REPRESENTATIONS OF FINITE GROUPS ON RIEemann-roCH
SPACES, II

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Abstract. If \( G \) is a finite subgroup of the automorphism group of a projective
curve \( X \) and \( D \) is a divisor on \( X \) stabilized by \( G \), then under the assumption
that \( D \) is nonspecial, we compute a simplified formula for the trace of the
natural representation of \( G \) on Riemann-Roch space \( L(D) \).

1. Introduction

Let \( X \) be a smooth projective (irreducible) curve over an algebraically closed
field \( k \) and let \( G \) be a finite subgroup of automorphisms of \( X \) over \( k \). We assume
throughout this paper that either char \( k = 0 \) or char \( k = p \) does not divide the order
of the group \( G \). If \( D \) is a divisor of \( X \) which \( G \) leaves stable then \( G \) acts on the
Riemann-Roch space \( L(D) \). We are interested in decomposing this representation
into irreducibles.

This question was originally addressed by Hurwitz, in the case where \( D \) was the
canonical divisor and \( G \) was cyclic, over \( k = \mathbb{C} \). Chevalley and Weil expanded this
result to any finite \( G \) [CW]. Since then further work has been done by Ellingsrud
and Lønsted [EL], Kani [Ka], Nakajima [N], Köck [K1 K2], and Borne [B]. In the
case where \( D \) is a nonspecial divisor, the character of \( L(D) \) has been computed in
the work of Borne [B]. We have computed a simpler formula for this character,
under a rationality criterion.

Theorem 1. Let \( D = \pi^*(D_0) \) be a nonspecial divisor on \( X \) which is a pullback of
a divisor \( D_0 \) on \( Y = X/G \) and assume that the (Brauer) character of \( L(D) \) is the
character of a \( \mathbb{Q}[G] \)-module. Then for each absolutely irreducible character of \( G \),
the multiplicity of the corresponding module \( W \) in \( L(D) \) is given by

\[
n = \dim(W)(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^{M}(\dim(W) - \dim(W^{H_\ell}))\frac{R_\ell}{2}.
\]

The sum is over all conjugacy classes of cyclic subgroups of \( G \), \( H_\ell \) is a representative
cyclic subgroup, \( W^{H_\ell} \) indicates the dimension of the fixed part of \( W \) under the action
of \( H_\ell \), and \( R_\ell \) denotes the number of branch points in \( Y \) where the inertia group is
conjugate to \( H_\ell \).

One motivation for seeking such a formula comes from coding theory. The con-
struction of AG codes uses the Riemann-Roch space \( L(D) \) of a divisor on a curve

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defined over a finite field. Automorphisms of $L(D)$ may provide more efficient encoding and storage of information, for some AG codes. See [JT] for more background on AG codes and automorphisms of Riemann-Roch spaces.

In Section 2 we will prove this theorem. In Section 3 we extend to the case that $D$ is not necessarily a pullback. In this case we use a formula due to Borne [B] which expresses $L(D)$ in terms of the equivariant degree of $D$ and the ramification module of the cover, which does not depend on $D$. Theorem 1 then gives us a simple formula for the ramification module when it obeys the rationality condition. This simple formula for the multiplicity of a $\mathbb{Q}[G]$-module in the ramification module has also been obtained by Köck [K2] using other methods. In Section 4, we give some examples.

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2. Proof of Theorem 1

We start with some definitions and notation.

Let $X$ be a smooth projective (irreducible) curve over an algebraically closed field $k$ and let $G$ be a finite subgroup of automorphisms of $X$ over $k$. We assume that either $\text{char } k = 0$ or $\text{char } k = p$ does not divide the order of the group $G$. For any point $P \in X(k)$, let $G_P$ be the inertia group at $P$ (i.e. the subgroup of $G$ fixing $P$). Our assumptions on char $k$ ensure that the quotient $\pi : X \rightarrow Y = X/G$ is tamely ramified, and this group $G_P$ is cyclic.

