Quasiperiodic tilings under magnetic field

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We study the electronic properties of a two-dimensional quasiperiodic tiling, the isometric generalized Rauzy tiling, embedded in a magnetic field. Its energy spectrum is computed in a tight-binding approach by means of the recursion method. Then, we study the quantum dynamics of wave packets and discuss the influence of the magnetic field on the diffusion and spectral exponents. Finally, we consider a quasiperiodic superconducting wire network with the same geometry and we determine the critical temperature as a function of the magnetic field.

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I. INTRODUCTION

For periodic systems such as the square, the triangular or the hexagonal lattice embedded in a magnetic field, it is well-known that commensurability effects between the characteristic magnetic length and the lattice period may lead to very interesting features. For example, if we consider a standard tight-binding model, the energy spectrum displays a complex multifractal structure when varying the magnetic flux per unit cell as illustrated by the famous Hofstadter butterfly [1, 2, 3]. However, this spectrum is strongly geometry-dependent since even for periodic tilings, some anomalies can appear leading to interesting quantum interference phenomena [4, 5]. This dependence has led several groups to analyze the interplay between the quasiperiodicity and the magnetic frustration [6, 7, 8, 9]. Nevertheless, in these studies, it is difficult to know the real contribution of the quasiperiodic order since the incommensurability between the tiles area is another source of aperiodicity in the spectrum [8].

We must indeed distinguish several classes of tilings depending on their structural order (periodic, quasiperiodic or disordered), their topology (trivial or nontrivial) and the ratio of the tiles area (rational or irrational). If we focus on quasiperiodically ordered structures, there is actually one type of tiling (those of codimension one) that has never been studied so far which consists in a nontrivial topology (several types of coordination numbers) with commensurate tile areas.

The aim of this work is to study such tilings by considering a modified version of the two-dimensional generalized Rauzy tilings (GRT) presented in Ref. [10] that are built with tiles having the same areas. Note that a great advantage of such systems is that all the physical quantities are periodic functions of the magnetic field so that we can restrict our analysis to a finite range of flux values.

This paper is organized as follows: in the next section, we briefly present the two-dimensional isometric generalized Rauzy tilings (iGRT) whose properties will always be compared to the square lattice ones. In Sect. II, we introduce the tight-binding Hamiltonian and we compute the energy spectrum of both tilings. We show that despite the quasiperiodic order, the spectrum of the iGRT has a fine nontrivial structure. To get more precise informations on the spectrum and the eigenstates, we compute in Sect. III the time evolution of wave packets for different fluxes per plaquette \( \phi \) and we focus on the average autocorrelation function and the mean square spreading. For the iGRT, a singular continuous spectrum and a sub-ballistic propagation is found for rational and irrational \( f = \phi / \phi_0 \) where \( \phi_0 \) is the flux quantum. By contrast, for the square lattice, the propagation is ballistic and the spectrum absolutely continuous for rational \( f \), whereas the spectrum is singular continuous and the spreading sub-ballistic for irrational \( f \). Sect. IV is devoted to the superconducting wire network with the iGRT geometry. After mapping the linearized Ginzburg-Landau equations onto a tight-binding model with nonconstant hopping terms, we compute the transition temperature as a function of the magnetic field and compare it with those of the square lattice. A cusp-like structure is clearly observed and shows the importance of the quasiperiodic order in this context.

II. THE ISOMETRIC GENERALIZED RAUZY TILINGS

The GRTs are codimension one quasiperiodic structures that can easily be built in any dimension \( D \) using the standard cut and project method [11]. These tilings have a complex topological structure with sites of coordination number ranging from \( D + 1 \) to \( 2D + 1 \). In this study, we focus on the two-dimensional GRT whose construction is based on the irrational solution \( \theta \approx 1.839 \) of the equation:

\[
x^3 = x^2 + x + 1.
\]  

Contrary to previous studies [11, 12, 13] concerning these tilings, we consider here an isometric version of the \( 2D \)
GRT where all the edge lengths are equal. This can be very easily done by projecting the selected sites of the 3D cubic lattice perpendicularly to the direction (1, 1, 1) instead of the Rauzy direction (1, θ, θ). Such a choice is motivated by the fact that since these tilings are codimension one tilings, equal edge lengths imply equal tile areas. This property which is only true for codimension one tilings, allows us to concentrate on the role played by the topological quasiperiodicity related to the connectivity matrix.

