On Fourier integral transforms for $q$-Fibonacci and $q$-Lucas polynomials

Natig Atakishiyev, Pedro Franco, Decio Levi and Orlando Ragnisco

1 Instituto de Matemáticas, Unidad Cuernavaca, Universidad Nacional Autónoma de México, CP 62251 Cuernavaca, Morelos, México
2 Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and INFN Sezione di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
3 Dipartimento di Fisica, Università degli Studi Roma Tre and INFN Sezione di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy

E-mail: natig@matcuer.unam.mx, pedro@matcuer.unam.mx, levi@Roma3.infn.it and Ragnisco@Roma3.infn.it

Received 9 December 2011, in final form 26 March 2012
Published 24 April 2012
Online at stacks.iop.org/JPhysA/45/195206

Abstract

We study in detail two families of $q$-Fibonacci polynomials and $q$-Lucas polynomials, which are defined by non-conventional three-term recurrences. They were recently introduced by Cigler and were then employed by Cigler and Zeng to construct novel $q$-extensions of classical Hermite polynomials. We show that both of these $q$-polynomial families exhibit simple transformation properties with respect to the classical Fourier integral transform.

PACS numbers: 02.30.Gp, 02.30.Tb, 02.30.Vv

1. Introduction

The Askey scheme of hypergeometric orthogonal polynomials and their $q$-analogues [15] accumulates current knowledge about a large number of these special functions. Depending on a number of parameters, associated with each polynomial family, they occupy different levels within the Askey hierarchy: for instance, the Hermite polynomials $H_n(x)$ are on the ground level, the Laguerre and Charlier polynomials $L_n^{(0)}(x)$ and $C_n(x; a)$ are one level higher and so on. All polynomial families in this scheme are characterized by a ‘canonical’ set of properties: they are solutions of differential or difference equations of the second order, can be generated by three-term recurrence relations, are orthogonal with respect to weight functions with finite or infinite supports, obey Rodrigues-type formulas and so on. Of course, many other polynomial families of interest arise both in pure and applied mathematics, which do not belong to the Askey $q$-scheme only because they lack some of the above-mentioned characteristic properties. So this paper is aimed at exploring in detail two particular $q$-polynomial families of this type, namely $q$-Fibonacci and $q$-Lucas polynomials, which are defined by non-conventional three-term recurrences. They were introduced in [6–8] and have been studied in detail in [9].
The Fibonacci and Lucas sequences and polynomials have many physical applications; for example, they appear in the study of diatomic chains [17], in dynamical systems and chaos theory [22], in Ising models [12], etc.

Our main result is to show that both of these \( q \)-polynomials exhibit simple transformation properties with respect to the classical Fourier integral transform.

Throughout this exposition we employ standard notations of the theory of special functions (see, for example, [1, 13–15]). In sections 2 and 3 we present some basic background facts in order to find explicit forms of Fourier integral transforms for these two families of \( q \)-polynomials. Section 5 contains the conclusions and a brief discussion of some further research directions of interest. Finally, the appendices conclude this work with the derivation of two transformation formulas for hypergeometric \( \, _2F_1 \)-polynomials, associated with the Chebyshev polynomials \( T_n(x) \) and \( U_n(x) \).

2. The Fibonacci and \( q \)-Fibonacci polynomials

Polynomials \( F_n(x, s) \) are defined by the three-term recurrence relation

\[
F_{n+1}(x, s) = xF_n(x, s) + sF_{n-1}(x, s), \quad n \geq 1,
\]

with initial values \( F_0(x, s) = 0 \) and \( F_1(x, s) = 1 \) [6, 7]. They are also given by the explicit sum formula

\[
F_{n+1}(x, s) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} s^{k},
\]

where \( \binom{n}{k} = n!/(k!(n-k)! \) is a binomial coefficient and \( [x] \) denotes the greatest integer in \( x \). The Fibonacci polynomials \( F_n(x, s) \) are normalized so that \( F_n(1, s) = \delta_n \) where \( \delta_n \) are the Fibonacci numbers \( \{ F_n \}_{n=1}^\infty = \{ 1, 1, 2, 3, 5, 8, 13, \ldots \} \), which ‘have been a source of delight to professional and amateur mathematicians for seven centuries’ [5] (see also [11, 21, 24]). The Fibonacci polynomials (2.2) (of degree \( n \) in \( x \) and of \( [n/2] \) in \( s \)) can also be represented as

\[
F_{n+1}(x, s) = (2\sqrt{3})^n p_n^{(F)} \left( \frac{x}{2\sqrt{3}} \right),
\]

\[
p_n^{(F)}(x) := x^s 2F_1 \left( \frac{n+1}{2}, -n; -n; -\frac{1}{x^2} \right),
\]

so that the fundamental properties of \( F_n(x, s) \) are basically defined by the monic polynomials \( p_n^{(F)}(x) \).\(^4\) Moreover, it turns out that the polynomials \( p_n^{(F)}(x) \) are essentially the Chebyshev polynomials of the second kind \( U_n(x) \) in an imaginary argument \( z = x \) (see p 449 in [16]),

\[
p_n^{(F)}(x) = (-i/2)^n U_n(ix).
\]

Observe that from the three-reccurrence relation

\[
2zU_n(z) = U_{n+1}(z) + U_{n-1}(z)
\]

\(^4\) We recall that an arbitrary polynomial \( p_n(x) = \sum_{k=0}^n \alpha_k x^k \) of degree \( n \) can be written in the monic form \( p_n^{(M)}(x) = c_n^{(-1)} p_n(x) = x^n + \sum_{k=0}^{n-1} c_n x^k \) just by changing its normalization.
for the Chebyshev polynomials of the second kind $U_n(z)$, it follows at once that
\begin{equation}
    p_{n+1}^{(F)}(x) = xp_n^{(F)}(x) + \frac{1}{4}p_{n-1}^{(F)}(x), \quad n \geq 1,
\end{equation}
with initial values $p_0^{(F)}(x) = 1$ and $p_1^{(F)}(x) = x$.

