Vortices in De Sitter Spacetimes

A.M. Ghezelbash† and R. B. Mann‡
††Department of Physics, University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada
†Department of Physics, Alzahra University,
Tehran 19834, Iran

January 8, 2022

Abstract

We investigate vortex solutions to the Abelian Higgs field equations in a four dimensional de Sitter spacetime background. We obtain both static and dynamic solutions with axial symmetry that are generalizations of the Nielsen-Olesen gauge vortices in flat spacetime. The static solution is located in the static patch of de Sitter space. We numerically solve the field equations in an inflationary (big bang) patch and find a time dependent vortex solution, whose effect to create a deficit angle in the spacetime. We show that this solution can be interpreted in terms of a renormalization group flow in accord with a generalized $c$-theorem, providing evidence in favour of a dS/CFT correspondence.
1 Introduction

De Sitter spacetimes are becoming of increasing interest in theoretical physics for a variety of reasons. They provide interesting arenas in which to study the classical no-hair conjecture first proposed by Ruffini and Wheeler [1], which states that after a given distribution of matter undergoes complete gravitational collapse, the only long range information of the resultant black hole is its electromagnetic charge, mass and angular momentum. The verification of this conjecture for a scalar field minimally coupled to gravity in asymptotically flat black-hole spacetimes has been extended to its de Sitter counterpart [2],[3]. While it is tempting to extend the no-hair theorem claim to all forms of matter, it is known that some long range Yang-Mills and/or quantum hair can be “painted” on a black hole [4]. Explicit calculations have been carried out which verify the existence of a long range Nielsen-Olesen vortex solution as a single stable hair for a Schwarzschild black hole in four dimensions [5], although it might be argued that this situation falls outside the scope of the classical no-hair theorem due to the non trivial topology of the string configuration.

Another motivation for studying de Sitter (dS) spacetimes is connected with the recently proposed holographic duality between quantum gravity on dS spacetime and a quantum field theory living on the past boundary of dS spacetime [6]. This proposed correspondence is undergoing extensive study in various directions [7]-[17]. So far this work suggests that the conjectured dS/CFT correspondence has a lot of similarity with the AdS/CFT correspondence, although some interpretive issues remain [13].

Motivated by these considerations, we pursue in this paper a study of vortex solutions in de Sitter spacetimes. We have already shown that the $U(1)$ Higgs field equations have a vortex solution in both a four dimensional AdS spacetime [18] and in a four dimensional Schwarzschild-AdS background [19]. Employing the well known AdS/CFT correspondence conjecture, the boundary conformal field theory can detect the presence of the vortex in the four dimensional AdS spacetime: the mass density of the vortex solution is encoded in the discontinuity of the two-point correlation function of the dual conformal operator [18]. We have also shown that vortex solutions exist in the background of rotating Kerr-AdS and charged Reissner-Nordstrom-AdS black holes [20], even in the extremal case. Concurrently, it has recently been shown that in an asymptotically AdS spacetime that a black hole can have scalar hair [21]. Indeed, insofar as the no-hair theorem is concerned it has been shown that there exists a solution to the $SU(2)$ Einstein-Yang-Mills equations which describes a stable Yang-Mills hairy black hole that is asymptotically AdS [4].

We therefore seek to learn if an analog of vortex holography discussed in [18] holds true for dS spacetime. In this article we take the first steps toward consideration of such a holographic phenomenon by searching for possible solutions of the Abelian Higgs field equations in a four dimensional dS background. Although an analytic solution to these equations appears to be intractable, we confirm by numerical calculation that vortex solutions do exist in dS spacetime. We find both static and time-dependent vortex solutions. The static solution corresponds to a core of vortex field energy located within the cosmological horizon in a static patch of dS spacetime. In the time-dependent solution, the energy of the vortex core is diluted by the cosmological expansion.
in an inflationary patch. To our knowledge this is the first construction of vortex solutions in asymptotically dS spacetimes.

We also consider the implications of our solutions for the recently conjectured dS/CFT correspondence [3]. We compute the renormalization group flow associated with the time-dependent vortex solution in the inflationary patch and find that the generalized \( c \)-function monotonically increases, in accord with the generalized \( c \)-theorem [17].

In section two, we solve numerically the Abelian-Higgs equations in the static dS background for different values of the cosmological constant. In section three, we compute the effect of the vortex solution on the dS spacetime. In section four, we solve numerically the same equations in a big bang patch of dS background. This is the first investigation of time dependent vortices in curved spacetime. In section five, we obtain the behaviour of the dS \( c \)-function of this solution and find that it increases (decreases) in an expanding (contracting) dS patch. We give some closing remarks in the final section.

