ELEMENTARY EVALUATION OF CONVOLUTION SUMS INVOLVING THE SUM OF DIVISORS FUNCTION FOR A CLASS OF POSITIVE INTEGERS

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ABSTRACT. We discuss an elementary method for the evaluation of the convolution sums
\[ \sum_{\substack{l, m \in \mathbb{N}^2 \atop \alpha l + \beta m = n}} \sigma(l)\sigma(m) \] for those \( \alpha, \beta \in \mathbb{N} \) for which \( \gcd(\alpha, \beta) = 1 \) and \( \alpha\beta = 2^v\mathcal{U} \), where \( v \in \{0, 1, 2, 3\} \) and \( \mathcal{U} \) is a finite product of distinct odd primes. Modular forms are used to achieve this result. We also generalize the extraction of the convolution sum to all natural numbers. Formulae for the number of representations of a positive integer \( n \) by octonary quadratic forms using convolution sums belonging to this class are then determined when \( \alpha \beta \equiv 0 \pmod{4} \) or \( \alpha \beta \equiv 0 \pmod{3} \). To achieve this application, we first discuss a method to compute all pairs \( (a, b), (c, d) \in \mathbb{N}^2 \) necessary for the determination of such formulae for the number of representations of a positive integer \( n \) by octonary quadratic forms when \( \alpha \beta \) has the above form and \( \alpha \beta \equiv 0 \pmod{4} \) or \( \alpha \beta \equiv 0 \pmod{3} \). We illustrate our approach by explicitly evaluating the convolution sum for \( \alpha \beta = 33 = 3 \cdot 11 \), \( \alpha \beta = 40 = 2^3 \cdot 5 \) and \( \alpha \beta = 56 = 2^3 \cdot 7 \), and by revisiting the evaluation of the convolution sums for \( \alpha \beta = 10, 11, 12, 15, 24 \). We then apply these convolution sums to determine formulae for the number of representations of a positive integer \( n \) by octonary quadratic forms. In addition, we determine formulae for the number of representations of a positive integer \( n \) when \((a, b) = (1, 1), (1, 3), (2, 3), (1, 9)\).

1. INTRODUCTION

We denote by \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers, respectively.

Suppose that \( k \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \). The sum of positive divisors of \( n \) to the power of \( k \), \( \sigma_k(n) \), is defined by
\[
\sigma_k(n) = \sum_{0 < \nu | n} \nu^k.
\]

It is obvious from the definition that \( \sigma_k(m) = 0 \) for all \( m \notin \mathbb{N} \). We write \( d(n) \) and \( \sigma(n) \) as a shorthand for \( \sigma_0(n) \) and \( \sigma_1(n) \), respectively.

Assume that the positive integers \( \alpha \leq \beta \) are given. Then the convolution sum \( W_{(\alpha, \beta)}(n) \) is defined by
\[
W_{(\alpha, \beta)}(n) = \sum_{\substack{(l, m) \in \mathbb{N}^2 \atop \alpha l + \beta m = n}} \sigma(l)\sigma(m).
\]

Let \( W_{(\beta, 1)}(n) \) stands for \( W_{(1, \beta)}(n) \). We set \( W_{(\alpha, \beta)}(n) = 0 \) if for all \((l, m) \in \mathbb{N}^2 \) it holds that \( \alpha l + \beta m \neq n \).

2010 Mathematics Subject Classification. 11A25, 11F11, 11F20, 11E20, 11E25, 11F27.

Key words and phrases. Sums of Divisors; Convolution Sums; Modular Forms; Dedekind eta function; Eisenstein forms; Cusp Forms; Octonary quadratic Forms; Number of Representations.
We give the values of $\alpha \beta$ for those convolution sums $W_{(\alpha, \beta)}(n)$ which have so far been evaluated in Table 1.

| Level $\alpha \beta$ | Authors                                                                 | References |
|----------------------|-------------------------------------------------------------------------|------------|
| 1                    | M. Besge, J. W. L. Glaisher, S. Ramanujan                             | [7, 11, 27]|
| 2, 3, 4              | J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams              | [12]       |
| 5, 7                 | M. Lemire & K. S. Williams, S. Cooper & P. C. Toh                      | [16, 9]    |
| 6                    | Ş. Alaca & K. S. Williams                                              | [6]        |
| 8, 9                 | K. S. Williams                                                         | [31, 30]   |
| 10, 11, 13, 14       | E. Royer                                                               | [28]       |
| 12, 16, 18, 24       | A. Alaca & Ş. Alaca & K. S. Williams                                   | [11, 13, 14]|
| 15                   | B. Ramakrishnan & B. Sahu                                             | [26]       |
| 10, 20               | S. Cooper & D. Ye                                                      | [19]       |
| 23                   | H. H. Chan & S. Cooper                                                 | [18]       |
| 25                   | E. X. W. Xia & X. L. Tian & O. X. M. Yao                               | [33]       |
| 27, 32               | Ş. Alaca & Y. Kesicioğlu                                               | [5]        |
| 36                   | D. Ye                                                                  | [34]       |
| 14, 26, 28, 30       | E. Ntienjem                                                            | [22]       |
| 22, 44, 52           | E. Ntienjem                                                            | [24]       |
| 48, 64               | E. Ntienjem                                                            | [23]       |

Table 1: Known convolution sums $W_{(\alpha, \beta)}(n)$ of level $\alpha \beta$

From the levels $\alpha \beta$ listed in Table 1, the following ones do not belong to the class of positive integers that we are handling in this paper: 9, 16, 18, 25, 27, 32, 36, 48 and 64.

We evaluate the convolution sum $W_{(\alpha, \beta)}(n)$ in the case where

$$\alpha \beta = 2^v \cdot 3^u$$

with $v \in \{0, 1, 2, 3\}$, and $3^u$ odd and squarefree positive integer.

The evaluation of the convolution sum for a class of natural numbers and especially for this class is new. We then apply the result for this class to determine the convolution sum for $\alpha \beta = 33 = 3 \cdot 11$, $\alpha \beta = 40 = 2^3 \cdot 5$ and $\alpha \beta = 56 = 2^3 \cdot 7$. Again, these explicit convolution sums have not been evaluated as yet. We revisit the evaluation of the convolution sums for $\alpha \beta = 10, 11, 12, 15, 24$. The re-evaluation of the convolution sums for $\alpha \beta = 10, 11, 12, 15, 24$ improves the previously obtained results.

Convolution sums are applied to establish explicit formulae for the number of representations of a positive integer $n$ by the octonary quadratic forms

$$a \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right) + b \left( x_5^2 + x_6^2 + x_7^2 + x_8^2 \right),$$

and

$$c \left( x_1^2 + x_1 x_2 + x_2^2 + x_3^2 + x_3 x_4 + x_4^2 \right) + d \left( x_5^2 + x_5 x_6 + x_6^2 + x_7 x_8 + x_8^2 \right),$$

respectively, where $(a, b), (c, d) \in \mathbb{N}^2$.

Known explicit formulae for the number of representations of $n$ by the octonary quadratic form Equation 1.4 are referenced in Table 2 and that for the octonary quadratic form Equation 1.5 in Table 3.

| (a, b)    | Authors         | References |
|-----------|-----------------|------------|
| (1, 2)    | K. S. Williams  | [31]       |
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Based on the structure of $\alpha$ and $\beta$, we provide a method to determine all pairs $(a, b) \in \mathbb{N}^2$ and $(c, d) \in \mathbb{N}^2$ that are necessary for the determination of the formulae for the number of representations of a positive integer by the octonary quadratic forms Equation 1.4 and Equation 1.5. Then we determine explicit formulae for the number of representations of a positive integer $n$ by the octonary quadratic forms Equation 1.4 and Equation 1.5 whenever $\alpha \beta$ has the above form and is such that $\alpha \beta \equiv 0 \pmod{4}$ or $\alpha \beta \equiv 0 \pmod{3}$. As an example, we determine formulae for the number of representations of a positive integer $n$ by octonary quadratic forms [Equation 1.4] and [Equation 1.5] using the convolution sums for $\alpha \beta = 33 = 3 \cdot 11$, $\alpha \beta = 40 = 2^3 \cdot 5$ and $\alpha \beta = 56 = 2^3 \cdot 7$, respectively.

This work is structured as follows. In Section 2, we discuss basic knowledge of modular forms, briefly define eta functions and convolution sums. The evaluation of the convolution sum for the above class of positive integers is discussed in Section 3. In Section 4, formulae for the number of representations of a positive integer by the octonary forms Equation 1.4 and Equation 1.5 are determined for this class of positive numbers. Examples to illustrate our method are then given in Section 5. The evaluated convolution sums for $\alpha \beta = 10, 11, 12, 15, 24$ are revisited in Section 6. We determine in Section 7 formulae for the number of representations of a positive integer $n$ for the illustrated examples and for $(a, b) = (1, 1), (1, 3), (2, 3), (1, 9)$. We then conclude in Section 8 with a brief outlook.

The results of this paper are obtained using Software for symbolic scientific computation. This software is composed of the open source software packages GiNaC, Maxima, REDUCE, SAGE and the commercial software package MAPLE.
2. Basic Knowledge

The upper half-plane, $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$, and $\Gamma = \text{SL}_2(\mathbb{R})$ the group of $2 \times 2$-matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ are considered in this paper.

Let $N \in \mathbb{N}$. Then

$$\Gamma(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \}$$

is a subgroup of $\Gamma$ and is known as the principal congruence subgroup of level $N$. If a subgroup $H$ of $\Gamma$ contains $\Gamma(N)$, then that subgroup is called a congruence subgroup of level $N$.

The following congruence subgroup of level $N$ is relevant for our purpose

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \}.$$ 

Let $N \in \mathbb{N}$, $\Gamma' \subseteq \Gamma$ be a congruence subgroup of level $N$, $k \in \mathbb{Z}$, $\gamma \in \text{SL}_2(\mathbb{Z})$ and $f^\gamma|k : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be defined by $f^\gamma|k(z) = (cz + d)^{-k} f(\gamma(z))$. The following definition is extracted from N. Koblitz’s book [14] p. 108].

Definition 2.1. Suppose that $N \in \mathbb{N}$, $k \in \mathbb{Z}$, $f$ is a meromorphic function on $\mathbb{H}$ and $\Gamma' \subseteq \Gamma$ is a congruence subgroup of level $N$. Let furthermore $\mathbb{N}_0^n = \{-n \mid n \in \mathbb{N}_0\}$ be the set of all negative and nonzero natural numbers.

(a) $f$ is a modular function of weight $k$ for $\Gamma'$ if
   (a1) for all $\gamma \in \Gamma'$ it holds that $f^\gamma|k = f$.
   (a2) for any $\delta \in \Gamma$ it holds that $f^\delta|k(z)$ can be expressed in the form $\sum a_n e^{\frac{2\pi i n z}{N}}$,
   wherein $a_n \neq 0$ for finitely many $n \in \mathbb{N}_0^n$.

(b) $f$ is a modular form of weight $k$ for $\Gamma'$ if
   (b1) $f$ is a modular function of weight $k$ for $\Gamma'$,
   (b2) $f$ is holomorphic on $\mathbb{H}$.
   (b3) for all $\delta \in \Gamma$ and for all $n \in \mathbb{N}_0^n$ it holds that $a_n = 0$.

(c) $f$ is a cusp form of weight $k$ for $\Gamma'$ if
   (c1) $f$ is a modular form of weight $k$ for $\Gamma'$,
   (c2) for all $\delta \in \Gamma$ it holds that $a_0 = 0$.

We denote the set of modular forms of weight $k$ for $\Gamma'$ by $\mathcal{M}_k(\Gamma')$, the set of cusp forms of weight $k$ for $\Gamma'$ by $\mathcal{E}_k(\Gamma')$ and the set of Eisenstein forms by $\mathcal{E}_k(\Gamma')$. The sets $\mathcal{M}_k(\Gamma')$, $\mathcal{E}_k(\Gamma')$ and $\mathcal{E}_k(\Gamma')$ are vector spaces over $\mathbb{C}$. Hence, $\mathcal{M}_k(\Gamma_0(N))$ is the space of modular forms of weight $k$ for $\Gamma_0(N)$, $\mathcal{E}_k(\Gamma_0(N))$ is the space of cusp forms of weight $k$ for $\Gamma_0(N)$, and $\mathcal{E}_k(\Gamma_0(N))$ is the space of Eisenstein forms. The decomposition of the space of modular forms as a direct sum of the space generated by the Eisenstein series and the space of cusp forms, i.e., $\mathcal{M}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N)) \oplus \mathcal{E}_k(\Gamma_0(N))$, is well-known; see for example W. A. Stein’s book (online version) [29] p. 81].

As noted in Section 5.3 of [29] p. 86], if the primitive Dirichlet characters are trivial and
2 $\leq k$ is even, then $E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, where $B_k$ are the Bernoulli numbers.

For the purpose of this paper we only consider trivial primitive Dirichlet characters and 2 $\leq k$ even. Theorems 5.8 and 5.9 in Section 5.3 of [29] p. 86] also hold for this special case.
2.1. Eta Quotients. The Dedekind eta function $\eta(z)$ is defined on the upper half-plane $\mathbb{H}$ by $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$. When we set $q = e^{2\pi iz}$, then we have

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q), \quad \text{where } F(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

Let $j, k \in \mathbb{N}$ and $e_j \in \mathbb{Z}$. According to G. Köhler [15] p. 31 an eta product or eta quotient, $f(z)$, is a finite product of Dedekind eta functions of the form

$$(2.1) \quad \prod_{j=1}^{K} \eta(jz)^{e_j}.$$  

Based on this definition there exists $N \in \mathbb{N}$ such that $N = \text{lcm}\{j \mid 1 \leq j \leq k\}$. We call such an $N$ the level of an eta product. An eta product will hence be understood as $\prod_{j \mid N} \eta(jz)^{e_j}$. An eta product $f(z)$ behaves like a modular form of weight $k$ on $\Gamma_0(N)$ with some multiplier system whenever $k = \frac{1}{2} \sum_{j=1}^{K} e_j$.

