APPLICATIONS OF GRÜNBAUM-TYPE INEQUALITIES

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Abstract. Let \(1 \leq i \leq k < n\) be integers. We prove the following exact inequalities for any convex body \(K \subset \mathbb{R}^n\) with centroid at the origin, and any \(k\)-dimensional subspace \(E \subset \mathbb{R}^n\):

\[
V_i(K \cap E) \geq \left( \frac{i + 1}{n + 1} \right)^i \max_{x \in K} V_i((K - x) \cap E),
\]

\[
\tilde{V}_i(K \cap E) \geq \left( \frac{i + 1}{n + 1} \right)^i \max_{x \in K} \tilde{V}_i((K - x) \cap E);
\]

\(V_i\) is the \(i\)th intrinsic volume, and \(\tilde{V}_i\) is the \(i\)th dual volume taken within \(E\). Our results are an extension of an inequality of M. Fradelizi, which corresponds to the case \(i = k\). Using the same techniques, we also establish extensions of “Grünbaum’s inequality for sections” and “Grünbaum’s inequality for projections” to dual volumes.

1. Introduction

A convex body \(K \subset \mathbb{R}^n\) is a convex and compact subset of \(\mathbb{R}^n\) with non-empty interior. The centroid of \(K\) is the affine covariant point

\[
g(K) := \frac{1}{\vol_n(K)} \int_K x \, dx \in \text{int}(K).
\]

Makai and Martini conjectured the following (Conjecture 3.3 in [5]): for integers \(1 \leq k < n\), any convex body \(K \subset \mathbb{R}^n\) with centroid at the origin, and any \(k\)-dimensional subspace \(E \in G(n,k)\),

\[
\vol_k(K \cap E) \geq \left( \frac{k + 1}{n + 1} \right)^k \max_{x \in \mathbb{R}^n} \vol_k((K - x) \cap E). \tag{1}
\]

Here, \(\vol_k\) denotes \(k\)-dimensional Lebesgue volume. They were able to prove [1] for \(k = 1, n - 1\); see Theorem 3.1 and Proposition 3.4 in [5]. Shortly thereafter, Fradelizi [1] proved the conjecture for all \(k\), including sharpness and a complete characterization of the equality conditions.

In this paper, we generalize [1] to intrinsic and dual volumes. We refer the reader to [4] for a nice summary of these concepts, but let us recall the basic definitions. For a convex and compact set \(L \subset \mathbb{R}^n\) and the \(n\)-dimensional Euclidean ball \(B^2_\mathbb{n}\) with unit radius, Steiner’s formula expands the volume of the Minkowski sum \(L + tB^2_\mathbb{n}\) into a polynomial of \(t\):

\[
\vol_n(L + tB^2_\mathbb{n}) = \sum_{i=0}^{n} \kappa_{n-i} V_i(L) t^{n-i} \quad \forall \ t \geq 0.
\]

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The coefficient $V_i(L)$ is the $i$th intrinsic volume of $L$, and $\kappa_{n-i}$ denotes the $(n-i)$-dimensional volume of $B_n^{n-i}$. We prove the following:

**Theorem 1.** Consider integers $1 \leq i \leq k < n$. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, and let $E \in G(n,k)$. Then

$$V_i(K \cap E) \geq \left(\frac{i + 1}{n + 1}\right)^i \max_{x \in K} V_i((K - x) \cap E).$$  \hspace{1cm} (2)

The constant in this inequality is the best possible.

**Remark.** When $i = k$, inequality (2) yields (1). When $i = k - 1$, our inequality gives a lower bound for the $k-1$ dimensional surface area of a $k$-dimensional section of $K$ through its centroid.

A star body $L \subset \mathbb{R}^n$ is a non-empty compact subset of $\mathbb{R}^n$ which is star-shaped with respect to the origin, and whose radial function

$$\rho_L(\xi) := \max\{a \geq 0 \mid a\xi \in L\}, \quad \xi \in S^{n-1},$$

is positive and continuous. The radial sum of the star body $L \subset \mathbb{R}^n$ with the ball $tB_2^n$ of radius $t > 0$ is the star body $L + tB_2^n$ whose radial function is equal to $\rho_L(\xi) + t$ for all $\xi \in S^{n-1}$. The dual Steiner’s formula expands the volume of $L + tB_2^n$ into a polynomial of $t$:

$$\text{vol}_n(L + tB_2^n) = \sum_{i=0}^n \binom{n}{i} \tilde{V}_i(L)t^{n-i} \quad \forall \ t \geq 0.$$  \hspace{1cm} (3)

The coefficient $\tilde{V}_i(L)$ is the $i$th dual volume of $L$. We prove the following:

**Theorem 2.** Consider integers $1 \leq i \leq k < n$. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, and let $E \in G(n,k)$. Then

$$\tilde{V}_i(K \cap E) \geq \left(\frac{i + 1}{n + 1}\right)^i \max_{x \in K} \tilde{V}_i((K - x) \cap E),$$  \hspace{1cm} (3)

where the dual volumes are taken within the $k$-dimensional subspace $E$. The constant in this inequality is the best possible.

**Remark.** When $i = k$, inequality (3) yields (1). Essentially, we prove Theorem 1 and Theorem 2 as consequences of “Grünam’s inequality for sections”: for integers $1 \leq k \leq n$, a convex body $K \subset \mathbb{R}^n$ with centroid at the origin, and $E \in G(n,k),$

$$\text{vol}_k(K \cap E \cap \xi^+) \geq \left(\frac{k}{n + 1}\right)^k \text{vol}_k(K \cap E) \quad \text{for all } \xi \in S^{n-1} \cap E.$$  \hspace{1cm} (4)

Here, $\xi^+ := \{x \in \mathbb{R}^n \mid \langle x, \xi \rangle \geq 0\}$. Inequality (4) was proved in [7]. The reader is also referred to the papers [2] and [6] for previous results on this topic. Inequality (4) implies “Grünam’s inequality for projections”,

$$\text{vol}_k((K|E) \cap \xi^+) \geq \left(\frac{k}{n + 1}\right)^k \text{vol}_k(K|E) \quad \text{for all } \xi \in S^{n-1} \cap E,$$  \hspace{1cm} (5)
which was proved earlier in \[8\] using a different method. The case \(k = n\) in both (4) and (5) is Grünbaum’s classic inequality \[3\], which states
\[
\text{vol}_n(K \cap \xi^+) \geq \left( \frac{n}{n+1} \right)^n \text{vol}_n(K) \quad \text{for all} \quad \xi \in S^{n-1}
\]
for every convex body with centroid at the origin.

