The nonmodal kinetic theory for the electrostatic instabilities of a plasma with a sheared Hall current.

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The kinetic theory for the instabilities driven by the Hall current with a sheared current velocity, which has the method of the shearing modes or the so-called non-modal approach as its foundation, is developed. The developed theory predicts that in the Hall plasma with the inhomogeneous electric field, the separate spatial Fourier mode of the perturbations is determined in the frame convected with one of the plasma components. Because of the different shearing of the ion and electron flows in the Hall plasma, this mode is perceived by the second component as the Doppler-shifted continuously sheared mode with time-dependent wave numbers. Due to this effect, the interaction of the plasma components forms the nonmodal time-dependent process, which should be investigated as the initial value problem. The developed approach is applied to the solutions of the linear initial value problems for the hydrodynamic modified two-stream instability and the kinetic ion-sound instability of the plasma with a sheared Hall current with a uniform velocity shear. These solutions reveal that the uniform part of the current velocity is responsible for the modal evolution of the instability, whereas the current velocity shear is the source of the development of the nonmodal instability with exponent growing with time as \( \sim (t - t_0)^3 \).

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I. INTRODUCTION

The crossed electric, $E$, and magnetic, $B$, fields configuration is frequently observed in fusion$^1$ and space$^2$ plasmas. This configuration is common to a large variety of the so-called $E \times B$ plasma devices$^3$, which includes the Hall-thrusters$^{4,5}$, cylindrical and planar magnetrons$^6$, and numerous other applications. Depending on the plasma and fields parameters, plasma response on these fields is very different.

The flow of the collisionless plasma with strongly magnetized electrons and ions forms in the spatially homogeneous crossed $E$, $B$ fields. The velocity of this flow, $V_0 = cE/B$, is the same for all particles such that this flow produces no current in a charge-neutral plasma and does not have any effects on the plasma stability. However, the sheared $E \times B$ poloidal rotation of the tokamak edge with spatially inhomogeneous electric field suppresses the instabilities in the drift - frequency range$^1$, which are responsible for the anomalous transport of plasma. This process is of critical importance for the formation and control of the high confinement mode of operation, or H-mode.

Contrary to the fusion plasmas, the Larmor radius of ions in the $E \times B$ devices is not small with respect to the dimensions of the system and the ions are considered as unmagnetized, while electrons are strongly magnetized. The relative motion of the unmagnetized ions and the strongly magnetized electrons drifting with velocity $V_0 = cE/B$ in such a plasma (generally referred to as the Hall plasma) forms the Hall current. This specific current, which is absent in plasmas with all magnetized species, is the source of numerous current-driven instabilities which have been observed experimentally$^7-11$ and in simulations$^{12-15}$, and were investigated analytically$^{16-28}$. The discovered instabilities are very dependent on the specific conditions and regions of a particular device and develops in a large range of frequencies and wavelengths which includes large scale low frequency 'rotating spokes'$^{14,29-31}$, the modified two-stream (MTS) instability and ion sound (IS) and lower hybrid instabilities with frequencies between the ion cyclotron and electron cyclotron frequencies, and the submillimeter electron cyclotron drift instabilities$^{28,32}$ in the MHz frequency range. Important set of the Hall plasma instabilities is the gradient-drift plasma instabilities$^{16-18,22-27}$ which develop in spatially inhomogeneous Hall plasma due to the combined effect of the Hall current and of the gradient-drift modes formed by the plasma density and temperature inhomogeneity. A typical example of these instabilities are the Simon-Hoh instabilities$^{16-18,23}$ which
have frequency much above the ion cyclotron frequency, but below the electron cyclotron frequency. It is generally believed that turbulence powered by the instabilities\textsuperscript{33–35} is responsible for anomalous transport of the Hall plasma and is considered as a source of the experimentally detected anomalous electron mobility\textsuperscript{36} in Hall plasma thrusters.

The stability theory of the Hall plasmas historically bases on the normal mode analysis. It successfully identifies the waves and instabilities in the Hall plasmas by employing the local approximation and modal approach for the homogeneous or weekly inhomogeneous plasma, which assumes that the plasma perturbations have a structure of a plane wave $\sim \exp (ikr - i\omega t)$. In the case of the spatially inhomogeneous plasma, the nonlocal analysis of plasma stability is performed\textsuperscript{24} assuming that the perturbations have a form $\sim \phi (x) \exp (ik_y y + ik_z z - i\omega t)$ in the nonuniform along coordinate $x$ plasma and solving the eigenfunction-eigenvalue problem for the mode structure $\phi (x)$ and frequency $\omega$.

Generally, the electric and magnetic fields in the $E \times B$ devices are spatially inhomogeneous, and the corresponding Hall current is spatially inhomogeneous and sheared. It was found in Ref.\textsuperscript{38}, where the hydrodynamic theory of the modified Simon-Hoh (MSH) instability of a plasma with a sheared Hall current was developed, that the local approximation, which admits the application of the modal plane wave approach to the stability analysis of the Hall plasma, should be revised when it applies to the sheared Hall current. The solution of the initial value problem in Ref.\textsuperscript{38}, instead of application of the spectral transform in time, discovered nonmodal exponential growth of the perturbations with time as $\sim \exp a (t - t_0)^3$ for this instability. This growth is missed in the normal mode analysis. It was found that this nonmodal growth dominates the normal mode growth when the current velocity shearing rate is above the growth rate of the MSH instability and includes also the nonmodal growth of the perturbations which are subcritical for the MSH instability of the plasma with uniform Hall current. This result confirms the general conclusion derived earlier in Refs.\textsuperscript{39–49}, that the nonmodal effects of the sheared flow as well as the derived in Ref.\textsuperscript{38} nonmodal effects of the sheared current, are missed completely in the usual normal modes analysis and the investigation of the stability of sheared flows and sheared currents needs more elaborated analysis grounded on the methodology of the sheared modes and solution of the corresponding initial value problems.

Kinetic effects, such as finite-Larmor-radius effects, the Landau and cyclotron damping, and existence of numerous kinetic instabilities, which are naturally not involved in the fluid
description of plasma shear flows, require the development of a kinetic theory of stability of the Hall plasma with a sheared Hall current. In this paper, we present an analytical nonmodal approach grounded on the methodology of shearing modes to the kinetic theory of instabilities driven by the sheared current. The governing equations of this theory for the electrostatic instabilities are derived in details in Sec. II. The application of the developed approach to the theory of the MTS and of the kinetic IS instabilities of a plasma with a sheared Hall current are given in Secs. III and IV. Conclusions are given in Sec. V.