Let $\langle G \rangle$ denote the set of conjugacy classes of cyclic subgroups of $G$. For each class in $\langle G \rangle$ choose a representative cyclic subgroup $H_\ell$, $\ell = 1 \ldots M$, and partially order them according to the order of the group so that $H_1$ is the trivial group. For each branch point of the cover $\pi : X \rightarrow Y$, the inertia groups at the ramification points $P$ above that branch point will be cyclic and conjugate to each other. For each $\ell$, let $R_\ell$ denote the number of branch points in $Y$ where the inertia groups are conjugate to $H_\ell$. ($R_1$ may be set to 0; it does not play a role in the formula).

Let $G^*_\mathbb{Q}$ denote the set of equivalence classes of irreducible $\mathbb{Q}[G]$-modules. By results in ([Sc], §13.1, §12.4), this set has the same number of elements, $M$, as $\langle G \rangle$. For each class in $G^*_\mathbb{Q}$, choose a representative irreducible $\mathbb{Q}[G]$-module $V_j$, $j = 1 \ldots M$, and denote its character by $\chi_j$. The character table of $G$ over $\mathbb{Q}$ is a square matrix with rows labelled by $G^*_\mathbb{Q}$ and columns labelled by $\langle G \rangle$. The rows are linearly independent (as $\mathbb{Q}$-class functions), so in fact the character table is an invertible matrix.

Let $F$ be a finite extension of $\mathbb{Q}$ such that every irreducible $F[G]$-module is absolutely irreducible (irreducible over $\mathbb{C}$), so that the character table of $G$ over $F$ is the same as the character table for $G$ over $\mathbb{C}$ ([Sc], p. 94). For each irreducible $\mathbb{Q}[G]$-module $V_j$, $V_j \otimes_{\mathbb{Q}[G]} F[G]$ decomposes into irreducible $F[G]$-modules. The Galois group of $F$ over $\mathbb{Q}$ permutes the components transitively, so each must have the same multiplicity (the Schur index of the representation $V_j$) and the same dimension. We write

$$V_j \otimes_{\mathbb{Q}[G]} F[G] \simeq m_j \bigoplus_{r=1}^{d_j} W_{j,r},$$

where $m_j$ is the Schur index, the $W_{j,r}$’s are irreducible $F[G]$-modules, and $\dim_F V_j = m_j d_j \dim_F W_{j,r}$ for each $r$. Let $\chi_{j,r}$ denote the character of $W_{j,r}$.
Theorem 1 is a consequence of the following.

**Theorem 2.** Let $D = \pi^*(D_0)$ be a nonspecial divisor on $X$ and assume that the (Brauer) character of $L(D)$ is the character of a $\mathbb{Q}[G]$-module $L(D)_\mathbb{Q}$. Then for each irreducible $\mathbb{Q}[G]$-module $V_j$, its multiplicity in $L(D)_\mathbb{Q}$ is given by

$$n_j = \frac{1}{m_j^2 d_j} \left( \dim(V_j)(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^M (\dim(V_j) - \dim(V_j^{H_\ell})) \frac{R_\ell}{2} \right).$$

**Proof:** The proof is similar to the proof of Theorem 2.3 in [K3]. We consider the quotients $X/H_\ell$ of $X$ by cyclic subgroups $H_\ell$. The morphism $\pi: X \to Y$ factors through this quotient, so on each $X/H_\ell$ there is a pullback divisor $D_\ell$ of $D_0$.

First, note that our assumption that $D$ is nonspecial means that for any quotient $X/H_\ell$, the pullback $D_\ell$ of $D_0$ to $X/H_\ell$ is also nonspecial. This is because

$$K_X - D = \pi^*_\ell(K_{X/H_\ell}) + \text{Ram}(X/H_\ell) - \pi^*_\ell(D_\ell) = \pi^*_\ell(K_{X/H_\ell} - D_\ell) + \text{Ram}(X/H_\ell)$$

where $\text{Ram}(X/H_\ell)$ is the ramification divisor of the covering $\pi_\ell: X \to X/H_\ell$. Any element of $L(K_{X/H_\ell} - D_\ell)$ would pull back to $X$ to give an element of $L(K_X - D - \text{Ram}(X/H_\ell))$. Since $\text{Ram}(X/H_\ell)$ is effective, this would also give an element of $L(K_X - D)$, contradicting our assumption that $D$ is nonspecial.

