PERIODIC PROBLEM FOR DOUBLY NONLINEAR EVOLUTION EQUATION

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Abstract

We are concerned with the time-periodic problem of some doubly nonlinear equations governed by differentials of two convex functionals over uniformly convex Banach spaces. Akagi–Stefanelli (2011) [4] considered Cauchy problem of the same equation via the so-called WED functional approach. Main purpose of this paper is to show the existence of the time-periodic solution under the same growth conditions on functionals and differentials as those imposed in [4]. Because of the difference of nature between Cauchy problem and the periodic problem, we can not apply the WED functional approach directly, so we here adopt standard compactness methods with suitable approximation procedures.

1 Introduction

Let $V$ be a uniformly convex real Banach space and $V^*$ be its uniformly convex dual. In this paper, we are concerned with the following time-periodic problem of doubly nonlinear evolution equation:

\begin{equation}
\begin{cases}
\frac{d\psi(u'(t)) + \partial \phi(u(t))}{dt} \ni f(t), & t \in (0, T) \text{ in } V^*, \\
u(0) = u(T),
\end{cases}
\end{equation}

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where \( d\psi \) and \( \partial \phi \) are Gâteaux differential and subdifferential of convex functionals \( \psi \) and \( \phi \) which are mapping from \( V \) into \((−\infty, +\infty]\). Here and henceforth, \( u' \) denotes the time derivative of \( u \) and \( f \) is a given external force belonging to \( L^{p'}(0, T; V^*) \), where \( p' := p/(p - 1) \) and \( p \in (1, \infty) \) (precise definitions and assumptions will be given in the next section). A typical example which can be reduced to (AP) is given by the following doubly nonlinear parabolic equation:

\[
\alpha(u'(x, t)) - \Delta_m u(x, t) = f(x, t),
\]

where \( \alpha : \mathbb{R} \to \mathbb{R} \) is a non-decreasing function and \( \Delta_m u := \nabla \cdot (|\nabla u|^{m-2} \nabla u) \) (so-called \( m \)-Laplacian).

Cauchy problem for (AP) in Hilbert space has been studied by Barbu [13], where the elliptic regularization technique by adding the term \(-\varepsilon (d_V \psi(u'))'\) is employed and by Arai [10], Senba [11], Colli–Visintin [21] via the standard relaxation procedure with the additional term \( \varepsilon u' \). Stefanelli [44] discussed a variational characterization of solution to gradient system relying on the Brézis–Ekland principle. Investigation of global solvability in Banach spaces began with Colli [20], where the time discretization and the polygonal chain approximation are exploited. In Akagi–Stefanelli [4], they adopted another approach based on the fact that the solution to Cauchy problem for (AP) with elliptic regularization can be characterized by the minimizer of some functional with the weight \( \exp(-t/\varepsilon) \), the so-called Weighted Energy-Dissipation (WED, for short) functional (see also [3] [7] [8] [22] [35] [36] [37] and references therein). By this procedure, they proved the existence of global solution to Cauchy problem assuming some growth conditions on \( \psi \) and \( \phi \) (see Remark 2.8 below).

We here comment on the following variant type of doubly nonlinear equation, studied more vigorously than (AP):

\[
(Au(t))' + Bu(t) \ni f(t),
\]

where \( A \) and \( B \) are maximal monotone operators. Cauchy problem of (2) is investigated in, e.g., [23] [27] [28] [34]. Akagi–Stefanelli attempt the WED functional approach to (2) in [6]. Moreover, papers [9] [16] [26] [30] [31] [39] [40] [46] [47] are devoted to the initial boundary value problem of specific parabolic PDEs such as \( \partial_t u - \Delta_m u^p = f \) and \( \partial_t u^p - \Delta_m u = f \).

Compared with Cauchy problem, there are a few results for the existence of time-periodic solution to doubly nonlinear equation. Periodic problem for (2) is considered in [2] [29] [32] [33] and concrete PDEs of the same type as (2) in [24] [25] [45] [48] [49] [50]. However, to the best of our knowledge, the investigation into other types of equations different from (2) can be found only in Akagi–Stefanelli [5]. They consider the solvability
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and structural stability of periodic problem of the following abstract equation:

\begin{equation}
A(u'(t)) + \partial \phi(u(t)) \ni f(t),
\end{equation}

where \( A \) is a possibly multi-valued maximal monotone operator.

In [5], they restrict their discussion to the Hilbert space framework and impose the linear growth condition on \( A \). These conditions seem to be more restrictive than those imposed for Cauchy problem in [4] when \( A = d\psi \). Main purpose of this paper is to show that we can discuss the periodic problem for (AP) under almost the same growth conditions as those in [4]. In the next section, we fix several notations, present a few auxiliary tools, and state our main results more precisely. Section 3 is devoted to our proofs. In Section 4, we show that estimates established in the previous section are immediately applicable to the study for some structural stability of (AP). Finally, we exemplify the applicability of our setting by dealing with doubly nonlinear parabolic equation (1).

Some parts of our arguments for estimates and convergence rely on [4]. However, our main strategy seems to be indispensable in order to cope with difficulties arising from the difference of nature between Cauchy problem and the time-periodic problem, for which one would perceive that it is hard to find a suitable variational structure to apply the WED functional approach by [4]. So here we depart from the WED functional setting and adopt the standard compactness method with a suitable approximation procedure (see (AP)_\varepsilon in Theorem 2.6 given later). In establishing a priori estimates for solutions of approximate equations, the coercivity of \( \partial \phi \) to be assumed plays an essential role, since \( u(0) \) is an unknown value which may possibly depend on approximation parameters for the periodic problem.

In proving our main results, we first deal with the easier case where \( \phi \) dominates \( \psi \) in Section 3.2–3.4 and the other case will be treated in Section 3.5.

2 Main Results

2.1 Preliminary

We first fix some notations and recall basic properties (see [14][15][18]), which will be used later.

Let \( E \) be a real Banach space and \( E^* \) be its dual. The norms of \( E \) and \( E^* \) are denoted by \( |\cdot|_E \) and \( |\cdot|_{E^*} \), respectively, and the duality pairing by \( \langle u^*, u \rangle_E \), where \( [u, u^*] \in E \times E^* \). We define the duality mapping \( F_E \) of \( E \) by

\[ F_E(u) := \{ u^* \in E^*; \quad \langle u^*, u \rangle_E = |u|^2_E = |u^*|^2_{E^*} \}. \]
By virtue of the Hahn-Banach theorem, $F_E(u)$ is non-empty for every $u \in E$. If $E^*$ is strictly convex, then $F_E$ is a single-valued semi-continuous mapping. We also obtain the following (see Prüss [38]):

**Proposition 2.1.** A Banach Space $E$ is uniformly convex if and only if for each $R > 0$ there exists a non-decreasing function $m_R : [0, \infty) \to [0, \infty)$ which satisfies $m_R(0) = 0$, $m_R(\rho) > 0$ for any $\rho > 0$, and

$$ \langle u_1^* - u_2^*, u_1 - u_2 \rangle_E \geq m_R(|u_1 - u_2|_E)|u_1 - u_2|_E,$$

for any $u_1, u_2 \in B_R$ and $u_1^* \in F_E(u_1)$, $u_2^* \in F_E(u_2)$,

where $B_R := \{u \in E; |u|_E \leq R\}$.

Let $A$ be a (possibly) multivalued operator from $E$ into $2^{E^*}$ (the power set of $E^*$) and we often identify $A$ with its graph $G(A)$, more precisely, we write $[u, u^*] \in A$ if $u \in D(A) := \{v \in E; Av \neq \emptyset\}$ and $u^* \in Au$. Operator $A$ is said to be monotone if

$$ \langle u_i^* - u_2^*, u_1 - u_2 \rangle_E \geq 0 \quad \forall [u_i, u_i^*] \in A \quad (i = 1, 2)$$

and monotone operator $A$ is said to be maximal monotone if there is no monotone operator which contains $A$ properly. If $E$ and $E^*$ are reflexive and strictly convex, maximality of $A$ is equivalent to $R(F_E + A) = E^*$ (see also Lemma A.2 below).

Let $\phi : E \to (-\infty, +\infty]$ be a proper (i.e., $\phi \not\equiv +\infty$) lower semi-continuous (l.s.c., for short) and convex functional. The set $D(\phi) := \{u \in E; \phi(u) < +\infty\}$ is called the effective domain of $\phi$ and the subdifferential operator $\partial \phi$ from $E$ into $2^{E^*}$ is defined by

\begin{equation}
\partial \phi : u \mapsto \{\eta \in E^*; \phi(v) \geq \phi(u) + \langle \eta, v - u \rangle_E \ \forall v \in D(\phi)\}.
\end{equation}

Then $\partial \phi$ becomes a maximal monotone operator from $E$ into $2^{E^*}$. Here we recall the following fundamental feature of closedness of maximal monotone operator (see Lemma 1.2 in Brézis–Crandall–Pazy [19]).

**Proposition 2.2.** Let $E$ be a real Banach space and $A : E \to 2^{E^*}$ be a maximal monotone operator. Assume that the sequences $\{u_n\}_{n \in \mathbb{N}} \subset E$ and $\{v_n\}_{n \in \mathbb{N}} \subset E^*$ satisfy $u_n \rightharpoonup u$ weakly in $E$, $v_n \rightharpoonup v$ weakly in $E^*$, $[u_n, v_n] \in A$, and either

$$ \limsup_{n,m \to \infty} \langle v_n - v_m, u_n - u_m \rangle_E \leq 0$$

or

$$ \limsup_{n \to \infty} \langle v_n, u_n \rangle_E \leq \langle v, u \rangle_E.$$

Then $[u, v] \in A$ and $\langle v_n, u_n \rangle_E \to \langle v, u \rangle_E$ as $n \to \infty$. 
Let $Z$ be another Banach space which is densely embedded in $E$ and $D(\phi) \subset Z$. Then we can consider the restriction of $\phi$ onto $Z$, denoted by $\phi_Z$ and regard $\phi_Z$ as a proper l.s.c. convex functional on $Z$. To distinguish between the two subdifferentials over $E$ and $Z$, we write

$$
\partial_E \phi(u) := \partial \phi(u) = \{ \eta \in E^*; \phi(v) \geq \phi(u) + \langle \eta, v - u \rangle_E \ \forall v \in D(\phi) \},
$$

$$
\partial_Z \phi_Z(u) := \{ \eta \in Z^*; \phi_Z(v) \geq \phi_Z(u) + \langle \eta, v - u \rangle_Z \ \forall v \in D(\phi_Z) \}.
$$

In general, we have $\partial_E \phi \subset \partial_Z \phi_Z$. More precisely it is shown that (see Proposition 2.1 in Akagi–Stefanelli [1])

$$
D(\partial_E \phi) = \{ u \in D(\partial_Z \phi_Z); \partial_Z \phi_Z(u) \cap E^* \neq \emptyset \},
$$

$$
\partial_E \phi(u) = \partial_Z \phi_Z(u) \cap E^* \ \forall u \in D(\partial_E \phi).
$$

We next define another functional derivative. Let $\psi$ be a functional defined over $E$. Then $\psi$ is said to be Gâteaux differentiable at $u \in E$ if there exists $\xi \in E^*$ such that

$$
\lim_{h \to 0} \frac{\psi(u + he) - \psi(u)}{h} = \langle \xi, e \rangle_E \ \forall e \in E.
$$

We call $\xi$ the Gâteaux derivative of $\psi$ at $u$ and write $\xi = d_E \psi(u)$. When there exists $d_E \psi(u)$ for every $u \in E$, $\psi$ is said to be Gâteaux differentiable over $E$. It is well known that if $\psi$ is proper l.s.c. convex and Gâteaux differentiable at $u$, then Gâteaux derivative $d_E \psi(u)$ coincides with the subdifferential $\partial_E \psi(u)$ (i.e., $\partial_E \psi(u) = \{ d_E \psi(u) \}$ holds).

For proper l.s.c. convex functional $\phi$ on $E$, we define

$$
\phi_\lambda(u) := \inf_{v \in E} \left\{ \frac{|u - v|^2}{2\lambda} + \phi(v) \right\} = \frac{|u - J_\lambda u|^2}{2\lambda} + \phi(J_\lambda u) \ \lambda > 0, \ u \in E,
$$

where $J_\lambda : E \to E$ stands for the resolvent of $\partial_E \phi$, i.e., $J_\lambda u$ is the unique solution of

$$
F_E(J_\lambda u - u) + \lambda \partial_E \phi(J_\lambda u) \ni 0.
$$

This functional $\phi_\lambda$, called the Moreau-Yosida regularization of $\phi$, is convex, continuous, and Gâteaux differentiable over $E$. We can show that the Gâteaux differential $d_E \phi_\lambda$ coincides with the Yosida approximation $(\partial_E \phi)_\lambda$ of the subdifferential $\partial_E \phi$, which is defined by

$$
(\partial_E \phi)_\lambda u := -\lambda^{-1} F_E(J_\lambda u - u).
$$

Moreover, we have $D(\phi_\lambda) = E$ and for any $u \in D(\phi)$

$$
d_E \phi_\lambda(u) \in \partial_E \phi(J_\lambda u), \ \phi(J_\lambda u) \leq \phi_\lambda(u) \leq \phi(u), \ \lim_{\lambda \to 0} \phi_\lambda(u) = \phi(u).
$$

We here present some useful tools, the chain rule and a formula concerning the integration by parts.
Proposition 2.3. (see Lemma 4.1 of Colli [20]) Let $E$ be a strictly convex reflexive Banach space and $\phi : E \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function. Assume that $u \in W^{1,p}(0,T; E)$ with $p \in (1, \infty)$, $\eta \in L^{p'}(0,T; E^*)$, and $\eta(t) \in \partial \phi(u(t))$ a.e. $t \in (0,T)$. Then $t \mapsto \phi(u(t))$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\phi(u(t)) = \langle \eta(t), u'(t) \rangle_E$$

for a.e. $t \in (0,T)$.

Proposition 2.4. (see Proposition 2.3 of Akagi–Stefanelli [4]) Let $m, p \in (1, \infty)$ and $E, Z$ be reflexive Banach spaces such that $Z \hookrightarrow E$ and $E^* \hookrightarrow Z^*$ are dense. Moreover, let $u \in L^m(0,T; Z) \cap W^{1,p}(0,T; E)$, $\xi \in L^{p'}(0,T; E^*)$, and its derivative of distribution $\xi'$ belong to $L^m(0,T; Z^*) + L^{p'}(0,T; E^*)$. Provided that $t_1, t_2 \in (0,T)$ are Lebesgue points of the function $t \mapsto \langle \xi(t), u(t) \rangle_E$, then it holds that

$$\langle \xi', u \rangle_{L^m(t_1,t_2;Z) \cap L^p(t_1,t_2;E)} = \langle \xi(t_2), u(t_2) \rangle_E - \langle \xi(t_1), u(t_1) \rangle_E - \langle \xi, u' \rangle_{L^p(t_1,t_2;E)}.$$

### 2.2 Assumptions

In order to formulate our results, we introduce a suitable Banach space $V$ and its subspace $X$.

(A.0) Both $V$ and its dual $V^*$ are uniformly convex real Banach spaces and $X$ is a subspace of $V$ such that $X$ and its dual $X^*$ are real reflexive Banach spaces. Furthermore the embeddings

$$X \hookrightarrow V, \quad V^* \hookrightarrow X^*$$

hold with densely defined compact canonical injections.

