Phase Transition Temperatures at Next-to-Leading Order

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Broken gauge symmetries are typically restored at high temperature, and the leading-order result for the critical temperature $T_c$ was found many years ago by Weinberg and by Dolan and Jackiw. I find a simple expression for the next-to-leading order correction to $T_c$, which is order $eT_c$ where $e$ is the gauge coupling. The result is a simple consequence of recent work on summing ring diagrams at high temperature in gauge theories. The result is valid when the Higgs self-coupling $\lambda$ is the same order as $e^2$, and it does not address the case of strongly first-order phase transitions, which arise when $\lambda \ll e^2$.

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In simple Higgs models of spontaneously broken gauge theories, the classical potential of the Higgs field has the usual form

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4.$$  (1)

At high temperatures, however, there is an additional effective mass term of order $e^2 T^2 \phi^2$ which is analogous to the Debye screening mass in an electromagnetic plasma. The effective potential at high temperature is given approximately by

$$V_{\text{eff}}(\phi) \approx \frac{1}{2}(-\mu^2 + c e^2 T^2)\phi^2 + \frac{1}{4!}\lambda\phi^4,$$  (2)

where the constant $c$ is model dependent.

In any given model, the critical temperature for symmetry restoration is then determined to leading order by the effective potential (2):

$$T_c \approx \frac{\mu}{\sqrt{c e}}.$$  (3)

I shall show that recent improvements to the effective potential yield simple corrections of order $eT_c$ to this formula for $T_c$. Explicit formulas will be presented for the Abelian Higgs model and for the weak sector of the minimal standard model.

The procedure is quite simple. In the next section, I review the ring-improved effective potential, which has recently been implemented by Carrington for gauge theories such as the Standard Model. In section 3, I then find the critical temperature by requiring that $d^2 V_{\text{eff}}/d\phi^2 = 0$ at $\phi = 0$. Finally, I argue that corrections to this result are higher-order than $eT_c$. In particular, finding $T_c$ by requiring the curvature of $V_{\text{eff}}$ to vanish at the origin is appropriate for a second-order phase transition but not for a first-order one; I show that the resulting discrepancy is higher order than $eT_c$ provided $\lambda \sim e^2$. Working in general Lorentz gauges, I check that the result is independent of the gauge parameter. I also find that the naive ring-improved potential would generate further corrections to $T_c$ at order $e^{3/2} T_c$. These corrections are a manifestation of the failure of the naive ring approximation at this order. A simple improvement of the approximation reduces the corrections to order $e^2 T_c$, at which point there is no known perturbative method for calculating them. In an Appendix, I discuss the subtleties of finding the same results in $R_\xi$ gauges.
1. Review of the Ring-Improved Effective Potential

1.1. Pure scalar theory

For simplicity, start by ignoring the gauge fields and focus on the simple theory of a single, real scalar field with the symmetry breaking potential \( \Phi^2 \). Diagrammatically, the screening mass discussed above comes from the quadratically divergent loop shown in fig. 1. After subtraction of zero-temperature counter-terms, the quadratic divergence is cut off by \( T \), so that such diagrams are order \( \lambda T^2 \). At high temperatures, this is the dominant interaction of the full one-loop potential. It has the effect of replacing \( -\mu^2 \) in the classical potential by

\[
-\mu^2_{\text{eff}} = -\mu^2 + \frac{1}{24} \lambda T^2,
\]

(4)
giving \( T_c^2 \approx 24\mu^2/\lambda \).

The full one-loop potential consists of all interactions generated at one-loop, such as fig. 2. For general theories, each species of particle gives a contribution to the finite-temperature piece of the one-loop potential that is simply the free energy of an ideal gas of such particles. Restricting attention to bosons:

\[
V_{\text{eff}}(T, \phi) = V_{\text{eff}}(0, \phi) + \sum_i n_i \Delta V_i(T, \phi),
\]

(5)

\[
\Delta V_i(T, \phi) = T \int \frac{d^3k}{(2\pi)^3} \ln \left\{ 1 - \exp[-\beta \sqrt{k^2 + m_i^2(\phi)}] \right\},
\]

(6)

where \( \beta \) is the inverse temperature, the sum is over all species \( i \), and \( n_i \) is the number of degrees of freedom associated with each species. \( m_i(\phi) \) is the effective mass of species \( i \) in the presence of a background scalar field \( \phi \). The high temperature limit \( T \gg m_i(\phi) \) yields:

\[
\Delta V(T, \phi) = \text{const.} + \frac{1}{24} m^2(\phi)T^2 - \frac{1}{12\pi} m^3(\phi)T + O(m^4 \ln T). \quad \text{(bosons)}
\]

(7)
The constant above is temperature dependent but \( \phi \) independent; it is not relevant to determining the critical temperature and shall be ignored.

