1. Introduction

Almost all lattice work on confinement has been carried out for an SU(2) gauge group. This is a good starting point. Confinement is caused by glue and every physicist believes that there is no essential physical difference in the confining properties of glue for SU(N), for arbitrary N. Despite this, in confinement by topological objects, questions arise for SU(3) which have no SU(2) analog. We discuss two of these. The first involves the fact that the formulation of the maximal abelian gauge (MAG) is more subtle for SU(3) than it is for SU(2). Calculations with the simplest form give poor results for the string tension (Sec.(3)). A generalized form which appears more natural is suggested (Sec.(4)). The second question has to do with the subgroup structure of monopoles, and their relation to P-vortices. There are strong arguments that monopoles should be associated with SU(2) subgroups of SU(3). There is also good evidence from our SU(3) lattice calculations that P-vortices pass through monopoles. This appears to be paradoxical at first, since P-vortices carry only center flux. We show how these two properties can coexist (Sec.(5)). We begin with a general discussion of gauge-fixing and projection (Sec.(2)).
2. Gauges, Maximal Gauges and Projection

The use of gauge-fixing followed by projection is very common in studies of confinement. The idea originates of course in 't Hooft’s famous 1981 paper [1]. Nevertheless, it is viewed with suspicion by many physicists. Projection implies the deletion of a large number of gauge degrees of freedom after choosing a particular gauge. Can this procedure ever be a controlled approximation? While no definitive answer will be attempted here, we give a discussion of some of the issues involved.

There is one very familiar example of gauge-fixing followed by projection where the reliability of the procedure is not in question. This is the Coulomb gauge in non-relativistic QED. The Coulomb gauge is a ‘maximal’ gauge, i.e. it seeks to squeeze the maximum physics into the timelike sector of the gauge field. This is accomplished by minimizing the spatial gauge degrees of freedom. The appropriate gauge functional is

$$G_{coul} = \int d^3 x A_k A_k$$

Minimizing $G_{coul}$ over the gauge group leads to the condition $\vec{\nabla} \cdot \vec{A} = 0$. The next stage is timelike projection; deleting the spatial components of the gauge field. The Coulomb gauge is clearly optimal for this projection. The truncated problem, say finding the energy levels of an atom, is then solved non-perturbatively. Finally, the effects of the spatial gauge field are brought back in and treated perturbatively.

This example shows that gauge-fixing followed by projection can be reliable. The key requirement is the presence of a small parameter. The Coulomb gauge exploits the fact that in atoms, electrons move slowly, with $v/c \sim \alpha$. We can say that non-relativistic QED obeys ‘timelike dominance’, justified by $v/c << 1$. Returning to the problem of confinement in $SU(N)$ Yang-Mills theory, no very small parameter like the fine structure constant is expected, but moderately small ratios can occur.

The maximal abelian gauge (MAG) followed by abelian projection has been the most intensely investigated gauge-fixing and projection scheme in non-abelian gauge theory. For $SU(2)$, if $A_3^\mu$ is chosen as the abelian field, then the MAG functional in the continuum is

$$G_{mag} = \int d^4 x [(A^1_\mu)^2 + (A^2_\mu)^2],$$

and at an extremum, the gauge conditions are

$$\partial_\mu A_\mu^\pm + i A_\mu^3 A_\mu^\pm = 0,$$

where $A^\pm_\mu = (A^1_\mu \pm i A^2_\mu)/\sqrt{2}$ are the charged fields. This similar in spirit to the Coulomb gauge example; to maximize the abelian sector of the gauge
field, the charged gauge degrees of freedom are minimized. After MAG

gauge-fixing, abelian projection is carried out; the charged degrees of free-

dom are deleted. A criterion which can give credence to abelian dominance

is a small value for the ratio $\xi_{\text{mon}}/\xi_\sigma$, where $\xi_{\text{mon}}$ is defined by the size

of the non-abelian core of a monopole, and $\xi_\sigma = 1/\sqrt{\sigma}$ is the correlation

length determined by the string tension, $\sigma$. There is some evidence from

lattice calculations that this ratio is in fact fairly small [2].

It is important for the generalization to $SU(3)$ to note that an alternate

way to write the $SU(2)$ MAG condition is to introduce a unit adjoint scalar

field $\Phi$, and write

$$G_{\text{higgs}} = \int d^4x D_\mu \Phi \cdot D_\mu \Phi.$$  (3)

An extremum of $G_{\text{higgs}}$ is found by holding the gauge field fixed, and varying

$\Phi$. Then the gauge transformation that takes $\Phi$ to the 3-axis or Higgs gauge,
takes the gauge fields to the MAG.

