Deflection of broken ice caused by an external load moving along a channel

K N Zavyalova1, K A Shishmarev1 and T I Khabakhpasheva2
1 Altai State University, Barnaul, Russia
2 Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk, Russia
E-mail: kristina-zavyalova-1996@mail.ru

Abstract. The unsteady problem of a load moving along a channel covered with broken ice is considered. Deflection of the broken ice is described by the equation of floating liquid. The channel has a rectangular cross-section. The fluid in the channel is inviscid and incompressible. The flow caused by deflections of broken ice is potential. The external load is modeled by a smooth localized pressure distribution which moves along the centre line of the channel at a constant speed. With the help of the Fourier transform along the channel the original problem was reduced to the problem of the wave profile across the channel, which was solved by the method of separating variables. The deflections of the broken ice are studied for large times. The solution is presented in the form of sum of local deflection near the load and infinite system of waves propagating from the load with the speed of the load. Dispersion relations, phase and group speeds of these waves are found. The formation of gravity waves in a channel covered with broken ice depending on the speed of the load is studied.

1. Introduction
The motion of an external load on a broken ice in a channel is considered. It is assumed that the ice in the channel was previously destroyed, but the resulting mass of ice particles still represents some surface covering fluid in the channel. The problems of the destruction of the ice cover were studied in, for example, [1, 2, 3]. In the absence of elastic forces in the broken ice waves can have large amplitudes both in the region near to the load and in the distance, in particular, on the channel walls and near river structures. This may affect safety transportation. The problem of the ice response to the external load moving on the frozen ice cover in the channel is well studied [2, 4, 5].

Periodic waves in a frozen channel are investigated in [6, 7]. Deflections of an unbounded thin ice plate by moving load are investigated in [1, 8]. Deflections of semi-infinite thin ice plate are investigated in [9, 10]. The problems were solved within the linear theory of hydroelasticity. It was shown that for channels or/and semi-infinite ice plates, the walls play an important role for ice deflections. The ice response to an underwater body moving in a channel is studied in [5, 11]. The effect of periodic load on the ice cover is investigated in [12].

There are two main approaches to solving considered problems. In the first approach, one deals with the numerical solution of a quasistationary problem in a coordinate system moving together with a load [2]. In such a system, the ice deflections and the velocity potential of the flow are time independent and are determined by numerical methods. The second approach is to solve the corresponding non-stationary problem, where the ice deflections parameters are
determined by asymptotic methods as the limiting characteristics of forced unsteady waves that are established for large times [4].

In the present paper, we consider the problem of a smooth localized pressure moving along broken ice in a channel. The boundary conditions for the deflection of broken ice are absent on the channel walls. The main attention is given to the broken ice deflections. The second approach described above is used.

2. Formulation of the problem
The deflections of broken ice caused by a load moving along a channel are considered. The channel is of rectangular cross section with depth $H$, $(-H < z < 0)$, width $2L$, $(-L < y < L)$ and of infinite extent in the $x$ direction. Here $(x, y, z)$ is the Cartesian coordinate system. The fluid in the channel is inviscid and incompressible with constant density $\ell$. The liquid is covered with the broken ice of constant thickness $h_i$. The broken ice is modelled by floating liquid with a density equal to the characteristic density of ice. Friction between the ice particles is not taken into account in this model. We assume the flow in the channel is potential. The velocity potential $\varphi$ satisfies the Laplace equation

$$\Delta \varphi(x, y, z, t) = 0 \quad (-\infty < x < \infty, -L < y < L, -H < z < 0),$$

boundary conditions

$$\varphi_y = 0 \quad (y = \pm L), \quad \varphi_z = 0 \quad (z = -H),$$

linearized kinematic condition and Bernoulli integral

$$\varphi_z(x, y, 0, t) = w_t(x, y, t), \quad p(x, y, 0, t) = -\rho_l g w(x, y, t) - \rho w_t(x, y, 0, t),$$

where $p(x, y, 0, t)$ is the hydrodynamic pressure on the broken ice/liquid interface, $g$ is the gravitational acceleration. The liquid is at rest,

$$\varphi(x, y, z, t) \to 0 \quad (|x| \to \infty),$$

in the far field at each finite time instant. Initially $\varphi = 0$ and $\varphi_t = 0$. The deflection of the broken ice $w(x, y, t)$ satisfies the equation of floating liquid [13]

