A UNIFORM RANDOM POINTWISE ERGODIC THEOREM

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ABSTRACT. Let \( a_n \) be the random increasing sequence of natural numbers which takes each value independently with decreasing probability of order \( n^{-\alpha} \), \( 0 < \alpha < 1/2 \). We prove that, almost surely, for every measure-preserving system \((X,T)\) and every \( f \in L^1(X) \) orthogonal to the invariant factor the modulated, random averages

\[
\sup_b \left| \frac{1}{N} \sum_{n=1}^N b(n) T^{a_n} f \right|
\]

converge to 0 pointwise almost everywhere, where the supremum is taken over a set of bounded functions with certain uniform approximation properties; good examples of such functions are given by

\[
A_{\delta, M} := \{e^{2\pi i t c} : m + \delta \leq c \leq m + 1 - \delta, 1 \leq m \leq M\}
\]

where \( M \geq 1 \) and \( 1/2 \geq \delta > 0 \) are arbitrary. This work improves upon previous work of the authors, in which a non-uniform statement was proven for some specific functions \( b \). Under further conditions on \( \{b\} \) we prove pointwise convergence to zero of the above averages for general \( f \in L^1(X) \); these conditions are met, for instance, by the sets of functions \( A_{\delta, M} \).

1. INTRODUCTION

Pointwise ergodic theory concerns the asymptotic pointwise behavior of the averages

\[
\frac{1}{N} \sum_{n \leq N} T^{a_n} f
\]

where \( \{a_n\} \subset \mathbb{N} \), and \( f \) is some \( L^p \) function in a measure-preserving system: a probability space \((X,\mu)\) equipped with a measure-preserving transformation, \( T : X \to X \). Birkhoff’s pointwise theorem [2] is simply the statement that, when \( a_n = n \), the averages in (1) converge pointwise \( \mu \)-a.e. for \( f \in L^1(X) \).

In his celebrated paper [3], Bourgain initiated a study of “random” pointwise ergodic theorems, where the subsequence \( \{a_n\} \) is randomly generated.

From now on \( \{X_n\} \) will denote a sequence of independent \( \{0,1\} \) valued random variables (on a probability space \( \Omega \)) with expectations \( \sigma_n \). The counting function

\[
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\]
\(a_n(\omega)\) is the smallest integer subject to the constraint 
\[X_1(\omega) + \cdots + X_{a_n(\omega)}(\omega) = n.\]

Bourgain established the following result.

**Theorem 1.2** ([3, Proposition 8.2]). Suppose 
\[\sigma_n = \frac{(\log \log n)^B_p}{n}, \quad B_p > \frac{1}{p-1}, \quad 1 < p \leq 2.\]

Then, almost surely, for each measure-preserving system, and each \(f \in L^p(X)\), the averages \(\frac{1}{N} \sum_{n \leq N} T^{a_n} f\) converge pointwise \(\mu\)-a.e.

Our main result is in the spirit of Bourgain’s uniform version [4] of the Wiener–Wintner theorem [12] (see Assani [1] for more results in this direction). It is uniform over the following classes of weights.

**Definition 1.3.** A collection of functions \(B := \{b\}, \ b : \mathbb{N} \to \mathbb{C}\) is approximable if for every \(\delta > 0\) there exists some \(\kappa = \kappa(\delta) > 0\) so that for every (sufficiently large) integer \(N\), there exist finite subsets \(B_N \subset B\) with the following two properties:

- \(|B_N| \leq C_\delta e^{N\kappa}\) for some constant \(C_\delta\) depending only on \(\delta\);
- for any \(b \in B\), there exists some \(b_0 \in B_N\) so that
  \[\sup_{t \leq N} |b(t) - b_0(t)| \leq C_\kappa N^{-\kappa}\]
  for some \(C_\kappa = C_\kappa(\delta)\).

Good examples of approximable sets of functions are
\[B_I := \{e(n^c) : c \in I\}, \quad e(t) := e^{2\pi i t},\]
for finite intervals \(I \subset \mathbb{R}\); note that finite unions of approximable sets remain approximable.

