EXTENDING THE PARKING SPACE
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Abstract. The action of the symmetric group $S_n$ on the set $\text{Park}_n$ of parking functions of size $n$ has received a great deal of attention in algebraic combinatorics. We prove that the action of $S_n$ on $\text{Park}_n$ extends to an action of $S_{n+1}$. More precisely, we construct a graded $S_{n+1}$-module $V_n$ such that the restriction of $V_n$ to $S_n$ is isomorphic to $\text{Park}_n$. We describe the $S_n$-Frobenius characters of the module $V_n$ in all degrees and describe the $S_{n+1}$-Frobenius characters of $V_n$ in extreme degrees. We give a bivariate generalization $V_n^{(\ell,m)}$ of our module $V_n$ whose representation theory is governed by a bivariate generalization of Dyck paths. A Fuss generalization of our results is a special case of this bivariate generalization.

1. Introduction

This paper is about extending the visible permutation action of $S_n$ on the space $\text{Park}_n$ spanned by parking functions of size $n$ to a hidden action of the larger symmetric group $S_{n+1}$. The $S_{n+1}$-module we construct will be a subspace of the coordinate ring of the reflection representation of type $A_n$ and will inherit the polynomial grading of this coordinate ring. Using statistics on Dyck paths, Theorem 2 will give an explicit combinatorial formula for the graded $S_n$-Frobenius character of our module. In Theorem 5 we will describe the extended $S_{n+1}$ action in extreme degrees.

As far as the authors know, this is the first example of an extension of the $S_n$-module structure on $\text{Park}_n$ to $S_{n+1}$ and the first proof that the $S_n$-module structure on $\text{Park}_n$ extends to $S_{n+1}$. Our main theorems should be thought of as parallel to several well known extensions, most notably the action of $S_{n+1}$ on the multilinear subspace of the free Lie algebra on $n+1$ symbols, which extends the regular representation of $S_n$ to $S_{n+1}$. See Section 4 for more on such questions.

2. Background and Main Results

A length $n$ sequence $(a_1, \ldots, a_n)$ of positive integers is called a parking function of size $n$ if its nondecreasing rearrangement $(b_1 \leq \cdots \leq b_n)$ satisfies

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either of the relations \( \lambda \subseteq \mu \text{ or the final nonzero entry in the vector difference } \text{Ferrers diagram of } \mu \text{ total order on partitions of fixed length } \lambda \).

We have that \( \text{Young's lattice} \) is a partition, we define a new partition \( \text{mult}(\lambda) \) whose parts are obtained by listing the (positive) part multiplicities in \( \lambda \) in weakly decreasing order. For example, we have that \( \text{mult}(4, 4, 3, 3, 1, 0) = (3, 2, 2, 1) \).

We will make use of two orders on partitions in this paper, one partial and one total. The first partial order is Young's lattice with relations given by \( \lambda \subseteq \mu \) if \( \lambda_i \leq \mu_i \) for all \( i \geq 1 \) (where we append an infinite string of zeros to the ends of \( \lambda \) and \( \mu \) so that these inequalities make sense). Equivalently, we have that \( \lambda \subseteq \mu \) if and only if the Ferrers diagram of \( \lambda \) fits inside the Ferrers diagram of \( \mu \). Graded reverse lexicographical (grevlex) order is the total order on partitions of fixed length \( n \) defined by \( \lambda \prec \mu \) if either \( |\lambda| < |\mu| \) or the final nonzero entry in the vector difference \( \lambda - \mu \) is positive. For example, if \( n = 6 \) we have \( (4, 2, 2, 1, 0) \prec (3, 3, 3, 1, 1, 0) \). In particular, either of the relations \( \lambda \subseteq \mu \) or \( \lambda \preceq \mu \) imply that \( |\lambda| \leq |\mu| \).

For a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n \), we let \( S_\lambda \) denote the Young subgroup \( S_{\lambda_1} \times \cdots \times S_{\lambda_k} \) of \( S_n \). We denote by \( M^\lambda \) the coset representation of \( S_n \) given by \( M^\lambda := \text{Ind}_{S_\lambda}^{S_n}(1_{S_\lambda}) \cong S_n \rtimes S_\lambda \) and we denote by \( S^\lambda \) the irreducible representation of \( S_n \) labeled by the partition \( \lambda \).

Let \( R_n \) denote the \( \mathbb{C} \)-vector space of class functions \( S_n \to \mathbb{C} \). Identifying modules with their characters, the set \( \{ S^\lambda : \lambda \vdash n \} \) forms a basis of \( R_n \). The graded vector space \( R := \bigoplus_{n \geq 0} R_n \) attains the structure of a \( \mathbb{C} \)-algebra.

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1. The terminology arises from the following situation. Consider a linear parking lot with \( n \) parking spaces and \( n \) cars that want to park in the lot. For \( 1 \leq i \leq n \), car \( i \) wants to park in the space \( a_i \). At stage \( i \) of the parking process, car \( i \) parks in the first available spot \( \geq a_i \), if any such spots are available. If no such spots are available, car \( i \) leaves the lot. The driver preference sequence \( (a_1, \ldots, a_n) \) is a parking function if and only if all cars are able to park in the lot.

2. Here we adopt the symmetric group multiplication convention that says, for example, \( (1, 2)(2, 3) = (1, 2, 3) \) so that this is a left action.
via the induction product $S^λ \circ S^\mu := \text{Ind}_{S_{n+m}}^{S_n \times S_m} (S^λ \otimes_C S^\mu)$, where $\lambda \vdash n$ and $\mu \vdash m$.

We denote by $\Lambda$ the ring of symmetric functions (in an infinite set of variables $X_1, X_2, \ldots$, with coefficients in $\mathbb{C}$). The $\mathbb{C}$-algebra $\Lambda$ is graded and we denote by $\Lambda_n$ the homogeneous piece of degree $n$. Given a partition $\lambda$, we denote the corresponding Schur function by $s_\lambda$ and the corresponding complete homogeneous symmetric function by $h_\lambda$.

The Frobenius character is the graded $\mathbb{C}$-algebra isomorphism $\text{Frob} : R \rightarrow \Lambda$ induced by setting $\text{Frob}(S^\lambda) = s_\lambda$. It is well known that we have $\text{Frob}(M^\lambda) = h_\lambda$. Generalizing slightly, if $V = \bigoplus_{k \geq 0} V(k)$ is a graded $S_n$-module, define the graded Frobenius character $\text{grFrob}(V; q) \in \Lambda \otimes_{\mathbb{C}} \mathbb{C}[q]$ to be the formal power series in $q$ with coefficients in $\Lambda$ given by $\text{grFrob}(V; q) := \sum_{k \geq 0} \text{Frob}(V(k))q^k$.

A Dyck path of size $n$ is a lattice path $D$ in $\mathbb{Z}^2$ consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(0, 0)$, ends at $(n, n)$, and stays weakly above the line $y = x$. A maximal contiguous sequence of vertical steps in $D$ is called a vertical run of $D$.

