SPACES OF EMBEDDINGS OF COMPACT POLYHEDRA INTO 2-MANIFOLDS

TATSUHIKO YAGASAKI

ABSTRACT. Let $M$ be a PL 2-manifold and $X$ be a compact subpolyhedron of $M$ and let $\mathcal{E}(X,M)$ denote the space of embeddings of $X$ into $M$ with the compact-open topology. In this paper we study an extension property of embeddings of $X$ into $M$ and show that the restriction map from the homeomorphism group of $M$ to $\mathcal{E}(X,M)$ is a principal bundle. As an application we show that if $M$ is a Euclidean PL 2-manifold and $\dim X \geq 1$ then the triple $(\mathcal{E}(X,M), \mathcal{E}_{\text{LIP}}(X,M), \mathcal{E}_{\text{PL}}(X,M))$ is an $(s,\Sigma,\sigma)$-manifold, where $\mathcal{E}_{\text{LIP}}(X,M)$ and $\mathcal{E}_{\text{PL}}(X,M)$ denote the subspaces of Lipschitz and PL embeddings.

1. Introduction

The investigation of the topology of the homeomorphism groups of compact 2-manifolds included the use of conformal mappings in order to develop some extension properties of embeddings of a circle into an annulus and proper embeddings of an arc into a disk. In this paper we establish a similar extension property of embeddings of trees into a disk. Since every graph can be decomposed into ads (cones over finite points) and arcs connecting them, this implies an extension property of embeddings of compact polyhedra into 2-manifolds.

Suppose $M$ is a PL 2-manifold and $K \subset X$ are compact subpolyhedra of $M$. Let $\mathcal{E}_K(X,M)$ denote the space of embeddings $f : X \hookrightarrow M$ with $f|_K = id$, equipped with the compact-open topology. An embedding $f : X \hookrightarrow M$ is said to be proper if $f(X \cap \partial M) \subset \partial M$ and $f(X \cap \text{Int} M) \subset \text{Int} M$. Let $\mathcal{E}_K(X,M)^*$ denote the subspace of proper embeddings in $\mathcal{E}_K(X,M)$, and let $\mathcal{E}_K(X,M)^*_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}_K(X,M)^*$. Our result is summarized in the next statement.

**Theorem 1.1.** For every $f \in \mathcal{E}_K(X,M)^*$ and every neighborhood $U$ of $f(X)$ in $M$, there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{E}_K(X,M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_{K\cup(M\setminus U)}(M)_0$ such that $\varphi(g)f = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = id_M$.

Let $\mathcal{H}_X(M)$ denote the group of homeomorphisms $h$ of $M$ onto itself with $h|_X = id$, equipped with the compact-open topology. Let $\mathcal{H}(M)_0$ denote the identity component of $\mathcal{H}(M)$. In the study of the homotopy type of $\mathcal{H}_X(M)_0$ and $\mathcal{E}_K(X,M)_0$ the restriction map $\pi : \mathcal{H}_K(M)_0 \to \mathcal{E}_K(X,M)^*_0$ plays an important role (cf. 3). The above extension maps yield local sections of this restriction map.

**Corollary 1.1.** For any open neighborhood $U$ of $X$ in $M$, the restriction map $\pi : \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \to \mathcal{E}_K(X,U)^*_0$, $\pi(f) = f|_X$, is a principal bundle with the fiber $\mathcal{G} \equiv \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \cap \mathcal{H}_X(M)$, where the subgroup $\mathcal{G}$ acts on $\mathcal{H}_{K\cup(M\setminus U)}(M)_0$ by right composition.

1991 Mathematics Subject Classification. 57N05, 57N20, 57N35.

Key words and phrases. Embeddings, Homeomorphism groups, 2-manifolds, Infinite-dimensional manifolds.
As an application of Extension Theorem 1.1 we can study the embedding space $E_K(X, M)$ from the viewpoint of infinite dimensional topology (see §4 for basic terminologies). In K. Sakai and R.Y. Wong showed the $(s, \Sigma, \sigma)$-stability property of triples of spaces of embeddings of compact polyhedra and subspaces of Lipschitz and PL embeddings, and posed the question whether these triples are $(s, \Sigma, \sigma)$-manifolds. The 1-dimensional case is studied in [15]. In this paper we will consider the 2-dimensional case and answer the question affirmatively.

Let $E_{PL}^K(X, M)$ denote the subspace of PL-embeddings. When $M$ is a Euclidean PL 2-manifold, let $E_{LIP}^K(X, M)$ denote the subspace of Lipschitz embeddings. The Extension Theorem enables us to reduce the ANR-property and the homotopy negligibility of embedding spaces to the ones of the homeomorphism groups. Using the characterization of $(s, \Sigma, \sigma)$-manifold [20] we have the following result.

**Theorem 1.2.** Suppose $M$ is a Euclidean PL 2-manifold and $K \subset X$ are compact subpolyhedra of $M$. If $\dim(X \setminus K) \geq 1$, then the triple $(E_K(X, M), E_{LIP}^K(X, M), E_{PL}^K(X, M))$ is an $(s, \Sigma, \sigma)$-manifold.

Further applications of Corollary 1.1 to the study of $H_X(M)$ and $E_K(X, M)$ will be given in a succeeding paper. We conclude this section with some remarks. In Section 2 we study the extension property of embeddings of a tree into a disk. Section 3 contains the proofs of Theorem 1.1 and Corollary 1.1. The final section 4 contains the proof of Theorem 1.2. Throughout the paper spaces are assumed to be separable and metrizable. A Euclidean PL $n$-manifold is a subpolyhedron of some Euclidean space $\mathbb{R}^m$ which is a PL-manifold with respect to the induced triangulation and is equipped with the metric induced from the standard metric of $\mathbb{R}^m$. When $M$ is an orientable manifold, $\mathcal{H}_+(M)$ denote the subspace of orientation preserving homeomorphisms of $M$. Finally $i_X : X \subset Y$ denotes the inclusion map.

2. **Extension property of embeddings of trees into disks**

In this section we will study some extension properties of embeddings of trees into disks. The proper embedding case is a consequence of a direct application of the conformal mapping theorem on simply connected domains (cf. [11]). Thus our interest is in the case of embeddings into the interior of a disk, where we need to apply the conformal mapping theorem on a doubly connected domain one boundary circle of which is collapsed to a tree.

Throughout the section we will work on the plane $\mathbb{C} (= \mathbb{R}^2)$ and use the following notations: For $z \in \mathbb{C}$ and $r > 0$, $D(z, r) = \{x \in \mathbb{C} : |z - x| \leq r\}$, $O(z, r) = \{x \in \mathbb{C} : |z - x| < r\}$, $C(z, r) = \{x \in \mathbb{C} : |z - x| = r\}$, and $D(r) = D(0, r)$, $O(r) = O(0, r)$, $C(r) = C(0, r)$. For $0 < r < s$, $A(r, s) = \{x \in \mathbb{C} : r \leq |x| \leq s\}$. For $A \subset \mathbb{C}$ and $\varepsilon > 0$, $O(A, \varepsilon) = \{x \in \mathbb{C} : |x - y| < \varepsilon$ for some $y \in A\}$ (the $\varepsilon$-neighborhood of $A$).

2.1. **Proper embeddings of trees into a disk.**

First we recall the conformal mapping theorem on simply connected domains normalized by the three points boundary condition. Consider the family $J = \{(J, w_1, w_2, w_3) : J$ is a simple closed curve
in $\mathbb{C}$ and $w_1, w_2, w_3 \in J$ are three distinct points lying on $J$ in counterclockwise order (with respect to the orientation induced from $\mathbb{C}$).} A sequence $\{A_n\}_{n \geq 1}$ of subsets of $\mathbb{C}$ is said to be uniformly locally connected if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $n \geq 1$ and any $x, y \in A_n$ with $|x - y| < \delta$ there exists an arc $\alpha$ in $A_n$ with connecting $x$ and $y$ and diam $\alpha < \varepsilon$.

