psi) \|_Z \leq \zeta,
\]
denoting by \(\mathcal{L}'\) the Fréchet derivative of \(\mathcal{L}\), we have
\[
C_L \| \mathcal{L}'(u, \phi) \|_{Z^*} \geq |\mathcal{L}(u, \phi) - \mathcal{L}(0, \psi)|^{1-\zeta}.
\]

The main result in this section is concerned with the convergence of trajectories to single equilibria (119).

Theorem 6.3. Let the assumptions of Proposition 7 hold. Suppose, in addition, that \(f\) is real analytic. For any given initial datum \((u, \phi) \in Z\), the corresponding solution \((u(t), \phi(t)) = \mathcal{S}_t(u_0, \phi_0)\) to problem (3.4), (3.10) converges to a single equilibrium \((0, \psi)\) in the topology of \(D(A_0) \times D(A_1)\), that is,
\[
\lim_{t \to +\infty} \left( |A_0 u(t)|_{L^2} + |A_1(\phi(t) - \psi)|_{L^2} \right) = 0.
\]
Moreover, there exist \(C \geq 0\) and \(\kappa \in (0, 1)\) depending on \((0, \psi)\) and \(\alpha > 0\) such that
\[
|A_0 u(t)|_{L^2} + |\phi(t) - \psi|_{D(A_1)} \leq C(1 + t)^{-\kappa},
\]
for all \(t \geq 0\).

Proof. We first observe that, if there exists \(t\) such that \(\mathcal{L}(u(t), \phi(t)) = \mathcal{L}_\infty\), then
\[
\phi(t) = \psi, \quad u(t) = 0, \quad \forall t \geq t_0.
\]
In this case, there is nothing to prove. Therefore, without loss of generality, suppose now that, for all \(t \geq t_0 \geq 0\), we have \(\mathcal{L}(u(t), \phi(t)) > \mathcal{L}_\infty\). We observe that, by Lemma 6.1 and Lemma 6.2, the functional \(\mathcal{L}\) satisfies the Lojasiewicz-Simon inequality (121) near every \((0, \psi) \in \omega(u_0, \phi_0)\). Since \(\omega(u_0, \phi_0)\) is compact in \(Z\), we can cover it by the union of finitely many balls \(B_j\) with centers \((0, \psi^j)\) and radii \(r_j\), where each radius is such that (121) holds in \(B_j\). Since \(\mathcal{L} = \mathcal{L}_\infty\) on \(\omega(u_0, \phi_0)\) (recall that, \(\mathcal{L}_\infty = \lim_{t \to +\infty} \mathcal{L}(u(t), \phi(t))\) exists), it follows from Lemma 6.2 that there exist uniform constants \(\xi \in (0, 1/2)\), \(C_L > 0\) (depending on \((0, \psi)\)) and a neighborhood \(\mathcal{U}\) of \(\omega(u_0, \phi_0)\) such that
\[
C_L \| \mathcal{L}'(u, \phi) \|_{Z^*} \geq |\mathcal{L}(u, \phi) - \mathcal{L}_\infty|^{1-\xi}, \quad \forall (u, \phi) \in \mathcal{U}.
\]
Recalling property (iii) of Lemma 6.1, we can find a time \(t_1 > 0\) such that \((u(t), \phi(t))\) belongs to \(\mathcal{U}\), for all \(t \geq t_1\). Set now \(t_2 \geq \max \{t_0, t_1\}\) so that (120) holds. Recalling (118), we obtain, for every \(t \geq t_2\),
\[
- \frac{d}{dt} (\mathcal{L}(u(t), \phi(t)) - \mathcal{L}_\infty)^{\xi} = \xi \left( - \frac{d}{dt} \mathcal{L}(u(t), \phi(t)) \right) (\mathcal{L}(u(t), \phi(t)) - \mathcal{L}_\infty)^{\xi-1}
\geq \frac{\xi C_L}{\mathcal{L}_\infty} \frac{\left( \nu_1/K \right) \left( \|u(t)\|^2 + \alpha^2 |A_0 u(t)|_{L^2}^2 \right) + \mu(t)^2}{\|\mathcal{L}'(u(t), \phi(t))\|_{Z^*}}.
\]
Using now Green’s formula on \(\mathcal{M}\), we obtain
\[
\langle \mathcal{L}'(u, \phi), (h, k) \rangle_{Z^*, Z} = \int_{\mathcal{M}} (-\nu_2 \Delta \phi + \epsilon f(\phi)) k dx + \int_{\mathcal{M}} u \cdot h \frac{dx}{K}.
\]
Hence, by using the Cauchy-Schwarz inequality and Poincaré’s inequality, we obtain
\[
\|L'(u, \phi)\|_{\mathcal{L}^2} = \sup_{\|\langle h, \phi \rangle\|_{\mathcal{L}^1}} \langle L'(u, \phi), (h, \psi) \rangle_{\mathcal{L}^2} 
\leq C_* \left( -\nu_2 \Delta \phi + \epsilon f(\phi) \|_{\mathcal{L}^2} + \sqrt{\nu_1/K} (\|u\| + \alpha |A_0 u|_{\mathcal{L}^2}) \right),
\]
where \(C_*\) depends on \(\nu_2\) and \(\mathcal{M}\), but is independent of time and initial data. Inserting now estimate (127) into estimate (125), we deduce

\[
- \frac{d}{dt} (L(u(t), \phi(t)) - \mathcal{L}'(\infty)) \geq C \left( \|\mu\|_{\mathcal{L}^2} + \sqrt{\nu_1/K} (\|u\| + \alpha |A_0 u|_{\mathcal{L}^2}) \right),
\]

since \(\mu = -\nu_2 \Delta \phi + \epsilon f(\phi)\). Here \(C\) is some positive constant depending on \(C_*, C_L\) and \(\xi\). By integrating this inequality over \([t_2, +\infty)\), and using the fact that \(L(u(t), \phi(t)) \to \mathcal{L}_{\infty}\) as \(t\) goes to \(+\infty\), we also infer that

\[
\mu \in L^1([t_2, +\infty) ; L^2(\mathcal{M})) \quad \text{and} \quad u \in L^1([t_2, +\infty) ; D(\mathcal{A}_0)),
\]