In this paper, we also prove an analogue of (4) and (5) for dual volumes.

**Theorem 3.** Consider integers \(1 \leq i \leq k \leq n\). Let \(K \subset \mathbb{R}^n\) be a convex body with centroid at the origin, and let \(E \in G(n,k)\). Then
\[
\tilde{V}_i(K \cap E \cap \xi^+) \geq \left( \frac{i}{n+1} \right)^i \tilde{V}_i(K \cap E)
\]
and
\[
\tilde{V}_i((K|E) \cap \xi^+) \geq \left( \frac{i}{n+1} \right)^i \tilde{V}_i(K|E)
\]
for all \(\xi \in S^{n-1} \cap E\), where the dual volumes are taken within the \(k\)-dimensional subspace \(E\). The constant in each inequality is the best possible.

**Remark.** When \(i = k\), inequality (7) yields (4) and inequality (8) yields (5).

Inequality (7) can also be written in the form
\[
\int_{S^{n-1} \cap E \cap \xi^+} \rho_K(u)^i \, du \geq \left( \frac{i}{n+1} \right)^i \int_{S^{n-1} \cap E} \rho_K(u)^i \, du;
\]
see (9) in Section 2.

The paper is organized as follows. We present some preliminaries in Section 2, prove Theorem 1 in Section 3, prove Theorem 2 in Section 4, and prove Theorem 3 in Section 5.

### 2. Preliminaries

According to Kubota’s integral formula (e.g. equation A.47 in \[4\]), the \(i\)th intrinsic volume \(V_i(L)\) of a convex and compact \(L \subset \mathbb{R}^n\) is essentially the average \(i\)-dimensional volume of the orthogonal projection \(L|F\) taken over all \(i\)-dimensional subspaces \(F \in G(n,i)\):
\[
V_i(L) = \frac{\kappa_n}{\kappa_i \kappa_{n-i}} \binom{n}{i} \int_{G(n,i)} \text{vol}_i(L|F) \, dF.
\]

Here, we are integrating with respect to the unique Haar probability measure on the Grassmannian \(G(n,i)\) of \(i\)-dimensional subspaces of \(\mathbb{R}^n\).

Similarly, the dual Kubota integral formula (e.g. Theorem A.7.2 in \[4\]) asserts that the \(i\)th dual volume \(\tilde{V}_i(L)\) of a star body \(L \subset \mathbb{R}^n\) is the average \(i\)-dimensional volume of the section \(L \cap F\) taken over all \(F \in G(n,i)\):
\[
\tilde{V}_i(L) = \frac{\kappa_n}{\kappa_i} \int_{G(n,i)} \text{vol}_i(L \cap F) \, dF.
\]

The dual volume \(\tilde{V}_i(L)\) can also be expressed as follows:
\[
\tilde{V}_i(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^i \, du = \frac{i}{n} \int_{L} |x|^{-n+i} \, dx.
\]
Proposition 4. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin. Let $1 \leq i \leq k \leq n$ be integers. For every $E \in G(n,k)$, $F \in G(n,i)$ with $F \subset E$, and $\xi \in S^{n-1} \cap F$, we have

$$vol_1 \left( \left( (K \cap E)|F \right) \cap \xi^+ \right) \geq \left( \frac{i}{n+1} \right)^i vol_1 \left( (K \cap E)|F \right)$$

(10)

and

$$vol_1 \left( (K|E) \cap F \cap \xi^+ \right) \geq \left( \frac{i}{n+1} \right)^i vol_1 \left( (K|E) \cap F \right).$$

(11)

These inequalities are exact. For example, there is equality in both inequalities when

$$K = \text{conv} \left( r_0 B_2^{i-1} - a \left( \frac{n-i+1}{n+1} \right) \xi, r_1 B_2^{n-i} + a \left( \frac{i}{n+1} \right) \xi \right);$$

(12)

$B_2^{i-1}$ is the unit Euclidean ball in $F \cap \xi^+$ centred at the origin, $B_2^{n-i}$ is the unit Euclidean ball in $F^\perp$ centred at the origin, and $a, r_0, r_1 > 0$ are any constants.

Proof. When $i = k$ or $k = n$, the inequalities (10) and (11) are exactly Grünbaum’s inequality for sections (4) and Grünbaum’s inequality for projections (5).

Assume $i < k < n$. Observe that

$$(K \cap E)|F = \left( K|(F \oplus E^\perp) \right) \cap F.$$ 

Let $K_1$ be the $(k-i)$-symmetral of $K$ parallel to $(F \oplus E^\perp)^\perp$, and let $K_2$ be the $(n-k)$-symmetral of $K$ parallel to $E^\perp$; i.e.

$$K_1 = \bigcup_{x \in K|(F \oplus E^\perp)} \left( \frac{\text{vol}_{k-i} \left( (K-x) \cap (F \oplus E^\perp)^\perp \right)}{\kappa_{k-i}} \right)^{\frac{1}{i}} B_2^{k-i} + x \bigcup_{x \in K|E} \left( \frac{\text{vol}_{n-k} \left( (K-x) \cap E^\perp \right)}{\kappa_{n-k}} \right)^{\frac{1}{k}} B_2^{n-k} + x,$$

where $B_2^{k-i}$ and $B_2^{n-k}$ are the Euclidean balls in $(F \oplus E^\perp)^\perp$ and $E^\perp$, respectively, with unit radius and centres at the origin. Now, $K_1$ and $K_2$ are convex bodies in $\mathbb{R}^n$ with

$$K|(F \oplus E^\perp) = K_1, (F \oplus E^\perp) = K_1 \cap (F \oplus E^\perp), \quad K|E = K_2, E = K_2 \cap E,$$

and centroids at the origin. Therefore,

$$(K \cap E)|F = (K|(F \oplus E^\perp) \cap F = K_1 \cap (F \oplus E^\perp) \cap F = K_1 \cap F$$

and

$$(K|E) \cap F = K_2 \cap E \cap F = K_2 \cap F.$$ 

We now see that (10) and (11) follow from an application of Grünbaum’s inequality for sections to $K_1$ and $K_2$, respectively, and the subspace $F$.

