Stability of hybrid Lévy systems

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Continuous-time stochastic systems have attracted a lot of attention recently, due to their wide-spread use in finance for modelling price-dynamics. More recently models taking into accounts shocks have been developed by assuming that the return process is an infinitesimal Lévy process. Lévy processes are also used to model the traffic in a telecommunication network. In this paper we focus on a particular technical problem: stability of time-varying stochastic systems driven or modulated by a Lévy process with discrete time interventions, such as parameter or state resetting. Such systems will be called hybrid Lévy systems. They are hybrid in the sense that jumps both in the dynamics may occur. The peculiarity of our systems is that the jump-times are defined by a more or less arbitrary point process, but there exists an asymmetry in the system dynamics. The novelty of our model relative to the theory of switching stochastic systems is two-fold. First, we allow slow time variation of the parameters, in a stochastic sense, without any statistical pattern, in the spirit of the classical stability result of Desoer, see [2]. Secondly, we allow certain jumps (resetting) in the system parameters almost without any a priori condition.

1 Introduction

Continuous-time stochastic systems have attracted a lot of attention recently, due to their wide-spread use in finance for modelling price-dynamics. A widely used model for continuous-time returns has been, since the works of L. Bachelier, Gaussian white noise with drift. More recently models taking into accounts shocks have been developed by assuming that the return process is an infinitesimal Lévy process. For long term modelling a more suitable model is a stochastic system with poles close to 1 driven by a Lévy process, see [4].

Lévy processes are also used to model the traffic in a telecommunication network. Other areas where Lévy processes are used in modeling: robotics, mechanical systems, biology [9],[11]. A further potential application is modelling the shocks received by the wheel of a car due to the irregularity of the road surface.

Stochastic processes driven or modulated by a Lévy process will be called a Lévy system. Description of real data in terms of Lévy systems is far from being settled. In this paper we focus on a particular technical problem that proved to be fundamental in the statistical analysis of continuous-time stochastic systems driven by Gaussian white noise, see [1]. Ultimately it is hoped that
this technical result may contribute to the development of a continuous-time recursive maximum likelihood method for finite dimensional linear stochastic Lévy systems, along the lines of [3].

The problem is the stability analysis of time-varying stochastic systems driven or modulated by a Lévy process with discrete time interventions, such as parameter or state resetting. Such systems will be called hybrid Lévy systems. They are hybrid in the sense that jumps both in the dynamics and the state may occur. The peculiarity of our systems is that the jump-times are defined by a more or less arbitrary point process, but there exists an asymmetry in the system dynamics, inasmuch jumps can occur only one-way, after a period of slow variation, namely, back to a fixed point.

We note that the well-developed theory of switching stochastic systems, see [5], does not cover the problem that we consider. The novelty of our model relative to the theory of switching stochastic systems is two-fold. First, we allow slow time variation of the parameters, in a stochastic sense, without any statistical pattern, in the spirit of the classical stability result of Desoer, see [2]. Secondly, we allow certain jumps (resetting) in the system parameters almost without any a priori condition.

The structure of the paper is as follows: in Section II we develop the basic technical tools, such as the geometric drift condition and the associated Lyapunov-function method, for the analysis of time-invariant Lévy systems, and provide estimates for higher order moments of the Lyapunov-function. In Section III we present the simplest version of an extension of Desoer’s theorem. In Sections IV we prove a stability result under parameter resetting.

A few basic notions related to Lévy processes will be given in the Appendix. Throughout the paper we use the following notations: $|\cdot|$ stands for the Euclidean norm, $\| \cdot \|$ stands for the induced matrix norm. $[X,Y]_t$ denotes the quadratic variation of semi-martingales $X_t$ and $Y_t$.

2 Preliminaries

Consider the time-invariant linear stochastic system

$$dX_t = AX_t \, dt + BdL_t + CdW_t$$

(1)

where $L_t$ is Lévy process that has finite variation and has no continuous part and $W_t$ is a standard Wiener process. Assume that $A$ and $B$ are time-independent constant matrices. $X_t \in \mathbb{R}^n$, $L_t \in \mathbb{R}^l$, $W_t \in \mathbb{R}^k$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{n \times k}$. We will denote the $i^{th}$ component of a vector $V$ with $V^{(i)}$. We will use the fact that if $L_t$ is of finite variation then it can be written as $L_t = L_0 + bt + \sum_{s \leq t} \Delta L_s$, and the quadratic covariance of two coordinates of such vector processes is of the form

