Resonant Activation in Asymmetric Potentials

Alessandro Fiasconaro

Mark Kac Complex Systems Research Center and Marian Smoluchowski Institute of Physics
Jagellonian University, Reymonta 4, 30-059 Kraków, Poland,

and

Dipartimento di Fisica e Tecnologie Relative† and CNISM, Università di Palermo
Viale delle Scienze, I-90128 Palermo, Italy

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Abstract

The resonant activation effect (RA) has been well studied in different ways during the last two decades. It consists in the presence of a minimum in the mean time spent by a Brownian particle to exit from a potential well in the presence of a fluctuating external force, as a function of the mean frequency (or the correlation time) of the latter. This work studies the role played by the asymmetry of a piecewise linear potential in the RA effect, and, in general, the behavior of the mean first passage time and the mean velocity of the particle crossing through the potential barrier. A strong dependence on the asymmetry of the potential has been found which can be put in relationship with the current in the ratchet whose the potential here used is an elementary module. In this case a current reversal as a function of the frequency of the switching potential occurs. Comparison of the calculations with the Doering-Gadua model have been performed, as well as comparison with smooth symmetrical potentials, by checking for the robustness of the resonant correlation time. The calculations have been done by solving numerically the Langevin equation in the presence of an uncorrelated Gaussian noise. The resonant mean first passage times show an unexpected behavior as a function of the thermal noise intensity. The related curves present for the different symmetries an unexpected inversion of their relative behavior beyond a certain threshold value of the noise. This means that the current reversal can only occur for weak noise intensities, lower than that threshold value.

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In the recent past years various theoretical works have been produced around the concept of Resonant Activation (RA) [1 2 3 4 5 6 7], which consists in the presence of a minimum of the mean escape time from a potential well of a Brownian particle when the system is subjected to a randomly switching force, as a function of the mean switching quantity. The RA effect has been also detected experimentally [8 9 10] and it is in principle involved in a wide branches of science from physics to biology. The occurrence of the RA together with other stochastic effects such as noise enhanced stability (NES) [11 10 12] and stochastic resonance (SR) [13] have been also investigated [9 14]. The article by Doering & Gadua [1] is considered as one of the most introductory work to the resonant activation phenomenon. They introduced a switching piecewise linear potential in a range [0, L] with fixed minima in x = 0 and x = L, and fluctuating amplitude of the maximum. They report the mean escape time for a Brownian particle in the case of fluctuations of the maximum of the potential between V0 and 0 (flat potential), and also between V0 and −V0 (well instead of barrier), showing the presence of the RA effect. A slightly different choice was made by Bier & Astumian [2] who used a potential fluctuating between V0 − a and V0 + a with a < V0, maintaining so the presence of the barrier in all the dynamics. Both the methods show the resonant activation effect, and have been analytically evaluated in some approximation [8]. Both the choices have in common that the potential is symmetrical in shape and maintains the same value at the two extrema in all the dynamics (V(0) = V(L)).

Aim of this work is to focus on the role played by the asymmetry of the potential on the resonant activation effect using a simple piecewise linear potential.

Many papers with both experimental and analytical investigation concerning the role played by the asymmetry of the potential in stochastic effects have been published during the last years. However, the investigations have been mainly devoted to the effects on the Stochastic resonance phenomenon, that is the presence of a noise induced regular oscillations in a system, which is revealed by means of a maximum in the signal to noise ratio of the output [15 16 17 18 19], while the relation between the RA and the shape of the potential has been previously performed using a single slope linear potential [20].

Comparison with the Bier-Astumian model have been performed, as well as comparison with smooth symmetrical polynomial potentials. The results obtained have been extended to the most elementary ratchet potential.
The potential \( V_\pm(x, t) \) is here given as the sum of a static potential \( V(x) \) (with, again, \( V(0) = V(L) \)) plus an additional time-dependent \( U(x, t) \) giving the two configuration 'up' and 'down' between which \( V_\pm(x, t) \) takes its values. The additional potential \( U(x, t) \) has to be necessarily a stochastic process to give rise to RA. It can be a smooth, continuous potential like a cosine or, instead, a stochastic potential related to RA \([4, 6]\). It can be a smooth, continuous potential. We use it like a cosine or, instead, a stochastic potential related to RA \([4, 6]\). It can be a smooth, continuous potential.