2 Abelian Higgs Vortex in static de Sitter Spacetime

In this section, we consider the abelian Higgs Lagrangian in the background of de Sitter spacetime

\[
\mathcal{L}(\Phi, A_\mu) = -\frac{1}{2}(\mathcal{D}_\mu \Phi)\mathcal{D}^\mu \Phi - \frac{1}{16\pi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \xi (\Phi^\dagger \Phi - \eta^2)^2
\]

where \( \Phi \) is a complex scalar Klein-Gordon field, \( \mathcal{F}_{\mu\nu} \) is the field strength of the electromagnetic field \( A_\mu \) and \( \mathcal{D}_\mu = \nabla_\mu + ieA_\mu \) in which \( \nabla_\mu \) is the covariant derivative. We employ Planck units \( G = \hbar = c = 1 \) which implies that the Planck mass is equal to unity. We use the following four dimensional static dS spacetime background

\[
ds^2 = -(1 - \frac{r^2}{l^2})dt^2 + \frac{1}{(1 - \frac{r^2}{l^2})}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

where the cosmological constant \( \Lambda \) is equal to \( \frac{3}{l^2} \). The horizon is at \( r = l \) and so the range of the coordinate \( r \) is bounded to \( 0 \leq r \leq l \). In this coordinate system \( \partial/\partial t \) is a future-pointing timelike Killing vector in only one diamond of the Penrose diagram [22] which generates the symmetry \( t \to t + t_0 \) for any constant \( t_0 \). In other diamonds of the Penrose diagram, this Killing vector is spacelike or else past-pointing timelike. After redefining the new fields \( X(x^\mu), P_\mu(x^\nu) \) by

\[
\Phi(x^\mu) = \eta X(x^\mu)e^{i\omega(x^\mu)} \\
A_\mu(x^\nu) = \frac{1}{\rho}(P_\mu(x^\nu) - \nabla_\mu \omega(x^\mu))
\]

and employing a suitable gauge, the equations of motion for a string with winding number \( N \) are

\[
(1 - \frac{\rho^2}{l^2}) \frac{d^2X}{d\rho^2} + \frac{1}{\rho} - \frac{4\rho}{l^2} \frac{dX}{d\rho} - \frac{1}{2}X(X^2 - 1) - \frac{N^2}{\rho^2}XP^2 = 0
\]

\[
(1 - \frac{\rho^2}{l^2}) \frac{d^2P}{d\rho^2} - \frac{dP}{d\rho}(\frac{1}{\rho} + \frac{2\rho}{l^2}) - \alpha PX^2 = 0
\]
Figure 1: $X(\rho)$ and $P(\rho)$ fields of a vortex, in dS spacetime for the different values of $l = 3, 5, 10$. These fields in the AdS spacetime with $l = 3, 5$ have similar behaviour.

where $\rho = r \sin \theta$ and $\alpha = \frac{4\pi e^2}{\xi}$. Note that by changing $l$ to $il$, equations (4) and (5) change to the equations of motion in an AdS background [18], or to the $m = 0$ equations discussed in [19]. We have solved the above equations numerically for dS spacetimes with $l = 3, 5, 10$ and unit winding number using the over-relaxation method [23].

Figure 1 shows the results for our calculation and also compares them with the vortex fields in the AdS spacetime [18]. For fixed $\rho \leq 3$, the $X$ field of the dS spacetime decreases with decreasing $l$, in contrast to the AdS spacetime which $X$ field increases with decreasing $l$.

In general the value of the $X$ field for AdS spacetime is always greater than its value for any dS spacetime for a given $\rho$. In other words, by increasing the cosmological constant $\Lambda$ from $-\infty$ to $+\infty$, the $X$ field decreases for fixed $\rho$. The $X$ field of flat spacetime is located between the dS and AdS cases where $|l| = \infty$. Over this same range of $\Lambda$ the value of the $P$ field increases. Physically the negative cosmological constant exerts additional pressure on the vortex, causing it to become thinner; a positive cosmological constant has the opposite effect, causing the core of the vortex to expand beyond the thickness it would have in flat spacetime. Moreover, we expect that the effect of the vortex on dS spacetime is to create a deficit angle in the metric (2) by replacing $\varphi \rightarrow \alpha \varphi$, which $\alpha$ is a constant. In the next section, we verify this point.

