The stagnation-point flow and heat transfer of nanofluid over a shrinking surface in magnetic field and thermal radiation with slip effects: a stability analysis

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Abstract. A numerical study is performed to evaluate the problem of stagnation-point flow and heat transfer towards a shrinking sheet with magnetic field and thermal radiation in nanofluid. The Buongiorno’s nanofluid model is used in this study along with slip effect at boundary condition. By using non-similar transformation, the governing equations are able to be reduced into an ordinary differential equation. Then, the ordinary differential equation can be solved by using the bvp4c solver in Matlab. A linear stability analysis shows that only one solution is linearly stable otherwise is unstable. Based on the numerical results obtained, the dual solutions do exist at certain ranges in this study. Then, the stability analysis is carried out to determine which one is stable between both of the solutions.

1. Introduction

The literature has shown the dual solutions is exist especially in solving shrinking sheet problem. Miklavčič and Wang [1] were the first who claimed that the solutions are non-unique for the shrinking sheet. In that case, a stability analysis is important to check which one is stable between the two solutions. Traditionally, Merkin [2] have shown that the stable one is the first solution while the second solution is not, by solving mixed convection in a porous medium problem. Then, there has been renewed interest in stability analysis as reported by Weidman et al. [3]. Recently, a considerable literature has grown up around the theme of stability analysis. Several studies investigating stability analysis have been carried out by Postelnicu and Pop [4], Roşca and Pop [5] and Mahapatra and Nandy [6]. The aim of the current study was to resolve the stability of the existing dual solutions in the stagnation-point flow over a shrinking sheet in magnetic field and thermal radiation with included slip effects in nanofluid.

2. Problem Formulation

Let we consider a viscous, incompressible and electrically conductive fluid on steady two-dimensional stagnation-point flow toward a shrinking sheet. Magnetic field $B_0$ with is uniform is applied into the sheet. $u_n(x) = cx$ is assumed as the velocity of stretching/shrinking and $U(x) = ax$ is represent the free stream velocity. In addition, the value of $a$ and $c$ are constant. To be noted, $c > 0$
and $c < 0$ is the stretching and shrinking sheet, respectively. The effect of velocity slip and viscous dissipation also included in this study. The continuity, momentum, energy and nanoparticles equations of nanofluids equations for current study are [7]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)
\]

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + v \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho} (U - u) \quad (2)
\]

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \nu \left( \frac{\partial u}{\partial y} \right)^2 - \frac{1}{\rho_f c_p} \frac{\partial q_r}{\partial y} + \frac{(\rho c)_f}{(\rho c)_p} \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_w} \left( \frac{\partial T}{\partial y} \right)^2 \right] \quad (3)
\]

\[
\frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = D_B \frac{\partial^2 \phi}{\partial y^2} + \frac{D_T}{T_w} \frac{\partial^2 T}{\partial y^2} \quad (4)
\]

suitable boundary conditions are

\[
v = v_w, \quad u = u_w(x) = c x + \frac{\partial u}{\partial y} x, \quad T = T_w, \quad \phi = \phi_w, \quad \text{at } y = 0
\]

\[
u \rightarrow 0(u(x)) = a x, \quad T \rightarrow T_w, \quad \phi \rightarrow \phi_w \quad \text{as } y \rightarrow \infty \quad (5)
\]

where $u$ is the velocity along the $x$-direction whereas $v$ is the velocity along the $y$-directions. $v_w$ is the mass transfer velocity, $\rho_f$ is the fluid density, $T$ is the fluid temperature, $\nu$ is the kinematic viscosity, $U(x)$ is the velocity of the external fluid, $\alpha_m$ is the thermal diffusivity, $\sigma$ is the electrical conductivity of the fluid, $\phi$ is the nanoparticle volume fraction, $q_r$ is the radiative heat flux, $D_B$ is the Brownian diffusion coefficient, $D_T$ is the thermophoresis diffusion coefficient, $c_p$ is the specific heat at constant pressure and $(\rho c)_p / (\rho c)_f$ is the ratio of the effective heat capacity of the nanofluid material to the heat capacity of the ordinary fluid. According to Rosseland approximation for radiation [8], we obtain

\[
q_r = -\frac{4\sigma^*}{3k_1} \frac{\partial T^4}{\partial y} \quad (6)
\]

where $\sigma^*$ is the Stefan – Boltzmann constant and $k_1$ is the mean absorption coefficient. By assuming the temperature variation $T^4$ can be expanded in a Taylor series about $T_w$ and the higher order terms is neglected, $T^4 \approx 4T_w^3T - 3T_w^4$ is obtain. Then, the updated of equation (3) is

\[
\frac{u}{\partial x} + v \frac{\partial T}{\partial y} + \alpha_m \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_w^3}{3k_1 \rho_f c_p} \frac{\partial T}{\partial y} + \frac{(\rho c)_p}{(\rho c)_f} \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_w} \left( \frac{\partial T}{\partial y} \right)^2 \right] \quad (7)
\]

