SPACE-CONVERGENT VERSION OF MINIMUM-CONTRAST ESTIMATOR FOR INFINITE-DIMENSIONAL FRACTIONAL ORNSTEIN-UHLENBECK PROCESS

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Abstract. A new modification of the minimum-contrast estimator of drift parameter in infinite-dimensional fractional Ornstein-Uhlenbeck process is introduced. Utilizing the self-similarity property, advantageous space-asymptotic properties (strong consistency and asymptotic normality) are achieved by setting appropriate weights to individual coordinate projections (reweighing technique). In this respect, this modification outperforms the standard (non-weighted) minimum contrast estimator, which is not space-convergent. The reweighing technique in non-diagonalizable setting is also studied.

1. Introduction

This paper is a contribution to the spectral approach in the theory of statistical inference for parabolic linear stochastic partial differential equations (SPDEs) with additive noise generated by a fractional Brownian motion (fBm), solutions of which can be interpreted as infinite-dimensional fractional Ornstein-Uhlenbeck processes. For more details on the spectral approach, consult the paper [10]. With respect to the drift parameter estimation in linear SPDEs, the following techniques have been studied:

• The maximum likelihood estimators (MLE), initiated in [6] for diagonalizable SPDEs driven by a cylindrical Wiener process and generalized for a cylindrical fBm with Hurst parameter $H \geq \frac{1}{2}$ in [3].
• The minimum contrast (MC) estimators, introduced in [7] for linear SPDEs with Wiener noise and studied in [8] and [11] for equations driven by a fBm.
• The least squares (LS) estimator, application of which to one-dimensional projections of solutions to linear SPDEs driven by regular fBm was studied in [12].
• The trajectory fitting estimator (TFE), introduced in the setting of parabolic diagonalizable linear SPDEs with Wiener noise in [2].

In this paper, a modification of the MC estimator (the weighted MC estimator) is introduced. It benefits from the self-similarity property of the coordinate projections and enables to fully utilize the information contained in the projections by setting appropriate weights to them. This approach significantly improves the space-asymptotic properties (ensures strong consistency and asymptotic normality) of the MC estimator. We believe it is potentially applicable also for other types of estimators, such as LS or TFE, and for different types of models (but still having

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the self-similarity property). To author’s best knowledge, this approach is new even in the basic case of parabolic diagonalizable equations with white noise (in space and time).

We also develop the reweighing technique for general (not necessarily diagonalizable) linear evolution equations, which goes beyond the standard setting considered in the spectral approach. In this case, where the trajectories of projections are not independent anymore, classical techniques like strong law of large numbers and central limit theorem must be replaced by more advanced techniques based on Malliavin calculus – the hypercontractivity property and the 4th moment theorem.

Properties of the MC estimator for real-valued fractional Ornstein-Uhlenbeck process have been intensively studied in last few years. We refer the reader to the articles [18] for continuous-time setting, [4] for discrete-time setting and [5] for comparison with LS estimator, to name just a few. These works benefit from the relation of Malliavin calculus and central limit theorems – a popular theory initiated in [15] and further developed by many authors (see e.g. [14] and references therein). These techniques were recently applied to the MC estimator in infinite-dimensional setting in [8] and are also utilized in the present work (especially in the non-diagonalizable setting).

This paper is organized as follows. In section 2, the setting for the weighted MC estimator is specified. In section 3, the weighted MC estimator for stationary solutions is derived and its consistency and asymptotic normality in space are proved. Discrete-time observations and continuous-time observations are studied separately. In section 4, the non-stationary solutions are considered. In section 5, the reweighing technique in non-diagonalizable setting is developed. Section 6 is devoted to the comparison of the weighted MC estimator to the (non-weighted) MC estimator, the MLE and the TFE.

2. Initial setting

Consider a linear stochastic evolution equation in a separable Hilbert space \( V \), which is driven by a fractional Brownian motion:

\[
\begin{align*}
\frac{dX(t)}{dt} &= \alpha AX(t) + \Phi dB_H(t), \\
X(0) &= X_0.
\end{align*}
\]

In this equation, \( \alpha > 0 \) is an unknown parameter, \( A : \text{Dom}(A) \subset V \to V \) and \( \Phi : \text{Dom}(\Phi) \subset V \to V \) are densely-defined self-adjoint linear operators and \( (B_H(t), t \in \mathbb{R}) \) is a standard two-sided cylindrical fractional Brownian motion on \( V \) with Hurst parameter \( H \in (0, 1) \), defined on a suitable probability space \( (\Omega, \mathcal{F}, P) \). Note that \( \Phi \) need not be bounded. The initial condition \( X_0 \) is assumed to be a random variable with values in an interpolation space \( V^\gamma \) (to be specified below) for some \( \gamma \in \mathbb{R} \).

Assume that the equation is diagonalizable, i.e. there is an orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \) of the space \( V \) consisting of common eigenfunctions of operators \( A \) and \( \Phi \):
Ae_k = -\theta_k e_k, \quad \text{with } \theta_k > 0, \text{ and }
\Phi e_k = \sigma_k e_k, \quad \text{with } \sigma_k > 0.

The standard cylindrical fractional Brownian motion \((B^H(t), t \geq 0)\) on \(\mathcal{V}\) can be understood in the weak sense as a functional acting on \(\mathcal{V}\) with

\[ \langle B^H(t), e_k \rangle = \beta^H_k(t), \quad \text{for } k = 1, 2, \ldots, \]

where \((\beta^H_k(t), t \geq 0)\) are mutually independent real-valued standard fractional Brownian motions and \(\langle B^H(t), x \rangle\) is the evaluation of \(B^H(t)\) at \(x\) (see e.g. [11] for more details). Note that in infinite-dimensional setting, \(B^H(t)\) does not take values in \(\mathcal{V}\).

Following the standard construction of the solution to diagonalizable stochastic parabolic equations (cf. [3]), we introduce a scale of Hilbert spaces \(\mathcal{V}^\gamma\) indexed by \(\gamma \in \mathbb{R}\) (also called the interpolation spaces). Take the strictly positive operator \(\Lambda = \sqrt{T - A}\). The powers of this operator are well-defined and

\[ \Lambda^{\gamma} e_k = (1 + \theta_k)^{\gamma/2} e_k. \]

For \(\gamma > 0\), we set \(\mathcal{V}^\gamma\) to be the domain of \(\Lambda^{\gamma}\) with the graph norm \(|.|_{\mathcal{V}^\gamma} = |\Lambda^{\gamma} .|_{\mathcal{V}}\). For \(\gamma = 0\), we set \(\mathcal{V}^0 = \mathcal{V}\). Finally, for \(\gamma < 0\) we define \(\mathcal{V}^\gamma\) as the completion of \(\mathcal{V}\) with respect to the graph norm \(|.|_{\mathcal{V}^\gamma} = |\Lambda^{\gamma} .|_{\mathcal{V}}\). The interpolation spaces can be represented via coordinate projections:

\[ \mathcal{V}^\gamma = \left\{ v = \sum_{k=1}^{\infty} v_k e_k : \sum_{k=1}^{\infty} (1 + \theta_k)^{\gamma} v_k^2 < \infty \right\}, \quad \forall \gamma \in \mathbb{R}, \]

with

\[ |v|_{\mathcal{V}^\gamma}^2 = \sum_{k=1}^{\infty} v_k e_k \quad \left(\begin{array}{c} \sum_{k=1}^{\infty} (1 + \theta_k)^{\gamma} v_k^2 \end{array}\right)^2 = \sum_{k=1}^{\infty} (1 + \theta_k)^{\gamma} v_k^2. \]

Recall that for \(\gamma_1 < \gamma_2\) the space \(\mathcal{V}^{\gamma_2}\) is continuously and densely embedded into \(\mathcal{V}^{\gamma_1}\) and for any \(\gamma > 0\), \(\mathcal{V}^{-\gamma}\) is the dual of \(\mathcal{V}^\gamma\) relative to the inner product in \(\mathcal{V}^\gamma\) with the dual pairing:

\[ \langle v_1 | v_2 \rangle = \langle \Lambda^{-\gamma} v_1, \Lambda^\gamma v_2 \rangle_{\mathcal{V}}, \quad v_1 \in \mathcal{V}^{-\gamma}, v_2 \in \mathcal{V}^\gamma. \]

Note that \(\{e_k\}_{k \in \mathbb{N}}\) is an orthogonal basis of \(\mathcal{V}^\gamma\) for each \(\gamma \in \mathbb{R}\) and for any \(v = \sum_{k=1}^{\infty} v_k e_k \in \mathcal{V}^\gamma\), the coordinates can be reconstructed by dual pairing:

\[ v_k = \langle v | e_k \rangle_{\gamma}. \]

**Definition 2.1.** The solution to the diagonalizable stochastic equation (1) with initial condition (2) is a process \((X(t) : t \geq 0)\) with values in \(\mathcal{V}^\gamma\) for some \(\gamma \in \mathbb{R}\) and with the expansion

\[ X_k(t) = \sum_{k=1}^{\infty} x_k(t) e_k, \]
where

\[ x_k(t) = x_k(0)e^{-\alpha \theta_k t} + \int_0^t e^{-\alpha \theta_k (t-s)} \sigma_k d\beta_k^H(s), \]

\[ x_k(0) = \langle X_0 | e_k \rangle_\gamma, \]

and the sum converges in $L_2(\Omega, V^\gamma)$ sense for some $\gamma \in \mathbb{R}$.

For each $k \in \mathbb{N}$, it is possible to find suitable initial condition so that are stationary fractional Ornstein-Uhlenbeck processes. Denote these processes ($z_k(t) : t \geq 0$) and build a stationary solution to the original equation as follows

\[ Z(t) := \sum_{k=1}^{\infty} z_k(t)e_k, \quad t \geq 0, \]

if the sum converges in $L_2(\Omega, V^\gamma)$ sense.

