ANALYTIC PARTIAL CROSSED PRODUCTS

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ABSTRACT. Partial actions of discrete abelian groups can be used to construct both groupoid C*-algebras and partial crossed product algebras. In each case there is a natural notion of an analytic subalgebra. We show that for countable subgroups of $\mathbb{R}$ and free partial actions, these constructions yield the same C*-algebras and the same analytic subalgebras.

We also show that under suitable hypotheses an analytic partial crossed product preserves all the information in the dynamical system in the sense that two analytic partial crossed products are isomorphic as Banach algebras if, and only if, the partial actions are conjugate.

1. Introduction

One way to obtain interesting non-self-adjoint subalgebras of C*-algebras is to restrict a crossed product to the subalgebra associated with a positive cone in the group. A significant limitation to this approach arises from the fact that very important C*-algebra contexts, for example, AF C*-algebras, do not appear as crossed products. In [8], Exel found a new construction, the partial crossed product, that gives many of these contexts, by using partial actions. Exel considered partial actions by $\mathbb{Z}$, the integers; in [11] McClanahan extended these ideas to discrete groups.

Finite dimensional C*-algebras and AF C*-algebras can be realized as partial actions of abelian ordered groups on an abelian C*-algebra (or, equivalently, by a partial action on the spectrum of the abelian C*-algebra). This allows us to specify “analytic” subalgebras of the partial crossed product by restricting to the subalgebra generated by the positive cone in the group. To indicate the range of algebras that may be obtained by this construction, we realize Power’s toroidal limit algebras [20] as analytic partial crossed products (Example 4.5). We should point out that the C*-envelopes of these algebras are the Bunce-Deddens C*-algebras.

AF C*-algebras are all groupoid C*-algebras; as such they also possess analytic subalgebras which are defined in terms of cocycles on the groupoid. This raises the question: how do analytic partial crossed products and analytic subalgebras of groupoid C*-algebras fit together? One of the main results in this paper, Theorem 5.1 addresses this issue: free partial actions of countable discrete subgroups of $\mathbb{R}$ on separable abelian C*-algebras yield the same C*-algebras as do locally compact r-discrete principal groupoids with second countable unit space and a locally constant cocycle. Most importantly, the analytic subalgebras in the two constructions coincide. From a slightly different perspective, partial actions lead to two constructs: a partial crossed product and a groupoid C*-algebra, each with a natural analytic subalgebra. Theorem 5.1 says that these two constructs, together with the analytic subalgebras, coincide. This makes available the theory of partial crossed products for the

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study of many non-self-adjoint algebras which have been primarily investigated via groupoid techniques.

One of the motivations for the study of analytic crossed products is that, unlike C*-algebra crossed products, they preserve the information contained in the dynamical systems which determine the crossed product. Results of this type can be found in [12, 2, 13, 12, 21].

In Section 6 we extend to partial actions of $\mathbb{Z}^+$ a theorem of Power [21] for analytic crossed products. Our result states that if $\alpha$ and $\beta$ are free partial actions of $\mathbb{Z}^+$ with the property that the domains of $\alpha_n$ and $\beta_n$ shrink as $n$ increases in $\mathbb{Z}^+$, then $C_0(X) \times_\alpha \mathbb{Z}^+$ and $C_0(X) \times_\beta \mathbb{Z}^+$ are isomorphic as Banach algebras if, and only if, the partial actions $\alpha$ and $\beta$ are conjugate in a natural sense.

While we are primarily interested in analytic crossed products of the form $C_0(X) \times_\alpha \Sigma$, where $\Sigma$ is a positive cone in a group $G$, these are defined in terms of the C*-algebra partial crossed product $C_0(X) \times_\alpha G$. As a first step in the direction of developing a theory which does not depend on the whole group and the C*-algebra, we modify the definition of partial action to a form suitable for a positive cone in a group. If our axioms for a partial action of $\Sigma$ are applied to a group, they simply form a redundant version of McClanahan’s definition. We show by example that these conditions are not redundant when applied to a cone $\Sigma$.

The question of the extension of a partial action from $\Sigma$ to the group $G$ arises immediately. This is solved easily when $\Sigma$ totally orders $G$. On the other hand, we give an example of a partial action of $(\mathbb{Z}^3)^+$ on a seven point space (or, equivalently, on a seven dimensional abelian C*-algebra) which has no extension to $\mathbb{Z}^3$. This material is presented in Section 3.

Theorem 5.1 in Section 5 makes substantial use of groupoids. For treatises with extensive treatment of groupoids, we direct the reader to [22, 14]. A helpful brief introduction can be found in [12]. For the convenience of the reader, we record a few definitions here. A groupoid is a set $\mathcal{G}$ with a partially defined multiplication and an inversion. If $a$ and $b$ can be multiplied, then $(a, b)$ is called a composable pair; $\mathcal{G}^2$ denotes the set of all composable pairs. The multiplication satisfies an associative law and inversion satisfies $(a^{-1})^{-1} = a$. Elements of the form $a^{-1}a$ and $aa^{-1}$ are called units. Units act as left and right identities when multiplied by elements with which they are composable.

We shall only be interested in principal groupoids; these are all equivalence relations. If $\mathcal{G}$ is an equivalence relation on a set $X$, then $(x, y)$ and $(w, z)$ are composable if, and only if, $y = w$. In this case, $(x, y)(y, z) = (x, z)$. Also $(x, y)^{-1} = (y, x)$, always. Groupoids have range and domain maps defined by $r(a) = aa^{-1}$ and $d(a) = a^{-1}a$. In the principal groupoid context, this gives $r(x, y) = (x, x)$ and $d(x, y) = (y, y)$. Since we can identify the diagonal $\{(x, x) \mid x \in X\}$ with $X$ in a natural way, we can view $r$ and $d$ as maps from $\mathcal{G}$ onto $X$; they are, in fact, just the coordinate projections. A subset $E \subseteq \mathcal{G}$ is said to be a $\mathcal{G}$-set if $r$ and $d$ are both one-to-one on $E$. In this case, $E$ is just the graph of a function from a subset of $X$ to some other subset.

All the groupoids which we consider will carry a locally compact topology; the groupoid operations will, of course, be continuous with respect to this topology. An $r$-discrete groupoid is one in which the set of units is open. In our context (principal groupoids) this means that the diagonal $\{(x, x) \mid x \in X\}$ is an open subset of $\mathcal{G}$. It will, in fact, be the case that the open $\mathcal{G}$-sets form a base for the topology. One consequence of note is that in $r$-discrete principal groupoids, each equivalence class is countable. Finally, a cocycle on $\mathcal{G}$ is a continuous map $c: \mathcal{G} \to \mathbb{R}$ such that $c(x, z) = c(x, y) + c(y, z)$, for all composable pairs $(x, y)$ and $(y, z)$ in $\mathcal{G}$.
2. Partial Actions

The following definition is an extension, suitable for use in the context of non-selfadjoint algebras, of the definition of a partial action of an abelian group on a C*-algebra. For the usual definition, see [8, 11].

2.1. Definition. Let $A$ be a C*-algebra, let $(G, +)$ be a discrete abelian group, and let $\Sigma$ be a subset of $G$ for which $\Sigma + \Sigma \subset \Sigma$. A partial action of $\Sigma$ on $A$ is a family of isomorphisms $\alpha = \{\alpha_t : D_{-t} \to D_t \ | \ t \in \Sigma\}$ between closed two-sided ideals of $A$ so that

1a: $\alpha_s(D_{-s} \cap D_t) = D_s \cap D_{s+t}$, if $s, t \in \Sigma$,

1b: $\alpha_t(D_{-t} \cap D_{-s-t}) = D_t \cap D_{-s}$, if $s, t \in \Sigma$,

2: $\alpha_{s+t}(x) = \alpha_s \circ \alpha_t(x)$, if $x \in D_{-t} \cap D_{-s-t}$, and

3: $D_0 = A$ and $\alpha_0 = \text{Id}_A$.

We will show below that Conditions 1a, 1b, and 2 are equivalent to a property which is usually much easier to work with. We shall refer to this property as the third arrow property.

2.2. Third Arrow Property. Any two of the following statements implies the third.

I: $x \in D_{-t}$ and $y = \alpha_t(x)$.

II: $y \in D_{-s}$ and $z = \alpha_s(y)$.

III: $x \in D_{-s-t}$ and $z = \alpha_{s+t}(x)$.

This can be indicated schematically by saying that if two of the arrows in the following diagram exist, then so does the third arrow.

\[
\begin{array}{ccc}
  x & \xrightarrow{\alpha_{s+t}} & z \\
  \alpha_t & & \alpha_s \\
  y & \xleftarrow{\alpha_s} & \\
\end{array}
\]

Another way of saying this is that in the order on the orbit of a point induced by the partial action, any two points are comparable. (The order is a total order.)

One consequence of the third arrow property (of I and II implies III, to be specific) is that for all $s, t \in \Sigma$, $\alpha_s \circ \alpha_t$ is a restriction of $\alpha_{s+t}$.

Moreover, when $\Sigma$ is the whole group, either of conditions 1a or 1b is also equivalent to another condition, which is weaker in the general case.

1′: $\alpha_s(D_{-s} \cap D_t) \subset D_{s+t}$, if $s, t \in \Sigma$.

Condition 1′ says exactly that statements I and II in the third arrow property imply statement III. In the case when $\Sigma$ is a group, all arrows can be reversed; so each of the three implications in the third arrow property implies the other two. Conditions 1′, 2 and 3 are exactly the definition of partial action of a group, as given by McClanahan in [11].

Proof of Equivalence of the Third Arrow Property. First, assume that the three conditions in the definition hold. We only show that Conditions II and III imply I; similar arguments apply to the other implications. Schematically

\[
\begin{array}{ccc}
  x & \xrightarrow{\alpha_{s+t}} & z \\
  \alpha_t & & \alpha_s \\
  y & \xleftarrow{\alpha_s} & \\
\end{array}
\]

Since $z \in D_s \cap D_{s+t}$, condition 1a implies that there exists $y' \in D_{-s} \cap D_t$ such that $\alpha_s(y') = z$. But $y \in D_{-s}$, $\alpha_s(y) = z$, and $\alpha_{-s}$ is injective on $D_{-s}$; this yields $y' = y$. Now we know that
$y \in D_t \cap D_{-s}$. Condition 1b gives the existence of $x' \in D_{-t} \cap D_{-s-t}$ such that $\alpha_t(x') = y$. By condition 2, $z = \alpha_s(y) = \alpha_s(\alpha_t(x')) = \alpha_{s+t}(x')$. Thus, $x, x' \in D_{-s-t}$ and $\alpha_{s+t}(x) = \alpha_{s+t}(x')$. Since $\alpha_{s+t}$ is injective on $D_{-s-t}$, $x = x'$. This yields $\alpha_t(x) = y$ and condition II.

For the converse, assume that the third arrow property holds. We first verify Condition 1b; the argument for Condition 1a is similar. Assume that $x \in D_{-t} \cap D_{-s-t}$. Then there exist $y \in D_t$ such that $\alpha_t(x) = y$ and $z \in D_{s+t}$ such that $\alpha_{s+t}(x) = y$. This is I and III, so II holds and $y \in D_s$. This gives $\alpha_t(D_{-t} \cap D_{-s-t}) \subseteq D_t \cap D_{-s}$.

Now let $y \in D_t \cap D_{-s}$. Then there exist $x \in D_{-t}$ such that $\alpha_t(x) = y$ and $z \in D_s$ such that $\alpha_s(y) = z$. This is I and II; III now gives $x \in D_{-s-t}$ and $y = \alpha_t(x) \in \alpha_t(D_{-t} \cap D_{-s-t})$. Thus $D_t \cap D_{-s} \subseteq \alpha_t(D_{-t} \cap D_{-s-t})$ and condition 1b is verified.

Condition 2 follows in a similar way from the statement that I and III imply II. □

Most of the time, we shall assume that $\Sigma \subset G$ is a cone, i.e., $\Sigma + \Sigma \subset \Sigma$ and $\Sigma - \Sigma = \{0\}$. We will be most interested in the context $A = C_0(X)$, where $X$ is a locally compact Hausdorff space and $G$ is a discrete subgroup of $(\mathbb{R}, +)$ with $\Sigma = G \cap [0, +\infty)$.

Partial actions on a locally compact Hausdorff space $X$ are defined in the same way as partial actions on $C^*$-algebras, except that the ideals are replaced by open subsets of $X$ and the *-isomorphisms between ideals are replaced by homeomorphisms between open sets. Often, it is convenient to "move" a partial action from $C_0(X)$ to $X$. This is routine, but here is a sketch of the procedure.

