AN ASYMPTOTIC COMPARISON OF TWO TIME-HOMOGENEOUS PAM MODELS

HYUN-JUNG KIM AND SERGEY VLADIMIR LOTOTSKY

Abstract. Both Wick-Itô-Skorokhod and Stratonovich interpretations of the Parabolic Anderson model (PAM) lead to solutions that are real analytic as functions of the noise intensity \( \varepsilon \), and, in the limit \( \varepsilon \to 0 \), the difference between the two solutions is of order \( \varepsilon^2 \) and is non-random.

1. Introduction

Let \( W = W(x), \ x \in [0, \pi] \) be a standard Brownian motion on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). With no loss of generality, we assume that all realizations of \( W \) are in \( C^{1/2-}((0, \pi)) \), that is, Hölder continuous of every order less than \( 1/2 \).

Consider the equations

\[
\frac{\partial u_\diamond(t, x; \varepsilon)}{\partial t} = \frac{\partial^2 u_\diamond(t, x; \varepsilon)}{\partial x^2} + \varepsilon u_\diamond(t, x; \varepsilon) \diamond \dot{W}(x), \ t > 0, \ 0 < x < \pi,
\]

\[ u_\diamond(t, 0; \varepsilon) = u_\diamond(t, \pi; \varepsilon) = 0, \ u_\diamond(0, x; \varepsilon) = \varphi(x), \tag{1.1} \]

and

\[
\frac{\partial u_\circ(t, x; \varepsilon)}{\partial t} = \frac{\partial^2 u_\circ(t, x; \varepsilon)}{\partial x^2} + \varepsilon u_\circ(t, x; \varepsilon) \circ \dot{W}(x), \ t > 0, \ 0 < x < \pi,
\]

\[ u_\circ(t, 0; \varepsilon) = u_\circ(t, \pi; \varepsilon) = 0, \ u_\circ(0, x; \varepsilon) = \varphi(x), \tag{1.2} \]

Equation (1.1) is the Wick-Itô-Skorokhod formulation of the parabolic Anderson model with potential \( \varepsilon \dot{W} \); equation (1.2) is the corresponding Stratonovich (or geometric rough path) formulation. These equations, with \( \varepsilon = 1 \), are studied in [1] and [2], respectively.

The objective of the paper is to show that

- The solutions of (1.1) and (1.2) are real-analytic functions of \( \varepsilon \): with suitable functions \( u_\diamond^{(n)} \) and \( u_\circ^{(n)} \), the equalities

\[
u_\diamond(t, x; \varepsilon) = u_\diamond(t, x; 0) + \sum_{n=1}^{\infty} \varepsilon^n u_\diamond^{(n)}(t, x) \tag{1.3}
\]

\[
u_\circ(t, x; \varepsilon) = u_\circ(t, x; 0) + \sum_{n=1}^{\infty} \varepsilon^n u_\circ^{(n)}(t, x) \tag{1.4}
\]
hold for all \( t > 0, \ x \in [0, \pi], \ \varepsilon > 0, \) and every realization of \( W; \)

\[ |u_\varepsilon(t, x; \varepsilon) - u_\varepsilon(t, x; \varepsilon)| = O(\varepsilon^2), \ \varepsilon \to 0, \]  

for all \( t > 0 \) and \( x \in [0, \pi], \) and every realization of \( W. \)

Equalities (1.3) and (1.4) are in the spirit of [5]. Equality (1.5) is similar to [9, Proposition 4.1]; see also [8].

The precise statement of the main result is in Section 2 and the proof is in Sections 3, 4, and 5.

2. The Main Result

Denote by \( p = p(t, x, y) \) the heat semigroup on \([0, \pi]\) with zero boundary conditions:

\[ p(t, x, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \sin(kx) \sin(ky), \ t > 0, \ x, y \in [0, \pi]. \]  

(2.1)

Let \( \varphi = \varphi(x) \) be a continuous function on \([0, \pi]\), and let \( u = u(t, x) \) be the solution of the heat equation

\[ \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2}, \ t > 0, \ 0 < x < \pi, \]  

\[ u(t, 0) = u(t, \pi) = 0, \ u(0, x) = \varphi(x), \]  

that is,

\[ u(t, x) = \int_0^\pi p(t, x, y)\varphi(y)dy. \]  

(2.3)

Next, define the function \( u = u(t, x) \) by

\[ u(t, x) = \int_0^t \int_0^\pi p(t - s, x, y)u(s, y) \, dW(y) \, ds. \]  

(2.4)

That is, \( u \) is the mild solution of

\[ \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x)\dot{W}(x), \ t > 0, \ 0 < x < \pi, \]  

\[ u(t, 0) = u(t, \pi) = u(0, x) = 0. \]  

(2.5)

Because \( u \) is non-random, no stochastic integral is required to define \( u \).