We also assume the following growth conditions:

(A.1) Functional $\phi : V \to (-\infty, +\infty]$ is proper lower semi-continuous convex and $\psi : V \to (-\infty, +\infty)$ is convex and Gâteaux differentiable over $V$. Assume that there exist a constant $C > 0$ satisfying

\begin{align*}
(7) & \quad |u|_V^p \leq C(\psi(u) + 1) \quad \forall u \in V, \\
(8) & \quad |d_V \psi(u)|_V^{p'} \leq C(|u|_V^p + 1) \quad \forall u \in V, \\
(9) & \quad |u|_X^m \leq C(\phi(u) + 1) \quad \forall u \in D(\phi), \\
(10) & \quad |\eta|_{X^*}^m \leq C(|u|_X^m + 1) \quad \forall [u, \eta] \in \partial_X \phi_X,
\end{align*}

with some exponents $p, m \in (1, \infty)$, where $p' := p/(p-1)$ and $m' := m/(m-1)$ (Hölder conjugate exponents).
Remark 2.5. i) From (7) and (8), we assume that \( \phi \geq 0 \) and \( \psi \geq 0 \) without loss of generality henceforth (replace \( \phi \) and \( \psi \) with \( \phi + 1 \) and \( \psi + 1 \), respectively).

ii) Combining (7) with (8), we can immediately see that

\begin{equation}
|d_{V\psi}(u)|^{p^\prime}_{V^*} \leq C(\psi(u) + 1) \quad \forall u \in V.
\end{equation}

Moreover, since \( \psi(u) \leq \psi(0) + 2\langle d_{V\psi}(u), u \rangle_{V^*} \) holds by the definition of the subdifferential, (8) leads to

\begin{align}
\psi(u) &\leq C(|u|^p_V + 1) \quad \forall u \in V, \\
|u|^p_V &\leq C(|d_{V\psi}(u)|^{p^\prime}_{V^*} + 1) \quad \forall u \in V.
\end{align}

From (9) and (10), we can derive analogous inequalities for \( \phi \):

\begin{align}
|\eta|^m_{X^*} &\leq C(\phi(u) + 1) \quad \forall [u, \eta] \in \partial_X \phi_X, \\
\phi(u) &\leq C(|u|^m_X + 1) \quad \forall u \in D(\partial_X \phi_X), \\
|u|^m_X &\leq C(|\eta|^m_{X^*} + 1) \quad \forall [u, \eta] \in \partial_X \phi_X.
\end{align}

2.3 Statements of Main Results

The following result is concerned with the solvability of the elliptic regularization problem for (AP).

Theorem 2.6. Assume (A.0) and (A.1) with \( m > p \). Then for any \( \varepsilon > 0 \) and \( f \in L^p(0, T; V^*) \), the time-periodic problem

\begin{align*}
\text{(AP)}_{\varepsilon} &\left\{ 
-\varepsilon(d_{V\psi}(u_\varepsilon(t)))' + d_{V\psi}(u_\varepsilon(t)) + \partial_X \phi_X(u_\varepsilon(t)) \\
&\quad + \varepsilon F_{\psi}(u_\varepsilon(t)) + \varepsilon d_{V\psi}(u_\varepsilon(t)) \geq f(t) \\
\right. \\
&\quad t \in (0, T) \quad \text{in } X^*, \\
u_\varepsilon(0) = u_\varepsilon(T), \\
d_{V\psi}(u'_\varepsilon(0)) = d_{V\psi}(u'_\varepsilon(T)),
\end{align*}

possesses at least one solution \( u_\varepsilon \) satisfying

\begin{align}
u_\varepsilon &\in W^{1,p}(0, T; V) \cap L^m(0, T; X), \\
d_{V\psi}(u_\varepsilon), d_{V\psi}(u'_\varepsilon), F_{\psi}(u_\varepsilon) &\in L^p(0, T; V^*), \\
\eta_\varepsilon &\in L^{m'}(0, T; X^*), \\
(d_{V\psi}(u'_\varepsilon))' &\in L^p(0, T; V^*) + L^{m'}(0, T; X^*),
\end{align}

and

\[ \int_0^T \langle d_{V\psi}(u'_\varepsilon(t)), u'_\varepsilon(t) \rangle_V dt \leq \int_0^T \langle f(t), u'_\varepsilon(t) \rangle_V dt. \]
where \( \eta_\varepsilon \) is the section of \( \partial_X \phi_X(u_\varepsilon) \) satisfying (AP), i.e., \( \eta_\varepsilon \) satisfies \( \eta_\varepsilon(t) \in \partial_X \phi_X(u_\varepsilon(t)) \) and 
\[-\varepsilon(d_V \psi(u'_\varepsilon(t)))' + d_V \psi(u'_\varepsilon(t)) + \eta_\varepsilon(t) + \varepsilon F_V(u_\varepsilon(t)) + \varepsilon d_V \psi(u_\varepsilon(t)) = f(t) \text{ for a.e. } t \in (0,T).
\]

Via this approximation, the solvability of original problem will be assured as follows:

**Theorem 2.7.** Assume (A.0) and (A.1). Then for every \( f \in L'p(0,T;V^*) \), (AP) possesses at least one solution \( u \) satisfying

\[
\begin{align*}
&u \in W^{1,p}(0,T;V) \cap L^\infty(0,T;X), \\
&\phi(u(t)) \text{ is absolutely continuous on } [0,T], \\
&d_V \psi(u'), \eta \in L'p(0,T;V^*),
\end{align*}
\]

where \( \eta \) is the section of \( \partial_V \phi(u) \) satisfying (AP), i.e., \( \eta \) satisfies \( \eta(t) \in \partial_V \phi(u(t)) \) and 
\( d_V \psi(u'(t)) + \eta(t) = f(t) \) for a.e. \( t \in (0,T) \).

**Remark 2.8.** i) In so far as \( A \) is given as a Gâteaux differential, i.e., \( A = d_V \psi \) in [3], we see that our result can cover that of Akagi–Stefanelli [5] by letting \( V = X \) be Hilbert spaces and \( p = 2 \).

ii) In Akagi-Stefanelli [4] (Cauchy problem), they assume almost the same growth condition as (A.1) and allow that constants \( C \) in [9] and [10] depend on \( |u|_V \). In order to reveal a difference of nature between Cauchy problem and the periodic problem, we here check how to derive a priori estimate of solution to (AP). Testing (AP) by \( u' \) and applying Proposition 2.3 we get

\[
\langle d_V \psi(u'(t)), u'(t) \rangle_V + \frac{d}{dt} \phi(u(t)) = \langle f(t), u'(t) \rangle_V.
\]

Integrating over \([0,t]\) and using (7), we have for every \( t \in [0,T] \)

\[
\int_0^t |u'(s)|_V^p ds + \phi(u(t)) \leq C \left(|f|_{L'p(0,T;V^*)}^p + \phi(u(0)) + T \right)
\]

with some suitable constant \( C > 0 \). Hence for Cauchy problem, since \( u(0) \) is given data, one can derive estimates of \( |u'|_{L^p(0,T;V)} \) and \( \sup_t \phi(u(t)) \) immediately. For the periodic problem, however, since \( u(0) \) is unknown, the boundedness of \( |u'|_{L^p(0,T;V)} \) can be assured but one can not get any information on \( \sup_t \phi(u(t)) \) from the procedure above.

## 3 Proofs of Main Results

### 3.1 Scheme of Proofs

Our proofs consist of the following four steps:
Step 1 We first deal with the case where \( m > p \) and show that

\[
(\text{AP})_\lambda^h \begin{cases}
-\varepsilon(d_V\psi(u_\lambda'(t)))' + \varepsilon d_V\psi(u_\lambda(t)) + \partial_X\phi_X(u_\lambda(t)) + \varepsilon F_V(u_\lambda(t)) \\
= f(t) + h(t) & t \in (0, T) \quad \text{in } V^*,
\end{cases}
\]

\( u_\lambda(0) = u_\lambda(T), \)

\( d_V\psi(u_\lambda'(0)) = d_V\psi(u_\lambda'(T)), \)

possesses a unique solution for every \( f, h \in L^p(0, T; V^*) \) and \( \lambda, \varepsilon > 0 \), where \( \phi_\lambda \) denotes the Moreau–Yosida regularization of \( \phi \) and \( \cdot' \) stands for the time derivative. By letting \( \lambda \to 0 \), we next show the following result.

Lemma 3.1. Let \( 1 < p < m \) and \( f, h \in L^p(0, T; V^*) \). Then for any \( \varepsilon > 0 \),

\[
(\text{AP})^h \begin{cases}
-\varepsilon(d_V\psi(u_h'(t)))' + \varepsilon d_V\psi(u_h(t)) + \partial_X\phi_X(u_h(t)) + \varepsilon F_V(u_h(t)) \\
\geq f(t) + h(t) & t \in (0, T) \quad \text{in } X^*,
\end{cases}
\]

\( u_h(0) = u_h(T), \)

\( d_V\psi(u_h'(0)) = d_V\psi(u_h'(T)), \)

has a unique solution satisfying

\[
(19) \quad u_h \in W^{1,p}(0, T; V) \cap L^m(0, T; X),
\]

\[
d_V\psi(u_h), d_V\psi(u_h'), F_V(u_h) \in L^p(0, T; V^*),
\]

\[
\eta_h \in L^{m'}(0, T; X^*),
\]

\[
(d_V\psi(u_h'))' \in L^p(0, T; V^*) + L^{m'}(0, T; X^*),
\]

where \( \eta_h \) is the section of \( \partial_X\phi_X(u_h) \) satisfying \((\text{AP})^h\), i.e., \( \eta_h(t) \in \partial_X\phi_X(u_h(t)) \) and

\[
-\varepsilon(d_V\psi(u_h'(t)))' + \varepsilon d_V\psi(u_h(t)) + \eta_h(t) + \varepsilon F_V(u_h(t)) = f(t) + h(t) \quad \text{for a.e. } t \in (0, T).
\]

Note that \( L^p(0, T; V^*) + L^{m'}(0, T; X^*) = L^{m'}(0, T; X^*) \) holds, since we here assume that \( p < m \). However, we still write as \( (19) \) for the sake of consistency with the notation in \([4]\).

Step 2 Let \( u_h \) be the unique solution of \((\text{AP})^h\). Define an operator \( \beta \) by

\[
(20) \quad \beta : h \mapsto u_h \mapsto -d_V\psi(u_h'),
\]

namely, \( \beta(h) := -d_V\psi(u_h') \). We apply the Schauder–Tychonoff fixed point theorem to \( \beta \) in \( \Xi := L^p(0, T; V^*) \) endowed with its weak topology. Let \( \bar{h} \) be the fixed point of \( \beta \), then \( u_{\bar{h}} \) gives a solution of \((\text{AP})_\varepsilon\) satisfying properties given in Theorem \([2,6]\).
Step 3  By establishing a priori estimates for solutions of \((AP)\varepsilon\) independent of the parameter \(\varepsilon\) and discussing the convergence of solutions, we show the solvability of \((AP)\) for \(p < m\).

Step 4  We deal with the case where \(m \leq p\).

### 3.2 Step 1 (Existence of Solutions of Auxiliary Equations)

Since our argument in this step is based on the procedure in Section 3 of Akagi–Stefanelli [4], one may find some duplications in this subsection. We first prepare the following result on some variational problem associated with \((AP)\lambda\).

**Lemma 3.2.** Let \(p < m\) and put \(\bar{p} := \max\{p, 2\}\), \(\Gamma := L^\bar{p}(0, T; V)\), \(\Gamma^* := L^{\bar{p}'}(0, T; V^*)\), \(\langle \cdot, \cdot \rangle_{\Gamma} := \int_0^T \langle \cdot, \cdot \rangle_V dt\), and let \(h, f \in \Gamma^*\). Then for any \(\varepsilon, \lambda > 0\) the functional over \(\Gamma\) defined by

\[
I_{\varepsilon, \lambda}(u) := \begin{cases} 
\int_0^T \left( \phi_\lambda(u(t)) + \varepsilon \psi(u'(t)) + \varepsilon \psi(u(t)) + \frac{\varepsilon}{2} |u(t)|^2_V - \langle f(t) + h(t), u(t) \rangle_V \right) dt \\
\quad \text{if } u \in W^{1,\bar{p}}(0, T; V), \ u(0) = u(T), \ \psi(u(\cdot)), \psi(u'(\cdot)) \in L^1(0, T), \\
\quad +\infty \quad \text{otherwise,}
\end{cases}
\]

admits a global minimizer \(u_\lambda\) on \(\Gamma\). Moreover, \(u_\lambda \in W^{1,\bar{p}}(0, T; V)\) is a unique solution to \((AP)^{h}_\lambda\) satisfying

\[
\begin{align*}
&u_\lambda(0) = u_\lambda(T), \ d_V \psi(u'_\lambda(0)) = d_V \psi(u'_\lambda(T)), \\
&d_V \psi(u'_\lambda) \in W^{1,\bar{p}'}(0, T; V^*), \ d_V \psi(u_\lambda), \ \partial_V \phi_\lambda(u_\lambda), \ F_V(u_\lambda) \in L^{\bar{p}'}(0, T; V^*).
\end{align*}
\]

**Proof.** It is easy to see that \(I_{\varepsilon, \lambda}\) is proper lower semi-continuous and strictly convex on \(\Gamma\). By assumptions (7) and (9), \(I_{\varepsilon, \lambda}\) is coercive and bounded from below. Then the standard argument guarantees the existence of global minimizer \(u_\lambda\) of \(I_{\varepsilon, \lambda}\).

Divide \(I_{\varepsilon, \lambda}\) into the following five parts:

\[
\begin{align*}
I^1_{\varepsilon, \lambda}(u) &:= \begin{cases} 
\varepsilon \int_0^T \psi(u'(t)) dt \quad \text{if } u \in D(I^1_{\varepsilon, \lambda}) := \{u \in W^{1,\bar{p}}(0, T; V); u(0) = u(T)\}, \\
+\infty \quad \text{otherwise,}
\end{cases} \\
I^2_{\varepsilon, \lambda}(u) &:= \int_0^T \phi_\lambda(u(t)) dt, \\
I^3_{\varepsilon, \lambda}(u) &:= \begin{cases} 
\varepsilon \int_0^T \psi(u(t)) dt \quad \text{if } \psi(u(\cdot)) \in L^1(0, T), \\
+\infty \quad \text{otherwise,}
\end{cases}
\end{align*}
\]
The subdifferential of $K$ is a proper indicator function of the closed linear subset $\{u \in \Lambda; u(0) = u(T)\}$ of $\Lambda$, the subdifferential of $K$ is well defined and $\langle v, e \rangle_\Lambda = 0$ holds for every $[u, v] \in \partial_\Lambda K$.
and $e \in \Lambda$ s.t. $e(0) = e(T)$. Since $D(d_{\Lambda}L) = \Lambda$ and $D(\partial_{\Lambda}K) = D(K)$, we can derive
\[
\partial_{\Lambda}I^1_{\lambda} = \partial_{\Lambda}(L + K) = d_{\Lambda}L + \partial_{\Lambda}K
\]
and
\[
D(\partial_{\Lambda}I^1_{\lambda}) = D(d_{\Lambda}L) \cap D(\partial_{\Lambda}K) = \{u \in \Lambda; u(0) = u(T)\}
\]
(see, e.g., Theorem 2.10 in Barbu [15]). Let $[u, v] \in \partial_T^1 I^1_{\varepsilon, \lambda}$ and $e \in \Lambda$ s.t. $e(0) = e(T)$. By the general relationship $\partial_T^1 I^1_{\varepsilon, \lambda} \subset \partial_{\Lambda}I^1_{\lambda}$, we have
\[
\langle v, e \rangle_{\Lambda} = \langle d_{\Lambda}L(u), e \rangle_{\Lambda} = \int_0^T \langle \varepsilon d_V \psi(u'(t)), e'(t) \rangle_V dt.
\]
In addition, since $\langle v, e \rangle_{\Lambda} = \langle v, e \rangle_{\Gamma} = \int_0^T \langle v(t), e(t) \rangle_V dt$ is verified by $v \in \Gamma^* \hookrightarrow \Lambda^*$ and $e \in \Lambda \hookrightarrow \Gamma$,
\[
\int_0^T \langle v(t), e(t) \rangle_V dt = \int_0^T \langle \varepsilon d_V \psi(u'(t)), e'(t) \rangle_V dt
\]
holds for every $e \in \Lambda$ with $e(0) = e(T)$. Hence by testing this by $e \in C^\infty_0((0, T); V)$, which is dense in $\Gamma$, we conclude that $v = -(\varepsilon d_V \psi(u'(\cdot)))' \in \Gamma^* = L([\rho])'(0, T; V^*)$. Combining this with the integration by parts, we obtain
\[
\langle d_V \psi(u'(T)), e(T) \rangle_V - \langle d_V \psi(u'(0)), e(0) \rangle_V = \langle d_V \psi(u'(T)) - d_V \psi(u'(0)), e(0) \rangle_V = 0.
\]
Therefore $d_V \psi(u'(T)) = d_V \psi(u'(0))$ in $V^*$ by the arbitrariness of $e(0)$ and then $\partial_T I^1_{\varepsilon, \lambda} \subset A$ is shown.

