THE \( L_p \) MINKOWSKI PROBLEM FOR POLYTOPES FOR NEGATIVE \( p \)

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ABSTRACT. Existence of solutions to the \( L_p \) Minkowski problem is proved for all \( p < 0 \).

For the critical case of \( p = -n \), which is known as the centro-affine Minkowski problem,
this paper contains the main result in \[72\] as a special case.

1. Introduction

A convex body in \( n \)-dimensional Euclidean space, \( \mathbb{R}^n \), is a compact convex set that has non-empty interior. If \( p \in \mathbb{R} \) and \( K \) is a convex body in \( \mathbb{R}^n \) that contains the origin in its interior, then the \( L_p \) surface area measure, \( S_p(K, \cdot) \), of \( K \) is a Borel measure on the unit sphere, \( S^{n-1} \), defined for each Borel \( \omega \subset S^{n-1} \) by

\[
S_p(K, \omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),
\]

where \( \nu_K : \partial' K \to S^{n-1} \) is the Gauss map of \( K \), defined on \( \partial' K \), the set of boundary points of \( K \) that have a unique outer unit normal, and \( \mathcal{H}^{n-1} \) is \( (n-1) \)-dimensional Hausdorff measure.

The \( L_p \) surface area measure was introduced by Lutwak [41]. The \( L_p \) surface area measure contains three important measures as special cases: the \( L_1 \) surface area measure is the classic surface area measure; the \( L_0 \) surface area measure is the cone-volume measure; the \( L_{-n} \) surface area measure is the centro-affine surface area measure. Today, the \( L_p \) surface area measure is a central notation in convex geometry analysis, and appeared in, e.g., [3, 8, 21, 28, 37, 52, 54, 56, 60, 65, 67].

The following \( L_p \) Minkowski problem that posed by Lutwak [41] is considered as one of the most important problems in modern convex geometry analysis.

\( L_p \) Minkowski problem: Find necessary and sufficient conditions on a finite Borel measure \( \mu \) on \( S^{n-1} \) so that \( \mu \) is the \( L_p \) surface area measure of a convex body in \( \mathbb{R}^n \).

The associated partial differential equation for the \( L_p \) Minkowski problem is the following Mong-Ampère type equation: For a given positive function \( f \) on the unit sphere, solve

\[
(1.1) \quad h^{1-p} \det(h_{ij} + h\delta_{ij}) = f,
\]

where \( h_{ij} \) is the covariant derivative of \( h \) with respect to an orthonormal frame on \( S^{n-1} \) and \( \delta_{ij} \) is the Kronecker delta.

The solutions of the \( L_p \) Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [70], Lutwak, Yang and Zhang [46], Ciachi, Lutwak,

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The solutions to the $L_p$ Minkowski problem are also related with some important flows (see, e.g., [12], [25–27]). The solutions to the $L_p$ Minkowski problem are related with some important flows (see, e.g., [1, 2, 61, 62]).

When $p = 1$, the $L_p$ Minkowski problem is the classical Minkowski problem. The existence and uniqueness for the solution of this problem was solved by Minkowski, Aleksandrov, and Fenchel and Jessen (see Schneider [57] for references). Regularity of the Minkowski problem was studied by e.g., Caffarelli [7], Cheng and Yau [10], Nirenberg [53] and Pogorelov [55].

For $p \neq 1$, the $L_p$ Minkowski problem was studied by, e.g., Lutwak [41], Lutwak and Oliker [12], Lutwak, Yang and Zhang [47], Chou and Wang [11], Guan and Lin [19], Hug, Lutwak, Yang and Zhang [22], Böröczky, Hug, Zhang, and Zhu [4], Böröczky, Lutwak, Yang and Zhang [5, 6], Chen [9], Dou and Zhu [14], Haberl, Lutwak, Yang and Zhang [22], Huang, Liu and Xu [30], Jian, Lu and Wang [32], Jian and Wang [34], Jiang, Wang and Wei [35], Lu and Wang [36], Stancu [61, 62], Sun and Long [63] and Zhu [71–73]. Analogous of the Minkowski problems were studied in, e.g., [13, 15, 16, 18, 20, 29, 68].

The uniqueness of solutions to the $L_p$ Minkowski for $p > 1$ can be shown by applying the $L_p$ Minkowski inequality established by Lutwak [41]. However, little is known about the $L_p$ Minkowski inequality for the case where $p < 1$. This is one of the main reasons that most of the previous work on the $L_p$ Minkowski problem was limited to the case where $p > 1$.

