SPREADING SPEED OF A DEGENERATE AND COOPERATIVE EPIDEMIC MODEL WITH FREE BOUNDARIES

MENG ZHAO AND WAN-TONG LI
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu, 730000, China

WENJIE NI
School of Science and Technology, University of New England
Armidale, NSW, 2351, Australia

(Communicated by Shigui Ruan)

Abstract. This paper deals with the spreading speed of the disease described by a partially degenerate and cooperative epidemic model with free boundaries. We show that the spreading speed is determined by a semi-wave problem. To find such a semi-wave solution, we prove the existence of a monotone solution to a reduced ODE by an upper and lower solution approach. And then we establish the uniqueness of the semi-wave solution via the sliding method. It is demonstrated that the precise asymptotic spreading speed is less than the minimal speed of traveling waves.

1. Introduction. The spatial spread of epidemics is an important subject in mathematical epidemiology. In order to model the cholera epidemic which spread in the European Mediterranean regions in 1973, Capasso and Paveri-Fontana [4] proposed the following model

\[
\begin{align*}
\frac{du}{dt} &= -au(t) + cv(t), \\
\frac{dv}{dt} &= -bv(t) + G(u(t)),
\end{align*}
\]

where \(a, b, c\) are all positive constants, \(u(t)\) and \(v(t)\) stand for the densities of the infectious agents and the infective human at time \(t\), the parameters \(a\) and \(b\) denote the unit decreasing rates of the infectious agents and the infective human in the environment respectively, \(cv\) is the growth rate of the infectious agents caused by the infective humans, and \(G(u)\) is the infection rate of human population under the assumption that the total susceptible human population is constant during the evolution of epidemic.

In 1981, Capasso and Maddalena [3] further considered the corresponding spatial diffusion problem. By assuming that the bacteria disperse randomly while the small mobility of the infective human population is neglected, they derived the following
diffusive model
\[
\begin{aligned}
&u_t = d\Delta u - au + cv, \quad (t, x) \in (0, +\infty) \times \Omega, \\
v_t = -bv + G(u), \quad (t, x) \in (0, +\infty) \times \Omega, \\
\frac{\partial u}{\partial n} + au = 0, \quad (t, x) \in (0, +\infty) \times \partial \Omega, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega,
\end{aligned}
\] (2)
where the function $G(\cdot)$ satisfies
\begin{align*}
&\text{(G1) } G \in C^2([0, \infty)), \quad G(0) = 0, \quad G'(z) > 0 \text{ for } \forall z \geq 0; \\
&\text{(G2) } \frac{G(z)}{z} \text{ is decreasing and } \limsup_{z \to +\infty} \frac{G(z)}{z} < \frac{ab}{c}.
\end{align*}

By introducing a threshold parameter $R_0^D = \frac{cG'(0)}{(a + d\Lambda_1)\bar{b}}$, they proved that the epidemic eventually tends to extinction if $0 < R_0^D < 1$, and a globally asymptotically stable endemic state appears if $R_0^D > 1$, where $\Lambda_1$ is the first eigenvalue of
\[
-\Delta \phi = \lambda \phi \text{ in } \Omega \text{ with } \frac{\partial \phi}{\partial n} + \alpha \phi = 0 \text{ on } \partial \Omega.
\]

It is well-known that traveling wave solutions can be considered as dispersal process of epidemic from outbreak to an endemic. To understand such a dispersal process, Zhao and Wang [27] considered corresponding traveling waves of (2) and obtained the following result.

**Proposition 1.** For the problem
\[
\begin{align*}
& s\Phi' - d\Phi'' = -a\Phi + c\Psi, \quad \xi \in \mathbb{R}, \\
& s\Psi' = -b\Psi + G(\Phi), \quad \xi \in \mathbb{R}, \\
& \Phi(-\infty) = 0, \quad \Phi'(\xi) > 0, \quad \Phi(+\infty) = u^*, \quad \xi \in \mathbb{R}, \\
& \Psi(-\infty) = 0, \quad \Psi'(\xi) > 0, \quad \Psi(+\infty) = v^*, \quad \xi \in \mathbb{R},
\end{align*}
\] (3)
there exists $s_0$ such that (3) has a solution when $s \geq s_0$ and it has no solution when $s < s_0$.

The epidemic always spread gradually, but the works mentioned above are hard to explain this gradual expanding process. In 2010, a different approach was proposed by Du and Lin [7] to understand the spreading of species, where the spreading front is represented by a free boundary, the deduction of such a free boundary condition can be found in [17, 2]. Since then, many problems with free boundaries have been investigated, see e.g. [6, 8, 9, 13, 14, 15, 16, 19, 20, 21, 22, 23] and their references. In particular, Ahn et al. [1] used the following model to describe the spreading of epidemic in (2):
\[
\begin{aligned}
&u_t = du_{xx} - au + cv, \quad t > 0, \quad g(t) < x < h(t), \\
v_t = -bv + G(u), \quad t > 0, \quad g(t) < x < h(t), \\
u(t, x) = v(t, x) = 0, \quad t > 0, \quad x \leq g(t) \text{ or } x \geq h(t), \\
g(0) = -h_0, \quad g'(t) = -\mu u_x(t, g(t)), \quad t > 0, \\
h(0) = h_0, \quad h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad -h_0 \leq x \leq h_0.
\end{aligned}
\] (4)
They proved that (4) admits a unique solution which is defined for all $t > 0$, and the bacteria will spread if
\[
h_\infty - g_\infty = \infty \quad \text{and} \quad \lim_{t \to \infty} (u, v) = (u^*, v^*) \text{ uniformly in any bounded subset of } \mathbb{R},
\]
or vanish if

$$h_{\infty} - g_{\infty} < h^*$$

and

$$\lim_{t \to \infty} (\|u(t, \cdot)\|_{C([g(t), h(t)])} + \|v(t, \cdot)\|_{C([g(t), h(t)])}) = 0,$$

where \((u^*, v^*)\) is the unique positive equilibrium of (1). Furthermore, by introducing the so-called spatial-temporal risk index

$$R_0^F(t) = \frac{G'(0)c}{a + d \left( \frac{\pi}{h(t) - g(t)} \right)^2},$$

they proved that: (i) if $$R_0 = \frac{cG'(0)}{ab} \leq 1$$, the epidemic will vanish; (ii) if $$R_0^F(0) \geq 1$$, the epidemic will spread; (iii) if $$R_0^F(0) < 1$$, epidemic will vanish for the small initial densities or small $$\mu$$; (iv) if $$R_0^F(0) < 1 < R_0$$, epidemic will spread for the large initial densities or large $$\mu$$. These results may seem to be more reasonable.

Besides the above important feature (spreading-vanishing dichotomy), it is well-known that the asymptotic spreading speed of the spreading fronts is another important subject in order to study the spreading of disease and biological invasions. Du and Lin [7] first considered the logistic equation with free boundary and estimated the precise asymptotic spreading speed. In particular, they showed the spreading speed is less than the minimal speed of traveling waves. The related results can refer to [9, 10].

Recently, there are some results on determining the precise spreading speed of reaction-diffusion systems. In 2017, Du et al. [11] first studied a competition diffusion system (an invasive species invades the territory of a native competitor) and obtained the precise asymptotic spreading speed of the invasive species for the weak-strong competition case, which is determined by a certain traveling wave type system, called semi-wave. After that, Wang et al. [24] considered the weak competition case and Du and Wu [12] investigated the spreading behavior of two invasive species modeled by a Lotka-Volterra diffusive competition system with two different free boundaries for the weak-strong competition case. By making use of the results in [11], Du and Wu [12] showed that two competing species can spread successfully with two different speeds and their masses tend to segregate under some suitable conditions. Very recently, Wang et al. [25] also determined the precise asymptotic spreading speed of the virus for a West Nile virus model (a cooperative system) with free boundary studied by Lin and Zhu [18].

Just as pointed out by Ahn et al. [1] that the study of the precise spreading speed of spreading fronts of (4) is an interesting question. The purpose of this paper is to consider it and show its relationship with the minimal speed $$s_0$$ of the traveling waves of (3). Since the epidemic will vanish when $$R_0 \leq 1$$, we just need to consider the case $$R_0 > 1$$, namely,

$$ab < cG'(0).$$

Throughout this paper, we always assume that the conditions (G1)-(G3) hold.

Now, we state our first main result as follows, which is the so-called semi-wave result.

**Theorem 1.1.** For any $$s \in (0, s_0)$$, the problem

$$\begin{cases}
    s\phi' - d\phi'' = -a\phi + c\psi, & \xi > 0, \\
    s\psi' = -b\psi + G(\phi), & \xi > 0, \\
    (\phi(0), \psi(0)) = (0, 0), & (\phi(+\infty), \psi(+\infty)) = (u^*, v^*)
\end{cases}$$

(5)
has a unique strictly increasing solution \((\phi_s, \psi_s)\) satisfying

\[
\phi_{s_1}'(0) > \phi_{s_2}'(0), \quad \phi_{s_1}(\xi) > \phi_{s_2}(\xi), \quad \psi_{s_1}(\xi) > \psi_{s_2}(\xi)
\]

for \(0 < s_1 < s_2 < s_0\). Furthermore, for any \(\mu > 0\), there exists a unique \(s_\mu \in (0, s_0)\) such that \(\mu \phi_{s_\mu}'(0) = s_\mu\).

The proof of Theorem 1.1 follows from that of [11, Theorem 1.3] and [25, Theorem 3.2]. However, there are some differences in the proof as a result of the disappearance of diffusion coefficient of \(v\) in (4).