\[
\dot{x} = -V'(x) + \eta(t) + \xi(t)
\]

where \( \xi(t) \) is the Gaussian white noise, with zero mean and correlation function \( \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') \). The intensity \( D \) is related to thermal bath and damping coefficient \( \gamma \) (here \( \gamma = 1 \)) by means of the relation \( D = \gamma k_B T \). The random force \( \eta(t) \) represents a dichotomous stochastic process, the random telegraph noise (RTN), taking the two values \( \{-a, a\} \) with an exponential correlation function \( \langle \eta(t)\eta(t') \rangle = (Q/\tau)e^{-(t-t')/\tau} \), where the intensity \( Q = a^2\tau \) and \( \tau \) is the correlation time of the process.

The potential \( V_\pm(x, t) \) is then defined as:

\[
V_\pm(x, t) = V(x) + U(x, t) = V(x) - x\eta(t).
\]  

(right) we see an example of the static potential with the asymmetry parameter \( k = -0.25 \).

The difference between the choice of the fluctuating potential here used \([Eqs. 2 and 3]\) and that by Bier-Astumian is visible in Fig. 1 where the potentials are drawn in the two cases. Here, with respect to the 'mean' static potential, the additional fluctuations \( -x\eta(t) \) give only two values ('up' and 'down' in fact) in the flipping, being the force \( \eta(t) \) uniform overall the \( x \)-range of the potential. In the Doering & Gradau model (as in the Bier-Astumian) two values of the force for each potential slope have to be considered to hold the minimum on the right at the same level \( V(L) = \text{const} \).

The potential \( V_\pm(x, t) \) here defined can be considered as a base modulus of the piecewise linear ratchet subjected to RTN widely used in literature \([21, 22, 23]\). The equation (1) has been solved numerically by using

![Figure 1: Piecewise linear potential used in the calculation. The position \( x_0 = 0 \) represents the starting point of the simulations and there is also put a reflecting boundary. With respect to Bier-Doering choice (left) the right extremum of the potential is not fixed (center) and the whole potential flips randomly between the two shapes \( \{V_+, V_-\} \) with a correlation time \( \tau \). The right draw shows an example of asymmetric piecewise linear static potential. The parameter \( k \) represents the position of the maximum \( x_m \) with respect to the position of the maximum in the symmetric case \( x_s \) (here \( x_s = 0.5 \), \( x_m = 0.25 \), \( k = -0.25 \)).](image1)

\[
V(x) = \begin{cases} 
& \frac{x}{L-x} & 0 \leq x < x_m \\
& \frac{L-x}{L-x_m} & x \geq x_m 
\end{cases}
\]

Here \( L = 1 \), \( h = 2 \), \( x_m = L/2 + k \), and \( k \) represents the asymmetry parameter, defined as the distance of the position of the maximum of the potential \( x_m \) from the position of the symmetrical maximum \( x_s \). In Fig. 2 MFPT showing the resonant activation effect for three values of the asymmetry parameter \( k \) of the piecewise linear potential \( \{k = -0.25, k = 0 \text{, such as symmetric potential, } k = 0.25\} \). The white noise intensity is here \( D = 0.18 \). The intensity of the dichotomous force is \( a = 1.2 \). The bottom/left inset show the mean velocity of the Brownian particle for the same asymmetries, as a function of the correlation time \( \tau \).

\[
dt = 10^{-3}
\]

and the averages have been performed over a sampling of \( N = 20,000 \) realizations. In the \( i \)-th realization the particle is put in the starting position \( x_0 = 0 \) and the time \( t_i \) to cross the position \( x = L \) is computed. A reflecting boundary is put in the left extremum of the potential while an absorbing boundary is present at the right extremum. The ensemble average of the \( t_i \) gives the Mean First Passage Time (MFPT), which presents, for all the cases here studied, the evidence of the RA effect, i.e. a well drawn minimum as a function of the correlation time \( \tau \). In fact, as well as the symmetric case, the MFPT obtained with the asymmetrical potentials show a resonant effect which is drawn in Fig. 2. We notice that for the three values of the asymmetry parameter \( k \), we find quite the same value of the resonant correlation time \( \tau_R \approx 10 \), but different values of the corresponding resonant MFPTs \( (T_R) \), which decrease by increasing the
We note that the resonant region shows an inversion in the behavior of the MFPT curves for the three potentials to both the low and high correlation times with respect to the intermediate one. In fact for \( \tau \) lower than \( \tau_{CL} \approx 10^{-1} \) the curves show a MFPT higher for positive asymmetry (\( k = 0.25 \)) and lower for negative asymmetry (\( k = -0.25 \)) and the same qualitative behavior is visible in the long correlation time region \( \tau \) higher than \( \tau_{CR} \approx 10^{3} \). In the intermediate region \( \tau \in [\tau_{CL}, \tau_{CR}] \), where we also find the resonant values, the situation is inverted: the highest \( T_{R} \) value corresponds to the negative asymmetry parameter and the lowest \( T_{R} \) to the positive one.

