Shift Theorem Involving the Exponential of a Sum of Non-Commuting Operators in Path Integrals

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Abstract

We consider expressions of the form of an exponential of the sum of two non-commuting operators of a single variable inside a path integration. We show that it is possible to shift one of the non-commuting operators from the exponential to other functions which are pre-factors and post-factors when the domain of integration of the argument of that function is from $-\infty$ to $+\infty$. This shift theorem is useful to perform certain integrals and path integrals involving the exponential of sum of two non-commuting operators.

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I. INTRODUCTION

In background field methods in quantum field theory one often encounters the exponential of the sum of non-commuting operators inside the path integration. A simple example of this type occurs when looking at pair production of charged scalars \[1\] \[2\] in the presence of a time dependent background electric field \(E(t)\) in the longitudinal \((z)\) direction. In the Axial gauge \(A_z = 0\) so that

\[A_0 = -E(t)z.\] (1)

The action can be written in the form \[3\]

\[S^{(1)} = -i \int_0^\infty \frac{ds}{s} \int_0^\infty dt < t | \int_{-\infty}^{+\infty} dx < x | \int_{-\infty}^{+\infty} dy < y | \int_{-\infty}^{+\infty} dz < z | \]
\[e^{-i\hat{p}_0 + e(E(t)z)^2 - \hat{p}_z^2 - \hat{p}_z^2 - m^2 - i\epsilon} - e^{-i\hat{p}_0 + e(\hat{p}_z^2 - m^2 - i\epsilon)} | z > | y > | x > | t > .\] (2)

Inserting complete set of states we get \(|p_T >\) states \(\int d^2p_T |p_T > < p_T| = 1\) we find (we use the normalization \(< q|p > = \frac{1}{\sqrt{2\pi}} e^{iqp}\))

\[S^{(1)} = \frac{-i}{(2\pi)^2} \int_0^\infty \frac{ds}{s} \int d^2x_T \int d^2p_T e^{i(p_T^2 + m^2 + i\epsilon)} | \int_{-\infty}^{+\infty} dt < t | \int_{-\infty}^{+\infty} dz < z | e^{-i\hat{p}_0 + e(\hat{p}_z^2 - m^2)} | z > | t > - \int dt \int dz \frac{1}{4\pi s}.\] (3)

Inserting \(|p_z >\) and \(|p_0 >\) complete set of states we get

\[S^{(1)} = \frac{-i}{(2\pi)^2} \int_0^\infty \frac{ds}{s} \int d^2x_T \int d^2p_T \int dp_0 \int dp_0' \int dp_z \int dp_z' e^{i(p_0^2 + m^2 + i\epsilon)} | \int_{-\infty}^{+\infty} dt e^{itp_0} \int_{-\infty}^{+\infty} dz e^{izp_z} e^{-i\hat{p}_0' + e(\hat{p}_z' - m^2)} e^{-i\hat{p}_0} - \int dt \int dz \frac{1}{4\pi s}.\] (4)

In the coordinate representation the operators \(\hat{p}_0 = \frac{i}{\hbar} \frac{\partial}{\partial t}\) and \(E(t)\) do not commute with each other. In order to evaluate this type of Path Integral it is quite useful to be able to shift the derivative operator from the exponential to pre-factor and post-factor functions that occur when we insert complete sets of states in order to evaluate the Path Integral.

In particular we would like to show that the following theorem is true:

\[\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y | e^{-(a(y)x + b \frac{d}{dy})^2 + b \frac{d^2}{dx^2} + c(y)} | x > | y > = \]
\[\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y | x - \frac{h}{a(y)} \frac{d}{dy} e^{-(a(y)x^2 + b \frac{d^2}{dx^2} + c(y)} | x - \frac{h}{a(y)} \frac{d}{dy} > | y >\] (5)
which after inserting complete set of states can be written as

\[
\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \ < y \ | < x| e^{-\left[(a(y)x + b \frac{dy}{dx})^2 + b \frac{d^2}{dx^2} + c(y)\right]} \ | x > | y > = \frac{1}{(2\pi)^2} \int dp_y \int dp_y' \int dp_x
\]

\[
\int dp_x' \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx e^{ipy} e^{ipy'} < p_y | < p_x | e^{-a^2(y)x^2 + b \frac{d^2}{dx^2} + c(y)} | p_x' > | p_y' >
\]

\[
e^{-ip_x' \left(x - a(y) \frac{dy}{dy}\right)} e^{-ip_y'}
\]

