Results from a study of the Nambu–Jona-Lasinio model on the lattice

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The main results of our analysis of the two flavor Nambu–Jona-Lasinio model with SU(2) × SU(2) chiral symmetry on the four–dimensional hypercubic lattice with naive and Wilson fermions are presented. Large $N$ techniques and numerical simulations are used to study various properties of the model. The scalar and pseudoscalar spectrum, the approach to the continuum and chiral limits, the size of the $1/N$ corrections, and the effects of the zero momentum fermionic modes on finite lattices are studied. Also, some interesting observations are made by viewing the model as an embedding theory of the Higgs sector.

1. Introduction

When the high frequency modes of the gauge and fermionic fields of QCD are integrated down to the energy scale $E$ corresponding to the correlation length of the gauge field ($E \approx$ glueball mass $\approx 1550$ MeV \textsuperscript{[1]}), the resulting effective theory will essentially be a theory of fermions with contact interactions and cutoff $\Lambda \lesssim 1550$ MeV. The resulting effective Lagrangian will maintain the original chiral symmetry but will be more complicated. At energies much smaller than $\Lambda$ it should be enough to keep in the Lagrangian the least irrelevant operator, namely the four–Fermi dimension six operator. This is one way \textsuperscript{[2]} to motivate the Nambu–Jona-Lasinio (NJL) model.

Unfortunately, by only keeping the four–Fermi operator, valuable information is lost and the NJL model does not confine the quarks. Furthermore, if, for example, we want to study the $\sigma$ particle, which on phenomenological grounds is believed to have mass $\approx 750$ MeV, then the separation of scales is probably not large enough to justify the neglect of operators with dimension higher than six. Nevertheless, the NJL model possesses the same chiral symmetry as QCD and it can realize this symmetry in the Goldstone mode. It is this feature that is most crucial in the understanding of the lightest hadrons. Furthermore, our interest in the model is not so much aimed at its quantitative predictability but rather on the qualitative insights that can provide as a low energy theory of QCD, as an embedding theory of the Scalar Sector of the Minimal Standard Model and as a four–dimensional interacting theory of fermions on the lattice.

The NJL model has been studied extensively for various cases with continuum type regularizations. For a comprehensive review the reader is referred to \textsuperscript{[3]} and references therein. Furthermore, the NJL model is a special case of Yukawa models that, under a different context, have been studied extensively with lattice regularization \textsuperscript{[4]}. The model has also been studied on the lattice \textsuperscript{[5]} in connection with the possible equivalence of the top quark condensate with the Higgs field \textsuperscript{[6]}. Also in this conference work was presented \textsuperscript{[7]} regarding the study of the NJL model on the lattice with staggered fermions in connection with QED.

In this work \textsuperscript{[8]} we consider the two flavor (up and down) NJL model with SU(2) × SU(2) chiral symmetry and SU(N) color symmetry, with scalar and pseudoscalar couplings \textsuperscript{[9]} on the four–dimensional hypercubic lattice and we consider both naive and Wilson fermions. We study the NJL model using large $N$ techniques and obtain analytical results both on finite and infinite volumes. The infinite volume results are obtained sufficiently close to the continuum limit using asymptotic expansions. We also study the model for $N = 2$ using an HMC numerical simulation \textsuperscript{[10]} with Conjugate Gradient and leap–frog algorithms.

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2. Results

The seven main results that stem from our analysis are presented below.

1) For naive fermions we calculate at large \(N\) and with \(M_f = 140\) MeV the \(\sigma\) mass (\(M_\sigma\)), the \(\sigma\) width (\(\Gamma_\sigma\)) and the constituent quark mass (\(M_q\)) in physical units as functions of the cutoff. By setting \(M_f = 310\) MeV, we find \(M_\sigma = 726\) MeV, \(\Gamma_\sigma = 135\) MeV, and \(\Lambda = \pi/a = 1150\) MeV. \(M_\sigma\) is consistent with phenomenological expectations and \(\Lambda\) is consistent with the expectation that the cutoff should be close and below the mass of the lightest glueball (1550 MeV). The width however is underestimated. The reason is traced to the fact that to leading order in large \(N\) the width receives contributions only from the quark bubble and not from the pion bubble because the pion bubble is of order \(1/N\). Because the phase space available for the \(\sigma\) to decay to two quarks is much smaller than the phase space to decay to two pions the pion loop contribution, although of order \(1/N\), is probably more important than the quark loop contribution.

