Three affine SL(2, 8)-unitals

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Abstract
SL(2, q)-unitals are unitals of order q admitting a regular action of SL(2, q) on the complement of some block. We introduce three non-classical affine SL(2, 8)-unitals and their full automorphism groups. Each of those three affine unitals can be completed to at least two non-isomorphic unital, leading to six pairwise non-isomorphic unitals of order 8.

Keywords Design · Unital · Affine unital · Non-classical unital · Automorphism

Mathematics Subject Classification 51E26 · 51A10 · 05B30

1 Preliminaries

One strategy to construct projective planes is to build an affine plane first and then to add points at infinity, namely a new point for each parallel class and a line containing all these new points. This strategy of constructing an affine part of a geometry first and then completing it by adding some objects at infinity can successfully be applied to other incidence structures than affine and projective planes. We apply such an approach to unital.

A unital of order n is a 2-(n^3 + 1, n + 1, 1) design, i.e. an incidence structure with n^3 + 1 points, n + 1 points on each block and unique joining blocks for any two points. We consider affine unitals, which arise from unitals by removing one block (and all the points on it) and can be completed to unitals via a parallelism on the short blocks. We give an axiomatic description:

Definition 1.1 Let n ∈ N, n ≥ 2. An incidence structure U = (P, B, I) is called an affine unital of order n if:

Most of the results in the present paper have been obtained in the author’s Ph.D. thesis Möhler (2020c), where detailed arguments can be found for some statements that we leave to the reader here.

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(AU1) There are $n^3 - n$ points.

(AU2) Each block is incident with either $n$ or $n + 1$ points. The blocks incident with $n$ points will be called **short blocks** and the blocks incident with $n + 1$ points will be called **long blocks**.

(AU3) Each point is incident with $n^2$ blocks.

(AU4) For any two points there is exactly one block incident with both of them.

(AU5) There exists a **parallelism** on the short blocks, meaning a partition of the set of all short blocks into $n + 1$ parallel classes of size $n^2 - 1$ such that the blocks of each parallel class are pairwise non-intersecting.

The existence of a parallelism as in (AU5) must explicitly be required (see Möhler 2020c, Example 3.10). An affine unital $U$ of order $n$ with parallelism $\pi$ can be completed to a unital $U^{\pi}$ of order $n$ as follows: For each parallel class, add a new point that is incident with each short block of that class. Then add a single new block $[\infty]^{\pi}$, incident with the $n + 1$ new points (see Möhler 2020c, Proposition 3.9). We call $U^{\pi}$ the $\pi$-**closure** of $U$. Note that the closure depends on the parallelism, which need not be unique. Given an affine unital $U$ with parallelisms $\pi$ and $\pi'$, the closures $U^{\pi}$ and $U^{\pi'}$ are isomorphic with $[\infty]^{\pi} \mapsto [\infty]^{\pi'}$ exactly if there is an automorphism of $U$ which maps $\pi$ to $\pi'$ (see Möhler 2020c, Proposition 3.12).

### 2 Affine $SL(2, q)$-unitals

From now on let $p$ be a prime and $q := p^e$ a $p$-power. We are interested in a special kind of affine unitals, namely affine $SL(2, q)$-unitals. The construction of those affine unitals is due to Grundhöfer et al. (2016). They consider translations of unitals, i.e. automorphisms fixing each block through a given point (the so-called center). Of special interest are unitals of order $q$ where two points are centers of translation groups of order $q$. In the classical (Hermitian) unital of order $q$, any two such translation groups generate a group isomorphic to $SL(2, q)$; see Grundhöfer et al. (2020), Main Theorem for further possibilities. The construction of (affine) $SL(2, q)$-unitals is motivated by this action of $SL(2, q)$ on the classical unital.

Let $S \leq SL(2, q)$ be a subgroup of order $q + 1$ and let $T \leq SL(2, q)$ be a Sylow $p$-subgroup. Recall that $T$ has order $q$ (and thus trivial intersection with $S$), that any two conjugates $T^h := h^{-1}Th$, $h \in SL(2, q)$, have trivial intersection unless they coincide and that there are $q + 1$ conjugates of $T$.

