A NEW APPROACH FOR WORST-CASE REGRET PORTFOLIO OPTIMIZATION PROBLEM

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Abstract. This paper considers the worst-case regret portfolio optimization problem when the distributions of the asset returns are uncertain. In general, the solution to this problem is NP hard and approximation methods that minimise the difference between the maximum return and the sum of each portfolio return are often proposed. Applying the duality of semi-infinite programming, the worst-case regret portfolio optimization problem with uncertain distributions can be equivalently reformulated to a linear optimization problem, and the established solution approaches for linear optimization can then be applied. An example of a portfolio optimization problem is provided to show the efficiency of our method and the results demonstrate that our method can satisfy the portfolio risk diversification property under the uncertain distributions of the returns.

1. Introduction. From Markowitz’s (1952) seminal work on modern portfolio theory, the portfolio optimization problem is defined as that which seeks to maximize returns by optimally allocating capital into a large number of assets under a minimization constraint of risk [9]. Since then, the portfolio optimization problem has been widely studied through many optimization methods, for example stochastic optimization, fuzzy optimization, and robust optimization [2, 8]. While these optimization methods assume uncertain asset returns, they are applied in different settings. For instance, stochastic optimization is often used when the probability of the uncertainties can be obtained. Fuzzy optimization is applied when the uncertainties can be treated as fuzzy variables. Robust optimization differs from the former two methods as it supposes that the uncertain returns belong to a compact set and the original optimization problem can be reformulated as a computationally tractable problem.

While many portfolio optimization models have been extensively proposed, most of them only focus on the solutions for a single investment which contradicts the portfolio risk diversification property. Therefore, recently, the regret portfolio optimization approach has been proposed to secure the diversification of the investment

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The regret portfolio optimization approach is defined as the minimization of the difference between the maximum return and the sum of each portfolio return which quantifies investor regret when the best possible portfolio return is different from the actual portfolio return. In general, investor regret can be classified into two categories. One regret refers to the case when an investor might regret having allocated too much to an asset which is later revealed to have a very low return. Another regret is when the investor might regret not investing enough into an asset which is later revealed to yield very high returns. Therefore, it makes sense to develop a regret portfolio optimization approach to handle investor regret and optimise the diversification of the investment. However, most of the prevailing regret portfolio optimization approaches are based on stochastic optimization and fuzzy optimization. There is little work on regret portfolio optimization using robust optimization. Ji et al. (2014) propose a worst-case discounted regret portfolio optimization model, in which the robust optimization approach is not used to deal with the uncertain return but the uncertain weight of the regret function [7]. In this paper, we extend this idea to present a robust regret portfolio optimization problem when the distributions of the return are unclear. Our method is different from this method in that we deal with the uncertain return.

In the deterministic portfolio optimization model, asset returns are often assumed. Here the uncertainties in the input data are not included. While both stochastic optimization and robust optimization have been extensively studied to cope with the uncertainties of the input data, they cannot deal with the case when the distributions of the returns are uncertain. Recently, distributional robust optimization or min-max stochastic programming approach has been proposed for the distributional uncertainty case, but the solution for these problems are often NP hard when the first-order and second-order information and the support set information can be obtained [4, 5]. Thus, many approximate methods are presented [1, 3]. However, in some cases, the approximation methods cannot generate satisfactory solutions [3]. In this paper, we extend this idea to the regret portfolio optimization problem with the distributions of the returns being uncertain and propose a linearized worst-case regret portfolio optimization approach. We suppose that the decision maker has some information of the distributions, for example the first-order, support set, and affine first-order information. By applying the duality of semi-infinite programming, the worst-case regret portfolio optimization problem with uncertain distributions can be equivalently reformulated as a linear optimization problem. From there, the established solution approaches for linear optimization can be applied and polynomial algorithms are available to obtain the solution to the worst-case regret portfolio optimization problem.