II. THE SHEARING MODES APPROACH TO THE THEORY OF THE INSTABILITIES DRIVEN BY THE SHEARED CURRENT

Our theory bases on the system of Vlasov equations for electrons and ions and the Poisson equation for the perturbed electrostatic potential. In this paper, we consider a plasma in the linearly changing electric field $E_0 (r) = E_0 (r = 0) + E'_0 x e_x$ with $E'_0 = \partial E_0 / \partial x = \text{const}$, directed across the uniform magnetic field $B = B e_z$ pointed along the coordinate $z$. The strength of the magnetic field is such that the ion Larmor radius $\rho_i$ is much larger than the characteristic plasma length scale $L$, whereas the electron Larmor radius $\rho_e$ is much less than $L$. We will consider the electrostatic perturbations with a frequency much above the ion cyclotron frequency $\omega_{ci}$. The evolution of the magnetized electrons is governed by the Vlasov equation for the electron distribution function $F_e (v, r, t)$,

$$
\frac{\partial F_e}{\partial t} + v \frac{\partial F_e}{\partial r} + \frac{e}{m_e} \left( (E_0 + E'_0 x) e_x + \frac{1}{c} [v \times B_0] - \nabla \varphi (r, t) \right) \frac{\partial F_e}{\partial v} = 0.
$$

(1)

The evolution of the unmagnetized ions is governed by the Vlasov equation for $F_i (v, r, t)$,

$$
\frac{\partial F_i}{\partial t} + v \frac{\partial F_i}{\partial r} + \frac{e}{m_i} \left( (E_0 + E'_0 x) e_x - \nabla \varphi (r, t) \right) \frac{\partial F_i}{\partial v} = 0.
$$

(2)

The potential $\varphi (r, t)$ in Eqs. (1) and (2) is determined by the Poisson equation,

$$
\Delta \varphi (r, t) = -4\pi \sum_{\alpha=i,e} e_\alpha n_\alpha (r, t)
= -4\pi \sum_{\alpha=i,e} e_\alpha \int f_\alpha (v, r, t) d\nu,
$$

(3)

where $f_\alpha (v, r, t) = F_\alpha (v, r, t) - F_{0\alpha} (v)$ is the perturbation of the electron ($\alpha = e$) and ion ($\alpha = i$) distribution functions. The simplest solutions to Eqs. (1), (2) may be obtained
applying the so called local approximation. Usually, when the local approximation supposed to apply to the Vlasov equation (see, for example, Ref. [37]) for the magnetized electrons, the transformation of Eq. (1) to the frame of references that moves with velocity \( \mathbf{V}_{e0}(r) = c\mathbf{E}_0(x) \times \mathbf{B}/B^2 \) in the electron velocity space, but unchanged in the configuration space, is employed. Also, the transformation to the frame of references that moves with velocity \( \mathbf{V}_{i0}(r) \approx (e/m_i)\mathbf{E}_0t \) in the ion velocity space, but unchanged in the configuration space, we employ to Eq. (2) for the unmagnetized ions. With new velocities \( \mathbf{v}_e = \mathbf{v} - \mathbf{V}_{e0}(r) \) and \( \mathbf{v}_i = \mathbf{v} - \mathbf{V}_{i0}(r) \) Eqs. (1) and (2) become

\[
\frac{\partial F_e}{\partial t} + \mathbf{V}_{e0}(x) \frac{\partial F_e}{\partial y} + \mathbf{v}_e \frac{\partial F_e}{\partial \mathbf{r}} + e \frac{\partial \varphi}{\partial \mathbf{v}_e} \left( \frac{1}{c} \mathbf{v}_e \times \mathbf{B} - \nabla \varphi(r, t) \right) = 0,
\]

\[
\frac{\partial F_i}{\partial t} + \mathbf{V}_{i0}(x) \frac{\partial F_i}{\partial y} + \mathbf{v}_i \frac{\partial F_i}{\partial \mathbf{r}} - e \frac{\partial \varphi}{\partial \mathbf{v}_i} \left( \frac{1}{c} \mathbf{v}_i \times \mathbf{B} - \nabla \varphi(r, t) \right) = 0,
\]

where it was assumed in Eq. (1) that the electron flow velocity shear \( V'_e = dV_{e0}(x)/dx = -cE_0/B = const \) is much less than the electron cyclotron frequency \( \omega_{ce} \). The local approximation grounds on the assumption that all modes being considered have wavelengths significantly shorter than the spatial scale length \( L_{V_e,i}(x) \) velocities inhomogeneities, i.e.

\[
L_{V_e,i}, k_x \gg 1.
\]

With local approximation, the solutions of both Vlasov equation for the perturbations \( f_e \) and \( f_i \) of the equilibrium electron distribution functions \( F_{e0} \) and \( F_{i0} \) are derived in the modal form of a plane wave \( \sim \exp(ikr - i\omega t) \), considering velocities \( V_{e0,i0}(x) \) as spatially homogeneous. The employment of the obtained solutions \( f_e(\mathbf{v}_e, k, \omega - k_yV_{e0}) \) and \( f_i(\mathbf{v}_i, k, \omega - k_yV_{i0}) \) in the Fourier transformed Poisson equation gives the well known local dispersion equation

\[
1 + \varepsilon_e(k, \omega - k_yV_{e0}(x)) + \varepsilon_i(k, \omega - k_yV_{i0}(x)) = 0
\]

as for the instabilities driven by the spatially uniform current with current velocity \( U = V_{e0} - V_{i0} \). We found, however, that the condition (6) of the local approximation is not sufficient for the application of the modal approach to the stability analysis of the plasma with a sheared Hall current. In this paper, we derive the solutions to the Vlasov-Poisson system (4), (5) for \( f_e, f_i \) and (3) as of the initial value problem without application of
the spectral transforms over time variable. On this way, we will find the above mentioned solutions of the modal type, and derive additional conditions which are necessary for the validity of the modal solutions. Also, we find the nonmodal solutions, which are missed in the conventional normal mode analysis.