Consider a collection $\mathcal{D}$ of subsets of $SL(2, q)$ such that each set $D \in \mathcal{D}$ contains $\mathbb{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that $\#D = q + 1$ for each $D \in \mathcal{D}$, and the following properties hold:

(Q) For each $D \in \mathcal{D}$, the map

$$(D \times D) \setminus \{(x, x) \mid x \in D\} \rightarrow SL(2, q), \quad (x, y) \mapsto xy^{-1},$$

is injective, i.e. the set $D^* := \{xy^{-1} \mid x, y \in D, \ x \neq y\}$ contains $q(q + 1)$ elements.

(P) The system consisting of $S \setminus \{\mathbb{I}\}$, all conjugates of $T \setminus \{\mathbb{I}\}$ and all sets $D^*$ with $D \in \mathcal{D}$ forms a partition of $SL(2, q) \setminus \{\mathbb{I}\}$.

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Set

\[ P := \text{SL}(2, q), \]
\[ \mathcal{B} := \{ Sg \mid g \in \text{SL}(2, q) \} \cup \{ T^h g \mid h, g \in \text{SL}(2, q) \} \cup \{ Dg \mid D \in \mathcal{D}, g \in \text{SL}(2, q) \} \]

and let the incidence relation \( I \subseteq P \times \mathcal{B} \) be containment.

Then we call the incidence structure \( \bigcup \mathcal{S}_D := (P, \mathcal{B}, I) \) an affine \( \text{SL}(2, q) \)-unital. Each affine \( \text{SL}(2, q) \)-unital is indeed an affine unital of order \( q \), see Möhler (2020c), Prop. 3.15. We call the sets \( \hat{D} := \{ Dd - 1 \mid d \in D \} \), \( D \in \mathcal{D} \), the hats of \( \bigcup \mathcal{S}_D \) and the blocks \( Dg \), \( D \in \mathcal{D} \) and \( g \in \text{SL}(2, q) \), the arcuate blocks of \( \bigcup \mathcal{S}_D \).

For the construction of an affine \( \text{SL}(2, q) \)-unital, we have to choose a subgroup \( S \leq \text{SL}(2, q) \) of order \( q + 1 \) and find a set \( \mathcal{D} \) of arcuate blocks through \( 1 \) such that (Q) and (P) hold.

**Example 2.1**

(a) For each prime power \( q \) we may choose \( S = C \) to be cyclic and \( \mathcal{H} \) a set of arcuate blocks through \( 1 \) such that \( \bigcup \mathcal{S}_C, \mathcal{H} \) is isomorphic to the affine part of the classical unital. We call \( \bigcup \mathcal{S}_C, \mathcal{H} \) the classical affine \( \text{SL}(2, q) \)-unital.

See Grundhöfer et al. (2016), Example 3.1 or Möhler (2020c), Section 3.2.2 for details.

(b) Grundhöfer et al. (2016) introduce a non-classical affine \( \text{SL}(2, q) \)-unital.

**Proposition 2.2**

Let \( p = 2 \) and let \( S \leq \text{SL}(2, q) \) be a subgroup of order \( q + 1 \). Then \( S \) is cyclic and unique up to conjugation.

**Proof**

For \( p = 2 \), we have \( \text{SL}(2, q) \cong \text{PSL}(2, q) \). Using Dickson’s list of subgroups of \( \text{PSL}(2, q) \) (see e.g. Huppert 1967, Hauptsatz II.8.27), we see that each subgroup of order \( q + 1 \) is cyclic. From Huppert (1967), Satz II.8.5, we get that there is exactly one conjugacy class of cyclic subgroups of \( \text{PSL}(2, q) \) of order \( q + 1 \).

**Remark 2.3**

Möhler (2020c), Proposition 2.5 gives a complete list of possible subgroups \( S \leq \text{SL}(2, q) \) of order \( q + 1 \). For \( p \not\equiv 3 \mod 4 \), the group \( S \) is cyclic. For \( p \equiv 3 \mod 4 \), \( S \) is cyclic or generalized quaternion and there is one exceptional case for \( q = 23 \) and one for \( q = 47 \).