Then, the following similarity variables is introduce to solve equations (1) – (4)

\[
\psi = \sqrt{\alpha v x f (\eta)}, \quad \eta = \sqrt{\frac{u}{v}}, \quad \theta(\eta) = \frac{T - T_w}{T_w - T_\infty}, \quad S(\eta) = \frac{\phi - \phi_w}{\phi_w - \phi}, \quad \text{where } \psi \text{ is known as stream function and always to be define as } u = \psi \frac{\partial \psi}{\partial x} \text{ and also } v = -\frac{\partial \psi}{\partial x}.
\]

Then, substituting equations (8) into equations (1) – (4), we obtained the equations as below

\[
f^m + f f^* - f^2 + 1 + M(1 - f^*) = 0 \quad (9)
\]

\[
\frac{1}{Pr} \left( 1 + 4 \frac{R}{3} \right) \theta^* + f \theta' + Ec f \theta^2 + Nb \theta' S' + Ni \theta^2 = 0 \quad (10)
\]
\[ S' + \text{Le} S + \frac{Nt}{Nb} \theta' = 0 \]  

(11)

where \( M = \sigma B_0^2 / \rho f a \) is the magnetic parameter, \( \text{Pr} = v / \alpha_m \) is the Prandtl number, \( R = 4\delta T_x^3 / k_1 \kappa \) is the thermal radiation parameter, \( Nt = (\rho c)_p / (\rho c)_f D_f (T_w - T_x) / v T_w \) is the thermophoresis parameter, \( Nb = (\rho c)_p / (\rho c)_f D_b (\phi_w - \phi_e) / v \) is the Brownian motion parameter, \( Ec = a^2 x^2 / c_p (T_w - T_x) \) is the Eckert number and \( Le = v / D_B \) is the Lewis number. Then, the boundary conditions (5) reduce to the forms:

\[
\begin{align*}
 f(0) &= s, \quad f'(0) = \alpha + \delta f^*(0), \quad \Theta(0) = 1, \quad S(0) = 1, \quad f'(\infty) \to 1, \quad \Theta(\infty) \to 0, \quad S(\infty) \to 0
\end{align*}
\]

(12)

with \( \delta = (a / v)^{1/2} L \) is the velocity slip parameter, \( \alpha \) is the ratio of the stretching/shrinking velocity and \( s = -v_w / \sqrt{av} \) is the suction parameter. The physical quantities of interest are the skin friction or the wall shear stress coefficient \( C_f \), the local Nusselt number \( N_u \) and local Sherwood number \( S_h \) are

\[
C_f = \frac{\mu}{\rho f U^2(x)} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad N_u = \frac{x}{T_w - T_x} \left( -\frac{\partial T}{\partial y} \right)_{y=0}, \quad S_h = \frac{x}{(\phi_w - \phi_e)} \left( -\frac{\partial \phi}{\partial y} \right)
\]

(13)

using variables from equation (8), which become

\[
\text{Re}_x^{1/2} C_f = f^*(0), \quad \text{Re}_x^{-1/2} N_u = -\theta'(0), \quad S_h \text{Re}_x^{-1/2} = -S(0)
\]

(14)

where \( \text{Re}_x = U(x) x / v \) is the local Reynolds number.

3. Stability analysis

There is a growing body of literature that recognizes the importance of stability analysis since the dual solutions exist. Motivated by paper Weidman et al. [3], a dimensionless time variable which is defined as \( \tau \) has to be introduced. Basically, the use of \( \tau \) is linked to an initial value problem. Then, we have

\[
\begin{align*}
 u &= a x f' (\eta, \tau), \quad v = -\sqrt{a x} f (\eta, \tau), \quad \psi = \sqrt{a x} f (\eta, \tau), \\
 \eta &= \sqrt{\frac{a}{v}} y, \quad \Theta(\eta, \tau) = \frac{T - T_x}{T_w - T_x}, \quad S(\eta, \tau) = \frac{\phi - \phi_e}{\phi_w - \phi_e}, \quad \tau = a t
\end{align*}
\]

