EQUIDISTRIBUTION OF ZEROS OF RANDOM HOLOMORPHIC SECTIONS

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Abstract. We study asymptotic distribution of zeros of random holomorphic sections of high powers of positive line bundles on projective homogenous manifolds. We work with a very general class of distributions that includes real and complex Gaussians. We prove that normalized simultaneous zero currents of i.i.d. random holomorphic sections, orthonormalized on a regular compact set, converges almost surely to the expected limit distribution. As a special case, we obtain that normalized zero measures of m i.i.d random polynomials, orthonormalized on a regular compact set $K \subset \mathbb{C}^m$, are almost surely asymptotic to the equilibrium measure of $K$.

1. Introduction

Study of asymptotic distribution of zeros of real Gaussian random polynomials goes back to Kac [23]. A classical result due to Hammersley [20] asserts that normalized zeros of complex Gaussian random polynomials of large degree tend to accumulate on the unit circle. More generally, Bloom [4] studied the zero distribution of Gaussian random polynomials orthonormalized on a regular compact set $K \subset \mathbb{C}$ with respect to a measure on $K$ satisfying Bernstein-Markov property (see section 1.1) and proved that normalized zeros are asymptotic to the equilibrium measure of $K$ (see also [5] for weighted case). On the other hand, there has been recent interest on distribution of zeros of random polynomials in higher dimensions. For a regular compact set $K \subset \mathbb{C}^m$ and a measure $\tau$ on $K$ satisfying the Bernstein-Markov property, Bloom and Shiffman [7] generalized the results of [4] to higher dimensions. Namely, for ensembles of random polynomials $P_n$ of degree $n$ endowed with Gaussian probability measure induced by $L^2(\tau)$ they proved that expected normalized zero currents of $k$ independent identically distributed Gaussian random polynomials orthonormalized on $K$ converges to an expected limit distribution. Moreover, Shiffman [24] established the almost everywhere convergence of normalized zero currents to the expected limit distribution. However, beyond the complex Gaussian coefficients case not much is known about the distribution of zeros of random polynomials in higher dimensions. More recently, Bloom and Levenberg [6] studied this problem under more general distributions and they generalized the result of [7]. Moreover, they posed the almost everywhere convergence of normalized zero currents of $k$ i.i.d random polynomials in $P_n$ to the expected distribution as an open problem. We address this question in the affirmative for a class of even more general distributions.

Our setting is as follows: Let $X$ be a projective manifold of complex dimension $m$ and $L \to X$ be a positive holomorphic line bundle. We also let $\omega$ be a smooth positive $(1,1)$ form representing $c_1(L)$, the first Chern class of $L$. Given a non-pluripolar compact set $K \subset X$ and a continuous weight function $q : K \to \mathbb{R}$ the weighted global extremal function $V_{K,q}$ of $K$ is defined to be usc regularization of

$$V_{K,q} := \{ \varphi \in L^1(X) : \omega + dd^c \varphi \geq 0 \quad \text{and} \quad \varphi(x) \leq q(x) \quad \text{for} \quad x \in K \}.$$ 

Throughout this work, we assume that $V_{K,q}$ is continuous and hence $V_{K,q} = V_{K,q}'$. Recall that a sufficient condition for continuity of $V_{K,q}$ is local regularity of $K$. Note that $T_{K,q} := \omega + dd^c V_{K,q}$ is a positive closed $(1,1)$ current representing the class $c_1(L)$ in $H^{1,1}(X,\mathbb{R})$. Since $T_{K,q}$ has locally

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bounded potentials, it follows from Bedford-Taylor theory \([1]\) that the exterior powers \(T^{k}_{K,q} := T_{K,q} \wedge T_{K,q} \wedge \cdots \wedge T_{K,q}\) are well defined positive closed \((k,k)\) currents for each \(1 \leq k \leq \dim X\). In particular, the top degree self-intersection defines a (positive) measure on \(X\).

Given a measure \(\tau\) on \(K\) satisfying the Bernstein-Markov property one can define an inner product on the space of sections \(H^0(X, L^{\otimes n})\) (see section \([2.4]\) for details). We fix an orthonormal basis \(\{S_{j}^{n}\}_{j=1}^{d_n}\) for \(H^0(X, L^{\otimes n})\) induced by \(L^2(\tau)\). Then each \(s_n \in H^0(X, L^{\otimes n})\) can be written as

\[
s_n = \sum_{j=1}^{d_n} a_j^{(n)} S_j^n.
\]

We assume that \(a_j^{(n)}\) are real or complex valued i.i.d random variables whose distribution law is of the form \(P := \phi(z) d\lambda(z)\) satisfying

\[
0 \leq \phi(z) \leq C \text{ for some } C > 0
\]

\[
\mathbb{P}\{z \in \mathbb{C} : |z| > R\} \leq \frac{C}{(\log R)^\rho} \text{ for sufficiently large } R > 0
\]

where \(\lambda\) is the Lebesgue measure on \(\mathbb{C}\) and \(\rho > \dim X + 1\). If \(\phi\) is a function defined on real numbers then we replace \(\phi(z) d\lambda\) by \(\phi(x) dx\). We remark that our setting includes standard real and complex Gaussian distributions of mean zero and variance one. Recall that, in the later case the tail decay of the integrals is of order \(e^{-n^2}\).

We identify \(S_n = H^0(X, L^{\otimes n})\) with \(\mathbb{C}^{d_n}\) where \(d_n := \dim(X, L^{\otimes n})\) and endow it with the product probability measure \(\mu_n\) induced by \(P\). We also consider \(S_\infty := \prod_{n=1}^{\infty} S_n\) as a probability space endowed with the product measure \(\mu := \prod \mu_n\). For a system \(S_n = (s_1^n, s_2^n, \ldots, s_k^n) \in S_k^n\) of i.i.d. random holomorphic sections with \(1 \leq k \leq m\) we denote the common zero locus by

\[
Z_{S_n} := \{x \in X : s_1^n(x) = s_2^n(x) = \cdots = s_k^n(x) = 0\}
\]

and define the normalized zero currents

\[
\bar{Z}_{S_n} := \frac{1}{n^k}[Z_{S_n}]
\]

where \([Z_{S_n}]\) denotes the current of integration along \(Z_{S_n}\). Then the expected zero current is defined by

\[
\langle E[\bar{Z}_{S_n}], \Phi \rangle := \int_{\bar{Z}_{S_n}} (\bar{Z}_{S_n}, \Phi) d\mu_n^k(S_n)
\]

where \(\Phi\) is a \((m-k,m-k)\) test form on \(X\) and \(\mu_n^k = \mu_n \times \cdots \times \mu_n\) is the \(k\)-fold product measure.

**Theorem 1.1.** Let \(L \rightarrow X\) be a positive holomorphic line bundle over a projective manifold \(X\) and \(K \subset X\) be a locally regular compact set with a continuous weight function \(q : K \rightarrow \mathbb{R}\) and endowed with a measure satisfying Bernstein-Markov property. Then

\[
E[\bar{Z}_{S_n}] \rightarrow T^k_{K,q}
\]

in the sense of currents as \(n \rightarrow \infty\). Moreover, if the ambient space \(X\) is complex homogeneous then almost surely

\[
\bar{Z}_{S_n} \rightarrow T^k_{K,q}
\]

in the sense of currents as \(n \rightarrow \infty\).

The proof of Theorem 1.1 is based on induction on bidegrees. To prove almost everywhere convergence for \(k = 1\), we use Kolmogorov’s strong law of large numbers which requires a variance estimate (Lemma 4.2). To obtain the variance estimate we make use of exponential estimates for \(\text{qpsh}\) functions which can be considered as a global version of uniform Skoda integrability theorem. Finally, we use extremal property of \(V_{K,q}\) to dominate quasi-potentials of limit points of random sequence of zero currents \(\{\bar{Z}_{S_n}\}_{n \geq 1}\). In higher bidegrees, we work with super-potentials of positive
closed currents. Recall that the super-potentials of positive closed currents were introduced by Dinh and Sibony [13] which extends the notion of quasi-potentials of positive closed (1, 1) currents.

Distribution of zeros of Gaussian systems of i.i.d. random holomorphic sections of high powers of a positive line bundle on a projective manifold was studied by Shiffman and Zelditch [25, 26]. More precisely, the randomization in [25] is obtained by endowing $\mathbb{P}H^0(X, L^{\otimes n})$ with the Fubini-Study volume form. In particular, if the ambient space is homogenous our results generalizes that of [25, 24]. Note that our proof is partly based on resolution of $\partial \bar{\partial}$-equations with qualitative estimates (see Theorem 2.1) which requires the ambient space to be homogenous. It would be interesting to know if one can prove such equidistribution results on arbitrary projective manifolds. More recently, Dinh and Sibony [12] studied this problem by endowing $\mathbb{P}H^0(X, L^{\otimes n})$ with a more general class of probability measures, particularly moderate measures. It follows from [16] that a measure is moderate if it is a Mongè-Ampère measure of a Hölder continuous function.

1.1. Distribution of zeros of random polynomials on $\mathbb{C}^m$. In the special case, let $K \subset \mathbb{C}^m$ be a non-pluripolar compact set, $q : K \to \mathbb{R}$ be a continuous function and $\tau$ be a measure on $K$ so that the triple $(K, q, \tau)$ satisfies the Bernstein-Markov property, that is

$$\max_{z \in K} |p(z)e^{-nq(z)}| \leq M_n\|p\|_{L^2(\tau)}$$

for all $p \in \mathcal{P}_n$ and $\lim \sup_{n \to \infty} \frac{\log |p(z)|}{n} = 1$. Recall that the weighted global extremal function $V_{K,q}^*$ is given by use regularization of

$$V_{K,q} := \sup \left\{ \frac{1}{\deg p} \log |p(z)| : p \in \cup_{n=1}^{\infty} \mathcal{P}_n \text{ and } \max_{z \in K} |p(z)e^{-nq(z)}| \leq 1 \right\}.$$

We assume that $V_{K,q}$ is continuous psh function on $\mathbb{C}^m$ which implies that $V_{K,q} = V_{K,q}^*$. Recall that if $K$ is locally regular then this is the case (see for instance [25 Prop. 2.13]).

Now, we let $\{P_j^n\}$ be an orthonormal basis for $\mathcal{P}_n$ with respect to the inner product

$$\langle p_1, p_2 \rangle := \int_K p_1(z)\overline{p_2(z)}e^{-2nq(z)}d\tau(z).$$

Then each $f_n \in \mathcal{P}_n$ can be written as

$$f_n(z) = \sum_{j=1}^{d_n} a_j^{(n)} P_j^{(n)}(z)$$

where we assume that $a_j^{(n)}$ are i.i.d. complex (or real) valued random variables whose distribution satisfies (1.1) and (1.2) with $\rho > m+1$. We denote simultaneous zero locus of $k$-tuples of i.i.d random polynomials by

$$Z_{F_n} := \{ z \in \mathbb{C}^m : f_1(z) = f_2(z) = \cdots = f_k(z) = 0 \}$$

where $1 \leq k \leq m$ and $F_n = (f_1, f_2, \ldots, f_k)$ and $f_j \in \mathcal{P}_n$. We also let $\tilde{Z}_{F_n} := \frac{1}{n^k}[Z_{F_n}]$ denote the normalized zero current. Note that by Bertini’s theorem for generic $F_n$ we have

$$\tilde{Z}_{F_n} = \frac{1}{n^k}dd^c \log |f_1(z)| \wedge \cdots \wedge dd^c \log |f_k(z)|.$$
in the sense of currents as \( n \to \infty \). Moreover, almost surely
\[
\bar{Z}_{F_n} \to (dd^c V_{K,q})^k
\]
in the sense of currents as \( n \to \infty \).

Finally, we remark that conditions (1.1) and (1.2) with \( \rho > 2 \) are not optimal in the case of \( X = \mathbb{P}^1 \) and \( K = S^1 \) the unit circle in \( \mathbb{C} \) with \( q \equiv 0 \). It was observed in [22] that the normalized zero measures \( \bar{Z}_{f_n} \) of i.i.d random polynomials \( f_n \in P_n \) converges almost surely to the Lebesgue measure \( \frac{1}{2\pi} d\theta \) if
\[
(1.3) \quad \int_{\mathbb{C}} \log(1 + |z|) \phi(z) d\lambda(z) < \infty.
\]
An easy computation shows that (1.3) holds if \( \rho > 1 \). Moreover, by the argument [22, pp. 6] the assumption (1.3) is a necessary condition for \( E[\bar{Z}_{f_n}] \to \frac{1}{2\pi} d\theta \) weak* as \( n \to \infty \).

