Quantum Intermittency in Almost-Periodic Lattice Systems Derived from their Spectral Properties

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This paper is dedicated to the memory of Professor Joseph Ford,
teacher, original researcher, and founder of Physica D

Abstract

Hamiltonian tridiagonal matrices characterized by multi-fractal spectral measures in the family of Iterated Function Systems can be constructed by a recursive technique here described. We prove that these Hamiltonians are almost-periodic. They are suited to describe quantum lattice systems with nearest neighbours coupling, as well as chains of linear classical oscillators, and electrical transmission lines.

We investigate numerically and theoretically the time dynamics of the systems so constructed. We derive a relation linking the long-time, power-law behaviour of the moments of the position operator, expressed by a scaling function \( \beta \) of the moment order \( \alpha \), and spectral multi-fractal dimensions, \( D_q \), via \( \beta(\alpha) = D_{1-\alpha}. \) We show cases in which this relation is exact, and cases where it is only approximate, unveiling the reasons for the discrepancies.

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1 Introduction

Usually, the study of almost/quasi-periodic systems starts by assigning a suitable rule for building a quantum Hamiltonian operator, and then proceeds to the determination of its spectral quantities \[ \text{(which is frequently a hard task)} \] and of the time dynamics it generates. In so doing, multi-fractal energy spectra have been frequently observed, and anomalous characteristics of the time evolution have been exhibited \[ 2, 3, 4, 5, 6. \]

These findings raise the question if multi-fractal spectra are typical in almost/quasi-periodic systems \[ 7 \], and, vice versa, if almost/quasi-periodicity is always associated with singular continuous spectral measures. The relations between this pair (spectral
multi-fractality and Hamiltonian almost-periodicity) and the time dynamics generated
via Schrödinger’s equation are also interesting, and intricate: do the former always imply
anomalous scaling relations of physical quantities like, for instance, the expectation value
of the position operator? Can we make any quantitative statement to this effect?

In this paper, we employ a new algorithm for deriving a Hamiltonian operator (called
a Jacobi matrix because of its mathematical nature) with a pre-assigned spectral measure
in the vast class of Iterated Function Systems (I.F.S.). This technique provides us with an
ideal patient for our surgical table, who can be fully dissected and analyzed. In particular,
we provide evidence that Jacobi matrices associated with I.F.S. are almost periodic, and
we argue that this is likely to be the typical case in a large class of measures with fractal
support.

The analysis which can be carried out in this example permits us to compute exactly
the asymptotic behaviour of the wave-function projections, for short and long times. By
introducing a renormalization approach in the theory of orthogonal polynomials, we also
derive a relation linking the asymptotic power-law growth of the moments of the position
operator and multi-fractal generalized dimensions. This theory explains the phenomenon
that we have termed quantum intermittency.

The specific properties of I.F.S. are crucial for our theory; Yet, since particular I.F.S.’s
can be found so to approximate arbitrarily well (in a technical sense) any “fractal”
measure [8, 9, 10, 11], the results obtained in the I.F.S. class may have a much wider
generality.

We shall present our results as follows: in Section II we introduce the general formal-
ism of I.F.S. and of Jacobi matrices, employed to solve the inverse problem of finding
an Hamiltonian with a given spectrum. This formalism is then applied in Sect. III to
derive a stable solution algorithm. The almost-periodic properties of the Hamiltonian so
determined are studied numerically in Sect. IV, and the intermittent quantum dynamics
it generates is then discussed in Sect. V and VI. The Conclusions summarize the work
and present some previews on further investigations.

2 I.F.S. and their Jacobi Matrices

Systems of linear iterated functions [12, 13, 14, 15] are finite collections of maps

\[ \phi_i(x) := \delta_i x + \beta_i, \quad i = 1, \ldots, M, \]  

(1)

where \( \delta_i, \beta_i \) are real constants, and where the contraction rates \( \delta_i \) have modulus less than
one. Without loss of generality, we may assume that each \( \phi_i \) maps \([0, 1]\) into itself, and
that \( \phi_1(0) = 0 \).

A probability, \( \pi_i \), is associated with each map: \( \pi_i > 0, \sum \pi_i = 1 \). Employing these
probabilities, a measure over \([0, 1]\) can be defined as the unique positive measure satisfying
the balance property

\[ \int_0^1 f \ d\mu = \sum_{i=1}^M \pi_i \int_0^1 (f \circ \phi_i) \ d\mu, \]  

(2)

for any continuous function \( f \). This measure is supported on \( A \), the subset of \([0, 1]\) which
solves the equation

\[ A = \bigcup_{i=1,\ldots,M} \phi_i(A), \]  

(3)

2
The set $A$ is invariant under the action of shrinking it to smaller copies of itself, and gluing them together. Because of eq. (3), the geometry of this set is typically fractal (except for special choices of the map parameters); In turn, the balance relation (2) is responsible for the multi-fractal properties of the measure $\mu$. In fact, let us consider a disconnected I.F.S., that is to say, one for which the sets $\phi_i(A)$ do not intersect each other. Under these circumstances, the multi-fractal properties of the balanced measure are easily computable: the spectrum of generalized dimensions $D_q$ follows from the equation

$$\sum_{j=1}^{M} \pi_j^q \delta_j^{-\tau} = 1,$$

whose unique real solution defines $\tau$ as a function of $q$, and leads to $D_q = \frac{\pi(q)}{q-1}$. In virtue of this relation, one can tune the map parameters to obtain various multi-fractal spectra.

