Solutions of the Hamilton–Jacobi equation for one component two dimensional Field Theories

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Abstract

The Hamilton–Jacobi formalism generalized to 2–dimensional field theories according to Lepage’s canonical framework is applied to several covariant real scalar fields, e.g. massless and massive Klein–Gordon, Sine–Gordon, Liouville and $\phi^4$ theories. For simplicity we use the Hamilton–Jacobi equation of DeDonder and Weyl.

Unlike mechanics we have to impose certain integrability conditions on the velocity fields to guarantee the transversality relations between Hamilton–Jacobi wave fronts and the corresponding families of extremals embedded therein.

Bäcklund Transformations play a crucial role in solving the resulting system of coupled nonlinear PDEs.

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1 Introduction

Varying the relativistic invariant action integral leads to the still covariant Euler–Lagrange formulation. However, in getting a canonical Hamiltonian description, one has to break the manifest covariance by distinguishing a time variable and regarding the other Minkowski variables as “indices” counting an infinite number of degrees of freedom. Treating field theories as one parametric “mechanical” Lagrangian systems hides a part of their rich geometrical structure.

If one would like to avoid this by handling all the Minkowski variables on the same footing following Cartan’s framework of forms as applied by Lepage\footnote{A reformulation of these ideas in the multisymplectic framework may be found e.g. in \cite{2,3}.} \cite{1} it turns out that a variety of geometrically distinct Hamiltonian formulations \cite{3} exists, but only in theories which involve more than one real field \cite{1,4}.

Nambu \cite{5} and Hosotani \cite{6} postulated a quantum theory for relativistic strings within this covariant formalism generalizing semiclassical approximations. In “conventional” semiclassical considerations, where probability currents associated with families of extremals are considered, transversality relations between solutions of the Hamilton–Jacobi and the canonical equations play a crucial role. But in contrast to mechanics the ability to embed extremals in a given family of wave fronts can be maintained in field theories only if the corresponding slope functions (velocity fields) satisfy certain integrability conditions (IC).

The simplest examples for such a Hamilton–Jacobi theory involves one dynamical field depending on two space–time (1+1) variables. In this case it is not necessary to distinguish the different canonical formulations mentioned above. We use the formalism of DeDonder and Weyl for simplicity.

We consider the Hamilton–Jacobi equation and the related integrability condition for several 2–dimensional free and selfinteracting models, e.g. massless and massive Klein–Gordon equations, Sine–Gordon, Liouville and $\phi^4$ theory. For the corresponding system of partial differential equations a family of solutions is constructed perturbatively in such a way that a given single extremal of interest (e.g. a soliton like a (anti-) kink or a bell solution) can be embedded in sets of wave fronts to be calculated from the Hamilton–Jacobi equation (HJE) and the corresponding integrability condition. The expansion of the wave fronts in powers of the field variable leads to a hierarchy of nonlinear PDEs that can be reduced to linear PDEs with nonconstant coefficients. By applying Bäcklund transformations they are reduced to wave or free Klein–Gordon equations. Remarkably, after solving only two linear PDEs the general solution for every order of the hierarchy is obtained by integration.

In this way one can construct a \(d = 1\)–parametric family of extremals from one single solution of the canonical equations (strong embedding).

\[ \text{\footnote{A reformulation of these ideas in the multisymplectic framework may be found e.g. in \cite{2,3}.}} \]
2 Lepage’s Canonical Formulation of Mechanics

We very briefly recall Lepage’s main idea of introducing the canonical formalism in mechanics for one configuration variable $q$. The general case is discussed in [1].

The initial canonical Lagrangian form $\omega = L(t, q, \dot{q}) \, dt$ is extended by the product of a Lagrangian multiplier $h(t, q, v)$ and the Pfaffian form $\varrho = dq - v\, dt$ vanishing on the tangent vectors of the extremals $C = C_0 := \{(t, q = q_0(t))\}$. Thus the action integral $A[C]$ over the path $C := \{(t, q(t))\}$ in the extended configuration space $M_{1+1} := \{(t, q)\}$ is modified $A[C] \to \tilde{A}[C]$ without changing the Euler–Lagrange equations and their solutions $C = C_0$:

$$A[C] = \int_C \omega = \int_C [L(t, q(t), \dot{q}(t)) \, dt \quad \Rightarrow \quad \tilde{A}[C] = \int_C \Omega = \int_C [L(t, q, v) \, dt + h(t, q, v) \varrho] .$$

