Orders generated by character values

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Abstract

Let $K := \mathbb{Q}(G)$ be the number field generated by the complex character values of a finite group $G$. Let $\mathbb{Z}_K$ be the ring of integers of $K$. In this paper we investigate the suborder $\mathbb{Z}[G]$ of $\mathbb{Z}_K$ generated by the character values of $G$. We prove that every prime divisor of the order of the finite abelian group $\mathbb{Z}_K/\mathbb{Z}[G]$ divides $|G|$. Moreover, if $G$ is nilpotent, we show that the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a proper divisor of $|G|$ unless $G = 1$. We conjecture that this holds for arbitrary finite groups $G$.

Keywords: finite groups, field of character values, orders, algebraic integers

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1 Introduction

It is well-known that the complex character values of a finite group $G$ are algebraic integers. We like to measure how “many” algebraic integers actually arise in this way. The field

$$K := \mathbb{Q}(G) := \mathbb{Q}(\chi(g) : \chi \in \text{Irr}(G), \ g \in G) \subseteq \mathbb{C}$$

of character values of $G$ is contained in $\mathbb{Q}_{\exp(G)}$ where $\exp(G)$ denotes the exponent of $G$ and $\mathbb{Q}_n$ is the cyclotomic field generated by the complex $n$-th roots of unity. Let $\mathbb{Z}_K$ be the ring of integers of $K$. The character values of $G$ also generate an order $\mathbb{Z}[G]$ contained in $\mathbb{Z}_K$ (here $\mathbb{Z}[G]$ is neither the group algebra nor the ring of generalized characters). The deviation of $\mathbb{Z}[G]$ from $\mathbb{Z}_K$ can be measured by the structure of the finite abelian group $\mathbb{Z}_K/\mathbb{Z}[G]$. If $G$ is a rational group for instance, then $K = \mathbb{Q}$ and $\mathbb{Z}[G] = \mathbb{Z} = \mathbb{Z}_K$. If $G$ is abelian, then $K = \mathbb{Q}_{\exp(G)}$ and $\mathbb{Z}_K = \mathbb{Z}[e^{2\pi \sqrt{-1}/\exp(G)}]$. In this case it is easy to see that $\mathbb{Z}[G] = \mathbb{Z}_K$ as well. On the other hand, we construct a group $G$ of order 240 such that

$$\mathbb{Z}_K/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_{14}^4$$

where $C_n$ denotes a cyclic group of order $n$. Nevertheless, our main theorems show that the structure of $\mathbb{Z}_K/\mathbb{Z}[G]$ is restricted by the order of $G$.

Theorem A. Let $G$ be a finite group and $K := \mathbb{Q}(G)$. Then the prime divisors of $|\mathbb{Z}_K/\mathbb{Z}[G]|$ divide $|G|$.

Theorem B. Let $G \neq 1$ be a nilpotent group and $K := \mathbb{Q}(G)$. Then the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a proper divisor of $|G|$. In particular, $|G|\mathbb{Z}_K \subseteq \mathbb{Z}[G]$.

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In the final section we exhibit many examples which indicate that Theorem B might be true without
the nilpotency hypothesis.

**Conjecture C.** Let $G \neq 1$ be a finite group and $K := \mathbb{Q}(G)$. Then the exponent of $\mathbb{Z}_K/\mathbb{Z}[G]$ is a
proper divisor of $|G|$.

## 2 Preliminaries

In addition to the notation introduced above, we define

\[
\begin{align*}
Q(g) & := Q(\chi(g) : \chi \in \text{Irr}(G)) \\
\mathbb{Z}[g] & := \mathbb{Z}[\chi(g) : \chi \in \text{Irr}(G)], \\
Q(\chi) & := Q(\chi(g) : g \in G) \\
\mathbb{Z}[\chi] & := \mathbb{Z}[\chi(g) : g \in G].
\end{align*}
\]

For number fields $K \subseteq L$ we denote the relative discriminant of $L$ with respect to $K$ by $d_{L|K} \in \mathbb{Z}_K$. If $K = \mathbb{Q}$ we write $d_L := d_{L|\mathbb{Q}}$ as usual. We make use of the following tools from algebraic number theory.

**Proposition 1.** The discriminant of any subfield of $\mathbb{Q}_n$ divides $n^{\varphi(n)}$.

**Proof.** If $n = p^m$ is a power of a prime $p$, then by [9, Lemma I.10.1] the discriminant $d_n$ of $\mathbb{Q}_n$ is $\pm p^{m-1(mp-m-1)}$, a divisor of $n^{\varphi(n)} = p^{mp^{m-1}(p-1)}$. For arbitrary $n$ we obtain $d_n \mid n^n$ from [9, Proposition I.2.11]. Now if $K \subseteq \mathbb{Q}_n$ is any subfield, then by [9, Corollary III.2.10] even $d_{K|\mathbb{Q}}$ divides $d_n$.

Although we only need a weak version of the following result, it seems worth stating a strong form.

**Proposition 2.** Let $K$ and $L$ be Galois number fields. Then

\[
gcd(d_K, d_L)\mathbb{Z}_{KL} \subseteq \frac{\gcd(d_K, d_L)}{d_{K\cap L}}\mathbb{Z}_{KL} \subseteq \mathbb{Z}_K\mathbb{Z}_L
\]

where $m := \min\{|K : L|, |KL : L|\}$. In particular, $\mathbb{Z}_{KL} = \mathbb{Z}_K\mathbb{Z}_L$ if $d_K$ and $d_L$ are coprime.