Now we decompose $L(D)_\mathbb{Q}$ as

$$L(D)_\mathbb{Q} \simeq \bigoplus_{j=1}^M n_j V_j.$$ 

For each $H_\ell$ in $\langle G \rangle$, consider the dimension of the piece of this module fixed by $H_\ell$. Since the elements of $L(D)$ fixed by $H_\ell$ are exactly the elements of $L(D_\ell)$, $\dim_{\mathbb{Q}} L(D)_\mathbb{Q}^{H_\ell} = \dim_k L(D)^{H_\ell} = \dim_k L(D_\ell)$ and we get an equation for each $\ell$:

$$\dim_k L(D_\ell) = \sum_{j=1}^M n_j \dim_{\mathbb{Q}}(V_j^{H_\ell}), \quad 1 \leq \ell \leq M.$$ 

This gives us a system of $M$ equations in the $M$ unknowns $n_j$. We need to show that the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is invertible, so this system has a unique solution, and that the above equation is the claimed solution.

First let us consider the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$. Each matrix entry is equal to the multiplicity of the trivial representation of $H_\ell$ in the restricted representation of $H_\ell$ on $V_j$. This is the inner product of characters $\langle \text{Res}_{H_\ell}^G \chi_j, 1 \rangle$, which is defined as

$$\dim V_j^{H_\ell} = \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \chi_j(a).$$

Thus each column of the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is a sum of columns of the character table of $G$ over $\mathbb{Q}$. Each element $a$ in $H_\ell$ generates either all of $H_\ell$ or a cyclic subgroup of lower order, hence earlier in the list $\langle G \rangle$. Thus if we write our matrix in terms of the basis of columns of this character table, we get a lower triangular
matrix with nonzero entries on the diagonal. This implies that our matrix is also invertible.

Now it remains to verify that our equation is the correct solution to (5).

Note that

\[
\dim L(D_\ell) = \frac{|G|}{|H|} \deg(D_0) + 1 - g(X/H_\ell),
\]

for \(1 \leq \ell \leq M\), by the Riemann-Roch theorem and the hypothesis that \(D_\ell\) is nonspecial.

We will now substitute (3) into (5) and verify that the result agrees with (7), for each \(1 \leq \ell \leq M\). The argument is similar to that in [Ks].

Plugging (3) into (5) gives

\[
\sum_{j=1}^{M} n_j \dim(V_{H_\ell j}) = (\deg(D_0) + 1 - gY) \sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_{H_\ell j}) \dim(V_j)
\]

\[
- \sum_{i=1}^{M} \left( \sum_{j=1}^{M} \frac{1}{m_j^2 d_j} [\dim(V_{H_\ell j}) \dim(V_j) - \dim(V_{H_\ell j}) \dim(V_{H_\ell j})] R_i \right)
\]

Note that

\[
\dim(V_{H_\ell j}) = \langle \text{Res}^G_{H_\ell} \chi_j, 1 \rangle = m_j \sum_{r=1}^{d_j} \langle \text{Res}^G_{H_\ell} \chi_{jr}, 1 \rangle = m_j \sum_{r=1}^{d_j} \langle \chi_{jr}, \text{Ind}^G_{H_\ell} 1 \rangle,
\]

using (2) and Frobenius reciprocity. This gives us

\[
\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_{H_\ell j}) \dim(V_j) = \sum_{j=1}^{M} \frac{\dim V_j}{m_j d_j} \sum_{r=1}^{d_j} \langle \chi_{jr}, \text{Ind}^G_{H_\ell} 1 \rangle
\]

\[
= \sum_{j=1}^{M} \sum_{r=1}^{d_j} \dim W_{jr} \langle \text{Res}^G_{H_\ell} \chi_{jr}, 1 \rangle
\]

\[
= \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e) \chi_{jr}(a)
\]

