COBORDISMS OF WORDS

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ABSTRACT. We introduce an equivalence relation, called cobordism, for words and produce cobordism invariants of words.

1. INTRODUCTION

Finite sequences of elements of a given set \( \alpha \) are called words in the alphabet \( \alpha \).

Words have been extensively studied by algebraic and combinatorial means, see [Lo1], [Lo2]. Gauss [Ga] used words to encode closed plane curves, viewed up to homeomorphism. For further work on Gauss words of curves, see [Ro1], [Ro2], [LM], [DT], [CE], [CR].

Words can be investigated using ideas and techniques from low-dimensional topology. The relevance of topology is suggested by the connection to curves and also by the phenomenon of linking of letters in words. A prototypical example is provided by the words \( abab \) and \( aabb \). The letters \( a, b \) are obviously linked in the first word and unlinked in the second one. A similar linking phenomenon for geometric objects, for instance knotted circles in Euclidean 3-space, is studied in knot theory.

A study of words, based on a transposition of topological ideas, was started by the author in [Tu2]. We begin by fixing an alphabet (a set) \( \alpha \) with involution \( \tau : \alpha \to \alpha \). The concept of generic curves, which may have only double self-intersections, leads to a notion of nanowords over \( \alpha \). Every letter appearing in a nanoword occurs in it exactly twice. Using an analogy with homological intersection numbers of curves on a surface, we associate with any nanoword over \( \alpha \) a certain pairing called \( \alpha \)-pairing. The concept of deformation of curves on a surface can be also transposed to the setting of words. One can view a deformation of a curve as a sequence of local transformations or moves following certain simple models. Similar homotopy moves can be defined for nanowords over \( \alpha \); they generate an equivalence relation of homotopy.

In this paper we introduce further transformations on nanowords called surgeries. In topology, surgery is an operation on manifolds consisting in cutting out a certain submanifold (with boundary) and gluing at its place another manifold with the same boundary. Various operations of this kind can be considered for words. We take here the following approach: a surgery on a nanoword deletes a symmetric subnanoword, i.e., a subnanoword isomorphic to its opposite. More generally, a surgery on a nanoword may delete a symmetric subnanophrase. Symmetry plays
here the role of the Poincaré duality for manifolds; symmetric nanophrases are moral analogues of manifolds.

Surgeries and homotopy moves generate an equivalence relation on the class of nanowords called cobordism. The main aim of the theory of cobordisms of words is to classify nanowords up to cobordism or, equivalently, to compute the set of cobordism classes of nanowords over $\alpha$. This set, denoted $\mathcal{N}_c = \mathcal{N}_c(\alpha, \tau)$, is a group with respect to concatenation of nanowords. We use $\alpha$-pairings and other homotopy invariants of nanowords introduced in [Tu2] to construct group homomorphisms from $\mathcal{N}_c$ to simpler groups. We prove that $\mathcal{N}_c$ is infinitely generated provided $\tau \neq \text{id}$ and is non-abelian provided $\tau$ has at least 3 orbits.

For non-cobordant nanowords, it is interesting to measure how far they are from being cobordant. In other terms we are interested in finding natural metrics on $\mathcal{N}_c$. The group structure on $\mathcal{N}_c$ allows us to derive such metrics from norms on $\mathcal{N}_c$.

We define two $\mathbb{Z}$-valued norms on $\mathcal{N}_c$: the length norm and the bridge norm. The length norm counts the minimal length of a nanoword in the given cobordism class. This corresponds to the topological notion of the minimal number of crossings of a curve. The bridge norm reflects the idea of a surface of minimal genus spanned in a 3-manifold $M$ by a loop in $\partial M$. To define the bridge norm we introduce so-called bridge moves on nanowords generalizing surgery. The metric on $\mathcal{N}_c$ induced by the bridge norm has a nice feature of being invariant under both left and right translations. We give an estimate from below for this metric involving a numerical invariant of $\alpha$-pairings, the so-called genus.

Adding cyclic permutations of nanowords to the list of moves, we obtain a notion of weak cobordism and also define a weak bridge pseudo-metric on $\mathcal{N}_c$. We estimate this pseudo-metric from below using the genus of $\alpha$-pairings.

A canonical procedure, called desingularization, transforms any word $w$ in the alphabet $\alpha$ into a nanoword over $\alpha$, see [Tu2]. The latter determines an element of $\mathcal{N}_c$, called the cobordism class of $w$. This allows one to apply the invariants, metrics, etc. introduced in this paper to usual words in the alphabet $\alpha$.

The main results of this paper are a construction of a homomorphism from $\mathcal{N}_c$ to a group of cobordism classes of $\alpha$-pairings (Theorem 7.3.1) and the estimates of the bridge metric and the weak bridge pseudo-metric via the genus (Corollary 9.4.3, Theorem 10.2.1).

The organization of the paper is as follows. The definitions of nanowords and nanophrases are given in Sect. 2. In Sect. 3 and 4 we introduce cobordisms of nanowords and discuss a simple cobordism invariant. Sect. 5 and 6 are concerned with the general theory of $\alpha$-pairings. In Sect. 7 and 8 we discuss the $\alpha$-pairings of nanowords and the associated cobordism invariants of nanowords. Sect. 9 – 11 are devoted to the bridge metric, the circular shift on nanowords, and the weak bridge pseudo-metric. In Sect. 12 – 14 we discuss connections between words and bridge moves on the one hand and loops on surfaces and surfaces in 3-manifolds on the other hand. We use these connections to prove two lemmas from Sect. 7 and 9. Note that Sect. 1 – 11 are written in a purely algebraic language while Sect. 12 – 14 use elementary topology.
This work is a sequel to [Tu1] – [Tu4] but a knowledge of these papers is not required. Throughout the paper, the symbol $\alpha$ denotes a fixed set endowed with an involution $\tau : \alpha \to \alpha$.

2. Nanowords and nanophrases

In this section we recall the basics of the theory of nanowords, see [Tu2].

2.1. Words and nanowords. For a positive integer $n$, set $\hat{n} = \{1, 2, \ldots, n\}$. A word of length $n$ in the alphabet $\alpha$ is a mapping $w : \hat{n} \to \alpha$. Such a word $w$ is encoded by the sequence $w(1)w(2)\cdots w(n)$. Writing the letters of $w$ in the opposite order we obtain the opposite word $w^- = w(n)w(n-1)\cdots w(1)$ in the same alphabet.

An $\alpha$-alphabet $A$ endowed with a mapping $A \to \alpha$ called projection. The image of $A \in A$ under this mapping is denoted $|A|$. An isomorphism of $\alpha$-alphabets $A_1, A_2$ is a bijection $f : A_1 \to A_2$ such that $|A_1| = |f(A)|$ for all $A \in A_1$.

A nanoword of length $n$ over $\alpha$ is a pair $(\alpha, w)$, a mapping $w : \hat{n} \to A$ such that each element of $A$ is the image of precisely two elements of $\hat{n}$. Clearly, $n = 2 \text{card}(A)$. By definition, there is a unique empty nanoword $\emptyset$ of length 0.

We say that nanowords $(A, w)$ and $(A', w')$ over $\alpha$ are isomorphic and write $w \approx w'$ if there is an isomorphism of $\alpha$-alphabets $f : A \to A'$ such that $w' = fw$.

The concatenation product of two nanowords $(A_1, w_1)$ and $(A_2, w_2)$ is defined as follows. Replacing if necessary $(A_1, w_1)$ with an isomorphic nanoword we can assume that $A_1 \cap A_2 = \emptyset$. Then the product of $w_1$ and $w_2$ is the nanoword $(A_1 \cup A_2, w_1w_2)$ where $w_1w_2$ is obtained from $w_1, w_2$ by concatenation. The nanoword $(A_1 \cup A_2, w_1w_2)$ is well defined up to isomorphism. Multiplication of nanowords is associative and has a unit $\emptyset$ (the empty nanoword).

A nanoword $w : \hat{n} \to A$ is symmetric if it is isomorphic to the opposite nanoword $w^- : \hat{n} \to A$, i.e., if there is a bijection $\iota : A \to A$ commuting with the projection to $\alpha$ and such that $\iota w = w^-$. The latter means that $\iota(w(i)) = w(n+i-1)$ for all $i \in \hat{n}$. Clearly, $\iota$ is uniquely determined by $w$ and $\iota^2 = \text{id}$. For example, the nanoword $ABBA$ with arbitrary $|A|, |B| \in \alpha$ is symmetric with $\iota = \text{id}$. The nanoword $ABAB$ is symmetric if and only if $|A| = |B|$.

2.2. Homotopy. There are three basic transformations of nanowords called homotopy moves. The first of them transforms a nanoword $(A, xAy)$ with $A \in A$ into the nanoword $(A - \{A\}, xy)$. The second homotopy move transforms a nanoword $(A, xAByBAz)$ where $A, B \in A$ with $|B| = \tau(|A|)$ into $(A - \{A, B\}, xyz)$. The third move transforms a nanoword $(A, xAByACzBCt)$ where $A, B, C \in A$ are distinct letters with $|A| = |B| = |C|$ into $(A, xBAYCAzCBt)$.

The homotopy moves and isomorphisms generate an equivalence relation of homotopy in the class of nanowords. Nanowords homotopic to $\emptyset$ are contractible. For example, for $a, b \in \alpha$ consider the nanoword $w_{a,b} = ABAB$ with $|A| = a, |B| = b$. A homotopy classification of such nanowords is given in [Tu2], Theorem
8.4.1: \( w_{a,b} \) is contractible if and only if \( a = \tau(b) \), two non-contractible nanowords \( w_{a,b}, w_{a',b'} \) are homotopic if and only if \( a = a' \) and \( b = b' \).

The third homotopy move has a more general version (see [Tu3]), but we shall not consider it here.

2.3. **Nanophrases.** A sequence of words \( w_1, \ldots, w_k \) in an \( \alpha \)-alphabet \( A \) is a nanophrase of length \( k \) (over \( \alpha \)) if every letter of \( A \) appears in \( w_1, \ldots, w_k \) exactly twice or, in other terms, if the concatenation \( w_1w_2 \cdots w_k \) is a nanoword.

We denote such a nanophrase by \( (A, (w_1 | \cdots | w_k)) \) or shorter by \( (w_1 | \cdots | w_k) \).

For a nanophrase \( \nabla = (A, (w_1 | \cdots | w_k)) \), define a function \( \varepsilon_\nabla : A \to \{0,1\} \) by \( \varepsilon_\nabla(A) = 0 \) if \( A \in A \) occurs twice in the same word of \( \nabla \) and \( \varepsilon_\nabla(A) = 1 \) if \( A \in A \) occurs in different words of \( \nabla \). Nanophrases of length 1 are just nanowords.

A nanophrase \( \nabla = (w_1 | \cdots | w_k) \) is symmetric if there is a bijection \( \iota : A \to A \) such that \( \omega_{A} = w_{A} \) for \( r = 1, \ldots, k \) and \( |\iota(A)| = \tau \varepsilon_\nabla(A) (|A|) \) for all \( A \in A \).

In the sequel we often write \( A' \) for \( \iota(A) \). The involution \( \iota \) transforms the \( i \)-th letter of \( w_i \) into the \((n_r + 1 - i)\)-th letter of \( w_r \) for all \( i, r \) where \( n_r \) is the length of \( w_r \).

Hence \( \iota = \iota_\nabla \) is determined by \( \nabla \) uniquely and \( \iota^2 = \text{id} \). For nanowords \( (k = 1) \), this notion of symmetry coincides with the one in Sect. 2A.

A nanophrase \( (w_1 | \cdots | w_k) \) is even if all the words \( w_1, \ldots, w_k \) have even length. Note that the sum of the lengths of \( w_1, \ldots, w_k \) is always even. All nanophrases of length 1 are even.

For example, the nanophrase \( (AB | BA) \) is even. It is symmetric if and only if \( |A| = \tau(|B|) \). The nanophrase \( (A | A) \) is not even. It is symmetric if and only if \( |A| = \tau(|A|) \).

2.4. **Remark.** If the involution \( \tau : \alpha \to \alpha \) is fixed-point-free, then any symmetric nanophrase \( \nabla \) over \( \alpha \) is even. Indeed, if \( \nabla \) contains a word \( w \) of odd length, then its central letter, \( A \), satisfies \( A' = A \) for \( \iota = \iota_\nabla \). If the second entry of \( A \) in \( \nabla \) also occurs in \( w \), then \( A \) occurs in \( w \) at least 3 times which contradicts the definition of a nanophrase. If the second entry of \( A \) occurs in another word of \( \nabla \), then \( \varepsilon_\nabla(A) = 1 \) and \( \tau(|A|) = |A'| = |A| \) which contradicts the assumption on \( \tau \).

3. **Surgery and cobordism**

3.1. **Surgery.** A nanophrase \( \nabla = (B, (v_1 | \cdots | v_k)) \) is a factor of a nanoword \( (A, w) \) if \( B \subset A \) and

\[
w = x_1v_1x_2v_2 \cdots x_kv_kx_{k+1}
\]

where \( x_1, x_2, \ldots, x_{k+1} \) are words in the \( \alpha \)-alphabet \( C = A - B \). It is understood that the projections \( B \to \alpha \) and \( C \to \alpha \) are the restrictions of the projection \( A \to \alpha \). Deleting \( v_1, \ldots, v_k \) from \( w \), we obtain a nanoword \( (C, x_1x_2 \cdots x_{k+1}) \). When \( \nabla \) is even and symmetric, the transformation \( (A, w) \to (C, x_1x_2 \cdots x_{k+1}) \) is called surgery. Thus, surgery deletes an even symmetric factor from a nanoword. The inverse transformation inserts an even symmetric factor.

For example, the first homotopy move deleting a factor \( AA \) is a surgery since the nanoword \( AA \) is symmetric. The second homotopy move deleting a factor \( (AB | BA) \) with \( |A| = \tau(|B|) \) is also a surgery since the nanophrase \( (AB | BA) \)
is even and symmetric. The third homotopy move is neither a surgery nor an inverse to a surgery.

Here are more examples of even symmetric factors: \((AB | AB)\) with \(|A| = \tau(|B|); (AB | CACDBD)\) with \(|A| = \tau(|B|), |C| = |D|; (AB | CDEF | DACFBE)\) with \(|A| = \tau(|B|), |C| = \tau(|F|), |D| = \tau(|E|)\).

3.2. Group \(\mathcal{N}_c\). We say that nanowords \(v, w\) (over \(\alpha\)) are cobordant and write \(v \sim_c w\) if \(v\) can be transformed into \(w\) by a finite sequence of moves from the following list:

(TR) isomorphisms, homotopy moves, surgeries, and the inverse moves.

**Lemma 3.2.1.** (i) Cobordism is an equivalence relation on the class of nanowords. Homotopic nanowords are cobordant.

(ii) If \(v \sim_c w\), then \(v^{-} \sim_c w^{-}\).

(iii) If \(v_1 \sim_c v_2\) and \(w_1 \sim_c w_2\), then \(v_1 w_1 \sim_c v_2 w_2\).

**Proof.** Claim (i) follows from the definitions. Consider a sequence of nanowords 
\(v = v_1, v_2, ..., v_n = w\) such that \(v_{i+1}\) is obtained from \(v_i\) by one of the moves (TR) for all \(i\). Then \(v_{i+1}\) is obtained from \(v_i^{-}\) by one of the moves (TR) for all \(i\). Therefore \(v^{-} \sim_c w^{-}\). To prove (iii), consider a sequence of moves (TR) transforming \(v_1\) into \(v_2\) (resp. \(w_1\) into \(w_2\)). Effecting these moves first on \(v_1\) and then on \(v_2\), we can transform \(v_1 w_1\) into \(v_2 w_2\). Hence \(v_1 w_1 \sim_c v_2 w_2\). \(\square\)

Nanowords cobordant to \(\emptyset\) are said to be slice. A symmetric nanoword is slice: being its own symmetric factor it can be deleted to give \(\emptyset\). Contractible nanowords are slice. A nanoword opposite to a slice nanoword is slice. The concatenation of two slice nanowords is slice.

The cobordism classes of nanowords form a group \(\mathcal{N}_c = \mathcal{N}_c(\alpha, \tau)\) with multiplication induced by concatenation of nanowords. The inverse to a nanoword \(w\) in \(\mathcal{N}_c\) is \(w^{-}\) since \(ww^{-}\) is symmetric and therefore slice.

3.3. The length norm. A \(\mathbb{Z}\)-valued norm on a group \(G\) is a mapping \(f: G \to \{0, 1, 2, \ldots\}\) such that \(f^{-1}(0) = 1, f(g) = f(g^{-1})\) for all \(g \in G\), and \(f(gg') \leq f(g) + f(g')\) for any \(g, g' \in G\). Such a norm \(f\) determines a metric \(\rho_f\) on \(G\) by \(\rho_f(g, g') = f(g^{-1} g')\). This metric is left-invariant: \(\rho_f(hg, hg') = \rho_f(g, g')\) for all \(g, g', h \in G\). We say that \(f\) is conjugation invariant if \(f(h^{-1} gh) = f(g)\) for all \(g, h \in G\). It is clear that if \(f\) is conjugation invariant, then \(\rho_f\) is right-invariant in the sense that \(\rho_f(g h, g'h) = \rho_f(g, g')\) for all \(g, g', h \in G\).

The length of nanowords determines a \(\mathbb{Z}\)-valued norm \(|| \cdot ||\) on \(\mathcal{N}_c\) called the length norm. Its value \(||w||_1\) on a cobordism class of a nanoword \(w\) is half of the minimal length of a nanoword cobordant to \(w\). The axioms of a norm are straightforward. In particular, \(||w||_1 = 0\) if and only if \(w\) is slice. Since all nanowords of length 2 are contractible, the length norm does not take value 1. Generally speaking, the length norm is not conjugation invariant. The associated left-invariant metric on \(\mathcal{N}_c\) is denoted \(\rho_l\) and called the length metric.
3.4. **Push-forwards and pull-backs.** Given another set with involution \((\overline{\alpha}, \overline{\tau})\) and an equivariant mapping \(f : \overline{\alpha} \to \alpha\), the induced **push-forward** transforms a nanoword \((A, w)\) over \(\overline{\alpha}\) in the same nanoword \((A, w)\) with projection \(A \to \overline{\alpha}\) replaced by its composition with \(f\). The push-forward is compatible with cobordism and induces a group homomorphism \(f_* : N_c(\overline{\alpha}, \overline{\tau}) \to N_c(\alpha, \tau)\). Clearly, \(||f_*(x)|| \leq ||x||\) for any \(x \in N_c(\overline{\alpha}, \overline{\tau})\).

For a \(\tau\)-invariant subset \(\beta\) of \(\alpha\), the **pull-back** to \(\beta\) transforms any nanoword \((A, w)\) over \(\alpha\) in the nanoword over \((\beta, \tau|_\beta)\) obtained by deleting from both \(A\) and \(w\) all letters \(a \in A\) with \(|A| \in \alpha - \beta\). This transformation is compatible with cobordism and induces a group homomorphism \(\varphi_\beta : N_c(\alpha, \tau) \to N_c(\beta, \tau|_\beta)\). Clearly, \(||\varphi_\beta(x)|| \leq ||x||\) for any \(x \in N_c(\alpha, \tau)\). Composing the push-forward \(i_* : N_c(\beta, \tau|_\beta) \to N_c(\alpha, \tau)\) induced by the inclusion \(i : \beta \hookrightarrow \alpha\) with \(\varphi_\beta\) we obtain the identity. Therefore \(i_*\) is injective and \(\varphi_\beta\) is surjective.

3.5. **Examples.** 1. For \(a, b \in \alpha\), consider the nanoword \(w_{a,b} = ABAB\) with \(|A| = a, |B| = b\). If \(a = b\), then \(w_{a,b}\) is symmetric and therefore slice. If \(a = \tau(b)\), then deleting the factor \((AB | AB)\) we obtain \(\emptyset\) so that \(w_{a,b}\) is slice (in fact \(w_{a,b}\) is contractible for \(a = \tau(b)\), see [Tu2], Lemma 3.2.2). If \(a, b\) belong to different orbits of \(\tau\), then \(w_{a,b}\) is not slice, see Sect. 3.1. Obviously, \(||w|| \leq 2\) and since \(||w|| \neq 0, 1\), we have \(||w|| = 2\).

2. Pick \(a, c \in \alpha\) and consider the nanoword \(w = ABACDCDB\) with \(|A| = |B| = a, |C| = |D| = c\). Deleting the symmetric nanoword \(CDACDB\) from \(w\), we obtain a symmetric nanoword \(ABAB\). Therefore \(w\) is slice. Note that \(w\) is not symmetric. If \(a, c\) belong to different orbits of \(\tau\) and \(a \neq \tau(a)\), then the pull-back of \(w\) to the orbit of \(a\) yields a non-contractible nanoword \(ABAB\). Therefore in this case \(w\) is not contractible.

3. The nanoword \(ABACDCDB\) with \(|A| = \tau(|B|)\) and \(|C| = |D|\) is slice since \(CADC\) is symmetric and \(ABAB\) is contractible. The nanoword \(ABCACDB\) with \(|A| = \tau(|B|), |C| = |D|\) is slice since the deletion of the even symmetric factor \((AB | CADCDB)\) gives \(\emptyset\).

4. **Homomorphism \(\gamma\)**

We construct a group homomorphism from \(N_c = N_c(\alpha, \tau)\) to a free product of cyclic groups. This allows us to show that, generally speaking, \(N_c\) is non-abelian.

4.1. **Group \(\Pi\) and homomorphism \(\gamma\).** Let \(\Pi\) be the group with generators \(\{z_a\}_{a \in \alpha}\) and defining relations \(z_a z_{\tau(a)} = 1\) for all \(a \in \alpha\). For a nanoword \((A, w) : \overline{\alpha} \to A\) over \(\alpha\), set \(\gamma(w) = \gamma_1 \cdots \gamma_n \in \Pi\) where \(\gamma_i = z_{w(i)}\) if \(i\) numerates the first entry of \(w(i)\) in \(w\) (that is if \(w(i) \neq w(j)\) for \(j < i\)) and \(\gamma_i = (z_{w(i)})^{-1}\) if \(i\) numerates the second entry of \(w(i)\) in \(w\). For example, for \(w = ABAB\) with \(|A| = a \in \alpha, |B| = b \in \alpha\), we have \(\gamma(w) = z_a z_b z_a^{-1} z_b^{-1}\).

**Lemma 4.1.1.** The element \(\gamma(w) \in \Pi\) is invariant under the moves \((TR)\) on \(w\). The formula \(w \mapsto \gamma(w)\) defines a group homomorphism \(\gamma : N_c \to \Pi\) and \(\gamma(N_c) = [\Pi, \Pi]\).
Proof. It is easy to check that $\gamma(w)$ is invariant under isomorphisms and homotopy moves on $w$. Let us check the invariance under surgery. It suffices to show that for any even symmetric factor $\nabla = (v_1 | \cdots | v_k)$ of $w$ and any $r \in \{1, \ldots, k\}$, we have $\gamma(v_r) = 1$. Fix $r$ and set $v = v_r$. Let $n \geq 0$ be the length of $v$. By definition, $\gamma(v) = \gamma_1 \cdots \gamma_n \in \Pi$ with $\gamma_i$ defined by the $i$-th letter of $v$ as above. We claim that $\gamma_i = (\gamma_{n+1-i})^{-1}$ for all $i$. This and the assumption that $n$ is even would imply that $\gamma(v) = 1$.

