A NOTE ON LIOUVILLIAN PICARD-VESSIOT EXTENSIONS

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ABSTRACT. In this paper, we prove a new characterization theorem for Picard-Vessiot extensions whose differential Galois groups have solvable identity components.

1. INTRODUCTION

Throughout this article, we fix a differential field\(^1\) \(F\) of characteristic zero with an algebraically closed field of constants \(C := \{x \in F \mid x' = 0\}\). Let \(E\) be a Picard-Vessiot extension of \(F\), \(K\) be a differential field intermediate to \(E\) and \(F\) and \(T(K|F)\) be the set of all solutions in \(K\) of all linear homogeneous differential equations over \(F\). It is known that \(T(E|F)\) is a finitely generated simple differential \(F\)-algebra whose field of fractions \(Q(T(E|F))\) equals the differential field \(E\). However, if \(r \in E\) and \(f, g \in T(E|F)\) are elements such that \(r = f/g\) then it is possible that neither \(f\) nor \(g\) belong to the differential field \(\langle r \rangle\), generated by \(F\) and \(r\). Thus \(Q(T(K|F))\) could be a proper subfield of \(K\). For example, consider the ordinary differential field \(\langle \mathbb{C}(x), \cdot \rangle\) of complex rational functions with derivation \(\cdot := d/dx\). Let \(E\) be a Picard-Vessiot extension of the Airy differential equation \(\mathcal{L}(Y) := Y'' - xy = 0\). Then the differential Galois group is isomorphic to \(SL(2, \mathbb{C})\) as algebraic groups. The differential field \(K\) fixed by the subgroup of upper triangular matrices in \(SL(2, \mathbb{C})\) is of the form \(K = \mathbb{C}(x)(w)\), where \(w\) is transcendental over \(\mathbb{C}(x)\) and \(w\) is a solution of the Ricatti equation \(w' = x - w^2\) and that \(T(K|\mathbb{C}(x)) = T(E|\mathbb{C}(x)) \cap K = \mathbb{C}(x)\) ([Mag99], pp. 86-87). Therefore, it is natural to ask for a characterization theorem of those Picard-Vessiot extensions whose intermediate differential fields \(K\) are the field of fractions of \(T(K|F)\).

A differential field extension \(E\) of \(F\) is called a liouvillian Picard-Vessiot extension if \(E\) is a liouvillian extension as well as a Picard-Vessiot extension of \(F\). Liouvillian Picard-Vessiot extensions are characterized by their differential Galois groups having a solvable identity component ([vdPS03], Theorem 1.43). Using this fact and the well-known structure theorem of \(T(E|F)\) ([Mag99], Theorem 5.12), we prove that if \(E\) is a Picard-Vessiot extension of \(F\) then \(Q(T(K|F)) = K\) for every differential field intermediate to \(E\) and \(F\) if and only if \(E\) is a liouvillian Picard-Vessiot extension of \(F\); in which case, we also show that \(T(K|F)\) is a finitely generated simple differential \(F\)-algebra (Theorem 3.1). If the differential Galois group of a liouvillian Picard-Vessiot extension \(E\) of \(F\) is connected then given any intermediate differential subfield \(K\), we find a tower of differential fields

\[
F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{m-1} \subseteq K_m = K,
\]

such that for each \(1 \leq i \leq m\), there is a \(t_i \in T(E|F)\) such that \(K_i = K_{i-1}(t_i)\) and \(t'_i = a_it_i + b_i\) for some \(a_i \in F\) and \(b_i \in K_{i-1}\) (Corollary 3.2).

We also prove that if \(E\) is an arbitrary Picard-Vessiot extension of \(F\) and \(K\) is an intermediate differential field then \(K\) contains a finitely generated simple differential \(F\)-subalgebra \(R\) such that \(Q(R) = K\) (Theorem 4.3). A structure theorem, similar to Corollary 3.2, for relatively algebraically closed intermediate...
differential subfields of liouvillian extensions can be found in [Sri20]. For fundamental results on Picard-Vessiot theory, we refer the reader to [Mag99] and [vdPS03].

2. PRELIMINARIES

In this section we record few definitions and results from Picard-Vessiot theory that are used in our proofs. Let \( F[\partial] \) be the ring of differential operators over \( F \) and \( \mathcal{L} \in F[\partial] \) be a monic operator of order \( n \). A Picard-Vessiot extension \( E \) of \( F \) for \( \mathcal{L} \) is a differential field extension of \( F \) having the same field of constants as \( F \) and satisfying the following conditions:

(a) The \( C \)-vector space \( V \) of all solutions of \( \mathcal{L}(Y) = 0 \) in \( E \) is of dimension \( n \).

(b) \( E = F(V) \), that is, the smallest differential field containing \( F \) and \( V \) is \( E \).