**Fact 2.1.** Let $z_1, z_2, z_3 \in C(1)$ be the fixed three points lying on $C(1)$ in counterclockwise order.

(i) (\cite[Corollary 2.7]{14}) For every $(J, w_1, w_2, w_3) \in J$ there exists a unique $\varphi = \varphi(J, w_1, w_2, w_3) \in E(D(1), \mathbb{C})$ such that $\varphi$ maps $O(1)$ conformally onto the interior of $J$, $\varphi(C(1)) = J$ and $\varphi(z_i) = w_i$ $(i = 1, 2, 3)$.

(ii) If a sequence $(J_n, w_1(n), w_2(n), w_3(n))$ $(n \geq 1)$ converges to $(J, w_1, w_2, w_3)$ in the following sense, then $\varphi(J_n, w_1(n), w_2(n), w_3(n))$ converges uniformly to $\varphi(J, w_1, w_2, w_3)$:

(*) $J_n$ converges to $J$ with respect to the Hausdorff metric, $\{J_n\}$ is uniformly locally connected, and $w_i(n) \to w_i$ $(i = 1, 2, 3)$.

For the statement (ii) we refer to the proof of \cite[Theorem 2.1, Proposition 2.3]{14} (also see the proof of Lemma 2.3).

**Lemma 2.1.** Suppose $D$ is a disk and $C = \partial D$.

(i) (cf. \cite[Lemma 3]{11}) There exists a map $\Phi : E(C, \mathbb{C}) \to E(D, \mathbb{C})$ such that $\Phi(f)|_C = f$ $(f \in E(C, \mathbb{C}))$.

(ii) (cf. \cite[Lemma 5]{11}) Suppose $T$ is a tree embedded into a disk $D$ such that $T \cap C$ coincides with the set of terminal vertices of $T$. Then there exists a map $\Psi : E_{T\cap C}(T, D)^* \to \mathcal{H}_\partial(D)$ such that $\Psi(f)|_T = f$ $(f \in E_{T\cap C}(T, D)^*)$ and $\Psi(i_T) = id_D$.

**Proof.** We may assume that $D = D(1)$. Let $z_1, z_2, z_3 \in C(1)$ be as in Fact 2.1.

(i) Let $E^{\pm} = \{f \in E(C(1), \mathbb{C}) : f$ preserves (reverses) orientation$\}$. If $f \in E^+(C(1), \mathbb{C})$, then $(f(C(1)), f(z_1), f(z_2), f(z_3)) \in J$ and by Fact 2.1 we obtain $\varphi(f) = \varphi(f(C(1)), f(z_1), f(z_2), f(z_3)) \in E(D(1), \mathbb{C})$. If $f_n \to f$ in $E^+$, then $(f(C(1)), f(z_1), f(z_2), f(z_3))$ converges to $(f(C(1)), f(z_1), f(z_2), f(z_3))$ in the sense $(\ast)$ of Fact 2.1(i). Hence the map $\varphi : E^+ \to E(D(1), \mathbb{C})$ is continuous. Let $c : \mathcal{H}(C(1)) \to \mathcal{H}(D(1))$ be the cone extension map and let $\gamma : C \to \mathbb{C}$ be the reflection $\gamma(z) = \overline{z}$. Then the extension map $\Phi$ is defined by $\Phi(f) = \varphi(f) c(\varphi(f)^{-1} f)$ for $f \in E^+$ and $\Phi(f) = \gamma \Phi(\gamma f)$ for $f \in E^-$.

(ii) The tree $T$ separates the disk $D(1)$ into subdisks $D_i$. By (i) each disk $D_i$ admits an extension map $\psi_i : E(\partial D_i, \mathbb{C}) \to E(D_i, \mathbb{C})$. Every $f \in E_{T\cap C(1)}(T, D(1))^*$ can be extended to $\overline{f} \in E_{C(1)}(T \cup C(1), D(1))$. The required extension map $\Psi$ is defined by $\Psi(f)|_{D_i} = \psi_i(\overline{f}|_{\partial D_i})$. To achieve $\Psi(i_T) = id_D$, replace $\Psi(f)$ by $\Psi(f)\Psi(i_T)^{-1}$.

In the proof of Theorem 1.1 we will apply the statement (ii) to the case where $T$ is an arc.

### 2.2. Embeddings of trees into the interior of a disk.

Suppose $T$ is a finite tree (\neq 1 pt) embedded into $O(2)$. We will use the following notation: For $a, b \in T$, let $E_T(a, b)$ denote the unique arc in $T$ connecting $a$ and $b$. Let $\{v_1, \ldots, v_n\}$ be the collection of end vertices of $T$. We can choose disjoint arcs $\alpha_1, \ldots, \alpha_n$ in $D(2)$ such that each $\alpha_i$ connects $v_i$ with a point $a_i$ in $C(2)$ and $\text{Int} \alpha_i \subset O(2) \setminus T$. We can arrange the ordering of $v_i$'s.
so that $a_1, \ldots, a_n$ lie on $C(2)$ in counterclockwise order. The labeling is unique up to the cyclic permutations. Note that $T$ does not meet the interior of the disk surrounded by the simple closed curve $\alpha_i \cup E_T(v_i, v_{i+1}) \cup \alpha_{i+1} \cup a_i a_{i+1}$, where $v_{n+1} = v_1$ and $a_{n+1} = a_1$.

**Lemma 2.2.** ([1] Ch. V, §1, Theorems 1.1, 1.2) There exists a unique real number $r$, $0 < r < 2$, and a unique map $h : A(r, 2) \rightarrow D(2)$ such that $h : \text{Int} A(r, 2) \rightarrow O(2) \setminus T$ is a conformal map and $h(2) = 2$. Furthermore, the map $h$ satisfies the following conditions: (i) $h$ maps $C(2)$ homeomorphically onto $C(2)$, (ii) $h(C(r)) = T$ and there exists a unique collection of points $\{u_1, \ldots, u_n\}$ lying on $C(r)$ in counterclockwise order such that $h$ maps each circular arc $u_i u_{i+1}$ homeomorphically onto the arc $E_T(v_i, v_{i+1})$, where $u_{n+1} = u_1$.

We refer to [4], Ch. 2. Theorem 2.1 for the extension to boundary and [4], Ch. 2, §1 Prime End Theorem, §§4, 5 and [3], p.40 for the correspondence between prime ends and boundary points. Let $E = E(T, O(2))$. For each $f \in E$ the image $f(T)$ is a tree in $O(2)$. Hence by Lemma 2.2 there exists a unique real number $r_f$, $0 < r_f < 2$, and a unique map $h_f : A(r_f, 2) \rightarrow D(2)$ such that $h_f : \text{Int} A(r_f, 2) \rightarrow O(2) \setminus f(T)$ is a conformal map and $h_f(2) = 2$. For $0 < r < 2$ let $\varphi_r : A(1, 2) \rightarrow A(r, 2)$ denote the radial map defined by $\varphi_r(x) = ((2 - r)(|x| - 1) + r)x/|x|$, and let $\mathcal{C}(A(1, 2), D(2))$ denote the space of continuous maps from $A(1, 2)$ to $D(2)$, with the compact-open topology. We have $h_f \varphi_r \in \mathcal{C}(A(1, 2), D(2))$.

**Lemma 2.3.** The map $\Psi : E(T, O(2)) \rightarrow \mathbb{R} \times \mathcal{C}(A(1, 2), D(2))$, $\Psi(f) = (r_f, h_f \varphi_{r_f})$, is continuous.

This continuity property can be verified using the length distortion under conformal mapping [4], Proposition 2.2]. When $L$ is a rectifiable (possibly open) curve in $\mathbb{R}^2$, we denote the length of $L$ by $\Lambda(L)$.

**Proof.** Suppose $f_n \rightarrow f$ in $E$. It suffices to show that the sequence $(r_n, h_n \varphi_{r_n}) \equiv (r_{f_n}, h_{f_n} \varphi_{r_{f_n}})$ has a subsequence $(r_{n_k}, h_{n_k} \varphi_{r_{n_k}})$ such that $r_{n_k} \rightarrow r_f$ and $h_{n_k} \varphi_{r_{n_k}}$ converges uniformly to $h_f \varphi_{r_f}$.

Let $R_0 > 2 (= \text{the radius of } D(2))$ and $\varepsilon(\rho) = 2\pi R_0/\sqrt{\log(1/\rho)}$ ($0 < \rho < 1$).