The problem of determining a Hamiltonian possessing $\mu$ as spectral measure can be solved [16] considering the set of associated orthonormal polynomials, $\{p_n\}$:

$$\int p_i(x) p_k(x) \, d\mu(x) = \delta_{i,k}. \tag{5}$$

In fact, any such set of polynomials is characterized by a three-terms recurrence relation which can be written

$$xp_j(x) = r_{j+1} p_{j+1}(x) + A_j p_j(x) + r_j p_{j-1}(x), \tag{6}$$

or, in matrix form

$$H p(x) = xp(x). \tag{7}$$

In the above, $p(x)$ is the vector whose components are the orthonormal polynomials evaluated at site $x$, and $H$ is the Jacobi matrix, which is constructed as the real, symmetric, tridiagonal matrix whose diagonal and outer diagonals are the vectors $A_j$ and $r_j$, respectively:

$$H_{i,i} = A_i, \quad H_{i+1,i} = H_{i,i+1} = r_{i+1}, \quad i = 0, 1, \ldots. \tag{8}$$

$H$ defines a nearest neighbours lattice system, with site energies $A_i$ and hopping constants $r_i$. Similarly, $H$ can describe a linear array of masses coupled by springs, and also an electrical transmission line, whose characteristics vary from one element to the next.

Standard theory proves that, letting the Jacobi matrix $H$ act in $l_2$ (the space of square summable sequences, whose canonical basis will be indicated by $\{e_0, e_1, \ldots\}$) the spectral measure of $H$ with respect to the vector $e_0$ (the local density of states of physical jargon) is precisely $\mu$: in fact, one has

$$(e_0, g(H)e_0) = \int g(x) \, d\mu(x), \tag{9}$$

for well-behaved functions $g$. This is the theoretical solution of the inverse problem we have proposed. In order to translate it into a practical solution, we need to compute the Jacobi matrix coefficients starting from the measure $\mu$, i.e. from the map parameters defining the I.F.S.
3 A Stable Technique for Computing I.F.S. Jacobi Matrices

The problem of constructing the Jacobi matrix associated with I.F.S. measures is hard, and the usual techniques of polynomial sampling \[17, 18, 19\] are plagued by exponentially increasing errors which allow only computation of very few Jacobi matrix coefficients \[20\]. Alternatively, the sole technique available so far has been an algebraic procedure programmed in MAPLE by Vrscay \[21\]. Yet, it is severely limited by memory and time requirements. To overcome these difficulties we have devised a direct algorithm applicable to I.F.S. measures.

We first observe that, for any \(n\),

\[ p_n(\phi_i(x)) = \sum_{l=0}^{n} \Gamma_{i,l}^{n} p_l(x), \quad i = 1, \ldots, M. \quad (10) \]

This is immediate, since \(p_n(\phi_i(x))\) is an \(n\)-th degree polynomials which can be expanded on the first \(n\) orthogonal polynomials. Less immediate is to derive a recursive rule for the coefficients \(\Gamma_{i,l}^{n}\), \(l = 0, \ldots, n\). It turns out that, at fixed \(n\), they can be determined from the map parameters, and from the Jacobi matrix entries \(A_j\), for \(j = 0, 1, \ldots, n - 1\), and \(r_m\), for \(m = 0, 1, \ldots, n\). In fact, (dropping for simplicity the map index \(i\)) we have that

\[ p_0(\delta x + \beta) = p_0(x), \quad \text{and hence} \quad \Gamma_{0}^{0} = 1. \]

Suppose now that \(\Gamma_{k}^{k}\) is known for \(k = 0, \ldots, n - 1\) and all relative \(i\)'s: from eq. \(8\) we obtain the complete decomposition of \(\Gamma_n^{(\delta x + \beta)}\) over \(p_l\), \(l = 1, \ldots, n\):

\[ r_n p_n(\delta x + \beta) = (\beta - A_{n-1}) \sum_{l=0}^{n-1} \Gamma_l^{n-1} p_l(x) + \delta \sum_{l=0}^{n-1} \Gamma_l^{n-1} (r_{l+1} p_{l+1}(x) + A_l p_l(x) + r_l p_{l-1}(x)) + \]

\[ -r_{n-1} \sum_{l=0}^{n-2} \Gamma_l^{n-2} p_l(x). \quad (11) \]

Equation \(11\) allows now the determination of the coefficients \(\Gamma_{i,l}^{n}\).