**Theorem 2.1.** Let

\[ \sum_{k=1}^{\infty} \frac{\sigma_k^2}{(1 + \theta_k)^\gamma} \left( 1 + \frac{1}{\theta_k} \right)^{2H} < \infty \]

for some $\gamma \in \mathbb{R}$. Then the equation admits a stationary solution $Z(t) : t \geq 0$ given by and $Z(t) \in L_2(\Omega, V^{2H-\gamma})$ for each $t \geq 0$.

In addition, if

\[ X_0 \in L_2(\Omega, V^{2H-\gamma}), \]

the equation with initial condition has a solution $X(t) : t \geq 0$ with $X(t) \in L_2(\Omega, V^{2H-\gamma})$ for each $t \geq 0$.

**Proof.** Recall (cf. for example)

\[ E z_k(t)^2 = \frac{\sigma_k^2}{(\alpha \theta_k)^{2H}} H \Gamma(2H) =: r_k(0), \quad \forall t \geq 0. \]

Consequently

\[ E \left[ \sum_{k=1}^{\infty} z_k(t)e_k \right]^2_{V^{2H-\gamma}} = E \sum_{k=1}^{\infty} (1 + \theta_k)^{2H-\gamma} z_k(t)^2 \]

\[ = \sum_{k=1}^{\infty} (1 + \theta_k)^{2H-\gamma} \frac{\sigma_k^2}{(\alpha \theta_k)^{2H}} H \Gamma(2H). \]

The condition then ensures the existence and integrability of the stationary solution.

For the solution with the initial condition, write

\[ x_k(t) = z_k(t) - e^{-\alpha \theta_k t} z_k(0) + e^{-\alpha \theta_k t} x_k(0). \]

Thus,

\[ E x_k(t)^2 \leq C_1 \frac{\sigma_k^2}{\theta_k^{2H}} + C_2 E x_k(0)^2. \]
To conclude the proof, calculate
\[
\mathbb{E} \left| \sum_{k=1}^{\infty} x_k(t) e_k \right|^2 \leq C_1 \sum_{k=1}^{\infty} (1 + \theta_k)^{2H - \gamma} \frac{\sigma_k^2}{\theta_k} + C_2 \sum_{k=1}^{\infty} (1 + \theta_k)^{2H - \gamma} \mathbb{E} x_k(0)^2.
\]

The first sum is finite due to (8) and the second sum due to (9).

Note that if \(\inf_k \{\theta_k\} > 0\), we can simplify the condition (8) in the form:
\[
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{(1 + \theta_k)^{\gamma}} < \infty.
\]

Example 2.1. Consider the following formal heat equation with distributed fractional noise and Dirichlet boundary condition:
\[
\begin{align*}
\frac{\partial f}{\partial t}(t, u) &= \alpha \Delta f(t, u) + \eta^H(t, u), \quad \text{for } (t, u) \in \mathbb{R}_+ \times \mathcal{O}, \\
f(t, u) &= 0, \quad \text{for } (t, u) \in \mathbb{R}_+ \times \partial \mathcal{O}, \\
f(0, u) &= X_0, \quad \text{for } u \in \mathcal{O},
\end{align*}
\]
where \(\Delta\) is Laplace operator, \(\mathcal{O} \subset \mathbb{R}^d\) is a bounded domain with smooth boundary \(\partial \mathcal{O}\), \(\alpha > 0\) is the unknown parameter (e.g., heat conductivity), \(X_0 \in L^2(\mathcal{O})\) is a deterministic initial condition and \((\eta^H(t, u) : t \geq 0, u \in \mathcal{O})\) is a noise, which is fractional in time with Hurst parameter \(H \in (0, 1)\) and white in space.

To give this formal equation rigorous meaning, reformulate it as a stochastic evolution equation (see (11))
\[
\begin{align*}
dX(t) &= \alpha AX(t)dt + \Phi dB(t), \\
X(0) &= X_0,
\end{align*}
\]
where \(\mathcal{V} = L^2(\mathcal{O})\), \(X_0 \in L^2(\mathcal{O})\), \(A = \Delta|_{\text{Dom}(A)}\) with \(\text{Dom}(A) = H^2(\mathcal{O}) \cap H^0_0(\mathcal{O})\) is Dirichlet Laplace operator defined on a standard Sobolev space (cf. [17]), \((B^H(t), t \geq 0)\) is a cylindrical fBm and \(\Phi\) is identity operator.

This equation is diagonalizable with eigenfunctions \(\{e_k\}_{k \in \mathbb{N}}\) of \(A\), which form an orthonormal basis of \(L^2(\mathcal{O})\). The corresponding eigenvalues can be arranged in a sequence meeting the following growth condition (cf. [17]):
\[
\theta_k \asymp k^{\frac{a}{2}},
\]
where \(a_k \asymp b_k\) means that there exist constants \(0 < c \leq C < \infty\) so that \(cb_k \leq a_k \leq Cb_k\) for all \(k = 1, 2, \ldots\). In view of (13), condition (8) is fulfilled if \(\gamma > \frac{d}{4}\) and (9) holds with \(2H - \gamma \leq 0\). Hence, for any \(t \geq 0\), we have the existence of the solution \(X_t \in L^2(\Omega, \mathcal{V}^{\text{min}(2H-\gamma,0)})\), with any \(\gamma > \frac{d}{4}\). In particular, should \(X_t \in L^2(\Omega, L^2(\mathcal{O}))\), the condition \(H > \frac{d}{4}\) must be satisfied.

3. Estimation in stationary case

In this section, the weighted MC estimator of \(\alpha\) for stationary solution is derived.
3.1. Preliminaries. Recall the following 4\textsuperscript{th} moment theorem (see e.g. [14] or references therein for details):

**Proposition 3.1.** Consider an isonormal Gaussian process \( X \) on a separable Hilbert space \( \mathcal{H} \). Let \( (F_n : n \in \mathbb{N}) \) be a sequence of random variables belonging to the \( q \)-th Wiener chaos of \( \mathcal{X} \) with \( \mathbb{E}F_n^2 = 1 \) and consider a normally distributed random variable \( U \sim \mathcal{N}(0, 1) \). Then

\[
d_{TV}(F_n, U) \leq \sqrt{\frac{4q - 4}{3q}} \sqrt{\mathbb{E}F_n^4} - 3 = \sqrt{\frac{4q - 4}{3q}} \kappa_4(F_n),
\]

where \( d_{TV} \) denotes the total-variation distance of measures (or distributions of random variables) and \( \kappa_4(F_n) = \mathbb{E}F_n^4 - 3 \) is the \( 4 \)-th cumulant of \( F_n \).

Next proposition (see e.g. [8] for proof) enables to handle asymptotic normality of the transformed random variables. It provides the upper bounds for the Kolmogorov distance localized on compacts.

**Proposition 3.2.** Consider \( U \sim \mathcal{N}(0, 1) \) and a random sequence \( (Y_n : n \in \mathbb{N}) \) with \( \mathbb{E}(Y_n) = \mu \) and a standardizing function converging to zero \( \sigma_N \to 0 \), such that

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \frac{Y_n - \mu}{\sigma_N} \leq z \right) - \mathbb{P}\left( U \leq z \right) \right| \leq \xi(N), \quad \forall N \in \mathbb{N},
\]

with \( \xi(N) \) being the upper bound for the Kolmogorov distance.

Next, consider a monotonous function \( g \in C^2(A) \), where \( \mathbb{P}(Y_n \in A) = 1 \) for all \( N \). Then for each \( K > 0 \) there exists a constant \( C_K \) such that

\[
\sup_{z \in [-K,K]} \left| \mathbb{P}\left( \frac{g(Y_n) - g(\mu)}{g'(\mu)\sigma_N} \leq z \right) - \mathbb{P}\left( U \leq z \right) \right| \leq C_K \max\{\xi(N), \sigma_N\}, \quad \forall N \in \mathbb{N}.
\]

In particular, if \( \frac{Y_n - \mu}{\sigma_N} \) is asymptotically normal with \( \sigma_N \to 0 \), then \( \frac{g(Y_n) - g(\mu)}{g'(\mu)\sigma_N} \) is asymptotically normal as well.

