Fixed-Parameter Tractability of the Weighted Edge Clique Partition Problem

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Abstract

We develop an FPT algorithm and a kernel for the Weighted Edge Clique Partition (WECP) problem, where a graph with \( n \) vertices and integer edge weights is given together with an integer \( k \), and the aim is to find \( k \) cliques, such that every edge appears in exactly as many cliques as its weight. The problem has been previously only studied in the unweighted version called Edge Clique Partition (ECP), where the edges need to be partitioned into \( k \) cliques. It was shown that ECP admits a kernel with \( k^2 \) vertices \([Mujuni and Rosamond, 2008]\), but this kernel does not extend to WECP. The previously fastest algorithm known for ECP had a runtime of \( 2^{O(k^2)} n^{O(1)} \) \([Issac, 2019]\). For WECP we develop a bi-kernel with \( 4k \) vertices, and an algorithm with runtime \( 2^{O(k^{3/2} \log k)} n^{O(1)} \), where \( w \) is the maximum edge weight. The latter in particular improves the runtime for ECP to \( 2^{\Theta(k^{3/2} \log k)} n^{O(1)} \). We also show that our algorithm necessarily needs a runtime of \( 2^{\Theta(k^{3/2} \log k)} n^{O(1)} \) to solve ECP.

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the problem is called fixed-parameter tractable (FPT) for parameter \( p \). For example, ECP is FPT for parameter treewidth, since it can be solved in \( O(4^t n) \) time [1], where \( t \) is the treewidth of the input graph.

In this paper we focus on FPT algorithms for the natural parameter \( k \), i.e., the number of cliques. Fleischer and Wu [7] show that on planar graphs, ECP can be solved in \( O^*((2^{96\sqrt{k}})) \) time\(^1\). On more general classes of graphs with degeneracy \( d \), ECP can be solved in \( O^*((2^k d^k)) \) time [7], i.e., the exponent is linear in \( k \) if \( d \) is constant (as for instance in planar graphs). For \( K_4 \)-free graphs, Mujuni and Rosamond [18] gave an algorithm with a runtime\(^2\) of \( O^*((d^{k+1})k)) = O^*((2^c k \log k)) \), which was improved by Fleischer and Wu [7] to \( O^*((\sqrt{k}/3)^k) \) and even \( O^*((64c)^k) \) for some large (unspecified) constant \( c \). Hence, also for these graphs an exponent linear in \( k \) is possible, albeit with a very large base. On the other hand, the algorithm of Mujuni and Rosamond [18] for \( K_4 \)-free graphs has been empirically shown [23] to be rather efficient, even though it “only” comes with a near-linear exponent of \( O(k \log k) \).

As originally shown by Mujuni and Rosamond [18], for general graphs ECP is also FPT for parameter \( k \), but no algorithms with (near-)linear dependence on \( k \) in the exponent are known. The fastest algorithm so far is given by Issac [12, Theorem 3.10] and runs in \( O^*(2^{k^2 + k \log_2 k + k}) \) time, i.e., the exponent is quadratic in \( k \). This algorithm is an adaptation of an algorithm by Chandran et al. [3] for the Bipartite Biclique Partition problem (where we want to partition the edge set of a bipartite graph into \( k \) bicliques). In contrast, the best runtime lower bound known for ECP only excludes a sub-linear dependence on \( k \) in the exponent, as there is no \( 2^{o(k)} n^{O(1)} \) time algorithm under the Exponential Time Hypothesis (ETH): this follows due to a \( 2^{o(n)} \) lower bound for 3-Dimensional Matching [13] under ETH, and a reduction from Exact 3-Cover (which is a generalization of 3-Dimensional Matching) to ECP by Ma et al. [16]. An obvious open problem arising here is to close the gap between the upper and lower bounds on the necessary runtime for ECP on general graphs.

Our main contribution is to show that for general graphs the exponent of the runtime of an FPT algorithm for ECP can be significantly lowered from quadratic to \( (k^{3/2} + O(k)) \log k \). In fact, our algorithm solves a more general problem, which we call the Weighted Edge Clique Partition problem (WECP): we are additionally given integer weights on the edges, and the task is to find a set of at most \( k \) cliques such that each edge appears in exactly as many cliques as its weight. Note that WECP is equivalent to ECP on a multigraph by taking the weights as the edge multiplicities. The WECP problem also has applications\(^3\) in computational biology, where gene pathways are inferred from gene co-expression data (here the pathways correspond to cliques). Thus developing efficient algorithms for WECP is of practical relevance. It was not known till now whether WECP is even FPT; in particular, the known results for ECP do not extend to WECP. We give an FPT algorithm for WECP that also implies an improved algorithm for ECP.

\textbf{Theorem 1.} Given a graph \( G \) on \( n \) vertices and integer edge weights with maximum weight \( w \), there is an algorithm that solves the WECP problem in \( (2e)^{(k^{3/2} + O(k)) \log w (k/w) + O(n^2 \log n)} \) time.

\textbf{Corollary 2.} Given a graph \( G \) on \( n \) vertices, there is an algorithm that solves the ECP problem in \( (2e)^{(k^{3/2} + O(k)) \log w (k) + O(n^2 \log n)} \) time.

\(^{1}\)the \( O^* \)-notation hides polynomial factors.

\(^{2}\)in [18] the runtime was mistakenly reported as \( O^*((k^{(k+3)/2})) \), cf. [7].

\(^{3}\)Blair Sullivan (University of Utah), personal communication, 2019.
Note that Theorem 1 implies an FPT algorithm for WECP when parameterized by k as \( w \leq k \) for any YES-instance. Corollary 2 follows from Theorem 1 by setting \( w = 1 \).

We also consider kernelizations, which is a topic tightly linked to FPT algorithms: given an instance \( I \) with parameter \( p \) to some problem, a kernelization algorithm computes an equivalent instance \( I' \) and parameter \( p' \), i.e., \( (I, p) \) is a YES-instance if and only if \( (I', p') \) is a YES-instance. The runtime of the algorithm is polynomial and the size of the new instance (called the kernel) is bounded in the input parameter, i.e., \( |I'| + p' \leq g(p) \) for some computable function \( g \). It is well-known [4] that a problem is FPT for some parameter, if and only if it admits a kernel for the same parameter. For ECP on general graphs, Mujuni and Rosamond [18] show that a kernel with \( k^2 \) vertices can be computed in \( O(n^2) \) time. Their kernelization algorithm is not generalizable to WECP, and no \( g(k) \)-size kernel was known for any computable function \( g \) for WECP. In this work, we show that for WECP a so-called bi-kernel with \( 4^k \) vertices can be computed in polynomial time. That is, the kernel is for an even more general problem we call the Annotated Weighted Edge Clique Partition (AWECP) problem. Here we are additionally given a vertex set \( W \) and integer vertex weights for the vertices in \( W \). The task is to find a set of \( k \) cliques such that every edge appears exactly as many cliques as its weight, and each vertex in \( W \) appears in as many cliques as its weight. Note that WECP is exactly the special case of AWECP when \( W \) is empty.