Our second main result is concerned with the asymptotic spreading speed of (4) when spreading happens by using the results of Theorem 1.1. If \(R_0 > 1\), then it follows from [1] that spreading will happen under the following cases:

- (S1) \(R_{0F}'(0) \geq 1\);
- (S2) \(R_{0F}'(0) < 1\), \(\|u_0(x)\|_{C([-h_0, h_0])}\) and \(\|v_0(x)\|_{C([-h_0, h_0])}\) are sufficiently large for given \(\mu\);
- (S3) \(R_{0F}'(0) < 1\), \(\mu\) is sufficiently large for given \(u_0(x), v_0(x)\).

Theorem 1.2. Assume that \(R_0 > 1\). If (S1), (S2) or (S3) holds, then we have

\[
\lim_{t \to \infty} \frac{g(t)}{t} = -s_\mu, \quad \lim_{t \to \infty} \frac{h(t)}{t} = s_\mu.
\]

The rest of this paper is organized as follows. We will discuss the semi-wave problem and prove Theorem 1.1 in Section 2. In Section 3, we give the precise estimation of asymptotic spreading speed when spreading occurs.

2. Semi-wave solutions. In this section, we mainly give the proof of Theorem 1.1, which divides into several lemmas.

By the second equation of (5), we have

\[
\psi(\xi) = e^{-\frac{b}{s}(\xi-\xi_0)}\psi(\xi_0) + \frac{1}{s} \int_{\xi_0}^{\xi} e^{-\frac{b}{s}(\xi-\tau)} G(\phi(\tau)) d\tau, \quad \forall \xi_0 \geq 0, \xi > \xi_0.
\]

By taking \(\xi_0 = 0\), we obtain

\[
\psi(\xi) = \frac{1}{s} \int_{0}^{\xi} e^{-\frac{b}{s}(\xi-\tau)} G(\phi(\tau)) d\tau, \quad \xi > 0.
\] (6)

Substituting (6) into the first equation of (5), we get

\[
\begin{cases}
  s\phi' - d\phi'' = -a\phi + \frac{1}{s} \int_{0}^{\xi} e^{-\frac{b}{s}(\xi-\tau)} G(\phi(\tau)) d\tau, \quad \xi > 0, \\
  \phi(0) = 0, \quad \phi(+\infty) = u^*.
\end{cases}
\] (7)

If \(\phi(\xi)\) is a strictly increasing solution of (7), then \(\psi(\xi)\) is strictly increasing solution of

\[
\begin{cases}
  s\psi' = -b\psi + G(\phi), \quad \xi > 0, \\
  \psi(0) = 0, \quad \psi(+\infty) = v^*.
\end{cases}
\] (8)

Namely, \((\phi, \psi)\) is a strictly increasing solution of (5). Hence, in order to prove the existence of solutions to (5), it is sufficient to consider the existence of solutions to (7).
Let $X$ be the space of all bounded and uniformly continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$ with the usual norm. Define a mapping $S: X \rightarrow X$ by

$$
S(\phi)(\xi) = \frac{c}{s d(\gamma_2 - \gamma_1)} \left[ \int_{\xi}^{\infty} \left( e^{\gamma_1(\xi - \theta)} - e^{\gamma_1 \xi - \gamma_1 \theta} \right) \int_{0}^{\theta} e^{-\frac{1}{2}(\theta - \tau)} G(\phi(\tau)) d\tau d\theta - \int_{\xi}^{\infty} \left( e^{\gamma_2(\xi - \theta)} - e^{\gamma_1 \xi - \gamma_2 \theta} \right) \int_{0}^{\theta} e^{-\frac{1}{2}(\theta - \tau)} G(\phi(\tau)) d\tau d\theta \right],
$$

where

$$
\gamma_1 = \frac{s - \sqrt{s^2 + 4da}}{2d} < 0 \text{ and } \gamma_2 = \frac{s + \sqrt{s^2 + 4da}}{2d} > 0
$$

are two different roots of

$$
d\gamma^2 - s\gamma - a = 0.
$$

By direct calculations, we see that $S(\phi(t))$ is the unique bounded solution on $\mathbb{R}^+$ to the following equation

$$
\begin{cases}
  du'' - su' - au + \frac{c}{s} \int_{0}^{\xi} e^{-\frac{1}{2}(\xi - \tau)} G(\phi(\tau)) d\tau = 0, \\
  u(0) = 0.
\end{cases}
$$

It is clear that any fixed point of $S$ in $X$ is a solution of (7).

Next, we introduce a definition for upper and lower solutions.

**Definition 2.1.** Assume that $\overline{\phi}$ and $\phi$ are twice continuously differentiable in $\mathbb{R}^+$ expect finite many points $\xi_i$ with $\overline{\phi}(\xi_i^+) \leq \overline{\phi}(\xi_i^-)(1 \leq i \leq m)$ and $\phi(\xi_j^+) \geq \phi(\xi_j^-)(1 \leq j \leq n)$, and satisfy

$$
\begin{cases}
  s\overline{\phi}'' - d\overline{\phi}'' \geq -a\overline{\phi} + \frac{c}{s} \int_{0}^{\xi} e^{-\frac{1}{2}(\xi - \tau)} G(\overline{\phi}(\tau)) d\tau, \ 0 < \xi < \infty, \ \xi \neq \xi_i, \ 1 \leq i \leq m, \\
  \overline{\phi}(0) = 0, \ \overline{\phi}(\infty) = u^*
\end{cases}
$$

and

$$
\begin{cases}
  s\phi'' - d\phi'' \leq -a\phi + \frac{c}{s} \int_{0}^{\xi} e^{-\frac{1}{2}(\xi - \tau)} G(\phi(\tau)) d\tau, \ 0 < \xi < \infty, \ \xi \neq \xi_j, \ 1 \leq j \leq n, \\
  \phi(0) = 0, \ \phi(\infty) < u^*
\end{cases}
$$

respectively. Then $\overline{\phi}$ and $\phi$ are called an upper solution and a lower solution of (7) respectively.

**Lemma 2.2.** The following statements hold:

(i) If $\phi_1 \leq \phi_2$, then $S(\phi_1)(\xi) \leq S(\phi_2)(\xi)$ for $s > 0$;

(ii) If $\phi$ is a monotone increasing function, then $S(\phi)$ is also monotone increasing.

**Proof.** In view of the definition of $S(\phi)$, we conclude that (i) holds. Next, we prove (ii). Let

$$
f(x) = \int_{0}^{x} e^{-\frac{1}{2}(x - \tau)} G(\phi(\tau)) d\tau,
$$

we have

$$
f(x + \vartheta) = \int_{0}^{x+\vartheta} e^{-\frac{1}{2}(x+\vartheta - \tau)} G(\phi(\tau)) d\tau
= \int_{-\vartheta}^{x} e^{-\frac{1}{2}(x-z)} G(\phi(z + \vartheta)) dz > f(x)
$$
for \( \vartheta > 0 \). Noting that \( \gamma_1 < 0 \) and \( \gamma_2 > 0 \), we obtain

\[
S(\phi)'(\xi) = \frac{c}{sd(\gamma_2 - \gamma_1)} \left( \gamma_1 \int_0^\xi e^{\gamma_1(\xi - \theta)} \int_0^\theta e^{-\frac{\theta}{s}(\theta - \tau)} G(\phi(\tau))d\tau d\theta \\
+ \gamma_2 \int_\xi^\infty e^{\gamma_2(\xi - \theta)} \int_0^\theta e^{-\frac{\theta}{s}(\theta - \tau)} G(\phi(\tau))d\tau d\theta \\
- \gamma_1 \int_0^\infty e^{\gamma_1(\xi - \gamma_2 \theta)} \int_0^\theta e^{-\frac{\theta}{s}(\theta - \tau)} G(\phi(\tau))d\tau d\theta \right)
\geq \frac{cf(\xi)}{sd(\gamma_2 - \gamma_1)} \left( \gamma_1 \int_0^\xi e^{\gamma_1(\xi - \theta)} d\theta + \gamma_2 \int_\xi^\infty e^{\gamma_2(\xi - \theta)} d\theta \right)
= \frac{cf(\xi)}{sd(\gamma_2 - \gamma_1)} e^{\gamma_1 \xi} > 0.
\]

Now we are ready to establish the existence of monotone solutions of (7).

**Theorem 2.3.** Suppose that (7) admits an upper solution \( \overline{\phi} \) and a lower solution \( \underline{\phi} \) such that

(i) \( \overline{\phi} \) is monotone increasing on \( \mathbb{R}^+ \);

(ii) \( \underline{\phi} \) is monotone increasing on \( \mathbb{R}^+ \) and \( \underline{\phi}(\xi) \leq \overline{\phi}(\xi), \forall \xi \in \mathbb{R}^+ \).

Then (7) has an increasing solution on \( \mathbb{R}^+ \).

**Proof.** The idea of this proof comes from [27, Theorem 2.1].

Let \( \phi_m = S^m(\overline{\phi}) \) for any \( m \geq 0 \). It follows from Definition 2.1 and Lemma 2.2 (i) that

\( \phi_m(\xi) \leq \phi_m(\xi) \leq \phi_{m-1}(\xi) \leq \overline{\phi}(\xi), \forall \xi > 0, m \geq 1 \).

For each \( \xi \), \( \{\phi_m(\xi)\} \) is decreasing in \( m \). Thus, \( \phi(\xi) = \lim_{m \to \infty} \phi_m(\xi) \) exists and

\( \phi(\xi) \leq \phi(\xi) \leq \overline{\phi}(\xi), \forall \xi > 0 \).

By Lemma 2.2 (ii), we see that \( \phi_m(\xi) \) is increasing in \( \xi \) for each \( m \). It follows that \( \phi(\xi) \) is increasing in \( \xi \) and

\( 0 = \phi(0) \leq \phi(\xi) \leq u^* \).