Finally, to show that inequalities (10) and (11) are sharp, suppose $K$ has the form (12). We first show that the centroid of $K$ is at the origin. Indeed, by symmetry, $g(K)$ must lie on the line $\mathbb{R} \xi$ passing through the origin and parallel to $\xi$. For $t \in \left[ -a \left( \frac{n-i+1}{n+1} \right), a \left( \frac{i}{n+1} \right) \right]$, the section $K \cap \{t \xi + \xi^+\}$ is the product of balls

$$\left( \tilde{r}_0(t) B_2^{i-1} \right) \times \left( \tilde{r}_1(t) B_2^{n-i} \right)$$
where
\[
\tilde{r}_0(t) = r_0 \left( \frac{i}{n+1} - \frac{t}{a} \right) \quad \text{and} \quad \tilde{r}_1(t) = r_1 \left( \frac{n-i+1}{n+1} + \frac{t}{a} \right).
\]

Applying Fubini’s Theorem and a change of variables,
\[
\int_K \langle x, \xi \rangle \, dx = \int_{-a(\frac{n+1}{n+1})}^{a(\frac{n+1}{n+1})} \int_{\tilde{r}_0(x_1)B_2^{i-1}}^{\tilde{r}_1(x_1)B_2^{i-1}} x_1 \, dx_n \cdots dx_2 \, dx_1
\]
\[
= r_0^{n-i+1}a^{n-i} \kappa_{i-1} \kappa_{n-i} \int_{-a(\frac{n+1}{n+1})}^{a(\frac{n+1}{n+1})} \left( \frac{i}{n+1} - \frac{x_1}{a} \right)^{i-1} \left( \frac{n-i+1}{n+1} + \frac{x_1}{a} \right)^{n-i} x_1 \, dx_1
\]
\[
= a^{2i-1}r_0^{n-i} \kappa_{i-1} \kappa_{n-i} \int_0^1 (1-t)^{i-1} t^{n-i} \left( t - \frac{n-i+1}{n+1} \right) dt.
\]
The last integral is equal to
\[
\int_0^1 t^{n-i+1} (1-t)^{i-1} \, dt = \frac{(n-i+1)}{(n+1)} \int_0^1 t^{n-i} (1-t)^{i-1} \, dt
\]
\[
= \frac{\Gamma(n-i+2) \Gamma(i)}{\Gamma(n+2)} - \frac{(n-i+1)}{(n+1)} \frac{\Gamma(n-i+1) \Gamma(i)}{\Gamma(n+1)}
\]
\[
= 0,
\]
using well-known identities for the Gamma function. Therefore, \( \langle g(K), \xi \rangle = 0 \), implying \( g(K) \) is the origin. Now, we have \( K \cap E = K|E \), so
\[
(K \cap E)|F = K|F \quad \text{and} \quad (K|E) \cap F = K \cap F.
\]
Finally, observing that
\[
K|F = K \cap F = \text{conv} \left( \tilde{r}_0 B_2^{i-1} - a \left( \frac{n-i+1}{n+1} \right), a \left( \frac{i}{n+1} \right) \xi \right)
\]
is an \( i \)-dimensional cone in \( F \) whose base is orthogonal to \( \xi \), a simple calculation verifies that \( K \) gives equality in (10) and (11).

\[\square\]

Remark 5. The inequalities in Proposition 4 are equivalent to
\[
\text{vol}_i \left( ((K \cap E)|F) \cap \xi^+ \right) \geq \frac{i^i}{(n+1)^i - i^i} \text{vol}_i \left( ((K \cap E)|F) \cap \xi^- \right)
\]
and
\[
\text{vol}_i \left( (K|E) \cap F \cap \xi^+ \right) \geq \frac{i^i}{(n+1)^i - i^i} \text{vol}_i \left( (K|E) \cap F \cap \xi^- \right),
\]
where \( \xi^- := \{ x \in \mathbb{R}^n \mid \langle x, \xi \rangle \leq 0 \} \).

3. INTRINSIC VOLUMES OF SECTIONS

Proof of Theorem 7. By the continuity and translation invariance of intrinsic volumes, there is an \( x_0 \in E^\perp \) such that
\[
V_i((K - x_0) \cap E) = \max_{x \in E^\perp} V_i((K - x) \cap E)
\]
\[
= \max_{x \in \mathbb{R}^n} V_i((K - x) \cap E) = \max_{x \in K} V_i((K - x) \cap E).
\]
Without loss of generality, \( x_0 \) is not the origin. Let \( \xi \in S^{n-1} \) be the unique unit vector parallel to \( x_0 \) and such that \(-t_0 := \langle \xi, x_0 \rangle < 0\). For any subspace \( F \) of \( \mathbb{R}^n \), define

\[
F_\xi := \text{span}(F, \xi).
\]

Let \( G(E, i) \) denote the Grassmannian of \( i \)-dimensional subspaces of \( E \). Note that

\[
(K - t\xi) \cap (K - t\xi + F) = \left( (K \cap E_\xi) \cap \{t\xi + F\} \right) - t\xi \quad \forall F \in G(E, i),
\]

because \( \xi \in E^\perp \). So, by Kubota’s integral formula, we have

\[
V_i((K - x_0) \cap E) = c_{k,i} \int_{G(E, i)} \text{vol}_i\left((K \cap E_\xi) \cap \{t_0 \xi + F\}\right) dF \tag{13}
\]

and

\[
V_i(K \cap E) = c_{k,i} \int_{G(E, i)} \text{vol}_i\left((K \cap E_\xi) \cap F\right) dF, \tag{14}
\]

where \( c_{k,i} > 0 \) is a constant depending on \( k \) and \( i \).