\[ \textbf{Theorem 3.} \text{Given a graph on} \ n \ \text{vertices with integer edge weights, a subset} \ W \ \text{of vertices with integer vertex weights, and an integer} \ k, \ \text{there is a kernelization algorithm for AWECP that runs in} \ O(n^2 \log n) \text{time and outputs a kernel on} \ 4^k \text{vertices that has encoding length} \ O(16^k \log k). \]

### 1.1 Our techniques

Our approach is based on the work of Chandran et al. [3], who solve the Bipartite Biclique Partition problem using linear algebraic techniques: we express AWECP as a low-rank matrix decomposition problem. For this we allow matrices to have wildcard entries in the diagonal that will be denoted by \( \ast \). For values \( a \) and \( b \), we write \( a \triangleq b \) if and only if either \( a = b \), or \( a = \ast \), or \( b = \ast \). For two matrices \( A \) and \( B \), we write \( A \triangleq B \) if and only if \( A_{i,j} \triangleq B_{i,j} \) for all \( i, j \). We say that a matrix \( B \) is a Binary Symmetric Decomposition (BSD) of matrix \( A \) if \( BB^T \triangleq A \) and \( B \) is binary. Now, the Binary Symmetric Decomposition with Diagonal Wildcards (BSD-DW) problem is: given an integer non-negative symmetric matrix \( A \in (\mathbb{Z}_{\geq 0} \cup \{\ast\})^{n \times n} \) such that the wildcards \( \ast \) appear only in the diagonal, and an integer \( k \), find a rank-\( k \) BSD of \( A \), i.e., a BSD \( B \in \{0,1\}^{n \times k} \) of \( A \).

We prove (in Lemma 4) that AWECP and BSD-DW are equivalent. Moreover, each column of \( B \) (solution to BSD-DW) corresponds to a clique (in the solution to AWECP), i.e. the rows that have a 1 in the \( j \)-th column correspond to the vertices that are in the \( j \)-th clique. Due to this, we will index the rows and columns of \( A \) with vertices, the rows of \( B \) with vertices and the columns of \( B \) with integers from \([k]\), that correspond to the \( k \) cliques. Moreover, we will be fluently switching between the contexts of edge partitionings of graphs (AWECP), and matrix decomposition (BSD-DW).

In Section 2 we will first prove that there is a kernel for AWECP with \( 4^k \) vertices. For this we group the vertices into equivalence classes (that we call blocks) of twin vertices (our notion of twins is slightly different than the usual one in literature). If a block has size more than \( 2^k \), we show that they can be reduced and represented by one vertex. For this reduction rule, we need to specify how often the representative vertex needs to be covered by cliques. Thus, even if the input instance belongs to WECP, the kernel we compute will be annotated, i.e., it
will be an instance of AWECP. The $4^k$ bound on the kernel size follows then by giving a $2^k$ upper bound on the number of blocks for a YES instance. Since the edge weights and vertex weights for vertices in $W$ cannot exceed $k$ if there is a solution with at most $k$ cliques, a kernel with at most $4^k$ vertices can be encoded using $O((\frac{k^2}{2}) \log k)$ bits, and so Theorem 3 follows.

To obtain Theorem 1, we first compute a kernel using Theorem 3 and then run an algorithm on the kernel. In particular, our algorithm actually solves the more general AWECP problem. As in [3] (where a different low-rank matrix decomposition problem is solved), the main idea of our algorithm is to guess a basis for a rank-$k$ BSD $B$, and to then fill the remaining rows of $B$. However we need to refine the techniques of Chandran et al. [3] in order to obtain our runtime improvement. In particular, there are two reasons why the algorithm in [3] has a quadratic dependence on $k$ in the exponent: first, to guess a basis of rank $k$, they need to guess $k$ binary vectors of length $k$ each, which takes $O(2k^2)$ time. But also, they need to guess which rows of the low-rank decomposition matrix are the $k$ basis vectors, for which there are $\binom{n}{k}$ possibilities if the matrix has $m$ rows. Since for Bipartite Biclique Partition there is a kernel where $m \leq 2^k$ [8], this adds another factor of $O(2k^2)$ to the runtime.

To circumvent these two runtime bottlenecks, in Section 3 we devise an algorithm that gets around guessing the rows for the basis in the solution matrix $B$. While this makes our algorithm more involved than the one by Chandran et al. [3], it means that the only bottleneck left is guessing the basis. For BSD-DW we can show that a basis with only $k^{3/2}w^{1/2} + k$ non-zero entries exists, which follows from the well-studied Zarankiewicz problem [19]. This bound on the structure of the basis then implies Theorem 1.

Since the only bottleneck, which prevents our algorithm from having near-linear dependence on $k$ in the exponent of the runtime, is the step that guesses the non-zero entries of a basis for the solution matrix $B$, a natural question is whether our upper bound of $k^{3/2}w^{1/2} + k$ is (asymptotically) tight. In Section 4 we answer this in the affirmative by constructing instances of ECP for which every basis has $\Omega(k^{3/2})$ ones. This implies that our algorithm necessarily takes at least $2^{\Theta(k^{3/2} \log k)}n^{O(1)}$ time, making our runtime bound essentially tight (at least in the unweighted case).

### 1.2 Related results

We now survey some results for ECP and related problems, apart from those mentioned above. For ECP it is also known that the problem is solvable in polynomial time on cubic graphs [7]. The problem of partitioning the vertices instead of the edges into $k$ cliques is equivalent to $k$-colouring on the complement graph, which is well-known to be NP-hard even for $k = 3$. Similarly, when the vertices need to be partitioned into bicliques or covered by bicliques, Fleischner et al. [8] proved NP-hardness for any constant $k \geq 3$.

Covering the edges of a graph by cliques or bicliques also turns out to be generally harder than partitioning the edges. For the Edge Clique Cover problem a kernel with $2^k$ vertices was shown to exist by Gramm et al. [10], which results in a double-exponential time FPT algorithm when solving the kernel by brute-force. Cygan et al. [5] showed that this is essentially best possible, as under ETH no $2^{o(k)}n^{O(1)}$ time algorithm exists for Edge Clique Cover and no kernel of size $2^{o(k)}$ exists unless $P = NP$. Similarly, for the Biclique Cover problem, where edges of a general graph need to be covered by bicliques, Fleischner et al. [8] gave a kernel with $3^k$ vertices, and for the Bipartite Biclique Cover problem they gave a kernel with $2^k$ vertices in each bipartition. These kernels naturally imply double-exponential time algorithms. Chandran et al. [3] proved that for Bipartite Biclique Cover, under ETH no $2^{o(k)}n^{O(1)}$ time algorithm exists, and unless $P = NP$ no kernel of size $2^{o(k)}$ exists.

Chalermsook et al. [2] showed that for the Biclique Cover problem, it is NP-hard to
compute an $n^{1-\varepsilon}$-approximation for any $\varepsilon > 0$. Edge Clique Cover is hard to approximate within $n^{0.5-\varepsilon}$ due to a reduction by Kou et al. [15]. In contrast, a PTAS exists for Edge Clique Cover on planar graphs [1].