If $D_t$ is an ideal in $C_0(X)$, then there is a closed subset $C_t$ of $X$ such that $D_t = \{ f \in C_0(X) \mid f \equiv 0 \text{ on } C_t \}$. Then $X_t = X \setminus C_t$ is an open subset of $X$ and $D_t \cong C_0(X_t)$. With this notation, given an isomorphism $\alpha_t : C_0(X_{-t}) \to C_0(X_t)$, there is a homeomorphism $\beta_t : X_{-t} \to X_t$ such that, for all $f \in C_0(X_{-t})$, $\alpha_t(f) = \beta_t^{-1} f$. It is straightforward to check that $\alpha$ is a partial action on $C_0(X)$ if, and only if, $\beta$ is a partial action on $X$.

The first two examples below are closely related to the standard and refinement triangular subalgebras of UHF $C^*$-algebras.

2.3. Example. Let $X$ be the Cantor set $\prod_{i=1}^{\infty} \{0, 1\}$ and let $\alpha_1$ be the partial map on $C(X)$ induced by the odometer map $\beta$: if $x \in X$ is not $(1, 1, \ldots)$, then $\beta(x)$ is given by adding 1 to the first coordinate of $x$, with carries to the right. Then $\alpha_n$ is defined to be $(\alpha_1)^n$ on the domain where this makes sense. This $\alpha$ defines a partial action of $\Sigma = \mathbb{Z}^+$ on $C(X)$.

2.4. Example. Let $G$ be the dyadic rationals $\{ k/2^n \mid k \in \mathbb{Z}, n \in \mathbb{N} \}$ and let $\Sigma = G \cap [0, +\infty)$. Let $X = \prod_{i=1}^{\infty} \{0, 1\}$, the Cantor set, where we associate elements of $X$ with base 2 representations of numbers in $[0, 1]$. Note that dyadic rationals in $[0, 1]$ have two expansions so, for example, $1/2$ becomes two numbers: $1/2^+ = .1000\ldots$ and $1/2^- = .0111\ldots$. For $s \in \Sigma$ we define $\beta_s(x)$ to be $x + s$, provided $x + s$ is in $[0, 1]$. Thus, $\beta_{1/2}$ is defined for $\{(x_i) \in X \mid x_1 = 0\}$ and sends $(0, x_2, x_3, \ldots)$ to $(1, x_2, \ldots)$. Then $\beta$ is a partial action of $\Sigma$ on $X$.

2.5. Example. Neither of conditions 1a and 1b implies the other. This is illustrated by modifications of a simpler version of Example 2.4. The modifications are not partial actions, of course.

Let $X = (0, 1)$ and $\Sigma = [0, \infty) \subseteq \mathbb{R}$. For each $t \in \Sigma$, let

$$X_{-t} = \begin{cases} (0, 1 - t), & \text{if } t < 1 \\ \emptyset, & \text{if } t \geq 1 \end{cases}$$
Define \( \beta_t : X_{-t} \to X_t \) by \( \beta_t(x) = x + t \). It is easy to see that \( \beta \) satisfies the third arrow property and Condition 3. If countable groups are desired, restrict \( \alpha \) to the intersection of \( \Sigma \) with any countable, dense subgroup of \( \mathbb{R} \).

We now modify \( \beta \) to obtain a system of partial homeomorphisms of \( X \) which satisfies condition 1a but not condition 1b. We take \( \gamma_0 = id_X (= \beta_0) \) and, for \( t \geq 1 \), \( \gamma_t = \emptyset (= \beta_t) \).

For \( 0 < t < 1 \), the values of interest, define \( \gamma \) as follows:

\[
\begin{align*}
\text{for } 0 < t \leq 1/2, & \quad \gamma_t \text{ is } \beta_t \text{ restricted to } (0, 1/2) = D_{-t} \setminus [1/2, 1 - t], \\
\text{for } 1/2 \leq t < 1, & \quad \gamma_t = \beta_t.
\end{align*}
\]

That is, \( \gamma \) is obtained from \( \beta \) by taking \( G = \{(x, y) \mid 0 < x \leq y < 1\} \), the union of the graphs of the \( \beta_t \) and deleting the subset \( \{(x, y) \in G \mid x \geq 1/2 \text{ and } x \neq y\} \).

This example satisfies I and II implies III and also II and III implies I of the third arrow property; it does not satisfy I and III implies II. To be specific, take \( x = 3/8, y = 5/8, z = 6/8, t = 2/8 \) and \( s = 1/8 \). Then

\[
\begin{align*}
\gamma_t(x) &= \gamma_{2/8}(3/8) = 5/8 = y \quad \text{and} \\
\gamma_{s+t}(x) &= \gamma_{3/8}(3/8) = 6/8 = z,
\end{align*}
\]

but \( y = 5/8 \) is not in the domain of \( \gamma_s = \gamma_{1/8} \), so we fail to have \( \gamma_s(y) = z \).

In terms of the conditions in the definition of a partial action, \( \beta \) satisfies 1a but not 1b. If instead, we define \( \gamma \) on the essential values by

\[
\begin{align*}
\text{for } 0 < t \leq 1/2, & \quad \gamma_t \text{ is } \beta_t \text{ restricted to } (1/2 - t, 1 - t) = D_{-t} \setminus (0, 1/2 - t], \\
\text{for } 1/2 \leq t < 1, & \quad \gamma_t = \beta_t,
\end{align*}
\]

then we obtain an example in which I and II implies III and I and III implies II but II and III do not imply I. (Condition 1b is satisfied but Condition 1a is not.) This example is obtained by deleting \( \{(x, y) \in G \mid y \leq 1/2 \text{ and } x \neq y\} \) from \( G \).

Finally, deletion of \( \{(x, y) \in G \mid y \leq 1/2 \text{ or } x \geq 1/2 \text{ and } x \neq y\} \) yields an example in which the only implication in the third arrow property to hold is I and II implies III. Conditions 1a and 1b in the definition both fail.

2.6. Definition. We say that a partial action is non-degenerate if the group generated by \( \{s \in \Sigma \mid D_s \neq \emptyset\} \) is \( G \). By replacing \( G \) with the subgroup generated by this set, we may always assume a partial action is non-degenerate.

2.7. Definition. A partial action satisfies the composition property if \( \alpha_{s+t} = \alpha_s \circ \alpha_t \) for all \( s, t \in \Sigma \). The partial action satisfies the domain ordering property if \( D_{-s-t} \subseteq D_{-t} \), for all \( s, t \in \Sigma \).

Remark. These two definitions are, in fact, equivalent. It is trivial to see that the composition property implies the domain ordering property. Now assume that the domain ordering condition holds. We have already seen that whenever \( \alpha \) is a partial action of \( \Sigma \), \( \alpha_s \circ \alpha_t \) is a restriction of \( \alpha_{s+t} \), so it remains only to show that \( \text{dom} \alpha_{s+t} \subseteq \text{dom} \alpha_s \circ \alpha_t \). Let \( x \in D_{-s-t} \). By the domain ordering property, we also have \( x \in D_{-t} \). Letting \( z = \alpha_{s+t}(x) \) and \( y = \alpha_t(x) \),
the third arrow property (I and III implies II) tells us that $y \in \mathcal{D}_{-s}$ and $z = \alpha_s(y)$. In particular, $x \in \text{dom} \alpha_s \circ \alpha_t$.

The next example gives a partial action which fails to satisfy these two equivalent conditions.

2.8. Example. Let $X$ be an arc in the unit circle; i.e., $X = \{e^{it} | a < t < b\}$, for some values of $a$ and $b$. Define a partial action of $\mathbb{Z}$ on $X$ as follows: for each $n$, let $X_{-n} = \{x \in X | e^{in}x \in X\}$ and let $\alpha_n(x) = e^{in}x$, for $x \in X_{-n}$. Let $\Sigma = \mathbb{Z}^+$. The restriction of $\alpha$ to $\Sigma$ is a partial action (the third arrow property is particularly easy to verify) which need not satisfy the composition condition. In particular, if $a = 0$ and $b = 1$ then $X_{-3} = \emptyset$ while $X_{-7} = \{e^{it} | 0 < t < 2\pi - 6 \approx 2.8\}$. Hence, $\alpha_4 \circ \alpha_3$ is a proper restriction of $\alpha_7$.

2.9. Example. Consider $X = \mathbb{R}$ and $\Sigma = \mathbb{Z}^+$. For $t$ odd, define $X_t$ to be $\bigcup_{a \in \mathbb{Z}}[a, a + 1/2]$ and $\beta_t$ to be translation by $t$. For $t$ even, define $X_t = \mathbb{R}$ and $\beta_t$ to be translation by $t$. Then the action is non-degenerate but $X_5 \subset X_6$. We can write this as a ‘direct sum’ of a non-degenerate action on $X_1$ and a degenerate action on $\mathbb{R} \setminus X_1$.

This example can be generalized by considering $\mathbb{N} \times [0, 1]$, $2\mathbb{N} \times [0, 1]$, $4\mathbb{N} \times [0, 1]$, and so on.

3. Extensions

Assume that $G$ is an abelian group and that $\Sigma$ is a subset of $G$ which satisfies:

(1) $\Sigma + \Sigma \subseteq \Sigma$.
(2) $\Sigma \cap -\Sigma = \{0\}$.
(3) $G = \Sigma - \Sigma$.

A subset which satisfies these properties will be referred to as a positive cone (or, sometimes, simply as a cone). The pair $(G, \Sigma)$ will be called a directed group. If, in addition, $G = \Sigma \cup (-\Sigma)$, then we say that $\Sigma$ totally orders $G$. The total ordering is given by $a \leq b$ if, and only if, $b - a \in \Sigma$.

Question. Suppose that $\alpha$ is a partial action of a positive cone $\Sigma$ acting on a $\mathbb{C}^*$-algebra $A$. Is there a (necessarily unique) extension of $\alpha$ to $G$?

This question is easy to settle in the totally ordered case:

3.1. Proposition. Assume that $G$ is totally ordered by a cone $\Sigma$. If $\alpha$ is a partial action of $\Sigma$ on a $\mathbb{C}^*$-algebra $A$, then $\alpha$ has a unique extension to a partial action of $G$ on $A$.

Proof. Since $G = \Sigma \cup (-\Sigma)$, we only need to define $\alpha$ appropriately on $-\Sigma$ and it is obvious how this must be done: for $t \in \Sigma$, take $\alpha_{-t} = \alpha_t^{-1}$. Condition 3 in the definition of partial action is trivially satisfied. Rather than verifying conditions 1a, 1b, and 2, it is easier to verify the third arrow property.

A “hand waving” proof is elementary: given a diagram consisting of three elements in suitable ideals and two arrows indexed by elements of $G$, we can, if necessary, reverse one or both of the arrows to obtain a diagram in which all arrows are indexed by elements of $\Sigma$. Since $\alpha$ satisfies the third arrow property on $\Sigma$, we obtain the third arrow in the new
diagram. Either this arrow or its inverse will yield the required third arrow for the original diagram.

Here is a more detailed argument showing that I and III implies II. The other two implications needed for the third arrow property can be verified in a similar fashion. We assume, then, that $x \in D_{-s-t}$, $x \in D_{-t}$, $z = \alpha_{s+t}(x)$, and $y = \alpha_t(x)$.

$$\begin{array}{ccc}
x & \xrightarrow{\alpha_{s+t}} & z \\
& \xleftarrow{\alpha_t} & y
\end{array}$$

We need to deal with the four possibilities regarding the membership of $s$ and $t$ in $\Sigma$ and $-\Sigma$.

First assume that $t$ and $s+t$ both lie in $\Sigma$. If $s$ is also in $\Sigma$, then the third arrow property for $\Sigma$ yields $y \in D_{-s}$ and $z = \alpha_s(y)$; thus II holds. If, instead, $s \in -\Sigma$, then view $\alpha_t$ as III and $\alpha_{s+t}$ as I; the third arrow property for $\Sigma$ yields $z \in D_s$ and $y = \alpha_{-s}(z)$. But then $y \in D_{-s}$ and $z = \alpha_s(y)$; once again II holds.

Now assume that $t \in -\Sigma$ and $s+t \in \Sigma$. This forces $s \in \Sigma$. Consider the diagram

$$\begin{array}{ccc}
x & \xrightarrow{\alpha_{s+t}} & z \\
& \xleftarrow{\alpha_{-t}} & y
\end{array}$$

in which both indices are in $\Sigma$. We can treat $\alpha_{-t}$ as I and $\alpha_{s+t}$ as II to obtain $z = \alpha_{-t+s+t}(y) = \alpha_s(y)$. In the original diagram, this is II, just what we need to deduce.

