NON-DEGENERACY AND EXISTENCE OF NEW SOLUTIONS FOR THE SCHRÖDINGER EQUATIONS

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Abstract. We consider the following nonlinear problem
\[-\Delta u + V(|y|)u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),\] (0.1)
where \(V(r)\) is a positive function, \(1 < p < \frac{N+2}{N-2}\). We show that the multi-bump solutions constructed in [20] is non-degenerate in a suitable symmetric space. We also use this non-degenerate result to construct new solutions for (0.1).

1. Introduction

Consider the following nonlinear elliptic problem
\[
\begin{cases}
-\Delta u + V(y)u = u^p, \quad u > 0, & \text{in } \mathbb{R}^N, \\
\lim_{|y| \to \infty} u(y) = 0,
\end{cases}
\] (1.1)
where \(N \geq 2, p \in (1, \frac{N+2}{N-2})\), and \(V\) is a continuous function which satisfies
\[
\lim_{|y| \to \infty} V(y) = 1.
\]
The existence of nontrivial solutions for (1.1), or the following nonlinear field equations in subcritical case
\[
\begin{cases}
-\Delta u + u = Q(x)u^p, \quad u > 0, & \text{in } \mathbb{R}^N, \\
\lim_{|y| \to \infty} u(y) = 0,
\end{cases}
\] (1.2)
attracts a lot of attentions in the last four decades. The readers can refer to [4]-[9], [15, 17] and the references therein. On the other hand, for results on the singularly perturbed problems corresponding to (1.1) and (1.2), we refer the readers to [1]-[3], [7, 8, 10]-[14], [18, 19].

The functional corresponding to (1.1) is given by
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(y)u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}, \quad u \in H^1(\mathbb{R}^N).
\]
It is well-known that the following problem has a unique solution \(U\)
\[
\begin{cases}
-\Delta u + u = u^p, \quad u > 0, & \text{in } \mathbb{R}^N, \\
u(y) \to 0, & \text{as } |y| \to \infty.
\end{cases}
\] (1.3)
satisfying $U(y) = U(|y|)$, $U' < 0$. We denote $U_{x_j}(y) = U(y - x_j)$.

For any integer $k > 0$ and $R > 0$ large, we define

$$D_{k,R} = \{(x_1, \cdots, x_k) : x_j \in \mathbb{R}^N, |x_j| \geq R, j = 1, \cdots, k, |x_i - x_j| \geq R, i \neq j\}.$$  

Then for any $(x_1, \cdots, x_k) \in D_{k,R}$, the function $\sum_{j=1}^k U_{x_j}$ is an approximate solution of (1.1). A natural question is whether we can make a small correction for this approximate solution to obtain a true solution for (1.1).

Direct computations give

$$I(\sum_{j=1}^k U_{x_j}) \approx kA + B_1 \sum_{j=1}^k (V(x_j) - 1) - B_2 \sum_{i \neq j} U(|x_i - x_j|),$$

where $B_1$ and $B_2$ are some positive constants, and $A = I(U)$.

To find a stable critical point for the function on the right hand side of (1.4) in $D_{k,R}$, the main difficulty is that the terms in this function may be of different order, depending on the location of each $x_j$. To avoid this difficulty, in [20], it assume that $V(y)$ is radial, and the following problem is studied

$$-\Delta u + V(|y|)u = u^p, u > 0 \text{ in } \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N).$$

We can see easily that if the function on the right hand side of (1.4) has a critical point in $D_{k,R}$, then $V(y) - 1$ can not be negative near the infinity. From this observation, in [20], the following condition is imposed.

(V): There are constants $a > 0$, $\alpha > 1$, and $\gamma > 0$, such that

$$V(r) = 1 + \frac{a}{r^\alpha} + O\left(\frac{1}{r^{\alpha+\gamma}}\right),$$

as $r \to +\infty$.

Let

$$x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0), \ j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$. Then, it is easy to calculate

$$I(\sum_{j=1}^k U_{x_j}) \approx kA + \frac{aB_1 k}{r^\alpha} - B_2 kU(|x_2 - x_1|).$$

For $k > 0$ large, the function $\frac{aB_1 k}{r^\alpha} - B_2 kU(|x_2 - x_1|)$ has a local maximum point $r_k \in [r_0 k \ln k, r_1 k \ln k]$, where $r_1 > r_0 > 0$ are some constants. Therefore, (1.5) has a solution with $k$ bumps near infinity provided $k$ is large enough. To be more precisely, we set
Define
\[ H_s = \{ u : u \text{ is even in } y_2, \cdots, y_N, \]
\[ u(r \cos \theta, r \sin \theta, y'') = u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'') \}. \]

Let
\[ W_r(y) = \sum_{j=1}^{k} U_{x_j}(y). \]

The following result is proved in [20].

**Theorem A.** Suppose that \( V(r) \) satisfies (1.6). Then there is an integer \( k_0 > 0 \) such that for any integer \( k \geq k_0 \), (1.5) has a solution \( u_k \) of the form
\[ u_k = W_{r_k}(y) + \omega_k, \]
where \( \omega_k \in H_s \cap H^1(\mathbb{R}^N) \), \( r_k \in [r_0 k \ln k, r_1 k \ln k] \) and as \( k \to +\infty \),
\[ \int_{\mathbb{R}^N} (|D\omega_k|^2 + \omega_k^2) \to 0. \]

If \( N \geq 4 \), for any large integer \( n > 0 \), we let
\[ p_j = \left( 0, 0, t \cos \frac{2(j-1)\pi}{n}, t \sin \frac{2(j-1)\pi}{n}, 0, \cdots, 0 \right), \quad j = 1, \cdots, n. \]

We are now interested in finding a new solution to (1.5) whose shape is, at main order,
\[ u \approx \sum_{j=1}^{k} U_{x_j} + \sum_{j=1}^{n} U_{p_j}. \tag{1.8} \]

We calculate
\[ I \left( \sum_{j=1}^{k} U_{x_j} + \sum_{j=1}^{n} U_{p_j} \right) \]
\[ \approx kA + \frac{aB_1 k}{r^\alpha} - B_2 k U(|x_2 - x_1|) \tag{1.9} \]
\[ + nA + \frac{aB_1 n}{l^\alpha} - B_2 k U(|p_2 - p_1|). \]

Similar to (1.4), if \( n >> k \), the terms on the right side hand of (1.9) are of different order. In other words, it is hard to see the contribution to the energy from the bumps \( U_{p_j} \). Therefore, it is very difficult to use a reduction argument directly to construct solutions of the form (1.8).

Following the idea in [16], for any fixed large integer \( k > 0 \), we will use \( u_k + \sum_{j=1}^{n} U_{p_j} \) as an approximate solution for (1.5). The key point that we can make a small correction
for $u_k + \sum_{j=1}^n U_j$, to obtain a true solution for (1.3) is to prove the non-degeneracy of the solution $u_k$ in $H_s \cap H^1(\mathbb{R}^N)$ in the sense that the following linearized operator

$$L_k \xi = -\Delta \xi + V(|y|) \xi - pu_k^{p-1} \xi,$$

has trivial kernel in $H_s \cap H^1(\mathbb{R}^N)$. To prove such non-degeneracy result, we need to impose the following conditions on $V$

$$V(r) = 1 + \frac{a_1}{r^\alpha} + \frac{a_2}{r^{\alpha+1}} + O\left(\frac{1}{r^{\alpha+2}}\right), \quad \text{(1.11)}$$

$$V'(r) = -\frac{a_1 \alpha}{r^{\alpha+1}} - \frac{a_2 (\alpha + 1)}{r^{\alpha+2}} + O\left(\frac{1}{r^{\alpha+3}}\right), \quad \text{(1.12)}$$

where $\alpha > \max\left(\frac{4}{p-1}, 2\right)$, $a_1 > 0$ and $a_2$ are some constants.