Proposition 2.1. If \( \varphi \in C([0, \pi]) \), then \( u \) is a continuous function of \( t \) and \( x \) for all \( t > 0 \) and \( x \in [0, \pi]. \)

Proof. This follows by the Kolmogorov continuity criterion: \( u \) is a Gaussian random field and direct computations show

\[ \mathbb{E}((u(t + \tau, x + h) - u(t, x))^2) \leq C(t)(\tau^2 + h^2)^{1/4} \max_{x \in [0, \pi]} |\varphi(x)|; \]

cf. [11, Sections 6 and 7].
Next, define the functions \( u^{(n)} = u^{(n)}(t, x) \), \( n = 0, 1, 2 \ldots \), \( t \geq 0, x \in [0, \pi] \), by 
\[
u^{(0)}(t, x) = u(t, x), \quad \text{and, for } n \geq 1, u^{(n)} \text{ is the mild solution of}
\]
\[
\frac{\partial u^{(n)}(t, x)}{\partial t} = \frac{\partial^2 u^{(n)}(t, x)}{\partial x^2} + u^{(n-1)}(t, x) \circ \dot{W}(x), \quad t > 0, \quad 0 < x < \pi,
\]
(2.6)

In other words,
\[
u^{(n)}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) u^{(n-1)}(s, y) \circ dW(y) ds, \quad n \geq 1,
\]
and, in particular, \( u^{(1)} = u \).

Similarly, define the functions \( u^{(n)} = u^{(n)}(t, x) \), \( n = 0, 1, 2 \ldots \), \( t \geq 0, x \in [0, \pi] \), by 
\[
u^{(0)}(t, x) = u(t, x), \quad \text{and, for } n \geq 1, u^{(n)} \text{ is the mild solution of}
\]
\[
\frac{\partial u^{(n)}(t, x)}{\partial t} = \frac{\partial^2 u^{(n)}(t, x)}{\partial x^2} + u^{(n-1)}(t, x) \circ \dot{W}(x), \quad t > 0, \quad 0 < x < \pi,
\]
(2.8)

In other words,
\[
u^{(n)}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) u^{(n-1)}(s, y) \circ dW(y) ds, \quad n \geq 1,
\]
and, in particular, \( u^{(1)} = u \).

The main result of the paper can now be stated as follows.

**Theorem 2.2.** Let \( \varphi \in C((0, \pi)) \). Then

1. Equality (1.3) holds with \( u^{(n)} \) from (2.7).
2. Equality (1.4) holds with \( u^{(n)} \) from (2.9).
3. Equality (1.5) holds and

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \left( u_\varepsilon(t, x; \varepsilon) - u_\varepsilon(t, x; \varepsilon) \right) = \int_0^\pi p^{(3)}(t, x, z) \varphi(z) dz,
\]
(2.10)

where
\[
p^{(3)}(t, x, z) = \int_0^\pi \int_0^t \int_0^s p(t-s, x, y) p(s-r, y, y) p(r, y, z) dr ds dy.
\]

The proof is carried out in the following three sections.

### 3. The Wick-Itô-Skorokhod Case

The objective of this section is the proof of (1.3).