The first step of our approach to the solution of the system (4), (5), and (3) is the transformation of the spatial coordinates r in Eqs. (4) for \( F_b \), determined in the laboratory frame, to the coordinates \( r_e \) determined in the frame moving with velocity \( \mathbf{V}_{e0}(x) = c \left( E_0 + E'_0 x \right) \mathbf{e}_y / B = (V_0 + V'_0 x) \mathbf{e}_y \) of a sheared equilibrium electron flow with uniform velocity shear \( V'_0 \). With coordinates \( x_e, y_e, z_e \) and velocities \( v_{ex}, v_{ey}, v_{ez} \) determined in the convected electron frame by the relations

\[
x = x_e, \quad y = y_e + V_{e0} t + V'_e x_e t, \quad z = z_e,
\]

\[
v_{ex} = v_{ex}, \quad v_y = v_{ey} + V_{e0} + V'_e x_e, \quad v_z = v_{ez},
\]

where it was assumed that the electric field \( \mathbf{E}_0 \) emerges at time \( t = 0 \), Eq. (4) becomes

\[
\frac{\partial F_e}{\partial t} + v_{ex} \frac{\partial F_e}{\partial x_e} + (v_{ey} - v_{ex} V'_e t) \frac{\partial F_e}{\partial y_e} + \omega_{ce} v_{ey} \frac{\partial F_e}{\partial v_{ex}} - \omega_{ce} v_{ex} \frac{\partial F_{e0}}{\partial v_{ey}} - e \frac{\partial \varphi}{\partial x_e} \left( \varphi_{e} V'_e t \right) \frac{\partial F_e}{\partial v_{ex}} + v_{ez} \frac{\partial F_e}{\partial z_e} = 0.
\]

The explicit spatial inhomogeneity introduced by the electric field \( \mathbf{E}_0 \) is absent in the Vlasov equation (9). With electron guiding center coordinates \( X_e, Y_e \), determined in the electron convective frame,

\[
x_e = X_e - \frac{v_{e \perp}}{\omega_{ce}} \sin (\phi_1 - \omega_{ce} t),
\]

\[
y_e = Y_e + \frac{v_{e \perp}}{\omega_{ce}} \cos (\phi_1 - \omega_{ce} t) + V'_e t (X_e - x_e),
\]

\[
z_e = z_{e1} + v_{ez} t,
\]

where \( \phi = \phi_1 - \omega_{ce} t, \quad v_{ex} = v_{e \perp} \cos \phi, \quad v_{ey} = v_{e \perp} \sin \phi \), Eq. (9) has the most simple form

\[
\frac{\partial F_e}{\partial t} + \frac{e}{m_e} \frac{\omega_{ce}}{v_{e \perp}} \left( \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F_e}{\partial v_{e \perp}} - \frac{\partial \varphi}{\partial v_{e \perp}} \frac{\partial F_e}{\partial \phi_1} \right) + \frac{e}{m_e} \frac{\omega_{ce}}{v_{e \perp}} \left( \frac{\partial \varphi}{\partial v_{e \perp}} \frac{\partial F_e}{\partial \phi_1} - \frac{\partial \varphi}{\partial \phi_1} \frac{\partial F_e}{\partial v_{e \perp}} \right) = 0.
\]

The potential \( \varphi \) in Eqs. (9), (11) is determined in the same electron convective-sheared coordinates (8) or (10), and may be presented by the Fourier transform as

\[
\varphi (x_e, y_e, z_e, t) = \int \varphi_e (k_{ex}, k_{ey}, k_{ez}, t) e^{ik_{ex} x_e + ik_{ey} y_e + ik_{ez} z_e} dk_{ex} dk_{ey} dk_{ez}
\]
follows, we consider the equilibrium distribution function \( F_e \) contains such inhomogeneity in convective coordinates (see Appendix 1 in Ref. 48). In what follows, we consider the equilibrium distribution function \( F_{e0} \) as a Maxwellian,

\[
F_{e0} = \frac{n_{e0}}{(2\pi v_{Te}^2)^{3/2}} \exp \left( -\frac{v_{e\perp}^2 + v_{ez}^2}{v_{Te}^2} \right).
\]

Therefore, the Vlasov equations (9) or (11) for \( f_e(\mathbf{v}_e, \mathbf{r}_e, t) \) does not contain the spatial inhomogeneity in the explicit form. These equations may be Fourier transformed over coordinates \( x_e, y_e, z_e \) with their conjugate wave numbers \( k_{ex}, k_{ey} \) and \( k_{ez} \) without any limitations imposed by the local approximation. Then, the equation for the separate spatial Fourier mode \( f_e(\mathbf{v}_e, \mathbf{k}_e, t) \) of the perturbation of the distribution function is derived as a function of the separate Fourier mode \( \varphi(\mathbf{k}_e, t) \) of the electrostatic potential. The solution to Eq. (11) for \( f_e(\mathbf{v}_e, \mathbf{k}_e, t) \) is calculated easily for any magnitudes of the velocity shear rate \( V'_e \) and is equal to

\[
f_e(t, \mathbf{k}_e, v_{e\perp}, \phi, v_{ez}) = \frac{i e}{m_e} \sum_{n=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \int_{t_0}^{t} dt_1 \varphi(t_1, \mathbf{k}_e) \times \exp \left( -i k_{ex}v_{ez}(t-t_1) + in(\phi_1 - \omega_{ce}t - \theta(t)) - in_1(\phi_1 - \omega_{ce}t_1 - \theta(t)) \right) \times J_n \left( \frac{k_{e\perp}(t)v_{e\perp}}{\omega_{ce}} \right) J_{n_1} \left( \frac{k_{e\perp}(t_1)v_{e\perp}}{\omega_{ce}} \right) \left[ k_{ey} \frac{\partial F_{e0}}{\partial X_e} + \omega_{ce}n_1 \frac{\partial F_{e0}}{\partial v_{e\perp}} + k_{ez} \frac{\partial F_{e0}}{\partial v_{ez}} \right],
\]

where \( t_0 \geq 0 \) is the initial time. In the electron convective coordinates, the Fourier transform \( n_e(\mathbf{k}_e, t) \) of the perturbed electron density is the separate spatial Fourier mode

\[
n_e(\mathbf{k}_e, t) = \int d\mathbf{r}_e n_e(\mathbf{r}_e, t) e^{-ik_e\mathbf{r}_e} = \int f_e(\mathbf{v}_e, \mathbf{k}_e, t) d\mathbf{v}_e,
\]

\[
(12)
\]

\[
(13)
\]

\[
(14)
\]

\[
(15)
\]

\[
(16)
\]
where the subscript in $n_e^{(e)}$ denotes the electron perturbed density, and superscript denotes that its Fourier transform is calculated in the electron convected frame. For the equilibrium Maxwellian electron distribution $[14]$, $n_e^{(e)} (k_e, t)$ is equal to

$$n_e^{(e)} (k_e, t) = -\frac{2\pi e n_0}{T_e} \sum_{n=-\infty}^{\infty} \int_{t_0}^{t} dt \varphi_e (k_e, t) I_n (k_{e\perp} (t) k_{e\perp} (t_1) \rho_e^2)$$

$$\times \exp \left[-\frac{\rho_e^2}{2} (k_{e\perp}^2 (t) + k_{e\perp}^2 (t_1)) - \frac{1}{2} k_{e\perp}^2 v_T^2 (t-t_1)^2 - i \omega_{ce} (t-t_1) - i n (\theta_e (t) - \theta_e (t_1)) \right]$$

$$\times (\omega_{ce} + k_{e\perp}^2 v_T^2 (t-t_1)).$$

(17)

