Abstract

The Fredrickson-Andersen 2-spin facilitated model on $\mathbb{Z}^d$ (FA-2f) is a paradigmatic interacting particle system with kinetic constraints (KCM) featuring dynamical facilitation, an important mechanism in condensed matter physics. In FA-2f a site may change its state only if at least two of its nearest neighbours are empty. The process, while reversible w.r.t. a product Bernoulli measure, has degenerate jumps rates because of the constraints and it is non-attractive, with an anomalous divergence of characteristic time scales as the density $q$ of the empty sites tends to zero. A natural random variable encoding the above features is $\tau_0$, the first time at which the origin becomes empty for the stationary process. Our main result is the sharp threshold

$$\tau_0 = \exp \left( \frac{d \cdot \lambda(d, 2) + o(1)}{q^{1/(d-1)}} \right) \text{ w.h.p.}$$

with $\lambda(d, 2)$ the sharp threshold constant for 2-neighbour bootstrap percolation on $\mathbb{Z}^d$, the monotone deterministic automaton counterpart of FA-2f. This is the first sharp result for a critical KCM and it compares with Holroyd’s 2003 result on bootstrap percolation and its subsequent improvements. It also settles various controversies accumulated in the physics literature over the last four decades. Furthermore, our novel techniques enable completing the recent ambitious program on the universality phenomenon for critical KCM and establishing sharp thresholds for other two-dimensional KCM.

Keywords. Kinetically constrained models, interacting particle systems, sharp threshold, bootstrap percolation, Glauber dynamics, Poincaré inequality
1 Introduction

Fredrickson-Andersen $j$-spin facilitated models (FA-$jf$) are a class of interacting particle systems that have been introduced by physicists in the 80’s [13] to model the liquid/glass transition, a major and still largely open, problem in condensed matter physics [2,6]. Later on, several models with different update rules have been introduced, and this larger class has been dubbed Kinetically Constrained Models (KCM) (see e.g. [15] and references therein). The key feature of KCM is that an update at a given vertex $x$ can occur only if a suitable neighbourhood of $x$ contains only holes, the facilitating vertices. The presence of this dynamical constraint gives rise to a mechanism dubbed dynamical facilitation [36] in condensed matter physics: motion on smaller scales begets motion on larger scales. Extensive numerical simulations indicate that indeed KCM can display a remarkable glassy behaviour, featuring in particular an anomalous divergence of characteristic time scales. As a good representative of a random variable whose law encodes the above behaviour one could take $\tau_0$, the first time the origin becomes a hole (or infected, in the jargon of the sequel). In the last forty years physicists have put forward several different conjectures on the scaling of $\tau_0$ as the equilibrium density of the holes goes to zero for FA-$jf$ models. However, to date a clear cut answer on the form of this scaling has proved elusive due to the very slow dynamics and large finite size effects intrinsic to its glassy dynamics.

From the mathematical point of view, the study of FA-$jf$ and KCM in general poses very challenging problems. This is largely due to the fact that these models do not feature an attractive dynamics (in the sense of [28, Chapter III]), and therefore many of the powerful tools developed to study attractive stochastic spin dynamics, e.g. monotone coupling or censoring, cannot be used. A central issue has been therefore that of developing novel mathematical tools to determine the long time behaviour of the stationary process and, more specifically, to find the scaling of $\mathbb{E}_{\mu_q}[\tau_0]$, the mean infection time of the origin for the stationary process, as the density $q$ of the empty sites (the facilitating ones) shrinks to zero.

With this motivation, an ambitious program was recently initiated in [31] to determine as accurately as possible the divergence of $\mathbb{E}_{\mu_q}[\tau_0]$ as $q \to 0$ for the FA-$jf$ models in any dimension and for general KCM in two dimensions. This program mirrors in some aspects the analogous program for general $\mathcal{U}$-bootstrap percolation cellular automata ($\mathcal{U}$-BP) launched by [9] and carried out in [3,8] and for $j$-neighbour bootstrap percolation [4,17,25,26]. Indeed $\mathcal{U}$-BP models and $j$-neighbour bootstrap percolation can be viewed as the monotone deterministic counterparts of generic KCM and FA-$jf$ models respectively. Despite the above analogy, the lack of monotonicity for KCM induces a much more complex behaviour and richer universality classes than BP [19–21,24,29,30].