**Theorem 1.4.** Let \(B\) be an approximable set of functions bounded in magnitude by 1, and suppose \(\sigma_n = n^{-\alpha}\) for some \(0 < \alpha < 1/2\). Then, almost surely, the following holds: For every measure-preserving system \((X, \mu, T)\), and every \(f \in L^1(X)\) orthogonal to the invariant factor,
\[\sup_{b \in B} \left| \frac{1}{N} \sum_{n \leq N} b(n)T^{a_n(\omega)} f \right| \to 0\]
\(\mu\)-a.e.

**Remark 1.5.** The restriction to the orthogonal complement of the invariant factor in Theorem 1.3 can be removed provided that
\[\sup_{b \in B} \left| \frac{1}{N} \sum_{n \leq N} b(n) \right| \to 0.\]
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The latter property holds for large classes of Hardy field functions, see [6, Theorem 2.10] (with \( f = 1_X \)); for an introduction to Hardy field functions and their properties, we refer the reader to e.g. [6, §2].

Note that by the strong law of large numbers, if \( \sigma_n = n^{-\alpha} \), almost surely there exists a constant \( C_\omega \) so that for all large \( n \),

\[
C_\omega^{-1} n^{1-\alpha} \leq a_n(\omega) \leq C_\omega n^{1-\alpha}.
\]

In particular, our sequence \( \{a_n(\omega)\} \) is asymptotically much denser than the sequence of squares.

This restriction appears in our proof because we exploit cancellation via a \( TT^* \) argument. If \( \sigma_n = n^{-1/2} \), then almost surely for large \( N \), \( a_n \) grows like \( n^{2} \), and fewer than a constant multiple of \( N^{1/2} \) elements of the interval \( \{1, \ldots, N\} \) generically appear in the sequence \( \{a_n\} \), so that their difference set fails to cover \( \{1, \ldots, N\} \) with a (generic) multiplicity which grows with \( N \). But, the \( TT^* \) argument we use is effective only when the generic behavior of

\[
\sum_{1 \leq n, n+h \leq N} X_{n+h}X_n, \ 1 \leq |h| \leq N
\]

concentrates strongly around its expected value, which leads to significant cancellation in the centered variant of the above random sum. This concentration occurs whenever \( \sigma_n = n^{-\alpha}, \ 0 < \alpha < 1/2, \) and LaVictoire’s maximal ergodic theorem [9] similarly exploits this concentration to show that the corresponding maximal operator has weak type \((1,1)\) under similar conditions.

The structure of this paper is as follows:
\begin{itemize}
  \item In §2 we introduce a few preliminary tools;
  \item In §3 we establish our key analytic inequality;
  \item Finally, we complete the proof of Theorem 1.3 in §4.
\end{itemize}

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2. Preliminaries

2.1. Notation and Tools. With \( X_n, \sigma_n \) as above, we let \( Y_n := X_n - \sigma_n \). We let

\[
S_N = \sum_{n=1}^{N} X_n \quad \text{and} \quad W_N = \sum_{n=1}^{N} \sigma_n
\]

so that \( W_N \) grows like \( N^{1-\alpha} \).

We will make use of the modified Vinogradov notation. We use \( X \lesssim Y \), or \( Y \gtrsim X \) to denote the estimate \( X \leq CY \) for an absolute constant \( C \). We use \( X \approx Y \) to mean
that both $X \lesssim Y$ and $Y \lesssim X$. If we need $C$ to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_\omega Y$ denotes the estimate $X \leq C_\omega Y$ for some $C_\omega$ depending on $\omega$.

We will require the following large deviation “martingale” inequality:

Let $\{Z_1, \ldots, Z_n\}$ be a sequence of scalar random variables with $|Z_i| \leq 1$ almost surely. Assume also that we have the martingale difference property

$$E(Z_i|Z_1, \ldots, Z_{i-1}) = 0$$

almost surely for all $1 \leq i \leq n$. Set

$$V_i := \text{Var}(Z_i|Z_1, \ldots, Z_{i-1}),$$

and $T_j := \sum_{i=1}^j V_i$. Note that in the case where the $\{Z_i\}$ are independent, $V_i = E|Z_i|^2$.

Then, we have the following large deviation inequality due to Freedman [7, Theorem 1.6].