The main result of this paper is the following.

**Theorem 1.1.** Suppose that $V(r)$ satisfies (1.11) and (1.12). Let $\xi \in H_s \cap H^1(\mathbb{R}^N)$ be a solution of $L_k \xi = 0$. Then $\xi = 0$.

Similar problem is studied for the following prescribed scalar curvature equation

$$-\Delta u = K(y) u^{2^* - 1}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N). \quad \text{(1.13)}$$

where $2^* = \frac{2N}{N-2}$, under the condition that $K(r)$ has a non-degenerate local maximum point $r_0 > 0$. There are some technical differences in the study of the non-degeneracy of solutions for the nonlinear Schrödinger equations and the prescribed scalar curvature equation. For the prescribed scalar curvature equation, the non-degeneracy of the local maximum point $r_0 > 0$ plays an essential role in using the local Pohozaev identities to kill the possible nontrivial kernel of the linear operator. For the nonlinear Schrödinger equations, we can regard condition (1.12) as $V$ has a critical point at infinity. But it is not clear that such an expansion implies certain non-degeneracy of $V$ at infinity. This difference, together with the exponential decay of the solution $U$ of (1.3), leads us to use the local Pohozaev identities in quite different ways.

A direct consequence of Theorem 1.1 is the following result for the existence of new solutions for (1.5).

**Theorem 1.2.** Suppose that $V(r)$ satisfies the assumptions in Theorem 1.1 and $N \geq 4$. Let $u_k$ be a solution in Theorem A and $k > 0$ is a large even number. Then there is an integer $n_0 > 0$, depending on $k$, such that for any even number $n \geq n_0$, (1.5) has a solution of the form (1.8) for some $t_n \to +\infty$.

This paper is organized as follows. In section 2, we will revisit the proof of Theorem A in order to obtains some extra estimates needed in the proof of Theorem 1.1. The main result on the non-degeneracy of the solutions will be proved in section 3, while the construction of the new solutions will be carried out in section 4.
2. REVISIT THE EXISTENCE PROBLEM

In this section, we will briefly revisit the existence problem for (1.5) and give a different proof of Theorem A, so that we can obtained extra estimates which are needed in the proof of the non-degeneracy result.

We denote

\[ Z_j = \frac{\partial U_{x_j}}{\partial r}, \quad j = 1, \ldots, k, \]

where \( x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0) \).

Let

\[ \tilde{H}_s = \{ u : u \text{ is even in } y_2; \ u(y', y'') = u(y', |y''|) \]

\[ u(r \cos \theta, r \sin \theta, y'') = u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y''), \} \]

\[ \tilde{E}_k = \{ v : v \in \tilde{H}_s \cap C(\mathbb{R}^N), \sum_{j=1}^k \int_{\mathbb{R}^N} U_{x_j}^{p-1} Z_j v = 0, \ j = 1, \ldots, k \}, \]

and

\[ S_k = \left[ \left( \frac{\alpha}{2\pi} - \beta \right) k \ln k, \left( \frac{\alpha}{2\pi} + \beta \right) k \ln k \right], \]  

(2.1)

where \( \alpha \) is the constant in the expansion for \( V \), and \( \beta > 0 \) is a small constant.

In this paper, we always assume that \( r \in S_k \).

Take a small constant \( \tau > 0 \). Define the norm

\[ \|u\|_* = \sup_{y \in \mathbb{R}^N} \frac{|u(y)|}{\sum_{j=1}^k e^{-\tau |y-x_j|}}. \]

Let

\[ L v = -\Delta v + V(|y|) v - p W_p^{p-1} v. \]

For \( f \in \tilde{H}_s \cap C(\mathbb{R}^N) \), we consider the following problem

\[ L v = f + b_k \sum_{j=1}^k U_{x_j}^{p-1} Z_j \]  

(2.2)

for some \( v \in \tilde{E}_k \) and \( a_k \in \mathbb{R} \). We have

**Lemma 2.1.** Assume that \((a_k, v_k)\) solves (2.2) for \( f = f_k \). If \( \|f_k\|_* \to 0 \) as \( k \to +\infty \), then \( \|v_k\|_* \to 0 \) as \( k \to +\infty \).

**Proof.** We argue by contradiction. Suppose that there are \( k_m \to +\infty, r_{k_m} \in S_{k_m} \), and \((b_{k_m}, v_{k_m})\) solving (2.2), such that \( \|f_{k_m}\|_* \to 0 \), \( \|v_{k_m}\|_* = 1 \). For simplicity of the notation, we drop the subscript \( k_m \).

We have
\[ v(x) = \int_{\mathbb{R}^N} G(y, x) \left( pW_r^{p-1}v + f + b \sum_{j=1}^k U_{x_j}^{p-1}Z_j \right), \quad (2.3) \]

where \( G(y, x) \) is the Green’s function of \(-\Delta + V(|y|)\) in \( \mathbb{R}^N \).

It is easy to prove

\[ |\int_{\mathbb{R}^N} G(y, x)f| \leq C\|f\|_* \sum_{j=1}^k \int_{\mathbb{R}^N} G(y, x)e^{-\tau|y-x_j|} dy \leq C\|f\|_* \sum_{j=1}^k e^{-\tau|x-x_j|}, \]

and for \( \tau > 0 \) small,

\[ |\int_{\mathbb{R}^N} G(y, x)W_r^{p-1}v| \leq C\|v\|_* \int_{\mathbb{R}^N} G(y, x) \left( \sum_{j=1}^k e^{-\tau|y-x_j|} \right)^{p-1} \left( \sum_{j=1}^k e^{-\tau|y-x_j|} \right) dy \leq C\|v\|_* \sum_{j=1}^k e^{-2\tau|x-x_j|}. \]

Moreover,

\[ |\int_{\mathbb{R}^N} G(y, x) \sum_{j=1}^k U_{x_j}^{p-1}Z_j| \leq C \sum_{j=1}^k e^{-|x-x_j|}. \]

On the other hand, from

\[ b \int_{\mathbb{R}^N} \sum_{j=1}^k U_{x_j}^{p-1}Z_j Z_1 = \int_{\mathbb{R}^N} (L v - f) Z_1, \]

we can prove that \( b \to 0 \). So, it holds

\[ |v(x)| \left( \sum_{j=1}^k e^{-\tau|y-x_j|} \right)^{-1} \leq o(1) + C\|v\|_* \sum_{j=1}^k e^{-\tau|y-x_j|} \left( \sum_{j=1}^k e^{-\tau|y-x_j|} \right)^{-1} \sum_{j=1}^k e^{-2\tau|x-x_j|}. \quad (2.4) \]

Since \( v \in \tilde{E} \), we can prove \( \|v\|_{L^\infty(B_{R}(x_j))} \to 0 \) for any \( R > 0 \). So we find from (2.4) that \( \|v\|_* \to 0 \). This is a contradiction.

Now we want to solve the following problem

\[ L\omega = l_k + R(\omega) + b_k \sum_{j=1}^k U_{x_j}^{p-1}Z_j, \quad (2.5) \]

for \( \omega \in E_k \) and \( b_k \in \mathbb{R} \), where
\[ l_k = \sum_{i=1}^{k} (V(|y|) - 1) U_{x_i} + (W^p_r - \sum_{i=1}^{k} U^p_{x_i}), \]

and

\[ R(\omega) = (W_r + \omega)^p - W^p_r - pW^{p-1}_r \omega. \]

We have

**Lemma 2.2.** There is a small \( \sigma > 0 \), such that

\[ \|l_k\| \leq C \frac{r^\alpha}{r^\alpha} + Ce^{-\min(\frac{\sigma}{2} - \tau, \omega)|x_2 - x_1|}, \]

where \( r = |x_i| \).