The critical case where $p = -n$ of the $L_p$ Minkowski problem is called the centro-affine Minkowski problem, which describes the centro-affine surface area measure. This problem is especially important due to the affine invariant of the partial differential equation (1.1). It is known that the centro-affine Minkowski problem has connections with several important geometric problems (see, e.g., Jian and Wang [34] for reference). The centro-affine Minkowski problem was explicitly posed by Chow and Wang [11]. Recently, the centro-affine Minkowski problem was studied by Lu and Wang [36] for rotationally symmetric case and was studied by Zhu [72] for discrete measures.

When $p < -n$, very few results are known for the $L_p$ Minkowski problem. So far as the author knows, in $\mathbb{R}^2$, the $L_p$ Minkowski problem for all $p < 0$ was studied by Dou and Zhu [14], Sun and Long [63]. It is the aim of this paper to study the $L_p$ Minkowski problem for all $p < 0$ and $n \geq 2$.

It is known that the Minkowski problem and the $L_p$ Minkowski problem (for $p > 1$) for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [31] or [57] pp. 392-393). This is one of the reasons why the Minkowski problem and the $L_p$ Minkowski problem for polytopes are of great importance.

A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive $n$-dimensional volume. The convex hull of a subset of these points is called a facet of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. Let $P$ be a polytope which contains the origin in its interior with $N$ facets whose outer unit normals are $u_1, ..., u_N$, and such that the facet with outer unit normal $u_k$ has area $a_k$ and distance $h_k$ from the origin for all $k \in \{1, ..., N\}$. Then,

$$S_p(P, \cdot) = \sum_{k=1}^{N} h_k^{1-p} a_k \delta_{u_k}(\cdot).$$

where $\delta_{u_k}$ denotes the delta measure that is concentrated at the point $u_k$. 
A finite subset $U$ of $S^{n-1}$ is said to be in general position if any $k$ elements of $U$, $1 \leq k \leq n$, are linearly independent.

In [72], the author solved the centro-affine Minkowski problem for polytopes whose outer unit normals are in general position:

**Theorem A.** Let $\mu$ be a discrete measure on the unit sphere $S^{n-1}$. Then $\mu$ is the centro-affine surface area measure of a polytope whose outer unit normals are in general position if and only if the support of $\mu$ is in general position and not concentrated on a closed hemisphere.

A linear subspace $X$ $(0 < \dim X < n)$ of $\mathbb{R}^n$ is said to be essential with respect to a Borel measure $\mu$ on $S^{n-1}$ if $X \cap \text{supp}(\mu)$ is not concentrated on any closed hemisphere of $X \cap S^{n-1}$.

Obviously, if the support of a discrete measure $\mu$ is in general position, then the set of essential subspaces of $\mu$ is empty. On the other hand, in $\mathbb{R}^n$ $(n \geq 3)$, one can easily construct a discrete measure $\mu$ such that $\mu$ does not have essential subspace but the support of $\mu$ is not in general position. Therefore, the set of discrete measures whose supports are in general position is a subset of the set of discrete measures that do not have essential subspaces.

It is the aim of this paper to solve the $L_p$ Minkowski problem for discrete measures that do not have essential subspaces. Obviously, the following main theorem of this paper contains Theorem A as a special case.

**Theorem 1.1.** Let $p < 0$ and $\mu$ be a discrete measure on the unit sphere $S^{n-1}$. Then $\mu$ is the $L_p$ surface area measure of a polytope whose $L_p$ surface area measure does not have essential subspace if and only if $\mu$ does not have essential subspace and not concentrated on a closed hemisphere.

## 2. Preliminaries

In this section, we standardize some notations and list some basic facts about convex bodies. For general references regarding convex bodies, see, e.g., [17, 57, 64].

The sets in this paper are subsets of the $n$-dimensional Euclidean space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the standard inner product of $x$ and $y$, $|x|$ for the Euclidean norm of $x$, and $S^{n-1}$ for the unit sphere of $\mathbb{R}^n$.

