HARMONIC ANALYSIS OF FRACTAL MEASURES INDUCED
BY REPRESENTATIONS OF A CERTAIN C\(^{*}\)-ALGEBRA

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Abstract. We describe a class of measurable subsets \( \Omega \) in \( \mathbb{R}^d \) such that \( L^2(\Omega) \) has an orthogonal basis of frequencies \( e_\lambda(x) = e^{i2\pi \lambda \cdot x} \) indexed by \( \lambda \in \Lambda \subset \mathbb{R}^d \). We show that such spectral pairs \((\Omega, \Lambda)\) have a self-similarity which may be used to generate associated fractal measures \( \mu \) with Cantor set support. The Hilbert space \( L^2(\mu) \) does not have a total set of orthogonal frequencies, but a harmonic analysis of \( \mu \) may be built instead from a natural representation of the Cuntz \( C^*\)-algebra which is constructed from a pair of lattices supporting the given spectral pair \((\Omega, \Lambda)\). We show conversely that such a pair may be reconstructed from a certain Cuntz-representation given to act on \( L^2(\mu) \).

1. Introduction

Let \( \Omega \) be a subset in \( d \) real dimensions (i.e., \( \Omega \subset \mathbb{R}^d \), \( d \geq 1 \)), and suppose that \( \Omega \) has finite positive \( d \)-dimensional Lebesgue measure. Let \( L^2(\Omega) \) be the corresponding Hilbert space with the usual inner product given by

\[
\langle f, g \rangle = m_d(\Omega)^{-1} \int_{\Omega} \overline{f(x)} \, g(x) \, dx
\]

where \( dx := dx_1 \cdots dx_d \), and \( m_d(\Omega) \) denoting the Lebesgue measure of \( \Omega \). Motivated by a problem of I. E. Segal and a paper by B. Fuglede [Fu], we considered in [JP1–3] the problem of deciding, for given \( \Omega \), when \( L^2(\Omega) \) may possibly have an orthogonal basis of frequencies: For \( \lambda \in \mathbb{R}^d \), let \( x \cdot \lambda = \sum_{j=1}^d x_j \lambda_j \) be the usual dot product, and set

\[
e_\lambda(x) = e^{i2\pi \cdot x \lambda}.
\]

We say that two vector frequencies \( \lambda, \lambda' \) in \( \mathbb{R}^d \) are orthogonal on \( \Omega \) if

\[
\int_{\Omega} e^{i2\pi(\lambda' - \lambda) \cdot x} \, dx = 0.
\]

When \( \Omega \) is further assumed open in \( \mathbb{R}^d \), this problem is directly connected (see [Fu, JP1]) with the problem of finding simultaneous commuting selfadjoint extension operators for the partial derivatives \( \sqrt{-1 \frac{\partial}{\partial x_j}} \) \( (1 \leq j \leq d) \) acting on \( C^\infty_c(\Omega) \) (= all smooth compactly supported functions in \( \Omega \)). In general, the problem may be given a group-theoretic formulation, and, in this form, we showed in [JP1] that it relates

1991 Mathematics Subject Classification. Primary 28A75, 42B10, 46L55.

Research supported by the NSF. The first author was partially supported by a University of Iowa Faculty Scholar Award and the UI (Oakdale Campus) Institute for Advanced Studies.

Received by the editors September 25, 1992

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0273-0979/93 $1.00 + $.25 per page
directly to a property of the representation ring generated by a certain induced representation. (See (2) below.)

2. Classical Examples

The most obvious examples of sets Ω with the basis property are measurable sets in \( \mathbb{R}^d \) which are **fundamental domains** of lattices (see [Fu, JP1]). Let \( \Gamma \) be a rank \( d \) lattice, and let \( \Gamma^0 \) be the dual lattice. (Recall \( \Gamma^0 = \{ \lambda \in \mathbb{R}^d : \lambda \cdot s \in \mathbb{Z}, \forall s \in \Gamma \} \).)

