Extended finite element methods for optimal control problems governed by Poisson equation in non-convex domains

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Abstract

This paper analyzes two eXtended finite element methods (XFEMs) for linear quadratic optimal control problems governed by Poisson equation in non-convex domains. We follow the variational discretization concept to discretize the continuous problems, and apply an XFEM with a cut-off function and a classic XFEM with a fixed enrichment area to discretize the state and co-state equations. Optimal error estimates are derived for the state, co-state and control. Numerical results confirm our theoretical results.

Keywords: extended finite element method, optimal control, non-convex domain, variational discretization concept.

1 Introduction

We consider the following linear quadratic optimal control problem:

\[
\min J(y, u) := \frac{1}{2} \int_{\Omega} (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_{\Omega} u^2 \, dx \tag{1.1}
\]

for \((y, u) \in H^1_0(\Omega) \times L^2(\Omega)\) subject to Poisson equation

\[
\begin{cases}
-\Delta y = u + f & \text{in } \Omega, \\
y = 0 & \text{on } \partial \Omega, 
\end{cases}
\tag{1.2}
\]

with the control constraint

\[
u_0 \leq u \leq u_1, \text{ a.e. on } \Omega, \tag{1.3}
\]

where \(\Omega\) is a bounded polygonal domain in \(\mathbb{R}^2\) with a single re-entrant corner of angle \(\pi/\beta\), \(\beta \in [\frac{1}{2}, 1]\). \(y_d \in L^2(\Omega)\) is the desired state to be achieved by controlling \(u\). \(\alpha\) is a positive constant and \(f, u_0, u_1 \in L^2(\Omega)\) with \(u_0 \leq u_1\) a.e. on \(\Omega\). For the sake of simplicity, we choose homogeneous boundary condition on \(\partial \Omega\).

In fact, we can obtain similar results for other boundary conditions.

For the Poisson problem (1.2) in non-convex domains, it is well-known that the weak solution \(y\) is generally not in \(H^2(\Omega)\), due to the singularities at the corner. The low regularity may lead to reduced accuracy for finite element approximations [9]. In literature there are two ways to improve the accuracy. The first way is to use graded meshes (cf. [29, 6, 34, 3]). The second way is to use some singular basis functions which characterize the singularity of the solution around the corner; see, for instance, the classic singular enrichment method [39], the dual singular function method [15], and the singular complement method [23]. Notice that when \(\Omega\) is a crack domain \((\beta = \frac{1}{2})\), all the above methods have to use body-fitted meshes. However, it is often difficult or expensive to construct such kinds of meshes, especially in time dependent problems.

In the past few decades, many numerical methods have been developed for the optimal control problem (1.1)-(1.3) in convex domains, see [16, 17, 42, 10, 25, 35, 13, 20, 32, 33, 31, 18, 25, 21, 22, 26, 44]. However, for optimal control problems in non-convex domains, there is only limited research work. In [2, 1, 4], finite element error estimates were derived on graded meshes.

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The extended finite element method (XFEM, also called generalized finite element method (GFEM)) is known to be a widely-used tool for the analysis of problems with singularities [11, 27, 40, 41, 12, 38, 8, 24, 7, 14, 5]. With additional basis functions characterizing the singularity added into the standard approximation space, XFEM does not need body-fitted meshes, and thus avoid complicated meshes.

In this paper, we consider an XFEM for the optimal control (1.1)-(1.3) problem in non-convex domains. We follow the variational discretization concept [21, 22] to discretize the continuous problem. Optimal error estimations are derived for the state, co-state and control. We apply the semi-smooth Newton method to solve the resultant nonlinear discrete system.

The rest of the paper is arranged as follows. Section 2 gives some notations, the optimality conditions, and regularity results for the optimal control problem. Section 3 introduces the XFEM, and shows several theoretical results associate with XFEM. Section 4 is devoted to the discrete optimal control problem, the discrete optimality conditions and error estimates for the state, co-state and control. Section 5 gives an iteration algorithm for the discrete system. Finally, Section 6 provides numerical results to verify the theoretical analysis.

2 Preliminary

Let $\Omega$ be a polygonal domain with a single re-entrant corner of angle $\frac{\pi}{\beta}$, $\beta \in [\frac{1}{2}, 1]$. And $(0,0)$ is the concave point of $\Omega$. For any nonnegative integer $m$, let $H^m(\Omega)$ and $H^m_0(\Omega)$ denote the standard Sobolev spaces on $\Omega$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$. In particular, $H^0(\Omega) = L^2(\Omega)$, with the standard $L^2$-inner product $(\cdot, \cdot)$.

For $u \in L^2(\Omega)$, the weak formulation of the state equation (1.2) is as follows: find $y \in H^1_0(\Omega)$ such that

\[
a(y, v) = (u + f, v), \quad \forall v \in H^1_0(\Omega),
\]

where $a(y, v) := (\nabla y, \nabla v)$.

Introduce a singular function

\[
S_\beta(r, \theta) = r^\beta \sin(\beta \theta).
\]

It is well known that the weak solution $y$ is generally not in $H^2(\Omega)$. According to [15, 19], $y$ has a singular part

\[
y_s = \chi(r)K_y S_\beta(r, \theta)
\]

such that

\[
y - y_s \in H^2(\Omega).
\]

Here $r$ and $\theta$ are polar coordinates with $0 \leq \theta \leq 2\pi$ (cf. Figure 1 for an L-shaped domain). $K_y$ is a positive constant depending on $u$ and can be regarded as a linear functional of $u + f$ (cf. [23, 37]).
\( \chi \in W^{2,\infty}(0, \infty) \) is a cut-off function defined by
\[
\chi(r) = \begin{cases} 
1, & \text{if } r < r_0, \\
0 < \chi(r) < 1, & \text{if } r_0 < r < r_1, \\
0, & \text{if } r_1 < r, 
\end{cases}
\]  
(2.4)
where \( r_0, r_1 \) are two constants with \( 0 < r_0 < r_1 \). And, throughout the paper, we use \( p \leq q \) to denote \( p \leq Cq \), where \( C \) is a generic positive constant independent of the solution \( u \) and the finite element mesh size \( h \).