The solution of (1.1) is defined as a chaos solution (cf. [6, Theorems 3.10]). It is a continuous in \((t, x)\) function (cf. [11, Sections 6 and 7]) and has a representation as a
series
\[ u_\alpha(t, x; \varepsilon) = \sum_{\alpha \in J} u_\alpha(t, x; \varepsilon) \xi_\alpha, \quad (3.1) \]

where
\[ J = \left\{ \alpha = (\alpha_k, k \geq 1) : \alpha_k \in \{0, 1, 2, \ldots\}, \sum_k \alpha_k < \infty \right\} , \]
\[ \xi_\alpha = \prod_k \left( \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right), \quad H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} , \]
\[ \xi_k = \int_0^\pi m_k(x) dW(x), \quad m_k(x) = \sqrt{2/\pi} \sin(kx) , \]
and, with \(|\alpha| = \sum_k \alpha_k, |(0)| = 0, |\varepsilon(k)| = 1,\]
\[ u(0)(t, x; \varepsilon) = u(t, x), \]
\[ u_\alpha(t, x; \varepsilon) = \varepsilon \sum_k \sqrt{\alpha_k} \int_0^t \int_0^\pi p(t-s, x, y) u_\alpha(s, y; 1) \xi_\alpha \cdot m_k(y) dy ds; \]

see [1, Section 3] for details. In particular,
\[ \sum_{|\alpha| = n} u_\alpha(t, x; 1) \xi_\alpha = \sum_{|\alpha| = n-1} \int_0^t \int_0^\pi p(t-s, x, y) u_\alpha(s, y; 1) \xi_\alpha \cdot dW(y) ds. \quad (3.2) \]

Comparing (2.7) with (3.2) shows that
\[ u_\alpha^{(n)}(t, x) = \sum_{|\alpha| = n} u_\alpha(t, x; 1) \xi_\alpha. \quad (3.3) \]

In other words, (1.3) is equivalent to (3.1).

Next,
\[ \mathbb{E}|u_\alpha^{(n)}(t, x)| = \mathbb{E} \left| \sum_{|\alpha| = n} u_\alpha(t, x; 1) \xi_\alpha \right| \leq \left( \mathbb{E} \left( \sum_{|\alpha| = n} u_\alpha(t, x; 1) \xi_\alpha \right)^2 \right)^{1/2} \]
\[ = \left( \sum_{|\alpha| = n} |u_\alpha(t, x; 1)|^2 \right)^{1/2} \leq C^n(t)n^{-n/4} \sup_{x \in (0, \pi)} |\varphi(x)|^{1/2} , \]

where the last inequality follows by [1, Theorem 4.1]. As a result,
\[ \sum_{n \geq 0} \varepsilon^n \mathbb{E}|u_\alpha^{(n)}(t, x)| < \infty , \]
that is, the series converges absolutely with probability one for all \( t > 0, x \in [0, \pi] \), and \( \varepsilon \in \mathbb{R} \).

This concludes the proof of (1.3).
4. The Stratonovich Case

The objective of this section is to prove (1.4). To simplify the presentation, we use the following notations:

\[ \Lambda = (-\Delta)^{1/2}, \quad H^\theta = \Lambda^{-\theta}(L_2((0, \pi))), \quad \| \cdot \| = \| \Lambda \cdot \|_{L_2((0, \pi))}, \quad \theta \in \mathbb{R}, \]

\[ p \ast g(t, s, x) = \int_0^\pi p(t - s, x, y)g(s, y)\, dy, \]

where \( \Delta \) is the Laplace operator on \((0, \pi)\) with zero boundary conditions and \( p \) is the heat kernel (2.1).

By direct computation,

\[ \| p \ast g \|_{\gamma, (t, s)} \leq C_{T, \theta} (t - s)^{-\theta/2} \| g \|_{\gamma - \theta, (s)}; \quad \theta > 0, \quad \gamma \in \mathbb{R}, \quad t \in (s, T); \quad (4.1) \]

cf. [3, Lemma 7.3].

Consider the equation

\[ \frac{\partial v(t, x)}{\partial t} = \frac{\partial^2 v(t, x)}{\partial x^2} + v(t, x) \circ \dot{W}(x) + f(t, x) \circ \dot{W}(x), \quad t > 0, \quad (4.2) \]

which includes (1.2) as a particular case. By definition, a solution (classical, mild, generalized, etc.) of (4.2) is a suitable limit, as \( \epsilon \to 0 \), of the corresponding solutions of

\[ \frac{\partial v_\epsilon(t, x)}{\partial t} = \frac{\partial^2 v_\epsilon(t, x)}{\partial x^2} + v_\epsilon(t, x)V_\epsilon(x) + f(t, x)V_\epsilon(x), \quad t > 0, \quad (4.3) \]

where \( V_\epsilon \), \( \epsilon > 0 \) are smooth functions on \([0, \pi]\) such that

\[ \sup_{\epsilon} \left\| \int_0^\pi V_\epsilon(y)\, dy - W(x) \right\|_{C^{1/2}} < \infty, \quad \lim_{\epsilon \to 0} \sup_{x \in [0, \pi]} \left| \int_0^x V_\epsilon(y)\, dy - W(x) \right| = 0. \]