For the simple scalar theory, there is only one species — the Higgs — and \( m^2(\phi) \) is the second derivative of the classical potential:

\[
m^2_{\text{cl}}(\phi) = -\mu^2 + \frac{1}{2} \lambda \phi^2.
\]  

However, we have already seen that the effective value of \( \mu^2 \) at finite temperature is quite different from the classical value at high temperature \( (T \gtrsim T_c) \). When studying temperatures of order \( T_c \), it is therefore important to make the replacement and use instead

\[
m^2_{\text{eff}}(\phi) = -\mu^2 + \frac{1}{24} \lambda T^2 + \frac{1}{2} \lambda \phi^2
\]

in the one-loop potential. This substitution corresponds to including the dominant contributions of the ring (also known as daisy) diagrams shown in fig. 3.

The leading (non-constant) term in the expansion simply reproduces the dominant \( \lambda T^2 \phi^2 \) interaction discussed earlier. The next term, of order \( m^3(\phi)T \), is the term that will generate the first correction to \( T_c \) in gauge theories. For the pure scalar theory, the ring-improved potential in this expansion is

\[
V_{\text{ring}}(\phi) = \frac{1}{2} \left( -\mu^2 + \frac{1}{24} \lambda T^2 \right) \phi^2 - \frac{T}{12\pi} \left( -\mu^2 + \frac{1}{24} \lambda T^2 + \frac{1}{2} \lambda \phi^2 \right)^{3/2} + \frac{1}{4!} \lambda \phi^4 + O(m^4 \ln T).
\]  

It will later be convenient to also view the ring-improved potential in the language of decoupling and the renormalization group. At high temperature, loops which are less than quadratically divergent (or non-divergent pieces of quadratically divergent ones) are dominated by their infrared behavior. In Euclidean space, this means loop momenta are dominated by \( k_0 = 0 \) and \( |\vec{k}| \sim m \). The dominant \( k_0 = 0 \) piece of the finite-temperature frequency sum \( T \sum_{k_0} \) gives such loops a linear, rather than quadratic, dependence on \( T \).
Taking $k_0 = 0$ in all such loops yields an effective three-dimensional theory whose squared coupling is $\lambda T$ and which may be viewed as an approximate effective theory at scales much smaller than $T$. The contributions from physics at scale $T$ will decouple like powers of $1/T$ except for possible renormalizations of masses and so forth. The replacement (4) is a statement of the relation between the renormalized mass $-\mu_{\text{eff}}^2$ in the effective three-dimensional theory and the renormalized mass $-\mu^2$ in the zero-temperature theory. Now computing and renormalizing the simple one-loop potential in the effective three-dimensional theory gives

\[ \Delta V(T, \phi) = \frac{T}{12\pi} (-\mu_{\text{eff}}^2 + \frac{1}{2}\lambda \phi^2)^{3/2}, \]

\[ V_{\text{eff}}(T, \phi) \approx -\frac{1}{2}\mu_{\text{eff}}^2 \phi^2 + \frac{1}{4!}\lambda \phi^4 + \Delta V(T, \phi) \]

which is equivalent, within my approximations, to the ring-improved result (10).

What is the size of corrections to the ring-improved one-loop potential from other diagrams? The squared coupling in the three-dimensional theory is $\lambda T$, so each loop added costs a factor of $\lambda T/m_{\text{eff}}$. The effective value of $m$ approaches zero as $T$ approaches $T_c$, and so the ring-improved loop expansion will break down very close to the phase transition, when $|m_{\text{eff}}| \lesssim \lambda T$. Eq. (4) implies this breakdown occurs when $|T - T_c| \lesssim \lambda T_c$, and so there is no simple way to compute corrections of order $\lambda T_c$ to $T_c$.