$$\sum_{s \leq t} \Delta L_s^{(i)} \Delta L_s^{(j)}.$$
Definition 2.1 We say that a vector \( L_t = (L_t^{(1)}, \ldots, L_t^{(l)}) \) with independent components that are Lévy processes satisfies the moment condition of order \( Q \) if \( \int_{\mathbb{R}} |x|^q \nu^{(i)}(dx) < \infty \) holds for all \( 1 \leq i \leq l \), and for \( 1 \leq q \leq Q \), where the Lévy measure of \( L_t^{(i)} \) is denoted by \( \nu^{(i)}(x) \).

The next definition is motivated by the geometric drift condition introduced in [1]. To estimate the moments of \( X_t \) we will use a quadratic Lyapunov function \( V_t \).

Definition 2.2 Let \( L_t^{(i)}, 1 \leq i \leq l \) be independent Lévy processes with finite variation. Let \( f \) be a polynomial with coefficients bounded uniformly in \( t \) and \( \deg f \leq Q \). Given a process \( V_t \) satisfying with some \( \epsilon > 0 \)

\[
dV_t = V_t - \left( u_t dt + dM_t \right) + V_t^{1-\epsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}),
\]

where \( \Delta L_t^{(i)} \) denotes the size of the jump of \( L^{(i)} \) at \( t \). We say that \( V_t \) satisfies the modified geometric drift condition of order \( Q \) if there exist \( \alpha, \gamma > 0 \), such that

\[
u_t \leq -\alpha \quad \text{and} \quad \frac{d|M_t|}{dt} \leq \gamma.
\]

Without loss generality may always assume that decomposition of \( L_t \) contains no drift term. Any possible drift term can be incorporated into \( u_t dt \). The following lemma will be used several times in the paper.

Lemma 2.1 Assume Condition 1, let \( D_0 \subset D \) compact and \( \theta_0 \in D_0 \). Then, there exists a smooth function \( P(\theta), \theta \in D \) and \( \alpha > 0 \) such that \( P(\theta) \geq P(\theta_0) \geq I \) for all \( \theta \in D_0 \) and \( P(\theta)A(\theta) + A^T(\theta)P(\theta) \leq -\alpha P(\theta) \), for all \( \theta \in D_0 \).

For its proof see [1].

The next two lemmas show that our Lyapunov function \( V_t \) and its \( q \)th power satisfy the modified geometric drift condition.

Lemma 2.2 Let \( X_t \) be defined via (1), and \( P \) given by Lemma 2.1. Define \( V_t = 1 + X_t^T P X_t \), then \( V_t \) satisfies the modified geometric drift condition of order two.

Lemma 2.3 Let \( X_t \) be defined via (1). Define \( V_t = 1 + X_t^T P X_t \), then \( V_t^q \) satisfies the modified geometric drift condition of order \( 2q \).

The result of the next Lemma will be used in the proof of Theorem 2.1.

Lemma 2.4 Let us suppose that \( V_t \) satisfies the modified geometric drift condition of order \( Q \), and suppose that \( L_t \) satisfies the moment condition of order \( Q \). Then

\[ \mathbb{E}[V_t] < \infty \]

holds.
Proof: Take a general process of the form (2). Then \( V_t \) satisfies

\[
\Delta V_t = V_{t-} Z_t,
\]

with

\[
Z_t = u_t dt + f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}).
\]

Using the Doleans-Dade exponential formula, see [12], for processes with finite variation yields the solution for \( V_t \):

\[
V_t = e^{Z^{(c)}_t - Z^{(c)}_0} \prod_{s \leq t} (1 + \Delta Z_s),
\]

where \( \cdot^{(c)} \) denotes the continuous part of a process. This \( V_t \) is also called as the stochastic exponential of \( U_t \).