![FIG. 1: A piece of the 2D isometric generalized Rauzy tiling. All the tiles have the same area.](image)

III. HAMILTONIAN AND BUTTERFLIES

In this study, we consider a standard tight-binding Hamiltonian given by:

$$H = - \sum_{\langle i,j \rangle} t_{ij} |i\rangle \langle j|$$

where $|i\rangle$ is a localized orbital on site $i$. The hopping term $t_{ij} = 1$ if $i$ and $j$ are nearest neighbors and 0 otherwise. In the presence of a magnetic field $\mathbf{B}$, $t_{ij}$ is multiplied by a phase factor $e^{i\gamma_{ij}}$ involving the vector potential $\mathbf{A}$:

$$\gamma_{ij} = \frac{2\pi}{\phi_0} \int_{\gamma_{ij}} A_d l,$$

where $\phi_0 = \hbar c/e$ is the flux quantum. In the following, we only consider the case of a uniform magnetic field $\mathbf{B} = B\mathbf{z}$ which can be obtained, for example, with the Landau gauge $\mathbf{A} = B(0, x, 0)$, and we denote by $\phi = B a^2 \sqrt{3}/2$ the magnetic flux through an elementary tile. It is also useful to introduce the dimensionless parameter $f = \phi/\phi_0$ known as the reduced flux.

The study of the zero-field spectrum of $H$ has been presented in Ref. [11, 12, 13] for the nonisometric GRTs, but for $B = 0$ the tile areas is an irrelevant parameter since $H$ is, in this case, simply proportional to the connectivity matrix. The spectral measure is absolutely continuous, the spectrum is gapless, and the weakly critical eigenstates, are responsible for an anomalous diffusion. To obtain the spectrum of $H$ for $B \neq 0$, we have used the recursion method [15] also known as Lanczos algorithm [10]. This powerful tool allows one to compute the local density of states (LDOS) of a given initial state. The main advantage of this method is that we can easily compute this LDOS for very large system size (about $10^6$ sites), by diagonalizing only a small half-chain of a few hundred sites. However, it does just give a local information but, if one considers an initial random phase state, its LDOS gives a fairly good approximation of the total density of states. To compute these spectra we have computed 500 recursion coefficients in a circular cluster of 785757 sites for the square lattice, and 460 coefficients in a circular cluster of 767604 sites for the iGRT. We must also mention an important point concerning the edge states. Indeed, since we compute the recursion coefficients up to a given order, there are obviously some spurious states that arise due to the fact that the recurrence cluster is finite with open boundary conditions. To get rid of them, we have selected eigenvectors for which the amplitude on the initial state of the recursion was greater than a given threshold (here $10^{-3}$).

As it can be seen in Fig. 2, the spectrum of the square lattice obtained here with the recursion scheme is in very good agreement with those computed by Hofstadter in 1976 using the mapping onto the Harper equation [10]. As it can be readily seen, both spectra display the symmetry $E \leftrightarrow -E$ related to the fact that both tilings are bipartite. In addition, since there is only one characteristic area, the Hamiltonian is a periodic function of $f$ with period 1. Finally, the symmetry with respect to $f = 1/2$ just reflects the symmetry $t_{ij} \leftrightarrow -t_{ij}$. At small fields, the emergence of Landau levels, expected in the continuum limit, is clearly observed. In the square lattice, it is well known that rational values of $f = p/q$ lead to an absolutely continuous spectrum made up of $q$ subbands that can touch each other as for $f = 1/2$. Of course, nothing similar is expected for quasiperiodic tilings since no periodicity exists. Nevertheless, it is interesting to note that for the iGRT, there is still a complex gap structure with notably a large gap that can be followed continuously from $f = 0$ (where it vanishes) till $f = 1/2$. We can also see that the iGRT spectrum display a fine subband structure that would require a detailed but difficult analysis since even in the zero field case, it is nontrivial.

