Abstract

Ensuring the privacy of training data is a growing concern since many machine learning models are trained on confidential and potentially sensitive data. Much attention has been devoted to methods for protecting individual privacy during analyses of large datasets. However, in many settings, global properties of the dataset may also be sensitive (e.g., mortality rate in a hospital rather than presence of a particular patient in the dataset). In this work, we depart from individual privacy to initiate the study of attribute privacy, where a data owner is concerned about revealing sensitive properties of a whole dataset during analysis. We propose definitions to capture attribute privacy in two relevant cases where global attributes may need to be protected: (1) properties of a specific dataset and (2) parameters of the underlying distribution from which dataset is sampled. We also provide two efficient mechanisms and one inefficient mechanism that satisfy attribute privacy for these settings. We base our results on a novel use of the Pufferfish framework to account for correlations across attributes in the data, thus addressing “the challenging problem of developing Pufferfish instantiations and algorithms for general aggregate secrets” that was left open by [KM14].

1 Introduction

Privacy in the computer science literature has generally been defined at the individual level, such as differential privacy [DMNS06], which protects the value of an individual’s data within analysis of a larger dataset. However, there are many settings where confidential information contained in the data goes beyond presence or absence of an individual in the data and instead relates to attributes at the dataset level. Global properties about attributes revealed from data analysis may leak trade secrets, intellectual property and other valuable information pertaining to the data owner, even if differential privacy is applied [Cor11].

In this paper, we are interested in privacy of attributes in a dataset, where an analyst must prevent global properties of sensitive attributes in her dataset from leaking during analysis. For example, insurance quotes generated by a machine-learned model might leak information about how many female and male drivers are insured by the company that trained the model; voice and facial recognition models may leak the distribution of race and gender among users in the training dataset [AMS+15, BG18]. Under certain circumstances, even releasing the distribution from which the data were sampled may be sensitive. For example, experimental findings by a pharmaceutical company measuring the efficacy of a new drug would be considered proprietary information. It is important to note that the problem we consider here departs from individual-level attribute privacy where one wishes to protect attribute value of a record (e.g., person’s race) as opposed to a function over all values of this attribute in the dataset (e.g., race distribution in a dataset).
Several recent attacks show that global properties about a dataset can indeed be leaked from machine learning model APIs \cite{SS20, MSCS19, ZTO20, GWY+18}. In fact, these works show that models learn sensitive attributes even when censorship is applied or when the attributes are deemed irrelevant for the actual learning task. Hence, the naive solution of removing sensitive attributes from the dataset is insufficient, as attributes are often correlated, and protected information can still be leaked by releasing non-sensitive information about the data. Though differential privacy can be used to protect sensitive attributes at the individual level \cite{DHP+12}, the study of attribute privacy at the dataset or distribution level is limited, both in terms of a framework for reasoning about it and mechanisms for protecting it.

\textbf{1.1 Our Contributions}

**Problem formulation** We initiate the study of \textit{attribute privacy} at the dataset and distribution level and establish the first formal framework for reasoning about these privacy notions. We identify two cases where information about global properties of a dataset may need to be protected: (1) properties of a specific dataset and (2) parameters of the underlying distribution from which dataset is sampled. We refer to the first setting as \textit{dataset attribute privacy}, where the data owner wishes to protect properties of her sample from a distribution, but is not concerned about revealing the distribution. For example, even though the overall prevalence of a disease may be known, a hospital may wish to protect the fraction of its patients with that disease. We refer to the second setting as \textit{distributional attribute privacy}, which considers the distribution parameter itself a secret. For example, demographic information of the population targeted by a company may reveal information about its proprietary marketing strategy. These two definitions distinguish between protecting a sample and protecting the distribution from which the dataset is sampled.

**Definitions of Attribute Privacy.** We propose definitions for capturing dataset and distributional attribute privacy by instantiating a general privacy framework called the Pufferfish framework \cite{KM14}. This framework was originally introduced to handle correlations across individual entries in a database. Instantiating this framework for attribute privacy is non-trivial as it requires reasoning about secrets and parameters at a dataset level.

For dataset attribute privacy, our definition considers the setting where individual records are independent of each other while correlations may exist between attribute values of each record. Then, to be able to capture general global properties of a dataset that need to be protected, we choose to express secrets as functions over attribute values across all records in a dataset. For example, this allows one to express that the average income of individuals in a dataset being below or above $50K is secret.

Our second definition also instantiates the Pufferfish framework while explicitly capturing the random variables used to generate attribute values of a record. Here, the parameters of the distribution of protected attributes are treated as confidential information. For example, in a dataset where records capture trials in a stochastic chemical environment, one can express that determining whether the probability with which a certain compound is added in each trial is 0.2 or 0.8 is a secret.

**Mechanisms to Protect Attribute Privacy.** Our definitions allow an analyst to specify secrets about global properties of a dataset that they wish to protect. In order to satisfy these definitions the analyst can use a general tool for providing Pufferfish privacy called the Wasserstein mechanism proposed by Song \textit{et al.} \cite{SWC17}. However, this mechanism is computationally expensive and may require computing an exponential number of pairwise Wasserstein distances, which is not feasible in most practical settings. To this end, we propose efficient mechanisms in the following two settings.

For dataset attribute privacy, we consider a special class of functions and attribute properties and propose a mechanism based on Gaussian noise. Though the nature of the noise is added from the same family of distributions as the differentially private Gaussian Mechanism, in Section 4 we articulate that the similarity between the two is based solely on the nature of the noise. In Section 4.1 we show that the mechanism can be applied to datasets where (1) attributes follow a multivariate Gaussian distribution and (2) the function
to be computed on the data and the attribute property to be protected are linear in the number of records in the dataset (e.g., mean). We note that with the help of variational auto-encoders (VAEs) \cite{KW14}, one can obtain a Gaussian representation of the data even if a dataset does not come from a Gaussian distribution. Moreover, such disentangled representations can be based on interpretable attributes \cite{HMP+17} that are easier for specifying which attributes require protection, particularly when the original data are complex (e.g., pixels on an image vs. the gender of the person in it). Nevertheless, we also consider the case where data may not follow Gaussian distribution. Specifically, in Section 4.2 we relax the Gaussian assumption and show that our mechanism can still provide dataset attribute privacy by leveraging Gaussian approximations.

For distributional attribute privacy, we consider a model where dependencies between the attributes form a Bayesian network. This model helps us capture the extent to which a sensitive attribute parameter affects parameters of attributes in the query, and we add noise proportional to this influence. Although our mechanism is inspired by the Markov Quilt mechanism \cite{SWC17}, the difference in settings prompts several changes, including a different metric for measuring influence between the variables.

Finally, we note that although \cite{KM14} identified that “there is little focus in the literature on rigorous and formal privacy guarantees for business data”, they leave “the challenging problem of developing Pufferfish instantiations and algorithms for general aggregate secrets” as future work.

1.2 Related work

Machine learning models have been shown to memorize and leak data used to train them, raising questions about the release and use of these models in practice. For example, membership attacks \cite{SSSS17} show that models can leak whether certain records (e.g., patient data) were part of the training dataset or not. Attribute (or feature) privacy attacks, on the other hand, consider leakage of attribute values at an individual level \cite{SS20, FJR15}, and property inference attacks show that global properties about datasets can be leaked \cite{MSCST19, GWY+18, AMS+15}.

Differential privacy (DP) \cite{DMNS06, DR14} guarantees individual-level privacy when publishing an output computed on a database, by bounding the influence of any record on the output and adding noise. Importantly, DP does not aim to protect population-level information, and was designed to learn global properties of a dataset without sacrificing individual privacy. DP does provide group privacy guarantees for groups of \(k\) correlated records, but these quantitative guarantees are only meaningful when \(k\) is small relative to the size of the dataset. Syntactically, DP guarantees that if any individual record were to be changed—including all attributes of that record—the result of the analysis would be approximately the same. For attribute privacy, we seek similar guarantees if an entire attribute of the dataset were to be changed—including all individuals’ values for that attribute.

The Pufferfish framework \cite{KM14} that we instantiate and describe in detail in the following sections, can be seen as a generalization of differential privacy that explicitly states the information that needs to be kept secret and the adversary’s background knowledge about the data. Blowfish privacy \cite{HMD14} also allows one to express secrets and publicly known information about the data, but expressed as constraints on the data rather than distributions over data. We adapt the Markov Quilt Mechanism from \cite{SWC17}, who also employ the Pufferfish framework \cite{KM14} for private analysis of correlated data, although they focus on individual-level privacy. Our focus instead on privacy of dataset properties and distributions leads to a substantially different instantiation of the Pufferfish framework where the secrets are defined over attribute values rather than individual records in the dataset.

Research on algorithmic fairness has proposed several definitions formalizing the idea that machine learning models should not exhibit discrimination based on protected attributes (e.g., gender or race). Demographic parity formalizes fairness by requiring that a classifier’s predicted label is independent of an individual’s protected attributes. Our notion of dataset attribute privacy is a general framework where one can specify what information about attributes need to be protected, with attribute independence being one such scenario. However, our attribute privacy definitions would not be useful for satisfying other fairness notions that...
explicitly incorporate protected attributes, such as affirmative action or fairness through awareness \cite{DHP+12}. Moreover, techniques proposed to obtain fair representations of the training data \cite{LSL+16,ZWS+13} have been shown to still leak sensitive attributes \cite{SS20} when applied in the privacy context.