We will associate two partitions to a Dyck path $D$ of size $n$. The vertical run partition $\lambda(D) \vdash n$ is obtained by listing the (positive) lengths of the vertical runs of $D$ in weakly decreasing order. For example, if $D$ is the Dyck path in Figure 1, then $\lambda(D) = (3, 2, 1)$. The area partition $\mu(D)$ is the partition of length $n$ whose Ferrers diagram is the set of boxes to the upper left of $D$ in the $n \times n$ square with lower left coordinate at the origin. For example, if $D$ is the Dyck path of size 6 in Figure 1, then $\mu(D) = (5, 1, 1, 1, 0, 0)$. The boxes in the Ferrers diagram of $\mu(D)$ are shaded. We define the area statistic\(^3\) on Dyck paths by $\text{area}(D) = |\mu(D)|$. For the Dyck path in our running example, $\text{area}(D) = 8$. By construction, we have that $\text{mult}(\mu(D)) = \lambda(D)$ for any Dyck path $D$ of size $n$.

Dyck paths of size $n$ can be used to obtain a decomposition of $\text{Park}_n$ as a direct sum of coset modules $M^\lambda$. In particular, let $D$ be a Dyck path of size $n$. A labeling of $D$ assigns each vertical run of $D$ to a subset of $[n] := \{1, 2, \ldots, n\}$ of size equal to the length of that vertical run such that every letter in $[n]$ appears exactly once as a label of a vertical run. Figure 1 shows an example of a labeled Dyck path of size 6, where the subsets labeling the vertical runs are placed just to the right of the runs.

The set of labeled Dyck paths of size $n$ carries an action of $S_n$ given by label permutation. There is an $S_n$-equivariant bijection from the set of labeled Dyck paths $D$ of size $n$ to parking functions $(a_1, \ldots, a_n)$ of size $n$ given by letting $a_i$ be one greater than the $x$-coordinate of the vertical run of $D$ labeled by $i$. For example, the labeled Dyck path in Figure 1 corresponds to the parking function $(2, 6, 1, 2, 1, 2) \in \text{Park}_6$. Since any fixed labeled Dyck path of size $D$ generates a cyclic $S_n$-module isomorphic to

\(^3\)Many authors instead define the area of a Dyck path $D$ to be the number of complete lattice squares between $D$ and the line $y = x$, so that our statistic would be the ‘coarea’.
M^λ(D), it is immediate that the parking space Park_n decomposes into coset representations as

\[ \text{Park}_n \cong S_n \bigoplus D \]

where the direct sum is over all Dyck paths D of size n. Equivalently, we have that the Frobenius character of Park_n is given by \( \text{Frob}(\text{Park}_n) = \sum_D h^\lambda(D) \).

For example, the 5 Dyck paths of size 3 shown in Figure 3 lead to the Frobenius character

\[ \text{Frob}(\text{Park}_3) = h^{(3)} + 3h^{(2,1)} + h^{(1,1,1)}. \]

The vector space underlying the \( S_{n+1} \)-module which will extend Park_n is a subspace of the polynomial ring \( \mathbb{C}[x_1, \ldots, x_{n+1}] \) in \( n+1 \) variables and first studied in the work of Postnikov and Shapiro [PoSh]. Let \( K_{n+1} \) denote the complete graph on the vertex set \( [n+1] \). Given an edge \( e = (i < j) \) in \( K_{n+1} \), we associate the polynomial weight \( p(e) := x_i - x_j \in \mathbb{C}[x_1, \ldots, x_{n+1}] \). A subgraph \( G \subseteq K_{n+1} \) (identified with its edge set) gives rise to the polynomial weight \( p(G) := \prod_{e \in G} p(e) \). Following Postnikov and Shapiro, we call a subgraph \( G \subseteq K_{n+1} \) slim if the complement edge set \( K_{n+1} - G \) is a connected graph on the vertex set \( [n+1] \).

**Definition 1.** Denote by \( V_n \) the \( \mathbb{C} \)-linear subspace of \( \mathbb{C}[x_1, \ldots, x_{n+1}] \) given by

\[ V_n := \text{span}\{p(G) : G \text{ is a slim subgraph of } K_{n+1}\}. \]

Let \( V_n(k) \) denote the homogeneous piece of \( V_n \) of polynomial degree \( k \); the space \( V_n(k) \) is spanned by those polynomials \( p(G) \) corresponding to slim subgraphs \( G \) of \( K_{n+1} \) with \( k \) edges.

While the set of polynomials \( \{p(G) : G \text{ is a slim subgraph of } K_{n+1}\} \) is linearly dependent in general, a basis for \( V_n \) can be constructed using standard matroid theoretic results [PoSh, Proposition 9.4]. Fix a total order on the edge set of \( K_{n+1} \). Given a spanning tree \( T \) of \( K_{n+1} \), the external activity \( \text{ex}(T) \) of \( T \) is the set of edges \( e \in K_{n+1} \) such that \( e \) is the minimal edge of the unique cycle in \( T \cup \{e\} \). A basis of \( V_n \) is given by

\[ \{p(K_{n+1} - (\text{ex}(T) \cup T)) : T \text{ is a spanning tree of } K_{n+1}\}. \]
Figure 2. The four slim subgraphs of $K_3$.

It follows immediately from Cayley’s theorem that $\dim V_n = (n + 1)^n - 1$. Aside from this dimension formula, we will make no further use of this basis (or, indeed, any explicit basis) of $V_n$ for the rest of the paper.

Since the slimness of a subgraph is preserved under the action of $S_{n+1}$ on the vertex set $[n+1]$ and $p(G)$ is homogeneous of degree equal to the number of edges in $G$, it follows that $V_n = \bigoplus_{k \geq 0} V_n(k)$ is a graded $S_{n+1}$-submodule of the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$. In fact, the space $V_n$ sits inside the copy of the coordinate ring of the reflection representation of type $A_n$ sitting inside $\mathbb{C}[x_1, \ldots, x_{n+1}]$ generated by $x_i - x_{i+1}$ for $1 \leq i \leq n$.

The following result was conjectured by the first author. We postpone its proof, along with the proofs of the other results in this section, to Section 3.

**Theorem 2.** Embed $S_n$ into $S_{n+1}$ by letting $S_n$ act on the first $n$ letters. We have that

$$\text{Res}_{S_n}^{S_{n+1}}(V_n(k)) \cong S_n \bigoplus_D M^{\lambda(D)},$$

where the direct sum is over all Dyck paths of size $n$ and area $k$. In particular, by Equation 2.2 we have that

$$\text{Res}_{S_n}^{S_{n+1}}(V_n) \cong S_n \text{ Park}_n.$$

**Example 3.** In the case $n = 2$, Figure 2 shows that four slim subgraphs of the complete graph $K_3$. From left to right, the corresponding polynomials are $1$, $x_1 - x_2$, $x_1 - x_3$, and $x_2 - x_3$. It follows that $V_2(0) = \text{span}\{1\}$ and $V_2(1) = \text{span}\{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$. Observe that the graded Frobenius character of $V_2$ is $\text{grFrob}(V_2; q) = s_{(3)}q^0 + s_{(2,1)}q^1$. By the branching rule for symmetric groups (see [Sag]), we have that $\text{grFrob}(\text{Res}_{S_2}^{S_3}(V_2); q) = s_{(2)}q^0 + (s_{(2)} + s_{(1,1)})q^1$. Setting $q = 1$ yields $\text{Frob}(\text{Res}_{S_2}^{S_3}(V_2)) = 2s_{(2)} + s_{(1,1)}$, which agrees with the Frobenius character of Park$_2$.