The rest of this paper is organized as follows. Section 2 provides the definition of the worst case regret optimization problem. Section 3 presents the reformulations for the worst case regret optimization problem with a polyhedral set and a conic support set respectively. Section 4 presents the numerical tests. Section 5 concludes.

2. Worst-case regret portfolio optimization problem. Suppose a capital market has a set of risky assets, each with uncertain returns. We now propose a worst-case regret portfolio optimization model. We first provide some notations. Let $x_i \geq 0$, $i = 1, \cdots, I$ denote the amount invested in the assets, where $x_i \geq 0$, indicates that short selling is not permitted. The uncertain returns of the investments $x := (x_1, \cdots, x_I)^T \in R^I$ in the assets are given by $u := (u_1, \cdots, u_I)^T \in R^I$. We
assume that the uncertain returns $u$ is defined on the probability measurable space $(\Omega, \mathcal{F})$, which is a continuous map from $\Omega$ to $R^l$, that is, $u : \omega \rightarrow R^l$, $\omega \in \Omega$. Different from the stochastic programming approach, instead of presuming knowledge of the actual joint distribution of $u$, we assume that the true distribution information $\mathbb{P}$ of the returns is contained in some family of distributions $\mathbb{F}$. Based on this, a worst-case regret portfolio optimization model or a distributionally robust portfolio optimization model is presented. Hence, we define the worst-case regret as follows,

$$ R(x) := \sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left[ \max_{1 \leq i \leq l} u_i - \sum_{i=1}^{l} u_i x_i \right]. $$ (1)

The worst-case regret portfolio optimization problem (WCRPOP) can be presented as,

$$ \min_{x \in S} R(x) = \sup_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left[ \max_{1 \leq i \leq l} u_i - \sum_{i=1}^{l} u_i x_i \right], $$ (2)

where $S$ is the constraint set for the invested amount $x$ and is represented as $S := \{ x \in R^l | \sum_{i=1}^{l} x_i = 1, \ x_i \geq 0 \}$.

It is well known that for describing the uncertain parameters, a fixed reference point is often introduced first and then a perturbation region around them is used to undertake the description, that is the uncertain parameters $u$ can be given as follows. Let $a \in R^l$ be the reference point of $u$ and $B \in R^{M \times l}$ be a coefficient matrix used to construct a perturbation region around $a$. Denote $z = (z_1, \cdots, z_M)$ as a vector of $M$ random variables defined on $\Omega$. Then we define the perturbation region around $u$ as follows,

$$ U = \{ u | u(\omega) = a + B^T z(\omega), \ \forall \omega \in \Omega \}. $$ (3)

The above method for defining the uncertain parameters has been well used in the robust optimization literature [1].

In this paper, we assume that the probability $\mathbb{P}$ of $z$ satisfies the following conditions (i) The support set of $z$ is $W \subseteq R^l$, that is $\mathbb{P}(z \in W) = 1$;

(ii) The mean value of $z$ is 0, that is $\mathbb{E}_z[z] = 0$;

(iii) The affine first-order information about $z$ is bounded by some constant, that is $\exists g_l \in R^M, h_l \in R, l = 1, \cdots, L$ and $\rho \in R$ such that $\mathbb{E}_z[\max_{1 \leq l \leq L} \{ (g_l)^T z + h_l \}] \leq \rho$.

Conditions (i) and (ii) give the support set and the first-order information about the uncertain parameter respectively which have been applied in the distributional robust optimization approach [4, 5]. Condition (iii) describes the affine first-order information as it can be easily obtained in many applications for example in inventory control and image processing [10, 12]. The most important consideration for the above assumptions is that the worst-case regret portfolio optimization problem based on the above three conditions is equivalent to a linear optimization problem. Hence, the well-established methods developed for linear optimization can be used in our setting. We denote the set of probability satisfying conditions (i)-(iii) as $\mathbb{F}_0(W, \rho)$. In the following section, we show that the worst-case regret portfolio optimization problem with $\mathbb{F}_0(W, \rho)$ is equivalent to a linear optimization problem.

