Threshold Singularities of the Spectral Shift Function for a Half-Plane Magnetic Hamiltonian

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Abstract

We consider the Schrödinger operator in constant magnetic field defined on the half-plane with Dirichlet boundary condition, $H_0$, and a decaying electric perturbation $V$. We analyze the spectral density near thresholds embedded in the absolutely continuous spectrum by studying the Spectral Shift Function (SSF) associated to the pair $(H_0 + V, H_0)$. For perturbations of fixed sign we estimate the SSF in terms of the eigenvalue counting function of certain compact operators and show that it is bounded away from the thresholds. If $V$ is power-like decaying at infinity there are singularities at the thresholds and we obtain the corresponding asymptotic behavior. Our technics give also results for the Neumann boundary condition.

Keywords: Magnetic Schrödinger operators; Dirichlet and Neumann boundary conditions; Spectral Shift Function.

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1 Introduction

Let $b > 0$ and $H_0$ be the self-adjoint operator generated in $L^2(\mathbb{R}_+ \times \mathbb{R})$ by the closure of the quadratic form

$$
\int_{\mathbb{R}_+ \times \mathbb{R}} \left| \frac{\partial u}{\partial x} \right|^2 + \left| i \frac{\partial u}{\partial y} + bxu \right|^2 dydx,
$$

defined originally on $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ ($\mathbb{R}_+:=(0,\infty)$). Thus $H_0$ is the Dirichlet realization in the half-plane of the operator

$$
-\frac{\partial^2}{\partial x^2} + \left( -i \frac{\partial}{\partial y} - bx \right)^2.
$$

This is the Hamiltonian of a 2D spinless nonrelativistic quantum particle moving in a half-plane, subject to a constant magnetic field of scalar intensity $b$.

The spectrum of $H_0$, $\sigma(H_0)$, is purely absolutely continuous and is given by

$$
\sigma(H_0) = \bigcup_{j \in \mathbb{N}} [b(2j-1), \infty) = [b, \infty).
$$

The Landau levels $\mathcal{E}_j := b(2j-1)$, $j \in \mathbb{N}$, play the role of thresholds in $\sigma(H_0)$.

Further, suppose that the electric potential $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function which satisfies

$$
|V(x,y)| \leq C(x)^{-\mu} (y)^{-\nu}, \quad (x,y) \in \mathbb{R}_+ \times \mathbb{R},
$$

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for some positive constant $C$ and $\mu, \nu > 1$. On the domain of $H_0$ introduce the operator

$$H := H_0 + V,$$

self-adjoint in $L^2(\mathbb{R}_+ \times \mathbb{R})$. Estimate (1.2) combined with the diamagnetic inequality in the half-plane (see [20]) imply that for any real $E_0 < \inf \sigma(H)$ the operator $|V|^{1/2}(H_0 - E_0)^{-1}$ is Hilbert–Schmidt, and hence the resolvent difference $(H - E_0)^{-1} - (H_0 - E_0)^{-1}$ is a trace-class operator. In particular the absolutely continuous spectrum of $H$ coincide with $[E_1, \infty)$. Furthermore, there exists a unique function $\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1})$, called the Spectral Shift Function (SSF) for the operator pair $(H, H_0)$, that satisfies the Lifshits-Kreın trace formula

$$\text{Tr}(f(H) - f(H_0)) = \int_\mathbb{R} \xi(E; H, H_0)f'(E)dE,$$

for each $f \in C_0^\infty(\mathbb{R})$, and vanishes identically in $(-\infty, \inf \sigma(H))$.

The SSF $\xi$ is an important object in the theory of perturbations. In scattering theory, it can be seen as the scattering phase of the operator pair $(H, H_0)$, namely we have the Birman-Kreın formula

$$\det(S(E)) = e^{-2\pi i \xi(E)}, \quad E \in [b, \infty) \text{ a.e.,}$$

where $S(E)$ is the scattering matrix of the operator pair $(H, H_0)$. In addition, for almost every $E < b = E_1$ the SSF coincide with the eigenvalue counting function of the operator $H$, i.e.

$$\xi(E; H, H_0) = -\text{Tr} 1_{(-\infty, E)}(H), \quad \text{ (1.3)}$$

where $1_{\Omega}$ denotes the characteristic function of the Borel set $\Omega \subset \mathbb{R}$.

In this article we investigate the properties of the SSF $\xi(E; H, H_0)$ for $V$ of definite sign. In Theorem 2.4 we give an estimate of the SSF whose proof permit us, in Corollary 2.5 to show that the SSF is bounded on compact sets not containing the thresholds $\{E_j\}$. We use Theorem 2.4 as well, in the proof of our main result Theorem 2.3 where for $V$ power-like decaying we obtain the “semiclassical” formula (2.13), which gives the asymptotic behavior of $\xi(E; H, H_0)$ as the energy $E$ approaches any of the singularities present at the spectral threshold $E_j$.

The asymptotic distribution of the discrete eigenvalues of 2D magnetic Hamiltonians has been well studied during the last 25 years (see for instance [30], [21], [14], [29], [34]). The study of the SSF proposed here is, to some extent, an extension of such problems to magnetic Hamiltonians having continuous spectrum (see (2.10) below). Although the asymptotic behavior of the SSF at the thresholds has been considered before for Magnetic Hamiltonians, this has been done mostly in the 3D case ([4], [5], [13], [32], [36]). A two dimensional Hamiltonian appears in [5], but that model presents some fundamental differences with the model considered in this article.

Let $\mathcal{F}$ be the partial Fourier transform with respect to $y$, i.e.

$$(\mathcal{F}u)(x, k) = (2\pi)^{-1/2} \int_\mathbb{R} e^{-iyk} u(x, y) dy, \quad (x, k) \in \mathbb{R}_+ \times \mathbb{R}.$$  

Then, we have the identity

$$\mathcal{F}H_0\mathcal{F}^* = \int_{\mathbb{R}} \oplus h(k) dk, \quad \text{ (1.4)}$$

where the operator $h(k)$ is the Dirichlet realization in $L^2(\mathbb{R}_+)$ of

$$\frac{d^2}{dx^2} + (bx - k)^2, \quad k \in \mathbb{R}.$$
Note that \( h(k), k \in \mathbb{R} \), is a Kato analytic family \([23]\). For each \( k \in \mathbb{R} \) the operator \( h(k) \) has a discrete and simple spectrum. Let \( \{E_j(k)\}_{j=1}^{\infty} \) be the increasing sequence of the eigenvalues of \( h(k) \). For \( j \in \mathbb{N} \), the function \( E_j(\cdot) \) is called the \( j \)-th band function. By Kato analytic perturbation theory \( E_j(k) \) is a real analytic functions of \( k \in \mathbb{R} \). Further, it is proved in \([11]\) (see also \([21\), Chapter 15.A\]), that for any \( j \in \mathbb{N} \), the band function \( E_j \) is strictly decreasing, and

\[
\lim_{k \to -\infty} E_j(k) = \infty, \quad \lim_{k \to \infty} E_j(k) = b(2j - 1) = \mathcal{E}_j. \tag{1.5}
\]

In the study of analytically fibered operators and its perturbations, as occurs in the present article, it is known that the extrema of the band functions play a significant role in the distribution of the eigenvalues or in the behavior of the Spectral Shift Function (see for example \([31\), \([13\), \([5\), \([32\), \([30\), \([4\]). For instance, the Schrödinger operator with constant magnetic field in \( \mathbb{R}^2 \) have band functions which are constants, and its spectrum is a discrete set (the Landau levels are eigenvalues of infinite multiplicity). In this particular case, the asymptotic behavior of the eigenvalues of the perturbed operator is governed by the eigenvalues distribution of some Toeplitz operators (see \([30\), \([33\), \([14\] for potential perturbations and \([28\), \([25\), \([16\] for obstacle perturbations).

In general, the band functions associated with a Schrödinger operator are not constant. For an energy level corresponding to a non degenerate extremum of the band function, there is a well known procedure to obtain effective Hamiltonians that allows us to describe the distribution of eigenvalues (as in \([31\), \([7\]) and the singularities of the SSF (see \([5\]) near this point.

In the present case, the minima of the band functions are not reached, they correspond to limit values at infinity. This is the source of one of the main technical difficulties that we have to deal with in order to describe the behavior of the SSF. We meet also such phenomena for a quantum Hall effect model (see \([9\]), for a Iwatsuka model (see \([24\]) and for the Neumann magnetic Schrödinger operator in the half-plane (see \([7\]). In all these works, only the counting functions of discrete eigenvalues was studied, although we believe that the results of this article can be extended to these models using the same type of techniques developed here.

### 2 Main results

For a compact self-adjoint operator \( A \), let us define the eigenvalue counting function

\[
n_{\pm}(s; A) := \text{Tr} \mathbf{1}_{(s, \infty)}(\pm A), \quad s > 0,
\]

and for an arbitrary compact linear operator \( A \) put

\[
n_*(s; A) := n_+(s^2; A^*A), \quad s > 0.
\]

Further, denote by \( \mathcal{S}_p, p \in [1, \infty) \), the Schatten – von Neumann class of compact operators, equipped with the norm

\[
\|T\|_p := \left( -\int_0^{\infty} r^p \, dn_*(r; T) \right)^{1/p}.
\]

**Reduction of the SSF to a counting function for a compact operator**

Fix \( k \in \mathbb{R} \) and \( j \in \mathbb{N} \). Denote by \( \pi_j(k) \) the one-dimensional orthogonal projection onto \( \text{Ker} (h(k) - E_j(k)) \). Then

\[
\pi_j(k) = |\psi_j(\cdot; k)\rangle \langle \psi_j(\cdot; k)| \tag{2.1}
\]
where \( \psi_j(x;k) \), \( x \in \mathbb{R}_+ \), is an eigenfunction of \( h(k) \) which satisfies
\[
h(k)\psi_j(\cdot;k) = E_j(k)\psi_j(\cdot;k), \quad \psi_j(0;k) = 0, \quad \|\psi_j(\cdot;k)\|_{L^2(\mathbb{R}_+)} = 1. \tag{2.2}
\]

Moreover, \( \psi_j(\cdot;k) \) could be chosen to be real-valued and analytic as functions of \( k \) in \( L^2(\mathbb{R}_+) \).

Fix \( j \in \mathbb{N} \). For \( z \in \mathbb{C}_+ := \{ \zeta \in \mathbb{C} | \text{Im } \zeta > 0 \} \) define
\[
T_j(z) := |V|^{1/2} \mathcal{F}^{*} \int_{\mathbb{R}} (E_j(k) - z)^{-1} \pi_j(k) \, dk \mathcal{F}|V|^{1/2}. \tag{2.3}
\]

By Proposition 5.3 below, the limit \( \lim_{\delta \downarrow 0} T_j(E + i\delta) := T_j(E) \) exists in the \( \mathcal{S}_1 \)-norm for energies \( E \in \mathbb{R}, E \neq E_j \).

**Theorem 2.1.** Assume that \( V \geq 0 \) satisfies (1.2), and write \( H_\pm = H_0 \pm V \). Fix \( j \in \mathbb{N} \). Then, for all \( r \in (0,1) \) we have
\[
n_+(1 + r; \text{Re} T_j(E_j + \lambda)) + O(1) \leq \pm \xi(E_j + \lambda; H_\pm, H_0) \leq n_+(1 - r; \text{Re} T_j(E_j + \lambda)) + O(1), \tag{2.4}
\]
as \( \lambda \to 0 \).