In spite of several important advances [11,21,24,29–31], the sharp estimates of the divergence of $\mathbb{E}_{\mu_q}[\tau_0]$ for KCM still remained a milestone open problem. Solving it requires
discovering the optimal infection/healing mechanism to reach the origin and crafting the mathematical tools to transform the knowledge of this mechanism into tight upper and lower bounds for \( \mathbb{E}_{\mu_q}[\tau_0] \). In this paper we solve this problem (see Section 1.5 for an account of our most prominent innovations) for the first time and we establish the sharp scaling for FA-2f models in any dimension (Theorem 1). In doing so, we also settle once and for all various unresolved controversies in the physics literature (see Section 1.4 for a detailed account).

In addition to leading to deeper results, our novel approach also extends in breadth. Indeed, it opens the way for accomplishing the final step [19] to complete the program of [31] for establishing KCM universality. Furthermore, it entails sharp results for a variety of KCM beyond FA-2f (see [19]), thus elevating the understanding of KCM to a level similar to the one currently available in BP.

### 1.1 Bootstrap percolation background

Let us start by recalling some background on \( j \)-neighbour bootstrap percolation. Let \( \Omega = \{0, 1\}^{Z^d} \) and call a site \( x \in Z^d \) infected (or empty) for \( \omega \in \Omega \) if \( \omega_x = 0 \) and healthy (or filled) otherwise. For fixed \( 0 < q < 1 \), we denote by \( \mu_q \) the product Bernoulli probability measure with parameter \( q \) under which each site is infected with probability \( q \). When confusion does not arise, we write \( \mu = \mu_q \). Given two integers \( 1 \leq j \leq d \) the \( j \)-neighbour BP model (\( jn \)-BP for short) on the \( d \)-dimensional lattice \( Z^d \) is the monotone cellular automaton on \( \Omega \) evolving as follows. Let \( A_0 \subset Z^d \) be the set of initially infected sites distributed according to \( \mu \). Then for any integer time \( t \geq 0 \) we recursively define

\[
A_{t+1} = A_t \cup \{ x \in Z^d, |N_x \cap A_t| \geq j \},
\]

where \( N_x \) denotes the set of neighbours of \( x \) in the usual graph structure of \( Z^d \). In other words, a site becomes infected forever as soon as its constraint becomes satisfied, namely as soon as it has at least \( j \) already infected neighbours.

**Remark 1.1.** The \( jn \)-BP is clearly monotone in the initial set of infection i.e. \( A_t \subset A_t' \) for all \( t \geq 1 \) if \( A_0 \subset A_0' \). Such a monotonicity will, however, be missing in the KCM models analysed in this work.

A key quantity for bootstrap percolation is the infection time of the origin defined as \( \tau_0^{BP} = \inf \{ t \geq 0, 0 \in A_t \} \). For \( j=1 \), trivially, \( \tau_0^{BP} \) scales as the distance to the origin of the nearest infected site and thus behaves w.h.p. as \( q^{-1/d} \). For \( j > 1 \), the typical value of \( \tau_0^{BP} \) w.r.t. \( \mu_q \) has been investigated in a series of works, starting with the seminal paper of Aizenman and Lebowitz [1] and Holroyd’s breakthrough [26] determining a sharp threshold for \( d = j = 2 \). We refer to [32] for an account of the field and only recall the more recent results, those that include second order corrections to the sharp threshold.
Here and throughout the paper, when using asymptotic notation we refer to \( q \to 0 \). For 2n-BP in \( d = 2 \), w.h.p. it holds [17, 25] that
\[
\tau^\text{BP}_0 = \exp \left( \frac{\pi^2}{18q} (1 - \sqrt{q} \cdot \Theta(1)) \right). \tag{1.1}
\]
For \( jn\text{-BP} \) for all \( d \geq j \geq 2 \), w.h.p. it holds [4, 39]
\[
\tau^\text{BP}_0 \geq \exp^{j-1} \left( \frac{\lambda(d, j)}{q^{1/(d-j+1)}} (1 - o(1)) \right), \tag{1.2}
\]
\[
\tau^\text{BP}_0 \leq \exp^{j-1} \left( \frac{\lambda(d, j)}{q^{1/(d-j+1)}} (1 - \Omega\left(q^{1/(2(d-j+1))}\right)) \right), \tag{1.3}
\]
where \( \exp^k \) denotes the exponential iterated \( k \) times and \( \lambda(d, j) \) are the positive constants defined explicitly in [5, (1)-(3)]. We recall that \( \lambda(2, 2) = \pi^2/18 \) [26, Proposition 5] and we refer the interested reader to [5, Table 1 and Proposition 4] for other values of \( d, j \).

