On the Connection Between Irrationality Measures and Polynomial Continued Fractions

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Abstract

Linear recursions with integer coefficients, such as the recursion that generates the Fibonacci sequence \( F_n = F_{n-1} + F_{n-2} \), have been intensely studied over millennia and yet still hide interesting undiscovered mathematics. Such a recursion was used by Apéry in his proof of the irrationality of \( \zeta(3) \), which was later named the Apéry constant. Apéry’s proof used a specific linear recursion that contained integer polynomials (polynomially recursive) and formed a continued fraction; such formulas are called polynomial continued fractions (PCFs). Similar polynomial recursions can prove the irrationality of other fundamental constants such as \( \pi \) and \( e \). More generally, the sequences generated by polynomial recursions form Diophantine approximations, which are ubiquitous in different areas of mathematics such as number theory and combinatorics. However, in general it is not known which polynomial recursions create useful Diophantine approximations and under what conditions they can be used to prove irrationality.

Here, we present general conclusions and conjectures about Diophantine approximations created from polynomial recursions. Specifically, we generalize Apéry’s work from his particular choice of PCF to any general PCF, finding the conditions under which a PCF can be used to prove irrationality or to provide an efficient Diophantine approximation. To provide concrete examples, we apply our findings to PCFs found by the Ramanujan Machine algorithms to represent fundamental constants such as \( \pi \), \( e \), \( \zeta(3) \), and the Catalan constant \( G \). For each such PCF, we demonstrate the extraction of its convergence rate and efficiency, as well as the bound it provides for the irrationality measure of the fundamental constant. We further propose new conjectures about Diophantine approximations based on PCFs.

Looking forward, our findings could motivate a search for a wider theory on sequences created by any linear recursions with integer coefficients. Such results can help the development of systematic algorithms for finding Diophantine approximations of fundamental constants. Consequently, our study may contribute to ongoing efforts to answer open questions such as the proof of the irrationality of the Catalan constant or of values of the Riemann zeta function (e.g., \( \zeta(5) \)).
1 Introduction

1.1 Apéry’s constant and his polynomial continued fraction (PCF)

In his paper [1,2], Apéry ingeniously presented a specific linear recursion with integer polynomial coefficients that proves the irrationality of $\zeta(3)$. This polynomial recursion generated two sequences $p_n, q_n$ (given different initial values) such that $p_n / q_n \xrightarrow[n \to \infty]{} 6 / \zeta(3)$, i.e., it constituted a Diophantine approximation of $\zeta(3)$. Apéry then showed that this specific sequence proved the irrationality of the number to which it converged. He also demonstrated [1] that the linear recursion was equivalent to the following polynomial continued fraction (PCF).

$$\frac{6}{\zeta(3)} = 5 - \frac{1}{117 - \frac{64}{535 - \frac{729}{1463 - \frac{n^6}{34n^3 + 51n^2 + 27n + 5}}}.$$  

Apéry’s paper inspired other researchers to apply related strategies to other problems in Diophantine approximations, to study irrationality measures of other constants, and to find applications in other fields [3-10].

Apéry’s result hints at a more general question: Which PCFs prove the irrationality of the number to which they converge? In other words: Which pairs of integer polynomials (such as $-n^6$ and $34n^3 + 51n^2 + 27n + 5$ in Apéry’s case) can be used to prove irrationality? This question is directly related to the intrinsic properties of PCFs, specifically, their rate of convergence and the properties of the Diophantine approximation sequences they create.
1.2 Polynomial continued fractions (PCF)

In their most general form, PCFs denote a generalized continued fraction in which $a_n = a(n)$ and $b_n = b(n)$, where $a$ and $b$ are polynomials with integer coefficients:

$$\text{PCF}[a_n, b_n] = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}$$

$a_n = a(n), b_n = b(n) \in \mathbb{Z}[n]$.

At each finite step $n$, the PCF is a rational number $p_n/q_n$, where $p_n$ and $q_n$ are the numerator and denominator of the $n$th convergent, respectively. Both $p_n$ and $q_n$ can be shown to satisfy the same recursion of depth 2:

$$u_n = a_n u_{n-1} + b_n u_{n-2}$$

with different initial conditions,

$$p_{-1} = 1, \quad p_0 = a_0$$

$$q_{-1} = 0, \quad q_0 = 1.$$

The limit of $p_n/q_n$ is the value of the PCF. There exist Möbius transformations with integer coefficients that transform between the limits of $p_n/q_n$ for different initial conditions – for any two pairs of rational, linearly-independent initial values [11].

PCFs appear in a wide range of fields of mathematics and are related to many special functions, including all trigonometric functions, exponentials, Bessel functions, generalized hypergeometric functions, and the Riemann zeta function, and many other important functions such as erf and log [12-14]. Moreover, any infinite sum can be
converted into a continued fraction (known as Euler’s continued fraction). The other direction is not correct – not every continued fraction can be converted into an infinite sum. The space of PCFs also contains all linear recursions of depth 2 with rational polynomial coefficients (and some of their generalizations). In his study, Apéry developed a linear recursion with rational polynomials, and since it was of depth 2, he was able to convert it to a PCF, using the standard definition above.

1.3 The goals of this paper

Looking at the bigger picture, it is interesting to generalize Apéry’s PCF. Consider an arbitrary linear integer recursion (of any order) used to create the numerators and denominators in a sequence of rational numbers. In other words, provided two sets of initial conditions, for the numerator and denominator, the linear recursion creates a Diophantine approximation sequence. Each such sequence may provide an efficient representation of the limit of the sequence. Intuitively, the efficiency is described by the rate (as a function of \( n \)) at which the sequence converges relatively to the sizes of the denominators. What can be said about the resulting sequence? What condition should the linear recursion fulfill for the generated sequence to prove that its limit is irrational? More generally, what bounds on irrationality measures does each linear recursion create?

In this paper, we describe the construction of a systematic method to find, for each PCF, the efficiency of its limit approximation, i.e., the lower bound it provides for the irrationality measure (we address a lower bound that simultaneously provides an upper bound [2]). We develop a criterion on the PCF for proving the irrationality of its limit. Specifically, **Theorem 2** states a formula for the irrationality bound for each PCF,
yielding \( \delta = \frac{\ln a - \ln |B| + \ln \lambda}{\ln a - \ln \lambda} \), where \( \alpha \) and \( B \) can be calculated directly from \( a_n \)'s and \( b_n \)'s coefficients and \( \lambda \) relates only to the growth rate of the greatest common divisor \( \text{GCD}[p_n, q_n] \) (specifically, \( \ln(\lambda) = \limsup \frac{1}{n} \ln \left( \frac{\text{GCD}[p_{n!\deg a_n}, q_{n!\deg b_n}]}{n^{\deg a_n}} \right) \)). Moreover, **Conjecture 1.1** states that, for the growth rate of the GCD to be sufficient for an irrationality proof, the polynomial \( b_n \) must be a product of two rational polynomials of equal degrees.

An important advantage of this approach is that it does not require the determination of the PCF limit or any knowledge of it. PCFs that yield efficient Diophantine approximations are in general also better for computing more quickly the numbers to which they converge. Consequently, the results of our study could be used to develop faster means for high precision calculations of fundamental constants, such as attempts to compute more digits and study the normality of such constants [15-22].

Any mathematical expression that can be converted to PCFs, such as infinite sums used for the computation of fundamental constants [15-17,20,21], could be analyzed with the approach that we present in this paper. The conjectures that arise from our study hint at a general theory that goes beyond PCFs to any polynomial recursion, and maybe eventually beyond it to any linear recursion with rational coefficients.

Some of the conclusions of our study presented below go beyond PCFs and beyond the motivation of irrationality proofs. In general, when given any linear recursive formula with integer coefficients, not necessarily one representing a PCF, it is interesting to study the GCD of two (or maybe more) sequences arising from the same recursion with different initial conditions. We find the solution for special cases of linear recursions,
showing the rate of growth of the GCD. We hope that our study will contribute to efforts toward finding the general rules for GCDs of arbitrary linear recursions.

1.4 Motivation and potential applications

Many of the PCF formulas that led us to the conjectures and proofs in this paper were originally found in the Ramanujan Machine project [10], which employs computer algorithms to find conjectured formulas for fundamental constants. Various algorithms are being developed as part of that project, and so far they all focus on formulas in the form of PCFs. Since the algorithms check candidate formulas by their numerical match to target constants, the results are always in the form of conjectures rather than proven theorems. The first algorithms succeeded in finding conjectured PCF formulas for \( \pi \), \( e \), values of the Riemann zeta function \( \zeta \), and the Catalan constant [10]. These latter formulas led to a new world record for the irrationality bound of the Catalan constant. The theorems and conjectures below can also help improve future algorithms that search for such conjectures.

We point to three interesting challenges that motivate this work, each having prospects in Diophantine approximations, as well as in experimental mathematics, i.e., computation-driven mathematical research (e.g., [10,23,24]):

1. Given the polynomials \( a_n, b_n \) of the PCF, determine whether the PCF provides a bound on the irrationality measure, and if so, then find the bound analytically from \( a_n, b_n \).

2. Estimate the efficiency of each PCF for computing fundamental constants to high precision.
Develop faster algorithms to compute PCFs; more generally, compute any polynomial recursion more efficiently.

1.5 The measure of irrationality of a number

The irrationality measure of a number $L$ is the largest $\delta$ for which there exists a sequence of rational numbers $p_n/q_n \neq L$ s.t.

$$\left| L - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{1+\delta}}.$$ 

This maximal $\delta$ is called the irrationality measure of the number $L$ [2,15], or the Liouville–Roth exponents. For irrational numbers, this maximum can be obtained by the regular continued fraction of $L$; however, its closed formula is often unknown (e.g., in the case of $\pi$). The Diophantine approximation is thought of as more efficient when $\delta$ is larger. Rational numbers have an irrationality measure 0, meaning that they cannot be approximated efficiently by other rational numbers. This property is part of the irrationality criterion: if there exists a sequence $p_n/q_n$ for which this inequality holds for some $\delta > 0$, then $L$ is irrational. Further, if there exists a sequence $p_n/q_n$ for which this inequality holds for some $\delta > 1$, then $L$ is transcendental by the Siegel–Roth theorem. Finally, if the inequality holds for arbitrarily large values of $\delta$, then $L$ is a Liouville number (infinite irrationality measure) [15]. Intuitively, for a sequence that proves irrationality, the growth of the denominator should be sufficiently slow in relation to the convergence rate of the PCF. For our purposes below, the sequences $p_n/q_n$ are generated by PCFs.
In the rest of the paper, we use the symbol \( \delta = \delta_{(p_n/q_n)} \) to denote the largest \( \delta \) that satisfies the inequality for a specific sequence of rationals \( \{p_n/q_n\} \) and almost all \( n \) values (also called an effective irrationality measure) [10,15]. For each sequence, there always exists a sequence for which \( \delta = 0 \) (for a rational \( L \)) or \( \delta \geq 1 \) (for an irrational \( L \)). However, the largest known \( \delta \) can be smaller or larger than 0. To find even one explicit sequence that reaches the maximal value is challenging. This challenge continues to motivate searches for new sequences \( \{p_n/q_n\} \) for constants, from which one can extract larger lower bounds for \( \delta \). Each constant for which the rationality or irrationality is still unknown, has all its known sequences with \( \delta \leq 0 \) (as in the case of the Catalan constant [19-22]). Then, finding one PCF for which \( \delta > 0 \) will directly prove irrationality. When \( \delta \) is known to be positive, as in \( \pi \), it is still interesting to find better PCFs with larger \( \delta \) values, because it improves the bounds on the constant’s irrationality measure (e.g., \( \pi \)'s upper bound [25] by Zeilberger and Zudilin). Therefore, it is of interest to find sequences for which \( \delta \) is as large as possible, even when the value is negative.

