HOPF MODULES AND NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

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Abstract. We define a new algebra of noncommutative differential forms for any Hopf algebra with an invertible antipode. We prove that there is a one to one correspondence between anti-Yetter-Drinfeld modules, which serve as coefficients for the Hopf cyclic (co)homology, and modules which admit a flat connection with respect to our differential calculus. Thus we show that these coefficient modules can be regarded as “flat bundles” in the sense of Connes’ noncommutative differential geometry.

1. Introduction

In this paper we define a new algebra of noncommutative differential forms over any Hopf algebra $H$ with an invertible antipode. The resulting differential calculus, denoted here by $K^*(H)$, is intimately related to the class of anti-Yetter-Drinfeld modules over $H$, herein called AYD modules, that were introduced in [7]. More precisely, we show that there is a one to one correspondence between AYD modules over $H$ and $H$–modules that admit a flat connection with respect to our differential calculus $K^*(H)$. In general terms, this gives a new interpretation of AYD modules as noncommutative analogues of local systems over the noncommutative space represented by $H$.

The introduction of AYD modules in [7] was motivated by the problem of finding the largest class of Hopf modules that could serve as coefficients for the cyclic (co)homology of Hopf algebras introduced by Connes and Moscovici [3, 4, 5] and its extensions and ramifications defined in [9, 10, 6, 8]. The question of finding an appropriate notion of local system, or coefficients, to twist the cyclic homology of associative algebras is an interesting open problem. The case of Hopf cyclic cohomology on the other hand offers an interesting exception in that it admits coefficients and furthermore, as we shall prove, these coefficient modules can be regarded as “flat bundles” in the sense of Connes’ noncommutative differential geometry [2].

Here is a plan of this paper. In Section 2 we recall the definitions of anti-Yetter-Drinfeld (AYD) and Yetter-Drinfeld (YD) modules over a Hopf algebra, and we give a characterization of the category of AYD modules over group algebras as a functor category. This will serve as our guiding example. Motivated by this, in Section 3 we define two natural differential calculi $K^*(H)$ and $\hat{K}^*(H)$ associated with an arbitrary
Hopf algebra $H$. Next, we give a complete characterization of the category of AYD and YD modules over a Hopf algebra $H$ as modules admitting flat connections over these differential calculi $\mathcal{K}^*(H)$ and $\hat{\mathcal{K}}^*(H)$, respectively. In the same section, we also investigate tensor products of modules admitting flat connections.

In Section 4, we place the differential calculi $\mathcal{K}^*(H)$ and $\hat{\mathcal{K}}^*(H)$ in a much larger class of differential calculi, unifying the results proved separately before. This last section is motivated by the recent paper [11]. After this paper was posted, T. Brzezinski informed us that he can also derive our differential calculus and flatness results from the theory of corings [1]. This would involve realizing AYD modules via a specific entwining structure, and using the general machinery of [1] that cast such structures as flat modules.

In this paper we work over a field $k$ of an arbitrary characteristic. All of our results, however, are valid over an arbitrary unital commutative ground ring $k$ if the objects involved (coalgebras, bialgebras, Hopf algebras and their (co)modules) are flat $k$–modules. We will assume $H$ is a Hopf algebra with an invertible antipode and all of our $H$–modules and comodules are left modules and comodules unless explicitly indicated otherwise.

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2. Preliminaries and a guiding example

Recall from [7] that a $k$–module $X$ is called a left-left anti-Yetter-Drinfeld (AYD, for short) module if (i) $X$ is a left $H$–module, (ii) $X$ is a left $H$–comodule, and (iii) one has a compatibility condition between the $H$–module and comodule structure on $X$ in the sense that:

$$\langle hx\rangle_{(-1)} \otimes \langle hx\rangle_{(0)} = h_{(1)}x_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}x_{(0)}$$

for any $h \in H$ and $x \in X$. The AYD condition [11] should be compared with the Yetter-Drinfeld (YD) compatibility condition:

$$\langle hx\rangle_{(-1)} \otimes \langle hx\rangle_{(0)} = h_{(1)}x_{(-1)}S(h_{(3)}) \otimes h_{(2)}x_{(0)}$$

for any $h \in H$ and $x \in X$.

A morphism of AYD modules $X \to Y$ is simply a $k$-linear map $X \to Y$ compatible with the $H$–action and coaction. The resulting category of AYD modules over $H$ is an abelian category. There is a similar statement for YD modules.
To pass from (1) to (2) one replaces $S^{-1}$ by $S$. This is, however, a nontrivial operation since the categories of AYD and YD modules over $H$ are very different in general. For example, the former is not a monoidal category in a natural way \[7\] while the latter is always monoidal. If $S^2 = id_H$, in particular when $H$ is commutative or cocommutative, then these categories obviously coincide.

To understand these compatibility conditions better we proceed as follows. Let $X$ be a left $H$-module. We define a left $H$-action on $H \otimes X$ by letting
\[
h(g \otimes x) := h_{(1)} g S^{-1}(h_{(3)}) \otimes h_{(2)} x
\]
for any $h \in H$ and $g \otimes x \in H \otimes X$.

**Lemma 2.1.** The formula given in (3) defines a left $H$-module structure on $H \otimes X$. Moreover, an $H$-module/comodule $X$ is an anti-Yetter-Drinfeld module iff its comodule structure map $\rho_X : X \to H \otimes X$ is a morphism of $H$-modules.