To go beyond this basic description, we have analyzed the nature of the spectrum for particular values of the reduced flux by studying the quantum dynamics of wave packets in both structures for remarkable values of the reduced flux.
For convenience, we have only considered rational values of \(0 < \phi/\phi < 1\) by approximating the evolution operator \(e^{-iHt}\). Practically, choosing a time \(t\) about one million sites. Contrary to exact diagonalizations, this method allows for the square lattice (top) and for the iGRT (bottom). Evidently, for a given initial state \(\langle \psi(0) \rangle\), to make the following approximation:

\[
|\psi(t + \Delta t)| = |\psi(t - \Delta t)| - 2i\Delta t |\hat{H}\psi(t)|. \tag{4}
\]

Contrary to exact diagonalizations, this method allows one to investigate the dynamics on very large systems (about one million sites). Practically, choosing a time step \(\Delta t = 0.05\) is sufficient to get a good accuracy and all our results have been obtained with this value. We have considered two quantities of interest: the averaged autocorrelation function

\[
C(t) = \frac{1}{t} \int_0^t dt' P(t') \sim t^{-\alpha}, \tag{5}
\]

where \(P(t) = \langle |\psi(0)| \psi(t)\rangle^2\) and the mean square spreading

\[
\Delta R^2(t) = \langle \psi(t)| \hat{R}^2 |\psi(t)\rangle - \langle \psi(t)| \hat{R} |\psi(t)\rangle^2 \sim t^{2\beta}, \tag{6}
\]

where \(\hat{R}\) is the position operator, the origin being taken at the center of the cluster. The spectral and diffusion exponents \(\alpha\) and \(\beta\) are defined from the long time behavior of \(C\) and \(\Delta R^2\) respectively. If \(\alpha = 1\), the spectrum is absolutely continuous, if \(\alpha = 0\), the spectrum is pure point, and if \(0 < \alpha < 1\) the spectrum is singular continuous. Concerning the diffusion exponent, \(\beta = 1\) corresponds to a ballistic propagation whereas \(0 < \beta < 1\) defines a sub-ballistic regime \((\beta = 1/2\) is the diffusive case).

For the square lattice, since all sites are equivalent, we have considered an initial state localized on a single (central) site. For the iGRT, the situation is more complicated since, as for the Octagonal tiling \([17]\), the exponents depends, a priori on the initial conditions as shown in Ref. \([14, 15]\). Of course, this is only true at short times, since in the asymptotic regime, we expect \(\alpha = \inf\{\alpha(E)\}, \beta = \sup\{\beta(E)\}\) for \(E\) in the spectrum. Here, we have built an initial random phase state localized on a circular cluster \((\sim 10^2\) sites) whose radius is about 1/100 of the cluster considered for the computation, and we have checked that our results were weakly sensitive to the precise random phase configuration chosen. However, we can never ensure that we have reached the asymptotic regime so that the exponents given here just give a qualitative idea of the wave packet spreadings.

We have considered several special values of the reduced flux: (i) \(f = 0\) for which the band width are maximum for both lattices; (ii) \(f = 1/2\) for which the band width of the iGRT is minimum; (iii) \(f = \sqrt{2}\) for which the band width of the square lattice is minimum; (iv) \(f = \theta\) which is a natural irrational number in this context. Our results are plotted in Fig.3 and Fig.4.