We are now ready to introduce the Fredrickson-Andersen model, a natural stochastic counterpart of \( jn\text{-BP} \) and the main focus of this work.

### 1.2 The Fredrickson–Andersen model and main result

For integers \( 1 \leq j \leq d \) the Fredrickson–Andersen \( j \)-spin facilitated model (FA-\(jf\)) is the interacting particle system on \( \Omega = \{0, 1\}^{2d} \) constructed as follows. Each site is endowed with an independent Poisson clock with rate 1. At each clock ring the state of the site is updated to an independent Bernoulli random variable with parameter \( 1 - q \) subject to the crucial constraint that if the site has fewer than \( j \) infected (nearest) neighbours currently, then the update is rejected. We refer to updates occurring at sites with at least \( j \) infected neighbours at the time of the update as legal.

**Remark 1.2.** Contrary to the \( jn\text{-BP} \) model, the FA-\(jf\) process is clearly non-monotone because of the possible recovery of infected sites with at least \( j \) infected neighbours. This feature is one of the major obstacles in the analysis of the process.

It is standard to show (see [28]) that the FA-\(jf\) process is well defined and it is reversible w.r.t. \( \mu_q \). As for \( jn\text{-BP} \) let
\[
\tau_0 = \inf\{t \geq 0, \omega_0(t) = 0\}
\]
be the first time the origin becomes infected. Our main goal is to quantify precisely \( \mathbb{E}_{\mu_q}[\tau_0] \), the average of \( \tau_0 \) w.r.t. the stationary process as \( q \to 0 \). In order to keep the
setting simple and the results more transparent, we will focus on the FA-2f model. Other models, including FA-\(j\f\) for all values of \(3 \leq j \leq d\), are discussed in Section 1.3. Recall the constants \(\lambda(d, 2)\) from (1.2), (1.3), so that \(\lambda(2, 2) = \pi^2/18\).

**Theorem 1.** As \(q \to 0\) the stationary FA-2f model on \(\mathbb{Z}^d\) satisfies:

\[\mathbb{E}_{\mu_q}[\tau_0] \geq \exp\left(\frac{\pi^2}{9q}(1 - \sqrt{q} \cdot O(1))\right),\]  
\[\mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{\pi^2}{9q}(1 + \sqrt{q} \cdot \log^{O(1)}(1/q))\right),\]  

if \(d = 2\), and

\[\mathbb{E}_{\mu_q}[\tau_0] \geq \exp\left(\frac{d \cdot \lambda(d, 2)}{q^{1/(d-1)}}(1 - o(1))\right),\]  
\[\mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{d \cdot \lambda(d, 2)}{q^{1/(d-1)}}(1 + q^{1/(2(d-1))}(\log(1/q))^{O(1)})\right),\]  

if \(d \geq 3\). Moreover, (1.4)-(1.7) also hold for \(\tau_0\) w.h.p.

In particular, recalling (1.2), (1.3), we have the following.

**Corollary 1.3.** W.h.p. \(\tau_0 = (\tau_{0}^{BP})^{d+o(1)}\).

The above are the first results that establish the sharp asymptotics of \(\log \mathbb{E}_{\mu_q}[\tau_0]\) within the whole class of “critical” KCM.

**Remark 1.4.** We will not provide an explicit proof of the case \(d \geq 3\) as it does not require any additional effort with respect to the case \(d = 2\). The only significant difference is that the lower bound from (1.1) is not available in higher dimensions, leading to a corresponding weakening of the lower bound (1.6) as compared to (1.4).