(i) Passing to a subsequence we may assume $r_n \rightarrow r_0$ for some $r_0, 0 \leq r_0 \leq 2$. First we will show that $0 < r_0 < 2$. (a) Suppose $r_0 = 2$. Take $\rho, 0 < \rho < 1$, such that $\varepsilon(\rho) < d(f(T), C(2))$. Choose $n \geq 1$ such that $\varepsilon(\rho) < d(f_n(T), C(2))$ and $|r_n - r_0| < \rho$. We can apply [4], Proposition 2.2] for any point $c \in C(2)$ (with $R = 2$) to find $\rho_0, \rho < \rho_0 < \sqrt{\rho}$, such that $\Lambda(h_n(L)) < \varepsilon(\rho)$, where $L$ is one of the two arc components of $C(c, \rho_0)$ \text{Int} $A(r_n, 2)$ which connects $C(r_n)$ and $C(2)$. This implies $d(f_n(T), C(2)) < \varepsilon(\rho)$, a contradiction. (b) Suppose $r_0 = 0$. Take $\rho, 0 < \rho < 1$, such that $\varepsilon(\rho) < \text{diam } f(T)$. Choose $n \geq 1$ such that $\varepsilon(\rho) < \text{diam } f_n(T)$ and $r_n < \rho$. By [4], Proposition 2.2] there exists $\rho_0, \rho < \rho_0 < \sqrt{\rho}$ such that $\Lambda(h_n(C(\rho_0))) < \varepsilon(\rho)$. Since $f_n(T)$ is contained in the interior of the circle $h_n(C(\rho_0))$, we have $\text{diam } f_n(T) < \varepsilon(\rho)$, a contradiction.

(ii) Next we will show that the sequence $h_n : A(r_n, 2) \rightarrow D(2)$ ($n \geq 1$) is equicontinuous, that is, for every $\varepsilon > 0$ there exists a $\rho > 0$ such that $|h_n(z) - h_n(w)| < \varepsilon$ for any $n \geq 1$ and $z, w \in A(r_n, 2)$ with $|z - w| < \rho$. Let $\varepsilon > 0$ be given. We may assume that $\varepsilon < d(C(2), f_n(T))$ for each $n \geq 1$. 

4
Proposition 2.1. (i) Let \( f_n(T) \) (respectively \( C(2) \)) connecting \( z \) and \( w \) and with \( \text{diam} A < \varepsilon/2 \). Choose \( \rho, 0 < \rho < 1, \) such that \( \varepsilon(\rho) < \delta \) and \( 2\sqrt{\rho} < 2 - \max_{n \geq 0} r_n \). Suppose \( z, w \in A(r_n, 2) \) and \( |z - w| < \rho. \) By Proposition 2.2] (with \( c = z \)) we have \( \rho_0, \rho < \rho_0 < \sqrt{\rho}, \) such that \( \Lambda(h_n(L)) < \varepsilon(\rho), \) where \( L = C(z, \rho_0) \cap A(r_n, 2). \) Since \( z, w \in D \equiv D(z, \rho_0) \cap A(r_n, 2), \) it suffices to show that \( \text{diam} h_n(D) < \varepsilon. \) By the choice of \( \rho, D(z, \rho_0) \) meet at most one of \( C(2) \) and \( C(r_n). \) If \( D(z, \rho_0) \subset A(r_n, 2) \) or \( D(z, \rho_0) \supset D(0, r_n), \) then \( L = C(z, \rho_0) \) and \( h_n(D) \) is a disk bounded by \( h_n(L), \) so \( \text{diam} h_n(D) < \varepsilon(\rho). \) Otherwise, \( L \) is an arc connecting two points \( P, Q \) with either (a) \( P, Q \in C(2) \) or (b) \( P, Q \in C(r_n). \) In both cases \( |h_n(P) - h_n(Q)| \leq \Lambda(h_n(L)) < \delta, \) hence by the choice of \( \delta, \) we have an arc \( A \subset C(2) \) (resp. \( f_n(T) \)) connecting \( h_n(P) \) and \( h_n(Q) \) and \( \text{diam} A < \varepsilon/2. \) In the case (a) \( h_n(L) \) separates \( D(2) \) into the subdisk \( h_n(D) \) and another subdisk. Since \( h_n(D) \cap f_n(T) = \emptyset \) and \( d(C(2), f_n(T)) > \varepsilon, \) the Jordan curve \( h_n(L) \cap A \) bounds the disk \( h_n(D), \) so \( \text{diam} h_n(D) < \varepsilon. \) In the case (b) the Jordan curve \( h_n(L) \cap A \) bounds a disk \( E \) in \( D(2) \) with \( \text{diam} E < \varepsilon. \) Since \( h_n(A(r_n, 2) \setminus (D \cup C(r_n))) \) is contained in the exterior of \( E \) and \( h_n(\text{Int } D) \cap \partial E = \emptyset, \) it follows that \( h_n(\text{Int } D) = \text{Int } E \cap f_n(T), \) so \( \text{diam} h_n(D) = E. \)

(iii) Since the sup-metric \( d(\varphi_{r_n}, \varphi_{r_0}) = |r_n - r_0| \to 0 \) (\( n \to \infty \)), the sequence \( h_n \varphi_{r_n} \) \((n \geq 1)\) is also equiuniform. By the Ascoli-Arzelà theorem, passing to a subsequence, we may assume that \( h_n \varphi_{r_n} \) converges to a map \( h_0 : A(1, 2) \to D(2). \) Set \( h_0 = h_0^{-1} \varphi_{r_0}. \) Then \( h_0(A(r_0, 2)) = D(2), \) \( h_0(C(2)) = C(2), \) \( h_0(C(r_0)) = f(T) \) and \( h_0(2) = 2. \) Since the sequence of univalent analytic maps \( h_n : \text{Int } A(r_n, 2) \to \mathbb{C} \) converges weakly uniformly to the map \( h_0 : \text{Int } A(r_0, 2) \to \mathbb{C} \) (i.e., for each compact subset \( K \) of \( \text{Int } A(r_0, 2), h_n|_K \) converges uniformly to \( h_0|_K) \) and \( h_0 \) is not constant, \( h_0 : \text{Int } A(r_0, 2) \to \mathbb{C} \) is also a univalent analytic map \([19], \text{Ch.3, Theorem 3.3.} \). It follows that \( h_0(\text{Int } A(r_0, 2)) = O(2) \setminus f(T) \) and \( h_0 : \text{Int } A(r_0, 2) \to O(2) \setminus f(T) \) is a conformal map, so \( (r_0, h_0) = (r_f, h_f) \) by the uniqueness in Lemma 2.2. This completes the proof.

Let \( i : T \to O(2) \) denote the inclusion and set \( \mathcal{E}_+ \equiv \mathcal{E}_+(T, O(2)) = \{ f \in \mathcal{E} : \text{there exists an } h \in \mathcal{H}_+(D(2)) \text{ with } hi = f \}, \) which is an open neighborhood of \( i \) in \( \mathcal{E}. \)

**Proposition 2.1.** (i) There exists a canonical map \( \Phi = \Phi_T : \mathcal{E}_+ \to \mathcal{H}_+(D(2)) \) such that \( \Phi(f)i = f \) \((f \in \mathcal{E}_+)\) and \( \Phi(i) = id. \)

(ii) There exists a neighborhood \( \mathcal{U} \) of \( i \) in \( \mathcal{E} \) and a map \( \varphi : \mathcal{U} \to \mathcal{H}_0(D(2)) \) such that \( \varphi(f)i = f \) \((f \in \mathcal{U})\) and \( \varphi(i) = id_D. \)

**Proof.** (i) Let \( f \in \mathcal{E}_+. \) Comparing two maps \( h_f \varphi_{r_f}, h_i \varphi_{r_i} : C(1) \to f(T), \) we obtain a unique map \( \Theta_0(f) \in \mathcal{H}_+(C(1)) \) such that \( h_f \varphi_{r_f} \Theta_0(f) = h_i \varphi_{r_i}. \) Extend \( \Theta_0(f) \) radially to \( \Theta(f) \in \mathcal{H}_+(A(1, 2)) \) by \( \Theta(f)(rz) = r\Theta_0(f)(z) \) \((z \in C(1), 1 \leq r \leq 2)\). The required map \( \Phi(f) \) is defined as the unique map \( \Phi(f) \in \mathcal{H}_+(D(2)) \) with \( h_f \varphi_{r_f} \Theta(f) = \Phi(f)h_i \varphi_{r_i}. \) In Claim below we will show that the map \( \Theta_0 \) is continuous. This implies the continuity of the map \( \Phi. \)
(ii) Since $\Phi(i) = id$, if we take a sufficiently small neighborhood $U$ of $i$, then $\Phi(f)|_{C(2)}$ is close to $id_{C(2)}$ for $f \in U$, and we can use a collar of $C(2)$ in $D(2)$ and a local contraction of a neighborhood of $id_{C(2)}$ in $\mathcal{H}(C(2))$ to modify the map $\Phi|_{U}$ to obtain the desired map $\varphi$. □

Claim. The map $\Theta_0 : \mathcal{E}_+ \to \mathcal{H}_+(C(1))$ is continuous.