**Proof.** First, it holds

\[
\left| \sum_{i=1}^{k} (V(|y|) - 1) U_{x_i} \right| \leq \frac{C}{1 + |y|^\alpha} \sum_{i=1}^{k} e^{-|y-x_i|} \leq \frac{C}{r^\alpha} \sum_{i=1}^{k} e^{-\tau|y-x_i|}.
\]

We define

\[ \Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{(y', 0)}{|y'|}, \frac{x_{k,j}}{|x_{k,j}|} \right\rangle \geq \cos \frac{\pi}{k} \right\}. \]

For any \( y \in \Omega_1 \), we have \( |y - x_j| \geq |y - x_1| \), \( j = 2, \ldots, k \). Thus, for any \( y \in \Omega_1 \), if \( p > 2 \), it holds

\[
|W^p - \sum_{i=1}^{k} U^p_{x_i}| \leq C \sum_{i=2}^{k} U^{p-1}_{x_1} U_{x_i} \leq Ce^{-(p-2)|y-x_1|} \sum_{i=2}^{k} e^{-|x_j-x_1|} \leq Ce^{-\tau|y-x_1|} \sum_{i=2}^{k} e^{-|x_j-x_1|},
\]

while if \( p \in (1, 2] \),

\[
|W^p - \sum_{i=1}^{k} U^p_{x_i}| \leq C \sum_{i=2}^{k} U^p_{x_1} U^p_{x_i} \leq Ce^{-\tau|y-x_1|} \sum_{i=2}^{k} e^{-\left(\frac{p}{2} - \tau\right)|x_j-x_1|}. \]

\[ \square \]
Using Lemmas 2.1 and the contraction mapping theorem, we can prove that there is an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), (2.5) has a solution \( \omega_k \in \tilde{E}_k \). Moreover,

\[
\|\omega_k\|_* \leq C \|l_k\|_* \leq \frac{C}{r^\alpha} + Ce^{-\min[\frac{r}{2}-\tau, 1]|x_2-x_1|}. \tag{2.6}
\]

Now we can calculate

\[
I(W_r + \omega_k) = kA + \frac{ab_1 r^k}{r^\alpha} - k(B_2 + o(1))U(|x_2 - x_1|) + kO\left(\frac{1}{r^{\alpha+\gamma}} + U^{1+\sigma}(|x_2 - x_1|)\right), \tag{2.7}
\]

where \( \sigma > 0 \) is a small constant. Noting that the function \( \frac{ab_1}{r^\alpha} - B_2 U(|x_2 - x_1|) \) has a maximum point in \( S_k \). Therefore, we can prove that (1.5) has a solution \( u_k \) of the form

\[
u_k = W_{r_k} + \omega_k,
\]

where \( \omega_k \in \tilde{E}_k \cap H^1(\mathbb{R}^N), r_k \in S_k \) and as \( k \to +\infty \),

\[
\|\omega_k\|_* \leq \frac{C}{r_k^\alpha} + Ce^{-\min[\frac{r}{2}-\tau, 1]|x_2-x_1|}. \tag{2.8}
\]

**Remark 2.3.** In [20], the reduction argument is carried out in \( H^1(\mathbb{R}^N) \cap H_s \). Thus, we only obtain the estimate for the error term \( \omega_k \) in the norm of \( H^1(\mathbb{R}^N) \). Estimate (2.8) is a point-wise estimate, which is needed when using the local Pohozaev identities.

Next, assuming that \( V \) satisfies (1.12), we derive a relation for \( r_k \), which is needed in the proof of the non-degeneracy result. From

\[
\int_{\mathbb{R}^N} (\nabla (W_{r_k} + \omega_k) \nabla \frac{\partial U_{x_1}}{\partial y_1} + V(y)(W_{r_k} + \omega_k) \frac{\partial U_{x_1}}{\partial y_1} - (W_{r_k} + \omega_k)^p \frac{\partial U_{x_1}}{\partial y_1}) = 0,
\]

we obtain

\[
\int_{\mathbb{R}^N} ((V(y) - 1)(W_{r_k} + \omega_k) \frac{\partial U_{x_1}}{\partial y_1} - \int_{\mathbb{R}^N} ((W_{r_k} + \omega_k)^p - \sum_{j=1}^{k} U^p_{x_j}) \frac{\partial U_{x_1}}{\partial y_1}) = 0.
\]

It is easy to check that

\[
\int_{\mathbb{R}^N} ((V(y) - 1)(W_{r_k} + \omega_k) \frac{\partial U_{x_1}}{\partial y_1} = \int_{\mathbb{R}^N} (V(y) - 1) U_{x_1} \frac{\partial U_{x_1}}{\partial y_1} + O\left(\frac{1}{r_k^{2\alpha}} + e^{-\min[p-2r,2]|x_2-x_1|}\right)
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^N} \frac{\partial V(y)}{\partial y_1} U^2_{x_1} + O\left(\frac{1}{r_k^{2\alpha}} + e^{-\min[p-2r,2]|x_2-x_1|}\right)
\]

\[
= \frac{a(\alpha + 1)x_1}{2r_k^{\alpha+2}} \int_{\mathbb{R}^N} U^2 + O\left(\frac{1}{r_k^{\alpha+2}} + \frac{1}{r_k^{2\alpha}} + e^{-\min[p-2r,2]|x_2-x_1|}\right).
\]

Moreover
\[ \int_{\mathbb{R}^N} \left( (W_{rk} + \omega_k)^p - \sum_{j=1}^{k} U_{x_j}^p \right) \frac{\partial U}{\partial y_1} \]

\[ = -p \int_{\mathbb{R}^N} U_{x_1}^{p-1} \sum_{j=2}^{k} U_{x_j} \frac{\partial U}{\partial y_1} + O\left( \frac{1}{r_k^{2\alpha}} + e^{-\min(p-\tau,2)|x_2-x_1|} \right) \]

\[ = \int_{\mathbb{R}^N} U_{x_1}^p \sum_{j=2}^{k} \frac{\partial U}{\partial y_1} + O\left( \frac{1}{r_k^{2\alpha}} + e^{-\min(p-\tau,2)|x_2-x_1|} \right) \]

\[ = (B + o(1)) U(|x_2 - x_1|) \left( \frac{(x_1 - x_2)_1}{|x_1 - x_2|} + \frac{(x_1 - x_k)_1}{|x_1 - x_k|} \right) + O\left( \frac{1}{r_k^{2\alpha}} + e^{-\min(p-\tau,2)|x_2-x_1|} \right). \]

Therefore, we have

\[ \frac{a(\alpha + 1)x_1}{2r_k^{\alpha+2}} \int_{\mathbb{R}^N} U^2 \]

\[ = (B + o(1)) U(|x_2 - x_1|) \left( \frac{(x_1 - x_2)_1}{|x_1 - x_2|} + \frac{(x_1 - x_k)_1}{|x_1 - x_k|} \right) \]

\[ + O\left( \frac{1}{r_k^{2\alpha}} + \frac{1}{r_k^{\alpha+2}} + e^{-\min(p-2\tau,2)|x_2-x_1|} \right). \]

Note that

\[ (x_1 - x_2)_1 = (x_1 - x_k)_1 = r(1 - \cos \frac{2\pi}{k}) = r \left( \frac{2\pi^2}{k^2} + O\left( \frac{1}{k^4} \right) \right). \]