Suppose $S$ is a subset of $\mathbb{R}^n$, then the positive hull, $\text{pos}(S)$, of $S$ is the set of all positive combinations of any finitely many elements of $S$. Let $\text{lin}(S)$ be the smallest linear subspace of $\mathbb{R}^n$ containing $S$. The diameter of a subset, $S$, of $\mathbb{R}^n$ is defined by

$$d(S) = \max\{|x - y| : x, y \in S\}.$$ 

The convex hull of a subset, $S$, of $\mathbb{R}^n$ is defined by

$$\text{Conv} (S) = \{\lambda x + (1 - \lambda) y : 0 \leq \lambda \leq 1 \text{ and } x, y \in S\}.$$ 

For convex bodies $K_1, K_2$ in $\mathbb{R}^n$ and $s_1, s_2 \geq 0$, the Minkowski combination is defined by

$$s_1K_1 + s_2K_2 = \{s_1x_1 + s_2x_2 : x_1 \in K_1, x_2 \in K_2\}.$$ 

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body $K$ is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$
Obviously, for \( s \geq 0 \) and \( x \in \mathbb{R}^n \),
\[
h(sK, x) = h(K, sx) = sh(K, x).
\]

If \( K \) is a convex body in \( \mathbb{R}^n \) and \( u \in S^{n-1} \), then the support set \( F(K, u) \) of \( K \) in direction \( u \) is defined by
\[
F(K, u) = K \cap \{ x \in \mathbb{R}^n : x \cdot u = h(K, u) \}.
\]

The Hausdorff distance of two convex bodies \( K_1, K_2 \) in \( \mathbb{R}^n \) is defined by
\[
\delta(K_1, K_2) = \inf \{ t \geq 0 : K_1 \subset K_2 + tB^n, K_2 \subset K_1 + tB^n \},
\]

where \( B^n \) is the unit ball.

Let \( \mathcal{P} \) be the set of polytopes in \( \mathbb{R}^n \). If the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere, let \( \mathcal{P}(u_1, \ldots, u_N) \) be the subset of \( \mathcal{P} \) such that a polytope \( P \in \mathcal{P}(u_1, \ldots, u_N) \) if the the set of the outer unit normals of \( P \) is a subset of \( \{ u_1, \ldots, u_N \} \). Let \( \mathcal{P}_N(u_1, \ldots, u_N) \) be the subset of \( \mathcal{P}(u_1, \ldots, u_N) \) such that a polytope \( P \in \mathcal{P}_N(u_1, \ldots, u_N) \) if, \( P \in \mathcal{P}(u_1, \ldots, u_N) \), and \( P \) has exactly \( N \) facets.

3. An extremal problem related to the \( L_p \) Minkowski problem

Suppose \( p < 0 \), \( \alpha_1, \ldots, \alpha_N > 0 \), the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere, and \( P \in \mathcal{P}(u_1, \ldots, u_N) \). Define the function, \( \Phi_P : \text{Int} \,(P) \to \mathbb{R} \), by
\[
\Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi \cdot u_k)^p.
\]

In this section, we study the extremal problem
\[
(3.1) \quad \sup \{ \inf_{\xi \in \text{Int} \,(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \}.
\]

The main purpose of this section is to prove that a dilation of the solution to problem (3.1) solves the corresponding \( L_p \) Minkowski problem.

**Lemma 3.1.** If \( p < 0 \), \( \alpha_1, \ldots, \alpha_N > 0 \), the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere and \( P \in \mathcal{P}(u_1, \ldots, u_N) \), then there exists a unique \( \xi(P) \in \text{Int} \,(P) \) such that
\[
\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int} \,(P)} \Phi_P(\xi).
\]

**Proof.** Since \( p < 0 \), the function \( f(t) = t^p \) is strictly convex on \((0, +\infty)\). Hence, for \( 0 < \lambda < 1 \) and \( \xi_1, \xi_2 \in \text{Int} \,(P) \),
\[
\lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) = \lambda \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda) \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_2 \cdot u_k)^p
\]
\[
= \sum_{k=1}^{N} \alpha_k [\lambda (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda)(h(P, u_k) - \xi_2 \cdot u_k)^p]
\]
\[
\geq \sum_{k=1}^{N} \alpha_k [h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k]^p
\]
\[
= \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2).
\]
Equality hold if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \ldots, N$. Since $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, \ldots, u_N\}$. Thus, $\xi_1 = \xi_2$. Hence, $\Phi_P$ is strictly convex on $\text{Int}(P)$.

From the fact that $P \in \mathcal{P}(u_1, \ldots, u_N)$, we have, for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$  

Thus, $\Phi_P(\xi) \to \infty$ whenever $\xi \in \text{Int}(P)$ and $\xi \to x$. Therefore, there exists a unique interior point $\xi(P)$ of $P$ such that

$$\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

\[\square\]

Obviously, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, \ldots, u_N)$,

$$\xi(\lambda P) = \lambda \xi(P),$$

and if $P_i \in \mathcal{P}(u_1, \ldots, u_N)$ and $P_i$ converges to a polytope $P$, then $P \in \mathcal{P}(u_1, \ldots, u_N)$.