By [2, Theorem 3.5],

- The generalized solution of (4.2) is the same as the generalized solution of the equation

  \[ v_t = \left( v_x + W(x)v + W(x)f \right)_x - W(x)v_x - W(x)f_x; \quad (4.4) \]

  the subscripts \( t \) and \( x \), as in \( f_x \), denote the corresponding partial derivatives;

- The mild solution of (4.2) is the solution of the integral equation

  \[ v(t, x) = \int_0^t p \ast ((f + v)W)_x(t, s, x)\, ds - \int_0^t p \ast ((f + v)W)(t, s, x)\, ds \]
  \[ + \int_0^\pi p(t, x, y)\varphi(y)\, dy. \quad (4.5) \]

On the one hand, mild and generalized solutions of (4.2) are the same: just use \( m_k \) as the test functions. On the other hand, different definitions of the solution lead to different regularity results.
By standard parabolic regularity, if \( \varphi \in H^0 \) and \( f \in L_2((0,T);H^\gamma) \), \( \gamma \in (1/2,1] \), then there is a unique generalized solution of (4.1) in the normal triple \((H^1,H^0,H^{-1})\) and
\[
v \in L_2((0,T);H^1) \cap C((0,T);H^0)
\] (4.6)
for every realization of \( W \); cf. [4] Theorem 3.4.1. Note that we cannot claim \( v \in C((0,T);H^\gamma) \) even if \( \varphi \in H^\gamma \). In fact, because \( W \in C^{1/2-} \) is a point-wise multiplier in \( H^\gamma \) for \( \gamma \in (-1/2,1/2) \) [3] Lemma 5.2, an attempt to find a traditional regularity result for equation (4.1) in a normal triple \((H^{r+1},H^r,H^{r-1})\) leads to an irreconcilable pair of restrictions on \( r \): to have \( Wf \in L_2((0,T);H^r) \) we need \( r < 1/2 \), whereas to have \( Wf_x \in L_2((0,T);H^{r-1}) \) we need \( r - 1 > -1/2 \) or \( r > 1/2 \).

Accordingly, to derive a bound on \( \|v\|_\gamma(t) \) for \( t > 0 \), we use the mild formulation (4.5).

**Proposition 4.1.** Let \( \gamma \in (1/2,1) \), \( f \in L_2((0,T);H^\gamma) \), \( \varphi \in H^0 \), and let \( v \) be the mild solution of (4.2) with \( v|_{t=0} = \varphi \). Then, for every \( T > 0 \) and every realization of \( W \), there exists a number \( C_0 \) such that
\[
\|v\|_\gamma(t) \leq C_0 \left( t^{-\gamma/2} \|\varphi\|_0 + \int_0^t (t-s)^{-\gamma} \|f\|_\gamma(s) \, ds \right).
\] (4.7)

**Proof.** Throughout the proof, \( C \) denotes a number depending on \( \gamma \), \( T \), and the norm of \( W \) in the space \( C^{1-\gamma} \). The value of \( C \) can change from one instance to another. With no loss of generality, we assume that \( \varphi \) and \( f \) are smooth functions with compact support.