(6)

where \( x \) integration from \(-\infty \) to \(+\infty \) must be performed for eqs. (5) and (6) to be true. Here \( h \) (which is equal to \( i \) in most of the physical examples, see eq. (4)) and \( b \) are constants and \( a, c \) are functions of single variable, such that the integration over \( x \) is well defined. In what follows we will assume that \( a(y) \) and \( c(y) \) are sufficiently differentiable, integrable etc. so that all the formal manipulations are valid. We have used the normalization

\[
<x|p_x> = \frac{1}{\sqrt{2\pi}} e^{ip_x}.
\]

(7)

It can be noted that eq. (6) can not be derived by replacing

\[
x \rightarrow x - \frac{h}{a(y)} \frac{d}{dy}
\]

(8)

directly in eq. (6). This is because \( a(y) \) and \( x \) commute with each other in the exponential whereas \( a(y) \) and \( \frac{d}{dy} \) do not commute with each other under this replacement. Hence we will use a similarity transformation technique to derive the above theorem which avoids this problem. The shift theorem leads to the special case

\[
W = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \ < y | < x| e^{-\left[(a(y)x + h \frac{dy}{dy})^2\right]} \ | x > | y >
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp_y' \int dp_x e^{ipy} e^{ipy'} < p_y | e^{-ip_x \left(x - a(y) \frac{dy}{dy}\right)} \ | p_x' > | p_y' >
\]

\[
< p_x | e^{-a^2(y)x^2} | p_x' > e^{ipy' \left(x - a(y) \frac{dy}{dy}\right)} | p_y' > e^{-ip_x' \left(x - a(y) \frac{dy}{dy}\right)}
\]

(9)

Let us evaluate the left and right hand side of the above equation separately. For the left hand side we find

\[
W = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \ < y | < x| e^{-\left[(a(y)x + h \frac{dy}{dy})^2\right]} \ | x > | y >
\]

\[
= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp_y' < y|p_y > < x|x > < p_y | e^{-\left[(a(y)x + h \frac{dy}{dy})^2\right]} | p_y' > | p_y' >
\]

(10)
Since \( < p_y e^{-[(a(y)x + h \frac{d}{dy})^2]} | p_y' > \) is independent of \( \frac{d}{dy} \) we can take \( < p_y' | y > \) to the left. We find

\[
W = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x < p_y' | y > < y | p_y > < x | p_x > < p_x | x > \\
= \frac{1}{2\pi} \int dp_x \int dp_y < p_y | \int_{-\infty}^{+\infty} dx e^{-(a(y)x + h \frac{d}{dy})^2} | p_y >
\]

(11)

Although eq. (11) is formally infinite, the \( x \)-integral inside \( W \)

\[
I(y) = \int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + h \frac{d}{dy})^2}
\]

(12)

is finite.

Now evaluating the right hand side of eq. (9) we find

\[
W = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x' \int dp_x'' e^{ixp_x} e^{ipy} < p_y | e^{-ip_x \frac{d}{dy}} \frac{d}{dy} | p_x' > \\
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x' \int dp_x'' \int dp_x'' e^{ipx} e^{ipy} < p_y | e^{-ip_x \frac{d}{dy}} \frac{d}{dy} | p_x' >
\]

(13)

It can be shown that

\[
< p_y | f(y) | p_y' > = \int dy' < p_y | f(y) | y' > < y' | p_y' > = \frac{1}{(2\pi)} \int dy' e^{iy(p-y-p'_y)} f(y')
\]

(14)

is independent of \( y \) and

\[
< p_y | f(y) \frac{d}{dy} | p'_y > = -i \int dy' < p_y | f(y) | y' > < y' | p'_y > \ p'_y' = -ip'_y \frac{1}{(2\pi)} \int dy' e^{iy(p-y-p'_y)} f(y')
\]

