MULTIPLE ENDEMIC STATES IN AGE-STRUCTURED SIR EPIDEMIC MODELS

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Abstract. SIR age-structured models are very often used as a basic model of epidemic spread. Yet, their behaviour, under generic assumptions on contact rates between different age classes, is not completely known, and, in the most detailed analysis so far, Inaba (1990) was able to prove uniqueness of the endemic equilibrium only under a rather restrictive condition.

Here, we show an example in the form of a $3 \times 3$ contact matrix in which multiple non-trivial steady states exist. This instance of non-uniqueness of positive equilibria differs from most existing ones for epidemic models, since it arises not from a backward transcritical bifurcation at the disease free equilibrium, but through two saddle-node bifurcations of the positive equilibrium. The dynamical behaviour of the model is analysed numerically around the range where multiple endemic equilibria exist; many other features are shown to occur, from coexistence of multiple attractive periodic solutions, some with extremely long period, to quasi-periodic and chaotic attractors.

It is also shown that, if the contact rates are in the form of a $2 \times 2$ WAIFW matrix, uniqueness of non-trivial steady states always holds, so that 3 is the minimum dimension of the contact matrix to allow for multiple endemic equilibria.

1. Introduction. SIR epidemic models are at the basis of research aiming at predicting the epidemic dynamics and assessing control measures for many kinds of infectious diseases, from measles, chickenpox and other childhood diseases, to influenza, and to emerging diseases. Either the models are exactly age-structured (generally with discrete age) SIR models or some variants of that, allowing for incubation periods or other complications.

A crucial point in the application of these models is the estimation of an adequate kernel that describes infection transmission between age classes. An approach going back to Anderson and May [2] is to divide ages into few ($n$) groups, so as to use a $n \times n$ matrix (often named WAIFW, “Who Acquires Infection From Whom”) to model the transmission between (and within) groups. This is the approach currently...
used in many publications, and several methods have been used for the estimation
of the WAIFW matrix, one of which [17] has required an extensive survey of daily
individual contacts in several European countries.

In contrast with the steady application of this type of models, lies the incomplete
theoretical analysis of their dynamic behaviour. To date, the most general results
have been obtained by Inaba [15] who, defining $R_0$ as the spectral radius of an
appropriate operator, proved that for $R_0 < 1$ the disease–free equilibrium is asymp-
totically stable and there are no positive stationary solutions, while for $R_0 > 1$, the
disease–free equilibrium is unstable and there exists at least one positive stationary
solution.

If transmission follows ‘separable mixing’ (precisely defined below), the conclu-
sion is sharper [7]: for $R_0 > 1$ there exists a unique positive stationary solution. The
same conclusion has been reached by Inaba [15] under hypothesis (H4) (see Section
3), which is however unlikely to hold for empirical transmission kernels. Thus, the
conjecture raised by Greenhalgh [11] that for $R_0 > 1$ there exists a unique positive
stationary solution, is still open in general. In this paper, we show that the con-
jecture is false by exhibiting a WAIFW matrix such that the age-structured SIR
model with that matrix as transmission kernel may have up to 3 positive equilibria
for appropriate parameter values.

More generally, we show a method to build WAIFW matrices with this property;
in the way, we prove that a $2 \times 2$ WAIFW matrix always leads to uniqueness of the
endemic equilibrium when $R_0 > 1$, while one obtain non-uniqueness with a different
$2 \times 2$ matrix-like structure with infectivity increasing with age.

A related issue concerns stability of the equilibria; while it is well known that
the disease-free equilibrium is stable for $R_0 < 1$ and unstable for $R_0 > 1$, little is
known about the stability of the endemic equilibria, except that general principles
about transcritical bifurcations guarantee that it will be asymptotically stable for
$R_0 > 1$ sufficiently close to 1. Thieme [18] showed that an endemic steady state
could be unstable, even with proportionate mixing. Andreasen [3] studied this
problem through perturbation methods under the assumption that the infectious
period is small compared to the individuals life span: he proved [3] that if the
disease transmission is independent of age then an endemic equilibrium is always
locally stable, while the introduction of age-dependent susceptibility may lead to
loss of local stability [4].

The numerical bifurcation study (based on the method in [5]) proposed in the
final section shows that much more complex bifurcation patterns are possible for
this type of model. This contrasts with the invariable convergence to the endemic
equilibrium occurring in all simulations (we are aware of) aiming at reproducing
empirical patterns. It is possible that the structure of realistic WAIFW matrices is
such that, for $R_0 > 1$, the endemic equilibrium is always unique and globally stable.
We hope that the current study helps in elucidating this feature.

2. The model. The population is divided into three compartments, the suscep-
tible, infective and immune individuals; as usual [1, 11, 15], we assume that the
total population is in a demographical stationary state. $n(a)$ is the stationary age
density of the total population, i.e. a stationary positive solution of the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) n(a,t) = -\mu(a)n(a,t), \ 0 < a < \omega, \ t > 0 \\
n(0,t) = \int_0^\omega b(a)n(a,t)da
\end{array} \right.
\end{align*}
\]
where \( b(a) \) is the fertility rate, \( \mu(a) \) is the death rate and a maximal age \( \omega \) is assumed; the assumption of demographical stationarity is equivalent ([14], chap.1) to the condition

\[
\int_0^\omega b(a)\pi(a) \, da = 1,
\]

where

\[
\pi(a) = \exp\{-\int_0^a \mu(s) \, ds\}
\]

represents the probability of surviving up to age \( a \).

We do not delve into the technical conditions [14] on the functions \( b \) and \( \mu \), because they are not actually needed for the analysis of the epidemic models. Note only that \( n(a) = K\pi(a) \) for some constant \( K > 0 \).

Let now \( S(a,t) \), \( I(a,t) \), \( R(a,t) \) be the age densities at time \( t \) of the susceptible, infective and immune individuals, respectively. Susceptible may become infected at rate \( \lambda(a,t) \) ("force of infection"); infectives become removed at rate \( \gamma \). Considering also deaths (at rate \( \mu(a) \)), the equations satisfied by \( S, I \) and \( R \) are

\[
\begin{align*}
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} S &= -(\lambda(a,t) + \mu(a))S \\
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} I &= \lambda(a,t)S - (\mu(a) + \gamma)I \quad 0 < a < \omega, \, t > 0 \\
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} R &= \gamma I - \mu(a)R.
\end{align*}
\]

(2)

Initial and boundary conditions have to be added to the equations. Neglecting transient maternal immunity, it is assumed that all newborns are susceptible, namely

\[
S(0,t) = \int_0^\omega b(a)n(a) \, da, \quad I(0,t) = 0, \quad R(0,t) = 0, \quad t \geq 0.
\]

(3)

In (3) it has been implicitly assumed \( S(a,t) + I(a,t) + R(a,t) = n(a) \). This is possible, since the sum \( S(a,t) + I(a,t) + R(a,t) \) satisfies equation (1), as long as the initial conditions satisfy

\[
S(a,0) + I(a,0) + R(a,0) = K\pi(a) \quad \text{for some} \, K > 0
\]

and then \( n(a) = K\pi(a) \).

Under these assumptions, it is convenient to change variables to the fractions \( x(a,t) = S(a,t)/n(a) \) and \( y(a,t) = I(a,t)/n(a) \) (it is not necessary to consider also \( z(a,t) = R(a,t)/n(a) \), since it can be obtained as \( z(a,t) = 1 - x(a,t) - y(a,t) \)). The system of equations satisfied by \( x \) and \( y \) is [15, p. 414]

\[
\begin{align*}
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} x(a,t) &= -\lambda(a,t)x(a,t) \quad 0 < a < \omega, \, t > 0 \\
\frac{\partial}{\partial t} + \frac{\partial}{\partial a} y(a,t) &= \lambda(a,t)x(a,t) - \gamma y(a,t) \\
x(0,t) &= 1, \quad y(0,t) = 0, \quad t > 0
\end{align*}
\]

(4)

with initial conditions

\[
x(a,0) = x_0(a), \quad y(a,0) = y_0(a), \quad 0 < a < \omega,
\]

with \( x_0(a), y_0(a) \geq 0 \) and \( x_0(a) + y_0(a) \leq 1 \).

To complete model (4), one still has to assign a constitutive rule for the force of infection \( \lambda(a,t) \). The standard assumption ([2], chap.9.2) is of a general linear functional on infective density

\[
\lambda(a,t) = \int_0^\omega \beta(a,s)I(s,t) \, ds = \int_0^\omega \beta(a,s)n(s)y(s,t) \, ds
\]

(5)
with $\beta(a, a')$ the contact coefficient between susceptible individuals aged $a$ and the infectives aged $a'$.