**Remark 1.5.** Despite the resemblance, our results are by no means a corollary of their 2n-BP counterpart (1.1). While the lower bounds (1.4) and (1.6) do indeed follow rather easily from (1.1) and (1.2) together with an improvement of the ”automatic” lower bound from [11, Theorem 6.9], the proof of (1.5) and (1.7) is much more involved. In particular, it requires guessing an efficient infection/healing mechanism to infect the origin, which has no counterpart in the monotone \(jn\)-BP dynamics (see Section 1.5).

### 1.3 Extensions

#### 1.3.1 FA-\(j\f\) with \(j \neq 2\)

For the sake of completeness, let us briefly discuss the FA-\(j\f\) model with other values of \(j\). The case \(j = 1\) is the simplest to analyse and behaves very differently: relaxation
is dominated by the motion of single infected sites and time scales diverge as $1/q^{\Theta(1)}$ (see [11,35] for the values of the exponent). For $d \geq j \geq 4$ minor modifications of the treatment of [31] along the lines provided by [5] can be used to prove that $\mathbb{E}_{\mu_q}[\tau_0]$ scales as $\tau_0^{BP}$ (see (1.2), (1.3)). The only remaining case, $d \geq j = 3$, requires some more work [23] still following the approach of [31]. Let us emphasise that all $d \geq j \geq 3$ can also be treated using the techniques of the present paper. However, the much faster divergence of the scaling involved allows the less refined technique of [31] to work, as there is a much larger margin for error, making those results easier.

### 1.3.2 More general update rules: $U$-KCM

The full power of the method developed in the present work is required to treat two-dimensional $U$-KCM, a very general class of interacting particle systems with kinetic constraints on $\mathbb{Z}^2$. These models and their bootstrap percolation counterpart, $U$-BP, are defined similarly to FA-$jf$ and $jn$-BP but with arbitrary local monotone constraints (or update rules) $U$ [11,31]. There is a class of constraints $U$ satisfying special symmetry assumptions for which the asymptotics of $\log \tau_0^{BP}$ is known exactly [12], with sometimes even the higher order corrections [10,18]. Our methods adapt to this setting to yield equally sharp results for $\mathbb{E}_{\mu_q}[\tau_0]$ of the corresponding $U$-KCM. In this general setting the outcome is again of the form $\mathbb{E}_{\mu_q}[\tau_0] \simeq (\tau_0^{BP})^2$ as for FA-2f. We leave this generalisation to a subsequent work by the first author [19].

We warn the reader that the exponent 2 in two dimensions relating $\mathbb{E}_{\mu_q}[\tau_0]$ to $\tau_0^{BP}$ is not general [20,21] and only holds for ‘isotropic’ models [19]. Nevertheless, developing the approach of the present work further, in [19] $\log \mathbb{E}_{\mu_q}[\tau_0]$ is determined up to a constant factor for all so-called “critical” KCM in two dimensions, matching the lower bounds established in [20] and establishing a richer KCM analogue of the BP universality result of [8].

### 1.4 Settling a controversy in the physics literature

Soon after the FA-$jf$ models were introduced, some conjectures in the physics literature predicted the divergence of $\mathbb{E}_{\mu_q}[\tau_0]$ at a positive critical density $q_c$ ([13,14,16]). These conjectures were subsequently ruled out in [11], the first contribution analysing rigorously FA-$jf$. After [11] and prior to the present work, the best known bounds on the infection time were

$$
\exp\left(\frac{\Omega(1)}{q^{1/(d-1)}}\right) \leq \mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{\log^{O(1)}(1/q)}{q^{1/(d-1)}}\right), \quad j = 2, \quad (1.8)
$$

$$
\exp^{j-1}\left(\frac{\lambda(d,j) - o(1)}{q^{1/(d-j+1)}}\right) \leq \mathbb{E}_{\mu_q}[\tau_0] \leq \exp^{j-1}\left(\frac{O(1)}{q^{1/(d-j+1)}}\right), \quad j \geq 3.
$$
The lower bounds follow from the general lower bound [31, Lemma 4.3] $\mathbb{E}_{\mu_q}[\tau_0] = \Omega(\text{median of } \tau_0^\text{BP})$ together with the $j$-nBP lower bounds (see Section 1.1) while the upper bounds were recently obtained by the second and third author in [31]. As such, the above results do not settle a controversy between several conjectures that were put forward in the physics literature.

