On cusp excursions of geodesics and Diophantine approximation

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Abstract

In this article we describe some new examples of correspondence between Diophantine approximation and homogeneous dynamics, by characterizing two kinds of exceptional orbits of geodesic flow associated with the Modular surface. The characterization uses a two parameter family of continued fraction expansion of endpoints of the lifts to the hyperbolic plane of the corresponding geodesics.

1 Introduction

A fundamental connection between Diophantine approximation and homogeneous dynamics is provided by Dani correspondence. This is a correspondence between certain Diophantine properties of real numbers and the behaviour of certain orbits under the acion of an one parameter subgroup of SL(2, R) inside the homogeneous space SL(2, Z)\SL(2, R), which is a non compact space with one non compact end, known as the cusp of the space. Let \{a_t\}_{t \in \mathbb{R}} be the one parameter subgroup of SL(2, \mathbb{R}) given by

\[
a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}.
\]

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It acts on the space \( SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R}) \) by right multiplication as follows:

\[(\Gamma g)a_t \mapsto (\Gamma)ga_t, \text{ for } g \in SL(2, \mathbb{R}),\]

where we denote \( SL(2, \mathbb{Z}) \) by \( \Gamma \). A real number \( x \) is called badly approximable if there exists some constant \( \delta > 0 \) such that \( \left| x - \frac{p}{q} \right| > \frac{\delta}{q^2} \), for all \( p \in \mathbb{Z} \) and all \( q \in \mathbb{N} \).

**Theorem 1.1.** (Dani correspondence, [3]) Given a real number \( x \), let

\[\Gamma_x = \Gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.\]

Then the orbit \( \{\Gamma_xa_t\}_{t \geq 0} \) is bounded (relatively compact) in \( \Gamma \setminus SL(2, \mathbb{R}) \) if and only if \( x \) is a badly approximable number. On the other hand \( \{\Gamma_xa_t\}_{t \geq 0} \) is divergent if and only if \( x \) is rational.

In particular the above correspondence implies that if the number \( x \) is not badly approximable (which is the case for almost all real numbers with respect to Lebesgue measure), then the orbit \( \{\Gamma_xa_t\}_{t \geq 0} \) visits any neighbourhood of the cusp infinitely often and if \( x \) is rational, then given any neighbourhood \( N(C) \) of the cusp, there exists \( t_0 > 0 \) such that \( \{\Gamma_xa_t\}_{t \geq t_0} \subset N(C) \). The above correspondence is true in more general settings and it has much wider applicability. But for this article we confine our attention to its simplest form and extend it in a sense to be made precise in a moment and while doing so we make use of an essential tool in number theory viz. the continued fraction expansion of real numbers. The most frequently used continued fraction in number theory is the classical one (also known as regular or simple continued fraction), which provides the best approximation to an irrational number by rational numbers (see Theorem 182 of [6]). The classical continued fraction expansion of a real number \( x \) is produced by the following algorithm. Let \( a_0 = [x] \), where \([x]\) denotes the greatest integer \( \leq x \) and denoting \( x \) by \( x_0, x_j = \frac{1}{x_{j-1} - a_{j-1}} \), \( a_j = [x_j] \) for \( j \geq 1 \). We write

\[x = [a_0, a_1, a_2, ...] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}, \tag{1}\]

with \( a_0 \in \mathbb{Z} \) and \( a_j \in \mathbb{N} \) for \( j \geq 1 \). If \( x \) is rational then the sequence \( \{a_j\} \) is a finite sequence, otherwise it is an infinite sequence. \( a_j \)'s are called the partial quotients and
\( \frac{p_j}{q_j} = [a_0, a_1, ..., a_j] \) is called the \( j \)th convergent, \( \frac{p_j}{q_j} \) converges to \( x \) with \( \{q_j\} \) being an increasing sequence of positive integers. It is well known (see [6] for instance) that a real number is badly approximable if and only if the partial quotients in its classical continued fraction expansion are bounded. So Theorem 1.1 can be restated in terms of classical continued fraction expansion of \( x \) as follows:

\[ \{\Gamma_x a_t\}_{t \geq 0} \text{ is a bounded orbit in } \Gamma \setminus \text{SL}(2,\mathbb{R}) \text{ if and only if the partial quotients in the continued fraction expansion of } x \text{ are bounded and } \{\Gamma_x a_t\}_{t \geq 0} \text{ is divergent if and only if } x \text{ has a finite continued fraction expansion.} \]

We extend this correspondence in the following sense. We relate the average behaviour of the partial quotients of the continued fraction expansion of \( x \), to the average behaviour of the orbit \( \{\Gamma_x a_t\}_{t \geq 0} \), in terms of spending time in a neighbourhood of the cusp. Let \( N(C) \) be a neighbourhood of the cusp and

\[ I_T = \frac{\{t \in [0, T] : \Gamma_x a_t \cap N(C) \neq \phi\}}{T}, \]

we say that \( \{\Gamma_x a_t\}_{t \geq 0} \) visits the cusp with frequency 0 if \( I_T \to 0 \) as \( T \to \infty \), for some neighbourhood \( N(C) \) of the cusp. We say \( \{\Gamma_x a_t\}_{t \geq 0} \) visits the cusp with frequency 1 if \( I_T \to 1 \) as \( T \to \infty \), for every neighbourhood \( N(C) \) of the cusp. If \( x \) is a real number and \( [a_0, a_1, a_2, ...] \) is the classical continued fraction expansion of \( x \), then for \( \xi > 1 \) and \( j \geq 1 \), we define the modified partial quotients as follows:

\[ a_{j}^{\xi} = \begin{cases} a_j, & \text{if } a_j > \xi, \\ 1, & \text{if } a_j \leq \xi. \end{cases} \]

**Theorem 1.2.** Let \( x \) be a real number with \( [a_0, a_1, a_2, ...] \) being its classical continued fraction expansion. For \( \xi > 1 \), let \( \{a_{j}^{\xi}\}_{j \geq 1} \) be the sequence of modified partial quotients as defined above. Also let \( A_N^\xi = \frac{1}{N} \sum_{j=1}^{N} \log a_j^{\xi} \) and \( A_N = \frac{1}{N} \sum_{j=1}^{N} \log a_j \). Then \( \{\Gamma_x a_t\}_{t \geq 0} \) visits the cusp with frequency 0 if and only if \( A_N^\xi \to 0 \) as \( N \to \infty \) for some \( \xi > 1 \). On the other hand \( \{\Gamma_x a_t\}_{t \geq 0} \) visits the cusp with frequency 1 if and only if \( A_N \to \infty \) as \( N \to \infty \).

Now let

\[ E_0 = \{x \in \mathbb{R} : \{\Gamma_x a_t\}_{t \geq 0} \text{ visits the cusp with frequency 0}\} \]
and 
\[ E_\infty = \{ x \in \mathbb{R} : \{ \Gamma_x a_t \}_{t \geq 0} \text{ visits the cusp with frequency } 1 \} . \]

Note that \( E_0 \) contains all the badly approximable numbers and \( E_\infty \) contains the rational numbers. It was shown by V. Jarnik ([7]) that the set of badly approximable numbers has full Hausdorff dimension (see \([\mathbb{H}]\) for the definition of Hausdorff dimension), from which it follows that \( \dim E_0 = 1 \) (\( \dim \) here stands for Hausdorff dimension).

For a real number \( x \) and \([a_0, a_1, a_2, ...]\) the classical continued fraction expansion of \( x \), the quantity 
\[ \gamma(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} \log a_j, \]
whenever the limit exists, is known as the Khintchin exponent of \( x \). For any non-negative real number \( \xi \) or \( \xi = \infty \), let 
\[ E'_\xi = \{ x \in [0, 1) : \gamma(x) = \xi \} . \]

The Khintchin spectrum \( \dim E'_\xi \) was studied in detail in [5] and in particular it was shown that \( \dim E'_\infty = \frac{1}{2} \). Since \( E_\infty = \bigcup_{j \in \mathbb{Z}} (j + E'_\infty) \), it follows that \( \dim E_\infty = \frac{1}{2} \).