**Lemma 3.2.** If $p < 0$, $\alpha_1, \ldots, \alpha_N > 0$, the unit vectors $u_1, \ldots, u_N$ are not contained in a closed hemisphere, $P_i \in \mathcal{P}(u_1, \ldots, u_N)$, and $P_i$ converges to a polytope $P$, then $\lim_{i \to \infty} \xi(P_i) = \xi(P)$ and

$$\lim_{i \to \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

**Proof.** Since $P_i$ converges to $P$ and $\xi(P_i) \in \text{Int}(P_i), \xi(P_i)$ is bounded. Let $\xi_0$ be the limit point of a subsequence, $\xi(P_{i_j})$, of $\xi(P_i)$. We claim that $\xi_0 \in \text{Int}(P)$. Otherwise, $\xi_0$ is a boundary point of $P$ with $\lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = \infty$, which contradicts the fact that

$$\lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P)) = \Phi(\xi(P)) < \infty.$$  

We claim that $\xi_0 = \xi(P)$. Otherwise,

$$\lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = \Phi_P(\xi_0)$$

$$> \Phi_P(\xi(P))$$

$$= \lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P)).$$

This contradicts the fact that

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi(P)).$$

Hence, $\lim_{i \to \infty} \xi(P_i) = \xi(P)$ and

$$\lim_{i \to \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

\[\square\]

**Lemma 3.3.** If $p < 0$, $\alpha_1, \ldots, \alpha_N > 0$, the unit vectors $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then

$$\sum_{k=1}^{N} \alpha_k \left[\frac{u_k}{h(P, u_k) - \xi(P) \cdot u_k}\right]^{1-p} = 0.$$
Proof. Define \( f : \text{Int}(P) \to \mathbb{R}^n \) by
\[
f(x) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - x \cdot u_k)^p.
\]

By conditions,
\[
f(\xi(P)) = \inf_{x \in \text{Int}(P)} f(x).
\]
Thus,
\[
\sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0,
\]
for all \( i = 1, \ldots, n \), where \( u_k = (u_{k,1}, \ldots, u_{k,n})^T \). Therefore,
\[
\sum_{k=1}^{N} \alpha_k \frac{u_k}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0.
\]
\[\square\]

Lemma 3.4. Suppose \( p < 0 \), \( \alpha_1, \ldots, \alpha_N > 0 \), the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere, and there exists a \( P \in \mathcal{P}_N(u_1, \ldots, u_N) \) with \( \xi(P) = o \), \( V(P) = 1 \) such that
\[
\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \right\}.
\]
Then,
\[
S_p(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot),
\]
where \( P_0 = \left( \sum_{j=1}^{N} \alpha_j h(P, u_j)^p/n \right)^{\frac{1}{p}} P \).

Proof. By conditions, there exists a polytope \( P \in \mathcal{P}_N(u_1, \ldots, u_N) \) with \( \xi(P) = o \) and \( V(P) = 1 \) such that
\[
\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \right\},
\]
where \( \Phi_Q(\xi) = \sum_{k=1}^{N} \alpha_k (h(Q, u_k) - \xi \cdot u_k)^p \).

For \( \tau_1, \ldots, \tau_N \in \mathbb{R} \), choose \(|t|\) small enough so that the polytope \( P_t \) defined by
\[
P_t = \bigcap_{i=1}^{N} \{ x : x \cdot u_i \leq h(P, u_i) + t\tau_i \}
\]
has exactly \( N \) facets. By \( \text{[57]} \) (Lemma 7.5.3),
\[
\frac{\partial V(P_t)}{\partial t} = \sum_{i=1}^{N} \tau_i a_i,
\]
where \( a_i \) is the area of \( F(P, u_i) \). Let \( \lambda(t) = V(P_t)^{-\frac{1}{n}} \), then \( \lambda(t)P_t \in \mathcal{P}_n^n(u_1, ..., u_N) \), \( V(\lambda(t)P_t) = 1 \) and

\[
\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \tau_i S_i.
\]

(3.4)

Define \( \xi(t) := \xi(\lambda(t)P_t) \), and

\[
\Phi(t) := \min_{\xi \in \lambda(t)P_t} \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi \cdot u_k)^p
\]

(3.5)

\[
= \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k)^p.
\]

