Abstract. We define the polar curves and the polar family associated to a projective web and obtain some results about the geometry of the generic element of this family. We also deal with the particular case of foliations and prove the constancy of the topological embedded type of the generic polar.

1. Introduction

Polar subvarieties of foliations have been extensively studied by many authors, cf. [5, 6, 13]. For example, in [6, 13] the authors bound the degree of an algebraic subvariety which is invariant by a foliation by using the polar classes of the foliation. Most recently, in [7] the authors obtained some more general results for projective webs of any dimension, again by using polar classes.

In [5] R. Mol made an study of the properties of the generic polar curve of a foliation on \( \mathbb{P}^2 \). He proves for instance that the generic polar curve of a holomorphic foliation is irreducible and computes its genus.

The aim of this work is to study polar curves of projective webs. Loosely speaking, a \( k \)-web is locally given by \( k \) foliations on the complement of a Zariski closed set. We obtain three main results, the first one is a characterization of the webs having a reducible generic polar, we prove that this is the case only when the web is decomposable or its degree is zero, see Theorem 3.1. Our second result says that the polar family characterizes the web, in other words, two webs having the same polar family are equal, see Theorem 3.2.

Finally, in the last section, we return to study polar curves of foliations, and prove that the topological embedded type of the generic polar is constant, see Theorem 4.1. This theorem says that the family of generic polars is a kind of globally equisingular family. This fact is well known in the local case (see [8]) and has been used in the study of curves and foliation singularities (see for example [9, 12]).

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2. Definitions and some properties

Roughly speaking, web geometry is the study of invariants for finite families of foliations. A $k$-web of codimension one on $\mathbb{P}^n$ is given by an open covering $\mathcal{U} = \{U_i\}$ of $\mathbb{P}^n$ and $k$-symmetric 1-forms $\omega_i \in \text{Sym}^k \Omega^1_{\mathbb{P}^n}(U_i)$ subject to the conditions:

(1) For each non-empty intersection $U_i \cap U_j$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}_{U_i \cap U_j}$ such that $\omega_i = g_{ij}\omega_j$.

(2) For every $i$ the zero set of $\omega_i$ has codimension at least two.

(3) For every $i$ and a generic $x \in U_i$, the germ of $\omega_i$ at $x$ is homogeneous polynomial of degree $k$ in the ring $\mathcal{O}_x[dx_1, ..., dx_n]$ is square-free.

(4) For every $i$ and a generic $x \in U_i$, the germ of $\omega_i$ at $x$ is a product of $k$ various 1-forms $\beta_1, \ldots, \beta_k$, where each $\beta_i$ is integrable.

The $k$-symmetric 1-forms $\{\omega_i\}$ patch together to form a global section $\omega = \{\omega_i\} \in H^0(\mathbb{P}^2, \text{Sym}^k \Omega^1_{\mathbb{P}^2} \otimes \mathcal{L})$ where $\mathcal{L}$ is the line bundle over $\mathbb{P}^n$ determined by the cocycle $\{g_{ij}\}$. The singular set of $\mathcal{W}$, denoted by $\text{Sing}(\mathcal{W})$, is the zero set of the twisted $k$-symmetric 1-form $\omega$. The degree of $\mathcal{W}$, denoted by $\deg(\mathcal{W})$, is geometrically defined as the degree of the tangency locus between $\mathcal{W}$ and a generic $\mathbb{P}^1$ linearly embedded in $\mathbb{P}^n$. If $i : \mathbb{P}^1 \to \mathbb{P}^n$ is the inclusion then the degree of $\mathcal{W}$ is the degree of the zero divisor of the twisted $k$-symmetric 1-form $i^*\omega \in H^0(\mathbb{P}^3, \text{Sym}^k \Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}|_{\mathbb{P}^1})$. Since $\Omega^1_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-2)$ it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{W}) + 2k)$. We refer to [11, Section 1.3] for more details and examples.

We say that $x \in \mathbb{P}^2$ is a smooth point of $\mathcal{W}$, for short $x \in \mathcal{W}_{\text{sm}}$, if $x \notin \text{Sing}(\mathcal{W})$ and the germ of $\omega$ at $x$ satisfies the conditions described in (3) and (4) above. For any smooth point $x$ of $\mathcal{W}$ we have $k$ distinct (not necessarily in general position) linearly embedded subspaces of dimension $n - 1$ passing through $x$. Each one of these subspaces will be called hyperplane tangent to $\mathcal{W}$ at $x$ and denoted by $T^1_x \mathcal{W}, ..., T^k_x \mathcal{W}$.