The proof of Theorem 2.1 is contained in Section 3.4.

As a result of the proof of this Theorem we have the following Corollary.

**Corollary 2.2.** Suppose that \( V \geq 0 \) satisfies (1.2). Then, on any compact set \( C \subset \mathbb{R} \setminus \{ E_j \}_{j=1}^\infty \),
\[
\sup_{E \in C} \xi(E;H_\pm,H_0) < \infty,
\]
i.e. the SSF is bounded away from the thresholds.

**Spectral asymptotics**

One consequence of Corollary 2.2 is that the only possible singularities of \( \xi(E;H,H_0) \) are present at the Landau Levels \( E_j \). In order to deduce from Theorem 2.1 the explicit asymptotic behavior of the SSF at these points, let us introduce more restrictive assumptions on the potential \( V \).

We suppose that \( V \) is the restriction onto \( \mathbb{R}_+ \times \mathbb{R} \) of a smooth function in \( \mathbb{R}^2 \), such that for any pair \( (\alpha,\beta) \in \mathbb{Z}_+^2 \) \( (\mathbb{Z}_+ := \{ 0, 1, 2, ... \} ) \), there exists a positive constant \( C_{\alpha,\beta} \) satisfying
\[
|\partial_\xi^\alpha \partial_\xi^\beta V(x,\xi)| \leq C_{\alpha,\beta} |x,\xi|^{-\nu \alpha - \beta - 2m} \quad \text{for all } (x,\xi) \in \mathbb{R}^2, \tag{2.5}
\]
where \( m > 2 \) is fixed, and \( (x,\xi) := (1 + x^2 + \xi^2)^{1/2} \). Clearly, this assumption implies (1.2) with \( \nu = \mu = m/2 \).

For \( a : \mathbb{R}^2 \to \mathbb{R} \) a measurable function and \( \lambda > 0 \), let us define the volume function
\[
N(\lambda, a) := \frac{1}{2\pi} \text{vol}\{ (x,\xi) \in \mathbb{R}_+ \times \mathbb{R}; a(x,\xi) > \lambda \}, \tag{2.6}
\]
where \( \text{vol} \) denotes the Lebesgue measure in \( \mathbb{R}^2 \). Note that if \( a \) satisfies (2.4), then \( N(\lambda, a) = O(\lambda^{-\frac{\nu}{2m}}) \), and if \( a \) is also compactly supported in one direction, then \( N(\lambda, a) = O(\lambda^{-\frac{\nu}{m/2}}) \), for \( \lambda \downarrow 0 \).
We introduce the assumption that for some positive constants $C$ and $\lambda_0$

$$N(\lambda, V) \geq C\lambda^{-2/m}, \quad 0 < \lambda < \lambda_0.$$  \quad (2.7)

Finally, we suppose that $N(\lambda, V)$ satisfies a homogeneity condition of the form

$$\lim_{\epsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \lambda^{2/m} (N(\lambda(1 - \epsilon), V) - N(\lambda(1 + \epsilon), V)) = 0.$$  \quad (2.8)

Conditions (2.7) and (2.8) are usually assumed in the study of the distribution of eigenvalues of some pseudodifferential operators \cite{10}, and are typically satisfied by $N(\lambda, V)$ if $V$ behaves like $(x, \xi)^{-m}$ at infinity.

In the following, for two functions $F$ and $G$ defined on some interval $I$, we will write $F(x) \approx G(x)$ if there exists positive constants $c_{\pm}$ such that

$$c_{-} G(x) \leq F(x) \leq c_{+} G(x),$$

for all $x$ in $I$.

Theorem 2.3. Let $V \geq 0$ and write $H_\pm = H_0 \pm V$. If $V$ satisfies (2.5) with $m > 2$ and $N(\lambda, \pm V)$ satisfies (2.7)-(2.8), the following asymptotic formula for the SSF

$$\xi(\mathcal{E}_j \pm \lambda; H_\pm, H_0) = \pm b N(\lambda, V)(1 + o(1)), \quad \lambda \downarrow 0,$$  \quad (2.9)

hold true for any fixed $j \in \mathbb{N}$. This implies in particular

$$\xi(\mathcal{E}_j \pm \lambda; H_\pm, H_0) \approx \lambda^{-2/m}, \quad \lambda \downarrow 0.$$

The proof of Theorem 2.3 can be found in Section 4.3.

Remarks

1. The results presented in Theorem 2.3 resemble those for the eigenvalue counting function of $H_{L,V} := H_L + V$, where $H_L$ is the Landau Hamiltonian in 2D \cite{30}. More explicitly, if $V \geq 0$ satisfies (1.2) we can define the function that counts the number of eigenvalues in the gaps of $\sigma_{ess}(H_{L,V})$ as

$$N_j(\lambda) = \text{Tr}(1(\mathcal{E}_j + \lambda, \mathcal{E}_{j+1})(H_{L,V})).$$

In this case, from (3.1) and the Birman-Schwinger principle one easily obtain that

$$N_j(\lambda) = \xi(\mathcal{E}_j + \lambda; H_{L,V}, H_L) + O(1),$$  \quad (2.10)

and therefore the study of the eigenvalue counting function of the perturbed Landau Hamiltonian $H_{L,V}$ is the same as the study of the Spectral Shift Function for the pair $(H_{L,V}, H_L)$. Moreover, under conditions similar to those in Theorem 2.3, the asymptotic behavior of $N_j(\lambda)$ is also given by a semiclassical formula of the form (2.9). In conclusion, one can say that, to some extent, our work extend the 2D results on the SSF of $\mathcal{H}_{0}$, to a case where the band functions of the unperturbed operator are not constant.

2. Related to the previous remark, we can mention the study of the SSF under compactly supported perturbations of $H_0$ (including obstacle perturbations), as a natural and important open problem (in particular from the physical point of view, see \cite{11, 19, 1, 15}). These cases present a different difficulty since the pseudodifferential analysis used here doesn’t work (there is no appropriate class of symbols for compactly supported potentials) and some non semiclassical asymptotics are expected (see \cite{33, 7}). In fact, using the effective Hamiltonian of our Theorem 2.1 together with ideas of \cite{6, 11}, one can show that for $V \geq 0$ of compact support

$$\xi(\mathcal{E}_j - \lambda; H_-, H_0) \approx |\ln \lambda|^{1/2}, \quad \lambda \downarrow 0.$$
3. The singularities of the SSF are naturally related to clusters of resonances (see for instance [4]). It would be interesting to analyze this phenomena, but the first difficulty to overcome is to define the resonances for our fiber operator. Due to the exponential decaying properties of the band functions, some non standard tools of complex analysis would be necessary.

Let us complete the results of Theorem 2.3 by other consequences of our analysis.

Corollary 2.4.

1. Assume that $V \geq 0$ satisfies (1.2). Then

$$\xi(\mathcal{E}_j - \lambda; H_+, H_0) = O(1), \quad \lambda \downarrow 0.$$  

2. On the other side, if $V \geq 0$ satisfies (2.5) with $m > 2$

$$\xi(\mathcal{E}_j + \lambda; H_-, H_0) = o(\lambda^{-2/m}), \quad \lambda \downarrow 0.$$  

The proof of part 1 of this Corollary is a direct consequence of Theorem 2.1, while part 2 can be proved using the same tools of pseudodifferential analysis used for the proof of Theorem 2.3.

Comparing Theorem 2.3 with Corollary 2.4 we can see that

$$\lim_{\lambda \downarrow 0} \xi(\mathcal{E}_j \pm \lambda; H_\pm, H_0) = 0.$$  

This result is different to the results obtained for 3D magnetic Hamiltonians for which the behavior of the SSF was studied at the thresholds (see [13], [32], [36]). For those models the corresponding limit (2.12) is a constant different from zero, at least for $H_-$, which gives a generalization of the Levinson’s formula. We can compare our situation again with the 2D constant magnetic field case since the eigenvalue counting function coincide with the SSF, up to uniformly bounded terms (see (2.10)). In that case, $\xi(\mathcal{E}_j \pm \lambda; H_{L,V}, H_L) = O(1)$ for $\mp V \geq 0$, then from the first Remark after Theorem 2.3 a result like (2.12) is also obtained for $(H_{L,V}, H_L)$.

At last, let us mention that following the proofs of our work, it is easy to obtain some analog results for the half-plane magnetic Schrödinger operator with a Neumann boundary condition.

Results for Neumann boundary conditions

Let us consider $H^N_0$, the Neumann realization of (1.1), namely the self-adjoint operator generated in $L^2(\mathbb{R}_+ \times \mathbb{R})$ by the closure of the quadratic form

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \left| \frac{\partial u}{\partial x} \right|^2 + \left| i \frac{\partial u}{\partial y} + bu \right|^2 dy dx,$$

defined originally on $C^\infty(\mathbb{R}_+ \times \mathbb{R})$.

A fibered decomposition of the form (1.4) holds true in this case as well. Thanks to [9] we known that each band function of the Neumann Hamiltonian is a decreasing function until a unique non degenerated minimum, and then increase converging to the corresponding Landau Level at infinity. Then, the minimum of each band is a threshold of the spectrum of $H^N_0$. Due to the non-degeneracy condition, it is well known how to study the behavior of the SSF at this points (see [5], [7]).
On the other side, for the extremal points at infinity, it can be shown that the behavior of the band functions and of the associated eigenfunctions are those of the Dirichlet operator (see Propositions 3.4 and 3.3 below). Thus, just like for the Dirichlet case we can justify that the main contribution of the SSF near a fixed Landau level \( E_j \) depends on the behavior of the corresponding band functions \( E_N^j \), the associated eigenfunction, and the interplay of this objects at infinity (which is given by the still valid relation (3.9)). The main difference comes from the fact that at infinity the band functions are below the corresponding Landau level, then, up to a change of sign of \( E_N^j - E_j \) (or equivalently of \( \lambda \) and of \( V \)), the above results remain true. More precisely, for \( H_N^\pm := H_0^N \pm V \), we have:

**Theorem 2.5.** \( \text{The statements of Theorems 2.1 and 2.3 hold true for the Neumann boundary conditions. Moreover the results of Corollary 2.4 have to be replaced by:} \)

1. If \( V \) satisfies (1.2), then \( \xi(E_j + \lambda; H^N_\pm, H_0^N) = O(1) \), as \( \lambda \downarrow 0 \).
2. If \( V \) satisfies (2.5) with \( m > 2 \), then \( \xi(E_j - \lambda; H^N_\pm, H_0^N) = \mathcal{O}(\lambda^{-2/m}) \), as \( \lambda \downarrow 0 \).

By the previous arguments it seems reasonable that these results could also be extended without mayor changes to other 2D magnetic models like (1.1) in the Half-plane with a Robin boundary condition, or to the Iwatsuka Hamiltonian [22].