1.2. Abelian Higgs Model

Now focus on the simplest example of a spontaneously broken gauge theory: the Abelian Higgs model given by

\[ \mathcal{L} = -\frac{1}{4} F^2 + |D\Phi|^2 - V(|\Phi|^2), \]

\[ V(|\Phi|^2) = -\mu^2 |\Phi|^2 + \frac{1}{3!}\lambda |\Phi|^4, \]

1
where $\Phi$ is a complex field and $D_\mu \Phi = (\partial_\mu - ieA_\mu)\Phi$. I shall typically express the potential in terms of $\phi = |\Phi|/\sqrt{2}$, so that it takes the form (13), and shall work in Lorentz gauges, where the gauge fixing term is

$$L_{g.f.} = \frac{1}{\xi} (\partial \cdot A)^2. \hspace{1cm} (14)$$

I shall assume $\lambda \sim e^2$ unless stated otherwise.

Now consider the one-loop effective potential (14). In Landau gauge ($\xi = 0$), the mass squared in the presence of a background scalar field $\phi$ is classically

$$M^2(\phi) = e^2 \phi^2 \quad (\text{vector})$$
$$m_1^2(\phi) = -\mu^2 + \frac{1}{2} \lambda \phi^2 \quad (\text{physical Higgs})$$
$$m_2^2(\phi) = -\mu^2 + \frac{1}{6} \lambda \phi^2 \quad (\text{unphysical Goldstone boson}) \hspace{1cm} (15)$$

The unimproved one-loop potential in Landau gauge is then

$$V_{\text{eff}}(\phi) \approx \frac{1}{2} \left[ -\mu^2 + \left( \frac{\lambda}{2} + \frac{\lambda}{6} + 3e^2 \right) \frac{T^2}{12} \right] \phi^2$$
$$- \frac{T}{12\pi} \left[ 3e^3 \phi^3 + (\frac{\mu^2}{2} + \frac{1}{2} \lambda \phi^2)^{3/2} + (\frac{\mu^2}{6} + \frac{1}{6} \lambda \phi^2)^{3/2} \right]$$
$$+ \frac{1}{4!} \lambda \phi^4. \hspace{1cm} (16)$$

To make the ring improvement, we need the leading finite-temperature contributions to the effective particle masses. For the Higgs boson, it can be read from the first term of (16) and corresponds to figs. 1 and 4. It is the same for the unphysical Goldstone boson:* 

$$m_1^2(\phi) \to -\mu^2 + \left( \frac{2}{3} \lambda + 3e^2 \right) \frac{T^2}{12} + \frac{1}{2} \lambda \phi^2$$
$$m_2^2(\phi) \to -\mu^2 + \left( \frac{2}{3} \lambda + 3e^2 \right) \frac{T^2}{12} + \frac{1}{6} \lambda \phi^2 \hspace{1cm} (17)$$

* Technically, these substitutions only make sense inside IR dominated loops, where the loop momentum $k$ is $\ll T$. The self-energies $\Pi(k)$ coming from the hard thermal loops of fig. 4 can then be approximated by $\Pi(0)$ when constructing ring-improved propagators. The substitutions are not valid in the $m^2(\phi)T^2$ term of (15) which, unlike the subsequent terms, arises from the quadratic divergence of loops, where the loop momentum $k$ is order $T$. In the case at hand, such worries only affect the $\phi$-independent constant terms, which I am ignoring. The linear terms in the effective potential found in Refs. [9] and [10], however, are the result of higher-order versions of such substitutions improperly made into the $m^2(\phi)T^2$ term.
The leading contribution to the thermal vector mass comes from the diagrams of fig. 5 and is momentum dependent. However ring graphs of the form of fig. 6 are dominated by their Euclidean infrared behavior, corresponding to momenta $k_0 = 0$ and $|\vec{k}| \ll T$. In this limit, the diagrams of fig. 5 generate a Debye screening mass of $eT/\sqrt{3}$ for $A_0$ (the longitudinal polarization) and nothing at the same order for $\vec{A}$. So, for computing the ring-improvement to the effective potential:

$$M_L^2(\phi) \rightarrow e^2\phi^2 + \frac{1}{3}e^2T^2 \quad \text{(longitudinal polarization)},$$
$$M_T^2(\phi) \rightarrow e^2\phi^2 \quad \text{(transverse polarizations)},$$