In Eq. (17), $I_n$ is the modified Bessel function of the first kind and order $n$, $k_{e\perp}^2 (t) = (k_{ex} - k_{ey} V_0^2)^2 + k_{ey}^2 + k_{ax}^2$, $\sin \theta (t) = k_{ey} / k_{e\perp} (t)$, $\rho_e = v_{Te} / \omega_{ce}$ is the thermal electron Larmor radius, $v_{Te}$ is the electron thermal velocity.

The model of Hall plasmas with unmagnetized ions and magnetized electrons is applicable to the processes whose temporal evolution is limited by the time much less then the period $\omega_{ci}^{-1}$ of the ion cyclotron Larmor rotation. At the time interval $t - t_0 \ll \omega_{ci}^{-1}$, the accelerated velocity $V_{i0} (r) \approx (e/m_i) E_0 t$ of the unmagnetized ion component in the electric field $E_0$ is much less than the electron flow velocity $V_0 (x)$. Therefore, for this time interval, we can neglect influence by the electric field $E_0$, as well as by the magnetic field, $B$, in the Vlasov equation for ions and identify the ion frame with a laboratory frame. The perturbed ion density $n_i (r, t) = \int f_i (v_i, r, t) dv_i$ is calculated in the ion (laboratory) frame employing the ion Vlasov equation for the perturbation $f_i (v_i, r, t)$ of the ion distribution function $F_{i0} (v_i)$,

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial r_i} = \frac{e}{m_i} \nabla \varphi (r_i, t) \frac{\partial F_{i0} (v_i)}{\partial v_i}.$$  

(18)

For the ion equilibrium Maxwell distribution $F_{i0} (v_i)$,

$$F_{i0} (v_i) = \frac{n_{i0}}{(2\pi v_{Ti}^2)^{3/2}} \exp \left(-\frac{v_i^2}{v_{Ti}^2} \right),$$

the ion density perturbation Fourier transformed over coordinate $r_i$ is

$$n_i^{(i)} (k_i, t) = -\frac{en_{i0}}{T_i} \int_{t_0}^{t} dt \varphi_i (k_i, t_1) k_i^2 v_{Ti}^2 (t - t_1) e^{-\frac{4k_i^2 v_{Ti}^2 (t-t_1)^2}{}}.$$  

(19)

The temporal evolution of the separate spatial harmonic of the potential $\varphi$ with Poisson equation $[3]$ may be investigated in the electron frame as the equation for $\varphi_e (k_e, t)$ by the Fourier transform of Eq. (3) over $r_e$, or as the equation for $\varphi_i (k_i, t)$ by the Fourier transform
of Eq. (3) over \( \mathbf{r}_i \). For the deriving the Fourier transformed Poisson equation (3) for \( \varphi_e (k_e, t) \) the Fourier transform over \( \mathbf{r}_e \) should be determined for \( n_i (\mathbf{r}_i, t) \) and for potential \( \varphi_i (\mathbf{r}_i, t_1) \). With coordinates transform (8) we obtain, that

\[
\int d\mathbf{r}_e n_i (\mathbf{r}_i, t) e^{-ik_e \mathbf{r}_e} = n_i^{(e)} (k_e, t)
\]

\[= n_i (k_{ex} - k_{ey} V_0' t, k_{ey}, k_{ez}, t) e^{ik_{ey} V_0 t}.
\]

(20)

where the superscript in \( n_i^{(e)} \) denotes that the Fourier transform of \( n_i \) is calculated in the electron convected frame.

This relation means that for obtaining the Fourier transform for the ion density perturbation \( n_i^{(i)} (k_i, t) \) over the coordinates \( \mathbf{r}_e \), it is necessary to multiple \( n_i^{(i)} (k_i, t) \) on \( e^{ik_{ey} V_0 t} \), that corresponds to the known Doppler effect for the frames which move with relative steady uniform velocity \( \mathbf{V}_0 \parallel e_y \), and to change the components of the wavevector \( k_i \) in \( n_i^{(i)} (k_i, t) \) on the components of the wavevector \( k_e \) as

\[
k_{ix} = k_{ex} - k_{ey} V_0' t, \quad k_{iy} = k_{ey}, \quad k_{iz} = k_{ez}.
\]

(21)

Equation (19) for \( n_i^{(i)} (k_i, t) \) includes the potential \( \varphi_i (k_i, t_1) \). Using the relation

\[
\varphi_i (k_i, t_1) = \int d\mathbf{r}_i \varphi_i (\mathbf{r}_i, t_1) e^{-ik_i \mathbf{r}_i}
\]

\[= \varphi_e (k_{ix} + k_{iy} V_0' t_1, k_{iy}, k_{iz}, t_1) e^{-ik_{iy} V_0 t_1},
\]

(22)

which follows from the identity \( \varphi_i (\mathbf{r}_i, t_1) = \varphi_e (\mathbf{r}_e, t_1) \), and relations (21), we find that potential \( \varphi_i (k_i, t_1) \) in Eq. (19) should be changed on the \( \varphi_e \) by employing the identity

\[
\varphi_i (k_i, t_1)
\]

\[= \varphi_e (k_{ex} - k_{ey} V_0' (t - t_1), k_{ey}, k_{ez}, t_1) e^{-ik_{ey} V_0 t_1}.
\]

(23)