In view of (2.2)-(2.3), we assume the following regularity on the solution \( y \) to the problem (2.1) (cf. [37]):
\[
K_y + \| y - y_s \|_2 \lesssim \| u \|_0 + \| f \|_0.
\]  
(2.5)

**Remark 2.1.** Set
\[
\tilde{y}_s := K_y S_{\beta}(r, \theta),
\]  
(2.6)
and let \( B(r) \) be a ball with center \( (0, 0) \) and radius \( r \). Then, for any given enrichment radius \( r_0 > 0 \), by (2.5) we easily get
\[
\| y - \tilde{y}_s \|_2 + \| \tilde{y}_s \|_1 + \| \tilde{y}_s \|_{2, \Omega \setminus B(r_0/2)} \lesssim \| \chi \|_0 + \| f \|_0.
\]  
(2.7)
In fact, from (2.5) it follows
\[
\| \tilde{y}_s \|_1 + \| \tilde{y}_s \|_{2, \Omega \setminus B(r_0/2)} = K_y (\| S_{\beta} \|_1 + \| S_{\beta} \|_{2, \Omega \setminus B(r_0/2)}) \lesssim \| u \|_0 + \| f \|_0.
\]

By the definition of the cut-off function \( \chi(r) \) we have
\[
\| (1 - \chi) \tilde{y}_s \|_2 = \| (1 - \chi) \tilde{y}_s \|_{2, \Omega \setminus B(r_0)} \lesssim \| \tilde{y}_s \|_{2, \Omega \setminus B(r_0)} \lesssim \| u \|_0 + \| f \|_0,
\]
which, together with the triangle inequality, yields
\[
\| y - \tilde{y}_s \|_2 \leq \| y - y_s \|_2 + \| (1 - \chi) \tilde{y}_s \|_2 \lesssim \| u \|_0 + \| f \|_0.
\]

Hence, the estimate (2.7) holds.

By Following the standard optimality technique in [43], we can easily get the optimality conditions of the optimal control problem (1.1)-(1.3).

**Lemma 2.1.** The optimal control problem (1.1)-(1.3) has a unique solution \( (y, p, u) \in H^1_0(\Omega) \times H^1_0(\Omega) \times U_{ad} \) such that
\[
a(u, v) = (u + f, v), \quad \forall v \in H^1_0(\Omega),
\]  
(2.8)
\[
a(v, p) = (y - y_d, v), \quad \forall v \in H^1_0(\Omega),
\]  
(2.9)
\[
(p + au, v - u) \geq 0, \quad \forall v \in U_{ad},
\]  
(2.10)
where
\[
U_{ad} := \{ v \in L^2(\Omega) : u_a \leq v \leq u_b \ a.e. \ in \ \Omega \}.
\]  
(2.11)

We note that \( p \) is called the co-state or adjoint state, and (2.9) is the co-state equation.

**Remark 2.2.** Let
\[
p_s := \chi(r) K_p S_{\beta}(r, \theta),
\]  
(2.12)
with \( K_p > 0 \), be the singular part of the solution \( p \) to the co-state equation (2.9) such that
\[
K_p + \| p - p_s \|_2 \lesssim \| y - y_d \|_0.
\]  
(2.13)
which, together with the fact \( \| y \|_1 \lesssim \| u \|_0 \), means
\[
K_p + \| p - p_s \|_2 \lesssim \| u \|_0 + \| y_d \|_0 + \| f \|_0.
\]  
(2.14)
Similar to Remark 2.1, set
\[
\tilde{p}_s := K_p S_{\beta}(r, \theta),
\]
then from (2.14) we have
\[
\| p - \tilde{p}_s \|_2 + \| \tilde{p}_s \|_1 + \| \tilde{p}_s \|_{2, \Omega \setminus B(r_0/2)} \lesssim \| u \|_0 + \| f \|_0 + \| y_d \|_0.
\]  
(2.15)
Remark 2.3. The variational inequality (2.10) means that

\[ u = P_{U_{ad}} \left( -\frac{1}{\alpha} p \right), \]  

(2.16)

where \( P_{U_{ad}} \) denotes the \( L^2 \) projection onto \( U_{ad} \).

3 XFEM for state and co-state equations

From Lemma 2.1, the state \( y \) and the co-state \( p \) can respectively be viewed as the solutions to the following two problems.

Find \( y \in H^1_0(\Omega) \) such that

\[ a(y, v) = (u + f, v), \quad \forall v \in H^1_0(\Omega). \]  

(3.1)

Find \( p \in H^1_0(\Omega) \) such that

\[ a(v, p) = (y - y_d, v), \quad \forall v \in H^1_0(\Omega). \]  

(3.2)

3.1 Formulations of XFEM

Let \( T_h \) be a shape-regular triangulation of \( \Omega \) consisting of open triangles with mesh size \( h = \max_{K \in T_h} h_K \), where \( h_K \) denotes the diameter of \( K \in T_h \). Denote by \( \Theta = \{ a_i : i \in 1, 2, \ldots, I \} \) the set of all the vertexes of all triangles in \( T_h \).

