Deformations of metabelian representations of knot groups into $SL(3, \mathbb{C})$

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Abstract

Let $K$ be a knot in $S^3$ and $X$ its complement. We study deformations of reducible metabelian representations of the knot group $\pi_1(X)$ into $SL(3, \mathbb{C})$ which are associated to a double root of the Alexander polynomial. We prove that these reducible metabelian representations are smooth points of the representation variety and that they have irreducible non metabelian deformations.

Introduction

Let $K$ be a knot in $S^3$. We let $X = \overline{S^3 \setminus V(K)}$ denote the knot complement where $V(K)$ is a tubular neighbourhood of $K$. Moreover we let $\pi = \pi_1(X)$ denote the fundamental group of $X$. Let $\mu$ be a meridian of $K$ and let $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ denote the Alexander polynomial of $K$. We associate to $\alpha \in \mathbb{C}^*$ a homomorphism

$$\eta_\alpha: \pi \to \mathbb{C}^*$$
$$\gamma \mapsto \alpha |\gamma|$$

with $|\gamma| = p(\gamma)$, where $p: \pi \to \pi/\pi' \simeq \mathbb{Z}$ denotes the canonical projection. Note that $\eta_\alpha(\mu) = \alpha$. We define $\mathbb{C}_\alpha$ to be the $\pi$-module $\mathbb{C}$ with the action induced by $\alpha$, i.e. $\gamma \cdot x = \alpha |\gamma| x$, for all $x \in \mathbb{C}$ and for all $\gamma \in \pi$. The trivial $\pi$-module $\mathbb{C}_1$ is simply denoted $\mathbb{C}$.

Burde and de Rham proved, independently, that when $\alpha$ is a root of the Alexander polynomial there exists a reducible metabelian, non abelian, representation $\phi: \pi \to GL(2, \mathbb{C})$ such that

$$\phi(\gamma) = \left( \begin{array}{cc} \alpha |\gamma| & z(\gamma) \\ 0 & 1 \end{array} \right).$$
Here \( z \) is a 1-cocycle in \( Z^1(\pi, \mathbb{C}_\alpha) \) which is not a coboundary (see [7] and [8]). The homomorphism \( \phi \) induces a representation into \( SL(2, \mathbb{C}) \) (Lemma [2]) given by
\[
\tilde{\rho}(\gamma) = \alpha^{-1/2}(\gamma)\phi(\gamma), \quad \forall \gamma \in \pi,
\]
where \( \alpha^{-1/2} : \pi \to \mathbb{C}^* \) is a homomorphism such that \((\alpha^{-1/2}(\gamma))^2 = \alpha^{-|\gamma|}\), for all \( \gamma \in \pi \).

This constitutes the starting point to the study of the problem of deformations of metabelian and abelian representations in \( SL(2, \mathbb{C}) \) or \( SU(2) \) that correspond to a simple zero of the Alexander polynomial (see [9] and [14]). The result of [9] is generalized in [11] and [12] by replacing the condition of the simple zero by a condition on the signature operator. Similar results are established in [19] in the case of cyclic torsion, but unfortunately none of these results has been published. In [2], [3] and [4], the authors considered the case of any complex connected reductive or real connected compact Lie group. They supposed that the \((t - \alpha)\)-torsion of the Alexander module is semisimple. A \( PSL(2, \mathbb{C}) \) version was given recently in [13].

Throughout this paper, we suppose that \( \alpha \in \mathbb{C}^* \) is a multiple root of the Alexander polynomial \( \Delta_K(t) \) and that \( \dim H^1(\pi, \mathbb{C}_\alpha) = 1 \). This means that the \((t - \alpha)\)-torsion of the Alexander module is cyclic of the form
\[
\tau_\alpha = \mathbb{C}[t, t^{-1}]/(t - \alpha)^r \quad \text{where} \quad r \geq 2.
\]

As particular generalization of Burde and de Rham’s result it is established in [15] that in this case there exists a reducible metabelian, non abelian, representation \( \rho_0 : \pi \to GL(3, \mathbb{C}) \) defined by
\[
\rho_0(\gamma) = \begin{pmatrix} \alpha^{|\gamma|} & z(\gamma) & g(\gamma) \\ 0 & 1 & h(\gamma) \\ 0 & 0 & 1 \end{pmatrix}.
\]

Here \( h : \pi \to (\mathbb{C}, +) \) is a non trivial homomorphism and \( g : \pi \to \mathbb{C}_\alpha \) is a 1-cochain verifying \( \delta g + z \cup h = 0 \). We normalize \( \rho_0 \) by considering
\[
\tilde{\rho} : \pi \to SL(3, \mathbb{C})
\]
\[
\gamma \mapsto \alpha^{-1/3}(\gamma)\rho_0(\gamma)
\]
where \( \alpha^{-1/3} : \pi \to \mathbb{C}^* \) is a homomorphism (Lemma [2]) such that
\[
(\alpha^{-1/3}(\gamma))^3 = \alpha^{-|\gamma|}, \quad \text{for all} \ \gamma \in \pi.
\]

The Lie algebra \( sl(3, \mathbb{C}) \) turns into a \( \pi \)-module via the adjoint action of the representation \( \tilde{\rho} \), \( Ad \circ \tilde{\rho} : \pi \to Aut(sl(3, \mathbb{C})) \). The aim of this paper is to
answer the following question: when can \( \tilde{\rho} \) be deformed into irreducible non metabelian representations?

We use the technical approach of [13] to prove the following result:

**Theorem 1** We suppose that the \((t - \alpha)\)-torsion of the Alexander module is cyclic of the form \( \mathbb{C}[t, t^{-1}]/(t - \alpha)^r \). If \( \alpha \) is a double root of the Alexander polynomial i.e. \( r = 2 \), then there exist irreducible non metabelian representations from \( \pi \) into \( SL(3, \mathbb{C}) \) which deform \( \tilde{\rho} \). Moreover, the representation \( \tilde{\rho} \) is a smooth point of the representation variety \( R(\pi, SL(3, \mathbb{C})) \); it is contained in an unique 10-dimensional component \( R_{\tilde{\rho}} \) of \( R(\pi, SL(3, \mathbb{C})) \).

The first example of classical knots whose Alexander polynomial has a double root \( \alpha \) such that the \((t - \alpha)\)-torsion of the Alexander module is cyclic is \( 8_{20} \).

This paper is organized as follows: In Section 1 the basic notation and facts are presented. The Section 2 includes the proof of Theorem 1. The cohomology computations are done in Section 3. The aim of Section 4 is to study the nature of the deformations of \( \tilde{\rho} \).

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## 1 Notation and facts

**Lemma 2** Let \( n \geq 2 \) and let \( \eta: \pi \to \mathbb{C}^* \) be a homomorphism. Then there exists \( \tilde{\eta}: \pi \to \mathbb{C}^* \) a homomorphism such that \((\tilde{\eta}(\gamma))^n = \eta(\gamma)\), for all \( \gamma \in \pi \).

**Proof.** Let \( \lambda: \pi \to \mathbb{C}^* \) be a map satisfying \((\lambda(\gamma))^n = \eta(\gamma)\), for all \( \gamma \in \pi \).

Then there exists a map \( \omega: \pi \times \pi \to U_n \) such that

\[
\lambda(\gamma_1 \gamma_2) = \lambda(\gamma_1) \lambda(\gamma_2) \omega(\gamma_1, \gamma_2), \quad \forall \gamma_1, \gamma_2 \in \pi,
\]

where \( U_n = \{ \xi \in \mathbb{C}^* \mid \xi^n = 1 \} \simeq \mathbb{Z}/n\mathbb{Z} \). Hence \( \lambda \) is unique up to multiplication by a \( n \)-th root of unity. It is not hard to check that \( \omega \) is a 2-cocycle in \( Z^2(\pi, \mathbb{Z}/n\mathbb{Z}) \). Since \( H^2(\pi, \mathbb{Z}/n\mathbb{Z}) = 0 \), there exists a 1-cochain \( d \) such that \( \omega = \delta d \) and we can easily verify that \( \lambda d \) is a homomorphism satisfying

\[
(\lambda(\gamma)d(\gamma))^n = \eta(\gamma), \quad \forall \gamma \in \pi.
\]

\( \square \)
1.1 Group cohomology

The general reference for this section is Brown’s book [6]. Let \( A \) be a \( \pi \)-module. We denote by \( C^*(\pi, A) \) the cochain complex. An element of \( C^n(\pi, A) \) can be viewed as a function \( f: \pi^n \to A \), i.e. as a function of \( n \) variables from \( \pi \) to \( A \). The coboundary operator \( \delta: C^n(\pi, A) \to C^{n+1}(\pi, A) \) is given by:

\[
\delta f(\gamma_1, \ldots, \gamma_{n+1}) = \gamma_1 \cdot f(\gamma_2, \ldots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i \gamma_{i+1}, \ldots, \gamma_{n+1}) + (-1)^{n+1} f(\gamma_1, \ldots, \gamma_n).
\]

Note that \( C^0(\pi, A) \simeq A \) and that, for \( a \in C^0(\pi, A) \), we have:

\[
\delta a(\gamma) = (\gamma - 1) \cdot a, \quad \forall \gamma \in \pi.
\]

The coboundaries (respectively cocycles, cohomology) of \( \pi \) with coefficients in \( A \) are denoted by \( B^*(\pi, A) \) (respectively \( Z^*(\pi, A) \), \( H^*(\pi, A) \)). For \( z \) a cocycle in \( Z^i(\pi, A) \), \( i \geq 1 \), the cohomology class in \( H^i(\pi, A) \) is denoted \( \{z\} \).