Next assume that $t \in \Sigma$ and $s+t \in -\Sigma$. Then we must have $s \in -\Sigma$. The appropriate diagram with indices in $\Sigma$ is

$$\begin{array}{ccc}
x & \xrightarrow{\alpha_{-s-t}} & z \\
& \xleftarrow{\alpha_t} & y
\end{array}$$

Use the third arrow property for $\Sigma$ (the I and II implies III component) to obtain $y = \alpha_{-s-t+t}(z) = \alpha_{-s}(z)$. But then $z = \alpha_s(y)$ and yet again II is verified for the original diagram.

Finally suppose that $t \in -\Sigma$ and $s+t \in -\Sigma$. This gives a diagram

$$\begin{array}{ccc}
x & \xrightarrow{\alpha_{-s-t}} & z \\
& \xleftarrow{\alpha_{-t}} & y
\end{array}$$

with indices in $\Sigma$. By the first part of the detailed proof, $y = \alpha_{-s}(z)$ and again $z = \alpha_s(y)$.

As mentioned earlier, the other two implications in the third arrow property can be verified with similar arguments. \qed

Returning to the general case, to extend $\alpha$ to all of $G$ it would be natural to define $\alpha_g$ to be the union of all compositions

$$\alpha_{t_n}^{-1} \circ \alpha_{s_n} \circ \cdots \circ \alpha_{t_1}^{-1} \circ \alpha_{s_1}$$

where $s_1, \ldots, s_n, t_1, \ldots, t_n$ are in $\Sigma$ and $g = -t_n + s_n - \cdots - t_1 + s_1$. The third arrow property for $\Sigma$ implies that we need only consider expressions for $g$ of this form, as we may compose adjacent elements with both indices in $\Sigma$ or both indices in $-\Sigma$.

We next find a sufficient condition for $\alpha_g$ to be well-defined. Suppose that $x \in A$ and $g \in G$, that $g$ has two expressions $g = -t_n + s_n \cdots - t_1 + s_1$ and $g = -v_k + u_k \cdots - v_1 + u_1$, and
that \( x \) is in the domain of both the associated compositions. If \( x' = \alpha_{t_n}^{-1} \circ \alpha_{s_n} \circ \cdots \circ \alpha_{t_1}^{-1} \circ \alpha_{s_1}(x) \), then \( x' \) is in the domain of the composition associated to

\[-v_k + u_k \cdots - v_1 + u_1 + t_n - s_n \cdots + t_1 - s_1.\]

This expression sums to the identity element of the group, so the associated composition must send \( x' \) to \( x' \). If this always holds, then \( \alpha_g \) is well-defined; it is easy to see that the third arrow property now holds.

The condition of the previous paragraph is clearly valid for a partial action of a group, so this condition is both necessary and sufficient to have an extension.

Summarizing, we have the following observation.

3.2. **Proposition.** Let \((G, \Sigma)\) be a directed group and let \( \alpha \) be a partial action of \( \Sigma \). Then \( \alpha \) extends to a partial action of \( G \) if and only if for all \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \) in \( \Sigma \) with \(-t_n + s_n - \cdots - t_1 + s_1 \) equal to the identity,

\[\alpha_{t_n}^{-1} \circ \alpha_{s_n} \circ \cdots \circ \alpha_{t_1}^{-1} \circ \alpha_{s_1}\]

is a restriction of the identity map.

The next example shows that there are partial actions which do not extend.

3.3. **Example.** Let \( X = \{1, 2, 3, 4, 5, 6, 7\} \) with the discrete topology and \( G = \mathbb{Z}^3 \) with the usual positive cone. Define \( \beta_{(1,0,0)} \) to have domain \( \{1, 5\} \) and send 1 to 2 and 5 to 4. Define \( \beta_{(0,0,1)} \) to have domain \( \{3, 5\} \) and send 3 to 2 and 5 to 6. Define \( \beta_{(0,0,1)} \) to have domain \( \{3, 7\} \) and send 3 to 4 and 7 to 6.

\[
\begin{array}{cccccc}
1 & 2 & \beta_{(1,0,0)} & \beta_{(0,0,1)} & 4 & \beta_{(0,1,0)}
\end{array}
\]

Since the domains of these maps are contained in \( \{1, 3, 5, 7\} \) and the ranges are contained in \( \{2, 4, 6\} \), all possible compositions are empty. Thus, for all other strictly positive elements of \( G \), we may define \( \beta_g \) to be the trivial map with empty domain. Of course, \( \beta_{(0,0,0)} \) is the identity map. The third arrow property holds trivially, since whenever there is a point in \( X \) that is in the domain of two different elements of \( \Sigma \), the two elements are incomparable.

Since \( \beta_{(0,0,1)}^{-1} \circ \beta_{(0,1,0)} \circ \beta_{(1,0,0)}^{-1} \circ \beta_{(0,0,1)} \circ \beta_{(0,1,0)}^{-1} \circ \beta_{(1,0,0)} \) sends 1 to 7, Proposition 3.2 is violated. Thus, this partial action does not extend to a partial action of \( \mathbb{Z}^3 \). Of course, if we replaced \( \mathbb{Z}^3 \) with the free group on three generators, then the partial action would extend.

3.4. **Example.** We give an example of an extension of a partial action from a positive cone which does not totally order the group. This example will play a fundamental role in Example 4.5, where we display Power’s toroidal limit algebras as an analytic partial crossed product.

Let \( X = \prod_i \{0, 1\} \). We could construct the same example on any finite set with an even number of points. For the purposes of this example, we could as well denote the elements
of the set with the integers from 1 to 2\(^2\), but binary notation will facilitate the discussion of the inverse system in example 4.1. The group in this example is \(\mathbb{Z}^2\) and the positive cone is \(\Sigma = \{(a, b) \mid a \geq 0, b \geq 0\}\). Write \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\). Let \(\omega\) be the finite odometer map on \(X\): \(\omega(1, \ldots, 1) = (0, \ldots, 0)\) and otherwise \(\omega\) adds 1 to the first entry with carries to the right. Of course, \(\omega\) is just a cyclic permutation of the group, viz. \((0, 0)\). Also, let \(D_0 = \{x \in X \mid x_1 = 0\}\) and \(D_1 = \{x \in X \mid x_1 = 1\}\).

Let \(\beta(0)\) be the identity map on \(X\); \(\beta(e_1)\), the restriction of \(\omega\) to \(D_0\) and \(\beta(e_2)\), the restriction of \(\omega^{-1}\) to \(D_0\). For all other \(s \in \mathbb{Z}^2\), \(\beta(s)\) is empty. In particular, we have

\[
\begin{align*}
D_0 & \xrightarrow{\beta(e_1)} D_1, \\
D_0 & \xrightarrow{\beta(e_2)} D_1.
\end{align*}
\]

It is easy to check that \(\beta\) is a partial action of \(\Sigma\) on \(X\).

We now wish to extend \(\beta\) to an action of \(\mathbb{Z}^2\) on \(X\). Proposition 3.2 gives guidance on how to proceed. For example, both \(\beta(e_1) \circ \beta(-e_2)\) and \(\beta(-e_2) \circ \beta(e_1)\) must be restrictions of \(\beta(e_1 - e_2)\). But the first composition is \(\omega^2\) restricted to \(D_0\) and the second is \(\omega^2\) restricted to \(D_1\). Thus, we have no choice but to take \(\beta(e_1 - e_2) = \omega^2\). In a similar vein, \(\beta(e_1) \circ \beta(-e_2) \circ \beta(e_1)\) must be a restriction of \(\beta(2e_1 - e_2)\). This is actually the only constraint, so we will be able to take \(\beta(2e_1 - e_2)\) equal to \(\omega^3\) restricted to \(D_0\). Considerations of this sort suggest that the following will be an extension of \(\beta\) from \(\Sigma\) to \(\mathbb{Z}^2\):

\[
\begin{align*}
\beta(ae_1 - ae_2) &= \omega^{2a}, \text{ for all } a \in \mathbb{Z}, \\
\beta((a + 1)e_1 - ae_2) &= \text{the restriction of } \omega^{2a+1} \text{ to } D_0, \text{ for all } a \in \mathbb{Z}, \\
\beta(ae_1 - (a + 1)e_2) &= \text{the restriction of } \omega^{2a+1} \text{ to } D_1, \text{ for all } a \in \mathbb{Z}, \\
\beta(s) &= \text{empty for all other } s \in \mathbb{Z}^2.
\end{align*}
\]

It is a routine matter to check that the extended \(\beta\) is a partial action of \(\mathbb{Z}^2\) on \(X\). Since we are dealing with a group, it is enough to check conditions 1', 2 and 3; the only sets which can be domains are \(X, D_0, D_1\) and \(\emptyset\) so the calculations are easy. Of course, the third arrow condition is also easy to check.

If we visualize \(\mathbb{Z}^2\) as the lattice points of a plane, and if we put \(\beta(ae_1 + be_2)\) at node \((a, b)\), we can “picture” the partial action as follows:

\[
\begin{array}{ccccccc}
\omega^{-4} & \omega^{-3}|_{D_0} & \emptyset & \emptyset & \emptyset \\
\omega^{-3}|_{D_1} & \omega^{-2} & \omega^{-1}|_{D_0} & \emptyset & \emptyset \\
\emptyset & \omega^{-1}|_{D_1} & \text{id}_X & \omega^1|_{D_0} & \emptyset \\
\emptyset & \emptyset & \omega^1|_{D_1} & \omega^2 & \omega^3|_{D_0} \\
\emptyset & \emptyset & \emptyset & \omega^3|_{D_1} & \omega^4
\end{array}
\]

4. **Analytic Partial Crossed Products**

We shall define an analytic partial crossed product as a subalgebra of a \(\mathcal{C}^*\)-partial crossed product. This applies to any partial action of a cone \(\Sigma\) which has a unique extension to a partial action of the whole group. In particular, the definition is available whenever \(\Sigma\) totally orders \(G\) (Proposition 3.1).

A brief review of the definition of a partial crossed product \(\mathcal{C}^*\)-algebra will provide notation and terminology. If \(\beta\) is a partial action of an abelian group \(G\) on a \(\mathcal{C}^*\)-algebra \(A\), we first
consider the set \( \mathcal{P} \) of all formal polynomials of the form \( \sum f_n U^n \), where \( f_n \in D_n \), for each \( n \) in some finite subset of \( G \). Define on this set a multiplication and an involution. The multiplication is a “twisted” convolution product; it is determined by specifying the product of two monomials:

\[
f U^n g U^m = \alpha_n(\alpha_{-n}(f)g) U^{n+m}.
\]

Here, \( \alpha_{-n}(f) \in D_{-n} \), so the product \( \alpha_{-n}(f)g \in D_{-n} \cap D_m \). From the definition of partial action, \( \alpha_n(\alpha_{-n}(f)g) \in D_{n+m} \). The involution is defined (on monomials) by

\[
(f U^n)^* = \alpha_{-n}(f) U^{-n}.
\]

Thus, \( \mathcal{P} \) is a \(*\)-algebra.

Typically, we think of \( U^m \) as the partial isometry implementing \( \alpha_m \). Notice, however, that if \( \alpha \) does not have the composition property, then the product of the partial isometries implementing \( \alpha_m \) and \( \alpha_n \) may be a proper restriction of the partial isometry implementing \( \alpha_{m+n} \).

Define a norm on \( \mathcal{P} \) by

\[
\| \sum f_n U^n \|_L = \sum \| f_n \|.
\]

A \( C^* \)-norm on \( \mathcal{P} \) is defined by

\[
\| x \| = \sup_{\pi} \| \pi(x) \|,
\]

where the supremum is taken over all representations which are continuous with respect to the \( L \)-norm. The partial crossed product \( A \times_\alpha G \) is the completion of \( \mathcal{P} \) with respect to the \( C^* \)-norm. (If, instead of polynomials, we had used all formal power series \( \sum_{n \in G} f_n U^n \) for which \( \sum_n \| f_n \| \) converges, then we would have obtained a Banach \(*\)-algebra whose enveloping \( C^* \)-algebra is \( A \times_\alpha G \). This is what appears in [8, 11].)

4.1. Definition. The analytic crossed product \( A \times_\alpha \Sigma \) is defined to be the closure in \( A \times_\alpha G \) of those polynomials \( \sum f_n U^n \) for which all \( n \) are in \( \Sigma \).

Since we are restricting attention to the case where \( A = C_0(X) \) is an abelian \( C^* \)-algebra we rephrase the definition of the multiplication in terms of the partial action \( \beta \) of \( \Sigma \) on \( X \), where \( \beta \) is associated with \( \alpha \):

\[
f U^n g U^m = [(f \circ \beta_n)g] \circ \beta_n^{-1} U^{n+m}.
\]

For future reference, note that if \( h \) denotes the coefficient function \([ (f \circ \beta_n)g] \circ \beta_n^{-1} \), then

\[
h(x) = \begin{cases} f(x)g(\beta_n^{-1}(x)), & \text{if } x \in X_n \cap X_{n+m}, \\ 0, & \text{otherwise.} \end{cases}
\]

When \( \beta \) is a partial action on \( X \) and \( \alpha \) is the dual action on \( C_0(X) \), we may write either \( C_0(X) \times_\alpha G \) or \( C_0(X) \times_\beta G \) for the partial crossed product.