For \( \forall a_i \in \Theta \), let \( \varphi_i \) be the corresponding nodal basis function of the continuous linear finite element method with respect to \( T_h \). Let \( r_s > 0 \) be a prescribed constant called enrichment radius. We define a vertex set

\[ \Theta_S := \{ a_i \in \Theta : \text{the distance between } a_i \text{ and the concave point is less than or equal to } r_s \}. \]

In particular, when \( \Omega \) is a cracked domain, i.e. \( \beta = \frac{1}{2} \), we set

\[ \Theta_H := \{ a_i \in \Theta : \text{the support of } \varphi_i \text{ is completely cut by the crack of } \Omega \}, \]

and define the Heaviside function \( H(x) \): for any \( x = (r \cos \theta, r \sin \theta) \in \Omega \),

\[ H(x) = \begin{cases} +1 & \text{if } x \cdot n \geq 0, \\ -1 & \text{if } x \cdot n < 0, \end{cases} \]

where \( n \) is a unit normal vector along the crack.

With the above notations, set

\[ W_1 := \text{span}\{ \varphi_i : a_i \in \Theta \} + \text{span}\{ \chi(r)S_{\beta} \}, \]

\[ W_1^* := \text{span}\{ \varphi_i : a_i \in \Theta \} + \text{span}\{ \varphi_i H : a_i \in \Theta_H \} + \text{span}\{ \chi(r)S_{\frac{1}{2}} \}, \]

\[ W_2 := \text{span}\{ \varphi_i : a_i \in \Theta \} + \text{span}\{ \varphi_i S_{\beta} : a_i \in \Theta_S \}, \]

\[ W_2^* := \text{span}\{ \varphi_i : a_i \in \Theta \} + \text{span}\{ \varphi_i H : a_i \in \Theta_H \} + \text{span}\{ \varphi_i S_{\beta} : a_i \in \Theta_S \}, \]

and \( V_0 := \{ v : v = 0 \text{ on } \partial \Omega \} \). Then we introduce the following two extended finite element spaces:

\[ V_h^1 := \begin{cases} W_1 \cap V_0 & \text{if } \frac{1}{2} < \beta \leq 1, \\ W_1^* \cap V_0 & \text{if } \beta = \frac{1}{2}, \end{cases} \]

\[ V_h^2 := \begin{cases} W_2 \cap V_0 & \text{if } \frac{1}{2} < \beta \leq 1, \\ W_2^* \cap V_0 & \text{if } \beta = \frac{1}{2}. \end{cases} \]

It is easy to observe that

\[ V_h^i \subset H^1_0(\Omega), \quad i = 1, 2. \]  

(3.3)
Take $V_h = V_h^1$ or $V_h^2$, then the XFEM formulations for the weak problems of the state $y$ and co-state $p$ read as follows:

Find $y^h \in V_h$ such that
\[
a(y^h, v_h) = (u + f, v_h), \quad \forall v_h \in V_h. \tag{3.4}\]

Find $p^h \in V_h$ such that
\[
a(v_h, p^h) = (y - y_d, v_h), \quad \forall v_h \in V_h. \tag{3.5}\]

\textbf{Remark 3.1.} The XFEM with $V_h = V_h^1$ was called an XFEM with a cut-off function [28], and the one with $V_h = V_h^2$ was called a classic XFEM with a fixed enrichment area [11].

### 3.2 Error estimates of XFEM

According to [28], it holds the following error estimates.

\textbf{Lemma 3.1.} Let $y, p$ be the solutions to the continuous problems (3.1) and (3.2) respectively such that the regularity conditions (2.5) and (2.14) hold, and $y^h, p^h$ be the solutions to the discrete schemes (3.4) and (3.5) respectively. Then the estimates
\[
||y - y^h||_1 \lesssim h ||y - y_s||_2 \lesssim h(||u||_0 + ||f||_0), \tag{3.6}
\]
\[
||p - p^h||_1 \lesssim h ||p - p_s||_2 \lesssim h (||u||_0 + ||f||_0 + ||y_d||_0) \tag{3.7}
\]
hold for $V_h = V_h^1$, and the estimates
\[
||y - y^h||_1 \lesssim h ((|y - y_s||_2 + ||\tilde{y}_s||_1 + ||\tilde{y}_s||_{2,\Omega \setminus B(r_s/2)}) \lesssim h(||u||_0 + ||f||_0), \tag{3.8}
\]
\[
||p - p^h||_1 \lesssim h ((|p - p_s||_2 + ||\tilde{p}_s||_1 + ||\tilde{p}_s||_{2,\Omega \setminus B(r_s/2)}) \lesssim h (||u||_0 + ||f||_0 + ||y_d||_0) \tag{3.9}
\]
hold for $V_h = V_h^2$.

Based on this lemma, we can follow standard duality arguments to derive $L^2$- estimates of the errors $y - y^h$ and $p - p^h$.

\textbf{Lemma 3.2.} Under the same conditions of Lemma 3.1, the following estimates hold:
\[
||y - y^h||_0 \lesssim h^2(||u||_0 + ||f||_0), \tag{3.10}
\]
\[
||p - p^h||_0 \lesssim h(||u||_0 + ||f||_0 + ||y_d||_0). \tag{3.11}
\]

\textbf{Proof.} We only show (3.10) for $V_h = V_h^1$, since the other cases follow similarly.

