Proofs of network quantum nonlocality aided by machine learning

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The study of nonlocality in scenarios that depart from the bipartite Einstein-Podolsky-Rosen setup is allowing to uncover many fundamental features of quantum mechanics. Recently, an approach to building network-local models based on machine learning lead to the conjecture that the family of quantum triangle distributions of [DOI:10.1103/PhysRevLett.123.140401] did not admit triangle-local models in a larger range than the original proof. We prove part of this conjecture in the affirmative. Our approach consists in reducing the family of original, four-outcome distributions to families of binary-outcome ones, which we prove not to admit triangle-local models by means of the inflation technique. In the process, we produce a large collection of network Bell inequalities for the triangle scenario with binary outcomes, which are of independent interest.

One of the most striking consequences of Bell’s theorem [1] and its violation by quantum mechanics is that nature cannot be modeled using only local variables. This phenomenon, known as quantum nonlocality, is now the basis of many protocols that process information encoded in quantum systems [2–4]. In the last decade, both theoretical advances and experimental improvements have allowed to shift the focus to multipartite scenarios, called networks, where many independent sources distribute physical systems among different, overlapping collections of parties [5, 6]. The study of nonlocality in these network scenarios is generating a very exciting research field. On one hand, it provides the framework needed to analyze the distribution of quantum information in complex networks (i.e., the quantum internet [7–9]). On the other, the consideration of independent sources departs from the traditional Einstein-Podolsky-Rosen scenario, enabling a deeper analysis on the foundations of quantum mechanics [10–14].

Nonlocality in networks is defined in a way analogous to the standard notion, namely by opposition to admitting a network-local model. Network-local correlations are those that can be generated by assuming that the sources in the network distribute independent classical shared randomness, and the parties process the randomness they receive (potentially depending on a private choice of input) in order to generate an outcome. Put as an example the so-called triangle scenario, whereby three bipartite sources distribute systems to three separate parties forming a triangle-shaped structure (see Fig. 1). Triangle-local correlations admit models of the form

$$P(a, b, c) = \int d\alpha d\beta d\gamma \mu_{BC}(\alpha)\mu_{AC}(\beta)\mu_{AB}(\gamma)\times P_A(a|\alpha, \beta)P_B(b|\beta, \gamma)P_C(c|\alpha, \beta),$$

where $\alpha$, $\beta$, and $\gamma$ denote sources of classical shared randomness distributed with densities $\mu_{BC}(\alpha)$, $\mu_{AC}(\beta)$ and $\mu_{AB}(\gamma)$, respectively, and $P_A$, $P_B$ and $P_C$ represent local operations on the variables received by each party. Note that in Eq. (1) the local operations do not depend on an input. These are, however, straightforward to insert if necessary. Any distribution that cannot be generated by a suitable choice of $\mu_{AB}$, $\mu_{BC}$, $\mu_{AC}$, $P_A$, $P_B$ and $P_C$ is thus termed triangle-nonlocal. For other networks, the definitions of network-local correlations and network nonlocality are analogous.

![FIG. 1. (Top left) Realization of the family of quantum genuine triangle nonlocal distributions $P_u(a, b, c)$ of Ref. [15]. Each of the sources distributes a maximally entangled state $|\psi^+\rangle \propto |01\rangle + |10\rangle$, and each of the parties performs the four-outcome measurement given by $\{0, |1\rangle, \bar{1}, \bar{0}\}$ described in the main text. (Top right) Implications from the existence of a triangle-local model for $P_u(a, b, c)$, in the notation used in this work (see Appendix B). Reference [15] used only condition (i) for asserting the nonlocality of $P_u(a, b, c)$. In this work we analyze when the condition (ii) is not true. (Bottom) The current standing of the proofs of nonlocality for the family.](image-url)
Many of the initial demonstrations of nonlocality in networks were based in standard bipartite Bell nonlocality. For instance, Ref. [17] showed that any entangled state that violates the CHSH inequality can be used to violate the bilocality inequality of Refs. [5, 18], and Ref. [19] demonstrated that bipartite nonlocality can be “disguised” in networks, leading to network-nonlocal distributions. Examples of this phenomenon motivated the search of stronger definitions for nonlocality in networks, that are in closer relation to the network structure [20] and give a more pivotal role to entangled measurements [21].