We observe that the highest order polynomial, \(p_n\), appears twice in the above equation, always in the form of the product \(r_n p_n\): hence, the coefficients in the expansion of the polynomial \(r_n p_n\) can be determined without knowing \(r_n\). Therefore, if we let \(\tilde{p}_n(x) = r_n p_n(x)\), a second decomposition can be written as

\[ \tilde{p}_n(\phi_i(x)) = \tilde{\Gamma}_{i,n}^{n} \tilde{p}_n(x) + \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,l}^{n} p_l(x), \quad (12) \]

where the coefficients \(\tilde{\Gamma}\) can be computed recursively from eq. \(11\), on the basis of the knowledge of only \(A_j, r_j\), for \(j = 0, 1, \ldots, n - 1\).

We can now compute the non-diagonal entries of the Jacobi matrix: from eq. \(8\) we write

\[ r_n^2 = \int \tilde{p}_n(x) x p_{n-1}(x) \, d\mu. \quad (13) \]

Hence, using the balance property \(3\) and eqs. \(10, 12\) this becomes

\[ r_n^2 = \sum_{i=1}^{M} \pi_i \int (\delta_i x + \beta_i) \left[ \sum_{m=0}^{n-1} \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,m}^{n,m} \Gamma_{i,l}^{n-1} p_m(x) p_l(x) + \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,n}^{n,n} \Gamma_{i,l}^{n-1} \tilde{p}_n(x) p_l(x) \right] \, d\mu. \quad (14) \]
Again, we can use the recurrence relations (8), to get

\[ r_n^2 = \sum_{i=1}^{M} \pi_i (B_i + C_i + D_i), \]  

(15)

where we have put:

\[ B_i = \sum_{l=0}^{n-1} (\beta_i + \delta_i A_l)\tilde{\Gamma}^n_{i,l}\Gamma^{n-1}_{i,l}, \]  

(16)

\[ C_i = \delta_i \sum_{l=0}^{n-2} r_{l+1}(\tilde{\Gamma}^n_{i,l}\Gamma^{n-1}_{i,l+1} + \tilde{\Gamma}^n_{i,l+1}\Gamma^{n-1}_{i,l}), \]  

(17)

and

\[ D_i = \delta_i \tilde{\Gamma}^n_{i,n}\Gamma^{n-1}_{i,n-1}r_n^2. \]  

(18)

Because of contractivity of the maps, \(|D_i|r_n^{-2} < 1\). Therefore, \(r_n^2\) (and hence \(r_n > 0\)) can be computed from eq. (15), on the basis of the knowledge of the coefficients in the expansions (14) of order \(n - 1\), of order \(n\) in (12), of the map parameters, and of the matrix entries \(A_j, r_j\), for \(j = 0, 1, \ldots, n - 1\).

A similar trick allows the computation of the diagonal entries \(A_n\); We use eqs. (2) and (3) (integrals are taken with respect to \(\mu\)):

\[ A_n = \int x p_n^2(x) = \sum_{i=1}^{M} \pi_i \int (\delta_i x + \beta_i) p_n^2(\delta_i x + \beta_i) = \sum_{i=1}^{M} \pi_i \int (\delta_i x + \beta_i) \sum_{m,l=0}^{n} \Gamma^n_{i,l} \Gamma^n_{i,m} p_l(x)p_m(x). \]  

(19)

Using the orthonormality properties of the sequence \(p_n\), and the recurrence relation, eq. (3), we get

\[ A_n = \sum_{i=1}^{M} \pi_i \left[ \sum_{m=0}^{n} (\Gamma^n_{i,m})^2 (\beta_i + \delta_i A_m) + \sum_{m=0}^{n-1} \Gamma^n_{i,m} \Gamma^n_{i,m+1} \delta_i (r_m + r_{m+1}) \right], \]  

(20)

thereby determining \(A_n\) as a function of the coefficients in eq. (10) of order \(n\) fixed, of the map parameters, and of the matrix entries \(A_j, r_j\), for \(j = 0, 1, \ldots, n - 1\), and \(r_m\), for \(m = 0, 1, \ldots, n\).

These results can be properly chained into an iterative construction of the Jacobi matrix \(H\): The algorithm is structured as follows:

- **Initialization.** At the first step, we have \(A_0 = \mu_1, r_0 = 0, \Gamma_0 = 1\). The first order moment of \(\mu, \mu_1\), can be simply computed from eq. (2).