1.3 Preliminaries

For an $m \times n$ matrix $A$, we use $A_{i,j}$ to denote the entry of $A$ at row $i$ and column $j$. We use $A_i$ to denote the row-vector given by the $i$-th row of $A$. For some $I \subseteq [m]$ and $J \subseteq [n]$, we use $A_{I,J}$ to denote the sub-matrix of $A$ when restricted to rows with indices in $I$ and columns with indices in $J$. Also, we use $A_{I}$ to denote the sub-matrix of $A$ when restricted to rows with indices in $I$.

Lemma 4 (*). AWECP is equivalent to BSD-DW.

Proof. Given an instance $(G,W,k)$ of AWECP we can construct an instance of $(A,k)$ of BSD-DW as follows. Let $V(G) = \{1, \ldots , n\}$; take the non-diagonal entries of $A$ as the corresponding entries of the weighted adjacency matrix of $G$, i.e., if there is an edge between two vertices $u$ and $v$ the entry $A_{u,v}$ is equal to the integer edge weight; for every vertex $v \in W$ take $A_{v,v}$ as the vertex weight of $v$; for every vertex $v \in V(G) \setminus W$, take $A_{v,v}$ as the wildcard $\star$. Note that the mapping is invertible, i.e., given a BSD-DW instance $(A,k)$ we get an AWECP instance $(G,W,k)$ as follows. Take $V(G) := \{1,2,\ldots,n\}$ where $n$ is the number of rows (and columns) of $A$. For distinct $u,v \in [n]$, if $A_{u,v}$ is non-zero, put an edge between $u$ and $v$ in $G$ with weight $A_{u,v}$. For each $v \in [n]$ such that $A_{v,v}$ is not a wildcard, put $v$ in $W$ and set its vertex weight to $A_{v,v}$.

It is clear that this mapping is a bijective mapping between AWECP and BSD-DW instances.

Now, we define a bijective mapping between candidate solutions of the two problems. Naturally, a candidate solution of AWECP is a set of $k$ cliques and a candidate solution of BSD-DW is an $n \times k$ matrix. Consider a candidate solution $C := \{C_1, C_2, \ldots, C_k\}$ of an AWECP instance $(G,w,k)$. We map it to a candidate solution $B \in \{0,1\}^{n \times k}$ of a BSD-DW instance $(A,k)$ as follows. Take the row $B_u$ as the characteristic vector of $u$ in the $k$ cliques, i.e., $B_{u,j} := 1$ if $u \in C_j$, and $B_{u,j} := 0$ otherwise. The inverse mapping then turns out to be as follows. Given a candidate solution $B \in \{0,1\}^{n \times k}$ of instance $(A,k)$ construct $k$ cliques where the $j$-th clique is $C_j := \{u \mid B_{u,j} = 1\}$. To see that $C_j$ is indeed a clique, consider any two vertices $u,v \in C_j$: since $B_{u,j} = B_{v,j} = 1$, we know that $A_{u,v} = B_u B_v^T \geq 1$, which implies that there is an edge between $u$ and $v$ in $G$.

It only remains to prove that $C$ is a solution of $(G,w,k)$ if and only if $B$ is a solution of $(A,k)$. First, suppose $C$ is a solution of $(G,w,k)$. We show that then $B$ is a solution of $(A,k)$. For this it is sufficient to prove that for all pairs $u,v \in [n]$, $B_u B_v^T \triangleq A_{u,v}$. First consider the case when $u$ and $v$ are distinct. Let $J$ denote the set of all $j$ such that both $u$ and $v$ appear together in $C_j$. Since $C$ is a solution of $(G,w,k)$, we have that $|J| = A_{u,v}$. By construction of $B$, we have that $J$ is exactly the set of indices $j$ where $B_{u,j} = B_{v,j} = 1$. Thus $B_u B_v^T = |J| = A_{u,v}$. Now consider the case when $u = v$. If $A_{u,u}$ is a $\star$ then clearly $B_u B_u^T \triangleq \star = A_{u,u}$. So, suppose $A_{u,u} \neq \star$. This means $u \in W$ implying that $u$ appears in exactly $A_{u,u}$ many cliques in $C$. Thus $B_u B_u^T = A_{u,u}$.

Now, let us prove the other direction. Assume $B$ is a solution of $(A,k)$. We will prove that then $C$ is a solution of $(G,w,k)$. For this it is sufficient to prove the following two statements: (1) every pair $u,v \in V(G)$ appears together in exactly $A_{u,v}$ many cliques in $C$ (2) each vertex $v \in W$ appears in $A_{v,v}$ many cliques in $C$. First we prove (1). We know $B_u B_v^T = A_{u,v}$. Since

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4The paper wrongly claims the same result also for Biclique Partition. The bug is acknowledged here: https://sites.google.com/site/parinyachalermsook/research?authuser=0
$B$ is binary, this means that there are exactly $A_{u,v}$ many indices $j$ such that $B_{u,j}$ and $B_{v,j}$ are both 1. Let $J$ be the set of those indices. Observe that the set of cliques where both $u$ and $v$ appear together are exactly $\{C_j : j \in J\}$. Thus, the edge $uv$ is in $|J| = A_{u,v}$ many cliques. Now we prove (2). Consider a vertex $v \in W$. We know $B_v B_v^T = A_{v,v}$. Since $B$ is binary, this means that there are exactly $A_{v,v}$ many ones in $B_v$. Thus, the vertex $v$ is in $A_{v,v}$ many cliques. □

2 Kernel

We will now give a kernel for AWECP and BSD-DW, thereby proving Theorem 3. Let $G$ be the input graph to AWECP and $A$ be the corresponding input matrix to BSD-DW obtained by the transformation as in the proof of Lemma 4. We may move seamlessly between the graph and matrix terminologies as both problems are equivalent. Whenever we say a solution in this section, we mean the solution to the BSD-DW instance i.e., a rank-$k$ BSD of $A$. We say two distinct vertices $u$ and $v$ are twins if they are adjacent and satisfy $A_u \equiv A_v$.

Lemma 5. For distinct vertices $u$, $v$ and $w$, suppose $u$ and $v$ are twins and $v$ and $w$ are twins. Then:
1. $u$ and $w$ are twins, and
2. all the entries of the submatrix $A_{\{u,v,w\},\{u,v,w\}}$ are the same except for wildcards.

Proof. First, let us prove the second statement. Let $A_{u,v} = \alpha$. Then we know $A_{u,w} = \alpha$ as $v$ and $w$ are twins. Then $A_{v,w} = \alpha$ as $u$ and $v$ are twins. Thus all the non-diagonal elements of $A_{\{u,v,w\},\{u,v,w\}}$ are equal to $\alpha$. If $A_{u,v} \neq \star$ then $A_{u,u} = A_{v,v} = \alpha$ as $u$ and $v$ are twins. Similarly, if $A_{v,w} \neq \star$ then $A_{v,v} = A_{w,w} = \alpha$ as $u$ and $v$ are twins. And, if $A_{w,u} \neq \star$ then $A_{w,w} = A_{v,v} = \alpha$ as $v$ and $w$ are twins.

