THE INFRARED LIMIT OF THE QCD DIRAC SPECTRUM

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The distribution of the low-lying QCD Dirac spectrum is analyzed by means of partial quenched chiral perturbation theory. We identify an energy scale below which the valence quark mass dependence of the QCD partition function is given by chiral Random Matrix Theory. This is the domain where the low-energy QCD partition function is dominated by the fluctuations of the mass term of the Goldstone bosons. In mesoscopic physics this domain is known as the ergodic domain and the corresponding energy scale is the Thouless energy.

1 Introduction

It has become clear, both from numerous lattice QCD simulations and hadronic phenomenology, that QCD at low energies is characterized by chiral symmetry breaking and confinement. Because of spontaneous breaking of chiral symmetry, the low-lying excitations are given by the associated Goldstone modes. Because of confinement, these are the only low-lying excitations. In this lecture, we wish to point out another important feature of the low energy QCD partition function, namely disorder. Ultimately we would like to answer the question whether confinement can occur if the classical motion of quarks in generic Yang-Mills fields is not chaotic?

One way to understand confinement is by analogy with impurity scattering. The argument starts from the semiclassical expression for the quark propagator in an external gauge field given by a sum over classical trajectories weighted by a phase. Its average over all gauge fields vanishes because of this phase. On the other hand, the pion propagator given by the absolute value of the quark propagator survives averaging over all gauge field configurations. We will analyze the quark propagator using a spectral representation of the Dirac operator. From the study of disordered systems we know that the eigenvalues of localized states with exponentially falling wave function are uncorrelated, whereas eigenvalues of extended states with a power-like fall-off of the wavefunction are correlated according to Random Matrix theory. Spectral correlations therefore constitute a sensitive measure for the tail of the wave functions. One of the questions we are interested in is whether the correlations of the Dirac eigenvalues support this picture of confinement.

The spectrum of the Dirac operator and the breaking chiral symmetry are
not unrelated. According to the Banks-Casher formula\footnote{4} the order parameter of the chiral phase transition is given by
\[ \Sigma = \frac{\pi \rho(0)}{V}, \tag{1} \]
where \( \rho(\lambda) \) is the average spectral density \( \rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle \), and \( V \) is the volume of space time. The average \( \langle \cdots \rangle \) is over the QCD action. With the smallest eigenvalues of the Dirac operator are spaced \( \Delta \lambda = \pi/\Sigma V \), it is natural to study this part of the spectrum by introducing the microscopic variable \( u = V \lambda \Sigma \), and the microscopic spectral density defined by
\[ \rho_s(u) = \lim_{V \to \infty} \frac{1}{V \Sigma} \langle \rho(\frac{u}{V \Sigma}) \rangle. \tag{2} \]

We expect that this limit converges to a universal function determined by the global symmetries of the QCD Dirac operator. Using partially quenched chiral QCD partition function we will show that \( \rho_s(u) \) is given by chiral Random Matrix Theory (chRMT) to be discussed in the next section. We will find the domain of validity of chRMT and will calculate the Dirac spectrum beyond this domain. An extensive discussion of the material of this talk with a complete list of references has appeared recently\footnote{7}, and several excellent recent reviews on Random Matrix Theory and disorder are available\footnote{8,9}.