For $r \geq l$, the coefficients of the $\frac{d^2X}{d\rho^2}$ and $\frac{d^2P}{d\rho^2}$ terms in equations (4) and (5) change sign and the numerical solution of the corresponding equations outside the cosmological horizon shows that the values of the $X$ and $P$ fields remain at their respective constant values of 1 and 0. Consequently the vortex is confined totally behind the cosmological horizon. In the next section we will find that in a different patch of dS spacetime the vortex solution exists everywhere in space and changes with time.
3 Vortex self gravity in static de Sitter space time

In this section, we consider the effect of the vortex on dS$_4$ spacetime described by the static metric (4). This requires finding the solutions of the coupled Einstein-Abelian Higgs differential equations in dS$_4$. This is a formidable problem – even for flat spacetime no exact solutions have yet been found. Using a thin-core approximation and numerical methods, we obtained in [19] the effect of the vortex on AdS$_4$. Here, we use the same method to obtain the effect of the vortex on dS$_4$ spacetime.

To obtain physical results, we make some approximations. First, we again assume the thin-core approximation, namely that the thickness of the vortex is much smaller that all other relevant length scales. Second, we assume that the gravitational effects of the string are weak enough so that the linearized Einstein-Abelian Higgs differential equations are applicable. For convenience, in this section we use the following form of the metric of dS$_4$:

\[
ds^2 = -\tilde{A}(r, \theta)^2 dt^2 + \tilde{B}(r, \theta)^2 d\varphi^2 + \tilde{C}(r, \theta)(\frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\theta^2) \tag{6}\]

In the absence of the vortex, we must have $A(r, \theta) = \sqrt{1 - \frac{r^2}{l^2}} = A_0(r, \theta)$, $B(r, \theta) = r \sin \theta = B_0(r, \theta)$, $C(r, \theta) = 1 = C_0(r, \theta)$, yielding the well known metric (4) of pure dS$_4$. Employing the two approximations concerning the thickness of the vortex core and its weak gravitational field, we solve numerically the Einstein field equations,

\[
G_{\mu\nu} + \frac{3}{l^2} g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{7}\]

to first order in $\varepsilon = -8\pi G$, where $T_{\mu\nu}$ is the energy-momentum tensor of the Abelian Higgs field in the dS background. By taking $g_{\mu\nu} \approx g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)}$, where $g_{\mu\nu}^{(0)}$ is the usual dS$_4$ metric and $g_{\mu\nu}^{(1)}$ is its first order correction, and writing

\[
\tilde{A}(r, \theta) = A_0(r, \theta)(1 + \varepsilon A(r, \theta))
\]
\[
\tilde{B}(r, \theta) = B_0(r, \theta)(1 + \varepsilon B(r, \theta))
\]
\[
\tilde{C}(r, \theta) = C_0(r, \theta)(1 + \varepsilon C(r, \theta)) \tag{8}\]

we obtain first-order corrections to the three functions $A_0(r, \theta), B_0(r, \theta)$ and $C_0(r, \theta)$ in (8). Hence in the first approximation the equations (7) become

\[
G_{\mu\nu}^{(1)} + \frac{3}{l^2} g_{\mu\nu}^{(1)} = T_{\mu\nu}^{(0)} \tag{9}\]

where $T_{\mu\nu}^{(0)}$ is the energy momentum tensor of the vortex field in the dS$_4$ background metric, and $G_{\mu\nu}^{(1)}$ is the correction to the Einstein tensor due to $g_{\mu\nu}^{(1)}$. The rescaled components of the energy momentum tensor of string in the background of dS$_4$ are given by

\[
T^t_{t}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 (1 - \frac{r^2}{l^2}) - \frac{1}{2} \frac{dP}{d\rho} (1 - \frac{r^2}{l^2}) - \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2
\]
\[
T^\varphi_{\varphi}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 (1 - \frac{r^2}{l^2}) + \frac{1}{2} \frac{dP}{d\rho} (1 - \frac{r^2}{l^2}) + \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2
\]
\[
(T^r_r^{(0)} + T^\theta_\theta^{(0)})(\rho) = - \frac{P^2 X^2}{\rho^2} - 2(X^2 - 1)^2 \tag{10}\]
Figure 2: $T_\phi^{(0)}(\text{solid}), T_t^{(0)}(\text{dash}), T_r^{(0)} + T_\theta^{(0)}$ (dash-dotted) curves of a vortex in dS spacetime with $l = 3$, versus $\rho$.