Let \( A_1 \), \( A_2 \) and \( A_3 \) be \( \pi \)-modules. The cup product of two cochains \( u \in C^p(\pi, A_1) \) and \( v \in C^q(\pi, A_2) \) is the cochain \( u \cup v \in C^{p+q}(\pi, A_1 \otimes A_2) \) defined by

\[
u(\gamma_1, \ldots, \gamma_{p+q}) := u(\gamma_1, \ldots, \gamma_p) \otimes \gamma_{p+1} \cdots \gamma_{p+q} \circ v(\gamma_{p+1}, \ldots, \gamma_{p+q}).
\]

Here \( A_1 \otimes A_2 \) is a \( \pi \)-module via the diagonal action. It is possible to combine the cup product with any bilinear map \( A_1 \otimes A_2 \to A_3 \). We are only interested by the product map \( \mathbb{C} \otimes \mathbb{C}_{\alpha \bar{\alpha}} \to \mathbb{C}_{\alpha \bar{\alpha}} \) and \( \mathbb{C}_{\alpha} \otimes \mathbb{C}_{\bar{\alpha}^{-1}} \to \mathbb{C} \).

A short exact sequence

\[
0 \to A_1 \xrightarrow{i} A_2 \xrightarrow{p} A_3 \to 0
\]

of \( \pi \)-modules gives rise to a short exact sequence of cochain complexes:

\[
0 \to C^*(\pi, A_1) \xrightarrow{\cdot i} C^*(\pi, A_2) \xrightarrow{\cdot p} C^*(\pi, A_3) \to 0.
\]

In the sequel we will make use of the corresponding long exact cohomology sequence:

\[
0 \to H^0(\pi, A_1) \to H^0(\pi, A_2) \to H^0(\pi, A_3) \xrightarrow{\delta^1} H^1(\pi, A_1) \to \cdots.
\]

In order to define the connecting homomorphism \( \delta^{n+1}: H^n(\pi, A_3) \to H^{n+1}(\pi, A_1) \) we let \( \delta_2 \) denote the coboundary operator of \( C^*(\pi, A_2) \). If \( z \in Z^n(\pi, A_3) \) is a cocycle then \( \delta^{n+1}(\{z\}) = \{i^{-1}_*(\delta_2(\tilde{z}))\} \) where the cochain \( \tilde{z} \in p^{-1}_*(z) \subset C^n(\pi, A_2) \) is any lift of \( z \).
1.2 Group cohomology and representation varieties

The set \( R_n(\pi) := R(\pi, SL(n, \mathbb{C})) \) of homomorphisms of \( \pi \) in \( SL(n, \mathbb{C}) \) is called the representation variety of \( \pi \) in \( SL(n, \mathbb{C}) \) and is a (not necessarily irreducible) algebraic variety.

**Definition 3** A representation \( \rho: \pi \to SL(n, \mathbb{C}) \) of the knot group \( \pi \) is called abelian (resp. metabelian) if the restriction of \( \rho \) to the first (resp. second) commutator subgroup of \( \pi \), denoted \( \pi' \) (resp. \( \pi'' \)) is trivial.

In this section, we present some results of [13] that we will use in the sequel. Let \( \rho: \pi \to SL(n, \mathbb{C}) \) be a representation. The Lie algebra \( sl(n, \mathbb{C}) \) turns into a \( \pi \)-module via \( Ad \circ \rho \). This module will be simply denoted by \( sl(n, \mathbb{C})^\rho \). A cocycle \( d \in Z^1(\pi, sl(n, \mathbb{C})^\rho) \) is a map \( d: \pi \to sl(n, \mathbb{C}) \) satisfying

\[
d(\gamma_1 \gamma_2) = d(\gamma_1) + Ad \circ \rho(\gamma_1)(d(\gamma_2)), \quad \forall \gamma_1, \gamma_2 \in \pi.
\]

It was observed by André Weil [21] that there is a natural inclusion of the Zariski tangent space \( T_{\rho}^{zar}(R_n(\pi)) \hookrightarrow Z^1(\pi, sl(n, \mathbb{C})^\rho) \). Informally speaking, given a smooth curve \( \rho_\epsilon \) of representations through \( \rho_0 = \rho \) one gets a 1-cocycle \( d: \pi \to sl(n, \mathbb{C}) \) by defining

\[
d(\gamma) := \left. \frac{d \rho_\epsilon(\gamma)}{d \epsilon} \right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \pi.
\]

It is easy to see that the tangent space to the orbit by conjugation corresponds to the space of 1-coboundaries \( B^1(\pi, sl(n, \mathbb{C})^\rho) \). Here, \( b: \pi \to sl(n, \mathbb{C}) \) is a coboundary if there exists \( x \in sl(n, \mathbb{C}) \) such that \( b(\gamma) = Ad \circ \rho(\gamma)(x) - x \). A detailed account can be found in [16].

Let \( dim_\rho R_n(\pi) \) be the local dimension of \( R_n(\pi) \) at \( \rho \) (i.e. the maximal dimension of the irreducible components of \( R_n(\pi) \) containing \( \rho \) [18, Ch. II]). In the sequel we will use the following lemmas from [13]:

**Lemma 4** Let \( \rho \in R_n(\pi) \) be given. If \( dim_\rho R_n(\pi) = dim Z^1(\pi, sl(n, \mathbb{C})^\rho) \) then \( \rho \) is a smooth point of the representation variety \( R_n(\pi) \) and \( \rho \) is contained in a unique component of \( R_n(\pi) \) of dimension \( dim Z^1(\pi, sl(n, \mathbb{C})^\rho) \).

**Lemma 5** Let \( A \) be a \( \pi \)-module and let \( M \) be any CW-complex with \( \pi_1(M) \cong \pi \). Then there are natural morphisms \( H_i(M, A) \to H_i(\pi, A) \) which are isomorphisms for \( i = 0, 1 \) and a surjection for \( i = 2 \). In cohomology there are natural morphisms \( H^i(\pi, A) \to H^i(M, A) \) which are isomorphisms for \( i = 0, 1 \) and an injection for \( i = 2 \).
Remark 6 Let $A$ be a $\pi$-module and $X$ a knot complement in $S^3$. The homomorphisms $H^*(\pi, A) \to H^*(X, A)$ and $H^*(\pi_1(\partial X), A) \to H^*(\partial X, A)$ are isomorphisms. This is a consequence of the asphericity of $X$ and $\partial X$. Moreover, the knot complement $X$ has the homotopy type of a 2-dimensional CW-complex which implies that $H^k(\pi, A) = 0$ for $k \geq 3$.

2 Deforming metabelian representations

The aim of the following sections is to prove that, when $\alpha$ is a root of the Alexander polynomial of order 2 then certain metabelian representations are smooth points of the representation variety.

In order to construct deformations of $\tilde{\rho}$ we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a deep theorem of Artin [1]. The formal deformations of a representation $\rho: \pi \to SL(3, \mathbb{C})$ are in general determined by an infinite sequence of obstructions (see [3] and [10]).

Given a cocycle in $Z^1(\pi, sl(n, \mathbb{C})_\rho)$ the first obstruction to integration is the cup product with itself. In general when the $k$-th obstruction vanishes, the obstruction of order $k+1$ is defined, it lives in $H^2(\pi, sl(n, \mathbb{C})_\rho)$.

Let $\rho: \pi \to SL(n, \mathbb{C})$ be a representation. A formal deformation of $\rho$ is a homomorphism $\rho_\infty: \pi \to SL(n, \mathbb{C}[[t]])$

$$\rho_\infty(\gamma) = \exp \left( \sum_{i=1}^{\infty} t^i u_i(\gamma) \right) \rho(\gamma)$$

where $u_i: \pi \to sl(n, \mathbb{C})$ are elements of $C^1(\pi, sl(n, \mathbb{C})_\rho)$ such that $ev_0 \circ \rho_\infty = \rho$. Here $ev_0: SL(n, \mathbb{C}[[t]]) \to SL(n, \mathbb{C})$ is the evaluation homomorphism at $t = 0$ and $\mathbb{C}[[t]]$ denotes the ring of formal power series. We will say that $\rho_\infty$ is a formal deformation up to order $k$ of $\rho$ if $\rho_\infty$ is a homomorphism modulo $t^{k+1}$.

An easy calculation gives that $\rho_\infty$ is a homomorphism up to first order if and only if $u_1 \in Z^1(\pi, sl(n, \mathbb{C})_\rho)$ is a cocycle. We call a cocycle $u_1 \in Z^1(\pi, sl(n, \mathbb{C})_\rho)$ integrable if there is a formal deformation of $\rho$ with leading term $u_1$.

