Invariant solutions in explicit form of the Boltzmann equation with a source term

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Abstract. This paper is devoted to applications of the group analysis method to the Boltzmann equation with a source function. Exact solutions of the nonlinear kinetic Boltzmann equation with a source function in the case of an isotropic distribution function and Maxwell model of isotropic scattering were constructed. An equivalence Lie group is used for the construction. One of the transformations of the equivalence Lie group uniquely singles out a class of source functions which allows us to find invariant solutions of the Bobylev-Krook-Wu type in an explicit form. In particular, some of these solutions have a meaningful physical interpretation.

1. Introduction
The classical Boltzmann equation of the gas kinetic theory from mathematical point of view is a homogeneous integro-differential equation [1]. However, in order to model some physical situations one needs to include additional source terms into the classical equation, in particular, for describing presence of an external source (drain) under initiation of high threshold process by injection of ‘hot’ particles or ‘running out’ high energetic molecules from a force field, removal events [2, 3, 4] such as nonelastic interactions with a background host medium, and others. In the present paper the Boltzmann equation with a source term depending on the independent and dependent variables is consider.

The first attempt of construction of invariant solutions [5] of the Boltzmann equation with a source by the group analysis method was in [6], where similar to [7] the Boltzmann equation was reduced to a nonlinear differential equation for a moment generating function. Complete group classification of this equation with respect to a source function was presented in [8]. However, a reduction of the obtained invariant solutions into solutions of the original Boltzmann equation is an extremely complicated problem.

Therefore, it is more perspective to consider a Fourier image of the Boltzmann equation for an isotropic distribution function for which the well-known invariant Bobylev-Krook-Wu (BKW) solution [9] was found. For this goal a variant of the preliminary group classification method was developed in [10, 11]. An idea of a preliminary group classification method was given in [12], and its further developing is in [13, 14]. An equivalence Lie group and its nonequivalent subgroups, in particular, elements of an optimal system of subgroups, are applied for a preliminary group classification. It is obvious that for the classification with respect to a source function, one can
use a Lie group and its optimal system of subgroups admitted by the homogeneous equation as an equivalence Lie group of the nonhomogeneous equation.

In [10], a group classification with respect to a source function which does not depend on a solution of the kinetic equation was studied. Directly solving the determining equation it was shown that the admitted infinitesimal generators of the nonhomogeneous equation are linear combinations of the generators of a complete admitted Lie algebra $L_4$ of the homogeneous equation which is obtained earlier in [15]. This result was expected, but its obtaining by a direct solving of the determining equation allowed stating that the obtained group classification is complete.

A more general and interesting from physics point of view case of source functions depending not only on independent but also a dependent variable was considered in [11]. Instead of solving a complicated determining equation, an admitted Lie group of a homogeneous equation was applied for a preliminary group classification of the nonhomogeneous equation. In this case the problem of completeness of the group classification is still an open problem. In the present paper it is shown that for equations with a source function depending on a solution of the kinetic equation it is not sufficient to exploit the generators used in [11]. An extension of the equivalence Lie group corresponding to $L_4$ allows obtaining invariant solutions of the Boltzmann equation with a source term depending on a solution in an explicit form.

2. The Boltzmann equation in isotropic case

It is considered the space homogeneous Boltzmann equation with a source term and Maxwell isotropic model of scattering for a distribution function $f(v, t)$ which is isotropic in the molecular velocity space. Its Fourier image in the velocity space $R^3(v)$ has the form [9]:

$$\frac{\partial \varphi(x, t)}{\partial t} + \varphi(x, t)\varphi(0, t) = \int_0^1 \varphi(xs, t)\varphi(x(1 - s), t) \, ds + G(\varphi(x, t), t, x). \tag{1}$$

Here the isotropic distribution function and its Fourier image are defined by the transformations:

$$f(v, t) = (2\pi)^{-3} \int dk \exp(ikv)\varphi(k, t), \quad \varphi(x, t) \equiv \varphi(k^2/2, t) = \int dv \exp(-ikv)f(v, t).$$