The Lagrangian multiplier $h(t, q, v)$ is fixed by the requirement that a Hamilton–Jacobi theory exists. Hence $\Omega$ has to be locally exact: $\Omega = dS(t, q)$ on every family of extremals covering $M_{1+1} = \{ (t, q) \}$ (or a part of it). The resulting condition: $d\Omega \in I[\varrho]$, namely

$$d\Omega = (\partial_v L - h) \, dv \wedge dt + (dh - \partial_q L \, dt) \wedge \varrho = (\partial_v L - h) \, dv \wedge dt + 0 \text{ (mod} I[\varrho]) \quad (2.1)$$

leads to the standard definition of the canonical momentum: $p := h^1 \partial_v L$, so that

$$\Omega = L \, dt + p \varrho = L \, dt + p \, (dq - v \, dt) = -(pv - L) \, dt + p \, dq = -H \, dt + p \, dq . \quad (2.2)$$

Thus the Legendre transformation $L \to H, v \to p$ can be implemented as a change of the basis in the cotangent bundle $T^*(M_{1+1})$, $\varrho \to dq, dt \to dt$.

The existence of a potential $S(t, q)$ for the basic differential form $\Omega$ yields the familiar Hamilton–Jacobi equation for $S(t, q)$ and the corresponding condition for the momentum:

$$\Omega = -H(t, q, p = \psi(t, q)) \, dt + \psi(t, q) \, dq \overset{!}{=} dS(t, q) = \partial_t S(t, q) \, dt + \partial_q S(t, q) \, dq . \quad (2.3)$$

Comparison of the coefficients yields:

$$\partial_t S(t, q) + H(t, q, p = \psi(t, q)) = 0 , \quad p = \psi(t, q) = \partial_q S(t, q) . \quad (2.4)$$

3 A Canonical Theory for Fields with one Component

We consider the Lagrangian $L$ as a function of one real scalar field $\varphi$ (depending on the variables $z = (x + t)/2, \bar{z} = (x - t)/2$) and the quantities $v, \bar{v}$ which on the extremals coincide with the derivatives of the fields: $v = \partial_z \varphi, \bar{v} = \partial_{\bar{z}} \varphi$.

\footnote{As to the variational principle it is preferable to consider the generalised velocity $v$ as an independent variable. The vanishing of the differential form $\varrho$ ensures the identification of $v(t)$ with $\dot{q}_0(t)$ on the extremals. $\varrho$ generates an ideal $I[\varrho]$ in the algebra $\Lambda$ of forms on the extended configuration space $M_{1+1} := \{(t, q)\}: \forall \theta \in \Lambda \forall \alpha \in I[\varrho] : \theta \wedge \alpha \in I[\varrho]$.}
As in mechanics the canonical 2–form \( \omega = \mathcal{L} \, dz \wedge d\bar{z} \) is extended by two Lagrangian parameters \( h(z, \bar{z}, \varphi) \), \( \bar{h}(z, \bar{z}, \varphi) \) and a 1–form \( \varrho = d\varphi - v \, dz - \bar{v} \, d\bar{z} \) that vanishes on the 2–dimensional extremals \( \varphi = \varphi_0(z, \bar{z}) \):

\[
\Omega = \mathcal{L} \, dz \wedge d\bar{z} + \bar{h} \, dz \wedge \varrho + h \, \varrho \wedge d\bar{z}.
\] (3.1)

The condition \( d\Omega \in I[\varrho] \) — resulting from the requirement that a Hamilton–Jacobi theory exists — is:

\[
d\Omega = (\partial_v \mathcal{L} - h) dv \wedge dz \wedge d\bar{z} + (\partial_p \mathcal{L} - \bar{h}) d\bar{v} \wedge dz \wedge d\bar{z} + 0 \, (\text{mod} I[\varrho]) \, \frac{1}{!} (\text{mod} I[\varrho]) .
\]

This leads to a determination of \( h, \bar{h} \): \( h = \partial_p \mathcal{L} \) and \( \bar{h} = \partial_v \mathcal{L} \). As before the Legendre transformation \( \mathcal{L} \rightarrow \mathcal{H}, v \rightarrow h, \bar{v} \rightarrow \bar{h} \) can be implemented as a change of the basis in the cotangent bundle \( \mathcal{T}^* (\mathcal{M}_{2+1}) \), \( \varrho \rightarrow d\varphi \), \( dz \rightarrow dz, d\bar{z} \rightarrow d\bar{z} \):

\[
\Omega = -\mathcal{H} \, dz \wedge d\bar{z} + \bar{h} \, dz \wedge d\varphi + h \, d\varphi \wedge d\bar{z} , \quad p := h, \quad \bar{p} := \bar{h} \quad \text{and} \quad \mathcal{H} = pv + \bar{p}\bar{v} - \mathcal{L} . \quad (3.2)
\]

