Quantum expression of electrical conductivity from massless quark matter to hadron resonance gas in presence of magnetic field

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We have gone through the study of classical and quantum expressions of electrical conductivity and their numerical estimation in presence of a magnetic field for hadron resonance gas (HRG) as well as massless quark matter. In earlier paper Phys.Rev.D 102 (2020) 1, 016016, classical results of transport coefficients of HRG matter in presence of magnetic field are well discussed by using standard relaxation time approximation of Boltzmann equation and kinetic theory. During the transition from without to with field picture, an isotropic to anisotropic transition of transportation and during the shifting from massless quark matter to HRG matter, an upper or Stephan-Boltzmann type limit to non-perturbative hadronic domain estimation of transportation have been shown by that reference. By quickly revisiting that part, which was already explored in earlier paper, present work has focused on classical to quantum transition of HRG transportation when one goes from high temperature and low magnetic field to low temperature and high magnetic field domain. We have also compared the quantum modification of HRG results with that of massless quark matter, where an opposite trends are found. Similar kind of opposite quantum impact is noticed between boson and fermion due to their different distribution functions, which have been checked for pion or other mesons and proton or other baryons. Though HRG carry both mesons and baryons, but its net magneto-thermodynamic phase-space through Landau quantization reveals meson or boson dominated quantum modification. That is why quantum modification of HRG results are showing opposite trend from massless quark matter, which face fermionic quantum modification.

PACS numbers:

I. INTRODUCTION

Quark gluon plasma (QGP) shows a number of interesting phenomena in presence of magnetic field\textsuperscript{[1]}\textsuperscript{[2]}. The magnitude of magnetic field produced at RHIC for Au-Au collisions at $\sqrt{s} = 200$ GeV is of the order of $10^{13}$ Gauss and for Pb-Pb collisions at LHC is of the order $eB \sim 10^{20}$ Gauss. This is much larger than the $\Lambda_{QCD}^2$, where $\Lambda_{QCD} \approx 0.25$ GeV is the strong interaction scale. Comparing the magnitude of magnetic field produced in colliders with the magnetic field produced in neutron stars and magnetars, i.e $10^{14} - 10^{15}$ Gauss, the value is very large.

Understanding the impact of this high magnetic field on different transport coefficients of QGP is recently appeared to be an important research topic within the community of heavy ion physics. Recent Refs.\textsuperscript{[3]}\textsuperscript{[14]} have gone through the calculations of different transport coefficients like shear viscosity\textsuperscript{[8]}\textsuperscript{[19]}, bulk viscosity\textsuperscript{[20]}\textsuperscript{[24]} and electrical conductivity\textsuperscript{[13]}\textsuperscript{[15]}\textsuperscript{[16]}\textsuperscript{[26]}\textsuperscript{[29]}\textsuperscript{[37]} in presence of magnetic field. Among them, electrical conductivity plays an important role in life time of magnetic field, produced in heavy ion collision (HIC). For a large value of electrical conductivity ($\sigma$) of RHIC or LHC matter, the produced magnetic field can exists for longer time\textsuperscript{[5]}.

From the microscopic calculations, studied in earlier Refs.\textsuperscript{[13]}\textsuperscript{[14]}\textsuperscript{[16]}\textsuperscript{[23]}\textsuperscript{[37]}, we can get temperature and magnetic field dependent values of the electrical conductivity of RHIC or LHC matter. If we analyze those investigations minutely, we can find mainly two classes of calculations, done in classical and quantum pictures. Among the Refs.\textsuperscript{[13]}\textsuperscript{[15]}\textsuperscript{[17]}\textsuperscript{[25]}\textsuperscript{[29]}\textsuperscript{[31]}\textsuperscript{[37]} Refs.\textsuperscript{[13]}\textsuperscript{[15]}\textsuperscript{[17]}\textsuperscript{[25]}\textsuperscript{[29]}\textsuperscript{[31]}\textsuperscript{[37]} have gone through the classical expressions of electrical conductivity, whose multi-component values clearly show the anisotropy in coordinate space. However, they have not considered the quantum aspects via Landau quantization. In the context of neutron star, Ref.\textsuperscript{[29]} has used the similar classical expression of conductivity by considering $10^{14}$ Gauss magnetic field, beyond which quantum effects should be considered, as done in Ref.\textsuperscript{[28]}. In quantum picture, the motion of electrons suffers Landau quantization in the plane perpendicular to the magnetic field. Refs.\textsuperscript{[25]}\textsuperscript{[27]}\textsuperscript{[28]}\textsuperscript{[32]}\textsuperscript{[34]}\textsuperscript{[36]} have considered this Landau quantization, where most of them\textsuperscript{[25]}\textsuperscript{[27]}\textsuperscript{[28]}\textsuperscript{[32]} have gone through the lowest Landau level (LLL) approximation, applicable for strong field limit. In LLL approximation, multi-components structure of conductivity is converted to only one component along the direction of magnetic field, known as longitudinal conductivity. The classical framework will be well applicable in weak field range, while LLL approximated quantum framework will be useful for strong field limit. During this weak to strong field transition, how does the multi-component classical expression change to one component quantum expression of conductivity? This question is attempted to reply in present work via numerical benchmark of ideal hadron...
resonance gas (HRG) model, which is quite famous as alternative description of QCD thermodynamics within hadronic temperature range \[39\].