In this paper we use eta function, eta quotient and eta product interchangeably as synonyms.

The eta function was systematically applied by M. Newman [20, 21] to construct modular forms for $\Gamma_0(N)$ and then to determine when a function $f(z)$ was a modular form for $\Gamma_0(N)$. That is partly explained above and leads to conditions (i)-(iii) in the following theorem. The order of vanishing of an eta function at the cusps of $\Gamma_0(N)$, which is condition (iv) or (iv') in Theorem 2.2, was determined by G. Ligozat [17].

In L. J. P. Kilford’s book [13, p. 99] and G. Köhler’s book [15, Cor. 2.3, p. 37] the following theorem is proved. We will use that theorem to determine eta quotients, $f(z)$, which belong to $\mathbb{M}_k(\Gamma_0(N))$, and especially those eta quotients which are in $\mathbb{S}_k(\Gamma_0(N))$.

Theorem 2.2 (M. Newman and G. Ligozat). Let $N \in \mathbb{N}$, $D(N)$ be the set of all positive divisors of $N$, $\delta \in D(N)$ and $r_5 \in \mathbb{Z}$. Let furthermore $f(z) = \prod_{\delta \in D(N)} \eta(z)^{\delta} \in \mathbb{M}_k(\Gamma_0(N))$, where $k = \frac{1}{2} \sum_{\delta \in D(N)} r_5$.

If the following four conditions are satisfied

(i) $\sum_{\delta \in D(N)} \delta r_5 \equiv 0 \pmod{24}$,  
(ii) $\prod_{\delta \in D(N)} \delta \delta^5$ is a square in $\mathbb{Q}$,

(iii) $0 < \sum_{\delta \in D(N)} r_5 \equiv 0 \pmod{4}$,  
(iv) $\forall d \in D(N)$ it holds $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta} r_5 \geq 0$,

then $f(z) \in \mathbb{M}_k(\Gamma_0(N))$, where $k = \frac{1}{2} \sum_{\delta \in D(N)} r_5$.

Moreover, the eta quotient $f(z)$ belongs to $\mathbb{S}_k(\Gamma_0(N))$ if (iv) is replaced by

(iv') $\forall d \in D(N)$ it holds $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta} r_5 > 0$.

2.2. Convolution Sums $W_{(\alpha, \beta)}(n)$. Suppose that $\alpha, \beta \in \mathbb{N}$ are such that $\alpha \leq \beta$. The convolution sum, $W_{(\alpha, \beta)}(n)$, is defined by Equation 1.2.

Now, suppose in addition that $\gcd(\alpha, \beta) = \delta > 1$. Therefore, there exist $\alpha_1, \beta_1 \in \mathbb{N}$ such that $\gcd(\alpha_1, \beta_1) = 1$, $\alpha = \delta \alpha_1$ and $\beta = \delta \beta_1$. Then

$$(2.2) \quad W_{(\alpha, \beta)}(n) = \sum_{(l, k) \in \mathbb{Z}^2_0} \sigma(l) \sigma(k) = \sum_{\delta \alpha_1 l + \delta \beta_1 k = n} \sigma(l) \sigma(k) = W_{(\alpha_1, \beta_1)} \left( \frac{n}{\delta} \right).$$
Therefore, we may simply assume that \( \gcd(\alpha, \beta) = 1 \) as does A. Alaca et al. [1]. We apply the formula proved by M. Besge, J. W. L. Glaisher, and S. Ramanujan [7, 11, 27] to Equation 2.2 to deduce that

\[
\forall \alpha \in \mathbb{N} \quad W_{\alpha, \alpha}(n) = W_{(1,1)}(\frac{n}{\alpha}) = \frac{5}{12} \sigma_3\left(\frac{n}{\alpha}\right) + \left(\frac{1}{12} - \frac{1}{2n}\right)\sigma\left(\frac{n}{\alpha}\right).
\]

Let \( q \in \mathbb{C} \) be such that \( |q| < 1 \). The Eisenstein series \( L(q) \) and \( M(q) \) are defined as follows:

\[
L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n,
\]

\[
M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.
\]

We state two relevant results for the sequel of this work. These two results generalize the extraction of the convolution sum to all natural numbers.

**Lemma 2.3.** Let \( \alpha, \beta \in \mathbb{N} \). Then

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha\beta)).
\]

**Proof.** If \( \alpha = \beta \), then trivially \( 0 = (\alpha L(q^\alpha) - \alpha L(q^\alpha))^2 \in \mathcal{M}_4(\Gamma_0(\alpha)) \) and there is nothing to prove. Therefore, we may suppose that \( \alpha \neq \beta > 0 \) in the sequel. We apply the result proved by W. A. Stein [29, Thms 5.8, 5.9, p. 86] to deduce \( L(q) - \alpha L(q^\alpha) \in \mathcal{M}_2(\Gamma_0(\alpha)) \subseteq \mathcal{M}_2(\Gamma_0(\alpha\beta)) \) and \( L(q) - \beta L(q^\beta) \in \mathcal{M}_2(\Gamma_0(\beta)) \subseteq \mathcal{M}_2(\Gamma_0(\alpha\beta)) \). Therefore,

\[
\alpha L(q^\alpha) - \beta L(q^\beta) = (L(q) - \beta L(q^\beta)) - (L(q) - \alpha L(q^\alpha)) \in \mathcal{M}_2(\Gamma_0(\alpha\beta))
\]

and so \( (\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha\beta)) \). \( \square \)

**Theorem 2.4.** Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \alpha < \beta \), and \( \alpha \) and \( \beta \) are relatively prime. Then

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left(240 \alpha^2 \sigma_3\left(\frac{n}{\alpha}\right) + 240 \beta^2 \sigma_3\left(\frac{n}{\beta}\right)
\]

\[
+ 48 \alpha (\beta - 6n) \sigma\left(\frac{n}{\alpha}\right) + 48 \beta (\alpha - 6n) \sigma\left(\frac{n}{\beta}\right) - 1152 \alpha \beta W_{(\alpha, \beta)}(n)\right)q^n.
\]

**Proof.** We first observe that

\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = \alpha^2 L^2(q^\alpha) + \beta^2 L^2(q^\beta) - 2\alpha \beta L(q^\alpha)L(q^\beta).
\]

J. W. L. Glaisher [11] has proved the following identity

\[
L^2(q) = 1 + \sum_{n=1}^{\infty} \left(240 \sigma_3(n) - 288 n \sigma(n)\right)q^n
\]

which we apply to deduce

\[
L^2(q^\alpha) = 1 + \sum_{n=1}^{\infty} \left(240 \sigma_3\left(\frac{n}{\alpha}\right) - 288 \frac{n}{\alpha} \sigma\left(\frac{n}{\alpha}\right)\right)q^n
\]

and

\[
L^2(q^\beta) = 1 + \sum_{n=1}^{\infty} \left(240 \sigma_3\left(\frac{n}{\beta}\right) - 288 \frac{n}{\beta} \sigma\left(\frac{n}{\beta}\right)\right)q^n.
\]
Since
\[
(\sum_{n=1}^{\infty} \frac{\sigma(n)}{\alpha} q^n)(\sum_{n=1}^{\infty} \frac{\sigma(n)}{\beta} q^n) = \sum_{n=1}^{\infty} \left( \sum_{\alpha k + \beta n = n} \sigma(k) \sigma(l) \right) q^n = \sum_{n=1}^{\infty} W(\alpha,\beta)(n) q^n,
\]
we conclude, when using the accordingly modified Equation 2.4, that
\[
L(q^\alpha)L(q^\beta) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)\sigma(n) q^n - 24 \sum_{n=1}^{\infty} \frac{\sigma(n)}{\beta} q^n + 576 \sum_{n=1}^{\infty} W(\alpha,\beta)(n) q^n.
\]
Therefore,
\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left( 240 \alpha^2 \sigma_3(n) + 240 \beta^2 \sigma_3(n) \right)
+ 48 \alpha (\beta - 6n) \sigma(n) + 48 \beta (\alpha - 6n) \sigma(n) - 1152 \alpha \beta W(\alpha,\beta)(n) q^n
\]
as asserted. \(\square\)

3. Evaluating \(W(\alpha,\beta)(n)\) for a Class of Natural Numbers \(\alpha\beta\)

Suppose that \(\alpha\) and \(\beta\) are positive integers which satisfy the following two conditions:

(i) \(\gcd(\alpha, \beta) = 1\)

(ii) \[\text{Equation 1.3} \]

We derive the formula for the convolution sum \(W(\alpha,\beta)(n)\) for all such \(\alpha\) and \(\beta\). Let in the sequel \(D(\alpha\beta)\) denote the set of all positive divisors of \(\alpha\beta\).

3.1. Bases for \(E_4(\Gamma_0(\alpha\beta))\) and \(S_4(\Gamma_0(\alpha\beta))\). The existence of a basis of the space of cusp forms of weight 2 \(\leq k\) even for \(\Gamma_0(\alpha\beta)\) when \(\alpha\beta\) is not a perfect square is discussed by A. Pizer [25]. We recall that the Dirichlet character \(\chi\) is assumed to be trivial, that is \(\chi = 1\).

According to the dimension formulae in T. Miyake’s book [19] Thrm 2.5.2, p. 60 or [29] Prop. 6.1, p. 91,

- and in addition due to the special form of \(\alpha\beta\), we deduce that

\[(3.1) \quad \dim(E_4(\Gamma_0(\alpha\beta))) = \sum_{\delta | \gcd(\delta, \alpha\beta)} \varphi(\delta) = \sum_{\delta | \gcd(\delta, \alpha\beta)} 1 = \sigma_0(\alpha\beta) = d(\alpha\beta),\]

where \(\varphi\) is the Euler’s totient function.

- we may assume that \(\dim(S_4(\Gamma_0(\alpha\beta))) = m_S \in \mathbb{N}\).

To determine as many elements of \(S_4(\Gamma_0(\alpha\beta))\) as possible for an explicitly given \(\alpha\beta\), we use an exhaustive search when we apply Theorem 2.2. We select from these determined elements of the space \(S_4(\Gamma_0(\alpha\beta))\) relevant ones for the purpose of the determination of a basis of this space.

The so-determined basis of the vector space of cusp forms is not unique. However, due to the change of basis which is an automorphism, it is sufficient to only consider this basis for our purpose.

**Theorem 3.1.**

(a) The set \(\mathcal{B}_E = \{ M(q^i) \mid i \in D(\alpha\beta) \}\) is a basis of \(E_4(\Gamma_0(\alpha\beta))\).

(b) Let \(1 \leq i \leq m_s\) be positive integers, \(\delta \in D(\alpha\beta)\) and \((r(i, \delta))_{i, \delta}\) be a table of the powers of \(\eta(\delta \zeta)\). Let furthermore \(\mathcal{B}_{\alpha\beta,i}(q) = \prod_{\delta | \gcd(\delta, \alpha\beta)} \eta^{r(i, \delta)}(\delta \zeta)\) be selected elements of \(S_4(\Gamma_0(\alpha\beta))\). Then the set \(\mathcal{B}_S = \{ \mathcal{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_s \}\) is a basis of \(S_4(\Gamma_0(\alpha\beta))\).
(c) The set $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ constitutes a basis of $\mathcal{M}_4(\Gamma_0(\alpha\beta))$.

**Remark 3.2.**  
(a) By Theorem 5.8 in Section 5.3 of W. A. Stein [29, p. 86] each $M(q')$ can be expressed in the form $\sum_{n=1}^{\infty} b_{\alpha\beta,i}(n)q^n$, where for each $n \geq 1$ the $b_{\alpha\beta,i}(n)$ are integers.

(b) Since $\mathcal{B}_E = \{1\}$, by Theorem 2.2, the set $\mathcal{B}_E$ constitutes a basis of $\mathcal{B}_{\alpha\beta}(\mathcal{S}_M)$. We prove the latter by induction on the elements of the set $\mathcal{B}_E$.

(c) The set $\mathcal{B}_S = \{\mathcal{B}_i\}_{i=1}^{m_S}$ constitutes a basis of $\mathcal{M}_4(\Gamma_0(\alpha\beta))$.

**Proof.**

(a) By Theorem 5.8 in Section 5.3 of W. A. Stein [29, p. 86] each $M(q')$ is in $\mathcal{M}_4(\Gamma_0(t))$, where $t \in D(\alpha\beta)$. Since $\mathcal{E}_4(\Gamma_0(\alpha\beta))$ has a finite dimension, it suffices to show that $M(q')$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ with $t \in D(\alpha\beta)$.

We prove this by induction on the elements of the set $D(\alpha\beta)$ which is assumed to be linearly ordered.

The case $t = 1 \in D(\alpha\beta)$ is obvious since comparing the coefficients of $q'$ on both sides of the equation $x_i M(q') = 0$ clearly gives $x_i = 0$.