This paper shows the conditions on \( a_n \) and \( b_n \) for which \( \text{PCF}[a_n, b_n] \) provides nontrivial \( \delta \) (larger than \(-1\)) and presents certain conjectures for the dependence of \( \delta \) on the choice of \( a_n, b_n \). To find these conditions and conjectures, we predict the PCF convergence rate \( |L - p_n/q_n| \) and the rate of growth of the reduced denominator \( \frac{q_n}{\gcd[p_n,q_n]} \).

We present a criterion for the growth rate of the greatest common divisor \( \gcd[n] \triangleq \gcd[p_n,q_n] \), which is necessary and sufficient for a nontrivial \( \delta \): \( \ln(\lambda) \triangleq \limsup \frac{1}{n} \ln \left( \frac{\gcd[n]}{n! \deg a_n} \right) > -\infty \). We use this criterion to calculate \( \delta \) and provide conjectures for its dependence on \( a_n, b_n \).
2 Results

2.1 Summary of the main results

Unless stated otherwise, we focus on “balanced-degree PCFs”, where \( \frac{\text{deg} b_n}{\text{deg} a_n} = 2 \). This PCF type is arguably the most common in the literature related to mathematical constants (see Appendix A, Ref. [10], and further references therein). We show that the growth rate (as a function of \( n \)) of the GCD is key to the analysis of PCFs of this type. We find special interest in cases of PCF\([a_n, b_n]\) for which the GCD grows so fast that it reduces most of the denominator \( q_n \): that is, while the denominator \( q_n \) grows as some power of \( n! \), the reduced denominator \( q_n / \text{GCD}[p_n, q_n] \) is of exponential order. We call this phenomenon \textbf{factorial reduction (FR)}. Below, we prove that for a PCF to provide a nontrivial \( \delta \) value, FR is necessary (\textbf{Theorem 1}). We also derive formulas for these \( \delta \)s (\textbf{Theorem 2}), which could help provide irrationality proofs. The other results of our work are conjectures which attempt to provide a complete characterization of PCFs with nontrivial \( \delta \). All the conjectures are backed with extensive, computer-based, numerical tests and await a formal proof.

Numerical tests show that FR is possible if and only if there exists a split of \( b_n \) into two equal degree rational polynomials (\textbf{Conjecture 1.1}). We further conjecture that for every such \( b_n \), the PCF\([a_n, b_n]\) has FR for infinitely many choices of polynomials \( a_n \) with \textit{rational} coefficients (\textbf{Conjecture 1.2}). Other experiments revealed that, for the special case of \( \text{deg} b_n = 2 \) and \( \text{deg} a_n = 1 \), each such splitable \( b_n \) has exactly two infinite families of integer polynomials \( a_n \) for which PCF\([a_n, b_n]\) has FR (\textbf{Conjecture
1.4 presents the formula for the \(a_n, b_n\) pairs. A necessary and sufficient condition on \(a_n, b_n\) of arbitrary degrees for PCF\([a_n, b_n]\) to have FR still awaits discovery and a proof.

Example: Apéry’s PCF

Let us explain how Apéry’s PCF appears as a special case in our study. First, his PCF provides nontrivial \(\delta\) because \(b_n = -n^6\) has only rational roots (a special case of Conjecture 1.1). Second, Apéry’s PCF has FR (a special case of Theorem 1), as he proved that a lower bound for the GCD size is \(n!/e^{n^d}\). Third, Apéry proved that \(\delta = \frac{\ln \alpha - 3}{\ln \alpha + 3}\) for \(\alpha = 17 + 12\sqrt{2}\), which exactly matches our general expression (see Theorem 2). Below, we generalize this process and conclusions to all PCFs, classify their different GCDs, and present a criterion for the PCF that allows us to prove irrationality.

2.2 Theorems about factorial reduction (FR)

We tested many PCFs for FR and identified a surprising phenomenon: despite the rarity of FR in an experimentally random PCF, we have so far found FR in every PCF that converges to a fundamental constant (we tested PCFs that converge to \(\pi\), \(e\), \(\zeta(3)\), \(\zeta(5)\), and the Catalan constant \(G\)). Specifically, we tested all the PCFs found so far in the Ramanujan Machine project [10] and many other PCF formulas. This relation between FR and PCFs of fundamental constants is surprising because the algorithmic search in [10] did not favor PCFs that have FR. This intriguing fact hints at an underlying structure of PCFs that is required for formulas that converge to certain mathematical constants.

Theorem 1 (The necessity of FR). For a balanced-degree PCF\([a_n, b_n]\) to prove irrationality, or to provide a nontrivial \(\delta\), its sequence \(\frac{p_n}{q_n}\) must have FR; i.e., GCD divided
by $n!^{\text{deg} a_n}$ is of exponential order: $\limsup \frac{1}{n} \ln \left( \frac{\text{GCD}}{n!^{\text{deg} a_n}} \right) > -\infty$. In other words, the reduced numerators $p_n/\text{GCD}$ and denominators $q_n/\text{GCD}$ are of exponential order rather than factorial.

**Proof.** See Appendix B. □

For any PCF, it is possible to define $\lambda$ such that $\ln \lambda = \limsup \frac{1}{n} \ln \left( \frac{\text{GCD}}{n!^{\text{deg} a_n}} \right)$, which, unless $\lambda = 0$, implies $\lambda^n = \frac{\text{GCD}}{n!^{\text{deg} a_n}}$ for some subsequence of indexes. FR means that $\lambda > 0$.

The notation $x_n \equiv y_n$ represents that $\lim_{n \to \infty} \frac{1}{n} \log \frac{x_n}{y_n} = 0$. That is, $x_n$ and $y_n$ agree in their exponentials but may still differ in slower than exponential portion (e.g., they may differ in polynomial pre-factors before their exponentials).

To set the ground for **Theorem 2**, we denote $\alpha$ to be the larger solution (in absolute value) of the equation

$$\alpha^2 = A \cdot \alpha + B,$$

where $A$ and $B$ are the leading coefficients of $a_n$ and $b_n$, respectively.

**Theorem 2** (*A formula for $\delta$*). For a PCF $[a_n, b_n]$ of the second type with FR, the effective irrationality measure $\delta$ is

$$\delta = \frac{\ln \alpha - \ln |B| + \ln \lambda}{\ln \alpha - \ln \lambda},$$

provided that $B > -A^2/4$.

**Proof.** See Appendix B. □
The case of $B \leq A^2/4$ remains for future study.

This formula connects the GCD to the value of $\delta$. Note that larger values of $\lambda$ imply larger values of $\delta$ (beneficial for proving irrationality). The maximum possible $\lambda$ value is $\lambda = \alpha$ (as $p_n = q_n = n^{\ell d} \cdot \alpha^n$), which implies a Liouville number, i.e., an infinite irrationality measure.

Examples: Different values of $\lambda$ and $\delta$

1. Apéry’s: $b_n = -n^6$ and $a_n = 34n^3 + 51n^2 + 27n + 5$, and therefore, $\alpha = \frac{34+\sqrt{34^2-4}}{2}$. Apéry showed [1] that $\left(\frac{n!}{\text{LCM}[n]}\right)^3|\text{GCD}$, where LCM$[n]$ is the least common multiple (LCM) of $1,2 \ldots n$, which satisfies LCM$[n] = e^n$. Therefore, $\text{GCD} \geq \left(\frac{n!}{\text{LCM}[n]}\right)^3$, and thus, $\lambda \geq \frac{1}{e^3}$. Our formula provides $\delta = \frac{\ln \alpha - \ln |B| + \ln \lambda}{\ln \alpha - \ln \lambda} \approx 0.0805$, which is exactly Apéry’s result.

2. Other irrational limits: For any integer $k > 0$, take $b_n = -n^2$ and $a_n = k(2n + 1)$; therefore, $\alpha = k + \sqrt{k^2 - 1}$. The proof in Appendix D shows that $\frac{n!}{\text{LCM}[n]}|(2^n \cdot \text{GCD})$. Hence, $\text{GCD} \geq \frac{n!}{2^n \text{LCM}[n]}$, and thus, $\lambda \geq \frac{1}{2e}$, providing $\delta \geq \frac{\ln(k + \sqrt{k^2 - 1}) - \ln 2 - 1}{\ln(k + \sqrt{k^2 - 1}) + \ln 2 + 1}$, which is positive for $k \geq 3$ and thus proves irrationality for all these PCFs' limits.

3. Additional values in Table 1

These examples emphasize the strength of our approach: the determination of an irrationality measure $\delta$ without a need to find a closed formula for a PCF sequence or to find the PCF limit.
2.3 Conditions for the existence of factorial reduction (FR)

Theorems 1 and 2 leave us with two important questions: (1) which PCFs have FR and if so, then (2) what are their exponential orders \( \lambda \). Following many computer tests, the following conjecture is an effort to answer the first question. The second will be discussed later.

**Conjecture 1.1.** Given a polynomial \( b_n \), there exists an \( a_n \) for which \( PCF[a_n, b_n] \) has FR if and only if \( b_n \) can be written as a multiplication of two polynomials over \( \mathbb{Q} \) with equal degrees.

In other words, if \( b_n = r_n s_n \) for some \( r_n, s_n \in \mathbb{Q}[n] \) where \( \deg(r_n) = \deg(s_n) \), then a polynomial \( a_n \) exists s.t. \( PCF[a_n, b_n] \) has FR.