**Remark 1.3.** \( E_\infty \) contains apart from rational numbers, some very well approximable numbers. A real number \( x \) is said to be very well approximable if there exists \( \varepsilon > 0 \), such that \( |x - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}} \) holds for infinitely many \( q \in \mathbb{N} \) and \( p \in \mathbb{Z} \). Now construct a real number \( x = [a_0, a_1, a_2, ...] \) using classical continued fraction with the choice of \( a_j \)'s as follows. Fix some \( \varepsilon > 0 \), choose \( a_0 \in \mathbb{Z} \) and \( a_1 \in \mathbb{N} \) arbitrarily and inductively choose \( a_{j+1} = [q_j^\varepsilon] + 1 \) for \( j \geq 1 \), where \( q_j \)'s are the denominator of the convergents as defined earlier. Then as \( \{q_j\}_{j \geq 1} \) is an increasing sequence, \( \{a_j\}_{j \geq 1} \) is also an increasing sequence. Then \( \frac{1}{N} \sum_{1}^{N} \log a_j \to \infty \) as \( N \to \infty \) and, therefore, \( x \in E_\infty \). On the other hand it follows from the construction of \( x \), that the sequence of convergents \( \frac{p_j}{q_j} \) satisfy the inequality \( |x - \frac{p_j}{q_j}| < \frac{1}{q_j^{2+\varepsilon}} \) for all \( j \geq 1 \), showing that \( x \) is a very well approximable number. Note that \( E_\infty \) can not contain all very well approximable numbers as the set of very well approximable numbers has Hausdorff dimension 1 ([7]) and \( E_\infty \) has Hausdorff dimension \( \frac{1}{2} \).

Our proof of Theorem 1.1 is based on an analysis of the geodesic flow associated with the modular surface. Let \( \mathbb{H} = \{ x + iy : y > 0 \} \) be the upper half plane.
endowed with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm I\}$ acts properly discontinuously on $\mathbb{H}$ by fractional linear transformations, giving rise to the modular surface $M = \mathbb{H}/\text{PSL}(2, \mathbb{Z})$, which is topologically a sphere with two singularities and one cusp. Let $T^1 \mathbb{H}$ be the unit tangent bundle of the hyperbolic plane which is the collection $\{(z, \zeta)\}$ with $z$ in $\mathbb{H}$ and $\zeta$ being a tangent vector of norm one at the point $z$. $\text{PSL}(2, \mathbb{Z})$ acts on $T^1 \mathbb{H}$ as well and the quotient space $T^1 \mathbb{H}/\text{PSL}(2, \mathbb{Z})$ can be identified with the unit tangent bundle of $M$, which we denote by $T^1 M$. We know that $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I\}$ can be identified with $T^1 \mathbb{H}$, where the identification is given by $g \mapsto g(i, \hat{i})$, for $g \in \text{PSL}(2, \mathbb{R})$, where $\hat{i}$ denotes the unit tangent vector based at the point $i$ and pointing upwards. Similarly $\text{PSL}(2, \mathbb{Z}) \setminus \text{PSL}(2, \mathbb{R}) \simeq \text{SL}(2, \mathbb{Z}) \setminus \text{SL}(2, \mathbb{R})$ can be identified with $T^1 M$ and the right action of the one parameter subgroup $\{a_t\}$ on $\text{SL}(2, \mathbb{Z}) \setminus \text{SL}(2, \mathbb{R})$ corresponds to the geodesic flow on $T^1 M$.

Studying the behaviour of geodesics on the modular surface using classical continued fraction goes back to E. Artin ([1]) and a more precise description can be found in the work of C. Series ([11]). In this article we are going to follow the approach of S. Katok and I. Ugarcovici which is developed more recently. They consider another class of continued fractions in which negative integers are also allowed as partial quotients and the continued fraction expansion is written using minus sign. For example one can rewrite the expression in (1) as

$$x = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots}}}$$

then it gives a particular continued fraction expansion of their kind. They actually consider a two parameter family of continued fraction known as $(a, b)$-continued fraction with $a, b \in \mathbb{R}$ satisfying a technical condition (see next section for more details about $(a, b)$-continued fractions). Using these continued fraction expansions of real numbers, they describe a coding of geodesics on the modular surface, which enables one to give a symbolic description of the geodesic flow associated with the modular surface. We use this symbolic description to obtain an analogous statement of Theorem 1.2 from which Theorem 1.2 follows as a corollary.

While considering $(a, b)$-continued fraction in this article we restrict our attention to the set of parameters
\[ \mathcal{P} = \{(a, b) \in \mathbb{R}^2 | -1 \leq a < 0 < b \leq 1, b - a \geq 1\}, \]

which exclude the possibilities \(a < -1\) and \(b > 1\), though in the work of Katok and Ugarcovici (\cite{9, 10}) those possibilities were also considered with \(-ab \leq 1\). Also let \(\mathcal{E}\) be the exceptional set discussed in \cite{9}, the elements of which do not satisfy the finiteness (see next section for the definition) condition and let \(\mathcal{S} = \mathcal{P} \setminus \mathcal{E}\). Now let \(\mathbb{H}_d = \{x + iy \in \mathbb{H} : y > d\}\) and \(\mathbb{H}_d \subset T^1 \mathbb{H}\) be given by \(\mathbb{H}_d = \mathbb{H}_d \times S^1\), \(S^1\) being the unit circle. Let \(\pi\) denote both the projections from \(\mathbb{H}\) to \(M\) and from \(T^1 \mathbb{H}\) to \(T^1 M\) and \(M_d = \pi(\mathbb{H}_d), \overline{M}_d = \pi(\mathbb{H}_d).\) Note that \(\overline{M}_d\) is a typical neighbourhood of the cusp in \(T^1 M\). We denote \((z, \zeta) \in T^1 M\) by \(v\). Any \(v \in T^1 M\) determines a unique geodesic in \(M\), if we consider the geodesic along with its tangent vector at each point, then it is the orbit of \(v\) under the geodesic flow. This orbit is denoted by \(\{g_t v\}\), where \(g_t\) denotes the geodesic flow on \(T^1 M\). We say that the orbit \(\{g_t v\}_{t \geq 0}\) visits the cusp with frequency 0 if there exists some \(d > 1\), such that \(\frac{1}{T} \int_0^T \chi_{M_d}(g_t v) dt \to 0\) as \(T \to \infty\) and an orbit \(\{g_t v\}_{t \geq 0}\) is said to visit the cusp with frequency 1 if for all \(d > 1\), \(\frac{1}{T} \int_0^T \chi_{M_d}(g_t v) dt \to 1\) as \(T \to \infty\), \(\chi_{M_d}\) here denotes the characteristic function of \(\overline{M}_d\). Now let \(x \in \mathbb{R}\) and \(x = [a_0, a_1, ...]_{a,b}\) be its \((a, b)\)-continued fraction expansion. Given \(\xi > 1\) and \(j \geq 0\), We define the modified partial quotients of the \((a, b)\)-continued fraction expansion of \(x\) as follows:

\[ a_0^\xi = a_j, \text{ if } |a_j| > \xi, \]
\[ a_j^\xi = 1, \text{ if } |a_j| \leq \xi. \]

\textbf{Theorem 1.4.} For a given \(v \in T^1 M\), let \(\gamma_v\) be the corresponding geodesic in \(M\) and let \(\tilde{\gamma}_v\) be one of its lifts to the hyperbolic plane. Let \(x\) be the attracting end point of \(\tilde{\gamma}_v\). For \((a, b) \in \mathcal{S}\), let the \((a, b)\)-continued fraction expansion of \(x\) be given by \(x = [a_0, a_1, a_2, ...]_{a,b}\). Also for \(\xi > 1\), let \(\{a_j^\xi\}_{j \geq 0}\) be the modified sequence of partial quotients as defined above. Let \(A_N^\xi = \frac{1}{N} \sum_{j=0}^{N-1} \log |a_j^\xi|\) and \(A_N = \frac{1}{N} \sum_{j=0}^{N-1} \log |a_j|\).