For each prime power \( q \), we may choose a cyclic subgroup \( C \leq \text{SL}(2, q) \) of order \( q + 1 \) as given in the following

**Remark 2.4**

Let \( a \in \mathbb{F}_q^\times \) such that \( X^2 - tX + d \) has no root in \( \mathbb{F}_q \), where \( t = 1 \) if \( q \) is even and \( t = 0 \) if \( q \) is odd. Then

\[ C := \left\{ \left( \begin{array}{cc} a & b \\ -db & a+tb \end{array} \right) \mid a^2 + tab + db^2 = 1 \right\} \]

is a cyclic subgroup of \( \text{SL}(2, q) \) of order \( q + 1 \). Note that \( C \) is the norm 1 group of the quadratic extension field

\[ \mathbb{F}_{q^2} := \left\{ \left( \begin{array}{cc} a & b \\ -db & a+tb \end{array} \right) \mid a, b \in \mathbb{F}_q \right\} . \]
We take a brief look on automorphisms of affine $\text{SL}(2, q)$-unitals, i.e. bijections of the point set such that the block set is invariant. On any affine $\text{SL}(2, q)$-unital $U_{S, \mathcal{D}}$, right multiplication with elements of $\text{SL}(2, q)$ obviously induces automorphisms. Let $R := \{ \rho_h \mid h \in \text{SL}(2, q) \} \leq \text{Aut}(U_{S, \mathcal{D}})$, where $\rho_h \in R$ acts on $U_{S, \mathcal{D}}$ by right multiplication with $h \in \text{SL}(2, q)$. Every automorphism of $\text{SL}(2, q)$ obviously induces a bijection of the point set of $U_{S, \mathcal{D}}$, but it need not leave the block set invariant. Let $\mathfrak{A}$ denote the permutation group given by all automorphisms of $\text{SL}(2, q)$.

We import a useful statement from Möhler (2020a):

**Theorem 2.5** (Möhler (2020a), Theorem 3.3) Let $q \geq 3$ and let $U_{S, \mathcal{D}}$ and $U_{S', \mathcal{D}'}$ be affine $\text{SL}(2, q)$-unitals.

(a) Let $\psi : U_{S, \mathcal{D}} \to U_{S', \mathcal{D}'}$ be an isomorphism. Then $\psi = \alpha \rho_h$ with $\rho_h \in R$ and $S \cdot \alpha = S'$.

(b) $\text{Aut}(U_{S, \mathcal{D}}) \leq \mathfrak{A}_S \rtimes R$. □

**Remark 2.6** The classical affine $\text{SL}(2, q)$-unital $U_{\mathcal{C}, \mathcal{H}}$ admits the whole group $\mathfrak{A}_C \rtimes R$ as automorphism group (see Möhler 2020c, Proposition 4.6). Hence, $\mathfrak{A}_S \rtimes R$ is a sharp upper bound for the automorphism group of any affine $\text{SL}(2, q)$-unital $U_{S, \mathcal{D}}$ of order $q \geq 3$.

### 3 Three affine $\text{SL}(2, 8)$-unitals

Let $q = 8$ and $\mathbb{F}_8^\times = \langle z \rangle$, with $z^3 = z + 1$. The polynomial $X^2 + X + 1$ has no root in $\mathbb{F}_8$ and the Frobenius automorphism

$$\varphi : \mathbb{F}_8 \to \mathbb{F}_8, \quad x \mapsto x^2,$$

has order 3. Since $q = 8$ is even, any subgroup $S \leq \text{SL}(2, 8)$ of order 9 is cyclic and we may hence choose

$$S := C = \left\{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \mid a, b \in \mathbb{F}_8, \quad a^2 + ab + b^2 = 1 \right\}.$$