Recently, a family of quantum distributions in the triangle scenario [see Eq. (2) for its definition] was proven not to admit triangle-local models [15]. The proof, based on on coarse-grains their respective outcomes—1, 1, and 1, into a single output, 1, admits an essentially unique triangle-local model with binary, uniformly random hidden variables. Then, when the full $P_u(a, b, c)$ admits a triangle-local model, this must be compatible with the former, and in particular with a new variable $t$ containing information about the random hidden variables used in the coarse-grained triangle-local model. This allows to introduce a distribution $g_u(i, j, k, t)$, with binary outcomes and related to $P_u(a, b, c)$ via linear constraints, whose existence is equivalent to $P_u(a, b, c)$ satisfying Eq. (1). We prove this statement in appendices A and B. Some of the marginals of this distribution are fixed by $P_u(a, b, c)$, leading to the form (derived explicitly in Appendix A)

$$q_u(i, j, k, t) = \frac{1}{27} \left[ 1 + (u^2 - v^2)^2 (ij + jk + ki) + (u^2 - v^2)(i + j + k)t + 8u^2v^2ijk + t(iF_{AB} + jF_{BC} + kF_{AC}) + ijk F_{ABC} \right].$$

Note that the expression contains four free parameters, $F_{AB}, F_{BC}, F_{AC}, F_{ABC} \in [-1, 1]$. The requirement that all probabilities in Eq. (3) are positive is the condition
that Ref. [15] uses to find that a triangle-local model for $P_u(a, b, c)$ does not exist for $u \geq \sqrt{\frac{-3+19+6\sqrt{3}}{2(9+6\sqrt{3})^{1/3}}} \equiv u_0 \approx 0.8860$ (see Fig. 1 and Appendix A). Indeed, for $u_0 < u < 1$, at least one of the probabilities of Eq. (3) is negative for any value of $F_{AB}, F_{BC}, F_{AC}, F_{ABC} \in [-1, 1]$.

In contrast, for $1/\sqrt{2} < u < u_0$, there always exist values of $F_{AB}, F_{BC}, F_{AC}$ and $F_{ABC}$ that make all probabilities in Eq. (3) positive. This does not mean, however, that the corresponding $P_u(a, b, c)$ admits a model of the form of Eq. (1). In Appendix B we show that this is the case if and only if there exist specific values for $F_{AB}, F_{BC}, F_{AC}$ and $F_{ABC}$ such that $q_u(i, j, k, t)$ is a proper probability distribution and the two conditionals $q_u(i, j, k) = q_u(i, j, k, t)$ admit triangle-local models of the form of Eq. (1) as well. Reversing the argument, $P_u(a, b, c)$ is proven to be triangle-nonlocal if no value of $F_{AB}, F_{BC}, F_{AC}$ and $F_{ABC}$ produces a positive $q_u(i, j, k, t)$ such that both $q_u(i, j, k)$, $q_u(i, j, k)$ admit triangle-local models. In the following, thus, we find the range of $u$ where $q_u(i, j, k)$ and $q_u(i, j, k)$ do not admit such models.

Proof for discrete values $u < u_0$.— We now show that, for $0.7504 \leq u \leq 0.8101$, at least one of the $q_u(i, j, k, t)$ for $t = \pm 1$ does not admit such a triangle-local model in the whole region where $q_u(i, j, k, t)$ is well defined. This region is described by the positivity of Eq. (3) for every tuple $(i, j, k, t)$. When the value of $u$ is fixed, the region is a polytope in $\mathbb{R}^4$, which we characterize explicitly in Appendix A. Let us denote this polytope $P_u$.

For $t = +1$, and at for $1/\sqrt{2} \leq u \leq 0.857$, there always exists a distribution in $P_u$ that admits a triangle-local model. We describe this distribution and its triangle-local model in Appendix C. For this reason, we focus on assessing the compatibility of $q^{-}(i, j, k)$. We do so via the inflation technique for causal compatibility [25].

Inflation provides a hierarchy of necessary conditions, characterized by linear programming problems, that are satisfied by correlations admitting network-local models [27]. Briefly, inflation is a technique by which the sources and measurement devices of a network are copied several times and arranged in different configurations. By studying the correlations that arise in these configurations, one can find necessary conditions for correlations to be generated in the original network. Note, however, that inflation is only a theoretical tool and does not mean that the actual network of interest is changed. The fact that, for a particular distribution under test, the necessary conditions are not satisfied at some level of the inflation hierarchy, constitutes a rigorous proof that such distribution does not admit a network local model.

In this work we consider compatibility with triangle-local models through the so-called web inflation of the triangle network (see Fig. 3 in Appendix D), and we relate distributions achievable in such inflation to $q_u^{-}(i, j, k)$ via three types of constraints (see Appendix D for their technical description and concrete examples). Firstly, we impose the hierarchy constraints. These are sufficient to prove the asymptotic convergence of inflation [27], and relate marginals in the inflation which completely reproduce the original network to the original distribution. Secondly, we impose the family termed higher-degree relations in Ref. [27]. These constraints, which are also used in earlier works (see, for instance, the discussion in [25, Example 2]) relate marginals in the inflation with polynomials of the original distribution of degree higher than the inflation level. Finally, we also impose linearized polynomial identification (LPI) constraints. These are, a priori, polynomial relations between the elements of the probability distribution in the inflation, which cannot be enforced in linear programming. However, once the remaining constraints have been imposed for a particular distribution to be tested, some of the factors are substituted by known probabilities. Thus, the relations become linear between two probabilities in the inflation, where the magnitude of the relation is given by the distribution under test. LPI constraints have been thoroughly used in the literature [10, 28] since they appear to constrain significantly the sets of compatible correlations. In Appendix D we describe the inflation employed and the different types of constraints that we use to relate the distribution in the inflation and $q_u^{-}(i, j, k)$.