$$V_{\text{ring}}(\phi) \approx \frac{1}{2}m^2_{\text{eff}}(T)\phi^2 - \frac{T}{12\pi} \left[ (e^2\phi^2 + \frac{1}{3}e^2T^2)^{3/2} + 2e^3\phi^3 + (m^2_{\text{eff}}(T) + \frac{1}{2}\lambda\phi^2)^{3/2} 
+ (m^2_{\text{eff}}(T) + \frac{1}{6}\lambda\phi^2)^{3/2} \right] + \frac{1}{4!}\lambda\phi^4,$$

where

$$m^2_{\text{eff}}(T) = -\mu^2 + \left( \frac{2}{3}\lambda + 3e^2 \right) \frac{T^2}{12}. \quad \text{(20)}$$

In general Lorentz gauge, one must include the unphysical polarization (the polarization proportional to the four-momentum) of the photon. When the background scalar field $\phi$ is non-zero, this polarization mixes with the unphysical Goldstone boson. Taking the one-loop potential from Ref. [3] and incorporating the ring improvement gives

$$V_{\text{ring}}(\phi) \approx \frac{1}{2}m^2_{\text{eff}}(T)\phi^2 - \frac{T}{12\pi} \left[ (e^2\phi^2 + \frac{1}{3}e^2T^2)^{3/2} + 2e^3\phi^3 + (m^2_{\text{eff}}(T) + \frac{1}{2}\lambda\phi^2)^{3/2} 
+ R^3_+ + R^3_- \right] + \frac{1}{4!}\lambda\phi^4,$$

where

$$R^2_\pm = \frac{1}{2}\bar{m}_2^2 \pm \frac{1}{2}\sqrt{\bar{m}_2^2(\bar{m}_2^2 - 4\xi e^2\phi^2)}, \quad \text{(22)}$$
$$\bar{m}_2^2 = m^2_{\text{eff}}(T) + \frac{1}{6}\lambda\phi^2. \quad \text{(23)}$$

** The exception is the ultraviolet piece of the quadratic interactions like figs. 1 and 4 with no mass insertions. These pieces give the $e^2T^2\phi^2$ interactions and are independent of the particle masses. The pieces of these and other diagrams which do depend on the particle masses, however, are dominated by their infrared behavior.
2. The Critical Temperature

With the ring-improved potential in hand, consider the computation of the critical temperature. The curvature of the effective potential (21) at $\phi = 0$ is†

$$V''_{\text{ring}}(0) = m_{\text{eff}}^2(T) - \frac{\sqrt{3}}{12\pi} e^3 T^2 - \left( \frac{\lambda}{6\pi} - \frac{\xi e^2}{4\pi} \right) m_{\text{eff}}(T) T. \quad (24)$$

Solving $V''_{\text{ring}}(0) = 0$ to order $eT_c$, I find that the last term in (24) is irrelevant. The result is

$$T_c^2 = \frac{\mu^2}{\frac{1}{12} \left( \frac{2}{3} \lambda + 3e^2 \right) - \frac{\sqrt{3}}{12\pi} e^3} + O(e^{3/2}T_c) \quad (25)$$

and is independent of the gauge parameter $\xi$ to this order. The source of the order $eT_c$ correction to the leading-order result is the photon Debye screening mass, which generated the second term in (24).

2.1. Validity of expansion

From the review of pure scalar theory, we know that the loop expansion breaks down when $|T - T_c| \lesssim e^2 T_c$, and so the critical temperature cannot be easily computed within $e^2 T_c$. To determine if the value (25) of $T_c$ is correct to order $eT_c$, one needs to know if the formula (24) is a good approximation when $|T - T_c| \sim eT_c$. For such temperatures, $m_{\text{eff}}^2 \sim e^3 T^2$. Unfortunately, the argument is slightly complicated because the naive ring-improved loop expansion is not controlled simply by $e^2 T/m$, as it was in the pure scalar case, because there are two soft scales when $\phi$ is near zero: $m$ and $M_L \sim eT \gg m$. (In order to avoid more proliferation of scales in the following discussion, I shall continue to focus on the effective potential close to $\phi = 0$.)