Since the Chebyshev polynomials of the second kind $U_n(z)$ can be expressed in terms of the hypergeometric $2F1$ polynomials as (see, for example, (9.8.36) in [15])
\begin{equation}
    U_n(z) = (n+1)_2F1 \left(-n, n+2; \frac{1-z^2}{2}\right),
\end{equation}
(2.6) is consistent with the second line in (2.2) only if the following transformation formula
\begin{equation}
    (n+1)_2F1 \left(-n, n+2; \frac{1-z^2}{2}\right) = (2z)_2F1 \left(\frac{n}{2}, -\frac{n}{2}; -n\right) z^2
\end{equation}
is valid. A direct proof of (2.7) is given in appendix A.

Cigler defined in [6, 7] a novel $q$-analogue of Fibonacci polynomials $F_n(x, s)$, which satisfy the three-term recursion
\begin{equation}
    F_{n+1}(x, s|q) = [1 + (q-1)s]F_n(x, s|q) + sF_{n-1}(x, s|q), \quad n \geq 1,
\end{equation}
where initial values are $F_0(x, s|q) = 0$ and $F_1(x, s|q) = 1$, and the Hahn $q$-difference operator $D_q$ is defined as
\begin{equation}
    D_qf(x) := \frac{f(x) - f(qx)}{(1-q)x}.
\end{equation}
Note that this type (2.8) of rather non-standard (since it does contain the $q$-difference operator $D_q$) three-term recurrence is usually referred to as a structure relation [18].

The $q$-Fibonacci polynomials $F_n(x, s|q)$ are explicitly given in the form
\begin{equation}
    F_{n+1}(x, s|q) = \sum_{k=0}^{[n/2]} q^{k(k+1)/2} \begin{pmatrix} n-k \\ k \end{pmatrix}_q x^{n-2k} \\
    = x^n q\phi_1 \left(q^{-n/2}, q^{-(1-n)/2}, -q^{-n/2}, -q^{1-n/2}; q^{-n}; q^{-q^2 s/x^2}\right), \quad n \geq 0,
\end{equation}
where $\begin{pmatrix} n \\ k \end{pmatrix}_q$ stands for the $q$-binomial coefficient
\begin{equation}
    \begin{pmatrix} n \\ k \end{pmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},
\end{equation}
and $(z; q)_n$ is the $q$-shifted factorial, that is, $(z; q)_0 = 1$, $(z; q)_n = \prod_{k=0}^{n-1} (1 - zq^k)$ for $n \geq 1$. The $q$-Fibonacci polynomials $F_n(x, s|q)$ are defined in such a way that in the limit as $q \to 1$ they reduce to the polynomials $F_n(x, s)$.
\begin{equation}
    F_n(x, s|1) = \lim_{q \to 1} F_n(x, s|q) = F_n(x, s).
\end{equation}

3. The Lucas and $q$-Lucas polynomials

Polynomials $L_n(x, s)$ for $n \geq 3$ ($L_0(x, s) = 1$) are defined by the same three-term recurrence relation as in (2.1), but with different initial values, $L_1(x, s) = x$ and $L_2(x, s) = x^2 + 2s$ [9].

They have the explicit sum formula
\begin{equation}
    L_n(x, s) = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \begin{pmatrix} n-k \\ k \end{pmatrix}_q x^n q^{2k} \\
    = x^n q\phi_1 \left(\frac{n}{2}, 1-n; 1-n; -4s/x^2\right), \quad n \geq 0.
\end{equation}
The Lucas polynomials $L_n(x, s)$ are normalized in such a way that $L_n(1, s) = l_n(x)$ (where $l_n(x)$ are the Lucas polynomials studied by Bicknell, see p 459 in [16]) and for the particular values of $x = s = 1$, the sequence $\{L_n(1, 1)\}_{n=1}^{\infty}$ reproduces Lucas numbers $\{L_n\} = \{1, 3, 4, 7, 11, 18, \ldots\}$ (see [24] for applications of Lucas numbers and [21] for some generalizations of Lucas polynomials).