The differential Galois group, denoted by \( \mathcal{G}(E|F) \), is the group of all field automorphisms of \( E \) that fixes the elements of \( F \) and commutes with the derivation of \( E \). The differential Galois group stabilizes \( V \) and thus it acts as a group of \( C \)-vector space (differential) automorphisms of \( V \). Since \( E = F(V) \), the induced map \( \phi : \mathcal{G}(E|F) \to GL(V) \) is a faithful representation of groups. In fact, \( \phi(\mathcal{G}(E|F)) \) can be shown to be a Zariski closed subgroup of \( GL(V) \) and in this sense, \( \mathcal{G}(E|F) \) is seen as a linear algebraic group. The fundamental theorem of Picard-Vessiot theory provides a bijective correspondence between differential subfields intermediate to \( E \) and \( F \) and the Zariski closed subgroups of \( \mathcal{G}(E|F) \) in a way that closely resembles the polynomial Galois theory. If \( \mathcal{H} \) is a closed subgroup of \( \mathcal{G}(E|F) \) and \( K \) is an intermediate differential field then the bijective correspondence is given by the maps

\[
K \mapsto \mathcal{G}(E|K) := \{ \sigma \in \mathcal{G}(E|F) | \sigma(u) = u \text{ for all } u \in K \} \\
\mathcal{H} \mapsto E^\mathcal{H} := \{ u \in E | \sigma(u) = u \text{ for all } \sigma \in \mathcal{H} \}.
\]

The field fixed by \( \mathcal{G}(E|F) \) is \( F \); that is, \( E^{\mathcal{G}(E|F)} = F \). Let \( K \) be a differential field intermediate to \( E \) and \( F \). Then \( K \) is a Picard-Vessiot extension of \( F \) if and only if \( \mathcal{G}(E|K) \) is a closed normal subgroup of \( \mathcal{G}(E|F) \) and in which case, the differential Galois group \( \mathcal{G}(K|F) \) is isomorphic to the quotient group \( \mathcal{G}(E|F)/\mathcal{G}(E|K) \).

If an intermediate differential field \( K \) is stabilized by the differential Galois group then \( \mathcal{G}(E|K) \) is a normal subgroup of \( \mathcal{G}(E|F) \) and consequently, \( K \) is a Picard-Vessiot extension of \( F \). The algebraic closure of \( E \) in a finite Galois extension, which we denote by \( F(x) \). Clearly, \( F(x) \) is stabilized by \( \mathcal{G}(E|F) \) and in fact, \( F(x) = E^{\mathcal{G}(E|F)^0} \), where \( \mathcal{G}(E|F)^0 \) is the connected component of \( \mathcal{G}(E|F) \). The quotient group \( \mathcal{G}(E|F)/\mathcal{G}(E|F)^0 \) coincides with the ordinary Galois group of \( F(x) \) over \( F \).

The differential \( F \)-algebra \( T(E|F) \), consisting of all solutions in \( E \) of linear homogeneous differential equations over \( F \), plays a very important role in Picard-Vessiot theory and it is well understood. The following facts on \( T(E|F) \) are well-known: The differential Galois group stabilizes \( T(E|F) \), \( Q(T(E|F)) = E \) and if the \( \mathcal{G}(E|F) \) orbit set of an element \( y \in E \) spans a finite dimensional \( C \)-vector space then \( y \in T(E|F) \). There is a structure theorem that describes \( T(E|F) \) in terms of the coordinate ring of \( \mathcal{G}(E|F) \) ([Mag99], Theorem 5.12): If \( \mathcal{T} \) is an algebraic closure of \( F \) then there is an \( \mathcal{T} \)-algebra isomorphism

\[
\mathcal{T} \otimes_F T(E|F) \to \mathcal{T} \otimes_C C[\mathcal{G}(E|F)].
\]

Furthermore, the above isomorphism respects the \( \mathcal{G}(E|F) \) action. Here \( \mathcal{G}(E|F) \) acts trivially on \( \mathcal{T} \) and acts as right translations on the coordinate ring \( C[\mathcal{G}(E|F)] \) of \( \mathcal{G}(E|F) \). When \( \mathcal{G}(E|F) \) is a connected solvable group, it is also known that

\[
T(E|F) \simeq F \otimes_C C[\mathcal{G}(E|F)],
\]

where again, the isomorphism is compatible with the action of \( \mathcal{G}(E|F) \) ([Mag99], Corollary 5.29).

In this article we will be studying the \( F \)-algebra \( T(K|F) \), where \( K \) is an intermediate differential field of a Picard-Vessiot extension of \( F \). In view of the fundamental theorem, if \( \mathcal{H} \) is a closed subgroup then \( \mathcal{H} = \mathcal{G}(E|K) \) for some intermediate differential field \( K \) and we have \( T(K|F) = T(E|F) \cap K = T(E|F)^\mathcal{H} \).