### 3 Proof of Theorem 2.1

#### 3.1 Pushnitski’s representation of the SSF

Let us start by recalling the representation of the SSF given by A. Pushnitski in [27] for perturbations of definite sign. For \( z \in \mathbb{C}^+ \) and \( V \geq 0 \), define

\[
T(z) := V^{1/2}(H_0 - z)^{-1}V^{1/2}.
\]

As shown in Lemma 3.8 below, the norm limits

\[
\lim_{\delta \to 0} T(E + i\delta) =: T(E + i0),
\]

exist for every \( E \in \mathbb{R} \), \( E \neq E_j, j \in \mathbb{N} \), provided that \( V \) satisfies (1.2). Moreover, \( T(E + i0) \) is a Hilbert-Schmidt operator, and \( 0 \leq \text{Im} T(E + i0) \).

**Theorem 3.1.** [27 Theorem 1.2] Assume that \( V \geq 0 \) satisfies (1.2). Then for almost all \( E \in \mathbb{R} \) we have

\[
\xi(E; H_\pm, H_0) = \pm \frac{1}{\pi} \int_{\mathbb{R}} n_\pm(1; \text{Re} T(E + i0) + t \text{Im} T(E + i0)) \frac{dt}{1 + t^2}. \tag{3.1}
\]

#### 3.2 Spectral properties of \( H_0 \)

The following well known results will be repeatedly used in the text. First, for \( r_1, r_2 > 0 \), we have the Weyl inequalities

\[
n_\pm(r_1 + r_2; T_1 + T_2) \leq n_\pm(r_1; T_1) + n_\pm(r_2; T_2), \tag{3.2}
\]

where \( T_j, j = 1, 2 \), are linear self-adjoint compact operators (see e.g. [3 Theorem 9.2.9]), as well as the Ky Fan inequality

\[
n_\ast(r_1 + r_2; T_1 + T_2) \leq n_\ast(r_1; T_1) + n_\ast(r_2; T_2), \tag{3.3}
\]
for compact but not necessarily self-adjoint $T_j$, $j = 1, 2$, (see e.g. [3 Subsection 11.1.3]). For $T$ compact

$$n_+(r; T) = n_+(r; T^*), \quad (3.4)$$

and if $T = T^*$$n_+(r; T) = n_+(r; T) + n_-(r; T). \quad (3.5)$$

Besides, for $T$ in the Schatten – von Neumann class $\mathfrak{S}_p$, we have the Chebyshev-type estimate

$$n_+(r; T) \leq r^{-p}\|T\|_p^p, \quad (3.6)$$

for any $r > 0$ and $p \in [1, \infty)$.

In order to prove our main results we need a description of some spectral properties of the unperturbed operator $H_0$, basically related to the behavior of the band functions $E_j$ and the eigenprojections $\pi_j$ at infinity.

Let $h_\infty(k)$ be the self-adjoint realization in $L^2(\mathbb{R})$ of

$$-\frac{d^2}{dx^2} + (bx - k)^2, \quad k \in \mathbb{R}.$$ 

The operator $h_\infty(k)$ has discrete spectrum and its eigenvalues are given by the sequence $\{E_j = b(2j - 1), j \in \mathbb{N}\}$. Denote by $\pi_{j,\infty}(k), k \in \mathbb{R}, j \in \mathbb{N}$, the orthogonal projection onto Ker $(h_\infty(k) - E_j)$. Then, similarly to (2.1)

$$\pi_{j,\infty}(k) = \langle \psi_{j,\infty}; k\rangle \langle \psi_{j,\infty}; k \rangle$$

where the eigenfunction $\psi_{j,\infty}(; k)$ satisfies

$$-\frac{d^2\psi_{j,\infty}(x; k)}{dx^2} + (bx - k)^2\psi_{j,\infty}(x; k) = E_j\psi_{j,\infty}(x; k), \quad \|\psi_{j,\infty}(; k)\|_{L^2(\mathbb{R})} = 1.$$ 

Again, $\psi_{j,\infty}(; k)$ could be chosen to be real-valued. Furthermore, the functions $\psi_{j,\infty}, j \in \mathbb{N}$, admit an explicit description. Namely, if we put

$$\varphi_j(x) := \frac{1}{(\sqrt{\pi}(j - 1)!2^{j-1})^{1/2}}H_{j-1}(x)e^{-x^2/2}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N}, \quad (3.7)$$

where

$$H_q(x) := (-1)^qe^{x^2}\frac{d^q}{dx^q}e^{-x^2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Hermite polynomials (see e.g. [2 Chapter I, Eqs. (8.5), (8.7)]), then the real-valued function $\varphi_j$ satisfies

$$-\varphi_j''(x) + x^2\varphi_j(x) = (2j - 1)\varphi_j(x), \quad \|\varphi_j\|_{L^2(\mathbb{R})} = 1,$$

and hence

$$\psi_{j,\infty}(x; k) = b^{1/4}\varphi_j(b^{1/2}x - b^{-1/2}k), \quad j \in \mathbb{N}, \quad x \in \mathbb{R}, \quad k \in \mathbb{R}. \quad (3.8)$$

Define the non-negative operator

$$\Lambda_k := h_\infty(k)^{-1} - (0_- \oplus h(k)^{-1}),$$

where $0_-$ is the zero operator in $L^2(-\infty, 0)$.

**Proposition 3.2.** [4 Propositions 3.4-3.5-3.6] Let $j \in \mathbb{N}$, then as $k \to \infty$:
1. \( \|A_k\| = \frac{3\sqrt{2}}{2} k^{-2}(1 + o(1)), \)
2. \( \|\pi_{j,\infty}(k) - (0_+ \oplus \pi_j(k))\| = O(\|\pi_{j,\infty}(k)\Lambda_k\|), \)
3. \( E_j(k) - \mathcal{E}_j = \mathcal{E}_j^2 \|\pi_{j,\infty}(k)\Lambda_k^{1/2}(k)\|^2 (1 + o(1)). \)

As a consequence of Proposition 3.2 we have

\[
\lim_{k \to \infty} (E_j(k) - \mathcal{E}_j)^{-1/2} \|\pi_{j,\infty}(k) - (0_+ \oplus \pi_j(k))\| = 0. \tag{3.9}
\]

A useful form of the second statement in Proposition 3.2, written in terms of eigenfunctions instead of projections, is given by the following proposition.

**Proposition 3.3.** For any \( j \in \mathbb{N} \), we have

\[
\int_{\mathbb{R}_+} |\psi_j(x; k) - \psi_{j,\infty}(x; k)|^2 \, dx = O\left(k^{-2}(E_j(k) - \mathcal{E}_j)\right), \quad k \to \infty.
\]

**Proof.** Define \( A(k) := \langle \psi_j(\cdot; k), \psi_{j,\infty}(\cdot; k) \rangle_{L^2(\mathbb{R}_+)} \), where \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)} \) is the scalar product in \( L^2(\mathbb{R}_+) \). Since \( ||(0_- \oplus \pi_j(k)) - \pi_{j,\infty||}|^2 = 2(1 - A_k^2) \), we have that \( A_k^2 \to 1 \), as \( k \to \infty \).

From the continuity of \( A_k \) we may assume from the beginning that \( A_k \to 1 \), as \( k \to \infty \).

Now,

\[
\int_{\mathbb{R}_+} |\psi_j(x; k) - \psi_{j,\infty}(x; k)|^2 \, dx = \int_{\mathbb{R}_+} \psi_j(x; k)^2 \, dx - 2 \int_{\mathbb{R}_+} \psi_j(x; k)\psi_{j,\infty}(x; k) \, dx + \int_{\mathbb{R}_+} \psi_{j,\infty}(x; k)^2 \, dx
\]

\[
= 2(1 - A_k) - \int_{\mathbb{R}_-} |\psi_{j,\infty}(x; k)|^2 \, dx
\]

\[
= \left|\|0_- \oplus \pi_j(k)\| - \pi_{j,\infty}\|^2\right| \times \frac{1}{1 + A_k} - \int_{\mathbb{R}_-} |\psi_{j,\infty}(x; k)|^2 \, dx.
\]

From the definition of \( \psi_{j,\infty} \), straightforward calculations show that

\[
\int_{\mathbb{R}_-} |\psi_{j,\infty}(x; k)|^2 \, dx = \frac{k^{2j-3}}{2b} e^{-b^{-1}k^2} \approx k^{-2}(E_j(k) - \mathcal{E}_j), \quad k \to \infty.
\]

Where we have used (3.10) below. Equalities 2. and 3. of Proposition 3.2 implies

\[
||0_- \oplus \pi_j(k)\| - \pi_{j,\infty}\|^2 = O(\|\pi_{j,\infty}(k)\Lambda_k\|^2) = O(||\Lambda_k||(E_j(k) - \mathcal{E}_j)), \quad k \to \infty. \tag{3.12}
\]

Then, putting together (3.10), (3.11), (3.12) and the fact that \((1 + A_k)^{-1}\) is bounded, we finish the proof.

**Proposition 3.4.** [27] Corollary 15.A.6] For any \( j \in \mathbb{N} \), and \( n \in \mathbb{Z}_+ \)

\[
(E_j(k) - \mathcal{E}_j)^{(n)} \lesssim k^{2j-1+n} e^{-b^{-1}k^2}, \quad k \to \infty,
\]

and

\[
E_j(k) = k^2(1 + o(1)), \quad k \to -\infty. \tag{3.14}
\]
Finally, since the function $E_j$ is strictly decreasing we can set $\varphi_j : (0, \infty) \to \mathbb{R}$ as the inverse function of $E_j - E_j$. Evidently, $\lim_{s \to 0} \varphi_j(s) = \infty$. Furthermore, using the preceding proposition, easy calculations yield:

\[
\begin{align*}
\varphi_j(s) &\asymp \ln |s|^{1/2}; \\
\varphi_j'(s) &\asymp -\frac{1}{s |\ln s|^{1/2}}; \\
\varphi_j''(s) &\asymp \frac{1}{s^2 |\ln s|^{1/2}}, \quad s \downarrow 0.
\end{align*}
\]  

\section{Analysis of $T$ on the real axis}

\begin{proposition}
Let $V \geq 0$ satisfy (12). Then, for all $j \in \mathbb{N}$, and all $z \in \mathbb{C}_+$, the operator $T_j(z)$, defined in (23), belongs to $\mathcal{S}_1$. Furthermore, for any $E \not\in \mathcal{E}_j$, the limit

\[
\lim_{\delta \downarrow 0} T_j(E + i\delta) := T_j(E),
\]

exists in the $\mathcal{S}_1$-norm and is continuous with respect to $E \in \mathbb{R} \setminus \{E_j\}$ in the standard operator-norm. Besides, for $E \not\in \mathcal{E}_j$ it satisfies

\[
\|\text{Re}(T_j(E))\|_1 = O(|\mathcal{E}_j - E|^{-1}).
\]  

\end{proposition}

\begin{proof}
Define the operator valued function $G_j : \mathbb{R} \to \mathcal{S}_2(L^2(\mathbb{R}^2 \times \mathbb{R}); \mathbb{C})$, by

\[
G_j(k)u := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}^+} e^{-iky} V^{1/2}(x,y)\psi_j(x;k)u(x,y) \, dx \, dy, \quad u \in L^2(\mathbb{R}^2 \times \mathbb{R}).
\]  

For any fixed $z$ with $\text{Im} \, z > 0$, let us prove that

\[
\|G_j^*(\cdot)G_j(\cdot) / (E_j(\cdot) - z)\|_{1} \in L^1(\mathbb{R}).
\]  

First note that by Lemma 3.5 below, the function $G_j(\cdot)^*G_j(\cdot)$ is locally Lipschitz. Then it is a measurable function.