where $T_\mu^{(0)} = \xi^{-1}\eta^{-4}T_\mu^{(0)}$. The Einstein equations (11) are

\begin{align*}
(1 - \frac{\rho^2}{l^2})\frac{d^2B}{d\rho^2} + 2\frac{dB}{d\rho}(\frac{1}{\rho} - \frac{2\rho}{l^2}) + \frac{1}{2}(1 - \frac{\rho^2}{l^2})\frac{d^2C}{d\rho^2} - \frac{\rho}{l^2}\frac{dC}{d\rho} + \frac{3C}{l^2} &= T_t^{(0)} \\
(1 - \frac{\rho^2}{l^2})\frac{d^2A}{d\rho^2} + \frac{4\rho}{l^2}\frac{dA}{d\rho} + \frac{1}{2}(1 - \frac{\rho^2}{l^2})\frac{d^2C}{d\rho^2} - \frac{2\rho}{l^2}\frac{dC}{d\rho} + \frac{3C}{l^2} &= T_\phi^{(0)} \\
(1 - \frac{\rho^2}{l^2})(\frac{d^2A}{d\rho^2} + \frac{d^2B}{d\rho^2}) + \frac{2}{\rho}(\frac{dA}{d\rho} + \frac{dB}{d\rho})(1 - 3\frac{\rho^2}{l^2}) + \frac{4C}{l^2} &= T_r^{(0)} + T_\theta^{(0)}
\end{align*}

We consider the case $l = 3$, for which the behaviour of $X$ and $P$ fields are given in figure 1. By using equations (11), we obtain figure 2 showing the behaviour of the stress tensor components inside the cosmological horizon as a function of $\rho$, whose minimum value is 0.1.

We note that outside the cosmological horizon, the stress tensor vanishes due to the constant values of the vortex fields. With this knowledge, we solve the coupled differential equations (11), inside and outside the cosmological horizon which gives the behaviour of the functions $A(\rho), B(\rho)$ and $C(\rho)$. The results are plotted in figure 3. These results emphasize that the functions $A(\rho) \approx 1, B(\rho) \approx 2$ and $C(\rho) \approx 0$ are constant over the entire range of $\rho$ to this order in $\varepsilon$. A very tiny deviation from the constant values in the curves $A(\rho), B(\rho)$ are due to the numerical calculation.

Hence by a redefinition of the time coordinate in (11) the metric can be rewritten as

\begin{equation}
\begin{aligned}
ds^2 &= -(1 - \frac{\rho^2}{l^2})dt^2 + \frac{1}{(1 - \frac{\rho^2}{l^2})}dr^2 + r^2(d\theta^2 + \alpha^2 \sin^2 \theta d\varphi^2)
\end{aligned}
\end{equation}

which is the metric of dS space with a deficit angle. So, the effect of the vortex on dS spacetime is to create a deficit angle in the metric (2) by replacing $\varphi \rightarrow \alpha \varphi$, where $\alpha \approx 1 + 2\varepsilon$ is a constant, since $\varepsilon < 0$. The above calculation for the effect of the vortex on dS spacetimes is not restricted to the special case $l = 3$, with other values of cosmological parameter $l$, the final result is the metric (12), with some other deficit angle $\alpha$. 

5
Figure 3: \( \mathcal{A} \) (dotted), \( \mathcal{B} \) (solid), \( \mathcal{C} \) (dashed) versus \( \rho \), inside and outside the cosmological horizon of the dS spacetime with \( l = 3 \).

4 Abelian Higgs Vortex in a Big Bang Patch of de Sitter Spacetime

In this section, we consider the abelian Higgs Lagrangian (1) in the following coordinate system

\[
ds^2 = -d\tau^2 + e^{2\tau/l} (dx^2 + dy^2 + dz^2)
\]  

(13)

where the coordinate \( \tau \in (-\infty, +\infty) \). After using the new fields \( X(\tau, r, \theta) \), \( P_\mu(\tau, r, \theta) \) given by (3), and applying the radial variable \( R = \sqrt{x^2 + y^2} = r \sin \theta \), the equations of motion for a string with winding number \( N \) become

\[
e^{-2\tau/l} \frac{\partial^2 X}{\partial R^2} + e^{-2\tau/l} \frac{\partial X}{\partial R} - \frac{\partial^2 X}{\partial \tau^2} - \frac{3}{l} \frac{\partial X}{\partial \tau} - \frac{N^2}{R^2} XP^2 - \frac{1}{2} X(X^2 - 1) = 0
\]