**Example 4.** Below we have the graded Frobenius character for $V_3$ and $V_4$.

$$\text{grFrob}(V_3) = s_{(4)} + s_{(3,1)}q + (s_{(4)} + s_{(3,1)} + s_{(2,2)})q^2 + (s_{(3,1)} + s_{(2,1,1)})q^3.$$
the module $V_n$ as the action of $S_n$ used to prove that the area and run partitions of the 5 Dyck paths of size 3 shown in Figure 3.

Equivalently, we have that $\text{grFrob}(\text{Res}^{S_{n+1}}_{S_n}(V_n); q) = \sum_D q^{\text{area}(D)} h_{\lambda(D)}$, where the sum is over all Dyck paths $D$ of size $n$. For example, computing the area and run partitions of the 5 Dyck paths of size 3 shown in Figure 3 shows that

$$\text{grFrob}(V_4) = s(5) + (s(5) + s(4,1) + s(3,2))q^2 + (s(5) + 2s(4,1) + s(3,2) + s(3,1,1))q^3 + (s(5) + 2s(4,1) + 2s(3,2) + s(3,1,1) + s(2,2,1))q^4 + (s(5) + 2s(4,1) + 2s(3,2) + 2s(3,1,1) + s(2,2,1))q^5 + (s(4,1) + s(3,2) + s(3,1,1) + s(2,2,1))q^6.$$

We leave it to the reader to check that the restrictions of these graded Frobenius characters yield the representations $\text{Park}_3$ and $\text{Park}_4$ of $S_3$ and $S_4$, respectively.

Postnikov and Shapiro showed that the dimension of the vector space $V_n$ is equal to $(n + 1)^{n-1}$, however the $S_n$-module structure of $V_n$ has remained unstudied. Indeed, Theorem 2 is the first description of the $S_n$-module structure of $V_n$.

It is natural to ask for an explicit description of the $S_{n+1}$-structure of $V_n$ or of its graded pieces $V_n(k)$. This problem is open in general, but we can describe the extended structure of $V_n(k)$ in the extreme degrees $k = 0, 1, \ldots, n - 1$ as well as $k = \binom{n}{2}$. Let $C_{n+1}$ be the cyclic subgroup of $S_{n+1}$ generated by the long cycle $c := (1, 2, \ldots, n+1)$ and let $\zeta$ be the linear representation of $C_{n+1}$ which sends $c$ to $e^{\frac{2\pi i}{n+1}}$. Mackey’s Theorem can be used to prove that the Lie representation $\text{Lie}_n := \text{Ind}_{C_{n+1}}^{S_{n+1}}(\zeta)$ of $S_{n+1}$ satisfies $\text{Res}^{S_{n+1}}_{S_n}(\text{Lie}_n) \cong S_n \mathbb{C}[S_n]$. Stanley proved that the Lie representation arises as the action of $S_{n+1}$ on the top poset cohomology of the lattice of set partitions of $[n + 1]$, tensored with the sign representation $[St]$.}

**Theorem 5.** The module $V_n(0)$ carries the trivial representation of $S_{n+1}$, the module $V_n(1)$ carries the reflection representation of $S_{n+1}$, and in general
Figure 4. A (2, 2)-Dyck path of size 3.

$V_n(k) = \text{Sym}^k(V_n(1))$ for $k < n$. The module $V_n(\binom{n}{2}) = V_n(\text{top})$ carries the Lie representation of $S_{n+1}$ tensor the sign representation.

The first part of this result is optimal in the sense that if $k \geq n$ then $V_n(k)$ is a proper subspace of $\text{Sym}^k(V_n(1))$.

We will prove a bivariate generalization of Theorem 2 which includes a ‘Fuss generalization’ as a special case. Given $\ell, m, n > 0$, define a $(\ell, m)$-Dyck path of size $n$ to be a lattice path $D$ in $\mathbb{Z}^2$ consisting of vertical steps $(1, 0)$ and horizontal steps $(0, 1)$ which starts at $(-\ell + 1, 0)$, ends at $(mn, n)$, and stays weakly above the line $y = \frac{x}{m}$. Taking $\ell = m = 1$, we recover the classical notion of a Dyck path of size $n$. Taking $\ell = 1$ and $m$ general, the $(1, m)$-Dyck paths are the natural Fuss extension of Dyck paths. As before, we define the vertical run partition $\lambda(D) \vdash n$ of an $(\ell, m)$-Dyck path $D$ of size $n$ to be the partition obtained by listing the lengths of the vertical runs of $D$ in weakly decreasing order. We also define the area partition $\mu(D)$ to be the length $n$ partition whose Ferrers diagram fits between $D$ and a $(\ell - 1 + mn) \times n$ rectangle with lower left hand coordinate $(-\ell + 1, 0)$. The area of $D$ is defined by $\text{area}(D) := |\mu(D)|$. We have that $\text{mult}(\mu(D)) = \lambda(D)$.

Figure 4 shows an example of a $(2, 2)$-Dyck path of size 3. The path $D$ starts at $(-1, 0)$, ends at $(6, 3)$, and stays above the line $y = \frac{x}{2}$. We have that $\lambda(D) = (2, 1) \vdash 3$, $\mu(D) = (5, 1, 1)$, and $\text{area}(D) = 7$.

Let $K_{n+1}^{(\ell,m)}$ be the multigraph on the vertex set $[n+1]$ with $m$ edges between $i$ and $j$ for all $1 \leq i < j \leq n$ and $\ell$ edges between $i$ and $n+1$ for all $1 \leq i \leq n$. We call a sub-multigraph $G$ of $K_{n+1}^{(\ell,m)}$ slim if the multi-edge set difference $K_{n+1}^{(\ell,m)} - G$ is a connected multigraph on $[n+1]$. We extend the polynomial weight $p(G) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to multigraphs $G$ in the obvious way.

**Definition 6.** Let $V_n^{(\ell,m)}$ be the $\mathbb{C}$-linear subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ given by the span

$$V_n^{(\ell,m)} := \text{span}\{p(G) : G \text{ is a slim sub-multigraph of } K_{n+1}^{(\ell,m)}\}.$$  

As in the case $m = \ell = 1$, the space $V_n^{(\ell,m)}$ is stable under the action of $S_n$, which respects the grading. When $\ell = m$, $V_n^{(\ell,\ell)}$ also has an $S_{n+1}$-action, which also respects the grading. Postnikov and Shapiro showed that
the dimension of $V_n^{(\ell,m)}$ is $\ell(mn + \ell)^{n-1}$ [PoSh]. Let $V_n^{(\ell,m)}(k)$ be the degree $k$ piece of $V_n^{(\ell,m)}$.