Equations (20) and (23) demonstrate that the separate spatial Fourier mode of the ion density perturbation \( n_i^{(i)} (k_i, t) \) and potential \( \varphi_i (\mathbf{r}_i, t_1) \) determined in the ion frame are detected in the electron frame as the Doppler-shifted continuously sheared modes with time-dependent wave numbers. Equation (20) displays that the time-dependent non-modal effect of the flow shear becomes important at time \( t \) for which \( |k_{ey} V_0' t| \geq |k_{ex}| \). For \( k_{ey} \sim k_{ex} \) and time \( t \sim \gamma^{-1} \), where \( \gamma \) is the growth rate of the considered modal instability, the separate spatial mode of the ion density perturbation is observed in the electron frame as a non-modal structure changed with time when \( V_0' \gtrsim \gamma \).
Employing connection relations (20) and (23) in Eq. (19) for $n_i^{(i)}(k_i, t)$, we obtain the equation governing the temporal evolution of the potential $\varphi_e(k_e, t)$ in the Hall plasma with a sheared Hall current,

$$K_e^2(t) \varphi_e(k_e, t) = -\frac{1}{\lambda_{De}^2} \sum_{n=-\infty}^{\infty} \int_{t_0}^{t} dt_1 \varphi_e(k_e, t_1) \left( in\omega_{ce} + k_{ez}^2 v_{Te}^2 (t - t_1) \right)$$

$$\times A_{en}(t, t_1) e^{-\frac{1}{2} k_{ez}^2 v_{Te}^2 (t-t_1)^2 - in\omega_{ce}(t-t_1) - in(\theta(t) - \theta(t_1))}$$

$$- \frac{1}{\lambda_{Di}^2} \int_{t_0}^{t} dt_1 \varphi_e(k_{ex} - k_{ey} V_0', t - t_1, k_{ey}, k_{ez}, t_1) e^{ik_{ey} V_0(t - t_1)}$$

$$\times K_e^2(t) v_{Ti}^2 (t - t_1) \left[ -\frac{1}{2} K_e^2(t) v_{Ti}^2 (t - t_1)^2 \right],$$

(24)

where

$$A_{en}(t, t_1) = I_n(k_{e\perp}(t) k_{e\perp}(t_1) \rho_e^2) e^{-\frac{1}{2} \rho_e^2 (k_{e\perp}(t) + k_{e\perp}(t_1))}$$

(25)

with $K_e^2(t) = (k_{ex} - k_{ey} V_0' t)^2 + k_{ey}^2 + k_{ez}^2$; $\lambda_{Di,e} = (T_{i,e}/4\pi n_{i,e} e^2)^{1/2}$ is the ion, electron Debye length. The counterpart of this equation - the equation for $\varphi_i(k_i, t)$ has a form

$$k_i^2 \varphi_i(k_i, t) = -\frac{1}{\lambda_{De}^2} \sum_{n=-\infty}^{\infty} \int_{t_0}^{t} dt_1 \varphi_i(k_{ix} + k_{iy} V_0', t - t_1) k_{iy}, k_{iz}, t_1)$$

$$\times e^{-ik_{iy} V_0(t-t_1) - in\omega_{ce}(t-t_1) - in(\theta(t) - \theta(t_1))} \left( in\omega_{ce} + k_{iz}^2 v_{Te}^2 (t - t_1) \right)$$

$$\times A_{en}(t, t_1) \exp \left[ -\frac{1}{2} k_{iz}^2 v_{Te}^2 (t - t_1)^2 \right]$$

$$- \frac{1}{\lambda_{Di}^2} \int_{t_0}^{t} dt_1 \varphi_i(k_i, t_1) k_i v_{Ti}^2 (t - t_1) e^{-\frac{1}{2} k_i^2 v_{Ti}^2 (t-t_1)^2}.$$  

(26)

in which the relations

$$n_i^{(i)}(k_i, t) = n_e^{(e)}(k_{ix} + V_0' k_{iy}, k_{iy}, k_{iz}, t) e^{-ik_{iy} V_0 t}$$

(27)

and

$$\varphi_e(k_e, t_1) = \varphi_i(k_{ix} + k_{iy} V_0' (t - t_1), k_{iy}, k_{iz}, t_1) e^{ik_{iy} V_0 t_1}$$

(28)

were used.
For the spatially homogeneous electric field $E_0$ for which $V_0' = 0$, Eqs. (20), (21), and (23) reproduce the known relations for the Doppler effect:

\[ n_i^{(e)}(k_e, t) = n_i^{(i)}(k_e, t) e^{ik_{ey}V_0 t}, \]
\[ \varphi_i(k_i, t_1) = \varphi_e(k_e, t_1) e^{-ik_{ey}V_0 t_1}. \]

In this case of the uniform Hall current, or when the local approximation (??) for the inhomogeneous current velocity is applied for which the inhomogeneous velocity is considered as almost uniform and the velocity shear does not distinguish, Eqs. (24) and (26) are the integral equations of the convolution type, which can be solved by using various kinds of integral transforms. In the $t_0 \to -\infty$ limit, Eqs. (24) and (26) have normal modes solutions for the Fourier transformed over time potentials in the form

\[ \varphi_e(k_e, \omega_e) (1 + \varepsilon_e(k_e, \omega_e) + \varepsilon_i(k_e, \omega_e + k_e V_0)) = 0 \]

for Eq. (24), and solution

\[ \varphi_i(k_i, \omega_i) (1 + \varepsilon_i(k_i, \omega_i) + \varepsilon_e(k_i, \omega_i - k_i V_0)) = 0 \]

for Eq. (26), where $\varepsilon_i$ and $\varepsilon_e$ are known ion and electron components of the electrostatic dielectric permittivity of the Hall plasma. In the cases of the sheared Hall current, the solutions to Eqs. (24), (26) can’t be presented in the modal forms (32) or (33), and the solution of Eqs. (24), (26) as the initial value problems are necessary.

It is obvious that the exact analytical solutions to integral equations (24) and (26) cannot be obtained explicitly. In this paper, we present the approximate nonmodal solutions of the integral equations (24) and (26) for two basic classes of instabilities: reactive (MTS instability) in Sec. III and kinetic (IS instability), in Sec. IV. The solutions are obtained for the case of the velocity shear $V_0'$ much less than the frequency $\omega_0$ of the corresponding modal instability of the shearless Hall plasma. The effect of such current shear is relatively small at the time interval $t - t_0$ for which

\[ \omega_0^{-1} \ll t - t_0 < (V_0')^{-1}, \]

The modal theory of the MTS instability of the Hall plasma with uniform current employs the approximation of the hydrodynamic ions and electrons which assumes that the thermal velocities of ion and electrons are much less than the phase velocity of the unstable
perturbations. In Sec. III, we present simple procedure for the deriving the analytical non-modal solution to Eq. (24) for the modified two-stream instability in this hydrodynamic approximation with accounting for the effect of the weak velocity shear.

Other model, of the hydrodynamic ions, but of adiabatic electrons the thermal velocity of which is larger than the phase velocity of the unstable perturbations, is employed for the modal IS kinetic instability of a plasma with uniform current. In Sec. IV, we derive for this model the solution to Eq. (26) for the nonmodal kinetic IS instability for the case (34) of the weak current velocity shear.