To begin, let us show that if \( V \) is the mild solution of
\[
\frac{\partial V(t,x)}{\partial t} = \frac{\partial^2 V(t,x)}{\partial x^2} + f(t,x) \circ \dot{W}(x), \ t > 0,
\]
\[V(0) = V(t, \pi) = 0, \ V|_{t=0} = \varphi,\]
then
\[
\|V\|_\gamma(t) \leq C t^{-\gamma/2} \|\varphi\|_0 + C \int_0^t (t-s)^{-\gamma} \|f\|_\gamma(s) \, ds.
\] (4.8)

Indeed, by (4.5),
\[
V(t,x) = \int_0^t p \ast (fW)_x(t,s,x) \, ds - \int_0^t p \ast (f_xW)(t,s,x) \, ds + \int_0^\pi p(t,x,y) \varphi(y) \, dy.
\]
Using (4.1) with \( \theta = \gamma \),
\[
\left\| \int_0^\pi p(t,\cdot,y) \varphi(y) \, dy \right\|_\gamma \leq C t^{-\gamma/2} \|\varphi\|_0.
\]

Then
\[
\|V\|_\gamma(t) \leq \int_0^t \|p \ast (fW)_x\|_\gamma(t,s) \, ds + \int_0^t \|p \ast (f_xW)\|_\gamma(t,s) \, ds + C t^{-\gamma/2} \|\varphi\|_0.
\] (4.9)

To estimate the first term on the right hand side of (4.9), we use (4.1) with \( \theta = 2\gamma \). Then
\[
\|p \ast (fW)_x\|_\gamma(t,s) \leq C (t-s)^{-\gamma} \|(fW)_x\|_{-\gamma}(s) \leq C (t-s)^{-\gamma} \|fW\|_{1-\gamma}(s),
\]
and, because $W \in C^{1/2-}((0, \pi))$ is a (point-wise) multiplier in $H^{1-\gamma}$,
\[
\|fW\|_{1-\gamma}(s) \leq CW\|f\|_{1-\gamma}(s);
\]
recall that $0 < 1 - \gamma < 1/2$. Finally, as $1 - \gamma < \gamma$,
\[
\|p * (fW)_x\|_{\gamma}(t, s) \leq C(t - s)^{-\gamma}\|f\|_{\gamma}(s). \tag{4.10}
\]
To estimate the second term on the right hand side of (4.9), we use (4.1) with $\theta = 1$. Then
\[
\|p * (fW)_x\|_{\gamma}(t, s) \leq \frac{C}{\sqrt{t - s}} \|fW\|_{\gamma-1}(s) \leq \frac{C}{\sqrt{t - s}} \|fW\|_{\gamma-1}(s),
\]
that is,
\[
\|p * (fW)_x\|_{\gamma}(t, s) \leq C(t - s)^{-1/2}\|f\|_{\gamma}(s). \tag{4.11}
\]
To establish (4.8), we now combine (4.9), (4.10), and (4.11), keeping in mind that $(t - s)^{-1/2} \leq C(t - s)^{-\gamma}$ because $\gamma > 1/2$.

Next, (4.8) applied to (4.2) implies
\[
\|v\|_{\gamma}(t) \leq C \int_0^t (t - s)^{-\gamma}\|v\|_{\gamma}(s) \, ds + C \int_0^t (t - s)^{-\gamma}\|f\|_{\gamma}(s) \, ds + C(t^{-\gamma/2}\|\varphi\|_0,
\]
and then a generalization of the Gronwall inequality (e.g. [10, Corollary 2]) leads to (4.7).

**Corollary 4.2.** If $\varphi \in H^0$ and $\gamma \in (1/2, 1)$, then, for every $T > 0$, $a > 0$, and every realization of $W$, there exists a number $\tilde{C}_0$ such that the mild solution of (1.2) satisfies
\[
\sup_{|\varepsilon| < \varepsilon_0} \|u_a(t, \cdot, \varepsilon)\|_{\gamma} \leq \tilde{C}_0 t^{-\gamma/2}\|\varphi\|_0, \ t \in (0, T]. \tag{4.12}
\]

Next, define the functions $u_0^{(n), \varepsilon} = u_0^{(n), \varepsilon}(t, x)$, $n = 0, 1, 2, \ldots$, $t \geq 0$, $x \in [0, 1]$, $\varepsilon \in \mathbb{R}$, by $u_0^{(0), \varepsilon}(t, \cdot) = u_0(t, \cdot, \varepsilon)$ and, for $n \geq 1$,
\[
\frac{\partial u_0^{(n), \varepsilon}(t, x)}{\partial t} = \frac{\partial^2 u_0^{(n), \varepsilon}(t, x)}{\partial x^2} + \varepsilon u_0^{(n), \varepsilon}(t, x) \circ \tilde{W}(x) + u_0^{(n-1), \varepsilon}(t, x) \circ \tilde{W}(x), \tag{4.13}
\]
and $u_0^{(n), \varepsilon}(t, 0) = u_0^{(n), \varepsilon}(t, \pi) = 0, u_0^{(n), \varepsilon}(0, x) = 0$.

