A NEW FROBENIUS EXACT STRUCTURE ON THE CATEGORY OF COMPLEXES

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Abstract. Let (1) be an automorphism on an additive category \( \mathcal{B} \), and let \( \eta: (1) \to \text{Id}_\mathcal{B} \) be a natural transformation satisfying \( \eta_X(1) = \eta_X(1) \) for any object \( X \) in \( \mathcal{B} \). We construct a new Frobenius exact structure on the category of complexes in \( \mathcal{B} \), which is associated to the natural transformation \( \eta \). As a consequence, the category introduced in Definition 2.4 of [J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39(1989), 436–456] has a Frobenius exact structure.

1. INTRODUCTION

Let \( \mathcal{B} \) be an additive category and \( C(\mathcal{B}) \) the category of complexes in \( \mathcal{B} \). It is well known that \( C(\mathcal{B}) \) has a Frobenius exact structure, where the conflations are given by chainwise split short exact sequences of complexes. The corresponding stable category coincides with the homotopy category \( K(\mathcal{B}) \) of complexes, which is a triangulated category.

Suppose that the category \( \mathcal{B} \) has a certain automorphism endofunctor \( (1) \) and that \( \eta: (1) \to \text{Id}_\mathcal{B} \) is a natural transformation, which satisfies \( \eta_X(1) = \eta_X(1) \) for any object \( X \) in \( \mathcal{B} \). Associated to the natural transformation \( \eta \), we construct a new Frobenius exact structure on \( C(\mathcal{B}) \), whose conflations are given by certain chainwise split short exact sequences of complexes; see Theorem 3.7. Consequently, the corresponding stable category \( K_\eta(\mathcal{B}) \) is a triangulated category by [2, Theorem 1.2.8]. To describe the morphisms in \( K_\eta(\mathcal{B}) \), we define a new homotopy relation on chain maps. In particular, the stable category \( K_\eta(\mathcal{B}) \) is different from \( K(\mathcal{B}) \), and thus is a new triangulated category.

This work is inspired by the construction of the functor \( \Phi \) in [6, Proposition 2.10], where the functor \( \Phi \) induces a triangle functor in [6, Proposition 2.11] and then gives rise to the famous derived Morita theory. More precisely, for an additive category \( \mathcal{A} \), Rickard introduces an additive category \( G(\mathcal{A}) \) in [6, Definition 2.4] and then defines \( \Phi \) as the composition of an operation \( (-)^{**} \) from a category of complexes to \( G(\mathcal{A}) \) and a “total” functor from \( G(\mathcal{A}) \) to a homotopy category of complexes. However, the operation \( (-)^{**} \) is not a functor. Recall that a homotopy relation on morphisms in \( G(\mathcal{A}) \) is introduced in [6, Definition 2.8]. Even up to this homotopy relation, the operation \( (-)^{**} \) is still not a functor.

Theorem 3.7 gives rise to a new homotopy relation on morphisms in \( G(\mathcal{A}) \), which makes the operation \( (-)^{**} \) a triangle functor up to homotopy. Consequently, the
triangle functor in \[6, \text{Proposition 2.11}\] can be realized as the composition of two triangle functors; see Proposition 4.9.

Indeed, we observe that \(G(A)\) is isomorphic to \(C(B)\) for some additive category \(B\), which is endowed with a natural transformation \(\eta: (1) \rightarrow \text{Id}_B\); see Proposition 4.4. Then by Theorem 3.7, \(G(A)\) admits a Frobenius exact structure associated to \(\eta\); moreover, we obtain a new homotopy relation on morphisms in \(G(A)\). Using this new homotopy relation, the operation \((-)^{**}\) induces a triangle functor.

This paper is organized as follows. In Section 2, we recall some basic facts on Frobenius categories. Section 3 deals with the construction of the new Frobenius exact structure on the category of complexes; see Theorem 3.7. In Section 4, we show that the category \(G(A)\) is isomorphic to a category of complexes in an additive category \(B\) endowed with a natural transformation \(\eta: (1) \rightarrow \text{Id}_B\) and give the new homotopy relation on morphisms by Theorem 3.7. Finally, we realize the triangle functor in \[6, \text{Proposition 2.11}\] as the composition of two triangle functors in Theorem 4.9.

Throughout this paper, all functors are additive functors.

2. Preliminaries on Frobenius categories

In this section we recall some basic facts on exact categories and Frobenius categories; see \[5, \text{Appendix A}\] for details.

Let \(C\) be an additive category. An exact pair of morphisms is a sequence \(X \xrightarrow{i} Y \xrightarrow{p} Z\), which satisfies that \(i = \text{Ker} \ p\) and \(p = \text{Coker} \ i\). Two exact pairs \((i, p)\) and \((i', p')\) are said to be isomorphic provided that there exist isomorphisms \(f: X \rightarrow X'\), \(g: Y \rightarrow Y'\) and \(h: Z \rightarrow Z'\) such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \\
\downarrow f & & \downarrow g & & \downarrow h \\
X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z'.
\end{array}
\]

Recall that an exact structure on an additive category \(C\) is a chosen class \(E\) of exact pairs in \(C\), which is closed under isomorphisms and satisfies the following axioms (Ex0), (Ex1), (Ex1)\(^{\text{op}}\), (Ex2) and (Ex2)\(^{\text{op}}\). An exact pair \((i, p)\) in the class \(E\) is called a conflation, \(i\) is called an inflation and \(p\) is called a deflation. The pair \((C, E)\) (or \(C\) for short) is called an exact category. The axioms of exact categories are listed as follows:

- (Ex0) the identity morphism of the zero object is a deflation;
- (Ex1) the composition of two deflations is a deflation;
- (Ex1)\(^{\text{op}}\) the composition of two inflations is an inflation;
- (Ex2) for a deflation \(p: Y \rightarrow Z\) and a morphism \(f: Z' \rightarrow Z\), there exists a pullback diagram such that \(p'\) is a deflation:

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & Z' \\
\downarrow f & & \downarrow \text{Id} \\
Y & \xrightarrow{p} & Z.
\end{array}
\]
(Ex2)$^{pp}$ for an inflation $i : X \to Y$ and a morphism $f : X \to X'$, there exists a pushout diagram such that $i'$ is an inflation:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{f} & & \downarrow{f'} \\
X' & \xrightarrow{i'} & Y'.
\end{array}
\]

Let $P$ be an object in $\mathcal{C}$. If for any deflation $p : Y \to Z$ and any morphism $f : P \to Z$, there exists a morphism $f' : P \to Y$ such that $f = pf'$, then we call that $P$ is a projective object in the exact category $\mathcal{C}$. For any object $X$ in $\mathcal{C}$, if there exists a deflation $p : P \to X$ with $P$ projective, then we say that exact category $\mathcal{C}$ has enough projective objects. Dually, one can define injective objects and having enough injective objects for $\mathcal{C}$. An exact category $\mathcal{C}$ is said to be Frobenius provided that it has enough projective and enough injective objects, and the class of projective objects coincides with the class of injective objects. By a Frobenius exact structure on an additive category $\mathcal{C}$, we mean a chosen class $\mathcal{E}$ such that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category.

In what follows, we recall the stable category $\mathcal{L}$ of a Frobenius category $\mathcal{C}$. The objects of $\mathcal{L}$ are the same as the ones in $\mathcal{C}$, and the set of morphisms are defined as

$$\text{Hom}_{\mathcal{L}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)/I(X, Y),$$

where $I(X, Y) = \{ f \in \text{Hom}_{\mathcal{C}}(X, Y) \mid f \text{ factors through some injective object} \}$ is the subgroup of $\text{Hom}_{\mathcal{C}}(X, Y)$, and the composition is defined naturally. For any object $X$, we fix a conflation $X \xrightarrow{i_X} I(X) \xrightarrow{p_X} S(X)$, where $I(X)$ is an injective object. For any morphism $f : X \to Y$, there exists a morphism $S(f) : S(X) \to S(Y)$ such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & I(X) \\
\downarrow{f} & & \downarrow{p_X} \\
Y & \xrightarrow{i_Y} & I(Y) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & S(X) \\
& \xrightarrow{i_X} & \downarrow{S(f)} \\
& & S(Y)
\end{array}
\]

Here, the existence the middle morphism is deduced from the injectivity of $I(Y)$. Note that $S(f)$ is not unique, but it is uniquely determined by $f$ in the stable category $\mathcal{L}$. This gives rise to the suspension functor $S : \mathcal{L} \to \mathcal{L}$; it is an auto-equivalence.

