THE STRONG LEFSCHETZ PROPERTY AND SIMPLE EXTENSIONS

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Abstract. Stanley [4] showed that monomial complete intersections have the strong Lefschetz property. Extending this result we show that a simple extension of an Artinian Gorenstein algebra with the strong Lefschetz property has again the strong Lefschetz property.

Introduction

Let \( K \) be a field, \( A \) be a standard graded Artinian \( K \)-algebra and \( a \in A \) a homogeneous form of degree \( k \). The element \( a \) is called a Lefschetz element if for all integers \( i \) the \( K \)-linear map \( a: A_i \rightarrow A_{i+k} \) (induced by multiplication with \( a \)) has maximal rank. One says that \( A \) has the weak Lefschetz property if there exists a Lefschetz element \( a \in A \) of degree 1. An element \( a \in A_1 \) for which all powers \( a^r \) are Lefschetz is called a strong Lefschetz element, and \( A \) is said to have the strong Lefschetz property if \( A \) admits a strong Lefschetz element. Note that the set of Lefschetz elements \( a \in A_1 \) form a Zariski open subset of \( A_1 \). The same holds true for the set of strong Lefschetz elements.

Assuming that the characteristic of \( K \) is zero and the defining ideal of \( A \) is generated by generic forms, it is conjectured that \( A \) has the strong Lefschetz property. Thus in particular, \( A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \) should have the strong Lefschetz property for generic forms \( f_1, \ldots, f_n \). Note that such an algebra is an Artinian complete intersection. It is expected that any standard graded Artinian complete intersection over a base field of characteristic 0 has the strong Lefschetz property. Stanley [4] and later J. Watanabe [5] proved this in case \( A \) is a monomial complete intersection. Stanley used the Hard Lefschetz Theorem to prove this result, while Watanabe used the representation theory of the Lie algebra \( sl(2) \).

As a main result of this paper we prove the following

Theorem: Let \( K \) be a field of characteristic 0, \( A \) be a standard graded Artinian Gorenstein \( K \)-algebra having the strong Lefschetz property, and let \( f \in A[x] \) be a monic homogeneous polynomial. Then the algebra \( B = A[x]/(f) \) has the strong Lefschetz property.

The proof only uses techniques from linear algebra. The result implies in particular Stanley’s theorem. More generally it implies that a complete intersection
\[ K[x_1,\ldots, x_n]/(f_1,\ldots, f_n) \text{ with } f_i \in K[x_1,\ldots, x_i] \text{ for } i = 1,\ldots, n \text{ has the strong Lefschetz property.} \]

1. **THE PROOF OF THE MAIN THEOREM**

Let \( A \) be a standard graded \( K \)-algebra and \( I \subset A \) a graded ideal. For convenience we will say that \( a \in A \) is Lefschetz for \( A/I \) if the residue class \( a + I \) is a Lefschetz element of \( A/I \).

In the proof of the main theorem we shall use the following two lemmata.

**Lemma 1.1.** Let \( A \) be a standard graded \( K \)-algebra, \( f, g \in A \) homogeneous elements which are nonzero divisors on \( A \). Then \( f \) is Lefschetz for \( A/(g) \) if and only if \( g \) is Lefschetz for \( A/(f) \).

**Proof.** Consider the long exact sequence for Koszul homology (see [1, Corollary 1.6.13])

\[ \cdots \rightarrow H_1(g; A) \rightarrow H_1(f, g; A) \rightarrow H_0(g; A) \xrightarrow{f} H_0(f, g; A) \rightarrow 0. \]

Since \( g \) is a non-zero divisor on \( A \) this yields the exact sequence

\[ 0 \rightarrow H_1(f, g; A) \rightarrow A/(g) \xrightarrow{f} A/(g) \rightarrow H_0(f, g; A) \rightarrow 0. \]

Similarly we obtain an exact sequence

\[ 0 \rightarrow H_1(f, g; A) \rightarrow A/(f) \xrightarrow{g} A/(f) \rightarrow H_0(f, g; A) \rightarrow 0. \]

Comparing this two exact sequences, the assertion follows. \( \square \)

**Lemma 1.2.** Let \( K \) be field of characteristic 0, \( A \) a standard graded Artinian \( K \)-algebra with strong Lefschetz property and \( f \in A[y] \) a monic homogeneous polynomial. Then for any strong Lefschetz element \( a \in A \) there exists a non-zero element \( c \in K \) such that \( f(a/c) \) is a Lefschetz element of \( A \).