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2. Preliminaries

2.1. Positive closed currents and super-potentials. Let \( X \) be a connected compact complex manifold and \( \text{Aut}(X) \) denote the group of holomorphic automorphisms of \( X \). Following Bochner and Montgomery [8], \( \text{Aut}(X) \) is a complex Lie group. We say that \( X \) is homogeneous if \( \text{Aut}(X) \) acts transitively on \( X \). In the sequel, we let \( (X, \omega) \) be a compact Kähler homogeneous manifold of dimension \( m \). It follows from [9] that \( X \) is a direct product of a complex torus and a projective rational manifold. In particular, complex projective space \( \mathbb{P}^m \) and \( (\mathbb{P}^1)^m \) are among the examples of such manifolds.

For \( 1 \leq k \leq m \), we let \( \mathcal{C}_k \) denote the set of all positive closed bidegree \((k, k)\) currents on \( X \) which are cohomologous to \( \omega^k \). This is a compact convex set. For a current \( T \in \mathcal{C}_k \), we denote its action on a test form \( \Phi \) by \( \langle T, \Phi \rangle \). For a smooth \((p, q)\) form \( \Phi \) denote by \( \|\Phi\|_{\omega^\alpha} \) the sum of \( \mathcal{C}^\alpha \)-norms of the coefficients in a fixed atlas. Following [13], for \( \alpha > 0 \) we define a distance function on \( \mathcal{C}_k \) by
\[
dist_\alpha(R, R') := \sup_{\|\Phi\|_{\omega^{\alpha}} \leq 1} \| (R - R', \Phi) \|
\]
where \( \Phi \) is a smooth \((m - k, m - k)\) form on \( X \). It follows from interpolation theory between Banach spaces [29] that
\[
dist_\beta \leq \dist_\alpha \leq C_{\alpha, \beta} \dist_\beta^{\frac{\alpha}{\beta}}
\]
for \( 0 < \alpha \leq \beta < \infty \) (see [13] Lem. 2.1.2 for the proof). Moreover, for \( \alpha \geq 1 \)
\[
dist_\alpha(\delta_a, \delta_b) \simeq \|a - b\|
\]
where \( \delta_a \) denotes the Dirac mass at \( a \) and \( \|a - b\| \) denotes the distance on \( X \) induced by the Kähler metric \( \omega \). We also remark that for \( \alpha > 0 \) topology induced by \( \dist_\alpha \) coincides with the weak topology on \( \mathcal{C}_k \) (cf. [13] Prop. 2.1.4]). In particular, \( \mathcal{C}_k \) is a compact separable metric space.

Let \( T \in \mathcal{C}_k \) with \( 1 \leq k \leq m \) then by \( dd^c \)-Lemma [13] there exists a real \((k - 1, k - 1)\) current \( U \), called a quasi-potential of \( T \) which satisfies the equation
\[
(2.1) \quad T = \omega^k + dd^c U
\]
where \( d = \partial + \bar{\partial} \) and \( dd^c := \frac{1}{2\pi} (\bar{\partial}(\partial - \bar{\partial})) \). In particular, if \( k = 1 \) a quasi-potential is nothing but a qpsh function. Note that two qpsh functions satisfying (2.1) differ by a constant. When \( k > 1 \) the quasi-potentials differ by \( dd^c \)-closed currents. For a real current \( U \) and \( 0 < r \leq \infty \), we denote the sum of \( L^r \) norms of its coefficients for a fixed atlas by \( \|U\|_{L^r} \). The quantity \( \langle U, \omega^{m-k+1} \rangle \) is called the mean of \( U \). The following result provides solutions to (2.1) with quantitative estimates.
Theorem 2.1. \cite{13} Let $X$ be a compact Kähler homogenous manifold and $T \in \mathcal{C}_k$ then there exists a negative quasi-potential $U$ of $T$ which depends linearly on $T$ such that for every $1 \leq r \leq \frac{m}{m-1}$ and $1 \leq s < \frac{2m}{m-1}$ we have

$$\|U\|_{L^s} \leq c_r \quad \text{and} \quad \|dU\|_{L^s} \leq c_s$$

where $c_r, c_s$ are positive constants independent of $T$. Moreover, $U$ depends continuously on $T$ with respect to the $L^s$ topology on $U$ and weak topology on $T$.

The quasi-potential $U$ is obtained in \cite{13} by using a kernel which solves $dd^c$-equation for the diagonal $\Delta$ of $X \times X$ (see also \cite{17} \cite{10}). More precisely, the current of integration $[\Delta]$ along the diagonal $\Delta \subset X \times X$ defines a positive closed $(m,m)$ current. It follows from Kinneth formula that $[\Delta]$ is cohomologous to a smooth real closed $(m,m)$ form $\Omega$ which is a linear combination of forms of type $\beta(z) \wedge \beta'(\zeta)$ where $\beta$ and $\beta'$ are closed real forms on $X$ of bidegree $(r,m-r)$ and $(m-r,r)$ respectively (cf. \cite{14} §2.1). Then by \cite{13} Proposition 2.3.2 there exists a negative $(m-1,m-1)$ form $K$ on $X \times X$, smooth away from $\Delta$ such that

$$dd^c K = [\Delta] - \Omega$$

satisfying

$$\|K(\cdot)\|_\infty \lesssim -\text{dist}(\cdot, \Delta)^{2(1-k)} \log \text{dist}(\cdot, \Delta) \quad \text{and} \quad \|\nabla K(\cdot)\|_\infty \lesssim \text{dist}(\cdot, \Delta)^{1-2k}$$

where $\|\nabla K\|_\infty$ denotes the sum $\sum_j |\nabla K_j|$ and $K_j$’s are the coefficients of $K$ for a fixed atlas of $X \times X$. This implies that for $T \in \mathcal{C}_k$, the $(k-1,k-1)$ current

$$U(z) := \int_{z \neq \zeta} (T(\zeta) - \omega^k(\zeta)) \wedge K(z, \zeta)$$

is well defined (cf. \cite{13} Theorem 2.3.1). Moreover,

$$dd^c U = T - \omega^k.$$ 

Indeed, let $\pi_i : X \times X \to X$ denote the projection on the $i$th coordinate with $i = 1, 2$. Note that

$$U = (\pi_1)_*(\pi_2^*(T - \omega^k) \wedge K)$$

and since $T - \omega^k$ is closed

$$dd^c U = (\pi_1)_*(\pi_2^*(T - \omega^k) \wedge dd^c K)$$

$$= (\pi_1)_*(\pi_2^*(T - \omega^k) \wedge [\Delta]) - (\pi_1)_*(\pi_2^*(T - \omega^k) \wedge \Omega)$$

$$= T - \omega^k$$

where the last equality follows from observing that the cohomology class $\{T - \omega^k\} = 0$ in $H^{1,1}(X, \mathbb{R})$ and $\Omega$ is a linear combination of forms of type $\beta \wedge \beta'$ with $\beta$ and $\beta'$ are closed.

Super-potentials of positive closed currents were introduced by Dinh and Sibony \cite{13} in the setting of complex projective space $\mathbb{P}^m$ (see also \cite{14}). The approach of \cite{13} can be easily extended to compact Kähler homogeneous manifolds. If $T$ is a smooth form in $\mathcal{C}_k$, super-potential of $T$ of mean $c$ is defined by

$$\mathcal{U}_T : \mathcal{C}_{m-k+1} \to \mathbb{R} \cup \{ -\infty \}$$

$$\mathcal{U}_T(R) = \langle T, U_R \rangle$$

where $U_R$ is a quasi-potential of $R$ of mean $c$. Then it follows that (see \cite{13} Lemma 3.1.1)

$$\mathcal{U}_T(R) = \langle U_T, R \rangle$$

where $U_T$ is a quasi-potential of $T$ of mean $c$. In particular, the definition of $\mathcal{U}_T$ in (2.3) is independent of the choice of $U_R$ of mean $c$. Note that super-potential of $T$ of mean $c'$ is given by $\mathcal{U}_{T+c'-c}$. More generally, for an arbitrary current $T \in \mathcal{C}_k$ super-potential of $T$ is defined by $\mathcal{U}_T(R)$ on smooth forms $R \in \mathcal{C}_{m-k+1}$ as in (2.3) where $U_R$ is smooth. Then the definition of super-potential can be extended
in a unique way to an affine use function on $\mathcal{C}_{m-k+1}$ with values in $\mathbb{R} \cup \{-\infty\}$ by approximation (see \cite{13} Proposition 3.1.6 and \cite{13} Corollary 3.1.7). Namely,

$$\mathcal{U}_T : \mathcal{C}_{m-k+1} \to \mathbb{R} \cup \{-\infty\}$$

$$\mathcal{U}_T (R) = \limsup_{R' \to R} \mathcal{U}_T (R')$$

where $R' \in \mathcal{C}_{m-k+1}$ is smooth.

**Remark 2.2.** It follows from Theorem 2.1 that for each $T \in \mathcal{C}_k$ there exists a negative super-potential $\mathcal{U}_T$ of $T$ such that its mean satisfies

$$|\mathcal{U}_T (\omega_{m-k+1})| \leq C$$

where $C > 0$ is independent of $T \in \mathcal{C}_k$.

Another feature of super-potentials is that for each $1 \leq k \leq m$, one can define a function

$$\mathcal{U}_k : \mathcal{C}_k \times \mathcal{C}_{m-k+1} \to \mathbb{R}$$

$$\mathcal{U}_k (T, R) := \mathcal{U}_T (R) = \mathcal{U}_R (T)$$

where $\mathcal{U}_T$ and $\mathcal{U}_R$ are super-potentials of $T$ and $R$ of the same mean. Moreover, $\mathcal{U}_k$ is u.s.c. (cf. \cite{13} Lemma 4.1.1):

**Lemma 2.3.** Let $T_n \in \mathcal{C}_k, R_n \in \mathcal{C}_{m-k+1}$ be sequences of positive closed currents such that $T_n \to T$ and $R_n \to R$ in the sense of currents as $n \to \infty$. Then

$$\limsup_{n \to \infty} \mathcal{U}_{T_n} (R_n) \leq \mathcal{U}_T (R).$$

Next result indicates that super-potentials determine the currents:

**Proposition 2.4.** Let $T, T'$ be currents in $\mathcal{C}_k$ with super-potentials $\mathcal{U}_T, \mathcal{U}_{T'}$ of mean $c$. If $\mathcal{U}_T = \mathcal{U}_{T'}$ on smooth forms in $\mathcal{C}_{m-k+1}$ then $T = T'$.

**Proof.** Let $\Phi$ be a smooth $(m-k, m-k)$ form. Then there exists $C > 0$ such that $C \omega_{m-k} + dd^c \Phi \geq 0$. Thus,

$$\mathcal{U}_T (C \omega_{m-k} + dd^c \Phi) = \mathcal{U}_{T'} (C \omega_{m-k} + dd^c \Phi)$$

which implies that

$$\langle T, \Phi \rangle = \langle T', \Phi \rangle.$$

\[\square\]

### 2.2. Currents with continuous super-potentials.

In this section we consider currents $T \in \mathcal{C}_k$ with continuous super-potentials. **The space $DSH^{m-k}(X)$:** Following \cite{12, 13}, a real $(m-k, m-k)$ current $\Phi$ of finite mass is called dsh if there exists positive closed currents $R^\pm$ of bidegree $(m-k+1, m-k+1)$ such that $dd^c \Phi = R^+ - R^-$. Then one can define

$$\|\Phi\|_{DSH} := \|\Phi\| + \min \|R^\pm\|$$

where $\|R^\pm\| := |\int_X R^\pm \wedge \omega^{k-1}|$. Note that since $R^+$ and $R^-$ are cohomologous we have $\|R^+\| = \|R^-\|$. We consider the space $DSH^{k-p}(X)$ with the weak topology: we say that $\Phi_n$ converges to $\Phi$ if

- $\Phi_n \to \Phi$ in the sense of currents
- $\|\Phi_n\|_{DSH}$ is bounded

It follows from Theorem 2.1 that if $R_n \to R$ weakly in $\mathcal{C}_k$ then there exists negative quasi-potentials $U_n, U$ of $R_n, R$ such that $U_n$ converges to $U$ in $DSH^{k-1}(X)$. A positive closed current $T \in \mathcal{C}_k$ is called PC if $T$ can be extended to a linear continuous form on $DSH^{m-k}(X)$. We denote the value of the extension by $(T, \Phi)$. Since smooth forms are dense in $DSH^{m-k}(X)$ the extension is unique. The following result is a consequence of Theorem 2.1 (see \cite{13} Proposition 3.3.1).
Proposition 2.5. A positive closed current is PC if and only if $T$ has continuous super-potentials.