Next, set $g_{j,\infty} : \mathbb{R} \to \mathcal{S}_2(L^2(\mathbb{R}^2 \times \mathbb{R}); \mathbb{C})$ as the operator valued function given by

\[
g_{j,\infty}(k)u := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}^+} e^{-iky} V^{1/2}(x,y)\psi_{j,\infty}(x;k)u(x,y) \, dx \, dy, \quad u \in L^2(\mathbb{R}^2 \times \mathbb{R}).
\]  

The function $\psi_{j,\infty}$ being defined in (3.8).

Obviously, for any $k \in \mathbb{R}$ the operators $G_j(k)^*G_j(k)$, $g_{j}^*(k)g_{j}(k) : L^2(\mathbb{R}^2 \times \mathbb{R}) \to L^2(\mathbb{R}^2 \times \mathbb{R})$ are of rank one, hence the Hilbert-Schmidt and trace norms coincide. Therefore, it is sufficient to use the estimate

\[
\frac{\|G_j(k)^*G_j(k)\|_1}{|E_j(k) - z|} \leq \|g_{j,\infty}(k)^*g_{j,\infty}(k)\|_2 + 2\|G_j(k)^*G_j(k) - g_{j,\infty}(k)^*g_{j,\infty}(k)\|_2.  
\]  

For the first term on the r.h.s. of (3.20) we can see that (12) implies

\[
\begin{align*}
\|g_{j,\infty}(k)^*g_{j,\infty}(k)\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} V(x,y)V(x',y') |\psi_{j,\infty}(x;k)\psi_{j,\infty}(x';k)|^2 \, dx \, dy \, dy' \\
&\leq \frac{1}{2\pi} \left( \int_{\mathbb{R}} \langle y \rangle^{-\nu} \, dy \right)^2 \left( \int_{\mathbb{R}} \langle x \rangle^{-\mu} \psi_{j,\infty}(x;k)^2 \, dx \right)^2.
\end{align*}
\]
Now, from (3.8)
\[
\int_{\mathbb{R}} |x|^{-\mu} \psi_{j,\infty}(x;k)^2 \, dx = b^{1/4} \left( \varphi_j^2 (b^{1/2} \cdot) * (\cdot)^{-\mu} \right) (b^{-1} k),
\]
with \( \varphi_j \) introduced in (3.7). Thus, Young’s inequality together with \( \mu > 1 \) imply that 
\[
\|g_{j,\infty}(k)^* g_{j,\infty}(k)\|_2 \in L^1(\mathbb{R}).
\]
Using that \((E_j(k) - z)^{-1}\) is bounded, we are done with the integrability of the first term in the r.h.s. of (3.20).

The second term of the r.h.s. of (3.20) is shown to be integrable in the following way. For \( k \) near minus infinity we use the integrability of \((E_j(k) - z)^{-1}\) (see (3.14)) and the bounds
\[
\|G_j(k)^* G_j(k)\|_2, \|g_{j,\infty}(k)^* g_{j,\infty}(k)\|_2 \leq \frac{1}{\sqrt{2\pi}} \sup_{x \in \mathbb{R}_+} \int_{\mathbb{R}} V(x,y) \, dy.
\]
On the other side
\[
\|G_j(k)^* G_j(k) - g_{j,\infty}(k)^* g_{j,\infty}(k)\|^2_2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} V(x,y) V(x',y') |\psi_j(x;k)\psi_j(x';k) - \psi_{j,\infty}(x;k)\psi_{j,\infty}(x';k)|^2 \, dx \, dx' \, dy \, dy'
\leq \frac{1}{2\pi} \left( \sup_{x \in \mathbb{R}_+} \int_{\mathbb{R}} V(x,y) \, dy \right)^2 \int_{\mathbb{R}_+} |\psi_j(x;k)\psi_j(x';k) - \psi_{j,\infty}(x;k)\psi_{j,\infty}(x';k)|^2 \, dx \, dx'
\leq \frac{1}{2\pi} \left( \sup_{x \in \mathbb{R}_+} \int_{\mathbb{R}} V(x,y) \, dy \right)^2 \| (0_- \oplus \pi_j(k)) - \pi_j,\infty(k) \|^2_2.
\]
(3.22)

Then by 2. of Proposition 3.2
\[
\|G_j(k)^* G_j(k) - g_{j,\infty}(k)^* g_{j,\infty}(k)\|_2 \leq C \|\Lambda_k \pi_{j,\infty}(k)\|
\]
for all sufficiently large \( k \), which together with 3. of Proposition 3.2 and Proposition 3.4 imply the integrability of \(\|G_j(k)^* G_j(k) - g_{j,\infty}(k)^* g_{j,\infty}(k)\|_2\) at infinity.

As a consequence of (3.18), the operator \( \int_{\mathbb{R}} G_j(k)^* G_j(k) \, dk \) is well defined in \( \mathcal{S}_1 \), where the integral is understood in the Bochner sense, and it is easy to see that for \( \text{Im} \, z > 0 \),
\[
T_j(z) = \int_{\mathbb{R}} \frac{G_j(k)^* G_j(k)}{E_j(k) - z} \, dk.
\]
Next, suppose that \( z = E_j + \lambda + i\delta \), with \( \lambda, \delta \in \mathbb{R} \). Recall that \( \varrho_j \) is the inverse function of \( E_j - E_j \). Hence, changing the variable \( E_j(k) - E_j, s \mapsto s \) we get
\[
T_j(E_j + \lambda + i\delta) = \int_{-\infty}^{\infty} \frac{G_j(k)^* G_j(k)}{E_j(k) - E_j - \lambda - i\delta} \, dk = \int_{0}^{\infty} \frac{G_j(\varrho_j(s))^* G_j(\varrho_j(s))}{\lambda - s + i\delta} \varrho_j'(s) \, ds \quad (3.23)
\]
If \( \lambda < 0 \) by Lebesgue dominated convergence theorem
\[
\lim_{\delta \downarrow 0} T_j(E_j + \lambda + i\delta) = \int_{-\infty}^{\infty} \frac{G_j(k)^* G_j(k)}{E_j(k) - E_j - \lambda} \, dk = \int_{0}^{\infty} \frac{\pi_j(k)}{E_j(k) - E_j - \lambda} \, dk \mathcal{F} V^{1/2}.
\]
in the \( \mathcal{S}_1 \) norm.
If \( \lambda > 0 \) write
\[
\int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{\lambda - s + i\delta} g_j'(s) \, ds
= \int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{(\lambda - s)^2 + \delta^2} (\lambda - s) g_j'(s) \, ds - i\delta \int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{(\lambda - s)^2 + \delta^2} g_j'(s) \, ds.
\]

Then, it is easy to see that
\[
\lim_{\delta \to 0} i\delta \int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{(\lambda - s)^2 + \delta^2} (\lambda - s) g_j'(s) \, ds = i\pi G_j(q_j(\lambda))^* G_j(q_j(\lambda)) g_j'(\lambda).
\]

Using the analyticity of \( q_j \) as well as (3.20) we can prove that
\[
\lim_{\delta \to 0} i\delta \int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{(\lambda - s)^2 + \delta^2} g_j'(s) \, ds = \operatorname{p.v.} \int_0^\infty \frac{G_j(q_j(s))G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds
= \int_0^{\lambda/2} \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds + \int_{3\lambda/2}^\infty \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds
+ \int_0^{\lambda/2} \frac{G_j(q_j(\lambda + s))^* G_j(q_j(\lambda + s))}{(\lambda - s)} g_j'(\lambda + s) \, ds - \int_0^{\lambda/2} \frac{G_j(q_j(\lambda - s))^* G_j(q_j(\lambda - s))}{(\lambda - s)} g_j'(\lambda - s) \, ds
\]

(3.25)

Thus, for \( \lambda > 0 \)
\[
T_j(\mathcal{E}_j + \lambda) = \operatorname{p.v.} \int_0^\infty \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds - i\pi G_j(q_j(\lambda))^* G_j(q_j(\lambda)) g_j'(\lambda).
\]

(3.26)

It is obvious that for \( \lambda < 0 \)
\[
T_j(\mathcal{E}_j + \lambda)^* = T_j(\mathcal{E}_j + \lambda),
\]
(3.27)

and for \( \lambda > 0 \)
\[
\operatorname{Re} T_j(\mathcal{E}_j + \lambda) = \operatorname{p.v.} \int_0^\infty \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds
\]
\[
\operatorname{Im} T_j(\mathcal{E}_j + \lambda) = -\pi G_j(q_j(\lambda))^* G_j(q_j(\lambda)) g_j'(\lambda).
\]

(3.28)

Then, the continuity property of \( T_j \) follows immediately from (3.24), (3.28), the continuity of \( q_j \) and from the continuity of \( G_j^* G_j \), which is given by Lemma 5.6.

Finally, we prove (3.10). For \( \lambda > 0 \), we take into account (3.24), (3.25), the inequalities
\[
\left\| \int_0^{\lambda/2} \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds \right\|_1 \leq \frac{2}{\lambda} \int_{\mathcal{E}_j(\lambda/2)}^\infty \| G_j(k) G_j(k) \|_1 \, dk = O(\lambda^{-1}),
\]
\[
\left\| \int_{3\lambda/2}^\infty \frac{G_j(q_j(s))^* G_j(q_j(s))}{(\lambda - s)} g_j'(s) \, ds \right\|_1 \leq \frac{C}{\lambda} \int_{\mathcal{E}_j(3\lambda/2)}^\infty \| G_j(k) G_j(k) \|_1 \, dk + O(1) = O(\lambda^{-1}).
\]

and, as a special case in the proof of Lemma 4.1 below:
\[
\left\| \operatorname{p.v.} \int_{\lambda/2}^{3\lambda/2} \frac{G_j(q_j(s))^* G_j(q_j(s)) g_j'(s)}{s - \lambda} \, ds \right\|_1 = O\left(\frac{\lambda^{-1}}{|\ln \lambda|^{1/2}}\right).
\]

For \( \lambda < 0 \), from (3.14), (3.27) and (3.24)
\[
\| \operatorname{Re} T_j(\lambda) \|_1 \leq C \sup_{k \in \mathbb{R}_+} |\mathcal{E}_j(k) - \mathcal{E}_j - \lambda|^{-1} \int_0^\infty \| G_j(k) G_j(k) \|_1 \, dk + O(1) = O(\lambda^{-1}).
\]

\( \square \)
Lemma 3.6. For any \( j \in \mathbb{N} \), the function \( G_j^* G_j : \mathbb{R} \to \mathfrak{S}_2(L^2(\mathbb{R} \times \mathbb{R})) \) is locally Lipschitz. Furthermore, there exist \( K_j \in \mathbb{R} \) such that
\[
\|G_j(k)^* G_j(k) - G_j(k')^* G_j(k')\|_1 \leq C_j |k - k'|, \tag{3.29}
\]
for a constant \( C_j \) independent of \( k, k' \geq K_j \).