Proof. Under the notations of Lemma 2.2, let $g_f = h_f \varphi_r f$ and $x_j(f) = \varphi_r^{-1}(u_j)$. For the inclusion $i : T \subset D(2)$, we abbreviate as $g = g_i$ and $x_j = x_j(i)$ let $L_j = x_j \cup x_{j+1}$ the (circular arc in $C(1)$). Also let $\tilde{f} = \Theta_0(f)$. Note that $g_f$ is continuous in $f$ (Lemma 2.3), $g_f \tilde{f} = g_f \tilde{f}(x_j) = x_j(f) = g_f^{-1}(f(v_j))$ and that $g_f$ maps $\tilde{f}(L_j)$ homeomorphically onto $f(E_T(v_j, v_{j+1}))$.

(1) First we will show the following statement:

(*) Suppose $f \in \mathcal{E}_+, U$ is any open neighborhood of $x_j(f)$ in $\mathbb{C}$ and $A_j$ is a small compact neighborhood of $x_j$ in $C(1)$ such that $g_f(\tilde{f}(A_j)) \cap g_f(A(1, 2) \setminus U) = \emptyset$ (hence $\tilde{f}(A_j) \subset U$). If $f'$ is sufficiently close to $f$, then $\tilde{f}'(A_j) \subset U$. In particular, $x_j(f) \in C(1)$ is continuous in $f$.

In fact, there exists an $\varepsilon > 0$ such that $O(g_f(A_j), \varepsilon) \cap O(g_f(A(1, 2) \setminus U), \varepsilon) = \emptyset$. If $f'$ is sufficiently close to $f$ then the sup-metric $d(f', f) < \varepsilon$ and $d(g_{f'}, g_f) < \varepsilon$. Hence, $f'g(A_j) = g_{f'} \tilde{f}'(A_j)$ does not meet $g_f(A(1, 2) \setminus U)$, so $g_{f'} \tilde{f}'(A_j) \subset U$.

(2) To show that $\tilde{f}$ is continuous in $f$, let $f \in \mathcal{E}_+$ and $\varepsilon > 0$ be given. It suffices to show that for each $j = 1, \ldots, n$ there exists a small neighborhood $U$ of $f$ in $\mathcal{E}_+$ such that $\tilde{f}$ and $\tilde{f}'$ are $\varepsilon$-close on $L_j$ for every $f' \in U$.

Set $U_j = O(x_j(f), \varepsilon/2)$ and $U_{j+1} = O(x_{j+1}(f), \varepsilon/2)$, and let $A_j$ and $A_{j+1}$ be small circular arc neighborhoods of $x_j$ and $x_{j+1}$ in $C(1)$ as in (1) with respect to $U_j$ and $U_{j+1}$ respectively. Set $K_j = d(L_j \setminus (A_j \cup A_{j+1}))$ and choose small circular arc neighborhoods $C_j$ and $C_{j+1}$ of $x_j(f)$ and $x_{j+1}(f)$ in $C(1)$ such that $g_f \tilde{f}(K_j)$ meets neither $g_f(C_j)$ nor $g_f(C_{j+1})$. Choose $\delta_1 > 0$ such that $O(g_f \tilde{f}(K_j), \delta_1) \cap O(g_f(C_j), \delta_1) \subset O(g_f \tilde{f}(x), \varepsilon)$.

By (1) there exists a neighborhood $U$ of $f$ in $\mathcal{E}_+$ such that if $f' \in U$, then $\tilde{f}'(A_j) \subset U_j, \tilde{f}'(A_{j+1}) \subset U_{j+1}, \tilde{f}'(x_j) \subset C_j, \tilde{f}'(x_{j+1}) \subset C_{j+1}$ and $d(f, f') < \delta, d(g_{f'}, g_f) < \delta$. Since $\tilde{f}'$ is orientation preserving, $\tilde{f}'(x_j) \subset C_j$ and $\tilde{f}'(x_{j+1}) \subset C_{j+1}$, it follows that $\tilde{f}'(L_j) \subset \tilde{f}(L_j) \cup C_j \cup C_{j+1}$. If $x \in A_j$, then $\tilde{f}'(x), \tilde{f}(x) \in U_j$ so that $d(\tilde{f}'(x), \tilde{f}(x)) < \varepsilon$. For each $x \in A_{j+1}$ we have the same conclusion. Suppose $x \in K_j$. Since $g_f \tilde{f}'(x) = f'g(x)$ is $\delta$-close to $f g(x) = g_f \tilde{f}(x) \in g_f(K_j)$ and $g_f' \tilde{f}'(C_j) \subset O(g_f(C_j), \delta)$, we have $\tilde{f}'(x) \subset C_j$. Similarly $\tilde{f}'(x) \subset C_{j+1}$, and we have $\tilde{f}(x) \subset C_j \cup C_{j+1}$. Since $g_f \tilde{f}(x) = f g(x)$ is $\delta$-close to $f'g(x) = g_{f'} \tilde{f}'(x)$ and the latter is also $\delta$-close to $g_f \tilde{f}'(x)$, we have $g_f \tilde{f}'(x) \in O(g_f \tilde{f}(x), 2\delta)$. Hence by the choice of $\delta$, $\tilde{f}'(x) \in O(\tilde{f}(x), \varepsilon)$. This completes the proof. □

Finally we will see a symmetry property of the map $\Phi_T$ in Proposition 2.1 (i). For $z \in C(1)$ let $\theta_z : \mathbb{C} \to \mathbb{C}$ denote the rotation $\theta_z(w) = z \cdot w$ and let $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection, $\gamma(x, y) = (x, -y)$.

Lemma 2.4. (i) $\Phi_T(\theta_z f) = \theta_z \Phi_T(f)$ ($f \in \mathcal{E}_+, z \in C(1)$).

(ii) $\Phi_T(\gamma f \gamma) = \gamma \Phi_T(f) \gamma$ ($f \in \mathcal{E}_+$). In particular, if $T$ is a segment in the $x$-axis, then $\Phi_T(\gamma f) = \gamma \Phi_T(f) \gamma$ ($f \in \mathcal{E}$).
Proof. (i) Let \( f \in \mathcal{E}_+, z \in C(1) \) and let \( w_0 \in C(2) \) be the unique point such that \( \theta_z h_f \theta^{-1}_z(w_0) = 2 \). Under Lemma 2.2, \( (r_f, \theta_z h_f \theta^{-1}_z \theta_w) \) corresponds to \( \theta_z f \), where \( w = w_0/2 \). Thus \( \Theta(\theta_z f) = \theta^{-1}_w \theta_z \Theta(f) \) and the conclusion follows from

\[
\Phi(\theta_z f) \varphi_i \varphi_j = (\theta_z h_f \theta^{-1}_z \theta_w) \varphi_i \varphi_j \Theta(\theta_z f) = (\theta_z h_f \theta^{-1}_z \theta_w) \varphi_i \varphi_j \varphi_i \Theta(f) = \theta_z h_f \varphi_i \varphi_j \Theta(f) = \theta_z \Phi(f) \varphi_i \varphi_j.
\]

(ii) Since \( (r_i, \gamma_h \gamma) \) corresponds to \( \gamma(T) \) and \( (r_f, \gamma h \gamma) \) corresponds to \( \gamma f(T) \), it follows that \( \Theta(\gamma h \gamma) = \gamma \Theta(T) \gamma \). The conclusion follows from

\[
(\gamma \Phi(f) \gamma)(\gamma h_i \gamma \varphi_j) = \gamma(\Phi(f) \varphi_i \varphi_j) \gamma = \gamma(h_f \varphi_i \varphi_j) \Theta(f) \gamma = (\gamma h_f \gamma \varphi_j)(\gamma \Theta(f) \gamma).
\]

\[ \square \]

3. Extension property of embeddings of compact polyhedra into 2-manifolds

In this section we prove Theorem 1.1 and Corollary 1.1. First we consider the case where \( M \) is compact.