Pick $i \in \{1, \ldots, n\}$. Consider first the case where the letter $v(i)$ occurs in $v$ twice. Then $\gamma_i = z_{|v(i)|}$ if $i$ numerates the first entry of $v(i)$ in $v$ and $\gamma_i = (z_{|v(i)|})^{-1}$ otherwise. Observe that if $i$ numerates the first (resp. the second) entry of $v(i)$, then by the symmetry of $\nabla$, the index $n + 1 - i$ numerates the second (resp. the first) entry of the letter $v(n + 1 - i) = \varepsilon_\nabla(v(i))$ in $v$. Also $\varepsilon_\nabla(v(i)) = 0$ and by the definition of a symmetric nanophrase, $|v(i)| = |\varepsilon_\nabla(v(i))| = |v(n + 1 - i)|$. Hence, $\gamma_i = (\gamma_{n+1-i})^{-1}$. If $v(i)$ occurs in $v = v_r$ only once, then by the symmetry, the same is true for $v(n + 1 - i)$. In particular, $v(i) \neq v(n + 1 - i)$. The symmetry implies also that the other entries of these two letters in $\nabla$ occur in the same word $v_{r'}$ where $r' \neq r$. Then $\gamma_i = (z_{|v(i)|})^\delta$ and $\gamma_{n+1-i} = (z_{|v(n+1-i)|})^\delta$ where $\delta = 1$ if $r' > r$ and $\delta = -1$ if $r' < r$. Observe that $\varepsilon_\nabla(v(i)) = 1$ and so $|v(i)| = \tau(|v(n + 1 - i)|)$. Hence $\gamma_i = (\gamma_{n+1-i})^{-1}$.

The second claim of the lemma follows from the definitions. The equality $\gamma(\mathcal{N}_c) = [\Pi, \Pi]$ follows from \cite[Lemma 4.1.1]{[ref]}.

The group $\Pi$ is a free product of the cyclic subgroups generated by $\{z_a\}$ and numerated by the orbits of the involution $\tau$. More precisely, $\Pi$ is a free product of $m$ infinite cyclic groups and $l$ cyclic groups of order 2 where $m$ is the number of free orbits of $\tau$ and $l$ is the number of fixed points of $\tau$. The commutator subgroup $[\Pi, \Pi]$ is a free group of infinite rank if $m \geq 2$ or $m = 1$ and $l \geq 1$. If $m = 1$ and $l = 0$, then $\Pi = \mathbb{Z}$ and $[\Pi, \Pi] = 0$. If $m = 0$, then $[\Pi, \Pi]$ is a free group of rank $2^{l-1}(l-2) + 1$. One can see it by realizing $\Pi$ as the fundamental group of the connected sum $X$ of $l$ copies of $RP^3$ and observing that the maximal abelian covering of $X$ has the same fundamental group as a connected graph with $2^{l-1}$ vertices and $2^l(l-1)$ edges. These computations and Lemma \cite[4.1.1]{[ref]} give the following information on the group $\mathcal{N}_c$.

**Theorem 4.1.2.** If $\tau$ has at least two orbits, then $\mathcal{N}_c$ is infinite. If $\tau$ has at least two orbits and $\tau \neq \text{id}$, then $\mathcal{N}_c$ is infinitely generated. If $\tau$ has at least three orbits or $\tau$ has two orbits and $\tau \neq \text{id}$, then $\mathcal{N}_c$ is non-abelian.

The free product structure on $\Pi$ allows us to detect easily whether two given elements of $\Pi$ are equal or not. As an application, consider a nanoword $w = A_1A_2 \cdots A_n$ such that $|A_i|, |A_{i+1}| \in \alpha$ do not lie in the same orbit of $\tau$ for $i = 1, \ldots, n - 1$. Then there are no cancellations in the expansion $\gamma(w) = \gamma_1 \cdots \gamma_n \in \Pi$. This implies that such $w$ is non-slice and moreover $||w||_l = n/2$. For instance, consider the nanoword $w = w_{a,b} = ABAB$ where $|A| = a \in \alpha, |B| = b \in \alpha$. By Example 3.3.1, if $a, b$ lie in the same orbit of $\tau$, then $w$ is slice. If $a, b$ do not lie in the same orbit of $\tau$, then by the criterion above, $w$ is non-slice and $||w||_l = 2$. 

\[ \square \]
Theorem 4.1.3. Two non-slice nanowords \( w = w_{a,b} \) and \( w' = w_{a',b'} \) with \( a, b, a', b' \in \alpha \) are cobordant if and only if \( a = a' \) and \( b = b' \).

Proof. If \( w \sim_c w' \), then
\[
(4.1.1) \quad z_a z_b z_a^{-1} z_b^{-1} = \gamma(w) = \gamma(w') = z_{a'} z_{b'} z_{a'}^{-1} z_{b'}^{-1}.
\]
The non-sliceness of \( w \) (resp. \( w' \)) implies that \( a, b \) (resp. \( a', b' \)) belong to different orbits of \( \tau \). Therefore there are no cancellations in the expansions for \( \gamma(w), \gamma(w') \) above. Formula (4.1.1) implies then that \( z_a = z_{a'} \) and \( z_b = z_{b'} \). Therefore \( a = a' \) and \( b = b' \).

4.2. Homomorphism \( \tilde{\gamma} \). The homomorphism \( \gamma \) admits the following refined version. Let \( \tilde{\Pi} \) be the group with generators \( \{ \tilde{z}_a \}_{a \in \alpha} \) and defining relations \( \tilde{z}_a \tilde{z}_{\tau(a)} \tilde{z}_b = \tilde{z}_b \tilde{z}_a \tilde{z}_{\tau(a)} \) for all \( a, b \in \alpha \). The formula \( \tilde{z}_a \mapsto z_a \) defines a projection \( \tilde{\Pi} \to \Pi \) which makes \( \tilde{\Pi} \) into a central extension of \( \Pi \). Replacing \( z \) with \( \tilde{z} \) in the definition of \( \gamma \), we obtain a lift of \( \gamma \) to a group homomorphism \( \tilde{\gamma} : \tilde{\mathcal{N}}_c \to \tilde{\Pi} \).

The homomorphisms \( \gamma \) and \( \tilde{\gamma} \) are not injective. For example, as we shall see in Sect. 7.3.1, the nanoword \( w = ABCBAC \) with \( |A| = |B| = |C| \neq \tau(|A|) \) is non-slice but obviously \( \tilde{\gamma}(w) = 1 \).

5. \( \alpha \)-PAIRINGS AND THEIR COBORDISM

We now turn to the main theme of this paper: a study of cobordisms of nanowords via a study of the linking properties of the letters. In this and the next sections we introduce a purely algebraic theory of \( \alpha \)-pairings; it will be applied to nanowords in later sections.

Fix an associative (possibly, non-commutative) ring \( R \) and a left \( R \)-module \( \pi \). The module \( \pi \) will be the target of all \( \alpha \)-pairings.

5.1. \( \alpha \)-PAIRINGS. An \( \alpha \)-pairing is a set \( S \) endowed with a distinguished element \( s \in S \) and mappings \( S - \{ s \} \to \alpha \) and \( e : S \times S \to \pi \). The conditions on \( S \) can be rephrased by saying that \( S \) is a disjoint union of an \( \alpha \)-alphabet \( S^0 = S - \{ s \} \) and a distinguished element \( s \). The image of \( A \in S^0 \) under the projection to \( \alpha \) is denoted \( |A| \). The pairing \( e : S \times S \to \pi \) uniquely extends to a bilinear form \( RS \times RS \to \pi \) where \( RS \) is the free \( R \)-module with basis \( S \). This form is denoted by \( \tilde{e} \) or, if it cannot lead to a confusion, simply by \( e \). Every \( A \in S \) determines a basis vector in \( RS \) denoted by the same symbol \( A \).

An isomorphism of \( \alpha \)-pairings \( (S_1, s_1, e_1), (S_2, s_2, e_2) \) is a bijection \( S_1 \to S_2 \) transforming \( s_1, e_1 \) into \( s_2, e_2 \), respectively, and inducing an isomorphism of \( \alpha \)-alphabets \( S_1^0 \to S_2^0 \). Isomorphism of \( \alpha \)-pairings is denoted \( \approx \).

For each \( \alpha \)-pairing \( p = (S, s, e) \), we have the opposite \( \alpha \)-pairing \( p^\sim = (S, s, e^\sim) \) where \( e^\sim(A, B) = -e(A, B) \) for \( A, B \in S \).

5.2. Hyperbolic \( \alpha \)-PAIRINGS. Consider an \( \alpha \)-pairing \( p = (S, s, e) \). A vector \( x \in RS \) is short (with respect to \( p \)) if \( x \in S^0 \subset S \subset RS \) or \( x = A + B \) for distinct \( A, B \in S^0 \) with \( |A| = |B| \) or \( x = A - B \) for distinct \( A, B \in S^0 \) with \( |A| = \tau(|B|) \). Note that if \( |A| = |B| = \tau(|B|) \) then both \( A + B \) and \( A - B \) are short.
A filling of $p$ is a finite family of vectors $\{\lambda_i \in RS\}_i$ such that one of the $\lambda_i$'s is equal to $s$, all the other $\lambda_i$ are short, and every element of $S^\circ$ occurs in exactly one of $\lambda_i$ with non-zero coefficient (this coefficient is then $\pm 1$). For example, the family $\{A\}_{A \in S}$ is a filling of $p$. It is called the tautological filling.

A filling $\{\lambda_i\}_i$ of $p$ is annihilating if $\bar{e}(\lambda_i, \lambda_j) = 0$ for all $i, j$. The $\alpha$-pairing $p$ is hyperbolic if it has an annihilating filling. Since the number of fillings of $p$ is finite, one can detect in a finite number of steps whether $p$ is hyperbolic or not.

If an $\alpha$-pairing is hyperbolic, then the opposite $\alpha$-pairing and all isomorphic $\alpha$-pairings are hyperbolic.

5.3. Summation of $\alpha$-pairings. The sum $p_1 \oplus p_2$ of $\alpha$-pairings $p_1 = (S_1, s_1, e_1)$ and $p_2 = (S_2, s_2, e_2)$ is the $\alpha$-pairing $(S = S_1^\circ \sqcup S_2^\circ \sqcup \{s\}, s, e)$ where $e : S \times S \to \pi$ is defined as follows. Consider the bilinear form $\tilde{e}_i : RS_i \times RS_i \to \pi$ extending $e_i$ for $i = 1, 2$. The direct sum $\tilde{e}_1 \oplus \tilde{e}_2$ is a bilinear form on $RS_1 \oplus RS_2$. Consider the $R$-linear embedding $f : RS \hookrightarrow RS_1 \oplus RS_2$ which extends the embeddings $S_i^\circ \hookrightarrow S_i \subset RS_i$ with $i = 1, 2$ and sends $s \in S$ to $s_1 \oplus s_2$. For $x, y \in S$, set

$$ e(x, y) = (\tilde{e}_1 \oplus \tilde{e}_2)(f(x), f(y)) \in \pi. $$

The values of $e$ can be computed explicitly: $e(S_i^\circ, S_j^\circ) = e(S_i^\circ, S_j^\circ) = 0$; $e|S_i^\circ = e_i|S_i^\circ$; $e(s, s) = e_1(s_1, s_1) + e_2(s_2, s_2)$; $e(A, s) = e_i(A, s_i)$, $e(s, A) = e_i(s_i, A)$ for $i = 1, 2$ and $A \in S_i^\circ$. It is clear that the bilinear extension $\tilde{e} : RS \times RS \to \pi$ of $e$ is obtained by pushing back $\tilde{e}_1 \oplus \tilde{e}_2$ along $f$. Observe that the projection $S^\circ = S_1^\circ \sqcup S_2^\circ \to \alpha$ is the disjoint union of the given projections $S_i^\circ \to \alpha$ and $S_i^\circ \to \alpha$. In these constructions we assume $S_1, S_2$ to be disjoint; if it is not the case, replace $p_1$ by an isomorphic $\alpha$-pairing and proceed as above.

We shall routinely describe fillings of $p_1 \oplus p_2$ in terms of their images under the embedding $f : RS \hookrightarrow RS_1 \oplus RS_2$. By abuse of the language, the image of a filling of $p_1 \oplus p_2$ under $f$ will sometimes be called a filling of $p_1 \oplus p_2$. A finite family of vectors $\lambda = \{\lambda_i\}_i \subset RS_1 \oplus RS_2$ is the image of a filling of $p_1 \oplus p_2$ if and only if it satisfies the following conditions: $\lambda$ consists of $s_1 + s_2$ and vectors of the form $A \in S_1^\circ \sqcup S_2^\circ$ or $A \pm B$, where $A, B$ are distinct elements of $S_1^\circ \sqcup S_2^\circ$ with $|A| \in \{|B|, \tau(|B|)\}$, and the sign in front of $B$ is necessarily $+$ if $|A| = |B| \neq \tau(|B|)$ and is necessarily $-$ if $|A| = \tau(|B|) \neq |B|$; every element of $S_1^\circ \sqcup S_2^\circ$ occurs in exactly one $\lambda_i$ with non-zero coefficient (equal then to $\pm 1$). It is clear that $\lambda$ corresponds to an annihilating filling of $p_1 \oplus p_2$ if and only if $(\tilde{e}_1 \oplus \tilde{e}_2)(\lambda_i, \lambda_j) = 0$ for all $i, j$.

The sum $p_1 \oplus p_2$ is well-defined up to isomorphism and $p_1 \oplus p_2 \approx p_2 \oplus p_1$. The sum of hyperbolic $\alpha$-pairings is hyperbolic.

5.4. Cobordism of $\alpha$-pairings. We say that two $\alpha$-pairings $p_1, p_2$ are cobordant and write $p_1 \simeq p_2$ if the $\alpha$-pairing $p_1 \oplus p_2^-$ is hyperbolic.

Lemma 5.4.1. Cobordism is an equivalence relation on the class of $\alpha$-pairings. Isomorphic $\alpha$-pairings are cobordant.
Proof. Consider an isomorphism $\varphi : S_1 \to S_2$ of $\alpha$-pairings $p_1 = (S_1, s_1, e_1), p_2 = (S_2, s_2, e_2)$. The set of vectors $\{A + \varphi(A)\}_{A \in S_1} \cup \{s\}$ is an annihilating filling of $p_1 \oplus p_2$. Therefore this pairing is hyperbolic and $p_1 \simeq_c p_2$. In particular, $p_1 \simeq_c p_1$.

If $p_1 \oplus p_2$ is hyperbolic, then so is its opposite $p'_1 \oplus p'_2 \simeq p_2 \oplus p'_1$. This implies the symmetry of cobordism.

Let us prove the transitivity. Let $p_1, p_2, p_3$ be $\alpha$-pairings such that $p_1 \simeq_c p_2$. We verify that $p_1 \simeq_c p_3$. Let $p_k = (S_k, s_k, e_k)$ for $i = 1, 2, 3$ and $p_2' = (S_2', s_2', e_2')$ be a copy of $p_2$ where $S_2' = \{A' | A \in S_2\}$, $s_2' = (s_2')'$, and $e_2'(A', B') = e_2(A, B)$ for all $A, B \in S_2$. Replacing the pairings by isomorphic ones, we can assume that the sets $S_1, S_2, S_2', S_3$ are disjoint. Let $A_1, A_2, A_3$ be free $R$-modules with bases $S_1, S_2, S_2', S_3$, respectively. Set $A = A_1 \oplus A_2 \oplus A_3 \oplus A_3$. There is a unique bilinear form $e = e_1 + e_2 + e_3 : A \times A \to \pi$ such that the sets $S_1, S_2, S_2', S_3 \subseteq \Lambda$ are orthogonal with respect to $e$ and the restrictions of $e$ to these sets are equal to $e_1, e_2, e_2', e_3$, respectively.

Let $\Phi$ be the submodule of $A_2 \oplus A_3$ generated by the vectors $\{A + A'\}_{A \in S_2}$. Set $L = A_1 \oplus \Phi \oplus A_3 \subseteq A$. The projection $q : L \to A_1 \oplus A_3$ along $\Phi$ transforms $e$ into $e_1 + e_3'$. Indeed, for any $A_1, B_1 \in S_1, A, B \in S_2, A_3, B_3 \in S_3$,

$$e(A_1 + A_1' + A_3, B_1 + B_1' + B_3) = e_1(A_1, B_1) - e_2(A, B) + e_3(A_2, B_3) = e_1(A_1, B_1) - e_3(A_3, B_3).$$

Pick a filling $\lambda = \{\lambda_i\}_i \subset A_1 \oplus A_2$ of $p_1 \oplus p_2$ and a filling $\mu = \{\mu_j\}_j \subset A_2 \oplus A_3$ of $p_2 \oplus p_3$. Consider the $R$-modules $V_\lambda \subset A_1 \oplus A_2$ and $V_\mu \subset A_2 \oplus A_3$ generated respectively by $\{\lambda_i\}_i$ and $\{\mu_j\}_j$. Below we construct a finite set $\psi \subset (V_\lambda + V_\mu) \cap L$ such that $q(\psi) \subset A_1 \oplus A_3$ is a filling of $p_1 \oplus p_3$. Choosing $\lambda, \mu$ to be annihilating fillings, we obtain that $e(V_\lambda, V_\lambda) = e(V_\mu, V_\mu) = 0$ and therefore $e(V_\lambda + V_\mu, V_\lambda + V_\mu) = 0$. Since $q : L \to A_1 \oplus A_3$ transforms $e$ into $e_1 + e_3'$, the filling $q(\psi)$ of $p_1 \oplus p_3$ is annihilating. Hence $p_1 \simeq_c p_3$.

To define $\psi$, we first derive from the filling $\lambda$ a 1-dimensional manifold $\Gamma_\lambda$. If $\lambda_i = A \pm B$ with distinct $A, B \in S_2^1 \cup S_2^2$, then $\lambda_i$ yields a component of $\Gamma_\lambda$ homeomorphic to $[0, 1]$ and connecting $A$ with $B$. (By the definition of a filling, $|A| \in \{\lfloor |A| \rfloor, \lceil |A| \rceil\}$.) If $\lambda_i = A \in S_2^1 \cup S_2^2$, then $\lambda_i$ yields a component of $\Gamma_\lambda$ homeomorphic to $[0, \infty)$ where 0 is identified with $A$. The vector $s_1 + s_2 \in \lambda$ does not contribute to $\Gamma_\lambda$. The definition of a filling implies that $\partial \Gamma_\lambda = S_2^1 \cup S_2^2$. The filling $\mu$ similarly gives rise to a 1-dimensional manifold $\Gamma_\mu$ with $\partial \Gamma_\mu = (S_2^1)^0 \cup S_2^2$. We can assume the manifolds $\Gamma_\lambda$ and $\Gamma_\mu$ to be disjoint. Gluing them along $S_2^2 \approx (S_2^2)^0$, we obtain a 1-dimensional manifold $\Gamma = \Gamma_\lambda \cup \Gamma_\mu$ with $\partial \Gamma = S_2^1 \cup S_2^3$.

Suppose with each component $K$ of $\Gamma$ with $\partial K \neq \emptyset$ a vector $\psi_K \in L$. Suppose first that $K$ is compact and let $A, B \in S_2^1 \cup S_2^2$ be its endpoints. The 1-manifold $K$ is glued from several components of $\Gamma_\lambda \cap \Gamma_\mu$ associated with certain vectors $\lambda_i, \mu_j$ (the components of $\Gamma_\lambda$ are intercalated in $K$ with the components of $\Gamma_\mu$). We define $\psi_K$ as an algebraic sum of these vectors $\sum_i \epsilon_i \lambda_i + \sum_j \eta_j \mu_j$, where the signs $\epsilon_i, \eta_j$ are determined by induction moving along $K$ from $A$ to $B$. For example, if $K$ is a union of a component of $\Gamma_\lambda$ connecting $A \in S_2^1$ to $C \in S_2^2$ and corresponding to $\lambda_i = A + C$ with a component of $\Gamma_\mu$ connecting $C$
to $B \in S_3^2$ and corresponding to $\mu_j = -C_i + B$, then $\psi_K = \lambda_i - \mu_j$. If $\mu_j = C_i - B$, then $\psi_K = \lambda_i - \mu_j$. In both examples $|A| = |C| = |C'| \in \{ |B|, \tau(|B|) \}$.

An easy inductive argument shows that $|A| \in \{ |B|, \tau(|B|) \}$ for any compact component $K$ of $\Gamma$ with endpoints $A, B$. We claim that $q(\psi_K) = A + B$ is short, i.e., that the sign $\pm$ satisfies the requirements in the definition of a short vector. If $\tau(|B|) = |B|$, then there is nothing to prove since both $+$ and $-$ satisfy these requirements. If $\tau(|B|) \neq |B|$, then this claim is obtained by a count of minuses in the sequence of vectors $\lambda_i, \mu_j$, corresponding to the components of $\Gamma \cap \Gamma_{\mu}$ forming $K$. Note that the vectors $\psi_K$ determined as above by moving along $K$ from $A$ to $B$ and from $B$ to $A$ may differ; we take any of them. If $K$ has only one endpoint $A$, then $\psi_K$ is similarly defined as an algebraic sum of the vectors associated with the components of $\Gamma \cap \Gamma_{\mu}$ forming $K$, where the signs are determined inductively from two conditions: $\psi_K \in L$ and $q(\psi_K) = A$. It follows from the definitions that in all cases $\psi_K \in (V_\lambda + V_\mu) \cap L$.

Set $\psi_0 = s_1 + s_2 + s_3 \in (V_\lambda + V_\mu) \cap L$ and set $\psi = \{ \psi_0 \} \cup \{ \psi_K \}$. Clearly, $\psi$ runs over the components of $\Gamma$ with non-void boundary. All vectors in the family $q(\psi)$, besides $q(\psi_0) = s_1 + s_3$ are short and all elements of $S_3^2 \cup S_3^3$ occur in exactly one of these vectors with non-zero coefficient. This means that $q(\psi)$ is a filling of $p_1 \oplus p_3$ as required.

5.5. The group $\mathcal{P}$. The cobordism classes of $\alpha$-pairings form an abelian group with respect to summation $\oplus$. This group is denoted $\mathcal{P} = \mathcal{P}(\alpha, \tau, \pi)$. The neutral element of $\mathcal{P}$ is the class of the trivial $\alpha$-pairing $(S = \{ s \}, s, e(s, s) = 0)$. The opposite in $\mathcal{P}$ to the class of an $\alpha$-pairing $p$ is the class of $p^{-}$.

An $\alpha$-pairing $p = (S, s, e)$ is skew-symmetric if $e(A, A) = 0$ and $e(A, B) = -e(B, A)$ for all $A, B \in S$. In particular, we must have $e(s, s) = 0$ and $e(s, B) = -e(B, s)$ for all $B \in S^o$. It is clear that the sum of skew-symmetric $\alpha$-pairings is skew-symmetric and the $\alpha$-pairing opposite to a skew-symmetric one is itself skew-symmetric. Therefore the cobordism classes of skew-symmetric $\alpha$-pairings form a subgroup of $\mathcal{P}$. It is denoted $\mathcal{P}_{sk} = \mathcal{P}_{sk}(\alpha, \tau, \pi)$.

5.6. Normal $\alpha$-pairings. Although we shall not need it in the sequel, we briefly discuss so-called normal $\alpha$-pairings. An $\alpha$-pairing $p = (S, s, e)$ is normal if $e(s, s) = 0$. The cobordism classes of normal $\alpha$-pairings form a subgroup of $\mathcal{P}$ denoted $\mathcal{P}_n$. Clearly, $\mathcal{P}_{sk} \subset \mathcal{P}_n$. The following lemma computes $\mathcal{P}$ from $\mathcal{P}_n$.

Denote by $\underline{R}$ the underlying additive group of $R$. For $r \in R$, denote by $i(r)$ the $\alpha$-pairing $(S = \{ s \}, s, e(s, s) = r)$.

Lemma 5.6.1. The formula $r \mapsto i(r)$ defines an injective group homomorphism $i : \underline{R} \to \mathcal{P}$ and $\mathcal{P} = \mathcal{P}_n \oplus i(\underline{R})$.