If such a split does not exist, then no \( a_n \) will create a \( PCF[a_n, b_n] \) that has FR.

Examples: The reducibility of \( b_n \) and the effect on the FR

In the case of \( b_n = n^2 - 2 \), which has two irrational roots, we found no \( a_n \) such that \( PCF[a_n, b_n] \) has FR. Similarly, we found no \( a_n \) for which there is FR in the case of \( b_n \) with two unreal roots such as \( b_n = n^2 + 1 \) (Appendix E).

On the other hand, we found that, for \( b_n = 8n^2 - 2 = 2(2n + 1)(2n - 1) \), there exist choices of \( a_n \) that provide PCFs with FR:

\[
PCF[a_n = 7n + 3, b_n = 8n^2 - 2] = 3 + \frac{6}{10} + \frac{30}{17} + \frac{70}{24} + \ldots
\]

Another example for \( b_n = -n^4 = -(n^2)(n^2) \) is
\[
\text{PCF}[a_n = 2n^2 + 2n + 13, b_n = -n^4] = 13 - \frac{1}{17 - \frac{16}{25 - \frac{81}{37 - \cdots}}}
\]

Families of \(a_n\)s for which the PCF has FR

The above \(a_n\) choices are part of infinite families (see the following examples). In fact, our computer tests always find the \(a_n\)s to belong to infinite families that all have FR, and we propose the following conjectures.

**Conjecture 1.2.** For each \(b_n\) that can be split, there exists at least one infinite family of \(a_n\)s for which every PCF\([a_n, b_n]\) has FR.

Such a family is presented on **Conjecture 1.3**.

This conjecture means that each PCF\([a_n, b_n]\) with FR could be generalized to an infinite family of PCFs with the same \(b_n\) and different \(a_n\)s. An interesting question is whether there would always exist members of this family that prove the irrationality of the constants to which they converge (see **Theorem 3** for more information).

**Example of a single PCF**

For \(b_n = -n^4\), and any integer \(k\) of the form \(k = m^2 - m + 1\), the following PCF has FR.

\[
\text{PCF}\left[a_n^{(k)} = 2n^2 + 2n + k, b_n = -n^4\right] = k - \frac{1}{4 + k - \frac{16}{12 + k - \frac{81}{24 + k - \cdots}}}
\]
Example of an infinite family of PCFs for any general $b_n$ (found empirically)

**Conjecture 1.3.** For any pair of polynomials $r_n, s_n \in \mathbb{Q}[n]$ with degrees $d$, the following PCF[$a_n, b_n$] has FR:

$$b_n = B \cdot r_n \cdot s_n$$

$$a_n = \frac{B}{m} \cdot r_{n+1} - m \cdot s_n$$

for any $m \in \mathbb{Q}\{0\}$, which does not satisfy $m^2 = |B|$ (since then $B = -A^2/4$, or $A = 0$).

The complete structure for $\deg b_n = 2$ and $\deg a_n = 1$

Having performed many numerical tests, we propose the next general conjecture for the $a_n$ families for which the PCF has FR for a given splittable $b_n$ of degree 2.

**Conjecture 1.4.** For every $b_n$ of the form $b_n = B(n-x_1)(n-x_2)$ with $x_1, x_2, B \in \mathbb{Q}$, PCF[$a_n, b_n$] has FR if and only if $a_n$ belongs to one (or more) of the families

$$a_n^{(k)} = \left(\frac{B}{m} - m\right)n + k$$

$$a_n^{(k)} = k(2n - x_1 - x_2 + 1),$$

where $k$ and $m$ are rationals and $m^2 \neq |B|$.

Important note. The above conjecture is formulated with rational parameters ($B, k,$ and $m$), yielding rational PCFs. Alternatively, an equivalent conjecture can be formulated using integer parameters and yielding integer PCFs. The equivalence is achieved by multiplying $a_n$ and $b_n$ by a constant (see inflation process in Appendix C). For the sake of simplicity and coherence, the examples below are chosen to be integers ($B \in \mathbb{Z}$, $k \in \mathbb{Z}$ or $k \in \frac{1}{2}\mathbb{Z}$, and $m|B$). However, the conjecture is presented in the most general form, using rational numbers.
Numerical experiments show that only these two families of PCFs have FR for the given $b_n$. Based on these numerical tests, we found exponentially tight ($\approx$) formulas for the GCD of some cases (Appendix E). Interestingly, the families share additional properties; e.g., their GCDs are always found to be closely related (we do not yet understand the general structure).

**Example**

For $b_n = 8n^2 - 2$, the first family has the following PCFs for $m = 1$ and $\forall k \in \mathbb{Z}$.

$$\text{PCF}[7n+k,8n^2-2] = 0 + k + \frac{8 \cdot 1^2 - 2}{7 + k + \frac{8 \cdot 2^2 - 2}{14 + k + \frac{8 \cdot 3^2 - 2}{\ldots}}}.$$  

And for $m = 2$ and $\forall k \in \mathbb{Z}$

$$\text{PCF}[2n+k,8n^2-2] = 0 + k + \frac{8 \cdot 1^2 - 2}{2 + k + \frac{8 \cdot 2^2 - 2}{4 + k + \frac{8 \cdot 3^2 - 2}{\ldots}}}.$$  

The second family contains $\forall k \in \mathbb{Z}$:

$$\text{PCF}[2n + 1,8n^2 - 2] = k \cdot (2 \cdot 0 + 1) + \frac{8 \cdot 1^2 - 2}{k \cdot (2 \cdot 1 + 1) + \frac{8 \cdot 2^2 - 2}{k \cdot (2 \cdot 2 + 1) + \frac{8 \cdot 3^2 - 2}{\ldots}}}.$$  

We tested many of these PCFs numerically and indeed they all have FR.

**2.4 Summary of our main conjectures regarding polynomial continued fractions (PCFs)**

The above examples summarize the four aspects of our conjectures so far:

1. FR of PCF[$\cdot$, $b_n$]: an $a_n$ exists if and only if $b_n$ can be split.
2. $a_n$ families: for every such $b_n$, there exist several infinite families of $a_n$ for which the PCF has FR.

3. Each $a_n$ belongs to (at least) one of the families.

4. PCFs of the same family have closely related GCDs.

A hint for the structure of arbitrary degrees

We expect similar results for any $b_n$ of order higher than 2 and explain why. For any $b_n$, from any degree, we always find $a_n$s with leading coefficients $\left(\frac{B}{m} - m\right)$ for some $m$, as for the first family. For example, PCF[$11n^2 - 2n - 1, 12n^4$], PCF[$4n^2 - 4n - 2, 12n^4$], and PCF[$n^2 - 6n - 3, 12n^4$] have FR, corresponding to $B = 12$ and $m = 1, 2, 3$, respectively. See Appendix E for more examples. We do not yet know what conditions must be satisfied by the other coefficients of the $a_n$s to have FR.

There exist other $a_n$s of a different and still unknown form, such as Apéry’s and the example of Conjecture 1.2 shown above. Furthermore, the form of the leading coefficient $A = \left(\frac{B}{m} - m\right)$ can be obtained by the following criterion: $A$ is valid if and only if the Diophantine equation $m^2 + Am - B = 0$ has a solution for some rational $m$, i.e., a rational root. It is interesting to try generalizing the above conjectures to discover the most general rules of this mathematical structure. Additional hints of the mathematical complexity of the yet unknown general structure are related to the existence of generalized Pythagorean triples (see Section 2.7 below).
2.5 Closed-form formula of the GCD and the effective irrationality measure $\delta$

The goal of considering the next sections is to predict exponentially tight formulas for the GCDs, i.e., up to a slower than exponential factor. For each PCF$[a_n, b_n]$, we aim to find both the exponential order $\lambda$ and the closed-form expression for the GCD that yields this $\lambda$. Representative examples are provided in Table 1 below. This table includes PCF examples of different types, some for which we found (conjectured) the exponentially tight formulas and others for which we did not.

| PCF        | $b_n$   | $a_n$  | $\lambda$  | $\delta$ | Remarks                                      |
|------------|---------|--------|-------------|----------|----------------------------------------------|
| $0 + \frac{2}{1} + \frac{5}{2 + \ldots}$ | $n^2 + 1$ | $n$    | $0$ - no FR | $-1$     | $b_n$ cannot be split into 2 equal degree rational polynomials |
| $-15 + \frac{250}{5 + \frac{750}{25 + \frac{1500}{\ldots}}}$ | $125n^2 + 125n$ | $4n^2 - 2n$ | $0.5370 \approx \frac{5}{2 \cdot \sqrt{3} \cdot e}$ | $-0.58$ | $5^n$ is due to inflation; see Appendix C |
| $2 + \frac{2}{5 + \frac{12}{8 + \ldots}}$ | $4n^2 - 2n$ | $3n + 2$ | $0.5413 \approx 2 \cdot \frac{2}{e^2}$ | $-0.31$ | Equivalent representation $\text{GCD} \doteq n! \cdot \left(\frac{2^n}{\text{LCM}[2n]}\right)$ |
| $5 + \frac{18}{14 + \frac{55}{23 + \frac{112}{\ldots}}}$ | $10n^2 + 7n + 1$ | $9n + 5$ | $0.2180 \approx ?$ | $-0.4$ | “Zebra” pattern (see below). 2, 5, and 11 are exponentially coprime to the GCD |
| $1 + \frac{4}{4 + \frac{28}{7 + \ldots}}$ | $9n^2 - 3n - 2$ | $3n + 1$ | $3$ | $1$ | This PCF is an inflation of the 
regular continued fraction of $\varphi$; see Appendix C |

| GCD        | The GCD growth rate is slower than a factorial | $\frac{5^n}{2^{n/2} \text{LCM}[n]}$ | $(2n + 1)!! \cdot \frac{2^n}{\text{LCM}[2n]}$ | $? \text{!! is double factorial}$ | $(3n + 1)!! \text{!! is triple factorial}$ |

*Table 1: PCF examples with different GCD formulas.* The presented GCD formulas differ from the exact GCD by sub-exponential factors.
The term “exponentially coprime to the GCD” generalizes the idea of a coprime and means that the highest powers of \( p \) dividing the GCD for the terms in the sequence increase sub-exponentially. This statement implies that a certain prime does not affect the reduction.