As noted in the introduction section, it can happen that \( T(K|F) = F \). The characterization theorem we
prove in this article says that \( \mathcal G(E|F)^0 \) is solvable if and only if \( T(K|F) \) has “enough elements” in the sense that \( Q(T(K|F)) = K \) for every intermediate differential field \( K \). For the proof of our theorem, we will rely on the structure theorem described in Equation 2.1 along with the following proposition.

**Proposition 2.1.** Let \( F \) be a differential field of characteristic zero with an algebraically closed field of constants. Let \( E \) be a Picard-Vessiot extension of \( F \) and \( F(x) \) be the algebraic closure of \( F \) in \( E \). Let \( K \) be a differential field intermediate to \( F \) and \( E \). Then

(a) \( T(K(x)|F(x)) = T(K(x)|F) \).

(b) \( T(K(x)|F) \) is an integral extension of \( T(K|F) \).

**Proof.** Every differential equation over \( F \) is also a differential equation \( F(x) \) and thus it is clear that \( T(K(x)|F) \subseteq T(K(x)|F(x)) \). Since \( F(x) \) is finite dimensional \( F \)-vector space, for any \( y \in F(x) \), there must be a nonnegative integer \( m \) such that \( y, y', \ldots, y^{(m)} \) are \( F \)-linearly dependent. Therefore, \( F(x) \subseteq T(K(x)|F) \). Now let \( y \in T(K(x)|F(x)) \setminus F(x) \) and \( \mathcal L = \partial^{(n)} + a_{n-1} \partial^{(n-1)} + \cdots + a_0 \in F(x)[\partial] \) be a monic operator of order \( n \geq 1 \) such that \( \mathcal L(y) = 0 \). Let \( V \) be the set of all solutions of \( \mathcal L \) in \( E \) and for any \( \sigma \in \mathcal G(E|F) \), let \( V_\sigma \) be the set of all solutions of \( \mathcal L_\sigma = \partial^{(n)} + \sigma(a_{n-1}) \partial^{(n-1)} + \cdots + \sigma(a_0) \) in \( E \). Observe that \( \sigma(V) = V_\sigma \). Since \( a_i \in F(x) \) for each \( i \) and \( E^G(E|F) = F(x) \), the orbit set of \( a_i \) under the action of \( \mathcal G(E|F) \) is a finite set for each \( i \). Therefore, there are only finitely many \( \mathcal L_\sigma \). Let \( \sigma_0 \in \mathcal G(E|F) \) be the identity and \( \mathcal L = \mathcal L_{\sigma_0}, \mathcal L_{\sigma_1}, \ldots, \mathcal L_{\sigma_k} \) be the distinct operators. Let \( W = V_{\sigma_0} + V_{\sigma_1} + \cdots + V_{\sigma_k} \). Clearly, \( W \) is a finite dimensional \( C \)-vector space. For any \( \sigma \in \mathcal G(E|F) \) and \( y \in V_{\sigma_i} \), we have \( \sigma(y) \in V_{\sigma_i} \subseteq W \). This implies that \( W \) is also a \( \mathcal G(E|F) \)-module. Thus, any \( y \in W \) must be a solution of some operator in \( F[\partial] \). That is, \( y \in T(K(x)|F) \). Hence

\[
T(E|F) = T(E|F(x)).
\]

(2.2)

Let \( r \in T(K(x)|F) \). Since \( F(x) \) is a finite Galois extension of \( F \), so is \( K(x) \) over \( K \). Then the ordinary Galois group \( \text{Aut}(K(x)|K) \) equals the differential Galois group \( \mathcal G(K(x)|K) \) ([vdPS03], Exercise 1.24). Let \( \{r = r_1, r_2, \cdots, r_m\} = \{\sigma(r) \mid \sigma \in \mathcal G(K(x)|K)\} \). Then for \( \sigma \in \mathcal G(K(x)|K) \), we have \( \sigma(r) \in T(K(x)|F) \) and thus \( r_i \in T(K(x)|F) \) for all \( i \). The coefficients of the monic irreducible polynomial of \( r \) over \( K \) are symmetric polynomials in \( r_1, \cdots, r_m \). Therefore the coefficients of the irreducible polynomial belong to \( T(K(x)|F) \cap K = T(K|F) \).

This shows that \( T(K(x)|F) \) is an integral extension of \( T(K|F) \).

\[\square\]

### 3. LIouvillian Picard-Vessiot Extensions

A differential field extension \( E \) of \( F \) is called a **liouvillian extension** of \( F \) if there exists a tower of fields

\[
E = E_n \supseteq E_{n-1} \supseteq \cdots \supseteq E_0 = F
\]

such that \( E_i = E_{i-1}(t_i) \) and that either \( t_i \) is algebraic over \( E_{i-1} \) or \( t'_i \in E_{i-1} \) or \( t_i \neq 0 \) and \( t'_i/t_i \in E_{i-1} \). We recall that a Picard-Vessiot extension \( E \) of \( F \) is called a **liouvillian Picard-Vessiot extension** if \( E \) is a liouvillian extension as well as a Picard-Vessiot extension and that the identity component of the differential Galois group of a liouvillian Picard-Vessiot extension is solvable.