**Proof.** Most textbooks only deal with the last claim. To prove the general case we follow [9, Proposition I.2.11]:

We consider the compositum $KL$ as an extension over $M := K \cap L$. Note that $\mathbb{Z}_{KL}$ ($\mathbb{Z}_K$, $\mathbb{Z}_L$ respectively) is the integral closure of $\mathbb{Z}_M$ in $KL$ ($K$, $L$ respectively). Let $b_1, \ldots, b_n$ be a $\mathbb{Z}_M$-basis of $\mathbb{Z}_K$ and let $c_1, \ldots, c_m$ be a $\mathbb{Z}_M$-basis of $\mathbb{Z}_L$. Then $\{b_ic_j : i = 1, \ldots, n, j = 1, \ldots, m\}$ is an $M$-basis of $KL$ as is well-known. Let $\alpha \in \mathbb{Z}_{KL}$ be arbitrary and write

\[
\alpha = \sum_{i,j} a_{ij}b_ic_j
\]

with $a_{ij} \in M$ for all $i, j$. Since $KL$ is a Galois extension over $\mathbb{Q}$, it is also a Galois extension over $K$ and over $L$. Thus, we may write $\text{Gal}(KL|K) = \{\sigma_1, \ldots, \sigma_m\}$ and $\text{Gal}(KL|L) = \{\tau_1, \ldots, \tau_n\}$. Then

\[
\text{Gal}(KL|M) = \{\sigma_i\tau_j : i = 1, \ldots, m, j = 1, \ldots, n\}
\]
and restriction yields isomorphisms \( \text{Gal}(KL|K) \to \text{Gal}(L|M) \) and \( \text{Gal}(KL|L) \to \text{Gal}(K|M) \). Let 
\[
D = (\tau_i(b_j))_{i,j=1}^n \in \mathbb{Z}_K^{n \times n}, \quad a = (\tau_1(\alpha), \ldots, \tau_n(\alpha)) \in \mathbb{Z}_M^n, \quad b := \left( \sum_{j=1}^m a_{ij}c_j \right)_{i=1}^n \in L^n.
\]

Then 
\[
\det(D)^2 = \det(D^tD) = \det((\text{Tr}_{K|M}(b_i b_j))_{i,j}) = \det((\text{Tr}_{K|M}(a_{ij} b_j))_{i,j}) = d_{K|M} \]
(here \( D^t \) denotes the transpose of \( D \) and \( \text{Tr}_{K|M} \) is the trace map of \( K \) with respect to \( M \)). Moreover, \( Db = a \). Denoting the adjoint matrix of \( D \) by \( D^* \in \mathbb{Z}_K^{n \times n} \) we obtain \( \det(D)b = D^*Db = D^*a \). The right hand side is an integral vector and so must be the left hand side. It follows that 
\[
d_{K|M}a_{ij} = \det(D)^2 a_{ij} \in \mathbb{Z}_M \subseteq \mathbb{Z}_K
\]
for all \( i, j \). Now by [9, Corollary III.2.10], we have 
\[
d_{K|M} \mid d_{K}:M \mid M (d_{K|M}) = d_{K}/d_{M} \mid M = d_{K}/d_{M} \mid M a_{ij} \in \mathbb{Z}_M
\]
and therefore \( \gcd(d_{K}, d_{L}) \mid d_{M} a_{ij} \in \mathbb{Z}_M \). Hence, we derive 
\[
\gcd(d_{K}, d_{L}) \mid d_{M} \alpha \in \mathbb{Z}_K \mathbb{Z}_L
\]
as desired. \( \square \)

It is well-known that \( \mathbb{Z}_{Q_n} = \mathbb{Z}[\zeta] \) for every primitive \( n \)-th root of unity \( \zeta \). We also need the following refinements.

**Proposition 3** (Leopoldt, see [12 Proposition 6.1]). Let \( K \) be a number field contained in \( Q_n \). Then \( \mathbb{Z}_K \) is generated as abelian group by the traces
\[
\sum_{\sigma \in \text{Gal}(K(\zeta)|K)} \sigma(\zeta)
\]
of \( n \)-th roots of unity \( \zeta \).