$$M w_{tt} = -P(x - Ut, y) + p(x, y, 0, t) \quad (-\infty < x < \infty, -L < y < L),$$

where the external pressure $P(x - Ut, y)$ moves along the ice in $x$-direction with constant speed $U$. Initial conditions and the condition at infinity for the ice elevation are

$$w(x, y, 0) = w_0(x, y), \quad w_t(x, y, 0) = 0, \quad w \to 0 \quad (|x| \to \infty),$$

where $w_0(x, y)$ is the solution of the stationary equation $\rho_l g w_0(x, y) - P(x, y) = 0$. Boundary value problem (1) – (6) describes the response of the broken ice to a given external load $P(x, y, t)$. Here $M = \rho_i h_i$ is the mass of the broken ice per unit area, $\rho_i$ is density of the broken ice. We restrict ourselves here to symmetric load, which is modeled by the following smooth and localized function

$$P(x - Ut, y) = P_0 P_1 \left( \frac{x}{L} \right) P_2 \left( \frac{y}{L} \right) \quad (-\infty < x < \infty, -L < y < L),$$

$$P_1(X/L) = \begin{cases} \frac{\cos(c_1 X/L) + 1}{2} & (c_1 |X|/L < 1), \\ 0 & (c_1 |X|/L \geq 1), \end{cases}$$
3. Method of the solution

The problem (1)–(7) is solved using the Fourier transform in the $x$ direction. We assume that the condition (6) is satisfied for any finite $t$, then applying the Fourier transform to the broken ice equation provides

$$Mw_{tt}^F = -\rho_\ell\varphi_t^F - \rho_\ell gw^F - P^F(\xi, y)e^{-i\xi Ut},$$

with initial conditions

$$w^F(x, y, 0) = w_0^F(\xi, y), \quad w_t^F(\xi, y, 0) = 0,$$

where

$$w^F(\xi, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x, y, t)e^{-ix} dx, \quad w(x, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w^F(\xi, y, t)e^{ix} dx,$$

and initial and boundary conditions

$$\varphi_{yy}^F + \varphi_{zz}^F = \xi^2 \varphi^F \quad (-L < y < L, -H < z < 0),$$

$$\varphi_{y}^F = 0 \quad (y = L, z = -H), \quad \varphi_{z}^F = 0 \quad (z = 0).$$

The Fourier variable $\xi$ plays role of a parameter here. We seek the deflection $w^F(\xi, y, t)$ by using the method of separating variables $y$ and $t$

$$w^F(\xi, y, t) = \sum_{n=0}^{\infty} a_n(\xi, t)\psi_n(y),$$

where $\psi_n(y) = \cos\left(\frac{\pi ny}{L}\right)$ and functions $a_n(\xi, t)$ are to be determined. The solution of the problem (10)–(11) is

$$\varphi^F(\xi, y, z, t) = \sum_{n=0}^{\infty} a_n(\xi, t)\psi_n(y)\cosh\left(\sqrt{\frac{\pi n}{L}} + \xi^2 (H + z)\right).$$

Substituting (12) and (13) to (8) gives

$$M \sum_{n=1}^{\infty} a_{n,tt}(\xi, t)\psi_n(y) = -\rho_\ell \sum_{n=1}^{\infty} \frac{a_n(\xi, t)\psi_n(y)\cosh\left(\sqrt{\frac{\pi n}{L}} + \xi^2 (H + z)\right)}{\sqrt{\pi n/L}^2 + \xi^2 \sinh(\sqrt{\pi n/L}^2 + \xi^2 H)} - \rho_\ell g \sum_{n=1}^{\infty} a_n(\xi, t)\psi_n(y) - P^F(\xi, y)e^{-i\xi Ut}.$$
Multiplying (14) by \( \psi_n(y) \) and then integrating the result in \( y \) from \(-L\) to \(L\) we arrive at the infinite system of ordinary differential equations of second order for \( a_n(\xi, t) \)

\[
\frac{d^2 a_n}{dt^2} + \omega_n^2(\xi) a_n = H_n(\xi)e^{-itU t} \quad n = 0, 1, 2, ... \tag{15}
\]

where \( H_n(\xi) = \frac{p_n(\xi)\sqrt{1+\xi^2}}{2L(M\sqrt{1+\xi^2}+\rho e\coth(\sqrt{1+\xi^2}H))} \). The initial conditions for the equation (15) follow from (9)

\[ a_n(\xi, 0) = \frac{H_n(\xi)}{\omega_n^2(\xi)}, \quad \frac{da_n}{dt}(\xi, 0) = 0. \]