(15)

is independent of \( y \) and \( \frac{d}{dy} \). Hence we can easily integrate over \( y \) in eq. (13). Also since

\[
< p_y'' | e^{ip'_y \frac{d}{dy}} \frac{d}{dy} | p_y'' > \]

is independent of \( \frac{d}{dy} \) we can bring it to the left. We find from eq. (13)

\[
W = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x' \int dp_x'' e^{ipx} e^{ipy} \int_{-\infty}^{+\infty} dx'
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x' \int dp_x'' \int_{-\infty}^{+\infty} dx' e^{ipx} e^{ipy} \int_{-\infty}^{+\infty} dx' e^{ipx} e^{ipy} \\
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx \int dp_y \int dp_x' \int dp_x'' \int_{-\infty}^{+\infty} dx' e^{ipx} e^{ipy} \\
< p_y'' | e^{ip'_y \frac{d}{dy}} \frac{d}{dy} | p_y'' > < p_y e^{-ip \frac{d}{dy}} \frac{d}{dy} | p_y' > < p_x' | p_x > < p_x | p_x > < p_x | p_x > < p_x' | p_x > | p_y'' >
\]

(16)
Since the \( x \) dependence is only in \( e^{ix(p_x - p'_x)} \) we can now easily integrate over \( x \) to find

\[
W = \frac{1}{(2\pi)^2} \int dp_y \int dp'_y \int dp_y'' \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dx' \cdot \frac{d}{dy} \frac{d}{dy} |p_y > < p_y| e^{-ip_y \frac{d}{dy}} |p_y' > < p_y'| e^{-a^2(y)x'^2} |p_y'' > \\
= \frac{1}{(2\pi)^2} \int dp_y \int dp_y'' \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dx' \cdot \frac{d}{dy} |p_y > < p_y| e^{-a^2(y)x'^2} |p_y' > \\
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' \cdot \frac{d}{dy} |p_y > < p_y| \int_{-\infty}^{+\infty} dx \cdot e^{-a^2(y)x^2} |p_y >
\]

(17)

Although eq. (17) is formally infinite, the \( x \)-integral inside \( W \)

\[
I(y) = \int_{-\infty}^{+\infty} dx \cdot e^{-a^2(y)x^2}
\]

(18)
is finite.

Hence from eqs. (11) and (17) we find

\[
I(y) = \int_{-\infty}^{+\infty} dx \cdot e^{-(a(y)x + h \frac{d}{dy})^2} = \int_{-\infty}^{+\infty} dx \cdot e^{-a^2(y)x^2}.
\]

(19)

This above theorem eq. (6) can be generalized to involve matrices as follows

\[
I_{ij}(y) = [\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y | e^{-(A(y)x + h \frac{d}{dy})^2 + B \frac{d^2}{dy^2} + C(y)} | x > | y >]_{ij} \\
= \frac{1}{(2\pi)^2} [\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_y \int dp'_y \int dp_x \int dp'_x \cdot e^{ip_y + \frac{d}{dy} e^{ip_y} e^{ip_x} < p_y | e^{-ip_x A(y) \frac{d}{dy}} | p'_y >} \\
< p_x | e^{-(A^2(y)x^2 + B \frac{d^2}{dy^2} + C(y))} | p'_x > e^{ip'_x \frac{d}{dy} + C(y)} | p'_y > e^{-ip'_y} e^{-ip'_y}]_{ij}.
\]

(20)

Here \( h \) and \( B \) are constants and \( A^{ij}(y), C^{ij}(y), \) are \((i, j)\) dimension matrices which do not commute with \( \frac{d}{dy} \); chosen that the integration over \( x \) is well defined. In what follows we will assume that \( A^{ij}(y) \) and \( C^{ij}(y) \) are sufficiently differentiable, integrable etc. so that all the formal manipulations are valid.