By [2] Theorem I.2.8], the stable category $\mathcal{L}$ of a Frobenius category $\mathcal{C}$ is a triangulated category, where the translation functor is given by $S$. The triangles in $\mathcal{L}$ are induced by conflations.

A classical example of Frobenius categories is the category of complexes in an additive category. Let $\mathcal{B}$ be an additive category, and let $\mathcal{C}(\mathcal{B})$ be the category of complexes in $\mathcal{B}$. The shift functor $[1]$ on $\mathcal{C}(\mathcal{B})$ is defined as follows: for any complex $X^\bullet$ in $\mathcal{C}(\mathcal{B})$, the complex $X^\bullet[1]$ is given by $(X^\bullet[1])^n = X^{n+1}$ and $d_{X[1]}^n = -d_X^{n+1}$; for any chain map $f^\bullet : X^\bullet \to Y^\bullet$, the morphism $f^\bullet[1]$ satisfies $(f^\bullet[1])^n = f_{n+1}^\bullet$. Then $[1]$ is an automorphism of $\mathcal{C}(\mathcal{B})$. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism. The mapping cone $\text{Con}(f^\bullet)$ of $f^\bullet$ is a complex defined by $(\text{Con}(f^\bullet))^n = X^{n+1} \oplus Y^n$, and the differential $d_{\text{Con}(f^\bullet)}^n = \begin{pmatrix} -d_{X}^{n+1} & 0 \\ 0 & d_{Y}^{n+1} \end{pmatrix}$. The category $\mathcal{C}(\mathcal{B})$ has a well-known exact structure $\mathcal{E}$ such that $(\mathcal{C}(\mathcal{B}), \mathcal{E})$ is a Frobenius category, where $\mathcal{E}$ consists of all chainwise split short exact sequences of complexes.
For a chain map $f^\bullet : X^\bullet \to Y^\bullet$, the sequence

$$
Y^\bullet \xrightarrow{(1)} \text{Con}(f^\bullet) \xrightarrow{(1\ 0)} X^\bullet[1]
$$

is called a standard exact pair associated to $f^\bullet$. In fact, each exact pair in $\mathcal{E}$ is isomorphic to some standard exact pair. Suppose that $X^\bullet \xrightarrow{\eta} Y^\bullet \xrightarrow{p} Z^\bullet$ is a conflation. Up to isomorphism, we can assume that $Y^n = Z^n \oplus X^n$, and $i^n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $p^n = (1\ 0)$. It follows that $d^n_Y = \begin{pmatrix} d^n_Z \\ h^{n+1} \\ d^n_X \end{pmatrix}$, and $h^*: Z^*[1] \to X^\bullet$ is a chain map. Then the exact pair $(i^*, p^*)$ is isomorphic to the standard exact pair associated to $h^*$, and the class of the chain map $h^*$ is unique up to homotopy and the homotopy class of $h^*$ is called a homotopy invariant of $(i^*, p^*)$; see [3] Definition 4.7.

3. A new Frobenius exact structure on the category of complexes

In this section, we construct a new Frobenius exact structure on the category of complexes, and then obtain a new triangulated category.

Let us fix the following notation. Let $\mathcal{B}$ be an additive category and $\mathcal{C}(\mathcal{B})$ the category of complexes in $\mathcal{B}$. The category $\mathcal{C}(\mathcal{B})$ has a well-known exact structure $\mathcal{E}$ such that $(\mathcal{C}(\mathcal{B}), \mathcal{E})$ is a Frobenius category, where $\mathcal{E}$ consists of all chainwise split short exact sequences of complexes.

Suppose that $(1)$ is an automorphism endofunctor on $\mathcal{B}$ and $\eta: (1) \to \text{Id}_\mathcal{B}$ is a natural transformation satisfying $\eta_{X(1)} = \eta_X(1)$ for any object $X$ in $\mathcal{B}$, where $\text{Id}_\mathcal{B}$ is the identity functor on $\mathcal{B}$. The inverse functor of $(1)$ is denoted by $(1^-1)$.

The automorphism $(1)$ and the natural transformation $\eta$ induce the ones on $\mathcal{C}(\mathcal{B})$, which are still denoted by $(1)$ and $\eta$. Here are some elementary properties of the automorphism $(1)$: $\mathcal{C}(\mathcal{B}) \to C(\mathcal{B})$ and the natural transformation $\eta: (1) \to \text{Id}_{\mathcal{C}(\mathcal{B})}$.

**Lemma 3.1.** For any objects $X^\bullet$, $Y^\bullet$ and any morphism $f^\bullet: X^\bullet \to Y^\bullet$ in $\mathcal{C}(\mathcal{B})$, we have

(i) $\eta_{X(1)}[1] = [1(1)];$

(ii) $\eta_{X^\bullet(1)} = \eta_{X^\bullet}(1), \eta_{X^\bullet[1]} = \eta_{X^\bullet}[1];$

(iii) $\text{Con}(f^\bullet(1)) = \text{Con}(f^\bullet)(1);$

(iv) $\eta_{\text{Con}(f^\bullet)} = \begin{pmatrix} \eta_{X^\bullet[1]} & 0 \\ 0 & \eta_{Y^\bullet} \end{pmatrix}.$

**Proof.** We give the proof only for (iv); the others are straightforward.

Recall that for any morphism $f^\bullet: X^\bullet \to Y^\bullet$, $(\text{Con}(f^\bullet))^n = X^{n+1} \oplus Y^n$, and $d^n_{\text{Con}(f^\bullet)} = \begin{pmatrix} -d^n_X \oplus 1 \\ d^n_Y \end{pmatrix}$. By definition, $\eta_{\text{Con}(f^\bullet)}: \text{Con}(f^\bullet)(1) \to \text{Con}(f^\bullet)$ is a chain map such that each component $(\eta_{\text{Con}(f^\bullet)})^n: X^{n+1}(1) \oplus Y^n(1) \to X^{n+1} \oplus Y^n$ is given by $\begin{pmatrix} \eta_{X^{n+1}} & 0 \\ 0 & \eta_{Y^n} \end{pmatrix}$. This proves (iv).

**Definition 3.2.** (1) Suppose that $f^\bullet: X^\bullet \to Y^\bullet$ is a chain map. If $f^\bullet$ factors through $\eta_{Y^\bullet}: Y^\bullet(1) \to Y^\bullet$, then

$$
Y^\bullet \xrightarrow{(1)} \text{Con}(f^\bullet) \xrightarrow{(1\ 0)} X[1]^\bullet
$$

is called a standard $\eta$-conflation.
(2) We denote by \( \mathcal{E}_\eta \) the subclass of \( \mathcal{E} \) consisting of all exact pairs that are isomorphic to some standard \( \eta \)-conflation. An exact pair \((i^\ast, p^\ast)\) in \( \mathcal{E}_\eta \) is called an \( \eta \)-conflation, where \( i^\ast \) is called an \( \eta \)-inflation, and \( p^\ast \) is called an \( \eta \)-deflation.

**Remark 3.3.** The chain map \( f^\ast: X^\ast \to Y^\ast \) factors through \( \eta_{X^\ast(-1)} \) if and only if \( f^\ast \) factors through \( \eta_{X^\ast(-1)} \) since \( \eta: (1) \to \text{Id}_{C(\mathcal{B})} \) is a natural transformation.

**Lemma 3.4.** The complex category \( C(\mathcal{B}) \) together with the class \( \mathcal{E}_\eta \) is an exact category.

**Proof.** The axiom (Ex0) is trivial. To show (Ex1), we assume that \( Y^\ast \xrightarrow{(1,0)} Z^\ast \) and \( W^\ast \xrightarrow{(1,0)} Y^\ast \) are \( \eta \)-deflations, where \( Y^\ast = \text{Con}(f^\ast) \) with the homotopy invariant \( f^\ast: Z^\ast[-1] \to X^\ast \), and \( W^\ast = \text{Con}(g^\ast) \) with the homotopy invariant \( g^\ast = (g_1^\ast, g_2^\ast): Y^\ast[-1] \to V^\ast \). By simple calculation, we have that \( g_2^\ast: X^\ast[-1] \to V^\ast \) and \( (f_1^\ast, g_2^\ast): Z^\ast[-1] \to \text{Con}(g_2^\ast) \) are chain maps, and \( W = \text{Con}(f_1^\ast) \).