After \( a_n(\xi, t) \) are found we substitute these functions into the series (12) and obtain the deflection \( w(x, y, t) \) in the moving coordinate system \((X, y, z)\) using the inverse Fourier transform

\[
w(X, y, t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_n(\xi, t) \cos(\pi n/L) e^{i\xi X} d\xi = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} w_n(X, y, t),
\]

\[
w_n(X, y, t) = \int_0^{\infty} H_n(\xi) \psi_n(y) \left( \frac{2\cos(\xi X)}{\omega_n^2 - \omega_n^2} + \frac{\omega_n^4 e^{i\xi U}}{2\omega_n} \left[ \frac{e^{i(\omega_n+U)\xi}}{\omega_n+U} - \frac{e^{-i(\omega_n-U)\xi}}{\omega_n-U} \right] - \frac{\rho e g}{M\sqrt{1+\xi^2}^2 + \xi^2 + \rho e \coth(\sqrt{1+\xi^2}H)} \right) d\xi,
\]

where \( \omega_n(\xi) \) are the dispersion relations for the broken ice (\( \xi \) is a wave number here). Within the considered model, the dispersion relations are obtained by solving the problem (1) – (7) in the form \( w(x, y, t) = F(y) \cos(\xi x - \omega t) \) without external load, \( P = 0 \). In the channel there are infinitely many dispersion relations. Frequencies \( \omega_n \) increase with increase of the number \( n \) for each fixed \( \xi \), \( \omega_0(\xi) < \omega_1(\xi) < \omega_2(\xi) < ... \). The equation for the \( \omega_n(\xi) \) is

\[
\omega_n^2(\xi) = \frac{\rho e g \sqrt{(\pi n/L)^2 + \xi^2}}{M\sqrt{(\pi n/L)^2 + \xi^2 + \rho e \coth(\sqrt{(\pi n/L)^2 + \xi^2}H)}}, \quad n = 0, 1, 2, ...
\]

The dispersion relations for one set of the parameters of the problem are shown in Figure 3 (a). Phase \( c_n \) and group \( c_n^g \) speeds are shown in Figure 3 (b). The phase speed are shown by solid lines. The group speeds by dashed lines. Only one frequency, \( \omega_0(\xi) \), equals to 0 for \( \xi = 0 \). It means that only one phase speed \( c_0 \) has a finite value at \( \xi = 0 \), where \( c_n \rightarrow +\infty \) for \( \xi \rightarrow 0, n = 1, 2, .... \). Note that denominators in (16) can be equal to 0 for some values of \( \xi \), if \( \omega_n - U \xi = 0 \). This condition corresponds to the condition \( c_n - U = 0 \). There can be two cases, when \( U \) intersect \( c_0 \) and when does not. The load speed for both cases is shown by black lines in Figure 3 (b). There are two common methods for solving the considered problem. In the first method the integrals (16) are calculated numerically for a given time. In the second method the behavior of integrals (16) are investigated for large times. In both cases, it is necessary to carefully calculate these improper integrals. We restrict ourselves with the second method here. Following [4] the deflection \( w_n(X, y, t) \) for large times \( t \) will be presented in the form

\[
w_n(X, y, t) = w_n^{loc}(X, y) - A_n \psi_n(y) \sin(\xi_n X) G_n(X),
\]

where \( w_n^{loc}(X, y) \) is the local deflection near the load and \( A_n \psi_n(y) \sin(\xi_n X) G_n(X) \) is the wave, propagating from the load with its speed. The equation for the amplitude \( A_n \) is

\[
A_n = \frac{2\pi H_n(\xi_n)}{(\omega_n^2(\xi_n) - U)\xi_n U},
\]

where \( \xi_n \) is the solution of the equation \( \omega(\xi) - \xi U = 0 \). Function \( G_n(X) \) is so called cut-off function and shows in which direction the wave propagates. Because the group speeds are smaller than phase speeds for the broken ice within the considered model, the waves will propagate behind the load. More detailed explanation of the cut-off function can be found in [4].
4. Numerical results

The problem of the broken ice deflection is studied for the parameters: \( \rho_i = 1024 \text{ kg/m}^3 \), \( \rho_l = 917 \text{ kg/m}^3 \), \( E = 4.2 \cdot 10^9 \text{ N/m}^2 \), \( L = 10 \text{ m} \), \( H = 5 \text{ m} \), \( h_i = 0.1 \text{ m} \), \( P_0 = 1000 \text{ N/m}^2 \). The size of the load is 2x2 meters, \( c_1 = c_2 = 10 \). The speed \( U \) is varied in our calculations.