For each \( \xi \), since \( \phi_m(\xi) = S(\phi_m-1)(\xi) \), by Lebesgue’s convergence theorem, we obtain \( \phi(\xi) = S(\phi)(\xi) \). This means that \( \phi \) is a monotone solution of (7). It remains to show that \( \phi(\infty) = u^* \). By [26, Lemma 2.2], it follows that \( \lim_{\xi \to \infty} \phi'(\xi) = 0 \) and \( \lim_{\xi \to \infty} \phi''(\xi) = 0 \). Letting \( z = \tau - \xi \), we have

\[
\frac{c}{s} \int_0^\xi e^{-\frac{\xi}{s}(\xi - \tau)} G(\phi(\tau)) d\tau = \frac{c}{s} \int_{-\xi}^0 e^{\frac{\xi}{s}z} G(\phi(z + \xi)) dz.
\]

By the definition of limit, we can show that

\[
\lim_{\xi \to \infty} \frac{c}{s} \int_{-\xi}^0 e^{\frac{\xi}{s}z} G(\phi(z + \xi)) dz = \frac{c}{s} \int_{-\infty}^0 e^{\frac{\xi}{s}z} G(\phi(\infty)) dz.
\]

Hence,

\[ a\phi(\infty) = \frac{c}{b} G(\phi(\infty)). \]

The uniqueness of positive equilibrium of (1) implies that \( \phi(\infty) = u^* \).
Define
\[ p_1(\alpha) = d\alpha^2 - s\alpha - a, \quad p_2(\alpha) = -s\alpha - b \]
and
\[ P_1(\alpha) = p_1(\alpha)p_2(\alpha) - cG'(0) = (d\alpha^2 - s\alpha - a)(-s\alpha - b) - cG'(0). \] (10)
Clearly, \( p_1(\alpha) \) has roots
\[ \alpha_1^\pm = \frac{s \pm \sqrt{s^2 + 4da}}{2d}, \]
and \( p_2(\alpha) \) has root
\[ \alpha_2^- = -\frac{b}{s}. \]

It follows from [27, Section 3] that the following results hold.

**Lemma 2.4.** There exists a unique \( s^* \) such that the following statements hold:
(i) For any \( s > s^* \), \( \alpha_i (i = 1, 2, 3) \) are three real roots of \( P_1(\alpha) \), and
\[ \alpha_1 < \min\{\alpha_1^-, \alpha_2^-\} < 0 < \alpha_2 < \alpha_3 < \alpha_1^+; \]
(ii) For \( s = s^* \), \( P_1(\alpha) \) has two real roots, \( \alpha_1 \) and \( \alpha_2 = \alpha_3; \)
(iii) For any \( 0 < s < s^* \), \( P_1(\alpha) \) has one negative root \( \alpha_1 \) and two complex roots with positive real parts.

We remark that it is easy to see that
\[ s^* = \inf\{\hat{s} > 0 : \text{all roots of } P_1(\alpha) = 0 \text{ are real for } s \geq \hat{s}\}. \]

In the following, we construct \( \overline{\phi}(x) \) satisfying
\[ s\overline{\phi}' - d\overline{\phi}'' \geq -a\overline{\phi} + \frac{c}{s} \int_0^x e^{-\frac{b}{s}(x-\tau)}G(\overline{\phi}(\tau))d\tau \] (11)
with the condition
(C) \( \overline{\phi}(0) = 0, \overline{\phi}(+\infty) = u^* \) and \( \overline{\phi}' \geq 0 \) except \( x = x_1 \), where \( x_1 \) will be given below.

**Lemma 2.5.** For \( s \in (0, s^*) \), the function \( \overline{\phi} \) is well defined.

**Proof.** Define
\[ \overline{\phi}(x) = \begin{cases} \sqrt{2}u^* \sin kx, & 0 \leq x \leq \frac{\pi}{4k}, \\ u^*, & \frac{\pi}{4k} < x < \infty. \end{cases} \]
Here we emphasize that \( x_1 = \frac{\pi}{4k} \). To check that \( \overline{\phi}(x) \) satisfies (11) with condition (C), we first estimate \( \overline{\phi}(x) \) defined as follows:
\[ \overline{\psi}(x) = \frac{1}{s} \int_0^x e^{-\frac{b}{s}(x-\tau)}G(\overline{\phi}(\tau))d\tau. \]

It is easy to see that \( \overline{\psi}(x) \) solves
\[ \begin{cases} s\overline{\psi}' = -b\overline{\psi} + G(\overline{\phi}(x)), & x > 0, \\ \overline{\psi}(0) = 0, \end{cases} \] (12)
and \( \overline{\psi}(x) \leq v^* \). Now, we define
\[ \overline{\phi}(x) = \begin{cases} \sqrt{2}v^* \sin kx, & 0 \leq x \leq \frac{\pi}{4k}, \\ v^*, & \frac{\pi}{4k} < x < \infty. \end{cases} \]
Then, we check that $\tilde{\psi}(x)$ is the super-solution of (12). For $0 \leq x \leq \frac{s}{4k}$, we have

$$s\tilde{\psi}'(x) + b\tilde{\psi} - G(\tilde{\phi}(x))$$

$$\geq \sqrt{2}sv^* k\cos kx + \sqrt{2}bv^* \sin kx - G'(0)\sqrt{2}u^* \sin kx$$

$$\geq \sin kx(\sqrt{2}sv^* k + \sqrt{2}bv^* - G'(0)\sqrt{2}u^*) > 0$$

provided that $k$ is chosen large enough such that

$$\sqrt{2}sv^* k + \sqrt{2}bv^* - G'(0)\sqrt{2}u^* > 0. \quad (13)$$

Moreover, $\tilde{\psi}(0) = 0$ and $\tilde{\psi}(\frac{s}{4k}) = v^* \geq \tilde{\psi}(\frac{s}{4k})$. Applying the comparison principle gives that $\tilde{\psi}(x) \leq \tilde{\psi}(x)$ for $0 \leq x \leq \frac{s}{4k}$.

Hence, for $0 \leq x \leq \frac{s}{4k}$,

$$s\tilde{\phi}' - d\tilde{\phi}'' + a\tilde{\phi} - \frac{c}{s} \int_0^x e^{-\frac{b}{2}(x-\tau)}G(\tilde{\phi}(\tau))d\tau$$

$$\geq \sqrt{2}sv^* k\cos kx + \sqrt{2}dv^* k^2 \sin kx + au^* \sin kx - cv^* \sin kx$$

$$\geq \sin kx(\sqrt{2}dv^* k^2 + au^* - cv^*) > 0$$

provided that $k$ is chosen large enough such that

$$\sqrt{2}dv^* k^2 + au^* - cv^* > 0. \quad (14)$$

For $x > \frac{s}{4k}$,

$$s\tilde{\phi}' - d\tilde{\phi}'' + a\tilde{\phi} - \frac{c}{s} \int_0^x e^{-\frac{b}{2}(x-\tau)}G(\tilde{\phi}(\tau))d\tau$$

$$\geq au^* - \frac{c}{s} \int_0^x e^{-\frac{b}{2}(x-\tau)}d\tau G(u^*)$$

$$= au^* - \frac{cG(u^*)}{b} \left(1 - e^{-\frac{b}{2}x}\right) > 0.$$

The condition (C) can be obtained by the construction of $\tilde{\phi}$. Hence, if we choose $k$ large enough such that both (13) and (14) hold, then $\tilde{\phi}$ will satisfy (11) with the condition (C).

Similarly, we can construct $\tilde{\phi}(x)$ satisfying

$$s\tilde{\phi}' - d\tilde{\phi}'' \leq -a\tilde{\phi} + \frac{c}{s} \int_0^x e^{-\frac{b}{2}(x-\tau)}G(\tilde{\phi}(\tau))d\tau$$

with the condition $\tilde{\phi}(0) = 0$.

**Lemma 2.6.** For $s \in (0, s^*)$, the function $\phi$ is well defined.

**Proof.** For $s \in (0, s^*)$,

$$P_1(\alpha) = (d\alpha^2 - sa - a)(-sa - b) - cG'(0)$$

has two complex roots with positive real parts. By the continuous dependence of the roots to the parameters, we know that the equation

$$(d\alpha^2 - sa - a)(-sa - b) - (c - \varepsilon)(G'(0) - \varepsilon) = 0$$

has still two complex roots denoted by $\alpha_2 = a_1 - b_1 i$ and $\alpha_3 = a_1 + b_1 i$ with $a_1, b_1 > 0$. We fix such $\varepsilon \in (0, \min\{c, G'(0)\})$, and consider the auxiliary system

$$\begin{cases}
    s\phi' - d\phi'' = -a\phi + (c - \varepsilon)\psi,
    
    s\psi' = -b\psi + (G'(0) - \varepsilon)\phi.
\end{cases} \quad (16)$$
By (G1), it follows that there exist $M$ if $x$ $\varphi$ It is easy to see that $(x$ For $Hence, By direct computation, the imaginary part of $(\phi, \psi)$ is $(\hat{\phi}(x), \hat{\psi}(x)) = (A \sin b_1 x + B \cos b_1 x, \sin b_1 x) e^{a_1 x}$ $\left(\sqrt{A^2 + B^2 \sin(b_1 x + \theta)}, \sin b_1 x\right) e^{a_1 x}$.