**Theorem 3.1.** Let \( F \) be a differential field with an algebraically closed field of constants and \( E \) be a Picard-Vessiot extension of \( k \). Then \( E \) is a liouvillian extension of \( F \) if and only if \( Q(T(K|F)) = K \) for any differential field \( K \) intermediate to \( E \) and \( F \); in which case \( T(K|F) \) is a finitely generated simple differential \( F \)-algebra.

**Proof.** Let \( E \) be a liouvillian extension of \( F \), \( K \) be an intermediate differential subfield and \( \mathcal H := \mathcal G(E|K) \). First we assume that \( \mathcal G(E|F) \) is connected. Then we have \( T(E|F) \cong F \otimes_C \mathcal G(E|F) = F[\mathcal G(E|F)] \). Since \( \mathcal G(E|F) \) is solvable, from [CPS77], Theorem 4.3, we have that closed subgroups of \( \mathcal G(E|F) \) are observable. Now from [ByBHM63], Theorem 3, we obtain \( Q(F[\mathcal G(E|F)]^\mathcal H) = Q(F[\mathcal G(E|F)])^\mathcal H \). Thus
Let $\mathcal{G}(E|F)$ be a differential field containing $F(x)$ such that $\mathcal{G}(E|F)$ is not connected. Then $E^\mathcal{G}(E|F)$ is a field containing $E$ and $E^\mathcal{G}(E|F)$ is not connected. From Proposition 2.1, $T(K(x)|F(x)) = T(K(x)|F)$ and thus $K(x) = Q(T(K(x)|F))$. We also know that $T(K(x)|F)$ is an integral extension of $T(K|F)$. Let $S = T(K|F) \setminus \{0\}$. Then $S^{-1}T(K(x)|F)$ is also an integral extension of $S^{-1}T(K|F)$. Since the latter is a field, so is the former. However, $K(x) = Q(T(K(x)|F))$ is the smallest field containing $T(K(x)|F)$ and thus $S^{-1}T(K(x)|F) = K(x)$. Now for any $r \in K$, we have $r = f/g$, where $f \in T(K(x)|F)$ and $g \in S = T(K|F) \setminus \{0\}$. Therefore, $f = gr \in T(K(x)|F) \cap K = T(K|F)$ and this proves that $Q(T(K|F)) = K$.

To prove the converse, we suppose that $E$ is not a liouvillian extension of $F$. Let $F(x) = E^\mathcal{G}(E|F)^0$ be the algebraic closure of $F$ in $E$. Then the identity component $\mathcal{G}(E|F)^0$ is not solvable and therefore it contains a nontrivial Borel subgroup $B$. Let $K = E^\mathcal{G}$ and $r \in T(E|F(x))^B$. Since $E$ is a Picard-Vessiot extension of $F(x)$ with Galois group $\mathcal{G}(E|F)^0$, the orbit set $O_r$ of $r$ under the action of $\mathcal{G}(E|F)^0$ is contained in a finite dimensional $C$-vector space which is also $\mathcal{G}(E|F)^0$-stable. Moreover, the quotient $\mathcal{G}(E|F)^0/B$ has the structure of a projective variety. Therefore, the induced map $\phi : \mathcal{G}(E|F)^0/B \to O_r$, given by $\phi(\sigma) = \sigma(r)$ for $\sigma \in \mathcal{G}(E|F)^0$, is a morphism from a projective variety into some affine space containing $O_r$. Thus $\phi$ must be a constant. That is, $r \in T(E|F(x))^B = F(x)$ and thus $F(x) = T(E|F(x))^B$. Note that

$$Q(T(K|F)) = Q(T(E|F)^B) = Q(T(E|F(x))^B) = F(x) \neq K.$$ 

This proves the converse.

Next, we shall show that $T(K|F)$ is a finitely generated differential $F$-algebra. First assume that $\mathcal{G}(E|F)$ is a connected solvable group. Let $\mathcal{H}$ be a closed subgroup of $\mathcal{G}(E|F)$ and $K := E^\mathcal{H}$. We have $T(E|F) \simeq F \otimes_C C[\mathcal{G}(E|F)]$ and therefore

$$T(K|F) = T(E|F)^\mathcal{H} \simeq (F \otimes_C C[\mathcal{G}(E|F)])^\mathcal{H} = F \otimes_C C[\mathcal{G}(E|F)]^\mathcal{H}.$$ 