In earlier Refs. \([13, 17, 33]\), HRG estimations of multicomponent transport coefficients in presence of magnetic field are done by using classical expressions but their quantum extensions are not done yet. Present work is going to fulfill the gap by exploring the electrical conductivity only, since mathematical extension of other transport coefficients are more or less same.

The article is organized as follows. Next section \([II]\) address all working formulas of multi-components of conductivity at finite magnetic field with quick description of kinetic theory formalism based on relaxation time approximation (RTA). This formalism section is divided by three subsections, where subsection \([IIA]\) is devoted for general RTA expressions of anisotropic conductivity tensors, which we call as classical expressions, then in subsection \([IIB]\), their quantum extensions are addressed and after that, in subsection \([IIC]\), HRG version of classical and quantum expressions both are addressed, which behave as working formulae of result section i.e. Sec. \([III]\). In result section, classical to quantum transition for HRG model are graphically sketched and discussed. At the end, Sec. \([IV]\) has summarized the investigation.

II. FORMALISM

Here, we will discuss about formalism part step by step so that we can see the changes from classical to quantum expressions and finally towards their HRG version expressions. We know that the conductivity tensors lose its isotropic nature in presence of magnetic field and we will get different values or expressions of its parallel and perpendicular components with a new component, called Hall conductivity. Using relaxation time approximation (RTA) based kinetic theory approach, one can obtain that anisotropic conductivity tensor \([13, 29, 40]\), whose expressions are considered here as classical (CL) expression as concept of Landau quantization is missing there. However, we are considering quantum aspects of statistical mechanics by using Fermi-Dirac (FD) and Bose-Einstein (BE) distribution functions. So, in that point of view, marking CL might not be completely exact, but still let us proceed with this informal notation. Describing this derivation of CL expressions in next subsection \([IIA]\), we have addressed their quantum (QM) expressions in subsection \([IIB]\) and finally, the HRG version of CL and QM expressions are formulated in subsection \([IIC]\). They are discussed below one by one.

A. Classical expressions of electrical conductivity

Reader can find the detailed derivation CL expressions of anisotropic conductivity tensor in Refs. \([13, 29, 40]\), but for our sake of completeness, let us quickly cover it here.

Let us consider an electric field \(\mathbf{E} = E_x \hat{x}\) is applied to a relativistic charged fermion/boson fluid, for which a current density is obtained along the same direction \(\mathbf{J} = J_x \hat{x}\). Hence, macroscopic Ohm’s law can be written as

\[
J_x = \sigma_{xx} E_x ,
\]

where \(\sigma_{xx}\) is the electrical conductivity. In microscopic picture of dissipation, equilibrium distribution function of fermion/boson,

\[
f_0 = \frac{1}{e^{\beta g} + 1} ,
\]

undergoes a small deviation

\[
\delta f \propto \left( \frac{\partial f_0}{\partial \omega} \right) ,
\]

\[
\delta f = -\phi \left( \frac{\partial f_0}{\partial \omega} \right) = -\alpha (\mathbf{k} \cdot \mathbf{E}) \left( \frac{\beta f_0}{\omega} \right) = \alpha (k_x E_x) \beta f_0 (1 \mp f_0) ,
\]

and therefore, one can express (dissipative) current density as \([13, 14, 29, 40]\)

\[
J_x = g \hat{e} \int \frac{d^3k}{(2\pi)^3} \frac{k_x}{\omega} \delta f = \left[ g \hat{e} \beta \int \frac{d^3k}{(2\pi)^3} \frac{k_x^2}{\omega} \alpha f_0 (1 \mp f_0) \right] E_x ,
\]

where \(\hat{e}\) is the degeneracy factor (excluding charge-flavor degeneracy), \(\hat{e}\) is electric charge and \(\omega = \{k_x^2 + m^2\}^{1/2}\) is energy of fermion/boson. To find out the constant \(\alpha\), we take help of relaxation time approximated-relativistic Boltzmann equation (RTA-RBE),

\[
-\hat{e} \mathbf{E} \cdot \nabla_k f_0 = -\delta f / \tau_c \Rightarrow \delta f = \tau_c \hat{e} \mathbf{E} \left[ \frac{k_x}{\omega} \frac{\partial f_0}{\partial \omega} \right] = \tau_c \hat{e} E_x \left( \frac{k_x}{\omega} \right) \left( \beta f_0 (1 \mp f_0) \right) ,
\]

and then comparing Eq. \((5)\) and \((3)\), we get \([13, 14, 29, 40]\)

\[
\alpha = \frac{\hat{e} \tau_c}{\omega} .
\]