† Instead of expanding around arbitrary background $\phi$ to compute $V_{\text{eff}}(\phi)$, I could have obtained $V_{\text{eff}}''(0)$ by simply expanding about $\phi = 0$ as usual and then evaluating the two-point function by ring-improving diagrams such as figs. 1 and 4. Having $V_{\text{eff}}(\phi)$, however, is useful for the later discussion of first vs. second order phase transitions and for the study of R$\xi$ gauge in the appendix.
Look at the corrections to the result for $T_c$ derived using the ring-improved one-loop potential. First note that $V_{\text{ring}}$ itself implied a gauge-dependent correction of order $e^{3/2}T_c$, which arises from the last term of (24). The cause of this correction is that $m_{\text{eff}}^2(T)$ as defined by (24) is order $e^{3}T_c^2$ (instead of zero) at $T_c$, and so the last term in (24) is suppressed by only $\sqrt{e}$ relative to the second term. The correction arises because the $m_{\text{eff}}^2(T)$ used in ring-improved Higgs propagators is not small enough when $|T - T_c| \lesssim eT_c$—it is a poor approximation to $V''_{\text{eff}}(0)$. One may fix the approximation by self-consistently replacing $m_{\text{eff}}^2(T)$ in the improved one-loop potential by the leading terms of $V''(0)$ from (24):

$$m_{\text{eff}}^2(T) \to -\mu^2 + \left(\frac{2}{3}\lambda + 3e^2\right)\frac{T^2}{12} - \frac{\sqrt{3}}{12\pi}e^3T^2.$$  (26)

Eq. (24) for $V''(0)$ then no longer produces an $O(e^{3/2}T_c)$ contribution to $T_c$. Diagrammatically, the redefinition of $m_{\text{eff}}(T)$ corresponds to using the dominant pieces of once-iterated daisy graphs such as fig. 7 for the Higgs-loop contribution to the potential. The smallest loops in fig. 7 are hard, with momenta of order $T$. The next-smallest loops are soft longitudinal photon loops, with momenta of order $M_L \sim eT$. Close to the critical temperature, the large Higgs loop is softer yet, with momenta of order $m_{\text{eff}} \ll eT_c$. Because of the hierarchy of scales, it is a good approximation at each level of fig. 7 to approximate resummed propagators $1/[p^2 + \Pi(p)]$ by $1/[p^2 + \Pi(0)]$.

For the purpose of understanding the size of other corrections to the effective potential, it is useful to restate the redefinition of $m_{\text{eff}}(T)$ in the language of decoupling. I earlier reviewed the effective three dimensional theory one obtains at scales below $T$. But when the Higgs mass gets sufficiently light as we approach $T_c$, there is then another heavy scale in the problem—the longitudinal photon mass of order $eT$—and a new effective theory may be obtained by integrating out its effects. The effects of this heavy scale will by suppressed except for renormalization of masses and so forth. The significant mass renormalization comes from the loop of fig. 8, which gives the $e^3T^2$ term incorporated into $m_{\text{eff}}^2$ in (26).
Effective interactions generated by the heavy contributions (all loop momenta of order $eT$) of higher-loop graphs will be suppressed by $e^2T/M_L \sim e$ and affect the derivation of $T_c$ only at order $e^2T_c$.

In the new effective theory, soft loops will be suppressed by $e^2T/m$ and the loop expansion is controlled when $m \gg e^2T$. As an example, the two-loop graph shown in fig. 9 is order $e^4T^2$, which affects $T_c$ only at order $e^2T_c$.

In conclusion, the error in the formula for (25) is order $e^2T_c$ rather than $e^{3/2}T_c$.

2.2. First vs. second-order transitions

The one-loop ring approximation (21) to the effective potential actually describes a first-order rather than second-order phase transition. As depicted in fig. 10, the critical temperature $T_c$ for a first-order phase transition is different from the temperature $T_0$ at which $V''(0) = 0$. The difference between these two temperatures is easily estimated. Working near the critical temperature, ignore the $R_{\pm}^3$ terms in the potential and consider the form of the potential for small values of $\phi$. Then

$$V_{\text{ring}}(\phi) \rightarrow \frac{1}{2} \left( m_{\text{eff}}^2(T) - \frac{\sqrt{3}}{12\pi} e^3T^2 \right) \phi^2 - \frac{1}{6\pi} e^3T\phi^3 + \frac{1}{4!}\lambda\phi^4,$$