The last part of this is summing over all irreducible \(F\)-characters of \(G\), so the last expression is in fact the inner product of two columns of the character table for \(G\) over \(F\). This inner product will be zero unless \(a = e\), so the sum becomes

\[
\frac{1}{|H_\ell|} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(e)^2 = \frac{|G|}{|H_\ell|}.
\]

We would like to do a similar simplification of

\[
\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_{H_\ell j}) \dim(V_{H_\ell j})
\]
using (8) twice. The induced representation \( \text{Ind}_{H_i}^G 1 \) is the action of \( G \) by permutations on the cosets of \( H_i \), and thus has a \( \mathbb{Q}[G] \)-module structure as well as an \( F[G] \)-module structure. It can be decomposed into irreducible \( F[G] \)-modules, such that for each \( j \) the multiplicities of the \( W_{jr} \)'s, \( \langle \chi_{jr}, \text{Ind}_{H_i}^G 1 \rangle \), are all equal. Using that fact, Frobenius reciprocity, and the definition of the Schur inner product, we have

\[
\sum_{j=1}^{M} \frac{1}{m_j d_j} \dim(V_j^{H_t}) \dim(V_j^{H_i}) \\
= \sum_{j=1}^{M} \frac{1}{d_j} \sum_{r=1}^{d_j} (\text{Res}^G_{H_t} \chi_{jr}, 1) \sum_{s=1}^{d_j} (\chi_{js}, \text{Ind}_{H_i}^G 1) \\
= \sum_{j=1}^{M} \sum_{r=1}^{d_j} (\text{Res}^G_{H_t} \chi_{jr}, 1) (\chi_{jr}, \text{Ind}_{H_i}^G 1) \\
= \sum_{j=1}^{M} \sum_{r=1}^{d_j} (\text{Res}^G_{H_t} \chi_{jr}, 1) (\text{Res}^G_{H_i} \chi_{jr}, 1) \\
= \frac{1}{|H_i||H_t|} \sum_{a \in H_t} \sum_{b \in H_i} \sum_{j=1}^{M} \sum_{r=1}^{d_j} \chi_{jr}(a) \chi_{jr}(b). 
\]

(12)

Again, this last is an inner product of columns of the character table of \( G \) over \( k \), so will be zero unless \( a \) and \( b \) are in the same conjugacy class. Let \( C_G(a) \) denote the conjugacy class of \( a \) in \( G \). We end up with

\[
\sum_{j=1}^{M} \frac{1}{m_j d_j} \dim(V_j^{H_t}) \dim(V_j^{H_i}) = \frac{1}{|H_i||H_t|} \sum_{a \in H_t} \#(H_t \cap C_G(a)) \sum_{j=1}^{M} \sum_{i=1}^{d_j} \chi_{jr}(a)^2 \\
= |H_t \backslash G/H_i| 
\]

the number of double cosets.

From this we get

\[
\sum_{j=1}^{M} n_j \dim V_j^{H_t} = (\deg(D_0) + 1 - g_Y) \frac{|G|}{|H_t|} - \sum_{i=1}^{M} \left( \frac{|G|}{|H_i|} - |H_i \backslash G/H_t| \right) \frac{R_i}{2} \\
= (\deg(D_0) + 1 - g_Y) \frac{|G|}{|H_t|} + 1 + \frac{|G|}{|H_t|} (g_Y - 1) - g_{X/H_t} \\
= \deg(D_0) \frac{|G|}{|H_t|} + 1 - g_{X/H_t}. 
\]

where the last equalities come from applying the Hurwitz formula to the cover \( X/H_t \to Y \) (see [Ks] for details). This is (7), as desired. □

**Proof of Theorem 1.** We use the decomposition (2) to compute the multiplicity of each \( W_{jr} \) in \( L(D)_Q \otimes F \). By our definition of \( F \), each absolutely irreducible character is the character of one of the \( W_{jr} \)'s, and the character of \( L(D) \) is the same as the character of \( L(D)_Q \otimes F \), so this will give us the correct answer.