3. **Stationary solutions.** A first problem concerning system (4)–(5) is the existence of stationary solutions (equilibria). For any values of parameters and functions involved, there exists the disease-free equilibrium $(x(a), y(a)) \equiv (1, 0)$. Concerning the existence of other equilibria, Greenhalgh [11] showed the equivalence of the problem of existence and uniqueness of the steady states of system (2) with the problem of existence and uniqueness of the solutions of the following nonlinear integral equation

$$
\lambda(a) = \int_0^\omega \lambda(\sigma)e^{-\int_0^\sigma \lambda(\tau)d\tau} \phi(a, \sigma)d\sigma,
$$

where the kernel $\phi(\cdot, \cdot)$ is given as

$$
\phi(a, \sigma) = \int_\sigma^\omega \beta(a, \xi)n(\xi)e^{-\gamma(\xi-\sigma)}d\xi.
$$

The most general result to date concerning this problem has been obtained by Inaba [15] under the following assumptions about the contact coefficient $\beta(\cdot, \cdot)$.

**H1** $\beta(a, \xi) \in L^\infty_+(\{0, \omega\} \times \{0, \omega\})$.

**H2** $\lim_\rho \int_0^\omega |\beta(a+h, \xi) - \beta(a, \xi)| da = 0$ uniformly for $\xi \in \mathbb{R}$, where $\beta$ is extended by $\beta(a, \xi) = 0$ for $a, \xi \in (-\infty, 0) \cup (\omega, +\infty)$.

**H3** There exist numbers $\alpha$ with $\omega > \alpha > 0$ and $\epsilon > 0$ such that $\beta(a, \xi) \geq \epsilon$ for a.e. $(a, \xi) \in (0, \omega) \times (\omega - \alpha, \omega)$.

He considered the positive nonlinear operator $\Phi : L^1_+(0, \omega) \rightarrow L^1_+(0, \omega)$ defined by

$$
(\Phi \psi)(a) = \int_0^\omega \psi(\sigma)e^{-\int_0^\sigma \psi(\tau)d\tau} \phi(a, \sigma)d\sigma, \quad a \in (0, \omega)
$$

with $\phi(\cdot, \cdot)$ given by (7). Solutions of (6) can be written as solutions $\psi$ of the nonlinear equation $\Phi \psi = \psi$, i.e. fixed points of the operator $\Phi$.

Let finally $T = \Phi'(0)$ be the positive linear operator given by the Frechet derivative of $\Phi$ in zero, $r(T)$ the spectral radius of $T$. Then

**Theorem 1.** (Inaba,1990) Assume (H1)–(H3); then

1. if $r(T) \leq 1$ then the only non-negative fixed point of $\Phi$ is $\psi = 0$;
2. if $r(T) > 1$, $\Phi$ has at least one non-zero fixed point (a positive solution of the equation $\Phi \psi = \psi$).

To prove uniqueness, Inaba added the assumption

**H4** For all $(a, \sigma) \in [0, \omega] \times [0, \omega]$ the inequality $\beta(a, s)n(s) - \gamma \phi(a, s) \geq 0$ holds.

He proved, using monotonicity methods, that under (H4) there exists at most one non-trivial positive solution of equation (6) [15, Prop. 4.10]; hence in case $r(T) > 1$ there exists one and only one non-trivial positive solution under Assumptions (H1)–(H4).

Another assumption under which is easy to prove uniqueness of positive solutions is the so-called separable mixing $\beta(a, s) = \varphi(a)q(s)$. In that case equation (6) can be reduced to a one-dimensional problem, and $r(T)$ can be explicitly computed.

The problem arises of understanding whether or not uniqueness of positive solutions of (6) holds in general. We will consider especially contacts that take the form...
of WAIFW matrices [2]: age is divided into \( n \) age groups \( I_j = (\alpha_{j-1}, \alpha_j), j = 1 \ldots n \) where \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \omega \). Then

\[
\beta(a, a') = \sum_{i,j=1}^{n} \beta_{ij} 1_{I_i}(a) 1_{I_j}(a')
\]  

(9)

where \( \beta_{ij} \geq 0 \forall i, j = 1, \ldots, n \).

In this case it is not difficult to see that, if \( \psi \in L^1_+(0, \omega) \) is a fixed point of \( \Phi \), it must be \( \psi = \sum_{i=1}^{n} x_i 1_{I_i} \) with \( (x_1, \ldots, x_n) \in \mathbb{R}_+^n \). Formally, one can define a map \( J : \mathbb{R}^n \to L^1 \) by

\[
J(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i 1_{I_i}
\]

so that \( \text{Im}(\Phi) \subset \text{Im}(J) \) and a map \( \tilde{\Phi} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) such that \( J \circ \tilde{\Phi} = \Phi \circ J \).

If \( \bar{x} \) is a fixed point of \( \tilde{\Phi} \), it is immediate that \( J(\bar{x}) \) is a fixed point of \( \Phi \). A similar reduction can be reached in the more general case of finite-dimensional mixing

\[
\beta(a, s) = \sum_{j=1}^{n} \varphi_j(a) q_j(s)
\]

for arbitrary functions \( \varphi_j \) and \( q_j \); this case will not be dealt with in this paper, although similar methods would apply.

Through simple computations, the map \( \tilde{\Phi} \) can be written explicitly as

\[
\tilde{\Phi}_i(x) = \sum_{j=1}^{n} \beta_{ij} g_j(x), \ i = 1, \ldots, n
\]

(10)

where

\[
g_1(x) = x_1 \int_{0}^{\alpha_1} n(\xi) \int_{0}^{\xi} e^{-\gamma(\xi-\tau)} e^{-\tau x_1} d\tau d\xi
\]

\[
g_j(x) = \sum_{k=1}^{j-1} x_k \int_{\alpha_{j-1}}^{\alpha_j} n(\xi) \exp\{- \sum_{h=1}^{k-1} x_h (\alpha_h - \alpha_{h-1})\} \int_{\alpha_{k-1}}^{\alpha_k} e^{-\gamma(\xi-\tau)-x_k(\tau-\alpha_{k-1})} d\tau d\xi
\]

\[
+ x_j \int_{\alpha_{j-1}}^{\alpha_j} n(\xi) \exp\{- \sum_{h=1}^{j-1} x_h (\alpha_h - \alpha_{h-1})\} \int_{\alpha_{j-1}}^{\xi} e^{-\gamma(\xi-\tau)-x_j(\tau-\alpha_{j-1})} d\tau d\xi.
\]

(11)

It is possible to prove the existence of non-zero fixed points of \( \tilde{\Phi} \) under weaker conditions than those of Theorem 1. Indeed, conditions (H1) and (H2) are automatic when (9) holds, and (H3) can be weakened, or dropped altogether allowing for nonnegative fixed points. Precisely, consider the assumption

\[
(H3') \sum_{j=1}^{n} \beta_{ij} > 0 \forall i = 1, \ldots, n \text{ and there exist } m \in \{1, \ldots, n-1\} \text{ and indices } i_1, \ldots, i_m, j_1, \ldots, j_m \text{ with } i_1 = 1, j_m = n, i_l \leq j_{l-1}, l = 2, \ldots, m, \text{ such that }
\]

\[
\beta_{i_1, j_1} \cdot \beta_{i_2, j_2} \cdots \beta_{i_m, j_m} > 0.
\]

Theorem 2. Assume (9). Then

(a) If \((H3')\) and \( r(T) > 1 \), there exists a positive fixed point of \( \tilde{\Phi} \), hence of \( \Phi \).
b) If \( r(T) > 1 \), there exists a nonnegative fixed point of \( \tilde{\Phi} \), hence of \( \Phi \).

The proof is in Appendix A where it is shown that (H3') is sufficient to exclude existence of non-trivial fixed points of \( \tilde{\Phi} \) on the boundary of \( \mathbb{R}_+^n \). The first part of (H3') ensures that \( \Phi_i(x) \), i.e. the force of infection in each age class \( I_i \), is positive for some \( x \in \mathbb{R}_+^n \). The second condition can be described as asking that, if there are some infective individuals in age-class \( n \), eventually the force of infection will be positive in age class 1. Only this condition is needed, because the passage of time ensures that if there are some infective individuals in age class \( i \), there will be infectives in age class \( j \) for each \( j > i \) at some later time.

It can be seen that (H3') is essentially necessary to exclude existence of fixed points of \( \tilde{\Phi} \) on the boundary of \( \mathbb{R}_+^n \) in the following sense. Without lack of generality, one can assume that there exists \( j \) such that \( \beta_{jn} > 0 \); otherwise, the first \( n-1 \) components of \( \tilde{\Phi} \) do not depend on \( x_n \) and one can restrict \( \tilde{\Phi} \) to \( \mathbb{R}^{n-1} \) (in biological terms, the \( n \)-th age class would be an epidemiological dead-end). Under this assumption, it is not difficult to see that, if (H3') is violated, the system \( \tilde{\Phi}(x) = x \) can be split into subsystems, so there exist matrices with the same sign structure that have non-trivial fixed points on the boundary of \( \mathbb{R}_+^n \).

In the course of the proof, we prove the following Proposition concerning the fixed point index \( i(\tilde{\Phi}, \cdot) \); it will also be used in later Sections, and may be of independent interest.