Generalizing from these examples and many more (Appendix E), a conjectured structure of the exact GCD forms (up to sub-exponential factors) is presented next. Note that there exist multiple equivalent ways to present some of the forms, for example using

\[
(2n - 1)!! \cdot \frac{2^n}{\text{LCM}[2n]} = n! \cdot \frac{\binom{2n}{n}}{\text{LCM}[2n]} \quad \text{(as shown in Table 1)}.
\]

**Conjecture 2 (The exact forms of the GCD). We can represent every GCD as a multiple of two parts, a factorial and an exponential expression:**

- **The factorial part in general appears in the form** \((un + v)^{u(v)} \cdot \deg(a_n)\), where \( u, v \in \mathbb{N} \) and \((\cdot)^{u(v)}\) is multifactorial of order \( u \).
  
  *(The special case of \( n! \) corresponds to \( u = 1, v = 0, \) and \( \deg(a_n) = 1 \).)*

- **The exponential part takes one of the following forms or their multiples:**
  
  - **Power of a prime** \( p^{\Theta(n)} \)
    
    o **In the numerator:** We found only integer or half-integer powers, such as \( 5^n \) and \( 7^{[n/2]} \). In some PCFs, these powers can be explained by inflation.
    
    o **In the denominator:** Only primes \( p \) raised to the power of \( \left\lfloor \frac{n}{p - 1} \right\rfloor \deg(a_n) \), such as \( 11^{[n/10]} \) and \( 2^n \). Note that this exponent \( \left\lfloor \frac{n}{p - 1} \right\rfloor \) conforms to the highest power of \( p \) that divides the factorial expression \((un + v)^{u(v)}\).
(when \( u \) and \( p \) are coprime). Hence, when this exponent appears in the denominator, the GCD is exponentially coprime with \( p \).

- When part of the expression is a “Zebra” (see below), we find more complicated fractional powers in the numerator, such as \( 2^{[3n/4]} \).

- \( \text{LCM}[f \cdot n] \) for some \( f \in \mathbb{N} \): For example, Apéry’s work has \( \text{LCM}[n] \), and Table 1 shows a case of \( \text{LCM}[2n] \). This is seen only in denominators.

- Zebra: There is an additional pattern for which we lack an explicit formula. We find this pattern in the denominators. We can identify this pattern in many PCFs but do not entirely understand it. The investigation of the Zebra pattern is left to future work.

For computational simplicity, most of our numerical analysis is focused on PCFs of \( \deg b_n = 2 \) and \( \deg a_n = 1 \). Based on this analysis and additional simulations, we conjecture that the above description captures any GCD sequence of a PCF, also of the higher order \( a_n, b_n \). Furthermore, we expect analogous mathematical structures to exist in the GCDs of any linear recursion with polynomial coefficients, the investigation of which remains for future work. Note that, for PCFs without FR, numerical analyses show that \( \Theta(n) \) primes in \([2, n]\) are exponentially coprime to GCD.

### 2.6 Fast calculation of PCFs using simplified recursion formulas and FR

In this section, we discuss an application of the ability to predict the exact formulas of FR and other forms of reduction. Provided we have a closed-form formula for the GCD, we can apply the reduction straight to the recursion, so that the computation is performed with smaller integer values. Such simplified recursions enable faster estimation of the
PCF limit. The computation advantage of such recursion is substantial with FR: it requires only manipulating sequences that grow exponentially with the PCF depth (instead of super-exponentially).

Example: A simple recursion for the reduced numerator and denominator

For $b_n = 2n^2 + n$, $a_n = n$, we find (numerically) that the GCD is $\frac{n!}{2^n}$ (up to a sub-exponential factor), and therefore, there exist integer sequences $p'_n$ and $q'_n$ such that

$$p_n = \frac{n!}{2^n} \cdot p'_n$$

$$q_n = \frac{n!}{2^n} \cdot q'_n.$$ 

In other words, $\text{GCD}[p_n, q_n] = \frac{n!}{2^n}$ and thus $\text{GCD}[p'_n, q'_n] = 1$, so that the majority of the original GCD is being used in this reduction. We can substitute in the recursion that $p_n$ and $q_n$ both uphold

$$u_n = a_n u_{n-1} + b_n u_{n-2},$$

and yield a recursion for $p'_n$ and $q'_n$ (after simple manipulations)

$$n(n-1)u'_n = 2n u'_{n-1} + 4u'_{n-2},$$

which has rational coefficients. This recursion generates the reduced numerator and denominator sequences. In fact, for any integer initial values, this recursion generates an integer sequence. Additional pairs of polynomials $a_n, b_n$ with the same properties are presented in Appendix E.
2.7 Hints for a deeper mathematical structure

This section provides additional examples of special mathematical properties that we found numerically and hint at a much wider theory that still awaits discovery.

For every \( b_n^{(x)} = x^2(2n^2 + n) \), \( x \in \mathbb{Z} \), there exist families of \( a_n \) for which \( \text{PCF}[a_n, b_n] \) has FR. These \( a_n \) families include

\[
a_n = z^{(x)} \cdot n + k,
\]

where \( k \) is an integer, but the options for integer \( z^{(x)} \) are finite. These \( z^{(x)} \)s are precisely the integers for which there exists a (-6)-Pythagorean triple \( (x, y, z) \) for some \( y \); i.e., the Diophantine equation \( z^2 = x^2 + y^2 + 6xy \) has a solution. This was discovered with the help of OEIS [27]. This result is a special case that coincides with the general structure we discovered for deg \( b_n = 2 \) (see Conjecture 1.4).

Apéry wrote two more pairs of polynomials, the PCFs of which prove the irrationality of \( \zeta(2) \) and \( \ln 2 \). After considering Conjecture 1.2 (infinite families for a given \( b_n \)), we looked for these families with the others \( a_n \)s. For \( \ln 2 \), where \( b_n = -n^2 \), we discovered \( a_n = k(2n + 1), k \in \mathbb{Z} \) as the particular structure for deg \( b_n = 2 \) predicts. Moreover, for odd \( k \)s, such as Apéry’s \( (k = 3) \), we get \( \text{GCD} \approx \frac{n!}{\text{LCM}[n]} \), and for even \( k \)s, \( \text{GCD} \approx \frac{n!}{2^n \text{LCM}[n]} \).

A generalized proof for this case, even without knowing the PCFs limits, is available in Appendix D. The theorem in the next section shows how almost any \( k \) constructs a PCF that proves the irrationality of its limit, although, apart from \( \ln 2 \), the identity of these irrational limits is still unknown to us.
2.8 Infinite $a_n$s that prove irrationality for a given $b_n$

The next section shows infinite families of PCFs that prove irrationality of certain numbers. Specifically, we conjecture that for any $b_n$, there exists an infinite set of $a_n$s such that each constructs a PCF that proves the irrationality of its limit.

**Theorem 3.** We consider families of PCFs of the second type ($\frac{\deg b_n}{\deg a_n} = 2$) that have FR and are created from a constant $b_n$, and $a_n^{(k)}$ that are multiples of a single polynomial, i.e., $a_n^{(k)} \in \{k \cdot a_n^{(1)} | k \in \mathbb{Z}\}$. Assuming that the exponential orders of the GCDs $\lambda$ is bounded as a sequence in $k$ (based on part 4 of **Conjecture 1**'s summary), we find

$$\lim_{k \to \infty} \delta = 1.$$

In particular, for large enough $k$s, the limit will be irrational since $\delta > 0$.

**Proof:** (Straightforward) If $k \to \infty$, then the leading coefficient of $a_n^{(k)}$ uphold $A_k \to \infty$ and the characteristic equation $\alpha^2 = \alpha A_k + B$ has a solution that certifies $\alpha_k \to \infty$. Substituting in **Theorem 2** with a constant $B$, while assuming $\ln \lambda$ is bounded, we have

$$\delta = \frac{\ln \alpha - \ln |B| + \ln \lambda}{\ln \alpha - \ln \lambda} \to \frac{\ln \alpha}{\ln \alpha} = 1. \quad \square$$

**Observation:** Combining this theorem and **Conjecture 1.4**, we expect that, for any $b_n$ of degree 2 with rational-only roots, there exists an infinite set of $a_n$s such that PCF[$a_n, b_n$] proves the irrationality of its limit. As for higher degrees, we conjecture the existence of similar structures.
2.9 Additional properties of the greatest common divisors

We investigate additional results that can help prove properties about \( \text{GCD}[p_n, q_n] \), for all PCFs cases, either with or without FR. Thus far in our paper we analyzed the growth rate of the sequence \( \text{GCD}[p_n, q_n] \) as a function of the PCF’s depth, \( n \). One property that we examined and believe could be useful for proving some of our conjectures is whether \( \text{GCD}[p_n, q_n] \) divides its consecutive \( \text{GCD}[p_{n+1}, q_{n+1}] \). We find that this does not hold for the definition of the GCD sequence. For example, in most of the fractions, we encountered prime factors of the GCD that do not divide the consecutive GCD, i.e., \( \text{GCD}_n \nmid \text{GCD}_{n+1} \).

The above observation motivates the study of the GCD of two consecutive numerators and denominators:

\[
\text{GCD}_2 \equiv \text{GCD}[p_n, q_n, p_{n-1}, q_{n-1}].
\]

By this definition and the recursion formula for \( p_n \) and \( q_n \), one can show that

\[
\text{GCD}_2 \mid \text{GCD}_{2n+1}.
\]

Since \( \text{GCD}_2 \mid \text{GCD}_n \), part of the reduction may be explained by the \( \text{GCD}_2 \). It remains to be seen what part of the FR and its exponential part is contained in \( \text{GCD}_2 \). Having inspected many PCFs numerically, with or without FR, we conjecture the following.

**Conjecture 3.** For any PCF:

\[
\text{GCD}_2 \equiv \text{GCD},
\]

i.e.,
\[
\gcd \left[ p_n, q_n, p_{n+1}, q_{n+1} \right] = \gcd \left[ p_n, q_n \right].
\]

The meaning of this exponentially tight equality is that all the theorems and conjectures presented here may apply also for \( \gcd_2 \). Specifically, if FR exists, then both the factorial and the exponential part of the GCD will exist in \( \gcd_2 \). The important consequence is that we can use either GCD or \( \gcd_2 \) for purposes of irrationality proofs, such as \textbf{Theorem 2}.

This conjecture enables us to treat the GCD as a growing product of some integer series and, at a given depth \( n \), calculate and reduce only one integer term: \( \frac{\gcd_2(n)}{\gcd_2(n-1)} \). For example, if \( \gcd_2 = \frac{n!}{\text{LCM}[n]} \), we can reduce the numerators and the denominators \textbf{at each depth} \( n \) by the factor \( n/p \) if \( n \) is a power of some prime number \( p \) and by \( n \) if it is not.