Thus, (2.9) can be written as

\[ \frac{a(\alpha + 1)}{2r_k^{\alpha+4}} \int_{\mathbb{R}^N} U^2 \]

\[ = (B + o(1)) U(|x_2 - x_1|) \frac{1}{k} + O\left( \frac{1}{r_k^{2\alpha}} + \frac{1}{r_k^{\alpha+2}} + e^{-\min(p-2\tau,2)|x_2-x_1|} \right). \]

Noting that \( e^{-|x_1 - x_2|} \sim \frac{1}{r_k^{\alpha+1}} \) and \( \alpha > \frac{4}{p-1} \), it holds

\[ e^{-\min(p-2\tau,2)|x_2-x_1|} \leq \frac{C}{r_k^{\alpha \min(p-2\tau,2)}} = o\left( \frac{1}{r_k^{\alpha+1}} \right) \]

3. **The non-degeneracy of the solutions**

Let \( u \) solve

\[ -\Delta u + V(|y|)u = u^p, \]

and let \( \xi \) solve
\[-\Delta \xi + V(|y|)\xi = pu^{p-1}\xi.\] (3.2)

**Lemma 3.1.** We have

\[- \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \xi}{\partial y_i} - \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\partial \Omega} \langle \nabla u, \nabla \xi \rangle \nu_i
+ \int_{\partial \Omega} V(|y|)u\xi \nu_i - \int_{\partial \Omega} w^p \xi \nu_i = \int_{\Omega} u\xi \frac{\partial V}{\partial y_i} \] (3.3)

**Proof.** We have

\[\int_{\Omega} (-\Delta u \frac{\partial \xi}{\partial y_i} + (-\Delta \xi) \frac{\partial u}{\partial y_i}) + \int_{\Omega} V(|y|) \left( u \frac{\partial \xi}{\partial y_i} + \xi \frac{\partial u}{\partial y_i} \right)
= \int_{\Omega} \left( w^p \frac{\partial \xi}{\partial y_i} + pu^{p-1}\xi \frac{\partial u}{\partial y_i} \right).\] (3.4)

It is easy to check that

\[\int_{\Omega} (w^p \frac{\partial \xi}{\partial y_i} + pu^{p-1}\xi \frac{\partial u}{\partial y_i}) = \int_{\Omega} \frac{\partial (w^p \xi)}{\partial y_i} = \int_{\partial \Omega} w^p \xi \nu_i, \] (3.5)

and

\[\int_{\Omega} V(|y|) \left( u \frac{\partial \xi}{\partial y_i} + \xi \frac{\partial u}{\partial y_i} \right) = \int_{\Omega} V(|y|) \frac{\partial (u\xi)}{\partial y_i}
= - \int_{\Omega} u\xi \frac{\partial V}{\partial y_i} + \int_{\partial \Omega} V(|y|)u\xi \nu_i.\] (3.6)

Moreover,

\[\int_{\Omega} (-\Delta u \frac{\partial \xi}{\partial y_i} + (-\Delta \xi) \frac{\partial u}{\partial y_i})
= - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \xi}{\partial y_i} + \int_{\Omega} \frac{\partial u}{\partial \nu} \frac{\partial^2 \xi}{\partial \nu \partial y_i} - \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\Omega} \frac{\partial \xi}{\partial y_i} \frac{\partial^2 u}{\partial y_i \partial \nu}
= - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \xi}{\partial y_i} - \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\partial \Omega} \frac{\partial u}{\partial y_i} \frac{\partial \xi}{\partial y_j} \frac{\partial u}{\partial y_j}
= - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \xi}{\partial y_i} + \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu} \frac{\partial u}{\partial y_i} + \int_{\partial \Omega} \langle \nabla u, \nabla \xi \rangle \nu_i.\] (3.7)

So we have proved \((3.3)\). □
Let \( u_k \) be a solution of
\[
-\Delta u + V(|y|)u = u^p,
\]
constructed in Theorem A.

We now prove Theorem 1.1, arguing by contradiction. Suppose that there are \( k_m \to +\infty \), satisfying \( \xi_m \in H_s, \|\xi_m\|_s = 1 \), and
\[
L_{k_m} \xi_m = 0.
\]

Let
\[
\tilde{\xi}_m(y) = \xi_m(y + x_{km,1}).
\]

**Lemma 3.2.** It holds
\[
\tilde{\xi}_m \to b \frac{\partial U}{\partial y_1},
\]
uniformly in \( C^1(B_R(0)) \) for any \( R > 0 \), where \( b \) is some constant.

**Proof.** In view of \( |\tilde{\xi}_m| \leq C \), we may assume that \( \tilde{\xi}_m \to \xi \) in \( C_{loc}(\mathbb{R}^N) \). Then \( \xi \) satisfies
\[
-\Delta \xi + \xi = p U^{p-1} \xi, \quad \text{in} \ \mathbb{R}^N.
\]
This gives
\[
\xi = \sum_{i=1}^{N} b_i \frac{\partial U}{\partial y_i}.
\]
Since \( \xi \) is even in \( y_j \) for \( j = 2, \ldots, N \), it holds \( b_2 = \cdots = b_N = 0 \).

We decompose
\[
\xi_m(y) = b_m \sum_{j=1}^{k} \frac{\partial U}{\partial x_j} + \xi^*_m,
\]
where
\[
\xi^*_m \in E_k = \{ v \in H_s \cap C(\mathbb{R}^N), \sum_{j=1}^{k} \int_{\mathbb{R}^N} U^{p-1}_{x_j} Z_j v = 0, \ j = 1, \cdots, k \}.
\]
By Lemma 3.2, \( b_m \) is bounded.

**Lemma 3.3.** It holds
\[
\|\xi^*_m\|_* \leq \frac{C}{|x_1|^\alpha} + C e^{-\min[\frac{\gamma}{2}-\tau,1]|x_2-x_1|}.
\]
Proof. It is easy to see that
\[ L_{k_m} \xi^*_m = b_m \left( V(|y|) - 1 \right) \sum_{j=1}^{k} \frac{\partial U_{x_j}}{\partial r} \]
\[ - b_m p \sum_{j=1}^{k} (u_{k_m}^{p-1} - U_{x_j}^{p-1}) \frac{\partial U_{x_j}}{\partial r}. \]

Similar to the proof of lemma 2.2 using (2.8), we can prove
\[ \| (V(|y|) - 1) \sum_{j=1}^{k} \frac{\partial U_{x_j}}{\partial r} \|_* \leq C |x_1|^\alpha, \]
and
\[ \| \sum_{j=1}^{k} (u_{k_m}^{p-1} - U_{x_j}^{p-1}) \frac{\partial U_{x_j}}{\partial r} \|_* \leq C e^{-\min\{\frac{r}{2} - \tau, 1\}|x_2 - x_1|}. \]

Moreover, from \( \xi^*_m \in E_k \), similar to Lemma 2.1 we can prove that there exists \( \rho > 0 \), such that
\[ \| \tilde{L}_{k_m} \xi^*_n \|_* \geq \rho \| \xi^*_n \|_* . \]
Thus, the result follows.

Lemma 3.4. It holds
\[ \bar{\xi}_m \to 0, \]
uniformly in \( C^1(B_R(0)) \) for any \( R > 0 \).

Proof. We apply the identities in Lemma 3.1 in the domain \( \Omega = B_{\frac{r}{2}|x_2 - x_1|}(x_1) \). For simplicity, we drop the subscript \( m \).