Consider the auxiliary problem
\[
\begin{cases}
-\Delta z &= y - y^h \quad \text{in } \Omega, \\
z &= 0 \quad \text{on } \partial \Omega,
\end{cases} \tag{3.12}
\]
which indicates
\[
a(z, v) = (y - y^h, v), \quad \forall v \in H^1_0(\Omega).
\]

Let $z^h \in V_h^1$ satisfy
\[
a(z^h, v_h) = (y - y^h, v_h), \quad \forall v_h \in V_h^1,
\]
and $z_s := \chi(r)K_2S_\beta(r, \theta)$ be the regular part of $z$ with
\[
K_2 + \|z - z_s\|_2 \lesssim ||y - y^h||_0.
\]

Then, similar to (3.6), it holds
\[
||z - z^h||_1 \lesssim h||z - z_s||_2 \lesssim h||y - y^h||_0.
\]
As a result, by the Galerkin orthogonality $a(z^h, y - y^h) = 0$ and (3.6) we have
\[
||y - y^h||_0^2 = a(z - z^h, y - y^h) \\
\phantom{||y - y^h||_0^2} \lesssim ||z - z^h||_1||y - y^h||_1 \\
\phantom{||y - y^h||_0^2} \lesssim h^2||y - y^h||_0(||u||_0 + ||f||_0),
\]
which yields (3.10).
4 Discrete optimal control problem

4.1 Discrete optimality conditions

In this subsection, we follow the variational discretization concept [21] to discretize the optimal control problem (1.1)-(1.3). The corresponding discrete optimal control problem is of the form

\[
\min_{(y_h, u_h) \in V_h \times U_{ad}} J_h(y_h, u) = \frac{1}{2} \int_{\Omega} (y_h - y_d)^2 ds + \frac{\alpha}{2} \int_{\Omega} u^2 ds,
\]

where \( y_h = y_h(u) \) satisfies

\[
a(y_h, v_h) = (u + f, v_h), \quad \forall v_h \in V_h.
\]

Similar to the continuous case, we have the following existence and uniqueness result and optimality conditions.

**Lemma 4.1.** The discrete optimal control problem (4.1)-(4.2) admits a unique solution \((y_h, u_h) \in V_h \times U_{ad}\), and its equivalent optimality conditions read: find \((y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}\) such that

\[
a(y_h, v_h) = (u + f, v_h), \quad \forall v_h \in V_h,
\]

\[
a(v_h, p_h) = (y_h - y_d, v_h), \quad \forall v_h \in V_h,
\]

\[
(p_h + \alpha u_h, v - u_h) \geq 0, \quad \forall v \in U_{ad}.
\]

**Remark 4.1.** We note that the optimal control \(u\) is not directly discretized in the objective functional (4.1), as \(U_{ad}\) is infinite dimensional. In fact, the variational inequality (4.5) means that the discrete control \(u_h\) is the \(L^2\)-projection of \(-\frac{p_h}{\alpha}\) onto \(U_{ad}\), i.e.

\[
u_h = P_{U_{ad}} \left(-\frac{p_h}{\alpha}\right).
\]

This is a key point of the variational discretization concept. In particular, if the functions \(u_0\) and \(u_1\) are well-defined at any \(x \in \Omega\), then (4.6) is equivalent to

\[
u_h = \min \left\{ u_1, \max \left\{ u_0, -\frac{p_h}{\alpha} \right\} \right\}.
\]

4.2 Error estimates

Recall that \(y^h \in V_h\) and \(p^h \in V_h\) are the solutions to the XFEM formulations (3.4) and (3.5), respectively. In what follows we first show that the errors between \((y, p, u)\) and \((y_h, p_h, u_h)\), which are the solutions of the continuous optimal control problem (2.8)-(2.10) and the discrete optimal control problem (4.2)-(4.5) respectively, are bounded from above by the errors between \((y, p)\) and \((y^h, p^h)\).

**Theorem 4.1.** Let \((y, p, u) \in H^1_0(\Omega) \times H^1_0(\Omega) \times U_{ad}\) and \((y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}\) be the solutions to the continuous problem (2.8)-(2.10) and the discrete problem (4.2)-(4.5), respectively. Then we have

\[
\alpha^{\frac{1}{2}} \|u - u_h\|_0 + \|y - y_h\|_0 \lesssim \|y - y^h\|_0 + \alpha^{-\frac{1}{2}} \|p - p^h\|_0,
\]

\[
\|p - p_h\|_0 \lesssim \|p - p^h\|_0 + \|y - y_h\|_0,
\]

\[
|y - y_h|_1 \lesssim |y - y^h|_1 + \|u - u_h\|_0,
\]

\[
|p - p_h|_1 \lesssim |p - p^h|_1 + \|y - y_h\|_0.
\]

**Proof.** We first show (4.8). From (3.4)-(3.5) and (4.2)-(4.4) it follows

\[
a(y_h - y^h, v_h) = (u_h - u, v_h), \quad \forall v_h \in V_h,
\]

\[
a(v_h, p_h - p^h) = (y_h - y, v_h), \quad \forall v_h \in V_h,
\]

which yields

\[
(y_h - y, y_h - y^h) = a(y_h - y^h, p_h - p^h) = (u_h - u, p_h - p^h).
\]
By (2.10) and (4.5) we get
\[(\alpha u + p, u_h - u) \geq 0, \quad (\alpha u_h + p_h, u - u_h) \geq 0,\]
which imply
\[(\alpha(u - u_h) + p - p_h, u_h - u) \geq 0.\]
This inequality, together with (4.14), indicates
\[\alpha\|u - u_h\|^2_0 \leq (u_h - u, p - p_h) \]
\[= (u_h - u, p - p^h) + (u_h - u, p^h - p_h) \]
\[\leq \frac{1}{2} \left( \alpha\|u - u_h\|^2_0 + \frac{1}{\alpha}\|p - p^h\|^2_0 \right) - (y - y_h, y - y_h) \]
\[\leq \frac{1}{2} \left( \alpha\|u - u_h\|^2_0 + \frac{1}{\alpha}\|p - p^h\|^2_0 \right) - \frac{1}{2}\|y - y_h\|^2_0 + \frac{1}{2}\|y - y^h\|^2_0,\]
which implies (4.8).