Clearly, $(\hat{\phi}(x), \hat{\psi}(x))$ is a solution of (16). Fix $\delta > 0$ and define $\frac{\phi(x)}{\psi(x)} = \begin{cases} \delta \hat{\phi}(x), & x \in \left(\frac{2\pi}{b_1}, \frac{3\pi}{b_1}\right), \\ 0, & \text{otherwise}, \end{cases}$ $\begin{cases} \delta \hat{\psi}(x), & x \in \left(\frac{2\pi}{b_1}, \frac{3\pi}{b_1}\right), \\ 0, & \text{otherwise}. \end{cases}$ If $x \in \left(\frac{2\pi}{b_1}, \frac{3\pi}{b_1}\right)$ $= \mathbb{X}$, then we have $\begin{cases} s \phi' - d \phi'' + a \phi - c \psi &= -\varepsilon \psi, \\ s \psi' + b \psi - G'(0) \hat{\phi} &= -\varepsilon \phi, \\ \psi(\frac{2\pi}{b_1}) &= 0. \end{cases}$

By (G1), it follows that there exist $M_1 > 0$ and small $\epsilon > 0$ such that $G(z) \geq G'(0) z - M_1 z^2$ for any $z \in [0, \epsilon]$. If $\delta$ is small enough such that $\delta < \min \left\{ \frac{\varepsilon}{M_1 \max_{x \in \mathbb{X}} \hat{\phi}(x)}, \frac{\epsilon}{\max_{x \in \mathbb{X}} \hat{\phi}(x)} \right\}$, then $s \psi' + b \psi - G(\hat{\phi}) \leq s \psi' + b \psi - G'(0) \hat{\phi} + M_1 \hat{\phi}^2$ $= \phi(-\varepsilon + M_1 \hat{\phi})$ $\leq \phi \left(-\varepsilon + M_1 \delta \max_{x \in \mathbb{X}} \hat{\phi}(x)\right) < 0.$ Hence, $\psi(x) \leq \frac{1}{s} \int_{\frac{2\pi}{b_1}}^{x} e^{-\frac{1}{s}(x-\tau)} G(\bar{\phi}(\tau)) d\tau.$

For $x \in \left(\frac{2\pi}{b_1}, \frac{3\pi}{b_1}\right)$, $s \phi' - d \phi'' + a \phi - \frac{\epsilon}{s} \int_{\frac{2\pi}{b_1}}^{x} e^{-\frac{1}{s}(x-\tau)} G(\bar{\phi}(\tau)) d\tau$ $\leq s \phi' - d \phi'' + a \phi - \frac{\epsilon}{s} \int_{\frac{2\pi}{b_1}}^{x} e^{-\frac{1}{s}(x-\tau)} G(\bar{\phi}(\tau)) d\tau$ $\leq \phi \left(-\varepsilon + \frac{c}{s} \right) \psi = -\varepsilon \psi \leq 0.$ For $x \in \left(\frac{2\pi-\theta}{b_1}, \frac{2\pi}{b_1}\right)$, $s \phi' - d \phi'' + a \phi - \frac{\epsilon}{s} \int_{\frac{2\pi-\theta}{b_1}}^{x} e^{-\frac{1}{s}(x-\tau)} G(\bar{\phi}(\tau)) d\tau$
\[ < s\phi' - d\phi'' + a\phi = \delta(c - \varepsilon)\tilde{\psi}_0 < 0. \]

**Lemma 2.7.** The problem (5) has a strictly increasing solution for any \( s \in (0, s^*) \).

**Proof.** Take \( \delta \) small enough such that \( \phi \leq \overline{\phi} \) for \( x > 0 \). It follows that \( \phi \) and \( \overline{\phi} \) are a pair of lower and upper solutions for (7). By Theorem 2.3, we know that (7) has an increasing solution denoted by \( p^*(\xi) \). Furthermore, \( (p^*)'(\xi) > 0 \) in \( \mathbb{R}^+ \) can be obtained by comparing \( p^*(\xi + \theta) \) with \( p^*(\xi) \) for \( \theta > 0 \). Let

\[ q^*(\xi) = \frac{1}{s} \int_0^\xi e^{-\frac{\delta}{s}(\xi - \tau)}G(p^*(\tau))d\tau. \]

We can check that the function \( q^*(\xi) \) is strictly increasing and solves (8). Hence, \((p^*(\xi), q^*(\xi))\) solves problem (5).

Next, we prove the uniqueness of solutions of (5). At first, we study the asymptotic behavior of \((\phi(\xi), \psi(\xi))\) as \( \xi \to +\infty \). The linearized system of problem (5) at \((u^*, v^*)\) is

\[ \begin{cases} -s\phi' + d\phi'' = a\phi - cv, \\ -s\psi' = b\psi - G'(u^*)\phi. \end{cases} \tag{17} \]

If \((p, q)^T e^{\beta \xi}\) solves (17), then

\[ A(\beta)(p, q)^T = 0 \quad \text{and} \quad P_2(\beta) = 0, \]

where

\[ A(\beta) = \begin{pmatrix} d\beta^2 - s\beta - a\beta & c \\ G'(u^*) & -s\beta - b \end{pmatrix} \tag{18} \]

and

\[ P_2(\beta) = P_{2,2}^\pm(\beta) = (d\beta^2 - s\beta - a)(-s\beta - b) - cG'(u^*). \tag{19} \]

Clearly,

\[ \beta^+_1 = \frac{s \pm \sqrt{s^2 + 4da}}{2d} \quad \text{and} \quad \beta^-_2 = -\frac{b}{s} \]

are respectively the roots of

\[ p_1(\beta) = d\beta^2 - s\beta - a \quad \text{and} \quad p_2(\beta) = -s\beta - b. \]

Let \( \tilde{f}(u) = \frac{G(u)}{u} - \frac{ab}{c} \) and \( f(u) = G(u) - \frac{ab}{c}u = u\tilde{f}(u) \) for \( u > 0 \). By conditions (G2) and (G3), we know that \( \tilde{f}(u) \) is decreasing in \( u \) and \( \lim_{u \to 0} \tilde{f}(u) > 0, \tilde{f}(u^*) = 0 \), \( \lim_{u \to +\infty} \tilde{f}(u) < 0 \). This implies that \( \tilde{f}(u^* - \varepsilon) > 0 \) and \( \tilde{f}(u^* + \varepsilon) < 0 \) for any \( \varepsilon > 0 \). Then, we have \( f(u^* - \varepsilon) > 0, f(u^*) = 0 \) and \( f(u^* + \varepsilon) < 0 \). Therefore, \( f'(u^*) < 0 \), namely, \( ab > cG'(u^*) \). It follows that \( P_2(0) > 0 \). Noting that \( \beta^-_1 < 0 \) and

\[ P_2(-\infty) = +\infty, \quad P(\beta^-_1) < 0, \quad P_2(+\infty) = -\infty, \]

we have \( P_2(\beta) \) has two negative real roots and one positive real root for all \( s > 0 \).

**Lemma 2.8.** Let \((\phi, \psi)\) be a solution of problem (5). Then there exist two positive constants \( \widehat{C}_1 \) and \( \widehat{C}_2 \) such that

\[ \phi(\xi) = u^* - \widehat{C}_1 e^{\beta_1 \xi}(1 + o(1)), \quad \psi(\xi) = v^* - \widehat{C}_2 e^{\beta_2 \xi}(1 + o(1)), \]

as \( \xi \to +\infty \).
Proof. Let \((\phi, \psi)\) be an arbitrary solution of problem (5). Since the real roots of \(P_2(\beta)\) have different signs, the first order ODE system satisfied by \((\phi, \phi', \psi)\) has a critical point \((u^*, v^*, 0)\), which is a saddle point. By standard stable manifold theory, we conclude that
\[(u^*, v^*) - (\phi(\xi), \psi(\xi)) \to (0, 0)\) exponentially \(\xi \to \infty\).
Let \(w = u^* - \phi\) and \(z = v^* - \psi\), then \((w, z)\) satisfies
\[
\begin{aligned}
-w' + dw'' &= aw - cz - au^* + cv^*, & \xi > 0, \\
-sz' &= bz - G'(u^*)w + G(u^* - w) - bv^* + G'(u^*)w, & \xi > 0, \\
(w(0), z(0)) &= (u^*, v^*), & (w(+\infty), z(+\infty)) &= (0, 0).
\end{aligned}
\tag{20}
\]
Note that
\[
G(u^* - w) - bv^* + G'(u^*)w = G(u^* - w) - G(u^* + G'(u^*)w
\]
 exponentially as \(\xi \to \infty\), where \(\theta(\xi) \in (u^* - w, u^*)\). Now we consider the corresponding linear system
\[
\begin{aligned}
-s\tilde{w}' + d\tilde{w}'' &= a\tilde{w} - c\tilde{z}, & \xi > 0, \\
-s\tilde{z}' &= b\tilde{z} - G'(u^*)\tilde{w}, & \xi > 0, \\
(\tilde{w}(0), \tilde{z}(0)) &= (u^*, v^*), & (\tilde{w}(+\infty), \tilde{z}(+\infty)) &= (0, 0).
\end{aligned}
\tag{21}
\]
By solving the ordinary differential equations (21), we can get the representation of \((\tilde{w}, \tilde{z})\) as follows
\[
(\tilde{w}(\xi), \tilde{z}(\xi)) = \sum_{i=1}^{3} C_i \begin{pmatrix} \tilde{p}_i \\ \tilde{q}_i \end{pmatrix} e^{\beta_i \xi},
\]
where, \(\beta_2 < \beta_1 < 0\) and \(\beta_3 > 0\) are three roots of \(P_2(\beta)\). By applying [5, Theorem 8.1 in Chapter 3] to system (20), we conclude that
\[
(w, z) = (\tilde{w}, \tilde{z})(1 + o(1)) \text{ as } \xi \to \infty.
\]
In view of \(w(+\infty) = 0\) and \(z(+\infty) = 0\), it follows that
\[
(w(\xi), z(\xi)) = (1 + o(1)) \sum_{i=1}^{2} C_i \begin{pmatrix} \tilde{p}_i \\ \tilde{q}_i \end{pmatrix} e^{\beta_i \xi} \text{ as } \xi \to \infty.
\]
We claim that \(C_1 \neq 0\). Otherwise, we have \(C_1 = 0\) and \(C_2 \neq 0\). Since the four elements of \(A(\beta_2)\) defined as (18) are positive, we have \(\tilde{p}_2\tilde{q}_2 < 0\), which implies \(w(\xi)z(\xi) < 0\) as \(\beta_2 \to \infty\). It is a contradiction. Therefore, we must have \(C_1 \neq 0\). It is easily checked that \(p_1(\beta_1), p_2(\beta_1) < 0\), and so we have \(\tilde{p}_1\tilde{q}_1 > 0\). For definiteness, we may assume that the constants \(\tilde{p}_1\) and \(\tilde{q}_1\) are positive. Due to \(\beta_2 < \beta_1 < 0\), we have
\[
w(\xi) = C_1\tilde{p}_1e^{\beta_1\xi}(1 + o(1)), \quad z(\xi) = C_1\tilde{q}_1e^{\beta_1\xi}(1 + o(1))
\]
as \(\xi \to +\infty\). Hence, for \(\xi \to +\infty\),
\[
\phi(\xi) = u^* - \tilde{C}_1e^{\beta_1\xi}(1 + o(1)), \quad \psi(\xi) = v^* - \tilde{C}_2e^{\beta_1\xi}(1 + o(1)),
\]
where \(\tilde{C}_1 = C_1\tilde{p}_1 > 0\) and \(\tilde{C}_2 = C_1\tilde{q}_1 > 0\). \(\Box\)