Proof. Since the operator \( G_j(k)^* G_j(k) \) is of rank one, we can use the different norms indistinctly. As in the proof of Proposition 3.5 we have
\[
\|G_j(k)^* G_j(k) - G_j(k')^* G_j(k')\|_2^2
\leq \frac{1}{2\pi} \left( \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} V(x, y) dy \right)^2 \int_{\mathbb{R}} \left| \psi_j(x; k) \psi_j(x'; k) - \psi_j(x; k') \psi_j(x'; k') \right|^2 dx dx'
= \frac{1}{2\pi} \left( \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}} V(x, y) dy \right)^2 \| \pi_j(k) - \pi_j(k') \|_2^2. \tag{3.30}
\]
The Lipschitz condition follows from (3.30) and the fact that the projection \( \pi_j \) depends analytically on \( k \).

Now, set \( K_j \) to be any number that satisfies \( E_{j-1}(K_j) < \varepsilon_j \). By (1.5) and the strict monotonicity of the bands functions, we can take a contour \( \gamma \) around \( E_j(k), E_j(k') \) such that no other eigenvalues of \( h(k) \) or \( h(k') \) lie inside the region enclosed by \( \gamma \) whenever \( k, k' \in (K_j, \infty) \). Then we can write the difference of the projections as
\[
\pi_j(k) - \pi_j(k') = \frac{1}{2\pi i} \oint_{\gamma} (h(k) - \omega)^{-1} - (h(k') - \omega)^{-1} d\omega
= \frac{1}{2\pi i} \oint_{\gamma} (h(k) - \omega)^{-1} (h(k') - h(k))(h(k') - \omega)^{-1} d\omega. \tag{3.31}
\]
The operators \( h(k) \) and \( h(k') \) have the same domains, moreover \( D(h(k)) = D(h(k')) \subset \{ f \in L^2(\mathbb{R}_+); xf(x) \in L^2(\mathbb{R}_+) \} \). Hence, we can split the difference \( h(k') - h(k) = (k' - k)((bx - k') + (bx - k)) \), obtaining
\[
\| \pi_j(k) - \pi_j(k') \|
\leq \frac{|k - k'|}{2\pi} \oint_{\gamma} \left\| (h(k) - \omega)^{-1} ((bx - k') + (bx - k))(h(k') - \omega)^{-1} \right\| d|\omega|. \tag{3.32}
\]
To finish the proof it is sufficient to find a uniform bound for the norms of the family of operators \( (h(k) - \omega)^{-1} ((bx - k') + (bx - k))(h(k') - \omega)^{-1} \), when \( \omega \) is in the contour \( \gamma \). By our choice of \( \gamma \) this is reduced to find a bound for \( \|(bx - k') (h(k') - \omega)^{-1} \| \).

For any \( g \in L^2(\mathbb{R}_+) \)
\[
\|(bx - k)(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R}_+)}
\leq \|(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R}_+)} + \|(bx - k)^2(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R}_+)},
\]
and the second term in the inequality satisfies
\[
\|(bx - k)^2(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R}_+)} \leq C\|(h(k) + 2b)(h(k) - \omega)^{-1}g\|_{L^2(\mathbb{R}_+)},
\]
for a constant \( C \) independent of \( g, \omega \) and \( k \) (see [12]). Due to our choice of \( \gamma \) it is easy to see that \( \|(h(k) - \omega)^{-1}\| \) and \( \|(h(k) + 2b)(h(k) - \omega)^{-1}\| \) are uniformly bounded in \( k \) and \( \omega \). \[\square\]
For $j \in \mathbb{N}$, let us introduce the projector
\[
P_j^+ := \mathcal{F}^* \int_\mathbb{R} \sum_{l>j} \pi_l(k) \, dk \mathcal{F},
\]
and for $z \in \mathbb{C}_+$
\[
T_j^+(z) := V^{1/2} P_j^+(H_0 - z)^{-1} P_j^+ V^{1/2} = V^{1/2} \mathcal{F}^* \int_\mathbb{R} \sum_{l>j} (E_l(k) - z)^{-1} \pi_l(k) \, dk \mathcal{F} V^{1/2},
\]
in both cases the infinite sums being understood in the strong sense.

**Lemma 3.7.** Let $j \in \mathbb{N}$. Then for $E \in (-\infty, \mathcal{E}_{j+1})$ the limit
\[
\lim_{\delta \to 0} T_j^+(E + i\delta) =: T_j^+(E),
\]
exists in the norm sense. Moreover, the function $T_j^+ : \mathbb{C}_+ \setminus [\mathcal{E}_{j+1}, \infty) \to \mathcal{S}_2$ is continuous, and for $E \in (-\infty, \mathcal{E}_{j+1})$
\[
\|T_j^+(E)\|_{\mathcal{S}_2} \leq C \mathcal{E}_{j+1} (\mathcal{E}_{j+1} - E)^{-1},
\]
where the constant $C$ is independent of $E$.

**Proof.** Due to the band structure \cite{14}, the spectrum of $P_j^+ H_0 P_j^+$ is $[\mathcal{E}_{j+1}, \infty)$, then the operator valued function $T_j^+(\cdot)$ is analytic in $\mathbb{C}_+ \setminus [\mathcal{E}_{j+1}, \infty)$. This implies in particular the existence of the limit \cite{33}.

Now, if $E \in (-\infty, \mathcal{E}_{j+1})$
\[
\|T_j^+(E)\|_2 = \|V^{1/2} P_j^+(H_0 - E)^{-1} V^{1/2}\|_2 \leq \|V^{1/2} P_j^+(H_0 - E)^{-1} H_0\| \|H_0^{-1} V^{1/2}\|_2 \leq \|V^{1/2}\|_\infty \|H_0^{-1} V^{1/2}\|_2 \frac{\mathcal{E}_{j+1}}{(\mathcal{E}_{j+1} - E)}.
\]
In the same way it can be shown that for $E_1, E_2 \in (-\infty, \mathcal{E}_{j+1})$
\[
\|T_j^+(E_1) - T_j^+(E_2)\|_2 \leq |E_1 - E_2| \|V^{1/2}\|_\infty \|H_0^{-1} V^{1/2}\|_2 \frac{\mathcal{E}_{j+1}}{(\mathcal{E}_{j+1} - E_1)(\mathcal{E}_{j+1} - E_2)},
\]
which implies the continuity of $T_j^+$ in the Hilbert-Schmidt norm. \hfill \Box

**Lemma 3.8.** Let $z = E + i\delta \in \mathbb{C}_+$. Then for all $E \in \mathbb{R} \setminus \{\mathcal{E}_l\}_{l \in \mathbb{N}}$ the norm limit
\[
\lim_{\delta \to 0} T(E + i\delta) = T(E + i0)
\]
exists in the Hilbert-Schmidt class. If $E \in (\mathcal{E}_{j-1}, \mathcal{E}_j) \cup (\mathcal{E}_j, \mathcal{E}_{j+1})$, for some $j \in \mathbb{N}$, then
\[
T(E + i0) = T_j^-(E) + T_j(E) + T_j^+(E),
\]
where $T_j^-(E) = \sum_{i=1}^{j-1} T_i(E)$. Moreover,
\[
\text{Re} \, T(E + i0) = \text{Re} \, T_j^-(E) + \text{Re} \, T_j(E) + T_j^+(E) (3.35)
\]
\[
\text{Im} \, T(E + i0) = \text{Im} \, T_j^-(E) + \text{Im} \, T_j(E), (3.36)
\]
and we have the continuous dependence, on $\mathbb{R} \setminus \{\mathcal{E}_l; l \in \mathbb{N}\}$, of $\text{Re} \, T(\cdot + i0)$ with the standard operator norm and of $\text{Im} \, T(\cdot + i0)$ in the trace class norm.
Proof. It suffices to use the representation
\[ T(E + i\delta) = V^{1/2} F^* \int_\mathbb{R} \sum_{l \in \mathbb{N}} (E_l(k) - (E + i\delta))^{-1} \pi_l(k) \, dk \, F V^{1/2}, \]
and apply Proposition 3.5 together with Lemma 3.7. From (3.2), if \( E \in (\mathcal{E}_j, \mathcal{E}_{j+1}) \), the imaginary part is just the finite rank operator
\[ \text{Im} T(E + i0) = -\pi \sum_{l \leq j} G_l(q_l(E - \mathcal{E}_l)) G_l(q_l(E - \mathcal{E}_l))' \]

\[ \lambda \]

Remark 3.9. The decomposition used in the above results was inspired by [6]. Related tools appear also in [26] and [15].

Remark 3.10. Following the arguments of Proposition 2.5 of [8], thanks to Lemma 3.8 it should be possible to prove that \( \xi(\cdot; H, H_0) \) is continuous on \( \mathbb{R} \setminus (\sigma_p(H) \cup \{ \mathcal{E}_l, l \in \mathbb{N} \}) \).

3.4 Proof of Theorem 2.1

From (3.1) we see that it is necessary to estimate \( n_\pm(1; \text{Re} T(\mathcal{E}_j + \lambda + i0) + t \text{Im} T(\mathcal{E}_j + \lambda + i0)) \). Since for \( \lambda \) small, \( \text{Im} T(\mathcal{E}_j + \lambda + i0) \) is of rank \( j \), by applying (3.2), we have the inequalities
\[ n_\pm(1; \text{Re} T(\mathcal{E}_j + \lambda + i0) - j) \leq n_\pm(1; \text{Re} T(\mathcal{E}_j + \lambda + i0) + t \text{Im} T(\mathcal{E}_j + \lambda + i0)) \leq n_\pm(1; \text{Re} T(\mathcal{E}_j + \lambda + i0)) + j, \]

(3.37)

for all \( t \in \mathbb{R} \).

Next, for the real part, Lemma 3.7 implies that
\[ V^{1/2} F^* \int_\mathbb{R} \sum_{l > j} (E_l(k) - \mathcal{E}_j)^{-1} \pi_l(k) \, dk \, F V^{1/2} \]

is the norm limit of the operator \( T_j^+(\mathcal{E}_j + \lambda) \), as \( \lambda \to 0 \). Therefore
\[ n_\pm(r; T_j^+(\mathcal{E}_j + \lambda)) = O(1), \quad \lambda \to 0, \]

and (3.5) together with Weyl inequalities (3.2) imply
\[ n_\pm(1 + r; \text{Re} T_j^-(\mathcal{E}_j + \lambda) + \text{Re} T_j(\mathcal{E}_j + \lambda)) + O(1) \leq n_\pm(1; \text{Re} T(\mathcal{E}_j + \lambda + i0)) \leq n_\pm(1 - r; \text{Re} T_j^- j (\mathcal{E}_j + \lambda) + \text{Re} T_j(\mathcal{E}_j + \lambda)) + O(1), \quad \lambda \to 0, \]

(3.38)

for any \( r \in (0, 1) \). To finish the proof we need to show that \( n_\pm(r; \text{Re} T_j^- (\mathcal{E}_j + \lambda)) \) remains bounded as \( \lambda \to 0 \) for all \( r > 0 \). From Proposition 3.3 Lebesgue dominated convergence theorem implies
\[ \lim_{\lambda \to 0} \text{Re} T_l(\mathcal{E}_j + \lambda) = \lim_{\lambda \to 0} \text{p.v.} \int_0^\infty G_l(q_l(s))^* G_l(q_l(s)) \frac{G_l(q_l(s))}{s - 2b(j - l) - \lambda} g_l(s) \, ds = \text{p.v.} \int_0^\infty G_l(q_l(s))^* G_l(q_l(s)) \frac{G_l(q_l(s))}{s - 2b(j - l)} g_l(s) \, ds. \]

(3.39)

Recalling that \( T_j^- (\mathcal{E}_j + \lambda) = \sum_{l=1}^{j-1} T_l(\mathcal{E}_j + \lambda) \), and putting together (3.11), (3.37), (3.38), (3.39), we obtain (2.4).