3.2. Discrete-time observations. First assume that the processes \( z_k \) are observed in discrete time instants, for simplicity let \( t = 1, 2, ...n \). Recall that the minimum-contrast estimator (see [8] or [11]) is based on the sample second moments, which take the following form in our setting:

\[
\frac{1}{n} \sum_{k=1}^{n} Z(t)^2 = \frac{1}{n} \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} z_k(t)^2 = \sum_{t=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} z_k(t)^2.
\]

Moreover, self-similarity of fractional Brownian motion implies that the distributions (on the space of trajectories) of the following two processes are same:

\[
\text{Law}\left( z_k(t) : t \in [0, T] \right) = \text{Law}\left( \frac{\sigma_k}{(\alpha \theta_k)^{\overline{H}}} z(\alpha \theta_k t) : t \in [0, T] \right), \quad \forall k \in \mathbb{N},
\]

where \( (z(t), t \geq 0) \) is the canonical fractional Ornstein-Uhlenbeck processes, which is the stationary solution to equation

\[
dz(t) = -z(t)dt + d\beta^H(t).
\]

Hence, the values of the processes \( z_k \) are scaled by \( \frac{\sigma_k}{(\alpha \theta_k)^{\overline{H}}} \) and the speed of their evolution by \( \alpha \theta_k \). To fully utilize the information about \( \alpha \) carried by each \( z_k \), offset the effect of different scales of values by appropriate weights. For finitely many
coordinates \( z_k(t), k = 1, \ldots N \) observed in finitely many time instants \( t = 1, \ldots n \), define

\[
Y_N := \frac{\sum_{k=1}^{N} \frac{1}{n} \sum_{t=1}^{n} \left( \frac{\theta_k}{\sigma_k} z_k(t) \right)^2}{N\Gamma(2H)} = \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^{2H}} \frac{1}{n} \sum_{t=1}^{n} z_k(t)^2}{N\Gamma(2H)}.
\]

Using (10), simple calculation yields

\[
E(Y_N) = \alpha^{-2H}.
\]

This motivates the definition of the weighted minimum-contrast estimator:

\[
\alpha_N^* := \left( Y_N \right)^{-\frac{1}{2H}} = \left( \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^{2H}} \frac{1}{n} \sum_{t=1}^{n} z_k(t)^2}{N\Gamma(2H)} \right)^{-\frac{1}{2H}}.
\]

**Remark 3.1.** Consider general weights \( w_1, \ldots w_N \) in:

\[
Y_N(w_1, \ldots, w_N) := \frac{\sum_{k=1}^{N} w_k \frac{1}{n} \sum_{t=1}^{n} z_k(t)^2}{\Gamma(2H)} \sum_{k=1}^{N} w_k \sigma_k^{2H}.
\]

Clearly \( E(Y_N(w_1, \ldots, w_N)) = \alpha^{-2H} \). Optimum weights minimizing the variance of \( Y_N \) depend on the unknown parameter \( \alpha \) and the speed of decay of the minimum variance fulfills

\[
\text{var}(Y_N(w_1^{(\text{opt})}, \ldots, w_N^{(\text{opt})})) \approx \frac{1}{N}, \text{ for } N \to \infty.
\]

On the other hand, for \( Y_N \) defined in (15), the same speed of decay \( \text{var}(Y_N) \approx \frac{1}{N} \) is shown in Theorem 3.1). Hence, the weights in (16) can be considered as optimal in terms of the speed of decay of variance of \( Y_N \) when \( N \to \infty \).

The so-called space asymptotics (number of coordinates \( N \) grows to infinity, number of time instants \( n \) remains fixed) of the weighted MC estimator is specified in the following theorem.

**Theorem 3.1.** Consider the weighted minimum-contrast estimator \( \alpha_N^* \) defined in (16). This estimator is strongly consistent in space, i.e.

\[
\alpha_N^* \xrightarrow{N \to \infty} \alpha \quad \text{a.s.,}
\]

and it is asymptotically normal in space, i.e.

\[
\frac{\alpha_N^* - \alpha}{\sqrt{\text{var}(Y_N)}} \xrightarrow{N \to \infty} U \sim N(0,1) \quad \text{in distribution,}
\]

with \( \text{var}(Y_N) \approx \frac{1}{N} \) for \( N \to \infty \).

Moreover, for each \( K > 0 \) there exists a constant \( C_K > 0 \) such that

\[
\sup_{z \in [-K, K]} \left[ \mathbb{P} \left( \frac{\alpha_N^* - \alpha}{\sqrt{\text{var}(Y_N)}} \leq z \right) - \mathbb{P} \left( U \leq z \right) \right] \leq C_K \frac{1}{\sqrt{N}}.
\]

Note that the asymptotic normality in space holds for any \( H \in (0,1) \). This contrasts the asymptotic normality in time \( (n \to \infty) \) of this type of estimators, which is violated for \( H > \frac{3}{4} \) (see e.g. [3], [18] or [5]) due to the strong long-range dependence.
Proof. Let us start with the strong consistency. Write

\[ Y_N - \mathbb{E}Y_N = \frac{1}{H\Gamma(2H)} \sum_{k=1}^{N} Q_k - \mathbb{E}Q_k, \]

where \( Q_k = \frac{\theta_k^{4H}}{\sigma_k^4} \frac{1}{n} \sum_{i=1}^{n} z_k(i)^2 \). Denote \( r_k(i) := \mathbb{E}z_k(t+i)z_k(t) \) and, in view of the Kolmogorov strong law of large numbers (denote SLLN, see e.g. [16] for details), calculate

\[ \sum_{k=1}^{\infty} \frac{\text{var}(Q_k)}{k^2} = \sum_{k=1}^{\infty} \frac{\theta_k^{4H}}{\sigma_k^4} \frac{2}{n} \sum_{i=-(n-1)}^{n-1} \left(1 - \frac{|i|}{n}\right) r_k(i)^2 \]

(21)

\[ \leq \sum_{k=1}^{\infty} \frac{\theta_k^{4H}}{\sigma_k^4} 4 r_k(0)^2 = \frac{4H^2\Gamma(2H)^2}{\alpha^{4H}} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \]

SLLN thus implies

\[ Y_N - \mathbb{E}Y_N \xrightarrow{N \to \infty} 0 \quad \text{a.s.}. \]

Since \( \mathbb{E}Y_N = \alpha^{-2H} \), the strong consistency is now immediate:

\[ \alpha^*_N = (Y_N)^{-\frac{1}{4H}} \xrightarrow{N \to \infty} (\alpha^{-2H})^{-\frac{1}{4H}} = \alpha \quad \text{a.s.}. \]

To explore asymptotic behavior of \( \text{var}(Y_N) \), start with calculation

\[ s_k^2 := \text{var} \left( \frac{1}{n} \sum_{i=1}^{n} z_k(t)^2 \right) = \frac{2}{n} \sum_{i=-(n-1)}^{n-1} \left(1 - \frac{|i|}{n}\right) r_k(i)^2 \approx \frac{\sigma_k^4}{\theta_k^{4H}}. \]

(22)

Consequently,

\[ \text{var}(Y_N) = \frac{1}{(N\Gamma(2H))^2} \sum_{k=1}^{N} \frac{\theta_k^{4H}}{\sigma_k^4} s_k^2 \asymp \frac{1}{N}. \]

Next step is to show the asymptotic normality of \( Y_N \) using the 4th moment theorem. Calculation of the corresponding 4th cumulant benefits from the independence of the coordinates:

\[ \kappa_4 \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}} \right) = \frac{1}{(\text{var}(Y_N))^2} \left( N\Gamma(2H) \right)^4 \sum_{k=1}^{N} \frac{\theta_k^{4H}}{\sigma_k^4} \frac{1}{n^4} \kappa_4 \left( \sum_{t=1}^{n} z_k(t)^2 - r_k(0) \right). \]

Next, use the upper bound for the 4th cumulant derived in [5]. In particular, equation (23) therein applied to the 1-dimensional Gaussian processes \( z_k \) (so that \( |Q(i)|_{L_2} \) is replaced by \( |r_k(i)| \)) yields

\[ \kappa_4 \left( \sum_{t=1}^{n} z_k(t)^2 - r_k(0) \right) \leq nC_2 \left( \sum_{i=-(n-1)}^{n-1} |r_k(i)|^2 \right)^3 \leq 8n^4C_2 \frac{\sigma_k^8}{(\alpha\theta_k)^{8H}} (\Gamma(2H))^4. \]

This results in

\[ \kappa_4 \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}} \right) \leq \frac{C}{N}, \]

(23)

for some constant \( C \) independent of \( N \).
Proposition 3.1 now provides the upper bound for the total-variation distance from the \( \mathcal{N}(0,1) \)-distributed random variable \( U \):

\[
    d_{TV} \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}}, U \right) \leq \frac{C}{\sqrt{N}}.
\]

To obtain the bound \([20]\) and the asymptotic normality, apply Proposition 3.2 with \( g(x) = x^{-\frac{H}{2}} \) and \( \text{var}(Y_N) \approx \frac{1}{N} \). \( \square \)

Remark 3.2. Observe that the self-similarity property \([14]\) implies

\[
    r_k(t) = \frac{\sigma^2_k}{(\alpha\theta_k)^{2H}} r(\alpha\theta_k t),
\]

where \( r(t) \) stands for the auto-covariance function of the canonical Ornstein-Uhlenbeck process. Now if \( \lim_{k \to \infty} \theta_k = \infty \), we can utilize the calculations from the previous proof of the strong consistency and extend them by employing the self-similarity property as follows

\[
    \text{var}(Y_N) = \frac{\sum_{k=1}^{N} \text{var}(Q_k)}{H^2\Gamma(2H)^2N^2} = \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^2} \frac{2}{n} \sum_{i=(n-1)}^{n-1} \left( 1 - \frac{|i|}{n} \right) r(i)^2}{H^2\Gamma(2H)^2N^2}
\]

\[
    = \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^2} \frac{2}{n} \sum_{i=(n-1)}^{n-1} \left( 1 - \frac{|i|}{n} \right) \frac{\sigma^2_i}{(\alpha\theta_k)^{2H}} r(\alpha\theta_k i)^2}{H^2\Gamma(2H)^2N^2}
\]

Since \( r(0) = H\Gamma(2H) \) and \( \lim_{t \to \infty} r(t) = 0 \), we have

\[
    \lim_{N \to \infty} N \text{var}(Y_N) = \frac{2}{n} \frac{1}{\alpha^{4H}}.
\]

This relation shows dependency of the constant in the speed of decay of \( \text{var}(Y_N) \) on the number of time instants \( n \) and the parameter \( \alpha \).

