COUNTING EQUIVALENCE CLASSES
OF IRREDUCIBLE REPRESENTATIONS

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Abstract. Let $n$ be a positive integer, and let $R$ be a (possibly infinite dimensional) finitely presented algebra over a computable field of characteristic zero. We describe an algorithm for deciding (in principle) whether $R$ has at most finitely many equivalence classes of $n$-dimensional irreducible representations. When $R$ does have only finitely many such equivalence classes, they can be effectively counted (assuming that $k[x]$ possesses a factoring algorithm).

1. Introduction

Let $n$ be a positive integer, fixed throughout. In [5] we observed that the existence of $n$-dimensional irreducible representations of finitely presented noncommutative algebras can be algorithmically decided. In this note we outline a procedure for effectively “counting” the number of such irreducible representations, up to equivalence, in characteristic zero. Our approach combines standard computational commutative algebra with results from [1] and [9].

1.1. Assume that $k$ is a computable field of characteristic zero, and that $\overline{k}$ is the algebraic closure of $k$.

Henceforth, let

$$R = k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t \rangle,$$

for some fixed choice of $f_1, \ldots, f_t$ in the free associative $k$-algebra $k\{X_1, \ldots, X_s\}$. In a slight abuse of notation, “$X_\ell$” will also denote its image in $R$, for $1 \leq \ell \leq s$.

By an $n$-dimensional representation of $R$ we will always mean a unital $k$-algebra homomorphism from $R$ into the $k$-algebra $M_n(\overline{k})$ of $n \times n$ matrices over $\overline{k}$. Representations $\rho, \rho' : R \rightarrow M_n(\overline{k})$ are equivalent if there exists a matrix $Q \in GL_n(\overline{k})$ such that

$$\rho'(X) = Q\rho(X)Q^{-1},$$

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for all $X \in R$.

We will say that the representation $\rho: R \to M_n(\overline{k})$ is irreducible when $\overline{k}\rho(R) = M_n(\overline{k})$ (cf. [1, §9]). Observe that $\rho$ is irreducible if and only if $\rho \otimes 1: R \otimes_k \overline{k} \to M_n(\overline{k})$ is surjective, if and only if $\rho \otimes 1$ is irreducible in the more common use of the term. (In particular, our approach below will use calculations over the computable field $k$ to study representations over the algebraically closed field $\overline{k}$.)

1.2. The existence of an $n$-dimensional representation of $R$ depends only on the consistency of a system of algebraic equations, over $k$, in $(t.n^2)$-many variables. Consequently, the existence of $n$-dimensional representations of $R$ is decidable (in principle) using Groebner basis methods. This idea is extended in [5] to give a procedure for deciding the existence of $n$-dimensional irreducible representations. On the other hand, possessing a nonzero finite dimensional representation is a Markov property, and so the existence – in general – of a finite dimensional representation of $R$ cannot be effectively decided, by [3].

We now state our main result; the proof will be presented in §2.

**Theorem.** Having at most most finitely many equivalence classes of irreducible $n$-dimensional representations is an algorithmically decidable property of $R$.

1.3. Assume that $k[x]$ is equipped with a factoring algorithm. If it has been determined that $R$ has at most finitely many equivalence classes of $n$-dimensional irreducible representations, these equivalence classes can (in principle) be effectively counted; see (2.9).

2. Proof of Theorem

2.1. (i) Set

$$B = k[x_{ij}(\ell) : 1 \leq i, j \leq n, 1 \leq \ell \leq s].$$

For $1 \leq \ell \leq s$, let $x_\ell$ denote the $n \times n$ generic matrix $(x_{ij}(\ell))$, in $M_n(B)$. For $g \in k\{X_1, \ldots, X_s\}$, let $g(x)$ denote the image of $g$, in $M_n(B)$, under the canonical map

$$k\{X_1, \ldots, X_s\} \xrightarrow{X_\ell \mapsto x_\ell} M_n(B).$$

Identify $B$ with the center of $M_n(B)$.