**Lemma 4.** Every subfield of \( \mathbb{Q}_{2^n} \) has the form \( \mathbb{Q}(\xi) \) where \( \xi \in \{\zeta, \zeta \pm \bar{\zeta}\} \) and \( \zeta \) is a \( 2^n \)-th root of unity. The inclusion of subfields is given as follows

```
Q(\zeta)
  / \nQ(\zeta + \bar{\zeta}) Q(\zeta^2) Q(\zeta - \bar{\zeta})
     / \     / \     / \\
Q(\zeta^2 + \bar{\zeta}^2) Q(\zeta^4) Q(\zeta^2 - \bar{\zeta}^2)
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Q(\sqrt{2}) Q(\sqrt{-1}) Q(\sqrt{-2})
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If $\xi = \zeta \pm \zeta$, then the elements $1$ and $\zeta^k + (\pm \zeta)^k$ with $k = 1, \ldots, 2^{n-1} - 1$ generate $\mathbb{Z}_K$ as an abelian group.

Proof. If $n \leq 2$, then $K \in \{\mathbb{Q}, \mathbb{Q}_4\}$ and the claim holds with $\xi = \zeta \in \{1, \sqrt{-1}\}$. Hence, let $n \geq 3$. By induction on $n$, we may assume that $K \not\subseteq \mathbb{Q}_{2n-1}$ and $\xi$ is a primitive $2^n$-th root of unity. The subfields of $\mathbb{Q}_{2^n}$ correspond via Galois theory to the subgroups of the Galois group

$$\mathcal{G} := \text{Gal}(\mathbb{Q}_{2^n}/\mathbb{Q}) \cong (\mathbb{Z}/2^n\mathbb{Z})^\times \cong C_2 \times C_{2^{n-2}}.$$ 

The involutions of $\mathcal{G}$ are $\alpha : \zeta \mapsto \zeta^{-1} = \overline{\zeta}$, $\beta : \zeta \mapsto \zeta^{-1} + 2^{n-1} = -\overline{\zeta}$ and $\gamma : \zeta \mapsto \zeta^{1+2^{n-1}} = -\zeta$. Since $K \not\subseteq \mathbb{Q}_{2n-1} = \mathbb{Q}_{2n}$, we must have $\text{Gal}(\mathbb{Q}_{2^n}/\mathbb{K}) \in \{\langle \alpha \rangle, \langle \beta \rangle\}$, i.e. $\mathbb{K} = \mathbb{Q}(\zeta \pm \overline{\zeta})$.

As remarked above, $1, \zeta, \ldots, \zeta^{2^n-1}$ is a $\mathbb{Z}$-basis of $\mathbb{Q}_{2^n}$. Hence, every $x \in \mathbb{Z}_K$ can be written in the form

$$x = \sum_{k=0}^{2^{n-1} - 1} a_k \zeta^k$$

with $a_0, \ldots, a_{2^{n-1} - 1} \in \mathbb{Z}$. Since $x$ is invariant under $\alpha$ or $\beta$, we obtain $a_k = (-1)^k a_{2^n - 1 - k}$ for $k = 1, \ldots, 2^{n-1} - 1$. Hence,

$$x = a_0 + \sum_{k=1}^{2^{n-2} - 1} a_k (\zeta^k + (\pm \zeta)^k)$$

and the second claim follows. \qed

**Proposition 5** ([8 Theorem 3.11]). Let $G$ be a finite group and $g \in G$. Then the natural map

$$N_G(\langle g \rangle)/C_G(g) \rightarrow \text{Gal}(\mathbb{Q}_{\langle g \rangle}/\mathbb{Q}(g))$$

is an isomorphism.

### 3 General results

We start our investigation with the “column fields” $\mathbb{Q}(g)$. Since products of characters are characters, we have $Z[g] = \sum_{\chi \in \text{Irr}(G)} Z\chi(g)$.

**Proposition 6.** For every finite group $G$ and $g \in G$ we have

$$|N_G(\langle g \rangle)/\langle g \rangle|Z_{\mathbb{Q}(g)} \subseteq Z[g].$$

Proof. Let $n := |\langle g \rangle|$ and $K := \mathbb{Q}(g) \subseteq \mathbb{Q}_n$. By Proposition 3, $Z_K$ is generated by the traces

$$\xi := \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{K})} \sigma(\zeta)$$

of $n$-th roots of unity $\zeta$. Let $\psi$ be a character of $\langle g \rangle$ such that $\psi(g) = \xi \in K$. Then by Proposition 5, it follows that

$$Z[g] \ni (\psi^G)(g) = \frac{1}{|\langle g \rangle|} \sum_{x \in N_G(\langle g \rangle)} \psi(g^x) = |N_G(\langle g \rangle)/\langle g \rangle|\xi.$$

This implies $|N_G(\langle g \rangle)/\langle g \rangle|Z_K \subseteq Z[g]$. \qed

The following consequence implies Theorem A.
Corollary 7. For every finite group $G$ there exists $e \in \mathbb{N}$ such that

$$|G|^e \mathbb{Z}_{Q(G)} \subseteq \mathbb{Z}[G].$$

Proof. Clearly, $Q(G) = \prod_{g \in G} Q(g)$. By Proposition 1 the discriminants of the fields $Q(g)$ for $g \in G$ divide $|G|^{|G|}$. Hence, Proposition 2 and Proposition 6 imply

$$|G|^e \mathbb{Z}_{Q(G)} \subseteq |G|^{|G|} \prod_{g \in G} \mathbb{Z}[g] \subseteq \mathbb{Z}[G]$$

for some (large) $e \in \mathbb{N}$. □

For specific groups one can estimate the exponent $e$ in Corollary 7 by using the full strength of Propositions 1 and 2. For nilpotent groups $G$ we will prove next that $e$ can be taken to be 1.

4 Nilpotent groups