Remark 3.3. In contrast to previous setting, fix now the number of observed coordinates \( N \) and consider the time (long-span) asymptotics of \( \alpha^*_N \) \( (n \to \infty, \text{fixed time step}) \). Observe that \( \alpha^*_N \) can be considered as the (non-weighted) minimum-contrast estimator constructed from the fractional Ornstein-Uhlenbeck process \( U^{(N)}(t) = \sum_{k=1}^{N} \frac{\theta_k^H}{\sigma_k} z_k(t)e_k \). Hence, we can directly use \([8]\) to see that:

- \( \alpha^*_N \) are strongly consistent as \( n \to \infty \).
- If \( H < \frac{1}{2} \), \( \alpha^*_N \) are asymptotically normal as \( n \to \infty \) with \( \text{var}(\alpha^*_N - \alpha) = O \left( \frac{1}{n} \right) \).

3.3 Continuous-time observations. Observations of processes \( z_k(t) \) in continuous-time window \( t \in [0, T] \) are considered in this section. Straightforward modification of the estimator \([16]\) (substituting sums by integrals) would preserve all properties specified in Theorem 3.1. However, if \( \theta_k \to \infty \), we can further improve the estimator. Recall the self-similarity equation \([14]\):

\[
    \text{Law} \left( z_k(t) : t \in [0, T] \right) = \text{Law} \left( \frac{\sigma_k}{(\alpha\theta_k)^H} z(\alpha\theta_k t) : t \in [0, T] \right).
\]

Change of variable leads to:

\[
    \text{Law} \left( \frac{1}{T} \int_0^T z_k(t)^2 dt \right) = \text{Law} \left( \frac{\sigma^2_k}{(\alpha\theta_k)^{2H}} \frac{1}{\alpha\theta_k T} \int_0^{\alpha\theta_k T} z(t)^2 dt \right).
\]
Thus, increasing $\theta_k$ changes not only the scale of values, but also increases the time horizon of the process $z$ (understood in law), which is $\alpha \theta_k T$. To make use of this increasing time horizon, weights should be growing faster compared to the discrete-time case.

To derive appropriate weights, recall the optimization problem in Remark 3.1. Consider weighted MC with general weights constructed from

$$ Y_N(w_1, \ldots, w_N) := \frac{\sum_{k=1}^{N} w_k \left( \int_0^T z_k(t)^2 dt \right)}{H \Gamma(2H) \sum_{k=1}^{N} w_k \frac{\sigma^2_k}{\theta_k^2}}. $$

Obviously $EY_N(w_1, \ldots, w_N) = \alpha^{-2H}$. Set the weights $w_1, \ldots, w_N$ in order to minimize the variance

$$ \text{var}(Y_N(w_1, \ldots, w_N)) = \frac{\sum_{k=1}^{N} w_k^2 \sigma^4_k}{\left( H \Gamma(2H) \sum_{k=1}^{N} w_k \frac{\sigma^2_k}{\theta_k^2} \right)^2}, $$

where

$$ s_k^2 = \text{var} \left( \frac{1}{T} \int_0^T z_k(t)^2 dt \right). $$

The optimum solution is

$$ w_k^{(\text{opt})} = \frac{\sigma^2_k}{\theta_k^{2H}} s_k^2, $$

and (using the self-similarity and change of variable)

$$ s_k^2 = \frac{\sigma^4_k}{(\alpha \theta_k)^{2H}} \frac{4}{\alpha \theta_k T} \int_0^{\alpha \theta_k T} r(s)^2 \left( 1 - \frac{s}{\alpha \theta_k T} \right) ds, $$

where $r(s) = \mathbb{E}z(s)z(0)$ is the auto-covariance function of the canonical fractional Ornstein-Uhlenbeck process. In [1], it is shown that

$$ r(s) = H(2H - 1)s^{2H-2} + O(s^{2H-4}), \quad s \to \infty. $$

Consequently, if $\theta_k \xrightarrow{k \to \infty} \infty$, the weighted MC estimator $\alpha_N^*$ takes the following forms:

- For $0 < H < \frac{3}{4}$ we have $s_k^2 \propto \frac{\sigma^4_k}{\theta_k^{2H}}$ as $k \to \infty$, $w_k = \frac{\theta_k^{2H-1}}{\sigma_k^2}$ and

$$ \alpha_N^* := Y_N^{\frac{1}{2H}} = \left( \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H+1}}{\sigma_k^2} \frac{1}{T} \int_0^T z_k(t)^2 dt}{H \Gamma(2H) \sum_{k=1}^{N} \theta_k} \right)^{-\frac{1}{2H}}. $$

- For $H = \frac{3}{4}$ we have $s_k^2 \propto \frac{\sigma^4_k}{\theta_k^{2H}} \ln(\theta_k T)$ as $k \to \infty$, $w_k = \frac{\theta_k^{2H-1}}{\sigma_k^2 \ln(\theta_k T)}$ and

$$ \alpha_N^* := Y_N^{\frac{1}{2H}} = \left( \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^4} \frac{1}{T} \int_0^T z_k(t)^2 dt}{\frac{1}{4} \Gamma(\frac{3}{4}) \sum_{k=1}^{N} \frac{\theta_k}{\ln(\theta_k T)}} \right)^{-\frac{1}{2H}}. $$
• For $\frac{3}{4} < H < 1$ we have $s_k^2 \sim \frac{\sigma_k^2}{\theta_k^2}$ as $k \to \infty$, $w_k = \frac{\theta_k^{4-2H}}{\sigma_k^2}$ and

$$\alpha^*_N := Y_N^{-1/m} = \left( \frac{\sum_{k=1}^{N} \frac{\theta_k^4}{\sigma_k^2} \left( \int_0^T z_k(t)^2 \, dt \right)}{\Gamma(2H) \sum_{k=1}^{N} \theta_k^{4-4H}} \right)^{-\frac{1}{2m}}. $$

Theorem 3.2. Let $\theta_k \to \infty$ and consider the weighted minimum-contrast estimators $\alpha^*_N$ defined in (30), (31) and (32). These estimators are strongly consistent in space, i.e.

$$\alpha^*_N \overset{N \to \infty}{\longrightarrow} \alpha \quad \text{a.s.,}$$

and for each $K > 0$ there exists a constant $C_K > 0$ such that we have local Berry-Esseen bound (consider $U \sim \mathcal{N}(0, 1)$)

$$\sup_{z \in [-K, K]} \left| \mathbb{P} \left( \frac{\alpha^*_N - \alpha}{\alpha + \mathbb{E}[z]} \sqrt{\text{var}(Y_N)} \right) \leq z \right| - \mathbb{P}(U \leq z) \leq C_K \sqrt{\zeta(N)},$$

where

$$\text{var}(Y_N) = \begin{cases} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } 0 < H < \frac{3}{4}, \\ \frac{1}{\sum_{k=m}^{N} \theta_k} & \text{for } H = \frac{3}{4}, \\ \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } \frac{3}{4} < H < 1, \end{cases}$$

and

$$\zeta(N) = \begin{cases} \frac{1}{\sum_{k=1}^{N} \theta_k} \cdot \left( \frac{1}{\sum_{k=1}^{N} \theta_k} \right)^{1-H} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } 0 < H < \frac{5}{8}, \\ \frac{1}{\sum_{k=1}^{N} \theta_k} \cdot \left( \frac{1}{\sum_{k=1}^{N} \theta_k} \right)^{1-H} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } H = \frac{5}{8}, \\ \frac{1}{\sum_{k=1}^{N} \theta_k} \cdot \left( \frac{1}{\sum_{k=1}^{N} \theta_k} \right)^{1-H} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } \frac{5}{8} < H < \frac{3}{4}, \\ \frac{1}{\sum_{k=1}^{N} \theta_k} \cdot \left( \frac{1}{\sum_{k=1}^{N} \theta_k} \right)^{1-H} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } H = \frac{3}{4}, \\ \frac{1}{\sum_{k=1}^{N} \theta_k} \cdot \left( \frac{1}{\sum_{k=1}^{N} \theta_k} \right)^{1-H} \frac{1}{\sum_{k=1}^{N} \theta_k} & \text{for } \frac{3}{4} < H < 1. \end{cases}$$

Proof. For strong consistency, apply SLLN to $Y_N$. In particular, for $H < \frac{3}{4}$, note that

$$\Gamma(2H) \sum_{k=1}^{N} \theta_k \not\to \infty \text{ as } N \to \infty$$

and

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}[z_k(t)^2]}{\Gamma(2H) \sum_{m=1}^{k} \theta_m} = \sum_{k=1}^{\infty} \frac{\theta_k^{4+2H}}{\sigma_k^2} S_k^2 \leq C \sum_{k=1}^{\infty} \frac{\theta_k^2}{\left( \sum_{m=1}^{k} \theta_m \right)^2} < \infty,$$

where the convergence of the last series follows from the fact that for $k \geq 2$:

$$\frac{\theta_k}{\left( \sum_{m=1}^{k} \theta_m \right)^2} \leq \frac{\theta_k}{\left( \sum_{m=1}^{k} \theta_m \right) \left( \sum_{m=1}^{k-1} \theta_m \right)} = \frac{1}{\sum_{m=1}^{k-1} \theta_m} - \frac{1}{\sum_{m=1}^{k} \theta_m}.$$
which leads to the telescopic series
\[
\sum_{k=1}^{\infty} \left( \frac{\theta_k}{\sum_{m=1}^{k} \theta_m} \right)^2 \leq \frac{1}{\theta_1} + \left( \frac{1}{\theta_1} - \frac{1}{\theta_1 + \theta_2} \right) + \left( \frac{1}{\theta_1 + \theta_2} - \frac{1}{\theta_1 + \theta_2 + \theta_3} \right) + \ldots = \frac{2}{\theta_1}.
\]

This verifies the assumptions of the SLLN and the almost-sure convergence of \( Y_N \) and strong consistency of \( \alpha_N^* \) are guaranteed.

If \( H = \frac{3}{4} \) or \( \frac{2}{3} < H < 1 \), strong consistency of \( \alpha_N^* \) can be proved similarly, with \( \theta_k \) being replaced with \( \frac{\theta_k}{m(\theta_k T)} \) or with \( \theta_k^{4-4H} \) in the conditions for SLLN above.