Suppose Ω is a measurable fundamental domain for \( \Gamma \). It is a simple matter to show then that \( \{ e_\lambda : \lambda \in \Gamma^0 \} \) is an orthogonal basis for \( L^2(\Omega) \). This elementary class of examples is in fact characterized by a multiplicative property (see [JP1, 2]), and they are called **multiplicative**. A pair—\((\Omega, \Lambda)\) such that \( 0 \in \Lambda \), and \( \{ e_\lambda : \lambda \in \Lambda \} \) is an orthogonal basis in \( L^2(\Omega) \)—is called a **spectral pair**, and the set \( \Lambda \) is called the **spectrum**. We further showed in [JP1] that every spectral pair \((\Omega, \Lambda)\) in \( d \) dimensions may be factored, \((\Omega, \Lambda) \simeq (\Omega', \Lambda') \times (\Omega'', \Lambda'')\), such that the factors each are spectral pairs in dimensions \( d' \), \( d'' \) respectively, \( d' + d'' = d \), \((\Omega', \Lambda')\) is multiplicative, and \((\Omega'', \Lambda'')\) is in “the other extreme”. Specifically, this second factor generates a representation ring which is a copy of the algebra of all \( q \times q \) complex matrices where \( q \) is a certain cover-multiplicity (see [JP2]), and \((\Omega'', \Lambda'')\) is called a **simple factor**.

3. Spectral pairs

In this paper, we shall consider the simple factors in more detail and show that they are associated with “fractals” in a sense which we proceed to describe. If \((\Omega, \Lambda)\) is a spectral pair in \( d \) dimensions, consider the group \( K = \Lambda^0 = \{ s \in \mathbb{R}^d : s \cdot \lambda \in \mathbb{Z}, \forall \lambda \in \Lambda \} \). We further showed in [JP1] that \( K \) is a rank \( d \) lattice and that there is a canonical embedding of \( \Omega \) into the torus \( \mathbb{R}^d/K \) such that the image \( \Omega' \) of \( \Omega \) on the torus again has the basis-property (relative to Haar measure on the torus) and the spectrum of \( \Omega' \) is the same set \( \Lambda \). We say that the pair \((\Omega', \Lambda')\) is in **reduced form**.

We have a second closed subgroup \( A \) in \( \mathbb{R}^d \) directly associated with some given spectral pair \((\Omega, \Lambda)\),

\[
A = \{ a \in \mathbb{R}^d : x + a \in \Omega + \Lambda^0 \text{ (a.e.) } x \in \Omega \}.
\]

Define a unitary representation \( U_t \) \((t \in \mathbb{R}^d)\), acting on \( L^2(\Omega) \), given by

\[
(2) \quad U_t e_\lambda = e^{i2\pi t \cdot \lambda} e_\lambda \quad (t \in \mathbb{R}^d, \lambda \in \Lambda),
\]

and note that \( A \) may be characterized alternatively as the group

\[
\{ t \in \mathbb{R}^d : U_t \text{ acts multiplicatively on } L^2(\Omega) \}.
\]

When \( t \in A \), then

\[
(3) \quad U_t f(x) = f(x + t), \text{ a.e. } x \in \Omega', \forall f \in L^2(\Omega')
\]

where the sum \( x + t \) is in the torus \( \mathbb{R}^d/K \). Hence, we get \( A \) acting as a group of torus-translations on \( \Omega' \).
We say that some given spectral pair \((\Omega, \Lambda)\) is multiplicative if \(A = \mathbb{R}^d\) and is a simple factor if \(A\) is a lattice in \(\mathbb{R}^d\). There is a sense in which simple factors may be generated by lattice systems, but we do not yet have a complete structure theorem which covers all simple factors. It is not known if, for a simple factor with associated lattices \(K\) and \(A\), the degenerate case \(K = A\) may occur. (We expect not!) In [JP1], we proved the following result (which will be needed below) about nondegenerate simple factors:

**Theorem 1** (see [JP1], Theorem 6.1). Let \((\Omega, \Lambda)\) be a spectral pair in \(\mathbb{R}^d\), and suppose that the group \(S\), given by
\[
S = \{ s \in \Lambda : s + \Lambda = \Lambda \}
\]
is a lattice. Let \(\Gamma = S^0\), and suppose

(i) \(A \subset \Gamma\), and

(ii) there is a section \(L\) for \(S\) in \(\Lambda\) such that \(A\) separates points on \(L\) (i.e., when \(\ell, \ell' \in L, \ell \neq \ell'\), then there is some \(a \in A\) s.t. \(e^{i2\pi \ell \cdot a} \neq e^{i2\pi \ell' \cdot a}\)).

Then it follows that every measurable section \(D'\) inside \(\Omega'\) (reduced form) for the action (3) of \(A\) by translation is a fundamental domain for \(\Gamma\) and, moreover, that

\[
\Omega' = \bigcup_{a \in A/K} (D' + a)
\]
and
\[
(D' + a_1) \cap (D' + a_2) = \emptyset
\]
for all $a_1 \neq a_2$ in $A/K$.

3.1. Spectral duality. In studying more general simple factors, we introduced in [JP3] an inductive limit construction which applies to the basic factors described in Theorem 1, and we found, as the limit object, the Hilbert space $L^2(\mu)$ where $\mu$ is a Hausdorff measure of fractional dimension (see [Fa, Hu, St1–3]). Such measures are known to be supported by Cantor type-sets, $C$, say (see [Hu]), but typically the Lebesgue measure of $C$ is zero. We now show that $C$ may be built by self-similarity from simple factors.

3.2. Let $(\Omega, \Lambda)$ be a spectral pair subject to the conditions in Theorem 1; let $K \subset A \subset \Gamma$ be the associated lattices; let $L$ be the section in $\Lambda$ (assume $0 \in L$); and finally, let $R$ be the inclusion matrix for $K \subset \Gamma$. (Let $\{u_i\}_{i=1}^d$ be generators for $K$ over $\mathbb{Z}$ and $\{v_i\}_{i=1}^d$ for $\Gamma$; then $R \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R})$ may be defined by $u_i = \sum_j R_{ij}v_j$. Recall $K = \{\sum_i n_iu_i : n_i \in \mathbb{Z}, \ 1 \leq i \leq d\}$, and similarly for $\Gamma$.) Since $L \subset K^0$, we may consider affine mappings, $s \mapsto Rs + \ell$, acting on the lattice $K^0$. This map will be denoted $\tau_\ell$, and the underlying lattice $K^0$ will be understood from the context. Consider the mapping $\tau_0(s) = Rs$ given by matrix-multiplication. When the bases $(u_i)$ for $K$ and $(v_i)$ for $\Gamma$ are given, let $(u_i^*)$ for $K^0$ and $(v_i^*)$ for $\Gamma^0$ be dual bases, i.e., $u_i^* \cdot v_j = \delta_{ij}, \ 1 \leq i, j \leq d$. For $s = \sum_i s_iu_i^*$ with integral coordinates, $s_i \in \mathbb{Z}$, we have

$$\tau_0(s) = \sum_i(Rs)_iu_i^* = \sum_i s_iu_i^*,$$

and $(Rs)_i = \sum_j R_{ij}s_j$. Note then that $\tau_0(K^0) \subset K^0$, and each $\tau_\ell, \ell \in L$, is affine on the lattice $K^0$. If $\Gamma^0$ is identified with a sublattice in $K^0$, then $\tau_0(K^0) = \Gamma^0$, and the matrix-transpose $R_{ij} = R_{ji}$ is the inclusion-matrix for the dual lattice-inclusion $\Gamma^0 \subset K^0$.