Consider the manifold $M = \mathbb{P}(T^*\mathbb{P}^n)$ which can be identified with the incidence variety of points and hyperplanes in $\mathbb{P}^n$, thus we have two natural projections $\pi$ and $\tilde{\pi}$ to $\mathbb{P}^n$ and $\mathbb{P}^n$ respectively. To any $k$-web $\mathcal{W}$ we can associate the subvariety $S_{\mathcal{W}}$ of $M$ defined as

$$S_{\mathcal{W}} = \{(x, H) \in M : x \in \mathcal{W}_{\text{sm}} \text{ and } \exists 1 \leq i \leq k, H = T^i_x \mathcal{W}\}.$$  

The discriminant of $\mathcal{W}$, denoted by $\Delta(\mathcal{W})$, is the projection of the ramification variety of $\pi|_{S_{\mathcal{W}}}$. Given a point $p \in \mathbb{P}^n$, we set $\mathcal{L}_p$ the family of hyperplanes passing through $p$. This set can be thought as an hyperplane in $\mathbb{P}^n$ and then one can take $\mathcal{H}_p = \tilde{\pi}^{-1}(\mathcal{L}_p) \subseteq M$. We define the polar set of $\mathcal{W}$ with center on $p$, denoted by $P_{\mathcal{W}}^p$, as

$$P_{\mathcal{W}}^p = \text{tang}(\mathcal{W}, \mathcal{L}_p) := \pi(S_{\mathcal{W}} \cap \mathcal{H}_p).$$

We also denote by $\hat{P}_{\mathcal{W}}^p = S_{\mathcal{W}} \cap \mathcal{H}_p$ which is a subvariety in $S_{\mathcal{W}}$ projecting onto $P_{\mathcal{W}}^p$.

In this work we shall focus on the case of $\mathbb{P}^2$. Observe that in this case $\mathcal{L}_p$ is the radial foliation with singularity at $p$ and $P_{\mathcal{W}}^p$ is the curve of tangencies between this foliation and the web.

In what follows we list some properties of the polar curve which are already known for foliations, cf. [3].

Proposition 2.1. Let $\mathcal{W}$ be a $k$ web of degree $d$ on $\mathbb{P}^2$, then
(1) The polar curve with center in \( p \) is the whole \( \mathbb{P}^2 \) if and only if \( \mathcal{W} \) can be written as the product \( \mathcal{W} = \mathcal{L}_p \boxtimes \mathcal{W'} \), for some \( (k-1) \) web \( \mathcal{W'} \).

(2) If \( P^W_p \subseteq \mathbb{P}^2 \), then \( \deg P^W_p = d + k \).

Proof.

(1) Let us assume that \( P^W_p = \mathbb{P}^2 \). Since \( \pi|_{\mathcal{H}_p} : \mathcal{H}_p \to \mathbb{P}^2 \) is birational one has \( \mathcal{H}_p \cap S_W = \mathcal{H}_p \) and then \( \mathcal{H}_p \) is an irreducible component of \( S_W \). The converse is clear.

(2) See Proposition 2.1 where is proved the same in the case of \( \mathbb{F}^n \).

□

Proposition 2.2. Let \( \mathcal{W}_1 = [\omega_1] \) and \( \mathcal{W}_2 = [\omega_2] \) be \( k \)-webs of degree \( d \) and \( p \in \mathbb{P}^2 \), then \( P^W_{\mathcal{W}_1} = P^W_{\mathcal{W}_2} \) if and only if the web given by \( \omega_2 - \omega_1 \) can be written as the product \( \mathcal{L}_p \boxtimes \mathcal{W}' \) for some \((k-1)\) web \( \mathcal{W}' \).