To show (35), combine equation (29) with formulas for \( w_k \) and asymptotic formulas for \( s_k^2 \), separately for the case \( 0 < H < \frac{3}{4} \), \( H = \frac{3}{4} \) and \( \frac{3}{4} < H < 1 \).

For local Berry-Esseen bound, start with calculations similar to the discrete-time case (using formula (27) from [8]):
\[
\kappa_4 \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}} \right) = \frac{1}{(\sum_{k=1}^{N} w_k^2 s_k^2)^2} \sum_{k=1}^{N} w_k^4 \kappa_4 \left( \frac{1}{T} \int_0^T z_k(t)^2 dt - r_k(0) \right) \leq \frac{1}{(\sum_{k=1}^{N} w_k^2 s_k^2)^2} \sum_{k=1}^{N} w_k^4 \frac{\tilde{C}}{T^3} \left( \int_{-T}^{T} r_k(t)^4 dt \right)^{\frac{3}{4}}.
\]

where \( \tilde{C} \) is a universal constant. To proceed further, we use the formula (23), the change-of-variable formula and the upper bound for the covariance function of the canonical fractional Ornstein-Uhlenbeck process (see e.g. Lemma 5.2 in [8]):
\[
|r(t)| \leq \min \{r(0), C |t|^{2H-2} \}.
\]

Calculations of the integrals of the resulting power functions then lead to the following upper bounds:
\[
\kappa_4 \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}} \right) \leq \begin{cases} 
\frac{C_1}{(\sum_{k=1}^{N} w_k^2 s_k^2)^2} \sum_{k=1}^{N} w_k^4 \left( \frac{\alpha \theta_k}{\alpha \theta_k T + 1} \right)^{\alpha \theta_k T + 1} & \text{for } 0 < H < \frac{5}{6}, \\
\frac{C_2}{(\sum_{k=1}^{N} w_k^2 s_k^2)^2} \sum_{k=1}^{N} w_k^4 \left( \frac{\alpha \theta_k}{\alpha \theta_k T + 1} \right)^{\alpha \theta_k T + 5} \ln^3 (\alpha \theta_k T) & \text{for } H = \frac{5}{6}, \\
\frac{C_3}{(\sum_{k=1}^{N} w_k^2 s_k^2)^2} \sum_{k=1}^{N} w_k^4 \left( \frac{\alpha \theta_k}{\alpha \theta_k T + 1} \right)^{\alpha \theta_k T + 5} (\alpha \theta_k T)^{8H-5} & \text{for } \frac{5}{6} < H < 1.
\end{cases}
\]

If we combine these bounds with the corresponding formulas for \( w_k \) and asymptotic formulas for \( s_k^2 \), we obtain
\[
\kappa_4 \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}} \right) \leq C \zeta(N),
\]
with \( \zeta(N) \) specified in (39). The Proposition 5.1 then yields the bound on the total-variation distance:
\[
d_{TV} \left( \frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}}, U \right) \leq C \sqrt{\zeta(N)}.
\]

The bound (32) is then the result of Proposition 5.2 with \( g(x) = x^{-\frac{3}{8}} \) and the fact that \( \zeta(N) \) dominates \( \text{var}(Y_N) \).
Remark 3.4. In discrete-time case, we demonstrated that var($Y_N$) $\propto 1/N$. Thus, continuous-time observations enable us to increase the speed of convergence of var($Y_N$) to zero if the weights are properly modified. This speed is given by (35).

Corollary 3.1. Consider the setting from Theorem 3.2. Let

- $H \leq 3/4$, or
- $3/4 < H < 1$ and $\theta_k \propto k^\beta$ for some $\beta > 0$.

Then $\zeta(N) \overset{N \to \infty}{\to} 0$ and the asymptotic normality of $\alpha_N^*$ holds.

Proof. Asymptotic normality for $H < 3/4$ follows easily from the condition $\theta_k \overset{k \to \infty}{\to} \infty$ combined with Hölder inequality.

For $H = 3/4$, write

$$\zeta(N) = \frac{1}{T^2 \alpha^8} \sum_{k=1}^N \frac{\theta_k^2}{\ln^2(\theta_k T)} \frac{1}{\ln^2(\theta_k T)} \left( \sum_{k=1}^N \frac{\theta_k}{\ln(\theta_k T)} \right)^2.$$

Observe that for an increasing positive sequence $\{a_k\}$ and a decreasing positive sequence $\{b_k\}$ the following inequality holds

$$\sum_{k=1}^N a_k b_k = \sum_{k=1}^N a_k \bar{b} + \sum_{k=1}^N a_k (b_k - \bar{b}) \leq \bar{b} \sum_{k=1}^N a_k,$$

where $\bar{b} = \frac{1}{N} \sum_{k=1}^N b_k$. Now apply this observation to the numerator:

$$\zeta(N) \leq \frac{1}{T^2 \alpha^8} \left( \sum_{k=1}^N \frac{\theta_k^2}{\ln^2(\theta_k T)} \right) \left( \frac{1}{N} \sum_{k=1}^N \frac{1}{\ln^2(\theta_k T)} \right) \left( \sum_{k=1}^N \frac{\theta_k}{\ln(\theta_k T)} \right)^2.$$

Hölder inequality then completes the proof:

$$\zeta(N) \leq \frac{1}{T^2 \alpha^8} \frac{1}{N} \sum_{k=1}^N \frac{1}{\ln^2(\theta_k T)} \overset{N \to \infty}{\to} 0.$$

For $H > 3/4$, we can prove the asymptotic normality by direct calculation using $\theta_k \propto k^\beta$.

Remark 3.5. Note that for $H > 3/4$, asymptotic normality was not proved in general. For example if $\theta_k = e^k$, $\zeta(N)$ will not converge to zero. In this case the weights grow so rapidly that the highest coordinates dominate in the estimator. This leads to insufficient mixing of (independent) coordinates. Moreover, recall that increasing $\theta_k$ acts (in law) as increasing time horizon (cf. (27)). This, together with strong long-range dependence in time, ruins the asymptotic normality.

Interestingly, the estimator (16), constructed for discrete-time observations, converges to normal distribution even in this example, because Theorem 3.1 does not impose any additional requirements on $\theta_k$. Compared to the continuous-time estimator, it exhibits lower speed of convergence (in terms of the variance), but it ensures asymptotic normality.
4. Estimation in non-stationary case

Space asymptotics of the weighted MC estimator calculated from a non-stationary solution to equation (14) with an initial condition (2) is studied in this section. Although the construction of the weighted MC estimator relies on the properties of the stationary solution, the acceleration of virtual time with growing $\theta_k$ (see the self-similarity property (14)) eliminates the effect of initial condition even in fixed time window. Thus, we can expect favorable asymptotic properties for wide range of initial conditions.

4.1. Discrete-time observations. Let $x_k(t)$, $k = 1, \ldots, N$ be the coordinates of a (non-stationary) solution as defined in Definition 2.1 and let these processes are observed in discrete time instants $t = 1, \ldots, n$. Consider the weighted minimum-contrast estimator

$$\alpha^*_N := \left( \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^2} \sum_{t=1}^{n} x_k(t)^2}{N \Gamma(2H)} \right)^{-1/N_H}.$$  

**Theorem 4.1.** If the following conditions hold:

(D1) $\theta_k \overset{k \to \infty}{\longrightarrow} \infty$, and

(D2) $e^{-2a_k \theta_k \frac{\sigma_k^2}{\sigma_k^2} E x_k^2(0)} \overset{k \to \infty}{\longrightarrow} 0$,

then $\alpha^*_N$ is weakly consistent in space, i.e. $\alpha^*_N \overset{N \to \infty}{\longrightarrow} \alpha$ in probability.

Let the conditions (D1), (D2) and (D3) $\sup_{k \in N} \left( e^{-4 \alpha_k \theta_k \frac{\sigma_k^2}{\sigma_k^2} E x_k^2(0)} \right) < \infty$ hold. Then $\alpha^*_N$ is strongly consistent in space, i.e. $\alpha^*_N \overset{N \to \infty}{\longrightarrow} \alpha$ almost surely.

Assume there are some constants $C > 0$ and $\beta < -1$ so that

(D1') $e^{-2a_k \theta_k} < C k^\beta$, and

(D2') $e^{-2a_k \theta_k \frac{\sigma_k^2}{\sigma_k^2} E x_k^2(0)} < C k^\beta$.

Then $\left( \frac{\alpha^*_N - \alpha}{N^{1/2} \sqrt{\beta}} \right) \overset{N \to \infty}{\longrightarrow} U \sim N(0, 1)$ in distribution.

Observe that (D1') $\Rightarrow$ (D1) and (D2') $\Rightarrow$ (D2).