(14)

\[
e^{-2\tau/l} \frac{\partial^2 P}{\partial R^2} - e^{-2\tau/l} \frac{\partial P}{\partial R} - \frac{\partial^2 P}{\partial \tau^2} - \frac{1}{l} \frac{\partial P}{\partial \tau} - \alpha PX^2 = 0
\]

(15)

By numerically solving equations (14,15), we are able to show that a vortex solution exists on a dS spacetime background for different values of the winding number \( N \) and cosmological parameter \( l \). As in the pure AdS case \([18]\) and asymptotically AdS spacetimes \([19], [20]\) the results indicate that increasing the winding number yields a greater vortex thickness. Furthermore, for a vortex with definite winding number, the string core increases with decreasing \( l \) at constant positive time. The \( X \) and \( P \) fields less rapidly approach their respective maximum and minimum values at larger distances as \( l \) decreases.

To obtain numerical solutions of (14) and (15), we must first select appropriate boundary conditions. At large distances physical considerations motivate a clear choice. Since in the limit \( l \to \infty \), the results must be in agreement with the results of flat spacetime, we demand that the solutions approach the solutions of the vortex equations in the flat spacetime given in ref. \([19]\).
This means that we demand $X \to 1$ and $P \to 0$ as $R$ goes to infinity. On the symmetry axis of the string, we take $X \to 0$ and $P \to 1$ for all time. Finally we take $X = 1$ and $P = 0$, everywhere (except on the symmetry axis) at the initial time $\tau = -\infty$. In the following, we consider the case $N = 1$. We can straightforwardly obtain similar results for other values of the winding number $N$.

We then employ a polar grid of points $(R_i, \tau_j)$, where $R$ goes from 0 to some large value $R_\infty$ which is much greater than $l$ and $\tau$ runs from a large negative number $-\tau_\infty$ to a large positive number $\tau_\infty$. Using finite difference methods, we rewrite the non linear partial differential equation (14) and (15) as

$$A_{ij}X_{i+1,j} + B_{ij}X_{i-1,j} + C_{ij}X_{i,j+1} + D_{ij}X_{i,j-1} + E_{ij}X_{i,j} = F_{ij}$$

(16)

$$A'_{ij}P_{i+1,j} + B'_{ij}P_{i-1,j} + C'_{ij}P_{i,j+1} + D'_{ij}P_{i,j-1} + E'_{ij}P_{i,j} = F'_{ij}$$

(17)

where $X_{ij} = X(R_i, \tau_j)$ and $P_{ij} = P(R_i, \tau_j)$. For the interior grid points, the coefficients $A_{ij}, ..., F'_{ij}$ can be straightforwardly determined from the corresponding continued differential equations (14) and (15). Using the well known successive overrelaxation method [2, 3] for the above mentioned finite difference equations, we obtain the values of $X$ and $P$ fields inside the grid. Initially, inside the grid points, we set the value of $X$ and $P$ fields 0 and 1 respectively which we denote them by $X^{(0)}$ and $P^{(0)}$. Then these values of $X$ and $P$ fields are used in the next step to obtain the values of $X$ and $P$ fields inside the grid which could be denoted by $X^{(1)}$ and $P^{(1)}$. Repeating this procedure, the value of the each field in the $n$-th iteration is related to the $(n-1)$-th iteration by

$$X^{(n)}_{ij} = X^{(n-1)}_{ij} - \omega \frac{\zeta^{(n-1)}_{ij}}{E^{(n-1)}_{ij}}$$

(18)

$$P^{(n)}_{ij} = P^{(n-1)}_{ij} - \omega \frac{\zeta^{(n-1)}_{ij}}{E^{(n-1)}_{ij}}$$

(19)

where the residual matrices $\zeta^{(n)}_{ij}$ and $\zeta^{(n)}_{ij}$ are the differences between the left and right hand sides of the equations (16) and (17) respectively, evaluated in the $n$-th iteration. $\omega$ is the overrelaxation parameter. The iteration is performed many times to some value $n = K$, such that $\sum_i \frac{X^{(K)}_{ij} - X^{(K-1)}_{ij}}{E^{(n-1)}_{ij}} < \epsilon$ and $\sum_i \frac{P^{(K)}_{ij} - P^{(K-1)}_{ij}}{E^{(n-1)}_{ij}} < \epsilon$ for a given error $\epsilon$. It is a matter of trial and error to find the value of $\omega$ that yields the most rapid convergence. Some typical results of this calculation are displayed in figures 4 and 5 for value of $l = 3$.