- **Iteration.** Suppose that \(A_l, r_l, \Gamma^l\) are known for \(l = 0, 1, \ldots, n - 1\). Then we:
  - **Compute** \(\tilde{\Gamma}^n\). We use equations (11 - 12).
  - **Compute** \(r_n\). We use eqs. (13 - 15).
  - **Compute** \(\Gamma^n\). This is immediate at this stage.
  - **Compute** \(A_n\). We use eqs. (19 - 20). Then we iterate the procedure.
Graphically:

\[
\left( \begin{array}{cc}
\Gamma_{n-1}, & r_0, \ldots, r_{n-1} \\
A_0, \ldots, A_{n-1}
\end{array} \right) \Rightarrow \tilde{\Gamma}^n \Rightarrow r_n \Rightarrow \Gamma^n \Rightarrow A_n \Rightarrow \left( \begin{array}{cc}
\Gamma^n, & r_0, \ldots, r_n \\
A_0, \ldots, A_n
\end{array} \right)
\] (21)

In a separate work \cite{21} we have analyzed the reasons of the failure of classical polynomial sampling \cite{17, 19} when applied to singular measures, and assessed the numerical stability of the recursive algorithm presented above. We have observed a polynomial error propagation with respect to matrix order for the recursive algorithm, while using the classical algorithms the error growth was found to be exponential.

4 Almost Periodicity of I.F.S. Jacobi Matrices

Having devised a stable solution of the Hamiltonian inverse problem, we can study the properties of large Jacobi matrices. Fig. 1 shows an I.F.S. measure, one of its orthogonal polynomials, and the beginning of the sequence of \(r_n\) coefficients. Let us focus our attention on the last.

We can clearly observe a zero frequency (the average value), a \(\pi\) frequency (flipping up and down), and clearly other frequencies are present in the sequence. A Fourier analysis is simply effected writing

\[
r_n = \sum_k F_k e^{i\omega_k}.
\] (22)

This sum may not converge in the usual sense, and it might have to be replaced by an integral in the case of a continuous component in the “spectrum” of the sequence \(r_n\). If the continuous component is absent, the system is almost-periodic. Within this case, if the set of frequencies \(\omega_k\) can be derived from a finite set of periods, the sequence \(r_n\) is quasi-periodic: that is, this is the case if there exist suitable \(\Omega_1, \ldots, \Omega_p\) such that for all \(k\) the frequency \(\omega_k\) can be written \(\omega_k = n_1\Omega_1 + \ldots + n_p\Omega_p\), for integer \(n_1, \ldots, n_p\).

A numerical, fast Fourier analysis of the sequence \(r_n\) is presented in Fig. 5, where peaks in the distribution of \(|F_k|^2\) with a clear hierarchical structure are observed. These peaks seem to suggest the presence of a point component in the spectrum of this sequence. Yet, care has always to be exerted to assess this fact numerically. To obtain a further piece of evidence we performed an analysis of the phase of \(F_k\) around these peaks, like that shown in Fig. 2, and found a \(\pi\) discontinuity, which indicates \cite{22} that they are indeed related to a point component. The sequence \(r_n\) is therefore almost periodic.

Since no simple rational relation among the peak sequences seems to hold, numerical evidence seems to suggest that the sequence is not quasi-periodic. Our numerical investigations have shown that these characteristics are typical in the class of Hamiltonian associated with I.F.S. measures, supported on Cantor sets. In view of the approximation properties of I.F.S. measures, this result is likely to be much more general: indeed, in the family of Jacobi matrices associated with real Julia sets \cite{23, 24}, which can be well approximated by I.F.S., limit periodicity of the sequence \(r_n\) has been proven directly \cite{25}. The problem of a formal proof is therefore open.
5 Quantum Dynamics of Almost Periodic Lattice Systems

Jacobi matrices generate a quantum dynamics in $l^2$ via Schrödinger’s equation,

$$i \frac{d\psi}{dt} = H\psi, \quad \psi(0) = e_0 := (1, 0, \ldots). \quad (23)$$

The initial state of the evolution, $e_0$, is the zeroth lattice state. In oscillator terms, this corresponds to a situation where the first mass is displaced from its equilibrium position, while all the other masses are at rest in their equilibria. In electrical terms, the current (or the voltage) is non-zero only in the first element of the transmission line described by the Jacobi matrix $H$.

The solution of Schrödinger equation can be formally obtained as \[26, 27\]

$$c_n(t) := (e_n, e^{-itH}e_0) = \int e^{-itx}p_n(x) \, d\mu(x), \quad (24)$$

where $c_n(t)$ is the component of $\psi(t)$ at the $n$-th lattice state. Equation (24) shows that this component is the Fourier transform of the orthogonal polynomial $p_n$ with respect to the spectral measure $\mu$. This fact allows us to derive important results.

Firstly, the asymptotic behaviour for small $t$ can be controlled as follows: $|c_n(t)|^2 \sim t^{2n}$. In fact,

$$c_n(t) = \int d\mu(x) \, p_n(x) \sum_{l=0}^{\infty} \frac{(-it)^l}{l!} x^l = \sum_{l=n}^{\infty} \frac{(-it)^l}{l!} \int d\mu(x)p_n(x)x^l. \quad (25)$$

Because of the orthogonality properties of the set $p_n$ this expansion begins with $l = n$, which proves the result.