Suppose now that the cardinality of the set $D(\alpha\beta)$ is greater than 1 and that $M(q')$ are linearly independent for all $t \in D(\alpha\beta)$ such that $t \leq t_1$ for a given $t_1$ with $1 < t_1 < \alpha\beta$. Let $C$ be the proper non-empty subset of $D(\alpha\beta)$ which contains all positive divisors of $\alpha\beta$ less than or equal to $t_1$. Note that all positive divisors of $t_1$ constitute a subset of $C$ and observe that each $t \in D(\alpha\beta)$ belongs to the class of positive integers defined by Equation 1.3. Let us consider the non-empty subset $C \cup \{t'\}$ of $D(\alpha\beta)$, wherein $t'$ is the next ascendant element of $D(\alpha\beta)$ which is greater than $t_1$ the greatest element of the set $C$. Then

$$\sum_{t \in C \cup \{t'\}} x_i M(q') = \sum_{t \in C} x_i M(q') + x_{t'} M(q') = 0.$$  

By the induction hypothesis it holds that $x_i = 0$ for all $t \in C$. So, we obtain from the above equation that $x_{t'} = 0$ when we compare the coefficient of $q'$ on both sides of the equation.

Hence, the solution of the homogeneous system of $d(\alpha\beta)$ linear equations is $x_i = 0$ for all $t \in D(\alpha\beta)$. Therefore, the set $\mathcal{B}_E$ is linearly independent and hence is a basis of $\mathcal{E}_4(\Gamma_0(\alpha\beta))$.

(b) Since $\mathcal{B}_{\alpha\beta}(\mathcal{S}_M)$ with $1 \leq i \leq m_S$ are obtained from an exhaustive search using Theorem 2.2, it holds that each $\mathcal{B}_{\alpha\beta,i}(\mathcal{S}_M)$ is in the space $\mathcal{E}_4(\Gamma_0(\alpha\beta))$.

Since the dimension of $\mathcal{E}_4(\Gamma_0(\alpha\beta))$ is $m_S \in \mathbb{N}$, it is sufficient to show that the set $\{\mathcal{B}_{\alpha\beta,i}(\mathcal{S}_M) \mid 1 \leq i \leq m_S\}$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ and

$$\sum_{i=1}^{m_S} x_i \mathcal{B}_{\alpha\beta,i}(\mathcal{S}_M) = 0.$$  

Then

$$\sum_{i=1}^{m_S} \sum_{n=1}^{\infty} x_i b_{\alpha\beta,i}(n) q^n = 0$$

which gives the following homogeneous system of $m_S$ linear equations in $m_S$ unknowns

$$\sum_{i=1}^{m_S} b_{\alpha\beta,i}(n) x_i = 0, \quad 1 \leq n \leq m_S.$$  

(3.2)
By Remark 3.2 (r2) we may consider without lost of generality two cases.

Case 1: For each $1 \leq i \leq m_S$ the smallest degree of $q$ in $\mathcal{B}_{\alpha\beta,i}(q)$ is $i$. It is then obvious that the $m_S \times m_S$ matrix which corresponds to this homogeneous system of linear equations is triangular with 1’s on the diagonal. Hence, the determinant of that matrix is 1 and so $x_i = 0$ for all $1 \leq i \leq m_S$.

Case 2: The set $\mathcal{B}_S$ does contain a subset, say $\mathcal{B}'_S = \{ \mathcal{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq u \}$ for some $1 \leq u < m_S$, for which the smallest degree of $q$ in $\mathcal{B}_{\alpha\beta,i}(q)$ is $i$. Since the smallest degree of $q$ is 1, $\mathcal{B}'_S$ is not empty. Let us consider $\mathcal{B}_S$ as an ordered set of the form $\mathcal{B}'_S \cup \mathcal{B}_S''$, where $\mathcal{B}_S'' = \{ \mathcal{B}_{\alpha\beta,i}(q) \mid u < i \leq m_S \}$. Let $A = (b_{\alpha\beta,i}(n))$ be the $m_S \times m_S$ matrix in Equation 3.2. In this matrix $i$ indicates the $i$-th column and $n$ indicates the $n$-th row. Note that applying case 1 the subset $\mathcal{B}'_S$ is linearly independent since the determinant of the corresponding $u \times u$ matrix is 1.

If $\det(A) \neq 0$, then $x_i = 0$ for all $1 \leq i \leq m_S$. Suppose now that $\det(A) = 0$. Then for some $u < k \leq m_S$ there exists $\mathcal{B}_{\alpha\beta,i}(q)$ which is causing the system of linear equations to be inconsistent. We substitute $\mathcal{B}_{\alpha\beta,i}(q)$ with, say $\mathcal{B}'_{\alpha\beta,i}(q)$, which does not occur in $\mathcal{B}_S$ and compute the determinant of the new matrix $A$. Since there are finitely many $\mathcal{B}_{\alpha\beta,i}(q)$ with $u < k \leq m_S$ that may cause the system of linear equations to be inconsistent and finitely many elements of $\mathcal{E}_4(\Gamma_0(\alpha\beta)) \setminus \mathcal{B}_S$, the procedure will terminate with a consistent system of linear equations. So, $x_i = 0$ for all $1 \leq i \leq m_S$.

Therefore, the set $\{ \mathcal{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_S \}$ is linearly independent and so is a basis of $\mathcal{E}_4(\Gamma_0(\alpha\beta))$.

Since $\mathcal{M}_4(\Gamma_0(\alpha\beta)) = \mathcal{E}_4(\Gamma_0(\alpha\beta)) \oplus \mathcal{E}_4(\Gamma_0(\alpha\beta))$, the result follows from (a) and (b).

The proof of Theorem 3.1 (b) provides an effective method to determine the basis of the space of cusp forms of level $\alpha\beta$ whenever $\alpha\beta$ belongs to this class of positive integers.

### 3.2. Evaluating the convolution sum $W_{(\alpha, \beta)}(n)$

#### Lemma 3.3. Let $\alpha, \beta \in \mathbb{N}$ be such that $\gcd(\alpha, \beta) = 1$. Let furthermore $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ be a basis of $\mathcal{M}_4(\Gamma_0(\alpha\beta))$. Then there exist $X_\delta \in \mathbb{C}$ and $Y_j \in \mathbb{C}$ with $\delta \in D(\alpha\beta)$ and $1 \leq j \leq m_S$ such that

$$
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = \sum_{\delta | \alpha\beta} X_\delta + \sum_{\delta | \alpha\beta} 1 \left( 240 \sum_{\delta | \alpha\beta} \sigma_1(\frac{n}{\delta}) X_\delta + \sum_{j=1}^{m_S} b_{\alpha\beta,j}(n) Y_j \right) q^n.
$$

#### Proof. That $(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha\beta))$ follows from Lemma 2.3. Hence, by Theorem 3.1 (c), there exist $X_\delta, Y_j \in \mathbb{C}, 1 \leq j \leq m_S$ and $\delta \in D(\alpha\beta)$, such that

$$
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = \sum_{\delta | \alpha\beta} X_\delta M(q^\delta) + \sum_{j=1}^{m_S} Y_j \mathcal{B}_{\alpha\beta,j}(q)
$$

or

$$
= \sum_{\delta | \alpha\beta} X_\delta + \sum_{\delta | \alpha\beta} 1 \left( 240 \sum_{\delta | \alpha\beta} \sigma_1(\frac{n}{\delta}) X_\delta + \sum_{j=1}^{m_S} b_{\alpha\beta,j}(n) Y_j \right) q^n.
$$
We compare the coefficients of $q^n$ on the right hand side of Equation 2.6, which yields

$$
\sum_{n=1}^{\infty} \left( 240 \sum_{\delta \neq \alpha \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta + \sum_{j=1}^{m_\delta} b_{\alpha \beta, j}(n) Y_j \right) q^n = \sum_{n=1}^{\infty} \left( 240 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) + 240 \beta^2 \sigma_3 \left( \frac{n}{\beta} \right) \\
+ 48 \alpha (\beta - 6n) \sigma \left( \frac{n}{\alpha} \right) + 48 \beta (\alpha - 6n) \sigma \left( \frac{n}{\beta} \right) - 1152 \alpha \beta W_{(\alpha, \beta)}(n) \right) q^n.
$$

We then take the coefficients of $q^n$ for which $n$ is in $D(\alpha \beta)$ and $1 \leq n \leq m_\delta$, but as many as the unknowns $X_\delta$ and $Y_j$. This results in a system of $d(\alpha \beta) + m_\delta$ linear equations whose unique solution determines the values of the unknown $X_\delta$ for all $\delta \in D(\alpha \beta)$ and the values of the unknown $Y_j$ for all $1 \leq j \leq m_\delta$. Hence, we obtain the stated result. □

In the following theorem, let $X_\delta$ and $Y_j$ stand for their values obtained in the previous theorem.

**Theorem 3.4.** Let $n$ be a positive integer. Then

$$
W_{(\alpha, \beta)}(n) = -\frac{5}{24} \alpha \beta \sum_{\delta \neq \alpha \beta} X_\delta \sigma_3 \left( \frac{n}{\delta} \right) + \frac{5}{24} \alpha \beta (\alpha^2 - X_\alpha) \sigma_3 \left( \frac{n}{\alpha} \right) \\
+ \frac{5}{24} \beta^2 - X_\beta) \sigma_3 \left( \frac{n}{\beta} \right) - \sum_{j=1}^{m_\delta} \frac{1}{1152} \alpha \beta Y_j b_{\alpha \beta, j}(n) \\
+ \left( \frac{1}{24} - \frac{1}{4 \beta} \right) \sigma \left( \frac{n}{\alpha} \right) + \left( \frac{1}{24} - \frac{1}{4 \alpha} \right) \sigma \left( \frac{n}{\beta} \right).
$$

**Proof.** We set the right hand side of Equation 3.3 with that of Equation 2.6 equal, which yields

$$
1152 \alpha \beta W_{(\alpha, \beta)}(n) = -240 \sum_{\delta \neq \alpha \beta} \sigma_3 \left( \frac{n}{\delta} \right) X_\delta - \sum_{j=1}^{m_\delta} b_{\alpha \beta, j}(n) Y_j + 240 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) \\
+ 240 \beta^2 \sigma_3 \left( \frac{n}{\beta} \right) + 48 \alpha (\beta - 6n) \sigma \left( \frac{n}{\alpha} \right) + 48 \beta (\alpha - 6n) \sigma \left( \frac{n}{\beta} \right).
$$

Then we solve for $W_{(\alpha, \beta)}(n)$ to obtain the stated result. □

**Remark 3.5.** Observe that the following part of Theorem 3.4

$$
\left( \frac{1}{24} - \frac{1}{4 \beta} \right) \sigma \left( \frac{n}{\alpha} \right) + \left( \frac{1}{24} - \frac{1}{4 \alpha} \right) \sigma \left( \frac{n}{\beta} \right)
$$

depends only on $n$, $\alpha$, and $\beta$ and not on the basis of the modular space $\mathcal{M}_4(\Gamma_0(\alpha \beta))$.

4. **Number of Representations of a Positive Integer for this Class of Positive Integer**

We discuss in this section the determination of formulae for the number of representations of a positive integer by the octonary quadratic forms Equation 1.4 and Equation 1.5, respectively.

4.1. **Representations of a Positive Integer by the Octonary Quadratic Form Equation 1.4** We restrict the general form of $\alpha \beta$ to $2^vU$ where $v \in \{2, 3\}$ and $U$ is odd square-free finite product of distinct odd primes; that is $\alpha \beta \equiv 0 \pmod{4}$. 
4.1.1. Determination of \((a, b) \in \mathbb{N}^2\). We carry out a method to determine all pairs \((a, b) \in \mathbb{N}^2\) necessary for the determination of \(N(a,b)(n)\) for a given \(a\beta \in \mathbb{N}\) which belongs to the above class.

Let \(\Lambda = \frac{a\beta}{3} = 2^{v-2}\mathfrak{U}\), \(P_4 = \{p_0 = 2^{v-2}\} \cup \bigcup_{j>1} \{p_j \mid p_j \text{ is a prime divisor of } \mathfrak{U}\}\) and \(\mathcal{P}(P_4)\) be the power set of \(P_4\). Then for each \(Q \in \mathcal{P}(P_4)\) we define \(\mu(Q) = \prod_{p \in Q} p\). We set \(\mu(Q) = 1\) if \(Q\) is an empty set. Let now

\[
\Omega_4 = \{ (\mu(Q_1), \mu(Q_2)) \mid \text{there exist } Q_1, Q_2 \in \mathcal{P}(P_4) \text{ such that } \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1)\mu(Q_2) = \Lambda \}.
\]

Observe that \(\Omega_4 \neq \emptyset\) since \((1, \Lambda) \in \Omega_4\).

To illustrate our method, suppose that \(a\beta = 2^3 \cdot 3\cdot 5\). Then \(\Lambda = 2^3 \cdot 5\cdot 3,\ P_4 = \{2, 3, 5\}\) and \(\Omega_4 = \{(1, 30), (2, 15), (3, 10), (5, 6)\}\).

**Proposition 4.1.** Suppose that \(a\beta\) has the above restricted form and suppose that \(\Omega_4\) is defined as above. Then for all \(n \in \mathbb{N}\) the set \(\Omega_4\) contains all pairs \((a, b) \in \mathbb{N}^2\) such that \(N(a,b)(n)\) can be obtained by applying \(W(a,b)(n)\) and some other evaluated convolution sums.

**Proof.** We prove this by induction on the structure of \(a\beta\).

Suppose that \(a\beta = 2^v p_2\), where \(v \in \{2, 3\}\) and \(p_2\) is an odd prime. Then by the above definitions we have \(\Lambda = 2^{v-2} p_2,\ P_4 = \{2^{v-2}, p_2\}\),

\[
\mathcal{P}(P_4) = \{\emptyset, \{2^{v-2}\}, \{p_2\}, \{2^{v-2}, p_2\}\},
\]

and \(\Omega_4 = \{(1, 2^{v-2} p_2), (2^{v-2} p_2)\}\).

We show that \(\Omega_4\) is the largest such set. Assume now that there exist another set, say \(\Omega_4'\), which results from the above definitions. Then there are two cases.