Moreover, this definition is advantageous because \( \gcd_2 \mid \gcd \), and it thus sorts out sub-exponential factors that have no effect on proving irrationality. This observation facilitates the numerical analysis and helps identify the exact formula for the GCD.

As a side note, \textbf{Conjecture 3} helps show that the GCD of PCFs that have FR always has a factorial term such as \((n!)^d\), rather than a term such as \( n^{d \cdot n} \) (which also grows like \((n!)^d\) up to an exponential factor by Stirling’s approximation). In fact, all the PCFs with FR that we encountered could be written as \((n!)^d \cdot \frac{S(n)}{R(n)}\) with \( S(n) \) and \( R(n) \) being integer sequences that grow exponentially. Some cases are more complex, such as when \( \gcd = (3n + 1)!!! \), but these do not contradict the above statement. It would be interesting to try to prove this phenomenon.
3 Discussion and Open Questions

3.1 Outlook and motivation

By their further development, the conjectures presented can provide useful tools for irrationality proofs, as well as for fast calculations of polynomial integer recursions of mathematical constants.

Specifically, the results related to FR can be applied to shrink the search space of the Ramanujan Machine algorithms [10]. By focusing on PCFs with FR, the algorithms would have a better chance to find new conjectures that are simultaneously of a relatively fast computation time and have nontrivial $\delta$s that we can extract. That is, removal of the cases that have no FR avoids all the hard-to-compute PCFs that also provide trivial $\delta$s.

Looking forward, we believe that by generalizing the mathematical structure of PCFs with FR, it would be possible to find universal structures in PCFs made from arbitrary-degree polynomials. As a more ambitious step, it is interesting to consider deeper linear recursions (beyond depth 2), which can also be harnessed to find new conjectures. One can search for analogous algebraic structures and ideas as presented above.

In the following, we present several ideas and open questions that arise from our mathematical experiments and from our conjectures. These open questions may be simple or complex, and we hope that they can engender more ideas for future research in different communities.
3.2 Implications of FR for a faster computation of PCFs

Once a closed formula for the GCD has been found, numerical calculations of PCFs will become easier and faster since the FR decreases considerably the numbers participating in the arithmetic operations. In particular, PCFs with FR benefit greatly from this reduction since \( p_n \) and \( q_n \) decrease from a super-exponentially (factorial) growth to exponential growth. In other words, finding the exact formula for the reduction enables one to construct a simpler recursion formula that directly gives the reduced numerators and denominators.

3.3 Families of PCFs

Following Conjecture 1.2, it is natural to try to generalize the families of \( a_n, b_n \) for higher degrees. What affects the number of families and subfamilies? Conjecture 1.4 claims that only two families exist for the discussed degrees, and one of them is branched into several subfamilies. In this case, the number of subfamilies depends merely on the number of divisors of \( B \) (the leading coefficient of \( b_n \)). We do not yet have a solid and more general conjecture that relates to all degrees. Another question regarding families of \( a_n \) or \( b_n \) is whether a relation exists between the limits of any sibling PCFs. For example, if this relation hints that the limits are equivalent, for proofs of irrationality, it will suffice to find just one limit and use Theorem 3 (infinite \( a_n \)s that proves irrationality).

3.4 Finding and proving the exact form of the GCD

We did not find the exact form of the GCD, but nevertheless tried to list the different types of expressions that comprise it. The motivation to find the closed form of the GCD is the possibility of writing a reduced recursion that yields the reduced numerators and
denominators, which can simplify any numerical calculation of the PCF. Moreover, a closed-form formula would also directly predict the effective irrationality measure $\delta$ given by the PCF.

We note that Apéry proved his case by finding an explicit expression for the PCF at each depth. As an example of taking a more general approach, in Appendix D we address a family of GCDs and bypass the need for an explicit expression. As examples that can promote future research, we present in Appendix E a set of unproven examples that yield precisely the same simplified recurrence relations.

3.5 Predicting the exponential order $\lambda$:

To search for conjectures in the form of PCFs that prove the irrationality of constants, it suffices to predict only the exponential order $\lambda$. Using this value, Theorem 2 calculates the effective irrationality measure $\delta$. It remains to find a direct relation from $a_n, b_n$ to $\lambda$. 
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5 Appendix

Appendix A: Classification of polynomial continued fractions (PCFs)

All PCFs can be split into three types by the ratio of the degrees of the polynomials $a_n, b_n$ (Table 2). This ratio determines the PCF’s convergence rate and the growth rate of $p_n$ and $q_n$. Part of the classification depends on the leading coefficients of $a_n$ and $b_n$, denoted by $A$ and $B$, respectively.

| Degree ratio | $\frac{\deg b_n}{\deg a_n} > 2$ | $\frac{\deg b_n}{\deg a_n} = 2$ | $\frac{\deg b_n}{\deg a_n} < 2$ |
|--------------|---------------------------------|---------------------------------|---------------------------------|
| Additional requirements | Converge: $B > 0$ and some cases of $B < 0^*$ | Converge: $B > -A^2/4$ and some cases of equality* | Always converge |
| Examples (from [10]) | $2 + \frac{\pi}{4} \pi = \frac{\pi}{4}$ $2 + \frac{\pi}{4}$ $\frac{\pi}{4}$ | $1 + \frac{1}{\pi} \frac{\pi}{3}$ $1 + \frac{1}{\pi}$ $\frac{\pi}{3}$ | $2 + \frac{1}{\pi} \frac{\pi}{6}$ $2 + \frac{1}{\pi} \frac{\pi}{6}$ |
| Does this PCF provide a nontrivial $\delta$? | Provides $\delta$ only if there is a reduction of $q_n$ after which $q_n/GCD$ is sub-exponential. Namely, it is necessary that $\text{GCD} \equiv q_n$ | Provides $\delta$ iff there is FR (see Theorem 1). $\delta$ depends on the reduction following Theorem 2 | Always provides $\delta = 1 - \frac{\deg b_n}{\deg a_n}$ or better if there is FR. Proves irrationality even without FR if $\deg b_n < \deg a_n$ (coincides with Tietze’s criterion [26]) |

Table 2: Summary of the three types of PCFs, partitioned by the ratio of the degrees of $a_n, b_n$. We show the conditions for each type that leads to a nontrivial effective irrationality measure $\delta$. Note the crucial role of FR in the second type (middle column), which is at the core of this manuscript. Proofs of part of the regimes of convergence can be found in [10]. *There are cases for which we do not know the conditions for convergence.
In most of this paper, we focused on the second PCF type, that of balanced-degree PCFs. PCFs of this type are those that prove the irrationality of $\zeta(3)$, $\zeta(2)$, and $\ln(2)$ in Apéry’s work [1,2] and many more mathematical constants. For PCFs of this type, Appendix B shows that $p_n \equiv q_n \equiv n! \deg a_n \cdot \alpha^n$, with $\alpha$ being the larger root (in absolute value) of the characteristic equation: $x^2 = Ax + B$. When there are no real roots or only one root, the PCF does not necessarily converge. This case may still create a useful Diophantine sequence, but such an investigation is beyond the scope of this work.
Appendix B: Proof for Theorems 1 and 2

For each PCF, we estimate the convergence ratio in relation to the denominator growth rate.

Step 1: Estimating the PCF’s convergence rate.

Denote the partial numerators and denominators of the PCF by $p_n$ and $q_n$; these answer the following recursion formula:

$$u_{n+1} = a_{n+1}u_n + b_{n+1}u_{n-1}.$$ 

For $L \overset{\text{def}}{=} \lim_{n \to \infty} \frac{p_n}{q_n}$, define the finite calculation error by $e_n \overset{\text{def}}{=} \frac{p_n}{q_n} - L$.

Now, from the determinant formula [10] we obtain

$$p_{n+1}q_n - p_nq_{n+1} = (-1)^n \prod_{i=1}^{n} b_i$$

$$e_{n+1} - e_n = \frac{p_{n+1}q_n - p_nq_{n+1}}{q_{n+1}q_n} = (-1)^n \frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n}$$

Here, we encounter an assumption that is justified below. If $\frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n}$ decreases exponentially or faster (super-exponentially), then one can write

$$e_n = \sum_{m=n}^{\infty} (e_m - e_{m+1}) = \sum_{m=n}^{\infty} (-1)^m \frac{\prod_{i=1}^{m} b_i}{q_{m+1}q_m},$$

$$e_n = O\left(\frac{\prod_{i=1}^{n} b_i}{q_{n+1}q_n}\right).$$
and since \( q_{n+1} \equiv q_n \), we use \( \prod_{i=1}^{n} b_i \). If the assumption is wrong (Case 1 below), then by numerical tests we conjecture \( e_n \) to be polynomial.

**Step 2: Exponentially tight estimation for \( a_n, b_n, \) and \( q_n \).**

Denoting \( d_a = \deg a_n \) and \( d_a = \deg a_n \), one can estimate the polynomials using their leading coefficients, \( A, B \), and some positive constant \( c \):

\[
|a_n - A \cdot n^{d_a}| \leq c \cdot A \cdot n^{d_a - 1}
\]

\[
|b_n - B \cdot n^{d_b}| \leq c \cdot B \cdot n^{d_b - 1}.
\]

Therefore,

\[
B \cdot n^{d_b} \left( \frac{n - c}{n} \right) \leq b_n \leq B \cdot n^{d_b} \left( \frac{n + c}{n} \right)
\]

\[
\frac{\text{const}}{(n + c)^c} \leq \prod_{i=1}^{n} \frac{i - c}{i} \leq \prod_{i=1}^{n} \frac{b_i}{B^n \cdot n^{d_b}} \leq \prod_{i=1}^{n} \frac{i + c}{i} \leq (n + c)^c \cdot \text{const}
\]

\[
\Rightarrow \prod_{i=1}^{n} b_i \equiv B^n \cdot n^{d_b}.
\]

Now, we need to split the fractions into three cases by the ratio \( \frac{d_b}{d_a} \). Each case differs by the growth rate of \( q_n \), which is affected by the significant element(s) in the next equation:

\[
q_{n+1} = a_n q_n + b_n q_{n-1}.
\]
Note that the split occurs because, in either case, \( q_n \doteq \alpha^n \cdot n!d^n \) for some \( d, \alpha \) and can be justified by estimating the multiplicative error as presented below for the second case only.

**First case**: \( \frac{d_b}{d_a} > 2 \), and therefore, the significant element is \( b_n \):

\[
q_{n+1} = B n^{d_b} q_{n-1} \doteq \frac{n}{2^d} \cdot n^{d_b}.
\]

**Second case**: \( \frac{d_b}{d_a} = 2 \), and therefore, both \( a_n \) and \( b_n \) are significant and the PCF is “balanced”.