Proof. Let us suppose that \( P^W_{\mathcal{W}_1} = P^W_{\mathcal{W}_2} \) and denote by \( S_{\mathcal{W}_1} = \{ F_i = 0 \} \). By hypothesis we have \( S_{\mathcal{W}_1} \cap \mathcal{H}_p = S_{\mathcal{W}_2} \cap \mathcal{H}_p \) so one can assume that \( (F_1 - F_2)|_{\mathcal{H}_p} = 0 \). Therefore \( \mathcal{H}_p \) is a component of the surface defined by \( F_1 - F_2 \). The converse part is clear.

Now we denote by \( C(r) = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)) \) the projective space of curves of degree \( r \) and, for a \( k \)-web \( \mathcal{W} \), by \( \mathcal{P} = \{ p_1, \ldots, p_m \} \) the set of centers of radial foliations tangent to \( \mathcal{W} \) (which could be empty). The **polar family** of \( \mathcal{W} \) is defined as the subvariety

\[
\mathcal{R}(\mathcal{W}) := \{ P^W_p : p \in \mathbb{P}^2 - \mathcal{P} \} \subseteq C(d + k).
\]

In other words, \( \mathcal{R}(\mathcal{W}) \) is the closure of the image of the rational map

\[
\mathcal{G} : \mathbb{P}^2 - \mathcal{P} \to C(d + k), \quad \mathcal{G}(p) = P^W_p.
\]

Remark 2.1. In the case \( \mathcal{W} = \mathcal{L}_p \) we have \( \mathcal{R}(\mathcal{W}) = \{ \text{lines through } p \} \), which is a line in \( C(1) = \mathbb{P}^2 \).

Proposition 2.3. If \( \mathcal{W} \) is a \( k \)-web of degree \( d \) different from \( \mathcal{L}_p \) then \( \dim \mathcal{R}(\mathcal{W}) = 2 \) and \( \deg \mathcal{R}(\mathcal{W}) = k^2 \).

Proof. If \( \mathcal{R}(\mathcal{W}) \) is an irreducible curve, then for a generic \( p \in \mathbb{P}^2 \) the inverse image \( \mathcal{G}^{-1}(P^W_p) \) is a curve. Therefore the point \( p \) belongs to \( P^W_q \) for infinitely many points \( q \in \mathcal{G}^{-1}(P^W_p) \). The only possibility is \( \mathcal{G}^{-1}(P^W_p) \) to be a line invariant by \( \mathcal{W} \). Since the only foliations by lines on \( \mathbb{P}^2 \) are the foliations of degree zero (lines through a point), \( \mathcal{G} \) defines a radial foliation \( \mathcal{L}_p \) tangent to \( \mathcal{W} \) for some point \( p \in \mathbb{P}^2 \). Fix now a generic point \( q \in \mathbb{P}^2 \); then it is clear that the leaf of \( \mathcal{L}_p \) passing through \( q \) is contained in \( P^W_q \). If the equality does not hold we can choose a regular point \( z \in P^W_q \) which is out of the leaf, but since all the points of this leaf has the same polar curve than \( q \) this is not possible. We have just proved that the polar centered in a generic point \( q \) is the line passing by \( p \) and \( q \). This shows that \( \mathcal{W} = \mathcal{L}_p \).

To find the degree we need just to do the intersection of \( \mathcal{R}(\mathcal{W}) \) with a generic plane of codimension 2 in \( C(d + k) \), which corresponds to the curves passing through two points. Then we observe that for generic \( p_1, p_2 \in \mathbb{P}^2 \), \( \{ p_1, p_2 \} \subseteq P^W_p \) if and only if \( x \in T_{p_1} \mathcal{W} \cap T_{p_2} \mathcal{W} \). We conclude by noting that the cardinal of \( T_{p_1} \mathcal{W} \cap T_{p_2} \mathcal{W} \) is \( k^2 \). □
Remark 2.2. If we set $B(R(W))$ the set of points which belong to every polar curve, then is easy to see that $B(R(W)) \subseteq \text{Sing}(W)$.

We conclude the section with an important property of the generic polar curve. The main ingredient for this result is the extended Bertini’s first theorem, cf. [4, Theorem 4.1].

Theorem 2.1. For a generic point $p \in \mathbb{P}^2$ we have

1. $\text{Sing}(P^W_p) \subseteq \Delta(W) \cup \{p\}.$
2. The are $k$ smooth and transversal branches of $P^W_p$ passing through $p$, tangent to the $k$ directions of $T_p W$.