**Theorem 7.** Under action of $S_n$ induced by permutation of vertex labels,
\begin{equation}
V_n^{(\ell,m)}(k) \cong_{S_n} \bigoplus_{D} M^{\lambda(D)},
\end{equation}
where the direct sum is over all $(\ell,m)$-Dyck paths of size $n$ and area $k$. The containment $V_n^{(\ell,m)}(k) \subseteq \text{Sym}^k(V_n^{(\ell,m)}(1))$ is an equality for $k < n$.

In the case $m = \ell$, the module $V_n^{(\ell,\ell)}(1)$ has $S_n$-structure given by the reflection representation of $S_{n+1}$, so that the equality $V_n^{(\ell,\ell)}(k) = \text{Sym}^k(V_n^{(\ell,\ell)}(1))$, $k < n$, describes the $S_n$-module structure completely. The top degree space $V_n^{(\ell,\ell)}(\text{top})$ is Lie$_n \otimes \langle \text{sign} \rangle \otimes \ell$.

**Example 8.** Take $n = 3, \ell = m = 2$ in Theorem 7, so that $V_3^{(2,2)}$ carries an $S_4$ action. We have that
\[
\text{grFrob}(\text{Res}_{S_4}^{S_3}(V_3^{(2,2)})) = h(3)q^0 + h(2,1)q^1 + 2h(2,1)q^2 + (h(3) + h(2,1) + h(1,1,1))q^3 + (3h(2,1) + h(1,1,1))q^4 + (3h(2,1) + 2h(1,1,1))q^5 + (2h(2,1) + 3h(1,1,1))q^6 + (2h(2,1) + 3h(1,1,1))q^7 + 3h(1,1,1)q^8 + h(1,1,1)q^9.
\]

3. Proofs

While Theorem 7 implies Theorem 2, the proof of Theorem 7 is a straightforward extension of the proof of Theorem 2 and it will be instructive to prove Theorem 2 first.

The first step in the proof of Theorem 2 is to relate the modules on both sides of the claimed isomorphism by associating a subgraph $G(D)$ of $K_{n+1}$ and a polynomial $p(D) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to any Dyck path $D$ of size $n$. We start by labeling the $1 \times 1$ box $b$ which is completely above the line $y = x$ with the edge $e(b) = (n - j, n - i)$ in $K_{n+1}$, where $(i, j)$ is the upper left coordinate of $b$. See Figure 5 for an example of this labeling in the case $n = 5$. We let $G(D)$ be the subgraph of $K_{n+1}$ consisting of those edges $e(b)$ for which the box $b$ is to the upper left of the path $D$. In Figure 5, the shaded boxes above the path $D$ each contribute an edge to the subgraph $G(D)$ and we have that $G(D) = \{1 - 6, 1 - 5, 1 - 4, 1 - 3, 2 - 6, 2 - 5, 3 - 6\}$.

**Lemma 9.** The subgraph $G(D)$ is slim for any Dyck path $D$.

**Proof.** The subgraph $G(D)$ contains none of the edges in the path $1 - 2 - \cdots - (n + 1)$. \hfill \Box

By Lemma 9, the polynomial $p(D) := p(G(D))$ is contained in $V_n$. For example, if $n = 5$ and $D$ is the Dyck path shown in Figure 5, we have that
\begin{equation}
p(D) = (x_1 - x_6)(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)(x_2 - x_6)(x_2 - x_5)(x_3 - x_6) \in V_5.
\end{equation}
By construction, for any Dyck path $D$ the polynomial $p(D)$ is homogeneous with degree equal to $\text{area}(D)$.

In order to prove the direct sum decomposition in Theorem 2, we will show that the polynomials $p(D)$ project nicely onto a certain subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$. Since Theorem 2 only concerns the restriction of $V_n$ to $S_n$, it is natural to consider a subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ which is closed under the action of $S_n$ but not of $S_{n+1}$.

Let $st_n := (n-1, n-2, \ldots, 1, 0)$ be the \textit{staircase partition} of length $n$. We call a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ \textit{sub-staircase} if $\lambda \subseteq st_n$ (observe that this definition has tacit dependence on $n$). For any Dyck path $D$ of size $n$, the partition $\mu(D)$ is sub-staircase.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we use the shorthand $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathbb{C}[x_1, \ldots, x_n]$. We call a monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ in the variables $x_1, \ldots, x_{n+1}$ \textit{sub-staircase} if there exists a permutation $w \in S_n$ and a sub-staircase partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ such that

$$x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = w.x^\lambda.$$  

In particular, the variable $x_{n+1}$ does not appear in any sub-staircase monomial. If the monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ is sub-staircase, the partition $\lambda$ is uniquely determined from the monomial; call this the \textit{exponent partition} of the monomial. More generally, if $x_1^{a_1} \cdots x_n^{a_n}$ is any monomial in $x_1, \ldots, x_n$, we call the rearrangement $(\lambda_1 \geq \cdots \geq \lambda_n)$ of $(a_1, \ldots, a_n)$ the \textit{exponent partition} of this monomial. Let $W_n \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the $\mathbb{C}$-linear span of all sub-staircase monomials. The subspace $W_n$ is closed under the action of $S_n$, but not under the action of $S_{n+1}$.

In the case $n = 3$, the $S_3$-orbits of the 16 sub-staircase monomials in $\mathbb{C}[x_1, \ldots, x_4]$ are shown in the following table, where the left column shows a representative from each orbit.
The $S_3$-orbits are parametrized by sub-staircase partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and each orbit contains a unique representative of the form $x^\lambda$. The staircase monomials form a linear basis of $W_3$ and the cyclic $S_3$-submodule of $W_3$ generated by $x^\lambda$ is isomorphic to $M^{\text{mult}}(\lambda)$. The natural bijection between exponent vectors and parking functions affords an isomorphism $W_3 \cong \text{Park}_3$. These observations generalize in a straightforward way to the following lemma, whose proof is left to the reader.

**Lemma 10.** The set of sub-staircase monomials forms a linear basis for $W_n$ and is closed under the action of $S_n$. The $S_n$-orbits are parametrized by sub-staircase partitions $\lambda$, and the orbit labeled by $\lambda$ has a unique monomial of the form $x^\lambda$. The cyclic $S_n$-submodule of $W_n$ generated by $x^\lambda$ is isomorphic to $M^{\text{mult}}(\lambda)$ and we have that $W_n \cong S_n \text{ Park}_n$.

With Lemma 10 in mind, we will construct a graded $S_n$-module isomorphism $V_n \sim W_n$. We define a graded $S_n$-module homomorphism $\phi : V_n \to W_n$ by the following composition:

$$\phi : V_n \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] \to \mathbb{C}[x_1, \ldots, x_n] \to W_n,$$

where the first map is inclusion, the second is the specialization $x_{n+1} = 0$, and the third linear map fixes the space $W_n$ pointwise and sends monomials which are not sub-staircase to zero.