Lemma 8. Let $G$ and $H$ be finite groups of coprime order. Let $K := Q(G)$ and $L := Q(H)$. Then $Q(G \times H) = KL$, $Z_{KL} = Z_K Z_L$ and $Z|G \times H| = Z[G]Z[H].$

Proof. Since $\text{Irr}(G \times H) = \text{Irr}(G) \times \text{Irr}(H)$, it is clear that $Q(G \times H) = KL$ and

$$Z[G \times H] = \left\{ \sum_{i=1}^{n} x_i y_i : n \in \mathbb{N}, x_1, \ldots, x_n \in Z[G], y_1, \ldots, y_n \in Z[H] \right\} = Z[G]Z[H].$$

Since $K \subseteq Q[G]$ and $L \subseteq Q[H]$, the discriminants $d_K$ and $d_L$ are coprime according to Proposition 1. By Proposition 2 we obtain $Z_{KL} = Z_K Z_L$. □

In the situation of Lemma 8 it is easy to determine $Z_{KL}/Z[G \times H]$ from the elementary divisors of $Z_K/Z[G]$ and $Z_L/Z[H]$. For instance, if $Z_K/Z[G]$ has elementary divisors 1, 2, 4 (in particular, $Z_K$ has rank 3) and $Z_L/Z[L]$ has elementary divisors 1, 3, then

$$Z_{KL}/Z[G \times H] \cong C_2 \times C_4 \times C_3 \times C_6 \times C_{12} \cong C_2 \times C_6 \times C_{12}^2.$$

The following is a special case of Theorem B.

Proposition 9. Let $G$ be a nilpotent group of odd order and let $p_1, \ldots, p_n$ be the prime divisors of $|G|$. Then

$$|G|Z_{Q(G)} \subseteq qZ[G]$$

where $q := \prod_{i=1}^{n} \min\{p_i^3, |G|_{p_i}\}$.

Proof. We may write $G = P_1 \times \ldots \times P_n$ with Sylow subgroups $P_1, \ldots, P_n$. By Lemma 8 it follows that

$$|G|Z_{Q(G)} = |P_1|Z_{Q(P_1)} \cdots |P_n|Z_{Q(P_n)}.$$

Thus, we may assume that $G$ is a non-abelian $p$-group for some odd prime $p$. In particular, $|G| \geq p^3$. The Galois group of $Q[G]$ (and therefore of every subfield) is cyclic. By Proposition 5, $\text{Gal}(Q[G]|Q(g))$ is a cyclic $p$-group for every $g \in G$. Hence, the fields $Q(g)$ are all cyclotomic and therefore they are totally ordered. In particular, there exists $g \in G$ such that $K := Q(G) = Q(g)$. By Proposition 6 it
follows that \( N\mathbb{Z}_K \subseteq \mathbb{Z}[G] \) where \( N := |\mathbb{N}_G(\langle g \rangle)|/\langle g \rangle \). If \( N \leq |G|/p^3 \), then we are done. So we may assume that \( N \geq |G|/p^2 \). If \( Q(G) = \mathbb{Q}_p \), then \( \mathbb{Z}_K = \mathbb{Z}[\lambda] \subseteq \mathbb{Z}[G] \) for any non-trivial linear character \( \lambda \in \text{Irr}(G) \). Therefore, we may assume that \( |G| \geq p^4 \). Since \( \langle g \rangle = p^2 \) and \( \mathbb{N}_G(\langle g \rangle) = C_G(g) = G \).

By Proposition 5, \( Q(g) = \mathbb{Q}(\langle g \rangle) = \mathbb{Q}(\zeta) \) for some root of unity \( \zeta \). Since the regular character of \( G \) is faithful, there exists \( \chi \in \text{Irr}(G) \) such that the restriction \( \chi_{\langle g \rangle} \) is faithful. Since \( g \in \mathbb{Z}(\chi) \), we have \( \chi(g) = \chi(1)\zeta^k \) for some integer \( k \) coprime to \( p \). Then for every \( l \geq 0 \) we also have \( \chi(g^p) = \chi(1)\zeta^{kp} \). This implies \( \chi(1)\mathbb{Z}_K \subseteq \mathbb{Z}[G] \). Since \( |G| \geq p^4 \) and \( \chi(1)^2 < |G| \), we obtain \( |G|\mathbb{Z}_K \subseteq p^3\mathbb{Z}[G] \).

The analysis of 2-groups \( G \) is more delicate, since it may happen that \( \mathbb{Q}(G) \neq \mathbb{Q}(g) \) for all \( g \in G \).

**Lemma 10.** Let \( G \) be a 2-group and \( g \in G \) such that \( \mathbb{Q}(g) \) is not a cyclotomic field. Then for every subfield \( K \) of \( \mathbb{Q}(g) \) there exists \( \chi \in \text{Irr}(G) \) such that \( K = \mathbb{Q}(\chi(g)) \).

**Proof.** We argue by induction on \( |G| \). We may assume that \( |\mathbb{Q}(g) : \mathbb{Q}| > 2 \). In particular, \( G \neq 1 \). By Lemma 4, the subfields of \( \mathbb{Q}(g) \) are totally ordered. In particular, there exists \( \chi \in \text{Irr}(G) \) such that \( \mathbb{Q}(\chi(g)) = \mathbb{Q}(g) \). Let \( Z \) be a central subgroup of \( G \) of order 2. Then \( \chi^2 \) is a character of \( G/Z \) and \( |\mathbb{Q}(\chi(g)) : \mathbb{Q}(\chi^2(g))| \leq 2 \).

Since \( \mathbb{Q}(gZ) = \mathbb{Q}(\psi(gZ) : \psi \in \text{Irr}(G/Z)) \subseteq \mathbb{Q}(g) \), we obtain \( |\mathbb{Q}(g) : \mathbb{Q}(gZ)| \leq 2 \). Since \( |\mathbb{Q}(g) : \mathbb{Q}| > 2 \), also \( \mathbb{Q}(gZ) \) is not a cyclotomic field. By induction, every proper subfield of \( \mathbb{Q}(g) \) has the form \( \mathbb{Q}(\psi(g)) \) for some \( \psi \in \text{Irr}(G/Z) \).