**Lemma 2.9.** For \(s \in (0, s^*)\), the monotone solution of problem (5) is unique.
Proof. Let \((\phi, \psi)\) be an arbitrary solution of (5). We will show that
\[
\phi(\xi) = p^*(\xi), \quad \psi(\xi) = q^*(\xi),
\]
where \(p^*(\xi)\) and \(q^*(\xi)\) are given in Lemma 2.7. We only prove \(\phi(\xi) \geq p^*(\xi)\) and \(\psi(\xi) \geq q^*(\xi)\), the same argument can be used to show \(\phi(\xi) \leq p^*(\xi)\) and \(\psi(\xi) \leq q^*(\xi)\). In the following, we will use the sliding method to prove the claim \(\phi(\xi) \geq p^*(\xi)\) and \(\psi(\xi) \geq q^*(\xi)\) as steps 3 and 4 in the proof of [11, Lemma 2.5].

By Lemma 2.8, we can find some large enough \(\xi_0\) and \(k_1 > 0\) such that
\[
\phi(\xi + k_1) \geq p^*(\xi), \quad \psi(\xi + k_1) \geq q^*(\xi) \quad \text{for} \quad \xi \geq \xi_0.
\]
Let \(k_0 = k_1 + \xi_0\). If \(k \geq k_0\), then
\[
\phi(\xi + k) \geq \phi(\xi + \xi_0 + k_1) \geq p^*(\xi + \xi_0) \geq p^*(\xi), \quad \psi(\xi + k) \geq q^*(\xi)
\]
as \(\xi \geq 0\). Define
\[
\phi_k(\xi) = \phi(\xi + k), \quad \psi_k(\xi) = \psi(\xi + k)
\]
and
\[
k^{-}\inf\{k_0 > 0 \mid \phi_k(\xi) \geq p^*(\xi), \quad \psi_k(\xi) \geq q^*(\xi) \text{ in } [0, \infty), \quad \text{for } k \geq k_0\}.
\]
Clearly, \(k^{-} \geq 0\) and \(\phi_k(\xi) \geq p^*(\xi)\) and \(\psi_k(\xi) \geq q^*(\xi)\) in \([0, \infty)\).

If \(k = 0\), the claim will be true. Otherwise, \(k > 0\). We will derive a contradiction.
Note that \(\phi_k(\xi)\) and \(p^*(\xi)\) satisfy
\[
\begin{align*}
s\phi_k'' - d\phi_k'' &= -a\phi_k + c\psi_k \geq -a\phi_k^* + cq^*, \\
dp'' - dp'' &= -ap^* + cq^*, \\
\phi_k(0) &= p^*(0), \quad \psi_k(\infty) = u^* = p^*(\infty).
\end{align*}
\]
It follows from [11, Lemma 2.1] and strong maximum principle that \(\phi_k(\xi) > p^*(\xi)\) in \([0, \infty)\). Similarly, we have \(\psi_k(\xi) > q^*(\xi)\) in \([0, \infty)\).

For \(\xi \leq M\), we can get \(\phi_k(\xi) \geq p^*(\xi)\) from above directly. Next, we will show that there exist small enough \(\varepsilon > 0\) and sufficiently large \(M\) such that
\[
\phi_{k-\varepsilon}(\xi) \geq p^*(\xi) \quad \text{for} \quad \xi > M. \quad (22)
\]
Define
\[
\chi(\xi) = \phi_k(\xi) - p^*(\xi), \quad \omega(\xi) = \psi_k(\xi) - q^*(\xi),
\]
then
\[
\begin{cases}
s\chi' - d\chi'' = -a\chi + c\omega, & \xi > 0, \\
s\omega' = -b\omega + G'(u^*)\chi + G(p_k) - G(p^*) - G'(u^*)\chi, & \xi > 0, \\
\chi(0) = 0, \quad \omega(0) = 0, \quad \chi(\infty) = \omega(\infty) = 0.
\end{cases}
\]
By Lemma 2.8, we have
\[
\chi(\xi) = C_\chi \varepsilon^{\beta_1}(1 + o(1)), \quad \omega(\xi) = C_\omega \varepsilon^{\beta_1}(1 + o(1)) \quad \text{as} \quad \xi \to +\infty,
\]
where \(C_\chi, C_\omega > 0\). While, as \(\xi \to +\infty\),
\[
\begin{align*}
\phi_k(\xi) &= u^* - C_\phi \varepsilon^{\beta_1}(1 + o(1)), \quad \psi_k(\xi) = v^* - C_\psi \varepsilon^{\beta_1}(1 + o(1)), \\
p^*(\xi) &= u^* - C_p \varepsilon^{\beta_1}(1 + o(1)), \quad q^*(\xi) = v^* - C_q \varepsilon^{\beta_1}(1 + o(1)),
\end{align*}
\]
which implies
\[
C_\chi = C_p - C_\phi \varepsilon^{\beta_1} > 0, \quad C_\omega = C_q - C_\psi \varepsilon^{\beta_1} > 0.
\]
Therefore, there exists \(\varepsilon_0\) small enough such that
\[
C_p - C_\phi \varepsilon^{\beta_1}(k^{-} - \varepsilon) \geq 0, \quad C_q - C_\psi \varepsilon^{\beta_1}(k^{-} - \varepsilon) \geq 0 \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0].
\]
In view of the above inequality, we have that \( \phi_{k-\varepsilon}^*(\xi) \geq p^*(\xi) \) and \( \psi_{k-\varepsilon}^*(\xi) \geq q^*(\xi) \) for large \( \xi > M \). Based on above arguments, \( \phi_{k-\varepsilon}^*(\xi) \geq p^*(\xi) \) and \( \psi_{k-\varepsilon}^*(\xi) \geq q^*(\xi) \) hold for all \( \xi > 0 \). By the monotonicity of \( \phi \), we have \( \phi_k(\xi) \geq p^*(\xi) \) and \( \psi_k(\xi) \geq q^*(\xi) \) for all \( k \geq k - \varepsilon \). Hence, \( k - \varepsilon \geq k \), which contradicts the definition of \( k \).

By above arguments, we know that (5) has a unique solution for \( s \leq s^* \). In following, we want to show \( s^* = s_0 \), where \( s_0 \) is given in Proposition 1.

**Remark 1.** In [27], Zhao and Wang proved the existence of solutions of problem

\[
\begin{cases}
  s\phi' = d\phi'' - a\phi + \xi \int_{-\infty}^{\xi} e^{-\frac{\xi}{s}(\xi - \tau)}G(\phi(\tau))d\tau, & \xi \in \mathbb{R}, \\
  \phi(-\infty) = 0, & \phi'(+\infty) = u^*.
\end{cases}
\]

(23)

In fact, linearizing (23) at \( \phi = 0 \), it is easy to obtain

\[
s\phi' = d\phi'' - a\phi + \frac{cG(0)}{s} \int_{-\infty}^{\xi} e^{-\frac{\xi}{s}(\xi - \tau)}\phi(\tau)d\tau.
\]

By substituting \( \phi(\xi) = e^{\alpha\xi} \), we have the characteristic equation

\[
da^2 - sa - \frac{cG(0)}{s(\alpha + \frac{b}{2})} = 0,
\]

(24)

which implies that

\[
s_0 = \inf\{\tilde{s} > 0 : \text{all roots of } (24) \text{ are real for } s \geq \tilde{s}\}.
\]

Zhao and Wang [27] proved that \( s_0 \) is the minimal speed of the traveling waves of (3). Actually, we can easily to see that \( s_0 = s^* \).