In the figure 4, time is running from $-\tau_\infty$ to 0 whereas in figure 6 the time runs from 0 to $\tau_\infty$. We notice that by increasing the time from $-\tau_\infty$ to 0, the string core increases rapidly to a well defined value at time $\tau = 0$. Then increasing the time to more positive values, the string core increases more and more such that at time $\tau = \tau_\infty$, the $X$ and $P$ fields approach the constant values 0 and 1 respectively. So, we conclude that by increasing the time from $-\tau_\infty$ to $\tau_\infty$, for which the constant time slices of the spacetime (13) become bigger and bigger, the string thickness becomes wider and wider. Moreover the energy density of the string spreads to bigger and bigger distances as time increases due to the larger size of the constant time slices.

The preceding calculation of vortex fields on dS spacetimes is not restricted to the special case of $l = 3$; with other values of the cosmological parameter $l$, we obtain similar results. For example, another typical results of our calculation are displayed in figures 4 and 5 for value of $l = 10$. 

7
Figure 4: $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 3$, for the different values of time. Time goes from $-\tau_{\infty}$ to 0, from left to right on the curves.

Figure 5: $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 3$, for the different values of time. Time goes from 0 to $\tau_{\infty}$, from left to right on the curves.
Figure 6: $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 10$, for the different values of time. Time goes from $-\tau_\infty$ to 0, from left to right on the curves.

Figure 7: $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 10$, for the different values of time. Time goes from 0 to $\tau_\infty$, from left to right on the curves.
For time slices near \( \tau = 0 \), the behaviour of the vortex fields do not change by changing the cosmological parameter \( l \). The physical reason is that at this special time the spacetime is flat and independent of the cosmological constant. However, if we consider a constant positive time slice, then by increasing the cosmological parameter \( l \), the string thickness decreases; for a constant negative time slice, by increasing the cosmological parameter \( l \), it increases. The asymmetric behaviour of string thickness for positive and negative times is due to the inflation function \( e^{2\tau/l} \) in (13). Hence by increasing \( l \), the density of vortex field time-constant contours concentrates mainly near the time constant \( \tau = 0 \) slice.

To obtain the effect of the vortex on the big bang patch of dS spacetime, we use the results of the preceding section, in which we found that vortex induces a deficit angle on the static spacetime metric given in (12). By using the following transformations,

\[
\begin{align*}
\tau &= -t + \frac{l}{2} \ln \left| 1 - \frac{r^2}{l^2} \right| \\
X &= \frac{r}{\sqrt{\left| 1 - \frac{r^2}{l^2} \right|}} \exp(t/l) \sin \theta \cos(\alpha \varphi) \\
Y &= \frac{-r}{\sqrt{\left| 1 - \frac{r^2}{l^2} \right|}} \exp(t/l) \sin \theta \sin(\alpha \varphi) \\
Z &= \frac{r}{\sqrt{\left| 1 - \frac{r^2}{l^2} \right|}} \exp(t/l) \cos \theta
\end{align*}
\]

the following metric

\[
ds^2 = -d\tau^2 + e^{2\tau/l}(dX^2 + dY^2 + dZ^2)
\]

(20)
can be written in the well known static dS spacetime with deficit angle \( \alpha \) given by (12). Hence the effect of the vortex on a big bang patch of dS spacetime is to create a deficit angle in the X-Y plane that is constant as the (locally) flat spatial slice evolves in time.

Similar calculations for larger winding numbers show that increasing the winding number yields a greater vortex thickness in each constant time slice relative to winding number \( N = 1 \). This tendency runs counter to that of increasing \( l \), for which the vortex thickness decreases on a constant positive time slice. As \( l \to \infty \) we can increase the winding number to larger and larger values, and we find that the resultant thickness increases. Hence the effect of increasing winding number to thicken the vortex dominates over the thinning of the vortex due to decreasing cosmological constant.

What we have found about the behaviour of the vortex in the big bang patch of dS spacetime can be straightforwardly generalized to the big crunch patch of the dS spacetime. The big crunch patch is given by replacing \( \tau \to -\tau \) in the metric (13). Increasing the time from \( -\tau_\infty \) to \( \tau_\infty \), for which the constant time slices of the big crunch spacetime become smaller and smaller, the string thickness becomes narrower and narrower. In fact, increasing the time, the energy density of the string concentrates to smaller and smaller distances due to the smaller size of the constant time slices.
5 De Sitter $c$-function

Obtaining evidence in support of (or against) a conjectured dS/CFT correspondence is somewhat harder to come by than its AdS/CFT counterpart. Although it is tempting to think of the former as a ‘Wick-rotation’ of the latter, a number of subtleties arise whose physical interpretation is not always straightforward [13].