## 2 Preliminaries

**Pufferfish Privacy.** The Pufferfish privacy framework \cite{KM14} consists of three components: a set of secrets $S$, a set of discriminative pairs $Q \subseteq S \times S$, and a class of data distributions $\Theta$. $S$ is a set of possible facts about the database that we might wish to hide. $Q$ is the set of secret pairs $(s_i, s_j)$, $s_i, s_j \in S$, that we wish to be indistinguishable, where $s_i$ and $s_j$ must be mutually exclusive. The class of data distributions $\Theta$ can be viewed as a set of conservative assumptions about the underlying distribution that generates the database.

**Definition 1** \((\epsilon, \delta)\)-Pufferfish Privacy \cite{KM14,SWC17}. A mechanism $M$ is \((\epsilon, \delta)\)-Pufferfish private in a framework \((S, Q, \Theta)\) if for all $\theta \in \Theta$ with $X \sim \theta$, for all secret pairs $(s_i, s_j) \in Q$ such that $P(s_i|\theta) \neq 0$ and $P(s_j|\theta) \neq 0$, and for all $T \subseteq \text{Range}(M)$, we have

$$P_{M,\theta}(M(X) \in T|s_i, \theta) \leq \exp(\epsilon)P_{M,\theta}(M(X) \in T|s_j, \theta) + \delta.$$  

The Wasserstein Mechanism proposed in \cite{SWC17} and defined formally in Section 6 is the first general mechanism for satisfying instantiations of Pufferfish privacy framework. It defines sensitivity of a function $F$ as the maximum Wasserstein distance between the distribution of $F(X)$ given two different realizations of secrets $s_i$ and $s_j$ for $(s_i, s_j) \in Q$. The mechanism then instantiates the Laplace mechanism by outputting $F(X)$ plus Laplace noise that scales with this sensitivity. Although this mechanism works in general for any instantiation of the Pufferfish framework, computing Wasserstein distance for all pairs of secrets is computationally expensive, and will typically not be feasible in practice.

\cite{SWC17} also gave the Markov Quilt Mechanism (Algorithm 5 in Appendix A) for some special structures of data dependence. It is more efficient than the Wasserstein Mechanism and also guarantees \((\epsilon, 0)\)-Pufferfish privacy. The Markov Quilt Mechanism of \cite{SWC17} assumes that the entries in the input database $Y$ form a Bayesian network, as defined below. These entries could either be: (1) the multiple attributes of a single record when the database contained only one record, or (2) the attribute values across multiple records for a single-fixed attribute when the database contained multiple attributes. Hence, the original Markov Quilt Mechanism could not accommodate correlations across multiple attributes in multiple records, as we study in this work. Full details of this algorithm are given in Appendix A.

**Definition 2** (Bayesian Networks). A Bayesian network is described by a set of variables $Y = \{Y_1, \ldots, Y_n\}$ and a directed acyclic graph $G = (Y, E)$ whose vertices are variables in $Y$. The probabilistic dependence on $Y$ included by the network can be written as: $Pr(Y_1, \ldots, Y_n) = \Pi_{i=1}^{n} Pr(Y_i|\text{parent}(Y_i))$.

## 3 Attribute Privacy Definitions

**Data model and representation.** The dataset $X$ contains $n$ records, where each record consists of $m$ attributes. We view the dataset $X$ as an $n \times m$ matrix. In this work, we are interested in privacy of the **columns**, which represent attributes that a data owner wishes to protect. Thus we refer to the matrix $X$ as $X = [X_1, \ldots, X_m]$, where $X_i$ is the column vector related to the $i$th attribute (column). In contrast, traditional differential privacy \cite{DMNS06,DRI14} is concerned with privacy of the **rows** of the dataset matrix. We let $X_{ij}$ denote $i$th attribute value for $j$th record.

Each record is assumed to be sampled i.i.d. from an unknown distribution, where attributes within a single record can be correlated (e.g., consider height and weight). We use $C \subseteq [m]$ to denote a set of indices of the sensitive attributes that require privacy protection (e.g., race and gender may be sensitive attributes; hair color may be non-sensitive). The data owner wishes to compute a function $F$ over her dataset and release the value (or estimate of the value) $F(X)$ while protecting some information about the sensitive attributes.

\footnote{The original definition \cite{KM14} and the one considered in \cite{SWC17} is \((\epsilon, 0)\)-Pufferfish. We extend the definition to \((\epsilon, \delta)\)-Pufferfish in the natural way.}
Privacy notions. We distinguish between three kinds of attribute privacy, corresponding to three different types of information the data owner may wish to protect.

Individual attribute privacy protects $X_i^j$ for sensitive attribute $i$ when $F(X)$ is released. Note that differential privacy provides individual attribute privacy simultaneously for all individuals and all attributes \cite{DMNS06}, but does not protect against individual-level inferences from population-level statistics \cite{Cor11}. For example, if a DP result shows a correlation between lung disease and smoking, one may infer that a known-smoker in the dataset has an elevated likelihood of lung disease.

Dataset attribute privacy is applicable when the owner wishes to reveal $F(X)$ while protecting the value of some function $g(X_i)$ for sensitive attribute $i \in C$ (e.g., whether there were more Caucasians or Asians present in the dataset).

Distribution attribute privacy protects privacy of a parameter $\phi_i$ that governs the distribution of $i$th sensitive attribute in the underlying population from which the data are sampled.

The last two notions are the ones put forward in this paper and studied in detail. The difference between them may be subtle depending on $g$ and $\phi$. For example, consider one setting where the sensitive attribute is binary and $g$ is the fraction of records where this attribute is 1, and another setting where the sensitive attribute is a Bernoulli random variable with parameter $\phi$. In this case, $g$ can be seen as an estimate of $\phi$ based on a sample. The difference becomes particularly relevant in settings where privacy is required for realizations of the dataset that are unlikely under the data distribution, or settings with small datasets where $g$ is a poor estimate of $\phi$.

Formal framework for attribute privacy. The standard notion of differential privacy is not directly applicable to our setting since we are interested in protecting population-level information. Instead, we formalize our attribute privacy definitions using the Pufferfish privacy framework of Definition 1 by specifying the three components $(S, Q, \Theta)$. The distributional assumptions of this framework are additionally useful for formalizing correlation across attributes.

Definition 3 (Dataset Attribute Privacy). Let $(X_1^j, X_2^j, \ldots, X_m^j)$ be a record with $m$ attributes that is sampled from an unknown distribution $D$, and let $X = [X_1, \ldots, X_m]$ be a dataset of $n$ records sampled i.i.d. from $D$ where $X_i$ denotes the (column) vector containing values of $i$th attribute of every record. Let $C \subseteq [m]$ be the set of indices of sensitive attributes, and for each $i \in C$, let $g_i(X_i)$ be a function with codomain $\mathcal{U}_i$.

A mechanism $M$ satisfies $(\epsilon, \delta)$-dataset attribute privacy if it is $(\epsilon, \delta)$-Pufferfish private for the following framework $(S, Q, \Theta)$:

Set of secrets: $S = \{s_i^a := 1[g_i(X_i) \in \mathcal{U}_i^a] : \mathcal{U}_i^a \subseteq \mathcal{U}_i, i \in C\}$.

Set of secret pairs: $Q = \{(s_i^a, s_j^b) \in S \times S, i \in C\}$.

Distribution: $\Theta$ is a set of possible distributions $\theta$ over the dataset $X$. For each possible distribution $D$ over records, there exists a $\theta_D \in \Theta$ that corresponds to the distribution over $n$ i.i.d. samples from $D$.

This definition defines each secret $s_i^a$ as the event that $g_i(X_i)$ takes a value in a particular set $\mathcal{U}_i^a$, and the set of secrets $S$ is the collection of all such secrets for all sensitive attributes. This collection may include all possible subsets of $\mathcal{U}_i$, or it may include only application-relevant events. For example, if all $\mathcal{U}_i^a$ are singletons, this corresponds to protecting any realization of $g_i(X_i)$. Alternatively, the data owner may only wish to protect whether $g_i(X_i)$ is positive or negative, which requires only $\mathcal{U}_i^a = (-\infty, 0)$ and $\mathcal{U}_i^b = [0, \infty)$. The set of secret pairs $Q$ that must be protected includes all pairs of the events on the same sensitive attribute. The Pufferfish framework considers distributions $\theta$ over the entire dataset $X$, whereas we require distributions $D$ over records. We resolve this by defining $\Theta$ to be the collection of distributions over datasets induced by the allowable i.i.d. distributions over records.

Determining which functions $g_i$ to consider is an interesting question. For example, in \cite{MG06} the authors show that it is tractable to check whether the output of certain classes of functions evaluated on a dataset reveals information about the output of another query evaluated on the same dataset. Hence, given a function $F$ whose output a data owner wishes to release, the owner may consider either those $g_i$’s about which $F$ reveals information, or those for which verifying perfect privacy w.r.t. $F$ is infeasible.
Definition 4 (Distributional Attribute Privacy). Let \((X_1^j, X_2^j, \ldots, X_m^j)\) be a record with \(m\) attributes that is sampled from an unknown distribution described by a vector of random variables \((\phi_1, \ldots, \phi_m)\), where \(\phi_i\) parameterizes the marginal distribution of \(X_i^j\) conditioned on the values of all \(\phi_k\) for \(k \neq i\). The \((\phi_1, \ldots, \phi_m)\) are drawn from a known joint distribution \(P\), and each \(\phi_i\) has support \(\Phi_i\). Let \(X = [X_1, \ldots, X_m]\) be a dataset of \(n\) records sampled i.i.d. from the distribution described by \((\phi_1, \ldots, \phi_m)\) where \(X_i\) denotes the (column) vector containing values of \(i\)th attribute of every record. Let \(C \subseteq [m]\) be the set of indices of sensitive attributes.