We define the following bilinear form
\[ L(u, \xi, \Omega) = - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \frac{\partial \xi}{\partial y_1} - \int_{\partial \Omega} \frac{\partial \xi}{\partial \nu} \frac{\partial u}{\partial y_1} + \int_{\partial \Omega} \langle \nabla u, \nabla \xi \rangle \nu_1 + \int_{\partial \Omega} u \xi \nu_1, \]
where \( \nu \) is the outward unit normal of \( \partial \Omega \). Then it follows Lemma 3.1 that
\[ L(u, \xi, \Omega) + \int_{\partial \Omega} (V(|y|) - 1) u \xi \nu_1 = \int_{\partial \Omega} u \xi \nabla V. \]

Let recall that
\[ L(u, \xi, \Omega) = \int_{\Omega} (-\Delta u + u - u^p) \frac{\partial \xi}{\partial y_1} + \int_{\Omega} (-\Delta \xi + \xi - pu^{p-1}\xi) \frac{\partial u}{\partial y_1} + \int_{\partial \Omega} u^p \xi \nu_1. \] (3.17)

Note that
\[ u_k = \sum_{j=1}^{k} U_{x_j}(y) + \omega_k, \]
and
\[ \xi_m(y) = b_m \sum_{j=1}^{k} \frac{\partial U_{x_j}}{\partial r} + \xi^*_m, \]

We have
\[ \frac{\partial U_{x_j}}{\partial r} = \frac{\partial U}{\partial r} \left( y - (r \cos \frac{2(j-1)}{k}, r \sin \frac{2(j-1)}{k}, 0) \right) \]
\[ = -U'(|y - x_j|) \left\langle \frac{y - x_j}{|y - x_j|}, (\cos \frac{2(j-1)}{k}, \sin \frac{2(j-1)}{k}, 0) \right\rangle, \] (3.18)
and
\[ -\Delta \frac{\partial U_{x_j}}{\partial r} + \frac{\partial U_{x_j}}{\partial r} - pU_{x_j}^{p-1} \frac{\partial U_{x_j}}{\partial r} = 0. \]

We now calculate
\[ L(U_{x_j}, \frac{\partial U_{x_j}}{\partial r}, \Omega) \]
\[ = \int_{\Omega} (-\Delta U_{x_j} + U_{x_j} - U_{x_j}^p) \frac{\partial U_{x_j}}{\partial y_1} + \int_{\Omega} (-\Delta \frac{\partial U_{x_j}}{\partial r} + \frac{\partial U_{x_j}}{\partial r} - pU_{x_j}^{p-1} \frac{\partial U_{x_j}}{\partial r}) \frac{\partial U_{x_j}}{\partial y_1} + \int_{\partial \Omega} U_{x_j}^p \frac{\partial U_{x_j}}{\partial r} \nu_1 \]
\[ = \int_{\Omega} U_{x_j}^p \frac{\partial U_{x_j}}{\partial r} \nu_1 = O(\exp(-\frac{|x_2 - x_1|}{2} \sigma)), \] (3.19)
where \( \sigma > 0 \) is any fixed small constant.

On the other hand, for \( i \neq j \), we have
\[
L(U_{x_i}, \frac{\partial U_{x_j}}{\partial r}, \Omega) = \int_{\Omega} \left(-\Delta \frac{\partial U_{x_j}}{\partial r} + \frac{\partial U_{x_j}}{\partial r} - pU_{x_i}^{p-1} \frac{\partial U_{x_i}}{\partial r} \right) \frac{\partial U_{x_i}}{\partial y_1} + O\left(e^{-\left(\frac{p+1}{2}\sigma\right)|x_2-x_1|}\right)
\]

Therefore, by (3.18), we have

\[
\int_{\Omega} U_{x_i}^{p-1} \frac{\partial U_{x_j}}{\partial r} \frac{\partial U_{x_i}}{\partial y_1} = O\left(e^{-\left(1+\frac{p-1}{2}\sigma\right)|x_1-x_2|}\right),
\]

and

\[
\int_{\Omega} U_{x_j}^{p-1} \frac{\partial U_{x_i}}{\partial r} \frac{\partial U_{x_j}}{\partial y_1} = O\left(e^{-\left(1+\frac{p-1}{2}\sigma\right)|x_1-x_2|}\right),
\]

where \(\sigma > 0\) is any fixed small constant.

Combining (3.19)–(3.22), we obtain

\[
L(\sum_{j=1}^{k} U_{x_j}; \sum_{i=1}^{k} \frac{\partial U_{x_i}}{\partial r}, \Omega) = p\int_{\Omega} U_{x_i}^{p-1} \frac{\partial U_{x_j}}{\partial y_1} \left(\frac{\partial U_{x_1}}{\partial y_1} + \frac{\partial U_{x_k}}{\partial y_1}\right) - \int_{\Omega} U_{x_i}^{p-1} \left(\frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_k}}{\partial r}\right) \frac{\partial U_{x_1}}{\partial y_1} + O\left(e^{-\left(\frac{p+1}{2}\sigma\right)|x_2-x_1|}\right)
\]

\[
= -p\int_{\Omega} U_{x_i}^{p-1} \frac{\partial U_{x_1}}{\partial y_1} \left(\frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_k}}{\partial y_1} + \frac{\partial U_{x_k}}{\partial y_1}\right) + O\left(e^{-\left(\frac{p+1}{2}\sigma\right)|x_2-x_1|}\right)
\]

\[
= \int_{\mathbb{R}^N} U_{x_i}^{p} \frac{\partial}{\partial y_1} \left(\frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_k}}{\partial y_1} + \frac{\partial U_{x_k}}{\partial y_1}\right) + O\left(e^{-\left(\frac{p+1}{2}\sigma\right)|x_2-x_1|}\right)
\]

\[
= (B + o(1)) \left. \frac{\partial}{\partial y_1} \left(\frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_2}}{\partial r} + \frac{\partial U_{x_k}}{\partial y_1} + \frac{\partial U_{x_k}}{\partial y_1}\right) \right|_{y=x_1} + O\left(e^{-\left(\frac{p+1}{2}\sigma\right)|x_2-x_1|}\right),
\]

where \(B > 0\) is a constant.

By (3.18), we have

\[
\frac{\partial U_{x_2}}{\partial y_1} + \frac{\partial U_{x_2}}{\partial r} = U'(|y-x_2|) \left\langle \frac{y-x_2}{|y-x_2|}, (1,0,0) - \left(\cos \frac{2\pi}{k}, \sin \frac{2\pi}{k}, 0\right) \right\rangle.
\]

Therefore,
\[
\frac{\partial}{\partial y_1} \left( \frac{\partial U_{x_2}}{\partial y_1} + \frac{\partial U_{x_2}}{\partial r} \right) = U''(y - x_2) \left( \frac{y - x_2}{|y - x_2|}, (1, 0, 0) - \left( \cos \frac{2\pi}{k}, \sin \frac{2\pi}{k}, 0 \right) \right) \\
+ U'(y - x_2) \left( (1, 0, 0) - \frac{(y - x_2)}{|y - x_2|}, (1, 0, 0) - \left( \cos \frac{2\pi}{k}, \sin \frac{2\pi}{k}, 0 \right) \right).
\]

We have

\[(1, 0, 0) - \left( \cos \frac{2\pi}{k}, \sin \frac{2\pi}{k}, 0 \right) = \left( \frac{2\pi^2}{k^2} + O\left( \frac{1}{k^4} \right), \frac{2\pi}{k} + O\left( \frac{1}{k^2} \right) \right),\]