Lemma 3.1. Suppose \( M \) is a compact PL 2-manifold and \( K \subset X \) are compact subpolyhedra of \( M \). Then there exists an open neighborhood \( U \) of \( i_X \) in \( \mathcal{E}_K(X, M)^* \) and a map \( \varphi : U \to \mathcal{H}_K(M) \) such that \( \varphi(f)|_X = f \) (\( f \in U \)) and \( \varphi(i_X) = \text{id}_M \).

Proof. We may assume that \( K = \emptyset \), since if \( \varphi \) satisfies the above condition in the case where \( K = \emptyset \) then we have \( \varphi(U \cap \mathcal{E}_K(X, M)^*) \subset \mathcal{H}_K(M) \) for any \( K \subset X \).

(1) The case when \( \partial M = \emptyset \): We fix a triangulation of \( X \) and let \( S_k \) \((k = 0, 1, 2)\) denote the set of \( k \)-simplices of this triangulation and \( X^{(1)} \) denote the 1-skeleton of \( X \). For each \( \sigma \in S_1 \) with ends \( v, w \) we choose two disjoint subarcs \( \sigma_v, \sigma_w \) of \( \sigma \) with \( v \in \sigma_v, w \in \sigma_w \) and a subarc \( e_{\sigma} \) of \( \text{Int} \sigma \) with \( \text{Int} e_{\sigma} \supset c l (\sigma \setminus (\sigma_v \cup \sigma_w)) \). For each \( \sigma \in S_0 \) set \( T_v = \{ v \} \cup (\cup_{v \in \sigma \in S_1} \sigma_v) \), which is an arc or a single point. We choose two disjoint families of closed disks \( \{ D_v \}_{v \in S_0} \) and \( \{ E_\sigma \}_{\sigma \in S_1} \) in \( M \) such that (i) \( T_v \subset \text{Int} D_v \) (\( v \in S_0 \)) and (ii) \( X^{(1)} \cap E_\sigma = e_{\sigma} \) and \( \text{Int} e_{\sigma} \subset \text{Int} E_\sigma \) (i.e., \( e_{\sigma} \) is a proper arc of \( E_\sigma \)).

![Figure 1.a](image-url)
Since $\lambda(i_X) = \text{id}_M$ and $\lambda(f)^{-1}f|_{T_v} = i_{T_v}$ ($v \in S_0$), if $\mathcal{U}$ is small enough, then $\lambda(f)^{-1}f$ is sufficiently close to $i_X$ so that $\lambda(f)^{-1}f|_{\epsilon_\sigma} \in \mathcal{W}_\sigma$. Hence we can define a map $\mu : \mathcal{U} \to \mathcal{H}(M)$ by

$$
\mu(f) = \begin{cases} 
\beta_\sigma(\lambda(f)^{-1}f|_{\epsilon_\sigma}) & \text{on } E_\sigma, \\
\text{id} & \text{on } M \setminus \cup_\sigma E_\sigma.
\end{cases}
$$

Then $\mu(i_X) = i_M$ and $\hat{f} \equiv \mu(f)^{-1}\lambda(f)^{-1}f$ is equal to the identity map on $X^{(1)}$ for each $f \in \mathcal{U}$. Since $\hat{f}(\sigma) = \sigma$ ($\sigma \in S_2$), we can define a map $\nu : \mathcal{U} \to \mathcal{H}(M)$ by $\nu(f)|_X = \hat{f}$ and $\nu(f)|_{M \setminus X} = \text{id}$. Since $\nu(i_X) = i_M$ and $\nu(f)^{-1}\mu(f)^{-1}\lambda(f)^{-1}f = i_X$, the map $\varphi : \mathcal{U} \to \mathcal{H}(M)$, $\varphi(f) = \lambda(f)\mu(f)\nu(f)$ ($f \in \mathcal{U}$) satisfies the desired conditions.

(2) The case when $\partial M \neq \emptyset$: We can use the double $N = M \cup \partial M$. Since $X$ is a subpolyhedron of $M$, $Y = X \cap \partial M$ is also a subpolyhedron of $\partial M$.

(i) By (1) (where $K \neq \emptyset$) we have a neighborhood $V_0$ of $i_X \cup \partial M$ in $\mathcal{E}_{\partial M}(X \cup \partial M, N)$ and an extension map $\psi_0 : V_0 \to \mathcal{H}_{\partial M}(N)$. We can extend every $f \in \mathcal{E}_Y(X, M)^*$ to an $f_0 \in \mathcal{E}_{\partial M}(X \cup \partial M, N)$ by the identity on $\partial M$. If $V$ is a small neighborhood of $i_X$ in $\mathcal{E}_Y(X, M)^*$, then for every $f \in V$ we have $f_0 \in V_0$, so $\psi(f_0)$ is defined and $\psi_0(f_0)(M) = M$. Thus we have an extension map $\psi : V \to \mathcal{H}_{\partial M}(M)$, $\psi(f) = \psi_0(f_0)|_M$.

(ii) Since $\mathcal{H}(\partial M)$ is locally contractible, using a collar of $\partial M$ in $M$, we have a neighborhood $W$ of $i_M$ in $\mathcal{H}(\partial M)$ and a map $F : W \to \mathcal{H}(M)$ such that $F(g)|_{\partial M} = g$ ($g \in W$) and $F(i_M) = i_M$. We can easily verify a 1-dimensional version of Lemma 3.1 and find a neighborhood $W_0$ of $i_Y$ in $\mathcal{E}(Y, \partial M)$ and an extension map $\lambda_0 : W_0 \to \mathcal{H}(\partial M)$. We may assume that $\lambda_0(W_0) \subset W$. Hence if $\mathcal{U}$ is a small neighborhood of $i_X$ in $\mathcal{E}(X, M)^*$, then we have a map $\lambda : \mathcal{U} \to \mathcal{H}(M)$, $\lambda(f) = F(\lambda_0(f)|_Y)$.

Lemma 3.2. If $M$ is a compact PL 2-manifold and $X$ is a compact subpolyhedron of $M$, then $\mathcal{H}_X(M)$ is an ANR.

Proof. Let $\pi : \mathcal{H}(M) \to \mathcal{E}(X, M)^*$, $\pi(h) = h|_X$, denote the restriction map. By Lemm 3.1 (with $K = \emptyset$) there exists an open neighborhood $\mathcal{U}$ of $i_X$ in $\mathcal{E}(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}(M)$ such that $\varphi(f)|_X = f$. Then $\Phi : \mathcal{U} \times \mathcal{H}_X(M) \cong \pi^{-1}(\mathcal{U})$, $\Phi(f, h) = \varphi(f)h$, is a homeomorphism with the inverse $\Phi^{-1}(k) = (k|_X, \varphi(k|_X)^{-1}k)$. Since $\mathcal{H}(M)$ is an ANR and $\pi^{-1}(\mathcal{U})$ is open in $\mathcal{H}(M)$, $\mathcal{H}_X(M)$ is also an ANR.

Proof of Theorem 1.1. Theorem 1.1 can be reduced to Lemma 3.1 by the following observations:

(i) Since there exists an $h \in \mathcal{H}_{K \cup (M \cup U)}(M)$ such that $hf$ is a PL embedding (cf. Appendix) we may assume that $f$ is a PL-embedding. Replacing $X$ by $f(X)$, we may assume that $f = i_X : X \subset M$.

