Relations between ranks of matrix polynomials

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Abstract

We show that the sum of ranks of two matrix polynomials is the same as the sum of the rank of the matrix obtained by applying the greatest common divisor of the polynomials, with the rank of the matrix obtained by applying the lowest common multiple of the polynomials. Many applications, for older or more recent problems, of this result are obtained.

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1. Introduction.

Let $(K, +, ·)$ be a field, let $K[X]$ be the ring of polynomials with coefficients in $K$ and $\mathcal{M}_n(K)$ the ring of square matrices of order $n \geq 2$ with coefficients in $K$. Let $D := (f, g)$ denote the greatest common divisor and $M := [f, g]$ denote the lowest common multiple of the polynomials $f, g$. The main result of this paper is following theorem.

**Theorem 1** (Sum of ranks of matrix polynomials). For any two polynomials $f, g \in K[X]$ and for any matrix $A \in \mathcal{M}_n(K)$ the following relation holds

$$\text{rank } f(A) + \text{rank } g(A) = \text{rank } D(A) + \text{rank } M(A).$$

From the above relation one can obtain other interesting relations, particular theorems that characterize some matrices as idempotent, tripotent, involutive matrices and other results which may be well known for experts. These are exposed in Section 3 where we reviewed some particular results obtained in [1], [2] and [3].

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2. Proof of Theorem \[ \text{I} \]

We prove the relation using the method of elementary transformations in block matrices which appears heavily in \([6]\). Other sources of methods for matrices theory and applications which we are using are from \([4]\), \([5]\) and \([7]\). The main result that we use is the following well-known lemma:

**Lemma 1.** If \( f, g \in K[X] \) then there are \( \varphi_1, \varphi_2 \in K[X] \) such that

\[
    f \cdot \varphi_1 + \varphi_2 \cdot g = D.
\]

**Proof.** (of Theorem \[ \text{I} \]) For the matrix \( A \) the relation \( (1) \) become

\[
    f(A) \cdot \varphi_1(A) + \varphi_2(A) \cdot g(A) = D(A)
\]

We start with the block matrix \( B \in \mathcal{M}_{2n}(K) \)

\[
    B = \begin{bmatrix}
    f(A) & 0 \\
    0 & g(A)
    \end{bmatrix}
\]

with rank \( B = \text{rank } f(A) + \text{rank } g(A) \) and we perform the following elementary transformations:

\[
\begin{align*}
    & \quad \begin{bmatrix}
    f(A) & 0 \\
    0 & g(A)
    \end{bmatrix} \xrightarrow{C_1} \begin{bmatrix}
    f(A) & f(A) \cdot \varphi_1(A) \\
    0 & g(A)
    \end{bmatrix} \xrightarrow{L_1} \\
    & \quad \begin{bmatrix}
    f(A) & D(A) \\
    0 & g(A)
    \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix}
    f(A) & D(A) \\
    -M(A) & 0
    \end{bmatrix} \xrightarrow{C_2} \\
    \rightarrow \quad \begin{bmatrix}
    0 & D(A) \\
    -M(A) & 0
    \end{bmatrix} = C,
\end{align*}
\]

with rank \( C = \text{rank } D(A) + \text{rank } M(A) \).

The matrices used for the elementary transformations on columns \( (C_1, C_2) \) and on rows \( (L_1, L_2) \) are

\[
\begin{align*}
    C_1 &= \begin{bmatrix}
    I_n & \varphi_1(A) \\
    0 & I_n
    \end{bmatrix}, \quad L_1 &= \begin{bmatrix}
    I_n & \varphi_2(A) \\
    0 & I_n
    \end{bmatrix}, \\
    C_2 &= \begin{bmatrix}
    I_n & 0 \\
    -\psi_2(A) & I_n
    \end{bmatrix}, \quad L_2 &= \begin{bmatrix}
    I_n & 0 \\
    -\psi_1(A) & I_n
    \end{bmatrix},
\end{align*}
\]

where \( g(A) = \psi_1(A) \cdot D(A) \) and \( f(A) = D(A) \cdot \psi_2(A) \). We used relation \( (2) \) in the third step and \( \psi_1, \psi_2 \) are polynomials obtained by dividing the given polynomials \( f, g \) to their greatest common divisor.

We obtain the relations

\[
    C = L_2 \cdot L_1 \cdot B \cdot C_1 \cdot C_2, \quad M \cdot D = f \cdot g,
\]

which concludes the proof.
3. Applications.

**Corollary 1.** $M(A) = 0$ if and only if $\text{rank } f(A) + \text{rank } g(A) = \text{rank } D(A)$.

**Corollary 2.** If $m_A$ is the minimal polynomial of the matrix $A$ then $m_A | M$ if and only if $\text{rank } f(A) + \text{rank } g(A) = \text{rank } D(A)$.

**Corollary 3.** The polynomials $f$ and $g$ are coprime polynomials if and only if for any matrix $A \in \mathcal{M}_n(K)$ the following relation hold

$$\text{rank } f(A) \cdot g(A) + n = \text{rank } f(A) + \text{rank } g(A).$$

The following corollary appears in [3].

**Corollary 4.** If $f$ and $g$ are coprime polynomials, then $f(A) \cdot g(A) = 0$ if and only if $\text{rank } f(A) + \text{rank } g(A) = n$.