The multiplicity of \( W_{jr} \) in \( V_j \) is \( m_j \), and \( \dim V_j = m_j d_j \dim W_{jr} \). Equation (8) and the fact that \( \text{Ind}_{H_i}^G 1 \) has a \( \mathbb{Q}[G] \)-module structure means that \( \dim W_{jr} \) is the same for each \( r \), so \( \dim V_j^{H_t} = m_j d_j \dim W_{jr}^{H_t} \). Thus we can factor \( m_j d_j \) out from the inside and multiply the whole thing by \( m_j \) to get formula (11). □

**Remark.** The rationality criterion is necessary for this formula to be accurate. If the character of \( L(D) \) is not the character of a \( \mathbb{Q}[G] \)-module, it will still be the character of an \( F \)-module \( L(D)_F \), and \( L(D)_F \) will decompose into irreducibles \( W_{jr} \). However in this case for each \( j \), the multiplicities of the \( W_{jr} \)'s may not be all the
same. The right hand side of equation (11) will then compute the average of these multiplicities:

\[ \frac{1}{d_j} \sum_{r=1}^{d_j} \langle \chi_{jr}, L(D) \rangle = \dim(W_{jr})(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^{M} (\dim(W_{jr}) - \dim(W_{H_{jr}})) \frac{R_{\ell}}{2}. \]

3. D IS NOT A PULLBACK

Now we wish to extend our results to the case where \( D \) is not necessarily the pullback of a divisor on \( Y = X/G \). For this we need to build on work previously done on this problem by Nakajima, Borne, Ellingsrud and Lønsted, Kock, Kani, and others. We refer to [B] for references. We start with two definitions: the ramification module of the cover \( X \to X/G \) and the equivariant degree of a divisor.

For any point \( P \in X(k) \), the inertia group \( G_P \) acts on the cotangent space of \( X(k) \) at \( P \) by a \( k \)-character \( \psi_P \). This character is the ramification character of \( X \) at \( P \). The ramification module is defined by

\[ \Gamma_G = \sum_{P \in X(k)_{ram}} \text{Ind}_{G_P}^{G} (\sum_{\ell=1}^{e_P} \ell \psi_P^\ell), \]

where \( e_P = |G_P| \). By Theorem 2 in [N], there is a unique \( G \)-module \( \Gamma_G^{\ast} \) such that

\[ \Gamma_G = |G| \Gamma_G^{\ast}. \]

In this paper we are only concerned with \( \Gamma_G \), so we abuse terminology and call \( \Gamma_G \) the ramification module.

Now consider a \( G \)-invariant divisor \( D \) on \( X(k) \). If \( D = \frac{1}{e_P} \sum_{g \in G} g(P) \) then we call \( D \) a reduced orbit. The reduced orbits generate the group of \( G \)-invariant divisors \( \text{Div}(X)^G \).

**Definition 3.** The **equivariant degree** is a map from \( \text{Div}(X)^G \) to the Grothendieck group \( R_k(G) = \mathbb{Z}[G_k^\ast] \) of virtual \( k \)-characters of \( G \),

\[ \text{deg}_{eq}: \text{Div}(X)^G \to R(G), \]

defined by the following conditions:

1. \( \text{deg}_{eq} \) is additive on \( G \)-invariant divisors of disjoint support,
2. If \( D = \frac{1}{e_P} \sum_{g \in G} g(P) \) is an orbit then

\[ \text{deg}_{eq}(D) = \begin{cases} \text{Ind}_{G_P}^{G} (\sum_{\ell=1}^{r} \psi_P^{-\ell}), & \text{if } r > 0, \\ -\text{Ind}_{G_P}^{G} (\sum_{\ell=0}^{-(r+1)} \psi_P^{\ell}), & \text{if } r < 0, \\ 0, & \text{if } r = 0, \end{cases} \]

where \( \psi_P \) is the ramification character of \( X \) at \( P \).