One of the benefits of formulating the compatibility problem as a linear program is that a negative answer provides, via Farkas’ lemma [29], a witness which applies to distributions other than that analyzed. In fact, this approach is standard for finding Bell inequalities from results on the inexistence of local models [20, 26, 30]. For the inflation used (recall Fig. 3 in Appendix D) and the distributions $q_u^{-}(i, j, k)$ created from conditioning Eq. (3), Farkas’ lemma provides a witness valid for fixed $u$. By discretizing the range of $u$ we find, for every point $0.7504 \leq u \leq 0.8101$, an infeasibility certificate that identifies every distribution in the corresponding $P_u$ as incompatible with a triangle-local model (an illustration of one such certificate appears in Fig. 5d in Appendix D). This certificate is obtained when analyzing the compatibility of the distribution $q_u^{-}(i, j, k)$ derived from Eq. (3) for $E_{AB} = E_{BC} = E_{AC} = 2v^2(u^2 - v^2)$, $E_{ABC} = 1 - 8u^2v^3$, which is a vertex of $P_u$ for $1/\sqrt{2} \leq u \leq 0.8457$. In the computational appendix [31] we provide computer codes that find such witnesses. Because of the relations of the binary-outcome family $q_u^{-}(i, j, k)$ with the four-outcome distribution $P_u(a, b, c)$, the existence of these witnesses implies the triangle nonlocality of $P_u(a, b, c)$ for the corresponding values of $u$.

Outside the range, it is still possible to find points within $P_u$ that do not admit a model. The associated witnesses, however, do not identify nonlocality for all distributions in $P_u$, and so they do not guarantee that $P_u(a, b, c)$ is triangle-nonlocal. Since inflation provides outer approximations of the sets of compatible correlations, and $P_u$ does not qualitatively change at the boundaries of the interval, it is possible that the cause of this behavior is of a purely computational nature. Thus, increasing the computational capabilities in order to con-
sider larger inflations is likely to push the boundaries of the interval where nonlocality can be proven.

**Extending to the continuum.** — The fact that we can identify that \( q_u(i,j,k) \) does not admit a triangle-local model for discrete values of 0.7504 ≤ \( u \) ≤ 0.8101 does not rule out the possibility of the existence of discrete points or small regions within that interval where a model exists. This is, in fact, a common problem that appears when using numerical techniques to characterize nonlocality. The conditions of Farkas’ lemma restrict the range of application of the witnesses obtained above to distributions with fixed single-party marginals when one considers LPI constraints in the web inflation (see Appendix E). This, in our case, implies that the witness obtained at a fixed value of \( u \) is unable to witness the nonlocality of distributions for \( u′ \neq u \). However, we show in Appendix E that it is possible to relax the conditions on Farkas’ lemma in such a way that a witness can apply to neighboring values of \( u \). Using this relaxation, in the computational appendix [31] we provide a computer program that, recursively, proves the triangle nonlocality of \( P_u(a,b,c) \) for a particular value of \( u \), computes the furthest value \( u′ \) that the associated witness is capable of identifying that \( P_{{u′}}(a,b,c) \) is triangle-nonlocal as well, and sets \( u = u′ \) to begin again. Starting at any point \( u \in [0.7504, 0.8101] \), the program halts at one of the extremes, producing along the way a collection of witnesses that cover the whole range from the starting point to the end. This collection proves that \( P_u(a,b,c) \) is triangle nonlocal in the whole interval 0.7504 ≤ \( u \) ≤ 0.8101.

**Discussion.** — In this work we rigorously prove a conjecture initially formulated by applying machine learning techniques to network nonlocality. Namely, that the family of quantum triangle distributions of Ref. [15] is also triangle-nonlocal below \( u_0 \approx 0.8860 \), calculated in the original proof. To our knowledge, this work constitutes the first successful use of inflation in a proof of quantum nonlocality in networks for distributions whose nonlocality could not be proved with alternative methods (e.g., those in Refs. [32–34]). The current standing, captured in Fig. 1, is that the family of distributions is triangle-nonlocal for \( u \in [0.7504, 0.8101] \cup \{u_0\} \). For the remaining region, namely \( u \in [1/\sqrt{2}, 0.7504) \cup (0.8101, u_0[\right) \), the question remains open. The intuitions developed via the machine learning approach of Ref. [16] point to a positive answer for the whole range \( u \in [1/\sqrt{2}, u_0[\right) \). Because the characterization of the set of triangle-local distributions offered by inflation is an outer approximation of the real one, and the polytope \( P_u \) used in our proof does not qualitatively change in the endpoints found, we expect that the conjecture is true for the full range and its proof can be asymptotically reached with increased computational capabilities in order to work with larger inflations.