A generator of $C$ is given by $g := \begin{pmatrix} z^2 & z^4 \\ z^4 & z \end{pmatrix}$. Let $f := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathfrak{A}_C = \text{Aut}(\text{SL}(2, 8))_C = \langle \gamma_g \rangle \rtimes \langle \gamma_f \cdot \varphi \rangle \cong C_9 \rtimes C_6,$$

where $\varphi$ acts entrywise on a matrix and $\gamma_x$ describes conjugation with $x$. Representatives of the conjugacy classes of minimal subgroups of $\mathfrak{A}_C$ are

$$F := \langle \gamma_f \rangle \cong \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cong C_2,$$

$$U := \langle \gamma_g \rangle \cong \langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cong C_3$$

and

$$L := \langle \varphi \rangle \cong C_3.$$
The Weihnachtsunital was discovered around Christmas 2017, whence the name. and the full automorphism group

\[
H_1 := \{ 1, \left( \begin{array}{cc}
z^5 & \frac{z^4}{z^2} \\
z^2 & z^5 
\end{array} \right), \left( \begin{array}{cc}
0 & \frac{z}{z^3} \\
z^3 & z^6 
\end{array} \right), \left( \begin{array}{cc}
z^2 & \frac{z}{z^6} \\
z^6 & z^3 
\end{array} \right), \left( \begin{array}{cc}
1 & \frac{z^2}{z} \\
z & z^4 
\end{array} \right), \left( \begin{array}{cc}
z^4 & \frac{z^3}{z} \\
z^3 & z 
\end{array} \right), \left( \begin{array}{cc}
z & \frac{z^5}{z^2} \\
z^{5} & 0 
\end{array} \right) \},
\]

\[
H_2 := H_1 \cdot \varphi, H_3 := H_1 \cdot \varphi^2,
\]

\[
H_4 := \{ 1, \left( \begin{array}{cc}
\frac{z^5}{z^6} & z^2 \\
z^6 & \frac{z^4}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
\frac{z}{z^6} & \frac{z^5}{z^4} \\
z^5 & \frac{z^3}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
0 & \frac{z}{z^2} \\
z^2 & \frac{z^5}{z^4} 
\end{array} \right), \left( \begin{array}{cc}
z^2 & \frac{z^3}{z} \\
z^3 & \frac{z^5}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
z^4 & \frac{z^3}{z} \\
z^3 & \frac{z^5}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
z^5 & \frac{z^2}{z} \\
z^2 & \frac{z^5}{z^4} 
\end{array} \right) \},
\]

\[
H_5 := H_4 \cdot \varphi, H_6 := H_4 \cdot \varphi^2
\]

and \( \mathcal{H} := \{ H_1, \ldots, H_6 \} \). Then \( U_{C, \mathcal{H}} \) is the classical affine unital of order 8. Recall that for \( H \in \mathcal{H} \), we denote by \( \bar{H} \) the set of arcuate blocks \( \{ Hh^{-1} \mid h \in H \} \). As indicated, \( \varphi \) acts on the set of hats \( \{ \bar{H} \mid H \in \mathcal{H} \} \) in two orbits of length 3. Conjugation by \( g \) stabilizes each \( \bar{H} \) and acts transitively on the blocks of each \( \bar{H} \). Conjugation by \( f \) also stabilizes each \( \bar{H} \) but fixes exactly one block per \( \bar{H} \).

**Theorem 3.2** (Weihnachtsunital) Let \( C := \langle g \rangle = \langle \left( \begin{array}{cc}
z^4 & \frac{z^3}{z} \\
z^3 & \frac{z^5}{z^2} 
\end{array} \right) \rangle \) as above and let

\[
D_1 := \left\{ 1, \left( \begin{array}{cc}
z^5 & \frac{z^4}{z^2} \\
z^2 & z^5 
\end{array} \right), \left( \begin{array}{cc}
0 & \frac{z}{z^3} \\
z^3 & z^6 
\end{array} \right), \left( \begin{array}{cc}
z^2 & \frac{z}{z^6} \\
z^6 & z^3 
\end{array} \right), \left( \begin{array}{cc}
1 & \frac{z^2}{z} \\
z & z^4 
\end{array} \right), \left( \begin{array}{cc}
z^4 & \frac{z^3}{z} \\
z^3 & z 
\end{array} \right), \left( \begin{array}{cc}
z & \frac{z^5}{z^2} \\
z^{5} & 0 
\end{array} \right) \},
\]