The Lucas polynomials (3.1) (of degree $n$ in $x$ and of degree $\lfloor n/2 \rfloor$ in $s$) can also be represented as

$$L_n(x, s) = s^{n/2}p^{(L)}_n \left( \frac{x}{\sqrt{s}} \right),$$

$$p^{(L)}_n(x) := x^s \, _2F_1 \left( -\frac{n}{2}, \frac{1-n}{2}; 1-n; -\frac{4}{x^2} \right).$$

so that the fundamental properties of $L_n(x, s)$ are basically defined by the monic polynomials $p^{(L)}_n(x)$. Moreover, it turns out that the polynomials $p^{(L)}_n(x)$ are in fact the Chebyshev polynomials of the first kind $T_n(x)$ in an imaginary argument $z$. Indeed,

$$L_0(x, s) = 1, \quad L_n(x, s) = 2(-i\sqrt{s})^n T_n \left( \frac{ix}{2\sqrt{s}} \right), \quad n \geq 1,$$

and, consequently, $p^{(L)}_0(x) = 1$, $p^{(L)}_n(x) = 2(-i)^n T_n(i\, x/2)$, $n \geq 1$. Let us observe that from (3.2), (3.3) and the three-term recurrence relation

$$2zT_n(z) = T_{n+1}(z) + T_{n-1}(z)$$

for the Chebyshev polynomials of the first kind $T_n(z)$, it follows at once that

$$p^{(L)}_{n+1}(x) = xp^{(L)}_n(x) + p^{(L)}_{n-1}(x), \quad n \geq 1,$$

with initial values $p^{(L)}_0(x) = 1$ and $p^{(L)}_1(x) = x$.

Since the Chebyshev polynomials of the first kind $T_n(z)$ can be written in terms of the hypergeometric $_2F_1$ polynomials as (see, for example, (9.8.35) in [15])

$$T_n(z) = _2F_1 \left( -n, n; 1/2; \frac{1-z}{2} \right),$$

(3.3) is consistent with the second line in (3.1) only if the transformation formula

$$\binom{z}{n} = 2^{n-1}\, \zeta^n \, _2F_1 \left( -\frac{n}{2}, \frac{1-n}{2}; 1-n; 1/z^2 \right), \quad n \geq 1,$$

is valid. A proof of this identity is given in appendix B.

A $q$-extension of the Lucas polynomials $L_n(x, s)$ was introduced in [7, 8] as

$$L_n(x, s|q) = L_n(x + (q-1)sD_q, s) \cdot 1,$$  \hspace{1cm} (3.8)

where $D_q$ is defined in (2.9). From (3.8) and the three-term recurrence relation for $L_n(x, s)$, it then follows that the $q$-Lucas polynomials $L_n(x, s|q)$ satisfy a non-standard three-term recurrence of the form

$$L_{n+1}(x, s|q) = [x + (q-1)sD_q]L_n(x, s|q) + sL_{n-1}(x, s|q), \quad n \geq 2,$$  \hspace{1cm} (3.9)

The initial values are $L_0(x, s|q) = 1$, $L_1(x, s|q) = x$ and $L_2(x, s|q) = x^2 + (1+q)s$. The $q$-Lucas polynomials $L_n(x, s|q)$ have an explicit sum formula (cf (3.1) and (2.10))

$$L_n(x, s|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(n-k-1)/2} \binom{n}{n-k}_q \sum_{l=k}^{n-k} \binom{n-k}{l}_q \binom{n-k}{l}_q \phi_1 \left( q^{-n/2}, q^{(1-n)/2}; q^{-n/2}, -q^{-n/2}, -q^{1-n/2}, q^{1-n}; q; \frac{q^2s}{x^2} \right),$$  \hspace{1cm} (3.10)
where \([n]_q := (1 - q^n)/(1 - q)\). In the limit as \(q \to 1\) they reduce to the polynomials \(L_n(x, s)\), given in (3.1). Since the polynomials \(F_n(x, s)\) and \(L_n(x, s)\) satisfy the same recurrence relation (2.1) but with different initial conditions, they are known to be interconnected by the relation

\[
L_n(x, s) = F_{n+1}(x, s) + sF_{n-1}(x, s)
\]

(cf the classical relation \(2T_n(x) = U_n(x) - U_{n-2}(x)\) for the Chebyshev polynomials). Similarly, a \(q\)-extension of this relation,

\[
L_n(x, s|q) = F_{n+1}(x, s|q) + sF_{n-1}(x, s|q),
\]

interconnects the two \(q\)-polynomial families \(F_n(x, s|q)\) and \(L_n(x, s|q)\). The relation (3.11) is readily verified by using the explicit forms (2.10) and (3.10) of the polynomials \(F_n(x, s)\) and \(L_n(x, s)\) and the identity

\[
\binom{n-k}{k}_q = \frac{1 - q^{n-k}}{1 - q^k} \binom{n-k-1}{k-1}_q
\]

for the \(q\)-binomial coefficient (2.11).

4. Fourier transforms of \(F_n(x, s|q)\) and \(L_n(x, s|q)\)

To derive an explicit form of the classical Fourier integral transform for the \(q\)-Fibonacci polynomials \(F_n(x, s|q)\), let us first define how this \(q\)-polynomial family changes under the transformation \(q \to 1/q\). Rewriting (2.10) as

\[
F_{n+1}(x, s|q) = \sum_{k=0}^{[n/2]} c_{n,k}^{(F)}(q)q^s x^{n-2k},
\]

we have

\[
F_{n+1}(x, s|1/q) = \sum_{k=0}^{[n/2]} c_{n,k}^{(F)}(1/q)q^{s} x^{n-2k}.
\]

From the definition of the \(q\)-binomial coefficient, it is not hard to derive a formula relating \([[n]_q, k]_q\) to \([[n]_q, k]_{1/q}\),

\[
\binom{n}{k}_q = q^{k(n-a)} \binom{n}{k}_{1/q}.
\]