The cyclic group \( G = \langle g \rangle \cong C_8 \) shows the assumption on \( \mathbb{Q}(g) \) in Lemma 10 is necessary.

**Lemma 11.** Let \( G \) be a 2-group and \( g \in G \) such that \( K := \mathbb{Q}(g) \) is not a cyclotomic field. Then

\[
M\mathbb{Z}_K \subseteq 2\mathbb{Z}[G]
\]

where \( M := \max\{\chi(1) : \chi \in \text{Irr}(G)\} \).

**Proof.** By Lemma 4, there exists a primitive \( 2^n \)-th root of unity \( \zeta \) such that \( K = \mathbb{Q}(\zeta \pm \zeta^2) \). Moreover, \( \mathbb{Z}_K \) is generated by the elements 1 and \( \xi_k := \zeta^k + (\pm \zeta)^k \) with \( k = 1, \ldots, 2^{n-2} - 1 \). For every such \( k \) there exists \( \chi \in \text{Irr}(G) \) such that \( \mathbb{Q}(\chi(g)) = \mathbb{Q}(\xi_k) \) by Lemma 10. It suffices to show that \( \chi(1)\xi_k \) is an integral linear combination of the Galois conjugates of \( 2\chi(g) \). To this end, we may assume that \( k = 1 \) and \( \xi := \xi_1 \).

Let \( d := \chi(1) \) and note that \( d > 1 \) since \( \mathbb{Q}(\chi(g)) = \mathbb{Q}(\xi) = K \) is not a cyclotomic field. There exist integers \( a_0, \ldots, a_{2^{n-1}-1} \) such that

\[
\chi(g) = \sum_{i=0}^{2^{n-1}-1} a_i \zeta^i = a_0 + \sum_{i=1}^{2^{n-2}-1} a_i \xi_i.
\]

Since \( \chi(g) \) is a sum of \( d \) roots of unity, \( |a_0| + \ldots + |a_{2^{n-1}-1}| \leq d \) (it may happen that other roots, even of higher order than \( 2^n \), cancel each other out). The Galois group \( G \) of \( \mathbb{Q}_2^n \) acts on \( K \) and on \( \{\psi(g) : \psi \in \text{Irr}(G)\} \). Let \( \sigma \in G \) such that \( \sigma(\xi) = \xi^{1+2^{n-1}} = -\zeta \). Then

\[
\omega := \sum_{i=0}^{s-1} b_i \xi_{2i+1} = \chi(g) - \sigma(\chi(g)) \in \mathbb{Z}[G]
\]
where \( s := 2^{n-3} \) and \( b_i := 2a_{2i+1} \) for \( i = 0, \ldots, s-1 \). Let \( \tau \in \mathcal{G} \) such that \( \tau(\zeta) = \zeta^5 \). Note that \( \tau^5(\xi) = \sigma(\xi) = -\xi \). We may relabel the elements \( b_i \) in a suitable order such that

\[
\omega = \sum_{i=0}^{s-1} b_i \tau^i(\xi).
\]

Next we consider

\[
\gamma := \sum_{i=0}^{s-1} b_i \zeta^{4i} \in \mathbb{Z}_{Q_{2s}}.
\]

It is known that the prime 2 is fully ramified in \( \mathbb{Q}_{2s} \). More precisely, \( (2) = (\zeta^4 - 1)^s \) and \( (\zeta^4 - 1) \) is a prime ideal (see [9, Lemma I.10.1]). Let \( e \) be the 2-part of \( \gcd(b_0, \ldots, b_{s-1}) \). Then \( \frac{1}{2^n} \gamma \) is an algebraic integer, but \( \frac{1}{2^n} \gamma \) is not. Hence, \( \frac{1}{2^n} \gamma = (\zeta^4 - 1)^t p \) where \( t < s \) and \( p \) is an ideal of \( \mathbb{Z}_{Q_{2s}} \), coprime to \( (\zeta^4 - 1) \). This implies the existence of some \( \delta \in \mathbb{Z}_{Q_{2s}} \) such that \( \gamma \delta = 2em \) where \( m \) is an odd integer. We write \( \delta = \sum_{i=0}^{s-1} c_i \zeta^{4i} \) with \( c_0, \ldots, c_{s-1} \in \mathbb{Z} \). Then

\[
2em = \gamma \delta = \sum_{i,j=0}^{s-1} b_i c_j \zeta^{4(i+j)}.
\]

Comparing coefficients yields

\[
\sum_{i+j=t} b_i c_j - \sum_{i+j=s-t} b_i c_j = \begin{cases} \frac{2em}{2} & \text{if } t = 0, \\ 0 & \text{if } 1 \leq t \leq s - 1. \end{cases}
\]

Finally we compute

\[
\sum_{j=0}^{s-1} c_j \tau^j(\omega) = \sum_{i,j=0}^{s-1} b_i c_j \tau^{i+j}(\xi) = \sum_{i=0}^{s-1} \left( \sum_{j=t} b_i c_j - \sum_{i+j=s-t} b_i c_j \right) \tau^t(\xi) = 2em \xi.
\]

Hence, \( 2em \xi \in \mathbb{Z}[G] \). By Proposition 6 we also have \( |G| \xi \in |G|Z_{Q(G)} \subseteq \mathbb{Z}[G] \). Therefore,

\[
2e \xi = \gcd(2em, |G|) \xi \in \mathbb{Z}[G].
\]

Note that

\[
e \leq \sum_{i=0}^{s-1} |b_i| = \sum_{i=0}^{2n-2-1} |a_{2i+1}| \leq \sum_{i=0}^{2n-1-1} |a_i| \leq d.
\]

(4.1)