Denoting \( d = d_a = \frac{d_b}{2} \), we show inductively that \( \forall \epsilon > 0, \exists K \in \mathbb{R}^+ \) s.t.

\[
K(1 - \epsilon)^n \leq \frac{q_n}{\alpha^n \cdot n!d} \leq K(1 + \epsilon)^n
\]

for all \( n \), implying

\[
q_n \doteq \alpha^n \cdot n!d
\]

by definition (which is \( \lim_{n \to \infty} \frac{1}{n} \log \frac{q_n}{\alpha^n \cdot n!d} = 0 \)).

Set \( \epsilon > 0 \), from some \( n_0 \) onward; these two inequalities hold

\[
1 + \frac{c}{n} \leq 1 + \epsilon
\]

\[
\frac{n}{n+1} \cdot \left(1 - \frac{c}{n}\right) \geq 1 - \epsilon.
\]

Choose \( K \) s.t. for all \( n \leq n_0 \)

\[
K(1 - \epsilon)^n \leq \frac{q_n}{\alpha^n \cdot n!d} \leq K(1 + \epsilon)^n.
\]
Now, assume inductively that the above inequality holds till some $n \geq n_0$, and for $n + 1$ write

$$q_{n+1} = a_n q_n + b_n q_{n-1}.$$ 

$$q_{n+1} \leq An^d \left(1 + \frac{c}{n}\right) \cdot \alpha^n \cdot n^d \cdot K(1 + \epsilon)^n + Bn^{2d} \left(1 + \frac{c}{n}\right) \cdot \alpha^{n-1} \cdot (n - 1)!^d \cdot K(1 + \epsilon)^{n-1}$$

$$\leq \alpha^{n-1} (\alpha A + B) \cdot (n + 1)!^d \cdot K \left(1 + \frac{c}{n}\right) (1 + \epsilon)^n \leq 1 + \epsilon (1 + \epsilon)^{n+1}.$$ 

$$q_{n+1} \geq An^d \left(1 - \frac{c}{n}\right) \cdot \alpha^n \cdot n^d \cdot K(1 - \epsilon)^n + Bn^{2d} \left(1 - \frac{c}{n}\right) \cdot \alpha^{n-1} \cdot (n - 1)!^d \cdot K(1 - \epsilon)^{n-1}$$

$$\geq \alpha^{n-1} (\alpha A + B) \cdot (n + 1)!^d \cdot K \left(1 - \frac{c}{n}\right) (1 - \epsilon)^n \geq 1 - \epsilon (1 - \epsilon)^{n+1}.$$ 

By combining these two, we obtain

$$K(1 - \epsilon)^{n+1} \leq \frac{q_{n+1}}{\alpha^{n+1} \cdot (n + 1)!^d} \leq K(1 + \epsilon)^{n+1},$$

as required for $q_{n+1}$.

**Third case:** $\frac{d_b}{d_a} < 2$, and therefore, the significant element is $a_n$.

$$q_{n+1} \doteq An^{da} q_n \doteq A^n \cdot n^{da}.$$
Step 3: Combining these results with the irrationality criterion.

First, without reducing \( q_n \),

\[
\left| \frac{p_n}{q_n} - L \right| < \frac{1}{q_n^{1+\delta}}
\]

\[
\downarrow
\]

\[
\frac{|\prod b_n|}{q_n^2} < \frac{1}{q_n^{1+\delta}}
\]

\[
\delta < \frac{\ln(q_n) - \ln(|\prod b_n|)}{\ln(q_n)}
\]

and second, when we reduce \( q_n \) by \( \text{GCD} = \gcd(p_n,q_n) \), we obtain better results. The finite calculation error remains the same since the reduced fraction represents the same number. However, the real denominator becomes smaller, and therefore, only the right hand side changes, and the inequality transforms to

\[
\left| \frac{p_n}{q_n} - L \right| < \frac{1}{(q_n/\text{GCD})^{1+\delta}}
\]

\[
\downarrow
\]

\[
\frac{|\prod b_n|}{q_n^2} < \frac{1}{(q_n/\text{GCD})^{1+\delta}}
\]

\[
\delta < \frac{\ln(q_n) - \ln(|\prod b_n|) + \ln(\text{GCD})}{\ln(q_n) - \ln(\text{GCD})}
\]
Notice that, as expected, the bigger the GCD, the bigger is $\delta$. We use the equality sign for the limit (inferior) as $n$ tend to infinity, as this value yields a lower bound on the irrationality measure.

For all cases, note that $\ln n! \in \Theta(n \ln n)$, $\prod_n b \doteq B^n \cdot n!^{d_b}$, and denote some sub-exponential pre-factors such as $E_n^q, E_n^b, E_n^G$, which are all $\doteq 1$.

First case:

$$\frac{|\prod_n b|}{q_n^2} \doteq \frac{|B|^n \cdot n!^{d_b}}{\left(B^x \cdot n!^{\frac{d_b}{2}}\right)^2} \doteq 1,$$

and therefore, the assumption is not justified, and the convergence rate is sub-exponential. For this reason, to provide nontrivial $\delta$, the GCD must be exponentially equal to $q_n$, so that both sides of the inequality will decrease sub-exponentially, and a more delicate analysis is required. In conclusion, the condition $\gcd \doteq q_n \doteq B^{n/2} \cdot n!^{d_b/2}$ is necessary but not sufficient for yielding nontrivial $\delta$.

Second case:

$$q_n \doteq \alpha^n \cdot n!^{d_a}, d_b = 2d_a$$

$$\frac{\prod_n b}{q_n^2} \doteq \frac{|B|^n \cdot n!^{d_b}}{(\alpha^n \cdot n!^{d_a})^2} = \left(\frac{|B|}{\alpha^2}\right)^n$$

and the assumption holds, except for $|B| = \alpha^2$. 
Without reduction,

\[
\frac{\ln(q_n) - \ln(\prod b)}{\ln(q_n)} = \frac{d_a n \ln n + n \ln \alpha + \ln(E_n^q) - 2d_a n \ln n - n \ln|B| - \ln(E_n^b)}{d_a n \ln n + n \ln \alpha + \ln(E_n^q)}
\]

\[
\to -1
\]

by regarding the largest terms (in bold), and therefore, we have trivial \(\delta\).

On the contrary, with reduction

\[
\frac{\ln(q_n) - \ln(\prod b) + \ln(\text{GCD})}{\ln(q_n) - \ln(\text{GCD})} = \frac{d_a n \ln n + n \ln \alpha + \ln(E_n^q) - 2d_a n \ln n - n \ln|B| - \ln(E_n^b) + \ln(\text{GCD}) + \ln(E_n^G)}{d_a n \ln n + n \ln \alpha + \ln(E_n^q) - \ln(\text{GCD}) - \ln(E_n^G)}
\]

Here comes the crucial part of the proof for **Theorem 1**: To have nontrivial \(\delta\), this expression must not converge to \(-1\) as \(n\) tends to infinity. Thus, the GCD must contain a super-exponential factor at the size \(n!d_a\) – i.e., factorial reduction. Note that a bigger super-exponential factor is not possible since \(p_n = q_n = n!d_a \cdot \alpha^n\).

Now, to prove **Theorem 2**, inserting \(\text{GCD} = n!d_a \cdot \lambda^n\) in the last equation gives

\[
\delta = \frac{d_a n \ln n + n \ln \alpha + \ln(E_n^q) - 2d_a n \ln n - n \ln|B| - \ln(E_n^b) + d_a n \ln n + n \ln \lambda + \ln(E_n^G)}{d_a n \ln n + n \ln \alpha + \ln(E_n^q) - d_a n \ln n - n \ln \lambda - \ln(E_n^G)}
\]

\[
\to \lim_{n \to \infty} \frac{\ln \alpha - \ln|B| + \ln \lambda}{\ln \alpha - \ln \lambda},
\]

which matches the expression in **Theorem 2**. Note that if \(\lambda = \alpha\) then \(\delta\) is not bounded (infinite irrationality measure) and the limit is a Liouville number.
Third case:

\[
\frac{\prod_n b}{q_n^2} = \frac{|B|^n \cdot n!^{d_b}}{(A^n \cdot n!^{d_a})^2} \approx \left( \frac{|B|}{A^2} \right)^n \cdot n!^{d_b - 2d_a},
\]

and since \(d_b - 2d_a < 0\), the assumption holds.

Without reduction

\[
\frac{\ln(q_n) - \ln(\prod_n b)}{\ln(q_n)} = \frac{d_a n \ln n + n \ln A + \ln(E_n^q) - d_b n \ln n - n \ln|B| - \ln(E_n^b)}{d_a n \ln n + n \ln A + \ln(E_n^q)} \quad \longrightarrow \quad \frac{d_a - d_b}{d_a},
\]

which is positive if \(d_a > d_b\) and proves irrationality.

With reduction: To change the limit above, the GCD must be of factorial order. If it is, and the factorial power is \(r\) (the exponential factors have no effect), then

\[
\frac{\ln(q_n) - \ln(\prod_n b) + \ln(\text{GCD})}{\ln(q_n) - \ln(\text{GCD})} = \frac{d_a n \ln n + n \ln A + \ln(E_n^q) - d_b n \ln n - n \ln|B| - \ln(E_n^b) + r n \ln n + \ln(E_n^G)}{d_a n \ln n + n \ln A + \ln(E_n^q) - r n \ln n - \ln(E_n^G)} \quad \longrightarrow \quad \frac{d_a - d_b + r}{d_a - r},
\]

which is better than the without reduction. Here, if \(r = d_a\), then \(\delta\) is arbitrarily large and the limit is a Liouville number.
Appendix C: Inflation and deflation of continued fractions

In his paper, Apéry showed a linear recursion of depth 2 with rational function coefficients (ratio of two polynomial) and a related PCF. The direct translation of Apéry’s recursion into a continued fraction has $a_n$ and $b_n$ as rational functions and not integer polynomials. However, they can be converted to a PCF form. To see the conversion, we multiply $a_n$ and $b_n$ by a non-zero sequence, thus converting them to integer polynomials without changing the limit. We call this process “inflation”. This process is also needed when some rational coefficient is used in Conjecture 1.4.

Conversely, any PCF that has been multiplied by a non-zero sequence can be simplified by removing that sequence. We call this process deflation. Deflating makes the PCFs’ $b_n$ and $a_n$ smaller (possibly of a lower degree), and most importantly, helps simplify the GCDs, despite not changing the induced $\delta$. This process can explain some powers of prime in Table 1.