Then the forward orbit \(\{g_t v\}_{t \geq 0}\) visits the cusp with frequency 0 if and only if \(A_N^\xi \to 0\) as \(N \to \infty\) for some \(\xi > 1\).

On the other hand \(\{g_t v\}_{t \geq 0}\) visits the cusp with frequency 1 if and only if \(A_N \to \infty\) as \(N \to \infty\).

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2 \ (a,b)\text{-continued fractions and geodesic flow}

Following S. Katok and I. Ugarovcici ([9]), for \((a,b) \in \mathcal{P}\), the \((a,b)\text{-continued fraction expansion of a real number can be defined using a generalized integral part function:}

\[
[x]_{a,b} = \begin{cases} 
[x-a] & \text{if } x < a \\
0 & \text{if } a \leq x < b \\
[x-b] & \text{if } x \geq b,
\end{cases}
\]

where \([x] = \lfloor x \rfloor + 1\). For \((a,b) \in \mathcal{P}\), every irrational number \(x\) can be expressed uniquely as an infinite continued fraction of the form (see [9] for details)

\[
x = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots}}}, \quad (a_j \in \mathbb{Z}, a_j \neq 0 \text{ for } j \geq 1),
\]

which we denote by \(x = [a_0, a_1, a_2, \ldots]_{a,b}\), where \(x_0 = x\), \(a_0 = [x_0]_{a,b}\) and \(x_j = -\frac{1}{x_{j-1} - a_{j-1}}\), \(a_j = [x_j]_{a,b}\) for \(j \geq 1\). As in the case of classical continued fraction, \(a_j\) is called the \(j\)th partial quotient and \(r_j = \frac{p_j}{q_j} = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_j}}}}\) is called the \(j\)th convergent. The sequence \(\{|q_j|\}\) is eventually increasing and \(r_j\) converges to \(x\). A particular case of \((a,b)\text{-continued fraction, viz. the } (-1,1)\text{-continued fraction (also called the alternating continued fraction) is closely related to the classical continued fraction. If } \{a_j\}_{j \geq 0} \text{ is the sequence of partial quotients in the classical continued fraction expansion of a real number } x, \text{ then } \{(-1)^ja_j\}_{j \geq 0} \text{ is the sequence of partial quotients in the } (-1,1)\text{-continued fraction expansion of } x. \text{ A similar relation holds between the nearest integer continued fraction (also known as Hurwitz’s continued fraction introduced by Hurwitz) expansion and } (-\frac{1}{2}, \frac{1}{2})\text{-continued fraction expansion of any real number.}

Let \(\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}\) and \(f_{a,b} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}\) be defined by

\[
f_{a,b}(x) = \begin{cases} 
x + 1 & \text{if } x < a \\
-\frac{1}{x} & \text{if } a \leq x < b \\
x - 1 & \text{if } x \geq b.
\end{cases}
\]

Note that \(f_{a,b}\) is defined using the standard generators \(T(x) = x + 1\) and \(S(x) = -\frac{1}{x}\).
of the modular group SL(2, \mathbb{Z}) and the continued fraction algorithm described above can be obtained using the first return map of \( f_{a,b} \) to the interval \([a, b)\).

The main object of study in [9] is a two dimensional realization of the natural extension map of \( f_{a,b} \), \( F_{a,b} : \mathbb{R} \setminus \Delta \to \mathbb{R} \setminus \Delta, \Delta = \{(x, y) \in \mathbb{R}^2 \mid x = y\} \), defined by

\[
F_{a,b}(x, y) = \begin{cases} 
(x + 1, y + 1) & \text{if } x < a \\
(1-x, -y) & \text{if } a \leq x < b \\
(x - 1, y - 1) & \text{if } x \geq b.
\end{cases}
\]

The following theorem is just a restatement of the main result of [9] for the restricted set of parameters \( \mathcal{P} \).

**Theorem 2.1.** ([9]) There exists an one-dimensional Lebesgue measure zero, uncountable set \( \mathcal{E} \) contained in \( \{(a, b) \in \mathcal{P} : b = a + 1\} \), such that for all \( (a, b) \in \mathcal{S} = \mathcal{P} \setminus \mathcal{E} \),

1. the map \( F_{a,b} \) has an attractor \( D_{a,b} = \bigcap_{n=0}^{\infty} F_{a,b}^n(\mathbb{R}^2 \setminus \Delta) \) on which \( F_{a,b} \) is essentially bijective.

2. The set \( D_{a,b} \) consists of two (or one in degenerate cases) connected components each having finite rectangular structure, i.e. bounded by non-decreasing step-functions with finitely many steps.

3. Almost every point \((x, y)\) of the plane \((x \neq y)\) is mapped to \( D_{a,b} \) after finitely many iterations of \( F_{a,b} \).

In [9], to deduce the above theorem, a crucial role in the arguments used, is played by the orbits of \( a \) and \( b \) under \( f_{a,b} \), viz. to \( a \), the upper orbit \( \mathcal{O}_u(a) \) (i.e. the orbit of \( Sa \)) and the lower orbit \( \mathcal{O}_l(a) \) (i.e. the orbit of \( Ta \)), and to \( b \), the upper orbit \( \mathcal{O}_u(b) \) (i.e. the orbit of \( T^{-1}b \)) and the lower orbit \( \mathcal{O}_l(b) \) (i.e. the orbit of \( Sb \)). It was proved in [9] that if \((a, b) \in \mathcal{P} \setminus \mathcal{E}\), then \( f_{a,b} \) satisfies the finiteness condition, which means that for both \( a \) and \( b \), their upper and lower orbits are either eventually periodic, or they satisfy the cycle property, i.e. they meet forming a cycle, in other words there exist integers \( k_1, m_1, k_2, m_2 \geq 0 \) such that

\[
f_{a,b}^{m_1}(Sa) = f_{a,b}^{k_1}(Ta) = c_a \quad \text{(respectively } f_{a,b}^{m_2}(T^{-1}b) = f_{a,b}^{k_2}(Sb) = c_b),
\]

where \( c_a \) and \( c_b \) are the ends of the cycles. If the products of transformations over the upper and lower sides of the cycle of \( a \) (respectively \( b \)) are equal, \( a \) (respectively \( b \)) is said to have strong cycle property, otherwise it has weak cycle property. Let
\[ L_a = \begin{cases} \mathcal{O}_l(a) & \text{if } a \text{ has no cycle property} \\ \text{lower part of } a\text{-cycle} & \text{if } a \text{ has strong cycle property} \\ \text{lower part of } a\text{-cycle} \cup \{0\} & \text{if } a \text{ has weak cycle property.} \end{cases} \]

\[ U_a = \begin{cases} \mathcal{O}_u(a) & \text{if } a \text{ has no cycle property} \\ \text{upper part of } a\text{-cycle} & \text{if } a \text{ has strong cycle property} \\ \text{upper part of } a\text{-cycle} \cup \{0\} & \text{if } a \text{ has weak cycle property.} \end{cases} \]

and \( L_b, U_b \) be defined similarly. Also let \( L_{a,b} = L_a \cup L_b \) and \( U_{a,b} = U_a \cup U_b \). So, \( f_{a,b} \) satisfies the finiteness condition means that both the sets \( L_{a,b} \) and \( U_{a,b} \) are finite, which is true when \((a,b) \in S\). In [9], first a set \( A_{a,b} \), having finite rectangular structure, was constructed (see Theorem 5.5 in [9]) using the values in the sets \( U_{a,b} \) and \( L_{a,b} \) and then it was shown (Theorem 6.4 in [9]) that \( A_{a,b} \) actually coincides with the attractor \( D_{a,b} \). The upper component of \( D_{a,b} \) is bounded by non-decreasing step functions with values in the set \( U_{a,b} \) and the lower component of \( D_{a,b} \) is bounded by non-decreasing step functions with values in the set \( L_{a,b} \).

Making use of the properties of the map \( F_{a,b} \) and the attractor \( D_{a,b} \), in a subsequent paper ([10]), S. Katok and I. Ugarcovici developed a general method of coding geodesics on the modular surface and gave a symbolic description of the geodesic flow associated with the modular surface. We first recall from [10] the notion of \((a,b)\)-reduced geodesics, which plays a crucial role in determining the cross-section for the geodesic flow needed for coding purposes.