Proof. Let us consider the linear system on $S_W$ formed by the curves $\tilde{P}^W_p = \tilde{\pi}|_{S_W}^{-1}(\tilde{p})$. It is clear that this system has no fixed component, therefore we can apply Bertini’s first theorem to conclude that the generic element is smooth outside of the base locus. Take now a generic $P^W_p = \pi(\tilde{P}^W_p)$ and $x \in \text{sing}(P^W_p)$. We have the following possibilities:

- $x$ comes from a ramification point of $\pi|_{S_W}$ and so is a point of $\Delta(W)$.
- $x$ comes from a singularity of $\tilde{P}^W_p$ and so comes from a base point.
- There are two regular points $z_1, z_2 \in \tilde{P}^W_p$, out of the ramification of $\pi|_{S_W}$ which project to $x$. If $x \neq p$, then $W$ decomposes in a neighborhood of $x$ and $L_p$ is tangent in $x$ just to one of the foliations of this decomposition.

Then there is just one point in $\tilde{P}^W_p$ over $x$, contradiction. Therefore $x = p$.

This concludes the proof of the first assertion.

In order to prove the second part, take local coordinates such that $p = (0, 0)$ and $W$ is given near to $p$ by the regular 1-forms $A_1 dx + B_1 dy, \ldots, A_k dx + B_k dy$. Then an equation for $P^W_p$ is $(xA_1 + yB_1) \ldots (xA_k + yB_k) = 0$. Since

$$xA_j + yB_j = xA_j(0) + yB_j(0) + h.o.t$$

we conclude that $P^W_p$ has $k$ smooth transversal branches through $p$. $\Box$

Remark 2.3. Observe that in the proof of (2) we only use $p \notin \Delta(W)$, in particular, for a foliation $F$, the polar curve $P^F_p$ is smooth at $p$ when $p$ is a regular point of $F$.

3. Irreducibility of the generic polar

The goal of this section is to classify the webs whose generic polar curve is decomposable. We begin with an easy observation.

Remark 3.1. If one considers a decomposable web $W = W_1 \boxtimes \ldots \boxtimes W_r$, then $P^W_p$ has $P^{W_1}_p, \ldots, P^{W_r}_p$ as components and therefore is decomposable.

Theorem 3.1. Let $W$ be an irreducible $k$-web with $k \geq 2$. Then $P^W_p$ is decomposable for generic $p \in \mathbb{P}^2$ if and only if $\text{deg } W = 0$.

Proof. By hypothesis the surface $S_W$ is irreducible, then we apply the second extended Bertini’s theorem (cf. [4, Theorem 5.3]) to conclude that the generic curve $P^W_p$ is reducible if and only if the system is a composite with a pencil. So if we suppose that the generic polar is decomposable then the image of $\tilde{\pi}|_{S_W}$ is a curve and this imply $\text{deg } W = 0$ (see [11, Proposition 1.4.2]). The converse part is clear. $\Box$
Corollary 3.1. For a $k$-web $W$ the generic polar is decomposable if and only if $W$ decomposable or $\text{deg } W = 0$ and $k \geq 2$.

An interesting fact about the polar curves is that they determine the web as we will see.

Lemma 3.1. For $i = 1, 2$ let $W_i$ be a $k_i$-web of degree $d_i$ and suppose $k_1 \geq 2$. If $\mathcal{R}(W_1) = \mathcal{R}(W_2)$ then $P_{p}^{W_1} = P_{p}^{W_2}$ for every $p \in \mathbb{P}^2$ and $(k_1, d_1) = (k_2, d_2)$.

Proof. We recall that $\mathcal{R}(W_i)$ is the image of the rational map $G_i : \mathbb{P}^2 \dashrightarrow C(d_i + k_i)$. Take now $P_{p}^{W_i} \in \mathcal{R}(W_1)$ = $\mathcal{R}(W_2)$ generic for $W_1$ and $W_2$ in the sense of Theorem 2.1 which is outside of $G_1(\Delta(W_2))$. Then $P_{p}^{W_1} = P_{q}^{W_2}$ for some point $q$. As $P_{q}^{W_2}$ has $k_1$ branches through $p \notin \Delta(W_2)$, Theorem 2.1 implies that $p = q$ and $k_1 = k_2$. To conclude we only need to point that $k_1 + d_1 = k_2 + d_2$. \hfill \square

Theorem 3.2. If $W_1$ and $W_2$ are $k$-webs of degree $d$, with $k \geq 2$, such that $\mathcal{R}(W_1) = \mathcal{R}(W_2)$, then $W_1 = W_2$.