Note that

\[|x_2 - x_1| = 2r_k \sin \frac{\pi}{k} = \frac{2\pi r_k}{k} + O\left( \frac{r_k}{k^3} \right),\]

\[(x_1 - x_2)_1 = r_k \left( 1 - \cos \frac{2\pi}{k} \right) = \frac{2\pi^2 r_k}{k^2} + O\left( \frac{r_k}{k^4} \right),\]

and

\[(x_1 - x_2)_2 = -r_k \sin \frac{2\pi}{k} = -\frac{2\pi r_k}{k} + O\left( \frac{r_k}{k^3} \right).\]

So we obtain

\[
\frac{\partial}{\partial y_1} \left( \frac{\partial U_{x_2}}{\partial y_1} + \frac{\partial U_{x_2}}{\partial r} \right) \bigg|_{y = x_1} = U''(|x_1 - x_2|) \left( \frac{x_1 - x_2)_1 (x_1 - x_2)_2}{|x_1 - x_2| (x_1 - x_2)} \left( -\sin \frac{2\pi}{k} \right) \right) \\
+ U'(|x_1 - x_2|) \left( \frac{1}{|x_1 - x_2|} \left( 1 - \cos \frac{2\pi}{k} \right) + \frac{(x_1 - x_2)_2 (x_1 - x_2)_1}{|x_1 - x_2|^3} \sin \frac{2\pi}{k} \right) + O\left( \frac{1}{k^3} \right) \\
= U(|x_1 - x_2|) \left( \frac{(x_1 - x_2)_1}{|x_1 - x_2|} \left( \sin \frac{2\pi}{k} \frac{1}{r_k} + \frac{1}{|x_1 - x_2|} \sin \frac{2\pi}{k} \right) \left( 1 + o(1) \right) \right) \\
= \frac{2\pi^2 U(|x_1 - x_2|)}{k^2} \left( 1 + o(1) \right).
\]

Similarly
\[
\frac{\partial}{\partial y_1} \left( \frac{\partial U_{x_k}}{\partial y_1} + \frac{\partial U_{x_k}}{\partial r} \right) \bigg|_{y=x_1} \\
= U''(|x_1 - x_k|) \left( \frac{(x_1 - x_k)_1}{|x_1 - x_k|} \right) \left( \frac{(x_1 - x_k)_2}{|x_1 - x_k|} \right) \left( - \sin \frac{2(k-1)\pi}{k} \right) \\
+ U'(|x_1 - x_k|) \left( \frac{1}{|x_1 - x_k|} \right) \left( 1 - \cos \frac{2(k-1)\pi}{k} \right) + \left( \frac{(x_1 - x_k)_2(x_1 - x_k)_1}{|x_1 - x_k|^3} \right) \sin \frac{2(k-1)\pi}{k} \\
+ O\left( \frac{1}{k^3} \right) \\
= \frac{2\pi^2 U(|x_1 - x_k|)}{k^2} \left( 1 + o(1) \right).
\]

So we have proved

\[
L \left( \sum_{j=1}^{k} U_{x_j}, \sum_{i=1}^{k} \frac{\partial U_{x_k}}{\partial r}, \Omega \right) = \left( B' + o(1) \right) \frac{U(|x_1 - x_k|)}{k^2} + O \left( e^{-\left(\frac{2\pi}{\sqrt{2} - \sigma} - \frac{1}{2}\right)|x_2 - x_1|} \right). \tag{3.24}
\]

Using (2.8) and Lemma 3.3, together with \(L^p\)-estimates for the elliptic equations, we can prove

\[
|\nabla \omega_{k,m}(y)|, \ |\nabla \xi_{m}^{*}(y)| \leq \frac{C}{|x_1|^\alpha} + C e^{-\min\{|\frac{p}{2} - \tau, 1\}|x_2 - x_1|}, \quad \forall y \in \partial \Omega,
\]

from which, we see

\[
L(\omega_{k,m}, \sum_{i=1}^{k} \frac{\partial U_{x_i}}{\partial r}, \Omega) = O\left( \frac{1}{|x_1|^\alpha} + C e^{-\min\{|\frac{p}{2} - \tau, 1\}|x_2 - x_1|} \right) e^{-\frac{1-\sigma}{2}|x_1 - x_2|}, \tag{3.25}
\]

and

\[
L\left( \sum_{j=1}^{k} U_{x_j}, \xi_{m}^{*}, \Omega \right) = O\left( \frac{1}{|x_1|^\alpha} + C e^{-\min\{|\frac{p}{2} - \tau, 1\}|x_2 - x_1|} \right) e^{-\frac{1-\sigma}{2}|x_1 - x_2|}. \tag{3.26}
\]

It is easy to check that

\[
\int_{\partial \Omega} (V(|y|) - 1) u_\xi \nu_1 = O \left( \frac{1}{k^\alpha} e^{-\left(\frac{2\pi}{\sqrt{2} - \sigma} \right)|x_1 - x_2|} \right), \tag{3.27}
\]

and

\[
\int_{\partial \Omega} u^p \xi \nu_1 = O \left( e^{-\left(\frac{2\pi}{\sqrt{2} - \sigma} \right)|x_1 - x_2|} \right). \tag{3.28}
\]

Combining (3.10), and (3.24)–(3.28), we obtain
Thus, by (1.12),

\[ h > \frac{1}{k^2}. \]

Therefore, by (2.8) and Lemma 3.3, we find

\[
\int_{\Omega} u \xi V'(|y|)|y|^{-1}y_1 = b_m \int_{\mathbb{R}^N} U_{x_1} \partial_{y_1} V'(|y|)|y|^{-1}y_1 + \frac{1}{|x_1|^{\alpha+1}} O\left( \frac{1}{|x_1|^{\alpha}} + e^{-\min(\frac{\alpha}{2}-\sigma, 1)|x_2-x_1|} \right). \tag{3.30}
\]

For any \( h > 1 \), it holds

\[
\frac{y_1 + x_{1,1}}{|y + x_1|^h} = \frac{x_{1,1}}{|x_1|^h} - (h - 1) \frac{y_1}{|x_1|} + O\left( \frac{1}{|x_1|^{h+1}} \right).
\]

Thus

\[
\int_{\mathbb{R}^N} U \partial_{y_1} y_1 + x_{1,1} = -(h - 1) \int_{\mathbb{R}^N} U \partial_{y_1} y_1 + O\left( \frac{1}{|x_1|^{h+1}} \right).
\]

Therefore, by (1.12),

\[
\int_{\mathbb{R}^N} U_{x_1} \partial_{y_1} V'(|y|)|y|^{-1}y_1 = \int_{\mathbb{R}^N+x_1} U \partial_{y_1} V'(|y + x_1|)|y + x_1|^{-1}(y_1 + x_{1,1})
\]

\[
= \int_{\mathbb{R}^N} U \partial_{y_1} \left(-\frac{\alpha a_1}{|y + x_1|^{\alpha+1}} \right) - \frac{(\alpha + 1) a_2}{|y + x_1|^{\alpha+2}} + O\left( \frac{1}{|y + x_1|^{m+3}} \right) \right) y_1 + x_{1,1}
\]

\[
= \frac{\alpha (\alpha + 1) a_1}{|r_{k,x}|^{\alpha+2}} \left( \int_{\mathbb{R}^N} U U'(|y|) y_1^2 + o(1) \right). \tag{3.31}
\]