We now check the remainders $I^i_{\varepsilon, \lambda} \cdot$ $(i = 2, 3, 4, 5)$. Obviously, these are proper lower semi-continuous convex functional defined on $\Gamma$. We first get
\[
I^2_{\varepsilon, \lambda}(u) \leq \int_0^T \left( \frac{|u(t) - v(t)|^2}{2\lambda} + \phi(v) \right) dt \forall u \in \Gamma \text{ and } \forall v \in D(\phi)
\]
from the definition of the Moreau–Yosida regularization and the definition of $\partial \psi$ and [8] yield
\[
I^3_{\varepsilon, \lambda}(u) \leq \varepsilon T \psi(v) + \varepsilon \int_0^T |d_V \psi(u(t))|_V \cdot |u(t) - v|_V dt
\]
\[
\leq \varepsilon T \psi(v) + \varepsilon C (|u(t)|^p_V + |v|^p_V + 1) \forall u \in \Gamma \text{ and } \forall v \in D(\psi),
\]
where $C$ is some suitable constant independent of $\varepsilon$ and $\lambda$. Hence $D(I^3_{\varepsilon, \lambda}) = \Gamma$ and $I^4_{\varepsilon, \lambda}, I^5_{\varepsilon, \lambda}$ are well defined on $\Gamma$. Moreover, by the standard argument (see Appendix I in [17]), we have
\[
\partial_T I^2_{\varepsilon, \lambda}(u) = \partial_V \phi_{\lambda}(u(t)), \quad \partial_T I^3_{\varepsilon, \lambda}(u) = \varepsilon d_V \psi(u(t)),
\]
\[
\partial_T I^4_{\varepsilon, \lambda}(u) = \varepsilon F_V(u(t)), \quad \partial_T I^5_{\varepsilon, \lambda}(u) = -f(t) - h(t),
\]
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for any \( u \in \Gamma \) and a.e. \( t \in (0, T) \). Hence we immediately have

\[
\partial_t I_{\epsilon, \lambda} = \partial_t (I_{\epsilon, \lambda}^1 + I_{\epsilon, \lambda}^2 + I_{\epsilon, \lambda}^3 + I_{\epsilon, \lambda}^4 + I_{\epsilon, \lambda}^5)
= \partial_t I_{\epsilon, \lambda}^1 + \partial_t I_{\epsilon, \lambda}^2 + \partial_t I_{\epsilon, \lambda}^3 + \partial_t I_{\epsilon, \lambda}^4 + \partial_t I_{\epsilon, \lambda}^5.
\]

Therefore, since the global minimizer \( u_\lambda \) satisfies \( 0 = \partial_t I_{\epsilon, \lambda}(u_\lambda) \), we conclude that \( u_\lambda \) is the unique solution to \((AP)^{\lambda}\).

**Remark 3.3.** If one tries to apply the WED approach to \( I_{\epsilon, \lambda}^1 \) by the same argument as that in [4], one gets a solution with condition \( d_{\epsilon, \lambda} \)

We next take the limit \( \lambda \to 0 \) in \((AP)^{\lambda}\). In this procedure, we note that if we multiply the equation of \((AP)^{\lambda}\) by \( u_\lambda \) as in [4], we get for any \( t_1 \) and \( t_2 \),

\[
\int_{t_1}^{t_2} \langle f(t) + h(t), u_\lambda(t) \rangle_V \, dt,
\]

where \( \psi^* \) is Legendre–Fenchel transform of \( \psi \) (precise treatment of the first term of L.H.S. will be given in (55) and (56) below). However, this inequality seems to be not useful for the periodic problem, since \( u_\lambda(0) \) is still an unknown value. So we here adopt a different manner to prove Lemma 3.1.

**Proof of Lemma 3.1** Let \( h, f \in L^p(0, T; V^*) \). Since \( L^p(0, T; V^*) \subset L^{(p)'}(0, T; V^*) \), we note that Lemma 3.2 assures the existence of the unique solution \( u_\lambda \) of \((AP)^{\lambda}\) satisfying \( u_\lambda \in W^{1,p}(0, \Gamma) \subset W^{1,p}(0, T; V) \). We first multiply the equation of \((AP)^{\lambda}\) by \( u_\lambda \). Then using the integration by parts and the periodicity of \( u_\lambda \), we have

\[
\varepsilon \int_0^T \langle d_{\psi}(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt + \varepsilon \int_0^T \langle d_{\psi}(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt
+ \int_0^T \langle \partial_V \phi(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt + \varepsilon \int_0^T |u_\lambda(t)|^2_V \, dt = \int_0^T \langle f(t) + h(t), u_\lambda(t) \rangle_V \, dt.
\]

The definition of subdifferential and (1) yield

\[
\int_0^T \langle d_{\psi}(v(t)), v(t) \rangle_V \, dt \geq \int_0^T (\psi(v(t)) - \psi(0)) \, dt \geq c_1 \left( |v|^p_{L^p(0, T; V)} - T \right),
\]

and hence by (24) with \( v = u_\lambda' \) and \( v = u_\lambda \), we get

\[
\varepsilon c_1 \int_0^T (|u_\lambda'(t)|^p_V + |u_\lambda(t)|^p_V) \, dt + \int_0^T \langle \partial_V \phi(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt + \varepsilon \int_0^T |u_\lambda(t)|^2_V \, dt
\]

\[
\leq \int_0^T \langle f(t) + h(t), u_\lambda(t) \rangle_V \, dt + 2\varepsilon C_1 T.
\]
Here $c_1$ and $C_1$ denote positive general constants independent of the parameters $\lambda$ and $\varepsilon$. From this and Young’s inequality, we can derive

$$
\varepsilon c_1 \int_0^T (|u'_\lambda(t)|_V^p + |u_\lambda(t)|_V^p) \, dt + \int_0^T \langle \partial V \phi_\lambda(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt + \varepsilon \int_0^T |u_\lambda(t)|_V^p \, dt
\leq C_\varepsilon \int_0^T |f(t) + h(t)|_V^p \, dt + 2\varepsilon C_1 T,
$$

where $C_\varepsilon$ is a general constant depending only on $\varepsilon$. Then we obtain the following estimate independent of $\lambda$:

$$
\int_0^T \langle \partial V \phi_\lambda(u_\lambda(t)), u_\lambda(t) \rangle_V \, dt + \varepsilon \int_0^T (|u'_\lambda(t)|_V^p + |u_\lambda(t)|_V^p + |u_\lambda(t)|_V^2) \, dt \leq C_\varepsilon.
$$

Hence for any fixed $v \in D(\phi) \subset X$, the definition of subdifferential and the fact that $\phi_\lambda(v) \leq \phi(v)$ yield

$$
0 \leq \int_0^T \phi_\lambda(u_\lambda(t)) \, dt \leq \int_0^T \langle \partial V \phi_\lambda(u_\lambda(t)), u_\lambda(t) - v \rangle_V \, dt + T\phi(v)
\leq C_\varepsilon + |v|_X \int_0^T |\partial V \phi_\lambda(u_\lambda(t))|_X \, dt.
$$

Then since $\partial V \phi(J_\lambda u_\lambda) \subset \partial X \phi_X(J_\lambda u_\lambda)$ and $\phi(J_\lambda u_\lambda) \leq \phi_\lambda(u_\lambda) \leq \phi(u_\lambda)$ (recall (6)), we have from (9) and (10)

$$
\int_0^T \phi_\lambda(u_\lambda(t)) \, dt \leq C_\varepsilon + C|v|_X \int_0^T (\phi(J_\lambda u_\lambda) + 1)^{1/m'} \, dt
\leq C_\varepsilon + CT^{1/m'}|v|_X \left( \int_0^T (\phi(u_\lambda) + 1) \, dt \right)^{1/m'},
$$

whence follows

$$
0 \leq \int_0^T \phi_\lambda(u_\lambda(t)) \, dt \leq C_\varepsilon.
$$

Hence, again by (6), (9) and (10), we obtain

$$
\int_0^T \left( |J_\lambda u_\lambda(t)|_X^p + |\partial V \phi_\lambda(u_\lambda(t))|_{X^*}^{m'} \right) \, dt \leq C_\varepsilon,
$$

which together with (27) yields

$$
|\partial V \phi_\lambda(u_\lambda)|_{L^{m'}(0,T;X^*)} + |u_\lambda|_{W^{1,p}(0,T;V)} + |u_\lambda|_{L^2(0,T;V)} + |J_\lambda u_\lambda|_{L^m(0,T;X)} \leq C_\varepsilon.
$$

Therefore (5) leads to

$$
\int_0^T |dV \psi(u_\lambda(t))|_V^p \, dt + \int_0^T |dV \psi(u'_\lambda(t))|_V^p \, dt \leq C_\varepsilon.
$$
and (12) implies

\[
\int_0^T \psi(u_\lambda(t))\,dt + \int_0^T \psi(u'_\lambda(t))\,dt \leq C_\varepsilon.
\]

Returning to (AP)\(^h_\lambda\), we get

\[
|\varepsilon(d_V\psi(u'_\lambda))'|_{L^{p'}(0,T;V^*)} + |\partial_V\phi_\lambda(u_\lambda)|_{L^{m'}(0,T;X^*)} + |\varepsilon F_V(u_\lambda)|_{L^{p'}(0,T;V^*)} + |f + h|_{L^{p'}(0,T;V^*)} \leq C_\varepsilon.
\]

Here we used the fact that \(W^{1,p}(0,T;V) \subset L^\infty(0,T;V)\) and \(|F_V(u_\lambda)|_{V^*} = |u_\lambda|_V\).

Then by (29), (30) and (32), we can extract a subsequence \(\{\lambda_n\}_{n \in \mathbb{N}}\) such that \(\lambda_n \to 0\) as \(n \to \infty\) and

\[
\begin{align*}
    u_{\lambda_n} & \rightharpoonup u & \text{weakly in } W^{1,p}(0,T;V), \\
    J_{\lambda_n}u_{\lambda_n} & \rightharpoonup v & \text{weakly in } L^m(0,T;X), \\
    F_V(u_{\lambda_n}) & \rightharpoonup w & \text{weakly in } L^{p'}(0,T;V^*), \\
    \partial_V\phi_\lambda(u_{\lambda_n}) & \rightharpoonup \eta & \text{weakly in } L^{m'}(0,T;X^*), \\
    d_V\psi(u_{\lambda_n}) & \rightharpoonup a & \text{weakly in } L^{p'}(0,T;V^*), \\
    d_V\psi(u'_{\lambda_n}) & \rightharpoonup \xi & \text{weakly in } L^{p'}(0,T;V^*), \\
    (d_V\psi(u'_{\lambda_n}))' & \rightharpoonup \xi' & \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*),
\end{align*}
\]

where \(\xi'\) stands for the derivative of \(\xi\) in the sense of distributions. Taking the limit in (AP)\(^h_\lambda\), we have

\[-\varepsilon \xi' + \varepsilon a + \eta + \varepsilon w = f + h \quad \text{in} \quad L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).\]

Now we are going to show that \(u\) gives a solution of (AP)\(^h\). We first note that \(\{u_\lambda\}_{\lambda > 0}\) forms an equi-continuous family in \(C([0,T];V)\), which is assured by \(|u'_\lambda|_{L^p(0,T;V)} \leq C_\varepsilon\).

Moreover, since \(u_\lambda \in C([0,T];V)\), there exists \(t_0 \in [0,T)\) such that

\[
T^{1/p}|u_\lambda(t_0)|_V = T^{1/p} \min_{t \in [0,T]} |u_\lambda(t)|_V \leq |u_\lambda|_{L^p(0,T;V)} \leq C_\varepsilon.
\]

Then for any \(t \in [t_0, t_0 + T]\), we get

\[
|u_\lambda(t)|_V \leq |u_\lambda(t_0)|_V + \int_{t_0}^t |u'_\lambda(s)|_V \, ds \leq |u_\lambda(t_0)|_V + T^{1/p'}|u'_\lambda|_{L^p(0,T;V)} \leq C_\varepsilon,
\]