Using above \(\alpha\) in Eq. \((4)\) and then comparing with Eq. \((1)\), we get expression of electrical conductivity which give rise to electric current in x direction as \([13, 14, 29, 40]\),

\[
\sigma_{xx} = g \hat{e}^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{k_x^2}{\omega^2} f_0 (1 \mp f_0) .
\]

Next, we will proceed to derive the electrical conductivity in presence of magnetic field \(\mathbf{B} = B \hat{z}\) \([13, 14, 29\).
So, final form of deviation becomes

\[ -\hat{\varepsilon}(E + \frac{k}{\omega} \times B) \cdot \nabla_k f_0 = -\frac{\delta f}{\tau_c} \]

\[ -\hat{\varepsilon}(E + \frac{k}{\omega} \times B) \cdot (\frac{k}{\omega} \frac{\partial f_0}{\partial \omega}) = -\frac{\delta f}{\tau_c}. \quad (8) \]

It is because of vector identity \((k \times B)k = B \cdot (k \times k) = 0\), the second term of the left hand side will be vanished, so we consider the \(\nabla_k(\delta f)\) term also in RTA-RBE,

\[ -\hat{\varepsilon}E \cdot \left(\frac{k}{\omega} \frac{\partial f_0}{\partial \omega} - \frac{k}{\omega} (k \times B) \cdot \nabla_k(\delta f) = -\delta f/\tau_c, \quad (9) \]

where we assume \(\delta f = -\phi \frac{\partial f_0}{\partial \omega}\) with \(\phi = k \cdot F\). Now, using the standard vector identity,

\[ (\frac{k}{\omega} \times B) \cdot \nabla_k(\delta f) = - (\frac{k}{\omega} \times B) \cdot \nabla_k(k \cdot F) \frac{\partial f_0}{\partial \omega} \]

Equating the coefficients of \(\hat{x}, \hat{z}\) and \(\hat{y}\) of Eq. \((13)\), we get

\[ A_z = 0 \]

\[ A_x = -\hat{\varepsilon} + \frac{\hat{\varepsilon}}{1 + (\tau_c/\tau_B)^2} \frac{\tau_c}{\omega} E_x \]

\[ A_y = -\hat{\varepsilon} \frac{\tau_c}{1 + (\tau_c/\tau_B)^2} \frac{\tau_c}{\omega} E_x \]

where \(\tau_B = \omega/(\varepsilon B)\) is inverse of synchrotron frequency. So, final form of deviation becomes

\[ \delta f = \frac{\varepsilon}{\omega} \left\{ \frac{\tau_c}{\tau_B} \left( \hat{x} + \frac{\tau_c}{\tau_B} \hat{y} \right) \right\} \frac{1}{1 + (\tau_c/\tau_B)^2} \frac{\partial f_0}{\partial \omega} \]

\[ = -\hat{\varepsilon} \tau_c \left( \frac{k_x}{\omega} + \frac{k_y}{\omega} \right) \frac{E_x}{1 + (\tau_c/\tau_B)^2} \beta f_0(1 \mp f_0) \]

Now, using this \(\delta f\) in matrix form of Ohm’s law,

\[ \left( \begin{array}{c} J_x \\ J_y \end{array} \right) = \left( \begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{array} \right) \left( \begin{array}{c} E_x \\ 0 \end{array} \right) \]

one can obtain

\[ \sigma_{xx} = g^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{1}{(1 + (\tau_c/\tau_B)^2) \frac{k_x^2}{\omega^2}} f_0(1 \mp f_0) \]

\[ = g^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{1}{1 + (\tau_c/\tau_B)^2} \frac{k_x^2}{\omega^2} f_0(1 \mp f_0) \]

\[ \sigma_{xy} = g^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{\tau_c}{1 + (\tau_c/\tau_B)^2} \frac{k_x^2}{\omega^2} f_0(1 \mp f_0) \]

in Eq. \((9)\), we will get

\[ \left( \frac{k}{\omega} \right) \cdot \left[ -\hat{\varepsilon}E + \hat{\varepsilon}(B \times F) \right] = k \cdot F/\tau_c. \quad (11) \]