There is of course a similar characterization of YD modules. The left action (3) should simply be replaced by the left action
\[
h(g \otimes x) := h_{(1)} g S(h_{(3)}) \otimes h_{(2)} x
\]

Let us give a characterization of AYD modules in a concrete example. Let $G$ be a, not necessarily finite, discrete group and let $H = k[G]$ be its groups algebra over $k$ with its standard Hopf algebra structure, i.e. $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$ for all $g \in G$. We define a groupoid $G \rtimes G$ whose set of morphisms is $G \times G$ and its set of objects is $G$. Its source and target maps $s : G \rtimes G \to G$ and $t : G \rtimes G \to G$ are defined by
\[
s(g, g') = g \quad \quad t(g, g') = gg'g^{-1}
\]
for any $(g, g')$ in $G \rtimes G$. It is easily seen that under group multiplication as its composition, $G \rtimes G$ is a groupoid.

**Proposition 2.2.** The category of Yetter-Drinfeld modules over $k[G]$ is isomorphic to the category of functors from $G \rtimes G$ into the category of $k$-modules.

**Proof.** Let $M$ be a $k[G]$-module/comodule. Denote its structure morphisms by $\mu : k[G] \otimes M \to M$ and $\rho : M \to k[G] \otimes M$. Since we assumed $k$ is a field, $M$ has a basis of the form $\{e^i\}_{i \in I}$ for some index set $I$. Since $M$ is a $k[G]$-comodule one has
\[
e^i_{(-1)} \otimes e^j_{(0)} = \sum_{j \in I} \sum_{g \in G} e^i_{j,g}(g \otimes e^j)
\]
where only finitely many $c_{j,g}$ is non-zero. One can choose a basis $\{m^\lambda\}_{\lambda \in \Lambda}$ for $M$ such that

$$m^\lambda_{(-1)} \otimes m^\lambda_{(0)} = \sum c_{\lambda,\alpha}(g^\alpha \otimes m^\alpha)$$

and since all comodules are counital and $k[G]$ has a counit $\varepsilon(g) = 1$ for any $g \in G$ we see that

$$m^\lambda = \sum c_{\lambda,\alpha}m^\alpha$$

implying $c_{\lambda,\alpha}$ is uniformly zero except $c_{\lambda,\lambda}$ which is 1. In other words, one can split $M$ as $\bigoplus_{g \in G} M_g$ such that $\rho(x) = g \otimes x$ for any $x \in M_g$. Now assume $M$ is an AYD module. Then since

$$(hx)_{(-1)} \otimes (hx)_{(0)} = hgh^{-1} \otimes hx$$

for any $x \in M_g$ and $h \in G$ one can see that $L_h : M_g \rightarrow M_{hgh^{-1}}$ where $L_h$ is the $k$–vector space endomorphism of $M$ coming from the left action of $h$. This observation implies that the category of AYD modules over $k[G]$ and the category of functors from $G \ltimes G$ into the category of $k$–modules are isomorphic. □

3. Differential calculi and flat connections

Our next goal is to find a noncommutative analogue of Proposition 2.2. To this end, we will replace the groupoid $G \ltimes G$ by a differential calculus $K^*(H)$ naturally defined for any Hopf algebra $H$. The right analogue of representations of the groupoid $G \ltimes G$ will be $H$–modules admitting flat connections with respect to the differential calculus $K^*(H)$.

Let us first recall basic notions of connection and curvature in the noncommutative setting from [2].

Let $A$ be a $k$–algebra. A differential calculus over $A$ is a differential graded $k$–algebra $(\Omega^*, d)$ endowed with a morphism of algebras $\rho : A \rightarrow \Omega^0$. The differential $d$ is assumed to have degree one. Since in our main examples we have $\Omega^0 = A$ and $\rho = id$, in the following we assume this is the case.

Assume $M$ is a left $A$–module. A morphism of $k$–modules $\nabla : M \rightarrow \Omega^1 \otimes_A M$ is called a connection with respect to the differential calculus $(\Omega^*, d)$ if one has a Leibniz rule of the form

$$\nabla(am) = a\nabla(m) + d(a) \otimes m_{\bar{A}}$$

for any $m \in M$ and $a \in A$. Given any connection $\nabla$ on $M$, there is a unique extension of $\nabla$ to a map $\hat{\nabla} : \Omega^* \otimes_A M \rightarrow \Omega^* \otimes_A M$ satisfying a graded Leibniz rule. It is given by

$$\hat{\nabla}(\omega \otimes m) = d(\omega) \otimes m_{\bar{A}} + (-1)^{\abs{\omega}}\omega \nabla(m)$$

for any $m \in M$ and $\omega \in \Omega^*$. A connection $\nabla : M \rightarrow \Omega^1 \otimes_A M$ is called flat if its curvature $R := \hat{\nabla}^2 = 0$. This is equivalent to saying $\Omega^* \otimes_A M$ is a differential graded $\Omega^*$–module with the extended differential $\hat{\nabla}$. 

Proof. For any $h \in H$ and $(x^0 \otimes \cdots \otimes x^n) \in X_0 \otimes \cdots \otimes X_n$. Checking the bimodule conditions is straightforward. We denote this bimodule by $X_0 \otimes \cdots \otimes X_n$.

Remark 3.2. We should remark that $X \otimes Y$ is not a monoidal product. In other words given any three $H$–bimodules $X$, $Y$, and $Z$, then $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are not isomorphic as left $H$–modules unless $H$ is cocommutative.