Let \( c \) and \( M \) be uniform bounds for \( u_t \) and for the coefficients of \( f \). Increasing both \( u_t \) and the coefficients of \( f \) and taking absolute value of the jumps we obtain a bound on the solution \( V_t \):

\[
V_t \leq e^{ct} \prod_{s \leq t} \left( 1 + M \sum_{0 \leq j_1 + \ldots + j_l \leq Q} \prod_{i=1}^l \left| \Delta L_s^{(i)} \right|^j_i \right) \leq e^{ct} \prod_{i=1}^l \prod_{s \leq t} \left( 1 + M \sum_{j=1}^Q \left| \Delta L_s^{(i)} \right|^j \right)
\]

Since \( L_t^{(i)} \)-s are independent processes it is sufficient show that

\[
\mathbb{E} \left[ \prod_{s \leq t} \left( 1 + M \sum_{j=1}^Q \left| \Delta L_s^{(i)} \right|^j \right) \right] < \infty.
\]

Since \( L_t \) satisfies the moment condition of order \( Q \)

\[
\mathbb{E} \left[ 1 + M \sum_{j=1}^Q \left| \Delta L_s^{(i)} \right|^j \right] < \infty.
\]

Hence applying Lemma 6.1, see Appendix, concludes the proof.

The next theorem implies the stability of \( X_t \) defined in (1).

**Theorem 2.1** Let us suppose that \( V_t \) satisfies the modified geometric drift condition of order \( Q \), and suppose that \( L_t \) satisfies the moment condition of order \( Q \). Then

\[
\sup_{t \geq 0} \mathbb{E} |V_t| < \infty
\]

holds for \( 1 \leq q \leq Q \).

The proof will be given in the Appendix.

**Corollary:** Since \( V_t = 1 + X_t^T P X_t \), satisfies the modified geometric drift condition so does \( V_t^q \). Hence, by Theorem 2.1 \( \sup_{t \geq 0} \mathbb{E} |V_t^q| < \infty \), which implies that \( \sup_{t \geq 0} \mathbb{E} |X_t^q| < \infty \) holds under conditions seen in the theorem above.
3 A stochastic Desoer’s Theorem

Consider a parametric family of linear stochastic state-space systems given by the state space equations:

\[ dX_t = A(\theta_t)X_t dt + B(\theta_t)dW_t + C(\theta_t)dL_t \]  

(6)

Condition 1: \( A(\theta) \) is stable for each \( \theta \in D \), where \( D \subset \mathbb{R}^p \) is an open set, and \( A(\theta), B(\theta) \) and \( C(\theta) \) are smooth in \( D \).

Definition 3.1 We say that \( \theta_t \) is slowly varying in a stochastic sense if

\[ d\theta_t = \beta_t dt + \sigma_t dW_t + \rho_t dL_t, \]  

(7)

with \( |\beta|_t^2 + ||\sigma||_t^2 + ||\rho||_t^2 < \delta \), for some \( \delta > 0 \) and all \( t \).

Theorem 3.1 Assume that \( L_t \) satisfies the moment condition of order \( Q \), \( \theta_t \) is slowly varying in the stochastic sense above, furthermore assume that \( \theta_t \in \mathbb{R}^p \) is an adapted process taking its values in a compact set \( D_0 \subset D \), and that Condition 1 holds. Then for a sufficiently small \( \delta \).

\[ \sup_{t \geq 0} \mathbb{E}[V_t^q] < \infty, \]  

(8)

for \( 1 \leq q \leq Q \).

Proof: The case when no Lévy terms are present in the dynamics of \( x_t \) and \( \theta_t \) has been settled in Theorem 1 of [1]. We may therefore assume that \( B(\theta) = 0 \) and \( \sigma_t = 0 \).

For a given \( \theta \), let \( P(\theta) \in C^2 \) be a symmetric, positive definite matrix that solves

\[ P(\theta)A(\theta) + A(\theta)^T P(\theta) \leq -\alpha P(\theta), \]  

(9)

with some \( \alpha > 0 \), and \( P(\theta) \geq I \). Let \( P_t = P(\theta_t) \), and consider \( V_t = (1 + X_t^T P_t X_t)^{t/2} \). It is enough to prove that \( V_t \) satisfies the modified geometric drift condition. By Lemma 2 we only need to check that \( 1 + X_t^T P_t X_t \) satisfies the modified geometric drift condition. We can write the dynamics of \( 1 + X_t^T P_t X_t = 1 + \text{Tr}(P_t Z_t) \), with \( Z_t = X_t X_t^T \) as

\[ d\text{Tr}(P_t Z_t) = \text{Tr}(P_t dZ_t) + \text{Tr}(dP_t Z_t) + \sum_{i,j} dP_{t,i} d[X^{(i)}, X^{(j)}]_t \]  

(10)

The first term can be handled using Lemma 2. The dynamics of \( P_t \) is given by

\[ dP_t = u_t dt + \Sigma_t dL_t, \]  

(11)

with \( ||u||_t^2 + ||\Sigma||_t^2 < c\delta \), with some \( c \). Thus, the second and the third term give drift terms that do not spoil the modified geometric drift condition. The typical
form of the contribution of the second term up to a bounded constant multiplier is
\[ X^{(i)} X^{(j)} dL^{(k)}, \]
and that of the third term is
\[ dL^{(k)} d[L^{(i)}, L^{(j)}]. \]
Hence, \( 1 + X^T P_t X_t \) indeed satisfies the modified geometric drift condition. Thus, applying Theorem 2.1 concludes the proof.