**Proposition 1.** Assume \( r(T) > 1 \) and (H3'). Then there exists a bounded open set \( G \subset \mathbb{R}_+^n \) such that

\[
i(\tilde{\Phi}, G) = 1
\]

and all fixed points \( x \in \mathbb{R}_+^n, x \neq 0 \), belong to \( G \).

Actually, the proof of Proposition 1 works even in the infinite dimensional case [15], using a generalization of the fixed point index [9, Ch. 20] needed because the positive cone of \( L^1(0, \omega) \) has empty interior.

4. **An example of non-uniqueness.** The example of non-uniqueness of stationary solutions has been obtained choosing coefficients \( \beta_{ij} \) that allowed for drastic simplifications. Specifically, in Appendix B we show the following

**Example 1.** Let \( \beta_{ij} = 0 \) for \((i, j) \notin \{(1, 3), (2, 2), (3, 1)\}\). \( i, j = 1, 2, 3 \), \( n(\xi) \equiv 1 \) on \((0, \omega)\), and let \( \Phi(\vartheta, x) \) be the map defined in (10), making it explicit its dependence on all parameters \( \vartheta = (\beta_{13}, \beta_{22}, \beta_{31}, \alpha_1, \alpha_2, \omega, \gamma) \) beyond \( x = (x_1, x_2, x_3) \).

Then there exist values \( \vartheta^* \in \mathbb{R}_+^7 \) and \( x^* \in \mathbb{R}_+^3 \) such that

\[
\Phi(\vartheta^*, x^*) = x^*
\]

\[
\det \left( I - \Phi_x(\vartheta^*, x^*) \right) = 0.
\]

More precisely, the construction of the example yields values of \( \gamma = 1 \), \( \beta_{13} \approx 1.8702 \cdot 10^5 \), \( \beta_{22} \approx 2.4633 \), \( \beta_{31} \approx 1.6835 \), \( \alpha_1 = 2 \) such that for each \( \omega \) large enough there exists \( \alpha_2 \) with \( \alpha_1 < \alpha_2 < \omega \) such that (13) and (14) are satisfied at \( x^* = (1, 1, 1) \). It is a constructive recipe that, for instance, for \( \omega = 100 \), yields \( \alpha_2 \approx 12.4342 \).

Condition (13) says that \( x^* \) is a fixed point of \( \tilde{\Phi} \) when \( \vartheta = \vartheta^* \), while (14) is the necessary condition for saddle-node bifurcations of fixed points of maps. Given a
saddle-node bifurcation, one can choose a one-dimensional parameter from $\theta$ (for instance $\alpha_2$, which has been used in the numerical computations) and obtain that for $\alpha_2$ close to $\alpha_2^*$ there exist two fixed points of $\Phi$ close to $x^*$ for $\alpha_2 > \alpha_2^*$ [or $\alpha_2 < \alpha_2^*$] and none for $\alpha_2 < \alpha_2^*$ [or $\alpha_2 > \alpha_2^*$]. This would suffice to prove that, for certain parameter values, equation $x = \Phi(x)$ has multiple positive solutions, hence (4) has multiple positive equilibria.

In order to prove the occurrence of a saddle-node bifurcation in $(\alpha_2^*, x^*)$, one would need to check the transversality conditions. This can be performed through very long computations. We are satisfied here with checking them numerically, and showing the bifurcation diagram obtained numerically through the use of MATLAB and MATCONT software (see Figures 1 and 2).

One may note that the choice of $\beta_{ij}$'s in Proposition 1 does not satisfy condition (H3) but only (H3'); thus it cannot be technically considered a counter-example to the conjecture of uniqueness of equilibria under (H1)-(H3); from the computations it is clear, however, that by continuity the results would be the same if a small $\varepsilon > 0$ were added to all values of $\beta_{ij}$.

5. Numerical results. First of all, using MATCONT, we computed, starting from the value found analytically (2.4633, 12.4342) the curve of points in the parameter space $(\beta_{22}, \alpha_2)$ at which the two conditions (13)–(14) occur at some point $x^* \in \mathbb{R}_+^3$. Other parameter values are $\beta_{13} = 1.8702 \cdot 10^4$, $\beta_{31} = 1.6835$, $\alpha_1 = 2$, $\omega = 100$, $\gamma = 1$.

It can be seen that the curve has a cusp around $(2.51, 12.3)$ from which two branches depart for $\beta_{22} < 2.51$: for such values two saddle-node bifurcations occur at $(\alpha_2^*, x^*)$ and $(\alpha_2^{**}, x^{**})$, which get further apart as $\beta_{22}$ decreases away from 2.51.

We then looked at a bifurcation diagram in the single parameter $\alpha_2$, setting $\beta_{22} \approx 1.83$, so that the two saddle-node bifurcation points are visibly away from each other. The overall bifurcation structure (Fig. 2) confirms that for $\alpha_2 \in (\alpha_2^*, \alpha_2^{**})$ equation (6) has actually 3 positive solutions, while only 1 (unless other solution branches exist) for $\alpha_2 < \alpha_2^*$ and $\alpha_2 > \alpha_2^{**}$. 

![Figure 1. The curve in parameter space $(\beta_{22}, \alpha_2)$ at which the two conditions (13)–(14) occur at some point $x^* \in \mathbb{R}_+^3$. Other parameter values are $\beta_{13} = 1.8702 \cdot 10^4$, $\beta_{31} = 1.6835$, $\alpha_1 = 2$, $\omega = 100$, $\gamma = 1$.](image-url)
Figure 2. Coordinate $x^*_1$ of the stationary positive solutions of system (4) against the value of $\alpha_2$. Other parameter values are $\beta_{13} \approx 1.8702 \cdot 10^4$, $\beta_{22} \approx 1.83$, $\beta_{31} \approx 1.6835$, $\alpha_1 = 2$, $\omega = 100$, $\gamma = 1$. The points at which roots of the characteristic equation cross the imaginary axis are also shown: moving on the curve from right to left dashed labels denote crossings left-to-right, solid labels denote crossing right-to-left, thick labels are for real crossings, normal labels are for complex crossings. Traits between bifurcation points are shown in different style: thick solid denotes stable equilibrium, dashed unstable equilibrium, solid periodic orbits. More details are in the text.

The stability of the corresponding stationary solutions of (4) has been studied numerically through the approximation of the eigenvalues of the linearisation at equilibria, as depicted in Fig. 2. The technique, proposed in [5] and applied similarly in [6], is based on the reduction to finite dimension of the infinitesimal generator of the semigroup associated to the model. Starting the description from the right, along the lower thick solid curve all eigenvalues have negative real part (and thus the equilibrium is asymptotically stable) for all $\alpha_2 > \alpha_2^* \approx 13.673273$, the point of saddle-node bifurcation (thick dashed label). Proceeding along the middle branch of the curve, in the dashed part there is one real positive eigenvalue, while both at $\alpha_2 \approx 14.2304$ and $\alpha_2 \approx 14.5766$ (dashed labels) a pair of complex conjugate roots crosses the imaginary axis from left to right, thus giving rise through Hopf bifurcations to periodic solutions that inherit the instability of the equilibrium they are bifurcating from (thin solid part). At $\alpha_2 = \alpha_2^{**} \approx 14.945014$ a second saddle-node bifurcation occurs with the real positive eigenvalue becoming negative (thick solid label); the upper solution branch starts however unstable (still thin solid), because two pairs of complex conjugate eigenvalues have positive real part. At $\alpha_2 \approx 14.9245$ and $\alpha_2 \approx 14.7972$ (dashed labels) two further pairs of complex conjugate eigenvalues cross the imaginary axis from left to right; thus for $\alpha_2$ to the left of 14.7972, the linearisation at the equilibrium on the upper solution branch has four pairs of complex conjugate
eigenvalues in the positive half-plane. These move back in the negative half-plane at the bifurcation points at $\alpha_2 \approx 12.8405, 9.8239, 8.9651$ and $8.70093$ (solid labels). For $\alpha_2 < 8.70093$ all eigenvalues have negative real part (thick solid curve), so that the equilibrium is asymptotically stable. Looking at the bifurcation diagram in the opposite direction, one sees that at $\alpha_2 = 8.70093$ a Hopf bifurcation occurs with the emergence of a periodic solution that, in the supercritical case, would inherit the asymptotical stability. Because of the complexity of the system, a numerical continuation of the branches of periodic solutions and the corresponding analysis of their stability was unfeasible. Instead we approximated the solutions of the PDE system (6) at selected values of $\alpha_2$, chosen in relation to the diagram of Fig. 2.