Proof. The additivity of $i$ follows from the definitions. For an $\alpha$-pairing $p = (S, s, e)$, set $r_p = e(s, s) \in R$ and consider the $\alpha$-pairing $p' = (S, s, e')$ where $e'(s, s) = 0$ and $e'(A, B) = e(A, B)$ for all pairs $A, B \in S$ distinct from the pair $(s, s)$. It follows from the definitions that $p \approx p' \oplus i(r_p)$. Therefore $\mathcal{P} = \mathcal{P}_n + i(\underline{R})$. Observe that if two $\alpha$-pairings $p_1 = (S_1, s_1, e_1), p_2 = (S_2, s_2, e_2)$ are cobordant,
then the vector $s_1 + s_2$ belongs to an annihilating filling of $p_1 \oplus p_2$ and therefore

$$0 = (e_1 + e_2^-)(s_1 + s_2, s_1 + s_2) = e_1(s_1, s_1) - e_2(s_2, s_2) = r_{p_1} - r_{p_2}.$$ 

Hence the formula $p \mapsto r_p$ defines a group homomorphism $r : \mathcal{P} \to \mathbb{R}$. Clearly, $r \circ i = \text{id}$ and $r(P_n) = 0$. Therefore $i$ is an injection and $\mathcal{P}_n \cap i(\mathbb{R}) = 0$. □

6. Cobordism invariants of $\alpha$-pairings

We give two constructions of cobordism invariants of $\alpha$-pairings. Fix as above a left module $\pi$ over a ring $R$.

6.1. The polynomial $u$. Let $I$ be the free $R$-module with basis $\{\delta_g\}_{g \in \pi \setminus \{0\}}$. Let $J$ be the submodule of $I$ generated by the vectors $\{\delta_g + \delta_{-g}\}_{g \in \pi \setminus \{0\}}$. For an $\alpha$-pairing $p = (S, s, e)$ and any $a \in \alpha$, set

$$[a]_p = \sum_{A \in S^\circ, |A| = a, e(A, s) \neq 0} \delta_{e(A, s)} \in I.$$ 

The $u$-polynomial $u^p$ of $p$ is the function on $\alpha$ defined as follows: for $a \in \alpha$ with $\tau(a) \neq a$,

$$u^p(a) = [a]_p - [\tau(a)]_p \pmod{J} \in I/J$$

and for $a \in \alpha$ with $\tau(a) = a$,

$$u^p(a) = [a]_p \pmod{J + 2I} \in I/(J + 2I).$$

Clearly, $u^p(a) = -u^p(\tau(a))$ for all $a \in \alpha$. The function $u^p$ was introduced in [112] without factorization by $J$. This factorization is needed here to ensure the following theorem.

**Theorem 6.1.1.** $u^p$ is an additive cobordism invariant of $p$.

**Proof.** It follows from the definitions that $u^p$ is additive and $u^{-p} = -u^p$ for all $p$. It remains to show that if $p$ is hyperbolic, then $u^p(a) = 0$ for all $a \in \alpha$. Pick an annihilating filling $\{\lambda_i \in RS\}_i$ of $p$ and let $\lambda_0$ be the vector of this filling equal to $s$. If $\lambda_i = A \in S^\circ$, then the equalities $e(A, s) = e(\lambda_i, \lambda_0) = 0$ imply that $A$ contributes 0 to $[b]_p$ for all $b \in \alpha$. Therefore $A$ contributes 0 to $u^p(a)$.

Consider a vector of this filling $\lambda_i = A \pm B$ with $A, B \in S^\circ$. Recall that $[A] \in \{|B|, \tau(|B|)\}$. The condition $e(\lambda_i, \lambda_0) = 0$ implies that $e(A, s) = \mp e(B, s)$. If $e(A, s) = 0$, then $e(B, s) = 0$ so that both $A$ and $B$ contribute 0 to $[b]_p$ for all $b \in \alpha$ and hence contribute 0 to $u^p(a)$. If $|A| \notin \{a, \tau(a)\}$, then $|B| \notin \{a, \tau(a)\}$ so that both $A$ and $B$ contribute 0 to $[a]_p$, $[\tau(a)]_p$ and $u^p(a)$. Suppose from now on that $e(A, s) \neq 0$ and $|A| \in \{a, \tau(a)\}$. If $a = \tau(a)$, then the equality $e(A, s) = \mp e(B, s)$ implies that $\delta_{e(A, s)} + \delta_{e(B, s)} \in J + 2I$ and therefore the pair $A, B$ contributes 0 to $u^p(a) = [a]_p \pmod{J + 2I}$. Suppose that $a \neq \tau(a)$. If $|A| = |B| = a$, then $\lambda_i = A + B$, $e(A, s) = -e(B, s)$, and $A, B$ contribute $\delta_{e(A, s)} + \delta_{e(B, s)}$ to $[a]_p$ and 0 to $[\tau(a)]_p$. Hence $A, B$ contribute 0 to $u^p(a)$. If $|A| = a, |B| = \tau(a)$, then $\lambda_i = A - B$, $e(A, s) = e(B, s)$ and $A, B$ contribute $\delta_{e(A, s)}$ to $[a]_p$ and $\delta_{e(B, s)}$ to $[\tau(a)]_p$. Hence $A, B$ contribute 0 to $u^p(a)$. The cases where $|A| = \tau(a)$ and $|B| = \tau(a)$ or $|B| = a$ are similar. Since every letter of $S^\circ$
appears in exactly one $\lambda_i$, summing up the contributions of all letters to $u^p(a)$ we obtain $u^p(a) = 0$. □

6.2. Genus of $\alpha$-pairings. Let $F$ be a commutative $R$-algebra without zero-divisors. Fix an $R$-module homomorphism $\varphi : \pi \to F$. For an $\alpha$-pairing $p = (S, s, e)$, we define a non-negative half-integer $\sigma_\varphi(p)$ as follows. Consider the bilinear pairing $e = \varphi e : RS \times RS \to F$. For a filling $\alpha = \{\lambda_i\}_i$ of $p$, the matrix $(\varphi e(\lambda_i, \lambda_j))_{i,j}$ is a square matrix over $F$. Let

$$\sigma_\varphi(\lambda) = (1/2) \text{rk}(\varphi e(\lambda_i, \lambda_j))_{i,j} \in \frac{1}{2}\mathbb{Z}$$

be half of its rank. The rank $\text{rk}$ for matrices and bilinear forms over $F$ is defined by extending $F$ to its quotient field and using the standard definitions for the latter. Set

$$\sigma_\varphi(p) = \min_{\lambda} \sigma_\varphi(\lambda),$$

where $\lambda$ runs over all fillings of $p$. The number $\sigma_\varphi(p)$ is called the $\varphi$-genus of $p$. It is obvious that this number is invariant under isomorphisms of $\alpha$-pairings and $\sigma_\varphi(p^-) = \sigma_\varphi(p)$. If $p$ is hyperbolic, then $\sigma_\varphi(p) > 0$. If $p$ is skew-symmetric, then the matrix $(\varphi e(\lambda_i, \lambda_j))_{i,j}$ is skew-symmetric, so that $\sigma_\varphi(\lambda) \in \mathbb{Z}$ for all $\lambda$ and $\sigma_\varphi(p) \in \mathbb{Z}$.

**Lemma 6.2.1.** For any $\alpha$-pairings $p_1, p_2, p_3$,

$$\sigma_\varphi(p_1 \oplus p_2^2) + \sigma_\varphi(p_2 \oplus p_3^2) \geq \sigma_\varphi(p_1 \oplus p_3^2). \tag{6.2.1}$$

**Proof.** We use notation introduced in the proof of Lemma 5.4.1. Pick a filling $\lambda = \{\lambda_i\}_i \subset \Lambda_1 \oplus \Lambda_2$ of $p_1 \oplus p_2$ such that $\sigma_\varphi(p_1 \oplus p_2^2) = \sigma_\varphi(\lambda)$. Pick a filling $\mu = \{\mu_j\}_j \subset \Lambda_2' \oplus \Lambda_3$ of $p_2' \oplus p_3'$ such that $\sigma_\varphi(p_2' \oplus p_3') = \sigma_\varphi(\mu)$. Consider the $R$-modules $V_\lambda \subset \Lambda_1 \oplus \Lambda_2$ and $V_\mu \subset \Lambda_2' \oplus \Lambda_3$ generated respectively by $\{\lambda_i\}_i$ and $\{\mu_j\}_j$. Recall the projection $q : L = \Lambda_1 \oplus \Phi \oplus \Lambda_3 \to \Lambda_1 \oplus \Lambda_3$ transforming $e = e_1 \oplus e_2' \oplus e_3' \oplus e_3$ into $e_1 \oplus e_3$. The proof of Lemma 5.4.1 yields a finite set $\psi \subset (V_\lambda + V_\mu) \cap L$ such that $q(\psi) \subset \Lambda_1 \oplus \Lambda_3$ is a filling of $p_1 \oplus p_3^2$. Denote by $V$ the $R$-submodule of $\Lambda_1 \oplus \Lambda_3$ generated by $q(\psi)$. Clearly,

$$\sigma_\varphi(p_1 \oplus p_3^2) \leq \sigma_\varphi(q(\psi)) = (1/2) \text{rk}(\varphi e_1 \oplus \varphi e_3^2)|_{\psi} = (1/2) \text{rk}(\varphi e_{l^{-1}(\psi)}).$$

Observe that $q^{-1}(V) \subset (V_\lambda + V_\mu) \cap L + \text{Ker } q$ and that $\text{Ker } q = \Phi$ lies in both the left and the right annihilators of $e|_L$. Therefore

$$(1/2) \text{rk}(\varphi e_{l^{-1}(\psi)}) \leq (1/2) \text{rk}(\varphi e_{|V_\lambda + V_\mu)} \leq (1/2) \text{rk}(\varphi e_{V_\lambda + V_\mu}) = (1/2) \text{rk}(\varphi e_{V_\lambda}) + (1/2) \text{rk}(\varphi e_{V_\mu}) = \sigma_\varphi(\lambda) + \sigma_\varphi(\mu) = \sigma_\varphi(p_1 \oplus p_2^2) + \sigma_\varphi(p_2' \oplus p_3') = \sigma_\varphi(p_1 \oplus p_2^2) + \sigma_\varphi(p_2 \oplus p_3^2).$$

□

**Theorem 6.2.2.** The $\varphi$-genus of an $\alpha$-pairing is a cobordism invariant.

**Proof.** If $p_1 \simeq e p_2$, then $p_1 \oplus p_2^2$ is hyperbolic and $\sigma_\varphi(p_1 \oplus p_3^2) = 0$. Applying the previous lemma to the triple $(p_1, p_2, p_3)$ where $p_3 = (\{s\}, s, e = 0)$ is the trivial $\alpha$-pairing, we obtain $\sigma_\varphi(p_2) \geq \sigma_\varphi(p_1)$. By symmetry, $\sigma_\varphi(p_1) = \sigma_\varphi(p_2)$. □
6.3. Remark. For any $\alpha$-pairings $p_1 = (S_1, s_1, e_1), p_2 = (S_2, s_2, e_2)$,

\[(6.3.1) \quad \sigma_\varphi(p_1) + \sigma_\varphi(p_2) \geq \sigma_\varphi(p_1 \oplus p_2).\]

This can be deduced from (6.2.1) by choosing there $p_2$ to be the trivial $\alpha$-pairing.

A direct proof of (6.3.1) goes by taking the union of a filling $\lambda^{(1)}$ of $p_1$ with a filling $\lambda^{(2)}$ of $p_2$ and replacing in this union the elements $s_1, s_2$ by $s_1 + s_2$. This gives a filling $\lambda$ of $p_1 \oplus p_2$ such that $\sigma_\varphi(\lambda^{(1)}) + \sigma_\varphi(\lambda^{(2)}) \geq \sigma_\varphi(\lambda)$. Hence (6.3.1).

In general, (6.3.1) is not an equality. This is clear already from the fact that the $\varphi$-genus takes only non-negative values and annihilates $p \oplus p^-$ for any $\alpha$-pairing $p$.

7. Homomorphism $N_c \to P_{sk}$

7.1. The group $\pi$. From now on, unless explicitly stated to the contrary, $\pi = \pi(\alpha, \tau)$ is the abelian group with generators $\{a\}_{a \in \alpha}$ and defining relations $a + \tau(a) = 0$ for all $a \in \alpha$. This group is the abelianization of the group $\Pi$ considered in Sect. 4. Clearly, $\pi$ is a direct sum of copies of $\mathbb{Z}$ numerated by the free orbits of $\tau$ and copies of $\mathbb{Z}/2\mathbb{Z}$ numerated by the fixed points of $\tau$. The group $\pi$, considered as a module over $R = \mathbb{Z}$, will be the target of $\alpha$-pairings. Note that in [Tu2] the group operation in $\pi$ is written multiplicatively rather than additively as here.

7.2. $\alpha$-pairings of nanowords. By [Tu2], each nanoword $(A, w : \hat{n} \to A)$ gives rise to a skew-symmetric $\alpha$-pairing $p(w) = (S = A \cup \{s\}, s, e_w : S \times S \to \pi)$ with target $\pi = \pi(\alpha, \tau)$. Here the projection $S^\circ = A \to \pi$ is determined by the structure of an $\alpha$-alphabet in $A$. Recall the definition of $e_w$. First, for any $A, B \in A$, we define an integer $n_w(A, B)$ to be $+1$ if $w = \cdots A \cdots B \cdots A \cdots B \cdots$, to be $-1$ if $w = \cdots B \cdots A \cdots B \cdots A \cdots$ and to be 0 in all other cases. Given $A \in A$, denote by $i_A$ (resp. $j_A$) the minimal (resp. the maximal) element of the 2-element set $w^{-1}(A) \subset \hat{n}$. For $A, B \in A$, set

\[A \circ_w B = \sum_{D \in A, i_A < i_D < j_A \text{ and } i_B < j_D < j_B} |D| \in \pi.\]

Set

\[e_w(A, B) = 2(A \circ_w B - B \circ_w A) + n_w(A, B) (|A| + |B|) \in \pi,\]

\[e_w(A, s) = -e_w(s, A) = \sum_{D \in A} n_w(A, D) |D| \in \pi\]

and $e_w(s, s) = 0$. All these expressions are sums over $D \in A$ of terms $m_D|D|$ with $m_D \in \mathbb{Z}$. The term $m_D|D|$ is the contribution of $D$. For instance, the contributions of $A, B$ to $e_w(A, B)$ are $n_w(A, B) |A|$ and $n_w(A, B) |B|$, respectively. The contribution of $A$ to $e_w(A, s)$ is 0. It is clear that a letter $D \neq A, B$ may contribute non-trivially to $e_w(A, B)$ only if (i) each entry of $D$ appears between the two entries of $A$ or between the two entries of $B$ (or both) and (ii) $D$ occurs at least once between the entries of $A$ and at least once between the entries of $B$.

It is useful to have more direct formulas for $e_w(A, B)$. For words $x, y$ in the alphabet $A$, set $(x, y) = \sum_D |D| \in \pi$ where $D$ runs over the letters in $A$ occurring
Theorem 7.3.1. The formula $w \mapsto p(w)$ defines a group homomorphism $p : \mathcal{N}_c \to \mathcal{P}_{sk}$.

The homomorphism $p : \mathcal{N}_c \to \mathcal{P}_{sk}$ is in general not injective. This is clear already from the fact that $\mathcal{N}_c$ may be non-commutative while $\mathcal{P}_{sk}$ is commutative.

We now reduce Theorem 7.3.1 to a lemma which will be proven in Sect. 12.3 using topological techniques.

7.4. Proof of Theorem 7.3.1 (modulo a lemma). The multiplicativity of $p : \mathcal{N}_c \to \mathcal{P}_{sk}$ follows from the definitions. We need to check only that $p$ is well-defined, i.e., that cobordant nanowords give rise to cobordant $\alpha$-pairings. A direct comparison shows that two nanowords related by the third homotopy move have isomorphic $\alpha$-pairings (cf. [1u2], proof of Lemma 7.6.1). Since the first and second homotopy moves are special instances of surgery, it remains to show that nanowords related by a surgery have cobordant $\alpha$-pairings.

We begin by fixing notation. Consider a nanoword $(A, w)$ and its even symmetric factor $\nabla = (B, (v_1 \mid \cdots \mid v_k))$. Thus, $B \subset A$ and $w = x_1v_1x_2v_2 \cdots x_kv_kx_{k+1}$ where $x_1, x_2, \ldots, x_{k+1}$ are words in the $\alpha$-alphabet $C = A - B$. Deleting $\nabla$ we obtain the nanoword $(C, x = x_1x_2 \cdots x_{k+1})$. Let $i = i_{\nabla} : B \to B$ and $\varepsilon = \varepsilon_{\nabla} : B \to \{0, 1\}$ be the involution and the mapping associated with $\nabla$ in Sect. 2.3.
We must prove that the $\alpha$-pairings $p(w)$ and $p(x)$ are cobordant. Replacing each letter $C \in \mathcal{C}$ by its copy $C'$, we obtain a nanoword $(C' = \{C'\}_{C \in \mathcal{C}}, x')$ isomorphic to $(C, x)$. It suffices to verify that the $\alpha$-pairing $p(w) \oplus p(x')$ has an annihilating filling. Let $p = p(w) = (\lambda \subset \mathcal{C}, s, e(w))$ and $p' = p(x') = (\lambda' \subset \mathcal{C}, s', e(x'))$ where $S = \mathcal{A} \cup \{s\}$ and $S' = C' \cup \{s'\}$. With every $C \in \mathcal{C}$ we associate the vector $\lambda_C = C + C' \in ZS \oplus ZS'$. With every $B \in \mathcal{B}$ we associate the vector $\lambda_B \in ZS \subset ZS \oplus ZS'$ equal to $B$ if $B \neq B'$ and equal to $\pm(1)^{\varepsilon(B)} B'$ if $B = B'$. (Note that $\lambda_B = (-1)^{\varepsilon(B)} \lambda_B$.) To proceed, pick a set $\mathcal{B} \subset \mathcal{B}$ meeting every orbit $\iota: \mathcal{B} \to \mathcal{B}$ in one element. The set of vectors $\{\lambda_C\}_{C \in \mathcal{C}} \cup \{\lambda_B\}_{B \in \mathcal{B}} \cup \{s + s'\}$ is a filling of $p \oplus (p')^-$. We claim that this filling is annihilating. Indeed, for $C \in \mathcal{C}$,

\[(e_w \oplus e_{x'}^-)(\lambda_C, s + s') = e_w(C, s) + (e_{x'}^-)(C', s') = e_w(C, s) - e_x(C, s) = 0.\]

The latter equality follows from two facts: for $D \in \mathcal{B}$ with $D = D'$, we have $n_w(D, C) = 0$ so that $D$ contributes to $e_w(C, s)$; for $D \in \mathcal{B}$ with $D \neq D'$, we have either $n_w(D, C) = n_w(D', C) = 0$ or $n_w(D, C) = n_w(D', C) = \pm 1$. In the latter case $\varepsilon(D) = 1$ and $|D'| = \tau(|D|)$. In all cases, the pair $D, D'$ contributes 0 to $e_w(C, s)$. Hence the total contribution of all $D \in \mathcal{B}$ to $e_w(C, s)$ is 0. Therefore $e_w(C, s) = e_x(C, s)$.

Similarly, for $C_1, C_2 \in \mathcal{C}$,

\[(e_w \oplus e_{x'}^-)(\lambda_{C_1}, \lambda_{C_2}) = e_w(C_1, C_2) - e_x(C_1, C_2) = 0.\]

Indeed, if $D \in \mathcal{B}$ contributes non-trivially to $e_w(C_1, C_2)$ then $\varepsilon(D) = 1, |D'| = \tau(|D|)$ and the sum of the contributions of $D, D'$ to $e_w(C_1, C_2)$ is 0. Here we use the evenness of $\nabla$ which implies that $\varepsilon(D) = 1 \Rightarrow D \neq D'$.

It remains to prove that for any $B, B_1, B_2 \in \mathcal{B}, C \in \mathcal{C}$,

\[(e_w \oplus e_{x'}^-)(\lambda_{B_1}, \lambda_{B_2}) = (e_w \oplus e_{x'}^-)(\lambda_B, \lambda_C) = (e_w \oplus e_{x'}^-)(\lambda_B, s + s') = 0.\]

Since $\lambda_B, \lambda_{B_1}, \lambda_{B_2} \in ZA$, these formulas are equivalent to

\[(7.4.1)\quad e_w(\lambda_{B_1}, \lambda_{B_2}) = e_w(\lambda_B, s + s') = 0.\]

In the rest of the proof, we denote by $\alpha_0$ the 2-letter alphabet $\{+, -\}$ with involution $\tau_0$ permuting $+ \leftrightarrow -$.

**Lemma 7.4.1.** Formula \((7.4.1)\) holds for $\alpha = \alpha_0$ and $\tau = \tau_0$.

Lemma \((7.4.1)\) will be proven in Sect. \([12]\) using topological techniques.

**Lemma 7.4.2.** Let $\varepsilon \in \pi = \pi(\alpha, \tau)$ be one of the expressions $e_w(\lambda_{B_1}, \lambda_{B_2}), e_w(\lambda_B, s)$ in Formula \((7.4.1)\), where $\alpha$ is an arbitrary alphabet with involution $\tau$. For any additive homomorphism $\varphi: \pi \to Z$ sending all the generators $a \in \alpha$ to $\{-1, +1\} \subset Z$, we have $\varphi(e) = 0$.

**Proof.** The homomorphism $\varphi$ induces a mapping $f: \alpha \to \alpha_0 = \{-1, +1\}$ sending each $a \in \alpha$ to $\varphi(a) \in \alpha_0$. The additivity of $\varphi$ implies that $f$ is equivariant with respect to the involutions $\tau: \alpha \to \alpha$ and $\tau_0: \alpha_0 \to \alpha_0$. Let $\omega_0$ be the nanoword over $\alpha_0$ obtained by pushing $w$ forward along $f$. The $\alpha_0$-pairing $(S, s, e_{w_0})$ of $w_0$ is obtained from the $\alpha$-pairing $(S, s, e_w)$ of $w$ by composing $e_w: S \times S \to \pi$ with $\varphi$. Thus $\varphi(e_w(\lambda_{B_1}, \lambda_{B_2})) = e_{w_0}(\lambda_{B_1}, \lambda_{B_2}), \varphi(e_w(\lambda_B, s)) = e_{w_0}(\lambda_B, s)$, and
\[ \varphi(\varepsilon_w(\lambda_B, s)) = \varepsilon_w(\lambda_B, s). \] Lemma 7.4.1 implies that the right-hand sides of these formulas are equal to 0. \hfill \square

**Lemma 7.4.3.** Formula 7.4.1 holds for any alphabet \( \alpha \) with fixed-point-free involution \( \tau \).

**Proof.** If \( \tau \) is fixed-point-free, then \( \pi = \pi(\alpha, \tau) \) is a free abelian group with basis \( \beta \) where \( \beta \) is any subset of \( \alpha \) meeting every orbit of \( \tau \) in one element. Let \( e \in \pi \) be one of the expressions \( e_w(\lambda_{B_1}, \lambda_{B_2}), e_w(\lambda_B, C), e_w(\lambda_B, s) \) in Formula 7.4.1.