Definition 3.3. For each $n \geq 0$, let $\mathcal{K}^n(H) = H^\otimes n + 1$. We define a differential $d : \mathcal{K}^n(H) \to \mathcal{K}^{n+1}(H)$ by

$$d(h^0 \otimes \cdots \otimes h^n) = -(1 \otimes h^0 \otimes \cdots \otimes h^n) + \sum_{j=0}^{n-1} (-1)^j (h^0 \otimes h^j_1 \otimes h^j_2 \otimes \cdots \otimes h^n)$$

$$+ (-1)^n (h^0 \otimes \cdots \otimes h^{n-1} \otimes h^1_1) S^{-1}(h^n_1) \otimes h^n_2).$$

We also define an associative graded product structure by

$$(x^0 \otimes \cdots \otimes x^n)(y^0 \otimes \cdots \otimes y^m)$$

$$= x^0 \otimes \cdots \otimes x^{n-1} \otimes x^{(1)}_1 y^0 S^{-1}(h^{(2m+1)}_1) \otimes \cdots \otimes x^{(m)}_m y^m S^{-1}(x^{(m+1)}_m) \otimes x^{(m+1)}_m y^m$$

for any $(x^0 \otimes \cdots \otimes x^n)$ in $\mathcal{K}^n(H)$ and $(y^0 \otimes \cdots \otimes y^m)$ in $\mathcal{K}^m(H)$.

Proposition 3.4. $\mathcal{K}^*(H)$ is a differential graded $k$–algebra.

Proof. For any $x \in \mathcal{K}^0(H)$ one has

$$d(x) = -(1 \otimes x) + (x_1) S^{-1}(x_3) \otimes x_2 = [x, (1 \otimes 1)]$$

and for $(y \otimes 1)$ in $\mathcal{K}^1(H)$ and $x \in \mathcal{K}^0(H)$ we see

$$d(x(y \otimes 1)) = d(x_1) y S^{-1}(x_3) \otimes x_2$$

$$= - (1 \otimes x_1) y S^{-1}(x_3) \otimes x_2 + (x_1) y_1 S^{-1}(x_5) \otimes x_2 y_2 S^{-1}(x_4) \otimes x_3$$

$$- (x_1) y S^{-1}(x_5) \otimes x_2 S^{-1}(x_4) \otimes x_3$$

$$=d(x)(y \otimes 1) + xd(y \otimes 1)$$
We also see for \((x \otimes y)\) in \(\mathcal{K}^1(H)\) the we have
\[
d((x \otimes 1)y) = d(x \otimes y) = -(1 \otimes x \otimes y) + (x(1) \otimes x(2) \otimes y) - (x \otimes y(1)S^{-1}(y(3)) \otimes y(2)) \\
= -(1 \otimes x \otimes 1)y + (x(1) \otimes x(2) \otimes 1)y - (x \otimes 1 \otimes 1)y + (x \otimes 1)(1 \otimes y) \\
= d(x \otimes 1)y - (x \otimes 1)d(y).
\]
Note that with the product structure on \(\mathcal{K}^*(H)\) one has
\[
(x^0 \otimes \cdots \otimes x^n) = (x^0 \otimes 1) \cdots (x^{n-2} \otimes 1)(x^{n-1} \otimes 1)x^n
\]
for any \(x^0 \otimes \cdots \otimes x^n\) in \(\mathcal{K}^n(H)\). Now, one can inductively show that
\[
d(\Psi \Phi) = d(\Psi)\Phi + (-1)^{|\Psi|}\Phi d(\Phi)
\]
for any \(\Psi\) and \(\Phi\) in \(\mathcal{K}^*(H)\). Since the algebra is generated by degree zero and degree one terms, all that remains is to show that for all \(x \in H\) we have \(d^2(x) = 0\) and \(d^2(x \otimes 1) = 0\). For the first assertion we see that
\[
d^2(x) = -d(1 \otimes x) + d(x(1)S^{-1}(x(3)) \otimes x(2)) \\
= (1 \otimes 1 \otimes x) - (1 \otimes 1 \otimes x) + (1 \otimes x(1)S^{-1}(x(3)) \otimes x(2)) - (1 \otimes x(1)S^{-1}(x(3)) \otimes x(2)) \\
+ (x(1)S^{-1}(x(3)) \otimes x(1)S^{-1}(x(3)) \otimes x(2)) - (x(1)S^{-1}(x(3)) \otimes x(2)S^{-1}(x(2)S^{-1}(x(2))) \otimes x(2)) \\
= 0
\]
for any \(x \in H\). For the second assertion we see
\[
d^2(x \otimes 1) = -d(1 \otimes x \otimes 1) + d(x(1) \otimes x(2) \otimes 1) - d(x \otimes 1 \otimes 1) \\
= -(1 \otimes 1 \otimes x \otimes 1) + (1 \otimes x(1) \otimes x(2) \otimes 1) - (1 \otimes x(1) \otimes x(2) \otimes 1) + (1 \otimes x \otimes 1 \otimes 1) \\
- (1 \otimes x(1) \otimes x(2) \otimes 1) + (x(1) \otimes x(2) \otimes x(3) \otimes 1) - (x(1) \otimes x(2) \otimes x(3) \otimes 1) + (x(1) \otimes x(2) \otimes 1 \otimes 1) \\
+ (1 \otimes x \otimes 1 \otimes 1) - (x(1) \otimes x(2) \otimes 1 \otimes 1) + (x \otimes 1 \otimes 1 \otimes 1) - (x \otimes 1 \otimes 1 \otimes 1) \\
= 0
\]
for any \((x \otimes 1)\) in \(\mathcal{K}^*(H)\). The result follows. □

Note that the calculus \(\mathcal{K}^*(H)\) is determined by (i) the \(H\)-bimodule \(\mathcal{K}^1(H) = H \otimes H\) (ii) the differential \(d_0 : H \to H \otimes H\) and \(d_1 : H \otimes H \to H \otimes H \otimes H\) and (iii) the Leibniz rule \(d(\Psi \Phi) = d(\Psi)\Phi + (-1)^{|\Psi|}\Phi d(\Phi)\).
Recall that for any coassociative coalgebra $C$, the category of $C$–bicomodules has a monoidal product called cotensor product, which is denoted by $\boxtimes_C$. The cotensor product is left exact and its right derived functors are denoted by $\text{Cotor}_C^\ast(\cdot, \cdot)$.