The following proposition relates continuity of quasi-potentials to that of super-potentials of a positive closed current.

Proposition 2.6. Let $T \in \mathcal{C}_1$

(1) $T$ is PC if and only if $T$ has continuous quasi-potentials.

(2) Let $R \in \mathcal{C}_k$ be a PC current and $T = \omega + dd^c u$ for some continuous qpsh function $u$. Then $T \wedge R$ is a PC current. In particular,

$$T^k := T \wedge \ldots \wedge T$$

is PC for $1 \leq k \leq m$.

Proof. (1) Let $T = \omega + dd^c u$ where $u$ is continuous and $\Phi$ be a current in $DSH^{m-1}(X)$. We write $dd^c \Phi = \nu^+ - \nu^-$ for some measures $\nu^\pm \in \mathcal{C}_m$. If $\Phi$ and $\nu^\pm$ are smooth then

$$\langle T, \Phi \rangle = \langle \omega, \Phi \rangle + \langle u, dd^c \Phi \rangle = \langle \omega, \Phi \rangle + \langle u, \nu^+ \rangle - \langle u, \nu^- \rangle$$

since right hand side is well-defined and depends continuously on $\Phi \in DSH^{m-1}(X)$ we conclude that $T$ extends to a continuous linear form on $DSH^{m-1}(X)$.

Conversely, for $x \in X$ let $\Phi_x$ be a $(m-1, m-1)$ current satisfying $dd^c \Phi_x = \delta_x - \omega^m$. Then using a regularization of $\Phi_x$ we see that

$$\langle T, \Phi_x \rangle = \langle \omega, \Phi_x \rangle - \langle u, \omega^m \rangle + u(x)$$

since $\Phi_x$ and $\langle T, \Phi_x \rangle$ depend continuously on $x$ we conclude that $u$ is continuous on $X$.

(2) Since $T$ has locally bounded potentials the current $T \wedge R$ is well-defined $\mathbb{I}$. Now, if $\Phi \in DSH^{m-k-1}(X)$ is a smooth form then one can define

$$\langle T \wedge R, \Phi \rangle := \langle R, \omega \wedge \Phi \rangle + \langle R, udd^c \Phi \rangle$$

When $\Phi$ is not smooth the right hand side is still well-defined and depends continuously on $\Phi$. Indeed, since $R$ is PC, for every positive closed $(m-k, m-k)$ current $S$ the measure $R \wedge S$ is well-defined and depends continuously on $S$. This is because for every smooth function $\varphi$ the current $\varphi S$ is DSH. Thus, we can define

$$\langle R \wedge S, \varphi \rangle := \langle R, \varphi S \rangle.$$

Hence, $T \wedge R$ extends to a continuous linear form on $DSH^{m-k-1}(X)$.

$\square$

2.2.1. Super-potentials of intersection products. Let $T_1$ and $T_2$ be two positive closed current of bidegree $(1, 1)$ and $(k, k)$ respectively and assume that $1 \leq k \leq m - 1$. We also let $T_1 = \omega + dd^c \varphi$ where $\varphi$ is a qpsh function and $\varphi_{T_1}$ denote the super-potential of $T_1$ of mean zero. Recall that the wedge product $T_1 \wedge T_2$ is well-defined in the sense of currents if and only if $\varphi \in L^1(T_2 \wedge \omega^{m-k})$ (see \[11\] Chapter 1). In this case, by \[13\] §4 the super-potential of $T_1 \wedge T_2$ of mean zero is given by

$$\varphi_{T_1 \wedge T_2} : \mathcal{C}_{m-k} \rightarrow \mathbb{R}$$

(2.5) \quad $\varphi_{T_1 \wedge T_2}(R) = \langle T_1 \wedge T_2, U_R \rangle = \langle T_2, \omega \wedge U_R \rangle + \varphi_{T_1}(T_2 \wedge R) - \varphi_{T_2}(T_2 \wedge \omega^{m-k})$

whenever $R$ is a smooth form and $U_R$ is a smooth quasi-potential of $R$ of mean zero.
2.3. Holomorphic sections. Let $X$ be a projective manifold of complex dimension $m$. Given a holomorphic line bundle $\pi : L \to X$ we can find an open cover $\{U_\alpha\}$ of $X$ and biholomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, trivializations of $\pi^{-1}(U_\alpha)$. Then the line bundle $L$ is uniquely determined (up to isomorphism) by the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$ defined by $g_{\alpha\beta}(z) = (\varphi_\alpha \circ \varphi_\beta^{-1})_z|_{\pi^{-1}(z)}$.

The functions $g_{\alpha\beta}$ are non-vanishing holomorphic functions on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ satisfying

$$\begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \end{cases}$$

Recall that a singular metric $h$ is given by a collection $\{e^{-\psi_\alpha}\}$ of functions $\psi_\alpha \in L^1(U_\alpha)$ which are called weight functions with respect to the trivializations $\varphi_\alpha$ satisfying the compatibility conditions $\psi_\alpha = \psi_\beta + \log |g_{\alpha\beta}|$. We say that the metric is positive if $\psi_\alpha \in psh(U_\alpha)$ and the metric is called smooth if $\psi_\alpha \in C^\infty(U_\alpha)$ for every $\alpha$. Recall that $L \to X$ is called pseudo-effective (psef for short) if $L$ admits a singular positive metric. Moreover, we say that $L \to X$ is semi-ample if the base locus of $L\otimes m$ is empty for some $n \in \mathbb{N}$. If $X$ is complex homogenous manifold then it follows from [21] that every positive closed $(1, 1)$ current $T$ can be approximated by smooth semi-positive forms $\theta_t$ which are cohomologous to $T$. In particular, this implies that every psef line bundle $L \to X$ on a projective homogenous manifold is semi-ample. In the sequel, we assume that $L$ admits a smooth positive metric $h = \{e^{-\psi_\alpha}\}$. Then its curvature form is locally defined by

$$\omega := dd^c\psi_\alpha$$

which is a globally well-defined smooth closed $(1, 1)$ positive form representing the class $c_1(L)$ where $c_1(L)$ is the image of the Chern class of $L$ under the mapping $i : H^2(X, \mathbb{Z}) \to H^{1,1}(X, \mathbb{R})$ induced by the inclusion $i : \mathbb{Z} \to \mathbb{R}$. In what follows we assume that $\omega$ is normalized so that the volume form $dV := \omega^m$ is a probability measure.

A global holomorphic section $s = \{s_\alpha\}$ of $L\otimes n$ is a collection of holomorphic functions satisfying the compatibility conditions $s_\alpha = s_\beta\cdot g_{\alpha\beta}^n$ on $U_{\alpha\beta}$. We denote the set of all global holomorphic sections by $H^0(X, L\otimes n)$.

The metric $h$ induces an inner product on $H^0(X, L\otimes n)$ which is defined by

$$\langle s_1(x), s_2(x) \rangle_{h_\alpha} := s_1(x)\overline{s_2(x)}e^{-2n\psi_\alpha(x)}$$

for $s_1, s_2 \in H^0(X, L\otimes n)$ and $x \in U_\alpha$. For $s \in H^0(X, L\otimes n)$ we also set

$$||s(x)||_{h_\alpha} := |s_\alpha(x)|e^{-n\psi_\alpha(x)}$$

on $U_\alpha$. By compatibility conditions this definition is independent of $\alpha$.

For $s \in H^0(X, L\otimes n)$ we let $[Z_s]$ denote the current of integration along the zero divisor of $s$. Then by Poincaré-Lelong formula, locally we can write

$$[Z_s] = dd^c\log |s_\alpha|$$

on $U_\alpha$ hence,

$$[Z_s] = n\omega + dd^c\log ||s||_{h_\alpha}$$

on $X$ where the equality follows from compatibility conditions. Thus, we conclude that $\tilde{Z}_s := \frac{1}{n}[Z_s]$ is a positive closed $(1, 1)$ current representing the class $c_1(L)$ that is $\tilde{Z}_s \in \mathcal{Q}_1$.

On the other hand, if $X$ is complex homogenous then for each $s \in H^0(X, L\otimes n)$ we let

$$\varphi_s(z) := \int_{z \neq \zeta} (\tilde{Z}_s(\zeta) - \omega(\zeta)) \wedge K(z, \zeta).$$

Then by Theorem 2.1 we have

$$\tilde{Z}_s = \omega + dd^c\varphi_s$$

where $\varphi_s \leq 0$ and

$$|\int_X \varphi_s dV| \leq C$$
where $C > 0$ independent of $s \in H^0(X, L^{\otimes n})$ and $n \in \mathbb{N}$. Note that since $\varphi_s \leq 0$ the $L^1(\omega^m)$-norms of $\varphi_s$ are uniformly bounded; thus, by Hartog’s lemma the set $\{\varphi_s : s \in \cup_{n=1}^\infty \mathcal{S}_n\}$ is pre-compact in $L^1(\omega^m)$.

In the sequel, we denote the super-potential of $\tilde{Z}_s$ by

$$
\mathcal{W}_{\tilde{Z}_s} : C_m \to [-\infty, 0]
$$

(2.7)

$$
\mathcal{W}_{\tilde{Z}_s}(\nu) = \int_X \varphi_s d\nu
$$

We remark that above definition of $\mathcal{W}_{\tilde{Z}_s}$ depends on the choice of the Kernel $K(z, \zeta)$. On the other hand, for smooth $\nu \in C_m$ and $U_\nu$ is a quasi-potential of $\nu$ of mean equal to $\int_X \varphi_s d\nu$ then by (2.1) we have

$$
\mathcal{W}_{\tilde{Z}_s}(\nu) = \langle \tilde{Z}_s, U_\nu \rangle.
$$

(2.8)

### 2.4. Bernstein-Markov property

Let $K \subset X$ be a non-pluripolar compact set, $q : K \to \mathbb{R}$ be a continuous function and $\tau$ be a measure on $K$. We say that the triple $(K, q, \tau)$ satisfies the Bernstein-Markov property if

$$
\max_{x \in K} (\|s(x)\|_{h_n e^{-nq(x)}}) \leq M_n \left(\int_K \|s(z)\|^2_{h_n e^{-2nq(z)}} d\tau\right)^{\frac{1}{2}}
$$

for all $s \in H^0(X, L^{\otimes n})$ and $\limsup_{n \to \infty} (M_n)^{\frac{1}{n}} = 1$.

It follows that such a triple induces a non-degenerate weighted inner product on $H^0(X, L^{\otimes n})$:

$$
\langle s_1, s_2 \rangle_{q, \tau} := \int_K \langle s_1(x), s_2(x) \rangle_{h_n e^{-2nq(z)}} d\tau(x)
$$

(2.9)

we also set

$$
\|s\|_{L^2_{q, \tau}} := \left(\int_K \|s\|^2_{h_n e^{-2nq(x)}} d\tau(x)\right)^{\frac{1}{2}}
$$

and we fix an orthonormal basis $\{S_j^{(n)}\}_{j=1}^{d_n}$ for $H^0(X, L^{\otimes n})$ with respect to the inner product defined by (2.9). Then a section $s_n \in H^0(X, L^{\otimes n})$ can be written as

$$
s_n(x) = \sum_{j=1}^{d_n} a_j^{(n)} S_j^{(n)}(x).
$$

where $d_n := \dim H^0(X, L^{\otimes n})$. In the sequel, we assume that $a_j^{(n)}$ are i.i.d. complex (or real) valued random variables whose distribution $P = \phi(z) d\lambda$ is absolutely continuous with respect to the Lebesgue measure $\lambda$ on $\mathbb{C}$ and its density function satisfies

$$
0 \leq \phi(z) \leq C \quad \text{for all } z \in \mathbb{C}
$$

$$
P\{z \in \mathbb{C} : |z| > R\} \leq \frac{C}{(\log R)^{\rho}} \quad \text{for sufficiently large } R > 0
$$

where $\rho > \dim X + 1$.