Some results have been selected out of many simulations and are shown in Figures 3 and 4 where $\alpha_2$ increases going from left to right and from top to bottom. As initial conditions, we choose, as described in the legends, either the final conditions of other simulations, or fractions of susceptible and infectives that would be at equilibrium if the force of infection were fixed at $(x_1, x_2, x_3) = (1, 1, 1)$. Precisely

$$x(a) = \begin{cases} e^{-x_1 a} & \text{if } 0 < a < \alpha_1 \\ e^{-x_1 a + x_2 (a - \alpha_1)} & \text{if } \alpha_1 < a < \alpha_2 \\ e^{-x_1 a + x_2 (\alpha_2 - \alpha_1) - x_3 (a - \alpha_2)} & \text{if } \alpha_2 < a < \omega \end{cases}$$

$$y(a) = \int_0^a (x_1 1_{(0, \alpha_1)}(\tau) + x_2 1_{(\alpha_1, \alpha_2)}(\tau) + x_3 1_{(\alpha_2, \omega)}(\tau)) e^{-\gamma (a - \tau)} x(\tau) d\tau.$$

From Fig. 3 it can be seen that, as expected, solutions converge to the (unique) equilibrium for $\alpha_2 = 8.5$ (first panel) and, passing the Hopf bifurcation, to a periodic solution (of period approximately 20 time-units) for $\alpha_2 = 9$ (second panel). Increasing furthermore $\alpha_2$, the attractive periodic solution acquires a more complex structure (at $\alpha_2 = 11$), then a parameter region ($\alpha_2$ between 11.59 and 11.67) emerges in which solutions appear chaotic, followed by a region (we show $\alpha_2 = 12$) where solutions converge to a periodic solutions of much shorter period (around 8 time units).

Looking at Fig. 4, obtained with larger values of $\alpha_2$, one sees that at $\alpha_2 = 13$, two different periodic solutions exist, one of long period (around 50 time units), the other one much shorter (around 9 time units). They appear to coexist for $\alpha_2$ up to around 13.5, where the period of the long-period solution approaches 80 time-units, while the solution of short period undergoes a Neimarck-Sacker bifurcation, giving rise to a quasi-periodic solution (fourth panel). This solution disappears into a chaotic region (not shown) for $\alpha_2$ around 13.6, while the long-period solution persists and, at $\alpha_2 = 13.66$, has period around 200 years. Increasing $\alpha_2$ further, the period lengthens but, as $\alpha_2$ passes the saddle-node value around 13.675, all solutions appear to converge to the stable equilibrium solution (the lower branch in Fig. 2). The suspicion arises that the stable periodic solution disappears through a homoclinic bifurcation at the same time as the saddle-node bifurcation occurs. The complexity of the system prevents us from reaching definite conclusions in this respect as well as in the apparent transitions to chaos.

6. **A uniqueness result.** The example of non-uniqueness of equilibria of (4) has been obtained assuming that the contact rates have the structure (9) with $n = 3$. Since uniqueness is obvious with $n = 1$, one may wonder whether it is possible to find an example of non-uniqueness with the structure (9) with $n = 2$. The answer is actually negative, as shown below.
Figure 3. The values of $x_1(t) = \beta_{13} \int_0^\infty n(s) y(s, t) \, ds$, $x_2(t) = \beta_{22} \int_0^{\alpha_2} n(s) y(s, t) \, ds$, $x_3(t) = \beta_{31} \int_0^{\alpha_1} n(s) y(s, t) \, ds$ obtained by approximating (4) with a finite-difference scheme. The value of $\alpha_2$ (from 8.5 to 12) is reported in each panel. All other parameter values as in Fig. 2. The initial conditions were, for the first three panels, the final conditions of the panel in Fig. 4a; in the last three panels, the final conditions of the third panel.
First note that, assuming (9) with \( n = 2 \) as contact rates, then the operator equation (6) takes the form \( x_i = \tilde{\Phi}_i(x) \), \( i = 1, 2 \), where \( \tilde{\Phi} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2 \) is the
operator defined in (10) with \( n = 2 \) and that we now rewrite as
\[
\tilde{\Phi}_i(x) = \beta_{i1} \int_0^\alpha n(\xi) e^{-\gamma \xi} x_1 \varphi_\alpha(x_1) d\xi + \beta_{i2} \int_\alpha^\infty n(\xi) e^{-\gamma \xi} \times \left( x_1 \varphi_\alpha(x_1) + x_2 e^{\alpha(x_2-x_1)} (\varphi_\alpha(x_2) - \varphi_\alpha(x_2)) \right) d\xi, \quad i = 1, 2
\]
where
\[
\varphi_s(t) = \int_0^s e^{\tau(g-t)} d\tau = \left\{ \begin{array}{ll}
\frac{e^{(\gamma-t)s} - 1}{\gamma-t} & \gamma \neq t \\
\frac{s}{\gamma-t} & \gamma = t \\
0 & \gamma = t, s \leq 0, t \geq 0
\end{array}\right.
\]
and now \( I_1 = (0, \alpha), I_2 = (\alpha, \omega) \).

We assume in what follows that the entries of the matrix \( \{\beta_{ij}\}_{i,j=1,2} \) satisfy the condition \((H3')\), that now is equivalent to the condition
\[
\beta_{12} > 0, \beta_{21} + \beta_{22} > 0.
\]

This assumption ensures that non-zero fixed points are in the interior of the positive cone \( \mathbb{R}_+^2 \).

The main result we obtain in this Section is

**Theorem 3.** Assume \((H3')\) and let \( \tilde{\Phi} \) be defined as in (16) and \( T = \tilde{\Phi}'(0) \). If \( r(T) > 1 \), then there exists a unique \( x^* \in \text{Int}(\mathbb{R}_+^2) \) s.t. \( x^* = \tilde{\Phi}(x^*) \).

The existence part was firstly proved on Theorem 1 by Inaba under assumption \((H3)\) \([15]\) and it is proved in Theorem 2, for the finite-dimensional case, under the more general assumption \((H3')\). Uniqueness is proved through a series of intermediate results. The crucial one is summarised as

**Proposition 2.** Assume \( x \in \text{Int}(\mathbb{R}_+^2) \) is a fixed point of (16). Then
\[
\det (I - \tilde{\Phi}_x(x)) > 0.
\]

Proposition 2 shows that the necessary condition (14) for a saddle-node bifurcation cannot occur for \( n = 2 \).

Proposition 2 is proved in Appendix C through a series of Lemmas.

In order to prove Theorem 3, first note that all nontrivial fixed points (that are in the interior of \( \mathbb{R}_+^2 \) because of \((H3')\)) are isolated since, by Proposition 2, \( \tilde{\Phi} \) is a local diffeomorphism in each fixed point \( x \). Then, the local fixed point index in \( x \) is defined \([19, \text{Ch. 12.3}]\) and given by
\[
i(\tilde{\Phi}, x) = \text{sign} \left( \det (I - \tilde{\Phi}_x(x)) \right) = 1.
\]

We can finally prove the main result of this Section.

**Proof of Theorem 3.** Consider the open set \( G \) defined in Proposition 1. The set of positive fixed points of \( \tilde{\Phi} \) is contained in \( G \). We already know that such fixed points are isolated. Hence their number is finite for \( G \) is bounded. Let \( x_1, \ldots, x_m \) be such points and \( B_\rho(x_i), i = 1, \ldots, m \), be disjoint neighborhoods of the \( x_i \), \( B_\rho(x_i) \subseteq G \). Then for the index sum Theorem \([19, \text{Prop. 12.6}]\) and Propositions 2 and 1 we have
\[
1 = i(\tilde{\Phi}, G) = \sum_{i=1}^m i(\tilde{\Phi}, B_\rho(x_i)) = \sum_{i=1}^m \text{sign}(\det (I - \tilde{\Phi}_x(x_i))) = m.
\]

Then we have \( m = 1 \). \(\square\)
We have seen above that, if the contact rates have the form (9) with \( n = 2 \), the positive equilibrium is unique above the threshold. It is instead possible to choose \( \beta(a, a') \) in a way such that the operator \( \Phi \) is essentially 2-dimensional, but the condition (14) for saddle-node bifurcation of equilibrium (thus giving rise to multiple positive equilibria) is satisfied.

Choose in fact
\[
\beta(a, \xi) = \sum_{i,j=1}^{2} \beta_{ij} I_i(a) I_j(\xi) n(\xi) e^{-\gamma \xi}. 
\]

With this choice of \( \beta(\cdot, \cdot) \), a positive fixed point \( \psi \) for \( \Phi \) must be
\[
\psi = \sum_{i=1}^{2} x_i I_i
\]
where \((x_1, x_2)\) is a solution of the equation on \( \mathbb{R}^2 \):
\[
x_i = \beta_{1i} x_1 \int_{0}^{\alpha} \varphi_\xi(x_1) d\xi + \beta_{12} x_1 \varphi_\alpha(x_1) (\omega - \alpha) 
+ \beta_{2i} x_2 e^{\alpha(x_2-x_1)} \int_{\alpha}^{\omega} (\varphi_\xi(x_2) - \varphi_\alpha(x_2)) d\xi, \quad i = 1, 2
\]
with \( \varphi \) given as in (17). Considering now \( \tilde{\Phi} : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \) defined as the right hand side of (20), one can find, with computations similar to those performed in Section 4, that there exist values of the parameters and of \( x^* \) such that \( \tilde{\Phi}(x^*) = x^* \) and
\[
\det \left( I - \tilde{\Phi}_x(\vartheta^*, x^*) \right) = 0.
\]

One such example is obtained at \( x^* = (1, 1) \) with
\[
\beta_{11} = \beta_{22} = 0, \quad \gamma = 1, \quad \alpha = 4, \quad \omega = 28 + \sqrt{720}, \quad \beta_{12} = \frac{2}{\omega^2 - \alpha^2}, \quad \beta_{21} = \frac{2}{\omega^2}. 
\]

It must be remarked that the choice (19) entails that \( \beta(a, \xi) \) is unbounded as \( \xi \) goes to \( \omega \) if, as often but not always assumed, \( \lim_{a \to \omega} n(a) = 0 \). This problem can be adjusted by modifying \( \beta(a, \xi) \) when \( \xi \) is in a suitably small neighbourhood of \( \omega \) while, by continuity, multiple solutions will continue to exist. However, the choice (19) appears anyway unrealistic, because with that infectiousness increases exponentially with age \( \xi \).