Now, for the first statement to hold, we only need to show that $A_{u,z} = A_{w,z}$ for all $z \notin \{u,v,w\}$. Indeed, $A_{u,z} = A_{v,z} = A_{w,z}$ where the first equality is because $u$ and $v$ are twins and the second is because $v$ and $w$ are twins. □

Thus we have that the relation twins is transitive. It is also symmetric, as easily seen from the definition. To make it also reflexive, we consider a vertex to be twin of itself. Thus, we can group the vertices into equivalence classes of twins. We call each equivalence class a block. Note that there can be blocks containing only a single vertex. The following lemma is a direct consequence of Lemma 5.

Lemma 6. For a block $D$, the entries of the sub-matrix $A_{D,D}$ are all same except for wildcards.

Fact 7. For values $a, b$ and $c$, if $a \equiv b$ and $b \equiv c$, and $b \neq \star$ then $a \equiv c$.

Lemma 8. Suppose we have a YES instance of AWECP without isolated vertices. Then there can be at most $2^k$ blocks.

Proof. Let $B$ be a rank-$k$ BSD of $A$. Note that $B$ exists as we have a YES instance. In order to prove the lemma, it is sufficient to show that if $u$ and $v$ are in different blocks, then $B_u$ and $B_v$ are distinct, because then there can only be $2^k$ distinct rows of $B$, as there are only $k$ columns in $B$ and $B$ is binary. We will prove the contrapositive, i.e., we will show that if $B_u = B_v$ then $u$ and $v$ are in the same block. Assume for the sake of contradiction that $B_u = B_v$ and $u$ and $v$ are in different blocks, i.e., they are not twins. Let $b := B_u B^T = B_v B^T$. We have $A_u \equiv B_u B^T = b$ and $A_v \equiv B_v B^T = b$. This implies $A_u \equiv A_v$ using Fact 7, as the vector $b$ contains no wildcards. Then, for $u$ and $v$ to be not twins, it should be the case that
u and v are not adjacent, i.e., $A_{u,v} = 0$. But then, $B_u B_u^T = 0$. Since $B_u = B_v$ by assumption, we have that $B_u = B_v = 0$ and hence $A_u = A_v = 0$. This means that $u$ and $v$ are isolated vertices, which is a contradiction.

The above lemma shows the soundness of our first reduction rule that is as follows.

**Reduction rule 1.** If the number of blocks is more than $2^k$, output that the instance is a NO instance.

Next, we prove the following lemma about twins that helps us to come up with a reduction rule that bounds the size of each block.

**Lemma 9.** Let $D := \{v_1, v_2, \ldots, v_t\}$ be a block of twins. For a YES instance, there exists a solution $B$ such that the rows $B_{v_1}, B_{v_2}, \ldots, B_{v_t}$ are either all pairwise distinct, or all same.

**Proof.** It is sufficient to prove the following statement: if there is a solution $B$ such that $B_{v_1} = B_{v_2}$, then there is also a solution $C$ such that $C_{v_1} = C_{v_2} = \cdots = C_{v_t}$. So, assume that $B_{v_1} = B_{v_2}$. Let $C$ be the matrix defined as $C_v := B_v$ for all $v \notin D$, and $C_v := B_{v_1} = B_{v_2}$ for all $v \in D$. We will prove that $C$ is also a solution. For this, it is sufficient to prove that $C_u C_u^T = A_{u,u'}$ for all $u, v \in V$ such that $A_{u,v} \neq \star$. If both $u$ and $v$ are not in $D$, then $C_u C_u^T = B_u B_u^T = A_{u,u'}$. So, without loss of generality assume that $u \in D$. We distinguish the following cases.

1. If $v \in V \setminus D$, then $C_u C_u^T = B_{v_1} B_{v_1}^T = A_{v_1,v_1} = A_{u,u'}$, where the last equality follows as $v_1$ and $u$ are twins.
2. If $v \in D \setminus \{u\}$, then $C_u C_u^T = B_{v_1} B_{v_2}^T = A_{v_1,v_2} = A_{u,u'}$, where the last equality follows from Lemma 6.
3. If $v = u$: if $A_{u,u} = \star$ then there is nothing to prove, so assume $A_{u,u} \neq \star$. Then $A_{u,u} = A_{v_1,v_2}$ by Lemma 6. Hence we get $C_u C_u^T = B_{v_1} B_{v_2}^T = A_{v_1,v_2} = A_{u,u'}$.

By Lemma 9, if we have a YES instance then there is a solution $B$ such that for every block $D$ either all corresponding rows of $B$ are the same, or they are all distinct. If there is a block $D$ of size more than $2^k$, then the corresponding rows of $B$ have to be all the same, since there are only $2^k$ possible distinct rows for $B$. In this case we only need to keep one representative vertex $v$ for all the vertices in $D$, and the entry $A_{u,v}$ is now forced to be $A_{v,u}$ where $u \in D$ was some other vertex in the same block. The solution can be extended by copying the row $B_v$ to the rows of other twins. Thus, we obtain the following reduction rule.

**Reduction rule 2.** Suppose there is a block $D$ with more than $2^k$ vertices. Pick any two arbitrary vertices $u, v \in D$. Set $A_{u,v} = A_{u,v}$. Delete all vertices in $D$ except $v$.

After the above rules are exhaustively applied, each block has size at most $2^k$ and the number of blocks is at most $2^k$. Thus we have the required kernel of size $4^k$. The time required for computing the kernel can be shown to be $O(n^2 \log n)$. This is achieved by using sorting to find blocks of twins. Since the edge weights and vertex weights for vertices in $W$ cannot exceed $k$ if there is a solution with at most $k$ cliques, a kernel with at most $4^k$ vertices can be encoded using $O((4^k) \log k)$ bits, and so Theorem 3 follows.

### 3 Algorithm

Here we give an algorithm for the BSD-DW problem. The algorithm also solves AWECPC due to the equivalence from Lemma 4. In particular, it solves WECP thereby proving Theorem 1.

We now give a description of the algorithm. The complete pseudocode is given in Algorithm 1. Our input is a symmetric matrix $A \in (\mathbb{Z}_{\geq 0} \cup \{\star\})^{n \times n}$ where wildcards $\star$ appear...
only in the diagonal. First we guess a matrix \( P \in \{0,1\}^{k \times k} \) such that for some \( r \leq k \), \( P_{[r],[r]} \) is a row basis of solution \( B \). We show that it is sufficient to enumerate \( k \times k \) binary matrices that satisfy a specific property defined as follows. Let \( w := ||A||_{\infty} \). We call a matrix \( w\text{-limited} \) if the dot-product of each pair of its rows is at most \( w \). The following fact shows that we only need to enumerate \( w\text{-limited} \) matrices in \( \{0,1\}^{k \times k} \) to guess \( P \).

\( \blacktriangleright \) Fact 10. If \( B \) is a BSD of matrix \( A \) and \( w = ||A||_{\infty} \), then any submatrix of \( B \) (including \( B \)) is \( w\text{-limited} \).

Note that guessing \( P \) is done in Loop 1 of Algorithm 1. We will later give a bound on the number of \( w\text{-limited} \) matrices in \( \{0,1\}^{k \times k} \) in the running time analysis, thereby bounding the number of iterations of Loop 1.