**Definition 2.2.** A geodesic in \( \mathbb{H} \) with real endpoints \( u \) and \( w \), \( w \) being the attracting and \( u \) being the repelling endpoints, is called \((a,b)\)-reduced if \((u,w) \in \Lambda_{a,b} \), where

\[ \Lambda_{a,b} = F_{a,b}(D_{a,b} \cap \{a \leq w < b\}) = S(D_{a,b} \cap \{a \leq w < b\}). \]

Given any geodesic \( \gamma' \) in \( \mathbb{H} \), one can obtain an \((a,b)\)-reduced geodesic PSL\((2,\mathbb{Z})\)-equivalent to \( \gamma' \) by using the reduction property (3rd assertion in Theorem 2.1) of the map \( F_{a,b} \). More precisely if \( \gamma' \) is a geodesic which is not \((a,b)\)-reduced and if \( w' = [a_0', a_1', a_2', ...]_{a,b} \) is the attracting end point of \( \gamma' \), then there exists some positive integer \( n \) such that \( ST^{-a'_n} ... ST^{-a'_1} ST^{-a'_0}(\gamma') \) is an \((a,b)\)-reduced geodesic (see [10] for details). Now let \( \gamma \) be an \((a,b)\)-reduced geodesic with attracting and repelling endpoint \( w \) and \( u \) respectively and \([a_0, a_1, a_2, ...]_{a,b} \) be the \((a,b)\)-continued fraction expansion of \( w \). Using the essential bijectivity of the map \( F_{a,b} \) one can extend the sequence \([a_0, a_1, a_2, ...]\) in the past as well to get a bi-infinite sequence \([..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...]\), called the coding sequence of \( \gamma \) and written as
Now we recall from [10] the description of the cross-section. Let $C = \{ z \in \mathbb{H} \mid |z| = 1, \text{Im} z \geq 0 \}$ be the upper half of the unit circle and let $F$ denote the standard fundamental domain for the action of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{H}$, given by

$$F = \{ z = x + iy \in \mathbb{H} \mid |z| \geq 1, |x| \leq \frac{1}{2} \}.$$

Using the definition of $(a, b)$-reduced geodesic it is easy see the following fact.

**Proposition 2.3.** ([10]) For $(a, b) \in S$, every $(a, b)$-reduced geodesic intersects $C$.

Given an $(a, b)$-reduced geodesic $\gamma$ with attracting and repelling endpoints $w$ and $u$ respectively, the cross-section point on $\gamma$ is the intersection point of $\gamma$ with $C$. Let $\phi : \Lambda_{a,b} \to T^1 \mathbb{H}$ be defined by

$$\phi(u, w) = (z, \zeta),$$

where $z \in \mathbb{H}$ is the cross-section point on the geodesic $\gamma$, joining $u$ and $w$ and $\zeta$ is the unit vector tangent to $\gamma$ at $z$. The map $\phi$ is clearly injective and after composing with the cannonical projection $\pi$ we obtain a map
\[ \pi \circ \phi : \Lambda_{a,b} \to T^1M. \]

Let \( C_{a,b} = \pi \circ \phi(\Lambda_{a,b}) \subset T^1M. \) Then \( C_{a,b} \) is a cross-section for the geodesic flow associated to the modular surface. The lift of \( C_{a,b} \) to \( T^1\mathbb{H} \), restricted to the unit tangent vectors having base point on the fundamental domain \( \mathcal{F} \), can be described as follows:

\[ \pi^{-1}(C_{a,b}) \cap (\mathcal{F} \times S^1) = P \cup Q_1 \cup Q_2 \] (see Figure 1),

where \( P \) consists of unit tangent vectors on the circular boundary of the fundamental region \( \mathcal{F} \) and pointing inward such that the corresponding geodesic \( \gamma \) on \( \mathbb{H} \) is \((a,b)\)-reduced; \( Q_1 \) consists of unit tangent vectors with base points on the right vertical boundary of \( \mathcal{F} \) and pointing inward such that if \( \gamma \) is the geodesic corresponding to one such unit vector, then \( TS\gamma \) is \((a,b)\)-reduced; \( Q_2 \) consists of unit tangent vectors with base points on the left vertical boundary of \( \mathcal{F} \) and pointing inward such that if \( \gamma \) is the geodesic corresponding to one such unit vector, then \( T^{-1}S\gamma \) is \((a,b)\)-reduced.

Now let \( v \in T^1M \) and \( \gamma_v \) be the corresponding geodesic in \( M \) and \( \tilde{\gamma}_v \) be an \((a,b)\)-reduced lift of it inside \( \mathbb{H} \). Also let \( \eta : T^1M \to M \) be the canonical projection of \( T^1M \) onto \( M \). The following theorem from [10] provides the base for coding geodesics on the modular surface using \((a,b)\)-continued fractions.

**Theorem 2.4.** ([10]) Let \( \gamma_v \) and \( \tilde{\gamma}_v \) be as above. Then each geodesic segment of \( \gamma_v \) between successive returns to \( \eta(C_{a,b}) \), while extended to a geodesic, produces an \((a,b)\)-reduced geodesic on \( \mathbb{H} \), and each \((a,b)\)-reduced geodesic \( \text{PSL}(2,\mathbb{Z}) \)-equivalent to \( \tilde{\gamma}_v \) is obtained in this way. The first return of \( \gamma_v \) to \( \eta(C_{a,b}) \) corresponds to a left shift of the coding sequence of \( \tilde{\gamma}_v \).

Now let \( \{g_tv\} \) be the orbit of the geodesic flow on \( T^1M \) corresponding to the geodesic \( \gamma_v \), i.e. \( \gamma_v(t) = \eta(g_tv) \) and let \( \gamma^j_v \) be the segment of the geodesic \( \gamma_v \) corresponding to the portion of the orbit \( \{g_tv\}_{t \geq 0} \) between \((j-1)\)th and \( j \)th returns to the cross-section \( C_{a,b} \). We call the segment \( \gamma^j_v \) the \( j \)th excursion of the geodesic \( \gamma_v \) into the cusp. Let \( w = [a_0, a_1, a_2, ...]_{a,b} \) be the attracting end point of \( \tilde{\gamma}_v \) and \( \tilde{\gamma}_{vj} = ST^{-a_j-1}...ST^{-a_1}ST^{-a_0}(\tilde{\gamma}_v) \). Then the segment of \( \tilde{\gamma}_{vj} \) between \( C \) and \( a_j + C \), denoted by \( \tilde{\gamma}^j_v \) is a lift of \( \gamma^j_v \) to the hyperbolic plane. Assuming the geodesics to be parametrized by arc length, the time between the \((j-1)\)th and the \( j \)th return of \( \{g_tv\} \) to the cross-section, called the \( j \)th return time, is given by \( t_j = h(\gamma^j_v) = h(\tilde{\gamma}^j_v) \), where \( h \) stands for the hyperbolic length of the geodesic segment. Let \( S' = S \setminus (-1,1) \). We will see shortly that for \((a,b)\)-continued fraction coding with \((a,b) \in S' \), each return
time \( t_j \) is at a bounded distance from \( 2 \log |a_j| \) for all \( j \geq 1 \), which relies on the fact that the cross-section \( C_{a,b} \) is contained inside a compact subset of \( T^1 M \).