7. Conclusions. The analysis presented in this paper shows that the age-structured SIR epidemic model may have rather unexpected dynamical behaviour, with multiple endemic equilibria, coexistence of multiple attractive periodic solutions, some with extremely long period, occurrence of quasi-periodic and chaotic attractors and, more generally, a rather complicated bifurcation structure, of which we believe we have captured only some partial information.

Although the contact rates have been chosen according to the widely used pattern of WAIFW matrices, it must be admitted that the parameter values employed do not aim at being realistic, but have been chosen purely out of mathematical simplicity. Indeed, it is difficult to conceive that individuals in the intermediate age class have contact only within themselves, while those in the youngest age class can be infected only by those in the oldest class and vice versa. Furthermore in the example shown in Sections 4 and 5 the strength of infection old-to-young is orders of magnitude higher than that of infection young-to-old. Thus, the astonishing pattern of periodic epidemics occurring every around 200 years (Fig. 4e), assuming an average infectious period of 1 year and expected life of 100 years, may look just a mathematical curiosity.

An intuitive explanation of the multiple equilibria appears appropriate. If there were no population fluxes among age classes, with the WAIFW matrix chosen in the example, there would be two separate epidemics: one involving youngest and oldest age class, and another one involving the intermediate age class. Since there
is instead population flux, an endemic infection in young and old age classes is such that (when parameter values are suitable) too few individuals arrive susceptible at the intermediate age class, so that the infection in that age class is not self-sustaining; this corresponds to the upper equilibrium branch in Fig. 2 that exists only if $\alpha_2$ is not too large (otherwise, the size of the intermediate class becomes large enough that the infection in that age class becomes self-sustaining); in the alternative equilibrium (lower equilibrium branch in Fig. 2), the infection is endemic in the intermediate age class and too few susceptibles are left for the older age class to be able to spread sufficiently the infection to the young age class and back.

A similar pattern can be seen in the periodic solutions, where infections in the intermediate age class on one side, and in the young and old age class, on the other side, are desynchronized. One can possibly conceive less extreme examples in which partial desynchronizations of epidemics in different age classes occur. The examples shown, though admittedly unrealistic, give insight on the mechanisms through which, when a dynamical state (such as age) affects epidemic transmission, complicated patterns may arise that are impossible when states are static; as well known, in multigroup SIR models, global convergence to a unique equilibrium occurs above threshold [13, 12]. The complex behaviour of this system may appear reminiscent of what found for the LPA model of Tribolium [8], where again there are 3 stages with simple chain transitions and nonlinear interactions.

One may finally wonder whether, if the contact matrices are ‘realistic’ in some sense, uniqueness of the endemic equilibrium occurs also in age-structured epidemic model. The question becomes which are the features of ‘realistic’ contact matrices: for instance, one may require that they are not too far from symmetry and that their diagonal elements are larger than the non-diagonal (as in the empirically identified matrices in POLYMOD [17]). Looking at the same question from a different perspective, one could aim at identifying which properties of contact matrices are required to obtain a saddle-node bifurcation, although computations appear prohibitive without simplifying assumptions of the type we used in Section 4.

Appendix A. Proof of Theorem 2. We now use the structure of $\tilde{\Phi}$ (10) to prove the following.

Lemma 1. If (H3’) holds, then $\tilde{\Phi}$ has no fixed points on the boundary of $\mathbb{R}_+^n$ but for 0.

Proof. One immediately see that, if $x \in \mathbb{R}_+^n$ with $x_i > 0$ then $g_j(x) > 0$ (11) for all $j \geq i$; in particular, if $x \in \mathbb{R}_+^n \setminus \{0\}$, then $g_n(x) > 0$.

Assume $x \in \mathbb{R}_+^n$, $x \neq 0$, is a fixed point of $\tilde{\Phi}$. Since $g_n(x) > 0$, from (H3’) $\tilde{\Phi}_{i_m}(x) > 0$, so that $x_{i_m} > 0$; similarly from $x_i > 0$, one obtains $g_{j_{i-1}}(x) > 0$, implying $x_{i_{i-1}} = \tilde{\Phi}_{i_{i-1}}(x) > 0$. Applying this recursively backwards with $l$ from $m$ to 2, one arrives at $x_i = x_1 > 0$. Then from (H3’), one gets $g_j(x) > 0$ for all $j$, so that $x_i = \tilde{\Phi}_i(x) > 0$ for all $i = 1, \ldots, n$.

Proof of Proposition 1. We observe that

$$M_i = \sup_{x \in \mathbb{R}_+^n} \tilde{\Phi}_i(x) < +\infty, \ i = 1, \ldots, n,$$

so we can choose $R > \sum_{i=1}^n M_i (\alpha_i - \alpha_{i-1})$ and define

$$G = \{x \in \mathbb{R}_+^n : x_i > 0 \forall i, \ \rho < \|J(x)\|_1 < R\},$$
with \( \rho > 0 \) to be selected below, \( \| \cdot \|_1 \) the \( L^1 \)-norm on \((0, \omega)\) and recalling the definition
\[
J(x) = \sum_{i=1}^{n} x_i 1_{I_i}
\]

Choose \( x_0 \) in the interior of \( \mathbb{R}^n_+ \) with \( \| J(x_0) \|_1 < R \), and let
\[
q(t) = \max \{ \ln \left( r((1-t)T) \right), t \| J(x_0) \|_1 \}
\]
for \( t \in [0,1] \). \( q \) is continuous since \( \ln \left( r((1-t)T) \right) = \ln \left( (1-t)r(T) \right) \) because of the spectral mapping theorem ([16, Chapt. 1.6]). We notice that \( q(t) > 0 \) for all \( t \in [0,1] \) because of the assumption on \( r(T) \) and the choice of \( x_0 \). Hence \( \min_{t \in [0,1]} q(t) > 0 \) and we can choose \( \rho \) with \( 0 < \rho \leq \min_{t \in [0,1]} q(t) \).

Now we define the homotopy
\[
H: \mathcal{G} \times [0,1] \rightarrow \mathbb{R}^n \quad \text{by} \quad H(x,t) = (1-t)\tilde{\Phi}(x) + tx_0.
\]
We need to show that \( H(x,t) \neq x \) for all \( x \) on the boundary of \( G \) and \( t \in [0,1] \). By construction
\[
\| J(H(x,t)) \|_1 = (1-t)\| J(\tilde{\Phi}(x)) \|_1 + t\| J(x_0) \|_1 < (1-t)\sum_{j=1}^{n} M_j(\alpha_j - \alpha_{j-1}) + tR < R.
\]
Furthermore, in Remark 6 of [10], it is shown that, if \( \tilde{\psi} \) is a fixed point in \( L^1_+(0,\omega) \) of a map \( \Phi(\psi) + u_0 \) with \( \Phi \) given by (8) for some kernel \( \beta(a,d') \), \( u_0 \in L^1_+(0,\omega) \) and \( r(\Phi'(0)) > 1 \), then \( \| \tilde{\psi} \|_1 > \ln(r(\Phi'(0))) \). Note that, when \( \beta \) has the structure (9), we do not need to distinguish between \( r(\Phi'(0)) \) and \( r(\Phi'(0)) \), because the two operators have the same eigenvalues.

Assume then that \( \bar{x} = (1-t)\tilde{\Phi}(\bar{x}) + tx_0 \) for some \( t \in [0,1] \) such that \( r((1-t)T) > 1 \) (this occurs for \( 0 \leq t < 1-1/r(T) \)). Applying the previous observation to \( (1-t)\Phi \) with \( u_0 = J(tx_0) \), one obtains \( \| J(\bar{x}) \|_1 > \ln(r((1-t)T)) \). Instead, if \( r((1-t)T) \leq 1 \), then \( \| J(\bar{x}) \|_1 > \ln(r((1-t)T)) \) is obvious.