\[
D_2 := D_1 \cdot \varphi, D_3 := D_1 \cdot \varphi^2,
\]

\[
D_4 := \left\{ 1, \left( \begin{array}{cc}
z^6 & \frac{z^5}{z^4} \\
z^5 & z^3 
\end{array} \right), \left( \begin{array}{cc}
\frac{z}{z^6} & \frac{z^5}{z^4} \\
z^5 & \frac{z^3}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
0 & \frac{z}{z} \\
z & \frac{z^2}{z} 
\end{array} \right), \left( \begin{array}{cc}
z^3 & \frac{z}{z^6} \\
z^6 & \frac{z^5}{z^4} 
\end{array} \right), \left( \begin{array}{cc}
z^4 & \frac{z^3}{z} \\
z^3 & \frac{z^5}{z^2} 
\end{array} \right), \left( \begin{array}{cc}
z^5 & \frac{z^2}{z} \\
z^2 & \frac{z^5}{z^4} 
\end{array} \right) \},
\]

\[
D_5 := D_4 \cdot \varphi, D_6 := D_4 \cdot \varphi^2
\]

and \( \mathcal{D} := \{ D_1, \ldots, D_6 \} \). Then \( \mathbb{W} := U_{C, \mathcal{D}} \) is an affine \( SL(2, 8) \)-unital and we call it Weihnachtsunital.\(^1\) The stabilizer of \( 1 \) in \( Aut(\mathbb{W}) \) is

\[
Aut(\mathbb{W})_1 = U \rtimes (F \times L) = \langle \gamma g^3 \rangle \rtimes \langle \gamma_f \cdot \varphi \rangle \cong C_3 \rtimes C_6
\]

and the full automorphism group

\[
Aut(\mathbb{W}) = Aut(\mathbb{W})_1 \rtimes R
\]

has index 3 in \( Aut(U_{C, \mathcal{H}}) = \mathfrak{A}_C \rtimes R \).

**Proof** The proof is basically computation (recall Theorem 2.5). Note that the given description already uses the automorphism \( \varphi \in Aut(\mathbb{W})_1 \). Conjugation by \( f \) stabilizes each hat with exactly one fixed block per hat. Conjugation by the generator \( g \) of \( C \) does not induce an automorphism of \( \mathbb{W} \), but conjugation by \( g^3 \) yields an automorphism of \( \mathbb{W} \) such that each hat is fixed. \( \square \)

Having computed the full automorphism group of \( Aut(\mathbb{W}) \), we know in particular that the Weihnachtsunital is not isomorphic to the classical affine \( SL(2, 8) \)-unital \( U_{C, \mathcal{H}} \). Another way to see that \( \mathbb{W} \) is not isomorphic to \( U_{C, \mathcal{H}} \) is via O’Nan configurations. An O’Nan configuration consists of four distinct blocks meeting in six distinct points:

\(^1\) The Weihnachtsunital was discovered around Christmas 2017, whence the name.
O’Nan observed that classical unitals do not contain such configurations (see O’Nan 1972, 507).

**Remark 3.3** In $\mathbb{W}U$, there are lots of O’Nan configurations, e.g.

$$C = \{\mathbb{1}, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\},$$

$$T := \left\{ \begin{array}{c}
\mathbb{1}, \left( \begin{array}{cc} 1 & 1 \\
1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & z^2 \\
1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & z^3 \\
1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & z^4 \\
1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & z^5 \\
1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & z^6 \\
1 & 0 \end{array} \right) \end{array} \right\},$$

$$D_2 \cdot \left( \begin{array}{c} z^4 \\
1 \end{array} \right) = \left\{ \left( \begin{array}{c} z^4 \\
1 \end{array} \right), \left( \begin{array}{c} 0 \ z^3 \\
1 \end{array} \right), \left( \begin{array}{c} 1 \ z^2 \\
1 \end{array} \right), \left( \begin{array}{c} z^2 \ z^3 \\
1 \end{array} \right), \left( \begin{array}{c} \ z \ 1 \ z^4 \\
1 \end{array} \right), \left( \begin{array}{c} 0 \ 1 \ z^5 \\
1 \end{array} \right), \left( \begin{array}{c} z^4 \ z^3 \\
1 \end{array} \right), \left( \begin{array}{c} z^3 \ 0 \ z^5 \\
1 \end{array} \right) \right\}$$