**Proof.** Let \( f = y^d + a_1 y^{d-1} + \cdots + a_d \), and \( s = \max\{i: A_i \neq 0\} \). We may assume that \( d \leq s \) because otherwise the statement is trivial. For \( c \in K \) we set \( f_c = y^d + \sum_{i=1}^d c^i a_i y^{d-i} \). Let \( a \in A_1 \) be a strong Lefschetz element. Then \( a^d \) is a Lefschetz element, that is, for all \( i \) the multiplication map \( f_0(a): A_i \rightarrow A_{i+d} \) has maximal rank.

Fix \( i \leq s - d \) and \( K \)-bases of the nonzero \( K \)-vector spaces \( A_i \) and \( A_{i+d} \), and let \( D_c \) be the matrix describing the \( K \)-linear map \( f_c(a): A_i \rightarrow A_{i+d} \). Note that the entries of \( D_c \) are polynomial expressions in \( c \) with coefficients in \( K \). Now \( P_c(a) \) has maximal rank if and only if one maximal minor \( M_j(D_c) \) of \( D_c \) does not vanish. In particular, \( M_j(0) \neq 0 \) for some \( j_0 \). Since \( M_j(0) \) is a (non-zero) polynomial expression in \( c \) with coefficients in \( K \), there exist only finitely many \( c \in K \) such that \( M_j(0) = 0 \). Thus, since \( K \) is infinite, we have \( M_j(D_c) \neq 0 \) for infinitely many \( c \in K \), and so \( f_c(a): A_i \rightarrow A_{i+d} \) has maximal rank for infinitely many \( c \in K \). Since \( A \) has only finitely many non-zero components, we can therefore find \( c \in K \), \( c \neq 0 \) such that \( f_c(a) \) has maximal rank for all \( i \). Then \( a/c \in A_1 \) has the desired property, since \( f(a/c) = f_c(a)/c^d \). \( \square \)
By induction on \( r \) we may assume that \( c \in C \) has a strong Lefschetz element. Now Lemma 1.2 implies that \( f(c) \) is a Lefschetz element. Thus it suffices to show that for each \( r \geq 1 \) there exists an element \( b_r \in B_1 \) such that \( b_r^r \) is a Lefschetz element.

By Lemma 1.2 we may choose an element \( a \in A_1 \) such that \( f(a) \) is a Lefschetz element of \( A \). It follows that \( f(x) \) is Lefschetz for \( A[x]/(a - x) \). Thus by Lemma 1.1 the element \( b_1 = a - x \) is Lefschetz for \( B \).

In case \( r > 1 \), we may view \( f(y) \) as a polynomial in \( C[y] \) where \( C = A[x]/(x^r) \). By our assumption \( C \) has a strong Lefschetz element. Now Lemma 1.2 implies that we can find a strong Lefschetz element \( c \in C_1 \) such that \( f(c) \) is a Lefschetz element of \( C \). Since the strong Lefschetz elements form a nonempty Zariski open set in \( C_1 \), we may assume that \( c = a + \lambda x \) with \( a \in A_1 \) and \( \lambda \in K \), \( \lambda \neq 0 \). Applying the substitution \( x \mapsto b_r = \lambda^{-1}(x-a) \) it follows that \( f(x) \) is Lefschetz for \( A[x]/b_r^r \). Thus by Lemma 1.2 the element \( b_r^r \) is Lefschetz for \( A[x]/(f) \).