We maintain partially filled matrices during the algorithm, i.e., we allow matrices to have null rows (this is different from wildcards). Think of the null rows as the rows that have not been filled yet. If each row of a matrix is either a binary row or a null row, we call it a binary matrix with possibly null rows. We denote by \( \mathbb{E}^{n \times k} \), the set of all \( n \times k \) binary matrices with possibly null rows.

We maintain a matrix \( \bar{B} \in \mathbb{E}^{n \times k} \) as a potential basis for our solution \( B \). In Line 8, we call \texttt{ExtendBasis} that checks whether the current \( \bar{B} \) can be extended to a full solution \( B \). Note that \texttt{ExtendBasis} does not try all possibilities to fill the remaining rows. It fills a row with the first binary vector that is compatible with the rows so far, where compatibility is defined as follows. For a matrix \( B \in \mathbb{E}^{n \times k} \), we say that a vector \( v \in \{0,1\}^k \) is \( i\text{-compatible} \) for \( B \) if \( v^T v \triangleq A_{i,i} \), and for all \( j \neq i \) such that \( B_j \) is not a null row, \( v^T B^T_j = A_{i,j} \). If \texttt{ExtendBasis} is able to fill all the rows with \( i\text{-compatible} \) binary vectors, then we are done and we return the resulting matrix (in Line 9). If not, then we claim that the row for which we are not able to fill can be added to the basis (in Claim 12). So we add one more row to the basis by copying the next row from \( P \) (in Line 7). Thus we increase the number of non-null rows in the basis \( \bar{B} \) by one and repeat. Since the basis can be at most of size \( k \), we need to repeat this at most \( k \) times.

### 3.1 Correctness of the algorithm

The algorithm outputs either through Line 9 or through Line 11. In the former case, we prove the following claim.

\( \blacktriangleright \) Claim 11. If output occurs through Line 9, then the matrix \( B \) that is output, is a rank-\( k \) BSD of \( A \).

**Proof.** If Line 9 is executed, then this means that the preceding \texttt{ExtendBasis} call on Line 8 returned \( i = n + 1 \). This implies that the return from \texttt{ExtendBasis} happened on Line 16. This in turn means that Loop 3 was exited, implying that the condition of the while loop in Line 12 was no longer true. This means the matrix \( B \) did not have any null rows at the time of return. Thus \( B \in \{0,1\}^{n \times k} \). The rows of \( B \) were each filled either in Line 7 (when it was \( \bar{B} \) before being passed to \texttt{ExtendBasis} or in Line 14. In both places, we filled each row \( i \) with a vector that was \( i\text{-compatible} \) at the time of filling. From the definition of \( i\text{-compatibility} \), it follows that \( BB^T \triangleq A \), and hence \( B \) is a rank-\( k \) BSD of \( A \). \( \blacktriangleleft \)

Consider a NO instance first. From Claim 11 it follows that the output does not occur through Line 9. Thus the output has to occur through Line 11 and hence we correctly output that \( A \) does not have a rank-\( k \) BSD. So it only remains to prove the correctness when \( A \) is a YES instance, i.e., when \( A \) has a rank-\( k \) BSD, which is the case we consider for the remainder of the proof. Let \( B^* \) be a fixed rank-\( k \) BSD of \( A \).
Thus for a matrix $A$, we say that a matrix $B$ is consistent with $A^*$ if $B_j = B^*_j$ for each $j$ such that $B_j$ is a non-null row.

\begin{algorithm}
\textbf{Algorithm 1: Algorithm for BSD-DW}
\begin{algorithmic}
\State \textbf{Input}: An $n \times n$ symmetric integer diagonal-wildcard matrix $A$
\State \textbf{Output}: If $A$ has a rank-$k$ BSD then output a $B \in \{0, 1\}^{n \times k}$ such that $BB^T \triangleq A$; otherwise report that $A$ has no rank-$k$ BSD
\State $w \leftarrow ||A||_\infty$
\State \textbf{foreach} $w$-limited $P \in \{0, 1\}^{k \times k}$ \textbf{do} \hspace{1cm} // Loop 1
\State Initialize $\tilde{B}$ to be an $n \times k$ matrix with all null rows
\State $b \leftarrow 1$
\State $i \leftarrow 1$
\State \textbf{while} $b \leq k$ and $P_b$ is $i$-compatible for $\tilde{B}$ \textbf{do} \hspace{1cm} // Loop 2
\State $\tilde{B}_i \leftarrow P_b$
\State $(B, i) \leftarrow \text{ExtendBasis}(A, \tilde{B})$
\State \textbf{if} $i = n + 1$ \textbf{then} output $B$ and \textbf{terminate} the algorithm
\State $b \leftarrow b + 1$
\State \textbf{output} that $A$ has no rank-$k$ BSD and \textbf{terminate} the algorithm
\State
\State \textbf{Function} $\text{ExtendBasis}(A, B)$:
\State \hspace{1cm} \textbf{while} there is a null row $i$ in $B$ \textbf{do} \hspace{1cm} // Loop 3
\State \hspace{1.5cm} \textbf{if} there is a $v \in \{0, 1\}^k$ such that $v$ is $i$-compatible for $B$ \textbf{then}
\State \hspace{2.2cm} $B_i \leftarrow v$
\State \hspace{1.5cm} \textbf{else} return $(B, i)$
\State \hspace{1cm} \textbf{return} $(B, n + 1)$
\end{algorithmic}
\end{algorithm}

Observe that $\tilde{B}$ changes as follows during each iteration of Loop 1: it is initialized to all null rows and each time the algorithm encounters Line 7 a null row is replaced with a binary row vector. We say that a matrix $B$ is consistent with $A^*$ if $B_j = B^*_j$ for each $j$ such that $B_j$ is a non-null row.

\begin{claim}
Consider a matrix $\tilde{B} \in \mathbb{B}^{n \times k}$ that is consistent with $A^*$. If $\text{ExtendBasis}(A, \tilde{B})$ returns $i \in [n]$ then $B^*_i$ is linearly independent from the non-null rows of $\tilde{B}$.
\end{claim}

\begin{proof}
For a matrix $M \in \mathbb{B}^{n \times k}$, we denote by $R(M)$ the set of indices of the non-null rows of $M$. Suppose for the sake of contradiction that $\text{ExtendBasis}(A, \tilde{B})$ returns $i \in [n]$ and $B^*_i$ is linearly dependent on the non-null rows of $\tilde{B}$. Then, we have $B^*_i = \sum_{\ell \in R(\tilde{B})} \lambda_\ell B^*_\ell$ for some $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$. Since $\tilde{B}$ is consistent with $A^*$, we can write $B^*_i = \sum_{\ell \in R(\tilde{B})} \lambda_\ell B^*_\ell$.

As $\text{ExtendBasis}$ returned $i$, we know that during that iteration of Loop 3 in which row $i$ was considered, no vector $v \in \{0, 1\}^k$ was $i$-compatible for $B$ (here $B$ is the matrix maintained by $\text{ExtendBasis}$ that was initialized to $\tilde{B}$ by the function call). In particular, $B^*_i \in \{0, 1\}^k$ was not $i$-compatible for $\tilde{B}$. Therefore, either there was some $j \in R(B)$ such that $B^*_i B^*_j B^*_j B^*_j T \neq A_{i,j}$, or $B^*_i (B^*_i)^T \neq A_{i,j}$. The latter cannot be true as $B^*$ is a rank-$k$ BSD of $A$. So there was a $j \in R(B)$ such that $B^*_i B^*_j B^*_j B^*_j T \neq A_{i,j}$.