Consequently, since \(|B_1(u, \phi)|_{\mathcal{L}^2} \leq c |A_0 u|_{\mathcal{L}^2} |\phi|_{H^1(\mathcal{M})}\), we also deduce, on account of (129) and the last equation of (3.10), that

\[
\partial_t \phi \in L^1([t_2, +\infty) ; L^2(\mathcal{M})).
\]

Furthermore, the following bounds are also consequences of standard Sobolev embeddings:

\[
\|B_0(u, u)\|_{D(\mathcal{A}_0)} \leq c |A_0 u|_{\mathcal{L}^2} \|u\|, \\
\|R_0(\nu_2 A_1 \phi, \phi)\|_{D(\mathcal{A}_0)} = \|R_0(\mu, \phi)\|_{D(\mathcal{A}_0)*} \leq c |\phi|_{H^1} |\mu|_{\mathcal{L}^2}.
\]

Consequently, employing these inequalities, on account of (129) and the first equation of (3.10), we also deduce that

\[
\partial_t u \in L^1([t_2, +\infty) ; (D(\mathcal{A}_0)^*)).
\]

We now recall that, due to Lemma 6.1, there exists an increasing unbounded sequence \(\{t_k\}\) and an element \((0, \psi) \in \mathcal{M}(u_0, \phi_0)\) such that \((u(t_k), \phi(t_k)) \to (0, \psi)\) in \(\mathcal{Z}\) as \(h\) goes to \(+\infty\). This fact combined with (130) and (131) imply that \((u(t), \phi(t)) \to (0, \psi)\) in \(D(\mathcal{A}_0)^* \times L^2(\mathcal{M})\) as \(t\) goes to \(+\infty\) and in \(D(\mathcal{A}_0) \times D(\mathcal{A}_1)\) as well, thanks to Proposition 7. Hence \(\omega(u_0, \phi_0) = \{0, \psi\}\) and (122) holds.

There remains to prove (123). For \(t \geq t_2\), it follows from (124) and (125) that

\[
\frac{d}{dt} (L(u(t), \phi(t)) - \mathcal{L}_{\infty}) \geq C (L(u(t), \phi(t)) - \mathcal{L}_{\infty})^{1 - \xi} \leq 0.
\]

Then, we deduce that

\[
L(u(t), \phi(t)) - \mathcal{L}_{\infty} \leq C (1 + t)^{-1/(1 - 2\xi)}, \quad \forall t \geq t_2.
\]

Thus, integrating (128) over \([t, +\infty)\), thanks to estimate (132), we get

\[
\int_t^{+\infty} \left( |\mu(s)|_{\mathcal{L}^2} + \sqrt{\nu_2/K} (|u(s)| + \alpha |A_0 u(s)|_{\mathcal{L}^2}) \right) ds \leq C (1 + t)^{-\xi/(1 - 2\xi)},
\]

for all \(t \geq t_2\). By properly adjusting the constant \(C\) in (133), from (130)-(131), we also deduce

\[
|\phi(t) - \psi|_{\mathcal{L}^2} \leq C (1 + t)^{-\xi/(1 - 2\xi)}, \quad \forall t \geq t_2, \\
\|u(t)\|_{D(\mathcal{A}_0)*} \leq C (1 + t)^{-\xi/(1 - 2\xi)}, \quad \forall t \geq t_2.
\]
Taking advantage of the above (lower order) convergence estimates and standard interpolation inequalities, we can prove the higher-order estimate (123). We omit the details, the arguments being the same as in [17, Theorem 5.7]. The proof is finished.

Remark 7. Let us assume that instead of (2.6), \( f \) obeys the following assumptions:

\[
\lim_{|s| \to +\infty} f'(s) > 0, \quad f(1) \geq 0 \quad \text{and} \quad f(-1) \leq 0,
\]

where the order parameter \( \phi \) is normalized in such a way that the two pure phases of the fluid are \(-1\) and \(+1\), respectively. Then, arguing as in the proof of [17, Theorem 6.1] by virtue of the maximum principle, it is not difficult to show that the set

\[
\mathcal{B}_\infty = \{ \phi \in L^\infty(\mathcal{M}) : -1 \leq \phi \leq 1, \ \text{a.e. in} \ \mathcal{M} \}
\]

is invariant for \((\mathbb{Z}, \mathbb{S}_\alpha(t))\), so that we can take \( V_1 \times (H^1(\mathcal{M}) \cap B_\infty) \) as phase space. In this case, the pure phase flows \((u, \pm 1)\) are admissible solutions for (3.4), (3.10). Moreover, the results in [23] can be also easily improved for nonlinearities which satisfy (136). In particular, one can show the existence of a finite-dimensional (connected) global attractor \( \mathcal{A} \subset V_1 \times (H^1(\mathcal{M}) \cap B_\infty) \) for the semigroup \( \mathbb{S}_\alpha(t) \). Finally, note that the derivative \( f(s) = s(s^2 - 1) \) of the double-well potential \( F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2 \) clearly satisfies (136).

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