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Proof of Corollary 2.2

Suppose that \( C \subset [0, E_0] \), for some \( l_0 \in \mathbb{N} \). If \( E \in C \), by Theorem 3.1 and \( \ref{eq:3.37} \)

\[ |\xi(E; H_\pm, H_0)| \leq n_\pm(1; \text{Re} T(E + i0)) + l_0. \]

From \( \ref{eq:3.36} \), \( \ref{eq:3.35} \) and \( \ref{eq:3.36} \)

\[ n_\pm(1; \text{Re} T(E + i0)) \leq 2 \sum_{l=1}^{l_0} \|\text{Re} T_l(E)\|_1 + 4\|T_{l_0}^+(E)\|^2_2. \]

Then, \( \ref{eq:3.10} \) and \( \ref{eq:3.31} \) give us the boundedness of the SSF on \( C \).

4 Proof of Theorem 2.3

The proof of \( \ref{eq:4.2} \) is based on equation \( \ref{eq:4.2} \) for both cases, \("+" \) and \("-" \). However, the \("+" \) case present more difficulties due to the presence of the principal value term on its effective Hamiltonian (compare \( \ref{eq:3.24} \) with \( \ref{eq:3.28} \)). In order to maintain the notation \("+" \) case i.e., for \( V \geq 0 \) and \( \lambda > 0 \) we will consider \( \xi(E_j + \lambda; H_+, H_0) \). The proof of the \("-" \) case can be performed using the same kind of ideas.

For \( (x, y) \in \mathbb{R}^2 \) define

\[ \overline{V}(x, y) := \begin{cases} 0 & \text{if } x \leq 0, \\ V(x, y) & \text{if } x > 0. \end{cases} \]

It is clear that the operator in \( L^2(\mathbb{R}^2) \),

\[ \overline{V}^{1/2} \mathcal{F} \int_{\mathbb{R}} (E_j(k) - E_j - \lambda)^{-1}(0_+ \oplus \pi_j(k)) \, dk \mathcal{F} \overline{V}^{1/2}, \quad (4.1) \]

has the same non zero eigenvalues that

\[ T_j(E_j + \lambda) = V^{1/2} \mathcal{F} \int_{\mathbb{R}} (E_j(k) - E_j - \lambda)^{-1} \pi_j(k) \, dk \mathcal{F} V^{1/2}, \]

which acts in \( L^2(\mathbb{R}_+ \times \mathbb{R}) \). In what follows we will consider the operator defined by \( \ref{eq:4.2} \), but by an abuse of notation we will denote it by \( T_j(z) \) as well.

4.1 First spectral estimates

Lemma 4.1. Let \( r > 0 \) and \( \epsilon_\lambda \in (0, \lambda) \). Then, as \( \lambda \) tends to 0

\[ n_\pm \left( r; \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_j(\varphi_j(s)) G_j(\varphi_j(s))}{\lambda - s} \varphi_j(s) \, ds \right) = O\left( \frac{\epsilon_\lambda}{\lambda^2 |\ln \lambda|^2} \right). \quad (4.2) \]

Proof. To prove \( \ref{eq:4.2} \) write

\[ \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_j(\varphi_j(s))}{\lambda - s} \varphi_j(s) \, ds \]

\[ = \int_0^{\epsilon_\lambda} (G_j(\varphi_j(\lambda + s)) - G_j(\varphi_j(\lambda - s))) \varphi_j(\lambda + s) \, ds \]

\[ + \int_0^{\epsilon_\lambda} G_j(\varphi_j(\lambda - s)) (\varphi_j(\lambda + s) - \varphi_j(\lambda - s)) \, ds \]

\[ = \mathcal{M}_1(\lambda) + \mathcal{M}_2(\lambda). \]

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Moreover, for Proof.

For a similar procedure we can see that

Using (3.2), (3.5) and (3.6) we get

$$n_\pm(r; \text{p.v.} \int_{\lambda - \epsilon_\lambda}^{\lambda + \epsilon_\lambda} \frac{G_j(\varrho_j(s)) G_j(\varrho_j(s))}{\lambda - s} \varrho_j'(s) \, ds) \leq \frac{1}{2r} \|M_1(\lambda)\|_1 + \frac{1}{2r} \|M_2(\lambda)\|_1.$$  

From Lemma 3.6 and (3.14)

$$\|M_1(\lambda)\|_1 \leq \int_0^{\epsilon_\lambda} \|G_j(\varrho_j(\lambda + s)) G_j(\varrho_j(\lambda - s)) - G_j(\varrho_j(\lambda - s)) G_j(\varrho_j(\lambda - s))\|_1 \varrho_j'(\lambda + s) \frac{ds}{s} \leq C \int_0^{\epsilon_\lambda} \frac{ds}{s} = O\left(\frac{\epsilon_\lambda}{\lambda^2 \ln \lambda}\right), \quad \lambda \downarrow 0.$$

By a similar procedure we can see that

$$\|M_2(\lambda)\|_1 \leq C \int_0^{\epsilon_\lambda} \frac{ds}{\lambda^2 \ln \lambda} = O\left(\frac{\epsilon_\lambda}{\lambda^2 \ln \lambda}\right), \quad \lambda \downarrow 0. \quad \blacksquare$$

For $\lambda \in \mathbb{R}$ and $I \subset (0, +\infty) \setminus \{\lambda\}$ let us introduce $S_{j,\infty}[\lambda, I] : L^2(\varrho_j(I)) \to L^2(\mathbb{R}^2)$ as the integral operator with kernel

$$(2\pi)^{-1/2} \tilde{V}(x, y)^{1/2} e^{iky} \psi_j, \infty(x; k)|E_j(k) - E_j - \lambda|^{-1/2}. \quad (4.3)$$

In the same way, define $S_j[\lambda, I] : L^2(\varrho_j(I)) \to L^2(\mathbb{R}^2)$ as the integral operator with kernel (4.3), but using $\psi_j$ instead of $\psi_j, \infty$.

**Lemma 4.2.** Let $r > 0$ and $\lambda_0 > 0$, then for any $j \in \mathbb{N}$ and $\lambda \in (0, \lambda_0)$,

$$n_* (r; S_j[\lambda, (\lambda_0, +\infty)]) = O(1), \quad n_* (r; S_j, \infty[\lambda, (\lambda_0, +\infty)]) = O(1). \quad (4.4)$$

Moreover, for $\lambda > 0$, $\epsilon_\lambda \in (0, \lambda)$ and $I = (0, \lambda - \epsilon_\lambda) \cup (\lambda + \epsilon_\lambda, +\infty)$

$$n_* (r; S_j[\lambda, I] - S_{j,\infty}[\lambda, I]) = O\left(\frac{|\ln \epsilon_\lambda|}{|\ln \lambda|^{3/2}}\right), \quad \lambda \downarrow 0. \quad (4.5)$$

**Proof.** For $k \in \varrho_j(\lambda_0, +\infty) = (-\infty, \varrho_j(\lambda_0))$ we have $E_j(k) - E_j \geq \lambda_0$, then the compact operator $S_j S^*_{j,\infty}[\lambda, (\lambda_0, +\infty)] = \tilde{V}^{1/2} \mathcal{F}^* \int_{(-\infty, \varrho_j(\lambda_0))} (E_j(k) - E_j - \lambda)^{-1} \pi_j(k) \, dk \mathcal{F} \tilde{V}^{1/2}$ admits a norm limit as $\lambda$ goes to $0$, yielding the estimate in (4.4). The same idea works for $S_{j,\infty}[\lambda, (\lambda_0, +\infty)]$.

In order to prove (4.5), using the Ky Fan inequality (3.3) and (4.4), it is sufficient to show the following estimates for any $\eta > 1$ and $\lambda_0$ small enough:

$$n_* (r; (S_j - S_{j,\infty})[\lambda, (0, \lambda - \epsilon_\lambda)]) = O\left(\frac{|\ln \epsilon_\lambda|}{|\ln \lambda|^{3/2}}\right), \quad (4.6)$$

$$n_* (r; (S_j - S_{j,\infty})[\lambda, (\lambda + \epsilon_\lambda, \eta \lambda)]) = O\left(\frac{|\ln \epsilon_\lambda|}{|\ln \lambda|^{3/2}}\right), \quad (4.7)$$

$$n_* (r; (S_j - S_{j,\infty})[\lambda, (\eta \lambda, \lambda_0)]) = 0. \quad (4.8)$$
First, in the proof of \[4.8\] we use that for \( k \in \mathcal{G}_j(\eta \lambda, \lambda_0) \), \( E_j(k) - \mathcal{E}_j \geq \eta \lambda \) and then

\[
|E_j(k) - \mathcal{E}_j - \lambda|^{-1/2} \leq \left( \frac{\eta}{\eta - 1} \right)^{1/2} |E_j(k) - \mathcal{E}_j|^{-1/2},
\]

which together with the min-max principle yield

\[
n_* \left( r; (S_j - S_{j,\infty})[\lambda, (\eta \lambda, \lambda_0)] \right) \leq n_* \left( r \left( \frac{\eta - 1}{\eta} \right)^{1/2} ; (S_j - S_{j,\infty})[0, (0, \lambda_0)] \right). \tag{4.9}
\]

Besides, by the definition of \( S_j \) and \( S_{j,\infty} \)

\[
\|(S_j - S_{j,\infty})[0, (0, \lambda_0)]\| \leq \|V\|^{1/2}_{L^{\infty}(\mathbb{R}^2)} \sup_{k > \eta_1(\lambda_0)} \left( |E_j(k) - \mathcal{E}_j|^{-1/2} \|\psi_j(\cdot, k) - \psi_{j,\infty}(\cdot, k)\|_{L^2(\mathbb{R}^2)} \right). \tag{4.10}
\]

Consequently, by Proposition 3.3 we can find \( \lambda_0 > 0 \) sufficiently small such that the operator norm \( \|(S_j - S_{j,\infty})[0, (0, \lambda_0)]\| < r \), and then \( n_* \left( r; (S_j - S_{j,\infty})[0, (0, \lambda_0)] \right) = 0. \)

To prove \((4.6)\) and \((4.7)\), we use \((3.6)\) to obtain

\[
n_* \left( r; (S_j - S_{j,\infty})[\lambda, I] \right) \leq \frac{1}{r} \|(S_j - S_{j,\infty})[\lambda, I]\|_1
\]

\[
= \frac{1}{r} \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} V(x, y) \int_{\eta_1(\lambda)} |E_j(k) - \mathcal{E}_j - \lambda|^{-1} |\psi_j(x; k) - \psi_{j,\infty}(x; k)|^2 \, dk \, dy \, dx \tag{4.11}
\]

\[
\leq C \int_{\mathbb{R}^+} \int_{\eta_1(\lambda)} |E_j(k) - \mathcal{E}_j - \lambda|^{-1} |\psi_{j,\infty}(x; k) - \psi_{j,\infty}(x; k)|^2 \, dk \, dx.
\]