2 Chiral Random Matrix Theory

The chiral random matrix partition function with the global symmetries of the QCD partition function is defined by
\[ Z^\nu_\beta(\mathcal{M}) = \int DW \prod_{f=1}^{N_f} \det \left( \begin{array}{cc} m_f & iW \\ iW^\dagger & m_f \end{array} \right) e^{-\frac{\beta}{4} \Sigma \text{Tr} W^\dagger W}, \tag{3} \]
where \( W \) is a \( n \times m \) matrix with \( \nu = |n - m| \) and \( N = n + m \). As is the case in QCD, we assume that the equivalent of the topological charge \( \nu \) does not exceed \( \sqrt{N} \), so that, to a good approximation, \( n = N/2 \). Then the parameter \( \Sigma \) can be identified as the chiral condensate and \( N \) as the dimensionless volume of space time (Our units are defined such that the density of the modes \( N/V = 1 \)). The matrix elements of \( W \) are either real (\( \beta = 1 \), chiral Gaussian Orthogonal Ensemble (chGOE)), complex (\( \beta = 2 \), chiral Gaussian Unitary Ensemble (chGUE)), or quaternion real (\( \beta = 4 \), chiral Gaussian Symplectic Ensemble (chGSE)). For QCD with three or more colors and quarks in the fundamental representation the matrix elements of the Dirac operator are complex and we
have $\beta = 2$. The ensembles with $\beta = 1$ and $\beta = 4$ are relevant in the case of two colors or adjoint fermions. The reason for choosing a Gaussian distribution of the matrix elements is its mathematically simplicity. It can be shown that the correlations of the eigenvalues on the scale of the level spacing do not depend on the details of the probability distribution.

3 Scales in the Dirac Spectrum

Of course, QCD is not chiral Random Matrix Theory. This implies the existence of a scale beyond which the correlations QCD Dirac spectra are no longer given by chRMT. We have studied such scale by means of the valence quark mass dependence of the chiral condensate defined by

$$\Sigma(m_v) = \langle \text{Tr} \frac{1}{m_v + D} \rangle. \quad (4)$$

Here, the Euclidean Dirac operator is denoted by $D$ and the average is over the Euclidean QCD action which includes a fermion determinant that depends on the sea-quark masses. The spectrum of $D$ is directly related to $\Sigma(m_v)$,

$$\rho(\lambda)/V = \frac{1}{2\pi} (\Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon)). \quad (5)$$

As is the case in standard chiral perturbation theory, an important scale is the mass where the Compton wave length of Goldstone bosons corresponding to the valence quark masses is equal to the linear size of the system, i.e. the mass scale $m_c$ given by

$$\frac{2m_c}{F^2} = \frac{1}{L^2}. \quad (6)$$

We have identified the range as the domain of validity of chRMT. In other words, below the scale $m_c$ the correlations of the QCD Dirac eigenvalues are completely determined by the global symmetries of the QCD partition function.

A second important scale is the average position of the smallest eigenvalue of the Dirac operator, which is approximately given by the average spacing of the eigenvalues near $\lambda = 0$. With spacing given by $1/\rho(0)$ and the Banks-Casher formula we find

$$\lambda_{\text{min}} = \frac{\pi}{\Sigma V}. \quad (8)$$
In the QCD Dirac spectrum we can thus distinguish three different scales,

$$\lambda_{\text{min}} \ll m_c \ll \Lambda_{\text{QCD}}.$$  \hspace{1cm} (9)

These scales separate four different domains. Similar domains occur in mesoscopic physics. The reason is that the transport of electrons in disordered samples is described by Goldstone modes. The classical propagation of Goldstone modes is given by a diffusion equation and therefore $\hbar/m_c$ can be interpreted as the time for an initially localized wave packet to diffuse over the length of the sample (see for example the book by Efetov). The domain $m_v \ll m_c$ is therefore known as the ergodic domain. In mesoscopic physics the scale corresponding to $m_c$ is known as the Thouless energy.

### 4 Partially Quenched Chiral Perturbation Theory

The scales in QCD Dirac spectra can be studied more precisely via the generating function of the chiral condensate. This is the partially quenched QCD partition function which, in addition to the usual quarks describing the fermion determinants, contains additional quarks and their bosonic super-partners which describe the quenched valence quark mass dependence of the chiral condensate. It is defined by

$$Z^{pq}(m_v, J) = \int [dA] \frac{\det(\mathcal{D} + m_v + J)}{\det(\mathcal{D} + m_v)} \prod_{f=1}^{N_f} \det(\mathcal{D} + m_s) e^{-S_{YM}[A]} , \hspace{1cm} (10)$$

where $\mathcal{D}$ is the Euclidean Dirac operator and $S_{YM}[A]$ is the Euclidean Yang-Mills action. The valence quark mass dependence of the chiral condensate is then given by

$$\Sigma(m_v) = \frac{1}{V} \left\langle \sum_{k} \frac{1}{m_v + i\lambda_k} \right\rangle = \frac{1}{V} \left. \frac{\partial f}{\partial J} \right|_{J=0} \log Z^{pq}(m_v, J). \hspace{1cm} (11)$$