This result implies the stability of the parameter varying system defined by (6).

**Corollary:** Under conditions seen in the previous theorem \( \sup_{t \geq 0} E[|X|^q_t] < \infty \) holds.

### 4 Jumps in the dynamics of the parameter

We now assume that the slowly parameter varying process \( \theta_t \) resets at random times defined by a point process with counting process \( N_t \).

\[ d\theta_t = \beta_t dt + \sigma_t dW_t + dL_t + (\theta_0 - \theta_{t-})dN_t, \]
where \( |\beta_t|^2 + ||\sigma_t||^2 < \delta. \)

**Theorem 4.1** Assume that Condition 1 holds, and that \( L_t \) satisfies the moment condition of order \( Q \), and let \( X_t \) be defined via (6). Then \( \sup_{t \geq 0} E[|X|^q_t] < \infty \) holds for \( 1 \leq q \leq Q \).

**Proof:** We may assume that there is no diffusion part in the dynamics of \( X_t \) and \( \theta_t \). Let \( P(\theta) \) be defined by Lemma 2.1 so that it attains its minimum on \( D \) in \( \theta_0 \). Define \( V_t = (1 + X^T P(\theta_t)X_t)^{q/2} \). Let \( \xi_t \) be the size of the jump at \( t \) induced by the jump of \( \theta \), ie.

\[ \xi_t = (1 + X^T P(\theta_0)X_t)^{q/2} - (1 + X^T P(\theta_t)X_t)^{q/2}, \]
using this notation the dynamics of \( V_t \) can be written as

\[ dV_t = V_{t-} U_t + \xi_t dN_t, \]
with

\[ U_t = u_t dt + \sum_{0 \leq j_1 + \ldots + j_l \leq Q} c_{j_1, \ldots, j_l} \prod_{i=1}^l \left( \Delta L^{(i)}_{t} \right)^{j_i}. \]

By the minimality of \( P(\theta_0) \), the jump term in (10) causes a non-positive jump in \( V_t \). Let \( \varphi_t \) be the stochastic exponential of \( U_t \), then

\[ V_t = \varphi_t V_0 + \int_0^t \varphi_s \varphi_t^{-1} \xi_s dN_s \leq \varphi_t V_0. \]

Since \( E[\varphi_t V_0] < \infty \) is implied by Theorem 2.1, we conclude the proof.
5 Discussion: State resetting for jump processes

Consider the hybrid linear system with jumps
\[ dX_t = AX_t + BdW_t + CdL_t + (X_0 - X_{t-})dN_t, \]  
where \( W_t \) is a Wiener process, and \( N_t \) is a counting process.

**Conjecture 5.1** Suppose that \( L_t \) satisfies the moment condition of order \( Q \), then for \( X_t \) defined by (18)
\[ \sup_{t \geq 0} E[|X|^q] < \infty, \]
holds for \( 1 \leq q \leq Q \).

Our future work will focus on proving this conjecture. Although the main ideas of the proof are established some technical issues are still to be proven.

6 Appendix

Lévy processes with finite variation are formally obtained via
\[ Z_t = \int_0^t \int_{\mathbb{R}^1} x N(ds, dx), \]  
where \( N(dt, dx) \) is a time-homogeneous, space-time Poisson point process, counting the number of jumps of size \( x \) at time \( t \). Lévy processes are characterized by their Lévy measures that can be defined via the intensity of \( N(dt, dx) \)
\[ \mathbb{E}[N(dt, dx)] = dt \cdot \nu(dx), \]
where \( \nu(dx) \) is the Lévy-measure. The quadratic variation of semimartingales \( X \) and \( Y \) is defined by the following process:
\[ d[X, Y]_t = d(XY)_t - X_{t-}dY_t - Y_{t-}dX_t. \]
If \( X = Y \) we get the quadratic variation of \( X \).