Recall that since $L$ is positive there exists a constant $C > 0$ such that $C^{-1} n^m \leq d_n \leq C n^m$ for $n \in \mathbb{N}$. In what follows, we consider the coefficients $a^{(n)} := \{a_j^{(n)}\}_j$ as points in $\mathbb{C}^{d_n}$ and $(\mathbb{C}^{d_n}, P_n)$ as a probability space where $P_n$ is the $d_n$-fold product measure induced by $P$. We also denote

$$
\|a^{(n)}\| := \left(\sum_{j=1}^{d_n} |a_j^{(n)}|^2\right)^{\frac{1}{2}}
$$

is the $\ell^2$ norm on $\mathbb{C}^{d_n}$. Finally, we define $\mathcal{C} := \prod_{n=1}^{\infty} \mathbb{C}^{d_n}$ endowed with the product measure $\mathbb{P} := \prod_{n=1}^{\infty} P_n$ and consider the probability space $(\mathcal{C}, \mathbb{P})$. The following lemma will be useful in the sequel:
Lemma 2.7. For \(\mathbb{P}\)-a.e. \(\{a^{(n)}\}_{n \geq 1} \in \mathcal{C}\)

\[
\lim_{n \to \infty} \frac{1}{n} \log \|a^{(n)}\| = 0.
\]

Proof. We fix \(\epsilon > 0\) such that \(\rho(1-\epsilon) > m+1\). First, we show that (1.2) implies that with probability one, \(\|a^{(n)}\| \leq d_n e^{n^{1-\epsilon}}\) for sufficiently large \(n\). Indeed,

\[
P\{a^{(n)} \in \mathbb{C} : |a^{(n)}| > e^{n^{1-\epsilon}}\} \leq \frac{C}{n^{(1-\epsilon)}\rho}
\]

which implies that

\[
P_n\{a^{(n)} \in \mathbb{C} : \|a^{(n)}\| > d_n e^{n^{1-\epsilon}}\} \leq \frac{Cd_n}{n^{(1-\epsilon)}\rho}
\]

where the later defines a summable sequence. Thus, the claim follows from Borel-Cantelli lemma. Since \(d_n = O(n^m)\) we conclude that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|a^{(n)}\| \leq 0
\]

with probability one.

On the other hand, since \(|\phi(z)| \leq C\)

\[
P\{a^{(n)} : |a^{(n)}| < \frac{1}{n}\} \leq C\lambda\{z \in \mathbb{C} : |z| < \frac{1}{n}\} = \frac{C\pi}{n^2}
\]

thus, again, by using Borel-Cantelli lemma we obtain

\[\|a^{(n)}\| \geq \frac{d_n}{n^2}\]

with probability one. Hence,

\[
\liminf_{n \to \infty} \frac{1}{n} \log \|a^{(n)}\| \geq 0.
\]

\(\square\)

2.5. Extremal function. Let \(K \subset X\) be a non-pluripolar compact set and \(q : K \to \mathbb{R}\) be a continuous function. Following [19] we define the weighted global extremal function \(V^*_{K,q}\) as usc regularization of

\[
V_{K,q} := \sup\{\varphi \in psh(X,\omega) | \varphi(x) \leq q(x) \text{ for } x \in K\}.
\]

It follows that \(V^*_{K,q} \in psh(X,\omega)\), that is, \(T_{K,q} := \omega + dd^cV^*_{K,q}\) defines a positive closed \((1,1)\) current. Throughout this paper we assume that \(V_{K,q}\) is continuous so that \(V^*_{K,q} = V_{K,q}\). It is well know that this is the case when \(K\) is locally regular [28 Proposition 2.13]. Moreover, we have the following result (see [19, §6], [2, §4]):

Theorem 2.8. Let \((K,q)\) be as above then

\[
V_{K,q} = \sup\{\frac{1}{n} \log \|s(x)\|_{h_n} : s \in \cup_{n=1}^\infty H^0(X, L^{\otimes n}) \text{ and } \max_{x \in K}(\|s(x)\|_{h_n} e^{-nq(x)}) \leq 1\}.
\]
2.6. Bergman kernel. A triple \((K, \tau, q)\) satisfying the Bernstein-Markov property induces an inner product on the space of global sections with values in \(L^{\otimes n}\). Recall that Bergman kernel for the Hilbert space \(H^0(X, L^{\otimes n})\) is the integral kernel of the orthogonal projection from \(L^2\)-space of global sections with values in \(L^{\otimes n}\) onto \(H^0(X, L^{\otimes n})\). It is well-known that it can be represented as a holomorphic section
\[
S(x, y) = \sum_{j=1}^{d_n} S_j^n(x) \otimes \overline{S}_j^n(y)
\]
of the line bundle \(L^{\otimes n} \otimes L^{\otimes n}\) over \(X \times X\). The point wise norm of restriction of \(S(x, y)\) to the diagonal is given by
\[
\|S_n(x, x)\|_{h_n}^2 = \sum_{j=1}^{d_n} \|S_j^n(x)\|_{h_n}^2.
\]
The following result is well-known in the \(C^m\) setting [7, Lemma 3.2]. Moreover, the argument in [7] can be extended to our setting, see also [2, Theorem 1.5].

**Proposition 2.9.** Let \(S_n(z, z)\) denote the Bergman kernel on the diagonal then
\[
\lim_{n \to \infty} \frac{1}{2n} \log \|S_n(x, x)\|_{h_n} = V_{K, q}(x)
\]
for every \(x \in X\). Moreover, the convergence is uniform if \(V_{K, q}\) is continuous.

The next observation will be useful in the sequel.

**Proposition 2.10.** For \(\mu\text{-a.e.}\ \{s_n\} \in \mathcal{S}_\infty\)
\[
\limsup_{n \to \infty} \frac{1}{n} \log \|s_n(x)\|_{h_n} \leq V_{K, q}(x)
\]
for every \(x \in X\).

**Proof.** We write \(s_n = \sum_{j=1}^{d_n} a_j^{(n)} S_j^{(n)}\) then by Cauchy-Schwarz inequality
\[
\frac{1}{n} \log \|s_n(x)\|_{h_n} \leq \frac{1}{2n} \log \sum_{j=1}^{d_n} |a_j^{(n)}|^2 + \frac{1}{2n} \log \sum_{j=1}^{d_n} \|S_j^n(x)\|_{h_n}^2
\]
thus the assertion follows from Lemma 2.7 and Proposition 2.9. \(\square\)

3. Expected distribution of zeros

In this section we prove the first part of Theorem 1.1. For this, we assume that \(X\) is merely projective and \(L \to X\) is a positive holomorphic line bundle. The following lemma is slightly improved version of [6, Proposition 7.2]:

**Lemma 3.1.** Let \(u \in \mathbb{C}^{d_n}\) be a unit vector then
\[
\int_{\mathbb{C}^{d_n}} |\log |\langle a, u \rangle||d\mathbf{P}_n(a) = O(n^{1-\epsilon})
\]
for some small \(\epsilon > 0\).

**Proof.** We fix \(\epsilon > 0\) such that \((\rho - 1)(1-\epsilon) \geq m\). First, we prove that
\[
(3.1) \quad \int_{\{\log |\langle a, u \rangle| > mn^{1-\epsilon}\}} |\log |\langle a, u \rangle||d\mathbf{P}_n(a) \leq C_m
\]
where \(C_m > 0\) is a constant which depends only on \(m\). Note that for \(k \in \mathbb{N}\)
\[
\{a \in \mathbb{C}^{d_n} : \log |\langle a, u \rangle| > kn^{1-\epsilon}\} \subset \{a \in \mathbb{C}^{d_n} : \|a\| > e^{kn^{1-\epsilon}}\}
\]
hence by (1.2) and independence of $a_j$'s where $a = (a_j)_{j=1}^d$

$$r_k := P_n \{ a \in \mathbb{C}^d : \log |\langle a, u \rangle| > kn^{1-\epsilon} \} \leq \frac{C_n^m}{k^\rho n^{(1-\epsilon)\rho}}$$

Now, letting

$$R_k := \{ a \in \mathbb{C}^d : kn^{1-\epsilon} < \log |\langle a, u \rangle| \leq (k + 1)n^{1-\epsilon} \}$$

then

$$P_n(R_k) = r_k - r_{k+1}$$

and we see that

$$\int_{\{\log |\langle a, u \rangle| > mn^{1-\epsilon}\}} \log |\langle a, u \rangle| dP_n(a) \leq \sum_{k=m}^{\infty} (k + 1)n^{1-\epsilon}(r_k - r_{k+1})$$

$$\leq n^{1-\epsilon}((m + 1)r_m + \sum_{k=m+1}^{\infty} r_k)$$

$$\leq n^{1-\epsilon}((m + 1)C_n^m n^\rho n^{(1-\epsilon)} + C_n^m n^\rho n^{(1-\epsilon)} \sum_{k=m+1}^{\infty} k^{-\rho})$$

$$\leq \frac{C_n^m}{n^\rho (1-\epsilon)} \left( \frac{m + 1}{m^\rho} + \sum_{k=m+1}^{\infty} k^{-\rho} \right)$$

for every $n \in \mathbb{N}$. Since $(\rho - 1)(1 - \epsilon) \geq m$ the claim follows.

Next, we show that

$$\int_{\{\log |\langle a, u \rangle| < -mn^{1-\epsilon}\}} \log |\langle a, u \rangle| dP_n(a) \leq C_m$$

where $C_m > 0$ is a constant which depends only on $m$. Note that

$$l_k := P_n \{ a \in \mathbb{C}^d : |\langle a, u \rangle| < e^{-kn^{1-\epsilon}} \} \leq C_d n e^{-2kn^{1-\epsilon}}.$$ 

where $C > 0$ constant as in (1.1), in particular, independent of $n$. Indeed, we may assume that $|u_1| \geq \frac{1}{\sqrt{d_n}}$ where $u = (u_1, u_1, \ldots, u_d)$. Following [6, Lemma 2.8] we apply the change of variables

$$\alpha_1 = \sum_{i=1}^{d_n} a_i u_i, \alpha_2 = a_2, \ldots, \alpha_{d_n} = a_{d_n} \text{ and we obtain}$$

$$l_k = \int_{C_d n^{-1}} \int_{|\alpha_1| < e^{-kn^{1-\epsilon}}} \frac{1}{|u_1|^2} \phi(\frac{\alpha_1 - \alpha_2 u_2 - \cdots - \alpha_{d_n} u_{d_n}}{u_1}) \phi(\alpha_2) \cdots \phi(\alpha_{d_n}) d\lambda(\alpha_1) \cdots d\lambda(\alpha_{d_n})$$

$$\leq C_d n e^{-2kn^{1-\epsilon}}.$$ 

Next, for $k \geq m$ we let

$$D_k := \{ a \in \mathbb{C}^d : e^{-(k+1)n^{1-\epsilon}} \leq |\langle a, u \rangle| < e^{-kn^{1-\epsilon}} \}$$

then

$$P_n(D_k) \leq P_n \{ a \in \mathbb{C}^d : |\langle a, u \rangle| < e^{-kn^{1-\epsilon}} \} \leq C n^m e^{-2kn^{1-\epsilon}}$$
and
\[
\int_{\{\log |\langle a, u \rangle| < -mn^{1-\varepsilon}\}} |\log |\langle a, u \rangle||d\mathbf{P}_n(a) = \sum_{k=m}^{\infty} \int_{D_k} |\log |\langle a, u \rangle||d\mathbf{P}_n(a) \\
\leq \sum_{k=m}^{\infty} (k+1)n^{m+1-\varepsilon}e^{-2kn^{1-\varepsilon}} \\
\leq n^{m+1-\varepsilon} \int_{m}^{\infty} (x+1)e^{-2xn^{1-\varepsilon}} dx \\
\leq Ce^{-2mn^{1-\varepsilon}}n^m
\]

where \(C > 0\) depends only on \(m\) which proves \(3.3\).