Lemma 7 Let $u_1, \ldots, u_k \in C^1(\pi, sl(n, \mathbb{C})_\rho)$ such that

$$\rho_k(\gamma) = \exp \left( \sum_{i=1}^{k} t^i u_i(\gamma) \right) \rho(\gamma)$$
is a homomorphism into $SL(n, \mathbb{C}[[t]])$ modulo $t^{k+1}$. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1, \ldots, u_k)} \in H^2(\pi, sl(n, \mathbb{C})_\rho)$ with the following properties:

(i) There is a cochain $u_{k+1}: \pi \to sl(n, \mathbb{C})_\rho$ such that

$$\rho_{k+1}(\gamma) = \exp \left( \sum_{i=1}^{k+1} t^i u_i(\gamma) \right) \rho(\gamma)$$

is a homomorphism modulo $t^{k+2}$ if and only if $\zeta_{k+1} = 0$.

(ii) The obstruction $\zeta_{k+1}$ is natural, i.e. if $f: \Gamma \to \pi$ is a homomorphism then $f^* \rho_k := \rho_k \circ f$ is also a homomorphism modulo $t^{k+1}$ and $f^*(\zeta_{k+1}^{(u_1, \ldots, u_k)}) = \zeta_{k+1}^{(f^* u_1, \ldots, f^* u_k)}$.

Proof. The proof is completely analogous to the proof of Proposition 3.1 in [14]. We replace $SL(2, \mathbb{C})$ (resp. $sl(2, \mathbb{C})$) by $SL(n, \mathbb{C})$ (resp. $sl(n, \mathbb{C})$).

Let $i: \partial X \to X$ be the inclusion. For the convenience of the reader, we state the following result which is implicitly contained in [13]:

**Theorem 8** Let $\rho \in R_n(\pi)$ be a representation such that $H^0(X, sl(n, \mathbb{C})_\rho) = 0$ i.e. the centralizer $Z(\rho) \subset SL(n, \mathbb{C})$ is finite.

If $\dim H^0(\partial X, sl(n, \mathbb{C})_\rho) = \dim H^2(X, sl(n, \mathbb{C})_\rho) = n - 1$ and if $\rho \circ i_\#$ is a smooth point of $R_n(\pi_1(\partial X))$, then $\rho$ is a smooth point of the representation variety $R_n(\pi)$; it is contained in a unique irreducible component of dimension $n^2 + n - 2 = (n + 2)(n - 1)$.

Proof. Recall that the Zariski tangent space of $R_n(\pi)$ at $\rho$ is contained in $Z^1(\pi, sl(n, \mathbb{C})_\rho)$ [21]. To prove the smoothness, we show that all cocycles in $Z^1(\pi, sl(n, \mathbb{C})_\rho)$ are integrable. Therefore, we prove that all obstructions vanish, by using the fact that the obstructions vanish on the boundary.

We consider the exact sequence in cohomology for the pair $(X, \partial X)$:

$$0 \to H^0(\partial X, sl(n, \mathbb{C})_\rho) \to H^1(X, \partial X, sl(n, \mathbb{C})_\rho) \to H^1(X, sl(n, \mathbb{C})_\rho) \to H^2(X, \partial X, sl(n, \mathbb{C})_\rho) \to H^2(X, sl(n, \mathbb{C})_\rho) \overset{i_*}{\to} H^2(\partial X, sl(n, \mathbb{C})_\rho) \to 0.$$ 

Poincaré duality implies that $\dim H^2(\partial X, sl(n, \mathbb{C})_\rho) = n - 1$ and the Poincaré–Lefschetz duality gives

$$H^3(X, \partial X, sl(n, \mathbb{C})_\rho) \simeq H^0(X, sl(n, \mathbb{C})_\rho)^* = 0.$$
Since \( i_1^* \) is surjective and \( \dim H^2(X, sl(n, \mathbb{C})_\rho) = \dim H^2(\partial X, sl(n, \mathbb{C})_\rho) \) we get \( H^2(X, sl(n, \mathbb{C})_\rho) \cong H^2(\partial X, sl(n, \mathbb{C})_\rho) \). From Lemma 5 we deduce that

\[
i^*: H^2(\pi, sl(n, \mathbb{C})_\rho) \rightarrow H^2(\pi_1(\partial X), sl(n, \mathbb{C})_\rho)
\]
is an isomorphism.

We will now prove that every element of \( Z^1(\pi, sl(n, \mathbb{C})_\rho) \) is integrable. Let \( u_1, \ldots, u_k: \pi \rightarrow sl(n, \mathbb{C}) \) be such that

\[
\rho_k(\gamma) = \exp \left( \sum_{i=1}^k t^i u_i(\gamma) \right) \rho(\gamma)
\]
is a homomorphism modulo \( t^{k+1} \). Then the restriction \( \rho_k \circ i_\# : \pi_1(\partial X) \rightarrow SL(n, \mathbb{C}[i]) \) is also a formal deformation of order \( k \). Since \( \rho \circ i_\# \) is a smooth point of the representation variety \( R_n(\mathbb{Z} \oplus \mathbb{Z}) \), the formal implicit function theorem gives that \( \rho_k \circ i_\# \) extends to a formal deformation of order \( k + 1 \) (see [14, Lemma 3.7]). Therefore, we have that

\[
0 = \zeta^\gamma_{k+1}(u_1, \ldots, u_k) = i^* \zeta^\gamma_{k+1}(u_1, \ldots, u_k)
\]

Now, \( i^* \) is injective and the obstruction vanishes.

Hence all cocycles in \( Z^1(\pi, sl(n, \mathbb{C})_\rho) \) are integrable. By applying Artin’s theorem [11] we obtain from a formal deformation of \( \rho \) a convergent deformation (see [14, Lemma 3.3] or [3, § 4.2]).

Thus \( \rho \) is a smooth point of the representation variety \( R_n(\pi) \). The Euler characteristic \( \chi(X) \) vanishes. Hence, \( \dim H^1(\pi, sl(n, \mathbb{C})_\rho) = \dim H^2(\pi, sl(n, \mathbb{C})_\rho) = n - 1 \). Since \( \dim B^1(\pi, sl(n, \mathbb{C})_\rho) = n^2 - 1 \), then \( \dim Z^1(\pi, sl(n, \mathbb{C})_\rho) = n^2 + n - 2 \). \( \square \)

Following [20, § 3.5] we call an \( A \in SL(n, \mathbb{C}) \) a regular element if the dimension of the centralizer \( Z(A) \) of \( A \) in \( SL(n, \mathbb{C}) \) is \( n - 1 \). Moreover note that \( A \) is regular iff \( Z(A) \) is abelian. The regular elements form an open dense set in \( SL(n, \mathbb{C}) \) (see [20, § 3.5] for details).

**Lemma 9** Let \( \rho \in R_n(\mathbb{Z} \oplus \mathbb{Z}) \) be a representation and let \( \mu \in \mathbb{Z} \oplus \mathbb{Z} \) be simple i.e. there exists \( \lambda \in \mathbb{Z} \oplus \mathbb{Z} \) such that \( (\mu, \lambda) \) is a basis.

If \( \rho(\mu) \in SL(n, \mathbb{C}) \) is a regular element then \( \rho \) is a smooth point of \( R_n(\mathbb{Z} \oplus \mathbb{Z}) \). It belongs to a \( (n+2)(n-1) \)-dimensional component of \( R_n(\mathbb{Z} \oplus \mathbb{Z}) \).

**Proof.** We have that

\[
H^0(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = sl(n, \mathbb{C})^{\mathbb{Z} \oplus \mathbb{Z}} = sl(n, \mathbb{C})^\rho(\mu) \cap sl(n, \mathbb{C})^\rho(\lambda).
\]
The regularity of $\rho(\mu)$ implies that $\dim sl(n, \mathbb{C})^{\rho(\mu)} = n - 1$ and $\dim \mathbb{C}[\rho(\mu)] = n$ where $\mathbb{C}[\rho(\mu)] \subset M(n, \mathbb{C})$ denotes the algebra generated by $\rho(\mu)$ (see [20] § 3.5]). On the other hand we have

$$\mathbb{C}[\rho(\mu)] \cap sl(n, \mathbb{C}) \subset sl(n, \mathbb{C})^{\rho(\mu)}$$

and the equality of the dimensions gives $\mathbb{C}[\rho(\mu)] \cap sl(n, \mathbb{C}) = sl(n, \mathbb{C})^{\rho(\mu)}$. Therefore we have for each $A \in Z(\rho(\mu))$ that

$$sl(n, \mathbb{C})^{\rho(\mu)} \subset sl(n, \mathbb{C})A$$

and the equality of the dimensions gives $\mathbb{C}[\rho(\mu)] \cap sl(n, \mathbb{C}) = sl(n, \mathbb{C})^{\rho(\mu)}$.

This gives $\dim H^0(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = n - 1$. Now, Poincaré duality implies that $\dim H^2(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = n - 1$ and since the Euler characteristic $\chi(\partial X)$ vanishes we obtain $\dim H^1(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = 2(n - 1)$. Thus $\dim B^1(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = n^2 - n$ and $\dim Z^1(\mathbb{Z} \oplus \mathbb{Z}, sl(n, \mathbb{C})_\rho) = n^2 + n - 2$.