With the probability set $\mathbb{F}_0(W, \rho)$, the worst-case regret (1) can be rewriten as

$$ R(x) = \sup_{\mathbb{P} \in \mathbb{F}_0} \mathbb{E}_{\mathbb{P}} \left[ \max_{1 \leq i \leq l} u_i - \sum_{i=1}^{l} u_i x_i \right] $$
\[
\sup_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[ \max_{1 \leq i \leq I} \left\{ u_i e^T x - u^T x \right\} \right] = \sup_{\mathbb{P}\in\mathcal{F}(W, \rho)} \mathbb{E}_{\mathbb{P}} \left[ \max_{1 \leq i \leq I} \left\{ \bar{a}_i^T x + z^T \bar{B}_i x \right\} \right] \tag{4}
\]

where \( e := (1, \cdots, 1)^T \in \mathbb{R}^I, \bar{a}_i := (a_i - a_1, \cdots, a_i - a_{i-1}, 0, a_i - a_{i+1}, \cdots, a_i - a_I)^T \in \mathbb{R}^I, \bar{B}_i := b_i e^T - B, b_i \) is the \( i \)-th column of matrix \( B \) and the second equality comes from \( e^T x = 1 \).

3. Linearization method for WCRPOP. We now discuss the linearization method for the WCRPOP with the support set \( W \) as polyhedral and conic representations respectively.

3.1. Linearization method with polyhedral support set. The simplest support set often used is the polyhedral support set. We first present a tractable reformulation for WCRPOP with the polyhedral support set as an equivalent linear optimization problem. We assume that \( W \) is a polytope and \( z^1, \cdots, z^J \) are the corresponding extreme points.

**Lemma 3.1.** Let the probability \( \mathbb{P} \) of \( z \) satisfy conditions (i)-(iii) and the support set \( W \) be the polytope with extreme points \( z^1, \cdots, z^J \). Then the worst-case regret (4) is equivalent to the following linear optimization problem,

\[
\inf_{\mu, \mu_0, s, r, \xi} \mu_0 + \rho s \\
\text{s.t. } \mu_0 - \bar{a}_i^T x + \xi \geq 0, \forall i, l \\
r \geq g_l^T z^j + h_l, \forall l, j \\
\xi \geq (\mu - x^T \bar{B}_i^T)^T z^j + sr, \forall i, j \\
s \geq 0,
\]

(5)

where \( i = 1, \cdots, I, l = 1, \cdots, L \) and \( j = 1, \cdots, J \).

**Proof.** Let \( f \) be the p.d.f. of random variable \( z \), then (4) is equivalent to the following problem

\[
\sup_{f()} \int_W \max_{1 \leq i \leq I} \left\{ \bar{a}_i^T x + z^T \bar{B}_i x \right\} f(z) dz \\
\text{s.t. } \int_W f(z) dz = 1 \\
\int_W z f(z) dz = 0 \\
\int_W \max_{1 \leq i \leq I} \left\{ (g_l)^T z + h_l \right\} f(z) dz \leq \rho, \\
f(z) \geq 0, \forall z \in W.
\]

Let \( \mu_0, \mu \) and \( s \) be dual variables to the first three constraints. Then, from the strong duality theorem proposed by [6], (6) is equivalent to the following problem

\[
\inf_{\mu_0, \mu, s} \mu_0 + \rho s \\
\text{s.t. } \mu_0 + \mu^T z + (s \max_{1 \leq i \leq L} \left\{ (g_l)^T z + h_l \right\}) \\
\geq \max_{1 \leq i \leq I} \left\{ \bar{a}_i^T x + z^T \bar{B}_i x \right\}, \forall z \in W \\
s \geq 0.
\]
The above problem is equivalent to
\[
\inf_{\mu, \mu_0, s} \mu_0 + \rho s \\
\text{s.t. } \mu_0 - \bar{a}_i^T x + \min_{z \in W} \left( (\mu - x^T \bar{B}_i^T) z + s \max_{1 \leq l \leq L} \{g_l^T z + h_l\} \right) \geq 0, \forall i \tag{7}
\]
\[s \geq 0.\]