**Case \(\Omega_4' \subseteq \Omega_4\):** There is nothing to show. So, we are done.

**Case \(\Omega_4 \subseteq \Omega_4'\):** Let \((e, f) \in \Omega_4 \setminus \Omega_4\). Since \(ef = 2^{v-2} p_2\) and \(\gcd(e, f) = 1\), we must have either \((e, f) = (1, 2^{v-2} p_2)\) or \((e, f) = (2^{v-2}, p_2)\). So, \((e, f) \in \Omega_4\). Hence, \(\Omega_4 = \Omega_4'\).

Suppose now that \(a\beta = 2^v p_2 p_3\), where \(v \in \{2, 3\}\) and \(p_2, p_3\) are distinct odd primes. Then by the induction hypothesis and by the above definitions we have essentially

\[
\Omega_4 = \{(1, 2^{v-2} p_2 p_3), (2^{v-2}, p_2 p_3), (2^{v-2} p_2, p_3), (2^{v-2} p_3, p_2)\}.
\]

Again, we show that \(\Omega_4\) is the largest such set. Suppose that there exist another set, say \(\Omega_4'\), which results from the above definitions. Two cases arise.

**Case \(\Omega_4' \subseteq \Omega_4\):** There is nothing to prove. So, we are done.

**Case \(\Omega_4 \subseteq \Omega_4'\):** Let \((e, f) \in \Omega_4 \setminus \Omega_4\). Since \(ef = 2^{v-2} p_2 p_3\) and \(\gcd(e, f) = 1\), we must have \((e, f) = (1, 2^{v-2} p_2 p_3)\) or \((e, f) = (2^{v-2}, p_2 p_3)\) or \((e, f) = (2^{v-2} p_2, p_3)\) or \((e, f) = (2^{v-2} p_3, p_2)\). So, \((e, f) \in \Omega_4\). Hence, \(\Omega_4 = \Omega_4'\).

\[\square\]

4.1.2. Formulae for the Number of Representations by \(\text{Equation 1.4}\) As an immediate application of \(\text{Theorem 3.4}\) a formula for the number of representations of a positive integer \(n\) by the octonary quadratic form \(\text{Equation 1.4}\) is determined for each \((a, b) \in \Omega_4\).

Let \(n \in \mathbb{N}_0\) and let the number of representations of \(n\) by the quaternary quadratic form \(x_1^2 + x_2^2 + x_3^2 + x_4^2\) be \(r_4(n) = \text{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2\})\). It follows
from the definition that \( r_4(0) = 1 \). For all \( n \in \mathbb{N} \), the following Jacobi’s identity is proved in K. S. Williams’ book \([52]\) Thrm 9.5, p. 83]

\[
(4.1) \quad r_4(n) = 8\sigma(n) - 32\sigma_4(n).
\]

Now, let the number of representations of \( n \) by the octonary quadratic form \([\text{Equation 1.4}]\) be

\[
N_{(a,b)}(n) = \text{card}\{ (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \},
\]

where \( a, b \in \mathbb{N} \). Let \( 1 < \lambda \in \mathbb{N} \) and \( \tau : \mathbb{N} \rightarrow \mathbb{N} \) be an injective function such that \( \tau(n) = \lambda \cdot n \) for each \( n \in \mathbb{N} \).

We then derive the following result:

**Theorem 4.2.** Let \( n \in \mathbb{N} \) and let \( (a, b) \in \Omega_4 \). Then

\[
N_{(a,b)}(n) = 8\sigma(n) - 32\sigma_4(n) + 8\sigma(n) - 32\sigma_4(n) + 64W_{(a,b)}(n) + 1024W_{(a,b)}(n)
\]

\[
- 256 \left( W_{(4a,b)}(n) + W_{(a,4b)}(n) \right).
\]

**Proof.** We have

\[
N_{(a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2} r_4(l)r_4(m) = r_4(n) + r_4(n) - r_4(n) + \sum_{(l,m) \in \mathbb{N}^2} r_4(l)r_4(m).
\]

We make use of \([\text{Equation 4.1}]\) to obtain

\[
N_{(a,b)}(n) = 8\sigma(n) - 32\sigma_4(n) + 8\sigma(n) - 32\sigma_4(n) + \sum_{(l,m) \in \mathbb{N}^2} (8\sigma(l) - 32\sigma_4(l))(8\sigma(m) - 32\sigma_4(m)).
\]

We know that

\[
(8\sigma(l) - 32\sigma_4(l))(8\sigma(m) - 32\sigma_4(m)) = 64\sigma(l)\sigma(m) - 256\sigma_4(l)\sigma(m) - 256\sigma_4(l)\sigma(m) + 1024\sigma_4(l)\sigma(m).
\]

In the sequel of this proof, we assume that the evaluation of

\[
W_{(a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m),
\]

\( W_{(4a,b)}(n) \) and \( W_{(a,4b)}(n) \) are known. We set \( \lambda = 4 \) in the sequel. When we use the function \( \tau \) with \( l \) as argument we derive

\[
W_{(4a,b)}(n) = \sum_{(l,m) \in \mathbb{N}^2} \sigma_4(l)\sigma(m) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m).
\]
When we apply the function \( \tau \) with \( m \) as argument we infer

\[
W_{(a,b)}(n) = \sum_{\{l,m\} \in \mathbb{N}^2 \atop al+bm=n} \sigma(l)\sigma\left(\frac{m}{4}\right) = \sum_{\{l,m\} \in \mathbb{N}^2 \atop al+4bm=n} \sigma(l)\sigma(m).
\]

We simultaneously apply the function \( \tau \) with \( l \) and \( m \) as arguments, respectively, to conclude

\[
\sum_{\{l,m\} \in \mathbb{N}^2 \atop al+bm=n} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) = \sum_{\{l,m\} \in \mathbb{N}^2 \atop al+4bm=n} \sigma(l)\sigma(m) = W_{(a,b)}\left(\frac{n}{4}\right).
\]

We finally put all these evaluations together to obtain the stated result for \( N_{(a,b)}(n) \).

\( \square \)

4.2. Representations of a Positive Integer by the Octonary Quadratic Form \textbf{Equation 1.5} In this case, the general form of \( \alpha \beta \) is restricted to \( 2^y \mathcal{U} \), where \( \mathcal{U} \equiv 0 \pmod{3} \).

4.2.1. Determination of \((c,d) \in \mathbb{N}^2\). The following method determine all pairs \((c,d) \in \mathbb{N}^2\) necessary for the determination of \( R_{(c,d)}(n) \) for a given \( \alpha \beta \in \mathbb{N} \) belonging to the above class. The following method is quasi similar to the one used in Subsection 4.1.1.

Let \( \Delta = \frac{\alpha \beta}{\gcd(\alpha,\beta)} = \frac{a}{\gcd(a,b)} \). Let \( P_3 = \{p \mid p = 2^y \} \cup \{p_j \mid p_j \text{ is a prime divisor of } \mathcal{U} \} \). Let \( \mathcal{P}(P_3) \) be the power set of \( P_3 \). Then for each \( Q \in \mathcal{P}(P_3) \) we define \( \mu(Q) = \prod_{p \in Q} p \). We set \( \mu(Q) = 1 \) if \( Q \) is an empty set. Let now \( \Omega_3 \) be defined in a similar way as \( \Omega_4 \) in Subsection 4.1.1, however with \( \Delta \) instead of \( \Lambda \), i.e.,

\[
\Omega_3 = \{(\mu(Q_1),\mu(Q_2)) \mid \text{there exist } Q_1, Q_2 \in \mathcal{P}(P_3) \text{ such that } \gcd(\mu(Q_1),\mu(Q_2)) = 1 \text{ and } \mu(Q_1)\mu(Q_2) = \Delta \}.
\]

Note that \( \Omega_3 \neq \emptyset \) since \((1,\Delta) \in \Omega_3 \). As an example, suppose again that \( \alpha \beta = 2^3 \cdot 3 \cdot 5 \). Then \( \Delta = 2^3 \cdot 5 \), \( P_3 = \{2,3,5\} \) and \( \Omega_3 = \{(1,40),(5,8)\} \).

Proposition 4.3. Suppose that \( \alpha \beta \) has the above restricted form and Suppose that \( \Omega_3 \) be defined as above. Then for all \( n \in \mathbb{N} \) the set \( \Omega_3 \) contains all pairs \((c,d) \in \mathbb{N}^2 \) such that \( R_{(c,d)}(n) \) can be obtained by applying \( W_{(a,b)}(n) \) and some other evaluated convolution sums.

Proof. Similar to the proof of Proposition 4.1 \( \square \)

4.2.2. Formulae for the Number of Representations by \textbf{Equation 1.5} We apply Theorem 3.4 to determine a formula for the number of representations of a positive integer \( n \) by the octonary quadratic form \textbf{Equation 1.5} for each \((c,d) \in \Omega_3 \).

Let \( n \in \mathbb{N}_0 \) and let \( s_4(n) \) denote the number of representations of \( n \) by the quaternary quadratic form \( x_1^2 + x_1x_2 + x_2^2 + x_2^2 + x_3x_4 + x_4^2 \), that is,

\[
s_4(n) = \text{card}(\{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \mid n = x_1^2 + x_1x_2 + x_2^2 + x_2^2 + x_3x_4 + x_4^2\}).
\]

It is obvious that \( s_4(0) = 1 \). J. G. Huard et al. \cite{12}, G. A. Lomadze \cite{18} and K. S. Williams \cite{52} Thrm 17.3, p. 225] have proved that for all \( n \in \mathbb{N} \)

\[
s_4(n) = 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right).
\]
Now, let the number of representations of $n$ by the octonary quadratic form be

$$R_{(c,d)}(n) = \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = c(x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2) + d(x_3^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\}.$$ 

Let $\lambda$ and $\tau$ be defined as in Section 4.1.2.

We infer the following result:

**Theorem 4.4.** Let $n \in \mathbb{N}$ and $(c,d) \in \Omega_3$. Then

$$R_{(c,d)}(n) = 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) + 144W_{(c,d)}(n) + 1296W_{(c,d)}\left(\frac{n}{3}\right) - 432\left(W_{(3c,d)}(n) + W_{(c,3d)}(n)\right).$$

**Proof.** It holds that

$$R_{(c,d)}(n) = \sum_{(l,m) \in \mathbb{N}^2} s_4(l)s_4(m) = s_4\left(\frac{n}{c}\right)s_4(0) + s_4(0)s_4\left(\frac{n}{d}\right) + \sum_{(l,m) \in \mathbb{N}^2} s_4(l)s_4(m).$$

We apply Equation 4.2 to derive

$$R_{(c,d)}(n) = 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) + \sum_{(l,m) \in \mathbb{N}^2} (12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)).$$

We know that

$$(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)) = 144\sigma(l)\sigma(m) - 432\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) - 36\sigma\left(\frac{m}{3}\right).$$

We assume that the evaluation of

$$W_{(c,d)}(n) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m),$$

$W_{(c,3d)}(n)$ and $W_{(3c,d)}(n)$ are known. We set $\lambda = 3$ in the sequel. We apply the function $\tau$ to $m$ to derive

$$\sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma\left(\frac{m}{3}\right) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m) = W_{(c,3d)}(n).$$

We make use of the function $\tau$ with $l$ as argument to conclude

$$\sum_{(l,m) \in \mathbb{N}^2} \sigma(m)\sigma\left(\frac{l}{3}\right) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m) = W_{(3c,d)}(n).$$

We simultaneously apply apply the function $\tau$ to $l$ and to $m$ as arguments, respectively, to infer

$$\sum_{(l,m) \in \mathbb{N}^2} \sigma\left(\frac{m}{3}\right)\sigma\left(\frac{l}{3}\right) = \sum_{(l,m) \in \mathbb{N}^2} \sigma(l)\sigma(m) = W_{(c,d)}\left(\frac{n}{3}\right).$$
Finally, we bring all these evaluations together to obtain the stated result for $R_{(c,d)}(n)$. □

5. Evaluation of the convolution sums when $\alpha \beta = 33, 40, 56$

In this section, we give explicit formulae for the convolution sum $W_{(\alpha, \beta)}(n)$ when $\alpha \beta = 33 = 3 \cdot 11, \alpha \beta = 40 = 2^3 \cdot 5$ and $\alpha \beta = 56 = 2^3 \cdot 7$

When we apply T. Miyake [19, Thrm 2.5.2, p. 60] or [29, Prop. 6.1, p. 91] to deduce that $\dim B_{(\alpha, \beta)} = 33$ we conclude that $\dim B_{(\alpha, \beta)} = 33$.

Let $\delta_1 \in D(33)$ and $(r(i, \delta_1))_{\delta_1}$ be the Table of the powers of $\eta(\delta_1 z)$.
Let $\delta_2 \in D(40)$ and $(r(j, \delta_2))_{\delta_2}$ be the Table of the powers of $\eta(\delta_2 z)$.
Let $\delta_3 \in D(56)$ and $(r(k, \delta_3))_{\delta_3}$ be the Table of the powers of $\eta(\delta_3 z)$.

Let furthermore

$$\mathcal{B}_{33, i}(q) = \prod_{\delta_1 \in D(33)} \eta^{r(i, \delta_1)}(\delta_1 z), \quad \mathcal{B}_{40, j}(q) = \prod_{\delta_2 \in D(40)} \eta^{r(j, \delta_2)}(\delta_2 z),$$

$$\mathcal{B}_{56, k}(q) = \prod_{\delta_3 \in D(56)} \eta^{r(k, \delta_3)}(\delta_3 z)$$

be selected elements of $\mathcal{S}_4(\Gamma_0(33)), \mathcal{S}_4(\Gamma_0(40))$ and $\mathcal{S}_4(\Gamma_0(56))$, respectively.