The first quantitative prediction for the scaling of $E_{\mu_q}[\tau_0]$ appeared in [33] where, based on numerical simulations\(^4\), a faster than exponential divergence in $1/q$ was conjectured for FA-2f in $d = 2$. For the latter, the first to claim an exponential scaling $\exp(\Theta(1)/q)$ was Reiter [34]. He argue that the infection process of the origin is dominated by the motion of macro-defects, i.e. rare regions having probability $\exp(-\Theta(1)/q)$ and size $\text{poly}(1/q)$ that move at an exponentially small rate $\exp(-\Theta(1)/q)$. Later Biroli, Fisher and the last author [38] considerably refined the above picture. They argued that macro-defects should coincide with the critical droplets of 2n-BP having probability $\exp(-\pi^2/(9q))$ and that the time scale of the relaxation process inside a macro-defect should be $\exp(c/\sqrt{q})$, i.e. sub-dominant with respect to the inverse of their density, in sharp contrast with the prediction of [34]. Based on this and on the idea that macro-defects move diffusively, the relaxation time scale of FA-2f in $d = 2$ was conjectured to diverge as $\exp(\pi^2/(9q))$ in $d = 2$ [38, Section 6.3]. Yet, a different prediction was later made in [37] implying a different scaling of the form $\exp(2\pi^2/(9q))$. Concerning the behaviour of FA-2f in higher dimensions, in [38] the relaxation time was predicted to diverge as $(\tau^\text{BP}_0)^d$, though the prediction was less precise than for the two dimensional case since the sharp results for 2n-BP in dimension $d > 2$ proved in [4] were yet to be established.

Theorem 1 settles the above controversy by confirming the scaling prediction of [34, 38] and by disproving those of [33, 37]. Moreover, our result on the characteristic time scale of the relaxation process inside a macro-defect (see Proposition 3.9) agrees with the prediction of [38] and disproves the one of [34].

1.5 Behind Theorem 1: high-level ideas

The main intuition behind Theorem 1 is that for $q \ll 1$ the relaxation to equilibrium of the stationary FA-2f process is dominated by the slow motion of unusually unlikely patches of infection, dubbed mobile droplets or just droplets. In analogy with the critical droplets of bootstrap percolation (see [26]), mobile droplets have a linear size which is polynomially increasing in $q$ (with some arbitrariness), i.e. they live on a much smaller scale than the metastable length scale $e^{\Theta(1/q^{1/(d-1)})}$ arising in 2n-BP percolation model. One of the main requirements dictating the choice of the scale of mobile droplets is the requirement that the typical infection environment around a droplet is w.h.p. such that the

\(^4\)Due to the sharp divergence of time scales involved, obtaining clear cut answers via numerical simulations is very challenging.
droplet is able to move under the FA-2f dynamics in any direction. Within this scenario the main contribution to the infection time of the origin for the stationary FA-2f process should come from the time it takes for a droplet to reach the origin.

In order to translate the above intuition into a mathematically rigorous proof, one is faced with two different fundamental problems:

1. a precise, yet workable, definition of mobile droplets;
2. an efficient model for their “effective” random evolution.

In [24, 30, 31] mobile droplets (dubbed “super-good” regions there) have been defined rather rigidly as fully infected regions of suitable shape and size and their motion has been modelled as a generalised FA-1f process on $\mathbb{Z}^2$ [30, Section 3.1]. In the latter process mobile droplets are freely created or destroyed with the correct heat-bath equilibrium rates but only at locations which are adjacent to an already existing droplet. The main outcome of these papers have been (upper) bounds on the infection time of the origin of the form $\tau_0 \leq 1/\rho_D^{\log \log (1/\rho_D)^{O(1)}}$ w.h.p.