Secondly, in the infinite time limit, denoting by $\overline{S}_n(T)$ the time average of $|c_n|^2$ up to time $T$,

$$\overline{S}_n(T) = \frac{1}{2T} \int_{-T}^{T} |c_n|^2(t) \, dt,$$

we have that

$$\overline{S}_n(T) \sim T^{-D_2} \quad (26)$$

for all $n$, a result which involves the correlation dimension $D_2$ of the fractal measure $\mu$. The case with $n = 0$ is implicitly contained in Bessis et al. \[28\], and was originally proposed in the present context by Ketzmerick et al. \[4\]. Successively, it has attracted a lot of attention, mainly from cultors of mathematical rigour. Our generalization has the advantage of requiring a simple proof, via the usage of the Mellin transform, as in \[28\]. In fact, we write

$$\overline{S}_n(T) = \int d\mu(x) \int d\mu(y) \sin \frac{(x-y)T}{(x-y)T} p_n(x)p_n(y). \quad (27)$$

To find the asymptotic behaviour of eq. (27), we take the Mellin transform, $M_n(z)$, of $\overline{S}_n(T)$:

$$M_n(z) = \int T^{z-1} \overline{S}_n(T) dT = G \times \int d\mu(x) \int d\mu(y) \frac{p_n(x)p_n(y)}{|x-y|^z} = G(z) \times E_n(z), \quad (28)$$
where \( G(z) = \Gamma(z - 1) \sin \left( \frac{\pi}{2}(z - 1) \right) \), and where \( E_n(z) \) is defined implicitly by the last equality. The dominating power law in the long time behaviour of \( S_n \) is determined by the divergence abscissa of \( M_n(z) \): that is to say, \( \Sigma_n(T) \sim T^{-w} \), where \( w \) is the largest real \( z \) for which \( M_n(z) \), hence \( E_n(z) \) converges. It is apparent from eq. (28) that the divergence of \( E_n \) is piloted by the small scale structure of the measure \( \mu \). Because the polynomials \( p_n \) are smooth functions, with bounded derivatives on the support of \( \mu \), the divergence abscissa of \( E_n \) is the same for all \( n \), and, in particular, it coincides with that of \( E_0 \). \( E_0(z) \) is known as the generalized electrostatic energy of the measure \( \mu \) and its divergence abscissa is known to be \( D_2 \) \([28]\), the correlation dimension of the measure \( \mu \).

It is important to remark that the domains of validity of the asymptotic expansions just derived are not uniform in \( n \). This adds to the difficulty of the problem to be discussed in the next Section.

### 6 Renormalization Theory of Quantum Intermittency

An important characteristic of the quantum motion introduced in the previous section is the way it spreads over the \( l_2 \) lattice basis, \( \{e_n\} \). In fact, in oscillator terms, spreading corresponds to energy transmission along the linear chain, be it mechanical or electrical. In quantum mechanical terms, it corresponds to unbounded motion of the lattice particle, of the kind treated only qualitatively by R.A.G.E. theorems. To gauge this phenomenon, we define the moments of the position operator \( \hat{n} \):

\[
\nu_\alpha(t) := (\psi(t), \hat{n}^\alpha \psi(t)) = \sum_n n^\alpha |c_n(t)|^2.
\] (29)

Their asymptotic behaviour follows a power law,

\[
\nu_\alpha(t) \sim t^\beta \alpha^\beta,
\] (30)

where \( \beta \) is a non-trivial function of the moment order \( \alpha \). In \([6, 27]\) we found that \( \beta \) is convex, non-decreasing, and non-constant even in the case of a one-scale Cantor set, characterized by trivial thermodynamics: this is what we call quantum intermittency. Corrections to eq. (30) can also be observed in the form of log-periodic oscillations of \( \nu_\alpha(t) \), super-imposed to its leading behavior. They can be explained by the Mellin-type analysis presented in the previous section.

We can estimate the function \( \beta(\alpha) \) on the basis of simple renormalization group considerations. For simplicity, let us consider an I.F.S. with \( M \) maps, of equal probability \( \pi_i = \frac{1}{M} \). Let this I.F.S. be non-overlapping. Then, let \( I \) be the smallest interval containing \( A \), the I.F.S. attractor, and let \( I_l \) be the image of \( I \) under the map \( \phi_l \). Clearly, \( I_l \cap I_m = \emptyset \) if \( l \neq m \), and the measure \( \mu \) restricted to \( I_l \) is a linearly rescaled copy of the original. Then, as a first approximation, we can assume that the orthogonal polynomials of the restricted measure are also obtained by linear rescaling of the original polynomials:

\[
p_{Mn}(\phi_l(x)) = \sum_{k=0}^{Mn} \Gamma_{l,k}^{Mn} p_k(x) \simeq \sigma_l^n p_n(x),
\] (31)

where \( \sigma_l^n = \pm 1 \). In other words, we assume a very simple form for the coefficients \( \Gamma_{l,k}^{Mn} \), which amounts to making a renormalization ansatz.
Let us now consider $S_{Mn}(T)$, as defined above. Because of the balance property (2), it can be written

$$S_{Mn}(T) = \sum_{l,m=1}^{M} \pi_l \pi_m \int d\mu(x) \int d\mu(y) \frac{\sin T(\phi_l(x) - \phi_m(y))}{T(\phi_l(x) - \phi_m(y))} p_{Mn}(\phi_l(x))p_{Mn}(\phi_m(y)).$$

(32)

In the previous equation, $\phi_l(x)$ and $\phi_m(y)$ belong to $I_l$ and $I_m$, respectively. If $l \neq m$, these intervals are separated by a finite gap. As $T$ tends to infinity, these contributions tend to zero as $T^{-1}$. We can therefore retain only the diagonal terms in eq. (32).