By the monotonicity of \( J \) and the \( L^1 \)-norm, one has \( J(\bar{x}) \geq J(tx_0) \), hence \( \| J(\bar{x}) \|_1 \geq \| J(tx_0) \|_1 = t\| J(x_0) \|_1 \). Thus \( \| J(\bar{x}) \|_1 \geq q(t) \) for each \( t \in [0,1] \), hence it must be \( \| J(\bar{x}) \|_1 > \rho \).

To conclude, we must exclude that there exists fixed points of \( H(\cdot,t) \) on the boundaries where \( x_i = 0 \) for some \( i = 1, \ldots, n \). If \( t = 0 \) this is impossible by the previous Lemma; if \( t > 0 \), this is prevented from \( \bar{x} > tx_0 \in \text{Int}(\mathbb{R}^n_+) \).

Hence \( H \) is an admissible homotopy; it follows for homotopy invariance [19, Chapt. 12.3]
\[
1 = i(x_0, G) = i(\tilde{\Phi}, G).
\]

\( \square \)

**Proof of Theorem 2.**
a) If (H3') holds, then \( i(\tilde{\Phi}, G) = 1 \), which implies that \( \tilde{\Phi} \) has at least one fixed point in \( G \) ([19, Theor. 12.4(2)]) , as claimed.

b) Returning to the Proof of Proposition 1, without condition (H3') we cannot exclude that \( \tilde{\Phi} \) has nonzero fixed points on the boundary of \( \mathbb{R}^n_+ \). However since \( x_0 \in \text{Int}(\mathbb{R}^n_+) \), \( H(\cdot,t) \) cannot have fixed points on the boundary of \( G \) for \( t > 0 \), so we can use the homotopy \( H(x,t) \) for \( t \in [\varepsilon,1] \) for any \( \varepsilon > 0 \) and conclude that \( H(x,\varepsilon) \) will have a fixed point, say \( \bar{x}_\varepsilon \) in \( G \) for each \( \varepsilon > 0 \).

By the boundedness of \( G \), we can find a sequence \( \varepsilon_k \) with \( \lim_{k \to \infty} \varepsilon_k = 0 \) such that \( \bar{x}_{\varepsilon_k} \) converges to some \( \bar{x} \in G \). Passing to the limit in
\[
\bar{x}_{\varepsilon_k} = (1-\varepsilon_k)\tilde{\Phi}(\bar{x}_{\varepsilon_k}) + \varepsilon_k x_0
\]
and using the continuity of \( \tilde{\Phi} \), we can conclude that \( \bar{x} = \tilde{\Phi}(\bar{x}) \).

\( \square \)
Appendix B. Construction of Example 1. Let us start by rewriting (10) with \( n = 3 \), as

\[
\hat{\Phi}_i(x) = \beta_{i1} \int_{0}^{\alpha_1} n(\xi) e^{-\gamma \xi} x_1 \varphi_{\xi}(x_1) d\xi \\
+ \beta_{i2} \int_{\alpha_1}^{\alpha_2} n(\xi) e^{-\gamma \xi} \left( x_1 \varphi_{\alpha_1}(x_1) + x_2 e^{\alpha_1(x_2-x_1)} (\varphi_{\xi}(x_2) - \varphi_{\alpha_1}(x_2)) \right) d\xi \\
+ \beta_{i3} \int_{\alpha_2}^{\alpha_3} n(\xi) e^{-\gamma \xi} \left( x_1 \varphi_{\alpha_3}(x_1) + x_2 e^{\alpha_1(x_2-x_1)} (\varphi_{\alpha_2}(x_2) - \varphi_{\alpha_1}(x_2)) + x_3 e^{\alpha_1(x_2-x_1)} e^{\alpha_2(x_2-x_3)} (\varphi_{\xi}(x_3) - \varphi_{\alpha_2}(x_3)) \right) d\xi , \quad i = 1, 2, 3,
\]

where \( \varphi_s(t), s \geq 0 \) and \( t \geq 0 \), is defined in (17).

If \( \beta_{ij} \) are chosen as in the Example 1 and \( x \in \text{Int}(\mathbb{R}_+^4) \) is a fixed point of (22), then

\[
\det \left( I - \hat{\Phi}_x(x) \right) = \beta_{13} \beta_{22} \beta_{31} (A_1 - A_2 - A_3)
\]

where

\[
A_1 = \int_{\alpha_2}^{\alpha_3} n(\xi) e^{-\gamma \xi} \left[ \left( \frac{1}{x_1} + \alpha_1 \right) x_3 e^{\alpha_1(x_2-x_1)} e^{\alpha_2(x_2-x_3)} (\varphi_{\xi}(x_3) - \varphi_{\alpha_2}(x_3)) + \left( \frac{1}{x_1} + \alpha_1 \right) x_2 e^{\alpha_1(x_2-x_1)} (\varphi_{\alpha_2}(x_2) - \varphi_{\alpha_1}(x_2)) - x_1 \varphi_{\alpha_1}(x_1) \right] d\xi \\
+ \left( \frac{1}{x_1} + \alpha_1 \right) x_2 e^{\alpha_1(x_2-x_1)} (\varphi_{\alpha_2}(x_2) - \varphi_{\alpha_1}(x_2)) - x_1 \varphi_{\alpha_1}(x_1) \right] d\xi \\
+ \left( \varphi_{\alpha_1}(x_2) - \varphi'_{\alpha_1}(x_2) \right) \right] d\xi \cdot \int_{0}^{\alpha_1} n(\xi) e^{-\gamma \xi} \frac{x_1}{x_3} \varphi_{\xi}(x_1) d\xi
\]

\[
A_2 = \int_{\alpha_2}^{\alpha_3} n(\xi) e^{-\gamma \xi} e^{\alpha_1(x_2-x_1)} e^{\alpha_2(x_2-x_3)} \left[ (1 + \alpha_2 x_3) (\varphi_{\xi}(x_3) - \varphi_{\alpha_2}(x_3)) + x_3 (\varphi'_{\xi}(x_3) - \varphi'_{\alpha_2}(x_3)) \right] d\xi \\n+ \left( \varphi'_{\alpha_1}(x_2) - \varphi'_{\alpha_2}(x_2) \right) \right] d\xi \cdot \int_{0}^{\alpha_1} n(\xi) e^{-\gamma \xi} \left( \varphi_{\xi}(x_1) + x_1 \varphi_{\xi}(x_1) \right) d\xi
\]
\[ A_3 = \int_0^\omega n(\xi)e^{-\gamma \xi} \left[ (\alpha_1 - \alpha_2) x_3 e^{\beta \alpha_1(x_2-x_1)} e^{\beta_2(x_3-x_2)} (\varphi_\xi(x_3) - \varphi_{\alpha_2}(x_3)) + e^{\beta \alpha_1(x_2-x_1)} \left( 1 + \alpha_1 x_2 \right) \left( \varphi_{\alpha_2}(x_2) - \varphi_{\alpha_1}(x_2) \right) \right] d\xi \]

\[ + e^{\beta \alpha_1(x_2-x_1)} \left( 1 + \alpha_1 x_2 \right) \left( \varphi_{\alpha_2}(x_2) - \varphi_{\alpha_1}(x_2) \right) \]

\[ \cdot \int_{\alpha_1}^{\alpha_2} n(\xi)e^{-\gamma \xi} \left[ \varphi_{\alpha_1}(x_1) + x_1 \varphi'_{\alpha_1}(x_1) - \alpha_1 x_2 e^{\beta \alpha_1(x_2-x_1)} (\varphi_\xi(x_2) - \varphi_{\alpha_1}(x_2)) \right] d\xi \]

\[ \cdot \int_0^{\alpha_1} n(\xi)e^{-\gamma \xi} \frac{x_1}{x_3} \varphi_\xi(x_1) d\xi \].

(25)

To make the computations as simple as possible, we choose
\[ \gamma = 1 , \ n(\xi) = 1 \ \forall \ \xi \in (0, \omega) , \]

and set the other parameters in such a way that \( x^* = (1, 1, 1) \) is a solution of (22). From (22) we have

\[ 1 = \beta_{13} \int_0^\omega e^{-\gamma \xi} d\xi = \beta_{13} \left( g_1(\omega) - g_1(\alpha_2) \right) \]

\[ 1 = \beta_{22} \int_0^{\alpha_2} e^{-\gamma \xi} d\xi = \beta_{22} \left( g_1(\alpha_2) - g_1(\alpha_1) \right) \]

\[ 1 = \beta_{31} \int_0^{\alpha_1} e^{-\gamma \xi} d\xi = \beta_{31} g_1(\alpha_1) \]

where

\[ g_1(\alpha) = \int_0^\alpha e^{-\gamma \xi} d\xi = 1 - e^{-\alpha} - \alpha e^{-\alpha} . \]

If we assume

\[ \beta_{13} = \frac{1}{g_1(\omega) - g_1(\alpha_2)} \]

\[ \beta_{22} = \frac{1}{g_1(\alpha_2) - g_1(\alpha_1)} \]

\[ \beta_{31} = \frac{1}{g_1(\alpha_1)} , \]

then indeed \( x^* = (1, 1, 1) \) is a solution of (22).