Respectively,

$$G(\varphi(x, t), t, x) = \int dv \exp(-ikv)\tilde{G}(f(v, t), t, v)$$

is the Fourier image of the source function in the velocity space:

$$v \in R^3(v), \quad k \in R^3(k), \quad v = |v|, k = |k|.$$ 

A Cauchy problem (relaxation) is formulated for equation (1):

$$\varphi(x, 0) = \varphi_0(x). \tag{2}$$

The Cauchy problem (1), (2) determines evolutionary laws of particle number density

$$n(t) \equiv \varphi(0, t) = \int dv f(v, t)$$

and energy

$$n(t)e(t) = \frac{\partial \varphi}{\partial x}(0, t),$$

which are written in the form

$$\frac{\partial \varphi(0, t)}{\partial t} = G(\varphi(0, t), t, 0), \tag{3}$$
\[
\frac{\partial \varphi}{\partial t}(0,t)e(t) + \frac{\partial e(t)}{\partial t}\varphi(0,t) = \frac{\partial G}{\partial x}(\varphi(0,t),t,0) + \frac{\partial G}{\partial \varphi}(\varphi(0,t),t,0).
\]

(4)

In the homogeneous case with \( G \equiv 0 \) the problem (1), (2) has the following conservation laws of the particle density and energy [9]:

\[
\varphi(0,t) = 1, \quad \frac{\partial \varphi}{\partial x}(0,t) = -1.
\]

(5)

In [15], it was shown that the complete Lie algebra \( L_4 \), admitted by the homogeneous equation (1), is defined by the generators

\[
X_0 = x\partial_x, \quad X_1 = x\varphi\partial_\varphi, \quad X_2 = \varphi\partial_\varphi - t\partial_t, \quad X_3 = \partial_t.
\]

The invariant BKW-solution [9] has the form

\[
\varphi_B(x,t) = (1 - y)\exp(y - x), \quad y = x \exp(\lambda t), \quad \lambda = -1/6, \quad 0 \leq \theta \leq 2/5.
\]

(6)

This solution is an invariant solution with respect to the generator \( X_0 + \lambda^{-1}X_3 \) [15].

3. Invariant solutions of the BKW-type

For the preliminary group classification of equation (1) in [11], admitted generators were sought in the form

\[
X = c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3.
\]

One can note that in all presented their cases an invariant solution corresponding to the BKW-type depends on the self-similar variable \( y = x \exp(-\lambda t) \) which is defined by the generator \( X_0 + \lambda^{-1}X_3 \):

\[
\varphi(x,t) = \Psi(y).
\]

This dependence contradicts to the presence in equation (1) a nonzero source function. However, the author of [2], assuming a representation of a solution similar to [16], found a generalized BKW-solution in an explicit form. In this case a source function in Fourier image had the form

\[
G = -C_R\varphi(0,t)\varphi(t,x).
\]

(7)

The principal difference of this source from the studies \( G(\varphi(x,t),t,x) \) in [17] is in the dependence of the functional:

\[
\varphi(0,t) = \int dv f(v,t).
\]

It brings an additional nonlocalness in a classifying equation. This nonlocalness could not be taken into account by only applying the generators of \( L_4 \) used as an equivalence Lie group in [17].