Secondly, let us prove (4.9). Since \(p_h - p^h \in V_h \subset H^1_0(\Omega)\) (cf. (3.3)), by (4.13) we have
\[\|p_h - p^h\|^2_0 \lesssim \|p_h - p^h\|_1^2 \]
\[= a(p_h - p^h, p_h - p^h) = (y_h - y, p_h - p^h) \]
\[\lesssim \|y_h - y\|\|p_h - p^h\|_0,\]
which, together with the triangle inequality, leads to
\[\|p - p_h\|_0 \leq \|p - p^h\|_0 + \|p^h - p_h\|_0 \]
\[\lesssim \|p - p^h\|_0 + \|y_h - y\|_0,\]
i.e. (4.9) holds.

Thirdly, let us derive (4.10). In view of (4.12), we obtain
\[\|y_h - y^h\|^2 = a(y_h - y^h, y_h - y^h) = (u_h - u, y_h - y^h) \]
\[\leq \|u - u_h\|\|y_h - y^h\|_0 \]
\[\lesssim \|u - u_h\|\|y_h - y^h\|_1,\]
which, together with the triangle inequality, indicates (4.10).

Finally, let us show (4.11). From (4.13) we get
\[\|p_h - p^h\|_1^2 = a(p_h - p^h, p_h - p^h) = (y_h - y, p_h - p^h) \]
\[\lesssim \|y_h - y\|\|p_h - p^h\|_0 \]
\[\lesssim \|y_h - y\|\|p_h - p^h\|_1,\]
which, together with the triangle inequality, yields (4.11).

Based on Theorem 4.1 and Lemmas 3.1-3.2, we immediately have the following optimal error estimates.

**Theorem 4.2.** Let \((y, p, u) \in H^1_0(\Omega) \times H^1_0(\Omega) \times U_{ad}\) and \((y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}\) be the solutions to the continuous problem (2.8)-(2.10) and the discrete problem (4.2)-(4.5) respectively such that the regularity conditions (2.5) and (2.14) hold. Then we have
\[\|u - u_h\|_0 + \|y - y_h\|_0 + \|p - p_h\|_0 \lesssim h^2 (\|u\|_0 + \|f\|_0 + \|y_d\|_0), \quad (4.15)\]
\[\|y - y_h\|_1 + \|p - p_h\|_1 \lesssim h (\|u\|_0 + \|f\|_0 + \|y_d\|_0). \quad (4.16)\]
5 Iteration algorithm

Notice that the optimal control problem (1.1)-(1.2) without the constraint (1.3) is a linear problem, and
the resultant discrete linear system is easy to solve. However, for the constrained optimal control problem
(1.1)-(1.3), the corresponding discrete optimal control problem (4.1)-(4.2) or its equivalent optimality
problem (4.3)-(4.5) is a nonlinear system, and we shall apply the semi-smooth Newton algorithm [22]
to solve it. To describe this iteration algorithm, we first show the matrix form of the discrete system
(4.3)-(4.5).

Let \{\varphi_i : i = 1, 2, \cdots, I\} be a set of basis functions of the XFE space \(V_h\) with \(I = \dim(V_h)\), and
\(Y_h, P_h\) be column vectors consisting of corresponding degrees of freedom of \(y_h, p_h\) respectively, such that
\[
y_h = (\varphi_1, \varphi_2, \cdots, \varphi_I) Y_h, \quad p_h = (\varphi_1, \varphi_2, \cdots, \varphi_I) P_h.
\]
Define matrices \(A, M \in \mathbb{R}^{I \times I}\) and vectors \(F_1, F_2 \in \mathbb{R}^I\) by
\[
A(i, j) = a(\varphi_i, \varphi_j), \quad M(i, j) = (\varphi_i, \varphi_j),
\]
for \(i, j = 1, 2, \cdots, I\). Then (4.3) and (4.4) are equivalent to the following matrix equations:
\[
AY_h = MU_h + F_1, \quad AP_h = MY_h - F_2.
\]  
\[
(5.1)
\]
\[
(5.2)
\]

In view of (4.7), i.e. \(u_h = \min \{u_1, \max \{u_0, -\frac{p_h}{\alpha}\}\}\), we define the in-active set \(\Omega_I\) and active set \(\Omega_A\)
as follows:
\[
\Omega_I := \{x \in \Omega : u_0 < -\frac{p_h}{\alpha} < u_1\},
\]
\[
\Omega_A = \Omega_{A0} \cup \Omega_{A1}, \quad \Omega_{A0} := \{x \in \Omega : -\frac{p_h}{\alpha} \leq u_0\}, \quad \Omega_{A1} := \{x \in \Omega : -\frac{p_h}{\alpha} \geq u_1\}.
\]
It is evident that
\[
u_h|_{\Omega_I} = -\frac{p_h}{\alpha}, \quad u_h|_{\Omega_{A0}} = u_0, \quad u_h|_{\Omega_{A1}} = u_1.
\]  
\[
(5.3)
\]
Note that in the algorithm to be given \(u_0, u_1\) will be replaced by their approximations.

Let \(V_h^*\) be the modified finite element space of \(V_h\),
\[
V_h^{1, *} := \begin{cases}
W_1 & \text{if } \frac{1}{2} < \beta \leq 1, \\
W_1^* & \text{if } \beta = \frac{1}{2},
\end{cases}
\]
\[
V_h^{2, *} := \begin{cases}
W_2 & \text{if } \frac{1}{2} < \beta \leq 1, \\
W_2^* & \text{if } \beta = \frac{1}{2}.
\end{cases}
\]
\(u_0^*, u_1^* \in V_h^*\) are \(L^2\)– projections of \(u_0, u_1\) onto \(V_h^*\), respectively. Let \(u_h^*\) be an approximation of \(u_h\) with
\[
u_h^*|_{\Omega_I} = u_h, \quad u_h^*|_{\Omega_{A0}} = u_0^*, \quad u_h^*|_{\Omega_{A1}} = u_1^*.
\]  
\[
(5.4)
\]
It is obvious that \(u_h^* \in \phi V_h^* + (1 - \phi)V_h^*\), where \(\phi\) is the characteristic function of \(\Omega_I\). Let \(U_{h,1}, U_{h,2} \in \mathbb{R}^I\) denote the column vectors consisting of corresponding degrees of freedom of \(\phi u_h^*\) and \((1 - \phi)u_h^*\), respectively, and define matrices \(M_1, M_2 \in \mathbb{R}^{I \times I}\) by
\[
M_1(i, j) := (\varphi_i, \varphi_j), \quad M_2(i, j) := (\varphi_i, (1 - \phi)\varphi_j)
\]
for \(i = 1, 2, \cdots, I\) and \(j = 1, 2, \cdots, I\). Then the matrix form (5.1) is modified as
\[
AY_h = M_1 U_{h,1} + M_2 U_{h,2} + F_1.
\]  
\[
(5.5)
\]
Based on (5.1)-(5.5), we can describe the semi-smooth newton algorithm as follows.
Semi-smooth newton algorithm