For the square lattice, everything is qualitatively known for these values. Indeed, when \(f\) is a rational number, the eigenstates are Bloch waves, the spectrum is absolutely continuous \((\alpha = 1)\) and the spreading is ballistic \((\beta = 1)\). For \(f = 0\), when starting from a single site, one can easily show that: \(P(t) = J_0(2\pi t)\) which behaves as \(t^{-2}\) at large time \([38]\) so that \(C(t) \sim t^{-1}\) and \(\Delta R^2(t) = 4t^2\). For irrational \(f\), the spectrum is known to be singular continuous \([1, 18, 21]\), and the propagation sub-ballistic \([23, 24]\). Nevertheless, we emphasize that existing results about the dynamics have been obtained by analyzing the Harper chain and do not reflect the dynamics on the square lattice. Indeed, the mapping of the square lattice onto the Harper chain results from a decoupling of the eigenstates given by \(\Psi(x, y) = \psi(x)\psi(y)\) which is due the special Landau gauge choice. Thus, the dynamics discussed in Refs. \([18, 21, 23, 24]\) only concerns the \(x\) direction (at fixed \(k_y\)) and thus do not give the exponent of the two-dimensional square lattice. For example, we have computed the dynamics for \(f = (1 + \sqrt{5})/2 = \tau\) and we have found \(\alpha = 0.50(1)\) in the square lattice instead of \(\alpha = 0.14(1)\) in the Harper chain \([18, 21]\).

Interestingly, for \(f = \sqrt{2}\), we have found a diffusive propagation with \(\beta = 0.50(1)\) whereas for \(f = \theta\), the problem is more complicated since there are clearly

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Energy spectrum as function of the reduced flux \(f = \phi/\phi_0\) for the square lattice (top) and for the iGRT (bottom). For convenience, we have only considered rational values of \(f = p/q\) for \(q \leq 37\).}
\end{figure}
\]
two regimes. If we consider only $t \gtrsim 250$ we obtain, $\beta = 0.94(2)$ (weakly sub-ballistic motion) but we cannot decide whether it is an intermediate regime or not. The exponents obtained by fitting our data are given in Table I.

For the iGRT, the situation is different since even for $f = 0$, the spectrum has some critical states responsible for a sub-ballistic spreading [10]. However, as shown in Ref. [12] the diffusion exponent is strongly energy-dependent and might even be very close to 1 at the band edges. In the asymptotic regime, a quasi-ballistic propagation should therefore be recovered. Contrary to the square lattice, a sub-ballistic spreading is found for $f = 1/2$ with $\beta = 0.61(1)$ which is not really surprising since even though $1/2$ is a rational number, the underlying quasiperiodicity of the iGRT does not allow for the construction extended eigenstates. In a sense, there are no commensurability effects for rational $f$. For $f = \sqrt{2}$, it is interesting to note that the propagation is faster in the iGRT ($\beta = 0.57(2)$) than in the square lattice $\beta = 0.50(1)$. This means that the quasiperiodic order can break the destructive quantum interference effects that slow the spreading down for incommensurate $f$ in periodic structures. Nevertheless, this is not always the case, since for $f = \theta$, the propagation is faster in the square lattice than in the iGRT. Concerning the nature of the spectrum, it is always found to be clearly singular continuous but for $f = 0$, we cannot be sure that $\alpha \neq 1$ in the asymptotic regime.

V. SUPERCONDUCTING WIRE NETWORKS

In this section, we analyze the role of the quasiperiodic order on the superconducting-normal transition temperature as a function of the magnetic field $T_c(f)$. As explained above, the main advantage of the iGRT is that all its elementary tiles have the same area. Thus, we can really analyze the role of the topological quasiperiodic order which has, in fact, never been investigated.