In the $SU(3)$ work presented here, we will start from the MAG. Al-

though usually taken to imply a monopole/dual superconductor picture of

confinement, center projection is still possible; i.e. the ‘indirect’ route to

the center is open. This gauge is known as the indirect maximal center
gauge (IMCG). In the IMCG, the relation between MAG monopoles and

P-vortices can be examined in detail. Advocates of center dominance of-

ten recoil at the use of the MAG, but capturing the confining degrees of

freedom in the maximal abelian subgroup does not preclude the possibility

that only the center is really relevant. For $SU(N)$, the center subgroup is

always a subgroup of the maximal abelian subgroup.

3. Simple $SU(3)$ MAG

For $SU(3)$, there are two abelian gauge fields, $A_\mu^3$ and $A_\mu^8$. The remaining six
gauge fields are all charged with respect to at least one of $A_\mu^3$, $A_\mu^8$. Treating
these charged fields democratically, the simplest maximal abelian gauge
condition for $SU(3)$ would minimize the continuum functional

$$G_{\text{mag}} = \int d^4x [(A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^4)^2 + (A_\mu^5)^2 + (A_\mu^6)^2 + (A_\mu^7)^2]$$  (4)

over the $SU(3)$ gauge group. The functional $G_{\text{mag}}$ is symmetric with re-
spect to each of the three $SU(2)$ subgroups of $SU(3)$. From this it is easy
to show that at an extremum of Eq.(4), there are three gauge conditions.
One is Eq.(2). The other two are similar and involve the charged combi-
nations $A_\mu^{\pm} = (A_\mu^1 \pm iA_\mu^5)/\sqrt{2}$, and $A_\mu^{\pm} = (A_\mu^6 \pm iA_\mu^7)/\sqrt{2}$. So with Eq.(4)
the $SU(3)$ MAG conditions are just $SU(2)$ MAG conditions with respect
to each of the three subgroups [4].
Minimizing the continuum functional $G_{\text{mag}}$ is equivalent to maximizing the lattice functional defined by [5]

$$G_{\text{mag}} = \sum_{x,\mu} \left( |(U_{\mu})_{11}|^2 + |(U_{\mu})_{22}|^2 + |(U_{\mu})_{33}|^2 \right).$$

(5)

| TABLE 1. SU(3) String Tensions |
|-----------------------------|
| $\beta$       | 5.90 | 6.0  |
| $SU(3)$       | 0.068(3) | 0.050(1) |
| $U(1) \times U(1)$ | 0.063(3) | 0.045(2) |
| mono          | 0.050(2) | 0.038(1) |
| $Z(3)$        | 0.060(3) | 0.040(2) |

We performed $SU(3)$ calculations at $\beta = 5.90$ on a $10^3 \times 16$ lattice and at $\beta = 6.0$ on a $16^4$ lattice. The details are fully discussed elsewhere [4]. The various string tensions are summarized in Table 1. All of the cases which involve gauge-fixing and projection made use of the MAG in the form of Eq.(5), followed by abelian projection to $U(1) \times U(1)$. There are four string tensions listed: full $SU(3)$, MAG $U(1) \times U(1)$, MAG monopoles, and IMCG $Z(3)$. The latter requires a second gauge-fixing applied to the abelian projected $U(1) \times U(1)$ links, followed by center projection. The data in the table are for one gauge fixing/configuration. The effect of gauge ambiguities has also been investigated, with the expected result. Namely, as gauge copies with higher values of the gauge functionals are used, the projected string tensions all decrease. For 10 copies/configuration, we found a roughly 10% decrease in the three string tensions in Table 1 which involve gauge-fixing and projection.

The projected string tensions of Table 1 are all smaller than the corresponding full $SU(3)$ values. This behavior is quite different from that found in $SU(2)$. For $SU(2)$, the MAG $U(1)$ string tension is larger than the $SU(2)$ string tension for one gauge copy/configuration. Taking gauge copies with higher functional values causes the $U(1)$ string tension to decrease and approach the full $SU(2)$ string tension. For $SU(3)$, the abelian projected $U(1) \times U(1)$ string tension is already low for one gauge copy/configuration. Taking account of ‘more maximal’ gauge copies reduces it still further below the $SU(3)$ string tension. The low results found for MAG monopole and IMCG $Z(3)$ string tensions are certainly caused in part by this difficulty. We conclude that either something is wrong with the whole idea of
the MAG and abelian projection for $SU(3)$, or that a different form of the MAG is needed. In the next section, we explore the latter possibility.