From the assumption on \( W \), the minimization problem about \( z \) in (7) can be equivalently transformed into
\[
\begin{align*}
\min_{r, \xi} & \quad \xi \\
\text{s.t.} & \quad r \geq g_l^T z_j + h_l, \forall l, j \\
& \quad \xi \geq (\mu - x^T \bar{B}_i^T) z_j + sr, \forall i, j.
\end{align*}
\tag{8}
\]
Therefore from (7) and (8), the conclusion holds.

It follows from the above lemma that WCRPOP with the worst case regret (4) is equivalent to a linear optimization problem set forth in the next theorem.

**Theorem 3.2.** Suppose the assumptions on the information of random variable \( z \) in Lemma 3.1 hold. Then, WCRPOP with the worst case regret (4) is equivalent to the following linear optimization problem

\[
\begin{align*}
(LP1) & \quad \min_{\mu, \mu_0, s, r, \xi, \tau, x} \tau \\
\text{s.t.} & \quad \tau \geq \mu_0 + \rho s \\
& \quad \mu_0 - \bar{a}_i^T x + \xi \geq 0, \forall i \\
& \quad r \geq g_l^T z_j + h_l, \forall l, j \\
& \quad \xi \geq (\mu - x^T \bar{B}_i^T) z_j + sr, \forall i, j \\
& \quad x \in S, \ s \geq 0.
\end{align*}
\tag{9}
\]

**Proof.** Clearly, WCRPOP is equivalent to the following optimization problem

\[
\begin{align*}
\min_{r, x} & \quad \tau \\
\text{s.t.} & \quad \tau \geq R(x), \ x \in S,
\end{align*}
\tag{10}
\]
which together with Lemma 3.1 imply that this theorem holds.

The above theorem gives the tractable reformulation about WCRPOP when all of the extreme points of the support set \( W \) can be enumerated. However, when \( W \) is defined by linear constraints, the number of extreme points of \( W \) increases exponentially with the growth of the dimension \( M \). It is difficult to enumerate all of the extreme points of \( W \). So the reformulation is different from the above theorem when the support set is given by inequality and equality constraints. This brings us to the next theorem.

**Theorem 3.3.** Let the probability \( P \) of \( z \) satisfy conditions (i)-(iii). Suppose that the nonempty support set \( W \) is defined as \( W := \{ z \in R^M \mid Az = a, \ Cz \geq b \} \) where \( A \in R^{K \times M}, \ C \in R^{P \times M}, \) where \( K \) and \( P \) are two positive integers. Then
the WCRPOP with the worst case regret (4) is equivalent to the following linear optimization problem,

\[
\begin{align*}
\min_{\mu, \mu_0, s, \tau, x', y', y''} & \tau \\
\text{s.t.} & \quad \tau \geq \mu_0 + ps \\
& \mu_0 - \bar{a}_i^T x + h_i^T y' + a_i^T y'' + b_i^T y''' \geq 0, \forall i \\
& \begin{bmatrix}
-\bar{g}_1^T & 1 \\
\vdots \\
-\bar{g}_L^T & 1 \\
A & 0 \\
C & 0
\end{bmatrix}
\begin{bmatrix}
y' \\
y'' \\
y'''
\end{bmatrix}
= \begin{bmatrix}
\mu - x^T \bar{B}_1^T \\
\vdots \\
\mu - x^T \bar{B}_M^T \\
s
\end{bmatrix}, \\
x \in S, \quad s \geq 0, \quad y' \geq 0, \quad y'' \geq 0
\end{align*}
\]

where \( h = (h_1, \cdots, h_L)^T \in R^L \).