which implies

\[
\sup_{t \in [0,T]} |u_\lambda(t)|_V \leq C_\varepsilon.
\]
Thanks to the general property $|J_{\lambda}u_{\lambda}|_V \leq C_1(|u_{\lambda}|_V + 1)$, we also get
\[
(35) \quad \sup_{t \in [0,T]} |J_{\lambda}u_{\lambda}(t)|_V \leq C_\varepsilon.
\]
We here recall that $J_{\lambda}u_{\lambda}$ is defined by the unique solution of $F_V(J_{\lambda}u_{\lambda} - u_{\lambda}) \in -\lambda \partial_{V} \phi(J_{\lambda}u_{\lambda})$. Then from the monotonicity of $\partial_{V} \phi$,
\[
\langle F_V(J_{\lambda}u_{\lambda}(t+h) - u_{\lambda}(t+h)) - F_V(J_{\lambda}u_{\lambda}(t) - u_{\lambda}(t)), J_{\lambda}u_{\lambda}(t+h) - J_{\lambda}u_{\lambda}(t) \rangle_V \leq 0
\]
holds for any $h \in \mathbb{R}$. On the other hand, (34) and (35) yield
\[
\langle F_V(J_{\lambda}u_{\lambda}(t+h) - u_{\lambda}(t+h)) - F_V(J_{\lambda}u_{\lambda}(t) - u_{\lambda}(t)), -u_{\lambda}(t+h) + u_{\lambda}(t) \rangle_V \\
\leq C_\varepsilon |u_{\lambda}(t+h) - u_{\lambda}(t)|_V.
\]
Then by adding two inequalities above, we obtain by Proposition 2.1
\[
m_{C_\varepsilon} |(J_{\lambda}u_{\lambda}(t+h) - u_{\lambda}(t+h)) - (J_{\lambda}u_{\lambda}(t) - u_{\lambda}(t))|_V \\
\times |(J_{\lambda}u_{\lambda}(t+h) - u_{\lambda}(t+h)) - (J_{\lambda}u_{\lambda}(t) - u_{\lambda}(t))|_V \\
\leq C_\varepsilon |u_{\lambda}(t+h) - u_{\lambda}(t)|_V,
\]
which implies that $\{J_{\lambda}u_{\lambda} - u_{\lambda}\}_{\lambda>0}$ forms an equi-continuous family in $C([0,T];V)$. Hence so does $\{J_{\lambda}u_{\lambda}\}_{\lambda>0}$. Therefore Theorem 3 of Simon [42] assures the relative compactness of $\{J_{\lambda}u_{\lambda}\}_{\lambda>0}$ in $C([0,T];V)$, i.e.,
\[
J_{\lambda_n}u_{\lambda_n} \to v \quad \text{strongly in} \quad C([0,T];V)
\]
(recall that $v$ is the weak limit $\{J_{\lambda_n}u_{\lambda_n}\}_{n \in \mathbb{N}}$ in $L^m(0,T;X)$). Since $J_{\lambda}u_{\lambda}(0) = J_{\lambda}u_{\lambda}(T)$, the limit $v$ is also time-periodic. Furthermore, by the definition of $\phi_{\lambda}$ and the fact that $\phi \geq 0$, we have
\[
\int_0^T |u_{\lambda}(t) - J_{\lambda}u_{\lambda}(t)|_V^2 dt \leq 2\lambda \int_0^T \phi_{\lambda}(u_{\lambda}(t)) dt \leq 2\lambda C_\varepsilon,
\]
which implies that $\{u_{\lambda} - J_{\lambda}u_{\lambda}\}_{\lambda>0}$ converges to 0 strongly in $L^2(0,T;V)$ and then $u = v$. Using the strong convergence of $\{J_{\lambda_n}u_{\lambda_n}\}_{n \in \mathbb{N}}$ and uniform boundedness (34) and (35), we also have
\[
(36) \quad u_{\lambda_n} \to u \quad \text{strongly in} \quad L^r(0,T;V) \quad \forall r \in [1,\infty)
\]
and then by the demiclosedness of $F_V$ and $d_V \psi$, we get
\[
(37) \quad w = F_V(u), \quad a = d_V \psi(u).
\]
Here we can extract a subsequence of $\{u_{\lambda_n}\}_{n \in \mathbb{N}}$ (we omit relabeling) such that
\[
(38) \quad u_{\lambda_n}(t) \to u(t) \quad \text{strongly in} \quad V \quad \text{for a.e.} \quad t \in (0,T).
\]
Since (30), (32), and compact embedding $V^* \hookrightarrow X^*$ holds, Theorem 3 of [42] is also applicable to the sequence \( \{d_V\psi(u_{\lambda_n}')\}_{n \in \mathbb{N}} \). Readily,

\[
(39) \quad d_V\psi(u_{\lambda_n}') \to \xi \quad \text{strongly in } C([0, T]; X^*).
\]

Remark that $\xi(0) = \xi(T)$ also can be deduced from the periodicity of $d_V\psi(u_{\lambda_n}')$.

In order to complete this step, we have to check $\xi = d_V\psi(u')$ and $\eta \in \partial_X\phi_X(u)$. We here define a subset $\Upsilon$ of $(0, T)$ by

\[
\Upsilon := \left\{ t \in (0, T) \mid \begin{array}{l}
t \text{ is a Lebesgue point of } t \mapsto \langle \xi(t), u(t) \rangle_V \text{ and} \\
\{\lambda_n\} \text{ has a subsequence } \{\lambda_n'\} \text{ tending to 0 as } n' \to \infty \\
\text{s.t. } \liminf_{n' \to \infty} \langle d_V\psi(u_{\lambda_n'}'), u'_{\lambda_n'}(t) \rangle_V \to \langle \xi(t), u(t) \rangle_V \text{ as } n' \to \infty.
\end{array} \right\}.
\]

Since $\langle \xi(\cdot), u(\cdot) \rangle_V \in L^1(0, T)$, almost every point of $(0, T)$ satisfy the first requirement. Moreover, thanks to Fatou’s lemma and (30), \( t \mapsto \liminf_{n \to \infty} |d_V\psi(u_{\lambda_n}(t))|^p_V \) belongs to $L^1(0, T)$, and then takes a finite value at a.e. $t \in (0, T)$. Hence from (38) and (39), the second requirement is also assured by a.e. $t \in (0, T)$. Therefore $\Upsilon$ has full Lebesgue measure in $(0, T)$.

Fix $t_1, t_2 \in \Upsilon$ with $t_1 < t_2$ arbitrarily. Since $\lambda \partial_V\phi_X(u_\lambda) = F_V(u_\lambda - J_\lambda u_\lambda)$ holds by the definition of the Yosida approximation, we have for every $\lambda > 0$

\[
\int_{t_1}^{t_2} \langle \partial_V\phi_X(u_\lambda(t)), J_\lambda u_\lambda(t) \rangle_X dt = \int_{t_1}^{t_2} \langle \partial_V\phi_X(u_\lambda(t)), J_\lambda u_\lambda(t) \rangle_V dt
\]

\[
= \int_{t_1}^{t_2} \langle \partial_V\phi_X(u_\lambda(t)), u_\lambda(t) \rangle_V dt - \lambda^{-1} \int_{t_1}^{t_2} |u_\lambda - J_\lambda u_\lambda|^2_V dt
\]

\[
\leq \int_{t_1}^{t_2} \langle \partial_V\phi_X(u_\lambda(t)), u_\lambda(t) \rangle_V dt.
\]

Repeating the same calculation as that for (23), i.e., multiplying the equation of (AP)$_\lambda$ by $u_\lambda$ and integrating over $(t_1, t_2)$, we get

\[
\varepsilon \int_{t_1}^{t_2} \langle d_V\psi(u_\lambda'(t)), u_\lambda'(t) \rangle_V dt - \varepsilon \langle d_V\psi(u_\lambda'(t_2)), u_\lambda(t_2) \rangle_V + \varepsilon \langle d_V\psi(u_\lambda'(t_1)), u_\lambda(t_1) \rangle_V
\]

\[
+ \varepsilon \int_{t_1}^{t_2} \langle d_V\psi(u_\lambda(t)), u_\lambda(t) \rangle_V dt + \int_{t_1}^{t_2} \langle \partial_V\phi_X(u_\lambda(t)), u_\lambda(t) \rangle_V dt
\]

\[
+ \varepsilon \int_{t_1}^{t_2} \langle F_V(u_\lambda(t)), u_\lambda(t) \rangle_V dt = \int_{t_1}^{t_2} \langle f(t) + h(t), u_\lambda(t) \rangle_V dt.
\]

Monotonicity of $d_V\psi$ in $L^p(t_1, t_2; V)$ yields

\[
(42) \quad \liminf_{n \to \infty} \int_{t_1}^{t_2} \langle d_V\psi(u_{\lambda_n}'(t)), u_{\lambda_n}'(t) \rangle_V dt \geq \int_{t_1}^{t_2} \langle \xi(t), u'(t) \rangle_V dt.
\]
Furthermore, in view of \((33), (36)\) and \((37)\), we get
\[
\lim_{n \to \infty} \int_{t_1}^{t_2} \langle d_V \psi(u_{\lambda_n}(t)) + F_V(u_{\lambda_n}), u_{\lambda_n}(t) \rangle_V dt = \int_{t_1}^{t_2} \langle d_V \psi(u(t)) + F_V(u(t)), u(t) \rangle_V dt.
\]
(43)

Thus choosing a suitable subsequence of \(\{\lambda_n\}\) in \((42)\), denoted by \(\{\lambda_k\}\), and taking the limit in \((41)\) with \(\lambda = \lambda_k \to 0\) as \(k \to \infty\), we obtain by \((40), (41), (42)\) and \((43)\)
\[
\lim_{k \to \infty} \sup \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t) \rangle_X dt \\
\leq -\varepsilon \int_{t_1}^{t_2} \langle \xi(t), u'(t) \rangle_V dt + \varepsilon \langle \xi(t_2), u(t_2) \rangle_V - \varepsilon \langle \xi(t_1), u(t_1) \rangle_V \\
- \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t) \rangle_V dt.
\]
(44)

From the fact that \(u = v \in W^{1,p}(0,T;V) \cap L^m(0,T;X), \xi \in L^{p'}(0,T;V^*), \) and \(\xi' \in L^{p'}(0,T;V^*) + L^m(0,T;X)^*\), we can apply Proposition \(2.4\) and derive
\[
\lim_{k \to \infty} \sup \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t) \rangle_X dt \\
\leq \varepsilon \langle \xi'(t), u(t) \rangle_{L^p(t_1,t_2;V) \cap L^m(t_1,t_2;X)} \\
- \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t) \rangle_V dt.
\]
(45)

Substituting \(\varepsilon \xi' = \varepsilon d_V \psi(u) + \varepsilon F_V(u) + \eta - f - h\) in \((45)\), we have
\[
\lim_{k \to \infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t) \rangle_X dt \leq \int_{t_1}^{t_2} \langle \eta(t), u(t) \rangle_X dt.
\]
(46)

Since \(\partial_V \phi_{\lambda_k}(u_{\lambda_k}) \in \partial_V \phi(J_{\lambda_k} u_{\lambda_k}) \subset \partial_X \phi_X(J_{\lambda_k} u_{\lambda_k})\) and \(\partial_X \phi_X\) is maximal monotone in \(L^m(t_1,t_2;X)\), then \((46)\) implies that (recall Proposition \(2.2\))
\[
\lim_{k \to \infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t) \rangle_X dt = \int_{t_1}^{t_2} \langle \eta(t), u(t) \rangle_X dt, \\
\eta(t) \in \partial_X \phi_X(u(t)) \quad \text{for a.e. } t \in (t_1, t_2).
\]
(47)

Moreover, since \(t_1, t_2\) is chosen arbitrarily in \(\Upsilon\) which has full Lebesgue measure in \((0,T)\), we see that \(\eta \in \partial_X \phi_X(u)\) in \(L^m(0,T;X)\).
Let $t_1, t_2 \in \mathcal{Y}$. Recall that (10) and (11) give us
\[
\varepsilon \int_{t_1}^{t_2} \langle d_V \psi(u'_\lambda(t)), u'_\lambda(t) \rangle_V \, dt \\
\leq \varepsilon \langle d_V \psi(u'_\lambda(t_2)), u_\lambda(t_2) \rangle_V - \varepsilon \langle d_V \psi(u'_\lambda(t_1)), u_\lambda(t_1) \rangle_V - \int_{t_1}^{t_2} \langle \partial_V \phi_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t) \rangle_X \, dt \\
- \int_{t_1}^{t_2} (\varepsilon d_V \psi(u_\lambda(t)) + \varepsilon F_V(u_\lambda(t)) - f(t) - h(t), u_\lambda(t))_V \, dt.
\]
Taking the limit $\lambda = \lambda_k \to 0$ in the inequality above and using (17), the definition of $\mathcal{Y}$, and Proposition 2.4 (integration by parts), we obtain
\[
\varepsilon \limsup_{k \to \infty} \int_{t_1}^{t_2} \langle d_V \psi(u'_{\lambda_k}(t)), u'_{\lambda_k}(t) \rangle_V \, dt \\
\leq \varepsilon \langle \xi(t_2), u(t_2) \rangle_V - \varepsilon \langle \xi(t_1), u(t_1) \rangle_V - \int_{t_1}^{t_2} \langle \eta(t), u(t) \rangle_X \, dt \\
- \int_{t_1}^{t_2} (\varepsilon d_V \psi(u(t)) + F_V(u(t)) - f(t) - h(t), u(t))_V \, dt \\
\leq \varepsilon \langle \xi', u \rangle_{L^p(t_1,t_2;X) \cap L^p(t_1,t_2;V')} + \varepsilon \int_{t_1}^{t_2} \langle \xi, u' \rangle_V \, dt - \int_{t_1}^{t_2} \langle \eta(t), u(t) \rangle_X \, dt \\
- \int_{t_1}^{t_2} (\varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t))_V \, dt.
\]
Substituting $\varepsilon \xi' - \eta - \varepsilon d_V \psi(u) - \varepsilon F_V(u) + f + h = 0$ in (48), we have
\[
\limsup_{k \to \infty} \int_{t_1}^{t_2} \langle d_V \psi(u'_{\lambda_k}(t)), u'_{\lambda_k}(t) \rangle_V \, dt \leq \int_{t_1}^{t_2} \langle \xi, u' \rangle_V \, dt.
\]
Hence we can repeat the same argument as that for $\eta \in \partial_X \phi_X(u)$ with aid of the maximal monotonicity of $d_V \psi$ and we can show that $\xi = d_V \psi(u)$ in $L^p(0,T;V^*)$. Therefore, we conclude that $u$ is the desired solution to (AP)$_h$ satisfying (19). The uniqueness of solution can be assured by the monotonicity of operators in (AP)$_h$ and Proposition 2.1.

3.3 Step 2 (Proof of Theorem 2.6)

Let $\Xi := L^p(0,T;V^*)$ with the weak topology. By Step 1, $\beta$ defined by the relationship (20) is well defined as an operator acting on $\Xi$. In order to apply the Schauder–Tychonoff fixed point theorem, we shall check that

1. Let $K_R := \{ h \in L^p(0,T;V^*) : |h|_{L^p(0,T;V^*)} \leq R \}$. Then there exists $R > 0$ such that $\beta$ maps $K_R$ into itself.

2. For every sequence $\{ h_n \}_{n \in \mathbb{N}}$ which weakly converges to $h$ in $L^p(0,T;V^*)$, $\{ \beta(h_n) \}_{n \in \mathbb{N}}$ weakly converges to $\beta(h)$ in $L^p(0,T;V^*)$.\[\square]
Then by virtue of Theorem 1 in Arino–Gautier–Penot [11], β possesses at least one fixed point \( h_0 \in L^{p'}(0, T; V^*) \). Obviously, \( u_{h_0} \) is a solution to \((AP)_\varepsilon\) satisfying regularities \([17]\) in Theorem 2.6.

In this subsection, \( c_2 \) and \( C_2 \) stand for general constants independent \( \varepsilon > 0 \) and \( h \).

(Proof of #1) Let \( \eta_h \) be the section of \( \partial_X \phi_X(u_h) \) satisfying \((AP)^h\). Repeating the same calculation as that for \([25]\), i.e., multiplying the equation of \((AP)^h\) by \( u_h \) and using

\[
\int_0^T \langle \eta_h(t), u_h(t) \rangle_X dt = \int_0^T \langle \eta_h(t), u_h(t) - v \rangle_X dt + \int_0^T \langle \eta_h(t), v \rangle_X dt
\]

(50)

\[
\geq \int_0^T \phi(u_h(t))dt - T\phi(v) - |v|_X \int_0^T |\eta_h(t)|_X dt \quad \forall \ v \in D(\phi)
\]

we obtain by Young’s inequality

\[
c_2 \varepsilon |u_h|_{W^{1,p}(0,T,V)}^p + c_2 |u_h|_{L^m(0,T,X)}^m + \varepsilon |u_h|_{L^2(0,T;V)}^2\]

(51)

\[
\leq \int_0^T \langle f(t) + h(t), u_h(t) \rangle_V dt + 2(\varepsilon + 1) C_2
\]

\[
\leq C_2 \left( |f|_{L^{p'}(0,T;V^*)}^p + 1 \right) + \delta |h|_{L^{p'}(0,T;V^*)}^p + C_\delta |u_h|_{L^p(0,T;V)}^p,
\]

where \( \delta > 0 \) is an arbitrarily fixed constant and \( C_\delta \) is a general constant determined only by \( \delta > 0 \). Since we assume that \( X \hookrightarrow V \) and \( m > p \), we get

\[
C_\delta |u_h|_{L^p(0,T;V)}^p \leq C_\delta |u_h|_{L^m(0,T;X)}^m \leq \delta |u_h|_{L^m(0,T;X)}^m + C_\delta.
\]

Hence for every \( \delta \leq c_2 \), we have

\[
c_2 \varepsilon |u_h|_{L^p(0,T;V)}^p \leq C_\delta \left( 1 + |f|_{L^{p'}(0,T;V^*)}^p \right) + \delta |h|_{L^{p'}(0,T;V^*)}^p.
\]

From \([8]\) we can derive

\[
c_2 \varepsilon |\beta(h)|_{L^{p'}(0,T,V^*)}^p = c_2 \varepsilon |d_V \psi(u_h)|_{L^{p'}(0,T;V^*)}^p
\]

(52)

\[
\leq C C_\delta \left( 1 + |f|_{L^{p'}(0,T;V^*)}^p \right) + C \delta |h|_{L^{p'}(0,T;V^*)}^p + C C_\delta T,
\]

where \( C \) is the constant appearing in \([8]\).