$$D_3 \cdot \left( \begin{array}{c} z \ z^3 \\
0 \ z^6 \end{array} \right) = \left\{ \left( \begin{array}{c} z \ z^3 \\
0 \ z^6 \end{array} \right), \left( \begin{array}{c} 1 \ 1 \ 0 \\
1 \ 1 \end{array} \right), \left( \begin{array}{c} z \ z^3 \ z \ z^4 \\
0 \ z^6 \end{array} \right), \left( \begin{array}{c} \ z \ z^3 \ z^4 \\
0 \ z^6 \end{array} \right), \left( \begin{array}{c} z \ z^3 \ z^5 \\
z \ z^5 \end{array} \right), \left( \begin{array}{c} z \ z^3 \ z^5 \\
0 \ z^6 \end{array} \right), \left( \begin{array}{c} z \ z^3 \ z^5 \\
0 \ z^6 \end{array} \right), \left( \begin{array}{c} z \ z^3 \ z^5 \\
0 \ z^6 \end{array} \right) \right\}.$$

**Theorem 3.4** (Osterunital and Pfingstunital\(^2\)) Let $C := \langle g \rangle = \langle \left( \begin{array}{cc} z^2 \ z^4 \\
1 \ z^5 \end{array} \right) \rangle$ as above.

\(^2\) The Osterunital and the Pfingstunital were discovered in 2018, you might guess the dates.
(a) Let
\[
D_1 := \left\{ I, \left( z^5 z^5, z^6 \right), \left( z^4 z^4, z^2 \right), \left( 1, z^5 \right), \left( 0, z^6 \right), \left( 1, z^2 \right), \left( z^3 z^3, 1 \right), \left( z^5 z^4, z^4 \right), \left( z^2 0, z^5 \right) \right\},
\]
\[
D_2 := D_1^g, \quad D_3 := D_1^g^2,
\]
\[
D_4 := \left\{ I, \left( z^5 0, z^2 \right), \left( z^5 z^2, z^5 \right), \left( z^3 z^4, z^5 \right), \left( 1, z^1 \right), \left( 1, z^1 \right), \left( z^2 z^2, z^5 \right), \left( z^0 z, z^6 \right) \right\},
\]
\[
D_5 := D_4^g, \quad D_6 := D_4^g^2
\]

and \( \mathcal{D} := \{ D_1, \ldots, D_6 \} \). Then \( \mathcal{OU} := \mathbb{U}_{C, \mathcal{D}} \) is an affine SL(2, 8)-unital and we call it Osterunital.

(b) Let \( f = \left( \begin{array}{ll} 0 & 1 \\ 1 & 0 \end{array} \right) \) as above and let
\[
D_1^f := D_1, \quad D_2^f := D_2, \quad D_3^f := D_3,
\]
\[
D_4^f := D_4^g, \quad D_5^f := (D_4^g)^g, \quad D_6^f := (D_4^g)^g^2
\]

and \( \mathcal{D}' := \{ D_1^f, \ldots, D_6^f \} \). Then \( \mathcal{PU} := \mathbb{U}_{C, \mathcal{D}'} \) is an affine SL(2, 8)-unital and we call it Pfingstunital.

We denote by \( C \) also the automorphism group \( C := \langle \gamma_8 \rangle \leq \mathbb{A}_C \). The full stabilizers of \( I \) in \( \text{Aut}(\mathcal{OU}) \) and \( \text{Aut}(\mathcal{PU}) \), respectively, are

\[
\text{Aut}(\mathcal{OU})_I = \text{Aut}(\mathcal{PU})_I = C \rtimes L = (\gamma_8) \rtimes (\varphi) \cong C_9 \rtimes C_3
\]

and the full automorphism groups

\[
\text{Aut}(\mathcal{OU}) = \text{Aut}(\mathcal{PU}) = (C \rtimes L) \rtimes R
\]

have index 2 in \( \text{Aut}(\mathbb{U}_{C, \mathcal{D}}) \).