**Proof.** The inner minimization problem about \( z \) in (7) with \( W \) can be equivalently transformed into

\[
\begin{align*}
\min_{z,r} & \ (\mu - x^T \bar{B}_1^T)^T z + sr \\
\text{s.t.} & \quad r \geq g_l^T z + h_l, \forall l \\
& \quad A z = a, C z \geq b.
\end{align*}
\]

From duality theory, the dual problem to (12) is

\[
\begin{align*}
\max_{y' \in R^L, y'' \in R^K, y''' \in R^P} & \ h_i^T y' + a_i^T y'' + b_i^T y''' \\
\text{s.t.} & \quad \begin{bmatrix}
-\bar{g}_1^T & 1 \\
\vdots \\
-\bar{g}_L^T & 1 \\
A & 0 \\
C & 0
\end{bmatrix}
\begin{bmatrix}
y' \\
y'' \\
y'''
\end{bmatrix}
= \begin{bmatrix}
\mu - x^T \bar{B}_1^T \\
\vdots \\
\mu - x^T \bar{B}_M^T \\
s
\end{bmatrix}, \\
y' \geq 0, \quad y''' \geq 0.
\end{align*}
\]

Therefore, it follows from the strong duality of linear optimization and (7), (10), (12), and (13) that this theorem is true.

**3.2. Linearization method with conic support set.** In many practical applications, the support sets are often described as second order cones which are a special case of the general convex cones. Hence, it is meaningful to study the WCRPOP with a conic support set which presents a general approach for representing the second order cone support sets other than polyhedral support sets. As such, we first discuss the reformulation with one conic support set, then generalize it to multiple conic support sets.

**3.2.1. WCRPOP with single conic support set.** We first treat the case where the support set \( W \) is defined as a conic representation

\[
W = \{ z : H z + D u \succcurlyeq_K b \text{ for some } u \in R^P \},
\]

where \( H z + D u \succcurlyeq_K b \) means that \( H z + D u - b \in K \), \( K \) is a closed convex cone in \( R^N \), \( H \in R^{N \times M} \) and \( D \in R^{N \times P} \) are two given matrices, and \( b \in R^N \) is a fixed vector.
Further, we assume that Slater’s condition holds, that is, there \( \exists \bar{z} \in R^M \) and \( \bar{u} \in R^P \) such that \( H\bar{z} + D\bar{u} \succ_K b \). We can then show that the WCRPOP with conic support set (14) is equivalent to a linear optimization problem with conic constraints.

**Theorem 3.4.** Let the probability \( \mathbb{P} \) of \( z \) satisfy conditions (i)-(iii). Suppose \( W \) is defined by (14). Then the WCRPOP with the worst case regret (4) is equivalent to the following linear optimization problem,

\[
\min_{\mu, \mu_0, s, r, x, \lambda, \nu} \tau \\
\text{s.t. } \tau \geq \mu_0 + \rho s
\]

\[(LP3) \quad H^T \lambda_i - \sum_{l=1}^{L} \nu_{l,i} g_l = \mu - x^T \bar{B}^T, \forall i
\]

\[
D^T \lambda_i = 0, \quad \sum_{l=1}^{L} \nu_{l,i} = s, \forall i
\]

\[
\lambda_i \succ_K \lambda^*, \nu_{l,i} \geq 0, \forall i, l
\]

\[
x \in S, \ s \geq 0
\]

where \( \lambda = (\lambda_1, \cdots, \lambda_n) \in R^{N \times I} \), \( \nu = (\nu_{l,i})_{L \times I} \in R^{L \times I} \) and \( K^* \) is the dual cone to \( K \).