A mechanism \(M\) satisfies \((\epsilon, \delta)\)-distributional attribute privacy if it is \((\epsilon, \delta)\)-Pufferfish private for the following framework \((S, Q, \Theta)\):

Set of secrets: \(S = \{s_a^i := \mathbb{1}[\phi_i \in \Phi_a^i] : \Phi_a^i \subset \Phi^i, i \in C\}\).

Set of secret pairs: \(Q = \{(s_a^i, s_b^j) \in S \times S, i, j \in C\}\).

Distribution: \(\Theta\) is a set of possible distributions \(\theta\) over the dataset \(X\). For each possible \(\phi = (\phi_1, \ldots, \phi_m)\) describing the conditional marginal distributions for all attributes, there exists a \(\theta_\phi \in \Theta\) that corresponds to the distribution over \(n\) i.i.d. samples from the distribution over records described by \(\phi\).

This definition naturally parallels Definition 3 with the attribute-specific random variable \(\phi_i\) taking the place of the attribute-specific function \(g_i(X_i)\). Although it might seem natural for \(\phi_i\) to define the marginal distribution of the \(i\)th attribute, this would not capture the correlation across attributes that we wish to study. Instead, \(\phi_i\) defines the conditional marginal distribution of the \(i\)th attribute given all other \(\phi_{\neq i}\), which does capture such correlation. This also allows the distribution \(\theta\) over datasets to be fully specified given these parameters and the size of the dataset.

More specifically, we model attribute distributions using standard notion of Bayesian hierarchical modeling. The \((\phi_1, \ldots, \phi_m)\) can be viewed as a set of hyper-priors of the distributions of the attributes, and \(P\) as hyper-priors of the hyperparameters. The distribution \(P\) is captured in \(\Theta\), and the distribution of attribute \(X_i\) is governed by a realization of the random variable \(\phi_i\). The \(\phi_i\) describes the conditional marginal distribution for attribute \(i\): it is the hyperparameter of the probability of \(X_i\) given hyperparameters of all other attributes \(P(X_i|\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_m)\). We make the “naive” conditional independence assumption that all attributes \(X_i\) are mutually independent conditional on the set of parameters \((\phi_1, \ldots, \phi_m)\), hence, \((\phi_1, \ldots, \phi_m)\) fully capture the distribution of a record. The “naive” conditional independence is a common assumption in probabilistic models, and naive Bayes is a simple example that employs this assumption.

Since both of our attribute privacy definitions are instantiations of the Pufferfish privacy framework, one could easily apply the Wasserstein Mechanism \cite{SWC17} to satisfy \((\epsilon, 0)\)-attribute privacy for either of our definitions. The Wasserstein distance metric has also been used to calibrate noise in prior work on distributional variants of differential (individual-level) privacy \cite{KM19, KM19a}. However, as described in Section 2 implementing this mechanism requires computing Wasserstein distance between the conditional distribution on \(F(X)\) for all pairs of secrets in \(Q\). Computing exact Wasserstein distance is known to be computationally expensive, and our settings may require exponentially many computations in the worst case.

In the remainder of the paper, we provide efficient algorithms that satisfy each of these privacy definitions, focusing on dataset attribute privacy in Section 4 and distributional attribute privacy in Section 5 before returning to the (inefficient) Wasserstein Mechanism in Section 6.

4 The Gaussian Mechanism for Dataset Attribute Privacy

In this section we consider dataset attribute privacy as introduced in Definition 3. In this setting, an analyst wants to publish a function \(F\) evaluated on her dataset \(X\), but is concerned about an adversary observing \(F(X)\) and performing a Bayesian update to make inferences about a protected quantity \(g_i(X_i)\).

We propose a variant of the Gaussian Mechanism \cite{DR14} that satisfies dataset attribute privacy when \(F(X)\) conditioned on \(g_i(X_i)\) follows a Gaussian distribution, with constant variance conditioned on \(g_i(X_i) = a\) for all \(a\). Although this setting is more restrictive, it is still of practical interest. For example, it can be
Algorithm 1 presents the Attribute-Private Gaussian Mechanism for answering a real-valued query \( F(X) \) while protecting the values of \( g_i(X_i) \) for \( i \in C \). Much like the Gaussian Mechanism for differential privacy [DR14], the Attribute-Private Gaussian Mechanism first computes the true value \( F(X) \), and then adds a Gaussian noise term with mean zero and standard deviation that scales with the sensitivity of the function. Hence, privacy also comes for free if the function of interest has low correlation with \( g_i(X_i) \), as the sampling noise can mask some of the correlation. However, sensitivity of \( F \) in the attribute privacy setting is defined with respect to each secret attribute \( X_i \) as,

\[
\Delta_i F = \max_{\theta \in \Theta} \max_{(s^a_i, s^b_i) \in \mathcal{Q}} \left| \mathbb{E}_s [F(X)|s^a_i, \theta] - \mathbb{E}_s [F(X)|s^b_i, \theta] \right|.
\]

This differs from the sensitivity notion used in differential privacy in two key ways. First, we are concerned with measuring changes to the value of \( F(X) \) caused by changing secrets \( s^a_i \) corresponding to realizations of \( g_i(X_i) \), rather than by changing an individual’s data. Second, we assume our data are drawn from an unknown underlying distribution \( \theta \), so \( F(X) \) is a random variable. Our attribute privacy sensitivity bounds the maximum change in posterior expected value of \( F(X) \) in the worst case over all distributions and pairs of secrets for each attribute. We note that if \( F(X) \) is independent of the protected attribute \( X_i \), then \( \Delta_i F = 0 \) and no additional noise is needed for privacy. The Attribute-Private Gaussian Mechanism of Algorithm 1 further benefits from the inherent randomness of the output \( F(X) \). In particular, it reduces the variance \( \sigma^2 \) of the noise added by the conditional variance of \( F(X) \) given \( g_i(X_i) \) and \( \theta \), as the sampling noise can mask some of the correlation. Hence, privacy also comes for free if the function of interest has low correlation with the protected attributes.

Algorithm 1 can be easily extended to handle vector-valued queries with \( F(X) \in \mathbb{R}^k \) and sensitive functions \( g_i \) over multiple attributes by changing \( \Delta_i F \) in Equation (1) to be the maximum \( \ell_2 \) distance rather than absolute value. Additionally, the noise adjustment for each attribute should be based on the conditional covariance matrix of \( F(X) \) rather than the conditional variance.

**Algorithm 1** Attribute-Private Gaussian Mechanism, APGM\((X, F; \{g_i\}, C; \{S, Q, \Theta\}, \epsilon, \delta)\) for dataset attribute privacy.

**Input:** dataset \( X \), query \( F \), functions \( g_i \) for protected attributes \( i \in C \), framework \( \{S, Q, \Theta\} \), privacy parameters \( \epsilon, \delta \).

Set \( \sigma^2 = 0 \), \( c = \sqrt{2\log(1.25/\delta)} \).

for each \( i \in C \) do

Set \( \Delta_i F = \max_{\theta \in \Theta} \max_{(s^a_i, s^b_i) \in \mathcal{Q}} \left| \mathbb{E}_s [F(X)|s^a_i, \theta] - \mathbb{E}_s [F(X)|s^b_i, \theta] \right| \).

if \( (c\Delta_i F/\epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(F(X)|g_i(X_i), \theta) \geq \sigma^2 \) then

Set \( \sigma^2 = (c\Delta_i F/\epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(F(X)|g_i(X_i), \theta) \).

else if \( \sigma^2 > 0 \) then

Sample \( Z \sim \mathcal{N}(0, \sigma^2) \).

Return \( F(X) + Z \).

else Return \( F(X) \).
The Attribute-Private Gaussian Mechanism

APGM\((X, F, \{g_i\}, C, \{S, \Theta, \epsilon, \delta\})\) is \((\epsilon, \delta)\)-dataset attribute private when \(F(X)|g_i(X_i)\) is Gaussian distributed for any \(\theta \in \Theta\) and \(i \in C\).

Privacy follows from the observation that the summation of \(F(X)\) and the Gaussian noise \(Z\) is Gaussian distributed conditioned on any secrets, and the probabilities of the output conditioned on any pairs of secrets have the same variance with mean difference \(\Delta, F\). Since we bound the ratio of the two probabilities caused by shifting this variable, the analysis reduces to the proof of Gaussian mechanism in differential privacy.

Proof. Fix any pair of secrets \((s_a^i, s_b^i)\) in \(Q\) for a fixed secret attribute \(X_i\) under any \(\theta \in \Theta\). Recall that \(s_a^i\) denotes the event that \(g_i(X_i) \in U^\epsilon\). Let \(Z \sim N(0, \sigma^2)\) denote the Gaussian noise added in Algorithm 1. We have \([M(X)|s_a^i, \theta] = [F(X) + Z]|s_a^i, \theta] = [F(X)|s_a^i, \theta] + Z\), because \(Z\) is independent of \(s_a^i\) and \(\theta\). Since we have assumed that \(F(X)|g_i(X_i)\) is Gaussian distributed and the summation of two Gaussians is Gaussian, \(M(X)|s_a^i\) follows a Gaussian distribution with mean \(\mathbb{E}[F(X)|s_a^i, \theta]\) and variance \(\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2\).