Combining (3.29) and (3.31), we obtain

\[
b_m \frac{\alpha (\alpha + 1) a_1}{|r_{k,x}|^{\alpha+2}} \left( \int_{\mathbb{R}^N} U U'(|y|) y_1^2 + o(1) \right) + \frac{1}{|x_1|^{\alpha+1}} O\left( \frac{1}{|x_1|^{\alpha}} + e^{-\min(\frac{\alpha}{2}-\sigma, 1)|x_2-x_1|} \right)
\]

\[
- b_m \left( B' + o(1) \right) \frac{U(|x_1 - x_k|)}{k^2} + O\left( \frac{1}{r_k^2} e^{-(1-\sigma)|x_1-x_2|} + e^{-\left(\frac{\alpha+1}{2}\right)-\sigma}|x_2-x_1|} \right). \tag{3.32}
\]

Since \( \alpha > \max\left(\frac{4}{p-1}, 2\right) \), we see that

\[
\frac{1}{r_k^2} e^{-(1-\sigma)|x_1-x_2|} + e^{-\left(\frac{\alpha+1}{2}\right)-\sigma}|x_2-x_1| = o\left( \frac{U(|x_1 - x_k|)}{k^2} \right).
\]

By (2.10),
\[ \frac{1}{r_k^{\alpha+1}} \sim U(|x_2 - x_1|) \frac{1}{k}. \]  

(3.33)

Hence

\[ \frac{U(|x_1 - x_k|)}{k^2} \sim \frac{1}{k} \frac{1}{r_k^{\alpha+1}} = \frac{r_k}{k} \frac{1}{r_k^{\alpha+1}} \gg \frac{1}{r_k^{\alpha+2}}, \]

since \( r_k \sim k \ln k \). Thus, \( (3.32) \) gives \( b_m \to 0 \). \( \square \)

**Proof of Theorem 1.** Let \( G(y, x) \) be the Green’s function of \(-\Delta u + V(|y|)u\) in \( \mathbb{R}^N \). Then

\[ 0 < G(z, y) \leq C e^{-|z-y|} \]

as \( |z - y| \to +\infty \). We have

\[ \xi_k(y) = p \int_{\mathbb{R}^N} G(z, y) u_k^{p-1}(z) \xi_k(z) \, dz. \]  

(3.34)

Now we estimate

\[ \left| \int_{\mathbb{R}^N} G(z, y) u_k^{p-1}(z) \xi_k(z) \, dz \right| \leq C \| \xi_k \|_* \int_{\mathbb{R}^N} G(z, y) u_k^{p-1}(z) \sum_{j=1}^k e^{-\tau|z-x_j|} \, dz \]

\[ \leq C \| \xi_k \|_* \int_{\mathbb{R}^N} G(z, y) \left( W^p_\tau(z) + |\omega|^{p-1} \right) \sum_{j=1}^k e^{-\tau|z-x_j|} \, dz \]

\[ \leq C \| \xi_k \|_* \int_{\mathbb{R}^N} G(z, y) \left( \sum_{j=1}^k e^{-(p-1-\sigma+\tau)|z-x_j|} + \|\omega\|^{p-1} \sum_{j=1}^k e^{-(\sigma-\tau)|z-x_j|} \right) \]

\[ \leq C \| \xi_k \|_* \sum_{j=1}^k e^{-(\sigma-\tau)|y-x_j|}, \]

where \( \sigma \in (0, \tau) \) is any fixed small constant. So we obtain

\[ \frac{|\xi_k(y)|}{\sum_{j=1}^k e^{-\tau|y-x_j|}} \leq C \| \xi_k \|_* \frac{\sum_{j=1}^k e^{-(\sigma-\tau)|y-x_j|}}{\sum_{j=1}^k e^{-\tau|y-x_j|}}. \]

Since \( \xi_k \to 0 \) in \( B_R(x_{j,k}) \) and \( \| \xi_k \|_* = 1 \), we know that \( \frac{|\xi_k(y)|}{\sum_{j=1}^k e^{-\tau|y-x_j|}} \) attains its maximum in \( \mathbb{R}^N \setminus \bigcup_{j=1}^k B_R(x_{j,k}) \). Thus

\[ \| \xi_k \|_* \leq o(1) \| \xi_k \|_* . \]

So \( \| \xi_k \|_* \to 0 \) as \( k \to +\infty \). This is a contradiction to \( \| \xi_k \|_* = 1 \). \( \square \)
4. Existence of new solutions

Let \( u_k \) be the solutions constructed in section 2, where \( k > 0 \) is a large even integer. Since \( k \) is even, \( u_k \) is even in each \( y_j, j = 1, \cdots, N \). Moreover, \( u_k \) is radial in \( y'' = (y_3, \cdots, y_N) \).

Let \( n \geq k \) be a large even integer. Set
\[
p_j = \left( 0, 0, t \cos \left( \frac{2(j-1)\pi}{n} \right), t \sin \left( \frac{2(j-1)\pi}{n} \right), 0 \right), \quad j = 1, \cdots, n,
\]
where \( t \in [t_0 n \ln n, t_1 n \ln n] \).

Define
\[
X_s = \left\{ u : u \in H, u \text{ is even in } y_h, h = 1, \cdots, N, \right. \\
u(y_1, y_2, t \cos \theta, t \sin \theta, y^*) = u(y_1, y_2, t \cos(\theta + \frac{2\pi j}{n}), t \sin(\theta + \frac{2\pi j}{n}), y^*) \left. \right\}
\]
where \( y^* = (y_5, \cdots, y_N) \).

Note that both \( u_k \) and \( \sum_{j=1}^{n} U_{p_j} \) belong to \( X_s \), while \( u_k \) and \( \sum_{j=1}^{n} U_{p_j, \lambda} \) are separated from each other. We aim to construct a solution for (1.5) of the form
\[
u = u_k + \sum_{j=1}^{n} U_{p_j, \lambda} + \xi,
\]
where \( \xi \in X_s \) is a small perturbed term. Since the procedure to construct such solutions are similar to that in [20], we just sketch it.

We define the linear operator
\[
Q_n \xi = -\Delta \xi + V(|y|)\xi - p(u_k + \sum_{j=1}^{n} U_{p_j})^{p-1} \xi, \quad \xi \in X_s.
\]
\[
\langle Q_n \xi, \phi \rangle = \int_{\mathbb{R}^N} \left( \nabla \xi \nabla \phi + V(|y|)\xi \phi - p(u_k + \sum_{j=1}^{n} U_{p_j})^{p-1} \xi \phi \right), \quad \xi, \phi \in X_s.
\]

We can regard \( Q_n \xi \) as a function in \( X_s \), such that
\[
\langle Q_n \xi, \phi \rangle = \int_{\mathbb{R}^N} \left( \nabla \xi \nabla \phi + V(|y|)\xi \phi - p(u_k + \sum_{j=1}^{n} U_{p_j})^{p-1} \xi \phi \right), \quad \xi, \phi \in X_s.
\]

Let
\[
\tilde{Z}_j = \frac{\partial U_{p_j}}{\partial t}, \quad j = 1, \cdots, k.
\]

Let \( h_n \in X_s \). Consider
\[
\left\{ \begin{aligned}
Q_n \xi_n &= h_n + c_n \sum_{j=1}^{n} \tilde{Z}_j, \\
\xi_n &\in X_s, \\
\sum_{j=1}^{n} \int_{\mathbb{R}^N} U_{p_j}^{p-1} \tilde{Z}_j \xi_n &= 0,
\end{aligned} \right. \tag{4.3}
\]
for some constants \( c_n \), depending on \( \xi_n \).
Let
\[ D_j = \left\{ y = (y', y_3, y_4, y^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4} : \left\langle \frac{(0, 0, y_3, y_4, \ldots, 0)}{|(y_3, y_4)|}, \frac{p_j}{|p_j|} \right\rangle \geq \cos \frac{\pi}{n} \right\}. \]

**Lemma 4.1.** Assume that \( \xi_n \) solve (4.3). If \( \| h_n \|_{H^1(\mathbb{R}^N)} \to 0 \), then \( \| \xi_n \|_{H^1(\mathbb{R}^N)} \to 0 \).