Remark. In place of the \(*\)-algebra \( \mathcal{P} \) used in defining a partial crossed product we can use a somewhat smaller algebra. Let \( \mathcal{P}_c \) be the subalgebra of \( \mathcal{P} \) consisting of all elements of \( \mathcal{P} \) whose coefficients have compact support. So, a polynomial \( \sum f_n U^n \) is in \( \mathcal{P}_c \) if, and only if, each \( f_n \in D_n \) and is compactly supported.

When \( \mathcal{P} \) is provided with the \( C^* \)-norm, \( \mathcal{P}_c \) is a dense subalgebra. This remark will be useful in Theorem 5.1.
We devote the remainder of this section to examples of analytic partial crossed product algebras. These include analytic limit algebras, instances of a general result we prove in the next section, and also a family of non-triangular algebras.

The partial actions of the groups in Examples 2.3 and 2.4 both yield $2\infty$ UHF $C^*$-algebras. The two analytic subalgebras are, respectively, the standard embedding TAF algebra and the refinement embedding TAF algebra. More generally, the standard $\mathbb{Z}$-analytic algebras of [15, 19], the locally constant cocycle nest algebras of [7], and the order preserving algebras of [5] are all analytic partial crossed product algebras. The validity of these assertions follows directly from Theorem 5.1.

Before constructing our non-triangular example, we need a preliminary result.

4.2. Proposition. Fix an abelian group $G$.

(1) Let $\beta$ and $\hat{\beta}$ be two partial actions of $G$ on locally compact Hausdorff spaces $X$ and $\hat{X}$. As usual, suppose that $\beta_g$ has domain $D_{-g}$ and $\hat{\beta}_g$ has domain $\hat{D}_{-g}$, for each $g$. If $\phi: \hat{X} \to X$ is a continuous surjection so that, for all $g \in G$ and all $x \in \hat{X}$, $\phi(x) \in D_g$ if, and only if, $x \in \hat{D}_g$ and

$$\beta_g(\phi(x)) = \phi(\hat{\beta}_g(x)),$$

then there is a continuous injection $\Phi: C_0(X) \times_\beta G \to C_0(\hat{X}) \times_{\hat{\beta}} G$.

When equation (1) holds, we will say that $\phi$ intertwines the two actions.

(2) Suppose that $(X_i, \phi_i)$ is an inverse system of locally compact Hausdorff spaces in which each $\phi_i: X_{i+1} \to X_i$ is a continuous surjection and is proper (in the sense that the inverse images of compact sets are compact). Assume further that there is a family of partial actions $\beta_i$ on the $X_i$ so that each $\phi_i$ intertwines $\beta_{i+1}$ and $\beta_i$. Let $\Phi_i: C_0(X_i) \times_{\beta_i} G \to C_0(X_{i+1}) \times_{\beta_{i+1}} G$ be the map induced by Part (1). If $X = \lim \leftarrow (X_i, \phi_i)$, then there is a partial action $\beta$ of $G$ on $X$ so that $C_0(X) \times_\beta G$ is isomorphic to $\lim \rightarrow (C_0(X_i) \times_{\beta_i} G, \Phi_i)$.

(3) Suppose further that $\Sigma$ is a positive cone in $G$. Then $\Phi_i$ maps $C_0(X_i) \times_{\beta_i} \Sigma$ into $C_0(X_{i+1}) \times_{\beta_{i+1}} \Sigma$ and $C_0(X) \times_\beta \Sigma = \lim \rightarrow C_0(X_i) \times_{\beta_i} \Sigma$.

Proof. We prove (1) first. As is well-known, the map $\tilde{\phi}: C_0(X) \to C_0(\hat{X})$ given by $f \mapsto f \circ \phi$ is a continuous injection. Denote the monomials of $C_0(X) \times_\beta G$ by $fU^m$ and those of $C_0(\hat{X}) \times_{\hat{\beta}} G$ by $fV^m$. We define $\Phi$ on the monomials of $C_0(X) \times_\beta G$ by $\Phi(fU^m) = \tilde{\phi}(f)V^m$ and then extend to polynomials. Since $\tilde{\phi}$ is an injection, so is $\Phi$. 


To see that $\Phi$ is a homomorphism, observe first that if $\alpha$ and $\hat{\alpha}$ are the dual partial actions on $C_0(X)$ and $C_0(\hat{X})$, then equation (11) implies that $\hat{\phi} \circ \alpha_m = \hat{\alpha}_m \circ \phi$. Thus,

$$
\Phi(fU^m)\Phi(gU^n) = \tilde{\phi}(f)V^n\tilde{\phi}(g)V^m
= \hat{\alpha}_m(\hat{\alpha}_m(\tilde{\phi}(f))\tilde{\phi}(g))V^{m+n}
= \hat{\alpha}_m(\tilde{\phi}(\alpha_m(f))\tilde{\phi}(g))V^{m+n}
= \hat{\alpha}_m(\tilde{\phi}(\alpha_m(f))g)V^{m+n}
= \tilde{\phi}(\alpha_m(\alpha_m(f))g)V^{m+n}
= \Phi(\alpha_m(\alpha_m(f))gU^{m+n})
= \Phi(fU^m gU^n).
$$

It remains only to show that $\Phi$ is continuous on the polynomials with the $C^*$-norms, so that we may extend by continuity to the crossed product. Clearly, $\Phi$ is contractive when the polynomials are equipped with the $L$-norm, since $\hat{\phi}$ is contractive on $C_0(X)$. Composing $\Phi$ with an $L$-norm continuous representation of the polynomials of $C_0(\hat{X}) \times_{\hat{\beta}} G$ gives an $L$-norm continuous representation of the polynomials of $C_0(X) \times_{\beta} G$, and so $\Phi$ is contractive with respect to the $C^*$-norms.

Turning to part (2), we can identify $X$ with the set of sequences $x = (x_1, x_2, \dotsc)$, where each $x_i \in X_i$ and $\phi_i(x_{i+1}) = x_i$ for all $i$. Of course, the topology is the relative product topology. For each $g \in G$, let $\text{dom} \beta(g) = \{x \mid x_i \in \text{dom} \beta_i(g) \text{ for all } i\}$ and define $\beta(g)x$ to be $(\beta_1(g)x_1, \beta_2(g)x_2, \dotsc)$. The intertwining condition implies that $\phi_i(\beta_{i+1}(g)x_{i+1}) = \beta_i(g)x_i$ for all $i$, and so $\beta(g)x \in X$.

For $w \in X_i$, let $J(w) = \{(y_j) \in X \mid y_i = w\}$. Note that $y_j$ for $j \leq i$ are determined by the condition $y_i = w$. If $x \in \text{dom} \beta_i(g)$ and $y \in J(w)$, then $y_j \in \text{dom} \beta_j(g)$, for all $j$, and $y \in \text{dom} \beta(g)$. Thus $J(w) \subseteq \text{dom} \beta(g)$ and $\beta(g)(J(w)) = J(\beta_i(g)w)$.

Let $C_i$ be the subalgebra of $C_0(X) \times_{\beta_i} G$ generated by $fU^m$ where $m \in G$ and $f \in C_0(X)$ is constant on the sets $J(w)$, $w \in X_i$. Then $C_i$ is isomorphic to $C_0(X_i) \times_{\beta_i} G$ and $C_i \subseteq C_{i+1}$, for all $i$. Under these isomorphisms and containments, the diagram

$$
\begin{array}{ccc}
C_i & \longrightarrow & C_{i+1} \\
\downarrow & & \downarrow \\
C_0(X_i) \times_{\beta_i} G & \xrightarrow{\Phi_i} & C_0(X_{i+1}) \times_{\beta_{i+1}} G
\end{array}
$$

commutes. To prove that $C_0(X) \times_{\beta} G$ is isomorphic to $\lim (C_0(X_i) \times_{\beta_i} G, \Phi_i)$ it is sufficient to show that the union of the $C_i$ is dense in $C_0(X) \times_{\beta} G$.

To this end, let

$$
F_i = \{f \in C_0(X) \mid f \text{ is constant on } J(w), \text{ for each } w \in X_i\}.
$$

Then $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dotsc$ and each $F_i$ is a $*$-subalgebra of $C_0(X)$. If we show that the subalgebra $\bigcup F_i$ is dense in $C_0(X)$, then the density of $\bigcup C_i$ in $C_0(X) \times_{\beta} G$ follows. To do this, all we need to show is that $\bigcup F_i$ separates points of $X$. 


Let $x$ and $y$ be two distinct points in $X$. Then there is an index $i$ such that $x_i \neq y_i$. Therefore, $J(x_i) \cap J(y_i) = \emptyset$. Let $g$ be a continuous function on $X_i$ such that $g(x_i) \neq g(y_i)$. The composition of $g$ with the canonical projection of $X$ onto $X_i$ is then an element of $F_i$ which separates $x$ from $y$. This completes the proof of the second part of the Proposition.

The proof of the third part is trivial. □

Proposition 4.2 provides a convenient framework for realizing limit algebras as analytic partial crossed products, as the following examples illustrate.

4.3. Example. To obtain the $2^\infty$-UHF algebra and the $2^\infty$-standard TUHF subalgebra as direct limits in a partial crossed product framework, let $X_i = \prod_i \{0,1\}$ and let $\beta_i$ be the finite odometer restricted to $X_i \setminus \{(1,\ldots,1)\}$. (The odometer map sends a tuple $(x_1,\ldots,x_i)$ to $(y_1,\ldots,y_i)$, where $1 + \sum x_k 2^{k-1} = \sum y_k 2^{k-1}$ (mod $2^i$).) The powers of $\beta_i$ give a partial action of $\mathbb{Z}$ on $X_i$. Let $\phi_i: X_{i+1} \to X_i$ be the map which deletes the last entry of elements of $X_{i+1}$. Then $C(X_i) \times_{\beta_i} \mathbb{Z} \cong M_{2^n}$, $C(X_i) \times_{\beta_i} \mathbb{Z}^+ \cong T_{2^n}$, and the embeddings $\Phi_i$ are the standard embedding maps. Thus, we recover the usual presentation of the standard TUHF algebra.

4.4. Example. As a second easy application, this time with (full) crossed products, let $\alpha_i$ be the odometer on $X_i = \prod_i \{0,1\}$ and let $\phi_i$ be the same as the Example 4.3. Then each $C(X_i) \times_{\alpha_i} \mathbb{Z}$ is isomorphic to $M_{2^n}(C(\mathbb{T}))$—$\mathbb{T}$ is unit circle—and, since the induced map $\Phi_i$ is the twice-around embedding, the direct limit is the $2\infty$-Bunce-Deddens $C^*$-algebra. The inverse limit action of the $\alpha_i$ on $X = \prod_{i} \{0,1\}$ is the odometer action on $X$. This gives the known result that $C(X) \times_{\alpha} \mathbb{Z}$ is isomorphic to $\varprojlim(M_{2^n}(C(\mathbb{T})), \Phi_i)$.

4.5. Example. We realize the toroidal limit algebras of [20] as analytic partial crossed products. Let $X_i = \prod_i \{0,1\}$ and once again let $\phi_i$ send $(x_1,\ldots,x_i,x_{i+1})$ to $(x_1,\ldots,x_i)$. For clarity, we confine ourselves to the $2\infty$ case; replacing each $\{0,1\}$ with $\{0,\ldots,n_i\}$, where $2 \mid n_i$, gives the general construction.

For each $i$, let $\beta_i$ be the partial action of $\mathbb{Z}^2$ and the positive cone $\Sigma$ described in Example 4.4. It is easy to check that the $\phi_i$ and the $\beta_i$ satisfy the hypotheses of Proposition 4.2.

For each $i$, $C(X_i) \times_{\beta_i} \Sigma$ is a $2^i$-cycle algebra, that is, the subalgebra of $M_{2^n}$ spanned by the diagonal matrix units, the matrix unit $e_{1,2^i}$, and the matrix units $e_{j,k}$ where $j$ is odd, $k$ is even, and $|j-k| = 1$. As $\phi_i$ is the usual double cover embeddings of $X_{i+1}$ into $X_i$, the induced map $\Phi_i$ wraps the $2^i$-cycle algebra twice around the $2^{i+1}$-cycle algebra. Thus, the direct system $(C(X_i) \times_{\beta_i} \Sigma, \Phi_i)$ is precisely the direct system given by Power [20, p. 51] and appealing to his Theorem 4.1, we have a generating subalgebra of the $2\infty$-Bunce-Deddens $C^*$-algebra.