On the other hand, calculation performed with the Bier-Astumian and Doering & Gadua model, that is using fixed extrema of the same asymmetric piecewise linear potentials, which fluctuates between the same highs (\( a = 0.6 \)) in all the asymmetries, give strongly different curves, as visible in Fig. 3, where we can see even a strong displacement of the resonant correlation time by changing the asymmetry, but no crosses are present between the MFPT curves. The inversion of the MFPTs curves behavior, and consequently the presence of the two crosses at approximatively \( \tau_{CL} \) and \( \tau_{CR} \), is so uniquely present in MFPTs calculated for asymmetrical potentials using uniform fluctuating force overall the range \([0, L] \), and it does not appear neither in the symmetrical ones with different shapes (See Fig. 4), nor in the asymmetrical ones with fixed extrema and fluctuating barriers (Fig. 3). In other words, the comparison between the results plotted in Figs. 2 and 3 put in evidence that the cross features of the MFPT curves occurs not merely because of the asymmetry of the potentials, but, instead, because of the presence of the asymmetry together with the uniformity in space of the fluctuating external force \( \eta(t) \) added to the system.

The main relevant feature in adding a uniform force in the range of the constant potential, lies in the fact that in this case the barrier high of the fluctuating potential takes different values for different positions of the maximum, i.e. as a function of the asymmetry parameter \( k \). In fact the resonant MFPT values \( T_{RS} \) depend mainly by the lower value taken by the potential (\( V(\cdot) \)), being proportional to \( (1/\sqrt{2})e^{V(\cdot)/D} \), and this value becomes lower and lower, by increasing the value of \( ks \). This means that at a first sight we can expect that the \( T_{R} \) values take a lower value for the positive asymmetry than for the negative ones. However, as we can see below in the text, this expectation holds only up to a certain threshold value of noise intensity \( (D_{T}) \) and the inverse behavior occurs for higher values \( (D > D_{T}) \).

The model here investigated presents interesting features in the MFPT: first of all it has a value of the resonant mean period \( \tau_{R} \) not too strongly dependent on the asymmetry parameter \( k \); then, it presents two period intervals, close to \( \tau_{CL} \approx 10^{-1} \), and close to \( \tau_{CR} \approx 10^{3} \) having approximatively the same MFPT for all the \( k \) parameters.

![Figure 3: MFPT showing the resonant activation effect for different asymmetrical piecewise linear potential with fixed extrema and equal potential excursion in the three cases. Differently from the results in Fig. 2 we don’t find any cross between the three curves and also the resonant correlation time \( \tau_{R} \) changes for the different asymmetries. The potential increase is here \( a = 0.6 \), as the one of the symmetric case in Fig. 2.](image)

![Figure 4: Resonant activation evolution for various noise intensities. We observe the disappearing of the crossings in the MFPT curves and the shift of the minima by increasing the noise intensity \( D \). These behaviors are shown in details Fig. 4 where the resonant frequency and resonant times are plotted as a function of \( D \).](image)
The presence of a resonant behavior, as well as the cross value at $\tau_{C_l}$, is also found in the plot of the mean velocity of the Brownian particle. Left inset of Fig. 2 shows, in fact, this measure as a function of the correlation time of the fluctuating dichotomous force, calculated as $\bar{v} = N^{-1} \sum_{i=1}^{N} L/t_i$. For all the asymmetry parameters, we see the presence, before the saturating behavior, of a weak maximum which corresponds to the resonant correlation time $\tau_R$. We can also see that for low values of the correlation times ($\tau < \tau_{C_l}$) the mean velocity is higher for negative asymmetry and lower for positive ones, while for ($\tau > \tau_{C_l}$) is the inverse. This feature gives rise to a reversal current in the ratchet, as predicted in other works [24, 25, 26] and whose occurrence has been also demonstrated experimentally [27].