From (4.2) and (4.3) it follows that

\[
c_{n,k}^{(F)}(q^{-1}) = q^{k(n-a)} c_{n,k}^{(F)}(q),
\]

which means that (cf (2.10))

\[
F_{n+1}(x, s|q^{-1}) = \sum_{k=0}^{[n/2]} c_{n,k}^{(F)}(q^{-1})q^s x^{n-2k} = \sum_{k=0}^{[n/2]} q^{k(n-a)} c_{n,k}^{(F)}(q)q^s x^{n-2k}
\]

\[
= x^n q^{s} \left( q^{-n/2}, q^{1-n/2}, -q^{-n/2}, -q^{1-n/2}, q^{-n}, 0, 0 \right| q^{-s} - x^2/q^2 \right).
\]

Taking into account the well-known Fourier transform

\[
\int_{\mathbb{R}} e^{ixy-x^2/2} \, dx = \sqrt{2\pi} \, e^{-y^2/2}
\]

for the Gauss exponential function \(e^{-x^2/2}\), we can determine the Fourier integral transform of the exponential function \(\exp \left[ i(n-2k) x - x^2/2 \right] \) as

\[
\int_{\mathbb{R}} e^{ixy+(n-2k)x}e^{-x^2/2} \, dx = \sqrt{2\pi} q^{-(y+(n-2k)x)^2/2}
\]

\[
= \sqrt{2\pi} q^{n^2/4} q^{k(n)} e^{-(n-2k)xy-y^2/2},
\]

where \(q = e^{-2k^2}\).
We are now in the position to formulate and prove the following theorem.

**Theorem 4.1.** The classical Fourier integral transform of the \( q \)-Fibonacci polynomials \( F_{n+1}(a \ e^{ix}, s \mid q) \) times the Gauss exponential function \( e^{-x^2/2} \) has the form

\[
\int_{\mathbb{R}} F_{n+1}(a \ e^{ix}, s \mid q) \ e^{iy-x^2/2} \ dx = \sqrt{2\pi} \ q^{n/4} F_{n+1}(a \ e^{-sy}, qs \mid q^{-1}) \ e^{-y^2/2},
\]

(4.7)

where \( a \) is an arbitrary constant factor.

**Proof.** By definition (2.10), the interrelation (4.4) between the coefficients \( c_{n,k}(F) \) and \( c_{n,k}(q^{-1}) \) and then the Fourier integral transform (4.6), one obtains

\[
\int_{\mathbb{R}} F_{n+1}(a \ e^{ix}, s \mid q) \ e^{iy-x^2/2} \ dx = \sum_{k=0}^{[n/2]} c_{n,k}^{(F)}(q) s^k a^{n-2k} \int_{\mathbb{R}} e^{iy+(n-2k)x-x^2/2} \ dx
\]

\[
= \sqrt{2\pi} \ q^{n/4} \ e^{-y^2/2} \sum_{k=0}^{[n/2]} q^{k(n-k)} c_{n,k}^{(F)}(q) s^k (a \ e^{-sy})^{n-2k}
\]

\[
= \sqrt{2\pi} \ q^{n/4} \ e^{-y^2/2} \sum_{k=0}^{[n/2]} c_{n,k}^{(F)}(q^{-1})(qs)^k (a \ e^{-sy})^{n-2k}
\]

\[
= \sqrt{2\pi} \ q^{n/4} F_{n+1}(a \ e^{-sy}, qs \mid q^{-1}) \ e^{-y^2/2}.
\]

\[\square\]

We turn now to determine an explicit form of the classical Fourier integral transform for the \( q \)-Lucas polynomials \( L_n(x, s \mid q) \). Let us rewrite the defining sum (3.1) for \( L_n(x, s \mid q) \) as

\[
L_n(x, s \mid q) = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(q) s^k x^{n-2k},
\]

(4.8)

where the coefficients in (4.8) are given by

\[
c_{n,k}^{(L)}(q) := q^{k(k-1)/2} \frac{[n]_q}{[n-k]_q} \binom{n-k}{k}_q.
\]

(4.9)

From the definition of the symbol \([n]_q\) in (3.10) it is not hard to show that

\[
[n]_{1/q} = q^{-1-n} [n]_q.
\]

(4.10)

Consequently, from (4.3) and (4.10) it follows that

\[
c_{n,k}^{(L)}(q^{-1}) = q^{k(k-n)} c_{n,k}^{(L)}(q).
\]

(4.11)

This means that (cf (3.10))

\[
L_n(x, s \mid q^{-1}) = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(q^{-1}) s^k x^{n-2k} = \sum_{k=0}^{[n/2]} q^{k(k-n)} c_{n,k}^{(L)}(q) s^k x^{n-2k}
\]

\[
= x^n q^{n/2} \sum_{r=0}^{[n/2]} q^{-r(n-r)/2} q^{r(n-r)/2} q^{-r/2} q^{-r/2} q^{n-2} \left| q^{-n-2} \left( q^{-n-2} \right)^{-1} \right| = x^n \left( q^{-n-2} \right) \left( q^{-n-2} \right)^{-1} = x^n \left( q^{n+2} \right).
\]

(4.12)

The next step is to take into account the Fourier integral transform (4.6) in order to prove the following theorem.