We can show this directly. Consider the element

$$y_i = \chi_{D^1} U^{e_1} + \chi_{D^0} U^{-e_2}$$

in $C(X_i) \times_{\beta_i} \mathbb{Z}^2$. ($D^0$ and $D^1$ are the domain and range sets for $\beta_i(e_1)$ and $\beta_i(e_2)$, as in Example 4.4.) We first observe that $y_i$ is unitary. The calculation uses the following facts (in which $e$ is either $e_1$ or $e_2$ and $\beta_i(-e) = \beta_i(e)^{-1}$):

$$\chi_{D^1} \circ \beta_i(e) = \chi_{D^0}, \quad \chi_{D^0} \circ \beta_i(e) = 0,$$

$$\chi_{D^0} \circ \beta_i(-e) = \chi_{D^1}, \quad \chi_{D^1} \circ \beta_i(-e) = 0.$$
We then have
\[ y_i^* = \chi_{D^1} \circ \beta_i(e_1) U^{-e_1} + \chi_{D^0} \circ \beta_i(-e_2) U^{e_2} = \chi_{D^0} U^{-e_1} + \chi_{D^1} U^{e_2} \]
and
\[ y_i y_i^* = (\chi_{D^1} U^{e_1} + \chi_{D^0} U^{-e_2})(\chi_{D^0} U^{-e_1} + \chi_{D^1} U^{e_2}) \]
\[ = \chi_{D^1} \chi_{D^0} \circ \beta_i(-e_1) U^{e_1+e_1} + \chi_{D^1} \chi_{D^1} \circ \beta_i(-e_1) U^{e_1+e_2} \]
\[ + \chi_{D^0} \chi_{D^0} \circ \beta_i(e_2) U^{-e_2+e_1} + \chi_{D^0} \chi_{D^1} \circ \beta_i(e_2) U^{-e_2+e_2} \]
\[ = \chi_{D^1} U^0 + \chi_{D_0} U^0 \]
\[ = \chi_{X_i} U^0 = I. \]

Similarly, \( y_i^* y_i = I. \)

Observe that \( y_i \) implements the action of the odometer on \( X_i \) (transferred to \( C(X_i) U^0 \)): for \( f \in C(X_i) \),
\[ y_i^* f U^0 y_i = (\chi_{D^0} U^{-e_1} + \chi_{D^1} U^{e_2}) f U^0 (\chi_{D^1} U^{e_1} + \chi_{D^0} U^{-e_2}) \]
\[ = \chi_{D^0} f \circ \beta_i(e_1) \chi_{D^0} U^0 + \chi_{D^1} f \circ \beta_i(-e_2) \chi_{D^1} U^0 \]
\[ = [\chi_{D^0} f \circ \omega|_{D^0} + \chi_{D^1} f \circ \omega|_{D^1}] U^0 \]
\[ = f \circ U^0. \]

We next claim that \( y_i \) and \( C(X_i) U^0 \) generate \( C(X_i) \times_{\beta_i} \mathbb{Z}^2 \). Let \( A \) be the subalgebra generated by \( y_i \) and \( C(X_i) U^0 \). Multiply \( y_i \) on the left by \( \chi_{D^1} U^0 \) to see that \( \chi_{D^1} U^{e_1} \in A \) and on the left by \( \chi_{D^0} U^0 \) to see that \( \chi_{D^0} U^{-e_2} \in A \). The adjoint of the latter, \( \chi_{D_1} U^{e_2} \) is therefore also in \( A \). The square of \( y_i \) is \( \chi_{X_i} U^{e_1-e_2} \), so this monomial is in \( A \). (We omit this and subsequent calculations, since they are similar to the ones done above.) For any positive integer \( a \), the \( a \)th power of \( \chi_{X_i} U^{e_1-e_2} \) is \( \chi_{X_i} U^{ae_1-ae_2} \); it follows that \( \chi_{X_i} U^{ae_1-ae_2} \in A \) for all integers \( a \). Multiplication of \( \chi_{X_i} U^{ae_1-ae_2} \) on the left by \( \chi_{D^1} U^{e_1} \) yields \( \chi_{D_1} U^{(a+1)e_1-ae_2} \in A \); multiplication on the left by \( \chi_{D^0} U^{-e_2} \) yields \( \chi_{D_0} U^{(a+1)e_2} \in A \). Now if \( s \) is an element of \( \mathbb{Z}^2 \) which is not of the form \( ae_1-ae_2 \) or \((a+1)e_1-ae_2 \) or \( ae_1-(a+1)e_2 \), then \( \beta_i(s) \) is empty; therefore, the only possible coefficient for \( U^s \) in the partial crossed product is 0. Thus, we have shown that every monomial \( \chi_{ran_\beta(s)} U^s \) is in \( A \). But these monomials and \( C(X_i) U^0 \) generate \( C(X_i) \times_{\beta_i} \mathbb{Z}^2 \), so \( A = C(X_i) \times_{\beta_i} \mathbb{Z}^2 \).

Let \( \Psi_i \) denote the inclusion of \( C(X_i) \times_{\beta_i} \mathbb{Z}^2 \) in \( C(X_{i+1}) \times_{\beta_{i+1}} \mathbb{Z}^2 \). As \( \Psi_i(y_i) = y_{i+1} \), we may let \( y \) denote the image in \( C(X) \times_{\beta} \mathbb{Z}^2 \) of \( y_i \). Since \( C(X_i) U^0 \) and \( y_i \) generate \( C(X_i) \times_{\beta_i} \mathbb{Z}^2 \) for each \( i \), it follows that \( C(X) U^0 \) and \( y \) generate \( C(X) \times_{\beta} \mathbb{Z}^2 \). Letting \( \alpha \) be the action of the odometer map on \( X \), it follows that \( C(X) \times_{\beta} \mathbb{Z}^2 \) is a quotient of \( C(X) \times_{\alpha} \mathbb{Z} \), the \( 2^\infty \) Bunce-Deddens C*-algebra. But the latter algebra is simple, so \( C(X) \times_{\beta} \mathbb{Z}^2 \) is the \( 2^\infty \) Bunce-Deddens algebra. By Proposition 1.24 \( C(X) \times_{\beta} \Sigma \) is Power’s toroidal limit algebra.

5. Analyticity and Coordinates

Theorem 5.1 relates partial crossed products to groupoid C*-algebras, with the analytic subalgebras in correspondence. Relevant aspects of the groupoid C*-algebra construction will be reviewed briefly in the course of the proof. For complete, systematic accounts, see Renault [22] or Paterson [14]. For a convenient summary, see Muhly and Solel [12].

Recall that a partial action is said to be free if \( \alpha_t(x) = x \) implies that \( t = 0 \).
5.1. Theorem. Let $G$ be a countable discrete subgroup of $\mathbb{R}$ with positive cone $\Sigma = G \cap [0, \infty)$. Let $\alpha$ be a free partial action of $G$ on a separable abelian $C^*$-algebra $A$. Then there is a locally compact $r$-discrete principal groupoid $\mathcal{G}$ with second countable unit space and a locally constant real valued cocycle on $\mathcal{G}$ such that the partial crossed product $A \times_\alpha G$ is $^*$-isomorphic to the groupoid $C^*$-algebra $C^*(\mathcal{G})$; the analytic subalgebra of $A \times_\alpha G$ associated with $\Sigma$ is carried by this isomorphism onto the analytic subalgebra of $C^*(\mathcal{G})$ determined by the cocycle.

Conversely, given a locally compact $r$-discrete principal groupoid $\mathcal{G}$ with second countable unit space and a locally constant real valued cocycle, there is a countable discrete subgroup $G$ of $\mathbb{R}$ and a partial action $\alpha$ of $G$ on a locally compact second countable Hausdorff space $X$ such that $C^*(\mathcal{G})$ is isomorphic to $C_0(X) \times_\alpha G$; again, the analytic subalgebra of $C^*(\mathcal{G})$ determined by the cocycle is carried by this isomorphism onto the analytic subalgebra of $C_0(X) \times_\alpha G$ determined by the positive cone $G \cap [0, \infty)$.

Proof. Let $A = C_0(X)$ be a separable abelian $C^*$-algebra and let $G$ be a discrete subgroup of $\mathbb{R}$ with a free partial action $\alpha$ on $X$. As a set, the groupoid $\mathcal{G}$ is:

$$\mathcal{G} = \{(x, \alpha_t(x)) \mid t \in G, x \in X_{-t}\}.$$ 

This is an equivalence relation on $X$ (transitivity follows from the definition of partial action, most trivially from the third arrow property), so the groupoid is principal.

For each $t \in G$ and each open subset $U \subseteq X_{-t}$, let $\mathcal{O}_{t,U} = \{(x, \alpha_t(x)) \mid x \in U\}$. Since the action is free, $\alpha_t(x) = \alpha_s(x)$ implies $t = s$; consequently, the family $\{\mathcal{O}_{t,U}\}$ is closed under finite intersections. (Either $\mathcal{O}_{t,U} \cap \mathcal{O}_{s,V} = \emptyset$ or $s = t$ and $\mathcal{O}_{t,U} \cap \mathcal{O}_{t,V} = \mathcal{O}_{t,U \cap V}$.) Give $\mathcal{G}$ the smallest topology in which all the $\mathcal{O}_{t,U}$ are open sets. The family of all $\mathcal{O}_{t,U}$ is a basis for this topology.

With this topology, $\mathcal{G}$ is a locally compact topological space. For each $t \in G$, the graph of $\alpha_t$, namely $\{(x, \alpha_t(x)) \mid x \in X_{-t}\}$ is an open subset of $\mathcal{G}$. Furthermore, this set (with the relative topology) is isomorphic to $X_{-t}$ (and $X_t$). Indeed, the range and domain maps on the groupoid when restricted to the graph of $\alpha_t$ yield isomorphisms with $X_{-t}$ and $X_t$, respectively.

If $g_n$ is a convergent sequence in $\mathcal{G}$, then for some $t \in G$, $g_n$ is eventually in the graph of $\alpha_t$. So, for all large $n$, $g_n = (x_n, \alpha_t(x_n))$ with $x_n \in X_{-t}$. Since $g_n$ is convergent, so is $x_n$. Thus, there is $x \in X_{-t}$ such that $x_n \to x$ and $\alpha_t(x_n) \to \alpha_t(x)$.

Since the set $\mathcal{O}_{t,U}$ corresponds to the set $\mathcal{O}_{t^{-1}, \alpha_t(U)}$ under the inverse map $x \to x^{-1}$, the inverse map is continuous (indeed, a homeomorphism of $\mathcal{G}$ onto itself).

To show that $\mathcal{G}$ is a topological groupoid it remains to verify that the multiplication is continuous (when $\mathcal{G}^2$ is given the relative product topology from $\mathcal{G} \times \mathcal{G}$). Suppose $g_n$ and $h_n$ are two convergent sequences in $\mathcal{G}$ and that, for each $n$, $g_n$ and $h_n$ are composable. It follows that (for all large $n$), there exist $t, s \in \mathcal{G}$ and $x_n \in X_{-t}$ such that $g_n = (x_n, \alpha_t(x_n))$ and $h_n = (\alpha_t(x_n), \alpha_s(\alpha_t(x_n)))$ and further, that $x_n$ is convergent in $X_{-t}$, $\alpha_t(x_n)$ is convergent in $X_t$, and $\alpha_s(\alpha_t(x_n))$ is convergent in $X_{s+t}$. Consequently, $g_nh_n = (x_n, \alpha_{s+t}(x_n))$ is convergent in $\mathcal{G}$. Thus, multiplication is continuous and $\mathcal{G}$ is a topological groupoid.

The unit space $\mathcal{G}^0$ for the groupoid is the graph of $\alpha_0$, i.e., $\{(x, x) \mid x \in X\}$ and hence is open. Thus $\mathcal{G}$ is a locally compact $r$-discrete principal groupoid.

Define a cocycle $c$ on $\mathcal{G}$ by $c(x, \alpha_t(x)) = t$, for all $(x, \alpha_t(x)) \in \mathcal{G}$. Since $\alpha$ is a partial action, $c$ satisfies the cocycle property. Clearly, $c$ is constant on each open set $\mathcal{O}_{t,U}$, so $c$ is a locally constant real valued cocycle.
We need to verify that the groupoid C*-algebra constructed from $\mathcal{G}$ is isomorphic with the partial crossed product C*-algebra induced by the partial action $\alpha$. The construction of $C^*(\mathcal{G})$ begins with the family $C_c(\mathcal{G})$ of continuous functions on $\mathcal{G}$ with compact support. Since the graphs of the $\alpha_t$ form a disjoint family of open sets, the support of a function in $C_c(\mathcal{G})$ intersects only finitely many graphs.