We expand \( e = \sum_{x \in \beta} k_x x \) with \( k_x \in \mathbb{Z} \). We claim that \( k_x = 0 \) for all \( x \). Indeed, pick any \( x \in \beta \). Consider the additive homomorphisms \( \varphi_+, \varphi_- : \pi \to \mathbb{Z} \) defined by \( \varphi_+(y) = 1 \) for all \( y \in \beta - \{x\} \) and \( \varphi_+(x) = \pm 1 \). By the previous lemma, \( \varphi_+(e) = 0 \). Therefore \( 2k_x = \varphi_+(e) - \varphi_-(e) = 0 \). Hence \( k_x = 0 \). \hfill \square

We can now prove Formula 7.4.1 in its full generality. We begin by associating with the nanoword \( (\mathcal{A}, w) \) another nanoword as follows. For \( i = 0, 1 \), set \( B_i = \{ B \in B | \varepsilon_\tau(B) = i \} \). It follows from the definition of the involution \( \iota = \iota_\tau \) on \( B \) that \( \iota(B_0) = B_0 \). Set \( X = B_0 / \iota \) and let \( X' = \{ x' \mid x \in X \} \) be a copy of \( X \). Similarly, let \( C' = \{ C' \mid C \in C \} \) be a copy of \( C = \mathcal{A} - B \). Set \( \overline{\tau} = X \cup B_1 \cup C \cup X' \cup C' \) where it is understood that the five sets on the right-hand side are disjoint. There is a unique involution \( \overline{\tau} \) on \( \overline{\tau} \) such that \( \tau(x) = x' \) for \( x \in X \), \( \tau(C) = C' \) for \( C \in C \) and \( \tau(B) = B' \) for \( B \in B_1 \). The projection \( B_0 \to X \) of the identity on \( B_1 \cup C \) form a mapping \( p : \mathcal{A} = B_0 \cup B_1 \cup C \to \overline{\mathcal{A}} \). This mapping makes \( \mathcal{A} \) into an \( \overline{\mathcal{A}} \)-alphabet. The pair \( (\mathcal{A}, w) \) becomes thus a nanoword over \( \overline{\mathcal{A}} \). We denote this nanoword by \( \overline{w} \).

The nanoword \( \overline{w} \) and the original nanoword \( (\mathcal{A}, w) \) over \( \alpha \) coincide as words in the alphabet \( \mathcal{A} \) and differ in the choice of the ground alphabet. The nanoword \( w \) is a push-forward of \( \overline{w} \) as follows. Define a mapping \( f : \overline{\mathcal{A}} \to \alpha \) by \( f(x) = \|B| \) for any \( x \in X, B \in p^{-1}(x) \subset B_0 \) and \( f(B) = \|B| \) for \( C \in C \). The mapping \( f \) is well-defined on \( X \subset \overline{\mathcal{A}} \) because \( |B'| = |B| \) for \( B \in B_0 \). The definition of \( \overline{\mathcal{A}} \) and the equality \( |B'| = \tau(|B|) \) for \( B \in B_1 \) imply that \( f \) is equivariant with respect to the involutions \( \tau : \overline{\mathcal{A}} \to \overline{\mathcal{A}} \) and \( \tau : \alpha \to \alpha \). Since the composition of \( f \) with \( p : \mathcal{A} \to \overline{\mathcal{A}} \) is the given projection \( \mathcal{A} \to \alpha, \mathcal{A} \to |\mathcal{A}| \), the nanoword \( w \) is the push-forward of \( \overline{w} \) along \( f \).

The phrase \( (\mathcal{B}, \overline{\varnothing}) \) with projection \( p_B : \mathcal{B} \to \overline{\mathcal{A}} \) is a nanophase over \( \overline{\mathcal{A}} \) denoted \( \overline{\varnothing} \). Clearly, \( \overline{\varnothing} \) a factor of \( \overline{w} \) since this property does not involve the ground alphabet. The nanophase \( \overline{\varnothing} \) is even because so is \( \varnothing \). Obviously, \( \iota_{\overline{\varnothing}} = \iota_{\overline{\varnothing}} = \iota : \mathcal{B} \to \mathcal{B} \) and \( \varepsilon_{\overline{\varnothing}} = \varepsilon_{\overline{\varnothing}} : \mathcal{B} \to \{0, 1\} \). Our definition of \( \overline{\mathcal{A}} \) ensures that \( \overline{\varnothing} \) is symmetric. Indeed, for any \( B \in \mathcal{B} \), if \( \varepsilon_{\overline{\varnothing}}(B) = 0 \), then \( B \in B_0 \) and \( p(B') = p(B) \in X \) by the definition of \( X, p \). If \( \varepsilon_{\overline{\varnothing}}(B) = 1 \), then \( B \in B_1 \) and \( p(B') = B' = \tau(B) \in \mathcal{A} \).

Observe that the involution \( \overline{\tau} \) is fixed-point-free. This follows from the fact that \( \iota = \iota_{\overline{\varnothing}} \) has no fixed points in \( B_1 \) which in its turn follows from the evenness of \( \overline{\varnothing} \). By Lemma 7.4.3

\[ e_{\overline{\mathcal{A}}}(\lambda_{B_1}, \lambda_{B_2}) = e_{\overline{\mathcal{A}}}(\lambda_B, C) = e_{\overline{\mathcal{A}}}(\lambda_B, s) = 0. \]

Since \( w \) is a push-forward of \( \overline{w} \), the latter formula implies Formula 7.4.1.
7.5. Examples and remarks. 1. Pick three (possibly coinciding) elements \( a, b, c \in \alpha \). Consider the nanoword \( w = ABCBAC \) with \(|A| = a, |B| = b, |C| = c\). If \( a = \tau(b) \), then \( w \) is contractible and therefore slice. We use Theorem 7.3.1 to verify that \( w \) is not slice for \( a \neq \tau(b) \). It suffices to verify that the \( \alpha \)-pairing \( p(w) = (S = \{s, A, B, C\}, s, e_w) \) is not hyperbolic. A direct computation from definitions shows that \( e_w \) is given by the matrix

\[
\begin{pmatrix}
0 & -c & -c & a + b \\
c & 0 & 0 & a + 2b + c \\
c & 0 & 0 & b + c \\
-a - b & -a - 2b - c & -b - c & 0
\end{pmatrix}
\]

where the rows and columns correspond to \( s, A, B, C \), respectively. An easy check shows that \( p(w) \) has no annihilating fillings. Indeed, the tautological filling formed by the vectors \( s, A, B, C \) is non-annihilating since \( c \neq 0 \) in \( \pi \). The vectors \( A - B \) and \( B - A \) cannot belong to a filling since \( a \neq \tau(b) \). A vector \( \lambda \) of the form \( A \pm C, B \pm C, C - A, C - B \) cannot belong to an annihilating filling since \( e_w(\lambda, s) \) is an algebraic sum of \( a, b, c \) and is non-zero in \( \pi \). It remains to consider the family of vectors \( \{s, A + B, C\} \) which is a filling if \( a = b \). We have \( e_w(A + B, s) = 2c \) and \( e_w(A + B, C) = a + 2c + 3b = 4a + 2c \). If this filling is annihilating, then \( 2c = 4a + 2c = 0 \) in \( \pi \) and therefore \( 4a = 0 \). This may happen only when \( a = \tau(a) \) which is excluded by \( a = b \). Thus, \( a \neq \tau(b) \), the pairing \( p(w) \) is not hyperbolic and the nanoword \( w \) is not slice. Note that if \( a, b, c \) belong to one orbit of \( \tau \), then \( \gamma(w) = 1 \). This shows that the \( \alpha \)-pairings may provide more information than the homomorphism \( \gamma \).

2. Consider the nanoword \( w = ABCADCBAD \) with \(|A| = \tau(|B|), |C| = |D|\). Direct computations show that \( \gamma(w) = 1 \) and \( p(w) \) is hyperbolic with annihilating filling \( \{s, A - B, C + D\} \). (A general construction producing such examples will be discussed in Sect. 10.1.) The author does not know whether \( w \) is slice except in the case where \(|A| = |D| \) (then \( w \) is symmetric and therefore slice).

3. Given a nanoword \( (A, w) \) over \( \alpha \) and a family \( H = \{H_a\}_{a \in \alpha} \) of subgroups of \( \pi \) such that \( H_a = H_{\tau(a)} \) for all \( a \), we define \( H \)-covering of \( w \) to be the nanoword \( (A^H, w^H) \) over \( \alpha \) obtained by deleting from both \( A \) and \( w \) all letters \( A \) such that \( e_w(A, s) \notin H_{|A|} \) (cf. [Tu2]). One may check that the \( H \)-coverings of cobordant nanowords are cobordant (we shall not use it). Moreover, the formula \( w \mapsto w^H \) defines a group endomorphism of \( N_c(\alpha, \tau) \).

8. Cobordism invariants of nanowords

8.1. The \( u \)-polynomial. The \( u \)-polynomial of a nanoword \( w \) is defined by \( u^w = u^{p(w)} \) where \( p(w) \) is the \( \alpha \)-pairing associated with \( w \) and \( u^{\beta(w)} \) is its \( u \)-polynomial. By Theorems 6.11 and 7.3.1, \( u^w \) is a cobordism invariant of \( w \).

Consider in more detail the case where \( \tau \) is fixed-point-free. Let \( k \) be the number of orbits of \( \tau \) and \( t_1, t_2, \ldots, t_k \in \alpha \) be representatives of the orbits so that each orbit contains exactly one \( t_i \). It is convenient to switch to the multiplicative notation for the group operation in \( \pi = \pi(\alpha, \tau) \). Thus, any \( g \in \pi \) expands uniquely as a monomial \( g = \prod_{i=1}^{k} t_i^{m_i} \) with \( m_1, \ldots, m_k \in \mathbb{Z} \). Identifying \( \delta \in \mathbb{I} \)
with this monomial, we identify the $\mathbb{Z}$-module $I$ from Sect. 6.1 with the additive group of Laurent polynomials over $\mathbb{Z}$ in the commuting variables $t_1, \ldots, t_k$ with zero free term. The $\mathbb{Z}$-submodule $J \subset I$ consists of those Laurent polynomials with zero free term which are invariant under the inversion of the variables $t_1 \mapsto t_1^{-1}, \ldots, t_k \mapsto t_k^{-1}$. The quotient $I/J$ is an infinitely generated free abelian group with basis $\prod_{i=1}^{k} t_i^{m_i} (\text{mod } J)$ where the tuple $(m_1, \ldots, m_k)$ runs over $k$-tuples of integers such that at least one of its entries is non-zero and the first non-zero entry is positive. The degree of such a basis monomial is the number $\sum_{i=1}^{k} |m_i| \geq 1$. For $x \in I/J$, we define its degree $\deg(x)$ to be the maximal degree of a basis monomial appearing in $x$ with non-zero coefficient. The number $\deg(x)$ does not depend on the choice of $t_1, \ldots, t_k$. It can be used to estimate the length norm of nanowords. It follows from the definitions that for any nanoword $w$ and any $a \in \alpha$, 

\begin{align}
||w||_t \geq \deg (u^w(a)) + 1.
\end{align}

**Theorem 8.1.1.** If $\tau \neq \text{id}$, then the group $\mathcal{N}_c = \mathcal{N}_c(\alpha, \tau)$ is infinitely generated.

**Proof.** Let $\beta \subset \alpha$ be a free orbit of $\tau$. Since the pull-back homomorphism $\mathcal{N}_c(\alpha, \tau) \rightarrow \mathcal{N}_c(\beta, \tau_{|\beta})$ is surjective, it suffices to prove the theorem in the case where $\alpha$ consists of two elements permuted by $\tau$. We give a more general argument working for all fixed-point-free $\tau$.

Pick representatives $t_1, t_2, \ldots, t_k \in \alpha$ of the orbits of $\tau$ as above. For $j = 1, \ldots, k$, define an additive homomorphism $\partial_j : I \rightarrow \mathbb{Z}$ by $\partial_j(\prod_{i=1}^{k} t_i^{m_i}) = m_j$. Clearly, $\partial_j(J) = 0$. The induced group homomorphism $I/J \rightarrow \mathbb{Z}$ is also denoted $\partial_j$.

Given a nanoword $w$, the function $u^w : \alpha \rightarrow I/J$ satisfies $u^w(a) = -u^w(\tau(a))$ and is therefore determined by its values on $t_1, \ldots, t_k$. Theorem 6.3.1 of [Tu2] implies that a sequence $f_1, \ldots, f_k \in I/J$ is realizable as a sequence $u^w(t_1), \ldots, u^w(t_k)$ for a nanoword $w$ if and only if $\partial_j(f_i) + \partial_i(f_j) = 0$ for all $i, j$. This gives $k(k+1)/2$ conditions so that realizable sequences form a subgroup of $(I/J)^k$ of corank $\leq k(k+1)/2$. Since $I/J$ is an infinitely generated free abelian group, the image of the homomorphism $\mathcal{N}_c \rightarrow (I/J)^k, w \mapsto (u^w(t_1), \ldots, u^w(t_k))$ is infinitely generated. Therefore $\mathcal{N}_c$ is infinitely generated. \hfill \square

### 8.2. The genus

Any commutative domain $F$ is a $\mathbb{Z}$-algebra in the usual way. Given an additive homomorphism $\varphi : \pi \rightarrow F$, the $\varphi$-genus of a nanoword $w$ is defined by $\sigma_{\varphi}(w) = \sigma_{\varphi}(p(w)) \in \mathbb{Z}$. By Theorems 6.2.2 and 7.3.4 this number is a cobordism invariant of $w$. For any nanowords $w, w_1, w_2$ we have $\sigma_{\varphi}(w) = \sigma_{\varphi}(w^-)$ and $\sigma_{\varphi}(w_1 w_2) \leq \sigma_{\varphi}(w_1) + \sigma_{\varphi}(w_2)$. The $\varphi$-genus can be used to estimate the length norm from below: for any non-slice $w$,

\begin{align}
||w||_t \geq \sigma_{\varphi}(w)/2 + 1,
\end{align}

see Sect. 9.4. If $w$ is slice, then $||w||_t = \sigma_{\varphi}(w) = 0$.

### 8.3. Example

Consider the nanoword $w = ABCBAC$ from Example 7.3.1. Suppose that $a = |A|, b = |B|, c = |C| \in \alpha$ are not fixed points of $\tau$. We show how to use the $u$-polynomial $u^w$ to compute $||w||_t$. Pushing back, if necessary, to the union of the orbits of $a, b, c$ we can assume that $\tau$ is fixed-point-free. We
have \( [c]_{p(w)} = \delta_{a-b} + r \delta_c \) where \( r \in \mathbb{Z} \) is zero unless \( c = a \) and/or \( c = b \). Since \( a \neq \tau(b) \), the monomials \( \delta_{a-b}, \delta_c \) have degrees 2 and 1, respectively. Therefore \( \deg u^w(c) = \deg u^{p(w)}(c) = 2 \) and by Formula \([4.1.1]\) we have \( ||w||_1 \geq 3 \). Since \( w \) is a nanoword of length 6, we have \( ||w||_1 = 3 \).

Assume additionally that \( a, b, c \) belong to different orbits of \( \tau \). Then the \( \alpha \)-pairing \( p(w) \) has only one filling \( \lambda \), the tautological one. A direct computation shows that for any homomorphism \( \varphi \) from \( \pi \) to a commutative domain such that \( \varphi(a+b) \neq 0 \) and \( \varphi(c) \neq 0 \), we have \( \sigma_\varphi(p(w)) = \sigma_\varphi(\lambda) = 2 \). Formula \([8.2.1]\) gives in this case \( ||w||_1 \geq 2 \) which is weaker than Formula \([8.1.1]\).

9. Bridges and the Bridge Norm

9.1. Bridges. A quasi-bridge in a nanoword \((A, w)\) is a pair consisting of a factor \( \nabla = (B, (v_1 | \cdots | v_k)) \) of \( w \) with \( k \geq 1 \) and an involutive permutation \( \kappa : \hat{k} \rightarrow \hat{k} \) of the set \( \hat{k} = \{1, 2, \ldots, k\} \) satisfying the following two conditions:

(a) the length of \( v_r \) is even for any \( r \in \hat{k} \) such that \( \kappa(r) = r \);
(b) there is a mapping \( \iota : B \rightarrow B \) such that \( v_r = v_{\kappa(r)}^{-1} \) for all \( r \in \hat{k} \).

Consider a quasi-bridge \((\nabla, \kappa)\). Let \( n_r = n_{\kappa(r)} \) be the length of \( v_r \) for \( r = 1, \ldots, k \). Any entry of a letter \( v_r \in B \) in \( w \) appears in some \( v_{\zeta(r)} \), say, on the \( j \)-th position where \( 1 \leq j \leq n_r \). By (b), the letter \( B^r = \iota(B) \) appears in \( v_{\kappa(r)} \) on the \((n_r+1-j)\)-th position. The latter entry of \( B^r \) in \( w \) is said to be symmetric to the original entry of \( B \) in \( w \). Thus, the mapping \( \iota = \iota_{\nabla, \kappa} \) is uniquely determined by \((\nabla, \kappa)\) and \( \iota^2 = \text{id} \).

For \( B \in B \), set \( \varepsilon_{\nabla, \kappa}(B) = 1 \) if the entry symmetric to the leftmost entry of \( B \) in \( w \) is the leftmost entry of \( B^r \) in \( w \). Otherwise, set \( \varepsilon_{\nabla, \kappa}(B) = 0 \). Clearly, \( \varepsilon_{\nabla, \kappa}(B) = \varepsilon_{\nabla, \kappa}(B^r) \). The quasi-bridge \((\nabla, \kappa)\) is a bridge if for all \( B \in B \),

\[
|B^r| = r^\varepsilon_{\nabla, \kappa}(B)(|B|).
\]

Given a bridge \((\nabla = (B, (v_1 | \cdots | v_k)), \kappa)\) in a nanoword \((A, w)\), we can delete all letters of the set \( B \subset A \) from \( A \) and \( w \). For \( w = x_1v_1x_2v_2 \cdots x_kv_kx_{k+1} \), the deletion yields the nanoword \((C = A - B, x_1x_2 \cdots x_{k+1})\). This transformation \((A, w) \rightarrow (C, x_1x_2 \cdots x_{k+1})\) is the bridge move determined by \((\nabla, \kappa)\). A bridge move is always associated with a specific bridge. Thus, two bridges \((\nabla, \kappa)\) and \((\nabla', \kappa')\) in \( w \) determine the same move if and only if \( \nabla = \nabla' \) and \( \kappa = \kappa' \).

For a bridge move \( m \) determined by a bridge \((\nabla, \kappa)\), the free (2-element) orbits of the involution \( \kappa : \hat{k} \rightarrow \hat{k} \), where \( k \) is the length of \( \nabla \), are called arches of \( m \). The number of arches of \( m \) is denoted \( g(m) \). Obviously, \( g(m) \leq [k/2] \). For the inverse move \( m^{-1} \), set \( g(m^{-1}) = g(m) \).

For \( \kappa = \text{id} \), a pair \((\nabla, \kappa)\) as above is a bridge if and only if \( \nabla \) is an even symmetric factor. This follows from the fact that in this case \( \varepsilon_{\nabla, \kappa} = \varepsilon_{\nabla} \) is the function introduced in Sect. 2.3. Therefore bridges generalize even symmetric factors. The latter are precisely the bridges with 0 arches. Surgeries are precisely the bridge moves determined by bridges with 0 arches.
9.2. Examples. 1. Given a nanoword \((A, w)\) and a letter \(A \in A\) we can split \(w\) uniquely as \(x_1Ax_2Ax_3\) where \(x_1, x_2, x_3\) are words in the alphabet \(A - \{A\}\). The factor \((A | A)\) of \(w\) endowed with transposition \(\kappa = (12)\) and the identity mapping \(\iota: \{A\} \to \{A\}\) is a bridge with one arch (here \(\iota\langle,\kappa(A) = 0\). Deleting this bridge, we obtain the nanoword \((A - \{A\}, x_1x_2x_3)\).

2. Consider a nanoword \((A, w = x_1ABx_2BAx_3)\) where \(A, B \in A\). The factor \((AB | BA)\) of \(w\) endowed with \(\kappa = (12)\) is a bridge. Its deletion gives the nanoword \((A - \{A, B\}, x_1x_2x_3)\).

3. Consider a nanoword \(ABCBCDAD\) with \(|A|, |B|, |C|, |D| \in \alpha\). The factor \((A | BC | A)\) of \(w\) with \(\kappa = (13)\) is a bridge. Its deletion gives \(DD\).

4. Consider a nanoword \(ADEBCDCAEB\) with \(|A| = |B| = |C| = |D|, |E| \in \alpha\). The factor \((A | BC | CA | B)\) of \(w\) with \(\kappa = (14)(23)\) is a bridge. Its deletion gives \(DEDE\).

9.3. The bridge norm. The list \((TR)\) of transformations on nanowords considered in Sect. 3.2 can be extended to the following wider list:

\((TR+)\) isomorphisms, homotopy moves, bridge moves, and the inverse moves.

Given two nanowords \(v, w\), a metamorphosis \(m: v \to w\) is a finite sequence \(m = (m_1, ..., m_n)\) of moves from the list \((TR+)\) transforming \(v\) into \(w\). The inverse metamorphosis \(m^{-1} = (m_n^{-1}, ..., m_1^{-1})\) transforms \(w\) into \(v\). Set \(g(m) = g(m_1) + \cdots + g(m_n)\) where \(g(m_i)\) is the number of arches of \(m_i\) if \(m_i\) is a bridge move or its inverse and \(g(m_i) = 0\) for all other moves. Clearly, \(g(m^{-1}) = g(m)\) and \(g(m) = 0\) if and only if all the bridge moves in \(m\) are surgeries or inverses of surgeries.

For any nanoword \(w\), there is a metamorphosis \(w \to \emptyset\). For instance, one can consecutively delete the letters of \(w\) as in Example 9.2.1. Set

\[|| w ||_{br} = \min_{m} g(m) \geq 0\]

where \(m\) runs over all metamorphoses \(w \to \emptyset\).

Lemma 9.3.1. The function \(w \mapsto || w ||_{br}\) induces a conjugation invariant \(Z\)-valued norm on \(N_c = N_c(\alpha, \tau)\).

Proof. If \(w \sim_c v\), then there is a metamorphosis \(m: w \to v\) with \(g(m) = 0\). Composing \(m\) with a metamorphosis \(M: v \to \emptyset\) we obtain a metamorphosis \(M': w \to \emptyset\) with \(g(M') = g(M)\). Therefore \(||w||_{br} \leq ||v||_{br}\). By symmetry, \(||w||_{br} = ||v||_{br}\). Therefore the formula \(w \mapsto ||w||_{br}\) defines a function \(N_c \to Z\). The latter satisfies all axioms of a \(Z\)-valued norm directly follows from the definitions. To show that it is invariant under conjugation, it is enough to show that \(|| v w v^{-1} ||_{br} \leq || w ||_{br}\) for all nanowords \(v, w\). Any metamorphosis \(m: w \to \emptyset\), extends by the identity on \(v, v^{-1}\) to a metamorphosis \(m': v w v^{-1} \to v \emptyset v^{-1} = v v^{-1}\) with \(g(m') = g(m)\). The symmetric nanoword \(v v^{-1}\) can be transformed into \(\emptyset\) by a single surgery. Therefore \(|| v w v^{-1} ||_{br} \leq || w ||_{br}\). □

The \(Z\)-valued norm on \(N_c\) provided by this lemma is denoted \(|| \cdot ||_{br}\) and called the bridge norm. A consecutive deletion of all but one letters of a nanoword \(w\)
yields a metamorphosis of $w$ into a contractible nanoword of type $AA$. Therefore if $w$ is non-slice, then
\begin{equation}
||w||_l \geq ||w||_{br} + 1.
\end{equation}

9.4. The bridge metric. The bridge norm induces a left- and right-invariant metric $\rho_{br}$ on $N_c$ by $\rho_{br}(w, v) = ||wv^-||_{br}$ for any nanowords $w, v$, cf. Sect. 3.3. Formula (9.3.1) implies that $\rho_l(w, v) \geq \rho_{br}(w, v) + 1$.

Lemma 9.4.1. For any nanowords $v, w$, 
\[
\rho_{br}(w, v) = \min_m g(m)
\]
where $m$ runs over all metamorphoses $w \rightarrow v$.