For a confining theory such as QCD, the low energy excitations of this theory are the Goldstone modes associated with the spontaneous breaking of chiral symmetry. To lowest order in the momenta, the effective action of these modes is completely determined by Lorentz invariance and chiral symmetry breaking. In agreement with the Vafa-Witten theorem and maximum breaking of chiral symmetry, we expect that the chiral symmetry is broken according to

$$\text{Gl}(N_f + 1|1) \times \text{Gl}(N_f + 1|1) \rightarrow \text{Gl}(N_f + 1|1). \hspace{1cm} (12)$$

In order to obtain convergent integrals, the effective partition function is then based on the maximum Riemannian submanifold of the symmetric superspace.
$Gl(N_f + 1|1)$ which will be denoted by $\hat{Gl}(N_f + 1|1)$. It contains a kinetic term determined by Lorentz invariance and chiral symmetry and a mass terms determined by the pattern of chiral symmetry breaking. An explicit axial $Gl(1|1)$ symmetry breaking is included as well

$$Z(\theta, \hat{M}) = \int_{U \in \hat{Gl}(N_f + 1|1)} dU \exp \int d^4 x \left[ \frac{F^2}{4} \text{Str}(\partial_{\mu} U \partial_{\mu} U^{-1}) + \frac{\Sigma}{2} \text{Str}(\hat{M}U + \hat{M}U^{-1}) + \frac{F^2 m_0^2}{12} (\sqrt{2} \Phi - \theta)^2 \right]. \quad (13)$$

In terms of the action of this partition function, the scale $m_c$ is the scale where the fluctuations of the mass term and the kinetic term are of equal order of magnitude. For valence quark masses $m_v \ll m_c$ the fluctuations of the kinetic term can be neglected and the calculation of the valence quark mass dependence of the chiral condensate is reduced to the calculation of zero dimensional super-integrals. Projecting onto fixed topological charge by Fourier decomposing the $\theta$-dependence the zero-dimensional effective partially quenched partition function is given by

$$Z_{\text{eff}}^{\nu}(\hat{M}) = \int_{U \in \hat{Gl}(N_f + 1|1)} dU \text{Sdet}^\nu U e^{\nu \frac{\Sigma}{2} \text{Str}(\hat{M}U + \hat{M}U^{-1})}. \quad (14)$$

As an illustration, we mention the parameterization of the integration manifold for $N_f = 0$. Then $U \in \hat{Gl}(1|1)$ is given by

$$U = \begin{pmatrix} e^{i\phi} & \alpha \\ \beta & e^s \end{pmatrix}, \quad (15)$$

where $\phi \in [0, 2\pi]$, $s \in (-\infty, \infty)$, and $\alpha$ and $\beta$ are Grassmann variables. In terms of these variables the integration measure is simply given by $dU = d\phi ds d\alpha d\beta$. The calculation of the integrals is straightforward in this case. For arbitrary $N_f$ and $\nu$, on the other hand, the integrals in (14) are technically quite involved, but we have succeeded to obtain an analytical result for the valence quark mass dependence of the chiral condensate,

$$\Sigma(x) = \frac{\Sigma(x)}{x} = x(I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)), \quad (16)$$

where $a = N_f + |\nu|$. It coincides with the result derived from chRMT. The microscopic spectral density follows from the discontinuity and agrees with the chRMT result as well.
Our effective partition function is amenable to chiral Perturbation Theory allowing us to go beyond the Random Matrix Theory limit of QCD. Taking into account the kinetic term to one-loop order we have obtained an exact analytical expression for the QCD Dirac spectrum in the domain where chiral perturbation theory applies, i.e. for \( m_v \ll \Lambda_{QCD} \). Among others, we have calculated the slope of the Dirac spectrum and shown that the spectral density diverges logarithmically in the quenched limit. The slope vanishes for two massless flavors, not only for the standard pattern of chiral symmetry breaking \( \Sigma \), but also for other patterns of chiral symmetry breaking relevant for QCD with two colors and fundamental fermions or QCD with adjoint fermions \( N_c = 3 \).