**Proof** Again this is basically computation. The given description already uses the automorphism \( \gamma_8 \) in both \( \text{Aut}(\mathcal{OU})_I \) and \( \text{Aut}(\mathcal{PU})_I \). The Frobenius automorphism \( \varphi \) acts as automorphism on \( \mathcal{OU} \) as well as on \( \mathcal{PU} \) in the same way as it does on \( \mathbb{U}_{C, \mathcal{D}} \) and on \( \mathbb{WU} \). The orbits of \( \varphi \) in \( \mathcal{D} \) are \( \{ D_1, D_2, D_3 \} \) and \( \{ D_4, D_5, D_6 \} \) and its orbits in \( \mathcal{D}' \) are \( \{ D_1^f, D_2^f, D_3^f \} \) and \( \{ D_4^f, D_5^f, D_6^f \} \). Conjugation by \( f \) induces no automorphism on neither \( \mathcal{OU} \) nor \( \mathcal{PU} \).

**Remark 3.5** Other than in the Weihnachtsunital, there is a difference between the action of \( \text{Aut}(\mathcal{OU})_I = \text{Aut}(\mathcal{PU})_I \leq \mathbb{A}_C \) on the set of hats of the Oster- and Pfingstunital, respectively, and its action on the set of hats of the classical affine SL(2, 8)-unital \( \mathbb{U}_{C, \mathcal{D}} \). In \( \mathbb{U}_{C, \mathcal{D}} \), conjugation by \( g \) fixes every hat, while on \( \mathcal{OU} \) and \( \mathcal{PU} \) it acts on the set of hats in two orbits of length 3.
Remark 3.6 As in the Weihnachtsunital, there are also many O’Nan configurations in $\mathbb{O}U$ and $\mathbb{P}U$, e.g.

\[
C = \{1, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\},
\]

\[
D_1 = \left\{ \begin{array}{c}
1, \\
(z^5, z^5), \\
(z^4, z^2), \\
(z^3, z^1), \\
(z^2, 1), \\
(z^1, 0), \\
(z^0, z^2), \\
(z^1, 0), \\
(z^2, 1)
\end{array} \right\},
\]

\[
D_2 \cdot g = \left\{ g, \left( \begin{array}{c}
z^7, z^5 \\
z^3, z^1 \\
z^6, z^2 \\
z^4, z^3 \\
z^5, z^4 \\
z^6, z^5 \\
z^4, z^1 \\
z^3, z^2 \\
z^5, z^3
\end{array} \right), \left( \begin{array}{c}
z^2, 0 \\
z^6, z^2 \\
z^4, z^1 \\
z^5, z^3 \\
z^1, 0 \\
z^4, z^2 \\
z^2, 1 \\
z^3, z^4 \\
z^3, z^5
\end{array} \right) \right\},
\]

\[
D_3 \cdot \left( \begin{array}{c}
z^7, z^5 \\
z^3, z^1 \\
z^6, z^2 \\
z^4, z^3 \\
z^5, z^4 \\
z^6, z^5 \\
z^4, z^1 \\
z^3, z^2 \\
z^5, z^3
\end{array} \right) = \left\{ \left( \begin{array}{c}
z^7, z^5 \\
z^3, z^1 \\
z^6, z^2 \\
z^4, z^3 \\
z^5, z^4 \\
z^6, z^5 \\
z^4, z^1 \\
z^3, z^2 \\
z^5, z^3
\end{array} \right), \left( \begin{array}{c}
z^2, 0 \\
z^6, z^2 \\
z^4, z^1 \\
z^5, z^3 \\
z^1, 0 \\
z^4, z^2 \\
z^2, 1 \\
z^3, z^4 \\
z^3, z^5
\end{array} \right) \right\}.
\]

Although they look quite similar, the Osterunital and the Pfingstunital are not isomorphic, as is shown in the following

Proposition 3.7 There is no isomorphism between $\mathbb{O}U$ and $\mathbb{P}U$.