**Theorem 4.2.** The Fourier integral transform of the \( q \)-Lucas polynomials \( L_n(b \ e^{ix}, s \mid q) \) times the Gauss exponential function \( e^{-x^2/2} \) has the form

\[
\int_{\mathbb{R}} L_n(b \ e^{ix}, s \mid q) \ e^{iy-x^2/2} \ dx = \sqrt{2\pi} q^{n/4} L_n(b \ e^{-sy}, s \mid q^{-1}) \ e^{-y^2/2},
\]

(4.13)

where \( b \) is an arbitrary constant factor.
Proof. Starting from formula (3.10) for the \(q\)-Lucas polynomials, the interrelation (4.11) between the coefficients \(c_{n,k}^{(L)}(q)\) and \(c_{n,k}^{(F)}(q^{-1})\) and using the Fourier integral transform (4.6) we obtain

\[
\int_{\mathbb{R}} L_n(b e^{ix}, s|q) e^{i\pi y s^2/2} dx = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(q) s^k b^{-2k} \int_{\mathbb{R}} e^{iyx(n-2k)s-x^2/2} dx = \sqrt{2\pi} q^{n/4} e^{-y^2/2} \sum_{k=0}^{[n/2]} q^{k(n-k)} c_{n,k}^{(L)}(q) s^k (b e^{-xy})^{n-2k}
\]

\[
= \sqrt{2\pi} q^{n/4} e^{-y^2/2} \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(q^{-1}) s^k (b e^{-xy})^{n-2k}
\]

\[
= \sqrt{2\pi} q^{n/4} L_n(b e^{-xy}, s|q^{-1}) e^{-y^2/2}.
\]

\[
\square
\]

5. Concluding remarks and outlook

We have studied the transformation properties with respect to Fourier transform of the \(q\)-Fibonacci and \(q\)-Lucas polynomials, governed by the non-conventional three-term recurrences (3.1) and (3.8), respectively, and proved that these families of \(q\)-polynomials exhibit a simple transformation behavior (4.7) and (4.13) under the classical Fourier integral transform. One may then use the Mehta–Dahlquist–Matveev techniques (see [20, 10, 19, 3, 4]) to show that the Fourier integral transformation formulas (4.7) and (4.13) in fact entail a similar behavior of the \(q\)-Fibonacci and \(q\)-Lucas polynomials under the discrete (finite) Fourier transform.

It is worthwhile to mention here that, apart from Cigler’s \(F_n(x, s|q)\) and \(L_n(x, s|q)\), there are other \(q\)-extensions of Fibonacci and Lucas polynomials of interest. For instance, the following two monic \(q\)-polynomial families

\[
r_n^{(F)}(x|q) = x^n \phi_1 \left( q^{-n}, q^{-1-n}; q^2, -\frac{1}{q x^2} \right),
\]

\[
r_n^{(L)}(x|q) = x^n \phi_1 \left( q^{-n}, q^{-1-n}; q^2, -\frac{q}{x^2} \right),
\]

represent very natural extensions of the Fibonacci and Lucas polynomials \(p_n^{(F)}(x)\) and \(p_n^{(L)}(x)\), defined in (2.4) and (2.16), respectively. Contrary to \(F_n(x, s|q)\) and \(L_n(x, s|q)\), these \(q\)-polynomial families do satisfy standard three-term recurrence relations of the form

\[
r_{n+1}^{(F)}(x|q) = x r_n^{(F)}(x|q) + q^{n-1} (1 + q^n) (1 + q^{n+1}) r_{n-1}^{(F)}(x|q).
\]

\[
r_{n+1}^{(L)}(x|q) = x r_n^{(L)}(x|q) + q^{n-1} (1 + q^n) (1 + q^{n-1}) r_{n-1}^{(L)}(x|q).
\]

Moreover, the \(q\)-polynomials \(r_n^{(F)}(x|q)\) and \(r_n^{(L)}(x|q)\) are associated with the monic \(q\)-polynomial families

\[
s_n^{(U)}(x|q) = i^{-n} r_n^{(F)}(ix|q), \quad s_n^{(T)}(x|q) = i^{-n} r_n^{(L)}(ix|q),
\]

respectively, which do satisfy the conditions of Favard’s characterization theorem (see, for example, p 176 in [13]); thus, they can be viewed as natural \(q\)-extensions of the Chebyshev
polynomials $U_n(x)$ and $T_n(x)$. It should be noted that the polynomials $s^{(U)}_n(x|q)$ and $s^{(T)}_n(x|q)$, explicitly given by

\[
s^{(U)}_n(x|q) = x^n \phi_1 \left( q^{-n}, q^{1-n}; q^{-2n}; q^n; \frac{1}{q^{1/2}} \right),
\]

\[
s^{(T)}_n(x|q) = x^n \phi_2 \left( q^{-n}, q^{1-n}; q^{2(1-n)}; q^n; \frac{q}{x^2} \right),
\]