Set $k = 0, \phi^{(0)} = 0, U_{h,1}^{(0)} = 0, U_{h,2}^{(0)} = 0$;

Do until convergence

1. Compute $y_h^{(k+1)} \in V^h$ by
   \[
   a(y_h^{(k+1)}, v_h) = \left( \phi^{(k)} u_h^{(k)} + (1 - \phi^{(k)}) u_h^{(k)} + f, v_h \right), \forall v_h \in V^h;
   \]
   or, equivalently, compute
   \[
   Y_h^{(k+1)} = A^{-1} \left( M_1^{(k)} U_{h,1}^{(k)} + M_2^{(k)} U_{h,2}^{(k)} + F_1 \right);
   \]

2. Compute $p_h^{(k+1)} \in V^h$ by
   \[
   a(v_h, p_h^{(k+1)}) = (y_h^{(k+1)} - y_d, v_h), \forall v_h \in V^h;
   \]
   or, equivalently, compute
   \[
   P_h^{(k+1)} = A^{-1} (MY^{(k+1)} - F_2) = A^{-1} \left( MA^{-1} \left( M_1^{(k)} U_{h,1}^{(k)} + M_2^{(k)} U_{h,2}^{(k)} + F_1 \right) - F_2 \right);
   \]

3. Compute
   \[
   \Omega_f^{(k+1)} := \left\{ x \in \Omega : u_0 < -\frac{p_h^{(k+1)}}{\alpha} < u_1 \right\},
   \]
   \[
   \Omega_{a0}^{(k+1)} := \left\{ x \in \Omega : -\frac{p_h^{(k+1)}}{\alpha} \leq u_0 \right\},
   \]
   \[
   \Omega_{a1}^{(k+1)} := \left\{ x \in \Omega : -\frac{p_h^{(k+1)}}{\alpha} \geq u_1 \right\},
   \]
   and the characteristic function, \( \phi^{(k+1)} \), of \( \Omega_f^{(k+1)} \);

4. Compute $u_{h,2}^{(k+1)}$ (or $U_{h,2}^{(k+1)}$) with
   \[
   u_{h,2}^{(k+1)}|_{\Omega_{a0}^{(k+1)}} = u_0^a, \quad u_{h,2}^{(k+1)}|_{\Omega_{a1}^{(k+1)}} = u_1^a, \quad u_{h,2}^{(k+1)}|_{\Omega_f^{(k+1)}} = 0;
   \]

5. Compute $U_{h,1}^{(k+1)}$ by
   \[
   U_{h,1}^{(k+1)} = -\frac{1}{\alpha} \left( A^{-1} \left( MA^{-1} \left( M_1^{(k+1)} U_{h,1}^{(k+1)} + M_2^{(k+1)} U_{h,2}^{(k+1)} + F_1 \right) - F_2 \right) \right),
   \]
   i.e. $U_{h,1}^{(k+1)} = -\frac{1}{\alpha} \left( I + \frac{1}{\alpha} A^{-1} MA^{-1} M_1^{(k+1)} \right)^{-1} \left( A^{-1} MA^{-1} M_1 U_{h,1}^{(k+1)} + A^{-1} MA^{-1} F_1 - A^{-1} F_2 \right);
   \]

6. $k = k + 1$;

end

It should be pointed out that in step 3, the active set can only be computed approximately for XFEM, even when $u_0, u_1$ are constants, since some basis functions of the XFE spaces are non-linear. In actual computation we just use their piecewise linear interpolations to replace the nonlinear basis functions so as to compute the approximate active set. We refer to [30] for an efficient method to compute the active set for high order finite element methods.

6 Numerical results

In this section, we shall provide several numerical examples to verify the performance of the proposed methods, i.e. the discrete schemes (4.1)-(4.2) or (4.3)-(4.5) with $V_h = V^1_h$ and $V_h = V^2_h$. We recall, cf. Remark 3.1, that $V^1_h$ and $V^2_h$ are corresponding to the XFEM with a cut-off function (abbr. cut XFEM) and the classic XFEM with a fixed enrichment area (abbr. classic XFEM).
Example 6.1. An unconstrained problem in a crack domain.

Take $\Omega = [-1, 1] \times [-1, 1]$ with a segment crack from the point $(-1, 0)$ to the crack-tip $(0, 0)$ (cf. Figure 2). We choose $\alpha = 0.01$, the enrichment radius $r_s = 0.5$ (cf. Remark 2.1), and the cut-off function $\chi(r)$ in (2.4) is a polynomial with $r_0 = 0.01$ and $r_1 = 0.99$. Let

$$y = \sqrt{\pi} \sin\left(\frac{\theta}{2}\right) - \frac{1}{4} r^2,$$

$$p = x_2^2(1 - x_1^2)(1 - x_1^2) + \frac{1}{2} \sqrt{\pi} \sin\left(\frac{\theta}{2}\right)(1 - x_1^2)(1 - x_2^2),$$

$$u = -\frac{p}{\alpha}$$

be the analytical state, co-state and control of the optimal control problem (1.1) subject to

$$\left\{ \begin{array}{l}
-\Delta y = u + f, \quad \text{in } \Omega \\
y = y_b, \quad \text{on } \partial \Omega.
\end{array} \right.$$

(6.1)

Note that in this case $U_{ad} = L^2(\Omega)$, and $y_d$ can be obtained by $-\Delta p = y - y_d$. In particular, the discrete equation (4.5) yields $u_h = -\frac{p_h}{\alpha}$, which means $u - u_h = -\frac{p - p_h}{\alpha}$.