Our construction provides a large family of polynomial, network Bell inequalities for binary-outcome distributions in the triangle network. Despite initially only applicable for the family of binary-outcome distributions under study, we have extended them to be valid for arbitrary binary-outcome distributions. Extending the range of applicability of Bell inequalities obtained from infeasibility certificates is a novel approach, which can help in making rigorous statements from numerical results. However, because the reduction from four-outcome to binary-outcome distributions is exclusive to the family of distributions in Ref. [15], these inequalities cannot be directly understood as noise-robust certificates of triangle nonlocality for the family \( P_u(a,b,c) \).

More generally, it is not known whether the derived inequalities can be violated in quantum mechanics. Finding such a violation in any of them would decide in the positive another long-standing open question in the field, namely whether the binary-outcome triangle scenario without inputs supports network nonlocality. In particular, it is interesting to test the distributions generated out of maximally network-nonlocal boxes of Ref. [35]. Moreover, a very interesting question that we now open is the existence of a family of network Bell inequalities, with the coefficients being continuous functions of \( u \), that identify the corresponding distribution as triangle-nonlocal. This family could provide insights into noise-robust witnesses of the nonlocality of the family of Ref. [15]. A potentially promising starting point is the distribution for \( u = 55/73 \approx 0.7534 \). Since (48, 55, 73) is a Pythagorean triple, the associated inflation problem (D7) and solution are formulated exactly in terms of rational numbers.

In a broader view, our results can be framed in the context of artificial intelligence augmentation [36], which understands that the ultimate use of machine learning in science will not be replacing humans in solving complicated problems, but becoming a tool for understanding where interesting problems lie. A paradigmatic example of this approach to physics problems is the automated analysis of phase diagrams, which has suggested the existence of new phases in various models [37, 38]. In this picture, the machine learning approach of Ref. [16] pointed to an interesting question, which in turn produced results that have increased our understanding of triangle nonlocality, and provided with new network Bell inequalities that can help us in advancing other problems in the field.

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Appendix A: Conditions on $q$ and the feasible polytope

As an introduction, let us begin by analyzing which are the necessary conditions for Eq. (3) to describe a valid probability distribution. A general four-partite, binary-outcome distribution with outcomes taking values $\pm A$ and $\pm B$ which fixes $M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, I. Šupić, J. Bowles, M.-O. Renou, A. Acín, and M. J. X. Coiteux-Roy, E. Wolfe, and M.-O. Renou, Phys. Rev. Lett. 128, 010403 (2022).

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depends only on the four parameters $F_{AB}$, $F_{BC}$, $F_{AC}$ and $F_{ABC}$ and reads

$$q_u(i, j, k, t) = \frac{1}{2^4} \left[ 1 + (u^2 - v^2)^2(ij + jk + ki) + (u^2 - v^2)(i + j + k)t + 8u^3v^3ijk + t(ijkF_{AB} + jkF_{BC} + kiF_{AC} + ijktF_{ABC}) \right].$$  \tag{A2}$$

In order for $g(i, j, k, t)$ to be well defined, all probabilities must be positive. These conditions give rise to a polytope in the space $(F_{AB}, F_{BC}, F_{AC}, F_{ABC})$, determined by the following sixteen linear inequalities:

1. $1 - F_{AB} - F_{AC} - F_{BC} + F_{ABC} + 6u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3a}
2. $1 + F_{AB} + F_{AC} + F_{BC} - F_{ABC} - 6u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3b}
3. $1 - F_{AB} + F_{AC} + F_{BC} - F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3c}
4. $1 + F_{AB} - F_{AC} - F_{BC} + F_{ABC} - 2u^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3d}
5. $1 + F_{AB} + F_{AC} - F_{BC} - F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3e}
6. $1 - F_{AB} - F_{AC} + F_{BC} - F_{ABC} - 2u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3f}
7. $1 - F_{AB} + F_{AC} - F_{BC} - F_{ABC} - 2u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3g}
8. $1 - F_{AB} - F_{AC} + F_{BC} - F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3h}
9. $1 + F_{AB} - F_{AC} - F_{BC} + F_{ABC} - 2u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3i}
10. $1 + F_{AB} + F_{AC} - F_{BC} + F_{ABC} - 2u^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3j}
11. $1 - F_{AB} - F_{AC} + F_{BC} + F_{ABC} - 2u^2(u^2 - v^2) - 8u^3v^3 \geq 0$, \tag{A3k}
12. $1 + F_{AB} - F_{AC} - F_{BC} + F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3l}
13. $1 - F_{AB} + F_{AC} - F_{BC} - F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3m}
14. $1 + F_{AB} - F_{AC} - F_{BC} + F_{ABC} + 2v^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3n}
15. $1 + F_{AB} + F_{AC} - F_{BC} - F_{ABC} + 6u^2(u^2 - v^2) + 8u^3v^3 \geq 0$, \tag{A3o}
16. $1 + F_{AB} + F_{AC} + F_{BC} + F_{ABC} + 6u^2(u^2 - v^2) + 8u^3v^3 \geq 0$. \tag{A3p}