are also of interest. It is well known that the Chebyshev polynomials $U_n(x)$ and $T_n(x)$ are special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with the parameters $\alpha = \beta = 1/2$ and $\alpha = \beta = -1/2$, respectively. Therefore it seems natural to expect that the continuous $q$-Jacobi polynomials $P_n^{q^{(\alpha,\beta)}}(x)$ (which evidently represent $q$-extensions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$) with the particular values of the parameters $\alpha = \beta = 1/2$ and $\alpha = \beta = -1/2$ could provide appropriate $q$-extensions of the Chebyshev polynomials $U_n(x)$ and $T_n(x)$, respectively. Under closer examination however, it turns out that the continuous $q$-Jacobi polynomials $P_n^{1/2,1/2}(x|q)$ and $P_n^{-1/2,-1/2}(x|q)$ are constant (but $q$-dependent) multiples of the Chebyshev polynomials $U_n(x)$ and $T_n(x)$. In other words, the continuous $q$-Jacobi polynomials $P_n^{1/2,1/2}(x|q)$ and $P_n^{-1/2,-1/2}(x|q)$ differ from the Chebyshev polynomials $U_n(x)$ and $T_n(x)$ only in the choice of the normalization constants; therefore, the former two polynomial families are just trivial $q$-extensions of the latter ones.\footnote{This curious ‘$q$-degeneracy’ of the continuous $q$-Jacobi polynomials $P_n^{q^{(\alpha,\beta)}}(x|q)$ for the values of the parameters $\alpha = \beta = 1/2$ and $\alpha = \beta = -1/2$ was first noted by Askey and Wilson in their seminal work [2]. We are grateful to Tom Koornwinder for reminding us of this fact.}

The polynomials $s^{(U)}_n(x|q)$ and $s^{(T)}_n(x|q)$ can be thus viewed as non-trivial compact $q$-extensions of the Chebyshev polynomials $U_n(x)$ and $T_n(x)$, which do not match with the continuous $q$-Jacobi polynomials $P_n^{1/2,1/2}(x|q)$ and $P_n^{-1/2,-1/2}(x|q)$. Both of them can be expressed in terms of the little $q$-Jacobi polynomials $p_n(x; a, b|q)$ as

\[
s^{(U)}_{2n}(x|q) = (-1)^n q^{n(n-1)} \left( \frac{q; q^2}{q_{2n+1}; q^2} \right)_n p_n(x^2; q^{-1}, q|q^2),
\]

\[
s^{(U)}_{2n+1}(x|q) = (-1)^n q^{n(n-1)} \left( \frac{q; q^2}{q_{2n+2}; q^2} \right)_n x p_n(x^2; q; q|q^2),
\]

and

\[
s^{(T)}_{2n}(x|q) = (-1)^n q^{n(n-1)} \left( \frac{q; q^2}{q^{2n}; q^2} \right)_n p_n(x^2; q^{-1}, q^2),
\]

\[
s^{(T)}_{2n+1}(x|q) = (-1)^n q^{n(n-1)} \left( \frac{q; q^2}{q^{2n+1}; q^2} \right)_n x p_n(x^2; q; q^{-1}|q^2),
\]

where $p_n(x; a, b|q) := \phi_1(q^{-n}, ab q^{n+1}; aq|q; qx)$ (see, for example, (14.12.1), p 482 in [15]). It would be of considerable interest to examine the properties of the polynomials $s^{(U)}_n(x|q)$ and $s^{(T)}_n(x|q)$ in more detail, including their transformation properties with respect to the Fourier integral transform. This project is beyond the subject of this paper and will be dealt with elsewhere.

Acknowledgments

Discussions with T H Koornwinder and K B Wolf are gratefully acknowledged. One of us (NA) would like to thank the Physics Department and Department of Electronic Engineering, University Roma Tre, Italy, for the hospitality extended to him during his visit in September–October 2011, when the final part of this work was carried out. The participation of NA in this
work has been supported by the DGAPA-UNAM IN105008-3 and SEP-CONACYT 79899 projects ‘Optica Matemática’. DL and OR have been partly supported by the Italian Ministry of Education and Research, 2010 PRIN ‘Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps’.

Appendix A. Proof of formula (2.7)

In order to give a direct proof of the transformation formula (2.7) which was stated in section 2, we start from the defining relation for the hypergeometric \( _2F_1 \)-polynomial on the left-hand side of (2.7)

\[
_2F_1 \left( -n, n + 2; 3/2 \middle| \frac{1 - z}{2} \right) := \sum_{k=0}^{n} \frac{(-n)_{k} (n + 2)_{k} (1 - z)^k}{(3/2)_k 2^k k!}.
\]

To obtain (A.1) we used the relation \(-n)_{k} = (-1)^k n!/(n - k)!\). We then reverse the order of summation in (A.1) with respect to the indices \( k \) and \( l \) and obtain

\[
_2F_1 \left( -n, n + 2; 3/2 \middle| \frac{1 - z}{2} \right) = \frac{\Gamma(3/2)}{n + 1} \sum_{l=0}^{n} \frac{\Gamma(2n + 2 - l) (z/2)^{n-l} (-l)_{k} (2n + 2 - k)_{k} 1}{l! (n - l + 3/2)_k} \frac{1}{2^l l!}.
\]

The sum over index \( k \) in (A.2) represents the hypergeometric polynomial

\[
_2F_1 (-l, 2n + 2 - l; n - l + 3/2 | x)
\]

for \( x = 1/2 \), which can be evaluated by Gauss’s second summation theorem (see, for example, (1.7.1.9) on p 32 in [23]) and gives

\[
_2F_1 (2a, 2b; a + b + 1/2 | 1/2) = \frac{\Gamma(1/2) \Gamma(a + b + 1/2)}{\Gamma(a + 1/2) \Gamma(b + 1/2)}, \quad a + b + 1/2 \neq -m, \quad m \geq 0,
\]

with \( a = -l/2 \) and \( b = n + 1 - l/2 \) (so that \( a + b + 1/2 = n - l + 3/2 \geq 3/2 \) for all \( 0 \leq l \leq n \)). The sum over index \( k \) in (A.2) thus reduces to

\[
_2F_1 (-l, 2n + 2 - l; n - l + 3/2 | 1/2) = \frac{\Gamma(1/2) \Gamma(n - l + 3/2)}{\Gamma((2n - l + 3)/2) \Gamma((1 - l)/2)}.
\]