This shift by derivative technique will be very useful when one studies particle production from arbitrary background fields via Schwinger-like mechanisms in QED and QCD [1, 2]. Quark and gluon production from arbitrary classical chromofields is expected to be an important ingredient in the production and equilibration of the quark-gluon plasma found at the RHIC and LHC [4, 5].
This paper is organized as follows. In section II we provide general derivations of eqs. (6), and (20) by using similarity transformation techniques. We verify eq. (19) by directly performing the integration, where we consider the integrals as a function of the variable \( h \) and assume that the integrals have a unique Taylor Series in \( h \). We present our conclusions in section IV.

II. SIMILARITY TRANSFORMATION APPROACH FOR DERIVING THE “SHIFT THEOREM”

In this section we provide a general derivation of eqs. (6) and (20) by using similarity transformations. Before giving such a derivation we consider here a simple case \( (a(y) = a=\text{constant}) \) to show how the derivative operator acts as a c-number when \( x \) integration is from \(-\infty \) to \(+\infty \). We find by using Fourier transformation technique

\[
\int_{-\infty}^{+\infty} dxe^{-(ax+h \frac{d}{dy})^2} f(y) = \int_{-\infty}^{+\infty} dxe^{-(ax+h \frac{d}{dy})^2} \int dp f(p) e^{yp} = \int dp \int_{-\infty}^{+\infty} dxe^{-(ax+ihp)^2} f(p) e^{yp} \\
= \int dp \int_{-\infty}^{+\infty} dx e^{-a^2x^2} f(p) e^{yp} = \int_{-\infty}^{+\infty} dx e^{-a^2x^2} f(y)
\]

(21)

In the above we have assumed that \( f(y) \) is well enough behaved so that it is legal to change the order in which the integrations are taken. In what follows we will assume that \( f(y) \) is sufficiently differentiable, integrable etc. so that all the formal manipulations are valid.

However, this Fourier transformation technique does not work if \( a(y) \) is not a constant. This is because \( a(y) \) and \( \frac{d}{dy} \) do not commute with each other in the exponential. For this purpose we use a similarity transformation technique to derive the shift theorem.

A. Shift Theorem Involving Non-Commuting Operators in the Exponential

Consider the following similarity transformations acting on \( x \):

\[
x \pm \frac{h}{a(y)} \frac{d}{dy} = e^{\pm \frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} x e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}}.
\]

(22)

Since \( e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} \) commutes with \( b \frac{d^2}{dx^2} \) we find

\[
(a(y)x + h \frac{d}{dy})^2 + b \frac{d^2}{dx^2} = e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} [(e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} a(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} x)^2 + b \frac{d^2}{dx^2}] e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}}.
\]

(23)
Hence
\[
F(y) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \quad < y < x \quad e^{-(a(y)x + b(x)y^2 + c(y))} \quad |x| > |y| > 0
\]
\[
= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y < x \quad e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}} - e\left(e\frac{-h(y)}{a(y) dy} \frac{d}{dx} a(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}x\right)^2 + b \frac{d^2}{dx^2} + e\left(\frac{-h(y)}{a(y) dy} \frac{d}{dx} c(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}\right) \quad |x| > |y| > 0.
\]  
\]  
(24)

Now inserting complete set of states we find
\[
F(y) = \int dp_y \int dp'_y \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y|p_y > < x|p_x > < p_y < < p_x |e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}} - e\left(e\frac{-h(y)}{a(y) dy} \frac{d}{dx} a(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}x\right)^2 + b \frac{d^2}{dx^2} + e\left(\frac{-h(y)}{a(y) dy} \frac{d}{dx} c(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}\right) \quad |x| > |y| > 0.
\]
\[
= \frac{1}{(2\pi)^2} \int dp_y \int dp'_y \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx e^{iy p_y} e^{i p_x} < x|p_x > < p_y < < p_x |e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}} - e\left(e\frac{-h(y)}{a(y) dy} \frac{d}{dx} a(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}x\right)^2 + b \frac{d^2}{dx^2} + e\left(\frac{-h(y)}{a(y) dy} \frac{d}{dx} c(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}\right) \quad |x| > |y| > 0.
\]  
\]  
(25)

Unlike the situation in eq. (8) we can now change the \(x\) integration variable to \(x'\) via
\[
x = x' - \frac{h}{a(y)} dy.
\]  
(26)

This is because (unlike the left hand side of eq. (24)), \(a(y)\) and \(x\) can not be interchanged in the right hand side of eq. (24). Hence we can change the \(x\) variable to \(x'\) via eq. (26) which involves a derivative. With the above change in integration variable the integration limits for \(x'\) remain \(\pm \infty\). Under this change of integration variable one also has \(dx = dx'\).