Combining \(3.1\) and \(3.3\) we conclude that
\[
\int_{\{\log |\langle a, u \rangle| > mn^{1-\varepsilon}\}} |\log |\langle a, u \rangle||d\mathbf{P}_n(a) \leq C_m
\]
since \(\mathbf{P}_n\) is a probability measure this finishes the proof. \(\square\)

**Theorem 3.2.** Let \(X\) be a projective manifold, \(L \to X\) be a positive holomorphic line bundle and \(K \subset X\) is a locally regular compact set together with a continuous weight function \(q : K \to \mathbb{R}\) then
\[
E[Z_s] \to T_{K,q}
\]
in the sense of currents as \(n \to \infty\).

**Proof.** We want to show that for every smooth \((m-1, m-1)\) form \(\Phi\) on \(X\)
\[
\int_{S_n} \langle Z_s, \Phi \rangle d\mu_n(s) \to \langle T_{K,q}, \Phi \rangle
\]
as \(n \to \infty\). We may assume that \(\text{supp}(\Phi) \subset U_\alpha\) for some \(\alpha\). The general case follows from covering the \(\text{supp}(\Phi)\) by \(U_\alpha\)‘s and using the compatibility conditions. Following [25], let \(e^n_L\) be a holomorphic frame for \(L^\otimes n\) over \(U_\alpha\) then for \(s \in H^0(X, L^\otimes n)\) we may write
\[
s = \sum_{j=1}^{d_n} a_j f_j e^n_L
\]
and denote
\[
\langle a, f \rangle := \sum_{j=1}^{d_n} a_j f_j
\]
where \(S^n_j = f_j e^n_L\) on \(U_\alpha\) and \(f = (f_1, f_2, \ldots, f_{d_n})\). Then by Poincare Lelong formula
\[
\overline{Z}_s = \frac{1}{n} dd^c \log |\langle a, f \rangle|
\]
on \(U_\alpha\) Evidently
\[
\log |\langle a, f \rangle| = \log |\langle a, u \rangle| + \log |f|
\]
where \(f = |f|u\) and \(|u| \equiv 1\) on \(U_\alpha\). Then it follows from Lemma \(3.1\)
\[
(3.5) \quad \int_{X \times \mathbb{C}^{d_n}} |\log |\langle a, u \rangle||dd^c \Phi|d\mathbf{P}_n(a) = \int_{\mathbb{C}^{d_n}} |\log \langle a, u \rangle||d\mathbf{P}_n(a) \int_{X} |dd^c \Phi| \leq Cn^{1-\varepsilon}||dd^c \Phi||_\infty
\]
Hence, by Fubini’s Theorem we obtain
\[
\int_{S_n} (\tilde{Z}_{s_n}, \Phi) d\mu_n(s) = \int_{\mathbb{C}d^n} \int_X \frac{1}{n} df \log |(a, f)| \wedge \Phi \ dP_n(a) = \int_X \int_{\mathbb{C}d^n} \frac{1}{n} \log |(a, u(x))| dP_n(a) \ df \Phi(x) + \int_X \frac{1}{2n} \log \sum_{j=1}^{d_n} |f_j|^2 df \Phi
\]
\[= I_1(n) + I_2(n) \]
Note that by (3.3) we have \(I_1(n) \to 0\) as \(n \to \infty\). On the other hand, by Proposition 2.9 we obtain \(I_2(n) \to \langle T_{K,q}, \Phi \rangle\) as \(n \to \infty\). This completes the proof. \(\square\)

We remark that if \(a_j(n)\) are standard complex Gaussians then the integral \(I_1(n)\) in the proof of Theorem 3.2 is equal to zero (see [25, Lemma 3.1]). In particular, the expected zero current

(3.6) \[E[\tilde{Z}_{s_n}] = \omega + \frac{dd^c}{2n} \log \|S_n(x, x)\|_{b_n} \]
in the Gaussian case. In our setting this is no longer the case. In fact, it follows from Theorem 3.2 that
\[E[\tilde{Z}_{s_n}] = \omega + \frac{1}{2n} dd^c \log \|S_n(x, x)\|_{b_n} + O(n^{-\epsilon}) \]
for some small \(\epsilon > 0\) where by \(O(n^{-\epsilon})\) we mean a real closed \((1, 1)\) current \(T_n\) such that
\[\|\langle T_n, \Phi \rangle\| \leq Cn^{-\epsilon}\|dd^c \Phi\|_{\infty} \]
where \(C > 0\) is independent of \(n\) and the smooth form \(\Phi\). If \(a_j(n)\) are standard complex Gaussian then it is classical that (see [26, 27, 7]) the identity (3.6), Proposition 2.9 and Theorem 3.2 implies that for each \(1 \leq k \leq m\)
\[E[\tilde{Z}_{s_{k_1}, \ldots, s_k}] = (\omega + \frac{1}{2n} dd^c \log \|S_n(x, x)\|_{b_n})^k \to T_{K,q}^k \]
in the sense of currents as \(n \to \infty\). We prove the analogue result in our setting. We utilize some arguments from [27] to prove the following:

**Corollary 3.3.** Let \(X\) be a projective manifold, \(L \to X\) be a positive holomorphic line bundle and \(K \subset X\) is a locally regular compact set together with a continuous weight function \(q : K \to \mathbb{R}\). We also let \(s_1, s_2, \ldots, s_k\) be i.i.d. random holomorphic sections in \(S_n\) with \(1 \leq k \leq m\). Then
\[E[\tilde{Z}_{s_{k_1}, \ldots, s_k}] = E[\tilde{Z}_{s_{k_1}}] \wedge E[\tilde{Z}_{s_{k_2}}] \wedge \cdots \wedge E[\tilde{Z}_{s_{k_k}}] \]
Moreover,
\[E[\tilde{Z}_{s_{k_1}, \ldots, s_k}] \to T_{K,q}^k \]
in the sense of currents as \(n \to \infty\).

**Proof.** We prove the assertion by induction on \(k\). Note that the case \(k = 1\) was proved in Theorem 3.2. Assume that the claim holds for \(k-1\). We fix \(s_1\) such that \(X' := Z_{s_1}\) is a smooth hypersurface in \(X\). We also denote \(s' := s|_{X'}\) for generic \(s \in S_n\) and define the restriction map \(\rho : S_n \to S'_{n}\) where \(S'_{n} = S_n|_{X'}\). We endow \(S'_{n}\) with the probability measure \(\mu'_n := \rho_* \mu_n\). Then by induction hypothesis applied on \(X' = Z_{s_1}\)
\[\int_{S'_{n-1}} \langle \tilde{Z}_{s_1, s_2, \ldots, s_k}, \Phi \rangle d\mu'_n(s_2) \cdots d\mu'_n(s_k) = \int_{S'_{n-1}} \langle \tilde{Z}_{s_1, s_2', \ldots, s_k}, \Phi \rangle d\mu'_n(s_2') \cdots d\mu'_n(s_k') = \langle E[s_2'] \wedge \cdots \wedge E[s_k'], \Phi \rangle \]
\[= \int_{Z_{s_1}} E[s_2] \wedge \cdots \wedge E[s_k] \wedge \Phi \]

Note that by (3.3) we have \(I_1(n) \to 0\) as \(n \to \infty\). On the other hand, by Proposition 2.9 we obtain \(I_2(n) \to \langle T_{K,q}, \Phi \rangle\) as \(n \to \infty\). This completes the proof. \(\square\)
then taking the average over \(s_1\) we obtain the first assertion.

To prove the second assertion, we let

\[
(3.7) \quad \alpha_n := \omega + \frac{1}{2n} dd^c \log \|S_n(x, x)\|_{h_n}
\]

and we claim that

\[
\langle E[\tilde{Z}_{s_1, \ldots, s_k}], \Phi \rangle = \langle \alpha_n, \Phi \rangle + C_{\Phi, n}
\]

where \(C_{\Phi, n}\) is the “error term” which satisfies the uniform estimate

\[
|C_{\Phi, n}| \leq C n^{-\epsilon} ||dd^c \Phi||_{\infty}
\]

where \(\epsilon > 0\) small as in Lemma 5.1 and \(C > 0\) is independent of smooth form \(\Phi\) and \(n\). Note that the case \(k = 1\) was proved in Theorem 3.2. Now, using the above notation and by applying induction hypothesis on \(X' = Z_{s_1}\)

\[
\int_{Z_{s_1}} (\tilde{Z}_{s_2, \ldots, s_k}, \Phi) d\mu_n(s_2) \ldots d\mu_n(s_k) \quad = \quad \langle E[\tilde{Z}_{s'_2, \ldots, s'_k}], \Phi|_{X'} \rangle
\]

\[
\quad \quad = \int_{Z_{s_1}} \alpha_n^{k-1} \land \Phi + C_{X', \Phi, n}
\]

where

\[
|C_{X', \Phi, n}| \leq C n^{-\epsilon} ||dd^c \Phi||_{X'} \leq C n^{-\epsilon} ||dd^c \Phi||_{\infty} \int_{X'} \omega^{m-1} = C n^{-\epsilon} ||dd^c \Phi||_{\infty}
\]

where the later equality comes from computing the integral in cohomology. Now, taking the average over \(s_1\) and using the estimate in proof of Theorem 3.2 we obtain

\[
\langle E[\tilde{Z}_{s_1, \ldots, s_k}], \Phi \rangle \quad = \quad \langle \alpha_n, \alpha_n^{k-1} \land \Phi \rangle + C_{\Phi, n} + \int_{S_n} C_{X', \Phi, n} d\mu_n(s_1)
\]

\[
\quad = \quad \langle \alpha_n, \Phi \rangle + C_{\Phi, n}
\]

where

\[
|C_{\Phi, n}| \leq |C_{\Phi, n}'| + \int_{S_n} |C_{X', \Phi, n}| d\mu_n(s_1) \leq C n^{-\epsilon} \|\alpha_n^{k-1} \land dd^c \Phi\|_{\infty} + C n^{-\epsilon} ||dd^c \Phi||_{\infty}
\]

Thus, the assertion follows from the above estimate and the uniform convergence of Bergman kernels to weighted global extremal function (Proposition 2.4) together with a theorem of Bedford and Taylor [1, §7] on convergence of Mongé-Ampére measures. \(\square\)

4. Almost everywhere convergence for bidegree (1,1)

In this section we prove the second assertion in Theorem 1.1 for the case \(k = 1\).

**Theorem 4.1.** Let \(X\) be a projective homogeneous manifold, \(L \to X\) be a positive holomorphic line bundle and \(K \subset X\) is a locally regular compact set with a continuous weight function \(q : K \to \mathbb{R}\) and endowed with a measure satisfying Bernstein-Markov property. Then for \(\mu\)-a.e. \(\{s_n\}_{n \geq 1} \in \mathcal{S}_\infty\)

\[
\hat{Z}_{s_n} \to T_{K, q}
\]

in the sense of currents as \(n \to \infty\).

**Proof.** By [13, Lemma 3.2.5] and Proposition 2.4, it is enough to show that for \(\mu\)-a.e. \(\{s_n\} \in \mathcal{S}_\infty\) the sequence of super potentials \(\{\mathcal{W}_{\hat{Z}_{s_n}}\}\) converges to the super-potential of \(T_{K, q}\) on smooth measures. To this end, for a fixed smooth measure \(\nu \in \mathcal{C}_m\) we define the sequence of random variables

\[
X_n : \mathcal{S}_\infty \to (-\infty, 0]
\]

\[
X_n(\{s_j\}_{j \geq 1}) = \mathcal{W}_{\hat{Z}_{s_n}}(\nu)
\]
where $\mathcal{U}_{\bar{Z}_{s_n}}$ denotes the super-potential of $\bar{Z}_{s_n}$ defined by (2.7). Thus, $\{X_n\}$ is a sequence of negative independent random variables (but they are not identically distributed). Note that since $\nu$ is smooth, $V_{K,q}$ is $\nu$-integrable and by Theorem 3.2
\[E[X_n] = \int_{S_n} \langle \bar{Z}_{s_n}, U_\nu \rangle d\mu_n(s_n) \rightarrow \langle T_{K,q}, U_\nu \rangle = \mathcal{U}_{T_{K,q}}(\nu)\]
as $n \rightarrow \infty$ where $U_\nu$ is the quasi potential of $\nu$ defined by (2.8). This in turn implies that
\[\lim_{n \rightarrow \infty} E\left[ \frac{1}{n} \sum_{k=1}^{n} X_k \right] = \mathcal{U}_{T_{K,q}}(\nu).\]

On the other hand, the variance of $X_n$ is given by
\[\text{Var}[X_n] = E[X_n^2] - (E[X_n])^2.\]

Note that the second term in the variance of $X_n$ is bounded by a constant independent of $n$. We will show that the first term is also bounded by a constant independent of $n$:

**Lemma 4.2.** Let $\nu$ and $X_n$ be as above. Then
\[\text{Var}[X_n] \leq C_{L,\nu}\]
where $C_{L,\nu} > 0$ depends only on $\nu$ and $L \rightarrow X$.