Consider any \( F \in G(E, i) \) for which

\[
\text{vol}_i\left((K \cap E_\xi) \cap \{t_0 \xi + F\}\right) > \text{vol}_i\left((K \cap E_\xi) \cap F\right).
\]

It is possible that

\[
\text{vol}_i\left((K \cap E_\xi) \cap \{t_0 \xi + F\}\right) < \max_{t \in \mathbb{R}} \text{vol}_i\left((K \cap E_\xi) \cap \{t\xi + F\}\right),
\]

but this does not matter to the following argument. Let \( \bar{K} \) be the \( i \)-symmetrical of \((K \cap E_\xi)\) in \( F_\xi \) parallel to \( F \); i.e.

\[
\bar{K} = \bigcup_{t \xi \in (K \cap E_\xi) \xi} \left( \frac{\text{vol}_i\left((K \cap E_\xi) \cap \{t\xi + F\}\right)}{\kappa_i} \right)^{\frac{1}{i}} \cdot B_i^j + t\xi,
\]

where \( B_i^j \) is the \( i \)-dimensional Euclidean ball in \( F \) with unit radius and centred at the origin. Note that \( \bar{K} \) is an \((i + 1)\)-dimensional convex body in \( F_\xi \). Let \( G \) be the unique \((i + 1)\)-dimensional cone in \( F_\xi \) with

\begin{itemize}
  \item \( i \)-dimensional base \( \bar{K} \cap \{-t_0 \xi + F\} \);
  \item \( i \)-dimensional cross-section \( G \cap F = \bar{K} \cap F \);
\end{itemize}

(15)

see Figure \[.\] Necessarily, the apex of the cone \( G \) is at \( t_1 \xi \) for some \( t_1 > 0 \). By convexity, we have that \( G \cap \xi^- \subset \bar{K} \cap \xi^- \) and \( \bar{K} \cap \xi^+ \subset G \cap \xi^+ \). Therefore,

\[
\text{vol}_{i+1}(G \cap \xi^-) \leq \text{vol}_{i+1}(\bar{K} \cap \xi^-) = \text{vol}_{i+1}\left((K \cap E_\xi) \cap \{t\xi + F\}\right) \leq \text{vol}_{i+1}(G \cap \xi^+),
\]

and

\[
\text{vol}_{i+1}\left((K \cap E_\xi) \cap \xi^+\right) = \text{vol}_{i+1}(\bar{K} \cap \xi^+) \leq \text{vol}_{i+1}(G \cap \xi^+),
\]

implying

\[
\frac{\text{vol}_{i+1}\left((K \cap E_\xi) \cap \xi^+\right)}{\text{vol}_{i+1}\left((K \cap E_\xi) \cap \xi^-\right)} \leq \frac{\text{vol}_{i+1}(G \cap \xi^+)}{\text{vol}_{i+1}(G \cap \xi^-)}.
\]

(16)
Define
\[ L_0 := \text{conv} \left( \frac{\text{vol}_1(\tilde{K} \cap \{ -t_0 \xi + F \})}{\kappa_i} \right)^{\frac{1}{2}} B_2^{n-i-1} - t_0 \xi, B_2^{n-i-1} + t_1 \xi \), \] (17)
where \( B_2^{n-i-1} \) is still the unit Euclidean ball in \( F = F_\xi \cap \xi^\perp \) centred at the origin, and \( B_2^{n-i-1} \) is the unit Euclidean ball in \( F_\xi \cap \xi^\perp \) centred at the origin. Note that \( L_0 \) is a convex body in \( \mathbb{R}^n \) with
\[ (L_0 \cap E_\xi) F_\xi = G \]
and centroid
\[ g(L_0) = -t_0 \xi + (t_1 + t_0) \left( \frac{n-i}{n+1} \right) \xi. \]

Consider the translate \( L := L_0 - g(L_0) \) with centroid at the origin. Assume \( \langle g(L_0), \xi \rangle < 0 \). In this case,
\[ \text{vol}_{i+1} \left( (L \cap E_\xi) F_\xi \right) \cap \xi^- < \text{vol}_{i+1} \left( G \cap \xi^- \right) \]
and
\[ \text{vol}_{i+1} \left( G \cap \xi^+ \right) < \text{vol}_{i+1} \left( (L \cap E_\xi) F_\xi \cap \xi^+ \right), \]
implying
\[ \frac{\text{vol}_{i+1} \left( G \cap \xi^+ \right)}{\text{vol}_{i+1} \left( G \cap \xi^- \right)} < \frac{\text{vol}_{i+1} \left( (L \cap E_\xi) F_\xi \cap \xi^+ \right)}{\text{vol}_{i+1} \left( (L \cap E_\xi) F_\xi \cap \xi^- \right)}. \] (18)
Observe that $L$ gives equality in Proposition 4. Considering Remark 5 and combining inequalities (16) and (18), we get

$$\frac{\text{vol}_{i+1}((K \cap E_\epsilon)\cap \xi^+)}{\text{vol}_{i+1}((K \cap E_\epsilon)\cap \xi^-)} \leq \frac{\text{vol}_{i+1}((L \cap E_\epsilon)\cap \xi^+)}{\text{vol}_{i+1}((L \cap E_\epsilon)\cap \xi^-)} = \frac{i^i}{(n+1)^i - i^i},$$

which contradicts inequality (10) in Proposition 4. Therefore, $\langle g(L_0), \xi \rangle \geq 0$, and consequently

$$\text{vol}_i\left((L \cap E_\epsilon)\cap F_\epsilon \right) \leq \text{vol}_i\left((K \cap E_\epsilon)\cap F_\epsilon \right),$$