The ratio of probabilities of seeing an output \(w\) on a pair of secrets \((s_a^i, s_b^i)\) in the worst case is as follows,

\[
\max_{(s_a^i, s_b^i) \in Q} \log \frac{\exp(-\frac{1}{2}(\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2)(w - \mathbb{E}[F(X)|s_a^i, \theta])^2)}{\exp(-\frac{1}{2}(\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2)(w - \mathbb{E}[F(X)|s_b^i, \theta])^2)} = \log \frac{\exp(-\frac{1}{2}(\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2)w^2)}{\exp(-\frac{1}{2}(\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2)(w + \Delta)^2)}.
\]

where \(\Delta = \max_{(s_a^i, s_b^i) \in Q} \mathbb{E}[F(X)|s_a^i, \theta] - \mathbb{E}[F(X)|s_b^i, \theta]\). Equation 2 follows from shifting the variable \(w\) to \(w + \mathbb{E}[F(X)|s_a^i, \theta]\). We observe that the probability ratio can be viewed as the probability ratio in the Gaussian Mechanism in differential privacy with noise draw from \(N(0, \text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2)\), and query sensitivity \(\max_{(s_a^i, s_b^i) \in Q} \mathbb{E}[F(X)|s_a^i] - \mathbb{E}[F(X)|s_b^i]\). Then, our analysis reduces to the proof of the Gaussian Mechanism in differential privacy. The Gaussian Mechanism for differential privacy with

\[
\text{Var}(F(X)|g_i(X_i), \theta) + \sigma^2 \\
\geq 2\log(1.25/\delta) \left(\max_{(s_a^i, s_b^i) \in Q} \mathbb{E}[F(X)|s_a^i, \theta] - \mathbb{E}[F(X)|s_b^i, \theta]\right)^2
\]

ensures that with probability at least \(1 - \delta\), we have

\[P(M(X) \in T|s_a^i, \theta) \leq \exp(\epsilon)P(M(X) \in T|s_b^i, \theta) + \delta,\]

for any \(T \subseteq \text{Range}(M)\). Equivalently, we have

\[
\sigma^2 \geq 2\log(1.25/\delta) \left(\max_{(s_a^i, s_b^i) \in Q} \mathbb{E}[F(X)|s_a^i, \theta] - \mathbb{E}[F(X)|s_b^i, \theta]\right)^2
\]

\[
= \text{Var}(F(X)|g_i(X_i), \theta).
\]

Taking the maximum over \(i\) for all secret attributes and over all \(\theta \in \Theta\), we require

\[
\sigma^2 \geq \max_{i \in C} \left[2\log(1.25/\delta)(\Delta_iF/\epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(F(X)|g_i(X_i), \theta)\right]
\]

which will ensure that the ratio of the probabilities that the algorithm \(M(X)\) outputs a query value in any subset \(T\) on any pair of secrets for any \(\theta \in \Theta\) is bounded by \(\epsilon\) with probability at least \(1 - \delta\). \(\square\)

High probability additive accuracy bounds on the output of Algorithm 1 can be derived using tail bounds on the noise term \(Z\) based on its variance \(\sigma^2\). The formal accuracy guarantee is stated in Theorem 2 which follows immediately from tail bounds of a Gaussian.
Theorem 2. The Attribute-Private Gaussian Mechanism
APGM(\(X, F, \{g_i\}, C, \{S, Q, \Theta\}, \epsilon, \delta\)) in Algorithm 1 is \((\alpha, \beta)\)-accurate for any \(\beta > 0\) and
\[
\alpha = \sqrt{\max \{0, \max_{i \in C} \left( (c \Delta_i F / \epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(F(X)|g_i(X_i), \theta) \right) \} \Phi^{-1}(1 - \frac{\beta}{2})},
\]
where \(c = \sqrt{2 \log(1.25/\delta)}\) and \(\Phi\) is the CDF of the standard normal distribution.

In general, if \(F(X)\) is independent of, or only weakly correlated with the protected functions \(g_i(X_i)\), then no noise is needed to preserve dataset attribute privacy, and the mechanism can output the exact answer \(F(X)\). On the other hand, if \(F(X)\) is highly correlated with \(g_i(X_i)\), we then consider a tradeoff between the sensitivity and the variance of \(F(X)\). If the variance of \(F(X)\) is relatively large, then \(F(X)\) is inherently private, and less noise is required. If the variance of \(F(X)\) is small and the sensitivity of \(F(X)\) is large, the mechanism must add a noise term with large \(\sigma^2\), resulting in low accuracy with respect to the true answer. To make these statements more concrete and understandable, Section 4.3 provides a concrete instantiation of Algorithm 1.

4.2 Privacy guarantees without Gaussian assumptions
A natural question is how we can apply the Attribute-Private Gaussian Mechanism when the distributional assumptions of Section 4.1 do not hold. That is, when the actual distribution of \(F(X)\) conditioned on \(g_i(X_i)\) is not Gaussian distributed. Our idea is based on using a collection of Gaussian distributions to approximate the actual distributions. In this section, we show that the Attribute-Private Gaussian Mechanism can still be applied in this case to achieve distributional attribute privacy.

We quantify this distributional closeness using \(\eta\)-approximate max-divergence. The max-divergence and approximate max-divergence are defined as follows.

Definition 5 (Max-Divergence). Let \(p\) and \(q\) be two distributions with the same support. The max-divergence \(D(p||q)\) between them is defined as:
\[
D(p||q) = \sup_{T \subset \text{Support}(p)} \log \frac{\Pr(p(x) \in T)}{\Pr(q(x) \in T)}.
\]

Definition 6 (\(\eta\)-Approximate Max-Divergence). Let \(p\) and \(q\) be two distributions. The \(\eta\)-approximate max-divergence between them is defined as:
\[
D^\eta(p||q) = \sup_{T \subset \text{Support}(p): \Pr[p(x) \in T] \geq \eta} \log \frac{\Pr(p(x) \in T) - \eta}{\Pr(q(x) \in T)}.
\]

We will consider the following variant of \(\eta\)-approximate max-divergence:
\[
D^\eta(f_{s_i, \theta}, \tilde{f}_{s_i, \theta}) := \max \{D^\eta(f_{s_i, \theta}||\tilde{f}_{s_i, \theta}), D^\eta(\tilde{f}_{s_i, \theta}||f_{s_i, \theta})\}.
\]  

Formally, let \(f_{s_i, \theta}\) denote the actual conditional distribution of \(F(X)\) given the secret \(s_i \in S\) and \(\theta \in \Theta\). For each \(f_{s_i, \theta}\), let \(\tilde{f}_{s_i, \theta}\) denote a Gaussian approximation to \(f_{s_i, \theta}\). For any \(\eta > 0\), let \(\lambda_\eta\) be the bound such that for every \(s_i\), the (variant) approximate max-divergence \(D^\eta(f_{s_i, \theta}, \tilde{f}_{s_i, \theta}) \leq \lambda_\eta\). That is, \(\lambda_\eta\) is a constant determined by \(\eta\). For any fixed \(\theta \in \Theta\) and \(i \in C\), the collection of \(\{\tilde{f}_{s_i, \theta}\}\) is chosen to have constant variance for any \(s_i \in S\). We denote this variance as \(\text{Var}(\tilde{f}_{s_i, \theta})\). The Attribute-Private Gaussian Mechanism for non-Gaussian data of Algorithm 2 allows the analyst to choose a set of Gaussian approximations \(\{\tilde{f}_{s_i, \theta}\}\) for each secret \(s_i \in S\) and \(\theta \in \Theta\), and use it instead of the actual conditional distribution of \(F(X)\) in the rest of the algorithm steps, which are the same as in the Attribute-Private Gaussian Mechanism of Algorithm 1.

\(^3\)The approximation parameter is typically named \(\delta\) in the literature; we use \(\eta\) instead to avoid confusion with the privacy parameter.
Algorithm 2 Attribute-Private Gaussian Mechanism for non-Gaussian data, 

\text{APGMnG}(X, F; \{g_i\}, C, \{S, Q, \Theta\}, \epsilon, \delta).

\textbf{Input}: dataset \(X\), query \(F\), functions \(g_i\) for protected attributes \(i \in C\), framework \(\{S, Q, \Theta\}\), privacy parameters \(\epsilon, \delta\).

\begin{algorithmic}
\State Set \(\sigma^2 = 0, c = \sqrt{2\log(1.25/\delta)}\).
\For {each \(i \in C\)}
\State Choose a Gaussian approximation \(\tilde{f}_{s_i, \theta}\) for each secret \(s_i \in S\) and \(\theta \in \Theta\).
\State Set \(\Delta_i F = \max_{\theta \in \Theta} \max_{(s_i, s_i') \in Q} \left| \mathbb{E}[\tilde{f}_{s_i, \theta}] - \mathbb{E}[\tilde{f}_{s_i', \theta}] \right|\).
\EndFor
\If {\((c\Delta_i F/\epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(\tilde{f}_{i, \theta}) \geq \sigma^2\)}
\State Set \(\sigma^2 = (c\Delta_i F/\epsilon)^2 - \min_{\theta \in \Theta} \text{Var}(\tilde{f}_{i, \theta})\).
\Else
\State Sample \(Z \sim \mathcal{N}(0, \sigma^2)\).
\State Return \(F(X) + Z\).
\EndIf
\EndAlgorithm

The following theorem states that if the Gaussian approximations in Algorithm 2 are close to the actual distributions of \(F(X)\) conditioned on \(g_i(X_i)\), then the loss in privacy is not too large. We note that Theorem 3 involves the variant definition of \(\eta\)-approximate max-divergence introduced in Definition 3, which does not require that the actual distribution has the same support as the Gaussian distribution; thus, it is also applicable for all discrete distributions or finite-support distributions.