(ii) Taking a compact PL-submanifold neighborhood $N$ of $X$ in $U$ and replacing $(M, X, K)$ by $(N, X \cup...
Fr_M N, K \cup Fr_M N), we may assume that M is compact and U = M.

(iii) If M is compact then \( H_K(M)_0 \) is open in \( H_K(M) \) by Lemma 3.2. Hence we can take a smaller U to attain \( \varphi(U) \subset H_K(M)_0. \)

**Proof of Corollary 1.1.** Let \( f \in E_k(X, U)^* \) and let \( U_f, \varphi_f \) be as in Theorem 1.1. If \( U_f \cap \text{Im} \pi \neq \emptyset \) then \( U_f \subset \text{Im} \pi. \) In fact, if \( h \in H_{K \cup (M \cup U)}(M)_0 \) and \( \pi(h) = h|_X \in U_f, \) then for any \( g \in U_f \) we have \( g = \pi(\varphi_f(g) \varphi_f(h|_X)^{-1} h). \) Hence \( \text{Im} \pi \) is clopen in \( E_k(X, U)^* \), so \( \text{Im} \pi = E_k(X, U)^*_0 \) and \( U_f \subset E_k(X, U)^*_0 \). Choose an \( h_f \in H_{K \cup (M \cup U)}(M)_0 \) with \( h_f|_X = f \) and define a local trivialization \( \Phi: U_f \times G \cong \pi^{-1}(U_f) \) by \( \Phi(g, h) = \varphi_f(g)h_fh. \)

By a similar argument we can also show the following statements.

**Corollary 3.1.** Suppose \( K \subset Y \subset X \) are compact subpolyhedra of a PL 2-manifold M.

(i) For any open neighborhood \( U \) of X in M the restriction map \( \pi: H_{K \cup (M \setminus U)}(M) \to \text{Im} \pi \subset E_k(X, U)^* \) is a principal bundle with the fiber \( H_{X \cup (M \cup U)}(M) \) and \( \text{Im} \pi \) is clopen in \( E_k(X, U)^*. \)

(ii) The restriction map \( p: E_k(X, M)^* \to \text{Im} p \subset E_k(Y, M)^* \) is locally trivial and \( \text{Im} p \) is clopen in \( E_k(Y, M)^*. \)

4. THE SPACES OF EMBEDDINGS INTO 2-MANIFOLDS

In this final section we will prove Theorem 1.2.

4.1. **Basic facts on infinite-dimensional manifolds.**

First we recall some basic facts on infinite-dimensional manifolds. As for the model spaces we follow the standard convention: \( s = (-\infty, \infty) \), \( \Sigma = \{(x_n) \in s : \sup_n |x_n| < \infty\}, \sigma = \{(x_n) \in s : x_n = 0 \text{ (almost all n)}\}. \) A triple \((X, X_1, X_2)\) means a triple of a space X and subspaces \( X_1 \supset X_2. \) A triple \((X, X_1, X_2)\) is said to be a \((s, \Sigma, \sigma)\)-manifold if each point of X admits an open neighborhood U in X and an open set V in s such that \((U, U \cap X_1, U \cap X_2) \cong (V, V \cap \Sigma, V \cap \sigma)\) (a homeomorphism of triples). In [20] we have obtained a characterization of \((s, \Sigma, \sigma)\)-manifolds in terms of some class conditions, a stability condition and the homotopy negligible complement condition. A space is \( \sigma-\text{(fd)}\)-compact if it is a countable union of (finite dimensional) compact subsets. A triple \((X, X_1, X_2)\) is said to be \((s, \Sigma, \sigma)\)-stable if \((X \times s, X_1 \times \Sigma, X_2 \times \sigma) \cong (X, X_1, X_2). \) We say that a subset Y of X has the homotopy negligible (h.n.) complement in X if there exists a homotopy \( \varphi_t: X \to X \) \((0 \leq t \leq 1)\) such that \( \varphi_0 = \text{id}_X \) and \( \varphi_t(X) \subset Y \) \((0 < t \leq 1)\). The homotopy \( \varphi_t \) is called an absorbing homotopy of X into Y.

**Fact 4.1.** (i) Y has the h.n. complement in X iff each point \( x \in X \) has an open neighborhood U and a homotopy \( \varphi: U \times [0, 1] \to X \) such that \( \varphi_0 = \text{id}_U : U \subset X \) and \( \varphi_t(U) \subset Y \) \((0 < t \leq 1)\).
(ii) If Y has the h.n. complement in X, then X is an ANR iff Y is an ANR by [10].
(iii) ([17]) Suppose X is an ANR. Then Y has the h.n. complement in X iff for any open set U of X the inclusion \( U \cap Y \subset U \) is a weak homotopy equivalence. Hence if both \( Y \subset X \) and \( Z \subset Y \) have the h.n. complement, then so does \( Z \subset X. \)
In (i) $U \cap Y$ has the h.n. complement in $U$ and local absorbing homotopies can be uniformized to a global one [3].

We will apply the following characterization of $(s, \Sigma, \sigma)$-manifolds [20].

**Proposition 4.1.** A triple $(X, X_1, X_2)$ is an $(s, \Sigma, \sigma)$-manifold iff

(i) $X$ is a separable completely metrizable ANR, $X_1$ is $\sigma$-compact and $X_2$ is $\sigma$-fd-compact,

(ii) $X_2$ has the h.n. complement in $X$,

(iii) $(X, X_1, X_2)$ is $(s, \Sigma, \sigma)$-stable.

We refer to [18] for related topics in infinite-dimensional topology.

4.2. **The spaces of embeddings into 2-manifolds.**

First we summarize the stability property and the class property of embedding spaces. Suppose $(X,d)$ and $(Y,\rho)$ are metric spaces. An embedding $f : X \to Y$ is said to be $L$-Lipschitz ($L \geq 1$) if $\frac{1}{L}d(x,y) \leq \rho(f(x),f(y)) \leq Ld(x,y)$ for any $x,y \in X$.

**Lemma 4.1.** ([6] Theorems 1.2]) Suppose $M$ is a Euclidean PL 2-manifold and $K \subset X$ are compact subpolyhedra of $M$. If $\dim(X \setminus K) \geq 1$, then the triples $(\mathcal{E}_K(X,M), \mathcal{E}_{\text{Lip}}^\text{PL}(X,M), \mathcal{E}_{\text{PL}}^\text{PL}(X,M))$ and $(\mathcal{E}_K(X,M)^*, \mathcal{E}_{\text{Lip}}^\text{PL}(X,M)^*, \mathcal{E}_{\text{PL}}^\text{PL}(X,M)^*)$ are $(s, \Sigma, \sigma)$-stable.

**Lemma 4.2.** (1) Suppose $X$ is a compact metric space, $K$ is a closed subset of $X$ and $Y$ is a locally compact, separable metric space. Then (i) $\mathcal{E}_K(X,Y)$ is separable, completely metrizable, and (ii) $\mathcal{E}_K^\text{PL}(X,Y)$ is $\sigma$-compact.

(2) ([3]) If $X$ is a compact polyhedron, $K$ is a subpolyhedron of $X$, and $Y$ is a locally compact polyhedron, then $\mathcal{E}_K^\text{PL}(X,Y)$ is $\sigma$-fd-compact.

**Proof.** (1) (i) $\mathcal{C}(X,Y)$ is completely metrizable by the sup-metric, and $\mathcal{E}(X,Y)$ is $G_\delta$ in $\mathcal{C}(X,Y)$.

(ii) For $L \geq 1$ let $\mathcal{E}_{\text{Lip}}^\text{PL}(L)(X,Y)$ denote the subspace of $L$-Lipschitz embeddings. If we write $Y = \bigcup_{n=1}^{\infty} Y_n$ ($Y_n$ is compact and $Y_n \subset \text{Int} \ Y_{n+1}, n \geq 1$), then $\mathcal{E}_{\text{Lip}}^\text{PL}(X,Y) = \bigcup_{n=1}^{\infty} \mathcal{E}_{\text{Lip}}^\text{PL}(n)(X,Y_n)$. Since $\mathcal{E}_{\text{Lip}}^\text{PL}(n)(X,Y_n)$ is equicontinuous and closed in $\mathcal{C}(X,Y_n)$, it is compact by Arzela-Ascoli Theorem ([2, Ch. XII. Theorem 6.4]). Hence $\mathcal{E}_{\text{Lip}}^\text{PL}(X,Y)$ is $\sigma$-compact.