**Proposition 2.1** (Freedman’s Martingale Inequality, Special Case). With the above notation, for any real numbers $a, b > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n Z_i\right| \geq a, T_n \leq b\right) \leq 2e^{-\frac{a^2}{2(b+a)}}.$$

### 3. A Key Proposition

The focus of this section is to prove an $\ell^2(\mathbb{Z})$ inequality for functions $f : \mathbb{Z} \to \mathbb{C}$. Here is the set-up.

Fix some constant of lacunarity $\rho > 1$, which we will think of as arbitrarily close to 1; henceforth, all upper case indices, $M, N,$ etc. will belong to the sequence

$$\{[\rho^k] : k \geq 0\}.$$

For each $N$, suppose that $B_N$ are a finite collection functions, all bounded in magnitude by 1, with

$$|B_N| \lesssim \delta e^{N\delta}$$

for any $\delta > 0$.

We will be interested in bounding the $\ell^2$-norm of the maximal functions

$$\mathcal{M}_N f(x) := \sup_{b \in B_N} \left| \frac{1}{N^{1-\alpha}} \sum_{n \leq N} Y_n b(S_{n-1}) f(x-n) \right|$$

with high probability. Here is our proposition.

**Proposition 3.1.** Suppose $0 < \alpha < 1/2$. Then, for some $\epsilon = \epsilon(\alpha) > 0$, $\omega$-almost surely we may estimate

$$\|\mathcal{M}_N f\|_{\ell^2} \lesssim_\omega N^{-\epsilon} \|f\|_{\ell^2}.$$
Proof. The proof is by linearization and $TT^*$. Specifically, for an appropriate disjoint partition of $\mathbb{Z}$, $\{E_b\}_{b \in B_N}$, we may express

$$\mathcal{M}_N f = |T_N f|,$$

where

$$T_N f(x) := \frac{1}{N^{1-\alpha}} \sum_{b \in B_N} 1_{E_b}(x) \sum_{n \leq N} Y_n b(S_{n-1}) f(x-n).$$

Then, $T_N T_N^* f(x)$ can be expressed as the sum of two terms

$$\frac{1}{N^{2-2\alpha}} \sum_{b,b'} \sum_{0 < |h| \leq N} K_N(h; b, b')(1_{E_b} f)(x+h),$$

where

$$K_N(h; b, b') := \sum_{1 \leq n, n+h \leq N} Y_{n+h} Y_n b(S_{n+h-1}) \overline{b}(S_{n-1})$$

and

$$\left( \frac{1}{N^{2-2\alpha}} \sum_{n \leq N} Y_n^2 \right) \cdot f(x).$$

The goal will now be to show that, $\omega$-almost surely

$$\|T_N T_N^* f\|_{l^2} \lesssim_\omega N^{-2\alpha} \|f\|_{l^2}.$$  \hfill (3.2)

Now, by Proposition 2.1, or more simply by Chernoff’s inequality, [11], and a Borel-Cantelli argument, we see that $\omega$-almost surely

$$\|(3)\|_{l^2} \lesssim_\omega N^{\alpha-1} \|f\|_{l^2},$$

so we will disregard it in what follows. We will also restrict attention in what follows to positive $1 \leq h \leq N$, as the case of negative $h$ can be handled by similar arguments.

We begin with the following observation, which we state in the form of the following lemma.

Lemma 3.3. For any $\delta > 0$, there exists an absolute constant $c$ so that

$$\mathbb{P} \left( \sup_{1 \leq h \leq N} \left| \sum_{n=1}^{N-h} \sigma_{n+h} \left( |Y_n|^2 - \mathbb{E}|Y_n|^2 \right) \right| \gtrsim N^{1-2\alpha} \right) \lesssim_\delta e^{-cN^{1-\alpha-\delta}}.$$  \hfill (4.1)

Sketch. The trivial union bound allows one to estimate the inner probability without the supremum, for a sub-exponential loss in $N$. The result then follows from Proposition 2.1, or more simply from Chernoff’s inequality, [11]. \qed
Let us consider the kernel
\[ K_N(h; b, b') := \sum_{1 \leq n, n+h \leq N} Y_{n+h} Y_n b(S_{n+h-1}) \overline{b}(S_{n-1}); \]

since \( Y_{n+h} \) is independent from all other random variables appearing in each summand, \( K_N(h; b, b') \) is a sum of martingale increments. Its conditional variance is given by
\[ T_N(h) := \sum_{n=1}^{N-h} \sigma_{n+h}^2 |Y_n|^2. \]