In order to complete the proof of the theorem it remains to be shown that if \( A \) is a standard graded Artinian Gorenstein \( K \)-algebra having the strong Lefschetz property, then \( A[x]/(x^q) \) has the strong Lefschetz property. We use Lemma 1.1 and show instead that if \( a \in A_1 \) is a strong Lefschetz element, then for all \( k \) the element \( x^q \) is Lefschetz for \( B = A[x]/(a + x)^k \).

In \( B \) we have

\[
x^k = - \sum_{j=0}^{k-1} \binom{k}{j} a^{k-j} x^j.
\]

By induction on \( r \) it follows that

\[
x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r-j-1}{r-k} \binom{r}{j} a^{r-j} x^j \quad \text{for} \quad r \geq k.
\]

Note that

\[
\binom{r-j-1}{r-k} \binom{r}{j} = \frac{k}{r-j} \binom{r}{j} \binom{k-1}{j},
\]

so that

\[
x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r}{k} \binom{k-1}{j} \frac{k}{r-j} a^{r-j} x^j
\]

for all \( r \geq k \). Thus for all \( r \geq 0 \) we have

\[
x^r = \sum_{j=0}^{k-1} c_{rj} a^{r-j} x^j
\]

for all \( \lambda \) and any \( \lambda - 1 \) in the field \( K \). This completes the proof of the theorem.
with
\[ c_{rj} = \begin{cases} \delta_{rj}, & \text{if } r \leq k - 1 \\ (-1)^{r-k-1} (\frac{r}{k}) (\frac{k-1}{j}) \frac{k}{r-j}, & \text{if } r \geq k, \end{cases} \]

where \( \delta_{rj} \) denotes the Kronecker symbol.

Now we show that the map \( \beta_i^q : B_i \to B_{i+q} \) given by multiplication with \( x^q \) has maximal rank.

We denote by \( \alpha_i^j : A_i \to A_{i+j} \) the \( K \)-linear map given by multiplication with \( a^j \). For each element \( ux^i \in A_{i-1}x^i \) we have

\[ x^q(ux^i) = \sum_{j=0}^{k-1} c_{q+i,j} a^{q+i-j} ux^j = \sum_{j=0}^{k-1} c_{q+i,j} \alpha_{t-i}^{q+i-j}(u)x^j. \]

Since for each \( j \) the \( K \)-vectorspace \( B_j \) has the direct sum decomposition

\[ B_j = \bigoplus_{i=0}^{k-1} A_{t-i}x^i, \]

the linear map \( \beta_i^q \) can be described by the following block matrix

\[ M = \begin{pmatrix} c_{q,0} \alpha_{t-0}^q & c_{q+1,0} \alpha_{t-1}^{q+1} & \cdots & c_{q+k-1,0} \alpha_{t-k+1}^{q+k-1} \\ c_{q,1} \alpha_{t-1}^{q-1} & c_{q+1,1} \alpha_{t-1}^q & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{q,k-1} \alpha_{t-k+1}^{q-k+1} & c_{q+1,k-1} \alpha_{t-k+1}^{q-k+2} & \cdots & c_{q+k-1,k-1} \alpha_{t-k+1}^q \end{pmatrix}. \]

Our aim is to show that \( M \) has maximal rank. Assume first that \( q < k \), then

\[ M = \begin{pmatrix} 0 & N \\ \text{id} & * \end{pmatrix}, \]

where \( N = (c_{q+j,i} \alpha_{t-j}^{q+j-i})_{i=0,\ldots,q-1, j=k-1,\ldots,k-1} \). It follows that \( M \) has maximal rank if and only if \( N \) has maximal rank. Thus the general case is treated if we can prove that for all \( q \) the matrix

\[ N = (c_{q+j,i} \alpha_{t-j}^{q+j-i})_{i=0,\ldots,s-1, j=r,\ldots,k-1} \quad \text{with} \quad s = \min\{q,k\} \quad \text{and} \quad r = \max\{q,k\} \]

has maximal rank.