We want to show that $\phi$ is an isomorphism. Postnikov and Shapiro showed that $\dim(W_n) = \dim(V_n) = (n + 1)^{n-1}$ [PoSh], so it is enough to show that $\phi$ is surjective. We will do this by analyzing the polynomials $\phi(p(D))$, where $D$ is a Dyck path of size $n$.

The next lemma states that the transition matrix between the set $\{\phi(p(D)) : D \text{ a Dyck path of size } n\}$ expands in the monomial basis of $W_n$ given by $\{x^\lambda : \lambda \text{ sub-staircase}\}$ in a unitriangular way with respect to grevlex order (where we associate $\phi(p(D))$ with the partition $\mu(D)$). The authors find it surprising that the corresponding unitriangularity statement is false when one considers linear extensions of the dominance order on partitions; grevlex order is primarily known for its utility in the efficient computation of Gröbner bases and is far less ubiquitous in combinatorial representation theory than dominance order.

**Lemma 11.** Let $D$ be a Dyck path of size $n$. There exist integers $c_{\lambda,w} \in \mathbb{Z}$ such that

$$\phi(p(D)) = x^{\mu(D)} + \sum_{\substack{\lambda < \mu(D) \\ |\lambda| = |\mu(D)| \\ w \in S_n}} c_{\lambda,w} w.x^\lambda.$$
Proof. By definition, we have that
\begin{equation}
(3.5) \quad p(D) = \prod_{e=(i<j)\in G(D)} (x_i - x_j).
\end{equation}

Up to sign, a typical monomial in this expansion can be obtained by choosing an orientation \(\mathcal{O}\) of the graph \(G(D)\), associating an oriented edge \(k \to \ell\) to the variable \(x_k\), and multiplying the corresponding variables together. The map \(\phi\) kills any monomial which contains the variable \(x_{n+1}\), so up to sign a typical monomial in \(\phi(p(D))\) is obtained by choosing an orientation \(\mathcal{O}\) of \(G(D)\) such that every edge which contains the vertex \(n+1\) is oriented away from \(n+1\). If we denote by \(\mathcal{O}_0\) the orientation of \(G(D)\) which directs every edge towards its smaller endpoint, we see that the monomial \(x^\mu(D)\) arises in this way (with coefficient 1). We argue that there does not exist an orientation \(\mathcal{O}\) of \(G(D)\) other than \(\mathcal{O}_0\) such that the monomial \(m\) corresponding to \(\mathcal{O}\) has exponent partition \(\lambda\) satisfying \(\lambda \succeq \mu(D)\).

Suppose for the sake of contradiction that there was an orientation \(\mathcal{O}\) of \(G(D)\) directing incident edges away from the vertex \(n+1\) other than the orientation \(\mathcal{O}_0\) which gave rise to a monomial \(m\) in the variables \(x_i\) with exponent partition \(\lambda\) where \(\mu(D) \not\succeq \lambda\). If \(\mu(D)\) were the empty partition, then \(G(D)\) would be the empty graph and \(\mathcal{O} = \mathcal{O}_0\), so we conclude that \(\mu(D)\) must have at least one nonempty column. For \(1 \leq i \leq n\), let \(\mu(D)^i\) denote the number of parts of the partition \(\mu(D)\) which are \(\geq i\). Then the number of edges of \(G(D)\) that contain \(n+1\) equals \(\mu(D)^1\). By the way we labeled our boxes, this means that the product \(x_1 \cdots x_{\mu(D)^1}\) divides \(m\) and that the exponent partition \(\lambda\) of \(m\) is componentwise \(\geq (1,1,\ldots,1,0,\ldots,0)\), where there are \(\mu(D)^1\) copies of 1 and \(n-\mu(D)^1\) copies of 0. If \(\mu(D)\) consists of a single nonempty column, we have \(m = x_1 \cdots x_{\mu(D)^1} = x^{\mu(D)}\), \(\mathcal{O} = \mathcal{O}_0\), and our contradiction. Otherwise, since \(D\) is a Dyck path, we have that \(n - \mu(D)^1 > 0\). In particular, the condition \(\mu(D) \not\succeq \lambda\) implies that none of the nonempty list of variables \(x_{\mu(D)^1+1}, x_{\mu(D)^1+2}, \ldots, x_n\) appear in \(m\). In particular, this means that the orientation \(\mathcal{O}\) directs every edge away from the vertex \(n\) (there are edges which are incident to \(n\) because of the way we labeled our boxes and the fact that \(\mu(D)\) has at least two nonempty columns).

In general, suppose that the orientations \(\mathcal{O}\) and \(\mathcal{O}_0\) do not agree and choose an edge \(\{i,j\}\) with \(i < j\) of \(G(D)\) such that \(\mathcal{O}\) and \(\mathcal{O}_0\) disagree on \(\{i,j\}\) but agree on all edges \(\{k,\ell\}\) with \(k < \ell\) such that \(\ell > j\) or \(\ell = j\) but \(k < i\). By the reasoning of the last paragraph, we must have \(j < n\). Moreover, by considering the edges of \(G(D)\) with larger vertex \(n+1, n, n-1, \ldots, j+1\), we see that \(m\) is divisible by the monomial
\[
m^j := (x_1 \cdots x_{\mu(D)^1})(x_1 \cdots x_{\mu(D)^2}) \cdots (x_1 \cdots x_{\mu(D)^{n-j+1}}).
\]

This means that the exponent partition \(\lambda\) is componentwise \(\geq\) the partition \((n-j+1, \ldots, n-j+1, n-j, \ldots, n-j, \ldots, 1, 0, \ldots, 0)\), where there are \(\mu(D)^{n-j+1}\) copies of \(n-j+1\), \(\mu(D)^{n-j} - \mu(D)^{n-j+1}\) copies of \(n-j\),
\[ \ldots, \mu(D)_{j+1} - \mu(D)_j \] copies of 1, and \( n - \mu(D)_{j+1} \) copies of 0. In particular, the subsequence of this partition after the initial string of \((n - j + 1)\)'s agrees with the partition \( \mu(D) \). The fact that \( D \) is a Dyck path implies that the list of variables \( x_{\mu(D)_{n-j+1+1}}, x_{\mu(D)_{n-j+1+2}}, \ldots, x_{j} \) is nonempty and the fact that \( \lambda \preceq \mu(D) \) implies that none of these variables divides the quotient \( \frac{m}{m'} \). But the fact that \( \mathcal{O} \) directs \( i \to j \) implies that \( x_j \) divides this quotient, which is a contradiction. We conclude that \( m = x^{\mu(D)} \) and \( \mathcal{O} = \mathcal{O}_0. \)

As an example of Lemma 11, consider the case \( n = 5 \) and let the Dyck path \( D \) be shown in Figure 5 with \( \mu(D) = (4, 2, 1, 0, 0) \). To calculate \( \phi(p(D)) \), we set \( x_0 = 0 \) in the product formula for \( p(D) \) given in Equation 3.1 and expand. The resulting polynomial is

\[ \phi(p(D)) = x_1(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)x_3 = x_1^2x_2x_3 + \cdots. \]

where the ellipsis denotes terms involving sub-staircase monomials with exponent partition \( \prec (4, 2, 1, 0, 0) \). We are ready to complete the proof of Theorem 2.