Suppose that \( d \xi \notin 2\mathbb{Z}[G] \). Then \( d \leq 2e \) (keep in mind that \( d \) and \( e \) are 2-powers). If the first inequality in (4.1) is strict, then \( 2e \leq \sum_{i=0}^{s-1} |b_i| \) since the right hand side is divisible by \( e \). Thus, in any case one of the inequalities in (4.1) is an equality. If \( e = \sum_{i=0}^{s-1} |b_i| \), then \( e = |b_i| \) and \( \omega = b_i \tau^i(\xi) \) for some \( i \in \{0, \ldots, s-1\} \). Then we obtain \( e \xi \in \mathbb{Z}[G] \). If, on the other hand, \( \sum_{i=0}^{s-2} |a_{2i+1}| = \sum_{i=0}^{2n-1} |a_i| \), then \( \omega = 2 \chi(g) \) and \( e \xi \in \mathbb{Z}[G] \) by the computation above. Hence in any case we deduce that \( d = e \). But now \( \chi(g) = a_{2i+1} \tau^i(\xi) \) and \( d = 2|a_{2i+1}| \). This implies \( d \xi \in 2\mathbb{Z}[G] \) as desired.

The next result is a restatement of Theorem B.

**Theorem 12.** For every nilpotent group \( G \neq 1 \) the exponent of \( \mathbb{Z}_{Q(G)}/\mathbb{Z}[G] \) is a proper divisor of \( |G| \).
Proof. By Proposition 9 and its proof, we may assume that $G$ is a 2-group. By Lemma 4, $Q(G) = Q(\xi)$ where $\xi \in \{\zeta, \zeta \pm \zeta\}$ and $\zeta$ is a primitive $2^n$-th root of unity. If there exists $g \in G$ such that $Q(G) = Q(g)$, then we obtain $|G| Z_{Q(G)} \subseteq Z[G]$ from Proposition 6. Otherwise we have $n \geq 3$, $Q(G) = Q(\zeta)$ and there exists $g \in G$ such that $K := Q(g) = Q(\zeta \pm \zeta)$. Moreover, there exist $h \in G$ and $\psi \in \text{Irr}(G)$ such that

$$\psi(h) = \sum_{i=0}^{2^n-1-1} a_i \zeta^i \notin K$$

where $a_0, \ldots, a_{2^n-1-1} \in \mathbb{Z}$. Lemma 11 shows that $MZ_K \subseteq 2\mathbb{Z}[G]$ where $M := \max \{\chi(1) : \chi \in \text{Irr}(G)\}$. It suffices to prove $|G|^k \zeta \in 2\mathbb{Z}[G]$ for every $k \in \mathbb{Z}$.

Let $\sigma$ be the Galois automorphism of $Q(\zeta)$ such that $\sigma(\zeta) = \pm \zeta$. Since $\psi(h) \notin K$, we have $\psi(h) \neq \sigma(\psi(h))$. We consider

$$\omega := \psi(h) - \sigma(\psi(h)) = \sum_{i=1}^{2^n-1} b_i \zeta^i \in \mathbb{Z}[G]$$

where $b_i := a_i \pm a_{2^n-1-i}$ if $i$ is odd and $b_i := a_i + a_{2^n-1-i}$ otherwise. Let $e$ be the 2-part of $\gcd(b_0, \ldots, b_{2^n-1-1})$. As in the proof of Lemma 11, there exists an odd integer $m$ such that $2em\omega^{-1}$ is an algebraic integer. Hence for every $k \in \mathbb{Z}$,

$$2em\frac{\zeta^k - \sigma(\zeta)^k}{\omega} \in Z_{Q(\zeta)} \cap Q(\zeta)^m = Z_K.$$ 

We conclude that

$$2em \zeta^k = em M(\zeta^k + \sigma(\zeta)^k) + em \frac{\zeta^k - \sigma(\zeta)^k}{\omega} \omega \in Z[G].$$

By Corollary 7, there exists $s \in \mathbb{N}$ such that $|G|^s \zeta^k \in Z[G]$. Hence,

$$2eMZ_{Q(G)} \subseteq \gcd(2emM, |G|^s) Z[\zeta] \subseteq Z[G].$$

If $b_i \neq 0$ for some $i \neq 2^{n-2}$, then $e \leq |b_i| \leq |a_i| + |a_{2^n-1-i}| \leq \psi(1)$. Otherwise, $\omega = 2a_{2^n-2} \sqrt{-1}$. If, in this case, there exists some $a_i \neq 0$ with $i \neq 2^{n-2}$, then $e \leq |b_{2^n-2}| < 2|a_{2^n-2}| + |a_i| \leq 2\psi(1)$. Since $e$ and $\psi(1)$ are 2-powers, we still have $e \leq \psi(1)$. Finally, let $\psi(h) = a_{2^n-2} \sqrt{-1} = \omega/2$. Then we may repeat the calculation above with $\psi(h)$ instead of $\omega$ in order to obtain $eMZ_{Q(G)} \subseteq Z[G]$ where $e \leq 2\psi(1)$. In summary,

$$2M\psi(1)Z_{Q(G)} \subseteq Z[G]$$

in every case. Since $|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2$, we have $2M\psi(1) \leq 2M^2 \leq |G|$. If $2M\psi(1) = |G|$, then $\psi(1) = M$ and $\psi$ is the only irreducible character of degree $M$. But then $\psi$ is rational and we derive the contradiction $\psi(h) \notin K$. Therefore, $2M\psi(1) < |G|$ and the claim follows. \qed

5 Examples

We show first that Proposition 9 is sharp in the following sense.

**Proposition 13.** For every prime $p$ and every integer $n \geq 1$ there exists a group $P$ of order $p^{2n+2}$ and exponent $p^2$ such that $K := Q(P) = Q_{p^2}$ and $Z_K/Z[P] \cong C_{p^n}^{(p-1)^2}$.
Proof. Let $P$ be the central product of an extraspecial group $E$ of order $p^{2n+1}$ (it does not matter which one) and a cyclic group $C = \langle c \rangle$ of order $p^2$. The irreducible characters of $P$ are those of $E \times C$ which agree on $Z(E) = \langle z \rangle$ and $\langle c^p \rangle$. It is well-known that $\text{Irr}(E)$ consists of $p^{2n}$ linear character and $p - 1$ faithful characters $\chi_1, \ldots, \chi_{p-1}$ of degree $p^n$ (see [3], Example 7.6(b)) for instance. Since $E/E'$ is elementary abelian, the linear character values of $E$ and also of $P$ generate $Q_p$. Let $\zeta$ be a primitive $p^2$-th root of unity. After relabeling, we may assume that $\chi_i(z) = p^n \zeta^i$ and $\chi_i(g) = 0$ for $g \in E \setminus Z(E)$ and $i = 1, \ldots, p - 1$. Hence, the non-linear character of $P$ take the values 0 and $p^n \zeta^i$ for $i \in \mathbb{Z}$. This shows $K = Q_{p^2}$ and