While rather powerful and robust, the solution proposed in [24, 30, 31] to (1) and (2) above has no chance to get the exact asymptotics of the infection time because of the rigidity in the definition of the mobile droplets and of the chosen model for their effective dynamics. Indeed, a mobile droplet should be allowed to deform itself and move to a nearby position like an amoeba, by rearranging its infection using the FA-2f moves. This “amoeba motion” between nearby locations should occur on a time scale much smaller than the global time scale necessary to bring a droplet from far away to the origin. In particular, it should not require to first create a new droplet from the initial one and only later destroy the original one (the main mechanism of the droplet dynamics under the generalised FA-1f process).

With this in mind we offer a new solution to (1) and (2) above which indeed leads to determining the exact asymptotics of the infection time. Concerning (1), our treatment in Section 3 consists of two steps. We first propose a sophisticated multiscale definition of mobile droplets which, in particular, introduces a crucial degree of softness in their microscopic infection’s configuration. The second and much more technically involved step is developing the tools necessary to analyse the FA-2f dynamics inside a mobile droplet. In particular, we then prove two key features (see Propositions 3.7 and 3.9 for the case $d = 2$):

1.a) to the leading order the probability $\rho_D$ of mobile droplets satisfies

$$\rho_D \geq \exp \left( -\frac{d\lambda(d, 2)}{q^{1/(d-1)}} - \frac{O(\log^2(1/q))}{q^{1/(2d-2)}} \right),$$

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5This construction is inspired by one suggested by P. Balister in 2017, which he conjectured would remove the spurious log-corrections in the bound (1.8) available at that time.
(1.b) the “amoeba motion” of mobile droplets between nearby locations occurs on a time scale $\exp\left(\frac{O(\log(1/q)^3)}{q^{1/(2d-2)}}\right)$ which is sub-leading w.r.t. the main time scale of the problem and only manifests in the second term of (1.5).

Property (1.a) follows rather easily from well known facts from bootstrap percolation theory, while proving property (1.b), one of the most innovative steps of the paper, requires a substantial amount of new ideas.

While properties (1.a) and (1.b) above are essential, they are not sufficient on their own for solving problem (2) above. In Section 5 we propose to model (admittedly only at the level of a Poincaré inequality, which however suffices for our purposes) the random evolution of mobile droplets as a symmetric simple exclusion process with two additional crucial add-ons: a coalescence part (when two mobile droplets meet they are allowed to merge) and a branching part (a single droplet can create a new one nearby as in the generalized FA-1f process). This model, which we call $g$-CBSEP, was studied for the purpose of its present application in the preparatory work [22]. Finally, the fact that $g$-CBSEP relaxes on a time scale proportional to the inverse density of mobile droplets (modulo logarithmic corrections) (see Proposition 5.1) yields the scaling of Theorem 1. We emphasise that modeling the large-scale motion of droplets by $g$-CBSEP instead of a generalized FA-1f process is an absolute novelty, also with respect to the physics literature.

2 Proof of Theorem 1: lower bound

In this section we establish the lower bounds (1.4) and (1.6) of Theorem 1. Our proof is actually a procedure to establish a general lower bound for $\mathbb{E}_{\mu_q}[\tau_0]$ based on bootstrap percolation which improves upon a previous general result [31, Lemma 4.3] which lower bounds $\mathbb{E}_{\mu_q}[\tau_0]$ with the mean infection time for the corresponding bootstrap percolation model.

We begin with an auxiliary statement. For a rectangle $R \subset \mathbb{Z}^d$ and $\eta \in \Omega_{\mathbb{Z}^d}$ we denote by $[\eta]_R$ the set of sites $x \in R$ which can become infected by legal moves only using infections in $R$. Note that $[\eta]_R$ is a union of disjoint cuboids with sides parallel to the lattice directions. For $x, y \in R$ we write $\{x \leftarrow R \rightarrow y\}$ for the event that $[\eta]_R$ contains a rectangle containing $x$ and $y$.

**Proposition 2.1.** Let $V = [-\ell, \ell]^d$ with $\ell = \ell(q)$ be such that

$$\mu(0 \in [\eta]_V) = o(1)$$

(2.1)

and let

$$\rho := \sup_{x \in V : d(x, V^c) = 1} \mu(\{x \leftarrow V \rightarrow 0\}).$$

(2.2)
Then
\[ \mathbb{E}_{\mu_q}[\tau_0] \geq \frac{\Omega(1)}{\rho |V|} \]
and \( \tau_0 \geq q/(|V|\rho) \) w.h.p.