In fact the difference between the mean velocities of the positive asymmetry and the negative one change sign at the $\tau_{C_l}$ value. In an asymmetrical ratchet this difference represents a net velocity flux, provided that the absence of any reflecting boundary in that case gives rise to changes in the values of the velocity. Both the presences of a maximum for $\tau \approx \tau_R$ and the cross at $\tau \approx \tau_{C_l}$ are in total agreement with the behavior of the MFPT.

This agreement fails, instead, for values of the correlation times higher than $\tau_R$. While the MFPT curves increase in a different way and join together at the second cross, the velocities decrease only a few, reaching a saturation value. This is because for high values of the correlation times, the particle tends to cross the potential barrier when it is in its lower high, so acquiring a relatively high speed because of the low travelling time. When the potential is in the high level, the particle takes a longer time to cross and so the contribution to the mean velocity becomes very low and relatively negligible. This means that, for high correlation times, $\bar{v}$ maintains a relatively high value which doesn’t change so strongly as the MFPT.

The results found above for the single barrier potentials, mirrors, of course, to the ratchet potential having the same asymmetric profile as elementary module. In
this respect a set of calculations has been performed with the aim to join together the results of the single barrier described above with the simplest ratchet case, such as a ratchet with two barriers only. Fig. 6 shows the results in such a case and the bottom/right inset shows the corresponding elementary ratchet. The system consists of two asymmetric barriers without the presence of any reflecting boundary. The MFPT presents again a resonant correlation value \( \tau_R \) which is the same for the single barrier case, as we can expect. In this system the MFPT is the mean time spent by the Brownian particle starting at \( x = 0 \) to reach the position \( x = 1 \) or \( x = -1 \), differently. The particle, of course will follow the easiest path, and the MFPT represents the minimum time of the two single barrier case seen above. This also means that the curve is lowered and the RA effect less pronounced. The mean velocity, plotted in the upper-left inset of the Fig. 6 shows again a maximum at the same resonant value \( \tau_R \). For very low correlation time the mean velocity has a weak negative velocity (right/top inset in Fig. 6). This means that a current reversal appears at a certain correlation time \( \tau_{rev} \). This features follows from the different behavior of the mean velocity in the two specular asymmetric single barrier potentials seen above (inset of Fig. 2), where the presence of the cross value \( \tau_{CL} \) indicates a current reversal as a function of \( \tau \). The difference in value between \( \tau_{rev} \) and \( \tau_{CL} \) as well as the difference in the absolute value of the mean velocity of the Brownian particle, have to be imputed to the presence of the reflecting boundary in the single barrier case which change the traveling times of the particle and, so, the related mean velocities.

As a last remark concerning the relationship between the resonant activation effect and the shape of the potential, some calculations have been performed using symmetrical smooth potentials. The static potentials used have the form:

\[
V_{2N}(x) = h 2^{2N} \frac{x^N}{N} \left(1 - \frac{x}{N}\right)^N
\]

where in our calculation \( h = 2 \). The values used are: \( N = 1, 2, 3 \), such as parabolic, quartic and 6th power potentials. As we can see in Fig. 7, the resonant mean time is quite the same for all the cases, again confirming that \( \tau_R \) is a robust value in the model investigated. Another remarkable and well visible feature is that the four curves of the MFPT differ each other of a constant quantity, at least for \( \tau \lesssim \tau_R \). This means that, in that region, their logarithmic distance is constant and so an exponential form factor has to be taken into account in order to estimate the MFPT for each potential shape.

Summarizing the results, the shapes of the potential (both symmetrical and asymmetrical ones) play a very important role in the evaluation of the RA effect and MFPT behaviors. With a spatially uniform random telegraph force, the resonant correlation time \( \tau_R \) appears to be a robust value independently on that shape, while this latter acts always in a strong way by modifying the resonant values of the mean first passage times \( T_{R8} \). In the context of uniform forces, the asymmetry of the potential is then responsible for the crosses of the MFPT curves in a certain range of low thermal noise intensities, giving an explanation for the appearance of the current reversal as a function of the correlation time of the fluctuating force in ratchet potentials. These crosses, and, consequently the current reversal in ratchet, are only present at weak noise intensity, as indicated by the presence of an upper noise intensity threshold \( D_T \).

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