The convergence of the amplitudes \( A_n \) is shown in the figure 4 (a) for three different speeds of the load \( U = 3, 4 \) and 5 m/s. It was found that for accurate calculation of the deflections of broken ice it is necessary to take into account from 20 to 30 terms in the series (12). There are infinitely many waves which propagate behind the load at any speed. In the channel there is an infinite number of dispersion relations, and the line of the speed of the load will always intersect the phase speeds, except for the first one for \( U > c_0(0) \). In contrast to the problem of a load moving along the solid ice cover in a channel [2], we need to account many terms in (12). The maximum value of \( A_n \) for \( n = 0, ..., 30 \) as a function of the speed of the load is shown in figure 4 (b). The maximum deflection of broken ice for the considered parameters is observed for the speed of the load \( U \approx 1.8 \text{ m/s} \). Note that the behavior of the amplitude \( A_0 \) differs from other amplitudes. We can relate this to the unique behavior of \( c_0(\xi) \) from other phase speeds.

The 3D shape of the broken ice deflection for \( U = 5 \text{ m/s} \) with 5 terms in the series 12 is shown in figure 3 (a) and with 20 terms in the series 12 Figure 3 (b). In general, both figures have a similar shape. The orders of amplitudes also coincide. The wave shape in the form of a wedge is clearly traced. Increase \( n \) details the waveform.
5. Conclusion
The mathematical model of unsteady waves in the broken ice caused by an external load moving along a channel has been considered. The formulation of the problem was presented and discussed. The broken ice was modeled by the floating liquid. The problem was solved by the method described in [4]. With the help of the Fourier transform along the channel the original problem was reduced to the problem of the wave profile across the channel, which was solved by the method of separating variables.

The problem was solved numerically. The solution is presented in the form of sum of local deflection near the load and infinite system of waves propagating from the load with the speed of the load. It has been shown that these waves propagate only behind the load within considered model. Dispersion relations, phase and group speeds of these waves are found. The amplitude of the waves is decay with the increase of their frequencies. In order to get accurate solution it is required to take into account from 20 to 30 number of the waves. The maximum amplitude was observed for the $U$ lesser than maximum of the phase speed for the wave with lowest frequency. Increase number of considered waves details the waveform, but for the rough estimation of the shape of the broken ice deflections and average amplitude small number of the waves can be considered. in this paper friction between ice particles is not taken into account. To determine the maximum amplitudes on the channel walls, it is required to consider a model including viscosity of ice.

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References
[1] Squire V, Hosking R., Kerr A and Langhorne P 1996 Moving Loads on Ice (Dordrecht: Kluwer Academic Publishers)
[2] Shishmarev K, Khabakhpasheva T and Korobkin A 2016 Appl. Ocean Res. 59 313–26
[3] Kozin V M 2007 Resonance Method of Breaking of Ice Cover. Inventions and Experiments (Penza: Akademija Estestvoznanija) (in Russian)
[4] Khabakhpasheva T, Shishmarev K and Korobkin A 2019 Appl. Ocean Res. 86 154–65
[5] Shishmarev K, Khabakhpasheva T and Korobkin A 2019 Appl. Ocean Res. 91 101877
[6] Korobkin A, Khabakhpasheva T and Papin A 2014 Eur. J. Mech. B-Fluid 47 166–75
[7] Batyaev E A and Khabakhpasheva T I 2015 Fluid Dyn. 50 775–88
[8] Zhetskaya V D 1999 J. Appl. Mech. Tech. Phys. 40 770–5
[9] Brocklehurst P 2012 Hydroelastic Waves and Their Interaction with Fixed Structures (Norwich: UEA)
[10] Starova I V and Tkacheva L A 2017 J. Phys.: Conf. Ser. 894 012092
[11] Savin A A and Savin A S 2013 Fluid Dyn. 48 303–9
[12] Tkacheva L A 2017 Fluid Dyn. 52 219–29
[13] Gabov S A and Sveshnikov A G 1991 J. Sov. Math. 54 979–1041