The set $C_c(\mathcal{G})$ is provided with an involution, given by the formula

$$f^*(x, \alpha_t(x)) = f(\alpha_t(x), x)$$

and a multiplication defined by

$$f \cdot g(x, \alpha_t(x)) = \sum f(x, \alpha_s(x))g(\alpha_s(x), \alpha_t(x)),$$

where the sum is taken over all $s$ for which $x \in X_{-s}$ and $\alpha_s(x) \in X_{s-t}$. Since $f$ and $g$ have compact support, only finitely many terms in the sum are non-zero. Furthermore, when $f$ and $g$ are supported on $\mathcal{G}$-sets, at most one term in this sum is non-zero.

$C_c(\mathcal{G})$ is provided with a norm (usually called the $I$-norm), in which the norm of $f$ is given by

$$\|f\|_I = \max \left\{ \sup_{x \in X} \sum_t |f(x, \alpha_t(x))|, \sup_{y \in X} \sum_t |f(\alpha_t(y), y)| \right\}$$

and then a C*-norm is obtained by defining $\|f\| = \sup_\pi \|\pi(f)\|$, where $\pi$ varies over all *-representations of $C_c(\mathcal{G})$ which are norm decreasing with respect to the $I$-norm.

Recall that the analytic subalgebra associated with the cocycle $c$ is the closure in the C*-norm of all the functions in $C_c(\mathcal{G})$ which are supported on $\{(x, y) \mid c(x, y) \geq 0\}$. For a discussion of analytic subalgebras, see [12].

We next define a *-isomorphism $\Phi$ between $P_c$ and $C_c(\mathcal{G})$. As a map from $(P_c, \|\|_L)$ to $(C_c(\mathcal{G}), \|\|_I)$, $\Phi$ will be norm decreasing but not norm preserving. But with respect to the C*-norms on $P_c$ and $C_c(\mathcal{G})$, $\Phi$ will be an isometry; its extension to the completions yields a *-isomorphism from $A \times_\alpha G$ onto $C^*(\mathcal{G})$.

The isomorphism (and its properties) is determined by its action on monomials in $P_c$. So, if $fU^n$ is a monomial in $P_c$, define $\Phi(fU^n)$ to be the function on $\mathcal{G}$ given by

$$\Phi(fU^n)(x, \alpha_t(x)) = \begin{cases} f(x), & \text{if } t = -n \text{ and } x \in X_n, \\ 0, & \text{otherwise}. \end{cases}$$

Since $f$ is continuous and has compact support in $X_n$, $\Phi(fU^n) \in C_c(\mathcal{G})$ and has support on the graph of $\alpha_{-n}$.

Since $(fU^n)^* = \overline{f \circ \alpha_n U^{-n}}$, we have

$$\Phi((fU^n)^*)(x, \alpha_t(x)) = \Phi(\overline{f \circ \alpha_n U^{-n}})(x, \alpha_t(x)) = \begin{cases} \overline{f(\alpha_n(x))}, & \text{if } t = n \text{ and } x \in X_{-n}, \\ 0, & \text{otherwise}. \end{cases}$$
while
\[
\Phi(f^n)(x, \alpha_t(x)) = \Phi(f^n)(\alpha_t(x), x)
\]
\[
= \begin{cases} 
  f(\alpha_n(x)), & \text{if } t = n \text{ and } x \in X_{-n}, \\
  0, & \text{otherwise.}
\end{cases}
\]

Thus \( \Phi \) is \(*\)-preserving on monomials; when \( \Phi \) is extended to \( \mathcal{P}_c \) by linearity, it remains \(*\)-preserving.

Now suppose that \( f^m \) and \( g^m \) are monomials in \( \mathcal{P}_c \). Recall that \( f^m g^m = h^{n+m} \), where
\[
h(x) = \begin{cases} 
  f(x)g(\alpha_n(x)), & \text{if } x \in X_n \cap X_{n+m}, \\
  0, & \text{otherwise.}
\end{cases}
\]

Therefore
\[
\Phi(f^m g^m)(x, \alpha_t(x)) = \Phi(h^{n+m})(x, \alpha_t(x))
\]
\[
= \begin{cases} 
  h(x), & \text{if } t = -n - m \text{ and } x \in X_{n+m}, \\
  0, & \text{otherwise,}
\end{cases}
\]
\[
= \begin{cases} 
  f(x)g(\alpha_{-n}(x)), & \text{if } t = -n - m \text{ and } x \in X_n \cap X_{n+m}, \\
  0, & \text{otherwise.}
\end{cases}
\]

On the other hand, since \( \Phi(f^n) \) is supported on the graph of \( \alpha_{-n} \) and \( \Phi(g^m) \) is supported on the graph of \( \alpha_{-m} \), \( \Phi(f^n) \cdot \Phi(g^m)(x, \alpha_t(x)) = 0 \) whenever \( t \neq -n - m \) and
\[
\Phi(f^n) \cdot \Phi(g^m)(x, \alpha_{-n-m}(x)) = \Phi(f^n)(x, \alpha_n(x))\Phi(g^m)(\alpha_{-n}(x), \alpha_{-m}(\alpha_{-n}(x)))
\]
\[
= \begin{cases} 
  f(x)g(\alpha_{-n}(x)), & \text{if } x \in X_n \cap X_{n+m}, \\
  0, & \text{otherwise.}
\end{cases}
\]

Thus \( \Phi(f^m g^m) = \Phi(f^n)\Phi(g^m) \). It follows immediately that \( \Phi(pq) = \Phi(p)\Phi(q) \) for all \( p, q \in \mathcal{P}_c \) and so \( \Phi \) is a \(*\)-isomorphism of \( \mathcal{P}_c \) onto \( C_c(G) \). (To see that \( \Phi \) is surjective, recall that an element \( f \) in \( C_c(G) \) is supported on only finitely many graphs; restrict \( f \) to each of these graphs. On each graph, the map \((x, \alpha_{-n}(x)) \mapsto x \) is a homeomorphism, so \( f \) can be transferred to a continuous function with compact support on \( X_n \). These are the coefficients of the polynomial in \( \mathcal{P}_c \) which is mapped by \( \Phi \) to \( f \).)

Let \( p = \sum f_n U^n \) be a polynomial in \( \mathcal{P}_c \). Each \( \Phi(f_n U^n) \) is supported on the graph of a single \( \alpha_t \) (viz., \( t = -n \)), so
\[
\Phi(p)(x, \alpha_t(x)) = \begin{cases} 
  f_{-t}(x), & \text{if } -t \text{ is an index in the sum for } p, \\
  0, & \text{otherwise.}
\end{cases}
\]

Therefore, for any \( x \),
\[
\sum_t |\Phi(p)(x, \alpha_t(x))| = \sum_n |f_n(x)| \leq \sum_n \|f_n\| = \|p\|_L.
\]
(The first sum is taken over those \( t \) for which \( x \in X_{-t} \). The second sum is similarly restricted, but the third sum is taken over all the indices in the expression for \( p \).) Similarly,
\[
\sum_t |\Phi(p)(\alpha_t(y), y)| \leq \|p\|_L, \text{ for each } y; \text{ it follows that } \|\Phi(p)\|_I \leq \|p\|_L.
\]
Although the map $\Phi$ is not norm preserving, as we show after the proof, the fact that it does satisfy $\| \Phi(p) \|_I \leq \| p \|_L$ for all $p \in \mathcal{P}_c$ is enough to imply that $\Phi$ is norm decreasing with respect to the $C^*$-norms on $\mathcal{P}_c$ and $C_c(\mathcal{G})$. Indeed, if $\rho$ is any $\| \cdot \|_I$-decreasing representation of $C_c(\mathcal{G})$, then $\rho \circ \Phi$ is a $\| \cdot \|_I$-decreasing representation of $\mathcal{P}_c$. It follows that $\rho \circ \Phi$ is norm decreasing with respect to the $C^*$-norm on $\mathcal{P}_c$: $\| \rho(\Phi(p)) \| \leq \| p \|$. Since this is true for all $\| \cdot \|_I$-decreasing representations, $\| \Phi(p) \| \leq \| p \|$.

We can, in fact, show that $\Phi$ is an isometry with respect to the $C^*$-norms on $\mathcal{P}_c$ and $C_c(\mathcal{G})$. This task is made simpler by the fact that the reduced partial crossed product (as defined by McClanahan in [11]) is isomorphic to $\mathcal{A} \ltimes_\alpha \mathcal{G}$. This means that we do not need to use all $\| \cdot \|_L$-decreasing representations of $\mathcal{P}_c$ to determine the $C^*$-norm on $\mathcal{A} \ltimes_\alpha \mathcal{G}$. Instead, we can restrict to a family of representations constructed from representations of $\mathcal{A}$ acting on a Hilbert space $H$ and the left regular representation of the group $\mathcal{G}$ acting on $\ell^2(\mathcal{G}, \mathcal{H})$. (In fact, it will suffice to consider only certain of these representations.)

Given a representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$, McClanahan constructs a “regular representation” $\tilde{\pi}$ of $\mathcal{A}$ acting on $\ell^2(\mathcal{G}, \mathcal{H})$. This is done as follows. For each $g \in \mathcal{G}$, a representation $\pi_g: D_g \to \mathcal{B}(\mathcal{H})$ is defined by $\pi_g(z) = \pi(\alpha_g(z))$. This is extended to a representation of all of $\mathcal{A}$ (in a unique way) by taking $\pi_g(z) = s\lim_v \pi_g(u_vz)$, where $u_v$ is an approximate identity in $D_g$ and $s\lim_v$ refers to the limit in the strong operator topology. Finally, $\tilde{\pi}$ is defined by $\tilde{\pi}(z)\xi(g) = \pi_g(z)\xi(g)$ for all $\xi \in \ell^2(\mathcal{G}, \mathcal{H})$. Let $\lambda$ denote the left regular representation of $\mathcal{G}$ acting on $\ell^2(\mathcal{G}, \mathcal{H})$. Then the representations which determine the reduced partial crossed product norm (and hence, in our case, the partial crossed product norm) are those of the form $\tilde{\pi} \times \lambda$. The action of $\tilde{\pi} \times \lambda$ is given on monomials by $\tilde{\pi} \times \lambda(f_iU^t) = \tilde{\pi}(f_i)\lambda_t$.

The situation is further simplified by Proposition 3.4 in [11], which tells us that if $\tilde{\pi}$ is a faithful representation of $\mathcal{A}$, then $\tilde{\pi} \times \lambda$ is a faithful representation of $\mathcal{A} \ltimes_\alpha \mathcal{G}$. We can obtain a faithful representation of $\mathcal{A}$ from the following family of representations: for each $x \in X$, let $\pi^x: \mathcal{A} = C_0(X) \to \mathcal{C}$ be defined by $\pi^x(f) = f(x)$; just let $\pi = \sum \pi^x$. McClanahan’s construction respects direct sums, so $\tilde{\pi} \times \lambda = \sum \pi^x \times \lambda$.

In order to prove that $\Phi$ is norm decreasing (with respect to the $C^*$-norms), it will suffice to show that for $p \in \mathcal{P}_c$, $\| (\tilde{\pi} \times \lambda)(p) \| \leq \| \Phi(p) \|_I$. From this, it follows that $(\tilde{\pi} \times \lambda) \circ \Phi^{-1}$ is continuous with respect to the $\| \cdot \|_I$-norm, and hence with respect to the $C^*$-norm. But $\tilde{\pi} \times \lambda$ is a faithful representation, so $\Phi^{-1}$ is $C^*$-norm decreasing; since the same is true for $\Phi$, $\Phi$ is an isometry from $\mathcal{P}_c$ onto $C_c(\mathcal{G})$. Therefore, $\Phi$ has a unique extension to a $^*$-isomorphism from $\mathcal{A} \ltimes_\alpha \mathcal{G}$ to $C^*(\mathcal{G})$. This extension clearly maps the analytic algebra associated with the positive cone in $\mathcal{G}$ onto the analytic algebra associated with the cocycle on $\mathcal{G}$.