**Theorem 3.** The Attribute-Private Gaussian Mechanism for non-Gaussian data \(\text{APGMnG}(X, F; \{g_i\}, C, \{S, Q, \Theta\}, \epsilon, \delta)\) in Algorithm 2 is \((\epsilon + 2\lambda, \exp(\lambda)\delta + \eta)\)-dataset attribute private, when the Gaussian approximation \(\tilde{f}_{s_i, \theta}\) satisfies \(D(\tilde{f}_{s_i, \theta}, f_{s_i, \theta}) \leq \lambda\eta\) for any \(\eta > 0\) for all \(s_i \in S\) and \(\theta \in \Theta\).

**Proof.** We first analyze the probability of the output conditioned on the event that \(\max_{i \in C} \max_{s_i \in S} \max\{D(\tilde{f}_{s_i, \theta} || f_{s_i, \theta}), D(\tilde{F}_{s_i, \theta} || f_{s_i, \theta})\} \leq \lambda\eta\). Fix any pair of secrets \((s_i^1, s_i^2) \in Q\) for a fixed sensitive attribute \(X_i\) under any \(\theta \in \Theta\). Let \(Z \sim \mathcal{N}(0, \sigma^2)\) denote the Gaussian noise added in Algorithm 2.

Let us partition \(\mathbb{R}\) as \(\mathbb{R} = R_1 \cup R_2\), where

\[
R_1 = \{ F(X) + Z \in \mathbb{R} : |F(X) + Z| \leq c\Delta_i F/\epsilon \},
\]

and

\[
R_2 = \{ F(X) + Z \in \mathbb{R} : |F(X) + Z| > c\Delta_i F/\epsilon \}.
\]

Fix any subset \(T \subseteq \mathbb{R}\), and define \(T_1 = T \cap R_1\) and \(T_2 = T \cap R_2\).

For any \(w \in T_1\), we can write the probability ratio of seeing the output \(w\) as follows:

\[
\frac{\Pr(F(X) + Z = w|s_i^1, \theta)}{\Pr(F(X) + Z = w|s_i^2, \theta)} = \frac{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^1, \theta})}{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^2, \theta})}, \frac{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^1, \theta})}{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^2, \theta})}, \frac{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^1, \theta})}{\Pr(F(X) + Z = w|F \sim \tilde{f}_{s_i^2, \theta})}.
\]

For any \(w \in T_1\), the Attribute-Private Gaussian Mechanism for non-Gaussian data ensures that the last ratio in Equation (4) is bounded above by \(\exp(\epsilon)\). For the first ratio in Equation (4), since the generation process
for $z$ is independent of the data, we have
\[
\frac{\Pr(F(X) + Z = w | F \sim f_{s_a^i, \theta})}{\Pr(F(X) + Z = w | F \sim \hat{f}_{s_a^i, \theta})} = \frac{\int_f \Pr(Z = w - f) \Pr(F(X) = f | F \sim f_{s_a^i, \theta})df}{\int_f \Pr(Z = w - f) \Pr(F(X) = f | F \sim \hat{f}_{s_a^i, \theta})df} \leq \frac{\Pr(F(X) = f | F \sim f_{s_a^i, \theta})}{\Pr(F(X) = f | F \sim \hat{f}_{s_a^i, \theta})}
\]

Similarly, we can bound the second ratio by $\exp(\lambda_\eta)$. Thus, Equation (4) is bounded by $\exp(\epsilon + 2\lambda_\eta)$, which is equivalent to
\[
\Pr(F(X) + Z \in T_1 | s_a^i, \theta) \leq \exp(\epsilon + 2\lambda_\eta) \Pr(F(X) + Z \in T_1 | s_b^i, \theta)
\]

We also bound the probability that the output belongs to the subset $T_2$ as follows:
\[
\Pr(F(X) + Z \in T_2 | s_a^i, \theta) \leq \exp(\lambda_\eta) \Pr(F(X) + Z \in T_2 | F \sim \hat{f}_{s_a^i, \theta}) \leq \exp(\lambda_\eta)\delta.
\]

Then, by (5) and (6), we have
\[
\Pr(F(X) + Z \in T | s_a^i, \theta) = \Pr(F(X) + Z \in T_1 | s_a^i, \theta) + \Pr(F(X) + Z \in T_2 | s_a^i, \theta) \leq \exp(\epsilon + 2\lambda_\eta) \Pr(F(X) + Z \in T_1 | s_b^i, \theta) + \exp(\lambda_\eta)\delta.
\]

We then analyze the probability of the output when $\max_{i\in C, s_b^i \in S} \max \{D(f_{s_a^i, \theta} \| \hat{f}_{s_a^i, \theta}), D(f_{s_a^i, \theta} \| f_{s_a^i, \theta})\}$ is bounded by $\lambda_\eta$ with probability at least $1 - \eta$, which is equivalent to the $\eta$-approximate max-divergence $D_\eta(f_{s_a^i, \theta}, \hat{f}_{s_a^i, \theta})$ is bounded above by $\lambda_\eta$. For any $\eta > \delta$, define this high probability event as follows:
\[
E_\eta := \{ \max_{i\in C} \max_{s_b^i \in S} \max \{D(f_{s_a^i, \theta} \| \hat{f}_{s_a^i, \theta}), D(f_{s_a^i, \theta} \| f_{s_a^i, \theta})\} \leq \lambda_\eta \}.
\]

Let $E_\eta^c$ denote the complement set. By the choice of $\lambda_\eta$, we have $\Pr[E_\eta^c] \leq \eta$. Then by (7) and the observation that $\Pr[E_\eta^c] \leq \eta$, we have that for any subset $T$,
\[
\Pr(F(X) + Z \in T | s_a^i, \theta) \leq \Pr(F(X) + Z \in T | s_a^i, \theta, E_\eta) \Pr[E_\eta] + \Pr[E_\eta^c] \\
\leq (\exp(\epsilon + 2\lambda_\eta) \Pr(F(X) + Z \in T | s_b^i, \theta, E_\eta) + \exp(\lambda_\eta)\delta) \Pr[E_\eta] \\
+ \Pr[E_\eta^c] \\
= \exp(\epsilon + 2\lambda_\eta) \Pr(F(X) + Z \in T | s_b^i, \theta, E_\eta) \Pr[E_\eta] \\
+ \exp(\lambda_\eta)\delta \Pr[E_\eta] + \Pr[E_\eta^c] \\
\leq \exp(\epsilon + 2\lambda_\eta) \Pr(F(X) + Z \in T | s_b^i, \theta) + \exp(\lambda_\eta)\delta + \eta,
\]

completing the proof.

4.3 Instantiation with Gaussian distributed data

In this section, we show an instantiation of our Attribute-Private Gaussian Mechanism when the joint distribution of the $m$ attributes is multivariate Gaussian. The privacy guarantee of this mechanism requires
that $F(X)|g_i(X_i)$ is Gaussian distributed, which is satisfied when $g_i$ and $F$ are linear with respect to the entries of $X$. For simplicity of illustration, we will choose both $F(X)$ and all $g_i(X_i)$ to compute averages.

As a motivating example, consider a dataset that consists of students’ SAT scores $X_s$, heights $X_h$, weights $X_w$, and their family income $X_f$. As a part of a wellness initiative, the school wishes to release the average weight of its students, so $F(X) = \frac{1}{n} \sum_{j=1}^{n} X_w^j$. The school also wants to prevent an adversary from inferring the average SAT scores or family income of their students, so $C = \{s, i\}$ and $g(X_i) = \frac{1}{n} \sum_{j=1}^{n} X_i^j$ for $i \in C$.

To instantiate our framework, let $s_a$ denote the event that $g(X_i) = a$, which means the average value of column $X_i$ is $a$. If $g(X_i)$ has support $\mathcal{U}$, then the set of secrets is $S = \{s_a : a \in \mathcal{U}, i \in C\}$, and the set of secret pairs is $Q = \{(s_a, s_b) : a, b \in \mathcal{U}, a \neq b, i \in C\}$. Each $\theta \in \Theta$ is a distribution over $n$ i.i.d. samples from an underlying multivariate Gaussian distribution with mean $(\mu_1, \ldots, \mu_m)^T$ and covariance matrix $(V_{ij})$, $i, j \in [m]$, where $V_{ij} = V_{ji}$ is the covariance between $X_i$ and $X_j$ if $i \neq j$, and $V_{ii}$ is the variance of $X_i$. We note that the variable heights, weights and SAT score may not be Gaussian distributed in practice. Hence, the choice of whether to use the Attribute-Private Gaussian Mechanism for Gaussian or non-Gaussian data should be determined by the practitioner.