With these changes we find from the equation (24)
\[
F(y) = \frac{1}{(2\pi)^2} \int dp_y \int dp'_y \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx e^{iy p_y} e^{i p_x(x - \frac{h}{a(y)} dy)} < p_y < p_x |e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}} - e\left(e\frac{-h(y)}{a(y) dy} \frac{d}{dx} a(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}x\right)^2 + b \frac{d^2}{dx^2} + e\left(\frac{-h(y)}{a(y) dy} \frac{d}{dx} c(y)e^{\frac{h(y)}{a(y) dy} \frac{d}{dx}}\right) \quad |x| > |y| > 0.
\]
\[
e^{-i p'_x \left(x - \frac{h}{a(y)} dy\right)} e^{-i y p'_y}.
\]  
(27)

Using eq. (22) for the similarity transformation of \((x - h/a(y) dy)\) we find
\[
\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y < x \quad e^{-(a(y)x + b(x)y^2 + c(y))} \quad |x| > |y| = \frac{1}{(2\pi)^2} \int dp_y \int dp'_y \int dp_x \int dp'_x \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx e^{iy p_y} e^{i p'_x \left(x - \frac{h}{a(y)} dy\right)} < p_y < p_x |e^{-a(y)x^2 + b \frac{d^2}{dx^2} + c(y)} |p'_x > |p'_y >
\]
\[
e^{-i p'_x \left(x - \frac{h}{a(y)} dy\right)} e^{-i y p'_y}.
\]  
(28)

This concludes our derivation of eq. (6), and consequently the special case (19).
B. Shift Theorem Involving Matrices and Non-Commuting Operators in the Exponential

We next consider the similarity transformation on the matrices \( x^\delta_{ij} \) as follows

\[
\delta_{ij} x \pm \left[ \frac{h}{A(y)} \frac{d}{dy} \right]^{ij} = \left[ e^{\pm \frac{h}{A(y)} \frac{d}{dy}} x e^{\mp \frac{h}{A(y)} \frac{d}{dy}} \right]^{ij}
\]

(29)

where \( A^{ij}(y) \) is \( y \)-dependent matrix.

Since \( e^{\frac{h}{A(y)} \frac{d}{dy}} \) commutes with \( B \delta_{ij} \frac{d^2}{dx^2} \) we find

\[
[(A(y)x + h[\frac{d}{dy}])^2]^{ij} + \delta^{ij} B \frac{d^2}{dx^2} = \left[ e^{\frac{h}{A(y)} \frac{d}{dy}} \right]^{im} \left[ \left( e^{-\frac{h}{A(y)} \frac{d}{dy}} A(y) e^{\frac{h}{A(y)} \frac{d}{dy}} \right) x \right]^2 + B \frac{d^2}{dx^2} \right]^{ml}
\]

\[
[ e^{-\frac{h}{A(y)} \frac{d}{dy}} ]^{lj}.
\]

(30)

Repeating the same logic as used previously, we obtain

\[
I_{ij}(y) = \left[ \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y | x | e^{-[(A(y)x + h \frac{d}{dy})^2 + B \frac{d^2}{dx^2} + C(y)]} | x > | y > \right]_{ij}
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int dp_x \int dp_y \int dp_x' \int dp_y' e^{ip_x} e^{-ip_x' A(y) \frac{d}{dy}} - e^{-ip_x' A(y) \frac{d}{dy}} | p_x' > e^{ip_y} e^{-ip_y' A(y) \frac{d}{dy}} | p_y' > \right]_{ij}.\]

(31)

Since this "derivation" is rather formal and relies on similarity transformations that are not very familiar, we will now give examples demonstrating the usefulness and validity of the special case eq. (19), assuming that the integrals define a function which is Taylor "expandable" in a series in \( h \).