4. Generalized Maximal Abelian Gauge

For $SU(2)$ the functional $G_{\text{higgs}}$ of Eq.(3) is merely a rewriting of the MAG functional, Eq.(1). However, the analogous statement is not true for $SU(3)$. Introduce a unit length adjoint scalar field for $SU(3)$. In Higgs gauge, this field is gauge transformed to the Cartan algebra,

$$\Phi = \hat{3} \cos(\chi) + \hat{8} \sin(\chi).$$

where the angle $\chi$ is $SU(3)$ gauge invariant. In this gauge, the $SU(3)$ version of the functional of Eq.(3) becomes

$$G_{\text{higgs}} = \int d^4x \left\{ \cos^2(\chi) \left[ (A_1^1)^2 + (A_2^2)^2 \right] + \cos^2\left(\frac{\pi}{3} - \chi\right) \left[ (A_4^1)^2 + (A_5^5)^2 \right] + \cos^2\left(\frac{\pi}{3} + \chi\right) \left[ (A_6^6)^2 + (A_7^7)^2 \right]\right\},$$

which is clearly different from Eq.(4). A single $SU(3)$ adjoint scalar field cannot give equal coefficients to the charged fields in the different $SU(2)$ subgroups. Formally, equal coefficients can be attained by averaging over the angle $\chi$, but this is in effect saying there is a continuous distribution of scalar fields with different $\chi$ angles. A continuous distribution is actually unnecessary; it suffices to have two fields, at angles $\chi$ and $\chi + \pi/2$ relative to 3. This way of obtaining the MAG functional of Eq.(4) suggests that Eq.(4) is itself rather unnatural. Minimizing this functional involves attempting to suppress the charged gauge fields with respect to (at least) two different directions in the Cartan algebra. These conflicting requirements may be the cause of the low string tensions found in Table 1.

The functional of Eq.(6) can still be regarded as defining a maximal abelian gauge since minimizing it will tend to suppress all the charged gauge fields. The equivalent lattice functional is easily written down;

$$G_{\text{higgs}} = \sum_{x,\mu} \text{tr}((\lambda_3 \cos \chi + \lambda_8 \sin \chi)U_{\mu}(x) (\lambda_3 \cos \chi' + \lambda_8 \sin \chi')U_{\mu}^\dagger(x)),$$

where $\chi = \chi(x)$ and $\chi' = \chi(x + \hat{\mu}a)$. Eq.(7) allows two distinct possibilities for a generalized MAG. The angle $\chi$ can be held fixed, or allowed to vary with $x$. (The latter possibility requires an extra term, $\partial_\mu \chi \partial_\mu \chi$, in the integrand of Eq.(6).) Either way, there is now just one effective Higgs field. We are at present actively exploring Eq.(7) both for the case where $\chi$ is held fixed and where it is allowed to vary. Whether $\chi$ is held fixed or allowed to vary, the functional describes a ‘maximal’ gauge, so there will be
gauge ambiguities. An inexpensive way to see if Eq.(7) leads to an improved abelian projection will be to calculate with one gauge-fixing/configuration. The result should be a $U(1) \times U(1)$ string tension larger than the full $SU(3)$ result, which would then decrease when gauge copies with higher functional values are used.

The overall message of this section is that Eq.(5) is based on a too-literal analogy with the $SU(2)$ case, and that more general possibilities for the $SU(3)$ MAG exist and are needed.

5. Monopoles and P-Vortices in $SU(3)$

There are interesting questions about monopoles and vortices and their relation to the degrees of freedom which control confinement. For example, how are the monopoles and P-vortices found on the lattice related to physical monopoles and center vortices? Is either of these degrees of freedom more fundamental than the other, and if so which? In this section, we discuss a question which first arises upon turning to an $SU(3)$ gauge group from $SU(2)$.