III. THE HYDRODYNAMIC NONMODAL MODIFIED TWO-STREAM INSTABILITY

In this section, we consider the temporal evolution of the MTS instability\textsuperscript{19,50} of the Hall plasma with a sheared Hall current. This investigation may be performed using any of Eqs. (24) and (26). Here we employ for this task Eq. (24).

The MTS instability is a long-wavelength, $k_{\perp} \rho_e \ll 1$, instability which develops in the intermediate-frequency range $\omega_{ci} \ll \omega \ll \omega_{ce}$ in plasmas with electrons drifting relative to ions across the magnetic field. The phase velocity across the magnetic field of the unstable waves of MTS instability is much above the ion thermal velocity, and the phase velocity along the magnetic field is much above the electron thermal velocity. The dispersion equation for this instability in the electron frame is

\begin{equation}
1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{k_{ez}^2 \omega_{pe}^2}{k_{ey}^2 \omega^2 (k_e)} - \frac{\omega_{pi}^2}{(\omega (k_e) + k_{ey} V_0)^2} = 0. \tag{35}
\end{equation}

The MTS instability is an example of a general class of instabilities which have been referred to as reactive\textsuperscript{51,52}. These instabilities occur when two wave modes couple at a critical frequency. The MTS instability develops due to the coupling\textsuperscript{19} the electron mode with the frequency $\omega_1 = \omega_{Lh} (k_z/k) (m_i/m_e)^{1/2}$ with Doppler-shifted lower hybrid wave $\omega_2 = \omega_{Lh} - k_y V_0$, where $\omega_{Lh}$ is the lower hybrid frequency,

\begin{equation}
\omega_{Lh}^2 = \omega_{pi}^2 \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\right)^{-1}.
\end{equation}

under conditions when the frequencies $\omega_1$ and $\omega_2$ are almost equal. The solution of Eq. (35),

\begin{equation}
\omega (k_e) = \frac{1}{2} \left(\omega_{Lh} - k_{ey} V_0 - \frac{k_{ez}}{k_e} \left(\frac{m_i}{m_e}\right)^{1/2}\right).
\end{equation}
\[ \pm \frac{1}{2} \left[ \left( \omega_L h - k_{ey} V_0 + \omega_L h \frac{k_{ez}}{k_e} \left( \frac{m_i}{m_e} \right)^{1/2} \right)^2 - \omega_L^2 h \frac{k_{ez}}{k_e} \left( \frac{m_i}{m_e} \right)^{1/2} \right]^{1/2}, \tag{36} \]

predicts the MTS instability development for \( k_{ez}/k_e \sim (m_e/m_i)^{1/2} \).

For the solution of the integral equation (24) for the instabilities which have the phase velocities larger than the thermal velocities of particles, the alternative form of Eq. (24), resulted from the integration by parts of Eq. (24), is more suitable,

\[ \left( K_e^2 (t) + \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} \right) \varphi_e (k_e, t) \]
\[ = \frac{1}{\lambda_{De}^2} \sum_{n=-\infty}^{\infty} \int_{t_0}^{t} dt_1 \frac{d}{dt_1} \left\{ \varphi_e (k_e, t_1) A_{en} (t, t_1) \right\} \]
\[ \times e^{-i \omega_{ce} (t-t_1)-i \omega_i (t-t_1)} \right\} e^{-\frac{1}{2} k_{ez}^2 v_T^2 (t-t_1)^2} \]
\[ - \frac{1}{\lambda_{De}^2} \sum_{n=-\infty}^{\infty} \omega_{ce} \int_{t_0}^{t} dt_1 \varphi_e (k_e, t_1) A_{en} (t, t_1) \]
\[ \times e^{-\frac{1}{2} k_{ez}^2 v_T^2 (t-t_1)^2-i \omega_{ce} (t-t_1)-i \omega_i (t-t_1)} \]
\[ + \frac{1}{\lambda_{Di}^2} \int_{t_0}^{t} dt_1 \frac{d}{dt_1} \left\{ \varphi_e (k_{ez} - k_{ey} V_0' (t-t_1), k_{ey}, k_{ez}, t_1) e^{i k_{ey} V_0 (t-t_1)} \right\} \]
\[ \times \exp \left[ -\frac{1}{2} K_e^2 (t) v_T^2 (t-t_1)^2 \right] + Q (k_e, t, t_0), \tag{37} \]

where

\[ Q (k_e, t, t_0) = \frac{1}{\lambda_{De}^2} \varphi_e (k_e, t_0) \frac{T_i}{T_e} \sum_{n=-\infty}^{\infty} A_{en} (t, t_0) \]
\[ \times e^{-\frac{1}{2} k_{ez}^2 v_T^2 (t-t_0)^2-i \omega_{ce} (t-t_0)-i \omega_i (t-t_0))} \]
\[ + \frac{1}{\lambda_{Di}^2} \varphi_e (k_{ez} - k_{ey} V_0' (t-t_0), k_{ey}, k_{ez}, t_0) e^{i k_{ey} V_0 (t-t_0)} \]
\[ \times \exp \left[ -\frac{1}{2} k_{ez}^2 v_T^2 (t-t_0)^2+k_{ey}^2+k_{ez}^2 \right] v_T^2 (t-t_0)^2 \tag{38} \]

determines the input from the \( t = t_0 \) limit of the integration of Eq. (24) by parts. The approximations

\[ \exp \left( -\frac{1}{2} k_{ez}^2 v_T^2 (t-t_1)^2 \right) \approx 1 - \frac{1}{2} k_{ez}^2 v_T^2 (t-t_1)^2, \]
\[ \exp \left( -\frac{1}{2} K_e^2 (t) v_T^2 (t-t_1)^2 \right) \approx 1 - \frac{1}{2} K_e^2 (t) v_T^2 (t-t_1)^2, \tag{39} \]
which corresponds to the weak electron and ion Landau damping (hydrodynamic approximation) strongly simplify the solution of Eq. (37). Accounting for the only term with \( n = 0 \) in Eq. (37) and using the approximation \( A_{\omega_0} (t, t_1) \approx 1 \) that is sufficient for a long-wavelength, \( k_\perp \rho_e \ll 1 \), MTS instability, which has the frequency and the growth rate much less than the electron cyclotron frequency \( \omega_{ce} \), we obtain the equation

\[
\left( K_e^2 (t) + \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} \right) \varphi_e (k_e, t) + \frac{1}{\lambda_{De}^2} \int_{t_0}^{t} dt_1 \left( 1 - \frac{1}{2} k_{ex}^2 v_T^2 (t - t_1)^2 \right) \frac{d}{dt_1} \varphi_e (k_e, t_1) + \frac{1}{\lambda_{Di}^2} \int_{t_0}^{t} dt_1 \frac{d}{dt_1} \left\{ \varphi_e (k_{ex} - k_{ey} V'_0 (t - t_1), k_{ey}, k_{ez}, t_1) e^{i k_{ey} V_0 (t - t_1)} \right\} \times \left( 1 - \frac{1}{2} K_e^2 v_T^2 (t - t_1)^2 \right) = Q (k_e, t, t_0). \tag{40}
\]