So to prove that $\rho$ is a smooth point of $R_n(\mathbb{Z} \oplus \mathbb{Z})$, we will verify that $\rho$ is contained in a $(n^2 + n - 2)$-dimensional component of $R(\mathbb{Z} \oplus \mathbb{Z})$. Since the set of regular elements of $SL(n, \mathbb{C})$ is open and since the dimension of the centralizer of a regular element is by definition $n - 1$ it follows easily that $\rho$ is contained in an $(n^2 - 1) + (n - 1)$ dimensional component.

From now on, $sl(3, \mathbb{C})$ is considered as a $\pi$-module via the action of $\text{Ad} \circ \tilde{\rho}$. Let’s recall that the image of a meridian is given by

$$\tilde{\rho}(\mu) = \alpha^{-1/3} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

which is a regular element of $SL(3, \mathbb{C})$.

We have:

$$H^0(\partial X, sl(3, \mathbb{C})) = sl(3, \mathbb{C})^{\tilde{\rho}(\mu)}$$

is two dimensional.

The next proposition will be proved in Section 3.

**Proposition 10** Let $K \subset S^3$ be a knot and suppose that the $(t - \alpha)$-torsion of the Alexander module of $K$ is of the form $\tau_\alpha = \mathbb{C}[t, t^{-1}] / (t - \alpha)^2$.

Then we have:

1. $H^0(\pi, sl(3, \mathbb{C})) = 0$.
2. $\dim H^1(\pi, sl(3, \mathbb{C})) = \dim H^2(\pi, sl(3, \mathbb{C})) = 2$. 

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First part of the proof of Theorem 1. Recall that dim $H^0(\partial X, sl(3, \mathbb{C})) = 2$. Now, we apply Theorem 8 using Lemma 9 and the fact that dim $H^2(\pi, sl(3, \mathbb{C})) = 2$ (Proposition 10), to prove that $\tilde{\rho}$ is a smooth point of $R^3(\pi)$. By Proposition 10 we have dim $H^0(\pi, sl(3, \mathbb{C})) = 0$ and dim $H^1(\pi, sl(3, \mathbb{C})) = 2$ which implies that dim $Z^1(\pi, sl(3, \mathbb{C})) = 10$. Hence the representation $\tilde{\rho}$ is contained in a 10-dimensional component $R_{\tilde{\rho}}$ of the representation variety (Lemma 4). In Theorem 20 and Corollary 21, we will see that the component $R_{\tilde{\rho}}$ contains irreducible non metabelian representations.

3 Cohomology of metabelian representations

Throughout this section we will suppose that the $(t - \alpha)$-torsion of the Alexander module of $K$ is of the form $\tau_\alpha = \mathbb{C}[t, t^{-1}]/(t - \alpha)^2$. This implies that dim $H^1(\pi, \mathbb{C}_\alpha) = 1$.

Recall that the Lie algebra $sl(3, \mathbb{C})$ is a $\pi$-module via the action of the adjoint representation $\text{Ad} \circ \tilde{\rho} = \text{Ad} \circ \rho_0$, where

$$\rho_0(\gamma) = \begin{pmatrix} \alpha |_\gamma & z(\gamma) & g(\gamma) \\ 0 & 1 & h(\gamma) \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The goal in this section is to prove Proposition 10. The calculation of the dimensions of $H^*(\pi, sl(3, \mathbb{C}))$ uses several long exact sequences in cohomology associated to the $\pi$-module $sl(3, \mathbb{C})$.

3.1 The setup

Denote by $(E_{ij})_{1 \leq i, j \leq 3}$ the canonical basis of $M(3, \mathbb{C})$ and let $D_1 = E_{22} - 2E_{11} + E_{33}$ and $D_2 = E_{11} - 2E_{33} + E_{22}$. Then $(D_1, D_2, E_{ij} \mid 1 \leq i \neq j \leq 3)$ is a basis of $sl(3, \mathbb{C})$.

It is not hard to check that:

$$H^0(\pi, \mathbb{C}) \simeq \mathbb{C} \quad ; \quad H^1(\pi, \mathbb{C}) \simeq \mathbb{C} \quad ; \quad H^2(\pi, \mathbb{C}) = 0$$
$$H^0(\pi, \mathbb{C}_\alpha) = 0 \quad ; \quad H^1(\pi, \mathbb{C}_\alpha) \simeq \mathbb{C} \quad ; \quad H^2(\pi, \mathbb{C}_\alpha) \simeq \mathbb{C}$$

For more details see [2].

We define $\mathbb{C}_+(3) := \langle E_{12}, E_{23}, E_{13} \rangle$ and $b_+ := \langle D_1, D_2, E_{12}, E_{23}, E_{13} \rangle$ the Borel subalgebra of upper triangular matrices. Then the action of $\text{Ad} \circ \rho_0$ on
\( \mathbb{C}_+(3) \) and \( b_+ \) is given by

\[
\begin{align*}
\gamma \cdot E_{12} &= \alpha^{\| \gamma \|} E_{12} - \alpha^{\| \gamma \|} h(\gamma) E_{13} \\
\gamma \cdot E_{13} &= \alpha^{\| \gamma \|} E_{13} \\
\gamma \cdot E_{23} &= E_{23} + z(\gamma) E_{13} \\
\gamma \cdot D_1 &= D_1 + 3z(\gamma) E_{12} - (3z(\gamma) h(\gamma) - 3g(\gamma)) E_{13} \\
\gamma \cdot D_2 &= D_2 - 3h(\gamma) E_{23} - 3g(\gamma) E_{13}
\end{align*}
\] (1)

Hence \( \mathbb{C}_+(3) \) and \( b_+ \) are \( \pi \)-submodules of \( \mathfrak{sl}(3, \mathbb{C}) \). Moreover \( \langle E_{13} \rangle \) is a \( \pi \)-submodule of \( \mathbb{C}_+(3) \) where the action is given by the multiplication by \( \alpha \).

We have the following short exact sequence

\[
0 \to \langle E_{13} \rangle \to \mathbb{C}_+(3) \to \mathbb{C}_+(3) / \langle E_{13} \rangle \to 0.
\] (2)

Note that \( \langle E_{13} \rangle \cong \mathbb{C}_\alpha \) and that \( \mathbb{C}_+(3) / \langle E_{13} \rangle \cong \mathbb{C} \oplus \mathbb{C}_\alpha \). The first isomorphism is induced by the projection \( p_{13} : \langle E_{13} \rangle \to \mathbb{C}_\alpha \) and the second isomorphism is induced by the projection \( pr_1 : \mathbb{C}_+(3) \to \mathbb{C} \oplus \mathbb{C}_\alpha, \; pr_1(M) = (p_{23}(M), p_{12}(M)). \)

Here \( p_{ij} : M(3, \mathbb{C}) \to \mathbb{C} \) denotes the projection onto the \( (i, j) \)-coordinates. Hence (2) gives the short exact sequence

\[
0 \to \mathbb{C}_\alpha \xrightarrow{i_1} \mathbb{C}_+(3) \xrightarrow{pr_1} \mathbb{C} \oplus \mathbb{C}_\alpha \to 0
\] (3)

where \( i_1 : \mathbb{C}_\alpha \to \mathbb{C}_+(3) \) is given by \( i_1(c) = cE_{13} \).

On the other hand, \( \mathbb{C}_+(3) \) is a \( \pi \)-submodule of \( b_+ \). Let us denote \( D_+ := b_+ / \mathbb{C}_+(3) \), so we have the short exact sequence

\[
0 \to \mathbb{C}_+(3) \xrightarrow{i_2} b_+ \to D_+ \to 0.
\] (4)

The projection of an element of \( b_+ \) onto its coordinates on \( D_1 \) and \( D_2 \) induces by (1) an isomorphism \( pr_2 : D_+ \to \mathbb{C} \oplus \mathbb{C} \). Hence (4) gives the short exact sequence

\[
0 \to \mathbb{C}_+(3) \xrightarrow{i_2} b_+ \xrightarrow{pr_2} \mathbb{C} \oplus \mathbb{C} \to 0.
\] (5)

We define \( \mathbb{C}_-(3) \) as \( sl(3, \mathbb{C}) / b_+ \). Then we have a short exact sequence of \( \pi \)-modules

\[
0 \to b_+ \to sl(3, \mathbb{C}) \to \mathbb{C}_-(3) \to 0.
\] (6)

The action of \( \text{Ad} \circ \rho_0 \) on the lower triangular matrices in \( sl(3, \mathbb{C}) \) is given by:

\[
\begin{align*}
\gamma \cdot E_{21} &= \alpha^{-\| \gamma \|} E_{21} \pmod{b_+} \\
\gamma \cdot E_{31} &= \alpha^{-\| \gamma \|} h(\gamma) E_{21} + \alpha^{-\| \gamma \|} E_{31} - \alpha^{-\| \gamma \|} z(\gamma) E_{32} \pmod{b_+} \\
\gamma \cdot E_{32} &= E_{32} \pmod{b_+}.
\end{align*}
\] (7)
Let $E_{ij} = E_{ij} + b_+$, for $1 \leq j < i \leq 3$. Equation (7) gives that $\langle E_{21}, E_{32} \rangle$ is a $\pi$-submodule of $\mathbb{C}_-(3)$ and that

$$\langle E_{21}, E_{32} \rangle \simeq \mathbb{C}_{\alpha} - 1 \oplus \mathbb{C}.$$ 