Here we fix \( R \) and \( \delta \) such that

\[
C C_\delta \left( 1 + |f|_{L^{p'}(0,T;V^*)}^p \right) + C C_\delta T = \frac{c_2 \varepsilon}{2} R^{p'}, \quad \delta = \min(c_2, \frac{c_2 \varepsilon}{2C}).
\]

Then by \([52]\), we get

\[
|\beta(h)|_{L^{p'}(0,T,V^*)}^p \leq \frac{1}{c_2 \varepsilon} \left( \frac{c_2 \varepsilon}{2} + C \delta \right) R^{p'} \leq R^{p'},
\]
which implies \( \beta(h) \in K_R \).

(Proof of \#2) Let \( \{h_n\}_{n \in \mathbb{N}} \) be a sequence such that \( h_n \to h \). Let \( u_n = u_{h_n} \) be the solution of \((AP)_{h_n}\) and denote by \( \eta_n \) the section of \( \partial_X \phi_X(u_n) \) satisfying \((AP)_{h_n}\). Moreover, let \( C' \) denote a general constant independent of \( n \), which may depend on \( \varepsilon \). Noting that \( |h_n|_{L^{p}(0, T; \mathcal{V}^*)} \leq C' \), we repeat the same procedure to establish a priori estimates as in the previous step. Indeed, multiplying the equation of \((AP)_{h_n}\) by \( u_n \), we have (see (51))

\[
|u_n|_{W^{1,p}(0, T; \mathcal{V})}^p + |u_n|_{L^2(0, T; \mathcal{V})}^2 + \int_0^T \langle \eta_n(t), u_n(t) \rangle_\mathcal{X} \, dt \leq C',
\]

which yields (see (50))

\[
|F_V(u_n)|_{L^{p'}(0, T; \mathcal{V}^*)} + \int_0^T \phi(u_n(t)) \, dt \leq C'.
\]

Then by (8), (2), and (14), we get

\[
|d_\psi u_n|_{L^{p'}(0, T; \mathcal{V}^*)} + |d_\psi u'_n|_{L^{p'}(0, T; \mathcal{V}^*)} + |u_n|_{L^m(0, T; \mathcal{X})} + |\eta_n|_{L^{m'}(0, T; \mathcal{X}^*)} \leq C'.
\]

From the boundedness of remainder terms in \((AP)_{h_n}\), we obtain

\[
|(d_\psi u'_n)|_{L^{p'}(0, T; \mathcal{V}^*)} + |a|_{L^{m'}(0, T; \mathcal{X}^*)} \leq C'.
\]

Arguments for the convergence given in the previous step can be repeated for this step. In fact, applying Theorem 3 of [12] to \( \{u_n\}_{n \in \mathbb{N}} \), which is uniformly bounded in \( W^{1,p}(0, T; \mathcal{V}) \cap L^m(0, T; \mathcal{X}) \), we can extract a subsequence denoted by \( \{u_t\}_{t \in \mathbb{N}} \) which strongly converges in \( C([0, T]; \mathcal{V}) \). Let \( u \) stands for its limit (remark \( u(0) = u(T) \) holds).

Furthermore, there exists a subsequence of \( \{u\}_{t \in \mathbb{N}} \) (we still employ the same index) such that

\[
F_V(u_t) \to \exists w \quad \text{weakly in } L^{p'}(0, T; \mathcal{V}^*),
\]

\[
\eta_t \to \exists \eta \quad \text{weakly in } L^{m'}(0, T; \mathcal{X}^*),
\]

\[
d_\psi u_t \to \exists a \quad \text{weakly in } L^{p'}(0, T; \mathcal{V}^*),
\]

\[
d_\psi u'_t \to \exists \xi \quad \text{weakly in } L^{p'}(0, T; \mathcal{V}^*),
\]

\[
(d_\psi u'_t)' \to \xi' \quad \text{weakly in } L^{p'}(0, T; \mathcal{V}^*) + L^{m'}(0, T; \mathcal{X}^*).
\]

By virtue of the demiclosedness of \( F_V \) and \( d_\psi \psi \), we can easily see that \( w = F_V(u) \) and \( a = d_\psi \psi(u) \). Therefore limit functions given above fulfill the equation

\[-\varepsilon \xi' + \varepsilon d_\psi \psi(u) + \eta + \varepsilon F_V(u) = f + h \quad \text{in } L^{p'}(0, T; \mathcal{V}^*) + L^{m'}(0, T; \mathcal{X}^*).
\]

By the same reasoning as that for (52), we can deduce the strong convergence of \( \{d_\psi \psi(u'_t)\}_{t \in \mathbb{N}} \) to \( \xi \) in \( C([0, T]; \mathcal{X}^*) \) and then \( \xi(0) = \xi(T) \).
Define \( \Upsilon_h \), which has full Lebesgue measure in \((0, T)\) by

\[
\Upsilon_h := \left\{ \begin{array}{l}
t \in (0, T) \quad \text{if } t \text{ is a Lebesgue point of } t \mapsto \langle \xi(t), u(t) \rangle_V \text{ and}
\{ l \} \text{ has a subsequence } \{ l_k^n \} \text{ tending to } \infty \text{ as } k \to \infty
s.t. \quad \left\langle d_V \psi(u'_k(t)), u''_k(t) \right\rangle_V \to \langle \xi(t), u(t) \rangle_V \text{ as } k \to \infty.
\end{array} \right. 
\]

Let \( t_1 < t_2 \) belong to \( \Upsilon \). Tracing the argument from (11) to (15), we can show that

\[
\limsup_{k \to \infty} \int_{t_1}^{t_2} \langle \eta_k(t), u_k(t) \rangle_X dt \leq \varepsilon \langle \xi(t), u(t) \rangle_{L^p(t_1, t_2; V) \cap L^m(t_1, t_2; X)} - \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t) \rangle_V dt
= \int_{t_1}^{t_2} \langle \eta(t), u(t) \rangle_X dt,
\]

where \( \{ l_k \}_{k \in \mathbb{N}} \) is a suitable subsequence of \( \{ l \} \). Hence from Proposition 2.2, we derive
\( \eta \in \partial_X \phi_X(u) \) in \( L^m(0, T; X) \). Moreover, repeating the same verification as that for (19), we can obtain

\[
\limsup_{k \to \infty} \int_{t_1}^{t_2} \langle d_V \psi(u_k'(t)), u_k'(t) \rangle_V dt \leq \int_{t_1}^{t_2} \langle \xi(t), u'(t) \rangle_V dt,
\]

which implies \( \xi = d_V \psi(u') \) in \( L^p'(0, T; V^*) \).

Since the uniqueness of solution to (AP)\(^{h}\) assures that the above argument does not depend on the choice of subsequences, the original sequence \( \{ d_V \psi(u'_n) \}_{n \in \mathbb{N}} = \{ \beta(h_n) \}_{n \in \mathbb{N}} \) also converges to \( d_V \psi(u') = \beta(h) \) weakly in \( L^p'(0, T; V^*) \).

**Remark 3.4.** In order to derive the uniqueness of the solution of (AP)\(^{v}\), by the standard argument, we need to handle the following terms concerning the difference of two solutions \( u_1 \) and \( u_2 '\):

\[
\langle d_V \psi(u'_1) - d_V \psi(u'_2), u_1 - u_2 \rangle_V, \quad \langle \partial \phi(u_1) - \partial \phi(u_2), u'_1 - u'_2 \rangle_V.
\]

Even for Cauchy problem, assumptions (7)-(10) are not enough to control these terms. The characterization of Cauchy problem by Euler–Lagrange equation of the WED functional copes with this difficulty arising in the doubly nonlinear evolution equations and enables us to assure the uniqueness of solution to approximation problem when either \( \phi \) or \( \psi \) is strictly convex (see Theorem 4.2 of Akagi–Stefanelli [4]). If one leaves variational approach, however, the uniqueness can not be assured only by such a natural condition especially in the case of the time-periodic problem, where the technical assumptions might be more aggravated than those in Cauchy problem in general.

In order to complete the proof of Theorem 2.6 we verify the following:
Lemma 3.5. There exist a solution $u_\epsilon$ to (AP)$_\epsilon$ which satisfies

$$
\int_0^T \langle d_V \psi(u'_\epsilon(t)), u'_\epsilon(t) \rangle_V \, dt \leq \int_0^T \langle f(t), u'_\epsilon(t) \rangle_V \, dt.
$$

Proof. We return to Step 1. Let $u_\lambda$ be the unique solution to (AP)$_\lambda$. Multiplying (AP)$_\lambda$ by $u'_\lambda$ and integrating over $(0, T)$ with respect to $t$, we have

$$
-\epsilon \int_0^T \langle (d_V \psi(u'_\lambda(t)))', u'_\lambda(t) \rangle_V \, dt - \int_0^T \langle h(t), u'_\lambda(t) \rangle_V \, dt = \int_0^T \langle f(t), u'_\lambda(t) \rangle_V \, dt.
$$

Here we used Proposition 2.3 (chain rule) and the periodicity of $u_\lambda$.

By formal calculation, we get

$$
- \int_0^T \langle (d_V \psi(u'_\lambda(t)))', u'_\lambda(t) \rangle_V \, dt
$$

$$
= - \langle d_V \psi(u'_\lambda(0)), u'_\lambda(T) - u'_\lambda(0) \rangle_V + \int_0^T \langle d_V \psi(u'_\lambda(t)), u''_\lambda(t) \rangle_V \, dt
$$

$$
\geq \psi(u'_\lambda(0)) - \psi(u'_\lambda(T)) + \int_0^T \frac{d}{dt} \psi(u'_\lambda(t)) \, dt = 0.
$$

Although we can not assure the existence of $u''_\lambda(t)$ in a proper way, we can rigorously justify the following:

$$
\int_0^T \langle (d_V \psi(u'_\lambda(t)))', u'_\lambda(t) \rangle_V \, dt \geq 0.
$$

Indeed, by the same procedure as that in the proof for Lemma A.1 in [4], one can see that

$$
\int_{t_1}^{t_2} \langle (d_V \psi(u'_\lambda(t)))', u'_\lambda(t) \rangle_V \, dt \leq \psi^*(d_V \psi(u'_\lambda(t_2))) - \psi^*(d_V \psi(u'_\lambda(t_1)))
$$

holds for every $t_1, t_2$ belonging to

$$
\Theta_\lambda := \left\{ t \in (0, T) \mid \begin{array}{l}
\text{t is a Lebesgue point of} \\
\text{t \mapsto \langle (d_V \psi(u'_\lambda(t)))', u'_\lambda(t) \rangle_V}, \\
\text{and } u_\lambda \text{ is differentiable at } t.
\end{array} \right\},
$$

where $\psi^*$ denotes the Legendre–Fenchel transform of $\psi$. Since $d_V \psi(u'_\lambda(t)) \in W^{1,p'}(0, T; V^*)$ and $u_\lambda \in W^{1,p}(0, T; V)$ (recall [21]), $\Theta_\lambda$ has full Lebesgue measure in $(0, T)$. Moreover, from the fundamental facts that $\psi^*$ is proper l.s.c. convex functional on $V^*$ and $[v, u] \in \partial \psi \psi^*$ is equivalent to $v = \partial \psi(u)$, we can derive the absolute continuity of $\psi^*(d_V \psi(u'_\lambda(t)))$ on $[0, T]$. Then letting $t_1 \to 0$ and $t_2 \to T$, we obtain [55].

Hence $u_\lambda$ satisfies

$$
\int_0^T \langle h(t), u'_\lambda(t) \rangle_V \, dt \leq \int_0^T \langle f(t), u'_\lambda(t) \rangle_V \, dt.
$$
Since we already showed that $u'_\lambda \to u'_h$ in $L^p(0, T; V)$ as $\lambda \to +0$, where $u_h$ stands for the unique solution to $(AP)^h$, (54) leads to

\begin{equation}
- \int_0^T \langle h(t), u_h'(t) \rangle_V dt \leq \int_0^T \langle f(t), u_h'(t) \rangle_V dt
\end{equation}

for every $h \in L^{p'}(0, T; V^*)$. Now let $h_0 \in L^{p'}(0, T; V^*)$ be a fixed point of $\beta$, i.e., $h_0 = \beta(h_0) := -d_V \psi(u_{h_0})$. Then substituting $h_0$ for $h$ in (58), we can assure that $u_{h_0}$ is one of solutions to $(AP)_\varepsilon$ satisfying (54).

\[\Box\]

### 3.4 Step 3 (Proof of Theorem 2.7: Case $p < m$)

Henceforth, let $u_\varepsilon$ be the solution of $(AP)_\varepsilon$ given in Theorem 2.6 and $C_3$ stand for a general constant independent of the parameter $\varepsilon \in (0, 1)$. We first note that the definition of the subdifferential and (7) yield

\[\langle d_V \psi(u'_\varepsilon(t)), u'_\varepsilon(t) \rangle_V \geq \psi(u'_\varepsilon(t)) - \psi(0) \geq \frac{1}{C} |u'_\varepsilon(t)|^p_V - 1 - \psi(0)\]

Hence by (54), we get

\[
\int_0^T \left| u'_\varepsilon(t) \right|^p_V dt \leq C_3 \left( \int_0^T \langle f(t), u'_\varepsilon(t) \rangle_V dt + 1 \right) \leq \frac{1}{2} \int_0^T \left| u'_\varepsilon(t) \right|^p_V dt + C_3 \left( |f|^{p'}_{L^{p'}(0, T; V^*)} + 1 \right),
\]

whence follows the a priori bound for $|u'_\varepsilon|_{L^p(0, T; V)}$. Then by (8) and (12), we obtain

\begin{equation}
\int_0^T \left| u'_\varepsilon(t) \right|^p_V dt + \int_0^T \left| d_V \psi(u'_\varepsilon(t)) \right|^p_{V^*} dt + \int_0^T \psi(u'_\varepsilon(t)) dt \leq C_3.
\end{equation}