We expand the foregoing out as
\[ N-h \sum_{n=1}^{N-h} \sigma_{n+h}^2 \mathbb{E}|Y_n|^2 + \sum_{n=1}^{N-h} \sigma_{n+h} (|Y_n|^2 - \mathbb{E}|Y_n|^2), \]
which we may bound, in light of the previous technical Lemma 3.2, by a constant multiple of \( N^{1-2\alpha} \) away from a set of probability \( \lesssim_{\delta} e^{-cN^{1-\alpha-\delta}} \).

We will now apply Proposition 2.1 to estimate the magnitudes of
\[ \{K_N(h; b, b') : 1 \leq |h| \leq N, b, b' \in B_N \}. \]

First, choose \( \epsilon = \epsilon(\alpha) > 0 \) so small that
\[ 1 - 2\alpha - 10\epsilon > 0, \]
and bound
\[ \mathbb{P}(|K_N(h; b, b')| \gtrsim N^{1-2\alpha-2\epsilon}) \]
by a constant multiple (determined by \( \epsilon = \epsilon(\alpha) \)) of
\[ \mathbb{P}(|K_N(h; b, b')| \gtrsim N^{1-2\alpha-2\epsilon}, |T_N(h)| \lesssim N^{1-2\alpha}) + e^{-cN^{1-\alpha-\epsilon}}; \]
Freedman’s Martingale inequality, Proposition 2.1, then allows us to bound the foregoing by a constant multiple of
\[ e^{-cN^{1-2\alpha-4\epsilon}}. \]

Using the crude union bound, and the cardinality estimate
\[ |B_N| \lesssim_{\epsilon} e^{N^\epsilon}, \]
we may pass to the estimate
\[ \mathbb{P} \left( \sup_{b, b' \in B_N} \sup_{1 \leq |h| \leq N} |K_N(h; b, b')| \gtrsim N^{1-2\alpha-2\epsilon} \right) \lesssim e^{-cN^{1-2\alpha-5\epsilon}}. \]

In particular, by a Borel-Cantelli argument, \( \omega \)-almost surely,
\[ \sup_{b, b' \in B_N} \sup_{1 \leq |h| \leq N} |K_N(h; b, b')| \lesssim_{\omega} N^{1-2\alpha-2\epsilon}. \]

We are now ready to quickly prove Proposition 3.1:
Up to (3), we may almost surely bound
\[ |TN_{N}^{*}f(x)| \lesssim \frac{1}{N^{2-2\alpha}} \sum_{b,b' \in B_{N}} 1_{E_{b}}(x)N^{1-2\alpha-2\epsilon} \sum_{1 \leq |b| \leq N} (1_{E_{b'}}f)(x + h) \]
\[ \lesssim_{\omega} N^{-2\epsilon} M_{HL}f(x), \]
where $M_{HL}$ is the standard Hardy-Littlewood maximal function. The result follows.

With this in mind, we are ready for our proof of Theorem 1.3.

4. THE PROOF OF THEOREM 1.3

We begin by using the almost-sure weak-type $1 - 1$ boundedness of the maximal function
\[ M_{LV}f := \sup_{N} \frac{1}{N} \sum_{n \leq N} T^{a_{n}(\omega)}|f| \]
\[ \equiv \sup_{N: S_{N} \neq 0} \frac{1}{S_{N}} \sum_{n \leq N} X_{n}(\omega)T^{n}|f| \]
\[ \approx_{\omega} \sup_{N} \frac{1}{W_{N}} \sum_{n \leq N} X_{n}(\omega)T^{n}|f| \]
[9] to replace general $L^{1}$ functions appearing in the statement of Theorem 1.3 by $f \in \{h - Th : h \in L^{\infty}(X)\}$; the full strength of Theorem 1.3 may be recovered by a standard density argument. Note the usage of the strong law of large numbers to conclude that, $\omega$-almost surely
\[ S_{n} \approx_{\omega} W_{n} \]
for all $n$ such that $S_{n} \neq 0$.