An illustration of the projection of the polytope for $u = 0.8090$ onto the subspace satisfying $F_{AB} = F_{BC} = F_{AC}$ is depicted in Fig. 2. A rather lengthy but straightforward calculation shows that the polytope is empty for $u > \sqrt{-3 + (9 + 6\sqrt{3})^2/3} / 2(9 + 6\sqrt{3})^{1/3} \equiv u_0$. This recovers the proof developed in Ref. [15]. Moreover, similar calculations allow to show that the polytope becomes a single point at $u = 1/\sqrt{2}$, where it is easy to see that $P_u(a, b, c)$ admits a model of the form of Eq. (1).
Appendix B: Equivalence between triangle-locality of $P_u(a, b, c)$ and existence of a suitable $q_u(i, j, k, t)$

Here, we prove that $P_u(a, b, c)$ is triangle-local if and only if there exists at least one distribution $q_u(i, j, k, t)$ satisfying the positivity conditions given in Appendix A and such that both $q_u^+(i, j, k)$ and $q_u^-(i, j, k)$ are triangle-local.

1. $P_u(a, b, c)$ is triangle-local $\Rightarrow \exists q_u(i, j, k, t)$ compatible such that $q_u^+(i, j, k)$ and $q_u^-(i, j, k)$ are triangle-local

Let us begin assuming that $P_u(a, b, c)$ is triangle-local and obtained with a model (or strategy) called $S$. This model determines a coarse-grained model, $S_{TC}$, when grouping the outputs $I_+, I_-$ into a single output, $I$, that reproduces the coarse-grained distribution $P_{TC}(a, b, c)$.

References [15, 22, 23] proved that there is essentially a unique way to obtain $P_{TC}(a, b, c)$ with a triangle-local model, called the Token-Counting strategy. In this strategy for $P_{TC}(a, b, c)$, each source distributes a uniformly binary random hidden variable (a token) and each party produces her output $\{0, 1, 2\}$ by counting the total number of tokens received.

This implies that in $S$, the sources $\alpha$, $\beta$, and $\gamma$, can be taken to each distribute a uniformly random bit (call them $T_\alpha(\alpha)$, $T_\beta(\beta)$ and $T_\gamma(\gamma)$) that ultimately determine whether a given party outputs 0, 2, or something else, and additional information that specify, in the case of outputting something else, whether that is 1, or 0.

| No. | Description                                                                                                           |
|-----|-----------------------------------------------------------------------------------------------------------------------|
| 1.  | Counts the total number of tokens $T_A = f(T_\beta, T_\gamma) \in \{0, 1, 2\}$ she received,                           |
| 2.  | Outputs $T_A$ if $T_A \in \{0, 2\}$,                                                                                   |
| 3.  | Otherwise (i.e., when $T_A = 1$), she looks at where the token is coming from. If it comes from the source $\beta$, she  |
|     | uses the values of $\beta_+$ and $\gamma_+$ to produce her output according to the strategy $S_+$. If, on the contrary,  |
|     | it comes from the source $\gamma$, she uses the values of $\beta_-$ and $\gamma_-$ to output according to $S_-$.           |
Bob and Charlie, on their ends, adopt analogous strategies.

The strategy outlined above, which we call $S$, is indeed a triangle-local model for $P_u(a, b, c)$. Consider first the probability to observe $a = 1$, $b = 2$, $c = 0$ in $S$. This is obtained when both $T_\alpha$ and $T_\beta$ sends their tokens to Bob, $T_\beta$ sends its token to Alice, and Alice plays strategy $S_+$ and obtains output $a = 1$. Hence, it is obtained with probability $\frac{1}{2} q_u(i|t=+1) = \frac{1}{2} q(i|t=+1) = P_u(\bar{1}, 2, \bar{0})$, where the last equality comes from the fact that $q(i, j, k, t)$ is compatible with the constraints given in Appendix A, and hence satisfies [15, Eq. (8)]. Similarly, $\frac{1}{2} q_u(i|t=-1) = P_u(1, 0, 2)$, $\frac{1}{2} q_u(j|t=-1) = P_u(0, \bar{1}, j, \bar{2})$, and so on.