In [18], a transformation of time and a distribution function was presented. This transformation allowed mapping a nonhomogeneous equation (1) into homogeneous for which one can derive a BKW-solution in new variables, and thus, to avoid cumbersome computations [2]. This transformation has the form

\[
\tilde{t} = \tau(t), \quad \tilde{x} = x, \quad \tilde{\varphi} = p(t)\varphi.
\]

(8)

One can consider a more general form of the source term in equation (1)

\[
G = G(\varphi(0,t),\varphi(x,t),t,x).
\]
In [17], several ways of obtaining conditions for the transformation (8) to be an equivalence transformation are given. These conditions are

\[ p(t)\tau'(t) = 1, \]

and

\[ G(\varphi(0,t), \varphi(x,t), t, x) = \alpha(\varphi(0,t), t)\varphi(x,t). \] (9)

In terms of particle number density \( n(t) = \varphi(0,t) \) the conditions for the equivalence transformations (8) have the form

\[ p'(t) + p(t)\alpha(t, n(t)) = 0, \quad \tau' = p^{-1}(t), \quad n'(t) = \alpha(t, n(t))n(t). \]

From the evolution law (3) one derives that \( \alpha(t, n(t)) = n'(t)/n(t) \). Finally, one obtains

\[ p = \frac{n_0}{n}, \quad \tau' = \frac{n}{n_0}, \quad \tau(t) = \int_0^t \frac{n(s)}{n_0} ds, \] (10)

where \( n_0 \) is an arbitrary constant.

Moreover, from the energy evolution law (4) one finds for a linear source (9) that

\[ \frac{\partial \varphi(0,t)}{\partial t} e(t) + \frac{\partial e(t)}{\partial t} \varphi(0,t) = \alpha(\varphi(0,t), t)e(t)\varphi(0,t). \]

Hence, from (3) one gets that the average energy (temperature) of particles is conserved in this case

\[ e(t) = e_0 = \text{const}. \]

Thus, the transformation (8) with (10) is an equivalence transformation of equation (1) with a source (9) which maps it into an equation of the same form with the zero right hand side:

\[ \tilde{\varphi}_t(x, \tau(t)) + n_0\tilde{\varphi}(x, \tau(t)) - \int_0^1 \tilde{\varphi}(x, s, \tau(t))\tilde{\varphi}(x(1-s), \tau(t)) ds = 0. \] (11)

Choosing the constants according to the conservation laws (5):

\[ n_0 = 1, \quad e_0 = -1, \]

one is able to rewrite the BKW-solution (6) of equation (11) in the transformed variables, and then return to the original variables. By this way one can construct generalized BKW-solutions for a class of sources (9).

According to the results of [15], the BKW-solution is determined by the generator

\[ x\partial_x + \lambda^{-1}(p\partial_t - p'\varphi \partial_\varphi), \]

and a corresponding representation of a generalized invariant BKW-solution has the form

\[ \varphi(x,t) = p^{-1}(t)\tau'(t)F(xe^{\lambda\tau(t)}). \]

Substitution of this representation into equation (1) with a source function from the class (9) leads to the reduced equation for the function \( F \) with the zero right hand side which gives a sought solution. Several examples of generalized BKW-solutions in an explicit form for some source functions of the class (9):

\[ G_1(v,t) = -C_R n(t)f(v,t), \quad G_2(v,t) = -C_v n^2(t)f(v,t), \quad G_3(v,t) = -C_v f(v,t), \]

which model some physical phenomena are presented in [17]. One can study a source function of a more general form, for example,

\[ G_k(v,t) = -(C_1 + C_2 n(t))^k f(v,t). \]

In particular, this form allows obtaining a combination of the presented above source functions.
4. Conclusion

Using the Lie algebra $L_4$ admitted by the Boltzmann equation in the case of an isotropic distribution function and a Maxwellian isotropic interaction potential an equivalence Lie group of equation with a source term is constructed. A class of source functions which vanish by the equivalence transformations is uniquely singled out. This allows finding generalized BKW-solutions in an explicit form.

At the same time there is a problem of further extension of equivalence Lie group. In particular, by transformations which can allow obtaining generalized BKW-solutions for other classes of source functions. A proof of absence of such transformations can be its alternative.

Acknowledgments

SVM and YuNG acknowledge RFFI (code of the project 17-01-00209a) for a partial financial support. AK thanks Walailak University, and Royal Thai Government Scholarship through the Ministry of Science and Technology of Thailand for financial support.

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