(27)

where I have assumed that $\lambda \gg e^4$ so that corrections to $\lambda\phi^4$ may be ignored. At the true $T_c$ depicted in fig. 10, all three terms above must be the same order of magnitude. Equating the magnitude of the last two terms gives $\phi_c \sim e^3T/\lambda$ and then equating with the first term gives

$$V''(0) \sim V''(\phi_c) \sim e^6/\lambda T^2, \quad T_c - T_0 \sim \frac{e^4}{\lambda} T_c.$$

(28)

For $\lambda \sim e^2$, the difference between the two temperatures is order $e^2T_c$ and so does not affect the earlier result for the order $eT_c$ correction to $T_c$. Note that $T_c$ is close enough to $T_0$ that $m_{\text{eff}}^2 \sim e^2T$ at $T_c$, and so the improved loop expansion has just started to break
down. Thus, one may not conclude based solely on the ring-improved effective potential that the phase transition is in fact first order when $\lambda \sim e^2$. The potential merely indicates that, if it is first-order, then the difference between $T_c$ and $T_0$ is smaller than order $eT_c$. Other arguments, given by Ginsparg,[11] suggest that the transition is indeed first order.

For $\lambda \ll e^2$, the first-order nature of the phase transition becomes strong enough that the effective potential can be trusted to distinguish between first and second-order transitions. In particular, $\lambda \sim e^3$ implies $T_c - T_0 \sim eT_c$, and the effective scalar mass is order $e^{3/2}T$ at $T_c$, giving a loop expansion controlled by $e^2T/m \sim \sqrt{e}$. My earlier calculation still gives $T_0$ to order $eT_c$, but now this is not an accurate calculation of $T_c$ to the same order. In this case, the formula for $T_c$ to order $eT_c$ is not simple, and $T_c$ is most easily found by evaluating the ring-improved potential numerically, as was done by Carrington.[7]

3. The Minimal Standard Model

The calculation of the previous section is easily generalized to the weak sector of the Minimal Standard Model with three families, where Carrington[7] has derived the ring-improved one-loop potential in Landau gauge. Expanding her result in the high-temperature limit gives

$$V_{\text{ring}}(\phi) \approx \frac{1}{2} m^2(T) \phi^2 - \frac{T}{12\pi} \left[ 3 \left( \frac{1}{4} g^2 \phi^2 + \frac{11}{6} g^2 T^2 \right)^{3/2} + 6 \left( \frac{1}{4} g^2 \phi^2 \right)^{3/2} \right.$$

$$+ \left( \frac{1}{4} g^2 T^2 \right)^{3/2} + 2 \left( \frac{1}{4} g^2 \phi^2 \right)^{3/2} + \left( m^2(T) + \frac{1}{2} \lambda \phi^2 \right)^{3/2} + 3 \left( m^2(T) + \frac{1}{6} \lambda \phi^2 \right)^{3/2} \left. \right] + \frac{1}{4!} \lambda \phi^4,$$

where

$$\tilde{m}^2(T) = -\mu^2 + \left( \lambda + \frac{9}{4} g^2 + \frac{3}{4} g^2 + 3 g_\gamma^2 \right) \frac{T^2}{12}.$$

(29)
$g_y$ is the top quark coupling, which is the only one I have treated as significant. The conventions for the coupling constants are that $D_\mu = \partial_\mu + gA_\mu \cdot \tau/2 + Y g'/B_m u/2$ for doublets, the hypercharge is normalized so that $Q = T_3 + Y/2$, and the Yukawa coupling is $g_y \bar{q}_L \cdot \Phi t_R$ where $\Phi$ is the full complex doublet, whose classical potential is of the form (13). The correction to $T_c$ is generated by the Debye screening masses of the gauge bosons, and proceeding as before gives

$$T_c^2 = \frac{\mu^2}{\frac{1}{12} (\lambda + \frac{9}{4} g^2 + \frac{3}{4} g'^2 + 3 g_y^2)} - \frac{1}{12\pi} \sqrt{\frac{11}{6} \left( \frac{9}{4} g^3 + \frac{3}{4} g'^3 \right)} + O(g^2 T_c). \quad (31)$$

In non-Abelian theories, a little more needs to be said about the convergence of the loop expansion than in the Abelian case. Because of the 3-point gauge coupling, it is possible to construct loops solely from the massless (at $\phi = 0$), transverse gauge bosons, such as contained in fig. 11. Such loops are generally infrared divergent. It is presumed\[12\] that such loops are cut-off by a non-perturbative magnetic screening mass of order $g^2 T$, for which the loop expansion parameter $g^2 T/M_T$ is then order 1. However, if we indeed cut off the infrared behavior of transverse gauge loops at order $g^2 T$, then their contributions to $V''(0)$, such as in fig. 11, are order $g^4 T^2$. So the incalculable contribution of such loops only affects $T_c$ at order $g^2 T_c$.