In general, we can consider

\[ F = (A_x \hat{x} + A_z \hat{z} + A_y (\hat{x} \times \hat{z})), \]

for which Eq. \((11)\) becomes

\[ = g^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{k_x^2}{\omega^2} f_0(1 \mp f_0) \]

where \(g^2 = 2 \times 3 \times 4 \times \left( \frac{4^2}{9} + \frac{8^2}{9} + \frac{8^2}{9} \right) = 8 \varepsilon^2\) for 3 flavor quark matter \([14, 34]\). For other relevant system or matter (e.g. HRG matter), we have to consider corresponding \(g^2\) values.

Similarly, \(\sigma_{yy}, \sigma_{xy}\) can be obtained by repeating same calculation for \(E = E_y \hat{y}\) and they are related as \(\sigma_{xx} = \sigma_{yy}, \sigma_{xy} = -\sigma_{yx}\). Longitudinal conductivity along z-axis will remain unaffected by magnetic field, because Lorentz force never work along the direction of magnetic field (at least classically). Hence, the classical expression of the longitudinal conductivity will be

\[ \sigma_{zz} = g^2 \beta \int \frac{d^3k}{(2\pi)^3} \tau_c \frac{k_x^2}{\omega^2} f_0(1 \mp f_0), \]

Regarding Hall component \(\sigma_{xx} = \sigma_{xy} = -\sigma_{yx}\) of quark matter, we have to understand one fact that additional term \(\tau_c/\tau_B\) in numerator will carry opposite sign for particle and anti-particle cases and so, net Hall component will be zero at vanishing quark/baryon chemical potential \((\mu = 0)\). In that context, the Eq. \((15)\) is bit of misleading as particle-anti-particle degeneracy factor 2 will not go to \(g^2\), rather we have to sum their contributions. Since our present focus on conductivity components of RHIC or LHC matter with almost zero quark/baryon chemical
potential, so we will focus only its parallel component \( \sigma_\parallel = \sigma_{xx} = \sigma_{yy} \) and perpendicular component \( \sigma_\perp = \sigma_{zz} \) only as Hall component will be zero.

### B. Quantum expressions of electrical conductivity

Now, this scenario will be changed in quantum description via Landau quantizations. The main modification will occur in energy \( \omega \) and phase space \( \int d^3k \) by the following replacements:

\[
\omega = (k^2 + m^2)^{1/2} \quad \rightarrow \quad \omega_l = (k_z^2 + m^2 + 2l|\vec{e}|B)^{1/2},
\]

\[
2 \int \frac{d^3k}{(2\pi)^3} \quad \rightarrow \quad \sum_{l=0}^{\infty} \alpha_l \frac{|\vec{e}|B}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi}.
\]

where spin degeneracy 2 in left hand side of last line will be converted to \( \alpha_l \), which will be 2 for all Landau levels \( l \), except lowest Landau level (LLL) \( l = 0 \), where \( \alpha_l = 1 \). In general, one can write \( \alpha_l = 2 - \delta_{l,0} \). Here, we also assume roughly, \( k_x^2 \approx k_y^2 \approx \frac{k_z^2 + k_{z,\perp}^2}{2} = \frac{2|\vec{e}|B}{3} \), then conductivities can be expressed as

\[
\sigma^{xx} = g e^2 \beta \sum_{l=0}^{\infty} \alpha_l \frac{|\vec{e}|B}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{1}{\omega_l^2} \frac{1}{\tau_c} \frac{1}{1 + (\tau_c/\tau_B)^2} f_0(\omega_l)[1 - f_0(\omega_l)],
\]

\[
\sigma^{zz} = g e^2 \beta \sum_{l=0}^{\infty} \alpha_l \frac{|\vec{e}|B}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{k_z^2}{\omega_l^2} \frac{1}{\tau_c} f_0(\omega_l)[1 - f_0(\omega_l)].
\]

Most of the earlier works [25, 27, 28, 32], dealing Landau quantization, considered only longitudinal conductivity \( \sigma^{zz} \) for \( l = 0 \), which is known as LLL approximation. At very high magnetic field this scenario might be achieved, where all medium constituents sit on \( l = 0 \) energy level. It means that perpendicular motion of medium constituents are completely disappeared as \( k_x \approx k_y \approx 0 \) at \( l = 0 \). Therefore, in this LLL case, \( \sigma^{xx} \approx \sigma^{yy} \approx 0 \) and \( \sigma^{zz} \neq 0 \). However, below that strong magnetic field, \( l > 0 \) energy levels might have some non-negligible contributions, so one might estimate all components to visualize the exact anisotropic picture in quantum case along with the classical one.