**Proof of Lemma 2.2**

**Proof:** This lemma is an extension of Lemma 8 in [1], where no Lévy processes are present in defining the dynamics of \( V_t \). Thus, for the sake of simplicity, we may omit the martingale \( M_t \) from (2).

Write \( V_t = 1 + X_t^T P X_t = 1 + \text{Tr}(PZ_t) \), where \( Z_t = X_t X_t^T \). The dynamic of \( Z_t \) can be written as
\[ dZ_t = X_t - dX_t^T + dX_t X_t^T + Bd[L_t, L_t] B^T, \]  
where \( d[L_t, L_t] \) is an \( l \times l \) matrix with entries representing quadratic covariances, that is \( d[L_t, L_t]_{ij} = d[L^{(i)}, L^{(j)}] \). Equation (20) reads as
\[ X_{t-} (X_t^TA_t dt + dL_t B^T) + (AX_t dt + BdL_t) X_t^T + Bd[L_t, L_t] B^T \]  
(21)
Thus the dynamics of $V_t = 1 + \Tr(PZ_t)$ can be written as

\begin{equation}
PX_t X_t^T A^T + PAX_t X_t^T \right) dt + \\
+ PX_t dL_t B^T + PBdL_t X_t^T + PBd[L, L]_t B^T.
\end{equation}

So the $dt$ terms in the dynamics of $V_t = 1 + \Tr(PZ_t)$ we can write

\begin{equation}
\Tr \left( X_t^T A^T PX_t - X_t^T PAX_t \right) \leq -\alpha X_t^T PX_t = -\alpha V_t + \alpha,
\end{equation}

for the terms having $dL_t$

\begin{align*}
\Tr \left( P(X_t dL_t B^T + BdL_t X_t^T) \right) = \\
\Tr \left( (P + P^T) BdL_t X_t^T \right) = \\
2X_t^T (P + P^T) BdL_t = \psi_t^T dL_t,
\end{align*}

with $|\psi_t|^2 = 4X_t^T (P + P^T) BB^T (P + P^T) X_t \leq 4KV_t$, with some fixed $K$.

Finally for the term with $d[L, L]_t$

\begin{equation}
\Tr \left( PBd[L, L]_t B^T \right) = \sum_{i,j=1}^l c_{i,j} d[L(i), L(j)]_t = \\
\sum_{i,j=1}^l c_{i,j} \Delta L_t^{(i)} \Delta L_t^{(j)},
\end{equation}

with some $c_{i,j}, 1 \leq i, j \leq l$ constants. It follows that the dynamic of $V_t$ can be written as

\begin{equation}
dV_t = V_t u_t dt + \\
V_t^{1/2} \left( \sum_{i=1}^l \frac{\psi_t^{(i)}}{V_t^{1/2}} \Delta L_t^{(i)} + \sum_{i,j=1}^l \frac{c_{i,j}}{V_t^{1/2}} \Delta L_t^{(i)} \Delta L_t^{(j)} \right),
\end{equation}

with uniformly bounded $u_t \frac{\psi_t^{(i)}}{V_t^{1/2}}, \frac{c_{i,j}}{V_t^{1/2}}$ for any $1 \leq i, j \leq l$.

**Proof of Lemma:**

The dynamics of $V_t^q$ can be written as

\begin{align*}
dV_t^q = qV_t^{q-1} dV_{t,(e)} + V_t^q - V_t^q = \\
qV_t^{q-1} u_t dt + (V_t + \Delta V_t)^q - V_t^q = \\
qV_t^{q-1} u_t/V_t - 1 dt + \sum_{k=1}^q \binom{q}{k} (\Delta V_t)^k V_t^{q-k},
\end{align*}

with $u_t/V_t < \alpha$. Using that

\[ \Delta V_t = V_t^{1-\varepsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}), \]

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we obtain that a typical jump term in \( V_{t-k}^q \) reads as up to constant multiplier
\[
V_{t-k}^q f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)})^k
\]
This implies that \( V_t^q \) satisfies the modified geometric drift condition of order \( 2q \).

The next two technical Lemmas will be used in the proof of Theorem 2.1.