The study of quasiperiodic wire networks has been initially motivated by several experiments [24, 25, 26, 27] in which the superconducting transition temperature as a function of the magnetic field was determined. Using a mapping between the linearized Ginzburg-Landau equations and the tight-binding model, the rich structure of this phase boundaries has been then computed by Nori and coworkers [28, 29]. The main idea of this correspondence developed by Alexander [30] and De Gennes [31] is to integrate the LGL equations on each wire and to obtain a set of equations that determines the order parameter $\psi_i$ at each nodes $i$. This can be simply achieved using the continuity of the order parameter and the current conservation at each node. Denoting by $l_{ij}$ the distance between the (connected) nodes $i$ and $j$ and by $\xi$ the coherence length, the current conservation at each node
TABLE I: Spectral and diffusion exponents for the square lattice and for the iGRT.

|           | square lattice | iGRT |
|-----------|----------------|------|
| $f = 0$  | $0.99(1)$      | $0.99(1)$ |
| $f = 1/2$| $0.99(1)$      | $0.99(1)$ |
| $f = \sqrt{2}$ | $0.55(5)$ | $0.57(2)$ |
| $f = \theta$ | $0.50(1)$      | $0.60(1)$ |

$i$ expresses as:

$$-\psi_i \sum_{(i,j)} \cot(l_{ij}/\xi) + \sum_{(i,j)} \psi_j e^{i\gamma_{ij}} \sin(l_{ij}/\xi) = 0,$$

where the elementary flux quantum involved on the definition (8) of $\gamma_{ij}$ is here $\phi_0 = \hbar c/2e$. The coherence length depends on the temperature:

$$\xi(T) = \xi(0) \left( \frac{T_c(0)}{T_c(0) - T} \right)^{1/2},$$

where $T_c(0)$ is the transition temperature at zero field. For a given value of the magnetic field, the transition temperature is thus given by the highest temperature for which a nontrivial solution of (8) exists. If we now consider a system where all the lengths are equal $l_{ij} = l$, and if we set $\phi_i = \sqrt{z_i} \psi_i$ where $z_i$ is the coordination of the node $i$, Eq. (8) can be rewritten as [29]:

$$\cos(l/\xi) \phi_i = \sum_{(i,j)} (z_i z_j)^{-1/2} e^{i\gamma_{ij}} \phi_j.$$

Setting with $t_{ij} = -(z_i z_j)^{-1/2}$ and $E = \cos(l/\xi)$, we obtain the secular system given by the Hamiltonian (9) and thus the field dependence of the transition temperature:

$$1 - \frac{T_c(f)}{T_c(0)} = \frac{\xi(0)^2}{l^2} \arccos^2(E_0(f))$$

where $E_0(f)$ is the highest eigenvalue of this modified Hamiltonian for a reduced flux $f = 2\pi \phi/\phi_0$.

VI. CONCLUSION

We have studied the spectrum and the quantum dynamics of an electron in a quasiperiodic tiling embedded in a magnetic field. As in periodic structures, its spectrum displays a complex energy pattern when the magnetic field is varied. We have shown that for several values of the reduced flux $f$, the wave packet spreading is sub-ballistic with a field-dependent diffusion exponent. For $f = \sqrt{2}$, we have even found a propagation which is faster in the quasiperiodic tiling than in the square lattice for which the motion is, in this case, diffusive. It would be very interesting to study the influence of the disorder (topological, geometrical, or Anderson-like) on these exponents since, even in the zero-field case, nontrivial behaviours have already been observed.

We have also determined the superconducting-normal transition line $T_c(f)$ that displays a cusp-like structure revealing the importance of the quasiperiodic order. However, the analysis of the cusp positions clearly requires further investigations. It would be also important to study the transition line for codimension one random tilings that also have one type of tile but no quasiperiodic order. In this case, the absence of...
cusp would confirm the particular role played by the quasiperiodicity.

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[37] For example, the Penrose (codimension 3) and Octagonal (codimension 2) tilings have incommensurate tile areas but equal edge lengths.
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