### C. Classical and quantum expression of electrical conductivity under the hardon resonance gas model

Next, we go to the HRG model calculation and see their transition from classical to quantum pictures. Massless case might be considered as non-interacting or Stefan Boltzmann (SB) limit type picture, while HRG calculation will map the interacting picture. In the magnetic field picture, we can classify hadrons into

1. charged mesons \( M \), which are basically bosons,
2. charged baryons \( B \), which are basically fermions,

which will be summed finally. Neutral hadrons don’t have any role in electrical conductivity. In classical picture (without considering Landau quantization), perpendicular and parallel components of electrical conductivity (Eqs. (17) and (19)), in the HRG model can be expressed as:

\[
\sigma^{xx} = \sum_{M,B} \frac{g e^2 \beta}{(2\pi)^3} \tau_c \frac{1}{1 + (\tau_c/\tau_B)^2} \frac{k^2}{3\omega^2} f_0(1 \pm f_0),
\]

\[
\sigma^{zz} = \sum_{M,B} \frac{g e^2 \beta}{(2\pi)^3} \tau_c \frac{k_z^2}{3\omega^2} f_0(1 \pm f_0).
\]
\begin{align}
\sigma^{xx} &= \sum_{M} g_1 e^2 \beta \frac{|e|B}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \left( l + \frac{1}{2} \right) \frac{|\bar{\epsilon}|B}{\omega_l^2} r_c \frac{1}{1 + (\tau_c/\tau_B)^2} f_0(\omega_l)[1 + f_0(\omega_l)] \\
+ \sum_{B} g_2 e^2 \beta \frac{|e|B}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{k_z^2}{\omega_l^2} r_c f_0(\omega_l)[1 + f_0(\omega_l)] \\
\sigma^{zz} &= \sum_{M} g_1 e^2 \beta \frac{|e|B}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{k_z^2}{\omega_l^2} r_c f_0(\omega_l)[1 + f_0(\omega_l)] \\
+ \sum_{B} g_2 e^2 \beta \frac{|e|B}{2\pi} \sum_{l=0}^{\infty} \alpha_l \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{k_z^2}{\omega_l^2} r_c f_0(\omega_l)[1 - f_0(\omega_l)].
\end{align}

For different spin particles expressions for energy \( \omega_l \) and spin degeneracy \( \alpha_l \) of hadrons in \( l \) Landau level are given in the Table (I) \[41,42\]. To make our life simple, we have considered only mesons with spin 0 and 1 and baryons with spin 1/2 and 3/2. Higher spin hadrons are not considered and that may be justified because their thermodynamic weight factors are quite less due to their heavy masses.

| Particle species | Spin | \( \omega_l \) | \( \alpha_l \) |
|------------------|------|----------------|----------------|
| Baryon           | 1/2  | \( (k_z^2 + m^2 + 2l|\bar{\epsilon}|B)^{1/2} \) | 2 - \( \delta_{10} \) |
| Baryon           | 3/2  | \( (k_z^2 + m^2 + 2l|\bar{\epsilon}|B)^{1/2} \) | 4 - 2\( \delta_{10} \) - \( \delta_{11} \) |
| Meson            | 0    | \( (k_z^2 + m^2 + (2l + 1)|\bar{\epsilon}|B)^{1/2} \) | 1 |
| Meson            | 1    | \( (k_z^2 + m^2 + (2l + 1)|\bar{\epsilon}|B)^{1/2} \) | 3 - \( \delta_{10} \) |

TABLE I: Particles energy and degeneracy

Being fermion, perpendicular momenta of baryons are considered as \( k_z^2 \approx k_y^2 \approx (k_x^2 + k_y^2) = \frac{2|\bar{\epsilon}|B}{2} \), whereas, due to bosonic property, perpendicular momenta of mesons will be \( k_x^2 \approx k_y^2 \approx \left( \frac{k_y^2 + k_z^2}{2} \right) = \frac{(2 + 1)|\bar{\epsilon}|B}{2} \). Owing to this fact, we will get an interesting difference between massless quark matter and HRG systems. For LLL case, perpendicular component will be vanished for quark matter but not for HRG systems. A non-zero values can be obtained from charge mesons. This fact will be discussed graphically in results section.

Here, for HRG system, we have to take care about the values of factors \( g_1 e^2 \) for different hadrons, which are basically multiplication of degeneracy factor \( g \) and their square of electric charge \( e^2 \). For example, \( \pi^+ \) meson with mass 140 MeV, spin 0 has \( g_1 e^2 = 1 \times e^2 \); \( \rho^+ \) meson with mass 770 MeV, spin 1 has \( g_1 e^2 = 3 \times e^2 \); \( \Delta^+ \) baryon with mass 1232 MeV, spin 3/2 has \( g_1 e^2 = 4 \times 4e^2 \) etc.