III. SOME SPECIAL CASES

In this section we would like to consider the special case

\[
A[h, y] = \int_{-\infty}^{+\infty} dx e^{-\alpha(y)x^2 + h \frac{d}{dy}} f(y)
\]

(32)

We would like to show that

\[
A[h, y] = \int_{-\infty}^{+\infty} dx e^{-\alpha(y)x^2} f(y) = A[h = 0, y].
\]

(33)

To do this we will need to assume that \( f(y) \) is such that \( A[h, y] \) has a unique Taylor expansion in the variable \( h \).
To obtain the Taylor series we will use a theorem for two non-commuting operators \( A, B \)
\[
e^{-(A+B)} = e^{-A} \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^{n} \left[ \int_{0}^{x_{i-1}} dx_i \ e^{x_i A} B e^{-x_i A} \right] \right]. \quad (34)
\]
Using eq. (34) in (32) we find
\[
e^{-(a(y)x+h \frac{dy}{dx})^2} f(y) = e^{-A} \left[ 1 - \int_{0}^{1} dx_1 e^{x_1 A} B e^{-x_1 A} \right. \]
\[+ \int_{0}^{1} dx_1 e^{x_1 A} B e^{-x_1 A} \int_{0}^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A} \]
\[ - \int_{0}^{1} dx_1 e^{x_1 A} B e^{-x_1 A} \int_{0}^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A} \int_{0}^{x_2} dx_3 e^{x_3 A} B e^{-x_3 A} + \ldots. \right] f(y) \quad (35)
\]
where
\[
(a(y)x+h \frac{dy}{dx})^2 = A + B,
\]
\[A = a^2(y)x^2 \]
\[B = 2x a(y) \ h \frac{dy}{dx} + x h \frac{da(y)}{dy} + h^2 \frac{d^2}{dy^2}. \quad (36)
\]
Integrating \( x \) from \(-\infty\) to \(+\infty\) in eq. (35) we write
\[
A[h, y] = \int_{-\infty}^{+\infty} dx \ e^{-A} \left[ f(y) - I_1[h, y] + I_2[h, y] - I_3[h, y] + I_4[h, y] + \ldots. \right] \quad (37)
\]
where
\[
I_n[h, y] = \frac{\int_{-\infty}^{+\infty} dx \ e^{-A} \left[ \prod_{i=1}^{n} \left[ \int_{0}^{x_{i-1}} dx_i \ e^{x_i A} B e^{-x_i A} \right] \right] f(y)}{\int_{-\infty}^{+\infty} dx \ e^{-A}} \quad (38)
\]
with \( x_0 = 1 \) and \( n=1,2,3...\) etc. The \( I_n \) consist of a finite number of terms in \( h \) from \( h^n \) up to \( h^{2n} \). Using the expressions for \( A \) and \( B \) from eq. (36) and performing the \( x \) and \( x_i \)'s integrations explicitly in eq (38) we can obtain explicit expressions for all \( I_n \). Then assuming that after we perform the integrations we can write
\[
A[h, y] = \sum_{n=0}^{\infty} A_n[y] h^n \quad (39)
\]
we will find that except for \( n = 0 \) all the coefficients in the Taylor series in \( h \) are zero.

A. Examples using simple function for \( a(y) \) and \( f(y) \)

First, we will consider two examples for \( a(y) \) and \( f(y) \) to demonstrate how eq. (32) works before giving the result for general \( a(y) \) and \( f(y) \). If we look at each power of \( h \)
in the expression of \( I_n \), each odd power of \( h \) formally vanishes because it contains an odd integration over \( x \) (see eq. \((36)\)).