(15)

so that (2) - (4) will be written as

\[
\begin{align*}
 \frac{\partial^3 f}{\partial \eta^3} + f \frac{\partial^2 f}{\partial \eta^2} - \left( \frac{\partial f}{\partial \eta} \right)^2 + M \left( 1 - \frac{\partial f}{\partial \eta} \right) - \frac{\partial^2 f}{\partial \eta \partial \tau} &= 0 \\
 \frac{1}{\text{Pr}} \left( 1 + \frac{4 R}{3} \right) \frac{\partial^4 \Theta}{\partial \eta^4} + f \frac{\partial^4 \Theta}{\partial \eta^2} + \frac{\partial \Theta}{\partial \eta} + Ec \left( \frac{\partial^2 f}{\partial \eta^2} \right)^2 + Nb \frac{\partial \Theta}{\partial \eta} + Nt \left( \frac{\partial \Theta}{\partial \eta} \right)^2 - \frac{\partial \Theta}{\partial \tau} &= 0 \\
 \frac{\partial^2 S}{\partial \eta^2} + \text{Le} f \frac{\partial S}{\partial \eta} + Nt \frac{\partial^2 \Theta}{\partial \eta^2} - \frac{\partial S}{\partial \tau} &= 0
\end{align*}
\]

(16)

(17)

(18)

boundary conditions are

\[
\begin{align*}
 f(0, \tau) &= s, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \alpha + \delta \frac{\partial^2 f}{\partial \eta^2}(0, \tau), \quad \Theta(0, \tau) = 1, \quad S(0, \tau) = 1
\end{align*}
\]
\[ \frac{\partial f}{\partial \eta}(\eta, \tau) \to 1, \quad \theta(\eta, \tau) \to 0, \quad S(\eta, \tau) \to 0 \text{ as } \eta \to \infty \quad (19) \]

In order to analyze the stability for the steady flow solution \( f(\eta) = f_0(\eta), \quad \theta(\eta) = \theta_0(\eta) \) and \( S(\eta) = S_0(\eta) \) comply with the boundary – value problem (9)-(12), we write \[ f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} F(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma \tau} G(\eta, \tau), \quad S(\eta, \tau) = S_0(\eta) + e^{-\gamma \tau} H(\eta, \tau) \quad (20) \]

where \( F(\eta, \tau) \) is small relative to \( f_0(\eta) \), \( G(\eta, \tau) \) is small relative to \( \theta_0(\eta) \), \( H(\eta, \tau) \) is small relative to \( S_0(\eta) \) and \( \gamma \) is an unknown eigenvalue. Substituting (20) into (16) - (18), the result is

\[ \frac{\partial^3 F}{\partial \eta^3} + f_0 \frac{\partial^2 F}{\partial \eta^2} + f_0' F + (\gamma - M - 2 f_0') \frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial \eta \partial \tau} = 0 \quad (21) \]

\[ \frac{1}{\text{Pr}} \left( 1 + \frac{4 R}{3} \right) \frac{\partial^2 G}{\partial \eta^2} + (f_0 + N b S_0' + 2 N t \theta_0') \frac{\partial G}{\partial \eta} + \theta_0' F + 2 E c f_0 \frac{\partial^3 F}{\partial \eta^3} + N b \theta_0' H + \gamma G - \frac{\partial \theta}{\partial \tau} = 0 \quad (22) \]

\[ \frac{\partial^2 H}{\partial \eta^2} + \text{Le} f_0 \frac{\partial H}{\partial \eta} + \text{Le} S_0' F + \frac{N t \partial^2 G}{N b} + \gamma G - \partial S \frac{\partial}{\partial \tau} = 0 \quad (23) \]

follows with the boundary conditions

\[ F(0, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(0, \tau) = \delta \frac{\partial^2 F}{\partial \eta^2}(0, \tau), \quad G(0, \tau) = 0, \quad H(0, \tau) = 0, \]

\[ \frac{\partial F}{\partial \eta}(\infty, \tau) \to 0, \quad G(\infty, \tau) \to 0, \quad H(\infty, \tau) \to 0 \quad (24) \]

The solutions \( f(\eta) = f_0(\eta), \quad \theta(\eta) = \theta_0(\eta) \) and \( S(\eta) = S_0(\eta) \) of the steady solutions (9) - (11) are obtained by setting \( \tau = 0 \). Therefore, \( F = F_0(\eta), \quad G = G_0(\eta) \) and \( H = H_0(\eta) \) in (21) - (23) identify initial increment of the solution (20). In this case, we have to deal with the linear eigenvalue problem

\[ F_0'' + f_0 F_0' + f_0' F_0 + (\gamma - M - 2 f_0') F_0 = 0 \quad (25) \]

\[ \frac{1}{\text{Pr}} \left( 1 + \frac{4 R}{3} \right) G_0'' + (f_0 + N b S_0' + 2 N t \theta_0') G_0' + \theta_0' F_0 + 2 E c f_0 F_0'' + N b \theta_0' H_0' + \gamma G_0 = 0 \quad (26) \]

\[ H_0'' + \text{Le} f_0 H_0' + \text{Le} S_0' F_0 + \frac{N b}{N t} G_0' + \gamma H_0 = 0 \quad (27) \]

and the boundary conditions will be

\[ F_0(0) = 0, \quad F_0'(0) = \delta F_0''(0), \quad T_0(0) = 0, \quad G_0(0) = 0, \quad F_0'(\infty) \to 0, \quad T_0(\infty) \to 0, \quad G_0(\infty) \to 0 \quad (28) \]