We branch into two cases: case 1 when $j \in R(\tilde{B})$ and case 2 when $j \in R(B) \setminus R(\tilde{B})$. In case 1, we have $B_j = \tilde{B}_j = B^*_j$ where the second equality is because $\tilde{B}$ and $B^*$ are consistent. Thus $B^*_i B^*_j B^*_j B^*_j T = A_{i,j}$, giving a contradiction.

In case 2, $B_j$ was added in Line 14 and hence $B_j$ was $j$-compatible for $B$ at this time, implying that $B_\ell B^*_\ell B^*_\ell B^*_\ell T = A_{i,j}$ for all $\ell \in R(\tilde{B})$. Since $B_\ell = \tilde{B}_\ell = B^*_\ell$ for $\ell \in R(\tilde{B})$, we have
that $B^*_i B^*_j^T = A_{i,j}$ for all $\ell \in R(\tilde{B})$. Then, we have

$$B^*_i B^*_j^T = \sum_{\ell \in R(\tilde{B})} \lambda_{\ell} B^*_i B^*_j^T \lambda_{\ell} A_{i,j} = \sum_{\ell \in R(\tilde{B})} \lambda_{\ell} A_{i,j} = B^*_i (B^*_j)^T = (A_{i,j})^T$$

This is a contradiction. ▲

For a matrix $X \in \{0,1\}^{k \times k}$, we say we are in iteration $(X,t)$ of the algorithm if we are in the iteration of Loop 1 with $P = X$ and the iteration of Loop 2 with $b = t$. We use $\tilde{B}(X,t)$ to denote the value of $\tilde{B}$ after the execution of Line 7 during iteration $(X,t)$.

◨ Claim 13. At any step of the algorithm, if $\tilde{B}$ is consistent with $B^*$ then the non-null rows of $\tilde{B}$ are linearly independent.

Proof. Consider the first time this is violated during the algorithm. This has to be during the addition of a new non-null row at Line 7. Let $(X,t)$ be the iteration in which this happens. Let $p$ be the index of the row that was added. Observe that $\tilde{B}(X,t)$ has only one additional non-null row compared to $\tilde{B}(X,t-1)$. Also, this additional non-null row is equal to $B^*_p$ as $\tilde{B}(X,t)$ is consistent with $B^*$. We know the rows of $\tilde{B}(X,t-1)$ are linearly independent as we assumed that the first violation of lemma happens in iteration $(X,t)$. Also, during iteration $(X,t-1)$, $\tilde{B}(X,t)$ has only one additional non-null row that is added during iteration $(X,t)$. This implies that $B^*_p$ is linearly independent from the non-null rows of $\tilde{B}(X,t-1)$ due to Claim 12. Hence the rows of $\tilde{B}(X,t)$ are linearly independent. ▲

◨ Claim 14. If the iteration $(X,k)$ occurs during the algorithm for some $X \in \{0,1\}^{k \times k}$ such that $\tilde{B}(X,k)$ is consistent with $B^*$ then the algorithm outputs through Line 9 in iteration $(X,k)$.

Proof. Consider the $i$ returned by $\text{ExtendBasis}(A,\tilde{B}(X,k))$. It is sufficient to prove that $i \notin [n]$. Suppose for the sake of contradiction that $i \notin [n]$. Then by Claim 12, $B^*_i$ is linearly independent from the non-null rows of $\tilde{B}(X,k)$. But by Claim 13, we have that the non-null rows of $\tilde{B}(X,k)$ are linearly independent and hence span the whole space, thus giving a contradiction. ▲

◨ Claim 15. Assume that the output of the algorithm does not occur through Line 9. If for some $Y \in \{0,1\}^{k \times k}$ and $t \leq k - 1$, iteration $(Y,t)$ occurs and $\tilde{B}(Y,t)$ is consistent with $B^*$, then there exists some $Z \in \{0,1\}^{k \times k}$ such that iteration $(Z,t + 1)$ occurs and $\tilde{B}(Z,t + 1)$ is consistent with $B^*$.

Proof. Since $\tilde{B}(Y,t)$ is consistent with $B^*$, we know that $Y_{[t]}$ is a sub-matrix of $B^*$. As the condition in Line 9 is false, we know that an $i \in [n]$ was returned in Line 8 in iteration $(Y,t)$. It is clear from the algorithm that $i$ is a null-row in $\tilde{B}(Y,t)$. Let $Z \in \{0,1\}^{k \times k}$ be such that $Z_{[t]} := Y_{[t]}$, $Z_{t+1} := B^*_{i}$, and $Z_{q} := 0$ for all $q \geq t + 1$. Observe that $Z_{[t+1]}$ is a submatrix of $B^*$ and hence is $w$-limited by Fact 10. Since adding zeroes does not destroy $w$-limitedness, we have that $Z$ is a $w$-limited $n \times k$ matrix. Thus there is some iteration of Loop 1 with $P = Z$. In this iteration the algorithm behaves similarly to the iteration with $P = Y$ for the first $t$ iterations of Loop 2 as the algorithm has seen only the first $t$ rows of $P$ up to then. Thus $\tilde{B}(Z,t) = \tilde{B}(Y,t)$ and $i$ is returned by Line 8 in iteration $(Z,t)$. ▲
Now in Line 7 of iteration \((Z, t + 1)\), \(\tilde{B}_i\) is assigned \(Z_{t+1}\). Note that \(Z_{t+1} = B^*_t\) is indeed \(i\)-compatible for \(\tilde{B}(Z, t)\) (as \(\tilde{B}(Z, t) = \tilde{B}(Y, t)\) and \(\tilde{B}(Y, t)\) is consistent with \(B^*\)) and that \(t + 1 \leq k\). Hence the loop condition of Loop 2 is true in iteration \((Z, t + 1)\). Thus, we have \((\tilde{B}(Z, t + 1))_i = Z_{t+1} = B^*_t\) and for all \(j \neq i\), we have \((\tilde{B}(Z, t + 1))_j = (\tilde{B}(Y, t))_j\). Since \(\tilde{B}(Y, t)\) is consistent with \(B^*\), it follows that \(\tilde{B}(Z, t + 1)\) is consistent with \(B^*\).

Let \(t\) be the largest number for which there exists a \(P \in \{0, 1\}^{k \times k}\) such that iteration \((P, t)\) happens and \(\tilde{B}(P, t)\) is consistent with \(B^*\). Due to Claim 15, we know that \(t = k\). Then the algorithm outputs through Line 9 according to Claim 14. Thus the algorithm outputs a correct solution \(B\) due to Claim 11.

### 3.2 Runtime Analysis

First, let us bound the number of iterations of Loop 1. For this it is sufficient to bound the number of \(w\)-limited matrices in \(\{0, 1\}^{k \times k}\).

**Lemma 16.** The number of binary \(w\)-limited \(k \times k\) matrices is at most \(2e\sqrt{\frac{k}{w}}^{k/2}w^{1/2} + k\).