Proof. Denote the right-hand side by $\eta$. Given a metamorphosis $m : w \rightarrow v$ we can extend it by the identity on $v^-$ to a metamorphosis $m' : wv^- \rightarrow vv^-$. Composing the latter with the surgery $vv^- \rightarrow \emptyset$, we obtain a metamorphosis $m'' : wv^- \rightarrow \emptyset$ with $g(m'') = g(m') = g(m)$. Therefore $||wv^-||_{br} \leq \eta$. Conversely, given a metamorphosis $M : wv^- \rightarrow \emptyset$, extend it by the identity on $v$ to a metamorphosis $M' : ww^- \rightarrow \emptyset$. Composing $M'$ with the inverse surgery $ww^- \rightarrow v$, we obtain a metamorphosis $M'' : w \rightarrow v$ with $g(M'') = g(M') = g(M)$. Therefore $||ww^-||_{br} \geq \eta$. Hence $\rho_{br}(w, v) = ||ww^-||_{br} = \eta$. \hfill \Box

We can estimate the bridge norm and the bridge metric via the genus. Recall the group $\pi = \pi(\alpha, \tau)$ from Sect. 7.1.

Theorem 9.4.2. Let $\varphi : \pi \rightarrow \mathbb{Z}$ be an additive homomorphism such that $\varphi(a) \in \{+1, -1\}$ for all $a \in \alpha$. For any nanoword $w$ over $\alpha$,
\[
||w||_{br} \geq \sigma_{\varphi}(p_w)/2,
\]
where $p_w = p(w)$ is the $\alpha$-pairing associated with $w$.

This theorem yields a computable a priori estimate from below for the total number of arches in any metamorphism $w \rightarrow \emptyset$. I do not know whether the assumption $\varphi(a) \in \{+1, -1\}$ for all $a \in \alpha$ is really necessary here.

Theorem 9.4.2 and inequality (9.3.1) directly imply inequality (8.2.1). Note also the following corollary.

Corollary 9.4.3. For any nanowords $w, v$ and any $\varphi$ as in Theorem 9.4.2,
\[
\rho_{br}(w, v) \geq \sigma_{\varphi}(p_w + p_v^-)/2.
\]

Indeed,
\[
\rho_{br}(w, v) = ||wv^-||_{br} \geq \sigma_{\varphi}(p_{wv^-})/2 = \sigma_{\varphi}(p_w + p_v^-)/2.
\]

We now deduce Theorem 9.4.2 from the following lemma whose proof, postponed to Sect. 14, uses topological techniques.

Lemma 9.4.4. For any bridge move $m : w \rightarrow x$ and any $\varphi$ as in Theorem 9.4.2
\begin{equation}
(9.4.1)
g(m) \geq \sigma_{\varphi}(p_w + p_x^-)/2.
\end{equation}
9.5. **Proof of Theorem 9.4.2** We claim that the inequality (9.4.1) holds for any move $m : w \to x$ from the list (TR+). If $m$ is a bridge move, then this is Lemma 9.4.4. If $m$ is an isomorphism or a homotopy move, then $p_w, p_x$ are cobordant and $p_w \oplus p_x$ is hyperbolic. Then $g(m) = 0 = \sigma(x(p_w \oplus p_x))$. If (9.4.1) holds for $m : w \to x$, then it holds for $m^{-1} : x \to w$ since
\[ g(m^{-1}) = g(m) \geq \sigma_x(p_w \oplus p_x)/2 = \sigma_x(p_x \oplus p_w)/2 = \sigma_x(p_x \oplus p_w)/2. \]

Consider a metamorphosis $m : w \to x$ that splits as a composition of two metamorphoses $m' : w \to v$ and $m'' : v \to x$ where $v$ is a nanoword. If $m'$ and $m''$ satisfy (9.4.1), then so does $m$ since, by (6.2.1),
\[ g(m) = g(m') + g(m'') \geq \sigma_x(p_w \oplus p_v)/2 + \sigma_x(p_v \oplus p_x)/2 \geq \sigma_x(p_w \oplus p_v)/2. \]

We conclude that (9.4.1) holds for all metamorphoses $m : w \to x$. For $x = \emptyset$, this gives $g(m) \geq \sigma_x(p_w)/2$. Taking the minimum over all $m : w \to \emptyset$, we obtain the claim of the theorem.

### 10. Circular shifts and a weak bridge metric

10.1. **Shifts.** The (circular) shift of a nanoword $(A, w)$ is the nanoword $(\hat{A}, \hat{w})$ obtained by moving the first letter $A = w(1)$ of $w$ to the end and applying $\tau$ to $|A| \in \alpha$. More precisely, $\hat{A} = (A - \{A\}) \cup \{\hat{A}\}$ where $\hat{A}$ is a “new” letter not belonging to $A$. The projection $\hat{A} \to \alpha$ extends the given projection $A-\{A\} \to \alpha$ by $|\hat{A}| = \tau(|A|)$. The word $\hat{w}$ in the alphabet $\hat{A}$ is defined by $\hat{w} = x_{\hat{A}}y_{\hat{A}}$ for $w = xAy$.

The $n$-th power of the shift transforms a nanoword of length $n$ into itself. Hence the inverse to the shift is a power of the shift.

Two nanowords are *weakly cobordant* if they can be related by a finite sequence of homotopy moves, surgeries, circular shifts and inverse moves. For example, for $a, b \in \alpha$, the shift transforms $w_{a,b}$ into $w_{b,\tau(a)}$. Therefore $w_{a,b}$ and $w_{b,\tau(a)}$ are weakly cobordant. If $a, b$ belong to different orbits of $\tau$, then these two nanowords are not cobordant.

A simple invariant of weak cobordism is provided by the conjugacy class of $\gamma$: if nanowords $w, v$ are weakly cobordant, then $\gamma(w), \gamma(v) \in \Pi$ are conjugate in $\Pi$. This follows from Lemma 4.1.1 and the identity $\gamma(w) = \gamma(w_1)^{-1}\gamma(w)|_{w(1)}$. In particular, if $\gamma(w) = 1$, then $\gamma(v) = 1$ for all nanowords $v$ weakly cobordant to $w$.

We will see below that the genera and the $u$-polynomial of nanowords are weak cobordism invariants. Here we note the following result.

**Lemma 10.1.1.** If the cobordism class of a nanoword $w$ lies in $\text{Ker}(p : N_c \to P_{sk})$, then all nanowords weakly cobordant to $w$ have the same property.

**Proof.** It suffices to verify that if $p_w$ is hyperbolic, then so is $p_{\hat{w}}$, where $\hat{w}$ is obtained from $w$ by the shift. Let $p_w = (S, s, e)$ and $A = w(1) \in S - \{s\}$. Then $p_{\hat{w}} = (\hat{S}, s, \hat{e})$ where $\hat{S} = (S - \{A\}) \cup \{\hat{A}\}$. A direct computation shows that $\hat{e} : \hat{S} \times \hat{S} \to \pi$ is the unique skew-symmetric pairing such that $\hat{e}|_{\hat{S} - \{\hat{A}\}} = e|_{S - \{A\}}$ and $\hat{e}(\hat{A}, B) = e(2s - A, B)$ for all $B \in S - \{A\}$. In particular, $e_A(\hat{A}, s) = -e(A, s)$. 

Observe now that any filling $\lambda = \{\lambda_i\}_i$ of $p_w$ yields a filling $\hat{\lambda}$ of $p_{\emptyset}$ by changing the unique vector $\lambda_{i_0}$ in which the letter $A = w(1)$ occurs: if $\lambda_{i_0} = A$, then it is replaced with $\overline{A}$; if $\lambda_{i_0} = \pm A + B$, then $\lambda_{i_0}$ is replaced with $\mp A + B$; if $\lambda_{i_0} = A - B$, then $\lambda_{i_0}$ is replaced with $\overline{A} + B$. It is easy to see that if $\lambda$ is an annihilating filling of $p_w$, then $\hat{\lambda}$ is an annihilating filling of $p_{\emptyset}$. Therefore if $p_w$ is hyperbolic, then so is $p_{\emptyset}$. \hfill $\square$

10.2. **The weak bridge pseudo-metric.** The list $(\text{TR}+)$ of moves on nanowords considered in Sect. 9.3 can be extended to the following wider list:

- $(\text{TR}++)$ isomorphisms, homotopy moves, bridge moves, circular shifts, and the inverse moves.

For nanowords $v, w$, a **circular metamorphosis** $m : w \to v$ is a finite sequence $m = (m_1, \ldots, m_n)$ of moves from the list $(\text{TR}++)$ transforming $w$ into $v$. Set $g(m) = g(m_1) + \cdots + g(m_n)$ where $g(m_i)$ is the number of arches of $m_i$ if $m_i$ is a bridge move or its inverse and $g(m_i) = 0$ for all other moves. Set

$$\rho_{\text{wbr}}(w, v) = \min_m g(m) \geq 0$$

where $m$ runs over all circular metamorphoses $w \to v$. The resulting function $\rho_{\text{wbr}}$ on $N_c \times N_c$ is a pseudo-metric, i.e., it is symmetric, non-negative, satisfies the triangle inequality, and $\rho_{\text{wbr}}(w, w) = 0$ for all $w$. Lemma 9.4.1 implies that $\rho_{\text{br}}(w, v) \geq \rho_{\text{wbr}}(w, v)$. The following theorem yields an estimate of $\rho_{\text{abr}}$ from below via the genus.

**Theorem 10.2.1.** For any nanowords $w, v$ and any $\varphi$ as in Theorem 9.4.2

$$\rho_{\text{abr}}(w, v) \geq (\sigma_{\varphi}(p_w) \oplus p_v) - 1)/2.$$  

In the next section, we deduce Theorem 10.2.1 from Lemma 9.4.4.

11. **Weak cobordism of $\alpha$-pairings**

Fix a ring $R$ and a left $R$-module $\pi$. In this section we study algebraic properties of $\alpha$-pairings and apply them to nanowords.

11.1. **Hyperbolic $\alpha$-pairings.** The theory of fillings and hyperbolic $\alpha$-pairings extends to tuples of $\alpha$-pairings (with values in $\pi$) as follows. Consider a tuple of $\alpha$-pairings $p_1 = (S_1, s_1, e_1), \ldots, p_r = (S_r, s_r, e_r)$ with $r \geq 1$. Replacing these $\alpha$-pairings by isomorphic ones, we can assume that the sets $S_1, \ldots, S_r$ are disjoint. Set $S = \cup_{i=1}^r S_i$ and $S^0 = \cup_{i=1}^r S_i^\circ = S - \{s_1, \ldots, s_r\}$. Let $\Lambda = RS$ be the free $R$-module with basis $S$. Let $\Lambda_\circ$ be the submodule of $\Lambda$ generated by the basis vectors $s_1, \ldots, s_r$. A vector $x \in \Lambda$ is **weakly short** if $x = A (\text{mod } \Lambda_\circ)$ for $A \in S^0$ or $x = A + B (\text{mod } \Lambda_\circ)$ for distinct $A, B \in S^0$ with $|A| = |B|$ or $x = A - B (\text{mod } \Lambda_\circ)$ for distinct $A, B \in S^0$ with $|A| = \tau(|B|)$. Removing the expression (mod $\Lambda_\circ$) in these formulas we obtain a notion of a short vector.

A **weak filling** of the tuple $p_1, \ldots, p_r$ is a finite family $\{\lambda_i\}_i$ of vectors in $\Lambda$ such that one of $\lambda_i$ is equal to $s_1 + s_2 + \ldots + s_r$, all the other $\lambda_i$ are weakly short, and every element of $S^0$ occurs in exactly one of $\lambda_i$ with non-zero coefficient (this coefficient is then $\pm 1$). The basis vectors $s_1, \ldots, s_r$ may appear in several $\lambda_i$ with
non-zero coefficients. For example, the families \( \{A\}_{A \in S^+} \cup \{s_1 + s_2 + ... + s_r\} \) and \( \{A + s_1\}_{A \in S^+} \cup \{s_1 + s_2 + ... + s_r\} \) are weak fillings of \( p_1, ..., p_r \).

The pairings \( \{e_i : S_i \times S_i \to \pi\}_{i=1}^r \) induce a bilinear form \( e = \oplus_i e_i : \Lambda \times \Lambda \to \pi \) such that \( e(S_i, S_{i'}) = 0 \) for \( i \neq i' \). A weak filling \( \{\lambda_i\}_i \) of \( p_1, ..., p_r \) is annihilating if \( e(\lambda_i, \lambda_j) = 0 \) for all \( i, j \). The tuple \( p_1, ..., p_r \) is hyperbolic if it has an annihilating weak filling. The hyperbolicity is preserved under permutations of \( p_1, ..., p_r \).

For \( r = 1 \), the notion of a weak filling is wider than the one of a filling, cf. Sect. 5.2. Any weak filling \( \{\lambda_i\}_i \) of \( p_1 \) can be transformed into a filling of \( p_1 \) by adding appropriate multiples of \( s_1 \) to all \( \lambda_i \neq s_1 \). Therefore, for \( r = 1 \), the notions of hyperbolicity introduced in this section and in Sect. 5.2 are equivalent.

For \( r = 2 \), the construction of Sect. 5.3 shows that each filling of the \( \alpha \)-pairing \( p_1 \oplus p_2 \) yields a weak filling of the pair \( (p_1, p_2) \). If the former is annihilating, then so is the latter. We conclude that if \( p_1 \oplus p_2 \) is hyperbolic, then so is the pair \( (p_1, p_2) \). The converse may be not true.

11.2. Weak cobordism. We say that \( \alpha \)-pairings \( p, q \) are weakly cobordant and write \( p \simeq_{wc} q \) if the pair \( (p, q^-) \) is hyperbolic. By the remarks above, if the \( \alpha \)-pairing \( p \oplus q^- \) is hyperbolic, then so is the pair \( (p, q^-) \). Therefore cobordant \( \alpha \)-pairings are weakly cobordant.

**Lemma 11.2.1.** Weak cobordism of \( \alpha \)-pairings is an equivalence relation.

**Proof.** It is clear that if a tuple of \( \alpha \)-pairings \( p_1, ..., p_r \) is hyperbolic, then so is the tuple of opposite \( \alpha \)-pairings \( p_1^-, ..., p_r^- \). Thus, if a pair \( (p, q^-) \) is hyperbolic, then so is the pair \( (p^-, q) \). This implies the symmetry of the weak cobordism.

The transitivity of the weak cobordism is proven similarly to the transitivity of cobordism in Lemma 6.4.1 and we indicate only the necessary changes. As \( \lambda = \{\lambda_i\}_i \) (resp. \( \mu = \{\mu_j\}_j \)), we take any weak filling of the pair \( (p_1, p_2^-) \) (resp. \( (p_2, p_2^-) \)). Before constructing \( \psi \), we modify \( \lambda \) as follows. Let \( \lambda_0 \) be the vector of \( \lambda \) equal to \( s_1 + s_2 \). Adding appropriate multiples of \( \lambda_0 \) to the other \( \lambda_i \), we can assume that the basis vector \( s_2 \in S_2 \) appears in all \( \{\lambda_i\}_{i \neq 0} \) with coefficient 0. This does not change the \( R \)-module \( \Lambda \) generated by \( \{\lambda_i\}_i \). Similarly, there is a vector \( \mu_0 \) of \( \mu \) equal to \( s_2' + s_2 \), and we can assume that \( s_2' \in S_2' \) appears in all \( \{\mu_j\}_{j \neq 0} \) with coefficient 0. In the rest of the proof instead of \( q(\psi_K) = A \pm B \) and \( q(\psi_K) = A \) it should be respectively \( q(\psi_K) = A \pm B \) \( \mod R s_1 + R s_3 \) and \( q(\psi_K) = A \) \( \mod R s_1 + R s_3 \). Instead of \( \lambda_i = A + C \) and \( \mu_j = C' + B \) it should be respectively \( \lambda_i = A + C \) \( \mod R s_1 \) and \( \mu_j = -C' + B \) \( \mod R s_3 \), etc. The word “short” should be replaced with “weakly short”.

11.3. Invariants. We can generalize the genus of \( \alpha \)-pairings to tuples as follows. Let \( F \) and \( \phi : \pi \to F \) be as in Sect. 5.2. For a tuple of \( \alpha \)-pairings \( p_1 = (S_1, s_1, e_1), ..., p_r = (S_r, s_r, e_r) \), set \( S = \bigcup_{i=1}^r S_i \) and let \( e = \oplus_i e_i : RS \times RS \to \pi \) as in Sect. 11.1. For a weak filling \( \lambda = \{\lambda_i\}_i \) of the tuple \( p_1, ..., p_r \), the matrix \( (\phi e(\lambda_i, \lambda_j))_{i,j} \) is a square matrix over \( F \). Let \( \sigma_\phi(\lambda) \in \mathbb{Q} \) be half of its rank and

\[
\sigma_\phi(p_1, ..., p_r) = \min_{\lambda} \sigma_\phi(\lambda) \geq 0
\]
Lemma 11.3.1. For any \( \alpha \)-pairings \( p_1, p_2, p_3 \),
\[
\sigma_\varphi(p_1, p_2^\cdot) + \sigma_\varphi(p_2, p_3^\cdot) \geq \sigma_\varphi(p_1, p_3^\cdot).
\]

Proof. Pick a weak filling \( \lambda \) of \((p_1, p_2^\cdot)\) such that \( \sigma_\varphi(p_1, p_2^\cdot) = \sigma_\varphi(\lambda) \). Pick a weak filling \( \mu \) of \((p_2^\cdot, p_3^\cdot)\) (where \( p_2^\cdot \) is a copy of \( p_2 \)) such that \( \sigma_\varphi(p_2^\cdot, p_3^\cdot) = \sigma_\varphi(\mu) \). We modify \( \lambda \) and \( \mu \) as in the proof of Lemma 11.2.1. This modification preserves the \( R \)-modules \( V_\lambda, V_\mu \) generated by these families of vectors and therefore preserves \( \sigma_\varphi(\lambda) \) and \( \sigma_\varphi(\mu) \). The rest of the argument goes as in the proof of Lemma 6.2.1.

Theorem 11.3.2. The \( \varphi \)-genus of \( \alpha \)-pairings is a weak cobordism invariant.

Proof. We need to prove that \( p_1 \simeq_{wc} p_2 \Rightarrow \sigma_\varphi(p_1) = \sigma_\varphi(p_2) \). The hyperbolicity of the pair \( p_1, p_2 \) implies that \( \sigma_\varphi(p_1, p_2^\cdot) = 0 \). Applying Lemma 11.3.1 to the triple \( p_1, p_2, p_3 \) where \( p_3 = (s, s, e = 0) \) is a trivial \( \alpha \)-pairing, we obtain the inequality \( \sigma_\varphi(p_2) \geq \sigma_\varphi(p_1) \). By symmetry, \( \sigma_\varphi(p_1) = \sigma_\varphi(p_2) \). □

Lemma 11.3.3. For any \( \alpha \)-pairings \( p_1, p_2 \),
\[
\sigma_\varphi(p_1 \oplus p_2) \geq \sigma_\varphi(p_1, p_2) \geq \sigma_\varphi(p_1 \oplus p_2) - 1.
\]

Proof. Let \( p_1 = (S_1, s_1, e_1) \) and \( p_2 = (S_2, s_2, e_2) \). By Sect. 5.3, every filling \( \lambda \) of \( p_1 \oplus p_2 \) yields a weak filling of the pair \( p_1, p_2 \). Therefore \( \sigma_\varphi(\lambda) \geq \sigma_\varphi(p_1, p_2) \). Taking minimum over all fillings \( \lambda \) of \( p_1 \oplus p_2 \), we obtain \( \sigma_\varphi(p_1 \oplus p_2) \geq \sigma_\varphi(p_1, p_2) \).

Conversely, any weak filling \( \mu \) of the pair \( (p_1, p_2) \) gives rise to a filling \( \mu' \) of \( p_1 \oplus p_2 \) by adding appropriate multiples of \( s_1, s_2 \) to all vectors of \( \mu \) distinct from \( s_1 + s_2 \).

Let \( V' \) be the submodules of \( RS_1 \oplus RS_2 \) generated respectively by \( \mu, \mu' \). Clearly \( V' \subset V + RS_1 + RS_2 = V + RS_1 \) (since \( s_1 + s_2 \in V \)). Therefore the rank of the pairing \( \varphi \circ (e_1 \oplus e_2) \) restricted to \( V' \) does not exceed the rank of this pairing restricted to \( V \) plus 2. For the half-ranks, we have \( \sigma_\varphi(\mu) \geq \sigma_\varphi(\mu') - 1 \geq \sigma_\varphi(p_1 \oplus p_2) - 1 \). Taking minimum over all \( \mu \), we obtain \( \sigma_\varphi(p_1, p_2) \geq \sigma_\varphi(p_1 \oplus p_2) - 1 \). □

11.4. Applications to nanowords.

Lemma 11.4.1. If nanowords \( w, v \) are weakly cobordant, then \( p(w) \simeq_{wc} p(v) \).

Proof. We begin by defining for any integer \( m \), a transformation of skew-symmetric \( \alpha \)-pairings called \( m \)-shift. Consider a skew-symmetric \( \alpha \)-pairing \( p = (S, s, e) \). Pick \( A \in S_0 \) and replace it with a “new” element \( \overline{A} \) such that \( [\overline{A}] = \tau([A]) \). Endow the resulting set \( S_A = (S - \{A\}) \cup \{\overline{A}\} \) with the unique skew-symmetric pairing \( e_A : S_A \times S_A \to \pi \) such that \( e_A|_{S_A - \{\overline{A}\}} = e|_{S - \{A\}} \) and \( e_A(\overline{A}, B) = e(ms - A, B) \).
for $B \in S - \{A\}$. In particular, $e_A(A, s) = -e(A, s)$. We say that the $\alpha$-pairing $p_A = (S_A, s, e_A)$ is obtained from $p$ by the $m$-shift at $A$. We claim that $p$ and $p_A$ are weakly cobordant. Consider a copy $p' = (S' = \{B'\}_{B \in S}, s', e')$ of $p$ and the weak filling of the pair $(p_A, (p')^-)$ formed by the vectors $\lambda_0 = s + s', \lambda_A = (\Lambda - ms) - A$ and $\{\lambda_B = B + B'\}_{B \in S - \{A\}}$. This weak filling is annihilating. In particular,

$$(e_A \oplus (e')^-)(\lambda_A, \lambda_0) = e_A(\Lambda - ms, s) + (e')^-(-A', s) = -e(A, s) + e'(A', s') = 0,$$

$$(e_A \oplus (e')^-)(\lambda_A, \lambda_B) = e_A(\Lambda - ms, B) + (e')^-(-A', B') = -e(A, B) + e'(A', B') = 0.$$  

Thus the pair $(p_A, (p')^-)$ is hyperbolic so that $p_A \cong_{\text{wc}} p' \cong p$.

To prove the lemma, we need only to show that $p(\tilde{w}) \cong_{\text{wc}} p(w)$, where $\tilde{w}$ is obtained from $w$ by the shift. The proof of Lemma [10.1.1] shows that $p(\tilde{w})$ is obtained from $p(w)$ by the 2-shift at $A = w(1)$. Hence $p(\tilde{w}) \cong_{\text{wc}} p(w)$. \square

**Theorem 11.4.2.** For any additive homomorphism $\varphi$ from $\pi = \pi(\alpha, \tau)$ to a commutative domain, the $\varphi$-genus of manifolds is a weak cobordism invariant.

This theorem follows from Theorem [11.3.2] and Lemma [11.4.1].