(19)

An explicit calculation gives

$$\text{vol}_i\left((L \cap E_\epsilon)\cap F_\epsilon \right) = \left(\frac{i+1}{n+1}\right)^i \max_{t \in \mathbb{R}}\text{vol}_i\left((L \cap E_\epsilon)\cap \{t \xi + F_\epsilon \} \right)$$

$$= \left(\frac{i+1}{n+1}\right)^i \text{vol}_i\left((K \cap E_\epsilon)\cap \{-t_0 \xi + F_\epsilon \} \right).$$

(20)

Inequality (2) in our theorem statement follows from inequalities (19) and (20), together with the integral expressions (13) and (14). We still need to show that the constant in inequality (2) is the best possible. To this end, assume there is a constant

$$C > \left(\frac{i+1}{n+1}\right)^i$$

such that

$$V_i(K \cap E) \geq C \cdot \max_{x \in E^+} V_i((K - x) \cap E)$$

for every convex body $K$ in $\mathbb{R}^n$ with centroid at the origin. Define the convex bodies

$$K_\epsilon = \text{conv}(B_{\xi}^i - \left(\frac{n-i}{n+1}\right) \xi, \epsilon B_{\xi}^n - \left(\frac{i+1}{n+1}\right) \xi)$$

for $\epsilon > 0$ and fixed $F \in G(E, i)$, with $B_{\xi}^2 \subset F$ and $B_{\xi}^n - i-1 \subset F_\epsilon^\perp$ as before. Note that $g(K_\epsilon) = 0$ for all $\epsilon > 0$. We have

$$\lim_{\epsilon \to 0^+} K_\epsilon \cap E = \left(\frac{i+1}{n+1}\right) B_{\xi}^2 \quad \text{and} \quad \lim_{\epsilon \to 0^+} K_\epsilon + \left(\frac{n-i}{n+1}\right) \xi \cap E = B_{\xi}^i,$$

where the convergence is with respect to the Hausdorff metric. Therefore, it follows from our assumption and the continuity of intrinsic volumes that

$$V_i\left(\left(\frac{i+1}{n+1}\right) B_{\xi}^2 \right) = \lim_{\epsilon \to 0^+} V_i(K_\epsilon \cap E)$$

$$\geq C \lim_{\epsilon \to 0^+} \max_{x \in E^+} V_i((K_\epsilon - x) \cap E)$$

$$\geq C \lim_{\epsilon \to 0^+} V_i\left((K_\epsilon + \left(\frac{n-i}{n+1}\right) \xi) \cap E \right) = C \cdot V_i(B_{\xi}^i).$$

This implies

$$\left(\frac{i+1}{n+1}\right)^i \geq C,$$
By the dual Kubota integral formula,

\[ x \]

Without loss of generality, there is an

\[ \text{Remark. It is often convenient to state inequalities with dimension-independent constants. For example, the classic Gr"unbaum inequality} \]

\[ \text{yields} \]

\[ \text{for any convex body} \ K \subset \mathbb{R}^n \text{ with centroid at the origin. The constant} \ e^{-1} \text{ is} \]

\[ \text{asymptotically sharp. Similarly, inequality} \ (1) \text{ for} \ k = n - 1 \text{ gives} \]

\[ \text{for every} \ (n - 1)\text{-dimensional subspace} \ E. \]

\[ \text{Our Theorem} \ 1 \text{ yields an analogue of the previous inequality for surface areas of} \]

\[ \text{sections with} \ (n - 1)\text{-dimensional subspaces} \ E: \]

\[ \text{The constant} \ e^{-2} \text{ is asymptotically sharp.} \]

\[ \]

4. Dual Volumes of Sections

\[ \text{Proof of Theorem} \ 2 \text{ The proof proceeds similarly to that of Theorem} \ 1 \text{ By continuity, there is an} \]

\[ \text{such that} \]

\[ \text{Without loss of generality,} \ x_0 \text{ is not the origin. Note that} \]

\[ \text{By the dual Kubota integral formula,} \]

\[ \text{where} \ c_{k,i} > 0 \text{ is a constant depending on} \ k \text{ and} \ i. \]

\[ \text{Consider any} \ F \in \mathcal{G}(E, i) \text{ for which} \]

\[ \text{Let} \ \xi \in S^{n-1} \text{ be the unique unit vector that is parallel to} \ x_0 \text{ and such that} \]

\[ \text{Let} \ \tilde{K} \text{ be the} \ i\text{-symmetral of} \ K \cap F_\xi \text{ in} \ F_\xi \text{ parallel to} \ F. \text{ Let} \ G \text{ be the} \]

\[ \text{Define} \ L_0 \text{ as in} \ (17), \text{ and} \ L := L_0 - g(L_0) \text{ as before. Following the argument in the proof of} \]

\[ \text{we find} \]

\[ \frac{\text{vol}_{i+1}((K \cap F_\xi) \cap \xi^+)}{\text{vol}_{i+1}((K \cap F_\xi) \cap \xi^-)} = \frac{\text{vol}_{i+1}((\tilde{K} \cap \xi^+)}{\text{vol}_{i+1}((\tilde{K} \cap \xi^-)} \leq \frac{\text{vol}_{i+1}(G \cap \xi^+)}{\text{vol}_{i+1}(G \cap \xi^-)}. \]
If \( \langle g(L_0), \xi \rangle < 0 \), we then have
\[
\frac{\text{vol}_{i+1}(G \cap \xi^+)}{\text{vol}_{i+1}(G \cap \xi^-)} < \frac{\text{vol}_{i+1}(L \cap F_\xi) \cap \xi^+}{\text{vol}_{i+1}(L \cap F_\xi) \cap \xi^-} = \frac{i^i}{(n+1)^i - i^i}.
\]
The two previous inequalities together contradict Grünbaum’s inequality for sections \( \mathbb{L} \) in light of Remark \( \mathbb{L} \) so necessarily \( \langle g(L_0), \xi \rangle \geq 0 \). Therefore,
\[
\text{vol}_i((K \cap F_\xi) \cap F) \geq \text{vol}_i((L \cap F_\xi) \cap F)
\]
\[
= \left( \frac{i+1}{n+1} \right)^i \max_{t \in \mathbb{R}} \text{vol}_i((L \cap F_\xi) \cap \{t \xi + F\})
\]
\[
= \left( \frac{i+1}{n+1} \right)^i \text{vol}_i((K \cap F_\xi) \cap \{-t_0 \xi + F\});
\]
this inequality, (21), and (22) imply inequality (3) in our theorem statement.