It follows from Lemma 3.3 that

\[
\sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k]^{1-p}} = 0,
\]

for \( i = 1, ..., n \), where \( u_k = (u_{k,1}, ..., u_{k,n})^T \). In addition, since \( \xi(P) \) is the origin,

\[
\sum_{k=1}^{N} \alpha_k \frac{u_k}{h(P, u_k)^{1-p}} = 0.
\]

(3.6)

Let \( F = (F_1, ..., F_n) \) be a function from an open neighbourhood of the origin in \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^n \) such that

\[
F_i(t, \xi_1, ..., \xi_n) = \sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})]^{1-p}}
\]

for \( i = 1, ..., n \). Then,

\[
\left. \frac{\partial F_i}{\partial t} \right|_{(t, \xi_1, ..., \xi_n)} = \sum_{k=1}^{N} \frac{(p-1)\alpha_k u_{k,i} [\lambda(t)h(P_t, u_k) + \lambda(t)\tau_k]}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})]^{2-p}},
\]

\[
\left. \frac{\partial F_i}{\partial \xi_j} \right|_{(t, \xi_1, ..., \xi_n)} = \sum_{k=1}^{N} \frac{(1-p)\alpha_k u_{k,i} u_{k,j}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})]^{2-p}},
\]

are continuous on a small neighbourhood of \((0, 0, ..., 0)\) with

\[
\left( \frac{\partial F}{\partial \xi} \right)_{(0, ..., 0)} \bigg|_{0 \times n} = \sum_{k=1}^{N} \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} u_k^T u_k^T,
\]

where \( u_k u_k^T \) is an \( n \times n \) matrix.
Since \( u_1, \ldots, u_N \) are not contained in a closed hemisphere, \( \mathbb{R}^n = \text{lin}\{u_1, \ldots, u_N\} \). Thus, for any \( x \in \mathbb{R}^n \) with \( x \neq 0 \), there exists a \( u_{i_0} \in \{u_1, \ldots, u_N\} \) such that \( u_{i_0} \cdot x \neq 0 \). Then,

\[
x^T \left( \sum_{k=1}^{N} \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} u_k u_k^T \right) x = \sum_{k=1}^{N} \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} (x \cdot u_k)^2 \\
\geq \frac{(1-p)\alpha_{i_0}}{h(P, u_{i_0})^{2-p}} (x \cdot u_{i_0})^2 > 0.
\]

Therefore, \( \left. \frac{\partial F}{\partial \xi} \right|_{(0, \ldots, 0)} \) is positive defined. By this, the fact that \( F_i(0, \ldots, 0) = 0 \) for all \( i = 1, \ldots, n \), the fact that \( \frac{\partial F}{\partial \xi} \) is continuous on a neighbourhood of \( (0, 0, \ldots, 0) \) for all \( 0 \leq i, j \leq n \) and the implicit function theorem, we have

\[
\xi'(0) = (\xi'_1(0), \ldots, \xi'_n(0))
\]

exists.

From the fact that \( \Phi(0) \) is an extreme value of \( \Phi(t) \) (in Equation (3.5)), Equation (3.4) and Equation (3.6), we have

\[
0 = \Phi'(0)/p \\
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} (\lambda'(0) h(P, u_k) + \tau_k - \xi'(0) \cdot u_k)
\]

\[
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} \left[ -\frac{1}{n} \left( \sum_{i=1}^{N} a_i \tau_i \right) h(P, u_k) + \tau_k \right] - \xi'(0) \cdot \left[ \sum_{k=1}^{N} \frac{\alpha_k}{h(P, u_k)^{1-p}} \frac{u_k}{h(P, u_k)^{1-p}} \right]
\]

\[
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} \tau_k - \left( \sum_{i=1}^{N} a_i \tau_i \right) \frac{\sum_{k=1}^{N} \alpha_k h(P, u_k)^p}{n}
\]

\[
= \sum_{k=1}^{N} \left( \alpha_k h(P, u_k)^{p-1} - \frac{\sum_{j=1}^{N} \alpha_j h(P, u_j)^p}{n} a_k \right) \tau_k.
\]

Since \( \tau_1, \ldots, \tau_N \) are arbitrary,

\[
\sum_{j=1}^{N} \alpha_j h(P, u_j)^p \frac{n}{h(P, u_k)^{1-p}} a_k = \alpha_k,
\]

for all \( k = 1, \ldots, N \). By letting

\[
P_0 = \left( \frac{\sum_{j=1}^{N} \alpha_j h(P, u_j)^p}{n} \right)^{\frac{1}{p}} P,
\]

we have

\[
S_p(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot).
\]

□
4. The proof of the main theorem

In this section, we prove the main theorem of this paper.