Proof According to Theorem 2.5, any isomorphism between $\mathbb{O}U$ and $\mathbb{P}U$ must be contained in $\mathbb{A}_C \rtimes \mathbb{R}$. But since the index of $\text{Aut}(\mathbb{O}U)$ in $\mathbb{A}_C \rtimes \mathbb{R}$ equals 2 and computation shows that $D_1$ is no block of $\mathbb{P}U$, the statement follows. \hfill \Box

In particular, the Oster- and Pfingstunital are two non-isomorphic affine $SL(2, q)$-unital with the same full automorphism group.

Remark 3.8 The Weihnachts-, Oster- and Pfingstunital were found by a computer search. In fact, we did an exhaustive search for affine $SL(2, 8)$-unital, where the groups $F$, $U$ and $L$ act in the same way as on the classical affine $SL(2, 8)$-unital. Those three affine unital were the only ones appearing through the search. See Möhler (2020c), Chapter 6 for details about the search.
4 Completion to unitals

Any affine unital can be completed to a unital by each of its parallelisms. In any affine SL(2, q)-unital, the set of short blocks is the set of all right cosets of the Sylow $p$-subgroups of SL(2, q). Note that each right coset $Tg$ is a left coset $gT^g$ of a conjugate of $T$. A parallelism as in (AU5) means a partition of the set of short blocks into $q + 1$ sets of $q^2 − 1$ pairwise non-intersecting cosets. For each prime power $q$, there are hence two obvious parallelisms, namely partitioning the set of short blocks into the sets of right cosets or into the sets of left cosets of the Sylow $p$-subgroups. We name those two parallelisms “flat” and “natural”, respectively, and denote them by the corresponding musical signs 

$$♭ := \{(Tg \mid g \in \text{SL}(2, q)) \mid T \in \mathbb{P}\}$$ and $$♯ := \{(gT \mid g \in \text{SL}(2, q)) \mid T \in \mathbb{P}\},$$

where $\mathbb{P}$ denotes the set of Sylow $p$-subgroups of SL(2, q).

Given an affine SL(2, q)-unital $\mathbb{U}_S$ with parallelism $\pi$, we call the $\pi$-closure an SL(2, q)-(π-)unital. Completing $\mathbb{W}U$, $\mathbb{O}U$ and $\mathbb{P}U$ with $♭$ and $♯$ each, we obtain six pairwise non-isomorphic SL(2, q)-unitals of order 8. Since they are all $♭$- or $♯$-closures of non-classical affine SL(2, q)-unitals of order 8 $\geq 3$, we know from Möhler (2020a), Proposition 3.11 and Theorem 3.16 that their full automorphism groups fix the block $[∞]$. Since the parallelisms $♭$ and $♯$, respectively, are preserved under the action of $Ω \rtimes R$, we get

$$\text{Aut}(\mathbb{U}_\pi) = \text{Aut}(\mathbb{U}_\pi)_{[∞]} = \text{Aut}(\mathbb{U})$$

for any $\mathbb{U} \in \{\mathbb{W}U, \mathbb{O}U, \mathbb{P}U\}$ and $\pi \in \{♭, ♯\}$.

**Remark 4.1** In any SL(2, q)-♯-unital, the Sylow $p$-subgroups act (via right multiplication) as translation groups of order $q$ with centers on the block $[∞]$. Hence, $\mathbb{W}U^♯$, $\mathbb{O}U^♯$ and $\mathbb{P}U^♯$ are examples of non-classical unitals of order $q$ where the translations generate SL(2, q).

**Remark 4.2** There might be more parallelisms on the short blocks of SL(2, 8)-unitals, leading to further closures. We already know a class of parallelisms for each odd order and one for square order [described in Möhler (2020b), Sections 3.1 and 3.2] and some parallelisms for order 4, leading to 12 new SL(2, 4)-unitals, the so-called Leonids unitals (see Möhler 2020b, Section 3.3 and Möhler 2020c, Section 6.2.2).

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