**Lemma 4.** (Borne’s formula) If \( D \) is a \( G \)-equivariant nonspecial divisor, then the (virtual) character of \( L(D) \) is given by

\[ \chi(L(D)) = (1 - g_Y)\chi(k[G]) + \text{deg}_{eq}(D) - \chi(\tilde{\Gamma}_G). \]

We derive the following from Borne’s formula and Theorem [B]. The notation is as in Section 1.
Proposition 5. If \( \tilde{\Gamma}_G \) has a \( \mathbb{Q}[G] \)-module structure, then it decomposes into irreducible \( \mathbb{Q}[G] \)-modules as

\[
\tilde{\Gamma}_G \simeq \bigoplus_j \frac{1}{n_j} \left( \sum_{\ell} (\dim(V_j) - \dim(V_j^{H_\ell})) \right) R_\ell V_j.
\]

Proof: The ramification module does not depend on the divisor, so we compare Theorem 1 with Borne’s formula in the case where \( D \) is a pullback. If \( D = \pi^*(D_0) \) is the pullback of a divisor \( D_0 \in \text{Div}(Y) \) then the equivariant degree \( \text{deg}_{eq}(D) \) has a very simple form. On each orbit, \( r \) is a multiple of \( e_P \), so every character of the cyclic group \( G_P \) appears. The equivariant degree on this orbit is induced from a multiple of the regular representation of \( G_P \). Thus we have

\[
(16) \quad \text{deg}_{eq}(D) = \text{deg}(D_0) \chi(k[G]),
\]

(This is also a special case of Corollary 3.10 in \([B]\).)

The first two terms of Borne’s formula then become

\[
(\text{deg } D_0 + 1 - g_Y) \chi(k[G]).
\]

This is clearly the character of a \( \mathbb{Q}[G] \)-module, so \( L(D) \) will have a \( \mathbb{Q}[G] \)-module structure if and only if \( \tilde{\Gamma}_G \) does. The rest of the proposition follows from Theorem 1. \( \square \)

Proposition 5 has also been proven by Köck \([K2]\), using a different method.

Corollary 6. Suppose that \( \tilde{\Gamma}_G \) has a \( \mathbb{Q}[G] \)-module structure. Let \( W \) be an irreducible \( F[G] \)-module. Then the multiplicity of the character of \( W \) in \( \tilde{\Gamma}_G \) is

\[
(17) \quad \sum_{\ell} (\dim(W) - \dim(W^{H_\ell})) \frac{R_\ell}{2}.
\]

Proof: The same as the proof of Theorem 1 from Theorem 1. \( \square \)

Remark. Again, the rationality criterion is necessary. If \( \tilde{\Gamma}_G \) does not have a \( \mathbb{Q}[G] \)-module structure, we get an average of multiplicities, similar to \( (14) \):

\[
(18) \quad \frac{1}{d_j} \sum \langle \chi_{jr}, \tilde{\Gamma}_G \rangle = \frac{M}{\sum_{\ell=1}^M (\dim(W_{jr}) - \dim(W_{jr}^{H_\ell}))} \frac{R_\ell}{2}
\]

with notation as in \([2]\).

4. Examples

Example 1. Consider the nonsingular projective curve \( X \) which is the closure of

\[
\{(x, y, t) \in \mathbb{C}^3 \mid y^2 = x(x - 2)(x - 4), \ t^2 = x + 4\}.
\]

This has an action of \( G = C_2 \times C_2 \) given by

\[
\begin{align*}
\alpha : (x, y, z) &\mapsto (x, -y, t), \\
\beta : (x, y, z) &\mapsto (x, y, -t), \\
\alpha \beta : (x, y, z) &\mapsto (x, -y, -t).
\end{align*}
\]
The quotient by $\beta$ is a degree two cover of an elliptic curve, ramified at the two points with $x = -4$, so $X$ has genus 2. The quotient $Y = X/G$ is the projective $x$-line.