(ii) Let $\text{Rel}(M_n(B))$ be the ideal of $M_n(B)$ generated by $f_1(x), \ldots, f_t(x)$.

(iii) Let $\text{Rel}(B)$ denote the ideal of $B$ generated by the entries of the matrices $f_1(x), \ldots, f_t(x) \in M_n(B)$. Note that

$$\text{Rel}(B) = \text{Rel}(M_n(B)) \cap B.$$

(iv) Let

$$A = k\{x_1, \ldots, x_s\},$$

the $k$-subalgebra of $M_n(B)$ generated by the generic matrices $x_1, \ldots, x_s$. Set

$$\text{Rel}(A) = \text{Rel}(M_n(B)) \cap A.$$
2.2. Every \( n \)-dimensional representation of \( R \) can be written in the form

\[
R \xrightarrow{X_\ell \mapsto x_\ell + \text{Rel}(A)} \left( \frac{A}{\text{Rel}(A)} \right) \xrightarrow{\text{inclusion}} \left( \frac{M_n(B)}{\text{Rel}(M_n(B))} \right) \rightarrow M_n(\overline{k}),
\]

and every \( k \)-algebra homomorphism

\[
M_n(B)/\text{Rel}(M_n(B)) \rightarrow M_n(\overline{k})
\]

is completely determined by the induced map

\[
B/\text{Rel}(B) \rightarrow \overline{k}.
\]

For each representation \( \rho: R \rightarrow M_n(\overline{k}) \), let \( \chi_\rho: B \rightarrow \overline{k} \) be the homomorphism (with \( \text{Rel}(B) \subseteq \ker \chi_\rho \)) given by this correspondence.

2.3. Let \( T \) be the \( k \)-subalgebra of \( B \) generated by the coefficients of the characteristic polynomials of elements in \( A \). (Since the characteristic of \( k \) is zero, \( T \) is in fact generated by the traces, as \( n \times n \) matrices, of the elements in \( A \).) Set

\[
\text{Rel}(T) = \text{Rel}(B) \cap T.
\]

Note, when \( \rho, \rho': R \rightarrow M_n(\overline{k}) \) are equivalent representations, that the restrictions of \( \chi_\rho \) and \( \chi_{\rho'} \) to \( T \) will coincide.

2.4. Let \( \text{simple}_n(R) \) denote the set of equivalence classes of irreducible \( n \)-dimensional representations of \( R \). By (2.3) there is a well-defined function

\[
\Phi: \text{simple}_n(R) \rightarrow V(\text{Rel}(T)),
\]

where \( V(\text{Rel}(T)) \) denotes the \( \overline{k} \)-affine algebraic set of points on which the polynomials in \( \text{Rel}(T) \) vanish. It follows from [1, pp. 558–559] that \( \Phi \) is injective.

2.5. (i) Recall the \( m \)th standard identity

\[
s_m = \sum_{\sigma \in S_m} (\text{sgn } \sigma) Y_{\sigma(1)} \cdots Y_{\sigma(m)} \in \mathbb{Z}\{Y_1, \ldots, Y_m\}.
\]

If \( \Lambda \) is a commutative ring, then the Amitsur-Levitzky Theorem ensures that \( M_n(\Lambda) \) satisfies \( s_m \) if and only if \( m \geq 2n \); see, for example, [6, 13.3.2, 13.3.3].