11.5. Proof of Theorem [10.2.1] We claim that for any manifolds $w, v$,

$$\rho_{\text{wbr}}(w, v) \geq \sigma_\varphi(p_w, p_v)/2.$$  

By Lemma [11.3.3] this will imply the theorem. By the definition of $\rho_{\text{wbr}}$, it suffices to prove that for any circular metamorphosis $m : w \to v$,

$$(11.5.1) \quad g(m) \geq \sigma_\varphi(p_w, p_v)/2.$$  

If $m$ is a bridge move, then this inequality directly follows from Lemma [9.4.4] and the left inequality in Lemma [11.3.3]. If $m$ is an isomorphism or a homotopy move or a circular shift, then $p_w, p_v$ are weakly cobordant so that $g(m) = 0 = \sigma_\varphi(p_w, p_v)$. If (11.5.1) holds for $m : w \to v$, then it holds for $m^{-1} : v \to w$ since $g(m^{-1}) = g(m)$ and $\sigma_\varphi(p_w, p_v) = \sigma_\varphi(p_v, p_w) = \sigma_\varphi(p_w, p_v)$. Finally, if a metamorphosis $m : w \to v$ splits as a composition of two metamorphoses $m' : w \to x$ and $m'' : x \to v$ satisfying (11.5.1), then Lemma [11.3.1] ensures that $m$ also satisfies (11.5.1):

$$g(m) = g(m') + g(m'') \geq \sigma_\varphi(p_w, p_v)/2 + \sigma_\varphi(p_x, p_v)/2 \geq \sigma_\varphi(p_w, p_v)/2.$$  

We conclude that (11.5.1) holds for all circular metamorphoses $m : w \to v$.

11.6. Remarks. 1. The results obtained above for the $\varphi$-genera of pairs extend to tuples as follows. Pick an arbitrary tuple $p_1, \ldots, p_r$ of $\alpha$-pairings with $r \geq 1$. Lemma [11.3.4] generalizes to the following claim: for any $1 \leq k \leq l \leq r$,

$$\sigma_\varphi(p_1, \ldots, p_l) + \sigma_\varphi(p_k, \ldots, p_{l+1}, \ldots, p_r) + l - k \geq \sigma_\varphi(p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_r).$$  

Setting here $k = l = r - 1$, we can deduce that $\sigma_\varphi(p_1, \ldots, p_r)$ depends only on the weak cobordism classes of $p_1, \ldots, p_r$. Lemma [11.3.3] generalizes to

$$\sigma_\varphi(p_1 \oplus \ldots \oplus p_r) \geq \sigma_\varphi(p_1, \ldots, p_r) \geq \sigma_\varphi(p_1 \oplus \ldots \oplus p_r) + 1 - r.$$
The arguments of Lemma 11.2.1 extend to show that if the tuples $p_1, ..., p_k$ and $p_{k+1}, p_{k+2}, ..., p_r$ are hyperbolic, then so is the tuple $p_1, ..., p_{k-1}, p_{k+1}, ..., p_r$. Taking $k = 1, r = 2$, we obtain that an $\alpha$-pairing weakly cobordant to a hyperbolic $\alpha$-pairing is itself hyperbolic.

2. It is easy to show that the $u$-polynomial of skew-symmetric (more generally, normal) $\alpha$-pairings is invariant under weak cobordism. Therefore the $u$-polynomial of nanowords is invariant under weak cobordism.

12. Words and loops

In this section we study nanowords over the 2-letter alphabet $\alpha_0 = \{+, -\}$ with involution $\tau_0$ permuting $+$ and $-$. These nanowords are shown to be disguised forms of generic loops on surfaces. As an application, we prove Lemma 7.4.1.

12.1. Loops. By a loop $f : S^1 \to \Sigma$, we mean a generic immersion of an oriented circle $S^1$ into an oriented connected surface $\Sigma$. A loop may have only a finite number of self-intersections which are all double and transversal. We shall sometimes use the term “loop” for the set $f(S^1)$. A loop is pointed if it is endowed with a base point (the origin) which is not a self-intersection. A loop $f : S^1 \to \Sigma$ is spinal if $\Sigma$ is a compact connected oriented surface that deformation retracts on the set $f(S^1)$. Two pointed spinal loops are homeomorphic if there is a an orientation preserving homeomorphism of the ambient surfaces mapping the first loop onto the second one keeping the origin and the orientation of the loop.

We associate with any pointed loop $f$ a nanoword over $\alpha_0$. To this end, label the self-intersections of $f$ by (distinct) letters $A_1, ..., A_m$ where $m$ is the number of self-intersections. Starting at the origin of $f$ and following along $f$ in the positive direction we write down the labels of all self-intersections until the return to the origin. Since every self-intersection is traversed twice, this gives a word $w$ in the alphabet $A = \{A_1, ..., A_m\}$ such that every $A_i$ appears in $w$ twice. The word $w$, called the Gauss word of $f$, was first constructed by Gauss [Ga]. We define a projection $A \to \alpha_0$ as follows. For $i = 1, ..., m$, we may speak about the first and second branches of $f$ appearing at the first and second passages of $f$ through the self-intersection labelled by $A_i$. Let $t^1_i$ (resp. $t^2_i$) be a positively oriented tangent vector of the first (resp. second) branch of $f$ at this self-intersection. Set $|A_i| = +$ if the pair $(t^1_i, t^2_i)$ is positively oriented and $|A_i| = -$ otherwise. This makes $(A, w)$ into a nanoword over $\alpha_0$ of length $2m$. It is well defined up to isomorphism and is called the underlying nanoword of $f$. Obviously, homeomorphic loops have isomorphic underlying nanowords.

**Theorem 12.1.1.** The map assigning to a pointed loop its underlying nanoword establishes a bijective correspondence between the set of homeomorphism classes of pointed spinal loops and the set of isomorphism classes of nanowords over $\alpha_0 = \{+, -\}$.

**Proof.** Given a nanoword $(A, w : \hat{a} \to A)$ over $\alpha_0$ we define a pointed spinal loop as follows. Let $S^1 = \mathbb{R} \cup \{\infty\}$ be the circle obtained by the compactification of the line $\mathbb{R}$ with right-handed orientation. Since every letter of $A$ appears in
w twice, the family \( \{ w^{-1}(A) \}_{A \in A} \) is a partition of the set \( \hat{n} \subset \mathbb{R} \subset S^1 \) into pairs. Identifying the elements of \( w^{-1}(A) \) for every \( A \in A \), we transform \( S^1 \) into a graph (i.e., a 1-dimensional CW-complex) \( \Gamma = \Gamma_w \). This graph has \( n \) edges, which we endow with orientation induced by the one in \( S^1 \), and \( n/2 \) four-valent vertices \( \{ V_A \}_{A \in A} \) where \( V_A \) is the image of \( w^{-1}(A) \) under the projection \( S^1 \to \Gamma \). Next, we thicken \( \Gamma \) to a surface \( \Sigma = \Sigma_w \). If \( n = 0 \) (so that \( w = \emptyset \)), then \( \Gamma = S^1 \subset \Sigma = S^1 \times [-1, +1] \). Assume that \( n \geq 2 \). A neighborhood of a vertex \( V_A \in \Gamma \) embeds into a copy \( d_A \) of the standard unit 2-disk \( \{(p, q) \in \mathbb{R}^2 | p^2 + q^2 \leq 1 \} \) as follows. Suppose that \( w^{-1}(A) = \{i, j\} \) with \( 1 \leq i < j \leq n \). Note that any point \( x \in S^1 \) splits its small neighborhood in \( S^1 \) into two oriented arcs, incoming and outgoing with respect to \( x \). A neighborhood of \( V_A \) in \( \Gamma \) consists of four arcs which can be identified with incoming and outgoing arcs of \( i, j \) on \( S^1 \). We embed this neighborhood into \( d_A \) so that \( V_A \) goes to the origin \((0, 0)\) and the incoming (resp. outgoing) arcs of \( i, j \) go to the intervals \([-1, 0] \times 0, 0 \times [-1, 0] \) (resp. \([0, 1] \times 0, 0 \times [0, 1]\)), respectively. We endow \( d_A \) with counterclockwise orientation if \( |A| = + \) and with clockwise orientation if \( |A| = - \). In this way the vertices of \( \Gamma \) are thickened to disjoint oriented copies of the unit 2-disk. An edge of \( \Gamma \) leads from a vertex, \( V_A \), to a vertex, \( V_B \), (possibly \( A = B \)). Its thickening is the union of \( d_A, d_B \) and a ribbon \( R_{A,B} \) connecting these 2-disks. The ribbon \( R_{A,B} \) is a copy of the rectangle \([0, 1] \times [-1/10, +1/10]\) endowed with counterclockwise orientation. The copies in \( R_{A,B} \) of the intervals \( 0 \times [-1/10, +1/10], 1 \times [-1/10, +1/10], [0, 1] \times 0 \) are called the left side, the right side, and the core of \( R_{A,B} \), respectively. It is understood that \( R_{A,B} \) meets \( \Gamma \) along its core and meets \( d_A \cup d_B \) along its sides. More precisely, the ribbon \( R_{A,B} \) is glued to the disk \( d_A \) (resp. \( d_B \)) along a length-preserving embedding of its left (resp. right) side into the boundary of the disk such that the orientations of this disk and \( R_{A,B} \) are compatible. Thickening in this way all the vertices and edges of \( \Gamma \), we embed \( \Gamma \) into a compact connected oriented surface \( \Sigma \). Composing the projection \( S^1 \to \Gamma \) with the inclusion \( \Gamma \to \Sigma \), we obtain a spinal loop \( f : S^1 \to \Sigma \) with origin \( f(0) \) for \( 0 \in \mathbb{R} \subset S^1 \). It is straightforward to see that the underlying nanoword of \( f \) is isomorphic to \( w \). Applying this construction to the underlying nanoword of a pointed spinal loop, we obtain a homeomorphic pointed loop. This proves the claim of the theorem. 

\[ \square \]

**Corollary 12.1.2.** There is a bijective correspondence between the set of homeomorphism classes of non-pointed spinal loops and the set of isomorphism classes of nanowords over \( \alpha_0 \) considered up to shifts.

It suffices to observe that when the base point of a loop is pushed along the loop across a self-intersection, the corresponding nanoword over \( \alpha_0 \) changes via the circular shift determined by \( \tau_0 \).

12.2. **Homological computations.** We analyze in more detail the relationships between a nanoword \( (A, w : \hat{n} \to A) \) over \( \alpha_0 \) and the corresponding pointed spinal loop \( f : S^1 \to \Sigma = \Sigma_w \) constructed in Theorem [12.1.1]. The orientation of \( \Sigma \) determines a skew-symmetric intersection pairing \( b : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z} \), where \( H_1(\Sigma) = H_1(\Sigma; \mathbb{Z}) \). By abuse of notation, the homological intersection number
of two loops $x, y$ in $\Sigma$ will be denoted $b(x, y)$. Thus, $b(x, y) = b([x], [y])$, where $[x], [y] \in H_1(\Sigma)$ are the homology classes of $x, y$, respectively. To compute $b(x, y)$, one deforms $x, y$ on $\Sigma$ so that they have only a finite number of intersections which are all transversal and distinct from the self-crossings of $x, y$. Then $b(x, y)$ is equal to the number of intersections where $x$ crosses $y$ from left to right minus the number of intersections where $x$ crosses $y$ from right to left.

For a letter $A \in \mathcal{A}$, we define a loop on $\Sigma$ as follows. Let $w^{-1}(A) = \{i, j\}$ with $1 \leq i < j \leq n$. Since $f(i) = f(j)$, the map $f$ transforms the interval $[i, j] \subset \mathbb{R} \subset S^1$, oriented from $i$ to $j$, into a loop on $\Sigma$ with origin $V_A = f(i) = f(j)$. This loop is denoted $f_A$.

Recall the abelian group $\pi = \pi(\alpha_0, \tau_0)$ generated by the elements of $\alpha_0$ subject to the relations $a + \tau_0(a) = 0$. The group homomorphism $\pi \to \mathbb{Z}$ sending $+\tau_0$ to $+1$ and $-\tau_0$ to $-1$ is an isomorphism, and we use it to identify $\pi$ with $\mathbb{Z}$. The $\alpha_0$-pairing $(S = A \cup \{s\}, s, e_w : S \times S \to \pi = \mathbb{Z})$ associated with $w$ is related to $b : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ as follows.

**Lemma 12.2.1.** For any $A \in \mathcal{A}$,

$$e_w(A, s) = b(f_A, f).$$

For any $A, B \in \mathcal{A}$,

$$e_w(A, B) = 2b(f_A, f_B).$$

**Proof.** We need an additional piece of notation. Let as above $A \in \mathcal{A}$ and $w^{-1}(A) = \{i, j\}$ with $1 \leq i < j \leq n$. Denote by $[j, i]$ the oriented interval in $S^1$ going from $j$ to $+\infty = -\infty$ and then from $-\infty$ to $i$. Thus, $[i, j] \cup [j, i] = S^1$ and $[i, j] \cap [j, i] = \{i, j\}$. The mapping $f$ transforms $[j, i]$ into a loop, $f_A$, on $\Sigma$ such that $[f_A] + [f_A] = [f]$. Drawing a picture of the loops $f_A, f_A$ in the disk neighborhood $d_A$ of their common origin $V_A = f(i) = f(j)$, one observes that a little deformation makes $f_A, f_A$ disjoint in $d_A$. Outside $d_A$, these loops meet transversely at the points $V_D$, where $D$ runs over letters in $\mathcal{A}$ such that either $w = \cdots A \cdots D \cdots A \cdots$ or $w = \cdots D \cdots A \cdots D \cdots A \cdots \cdots$. The intersection sign of $f_A, f_A$ at $V_D$ is $|D| \in \alpha_0 = \{\pm\}$ in the first case and $-|D|$ in the second case. Therefore

$$b(f_A, f_A) = \sum_{D \in \mathcal{A}} n_w(A, D) |D| = e_w(A, s).$$

This implies Formula (12.2.1):

$$b(f_A, f) = b(f_A, f_A) + b(f_A, f_A) = b(f_A, f_A) = e_w(A, s).$$

Let us prove Formula (12.2.2). If $A = B$, then both sides are equal to 0. Assume that $A \neq B$. Let $w^{-1}(A) = \{i, j\}$ with $i < j$ and $w^{-1}(B) = \{\mu, \nu\}$ with $\mu < \nu$. Note that the numbers $i, j, \mu, \nu$ are pairwise distinct. By the skew-symmetry of $e_w$ and $b$, if Formula (12.2.2) holds for $A, B$, then it also holds for $B, A$. Permuting if necessary $A$ and $B$, we can assume that $i < \mu$. We distinguish three cases depending on the order of $j, \mu, \nu$.

**Case** $i < j < \mu < \nu$. Then $w = x'AxA'BzBz'$ where $x, y, z, x', z'$ are words in the alphabet $\mathcal{A}$. Observe that the intervals $[i, j]$ and $[\mu, \nu]$ are disjoint. Therefore
the loops \( f_A, f_B \) meet transversely at the points \( \{V_D\} \) where \( D \) runs over letters in \( \mathcal{A} \) which appear once between the entries of \( A \) and once between the entries of \( B \). The intersection sign of \( f_A, f_B \) at \( V_D \) is \(|D|\). Therefore, in the notation of Sect. 7.2, \( b(f_A, f_B) = \langle x, z \rangle \). Formula 7.2.1 implies that \( e_w(A, B) = 2b(f_A, f_B) \).

Case \( i < \mu < \nu < j \). Then \( w = x'AxByBzAz' \) where \( x, y, z, x', z' \) are words in the alphabet \( \mathcal{A} \). Observe that the intervals \([j, i] \) and \([\mu, \nu] \) on \( S^1 \) are disjoint. Therefore the loops \( f_A' A, f_B \) meet transversely at the points \( \{V_D\} \) where \( D \) runs over letters in \( \mathcal{A} \) which appear once between the entries of \( B \) and once before the first entry of \( A \) or after the last entry of \( A \). The intersection sign of \( f_A, f_B \) at \( V_D \) is \(|D|\) in the first case and \(-|D|\) in the second case. Therefore

\[
b(f_A, f_B) = \langle x', y \rangle - \langle y, z' \rangle.
\]

As we know,

\[
b(f, f_B) = -b(f, f) = -e_w(B, s) = \langle x', y \rangle + \langle x, y \rangle - \langle y, z \rangle - \langle y, z' \rangle.
\]

Then

\[
b(f_A, f_B) = b(f, f_B) - b(f_A', f_B) = \langle x, y \rangle - \langle y, z \rangle.
\]

Now, Formula 7.2.2 implies that \( e_w(A, B) = 2b(f_A, f_B) \).

Case \( i < \mu < j < \nu \). Then \( w = x'AxByBzAz' \) where \( x, y, z, x', z' \) are words in the alphabet \( \mathcal{A} \). This case is more involved since neither the loops \( f_A, f_B \) nor the complementary loops are transversal. Note that composing the projection \( \mathcal{A} \to \mathcal{A}_0 \) with \( \tau_0 : \mathcal{A}_0 \to \mathcal{A}_0 \) we obtain a new nanoword \( (A, w') \) over \( \mathcal{A}_0 \) such that \( e_w(A, B) = -e_w(A, B) \). The spinal loop corresponding to \( w' \) is obtained from \( f \) by reversing orientation in the ambient surface; the associated intersection form is \(-b\). Therefore, replacing if necessary \( w \) by \( w' \), we can assume that \(|B| = +\).

Choose coordinates \((p, q)\) in the disk neighborhood \( d_B \subset \Sigma \) of the point \( V_B = f(\mu) = f(\nu) \) so that \( f_A \cap d_B \) is the line \( q = 0 \), \( f_B \cap d_B \) is the union of half-lines \( p = 0, q \leq 0 \) and \( q = 0, p \geq 0 \), and the orientation on \( f_A, f_B \) is right-handed on the latter half-line. Since \(|B| = +\), the coordinates \((p, q)\) determine the orientation of \( \Sigma \). Pushing \( f_B \) slightly to its left in \( \Sigma \), we obtain a “parallel” loop, \( f_B' \), transversal to \( f_A \). We can assume that \( f_B' \cap d_B \) is the union of half-lines \( p = -1, q \leq 1 \) and \( q = 1, p \geq -1 \).

To compute \( b(f_A, f_B) = b([f_A], [f_B]) = b(f_A, f_B') \), we split the set \( f_A \cap f_B' \) into five disjoint subsets. The first of them consists of the single intersection of \( f_A \) and \( f_B' \) in \( d_B \), given in the coordinates above by \( p = -1, q = 0 \). The intersection sign of \( f_A \cap f_B' \) at this point is \(+1\). The second subset of \( f_A \cap f_B' \) consists of the intersections of \( f_A \) and \( f_B' \) in the disk neighborhood \( d_A \) of \( V_A = f(i) = f(j) \). An inspection shows that if \(|A| = +1 \in \mathbb{Z} \), then \( f_A \) and \( f_B' \) do not meet in \( d_A \) and if \(|A| = -1 \in \mathbb{Z} \), then \( f_A \) and \( f_B' \) meet transversely in one point in \( d_A \) and their intersection sign at this point is \(-1\). The joint contribution of the first and second sets to \( b(f_A, f_B') \) is equal to \((|A|+1)/2 = (|A|+|B|)/2 \). The third subset of \( f_A \cap f_B' \) is \( f([i, \mu]) \cap f_B' \); its points \( \{V_D\} \) are numerated by letters \( D \) which appear once between the first entry of \( A \) and the first entry of \( B \) and once between the entries of \( B \). The intersection sign of \( f_A, f_B \) at such \( V_D \) is \(|D|\). The contribution of these crossings to \( b(f_A, f_B) \) is equal to \( \langle x, y \rangle + \langle x, z \rangle \). The fourth subset of \( f_A \cap f_B' \) is
numerated by the crossings of \( f([\mu,j]) \) with the part of \( f_B' \) obtained by pushing \( f([j,\nu]) \subset f_B \) to the left; they are numerated by letters \( D \) which appear once in \( y \) and once in \( z \). These crossings contribute \( \langle y,z \rangle \) to \( b(f_A,f_B) \). The remaining subset of \( f_A \cap f_B' \) is numerated by the self-crossings of \( f([\mu,j]) \): each of them gives rise to two points of \( f_A \cap f_B \) with opposite intersection signs. This subset contributes 0 to \( b(f_A,f_B) \). Summing up these contributions we obtain

\[
b(f_A,f_B) = b(f_A,f_B') = (|A| + |B|)/2 + \langle x,y \rangle + \langle x,z \rangle + \langle y,z \rangle.
\]

Now, Formula 7.2.3 implies that \( c_w(A,B) = 2b(f_A,f_B) \). \( \square \)

12.3. Proof of Lemma 7.4.1. The idea of the proof is as follows. Let \( f : S^1 \to \Sigma \) be the pointed spinal loop constructed from \( w \) in Theorem 12.1.1. Lemma 12.2.1 allows us to interpret the expression \( c_w(\lambda_{B_1},\lambda_{B_2}) \) in Formula 7.4.1 as an intersection number of certain loops on \( \Sigma \) associated with \( B_1,B_2 \). We construct an oriented 3-dimensional manifold \( M \), depending on the nanophrase \( \nabla \), such that \( \Sigma \subset \partial M \) and the loops on \( \Sigma \) associated with all \( B \in \mathcal{B} \) are homologically trivial in \( M \). This implies that the intersection number of two such loops, \( c_w(\lambda_{B_1},\lambda_{B_2}) \), is equal to 0. Other equalities in Formula 7.4.1 are proven similarly. The construction of \( M \) needs a few preliminaries which we now discuss.

We keep notation introduced in the second paragraph of Sect. 7.4 and in the proof of Theorem 12.1.1. Thus, each letter \( A \in \mathcal{A} \) gives rise to a self-intersection of \( f \) and to its disk neighborhood \( d_A \) which is a copy of the unit 2-disk \( \{(p,q) \in \mathbb{R}^2 : p^2 + q^2 \leq 1\} \). The curve \( f \) traverses \( d_A \) first time along \([-1,+1] \times 0 \) and second time along \([0 \times [-1,+1] \), both times from \(-1 \) to \(+1 \). We call the points \((-1,0),(1,0),(0,-1),(0,1) \in \partial d_A \), respectively, the first input, the first output, the second input, and the second output of \( d_A \). Each consecutive pair of letters \( A,B \) in \( w \) gives rise to a ribbon \( R_{A,B} \subset \Sigma \) which is a copy of the rectangle \( \{(p,q) \in \mathbb{R}^2 : p \in [0,1],q \in [-1/10,+1/10]\} \) endowed with counterclockwise orientation. The curve \( f \) traverses the ribbon \( R_{A,B} \) once along its core \([0,1] \times 0 \) in the direction from 0 to 1. Warning: the notation \( R_{A,B} \) may be misleading since this ribbon depends not only on \( A,B \) but on the exact position of \( AB \) in \( w \): if the sequence of two consecutive letters \( AB \) occurs in \( w \) twice, then it gives rise to two distinct ribbons. In our arguments it will be always clear which sequence \( AB \) is implied. One more ribbon \( R_{w(n),w(1)} \) in \( \Sigma \) is obtained by thickening the interval \([n,1] \subset S^1 \), where \( n \) is the length of \( w \). This ribbon connects \( d_{w(n)} \) to \( d_{w(1)} \). Each ribbon \( R_{A,B} \) meets \( d_A,d_B \) along its sides; otherwise these \( n \) ribbons and \( n/2 \) disks are disjoint.

Let \( \Sigma_1 \) be the compact subsurface of \( \Sigma \) formed by the disks \( \{d_B\}_{B \in \mathcal{B}} \) and the ribbons \( R_{B_1,B_2} \) associated with pairs of consecutive letters \( B_1,B_2 \) in \( w \) contained in one of the words \( v_1,\ldots,v_k \) forming the nanophrase \( \nabla \). (The number of such pairs is equal to \( 2\text{card}(\mathcal{B}) - k \). Note that if there are no letters in \( w \) between \( v_{i-1} \) and \( v_i \), then the pair consisting of the last letter of \( v_{i-1} \) and the first letter of \( v_i \) does not contribute to \( \Sigma_1 \).) The orientation of \( \Sigma \) induces an orientation of \( \Sigma_1 \).