Moreover, the quotient $\mathbb{C}_-(3)/\langle E_{21}, E_{32} \rangle$ is isomorphic to $\mathbb{C}_{\alpha} - 1$. This isomorphism is simply induced by the projection $p_{31}$. Hence we obtain a short exact sequence

$$0 \to \mathbb{C}_{\alpha} - 1 \oplus \mathbb{C} \to \mathbb{C}_-(3) \to \mathbb{C}_{\alpha} - 1 \to 0. \quad (8)$$

### 3.2 The computations

**Lemma 11** Same assumptions as in Proposition 10.

For the cohomology groups $H^k(\pi, \mathbb{C}_+(3))$ the following holds:

$$H^k(\pi, \mathbb{C}_+(3)) = 0, \text{ if } k \neq 1, 2,$$

$$(pr_1)_*: H^1(\pi, \mathbb{C}_+(3)) \xrightarrow{=} H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_\alpha)$$

is an isomorphism and there is a short exact sequence

$$0 \to H^2(\pi, \mathbb{C}_\alpha) \xrightarrow{(\delta_1)_*} H^2(\pi, \mathbb{C}_+(3)) \xrightarrow{(pr_1)_*} H^2(\pi, \mathbb{C}_\alpha) \to 0.$$

In particular, $\dim H^1(\pi, \mathbb{C}_+(3)) = \dim H^2(\pi, \mathbb{C}_+(3)) = 2$.

**Proof.** The long exact cohomology sequence associated to (3) gives:

$$0 \to H^0(\pi, \mathbb{C}_+(3)) \to H^0(\pi, \mathbb{C}) \xrightarrow{\delta^1} H^1(\pi, \mathbb{C}_\alpha) \to H^1(\pi, \mathbb{C}_+(3))$$

$$\to H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_\alpha) \xrightarrow{\delta^2} H^2(\pi, \mathbb{C}_\alpha) \to H^2(\pi, \mathbb{C}_+(3)) \to H^2(\pi, \mathbb{C}_\alpha) \to 0.$$

In order to calculate $\delta^1: H^0(\pi, \mathbb{C}) \to H^1(\pi, \mathbb{C}_\alpha)$ let $\delta$ denote the coboundary operator of $C^*(\pi, \mathbb{C}_+(3))$.

If $c \in Z^0(\pi, \mathbb{C}) = H^0(\pi, \mathbb{C})$ then $cE_{23} \in (pr_1)^{-1}(c, 0) \subset C^0(\pi, \mathbb{C}_+(3))$ and by (1) we obtain:

$$\delta(cE_{23})(\gamma) = c(\gamma - 1) \cdot E_{23} = cz(\gamma)E_{13}.$$

Therefore $\delta^1(c) = c\{z\} \in H^1(\pi, \mathbb{C}_\alpha)$. Since $\{z\} \neq 0$ in $H^1(\pi, \mathbb{C}_\alpha)$, $\delta^1$ is injective and hence an isomorphism (recall that $\dim H^0(\pi, \mathbb{C}) = \dim H^1(\pi, \mathbb{C}_\alpha) = 1$). This implies that $H^0(\pi, \mathbb{C}_+(3)) = 0$ and the long exact sequence in cohomology becomes

$$0 \to H^1(\pi, \mathbb{C}_+(3)) \to H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_\alpha) \xrightarrow{\delta^2} H^2(\pi, \mathbb{C}_\alpha) \to$$

$$H^2(\pi, \mathbb{C}_+(3)) \to H^2(\pi, \mathbb{C}_\alpha) \to 0. \quad (9)$$
Next we consider \( \delta^2 : H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_\alpha) \to H^2(\pi, \mathbb{C}_\alpha) \). For \( h' \in Z^1(\pi, \mathbb{C}) = H^1(\pi, \mathbb{C}) \), we have that \( h'E_{23} \in (pr_1)_*(h',0) \subset C^1(\pi, \mathbb{C}_+)(3) \) and (1) gives:

\[
\delta(h'E_{23})(\gamma_1, \gamma_2) = \gamma_1 \cdot (h'(\gamma_2)E_{23}) - h'(\gamma_1\gamma_2)E_{23} + h'(\gamma_1)E_{23}.
\]

Hence \( \delta^2(h') = \{ z \cup h' \} \) and a similar computation for \( z' \in Z^1(\pi, \mathbb{C}_\alpha) \) gives \( \delta^2(\{ z' \}) = \{-h \cup z' \} \).

So \( \delta^2(h' + z') = \{ z \cup h' \} - \{ h \cup z' \} \). Since \( \alpha \) is not a simple root of the Alexander polynomial it follows that \( \delta^2 \equiv 0 \) (see [4] Theorem 3.2). We obtain, from (2) the following exact sequences:

\[
0 \to H^1(\pi, \mathbb{C}_+)(3) \xrightarrow{(pr_1)_*} H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_\alpha) \to 0
\]

\[
0 \to H^2(\pi, \mathbb{C}_\alpha) \xrightarrow{(i_1)_*} H^2(\pi, \mathbb{C}_+)(3) \xrightarrow{(pr_1)_*} H^2(\pi, \mathbb{C}_\alpha) \to 0
\]

from which we deduce that \( \dim H^1(\pi, \mathbb{C}_+)(3) = \dim H^2(\pi, \mathbb{C}_+)(3) = 2 \). \( \square \)

In the following lemma, we will compute the dimensions of \( H^*(\pi, b_+) \).

**Lemma 12** Same assumptions as in Proposition 10.

We have \( \dim H^0(\pi, b_+) = 0 \) and \( \dim H^1(\pi, b_+) = \dim H^2(\pi, b_+) = 1 \). Moreover, we have \( \text{Ker}(i_2)_* = \text{Ker}(pr_1)_* \) where

\[(i_2)_* : H^2(\pi, \mathbb{C}_+)(3) \to H^2(\pi, b_+) \quad \text{and} \quad (pr_1)_* : H^2(\pi, \mathbb{C}_+)(3) \to H^2(\pi, \mathbb{C}_\alpha) .
\]

**Proof.** The short exact sequence (5) gives the following long exact cohomology sequence

\[
0 \to H^0(\pi, b_+) \to H^0(\pi, \mathbb{C}) \oplus H^0(\pi, \mathbb{C}) \xrightarrow{\delta^1} H^1(\pi, \mathbb{C}_+)(3) \to H^1(\pi, b_+)
\]

\[
\to H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}) \xrightarrow{\delta^2} H^2(\pi, \mathbb{C}_+)(3) \to H^2(\pi, b_+) \to 0 .
\]

A calculation similar to the one in the last proof gives that \( \delta^1 \) is injective. Thus \( H^0(\pi, b_+) = 0 \) and

\[
0 \to H^1(\pi, b_+) \to H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}) \xrightarrow{\delta^2} H^2(\pi, \mathbb{C}_+)(3) \to H^2(\pi, b_+) \to 0
\]

is exact.

Now we are interested in \( \delta^2 : H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}) \to H^2(\pi, \mathbb{C}_+)(3) \). The element \( h'D_1 \in C^1(\pi, b_+) \) projects via \( (pr_2)_* \) onto \( (h',0) \in Z^1(\pi, \mathbb{C}) \oplus Z^1(\pi, \mathbb{C}) \). Moreover:

\[
\delta(h'D_1)(\gamma_1, \gamma_2) = \gamma_1 \cdot (h'(\gamma_2)D_1) - h'(\gamma_1\gamma_2)D_1 + h'(\gamma_1)D_1
\]

\[
= z \cup h'(\gamma_1, \gamma_2)E_{12} - 3((zh - g) \cup h'(\gamma_1, \gamma_2))E_{13} ,
\]

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where $\delta$ denotes the coboundary operator of $C^*(\pi, b_+)$. Here $zh \in C^1(\pi, \mathbb{C}_\alpha)$ is simply defined by $zh(\gamma) := z(\gamma) h(\gamma)$, for $\gamma \in \pi$. Similarly, $h''D_2 \in C^1(\pi, b_+)$ projects onto $(0, h'') \in Z^1(\pi, \mathbb{C}) \oplus Z^1(\pi, \mathbb{C})$ and

$$\delta(h''D_2)(\gamma_1, \gamma_2) = \gamma_1 \cdot (h''(\gamma_2)D_2) - h''(\gamma_1\gamma_2)D_2 + h''(\gamma_1)D_2$$

$$= -3h \cup h''(\gamma_1, \gamma_2)E_{23} - 3g \cup h''(\gamma_1, \gamma_2)E_{13}.$$

We know that $\{ h \cup h'' \} = 0$ since $H^2(\pi, \mathbb{C}) = 0$. So let $h_2 : \pi \to \mathbb{C}$ be a 1-cochain such that $\delta h_2 + h \cup h'' = 0$. Then $h''D_2 + 3h_2E_{23} \in C^1(\pi, b_+)$ projects also via $(pr_2)_*$ onto $(0, h'') \in Z^1(\pi, \mathbb{C}) \oplus Z^1(\pi, \mathbb{C})$ and

$$\delta(h''D_2 + 3h_2E_{23})(\gamma_1, \gamma_2) = 3(z \cup h_2(\gamma_1, \gamma_2) + g \cup h''(\gamma_1, \gamma_2))E_{13}.$$

Hence

$$\delta^2(0, h'') = 3\{ z \cup h_2 + g \cup h'' \} E_{13} \in (i_1)_*(H^2(\pi, \mathbb{C}_\alpha)) \subset H^2(\pi, \mathbb{C}_+(3)).$$

Moreover, we know that $\{ z \cup h_2 + g \cup h'' \} \neq 0$ in $H^2(\pi, \mathbb{C}_\alpha)$ (see [15, Theorem 1] which implies $\text{rk } \delta^2 \geq 1$.