For the proper PL-embedding case we need some basic facts:

**Fact 4.2.** (1) Suppose $A$ is a PL disk (or a PL arc) and $a \in \text{Int} \ A$. Then there exists a map $\varphi : \text{Int} \ A \to \mathcal{H}^\text{PL}_{\partial A}(A)$ such that $\varphi_x(a) = x \ (x \in \text{Int} \ A)$ and $\varphi_a = \text{id}_A$.

(2) Suppose $N$ is a PL 1-manifold with $\partial N = \emptyset$, $Y$ is a compact subpolyhedron of $N$, $U$ is an open neighborhood of $Y$ in $N$. Then there exists an open neighborhood $U$ of $i_Y$ in $\mathcal{E}^\text{PL}(Y,N)$ and a map $\varphi : U \to \mathcal{H}^\text{PL}_{N,Y}(N)$ such that $\varphi(f)|_Y = f$ and $\varphi(i_Y) = \text{id}_N$.

(3) Suppose $M$ is a PL 2-manifold, $N$ is a compact 1-submanifold of $\partial M$ and $U$ is an open neighborhood of $N$ in $M$. Then there exists an open neighborhood $U$ of $id_{\partial M}$ in $\mathcal{H}^\text{PL}_{\partial M,N}(\partial M)$ and a map $\varphi : U \to \mathcal{H}^\text{PL}_{M,U}(M)$ such that $\varphi(f)|_{\partial M} = f$ and $\varphi(id_{\partial M}) = \text{id}_M$. 


(4) Suppose $M$ is a PL 2-manifold, $Y$ is a compact subpolyhedron of $\partial M$ and $U$ is an open neighborhood of $Y$ in $M$. Then there exists an open neighborhood $V$ of $i_Y$ in $E^{PL}(Y, \partial M)$ and a map $\varphi : V \to H^{PL}_{M \setminus U}(M)$ such that $\varphi(g)|_Y = g$ and $\varphi(i_Y) = id_M$.

Comment. (3) Using a PL-collar of $\partial M$ in $M$, the assertion follows from the following remarks:

(3-i) If $A$ is a PL arc (or a PL open arc), then there exists a map $\varphi : H^A_+(\partial M) \to H^A_+(\partial M \times [0, 1])$ such that $\varphi(f)$ is an isotopy from $f$ to $id_A$ (i.e. $\varphi(f)(x,t) = (x,t)$, $\varphi(f)(x,0) = f(x)$ and $\varphi(f)(x,1) = (x,1)$) for each $f \in H^A_+(A)$ and $\varphi(id_A) = id_A \times [0,1]$.

(3-ii) Suppose $S$ is a PL circle. Then there exists an open neighborhood $U$ of $id_S$ in $H^PL(S)$ and a map $\varphi : U \to H^PL(S \times [0, 1])$ such that $\varphi(f)$ is an isotopy from $f$ to $id_S$ for each $f \in U$ and $\varphi(id_S) = id_S \times [0,1]$.

In (3-i) we may assume that $A = [0,1]$ (or $A = \mathbb{R}$). Then $\varphi(f)$ is defined as the linear isometry $\varphi(f)(x,t) = ((1-t)f(x) + tx, t)$.

(4) This follows from (2) and (3).

\[\square\]

**Lemma 4.3.** If $M$ is a PL 2-manifold and $K \subset X$ are compact subpolyhedra of $M$, then (i) $E_K(X, M)^*$ is completely metrizable and (ii) $E_K^{PL}(X, M)^*$ is $\sigma$-fd-compact.

**Proof.** (i) $E_K(X, M)^*$ is $G_\delta$ in $E_K(X, M)$.

(ii) We may assume that $K = \emptyset$. It suffices to show that each $f \in E^{PL}(X, M)^*$ has a $\sigma$-fd-compact neighborhood. Since $E^{PL}(X, M)^* \cong E^K(X, M)^*$, we may assume that $f = i_X$. Choose a sequence of small collars $C_n$ of $\partial M$ in $M$ pinched at $Y = X \cap \partial M$ such that $C_n$ becomes thinner and thinner and also the angle between $Fr_M C_n$ and $\partial M$ at $Fr_M Y$ becomes smaller and smaller as $n \to \infty$. Let $M_n = cl(M \setminus C_n)$. Then $E^{PL}(X, M)^* = \cup_n E^{PL}(X, M_n)$ and $E^{PL}(Y, \partial M)$ are $\sigma$-fd-compact by [3].

By Fact 4.2.(4) there exists an open neighborhood $V$ of $i_Y$ in $E^{PL}(Y, \partial M)$ and a map $\varphi : V \to H^{PL}(M)$ such that $\varphi(g)|_Y = g$ and $\varphi(i_Y) = id_M$. Let $\psi : E^{PL}(X, M)^* \to E^{PL}(Y, \partial M)$ be the restriction map, $\psi(f) = f|_Y$ and let $U = \psi^{-1}(V)$. Then $\Phi : V \times E^{PL}(X, M)^* \to U$, $\Phi(g, h) = \varphi(g)h$, is a homeomorphism with the inverse $\Phi^{-1}(f) = (f|_Y, \varphi(f|_Y)^{-1}f)$. Hence $U$ is also $\sigma$-fd-compact. This implies the conclusion.

\[\square\]

Next we verify the ANR-condition and the h.n. complement condition.

**Fact 4.3.** (4, 3) Suppose $M$ is a compact PL 2-manifold and $X$ is a compact subpolyhedron of $M$. Then $H^{PL}_X(M)$ has the h.n. complement in $H_X(M)$.

**Comment.** By [4, p10] (a comment on a relative version) $H^{PL}_X(M)$ is (uniformly) locally contractible. Since $H_X(M)$ is an ANR, by [3] $H^{PL}_X(M)$ has the h.n. complement in $H_X(M)$. Note that in dimension 2, the local contractibility of $H^{PL}_X(M)$ at $id_M$ simply reduces to the case where $X = \emptyset$ by the following splitting argument:

(1) We may assume that $X$ has no isolated points in $\text{Int} M$. If $X$ has the isolated points $x_i$ ($i = 1, \cdots, n$) in $\text{Int} M$, then we can choose mutually disjoint PL disk neighborhood $D_i$ of $x_i$ in $\text{Int} M \setminus X_0$, where $X_0 = X \setminus \{x_1, \cdots, x_n\}$. By Fact 4.2.(1) there exists a map $\varphi : \prod_{i=1}^n \text{Int} D_i \to H^{PL}_{X_0}(M)$
such that \( \varphi(y_1, \cdots, y_n)(x_i) = y_i \) and \( \varphi(x_1, \cdots, x_n) = id_M \). Then \( U = \{ f \in \mathcal{H}^{PL}_X(M) : f(x_i) \in Int D_i (i = 1, \cdots, n) \} \) is an open neighborhood of \( id_M \) in \( \mathcal{H}^{PL}_X(M) \) and \( \Phi : (\prod \text{Int} D_i) \times \mathcal{H}^{PL}_X(M) \to U, \Phi(y_1, \cdots, y_n, g) = \varphi(y_1, \cdots, y_n)g \), is a homeomorphism with the inverse \( \Phi^{-1}(f) = (f(x_1), \cdots, f(x_n), \varphi(f(x_1), \cdots, f(x_n))^{-1}f) \). Hence if \( \mathcal{H}^{PL}_X(M) \) is locally contractible, then \( \mathcal{H}^{PL}_X(M) \) is also locally contractible.

(2) Cutting \( M \) along \( \text{Fr}_M \) \( X \) we may assume that \( X \subset \partial M \).