The microscopic spectral density near zero virtuality was first studied in terms of the valence quark mass dependence of the chiral condensate. The lattice data for \( \Sigma(m_v) \) were obtained by the Columbia group \(^{22}\) for two dynamical flavors with sea-quark mass \( m_a = 0.01 \) and \( N_c = 3 \) on a \( 16^3 \times 4 \) lattice. In Fig. 4 we plot the ratio \( \Sigma(m_v)/\Sigma \) as a function of \( x = m_v V \Sigma \) (the ‘volume’ \( V \) is equal to the total number of Dirac eigenvalues), and compare the results with the universal curves obtained from pq-QCD and chRMT (see eq. (16)). We observe that the lattice data for different values of \( \beta \) fall on a single curve. Moreover, in the mesoscopic range this curve coincides with the random matrix prediction for \( N_f = \nu = 0 \).

![Figure 1: The valence quark mass dependence of the chiral condensate \( \Sigma(m) \) plotted as \( \Sigma(m)/\Sigma \) versus \( m_v V \Sigma \). The dots and squares represent lattice results by the Columbia group \(^{22}\) for values of \( \beta \) as indicated in the label of the figure.](image)
mass, the fermion determinant has no bearing on the Dirac spectrum and we have effectively \( N_f = 0 \). At finite lattice spacing, the Kogut-Susskind action does not have the axial \( U(1) \) symmetry and the effects of topology become only visible in the continuum limit. We observe a plateau in the valence quark mass dependence of the chiral condensate. The increase at larger masses is also found in partially quenched chiral perturbation theory and is due to the effects of the kinetic term. The scale \( m_c \) (see eq. (6)) is roughly located at the center of the plateau. Meanwhile, the valence quark mass dependence of the chiral condensate and the scalar susceptibility have also been investigated for \( SU(2) \) gauge field configurations and different types of fermions. In all cases agreement between lattice QCD spectra and the universal result (16) has been found. A detailed analysis of scalar susceptibilities in terms of quenched chiral perturbation theory was recently performed by Berbenni-Bich et al. The microscopic spectral density can be calculated directly and a similar type of agreement with chRMT has been observed.

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![Figure 2: The number variance \( \Sigma_2(n) \) versus \( n \) approximation for an interval starting at \( \lambda = 0 \). The total number of instantons is denoted by \( N \).](image)

Another interesting observable in the number variance, \( \Sigma_2(n) \), defined as the variance of the number of eigenvalues in an interval containing \( n \) eigenvalues on average. It can be obtained in partially quenched chiral perturbation theory from the double discontinuity of the pseudo-scalar susceptibility (A recent discussion of the relation between the pion propagator and the number variance can be found in [50]). The number variance has been calculated both by means of lattice QCD [47, 48] and instanton liquid simulations. In Fig. 1, we show \( \Sigma_2(n) \) versus \( n \) for eigenvalues obtained from the Dirac operator in the background of instanton liquid gauge field configurations. The chRMT result,
given by the solid curve, is reproduced up to about two level spacings. In units of the average level spacing, \( \Delta = 1/\rho(0) = \pi/\Sigma \), the energy \( E_c \) is given by \( n_c \equiv E_c/\Delta = F^2 L^2/\pi \). For an instanton liquid with instanton density \( N/V = 1 \) we find that \( n_c \approx 0.07 \sqrt{N} \). We conclude that chRMT appears to describe the eigenvalue correlations up to the predicted scale.

5 Conclusions

We have found that for mass scales where chiral perturbation theory is valid the fluctuations of the QCD Dirac eigenvalues can be obtained analytically from the partially quenched effective chiral Lagrangian. For mass scales below \( F^2/\Sigma L^2 \) the eigenvalue correlations are given by chRMT. A possible semiclassical interpretation is that quarks undergo a chaotic-diffusive motion in four Euclidean dimensions plus one artificial time dimension. This observation is consistent with a picture of confinement based on impurity scattering.

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