Finally, consider the probability to observe $a = 1$, $b = 1$, and $c = 1$. This is obtained when either $T_\alpha$ sends its token to Bob, $T_\beta$ sends its token to Charlie, $T_\gamma$ sends its token to Alice (which is labelled as $t = +1$) and all parties play strategy $S_+$, or when $T_\alpha$ sends its token to Charlie, $T_\beta$ sends its token to Alice, $T_\gamma$ sends its tokens to Bob (which is labelled as $t = -1$) and all parties play strategy $S_-$. Hence, the outcome $a = 1$, $b = 1$, $c = 1$ is obtained with probability $\frac{1}{2} [q_u(i, j, k|t=+1) + q_u(i, j, k|t=-1)] = \frac{1}{2} [q_u(i, j, k|t=+1) + q_u(i, j, k|t=-1)] = \frac{1}{2} q_u(i, j, k) = P_u(\bar{1}, 1, j, 1)$, where the last equality comes from the fact that $q(i, j, k, t)$ is compatible with the constraints given in Appendix A, hence satisfies [15, Eq. (7)]. The remaining probabilities are zero in $P_u(a, b, c)$ and in the strategies $S_+$ and $S_-$, and thus the proof concludes.

Appendix C: A triangle-local model for $q_u^\pm(i, j, k)$

Here we prove that, in a large range of $u$, there exists one choice of $F_{AB}, F_{BC}, F_{AC}$ and $F_{ABC}$ within the polytope of Eq. (A3) for which $q_u^\pm(i, j, k) := q_u(i, j, k|t=\pm1)$ admits a triangle-local model. This choice is the intersection of Eqs. (A3b), (A3h), (A3l) and (A3n), which is described by $F_{AB} = F_{BC} = F_{AC} = 2v^3(u^2 - v^2)$ and $F_{ABC} = 1 - 8w^3v^3$, and gives rise to the distribution

\[
q_u^+(+1, +1, -1) = \frac{v^2}{2}, \tag{C1a}
\]

\[
q_u^+(-1, -1, -1) = u^2 - \frac{v^2}{2}, \tag{C1b}
\]

plus permutations of parties, and all remaining probabilities being zero. This distribution can be realized in a triangle-local model as follows: set the hidden variables $\alpha, \beta$ and $\gamma$ to be binary, with equal probability distributions where any one value is set to be $(1 + \sqrt{u^2 - v^2})/2$. If every party outputs $-1$ when the received hidden variables are equal and $+1$ otherwise, the distribution (C1) is recovered.

The point $(F_{AB}, F_{BC}, F_{AC}, F_{ABC})$ described by the intersection of Eqs. (A3b), (A3h), (A3l) and (A3n) is part of the polytope for all $1/\sqrt{2} \leq u \leq 1/2 \sqrt{1 + \sqrt{3} + 3^{2/3} - 5} + \sqrt{\frac{20}{\sqrt{3}} + 3^{2/3} - 5} - 10 - 3^{4/3} - 3^{2/3} \approx 0.8457$, so at least for this whole range $q_u^\pm(i, j, k)$ cannot be used for proving the triangle nonlocality of $P_u(a, b, c)$.

Appendix D: Inflation used and constraints imposed

Inflation, introduced by Wolfe et al. [25], is a powerful concept that enables the analysis of correlations in arbitrary causal structures, therefore including networks. Its underlying mechanism is proving through contradiction: in the context of networks, if a distribution can be generated between some parties by using certain sources, then one can consider which kinds of distributions one would be able to generate if given access to multiple copies of such parties and sources. The networks that are generated by arranging those copies are called inflations of the original network.

Let us now use inflation arguments to derive necessary conditions for a distribution to admit a model of the type given by Eq. (1). We begin assuming that such model exists, so we know the local variables $\alpha$, $\beta$ and $\gamma$, characterized by the distributions $\mu_{BC}(\alpha)$, $\mu_{AC}(\beta)$ and $\mu_{AB}(\gamma)$, respectively, and the responses $P_\alpha(a|\beta, \gamma)$, $P_\beta(b|\alpha, \gamma)$ and $P_\gamma(c|\alpha, \beta)$ of Alice, Bob and Charlie, respectively, which all combined give rise to $P(a, b, c)$. Having access to this information means that we can imagine duplicating the local variables and responses (by going to the manufacturer and buying new sources and measurement devices), and also cloning the information that the sources send to the parties. Then, one could imagine arranging the available sources and parties in the inflated network depicted in Fig. 3. The question is now, what are the properties of the distribution $p_{inf}(a^{\infty}, b^{\infty}, c^{\infty})$ that is generated in this new network?
shared variables. Therefore, there are four copies of each of the parties, each of which receives one different pair of copies of the corresponding

the web inflation in Refs. [25, 26], or as the hexagon web inflation. In this inflation there are two copies of each source, and

FIG. 3. Two depictions of the second level in the inflation hierarchy of Ref. [27] for the triangle scenario. It is also known as

probability distribution. It is possible also to consider more convoluted relations, where the right-hand side contains