We show this by applying certain block row and block column operations in order to simplify the matrix without changing its rank. The kind of operations we will apply are the following:

(i) multiplication of a block row or a block column of \( N \) with a non-zero rational number;

(ii) for \( d \in \mathbb{Q}, d \neq 0 \) and \( j < i \) compose \( d\alpha_{t-i}^{i-j} \) with each block \( c_{jt} \alpha_{t-j}^{j-t} \) of the \( j \)-th block column of \( N \) to obtain a \( j \)-th block column whose blocks are \( dc_{jt} \alpha_{t-j}^{j-t} \circ \alpha_{t-j}^{j-t} = dc_{jt} \alpha_{t-j}^{j-t} \), and subtract this new block column from the \( i \)-th block column to obtain the new \( i \)-th block column whose blocks are

\[ (c_{il} - dc_{jt}) \alpha_{t-l}^{i-l}, \quad l = 0, \ldots, s-1. \]
These operations only change the coefficients $c_{ji}$ of the block entries, that is, the matrix $N = (c_{q+j,i}q^{+q+1-i})_{j=0,...,s-1}^{j=0,...,s-1}$ will be transformed into a matrix of the form $N' = (c'_{q+j,i}q^{+q+1-i})_{j=0,...,s-1}^{j=0,...,s-1}$ with certain new coefficients $c'_{q+j,i} \in \mathbb{Q}$.

Consider the “coefficient matrix” $L = (c_{q+j,i})_{j=0,...,s-1}^{j=0,...,s-1}$ of $N$. Then the coefficient matrix $L'$ of $N'$ is obtained from $L$ by the following row and column operations:

(i) multiplication or division of a row or a column with a non-zero rational number;

(ii) subtraction of a multiple of the $j$th column from the $i$th column where $j < i$.

Next we intend to show that by these operations $L$ can be transformed into a matrix $L'$ such that all entries of $L'$ on the anti-diagonal are non-zero, while the entries below the anti-diagonal are all zero.

We first simplify $L$ by dividing each $j$th column by $(-1)^{i-k-1}(q^i)$ and each $i$th row by $(k-i)$. The result of these operations is the matrix

$$
\begin{pmatrix}
1/r & 1/(r+1) & \cdots & 1/(r+s-1) \\
1/(r-1) & 1/r & \cdots & 1/(r+s-2) \\
\vdots & \vdots & \ddots & \vdots \\
1/(r-s+1) & 1/(r-s+2) & \cdots & 1/r
\end{pmatrix},
$$

which we again denote by $L$.

We will use the following simple fact from linear algebra: suppose $F = (f_{ij})_{i,j=1,...,n}$ is an $n \times n$-matrix with coefficients in a field $K$. Then the following conditions are equivalent:

(a) the matrix $F$ can be transformed by operations of type (ii) into a matrix $F'$ with $f'_{ij} \neq 0$ for $i+j = n+1$, and $f'_{ij} = 0$ for $i+j > n+1$;

(b) $\det(F_i) \neq 0$ for $i = 1, \ldots, n$ where $F_i = (f_{kl})_{k=1,...,n}^{l=1,...,i}$.

Indeed, it is clear that (a) $\Rightarrow$ (b). Conversely, assuming (b) we have $\det(F_n) = f_{n1} \neq 0$. Thus by subtracting suitable multiples of the first column from the other columns we obtain a matrix $G = (g_{ij})$ with $g_{ni} = 0$ for $i = 2, \ldots, n$, and such that $\det(F_i) = \det(G_i)$ for $i = 1, \ldots, n$, where $G_i = (g_{kl})_{k=1,...,n}^{l=1,...,i}$. Applying the induction hypothesis to the matrix $G' = (g_{kl})_{k=1,...,n-1}^{l=2,...,n}$, the assertion follows.

Applying this result from linear algebra, we see that $L$ can be transformed by operations of type (ii) into the matrix $L'$ of the desired form if for all integers $0 \leq t < s$ the matrices of the shape

$$S = (1/(r-i+j))_{i,j=0,...,t}
$$

are non-singular. It is an easy exercise in linear algebra to show that this is indeed the case.

After all these operations our matrix $N$ is transformed into the matrix $N'$ whose anti-diagonal has the block entries

$$c'_1a_{q-r}, c'_2a_{q-r-1}, \ldots, c'_{s-1}a_{q-s-1}$$
with non-zero rational coefficients $c_i'$, and whose block entries below the anti-diagonal are all zero.