We will find the solution of Eq. (40) under condition (34) in the WKB-like form\(^\text{28}\)

\[
\varphi (k_e, t_1) = \Phi_e (k_e) e^{-i \int_{t_0}^{t_1} \omega (k_e, t_2) dt_2}, \tag{41}
\]

where \( \Phi (k_e) = \int_{-\infty}^{\infty} e^{-i k \cdot r_e} \varphi (k_e, t_0) d k_e \) is the Fourier transform of the initial perturbation of \( \varphi (r_e, t_1) \) at \( t_1 = t_0 \). The equation for \( \omega (k_e, t) \) is derived iteratively by integration by parts of Eq. (40) in the form of a power series expansion in powers of \( |V'_0 / \omega_0| < 1 \),

\[
\Phi_e (k_e) e^{-i \int_{t_0}^{t_1} \omega (k_e, t_1) dt_1} \left[ 1 + \frac{\omega_{pe}^2}{\omega_{ee}^2} - \frac{k_{ex}^2}{k_{ey}^2} \frac{\omega_{pe}^2}{\omega_{ee}^2 (k_e)} \left( 1 + \frac{3i}{\omega_0^2 (k_e)} \frac{d \omega (k_e, t)}{dt} \right) \right. \\
\left. - \frac{\omega_{pi}^2}{(\omega_0 (k_e) + k_{ey} V_0')^2} \left( 1 - \frac{2 k_{ey} V'_0}{(\omega_0 (k_e) + k_{ey} V_0')} \frac{\partial \omega_0 (k_e)}{\partial k_{ex}} (t - t_0) \right) \right] = Q (k_e, t, t_0), \tag{42}
\]

where \( \omega_0 (k_e) \) is the solution (36) of Eq. (35). For the potential exponentially growing with time, we can neglect by \( Q (k_e, t, t_0) \) in Eq. (40) and obtain the solution for the exponential of Eq. (41),

\[
-i \int_{t_0}^{t} \omega (k_e, t_1) dt_1 = -i \omega_0 (k_e) (t - t_0)
\]

\[+ \frac{1}{9} \omega_{Lh}^2 \left( \frac{m_i}{m_e} \right)^{3/2} \left( \frac{k_{ex}}{k_e} \right)^3 k_{ex} k_{ey} V'_0 (t - t_0)^3. \tag{43}\]

This solution predicts fast nonmodal growth as \( \exp a (t - t_0)^3 \) of the potential \( \varphi_e (k_e, t) \) for the perturbations with \( k_{ey} V'_0 > 0 \).
Equation (43) displays that the uniform and the sheared components of the current velocity are the independent sources of the current driven instabilities. We found that the uniform part, $V_0$, of the current velocity is responsible for the modal type of the instability development, whereas the current velocity shear $V'_0$ is the source of the free energy for the development of the instability of the nonmodal type.

IV. THE NONMODAL KINETIC ION-SOUND INSTABILITY

For $k_{ex}/k_e > (m_e/m_i)^{1/2}$ the electron mode does not couple with the Doppler-shifted lower hybrid wave. In this case, the lower hybrid wave goes into the IS wave\textsuperscript{19,20}. The dispersive properties of the long wavelength, $k_\perp \rho_e \ll 1$, IS instability of a plasma with a uniform Hall current are determined by the equation

$$
\varepsilon_0(k, \omega) = 1 - \frac{\omega^2}{\omega_0^2} + \frac{1}{k^2 \lambda^2_{De}} + \frac{1}{k^2 \lambda^2_{De}} \sqrt{2} z_e W(z_e) = 0, \quad (44)
$$

where $W(z_e) = e^{-z^2_e} \left(1 + (2i/\sqrt{\pi}) \int_0^{z_e} e^{t^2} dt\right)$ is the complex error function (also known as the Faddeeva function\textsuperscript{53}) with argument $z_e = (\omega - k_y V_0) / \sqrt{2} k_z v_T e$. The solution of Eq. (44) for the adiabatic electrons ($|z_e| \ll 1$) is $\omega(k) = \omega_{IS} + \delta \omega(k)$, where $\omega_{IS}(k)$ is the frequency of the ion sound wave, $\omega^2_{IS}(k) = k^2 v_s^2 \left(1 + k^2 \lambda^2_{De} \right)^{-1}$, $v_s^2 = T_e/m_i$, and

$$
\delta \omega(k) = -\frac{i}{2} \omega_{IS} z_{e0} W(z_{e0}) \ll \omega_{IS}(k) \quad (45)
$$

with $z_{e0} = (\omega_{IS}(k) - k_y V_0) / \sqrt{2} k_z v_T e$. The IS instability develops when $k_y V_0 > k v_s$ with the growth rate $\gamma_{IS}(k) = \text{Im} \delta \omega(k)$ and with $|z_{e0}| < 1$ when $k_z/k > (m_e/m_i)^{1/2}$.

Now we consider the temporal evolution of the IS instability in a plasma with a sheared Hall current. For this goal, we obtain the solution to Eq. (26) for the potential $\varphi_i(k_i, t)$ determined in the ion frame. This solution we shall find under condition (34) in the WKB form

$$
\varphi(k_i, t) = \Phi_i(k_i) e^{-i \int_{t_0}^t \omega(k_i, t_1) dt_1}, \quad (46)
$$