Since the gamma function \( \Gamma(z) \) has poles at the points \( z = -n, \ n \geq 0 \), the right-hand side of (A.4) vanishes for all odd values of the index \( l \) due to the presence of the factor \( \Gamma((1 - l)/2) \) in its denominator. This means that in the sum over \( l \) in (A.2) only the terms with even \( l \)’s give non-zero contribution

\[
_2F_1 \left( -n, n + 2; 3/2 \middle| \frac{1 - z}{2} \right) = \frac{\pi}{2(n + 1)} \sum_{m=0}^{[n/2]} \frac{\Gamma(2n + 2 - 2m)}{\Gamma(n - m + 3/2) \Gamma(1/2 - m) (n - 2m)!} \frac{(z/2)^{n-2m}}{m!}.
\]
In (A.5) we used the duplication formula
\[ \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + 1/2) \] (A.6)
for the gamma function \( \Gamma(z) \) and the relation \( \Gamma(m+1/2) \Gamma(1/2-m) = \pi / \cos \pi m = (-1)^m \pi \) .
Finally, it remains only to use the identity \( \Gamma(z + 1 - n) = (-1)^n \Gamma(z + 1) / (-z)_n \) and again the duplication formula (A.6) in order to show that
\[
\begin{align*}
&\binom{2}{n, n+2; \frac{3}{2}} \frac{1 - z}{2} = \frac{\sqrt{\pi}}{n + 1} \frac{\Gamma(n + 1)}{\Gamma(1/2)} \sum_{m=0}^{\lfloor n/2 \rfloor} \left( \frac{-1}{m} \right) \frac{(\frac{1 - z}{2})_m}{(-z)_m} \frac{\pi}{m!} \\
&= \frac{2^n}{n + 1} \sum_{m=0}^{\lfloor n/2 \rfloor} \left( \frac{-1}{m} \right) \frac{(\frac{1 - z}{2})_m}{(-z)_m} \frac{\pi}{m!} = (2z)^n \frac{\Gamma(n + 1)}{\Gamma(1/2)} \binom{2}{n, -1; -n} \left( \frac{1 - z}{2} \right).
\end{align*}
\] (A.7)

Appendix B. Proof of formula (3.7)

The second transformation (3.7), stated in section 3, is proved by using first the defining relation for the hypergeometric \( \binom{2}{n} \) polynomial appearing on the left-hand side of (3.7). Thus, for \( n \geq 1 \), we obtain
\[
\binom{2}{n, n; 1/2} \frac{1 - z}{2} = \sum_{k=0}^{n} \left( \frac{-n}{k} \right) \frac{(1 - z)^k}{(1/2)^k} k!
\]
\[
= \sum_{k=0}^{n} \left( \frac{-1}{k} \right) \frac{(n - k)!}{(1/2)^k} \sum_{l=0}^{k} \left( \frac{k}{l} \right) (-z)^l.
\] (B.1)

To obtain (B.1) we have employed the relation \( (-n)_k = (-1)^k n!/(n - k)! \). The next step is to reverse the order of summation in (B.1) with respect to the indices \( k \) and \( l \). In this way we obtain
\[
\binom{2}{n, n; 1/2} \frac{1 - z}{2} = n \Gamma(1/2) \sum_{l=0}^{n} \Gamma(2n - l) \frac{(z/2)^{n-l}}{(n - l)!} \frac{1}{(n - l + 1/2)!} \frac{1}{(n - l + 1/2)!} \frac{1}{(n + 1/2)!}.
\] (B.2)

The sum over index \( k \) in (B.2) represents the hypergeometric polynomial
\[ \binom{2}{-l, 2n - l; n - l + 1/2} \]
calculated for \( x = 1/2 \), which gives (A.3). The sum over index \( k \) in (B.2) thus reduces to
\[
\binom{2}{-l, 2n - l; n - l + 1/2} = \frac{\Gamma(1/2) \Gamma(n - l + 1/2)}{\Gamma(2n - l + 1/2) \Gamma(1/2 + n - l)}.
\] (B.3)

Since the gamma function \( \Gamma(z) \) has poles at the points \( z = -n, n \geq 0 \), the right-hand side of (B.3) vanishes for all odd values of the index \( l \) due to the presence of the factor \( \Gamma(1/2 + n - l) \) in its denominator. This means that only terms with even \( l \)'s give non-zero contribution to the sum over \( l \) in (B.2), and, using the duplication formula (A.6) for the gamma function \( \Gamma(z) \) and the relation \( \Gamma(m + 1/2) \Gamma(1/2 - m) = \pi / \cos \pi m = (-1)^m \pi \), we obtain
\[
\binom{2}{-l, 2n - l; n - l + 1/2} = \frac{\pi}{\Gamma(1/2) \Gamma(n - l + 1/2)} \frac{(z/2)^{n-l}}{(n - l)!} \frac{1}{(n - l + 1/2)!} \frac{1}{(n + 1/2)!} \frac{1}{(n + 1/2)!}.
\] (B.4)
Finally, using the identity \( \Gamma(z + 1 - n) = (-1)^n \Gamma(z + 1)/(-z)_n \) and the duplication formula (A.6) we obtain the final result