The divisor

$$D = (0, 0, 2) + (0, 0, -2) + (-4, 8\sqrt{3}, 0) + (-4, -8\sqrt{3}, 0)$$

is $G$-equivariant, and $2D$ is the pullback of the divisor $D_0 = x = 0, x = -4$ on $Y$. From the Riemann-Roch theorem we know that $\dim L(2D) = 7$.

First, let us use Theorem 1 to decompose $L(2D)$ into irreducibles. The cyclic subgroups of $G$ are the trivial group, $H_1$ and each of the two-element subgroups generated by $\alpha$, $\beta$, and $\alpha\beta$. Let us call the last three $H_\alpha$, $H_\beta$, and $H_{\alpha\beta}$. Each is in its own conjugacy class.

The cover $X \to X/G$ has 5 branch points: three with inertia group $H_\alpha$ (at $x = 0, 2, 4$), one with inertia group $H_\beta$ (at $x = -4$), and one with inertia group $H_{\alpha\beta}$ (at $x = \infty$). This means

$$R_\alpha = 3, \quad R_\beta = 1, \quad R_{\alpha\beta} = 1.$$ 

The group $G$ has character table

|     | 1     | $\alpha$ | $\beta$ | $\alpha\beta$ |
|-----|-------|----------|---------|--------------|
| $\chi_1$ | 1     | 1        | 1       | 1            |
| $\chi_2$ | 1     | 1        | -1      | -1           |
| $\chi_3$ | 1     | -1       | 1       | -1           |
| $\chi_4$ | 1     | -1       | -1      | 1            |

Each irreducible representation is one dimensional, and every $\mathbb{C}[G]$-module is a $\mathbb{Q}[G]$-module, so $d_j$ and the Schur index $m_j$ are both 1. The dimension $\dim(V_j^{H_\ell})$ is 1 if the character of $V_j$ is 1 on the generator and 0 otherwise. From this we get:

$$n_1 = (2 + 1 - 0) - 0 = 3$$
$$n_2 = (2 + 1 - 0) - \frac{1}{2}(R_\beta + R_{\alpha\beta}) = 3 - 1 = 2$$
$$n_3 = (2 + 1 + 0) - \frac{1}{2}(R_\alpha + R_{\alpha\beta}) = 3 - 2 = 1$$
$$n_4 = (2 + 1 + 0) - \frac{1}{2}(R_\alpha + R_\beta) = 3 - 2 = 1.$$ 

Thus the character of $L(2D)$ is $3\chi_1 + 2\chi_2 + \chi_3 + \chi_4$.

Now let us consider $L(D)$. The Riemann-Roch theorem tells that that this will be a three dimensional space. Since $D$ is not a pullback from $Y$, we cannot use Theorem 1. However, the ramification module does have a $\mathbb{Q}[G]$-module structure, so we can use Proposition 3 with Borne’s formula. The calculations above tell us that the ramification module has character $\chi_2 + 2\chi_3 + 2\chi_4$.

Now we need to calculate the equivariant degree of $D$. The divisor consists of two reduced orbits, the orbit of $(0, 0, 2)$ and the orbit of $(-4, 8\sqrt{3}, 0)$. At the first point the inertia group is $H_\alpha$, and at the second point the inertia group is $H_\beta$. In both cases the ramification character is the nontrivial character of $C_2$. Adding the induced characters of $G$ gives us $\deg_{eq}(D) = \chi_2 + \chi_3 + 2\chi_4$. 

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Adding the pieces of Borne’s formula, we get the character of $L(D)$ to be $\chi_1 + \chi_2 + \chi_4$. In fact, one can check that the three functions $\{1, \frac{1}{2}, \frac{1}{2} \}$ form a basis for $L(D)$, and $G$ acts on the three basis elements by the three respective characters. □

Example 2. Let $X = \mathbb{P}^1$ and let $G$ be a cyclic group of prime order $q$. Let $a$ be a generator of $G$, and let $a$ act on $X$ by $z \mapsto \zeta z$, where $\zeta$ is a primitive $q$th root of unity. The cyclic subgroups of $G$ are the trivial group and $G$ itself; the irreducible representations of $G$ over $\mathbb{Q}$ are the one-dimensional trivial representation and a $q - 1$ dimensional representation $V$. Let $\psi$ be the character of $G$ over $\mathbb{C}$ whose value on $a$ is $\zeta$; then the irreducible characters of $G$ over $\mathbb{C}$ are the tensor powers $\psi, \psi \otimes \zeta, \ldots, \psi \otimes \zeta^{q-1}, \psi \otimes \zeta^q = 1$. The character of $V$ is $\psi + \psi^2 + \ldots + \psi^{q-1}$.