Further, the specific values of parameter involved such as \( s, \quad N b, \quad N t, \quad M, \quad R, \quad E c, \quad \text{Le} \) and \( \text{Pr} \), the stability of the steady flow solutions for \( f_0(\eta), \quad \theta_0(\eta) \) and \( S_0(\eta) \) can be tested via the smallest eigenvalue \( \gamma \). As it has been recommended by Harris et al [9], as a result of relaxing a boundary condition on \( F_0(\eta), \quad G_0(\eta) \) or \( H_0(\eta) \), we can determine the possible range of eigenvalues. Therefore, in this study, the condition is selected to be relaxed is \( F_0' \to 0 \) as \( \eta \to \infty \) and then solved the equations (25) - (27) through the update boundary conditions as \( F_0'(0) = 1 \).
4. Results and discussions

Governing ordinary differential equations (9) - (11) with the boundary conditions, (12) can be solved numerically by using bvp4c solver in Matlab. By guessing the difference between initial values for $f''(0)$, $-\theta'(0)$ and $-S'(0)$, the dual solutions are obtained where both profiles satisfy the boundary condition (12). The present numerical results have shown in good agreement with previous work by Nandy and Pop [7] when absent the velocity slip in and viscous dissipation. In this study, the solution is unique when $\alpha > -1.0$, dual solutions when $\alpha_c < \alpha \leq -1$ and no solution for $\alpha < \alpha_c$ where $\alpha_c$ is the critical value of $\alpha$.

![Figure 1](image1.png)

**Figure 1.** Velocity profiles for certain values of $\alpha$ with $s = M = 0.1$.

![Figure 2](image2.png)

**Figure 2.** Temperature profiles for certain values of $\alpha$ with $Pr = 1$, $Nb = Nt = R = 0.3$ and $Ec = 0.1$.

![Figure 3](image3.png)

**Figure 3.** Concentration profiles for certain values of $\alpha$ with $Nb = Nt = 0.3$ and $Le = 1$.

Figure 1 – 3 provides the results obtained from the velocity, temperature and concentration profile. As illustrates in all figures, the chosen certain value of $\alpha$ has clearly shown non-unique solutions. Then, the stability analysis is required. It is also essential to mention that, the flow is declared as stable when the smallest $\gamma$ eigenvalue is a positive and vice versa. Table 1 displays the smallest eigenvalue $\gamma$ for the variable values of $\alpha$ with different value of $\delta$. The numerical results have indicated that the eigenvalue $\gamma$ is real and positive for the first solution while is real and negative for the second solution.
Table 1. Smallest eigenvalues of $\gamma$ for several values of $\alpha$ when, $Le = Pr = 1.0$, $s = M = Ec = 0.1$ and $Nb = Nt = 0.3$.

| $\delta$ | $\alpha$ | First solution | Second solution |
|----------|----------|----------------|-----------------|
| 0.1      | -1.45    | 0.23811        | -0.22893        |
|          | -1.40    | 0.64789        | -0.58171        |
|          | -1.35    | 0.89199        | -0.76642        |
| 0.2      | -1.55    | 0.08296        | -0.08192        |
|          | -1.52    | 0.45591        | -0.42558        |
|          | -1.43    | 0.91673        | -0.79464        |
| 0.3      | -1.65    | 0.23151        | -0.22424        |
|          | -1.61    | 0.55493        | -0.51392        |
|          | -1.50    | 1.01670        | -0.87808        |

5. Conclusion
This research extends our knowledge of stability analysis for the problem of the stagnation-point flow and heat transfer in a magnetic field and thermal radiation over a shrinking surface in nanofluid with slip effects. This study has shown that the first solution is linearly stable while the second solution is linearly unstable. Therefore, these results support the previous research into the stability analysis area.

Acknowledgments
The first author wishes to express a ton of gratitude to the Ministry of Education Malaysia through MyBrain (MyPhD) and the financial support received in the form of an FRGS research grant from the Ministry of Higher Education, Malaysia.

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