For once, the distribution is well defined, i.e.,

\[ \sum_{\{a^{i,j}\}} \sum_{\{b^{k,l}\}} \sum_{\{c^{m,n}\}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) \geq 0 \quad \forall a^{i,j}, b^{k,l}, c^{m,n}, i, j, k, l, m, n, \quad (D1) \]

\[ \sum_{\{a^{i,j}\}} \sum_{\{b^{k,l}\}} \sum_{\{c^{m,n}\}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) = 1. \quad (D2) \]

The assumption that the sources and parties are all indistinguishable copies implies that the distribution is invariant

under a number of permutations of its elements. As an illustration, performing the swap \(\sigma_1 \leftrightarrow \sigma_2\) in Fig. 3, which
corresponds to the transformations \(b^{1,1} \leftrightarrow b^{1,2}\) and \(c^{1,n} \leftrightarrow c^{2,n}\), leaves the distribution invariant. More generally, if \(\pi\),
\(\pi'\) and \(\pi''\) are three independent permutations, we have that \(p_{\text{inf}}\) must satisfy

\[ p_{\text{inf}}(\{a^{\pi(i)}, \pi'(j)\}, \{b^{\pi'(k)}, \pi''(l)\}, \{c^{\pi''(m)}, \pi(n)\}) - p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) = 0 \quad \forall a^{i,j}, b^{k,l}, c^{m,n}, i, j, k, l, m, n, \pi, \pi', \pi''. \quad (D3) \]

Moreover, given that the sources and response functions in the inflation are copies of the elements in the original

model assumed, there exist marginals of \(p_{\text{inf}}\) that can be associated to elements of the original probability distribution
\(P(a, b, c)\). Firstly, note that if one marginalizes over the copies that have different indices on the left and on the right,
one is left with two copies of the original triangle scenario. This means that

\[ \sum_{a^{1,2}, b^{1,2}, b^{2,1}} \sum_{c^{1,2}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) = P(a^{1,1}, b^{1,1}, c^{1,1}) P(a^{2,2}, b^{2,2}, c^{2,2}) \quad \forall a^{1,1}, a^{2,2}, b^{1,1}, b^{2,2}, c^{1,1}, c^{2,2}. \quad (D4) \]

Similarly, another constraint satisfied is

\[ \sum_{a^{1,1}, b^{1,2}, b^{2,1}, c^{1,2}} \sum_{c^{1,2}} \sum_{c^{2,2}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) = P(a^{1,2}) P(b^{1,2}) P(c^{1,2}) \quad \forall a^{1,2}, b^{1,2}, c^{1,2}, \quad (D5) \]

where the right-hand side represents a product of single-party marginals of the original distribution \(P(a, b, c)\). Note

that Eqs. (D4-D5) relate combinations of probabilities in the inflation to products of the elements of the original

probability distribution. It is possible also to consider more convoluted relations, where the right-hand side contains

elements of the distribution in the inflation as well. Namely, one can consider the constraints

\[ \sum_{a^{1,1}} \sum_{b^{1,2}} \sum_{c^{1,1}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) - P(a^{1,1}) \sum_{a^{1,1}} \sum_{b^{1,1}} \sum_{c^{1,1}} p_{\text{inf}}(\{a^{i,j}\}, \{b^{k,l}\}, \{c^{m,n}\}) = 0 \quad \forall a^{1,1}, a^{2,1}, b^{2,1}, b^{2,2}, c^{1,2}, c^{2,2}, \quad (D6) \]

plus the equivalent under cyclic permutation of parties.

The constraints in Eq. (D4) are sufficient to define a hierarchy of inflations that asymptotically converges to the

characterization of all \(P(a, b, c)\) compatible with Eq. (1) [27]. However, the additional constraints (D5) and (D6)
have proved useful for further constraining the characterizations offered by inflation under restricted computational resources. The constraints in Eq. (D5), denoted as higher-degree relations in Ref. [27], allow to identify the $W$ distribution as incompatible with Eq. (1) [25]. In turn, Eq. (D6) are the linearized polynomial identification constraints. These constraints have been used in Ref. [10] for providing tighter characterizations of distributions compatible with the triangle scenario, and a comparison between the characterizations obtained with and without these constraints can be found in Ref. [28].

Recall that the conditions (D1-D6) are consequences of the correlations admitting a model of the type of Eq. (1). Thus, in order to see whether a distribution $P(a,b,c)$ admits such a model, one can consider the problem

$$\text{find } p_{\text{inf}} \text{ such that (D1), (D2), (D3), (D4), (D5), (D6)},$$

which, since all constraints are linear once $P(a,b,c)$ is defined, can be formulated as a linear program. If a solution to (D7) does not exist, then it is certified that the premise, namely that $P(a,b,c)$ satisfies Eq. (1), is false. The absence of a solution for Eq. (D7) can be demonstrated, via Farka’s lemma, in terms of a certificate of infeasibility. In Fig. 5 we show an exemplification of the certificates obtained when the different types of constraints are added to the linear program.