The following lemmas will be needed.

**Lemma 4.1.** Let \( \{h_{1j}\}_{j=1}^{\infty}, \ldots, \{h_{Nj}\}_{j=1}^{\infty} \) be \( N \) (\( N \geq 2 \)) sequences of real numbers. Then, there exists a subsequence, \( \{j_n\}_{n=1}^{\infty} \), of \( \mathbb{N} \) and a rearrangement, \( i_1, \ldots, i_N \), of \( 1, \ldots, N \) such that

\[
h_{i_1j_n} \leq h_{i_2j_n} \leq \ldots \leq h_{i_Nj_n},
\]
for all \( n \in \mathbb{N} \).

**Proof.** For each fixed \( j \), the number of the possible order (from small to big) of \( h_{1j}, \ldots, h_{Nj} \) is \( N! \). Therefore, there exists a subsequence, \( \{j_n\}_{n=1}^{\infty} \), and a rearrangement, \( i_1, \ldots, i_N \), of \( 1, \ldots, N \) such that

\[
h_{i_1j_n} \leq h_{i_2j_n} \leq \ldots \leq h_{i_Nj_n},
\]
for all \( n \in \mathbb{N} \). \( \square \)

**Lemma 4.2.** Suppose the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere, and for any subspace, \( X \), of \( \mathbb{R}^n \) with \( 1 \leq \dim X \leq n - 1 \), \( \{u_1, \ldots, u_N\} \cap X \) is concentrated on a closed hemisphere of \( S^{n-1} \cap X \). If \( P_m \) is a sequence of polytopes with \( V(P_m) = 1 \), \( o \in \text{Int}(P_m) \) and \( P_m \in \mathcal{P}(u_1, \ldots, u_N) \), then \( P_m \) is bounded.

**Proof.** We only need to prove that if the diameter, \( d(P) \), of \( P \) is not bounded, then there exists a subspace, \( X \), of \( \mathbb{R}^n \) with \( 1 \leq \dim(X) \leq n - 1 \) and \( \{u_1, \ldots, u_N\} \cap X \) is not concentrated on a closed hemisphere of \( S^{n-1} \cap X \).

Let \( \mu \) be a discrete measure on the unit sphere such that \( \text{supp}(\mu) = \{u_1, \ldots, u_N\} \), \( \mu(u_i) = \alpha_i > 0 \) for \( 1 \leq i \leq N \). Obviously, we only need to prove the lemma under the condition that \( \xi(P_m) = o \) for all \( m \in \mathbb{N} \).

By Lemma 4.1, we may assume that

\[
h(P_m, u_1) \leq \ldots \leq h(P_m, u_N).
\]

By this and the condition that \( V(P_m) = 1 \) and \( \lim_{m \to \infty} d(P_m) = \infty \),

\[
\lim_{m \to \infty} h(P_m, u_1) = 0 \quad \text{and} \quad \lim_{m \to \infty} h(P_m, u_N) = \infty.
\]

By this and (4.0), there exists an \( i_0 \) (\( 1 \leq i_0 \leq N \)) such that

\[
\lim_{m \to \infty} \frac{h(P_m, u_{i_0})}{h(P_m, u_1)} = \infty,
\]
and for \( 1 \leq i \leq i_0 - 1 \)

\[
\lim_{m \to \infty} \frac{h(P_m, u_i)}{h(P_m, u_1)}
\]
exists and equals to a positive number.