Proof of Theorem 2. By Lemma 10, the set of sub-staircase monomials forms a linear basis of \( W_n \), so Lemma 11 implies that the \( S_n \)-module homomorphism \( \phi : V_n \to W_n \) is surjective. Since \( \dim(V_n) = \dim(W_n) \), this implies that \( \phi \) is also injective and gives an isomorphism \( \text{Res}^{S_{n+1}}_{S_n} (V_n) \cong S_n \) Parkn. To prove the graded isomorphism in Theorem 2, it is enough to observe that \( \text{mult}(\mu(D)) = \lambda(D) \) for any Dyck path \( D \) and apply Lemmas 10 and 11 together with the fact that \( \phi \) is graded.

It may be tempting to guess that \( p(D) \) generates a cyclic \( S_n \)-submodule of \( V_n \) isomorphic to \( M^{\lambda(D)} \), but this is false in general. The reason for this is that while the ‘leading term’ in the expansion of \( \phi(p(D)) \) in Lemma 11 generates the submodule \( M^{\lambda(D)} \) under the action of \( S_n \), the other terms in this expansion can cause \( \phi(p(D)) \) to generate a different cyclic submodule.

We are ready to prove the claimed \( S_{n+1} \)-structure of the extreme degrees of the graded module \( V_n(k) \).

Proof of Theorem 5. It is clear from the definitions that \( V_n(0) \) carries the trivial representation of \( S_{n+1} \). The space \( V_n(1) \) has basis given by the polynomials \( x_1 - x_2, x_2 - x_3, \ldots, x_n - x_{n+1} \) and hence carries the reflection representation of \( S_{n+1} \) (i.e., the irreducible \( S_{n+1} \)-module corresponding to the partition \((n, 1))\). Since \( V_n \subseteq \text{Sym}(V_n(1)) \) we are claiming that in degree \( k < n \) this is an equality. The Hilbert series of \( V_n \) is the Tutte polynomial evaluation \( q^{\frac{(n+1)}{2}} T_{K_{n+1}}(1, 1/q) \) and so we must prove that the first \( n - 1 \) terms of this sum are the binomial coefficients \((n+k-1) \). There is nothing special about \( K_{n+1} \) in this claim and we will prove a more general statement in Lemma 12.
To prove that $V_n$(top) is isomorphic to $\text{Lie}_{n+1} \otimes \text{sign}$ we reason as follows.

The space $V_n$(top) is spanned by those $p(G)$ where the complementary subgraph $K_{n+1} \setminus G$ is connected and has $n$ edges.

Let $A_n$ denote the braid arrangement in $\mathbb{C}^{n+1}$, which is the union of those hyperplanes with at least two coordinates equal. Let $H^r(\mathbb{C}^{n+1} \setminus A_n; \mathbb{C})$ denote the (complexified) de Rham cohomology ring of its complement. Consider, now, the linear map $c : V_n$(top) $\rightarrow H^n(\mathbb{C}^{n+1} \setminus A_n)$ that sends

\[ p(G) \mapsto p(G) \cdot d(x_1 - x_2) \wedge d(x_2 - x_3) \wedge \cdots \wedge d(x_n - x_{n+1})/ \prod_{1 \leq i < j \leq n} (x_i - x_j). \]

This is an isomorphism of vector spaces, since it is division by the Vandermonde product, followed by multiplication by the $n$-form. To see that $c$ is equivariant notice that $V_n$V_n(1) carries the sign representation of $S_{n+1}$, because it is 1 dimensional and non-trivial. Likewise does the one dimensional representation spanned by the Vandermonde product. It follows that the signs introduced by multiplication by the $n$-form and division by the Vandermonde cancel, and $c$ is equivariant.

Finally, the top degree cohomology of the complement $\mathbb{C}^{n+1} \setminus A_n$ is known to be isomorphic to the top degree Whitney homology of its lattice of flats [Bj, Theorem 7.2.10], and this correspondence is at once seen to be $S_{n+1}$-equivariant. The lattice of flats of $A_n$ is the partition lattice $\Pi_{n+1}$ and by a result of Stanley [St] (beautifully recounted by Wachs in [Wa, Section 4.4]), the top degree Whitney homology of the partition lattice $\Pi_{n+1}$ is $\text{Lie}_{n+1} \otimes \text{sign}$. □

\section*{Lemma 12}

Let $G$ be a connected graph on $v$ vertices with $e$ edges. Denote the Tutte polynomial of $G$ by $T_G(x, y)$. Then, the polynomial $q^{e-v+1}T_G(1, 1/q)$ takes the form,

\[ 1+(v-1)q+ \binom{v}{2} q^2 + \binom{v+1}{3} q^3 + \cdots + \left( \frac{(v-1) + (v-2) - 1}{v-2} \right) q^{v-2} + O(q^{v-1}). \]

\begin{proof}

We write $T_G(x, y)$ in terms of the two variable coboundary polynomial, $\overline{\chi}_G(\lambda, \nu)$. This is the sum

\[ \overline{\chi}_G(\lambda, \nu) = \frac{1}{\lambda} \sum_{i=0}^{e} c_i(G; \lambda) \nu^i \]

where $c_i(G; \lambda)$ is the number of ways to color the vertices of $G$ with $\lambda$ colors and exactly $i$ monochromatic edges. It is a fact that this is a polynomial in $\lambda$ and $\nu$. Now by [BrO, Proposition 6.3.26],

\[ q^{e-v+1}T_G(1, 1/q) = \frac{q^e}{(1-q)^{v-1}} \overline{\chi}_G(0, 1/q). \]

Thus, to prove the first part of the lemma we will show that $c_i(G; \lambda) = 0$ for $e - v + 1 < i < e$, and that $c_e(G; \lambda) = \lambda$. Suppose that we have colored the vertices of $G$ and we have more than $e - v + 1$ monochromatic edges. Then the collection of monochromatic edges forms a connected subgraph of
Figure 6. A (3, 2)-Dyck path $D$ of size 4 and the associated
sub-multigraph $G(D)$ of $K^{(3, 2)}_4$.

It follows that all vertices of $G$ are colored the same and hence all edges
of $G$ are monochromatic. This means that $c_i(G; \lambda) = 0$ unless $i = e$. That
$c_e(G; \lambda) = \lambda$ is clear.

The proof of Theorem 7 is a straightforward extension of the proof of
Theorem 2. We will be somewhat brief.

Proof of Theorem 7. Given any $(\ell, m)$-Dyck path $D$ of size $n$ we associate
a sub-multigraph $G(D)$ of $K^{(\ell, m)}_n$ by letting every box which contributes to
area($D$) correspond to a single edge in the multigraph $G(D)$; the labeling
which accomplishes this is shown in Figure 6 in the case $(\ell, m) = (3, 2)$ and
$n = 4$. For general $\ell, m$, and $n$, we label the boxes in the $i$th row from the
top from left to right with $(\ell + m - 2)$ copies of the edge $i - (n + 1)$, $m$ copies
of the edge $i - n$, $m$ copies of the edge $i - (n - 1)$, . . . , $m$ copies of the edge
$i - (i + 2)$, and $(m - 1)$ copies of the edge $i - (i + 1)$.