We define an orientation reversing involution \( I : \Sigma_1 \to \Sigma_1 \). We begin by defining it on \( \cup_{B \in \mathcal{B}} d_B \). For \( B \in \mathcal{B} \), let \( I_B : d_B \to d_B^\perp \) be the homeomorphism acting as follows: a point on \( d_B \) with coordinates \((p,q)\) goes to the point on \( d_B^\perp \).
the points of $\partial d$ if $\varepsilon(B) = 1$ and with coordinates $(-q, -p)$ if $\varepsilon(B) = 0$. Recall that $d_B$ is oriented counterclockwise (with respect to the coordinates $p, q$) if $|B| = +$ and clockwise if $|B| = -$. That $I_B$ is orientation reversing follows from the assumption that $|B| = \tau_0^{\varepsilon(B)}(|B'|)$. The equality $\varepsilon(B) = \varepsilon(B')$ implies that $I_B, I_B = \text{id}$. Note that $I_B$ transforms the outputs into the inputs and vice versa. More precisely, if $\varepsilon(B) = 1$, then $I_B$ sends the $i$-th output of $d_B$ to the $i$-th input of $d_B$ for $i = 1, 2$. If $\varepsilon(B) = 0$, then $I_B$ sends the $i$-th output of $d_B$ to the $(3 - i)$-th output of $d_B$. for $i = 1, 2$.

We define a similar involution on the ribbons forming $\Sigma_1$. Consider the ribbon $R_{B_1, B_2} \subset \Sigma_1$ arising from a 2-letter segment $B_1 B_2$ in $v_r$ where $1 \leq r \leq k$. Let $B'_1 B'_2$ be the symmetric segment in $v_r$; if $B_1, B_2$ appear on the $j$-th and $(j + 1)$-th positions in $v_r$ and the length of $v_r$ is $n_r$, then the symmetric segment is formed by the letters appearing on the $(n_r - j)$-th and $(n_r + 1 - j)$-th positions in $v_r$. We define a homeomorphism $I_{B_1, B_2} : R_{B_1, B_2} \rightarrow R_{B'_1, B'_2}$ using the coordinates $(p, q)$ on these ribbons: a point on $R_{B_1, B_2}$ with coordinates $(p, q)$ goes to the point on $R_{B'_1, B'_2}$ with coordinates $(1 - p, q)$. This homeomorphism is orientation reversing and exchanges the sides left $\leftrightarrow$ right of the ribbons. We claim that $I_{B_1, B_2}$ coincides with $I_{B_1} : d_{B_1} \rightarrow d_{B'_1}$ on $R_{B_1, B_2} \cap d_{B_1}$, i.e., on the left side of $R_{B_1, B_2}$. Since both these homeomorphisms are orientation reversing and length preserving, it suffices to check that $I_{B_1}$ sends the output of $d_{B_1}$ lying on the left side of $R_{B_1, B_2}$ (in its metric center) into the input of $d_{B_1}$ lying on the right side of $R_{B'_1, B'_2}$ (again in its metric center). If $\varepsilon(B_1) = 0$ and the entry of $B_1$ in question is its $i$-th entry in $v_r$ with $i = 1, 2$, then the entry of $B'_1$ in question is its $(3 - i)$-th entry in $v_r$. Thus, $R_{B_1, B_2}$ is incident to the $i$-th output of $d_{B_1}$ and $R_{B'_1, B'_2}$ is incident to the $(3 - i)$-th input of $d_{B'_1}$. As observed above, these output and input are related by $I_{B_1}$. Similarly, if $\varepsilon(B_1) = 1$ and the entry of $B_1$ in question is its $i$-th entry in $w$ with $i = 1, 2$, then the entry of $B'_1$ in question is also its $i$-th entry in $w$. Thus $R_{B_1, B_2}$ is incident to the $i$-th output of $d_{B_1}$ and $R_{B'_1, B'_2}$ is incident to the $i$-th input of $d_{B'_1}$. These output and input are related by $I_{B_1}$. A similar argument shows that the homeomorphism $I_{B_1, B_2}$ is compatible with $I_{B_2} : d_{B_2} \rightarrow d_{B'_2}$; indeed the latter sends the input of $d_{B_2}$ lying on the right side of $R_{B_1, B_2}$ to the output of $d_{B'_2}$ lying on the left side of $R_{B'_1, B'_2}$.

We conclude that the homeomorphisms $\{I_B\}$ and $\{I_{B_1, B_2}\}$ extend to an orientation reversing homeomorphism $I : \Sigma_1 \rightarrow \Sigma_1$. Clearly, $I^2 = \text{id}$. We describe the set of fixed points $\text{Fix}(I)$ of $I$. If $B \neq B'$, then $I|_{d_B} = I_B : d_B \rightarrow d_B'$ has no fixed points. If $B = B'$, then $\varepsilon(B) = 0$ and $I|_{d_B} = I_B : d_B \rightarrow d_B$ is defined by $(p, q) \mapsto (-q, -p)$. The set $\text{Fix}(I) \cap d_B$ is then the interval $p + q = 0$ connecting the points of $\partial d_B$ with coordinates $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$. Both these points lie on $\partial \Sigma_1$. Similarly, the homeomorphism $I_{B_1, B_2} : R_{B_1, B_2} \rightarrow R_{B'_1, B'_2}$ may have fixed points if and only if the 2-letter segment $B_1 B_2$ in $v_r$ lies precisely in the center of $v_r$. If it is the case, then $I_{B_1, B_2}$ is an involution on $R_{B_1, B_2}$ given by $(p, q) \mapsto (1 - p, q)$. Its set of fixed points is the interval $(1/2) \times [-1/10, 1/10] \subset
appears once in
implies \((\rho f)\). This implies \((i, j)\). If \(i < j < \mu < \nu\), then \(f(i, j) \subset \Sigma_1\). For all \(u = 0, 1, \ldots, j - i - 1\), we have \(I(f(i + u)) = f(j - u)\) and \(I\) maps the arc \(f([i + u, i + u + 1])\) bijectively onto \(f([j - u - 1, j - u])\) (reversing orientation). This shows that each path \(f([i + u, j - u])\) is folded in two in the quotient \(\Sigma_1/I\), that is it becomes a loop of type \(\delta\delta^{-1}\), where \(\delta\) is a path in \(\Sigma_1/I\) and \(\delta^{-1}\) is the inverse path. Such a loop is contractible in \(\Sigma_1/I\).

Let \(M\) be the topological space obtained from the cylinder \(\Sigma \times [0, 1]\) by the identification \(a \times 1 = I(a) \times 1\) for all \(a \in \Sigma\). An inspection of neighborhoods of points shows that \(M\) is a 3-manifold. The fact that \(I\) is orientation-reversing implies that \(M\) is orientable. We identify \(\Sigma\) with \(\Sigma\) lying in the kernel of the inclusion homomorphism \(H_1(\Sigma) \to H_1(M)\).

We can now prove that \(e_w(\lambda_{B_1}, \lambda_{B_2}) = 0\) for all \(B_1, B_2 \in B\). Define an additive homomorphism \(\rho : \mathbb{Z}A \to H_1(\Sigma)\) by \(\rho(A) = [f_A]\) for \(A \in A\), where \(f_A\) is the loop introduced in Sect. 12.2. By Lemma 12.2.1, \(e_w(\lambda_{B_1}, \lambda_{B_2}) = 2\rho(\lambda_{B_1}, \rho(\lambda_{B_2}))\).

To prove the equality \(e_w(\lambda_{B_1}, \lambda_{B_2}) = 0\) it suffices to prove the following claim:

(*) for all \(B \in B\), the homology class \(\rho(\lambda_B) \in H_1(\Sigma)\) is homologically trivial in \(M\).

Suppose first that \(B' = B\) so that \(B\) appears twice in the same word \(v_r\) on symmetric spots. The loop \(f_B = f_B \times 0\) on \(\Sigma = \Sigma \times 0\) is obviously homotopic to the loop \(f_B \times 1\) in \(\Sigma \times [0, 1]\). By the argument above, the latter loop lies on \(\Sigma_1 \times 1\) and projects to a contractible loop in \((\Sigma_1 \times 1)/I \subset M\). Therefore the loop \(f_B\) is contractible in \(M\). Hence \(\rho(\lambda_B) = \rho(B) = [f_B]\) is homologically trivial in \(M\).

Suppose now that \(B' \neq B\). Let \(i < j\) (resp. \(\mu < \nu\)) be the numbers numerating the entries of \(B\) (resp. of \(B'\)) in \(v\). Exchanging if necessary \(B, B'\) and using that \(\lambda_{B'} = \pm \lambda_B\), we can assume that \(i < \mu\).

Consider first the case where \(\varepsilon(B) = 0\). Then \(\lambda_B = B + B'\) and \(B, B'\) appear twice in the same word \(v_r\) with \(r = 1, \ldots, k\). The definition of \(i\) implies that either \(i < j < \mu < \nu\) or \(i < \mu < j < \nu\). If \(i < j < \mu < \nu\), then the path \(f_{[i, \nu]}\) is the product of the loop \(f_B = f_{[i, j]}\), the path \(f_{[j, \mu]}\), and the loop \(f_{B'} = f_{[\mu, \nu]}\). Therefore \(\rho(\lambda_B) = [f_B] + [f_{B'}]\) is the homology class of the loop \((f_{[i, \nu]} f_{[j, \mu]})^{-1}\) in \(\Sigma_1\). Both paths forming the letter loop project to contractible loops in \(\Sigma_1/I\). This implies (*). If \(i < \mu < j < \nu\), then the path \(f_{[i, \nu]}\) is the product of the loop \(f_B = f_{[i, j]}\), the path \((f_{[j, \mu]})^{-1}\), and the loop \(f_{B'} = f_{[\mu, \nu]}\). Therefore \(\rho(\lambda_B) = [f_B] + [f_{B'}]\) is the homology class of the loop \((f_{[i, \nu]} f_{[j, \mu]})^{-1}\). As above, this implies (*).

Consider the case where \(\varepsilon(B) = 1\). Then \(\lambda_B = B - B'\) and both \(B\) and \(B'\) appear once in \(v_r\) and in \(v_r\) with \(r < r'\). We have either \(i < j < \mu < \nu\) or \(i < \mu < \nu < j\). In the first case \(f_B = f_{[i, j]} = f_{[i, \nu]} f_{[j, \mu]}\) and \(f_{B'} = f_{[\mu, \nu]} = f_{[\mu, \nu]} f_{[j, \nu]}\). Therefore \(\rho(\lambda_B) = [f_B] - [f_{B'}]\) is the homology class of the loop
$f_{[i,j]}^{-1}$. Both paths forming the letter loop project to contractible loops in $\Sigma_1/I$. This implies $(\ast)$. If $i < \mu < \nu < j$, then \( f_B = f_{[i,j]} \) is the product of the paths $f_{[i,\mu]}, f_{[\mu,\nu]},$ and $f_{[\nu,j]}$. Therefore $\rho(\lambda_B) = \{f_B\} - \{f_{B'}\}$ is the homology class of the loop $f_{[i,\mu]}f_{[\mu,\nu]}f_{[\nu,j]}$. As above, this implies $(\ast)$.

To prove the remaining equalities $e_w(\lambda_B, C) = e_w(\lambda_B, s) = 0$, we need more notation. For every $r = 1, \ldots, k$, we define two points $F_r, G_r \in f(S^1) \cap \partial \Sigma_1$. Let the first and the last letters of $v_r$ be numerated by $i = i(r), j = j(r) \in \tilde{n}$ with $i < j$. Then $F_r$ is the input of $d_{w(i)}$ lying on the right side of $R_{w, (i-1), w(i)}$ and $G_r$ is the output of $d_{w(j)}$ lying on the left side of $R_{w, (j), w(j+1)}$. (If $i = 1$, then $i - 1$ should be replaced with $n$, and if $j = n$, then $j + 1$ should be replaced with 1.) The sub-path of $f$ leading from $F_r$ to $G_r$ lies in $\Sigma_1$, and the sub-path of $f$ leading from $G_r$ to $F_{r+1}$ lies in $\Sigma_2 = \Sigma - \Sigma_1 \subset \Sigma$. Clearly, $\Sigma_2$ is a compact (possibly, disconnected) surface. We endow $\Sigma_2$ with the orientation induced by the one in $\Sigma$. The set $Y = \Sigma_1 \cap \Sigma_2 = \partial \Sigma_1 \cap \partial \Sigma_2$ consists of $2k$ disjoint closed intervals each meeting $f(S^1)$ transversely in one of the points $\{F_1, G_1, \ldots, F_k, G_k\}$. The involution $I$ on $\Sigma_1$ satisfies $I(F_r) = G_r$ for all $r$ and sends the interval in $Y$ containing $F_r$ to the interval in $Y$ containing $G_r$. Hence $I(Y) = Y$. It is clear that the involution $I|_Y$ inverts the orientation on $Y$ induced from the one on $\Sigma_2$. Let $\Psi$ be the compact oriented surface obtained from $\Sigma_2$ by identifying each point $y \in Y \subset \partial \Sigma_2$ with $I(y) \in Y$. The embedding $\Sigma_2 \times 1 \mapsto \Sigma \times 1$ induces an embedding $\Psi \rightarrow \partial M$ whose image is disjoint from $\Sigma = \Sigma \times 0 \subset \partial M$. The projection $\eta : \Sigma \times [0, 1] \rightarrow M$ maps $\Sigma_2 \times 1$ onto $\Psi$ and maps $(\Sigma_1 - \partial \Sigma_1) \times 1$ to $M - \partial M$.

We define a loop $g$ on $\Psi \subset \partial M$. It starts in $f(0)$ and goes along $f$ in $\Sigma_2$ until hitting $F_1$, then it switches to $I(F_1) = G_1$ and goes along $f$ in $\Sigma_2$ until hitting $F_3$, then it switches to $G_2$, etc., until finally returning to $f(0)$. The loop $g$ is continuous since the points $F_r, G_r$ are identified in $\Psi$ for all $r$. In a sense, $g$ is obtained by cutting out from $f$ the $k$ sub-paths lying on $\Sigma_1$ and corresponding to $v_1, \ldots, v_k$. Since these sub-paths project to contractible loops in $\Sigma_1/I$, the loops $g$ and $\eta(f \times 1)$ are homotopic in $M$. The loop $f \times 1$ being homotopic to $f = f \times 0$ in $\Sigma \times [0, 1]$, we can conclude that $g$ is homotopic to $f$ in $M$. Therefore the homology class $[f] - [g] \in H_1(\partial M)$ lies in the kernel of the inclusion homomorphism $H_1(\partial M) \rightarrow H_1(M)$. Since $\rho(\lambda_B) = \{f_B\} \pm \{f_{B'}\}$ also lies in this kernel, its intersection number with $[f] - [g]$ is equal to 0. On the other hand, this number is equal to $b(\rho(\lambda_B), \{f\})$ since the loops $f_B, f_{B'}$ do not meet $g$ (they lie in disjoint subsurfaces of $\partial M$). Thus $b(\rho(\lambda_B), \{f\}) = 0$. By Lemma 12.2.1, $e_w(\lambda_B, s) = 0$.

If $C \subset C$, then the loop $f_C$ on $\Sigma$ intersects $\partial \Sigma_2$ in the points $\{F_r, G_r\}$, where $r$ runs over all indices $1, \ldots, k$ such that the word $v_r$ lies between the two entries of $C$ in $w$. Cutting out from $f_C$ the sub-paths in $\Sigma_1$ corresponding to all such $v_r$, we obtain a loop $g_C$ in $\Psi \subset \partial M$ homotopic to $\eta(f_C \times 1)$ in $M$. Since the loop $f_C \times 1$ is homotopic to $f_C = f_C \times 0$, we conclude that $g_C$ is homotopic to $\eta f_C$ in $M$. The same argument as in the previous paragraph shows that $e_w(\lambda_B, C) = b(\rho(\lambda_B), \{f_C\}) = 0$. \qed
12.4. Remarks. 1. The geometric interpretation of nanowords over $\alpha_0$ may be extended to nanowords over an arbitrary alphabet $\alpha$. One possibility is to consider equivariant mappings $\alpha \rightarrow \alpha_0$ and the corresponding push-forward of nanowords. In this way any nanoword over $\alpha$ determines a family of pointed spinal loops on surfaces parametrized by the equivariant mappings $\alpha \rightarrow \alpha_0$. Another geometric interpretation of nanowords may be obtained by considering loops with additional data in the self-intersections. This data may be an over/under-crossing information or a label. For more on this, see [143].

2. Consider a nanoword $(\mathcal{A}, w)$ over $\alpha_0$ and the tautological filling $\lambda = \{\lambda_i\}_i$ of the associated $\alpha_0$-pairing $e_w$. Let $\varphi: \pi(\alpha_0, \tau_0) \rightarrow \mathbb{Z}$ be the identification isomorphism. By Lemma 12.2.1, the matrix $(\varphi e_w(\lambda_i, \lambda_j))_{i,j}$, considered up to multiplication of rows and columns by 2, is the matrix of homological intersections of the loops $\{f_A\}_{A \in \mathcal{A}}, f$ on the surface $\Sigma_w$ associated with $w$. Since the homological classes of these loops generate $H_1(\Sigma_w)$, the rank of this matrix is equal to $2g(\Sigma_w)$, where $g(\Sigma_w)$ is the genus of $\Sigma_w$. Hence $\sigma_\varphi(\lambda) = g(\Sigma_w)$. This equality prompted the term “genus” for $\sigma_\varphi$. We can conclude that $\sigma_\varphi(w) \leq g(\Sigma_w)$.

13. Surfaces in 3-manifolds

We discuss properties of surfaces in 3-manifolds needed in the next section to prove Lemma 9.4.4.

13.1. Simple surfaces. Let $F$ be a compact subspace of a 3-manifold $N$. A point $a \in F$ is a branch point if it lies inside a closed 3-ball $D^3 \subset N$ such that $F \cap D^3$ is the cone over a figure eight loop in $S^2 = \partial D^3$ with cone point $a \in \text{Int} D^3$. Here a figure eight loop in $S^2$ is a loop with one transversal self-intersection. The set of branch points of $F$ is denoted $\text{Br}(F)$. Clearly, $\text{Br}(F) \subset \text{Int} N = N - \partial N$. We call $F$ a simple surface in $N$ if any point of $F - \text{Br}(F)$ has an open neighborhood $V \subset N$ such that the pair $(V, V \cap F)$ is homeomorphic to either $(\mathbb{R}^3, \mathbb{R} \times 0), (\mathbb{R}^3, \mathbb{R} \times 0 \cup 0 \times \mathbb{R}^2)$, or $(\mathbb{R}^3, \mathbb{R} \times 0 \cup 0 \times \mathbb{R}^2)$, or $(\mathbb{R}^2, \mathbb{R} \times 0) \times \mathbb{R}_+$, or $(\mathbb{R}^2, \mathbb{R} \times 0 \cup 0 \times \mathbb{R}) \times \mathbb{R}_+$, where $\mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$. Points of $F - \text{Br}(F)$ having neighborhoods of the first or third type are flat. Non-flat points of $F - \text{Br}(F)$ are called double point of $F$. They form a 1-manifold $d(F)$ with boundary $d(F) \cap \partial N$. The closure $\bar{d}(F) = d(F) \cup \text{Br}(F)$ is a compact 1-manifold with boundary $\partial d(F) \cup \text{Br}(f)$.

A simple surface $F$ in $N$ can be parametrized by an abstract surface $\bar{F}$ obtained by blowing up the double points of $F$. More precisely, cutting out $F$ along $d(F)$ we obtain a compact surface $F_{\text{cut}}$ and a projection $p: F_{\text{cut}} \rightarrow F$. For $a \in d(F)$, the set $\bar{p}^{-1}(a) \subset \partial F_{\text{cut}}$ consists of 4 points adjacent to 4 branches of $F - d(F)$ near $a$. Moving around $d(F)$ in a neighborhood of $a$ in $N$ we can cyclically numerate these branches - and the corresponding points of $\bar{p}^{-1}(a)$ - by the numbers 1,2,3,4. The permutation $1 \leftrightarrow 3, 2 \leftrightarrow 4$ defines an involution on $\bar{p}^{-1}(a)$. This gives a free involution on $\bar{p}^{-1}(d(F)) \subset \partial F_{\text{cut}}$ commuting with $p$. Identifying every point of $\bar{p}^{-1}(d(F))$ with its image under this involution, we transform $F_{\text{cut}}$ into a compact surface, $\bar{F}$. The mapping $p$ induces a mapping $\bar{F} \rightarrow F$ denoted $\omega$. This is a parametrization of $\bar{F}$ in the sense that the pre-image of each double point of $F$
under \( \omega \) consists of 2 points and the restriction of \( \omega \) to the complement of this pre-image is a homeomorphism onto \( F - d(F) \).

Suppose from now on that \( N \) and \( \tilde{F} \) are oriented and provide \( \partial N, \partial \tilde{F} \) with induced orientations. Suppose also that \( \partial \tilde{F} \) is homeomorphic to a circle. The mapping \( h = \omega|_{\partial \tilde{F}} : \partial \tilde{F} \to F \subset N \) is a (generic) loop on \( \partial N \) and \( F \cap \partial N = h(\partial \tilde{F}) \).

Let \( \mathfrak{X} \subset \partial N \) be the set of double points of \( h \). We define an involution \( \nu \) on \( \mathfrak{X} \) as follows. Each point \( x \in \mathfrak{X} \) is an endpoint of a component of \( d(F) \). If this component is compact, then it has another endpoint, \( y \in \mathfrak{X} \), and \( \nu(x) = y \). Otherwise, \( \nu(x) = x \).

Fix a base point \( \ast \in \partial \tilde{F} \) such that \( h(\ast) \notin \mathfrak{X} \). For any \( x \in \mathfrak{X} \), consider the path \( h_x : [0, 1] \to \partial N \) beginning at \( x \), following along \( h(\partial \tilde{F}) \) until the first return to \( x \) and not passing through \( h(\ast) \). Set \( \text{sign}(x) = + \) if the pair of tangent vectors \((h_x'(0), h_x'(1))\) is positively oriented in the tangent space of \( x \) in \( \partial N \) and \( \text{sign}(x) = - \) in the opposite case. The path \( h_x \) determines a loop \( S^1 \to \partial N \) whose homology class in \( H_1(\partial N) \) is denoted \([h_x]\).

For a subset \( X \) of \( \mathfrak{X} \), set

\[
[X] = \sum_{x \in X} \text{sign}(x) [h_x] \in H_1(\partial N).
\]

**Lemma 13.1.1.** Let \( \iota : H_1(\partial N) \to H_1(N) \) be the inclusion homomorphism. For any orbit \( X \) of the involution \( \nu \) on \( \mathfrak{X} \), we have \( \iota([X]) \in \omega_\ast(H_1(\tilde{F})) \subset H_1(N) \).

**Proof.** We first compute \( \iota([X]) \) as follows. For any \( x \in \mathfrak{X} \), consider the path \( \omega_x : [0, 1] \to \partial N \) beginning at \( x \), following along \( \omega(\partial \tilde{F}) = h(\partial \tilde{F}) \) until the first return to \( x \) and such that the pair of tangent vectors \((\omega_x'(0), \omega_x'(1))\) is positive in the tangent space of \( x \) in \( \partial N \). In contrast to \( h_x \), the path \( \omega_x \) does not depend on the choice of the base point \( \ast \). The path \( \omega_x \) determines a loop \( S^1 \to \partial N \) whose homology class \([\omega_x] \in H_1(\partial N)\) is equal to \([h_x]\) if \( \text{sign}(x) = + \) and to \([h] - [h_x]\) if \( \text{sign}(x) = - \). Therefore \([X] = \sum_{x \in X} [\omega_x] \) modulo \([h]\). Since \( h : \partial \tilde{F} \to \partial N \) extends to a mapping of \( \tilde{F} \) to \( N \), we have \( \iota([h]) = 0 \). Therefore \( \iota([X]) = \sum_{x \in X} \iota([\omega_x]) \).