One way of making progress in this area is via consideration of the UV/IR correspondence. In both the AdS and dS cases there is a natural correspondence between phenomena occurring near the boundary (or in the deep interior) of either spacetime and UV (IR) physics in the dual CFT. Solutions that are asymptotically dS lead to an interpretation in terms of renormalization group flows and an associated generalized dS $c$-theorem. This theorem states that in a contracting patch of dS spacetime, the renormalization group flows toward the infrared and in an expanding spacetime, it flows toward the ultraviolet [17]. More precisely, the $c$-function in $(n+1)$-dimensional inflationary dS is given by,

$$c = (G_{\mu\nu}n^\mu n^\nu)^{-(n-1)/2} = \frac{1}{\varrho^{n-2}}$$  \hspace{1cm} (22)

where $n^\mu$ is the unit normal vector to a constant time slice and $\varrho$ is the energy density on a constant time hypersurface.

It is natural to consider the time-dependent solution of the previous section in this context. Although we have not solved the Einstein-Abelian-Higgs equations exactly, we have shown to leading order that the vortex solution we obtained induces a deficit angle into de Sitter spacetime, and so to this order our solution is (locally) asymptotically de Sitter. Furthermore, the gauge field values of our solution asymptotically approach the constant values expected in a de Sitter vacuum, and so at large distances (near the boundary) we expect that our solutions will in general be asymptotically de Sitter, at least locally.

So we want to examine the behaviour of the (22) in the big-bang patch (13) in the presence of a vortex. The energy density of the vortex is given by:

$$\varrho = -\frac{1}{2e^{2\tau}} \left( \frac{\partial X}{\partial R} \right)^2 - \frac{1}{2R^2e^{4\tau}} \left( \frac{\partial P}{\partial R} \right)^2 - (X^2 - 1)^2 - \frac{X^2P^2}{2R^2e^{2\tau}}$$  \hspace{1cm} (23)

Using the curves of the vortex fields we obtained for different time slices in figures (4) and (5), we get figure (8) for the $c$-function in terms of time at a fixed point $R = 2$ of space.

We find that by increasing the time from $-\tau_\infty$ to $\tau_\infty$, the $c$-function monotonically increases as the universe expands. We emphasize that this monotonic increase is not restricted to this special value of $R$: by changing the value of $R$ we find the same increasing behaviour of the $c$-function as time evolves.

Similar calculations show that in the big-crunch patch of dS spacetime, the $c$-function decreases monotonically as time evolves from $-\tau_\infty$ to $\tau_\infty$.

6 Conclusion

We have solved the Nielsen-Olesen equations in a static dS$_4$ background, and found that the Higgs and gauge fields are axially symmetric, with non-zero winding number. Our solution in the limit of...
large $l$ (small cosmological constant) reduces to the well known flat-space solution. The solution (to leading order in the gravitational coupling) induces a deficit angle in dS$_4$. We find in the static patch that an increasingly positive cosmological constant tends to make a thicker vortex solution due to cosmological expansion.

We solved the same equations in the big bang patch of the dS$_4$ background, and found that by increasing the time from early times at $\tau \to -\infty$ to $\tau = 0$, the string thickness increases from an infinitesimally small value to a finite size. Increasing the time to vary large values $\tau \to +\infty$, the string thickness grows more and more, so that at future infinity, the string fields approach constant values everywhere. At past infinity, which is in fact the dS$_4$ horizon, although the magnitude of the magnetic field goes to infinity near the string axis, the spatial dimensions shrink relative to spatial dimensions at other times, yielding a constant value for the magnetic flux of the vortex. In contrast to past infinity, at future infinity the magnetic field is very tiny over a big range of spatial coordinate, giving rise to the same value of the magnetic flux of the vortex at past infinity. The presence of a vortex in asymptotically dS spacetime technically violates the cosmic no-hair theorem in that the spacetime is not pure de Sitter; rather it is only locally dS due to the deficit angle induced from the non trivial topology of the vortex solution. Violation of the cosmic no hair theorem has also been observed in Einstein-Maxwell-Dilaton theory with a positive cosmological constant [24].