Then the sets

$$\mathcal{B}_{33, i} = \{ \mathcal{B}_{33, i}(q) | 1 \leq i \leq 10 \}, \quad \mathcal{B}_{40, j} = \{ \mathcal{B}_{40, j}(q) | 1 \leq j \leq 14 \},$$

$$\mathcal{B}_{56, k} = \{ \mathcal{B}_{56, k}(q) | 1 \leq k \leq 20 \}$$

are bases of $\mathcal{S}_4(\Gamma_0(33)), \mathcal{S}_4(\Gamma_0(40))$ and $\mathcal{S}_4(\Gamma_0(56))$, respectively.

The sets $\mathcal{B}_{56} = \mathcal{B}_{33} \cup \mathcal{B}_{40}$ constitute bases of $\mathcal{M}_4(\Gamma_0(33)), \mathcal{M}_4(\Gamma_0(40))$ and $\mathcal{M}_4(\Gamma_0(56))$, respectively.
By Remark 3.2 (r1), $\mathcal{B}_{33,2}(q)$, $\mathcal{B}_{40,1}(q)$ and $\mathcal{B}_{56,6}(q)$ can be expressed in the form
$$\sum_{n=1}^{\infty} b_{33,2}(n) q^n, \sum_{n=1}^{\infty} b_{40,1}(n) q^n \text{ and } \sum_{n=1}^{\infty} b_{56,6}(n) q^n,$$ respectively.

We observe that
- by Equation 5.1 the basis element $\mathcal{B}_{33,2}(q)$ is in $\mathcal{S}_4(\Gamma_0(11))$ and is the only one. In addition, $\mathcal{B}_{33,6}(q) = \mathcal{B}_{33,2}(q^2)$. Hence, $b_{33,6}(n) = b_{33,2}(\frac{n}{2})$.
- the basis elements of $\mathcal{S}_4(\Gamma_0(40))$ have been determined almost with respect to the inclusion relation Equation 5.2 except that $\mathcal{B}_{40,5}(q)$ results from the basis element of $\mathcal{S}_4(\Gamma_0(8))$ according to Equation 5.3.
- there is no element of $\mathcal{S}_4(\Gamma_0(7))$ which occurs as a basis element of $\mathcal{S}_4(\Gamma_0(56))$. This indicates that an element of $\mathcal{S}_4(\Gamma_0(7))$ cannot be determined when using Theorem 2.2. Other than that, the inclusion relation Equation 5.4 and Equation 5.5 preserve the bases.

Proof. It follows immediately from Theorem 3.1.

In case (a): the result is obtained when we set $n = 1, 3, 11, 33, n = 1, 2, 4, 5, 8, 10, 20, 40$ and $n = 1, 2, 4, 7, 8, 14, 28, 56$, respectively.

In case (b): the linear independence of the sets $\mathcal{B}_{5,33}$ and $\mathcal{B}_{5,56}$ is proved by applying case 2 in the proof of Theorem 3.1 and by taking $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and $n = 1, 2, 3, \ldots, 13, 14$, respectively. Finally $\mathcal{B}_{5,40}$ is linearly independent by case 1 in the proof of Theorem 3.1 and by taking $n = 1, 2, 3, \ldots, 19, 20$.

Therefore, we obtain the stated result. \hfill \Box

5.2. Evaluation of $W_{(\alpha, \beta)}(n)$ when $\alpha \beta = 33, 40, 56$.

Corollary 5.2. We have

\begin{align*}
(5.6) & \quad (L(q) - 33L(q^{33}))^2 = 1024 + \sum_{n=1}^{\infty} \left( \frac{2300736}{1271} \sigma_3(n) - \frac{59459328}{77531} \sigma_3\left(\frac{n}{3}\right) + \frac{6201427936}{77531} b_{33,1}(n) - \frac{14117760}{1271} b_{33,2}(n) \right) q^n, \\
& \quad + \frac{271016064}{1271} \sigma_3\left(\frac{n}{11}\right) - \frac{5752629808}{77531} \sigma_3\left(\frac{n}{33}\right) - \frac{348480}{1271} b_{33,1}(n) - \frac{14117760}{1271} b_{33,2}(n) \\
& \quad - \frac{6573339072}{77531} b_{33,3}(n) - \frac{2682366448}{77531} b_{33,4}(n) - \frac{6201427936}{77531} b_{33,5}(n) \\
& \quad - \frac{97134678144}{77531} b_{33,6}(n) - \frac{83738566400}{77531} b_{33,7}(n) - \frac{7428008448}{1271} b_{33,8}(n) \\
& \quad + \frac{4447872}{1271} b_{33,9}(n) - \frac{4444782}{1271} b_{33,10}(n) q^n, \\
(5.7) & \quad (3L(q^3) - 11L(q^{11}))^2 = 64 + \sum_{n=1}^{\infty} \left( - \frac{348480}{1271} \sigma_3(n) + \frac{106313472}{77531} \sigma_3\left(\frac{n}{3}\right) \right) q^n, \\
& \quad - \frac{34793088}{1271} \sigma_3\left(\frac{n}{11}\right) + \frac{8087561232}{77531} \sigma_3\left(\frac{n}{33}\right) + \frac{348480}{1271} b_{33,1}(n) \\
& \quad + \frac{3136320}{1271} b_{33,2}(n) + \frac{1346173632}{77531} b_{33,3}(n) + \frac{5361497604}{77531} b_{33,4}(n) \\
& \quad + \frac{11895235776}{77531} b_{33,5}(n) + \frac{17925551424}{77531} b_{33,6}(n) + \frac{15428171520}{77531} b_{33,7}(n) \\
& \quad + \frac{127847808}{1271} b_{33,8}(n) - \frac{444352}{1271} b_{33,9}(n) + \frac{4444782}{1271} b_{33,10}(n) q^n,
\end{align*}
(5.8) \[(L(q) - 40L(q^{40}))^2 = 1521 + \sum_{n=1}^{\infty} \left( \frac{26800}{117} \sigma_3(n) + \frac{43520}{117} \sigma_3(n) \cdot \frac{n}{2} \right) + \frac{245120}{39} \sigma_3(n) \cdot \frac{n}{4} - \frac{26800}{117} \sigma_3(n) \cdot \frac{n}{5} - \frac{1766400}{13} \sigma_3(n) \cdot \frac{n}{8} - \frac{127760}{117} \sigma_3(n) \cdot \frac{n}{10} \]
\[-\frac{357440}{39} \sigma_3(n) \cdot \frac{n}{20} + \frac{6558720}{13} \sigma_3(n) \cdot \frac{n}{40} + \frac{192224}{117} b_{40,1}(n) + \frac{439744}{117} b_{40,2}(n) \]
\[+ \frac{304832}{39} b_{40,3}(n) + \frac{1061120}{39} b_{40,4}(n) + \frac{41840}{3} b_{40,5}(n) - 15360 b_{40,6}(n) \]
\[-\frac{24320}{3} b_{40,7}(n) + \frac{1688320}{39} b_{40,8}(n) + 116800 b_{40,9}(n) - \frac{128000}{3} b_{40,10}(n) \]
\[-\frac{485120}{3} b_{40,11}(n) - \frac{1130240}{3} b_{40,12}(n) - \frac{121280}{3} b_{40,13}(n) - 69120 b_{40,14}(n) \) \right) q^n .

(5.9) \[(5L(q^5) - 8L(q^8))^2 = 9 + \sum_{n=1}^{\infty} \left( \frac{59200}{117} \sigma_3(n) - 76000 \sigma_3(n) \right) + \frac{16960}{39} \sigma_3(n) \cdot \frac{n}{4} - \frac{668000}{117} \sigma_3(n) \cdot \frac{n}{5} + \frac{721920}{13} \sigma_3(n) \cdot \frac{n}{8} - \frac{8240}{117} \sigma_3(n) \cdot \frac{n}{10} \]
\[-\frac{95360}{39} \sigma_3(n) \cdot \frac{n}{20} - \frac{721920}{13} \sigma_3(n) \cdot \frac{n}{40} - \frac{5920}{117} b_{40,1}(n) + \frac{22720}{117} b_{40,2}(n) \]
\[-\frac{39200}{39} b_{40,3}(n) + \frac{12800}{39} b_{40,4}(n) - \frac{38800}{3} b_{40,5}(n) + 6780 b_{40,6}(n) \]
\[-\frac{47360}{3} b_{40,7}(n) - \frac{505080}{39} b_{40,8}(n) - 67520 b_{40,9}(n) - \frac{12800}{3} b_{40,10}(n) \]
\[+ \frac{113920}{3} b_{40,11}(n) + \frac{298240}{3} b_{40,12}(n) + \frac{63040}{3} b_{40,13}(n) + 69120 b_{40,14}(n) \) \right) q^n .

(5.10) \[(L(q) - 56L(q^{56}))^2 = 3025 + \sum_{n=1}^{\infty} \left( \frac{1284}{5} \sigma_3(n) - 420 \sigma_3(n) \cdot \frac{n}{2} + \frac{31584}{5} \sigma_3(n) \right) \]
\[-\frac{1764}{5} \sigma_3(n) \cdot \frac{n}{7} - \frac{32256}{5} \sigma_3(n) \cdot \frac{n}{8} - \frac{51744}{5} \sigma_3(n) \cdot \frac{n}{14} - \frac{1140216}{5} \sigma_3(n) \cdot \frac{n}{56} \]
\[+ \frac{11916}{5} b_{56,1}(n) + \frac{92604}{5} b_{56,2}(n) + \frac{29568 b_{56,3}(n)}{5} + \frac{1140216}{5} b_{56,4}(n) \]
\[-\frac{411936 b_{56,5}(n)}{5} + \frac{2557632}{5} b_{56,6}(n) + 223608 b_{56,7}(n) + 3998400 b_{56,8}(n) \]
\[+ \frac{4042752 b_{56,9}(n)}{5} + \frac{145152}{5} b_{56,10}(n) - 8064 b_{56,11}(n) - 48384 b_{56,12}(n) \]
\[+ \frac{532224}{5} b_{56,14}(n) + 161280 b_{56,15}(n) + \frac{225792}{5} b_{56,16}(n) + 129024 b_{56,17}(n) \]
\[+ \frac{2515968}{5} b_{56,18}(n) + 1354752 b_{56,19}(n) - 225792 b_{56,20}(n) \) \right) q^n .
These identities follow immediately on taking $X$ Corollary 5.3.

We are now prepared to state and to prove our main result of this section.

**Proof.** These identities follow immediately on taking $(\alpha, \beta) = (1, 33)$, $(3, 11)$, $(1, 40)$, $(5, 8)$, $(1, 56)$, $(7, 8)$ in Lemma 3.3. In case $\alpha \beta = 40$ we take all $n$ in $\{1, 2, \ldots, 20, 40, 80\}$ to obtain a system of 22 linear equations with unknowns $\lambda_5$ and $\lambda_j$, where $\delta \in D(40)$ and $1 \leq j \leq 14$.

We are now prepared to state and to prove our main result of this section.

**Corollary 5.3.** Let $n$ be a positive integer. Then

$$W_{(1,33)}(n) = -\frac{13859}{335544} \sigma_3(n) + \frac{51614}{2558823} \sigma_3 \left(\frac{n}{3}\right) - \frac{7129}{1271} \sigma_3 \left(\frac{n}{11}\right) + \frac{60271327}{1860744} \sigma_3 \left(\frac{n}{33}\right) + \left(\frac{1}{24} - \frac{1}{132} n\right) \sigma(n) + \left(\frac{1}{12} - \frac{1}{4} n\right) \sigma \left(\frac{n}{3}\right) + \frac{11412047}{7626} \sigma_3,1(n) + \frac{517046}{1705682} \sigma_3,3(n) + \frac{517046}{1705682} \sigma_3,4(n) + \frac{5888837}{7626} \sigma_3,5 \left(\frac{n}{11}\right) + \frac{852841}{13981} \sigma_3,6(n) + \frac{852841}{13981} \sigma_3,7(n) + \frac{852841}{13981} \sigma_3,8(n) + \frac{7}{7626} \sigma_3,9(n) + \frac{117}{1271} \sigma_3,10(n),$$

(5.12)

$$W_{(3,11)}(n) = \frac{55}{7626} \sigma_3(n) + \frac{12869}{620248} \sigma_3 \left(\frac{n}{3}\right) + \frac{15089}{10168} \sigma_3 \left(\frac{n}{11}\right) - \frac{6382231}{232593} \sigma_3 \left(\frac{n}{33}\right) + \left(\frac{1}{24} - \frac{1}{44} n\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{12} n\right) \sigma \left(\frac{n}{3}\right) - \frac{55}{7626} \sigma_3,1(n) - \frac{165}{2542} \sigma_3,3(n) - \frac{1551359}{852841} \sigma_3,4(n) - \frac{6883817}{1705682} \sigma_3,5(n) - \frac{940353}{852841} \sigma_3,6(n) - \frac{4464170}{3363} \sigma_3,7(n) - \frac{155062}{1271} \sigma_3,8(n) + \frac{7}{7626} \sigma_3,9(n) - \frac{117}{1271} \sigma_3,10(n),$$