If we now employ the approximate estimate (31) in the r.h.s. of eq. (32) we can write

$$S_{Mn}(T) = \sum_{l=1}^{M} \pi_l^2 S_n(\delta_l T).$$

(33)

This too is a sort of renormalization equation which links the wave-function component at site $Mn$ and time $T$ to the component at site $n$ and at shorter times $\delta_l T$. When inserted in eqs. (29), eq. (33) implies that the growth exponent $\beta$ associated with the averaged moments $\nu_\alpha$ via eq. (30) must satisfy the relation

$$1 = M^{\alpha-1} \sum_{l=1}^{M} \delta_l^{\alpha \beta}.$$  

(34)

Comparing this result with eq. (4) we obtain the crucial equation

$$\beta(\alpha) = D_{1-\alpha},$$

(35)

which links multi-fractal properties and time dynamics. In particular, eq. (35) implies that $\beta(0) = D_1$, which is consistent with the rigorous result $\beta(0) \geq D_1$ [5]. Notice that $\beta(0)$ can be defined by a limiting procedure on $\beta(\alpha)$, or by the evolution of the logarithmic moment. We have also $\beta(1) = D_0$.

Because of the rough approximation involved in eq. (31), and because for the validity of eq. (33) both $c_{Mn}$ and $c_n$ need to be in their asymptotic regimes, we do not expect eq. (33) to be always exact. Indeed, in Fig. 4 we have considered a family of I.F.S. measures, characterized by $M = 2$, $\delta_2 = \frac{2}{3}$, $\beta_1 = 0$, $\beta_2 = \frac{2}{5}$, $\pi_1 = \frac{3}{5}$, and $\pi_2 = \frac{2}{5}$. The contraction rate $\delta_l$ is allowed to vary in the range $[1, 2]$, which implies a significant variation both in the structure of the support of the balanced measure and in its multi-fractal properties. Plotted in Fig. 4 are the scaling exponents $\beta(0)$ and $\beta(1)$, compared with the multi-fractal dimensions $D_1$ and $D_0$, respectively. We observe a substantial agreement between the two data sets, dynamical and multi-fractal. Numerically, the discrepancy is always less than five percent. We can therefore conclude that the relation (33) catches some essential part of the physics. Yet, the situation is more complicated, as the following pair of examples show.

The first is a magnificent counter-example. Let us consider a new class of I.F.S. measures (and related Hamiltonians) characterized by $M = 2$ and by a particular choice of the weights:

$$\pi_j = \delta_j^D, \quad j = 1, 2$$

(36)

where $D$ is the (constant) value $\frac{\log 2}{\log 5 - \log 2}$. This choice originates what is called a uniform Gibbs measure. The first of such I.F.S. is that of Figs. 1 to 3, and $D$ is its fractal
dimension. Indeed, all I.F.S. with the property (36) are characterized by the same flat thermodynamic function $D_q = D$. Clearly, because of eq. (36), and because $\pi_1 + \pi_2 = 1$, only one parameter among the map weights and contraction rates is left free. By varying this parameter we can construct different I.F.S. measures, with the same flat thermodynamics. What are then the corresponding dynamical exponents $\beta(\alpha)$? The approximate relation (35) predicts $\beta(\alpha) \simeq D$ for all $\alpha$.

In Fig. 5 we have considered: a: The I.F.S. with $\delta_1 = \delta_2 = \frac{2}{5}$, $\pi_1 = \pi_2 = \frac{1}{2}$, which is a “pure” Cantor Set. b: The I.F.S. with $\delta_1 = .5090$, $\delta_2 = .2978$, and $\pi_1 = \frac{3}{5}$. c: The I.F.S. with $\delta_1 = .5293$, $\delta_2 = .2802$, and $\pi_1 = .6180$. d: The I.F.S. with $\delta_1 = .6033$, $\delta_2 = .2196$, and $\pi_1 = .6823$. The first observation we can draw from this figure is that $\beta$ is not flat, as shown in [27], even if the intermittency range in the $[0, 5]$ interval is very narrow. The second, is that the prediction $\beta = D = .7565$ is correct within two percent at $\alpha = 0$ and about five percent at $\alpha = 5$. The third, and most important, that the scaling function $\beta$ is roughly invariant from case to case.