**Lemma 2.10.** For \( s \geq s_0 \), problem (5) has no solution.

Proof. Suppose on the contrary that (5) has a solution \( (\phi(\xi), \psi(\xi)) \) for some given \( s \geq s_0 \). It follows that \( (\phi, \psi) \leq (u^*, v^*) \). Next, we show that \( \phi', \phi'', \psi' \) are uniformly bounded on \( \mathbb{R}^+ \). By direct computation,

\[
|\phi'(\xi)| = \frac{c}{sd(\gamma_2 - \gamma_1)} \left| \gamma_1 \int_{\xi}^{\infty} e^{\gamma_1(\xi - \theta)} \int_{0}^{\theta} e^{-\frac{\xi}{s}(\theta - \tau)}G(\phi(\tau))d\tau d\theta \\
+ \gamma_2 \int_{\xi}^{\infty} e^{\gamma_2(\xi - \theta)} \int_{0}^{\theta} e^{-\frac{\xi}{s}(\theta - \tau)}G(\phi(\tau))d\tau d\theta \\
- \gamma_1 \int_{0}^{\infty} e^{\gamma_1\xi - \gamma_2\theta} \int_{0}^{\theta} e^{-\frac{\xi}{s}(\theta - \tau)}G(\phi(\tau))d\tau d\theta \right|
\leq \frac{cG(u^*)}{d(\gamma_2 - \gamma_1)b} \left[ \gamma_1 \int_{0}^{\xi} e^{\gamma_1(\xi - \theta)}d\theta + \gamma_2 \int_{\xi}^{\infty} e^{\gamma_2(\xi - \theta)}d\theta - \gamma_1 \int_{0}^{\infty} e^{\gamma_1\xi - \gamma_2\theta}d\theta \right]
\leq \frac{cG(u^*)}{d(\gamma_2 - \gamma_1)b} \left( 2 - e^{\gamma_1\xi - \gamma_2\xi} \right) \leq \frac{3cG(u^*)}{2\sqrt{adb}} := C_1
\]

for \( \forall \xi > 0 \). By the first and second equations of (5), we have

\[
|\phi''| = \frac{1}{d}|s\phi' + a\phi - c\psi| \leq \frac{1}{d}(sC_1 + au^* + cv^*) := C_2,
\]

\[
|\psi'| = \frac{1}{s} |b\psi + G(\phi)| \leq \frac{1}{s_0}(bv^* + G(u^*)) := C_3.
\]
Hence,
\[ \int_0^\infty \phi^{(i)}(\xi)e^{-\alpha_2 \xi}d\xi < \infty \quad (i = 0, 1, 2) \]
and
\[ \int_0^\infty \psi^{(j)}(\xi)e^{-\alpha_2 \xi}d\xi < \infty \quad (j = 0, 1), \]
where we use \( \phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \psi^{(0)}, \psi^{(1)} \) to denote \( \phi, \phi', \phi'', \psi, \psi' \) respectively. By
multiplying the equations in (5) by \( e^{-\alpha_2 \xi} \) and integrating from 0 to \( \infty \), we obtain
\[ p_1(\alpha_2) \int_0^\infty \phi(\xi)e^{-\alpha_2 \xi}d\xi + c \int_0^\infty \psi(\xi)e^{-\alpha_2 \xi}d\xi = d\phi'(0) > 0, \]
\[ p_2(\alpha_2) \int_0^\infty \psi(\xi)e^{-\alpha_2 \xi}d\xi = - \int_0^\infty G(\phi)(\xi)e^{-\alpha_2 \xi}d\xi > -G'(0) \int_0^\infty \phi(\xi)e^{-\alpha_2 \xi}d\xi. \]
By \( P_1(\alpha_2) = 0 \), we get
\[
\frac{p_1(\alpha_2)}{G'(0)} \left( \frac{cG'(0)}{p_1(\alpha_2)} \int_0^\infty \psi(\xi)e^{-\alpha_2 \xi}d\xi + G'(0) \int_0^\infty \phi(\xi)e^{-\alpha_2 \xi}d\xi \right) = \frac{p_2(\alpha_2)}{G'(0)} \int_0^\infty \psi(\xi)e^{-\alpha_2 \xi}d\xi + G'(0) \int_0^\infty \phi(\xi)e^{-\alpha_2 \xi}d\xi,
\]
which implies that \( p_1(\alpha_2) > 0 \). But we know \( p_1(\alpha_2) < 0 \) by \( \alpha_2 \in (\alpha^-_1, \alpha^+_1) \). This is a
contradiction. \( \square \)

**Lemma 2.11.** Let \( (\phi_s, \psi_s) \) denote the unique strictly increasing solution of (5) with \( s \in (0, s_0) \). Then, for \( 0 < s_1 < s_2 < s_0 \), we have
\[
\phi'_{s_1}(0) > \phi'_{s_2}(0), \quad \phi_{s_1}(\xi) > \phi_{s_2}(\xi), \quad \psi_{s_1}(\xi) > \psi_{s_2}(\xi).
\]
**Proof.** Since \( 0 < s_1 < s_2 \) and \( \phi_{s_2}' > 0, \psi_{s_2}' > 0 \), we have
\[
\begin{cases}
s_1 \phi_{s_2}' - d\phi_{s_2}'' < -aw_{s_2} + c\psi_{s_2}, & \xi > 0, \\
s_1 \psi_{s_2}' < -b\psi_{s_2} + G(\phi_{s_2}), & \xi > 0
\end{cases}
\]
with \( (\phi_{s_1}(0), \psi_{s_1}(0)) = (0, 0) \) and \( (\phi_{s_1}(\infty), \psi_{s_1}(\infty)) = (u^*, v^*) \) for \( i = 1, 2 \). By the comparison principle, we have \( \phi_{s_1}(\xi) \geq \phi_{s_2}(\xi) \) and \( \psi_{s_1}(\xi) \geq \psi_{s_2}(\xi) \). Moreover, applying strong maximum principle implies that \( \phi_{s_1}(\xi) > \phi_{s_2}(\xi) \) and \( \psi_{s_1}(\xi) > \psi_{s_2}(\xi) \).

Let \( w(\xi) = \phi_{s_1}(\xi) - \phi_{s_2}(\xi) \) and \( z(\xi) = \psi_{s_1}(\xi) - \psi_{s_2}(\xi) \), we have
\[
\begin{cases}
s_1 w' - dw'' > -aw + cz, & \xi > 0, \\
s_1 z' > -bq + G(\phi_{s_1}) - G(\phi_{s_2}), & \xi > 0, \\
w(0) = z(0) = 0.
\end{cases}
\] (25)

From the second of (25), we have
\[ s_1 z' > -bq \text{ with } z(0) = 0. \]
Then \( z(\xi) > 0 \) for \( \xi > 0 \). Thus,
\[ s_1 w' - dw'' > -aw \text{ for } \xi > 0, \quad w(0) = 0. \]
By Hopf boundary lemma, we have \( \omega'(0) > 0 \), namely, \( \phi'_{s_1}(0) > \phi'_{s_2}(0) \). \( \square \)
Lemma 2.12. The operator

\[ s \mapsto (\phi_s, \psi_s) : (0, s_0) \to C^2_{loc}([0, +\infty)) \times C^1_{loc}([0, +\infty)) \]

is continuous. Moreover,

\[ \lim_{s \to s_0} (\phi_s, \psi_s) = (0, 0) \text{ in } C^2_{loc}([0, +\infty)) \times C^1_{loc}([0, +\infty)). \]

Proof. For \( \hat{s} \in [0, s_0) \), let \( s_1 \in (\hat{s}, s_0) \) and suppose \( \{s_n\} \) is a sequence in \( [\hat{s}, s_1) \) such that \( s_n \to \hat{s} \) as \( n \to \infty \). Let \( (\phi_{s_n}, \psi_{s_n}) \) be the solution of (5) with \( s = s_n \). We will prove that \( (\phi_{s_n}, \psi_{s_n}) \) has a subsequence that converges to \( (\phi_{\hat{s}}, \psi_{\hat{s}}) \). It follows that the map \( s \mapsto (\phi_s, \psi_s) \) is continuous.

Noting that \( (\phi_{s_n}, \psi_{s_n}) \leq (u^*, v^*) \). Similarly to Lemma 2.10, we can have that \( \phi''_{s_n}, \psi''_{s_n}, \psi'_{s_n} \) are uniformly bounded on \( \mathbb{R}^+ \). Differentiating both sides of (5) with respect to \( \xi \), we can get

\[
\begin{cases}
    s_n \phi''_{s_n} - d\phi'''_{s_n} = -a\phi'_{s_n} + c\psi'_{s_n}, & \xi > 0, \\
    s_n \psi''_{s_n} = -b\psi'_{s_n} + G'(\phi_{s_n})\phi'_{s_n}, & \xi > 0.
\end{cases}
\]

Applying the boundedness of \( \phi', \phi'', \psi', \psi'' \) gives that \( \phi'''(\xi), \psi'''(\xi) \) are bounded for \( \forall \xi > 0 \). Consequently, \( \phi_{s_n}, \phi''_{s_n}, \psi_{s_n}, \psi''_{s_n} \) are equi-continuous and uniformly bounded on \( \mathbb{R}^+ \). Using Arzelà-Ascoli’s theorem, a nested subsequence argument, it follows that there exists a subsequence \( \{s_{n_j}\} \) of \( \{s_n\} \) such that \( (s_{n_j}, \phi_{n_j}, \psi_{n_j}) \to (\hat{s}, \hat{\phi}, \hat{\psi}) \) uniformly as \( j \to \infty \) on any compact subset of \( \mathbb{R}^+ \). Moreover, \( (\hat{\phi}, \hat{\psi}) \) solves the equations of (5) with \( s = \hat{s} \). Hence, \( \hat{\phi} \) solves the equation of (7). By the monotonicity of \( \phi_{s_n} \) in \( s \), we have \( \phi_{a_1} \leq \hat{\phi} \leq \phi_{s_1} \), where \( \phi \) is given in Lemma 2.5. Then we have \( \hat{\phi}(0) = 0 \) and \( \hat{\phi}(\infty) = u^* \) by the value of \( \phi_{s_1} (\xi) \) and \( \overline{\phi}(\xi) \) at \( \xi = 0, \infty \). Comparing \( \hat{\phi}(\xi + \delta) \) with \( \hat{\phi}(\xi) \) gives that \( \hat{\phi}(\xi + \delta) > \hat{\phi}(\xi) \), which implies that \( \hat{\phi}'(\xi) > 0 \). It follows from the representation of \( \hat{\psi}(\xi) \) that \( \hat{\psi}(0) = 0, \hat{\psi}(\infty) = v^* \) and \( \hat{\psi}' > 0 \). Hence, \( (\hat{\phi}, \hat{\psi}) \) is the solution of (5) with \( s = \hat{s} \). By uniqueness, we have \( (\phi_{\hat{s}}, \psi_{\hat{s}}) = (\phi_{\hat{s}}(\xi), \psi_{\hat{s}}(\xi)) \).