(ii) Let \( S \) denote the finite subset of \( T (\subseteq B) \) comprised of

\[
\text{trace } \left( M_0 \cdot s_{2(n-1)}(M_1, \ldots, M_{2(n-1)}) \right),
\]

for all monic monomials \( M_0, \ldots, M_{2(n-1)} \), in the generic matrices \( x_1, \ldots, x_s \), of length less than

\[
p = n\sqrt{2n^2/(n-1) + 1/4 + n/2 - 2}.
\]

(The choice of \( p \) will follow from [7]; see [5, 2.2].) Let \( \rho: R \rightarrow M_n(\overline{k}) \) be a representation. It now follows from [5, §2] that \( \rho \) is irreducible if and only if

\[
S \not\subseteq \ker \chi_\rho.
\]

(Other sets of polynomials can be substituted for \( S \); see [5, 2.6vi,vii].)
2.6. (i) Set
\[ W = V(\text{Rel}(T)) \setminus V(S). \]
Combining (2.4) with (2.5ii), we obtain a bijection
\[ \Phi : \text{simple}_n(R) \rightarrow W. \]

(ii) Set
\[ J = \text{ann}_B \left( \frac{\text{Rel}(B) + B.S}{\text{Rel}(B)} \right), \quad \text{and} \quad I = J \cap T = \text{ann}_T \left( \frac{\text{Rel}(T) + T.S}{\text{Rel}(T)} \right). \]

A finite generating set for $J$ can be specified, using standard methods, and we can identify $T/I$ with its image in $B/J$. Since $V(I)$ is the Zariski closure of $W$, to prove the theorem it suffices to find an effective procedure for determining whether or not $T/I$ is finite dimensional. (When not indicated otherwise, “dimension” refers to “dimension as a $k$-vector space.”)

2.7. (i) For the generic matrices $x_1, \ldots, x_s$, set Trace =
\[ \{ \text{trace}(y_1 y_2 \cdots y_u) : y_1, \ldots, y_u \in \{x_1, \ldots, x_s\} \text{ and } 1 \leq u \leq n^2 \}. \]

In [9] (cf. [4, p. 54]) it is shown that $T = k[\text{Trace}]$. (A larger finite generating set for $T$ was established in [8].)

(ii) By (2.6ii), to prove the theorem it remains to find an algorithm for deciding whether the monomials in Trace ($\subseteq B$) are algebraic over $k$, modulo $J$. We accomplish this task using a variant of the subring membership test (cf., e.g., [2, p. 270]): Let $C$ be a commutative polynomial ring, over $k$, in $m$ variables. Let $L$ be an ideal – equipped with an explicitly given list of generators – in $C$. Choose $f \in C$. Observe that $f$ is algebraic over $k$, modulo $L$, if and only if $L \cap k[f] \neq \{0\}$. Next, embed $C$, in the obvious way, as a subalgebra of the polynomial ring $C' = k(t) \otimes_k C$. Observe that $L \cap k[f] \neq \{0\}$ if and only if 1 is contained in the ideal $(t-f).C' + L.C'$ of $C'$. Hence, the decidability of ideal membership in $C'$ implies the decidability of algebraicity modulo $L$ in $C$.

The proof of the theorem follows.

2.8. Roughly speaking, the complexity of the procedure described in (2.1 – 2.7) varies according to the degrees of the polynomials involved in deciding the algebraicity of Trace modulo $J$. Note, for example, that the degrees of the members of $S$ can be as large as $p^{2n-1}$, for $p$ as in (2.5ii).

2.9. Assume that it has already been determined that the number (equal to $|W|$) of equivalence classes of irreducible $n$-dimensional representations of $R$ is finite. Further assume that $k[x]$ is equipped with a factoring algorithm. We conclude our study by sketching a procedure for calculating – in principal – this number.

Set $D = T/I$, and identify $D$ with the (finite dimensional) $k$-subalgebra of $B/J$ generated by the image of Trace. Since $B/J$ can be given a specific finite presentation, finding
a $k$-basis $E$ for $D$ amounts to solving systems of polynomial equations in $B$, and this task can be accomplished employing elimination methods. Next, using the regular representation of $D$, and the finite presentation of $B/J$, we can algorithmically specify $E$ as a set of commuting $m \times m$ matrices over $k$, for some $m$. Furthermore, the nilradical $N(D)$ will be precisely the set of elements of $D$ whose traces, as $m \times m$ matrices, are zero. Consequently, we can effectively compute the dimension of $D/N(D)$. This dimension is equal to $|W|$.

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