For the deriving the equation for the frequency $\omega(k_i, t_1)$ with a weak time-dependence resulted from the current velocity shearing, we perform the integration by parts in the ion term using the relation,

$$
e^{-i \int_{t_0}^t \omega(k_i, t_1) dt_1} = \frac{i}{\omega(k_i, t)} \frac{d}{dt} \left( e^{-i \int_{t_0}^t \omega(k_i, t_1) dt_1} \right), \quad (47)$$
and derive the expansion of the ion term in the form of the power series of $k_i v_{Te}/\omega (k_i, t)$. In the electron term of Eq. (26), we employ the expansion

$$\varphi_i (k_{ix} + k_{iy} V'_0 (t - t_1), k_{iy}, k_{iz}, t_1)$$

$$= \varphi_i (k_{ix}, t_1) + k_{iy} V'_0 (t - t_1) \frac{\partial \varphi_i}{\partial k_{ix}},$$

which is valid under condition (34). For the adiabatic electrons, for which $\omega \ll k_z v_{Te}$, the main input into an integral over time $t_1$ in the electron term of Eq. (26) gives the time interval $|t - t_1| \lesssim (k_z v_{Te})^{-1}$, that defines the validity of the approximation

$$e^{-i \int_{t_1}^{t} \omega (k_i, t_2) dt_2} \approx e^{i \omega (k_i, t)(t-t_1)}.$$  \hspace{1cm} (49)

Then, for the exponential term, exp \left( -i \int_{t_0}^{t} \omega (k_i, t_1) dt_1 \right), growing with time, we obtain the equation

$$e^{-i \int_{t_0}^{t} \omega (k_i, t_1) dt_1} \left[ \omega_{IS} (k_i) + \delta \omega (k_i) - 3 i \frac{\omega^2_{pi}}{\omega^4_{IS} (k_i)} \frac{d \omega (k_i, t)}{d t} \right]$$

$$+ \frac{1}{k^2_i \lambda^2_{De}} \frac{(\omega_{IS} (k_i) - k_{iy} V'_0)}{k_{iz} v_{Te}} \frac{k_{iy} V'_0}{k_{iz} v_{Te}} \left( \frac{\partial \ln \Phi (k_i)}{\partial k_{ix}} (t - t_0) \right) = 0.$$  \hspace{1cm} (50)

The solution of this equation is straightforward and gives the following exponential for solution (46):

$$- i \int_{t_0}^{t} \omega (k_i, t_1) dt_1 = -i (\omega_{IS} (k_i) + \delta \omega (k_i)) (t - t_0)$$

$$- \frac{1}{6 k^2_i \lambda^2_{De}} \frac{\omega^4_{0} (k_i)}{\omega^2_{pi}} \left( 1 - k^2_{i \perp} \rho^2_{e} \right) \frac{(\omega_{IS} (k_i) - k_{iy} V'_0)}{k_{iz} v_{Te}} \frac{k_{iy} V'_0}{k_{iz} v_{Te}}$$

$$\times \left( \frac{\partial \ln \Phi (k_i)}{\partial k_{ix}} (t - t_0)^2 + \frac{1}{3} \frac{\partial \omega_{IS} (k_i)}{\partial k_{ix}} (t - t_0)^3 \right).$$  \hspace{1cm} (51)

The first term in the right part of Eq. (51) corresponds to the modal IS instability evolution with growth rate $\gamma_{IS} (k)$. The second term proportional to $V'_0$ describes the nonmodal instability. The nonmodal growth is determined by the relation

$$\int_{t_0}^{t_2} \gamma_{nm} (k_i, t_2) dt_2 = \frac{1}{18} k^2_i v^2_s \left( \frac{m_e}{m_i} \right)^{1/2} \left( 1 - k^2_{i \perp} \rho^2_{e} \right) \frac{k_{iy} k_{iz}}{k_{iz} k_{ix}}$$

$$\times \frac{(k_{iy} V_0 - \omega_{IS} (k_i))}{k_{iz} v_{Te}} V'_0 (t - t_0)^3.$$  \hspace{1cm} (52)
Equation (52) displays that the nonmodal growth of the IS perturbations due to the shearing of the Hall current occurs for the IS perturbations with \( k_{iy}V_0 > \omega_{IS}(k_i) \) and \( (k_{ix}/k_{iz}) k_{iy}V'_0 > 0 \). The nonmodal growth is independent process from the development of the modal instability and accompanies it. In the general case, the instability driven by the current with current velocity shear includes modal and nonmodal growth and the net effect of the instability development is determined as a balance between them. Eq. (51) predicts that the nonmodal growth dominates over the modal growth when

\[
\omega_{IS}(k_i)V'_0(t-t_0)^2 > \frac{k_{iz}}{k_i} \left( \frac{m_i}{m_e} \right)^{1/2} \frac{k_i^2}{k_{iy}k_{ix}}.
\]  

The nonmodal growth occurs also for the subcritical perturbations for which \( (k_{ix}/k_{iz}) k_{iy}V'_0 < 0 \) with \( k_{iy}V_0 < \omega_{IS}(k_i) \) including the case when \( V'_0 = 0 \). These perturbations are suppressed with damping rate determined by Eq. (45), but become growing with time when condition (53) holds. The evolution of the kinetic instability at longer time at which condition (34) does not hold continues to be nonmodal. However, this evolution can be investigated only by the numerical solution of Eqs. (26) or (24) as the initial value problems.

V. CONCLUSIONS

In this paper, the basic equations (Eqs. (24) and (26)) of the kinetic theory of the electrostatic instabilities driven by the Hall current with a sheared current velocity were derived employing the shearing modes approach. These equations were obtained for the case of a Hall current with uniform current velocity shear, \( V'_0 = \text{const} \), without application of the local approximation and without imposing on the perturbations the requirement to have a static structure of the plane wave \( \sim \exp(ikr - i\omega t) \) with prescribed exponential time dependence of the canonical modal form. The developed theory predicts that the separate spatial Fourier mode of the perturbations in the Hall plasma with the inhomogeneous electric field is determined in the frame convected with one of the plasma components. The relations (20) - (23), which are the generalization on the sheared current velocity the relations (20) - (31) of the Doppler effect for the uniform current velocity, display that due to the different shearing of the ion and electrons flows in the Hall plasma, this mode is detected by the second component as the Doppler - shifted continuously sheared mode with time - dependent wave numbers. This effect of the mode shearing grows continuously with time and the interaction
of the plasma components forms the nonmodal time-dependent process which should be investigated as the initial value problem.

The nonmodal solutions of the integral equations (24) and (26) for two basic classes of instabilities: reactive (MTS instability) and kinetic (IS instability), are obtained in Secs. [III] and [IV] for the case of the weak uniform current velocity shear (34) as the solutions of the linear initial value problems. These solutions reveal that the uniform part of the current velocity, $V_0$, is responsible for the modal evolution of the instability, whereas the current velocity shear, $V'_0$, is the source of the development of the nonmodal instability with exponent growing with time as $\sim (t - t_0)^3$. This time-dependence, which is the same as in the solution$^{38}$ of the linear initial value problem for the Simon-Hoh instability, seems to be common for the plasma instabilities driven by the sheared Hall current with uniform shear.

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