**Proof.** With the support set defined as (14), the inner minimization problem about \( z \) in (7) is equivalent to

\[
\min_{z,r} (\mu - (\bar{B}x)^T z + sr) \\
\text{s.t. } r \geq g^T z + h_l, \forall l
\]

\[
Hz + Du \succ_K b \text{ for some } u.
\]

The corresponding dual problem is

\[
\max_{\lambda_i, \nu_i} \lambda_i^T b + \sum_{l=1}^{L} h_l \nu_{l,i}
\]

\[
\text{s.t. } H^T \lambda_i - \sum_{l=1}^{L} \nu_{l,i} g_l = \mu - x^T \bar{B}^T, \forall i
\]

\[
D^T \lambda_i = 0, \quad \sum_{l=1}^{L} \nu_{l,i} = s,
\]

\[
\lambda_i \succ_K \lambda^*, \nu_{l,i} \geq 0, \forall i, l
\]

where \( K^* \) is the dual cone to \( K \), \( \nu_i = (\nu_{1,i}, \cdots, \nu_{L,i})^T \in R^L \) and \( i \in \{1, \cdots, I\} \). It follows from Slater’s condition that strong duality holds [1]. Thus, the above primal and dual problems have the same optimal value and the theorem holds.

3.2.2. **WCRPOP with multiple conic support sets.** We now give the reformulation for WCRPOP with multiple conic support sets. Consider the support set \( W \) defined as the direct product of \( Q \) simpler cones \( K^1, \cdots, K^Q \) with

\[
W = \{ z : H^T z + D^T u^\gamma \succ_K \gamma b^\gamma, \gamma = 1, \cdots, Q, \text{ for some } u^\gamma \in R^P \}
\]
where $K^\gamma$, $\forall \gamma$ are all closed convex cones in $R^N$, $H^\gamma \in R^{N \times M}$ and $D^\gamma \in R^{N \times P}$, $\forall \gamma$ are the given matrices, and $h^\gamma \in R^N$, $\forall \gamma$ are fixed vectors.

Similar to Theorem 3.4, suppose Slater’s condition holds, that is, $\exists \bar{z} \in R^M$ and $\bar{u}^\gamma \in R^P$ s.t. $H^\gamma \bar{z} + D^\gamma \bar{u}^\gamma \succ K^\gamma b^\gamma$, $\gamma = 1, \cdots, Q$, then from the proof of Theorem 3.4, the WCRPOP with multiple conic support sets (18) is equivalent to a linear optimization problem with conic constraints as in the following corollary.

**Corollary 1.** Let the probability $P$ of $z$ satisfy conditions (i)-(iii) and $W$ is defined by (18). Then the WCRPOP with the worst case regret (4) is equivalent to the following linear optimization problem

$$
\min_{\mu, \mu_0, \bar{u}, x, (\lambda^\gamma)_\gamma \in R^{L \times I}, \nu} \tau
\text{ s.t. } \tau \geq \mu_0 + \rho s
\mu_0 - \bar{a}_i^T x + \sum_{\gamma=1}^Q (\lambda^\gamma_i)^T \nu_i g_i = \mu - x^T B^*_i, \forall i
\sum_{\gamma=1}^Q (D^\gamma)^T \lambda^\gamma_i = 0, \sum_{i=1}^L \nu_i s_i = s, \forall i, \gamma
\lambda^\gamma_i \succ K^\gamma, 0, \nu_i \geq 0, \forall i, l, \gamma
x \in S, s \geq 0
$$

(19)

where $\lambda^\gamma = (\lambda^\gamma_1, \cdots, \lambda^\gamma_i) \in R^{N \times I}$, $\forall \gamma$, $\nu = (\nu_i)_i \in R^{L \times I}$ and $K^\gamma*$ is the dual cone to $K^\gamma$.

Corollary 3.5 gives the result when the support set is represented as the direct product of simpler cones. However, in some cases, this representation is invalid, for example, when decision makers have multiple different choices to cluster around different regions and it can be given as separate cones. In this case, the cone $W$ can be described as $\bigcup \gamma K^\gamma$, that is

$$W = \{ z : \exists \gamma \in \{1, \cdots, Q\}, \text{ s.t. } H^\gamma z + D^\gamma u^\gamma \succ K^\gamma b^\gamma, \text{ for some } u^\gamma \in R^P \}. \quad (20)$$

Again, suppose Slater’s condition holds, i.e. $\exists \bar{z} \in R^M$ and $\bar{u}^\gamma \in R^P$ s.t. for some given $\gamma \in \{1, \cdots, Q\}$, $H^\gamma \bar{z} + D^\gamma \bar{u}^\gamma \succ K^\gamma b^\gamma$. So based on this assumption, similar to the proof of Theorem 3.4, we can show that WCRPOP with multiple conic support sets (18) is equivalent to a linear optimization problem with conic constraints as in the following corollary.