The choice of the Hamilton–Jacobi potentials \( S, \bar{S} \) for making \( \Omega \) exact is no longer unique. In the case of DeDonder and Weyl it is the following \([10]\):

\[
\Omega = d \{ S(z, \bar{z}, \varphi) \, d\bar{z} - \bar{S}(z, \bar{z}, \varphi) \, dz \} = dS \wedge d\bar{z} - d\bar{S} \wedge dz . \quad (3.3)
\]

Comparing this expression with equation \((3.2)\) we obtain the Hamilton–Jacobi equation and the conditions for the momenta for one component fields in two dimensions:

\[
\partial_z S + \partial_{\bar{z}} \bar{S} = -\mathcal{H} , \quad \partial_\varphi S = p , \quad \partial_{\varphi} \bar{S} = \bar{p} . \quad (3.4)
\]

In mechanics it is possible to construct wave fronts for 1–parametric families of extremals \( q_0(t) \) that cover a certain region of the configuration space \([1]\) \textit{and} vice versa. Given a solution of the Hamilton–Jacobi equation (HJE), the so–called “slope function”

\[
\Phi(t, q) = \partial_p H(t, q, p=\partial_q S(t, q))
\] (3.5)

determine the corresponding extremals \( q_0(t) \) from the differential equation \( \dot{q}(t) = \Phi(t, q(t)) \) the 1–parametric solution of which exists at least locally.

In general this is \textit{not} true for field theories; the ability to embed extremals \( \varphi_0(z, \bar{z}) \) in a given wave front can be maintained only if the slope functions \( v = \Phi(\varphi, z, \bar{z}), \bar{v} = \bar{\Phi}(\varphi, z, \bar{z}) \) obtained from the inverse Legendre transformation

\[
v = v(p = \partial_\varphi S, \bar{p} = \partial_{\varphi} \bar{S}, z, \bar{z}, \varphi) , \quad \bar{v} = \bar{v}(p = \partial_\varphi S, \bar{p} = \partial_{\varphi} \bar{S}, z, \bar{z}, \varphi)
\] (3.6)

satisfy the integrability condition

\[
\frac{d}{d\bar{z}} \Phi(z, \bar{z}, \varphi(z, \bar{z})) := \partial_z \Phi + \Phi \cdot \partial_{\varphi} \Phi = \frac{d}{dz} \bar{\Phi}(z, \bar{z}, \varphi(z, \bar{z})) := \partial_{\bar{z}} \bar{\Phi} + \bar{\Phi} \cdot \partial_{\varphi} \bar{\Phi} .
\] (3.7)

\(^6\mathcal{M}_{2+1}\) denotes the extended configuration space of two dimensional field theory: \( \mathcal{M}_{2+1} := \{(z, \bar{z}, \varphi)\} \).
4 Hamilton–Jacobi theory for one real field

We here restrict ourselves to Lagrangian densities of the following type: \( L = \partial_z \varphi \partial_{\bar{z}} \varphi - V(\varphi) \). The potential \( V(\varphi) \) is an analytic function. Here we have the canonical momenta \( p = \bar{v}, \quad \bar{p} = v \), the Hamiltonian density \( H = p\bar{p} + V \) and the slope functions \( \Phi = \partial_{\varphi} S, \quad \bar{\Phi} = \partial_{\bar{\varphi}} S \). We have the Hamilton–Jacobi equation

\[
\partial_z S + \partial_{\bar{z}} \bar{S} = \partial_{\varphi} S \partial_{\bar{\varphi}} \bar{S} + V(\varphi)
\]  

(4.1)

and the related integrability condition

\[
\partial_z \partial_{\varphi} S - \partial_{\bar{z}} \partial_{\bar{\varphi}} \bar{S} = \partial_{\varphi} S \partial_{\bar{\varphi}} \bar{S} - \partial_{\varphi} \bar{S} \partial_{\bar{\varphi}} S.
\]  

(4.2)

Knowing solutions \( S \) and \( \bar{S} \) of the equations (4.1) and (4.2) a family of embedded extremals \( \varphi = \bar{\varphi}(z, \bar{z}) \) is determined by a system of first order PDEs:

\[
\partial_z \bar{\varphi}(z, \bar{z}) = \bar{\Phi} = \partial_{\bar{z}} \bar{S}(z, \bar{z}, \varphi = \bar{\varphi}) , \quad \partial_{\bar{z}} \bar{\varphi}(z, \bar{z}) = \Phi = \partial_z S(z, \bar{z}, \varphi = \bar{\varphi}).
\]  

(4.3)