Next multiplying $(AP)_\varepsilon$ by $u_\varepsilon$ and repeating the same argument as that for (54), we have

\[
\varepsilon |u_\varepsilon|^{p}_{W^{1,p}(0,T;V)} + |u_\varepsilon|^{m}_{L^m(0,T;X)} + \varepsilon |u_\varepsilon|^{2}_{L^2(0,T;V)} \
\leq C_3 \left( |f|^{p'}_{L^{p'}(0,T;V^*)} + |d_V \psi(u'_\varepsilon(t))|^{p'}_{L^{p'}(0,T;V^*)} + |u_\varepsilon|^{p}_{L^p(0,T;V)} + 1 \right) \
\leq C_3 + \frac{1}{2} |u_\varepsilon|^{m}_{L^m(0,T;X)},
\]

that is,

\begin{equation}
\varepsilon |u_\varepsilon|^{p}_{W^{1,p}(0,T;V)} + |u_\varepsilon|^{m}_{L^m(0,T;X)} + \varepsilon |u_\varepsilon|^{2}_{L^2(0,T;V)} \leq C_3.
\end{equation}

Hence the canonical embedding $L^m(0,T;X) \hookrightarrow L^p(0,T;V)$ and (59) yield

\begin{equation}
|u_\varepsilon|^{p}_{W^{1,p}(0,T;V)} \leq C_3.
\end{equation}

Moreover, from (8) and (10), we can derive

\begin{equation}
\int_0^T |\eta_\varepsilon(t)|_{X^*} dt + \int_0^T |d_V \psi(u_\varepsilon(t))|^{p'}_{V^*} dt \leq C_3,
\end{equation}
where $\eta_\varepsilon$ is the section of $\partial_X \phi_X(u_\varepsilon(t))$ satisfying (AP)$_\varepsilon$. Hence by the equation of (AP)$_\varepsilon$, we also have

$$\tag{63} |\varepsilon(d_V \psi(u_\varepsilon'))|_{L^p'(0,T;V^*)} + |\varepsilon F_V(u_\varepsilon)|_{L^p'(0,T;V^*)} \leq C_3.$$  

By using (59)–(63), we discuss the convergence of $u_\varepsilon$. To begin with, (60) and (61) enable us to apply Theorem 3 of [42] and extract a subsequence $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}}$ which converges strongly in $C([0,T];V)$. Its limit, denoted by $u$, clearly satisfies $u(0) = u(T)$. We are going to show that $u$ is the desired periodic solution of (AP). By (61) and (62), we get

$$|\varepsilon d_V \psi(u_\varepsilon)|_{L^{p'}(0,T;V^*)} + |\varepsilon F_V(u_\varepsilon)|_{L^{p'}(0,T;V^*)} \leq \varepsilon C_3,$$

which implies that $\{\varepsilon_n d_V \psi(u_{\varepsilon_n})\}_{n \in \mathbb{N}}$ and $\{\varepsilon_n F_V(u_{\varepsilon_n})\}_{n \in \mathbb{N}}$ converge to zero strongly in $L^{p'}(0,T;V^*)$. Furthermore, there exists a subsequence, still denoted by $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}}$, such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^m(0,T;X),$$

$$\eta_{\varepsilon_n} \rightharpoonup \exists \eta \quad \text{weakly in } L^{m'}(0,T;X^*),$$

$$d_V \psi(u_{\varepsilon_n})' \rightharpoonup \exists \xi \quad \text{weakly in } L^{p'}(0,T;V^*),$$

$$\varepsilon (d_V \psi(u_{\varepsilon_n}'))' \rightharpoonup \exists \zeta \quad \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).$$

For every $v \in W^{1,m}(0,T;X)$ with $v(0) = v(T)$, we get by (59)

$$\langle \varepsilon_n (d_V \psi(u_{\varepsilon_n}'))', v \rangle_{L^{p'}(0,T;V^*)} = -\varepsilon_n \int_0^T \langle d_V \psi(u_{\varepsilon_n}(t))', v'(t) \rangle_V dt \rightarrow 0$$

as $n \rightarrow \infty$ (use Proposition 2.4). Then $\xi = 0$ holds by the density. Therefore, we obtain $\xi + \eta = f$. Immediately, $\eta \in L^{p'}(0,T;V^*)$, since the remainders $\xi$ and $f$ are both members of $L^{p'}(0,T;V^*)$.

Multiplying (AP)$_{\varepsilon_n}$ by $u_{\varepsilon_n}$, we have

$$\int_0^T \langle \eta_{\varepsilon_n}(t), u_{\varepsilon_n}(t) \rangle_X dt = -\varepsilon_n \int_0^T \langle d_V \psi(u_{\varepsilon_n}(t)), u_{\varepsilon_n}(t) \rangle_V dt - \varepsilon_n \int_0^T \langle d_V \psi(u_{\varepsilon_n}(t)), u_{\varepsilon_n}(t) \rangle_V dt$$

$$- \varepsilon_n |u_{\varepsilon_n}|^2_{L^2(0,T;V)} - \int_0^T \langle d_V \psi(u_{\varepsilon_n}(t)), u_{\varepsilon_n}(t) \rangle_V dt + \int_0^T \langle f(t), u_{\varepsilon_n}(t) \rangle_V dt.$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \int_0^T \langle \eta_{\varepsilon_n}(t), u_{\varepsilon_n}(t) \rangle_X dt = -\int_0^T \langle \xi(t), u(t) \rangle_V dt + \int_0^T \langle f(t), u(t) \rangle_V dt$$

$$= \int_0^T \langle \eta(t), u(t) \rangle_V dt = \int_0^T \langle \eta(t), u(t) \rangle_X dt,$$
which implies \( \eta \in \partial_X \phi_X(u) \) thanks to Proposition 2.2. Hence by virtue of (5) together with the fact that \( \eta \in L^{\prime}(0; V^*) \), we can conclude

\[
\eta \in \partial_V \phi(u).
\]  

Finally, letting \( \varepsilon_n \to 0 \) in (54) of Lemma 3.5, we can see that

\[
\limsup_{n \to \infty} \int_0^T \langle d_V \psi(u'_\varepsilon_n(t)), u'_\varepsilon_n(t) \rangle_V dt \leq \int_0^T \langle f(t), u'(t) \rangle_V dt = \int_0^T \langle \xi(t), u'(t) \rangle_V dt.
\]

Here we used Proposition 2.3 and (64). Then maximal monotonicity of \( d_V \psi \) leads to

\[
\xi = d_V \psi(u') \in L^{\prime}(0; V^*) \text{, hence it follows that } u \text{ is a solution to (AP)}.
\]

Furthermore Proposition 2.3 together with (61) and (64) assures that \( \phi(u(t)) \) is absolutely continuous on \([0, T]\) and hence (9) assures that \( u \in L^\infty(0, T; X) \).

### 3.5 Step 4 (Proof of Theorem 2.7: Case \( m \leq p \))

In this subsection, we consider the excluded case, i.e., the case where \( m \leq p \). Put

\[
\Phi := \phi + \frac{\mu}{1 + \alpha} \phi^{1+\alpha},
\]

where \( \mu \in (0, 1) \) and \( \alpha \) is some fixed exponent with \( \alpha > p/m - 1 \). It is easy to see that \( \Phi \) is proper l.s.c. convex functional over \( V \). Since \( \phi \) satisfies (9),

\[
\Phi(u) \geq c_\mu |u|^m_X(1 + |u|^m_\alpha) - C_\mu \geq c_\mu |u|^m_X^{(\alpha+1)} - C_\mu
\]

holds for every \( u \in D(\Phi) = D(\phi) \) with some constants \( c_\mu, C_\mu > 0 \) (which may depend on the parameter \( \mu \)). Moreover, since

\[
\partial_V \Phi = \partial_V \phi + \mu \phi^\alpha \partial_V \phi, \quad \partial_X \Phi_X = \partial_X \phi_X + \mu \phi^\alpha_X \partial_X \phi_X
\]

(whose proof will be given in Appendix), then for any \([u, \eta] \in \partial_X \Phi_X \) there exist some \( C > 0 \) independent of \( \mu \) such that

\[
|\eta|_X^{m'}(\frac{\alpha+1}{m' \alpha + 1}) \leq C \left[ |u|^m_X \left( 1 + |u|^m_\alpha \right) \right]^{\frac{\alpha+1}{m' \alpha + 1}} \leq C(|u|^m_X^{(\alpha+1)} + 1),
\]

since \( \phi \) satisfies (10) and (15). Hence \( \Phi \) fulfills (9) and (10) with \( \tilde{m} := m(\alpha + 1) > p \) (note that the Hölder conjugate of \( \tilde{m} \) coincides with \( \tilde{m}' = m'(\alpha + 1)/(m' \alpha + 1) \)). Then we can
carry out the same argument given above with \( \phi \) replaced by \( \Phi \). That is to say, for every \( \mu \in (0, 1) \) and \( f \in L^{p'}(0, T; V^*) \), time-periodic problem

\[
(\text{AP})^*_\mu \begin{cases}
d_V \psi(u'_\mu) + (1 + \mu \phi^\alpha(u_\mu)) \partial_V \phi(u_\mu) \ni f, \\
u_\mu(0) = u_\mu(T),
\end{cases}
t \in (0, T) \quad \text{in } V^*,
\]

possesses at least one solution satisfying

\[
u_\mu \in W^{1,p}(0, T; V) \cap L^m(0, T; X),
\]

\[
d_V \psi(u'_\mu), (1 + \mu \phi^\alpha(u_\mu))\eta_\mu \in L^{p'}(0, T; V^*),
\]

where \( \eta_\mu \in \partial_V \phi(u_\mu) \).

We shall establish a priori estimates of \( u_\mu \) independent of \( \mu \) by repeating the same calculation in Step 3. Let \( c_4, C_4 > 0 \) denote general constants independent of \( \mu \in (0, 1) \). Multiplying \( (\text{AP})^*_\mu \) by \( u'_\mu \), using the chain rule \( \Phi'(u(t)) = \langle (1 + \mu \phi^\alpha(u_\mu(t)))\eta_\mu, u'_\mu(t) \rangle_V \), and integrating over \([0, T] \), we have by (71), (8) and (12) (see (24))

\[
\int_0^T |u'_\mu(t)|^p_V \, dt + \int_0^T |d_V \psi(u'_\mu(t))|^p_{V^*} \, dt + \int_0^T \psi(u'_\mu(t)) \, dt \leq C_4.
\]

By the equation of \( (\text{AP})^*_\mu \), we obtain

\[
|\eta_\mu|_{L^{p'}(0, T; V^*)} + |\mu \phi^\alpha(u_\mu)\eta_\mu|_{L^{p'}(0, T; V^*)} \leq |d_V \psi(u'_\mu)|_{L^{p'}(0, T; V^*)} + |f|_{L^{p'}(0, T; V^*)} \leq C_4.
\]

From (15), (16) and the canonical embedding \( |\eta_\mu|_{V^*} \geq c_4 |\eta_\mu|_{X^*} \), we can derive

\[
\int_0^T \phi^{p'/m'}(u_\mu(t)) \, dt \leq C_4.
\]

By Proposition 2.3, \( \phi(u_\mu(\cdot)) \) is absolutely continuous on \([0, T]\) and hence there is \( t_0 = t_0^\mu \in [0, T] \) at which \( \phi(u_\mu(\cdot)) \) attains its minimum. Immediately, (70) implies \( \phi(u_\mu(t_0)) \leq C_4 \). Hence testing \( (\text{AP})^*_\mu \) by \( u'_\mu \) again and integrating over \([t_0, t]\) with \( t \in [t_0, t_0 + T] \), we obtain

\[
\sup_{0 \leq t \leq T} \phi(u_\mu(t)) \leq \sup_{0 \leq t \leq T} \Phi(u_\mu(t)) \leq C_4.
\]

Combining (71) with (9) and (10), we can show that

\[
c_4 \sup_{0 \leq t \leq T} |u_\mu(t)|_V \leq \sup_{0 \leq t \leq T} |u_\mu(t)|_X + \sup_{0 \leq t \leq T} |\eta_\mu(t)|_{X^*} \leq C_4.
\]

Note that (68), (69) and (72) enable us to follow the same convergence argument as that developed in Step 3. Indeed, (69) and (71) yield

\[
|\mu \phi^\alpha(u_\mu)\eta_\mu|_{L^{p'}(0, T; V^*)} \leq \mu C_4 \to 0 \quad \text{as } \mu \to 0.
\]
Applying Theorem 3 of Simon [42] and standard argument, we can extract a subsequence of \( \{u_{\mu}\}_{\mu \in \mathbb{N}} \) (we skip relabeling) such that

\[
\begin{align*}
 u_{\mu} & \to u \quad \text{strongly in } C([0,T];V), \\
 & \quad \text{weakly in } W^{1,p}(0,T;V), \\
 d_V \psi(u'_{\mu}) & \to \xi \quad \text{weakly in } L^p(0,T;V^*), \\
 \eta_{\mu} & \to \eta \quad \text{weakly in } L^p(0,T;V^*),
\end{align*}
\]

and \( \xi + \eta = f \) in \( L^p(0,T;V^*) \). Using (73), we have

\[
\limsup_{\mu \to 0} \int_0^T \langle \eta_{\mu}(t),u_{\mu}(t) \rangle_V dt = -\int_0^T \langle \xi(t),u(t) \rangle_V dt + \int_0^T \langle f(t),u(t) \rangle_V dt
\]

which together with Proposition 2.2 implies \( \eta \in \partial_V \phi(u) \). Furthermore, since

\[
\int_0^T \langle d_V \psi(u'_{\mu}(t)),u'_{\mu}(t) \rangle_V dt = \Phi(u_{\mu}(0)) - \Phi(u_{\mu}(T)) + \int_0^T \langle f(t),u'_{\mu}(t) \rangle_V dt \\
\to \phi(u(0)) - \phi(u(T)) + \int_0^T \langle f(t),u'(t) \rangle_V dt \quad \text{as } \mu \to 0,
\]

we obtain

\[
\limsup_{\mu \to 0} \int_0^T \langle d_V \psi(u'_{\mu}(t)),u'_{\mu}(t) \rangle_V dt = -\int_0^T \langle \eta(t),u'(t) \rangle_V dt + \int_0^T \langle f(t),u'(t) \rangle_V dt
\]

which together with Proposition 2.2 leads to \( \xi = d_V \psi(u') \). Thus it is shown that \( u \) gives a solution of (AP) satisfying (18).

\[
\square
\]

4 Structural stability

In this section, we show that the method developed in the previous section is applicable to the study for the structural stability for solution to (AP). More precisely, we here consider some perturbation to the functionals \( \phi \) and \( \psi \) in the following sense (see Definition 3.17 and Proposition 3.19 in Attouch [12]).

**Definition 4.1.** Let \( Z \) be a reflexive Banach space and let \( \varphi \) and \( \varphi_n \) \((n \in \mathbb{N})\) be proper lower semi-continuous convex functions from \( Z \) to \(( -\infty, +\infty ] \). Then it is said that \( \varphi_n \) converges to \( \varphi \) in the sense of Mosco, if the following two conditions (a) and (b) hold.
Periodic Problem for Doubly Nonlinear Equation

(a) (Liminf condition) If \( u_n \rightharpoonup u \) weakly in \( Z \), then \( \lim \inf_{n \to \infty} \varphi_n(u_n) \geq \varphi(u) \) holds.

(b) (Existence of recovery sequence) For every \( u \in D(\varphi) \), there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \) such that \( u_n \to u \) strongly in \( Z \) and \( \varphi_n(u_n) \to \varphi(u) \).

We here present the following fact (see Theorem 3.66 and Proposition 3.59 in [12]), which gives a generalization of Proposition 2.2.