We now show that inequality (3) is sharp. Assume \( i < k \); the equality conditions when \( i = k \) are described by Fradelizi \( \mathbb{L} \). Let \( E \) be a \( k \)-dimensional subspace of \( \mathbb{R}^n \), let \( F \) be an \( i \)-dimensional subspace of \( E \), and let \( \xi \in S^{n-1} \cap E \perp \). For each \( \epsilon > 0 \), define the convex body
\[
K_\epsilon := \text{conv} \left( \epsilon B_2^i - \left( \frac{n-i}{n+1} \right) \xi, \right. \left. B_2^{n-i-1} + \left( \frac{i+1}{n+1} \right) \xi \right)
\]
in \( \mathbb{R}^n \). Here, \( B_2^i \) and \( B_2^{n-i-1} \) are the Euclidean balls in \( F \) and \( F_\xi \), respectively, with unit radius and center at the origin. The centroid of \( K_\epsilon \) is at the origin for all \( \epsilon > 0 \).

Fix
\[
-\frac{n-i}{n+1} < t < \frac{i+1}{n+1}.
\]
We want to calculate the following limit of \( i \)th order dual volumes taken within \( E \):
\[
\lim_{\epsilon \to 0+} \frac{\bar{V}_i(K_\epsilon \cap E)}{\bar{V}_i((K_\epsilon - t \xi) \cap E)}.
\]
Notice that \( (K_\epsilon - t \xi) \cap E \) is a Cartesian product of balls. That is,
\[
(K_\epsilon - t \xi) \cap E = ((a \epsilon) B_2^i) \times (b B_2^{k-i})
\]
where
\[
a = a(t) := \frac{i+1}{n+1} - t \quad \text{and} \quad b = b(t) := \frac{n-i}{n+1} + t.
\]
Using Fubini’s theorem and passing to polar coordinates in the balls \( B_2^i \) and \( B_2^{k-i} \), we have
\[
\bar{V}_i((K_\epsilon - t \xi) \cap E) = \frac{i}{k} \int_{(K_\epsilon - t \xi) \cap E} |x|^{-k+i} \, dx
\]
\[
= \frac{i}{k} \int_{a B_2^i} \int_{b B_2^{k-i}} (x_1^2 + \cdots + x_k^2)^{(-k+i)/2} \, dx_1 \cdots dx_k
\]
\[
= \frac{i}{k} \omega_i \omega_{k-i} \int_{r_1}^{\epsilon a} r_1^{i-1} \int_0^b r_2^{k-i-1}(r_1^2 + r_2^2)^{(-k+i)/2} \, dr_2 \, dr_1.
\]
The notation $\omega_i$ gives the surface area of the $B^i_2$. Denoting

$$a_0 = \frac{i + 1}{n + 1} \quad \text{and} \quad b_0 = \frac{n - i}{n + 1},$$

we obtain

$$\frac{\tilde{V}_i(K \cap E)}{\tilde{V}_i((K - t\xi) \cap E)} = \frac{\int_0^{r_1} r_1^{i-1} \int_0^{b_0} r_2^{k-i-1}(r_1^2 + r_2^2)^{-(k+i)/2} dr_2 dr_1}{\int_0^{a_0} r_1^{i-1} \int_0^{a_0} r_2^{k-i-1}(r_1^2 + r_2^2)^{-(k+i)/2} dr_2 dr_1}$$

An application of the Dominated Convergence Theorem verifies that the numerator and denominator above approach zero as $\epsilon$ tends to zero. Thus, with L'Hôpital’s rule we obtain

$$\lim_{\epsilon \to 0^+} \frac{\tilde{V}_i(K \cap E)}{\tilde{V}_i((K - t\xi) \cap E)} = \lim_{\epsilon \to 0^+} \frac{\epsilon^{i-1} a^i_0 \int_0^{b_0} r_2^{k-i-1}((\epsilon a_0)^2 + r_2^2)^{-(k+i)/2} dr_2}{\epsilon^{i-1} a^i \int_0^{b_0} r_2^{k-i-1}((\epsilon a)^2 + r_2^2)^{-(k+i)/2} dr_2}.$$

The integrals in the numerator and denominator both approach infinity as $\epsilon$ tends to zero. We will show that their ratio approaches one. To see this, write the integral in the denominator as the sum of the integrals: from 0 to $a$ and from $a$ to $b$. As $\epsilon$ approaches zero, the integral from $a$ to $b$ approaches some constant, and so we will disregard it when computing the limit. The same argument applies to the integral in the numerator. Therefore,

$$\lim_{\epsilon \to 0^+} \frac{\tilde{V}_i(K \cap E)}{\tilde{V}_i((K - t\xi) \cap E)} = \lim_{\epsilon \to 0^+} \frac{a^i_0 \int_0^a r_2^{k-i-1}((\epsilon a_0)^2 + r_2^2)^{-(k+i)/2} dr_2}{a^i \int_0^a r_2^{k-i-1}((\epsilon a)^2 + r_2^2)^{-(k+i)/2} dr_2} = a^i_0 = \left(\frac{i + 1}{n + 1} - t\right)^i,$$

since the integrals in the numerator and denominator are the same, which can be seen using an obvious change of variables.