In particular,
\[
u_0^{(n), 0}(t, x) = u_0^{(n)}(t, x).
\]

Note that all equations in (4.13) are of the form (4.2).

**Proposition 4.3.** If $\varphi \in H^0$, then, for every $\gamma \in (1/2, 2/3)$ and every realization of $W$,
\[
\lim_{\varepsilon \to \varepsilon_0} \frac{1}{(\varepsilon - \varepsilon_0)^n} \left\| u_0(t, \cdot, \varepsilon) - \sum_{k=0}^n (\varepsilon - \varepsilon_0)^k u_0^{(k), \varepsilon_0}(t, \cdot) \right\|_{\gamma} = 0, \tag{4.14}
\]
for $n \geq 0, \varepsilon_0 \in \mathbb{R}, t \geq 0$.  

Proof. Throughout the proof, $C$ denotes a number depending on $\gamma$, $T$, $\varepsilon_0$, and the norm of $W$ in the space $C^{1-\gamma}$. Define
\begin{equation}
\tilde{v}_\varepsilon^{(n)}(t, x) = \frac{1}{(\varepsilon - \varepsilon_0)^n} \left( u_\varepsilon(t, x; \varepsilon) - \sum_{k=0}^{n} (\varepsilon - \varepsilon_0)^k u_\varepsilon^{(k, \varepsilon_0)}(t, x) \right). \tag{4.15}
\end{equation}

By (4.2),
\begin{equation}
\sum_{k=0}^{n} (\varepsilon - \varepsilon_0)^k \frac{\partial u_\varepsilon^{(k, \varepsilon_0)}(t, x)}{\partial t} = \sum_{k=0}^{n} (\varepsilon - \varepsilon_0)^k \frac{\partial^2 u_\varepsilon^{(k, \varepsilon_0)}(t, x)}{\partial x^2} + (\varepsilon - \varepsilon_0)^k u_\varepsilon^{(k, \varepsilon_0)}(t, x) \circ \tilde{W}(x) \tag{4.16}
\end{equation}
so that
\begin{equation}
\frac{\partial \tilde{v}_\varepsilon^{(0)}(t, x)}{\partial t} = \frac{\partial^2 \tilde{v}_\varepsilon^{(0)}(t, x)}{\partial x^2} + \varepsilon_0 \tilde{v}_\varepsilon^{(0)}(t, x) \circ \tilde{W}(x) + (\varepsilon - \varepsilon_0) u_\varepsilon(t, x; \varepsilon) \circ \tilde{W}(x), \tag{4.17}
\end{equation}
\begin{equation}
\frac{\partial \tilde{v}_\varepsilon^{(n)}(t, x)}{\partial t} = \frac{\partial^2 \tilde{v}_\varepsilon^{(n)}(t, x)}{\partial x^2} + \varepsilon_0 \tilde{v}_\varepsilon^{(n)}(t, x) \circ \tilde{W}(x) + \tilde{v}_\varepsilon^{(n-1)}(t, x) \circ \tilde{W}(x), \ n \geq 1, \tag{4.18}
\end{equation}
\begin{equation}
\tilde{v}_\varepsilon^{(n)}(0, x) = 0, \ n \geq 0, \text{ and (4.14) becomes}
\lim_{\varepsilon \to \varepsilon_0} \| \tilde{v}_\varepsilon^{(n)} \|_{\gamma}(t) = 0, \ \ n \geq 0, \ \ t \geq 0, \ \varepsilon_0 \in \mathbb{R}. \tag{4.19}
\end{equation}