**Proof.** It is enough to show that $E[X_n^2] \leq C_{L,\nu}$. Indeed, since $\nu$ is smooth, $\mathcal{U}_{\nu}$ is Lipschitz on $C_1$ with respect to $dist_\alpha$. Then by [12, Proposition A.3] and [15, Theorem 3.9] (see also [16, Theorem 1.1]) there exists constants $\alpha > 0, C > 0$ (depending only on $L \rightarrow X$ and $\nu$) such that for every $s_n \in H^0(X, L^{\otimes n})$ and $n \geq 1$
\[\int_X \exp(-\alpha \varphi_{s_n}) d\nu \leq C\]
where $\varphi_{s_n}$ is the quasi potential of $\bar{Z}_{s_n}$ defined by (2.6). Now, by using $e^x \geq \frac{x^2}{2!}$ for $x \geq 0$ we conclude that
\[\|\varphi_{s_n}\|_{L^2(\nu)} \leq C_{L,\nu}\]
for some constant $C_{L,\nu} > 0$ which depends only on $L \rightarrow X$ and $\nu$ but independent of $s_n$. Thus, by Jensen’s inequality we obtain
\[E[X_n]^2 = \int_{S_n} X_n^2 d\mu_n \leq \int_{S_n} (\mathcal{U}_{\bar{Z}_{s_n}}(\nu))^2 d\mu_n \leq C_{\omega,\nu}\]

Hence, by Lemma 4.2 Kolmogorov’s strong law of large numbers [3] and (4.1) we obtain that for $\mu$-a.e. $\{s_n\} \in S_\infty$
\[\frac{1}{n} \sum_{k=1}^{n} \mathcal{U}_{\bar{Z}_{s_k}}(\nu) \rightarrow \mathcal{U}_{T_{K,q}}(\nu)\]
as $n \rightarrow \infty$. Note that since $\nu$ is a probability measure, $L^2(\nu)$ norm dominates $L^1(\nu)$ norm. In particular $X_n$'s are bounded. Next, we use the following lemma:
Lemma 4.3. \cite{30} Theorem 1.20] Let \( \{b_j\} \) be a bounded sequence of negative real numbers. TFAE:

1. There exists a subsequence \( \{b_{j_k}\} \) of relative density one (i.e. \( \frac{k}{j_k} \to 1 \) as \( k \to \infty \)) such that \( b_{j_k} \to 0 \).
2. \( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} b_j = 0 \).

Thus, we conclude that \( \mu \)-a.e. \( \{s_n\} \in \mathcal{S}_\infty \) has a subsequence \( \{s_{n_j}\} \) of density one such that

\[
\mathcal{U}_{Z_{s_{n_j}}} (\nu) \to \mathcal{U}_{T_{K,q}} (\nu).
\]

We will show that in the above case, in fact, the whole sequence \( \{\mathcal{U}_{Z_{s_{n_j}}} (\nu)\} \) converges to \( \mathcal{U}_{T_{K,q}} (\nu) \).

Indeed, since \( \mathcal{U}_{Z_{s_{n_j}}} \leq 0 \) by a variation of Hartog’s Lemma \cite{13, Prop 3.2.6} either \( \mathcal{U}_{Z_{s_{n_j}}} \) converges uniformly to \( -\infty \) or there exists a subsequence \( Z_{s_{n_{k}}} \) such that \( Z_{s_{n_{k}}} \to T \) weakly for some \( T \in \mathcal{C}_1 \) and \( \mathcal{U}_{Z_{s_{n_{k}}}} \to \mathcal{U}_T \) on smooth measures. However, by Remark \ref{2.2} we know that the means \( \mathcal{U}_{Z_{s_{n}}} (\omega^m) \) are uniformly bounded. Hence, the later occurs. Next, we prove the following lemma:

Lemma 4.4. For \( \mu \)-a.e. \( \{s_n\} \in \mathcal{S}_\infty \)

\[
\limsup_{n \to \infty} \mathcal{U}_{Z_{s_{n}}} (\sigma) \leq \mathcal{U}_{T_{K,q}} (\sigma)
\]

for every smooth \( \sigma \in \mathcal{C}_m \). In particular, for \( \mu \)-a.e. \( \{s_n\} \in \mathcal{S}_\infty \), if \( Z_{s_{n_{k}}} \to T \) in the sense of currents then

\[
\mathcal{U}_T \leq \mathcal{U}_{T_{K,q}}
\]

on smooth probability measures in \( \mathcal{C}_m \) where \( \mathcal{U}_T \) and \( \mathcal{U}_{T_{K,q}} \) are super potentials of \( T \) and \( T_{K,q} \) respectively.

Proof. For smooth \( \sigma \in \mathcal{C}_m \) by \ref{2.8} we have

\[
\mathcal{U}_{Z_{s_{n}}} (\sigma) = \langle Z_{s_{n}}, U_\sigma \rangle = \langle \omega + \frac{1}{n} dd^c \log \|s_n\|_h, U_\sigma \rangle = \langle \omega, U_\sigma \rangle + \langle \frac{1}{n} \log \|s_n\|_h, dd^c U_\sigma \rangle
\]

Since \( dd^c U_\sigma = \sigma - \omega^m \) by Proposition \ref{2.10} and Fatou’s lemma we obtain

\[
\limsup_{n \to \infty} \mathcal{U}_{Z_{s_{n}}} (\sigma) \leq \langle \omega, U_\sigma \rangle + \langle V_{K,q}, dd^c U_\sigma \rangle = \langle T_{K,q}, U_\sigma \rangle = \mathcal{U}_{T_{K,q}} (\sigma)
\]

Now, by Proposition \ref{2.4} the super-potential \( \mathcal{U}_{T_{K,q}} \) is continuous on \( \mathcal{C}_m \). If, \( \mathcal{U}_T (\nu) \neq \mathcal{U}_{T_{K,q}} (\nu) \) then by \cite{13} Proposition 3.2.2

\[
\mathcal{U}_{Z_{s_{n_k}}} (\nu) < \mathcal{U}_{T_{K,q}} (\nu)
\]

for large \( k \). Since \( \mathcal{U}_{Z_{s_{n_k}}} \) are negative this contradicts \ref{4.2}. Thus, \( \mathcal{U}_T \) and \( \mathcal{U}_{T_{K,q}} \) agrees on smooth measures. Hence, \( T = T_{K,q} \) by Proposition \ref{2.4}

So far, we have proved that for every smooth measure \( \nu \in \mathcal{C}_m \) there exists a set \( \mathcal{S}_\nu \subset \mathcal{S}_\infty \) of probability one such that for every \( \{s_n\} \in \mathcal{S}_\nu \)

\[
\mathcal{U}_{Z_{s_{n}}} (\nu) \to \mathcal{U}_{T_{K,q}} (\nu)
\]
as \( n \to \infty \). Now, we fix a countable dense subset of smooth measures \( \{ \nu_j \}_{j \in \mathbb{N}} \subset \mathcal{C}_m \) with respect to the \( \text{dist}_\alpha \) for some fixed \( \alpha > 0 \) and define
\[
\mathcal{S} := \cap_{j=1}^{\infty} \mathcal{S}_{\nu_j}.
\]
Note that \( \mathcal{S} \) has probability one. We claim that for every smooth \( \nu \in \mathcal{C}_m \)
\[
U_{T_{K,q}}(\nu) = \lim_{n \to \infty} U_{\bar{Z}_{s_n}}(\nu)
\]
for every \( \{ s_n \} \in \mathcal{S} \). Indeed, let \( \nu' \to \nu \) in \( (\mathcal{C}_m, \text{dist}_\alpha) \) with \( \nu' \in \{ \nu_j \}_{j} \) then
\[
|U_{\bar{Z}_{s_n}}(\nu) - U_{T_{K,q}}(\nu)| \leq |U_{\bar{Z}_{s_n}}(\nu - \nu')| + |U_{\bar{Z}_{s_n}}(\nu') - U_{T_{K,q}}(\nu')| + |U_{T_{K,q}}(\nu') - U_{T_{K,q}}(\nu)|
\]
where the second term tends to 0 by above argument and the third terms tends to 0 by Proposition 2.3. Finally, since \( \nu \) and \( \nu' \) are smooth the first term can be bounded
\[
|U_{\bar{Z}_{s_n}}(\nu - \nu')| = |\langle \phi_{s_n}, \nu - \nu' \rangle| \leq | \int \phi_{s_n} f d\nu |
\]
where \( f := f_{\nu', \nu} \) is a smooth function with \( \|f\|_{\infty} \to 0 \) as \( \nu' \to \nu \). Since \( | \int \phi_{s_n} d\nu | \leq C \) for every \( s_n \in \mathcal{S}_n \) and \( n \in \mathbb{N} \) this finishes the proof. \( \square \)

**Remark 4.5.** Finally, we stress that one can work with quasi-potentials of positive closed \((1,1)\) currents rather than super-potentials to prove Theorem 4.1. In particular, the assertion of Theorem 4.7 is still valid if \( X \) is merely projective manifold but not homogenous. We refer the reader to [6] for such a treatment.

## 5. Almost everywhere convergence for bidegree \((k,k)\)

Let \( S_n = (s^1_n, s^2_n, \ldots, s^n_k) \) be a \( k \)-tuple of i.i.d random holomorphic sections \( s^j_n \in \mathcal{S}_n \) for \( j = 1, 2, \ldots, k \) where \( 1 \leq k \leq m \). We are interested in distribution of simultaneous zeros:
\[
Z_{S_n} := \{ x \in X : s^1_n(x) = s^2_n(x) = \cdots = s^n_k(x) = 0 \}
\]
We denote set of all such \( k \)-tuples (respectively sequences of \( k \)-tuples) by \( \mathcal{S}_n^k := \prod_{j=1}^{k} \mathcal{S}_n \) (respectively by \( \mathcal{S}_\infty^k := \prod_{j=1}^{k} \mathcal{S}_\infty \)) endowed with \( k \)-fold the product measure \( \mu^k_n \) (respectively \( \mu^k \)) induced by \( \mu_n \) (respectively \( \mu \)). By Bertini’s theorem [15, pp 137] with probability one the zero sets of \( Z_{s^j_n} \) are smooth and intersect transversally. In particular, for generic \( S_n \in \mathcal{S}_n^k \) the zero set \( Z_{S_n} \) is a complex submanifold of codimension \( k \). Moreover, almost surely the current of integration along the set \( Z_{S_n} \) is given by
\[
[Z_{S_n}] = [Z_{s^1_n}] \wedge [Z_{s^2_n}] \wedge \cdots \wedge [Z_{s^n_k}]
\]
thus, we may write
\[
\bar{Z}_{S_n} := \frac{1}{\mu^k_n}[Z_{S_n}] = \omega^k + dd^c U_{S_n}
\]
where
\[
U_{S_n}(z) = \int_{z \neq \zeta} (\bar{Z}_{S_n}(\zeta) - \omega^k(\zeta)) \wedge K(z, \zeta)
\]
is the negative \((k-1, k-1)\) current given by Theorem 2.1.

**Theorem 5.1.** Let \( X \) be a projective homogeneous manifold, \( L \to X \) be a positive holomorphic line bundle and \( K \subset X \) is a locally regular compact set with a continuous weight function \( q : K \to \mathbb{R} \) endowed with a measure satisfying Bernstein-Markov property. Then for \( \mu^k \)-a.e. \( \{ S_n \} \in \mathcal{S}_n^k \)
\[
\bar{Z}_{S_n} \to T^k_{K,q}
\]
in the sense of currents as \( n \to \infty \).
Proof. We will prove the theorem by induction on $k$. Note that the case $k = 1$ was proved in Theorem 4.1. Let’s assume that the assertion holds for $k - 1$.