**Proposition 4.2.** Let \( \{\varphi_n\}_{n \in \mathbb{N}} \) be a sequence of proper l.s.c. convex functions on a reflexive Banach space \( Z \) which converges to a proper l.s.c. convex function \( \varphi : Z \to (-\infty, +\infty] \) on \( Z \) in the sense of Mosco. Assume that \( [u_n, v_n] \in \partial_Z \varphi_n \) satisfies \( u_n \rightharpoonup u \) weakly in \( Z \), \( v_n \rightharpoonup v \) weakly in \( Z^* \), and

\[
\limsup_{n \to \infty} \langle v_n, u_n \rangle_Z \leq \langle v, u \rangle_Z.
\]

Then \( [u, v] \in \partial_Z \varphi \) and \( \langle v_n, u_n \rangle_Z \to \langle v, u \rangle_Z \) as \( n \to \infty \).

To state our result, we introduce the following growth conditions on \( \phi_n \) and \( \psi_n \), which is similar to (A.1):

(A.2) \( \{\phi_n\}_{n \in \mathbb{N}} \) and \( \{\psi_n\}_{n \in \mathbb{N}} \) are sequences of proper lower semi-continuous convex functionals and Gâteaux differentiable convex functionals over \( V \), respectively such that \( \phi_n \) and \( \psi_n \) converge to \( \phi \) and \( \psi \) in the sense of Mosco on \( V \), respectively. Furthermore there exist some constants \( C > 0 \) independent of the index \( n \) satisfying

\[
(74) \quad |u|_V^p \leq C(\psi_n(u) + 1) \quad \forall u \in V,
\]

\[
(75) \quad |d_V \psi_n(u)|_{V^*}^p \leq C(|u|_V^p + 1) \quad \forall u \in V,
\]

\[
(76) \quad |u|_X^m \leq C(\phi_n(u) + 1) \quad \forall u \in D(\phi_n),
\]

\[
(77) \quad |\eta|_{X^*}^{m'} \leq C(|u|_X^m + 1) \quad \forall [u, \eta] \in \partial_X(\phi_n)_X,
\]

where \( p, m \in (1, \infty) \) and \( (\phi_n)_X \) stands for the restriction of \( \phi_n \) onto \( X \).

Then our result is stated as follows.

**Theorem 4.3.** Assume (A.0), (A.1) and (A.2) and let \( u_n \) be solutions of

\[
(AP)_n \begin{cases}
  d\psi_n(u'_n(t)) + \partial\phi_n(u_n(t)) \ni f_n(t), & t \in (0, T) \text{ in } V^*, \\
  u_n(0) = u_n(T),
\end{cases}
\]

where \( \{f_n\}_{n \in \mathbb{N}} \) and \( f \) satisfy either of

i) \( f_n \to f \) strongly in \( L^p'(0, T; V^*) \),
ii) \( f_n \rightharpoonup f \) weakly in \( W^{1,p'}(0,T;V^*) \) and \( f(0) = f(T) \), \( f_n(0) = f_n(T) \) hold for any \( n \).

Then there exist a subsequence \( \{u_j\}_{j \in \mathbb{N}} := \{u_{n_j}\}_{j \in \mathbb{N}} \) and its limit \( u \) such that

\[
u_j \to u \quad \text{strongly in } C([0,T];V),
\]

\[
u_j \to u \quad \text{weakly in } W^{1,p}(0,T;V),
\]

\[
u_j \to u \quad \ast -\text{weakly in } L^\infty(0,T;X),
\]

\[
d_V\psi_j(u_j) \to d_V\psi(u) \quad \text{weakly in } L^p(0,T;V^*),
\]

\[
\eta_j \to \eta \quad \text{weakly in } L^p(0,T;V^*),
\]

and \( u \) is a solution to (AP). Here \( \eta \) and \( \eta_j = \eta_{n_j} \) denote the sections of \( \partial \phi(u) \) and \( \partial \phi_{n_j}(u_{n_j}) \) satisfying the equation of (AP) and (AP)\( _{n_j} \), respectively.

**Proof.** It is clear that assumptions \((74)\) and \((75)\) yield \((11)\)–\((13)\) with \( \psi \) replaced by \( \psi_n \) and \( C \) independent of \( n \), and \((14)\) with \( \phi \) replaced by \( \phi_n \) is a direct consequence of \((76)\) and \((77)\). Moreover, \((15)\)–\((16)\) also hold with \( \phi \) replaced by \( \phi_n \) by virtue of (b) of Definition \(4.1\). Indeed, let \( v \in D(\phi) \), then there exists a recovery sequence \( \{v_n\}_{n \in \mathbb{N}} \) which satisfies \( v_n \to v \) in \( V \) and \( \phi_n(v_n) \to \phi(v) \).

Remark that \((76)\) implies uniform boundedness of \( |v_n|_X \). By the definition of subdifferential,

\[
\phi_n(u) \leq \phi_n(v_n) + \langle \eta, u - v_n \rangle_X \leq \phi_n(v_n) + |\eta|_{X^*}(|u|_X + |v_n|_X)
\]

\[\leq C + |\eta|_{X^*}(|u|_X + C),\]

where \([u, \eta] \in \partial_X(\phi_n)_X \) and \( C \) is independent of \( n \). Hence \( \phi_n \) fulfills \((15)\) by \((77)\) and \((16)\) by \((76)\).

Then repeating exactly the same manipulations as those for \((68)\)–\((72)\) in Step 4 of the previous section, we can establish the following estimates for solutions \( u_n \) of (AP)\( _{n} \):

\[
|u'_n|_{L^p(0,T;V)} + |d_V\psi_n(u'_n)|_{L^p(0,T;V)} + \int_0^T \psi_n(u'_n(t))dt \leq C_5,
\]

\[
\sup_{0 \leq t \leq T} |u_n(t)|_X + \sup_{0 \leq t \leq T} |\eta_n(t)|_{X^*} + \sup_{0 \leq t \leq T} \phi_n(u_n(t)) \leq C_5,
\]

where \( C_5 > 0 \) is a general constant independent of \( n \). Therefore we can reprise the same discussion of convergence as that in Step 4 of the previous section and obtain

\[
\limsup_{n \to \infty} \int_0^T \langle \eta_n, u_n(t) \rangle_V dt \leq \int_0^T \langle \eta(t), u(t) \rangle_V dt,
\]

\[
\limsup_{n \to \infty} \int_0^T \langle d_V\psi_n(u'_n(t)), u'_n(t) \rangle_V \leq \int_0^T \langle \xi(t), u'(t) \rangle_V dt,
\]

where \( \eta \) and \( \xi \) are limits of (a suitable subsequence of) \( \{\eta_n\}_{n \in \mathbb{N}} \) and \( \{d_V\psi_n(u'_n(t))\}_{n \in \mathbb{N}} \), respectively. Here assumption i) on \( f_n \) is used to assert

\[
\int_0^T \langle f_n(t), u'_n(t) \rangle_V dt \to \int_0^T \langle f(t), u'(t) \rangle_V dt.
\]
as \( n \to \infty \) and obtain (78). Assumption ii) also leads to
\[
\int_0^T \langle f_n(t), u_n'(t) \rangle_V dt = \langle f_n(T), u_n(T) \rangle_V - \langle f_n(0), u_n(0) \rangle_V - \int_0^T \langle f_n'(t), u_n(t) \rangle_V dt \\
\to - \int_0^T \langle f'(t), u(t) \rangle_V dt = \int_0^T \langle f(t), u'(t) \rangle_V dt.
\]
Therefore, it follows our result with the aid of Proposition 4.2. \( \square \)

5 Application

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with sufficiently smooth boundary \( \partial \Omega \). As in Akagi–Stefanelli [4], we consider the following doubly nonlinear parabolic equation:

\[
(DNP) \begin{cases}
\alpha(u'(x,t)) - \Delta_m^a u(x,t) = f(x,t) & (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0 & (x,t) \in \partial \Omega \times (0,T),
\end{cases}
\]

where
\[
\Delta_m^a u(x) := \nabla \cdot \left( a(x) |\nabla u(x)|^{m-2} \nabla u(x) \right), \quad x \in \Omega, \ m \in (1,\infty), \ a : \Omega \to \mathbb{R}.
\]

In [4], it is assumed that \( a, \alpha \), and the exponent \( m \) satisfy

(a.1) There exist some constants \( a_1, a_2 > 0 \) such that \( a_1 \leq a(x) \leq a_2 \) for a.e. \( x \in \Omega \).

(a.2) \( \alpha : \mathbb{R} \to \mathbb{R} \) is a single-valued maximal monotone function with \( D(\alpha) = \mathbb{R} \). Moreover, there exist \( p \in (1,\infty) \) and constants \( c, C > 0 \) such that
\[
c |s|^p - \frac{1}{c} \leq A(s), \ |\alpha(s)|^p' \leq C(|s|^p + 1) \quad \forall s \in \mathbb{R},
\]

where \( A(s) := \int_0^s \alpha(\sigma)d\sigma \) (primitive function of \( \alpha \)).

(a.3) \( p < m^* := \frac{Nm}{(N - m)_+} \).

To reduce (DNP) to (AP), set
\[
V := L^p(\Omega), \ X := W^{1,m}_0(\Omega)
\]
and define functionals \( \psi \) and \( \phi \) on \( V \) by
\[
\psi(u) := \int_\Omega A(u(x))dx,
\]

\[
\phi(u) := \begin{cases}
\frac{1}{m} \int_\Omega a(x)|\nabla u(x)|^m dx & \text{if } u \in W^{1,m}_0(\Omega), \\
+ \infty & \text{otherwise}.
\end{cases}
\]

(79)
Note that $V$, $X$, and $V^* = L^p(\Omega)$ are uniformly convex and the embedding $X \hookrightarrow V$ is compact by (a.3). From the growth condition assumed in (a.2), we have $D(\psi) = L^p(\Omega)$. It is easy to obtain the convexity and differentiability of $\psi$ on $V$ and see that its derivative coincides with $d_V \psi(u) = \alpha(u)$. Since $\alpha$ is assumed to be a non-degenerate coefficient on $\Omega$, we can show that $\phi_X$ is differentiable on $X$ and $\partial_X \phi_X(u) = -\Delta_m^a u$ by the standard variational argument (immediately $\partial_V \phi(u) = -\Delta_m^a u$ by $\partial_V \phi \subset \partial_X \phi_X$). Moreover, assumptions (a.1) and (a.2) lead to growth conditions (7)–(10).

Therefore, (A.0) and (A.1) are verified for (DNP), so Theorem 2.7 assures that the following result holds.

**Theorem 5.1.** Assume (a.1)–(a.3). Then for every $f \in L^p(0, T; L^p(\Omega))$, (DNP) possesses at least one time-periodic solution satisfying

$$u \in W^{1,p}(0, T; L^p(\Omega)) \cap C([0, T]; W_0^{1,m}(\Omega)),
\alpha(u'), \Delta_m^a u \in L^p(0, T; L^p(\Omega)).$$

**Proof.** The assertions above follow from the direct application of Theorem 2.7 except $u \in C([0, T]; W_0^{1,m}(\Omega))$. To verify this, we first note that (a.1) assures that $L_0^n(\Omega)$ with norm $\|w\|_{L_0^n} := (\int_\Omega a |w|^m dx)^{1/m}$ is uniformly convex and $m \phi(u(t)) = |\nabla u(t)|_{L_0^n}^m$ is (absolutely) continuous on $[0, T]$. Therefore we easily derive $\nabla u \in C([0, T]; (L_0^n(\Omega))^{d})$, which together with (a.1) assures $u \in C([0, T]; W_0^{1,m}(\Omega))$. \qed

We next consider the structural stability of (DNP). For this purpose, we introduce the following conditions:

(a.4) \{${a}_n$\}$_{n \in \mathbb{N}}$ is a sequence of functions $a_n : \Omega \rightarrow \mathbb{R}$ such that $a_n(x) \rightarrow a(x)$ as $n \rightarrow \infty$ for a.e. $x \in \Omega$. In addition, (a.1) with $a$ replaced by $a_n$ holds for all $n$.

(a.5) \{${\alpha}_n$\}$_{n \in \mathbb{N}}$ is a sequence of functions $\alpha_n : \mathbb{R} \rightarrow \mathbb{R}$ such that the sequence of primitive functions $A_n(s) := \int_0^s \alpha_n(\sigma)d\sigma$ converges to $A(s) := \int_0^s \alpha(\sigma)d\sigma$ in the following sense:

i) If $s_n \rightarrow s$ in $\mathbb{R}$, then $\liminf_{n \rightarrow \infty} A_n(s_n) \geq A(s)$.

ii) For every $s \in \mathbb{R}$, there exists a sequence \{${s}_n$\}$_{n \in \mathbb{R}}$ such that $A_n(s_n) \rightarrow A(s)$ as $n \rightarrow \infty$.

In addition, (a.2) holds with functions $\alpha$ and $A$ replaced by $\alpha_n$ and $A_n$, respectively by the same $c, C > 0$ (independent of $n$).

Define $\psi_n$ and $\phi_n$ by (19) with $A(s)$ and $a(x)$ replaced by $A_n(s)$ and $a_n(x)$, respectively. Then according to (14) and standard facts in (12), (a.4) and (a.5) assure Mosco convergence.
of \( \{\psi_n\}_{n \in \mathbb{N}} \) and \( \{\phi_n\}_{n \in \mathbb{N}} \) to \( \psi \) and \( \phi \) on \( V \), respectively. Hence it follows from Theorem 4.3 that

**Theorem 5.2.** Assume (a.1)–(a.5) and \( \{f_n\}_{n \in \mathbb{N}} \) and \( f \) satisfy either of

i) \( f_n \to f \) strongly in \( L^{p'}(0, T; L^{p'}(\Omega)) \),

ii) \( f_n \rightharpoonup f \) weakly in \( W^{1,p'}(0, T; L^{p'}(\Omega)) \) and \( f(0) = f(T) \), \( f_n(0) = f_n(T) \) for any \( n \).

Let \( u_n \) be a solution of

\[
\begin{align*}
\alpha_n(u'_n(x, t)) - \Delta^\alpha_m u_n(x, t) &= f_n(x, t), & (x, t) &\in \Omega \times (0, T), \\
u_n(x, t) &= 0, & (x, t) &\in \partial \Omega \times (0, T), \\
u_n(x, 0) &= u_n(x, T), & x &\in \Omega,
\end{align*}
\]

(DNP)

whose existence is assured by Theorem 5.1. Then there exist a subsequence \( \{u_j\}_{j \in \mathbb{N}} := \{u_{n_j}\}_{j \in \mathbb{N}} \) and its limit \( u \) such that

\[
u_j \to u \text{ strongly in } C([0, T]; L^p(\Omega)),
\]

\[
u_j \rightharpoonup u \text{ weakly in } W^{1,p}(0, T; L^p(\Omega)),
\]

\[
u_j \rightharpoonup_* \nu \text{ weakly in } L^\infty(0, T; W^{1,m}_0(\Omega)),
\]

\[
\alpha(u'_j) \to \alpha(u') \text{ weakly in } L^{p'}(0, T; L^{p'}(\Omega)),
\]

\[
\Delta^\alpha_m u_j \to \Delta^\alpha_m u \text{ weakly in } L^{p'}(0, T; L^{p'}(\Omega)),
\]

and \( u \) is a solution to (DNP).