Since $\mathcal{G}(E|F)$ is solvable, the homogeneous space $\mathcal{G}(E|F)/\mathcal{H}$ is affine ([CPS77], Theorem 4.3 and Corollary 4.6) and we obtain $C[\mathcal{G}(E|F)]^\mathcal{H} = C[\mathcal{G}(E|F)/\mathcal{H}]$ is a finitely generated $C$-algebra. This in turn implies $T(K|F) \simeq F \otimes_C C[\mathcal{G}(E|F)]^\mathcal{H}$ is a finitely generated $F$-algebra. Now assume that only $\mathcal{G}(E|F)^0$ is solvable. Let $F(x) = E^\mathcal{G}(E|F)^0$ and observe that $\mathcal{G}(E|F(x)) = \mathcal{G}(E|F)^0$ is connected. Then we know $T(K(x)|F(x))$ is a finitely generated $F$-algebra and it follows that $T(K(x)|F(x))$ is a finitely generated $F$-algebra as well. Since $T(K(x)|F(x)) = T(K(x)|F)$ is an integral extension of $T(K|F)$, by Artin-Tate Theorem ([Eis95], p.143) we obtain that $T(K|F)$ is a finitely generated $F$-algebra.

Now it only remains to show that $T(K|F)$ is a simple differential $F$-algebra. As done earlier, we shall first prove simplicity when $\mathcal{G}(E|F)$ is connected. Let $I$ be a nonzero differential ideal of $T(K|F)$ and choose $0 \neq y \in I$ so that $\mathcal{L}(y) = 0$ for some $\mathcal{L} \in F[\partial]$ of smallest positive order $n$. Since the Galois group is connected, $\mathcal{L} = \mathcal{L}_{n-1}\mathcal{L}_1$ for $\mathcal{L}_{n-1}, \mathcal{L}_1 \in F[\partial]$ of order $n - 1$ and 1 ([Kol48], p.38). Let $\mathcal{L}_1 = \partial - a$ for $a \in F$ and observe that $\mathcal{L}_1(y) = y' - ay \in I$. Now since $0 = \mathcal{L}(y) = \mathcal{L}_{n-1}(\mathcal{L}_1(y))$, from the choice of $n$, we obtain that $y' - ay = b \in F$. Thus $b = \mathcal{L}_1(y) \in I$. If $b \neq 0$ then $I = T(K|F)$. On the other hand if $b = 0$ then $y' = ay$ and therefore $(1/y)' = -a/1(y)$. Thus $1/y \in T(K|F)$ and we again obtain $I = T(K|F)$.

This completes the proof when $\mathcal{G}(E|F)$ is connected. For an arbitrary liouvillian Picard-Vessiot extension $E$ of $F$, we have $T(K(x)|F(x)) = T(K(x)|F)$ to be a finitely generated simple differential $F$-algebra, where $F(x)$ is the algebraic closure of $F$ in $E$. Suppose that $T(K|F)$ is not simple and let $I \neq T(K|F)$ be a differential ideal that is maximal among all differential ideals not intersecting $\{I\}$. Then $I$ is known to be a prime ideal. Let $I^e$ be the extension ideal in $T(K(x)|F)$. It is easy to see that $I^e$ is a differential ideal and therefore $I^e = T(K(x)|F)$. Since $T(K(x)|F)$ is integral over $T(K|F)$ and that $I$ is prime, there must exist a prime ideal of $T(K(x)|F)$ that contracts to $I$. But any such prime ideal must contain $I^c = T(K(x)|F)$, a contradiction.

A liouvillian Picard-Vessiot extension $E$ of $F$ is known to have the following structure ([Mag99], Proposition 6.7): Let $E^\mathcal{G}(E|F)^0 = F(x)$ and $\mathcal{V}$ be the unipotent radical of $\mathcal{G}(E|F)$. Then $\mathcal{V} \subseteq \mathcal{G}(E|F)^0$ and that
have found an element \( E \) such that \( E \) is a Picard-Vessiot extension of \( F(x) \) with a differential Galois group isomorphic to a maximal torus of \( \mathcal{G}(E[F])^0 \).

From the inverse problem for tori ( [Mag99], p.99 or [vdPS03], Exercise 1.41), one can assume further that the \( F(x) \)-algebraically independent \( \xi_i \) are chosen so that \( \xi_i/\xi_i \in F(x) \) (as opposed to \( \xi_i/\xi_i \in F(x)(\xi_1, \ldots, \xi_{i-1}) \)). Using this description of \( E \), in the next corollary, we shall resolve \( K \) into a tower of differential fields such that each differential field in the tower is obtained from its predecessor by adjoining a solution of a first order equation of a special kind.

**Corollary 3.2.** Let \( F \) be a differential field of characteristic zero with an algebraically closed field of constants. Let \( E \) be a liouvillian Picard-Vessiot extension of \( F \) and \( \mathcal{G}(E[F]) \) be connected. Let \( K \) be a differential field intermediate to \( E \) and \( F \). Then \( K = F(t_1, \ldots, t_n) \), where for each \( i \), \( t_i \in T(K|F) \), \( t_i = a_i t_i + b_i \) for \( a_i \in k \) and \( b_i \in F(t_1, \ldots, t_{i-1}) \). Furthermore, if \( \mathcal{G}(E[F]) \) is a unipotent algebraic group then each \( a_i \) can be taken to be zero and if \( \mathcal{G}(E[F]) \) is a torus then each \( b_i \) can be taken to be zero.