Similarly, there exists $g' : \pi \to \mathbb{C}_\alpha$ a 1-cochain satisfying $\delta g' + z \cup h' = 0$ and $h'D_1 + 3g'E_{12} \in C^1(\pi, b_+)$ projects also onto $(h', 0) \in Z^1(\pi, \mathbb{C}) \oplus Z^1(\pi, \mathbb{C})$. We obtain:

$$\delta(h'D_1 + 3g'E_{12})(\gamma_1, \gamma_2) = -3(h \cup g'(\gamma_1, \gamma_2) + (zh - g) \cup h'((\gamma_1, \gamma_2))E_{13}.$$

Note that $\delta(zh - g) + h \cup z = 0$.

This gives $\text{Im } \delta^2 \subset (i_1)_*(H^2(\pi, \mathbb{C}_\alpha)) \subset H^2(\pi, \mathbb{C}_+(3))$. In particular, $\text{rk } \delta^2 \leq 1$ and hence $\text{rk } \delta^2 = 1$. Moreover we have $\text{Im } \delta^2 = \text{Im } (i_1)_*$ and hence $\text{Ker } (i_2)_* = \text{Ker } (pr_1)_*$.

The long exact sequence in cohomology gives $\dim H^1(\pi, b_+) = \dim H^2(\pi, b_+) = 1$. \hfill \Box

**Lemma 13** The short exact sequence [8] implies that $H^0(\pi, \mathbb{C}_-(3)) \simeq H^0(\pi, \mathbb{C})$ and gives the following exact sequences:

$$0 \to H^1(\pi, \mathbb{C}_{\alpha-1}) \oplus H^1(\pi, \mathbb{C}) \to H^1(\pi, \mathbb{C}_-(3)) \to H^1(\pi, \mathbb{C}_{\alpha-1}) \to 0$$

$$0 \to H^2(\pi, \mathbb{C}_{\alpha-1}) \to H^2(\pi, \mathbb{C}_-(3)) \to H^2(\pi, \mathbb{C}_{\alpha-1}) \to 0.$$  

In particular, we have $\dim H^0(\pi, \mathbb{C}_-(3)) = 1$, $\dim H^1(\pi, \mathbb{C}_-(3)) = 3$, and $\dim H^2(\pi, \mathbb{C}_-(3)) = 2$. 

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Proof. The long exact cohomology sequence associated to the short exact sequence (8) and the fact that $H^0(\pi, \mathbb{C}_\alpha^\pm) = 0$ gives the isomorphism

$$0 \to H^0(\pi, \mathbb{C}) \overset{\cong}{\to} H^0(\pi, \mathbb{C}_{-}(3)) \to 0$$

and the exact sequence

$$0 \xrightarrow{\delta^1} H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_{\alpha-1}) \to H^1(\pi, \mathbb{C}_{-}(3)) \to$$

$$H^1(\pi, \mathbb{C}_{\alpha-1}) \xrightarrow{\delta^2} H^2(\pi, \mathbb{C}_{\alpha-1}) \to H^2(\pi, \mathbb{C}_{-}(3)) \to H^2(\pi, \mathbb{C}_{\alpha-1}) \to 0.$$

Now, by similar computation as before and by using the fact that $H^2(\pi, \mathbb{C}) = 0$ we obtain

$$\delta^2(\{z_-\}) = \{h \cup z_-\} \in H^2(\pi, \mathbb{C}_{\alpha-1}).$$

Since $\alpha$ is a double root of the Alexander polynomial, $\{h \cup z_-\} = 0$ (see [13] or [15]), so $\delta^2 \equiv 0$ and lemma follows.

Proof of Proposition [14] The short exact sequence (6) of $\pi$-modules gives the following long exact cohomology sequence

$$0 \to H^0(\pi, sl(3, \mathbb{C})) \to H^0(\pi, \mathbb{C}_{-}(3)) \xrightarrow{\delta^1} H^1(\pi, b_+) \to H^1(\pi, sl(3, \mathbb{C}))$$

$$\to H^1(\pi, \mathbb{C}_{-}(3)) \xrightarrow{\delta^2} H^2(\pi, b_+) \to H^2(\pi, sl(3, \mathbb{C})) \to H^2(\pi, \mathbb{C}_{-}(3)) \to 0.$$

An explicit calculation gives:

$$H^0(\pi, sl(3, \mathbb{C})) = \{ A \in sl(3, \mathbb{C}) \mid \gamma \cdot A = A, \forall \gamma \in \pi \} = \{0\}$$

which implies that $\delta^1$ is injective. Since $\dim H^0(\pi, \mathbb{C}_{-}(3)) = \dim H^1(\pi, b_+)$ (Lemmas [12] and [13]), $\delta^1$ is an isomorphism. So, we obtain

$$0 \to H^0(\pi, \mathbb{C}_{-}(3)) \overset{\cong}{\to} H^1(\pi, b_+) \to 0$$

and

$$0 \to H^1(\pi, sl(3, \mathbb{C})) \to H^1(\pi, \mathbb{C}_{-}(3)) \xrightarrow{\delta^2} H^2(\pi, b_+) \to$$

$$H^2(\pi, sl(3, \mathbb{C})) \to H^2(\pi, \mathbb{C}_{-}(3)) \to 0.$$

Now, $H^1(\pi, \mathbb{C}) \oplus H^1(\pi, \mathbb{C}_{\alpha-1})$ injects in $H^1(\pi, \mathbb{C}_{-}(3))$ (Lemma [13]), so to understand the map $\delta^2$, we do the following calculations:

$$\gamma \cdot E_{32} = E_{32} + \frac{1}{3} h(\gamma) (2D_2 + D_1) + g(\gamma) E_{12} - h^2(\gamma) E_{23} - g(\gamma) h(\gamma) E_{13}.$$
Hence for $h' \in Z^1(\pi, \mathbb{C}) \simeq \text{Hom}(\pi, \mathbb{C})$, we have $h'\mathcal{E}_{32} \in Z^1(\pi, \mathbb{C}_-(3))$ and $h'E_{32} \in C^1(\pi, \text{sl}(3, \mathbb{C}))$ projects onto $h'\mathcal{E}_{32}$. Moreover,
\[
\delta(h'E_{32}) = \frac{1}{3} h \cup h'(2D_2 + D_1) + g \cup h'E_{12} - h^2 \cup h'E_{23} - gh \cup h'E_{13}.
\]
Here we let $\delta$ denote the coboundary operator of $C^*(\pi, \text{sl}(3, \mathbb{C}))$.

Let $h_2: \pi \to \mathbb{C}$ be a 1-cochain such that $\delta h_2 + h \cup h' = 0$, then
\[
h'E_{32} + \frac{1}{3} h_2(2D_2 + D_1) \in C^1(\pi, \text{sl}(3, \mathbb{C}))
\]
projects also onto $h'\mathcal{E}_{32}$ and
\[
\delta(h'E_{32} + \frac{1}{3} h_2(2D_2 + D_1))(g_1, g_2) = (g \cup h' + z \cup h_2)(g_1, g_2)E_{12}
\]
\[- (h^2 \cup h' + 2h \cup h_2)(g_1, g_2)E_{23} - (gh \cup h' + (zh + g) \cup h_2)(g_1, g_2)E_{13}.
\]
This gives that
\[
\delta(h'E_{32} + \frac{1}{3} h_2(2D_2 + D_1)) \in \text{Im} \left((i_2)_*: H^2(\pi, \mathbb{C}_+(3)) \to H^2(\pi, b_+)\right).
\]
Moreover,
\[
(pr_1)_*(\delta(h'E_{32} + \frac{1}{3} h_2(2D_2 + D_1))) = g \cup h' + z \cup h_2.
\]
Since $\{g \cup h' + z \cup h_2\} \neq 0$ in $H^2(\pi, \mathbb{C}_0)$ (see [15, Theorem 1]) we obtain from $\text{Ker}(pr_1)_* = \text{Ker}(i_2)_*$ that $\delta^2(h'E_{32}) \neq 0$ and hence $\text{rk} \delta^2 \geq 1$. Moreover, $\dim H^2(\pi, b_+) = 1$, so $\text{rk} \delta^2 = 1$ and the long exact sequence enables us to conclude that
\[
0 \to H^1(\pi, \text{sl}(3, \mathbb{C})) \to H^1(\pi, \mathbb{C}_-(3)) \to H^2(\pi, b_+) \to 0
\]
is exact and that
\[
H^2(\pi, \text{sl}(3, \mathbb{C})) \xrightarrow{\sim} H^2(\pi, \mathbb{C}_-(3))
\]
is an isomorphism. In particular, using Lemmas [11, 12 and 13] we have $\dim H^1(\pi, \text{sl}(3, \mathbb{C})) = \dim H^2(\pi, \text{sl}(3, \mathbb{C})) = 2$.