**Proposition 2.5.** If \((a, b) \in S'\), then \( C_{a,b} \) is contained inside a compact subset of \( T^1 M \).

**Proof.** The structure of \( D_{a,b} \) is discussed in detail in Theorem 5.5 of [9]. \( D_{a,b} \) has two connected components, the lower one we denote by \( D^l_{a,b} \) and the upper one we denote by \( D^u_{a,b} \). Both the sets \( D^l_{a,b} \) and \( D^u_{a,b} \) have finite rectangular structure i.e. bounded by non-decreasing step functions with finite number of steps. For \( D^l_{a,b} \) the values of the step function are given by the set \( \mathcal{L}_{a,b} \) and for \( D^u_{a,b} \) the values of the step function are given by the set \( \mathcal{U}_{a,b} \). The structure of the boundary (see Figure 2 for a typical picture of \( D_{a,b} \)) of \( D_{a,b} \) consists of finite number of horizontal segments at different points of the set \( \mathcal{L}_{a,b} \), called the different levels of \( D^l_{a,b} \) and consecutive levels are joined by vertical segments, where the highest level is \( y = a + 1 \). \( D^u_{a,b} \) has a similar description with the lowest level being \( y = b - 1 \). Let \( x^-_a \) be the \( x \)-coordinate of the vertical segment joining two consecutive levels \( y^-_a \) and \( y^+_a \) of \( D^l_{a,b} \) with \( y^-_a \leq a < y^+_a \) and \( x^+_a \) be the \( x \)-coordinate of the vertical segment joining two consecutive levels \( y^-_a \) and \( y^+_a \) with \( y^-_a \leq 0 < y^+_a \). Similarly, let \( x^-_b \) be the \( x \)-coordinate of the vertical segment joining two consecutive levels \( y'_- \) and \( y'_+ \) of \( D^u_{a,b} \) with \( y'_- < 0 \leq y'_+ \) and \( x^+_b \) be the \( x \)-coordinate of the vertical segment joining two consecutive levels \( y^-_b \) and \( y^+_b \) with \( y^-_b < b \leq y^+_b \). Also let \( y_l \) be the level above \( Sb \) and next to \( Sb \); \( y_u \) be the level below \( Sa \) and next to \( Sa \).

It follows from these assertions and the definition of \( \Lambda_{a,b} \), that a geodesic \( \tilde{\gamma}_v \) with attracting and repelling endpoints \( w \) and \( u \) respectively and \( w > 0 \), is \((a, b)\)-

![Figure 2: Structure of \( D_{a,b} \)](image.png)
reduced if and only if \((u, w) \in [-\frac{1}{x_a}, 0) \times [-\frac{1}{a}, \infty) \cup (0, -\frac{1}{b-1}] \times [-\frac{1}{b-1}, \infty)\). On the other hand if \(w < 0\), then \(\tilde{\gamma}_v\) is \((a, b)\)-reduced if and only if \((u, w) \in (0, -\frac{1}{x_b}] \times (-\infty, -\frac{1}{b}] \cup [-\frac{1}{x_a} , 0) \times (-\infty, -\frac{1}{a+1}].\) We shall show that \(x_a^-, x_a^+ > 1\) and \(x_b^-, x_b^+ < -1\).

For \((a, b) \in S',\) let \(m_a\) and \(m_b\) be positive integers such that \(a \leq T^{m_a} ST a < a + 1\) and \(a \leq T^{m_b} S b < a + 1\). Let \(m_a, m_b \geq 3\), then the the proof of Lemma 5.6 of \([9]\) shows that the vertical segment joining \(Sb\) and \(y_t\) has \(x\)-coordinate greater than 1 and the vertical segment joining \(y_u\) and \(Sa\) has \(x\)-coordinate less than \(-1\). Therfore, in these cases we have \(x_a^-, x_a^+ > 1\) and \(x_b^-, x_b^+ < -1\). Now we consider the situation when \(m_a, m_b \leq 2\). Note that \(m_a\) can never be 1, for if \(m_a = 1\), then \(a = 0\), since \(a > -1\), but we have assumed that \(a < 0\). So, \(m_a \geq 2\). Now if either \(m_a\) or \(m_b\) is 2, then from the explicit cycle description of \(a\) and \(b\) discussed in \([9]\), we see that there is always one level between \(y_t\) and \(a\); similarly there is always one level between \(b\) and \(y_u\). As the statement of Lemma 5.6 of \([9]\) guarantees that the vertical segment joining \(Sb\) and \(y_t\) has \(x\)-coordinate greater than or equal to 1 and the vertical segment joining \(y_u\) and \(Sa\) has \(x\)-coordinate less than or equal to \(-1\), it follows that \(x_a^-, x_a^+ > 1\) and \(x_b^-, x_b^+ < -1\) in these cases as well.

From the discussion above we have, \(-\frac{1}{x_a} < 1\) and \(-\frac{1}{x_b} > -1\). Now let \(\mu_a^+\) be the intersection point of the geodesic joining 0 and \(-\frac{1}{a}\), and \(C; \mu_b^+\) be the intersection point of the geodesic joining \(-\frac{1}{x} \) and \(-\frac{1}{b-1}\), and \(C\). We choose one of \(\mu_a^+\) and \(\mu_b^+\),
which has $y$-coordinate less than or equal to the other and denote it by $\mu_p^+$. Also let $\mu_p^-$ be the intersection point of $C$ and the vertical geodesic based at the point $-\frac{1}{x_a}$.

Then any $(a, b)$-reduced geodesic $\tilde{\gamma}_w$ having attracting endpoint $w > 0$, intersects the segment joining $\mu_p^-$ and $\mu_p^+$ of $C$ (see Figure 3). Consequently the cross-section point for any $(a, b)$-reduced geodesic having positive attracting endpoint, has $y$-coordinate uniformly bounded away from 0. The same is true for any $(a, b)$-reduced geodesic with negative attracting endpoint as well, which can be shown similarly by using the fact that $-\frac{1}{x_a} < 1$ and $-\frac{1}{x_b} > -1$. This completes the proof of the proposition.

For $(a, b) = (-1, 1)$ the cross-section $C_{-1,1}$ does not have a compact closure (see next section for details) and therefore needs to be treated separately. In the next section we discuss the cusp excursion of geodesics using $(a, b)$-continued fraction for $(a, b) \in S$. We first deal with the case $(a, b) \in S'$ and then the particular case $(a, b) = (-1, 1)$.

3 Cusp excursion with extreme frequency

3.1 $(a, b) \in S'$

While considering $(a, b)$-continued fractions in this subsection, we assume $(a, b) \in S'$ unless otherwise stated. Let $v, \gamma_v, \tilde{\gamma}_v, w = [a_0, a_1, a_2, ...]_{a,b}$ be as in the previous section. Once we know that $C_{a,b}$ is contained inside a compact set, it is easy to see that the partial quotients of the continued fraction expansion of $w$ determine how much further the orbit $\{g_t v\}$ of the geodesic flow goes into a typical neighbourhood of the cusp in its various portions between the cross-section. A more precise statement is contained in the following lemma.

**Lemma 3.1.** Let $j$ be a positive integer. Suppose $a_j > 0$, then if $a_j < 2d - b - \frac{1}{x_a}$, then $\tilde{\gamma}_{vj}$ does not intersect $\mathbb{H}_d$ and if $a_j > 2d - a - \frac{1}{x_b}$, then $\tilde{\gamma}_{vj}$ intersects $\mathbb{H}_d$. On the other hand suppose $a_j < 0$, then if $|a_j| < 2d + a + \frac{1}{x_b}$, $\tilde{\gamma}_{vj}$ does not intersect $\mathbb{H}_d$ and if $|a_j| > 2d + b + \frac{1}{x_a}$, then $\tilde{\gamma}_{vj}$ intersects $\mathbb{H}_d$.

**Proof.** If $a_j > 0$, then the attracting endpoint $w_j$ of $\tilde{\gamma}_{vj}$ lies in the interval $[a_j + a, a_j + b)$ and the repelling endpoint $u_j$ is contained in the interval $(-\frac{1}{x_a}, -\frac{1}{x_b})$.

So $\tilde{\gamma}_{vj}$ will be above the geodesic $\tilde{\gamma}_{a_j}$, joining $-\frac{1}{x_b}$ and $a_j + a$ and will be below the
geodesic $\tilde{\gamma}_{a_j}$, joining $-\frac{1}{x_a}$ and $a_j + b$. So if the radius of $\tilde{\gamma}_{a_j}$ is less than $d$, then $\tilde{\gamma}_{a_j}$ does not intersect $H_d$; on the other hand if the radius of $\tilde{\gamma}_{a_j}$ is greater than $d$, then $\tilde{\gamma}_{a_j}$ does intersect $H_d$. Now a simple calculation gives the assertion of the lemma in the case $a_j > 0$. If $a_j < 0$, then $\tilde{\gamma}_{a_j}$ lies above the geodesic joining $a_j + b$ and $-\frac{1}{x_a}$ and lies below the geodesic joining $a_j + a$ and $-\frac{1}{x_b}$. Again a simple calculation gives the assertion of the lemma in the case $a_j < 0$.