Next, we consider the case \( \hat{s} = s_0 \). Repeating the above arguments, we have

\[ (s_{n_j}, \phi_{s_{n_j}}, \psi_{s_{n_j}}) \to (s_0, \phi^0, \psi^0) \text{ as } j \to \infty, \]

where \( (\phi^0, \psi^0) \) satisfies

\[
\begin{cases}
    s_0(\phi^0)' - d(\phi^0)'' = -a\phi^0 + c\psi^0, & \xi > 0, \\
    s_0(\psi^0)' = -b\psi^0 + G(\phi^0), & \xi > 0, \\
    0 \leq \phi^0(\xi) \leq \phi_0(\xi), 0 \leq \psi^0(\xi) \leq \psi_0(\xi), \\
    (\phi^0)' \geq 0, (\psi^0)' \geq 0.
\end{cases}
\]

In the following, we will show that \( (\phi^0, \psi^0) \equiv (0, 0) \). If \( \phi^0 \neq 0 \), the strong maximum principle gives that \( \phi^0(\xi) > 0 \) for \( \xi \in \mathbb{R}^+ \), and \( \psi^0(\xi) > 0 \) can be derived by (6) with \( \phi = \phi^0 \). Using the boundedness and monotonicity of \( (\phi^0, \psi^0) \) on \( \mathbb{R}^+ \), combining with the fluctuation lemma [26, Lemma 2.2], we have that \( (\phi^0, \psi^0)(+\infty) = (u^*, v^*) \). This contradicts Lemma 2.10. Hence, \( \phi^0 \equiv 0 \) on \( \mathbb{R}^+ \). It follows that \( \psi^0 \equiv 0 \) on \( \mathbb{R}^+ \).

Lemma 2.13. For any \( \mu > 0 \), there exists a unique \( s_\mu \in (0, s_0) \) such that \( \mu \phi'_{s_\mu}(0) = s_\mu \).

Proof. Let \( \vartheta(s) = \mu \phi'_s(0) - s \). It is obvious that \( \vartheta(0) > 0 \). It follows from Lemma 2.12 that \( \vartheta(s_0) < 0 \). Since \( \vartheta'(s) < 0 \) by Lemma 2.11, we have that there exists a unique \( s_\mu \in (0, s_0) \) such that \( \vartheta(s) = 0 \).
3. Spreading speed. In this section, we will prove Theorem 1.2.

Proof of Theorem 1.2. The proof can be divided into the following two steps:

**Step 1.** We show
\[
\lim_{t \to \infty} \frac{g(t)}{t} \geq -s_\mu, \quad \lim_{t \to \infty} \frac{h(t)}{t} \leq s_\mu.
\]
By the comparison principle, we easily see that
\[
u(t, x) \leq U(t), \quad v(t, x) \leq V(t), \quad \text{for } t > 0 \text{ and } g(t) \leq x \leq h(t),
\]
where \((U, V)\) solves
\[
\begin{align*}
U_t &= -a U + c V, & t &> 0, \\
V_t &= -b V + G(U), & t &> 0, \\
U(0) &= \|u_0\|_\infty, & V(0) &= \|v_0\|_\infty.
\end{align*}
\]
Since \(\frac{cG'(0)}{ab} > 1\), we have \(\lim_{t \to \infty} (U, V) = (u^*, v^*)\).

For any given \(\delta_1, \delta_2 > 0\), we choose the positive constants \(\varepsilon_1\) and \(\varepsilon_2\) such that \((u^* + 2\varepsilon_1, v^* + 2\varepsilon_2)\) is the solution of
\[
\begin{align*}
-(a - \delta_1) \phi + c \psi &= 0, \\
-(b - \delta_2) \psi + G(\phi) &= 0.
\end{align*}
\]
It is easy to see that \((\varepsilon_1, \varepsilon_2) \to 0\) as \((\delta_1, \delta_2) \to 0\). Now, we consider the auxiliary problem
\[
\begin{alignat*}{2}
s\phi' - d \phi'' &= -(a - \delta_1) \phi + c \psi, & \quad && \xi > 0, \\
s\psi' &= -(b - \delta_2) \psi + G(\phi), & \quad && \xi > 0, \\
\phi(0) &= 0, \quad \phi'(\xi) > 0 & \quad (\xi \geq 0), \quad && \phi(+\infty) = u^* + 2\varepsilon_1, \\
\psi(0) &= 0, \quad \psi'(\xi) > 0 & \quad (\xi \geq 0), \quad && \psi(+\infty) = v^* + 2\varepsilon_2.
\end{alignat*}
\]  
(26)

Denote \(\delta = (\delta_1, \delta_2)\) and \(\varepsilon = (\varepsilon_1, \varepsilon_2)\). It follows from Theorem 1.1 that there exists a unique \(s_{\mu}^{\delta, \varepsilon}\) such that (26) has a unique solution \((\phi_{\delta, \varepsilon}, \psi_{\delta, \varepsilon})\) and
\[
\begin{align*}
\mu(\phi_{\delta, \varepsilon}'(0) &= s_{\mu}^{\delta, \varepsilon}, \\
\lim_{\delta \to 0, \varepsilon \to 0} s_{\mu}^{\delta, \varepsilon} = s_\mu.
\end{align*}
\]

For such small \(\varepsilon_1, \varepsilon_2 > 0\), we can find some \(T_1(\varepsilon_1, \varepsilon_2) \gg 1\) such that
\[
(U(t), V(t)) \leq (u^* + \varepsilon_1, v^* + \varepsilon_2) \quad \text{for } t \geq T_1.
\]
Hence,
\[
(u(t, x), v(t, x)) \leq (u^* + \varepsilon_1, v^* + \varepsilon_2) \quad \text{for } t \geq T_1 \text{ and } g(t) \leq x \leq h(t). \tag{27}
\]
In view of \((\phi_{\delta, \varepsilon}(+\infty), \psi_{\delta, \varepsilon}(+\infty)) = (u^* + 2\varepsilon_1, v^* + 2\varepsilon_2)\), there exists \(R\) large enough such that \(R > \max \{h(T_1), -g(T_1)\}\) and
\[
(\phi_{\delta, \varepsilon}(R - h(T_1)), \psi_{\delta, \varepsilon}(R - h(T_1))) > (u^* + \varepsilon_1, v^* + \varepsilon_2). \tag{28}
\]
Define
\[
\begin{align*}
T(t) &= s_{\mu}^{\delta, \varepsilon} t + R, \quad \overline{g}(t) = -s_{\mu}^{\delta, \varepsilon} t - R, \\
\tau(t, x) &= \begin{cases} \\
\phi_{\delta, \varepsilon}(\overline{h}(t) - x), & 0 < x < \overline{h}(t), \\
\phi_{\delta, \varepsilon}(x - \overline{g}(t)), & \overline{g}(t) < x < 0,
\end{cases} \\
\tau(t, x) &= \begin{cases} \\
\psi_{\delta, \varepsilon}(\overline{h}(t) - x), & 0 < x < \overline{h}(t), \\
\psi_{\delta, \varepsilon}(x - \overline{g}(t)), & \overline{g}(t) < x < 0.
\end{cases}
\end{align*}
\]
By (27) and (28), we have
\[ \pi(0, x) = \phi^{\delta, \varepsilon}(R - x) > \phi^{\delta, \varepsilon}(R - h(T_1)) > u^* + \varepsilon_1 > u(T_1, x) \text{ for } 0 < x < h(T_1), \]
\[ \pi(0, x) = \phi^{\delta, \varepsilon}(x + R) > \phi^{\delta, \varepsilon}(g(T_1) + R) > u^* + \varepsilon_1 > u(T_1, x) \text{ for } g(T_1) < x < 0. \]
Similarly, we have \( \pi(0, x) > v(T_1, x) \) for \( g(T_1) < x < h(T_1) \). Clearly,
\[
\begin{align*}
\pi_t - a \pi_{xx} &\geq -a \pi + c \pi, \\
\pi_t &\geq -b \pi + G(\pi), \\
\bar{h}'(t) &= s_{\mu, \varepsilon} = \mu(\phi^{\delta, \varepsilon})'(0) = -\mu \pi_x(t, \bar{h}(t)), \\
\bar{g}'(t) &= -s_{\mu, \varepsilon} = -\mu(\phi^{\delta, \varepsilon})'(0) = -\mu \pi_x(t, \bar{g}(t)), \\
\bar{h}(0) &= R > h(T_1), \quad \bar{g}(0) = -R > g(T_1), \\
\pi(t, x) &= \pi(t, x) = 0 \text{ for } x \in \{\bar{g}(t), \bar{h}(t)\}. 
\end{align*}
\]
Hence, we have
\[ u(t + T_1, x) \leq \pi(t, x), \quad v(t + T_1, x) \leq \pi(t, x), \quad g(t + T_1) \geq \bar{g}(t), \quad h(t + T_1) \leq \bar{h}(t) \]
for \( t \geq 0 \) and \( g(t) \leq x \leq h(t) \). Then we have
\[
\lim_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\bar{h}(t - T_1)}{t} = s_{\mu, \varepsilon}.
\]
It follows that \( \lim_{t \to \infty} \frac{h(t)}{t} \leq s_{\mu} \) by letting \( \delta \to 0 \) and \( \varepsilon \to 0 \). Similarly, \( \lim_{t \to \infty} \frac{g(t)}{t} \geq -s_{\mu} \).