III. RESULTS

Let us start our discussion with massless quark matter first, which might be considered as non-interacting or Stefan Boltzmann (SB) limit type estimations. Since gluon will not take part in conductivity, so we will not consider it. Massless quark matter means that we are considering all degeneracy factors like 3 color, 2 spin, 2 particle-anti-particle and 3 flavor (with appropriate electric charge) degeneracy factors. Though massless s quark consideration like massless u and d quarks is bit of hard approximation but at very high temperature limit, QGP can reach this massless or SB limits. After the estimation of massless quark matter case, we will go for HRG calculations, which may be considered as the interacting QCD estimations as its thermodynamics matches with lattice quantum chromodynamics (LQCD) thermodynamics data within hadron temperature range. Similar to thermodynamical quantities like pressure, energy density, entropy density, whose massless or non-interacting or SB limit estimations are considered as upper reference values, here also normalized conductivity components of massless quark matter will be considered as that reference point. Next, when we go for HRG estimations of conductivity components, these will represent their interacting estimation, which are expected to be suppressed from their massless limits within hadronic temperature domain like thermodynamical quantities. Using classical expressions of massless quark matter and HRG models, given in Eqs. \[17\], \[19\] and Eqs. \[23\], \[24\] respectively, this non-interacting to interacting estimations transformation of conductivity tensors are well discussed.
horizontal line in Figs. 1 and 2, which will act as our reference point like SB lines for thermodynamical quantities [13, 14]. Next, using QM Eq. (22) of $\sigma_\parallel$, we will get its deviation from horizontal line in the high $B$ and low $T$ domain. QM curve of $\sigma_\parallel$ - grey double-dash-dotted lines in Figs. 1 and 2 are exploring this fact. Due to Landau quantization, the transition from integration to Landau level summation, denoted in Eq. (21), we are getting this deviation, which increases towards low $T$ and high $eB$ domain. On the other hand, due to equivalence between integration and Landau level summation in high $T$ and low $eB$ range, QM and CL curves of $\sigma_\parallel$ are merging. Similar enhancement in low $T$ domain is noticed in the left panel of Fig. (10) of Ref. [13], when they draw normalized entropy density (also valid for other thermodynamical quantities) for SB limits. So, SB or massless limits of normalized entropy density ($s/T^3$) or other thermodynamical quantities and normalized longitudinal (parallel) conductivity ($\sigma_\parallel/\tau_c T^2$) exhibit same pattern - their horizontal lines are deviated (enhanced) in low $T$ domain due to Landau quantization. This Landau quantization effect is visible in low $T$ and high $eB$ domain, which is quite general fact and can be found for all thermodynamical quantities and transport coefficients. Based on that fact, we can grossly assume low $T$, high $eB$ as quantum domain and high $T$, low $eB$ as classical domain. Interestingly, these normalized quantities ($s/T^3, \sigma_\parallel/\tau_c T^2$) of massless quark matter, we basically find an enhancement of magneto-thermodynamical phase-space in low $T$ and high $eB$ domain.

We have also generated the lowest Landau level (LLL) estimation of $\sigma_\parallel/\tau_c T^2$ by imposing $l = 0$ in Eq. (22) and drawn by cyan dash-double dotted line in Figs. 1 and 2. One can notice the large deviation between full Landau level summation curve (grey double-dash-dotted lines) and LLL curve (cyan dash-double dotted). They are tending to merge towards high $eB$ and low $T$ domain as expected. However, within the RHIC/LHC matter coverage, say $T = 0.1-0.4$ GeV and $eB = 1-10 m_π^2$, this merging between LLL approximation and actual QM estimation is not probably possible, according to our graphs. Analyzing Figs. 2, we can say for massless QGP at $T = 0.150$ GeV that Landau quantization effect can be well noticed beyond $eB = m_π^2$ and LLL will not be a good approximation within $eB = (1-10)m_π^2$. So, neither CL expressions nor extreme QM expressions i.e. LLL expressions will be valid framework of conductivity for RHIC/LHC phenomenology, where $eB = 1-10 m_π^2$ magnetic field are expected. Present work is intended to communicate this message.

Next, massless quark matter case of perpendicular conductivity $\sigma_\perp$ from Eq. (17) can be written in simple form

$$\sigma_\perp = 12 \frac{\zeta(2)}{3 \pi^2} \tau_c T^2 \sum_{f=u,d,s} \frac{e_f^2}{1 + (\tau_c e_f B/3T)^2},$$

where $e_f = +\frac{2}{3}e, -\frac{1}{3}e, -\frac{1}{3}e$ for $f = u, d, s$ and average energy of massless quark is $3T$.
enhance probability in quantum domain (low Landau quantization of FD distribution always provide suppressions of baryons and mesons, these different statistical values in quantum domain.