\[
\begin{align*}
2F1\left(-n, n; \frac{1}{2}; \frac{1 - z}{2}\right) &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(n + 1)}{\Gamma\left(\frac{z + 1}{2}\right)} \frac{\left(\frac{n}{2}\right)_m \left(\frac{1}{2}\right)_m}{(1 - n)_m} z^{n-2m} m! \\
&= 2^{n-1} \left(\frac{n}{2}\right)_m \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(1 - n)_m}{(1 - n)_m} z^{n-2m} m! = 2^{n-1} z^n 2F1\left(-n, \frac{1}{2}; 1; 1 - n \left| 1/z^2 \right. \right).
\end{align*}
\]  

(B.5)

References

[1] Andrews G E, Askey R and Roy R 1999 *Special Functions (Encyclopedia of Mathematics and Its Applications* vol 71) (Cambridge: Cambridge University Press)

[2] Askey R and Wilson J A 1985 Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials *Mem. Am. Math. Soc.* 54 1–55

[3] Atakishiyev N M 2006 On \( q \)-extensions of Mehta’s eigenvectors of the finite Fourier transform *Int. J. Mod. Phys. A* 21 4993–5006

[4] Atakishiyev N M, Rueda J P and Wolf K B 2007 On \( q \)-extended eigenvectors of the integral and finite Fourier transforms *J. Phys. A: Math. Theor.* 40 12701–7

[5] Bers L and Karal F 1976 *Calculus* (New York: Holt, Rinehart and Winston)

[6] Cigler J 2003 *New Class of \( q \)-Fibonacci polynomials* *Fibonacci Q.* 41 31–40

[7] Cigler J 2003 A new class of \( q \)-Fibonacci polynomials *Electron. J. Comb.* 10 1–15

[8] Cigler J 2009 \( q \)-Lucas polynomials and associated Rogers–Ramanujan type identities arXiv:0907.0165v1

[9] Cigler J and Zeng J 2011 A curious \( q \)-analogue of Hermite polynomials *J. Comb. Theory A* 118 9–26

[10] Dahlquist G 1993 A ‘multigrid’ extension of the FFT for the numerical inversion of Fourier and Laplace transforms *Behav. Inf. Technol.* 33 85–112

[11] Dunlap R A 2006 *The Golden Ratio and Fibonacci Numbers* (Singapore: World Scientific)

[12] Gálvez F J and Dehesa J S 1985 Novel properties of Fibonacci and Lucas polynomials *Math. Proc. Camb. Phil. Soc.* 97 159–64

[13] Gasper G and Rahman M 2004 *Basic Hypergeometric Functions (Encyclopedia of Mathematics and Its Applications* vol 96) 2nd edn (Cambridge: Cambridge University Press)

[14] Ismail M E H 2005 *Classical and Quantum Orthogonal Polynomials in One Variable (Encyclopedia of Mathematics and Its Applications* vol 98) (Cambridge: Cambridge University Press)

[15] Koekoek R, Lesky P A and Swarttouw R F 2010 *Hypergeometric Orthogonal Polynomials and Their \( q \)-Analogues* (Springer Monographs in Mathematics) (Berlin: Springer)

[16] Koshy T 2001 *Fibonacci and Lucas Numbers with Applications* (New York: Wiley)

[17] Lang W 1992 On the characteristic polynomials of Fibonacci chains *J. Phys. A: Math. Gen.* 25 5395–413

[18] Maroni P 1991 Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques *Orthogonal Polynomials and Their Applications (IMACS Ann. Comput. Appl. Math.* vol 9) (Basel, Switzerland: Baltzer) pp 95–130

[19] Matveev V B 2001 Intertwining relations between the Fourier transform and discrete Fourier transform, the related functional identities and beyond *Inverse Problems* 17 633–57

[20] Mehta M L 1987 Eigenvalues and eigenvectors of the finite Fourier transform *J. Math. Phys.* 28 781–5

[21] Nalli A and Hakunaken P 2009 On generalized Fibonacci and Lucas polynomials *Chaos Solitons Fractals* 42 3179–86

[22] Schroeder M 1991 *Fractals, Chaos and Power Laws* (New York: W H Freeman Press)

[23] Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)

[24] Stakhov A and Aranson S 2011 Hyperbolic Fibonacci and Lucas functions, ‘golden’ Fibonacci goniometry, Bodnar’s geometry and Hilbert’s fourth problem: Part I. Hyperbolic Fibonacci and Lucas functions and ‘golden’ Fibonacci goniometry *Appl. Math.* 2 74–84

Stakhov A and Aranson S 2011 Hyperbolic Fibonacci and Lucas functions, ‘golden’ Fibonacci goniometry, Bodnar’s geometry and Hilbert’s fourth problem: Part II. A new geometric theory of phyllotaxis (Bodnar’s geometry) *Appl. Math.* 2 181–8

Stakhov A and Aranson S 2011 Hyperbolic Fibonacci and Lucas functions, ‘golden’ Fibonacci goniometry, Bodnar’s geometry and Hilbert’s fourth problem: Part III. An original solution of Hilbert’s fourth problem *Appl. Math.* 2 283–93