3.3. The Fractal Measure. Also consider the affine maps $S_b$ on $\mathbb{R}^d$ given by

$$S_b x = R^{-1}x + b, \quad x \in \mathbb{R}^d. \tag{5}$$

In formula (5), the term $R^{-1}x$ is really $\tau_0^{-1}(x)$, which is to say that the matrix-product $R^{-1}x$ must refer to the same basis $(u_i^*)$ for $K^0$ that was used in calculating $\tau_0$ above. (In some different basis, of course, the matrix will change, i.e., $R$ becomes $ARA^{-1}$ with $A$ denoting the associated transform matrix.)

Let $N$ be the cardinality of $L$; by Theorem 1, it is also the order of the group $A/K$. Pick a subset $B \subset A, 0 \in B$, representing the elements in $A/K$, equivalently a section for the quotient; and let the affine maps $S_b$ be indexed by $b \in B$. By Hutchinson’s theorem (see [Hu, St1–2]) there is self-similar probability measure $\mu$ on $\mathbb{R}^d$ such that $\mu = \frac{1}{N} \sum_{b \in B} \mu \circ S_b^{-1}$, or, equivalently,

$$\int f(x) \, d\mu(x) = \frac{1}{N} \sum_{b \in B} \int f(S_b x) \, d\mu(x)$$

for measurable functions $f$ on $\mathbb{R}^d$. We show in [JP3] that there is a “Cantor set” $C \subset \mathbb{R}^d$, which is built from iteration of the decomposition (4) and self-similarity and which supports $\mu$, i.e., $\mu(C) = 1$. We let $L^2(\mu)$ be the corresponding Hilbert space.
3.4. The Cuntz Algebra. Our two theorems below connect the classical harmonic analysis of \((\Omega, \Lambda)\) to the associated fractal measure \(\mu\):

**Theorem 2.** Let \((\Omega, \Lambda)\) be a nondegenerate simple factor given by the conditions in Theorem 1 with matrix \(R\) for the lattice inclusion \(K \subset \Gamma\), and section \(L\) for \(\Lambda\) such that \(0 \in L\) and \(\Lambda = L + \Gamma^0\), and finally let \(\mu\) be the associated Hutchinson measure with support \(C\). Then it follows that

(i) \(\{e_s : s \in K^0\}\) separates points in \(C\), i.e., for \(x \neq x'\) in \(C\), \(\exists s \in K^0\) s.t. \(e_s(x) \neq e_s(x')\).

(ii) For each \(\ell \in L\), an isometry \(T_\ell\) acting on \(L^2(\mu)\) is well defined by

\[
T_\ell e_s = e_{\tau_\ell(s)} \quad \forall s \in K^0.
\]

(iii) As operators on \(L^2(\mu)\), the isometries \(T_\ell\) satisfy

\[
T_\ell^* T_{\ell'} = \begin{cases} 0 & \text{if } \ell \neq \ell' \text{ in } L, \\ I & \text{if } \ell = \ell', \end{cases}
\]

and \(\sum_{\ell \in L} T_\ell T_\ell^* = I\), where \(I\) denotes the identity operator on \(L^2(\mu)\).

(iv) The representation of the Cuntz \(C^*\)-algebra \(O(L)\) generated by the isometries in (iii) (see [Cu]) has a canonical factor decomposition associated with the triple \(K \subset A \subset \Gamma\) of lattices and the (dual) fractal measure \(\mu\) may be reconstructed directly from the associated factor state on \(O(L)\) of the decomposition. (Note that the decomposition is orthogonal, and in the category of representations of \(C^*\)-algebras; see [BR]).

(v) The cyclic \(e_o\)-representation of \(O(L)\) by the \(T_\ell\) isometries is the GNS representation (see [BR]) of the factor state \(\omega\) on \(O(L)\) which is determined by the relations in (iii), \(\omega(I) = 1\), and \(\omega(T_\ell T_\ell^*) = 1 = \omega(T_o)\).

(vi) The set of all vectors

\[
\{e_o\} \cup \bigcup_{n=1}^{\infty} \{T_{\ell_1} \cdots T_{\ell_n} e_o : \ell_i \in L\}
\]

is maximal \(\mu\)-orthogonal and spans a closed subspace in \(L^2(\mu)\) with infinite-dimensional orthogonal complement.