Now, given $Z_{S_n} \in S^k_n$ let $U_{\tilde{Z}_{S_n}}$ be as in (5.2) then by Theorem 2.1

\begin{equation}
|\langle U_{\tilde{Z}_{S_n}}, \omega^{m-k+1} \rangle| \leq C
\end{equation}

where $C > 0$ is independent of $S_n$ and $n \in \mathbb{N}$. We denote the super-potential of $\tilde{Z}_{S_n}$ by $U_{\tilde{Z}_{S_n}}(R) = \langle U_{\tilde{Z}_{S_n}}, R \rangle$ whenever $R$ is a smooth form in $\mathcal{C}_{m-k+1}$. Note that by \[13, Lemma 3.2.5\] and Proposition 2.4 it is enough to show that with probability one, $U_{\tilde{Z}_{S_n}}(R)$ converges to $\mathcal{U}_{T_{K,q}}(R)$ for every smooth form $R \in \mathcal{C}_{m-k+1}$. To this end we fix a smooth form $R \in \mathcal{C}_{m-k+1}$ and define the random variables

$X_n : S^k_n \to (-\infty, 0]$

$X_n(\{S_j\}_{j \geq 1}) = U_{\tilde{Z}_{S_n}}(R)$

Note that $X_n$ are independent (but not identically distributed) random variables. We will need the following lemma, proof of which is deferred until the end of this section.

**Lemma 5.2.** Let $X_n$ be as above then the variance of $X_n$ satisfies

\begin{equation}
\text{Var}[X_n] \leq C_{L,R} n^{-\epsilon}
\end{equation}

where $\epsilon > 0$ small and $C_{L,R} > 0$ depends only on the Chern class of the line bundle $L \to X$ and the smooth form $R$.

Then by Kolmogorov’s strong law of large numbers we conclude that for $\sigma_k$-a.e. $S \in S^k_n$

\[
\frac{1}{n} \sum_{j=1}^{n} X_j(S) - E[\frac{1}{n} \sum_{j=1}^{n} X_j] \to 0
\]

as $n \to \infty$. On the other hand, by Corollary 3.3

\[
E[\tilde{Z}_{S_n}] \to T_{k,q}^k
\]

in the sense of currents as $n \to \infty$. Thus, we infer that with probability one

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} U_{\tilde{Z}_{S_j}}(R) = \mathcal{U}_{T_{K,q}}(R).
\end{equation}

Note that since $R$ is smooth by (5.3)

\[
|X_n(\{S_j\}_{j \geq 1})| \leq C_R
\]

where $C_R > 0$ depends on $R$ but independent of $S_n \in S^k_n$ and $n \in \mathbb{N}$. Thus, by Lemma 4.3 we conclude that $\sigma_k$-a.e. $\{S_n\}$ has a subsequence $\{S_{n_j}\}$ of density one such that

\[
\mathcal{U}_{\tilde{Z}_{S_{n_j}}}(R) \to \mathcal{U}_{T_{k,q}}(R)
\]

as $j \to \infty$. Next, we will show that in this case in fact the whole sequence $\{\mathcal{U}_{\tilde{Z}_{S_n}}(R)\}_{n \geq 1}$ converges. Indeed, since $\mathcal{U}_{\tilde{Z}_{S_n}} \leq 0$ on $\mathcal{C}_{m-k+1}$ by \[13, Prop 3.2.6\] either $\{\mathcal{U}_{\tilde{Z}_{S_n}}\}_{n \geq 1}$ converges uniformly to $-\infty$ or $\{S_n\}$ has a subsequence $S_{n_1}$ such that

\[
\tilde{Z}_{S_{n_1}} \to T
\]

for some $T \in \mathcal{C}_k$ in the sense of currents as $n_i \to \infty$ and

\[
\mathcal{U}_{\tilde{Z}_{S_{n_1}}} \to \mathcal{U}_T
\]
on smooth forms in $\mathcal{C}_{m-k+1}$ where $\mathcal{U}_T$ is the super-potential of $T$ of mean $m := \lim_{n_i \to \infty} \langle U_{\tilde{Z}_{S_{n_i}}}^1, \omega^{m-k+1} \rangle$. However, the former is not possible as the means $\{ \langle U_{\tilde{Z}_{S_{n_i}}}^1, \omega^{m-k+1} \rangle \}_{n \geq 1}$ are uniformly bounded by \([5, 3]\). Hence, the later occurs. Next, we prove the following lemma:

**Lemma 5.3.** For $\sigma_k$-a.e. $\{ S_n \} \in \mathcal{S}_\infty^k$,

$$\limsup_{n \to \infty} \mathcal{U}_{\tilde{Z}_{S_{n_i}}}^1 (\Phi) \leq \mathcal{U}_{T_{k,q}^m}^1 (\Phi)$$

for every smooth form $\Phi \in \mathcal{C}_{m-k+1}$. In particular, for $\sigma_k$-a.e. $\{ S_n \} \in \mathcal{S}_\infty^k$, if $\tilde{Z}_{S_{n_i}} \to T$ for some $T \in \mathcal{C}_k$ in the sense of currents then

$$\mathcal{U}_T \leq \mathcal{U}_{T_{k,q}^m}^1$$

on smooth forms in $\mathcal{C}_{m-k+1}$.

**Proof.** We let $\mathcal{U}$ denote the super-potential of mean zero. We prove the lemma by induction on $k$. Note that the case $k = 1$ was proved in Lemma 1.4. Assume that the assertion holds for $k - 1$. By Bertini’s theorem 1.8 for generic $S_n = (S'_n, s_n^k)$ we may write

$$\tilde{Z}_{S_{n_i}} = \tilde{Z}_{S_n} \wedge \tilde{Z}_{s_n^k}$$

where $S_n \in S_{\infty}^k$. Then by (2.3) almost surely the super potential of $\tilde{Z}_{S_{n_i}}$ of mean zero is given by

$$\mathcal{U}_{\tilde{Z}_{S_{n_i}}}^1 (\Phi) = \langle \tilde{Z}_{S_n}' \wedge U_\Phi, \omega \rangle + \mathcal{U}_{\tilde{Z}_{s_n^k}^1} (\tilde{Z}_{S_n}' \wedge \dd^c U_\Phi)$$

where $U_\Phi$ is a smooth quasi-potential of $\Phi \in \mathcal{C}_{m-k+1}$ of mean zero. Moreover, by induction hypothesis for generic sequences $\tilde{Z}_{S_n} \to T_{k,q}^{k-1}$ and $\tilde{Z}_{s_n^k}^1 \to T_{k,q}^{k-1}$ in the sense of currents. Then by Lemma 2.3 we have

$$\limsup_{n \to \infty} \mathcal{U}_{\tilde{Z}_{S_{n_i}}}^1 (\tilde{Z}_{S_n}' \wedge \dd^c U_\Phi) \leq \mathcal{U}_{T_{k,q}^{k-1}} (T_{k,q}^{k-1} \wedge \dd^c U_\Phi)$$

On the other hand, by induction hypothesis again

$$\limsup_{n \to \infty} \mathcal{U}_{S_n^k}^1 (\omega \wedge \Phi) = \limsup_{n \to \infty} \langle \tilde{Z}_{S_n}' \wedge \omega, U_\Phi \rangle \leq \mathcal{U}_{T_{k,q}^{k-1}} (\omega \wedge \Phi) = \langle T_{k,q}^{k-1}, \omega \wedge U_\Phi \rangle$$

Thus, combining these we conclude that

$$\limsup_{n \to \infty} \mathcal{U}_{\tilde{Z}_{S_{n_i}}}^1 (\Phi) \leq \mathcal{U}_{T_{k,q}^{k-1}} (\Phi)$$

for every smooth form $\Phi$ in $\mathcal{C}_{m-k+1}$. □

Now, by Proposition 2.6, the super-potential $\mathcal{U}_{T_{k,q}^m}$ is continuous on $\mathcal{C}_{m-k+1}$. If $\mathcal{U}_T (R) \neq \mathcal{U}_{T_{k,q}^m} (R)$ then

$$\mathcal{U}_T (R) < \mathcal{U}_{T_{k,q}^m} (R)$$

thus, by [13] Prop. 3.2.2

$$\mathcal{U}_{Z_{S_{n_j}}} (R) < \mathcal{U}_{T_{k,q}^m} (R)$$

for large $n_j$ but this contradicts (5.3) as $\mathcal{U}_{Z_{S_{n_j}}}^1$ are negative. Hence, we conclude that for every smooth $R \in \mathcal{C}_{m-k+1}$ there exists a set $S_R \subset S_{\infty}^k$ of probability one such that

$$\mathcal{U}_{T_{k,q}^{m}} (R) = \lim_{n \to \infty} \mathcal{U}_{Z_{S_{n_i}}} (R)$$

for every $\{ S_n \}_{n \geq 1} \in S_R$. Finally, applying the density argument given at the end of the Proof of Theorem 1.1 completes the proof. □

Next, we prove Lemma 5.2. The proof is based on induction on bidegree. For $k = 1$, we provide a different proof than the one given in Lemma 1.2 which provides some precision on the bound of the variance of $X'_S$. We utilize some ideas from 5.2, 24.
Proof of Lemma 5.2. We will prove the assertion by induction on \( k \). First, we prove the case \( k = 1 \): we use the same notation as in Lemma 3.1 and Theorem 3.2. Note that

\[
\text{Var}[X_n] = E[X_n^2] - (E[X_n])^2
\]

and by Theorem 3.2 it is enough to bound the term

\[
E[X_n^2] = \int_{\mathbb{C}_n} (\tilde{Z}_s, U_\nu)^2 d\mu_n(s)
\]

where \( \nu \) is a smooth measure in \( \mathcal{C}_m \) and \( U_\nu \) is a smooth quasi-potential as in (2.8). We claim that

\[
E[X_n^2] = \langle \alpha_n, U_\nu \rangle^2 + O(n^{-\epsilon}\|dd^c U_\nu\|_{\infty})
\]

where \( \alpha_n \) as in (5.7) and \( \epsilon > 0 \) as in Lemma 3.1. On the other hand, by Theorem 3.2

\[
E[X_n] = \langle \alpha_n, U_\nu \rangle + O(n^{-\epsilon}\|dd^c U_\nu\|_{\infty})
\]

to hence, the assertion follows. Next, we prove the claim. Following [25] we write

\[
E[X_n^2] = \frac{1}{n^2} \int_X \int_X dd^c U_\nu(z) dd^c U_\nu(w) \int_{\mathbb{C}^{4n}} \log |\langle a, f(z) \rangle| \log |\langle a, f(w) \rangle| dP_n(a)
\]

The later integrant can be written as

\[
\log |\langle a, f(z) \rangle| \log |\langle a, f(w) \rangle| = \log |f(z)| \log |f(w)| + \log |f(z)| \log |\langle a, u(w) \rangle| + \log |f(w)| \log |\langle a, u(z) \rangle| + \log |\langle a, u(z) \rangle| \log |\langle a, u(w) \rangle|
\]

where \( f \) and \( u \) as in the proof of Theorem 3.2. Thus, we may write

\[
E[X_n^2] =: I_1(n) + I_2(n) + I_3(n) + I_4(n).
\]

It follows from Theorem 3.2 that

\[
I_1(n) = \langle \alpha_n, U_\nu \rangle^2
\]

and for \( j = 2, 3 \)

\[
|I_j(n)| \leq C n^{-\epsilon} \|\alpha_n, U_\nu\|_{\infty}\|dd^c U_\nu\|_{\infty}.
\]

Finally, we claim that

\[
|I_4(n)| \leq C_m n^{-2\epsilon} \|dd^c U_\nu\|_{\infty}^2
\]

where \( C_m > 0 \) is independent of \( n \). Indeed, by Cauchy-Schwarz inequality

\[
|I_4(n)| \leq \|dd^c U_\nu\|_{\infty}^2 \sup_{z \in X} \int_{\mathbb{C}^{4n}} (\log |\langle a, u(z) \rangle|)^2 dP_n(a)
\]

thus, it is enough to show that for every unit vector \( u \in \mathbb{C}^d \)

\[
\int_{\{ |\langle a, u \rangle|^{2e} > m^{n^{2-2\epsilon}} \}} (\log |\langle a, u \rangle|)^2 dP_n(a) \leq C_m n^{1-\epsilon}.
\]

where \( C_m \) depends only on \( m \). We let

\[
L_k := \{ a \in \mathbb{C}^d : kn^{2-2\epsilon} \leq (\log |\langle a, u \rangle|)^2 < (k+1)n^{2-2\epsilon} \}.
\]