Let

\[
\Sigma = \text{pos}\{u_1, \ldots, u_{i_0-1}\}
\]
and

\[
\Sigma^* = \{x \in \mathbb{R}^n : x \cdot u_i \leq 0 \text{ for all } 1 \leq i \leq i_0 - 1\}.
\]
Let \( 1 \leq j \leq i_0 - 1 \) and \( x \in \Sigma^* \cap S^{n-1} \). From the condition that \( \xi(P_m) \) is the origin and Lemma 3.3, we have
\[
\sum_{i=0}^{N} \frac{\alpha_i (x \cdot u_i)}{[h(P_m, u_i)]^{1-p}} = 0.
\]
By this and the fact that \( x \in \Sigma^* \cap S^{n-1} \),
\[
0 \geq \alpha_j (x \cdot u_j)
\]
\[
= - \sum_{i \neq j} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i (x \cdot u_i)
\]
\[
\geq - \sum_{i \geq i_0} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i (x \cdot u_i)
\]
By this, (4.0), (4.1) and (4.2), \( \alpha_j (x \cdot u_j) \) is no bigger than 0 and no less than any negative number. Hence,
\[
x \cdot u_j = 0
\]
for all \( j = 1, \ldots, i_0 - 1 \) and \( x \in \Sigma^* \cap S^{n-1} \). Thus,
\[
\Sigma^* \cap \text{lin}\{u_1, \ldots, u_{i_0-1}\} = \{0\}.
\]
Obviously, \( \{u_1, \ldots, u_{i_0-1}\} \) is not concentrated on a closed hemisphere of \( S^{n-1} \cap \text{lin}\{u_1, \ldots, u_{i_0-1}\} \). Otherwise, there exists an \( x_0 \in \text{lin}\{u_1, \ldots, u_{i_0-1}\} \) with \( x_0 \neq 0 \) such that \( x_0 \cdot u_i \leq 0 \) for all \( 1 \leq i \leq i_0 - 1 \). This contradicts with (4.3).

We next prove that
\[
\text{lin}\{u_1, \ldots, u_{i_0-1}\} \neq \mathbb{R}^n.
\]
Otherwise, from the fact that \( u_1, \ldots, u_{i_0-1} \) are not concentrated on a closed hemisphere of
\[
\text{lin}\{u_1, \ldots, u_{i_0-1}\} \cap S^{n-1},
\]
we have, the convex hull of \( \{u_1, \ldots, u_{i_0-1}\} \) (denoted by \( Q \)) is a polytope in \( \mathbb{R}^n \) and contains the origin as an interior. Let \( F \) be a facet of \( Q \) such that \( \{su_{i_0} : s > 0\} \cap F \neq \emptyset \). Since \( F \) is the union of finite \((n-1)\)-dimensional simplexes and the vertexes of these simplexes are subsets of \( \{u_1, \ldots, u_{i_0-1}\} \), there exists a subset, \( \{u_{i_1}, \ldots, u_{i_n}\} \), of \( \{u_1, \ldots, u_{i_0-1}\} \) such that
\[
u_{i_0} \in \text{pos}\{u_{i_1}, \ldots, u_{i_n}\}.
\]
Since \( o \in \text{Int} (Q) \), there exists \( r > 0 \) such that \( rB^n \subset Q \). Choose \( t > 0 \) such that \( tu \in F \cap \text{pos}\{u_{i_1}, \ldots, u_{i_n}\} \). Then,
\[
tu = \beta_{i_1} u_{i_1} + \ldots + \beta_{i_n} u_{i_n},
\]
where \( \beta_{i_1}, \ldots, \beta_{i_n} \geq 0 \) with \( \beta_{i_1} + \ldots + \beta_{i_n} = 1 \). If we let \( a_{ij} = \beta_{ij} / t \) for \( j = 1, \ldots, n \), we have
\[
u = a_{i_1} u_{i_1} + \ldots + a_{i_n} u_{i_n}.
\]
Obviously, \( a_{ij} \geq 0 \) with
\[
a_{ij} = \beta_{ij} / t \leq 1 / r
\]
for all $j = 1, ..., n$. Hence,
\[
  h(P_m, u_i) = h(P_m, a_i u_i + ... + a_n u_n) \\
  \leq a_i h(P_m, u_i) + ... + a_n h(P_m, u_n) \\
  \leq \frac{1}{r} [h(P_m, u_i) + ... + h(P_m, u_n)],
\]
for all $m \in \mathbb{N}$. This contradicts (4.1) and (4.2). Therefore,
\[
  \text{lin}\{u_1, ..., u_{i_0-1}\} \neq \mathbb{R}^n.
\]

Let $X = \text{lin}\{u_1, ..., u_{i_0-1}\}$. Then, $1 \leq \dim X \leq n-1$ but $\{u_1, ..., u_N\} \cap X = \{u_1, ..., u_{i_0-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap X$, which contradicts the conditions of this lemma. Therefore, $d(P_m)$ is bounded. \hfill \Box

The following lemmas will be needed (see, e.g., [73]).

**Lemma 4.3.** If $P$ is a polytope in $\mathbb{R}^n$ and $v_0 \in S^{n-1}$ with $V_{n-1}(F(P, v_0)) = 0$, then there exists a $\delta_0 > 0$ such that for $0 \leq \delta < \delta_0$
\[
  V(P \cap \{x : x \cdot v_0 \geq h(P, v_0) - \delta\}) = c_n \delta^n + ... + c_2 \delta^2,
\]
where $c_n, ..., c_2$ are constants that depend on $P$ and $v_0$.