Proof. Let $p \in \mathbb{P}^2$ a generic point. By the previous lemma $P_{p}^{W_1} = P_{p}^{W_2}$, so the branches by $p$ are the same and by the second part of Theorem 2.1 we have $T_pW_1 = T_pW_2$. \hfill \square

Remark 3.2. The proof of Lemma 3.1 does not hold for foliations. Actually, in [2] we can find different foliations of degree one with the same polar family. However, in the same work the authors showed that Theorem 3.2 remains true for foliations of degree $d \geq 2$.

4. Polar curves of foliations

Now we focus in the case of foliations. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree $d \geq 1$, in this case the polar family $\mathcal{R}(\mathcal{F})$ is a net. We recall from [10] the definition of the inflexion divisor of $\mathcal{F}$, denoted by $\mathcal{E}(\mathcal{F})$, as the curve formed (out of the singularities) by the inflexion points of the leaves of $\mathcal{F}$. If the foliation is given locally by the vector field $X = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$, then

$$\mathcal{E}(\mathcal{F}) = \{B^2 A_y + A B A_x - A^2 B_x - A B B_y = 0\}.$$

Lemma 4.1. For every point $p \in \mathbb{P}^2$ we have $\text{Sing}(P_p^\mathcal{F}) \subseteq \mathcal{E}(\mathcal{F})$.

Proof. It is a straightforward computation to show that a point $q \in P_p^\mathcal{F}$ which does not belong to $\mathcal{E}(\mathcal{F})$ cannot be a singularity of $P_p^\mathcal{F}$. \hfill \square

We say that a singular point $q \in \text{Sing}(\mathcal{F})$ is quasi-radial if after doing a blow up at $q$ the exceptional divisor is not invariant by the strict transform of $\mathcal{F}$.

Proposition 4.1. For generic $p \in \mathbb{P}^2$ the polar curve $P_p^\mathcal{F}$ has the following properties:

1. If $q \in \text{Sing}(\mathcal{F})$ is not quasi-radial, the tangent cone of $P_p^\mathcal{F}$ at $q$ does not contain the line $\overline{pq}$,
2. If $q \in \text{Sing}(\mathcal{F})$ is quasi-radial, the tangent cone of $P_p^\mathcal{F}$ at $q$ contains the line $\overline{pq}$ exactly one time.
Proof. Take local coordinates such that \( q = (0, 0) \) and \( \mathcal{F} \) is given locally by the vector field \( X = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} = (A_k + A_{k+1} + \ldots) \frac{\partial}{\partial x} + (B_k + B_{k+1} + \ldots) \frac{\partial}{\partial y} \), where \( A_j \) and \( B_j \) are homogeneous polynomials of degree \( j \). Take now a generic point \( p = (a, b) \), then an equation for \( P_p^\mathcal{F} \) is

\[
(x - a)B - (y - b)A = bA_k - aB_k + h.o.t = 0.
\]

Thus the tangent cone of \( P_p^\mathcal{F} \) at \( q \) contains the line \( pq \) if and only if \( bA_k(a, b) - aB_k(a, b) = 0 \). If that holds for \( p \) generic, it is equivalent to have \( yA_k(x, y) - xB_k(x, y) \equiv 0 \), which says that \( q \) is a quasi-radial singularity of \( \mathcal{F} \). In this case one writes \( A_k = xP \) and \( B_k = yP \) for some polynomial \( P \) and then \( bA_k - aB_k = (bx - ay)P \). Because \( p \) is generic we can assume that \( bx - ay \) is not a factor of \( P \).

The precedent proposition says that, for \( p \) generic, a quasi-radial singularity of \( \mathcal{F} \) gives exactly one intersection between \( \hat{P}_p^\mathcal{F} \) and \( L_p \) (the duals of \( P_p^\mathcal{F} \) and \( \mathcal{L}_p \) respectively) and a non quasi-radial singularity gives no intersections between \( \hat{P}_p^\mathcal{F} \) and \( L_p \). On the other hand, the point \( p \) give exactly one intersection between \( \hat{P}_p^\mathcal{F} \) and \( L_p \), as the following lemma shows.