Their collective impact ultimately provide suppressed values in thermodynamical quantities in LQCD calculations are realized as non-perturbative source of QCD, which is well mapped alternatively by HRG model. Hence, reader may consider grossly Figs. (1) and (3) as pQCD/massless limits and non-pQCD values of normalized conductivity, expected in high temperature and low (hadronic) temperature domain respectively. This fact, using CL expressions of massless and HRG systems are well analyzed in Ref. [15]. For present work, we should focus on its QM modification only and some interesting findings are as follows. For \( \sigma_\parallel / (\tau_e T^2) \), QM > CL in low \( T \) and high \( eB \) domain for massless case but QM < CL for HRG case. Since HRG estimation is bit of complex due to mixture of bosons and fermions, so their individual investigations will be helpful to understand the reason of suppressed QM values. They will be discussed latter.

To understand the detail cyclotron motions of different charge hadrons, we have calculated their average cyclotron time (inverse of cyclotron frequency)

\[
\tau_B(m) = \frac{\omega_{av}}{|e|B} = \frac{1}{|e|B} \int d^3p \omega f_0(m)
\]

and then plotted it against mass \( m \) axis. Different points represent differen hadrons in their respective masses. It is quite straight forward that \( \tau_B \) will increase with \( m \) and decrease with \( |e| \) as reflecting in the graphs. Most of the charge hadrons, having charge \( |e| \) = \( |e| \), remain within a particular slope of \( \tau_B \) vs \( m \) but few are deviated due to their charges \( |e| \). So, light mass and heavy charge hadrons have smaller \( \tau_B \), which will take important role to suppress perpendicular conductivity component due to the anisotropic factor \( \tau/\tau_B \).
FIG. 6: Perpendicular component of electrical conductivity as a function of magnetic field for pion and proton.

In Fig. 6 we have plotted $\perp$ (xx) and $\parallel$ (zz) components of $\sigma$ for cases of classical (Cl), quantum (QM), and lowest Landau level (LLL) of HRG system, as a function of magnetic field, at $T = 0.150$ GeV and fixed relaxation time $\tau_c = 1$ fm. Now if we collectively take a glance on Figs. (3) and (5) for HRG estimations and compare with Figs. (1) and (2) for massless quark matter estimations, then our focal findings (which are more visible in quantum domain i.e. at low $T$ and high $eB$) can be summarized briefly in Table (II). We should be confined within hadronic temperature domain $T \approx 0.100-0.170$ GeV and experimentally expected magnetic field range $eB \approx (0-10)m_\pi^2$ for the tabulated outcomes. At a glance, massless quark matter and HRG matter outcomes are quite opposite to each other but the fact will be more clear, when we individually analyze proton and pion matter contributions. These are discussed in next paragraphs for perpendicular conductivity component only, although Table (III) carry the gross findings for both components.

From the tabulated outcomes, one can guess the opposite quantum impact on meson ($\pi$) and baryon ($P$) due to their bosonic and fermionic properties respectively. Another point is that qualitative outcomes of HRG and pion matter same, which indicate mesons dominating role over baryons in HRG estimation. Larger thermodynamical probability of mesons due to their low masses are expected reason for their dominance in HRG estimations. Let us try to explore this fact graphically for one of the component, say $\perp$ (or xx) component. In Fig. 6, we have plotted $\sigma_{\perp}/(\tau_c T^2)$ as a function of $eB/m_\pi^2$ for a pion (upper panel) and a proton (lower panel) for $T = 0.150$ GeV and $\tau_c = 1$ fm. Here, conductivity for $\pi^+$ is much higher than that of p, which is because of small mass of pion. We notice that for pion case, QM results (red dash line) are larger than CL results (black solid line) but difference is too small to distinguish. Whereas for proton case, opposite ranking is distinctly observed. Being boson and fermion, magneto-thermodynamical phase space of pion and proton are enhanced and suppressed due to their (Landau) quantized BE and FD distribution functions respectively.