Note that all equations in (4.17) are of the form (4.2), and (4.18) trivially holds for $t = 0$. Accordingly, combining the second equation in (4.17) with (4.17),
\begin{equation}
\| \tilde{v}_\varepsilon^{(n)} \|_{\gamma}(t) \leq C \int_0^t (t - s)^{-\gamma} \| \tilde{v}_\varepsilon^{(n-1)} \|_{\gamma}(s) \, ds, \ n \geq 1, \end{equation}
and then, for $t > 0$, (4.18) follows by induction: for $n = 0$, (4.12) yields
\begin{equation}
\| \tilde{v}_\varepsilon^{(0)} \|_{\gamma}(t) \leq |\varepsilon - \varepsilon_0| C \int_0^t (t - s)^{-\gamma} \| u_\varepsilon(s, \cdot; \varepsilon) \|_{\gamma} \, ds \leq C |\varepsilon - \varepsilon_0| \| \tilde{v}_0 \|_{0, t^{-1-(3/2)\gamma}} \to 0, \ \varepsilon \to \varepsilon_0;
\end{equation}
similarly, for $n \geq 1$,
\begin{equation}
\| \tilde{v}_\varepsilon^{(n)} \|_{\gamma}(t) \leq C^{(n)} |\varepsilon - \varepsilon_0| \| \tilde{v}_\varepsilon \|_{0},
\end{equation}
because $1 - (3/2)\gamma > 0$. \hfill \square

**Proposition 4.4.** If $\varphi \in H^0$ and $\gamma \in (1/2, 1)$, then
\begin{equation}
\lim_{n \to \infty} c^n \sup_{|\varepsilon| < c} \| u_\varepsilon^{(n)} \|_{\gamma}(t) = 0, \ t \geq 0, \tag{4.19}
\end{equation}
for all $c > 0, \ a > 0$, and every realization of $W$. 

Proof. Throughout this proof, $C$ denotes a number depending on $\gamma, T, a$, and the norm of $W$ in the space $C^{1-\gamma}$.

Combining (4.13) and (4.7),

$$\|u^{(n),\varepsilon}\|_{\gamma}(t) \leq C \int_0^t (t-s)^{r-1}\|u^{(n-1),\varepsilon}\|_{\gamma}(s) \, ds.$$  

By iteration and (4.12), with $r = 1 - \gamma > 0$,

$$\sup_{|\varepsilon| < a} \|u^{(n),\varepsilon}\|_{\gamma}(t) \leq C^n \|\varphi\|_0 \times \int_0^t \int_0^{s_n-1} \cdots \int_0^{s_2} (t-s_n)^{r-1}(s_n-s_{n-1})^{r-1} \cdots (s_2-s_1)^{r-1} s_1^{-\gamma/2} ds_1 \cdots ds_n \leq C^n \|\varphi\|_0 \frac{(\Gamma(r))^n \Gamma(1-(\gamma/2))}{\Gamma(nr + 1)} t^{nr-(\gamma/2)},$$  

where $\Gamma$ is the Gamma function

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt.$$  

Then (4.19) follows by the Stirling formula. \qed

Equality (1.4) now follows:

- By the Sobolev embedding theorem, every element, or equivalence, class from $H^\gamma$, $\gamma > 1/2$, has a representative that is a continuous function on $[0, \pi]$;
- By Proposition 4.3 and the Taylor formula,

$$u(t, x) = u^{(1)}(t, x) = u^{(1)} + \sum_{k=1}^n u^{(k)}(t, x) \varepsilon^k + R_n(t, x);$$  

- By Proposition 4.4

$$\lim_{n \to \infty} R_n(t, x) = 0.$$  

This concludes the proof of (1.4).

5. THE CORRECTION TERM

The objective of this section is the proof of (2.10).

Using (1.3) and (1.4), and remembering that $u^{(1)} = u^{(1)} = u$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \left( u^{(1)}(t, x; \varepsilon) - u^{(1)}(t, x; \varepsilon) \right) = u^{(2)}(t, x) - u^{(2)}(t, x) = \int_0^t \int_0^\pi p(t-s, x, y) \left( u(s, y) \circ dW(y) - u(s, y) \circ dW(y) \right) ds.$$  

By definition,

$$\xi_k \circ \xi_n = \begin{cases} \xi_k \xi_n, & k \neq n, \\ \xi_n^2 - 1, & k = n, \end{cases}$$  

where $\xi_k$ are the eigenvalues of $D$.
and therefore
\[ \xi_k \xi_n - \xi_k \diamond \xi_n = \begin{cases} 0 & k \neq n, \\ 1, & k = n, \end{cases} \tag{5.2} \]