We will show that for $i = 0, \ldots, s - 1$ either all $\alpha_{t+q-r-i}^{r-s+2i+1}$ are injective maps, or else all $\alpha_{t+q-r-i}^{r-s+2i+1}$ are surjective maps. Then clearly $N'$ has maximal rank, and consequently $N$ has maximal rank.

For all integers $i$ and $j$ with $0 \leq i < j$ the maps

$$\alpha_{i}^{j-i}: A_i \rightarrow A_j$$

have maximal rank, by assumption. In particular, $\alpha_{i}^{j-i}$ is injective if $\dim A_i \leq \dim A_j$ and surjective if $\dim A_i \geq \dim A_j$.

Let $\sigma = \max\{i: A_i \neq 0\}$. Then, since $A$ is Gorenstein, the Hilbert function of $A$ is symmetric (see e.g. [1, Corollary 4.4.6, Remark 4.4.7]), that is,

$$\dim A_i = \dim A_{\sigma-i} \text{ for all } i,$$

and since $A$ has the weak Lefschetz property ($A$ even has the strong Lefschetz property), the Hilbert function of $A$ is unimodal (see e.g. [2, Remark 3.3]). It then follows that

$$\dim A_i \leq \dim A_j \text{ if and only if } i \leq \sigma - j.$$

Thus we conclude that

$$\alpha_{i}^{j-i} \text{ is } \begin{cases} \text{injective} & \text{if } i \leq \sigma - j, \\ \text{surjective} & \text{if } i \geq \sigma - j. \end{cases}$$

Thus in case of the maps $\alpha_{t+q-r-i}^{r-s+2i+1}$, we have to compare the size of the numbers $t + q - r - i$ and $\sigma - [(r-s+2i+1) + (t + q - r - i)] = \sigma - t - q + s - i - 1$. Since it does not depend on $i$ which of the two numbers is less than or equal to other, it follows $\alpha_{t+q-r-i}^{r-s+2i+1}$ is injective for all $i$, or $\alpha_{t+q-r-i}^{r-s+2i+1}$ is surjective for all $i$, as desired.

2. Some comments

As an immediate consequence of our main theorem we obtain

**Corollary 2.1.** Let $K$ be field of characteristic 0, and $A$ be an Artinian Gorenstein $K$-algebra. For $i = 1, \ldots, n$ let $f_i \in A[x_1, \ldots, x_i]$ be a homogeneous polynomial which is monic in $x_i$. Then the $K$-algebra

$$A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

has the strong Lefschetz property.

The result implies in particular that $K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ has the strong Lefschetz property, if for $i = 1, \ldots, n$, $f_i \in K[x_1, \ldots, x_i]$ is a homogeneous and monic polynomial in $x_i$. In the special case that $f_i = x_i^{a_i}$ for $i = 1, \ldots, n$, we obtain the theorem of Stanley [4]. The slightly more general result with the $f_i$ as described before, can also be deduced directly from Stanley’s theorem using the following result of Wiebe [5, Proposition 2.9]: let $I \subset K[x_1, \ldots, x_n]$ be a graded ideal, and assume that $K[x_1, \ldots, x_n]/\text{in}(I)$ has the strong Lefschetz property, where $\text{in}(I)$ is the initial ideal with respect to some term order. Then $K[x_1, \ldots, x_n]/I$ has the strong Lefschetz property.
In the above situation we have \( \text{in}(f_i) = x_i^{\deg f_i} \) for \( i = 1, \ldots, n \), if we choose the lexicographical order induced by \( x_n > x_{n-1} > \cdots > x_1 \). Since the initial terms of the generators form a regular sequence it follows that
\[
\text{in}(I) = (\text{in}(f_1), \ldots, \text{in}(f_n)) = (x_1^{a_1}, \ldots, x_n^{a_n}).
\]

In case the \( K \)-algebra \( A \) is not Gorenstein, the proof of our main theorem yields the following weaker result.