(vii) The Fourier transform

\[
\hat{\mu}(t) = \int_C e_i(x) \, d\mu(x)
\]

satisfies the functional transformation law

\[
\hat{\mu}(Rt) = B(Rt) \hat{\mu}(t) \quad \forall t \in \mathbb{R}^d,
\]

where

\[
B(t) = \frac{1}{N} \sum_{b \in \mathcal{B}} e^{i2\pi b \cdot t}
\]

and \(\hat{\mu}(\cdot)\) has an associated infinite product-formula.

**Remark.** We view the representation (6) as a substitute for an orthogonal harmonic analysis for \(L^2(\mu)\), with \(\mu\) fractal, and note that the relations in (iii) above have the flavor of an orthogonal double-decomposition but not an orthogonal expansion in the classical sense of Fourier integrals (or series). Indeed, Strichartz [St2] showed that there is not a direct way of making an exact classical Fourier decomposition for \(L^2(\mu)\) when \(\mu\) is fractal.
4. Orthogonal frequencies in $L^2(\mu)$

Note that in (vi) the vectors from (7) are represented by orthogonal frequencies $e_\xi$ of the form (1) where $\xi$ is in the subset $\mathcal{L}(L) \subset \mathbb{R}^d$ of all affine sums (with $n$ variable):

$$\sum_{k=1}^{n} R^{k-1} \ell_k = \tau_{\ell_1} \tau_{\ell_2} \cdots \tau_{\ell_n}(0),$$

$\forall \ell_k \in L$, and the $n = 0$ term corresponding (by definition) to $\xi = 0$.

**Theorem 3** (details [JP3]). (i) \( \{ e_\xi : \xi \in \mathcal{L}(L) \} \) is maximally orthogonal in $L^2(\mu)$.

(ii) None of the functions $e_s(x) = e^{i2\pi s \cdot x} (x \in \mathbb{R}^d)$ for $s \in \mathbb{R}^d \setminus \mathcal{L}(L)$ is in the $L^2(\mu)$-closed linear span of the pure frequencies of $\mathcal{L}(L)$. That is,

$$\sigma_L(s) := \sum_{\xi \in \mathcal{L}(L)} |\hat{\mu}(s - \xi)|^2 < 1$$

when $s$ is in $\mathbb{R}^d \setminus \mathcal{L}(L)$.

However, computer-calculations (Mathematica) show that

$$\sigma_L(s) = \| P_{\mathcal{L}(L)} e_s \|_{L^2(\mu)}^2$$

is close to 1 (within third decimal place) when $s = (s_1, \ldots, s_d) \in \mathbb{K}^0 \setminus \mathcal{L}(L)$ and $s_i > 0, 1 \leq i \leq d$.

5. Returning to $(\Omega, \Lambda)$

Our final result shows that the system $(\Omega, \Lambda)$ may be reconstructed from a given Cuntz-representation acting on $L^2(\mu)$.

**Theorem 4.** Let $\mu$ be a probability measure on $\mathbb{R}^d$ with compact support, and let $K \subset \Gamma$ be a rank $d$ lattice system, with inclusion matrix $R$. Suppose a subset $L$ s.t. $0 \in L \subset \mathbb{K}^0$ induces operators $\{ T_\ell \}_{\ell \in L}$ by (6), acting isometrically on $L^2(\mu)$ and satisfying the Cuntz-relations (iii) in Theorem 2. Then it follows that $\mu$ is a fractal measure which is generated by self-similarity from some spectral pair $(\Omega, \Lambda)$ in $\mathbb{R}^d$ satisfying the conditions in Theorem 1 for nondegenerate simple factors.

**Acknowledgment**

The authors gratefully acknowledge helpful correspondence from Professor R. S. Strichartz on the connection of our earlier paper [JP1] to the harmonic analysis of fractal measures.

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