**Lemma 6.1** Let \( L_t \) be a Lévy process with Lévy measure \( \nu \). Suppose that a function \( f \) satisfies
\[
\int_{\mathbb{R}} f(x) \nu(dx) < \infty,
\]
then
\[
\mathbb{E} \left[ \prod_{s \leq t} (1 + f(\Delta L_s)) \right] = e^{\int_{\mathbb{R}} f(x) \nu(dx)}
\]
holds for any \( t \).

**Proof:** First suppose that \( L_t \) is a compound Poisson process with rate \( \lambda \), then the expected value of
\[
\psi_t = \prod_{s \leq t} f(\Delta L_s)
\]
can be estimated by conditioning on the number of jumps of \( L_t \). Let \( N_t, J_t \) denote the number of jumps of \( L_t \) on \([0, t]\), and the set of time indices when \( L \) jumps on \([0, t]\). Define \( D_T^n = \{(t_1, \ldots, t_n) : 0 \leq t_i \leq t, \text{ for all } 1 \leq i \leq n\} \).

\[
\mathbb{E} [\psi_t] = \sum_{n=0}^{\infty} \mathbb{E} [\psi_t | N_t = n] \mathbb{P} (N_t = n) = \\
\int_{D_T^n} \mathbb{E}[\psi_t | N_t = n, J_t = \{t_1, \ldots, t_n\}] P(N_t = N) dt_1 \ldots dt_n = \\
\sum_{n=0}^{\infty} (m + 1)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{\lambda t},
\]
where \( m = \mathbb{E} [f(\Delta L_t) | L \text{ jumps at } t] \), and \( P \) is the joint probability density of the jump times. For the general case define the truncated Lévy measure
\[
\nu^\varepsilon(x) = \frac{1}{\int_{|x| > \varepsilon} \nu(dx)} \nu(x)
\]
and let \( L_t^\varepsilon \) be the Lévy process with Lévy measure \( \nu^\varepsilon \). Then \( L_t \) is the weak limit of \( L_t^\varepsilon \) as \( \varepsilon \) tends to zero.

\[
m_f^\varepsilon = \int f(x) \nu^\varepsilon(dx) \quad (29)
\]
\[
m_f^\varepsilon = \mathbb{E} [f(\Delta L_t^\varepsilon) | L^\varepsilon \text{ jumps at } t] \quad (30)
\]
\[
\lambda^\varepsilon = \int_{|x| > \varepsilon} \nu(dx) \quad (31)
\]
writing (6) for \( L \) yields

\[
\mathbb{E} \left[ \prod_{s \leq t} (1 + f(\Delta L_s^\varepsilon)) \right] = e^{\lambda \varepsilon (m^* - 1) t}
\]

Note that \( \lambda m^*_k = \int_{|x| > \varepsilon} f(x) \nu(dx) \), it follows that \( e^{\lambda \varepsilon m^* t} \) has finite limit as \( \varepsilon \to 0^+ \) provided \( \int_{\mathbb{R}} f(x) \nu(dx) < \infty \) which is the case. Hence, \( \mathbb{E} [\psi_t] = e^{\lambda t} \int_{\mathbb{R}} f(x) \nu(dx) \) follows.

**Lemma 6.2** Let the one dimensional process \( L_t \) with Lévy measure \( \nu \) satisfy the moment condition of order \( Q \). Let \( f \) be a polynomial with \( \deg f \leq Q \), and \( f(0) = 0 \). Then

\[
\mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} f(\Delta L_s^\varepsilon) \right] = 1 - e^{-\alpha t} \int_{\mathbb{R}} f(x) \nu(dx)
\]

**Proof:** First consider the case when \( L_t \) is a compound Poisson process with intensity \( \lambda \). Let \( N_t, J_t \) denote the number of jumps of \( L_t \) on \([0, t]\), and the set of time indices when \( L \) jumps on \([0, t]\). Define \( D^n_t = \{(t_1, \ldots, t_n) : 0 \leq t_i \leq t, \text{ for all } 1 \leq i \leq n\} \).

\[
\mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} f(\Delta L_s^\varepsilon) | N_t = n, J_t = \{t_1, \ldots, t_n\} \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} f(\Delta L_s^\varepsilon) | N_t = n \right] P(N_t = n).
\]