**Proof.** The proof is based on exploring the difference between stationary solutions $z_k(t)$ and non-stationary solutions $x_k(t)$. Denote

$$Y_N^{(z)} = \frac{\sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^2} \sum_{t=1}^{n} z_k^2(t)}{N \Gamma(2H)}$$

and observe

$$\mathbb{E}|Y_N^{(z)} - Y_N^{(x)}| \leq \frac{1}{N \Gamma(2H)} \sum_{k=1}^{N} \frac{\theta_k^{2H}}{\sigma_k^2} \sum_{t=1}^{n} \mathbb{E}|x_k^2(t) - z_k^2(t)|.$$  

Clearly

$$x_k(t) - z_k(t) = e^{-\alpha_k t}(x_k(0) - z_k(0)).$$
and
\[ \mathbb{E}|x_k^2(t) - z_k^2(t)| = \mathbb{E} \left| \left( x_k(t) - z_k(t) \right) \left( 2z_k(t) + (x_k(t) - z_k(t)) \right) \right| \]
\[ \leq 2 \sqrt{\mathbb{E}(x_k(t) - z_k(t))^2} \sqrt{\mathbb{E}z_k^2(t)} + \mathbb{E}(x_k(t) - z_k(t))^2. \]

Continue with
\[ \mathbb{E}(x_k(t) - z_k(t))^2 \leq e^{-2\alpha \theta_k} \mathbb{E}(x_k(0) - z_k(0))^2 \]
\[ \leq e^{-2\alpha \theta_k} 4 \left( \mathbb{E}x_k^2(0) + \frac{\sigma_k^2}{\alpha \theta_k^{2H}} \Gamma(2H) \right), \]
\[ \mathbb{E}z_k^2(t) = \mathbb{E}z_k^2(0) = \frac{\sigma_k^2}{\alpha \theta_k^{2H}} \Gamma(2H). \]

These auxiliary calculations yield
\[ \mathbb{E}|Y^{(x)}_N - Y^{(z)}_N| \leq \frac{1}{H \Gamma(2H)} \frac{1}{N} \sum_{k=1}^{N} \left( D_k + 2 \sqrt{\frac{H \Gamma(2H)}{\alpha^{2H}}} \sqrt{D_k} \right), \]
where
\[ D_k = 4 e^{-2\alpha \theta_k} \frac{\theta_k^{2H}}{\sigma_k^2} \mathbb{E}x_k^2(0) + 4 e^{-2\alpha \theta_k} \frac{H \Gamma(2H)}{\alpha^{2H}}. \]

Conditions (D1) and (D2) guarantee the convergence $D_k \to 0$ and, consequently, $\mathbb{E}|Y^{(x)}_N - Y^{(z)}_N| \xrightarrow{N \to \infty} 0$. This, together with the weak consistency of $Y^{(z)}_N$, guarantees the weak consistency of $\alpha_N^\ast$.

To prove strong consistency, write
\[ Y^{(x)}_N - Y^{(z)}_N - \mathbb{E}(Y^{(x)}_N - Y^{(z)}_N) = \frac{1}{NH \Gamma(2H)} \sum_{k=1}^{N} (Q_k - \mathbb{E}Q_k), \]
where
\[ Q_k = \frac{\theta_k^{2H}}{\sigma_k^2} \frac{1}{n} \sum_{t=1}^{n} (x_k^2(t) - z_k^2(t)). \]

By similar calculations, see that (D1) and (D3) imply
\[ \sup_{k \in \mathbb{N}} \left( \text{var}(Q_k) \right) < \infty. \]

Kolmogorov SLLN then ensures $Y^{(x)}_N - Y^{(z)}_N - \mathbb{E}(Y^{(x)}_N - Y^{(z)}_N) \xrightarrow{N \to \infty} 0$ almost surely.

Conditions (D1) and (D2) guarantee $\mathbb{E}(Y^{(x)}_N - Y^{(z)}_N) \to 0$, which leads to
\[ Y^{(x)}_N - Y^{(z)}_N \xrightarrow{N \to \infty} 0 \quad \text{almost surely}. \]

Strong consistency of $\alpha_N^\ast$ now easily follows.

Let us conclude with asymptotic normality. Observe
\[ \frac{Y^{(x)}_N - \alpha^{-2H}}{\sqrt{\text{var}(Y^{(x)}_N)}} = \frac{Y^{(z)}_N - \alpha^{-2H}}{\sqrt{\text{var}(Y^{(z)}_N)}} + \frac{Y^{(x)}_N - Y^{(z)}_N}{\sqrt{\text{var}(Y^{(z)}_N)}}. \]
The first term is asymptotically normal and for the second term, utilize previous calculations and asymptotic behavior of \( \text{var}(Y_N) \) specified in \((26)\) to see:

\[
\mathbb{E} \left| \frac{Y_N^{(x)} - Y_N^{(z)}}{\text{var}(Y_N^{(z)})} \right| \leq \frac{1}{\sqrt{\text{var}(Y_N^{(z)})}} \left( D_k + 2\sqrt{\frac{H T(2H)}{\alpha^2 H}} \sqrt{D_k} \right) \leq C \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sqrt{D_k}.
\]

Conditions \((D1')\) and \((D2')\) ensure \( D_k \leq C k^\beta \) for some constants \( C > 0 \) and \( \beta < -1 \). Hence,

\[
C \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sqrt{D_k} \xrightarrow{N \to \infty} 0.
\]

This guarantees asymptotic normality of \( Y_N^{(x)} \). Asymptotic normality of \( \frac{\alpha_N - \alpha}{\sqrt{2\beta^2 \text{var}(Y_N^{(z)})}} \) then follows from Proposition 3.2. Finally, the denominator can be expressed explicitly by applying \((26)\) again. \(\blacksquare\)

**Remark 4.1.** Although the conditions in Theorem 4.1 are rather technical, they are not much restrictive. For example, if

- \( \theta_k \xrightarrow{m(k)} \infty \),
- \( \inf_k \sigma_k > 0 \), and
- \( \sup_k \mathbb{E} x_k^2(0) < \infty \),

conditions \((D1), (D2), (D1')\) and \((D2')\) are satisfied and \( \alpha_N^* \) is weakly consistent and asymptotically normal.

For strong consistency of \( \alpha_N^* \) (condition \((D3)\)), it suffices to replace \( \sup_k \mathbb{E} x_k^2(0) < \infty \) with stronger condition \( \sup_k \mathbb{E} x_k^2(0) < \infty \).

### 4.2. Continuous-time observations

In this section, observation of coordinates of a (non-stationary) solution \( x_k(t), k = 1, \ldots, N \) in a fixed time-window \( t \in [0, T] \) is considered. To let the accelerating time in higher coordinates eliminate the effect of the initial condition, we must leave a certain initial period of time idle (denote its length \( \delta > 0 \)). Define the weighted MC estimator (compare to \((30), (31)\) and \((32)\))

\[
\alpha_N^* := \begin{cases} 
\frac{\sum_{k=1}^{N} \frac{1}{H T(2H)} \var{f_k}(t)}{\sum_{k=1}^{N} \theta_k} & \text{for } 0 < H < \frac{3}{4}, \\
\frac{\sum_{k=1}^{N} \frac{1}{\theta_k} \var{f_k}(t)}{\Gamma\left(\frac{5}{4}\right) \sum_{k=1}^{N} \theta_k} & \text{for } H = \frac{3}{4}, \\
\frac{\sum_{k=1}^{N} \frac{1}{H T(2H)} \var{f_k}(t)}{\sum_{k=1}^{N} \theta_k^{3-2H}} & \text{for } \frac{3}{4} < H < 1.
\end{cases}
\]

**Theorem 4.2.** If \( H \leq \frac{3}{4} \) and

\((C1)\) \( \theta_k \xrightarrow{m(k)} \infty \), and
Proposition 3.2. Note that in case 

\[ \alpha \]

together with condition \((C)\) to \((C)\) Using formulas for \(w\) by direct calculation

\[ \frac{1}{2\pi} \sqrt{\text{var}(Y_N)} \]

then

\[ \frac{\alpha - \alpha}{\sqrt{\text{var}(Y_N)}} \overset{N \to \infty}{\longrightarrow} U \sim N(0,1) \]

in distribution.

If \(H > \frac{3}{4}\) and

\((C1')\) \(\theta_k \asymp k^\beta\) for some \(\beta > 0\), and

\((C2')\) \(\sup_k \frac{\alpha - \alpha}{\sqrt{\text{var}(Y_N)}} < \infty\),

then

\[ \frac{\alpha - \alpha}{\sqrt{\text{var}(Y_N)}} \overset{N \to \infty}{\longrightarrow} U \sim N(0,1) \]

in distribution.

Let \((C1)\) and \((C2)\) hold. Then \(\alpha_N \overset{N \to \infty}{\longrightarrow} \alpha\) almost surely for all \(H \in (0,1)\).

Note that \(\text{var}(Y_N)\), whose square root determines the speed of convergence of the estimators, is specified in \((35)\) and \((C1')\) \(\Rightarrow (C1)\) and \((C2')\) \(\Rightarrow (C2)\).

Proof. Proceed similarly to proof of Theorem 4.1. Consider

\[ Y_N^{(z)} = \sum_{k=1}^{N} w_k \frac{1}{\sqrt{\pi}} \int_{-\delta}^{\delta} x_k(t)^2 dt, \quad Y_N^{(x)} = \sum_{k=1}^{N} w_k \frac{1}{\sqrt{\pi}} \int_{-\delta}^{\delta} x_k(t)^2 dt, \]

with weights as in \((38)\). Employ \((29)\) to calculate

\[ \mathbb{E} \left| \frac{Y_N^{(z)} - Y_N^{(z)}}{\sqrt{\text{var}(Y_N)}} \right| \leq C \frac{1}{\sqrt{\sum_{k=1}^{N} w_k^2 s_k^2}} \sum_{k=1}^{N} \left( D_k + \sqrt{\frac{w_k \sigma_k^2}{(\alpha \theta_k)^{2H}}} \right) \]

where

\[ D_k = w_k e^{-2\alpha \theta_k \delta} \left( \mathbb{E} x_k^2(0) + \frac{\sigma_k^2}{(\alpha \theta_k)^{2H}} \right). \]

Using formulas for \(w_k\) and for asymptotic behavior of \(s_k\) (cf. \((30)\), \((31)\) and \((32)\) together with condition \((C2)\), gets (in all three cases):

\[ D_k + \sqrt{\frac{w_k \sigma_k^2}{(\alpha \theta_k)^{2H}}} \overset{N \to \infty}{\longrightarrow} \leq C e^{-\frac{\theta_k \delta}{4}} \]

for some constant \(C > 0\).