**Lemma 4.2.** Let \( p \) be a regular point of \( \mathcal{F} \). Then \( p \) is an inflexion point of \( P_p^\mathcal{F} \) if and only if \( p \in \mathcal{E}(\mathcal{F}) \).

Proof. Take local coordinates such that \( p = (0, 0) \) and \( \mathcal{F} \) is given by the vector field \( \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} \) with \( B(0, 0) = 0 \). Therefore \( P_p^\mathcal{F} = \{ y - xB = 0 \} \) has inflexion at \( p \) if and only if the Hessian of \( y - xB \) vanishes on the tangent direction of \( P_p^\mathcal{F} \) at \( p \). By a straightforward computation this is equivalent to have \( \frac{\partial B}{\partial x}(0, 0) = 0 \). On the other hand, the static curve \( \mathcal{E}(\mathcal{F}) \) pass through \( p \) if and only if \( \frac{\partial B}{\partial x}(0, 0) = 0 \).

Putting all this together as in [3, Proposition 4.1] we obtain a bound for the number of quasi-radial singularities:

\[ \#\text{Sing}_{QR}(\mathcal{F}) \leq \deg(\hat{P}_p^\mathcal{F}) - 1. \]

We recall that R. Mol obtains this inequality for quasi-radial singularities of multiplicity one.

Now we state the main result of this section, which establish the invariance of the topological type of the generic polar.

**Theorem 4.1.** For a foliation \( \mathcal{F} \), the generic polar has constant topological embedded type in \( \mathbb{P}^2 \), that is: if \( \mathcal{P}_{p_1}^\mathcal{F} \) and \( \mathcal{P}_{p_2}^\mathcal{F} \) are generic polar curves, then there exists a homeomorphism \( H : \mathbb{P}^2 \to \mathbb{P}^2 \) such that \( H(\mathcal{P}_{p_1}^\mathcal{F}) = \mathcal{P}_{p_2}^\mathcal{F} \). In particular, the genus of the generic polar is constant.

Proof. It is sufficient to prove that the generic polar has locally constant topological embedded type in \( \mathbb{P}^2 \). The polar net defines a local linear system on each singularity \( q_1, \ldots, q_l \) of \( \mathcal{F} \). For generic \( p \in \mathbb{P}^2 \), the family \( (\mathcal{P}_p^\mathcal{F}, q_j) \) is equisingular \((j = 1, \ldots, l)\) and each curve is reduced (see [1] chapter 7). Moreover, for each \( j = 1, \ldots, l \), there exists a finite sequence of blow-ups \( \pi_j \) at \( q_j \) with the following properties:

1. The sequence of blow-ups \( \pi_j \) desingularizes the generic curve \( (\mathcal{P}_p^\mathcal{F}, q_j) \).
(2) There is a list \(D_j^1, \ldots, D_j^{N_j}\) of components (not necessarily different) of the exceptional divisor of \(\pi_j\) such that the strict transform of the generic curve \((P^*_p, q_j)\) has exactly \(N_j\) branches \(\gamma_1(p), \ldots, \gamma_{N_j}(p)\) passing (transversely) through \(D_j^1, \ldots, D_j^{N_j}\) respectively.

Let \(\zeta^j_k(p)\) be the intersection between \(D^k_j\) and \(\gamma^j_k(p)\). Denote by \(M\) the surface resulting by the composition \(\pi\) of all the blow-ups involved in \(\pi_1, \ldots, \pi_l\), by \(E\) the total exceptional divisor and by \(P^*_p \subseteq M\) the strict transform of \(P^*_F\). Observe that \(P^*_p\) is smooth and intersects the exceptional divisor at the points \(\{\zeta^j_k(p) : j = 1, \ldots, l; \ k = 1, \ldots, N_j\}\).

**Lemma 4.3.** For any \(z \in M\), there is a neighborhood \(U\) of \(z\) and an analytic family \(\{f_p^U\}\) of functions on \(U\) such that \(P^*_p \cap U = \{f_p^U = 0\}\) for generic \(p\).