As an example of the numeric size of the order $g T_c$ corrections, consider the effect of including or eliminating the $g^3$ and $g'^3$ terms in the denominator of (31). The effect of the cubic terms is largest when the Higgs and top masses are small; in the limit that these masses are negligible (in which case (31) may no longer be valid), the inclusion of the cubic terms increases the result for $T_c$ by 37%. For $m_H = m_t = 100$ GeV, including the cubic terms increases the result for $T_c$ by 13%.

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Appendix A. Results in $R_\xi$ gauges

Discussions of the effective potential in the literature sometimes employ a generalized version of $R_\xi$ gauge. In this appendix, I discuss how the previous analysis of $T_c$ works in these gauges. For simplicity, I shall work in the Abelian Higgs model.

The generalized $R_\xi$ gauge of Dolan and Jackiw is fixed by

$$\mathcal{L}_{g.t.} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi e v \phi_2)^2 - \bar{\eta}(\partial^2 + \xi e^2 v \phi_1)\eta,$$  \hspace{1cm} (A.1)

where the complex Higgs field is decomposed as $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $\eta$ is the ghost field, and $v$ is an additional, arbitrary gauge parameter. In the usual definition of $R_\xi$ gauge, $v$ is set to the classical value $\phi^{cl}$ about which one expands $\phi_1$; this definition eliminates mixing of the scalars with the gauge field. This definition is unacceptable for computing the effective potential $V_{\text{eff}}(\phi^{cl})$, however, because the effective potential is gauge dependent. With the usual definition of $v$, varying $\phi^{cl}$ corresponds to changing gauge and so $V_{\text{eff}}$ will not be consistently computed in a single gauge. This pathology destroys the equality between the derivatives of $V_{\text{eff}}$ and zero-momentum tadpoles, self-energies and so forth. Following Dolan and Jackiw, I shall instead fix $v$ independent of $\phi^{cl}$ at the cost of having mixing between $\phi$ and $A_\mu$. Note that $v = 0$ is the previously treated case of Lorentz gauges.

Rather than studying the full potential $V(\phi_1^{cl}, \phi_2^{cl})$, I shall simplify the discussion by restricting attention to $\phi_2^{cl} = 0$. With this restriction, mixing occurs between the unphysical polarization (proportional to $k_\mu$) of $A_\mu$ and the unphysical Goldstone boson, but there is no mixing with the physical Higgs. I shall also generally drop subscripts and so forth to write $V(\phi_1^{cl}, 0)$ as $V(\phi)$.

As discussed in Ref. [14], it is important that gauge parameters not take extreme values (such as $\xi \gtrsim 1/e^2$); otherwise, the loop expansion for the effective potential breaks down in a non-trivial manner, making it difficult to compute $T_c$ even to leading-order in such gauges. For this discussion, I shall assume $\xi \sim 1$. Requiring that the loop expansion
be well-behaved then also puts constraints on how large \( v \) can be. In particular, consider the diagonal element, of the non-diagonal propagator, that propagates the unphysical polarization of \( A_\mu \) into itself. Expanding about \( \phi = 0 \) and taking the limit \( v \to \infty \), this component of the propagator turns out to be

\[
\frac{p_\mu p_\nu}{p^2} \frac{\xi^2 e^2 v^2}{p^2(p^2 + m^2)}
\]

(A.2)

where \( m \) is the effective Higgs mass. Loops involving this propagator will then be enhanced by factors of \( \xi^2 e^2 v^2 / m^2 \) if \( v \) is large. To avoid enhancement of higher-loop graphs, which would invalidate the improved one-loop approximation to the potential, I shall restrict attention to gauges with \( v \lesssim m/e \).