Using \( x^* = (1, 1, 1) \) and (26) in (23)–(24)–(25), the expressions simplify greatly yielding

\[ A_1 = \int_{\alpha_2}^\omega e^{-\gamma \xi} \left( \frac{\alpha_2^2}{2} + (1 + \alpha_1)(\xi - \alpha_1) \right) d\xi \]

\[ \cdot \int_{\alpha_1}^{\alpha_2} e^{-\gamma \xi} \left( \alpha_1 + \frac{(\xi - \alpha_1)^2}{2} \right) d\xi \cdot \int_0^{\alpha_1} e^{-\gamma \xi} d\xi \]

\[ A_2 = \int_{\alpha_2}^\omega e^{-\gamma (\xi - \alpha_2)} \left( 1 - \frac{\xi - \alpha_2}{2} \right) d\xi \cdot \int_{\alpha_1}^{\alpha_2} e^{-\gamma \xi} \left( \alpha_1 + \frac{(\xi - \alpha_1)^2}{2} \right) d\xi \cdot \int_0^{\alpha_1} e^{-\gamma \xi} \left( \xi \frac{1}{2} \xi^2 \right) d\xi \]
We set $\alpha = 594$ ANDREA FRANCESCHETTI, ANDREA PUGLIESE AND DIMITRI BREDA

uniformly for $D$ on the domain $\omega$

there exists $H \eta$ exists $\eta = 1$

Moreover, we can find values $\omega \omega = 2$, so that $\alpha_2 = \alpha_2^2 = 0$ and the second integral in $A_3$ becomes simpler.

Suitable values for $\alpha_2$ and $\omega$ are now to be found; we write $A_i = A_i(\omega, \alpha_2)$, $i = 1, 2, 3$, and compute

$$
\lim_{\omega \rightarrow +\infty} A_1(\omega, \alpha_2) = e^{-\alpha_2} (3\alpha_2 - 1) (3 e^{-2} - e^{-\alpha_2} (3 - \alpha_2 + \frac{\alpha_2^2}{2})) (1 - 3 e^{-2})
$$

$$
\lim_{\omega \rightarrow +\infty} A_2(\omega, \alpha_2) = 0
$$

$$
\lim_{\omega \rightarrow +\infty} A_3(\omega, \alpha_2) = e^{-\alpha_2} (\alpha_2 - 2)^2 (3 e^{-2} + e^{-\alpha_2} - \alpha_2 e^{-\alpha_2}) (1 - 3 e^{-2})
$$

uniformly for $\alpha_2 > 2$.

We set $\eta = \frac{1}{\omega}$ and consider the function

$$
H(\eta, \alpha_2) = \left\{ \begin{array}{ll}
A_1 \left( \frac{1}{\eta}, \alpha_2 \right) - A_2 \left( \frac{1}{\eta}, \alpha_2 \right) - A_3 \left( \frac{1}{\eta}, \alpha_2 \right) & \text{if } \eta > 0, \alpha_2 > 2

\lim_{\omega \rightarrow +\infty} (A_1(\omega, \alpha_2) - A_2(\omega, \alpha_2) - A_3(\omega, \alpha_2)) & \text{if } \eta = 0, \alpha_2 > 2
\end{array} \right.
$$

on the domain $D = \left\{ (\eta, \alpha_2) : \alpha_2 > \alpha_1, 0 \leq \eta < \frac{1}{\alpha_2} \right\}$. $H$ is continuous in $D$ and

$$
H(0, \alpha_2) = e^{-\alpha_2} (\alpha_2 - 2) (1 - 3 e^{-2}) (f_1(\alpha_2) - f_2(\alpha_2))
$$

where

$$
f_1(a) = \frac{(3a - 1)}{a - 2} \left( 3 - e^{-(a-2)} (3 - a + \frac{a^2}{2}) \right)
$$

$$
= \left( 3 + \frac{5}{a - 2} \right) \left( 3(1 - e^{-(a-2)}) - \frac{a}{2} e^{-(a-2)}(a - 2) \right)
$$

$$
f_2(a) = (a - 2) \left( 1 + e^{-(a-2)} - ae^{-(a-2)} \right).
$$

Since

$$
\lim_{a \rightarrow +\infty} f_1(a) = 10 \quad \lim_{a \rightarrow +\infty} f_1(a) = 9
$$

$$
\lim_{a \rightarrow +\infty} f_2(a) = 0 \quad \lim_{a \rightarrow +\infty} f_2(a) = +\infty
$$

there exists $\alpha_2^* > 2$ solution of the equation $f_1(a) - f_2(a) = 0$. Solving numerically this equation, we find the value $\alpha_2^* \approx 12.4342$ which gives $H(0, \alpha_2^*) = 0$.

Moreover, we can find values $\alpha_2^+ < \alpha_2^*$ such that $f_1(\alpha_2^+) > f_2(\alpha_2^+)$ and thus $H(0, \alpha_2^+) > 0$, while $f_1(\alpha_2^-) < f_2(\alpha_2^-)$ and thus $H(0, \alpha_2^-) < 0$. By continuity there exists $\eta_0 > 0$ such that $H(\eta, \alpha_2^+) > 0$ and $H(\eta, \alpha_2^-) < 0$ for each $\eta \in (0, \eta_0)$. Hence, given $\eta^* = \frac{1}{\omega^*} \in (0, \eta_0)$, there exists $\alpha_2^{**} \in (\alpha_2^+, \alpha_2^-)$ such that $H(\eta^*, \alpha_2^{**}) = 0$. So
if \( \alpha_1 = 2, \alpha_2 = \alpha_2^*, \omega = \frac{1}{\eta} \) we have \( A_1 - A_2 - A_3 = 0 \), from which it follows that \( \det (I - \Phi_x(x^*)) = 0 \).

Finally setting \( \beta_{13}, \beta_{22}, \beta_{31} \) from (27), we obtain \( \Phi(x^*) = x^* \). For instance, if we take \( \omega = 100 \), i.e. \( \eta^* = \frac{1}{100} \), then \( \alpha_2^* \approx 12.4342 \) and the corresponding values are \( \beta_{13} \approx 1.8702 \cdot 10^4, \beta_{22} \approx 2.4633, \beta_{31} \approx 1.6835 \).

Appendix C. Proofs of Section 6.

Lemma 2. Let

\[
\begin{align*}
    d_{11}(x) &= \int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi x_1}\varphi_\xi(x_1)d\xi; \\
    d_{21}(x) &= \int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi x_1}\varphi_\alpha(x_1)d\xi; \\
    d_{22}(x) &= \int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi x_2}e^{\alpha(x_2-x_1)}(\varphi_\xi(x_2) - \varphi_\alpha(x_2))d\xi;
\end{align*}
\]

so that

\[
    g_1(x) = d_{11}(x), \quad g_2(x) = d_{21}(x) + d_{22}(x).
\]

Then, the following inequalities hold:

\[
\frac{\partial g_i(x)}{\partial x_i} > 0, \quad i = 1, 2; \tag{28}
\]

\[
d_{ij}(x) > 0, \quad i = 1, 2, 1 \leq j \leq i \tag{29}
\]

\[
\frac{d_{ij}(x)}{x_j} - \frac{\partial g_i(x)}{\partial x_j} > 0, \quad i = 1, 2, 1 \leq j \leq i \tag{30}
\]

\( \forall x \in \text{Int}(\mathbb{R}^2_+), \forall 0 < \alpha < \omega, \forall \gamma > 0. \)

Proof. The derivative of (17) with respect to \( t \) is given by

\[
    \varphi_\gamma'(t) = \int_0^s \tau e^{(\gamma-\tau)d\tau} = \begin{cases} \frac{-1+\tau(t)e^{(\gamma-\tau)}-e^{(\gamma-\tau)}}{(\gamma-\tau)^2} & \gamma \neq t \\ -1/2 & \gamma = t. \end{cases}
\]

Then

\[
    \frac{\partial g_{11}(x)}{\partial x_1} = \int_{\alpha}^{\omega} n(\xi)e^{-\gamma\xi}(\varphi_\xi(x_1) + x_1\varphi_\xi'(x_1))d\xi.
\]

We note that

\[
    \int_0^{+\infty} e^{-\gamma \xi}(\varphi_\xi(t) + t\varphi_\xi'(t))d\xi = 0 \quad \forall \gamma > 0, \forall t > 0; \quad \text{for we have}
\]

\[
    \int_0^{+\infty} e^{-\gamma \xi}\varphi_\xi(t)d\xi = \frac{1}{\gamma t}, \quad \int_0^{+\infty} e^{-\gamma \xi}\varphi_\xi'(t)d\xi = -\frac{1}{\gamma t^2}. \tag{31}
\]
We now prove that
\[ \int_0^\alpha e^{-\gamma \xi} \left( \varphi_\xi(x_1) + x_1 \varphi'_\xi(x_1) \right) d\xi > 0 \]
\( \forall \alpha > 0, \forall \gamma > 0. \)