**Example I:** \( a(y) = y, \quad f(y) = y \).

Using \( a(y) = y \) and \( f(y) = y \) in eq. \((38)\) we find

\[
-I_1 = h^2 \frac{1}{2y}, \quad I_2 = -h^2 \frac{1}{2y} - h^4 \frac{19}{24y^3}
\]

which gives \(-I_1 + I_2 = -h^4 \frac{19}{24y^3} \), independent of terms containing two powers of \( h \). Similarly we find

\[
-I_3 = h^4 \frac{19}{12y^3} + h^6 \frac{35}{48y^5}, \quad I_4 = -h^4 \frac{19}{24y^3} + h^6 \frac{123}{48y^5} + h^8 \frac{4199}{640y^7}
\]

which gives \(-I_1 + I_2 - I_3 + I_4 = h^6 \frac{79}{24y^5} + h^8 \frac{4199}{640y^7} \), independent of terms containing four powers of \( h \). This process can be repeated and we find \(-I_1 + I_2 - ... - I_n \) is independent of terms containing up to \( n \) powers of \( h \). Thus if we assume that our series in \( B \) gives us the unique Taylor Series in \( h \), then we find that the answer is independent of \( h \) which is what we wished to show.

**Example II:** \( a(y) = \frac{1}{y^2}, \quad f(y) = e^{-y} \).

Using \( a(y) = \frac{1}{y^2} \) and \( f(y) = e^{-y} \) in eq. \((38)\) we find

\[
-I_1 = e^{-y} h^2 \left[-1 + \frac{2}{y} + \frac{1}{y^2}\right], \quad I_2 = e^{-y} \left\{ h^2 \left[1 - \frac{2}{y^2} - \frac{1}{y^2}\right] + h^4 \left[\frac{1}{2} - \frac{3}{y} + \frac{5}{6y^2} - \frac{5}{6y^2} + \frac{5}{6y^2}\right]\right\}, \quad (42)
\]

which gives \(-I_1 + I_2 = e^{-y} h^4 \left[\frac{1}{2} - \frac{5}{y} + \frac{5}{6y^2}\right]\), independent of terms containing two powers of \( h \) (or \( \frac{d}{dy} \)). Similarly we find

\[
-I_3 = e^{-y} \left\{ h^4 \left[-1 + \frac{4}{y} + \frac{20}{3y^2} + \frac{4}{3y^2}\right] + h^6 \left[-\frac{1}{6} + \frac{1}{y} + \frac{7}{6y^2} - \frac{7}{2y^4} - \frac{7}{y^5} - \frac{29}{6y^6}\right]\right\},
\]

\[
I_4 = e^{-y} \left\{ h^4 \left[\frac{1}{2} - \frac{2}{y} + \frac{5}{2y^2} + \frac{35}{6y^4}\right] + h^6 \left[\frac{1}{2} - \frac{3}{y} + \frac{49}{6y^2} + \frac{91}{2y^4} + \frac{91}{y^5} + \frac{101}{2y^6}\right]\right\}
\]

\[
+ h^8 \left[\frac{1}{24} - \frac{1}{3y} - \frac{1}{2y^2} + \frac{15}{4y^4} + \frac{15}{y^5} + \frac{1313}{42y^6} + \frac{736}{21y^7} + \frac{2855}{168y^8}\right]\right\}
\]

which gives

\[
-I_1 + I_2 - I_3 + I_4 = e^{-y} \left\{ h^6 \left[\frac{1}{3} - \frac{2}{y} - \frac{7}{y^2} + \frac{42}{y^4} + \frac{84}{y^5} + \frac{137}{3y^6}\right]\right\}
\]

\[
+ h^8 \left[\frac{1}{24} - \frac{1}{3y} - \frac{1}{2y^2} + \frac{15}{4y^4} + \frac{15}{y^5} + \frac{1313}{42y^6} + \frac{736}{21y^7} + \frac{2855}{168y^8}\right]\right\}
\]

(44)
independent of terms containing four powers of \( h \). So continuing this reasoning to larger \( n \) we find again naively the result is independent of \( h \). Of course for the above choices of \( a(y) \), if one wants to also integrate over \( y \) one must exclude the origin in further integration over \( y \) for this result to make sense.