2) The above result is relevant not only for the low energy QCD but also for the Higgs sector. It is well known that there is an equivalence between the \(\sigma-\pi\) sector of QCD with the scalar sector of the Minimal Standard Model. In the former the scale is set by the pion decay constant \((F_\pi = 93\) MeV) and in the latter by the weak scale \((F_\pi = 246\) GeV\). As mentioned above we find that in accordance with phenomenological expectations in the \(\sigma-\pi\) QCD sector \(M_\sigma/F_\pi \approx 8\), but in the Higgs sector all previous analysis predicts a triviality bound of the Higgs mass with \(M_\sigma/F_\pi \lesssim 2.8\) (see for example [1]). In the past this has been a reason for concern since it could imply that the Higgs mass bound may be underestimated. Our analysis suggests that this apparent discrepancy appears because the Higgs mass bounds are traditionally obtained for \(m_\sigma \lesssim 0.5\) while the \(M_\sigma/F_\pi \approx 8\) ratio is obtained for \(m_\sigma \approx 2\) and it should therefore be accompanied by large deviations from the low energy behavior. Nevertheless, this is only a suggestion since we have not calculated deviations from the low energy behavior of a physical process that would enable us to make exact statements. However, the value of the width serves as an indication of the size of such deviations. In a way, departure from low energy behavior will be signaled by an increasing width of the \(\sigma\) to two quark decay. At \(m_\sigma \approx 2\) the width is already fairly large.

3) If the Higgs sector is the low energy effective field theory of a NJL model (which in turn is an effective theory of a high energy QCD–like theory) with \(N_c = 3, n_f = 2\) and \(M_f = 0\), then if we set the fermion mass to \(\frac{M_f}{c} \approx \frac{93}{246}\), as is the case for the low energy sector of QCD, the Higgs mass will be \(M_\sigma = 1915\) GeV. This corresponds to \(m_\sigma = 2\) and at this point one would expect very large deviations from the low energy behavior of scattering cross sections. This suggests that as the CM energy is turned up, first the deviations from the low energy behavior will become large signaling the onset of new physics, and later on the Higgs particle would be observed.

4) With Wilson fermions we obtain at large \(N\) analytical expressions of the pion mass \((m_\pi)\) and constituent quark mass \((m_q)\) in lattice units as functions of the bare parameters of the model. We are then able to make exact statements regarding the approach to the continuum and chiral limits. We draw the “phase diagram” and identify the single point where a continuum chiral limit \((m_\pi \rightarrow 0, m_q \rightarrow 0)\) can be achieved. This may provide an insight on how the retrieval of the continuum chiral limit is achieved in QCD.

5) At large \(N\) and for Wilson fermions the \(\sigma\) particle has mass proportional to the cutoff. Our analysis traces this to two related reasons. First, although the Wilson term has raised the masses of the doublers to the cutoff, it has not decoupled them from the theory. Through vacuum polarization these contribute to the \(\sigma\) self energy and raise its mass. Second, although the Wilson term has not altered the low frequency behavior of the propagating quark, it has, however, altered its high frequency behavior. Again, through vacuum polarization the high frequency modes contribute to the \(\sigma\) self energy and also raise its mass. Such a phenomenon may also be responsible for the difficulty in observing a \(\sigma\) particle in numerical simulations of QCD with dynamical Wilson fermions.
The numerical simulation is performed on finite lattices. For naive fermions one would expect to be able to see some indication of the chiral phase transition as well as a $\sigma$ particle. However, by simply looking at the graph of the vacuum expectation value vs. the coupling on an $8^3 \times 16$ lattice one can not see an indication of a phase transition. Also the $\sigma$ particle in a $16^4$ lattice is either non existent or too heavy to be measured. Both of these unexpected results can be explained at large $N$. The reason is traced to the existence of zero quark momentum modes that on a finite lattice are not sufficiently suppressed. The zero modes besides obscuring some of the physics are also probably partially responsible for the large inversion times in the HMC algorithm. To leading order at large $N$ the smallest eigenvalue of the matrix that has to be inverted is $m_q^2$ and corresponds to the zero quark momentum mode. For small $m_q$, the condition number of the matrix is $4/m_q^2$ for $r = 0$ and $64r/m_q^2$ for $r = 1$ and it is clear that it depends strongly on the presence of the zero modes. A large condition number will make the inversion of $M^\dagger M$ very slow. Furthermore the spacing of the smallest eigenvalues behaves like $1/L^2$ and for larger lattices the inversion times will rapidly get worse. An important observation can be made by noticing the dependence of the condition number on $r$. This suggests that performing the simulation with smaller $r$ will yield a quite faster inversion. It is possible that this may also be the case for QCD.

The observables measured in the numerical simulation (chiral condensate, vacuum expectation value, pion wave function renormalization constant, and pion mass) have values that are in good agreement with leading order large $N$ predictions. In agreement with the large $N$ predictions discussed in (5) and (6) above, the $\sigma$ mass was very heavy to give a good signal and was not measured. Also, measurements of the sigma width were not performed, but, as discussed in (1) above, we expect the $1/N$ corrections to the width to be large.

There are some interesting issues relevant to lattice work that have not been considered in this paper. It would be important to calculate the three and four point vertices and therefore be able to calculate scattering amplitudes and their departure from low energy behavior as well as the $1/N$ corrections to the width. It would also be interesting to study the NJL model at finite temperature. Finally, it would be important to include vector meson couplings (see, for example, [9], [10]) and confirm that for the case of Wilson fermions the vector meson masses scale appropriately and do not become of the order cutoff as the $\sigma$ particle does.

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