We finally have

$$-\frac{n + 1}{n + 1} \left(\lim_{\epsilon \to 0^+} \frac{\tilde{V}_i(K \cap E)}{\tilde{V}_i((K - t\xi) \cap E)}\right)^i = \left(\frac{i + 1}{n + 1}\right)^i,$$

which proves that the constant in (3) is the best possible. \hfill \square

5. Grünbaum’s Inequality for Dual Volumes

Proof of Theorem 3 Assume through the proof that $i < k$, as Theorem 3 reduces to (4) and (5) for $i = k$.

For a $k$-dimensional subspace $E \subset \mathbb{R}^n$, $\xi \in S^{n-1} \cap E$, and any $i$-dimensional subspace $F \subset E$, Grünbaum’s inequality for sections (4) gives

$$\text{vol}_i\left((K \cap E \cap \xi^+) \cap F\right) = \text{vol}_i(K \cap F \cap \xi^+) \geq \left(\frac{i}{n + 1}\right)^i \text{vol}_i(K \cap F) = \left(\frac{i}{n + 1}\right)^i \text{vol}_i((K \cap E) \cap F),$$

and inequality [11] of Proposition 4 gives

$$\text{vol}_i\left((K|E) \cap \xi^+ \cap F\right) = \text{vol}_i((K|E) \cap F \cap \xi^+) \geq \left(\frac{i}{n + 1}\right)^i \text{vol}_i((K|E) \cap F).$$
Integrating these inequalities over \(G(E, i)\), and applying the dual Kubota formula, we respectively get

\[
\tilde{V}_i(K \cap E \cap \xi^+) \geq \left( \frac{i}{n+1} \right)^i \tilde{V}_i(K \cap E)
\]

and

\[
\tilde{V}_i((K|E) \cap \xi^+) \geq \left( \frac{i}{n+1} \right)^i \tilde{V}_i(K|E).
\]

The dual volumes are taken within \(E\).

We now prove that the constant in the above inequality is the best possible. Fix a \(k\)-dimensional subspace \(E \subset \mathbb{R}^n\), an \(i\)-dimensional subspace \(F \subset E\), and \(\xi \in S^{n-1} \cap F\). Consider the family of bodies

\[
K_\varepsilon = \text{conv} \left( \varepsilon B_2^{i-1} - \varepsilon \left( \frac{n+i+1}{n+1} \right) \xi, B_2^{n-i} + \varepsilon \left( \frac{i}{n+1} \right) \xi \right)
\]

for small \(\varepsilon > 0\). Here, \(B_2^{i-1}\) is the unit ball in \(F \cap \xi^\perp\), and \(B_2^{n-i}\) is the unit ball in \(F^\perp\). Interpret \(\varepsilon B_2^{i-1}\) as the origin when \(i = 1\). Note that the centroid of \(K_\varepsilon\) is at the origin. It is sufficient to show that

\[
\lim_{\varepsilon \to 0^+} \frac{\tilde{V}_i(K_\varepsilon \cap E \cap \xi^+)}{\tilde{V}_i(K_\varepsilon \cap E)} \leq \left( \frac{i}{n+1} \right)^i,
\]

because \(K_\varepsilon \cap E = K_\varepsilon|E\).

First consider the case \(2 \leq i \leq k - 1\). For each \(t \in \left[ -\varepsilon \left( \frac{n+i+1}{n+1} \right), \varepsilon \left( \frac{i}{n+1} \right) \right]\), the section \(K_\varepsilon \cap E \cap \{ t\xi + \xi^\perp \}\) is the product of two balls

\[
(a(t)B_2^{i-1}) \times (b(t)B_2^{k-i})
\]

where

\[
a(t) = \varepsilon \frac{i}{n+1} - t \quad \text{and} \quad b(t) = \frac{n-i+1}{n+1} + \frac{t}{\varepsilon}.
\]

Using Fubini’s Theorem and passing to polar coordinates in the balls \(B_2^{i-1}\) and \(B_2^{k-i}\), we have

\[
\tilde{V}_i(K_\varepsilon \cap E \cap \xi^+) = \frac{i}{k} \int_{K_\varepsilon \cap E \cap \xi^+} |x|^{-k+i} dx
\]

\[
= \frac{i}{k} \int_0^{\varepsilon \left( \frac{i}{n+1} \right)} \int_{a(x_i)B_2^{i-1}} \int_{b(x_i)B_2^{k-i}} (x_1^2 + \cdots + x_k^2)^{(-k+i)/2} dx_k \cdots dx_1
\]

\[
= \frac{i}{k} \omega_{i-1} \omega_{k-i} \int_0^{\varepsilon \left( \frac{i}{n+1} \right)} \int_0^{a(x_i)} r_1^{i-2} \int_0^{b(x_i)} r_2^{k-i-1} (x_1^2 + x_2^2)^{(-k+i)/2} dr_2 dr_1 dx_1
\]

Making the change of variables \(x_1 = \epsilon u, r_1 = \epsilon v, r_2 = w\), we get

\[
\tilde{V}_i(K_\varepsilon \cap \xi_1^+) = \frac{i}{k} \omega_{i-1} \omega_{k-i} \epsilon^i
\]

\[
\times \int_0^{\varepsilon \left( \frac{i}{n+1} \right)} \int_0^{\epsilon \left( \frac{i}{n+1} \right)} \int_0^{\epsilon \left( \frac{i}{n+1} \right)} u^{i-2} w^{k-i-1} (\epsilon^2 u^2 + \epsilon^2 v^2 + w^2)^{(-k+i)/2} dw dv du.
\]
Denoting the latter triple integral by $I$, and denoting by $II$ the triple integral below

$$II = \int_{-\frac{n+1}{n+1}}^{\frac{i+1}{n+1}} \int_{0}^{\frac{n-i+1}{n+1}-u} v^{i-2} \int_{0}^{\frac{n-i+1}{n+1}+u} w^{k-i-1}(\epsilon^2 u^2 + \epsilon^2 v^2 + w^2)^{(-k+i)/2} \, dw \, dv \, du,$$

we see that

$$\tilde{V}_i(K_\epsilon \cap E \cap \xi^+) = I/II.$$

We estimate $I$ from above by

$$I \leq \int_{0}^{\frac{i+1}{n+1}} \int_{0}^{\frac{i+1}{n+1}-u} v^{i-2} \int_{0}^{\frac{n-i+1}{n+1}+u} w^{k-i-1}(\epsilon^2 u^2 + \epsilon^2 v^2 + w^2)^{(-k+i)/2} \, dw \, dv \, du.$$