Condition \((C1)\) then ensures summability of the corresponding series. Verify further by direct calculation

\[ \sum_{k=1}^{N} w_k^2 s_k^2 \overset{N \to \infty}{\longrightarrow} \infty. \]

As a consequence,

\[ \mathbb{E} \left| \frac{Y_N^{(z)} - Y_N^{(z)}}{\sqrt{\text{var}(Y_N)}} \right| \overset{N \to \infty}{\longrightarrow} 0. \]

Asymptotic normality of \(\alpha_N^*\) follows easily from asymptotic normality of \(Y_N^{(z)}\) and Proposition \((32)\). Note that in case \(H > \frac{3}{4}\) the condition \((C1)\) must be strengthen to \((C1')\) to ensure asymptotic normality of \(Y_N^{(z)}\) (see Corollary \((3.1)\).
For strong consistency, write
\[ Y_N^{(x)} - Y_N^{(z)} - \mathbb{E}(Y_N^{(x)} - Y_N^{(z)}) = \frac{1}{\sum_{k=1}^{N} w_k H \Gamma(2H) \frac{\sigma_k^2}{\theta_k^2}} \sum_{k=1}^{N} (Q_k - \mathbb{E}Q_k), \]
where
\[ Q_k = w_k \frac{1}{T - \delta} \int_{\delta}^{T} (x_k^2(t) - z_k^2(t)) dt. \]
By similar calculations the conditions \((C1)\) and \((C2')\) imply
\[ \sup_{k \in \mathbb{N}} \left( \text{var}(Q_k) \right) < \infty. \]
Moreover,
\[ w_k H \Gamma(2H) \frac{\sigma_k^2}{\theta_k^2} \overset{k \to \infty}{\to} \infty. \]
Hence, application of Kolmogorov SLLN leads to
\[ Y_N^{(x)} - Y_N^{(z)} \overset{\mathbb{P}}{\to} 0 \]
almost surely. The convergence \( \mathbb{E}(Y_N^{(x)} - Y_N^{(z)}) \to 0 \) follows from the proof of asymptotic normality. In result
\[ Y_N^{(x)} - Y_N^{(z)} \to 0 \]
almost surely.
Strong consistency of \( \alpha_N^* \) is a direct consequence.

**Example 4.1.** The performance of the weighted MC estimator can be illustrated on the stochastic heat equation on \( d \)-dimensional domain, with distributed fractional noise, Dirichlet boundary condition and deterministic initial condition, as introduced in Example 2.1. It can be interpreted as a diagonalizable stochastic evolution equation with eigenvalues \( \theta_k \approx k^rac{2}{d} \) and \( \sigma_k = 1 \) for \( k = 1, 2, \ldots \).

If first \( N \) coordinate projections of the solution (see Definition 2.1) in discrete time-instants are observed, the weighted minimum-contrast estimator in the form (37) can be used. Theorem 4.1 (see Remark 4.1 for verification of its assumptions) provides the strong consistency and the asymptotic normality (as \( N \to \infty \)) of the estimator with the rate of convergence \( \frac{1}{\sqrt{N}} \).

Next, consider the observations of first \( N \) coordinates in continuous time-window \( t \in [0, T] \) are available. Since \( \theta_k \to \infty \), the continuous-time version of weighted MC estimator (see (38)) can be applied. Because all conditions in Theorem 4.2 hold, the estimator is strongly consistent and asymptotically normal (as \( N \to \infty \)) with the rate of convergence
\[ \sqrt{\text{var}(Y_N)} \approx \begin{cases} \frac{1}{\sqrt{N^{1+\frac{2}{d}}}} & \text{for } 0 < H < \frac{3}{4}, \\ \frac{1}{\sum_{k=1}^{N} \frac{k^\frac{2}{d}}{1+\frac{2}{d}}} & \text{for } H = \frac{3}{4}, \\ \frac{1}{\sqrt{N^{1+\frac{2}{d}}}} & \text{for } \frac{3}{4} < H < 1. \end{cases} \]

Speed of convergence of \( \alpha_N^* \) to \( \alpha \) in case of the continuous-time weighted MC estimator is obviously faster compared to its discrete-time version.
5. Non-diagonalizable equations

The presented reweighing technique can be applied also in more general, non-diagonalizable setting. Consider the equation (1), but without the diagonality assumption (3). Under suitable conditions (specified e.g. in [11]), there exists an initial condition $Z_0 \in L^2(\Omega, \mathcal{V})$ so that the corresponding mild solution

(40) \[ Z(t) = S_\alpha(t)Z_0 + \int_0^t S_\alpha(t-s)\Phi dB^H(s) \]

is a strictly stationary $\mathcal{V}$-valued process, where $(S_\alpha(t), t \geq 0)$ is an analytic semigroup generated by a densely defined, closed operator $\alpha A$. Note that the process $(Z(t), t \geq 0)$ is a centered Gaussian process with covariance operator $Q_\alpha$ and

(41) \[ Q_\alpha = \frac{1}{\alpha^{2H}} Q, \]

where $Q$ is the covariance operator corresponding to the case $\alpha = 1$ (cf. [11]).

Take an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of the space $\mathcal{V}$ and consider the first $N$ projections of $Z(t)$:

$$ Z^{(N)}(t) := \begin{bmatrix} z_1(t) \\ \vdots \\ z_N(t) \end{bmatrix} := \begin{bmatrix} \langle Z(t), e_1 \rangle_{\mathcal{V}} \\ \vdots \\ \langle Z(t), e_N \rangle_{\mathcal{V}} \end{bmatrix}. $$

It follows that $Z^{(N)}(t), t = 1, 2, \ldots$ is a strictly stationary $\mathbb{R}^N$-valued Gaussian process with covariance matrix

(42) \[ Q_\alpha^{(N)} = \mathbb{E} z_i(t) z_j(t)_{i,j=1,\ldots,N} = \begin{bmatrix} \langle Q_\alpha e_i, e_j \rangle_{\mathcal{V}} \end{bmatrix}_{i,j=1,\ldots,N}. \]

Consequently,

(43) \[ Q_\alpha^{(N)} = \frac{1}{\alpha^{2H}} Q^{(N)}, \]

where $Q^{(N)}$ corresponds to the case $\alpha = 1$.

Assume that $Q^{(N)}$ is regular. The reweighing analogous to (15) can be achieved by taking

(44) \[ Y_N := \frac{1}{N} \sum_{t=1}^n \| (Q^{(N)})^{-1/2} \{Z^{(N)}(t)\} \|^2 = \frac{1}{N} \sum_{t=1}^n \| U^{(N)}(t) \|^2, \]

where $U^{(N)}(t)$ is a stationary sequence of centered Gaussian vectors with diagonal covariance matrix

(45) \[ U^{(N)}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix} \sim \mathcal{N} \left( 0, \frac{1}{\alpha^{2H}} \mathbb{I}_N \right), \]

with $\mathbb{I}_N$ being $N \times N$ identity matrix. If follows immediately

$$ \mathbb{E} Y_N = \alpha^{-2H}. $$
Using Isserlis’ theorem for calculation of expectation of the product of four Gaussian random variables leads to

\[ \mathbb{E} \left( Y_N - \alpha^{-2H} \right)^2 = \frac{2}{N^2 n^2} \left[ \sum_{i=1}^{n} \sum_{k=1}^{N} \alpha^{-4H} + \sum_{i,s=1}^{n} \sum_{k,l=1}^{N} (\mathbb{E} u_k(t) u_l(s))^2 \right]. \]

Omitting the second (non-negative) summand on one hand and application of Cauchy-Schwartz inequality on the other hand yields (compare with (26))

\[ (46) \quad 2 \frac{\alpha^4 H}{N n} \leq \mathbb{E} \left( Y_N - \alpha^{-2H} \right)^2 \leq 2 \frac{\alpha^4 H}{N}. \]

Following construction of isonormal Gaussian process for an infinite-dimensional stationary Gaussian sequence in [8] (see Section 4.2. therein), it is possible to consider the variables \( Y_N - \alpha^{-2H} \) as elements of the second Wiener chaos.

**Lemma 5.1.** Let \( F_N, N = 1, 2, \ldots \) be a sequence of random variables from a fixed Wiener chaos of an isonormal Gaussian process and let for some constants \( c > 0 \) and \( \beta > 0 \)

\[ \mathbb{E} F_N^2 \leq \frac{c}{N^\beta} \quad \forall N = 1, 2, \ldots. \]

Then

\[ F_N \xrightarrow{N \to \infty} 0 \text{ a.s..} \]

**Proof.** Consider parameters \( \gamma \) and \( \delta \) so that \( 0 < \gamma < \beta/2 \) and \( \delta > \frac{1}{\beta/2 - \gamma} \). Denote by \( C \) a positive constant (independent of \( N \)), which may change from line to line and calculate

\[ \mathbb{P}(|F_N| > N^{-\gamma}) \leq \frac{\mathbb{E}|F_N|^\delta}{N^{-\gamma \delta}} \leq C \frac{(\mathbb{E} F_N^2)^{\delta/2}}{N^{-\gamma \delta}} \leq C \frac{1}{N^{\delta(\beta/2 - \gamma)}}, \]

where Chebyshev’s inequality and hypercontractivity property on the fixed Wiener chaos (see e.g. [13], Theorem 2.7.2) were used. Application of Borel–Cantelli lemma yields the almost-sure convergence. \( \Box \)

Note that Lemma 5.1 substitutes the SLLN. Although the projections in a fixed time-instant \( u_k(t), k = 1, \ldots, n \) are independent, the whole trajectories \( \{u_k(t) : t = 1, \ldots, n\} \) for \( k = 1, \ldots, N \) need not be independent and SLLN is not applicable. Using this lemma, (46) implies the convergence

\[ (47) \quad Y_N \xrightarrow{N \to \infty} \alpha^{-2H} \text{ a.s..} \]

This motivates the definition of the weighted MC estimator for non-diagonalizable equations

\[ \alpha_N^* := (Y_N)^{-\frac{1}{2\pi}} = \left( \frac{1}{n} \sum_{t=1}^{n} \|(Q^{(N)})^{-1/2} Z^{(N)}(t))\|^2 \right)^{-\frac{1}{2\pi}}. \]

Note that the estimator (46) is a special case of (48) considered in diagonalizable setting.