With respect to CL results, QM enhancement or suppression of pion and proton matter are well described in Fig. 7, where $\left(\frac{\sigma_{\perp}^{\text{QM}} - \sigma_{\perp}^{\text{Cl}}}{\sigma_{\perp}^{\text{Cl}}}\right) \times 100\%$ as a function of $eB/m_\pi^2$ for HRG (black solid line), $\pi^+$ (red dotted line), and $P$ (blue dash line) matters. Notice that below $eB = 1m_\pi^2$,
deviation is less than 1%. As the magnetic field increase, deviation in HRG model first increase slowly, reach a maximum of \( \sim 3\% \) then decrease and reach 0% deviation at \( eB \approx 10m_n^2 \), beyond which enhance to suppression trend will start. Although, we are confining our discussion within \( eB \approx 0-10m_n^2 \). By seeing the QM enhancement and suppression of pion and proton cases respectively, one may expect all mesons and baryons follow similar trends and their collective effect will be reflected in HRG estimation. During this competition between meson and baryon contribution, former become dominant over later within \( eB \approx 0-10m_n^2 \) at \( T = 0.150 \text{ GeV} \). Hence, being meson dominated, HRG results are showing effectively bosonic quantum modification and that is why it is opposite to the results of massless quark matter, which face fermionic quantum modification.

At the end, in a single portrait, Fig. \( \text{S} \) has summarized the classical outcomes with isotropic to anisotropic transition and massless quark matter to HRG matter transition, addressed in earlier Ref. \[15\] and their quantum outcomes, addressed in present article. It covers the typical range of temperature \( T = 0.1-0.4 \text{ GeV} \) and magnetic field \( eB = 0-10m_n^2 \), expected in LHC/RHIC experiments. Although reader should take this range very roughly. A gross trajectory line in \( B-T \) plane from high \( T, B \) (massless) quark matter to low \( T, B \) HRG matter is expected as shown in the Fig. \( \text{S} \). Now, based on the studies of Ref. \[15\] and present work, reader can identify the high \( T \), low \( B \) zone as isotropic and classical transportation domain and low \( T \), high \( B \) zone as anisotropic and quantum transportation domain. However, for phenomenological purpose, one should go through actual LHC/RHIC trajectory in \( B-T \) plane and accordingly find the all three possible transitions - isostropic to anisotropic, classical to quantum and massless quark matter to HRG matter.

**IV. SUMMARY**

In summary, we have explored comparative estimations of classical and quantum expressions of electrical conductivity in presence of magnetic field. First, we have obtained the massless case results for quark gluon plasma, where gluons do not take part in electrical conduction, and then for interacting QCD results, we have adopted HRG model calculations. With respect to magnetic field three will be three components of conduction - parallel, perpendicular and Hall components. At zero quark/baryon chemical potential, medium carry equal number of opposite electrical charges, so Hall conditions will disappear but one can find a non-zero Hall conduction in dense matter with non-zero quark/baryon chemical potential. In classical (without considering Landau quantization) and quantum both picture, parallel and perpendicular conductivity become different when external magnetic field will be applied. Hence, from low to high magnetic field, isotropic to anisotropic conduction is established in both picture. Classical estimation of conductivity tensors for massless quark matter and HRG matter are well explored in Refs. \[15\]. Present work is aimed to find their quantum extension by introducing the Landau quantization technique. Our estimation is focused within hadronic temperature domain \( T = 0.100-0.170 \text{ GeV} \) and magnetic field domain \( eB = 0-10m_n^2 \), within which our QM-version findings can be summarized in bullet points:

- **QM enhancement is observed in parallel/longitudinal conductivity for massless quark matter, proton (baryonic) matter in presence of magnetic field.**
- **QM suppression is observed in perpendicular/transverse conductivity for massless quark matter, proton (baryonic) matter in presence of magnetic field.**
- **QM suppression is observed in parallel/longitudinal conductivity for pionic (mesonic) matter, HRG matter in presence of magnetic field.**
- **QM enhancement is observed in perpendicular/transverse conductivity for pionic (mesonic) matter, HRG matter in presence of magnetic field.**

This opposite QM effect (suppression and enhancement) between bosons and fermions are coming due to their respective BE and FD distribution functions, through which Landau level summation is revealing in two different ways. Hence, present work indicate (probably first time) about non-negligible quantum effect in transport coefficients estimation for HRG matter, which might be applicable to other HRG-based phenomenology. We also have shown that lowest Landau level approximation of HRG estimation is not at all a good approximation within our focal temperature and magnetic field ranges. Full Landau level summation is recommended for quantum version of HRG estimations in presence of magnetic field. Present work has highlighted for transport coefficient like electrical conductivity but we think that it will be valid for other transport coefficients like shear viscosity, bulk viscosity, thermal conductivity etc as well as for other HRG related phenomenology, which might be investigated in future.

**Acknowledgment:** S. Satapathy acknowledge to the fellowship, funded by DST INSPIRE Faculty scheme of Research Project (IFA18-PH220) and to principle investigator of that project (Dr. Sudipan De).

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