**Proof.** For simplicity, we will use \( \| u \| \) to denote \( \| u \|_{H^1(\mathbb{R}^N)} \).

We argue by contradiction. Suppose that there are \( p_{n,j}, h_n \) and \( \xi_n \), satisfying (4.3), \( \| h_n \| \to 0 \) and \( \| \xi_n \| \geq c > 0 \). We may assume \( \| \xi_n \| = n \). Then \( \| h_n \|^2 = o(n) \).

For simplicity, we will use \( \| u \| \) to denote \( \| u \|_{H^1(\mathbb{R}^N)} \).

We have
\[ c_n \sum_{j=1}^{n} \int_{\mathbb{R}^N} U_{p_{n,j}}^{-1} \tilde{Z}_j \tilde{Z}_1 = \left\langle Q_n \xi_n - h_n, \tilde{Z}_1 \right\rangle, \]
from which we can proved that \( c_n = o(1) \).

On the other hand, from \( \int_{\mathbb{R}^N} U_{p_{n,j}}^{2^{*}-2} Z_j \xi_n = 0 \), it is standard to prove that
\[ \xi_n (y + p_{n,j}) \to 0, \text{ in } H^1_{loc}(\mathbb{R}^N). \]

Moreover, since \( \frac{1}{\sqrt{n}} \xi_n \) is bounded in \( H^1(\mathbb{R}^N) \), we can assume that
\[ \frac{1}{\sqrt{n}} \xi_n \rightharpoonup \xi, \text{ weakly in } H^1(\mathbb{R}^N), \]
and
\[ \frac{1}{\sqrt{n}} \xi_n \to \xi, \text{ strongly in } L^2_{loc}(\mathbb{R}^N). \]

Thus, \( \xi \) satisfies
\[ -\Delta \xi + V(|y|)\xi - pu_k^{p-1} \xi = 0, \text{ in } \mathbb{R}^N. \]

By Theorem 1.1, \( \xi = 0 \). Therefore,
\[ \int_{\mathbb{R}^N} (u_k + \sum_{j=1}^{n} U_{p_{n,j}, \lambda_n})^{p-1} \xi_n^2 = o(n). \]

We also have
\[ \left\langle h_n, \xi_n \right\rangle = o(n), \]
and
\[ \left\langle \sum_{j=1}^{n} \tilde{Z}_j, \xi_n \right\rangle = O(n). \]

So we obtain
\[ \int_{\mathbb{R}^N} |\nabla \xi_n|^2 = O\left( \int_{\mathbb{R}^N} (u_k + \sum_{j=1}^{n} U_{p_{n,j}, \lambda_n})^{p-1} \xi_n^2 \right) + \left\langle h_n, \xi_n \right\rangle + \int_{\mathbb{R}^N} c_n \left\langle \sum_{j=1}^{n} \tilde{Z}_j, \xi_n \right\rangle \]
\[ = o(n). \]
This is a contradiction. □

From now on, we assume that \( N \geq 4 \). We want to construct a solution \( u \) for (1.5) with

\[
\begin{align*}
  u = u_k + \sum_{j=1}^{n} U_{p_j, \lambda} + \xi,
\end{align*}
\]

where \( \xi \in X_s \) is a small perturbed term, satisfying

\[
\sum_{j=1}^{n} \int_{\mathbb{R}^N} U_{p_j}^{p-1} \tilde{Z}_j \xi = 0.
\]

Then \( \xi \) satisfies

\[
Q_n \xi = l_n + R(\xi),
\]

where

\[
l_n = \sum_{j=1}^{n} (V(|y|) - 1) U_{p_j} + \left( u_k + \sum_{j=1}^{n} U_{p_j} \right)^p - u_k^p - \sum_{j=1}^{n} U_{p_j}^p,
\]

and

\[
R_n(\xi) = \left( u_k + \sum_{j=1}^{n} U_{p_j} + \xi \right)^p - \left( u_k + \sum_{j=1}^{n} U_{p_j} \right)^p - p \left( u_k + \sum_{j=1}^{n} U_{p_j} \right)^{p-1} \xi.
\]

We have the following estimate for \( \|l_n\| \).

**Lemma 4.2.** There is a small \( \sigma > 0 \), such that

\[
\|l_n\| \leq C \left| \frac{1}{p_1} \right|^\alpha + C e^{-\min(\sigma,1)|p_2-p_1|},
\]

where \( \sigma > 0 \) is any fixed small constant.

**Proof.** The proof of this lemma is quite standard. We thus omit it. □

It is also standard to prove the following lemma.

**Lemma 4.3.**

\[
\|R_n(\xi)\| \leq C \|\xi\|^{\min(p,2)}.
\]

Moreover,

\[
\|R_n(\xi_1) - R_n(\xi_2)\| \leq C \left( \|\xi_1\|^{\min(p-1,1)} + \|\xi_2\|^{\min(p-1,1)} \right) \|\xi_1 - \xi_2\|.
\]
We consider the following problem:
\[
\begin{aligned}
Q_n \xi_n &= l_n + R_n(\xi_n) + C_N \sum_{j=1}^{n} \tilde{Z}_j, \\
\xi_n &\in X_s, \\
\sum_{j=1}^{n} \int_{\mathbb{R}^N} U_{p_j}^{p-1} \tilde{Z}_j \xi_n &= 0.
\end{aligned}
\]

(4.7)

Using Lemmas 4.1, 4.2 and 4.3, we can prove the following proposition in a standard way.

**Proposition 4.4.** There is an integer \( n_0 > 0 \), such that for each \( n \geq n_0 \) and \( t \in [t_0 n \ln n, t_1 n \ln n] \), (4.7) has a solution \( \xi_n \) for some constants \( c_n \). Moreover, \( \xi_n \) is a \( C^1 \) map from \([t_0 n \ln n, t_1 n \ln n]\) to \( X_s \), and

\[
\|\xi_n\| \leq C \left| \frac{p}{p_1} \right| + Ce^{-\min\left(\frac{p - 2 \sigma}{2}, 1\right)|p_2 - p_1|}
\]

for any fixed small constant \( \sigma > 0 \).

Let

\[
F(t) = I(u_k + \sum_{j=1}^{n} U_{p_j} + \xi_n).
\]

To obtain a solution of the form \( u_k + \sum_{j=1}^{n} U_{p_j, \lambda} + \xi_n \), we just need to find a critical point for \( F(t) \) in \([t_0 n \ln n, t_1 n \ln n]\).

**Proof of Theorem 1.1** We have

\[
F(t, \lambda) = I\left(\sum_{j=1}^{n} U_{p_j, \lambda}\right) + I(u_k) + nO\left(\frac{1}{t^{2 \alpha}} + e^{-\min(p-2\sigma,2)|p_2-p_1|}\right)
\]

\[
= I(u_k) + nA + n\left(\frac{B_1}{t^{\alpha}} - (B_2 + o(1))U(|p_2 - p_1|)\right) + nO\left(\frac{1}{t^{\alpha+1}} + e^{-\min(p-2\sigma,2)|p_2-p_1|}\right),
\]

(4.8)

where \( A = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1}, \) \( B_1 \) and \( B_2 \) are some positive constants, and \( \sigma > 0 \) is any fixed small constant.

Now to find a critical point for \( F(t) \), we just need to proceed exactly as in [20]. □

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