By definition, we assume that \( f_1^\ast = \eta_X^\ast \alpha^\ast \) and \( g_2^\ast = \eta_V^\ast \beta^\ast \), where \( \alpha^\ast: Z^\ast[-1] \to X^\ast(1) \) and \( \beta^\ast = (\beta_1^\ast, \beta_2^\ast): Y^\ast[-1] \to V^\ast(1) \) are chain maps. Observe that \( (\beta_1^\ast, \beta_2^\ast): Z^\ast[-1] \to \text{Con}(g_2^\ast)(1) \) is a chain map. By Lemma 3.3(iv), we have

\[
\left( f_1^\ast \right) = \left( \eta_X^\ast 0 \eta_Y^\ast \right) \left( \begin{array}{c} \alpha^\ast \\ \beta^\ast \end{array} \right) = \eta_{\text{Con}(g_2^\ast)} \left( \begin{array}{c} \alpha^\ast \\ \beta^\ast \end{array} \right).
\]

It follows that the chain map \( \left( f_1^\ast \right): Z^\ast[-1] \to \text{Con}(g_2^\ast) \) factors through \( \eta_{\text{Con}(g_2^\ast)} \). Therefore, the composition \( W^\ast \to Y^\ast \) and \( Y^\ast \to Z^\ast \) is an \( \eta \)-deflation and then (Ex1) holds.

To show (Ex2), for any chain map \( h^\ast: Z^\ast \to Z^\ast \), we have the following pullback diagram

\[
\begin{array}{ccc}
\text{Con}(f^\ast h^\ast[-1]) & \xrightarrow{(1,0)} & Z^\ast \\
(\begin{array}{c} h^\ast \\ 0 \end{array}) & \downarrow & \downarrow h^\ast \\
Y^\ast & \xrightarrow{(1,0)} & Z^\ast,
\end{array}
\]

where \( Y^\ast = \text{Con}(f^\ast) \) with the homotopy invariant \( f^\ast: Z^\ast[-1] \to X^\ast \). Since \( f^\ast h^\ast[-1] = \eta_X^\ast \alpha^\ast h^\ast[-1] \) factors through \( \eta_X^\ast \), we immediately have that (Ex2) holds.

Dually, (Ex1)\textsuperscript{op} and (Ex2)\textsuperscript{op} hold since the chain map \( f^\ast: X^\ast \to Y^\ast \) factors through \( \eta_Y^\ast \) if and only if \( f^\ast \) factors through \( \eta_{X^\ast(-1)} \). This finishes the proof. \( \square \)

**Lemma 3.5.** Let \( V^\ast \) be an object in \( C(\mathcal{B}) \). Then the mapping cone \( \text{Con}(\eta_V^\ast) \) is both projective and injective in \( (C(\mathcal{B}), \mathcal{E}_\eta) \).

**Proof.** By definition, we have

\[
(\text{Con}(\eta_V^\ast))^n = V^{n+1}(1) \oplus V^n, \text{ and } d^n_{\text{Con}(\eta_V^\ast)} = \begin{pmatrix} -d^{n+1} & (1) \\ \eta_{V^{n+1}}(1) & d^n \end{pmatrix}.
\]

Let \( Y^\ast \xrightarrow{(1)} Z^\ast \) be any \( \eta \)-deflation in \( \mathcal{E}_\eta \), where \( Y^\ast = \text{Con}(f^\ast) \) with the homotopy invariant \( f^\ast: Z^\ast[-1] \to X^\ast \). By definition, we assume that \( f^\ast = \eta_X^\ast \alpha^\ast \), where \( \alpha^\ast: Z^\ast[-1] \to X^\ast(1) \) is a chain map.
To show that $\text{Con}(\eta \nu \nu)$ is a projective object, it suffice to prove that $Y^\bullet \xrightarrow{(1)} Z^\bullet$ factors through any chain map $(g_1^\bullet, g_2^\bullet) : \text{Con}(\eta \nu \nu) \to Z^\bullet$.

Indeed, we have the following commutative diagram

$$
\begin{array}{ccc}
\text{Con}(\eta \nu \nu) & \xrightarrow{(g_1^\bullet, g_2^\bullet)} & Z^\bullet \\
Y^\bullet \xrightarrow{(1,0)} & \xrightarrow{\alpha^*(-1)g_1^\bullet[-1](-1)} & \xrightarrow{(g_1^\bullet, g_2^\bullet)} \\
& & \text{Y}^\bullet \\
\end{array}
$$

and it remains to check that $(g_1^\bullet, g_2^\bullet) : \text{Con}(\eta \nu \nu) \to Y^\bullet$ is a chain map. Since $(g_1^\bullet, g_2^\bullet)$ is a chain map, we have

\begin{equation}
\begin{aligned}
g_2^{n+1} \eta_{\nu^{n+1}} & = g_1^{n+1} d_{\nu}^{n+1}(1) + d_2^n g_2^n, \\
g_2^{n+1} d_{\nu}^{n} & = d_2^n g_2^n 
\end{aligned}
\end{equation}

for each $n \in \mathbb{Z}$. Notice that $g_2^n$ is a chain map, but $g_1^n$ may not be a chain map. By the following commutative diagram

$$
\begin{array}{ccc}
\text{Con}(\eta \nu \nu) & \xrightarrow{(g_1^\bullet, g_2^\bullet)} & Z^\bullet \\
\eta \nu \nu \xrightarrow{(0, \eta \nu \nu(-1))} & \xrightarrow{(g_1^\bullet, g_2^\bullet)} & \eta \nu \nu(-1) \\
\eta \nu \nu(-1) & \xrightarrow{(g_1^\bullet, g_2^\bullet)} & Z^\bullet(-1) \\
\end{array}
$$

we have $g_2^n(-1) \eta_{\nu^{n+1}} = \eta_{Z^n(-1)} g_2^n$. Therefore,

\begin{equation}
\begin{aligned}
\alpha^{n+1}(-1) g_1^n(-1) \eta_{\nu^{n+1}} & = \alpha^{n+1}(-1) \eta_{Z^n(-1)} g_1^n = \eta_{X^n \nu^n} \alpha^{n+1} g_1^n = f^{n+1} g_1^n,
\end{aligned}
\end{equation}

where the second equality is deduced from the naturality of $\eta$ since $\alpha^* : Z^*(-1) \to X^*(1)$ is a chain map. On the other hand, we have

\begin{equation}
\begin{aligned}
d_3^n \alpha^n(-1) g_1^{n-1} & + f^{n+1} g_2^n \\
= & \alpha^{n+1}(-1)(-d_{Z}^{n-1}(-1)) g_1^{n-1}(-1) + \eta_{X^n \nu^n} \alpha^{n+1} g_2^n \\
= & \alpha^{n+1}(-1)(-d_{Z}^{n-1}(-1)) g_1^{n-1}(-1) + \alpha^{n+1}(-1) \eta_{Z^n(-1)} g_2^n \\
= & \alpha^{n+1}(-1)(-d_{Z}^{n-1}(-1)) g_1^{n-1}(-1) + \alpha^{n+1}(-1) g_2^n(-1) \eta_{\nu^n(-1)} \\
= & \alpha^{n+1}(-1)(-d_{Z}^{n-1}(-1)) g_1^{n-1}(-1) + \alpha^{n+1}(-1) g_2^n(-1) \eta_{\nu^n(-1)} \\
= & \alpha^{n+1}(-1)(-d_{Z}^{n-1}(-1)) g_1^{n-1}(-1) + \alpha^{n+1}(-1) g_2^n(-1) \eta_{\nu^n(-1)} \\
\end{aligned}
\end{equation}

where the second and the third equalities are deduced from the naturality of $\eta$, and the fifth equality is a consequence of the equality (3.1). From the equalities (3.2), (3.3) and (3.4), it follows that $(g_1^\bullet, g_2^\bullet) : \text{Con}(\eta \nu \nu) \to Y^\bullet(1)$ is a chain map, and then $\text{Con}(\eta \nu \nu)$ is a projective object in $(C(B), E_\eta)$.