$$Z[G] = Z[\zeta^p, p^n \zeta^i : \gcd(i, p) = 1].$$

Since the elements $1, \zeta, \zeta^2, \ldots, \zeta^{p(p-1)}$ form a $\mathbb{Z}$-basis of $Z_K$, the claim follows easily. \[\square\]

Proposition 13 already shows that neither $|\langle g \rangle| Z_{Q(g)} \subseteq Z[G]$ nor $\exp(G) Z_{Q(G)} \subseteq Z[G]$ is true in general. Also the dual statements, motivated by Lemma 11, $\chi(1) Z_{Q(\chi)} \subseteq Z[G]$ and

$$\text{lcm}\{\chi(1) : \chi \in \text{Irr}(G)\} Z_{Q(G)} \subseteq Z[G]$$

do not always hold. Using GAP [5] and MAGMA [1] we computed the following example: The group

$$G = \text{SmallGroup}(48, 3) \cong C_4^2 \rtimes C_3$$
gives $K := Q(G) = Q_{12}$ and $Z[G] = Z[2\sqrt{-1}, \zeta]$ where $\zeta$ is a primitive third root of unity. Hence, $Z_K/Z[G] \cong C_2^5$, but $\text{lcm}\{\chi(1) : \chi \in \text{Irr}(G)\} = 3$.

For a single entry $\omega = \chi(g)$ of the character table of $G$ the group $Z_{Q(\omega)}/Z[\omega]$ usually has nothing to do with $G$. For instance, $G = D_{26} \times C_3$ has a character value $\omega$ such that $Z_{Q(\omega)}/Z[\omega]$ is cyclic of order $5^2 \cdot 157 \cdot 547$. It is not hard to show that every algebraic integer of an abelian number field occurs in the character table of some finite group (see proof of [3], Theorem 6).

For 2-groups the gap between $G$ and $Z_K/Z[G]$ can get even bigger than in Proposition 13. The exponent and the largest character degree of $G = \text{SmallGroup}(2^9, 6480850)$ is 8, but

$$Z_K/Z[G] \cong C_64 \times C_8 \times C_4.$$

Similarly, the group $G = \text{SmallGroup}(2^9, 60860)$ yields $|Z_K/Z[G]| = 2^{33}$.

For non-nilpotent groups, the arguments from the last section drastically fail as our next example shows. Let

$$G = \text{SmallGroup}(240, 13) \cong C_{15} \rtimes D_{16}$$

where the dihedral group $D_{16}$ acts with kernel $D_{16}'$ (commutator subgroup) on $C_{15}$. Then $K = Q_{120}$ and $2Z_{Q(g)} \subseteq Z[G]$ for all $g \in G$, but

$$Z_K/Z[G] \cong C_{120}^2 \times C_60^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}.$$

Now we consider some simple groups which support Conjecture C.

Proposition 14.

(i) Let $G = \text{PSL}(2, q)$ for some prime power $q \neq 1$. Then $Z_{Q(G)} = Z[G]$.

(ii) Let $G = \text{Sz}(q)$ for $q \geq 8$ an odd power of 2. Then $Z_{Q(G)}/Z[G] \cong C_2^a$ where $a = \varphi((q^2 + 1)(q - 1))/32$. 

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Proof.

(i) Assume first that \( q \geq 5 \) is odd. Then \( G \) has two irreducible characters taking only rational values and three families \( \chi_i, \theta_j, \eta_k \) taking (potentially) irrational values (see [3 Theorem 38.1] for instance). Let \( \zeta \) be a primitive \( n \)-th root of unity and let \( \epsilon := (-1)^{(q-1)/2} \). Set \( r := (q-1)/2 \) and \( s := (q+1)/2 \). Then the values of the \( \chi_i \) lie in \( G := \mathbb{Q}(\zeta + \zeta^r) \) and they contain the integral basis from Lemma 1. Similarly the values of the \( \theta_j \) generate the ring of integers of \( L := \mathbb{Q}(\zeta + \zeta^r) \). Finally, the values of the \( \eta_k \) generate the ring of integers of \( M := \mathbb{Q}(\sqrt{q}) \). The discriminants of \( K, L \) and \( M \) are pairwise coprime by Proposition 1. Hence, by Proposition 2 we have

\[
\mathbb{Z}[G] = \mathbb{Z}_K \mathbb{Z}_L \mathbb{Z}_M = \mathbb{Z}_{KLM} = \mathbb{Z}_{\mathbb{Q}(G)}.
\]

For \( q \) a power of 2, the result follows for \( \mathrm{PSL}(2, q) = \mathrm{SL}(2, q) \) with a similar argument from [3 Theorem 38.2].