Suppose we want to guarantee $(\epsilon, \delta)$-dataset attribute privacy through the Attribute-Private Gaussian Mechanism. Then we need to first compute $\mathbb{E}[F(X)|s_a]$ and $\text{Var}(F(X)|s_a)$ for each $i \in C$. Let $j$ denote the index of the attribute averaged in $F(X)$. By the properties of a multivariate Gaussian distribution, the distribution of $F(X)$ conditional on $g(X_i) = a$ is Gaussian $\mathcal{N}(\mu_a, \bar{V})$, where $\bar{V} = \frac{1}{n}(V_{jj} - \frac{V_{ii}^2}{V_{ii}})$. We define the diameter of $\mathcal{U}$ as $d(\mathcal{U}) = \max_{a, b \in \mathcal{U}} |a - b|$. The sensitivity is: $\Delta_i F = \max_{(s_a, s_b) \in Q} |\mu_a - \mu_b| = \frac{V_{ii}}{V_{ii}} \max_{a, b \in \mathcal{U}} |a - b| = \frac{V_{ii}}{V_{ii}} d(\mathcal{U})$. To ensure $(\epsilon, \delta)$-dataset attribute privacy for protected attribute $X_i$, the variance of the Gaussian noise must be at least $(\epsilon V_{ii}^2 d(\mathcal{U}))^2 - \frac{1}{n}(V_{jj} - \frac{V_{ii}^2}{V_{ii}})$ for $c = \sqrt{2\log(1.25/\delta)}$ as in Algorithm[4]. Adding Gaussian noise with variance $\sigma^2 = \max_{i \in C} \{c (\epsilon V_{ii}^2 d(\mathcal{U}))^2 - \frac{1}{n}(V_{jj} - \frac{V_{ii}^2}{V_{ii}})\}$ will provide $(\epsilon, \delta)$-dataset attribute privacy for all protected attributes.

We note that $\sigma^2$ is monotonically increasing with respect to $V_{jj}$. That is, our Attribute-Private Gaussian Mechanism will add less noise to the output if the query $F$ is about an attribute which has a low correlation with the protected attributes.

So far we have discussed about the case when $\Theta$ only consists of one distribution, in order to show the impact of $V_{jj}$. For the general case, the sensitivity $\Delta_i F$ is $\max_{\theta \in \Theta} \frac{V_{ii}}{V_{ii}} d(\mathcal{U})$, and the noise is scaled with variance $\sigma^2 = \max_{i \in C} \{c \max_{\theta \in \Theta} \frac{V_{ii}}{V_{ii}} d(\mathcal{U})\}^2 - \min_{\theta \in \Theta} \frac{1}{n} (V_{jj} - \frac{V_{ii}^2}{V_{ii}})$.

5 The Markov Quilt Mechanism for Distributional Attribute Privacy

In this section we consider distributional attribute privacy, as introduced in Definition[3] and develop a mechanism that satisfies this privacy definition. Recall that in this setting, an analyst aims to release $F(X)$ while protecting the realization of a random parameter $\phi_i$, which describes the conditional marginal distribution of the $i$th attribute, given the realization of all $\phi_k$ for $k \neq i$ for all other attributes. This formalization implies that all (column) attribute vectors $X_i$ are mutually independent, conditional on the set of parameters ($\phi_1, \ldots, \phi_m$).

5.1 Attribute-Private Markov Quilt Mechanism

We base our mechanism on the idea of a Markov Quilt, which partitions a network of correlated random variables into those which are “near” ($X_N$) a particular variable $X_i$, and those which are “remote” ($X_R$). Intuitively, we will use this to partition attributes into those which are highly correlated ($X_N$) with our sensitive attributes, and those which are only weakly correlated ($X_R$).
\textbf{Definition 7} (Markov Quilt). A set of nodes $X_Q$ in a Bayesian network $G = (X, E)$ is a Markov Quilt for a node $X_i$ if deleting $X_Q$ partitions $G$ into parts $X_N$ and $X_R$ such that $X_i \in X_N$ and $X_R$ is independent of $X_i$ conditioned on $X_Q$.

We quantify the effect that changing the distribution parameter $\phi_i$ of a sensitive attribute $X_i$ has on a set of distribution parameters $\phi_A$ (corresponding to a set of attributes $X_A$) using the \textit{max-influence}. Since attributes are mutually independent conditioned on the vector $(\phi_1, \ldots, \phi_m)$, the max-influence is sufficient to quantify how much a change of all values in attribute $X_i$ will affect the values of $X_A$. If $\phi_i$ and $\phi_A$ are independent, then $X_A$ and $X_i$ are also independent, and the max-influence is 0.

\textbf{Definition 8}. The max-influence of an attribute $X_i$ on a set of attributes $X_A$ under $\Theta$ is:

$$e_\Theta(X_A|X_i) = \sup_{\theta \in \Theta} \max_{\phi_i, \phi'A_\neq i} \max_{\phi_A \in \Phi_A} \log \frac{P(\phi_A|\phi_i^*; \theta)}{P(\phi_A|\phi_i^*; \theta)}.$$ 

The sensitivity of $F$ with respect to a set of attributes $A \subseteq [m]$, denoted $\Delta_A F$, is defined as the maximum change that the value of $F(X_i)$ caused by changing all columns $X_A$. Formally, we say that two datasets $X, X'$ are $A$-column-neighbors if they are identical except for the columns corresponding to attributes in $A$, which may be arbitrarily different. Then $\Delta_A F = \max_{X, X': A\text{-column-neighbors}} |F(X) - F(X')|$. Although changing $X_A$ may lead to changes in other columns, these changes are governed by the max influence, and will not affect attributes that are nearly independent of $X_i$.

Observe that the event that $X_R$ and $X_i$ are independent conditional on $X_N$ is equivalent to the event when $\phi_R$ and $\phi_i$ are independent conditional on $\phi_N$, which is why we can define the Markov Quilt based on $X_i$. However, since the distribution of $X_i$s are governed by $\phi_i$s, the max-influence score must be computed using $\phi_i$s rather than $X_i$s.

\textbf{The mechanism}. We extend the idea of the Markov Quilt Mechanism in [SWC17] to the attribute privacy setting as follows. Let $A \subseteq [m]$ be a set of attributes over which $F$ is computed. For example, $F$ may compute the average of a particular attribute or a regression on several attributes. At a high level, we add noise to the output of $F$ scaled based on the sensitivity of $F$ with respect to $X_N$'s. However, when computing the sensitivity of $F$ we only need to consider sensitivity of $F$ with respect to $A \cap N$, i.e., the queried set of attributes $A$ that are in the “nearby” set of the protected attribute. If the query $F$ is about attributes that are all in the “remote” set $X_R$, and the max-influence on the corresponding Markov quilt is less than the privacy parameter $\epsilon$, then $\Delta_{A \cap N} F$ is simply 0 and the mechanism will not add noise to the query answer.

\textbf{Algorithm 3} Attribute-Private Markov Quilt Mechanism, APMQM($X, F, A, C, \{S, Q, \Theta\}, \epsilon$) for distributional attribute privacy.

\begin{algorithmic}
\Functions
\Procedure{APMQM}{$X, F, A, C, \{S, Q, \Theta\}, \epsilon$}
\For{each $i \in C$}
\State Set $b_i = \Delta_A F/\epsilon$.
\State Set $G_i := \{(X_Q, X_N, X_R) : e_\Theta(X_Q|X_i) \leq \epsilon\}$ to be all possible Markov quilts of $X_i$ with max-influence less than $\epsilon$.
\If{$G_i \neq \emptyset$}
\For{each $(X_Q, X_N, X_R) \in G_i$}
\If{$\Delta_{A \cap N} F/(\epsilon - e_\Theta(X_Q|X_i)) \leq b_i$}
\State Set $b_i = \Delta_{A \cap N} F/(\epsilon - e_\Theta(X_Q|X_i))$.
\EndIf
\EndFor
\EndIf
\EndFor
\State Sample $Z \sim \text{Lap}(\max_{i \in C} b_i)$.
\State Return $F(X) + Z$.
\EndProcedure
\end{algorithmic}
Theorem 4. The Attribute-Private Markov Quilt Mechanism \( APMQM(X,F,A,C,\{S,Q,\Theta\},\epsilon) \) in Algorithm \( 3 \) is \((\epsilon,0)\)-distributional attribute private.

Proof. First, consider the case when \( G = \emptyset \), which means there are no Markov quilt partitions such that the max-influence score is less than \( \epsilon \). In this case, for a fixed secret attribute \( X_i \), the mechanism will simply add Laplace noise scaled with \( \Delta F \). This privacy follows from Laplace Mechanism in differential privacy.

Let us consider the case when \( G \neq \emptyset \). Below we bound the probability distribution for a single outcome \( w \). For the set of outcomes \( T \), the proof can be extended to bound the integral of the probability distribution over the set \( T \). We fix any pair of secrets \( (s^i_h,s^i_g) \in Q \) for a fixed secret attribute \( X_i \) under any \( \theta \in \Theta \). Here \( s^i_h \) denotes the event that \( s_i = a \), and we consider the more general case later. Let \( Z \) be the Laplace noise as generated in Algorithm \( 3 \). If there exists a Markov quilt for \( X_i \), \( i \in C \), with max-influence score less than \( \epsilon \), then for any attribute \( X_j \in A \):

\[
\max_{a,b} \frac{P(M(X) = w|\phi_i = a,\theta)}{P(M(X) = w|\phi_i = b,\theta)} = \max_{a,b} \frac{P(F(X) + Z = w|\phi_i = a,\theta)}{P(F(X) + Z = w|\phi_i = b,\theta)} = \max_{a,b} \max_{R_{\text{RUQ}}} \frac{P(F(X) + Z = w|\phi_i = a,\phi_{R_{\text{RUQ}}} = v,\theta)}{P(F(X) + Z = w|\phi_i = b,\phi_{R_{\text{RUQ}}} = v,\theta)} \cdot \frac{P(\phi_{R_{\text{RUQ}}} = v|\phi_i = a,\theta)}{P(\phi_{R_{\text{RUQ}}} = v|\phi_i = b,\theta)}
\]