**Proof.** Note that no \(w\)-limited matrix can have a \(2 \times (w + 1)\)-sub-matrix having all ones. The number of ones in such a matrix is a special case of the well-studied Zarankiewicz problem and is known [19] to be at most \(k^{3/2}w^{1/2} + k\). Hence it follows that the number of binary \(w\)-limited \(k \times k\) matrices is at most \(2k^{3/2}w^{1/2} + k\cdot\left(\frac{k^2}{w}w^{1/2} + k\right)\) by choosing the positions of the at most \(k^{3/2}w^{1/2} + k\) potential ones in the matrix and then choosing which of them are actually ones. The bound follows easily by using that \(\binom{n}{k} \leq \left(\frac{en}{k}\right)^k\).

Next, let us analyse the runtime of the function \texttt{ExtendBasis}. Loop 3 has at most \(n\) iterations. In Line 13, we need to check at most \(2^k\) vectors \(v \in \{0, 1\}^k\). The checking for \(i\)-compatibility of each vector takes \(O(nk)\) time. Hence \texttt{ExtendBasis} takes \(O(k^2n^2)\) time.

Now, we are ready to calculate the total run time. Due to Lemma 16, Loop 1 has at most \((2e\sqrt{\frac{k}{w}})^{k/2}w^{1/2} + k\) iterations. Line 3 takes \(O(nk)\) time. Loop 2 has at most \(k\) iterations. Line 7 takes at most \(O(k)\) time. The call to \texttt{ExtendBasis} in Line 8 takes at most \(O(k^2n^2)\) time as we already calculated. Any other step takes only constant time. Thus the total runtime is bounded by \(O\bigg((2e\sqrt{\frac{k}{w}})^{k/2}w^{1/2} + k\bigg) + 2^k(k + k^k) = O\bigg((2e\sqrt{\frac{k}{w}})^{k/2}w^{1/2} + k\bigg)\cdot k^22^k\). We may run our algorithm on the kernel provided by Theorem 3, which means we may set \(n = 4^k\) in the above expression. The total runtime is

\[
O\left((2e\sqrt{\frac{k}{w}})^{k/2}w^{1/2} + k\cdot 2^k + 2\cdot 2^k\log n\right) = O\left((2e)^{k/2}w^{1/2} + \log_{2^k}(k/w) + n^2\log n\right).
\]

### 4 A runtime lower bound

In this section we answer the question of whether our bound on the number of iterations of Loop 1 as given by Lemma 16 is tight. We will construct instances to BSD-DW for which every basis of every solution matrix \(B\) has \(\Omega(k^{3/2})\) non-zero entries. Using the same calculations as for the proof of Lemma 16, this implies that our algorithm necessarily needs \(2\Theta(k^{3/2}\log k)\) iterations of Loop 1 on these instances. This implies that our runtime bound given in Corollary 2 for Algorithm 1 is tight.

We obtain such instances via Finite Projective Planes (FPPs), which are defined by a set system \(\mathcal{S}\) over a universe \(U\) of elements such that

1. for each \(e, e' \in U\) there is exactly one \(S \in \mathcal{S}\) containing both of them,
2. for each \( S, S' \in \mathcal{S} \) there is exactly one \( e \in U \) such that \( e \in S \cap S' \), and
3. there is a set of 4 elements in \( U \) such that no three of them are in any \( S \in \mathcal{S} \).

It is known [17] that the definition implies that both the number of elements and the number of sets are equal to \( N^2 + N + 1 \) for some \( N \geq 2 \). Here \( N \) is called the order of the FPP. It also follows that for an FPP of order \( N \), each set has exactly \( N + 1 \) elements and each element is contained in exactly \( N + 1 \) sets. An FPP exists for every prime power \( N \). Given an FPP of order \( N \), in the following we will denote the characteristic incidence matrix of elements and sets by \( F \in \{0, 1\}^{(N^2 + N + 1) \times (N^2 + N + 1)} \), where rows are elements and columns are sets.

We now give a reduction from the problem of finding an FPP of order \( N \) to ECP. For this, consider a vertex set \( V \) with \( N^2 + N + 1 \) vertices. Let \( I \) be a subset of \( N + 1 \) vertices in \( V \). Let \( G_N \) be the graph defined as the clique over \( V \) minus the clique over \( I \), i.e., every pair of vertices in \( V \) is adjacent except when both are from \( I \). In other words, if \( X := V \setminus I \), then \( G_N \) is a split graph with \( X \) as the clique and \( I \) as the independent set, where all the adjacencies are present between \( X \) and \( I \). The following lemmas show that \( G_N \) has a small ECP solution if and only if an FPP of order \( N \) exists.

**Lemma 17.** If a finite projective plane \( S \) of order \( N \) exists, then \( G_N \) has a clique partition \( \mathcal{C} \) into \( |\mathcal{C}| \leq N^2 + N \) cliques.

**Proof.** Let \( S \) be an FPP of order \( N \) over a universe \( U \), and fix one of its sets \( S \in \mathcal{S} \). We identify this set with the independent set of \( G_N \), i.e., \( S = I \). After fixing the elements of \( S \), all other elements in \( U \setminus S \) are arbitrarily identified with the other vertices in \( X \). We claim that the remaining sets in \( S \setminus \{S\} \) form a clique partition, i.e., if \( C_{S'} = \{uv \in E(G_N) \mid u, v \in S'\} \) then the set \( \mathcal{C} = \{C_{S'} \mid S' \in S \setminus \{S\}\} \) partitions the edge set of \( G_N \) into cliques. From Property 1 of an FPP, for any edge \( uw \) (i.e., at least one of \( u \) and \( v \) is in \( X \)) there is exactly one set \( S' \in S \setminus \{S\} \) such that \( u, v \in S' \). This means that the subgraphs in \( \mathcal{C} \) are in the edge set. Furthermore, by Property 2 no \( S' \in S \setminus \{S\} \) intersects in more than one vertex with the independent set \( I \). Thus every subgraph of \( \mathcal{C} \) is a clique. Moreover, any FPP of order \( N \) has exactly \( N^2 + N + 1 \) sets, and so there are \( N^2 + N \) cliques in \( \mathcal{C} \).

To prove the other direction, i.e., that a small ECP solution implies the existence of an FPP, we need the following claim.

**Claim 18.** If \( \mathcal{C} \) is a set of cliques that partition the edges of \( G_N \) and \( |\mathcal{C}| \leq N^2 + N \), then for each \( C \in \mathcal{C} \), \( |V(C)| = N + 1 \).

**Proof.** First let us prove that \( |V(C)| \leq N + 1 \). Suppose for the sake of contradiction that \( |V(C)| \geq N + 2 \). Note that \( C \) contains at most one vertex from \( I \), as a clique and independent set can intersect on at most one vertex. Let \( C' := V(C) \setminus I \) and \( I' := I \setminus V(C) \). Clearly \( |C'| \geq N + 1 \) and \( |I'| \geq N \) (recall that \( |I| = N + 1 \)). Note that every edge in \( C' \times I' \) has to be covered by a distinct clique in \( C \setminus \{C\} \): any two edges that have different endpoints in \( I \) cannot be in the same clique, since there is no edge between these endpoints, while any two edges with different endpoints in \( C \) cannot be in the same clique, since the only edge between these endpoints is already covered by \( C \). But there are \( |C'| |I'| \geq N^2 + N \) such edges whereas there are only \( N^2 + N - 1 \) cliques in \( C \setminus \{C\} \). Thus we have a contradiction.