These results are intriguing: the coincidence of the curves in Fig. 5 suggests that the spectrum of generalized dimensions $D_q$ must play some rôle in determining $\beta(\alpha)$: the fractal measures a – d seem to have little in common beyond having the same flat thermodynamics. Nevertheless, precisely because in these cases $D_q$ is flat, neither eq. (35), nor any general relation of the kind $\beta(\alpha) = D_q(\alpha)$, with $q$ an as yet unknown function of $\alpha$ can hold rigorously.

Let us now come to a favourable example: we can construct a class of measures for which the renormalization eq. (31) is exact: these are the equilibrium measures of the Julia sets generated by the polynomials

$$P(z) = z^2 - \lambda,$$

where $\lambda \geq 2$ is a real constant. As we have already remarked, the Jacobi matrices for these problems can be constructed by a stable recursion algorithm [23]. [24]. Non-linearity of the I.F.S. maps stemming from eq. (37) as inverse branches of $P(z)$ can be treated by considering sufficiently high iterations $P^{(l)}$, and a theory perfectly analogous to (30-35) can be carried out, with the same result.

In fig. 6 we make the usual comparison between the moment scaling function, $\beta(\alpha)$, and thermodynamics [29], $D_{1-\alpha}$: the curves coincide within numerical precision! Therefore, one can conclude that discrepancies from eq. (35) are due to the non-exactness of eq. (31) when a spectral measure is approximated by I.F.S., except for the case of Julia measures, which are known to have strong algebraic properties.

Incidentally, we note that for Julia sets, the invariant measure coincides with the measure of the asymptotic distribution of the zeros of the associated orthogonal polynomials, the latter being also the physicists’ global density of states. Might it be that the correct quantity entering eq. (35) is this second measure? The analysis of the I.F.S. data presented here seems to exclude this case, although we cannot exclude that this rôle is played by yet another spectral measure still to be determined.

7 Conclusions

We have presented a stable algorithm for the determination of lattice Hamiltonian operators possessing a given spectral measure, in the class of linear I.F.S. This algorithm
consists of a recursive determination of the associated Jacobi matrix, in the framework of the theory of orthogonal polynomials.

The Hamiltonian operators determined in this way are characterized by almost periodic coefficients: since I.F.S. measures approximate arbitrary well any measure supported on a Cantor set, this fact might lead to a proof that almost periodicity is always associated with this kind of spectra.

In a quantum mechanical context, the Jacobi matrices studied here can be employed as models of almost-periodic systems: the dynamical properties of such systems can be studied in their essence, having extracted the crucial information on the related spectral measures. We have shown that connections between spectral properties and dynamics go far beyond the conventional RAGE theorems: in particular, delocalization of particle’s position along the lattice basis can be described by a scaling function $\beta$ governing the moments of order $\alpha$ of the position operator. Non-constancy of this function translates mathematically the phenomenon of quantum interference.

We have derived an intriguing relation, $\beta(\alpha) = D_{1-\alpha}$, linking dynamics and the thermodynamical properties of the spectral measure: considering the Jacobi matrices associated with Julia sets we have constructed a family of quantum systems for which the relation is exact, and we have discussed the reasons for the discrepancies present in the general case. We believe that a further refinement of the results presented in this paper will lead to a profound understanding of the mathematical and physical properties of almost-periodic quantum systems.

Finally, we remark that the Jacobi Hamiltonians considered in this paper are not simple exotic curiosities, but can also describe time-resolved energy absorption in externally perturbed quantum systems, as well as electron dynamics in solid-state eterostructures like super-lattices [30], where by varying an alloy concentration along a deposition axis different spectral structures can be found [31]. Here, our results may become relevant in several problems, like –for instance– the design of lasers and radiation detectors.
Figure Captions

Fig. 1.
Orthogonal polynomial $p_8(x)$ of the I.F.S. measure with maps $(\delta_i, \beta_i, \pi_i) = (\frac{2}{3}, 0, \frac{1}{2}), (\frac{2}{3}, \frac{2}{3}, \frac{1}{2})$, with a finite-resolution representation of the support of the measure obtained by plotting a large number of points on the attractor. Because of the finite size of points, this latter appears as a sequence of dashes. Only the symmetrical half is shown. In the inset, the beginning of the sequence of $r_n$. The vertical scale ranges from zero to $\frac{1}{2}$. Lines are merely to guide the eye.

Fig. 2.
Discrete Fourier transform of the $r_n$ sequence $(n = 1, \ldots, 2^{13})$ associated with the I.F.S. of Fig. 1. The constant and $\pi$ frequencies exceed the vertical scale, and are not reported.

Fig. 3.
Plot of the phase $\Phi$ of the discrete Fourier transform of the sequence $r_n$ associated with the I.F.S. of Fig. 1 and 2, to show the $\pi$ discontinuity close to the value of the main peak of Fig. 2.