**A Appendix**

**Lemma A.1.** Let \( E \) be a real reflexive Banach space and \( \varphi : E \to [0, \infty] \) be a proper lower semi-continuous convex functional. Define

\[
\Phi(u) := \varphi(u) + \frac{\mu}{1+\alpha} \varphi^{1+\alpha}(u) \quad u \in E
\]

for some \( \alpha, \mu > 0 \). Then the followings hold:

\[
D(\Phi) = D(\varphi), \quad D(\partial_E \Phi) = D(\partial_E \varphi), \quad \partial_E \Phi(u) = (1 + \mu \varphi(u)\alpha) \partial_E \varphi(u) \quad \forall u \in D(\partial_E \varphi).
\]

**Proof.** First, \( D(\Phi) = D(\varphi) \) is trivial since \( \varphi \geq 0 \). Moreover, we can easily see that

\[
\partial_E \varphi \subset D(\partial_E \varphi) \subset E \Phi.\quad \text{Indeed, for every } [u_1, v_1] \in \partial_E \varphi \text{ and } u_2 \in D(\Phi) = D(\varphi), \text{ we have}
\]

\[
\langle (1 + \mu \varphi^{\alpha}(u_1))v_1, u_1 - u_2 \rangle_E \\
\geq (1 + \mu \varphi^{\alpha}(u_1))(\varphi(u_1) - \varphi(u_2)) \\
\geq \varphi(u_1) - \varphi(u_2) + \mu \varphi^{\alpha+1}(u_1) - \mu \left( \frac{\alpha}{\alpha + 1} \varphi^{\frac{\alpha+1}{\alpha}}(u_1) + \frac{1}{\alpha + 1} \varphi^{\alpha+1}(u_2) \right) \\
\geq \Phi(u_1) - \Phi(u_2).
\]
Hence we only have to check the maximality of $A := (1 + \mu \varphi^\alpha) \partial_E \varphi$.

To this end, we rely on the following criterion (see Theorem 10.6 of [43]).

**Lemma A.2.** Let $E$ be a reflexive Banach space and $A : E \to 2^{E^*}$ be a monotone operator. Then $A$ is maximal monotone if and only if
\begin{equation}
G(A) + G(-F_E) = E \times E^*,
\end{equation}
i.e., $\forall (w, w^*) \in E \times E^*$, $\exists (u, u^*) \in G(A)$ such that $w^* \in u^* + F_E(u - w)$.

According to this lemma, it suffices to show that for any $(w, w^*) \in E \times E^*$ there exists $u \in D(A)$ such that
\begin{equation}
F_E(u - w) + (1 + \mu \varphi^\alpha(u)) \partial_E \varphi(u) \ni w^*.
\end{equation}
To see this, for any $(w, w^*) \in E \times E^*$ and $\lambda \geq 0$, we consider the following auxiliary equation:
\begin{equation}
F_E(u_\lambda - w) + (1 + \lambda) \partial_E \varphi(u_\lambda) \ni w^*.
\end{equation}
Since $(1 + \lambda) \partial_E \varphi$ is maximal monotone in $E \times E^*$, Lemma A.2 assures that (82) admits at least one solution. In our setting, however, the uniqueness of solution is not ensured.

Define the set of all solutions of (82) by $X_\lambda \subset E$ and a set-valued mapping $\gamma : [0, \infty) \to 2^{[0, \infty)}$ by
\[ \gamma(\lambda) := \{ \mu \varphi^\alpha(u_\lambda) ; u_\lambda \in X_\lambda \}. \]
We are going to show that $\gamma$ has a fixed point $\lambda_0$. It is easy to see that $u = u_{\lambda_0}$ gives a solution of (81), which completes the proof.

In order to show the existence of the fixed point of $\gamma$, we rely on Kakutani’s fixed-point theorem. To do this, it suffices to verify the following facts:

#1 There exists $L > 0$ such that $\gamma([0, L]) \subset [0, L]$.

#2 The graph of $\gamma$ is closed, i.e., if $\lambda_n \to \lambda$ and $y_n \to y$ with $y_n \in \gamma(\lambda_n)$, then $y \in \gamma(\lambda)$ holds true.

#3 $\gamma(\lambda)$ is convex and non-empty for each $\lambda \in [0, \infty)$.

We first establish a priori estimates for $u_\lambda$ and $\varphi(u_\lambda)$. Let $g_\lambda$ and $h_\lambda$ be the sections of $\partial_E \varphi(u_\lambda)$ and $F_E(u_\lambda - w)$ satisfying (82), namely,
\begin{equation}
g_\lambda := \frac{w^* - h_\lambda}{1 + \lambda} \in \partial_E \varphi(u_\lambda).
\end{equation}
By the definition of subdifferential,
\begin{equation}
\varphi(v) - \varphi(u_\lambda) \geq \langle g_\lambda, v - u_\lambda \rangle_E = \left\langle \frac{w^* - h_\lambda}{1 + \lambda}, v - u_\lambda \right\rangle_E.
\end{equation}
Periodic Problem for Doubly Nonlinear Equation

holds for any fixed $v \in D(\varphi)$. Hence $u_\lambda$ satisfies

$$2(1 + \lambda)\varphi(v) + |w - v|^2_E + 2|w^*|_{E^*}(|v|_E + |u_\lambda|_E) \geq 2(1 + \lambda)\varphi(u_\lambda) + |u_\lambda - w|^2_E,$$

which implies the uniform boundedness of $|u_\lambda|_E$ and $\varphi(u_\lambda)$ with respect to $\lambda$.

Let $u_\nu \in X_\nu$, i.e., $u_\nu \in D(\partial_E\varphi)$ be a solution of (82) with $\lambda$ replaced by $\nu$. Substituting $u_\nu$ with $v$ in (84), we get

$$2(1 + \lambda)\varphi(u_\nu) - (1 + \lambda)\varphi(u_\lambda) \geq \langle w^* - h\lambda, u_\nu - u_\lambda \rangle_E.$$

Reversing the roles of $u_\lambda$ and $u_\nu$, we have

$$2(1 + \nu)\varphi(u_\lambda) - (1 + \nu)\varphi(u_\nu) \geq \langle w^* - h\nu, u_\lambda - u_\nu \rangle_E.$$

By adding (86) and (87), we obtain

$$\langle (\nu - \lambda)\varphi(u_\lambda) - (\nu - \lambda)\varphi(u_\nu) \geq \langle h\lambda - h\nu, u_\lambda - u_\nu \rangle_E.$$

Monotonicity of $F_E$ and (88) yield

$$\varphi(u_0) \geq \varphi(u_\lambda) \geq \varphi(u_\nu) \quad \text{if} \quad \nu \geq \lambda \geq 0,$$

where $u_0$, $u_\lambda$, and $u_\nu$ are arbitrary elements of $X_0$, $X_\lambda$, and $X_\nu$, respectively. Then (88) also gives us

$$\langle h\lambda - h\nu, u_\lambda - u_\nu \rangle_E \leq \varphi(u_0)\lambda - \nu \quad \text{for any} \lambda, \nu \geq 0.$$

Multiplying the difference of two equations (82) with $\lambda = \lambda$ and $\lambda = \nu$ by $u_\lambda - u_\nu$ and using the monotonicity of $F_E$, we have

$$\langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E + \langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E \leq 0 \quad \text{for any} \lambda, \nu \geq 0.$$

Here, if $\lambda \geq \nu$,

$$-\langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E = -\langle \lambda - \nu \rangle \langle g_\lambda, u_\lambda - u_\nu \rangle_E - \nu \langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \leq (\lambda - \nu)(\varphi(u_\nu) - \varphi(u_\lambda)),$$

and if $\nu \geq \lambda$,

$$-\langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E = -\langle \lambda - \nu \rangle \langle g_\nu, u_\lambda - u_\nu \rangle_E - \lambda \langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \leq (\nu - \lambda)(\varphi(u_\lambda) - \varphi(u_\nu)).$$

Hence

$$\langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \leq \varphi(u_0)\lambda - \nu \quad \text{for any} \lambda, \nu \geq 0.$$
Now from (85), we can assure #1 with

\[ L := \mu \left( \varphi(v) + \frac{1}{2} |w - v|^2_E + |w^*|_{E^*} |v|_E + |w|_E + |w^*|_{E^*} \right)^\alpha, \]

where \( v \) is an arbitrary element in \( D(\varphi) \).

In order to see #2, let \( \{\lambda_n\}_{n \in \mathbb{N}} \) be a sequence of non-negative numbers which converges to \( \lambda \geq 0 \) and let \( y_n \to y \) with \( y_n = \mu \varphi(u_{\lambda_n})^\alpha \in \gamma(\lambda_n) \). By (83) and (85), we find that

\[ |u_{\lambda_n}|_E, |h_{\lambda_n}|_{E^*}, |g_{\lambda_n}|_{E^*} \]

are all uniformly bounded. Hence there exists a subsequence of \( \{\lambda_n\} \), denoted again by the same symbol, such that

\[ u_{\lambda_n} \rightharpoonup u_\lambda \quad \text{weakly in } E, \]
\[ h_{\lambda_n} \rightharpoonup h_\lambda \quad \text{weakly in } E^*, \]
\[ g_{\lambda_n} \rightharpoonup g_\lambda = \frac{w^* - h_\lambda}{1 + \lambda} \quad \text{weakly in } E^*. \]

Thanks to Proposition 2.2, (90) and (91) with \( \lambda = \lambda_n \) and \( \nu = \lambda_m \) imply that

\[ h_\lambda \in F_E(u_\lambda), \quad g_\lambda = \frac{w^* - h_\lambda}{1 + \lambda} \in \partial_E \varphi(u_\lambda) \]
\[ \langle h_{\lambda_n}, u_{\lambda_n} \rangle_E \to \langle h_\lambda, u_\lambda \rangle_E \quad \text{as } n \to \infty. \]

Hence \( u_\lambda \in X_\lambda \), i.e., \( u_\lambda \) is a solution of (82). Furthermore from (86) and (87) with \( \lambda = \lambda, \nu = \lambda_n \) together with (92), we can derive

\[ |\varphi(u_{\lambda_n}) - \varphi(u_\lambda)| \leq |\langle h_\lambda - w^*, u_{\lambda_n} - u_\lambda \rangle_E| + |\langle h_{\lambda_n} - w^*, u_\lambda - u_{\lambda_n} \rangle_E| \]
\[ \to 0 \quad \text{as } n \to \infty, \]

which implies that \( y = \mu \varphi(u_\lambda)^\alpha \in \gamma(\lambda) \). Thus #2 is verified.

To show #3, we prepare the following lemma.

**Lemma A.3.** For each \( \lambda \in [0, \infty) \), \( X_\lambda \) forms a non-empty closed convex subset of \( E \). Moreover for any \( u_\lambda, \bar{u}_\lambda \in X_\lambda \), it holds that

\[ \varphi(\tau u_\lambda + (1 - \tau) \bar{u}_\lambda) = \tau \varphi(u_\lambda) + (1 - \tau) \varphi(\bar{u}_\lambda) \quad \forall \tau \in (0, 1). \]

**Proof of Lemma A.3.** We first note that \( X_\lambda \) is not empty, since (82) admits at least one solution. We put

\[ \varphi_1(u) := \frac{1}{2} |u - w|^2_E + (1 + \lambda) \varphi(u) \quad u \in D(\varphi_1) := D(\varphi). \]

Obviously, \( \varphi_1 \) is a lower semi-continuous convex function and

\[ \partial_E \varphi_1(u) = F_E(u - w) + (1 + \lambda) \partial_E \varphi(u) \quad u \in D(\partial_E \varphi_1) = D(\partial_E \varphi). \]
Then \( u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda \) is equivalent to \([u_\lambda, w^*], [\bar{u}_\lambda, w^*] \in \partial_E \varphi_1 \), that is,

\[
\varphi_1(v) - \varphi_1(u_\lambda) \geq \langle w^*, v - u_\lambda \rangle_E, \quad \varphi_1(v) - \varphi_1(\bar{u}_\lambda) \geq \langle w^*, v - \bar{u}_\lambda \rangle_E
\]

for any \( v \in D(\varphi_1) \). These and convexity of \( \varphi_1 \) yield

\[
\varphi_1(v) - \varphi_1(\tau u_\lambda + (1 - \tau) \bar{u}_\lambda) \geq \varphi_1(v) - \tau \varphi_1(u_\lambda) - (1 - \tau) \varphi_1(\bar{u}_\lambda)
\]

which means that \( \tau u_\lambda + (1 - \tau) \bar{u}_\lambda \in \mathcal{X}_\lambda \) for every \( u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda \), i.e., \( \mathcal{X}_\lambda \) is convex.

Let \( u^n_\lambda \in \mathcal{X}_\lambda \) converge to \( u_\lambda \) strongly in \( E \), then letting \( n \to \infty \) in

\[
\varphi_1(v) - \varphi_1(u^n_\lambda) \geq \langle w^*, v - u^n_\lambda \rangle_E \quad \forall v \in D(\varphi_1),
\]

we have

\[
\varphi_1(v) - \varphi_1(u_\lambda) \geq \langle w^*, v - u_\lambda \rangle_E \quad \forall v \in D(\varphi_1),
\]

whence follows \( u_\lambda \in \mathcal{X}_\lambda \), i.e., \( \mathcal{X}_\lambda \) is closed in \( E \).

Let \( u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda \) and let \( g_\lambda, \bar{g}_\lambda \) be the sections of \( \partial_E \varphi(u_\lambda), \partial_E \varphi(\bar{u}_\lambda) \) satisfying (82). Then (91) with \( \nu = \lambda, u_\nu = \bar{u}_\lambda, g_\nu = \bar{g}_\lambda \) implies

\[
\langle g_\lambda - \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda \rangle_E = 0, \text{ i.e., } \langle \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda \rangle_E = \langle g_\lambda, u_\lambda - \bar{u}_\lambda \rangle_E.
\]

Therefore,

\[
\tau (\varphi(u_\lambda) - \varphi(\bar{u}_\lambda)) = \tau \varphi(u_\lambda) + (1 - \tau) \varphi(\bar{u}_\lambda) - \varphi(\bar{u}_\lambda)
\]

\[
\geq \varphi(\tau u_\lambda + (1 - \tau) \bar{u}_\lambda) - \varphi(\bar{u}_\lambda)
\]

\[
\geq \langle \bar{g}_\lambda, \tau u_\lambda + (1 - \tau) \bar{u}_\lambda - \bar{u}_\lambda \rangle_E
\]

\[
= \tau \langle \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda \rangle_E
\]

\[
= \tau \langle g_\lambda, u_\lambda - \bar{u}_\lambda \rangle_E
\]

\[
\geq \tau (\varphi(u_\lambda) - \varphi(\bar{u}_\lambda)),
\]

whence follows (93).

We here claim that \( \Gamma_\lambda := \{ \varphi(u_\lambda) ; u_\lambda \in \mathcal{X}_\lambda \} \) is bounded, closed and convex in \( \mathbb{R} \). The boundedness is obvious from (89) and the closedness can be derived from the same arguments for the verification of #2 above with \( \lambda_n \equiv \lambda \). For each \( y_1, y_2 \in \Gamma_\lambda \), there exist \( u_1, u_2 \in \mathcal{X}_\lambda \) such that \( y_i = \varphi(u_i) \) \( (i = 1, 2) \). Then from (93), we have

\[
\tau y_1 + (1 - \tau) y_2 = \tau \varphi(u_1) + (1 - \tau) \varphi(u_2) = \varphi(\tau u_1 + (1 - \tau) u_2).
\]

Hence \( \tau y_1 + (1 - \tau) y_2 \in \Gamma_\lambda \) follows by Lemma A.3. Thus \( \Gamma_\lambda \) is convex and there exist \( -\infty < a_\lambda \leq b_\lambda < +\infty \) such that

\[
\Gamma_\lambda = [a_\lambda, b_\lambda].
\]
Immediately, we can conclude

\[ \gamma(\lambda) = [\mu a^\alpha_\lambda, \mu b^\alpha_\lambda], \]

whence follows #3.

\[ \square \]

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