Since $\tilde{\pi} \times \lambda = \sum \pi^x \times \lambda$, we can complete this direction of the proof by showing that, for any $x \in X$ and $p \in \mathcal{P}_c$, $\| (\pi^x \times \lambda)(p) \| \leq \| \Phi(p) \|_I$. Now, for each $t \in \mathcal{G}$, $\pi^x_t: C_0(X_t) \to \mathcal{B}(\mathcal{C}) \cong \mathcal{C}$ is given by

$$
\pi^x_t(f) = \begin{cases} 
\alpha_{-t}(f)(x) = f(\alpha_t(x)), & \text{if } x \in X_{-t}, \\
0, & \text{otherwise.}
\end{cases}
$$

Hereafter, $f(\alpha_t(x))$ will be understood to designate 0 whenever $x \notin X_{-t}$. The unique extension of $\pi^x_t$ from $C_0(X_t)$ to all of $A = C_0(X)$ is given by exactly the same formula. The regular representation $\pi^x$ determined by $\pi^x$ maps $A$ into $\mathcal{B}(\ell^2(\mathcal{G}, \mathcal{C}))$ and is given by the formula $(\pi^x(f)\xi)(t) = \pi^x_t(f)\xi(t) = f(\alpha_t(x))\xi(t)$. We can more conveniently describe $\pi^x$ in...
terms of its action on the canonical basis \( \{ \delta_t \} \) for \( \ell^2(G, \mathbb{C}) \):
\[
\tilde{\pi}^x(f) \delta_t = f(t) \delta_t.
\]
The left regular representation \( \lambda \), on the other hand, is given by
\[
\lambda_t \delta_s = \delta_{t+s}.
\]
Thus, the matrix which represents \( \tilde{\pi}^x \) with respect to the canonical basis is a diagonal matrix and the matrix which represents \( \lambda \) is supported on a single sub or super diagonal (main diagonal when \( t = 0 \)), where all the entries are 1’s.

If \( f_t U^t \) is a monomial in \( \mathcal{P}_c \), then
\[
(\tilde{\pi}^x \times \lambda)(f_t U^t) = \tilde{\pi}^x(f_t) \lambda_t.
\]
This is an operator in \( \mathcal{B}(\ell^2(G, \mathbb{C})) \) whose matrix has non-zero entries only in the ‘diagonal’ determined by \( t \); in fact,
\[
(\tilde{\pi}^x \times \lambda)(f_t U^t)(\delta_s) = \tilde{\pi}^x(f_t) \lambda_t \delta_s
\]
\[
= \tilde{\pi}^x(f_t) \delta_{t+s} = f_t(\alpha_{t+s}(x)) \delta_{t+s}.
\]
If we let \( k \in G \), the single entry in the \( k \)-column of \( \tilde{\pi}^x(f_t) \lambda_t \) is \( f_t(\alpha_{t+k}(x)) \) and the single entry in the \( k \)-row is \( f_t(\alpha_k(x)) \).

Let \( p = f_{t_1} U^{t_1} + \cdots + f_{t_n} U^{t_n} \) be an arbitrary polynomial in \( \mathcal{P}_c \). Then
\[
(\tilde{\pi}^x \times \lambda)(p) = \tilde{\pi}^x(f_{t_1}) \lambda_{t_1} + \cdots + \tilde{\pi}^x(f_{t_n}) \lambda_{t_n}
\]
has finitely many entries in each row and in each column. The \( \ell^1 \)-norm of the \( k \)-column is
\[
c_k = |f_{t_1}(\alpha_{t_1+k}(x))| + \cdots + |f_{t_n}(\alpha_{t_n+k}(x))|
\]
and the \( \ell^1 \)-norm of the \( k \)-row is
\[
r_k = |f_{t_1}(\alpha_k(x))| + \cdots + |f_{t_n}(\alpha_k(x))|.
\]
Therefore, \( \|(\tilde{\pi}^x \times \lambda)(p)\| \leq \sup_k \{c_k, r_k\} \). We can complete the argument by showing that \( \sup \{c_k, r_k\} \leq \|\Phi(p)\|_1 \).

Since \( \Phi(p) \) is supported on the graphs of \( \alpha_{t_1}, \ldots, \alpha_{t_n} \) and since each term \( \Phi(f_{t_k} U^{t_k}) \) is supported on the graph of \( \alpha_{t_k} \) alone,
\[
\sum_t |\Phi(p)(z, \alpha_t(z))| = |\Phi(p)(z, \alpha_{t_1}(z))| + \cdots + |\Phi(p)(z, \alpha_{t_n}(z))|
\]
\[
= |f_{t_1}(z)| + \cdots + |f_{t_n}(z)|.
\]
It follows that
\[
\sup \{r_k\} \leq \sup \sum_{z \in X} |\Phi(p)(z, \alpha_{t}(z))|.
\]
Also,
\[
\sum_t |\Phi(p)(\alpha_t(z), z)| = \sum_t |\Phi(p)(\alpha_t(z), \alpha_{t}(\alpha_t(z)))|
\]
\[
= |f_{t_1}(\alpha_{t_1}(z))| + \cdots + |f_{t_n}(\alpha_{t_n}(z))|.
\]
Hence,
\[ \sup \{ c_k \} \leq \sup_{z \in X} \sum_t |\Phi(p)(z, \alpha_t(z))|. \]

This yields \( \| (\pi^x \times \lambda)(p) \| \leq \| \Phi(p) \|_I \) and the proof of this direction of the theorem is complete.

The converse direction in the theorem remains to be considered. Start with an \( r \)-discrete principal groupoid \( G \) (based on a locally compact, second countable, Hausdorff space \( X \)) with a locally constant cocycle \( c \). Let \( G \) be the range of the cocycle. Since \( X \) is second countable and \( c \) satisfies the cocycle property, \( G \) is a countable subgroup of \( \mathbb{R} \).

For \( t \in G \), the set \( U_t = \{(x, y) \mid c(x, y) = t\} \) is an open \( G \)-set in \( G \). With \( \pi_1 \) and \( \pi_2 \) the natural projections of \( X \times X \) onto \( X \) (first and second coordinate projections), let \( X_{-t} = \pi_1(U_t) \) and \( X_t = \pi_2(U_t) \). Then \( U_t \) is the graph of a homeomorphism, which we denote by \( \alpha_t \), of \( X_{-t} \) onto \( X_t \). The system \( \alpha \) satisfies all the requirements for a free partial action of \( G \) on \( X \). (Freeness follows from the fact that \( U_t \cap U_s = \emptyset \) whenever \( t \neq s \).)

Since the groupoid \( G \) is exactly the groupoid obtained from the partial action \( \alpha \) in the first part of the proof, the groupoid \( C^* \)-algebra and the partial crossed product \( C^* \)-algebra are isomorphic (with appropriate correspondence of analytic subalgebras).

\[ \square \]

Remark. (1) The map \( \Phi \) is not norm preserving with respect to the \( I \)-norm and the \( L \)-norm, as the following example shows. Let \( X = \{1, \ldots, n\} \). Let \( \beta_1 \) be the map \( k \mapsto k + 1 \) on the natural domain; \( \beta_1 \) generates a partial action of \( \mathbb{Z} \) on \( X \). The partial crossed product algebra \( (\mathcal{P}_\alpha) \) is isomorphic to \( M_n \). The associated groupoid is \( X \times X \), the full equivalence relation and the groupoid \( C^* \)-algebra is again isomorphic to \( M_n \). If we identify both the partial crossed product algebra and the groupoid \( C^* \)-algebra with \( M_n \) then we can describe the \( I \)-norm and the \( L \)-norm as follows: for \( a = (a_{ij}) \)

\[ \|a\|_I = \max_i \sum_j |a_{ij}|, \quad \|a\|_L = \sum_{t=-n+1}^{n-1} \max_{j-i=t} |a_{ij}|. \]

In other words, to compute the \( I \)-norm, compute the \( \ell^1 \)-norm of each row and of each column in \( a \) and take the largest value. To compute the \( L \)-norm, compute the \( \ell^\infty \)-norm of each diagonal (determined by fixing values for \( j-i \)) and add all these numbers.

If we now take a matrix which is zero except for entries on the counter diagonal, we obtain different values for the two norms. To be specific, suppose

\[ a_{ij} = \begin{cases} 1, & \text{if } i + j = n + 1, \\ 0, & \text{otherwise}. \end{cases} \]

Then \( \|a\|_I = 1 \) and \( \|a\|_L = n \). This disparity makes it clear that for infinite dimensional algebras, the two norms need not be equivalent.

(2) The beginning of the proof of Theorem 5.1 outlines the passage from a partial action of a discrete abelian group on a locally compact Hausdorff topological space to a groupoid \( C^* \)-algebra. (The groupoid is the union of the graphs of the partial homeomorphisms.) This connection has been studied for \( \mathbb{Z} \)-partial actions by Peters and Poon in [17] and for partial actions by countable discrete abelian groups by Peters and Zerr in [18]. These papers use
the language of partial dynamical systems rather than partial actions, but there is little substantive difference. In particular, the question of when the associated groupoid $C^*$-algebra is AF is addressed. Peters and Zerr give necessary and sufficient conditions on a partial dynamical system for the associated groupoid $C^*$-algebra to be AF. These conditions are rather involved, so we won’t restate them here, but merely refer the reader to [18]; in view of Theorem 5.1 these conditions characterize when a partial crossed product by a discrete countable subgroup of $\mathbb{R}$ is an AF $C^*$-algebra.

(2) The order preserving normalizer of a TAF algebra plays an important role in the study of the ideal structure of TAF algebras and in the study of automatic continuity for algebraic isomorphisms [6 4 3]. In particular, it is useful to know when an algebra is generated by its order preserving normalizer. The earliest result about when the order preserving normalizer generates an algebra appears in [13], where it is shown that if $\mathcal{G}$ is an $r$-discrete principal groupoid with a continuous cocycle onto a discrete ordered group, if $A$ is the analytic subalgebra of $C^*(\mathcal{G})$ associated with the cocycle, and if $P \subset \mathcal{G}$ is the spectrum of $A$, then $P$ is the union of the monotone $\mathcal{G}$-sets which it contains. Now, the (compact, open) $\mathcal{G}$-sets correspond to the partial isometries in $C^*(\mathcal{G})$ which normalize the diagonal $A \cap A^*$ and a $\mathcal{G}$-set is monotone if, and only if, the normalizing partial isometry is order preserving. Furthermore, the monotone $\mathcal{G}$ sets cover $P$ if, and only if, $A$ is generated by its order preserving normalizer. Since locally constant cocycles are necessarily continuous, the following corollary to Theorem 5.1 is immediate.

5.2. Corollary. If a TAF algebra is an analytic partial crossed product, then it is generated by its order preserving normalizer.

6. Conjugacy Results

A major theme in the study of analytic crossed products is that $C(X) \times_\alpha \mathbb{Z}^+$ contains all of the dynamical information about $\alpha: X \to X$, in the sense that two homeomorphisms of $X$ are conjugate if, and only if, the associated analytic crossed products are isomorphic. See, for example, [1 2 16 11 9 21]. In this section, we show that an analytic partial crossed product $C(X) \times_\alpha \mathbb{Z}^+$ similarly contains all of the dynamical information about the partial action $\alpha$.

There are several related results for limit algebras. Power, in [20, Theorem 3.4], used inverse systems of simplicial complexes to construct both operator algebras and dynamical systems; he showed that conjugacy of the dynamical systems is equivalent to isometric isomorphism of the operator algebras and to isomorphism of the associated coordinates. Poon and Wagner, in [13, Theorem 4.1], show that for a family of subalgebras of crossed products with a distinguished point, the algebras are isomorphic exactly when there is a conjugacy of the dynamical systems that sends one distinguished point to the other. Precisely, their systems are obtained from an essentially minimal homeomorphism, i.e., one possessing unique minimal closed invariant set, acting on a Cantor set; fixing a point $x$ in this unique minimal set, they consider the subalgebra generated by the diagonal, $C(X)$, and the set $\{Uf \mid f(x) = 0, f \in C(X)\}$, where $U$ is canonical unitary implementing the homeomorphism.

The composition of two partial homeomorphisms of a space $X$, say $\alpha: A \to B$ and $\beta: C \to D$, is the partial homeomorphism $\alpha \circ \beta$ on the domain where this makes sense, i.e., on $C \cap \beta^{-1}(A)$.
6.1. **Definition.** We say two partial homeomorphisms $\alpha: A \to B$ and $\beta: C \to D$ with $A, B \subseteq X$ and $C, D \subseteq Y$ are *conjugate* if there is a homeomorphism $\tau: X \to Y$ such that $\tau$ maps the domain of $\alpha$ onto the domain of $\beta$, the range of $\alpha$ onto the range of $\beta$ and $\beta \circ \tau = \tau \circ \alpha$, as partial homeomorphisms.

Two partial actions, $\alpha$ and $\beta$, of a positive cone $\Sigma$ acting on spaces $X$ and $Y$ are *conjugate* if there is a homeomorphism $\tau: X \to Y$ such that $\tau$ induces a conjugacy between $\alpha_s$ and $\beta_s$, for all $s \in \Sigma$.