**Corollary 2.** Let the probability $P$ of $z$ satisfy conditions (i)-(iii) and $W$ is defined by (18). Then the WCRPOP with the worst case regret (4) is equivalent to the following linear optimization problem,

$$
\min_{\mu, \mu_0, \bar{u}, x, (\lambda^\gamma)_\gamma \in R^{L \times I}, \nu, \xi} \tau
\text{ s.t. } \tau \geq \mu_0 + \rho s
\mu_0 - \bar{a}_i^T x + \sum_{i=1}^L h_i \nu_i \geq 0, \forall i
(H^\gamma)^T \lambda^\gamma_i - \sum_{i=1}^L \nu_i g_i = \mu - x^T B^*_i, \forall i, \forall \gamma
(D^\gamma)^T \lambda^\gamma_i = 0, \sum_{i=1}^L \nu_i s_i = s, \forall i, \gamma
(\lambda^\gamma_i)^T b^\gamma \leq \xi, \forall i, \gamma
$$

(21)


\[ \lambda_i^* \succ K\gamma, \ 0, \ \nu_i, l, \gamma \succ 0, \ \forall i, l, \gamma \]

\[ x \in S, \ s \geq 0. \]

From the above discussion, the worst-case regret portfolio optimization problem with uncertain distributions can be equivalently reformulated to a linear optimization problem which is different from the distributional robust approaches with the nonlinear and nonconvex reformulation [14, 13, 11].

4. Numerical tests. We now present the numerical tests for the above methods. The codes are written in Matlab 7.0 and the tests are conducted on a DELL computer with Intel(R) Core(TM)i5-2400 processor (3.10 GHz+3.10 GHz) and 4.00 GB of memory. We first assume that all the elements in vector \( a \) and matrix \( B \) in (5) are stochastically generated in \([-1, 1]\). Next, the affine first-order information about \( z \) is s.t. \( g_l \) and \( h(l) \), \( l = 1, \cdots, 5 \) and \( \rho \) are stochastically generated in \([-1, 1]\). We note that the statistics generated in this section are uniformly distributed. For simplicity, we assume that the support set is given by

\[ W := \{z : \|z - \bar{z}\|_2 \leq \hat{z}\}, \]

where the elements of \( \bar{z} \) are stochastically generated in \([-1,1]\) and \( \hat{z} \) is stochastically generated in \([1, 2]\).

To show the efficiency of the proposed methods, we compare their numerical performance with the discounted robust regret method proposed by [7]. For the discounted robust regret approach, we assume that the setting of the uncertain returns is same as the method in this paper and the weight set is defined as \( \{w_i = 1, \ w_i \geq 0, \ i = 1, \cdots, I\} \). We note that every problem is solved ten times for these two approach respectively. The column headings in the tables ‘\( D^1 \)’ and ‘\( D^2 \)’ stand for the dimension of the vector \( x \) and \( z \) respectively; ‘Iter’ represents the iterations for the problem; ‘Time’ is the CPU time needed (second); Wrv and Wdrv represent the corresponding regret values with the approaches in this paper and the paper of [7] respectively. In order to obtain the reduction in investor regret, we define the index \( Ix := (Wdrv - Wrv)/Wdrv \). The last column presents the reduction in regret in each experiment, which ranges from \(-23.18\%\) to \(25.05\%\), with an average reduction of \(24.52\%\). This implies that the method in this paper is effective in averagely reducing the expected regret. Further, the method is also more effective in iteration and computation time.