FIG. 5. Example of regions of the projection of the feasible polytope, $P_{\text{inf}}$, in the region defined by $(F_{AB} = F_{BC} = F_{AC} \equiv F_3, F_{ABC} \equiv F_3)$, for which the family of distributions $q_{\text{inf}}(i,j,k)$ is identified not to admit a triangle-local model of the form of Eq. (1) by the inflation linear programs with different types of constraints. (a) depicts $P_{0.8090}$ in blue. The orange region in (b) represents the distributions proved not to admit a model based on the standard inflation hierarchy, given by Eqs. (D1-D4). The green region in (c) represents the triangle-nonlocal distributions identified when the higher-order constraints of Eq. (D5) are added to the former. The red region in (d) represents the distributions that are proved not to admit a triangle-local model after adding the LPI constraints of Eq. (D6). Note that the linear program (D7) is infeasible for every point inside $P_{0.8090}$. Following the arguments in the main text, this is a proof that the distribution in Eq. (2) is triangle nonlocal for $u = 0.8090$. 

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FIG. 4. Types of marginals of the probability distribution in the inflation, $p_{\text{inf}}$, that can be associated to the original distribution. (a) represents the marginals used in the hierarchy constraints of Eq. (D4). These are sufficient to describe a convergent inflation hierarchy, but the remaining provide tighter characterizations at a fixed inflation level. (b) represents the marginals used in the higher-order relations of Eq. (D5). (c) represents the marginals used in the linearized polynomial identification constraints of Eq. (D6). Note that, in the latter, the whole dashed marginal is factorized in two: the stronger one, which is substituted by a known marginal of the original probability distribution, and the lighter one, which is a marginal of the inflation distribution by itself.
Appendix E: Relaxing Farkas’ lemma

In order to extend our derivation to the continuum, and thus prove that $P_u(a, b, c)$ is triangle-nonlocal in the whole range $0.7504 \leq u \leq 0.8101$, let us state explicitly Farkas’ lemma [29]:

$$
\exists \mathbf{x} \text{ s.t. } A \cdot \mathbf{x} \geq \mathbf{b} \iff \exists \mathbf{y} \text{ s.t. } \begin{cases} 
\mathbf{y} \geq \mathbf{0}, \\
\mathbf{y} \cdot A = \mathbf{0}, \\
\mathbf{y} \cdot \mathbf{b} > 0.
\end{cases}
$$

This is, if a linear program \[\text{find } \mathbf{x} \text{ s.t. } A \cdot \mathbf{x} \geq \mathbf{b}\] does not admit any feasible solution $\mathbf{x}$, Farkas’ lemma guarantees the existence of a vector $\mathbf{y} \geq \mathbf{0}$ that satisfies $\mathbf{y} \cdot A = \mathbf{0}$ and $\mathbf{y} \cdot \mathbf{b} > 0$. In the programs we consider [see Eq. (D7) in Appendix D], the vector $\mathbf{x}$ contains the probability elements of the distribution in the inflation, the vector $\mathbf{b}$ is built from the right-hand sides of Equations (D1)-(D6), and the matrix $A$ contains the coefficients accompanying the probability elements in the left-hand sides of Equations (D1)-(D6). Note that all the elements in $A$ are independent of $q_t u$, except for those encoding the LPI constraints. For these, some coefficients in the corresponding rows of $A$ are $c_u^t$ or $1 - c_u^t$, where $c_u^t = \sum_{j,k} q_u^t(1, j, k) = \sum_{i,j} q_u^t(i, j, 1) = \sum_{i,k} q_u^t(i, 1, k) = (1 - t + 2u^2)/2$. This is the reason behind the witnesses obtained being only applicable to a fixed $u$, since the condition $\mathbf{y}_u \cdot A(u) = \mathbf{0}$ is not robust to perturbations in $u$.

However, one does not need such strict conditions in order to have a guarantee of infeasibility. In a general setting, the vector $\mathbf{y}$ signals that no feasible solution $\mathbf{x}$ exists as long as $\max(\mathbf{y} \cdot A) < \mathbf{y} \cdot \mathbf{b}$, which is a condition robust to perturbations of both $A$ and $\mathbf{b}$. This is, the witness obtained at a given $u$ is also a witnesses for the distributions that satisfy $\max[\mathbf{y}_u \cdot A(u')] < \mathbf{y}_u \cdot \mathbf{b}(u')$. The procedure that we implement in the computational appendix [31] is based in this observation. Recursively, the program proves the triangle nonlocality of $P_u(a, b, c)$ for a particular value of $u$ by finding a Farkas’ certificate $\mathbf{y}_u$, computes the furthest value $u'$ for which the condition $\max[\mathbf{y}_u \cdot A(u')] < \mathbf{y}_u \cdot \mathbf{b}(u')$ guarantees that $P_{u'}(a, b, c)$ is triangle-nonlocal as well, and sets $u = u'$ to begin again.