Now we can identify the homology of the calculus $K^\ast(H)$ as follows:

**Proposition 3.5.** $H_\ast(K^\ast(H))$ is isomorphic to $\text{Cotor}_H^\ast(k, H^\text{coad})$ where $k$ is considered as an $H$–comodule via the unit and $H^\text{coad}$ is the coadjoint corepresentation over $H$, i.e. $\rho^\text{coad}(h) = h_{(1)}S^{-1}(h_{(3)}) \otimes h_{(2)}$ for any $h \in H$.

**Theorem 3.6.** The category of AYD modules over $H$ is isomorphic to the category of $H$–modules admitting a flat connection with respect to the differential calculus $K^\ast(H)$.

**Proof.** Assume $M$ is a $H$–module which admits a morphism of $k$–modules of the form $\nabla : M \to K^1_H(H) \otimes M \cong H \otimes M$. Define $\rho^M(m) = \nabla(m) + (1 \otimes m)$ and denote $\rho^M(m)$ by $(m_{(-1)} \otimes m_{(0)})$ for any $m \in M$.

First we see that

$$\nabla(hm) = (hm)_{(-1)} \otimes (hm)_{(0)} - (1 \otimes hm)$$

and also

$$d(h \otimes m + h\nabla(m)) = -(1 \otimes hm) + (h_{(1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m)$$

$$+ (h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}) - (h_{(1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m)$$

$$=(h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}) - (1 \otimes hm)$$

for any $h \in H$ and $m \in M$. This means $\nabla$ is a connection iff the $H$–module $M$ together with $\rho^M : M \to H \otimes M$ satisfy the AYD condition. The flatness condition will hold iff for any $m \in M$ one has

$$\hat{\nabla}^2(m) = d(m_{(-1)} \otimes 1 \otimes m_{(0)}) - (m_{(-1)} \otimes 1)\nabla(m_{(0)}) - d(1 \otimes 1)m + (1 \otimes 1)\nabla(m)$$

$$=(m_{(-1)}(1) \otimes m_{(-1)(2)} \otimes m_{(0)}) - (m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)}) = 0,$$

meaning $\nabla$ is flat iff $\rho^M : M \to H \otimes M$ defines a coassociative coaction of $H$ on $M$. $\square$

**Definition 3.7.** Let $X$ be an AYD module over $H$. Define $K^\ast(H, X)$ as the graded $k$–module $K^\ast(H) \otimes X$ equipped with the connection $\nabla_X(x) = \rho_X(x) - (1 \otimes x)$ as its differential.

Recall from [7] that an $H$–module/comodule $X$ is called stable if the composition $X \xrightarrow{\rho_X} H \otimes X \xrightarrow{\mu_X} X$ is $id_X$ where $\rho_X$ and $\mu_X$ denote the $H$–comodule and $H$–module structure maps respectively. Explicitly one has $x_{(-1)}x_{(0)} = x$ for any $x \in X$. 

**Theorem 3.8.** For an arbitrary AYD module $X$ one has $H_\ast(K^\ast(H, X)) \cong \text{Cotor}^\ast_H(k, X)$. Moreover, if $X$ is also stable, then $K^\ast(H, X)$ is isomorphic (as differential graded $k$–modules) to the Hochschild complex of the Hopf-cyclic complex of the Hopf algebra $H$ with coefficients in $X$.

**Proof.** The first part of the Theorem follows from the observation that $K^\ast(H, X)$, viewed just as a differential graded $k$–module, is really $B^\ast(k, H, X)$ the two sided cobar complex of the coalgebra $H$ with coefficients in $H$–comodules $k$ and $X$. The second assertion follows from Remark 3.13 and Theorem 3.14 of [8]. □

**Proposition 3.9.** Let $X$ be an AYD module over $H$. If $H$ is cocommutative, then $K^\ast(H, X)$ is a differential graded left $H$–module with respect to the AYD module structure on $K^\ast(H, X)$.

Instead of the AYD condition, one can consider the YD condition and form a differential calculus $\hat{K}^\ast(H)$ using the YD condition.

**Definition 3.10.** As before, assume $H$ is a Hopf algebra, but this time we do not require the antipode to be invertible. We define a new differential calculus $\hat{K}^\ast(H)$ over $H$ as follows: let $\hat{K}^n(H) = H \otimes H \otimes \cdots \otimes H$ and define the differentials as

$$d(x^0 \otimes \cdots \otimes x^n) = -(1 \otimes x^0 \otimes \cdots \otimes x^n) + \sum_{j=0}^{n-1} (-1)^j (x^0 \otimes \cdots \otimes x^j \otimes x^{j+1} \otimes \cdots \otimes x^n)$$

$$+ (-1)^n(x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n) S(x^0 \otimes \cdots \otimes x^n)$$

for any $x^0 \otimes \cdots \otimes x^n$ in $\hat{K}^n(H)$. The multiplication is defined as

$$(x^0 \otimes \cdots \otimes x^n)(y^0 \otimes \cdots \otimes y^m)$$

$$= x^0 \otimes \cdots \otimes x^{n-1} \otimes x^n y \otimes S(x_{(m+1)} y_{(2m+1)} \otimes \cdots \otimes x_{(m)} y_{(m+1)} S(x^n_{(m+1)} \otimes x_{(m+1)} y^m)$$

for any $x^0 \otimes \cdots \otimes x^n$ and $y^0 \otimes \cdots \otimes y^m$ in $\hat{K}^\ast(H)$.