Set \( T = \omega^{-1}(d(F) \cup Br(F)) \subset \tilde{F} \). A local inspection shows that \( T \) is an embedded 1-manifold in \( \tilde{F} \) with \( \partial T = T \cap \partial \tilde{F} = \omega^{-1}(\mathfrak{X}) \). For any point \( a \in \omega^{-1}(d(F)) \subset T \) there is exactly one other point \( b \in \omega^{-1}(d(F)) \) such that \( \omega(a) = \omega(b) \). The correspondence \( a \leftrightarrow b \) extends by continuity to an involution \( \Delta \) on \( T \) with fixed-point set \( \omega^{-1}(Br(F)) \). The mapping \( \omega \) defines a homeomorphism \( \partial T/\Delta = \mathfrak{X} \). We identify these two sets along this homeomorphism.

For \( a \in \partial T \), let \( I_a \) be the component of \( T \) with endpoint \( a \) and let \( \mu(a) \in \partial T \) be its other endpoint. The formula \( a \mapsto \mu_a \) defines a fixed-point-free involution \( \mu \) on \( \partial T \). We claim that \( \mu \) commutes with \( \Delta|_{\partial T} \). Indeed, if \( \Delta(I_a) = I_a \), then \( \Delta \) exchanges the endpoints of \( I_a \), so that \( \Delta = \mu \) on \( \partial I_a \). (In this case \( \Delta \) must have a unique fixed point inside \( I_a \).) If \( \Delta(I_a) \neq I_a \), then \( \Delta(I_a) \) has the endpoints \( \Delta(a), \Delta(\mu(a)) \) so that \( \mu(\Delta(a)) = \Delta(\mu(a)) \). Since \( \Delta|_{\partial T} \) and \( \mu \) commute, \( \mu \) induces an involution on \( \partial T/\Delta \). Under the identification \( \partial T/\Delta = \mathfrak{X} \), the latter involution coincides with \( \nu \).

We now verify that \( \text{sign}(\iota([X])) \in \omega_\ast(H_1(\tilde{F})) \) for any orbit \( X \) of \( \nu \). Pick \( x \in X \). The path \( \omega_x \) in \( \partial N \) defined above is obtained (up to reparametrization) by restricting
ω to an arc γx ⊂ ∂F leading from a point a to a point b, where \{a, b\} = ω^{-1}(x) ⊂ ∂T ⊂ ∂F.

Assume first that ν(x) = x. Then μ(a) ∈ \{a, b\} and since μ(a) ≠ a, we have μ(a) = b. By the definition of ∆, we have ∆(a) = b and ∆(b) = a. Since ∆ : T → T preserves the set ∂Ia = \{a, b\}, we have ∆(Ia) = Ia. Observe that the product of the path γx with the interval Ia ⊂ F oriented from b to a is a loop, ρ, in F. The loop ω(ρ) in N is a product of ω(γx) = ωx with the loop ω|Ia. The latter loop is contractible in N because it has the form δδ^{-1} where δ is the path in N obtained by restricting ω to the arc in Ia leading from b to the fixed point of ∆ on Ia. Hence \text{in}([X]) = \text{in}([ωx]) = [ω(ρ)] ∈ ω_*(H_1(F)).

Suppose that ν(x) ≠ x. Inspecting the orientations of the sheets of F meeting along ω(Ia), we observe that the path γ_{ν(x)} begins at μ(b) and terminates at μ(a) (this was first pointed out by Carter \cite{Ca}). Consider the loop ρ = γxIbγ_{ν(x)}(Ia)^{-1} in F beginning and ending at a. Here the intervals Ib, Ia are oriented from b to μ(b) and from a to μ(a), respectively. Then ω(ρ) is the product of the loop ωx beginning and ending at x, the path ω(Ib) leading from x to ν(x), the loop ω_{ν(x)} beginning and ending at ν(x), and the path (ω(Ia))^{-1} leading from ν(x) to x. The paths ω(Ib), (ω(Ia))^{-1} are mutually inverse since Ib = ∆(Ia) and ω∆ = ω. Hence \text{in}([X]) = \text{in}([ωx] + [ω_{ν(x)}]) = [ω(ρ)] ∈ ω_*(H_1(F)).

\□

Lemma 13.1.2. Let b be the intersection form H_1(∂N) × H_1(∂N) → Z. Let X_1, ..., X_t be the orbits of the involution ν : X → X. Set c_i = [X_i] ∈ H_1(∂N) for i = 1, ..., t and c_0 = [b(∂F)] = [ω(∂F)] ∈ H_1(∂N). Then the rank of the \((t + 1) \times (t + 1)\)-matrix \((b(c_i, c_j))_{i,j=0,1,\ldots,t}\) is smaller than or equal to 4g where g = g(F) is the genus of F.

Proof. The group H = H_1(F) is isomorphic to \(\mathbb{Z}^{2g}\). Set \(L = \text{in}^{-1}(\omega_*(H)) \subset H_1(∂N)\). Since the intersection form b annihilates the kernel of \text{in},

\[(13.1.1) \quad \text{rk}(b|_L : L \times L \to \mathbb{Z}) ≤ 2 \text{rk} \omega_*(H) ≤ 2 \text{rk} H = 4g\]

where \text{rk} is the rank of a bilinear form or of an abelian group. By Lemma 13.1.1 \(c_i \in L\) for \(i = 1, ..., t\). Also \(c_0 \in L\) since \(\text{in}(c_0) = 0\). The claim of the lemma now follows from Formula (13.1.1). \□

13.2. Remark. Generic surfaces in 3-manifolds are defined as simple surfaces but additionally allowing triple points where the surface looks like the union of three coordinate planes in \(\mathbb{R}^3\). Although we shall not need it, note that Lemmas 13.1.1 and 13.1.2 extend to generic surfaces, cf. \cite{Tu1}.

14. Proof of Lemma 9.4.4

14.1. Notation. Consider a bridge in a nanoword \((A, w)\) over α formed by a factor \(\nabla = (B, (v_1 | \cdots | v_k))\) and an involution \(κ : \hat{k} → \hat{k}\) on \(\hat{k} = \{1, 2, ..., k\}\). Thus, \(B ⊂ A\) and \(w = x_1v_1x_2v_2 \cdots x_kv_kv_{k+1}\) where \(x_1, x_2, ..., x_{k+1}\) are words in
the \( \alpha \)-alphabet \( \mathcal{C} = \mathcal{A} - \mathcal{B} \). The associated bridge move \( m \) transforms \( w \) in the nanoword \( (\mathcal{C}, x = x_1 x_2 \cdots x_k) \). Set \( \iota = \iota_{\mathcal{C}, \kappa} : \mathcal{B} \to \mathcal{B} \) and \( \varepsilon = \varepsilon_{\mathcal{C}, \kappa} : \mathcal{B} \to \{0, 1\} \).

Replacing each letter \( C \in \mathcal{C} \) by its copy \( C' \) we obtain a nanoword \( (C', x') \) isomorphic to \( (\mathcal{C}, x) \). The nanoword \( wx^- \) is isomorphic to \( w(x')^- \). Consider the \( \alpha \)-pairing of the latter nanoword

\[
p_{w(x')^-} = (S = \mathcal{A} \cup C' \cup \{s\}, s, e_{w(x')^-} : S \times S \to \pi(\alpha, \tau)).
\]

For \( C \in \mathcal{C} \), set \( \lambda_C = C + C' \in \mathcal{ZS} \). For \( B \in \mathcal{B} \), consider the vector \( \lambda_B \in \mathcal{ZS} \) equal to \( B \) if \( B = B' \) and equal to \( B + (-1)^{\varepsilon(B)} B' \) if \( B \neq B' \). (Note that \( \lambda_{B'} = (-1)^{\varepsilon(B)} \lambda_B \).) Pick a set \( \mathcal{B}_+ \subset \mathcal{B} \) meeting every orbit of \( \iota : \mathcal{B} \to \mathcal{B} \) in one element. The set of vectors

\[
(14.1.1) \quad \lambda = \{\lambda_A\}_{A \in \mathcal{B}_+ \cup C} \cup \{s + s'\}
\]

is a filling of \( p_{w(x')^-} \). It plays a crucial role in the next lemma.

**Lemma 14.1.1.** If \( \alpha = \alpha_0 = \{+, -\}, \tau = \tau_0 \), and \( \varphi_0 : \pi(\alpha_0, \tau_0) \to \mathcal{Z} \) is the canonical isomorphism, then \( g(m) \geq \sigma_{\varphi_0}(\lambda)/2 \).

**Proof.** Applying the constructions of Theorem [12.1.1] to \( w \), we obtain a pointed spinal loop \( f : S^1 \to \Sigma \) where \( S^1 = \mathbb{R} \cup \{\infty\} \) and the origin of \( f \) is the point \( f(0) \). The self-crossings of \( f \) are labelled by elements of \( \mathcal{A} \) bijectively. The first part of the proof of Lemma [7.4.1] (till the formula \( I^2 = \text{id} \)) applies word for word, though here \( \iota = \iota_{\mathcal{C}, \kappa} : \mathcal{B} \to \mathcal{B} \) and \( \varepsilon = \varepsilon_{\mathcal{C}, \kappa} : \mathcal{B} \to \{0, 1\} \). This gives a surface \( \Sigma_1 \subset \Sigma \) and an orientation-reversing involution \( I : \Sigma_1 \to \Sigma_1 \). This involution maps the sub-path of \( f \) corresponding to \( v_\tau \) onto itself for all \( r = 1, \ldots, k \) such that \( \kappa(r) = r \). If \( \kappa(r) \neq r \), then \( I \) maps the sub-path of \( f \) corresponding to \( v_\tau \) onto the sub-path of \( f \) corresponding to \( v_{\kappa(r)} \) with reversed direction.

We can define surfaces \( \Sigma_2 = \Sigma - \Sigma_1 \) and \( \Psi \) as well as a 3-manifold \( M \) as in the proof of Lemma [7.4.1]. However, \( \Psi \) and \( M \) are inadequate for our aims. The problem is that the pieces of \( f(S^1) \) lying on \( \Sigma_2 \) may not form a single loop in \( \Psi \). For example, for \( w = x_1 v_1 x_2 v_2 x_3 \) and \( \kappa = (12) \), this procedure gives two loops: one is glued from the paths arising from \( x_1, x_3 \) (the involution \( I \) maps the head of the first path to the tail of the second path) and another loop is the image of the path arising from \( x_2 \) (the involution \( I \) permutes its endpoints). To circumvent this problem, we modify our constructions as follows.

Pick a small positive number \( \delta < 1/10 \). Let \( \mathcal{R} \) denote the set of all \( r = 1, \ldots, k \) such that \( r < \kappa(r) \). For \( r \in \mathcal{R} \), consider the ribbon \( R_{A,B} \subset \Sigma \) where \( B' \in \mathcal{B} \) is the first letter of \( v_r \) and \( A \in \mathcal{A} \) is the preceding letter in \( w \) (we may have \( A \in \mathcal{B} \) if there are no letters between \( v_{r-1} \) and \( v_r \); if \( r = 1 \), then \( A \) is the last letter of \( w \)). Let \( D_r \subset R_{A,B} \) be the rectangle defined in the coordinates \((p, q)\) by

\[
D_r = [3/4 - \delta, 3/4 + \delta] \times [-1/10, 1/10].
\]

This rectangle meets \( f(S^1) \) along the arc \([3/4 - \delta, 3/4 + \delta] \times 0 \). Consider also the ribbon \( R_{B', A'} \subset \Sigma \) where \( B' \in \mathcal{B} \) is the last letter of \( v_r \) and \( A' \in \mathcal{A} \) is the (cyclically) next letter of \( w \). Let \( D_r' \subset R_{B', A'} \) be the rectangle defined in the coordinates \((p, q)\) by

\[
D_r' = [1/4 - \delta, 1/4 + \delta] \times [-1/10, 1/10].
\]

This rectangle meets \( f(S^1) \) along the arc \([1/4 - \delta, 1/4 + \delta] \times 0 \). Set

\[
D = \bigcup_{r \in \mathcal{R}} (D_r \cup D'_r) \subset \Sigma - \Sigma_1.
\]

We choose \( \delta \) small enough so that the origin \( f(0) \)
of \( f \) does not belong to \( D \) and moreover, the arc \( f([-\delta,0]) \) is disjoint from \( \Sigma_1 \) and from \( D \). (To ensure these properties one may need to deform the coordinate \( p \) on the ribbon containing \( f(0) \).)

We extend the involution \( I \) on \( \Sigma_1 \) to an orientation reversing involution \( I' \) on the (disconnected) surface \( \Sigma_3 = \Sigma_1 \cup D \) which sends a point with coordinates \((p,q)\) on \( D_r \) to the point with coordinates \((1-p,q)\) on \( D'_{r} \) for all \( r \in \mathcal{R} \). Clearly, \( \Sigma_4 = \Sigma - \Sigma_3 \) is a compact oriented surface. The set \( Y' = \Sigma_3 \cap \Sigma_4 = \partial \Sigma_3 \cap \partial \Sigma_4 \) consists of \( 2k+4 \) card(\( \mathcal{R} \)) disjoint closed intervals, each meeting \( f(S^1) \) transversely in one point. It is easy to see that \( I'(Y') = Y' \) and the restriction of \( I' \) to \( Y' \) inverts the orientation on \( Y' \) induced from the one on \( \Sigma_4 \). Let \( \Psi' \) be the compact oriented surface obtained from \( \Sigma_4 \) by identifying each point \( y \in Y' \) with \( I'(y) \in Y' \). One may check that \( \Psi' \) is obtained from \( \Psi \) by adding card(\( \mathcal{R} \)) one-handles.

The pieces of \( f(S^1) \) lying on \( \Sigma_4 \) glue together into a single loop \( g' : S^1 \to \Psi' \). The point \( f(0) \in \text{Int} \Sigma_4 \) serves as the origin of \( g' \). Note that \( f \) and \( g' \) have the same germ in their common origin \( f(0) = g'(0) \). The self-crossings of \( g' \) are precisely the self-crossings of \( f \) labelled by the elements of \( \mathcal{C} = A - B \). We prefer to call the self-crossings of \( g' \) with elements of \( \mathcal{C}' \) rather than the corresponding elements of \( \mathcal{C} \). The underlying nanoword of \( g' \) is then the copy \((\mathcal{C}', x')\) of \((\mathcal{C}, x)\).

Let \( N \) be the oriented 3-manifold obtained from \( \Sigma \times [0,1] \) by the identification \( a \times 1 = I'(a) \times 1 \) for all \( a \in \Sigma_3 \). Denote the projection \( \Sigma \times [0,1] \to N \) by \( \eta' \). The embedding \( \Sigma_4 \times 1 \hookrightarrow \Sigma \times 1 \) composed with \( \eta' \) yields an inclusion \( \Psi' \hookrightarrow \partial N \) whose image is disjoint from \( \Sigma = \Sigma \times 0 \subset \partial N \). It is easy to check that \( F = \eta'(f(S^1) \times [0,1]) \) is a simple surface in \( N \) in the sense of Sect. 13.1. Its set of branch points is \( \{ \eta'(V_B \times 1) \}_{B} \), where \( B \) runs over the letters in \( \mathcal{B} \) such that \( B = B^\circ \), and \( V_B \) denotes the self-crossing of \( f \) labelled by \( B \). The double points of \( F \) are the points of type \( \eta'(V_B,t) \), where \( B \in \mathcal{B} \) and \( t \in [0,1] \). The set \( F \cap \partial N \) consists of two loops \( f(S^1) \subset \Sigma \) and \( g'(S^1) \subset \Psi' \).

We now modify \( N \) and \( F \) to obtain a simple surface with connected boundary. Consider a 2-disk \( D_0 \subset \text{Int} \Sigma_4 \) meeting \( f(S^1) \) along the arc \( f([-\delta,0]) \). Set
\[
N_0 = N - \eta'(\text{Int} D_0 \times [0,1]).
\]

Then \( N_0 \) is a compact oriented 3-manifold with \( \partial N_0 \supset \Sigma \# (-\Psi') \), where \( \# \) is the connected sum of surfaces, and the sign \(-\) reflects the fact that the orientation of \( \Psi' \) induced from \( N \) is opposite to the one induced from \( \Sigma \). The set \( F_0 = F \cap N_0 \) is obtained from \( F \) by removing an embedded band joining two components of \( F \cap \partial N \) in the complement of branch points and double points. Clearly, \( F_0 \) is a simple surface in \( N_0 \) with the same branch points and double points as \( F \). Blowing up the double points of \( F_0 \), we obtain a parametrization \( \omega : F_0 \to F_0 \) by an abstract surface \( \vec{F}_0 \). The construction of \( F, F_0 \) implies that \( \vec{F}_0 \) is a compact connected orientable surface with boundary homeomorphic to \( S^1 \). The genus of \( \vec{F}_0 \) is easily seen to be equal to the number of arches \( g(m) = \text{card}(\mathcal{R}) \) of \( m \).

The loop \( h = \omega|_{\partial \vec{F}_0} : \partial \vec{F}_0 \to \partial N_0 \) starts at \( f(0) \) (which serves as the origin) and goes along \( f \) in \( \Sigma \) till \( f(-\delta) \), then along \( \eta'(f(-\delta) \times [0,1]) \) to \( \eta'(f(-\delta) \times 1) \), then along \( (g')^{-1} \) in \( \Psi' \) till \( \eta'(f(0) \times 1) \), and finally down to \( f(0) \) along \( \eta'(f(0) \times [0,1]) \). The self-crossings of \( h \) are those of \( f \) and those of \( g' \). They are bijectively labelled.
by elements of \( A \cup C' \). The self-crossing of \( h \) labelled by a letter \( A \in A \cup C' \) is denoted \( V_A \). The underlying nanoword of \( h \) is \( (A \cup C', w(x'))^- \).

We apply to \( F_0 \) and \( h \) the definitions of Sect. 13.1. The involution \( \nu \) on the set of self-crossings of \( h \) permutes \( V_C, V_{C'} \) for all \( C \in C \) and sends \( V_B \) to \( V_{B'} \) for \( B \in B \). By Lemma 13.1.2 the matrix, \( K \), of the intersection form on \( H_1(\partial N_0) \) computed on the vectors \( [h], \{[X]\}_X \in H_1(\partial N_0) \), where \( X \) runs over the orbits of \( \nu \), satisfies

\[
\text{rk}K \leq 4 g(\tilde{F}) = 4 g(m).
\]

We now compute the vectors \([X]\). We shall write \([h_A]\) for the homology class \([h_{V_A}] \in H_1(\partial N_0)\), where \( A \in A \cup C' \). Note that \( \text{sign}(V_A) = |A| \) for \( A \in A \) and \( \text{sign}(V_{C'}) = |C| \) for \( C \in C \); the latter equality follows from the fact that \( h \) goes along \((g')^-\) on \( \Psi' \) and that the orientation on \( \Psi' \) induced from \( N \) is opposite to the one induced from \( \Sigma \). For the orbit \( X = \{V_C, V_{C'}\} \) of \( \nu \) with \( C \in C \),

\[
(X) = \pm([h_C] + [h_{C'}]).
\]

For the orbit \( X = \{V_B, V_{B'}\} \), where \( B \in B \) and \( B'^{\dagger} \neq B \),

\[
[X] = |B|[h_B] + |B'|[h_{B'}] = \pm([h_B] + (-1)^{c(B)}[h_{B'}]).
\]

For the orbit \( X = \{V_B\} \), where \( B = B'^{\dagger} \in B \),

\[
[X] = \pm[h_B].
\]

Consider now the filling \( \lambda = \{\lambda_i\}_i \) of \( p_{w(x')^-} \) given by (14.1.1). Here \( i \) runs over the subset \( B_+ \cup C \) of \( A \) plus one additional index numerating \( s+s' \). We can apply Lemma 12.2.1 to the loop \( h \) representing the nanoword \( w(x')^- \). This lemma computes the matrix \((\varphi_0 e_{w(x')^-}(\lambda_i, \lambda_j))_{i,j}\) in terms of the intersection numbers of (formal linear combinations of) loops on \( \partial N_0 \). The loops in question are \( h \) and the formal linear combinations of loops appearing on the right-hand sides of Formulas 14.1.2 - 14.1.3. Therefore the matrix \((\varphi_0 e_{w(x')^-}(\lambda_i, \lambda_j))_{i,j}\) a submatrix of the matrix \( K \), at least up to multiplication of rows and columns by \( \pm 2 \) and \( \pm 1 \). Therefore the half-rank \( \sigma_{\varphi_0}(\lambda) \) of the former matrix can not exceed \((\text{rk}K)/2 \leq 2 g(m) \). Hence, \( \sigma_{\varphi_0}(\lambda)/2 \leq g(m) \).

\subsection{Proof of Lemma 9.4.4} It suffices to verify that \( g(m) \geq \sigma_{\varphi}(\lambda)/2 \), where \( \lambda \) is the filling (14.1.1) of \( p_{w(x')^-} \). By assumption, \( \varphi(\alpha) \subset \alpha_0 \). Pushing forward a nanoword \( v \) over \( \alpha \) along \( \varphi|_{\alpha} : \alpha \to \alpha_0 \), we obtain a nanoword over \( \alpha_0 \) denoted by \( v_0 \). Every filling \( \mu \) of the \( \alpha \)-pairing \( p_v \) induces a filling \( \mu_0 \) of the \( \alpha_0 \)-pairing \( p_{v_0} \) (actually, \( \mu_0 = \mu \) as sets of vectors). By Section 7.2, \( \sigma_\varphi(\mu) = \sigma_{\varphi_0}(\mu_0) \), where \( \varphi_0 : \pi(\alpha_0, \tau_0) \to \mathbb{Z} \) is the canonical isomorphism. We apply this observation to \( v = w(x')^- \) and the filling \( \mu = \lambda \) of \( p_v \). Here \( v_0 = w_0(x')^- \) and the induced filling \( \lambda_0 \) of \( p_{v_0} \) is also given by Formula 14.1.1. The bridge move \( m : w \to x \) induces a bridge move \( m_0 : w_0 \to x_0 \) with the same number of arches. By Lemma 14.1.4

\[
g(m) = g(m_0) \geq \sigma_{\varphi_0}(\lambda_0)/2 = \sigma_\varphi(\lambda)/2.
\]
15. Further directions and open problems

1. Give a combinatorial proof of Formula 7.4.1 and Lemma 9.4.4. An incomplete combinatorial approach to Formula 7.4.1 is discussed in the first version of this paper available in arXiv:math/0511513. A combinatorial proof of Lemma 9.4.4 might enable one to extend Theorem 9.4.2 to other $\varphi$.

2. Compute the image of the homomorphism $p : N_c \to \mathcal{P}_{sk}$ from Sect. 7.3.

3. Find further cobordism invariants of nanowords.

4. Is it true that for $\alpha$ consisting of only one element, $N_c = 1$? At the moment of writing, nothing contradicts the stronger conjecture that any two nanowords over a 1-letter alphabet are homotopic.

5. Classify words of small length, say $\leq 10$, up to cobordism.

6. A metamorphosis of nanowords over $(\alpha_0, \tau_0)$ gives rise to a generic surface in a 3-manifold $N$ interpolating between two disjoint loops in $\partial N$. (Besides the constructions above, one should observe that the third homotopy move naturally gives rise to a generic surface with one triple point in a 3-dimensional cylinder.) This defines a functor from the category of nanowords over $(\alpha_0, \tau_0)$ and their metamorphoses to the category of spinal loops on surfaces and interpolating surfaces in 3-manifolds. In what sense this is an equivalence of categories?

7. One can model homotopy (resp. cobordism) of surfaces in 3-manifolds to define homotopy (resp. cobordism) for metamorphoses of nanowords. Are their interesting invariants of metamorphoses preserved under these relations?

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