6.2. **Theorem.** Let $\alpha$ and $\beta$ be two partial actions of $\mathbb{Z}^+$ on locally compact Hausdorff spaces $X$ and $Y$, respectively. Assume that each action satisfies the domain ordering property and that each action is free. Then there is a continuous isomorphism of $C_0(X) \times_\alpha \mathbb{Z}^+$ onto $C_0(Y) \times_\beta \mathbb{Z}^+$ if, and only if, $\alpha$ and $\beta$ are conjugate.

**Proof.** The proof is a modification of the proof of Theorem 1 in [21]. First consider the algebra $A = C_0(X) \times_\alpha \mathbb{Z}^+$ associated with the action $\alpha$. Following the notation set up in Section 5, $A$ is the closure of polynomials $f_0 + f_1U^1 + f_2U^2 + \cdots + f_nU^n$, where each $f_k \in C_0(X_k)$. Let $\mathcal{C}_1$ be the closed commutator ideal in $A$ and, for each $k \geq 2$, let $\mathcal{C}_k$ be the closed ideal generated by $k$-fold products of elements of $\mathcal{C}_1$. For $k \geq 1$, let $\mathcal{B}_k$ be the closed ideal generated by all polynomials of the form $f_kU^k + \cdots f_nU^n$, $n \geq k$. Our first task is to show that $\mathcal{C}_k = \mathcal{B}_k$, for all $k$.

Let $f_nU^n$ and $g_mU^m$ be two monomials in $\mathcal{P}$. Then

$$f_nU^n g_mU^m = [(f_n \circ \alpha_n)g_m] \circ \alpha_{-n}U^{n+m}$$

and

$$g_mU^m f_nU^n = [(g_m \circ \alpha_m)f_n] \circ \alpha_{-m}U^{m+n}.$$  

(The coefficients are supported in $X_n \cap X_{n+m}$ and $X_m \cap X_{n+m}$, but each of these sets is just $X_{n+m}$ since $\alpha$ satisfies the domain ordering property.)

If $n = m = 0$, then $f_0g_0 - g_0f_0 = 0$, since $C_0(X)$ is abelian. If either $m \neq 0$ or $n \neq 0$, then the commutator $[f_nU^n, g_mU^m]$ lies in $\mathcal{B}_1$. By linearity, this is true for all elements of $\mathcal{P}$; hence $\mathcal{C}_1 \subseteq \mathcal{B}_1$.

If $f_0 \in C_0(X)$ and $g_m \in C_0(X_m)$ with $m \geq 1$, then

$$[f_0, g_mU^m] = [f_0g_m - [(g_m \circ \alpha_m)f_0] \circ \alpha_{-m}]U^m.$$  

Letting $h = f_0g_m - [(g_m \circ \alpha_m)f_0] \circ \alpha_{-m}$, we have $h(x) = 0$ when $x \notin X_m$ and

$$h(x) = f_0(x)g_m(x) - g_m(x)f_0(\alpha_{-m}(x))$$

$$= [f_0(x) - f_0(\alpha_{-m}(x))]g_m(x)$$

when $x \in X_m$.

Since $\alpha_{-m}(x) \neq x$, for all $x$, we can, for each pair of distinct points $x, y \in X_m$, choose $f_0 \in C_0(X)$ and $g_m \in C_0(X_m)$ so that $h(y) \neq h(x) \neq 0$. Let $\mathcal{E} = \{g \in C_0(X_m) \mid gU^m \in \mathcal{C}_1\}$. Then $\mathcal{E}$ is a closed *-subalgebra of $C_0(X_m)$ and, from what we have just observed, $\mathcal{E}$ separates points of $X_m$ and separates points from $\infty$. Hence, by the Stone-Weierstrass theorem, $\mathcal{E} = C_0(X_m)$. As $m \geq 1$ is arbitrary, $\mathcal{C}_1 = \mathcal{B}_1$.

If $p$ and $q$ are two polynomials in $\mathcal{C}_1 = \mathcal{B}_1$, then $pq$ is a polynomial in $\mathcal{B}_2$; thus $\mathcal{C}_2 \subseteq \mathcal{B}_2$.

Let $f_1U^1$ and $g_mU^m$ be two monomials (with $m \geq 1$). Then

$$f_1U^1 g_mU^m = [(f_1 \circ \alpha_1)g_m] \circ \alpha_{-1}U^{m+1}.$$
Letting $h = [(f_1 \circ \alpha_1)g_m] \circ \alpha^{-1}$, we have $h(x) = 0$ when $x \notin X_{m+1}$ and $h(x) = f_1(x)g_m(\alpha^{-1}(x))$ when $x \in X_1 \cap X_{m+1} = X_{m+1}$. (Since $\alpha$ satisfies the domain ordering property, $X_{m+1} \subseteq X_1$.)

The freedom to choose $f_1$ and $g_m$ arbitrarily in $C_0(X_1)$ and $C_0(X_m)$ means that the set of all coefficient functions $h$ which arise in this fashion separates points of $X_m$ and separates points from $\infty$. Again, the Stone-Weierstrass theorem shows that $C_2$ contains all polynomials of the form $f_2U^2 + \cdots + f_nU^n$; hence $B_2 \subseteq C_2$.

Once we have shown that $C_{k-1} = B_{k-1}$, the same argument yields $C_k = B_k$ (since $C_k$ is the closed ideal generated by products of elements in $C_1$ and $C_{k-1}$); thus $C_k = B_k$ for all $k \geq 1$.

From the characterization of $C_1$ given above, it is evident that $A/C_1 \cong C_0(X)$.

Let $k \geq 1$. We need to describe the ideals which lie between $C_{k+1}$ and $C_k$ and are maximal in this class. For each $x \in X_k$, let

$$I_x = \{f \in C_0(X) \mid f(x) = 0 \text{ and supp } f \subseteq X_k\}$$

and for each $y \in X_{-k}$, let

$$J_y = \{f \in C_0(X) \mid f(y) = 0 \text{ and supp } f \subseteq X_{-k}\}$$

Consider $I_xC_k + C_{k+1}$, the closure of all elements of the form $fp + q$, where $f \in I_x, p \in C_k$, and $q \in C_{k+1}$. Clearly, $C_{k+1} \subseteq I_xC_k + C_{k+1} \subseteq C_k$. Furthermore, $I_xC_k + C_{k+1}$ is an ideal. (Multiplication of an element of $I_xC_k + C_{k+1}$ by a monomial of the form $f_nU^n$ with $n \geq 1$ clearly yields an element in $C_{k+1}$. Multiplication by an element of $C_0(X)$ (viewed as an element of $P$) produces a product again in $I_xC_k + C_{k+1}$). It follows that $I_xC_k + C_{k+1}$ satisfies the ideal property.

We can easily identify $I_xC_k + C_{k+1}$ as the closed ideal generated by polynomials of the form $f_kU^k + \cdots + f_nU^n$ where $f_k(x) = 0$.

If $K$ is any ideal with $I_xC_k + C_{k+1} \subseteq K \subseteq C_k$, then all elements of $K$ can be written in the form $gU^k + p$, where $p \in C_{k+1}$. For some element of $K$, $g(x) \neq 0$. It now follows that the coefficients $g$ of $U^k$ in elements of $K$ separate points, whence $K = C_k$. Thus, each ideal $I_xC_k + C_{k+1}$ is maximal between $C_{k+1}$ and $C_k$.

On the other hand, let $K$ be a maximal ideal between $C_{k+1}$ and $C_k$. Again, any element of $K$ can be written in the form $gU^k + p$ with $p \in C_{k+1}$ and there must be some element $x \in X_k$ such that $g(x) = 0$ for all such $g$. (Otherwise, the Stone Weierstrass theorem again implies that $K = C_k$.) So $K \subseteq I_xC_k + C_{k+1}$.

Thus, $\{I_xC_k + C_{k+1} \mid x \in X_k\}$ is exactly the family of maximal ideals which lie between $C_{k+1}$ and $C_k$. Essentially the same argument also shows that $\{C_{k+1}J_y + C_{k+1}\}$ is also the family of maximal ideals which lie between $C_{k+1}$ and $C_k$. In particular, for each $x \in X_k$ there is one, and only one, $y \in X_{-k}$ such that $I_xC_k + C_{k+1} = C_kJ_y + C_{k+1}$. We next show that for such a pair $x$ and $y$, $x = \alpha_k(y)$.

Indeed, let $f_k \in C_0(X_k)$ and $f \in J_y$. Then $f_kU^kf = [(f_k \circ \alpha)g] \circ \alpha^{-1}f_k$. For $z \in X_k$, $[(f_k \circ \alpha)g] \circ \alpha^{-1}(z) = f_k(z)g(\alpha^{-1}(z))$. This vanishes if $\alpha^{-1}(z) = y$, i.e., if $z = \alpha(y)$, showing that all such terms $f_kU^k \in I_{\alpha_k}(y)$. Consequently, $I_xC_k + C_{k+1} = C_kJ_y + C_{k+1}$ if, and only if, $x = \alpha_k(y)$.

Now let $\alpha$ be a partial action of $\mathbb{Z}^+$ acting on $X$ and $\beta$ a partial action acting on $Y$. Let $A = C_0(X) \times_\alpha \mathbb{Z}^+$ and $B = C_0(Y) \times_\beta \mathbb{Z}^+$. Suppose that $\psi: A \to B$ is a continuous isomorphism. Let $C_k$, $k = 1, 2, \ldots$ be the ideals considered above for $A$ and let $D_k$ be the corresponding ideals for $B$. Since $\psi$ is a continuous isomorphism, $\psi(C_k) = D_k$, for all $k$. 

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Now, $\psi$ induces an isomorphism of $A/\mathcal{C}_1$ onto $B/D_1$ by $p + \mathcal{C}_1 \mapsto \psi(p) + D_1$. But $A/\mathcal{C}_1 \cong C_0(X)$ and $B/D_1 \cong C_0(Y)$, so $\psi$ induces an isomorphism of $C_0(X)$ onto $C_0(Y)$. In fact, we can view $C_0(X)$ as a subalgebra of $A$ and $C_0(Y)$ as a subalgebra of $B$ in a natural way and the isomorphism of $C_0(X)$ onto $C_0(Y)$ is just $\psi$ restricted to these subalgebras. Consequently, there is a homeomorphism $\tau: X \to Y$ such that $\psi(f) = f \circ \tau^{-1}$, for all $f \in C_0(X)$.

For each $k$, the isomorphism $\psi$ maps $\mathcal{C}_k$ onto $\mathcal{D}_k$. It follows that $\psi$ carries the ideals which are maximal between $\mathcal{C}_{k+1}$ and $\mathcal{C}_k$ to the ideals which are maximal between $\mathcal{D}_{k+1}$ and $\mathcal{D}_k$. Suppose that $\psi(I_x \mathcal{C}_k + \mathcal{C}_{k+1}) = I_x \mathcal{D}_k + \mathcal{D}_{k+1}$, where $x \in X_k$ and $z \in Y_k$.

If $f \in C_0(X_k)$ with $f \in I_x$, $p \in \mathcal{C}_k$, and $q \in \mathcal{C}_{k+1}$, then

$$\psi(f p + q) = \psi(f) \psi(p) + \psi(q) = f \circ \tau^{-1} \psi(p) + \psi(q)$$

with $\psi(p) \in \mathcal{D}_k$ and $\psi(q) \in \mathcal{D}_{k+1}$. It follows that $f(\tau^{-1}(z)) = 0$ for all $f \in I_x$, so $\tau^{-1}(z) = x$. Thus

$$\psi(I_x \mathcal{C}_k + \mathcal{C}_{k+1}) = I_{\tau(x)} \mathcal{D}_k + \mathcal{D}_{k+1}.$$

In particular, $\tau(X_k) = Y_k$. Similarly, $\tau(X_{-k}) = Y_{-k}$ and $\psi(\mathcal{C}_k J_y + \mathcal{C}_{k+1}) = \mathcal{D}_k J_{\tau(y)} + \mathcal{D}_{k+1}$.

But $\mathcal{C}_k J_y + \mathcal{C}_{k+1} = I_{\alpha_k(y)} \mathcal{C}_k + \mathcal{C}_{k+1}$ and $\mathcal{D}_k J_{\tau(y)} + \mathcal{D}_{k+1} = I_{\beta_k(y)} \mathcal{D}_k + \mathcal{D}_{k+1}$. Since $\psi$ maps the first of these ideals onto the second, we have $\tau(\alpha_k(y)) = \beta_k(\tau(y))$, for all $y \in X_{-k}$.

Thus, $\tau \circ \alpha_k = \beta_k \circ \tau$ as a partial homeomorphism; i.e., the diagram

$$\begin{array}{ccc}
X_{-k} & \xrightarrow{\tau} & Y_{-k} \\
\downarrow & & \downarrow \\
X_k & \xrightarrow{\tau} & Y_k
\end{array}$$

commutes. This shows that $\alpha$ and $\beta$ are conjugate.

The converse is routine. □

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