The proofs of the following facts are similar to the corresponding statements for the differential calculus $K^\ast(H)$ and AYD modules.

**Proposition 3.11.** $\hat{K}^\ast(H)$ is a differential graded $k$–algebra.

**Theorem 3.12.** The category of YD modules over $H$ is isomorphic to the category of $H$–modules admitting a flat connection with respect to the differential calculus $\hat{K}^\ast(H)$.

**Definition 3.13.** Let $X$ be a Yetter-Drinfeld module $X$ over $H$ with the structure morphisms $\mu_X : H \otimes X \to X$ and $\rho_X : X \to H \otimes X$. Define $\hat{K}^\ast(H, X)$ as the (differential) graded $k$–module $\hat{K}^\ast(H) \otimes \hat{K}^\ast(X)$ with the connection $\nabla_X(x) = \rho_X(x) - (1 \otimes x)$ defined for any $x \in X$. 
Proposition 3.14. For an arbitrary YD module $X$ one has $H_*(\hat{K}^*(H, X)) \cong \text{Cotor}_H^*(k, X)$.

Proposition 3.15. Assume $X$ is an arbitrary YD module over $H$. If $H$ is cocommutative, then $\hat{K}^*(H, X)$ is a differential graded left $H$–module with respect to the YD module structure on $\hat{K}^*(H)$.

Our next goal is to study tensor products of AYD and YD modules in our noncommutative differential geometric setup. It is well known that the tensor product of two flat vector bundles over a manifold is again a flat bundle. Moreover, from the resulting monoidal, in fact Tannakian, category one can recover the fundamental group of the base manifold. The situation in the noncommutative case is of course far more complicated and we only have some vestiges of this theory.

Assume $H$ is a Hopf algebra with an invertible antipode. Let $X$ be a $H$–module admitting a flat connection $\nabla$ with respect to the calculus $\mathcal{K}^*(H)$. We define a switch morphism $\sigma : X \otimes H \to H \otimes X$ by letting

$$\sigma(x \otimes h) = x_{(-1)} h \otimes x_{(0)} + h \otimes x,$$

for any $x \otimes h$ in $X \otimes H$ where we used a Sweedler notation to denote the connection: $\nabla(x) = x_{(-1)} \otimes x_{(0)}$. Note that $\sigma$ is a perturbation of the standard switch map.

Proposition 3.16. Let $X$ and $X'$ be two $H$–modules with flat connections $\nabla_X$ and $\nabla_{X'}$ with respect to the differential calculi $\hat{K}^*(H)$ and $\mathcal{K}^*(H)$ respectively. Then $\nabla_{X \otimes X'} : X \otimes X' \to H \otimes X \otimes X'$ given by

$$\nabla_{X \otimes X'}(x \otimes x') = \nabla_X(x) \otimes x' + (\sigma \otimes \text{id}_{X'}) (x \otimes \nabla_{X'}(x'))$$

for any $x \otimes x'$ in $X \otimes X'$ defines a flat connection on the $H$–module $X \otimes X'$ with respect to $\mathcal{K}^*(H)$.

Proof. Recall from Theorem 3.6 and Theorem 3.12 that the category of AYD (resp. YD) modules and the category of $H$–modules admitting flat connections with respect to the differential calculus $\mathcal{K}^*(H)$ (resp. $\hat{K}^*(H)$) are isomorphic. Then given two $H$–modules with flat connections $(X, \nabla_X)$ over $\hat{K}^*(H)$ and $(X', \nabla_{X'})$ over $\mathcal{K}^*(H)$ one can extract $H$–comodule structures by letting $x_{(-1)} \otimes x_{(0)} := \rho_X(x) := \nabla_X(x) + (1 \otimes x)$ and $x'_{(-1)} \otimes x'_{(0)} := \rho_{X'}(x') := \nabla_{X'}(x') + (1 \otimes x')$. Then we get

$$\nabla_{X \otimes X'}(x \otimes x') = \nabla_X(x) \otimes x' + (\sigma \otimes \text{id}_{X'}) (x \otimes \nabla_{X'}(x'))$$

$$= - (1 \otimes x \otimes x') + (x_{(-1)} \otimes x_{(0)} \otimes x') - (x_{(-1)} \otimes x_{(0)} \otimes x') + (x_{(-1)}x'_{(-1)} \otimes x_{(0)} \otimes x'_{(0)})$$

$$= - (1 \otimes x \otimes x') + (x_{(-1)}x'_{(-1)} \otimes x_{(0)} \otimes x'_{(0)})$$
for any \((x \otimes x')\) in \(X \otimes X'\). One can easily check that \(\rho_{X \otimes X'}(x \otimes x') = x_{(-1)}x'_{(-1)} \otimes x_{(0)} \otimes x'_{(0)}\) is a coassociative \(H\)–coaction on \(X \otimes X'\). Moreover, for any \(h \in H\) and \(x \otimes x'\) in \(X \otimes X'\) we have