We use $N \times N$ uniform triangular meshes (cf. Figure 2). Tables 1-2 show results of the relative errors between $(y_h, p_h)$ and $(y, p)$ in $H^1$ semi-norm and $L^2$ norm, and Figure 6 shows the relative errors against the mesh size $h = 2/N$. We can see that the proposed methods yield optimal convergence orders, i.e. first order rates of convergence for $|y - y_h|_1$ and $|p - p_h|_1$, and second order rates of convergence for $|y - y_h|_0$ and $|p - p_h|_0$. This is consistent with our theoretical results in Theorem 4.2.

For comparison we also show in Table 3 and Figure 4 the relative errors of $(y_h, p_h)$ for the standard linear element method ($P_1$ FEM) on body fitted meshes (cf. Figure 2). We can see that $P_1$ FEM yields only about 0.5 order convergence rates for $|y - y_h|_1$ and first order convergence rates for $|y - y_h|_0$ and $|p - p_h|_0$. This is conformable to the theoretical results in [9].

![Figure 2: Domain $\Omega$ (the red line is the crack) and meshes for Example 6.1: $9 \times 9$ mesh for XFEMs (left) and $10 \times 10$ mesh for $P_1$ FEM.](image)

| Table 1: Relative errors of the cut XFEM for Example 6.1. |
|---|---|---|---|---|---|---|
| $N$ | $|y - y_h|_1$ | order | $|y - y_h|_0$ | order | $|p - p_h|_1$ | order | $|p - p_h|_0$ | order |
| 39 | 0.1153 | 0.0235 | order | 0.1457 | 0.0092 | order |
| 49 | 0.0912 | 1.03 | 0.0148 | 2.02 | 0.1162 | 0.99 | 0.0058 | 1.99 |
| 59 | 0.0756 | 1.01 | 0.0101 | 2.06 | 0.0966 | 0.99 | 0.0040 | 2.00 |
| 69 | 0.0647 | 0.99 | 0.0073 | 2.10 | 0.0827 | 0.99 | 0.0029 | 2.01 |
| 79 | 0.0566 | 0.99 | 0.0054 | 2.14 | 0.0723 | 0.99 | 0.0022 | 2.02 |

Example 6.2. A constrained problem in a crack domain.
Table 2: Relative errors of the classic XFEM for Example 6.1.

| N  | $y - y_h$ | order | $y - y_{h,0}$ | order | $p - p_h$ | order | $p - p_{h,0}$ | order |
|----|-----------|-------|---------------|-------|-----------|-------|---------------|-------|
| 39 | 0.0632    | 0.0192| 0.1380        | 0.0084|           |       |               |       |
| 49 | 0.0491    | 1.11  | 0.0124        | 1.91  | 0.1103    | 0.98  | 0.0053        | 1.98  |
| 59 | 0.0403    | 1.06  | 0.0087        | 1.93  | 0.0919    | 0.99  | 0.0037        | 1.98  |
| 69 | 0.0344    | 1.02  | 0.0064        | 1.94  | 0.0787    | 0.99  | 0.0027        | 1.99  |
| 79 | 0.0288    | 1.29  | 0.0048        | 2.11  | 0.0685    | 1.02  | 0.0021        | 2.02  |

Figure 3: Convergence history of the cut XFEM (left) and classic XFEM (right) for Example 6.1.

Table 3: Relative errors of $P_1$ FEM for Example 6.1.

| N  | $y - y_{h,0}$ | order | $y - y_{h,0}$ | order | $p - p_{h,0}$ | order |
|----|---------------|-------|---------------|-------|---------------|-------|
| 40 | 0.2787        | 0.0811| 1.16          | 0.1779| 0.0206        |       |
| 50 | 0.2432        | 0.61  | 0.0627        | 1.16  | 0.1492        | 0.79  | 0.0159        | 1.16  |
| 60 | 0.2186        | 0.59  | 0.0511        | 1.13  | 0.1299        | 0.76  | 0.0130        | 1.11  |
| 70 | 0.2002        | 0.57  | 0.0431        | 1.11  | 0.1158        | 0.74  | 0.0110        | 1.09  |
| 80 | 0.1857        | 0.56  | 0.0372        | 1.09  | 0.1051        | 0.72  | 0.0095        | 1.07  |

Figure 4: Convergence history of $P_1$ FEM for Example 6.1.
Let the domain $\Omega$, the enrichment radius $r_s$, and the cut-off function $\chi(r)$ be the same as in Example 6.1. We take $\alpha = 1$, $u_0 = -\frac{1}{5}$, $u_1 = \frac{1}{5}$, and let
\[
y = \sqrt{r} \sin \left( \frac{\theta}{2} \right) - \frac{1}{4} r^2
\]
\[
p = x_2^2 (1 - x_1^2) (1 - x_1^2) + \frac{1}{2} \sqrt{r} \sin \left( \frac{\theta}{2} \right) (1 - x_1^2) (1 - x_1^2)
\]
\[
u = \min \left\{ \frac{1}{5}, \max \left\{ -\frac{p}{\alpha} - \frac{1}{5} \right\} \right\}
\]
be the analytical state, co-state, and control of the optimal control problem (1.1) subject to (6.1). We use the same meshes as in Example 6.1.