**Step 2.** We show
\[
\lim_{t \to \infty} \frac{g(t)}{t} \leq -s_{\mu}, \quad \lim_{t \to \infty} \frac{h(t)}{t} \geq s_{\mu}.
\]
For any given \( \delta_3, \delta_4 > 0 \), we choose the positive constants \( \varepsilon_3 \) and \( \varepsilon_4 \) such that \((u^* - 2\varepsilon_3, v^* - 2\varepsilon_4)\) is the solution of
\[
\begin{align*}
-(a + \delta_3) \phi + c \psi &= 0, \\
-(b + \delta_4) \psi + G(\phi) &= 0.
\end{align*}
\]
It is easy to see that \((\varepsilon_3, \varepsilon_4) \to 0\) as \((\delta_3, \delta_4) \to 0\). Now, we consider the auxiliary problem
\[
\begin{align*}
\phi'(t) - d \phi'' &= -(a + \delta_3) \phi + c \psi, \\
\psi'(t) &= -(b + \delta_4) \psi + G(\phi), \\
\phi(0) &= 0, \quad \phi'(\xi) > 0 \quad (\xi \geq 0), \quad \phi(+\infty) = u^* - 2\varepsilon_3, \\
\psi(0) &= 0, \quad \psi'(\xi) > 0 \quad (\xi \geq 0), \quad \psi(+\infty) = v^* - 2\varepsilon_4.
\end{align*}
\]
Denote \( \bar{\delta} = (\delta_3, \delta_4) \) and \( \bar{\varepsilon} = (\varepsilon_3, \varepsilon_4) \). It follows from Theorem 1.1 that there exists a unique \( s_{\mu, \bar{\varepsilon}} \) such that (29) has a unique solution \((\phi^{\delta, \varepsilon}, \psi^{\delta, \varepsilon})\)
\[
\mu(\phi^{\delta, \varepsilon})'(0) = s_{\mu, \bar{\varepsilon}}, \quad \lim_{\delta \to 0, \varepsilon \to 0} s_{\mu, \bar{\varepsilon}} = s_{\mu}.
\]
In view of
\[
(\phi^{\delta, \varepsilon}(+\infty), \psi^{\delta, \varepsilon}(+\infty)) = (u^* - 2\varepsilon_3, v^* - 2\varepsilon_4) \text{ and } (\phi^{\delta, \varepsilon})'(\xi) > 0, \quad (\psi^{\delta, \varepsilon})'(\xi) > 0,
\]
we have
\[
(\phi^{\delta, \varepsilon}(x), \psi^{\delta, \varepsilon}(x)) < (u^* - 2\varepsilon_3, v^* - 2\varepsilon_4) \text{ for all } x > 0.
\]
If spreading happens, then we have
\[
\lim_{t \to \infty} (u(t, x) , v(t, x)) = (u^*, v^*)
\]
uniformly in any compact subset of \(\mathbb{R}\).

Then, for such small \(\varepsilon_3, \varepsilon_4 > 0\), we can find some \(T_2(\varepsilon_3, \varepsilon_4) \gg 1\) such that
\[
(u(t, x), v(t, x)) \geq (u^* - \varepsilon_3, v^* - \varepsilon_4) \text{ for } t \geq T_2 \text{ and } 0 \leq x \leq L,
\]
(31)

where \(L < h(T_2)\). Define
\[
\bar{h}(t) = s_{\mu}^0 \varepsilon t + L, \\
\bar{u}(t, x) = \phi^0 \xi (\bar{h}(t) - x), \quad 0 < x < \bar{h}(t), \\
\bar{v}(t, x) = \psi^0 \xi (\bar{h}(t) - x), \quad 0 < x < \bar{h}(t).
\]

By (30) and (31), we have
\[
u(0, x) = \phi^0 \xi (L - x) < u^* - \varepsilon_3 < u(T_2, x) \text{ for } 0 < x < L,
\]
\[
u(t, 0) = \phi^0 \xi (\bar{h}(t)) < u^* - \varepsilon_3 < u(t, 0), \quad \nu(t, \bar{h}(t)) = 0 \text{ for } t > T_2.
\]
Similarly, we have
\[
u(0, x) < v(T_2, x) \text{ for } 0 < x < L,
\]
\[
u(t, 0) < v(t, 0), \quad \nu(t, \bar{h}(t)) = 0 \text{ for } t > T_2.
\]

Clearly,
\[
u_t - du_{xx} \leq -au + cv, \\
\bar{v}_t \leq -bv + G(u), \\
\bar{h}'(t) = s_{\mu}^0 \varepsilon = \mu(\phi^0 \xi)'(0) = -\mu u_x(t, \bar{h}(t)), \\
\bar{h}(0) = L < h(T_2).
\]

Hence, we have
\[
u(t + T_2, x) \geq \nu(t, x), \quad v(t + T_2, x) \geq v(t, x), \quad h(t + T_2) \geq \bar{h}(t)
\]
for \(t \geq 0\) and \(g(t) \leq x \leq h(t)\). Then we have
\[
\lim_{t \to \infty} \frac{h(t)}{t} \geq \lim_{t \to \infty} \frac{h(t - T_2)}{t} = s_{\mu}^0 \varepsilon.
\]
It follows that \(\lim_{t \to \infty} \frac{h(t)}{t} \geq s_{\mu}\) by letting \(\delta \to 0\) and \(\varepsilon \to 0\). Similarly, \(\lim_{t \to \infty} \frac{g(t)}{t} \leq -s_{\mu}\)

**Acknowledgments.** The authors are very grateful to the anonymous referee for his/her valuable comments helping us to improve the original manuscript. Research of M. Zhao was partially supported by the FRFCU (lzujbky-2017-it55). Research of W.T. Li was partially supported by NSF of China (11731005,11671180).

**REFERENCES**

[1] I. Ahn, S. Beak and Z. Lin, The spreading fronts of an infective environment in a man-environment-man epidemic model, *Appl. Math. Model.*, 40 (2016), 7082–7101.
[2] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: analysis of a free boundary model, *Netw. Heterog. Media*, 7 (2012), 583–603.
[3] V. Capasso and L. Maddalena, Convergence to equilibrium states for a reaction-diffusion system modeling the spatial spread of a class of bacterial and viral diseases, *J. Math. Biol.*, 13 (1981/82), 173–184.
[4] V. Capasso and S. L. Paveri-Fontana, A mathematical model for the 1973 cholera epidemic in the European Mediterranean region, *Rev. d’Epidemiol. Sante Publique*, 27 (1979), 32–121.
A DEGENERATE COOPERATIVE EPIDEMIC MODEL WITH FREE BOUNDARIES

[5] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
[6] Y. Du, Z. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, J. Funct. Anal., 265 (2013), 2089–2142.
[7] Y. Du and Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42 (2010), 377–405.
[8] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 3105–3132.
[9] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc., 17 (2015), 2673–2724.
[10] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal., 46 (2014), 375–396.
[11] Y. Du, M. Wang and M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, J. Math. Pures Appl., 107 (2015), 253–287.
[12] Y. Du and C. H. Wu, Spreading with two speeds and mass segregation in a diffusive competition system with free boundaries, Calc. Var. Partial Differential Equations, 57 (2018), Art. 52, 36 pp.
[13] J. Ge, K. I. Kim, Z. Lin and H. Zhu, A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, J. Differential Equations, 259 (2015), 5486–5509.
[14] H. Gu, B. Lou and M. Zhou, Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries, J. Funct. Anal., 269 (2015), 1714–1768.
[15] J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, J. Dynam. Differential Equations, 24 (2012), 873–895.
[16] K. I. Kim, Z. Lin and Q. Zhang, An SIR epidemic model with free boundary, Nonlinear Anal. Real World Appl., 14 (2013), 1992–2001.
[17] Z. Lin, A free boundary problem for a predator-prey model, Nonlinearity, 20 (2007), 1883–1892.
[18] Z. Lin and H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, J. Math. Biol., 75 (2017), 1381–1409.
[19] J. Wang and L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment, J. Math. Anal. Appl., 423 (2015), 377–398.
[20] M. Wang, On some free boundary problems of the prey-predator model, J. Differential Equations, 256 (2014), 3365–3394.
[21] M. Wang and J. Zhao, Free boundary problems for a Lotka-Volterra competition system, J. Dynam. Differential Equations, 26 (2014), 655–672.
[22] M. Wang and J. Zhao, A free boundary problem for the predator-prey model with double free boundaries, J. Dynam. Differential Equations, 29 (2017), 957–979.
[23] M. Wang and Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, J. Differential Equations, 264 (2018), 3527–3558.
[24] Z. Wang, H. Nie and Y. Du, Asymptotic spreading speed for the weak competition model with a free boundary, Discrete Contin. Dyn. Syst., 39 (2019), 5233–5262.
[25] Z. Wang, H. Nie and Y. Du, Spreading speed for a West Nile virus model with free boundary, J. Math. Biol., 79 (2019), 433–466.
[26] J. Wu and X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, J. Dynam. Differential Equations, 13 (2001), 651–687.
[27] X. Q. Zhao and W. Wang, Fisher waves in an epidemic model, Discrete Contin. Dyn. Syst. Ser. B, 4 (2004), 1117–1128.

Received January 2019; revised February 2019.

E-mail address: zhaom13@lzu.edu.cn (M. Zhao)
E-mail address: wtli@lzu.edu.cn (W.T. Li) (Corresponding author)
E-mail address: nwj10331598320163.com (W. Ni)