(5.13)
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\[ W_{(1,40)}(n) = \frac{1}{4212} \sigma_3(n) - \frac{17}{2106} \sigma_3\left(\frac{n}{2}\right) - \frac{383}{2808} \sigma_3\left(\frac{n}{4}\right) + \frac{335}{67392} \sigma_3\left(\frac{n}{5}\right) + \frac{115}{39} \sigma_3\left(\frac{n}{8}\right) \\
+ \frac{1597}{67392} \sigma_3\left(\frac{n}{10}\right) + \frac{1117}{5616} \sigma_3\left(\frac{n}{20}\right) - \frac{34}{13} \sigma_3\left(\frac{n}{40}\right) + \left(\frac{1}{24} - \frac{1}{160}\right)n\sigma(n) \\
+ \left(\frac{1}{24} - \frac{1}{4}\right)n\sigma\left(\frac{n}{40}\right) - \frac{6007}{168480} b_{40,1}(n) - \frac{6871}{84240} b_{40,2}(n) - \frac{4763}{28080} b_{40,3}(n) \\
- \frac{829}{1404} b_{40,4}(n) - \frac{523}{1728} b_{40,5}(n) + \frac{1}{3} b_{40,6}(n) + \frac{19}{108} b_{40,7}(n) \\
- \frac{1319}{1404} b_{40,8}(n) - \frac{365}{144} b_{40,9}(n) + \frac{25}{27} b_{40,10}(n) + \frac{37}{108} b_{40,11}(n) \\
+ \frac{883}{108} b_{40,12}(n) + \frac{379}{432} b_{40,13}(n) + \frac{3}{2} b_{40,14}(n), \quad (5.14) \]

\[ W_{(5,8)}(n) = -\frac{37}{33696} \sigma_3(n) + \frac{475}{33696} \sigma_3\left(\frac{n}{2}\right) + \frac{53}{5616} \sigma_3\left(\frac{n}{4}\right) + \frac{425}{67392} \sigma_3\left(\frac{n}{5}\right) - \frac{34}{39} \sigma_3\left(\frac{n}{8}\right) \\
+ \frac{103}{67392} \sigma_3\left(\frac{n}{10}\right) + \frac{149}{2808} \sigma_3\left(\frac{n}{20}\right) + \frac{47}{39} \sigma_3\left(\frac{n}{40}\right) + \left(\frac{1}{24} - \frac{1}{32}\right)n\sigma\left(\frac{n}{5}\right) \\
+ \left(\frac{1}{24} - \frac{1}{20}\right)n\sigma\left(\frac{n}{8}\right) + \frac{37}{33696} b_{40,1}(n) - \frac{71}{16848} b_{40,2}(n) + \frac{185}{5616} b_{40,3}(n) \\
- \frac{5}{702} b_{40,4}(n) + \frac{485}{1728} b_{40,5}(n) - \frac{1}{6} b_{40,6}(n) + \frac{37}{108} b_{40,7}(n) + \frac{1973}{7020} b_{40,8}(n) \\
+ \frac{211}{144} b_{40,9}(n) + \frac{5}{54} b_{40,10}(n) - \frac{89}{108} b_{40,11}(n) - \frac{37}{108} b_{40,12}(n) \\
- \frac{197}{432} b_{40,13}(n) - \frac{3}{2} b_{40,14}(n), \quad (5.15) \]

\[ W_{(1,56)}(n) = -\frac{1}{3840} \sigma_3(n) + \frac{5}{768} \sigma_3\left(\frac{n}{2}\right) - \frac{47}{480} \sigma_3\left(\frac{n}{4}\right) + \frac{7}{1280} \sigma_3\left(\frac{n}{7}\right) + \frac{1}{10} \sigma_3\left(\frac{n}{8}\right) \\
+ \frac{7}{768} \sigma_3\left(\frac{n}{14}\right) + \frac{77}{480} \sigma_3\left(\frac{n}{28}\right) + \frac{30}{30} \sigma_3\left(\frac{n}{56}\right) + \left(\frac{1}{24} - \frac{1}{224}\right)n\sigma(n) \\
+ \left(\frac{1}{24} - \frac{1}{4}\right)n\sigma\left(\frac{n}{56}\right) - \frac{331}{8960} b_{56,1}(n) - \frac{7717}{26880} b_{56,2}(n) - \frac{11}{24} b_{56,3}(n) \\
- \frac{6787}{1920} b_{56,4}(n) + \frac{613}{96} b_{56,5}(n) - \frac{1903}{240} b_{56,6}(n) - \frac{1331}{384} b_{56,7}(n) \\
- \frac{2975}{48} b_{56,8}(n) - \frac{9}{20} b_{56,9}(n) - \frac{1}{8} b_{56,10}(n) + \frac{3}{4} b_{56,12}(n) \\
- \frac{33}{20} b_{56,14}(n) - \frac{5}{2} b_{56,15}(n) - \frac{7}{10} b_{56,16}(n) - 2 b_{56,17}(n) - \frac{39}{5} b_{56,18}(n) \\
- 21 b_{56,19}(n) + \frac{7}{2} b_{56,20}(n), \quad (5.16) \]
\[ W_{(7,8)}(n) = \frac{11}{57600} \sigma_3(n) - \frac{67}{57600} \sigma_3\left(\frac{n}{2}\right) + \frac{179}{7200} \sigma_3\left(\frac{n}{4}\right) + \frac{289}{57600} \sigma_3\left(\frac{n}{6}\right) - \frac{7}{450} \sigma_3\left(\frac{n}{8}\right) \\
+ \frac{967}{57600} \sigma_3\left(\frac{n}{14}\right) + \frac{271}{7200} \sigma_3\left(\frac{n}{28}\right) + \frac{157}{450} \sigma_3\left(\frac{n}{56}\right) + \left(\frac{1}{24} - \frac{1}{28}\right) \sigma_3\left(\frac{n}{8}\right) - \frac{11}{57600} b_{56,1}(n) - \frac{29}{19200} b_{56,2}(n) - \frac{13}{1800} b_{56,3}(n) \\
- \frac{2167}{28800} b_{56,4}(n) + \frac{151}{2400} b_{56,5}(n) - \frac{643}{3600} b_{56,6}(n) + \frac{523}{9600} b_{56,7}(n) \\
- \frac{11411}{8400} b_{56,8}(n) - \frac{1581}{1400} b_{56,9}(n) - \frac{263}{7200} b_{56,10}(n) - \frac{11}{24} b_{56,11}(n) \\
- \frac{53}{60} b_{56,12}(n) + \frac{11}{12} b_{56,13}(n) + \frac{1969}{2100} b_{56,14}(n) - \frac{11}{12} b_{56,15}(n) \\
- \frac{13133}{3150} b_{56,16}(n) - \frac{11}{6} b_{56,17}(n) - \frac{1579}{1575} b_{56,18}(n) - \frac{20}{3} b_{56,19}(n) \\
- \frac{5}{6} b_{56,20}(n). \]

Proof. These identities follow from Theorem 3.4 when we set \((\alpha, \beta) = (1, 33), (3, 11), (1, 40), (5, 8), (1, 56), (7, 8)\). \(\square\)

6. RE-EVALUATION OF THE CONVOLUTION SUMS FOR \(\alpha \beta = 10, 11, 12, 15, 24\)

We revisit the convolution sums established by

- E. Royer [28] Thrm 1.1, and S. Cooper and D. Ye [10] Thrm 2.1 for \(\alpha \beta = 10\)
- E. Royer [28] Thrm 1.3 for \(\alpha \beta = 11\)
- A. Alaca et al. [1] [5] for \(\alpha \beta = 12, 24\)
- B. Ramakrishnan and B. Sahu [29] for \(\alpha \beta = 15\)

using modular forms. The obtained results in each case are immediate corollaries of Theorem 3.4 and improve the previous ones since we use the exact number of basis elements of the space of cusp forms in case of \(\alpha \beta = 12, 24\). The improvement of the previous results in case of \(\alpha \beta = 11, 15\) is obvious.

Since \(\alpha \beta = 10 = 2 \cdot 5\) and because of Equation 5.2 it holds that \(\mathcal{B}_{40,1}(q) = \mathcal{B}_{40,1}(q^2)\), and therefore \(\mathcal{B}_{40,2}(n) = b_{40,1}(\frac{n}{2})\). Our third basis element of the space \(\mathcal{S}_4(\Gamma_0(10))\) is different from the one used by S. Cooper and D. Ye [10], which explains the difference in the two results. However, since the change of basis is an automorphism, both results are the same.

In addition to the basis element \(\mathcal{B}_{33,2}(q)\) of the space \(\mathcal{S}_4(\Gamma_0(11))\), we use \(\mathcal{B}_{33,1}(q) = \eta^2(z)\eta^2(11z) = \sum_{n=1}^{\infty} b'_{33,1}(n)q^n\) which is a basis element of \(\mathcal{S}_2(\Gamma_0(11))\).

B. Ramakrishnan and B. Sahu achieve the evaluation of the convolution sums for \(\alpha \beta = 15\) using a basis which contains one cusp form of weight 2. We consider the following \(\eta\)-quotients as basis elements of the space \(\mathcal{S}_4(\Gamma_0(15))\)

\[ \mathcal{B}_{15,1}(q) = \eta^4(z)\eta^4(5z) \quad \mathcal{B}_{15,2}(q) = \eta^2(z)\eta^2(3z)\eta^2(5z)\eta^2(15z) \]
\[ \mathcal{B}_{15,3}(q) = \eta^4(3z)\eta^4(15z) \quad \mathcal{B}_{15,4}(q) = \frac{\eta^3(z)\eta^3(3z)\eta^3(15z)}{\eta^3(5z)}. \]

The following \(\eta\)-quotients build a basis of \(\mathcal{S}_4(\Gamma_0(24))\).

\[ \mathcal{B}_{24,1}(q) = \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) \]
Corollary 6.1. It holds that \( b(6.4) \) and 15 - \( 2 \times 12 \times 5^2 \times 10 + 524320 \)). Therefore \( b_{15,5}(n) = b_{15,1}(n) + b_{15,1}(n) \) and \( b_{24,4}(n) = b_{24,1}(n) + b_{24,1}(n) \) and \( b_{24,6}(n) = b_{24,3}(n) \).

Corollary 6.1. We have

\[
\begin{align*}
(6.1) \quad (L(q) - 10L(q^{10}))^2 &= 81 + \sum_{n=1}^{\infty} \left( \frac{2640}{13} \sigma_3(n) - \frac{1920}{13} \sigma_3\left(\frac{n}{2}\right) - \frac{12000}{13} \sigma_3\left(\frac{n}{5}\right) \right) q^n, \\
&\quad + \frac{264000}{13} \sigma_3\left(\frac{n}{10}\right) + \frac{2976}{13} \sigma_4,1(n) + \frac{14400}{13} \sigma_4,2(n) - 960 \sigma_4,3(n) q^n,
\end{align*}
\]

\[
\begin{align*}
(6.2) \quad (2L(q^2) - 5L(q^5))^2 &= 9 + \sum_{n=1}^{\infty} \left( -\frac{480}{13} \sigma_3(n) + \frac{10560}{13} \sigma_3\left(\frac{n}{2}\right) + \frac{66000}{13} \sigma_3\left(\frac{n}{5}\right) \right) \\
&\quad - \frac{48000}{13} \sigma_3\left(\frac{n}{10}\right) + \frac{480}{13} \sigma_4,1(n) - \frac{576}{13} \sigma_4,2(n) + 960 \sigma_4,3(n) q^n,
\end{align*}
\]

\[
\begin{align*}
(6.3) \quad (L(q) - 11L(q^{11}))^2 &= 100 + \sum_{n=1}^{\infty} \left( \frac{6240}{49} \sigma_3(n) + \frac{5524320}{49} \sigma_3\left(\frac{n}{10}\right) + \frac{17280}{49} \sigma_3\left(\frac{n}{11}\right) \right) q^n, \\
&\quad + \frac{77184}{49} \sigma_3\left(\frac{n}{11}\right) q^n,
\end{align*}
\]

\[
\begin{align*}
(6.4) \quad (L(q) - 12L(q^{12}))^2 &= 121 + \sum_{n=1}^{\infty} \left( \frac{1056}{5} \sigma_3(n) - \frac{432}{5} \sigma_3\left(\frac{n}{2}\right) - \frac{1296}{5} \sigma_3\left(\frac{n}{3}\right) \right) q^n, \\
&\quad - \frac{2304}{5} \sigma_3\left(\frac{n}{4}\right) - \frac{3888}{5} \sigma_3\left(\frac{n}{6}\right) + \frac{152064}{5} \sigma_3\left(\frac{n}{12}\right) + \frac{1584}{5} \sigma_3\left(\frac{n}{24}\right) + \frac{4896}{5} \sigma_3\left(\frac{n}{24}\right) q^n \\
&\quad + 864 \sigma_3\left(\frac{n}{24}\right) q^n,
\end{align*}
\]
\[(6.5) \quad (3L(q^3) - 4L(q^4))^2 = 1 + \sum_{n=1}^{\infty} \left( -\frac{144}{5} \sigma_3(n) - \frac{432}{5} \sigma_3\left(\frac{n}{2}\right) + \frac{9504}{5} \sigma_3\left(\frac{n}{3}\right) + \frac{16896}{5} \sigma_3\left(\frac{n}{4}\right) - \frac{3888}{5} \sigma_3\left(\frac{n}{6}\right) - \frac{20736}{5} \sigma_3\left(\frac{n}{12}\right) + \frac{144}{5} b_{24,1}(n) + \frac{2016}{5} b_{24,2}(n) - 864 b_{24,3}(n) \right) q^n,\]

\[(6.6) \quad (L(q) - 15L(q^{15}))^2 = 196 + \sum_{n=1}^{\infty} \left( \frac{2976}{13} \sigma_3(n) - \frac{3456}{13} \sigma_3\left(\frac{n}{3}\right) - \frac{144000}{13} \sigma_3\left(\frac{n}{5}\right) + \frac{756000}{13} \sigma_3\left(\frac{n}{15}\right) + \frac{5760}{13} b_{15,1}(n) + \frac{2304}{13} b_{15,2}(n) + \frac{48384}{13} b_{15,3}(n) - 3456 b_{15,4}(n) \right) q^n,\]