The following two lemmas crucial to the arguments to follow, can be proved easily using the fact that the cross-section $C_{a,b}$ is contained inside a compact set in $T^1M$. The proof of similar statements for the particular case $(a, b) = (-\frac{1}{2}, \frac{1}{2})$ is contained in [2] (Proposition 3.4 and Proposition 3.5 respectively) and the same proofs work for any $(a, b) \in S'$ as well.

**Lemma 3.2.** Let $d > 1$ be such that $\overline{M_d} \cap C_{a,b} = \emptyset$, then if $\gamma_d^j \cap M_d$ is nonempty, $\tilde{\gamma}_d^j \cap H_d$ is the only connected component of $\pi^{-1}(\gamma_d^j \cap M_d)$.

**Lemma 3.3.** Let $v \in T^1M$, $\gamma_v$ be the corresponding geodesic in $M$ and $\tilde{\gamma}_v$ be an $(a,b)$-reduced lift of $\gamma_v$ inside $H$. Let $w = [a_0, a_1, a_2, ...]_{a,b}$ be the attracting end point of $\tilde{\gamma}_v$ and $t_j$ be the $j$th return time for the corresponding orbit $\{g_tv\}$ of the geodesic flow. Then there exist a constant $\kappa > 0$ such that

$$|t_j - 2 \log |a_j|| \leq \kappa, \quad \forall j \geq 0.$$  (4)

**Remark 3.4.** The asymptotic estimates for values of binary quadratic forms at integer points were provided in [2] in terms of $(-\frac{1}{2}, \frac{1}{2})$-continued fraction expansion of the coefficients of the quadratic forms and the $(-\frac{1}{2}, \frac{1}{2})$-continued fraction coding of geodesics on the modular surface was used to obtain the estimates. The fact that the cross-section for geodesic flow corresponding to the $(-\frac{1}{2}, \frac{1}{2})$-continued fraction coding, is contained inside a compact subset of $T^1M$ and the return times can be bounded uniformly by the partial quotients, were used crucially to obtain those estimates. Since the above two properties hold for $(a,b)$-continued fraction coding as well for $(a,b) \in S'$, one can obtain similar estimates as in [2] for values of binary quadratic forms at integer points in terms of the $(a,b)$-continued fraction expansions of its coefficients as well.

Given $d > 1$, let $d^+ = 2d - b - \frac{1}{x_a}$, $d^- = 2d + a + \frac{1}{x_b}$ and

$$j_d^N = \#(0 \leq j < N : \text{either } a_j > d^+ \text{ if } a_j > 0 \text{ or } a_j < -d^- \text{ if } a_j < 0).$$
Let $\bar{d}_+ = 2d - a - \frac{1}{x_b}$, $\bar{d}_- = 2d + b + \frac{1}{x_a}$ and

$$j_d^N = \#(0 \leq j < N : \text{either } a_j > \bar{d}_+ \text{ if } a_j > 0 \text{ or } a_j < -\bar{d}_- \text{ if } a_j < 0).$$

Let $S_N = t_1 + t_2 + \ldots + t_N$, we denote the integral $\frac{1}{S_N} \int_0^{S_N} \chi_d(g,v)dt$ by $I_N^d$ and $\frac{1}{T} \int_0^{T} \chi_d(g,v)dt$ by $I_T^d$, where $\chi_d$ denotes the characteristic function of the neighbourhood $M_d$ of the cusp and $v \in T^1M$.

It is evident from Lemma 3.1, Lemma 3.2 and Lemma 3.3 that the $j$th excursion of the geodesic goes more and more into the cusp as the value of $|a_j|$ gets bigger and bigger and vice versa. The following proposition uses this fact to characterize those orbits of geodesic flow which visit the cusp with full frequency. We will consider $I_N^d$ only as it is easy to see that to conclude about the extreme behaviour of $I_T^d$ it is enough to consider $I_N^d$.

**Proposition 3.5.** Let $v \in T^1M$, $\gamma_v$ be the corresponding geodesic on $M$ and $\tilde{\gamma}_v$ be an $(a,b)$-reduced lift of $\gamma_v$ in $\mathbb{H}$. Let $w = [a_0, a_1, a_2, \ldots]_{a,b}$ be the attracting endpoint of $\tilde{\gamma}_v$. Then $I_N^d = \frac{1}{S_N} \int_0^{S_N} \chi_d(g,v)dt \to 1$ as $N \to \infty$ for all $d > 1$, if and only if

$$\frac{1}{N}(\log |a_0| + \log |a_2|, ..., \log |a_{N-1}|) \to \infty \text{ as } N \to \infty.$$  

**Proof.** We enumerate those $j$ for which either $a_j > \bar{d}_+$ for $a_j > 0$ or $a_j < -\bar{d}_-$ for $a_j < 0$, by the subsequence $\{j_k\}$ and by $\sum_{k=1}^{i_d^N} \log |a_{j_k}|$ we mean the sum

$$\sum_{0 \leq j \leq N-1} \log |a_j|, \quad a_j > \bar{d}_+ \text{ or } a_j < -\bar{d}_-$$

On the other hand we enumerate those $j$ for which $a_j \leq \bar{d}_+ \text{ if } a_j > 0 \text{ or } a_j \geq -\bar{d}_-$ if $a_j < 0$, by the subsequence $\{j_l\}$ and by $\sum_{l=1}^{N-i_d^N} \log |a_{j_l}|$ we mean the sum

$$\sum_{0 < a_j \leq \bar{d}_+ \text{ or } -\bar{d}_- < a_j < 0} \log |a_j|.$$
Now suppose \( \frac{1}{N} \sum_{j=0}^{N-1} \log |a_j| \to \infty \) as \( N \to \infty \) which implies by Lemma 3.3, that

\[
\frac{1}{N} \sum_{j=1}^{N} t_j \to \infty \text{ as } N \to \infty.
\]

Let \( d_{a,b} > 0 \) be such that \( M_{d_{a,b}} \cap C_{a,b} = \emptyset \). Now for any \( d > d_{a,b} \), let \( c_{j_k} = h(M_d \setminus \gamma^{t_j}) \). Then

\[
I_d^N = \frac{1}{SN} \int_0^{SN} \chi_d(gtv) dt \geq \frac{\sum_{k=1}^{N} (t_{j_k} - c_{j_k})}{\sum_{j=1}^{N} t_j} \quad \text{(5)}
\]

\[
= 1 - \frac{1}{N} \sum_{l=1}^{N - j_d^N} t_{j_l} - \frac{1}{N} \sum_{k=1}^{j_d^N} c_{j_k} - \frac{1}{N} \sum_{j=1}^{N} t_j \quad \text{(6)}
\]

As both the quantities \( \frac{1}{N} \sum_{l=1}^{N - j_d^N} t_{j_l} \) and \( \frac{1}{N} \sum_{k=1}^{j_d^N} c_{j_k} \) are bounded and \( \frac{1}{N} \sum_{j=1}^{N} t_j \) goes to \( \infty \) as \( N \to \infty \), it follows that \( I_d^N \to 1 \) as \( N \to \infty \).