\[\square\]

**Remark 14** If we consider the exact sequence in cohomology for the pair $(X, \partial X)$, we have:
\[
H^1(X, \partial X, \text{sl}(3, \mathbb{C})) \to H^1(X, \text{sl}(3, \mathbb{C})) \xrightarrow{i^*} H^1(\partial X, \text{sl}(3, \mathbb{C})).
\]
Applying the Poincaré duality, we obtain $\text{rk} i^* = \frac{1}{2} \dim H^1(\partial X, \text{sl}(3, \mathbb{C})) = 2$. So $\dim H^1(X, \text{sl}(3, \mathbb{C})) = \dim H^1(\pi, \text{sl}(3, \mathbb{C})) \geq 2$.

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4 The nature of the deformations

Throughout this section we will suppose that the \((t - \alpha)\)-torsion of the Alexander module of \(K\) is of the form \(\tau_\alpha = \mathbb{C}[t, t^{-1}]/(t - \alpha)^2\).

A representation \(\rho: \pi \to SL(n, \mathbb{C})\) is called reducible if there exists a proper subspace \(V \subset \mathbb{C}^n\) such that \(\rho(\pi)\) preserves \(V\). Otherwise \(\rho\) is called irreducible. By Burnside’s theorem, a representation \(\rho\) is irreducible if and only if the image \(\rho(\pi)\) generates the full matrix algebra \(M(n, \mathbb{C})\). The orbit of a representation \(\rho\) is the subset \(\mathcal{O}(\rho) = \{Ad_A \circ \rho \mid A \in SL(n, \mathbb{C})\} \subset R_n(\pi)\).

Note that the orbit of an irreducible representation is closed. The orbit of the representation \(\tilde{\rho}\) is not closed. This might be seen by looking at the one parameter subgroup \(\lambda: \mathbb{C}^* \to SL(3, \mathbb{C})\) given by \(\lambda(t) = diag(t, 1, t^{-1})\).

It follows that \(\rho_\alpha(\gamma) := \lim_{t \to 0} \lambda(t) \tilde{\rho}(\gamma) \lambda(t)^{-1}\)

is a diagonal representation \(\rho_\alpha: \pi \to SL(3, \mathbb{C})\) given by \(\rho_\alpha(\mu) = \alpha^{-1/3} diag(\alpha, 1, 1)\). Note that the orbit \(\mathcal{O}(\rho_\alpha)\) is closed and 4-dimensional. It is contained in the closure \(\overline{\mathcal{O}(\rho)}\) which is 8-dimensional.

**Definition 15** A representation \(\rho \in R_n(\pi)\) is called stable if its orbit \(\mathcal{O}(\rho)\) is closed and if the isotropy group \(Z(\rho)\) is finite. We denote by \(S_n(\pi) \subset R_n(\pi)\) the set of stable representations.

**Remark 16** By a result of Newstead [17, Proposition 3.8], the set \(S_n(\pi)\) is Zariski open in \(R_n(\pi)\). However, \(S_n(\pi)\) might be empty.

Next we will see that there are stable deformations of \(\tilde{\rho}\). In order to proceed we will assume that there is a Wirtinger generator \(S_1\) of \(\pi\) such that \(z(S_1) = 0 = g(S_1)\). This can always be arranged by adding a coboundary to \(z\) and \(g\) i.e. by conjugating the representation \(\tilde{\rho}\).

**Lemma 17** Let \(\rho_t: \pi \to SL(3, \mathbb{C})\) be a curve in \(R(\pi)\) with \(\rho_0 = \tilde{\rho}\). Then there exists a curve \(C_t\) in \(SL(3, \mathbb{C})\) such that \(C_0 = I_3\) and

\[
Ad_{C_t} \circ \rho_t(S_1) = \begin{pmatrix}
    a_{11}(t) & 0 & 0 \\
    0 & a_{22}(t) & a_{23}(t) \\
    0 & a_{32}(t) & a_{33}(t)
\end{pmatrix}
\]

for all sufficiently small \(t\).

**Proof.** Let \(A(t) := \rho_t(S_1)\). Note that \(\alpha^{2/3}\) is a simple root of the characteristic polynomial of \(A(0)\). Hence there is a simple eigenvalue \(a_{11}(t)\) of \(A(t)\)
which depends analytically on $t$. Note that the corresponding eigenvector $v_t$ can be chosen to depend also analytically on $t$ and such that $v_0$ is the first canonical basis vector $e_1$ of $\mathbb{C}^3$. Hence $(v_t, e_2, e_3)$ is a basis for all sufficiently small $t$ and $A(t)$ takes the form

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ 0 & a_{22}(t) & a_{23}(t) \\ 0 & a_{32}(t) & a_{33}(t) \end{pmatrix}.$$ \hspace{1cm}

Next observe that the matrix $(A_{11}(t) - a_{11}(t)I_2)$ is invertible for sufficiently small $t$. Here $A_{11}$ denotes the minor obtained from $A$ by eliminating the first row and the first column. Hence the system

$$(a_{12}(t), a_{13}(t)) + (x(t), y(t))(A_{11}(t) - a_{11}(t)I_2) = 0$$

has a unique solution and for

$$P(t) = \begin{pmatrix} 1 & x(t) & y(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the matrix $P(t)A(t)P(t)^{-1}$ has the desired form. \hfill \Box

For the next step we choose a second Wirtinger generator $S_2$ of $\pi$ such that $z(S_2) = b \neq 0 = z(S_1)$. This is always possible since $z$ is not a coboundary. Hence

$$\tilde{\rho}(S_1) = \begin{pmatrix} \alpha^{2/3} & 0 & 0 \\ 0 & \alpha^{-1/3} & \alpha^{-1/3} \\ 0 & 0 & \alpha^{-1/3} \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(S_2) = \begin{pmatrix} \alpha^{2/3} & b & c \\ 0 & \alpha^{-1/3} & \alpha^{-1/3} \\ 0 & 0 & \alpha^{-1/3} \end{pmatrix}$$

where $b \neq 0$.

**Proposition 18** Let $A(t)$ and $B(t) = (b_{ij}(t))_{1 \leq i,j \leq 3}$ be matrices depending analytically on $t$ such that

$$A(t) = \begin{pmatrix} a_{11}(t) & 0 & 0 \\ 0 & a_{22}(t) & a_{23}(t) \\ 0 & a_{32}(t) & a_{33}(t) \end{pmatrix}, \quad A(0) = \tilde{\rho}(S_1) \quad \text{and} \quad B(0) = \tilde{\rho}(S_2).$$

If the first derivative $b'_{31}(0) \neq 0$ then for sufficiently small $t$, $t \neq 0$, the matrices $A(t)$ and $B(t)$ generate the full matrix algebra.
Proof. We denote by $\mathcal{A}_t \subset M(3, \mathbb{C})$ the algebra generated by $A(t)$ and $B(t)$. Let $\chi_{A_{11}}(X)$ denote the characteristic polynomial of $A_{11}(t)$. It follows that $\chi_{A_{11}}(a_{11}(t)) \neq 0$ for small $t$ and hence

$$\frac{\chi_{A_{11}}(A(t))}{\chi_{A_{11}}(a_{11}(t))} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}[A(t)] \subset \mathcal{A}_t.$$

In the next step we will prove that

$$\mathcal{A}_t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbb{C}^3 \text{ and } (1 \ 0 \ 0) \mathcal{A}_t = \mathbb{C}^3 \text{, for small } t \in \mathbb{C}^*.$$

It follows from this that $\mathcal{A}_t$ contains all rank one matrices since a rank one matrix can be written as $v \otimes w$ where $v$ is a column vector and $w$ is a row vector. Note also that $A(v \otimes w) = (Av)w$ and $(v \otimes w)A = v \otimes (wA)$. Since each matrix is the sum of rank one matrices the proposition follows.

The vectors $(1 \ 0 \ 0)A(0), (1 \ 0 \ 0)B(0)$ and $(1 \ 0 \ 0)B(0)^2$ form a basis of the space of row vectors. Hence $(1 \ 0 \ 0)\mathcal{A}_t$ is the space of row vectors for sufficiently small $t$.