Note that

\[
L_k \subset R_{\sqrt{\tau}} \cup D_{\sqrt{\tau}}
\]

where \( R_{\sqrt{\tau}} \) and \( D_{\sqrt{\tau}} \) as in the proof of Lemma 3.1. Then by (3.2) and (3.3) we have

\[
P_n(L_k) \leq r_{\sqrt{\tau}} - r_{\sqrt{\tau} + 1} + C_m n^m e^{-2\sqrt{k}n^{1-\epsilon}}.
\]
Using \( (22) \) TURGAY BAYRAKTAR 
\[
\int_{\{(\log |\langle a, u \rangle|)^2 > mn^{2-2\epsilon} \}} (\log |\langle a, u \rangle|)^2 dP_n(a) \leq \sum_{k=m}^{\infty} (k+1)n^{2-2\epsilon}\left[ (r_{\sqrt{k}} - r_{\sqrt{k+1}} + C_m n^m e^{-2\sqrt{k}n^{1-\epsilon}} \right] \\
\leq n^{2-2\epsilon}(m+1)r_{\sqrt{m}} + \sum_{k=m+1}^{\infty} r_{\sqrt{k}} + C_m n^m \sum_{k=m}^{\infty} (k+1)e^{-2\sqrt{k}n^{1-\epsilon}} \\
\leq C_m n^{m+2-2\epsilon}\left[ 1/n(1-\epsilon) + \sum_{k=m+1}^{\infty} 1/k n^{(1-\epsilon)} + \sum_{k=m+1}^{\infty} (k+1)e^{-2\sqrt{k}n^{1-\epsilon}} \right]
\]

Thus, using \((\rho-1)(1-\epsilon) \geq m\) we obtain 
\[
\int_{\{(\log |\langle a, u \rangle|)^2 > mn^{2-2\epsilon} \}} (\log |\langle a, u \rangle|)^2 dP_n(a) \leq C_m n^{1-\epsilon}.
\]

Since, \( P_n \) is a probability measure we conclude that 
\[
\int_{C_n^e} (\log |\langle a, u \rangle|)^2 dP_n(a) \leq C_m n^{2-2\epsilon}
\]

which proves (5.5) and this completes the proof of the case \( k = 1 \).

Now, we assume that the assertion holds for \( k - 1 \). We denote \( S_n = (S_n', s_k) \) where \( S_n' = (s_1, \ldots, s_{k-1}) \in S^{k-1}_n \). Then by Corollary 3.3 
\[
E[\bar{Z}_{S_n}] = E[\bar{Z}_{S'_n}] \wedge E[\bar{Z}_{s_k}].
\]

and by \((24)\) 
\[
\mathcal{W}_{\bar{Z}_{S_n}}(R) = (\bar{Z}_{S'_n} \wedge \bar{Z}_{s_k}, U_R)
\]

where \( U_R \) is a smooth quasi-potential of the smooth form \( R \in \mathcal{C}_{m-k+1} \) of mean \( \langle U_{S_n}, \omega^{m-k+1} \rangle \).

Note that 
\[
\text{Var}[X_n] = E[X_n^2] - (E[X_n])^2 = \int_{S^k_n} (\langle \bar{Z}_{S_n}, U_R \rangle)^2 d\mu_n^k - (E[\bar{Z}_{S'_n}] \wedge E[\bar{Z}_{s_k}], U_R)^2
\]

We define \( J_1 \) and \( J_2 \) such that 
\[
J_1 + J_2 := \langle \bar{Z}_{S_n}, U_R \rangle^2 - (E[\bar{Z}_{S'_n}] \wedge E[\bar{Z}_{s_k}], U_R)^2
\]

where 
\[
J_1(S_n', s_k) = \langle \bar{Z}_{S_n}, U_R \rangle^2 - (\bar{Z}_{S'_n} \wedge E[\bar{Z}_{s_k}], U_R)^2
\]
\[
J_2(S_n') = \langle \bar{Z}_{S'_n} \wedge E[\bar{Z}_{s_k}], U_R \rangle^2 - (E[\bar{Z}_{S'_n}] \wedge E[\bar{Z}_{s_k}], U_R)^2
\]

which are well-defined (see proof of Corollary 3.3). Note that 
\[
\text{Var}[X_n] = E[J_1] + E[J_2]
\]

For a generic \( S'_n \in S^{k-1}_n \) the set \( V := \{ S'_n = 0 \} \) is a complex submanifold of codimension \( k - 1 \). Moreover, \( \bar{Z}_{S'_n} \wedge \bar{Z}_{s_k} = Z_{s_k}|_V \) for generic \( s_k \). Thus, 
\[
\mathcal{W}_{\bar{Z}_{S_n}}(R) = \langle \bar{Z}_{s_k}|_V, U_R|_V \rangle = \mathcal{W}_{\bar{Z}_{s_k}}(R|_V)
\]
Then applying induction hypothesis with $k = 1$ to $(V, \mu_{\nu} | V)$ and $\mu'_n$ in place of $(X, \nu)$ and $\mu_n$ where $\rho : S_n \to S'_n$ is the restriction map and $\rho^* \mu_n = \mu'_n$, we obtain
\[ \int_{S_n} J_1(S_n', s^k_n) d\mu_n(s^k_n) = \int_{S_n} \langle \mathcal{Z}_{s^k_n} | V, U_R | V \rangle^2 d\mu'_n(s^k_n) - \langle E[\mathcal{Z}_{s^k_n} | V], U_R | V \rangle^2 \leq C \| R | V \|^2 n^{-\epsilon} \| dd^c U_R | V \|_{\infty} \leq C_L \| R \|^2 n^{-\epsilon} \| dd^c U_R \|_{\infty} \]
where $\| R | V \|$ (respectively $\| R \|$) denotes the mass of restriction of $R$ on $V$ (respectively the mass of $R$ on $X$) with respect to $\omega | V$ (respectively $\omega$). Thus, taking the average over $S'_n$ we obtain
\[ E[J_1] = \int_{S'_n} \int_{S_n} J_1(S_n', s^k_n) d\mu_n(s^k_n) d\mu^{k-1}_n(S'_n) \leq C_{L,R} n^{-\epsilon}. \]
On the other hand, for a random variable $Y_n$ of mean $m$ we have $E[(Y_n - m)^2] = E[(Y_n)^2] - m^2$. Applying this argument to the random variables
\[ Y_n(\{S'_n\}_{j \geq 1}) := \langle \mathcal{Z}_{S'_n} \wedge E[\mathcal{Z}_{s^k_n}], U_R \rangle \]
we obtain
\[ E[J_2] = \int_{s^k_n = 1} \langle \mathcal{Z}_{S'_n} \wedge E[\mathcal{Z}_{s^k_n}], U_R \rangle^2 d\mu^{k-1}_n(S'_n) - \langle E[\mathcal{Z}_{S'_n}] \wedge E[\mathcal{Z}_{s^k_n}], U_R \rangle^2 \]
\[ = \int_{s^k_n = 1} \langle \mathcal{Z}_{S'_n} \wedge E[\mathcal{Z}_{s^k_n}] - E[\mathcal{Z}_{S'_n}] \wedge E[\mathcal{Z}_{s^k_n}], U_R \rangle^2 d\mu^{k-1}_n(S'_n) \]
\[ = \int_{s^k_n = 1} \left[ \left( \int_{S_n} \langle \mathcal{Z}_{S'_n} - E[\mathcal{Z}_{S'_n}] \rangle \wedge \mathcal{Z}_{s^k_n}, U_R \rangle d\mu_n(s^k_n) \right)^2 \right] d\mu^{k-1}_n(S'_n) \]
\[ \leq \int_{s^k_n = 1} \int_{S_n} \langle \mathcal{Z}_{S'_n} - E[\mathcal{Z}_{S'_n}] \rangle \wedge \mathcal{Z}_{s^k_n}, U_R \rangle^2 d\mu^{k-1}_n(S'_n) d\mu_n(s^k_n) \]
where the last inequality follows from Jensen’s inequality and Fubini’s theorem. Now, letting $W := \{s^k_n = 0\}$ since $W$ is a smooth hypersurface for generic $s^k_n$ we have
\[ \langle \mathcal{Z}_{S'_n} - E[\mathcal{Z}_{S'_n}] \rangle \wedge \mathcal{Z}_{s^k_n}, U_R \rangle = \langle (Z_{S'_n} - E[Z_{S'_n}]) | W, U_R | W \rangle \]
and applying the induction hypothesis for $k - 1$ on $W$ we obtain that
\[ E[J_2] \leq C_{L,R} n^{-\epsilon} \| dd^c U_R \|_{\infty} \]
this finishes the proof. \qed

**References**

[1] E. Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. *Acta Math.*, 149(1-2):1–40, 1982.

[2] R. J. Berman. Bergman kernels and equilibrium measures for line bundles over projective manifolds. *Amer. J. Math.*, 131(5):1485–1524, 2009.

[3] P. Billingsley. *Probability and measure*, volume 939. Wiley, 2012.

[4] T. Bloom. Random polynomials and Green functions. *Int. Math. Res. Not.*, (28):1689–1708, 2005.

[5] T. Bloom. Random polynomials and (pluri)potential theory. *Ann. Polon. Math.*, 91(2-3):131–141, 2007.

[6] T. Bloom and N. Levenberg. Random polynomials and pluripotential-theoretic extremal functions. preprint arXiv:1304.4529, 2013.

[7] T. Bloom and B. Shiffman. Zeros of random polynomials on $\mathbb{C}^m$. *Math. Res. Lett.*, 14(3):469–479, 2007.

[8] S. Bochner and D. Montgomery. Groups on analytic manifolds. *Ann. of Math.* (2), 48:659–669, 1947.

[9] A. Borel and R. Remmert. Über kompakte homogene Kählersche Mannigfaltigkeiten. *Math. Ann.*, 145:429–439, 1961/1962.

[10] J.-B. Bost, H. Gillet, and C. Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994.

[11] J.-P. Demailly. *Complex analytic and differential geometry*. http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, 2009.

[12] T.-C. Dinh and N. Sibony. Distribution des valeurs de transformations méromorphes et applications. *Comment. Math. Helv.*, 81(1):221–258, 2006.
[13] T.-C. Dinh and N. Sibony. Super-potentials of positive closed currents, intersection theory and dynamics. *Acta Math.*, 203(1):1–82, 2009.

[14] T.-C. Dinh and N. Sibony. Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms. *J. Algebraic Geom.*, 19(3):473–529, 2010.

[15] T.C. Dinh and V.A. Nguyen. Characterization of monge-ampere measures with holder continuous potentials. *arXiv preprint arXiv:1204.4883*, 2012.

[16] T.C. Dinh, V.A. Nguyễn, and N. Sibony. Exponential estimates for plurisubharmonic functions. *Journal of Differential Geometry*, 84(3):465–488, 2010.

[17] H. Gillet and C. Soulé. Arithmetic intersection theory. *Inst. Hautes Études Sci. Publ. Math.*, (72):93–174 (1991), 1990.

[18] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[19] V. Guedj and A. Zeriahi. Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.*, 15(4):607–639, 2005.

[20] J. M. Hammersley. The zeros of a random polynomial. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II*, pages 89–111, Berkeley and Los Angeles, 1956. University of California Press.

[21] A. Huckleberry. Subvarieties of homogeneous and almost homogeneous manifolds. In *Contributions to complex analysis and analytic geometry*, Aspects Math., E26, pages 189–232. Vieweg, Braunschweig, 1994.

[22] I. Bragimov and D. Zaporozhets. On distribution of zeros of random polynomials in complex plane. In *Prokhorov and Contemporary Probability Theory*, pages 303–323. Springer, 2013.

[23] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943.

[24] B. Shiffman. Convergence of random zeros on complex manifolds. *Sci. China Ser. A*, 51(4):707–720, 2008.

[25] B. Shiffman and S. Zelditch. Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.*, 200(3):661–683, 1999.

[26] B. Shiffman and S. Zelditch. Random polynomials with prescribed Newton polytope. *J. Amer. Math. Soc.*, 17(1):99–108 (electronic), 2004.

[27] B. Shiffman and S. Zelditch. Number variance of random zeros on complex manifolds. *Geometric and Functional Analysis*, 18(4):1422–1475, 2008.

[28] J. Siciak. Extremal plurisubharmonic functions in ${\mathbb{C}^n}$. *Ann. Polon. Math.*, 39:175–211, 1981.

[29] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.

[30] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

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