To prove the converse statement we will show that if \( \frac{1}{N} \sum_{j=0}^{N-1} \log |a_j| \not\to \infty \) as \( N \to \infty \), then there is some \( d > 1 \) such that \( I_d^N \) can not go to 1 as \( N \to \infty \). Now \( \frac{1}{N} \sum_{j=1}^{N} \log |a_j| \to \infty \) as \( N \to \infty \) means that there is a subsequence \( \{N_s\} \) and \( m > 0 \) such that \( \frac{1}{N_s} \sum_{j=1}^{N_s-1} \log |a_j| < m \) for all \( s \in \mathbb{N} \), which again means by Lemma 3.3
that $\frac{1}{N_s} \sum_{j=1}^{N_s} t_j < \tilde{m}$ for some $\tilde{m} > 0$ and for all $s \in \mathbb{N}$. Since $\frac{1}{N} j_d^N \to 1$ as $N \to \infty$ for all $d > 1$ implies $\frac{1}{N} \sum_{j=1}^{N} t_j \to \infty$ as $N \to \infty$, which again by Lemma 3.3 implies $\frac{1}{N} \sum_{j=0}^{N-1} \log |a_j| \to \infty$ as $N \to \infty$, we may assume that there exists some $r > 0$ and $d > 1$ such that (if needed by considering a subsequence of $\{N_s\}$ and denoting it again by $\{N_s\}$) $\frac{1}{N} j_d^{N_s} < 1 - r$ for all $s \in \mathbb{N}$. Now

$$I_{N_s}^d \leq 1 - \frac{\frac{1}{N_s} \sum_{j=1}^{N_s} t_j}{\frac{1}{N_s} \sum_{j=1}^{N_s} t_j}.$$  \hspace{1cm} (7)

Since the cross-section point for any $(a,b)$-reduced geodesic, is uniformly bounded away from the real line, it follows that $t_j$ has a uniform lower bound, i.e. $t_j > t$ for some $t > 0$ and all $j \geq 0$. Since $\frac{1}{N_s} (N_s - j_d^{N_s}) > r$ and $\frac{1}{N_s} \sum_{j=1}^{N_s} t_j < \tilde{m}$ for all $s$, it follows that $I_{N_s}^d \leq 1 - \frac{rt}{\tilde{m}} < 1$, for all $s$. Hence $I_N^d \not\to 1$ as $N \to \infty$, a contradiction. \quad \Box

Let us now concentrate on those orbits whose frequency of visiting the cusp is zero. A complete characterization of such orbits is given by the following proposition.

**Proposition 3.6.** If $\frac{1}{N} (\log |a_{j_1}| + \log |a_{j_2}| + \ldots + \log |a_{j_d}|_N) \to 0$ as $N \to \infty$ for some $d > 1$, then $I_N^d = \frac{1}{S^d} \int_{0}^{S_N} \chi_{d'}(g_t v) dt \to 0$ as $N \to \infty$ for all $d' > d$. On the other hand if $I_N^d \to 0$ as $N \to \infty$ for some $d' > 1$, then $\frac{1}{N} (\log |a_{j_1}| + \log |a_{j_2}| + \ldots + \log |a_{j_d}|_N) \to 0$ as $N \to \infty$ for all $d > d'$. 

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Proof. From Lemma 3.3, we have

$$\frac{1}{N} \sum_{k=1}^{j_d^N} 2 \log |a_{jk}| - \frac{1}{N} j_d^N \kappa \leq \frac{1}{N} \sum_{k=1}^{j_d^N} t_{jk} \leq \frac{1}{N} \sum_{k=1}^{j_d^N} 2 \log |a_{jk}| + \frac{1}{N} j_d^N \kappa$$  \hspace{1cm} (8)

where $\kappa$ is as in that Lemma. Note that $\frac{1}{N} \sum_{k=1}^{j_d^N} 2 \log |a_{jk}| \rightarrow 0$ implies that $\frac{1}{N} j_d^N \rightarrow 0$ as $N \rightarrow \infty$. Then from (8) we conclude that $\frac{1}{N} \sum_{k=1}^{j_d^N} t_{jk} \rightarrow 0$ as $N \rightarrow \infty$.

Now for any $d' > d$,

$$I_d' \leq I_d = \frac{1}{S_N} \int_0^{S_N} \chi_d(g_tv) dt \leq \frac{1}{N} \sum_{k=1}^{j_d^N} t_{jk},$$

which tends to 0 as $N \rightarrow \infty$ since $\frac{1}{N} \sum_{j=1}^{N} t_j$ is bounded below by $t$.

To prove the converse statement let us assume that $I_d' \rightarrow 0$ as $N \rightarrow \infty$ and $d > d'$. Suppose $\frac{1}{N} \sum_{j=1}^{N} t_{jk} \rightarrow r$ for all $s \in \mathbb{N}$. Note that as $I_d'_{N_s} \leq I_d^{r}_{N_s} \rightarrow 0$, when $s \rightarrow \infty$, we have $\frac{1}{N_s} j_d^{N_s} \rightarrow 0$ as $s \rightarrow \infty$. Because $\frac{1}{N_s} j_d^{N_s} > \tilde{r}$ for some $\tilde{r} > 0$ and for all $s \in \mathbb{N}$ would imply $\frac{1}{N_s} \sum_{k=1}^{j_d^{N_s}} (t_{jk} - c_{jk}) > \tilde{r} c_1$, which in turn implies that $I_d^{r}_{N_s} > \tilde{r} c_1$ for all $s \geq 1$, where $t_{jk} - c_{jk} > c_1 > 0$.

Now let $h_{d,a,b}$ denote the least upper bound of the distances from the cross-section
point on \( C \) to the horizontal line \( y = d \) for all \( (a, b) \)-reduced geodesics. Then

\[
I_{N_s}^d \geq \frac{1}{N_s} \sum_{k=1}^{N_s} (t_{j_k} - c_{j_k}) \geq \frac{1}{N_s} \sum_{j=1}^{N_s} t_j \geq \frac{1}{N_s} \sum_{k=1}^{N_s} t_{j_k} - \frac{1}{N_s} N_s 2h_{a,b}^d \geq \frac{1}{N_s} \sum_{j=1}^{N_s} t_j \quad (9)
\]

As \( \frac{1}{N_s} N_s \to 0 \) as \( s \to \infty \), it follows that there is some \( m > 0 \), such that

\[
\frac{1}{N_s} \sum_{j=1}^{N_s} t_j < m, \text{ for all } s \in \mathbb{N}. \]

Therefore, from (9) and (10) we conclude that there exists some \( 0 < r_1 < r \), such that \( I_{N_s}^d > \frac{r_1}{m} \) for sufficiently large \( s \). Which is a contradiction to the assumption that \( I_{N_s}^d \to 0 \) as \( N \to \infty \). This completes the proof of the proposition.

Now the proof of Theorem 1.4 for \( (a, b) \in S' \) follows from Proposition 3.5 and Proposition 3.6.

**Remark 3.7.** Note that in Proposition 3.5 and Proposition 3.6, we have considered an \((a, b)\)-reduced lift of \( \gamma_0 \), whereas in Theorem 1.4 we have considered any lift of \( \gamma_0 \) to \( \mathbb{H} \). This does not lead to any ambiguity because if we obtain an \((a, b)\)-reduced geodesic \( \gamma \) with attracting end point \( w = [a_0, a_1, a_2, ...]_{a,b} \), from a geodesic \( \gamma' \) with attracting end point \( w' = [a'_0, a'_1, a'_2, ...]_{a,b} \), then \( a_j = a'_{j+n} \) for some \( n \in \mathbb{N} \).