The cover $X \to X/G$ is totally ramified at 0 and $\infty$. The ramification module in this case is a $\mathbb{Q}[G]$-module, so we can use either Proposition 5 or Corollary 6 to find that

$$\tilde{\Gamma}_G = \psi + \psi^2 + \ldots + \psi^{q-1} = V.$$ □

The following example illustrates what can happen when the rationality condition is not met.

Example 3. Let $X$ be the Klein quartic

$$\{(x, y, z) \in \mathbb{P}^2 \mid x^3 y + y^3 z + z^3 x = 0 \}.$$ We assume that $k$ contains both cube roots of unity and 7th roots of unity; let $\omega$ be a primitive cube root of unity and $\zeta$ be a primitive seventh root of unity. Let $G$ be the group generated by

$$\begin{align*}
\sigma : (x : y : z) &\mapsto (y : z : x) \\
\tau : (x : y : z) &\mapsto (\zeta x : \zeta^4 y : \zeta^2 z)
\end{align*}$$

The group $G$ of automorphisms generated by these two actions is the semi-direct product $C_3 \rtimes C_7$. (This is not the full automorphism group of this curve.) $X$ has genus 2, and the quotient $Y = X/G$ has genus 0 [E].

The group $G$ has character table:

$$\begin{array}{c|cccccc}
\chi & e & \sigma & \tau & \sigma^{-1} & \tau^{-1} \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & \omega^2 & 1 & \omega & 1 \\
\chi_3 & 1 & \omega & 1 & \omega^2 & 1 \\
\chi_4 & 3 & 0 & \zeta^3 + \zeta^5 + \zeta^6 & 0 & \zeta + \zeta^2 + \zeta^4 \\
\chi_5 & 3 & 0 & \zeta + \zeta^2 + \zeta^4 & 0 & \zeta^3 + \zeta^5 + \zeta^6
\end{array}$$

There are two conjugacy classes of nontrivial cyclic subgroups, with representatives generated by $\sigma$ and $\tau$. Let $H_3 = \langle \sigma \rangle$ and $H_7 = \langle \tau \rangle$. The irreducible representations over $\mathbb{Q}$ have characters $\chi_1, \chi_2 + \chi_3,$ and $\chi_4 + \chi_5$. Each has Schur index 1.

The points of $X$ fixed by $H_7$ are $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0),$ and $P_3 = (0, 0, 1)$. These form one orbit under $G$, so $R_7 = 1$. There are seven points in the orbit of $(1 : \omega : \omega^2)$ and seven points in the orbit of $(1 : \omega^2 : \omega)$, all fixed by cyclic groups of order 3. Since these form two orbits, we have $R_3 = 2$.

This was obtained using [Gap]. Incidentally, there is only one non-cyclic group of order 21, up to isomorphism.
We now compute
\[
\sum_{\ell=1}^{M}(\dim(W) - \dim(W^{H_{\ell}}))\frac{R_{\ell}}{2},
\]
as in (18), for the irreducible representations over \( \mathbb{C} \). We find that
\[
\frac{1}{2} \langle \chi_2 + \chi_3, \tilde{\Gamma}_G \rangle = 1,
\]
\[
\frac{1}{2} \langle \chi_4 + \chi_5, \tilde{\Gamma}_G \rangle = \frac{7}{2}.
\]
These give the average multiplicities. In fact one can compute directly that \( \tilde{\Gamma}_G = \chi_2 + \chi_3 + 3\chi_4 + 4\chi_5 \).

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