For any $(\ell, m)$-Dyck path $D$ of size $n$, the multigraph complement of $G(D)$
within $K^{(\ell, m)}_n$ contains each of the edges in the path $1 - 2 - \cdots - n - (n + 1)$
with multiplicity at least one. Therefore, the sub-multigraph $G(D)$ is slim
and the polynomial $p(D) := p(G(D))$ is contained in $V^{(\ell, m)}_n$.

We say that a partition $\lambda$ with $n$ parts is sub-$(\ell, m)$-staircase if in Young’s
lattice we have the relation $\lambda \subseteq (\ell - 1 + m(n - 1), \ell - 1 + m(n - 2), \ldots, \ell - 1)$. A monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ is sub-$(\ell, m)$-staircase if there exists $w \in S_n$ and a
sub-$(\ell, m)$-staircase partition $\lambda$ such that

\begin{equation}
x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = x_1^{\lambda_{w(1)}} \cdots x_n^{\lambda_{w(n)}}.
\end{equation}

In particular, the variable $x_{n+1}$ does not appear in any sub-$(\ell, m)$-staircase
monomial.

Let $W^{(\ell, m)}_n$ be the subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ spanned by the set of all
sub-$(\ell, m)$-staircase monomials. The space $W^{(\ell, m)}_n$ carries a graded action of
$S_n$. The argument used to prove Lemma 10 extends to show that the degree
$k$ homogeneous piece of $W_{n}^{(\ell,m)}$ is isomorphic as an $S_n$-module to the direct sum on the right hand side of the isomorphism asserted in Theorem 7.

The isomorphism in Theorem 7 is proven by showing that the graded $S_n$-module homomorphism $\phi^{(\ell,m)}: V_n^{(\ell,m)} \rightarrow W_n^{(\ell,m)}$ given by the composite

$$\phi^{(\ell,m)}: V_n^{(\ell,m)} \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n] \rightarrow W_n^{(\ell,m)}$$

is an isomorphism, where the first map is inclusion, the second is the evaluation $x_{n+1} = 0$, and the third fixes $W_n^{(\ell,m)}$ pointwise and sends every monomial which is not sub-$((\ell,m))$-staircase to zero.

Postnikov and Shapiro proved that the vector space $V_n^{(\ell,m)}$ has dimension $\ell(\ell + mn)^{n-1}$ [PoSh]. Pitman and Stanley [PiSt] and Yan [Y] showed that the number of (exponent vector of) sub-$((\ell,m))$-staircase monomials equals $\ell(\ell + mn)^{n-1}$. Since these monomials form a basis for $W_n^{(\ell,m)}$, in order to prove that $\phi^{(\ell,m)}$ is an isomorphism of $S_n$-modules, it is enough to show that $\phi^{(\ell,m)}$ is surjective.

The fact that $\phi^{(\ell,m)}$ is surjective follows from the following triangularity result which generalizes Lemma 11. Recall that $\mu(D)$ is the partition whose Ferrers diagram lies to the northwest of an $(\ell,m)$-Dyck path $D$ (for example, if $D$ is the $(3,2)$-Dyck path appearing on the left in Figure 6, then $\mu(D) = (5,3,3,1)$. The proof of Lemma 13 is almost identical to the proof of Lemma 11 and is left to the reader.

**Lemma 13.** Let $D$ be an $(\ell,m)$-Dyck path of size $n$. The monomial expansion of $\phi^{(\ell,m)}(p(D))$ has the form

$$(3.8) \quad \phi^{(\ell,m)}(p(D)) = x^{\mu(D)} + \cdots,$$

where the ellipsis denotes terms involving monomials whose exponent partitions are $\prec \mu(D)$.

For example, if $D$ is the $(3,2)$-Dyck path in Figure 6, then

$$p(D) = (x_1 - x_4)^2(x_1 - x_5)^3(x_2 - x_3)^3(x_3 - x_5)^3(x_4 - x_5)$$

and

$$\phi^{(3,2)}(p(D)) = x_1^5x_2^3x_3^3x_4^1 + \text{terms whose exponent partitions are } \prec (5,3,3,1).$$

Lemma 13 implies that $\phi^{(\ell,m)}$ is surjective, and dimension counting implies that $\phi^{(\ell,m)}$ is a graded $S_n$-module homomorphism. The $S_n$-isomorphism in Theorem 7 follows.

To prove the remainder of Theorem 7, we need to show that $V_n^{(\ell,m)}(k) = \text{Sym}^k(V_n^{(\ell,m)}(1))$ for $k < n$. This follows at once from Lemma 12, since the Hilbert series of $V_n^{(\ell,m)}$ is the Tutte polynomial evaluation

$$q^{(\ell,n)+mn-n}T^{(\ell,m)}_{n+1}(1,1/q).$$

For this see [PoSh]. For the statement about the top degree, we take the elements in $V_n^{(\ell,\ell)}(\text{top})$, divide them all by $\prod_{1 \leq i < j \leq n+1}(x_i - x_j)^{\ell-1}$, which
yields an equivariant isomorphism with $V_n^{\text{top}} \otimes \text{sign} \otimes (\ell - 1)$. By Theorem 5 this is $\text{Lie}_n \otimes \text{sign} \otimes \ell$. The remainder of Theorem 7 is now proved. □

4. Concluding Remarks

In this paper we constructed a graded $S_{n+1}$-module $V_n$ which satisfies $\text{Res}_{S_{n+1}}^{S_n}(V_n) \cong S_n \text{Park}_n$. While we know the $S_{n+1}$-structure of $V_n$ in extreme degrees, the full $S_{n+1}$-structure remains unknown.

**Problem 14.** Give a nice expression for the graded $S_{n+1}$-Frobenius character of $V_n$.

Problem 14 may have an answer in terms of free Lie algebras. Let $\text{Lie}_{n+1}$ be the free Lie algebra on the generators $x_1, \ldots, x_{n+1}$. The group $S_{n+1}$ acts on $\text{Lie}_{n+1}$ by subscript permutation. By keeping track of the multiplicities of the $x_i$, the $S_{n+1}$-module $\text{Lie}_{n+1}$ carries the structure of an $\mathbb{N}^{n+1}$-graded vector space. The $(1, \ldots, 1)$-component of this vector space is stable under the action of $S_{n+1}$ and is known to carry the Lie representation, or $V_n^{\text{top}} \otimes \text{sign}$. Lower degrees of $V_n$ may also embed naturally inside free Lie algebras.

In this paper we showed that the action of $S_n$ on Park$_n$ extends to a graded module $V_n(k)$ of $S_{n+1}$. We identified the top degree $V_n^{\text{top}}$ of this extended action with the twisted Lie representation $\text{Lie}_n \otimes \text{sign}$ of $S_{n+1}$. Whitehouse [W] proved that the representation $\text{Lie}_n$ extends to $S_{n+2}$. This suggests the following problem.