Let $X^\bullet \xrightarrow{(\gamma_1)} Y^\bullet$ be any $\eta$-inflation, where $Y^\bullet = \text{Con}(f^\bullet)$ with $f^\bullet : Z^\bullet(-1) \to X^\bullet$ factors through $\beta^\bullet : Z^\bullet(-1) \to X^\bullet(1)$. Dually, for any chain map $(g_1^\bullet, g_2^\bullet) : X^\bullet \to \text{Con}(\eta \nu \nu)$, we have the following commutative diagram of complexes

$$
\begin{array}{ccc}
X^\bullet & \xrightarrow{(\gamma_1)} & Y^\bullet \\
\xrightarrow{(g_1^\bullet, g_2^\bullet)} & \xrightarrow{\alpha^*(-1)g_1^\bullet[-1](-1)} & \xrightarrow{(g_1^\bullet, g_2^\bullet)} \\
\text{Con}(\eta \nu \nu). & & \\
\end{array}
$$
Consequently, $\text{Con}(\eta_\bullet \eta \bullet)$ is also an injective object.

\begin{proof}

Lemma 3.6. Let $X^\bullet \xrightarrow{\eta} Y^\bullet \xrightarrow{\eta} Z^\bullet$ be an exact pair in $\mathcal{E}$. Then the following statement are equivalent:

1. $(i^\bullet, p^\bullet)$ is an $\eta$-conflation;
2. there exists a homotopy invariant $h^\bullet : Z^\bullet[-1] \to X^\bullet$ of $(i^\bullet, p^\bullet)$ which factors through $\eta_{X^\bullet}$;
3. for any object $V^\bullet$,
   \[ \text{Hom}(\text{Con}(\eta_{V^\bullet}), Y^\bullet) \xrightarrow{i^\bullet} \text{Hom}(\text{Con}(\eta_{V^\bullet}), Y^\bullet) \xrightarrow{p^\bullet} \text{Hom}(\text{Con}(\eta_{V^\bullet}), Z^\bullet) \]
   is short exact;
4. for any object $V^\bullet$,
   \[ \text{Hom}(Y^\bullet, \text{Con}(\eta_{V^\bullet})) \xrightarrow{p^\bullet} \text{Hom}(Y^\bullet, \text{Con}(\eta_{V^\bullet})) \xrightarrow{i^\bullet} \text{Hom}(X^\bullet, \text{Con}(\eta_{V^\bullet})) \]
   is short exact.

The equivalence of (1) and (2) follows immediately from the definition of $\eta$-conflations. By the projectivity (resp. injectivity) of $\text{Con}(\eta_{V^\bullet})$ in the Frobenius category $(\mathcal{C}(\mathcal{B}), E_\eta)$ for any object $V^\bullet$, we know that (2) implies (3) (resp. (4)).

Suppose that $Y^n = Z^n \oplus X^n$ and the differential $d^n_Y = \begin{pmatrix} d^n_Z & 0 \\ h_{n+1} & d^n_X \end{pmatrix}$ with the homotopy invariant $h^\bullet : Z^\bullet[-1] \to X^\bullet$, and $i^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for each $n \in \mathbb{Z}$.

“(3) $\implies$ (2)” Take $V^\bullet = Z^\bullet[-1][-1]$, and we have that

\[ \text{Hom}(\text{Con}(\eta_{Z^\bullet[-1][-1]}), Y^\bullet) \xrightarrow{p^\bullet} \text{Hom}(\text{Con}(\eta_{Z^\bullet[-1][-1]}), Z^\bullet) \]

is an epimorphism of abelian groups. It follows that there exists a chain map \( \begin{pmatrix} 1 & 0 \\ \alpha \beta \end{pmatrix} : \text{Con}(\eta_{Z^\bullet[-1][-1]}) \to Y^\bullet \) such that the following diagram commutes

\[
\begin{array}{ccc}
Y^\bullet & \xrightarrow{1 \ 0} & \text{Con}(\eta_{Z^\bullet[-1][-1]}) \\
\alpha \beta & \xrightarrow{(1,0)} & Z^\bullet \\
\end{array}
\]

Since \( \begin{pmatrix} 1 & 0 \\ \alpha \beta \end{pmatrix} \) is a chain map, we have
\[
h^n + d^n_X \cdot \alpha^{n-1} = \alpha^n \cdot d^n_Z + \beta^n \cdot \eta_{Z^\bullet[-1][-1]}
\]

Set \( h^n = h^n + d^n_X \cdot \alpha^{n-1} + \alpha^n \cdot (-d^n_Z) \), which is homotopic to $h^\bullet : Z^\bullet[-1] \to X^\bullet$. Then we have $Y^\bullet = \text{Con}(\tilde{h}^\bullet)$ and $h^\bullet = \beta^\bullet \cdot \eta_{Z^\bullet[-1][-1]} \cdot \eta_{X^\bullet}$ factors through $\eta_{Z^\bullet[-1][-1]}$. It follows from the naturality of $\eta$ that $\tilde{h}^\bullet$ factor through $\eta_{X^\bullet}$.

“(4) $\implies$ (2)” can be proved dually and we omit the proof.

\end{proof}

Theorem 3.7. The exact category $(\mathcal{C}(\mathcal{B}), E_\eta)$ is a Frobenius category.

\begin{proof}

Let $X^\bullet$ be any object in $\mathcal{C}(\mathcal{B})$. Since $X^\bullet \xrightarrow{(1 \ 0)} \text{Con}(\eta_{X^\bullet})$ is an $\eta$-inflation, and $\text{Con}(\eta_{X^\bullet[-1][-1]}) \xrightarrow{(1 \ 0)} X^\bullet$ is an $\eta$-deflation, it follows from Lemma 3.5 that $(\mathcal{C}(\mathcal{B}), E_\eta)$ has enough projective and injective objects.
\end{proof}
On the other hand, an object $X^\bullet$ is injective if and only if $X^\bullet$ is a direct summand of $\text{Con}(\eta_{X^\bullet})$. By Lemma 3.5, $\text{Con}(\eta_{X^\bullet})$ is a projective object. It follows that $X^\bullet$ is projective. Dually, a projective object is also an injective object. □

We denote $K_\eta(B)$ by the stable category of the Frobenius category $(\mathcal{C}(B), \mathcal{E}_\eta)$. Recall that $K_\eta(B)$ has the same objects as $\mathcal{C}(B)$ and a morphism in $K_\eta(B)$ is the equivalence class $[f^\bullet]$ of $f^\bullet: X^\bullet \to Y^\bullet$ of $\mathcal{C}(B)$ modulo the subgroup of morphisms factoring through some injective object in $(\mathcal{C}(B), \mathcal{E}_\eta)$.

By [2, Theorem I.2.8], we know that $K_\eta(B)$ is a triangulated category and $(1)[1]$ is the corresponding translation functor. Each chain map $f^\bullet: X^\bullet \to Y^\bullet$ induces a standard triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{(1)} \text{Con}(\eta_{X^\bullet}) \xrightarrow{(0)} X^\bullet[1].$$

Next we give the homotopy relation on morphisms in $\mathcal{C}(B)$ given by the stable category $(\mathcal{C}(B), \mathcal{E}_\eta)$.

**Definition 3.8.** Let $X^\bullet$ and $Y^\bullet$ be complexes in $\mathcal{B}$. The chain maps $f^\bullet, g^\bullet: X^\bullet \to Y^\bullet$ are said to be $\eta$-homotopic, denoted by $f^\bullet \sim_\eta g^\bullet$, provided that there is a family of morphisms $\{s^n: X^n(1) \to Y^{n-1}\}_{n \in \mathbb{Z}}$ in $\mathcal{B}$ such that

$$(f^n - g^n) \eta_{X^n} = s^{n+1}(1) + d_V^{n-1} s^n$$

for each $n \in \mathbb{Z}$. A chain map $f^\bullet: X^\bullet \to Y^\bullet$ is said to be $\eta$-null-homotopic if $f^\bullet \sim_\eta 0$, where 0 is the zero chain map.

Let $f^\bullet, g^\bullet: X^\bullet \to Y^\bullet$ be morphisms in $\mathcal{C}(B)$, then $f^\bullet \sim_\eta g^\bullet$ if and only if $f^\bullet \eta_{X^\bullet} \sim_\eta g^\bullet \eta_{X^\bullet}$, where $\sim_\eta$ is the homotopy relation on morphisms given by the stable category $(\mathcal{C}(B), \mathcal{E})$.

**Proposition 3.9.** A chain map $f^\bullet: X^\bullet \to Y^\bullet$ is $\eta$-null-homotopic if and only if $\overline{f^\bullet} = 0$ in the stable category $K_\eta(B)$.