### 3.2 \((a, b) = (-1, 1)\)

Now we concentrate on the special case \( (a, b) = (-1, 1) \). Recall that the coding of geodesics on the modular surface using this particular continued fraction expansion of real numbers is discussed in detail in \[3\], where it is called the alternating
continued fraction coding. The name alternating continued fraction comes from the fact that the partial quotients of the \((-1, 1)\)-continued fraction expansion of a real number has alternate signs. This particular coding procedure does not provide with a cross-section contained in a compact subset of \(T^1 M\). Recall from \([8]\) that a geodesic in \(\mathbb{H}\) is called \(A\)-reduced (\((-1, 1)\)-reduced with our convention), if its attracting endpoint \(w\) and repelling endpoint \(u\) satisfy \(|w| > 1\) and \(-1 < sgn(w)u < 0\) respectively. So the cross-section point for an \(A\)-reduced geodesic can be as close to the real line as one wants, showing that the cross-section is not contained inside a compact set in \(T^1 M\). So the \(j\)th return time may not be at a bounded distance from \(2 \log |a_j|\). But \(t_j\) can be controled using couple of preceding and couple of succeeding entries in the sequence of partial quotients. We recall from \([8]\), the formula for the \(j\)th return time

\[
t_j = 2 \log |w_j| + \log \frac{|w_j - u_j|\sqrt{w_j^2 - 1}}{w_j^2\sqrt{1 - u_j^2}} - \log \frac{|w_{j+1} - u_{j+1}|\sqrt{w_{j+1}^2 - 1}}{w_{j+1}^2\sqrt{1 - u_{j+1}^2}}
\]

Now assume that \(w_j > 0\), then it follows from the definition of \(A\)-reduced geodesics that \(u_j < 0\). Since the partial quotients have alternate signs, we also have \(w_{j+1} < 0\) and consequently \(u_{j+1} > 0\). Then,

\[
t_j \leq 2 \log |w_j| + \left| \log \frac{1 - u_j}{w_j} \right| + \frac{1}{2} \left| \log \left(1 - \frac{1}{w_j}\right) + \log(1 + u_j) + \log \left(1 + \frac{1}{w_{j+1}}\right) + \left| \log(1 - u_{j+1}) \right| + \frac{1}{2} \left| \log \frac{1 + \frac{1}{w_j}}{(1 - u_j)\left(1 - \frac{1}{w_{j+1}}\right)} \right| \right|.
\]

Now using the assumption that \(a_j > 0\) and consequently \(a_{j+1} < 0\), we see that

\[
1 - \frac{1}{w_j} \geq 1 - \frac{1}{1 + \frac{1}{|a_{j+1}|}}
\]

which implies that,

\[
\left| \log \left(1 - \frac{1}{w_j}\right) \right| \leq \log |a_{j+1}| + \log 2.
\]
Similarly, $1 + u_j \geq 1 - \frac{1}{1 + \frac{1}{|a_{j-2}|}}$, from which it follows that,

$$|\log(1 + u_j)| \leq \log |a_{j-2}| + \log 2.$$  

Using the continued fraction expansions for $w_{j+1}$ and $u_{j+1}$, we obtain similar estimates for other quantities in the above inequality involving $t_j$. The case $w_j < 0$ can be treated similarly and we get the following estimate for the return time $t_j$,

$$t_j \leq 2 \log |a_j| + 2 \max\{\log |a_{j+1}| + \log |a_{j-1}| + \log |a_{j+2}| + \log |a_{j-2}|\} + c,$$  

where $c$ is some constant independent of $j$. On the other hand, considering the definition of $A$-reduced geodesics and the fact that the length of the geodesic segment joining the point $i$ and $k + i$ is at a bounded distance from $2 \log |k|$, independent of $k \in \mathbb{Z}$, it is easy to see that

$$t_j \geq 2 \log |a_j| - c',$$  

where $c'$ can be taken as the hyperbolic length of the segment of the unit circle joining the point $i$ and $\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Also note that in this special case, whenever $\gamma_i \cap M_d$ is non-empty, the number
of connected components of $\pi^{-1}(\gamma_v^j \cap M_d)$ can be more than one, in fact it can be at most three. One component is $\tilde{\gamma}_v^j \cap \mathbb{H}_d$; one of the other two may be the segment starting from the cross-section point up to the intersection point of $\tilde{\gamma}_v^j$ with the horocycle $H_d$, where $H_d$ is the image of the horocycle $y = d$ under $T^{-1}S$ as shown in Figure 4; the third component may be a similar one coming from near the other end of $\tilde{\gamma}_v^j$. In Figure 4, the geodesic $\gamma_1$ is the geodesic which is tangent to the horizontal line $y = d$ and passes through the intersection point of the vertical line based at $-1$ and the horocycle $H_d$. Let $h_u^d$ be the hyperbolic length of the segment of $\gamma_1$ joining the pair of points where it cuts the horocycle $H_d$ and where it touches the line $y = d$. Then $h(\gamma_v^j \setminus M_d) \leq 2h_u^d$. On the other hand let $h_i^d$ denote the hyperbolic distance between the points $i$ and the horizontal line $y = d$. Then if $\gamma_v^j \cap M_d \neq \phi$, $h(\gamma_v^j \setminus M_d) \geq 2h_i^d$. Using these observations, the following two propositions, crucial in proving Theorem 1.4 in this case, can be proved by following a similar strategy as in the proof of Proposition 3.5 and Proposition 3.6 respectively.

**Proposition 3.8.** Let $v \in T^1 M$, $\gamma_v$ be the corresponding geodesic on $M$ and $\tilde{\gamma}_v$ be an $A$-reduced lift of $\gamma_v$ in $\mathbb{H}$. Let $w = [a_0, a_1, a_2, \ldots]_{(-1,1)}^{(0,1)}$ be the attracting endpoint of $\tilde{\gamma}_v$. Then $I_N^d = \frac{1}{S_N} \int_0^{S_N} \chi_d(g_t v) dt \to 1$ as $N \to \infty$ for all $d > 1$ if and only if

$$\frac{1}{N} \sum_{j=1}^{N} t_j \to \infty \text{ as } N \to \infty.$$ 

**Proposition 3.9.** If $\frac{1}{N} \sum_{j=1}^{N} t_j \to 0$ as $N \to \infty$ for some $c > 0$, then there exists $d > 1$ such that $I_N^d \to 0$ as $N \to \infty$ for all $d' > d$. On the other hand if $I_N^d \to 0$ as $N \to \infty$ for some $d > 1$, then there exist $c > 0$ such that $\frac{1}{N} \sum_{j=1}^{N} t_j \to 0$ as $N \to \infty$ for all $c' > c$.

**Proof of Theorem 1.4 in the case of $(-1,1)$-continued fraction.**

It follows easily from (11) and (12) that $\frac{1}{N} \sum_{j=0}^{N-1} \log |a_j| \to \infty$ as $N \to \infty$ is equivalent to $\frac{1}{N} \sum_{j=1}^{N} t_j \to \infty$ as $N \to \infty$. Now let

$$j_N^d = \# \{ 1 \leq j \leq N : \gamma_v^j \cap M_d \neq \phi \}.$$
Then for \( d > 1 \), from (11) we get
\[
\sum_{t_j > 10 \log d, 1 \leq j \leq N} t_j \leq \sum_{|a_j| > d, -2 \leq j \leq (N + 2)} 10 \log |a_j| + i_d^N c. \tag{13}
\]
Therefore, if \( \frac{1}{N} \sum_{|a_j| > d, 0 \leq j \leq N - 1} \log |a_j| \to 0 \) as \( N \to \infty \), for some \( d > 1 \),
which also implies \( \frac{1}{N} i_d^N \to 0 \) as \( N \to \infty \), it follows from (13), that
\[
\frac{1}{N} \sum_{t_j > d', 1 \leq j \leq N} t_j \to 0 \quad \text{as} \quad N \to \infty \quad \text{for all} \quad d' > 10 \log d. \quad \text{On the other hand,}
\]
it follows easily from (12), that if \( \frac{1}{N} \sum_{t_j > d, 1 \leq j \leq N} t_j \to 0 \) for some \( d > 0 \),
which also implies \( \frac{1}{N} i_d^N \to 0 \) as \( N \to \infty \), then there exists \( d'' > 1 \), such that
\[
\frac{1}{N} \sum_{|a_j| > d'', 0 \leq j \leq N - 1} \log |a_j| \to 0 \quad \text{as} \quad N \to \infty \quad \text{for all} \quad d'' > e^{d'/2}. \quad \text{With these}
\]
observations, now the proof of Theorem 1.4 in the case of \((-1, 1)\)-continued fraction, follows from Proposition 3.8 and Proposition 3.9. \( \square \)

Now let \( x \) be a real number and \([a_0, a_1, a_2, \ldots]\) be its classical continued fraction expansion, then as we have mentioned earlier the same partial quotients \( a_j \)'s with alternate signs give rise to the alternating continued fraction expansion of \( x \), i.e.,
\( x = [a_0, -a_1, a_2, \ldots]_{(-1,1)} \). Then considering the identification of \( T^1 M \) with
\( \text{SL}(2, \mathbb{Z})/\text{SL}(2, \mathbb{R}) \), Theorem 1.2 follows from Theorem 1.4.

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