\[
\rho_{X \otimes X'}(h(x \otimes x')) = \rho_{X \otimes X'}(h(1)(x) \otimes h(2)(x')) \\
= h(1)x_{(-1)}S(h(3))h(4)x'_{(-1)}S^{-1}(h(6)) \otimes h(2)(x(0)) \otimes h(5)x'(0) \\
= h(1)x_{(-1)}x'_{(-1)}S^{-1}(h(3)) \otimes h(2)(x(0)) \otimes x'(0)
\]

meaning \(X \otimes X'\) is an AYD module over \(H\). In other words \(\nabla_{X \otimes X'}\) is a flat connection on \(X \otimes X'\) with respect to the differential calculus \(\mathcal{K}^*(H)\). The result follows. \(\square\)

4. Equivariant differential calculi

The concepts of AYD and YD modules have recently been extended in [11]. In this section we give a further extension of this new class of modules and show that they can be interpreted as modules admitting flat connections with respect to a differential calculus. In this section we assume \(B\) is a bialgebra and \(B^{op,\cop}\) is the bialgebra \(B\) with the opposite multiplication and comultiplication. Assume \(\alpha : B \to B\) and \(\beta : B \to B^{op,\cop}\) are two morphisms bialgebras. Also in this section we fix a \(B\)–bimodule coalgebra \(C\). In other words \(C\) is a \(B\)–bimodule and the comultiplication \(\Delta_C : C \to C \otimes C\) is a morphism of \(B\)–bimodules where we think of \(C \otimes C\) as a \(B\)–bimodule via the diagonal action of \(B\). Equivalently, one has

\[
(bcb')_{(1)} = b_{(1)}c_{(1)}b'_{(1)} \otimes b_{(2)}c_{(2)}b'_{(2)}
\]

for any \(b, b' \in B\) and \(c \in C\). We also assume \(C\) has a grouplike element via a coalgebra morphism \(I : k \to C\).

We do not impose any condition on \(\varepsilon(\mathbb{1})\).

**Definition 4.1.** Define a graded \(B\)–bimodule \(\mathcal{K}^*_{(\alpha, \beta)}(C, B)\) by letting \(\mathcal{K}^*_{(\alpha, \beta)}(C, B) = C^\otimes n \otimes B\). The right action is defined by the right regular representation of \(B\) on itself, i.e.

\[
(e^1 \otimes \cdots \otimes e^n \otimes b')b = e^1 \otimes \cdots \otimes e^n \otimes b'\cdot b
\]

and the left action is defined by

\[
b(e^1 \otimes \cdots \otimes e^n \otimes b') = \alpha(b_{(1)})e^1\beta(b_{(2n+1)}) \otimes \cdots \otimes \alpha(b_{(n)})e^n\beta(b_{(n+2)}) \otimes b_{(n+1)}b'
\]

for any \(e^1 \otimes \cdots \otimes e^n \otimes b'\) in \(\mathcal{K}^n_{(\alpha, \beta)}(C, B)\) and \(b \in B\).

**Proposition 4.2.** \(\mathcal{K}^*_{(\alpha, \beta)}(C, B)\) is a differential graded \(k\)–algebra.
Proof. First we define a product structure. We let

\[(c^1 \otimes \cdots \otimes c^n \otimes b)(c^{n+1} \otimes \cdots \otimes c^{n+m} \otimes b')\]

\[= c^1 \otimes \cdots \otimes c^n \otimes \alpha(b_1(1))c^{n+1}b(b_{2m+1}) \otimes \cdots \otimes \alpha(b_m)\alpha(b_{m+1})c^{n+m}b(b_{m+2}) \otimes b_{m+1}b'\]

for any \(c^1 \otimes \cdots \otimes c^n \otimes b\) and \(c^{n+1} \otimes \cdots \otimes c^{n+m} \otimes b'\) in \(K^*_{(\alpha, \beta)}(C, B)\). Note that

\[c^1 \otimes \cdots \otimes c^n \otimes b = (c^1 \otimes 1) \cdots (c^n \otimes 1)b.\]

So it is enough to check associativity only for degree 1 terms. Then

\[((c^1 \otimes b^1)(c^2 \otimes b^2))(c^3 \otimes b^3) = (c^1 \otimes \alpha(b_1(1))c^2\beta(b_{31}) \otimes b^{12}_2b^2)(c^3 \otimes b^3)\]

\[= c^1 \otimes \alpha(b_1(1))c^2\beta(b_{31}) \otimes \alpha(b_2(1))\alpha(b_1(1))c^3\beta(b_{32}) \otimes b_3 \otimes b^2b^3 \]

\[= (c^1 \otimes b^1)(c^2 \otimes \alpha(b_1(1))c^3\beta(b_{32}) \otimes b^1(b_2b^3))\]

\[= (c^1 \otimes b^1)(c^2 \otimes b^2)(c^3 \otimes b^3)\]

for any \((c^i \otimes b^i)\) for \(i = 1, 2, 3\) as we wanted to show. Define the differentials as

\[d(c^1 \otimes \cdots \otimes c^n \otimes b) = -(\mathbb{1} \otimes c^1 \otimes \cdots \otimes c^n \otimes b) + \sum_{j=1}^{n} (-1)^{j-1}(c^1 \otimes \cdots \otimes c^j_{(1)} \otimes c^j_{(2)} \otimes \cdots \otimes c^n \otimes b)\]