Tables 4-5 show results of the relative errors between $(y_h, p_h, u_h)$ and $(y, p, u)$, and Figure 5 shows the relative errors against the mesh size $h = 2/N$. We can see that the proposed methods yield optimal convergence orders, i.e. first order rates of convergence for $|y - y_h|_1$ and $|p - p_h|_1$, and second order rates of convergence for $|y - y_h|_0$, $|p - p_h|_0$, and $|u - u_h|_0$. This is consistent with the theoretical results in Theorem 4.2.

### Table 4: Relative errors of the cut XFEM for Example 6.2.

| $N$ | $|y - y_h|_1$ | order | $|y - y_h|_0$ | order | $|p - p_h|_1$ | order | $|p - p_h|_0$ | order | $|u - u_h|_0$ | order |
|-----|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|
| 39  | 0.01089      | 0.0067| 0.1456       | 0.0106| 0.0108       |       | 0.0108       |       |
| 49  | 0.00880      | 0.0044| 1.1620       | 0.0067| 2.00         | 0.0068| 2.02         |       |
| 59  | 0.00738      | 0.0031| 1.89         | 0.0066| 2.02         | 0.0047| 1.97         |       |
| 69  | 0.00636      | 0.95  | 0.0023       | 1.89  | 2.04         | 0.0034| 2.05         |       |
| 79  | 0.00559      | 0.96  | 0.0018       | 1.92  | 2.06         | 0.0026| 2.00         |       |

### Table 5: Relative errors of the classic XFEM for Example 6.2.

| $N$ | $|y - y_h|_1$ | order | $|y - y_h|_0$ | order | $|p - p_h|_1$ | order | $|p - p_h|_0$ | order | $|u - u_h|_0$ | order |
|-----|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|
| 39  | 0.00545      | 0.0024| 0.1360       | 0.0063| 0.0003       | 0.0009| 1.97         | 0.0062| 2.02         |       |
| 49  | 0.00445      | 0.0013| 0.1113       | 0.0059| 0.0041       | 0.0004| 1.97         | 0.0043| 1.96         |       |
| 59  | 0.00377      | 0.0011| 0.1091       | 0.0078| 0.0030       | 0.0031| 1.98         | 0.0021| 2.02         |       |
| 69  | 0.00327      | 0.0008| 0.0987       | 0.0065| 0.0023       | 0.0024| 1.99         |       |               |       |
| 79  | 0.00277      | 0.0006| 0.0865       | 0.102 | 0.0023       | 2.04  | 0.0024       | 1.99  |               |       |

Figure 5: Convergence history of the cut XFEM (left) and classic XFEM (right) for Example 6.2.

**Example 6.3.** A constrained problem in a non-convex domain.

Let $\Omega$ be a unit circle (Figure 6), and take $\alpha = 0.01$. We consider the optimal control problem (1.1) subject to
\[
\left\{ \begin{array}{ll}
-\Delta y + y = u + f, & \text{in } \Omega, \\
y = 0, & \text{on } \partial \Omega
\end{array} \right.
\]

(6.2)
with the control constraint

\[-0.3 \leq u \leq 1, \text{ a.e. on } \Omega.\]  

(6.3)

Set \(r_s = 0.5\), and let \(\chi(r)\) be a polynomial with \(r_0 = 0.01, r_1 = 0.99\), and let

\[y = (r^2 - r^2)\sin(\lambda \theta),\]

\[p = \alpha (r^2 - r^2)\sin(\lambda \theta),\]

\[u = \min \left\{ 1, \max \left\{ -\frac{p}{\alpha}, -0.3 \right\} \right\}\]

be the analytical state, co-state and control, respectively. We note that the co-state \(p\) satisfies

\[-\Delta p + p = y - y_d \quad \text{in } \Omega,\]

\[p = 0, \quad \text{on } \partial \Omega.\]

We apply the MATLAB mesh generator \(\text{distmesh2d}([30])\) to generate quasi-uniform triangular meshes (Figure 6): for \(h = 1/4, 1/8, 1/12, 1/16, \cdots\),

\[fd = @(q)(\text{sqrt}(\text{sum}(q^2)) - 1) + 1 * (q(:,1) > 0 + \text{eps} \& \& q(:,2) < 0 - \text{eps});\]

\([q, t] = \text{distmesh2d}(fd, \text{@huniform}, h, [-1, -1; 1, 1];,[0 : h : 1]', (0 : h : 1)'*0; (0 : h : 1)'*0, (-1 : h : 0)');\]

Tables 6-7 show results of the relative errors between \((y_n, p_n, u_n)\) and \((y, p, u)\), and Figure 5 shows the relative errors against the number of the mesh nodes, \(ND\). It is known that the optimal convergence orders of the errors (against \(ND\)) in \(H^1\) semi-norm and \(L^2\) norm are 1/2, 1 respectively. We can see that the proposed methods yield optimal convergence rates. We note that in [2], graded meshes and a post-processing procedure were used to acquire optimal convergence for the \(P_1\) element.

In Figures 8-10, we also show the cut XFEM solutions of the state, control and boundary of the active set at the mesh with \(h = 1/8\).
Figure 7: Convergence history of the cut XFEM (left) and classic XFEM (right) for Example 6.3.

Figure 8: The exact control $u$ (left) and the discrete control $u_h$ by the cut XFEM (right) for Example 6.3.

Figure 9: The exact state $y$ (left) and the discrete state $y_h$ by the cut XFEM for Example 6.3.
Figure 10: The active and inactive sets of the exact control $u$ (left) and the sets of the discrete control $u_h$ by the cut XFEM with $h = 1/8$: the red line is the boundary of active set for Example 6.3.

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