Let us consider the functions \( u : [0, +\infty) \rightarrow \mathbb{R} \) defined as
\[ u(\alpha) = \int_0^\alpha e^{-\gamma \xi} \left( \varphi_\xi(x_1) + x_1 \varphi'_\xi(x_1) \right) d\xi \]
and \( v : [0, +\infty) \rightarrow \mathbb{R} \) defined as
\[ v(\xi) = \varphi_\xi(x_1) + x_1 \varphi'_\xi(x_1). \]

We have \( v'(\xi) = e^{\xi(\gamma - x_1)} (1 - \xi x_1) \), hence \( v(\xi) \) is monotone increasing on \((0, 1/x_1)\) and decreasing on \((1/x_1, +\infty)\); moreover \( v(\xi) \) is such that \( v(0) = 0 \) and
\[ \lim_{\xi \to +\infty} v(\xi) = \begin{cases} -\infty & \text{if } \gamma \geq x_1 \\ \frac{1}{\gamma x_1} & \text{if } 0 < \gamma < x_1 \end{cases} \]

Then
\[ \exists \alpha_0 > 1/x_1 \text{ s.t. } v(\xi) > 0 \text{ on } (0, \alpha_0), \ v(\xi) < 0 \text{ on } (\alpha_0, +\infty). \]

Hence we have \( u(\alpha) \) monotone increasing on \((0, \alpha_0)\) and decreasing on \((\alpha_0, +\infty)\); furthermore, \( u(0) = 0 \) and, because of (31), \( \lim_{\alpha \to +\infty} u(\alpha) = 0 \). We can then conclude that \( u(\alpha) > 0 \ \forall \alpha > 0, \forall \gamma > 0. \)

\( n(\xi) \), the stationary age profile of the population, is a monotone decreasing function (see for example [15]). If \( \alpha \leq \alpha_0 \), then
\[ \frac{\partial q_1}{\partial x_1}(x) = \int_0^\alpha n(\xi) e^{-\gamma \xi} v(\xi) d\xi > n(\alpha) \int_0^\alpha e^{-\gamma \xi} v(\xi) d\xi = n(\alpha) u(\alpha) > 0. \]

Otherwise if \( \alpha > \alpha_0 \)
\[ \frac{\partial q_1}{\partial x_1}(x) = \int_0^\alpha n(\xi) e^{-\gamma \xi} v(\xi) d\xi = \int_0^{\alpha_0} n(\xi) e^{-\gamma \xi} v(\xi) d\xi + \int_{\alpha_0}^\alpha n(\xi) e^{-\gamma \xi} v(\xi) d\xi \]
\[ > n(\alpha_0) \int_0^{\alpha_0} e^{-\gamma \xi} v(\xi) d\xi + n(\alpha_0) \int_{\alpha_0}^\alpha e^{-\gamma \xi} v(\xi) d\xi = n(\alpha_0) u(\alpha) > 0. \]

This proves the first inequality in (28). Since we have
\[ \frac{\partial q_2}{\partial x_2}(x) = \int_0^\omega n(\xi) e^{-\gamma \xi} \left( (1 + \alpha x_2) (\varphi_\xi(x_2) - \varphi_\alpha(x_2)) + x_2 (\varphi'_\xi(x_2) - \varphi'_\alpha(x_2)) \right) d\xi \]
\[ = e^{-\alpha x_2} \int_0^{\omega - \alpha} n(\xi + \alpha) e^{-\gamma \xi} \left( \varphi_\xi(x_2) + x_2 \varphi'_\xi(x_2) \right) d\xi, \]
the second inequality in (28) is proved in the same way.

To prove (29) and (30) we note that
\[ \varphi_\xi(t) = \int_0^\xi e^{\tau(\gamma - t)} d\tau > 0 \ \forall \xi > 0, \forall t > 0, \forall \gamma > 0; \quad (32) \]
\[ \varphi'_\xi(t) = -\int_0^\xi \tau e^{\tau(\gamma - t)} d\tau < 0 \ \forall \xi > 0, \forall t > 0, \forall \gamma > 0. \quad (33) \]

Then (29) follows from (32).
Let us write the inequalities in (30) explicitly:

\[
\begin{align*}
\frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x) &= -x_1 \int_0^\alpha n(\xi) e^{-\gamma \xi} \varphi'_\xi(x_1) d\xi > 0; \\
\frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) &= -x_1 \varphi'_\alpha(x_1) \int_\alpha^\omega n(\xi) e^{-\gamma \xi} d\xi \\
&+ \alpha x_2 e^{\alpha(\gamma - x_1)} \int_0^\omega (\xi + \alpha) e^{-\gamma \xi} \varphi'_\xi(x_2) d\xi > 0; \\
\frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) &= -x_2 e^{\alpha(\gamma - x_1)} \int_0^\omega (\xi + \alpha) e^{-\gamma \xi} \varphi'_\xi(x_2) d\xi > 0
\end{align*}
\]

\[\forall x \in \text{Int}(\mathbb{R}_+^2), \forall 0 < \alpha < \omega, \forall \gamma > 0 . \] So the Lemma is proved. \(\square\)

**Proof of Proposition 2.** If \(x = (x_1, x_2) \in \text{Int}(\mathbb{R}_+^2)\) is a fixed point of \(\tilde{\Phi}\), the entries of the matrix \(I - \tilde{\Phi}_x(x)\) are as follows, noting that \(\frac{\partial d_{ii}}{\partial x_i}(x) = 0\) for \(x \in \mathbb{R}_+^2, i = 1, 2,\)

\[
\begin{align*}
(I - \tilde{\Phi}_x(x))_{11} &= 1 - \frac{\partial \tilde{\Phi}_1}{\partial x_1}(x) = \frac{\tilde{\Phi}_1(x)}{x_1} - \frac{\partial \tilde{\Phi}_1}{\partial x_1}(x) \\
&= \beta_{11} \left( \frac{d_{11}(x)}{x_1} - \frac{\partial d_{11}}{\partial x_1}(x) \right) + \beta_{12} \left( \frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1} \right); \\
(I - \tilde{\Phi}_x(x))_{12} &= -\frac{\partial \tilde{\Phi}_1}{\partial x_2}(x) = -\beta_{12} \frac{\partial g_2}{\partial x_2}(x); \\
(I - \tilde{\Phi}_x(x))_{21} &= -\frac{\partial \tilde{\Phi}_2}{\partial x_1}(x) = -\beta_{21} \frac{\partial g_1}{\partial x_1}(x) - \beta_{22} \frac{\partial g_2}{\partial x_1}(x); \\
(I - \tilde{\Phi}_x(x))_{22} &= 1 - \frac{\partial \tilde{\Phi}_2}{\partial x_2}(x) = \frac{\tilde{\Phi}_2(x)}{x_2} - \frac{\partial \tilde{\Phi}_2}{\partial x_2}(x) \\
&= \beta_{21} \frac{d_{11}(x)}{x_2} + \beta_{22} \left( \frac{d_{21}(x)}{x_2} + \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) \right).
\end{align*}
\]

We notice that we can write \(\det (I - \tilde{\Phi}_x(x))\) as the sum of four terms,

\[\det (I - \tilde{\Phi}_x(x)) = A_1 + A_2 + A_3 + A_4\]
where
\[
A_1 = \beta_{11}\beta_{21}\left( \frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x) \right) \frac{d_{11}(x)}{x_2}; \\
A_2 = \beta_{11}\beta_{22}\left( \frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x) \right) \left( \frac{d_{21}(x)}{x_2} + \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) \right); \\
A_3 = \beta_{12}\beta_{21}\left( \frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1} \right) \frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_1}(x) \frac{\partial g_1}{\partial x_2}(x) \\
- \frac{\partial g_2}{\partial x_1}(x) \frac{\partial g_2}{\partial x_2}(x). \\
\]

\(A_1\) is positive as we have
\[
\left( \frac{d_{11}(x)}{x_1} - \frac{\partial d_{11}}{\partial x_1}(x) \right) \frac{d_{11}(x)}{x_2} > 0
\]

because of \((29)\) and \((30)\); in the same way it is proved that \(A_2\) is positive.\(A_3\) is positive as we have:
\[
\left( \frac{d_{21}(x)}{x_1} - \frac{\partial g_2}{\partial x_1}(x) + \frac{d_{22}(x)}{x_1} \right) \frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) \frac{\partial g_1}{\partial x_1}(x) \\
> \frac{d_{22}(x)}{x_1} \frac{d_{11}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) \frac{\partial g_1}{\partial x_1}(x) \\
= \frac{d_{11}(x)}{x_1} \left( \frac{d_{22}(x)}{x_2} - \frac{\partial g_2}{\partial x_2}(x) \right) + \frac{\partial g_2}{\partial x_2}(x) \left( \frac{d_{11}(x)}{x_1} - \frac{\partial g_1}{\partial x_1}(x) \right) > 0
\]

\(\forall 0 < \alpha < \omega, \forall \gamma > 0\) because of the inequalities \((28),(29), (30)\).\(\square\)

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