Identity 1 (Inflation of continued fractions). Let $c_n$ be a general sequence of non-zero complex numbers. It is straightforward to show that

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \ldots}}} = a_0 + \frac{c_1 b_1}{c_1 a_1 + \frac{c_1 c_2 b_2}{c_2 a_2 + \frac{c_2 c_3 b_3}{c_3 a_3 + \ldots}}},$$

since these two continued fractions give the same value at any finite depth; i.e., if $p_n$ and $q_n$ are the numerator and denominator of the right hand side and $p'_n$ and $q'_n$ are those of the left hand side, then for every $n$,
\[
\frac{p_n}{q_n} = \frac{p'_n}{q'_n}.
\]

Example: the relation between Apéry's recursion and his PCF

Setting \(a_n = \frac{34n^3 + 51n + 27n + 5}{(n+1)^3}\) and \(b_n = -\frac{n^3}{(n+1)^3}\), Apéry used the recursion

\[
u_{n+1} = a_{n+1}u_n + b_{n+1}u_{n-1}
\]

with the initial conditions

\[
p_{-1} = 1, \quad p_0 = 5
\]

\[
q_{-1} = 0, \quad q_0 = 1,
\]

which generates the following rational continued fraction:

\[
\frac{6}{\zeta(3)} = 5 - \frac{1}{8 - \frac{8}{27 - \frac{27}{64 - \frac{n^3}{(n+1)^3}}}}.
\]

Applying Identity \textbf{1}, we can inflate this continued fraction using the denominators of \(a_n\) and \(b_n\), i.e., the sequence \(c_n = (n + 1)^3\), and obtain the PCF
\[
\frac{6}{\zeta(3)} = 5 - \frac{1}{8 \cdot 2^3} - \frac{8}{27 \cdot 2^3 \cdot 3^3} - \frac{27}{64 \cdot 4^3 \cdot 3^3} - \frac{n^3}{(n + 1)^3 \cdot n^3(n + 1)^3} - \frac{34n^3 + 51n + 27n + 5}{(n + 1)^3 \cdot n^6} 
\]

\[
= 5 - \frac{1}{117 - \frac{535}{27 - \frac{1463}{64 \cdot 4^3 - \cdots}} - \frac{1}{34n^3 + 51n^2 + 27n + 5}} 
\]

that is, Apéry’s PCF, which is presented in our introduction.

The effect of these processes on the GCD remains to be seen. For this reason, the following theorem is presented.

**Theorem 4.** Consider the recursions

\[
u'_{n+1} = a'_{n+1}u'_n + b'_{n+1}u'_{n-1}
\]

\[
u_{n+1} = a_{n+1}u_n + b_{n+1}u_{n-1},
\]

where \(a'_n = c_n \cdot a_n\) and \(b'_n = c_n c_{n-1} \cdot b_n\) for some non-zero sequence \(c_n\). For the same initial values

\[
u'_n = \left(\prod_{i=1}^{n} c_i\right) \cdot u_n.
\]

**Proof.** The base cases, \(n = -1, 0\), are trivial since the product is empty. Assume this for all \(k \leq n\), and write for \(n + 1\)
\[ u'_{n+1} = a'_{n+1} u_n + b'_{n+1} u'_{n-1} = c_{n+1} a_{n+1} u_n + c_{n+1} c_n b_{n+1} u_{n-1} \]

\[ \begin{align*}
    c_{n+1} a_{n+1} \left( \prod_{i=1}^{n} c_i \right) u_n + c_{n+1} c_n b_{n+1} \left( \prod_{i=1}^{n-1} c_i \right) u_{n-1} = \\
    \left( \prod_{i=1}^{n+1} c_i \right) \left( a_{n+1} u_n + b_{n+1} u_{n-1} \right) = \left( \prod_{i=1}^{n+1} c_i \right) u_{n+1}.
\end{align*} \]

**Corollary 1.** We obtain the proof for the identity since

\[ \frac{p'_n}{q'_n} = \frac{(\prod_{i=1}^{n} c_i)p_n}{(\prod_{i=1}^{n} c_i)q_n} = \frac{p_n}{q_n} \]

**Corollary 2.** For an inflated PCF, or any inflated rational GCF

\[ \text{GCD}[p'_n, q'_n] = \text{GCD} \left[ \left( \prod_{i=0}^{n} c_i \right) p_n, \left( \prod_{i=0}^{n} c_i \right) q_n \right] = \left( \prod_{i=0}^{n} c_i \right) \text{GCD}[p_n, q_n]. \]

**Example:**

Considering the following PCF from Table 1:

\[ \text{PCF}[a'_n, b'_n] = 1 + \frac{3}{3 + \frac{15}{5 + \frac{35}{7 + \frac{(3n+1)(3n-2)}{3n+1}}} \right], \]

and observe that it is inflated by the sequence \( c_n = 3n + 1. \) By deflating it, we obtain the

*regular* continued fraction of the golden ratio \( \varphi: \)
\[
1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]

which upholds \( p_n = F_{n+2}, q_n = F_{n+1} \), where \( F_n \) is the \( n \)th Fibonacci number. Therefore, for the original PCF we obtain

\[
\text{GCD}[p'_n, q'_n] = \left( \prod_{i=0}^{n} c_i \right) \text{GCD}[p_n, q_n] = \prod_{i=0}^{n} (3i + 1) = (3n + 1)!!
\]

since consecutive Fibonacci numbers are coprime.

**Inflation by “\( \sqrt{p} \)”:**

An interesting deflation always exists when \( p \mid a_n, p \mid b_n \) but \( p^2 \nmid b_n \) for some \( p \). In this case, GCD contains powers of \( p \) of the forms \( p^{\lfloor n/2 \rfloor} \). For instance, \( \text{GCD} = n! \cdot \sqrt{3}^n \) for PCF\([3n + 6, 3n^2 + 9n]\). We call this case deflation with \( \sqrt{p} \).
Appendix D: Analysis and proof of the GCD formula for a family of PCFs

This section focuses on a family of PCFs for which we found part of the GCD explicitly and proved it. This proof is presented here. We hope that this section will promote future research of additional proofs.

**Theorem 5.** If for all $n$ the PCF’s polynomials satisfy

\[ (*) \quad a_n + a_{-1} = b_n - b_{-n} = b_0 = 0, \]

then the GCD of the corresponding PCF is divisible by $\frac{n!}{2^{\ell} \text{LCM}[n]}$ for some integer $\ell$.

Note that condition (*) involves negative-indexed coefficients that are not used (or defined) by the PCF. Nevertheless, since the coefficients are given by polynomials, we can use the polynomials to extend the sequences to all values of $n$.

**Proof:** Recall that both $p_n$ and $q_n$ satisfy the recursion

\[ (**) \quad u_{n+1} = a_{n+1}u_n + b_{n+1}u_{n-1}. \]

We first prove that for any odd prime $r$, $u_n$ is divisible by $r$ for all $n \geq 2r$. We do so by analyzing the sequence $u_n$ modulo any odd prime $r$.

Let $r = 2h + 1$. We now prove by induction that for all $-1 \leq m < h$

\[ u_{h+m} = u_{h-m-2} \cdot b_h \cdot b_{h-1} \cdot \ldots \cdot b_{h-m} \pmod{r}. \]

Initializing the induction at $m = -1$ is trivial since $u_{h-1} = u_{h-1}$.

Initializing the induction at $m = 0$ requires $u_h = u_{h-2}b_h$ modulo $r$. To prove that, we start by using (*) at $n = h$: 
\[ a_h + a_{-1-h} = 0. \]

Note that for modulo \( r \), every polynomial has a period of \( r = 2h + 1 \), namely

\[ a_{-1-h} = a_h = 0 \pmod{r}. \]

Substituting this into the sequence recursion \((**)\) at \( n = h - 1 \), we obtain the initialization of the induction:

\[ u_h = a_h u_{h-1} + b_h u_{h-2} = b_h u_{h-2} \pmod{r}. \]

To prove the induction at \( m + 1 \), write \((*)\) for \( n = h + m + 1 \)

\[ a_{h+m+1} + a_{-h-m-2} = b_{h+m+1} - b_{-h-m-1} = 0. \]

Using periodicity modulo \( r = 2h + 1 \),

\[ a_{-h-m-2} = a_{h-m-1} \pmod{r} \quad \text{and} \quad b_{-h-m-1} = b_{h-m} \pmod{r}. \]

Combining this with the symmetries, we have

\[ a_{h+m+1} = -a_{h-m-1} \pmod{r} \quad \text{and} \quad b_{h+m+1} = b_{h-m} \pmod{r}. \]

Now, substitute these into the recursion \((**)\) at \( n = h + m \):

\[ u_{h+m+1} = a_{h+m+1} u_{h+m} + b_{h+m+1} u_{h+m-1} = -a_{h-m-1} u_{h+m} + b_{h-m} u_{h+m-1} \pmod{r}. \]

Using the induction assumption at \( m \) and \( m - 1 \),

\[ u_{h+m+1} = -a_{h-m-1} u_{h-m-2} b_h b_{h-1} \ldots b_{h-m} + b_{h-m} u_{h-m-1} b_h b_{h-1} \ldots b_{h-m+1} \pmod{r}. \]

Rearranging, we obtain

\[ u_{h+m+1} = (-a_{h-m-1} u_{h-m-2} + u_{h-m-1}) b_h b_{h-1} \ldots b_{h-m} \pmod{r}. \]
By substituting the recursion (**) at \( n = h - m \), we obtain the induction at \( m + 1 \):

\[
u_{h+m+1} = u_{h-m-3} b_h b_{h-1} \ldots b_{h-m-1} \pmod{r},
\]

completing the proof of the induction.

Using the periodicity of the coefficients modulo \( r \), the relation

\[
u_{h+m} = u_{h-m-2} \cdot b_h \cdot b_{h-1} \cdot \ldots \cdot b_{h-m} \pmod{r}
\]

holds after shifting by \( r \):

\[
u_{r+h+m} = u_{r+h-m-2} \cdot b_{r+h} \cdot b_{r+h-1} \cdot \ldots \cdot b_{r+h-m} \pmod{r}.
\]

Now, set \( h \leq m < r + h \). Then \( b_r \) exists among the multipliers of the right hand side. Combining periodicity and (*), \( b_r = b_0 = 0 \pmod{r} \), and therefore, we finally have \( u_{r+h+m} = 0 \pmod{r} \); in other words, \( r \) divides \( u_n \) starting at \( n = r + 2h = 2r - 1 \).