**Proof.** By definition, $\overline{f^\bullet} = 0$ in the stable category $K_\eta(B)$ if and only if $f^\bullet$ factors through $X^\bullet \xrightarrow{(1)} \text{Con}(\eta_{X^\bullet})$, or equivalently, there exists a map $s^\bullet: X^\bullet[1](1) \to Y^\bullet$ such that $(s^\bullet, f^\bullet): \text{Con}(\eta_{X^\bullet}) \to Y^\bullet$ is a chain map. Explicitly, there exists $s^n: X^n(1) \to Y^{n-1}$ such that

$$f^n \eta_{X^n-1} = s^{n+1} d_X^n(1) + d_V^{n-1} s^n$$

for any $n \in \mathbb{Z}$. □

**Example 3.10.** (1) If $\eta$ is a natural isomorphism, then $\mathcal{E}_\eta$ consists of all chainwise split short exact sequences of complexes, that is, $\mathcal{E}_\eta = \mathcal{E}$.

(2) If $\eta$ is a zero natural transformation, then $\mathcal{E}_\eta$ consists of all split short exact sequences of complexes and the corresponding stable category is trivial.

**Remark 3.11.** Let $(\mathcal{C}, \mathcal{E})$ be an exact category, with enough projective objects and enough injective objects. Suppose that $\mathcal{P}'$ and $\mathcal{I}'$ are two full subcategories of $\mathcal{C}$. In [1, Theorem 5.1] for detail. Chen proved that the pair $(\mathcal{C}, \mathcal{E}')$ is a Frobenius category, where $\mathcal{E}'$ is the class of left $\mathcal{P}'$-acyclic conflations provide that $\mathcal{P}'$ and $\mathcal{I}'$ satisfy some certain conditions.

Take $\mathcal{C} = \mathcal{C}(B)$ together with the Frobenius exact structure $\mathcal{E}$ given by chainwise split short exact sequences of complexes, and $\mathcal{P}' = \mathcal{I}'$ is the smallest full additive subcategory of $\mathcal{C}(B)$ which contains $\text{Con}(\eta_{X^\bullet})$ for all $X^\bullet \in \mathcal{C}(B)$ and is closed.
under direct summands and under isomorphisms. From Lemma 4.6 it immediately follows that the Frobenius category $(\mathcal{C}(\mathcal{B}), \mathcal{E}')$ coincides with $(\mathcal{C}(\mathcal{B}), \mathcal{E}_n)$.

**Remark 3.12.** Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$, closed under isomorphisms, direct sums, and direct summands, which is both preenveloping and precovering. Denoted by $\mathcal{T}_X$ the stable category of $\mathcal{T}$ associated to $\mathcal{X}$. In [4], Jørgensen proved that $\mathcal{T}_X$ is a pretriangulated category. Take $\mathcal{T} = K(\mathcal{B})$ and $\mathcal{X}$ is the smallest full additive subcategory of $\mathcal{K}(\mathcal{B})$ which contain $\text{Con}(\mathcal{H})$ for all $\mathcal{X} \in K(\mathcal{B})$ and is closed under taking direct summands. One can check that $K(\mathcal{B})_X$ is exactly a triangulated category and coincides with the triangulate category $K_\eta(B)$.

**Remark 3.13.** For any $m \geq 1$, we consider the composition of $m$ copies of the functor (1), denoted by $(m)$. Clearly, $\eta^m_X = \eta_X(1) \cdots \eta_X(m-1)$ gives a natural transformation from $(m)$ to $\text{Id}_{\mathcal{B}}$. Repeated application of Theorem 3.7 enable us to obtain the Frobenius exact structure $\mathcal{E}_{\eta^m}$ on $\mathcal{C}(\mathcal{B})$. Denote by $\mathcal{P}_{\eta^m}$ the full subcategory of projective objects in the Frobenius category $(\mathcal{C}(\mathcal{B}), \mathcal{E}_{\eta^m})$. Then we have the following observations:

$$\mathcal{E} \supset \mathcal{E}_\eta \supset \cdots \supset \mathcal{E}_{\eta^m} \supset \cdots,$$

and $\mathcal{P} \subset \mathcal{P}_\eta \subset \cdots \subset \mathcal{P}_{\eta^m} \subset \cdots$.

where $\mathcal{P}$ is the full subcategory of projective-injective objects in the Frobenius category $(\mathcal{C}(\mathcal{B}), \mathcal{E})$.

4. An application

In this section, we introduce a new category $\mathcal{Z}(+) \mathcal{A}$ for any additive category $\mathcal{A}$ endowed with an automorphism endofunctor (1) and a natural transformation $\eta: (1) \to \text{Id}$ satisfying $\eta_X(1) = \eta_X(1)$ for any object $X$ in $\mathcal{Z}(+) \mathcal{A}$. The category $\mathcal{C}(\mathcal{Z}(+) \mathcal{A})$ of complexes in $\mathcal{Z}(+) \mathcal{A}$ is isomorphic to the category $\mathcal{G}(\mathcal{A})$ introduced in [6] Definition 2.4. By Theorem 3.7, the categories $\mathcal{C}(\mathcal{Z}(+) \mathcal{A})$ and $\mathcal{G}(\mathcal{A})$ admit a new Frobenius exact structure and the corresponding stable categories are triangulated categories. Moreover, using the $\eta$-homotopy relation in Proposition 4.3, we will realize the functor in [6] Proposition 2.11 as the composition of two triangle functors.

4.1. The category $\mathcal{Z}(+) \mathcal{A}$.

**Definition 4.1.** Let $\mathcal{A}$ be an additive category. The category $\mathcal{Z}(+) \mathcal{A}$ is defined as follows:

1. its objects are $\mathcal{Z}$-graded objects $X^\bullet$ with $X^j$ in $\mathcal{A}$ for each $j \in \mathcal{Z}$,
2. a morphism from $X^\bullet$ to $Y^\bullet$ is a collection $f = \{f_0, f_1, \cdots, f_n, \cdots\}$ of homogeneous maps from $X^\bullet$ to $Y^\bullet$, where $f_n$ has degree $n$ with $f_n^j: X^j \to Y^{j+n}$ in $\mathcal{A}$ for all $n \geq 0, j \in \mathcal{Z}$,
3. the composition of morphisms $f: X^\bullet \to Y^\bullet$ and $g: Y^\bullet \to Z^\bullet$ is defined by

$$(gf)^n_j = g^n_n f^n_0 + g_{n-1}^{n+1} f^n_1 + \cdots + g_0^{n+n} f^n_n$$

for $n \geq 0, j \in \mathcal{Z}$. The identity map is $\{\text{Id}, 0, 0, \cdots\}$.

We consider the functor $(1): \mathcal{Z}(+) \mathcal{A} \to \mathcal{Z}(+) \mathcal{A}$ given by $(X^\bullet(1))^j = X^{j+1}$, and $(f^\bullet(1))^j_n = f^{j+1}_n$ for any object $X^\bullet$ and any morphism $f: X^\bullet \to Y^\bullet$ in $\mathcal{Z}(+) \mathcal{A}$. Clearly, the functor $(1)$ is an automorphism with the inverse given by $(X^\bullet(-1))^j = \cdots$.
$X_j^{-1}$, and $(f(-1))_j^n = f_j^{n-1}$ for any object $X^\bullet$ and any morphism $f: X^\bullet \rightarrow Y^\bullet$ in $\mathbb{Z}_+A$. There exists a natural transformation $\eta: (1) \rightarrow \text{Id}_{\mathbb{Z}_+A}$ given by

$$\eta_{X^\bullet} = \{0, \text{Id}, 0, 0, \cdots\}: X^\bullet(1) \rightarrow X^\bullet$$

for any object $X^\bullet$ in $\mathbb{Z}_+A$, where $\text{Id}: (X^\bullet(1))^j \rightarrow X^{j+1}$ ($j \in \mathbb{Z}$) is the identity map in $A$. It is easily seen that $\eta_{X^\bullet(1)} = \eta_{X^\bullet}(1)$ for any object $X^\bullet$ in $\mathbb{Z}_+A$. From Theorem 3.7 it follows that the category of complexes $C(\mathbb{Z}_+A)$ has a new Frobenius exact structure $\mathcal{E}_n$.