**Proof.** To avoid triviality, we shall assume \( F \neq K \) and \( K \neq E \). Let \( M \) be any differential field such that \( F \subseteq K \subseteq \mathcal{G}(E[F]) \). We claim that there is a \( y \in T(K|F) \setminus M \) such that \( y' = a y + b \) for some \( a \in F \) and \( b \in M \) and that \( a \) can be taken to be zero if \( \mathcal{G} \) is unipotent and that \( b \) can be taken to be zero if \( \mathcal{G}(E[F]) \) is a torus. To prove this claim, we first observe from Theorem 3.1 that \( T(K|F) \setminus M \neq \emptyset \). Choose \( y \in T(K|F) \setminus M \) and \( \mathcal{L} \in F[\partial] \) of smallest positive degree \( m \) such that \( \mathcal{L}(y) = 0 \). Since \( \mathcal{G}(E[F]) \) is connected, \( \mathcal{L} = \mathcal{L}_{m-1} \mathcal{L}_1 \), where \( \mathcal{L}_{m-1}, \mathcal{L}_1 \in F[\partial] \) are of order \( m - 1 \) and \( 1 \) respectively. Let \( \mathcal{L}_1 = \partial - a \), \( a \in F \). Observe that \( \mathcal{L}_1(y) \in T(K|F) \) and \( \mathcal{L}_{m-1}(\mathcal{L}_1(y)) = 0 \). Therefore, from our choice of \( m \), \( \mathcal{L}_1(y) \in T(M|F) \subset M \). Thus we have found an element \( y \in T(K|F) \setminus M \) such that \( y' = a y + b \) where \( a \in F \) and \( b \in M \). If \( \mathcal{G}(E[F]) \) is unipotent then \( E = F[\xi_1, \ldots, \xi_s] \) where \( \xi_i' / \xi_i \in F \) for each \( i \). If \( b \neq 0 \) then apply [Sri20], Proposition 2.2. Thus, in this case, we may choose \( \mathcal{L}_1 = \partial - (a'/a) \) and obtain an element \( \alpha y / \alpha \in T(K|F) \) such that \( (\alpha y / \alpha)' = b / \alpha \in M \). Finally suppose that \( \mathcal{G}(E[F]) \) is a torus. Then \( E = F[\xi_1, \ldots, \xi_s] \), where \( \xi_i' / \xi_i \in F \) for each \( i \). If \( b = 0 \) then apply [Sri20], Proposition 2.2 to the extension \( M(\xi_1, \ldots, \xi_s) \) of \( M \) with \( \mathcal{L}(y) = y' - a y = b \) and obtain \( \alpha \in M \) such that \( a' = a \alpha = b \). Now \( y - \alpha \in T(K|F) \setminus M \) and \( (y-\alpha)'/(y-\alpha) = a / \in F \). This proves the claim. Now taking \( M = F \) one finds \( t_i \) and taking \( M = F(t_i, \ldots, t_{i-1}) \), one finds \( t_i \in T(K|F) \setminus M \), with the desired properties. Since \( K \), as a field, is finitely generated over \( F \), there must be an \( n \) such that \( K = F(t_1, \ldots, t_n) \). \( \square \)

**Remark 3.3.** In the above corollary, the hypothesis that \( \mathcal{G}(E[F]) \) is connected allowed us to factor the differential operator \( \mathcal{L} \) over \( F[\partial] \), which was a crucial step in the proof. In fact, the assumption that \( \mathcal{G}(E[F]) \) is connected cannot be dropped. For example, consider the liouvillian extension \( E = \mathbb{C}(x)(\sqrt{\mathbb{C}}, e^{\sqrt{\mathbb{C}}}) \), where the derivation is \( \prime := d/dx \). Then \( E \) is a liouvillian Picard-Vessiot extension of \( \mathbb{C}(x) \) for the differential equation

\[
\mathcal{L}(Y) = Y'' + \frac{1}{2x} Y' - \frac{1}{4x} Y = 0.
\]

The set \( V := \text{span}_e \{ e^{\sqrt{\mathbb{C}}}, e^{-\sqrt{\mathbb{C}}} \} \) is the set of all solutions of \( \mathcal{L}(Y) = 0 \) in \( E \). Since \( E \) contains the algebraic extension \( \mathbb{C}(x)(\sqrt{\mathbb{C}}), \mathcal{G}(E[F]) \) is not connected\(^2\). One can show that the intermediate differential field \( K := \mathbb{C}(x)(e^{\sqrt{\mathbb{C}}}, e^{-\sqrt{\mathbb{C}}}) \) contains no elements satisfying a first order equation over \( \mathbb{C}(x) \) other than the elements of \( \mathbb{C}(x) \) itself ( [Sri20], p.376).