Thus, regarding Proposition 5.3 and (3.15), for \( I = (0, \lambda - \epsilon \lambda) \) or \( I = (\lambda + \epsilon \lambda, \lambda \eta) \), as \( \lambda \downarrow 0 \), we have

\[
n_* \left( r; (S_j - S_{j,\infty})[\lambda, I] \right) \leq C \int_{\eta_1(\lambda)} |E_j(k) - \mathcal{E}_j - \lambda|^{-1} (E_j(k) - \mathcal{E}_j) \frac{dk}{k^2} = C \int_I \frac{|\partial^j(s)|}{|\lambda - s|} \, ds = O \left( \frac{1}{|\ln \lambda|^{3/2}} \int_I \frac{1}{|\lambda - s|} \, ds \right) = O \left( \frac{1}{|\ln \lambda|^{3/2}} \right). \tag{4.12}
\]

\[\square\]

4.2 Some pseudodifferential analysis

In this section we introduce tools of the pseudodifferential calculus and some results that we will use in the proof of Theorem 243. First of all we will need the following class of symbols. For \((x, \xi) \in \mathbb{R}^2\) set the quadratic form in \(\mathbb{R}^2\)

\[
g_{x,\xi}(y, \eta) = |y|^2 + \frac{|\eta|^2}{(x, \xi)^2},
\]

and for \( p, q \in \mathbb{R} \), define the weight \( w_{p,q} := |x|^{-p} |x, \xi|^{-q} \). Then, according to \[18\] Definition 18.4.6], consider the class of symbols \( \mathcal{S}^p_q := \mathcal{S}(w_{p,q}, g) \). A symbol \( a \) is in \( \mathcal{S}^p_q \), if for any \((\alpha, \beta) \in \mathbb{Z}_+^2 \), the quantity

\[
n^{p,q}_{\alpha,\beta}(a) := \sup_{(x, \xi) \in \mathbb{R}^2} |(x)^p (x, \xi)^{q+\alpha} \partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \tag{4.12}
\]

is finite.
For \( a \in S^0_p \) we define the operator \( \text{Op}^W(a) \) using the Weyl quantization
\[
(\text{Op}^W(a)u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} a \left( \frac{x+y}{2}, \xi \right) e^{-i(x-y)\xi} u(y) \, dy \, d\xi,
\]
for \( u \) in the Schwartz space \( S(\mathbb{R}) \).

For general properties of pseudodifferential operators like: composition, selfadjointness, norm estimates, compact properties, etc. we refer to [13]. Here we will consider more specifically selfadjoint pseudodifferential operators of negative order (i.e. \( a \in S^0_p \) with \( q > 0 \) and \( p \geq 0 \)) which are known to be compact.

Set \( \tilde{N}(\lambda, a) \) as the volume function in \( \mathbb{R}^2 \)
\[
\tilde{N}(\lambda, a) := \frac{1}{2\pi} \text{vol}\{(x, \xi) \in \mathbb{R}^2; a(x, \xi) > \lambda\}. \tag{4.13}
\]

**Lemma 4.3.** Let \( a \in S^0_p \) be a real valued symbol, with \( q > 0 \) and \( p \geq 0 \). Then, the operator \( \text{Op}^W(a) \) is essentially self-adjoint, compact and its eigenvalue counting functions satisfy
\[
n_+(\lambda, \text{Op}^W(a)) + n_-(\lambda, \text{Op}^W(a)) \leq C \tilde{N}(\lambda, \langle x \rangle^{-p}(x, \xi)^{-q}),
\]
where \( C \) depends only on a finite number of semi-norms \( n_{\alpha, \beta}^{p,q}(a) \).

**Proof.** Applying [10] Lemma 4.7 we have
\[
n_+(\lambda, \text{Op}^W(a)) + n_-(\lambda, \text{Op}^W(a)) \leq C \tilde{N}(\lambda, \langle x \rangle^{-p}(x, \xi)^{-q}),
\]
where \( C \) depends only on a finite number of semi-norms \( n_{\alpha, \beta}^{p,q}(a) \). Using that \( \Omega_\mu^q(\lambda) := \{(x, \xi) \in \mathbb{R}^2; \langle x \rangle^{-p}(x, \xi)^{-q} \geq \lambda\} \) satisfies
\[
\Omega_\mu^q(\lambda) = \{(x, \xi) \in \mathbb{R}^2; \langle x \rangle^2 + \xi^2 \leq \lambda^{-\frac{1}{q}} \langle x \rangle^{\frac{p}{q}} \} \subset \{(x, \xi); \langle x \rangle^{\frac{1}{1+\frac{q}{p}}} \leq \lambda^{-\frac{1}{q}}; |\xi| \leq \lambda^{-\frac{1}{q}} \langle x \rangle^{-\frac{q}{p}}\},
\]
we obtain
\[
\tilde{N}(\lambda, \langle x \rangle^{-p}(x, \xi)^{-q}) \leq 2\lambda^{\frac{1}{q}} \int_{|x| \leq \lambda^{-1/(p+q)}} \langle x \rangle^{-\frac{q}{p}} \, dx,
\]
and the claimed estimates follow. \( \square \)

Let \( W \geq 0 \) be a function in \( L^1(\mathbb{R}^2) \). Set \( Q_j(W) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) as the integral operator with kernel
\[
(2\pi)^{-1/2} W(x, y)^{1/2} e^{iky} \psi_j, \infty(x; k), \tag{4.14}
\]
and define the operator \( W_j : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by
\[
W_j = Q_j^*(W) Q_j(W). \tag{4.15}
\]

**Proposition 4.4** ([35], Lemma 5.1). Suppose that \( W \geq 0 \) satisfies (2.14). Then, for any \( j \in \mathbb{N} \) we have
\[
W_j = \text{Op}^W(w_j),
\]
where the symbol \( w_j \) is in \( S^0_m \). Further
\[
w_j(x, \xi) = W(x, -\xi) + R_1(x, \xi), \tag{4.16}
\]
where \( R_1 \in S^0_{m+1} \).
Lemma 4.5. Let $W \geq 0$ be a function satisfying (2.5) and $\text{supp } W \subset [K_0, \infty) \times \mathbb{R}$, for some $K_0 \in \mathbb{R}$. Then

$$n_+(\lambda; 1_{(-\infty,K_0-1)}(\cdot)) W_j 1_{(-\infty,K_0-1)}(\cdot) = O(\lambda^{-\frac{N}{2}}), \quad \lambda \downarrow 0.$$  

Proof. Consider a smooth function $\chi$ such that $0 \leq \chi \leq 1$, $\chi(k) = 1$ for $k \in (-\infty, K_0 - 1]$ and $\chi(k) = 0$ for $k \in [K_0 - \frac{1}{2}, \infty)$. We clearly have

$$n_+(\lambda; 1_{(-\infty,K_0-1)}(\cdot)) W_j 1_{(-\infty,K_0-1)}(\cdot) \leq n_+(\lambda; \chi W_j \chi(\cdot)).$$  

(4.17)

It is sufficient to prove that $\chi W_j \chi$ is a pseudo-differential operator in $\text{Op}^W(S_n^m)$ for any $p \geq 0$ and then apply Lemma 4.4. Moreover, since any derivative of $\chi$ is still supported in $(-\infty, K_0)$, using composition theorems, we are reduced to prove that for any $p \geq 0$ and any $(\alpha, \beta) \in \mathbb{N}^2$ there exists a positive constant $C(m, p, \alpha, \beta)$ such that

$$\forall (x, \xi) \in (-\infty, K_0 - \frac{1}{2}) \times \mathbb{R}, \quad |\langle x \rangle^p (x, \xi)^m \partial^\beta_x \partial^\alpha_x W_j(x, \xi)| \leq C(m, p, \alpha, \beta),$$  

(4.18)

where $w_j(x, \xi)$ is the Weyl’s symbol of $W_j$ given in Proposition 4.3.

Suppose first that $W$ is in the Schwartz space $S(\mathbb{R}^2)$, then $w_j(x, \xi)$ is given by

$$w_j(x, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}^3} e^{-izy} W(x', z - \xi) \psi_{j,\infty}(x'; x - \frac{y'}{2}) \psi_{j,\infty}(x'; x + \frac{y'}{2}) \, dx' \, dy' \, dz,$$

and satisfies

$$|\langle x, \xi \rangle^{m+\alpha} \partial^\beta_x \partial^\alpha_x w_j(x, \xi)| \leq C(x, \xi)^{m+\alpha}$$

$$\times \left| \int_{\mathbb{R}^3} \partial^\beta_x W(x', z - \xi) |z|^{-2N} \partial^\alpha_x (D_{y'})^{2N} \left( \psi_{j,\infty}(x'; x - \frac{y'}{2}) \psi_{j,\infty}(x'; x + \frac{y'}{2}) \right) \, dx' \, dy' \, dz \right|$$

$$\leq C \left| \left( x - bx' \right)^{m+\alpha} \partial^\beta_x (D_{y'})^{2N} \left( \psi_{j,\infty}(x'; x - \frac{y'}{2}) \psi_{j,\infty}(x'; x + \frac{y'}{2}) \right) \right| \, dx' \, dy' \, dz,$$

where in the last inequality we have used that $\langle x, \xi \rangle \leq C |z| |x - bx'| (x', z - \xi)$, for some positive constant $C$. Choosing $N$ sufficiently large and using that $W$ is supported on $[K_0, +\infty) \times \mathbb{R}$, we deduce

$$|\langle x, \xi \rangle^{m+\alpha} \partial^\beta_x \partial^\alpha_x w_j(x, \xi)| \leq C n^0_{0,0}(W)$$

$$\times \left| \int_{[K_0, +\infty) \times \mathbb{R}} (x - bx')^{m+\alpha} \partial^\beta_x (D_{y'})^{2N} \left( \psi_{j,\infty}(x'; x - \frac{y'}{2}) \psi_{j,\infty}(x'; x + \frac{y'}{2}) \right) \, dx' \, dy' \right|.$$  

Furthermore, straightforward calculations show that

$$\partial^\beta_y \psi_{j,\infty}(x', x - \frac{y'}{2}) \psi_{j,\infty}(x', x + \frac{y'}{2}) = P(b^{1/2}x' - b^{-1/2}x, y') e^{-\langle b^{1/2}x' - b^{-1/2}x \rangle^2 - \frac{1}{4}y^2},$$

where $P$ is a polynomial. This finally gives that there exists $M > 0$ and $C_{m, p, \alpha, \beta}$ (depending on $n^0_{0,0}(W)$) such that

$$|\langle x, \xi \rangle^{m+\alpha} \partial^\beta_x \partial^\alpha_x w_j(x, \xi)| \leq C_{m, p, \alpha, \beta} \int_{[K_0, +\infty)} (x - bx')^M e^{-\langle bx' - x \rangle^2 b^{-1}} \, dx'.$$  

(4.19)

The r.h.s. of (4.19) is clearly exponentially decreasing with respect to $x < K_0 - 1$, implying (4.18) for $W$ in Schwartz space. Using a limiting argument we deduce (4.18) for $W \in S_0^m$, and finish the proof of the Lemma.