More precisely, a complex in the category $\mathbb{Z}_+A$ can be written as a bigraded object $X^{**}$ together with the differential $d^{**}$, where $X^{ij}$ is an object of $A$ and $d^{**}$ is a collection of morphisms $\{d_{n}^{ij}: X^{ij} \rightarrow X^{i+1,j+n}, n \geq 0, i, j \in \mathbb{Z}\}$ of $A$ satisfying

$$(d^{**})^{i,j}d_{n}^{ij} + d_{n}^{i+1,j+n-1}d^{ij} + \cdots + d_{n}^{i+1,j+n}d_{0}^{ij} = 0$$

for any $n \geq 0$ and $i, j \in \mathbb{Z}$. For simplicity of notation, we sometimes write a complex $(X^{**}, d^{**})$ instead of $X$. A morphism $f: X \rightarrow Y$ is a collection of morphisms $\{f^{ij}_{n}: X^{ij} \rightarrow Y^{i+1,j+n}, n \geq 0\}$ in $A$ satisfying

$$(f^{**})^{i,j}d_{n}^{ij} + f^{i+1,j+n-1}_{n-1}d^{ij} + \cdots + f^{i+1,j+n}_{n}d_{0}^{ij} = 0$$

for any $n \geq 0$ and $i, j \in \mathbb{Z}$. The composition $gf$ of two morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is given by

$$(gf)^{i,j}_{n} = g^{ij}_{n}f^{ij} + g^{i,j+1}_{n-1}f^{ij}_{1} + \cdots + g^{i,j+n}_{n}f^{ij}_{n}$$

for any $n \geq 0, i, j \in \mathbb{Z}$.

Let $X$ be an object and $f: X \rightarrow Y$ be a morphism in $C(\mathbb{Z}_+A)$. We observe that $(X^{**}, d^{**}_{X,0})$ is a complex in $A$ for any $j$, and $f^{**}_{0}: (X^{**}, d^{**}_{X,0}) \rightarrow (Y^{**}, d^{**}_{Y,0})$ is a chain map in $C(A)$.

By Definition 3.8 and Proposition 3.9 we immediately get the $\eta$-homotopy relation on $C(\mathbb{Z}_+A)$.

**Corollary 4.2.** Let $f = \{f_{0}, f_{1}, \cdots, f_{n}, \cdots\}: X \rightarrow Y$ be a chain map in $C(\mathbb{Z}_+A)$. Then $f$ vanishes in $K_{\eta}(\mathbb{Z}_+A)$ if and only if there exist $s^{ij}_{n}: X^{ij} \rightarrow Y^{i+1,j+n-1}$ such that

$$0 = s^{i+1,j}_{0}d^{ij}_{X,0} + s^{i,j-1}_{0}d^{ij}_{Y,0} + \sum_{p+q=n+1} s^{i+1,j+q}_{p}d^{ij}_{X,q} + d^{i+1,j+q-1}_{Y,p}d^{ij}_{0}$$

for any $i, j \in \mathbb{Z}, n \geq 0$.

Rickard introduced a category $G(A)$ in [6, Definition 2.4] for any additive category $A$, which plays an important role in the Rickard’s proof of derived Morita theory.

**Definition 4.3 (6).** Let $A$ be an additive category. The category $G(A)$ is defined as follows. The objects of $G(A)$ are systems $\{X^{**}, d_{n}: n = 0, 1, \cdots\}$, where $X^{**}$ is a bigraded object of $A$, and $d_{n}$ is a graded endomorphism of $X^{**}$ of degree $(1-n, n)$, satisfying

$$d_{0}d_{n} + d_{1}d_{n-1} + \cdots + d_{n}d_{0} = 0, (n \geq 0).$$

The morphisms from $\{X^{**}, d_{X,n}\}$ to $\{Y^{**}, d_{Y,n}\}$ are collections $\alpha = \{\alpha_{n}: n = 0\}$ of graded maps $X^{**} \rightarrow Y^{**}$, such that $\alpha_{n}$ has degree $(-n, n)$ and

$$\alpha_{n}d_{X,n} + \alpha_{1}d_{X,n-1} + \cdots + \alpha_{n}d_{X,0} = d_{Y,0}\alpha_{n} + d_{Y,1}d_{n-1} + \cdots + d_{Y,n}\alpha_{0},$$
for each \( n \). The composition of maps \( \beta = \{ \beta_n : n \geq 0 \} : X^{**} \to Y^{**} \) and \( \alpha = \{ \alpha_n : n \geq 0 \} : Y^{**} \to Z^{**} \) is defined by
\[
(\alpha \beta)_n = \alpha_0 \beta_n + \alpha_1 \beta_{n-1} + \cdots + \alpha_n \beta_0.
\]

**Proposition 4.4.** Let \( \mathcal{A} \) be an additive category. Then the categories \( G(\mathcal{A}) \) and \( C(\mathbb{Z}_+ \mathcal{A}) \) are isomorphic.

**Proof.** We define a functor \( \Psi : G(\mathcal{A}) \to C(\mathbb{Z}_+ \mathcal{A}) \) as follows: for any object \( X = \{ X^i, d^{i*} : i = 0, 1, \cdots \} \) in \( G(\mathcal{A}) \),
\[
(\Psi(X))_{ij} = X^{i-j},
\]
and for any morphism \( \alpha = \{ \alpha_n^* : n \geq 0 \} : X \to Y \),
\[
(\Psi(\alpha))_{ij} = \alpha_{i-n}^* : (\Psi(X))_{ij} \to (\Psi(Y))_{ij}.
\]

By direct calculation, we have that \( \Psi \) is an isomorphic functor with the inverse functor \( \Psi^{-1} : C(\mathbb{Z}_+ \mathcal{A}) \to G(\mathcal{A}) \) given by
\[
(\Psi^{-1}(X))_{ij} = X^{i+j},
\]
and for any morphism \( \beta = \{ \beta_n : n \geq 0 \} : X \to Y \),
\[
(\Psi^{-1}(\beta))_{ij} = \beta_{i-n} : (\Psi^{-1}(X))_{ij} \to (\Psi^{-1}(Y))_{ij}.
\]

By the above isomorphism and Theorem 3.2, we know that the category \( G(\mathcal{A}) \) admits a Frobenius exact structure and the corresponding stable category is a triangulated category.

**Remark 4.5.** Rickard studies a homotopy relation on \( G(\mathcal{A}) \), see [6, Definition 2.8] for more details. It coincides with the homotopy relation corresponding to the stable category of \( (C(\mathbb{Z}_+ \mathcal{A}), \mathcal{E}) \), rather than the \( n \)-homotopy relation introduced in Definition 3.8.

### 4.2. Two functors

We denote by \( \mathbb{Z}_+^b \mathcal{A} \) the full additive subcategory of \( \mathbb{Z}_+ \mathcal{A} \) consisting of all objects \( X^i \) with finitely many nonzero \( X^i \) in \( \mathcal{A} \). Clearly, \( \mathbb{Z}_+^b \mathcal{A} \) is closed under the action of the functor (1). Replacing \( \mathbb{Z}_+ \mathcal{A} \) by \( \mathbb{Z}_+^b \mathcal{A} \), we get a new Frobenius exact structure \( \mathcal{E}_q \) on the category \( C(\mathbb{Z}_+^b \mathcal{A}) \).

We consider the functor \( \Xi : \mathbb{Z}_+^b \mathcal{A} \to \mathcal{A} \) defined as follows. For any object \( X^\bullet \) and any morphism \( f = \{ f_0, f_1, \cdots \} : X^\bullet \to Y^\bullet \) in \( \mathbb{Z}_+^b \mathcal{A} \),
\[
\Xi(X^\bullet) = \bigoplus_j X^j,
\]
\[
\Xi(f) = \begin{pmatrix}
  f_0 \\
  f_1 & f_0 \\
  \vdots & \vdots & \ddots \\
  f_n & f_{n-1} & \cdots & f_0 \\
  \vdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]
Clearly, each column and each row in \( \Xi(f) \) have only finitely many nonzero entries for any \( f : X^\bullet \to Y^\bullet \) in \( \mathbb{Z}_+ \mathcal{A} \). Indeed, if the additive category \( \mathcal{A} \) admits infinite direct sums, then one can similarly define the functor \( \Xi : \mathbb{Z}_+ \mathcal{A} \to \mathcal{A} \).
Proposition 4.6. The functor Ξ induces a triangle functor from $K_\eta(\mathbb{Z}_+^b\mathcal{A})$ to $K(\mathcal{A})$.