**Proof.** The proposition is obvious if \(z \notin E\). Let us take \(z \in E\) and suppose first that \(z\) is not a corner point. In a neighborhood of \(\pi(z) \in \mathbb{P}^2\) the polar family is defined by an analytic family of equations \(\{h_p\}\). Let \(g = 0\) be a local reduced equation of \(E\) at \(z\). Then, since the resolution of the family \(P^*_p\) (\(p\) generic) is achieved with the same sequence of blow-ups, there exists \(n \in \mathbb{N}\) such that, in a neighborhood of \(z\), the curve \(P^*_p\) is defined by the equation \(f_p = \frac{h_p \circ \pi}{g^n} = 0\) for generic \(p\). In the case of a corner, the proof is similar. □

Fix now a generic \(p_0 \in \mathbb{P}^2\) and consider the curve \(P = P^*_{p_0}\). According with the tubular neighborhood theorem, the curve \(P\) admits a neighborhood \(T\) fibered by smooth two dimensional disks transverse to \(P\). This fibration can be taken such that the intersection of \(E\) with \(T\) is invariant by the fibration, i.e. \(E \cap T\) is a finite union of disks.

From Lemma 4.3 it is easy to prove that for \(p\) close enough to \(p_0\), the curve \(P^*_p\) is contained in \(T\). We will use the following elementary lemma.

**Lemma 4.4.** Let \(f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\) be a \(C^\infty\) submersion such that \(f^{-1}(0) = \mathbb{D} \times \{0\}\). If \(h : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\) is close enough to \(f\) in the \(C^1\) topology, then \(h^{-1}(0)\) is the graph of a function \(\psi : \mathbb{D} \to \mathbb{D}\).

Let \(f_{U_1}^{p_0}, \ldots, f_{U_r}^{p_0}\) be a finite set of defining functions of \(P\) given by Lemma 4.3 \(\mathbb{P} \subset U_1 \cup \ldots \cup U_r\). Since the generic polar is reduced and \(\mathbb{P}\) is smooth, the functions \(f_{U_1}^{p_0}, \ldots, f_{U_r}^{p_0}\) are holomorphic submersions at points of \(\mathbb{P}\). Thus, by reducing \(T\) and the open sets \(U_1, \ldots, U_r\) if necessary we may assume that

1. The functions \(f_{U_1}^{p_0}, \ldots, f_{U_r}^{p_0}\) are submersions
(2) Any $z \in \mathcal{P}$ has a neighborhood $\Omega$ in $\mathcal{P}$ with the following property: If $D_\zeta$ denote the disc of the fibration on $T$ passing through $\zeta \in \mathcal{P}$, the closure of the set $\bigcup_{\zeta \in \Omega} D_\zeta$ is contained in some $U_i$.

Given $z \in \mathcal{P}$, take a neighborhood $\Omega$ of $z$ in $\mathcal{P}$ as above and such that $\Omega$ is diffeomorphic to $\mathbb{D}$. Let $T_\Omega$ be the restriction to $\Omega$ of the fibration on $T$. Clearly $T_\Omega$ admits $C^\infty$-coordinates $(x, y) \in \mathbb{D} \times \mathbb{D}$ such that $\mathcal{P}$ is given by $\mathbb{D} \times \{0\}$ and fibers are given by the sets $\{x\} \times \mathbb{D}$. Take some $f^U_{p_0}$ such that $T_\Omega$ is contained in $U_i$. Then $f^U_{p_0}$ is a submersion on $T_\Omega$ and by Lemma 4.4 we see that for $p$ generic and close to $p_0$ the curve $\mathcal{P}_p$ intersects once each fiber of $T_\Omega$. By compactness we conclude that for $p$ generic and close to $p_0$ the curve $\mathcal{P}_p$ is contained in $T$ and intersects once each fiber of $T$. Then it is easy to construct a homeomorphism $H : T \to T$ with the following properties:

(1) $H$ fix the fibers
(2) $H$ maps $\mathcal{P}_p$ onto $\mathcal{P}$
(3) $H$ is the identity when restricted to $\partial T$.

Clearly $H$ extends as a homeomorphism $H : M \to M$ by making $H(z) = z$ for $z \notin T$ and it is easy to see that $H(E) = E$. Then $H$ induces a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(P_F^p) = P_F^{p_0}$.

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