A solution is obtained by expanding \( S(z, \bar{z}, \varphi) \) and \( \bar{S}(z, \bar{z}, \varphi) \) in powers of the difference \( y = \varphi - \varphi_0 \) between \( \varphi \) and a known extremal \( \varphi_0(z, \bar{z}) \):

\[
S(z, \bar{z}, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(z, \bar{z}) y^n , \quad \bar{S}(z, \bar{z}, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{A}_n(z, \bar{z}) y^n.
\]  

(4.4)

Naturally \( \varphi_0 \) has to satisfy the relation (4.3), which fixes the functions \( A_1 = \partial_z \varphi_0 \) and \( \bar{A}_1 = \partial_{\bar{z}} \varphi_0 \) only, without influencing the remaining coefficients. \( A_0 \) and \( \bar{A}_0 \) are only affected by the HJE of zeroth order in \( y \): \( \partial_z A_0 + \partial_{\bar{z}} \bar{A}_0 = \mathcal{L}|_{\varphi=\varphi_0} = \partial_z \varphi_0 \partial_{\bar{z}} \varphi_0 - 2V(\varphi_0) \).

Thus one of them can be chosen arbitrarily.

The IC of first order \( \partial_z A_2 = \partial_{\bar{z}} \bar{A}_2 \) permits to reduce these two functions at least locally to one generating potential function: \( A_2 = \partial_z \ln(\theta) \) and \( \bar{A}_2 = \partial_{\bar{z}} \ln(\theta) \). This logarithmic substitution linearizes the HJE of order \( y^2 \):

\[
\partial_z \partial_{\bar{z}} \theta(z, \bar{z}) + \frac{1}{2} \left\{ \partial_{\varphi}^2 V(\varphi = \varphi_0(z, \bar{z})) \right\} \theta(z, \bar{z}) = 0.
\]  

(4.5)

Though its coefficient is nonconstant the general solution of this PDE can be obtained by employing Bäcklund transformations.

The 1–parametric family of extremals in the vicinity \( (y = \epsilon Y, \quad \epsilon \ll 1) \) of the original extremal \( \varphi_0(z, \bar{z}) \) is determined by integrating the PDEs (4.3):

\[
\bar{\varphi}(z, \bar{z}, c_0) = \varphi_0(z, \bar{z}) + \epsilon c_0 \theta(z, \bar{z}).
\]  

(4.6)

The result is equivalent to that obtained by a second variation of the action with respect to \( \varphi \), where \( \epsilon \) parametrizes a family of extremals.

In general the coefficients \( A_n, \bar{A}_n \) are determined by the \( n \)-th order of the HJE and the \((n-1)\)th order of the IC:

\[
A_n = \theta^{1-n} \partial_z \left[ \frac{\chi_n(z, \bar{z})}{\theta} - \bar{\chi}_n(z, \bar{z}) \right] , \quad \bar{A}_n = \theta^{1-n} \partial_{\bar{z}} \left[ \frac{\chi_n(z, \bar{z})}{\theta} + \bar{\chi}_n(z, \bar{z}) \right] , \quad n \geq 3 ,
\]

\(^7\)Obviously \( A_1 \) and \( \bar{A}_1 \) satisfy the first order of the HJE and the zeroth order of the IC.

\(^8\)For details see [9] and [12].
where the infinite hierarchy of functions $\chi_n(z, \bar{z}), \bar{\chi}_n(z, \bar{z})$ has to fulfill only two PDEs of second order:
\[
\partial_z \partial_{\bar{z}} \chi_n(z, \bar{z}) = \text{Inhomogeneity}, \quad (4.7)
\]
\[
\partial_z \partial_{\bar{z}} \bar{\chi}_n(z, \bar{z}) + \frac{1}{2} \left( \partial^2_{\varphi} V(\varphi_0) \right) \chi_n(z, \bar{z}) = \text{Inhomogeneity} \quad (4.8)
\]
with two inhomogeneities which depend on the functions $A_i, \bar{A}_i, i=1,...n-1$.