Hence we established \( |V(C)| \leq N + 1 \). Now suppose for the sake of contradiction \( |V(C)| < N + 1 \). Using the fact that every clique of \( \mathcal{C} \) has size at most \( N + 1 \), the total number of edges covered by \( \mathcal{C} \) is strictly less than \( |C| \binom{N + 1}{2} \leq (N^2 + N) \binom{N + 1}{2} = N^2(N + 1)^2/2 \). However, since \( |I| = N + 1 \) and consequently \( |X| = N^2 \), the total number of edges of \( G_N \) is \( \binom{N^2}{2} + N^2 \cdot (N + 1) = N^2(N + 1)^2/2 \). Thus, we have a contradiction.
Lemma 19. Let \( N \geq 2 \). If \( C \) is a set of cliques that partition the edges of \( G_N \) such that \(|C| \leq N^2 + N\), then \( \mathcal{S} = \{V(C) \mid C \in C\} \cup \{I\} \) is an FPP of order \( N \) over \( V \). Moreover, the incidence matrix \( F \) of \( \mathcal{S} \) after removing the column for \( I \) is the BSD of the adjacency matrix of \( G_N \) that corresponds to \( C \).

Proof. We will prove that \( \mathcal{S} = \{V(C) \mid C \in C\} \cup \{I\} \) satisfies the three properties in the definition of an FPP, which then has order \( N \) by Claim 18. Property 1 follows easily from the definition of an edge clique partition: for each pair of adjacent vertices there is exactly one clique covering their edge, while any pair of non-adjacent vertices only appear in \( I \).

Let us now prove Property 2. For any \( S, S' \in \mathcal{S} \), it follows easily from the definition of an edge clique partition that \(|S \cap S'| \leq 1\) (otherwise some edge is contained in two cliques). Also, for any \( S \in \mathcal{S} \), it is true that \(|S \cap I| \leq 1\) (otherwise some clique would contain a non-edge). Assume there are \( S, S' \in \mathcal{S} \) with \( S \cap S' = \emptyset \). By Claim 18, we have \(|S| = |S'| = N + 1\), and so all the \((N + 1)^2\) edges of \( S \times S' \) have to be covered by distinct cliques (otherwise some clique would contain an edge already covered by one of the cliques induced by \( S \) or \( S' \)). But we do not have so many cliques as \(|C| \leq N^2 + N\). Thus we have \(|S \cap S'| = 1\) for any \( S, S' \in \mathcal{S} \), and so Property 2 is satisfied.

Let us now prove Property 3. Consider any arbitrary clique \( C \in \mathcal{C} \). Pick two vertices from \( V(C) \setminus I \) and two vertices from \( I \setminus V(C) \). Note that \(|V(C) \setminus I| = |I \setminus V(C)| \geq N + 1 - 1 = N \geq 2\), and hence two vertices can be picked from the sets. It is easy to see that out of these four vertices at most two are in any set in \( \mathcal{S} \).

Moreover, note that the incidence matrix \( F \) of \( \mathcal{S} \) after removing the column for \( I \) is the BSD of the adjacency matrix of \( G_N \) that corresponds to the clique partition \( C \).

From the above statements we can conclude that there are ECP instances for which every basis of the corresponding rank-\( k \) BSD is dense, as we show next. This essentially follows from the fact that the incidence matrix of any FPP is known to have full rank [22].

Theorem 20. For any \( N \) such that an FPP of order \( N \) exists, every basis of any binary symmetric decomposition of the BSD-DW instance corresponding to \( G_N \) has rank \( N^2 + N \) and contains \( \Theta(N^3) \) non-zero entries.

Proof. From Lemma 17 we obtain that, if an FPP of order \( N \) exists, then there is a clique partition of \( G_N \) with at most \( N^2 + N \) cliques. Now, let \( \tilde{B} \) be any basis of any binary symmetric decomposition of the matrix \( A \) corresponding to \( G_N \) (cf. Lemma 4). As seen in Algorithm 1 (due to the ExtendBasis function and Claim 12), \( \tilde{B} \) can be extended to a rank-\( k \) BSD \( B \), where \( k \) is the rank of \( \tilde{B} \). By Lemma 4, \( B \) is the clique-edge incidence matrix of a clique partition \( C \), which means that \( k = |C| \leq N^2 + N \). By Lemma 19, \( \mathcal{S} = \{V(C) \mid C \in C\} \cup \{I\} \) is an FPP of order \( N \) for which \( B \) is equal to the the incidence matrix \( F \) of \( \mathcal{S} \) after removing the column corresponding to \( I \). The incidence matrix of any FPP is known to have full rank [22], and so \( \tilde{B} \) has rank \( k = N^2 + N \), as \( F \) has \( N^2 + N + 1 \) rows and columns. This also implies that there are \( k = N^2 + N \) columns in the basis \( \tilde{B} \) of \( B \), and each corresponds to a clique of \( C \) containing \( N + 1 \) vertices by Claim 18. Hence the number of 1’s in the binary matrix \( \tilde{B} \) is \( \Theta(N^3) \).

Since for every prime power \( N \) there is an FPP of that order [17], by Theorem 20 there exist infinitely many instances of ECP such that, setting \( k = N^2 + N \), every basis of any rank-\( k \) BSD matrix \( B \) has \( \Theta(N^3) = \Theta(k^{3/2}) \) non-zero entries.
5 Open problems

We showed that AWECMP admits a kernel with $4^k$ vertices, and an algorithm with a runtime of $2^{O(k^{3/2} w^{1/2} \log(kw))} n^{O(1)}$, which implies that ECP can be solved in $2^{O(k^{3/2} \log k)} n^{O(1)}$ time. This leaves the following open questions.

- Close the gap further between the upper and lower bounds on the running time for ECP that are currently $2^{O(k^{3/2} \log k)} n^{O(1)}$ and $2^{\Omega(k)} n^{O(1)}$ respectively.
- Seeing that ECP has a kernel with $k^2$ vertices [18], does WECP admit a polynomial sized kernel as well?
- The algorithm of Chandran et al. [3] for Bipartite Biclique Partition with runtime $2^{O(k^3)} n^{O(1)}$ is also based on guessing the basis of a binary decomposition $A = BC$. If we can show that in any solution at least one of $B$ and $C$ has a row basis (column basis in case of $C$) with at most $g(k)$ ones, then we get a running time $2^{O(g(k) \log k)} n^{O(1)}$ using a similar algorithm as we gave for ECP. Similar to Section 4, what is the minimum value of $g(k)$ possible?
- Can we obtain a similar lower bound on the runtime of our algorithm for WECP as we showed for ECP in Section 4, i.e., can we construct instances of WECP for which all bases have $\Omega(k^{3/2} w^{1/2})$ non-zero entries?

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