Then (5.2) and [7, Theorem 3.1.2] imply that, for a function \( f = f(x) \) of the form
\[ f(x) = \sum_{k=1}^{\infty} f_k(x) \xi_k, \]
with \( f_k \) non-random and satisfying
\[ \sum_k \int_0^\pi |f_k(x)| \, dx < \infty, \tag{5.3} \]
the following equality holds:
\[ \int_0^\pi f(x) \circ dW(x) - \int_0^\pi f(x) \diamond dW(x) = \sum_{k=1}^{\infty} \left( \int_0^\pi f_k(x) m_k(x) \, dx \right). \tag{5.4} \]

Condition (5.3) ensures that the sum on the right-hand side of (5.4) converges absolutely.

Next, recall that, by (2.4),
\[ u(s, y) = \sum_{k=1}^{\infty} \left( \int_0^\pi \int_0^{s} p(s-r, y, z) u(r, z) m_k(z) \, dr \, dz \right) \xi_k. \]

For fixed \( s \in [0, T] \) and \( y \in [0, \pi] \), define
\[ g(z) = \int_0^{s} p(s-r, y, z) u(r, z) \, dr, \quad g_k = \int_0^\pi g(z) m_k(z) \, dz. \]

Then
\[ u = \sum_{k=1}^{\infty} g_k \xi_k, \]
and (5.3) in this case will follow from uniform, in \( (s, y) \) convergence of
\[ \sum_{k=1}^{\infty} |g_k|, \]
which, by Bernstein’s theorem [11, Theorem VI.3-1], will, in turn, follow from
\[ |g(z+h) - g(z)| \leq Ch^\delta \]
with \( \delta \in (1/2, 1) \) and \( C \) independent of \( s, y, z \).

Recall that
\[ u(r, z) = \sum_{k=1}^{\infty} \varphi_k e^{-k^2r} m_k(z), \quad \varphi_k = \int_0^\pi \varphi(x) m_k(x) \, dx, \]
and, by integral comparison,
\[ \sum_{k=1}^{\infty} k^p e^{-k^2t} \leq \frac{C(p)}{p^{(1+p)/2}}, \quad p \geq 0. \]
Also,
\[ |\sin(k(z + h)) - \sin(kz)| \leq k^\delta h^\delta, \quad \delta \in (0, 1), \]
and the maximum principle implies \(|u(r, z)| \leq C\). Then
\[ p(s - r, y, z) \leq \frac{C}{\sqrt{s - r}}, \quad |p(s - r, y, z + h) - p(s - r, y, z)| \leq \frac{Ch^\delta}{(s - r)^{(1+\delta)/2}}, \]
\[ |u(r, z + h) - u(r, z)| \leq \frac{Ch^\delta}{r^{(1+\delta)/2}}, \]
and (5.3) follows because
\[ \int_0^s \frac{dr}{(r(s - r))^{(1+\delta)/2}} < \infty \]
for \(\delta \in (1/2, 1)\).

We now apply (5.4) to (5.1):
\[
\int_0^t \int_0^\pi p(t - s, x, y)(u(s, y) \diamond dW(y) - u(s, y) \diamond dW(y)) \, ds
\]
\[
= \int_0^t \int_0^\pi \sum_{n=1}^\infty \left( \int_0^\pi \left( \int_0^s p(s - r, y, z)u(r, z) \, dr \right) m_n(z) \, dz \right) m_n(y)p(t - s, x, y) \, dy \, ds
\]
\[
= \int_0^\pi \int_0^t \int_0^s p(t - s, x, y)p(s - r, y, y)u(r, y) \, dr \, ds \, dy,
\]
which, in view of (2.3) and the Fubini theorem, is the same as (2.10).

This concludes the proof of (2.10).

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*Current address*, H.-J. Kim: Department of Mathematics, USC, Los Angeles, CA 90089

*E-mail address*, H.-J. Kim: hyunjungmath@gmail.com

*URL*: https://hyunjungkim.org/

*Current address*, S. V. Lototsky: Department of Mathematics, USC, Los Angeles, CA 90089

*E-mail address*, S. V. Lototsky: lototsky@math.usc.edu

*URL*: http://www-bcf.usc.edu/~lototsky