**Application 1.** ([1], [2]) The matrix $A \in \mathcal{M}_n(K)$ is idempotent ($A^2 = A$) if and only if

$$\text{rank } A + \text{rank } (I_n - A) = n.$$

*Proof.* Take the polynomials $f(x) = x, g(x) = 1 - x$ and then we have $D(x) = 1, \ M(x) = x - x^2$. Now we apply Corollary 1 or Corollary 4. 

**Application 2.** ([1], [2]) The matrix $A \in \mathcal{M}_n(K)$ is involutive ($A^2 = I_n$) (the field $K$ of characteristic different than 2) if and only if

$$\text{rank } (I_n - A) + \text{rank } (I_n + A) = n.$$

*Proof.* We apply Corollary 4 for the polynomials $f(x) = 1 - x, g(x) = 1 + x$. 

**Application 3.** ([2]) Using the hypothesis that the field $K$ does not have the characteristic 2, the following statements are equivalent:

1) The matrix $A$ is tripotent ($A^3 = A$).

2) $\text{rank } A + \text{rank } (I_n - A^2) = n$.

3) $\text{rank } (I_n - A) + \text{rank } (A + A^2) = n$.

4) $\text{rank } A + \text{rank } (I_n - A) + \text{rank } (I_n + A) = 2n$. 

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Proof. "1) ⇔ 2)". We apply Corollary 3 for the polynomials \( f(x) = x, \quad g(x) = 1 - x^2. \)
"1) ⇔ 3)". We apply Corollary 3 for the polynomials \( f(x) = 1 - x, \quad g(x) = x + x^2. \)
"2) ⇔ 4)". It is enough to prove the relation

\[
\text{rank}(I_n - A^2) = \text{rank}(I_n - A) + \text{rank}(I_n + A),
\]

which follows from Corollary 3 for \( f(x) = 1 - x \) and \( g(x) = 1 + x. \)

**Application 4.** Let \( A \in M_n(K) \) and \( f_A \in K[X] \) be its characteristic polynomial, which we decompose in a product of polynomials that are coprime two by two

\[
f_A = f_1 \cdot f_2 \cdots f_k.
\]

Then we have the relation

\[
\text{rank} f_1(A) + \text{rank} f_2(A) + \cdots + \text{rank} f_k(A) = (k - 1)n.
\]

**Proof.** We apply Corollary 3 for \( f = f_1 \cdot f_2 \cdots f_{k-1}, \quad g = f_k \) and we obtain

\[
\text{rank} f(A) + \text{rank} f_k(A) = n.
\]

We apply now Corollary 3 for \( f = f_1 \cdot f_2 \cdots f_{k-2}, \quad g = f_{k-1} \) and we obtain

\[
\text{rank} (f_1 \cdot f_2 \cdots f_{k-2}) + \text{rank} f_{k-1}(A) + \text{rank} f_k(A) = 2n.
\]

By induction, we obtain the result.

**Application 5.** For any matrix \( A \in M_n(K) \) the following relation is verified

\[
\text{rank} (A + A^2) + \text{rank} (A - A^2) = \text{rank} A + \text{rank} (A - A^3).
\]

**Proof.** We apply Theorem 1 with

\[
f(x) = x + x^2, \quad g(x) = x - x^2, \quad D(x) = x, \quad M(x) = x - x^3.
\]

**Application 6.** (2) For a matrix \( A \in M_n(K) \) the following statements are equivalent:

1) \( A^3 = A^5. \)
2) \( \text{rank} A^3 + \text{rank} (I_n - A^2) = n. \)
3) \( \text{rank} (I_n - A) + \text{rank} (A^3 + A^4) = n. \)
4) \( \text{rank} (I_n + A) + \text{rank} (A^3 - A^4) = n. \)

5) \( \text{rank} A^3 + \text{rank} (I_n - A) + \text{rank} (I_n + A) = 2n. \)

6) \( \text{rank} (A - A^2) + \text{rank} (A^3 + A^4) = \text{rank} A. \)

7) \( \text{rank} (A + A^2) + \text{rank} (A^3 - A^4) = \text{rank} A. \)

8) \( \text{rank} (A^3 + A^4) + \text{rank} (A^3 - A^4) = \text{rank} A^3. \)

References

[1] V. Pop, The characterization of same linear maps using the rank, *Journal of Science and Arts (JOSA)*, 3 (36) (2016), 225–228.

[2] V. Pop., The method of elementary operations in block matrices for the determination of the annihilator polynomials of some matrices, *Journal of Science and Arts (JOSA)*, 4 (37) (2016), 295–302.

[3] V. Pop, The decomposition of some annihilator polynomials of linear maps in coprime polynomials, *Journal of Science and Arts (JOSA)*, 4 (41) (2017), 647–650.

[4] I. Tyan, G. Styan, Linear Algebra and its Applications 135, 101 (2001).

[5] I. Tyan, G. Styan, Journal of Computational and Applied Mathematics, 191, 77 (2006).

[6] F. Zhang, Matrix Theory – Basic Results and Techniques, Springer, New York 1999.

[7] D. S. Bernstein, Matrix Mathematics. Theory, Facts and Formulas, Princeton Univ. Press 2009.