Proof. By [2] Lemma I.2.8], it suffices to show that the functor $\Xi: C(\mathbb{Z}_+^b\mathcal{A}) \to C(\mathcal{A})$ is exact and preserves projective objects, where $C(\mathbb{Z}_+^b\mathcal{A})$ is the category of complexes in $\mathbb{Z}_+^b\mathcal{A}$ together with the Frobenius exact structure $\mathcal{E}_\eta$, and $C(\mathcal{A})$ is the complex category of $\mathcal{A}$ together with the Frobenius exact structure given by chainwise split short exact sequences of complexes.

Let $X^{**} \to Y^{**} \to Z^{**}$ be any $\eta$-conflation. Since it is also a conflation in $\mathcal{E}$, we have that $X^{i,**} \to Y^{i,**} \to Z^{i,**}$ is split in $\mathbb{Z}_+^b\mathcal{A}$ for any $i \in \mathbb{Z}$. By definition, $\Xi(X^{i,**}) \to \Xi(Y^{i,**}) \to \Xi(Z^{i,**})$ is split in $\mathcal{A}$, and then $\Xi(X^{**}) \to \Xi(Y^{**}) \to \Xi(Z^{**})$ is a conflation in $C(\mathcal{A})$.

For any object $V^{**}$ in $C(\mathbb{Z}_+^b\mathcal{A})$, $\text{Con}(\eta_{V^{**}})$ is a projective object in $(C(\mathbb{Z}_+^b\mathcal{A}), \mathcal{E}_\eta)$. Observe that $\Xi(\text{Con}(\eta_{V^{**}})) = \text{Con}(\text{Id}_{\Xi(\eta_{V^{**}})})$, which is a projective object in $C(\mathcal{A})$. For any projective object $P^{**}$ in $(C(\mathbb{Z}_+^b\mathcal{A}), \mathcal{E}_\eta)$, $P^{**}$ is a direct summand of some $\text{Con}(\eta_{V^{**}})$ and hence $\Xi(P^{**})$ is a direct summand of $\text{Con}(\text{Id}_{\Xi(\eta_{V^{**}})})$. It follows that $\Xi$ preserves projective objects. \qed

Recall that the construction of the functor $\Phi$ in [6] Proposition 2.11. Let $\Lambda$ be a ring with identity, and let $\text{Proj-}\Lambda$ be the category of projective modules and $\text{proj-}\Lambda$ be the category of finitely generated projective modules over $\Lambda$. For an additive category $\mathcal{A}$, we denote by $K(\mathcal{A})$ (resp. $K^b(\mathcal{A})$) the homotopy category of unbounded (resp. bounded) complexes in $\mathcal{A}$. In [6], Rickard denotes $G(\text{Proj-}\Lambda)$ by $G(\Lambda)$. We also use this notation later.

Recall that a tilting complex $T$ means an object of $K^b(\text{proj-}\Lambda)$ satisfying

(i) $\text{Hom}_{K^b(\text{proj-}\Lambda)}(T, T[n]) = 0$ for any $n \neq 0$;
(ii) add-$T$ generates $K^b(\text{proj-}\Lambda)$ as a triangulated category,

where add-$T$ is the full subcategory of $K^b(\text{proj-}\Lambda)$ consisting of all direct summands of finite direct sums of copies of $T$, see [6] Definition 6.5].

Let $X$ be a complex in $C(\text{Sum-T})$. To be precise, $X = (X^{**}, \delta_0^{**, \ast}, \delta^{**})$, where $X^{**}$ together with $\delta_0^{**, \ast} = (\delta_0^{ij} : X^{i,j} \to X^{i+1,j})_{i \in \mathbb{Z}}$ is an object in $\text{Sum-T}$ for each $j \in \mathbb{Z}$, and the differential $\delta^{ij} : X^{i,j} \to X^{i+1,j+1}$ satisfies $\delta_1 \delta_0 = \delta_0 \delta_1$ and $\delta_1 \delta_1 : X^{i,j} \to X^{i,j+2}$ is null-homotopic since $\text{Sum-T}$ is a full subcategory of $K(\text{Proj-}\Lambda)$.

We define the operation $\Theta : C(\text{Sum-T}) \to G(\Lambda)$ as follows: by Proposition 2.6 and Proposition 2.7 in [6] and the isomorphism in Proposition 1.4, for any object $X$ in $C(\text{Sum-T})$, there exists an object $(\Theta(X), d_n : n = 0, 1, \cdots)$ in $C(\mathbb{Z}_+\text{Proj-}\Lambda)$ such that $(\Theta(X))^{ij} = X^{i-j,j}$, $d_0^{ij} = \delta_0^{i-j,j}$ and $d_1^{ij} = (-1)^j \delta_1^{i-j,j}$ for any $i, j \in \mathbb{Z}$, and for any morphism $\alpha : X \to Y$ in $C(\text{Sum-T})$, there exists a morphism $\Theta(\alpha) = \{f_0, f_1, \cdots, f_n, \cdots\} : \Theta(X) \to \Theta(Y)$ such that $f_i^{ij} = \alpha^{i-j,j}$ for any $i, j \in \mathbb{Z}$.

By the $\eta$-homotopy relation introduced in Proposition 4.3, we show that the operation $\Theta$ induces a triangle functor from $K(\text{Sum-T})$ to $K_\eta(\mathbb{Z}_+\text{Proj-}\Lambda)$. For this, we need the following lemma.
Lemma 4.7. Let \( f = \{ f_0, f_1, \ldots, f_n, \ldots \} : X \to Y \) be a morphism in \( C(\mathbb{Z}_+, \text{Proj}-\Lambda) \), where \( X \) and \( Y \) are in the image of \( \Theta \). Then \( f \) is \( n \)-null homotopic if and only if there exist \( s_0 = \{ s_0^{(j)} : X^{i,j} \to Y^{i+1,j-1} \} \) and \( s_1 = \{ s_1^{(j)} : X^{i,j} \to Y^{i-1,j+1} \} \) such that

\[
0 = d_{Y,0}^{-1,j-1} s_0^{(j)} + s_0^{i+1,j} d_{X,0}^{(j)} - f_0 = d_{Y,1}^{-1,j-1} s_1^{(j)} + s_0^{i+1,j+1} d_{X,1}^{(j)} + s_1^{i+1,j} d_{X,0}^{(j)}.
\]

In particular, if \( f_0 = 0 \), then \( f \sim_0 0 \).

Proof. By the equality \( [4.30] \) the “only if” is obvious, and we use the induction to prove the “if” by Corollary 4.2. For convenience, we omit all superscripts. Suppose that there exist \( s_0 \) and \( s_1 \) such that

\[
0 = d_{Y,0} s_0 + s_0 d_{X,0},
\]

\[
f_0 = d_{Y,1} s_0 + d_{Y,0} s_1 + s_0 d_{X,1} + s_1 d_{X,0}.
\]

Proceeding by induction assume that there exists maps \( s_0, s_1, \ldots, s_k \) such that

\[
f_n = s_{n+1} d_{X,0} + s_n d_{X,1} + \cdots + s_0 d_{X,n+1} + d_{Y,n+1} s_0 + d_{Y,n} s_1 + \cdots + d_{Y,0} s_{n+1}
\]

for \( n < k \). By

\[
f_k d_{X,0} + f_{k-1} d_{X,1} + \cdots + f_0 d_{X,k} = d_{Y,k} f_0 + d_{Y,k-1} f_1 + \cdots + d_{Y,0} f_k,
\]

and the inductive hypothesis, we have

\[
d_{Y,0} (f_k - s_0 d_{X,k+1} - s_1 d_{X,k} - \cdots - s_k d_{X,1} - d_{Y,k+1} s_0 - d_{Y,k} s_1 - \cdots - d_{Y,1} s_k) = (-d_{Y,1} f_{k-1} - \cdots - d_{Y,k} f_0 + f_0 d_{X,k} + \cdots + f_0 d_{X,0}) + s_0 d_{X,0} d_{X,k+1}
\]

\[
+ (d_{Y,1} s_0 + s_0 d_{X,1} + s_1 d_{X,0} - f_0) d_{X,k} + \cdots
\]

\[
+ (d_{Y,1} s_1 + \cdots + d_{Y,k} s_0 + s_0 d_{X,k} + \cdots + s_k d_{X,0} - f_{k-1}) d_{X,1} + \cdots
\]