where the final equality Equation \( 3 \) follows from the independence between \( X_{R_{\text{RUQ}}} \) and \( \phi_i \) given \( \phi_{R_{\text{RUQ}}} = v \), and \( x_{R_{\text{RUQ}}} \) denotes a realization of the \( X_{R_{\text{RUQ}}} \) columns. For a fixed \( X_{R_{\text{RUQ}}} \), \( F(X) \) can vary by at most \( \Delta_{\text{RUQ}} F \), and therefore, the first ratio is bounded by \( \exp(\epsilon - \exp(e_\theta(X_i|X_i))) \). The second ratio in Equation \( 6 \) is 1, and the third ratio in Equation \( 8 \) is bounded by \( \exp(e_\theta(X_i|X_i)) \). Then Equation \( 3 \) is bounded above by \( \exp(\epsilon - \exp(e_\theta(X_i|X_i))) \exp(e_\theta(X_i|X_i)) = \exp(\epsilon) \). For the general case when \( s^i_h := \mathbb{1}[\phi_i \in \Phi^i_a]: \Phi^i_a \subset \Phi^i \), similarly, for any \( \Phi^i_a \) and \( \Phi^i_b \), we have,

\[
\max_{a,b} \frac{P(M(X) = w|\phi_i \in \Phi^i_a,\theta)}{P(M(X) = w|\phi_i \in \Phi^i_b,\theta)} \leq \max_{R_{\text{RUQ}}} \frac{P(F(X) + Z = w|X_{R_{\text{RUQ}}} = x_{R_{\text{RUQ}},\phi_i \in \Phi^i_a,\theta}) P(\phi_{R_{\text{RUQ}}} = v|\phi_i \in \Phi^i_a,\theta)}{P(F(X) + Z = w|X_{R_{\text{RUQ}}} = x_{R_{\text{RUQ}},\phi_i \in \Phi^i_b,\theta}) P(\phi_{R_{\text{RUQ}}} = v|\phi_i \in \Phi^i_b,\theta)} \leq \max_{a,b} \max_{R_{\text{RUQ}} \in \Phi^i_a \cup \Phi^i_b} \frac{P(F(X) + Z = w|X_{R_{\text{RUQ}}} = x_{R_{\text{RUQ}},\phi_i = a,\theta})}{P(F(X) + Z = w|X_{R_{\text{RUQ}}} = x_{R_{\text{RUQ}},\phi_i = b,\theta})} \max_{a,b} \max_{R_{\text{RUQ}} \in \Phi^i_a \cup \Phi^i_b} \frac{P(\phi_{R_{\text{RUQ}}} = v|\phi_i = a,\theta)}{P(\phi_{R_{\text{RUQ}}} = v|\phi_i = b,\theta)}
\]

The first ratio in Equation \( 10 \) is bounded by \( \exp(\epsilon - \exp(e_\theta(X_i|X_i))) \) and the second ratio is bounded by \( \exp(e_\theta(X_i|X_i)) \), so Equation \( 10 \) is bounded above by \( \exp(\epsilon) \), and the theorem follows.

\[\square\]

Example 1. Consider a dataset that consists of students’ SAT scores \( X_s \), heights \( X_h \), weights \( X_w \), gender \( X_g \), and their family income \( X_i \), where these variables form a Bayesian network as in Figure \( 7 \). The school wishes to release the number of students that are taller than \( 5'6'' \), while protecting the distribution of family income of their students with privacy parameter \( \epsilon \). In this case, \( C = \{i\} \), \( A = \{h\} \) and \( F(X) = \sum_{j=1}^n \mathbb{1}[X_{i,j} > 5'6''] \). Consider a Markov quilt for \( X_i \): \( Q = \{g\}, N = \{i,s\}, R = \{h,w\} \). Then \( A \cap N = \emptyset \), so we can safely release \( F(X) = \sum_{j=1}^n \mathbb{1}[X_{i,j} > 5'6''] \) without additional noise.

Next consider the case when the school wishes to release the number of students that are taller than \( 5'6'' \) and have SAT score > 1300. Then, \( F(X) = \sum_{j=1}^n \mathbb{1}[(X_{i,j} > 5'6'') \land (X_{j} > 1300)] \) and \( A = \{h,s\} \). In this case we can still use the same Markov quilt as before, but now \( A \cap N = \{s\} \). The mechanism will add Laplace noise scaled with \( \Delta_{\{s\}} F/\epsilon - e_\theta(X_{g|X_i}) \).
It is instructive to contrast the above mechanism to the Markov Quilt Mechanism of [SWC17], presented fully in Appendix A. The most important difference is that the mechanism in [SWC17] was not designed to guarantee attribute privacy. It provides privacy of the values $X_i$ but does not protect the distribution from which $X_i$ is generated. This difference in high-level goals leads to three key technical differences. Firstly, the definition of max-influence in [SWC17] measures influence of a \textit{variable value on values of other variables}. This is insufficient when one wants to protect distributional information, as $X_i$ may take a range of values while still following a particular distribution (e.g., hiding the gender of an individual in a dataset vs. hiding the proportion of females to males in this dataset.) Secondly, while it is natural to consider $L$-Lipschitz functions to bound sensitivity when one value changes (as is done in [SWC17]), this is not applicable to settings where the distribution of data changes, since this may change all values in a column. As a result, we do not restrict $F$ in this way. Finally, the mechanisms themselves are different as [SWC17] consider answering query $F$ over all attributes of an individual. As a result, they need to consider sensitivity of a function to all the “nearby” attributes. In contrast, we only consider sensitivity of those “nearby” attributes that happen to be in the query (i.e., those in $A$).

6 The Wasserstein Mechanism for General Attribute Privacy

The Wasserstein Mechanism [SWC17] (Algorithm 4) is a general mechanism for satisfying Pufferfish privacy; Algorithm 4 is $(\epsilon, 0)$-Pufferfish private for any instantiation of the Pufferfish framework [SWC17]. It defines sensitivity of a function $F$ as the maximum Wasserstein distance $W_\infty$ between the distributions of $F(X)$ under two different realizations of secrets $s_i$ and $s_j$ for $(s_i, s_j) \in Q$, and then outputs $F(X)$ plus Laplace noise that scales with this sensitivity. The distance metric $W_\infty$ denotes the $\infty$-Wasserstein distance between two probability distributions, formally defined below.

**Definition 9** ($\infty$-Wasserstein distance, $W_\infty$). Let $\mu, \nu$ be two probability distributions on $\mathbb{R}^d$ and let $\Gamma(\mu, \nu)$ be the set of all joint distributions with marginals $\mu$ and $\nu$. The $\infty$-Wasserstein distance between $\mu$ and $\nu$ is defined as:

$$W_\infty(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \max_{(x, y) \in \text{support}(\gamma)} |x - y|.$$  (11)

The $\infty$-Wasserstein distance is closely related to optimal transportation. Each $\gamma \in \Gamma(\mu, \nu)$ can be viewed as a way to shift probability mass between $\mu$ and $\nu$, and the cost is $\max_{(x, y) \in \text{support}(\gamma)} |x - y|$. For discrete distributions, the $\infty$-Wasserstein distance is the minimum of the maximum distance that any probability mass moves to transform $\mu$ to $\nu$.

**Algorithm 4** Wasserstein Mechanism $(X, F, \{S, Q, \Theta\}, \epsilon)$ [SWC17]

\begin{verbatim}
Input: dataset $X$, query $F$, Pufferfish framework $\{S, Q, \Theta\}$, privacy parameter $\epsilon$
for all $(s_i, s_j) \in Q$ and all $\theta \in \Theta$ such that $P(s_i|\theta) \neq 0$, and $P(s_j|\theta) \neq 0$
    do
        Set $\mu_{i, \theta} = P(F(X)|s_i, \theta)$, $\mu_{j, \theta} = P(F(X)|s_j, \theta)$.
        Calculate $W_\infty(\mu_{i, \theta}, \mu_{j, \theta})$.
    end for
Set $W = \sup_{(s_i, s_j) \in Q, \theta \in \Theta} W_\infty(\mu_{i, \theta}, \mu_{j, \theta})$.
Sample $Z \sim \text{Lap}(W/\epsilon)$.
Return $F(X) + Z$.
\end{verbatim}

\[\text{In general, Wasserstein distance can be defined on any metric space. We will use it only over the real numbers with the Euclidean metric.}\]
Since our framework is an instantiation of Pufferfish privacy, the Wasserstein Mechanism provides a general way to protect either dataset attribute privacy or distributional attribute privacy, when instantiated with the appropriate Pufferfish framework \((S, \mathcal{Q}, \Theta)\). This is stated formally in Theorem 5 and illustrated in Examples 2 and 3 below.

**Theorem 5.** The Wasserstein Mechanism \((X, F, \{S, \mathcal{Q}, \Theta\}, \epsilon)\) in Algorithm 4 is \((\epsilon, 0)\)-dataset attribute private and \((\epsilon, 0)\)-distributional attribute private.

Despite the general purpose nature of the Wasserstein Mechanism for achieving attribute privacy, it is known that computing Wasserstein distance is computationally expensive \([AM19]\). Instantiating Algorithm 4 to satisfy attribute privacy may require computing Wasserstein distance for exponentially many pairs of secrets, one for each subset of values of \(g_i(X_i)\) or \(\phi_i\). This motivates our study of the Attribute-Private Gaussian Mechanism (Algorithm 1) and the Attribute-Private Markov Quilt Mechanism (Algorithm 5), which are both computationally efficient for practical use.