**B. General \( a(y) \) and \( f(y) \)**

For general \( a(y) \) and \( f(y) \) we find from eq. (48)

\[
-I_1 = h^2 \left( -\frac{f[y]a'[y]^2}{2a[y]^2} + \frac{a'[y]f'[y]}{a[y]} + \frac{f[y]a''[y]}{2a[y]} - f''[y] \right) \tag{45}
\]

and

\[
I_2 = h^2 \left( \frac{f[y]a'[y]^2}{2a[y]^2} - \frac{a'[y]f'[y]}{a[y]} - \frac{f[y]a''[y]}{2a[y]} + f''[y] \right) + h^4 \left( \frac{5f[y]a'[y]^4}{24a[y]^4} - \frac{a'[y]^3f'[y]}{a[y]^3} - \frac{f[y]a'[y]^2a''[y]}{a[y]^3} - \frac{2a'[y]f'[y]a''[y]}{a[y]^2} - \frac{3f[y]a''[y]^2}{8a[y]^2} + \frac{4a'[y]^2f''[y]}{3a[y]^2} - \frac{7a''[y]f''[y]}{6a[y]} + \frac{7f[y]a'[y]a^{(3)}[y]}{12a[y]^2} - \frac{2f'[y]a^{(3)}[y]}{3a[y]} - \frac{a'[y]f^{(3)}[y]}{a[y]} - \frac{f[y]a^{(4)}[y]}{6a[y]} + \frac{1}{2}f^{(4)}[y] \right) \tag{46}
\]

By adding eqs. (45) and (46) we find

\[
-I_1 + I_2 = h^4 \left( \frac{5f[y]a'[y]^4}{24a[y]^4} - \frac{a'[y]^3f'[y]}{a[y]^3} - \frac{f[y]a'[y]^2a''[y]}{a[y]^3} - \frac{2a'[y]f'[y]a''[y]}{a[y]^2} - \frac{3f[y]a''[y]^2}{8a[y]^2} + \frac{4a'[y]^2f''[y]}{3a[y]^2} - \frac{7a''[y]f''[y]}{6a[y]} + \frac{7f[y]a'[y]a^{(3)}[y]}{12a[y]^2} - \frac{2f'[y]a^{(3)}[y]}{3a[y]} - \frac{a'[y]f^{(3)}[y]}{a[y]} - \frac{f[y]a^{(4)}[y]}{6a[y]} + \frac{1}{2}f^{(4)}[y] \right) \tag{47}
\]

which is independent of terms containing two powers of \( h \).

Similarly evaluating \( I_3 \) and \( I_4 \) we find that \( -I_1 + I_2 - I_3 + I_4 \) does not contain terms up to four powers of \( h \).

This process can be repeated up to arbitrary powers of \( h \) so we find that the coefficients of \( h^n \) for \( n=1,2,3,\ldots \) vanish.

Thus if there is a unique Taylor series for \( A[h,y], \) then we obtain

\[
A[h, y] = A[h = 0, y] \tag{48}
\]

which is what we wanted to show.
IV. CONCLUSIONS

To conclude, we have shown that, remarkably, inside of integrals over the entire real line one can shift the non-commuting derivative operator (not depending on the integration variable) which occurs in exponentials just as if it were a constant. In particular we have shown that

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < y | x > | y > = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx < x - \frac{h}{a(y)} \frac{d}{dy} | e^{-[(a(y)x + h \frac{d}{dy})^2 + b \frac{d^2}{dx^2} + c(y)]} | x - \frac{h}{a(y)} \frac{d}{dy} > | y >$$

(49)

as well as the extension to Matrix functions where $x$ integration from $-\infty$ to $+\infty$ must be performed for the above equation to be true. This equation leads to the special case

$$I(y) = \int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + h \frac{d}{dy})^2} \ f(y) = \int_{-\infty}^{+\infty} dx \ e^{-a^2(y)x^2} \ f(y)$$

(50)

where $h$ and $b$ are constants and $f, a, b, c$ are functions of single variable chosen that the integration over $x$ is well defined. This shift theorem should prove useful in the evaluation of Path Integrals that occur when utilizing the background field method.

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