\[
+ (d_{Y,1} d_{Y,k} + d_{Y,2} d_{Y,k-1} + \cdots + d_{Y,k+1} d_{Y,0}) s_0 + \cdots
\]

\[
+ (d_{Y,1} d_{Y,1} + d_{Y,2} d_{Y,0}) s_{k-1} + d_{Y,1} d_{Y,0} s_k
\]

\[
= d_{Y,1} (-f_{k-1} - s_0 d_{X,k} + \cdots + s_k d_{X,1} + d_{Y,k} s_0 + \cdots + d_{Y,0} s_k)
\]

\[
+ f_{k-2} (s_0 d_{X,k-1} + \cdots + s_k d_{X,1} + d_{Y,k-1} s_0 + \cdots + d_{Y,0} s_{k-1}) + \cdots
\]

\[
+ f_0 (-f_0 + d_{Y,1} s_0 + d_{Y,0} s_1 + s_0 d_{X,1})
\]

\[
+ s_0 (d_{X,0} d_{X,k+1} + d_{X,1} d_{X,k} + \cdots + d_{X,k} d_{X,1}) + \cdots + s_k d_{X,0} d_{X,1} + f_0 d_{X,0}
\]

\[
= (f_k - s_0 d_{X,k+1} - s_1 d_{X,k} - \cdots - s_k d_{X,1} - d_{Y,k+1} s_0 - d_{Y,k} s_1 - \cdots - d_{Y,1} s_k) d_{X,0}.
\]

It follows that

\[
f_k - s_0 d_{X,k+1} - s_1 d_{X,k} - \cdots - s_k d_{X,1} - d_{Y,k+1} s_0 - d_{Y,k} s_1 - \cdots - d_{Y,1} s_k
\]

is a morphism in \( \text{Hom}_{K(\text{Proj}-\Lambda)}(X^{\bullet,n+j}, Y^{\bullet,j}) \). By the definition of tilting complexes, we have

\[
\text{Hom}_{K(\text{Proj}-\Lambda)}(X^{\bullet,n+j}, Y^{\bullet,j}) = 0.
\]

So there exists \( s_{k+1} \) such that

\[
f_k - (s_0 d_{X,k+1} + \cdots + s_k d_{X,1} + d_{Y,k+1} s_0 + \cdots + d_{Y,1} s_k) = d_{Y,0} s_{k+1} + s_{k+1} d_{X,0},
\]

and in consequence,

\[
f_k = s_0 d_{X,k+1} + s_1 d_{X,k} + \cdots + s_{k+1} d_{X,0} + d_{Y,k+1} s_0 + d_{Y,k} s_1 + \cdots + d_{Y,0} s_{k+1}.
\]

It follows from Corollary 4.2 that \( f \) vanishes in \( K_n(\mathbb{Z}_+, \text{Proj}-\Lambda) \). \( \square \)

Proposition 4.8. The operation \( \Theta \) induces a triangle functor from \( K(\text{Sum}-\Lambda) \) to \( K_n(\mathbb{Z}_+, \text{Proj}-\Lambda) \), still denoted by \( \Theta \).
Proof. We first prove that $\Theta: K(\text{Sum-}T) \to K_\eta(\mathbb{Z}_+\text{Proj-}\Lambda)$ is a functor. By the construction of $\Theta$, it suffices to show that $\Theta(\alpha)$ is $\eta$-null homotopy if $\alpha: X \to Y$ is null-homotopy in $C(\text{Sum-}T)$, where $X = (X^{**}, \delta_{X,0}^{**}, \delta_{X,1}^{**})$ and $Y = (Y^{**}, \delta_{Y,0}^{**}, \delta_{Y,1}^{**})$ are complexes in $\text{Sum-}T$.

Indeed, if $\alpha = 0$ in $K(\text{Sum-}T)$, then there exists a morphism $s_{ij}^0: X^{**} \to Y^{**}$ in $\text{Sum-}T$ such that $\alpha^{ij} = s_0^{ij} + \delta_{Y,0}^{ij} s_0^{ij}$ in $K(\text{Sum-}T)$ for each $j \in \mathbb{Z}$. It follows that

$$\delta_{ij}^{ij} - \delta_{ij}^{ij} s_0^{ij} + s_0^{ij+1} \delta_{ij}^{ij} = 0,$$

and there exists $s_{ij}^1: X^{ij} \to Y^{ij}$ such that

$$\alpha^{ij} = s_0^{ij+1} \delta_{ij}^{ij} + \delta_{ij}^{ij-1} s_0^{ij} + s_1^{ij+1} \delta_{ij}^{ij} + \delta_{ij}^{ij-1} s_1^{ij}$$

for any $i, j \in \mathbb{Z}$. By Lemma 4.7, we have that $\Theta(\alpha): \Theta(X) \to \Theta(Y)$ vanishes in $K_\eta(\mathbb{Z}_+\text{Proj-}\Lambda)$, and $\Theta: K(\text{Sum-}T) \to K_\eta(\mathbb{Z}_+\text{Proj-}\Lambda)$ is exactly a functor.

Suppose that $X \xrightarrow{f_0} \Theta (\{1\}) \xrightarrow{f_1} [1]$ is a standard triangle in $K(\text{Sum-}T)$. The mapping cone $\text{Con}(\alpha)$ is given by $\text{Con}(\alpha) = X^{ij+1} \oplus Y^{ij}$, and

$$\delta_{ij}^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}: X^{i,j+1} \oplus Y^{ij} \to X^{i,j+1} \oplus Y^{i+1,j},$$

or

$$\delta_{ij}^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}: X^{i,j+1} \oplus Y^{ij} \to X^{i+1,j} \oplus Y^{i+1,j+1}.$$
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can choose a morphism \( \{ \alpha_n : n \geq 0 \} : X^{**} \to Y^{**} \) in \( G(\Lambda) \) such that \( \alpha_0 = \alpha \).

Defines \( \Phi(X) \) as the “total complex” of \( X^{**} \) and \( \Phi(\alpha) \) as the map in \( K(\text{Proj-}\Lambda) \) obtained by applying the total complex functor to \( \{ \alpha_n : n \geq 0 \} \).

We observe that the operation \((-)^{**}\) from \( C(\text{Sum-T}) \) to \( G(\Lambda) \) is not a functor. Even up to the homotopy relation introduced in [6, Definition 2.8], the operation is still not a functor. By the isomorphism in Proposition 4.4, the above observation means that the operation \( \Theta \) cannot induce a functor from \( K(\text{Sum-T}) \) to \( K(\text{Proj-}\Lambda) \). The following result shows that the functor \( \Phi : K(\text{Sum-T}) \to K(\text{Proj-}\Lambda) \) can be realized as the composition of two triangle functors.

**Theorem 4.9.** The functor \( \Phi : K(\text{Sum-T}) \to K(\text{Proj-}\Lambda) \) is the composition of two functors \( \Theta : K(\text{Sum-T}) \to K_{\eta}(\mathbb{Z}+ \text{Proj-}\Lambda) \) and \( \Xi : K_{\eta}(\mathbb{Z}+ \text{Proj-}\Lambda) \to K(\text{Proj-}\Lambda) \).

**Proof.** By Definition, the functor \( \Theta : K(\text{Sum-T}) \to K_{\eta}(\mathbb{Z}+ \text{Proj-}\Lambda) \) is induced by the composition of the operation \((-)^{**} : C(\text{Sum-T}) \to G(\Lambda) \) and the isomorphism \( \Psi^{-1} : G(\Lambda) \to C(\mathbb{Z}+ \text{Proj-}\Lambda) \). On the other hand, taking \( \mathcal{A} = \text{Proj-}\Lambda \) and we have that the functor \( \Xi : K_{\eta}(\mathbb{Z}+ \text{Proj-}\Lambda) \to K(\text{Proj-}\Lambda) \) is induced by the composition of the isomorphism \( \Psi : C(\mathbb{Z}+ \text{Proj-}\Lambda) \to G(\Lambda) \) in the proof of Proposition 4.4 and the “total” functor from \( G(\Lambda) \) to \( C(\text{Proj-}\Lambda) \) introduced in [6, Section 2, Remark]. By the construction of the functor \( \Phi : K(\text{Sum-T}) \to K(\text{Proj-}\Lambda) \), we immediately have that \( \Phi = \Xi \circ \Theta \).

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