**Lemma 4.4.** Suppose $p < 0$, $\alpha_1, ..., \alpha_N > 0$, and the unit vectors $u_1, ..., u_N$ are not concentrated on a hemisphere. If for any subspace $X$ with $1 \leq \dim X \leq n-1$, $\{u_1, ..., u_N\} \cap X$ is always concentrated on a closed hemisphere of $S^{n-1} \cap X$, then there exists a $P \in \mathcal{P}(u_1, ..., u_N)$ such that $\xi(P) = o$, $V(P) = 1$, and
\[
  \Phi_P(o) = \sup \{ \inf_{Q \in \mathcal{P}(u_1, ..., u_N)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \},
\]
where $\Phi_Q(\xi) = \sum_{k=1}^N \alpha_k (h(Q, u_k) - \xi \cdot u_k)^p$.

**Proof.** Obviously, for $P, Q \in \mathcal{P}(u_1, ..., u_N)$, if there exists a $x \in \mathbb{R}^n$ such that $P = Q + x$, then
\[
  \Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).
\]
Thus, we can choose a sequence of polytopes $P_i \in \mathcal{P}(u_1, ..., u_N)$ with $\xi(P_i) = o$ and $V(P_i) = 1$ such that $\Phi_{P_i}(o)$ converges to
\[
  \sup \{ \inf_{Q \in \mathcal{P}(u_1, ..., u_N)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \}.
\]

By the conditions of this lemma and Lemma 4.2, $P_i$ is bounded. From the Blaschke selection theorem, there exists a subsequence of $P_i$ that converges to a polytope $P$ such that $P \in \mathcal{P}(u_1, ..., u_N)$, $V(P) = 1$, $\xi(P) = o$ and
\[
  \Phi_P(o) = \sup \{ \inf_{Q \in \mathcal{P}(u_1, ..., u_N)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \}.
\]
We claim that $F(P, u_i)$ are facets for all $i = 1, ..., N$. Otherwise, there exists an $i_0 \in \{1, ..., N\}$ such that
\[
  F(P, u_{i_0})
\]
is not a facet of $P$.

Choose $\delta > 0$ small enough so that the polytope
\[
  P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, ..., u_N),
\]
and (by Lemma 4.3)

\[ V(P_\delta) = 1 - (c_n \delta^n + \ldots + c_2 \delta^2), \]

where \(c_n, \ldots, c_2\) are constants that depend on \(P\) and direction \(u_{i_0}\).

From Lemma 3.2, for any \(\delta_i \to 0\) it always true that \(\xi(P_\delta) \to o\). We have,

\[ \lim_{\delta \to 0} \xi(P_\delta) = o. \]

Let \(\delta\) be small enough so that \(h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta\) for all \(k \in \{1, \ldots, N\},\) and let

\[ \lambda = V(P_\delta)^{-\frac{1}{\pi}} = (1 - (c_n \delta^n + \ldots + c_2 \delta^2))^{-\frac{1}{\pi}}. \]

From this and Equation (3.2), we have

\[ (4.5) \]

\[ \Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) = \sum_{k=1}^{N} \alpha_k \left( h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta) \cdot u_k \right)^p \]

\[ = \lambda^p \sum_{k=1}^{N} \alpha_k \left( h(P_\delta, u_k) - \xi(P_\delta) \cdot u_k \right)^p \]

\[ = \lambda^p \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p - \alpha_{i_0} \lambda^p \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \]

\[ + \alpha_{i_0} \lambda^p \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p \]

\[ = \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p + (\lambda^p - 1) \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \]

\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right] \]

\[ = \Phi_P(\xi(P_\delta)) + B(\delta), \]

where

\[ B(\delta) = (\lambda^p - 1) \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \right) \]

\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right] \]

\[ = \left[ 1 - (c_n \delta^n + \ldots + c_2 \delta^2) \right]^{-\frac{p}{\pi}} - 1 \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \right) \]

\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right]. \]

From the facts that \(d_0 = d(P) > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta > 0,\)

\(p < 0\) and the fact that \(f(t) = t^p\) is convex on \((0, \infty)\), we have

\[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p > (d_0 - \delta)^p - d_0^p > 0. \]
Theorem 4.5 can be directly got by Lemma 3.4 and Lemma 4.4.

Proof. Theorem 4.5 can be directly got by Lemma 3.4 and Lemma 4.4. \qed
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