\[(-1)^n(c^1 \otimes \cdots \otimes c^n \otimes \alpha(b_1(1))\beta(b_{31}) \otimes b_{2})\]

for any \((c^1 \otimes \cdots \otimes c^n \otimes b)\) in \(K^*_{(\alpha, \beta)}(C, B)\) and one can check that

\[d(b) = -(\mathbb{1} \otimes b) + (\alpha(b_1)\beta(b_{31}) \otimes b_{2}) \quad d(c^1 \otimes 1) = -(\mathbb{1} \otimes c \otimes 1) + (c_{(1)} \otimes c_{(2)} \otimes 1) - (c \otimes \mathbb{1} \otimes 1)\]

for any \(b \in B, c \in C\). In order to prove that \(K^*_{(\alpha, \beta)}(C, B)\) is a differential graded \(k\)-algebra, we must prove that the Leibniz rule holds. Since \(K^*_{(\alpha, \beta)}(C, B)\) as an algebra is generated by degree 1 terms, it is enough to check the Leibniz rule for degree 0 and 1 terms. We see that

\[d(b') = -(\mathbb{1} \otimes b') + (\alpha(b'_{(1)})\beta(b_{31}) \otimes b_{2}(b_{3}) \otimes b_{2}(b_{2}))\]

\[= -((\mathbb{1} \otimes b') + (\alpha(b'_{(1)})\beta(b_{31}) \otimes b_{2}(b_{2}))\]

\[-(\alpha(b'_{(1)})\beta(b_{31}) \otimes b_{2}(b_{2})b) + (\alpha(b'_{(1)})\beta(b_{31}) \otimes b_{2}(b_{2}))\]

\[= d(b')b + b'd(b)\]
for any \( b, b' \in B \). Moreover,

\[
d((c \otimes b')b) = d(c \otimes b')b = -(I \otimes c \otimes b')b + (c_{(1)} \otimes c_{(2)} \otimes b')b - (c \otimes \alpha(b'_{(1)})\alpha(b_{(1)}) \beta(b_{(3)}) \otimes b'_{(2)}b_{(2)})
\]

\[
= -(I \otimes c \otimes b')b + (c_{(1)} \otimes c_{(2)} \otimes b')b - (c \otimes \alpha(b'_{(1)}) \beta(b'_{(3)}) \otimes b'_{(2)}b_{(2)})
\]

\[
= d(c \otimes b')b - (c \otimes b')d(b)
\]

for any \((c \otimes b')\) in \( \mathcal{K}^1_{(\alpha, \beta)}(C, B) \) and \( b \in B \). Then, again for the same elements

\[
d(b(c \otimes b')) = d(\alpha(b_{(1)})c\beta(b_{(3)}) \otimes b_{(2)})
\]

\[
= -(I \otimes \alpha(b_{(1)})c\beta(b_{(3)}) \otimes b_{(2)}b') + (\alpha(b_{(1)}))\beta(b_{(3)}) \otimes \alpha(b_{(2)})c\beta(b_{(3)}) \otimes b_{(3)}b')
\]

\[
+ (\alpha(b_{(1)})\beta(b_{(3)}) \otimes \alpha(b_{(2)}))c\beta(b_{(3)}) \otimes b_{(3)}b')
\]

\[
= d(b)(c \otimes b') + bd(c \otimes b')
\]

And finally for \((c \otimes 1) \) and \((c' \otimes 1) \) in \( \mathcal{K}_{(\alpha, \beta)}^*(C, B) \) we see that

\[
d((c \otimes 1)(c' \otimes 1)) = d(c \otimes c' \otimes 1)
\]

\[
= -(I \otimes c \otimes c' \otimes 1) + (c_{(1)} \otimes c_{(2)} \otimes c' \otimes 1) - (c \otimes c'_{(1)} \otimes c'_{(2)} \otimes 1) + (c \otimes c' \otimes I \otimes 1)
\]

\[
= -(I \otimes c \otimes c' \otimes 1) + (c_{(1)} \otimes c_{(2)} \otimes c' \otimes 1) - (c \otimes I \otimes c' \otimes 1)
\]

\[
+ (c \otimes I \otimes c' \otimes 1) - (c \otimes c'_{(1)} \otimes c'_{(2)} \otimes 1) + (c \otimes c' \otimes I \otimes 1)
\]

\[
= d(c \otimes 1)(c' \otimes 1) - (c \otimes 1)d(c' \otimes 1)
\]

Now, one can inductively prove that

\[
d(\Psi \Phi) = d(\Psi)\Phi + (-1)^{|\Psi|}\Psi d(\Phi)
\]

for any \( \Psi \) and \( \Phi \) in \( \mathcal{K}_{(\alpha, \beta)}^*(C, B) \) proving \( \mathcal{K}_{(\alpha, \beta)}^*(C, B) \) is a differential graded \( k \)-algebra. \( \square \)

**Corollary 4.3.** \( H_*(\mathcal{K}_{(\alpha, \beta)}^*(C, B)) \) is a graded algebra.

Now we can identify the homology of the \((\alpha, \beta)\)-equivariant differential calculus:

**Proposition 4.4.** \( B \) is a \( C \)-comodule and \( H_*(\mathcal{K}_{(\alpha, \beta)}^*(C, B)) \) is isomorphic to \( \text{Cotor}_C^*(k, B) \).