$\square$
Lemma 4.6. Let $W \geq 0$ be a function satisfying (2.5) and $\supp W \subset [K_0, \infty) \times \mathbb{R}$, for some $K_0 \in \mathbb{R}$. Besides, let $\epsilon_\lambda \in (\lambda^\theta, \delta \lambda)$, $\theta > 1$, $\delta \in (0, 1)$. Then, for any $A > 0$ and $\nu > 0$

$$n_+ \left( \epsilon_\lambda; 1_{(-\infty, \phi_j(A\lambda))}(\cdot) W_j 1_{(-\infty, \phi_j(A\lambda))}(\cdot) \right) = o(\epsilon_\lambda^{-\frac{\nu}{\theta}} \lambda^{-\nu}), \quad \lambda \downarrow 0. \quad (4.20)$$

Proof. Thanks to (4.15) and the Ky Fan inequality

$$n_+ \left( \epsilon_\lambda; 1_{(-\infty, \phi_j(A\lambda))}(\cdot) W_j 1_{(-\infty, \phi_j(A\lambda))}(\cdot) \right) \leq n_+ \left( \epsilon_\lambda; 1_{(-\infty, K_0-1)}(\cdot) W_j 1_{(-\infty, K_0-1)}(\cdot) \right) + n_+ \left( \epsilon_\lambda; 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) W_j 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) \right). \quad (4.21)$$

For the first term in the r.h.s. of (4.21) use Lemma 4.6 with $\epsilon_\lambda$ instead of $\lambda$. In order to estimate the second term, we will need the following inequality (in the sense of quadratic forms). When $\phi_j(I)$ is a bounded set, for any $p \geq 0$

$$Q_j(W) 1_{\phi_j(I)}(k) Q_j(W)^* \leq \left( \sup_{k \in \phi_j(I)} \langle k \rangle^p \right) Q_j(W) \langle \cdot \rangle^{-p} Q_j(W)^*.$$  \quad (4.22)

Then, using (4.14), that the postive eigenvalues of $TT^*$ coincide with those of $T^*T$ (for $T = Q_j(W) 1_{(K_0-1, \phi_j(A\lambda))}(\cdot)$ and for $T = Q_j(W) \langle \cdot \rangle^{-p/2}$), and the min-max principle, we have

$$n_+ \left( \epsilon_\lambda; 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) W_j 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) \right) = n_+ \left( \epsilon_\lambda; Q_j(W) 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) Q_j(W)^* \right) \leq n_+ \left( \epsilon_\lambda c_\lambda; Q_j(W) \langle \cdot \rangle^{-p} Q_j(W)^* \right) = n_+ \left( \epsilon_\lambda c_\lambda; \langle \cdot \rangle^{-p/2} W_j \langle \cdot \rangle^{-p/2} \right), \quad (4.23)$$

where $c_\lambda = 1/\sup\{\langle k \rangle^p; k \in (K_0 - 1, \phi_j(A\lambda))\}$. Since $\langle k \rangle^{-p} \in S^m_p$, Proposition 4.4 together with the composition formula give us that $\langle \cdot \rangle^{-p/2} W_j \langle \cdot \rangle^{-p/2} = O_p^W(a)$, where the symbol $a \in S^m_p$, for any $p \geq 0$. By using Lemma 4.3 with $p$ sufficiently large, we obtain that $n_+(\lambda, \langle \cdot \rangle^{-p/2} W_j \langle \cdot \rangle^{-p/2}) = O(\lambda^{-\frac{\nu}{\theta}})$, as $\lambda \downarrow 0$. Thus, from (4.23) we deduce

$$n_+ \left( \epsilon_\lambda; 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) W_j 1_{(K_0-1, \phi_j(A\lambda))}(\cdot) \right) = O((\epsilon_\lambda c_\lambda)^{-\frac{\nu}{\theta}}). \quad (4.24)$$

Now, regarding (3.15), $c_\lambda^{-\frac{\nu}{\theta}} = o(\lambda^{-\nu})$ for any $\nu > 0$. \hfill \Box

Proposition 4.7. Let $W \geq 0$ be a function satisfying (2.5) and $\supp W \subset [K_0, \infty) \times \mathbb{R}$ for some $K_0 \in \mathbb{R}$. Let $r > 0$ and $r_+ > 0$ such that $r_- < r < r_+$. Then for any $\nu > 0$

$$n_+(r_+ \lambda, W_j) = o(\lambda^{-\frac{\nu}{\theta}} \lambda^{-\nu}) \leq n_+ \left( r_+; Q_j(W) 1_{(\phi_j(\lambda-\epsilon_\lambda), \infty)}(\cdot) |E_j(\cdot) - E_j - \lambda|^{-1/2} \right) \quad (4.25)$$

$$\leq n_+(r_- \lambda, W_j) + o(\epsilon_\lambda^{-\frac{\nu}{\theta}} \lambda^{-\nu}), \quad \lambda \downarrow 0,$$

for $\epsilon_\lambda \in (\lambda^\theta, \delta \lambda)$, $\theta > 1$ and $\delta \in (0, 1)$. 

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Proof. For the upper bound, consider the inequalities \(|E_j(k) - \mathcal{E}_j - \lambda| \geq (1 - \delta)\lambda\), valid for \(k \in (\varrho_j(\delta\lambda), \infty)\), and \(|E_j(k) - \mathcal{E}_j - \lambda| \geq \epsilon_j\), valid for \(k \in (\varrho_j(\lambda - \epsilon\lambda), \varrho_j(\delta\lambda))\). Then, for \(r_1, r_2 > 0\) the Ky Fan inequality and the min-max principle imply

\[
| \Re T_j(\mathcal{E}_j + \lambda) | = \text{p.v.} \int_{\lambda - \epsilon\lambda}^{\lambda + \epsilon\lambda} \frac{G_j(\varrho_j(s)) G_j(\varrho_j(s))^*}{\lambda - s} \varrho_j(s) ds
\]

\[
+ S_j S_j^* [\lambda, (\lambda + \epsilon\lambda, \infty)] - S_j S_j^* [\lambda, (0, \lambda - \epsilon\lambda)].
\]

Then the Weyl inequalities (3.2), Lemma 4.1 and Lemma 4.2 imply that for any \(\rho \in (0, 1)\)

\[
n_+ (r(1 + \rho), S_j S_j^* [\lambda, (0, \lambda - \epsilon\lambda)]) + O \left( \frac{\epsilon_j}{\lambda^2 |\ln \lambda|^2} \right)
\]

\[
- n_+ (r\rho/2, S_j S_j^* [\lambda, (\lambda + \epsilon\lambda, \infty)])
\]

\[
\leq n_-(r, \Re T_j(\mathcal{E}_j + \lambda))
\]

\[
\leq n_+ (r(1 - \rho), S_j S_j^* [\lambda, (0, \lambda - \epsilon\lambda)]) + O \left( \frac{\epsilon_j}{\lambda^2 |\ln \lambda|^2} \right).
\]

4.3 Proof of Theorem 2.3

Using the above notations, for \(\lambda > 0, \epsilon\lambda \in (\lambda^\theta, \delta\lambda), \theta > 1\) and \(\delta \in (0, 1)\) we can write

\[
\Re T_j(\mathcal{E}_j + \lambda) = \text{p.v.} \int_{\lambda - \epsilon\lambda}^{\lambda + \epsilon\lambda} \frac{G_j(\varrho_j(s)) G_j(\varrho_j(s))^*}{\lambda - s} \varrho_j(s) ds
\]

\[
+ S_j S_j^* [\lambda, (\lambda + \epsilon\lambda, \infty)] - S_j S_j^* [\lambda, (0, \lambda - \epsilon\lambda)].
\]
Note that for $\lambda \notin I$, the operator $S_{j,\infty}[\lambda, I]$ is related to $Q_j(\tilde{V})$ by the expression

$$S_{j,\infty}[\lambda, I] = Q_j(\tilde{V}) \mathbf{1}_{E_j(\cdot)}(\cdot) |E_j(\cdot) - \lambda|^{-1/2}. \quad (4.32)$$

Then, exploiting that for $k \in \mathcal{G}_j(\lambda + \epsilon_\lambda, \infty)$, $|E_j(k) - \mathcal{E}_j - \lambda| \geq \epsilon_\lambda$, and following the proof of Lemma 4.4, we obtain

$$n_+(r, S_{j,\infty} S_{j,\infty}^* [\lambda, (\lambda + \epsilon_\lambda, \infty)]) \leq n_+ \left( \epsilon_{\lambda^\prime} r; \mathbf{1}_{(\epsilon_{\lambda^\prime})} - \mathbf{1}_{(\epsilon_{\lambda})} \right) \lambda \downarrow 0. \quad (4.33)$$

for any $\nu > 0$.

Now, consider two smooth functions $V^\pm$, defined on $\mathbb{R}^2$, such that they satisfy $2.5$, $0 \leq V^- \leq \tilde{V} \leq V^+$ and the differences $\tilde{V} - V^\pm$ are compactly supported in the $x$-direction. Under this conditions it is easy to prove that

$$\tilde{N}(\lambda, V^\pm) = \tilde{N}(\lambda, \tilde{V}) + O(\lambda^{-\frac{1}{3}}) = N(\lambda, V) + O(\lambda^{-\frac{1}{2}}). \quad (4.34)$$

In particular $\tilde{N}(\lambda, V^\pm)$ satisfy (2.7) and (2.8), and therefore for any $\rho \in (0,1)$

$$\lim_{\rho \downarrow 0} \frac{\tilde{N}(\lambda(1 \pm \rho), V^\pm)}{N(\lambda, V^\pm)} = 1. \quad (4.35)$$

Besides,

$$V^-_j \leq V_j \leq V^+_j, \quad (4.36)$$

where $V^\pm_j := Q_j^+(V^\pm) Q_j(V^\pm)$, $V_j := Q_j^+(\tilde{V}) Q_j(\tilde{V})$ are operators defined as in (4.13).

In consequence, taking into account Theorem 2.3, Propositions 4.7, 4.13, and choosing $\epsilon_\lambda = \lambda^0$, with $\theta = 2 - 2/m$, we obtain that for all $\rho \in (0,1)$,

$$n_+(\lambda(1 + \rho); V^-_j) + o(\lambda^{-2/m}) \leq \pm \xi(E_j \pm \lambda; H_+, H_0) \leq n_+(\lambda(1 - \rho); V^+_j) + o(\lambda^{-2/m}), \quad (4.37)$$

as $\lambda \downarrow 0$. To finish he proof of Theorem 2.3 we need the following result.

**Proposition 4.8.** ([10], Theorem 1.3) Let $a \in S^m_0$ be a real valued symbol, with $m > 0$. Assume that the volume functions $\tilde{N}(\lambda, \pm a)$ defined by (4.13) satisfies (2.7) and (2.8). Then, there exists $\nu > 0$ such that, as $\lambda \downarrow 0$, the counting functions satisfy

$$n_+(\lambda, Op^W(a)) = \tilde{N}(\lambda, \pm a)(1 + O(\lambda^{\nu})).$$

Hence, putting together (4.37), Propositions 4.4, 4.8, the Weyl inequalities, Lemma 4.3 (4.34) and (4.35) we obtain (2.4).

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