The ring-improved one-loop potential is

\[
V_{\text{ring}}(\phi) \approx \frac{1}{2} m_{\text{eff}}^2(T) \phi^2 - \frac{T}{12\pi} \left[ (e^2 \phi^2 + \frac{1}{3} e^2 T^2)^{3/2} + 2e^3 \phi^3 + (m_{\text{eff}}^2(T) + \frac{1}{2} \lambda \phi^2)^{3/2} \right. \\
\left. + R_+^3 T + R_-^3 T - 2(\xi e^2 v \phi)^{3/2} \right] + \frac{1}{4!} \lambda \phi^4,
\]

(A.3)

where the last term inside the brackets is from the ghost contribution and where

\[
R_\pm^2 = \frac{1}{2} (\bar{m}_2^2 + 2\xi e^2 v \phi) \pm \frac{1}{2} \sqrt{\bar{m}_2^2(\bar{m}_2^2 - 4\xi e^2(\phi - v)\phi)},
\]

(A.4)

\[
\bar{m}_2^2 = m_{\text{eff}}^2(T) + \frac{1}{6} \lambda \phi^2.
\]

(A.5)

Taking \( V''(0) \) in this approximation gives an infinite result. This is not disastrous because \( \phi = 0 \) is not, in fact, the symmetric state in this gauge. Expanding \( V_{\text{ring}}(\phi) \) about \( \phi = 0 \) gives

\[
V_{\text{ring}}(\phi) \to \text{const.} - \left( \frac{1}{4\pi} \xi e^2 v T m_{\text{eff}}(T) \right) \phi \\
+ \frac{1}{2} \left[ m_{\text{eff}}^2(T) - \frac{\sqrt{3}}{12\pi} e^3 T^2 - \left( \frac{\lambda}{6\pi} - \frac{\xi e^2}{4\pi} \right) \right] \phi^2 \\
+ \cdots.
\]

(A.6)

There is a linear term which above \( T_c \) shifts the ground state away from \( \phi = 0 \). This is not surprising because the \( R_\xi \) gauge choice \( \text{(A.1)} \) explicitly breaks the global isospin rotation
and $\phi \rightarrow -\phi$ symmetries which distinguish $\phi = 0$ from other values of $\phi$; the symmetric vacuum need not be at $\phi = 0$.*

If higher-order terms in (A.6) are ignored, then $V''(\phi)$ in the shifted vacuum is just the coefficient of the quadratic term in (A.6). The analysis is then the same as it was for the earlier case of Lorentz gauges, and one finds the same result for $T_c$ to order $eT_c$. In this approximation, the ground state above $T_c$ is at $\langle \phi \rangle$ of order $e^2vT/m_{\text{eff}}$. The correction from higher-order terms in (A.6) is suppressed by $e^2v\langle \phi \rangle/m_{\text{eff}}^2$ which, using my restriction that $v \lesssim m_{\text{eff}}/e$ is a suppression by at least the usual expansion parameter $e^2T/m_{\text{eff}}$.

---

* Above $T_c$, there should be no magnetic screening in this model. The reader may wonder how a $\phi \neq 0$ vacuum can give a zero photon mass. The $e\langle \phi \rangle$ contribution to the photon mass is the same order as 2-loop corrections to that mass, against which it presumably cancels. I have not checked this cancellation explicitly.
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Figure Captions

Fig. 1. The order $\lambda T^2$ contribution to the squared effective mass in a simple scalar theory.

Fig. 2. A generic contribution to the full one-loop effective potential.

Fig. 3. A generic contribution to the ring-improved one-loop effective potential. The small loops are hard with loop momenta $\sim T$; the large loop is soft, except for the special case of fig. 1.

Fig. 4. The order $e^2 T^2$ contribution to the squared effective scalar mass.

Fig. 5. Order $e^2 T^2$ contributions to the squared thermal vector mass.

Fig. 6. A generic daisy graph contribution to the effective potential in the Abelian Higgs model. The main loop is dominated by its infrared behavior.

Fig. 7. A generic once-iterated daisy graph for the Higgs loop contribution to the effective potential. The medium-size loops are longitudinal photons.

Fig. 8. The dominant scalar mass term induced by heavy ($eT$) longitudinal photons.

Fig. 9. An example of a two-loop contribution to the scalar mass.

Fig. 10. The ring-improved effective potential shown qualitatively at (1) the critical temperature $T_c$ and (2) the temperature $T_0$ where $V''_{\text{eff}}(0) = 0$. The curves have been arbitrarily normalized so that $V_{\text{eff}}(0) = 0$.

Fig. 11. A two-loop contribution from transverse gauge bosons to the effective scalar mass.