From the general ansatz for the Bäcklund transformation [7]
\[
\partial_z \hat{\theta} = F_1 \{ z, \bar{z}, \theta, \hat{\theta}, \partial_z \hat{\theta} \}, \quad \partial_{\bar{z}} \hat{\theta} = F_2 \{ z, \bar{z}, \theta, \hat{\theta}, \partial_{\bar{z}} \hat{\theta} \} \quad (4.9)
\]
and the requirement that $\hat{\theta}$ fulfills a free wave or Klein–Gordon equation and the integrability condition $\partial_z \partial_{\bar{z}} \hat{\theta} = \partial_{\bar{z}} \partial_z \hat{\theta} \Rightarrow dF_1/d\bar{z} = dF_2/dz$, we infer the relations
\[
\partial_z \hat{\theta} = + \partial_z \theta + (\partial_z \psi)(\hat{\theta} + \theta), \quad \partial_{\bar{z}} \hat{\theta} = - \partial_{\bar{z}} \theta + (\partial_{\bar{z}} \psi)(\hat{\theta} - \theta). \quad (4.10)
\]
Here $\psi = \psi(z, \bar{z})$ is required to be a special solution of
\[
\partial_z \partial_{\bar{z}} \psi - (\partial_z \psi)(\partial_{\bar{z}} \psi) - \frac{1}{2} \partial^2_{\varphi} V(\varphi_0) = 0, \quad \partial_z \partial_{\bar{z}} \psi + (\partial_{\bar{z}} \psi)(\partial_z \psi) - m^2 = 0. \quad (4.11)
\]
Knowing $\psi$ and $\hat{\theta}$ we can calculate $\theta$ by integrating the linear BTs (4.10).

### 5 Applications

The Hamilton–Jacobi theory for a free scalar field leads to the wave or the Klein–Gordon equation (1.3) without need of implementing a Bäcklund transformation (BT). Therefore we address to the more interesting case of self-interacting fields. For more details of the following results see [9] and [12].

- Applying our formalism to Liouville’s theory $L = \partial_z \varphi \partial_{\bar{z}} \varphi + 4 \exp(\varphi)$ and using an arbitrary solution of the corresponding equation of motion for which the general expression is known [7],
  \[
  \varphi_0 = \ln \left\{ \frac{2(\partial_z s(z))(\partial_{\bar{z}} s(\bar{z}))}{(s+\bar{s})^2} \right\}, \quad (5.1)
  \]
  the relation (4.3) yields
  \[
  \partial_z \partial_{\bar{z}} \theta - 2 \frac{(\partial_z s)(\partial_{\bar{z}} s)}{(s+\bar{s})^2} \theta = 0, \quad \text{implying} \quad \partial_s \partial_{\bar{s}} \hat{\theta}(s, \bar{s}) = 0 \quad (5.2)
  \]
  by using the transformation $z \rightarrow s(z), \bar{z} \rightarrow \bar{s}(\bar{z})$ and one BT $\psi = \ln(s+\bar{s})$.

- For the Sine–Gordon model $L = \partial_z \varphi \partial_{\bar{z}} \varphi + 4[1 - \cos(\varphi)]$ and starting from the (anti-) kink solution $\varphi_0 = \pm 4 \arctan[\exp(z+\bar{z})]$ we obtain:
  \[
  \partial_z \partial_{\bar{z}} \theta - \{2 \tanh^2(z + \bar{z}) - 1\} \theta = 0. \quad (5.3)
  \]
  This can be reduced by one BT $\psi = \ln[\cosh(z + \bar{z})]$ to a Klein–Gordon equation $\partial_z \partial_{\bar{z}} \hat{\theta} = \hat{\theta}$. Then, using a Fourier transformation and integrating the linear BT we obtain the general solution of (4.5).
Contrary to the two models above the following ones can only be solved by at least two BTs: the \(\phi^4\)-theories I) \(\mathcal{L} = \partial_z \phi \partial_{\bar{z}} \phi - 4\phi^2 + 2\phi^4\) and II) \(\mathcal{L} = \partial_z \phi \partial_{\bar{z}} \phi + 2\phi^2 - 2\phi^4\). In both cases soliton solutions are considered. We choose the (anti-) kink \(\varphi_0 = \pm \tanh(z + \bar{z})\) and the bell solutions \(\varphi_0 = \pm \cosh^{-1}(z + \bar{z})\), respectively. Then, we get the two relations

I) \(\partial_z \partial_{\bar{z}} \theta - \{6 \tanh^2(z + \bar{z}) - 2\} \theta = 0\),

II) \(\partial_z \partial_{\bar{z}} \theta_2 - \{6 \tanh^2(z + \bar{z}) - 5\} \theta_2 = 0\).

Both of them can be reduced to a free Klein–Gordon equation. The first one by the two successive BTs \(\psi_1 = 2 \ln[cosh(z + \bar{z})], \psi_2 = \ln[cosh(z + \bar{z})] + i\sqrt{3}(\bar{z} - z)\) and the second one by using the BTs \(\psi_3 = 2 \ln[cosh(z + \bar{z})] + \sqrt{3}(\bar{z} - z)\) and \(\psi_4 = \ln[cosh(z + \bar{z})]\).

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