Fig. 4.
Multi-fractal dimensions $D_0$ (full diamonds) and $D_1$ (full squares) and dynamical exponents $\beta(0)$ (open squares) and $\beta(1)$ (open diamonds) versus contraction rate $\delta_1$, for the family of I.F.S. described in the text.

Fig. 5.
Scaling functions $\beta(\alpha)$ for the four I.F.S.’s a - d described in the text: a: circles; b: squares; c: triangles; d: diamonds.

Fig. 6.
Scaling function $\beta(\alpha)$ for the Julia set measure with $\lambda = 2.2$ (diamonds) and thermodynamical dimensions $D_{1-\alpha}$ (crosses).

References

[1] P.G.Harper, *Proc.Roy.Soc.Lon.* A68, (1955) 874; M.Ya.Az’bel, *Sov.Phys.JETP* 19 (1964) 634; D.R.Hofstadter, *Phys. Rev. B* 14 (1976) 2239;

[2] C.Tang and M.Kohmoto, *Phys.Rev. B* 34 (1986) 2041.

[3] H.Hiramoto and S.Abe, *J.Phys.Soc. Japan* 57 (1988) 230; *ibid.*, (1988) 1365.

[4] T.Geisel, R.Ketzmerick, and G.Petschel, *Phys.Rev.Lett.* 66,1651(1991); *ibid.*, 67 (1991) 3635; R.Ketzmerick, G.Petschel and T.Geisel, *Phys.Rev.Lett.* 69 (1992) 695.

[5] I.Guarneri, *Europhys.Lett.* 10, 95(1989); *ibid.*, 21, 729 (1993).

[6] I.Guarneri and G.Mantica, *Ann. Inst. H. Poincaré* 61 (1994) 369.

[7] R.del Rio, S.Jitomirskaya, N.Makarov and B.Simon, *Singular Continuous Spectrum is generic*, preprint 1994.

[8] C.R. Handy and G. Mantica, *Physica D* 43, (1990) 17-36.

[9] G. Mantica and A. Sloan, *Complex Systems* 3, (1989) 37-62.
[10] E.R. Vrscay and C.J. Roehrig, *Iterated Function Systems and the Inverse Problem of Fractal Construction Using Moments*, in *Computers and Mathematics*, E. Kaltofen and S.M. Watt Eds., Springer (Berlin, 1989).

[11] D. Bessis and G. Mantica, *Phys. Rev. Lett.* **66**, (1991), 2939-2942.

[12] J. Hutchinson, *Indiana J. Math.* **30** (1981) 713-747.

[13] P. Diaconis, M. Shahshahani, *Contemporary Mathematics* **50** (1986) 173-182.

[14] M.F. Barnsley and S.G. Demko, *Proc. R. Soc. London A* **399** (1985) 243-275.

[15] M.F. Barnsley, *Fractals Everywhere*, Academic Press, (New York 1988).

[16] M. Case and M. Kac, *J. Math. Phys.* **14** (1973) 594.

[17] W. Gautschi, *Math. Comp.* **24** (1970) 245-260.

[18] D. Bessis and S. Demko, *Physica* **D 47** (1991) 427-438.

[19] W. Gautschi, in *Orthogonal Polynomials*, P. Nevai Ed., Kluwer (Dordrecht NL 1990), 181-216.

[20] G. Mantica, *A Stieltjes Technique for Computing Jacobi Matrices Associated With Singular Measures*, to appear in *Constructive Approximations*, (1995).

[21] E.R. Vrscay, *I.F.S. Theory and Applications and the Inverse Problem*, in *Fractal Geometry and Analysis*, J. Bélair and S. Dubuc Eds., Kluwer, (Dordrecht, NL 1992) 405-468.

[22] G. Mantica and G.A. Mezincescu, in preparation.

[23] J. Bellissard, D. Bessis, and P. Moussa, *Phys. Rev. Lett.* **49** (1982) 702-704.

[24] M.F. Barnsley, J.S. Geronimo, and A.N. Harrington, *Proc. Am. Math. Soc.* **88** # 4, (1983) 625-630.

[25] G.A. Baker, D. Bessis, and P. Moussa, *Physica A* **124** (1984) 61-77.

[26] D. Bessis and G. Mantica, *J. Comp. Appl. Math.* **48** (1993) 17-32.

[27] I.Guarneri and G.Mantica, *Phys. Rev. Lett.* **73** (1994) 3379.

[28] D. Bessis, J.D. Fournier, G. Servizi, G. Turchetti, and S. Vaienti, *Phys. Rev. A* **36**, 920-928 (1987).

[29] G. Servizi, G. Turchetti, and S. Vaienti, *Nuovo Cim.* **101 B**, 285-307 (1988).

[30] *Heterojunction band Discontinuities: Physics and Device Applications*, F. Capasso and G. Margaritondo Eds., Elsevier, B.V. (1987).

[31] G. Mantica and S. Mantica, *Phys. Rev. B* **46**, 7037-7045 (1992).