For verification of asymptotic normality, denote

\[ W_n^{(N)} := \sqrt{N}(Y_N - \alpha^{-2H}) \]
and utilize the upper bound for the $4^{th}$ cumulant presented in \cite{8} (see proof of Lemma 8.1. therein)

\begin{equation}
\kappa_4(W_n^{(N)}) \leq \frac{C}{N^2 n^4} \sum_{n,s'=1}^{n} \sum_{t,t'=1}^{n} \text{Tr}\left(Q_U^{(N)}(s-s')Q_U^{(N)}(t-s)Q_U^{(N)}(t'-t)Q_U^{(N)}(s'-t')\right),
\end{equation}

where $Q_U^{(N)}(t-s) = \left[\mathbb{E}u_i(t)u_j(s)\right]_{i,j=1,...,N}$ and $C$ is a universal constant.

Denote by $\|\cdot\|_{op}$ the operator norm of an $N \times N$ matrix with respect to the Euclidian norm in $\mathbb{R}^N$ (for simplicity, we omit $N$ from notations of norms below).

\begin{lemma}
\|Q_U^{(N)}(t)\|_{op} \leq \frac{1}{\alpha^{2H}}, \text{ for any integer } t.
\end{lemma}

\begin{proof}
Take arbitrary $v \in \mathbb{R}^N$ with $\|v\| = 1$ and denote $w := Q_U^{(N)}(t)v$. Calculate

\begin{align*}
\|Q_U^{(N)}(t)v\| &= \sqrt{\langle Q_U^{(N)}(t)v,v \rangle} = \sqrt{\mathbb{E}\langle U^{(N)}(t),v \rangle\langle U^{(N)}(0),w \rangle} \\
&\leq \left(\mathbb{E}\langle U^{(N)}(t),v \rangle^2 \mathbb{E}\langle U^{(N)}(0),w \rangle^2\right)^{1/4} = \left(\frac{1}{\alpha^{2H}} \|v\|^2 \frac{1}{\alpha^{2H}} \|w\|^2\right)^{1/4} \\
&= \frac{1}{\alpha^{H}} \sqrt{\|Q_U^{(N)}(t)v\|},
\end{align*}

where \cite{15} was applied. In result

\begin{equation}
\sqrt{\|Q_U^{(N)}(t)v\|} \leq \frac{1}{\alpha^H},
\end{equation}

which concludes the proof.
\end{proof}

Continue with

\begin{align*}
\text{Tr}\left(Q_U^{(N)}(s-s')Q_U^{(N)}(t-s)Q_U^{(N)}(t'-t)Q_U^{(N)}(s'-t')\right) \\
&\leq N\|Q_U^{(N)}(s-s')\|_{op} \|Q_U^{(N)}(t-s)\|_{op} \|Q_U^{(N)}(t'-t)\|_{op} \|Q_U^{(N)}(s'-t')\|_{op} \\
&\leq \frac{N}{\alpha^{8H}}.
\end{align*}

Combine this with \cite{14} to obtain

\begin{equation}
\kappa_4(W_n^{(N)}) \leq \frac{C}{N^2 n^4} \frac{n^4 N}{\alpha^{8H}} = \frac{C}{N \alpha^{8H}}.
\end{equation}

This and \cite{16} result in (compare to \cite{23})

\begin{equation}
\kappa_4 \left(\frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}}\right) \leq \frac{c}{N},
\end{equation}

for some constant $c$ independent of $N$. Proposition \cite{5.1} provides (compare to \cite{24})

\begin{equation}
d_{TV} \left(\frac{Y_N - \alpha^{-2H}}{\sqrt{\text{var}(Y_N)}}, U\right) \leq \frac{c}{\sqrt{N}},
\end{equation}

where $U$ is a $\mathcal{N}(0,1)$-distributed random variable.
Theorem 5.1. Consider $\alpha^*_N$ defined in (18). If $Q^{(N)}$ is regular for each $N = 1, 2, \ldots$, this estimator has asymptotic properties stated in Theorem 3.1. Namely, properties (18) – strong consistency in space and (19) – asymptotic normality in space with local Berry-Esseen bounds given in (20) hold true.

Proof. With reference to (47) and (52), same techniques as in the proof of Theorem 3.1 can be applied. □

Remark 5.1. As regards non-stationary solutions, calculations similar to those presented in section 4 with projections onto the orthonormal vectors $e_k$ can be performed. However, $e_k$ are generally not eigenvectors of operators $A$ and $\Phi$, which makes the calculations and the resulting conditions for strong consistency and asymptotic normality (technically) complicated. For sake of simplicity and readability, they are not included in this article.

6. Comparison to other estimators

6.1. Minimum-contrast estimator. Recall the standard (non-weighted) minimum-contrast (MC) estimator, defined in [7] and further studied in [8] and [11]. In diagonalizable case, the estimator can be written as follows:

$$\hat{\alpha} = (\hat{Y}_\infty)^{-1} \frac{1}{H} \left( \frac{1}{\sum_{k=1}^\infty \frac{1}{H} \sum_{t=1}^n x_k^2(t)} \sum_{k=1}^\infty \frac{1}{\sigma_k^2} \theta_k H \Gamma(2H) \right)^{-1/2},$$

for discrete-time observations and similarly for continuous-time observations. If only first $N$ coordinates are available, the modification is straightforward and it verifies:

$$\text{var}(\hat{Y}_N) = \frac{\sum_{k=1}^N \text{var} \left( \frac{1}{H} \sum_{t=1}^n x_k^2(t) \right)}{H^2 \Gamma^2(2H) \left( \sum_{k=1}^N \frac{1}{\sigma_k^2} \theta_k^2 \right)^2}.$$

If space asymptotics is considered ($N \to \infty$), the numerator of $\text{var}(\hat{Y}_N)$ is growing, whereas the denominator converges to a finite sum. In result, $\text{var}(\hat{Y}_N)$ does not converge to zero with $N \to \infty$ and the estimator is not consistent in space. It was shown in [8] that this MC estimator is consistent and asymptotically normal in time (i.e. $n \to \infty$), without assuming diagonality.

By simple reweighing of the coordinates (the weighted MC estimator), the poor space-asymptotic properties of the MC estimator are significantly improved.

6.2. Maximum likelihood estimator. The MLE for diagonalizable parabolic SPDEs was studied in [6] and [3]. Both works consider continuous-time observations, fixed time-window and increasing number of coordinates. The actual formula for the MLE is rather complicated and it can be found in [3], formula (3.8).

If considered in the setting of this paper and assuming $\sigma_k = 1$ for all $k = 1, 2, \ldots$, the MLE is strongly consistent in space ($N \to \infty$) if and only if

$$\sum_{k=1}^\infty \theta_k = \infty.$$
If this holds, the MLE is also asymptotically normal in space with speed of convergence given by

\[
\frac{1}{\sqrt{\sum_{k=1}^{N} \theta_k}}.
\]

If compared with the weighted MC estimator (its speed of convergence is given by the square root of (55), the case \( H < \frac{1}{2} \) is covered only by the weighted MC estimator, in case of \( \frac{1}{2} \leq H < \frac{3}{4} \), both estimators have the same speed of convergence and if \( \frac{3}{4} \leq H < 1 \), the MLE converges faster than the weighted MC estimator.

The implementation of the MLE is rather complicated (for details, see discussion at the end of [3]), in contrast to the simplicity of the weighted MC estimator. On the other hand, better performance of MLE in case of non-stationary solution can be expected, if only few coordinates are observed.

6.3. **Trajectory fitting estimator.** This estimator was first introduced in [9] in finite-dimensional setting and recently applied for continuous projections of the solution to diagonalizable parabolic SPDEs driven by a (cylindrical) Wiener process in [2]. The explicit expression for the TFE can be found in formula (2.10) therein and it does not contain any stochastic integration (integration with respect to a random process).

If considered in the setting of Example [21] (the heat equation on \( d \)-dimensional domain with distributed white noise and Dirichlet boundary condition), the TFE (denote \( \alpha^{(T FE)}_N \)) is strongly consistent in space. Moreover, if \( d \geq 2 \), it is also asymptotically normal in the following sense

\[
\frac{\alpha^{(T FE)}_N - \alpha + a_N}{b_N} \overset{d}{\to} N(0, 1),
\]

where

\[
b_N \asymp \frac{1}{\sqrt{N^{1+\frac{1}{d}}}}, \quad \text{and} \quad a_N \asymp \frac{1}{N^{\frac{1}{2}}},
\]

Note the bias term and its asymptotic behavior \( \frac{a_N}{b_N} \asymp N^{\frac{1}{2}-\frac{1}{d}} \). It diverges if \( d > 2 \).

In contrast, the weighted MC estimator is strongly consistent and asymptotically normal without any restriction on the dimension, it has no bias term and the speed of convergence in this example (assuming \( H = \frac{1}{2} \)) is \( \frac{1}{\sqrt{N^{1+\frac{1}{d}}}} \).

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