(ii) The character table of the group \( G = \mathrm{Sz}(q) \) was determined by Suzuki in [11 Theorem 13]. We use the names of characters in that theorem. Set \( r := q - 1, s := q + \sqrt{2q} + 1 \) and \( t := q - \sqrt{2q} + 1 \) and note that these odd numbers are pairwise coprime. Observe that \( \mathbb{Q}(G) = KLMN \), the composita of the fields \( K = \mathbb{Q}(X_1) = \mathbb{Q}(\zeta, \zeta^r) \), \( L = \mathbb{Q}(Y_1) = \mathbb{Q}(\zeta^s, \zeta^{s^2} + \zeta^{s^4}) \), \( M = \mathbb{Q}(Z_1) = \mathbb{Q}(\zeta^s, \zeta^t, \zeta^{t^2} + \zeta^{t^4}) \) and \( N = \mathbb{Q}(W_1) = \mathbb{Q}(\sqrt{1}) \), which have pairwise coprime discriminant by Proposition 1. Now \( \mathbb{Z}_K = \mathbb{Z}[\zeta + \zeta^r] = \mathbb{Z}[X_1] \) and \( \mathbb{Z}_L = \mathbb{Z}[\zeta + \zeta^s + \zeta^{s^2} + \zeta^{s^4}] = \mathbb{Z}[Y_1] \) and similarly for \( \mathbb{Z}_1 \). Further \( \mathbb{Z}[W_1] = \mathbb{Z}[W_2] = \mathbb{Z}[\sqrt{-1}] \), hence \( \mathbb{Z}_N/\mathbb{Z}[W_1] \) has elementary divisors 1 and 2. Similarly to the remark following Lemma 8 we can conclude that \( \mathbb{Z}_{KLMN}/\mathbb{Z}[G] \) has elementary divisors 1 and 2 each with multiplicity

\[
[KLM : \mathbb{Q}] = \frac{\varphi(r) \varphi(s) \varphi(t)}{4} = \frac{\varphi((q^2 + 1)(q - 1))}{32}.
\]

A minimal simple group (i.e. a simple group with all proper subgroups solvable) is isomorphic to some \( \mathrm{PSL}(2, q) \), to some \( \mathrm{Sz}(2^{2f+1}) \) or to \( \mathrm{PSL}(3, 3) \). For the last group one can check easily that \( \mathbb{Z}_{\mathbb{Q}(G)} = \mathbb{Z}[G] \). Hence, for minimal simple groups \( G \), the exponent of \( \mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G] \) is at most 2.

Finally we compute \( \mathbb{Z}[G] \) for the alternating group \( G = A_n \) of (small) degree \( n \). Let \( g \in G \) be non-rational. Then there exists a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n \) into pairwise distinct odd parts such that

\[
\mathbb{Z}[g] = \mathbb{Z}[(1 + \sqrt{d})/2]
\]

where \( d = (-1)^{(n-k)/2}\lambda_1 \cdots \lambda_k \equiv 1 \pmod{4} \) (see [7 Theorem 2.5.13] for instance). We may write \( \sqrt{d} = e \sqrt{d'} \) such that \( d' \) is squarefree. Let \( K := \mathbb{Q}(g) = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{d'}) \). Then

\[
\mathbb{Z}_K = \mathbb{Z}[(1 + \sqrt{d'})/2]
\]

and we obtain \( e \mathbb{Z}_K \subseteq \mathbb{Z}[g] \). Note that \( e^2 \mid d \mid n! = 2|G| \). Since the discriminant of \( K \) is \( d' \equiv 1 \pmod{2} \), it follows that \( |\mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G]| \) is odd by Proposition 2. It seems fairly difficult to determine the precise structure of \( \mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G] \). For \( n \geq 25 \), a theorem by Robinson–Thompson [10] states that

\[
\mathbb{Q}(G) = \mathbb{Q}(\sqrt{p^*} : p \text{ odd prime}, n - 2 \neq p \leq n)
\]

where \( p^* := (-1)^{(n-2)/2} p \). By Proposition 2 \( \mathbb{Z}_{\mathbb{Q}(G)} \) is generated as abelian group by all products of the elements \((1 + \sqrt{p^*})/2 \) with \( p \) as above. The following table lists the (non-trivial) elementary divisors of \( \mathbb{Z}_{\mathbb{Q}(G)}/\mathbb{Z}[G] \) for \( n \leq 31 \). In every case Conjecture C is fulfilled.
\begin{tabular}{|c|c|}
\hline
$n$ & $Z_{\mathbb{Q}(A_n)}/Z[A_n]$ \\
\hline
$\leq 11$ & 1 \\
12, 13, 14 & $3^4$ \\
15 & $3^4 \times 15^4 \times 45^4$ \\
16 & $3^4 \times 15^4$ \\
17 & $3^{12} \times 9^4 \times 45^4 \times 135^4$ \\
18 & $3^8 \times 15^8 \times 45^8$ \\
19 & $3^8 \times 15^8$ \\
20 & $3^{36} \times 9^{12} \times 45^{32} \times 10395^{28} \times 31185^4$ \\
21 & $3^{36} \times 105^4 \times 315^{12}$ \\
22 & $3^{52} \times 105^8 \times 315^{52} \times 945^4$ \\
23 & $3^{64} \times 4095^{32}$ \\
24 & 1 \\
25 & $3^{32} \times 15^{32} \times 315^{32}$ \\
26 & $3^{38} \times 15^{40} \times 45^{40} \times 315^{56} \times 945^8$ \\
27 & $3^{112} \times 9^{112} \times 27^{16}$ \\
28 & $3^{96} \times 15^{80} \times 45^{48}$ \\
29 & $3^{224} \times 15^{128}$ \\
30 & $3^{128} \times 105^{128}$ \\
31 & $3^{256}$ \\
\hline
\end{tabular}

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