**Proposition 2.2.** Let \( K \) be a field, \( A \) a standard graded Artinian \( K \)-algebra having the strong Lefschetz property, and let \( f \in A[x] \) be a monic homogeneous polynomial. Then the algebra \( B = A[x]/(f) \) has the weak Lefschetz property.

**Proof.** Recall the following step in the proof of the main theorem: by Lemma \ref{lem:randomization} we may choose an element \( a \in A_1 \) such that \( f(a) \) is a Lefschetz element of \( A \). It follows that \( f(x) \) is Lefschetz for \( A[x]/(a - x) \). Thus by Lemma \ref{lem:lefschetz} \( b = a - x \) is Lefschetz for \( B \). \( \square \)

Analyzing the arguments in the proof of our main theorem we see that all results remain valid if the characteristic of the base field is large enough. More precisely we have

**Corollary 2.3.** Let \( K \) be a field and \( A \) an Artinian Gorenstein \( K \)-algebra with socle degree \( \sigma = \max \{ t : A_t \neq 0 \} \) and multiplicity \( e(A) = \sum_{t=0}^\sigma \dim_K A_t \). Let \( f \in A[x] \) be a homogeneous monic polynomial of degree \( q \). Then \( B = A[x]/(f) \) has the strong Lefschetz property if
\[
\text{char } K \geq \begin{cases} 2q + \sigma - 1, & \text{and } f = x^q, \\ \max \{ e(A), 2q + \sigma - 1 \}, & \text{otherwise.} \end{cases}
\]

**Proof.** In case \( f = x^q \) we must make sure that all the binomials in the expression
\[
x^r = (-1)^{r-k-1} \sum_{j=0}^{k-1} \binom{r-j-1}{r-k} a^{-j} x^j
\]
are units in the field \( K \), and this must be satisfied for all \( r = q + k \) where are less than or equal the socle degree of \( A[x]/(x^q) \). Since the socle degree of \( A[x]/(x^q) \) is equal to \( q + \sigma - 1 \), we therefore need that \( \text{char } K \) does not divide any prime number \( \leq 2q + \sigma - 1 \).

In the general case we had to apply Lemma \ref{lem:randomization}. For the proof of this lemma it was necessary that the field \( K \) has enough elements, so that for all the polynomials in \( c \) defined by the maximal minors considered in the proof we find a common element \( c \in K \) for which these polynomials do not vanish. This is possible if \( \text{char } K > e(A) \).

\( \square \)

We conclude this note with the following

**Example 2.4.** If \( A \) has the strong Lefschetz property and \( f \in A \) is a generic form. One expect that \( A/(f) \) has again the strong Lefschetz property. However, in general this is no the case. Indeed, let \( A = K[x_1, \ldots, x_5]/(x_1^4, x_2^4, x_3^4, x_4^4, x_5^2) \). Then \( A \) has the strong Lefschetz property. Let \( f \in A \) be a generic form of degree 8 and set \( B = A/(f) \). (We use the “Randomized” command of CoCoA to produce generic forms.)
The Hilbert series of $B$ is given by
\[
\text{Hilb}_B(t) = 1 + 5t + 14t^2 + 30t^3 + 51t^4 + 71t^5 + 84t^6 + 84t^7 + 70t^8 + 46t^9 + 16t^{10}.
\]
Let $b \in B$ be a generic linear form, and set $C = B/(b^9)$. Then
\[
\text{Hilb}_C(t) = 1 + 5t + 14t^2 + 30t^3 + 51t^4 + 71t^5 + 84t^6 + 84t^7 + 70t^8 + 45t^9 + 12t^{10}.
\]
It follows that the map $B_1 \xrightarrow{b^9} B_{10}$ is not surjective but also not injective because $\dim_K B_1 + \dim_K C_{10} = 5 + 12 > 16 = \dim_K B_{10}$. Thus $B$ does not have the strong Lefschetz property.

On the other hand it can be checked that $B$ has the maximal rank property, that is, any generic form in $B$ has maximal rank. Such an example seems to be new, see [3].

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