**Proof.** The \( C \)-comodule structure is given by \( \rho_B(b) = \alpha(b_{(1)})I\beta(b_{(3)}) \otimes b_{(2)} \) for any \( b \in B \). Now, one should observe that \( \mathcal{K}_{(\alpha, \beta)}^*(C, B) \) is the two sided cobar complex \( B_*(k, C, B) \) of \( C \) with coefficients in \( k \) and \( B \). \( \square \)
Definition 4.5. A $k$–module $X$ is called an $(\alpha, \beta)$–equivariant $C$–comodule if (i) $X$ is a left $C$–comodule via a structure morphism $\rho_X : X \to C \otimes X$ (ii) $X$ is a left $B$–module via a structure morphism $\mu_X : B \otimes X \to X$ (iii) one has

$$\rho_X(bx) = \alpha(b(1))x_{(-1)}\beta(b(3)) \otimes b(2)x_{(0)}$$

for any $b \in B$ and $x \in X$.

Theorem 4.6. The category of $(\alpha, \beta)$–equivariant $C$–comodules is equivalent to the category of $B$–modules admitting a flat connection with respect to the differential calculus $\mathcal{K}_{(\alpha, \beta)}^*(C, B)$.

Proof. Assume we have a $k$–module morphism $\rho_X : X \to C \otimes X$ and define $\rho_X(x) := \nabla(x) + (1 \otimes x)$ for any $x \in X$ where we denote $\rho_X(x)$ by $x_{(-1)} \otimes x_{(0)}$. Now for any $b \in B$ and $x \in X$ we have

$$\nabla(bx) = - (\mathbb{I} \otimes bx) + (bx)_{(-1)} \otimes (bx)_{(0)}$$

$$= - (\mathbb{I} \otimes bx) + (\alpha(b(1))\mathbb{I}\beta(b(3))) \otimes b(2)x - (\alpha(b(1))\mathbb{I}\beta(b(3))) \otimes b(2)x + (\alpha(b(1))x_{(-1)}\beta(b(3))) \otimes b(2)x_{(0)}$$

$$= d(b) \otimes x + b\nabla(x)$$

iff $(bx)_{(-1)} \otimes (bx)_{(0)} = \alpha(b(1))x_{(-1)}\beta(b(3)) \otimes b(2)x_{(0)}$. In other words $\nabla : X \to C \otimes X$ is a connection iff the morphism of $k$–modules $\rho_X : X \to C \otimes X$ satisfies the $(\alpha, \beta)$–equivariance condition. Moreover, if we extend $\nabla$ to $\widehat{\nabla} : C \otimes X \to C \otimes C \otimes X$ by letting

$$\widehat{\nabla}(c \otimes x) = d(c \otimes 1_B) \otimes x - (c \otimes 1_B) \otimes d(x)$$

for any $(c \otimes x) \in C \otimes X$, then we have

$$\widehat{\nabla}^2(x) = - d(\mathbb{I} \otimes 1_B) \otimes x + (\mathbb{I} \otimes 1_B) \otimes \nabla(x) + d(x_{(-1)} \otimes 1_B) \otimes x_{(0)} - (x_{(-1)} \otimes 1_B) \otimes \nabla(x_{(0)})$$

$$= (\mathbb{I} \otimes \mathbb{I} \otimes x) - (\mathbb{I} \otimes \mathbb{I} \otimes x) + (\mathbb{I} \otimes \mathbb{I} \otimes x) - (\mathbb{I} \otimes x_{(-1)} \otimes x_{(0)})$$

$$- (\mathbb{I} \otimes x_{(-1)} \otimes x_{(0)}) + (x_{(-1)}(1) \otimes x_{(-1)(2)} \otimes x_{(0)}) - (x_{(-1)} \otimes \mathbb{I} \otimes x_{(0)})$$

$$+ (x_{(-1)} \otimes \mathbb{I} \otimes x_{(0)}) - (x_{(-1)} \otimes x_{(0)(-1)} \otimes x_{(0)(0)})$$

$$= (x_{(-1)(1)} \otimes x_{(-1)(2)} \otimes x_{(0)}) - (x_{(-1)} \otimes x_{(0)(-1)} \otimes x_{(0)(0)}) = 0$$

iff $\rho_X : X \to C \otimes X$ is a coassociative coaction of $C$. The result follows. \hfill \square

Definition 4.7. Let $X$ be an $(\alpha, \beta)$–equivariant $C$–comodule, i.e. $X$ admits a flat connection $\nabla : X \to C \otimes X$ with respect to the differential calculus $\mathcal{K}_{(\alpha, \beta)}^*(C, B)$. Define $\mathcal{K}_{(\alpha, \beta)}^*(C, B, X)$ as $\mathcal{K}_{(\alpha, \beta)}^*(C, B) \otimes X$ with the extended connection $\widehat{\nabla}$ as its differential.

Proposition 4.8. For any $(\alpha, \beta)$–equivariant module $X$ one has $H_*(\mathcal{K}_{(\alpha, \beta)}^*(C, B, X)) \cong \text{Cotor}_C^*(k, X)$ where we think of $k$ as a $C$–comodule via the grouplike element $\mathbb{I} : k \to C$. 


Remark 4.9. Assume $B$ is a Hopf algebra. If $\alpha = \text{id}_B$, $\beta = S$, and $C = B$, then the differential calculus is $\hat{K}(H)$ and $(\alpha, \beta)$–equivariant $C$–comodules are Yetter-Drinfeld modules. In case $\alpha = \text{id}_B$, $\beta = S^{-1}$, and $C = B$, then the differential calculus is $K(H)$ and $(\alpha, \beta)$–equivariant $C$–comodules are anti-Yetter-Drinfeld modules.

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