Consider the three column vectors

$$a(t) := A(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b(t) := B(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } c(t) := A(t)b(t)$$

and define the function $f(t) := \det(a(t), b(t), c(t))$. It follows that $f(t) = a_{11}(t)g(t)$ where $g(t)$ is given by

$$g(t) = \begin{vmatrix} b_{21}(t) & a_{22}(t)b_{21}(t) + a_{23}(t)b_{31}(t) \\ b_{31}(t) & a_{32}(t)b_{21}(t) + a_{33}(t)b_{31}(t) \end{vmatrix}.$$

Now it is easy to see that $g(0) = g'(0) = 0$ but $g''(0) = -\alpha^{-1/3}(b_{31}(0))^2$. Hence $g(t) \neq 0$ for sufficiently small $t$, $t \neq 0$.

□

Lemma 19 Let $z_+ \in Z^1(\pi, \mathbb{C}_{\alpha \pm 1})$ be nontrivial cocycles such that $z_+(S_1) = z_-(S_1) = 0$. If $z_+(S_2) \neq 0$ then $z_-(S_2) \neq 0$.

Proof. We define $a := \alpha + \alpha^{-1}$. The number $a$ is defined over $\mathbb{Q}$ since the Alexander polynomial is symmetric. Now we have an extension of degree two $\mathbb{Q}(a) \subset \mathbb{Q}(\alpha)$. The defining equation is simply $x^2 - ax + 1 = 0$ and
we obtain a Galois automorphism $\tau: \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$ of order two with fixed field $\mathbb{Q}(a)$ and $\tau(\alpha) = \alpha^{-1}$.

By fixing a Wirtinger presentation $\pi = \langle S_1, \ldots, S_n \mid R_1, \ldots, R_{n-1} \rangle$, each cocycle $z_\pm$ corresponds to a solution of a linear system $J(\alpha)z = 0$ where $J \in M_{n-1,n}(\mathbb{Z}[t, t^{-1}])$ is the Jacobi matrix of the presentation (see \cite[p. 976]{13}). More precisely, if $z = (s_1, \ldots, s_n)$ is a solution of the system $J(\alpha \pm 1)z = 0$ then the corresponding cocycle is given by $z_\pm(S_i) = s_i$, for $1 \leq i \leq n$.

If $z_+$ is a solution of $J(\alpha)z = 0$ with $s_1 = 0$ and $s_2 \neq 0$ then $\tau(z_+)$ is a solution of $J(\alpha^{-1})z = 0$ since $\tau$ is an automorphism it follows $\tau(s_1) = 0$ and $\tau(s_2) \neq 0$. Let $\tilde{z}_-$ denote the cocycle given by $\tilde{z}_-(S_i) = \tau(s_i)$. Note that $\tilde{z}_-$ is nontrivial since $\tilde{z}_-(S_1) \neq \tilde{z}_-(S_2)$.

It follows from Blanchfield duality that $\alpha^{-1}$ is a double root of the Alexander polynomial and that $\dim H^1(\pi, \mathbb{C}_{\alpha^{-1}}) = 1$ (see \cite[Proposition 4.7]{13}). Hence if $z_-$ is any nontrivial cocycle then there exists $t \in \mathbb{C}^*$ and $b \in \mathbb{C}$ such that $z_-(S_1) = t\tilde{z}_-(S_1) + (\alpha^{-1} - 1)b$. Now $z_-(S_1) = 0$ implies that $b = 0$ and hence $z_-(S_2) = t\tilde{z}_-(S_2) \neq 0$. 

Recall from the proof of Proposition \cite{10} that the projection $sl(3, \mathbb{C}) \to sl(3, \mathbb{C})/b_+ \cong \mathbb{C}_{*}(3)$ induces an isomorphism

$$
\Phi: H^1(\pi, sl(3, \mathbb{C})) \cong \text{Ker}(H^1(\pi, \mathbb{C}_{*}(3)) \xrightarrow{\delta^2} H^2(\pi, b_+)).
$$

Moreover, recall from Lemma \cite{13} that there is a short exact sequence

$$
0 \to H^1(\pi, \mathbb{C}_{*}(\alpha^{-1})) \oplus H^1(\pi, \mathbb{C}) \to H^1(\pi, \mathbb{C}_{*}(3)) \to H^1(\pi, \mathbb{C}_{*}(\alpha^{-1})) \to 0.
$$

In the sequel we will fix a non trivial cocycle $z_- \in Z^1(\pi, \mathbb{C}_{\alpha^{-1}})$ such that $z_-(S_1) = 0$. It follows from the preceding lemma that $z_-(S_2) \neq 0$. Moreover we have that the two cocycles $h \cup z_- \in Z^2(\pi, \mathbb{C}_{\alpha^{-1}})$ and $z \cup z_- \in Z^2(\pi, \mathbb{C})$ are coboundaries. We will also fix cochains $g_-: \pi \to \mathbb{C}_{\alpha^{-1}}$ and $g_0: \pi \to \mathbb{C}$ such that

$$
\delta g_- + h \cup z_- = 0 \text{ and } \delta g_0 + z \cup z_- = 0.
$$

From Equation (7) and the above sequence we obtain that $H^1(\pi, \mathbb{C}_{*}(3))$ is a three dimensional vector space with basis

$$
\tilde{z}_1 = z_2 \mathbf{E}_{21}, \quad \tilde{z}_2 = h\mathbf{E}_{32} \text{ and } \tilde{z}_3 = z_3 \mathbf{E}_{31} - g_0 \mathbf{E}_{32} + g_+ \mathbf{E}_{21}.
$$

Hence every $z \in Z^1(\pi, sl(3, \mathbb{C}))$ has the form

$$
z = \begin{pmatrix}
* & * & * \\
* & t_1z_- + t_3g_- + \delta b_1 & * \\
t_3z_- + \delta b_3 & t_2h - t_3g_0 & *
\end{pmatrix}
$$

where $t_i \in \mathbb{C}$.
Theorem 20  There exist deformations $\rho_t: \pi \to SL(3, \mathbb{C})$ such that $\rho_0 = \tilde{\rho}$, with the property that $\rho_t$ is stable for all sufficiently small $t$, $t \neq 0$.

Proof. Note that $Z(\rho)$ is finite if and only if $H^0(\pi, sl(3, \mathbb{C})_{\rho}) = 0$. Moreover, the condition $H^0(\pi, sl(3, \mathbb{C})_{\rho}) = 0$ is an open condition on the representation variety. Hence all representation sufficiently close to $\tilde{\rho}$ have finite stabilizer.

Let $z \in Z^1(\pi, sl(3, \mathbb{C}))$ be a cocycle such that $\Phi(z)$ is an open condition on the representation variety. Hence all representation sufficiently close to $\tilde{\rho}$ have finite stabilizer.

Let $\rho_t$ be a deformation of $\tilde{\rho}$ with leading term $z$. We apply Lemma 17 to this deformation for $A(t) = \rho_t(S_1)$ and $B(t) = \rho_t(S_2)$. Since $a_{31}(t) \equiv 0$ it follows that

$$a'_{31}(0) = \alpha^{2/3}(t_3z_-(S_1) + (\alpha^{-1} - 1)b_3) = 0$$

and hence $b_3 = 0$. By Lemma 19 we obtain $b'_{31}(0) = \alpha^{2/3}t_3z_-(S_2) \neq 0$.

Hence we can apply Proposition 18 and obtain that $\rho_t$ is irreducible for sufficiently small $t \neq 0$.

Corollary 21  There exist irreducible, non metabelian deformations of $\tilde{\rho}$.

Proof. Let $\rho_t$ be a deformation of $\tilde{\rho}$ such that $\rho_t$ is irreducible. Then for sufficiently small $t$ we have that $\text{tr} \rho_t(\mu)$ is close to $\text{tr} \tilde{\rho}(\mu) = \alpha^{-1/3}(\alpha + 2)$. Moreover we have $\text{tr} \tilde{\rho}(\mu) \neq 0$ since $-2$ is not a root of the Alexander polynomial: $(x + 2) | \Delta_K(x)$ implies $3 | \Delta_K(1) = \pm 1$ which is impossible.

By Theorem 1.2 of [5], we have for every irreducible metabelian representation $\rho: \pi \to SL(3, \mathbb{C})$ that $\text{tr} \rho(\mu) = 0$. Hence $\rho_t$ is irreducible non metabelian for sufficiently small $t$.

Remark 22  Let $\rho_\alpha: \pi \to SL(3, \mathbb{C})$ be the diagonal representation given by $\rho_\alpha(\mu) = \alpha^{-1/3} \text{diag}(\alpha, 1, 1)$. The orbit $O(\rho_\alpha)$ is contained in the closure $\overline{O(\tilde{\rho})}$. Hence $\tilde{\rho}$ and $\rho_\alpha$ project to the same point $\chi_\alpha$ of the variety of characters $X_3(\pi) = R_3(\pi) \parallel SL(3, \mathbb{C})$.

It would be natural to study the local picture of the variety of characters $X_3(\pi) = R_3(\pi) \parallel SL(3, \mathbb{C})$ at $\chi_\alpha$ as done in [13, § 8]. Unfortunately, there are much more technical difficulties since in this case the quadratic cone $Q(\rho_\alpha)$ coincides with the Zariski tangent space $Z^1(\pi, sl(3, \mathbb{C})_{\rho_\alpha})$. Therefore the third obstruction has to be considered.
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