(3) By Fact 4.2.(4) there exists an open neighborhood \( V \) of \( i_X \) in \( E^{PL}(X, \partial M) \) and a map \( \varphi : V \to \mathcal{H}^{PL}(M) \) such that \( \varphi(g)|_X = g \) and \( \varphi(i_X) = id_M \). Let \( \psi : \mathcal{H}^{PL}(M) \to E^{PL}(X, \partial M) \) be the restriction map, \( \psi(f) = f|_X \) and let \( U = \psi^{-1}(V) \). Then \( U \) is an open neighborhood of \( id_M \) in \( \mathcal{H}^{PL}(M) \) and \( \Phi : V \times \mathcal{H}^{PL}(X, \partial M) \to U, \Phi(g,h) = \varphi(g)h \), is a homeomorphism with the inverse \( \Phi^{-1}(f) = (f|_X, \varphi(f|_X)^{-1}f) \). Since \( \mathcal{H}^{PL}(M) \) is locally contractible \( \square \), \( \mathcal{H}^{PL}_X(M) \) is also locally contractible.

Suppose \( M \) is a PL 2-manifold and \( K \subset X \) are compact subpolyhedra of \( M \).

Lemma 4.4. (1) (i) \( E_K(X, M)^* \) is an ANR and (ii) \( E_K^{PL}(X, M)^* \) has the h.n. complement in \( E_K(X, M)^* \).

(2) (i) \( E_K(X, M) \) is an ANR and (ii) \( E_K^{PL}(X, M) \) has the h.n. complement in \( E_K(X, M) \).

Proof. (1)(i) For every \( f \in E_K(X, M)^* \), take a compact PL 2-submanifold neighborhood \( N \) of \( f(X) \) in \( M \) and consider the map \( \pi : \mathcal{H}^{PL}_K(M \setminus \text{Int}_M N) (M) \to E_K(X, M)^* \), \( \pi(h) = hf \). By Theorem 1.1 there exists an open neighborhood \( U \) of \( f \) in \( E_K(X, M)^* \) and a map \( \varphi : U \to \mathcal{H}^{PL}_K(M \setminus \text{Int}_M N)(M) \) such that \( \pi \varphi(g) = g \ (g \in U) \). Since \( \mathcal{H}^{PL}_K(M \setminus \text{Int}_M N)(M) \cong \mathcal{H}^{PL}_K(M \setminus \text{Fr}_M N)(N) \) is an ANR by Lemma 3.2, so is \( U \). Hence \( E_K(X, M)^* \) is an ANR.

(ii) By Fact 4.1.(i) it suffices to show that every \( f \in E_K(X, M)^* \) admits a neighborhood \( U \) and a homotopy \( F_t : U \to E_K(X, M)^* \) such that \( F_0 = id_U \) and \( F_t(g) \subset E_K^{PL}(X, M)^* \ (0 < t \leq 1) \). Take a compact PL 2-submanifold \( N \) of \( M \) with \( f(X) \subset U \equiv \text{Int}_M N \). Let \( \varphi : U \to \mathcal{H}^{PL}_K(M \setminus \text{Int}_M U)(M) \) be given by Theorem 1.1. Since \( \mathcal{H}^{PL}_K(M \setminus \text{Int}_M U)(M) \cong \mathcal{H}^{PL}_K(M \setminus \text{Fr}_M U)(N) \) by Fact 4.2 we have an absorbing homotopy \( \chi_t \) \( \mathcal{H}^{PL}_K(M \setminus \text{Fr}_M U)(N) \) into \( \mathcal{H}^{PL}_K(M \setminus \text{Fr}_M U)(M) \). There exists a \( h \in \mathcal{H}^{PL}_K(M \setminus \text{Fr}_M U)(M) \) such that \( hf \in E_K^{PL}(X, M)^* \). Define \( F_t \) by \( F_t(g) = \chi_t(\varphi(g)h^{-1})hf \ (g \in U) \).

(2) There exists an \( f \in E_K^{PL}(X, M) \) with \( f(X \setminus K) \subset \text{Int} M \). It induces a homeomorphism \( (E_K(f(X), M), E_K^{PL}(f(X), M)) \cong (E_K(X, M), E_K^{PL}(X, M)) : g \to gf \). Hence we may assume that \( X \setminus K \subset \text{Int} M \). Pushing towards \( \text{Int} M \) using a collar of \( \partial M \) pinched on \( \partial M \cap K \), it follows that \( E_K(X, M)^* \) has the h.n. complement in \( E_K(X, M) \). Thus (i) follows from (1)(i) and Fact 4.1.(ii), and (ii) follows from (1)(ii), Fact 4.1.(iii) and \( E_K^{PL}(X, M)^* \subset E_K^{PL}(X, M) \). \( \square \)

Theorem 1.2 follows from Proposition 4.1 and the above lemmas. For the proper embeddings we have a pair version.

Proposition 4.2. If \( \dim (X \setminus K) \geq 1 \), then \( (E_K(X, M)^*, E_K^{PL}(X, M)^*) \) is an \((s, \sigma)\)-manifold.

Remark 4.1. In general, \( E_K^{LIP}(X, M)^* \) is not \( \sigma \)-compact. For example, suppose \( X \) is a proper arc in \( M \) and \( K = \partial X \). If \( E_K^{LIP}(X, M)^* = \cup_{i \geq 1} F_i \), \( F_i \) is compact, then \( F_i = \{ f(x) : f \in F_i, x \in X \} \)
is a compact subset of $M$ with $F_i \cap \partial M = K$. By a simple diagonal argument we can define an $f \in E_{K}^{\text{LIP}}(X,M)$ such that $f(X) \not\subset F_i$ for each $i \geq 1$. Figure 2 indicates how to define such an $f$ near an end point of $X$.

**Figure 2.**

---

**References**

[1] Courant, R., *Dirichlet’s principle, conformal mapping, and minimal surfaces*, Pure and Applied Math., Interscience Publishers Inc., New York, 1950.

[2] Dugundji, J., *Topology*, Allyn and Bacon Inc., Boston, 1966.

[3] Epstein, D. B. A., Curves on 2-manifolds and isotopies, *Acta Math.*, 155 (1966) 83 - 107.

[4] Gauld, D. B., Local contractibility of spaces of homeomorphisms, *Compositio Math.*, 32 (1976) 3 - 11.

[5] Geoghegan, R., On spaces of homeomorphisms, embeddings, and functions, II: The piecewise linear case, *Proc. London Math. Soc.*, (3) 27 (1973) 463 - 483.

[6] Geoghegan, R. and Haver, W. E., On the space of piecewise linear homeomorphisms of a manifold, *Proc. of Amer. Math. Soc.*, 55 (1976) 145 - 151.

[7] Goluzin, G. M., *Geometric Theory of Functions of A Complex Variable*, Translations of Mathematical Monographs 26, Amer. Math. Soc., 1969.

[8] Hamstrom, M. E., Homotopy groups of the space of homeomorphisms on a 2-manifold, *Illinois J. Math.*, 10 (1966) 563 - 573.

[9] Luke, R. and Mason, W. K., The space of homeomorphisms on a compact two-manifold is an absolute neighborhood retract, *Trans. Amer. Math. Soc.*, 164 (1972), 275 - 285.

[10] Michael, E. A., Local properties of topological spaces, *Duke Math. J.*, 21 (1954) 163 - 172.

[11] Pommerenke, Ch., *Boundary Behaviour of Conformal Maps*, GMW 299, Springer-Verlag, New York, 1992.

[12] Sakai, K., An embedding space triple of the unit interval into a graph and its bundle structure, *Proc. Amer. Math. Soc.*, 111 (1991), 1171 - 1175.

[13] Toruńczyk, H., Concerning locally homotopy negligible sets and characterizing of $\mathcal{L}_2$-manifolds, *Fund. Math.*, 101 (1978) 93 - 110.

[14] van Mill, J., *Infinite-Dimensional Topology: Prerequisites and Introduction*, North-Holland, Amsterdam, 1989.

[15] Veech, W. A., *A second Course in Complex Analysis*, W.A. Benjamin Inc., New York, 1967.

[16] Yagasaki, T., Infinite-dimensional manifold tuples of homeomorphism groups, *Topology Appl.*, 76 (1997) 261 - 281.

---

Department of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyoku, Kyoto 606, Japan

*E-mail address: yagasaki@ipc.kit.ac.jp*