We now explain why this result suffices to prove the theorem. We proved that for \( n \geq 2r - 1 \), \( r \) divides both \( p_n \) and \( q_n \), and therefore, it divides the GCD, in line with the fact that \( r \) divides \( \frac{n!}{2^i \text{LCM}[n]} \) for \( n \geq 2r \). Furthermore, since \( u_n \) is divisible by \( r \) for \( n \geq 2r - 1 \), we can set a new sequence

\[
u_n^{(1)} = \frac{u_{n-2r}}{r} \pmod{r},
\]

which is well defined for \( n \geq -1 \). Since the sequence is obtained by scaling and shifting the original sequence, it satisfies the same recursion starting at \( n = 1 \). However, unlike
the original sequence, \( u_n^{(1)} \) in fact satisfies the recursion also at \( n = 0 \). This follows from

the fact that \( b_0 = 0 \), and therefore, the recursion at \( n = 0 \) does not involve \( u_{-2}^{(1)} \). Note that

the original sequence \( u_n \) has an arbitrary initial condition at \( u_0 \) and \( u_{-1} \) that may not

satisfy the recursion. This is not the case for \( u_n^{(1)} \) since it was generated by the recursion

even at \( n = 0 \).

We can thus apply the above induction result to prove that \( r \) divides \( u_n^{(1)} \) starting at

\( n = r - 1 \) (instead of \( n = 2r - 1 \) as in the original sequence). In other words, \( r^2 \) divides

the original \( u_n \) starting at \( n = 3r - 1 \). In general, \( r^k \) divides \( u_n \) starting at \( n = (k + 1)r - 1 \). Note that for \( r > \sqrt{n} \), \( r^k \) divides \( \frac{n!}{2^\ell \text{LCM}[n]} \) starting at \( n = (k + 1)r \), and

therefore, the requirement is met. For \( n = k \cdot r^2 \), note that \( n! \) obtains an additional \( r \)

factor. At \( k = 1 \), this factor is canceled by the denominator’s LCM[\( n \)], but for \( k > 1 \) we

must prove this additional factor of the GCD. To do that, note that the original proof that

\( r \) divides \( p_n \) and \( q_n \) for \( n \geq 2r - 1 \) was valid also when substituting prime powers for \( r \).

Thus, at \( n = 2r^2 - 1 \), the sequences obtain a factor of \( r^2 \) instead of the expected factor

of \( r \), as required by the theorem.
Appendix E: Additional examples of PCFs with FR

One may wonder whether the conjectures discovered in this study are indeed mathematical truth or merely mathematical coincidences that break down at higher degrees or larger coefficients. However, the method employed in this study makes it fairly unlikely that the conjectures will break down. Nevertheless, such an assumption does not replace the need for a formal proof. We believe that many (if not all) of the new conjectures are indeed truths awaiting a rigorous proof, not only because of vast search spaces examined in this work, but also by virtue of the aesthetic nature of the conjectures. To strengthen our conjectures, we give additional examples abundantly. Moreover, this appendix addresses three further causes:

1. Visualizing the results
2. Present a piece of the yet unknown general structure toward reveling it whole
3. Lay the groundwork and provide more data for proofs or additional conjectures

FR and rational roots:

**Conjecture 1.1** states that, for a given $b_n$, there exists an $a_n$ s.t. $\text{PCF}[a_n,b_n]$ has FR if and only if $b_n$ can be split into two rational equal-degree polynomials. Table 3 shows a classification, by numerical tests, of all the $b_n$ polynomials from degree 2 and with integer coefficients between 1 and 4. For each such $b_n$, we search for $a_n$ polynomials in the integer coefficient range 1 to 5.
| $b_n$          | $b_n$’s roots | $a_n$ example | $b_n$          | $b_n$’s roots | $a_n$ example |
|---------------|---------------|---------------|---------------|---------------|---------------|
| $n^2 + 2n + 1$| $-1$ $-1$     | $3 + 2n$      | $n^2 + 4n + 2$| $-2 - \sqrt{2}$ | none          |
|               |               |               | $n^2 + 4n + 2$| $-2 + \sqrt{2}$ | found         |
| $2n^2 + 3n + 1$| $-1$ $-1/2$   | $n + 1$       | $n^2 + 4n + 1$| $-2 - \sqrt{3}$ | none          |
|               |               |               |               | $-2 + \sqrt{3}$ | found         |
| $3n^2 + 4n + 1$| $-1$ $-1/3$   | $1 + 2$       | $n^2 + 1$     | $i$ $-i$      | none          |
|               |               |               |               |               | found         |
| $4n^2 + 4n + 1$| $-1/2$ $-1/2$ | $n + 1$       | $n^2 + 2n + 2$| $-1 - i$ $-1 + i$ | none          |
|               |               |               |               |               | found         |
| $n^2 + 3n + 2$| $-2$ $-1$     | $2n + 4*$     | $4n^2 + 4n + 2$| $-1/2 - i/2$ $-1/2 + i/2$ | none          |
|               |               |               |               |               | found         |
| $2n^2 + 4n + 2$| $-1$ $-1$     | $n + 1$       | $n^2 + 2n + 3$| $-1 - i\sqrt{2}$ | none          |
|               |               |               |               | $-1 + i\sqrt{2}$ | found         |
| $n^2 + 4n + 3$| $-3$ $-1$     | $5 + 2n$      | everything else| not rational | none          |
|               |               |               |               |               | found         |
| $n^2 + 4n + 4$| $-2$ $-2$     | $2n + 5$      | ...           | ...           | ...           |
|               |               |               |               |               |               |

Table 3: $b_n$ from degree 2 and coefficients 1 to 4, classified by the existence of FR. We can see that, for degree 2, only the $b_n$ polynomials with FR have all-rational roots and thus decompose, and vice versa. Further, note that all the $a_n$ examples belong to the conjectured complete structure for $\deg b_n = 2, \deg a_n = 1$ (Conjecture 1.4). *This PCF is an inflation of the regular continued fraction of $\sqrt{2} + 1$. 

A piece of structure for all degrees

In this section, we urge a generalizing of Conjecture 1.4 that deals only with $b_n$'s of degree 2, by demonstrating families of PCFs from many degrees with FR.

| degrees 2,1 | degrees 4,2 | degrees 4,2 | degrees 4,2 | degrees 6,3 | degrees 6,3 |
|------------|------------|------------|------------|------------|------------|
| $-n^2$     | $b_n = -n^4$ | $b_n = -4n^4 - 6n^3$ | $b_n = -n^2 \cdot \frac{1}{(n+2)(2n-3)}$ | $b_n = 4n^6 - 2n^5$ | $-n^6$ |
| $a_n$'s    | $a_n$'s    | $a_n$'s    | $a_n$'s    | $a_n$'s    | $a_n$'s    |
| $2n + 1$   | $2n^2 + 2n + 1$ | $4n^2 + 7n + 2$ | $3n^2 + 5n - 3$ | $3n^3 - 5n^2 - 3n - 1$ | $2n^3 + 3n^2 + 3n + 1$ |
| $8n + 4$   | $2n^2 + 2n + 3$ | $4n^2 + 7n + 3$ | $3n^2 + 3n - 1$ | $3n^3 + 10n^2 + 8n + 2$ | $6n^3 + 9n^2 + 5n + 1$ |
| $4n + 2$   | $2n^2 + 2n + 7$ | $4n^2 + 7n + 5$ | $3n^2 + n + 3$ | $2n^3 + 3n^2 + 11n + 5$ | |
| $6n + 3$   | $2n^2 + 2n + 13$ | $4n^2 + 7n + 8$ | $3n^2 + 7n + 3$ | | |
| $10n + 5$  | $5n^2 + 8n + 1$ | $3n^2 + 3n - 5$ | | | |
| $12n + 6$  | $4n^2 + 7n + 12$ | $3n^2 + 15n + 13$ | | | |
| $14n + 7$  | $5n^2 + 14n + 10$ | $3n^2 + 15n + 17$ | | | |
| $16n + 8$  | $4n^2 + 7n + 17$ | | | | |
| $18n + 9$  | | | | | |

Table 4: $b_n$ polynomials and found $a_n$'s for which PCF[$a_n, b_n$] has FR. For the first example, where deg $b_n = 2$, we conjecture a structure in Conjecture 1.4 However, we still do not possess a solid conjecture for higher degrees.

More rational recurrence relations that yield integer sequences

As follows from Section 2.4, a conjectured GCD sequence can be proven to be true if the following recurrence yields integer sequences for any pairs of initial values:

$$u'_{n+2} = \frac{GCD[n+1]}{GCD[n+2]}a_{n+1}'u'_{n+1} + \frac{GCD[n]}{GCD[n+2]}b_{n+1}'u'_{n+2}.$$
We address the community to prove this property for simple GCDs with the following examples. Furthermore, we request a general conditions on $a_n$ and $b_n$ so that this property holds, either for the presented special cases or hopefully for other cases (of Conjecture 2).

Table 5: $a_n, b_n$ examples for special simplified recurrence relations yielding integer sequences. A proof of this property also proves the formula for the GCD. Note that these are special cases of Section 2.4 and can be generalized to all GCD forms.

| $a_n$ | $b_n$ | $a_n$ | $b_n$ | $a_n$ | $b_n$ | $a_n$ | $b_n$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $3n + 1$ | $10n^2 + 20n$ | $15n + 1$ | $16n^2 + 2n$ | $-n - 2$ | $6n^2 + 3n$ | $6n - 1$ | $16n^2 + 2$ |
| $n - 1$ | $12n^2 + 6n$ | $15n + 3$ | $16n^2 + 4n$ | $n - 2$ | $2n^2 - n$ | $-2n + 1$ | $8n^2 + 2n$ |
| $n - 1$ | $6n^2 + 12n$ | $8n + 2$ | $9n^2 + 3n$ | $n - 2$ | $12n^2 + 3n$ | $3n + 1$ | $-2n^2 - 1$ |
| $-n + 1$ | $12n^2 + 6n$ | $n + 3$ | $2n^2 + 4n$ | $3n - 2$ | $4n^2 - n$ | $4n + 1$ | $12n^2 + 6n$ |
| $-n + 1$ | $6n^2 + 12n$ | $7n + 3$ | $8n^2 + 4n$ | $7n - 2$ | $8n^2 - n$ | $5n + 1$ | $6n^2 + 9n$ |
| $3n + 1$ | $4n^2 + 2n$ | $17n + 5$ | $18n^2 + 6n$ | $15n - 2$ | $16n^2 - n$ | $-n + 2$ | $2n^2 - n$ |
| $7n + 1$ | $8n^2 + 2n$ | $2n + 4$ | $3n^2 + 9n$ | $2n - 1$ | $8n^2 + 2n$ | $-n + 2$ | $12n^2 + 3n$ |