Our results are in accord with the generalized dS $c$-theorem, providing further evidence in favor of a conjectured dS/CFT correspondence. An interesting future challenge is that of obtaining a holographic description of a vortex solution in dS spacetimes. The boundary field theory will have to be on a sphere with the same deficit angle as that induced by the vortex. In the AdS case, the conformal two-point correlation function is obtained by evaluating the bulk propagator of a scalar field between two points on the boundary by integrating over all spacelike geodesics paths between the points, and the presence of a vortex is signified by a discontinuity in this correlation function [18]. However in the dS case the geodesic paths will be timelike, and will necessarily have to penetrate the cosmological horizon to detect a vortex in the static patch. Not all of $\mathcal{I}^+ (\mathcal{I}^-)$
is causally connected to the vortex in the big bang (big crunch) patch, raising issues of causality reminiscent of those considered in the case of particles forming black holes in (2 + 1) dimensions [23], and their resolution is far from clear.

We close by commenting on the physical relevance of our solutions. Since we employ Planck units $G = \hbar = c = 1$, the constant $l$ is in units of the Planck length. We have found that only for $l \lesssim 10$ do our solutions appreciably differ from the flat space case. In this regime the radius of curvature of the de Sitter spacetime becomes comparable to the Planck length, equalling it at $l = 1$, and one might be concerned that our results will be substantively modified due to quantum gravitational effects. Nevertheless we maintain that our results have both utility and validity for the following reasons. First, the qualitative behaviour of our solutions for large $l$ is the same as for $l \lesssim 10$; for example figure (8) will retain the same qualitative features for all finite values of $l$, and we have numerically checked this for $l = 100$. However smaller values of $l$ more clearly highlight the significance of the features of these solutions, which is why we have presented them here. Second, it is at the very least useful to know what the classical physics is that underlies any quantum effects, and our solutions furnish will such knowledge when it becomes available. Finally, quantum gravitational effects will typically induce quantitative corrections to our results of order $l^{-2}$, which is still only about 10% even for $l = 3$. Hence we expect the qualitative features of our solutions to remain even when quantum gravity is taken into account.

Inclusion of quantum gravitational effects is, of course, important. Other problems include the generalization of our solutions to asymptotically dS spacetimes with black holes and the possible dS/CFT correspondence of these solutions. Work on these problems is in progress.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada. We would like to thank F. LeBlond for discussions and comments on an earlier version of this manuscript.

References

[1] R. Ruffini and J.A. Wheeler. Phys. Today 24, 30 (1971).

[2] D. Sudarsky, Class. Quant. Grav. 12, 579 (1995).

[3] T. Torii, K. Maeda and M. Narita, Phys. Rev. D59, 064027 (1999).

[4] E. Winstanley, Class. Quant. Grav. 16, 1963 (1999).

[5] A. Achucarro, R. Gregory and K. Kuijken, Phys. Rev. D52, 5729 (1995).

[6] A. Strominger, J.H.E.P. 0110 034 (2001) (hep-th/0106113); J.H.E.P. 0111 049 (2001) (hep-th/0110087).

[7] S. Nojiri and S.D. Odintsov, hep-th/0106191.

[8] D. Klemm, hep-th/0106247.
[9] A.J. Tolley and N. Turok, *hep-th/0108119*.
[10] T. Shiromizu, D. Ida and T. Torii, *hep-th/0109057*.
[11] S. Ogushi, *hep-th/0111008*.
[12] R.-G. Cai, *hep-th/0111093*.
[13] A.M. Ghezelbash and R.B. Mann, JHEP 0201 (2002) 005.
[14] E. Halyo, *hep-th/0112093*.
[15] M. Spradlin and A. Volovich, *hep-th/0112223*.
[16] A.J.M. Medved, *hep-th/011226*.
[17] F. Leblond, D. Marolf and R.C. Myers, *hep-th/0202094*.
[18] H. Dehghani, A.M. Ghezelbash and R.B. Mann, *Nucl. Phys. B625*, 389 (2002), *hep-th/0105134*.
[19] M.H. Dehghani, A.M. Ghezelbash and R.B. Mann, *Phys. Rev. D65*, 044010 (2002).
[20] A.M. Ghezelbash and R.B. Mann, *hep-th/0110001*, to be published in Phys. Rev. D.
[21] T. Torii, K. Maeda and M. Narita, *Phys. Rev. D64*, 044007 (2001).
[22] M. Spradlin, A. Strominger and A. Volovich, *hep-th/0110007*.
[23] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *"Numerical Recipes in FORTRAN"*, Cambridge University Press (1992).
[24] T. Maeda, T. Torii and M. Narita, *Phys. Rev. D61*, 064012 (2000).
[25] J. Louko, D. Marolf and S.F. Ross, *Phys. Rev. D62*, 044041 (2000).