Now, by the boundedness of $f$ and Rosenblatt, Wierdl [10, Lemma 1.5], it is enough to restrict attention to lacunary sequences $N \in \{[\rho^{k}] : k \geq 0\}$, where $\rho$ is taken from a countable sequence converging to 1. We will fix some $\rho > 1$ throughout, and the averaging parameters are assumed to belong to $\{[\rho^{k}] : k \geq 0\}$.

By definition, it is enough to prove the stated convergence for
\[ \sup_{b \in B} \left| \frac{1}{S_{N}} \sum_{n \leq N} X_{n}b(S_{n})T^{n}f \right| ; \]

since
\[ X_{n}b(S_{n}) = X_{n}b(S_{n-1} + 1) \]
for any function $b$, it is enough to prove pointwise convergence to zero (along lacunary times) for

$$
\sup_{b \in B} \left| \frac{1}{S_N} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f \right|.
$$

By the strong law of large numbers (or by an easy application of Chernoff’s inequality [11]), we know that almost surely $\frac{S_N}{W_N} \to 1$; consequently we may instead prove our convergence result for

$$
\sup_{b \in B} \left| \frac{1}{W_N} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f \right|.
$$

By the definition of approximability, for any $b \in B$ and $N \in \lfloor \rho^N \rfloor$, there exists a $b_0 \in B_N \subset B$ so that

$$
\left| \frac{1}{W_N} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f - \frac{1}{W_N} \sum_{n \leq N} X_n b_0(S_{n-1} + 1) T^n f \right| \lesssim \frac{1}{W_N} \sum_{n \leq N} X_n \cdot \left( \sup_{t \leq N} |b(t) - b_0(t)| \right) \cdot T^n |f|
$$

$$
\lesssim_\omega N^{-\kappa} M_{LV} f,
$$

for some $\kappa > 0$. In particular, almost surely, for each $N$ we may bound

$$
\sup_{b \in B} \left| \frac{1}{W_N} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f \right| \lesssim_\omega \sup_{b \in B_N} \left| \frac{1}{W_N} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f \right| + N^{-\kappa} M_{LV} f;
$$

since almost surely this latter functions tends to zero pointwise $\mu$-a.e. it suffices to consider the first term on the right. Since $W_N = c_0 N^{1-\alpha} + O(1)$, we may replace the normalizing factor $W_N$ with $N^{1-\alpha}$. At this point, we have reduced the problem to showing that, under the above hypotheses, $\omega$-almost surely, for any measure-preserving system $(X, \mu, T)$,

$$
\lim_{N \to \infty, N \in \lfloor \rho^N \rfloor} \sup_{b \in B_N} \left| \frac{1}{N^{1-\alpha}} \sum_{n \leq N} X_n b(S_{n-1} + 1) T^n f \right| = 0
$$

$\mu$-a.e. for each $f \in L^1(X) \cap L^\infty(X)$.

By Proposition 3.1, Calderón’s transference principle [5], and a Borel-Cantelli argument, we know that, almost surely,

$$
\sup_{b \in B_N} \left| \frac{1}{N^{1-\alpha}} \sum_{n \leq N} Y_n b(S_{n-1} + 1) T^n f \right| \to 0
$$
\(\mu\)-a.e. along lacunary times. This is since, \(\omega\)-almost surely,

\[
\mu \left( \limsup_{N} \sup_{b \in B_{N}} \frac{1}{N^{1-\alpha}} \sum_{n \leq N} Y_n b(S_{n-1} + 1) T^n f \geq t \right)
\]

\[
\leq \lim_{M} \sum_{N \geq M} \mu \left( \sup_{b \in B_{N}} \left| \frac{1}{N^{1-\alpha}} \sum_{n \leq N} Y_n b(S_{n-1} + 1) T^n f \right| \geq t \right)
\]

\[
\lesssim_{\omega} \lim_{M} \sum_{N \geq M} t^{-2} N^{-2\varepsilon} \|f\|_{L^2(X)}^2
\]

\[
\lesssim_{\omega} \lim_{M} N^{-2\varepsilon} t^{-2} \|f\|_{L^2(X)}^2 = 0;
\]

we were able to estimate \(\sum_{N \geq M} N^{-2\varepsilon} \lesssim M^{-2\varepsilon}\) since our averaging parameters are restricted to a lacunary sequence. The upshot is that we have reduced matters to proving the following lemma.