**Problem 15.** What is the maximum value of $r$ for which the action of $S_n$ on Park$_n$ extends to an action of $S_{n+r}$? For fixed $n$ and $k$, what is the maximum value of $r$ for which the action of $S_{n+1}$ on $V_n(k)$ extends to $S_{n+r}$?

We have some computer evidence (see [B] for the relevant Mathematica code) that the value of $r$ in the first question may be greater than 1 for any $n$. For $n = 5, 6, 7$, the action of $S_n$ on Park$_n$ extends to an action of $S_{10}$. However, the action of $S_5$ on Park$_5$ does not extend to an action of $S_{11}$.

Any answer to the second question will depend on both $n$ and $k$. By Whitehouse’s result, for any $n > 0$, the $k$-value $k = \binom{n}{2}$ gives rise to an extension degree $r$ of at least two. Also, since $V_n(0)$ is the trivial representation of $S_{n+1}$ for any $n$, if $k = 0$ one can take $r = \infty$. On the other hand, if $k = 1$ we have that $V_n(1)$ is the reflection representation of $S_{n+1}$. For $n > 3$, this representation is not the restriction of any $S_{n+2}$ module.

The results of this paper and that of Whitehouse [W] motivate the following problem which in the opinion of the authors has received surprisingly little attention.

**Problem 16.** Let $M$ be an $S_n$-module. Give a nice criterion for deciding whether $M$ extends to a representation of $S_{n+1}$.

Very few irreducible representations of $S_n$ extend to $S_{n+1}$. Indeed, if $S^\lambda$ is the irreducible representation of $S_n$ labeled by a partition $\lambda \vdash n$, then
$S^\lambda$ extends to $S_{n+1}$ if and only if $\lambda$ is a ‘near rectangle’, i.e. a rectangular partition with $n+1$ boxes minus its outer corner.

On the other hand, an ‘asymptotically nonzero fraction’ of $S_n$-modules extend to $S_{n+1}$. More precisely, recall that the $\mathbb{Z}$-module $R_n$ of class functions on $S_n \to \mathbb{C}$ has basis given by the set of irreducible characters $\{S^\lambda : \lambda \vdash n\}$ (where we identify modules with characters). The $\mathbb{Z}$-linear map $\psi : R_{n+1} \to R_n$ induced by restriction is surjective. Indeed, if $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, form a new partition $\lambda^+ := (\lambda_1 + 1, \lambda_2, \ldots, \lambda_k) \vdash n+1$ by increasing the first part of $\lambda$ by one. By the branching rule for symmetric groups, the restriction $\text{Res}_{S_n}^{S_{n+1}}(S^{\lambda^+})$ has the form

$$\text{Res}_{S_n}^{S_{n+1}}(S^{\lambda^+}) \cong S_n S^{\lambda} \oplus \cdots,$$

where the ellipsis denotes a direct sum of irreducibles corresponding to partitions $> \lambda$ in the dominance order. The surjectivity of $\psi$ follows.

On the level of representations, the fact that $\psi$ is surjective means that the set $C_n$ of $S_n$-modules which extend to $S_{n+1}$ forms a full rank cone within the integer cone of representations of $S_n$. One way to interpret Problem 16 would be to describe the extremal rays and/or facets of $C_n$. Identifying representations of $S_n$ with points in $\mathbb{N}^{p(n)}$, where $p(n) = \#\{\lambda : \lambda \vdash n\}$ is the partition number, we could also ask for the size of $C_n$ by asking for the limit $\lim_{m \to \infty} \frac{\#(C_n \cap \{0, 1, \ldots, m\}^{p(n)})}{\binom{m+1}{p(n)}}$. The fact that $\psi$ is surjective means that this limit is nonzero, but we have no conjecture as to its value.

Since $\psi$ is surjective, every representation of $S_n$ is a restriction of a virtual $S_{n+1}$-module. In a slightly different direction, one could ask that a sufficiently large multiple of a representation extend. For arbitrary $m \leq n$, there is some integer $M$ such that $\mathbb{C}[S_m]^{\oplus M}$ extends to an $S_{n+1}$-module, by a result of Donkin [Do]. Indeed, take the canonical embedding

$$S_m \subset S_{n+1} \subset GL_{n+1}(\mathbb{C})$$

in the canonical way. Donkin asserts that there is a finite dimensional rational $GL_{n+1}(\mathbb{C})$-module $V$ such that $\text{Res}_{S_n}^{GL_{n+1}(\mathbb{C})} V \cong \mathbb{C}[S_n]^{\oplus m}$. It follows that $\text{Res}_{S_{n+1}}^{GL_{n+1}(\mathbb{C})} V$ is the desired extension.

Problem 16 is unsolved even for the coset representations $M^\lambda$. Not all of these representations extend (if they did, then any direct sum of coset representations such as the parking representation would extend automatically). For example, the representation $M^{(3,2,2)}$ of $S_7$ does not extend to a representation of $S_8$ (as can be checked by computer using [B]). However, the representation $M^\lambda$ of $S_n$ extends to a representation of $S_{n+1}$ for all $\lambda \vdash n$ and $0 \leq n \leq 6$.

Many variations on Problem 16 are possible. One could ask for a nice way of determining the greatest integer $k$ such that an $S_n$-module $M$ extends to $S_{n+k}$.

Also, one could ask whether a given permutation representation of $S_n$ extends to $S_{n+1}$. For $n \leq 5$, the permutation action of $S_n$ on $\text{Park}_n$ extends
to a permutation action of $S_{n+1}$; we are not sure whether this permutation action extends in general. A seeming difficulty with this question is that one would \textit{a priori} need to consider restrictions to $S_n$ of the action of $S_{n+1}$ on the set of cosets $S_{n+1}/G$ for \textit{any} subgroup $G$ of $S_{n+1}$.

A more combinatorial ‘permutation’ version of Problem 16 can be obtained by asking which permutation representations of $S_n$ are restrictions of permutation representations of $S_{n+1}$ which are direct sums of the modules $\{M^\lambda : \lambda \vdash n+1\}$. The parking representation Park$_4$ does not satisfy this property. To see this, one uses the fact that for $\lambda \vdash n + 1$, $\text{Res}^{S_{n+1}}_{S_n}(M^\lambda) = \bigoplus_{\mu} M^\mu$, where $\mu$ ranges over all partitions obtained by subtracting 1 from any nonzero part of $\lambda$ and sorting the resulting sequence into weakly decreasing order. It is a direct computation that Park$_4 = M^{(1,1,1,1)} \oplus 6 M^{(2,1,1)} \oplus 2 M^{(2,2)} \oplus 4 M^{(3,1)} \oplus M^{(4)}$ is not a $\mathbb{N}$-linear combination of the seven modules $\{\text{Res}^{S_{n+1}}_{S_n}(M^\lambda) : \lambda \vdash 5\}$.

Problem 16 could also be interesting in positive characteristic or for towers of linear or Weyl groups other than symmetric groups.

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