### 4. Intermediate Differential Subfields of Picard-Vessiot Extensions

Let \((\mathbb{C}(x), d/dx)\) be the ordinary differential field of complex rational functions with derivation \( \prime := d/dx \). Let \( E \) be a Picard-Vessiot extension of the Airy differential equation \( \mathcal{L}(Y) := Y'' - x Y = 0 \). As noted in Section 1, for the differential field \( K = \mathbb{C}(x)(w) \), where \( w \) satisfies the Ricatti equation \( w' = x - w^2 \), we

\(^2\)The differential Galois group \( \mathcal{G}(E[C(x)]) \) is isomorphic to \( G_n \times \mathbb{Z}_2 \).
have $T(K|\mathbb{C}(x)) = \mathbb{C}(x)$. Nonetheless, the differential ring $\mathbb{C}(x)[w]$ is in fact a (finitely generated) simple differential $F$–algebra whose field of fractions is $K$. To see this, it is enough to show that the differential ring $\mathbb{C}(x)[w]$ is simple. Suppose that $I$ is a differential ideal of $\mathbb{C}(x)[w]$. Then $I$ must be a principal ideal, say $I = (v)$ for some monic irreducible polynomial $v \in \mathbb{C}(x)[w]$. Write $v = \prod_{i=1}^{m} w - \alpha_i$, for distinct algebraic elements $\alpha_i$ of $\mathbb{C}(x)$. We have

\[ v' = \sum_{j} (w' - \alpha_j') \prod_{i\neq j} (w - \alpha_i). \]

Since $v$ divides $v'$, $w - \alpha_i$ must divide $w' - \alpha_i'$ and it follows that $\alpha_i' = x - \alpha_i^2$. This contradicts the fact that the Ricatti equation $w' = x - w^2$ has no solutions algebraic over $\mathbb{C}(x)$. Thus $\mathbb{C}(x)[w]$ is a (finitely generated) simple differential $F$–algebra. This example motivates us to ask whether intermediate differential fields $K$ of arbitrary Picard-Vessiot extensions can be obtained as field of fractions of some finitely generated simple differential $F$–subalgebras of $K$? In Theorem 4.3, we shall answer this question affirmatively.

**Proposition 4.1.** Let $K$ be a finitely generated differential field extension of $F$. Then $K$ contains a finitely generated differential $F$–algebra whose field of fractions is $K$.

**Proof.** Let $y_1, \ldots, y_{n-1}$ be a transcendence base of $K$ over $F$ and $F(y_1, \ldots, y_{n-1})[y_1] = K$, for $y_i \in K$ algebraic over $F(y_1, \ldots, y_{n-1})$. For each $y_i$, we shall construct a finitely generated differential $F$–algebra $R_i$ whose field of fractions is $F(y_i) = F(y_i, y_i', \ldots)$. Then the smallest $F$–algebra $R$ containing $R_1, \ldots, R_t$ will be a finitely generated differential $F$–algebra whose field of fractions is $K$.

Let $y \in \{y_1, \ldots, y_{n-1}\}$. Consider the differential field $F(y)$. Let $n_1$ be the smallest integer so that $y, y', \ldots, y^{(n_1-1)}$ are algebraically independent over $F$ and that $y^{(n_1)}$ be algebraic over the subalgebra $F[y, y', \ldots, y^{(n_1-1)}]$ of $K$. Let

\[ P(X) := \sum_{i=0}^{m} a_i X^i \in F[y, y', \ldots, y^{(n_1-1)}][X] \]

be a minimal polynomial of $y^{(n_1)}$ with $a_m \neq 0$. Now $P(y^{(n_1)}) = 0$ implies

\[ \sum_{i=0}^{m} a_i (y^{(n_1)})^i + \left( \sum_{i=0}^{m} ia_i (y^{(n_1)})^{i-1} \right) y^{(n_1+1)} = 0. \]

From the minimality of $n_1$, we have $r := \sum_{i=0}^{m} ia_i (y^{(n_1)})^{i-1} \neq 0$ and therefore

\[ y^{(n_1+1)} = -\sum_{i=0}^{m} a_i (y^{(n_1)})^i \in F[y, y', \ldots, y^{(n_1)}][r^{-1}], \]

Since $y^{(n_1+1)} \in R_y := F[y, y', \ldots, y^{(n_1)}]$, it is clear for Equation 4.1 that $R_y$ contains all the derivatives of $y$. Also $(1/r)' = -r'/r^2 \in R_y$ and thus $R_y$ is a finitely generated differential $F$–algebra whose field of fractions is $F(y)$. \hfill \square

**Proposition 4.2.** Let $F$ be a differential field of characteristic zero with an algebraically closed field of constants. Let $E$ be a Picard-Vessiot extension of $F$ and $S$ be a differential $F$–subalgebra of $E$ such that $T(E|F) \subseteq S$. Then $S$ is a simple differential $F$–algebra.