In an abelian or Higgs gauge, outside the core of a monopole, its fields are abelian. For an $SU(2)$ gauge group, this means the long range gauge field is $\sim A_3^\mu \tau_3/2$, where $\tau_3$ is a Pauli matrix. For an $SU(3)$ gauge group, a generic abelian field is of the form $\sim (A_3^\mu \lambda_3 + A_8^\mu \lambda_8)/2$. In his original paper, 't Hooft argued that for $SU(N)$, monopoles should be associated with $SU(2)$ subgroups [1]. Studying monopole solutions, E. Weinberg and P. Yi came to the same conclusion [10]. These results imply that in $SU(3)$, the long range field of a monopole is more specific than the generic abelian field. Namely, the field should always be ‘$\lambda_3$-like’, and there should be no purely ‘$\lambda_8$-like’ monopoles. We find support for this in our $SU(3)$ lattice calculations. The color magnetic current we find is generally of the $\lambda_3$-like forms $(1,-1,0), (1,0,-1), (0,1,-1)$. The $\lambda_8$-like forms $(1,1,-2), (1,-2,1), (-2,1,1)$ basically never happen.

If it is granted that monopoles are associated with $SU(2)$ subgroups, it is still possible that the net effect of superposing monopoles from different subgroups could be fields which mainly connect to the group center. However, from one monopole, the field is alive in only two out of three colors, whereas center flux is the same for all three. This would seem to argue against a detailed connection on the lattice between MAG monopoles and IMCG P-vortices.

It is well-established in $SU(2)$ lattice calculations that in the IMCG, P-vortices pass through MAG monopoles [8, 9]. This can be pictured as a squeezing of the monopole flux into $Z(2)$ Dirac strings. For $SU(2)$, the monopole flux $g$ comes in Schwinger units, $g = 4\pi/e$, where $e$ is the non-
abelian gauge coupling. If the monopole flux is squeezed into two strings, each carrying flux \( g/2 \), the strings are visible to fundamental (I=1/2) quarks, and behave like P-vortices.

In our \( SU(3) \) lattice calculations, using MAG monopoles and IMCG P-vortices, it was found that again there is an intimate connection between monopole and vortex degrees of freedom. If a link has a non-zero magnetic current, the cube dual to this link has faces pierced by the \( Z(3) \) flux of P-vortices over 80% of the time. Since methods of locating monopoles and P-vortices are not precise, the actual number could be 100%. The \( SU(3) \) case allows an odd number of P-vortices to meet at a monopole, but otherwise there appears to be little essential difference in lattice numerical calculations between \( SU(2) \) and \( SU(3) \).

To see how \( SU(3) \) P-vortices carrying only center flux can fit together with monopoles associated with \( SU(2) \) subgroups, consider a specific example. Put a monopole at the origin, and let it be associated with the 'I-spin' subgroup generated by \( \lambda_1, \lambda_2, \lambda_3 \). Imitate the \( SU(2) \) situation by squeezing the monopole flux into two strings on the \( z \)-axis, one on the +\( z \)-axis carrying upward flux \( g/2 \); the other on the −\( z \)-axis carrying upward flux −\( g/2 \). Now suppose that there is an \( SU(3) \) P-vortex on the \( z \)-axis. A P-vortex has flux quantized in units \( \Phi_P = g/\sqrt{3} \). For \( z > 0 \), we can represent the vortex in terms of the \( \lambda_8 \)-like matrix associated with V-spin, \( \lambda'_8 = (-\lambda_8 + \sqrt{3} \lambda_3)/2 \). When a fundamental quark goes around this part of the P-vortex, it picks up a phase factor

\[
\exp(ie\Phi_P \frac{\lambda'_8}{2}) \subset Z(3).
\]

For \( z < 0 \), we can represent the vortex in terms of the \( \lambda_8 \)-like matrix associated with U-spin, \( \lambda''_8 = (-\lambda_8 - \sqrt{3} \lambda_3)/2 \), giving the same phase factor

\[
\exp(ie\Phi_P \frac{\lambda''_8}{2}) \subset Z(3).
\]

We note that for \( z > 0 \) the part of the flux involving \( \lambda_3/2 \) is \( \Phi_P \sqrt{3}/2 = g/2 \), while that for \( z < 0 \) involving \( \lambda_3/2 \) is \( -\Phi_P \sqrt{3}/2 = -g/2 \). These are exactly as they should be for an I-spin monopole at the origin. The flux of the P-vortex has a different distribution over colors on opposite sides of the monopole. This flux difference is not visible in the phase factor experienced by a quark encircling the P-vortex. All this is entirely analogous to the \( SU(2) \) situation.

We conclude that in \( SU(3) \) there is no conflict between the following three facts: (1) Monopoles are associated with \( SU(2) \) subgroups. (2) P-vortices carry only center flux. (3) The magnetic current of monopoles is a world line located on the world sheet of P-vortices. The question of which
of MAG monopoles or IMCG P-vortices is most closely tied to physical topological objects remains unanswered.

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