| \( D^1 \) | \( D^2 \) | Iter | Time | Wrv | Time | Wdrv | Ix  |
|---------|---------|------|------|------|------|------|-----|
| 50      | 10      | 6    | 0.012| 0.2024| 0.017| 0.2255| 0.1024|
| 50      | 25      | 6    | 0.013| 0.2133| 0.019| 0.2371| 0.1003|
| 100     | 25      | 21   | 0.11 | 0.2978| 0.174| 0.2494| -0.1941|
| 100     | 50      | 23   | 0.129| 0.2347| 0.285| 0.1936| 0.1914|
| 200     | 50      | 37   | 0.930| 0.5484| 1.708| 0.5195| -0.0556|
| 200     | 100     | 37   | 1.288| 0.3922| 2.502| 0.3184| -0.2318|
| 300     | 100     | 49   | 3.577| 0.5266| 4.81 | 0.7026| 0.2505|
| 300     | 200     | 78   | 9.563| 0.4486| 11.84| 0.4887| 0.0821|

**TABLE 1. Numerical Results**
5. **Conclusion.** In this paper, we consider a portfolio optimization problem that the distributions of the asset returns are uncertain. A portfolio optimization model is then proposed with a worst-case regret as the risk measure. When the information on the distributions, for example the first-order, support set, and affine first-order information, is available, the established worst-case regret portfolio optimization problem can be equivalently reformulated as a linear optimization problem. Then, the original problem can be solved by using the well-established algorithm for linear optimization problems. The numerical tests show the efficiency of the proposed method and implies that our methods are more efficient for solving medium-scale problems. Therefore, to explore how to solve the large scale problems is our further studies. Another extension is to study other practical problems by using the proposed method.

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**REFERENCES**

[1] A. Ben Tal and A. Nemirovski, Robust convex optimizaion, *Math Oper Res*, 23 (1998), 769–805.
[2] L. Chen, S. He and S. Zhang, Tight bounds for some risk measures with applications to robust portfolio selection, *Operations Research*, 59 (2011), 847–855.
[3] X. Chen, M. Sim and P. Sun, A robust optimization perspective on stochastic programming, *Operational Research*, 55 (2007), 1058–1071.
[4] E. Delage and Y. Y. Ye, Distributionally robust optimization under moment uncertainty with application data-driven problems, *Operational Research*, 58 (2010), 595–612.
[5] J. Goh and M. Sim, Distributionally robust optimization and its tractable approximations, *Operational Research*, 58 (2010), 902–917.
[6] K. Isii, On the sharpness of Chebyshev-type inequalities, *Annals of the Institute of Statistical Mathematics*, 14 (1962/1963), 185–197.
[7] Y. Ji, T. N. Wang and M. Goh, The worst-case discounted regret portfolio optimization problem, *Applied Mathematics and Computation*, 239 (2014), 310–319.
[8] X. Li, B. Y. Shou and Z. F. Qin, An expected regret minimization portfolio selection model, *European Journal of Operational Research*, 218 (2012), 484–492.
[9] H. M. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1952), 77–91.
[10] D. Y. Meng, Q. Zhao and Z. B. Xu, Improve robustness of sparse PCA by $L_1$-norm maximization, *Pattern Recognit*, 45 (2012), 487–497.
[11] X. J. Tong and F. Wu, Robust reward-risk ratio optimization with application in allocation of generation asset, *Optimization*, 63 (2014), 1761–1779.
[12] M. R. Wagner, Fully distribution-free profit maximization: The inventory management case, *Math Operation Reserch*, 35 (2010), 728–741.
[13] W. Wiesemann, D. Kuhn and M. Sim, Distributionally robust convex optimization, *Operations Research*, 62 (2014), 1358–1376.
[14] S. Zymler, D. Kuhn and B. Rustem, Distributionally robust joint chance constraints with second-order moment information, *Mathematical Programming*, 137 (2013), 167–198.

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