**Proof.** Let $I$ be a nonzero differential ideal of $S$ and $0 \neq a \in I$. Then $a = f/g$ for $f, g \in T(E|F) \subseteq S$ and we have $ga = f \in I^* = T(E|F) \cap I$. Since the contraction ideal $I^*$ is a differential ideal and that $T(E|F)$ is a simple differential ring, we have $1 \in I^* \subseteq I$. This implies that $S$ is a simple differential ring. \hfill \square

**Theorem 4.3.** Let $F$ be a differential field of characteristic zero with an algebraically closed field of constants. Let $E$ be a Picard-Vessiot extension of $F$ and let $F \subseteq K \subseteq E$ be an intermediate differential field. Then $K$ contains a finitely generated simple differential $F$–algebra whose field of fractions is $K$.

**Proof.** Since Picard-Vessiot extensions are finitely generated field extensions, we apply Proposition 4.1 and obtain a finitely generated differential $F$–algebra $R$, whose field of fractions is $K$. Let
We first enlarge $T(E|F)$ to a finitely generated simple differential $F$–algebra $S$ so that $R$ becomes a subalgebra of $S$. To do so, let $S$ be the subring of $E$ generated by $T(E|F)$ and the set \( \{ \frac{1}{y_i} \mid 1 \leq i \leq n \} \). Since $T(E|F) \subseteq S$, from Proposition 4.2 it follows that $S$ is a finitely generated simple differential algebra.

Next, we shall find a suitable candidate for $r$. Let $E$ be an algebraic closure of $E$ and $\overline{E}$ be the algebraic closure of $F$ in $\overline{E}$. Note that $\overline{E}$ is an algebraically closed field. Let $R$ and $S$ be the rings generated by $R$ and $S$ over $\overline{E}$, respectively. Clearly $R \subseteq S$. $R \subseteq S$ and that $\overline{R}$ and $\overline{S}$ are integral extensions of $R$ and $S$ respectively. The domains $\overline{R}$ and $\overline{S}$ are finitely generated $\overline{E}$–algebras and therefore they are coordinate rings of some irreducible affine varieties $X$ and $Y$. Let $\phi : Y \rightarrow X$ be the morphism induced by the inclusion $\overline{R} \subseteq \overline{S}$. Then $\phi$ is dominant and therefore $\phi(Y)$ must contain an open set $U$ of $X$. Choose $f \in \overline{R}$ so that $X_f := \{ x \in X \mid f(x) \neq 0 \} \subseteq U$. Since $f$ must be integral over the domain $R$, there is a monic polynomial $P(X) = X^n + r_{n-1}X^{n-1} + \cdots + r \in R[X]$ such that $P(f) = 0$ and $r \neq 0$. Then $(f^{n-1} + r_{n-1}f^{n-2} + \cdots + r) = -r$ and we have $X_f \subseteq X_f \subseteq U \subseteq \phi(Y)$. Thus $\phi$ naturally restricts to a surjective morphism from $X_f$ to $X_f$. Observe that $\phi^{-1}(Y)$ is a non-empty subset of $X_f$. Hence there is a prime ideal $r$ of $\overline{R}[1/r]$. Since $X_f \subseteq \phi(Y)$, we obtain that $\phi^{-1}(\phi^{-1}(Y))$ is a non-empty subset of $Y_f$. Then $\phi^{-1}(\phi^{-1}(Y)) \subseteq Y_f$. Let $I'$ be the extension of $I$ in $\overline{S}[1/r]$ and therefore $I'$ is also a proper ideal of $\overline{S}[1/r]$.

Now we shall prove that $R[1/r]$ is a simple differential ring. Suppose that $a$ is a nonzero proper differential ideal of $R[1/r]$. Since every differential ideal is contained in a maximal differential ideal and maximal differential ideals are prime, we have a nonzero prime differential ideal $p$ containing $a$. But $\overline{R}[1/r]$ is an integral extension of $R[1/r]$ and therefore there is a prime ideal $q$ in $\overline{R}[1/r]$ such that $q \cap R[1/r] = p$. Let $p'$ be the extension ideal of $p$ in $\overline{R}[1/r]$. Clearly $p' \subseteq q$ and therefore $p'$ is a proper ideal of $\overline{R}[1/r]$. Let $b$ be the extension ideal of $p'$ in $\overline{S}[1/r]$. Then from our earlier observation, $b$ is a proper ideal. Since $p$ is a differential ideal, $b$ is also a differential ideal.

\[ \begin{align*}
R & := F[x_1, y_1, x_2, y_2, \ldots, x_n, y_n], \text{ where } x_i, y_i \in T(E|F). \text{ We shall find an element } r \in R \text{ so that } R[1/r] \text{ is a simple differential } F–\text{algebra and this would complete the proof.}
\end{align*} \]

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