**Lemma 4.1.** \(\omega\)-almost surely, for any measure-preserving system, and each simple \(f = h - Th, h \in L^\infty(X)\)

\[
\sup_{b \in B_{N}} \left| \frac{1}{N^{1-\alpha}} \sum_{n \leq N} \sigma_n b(S_{n-1} + 1) T^n f \right| \to 0
\]

\(\mu\)-a.e.

**Proof of Lemma 4.1.** Substituting \(f = h - Th\) and summing by parts lets us bound the above maximal function by a constant multiple of

\[
\frac{\|h\|_{\infty}}{N^{1-\alpha}} + \sup_{b \in B} \frac{1}{N^{1-\alpha}} \sum_{2 \leq m \leq N} \left| \sigma_m b(S_{m-1} + 1) - \sigma_{m-1} b(S_{m-2} + 1) \right| T^m h,
\]

which is in turn bounded by a constant multiple of

\[
\frac{\|h\|_{\infty}}{N^{1-\alpha}} + \frac{1}{N^{1-\alpha}} \sum_{2 \leq m \leq N} (m-1)^{-\alpha-1} \|h\|_{\infty} + \frac{1}{N^{1-\alpha}} \sum_{2 \leq m \leq N} (m-1)^{-\alpha} X_{m-1} \|h\|_{\infty}.
\]

The first two terms in the sum clearly tend to zero as \(N \to \infty\). For the third term, we proceed as follows. By the strong law of large numbers, \(\omega\)-almost surely, for any dyadic \(2 \leq K \leq N\)

\[
\sum_{K/2 < m \leq K} (m-1)^{-\alpha} X_{m-1} \lesssim K^{-\alpha} S_K \lesssim_{\omega} K^{1-2\alpha},
\]

and so we may almost surely bound

\[
\frac{1}{N^{1-\alpha}} \sum_{2 \leq m \leq N} (m-1)^{-\alpha} X_{m-1} \|h\|_{\infty} \lesssim N^{-\alpha} \|h\|_{\infty} \to 0
\]
as well. This completes the proof of Lemma 4.1, and with it, the proof of Theorem 1.3.

□

References

[1] I. Assani, Wiener Wintner Ergodic Theorems, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
[2] G. D. Birkhoff. Proof of the ergodic theorem, Proc. Natl. Acad. Sci. USA 17 (1931), 656–660.
[3] J. Bourgain. On the maximal ergodic theorem for certain subsets of the integers, Israel J. Math. 61 (1988), no. 1, 39–72.
[4] J. Bourgain, Double recurrence and almost sure convergence, J. Reine Angew. Math. 404 (1990), 140–161.
[5] A. Calderón, Ergodic theory and translation invariant operators, Proc. Nat. Acad. Sci., USA 59 (1968), 349–353.
[6] T. Eisner, B. Krause. (Uniform) Convergence of Twisted Ergodic Averages. Preprint, http://arxiv.org/pdf/1407.4736.pdf.
[7] D. Freedman. On tail probabilities for martingales. Ann. Probability 3 (1975), 100-118.
[8] B. Krause, P. Zorin-Kranich. A random pointwise ergodic theorem with Hardy field weights. Illinois J. Math. 59 (2015), no. 3, 663674.
[9] P. LaVictoire. An $L^1$ ergodic theorem for sparse random subsequences, Math. Res. Lett. 16 (2009), no. 5, 849859.
[10] J. Rosenblatt, M. Wierdl. Pointwise ergodic theorems via harmonic analysis. Ergodic theory and its connections with harmonic analysis (Alexandria, 1993). London Math. Soc. Lecture Note Ser., 205, Cambridge Univ. Press, Cambridge, (1995), 3–151.
[11] T. Tao and V. H. Vu. Additive combinatorics. Vol. 105. Cambridge Studies in Advanced Mathematics. Paperback edition [of MR2289012]. Cambridge University Press, Cambridge, 2010, pp. xviii+512. ISBN: 978-0-521-13656-3.
[12] N. Wiener; A. Wintner, Harmonic analysis and ergodic theory, Amer. J. Math. 63 (1941), 415–426.

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