If $i = k - 1$, then the integral with respect to $w$ can be computed directly and it equals $\ln(1 + \sqrt{1 + \epsilon^2 u^2}) - \ln(\epsilon u)$. If $i \leq k - 2$, then

$$\frac{w^{k-i-1}}{(\epsilon^2 u^2 + w^2)^{\frac{k-i}{2}}} \leq \frac{w}{\epsilon^2 u^2 + w^2},$$

and so

$$I \leq \int_{0}^{\frac{i+1}{n+1}} \int_{0}^{\frac{i+1}{n+1}-u} v^{i-2} dv \int_{0}^{\frac{n-i+1}{n+1}+u} w(\epsilon^2 u^2 + w^2)^{-1} \, dw \, du$$

$$= \frac{1}{2(i-1)} \int_{0}^{\frac{i+1}{n+1}} \left( \frac{i}{n+1} - u \right)^{i-1} \left( \ln(\epsilon^2 u^2 + 1) - 2 \ln(\epsilon) - 2 \ln(u) \right) \, du$$

$$= o(1) - \frac{\ln \epsilon}{i-1} \int_{0}^{\frac{i+1}{n+1}} \left( \frac{i}{n+1} - u \right)^{i-1} \, du - \int_{0}^{\frac{i+1}{n+1}} \left( \frac{i}{n+1} - u \right)^{i-1} \ln(u) \, du$$

$$= - \frac{\ln \epsilon}{i(i-1)} \left( \frac{i}{n+1} \right)^i (1 + o(1)).$$

Note that in the case $i = k - 1$, we get the same bound.

Now we estimate $II$ from below. Since $|u| \leq 1$, $|v| \leq 1$, we have

$$II \geq \int_{-\frac{n+1}{n+1}}^{\frac{i+1}{n+1}} \int_{0}^{\frac{n-i+1}{n+1}+u} w^{k-i-1}(2\epsilon^2 + w^2)^{(-k+i)/2} \, dw \, dv \, du$$

$$= \frac{1}{i-1} \int_{-\frac{n+1}{n+1}}^{\frac{i+1}{n+1}} \left( \frac{i}{n+1} - u \right)^{i-1} \int_{0}^{\frac{n-i+1}{n+1}+u} w^{k-i-1}(2\epsilon^2 + w^2)^{(-k+i)/2} \, dw \, du.$$
Using the change of variable $z = \frac{n+i+1}{n+1} + u$ and then integrating by parts, we get
\[
= \frac{1}{i-1} \int_0^1 (1-z)^{i-1} \int_0^z w^{k-1} (2e^2 + w^2)^{(-k+i)/2} \, dw \, dz \\
= \frac{1}{i(i-1)} \int_0^1 (1-z)^i z^{k-1} \int_0^z (2e^2 + z^2)^{(-k+i)/2} \, dz \\
\geq \frac{1}{i(i-1)} \int_\epsilon^1 (1-z)^i z^{k-1} \int_\epsilon^z (2e^2 z^{-2} + 1)^{(-k+i)/2} \, dz \\
= \frac{1}{i(i-1)} \int_\epsilon^1 (1-z)^i z^{-1} (1 + (-k+i)z^{-2}) \, dz,
\]
where we used the inequality $(1 + x)^p \geq 1 + px$ with $p < 0$ and $x \geq 0$. Note that
\[
(k - i) e^2 \int_\epsilon^1 (1-z)^i z^{-3} \, dz
\]
is positive, and bounded above by a constant $C > 0$ for small enough $\epsilon > 0$. Thus,
\[
II \geq \frac{1}{i(i-1)} \int_\epsilon^1 (1-z)^i z^{-1} \, dz - C \\
= \frac{1}{i(i-1)} \int_\epsilon^1 \left( z^{-1} + \sum_{j=1}^i \binom{i}{j} (-1)^j z^{j-1} \right) \, dz - C \\
= - \frac{\ln \epsilon}{i(i-1)} (1 + o(1)).
\]
Comparing the bounds for $I$ and $II$, we get (24).

We now consider the case $i = 1$, in which $K_\epsilon$ is a cone. The section
\[
K_\epsilon \cap E \cap \{t \xi + \xi^+\}
\]
is a ball $b(t)B^{n-1}$ for each $t \in \left[ -\epsilon \left( \frac{n}{n+1} \right), \epsilon \left( \frac{1}{n+1} \right) \right]$, with
\[
b(t) = \frac{n}{n+1} + \frac{t}{\epsilon}.
\]
Using Fubini’s Theorem, polar coordinates, and a change of variables, we find
\[
\frac{\tilde{V}_1(K_\epsilon \cap E \cap \xi^+)}{\tilde{V}_1(K_\epsilon \cap E)} = \frac{\int_0^{\pi/\epsilon} \int_0^{\pi/\epsilon + u} w^{k-2} (e^2u^2 + w^2)^{(-k+1)/2} \, dw \, du}{\int_{\pi/\epsilon}^{\pi/\epsilon + u} \int_0^{\pi/\epsilon + u} w^{k-2} (e^2u^2 + w^2)^{(-k+1)/2} \, dw \, du}.
\]
The numerator and denominator can again be bounded using the previous methods, so that we obtain (24).

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APPLICATIONS OF GRÜNBAUM-TYPE INEQUALITIES

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