In some special cases, the Attribute-Private Wasserstein Mechanism may be a feasible option. For example, when computing \(\infty\)-Wasserstein distance between a pair of 1-dimensional distributions as in Definition 0, then \(W_\infty\) can be computed efficiently \([LLL18]\). While there exist efficient approximations to \(\infty\)-Wasserstein distance, any approximation used in the Attribute-Private Wasserstein Mechanism must always be an overestimate of \(W_\infty\) to ensure that sufficient noise is added to guarantee privacy. Efficiently computable upper bounds on \(W_\infty\) exist under certain technical conditions on the distributions \([GG20]\). In both the case of 1-dimensional distributions and approximations of \(\infty\)-Wasserstein distance, if the instantiation of Pufferfish privacy requires computing \(W_\infty\) over a polynomial number of distributions (i.e., polynomially many pairs of secrets and polynomially many possible distributions \(\theta\)), then the Attribute-Private Wasserstein Mechanism (or an approximate version of the mechanism) can be run efficiently.

**Example 2 (Wasserstein Mechanism for Dataset Attribute Privacy).** Consider two binary attributes \(X_1\) and \(X_2\), where \(X_1\) is the non-sensitive attribute and \(X_2\) is the sensitive attribute. Suppose the dataset contains data from four people, and let the underlying distribution and dependence between \(X_1\) and \(X_2\) is characterized by the following probability distributions:

\[
P(X_1 = 1|X_2 = 1) = p_1 \text{ and } P(X_1 = 1|X_2 = 0) = p_2,
\]

Suppose \(0.4 \leq p_1 \leq 0.6\) and \(0.4 \leq p_2 \leq 0.6\). The analyst wishes to release the summation of \(X_1\), \(F(X) = \sum_{j=1}^{4} X_1^j\), while protecting the summation of \(X_2\), so \(g(X_2) = \sum_{j=1}^{4} X_2^j\). To instantiate our framework, let \(S_a\) denote the event that \(g(X_2) = a\). The support of \(g(X_2)\) is \(\mathcal{U} = \{0, 1, 2, 3, 4\}\). Then the set of secrets is \(S = \{s_a^2 : a \in \mathcal{U}\}\), and the set of secret pairs is \(\mathcal{Q} = \{(s_a^2, s_b^2) : a, b \in \mathcal{U}, a \neq b\}\). Each \(\theta \in \Theta\) is a certain pair of \(p_1\) and \(p_2\) such that \(0.4 \leq p_1 \leq 0.6\) and \(0.4 \leq p_2 \leq 0.6\).

Consider the pair of conditional probabilities \(\mu_{a,b} = P(F(X) = \cdot|s_a^2, \theta)\) and \(\mu_{b,a} = P(F(X) = \cdot|s_b^2, \theta)\). The worst case Wasserstein distribution between the pair of conditional probability distributions is reached when \(p_1 = 0.4, p_2 = 0.6, a = 0, b = 4\) when the two conditional probabilities differ the most. We list the conditional probabilities for this case in Table 7.

| \(j\) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \(P(F(X) = j|g(X_2) = 0)\) | 0.0256 | 0.1536 | 0.3456 | 0.3456 | 0.1296 |
| \(P(F(X) = j|g(X_2) = 4)\) | 0.1296 | 0.3456 | 0.3456 | 0.1536 | 0.0256 |

Table 1: Conditional probability distributions under two extreme secrets for dataset attribute privacy

Here, the Wasserstein distance \(W_\infty(P(F(X)|g(X_2 = 0)), P(F(X)|g(X_2 = 4)) = 1\), since the optimal transportation is moving the mass from 1 to 2 and 4 to 3, and the Wasserstein mechanism will add \(\text{Lap}(1/\epsilon)\) noise to \(F(X)\).

The mechanism with group differential privacy would add \(\text{Lap}(4/\epsilon)\), which gives worse utility. We note that the noise we add depends largely on the underlying distribution class \(\Theta\). For example, when
Example 3 (Wasserstein Mechanism for Distributional Attribute Privacy). Consider the same setting as Example 2, a dataset of four people with two binary attributes $X_1$ and $X_2$, where $X_1$ is non-sensitive and $X_2$ is sensitive. Let the underlying distribution and dependence between realized attributes $X_1$ and $X_2$ still be governed by \(\Theta = \{0, 1\}\), and for simplicity fix $p_1 = 0.4$ and $p_2 = 0.6$. In the setting of distributional attribute privacy, we are interested in the conditional marginal distribution parameters of $X_i$ given the parameter for $X_j$, rather than the realization of $X_i$. We denote the Bernoulli distribution parameter for $X_1$ and $X_2$ as $\phi_1$ and $\phi_2$, respectively. According to \[\phi_1 = 0.4\phi_2 + 0.6(1 - \phi_2) = 0.6 - 0.2\phi_2.\]

The analyst wishes to release the summation of $X_1$, $F(X) = \sum_{j=1}^{4} X_j$, while protecting the distributional parameter $\phi_2$ for $X_2$. To instantiate our framework, we let $s_a^2$ denote the event that $\phi_2 = a$, and we suppose the support of $\phi_2$ is $\Phi^2 = [0.2, 0.8]$. The set of secrets is $S = \{s_a^2 : a \in \Phi^2\}$, and the set of secret pairs is $Q = \{(s_a^2, s_b^2) : a, b \in \Phi^2, a \neq b\}$. Each $\theta \in \Theta$ is a certain pair of $\phi_1$ and $\phi_2$ such that $\Phi^2 = [0.2, 0.8]$ and $\phi_1 = 0.6 - 0.2\phi_2$.

In this case, the support for $\phi_1$ is $[0.44, 0.56]$. Consider the pair of conditional probabilities $\mu_{a,\theta} = P(F(X) = |s_a^2, \theta)$ and $\mu_{b,\theta} = P(F(X) = |s_b^2, \theta)$. The worst case Wasserstein distribution between the pair of conditional probability distributions is reached when $a = 0.8$ and $b = 0.2$ when the two conditional probabilities differ the most. We list the conditional probabilities for this case in Table 2.

| $j$ | $|\phi_2 = 0.8|$ | $|\phi_2 = 0.2|$ |
|-----|----------------|----------------|
| 0   | 0.0983         | 0.0375         |
| 1   | 0.3091         | 0.1908         |
| 2   | 0.3643         | 0.3643         |
| 3   | 0.1908         | 0.3091         |
| 4   | 0.0983         | 0.0375         |

Table 2: Conditional probability distributions under two extreme secrets for distributional attribute privacy

The Wasserstein distance $W(\infty)(P(F(X)|\phi_2 = 0.8), P(F(X)|\phi_2 = 0.2)) = 1$, since the optimal transportation is moving the mass from 1 to 2 and 4 to 3, and the Wasserstein mechanism will add Lap(1/\(\epsilon\)) noise to $F(X)$.

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A Additional Preliminaries

In this appendix we review the Markov Quilt Mechanism of [SWC17] for satisfying Pufferfish privacy. This algorithm assumes that the entries in the input database $Y$ form a Bayesian Network (Definition 2). These entries could either be: (1) the multiple attributes of a single record when the database contained only one record, or (2) the attribute values across multiple records for a single-fixed attribute when the database contained multiple attributes. Hence, the original Markov Quilt Mechanism could not accommodate correlations across multiple attributes in multiple records, as we study in this work.

The following definition measures influence of a variable value on values of other variables. One can compare this to Definition 8 used in the Attribute-Private Markov Quilt Mechanism, which instead measures max-influence of a parameter of the probability distribution of a variable on the distribution of parameters of other variables, as is needed in the attribute privacy setting.

**Definition 10** (Variable-Max-Influence [SWC17]). The maximum influence of a variable $Y_i$ on a set of variables $Y_A$ under $\Theta$ is:

$$e^\Theta(Y_A|Y_i) = \sup_{\theta \in \Theta} \max_{a,b \in \mathcal{Y}} \max_{y_A \in \mathcal{Y}^{\text{var}(Y_A)}} \log \frac{P(Y_A = y_A|Y_i = a, \theta)}{P(Y_A = y_A|Y_i = b, \theta)}.$$

Recall the definition of a Markov Quilt (Definition 7), which is used in this mechanism.

**Algorithm 5** Markov Quilt Mechanism ($Y, F, \{S, Q, \Theta\}, \epsilon$) [SWC17]

**Input:** database $Y$, $L$-Lipschitz query $F$, Pufferfish framework $\{S, Q, \Theta\}$, privacy parameter $\epsilon$.

**for each** $Y_i$ **do**

Let $G_i := \{(Y_Q, Y_N, Y_R) : Y_Q$ is a Markov Quilt of $Y_i\}$

**for all** $Y_Q$ (with $Y_N, Y_R$) in $G_i$ **do**

if $e^\Theta(Y_Q|Y_i) < \epsilon$ **then**

Set $b(Y_Q) = \frac{|Y_N|}{\epsilon - e^\Theta(Y_Q|Y_i)}$.

**else**

Set $b(Y_Q) = \infty$.

**end if**

**end for**

Set $b_i = \min_{Q \in G_i} b(Y_Q)$.

**end for**

Set $b_{\text{max}} = \max_i b_i$.

Sample $Z \sim \text{Lap}(L \cdot b_{\text{max}})$.

Return $F(Y) + Z$.

The Markov Quilt Mechanism [SWC17] given in Algorithm 5 guarantees $(\epsilon, 0)$-Pufferfish Privacy.