\[(6.7) \quad (3L(q^3) - 5L(q^5))^2 = 4 + \sum_{n=1}^{\infty} \left( -\frac{576}{13} \sigma_3(n) + \frac{25056}{13} \sigma_3\left(\frac{n}{3}\right) + \frac{204000}{13} \sigma_3\left(\frac{n}{5}\right) - \frac{216000}{13} \sigma_3\left(\frac{n}{15}\right) + \frac{576}{13} b_{15,1}(n) + \frac{576}{13} b_{15,2}(n) + \frac{8640}{13} b_{15,3}(n) + 3456 b_{15,4}(n) \right) q^n,\]

\[(6.8) \quad (L(q) - 24L(q^{24}))^2 = 529 + \sum_{n=1}^{\infty} \left( 672 \sigma_3(n) + \frac{33264}{5} \sigma_3\left(\frac{n}{2}\right) - 576 \sigma_3\left(\frac{n}{3}\right) - 36576 \sigma_3\left(\frac{n}{4}\right) - 35424 \sigma_3\left(\frac{n}{6}\right) - 4608 \sigma_3\left(\frac{n}{8}\right) - 27936 \sigma_3\left(\frac{n}{12}\right) + \frac{649728}{5} \sigma_3\left(\frac{n}{24}\right) + 432 b_{24,1}(n) - \frac{44064}{5} b_{24,2}(n) - 8640 b_{24,3}(n) - \frac{508608}{5} b_{24,4}(n) - 55296 b_{24,5}(n) - 316224 b_{24,6}(n) - 276480 b_{24,7}(n) - 857088 b_{24,8}(n) \right) q^n,\]

\[(6.9) \quad (3L(q^3) - 8L(q^8))^2 = 25 + \sum_{n=1}^{\infty} \left( \frac{864}{5} \sigma_3(n) + 2016 \sigma_3\left(\frac{n}{3}\right) - \frac{2016}{5} \sigma_3\left(\frac{n}{4}\right) - \frac{3024}{5} \sigma_3\left(\frac{n}{6}\right) - \frac{72192}{5} \sigma_3\left(\frac{n}{8}\right) - \frac{6624}{5} \sigma_3\left(\frac{n}{12}\right) - \frac{41472}{5} \sigma_3\left(\frac{n}{24}\right) + \frac{864}{5} b_{24,2}(n) - 1296 b_{24,3}(n) - \frac{7488}{5} b_{24,4}(n) - 15552 b_{24,5}(n) - 27648 b_{24,6}(n) \right) q^n.\]

In the case of the evaluation of $W_{(1,1)}(n)$, we observe, using Lemma 2.3, that for all $\alpha, \beta \in \mathbb{N}$ it holds that

\[(6.10) \quad 0 = (\alpha L(q^\alpha) - \alpha L(q^\beta))^2 \in \mathcal{M}_4(\Gamma_0(\alpha \beta)).\]

**Corollary 6.2.** Let $n$ be a positive integer. Then

\[(6.11) \quad \forall \alpha \in \mathbb{N} \quad W_{(\alpha,0)}(n) = \frac{5}{12} \sigma_3\left(\frac{n}{\alpha}\right) + \left( \frac{1}{12} - \frac{1}{2\alpha} \right) n \sigma_3\left(\frac{n}{\alpha}\right),\]

\[(6.12) \quad W_{(1,10)}(n) = -\frac{1}{312} \sigma_3(n) + \frac{1}{78} \sigma_3\left(\frac{n}{2}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{3}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{5}\right) + \left( \frac{1}{24} - \frac{1}{40} \right) n \sigma_3(n) + \left( \frac{1}{24} - \frac{1}{60} \right) n \sigma_3\left(\frac{n}{10}\right) - \frac{31}{1560} b_{40,1}(n) - \frac{5}{52} b_{40,1}\left(\frac{n}{2}\right) + \frac{1}{12} b_{40,3}(n),\]
\[ W_{(2,5)}(n) = \frac{1}{312} \sigma_3(n) + \frac{1}{78} \sigma_3\left(\frac{n}{2}\right) + \frac{25}{312} \sigma_3\left(\frac{n}{5}\right) + \frac{25}{78} \sigma_3\left(\frac{n}{10}\right) + \left(\frac{1}{24} - \frac{1}{20}\right) \sigma_3\left(\frac{n}{2}\right) \]
\[ + \left(\frac{1}{24} - \frac{1}{8}\right) \sigma_3\left(\frac{n}{5}\right) - \frac{1}{312} b_{40,1}(n) + \frac{1}{260} b_{40,1} \left(\frac{n}{2}\right) - \frac{1}{12} b_{40,3}(n), \]

\[ W_{(1,11)}(n) = \frac{5}{1464} \sigma_3(n) + \frac{605}{1464} \sigma_3\left(\frac{n}{11}\right) + \left(\frac{1}{24} - \frac{1}{44}\right) \sigma(n) \]
\[ + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma_3\left(\frac{n}{11}\right) - \frac{14615}{386496} b_{33,1}(n) - \frac{90493}{386496} b_{33,2}(n), \]

\[ W_{(1,12)}(n) = \frac{1}{480} \sigma_3(n) + \frac{1}{160} \sigma_3\left(\frac{n}{2}\right) + \frac{3}{160} \sigma_3\left(\frac{n}{3}\right) + \frac{1}{30} \sigma_3\left(\frac{n}{4}\right) + \frac{9}{160} \sigma_3\left(\frac{n}{6}\right) \]
\[ + \frac{3}{10} \sigma_3\left(\frac{n}{12}\right) + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma_3\left(\frac{n}{12}\right) - \frac{17}{240} b_{24,2}(n) - \frac{1}{16} b_{24,3}(n), \]

\[ W_{(3,4)}(n) = \frac{1}{480} \sigma_3(n) + \frac{1}{160} \sigma_3\left(\frac{n}{2}\right) + \frac{3}{160} \sigma_3\left(\frac{n}{3}\right) + \frac{1}{30} \sigma_3\left(\frac{n}{4}\right) + \frac{9}{160} \sigma_3\left(\frac{n}{6}\right) \]
\[ + \frac{3}{10} \sigma_3\left(\frac{n}{12}\right) + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma_3\left(\frac{n}{12}\right) - \frac{7}{240} b_{24,2}(n) + \frac{1}{16} b_{24,3}(n), \]

\[ W_{(1,15)}(n) = \frac{1}{1560} \sigma_3(n) + \frac{1}{65} \sigma_3\left(\frac{n}{3}\right) + \frac{25}{39} \sigma_3\left(\frac{n}{5}\right) - \frac{25}{104} \sigma_3\left(\frac{n}{15}\right) + \left(\frac{1}{24} - \frac{1}{60}\right) \sigma(n) \]
\[ + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma_3\left(\frac{n}{15}\right) - \frac{1}{39} b_{15,1}(n) - \frac{2}{15} b_{15,2}(n) - \frac{14}{65} b_{15,3}(n) + \frac{1}{5} b_{15,4}(n), \]

\[ W_{(3,5)}(n) = \frac{1}{390} \sigma_3(n) + \frac{7}{520} \sigma_3\left(\frac{n}{3}\right) - \frac{175}{312} \sigma_3\left(\frac{n}{5}\right) + \frac{25}{26} \sigma_3\left(\frac{n}{15}\right) + \left(\frac{1}{24} - \frac{1}{20}\right) \sigma(n) \]
\[ + \left(\frac{1}{24} - \frac{1}{8}\right) \sigma_3\left(\frac{n}{5}\right) - \frac{1}{390} b_{15,1}(n) - \frac{1}{30} b_{15,2}(n) - \frac{1}{26} b_{15,3}(n) - \frac{1}{5} b_{15,4}(n), \]

\[ W_{(1,24)}(n) = -\frac{1}{64} \sigma_3(n) - \frac{77}{320} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{48} \sigma_3\left(\frac{n}{3}\right) + \frac{127}{480} \sigma_3\left(\frac{n}{4}\right) + \frac{41}{160} \sigma_3\left(\frac{n}{6}\right) \]
\[ + \frac{1}{30} \sigma_3\left(\frac{n}{8}\right) - \frac{97}{480} \sigma_3\left(\frac{n}{12}\right) + \frac{3}{10} \sigma_3\left(\frac{n}{24}\right) + \left(\frac{1}{24} - \frac{1}{96}\right) \sigma(n) \]
\[ + \left(\frac{1}{24} - \frac{1}{4}\right) \sigma(n) - \frac{1}{64} b_{24,1}(n) + \frac{51}{160} b_{24,2}(n) + \frac{5}{16} b_{24,3}(n) \]
\[ + \frac{883}{240} b_{24,4}(n) + 2 b_{24,5}(n) + \frac{183}{16} b_{24,6}(n) + 10 b_{24,7}(n) + 31 b_{24,8}(n), \]

\[ W_{(3,8)}(n) = -\frac{1}{160} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{192} \sigma_3\left(\frac{n}{3}\right) + \frac{7}{480} \sigma_3\left(\frac{n}{4}\right) + \frac{7}{320} \sigma_3\left(\frac{n}{6}\right) \]
\[ + \frac{1}{30} \sigma_3\left(\frac{n}{8}\right) + \frac{23}{480} \sigma_3\left(\frac{n}{12}\right) + \frac{3}{10} \sigma_3\left(\frac{n}{24}\right) + \left(\frac{1}{24} - \frac{1}{96}\right) \sigma_3\left(\frac{n}{2}\right) \]
\[ + \left(\frac{1}{24} - \frac{1}{12}\right) \sigma\left(\frac{n}{8}\right) + \frac{1}{160} b_{24,2}(n) + \frac{3}{64} b_{24,3}(n) + \frac{13}{240} b_{24,4}(n) \]
\[ + \frac{9}{16} b_{24,6}(n) + b_{24,8}(n). \]
For example, Equation 6.11 is easily proved as follows. Due to Equation 6.10 and applying Equation 2.6 we have
\[ 0 = -1152 \alpha^2 W_{(\alpha,\alpha)}(n) + 480 \alpha^2 \sigma_3 \left( \frac{n}{\alpha} \right) + 96 \alpha (\alpha - 6 n) \sigma \left( \frac{n}{\alpha} \right). \]
Therefore, we obtain Equation 2.3. By setting \( \alpha = 1 \), one gets the result obtained by M. Besge [7], J. W. L. Glaisher [11] and S. Ramanujan [27].

7. Formulae for the Number of Representations of a Positive Integer

We make use of the convolution sums evaluated in Section 5 among others to determine explicit formulae for the number of representations of a positive integer \( n \) by the octonary quadratic forms Equation 1.4 and Equation 1.5 respectively.

7.1. Number of Representations of a Positive Integer Applying Illustrated Convolution Sums.

7.1.1. Representations by the Octonary Quadratic Forms [Equation 1.4] We determine formulae for the number of representations of a positive integer \( n \) by the Octonary Quadratic Form Equation 1.5 when we mainly apply the evaluation of the convolution sums \( W_{(1,33)}(n) \) and \( W_{(3,11)}(n) \). In order to do that, we recall that 33 = 3 \cdot 11 is of the restricted form in Section 4.2. Hence, from Proposition 4.3 we derive that \( \Omega_3 = \{(1,11)\} \). We then deduce the following result:

**Corollary 7.1.** Let \( n \in \mathbb{N} \). Then
\[
R_{(1,11)}(n) = 12\sigma(n) - 36\sigma \left( \frac{n}{3} \right) + 12\sigma \left( \frac{n}{11} \right) - 36\sigma \left( \frac{n}{33} \right) + 144W_{(1,11)}(n) + 1296W_{(1,11)} \left( \frac{n}{3} \right) - 432 \left( W_{(1,11)}(n) + W_{(3,11)}(n) \right).
\]

**Proof.** It follows immediately from Theorem 4.2 with \((c,d) = (1,11)\). One can then make use of Equation 6.14, Equation 5.12 and Equation 5.13 to simplify this formula. \( \square \)

7.1.2. Representations by Octonary Quadratic Forms [Equation 1.5] We give formulae for the number of representations of a positive integer \( n \) by the Octonary Quadratic Form Equation 1.5 by mainly applying the evaluation of the convolution sums \( W_{(2,40)}(n) \), \( W_{(1,56)}(n) \), \( W_{(5,8)}(n) \) and \( W_{(7,8)}(n) \). To achieve that, we recall that 40 = 2 \cdot 5 and 56 = 2 \cdot 7 are of the restricted form in Section 4.1. Therefore, we apply Proposition 4.1 to conclude that \( \Omega_4 = \{(1,10),(2,5)\} \) in case \( \alpha\beta = 40 \) and \( \Omega_4 = \{(1,14),(2,7)\} \) in case \( \alpha\beta = 56 \).

**Corollary 7.2.** Let \( n \in \mathbb{N} \). Then
\[
N_{(1,10)}(n) = 8\sigma(n) - 32\sigma \left( \frac{n}{4} \right) + 8\sigma \left( \frac{n}{10} \right) - 32\sigma \left( \frac{n}{40} \right) + 64W_{(1,10)}(n) + 1024W_{(1,10)} \left( \frac{n}{4} \right) - 256 \left( W_{(2,5)}(\frac{n}{2}) + W_{(1,40)}(n) \right),
\]
\[
N_{(2,5)}(n) = 8\sigma \left( \frac{n}{2} \right) - 32\sigma \left( \frac{n}{8} \right) + 8\sigma \left( \frac{n}{5} \right) - 32\sigma \left( \frac{n}{20} \right) + 64W_{(2,5)}(n) + 1024W_{(2,5)} \left( \frac{n}{4} \right) - 256 \left( W_{(5,8)}(n) + W_{(1,10)} \left( \frac{n}{2} \right) \right),
\]

Revisited Formulae for the Number of representations of a positive integer.

When we set from Theorem 4.2.