Calculating one term in the sum above

\[
\int_{D^n_t} \mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} f(\Delta L_s^\varepsilon) | N_t = n, J_t = \{t_1, \ldots, t_n\} \right] P(N_t = n) \frac{dt_1 \ldots dt_n}{t^n} = \int_{D^n_t} e^{-\lambda t} (\lambda t)^n \frac{dt_1 \ldots dt_n}{n!} \frac{e^{-\alpha t} \mathbb{E} [f(\Delta L_s^\varepsilon) | t_1 \in J_t] e^{-\lambda t} (\lambda t)^n}{n!} dt_1 \ldots dt_{n-1}
\]

\[
= n \int_{D^{n-1}_t} e^{-\alpha(t-t_n)} \mathbb{E} [f(\Delta L_s^\varepsilon) | t_1 \in J_t] e^{-\lambda t} (\lambda t)^n \frac{dt_1 \ldots dt_{n-1}}{t^{n-1}}
\]

\[
= ne^{-\lambda t} (\lambda t)^n \mathbb{E} [f(\Delta L_s^\varepsilon) | t_1 \in J_t] \frac{1 - e^{-\alpha t}}{\alpha t}.
\]
Now using this result in (32) yields
\[
1 - e^{-\alpha t} \frac{1}{\alpha t} \mathbb{E} \left[ f(\Delta L_s)|t_1 \in J_t \right] \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} =
\]
(34)
\[
1 - e^{-\alpha t} \lambda \mathbb{E} \left[ f(\Delta L_s)|t_1 \in J_t \right]
\]
For the general case define like in the proof of Lemma 6.1 process \( L^\varepsilon_t \) and its Lévy measure \( \nu^\varepsilon(dx) \), and \( m^\varepsilon = \mathbb{E} \left[ f(\Delta L_s)|t_1 \in J_t \right] \). Writing (34) for \( L^\varepsilon_t \) we obtain
\[
\mathbb{E} \left[ \int_0^t e^{-\alpha(t-s)} f(\Delta L^\varepsilon_s) \right] =
\]
(35)
\[
1 - e^{-\alpha t} \int_{|x|>\varepsilon} f(x) \nu(dx).
\]
Since \( L_t \) is the weak limit of \( L^\varepsilon_t \) as \( \varepsilon \) tends to zero, allowing \( \varepsilon \to 0^+ \) concludes the proof.

Proof of Theorem 2.1:

Proof: Let \( V_t \) satisfy the modified geometric drift condition
\[
dV_t = u_t dt + V_t^{1-\varepsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}).
\]
(36)
In the presence of a martingale in the dynamics of \( V_t \) one can apply methods shown in [1]. Applying Cauchy formula gives
\[
V_t = \int_0^t e^{-\alpha(t-s)} V_s^{1-\varepsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)})
\]
(37)
Now we estimate the expected values of \( V_t \) using \( V^*_t = \sup_{0 \leq s \leq T} \mathbb{E}[V_s] \)
\[
\mathbb{E}[V_t] = \int_0^t e^{-\alpha(t-s)} V_s^{1-\varepsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)})
\]
(38)
\[
\int_0^t e^{-\alpha(t-s)} \mathbb{E}[V_s^{1-\varepsilon}] f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}) \leq
\]
(39)
\[
\int_0^t e^{-\alpha(t-s)} \mathbb{E}[V_s]^{1-\varepsilon} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}) \leq
\]
(40)
\[
(V^*_t)^{1-\varepsilon} \int_0^t e^{-\alpha(t-s)} f(\Delta L_t^{(1)}, \ldots, \Delta L_t^{(l)}) \leq
\]
(41)
\[
(V^*_t)^{1-\varepsilon} \prod_{i=1}^l \int_0^t e^{-\alpha(t-s)} \mathbb{E}[g_i(\Delta L_t^{(i)})]
\]
(42)
In (38) we used Fubini’s theorem and the independency of \( V_s \) and \( \Delta L_s^{(i)} \). In (40) Hölder inequality was applied. Finally, in (42) we estimate \( f \) with products
of $g_i$ polynomials as we did in (5), clearly $\deg g_i \leq Q$ holds for all $i$. Applying Lemma 6.2 gives for (42)

$$ (V_T^*)^2 \leq \prod_{i=1}^{t} \frac{1 - e^{-\alpha t}}{\alpha} \int_{\mathbb{R}} g_i(x) \nu_i(dx), $$

where $\nu_i$ is the Lévy measure of $L^{(i)}_t$. Since $1 - e^{-\alpha t} < 1$ we obtained a bound on $V_T^*$ that do not depend on $T$. Hence, $\sup_{0 \leq t} \mathbb{E}[V_t] < \infty$, which concludes the proof.

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