**Proof.** These formulae follow immediately from Theorem 4.2 when we set $n$ to Proposition 4.1. We rather consider Equation 6.15, Equation 6.16, Equation 5.19 and Equation 5.14 for the sake of simplification in case of $N_{10}$ and $N_{25}$.

- S. Cooper and D. Ye [10] Thrm 2.1, Equation 6.12 and Equation 5.14 and Equation 5.15 for the sake of simplification in case of $N_{10}$ and $N_{25}$.
- E. Royer [28] Thms 1.7, E. Ntienjem [22] Thrm 3.2.1, Equation 5.16 and Equation 5.17 to simplify the formulae in case of $N_{114}$ and $N_{27}$.

\[ N_{(1,4)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{14}\right) - 32\sigma\left(\frac{n}{56}\right) + 64\sigma(\frac{n}{14}) + 1024\sigma(\frac{n}{4}) - 256\left(W_{(2,7)}\left(\frac{n}{2}\right) + W_{(1,56)}(n)\right), \]

\[ N_{(2,7)}(n) = 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) + 8\sigma\left(\frac{n}{7}\right) - 32\sigma\left(\frac{n}{28}\right) + 64\sigma(\frac{n}{27}) + 1024\sigma(\frac{n}{4}) - 256\left(W_{(7,8)}(n) + W_{(1,14)}\left(\frac{n}{2}\right)\right). \]

**Proof.** These formulae follow immediately from Theorem 4.2 when we set $(a, b) = (1, 10), (2, 5), (1, 14), (2, 7)$, respectively. One can then use the result of

- S. Cooper and D. Ye [10] E. Royer [28] Thms 1.7, E. Ntienjem [22] Thrm 3.2.1, Equation 5.16 and Equation 5.17 to simplify the formulae in case of $N_{114}$ and $N_{27}$.

7.2. Revisited Formulae for the Number of representations of a positive integer. In the following subsection, formulae for the number of representations of a positive integer $n$, $N_{(a, b)}(n)$, for $(a, b) = (1, 1), (1, 3), (2, 3), (1, 9)$, are determined as applications of the evaluation of the convolution sums $W_{(1,4)}(n)$ by J. G. Huard et al. [12], $W_{(1,12)}(n)$, $W_{(3,4)}(n)$, $W_{(1,24)}(n)$ and $W_{(3,8)}(n)$ by A. Alaca et al. [13], and $W_{(1,36)}(n)$ and $W_{(4,9)}(n)$ by D. Ye [34]. These numbers of representations of a positive integer $n$ are discovered due to Proposition 4.1. We rather consider Equation 6.15, Equation 6.16, Equation 6.19 and Equation 6.20 in the following result.

**Corollary 7.3.** Let $n \in \mathbb{N}$. Then

\[ N_{(1,1)}(n) = 16\sigma(n) - 64\sigma\left(\frac{n}{4}\right) + 64\sigma(\frac{n}{11}) + 1024\sigma(\frac{n}{14}) - 512\sigma(\frac{n}{14}) = 16\sigma_{3}(n) - 32\sigma_{3}\left(\frac{n}{2}\right) + 256\sigma_{3}(\frac{n}{4}), \]

\[ N_{(1,3)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) + 64\sigma(\frac{n}{13}) + 1024\sigma(\frac{n}{4}) - 256\left(W_{(3,4)}(n) + W_{(1,12)}(n)\right), \]

\[ N_{(2,3)}(n) = 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) + 64\sigma(\frac{n}{13}) + 1024\sigma(\frac{n}{4}) - 256\left(W_{(3,8)}(n) + W_{(1,12)}(n)\right), \]

\[ N_{(1,9)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{9}\right) - 32\sigma\left(\frac{n}{36}\right) + 64\sigma(\frac{n}{19}) + 1024\sigma(\frac{n}{4}) - 256\left(W_{(4,9)}(n) + W_{(1,36)}(n)\right). \]

**Proof.** When we set $(a, b) = (1, 1), (1, 3), (2, 3), (1, 9)$, these formulae follow immediately from Theorem 4.2.
8. Concluding Remark

To evaluate the convolution sum for $\alpha\beta$ that belongs to this class of positive integers, it now suffices to determine a basis of the space of cusp forms of weight 4 for $\Gamma_0(\alpha\beta)$. It is straightforward from Theorem 3.1 (a), that a basis of the space of Eisenstein forms of weight 4 for $\Gamma_0(\alpha\beta)$ is already given.

The determination of a basis of the space of cusp forms when $\alpha\beta$ is large is tedious. A future work is to carry out an effective and efficient method to build a basis of a space of cusp forms of weight 4 for $\Gamma_0(\alpha\beta)$ when $\alpha\beta$ is large.

A natural number can be expressed as a finite product of distinct primes to the power of some positive integers. The form for $\alpha\beta$ that we have considered falls under such an expression; however that form does not cover all natural numbers. The consideration of the class of natural numbers which is not discussed in this paper is a work in progress.

Competing interests

The authors declare that they have no competing interests.

Acknowledgments

I am indebtedly thankful to Prof. Emeritus Kenneth S. Williams for fruitful comments and suggestions on a draft of this paper.

References

[1] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3+4m=n} \sigma(l)\sigma(m)$. Adv Theor Appl Math, 1(1):27–48, 2006.
[2] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{l+11m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$. Int Math Forum, 2(2):45–68, 2007.
[3] A. Alaca, S. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{2l+1m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+n} \sigma(l)\sigma(m)$. Math J Okayama Univ, 49:93–111, 2007.
[4] A. Alaca, S. Alaca, and K. S. Williams. The convolution sum $\sum_{m \in \mathbb{N}} \sigma(m)\sigma(n-16m)$. Canad Math Bull, 51(1):3–14, 2008.
[5] S. Alaca and Y. Kesicioğlu. Evaluation of the convolution sums $\sum_{l+27m=n} \sigma(l)\sigma(m)$ and $\sum_{l+32m=n} \sigma(l)\sigma(m)$. Int J Number Theory, 12(1):1–13, 2016.
[6] S. Alaca and K. S. Williams. Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$. J Number Theory, 124(2):490–510, 2007.
[7] M. Besge. Extrait d’une lettre de M Besge à M Liouville. J Math Pure Appl, 7:256, 1885.
[8] H. H. Chan and S. Cooper. Powers of theta functions. Pac J Math, 235:1–14, 2008.
[9] S. Cooper and P. C. Toh. Quintic and septic Eisenstein series. Ramanujan J, 19:163–181, 2009.
[10] S. Cooper and D. Ye. Evaluation of the convolution sums $\sum_{l+25m=n} \sigma(l)\sigma(m)$, $\sum_{2l+3m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+2m=n} \sigma(l)\sigma(m)$. Int J Number Theory, 10(6):1386–1394, 2014.
[11] J. W. L. Glaisher. On the square of the series in which the coefficients are the sums of the divisors of the exponents. Messenger Math, 14:156–163, 1862.
[12] J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams. Elementary evaluation of certain convolution sums involving divisor functions. Number Theory Millenium, 7:229–274, 2002. A K Peters, Natick, MA.
[13] L. J. P. Kilford. Modular forms: A classical and computational introduction. Imperial College Press, London, 2008.
[14] N. Koblitz. Introduction to Elliptic Curves and Modular Forms, volume 97 of Graduate Texts in Mathematics. Springer Verlag, New York, 2nd edition, 1993.

[15] G. Köhler. Eta Products and Theta Series Identities, volume 3733 of Springer Monographs in Mathematics. Springer Verlag, Berlin Heidelberg, 2011.

[16] M. Lemire and K. S. Williams. Evaluation of two convolution sums involving the sum of divisors function. Bull Aust Math Soc, 73:107–115, 2006.

[17] G. Ligozat. Courbes modulaires de genre 1. Bull Soc Math France, 115:5–80, 1975.

[18] G. A. Lomadze. Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$. Acta Arith, 54(1):9–36, 1989.

[19] T. Miyake. Modular Forms. Springer monographs in Mathematics. Springer Verlag, New York, 1989.

[20] M. Newman. Construction and application of a class of modular functions. Proc Lond Math Soc, 7(3):334–350, 1957.

[21] M. Newman. Construction and application of a class of modular functions II. Proc Lond Math Soc, 9(3):373–387, 1959.

[22] E. Ntienjem. Evaluation of the convolution sums $\sum_{\alpha+i+\beta m=n} \sigma(l)\sigma(m)$, where $(\alpha, \beta)$ is in \{(1, 14), (2, 7), (1, 26), (2, 13), (1, 28), (4, 7), (1, 30), (2, 15), (3, 10), (5, 6)\}. Master’s thesis, School of Mathematics and Statistics, Carleton University, 2015.

[23] E. Ntienjem. Evaluation of the Convolution Sum involving primitive Dirichlet Characters for 48 and 64. Integers, 17, 2017. Accepted for publication.

[24] E. Ntienjem. Evaluation of the Convolution Sum involving the Sum of Divisors Function for 22, 44 and 52. Open Mathematics, 15(1):446–458, 2017.

[25] A. Pizer. The representability of modular forms by theta series. J Math Soc Japan, 28(4):689–698, 1976.

[26] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[27] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[28] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[29] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[30] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[31] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[32] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[33] E. Royer. Evaluating convolution sums of divisor function by quasimodular forms. Int J Number Theory, 3(2):231–261, 2007.

[34] D. Ye. Evaluation of the convolution sums $\sum_{l+36m=n} \sigma(l)\sigma(m)$ and $\sum_{4l+9m=n} \sigma(l)\sigma(m)$. Int J Number Theory, 11(1):171–183, 2015.

| $\sigma(m)$ | $\sigma(l)$ | $\sigma(n - 9m)$ |
|----------|----------|-----------------|
| 1        | 3        | 11              |
| 2        | 4        | 0               |
| 3        | 3        | 1               |
| 4        | 2        | 2               |
| 5        | 1        | 3               |
| 6        | 0        | 4               |
| 7        | 1        | -1              |
| 8        | 2        | -2              |
| 9        | 6        | 0               |
| 10       | 4        | 0               |

### Tables

**Tables**
Table 6: Power of $\eta$-quotients being basis elements of $\mathcal{S}_4(\Gamma_0(33))$

|   | 1  | 2  | 4  | 5  | 8  | 10 | 20 | 40 |
|---|----|----|----|----|----|----|----|----|
| 1 | 4  | 0  | 0  | 4  | 0  | 0  | 0  | 0  |
| 2 | 0  | 4  | 0  | 0  | 0  | 4  | 0  | 0  |
| 3 | 2  | 0  | 0  | -2 | 0  | 8  | 0  | 0  |
| 4 | 0  | 0  | 4  | 0  | 0  | 0  | 4  | 0  |
| 5 | 0  | 0  | 0  | 0  | 0  | 4  | 4  | 0  |
| 6 | 0  | 2  | 0  | 0  | 0  | -2 | 8  | 0  |
| 7 | 2  | -2 | 0  | -2 | 0  | 2  | 8  | 0  |
| 8 | 0  | 0  | 0  | 0  | 4  | 0  | 0  | 4  |
| 9 | 0  | 0  | 0  | 0  | 2  | 4  | -4 | 6  |
| 10| 2  | -2 | 2  | 2  | -2 | 0  | 0  | 6  |
| 11| 1  | 0  | 0  | -1 | 1  | 2  | -2 | 7  |
| 12| 0  | 0  | 2  | 0  | 0  | 0  | -2 | 8  |
| 13| 0  | 4  | 0  | 0  | -2 | 0  | -4 | 10 |
| 14| 0  | 2  | -2 | 0  | 0  | -2 | 2  | 8  |

Table 7: Power of $\eta$-quotients being basis elements of $\mathcal{S}_4(\Gamma_0(40))$

|   | 1  | 2  | 4  | 7  | 8  | 14 | 28 | 56 |
|---|----|----|----|----|----|----|----|----|
| 1 | 5  | -1 | 0  | 5  | 0  | -1 | 0  | 0  |
| 2 | 2  | 2  | 0  | 2  | 0  | 2  | 0  | 0  |
| 3 | 6  | -2 | 0  | -2 | 0  | 6  | 0  | 0  |
| 4 | 0  | 2  | 2  | 0  | 0  | 2  | 2  | 0  |
| 5 | 0  | 0  | 2  | 0  | 0  | 4  | 2  | 0  |
| 6 | 0  | 6  | -2 | 0  | 0  | -2 | 6  | 0  |
| 7 | 0  | 4  | -2 | 0  | 0  | 0  | 6  | 0  |
| 8 | 1  | 1  | 0  | 1  | 0  | -3 | 8  | 0  |
| 9 | 0  | 1  | 1  | 0  | 0  | -3 | 9  | 0  |
| 10| 0  | 0  | 0  | 0  | 2  | 0  | 4  | 2  |
| 11| 0  | -2 | 8  | 0  | -2 | 2  | -4 | 6  |
| 12| 0  | 0  | 6  | 0  | -2 | 0  | -2 | 6  |
| 13| 0  | 0  | 3  | 0  | -1 | 4  | -5 | 7  |
| 14| 0  | 0  | 4  | 0  | -2 | 0  | 0  | 6  |
| 15| 0  | 2  | 2  | 0  | -2 | -2 | 2  | 6  |
| 16| 0  | 1  | 1  | 0  | 0  | 1  | -3 | 8  |
| 17| 0  | 3  | -1 | 0  | 0  | -1 | -1 | 8  |
| 18| 0  | 0  | 1  | 0  | 1  | 0  | -3 | 9  |
| 19| 0  | 1  | 0  | 0  | -1 | -3 | 4  | 7  |
| 20| -2 | 5  | -3 | 2  | 0  | -5 | 7  | 4  |

Table 8: Power of $\eta$-quotients being basis elements of $\mathcal{S}_4(\Gamma_0(56))$
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