**Proof.** Let \( \{\eta_t\}_{t \geq 0} \) denote the stationary KCM on \( \mathbb{Z}^d \) and let \( \mathcal{I} = \{\omega : 0 \in [\omega]_V\} \). By assumption \( \mu_q(\mathcal{I}) = o(1) \). Let \( \tau = \inf\{t \geq 0, \eta_t \in \mathcal{I}\} \) and observe that \( \tau \leq \tau_0 \) and that \( \mathbb{P}_{\mu_q}(\tau > 0) = 1 - o(1) \). Notice also that at \( t = \tau \) a flip at a site \( x \in V \) with \( d(x, V^c) = 1 \) which is pivotal for \( \mathcal{I} \) must occur. In particular, \( \eta_\tau \in \{x \rightarrow 0\} \). For \( s > 0 \) let \( N_V(s) \) be the number of clock rings in \( V \) up to time \( s \) as defined in Section 1.2. By stationarity, at each of those updates the KCM configuration is distributed according to \( \mu_q \). Thus,
\[ \mathbb{P}_{\mu_q}(\tau \leq s) = \mathbb{E}(\mathbb{P}_{\mu_q}(\tau \leq s \mid N_V(s))) \leq \mathbb{P}_{\mu_q}(\tau = 0) + \mathbb{E}(N_V(s)) \rho \leq o(1) + s|V|\rho. \]

Above we used a union bound to write
\[ \mathbb{P}_{\mu_q}(0 < \tau \leq s \mid N_V(s)) \leq N_V(s) \rho, \]
together with \( \mathbb{E}(N_V(s)) = s|V| \). In conclusion,
\[ \lim_{\varepsilon \to 0} \limsup_{q \to 0} \mathbb{P}_{\mu_q}(\tau_0|V|\rho < \varepsilon) = 0, \]
which concludes the proof by Markov’s inequality.

We can now easily deduce the lower bounds of Theorem 1 from Proposition 2.1 and bootstrap percolation results.

**Proof of the lower bounds (1.4) and (1.6) in Theorem 1.** Let \( d = 2 \) and \( \ell = \frac{\log(1/q)}{4q} \). 2n-BP results imply that condition (2.1) of Proposition 2.1 holds [1, (5.11)]. Then [25, Theorem 6.1, Lemma 3.9] gives \( \rho \leq \exp(-\frac{\pi^2}{9q} + \frac{O(1)}{\sqrt{q}}) \) and (1.4) follows.

Assume \( d \geq 2 \) and chose \( \ell = cq^{-1/(d-1)} \) for \( c \) large enough. Again, BP results [1, (5.11)] imply the validity of condition (2.1), while the upper bound on \( \rho \) leading to (1.6) follows from [4, Theorem 17].

## 3 Mobile droplets

This section, which represents the core of the paper, is split into two distinct parts:

- the definition of mobile droplets together with the choice of the mesoscopic critical length scale \( L_D \) characterising their linear size;
- the analysis of two key properties of mobile droplets namely:
Mobile droplets are defined as boxes of suitable linear size in which the configuration of infection is super-good (see Definition 3.6). In turn, the super-good event (see Section 3.2) is constructed recursively via a multi-scale procedure on a sequence of exponentially increasing length scales \((\ell_n)_{n=1}^N\) (see Definition 3.3). While clearly inspired by the classical procedure used in bootstrap percolation [26], an important novelty in our construction is the freedom that we allow for the position of the super-good core of scale \(\ell_n\) inside the super-good region of scale \(\ell_{n+1}\). The final scale \(\ell_N\) corresponds to the critical scale \(L_D\) mentioned above and a convenient choice is \(L_D \sim q^{-17/2}\) (see (3.4)). There is nothing special in the exponent \(17/2\): as long as we choose a sufficiently large exponent our results would not change. The choice of \(L_D\) is in fact only dictated by the requirement that w.h.p. there exist no \(L_D\) consecutive lattice sites at distance \(e^{O(\log(1/q))}/q\) from the origin which are healthy and \(L_D \ll e^{o(1/q)}\). Finally, similarly to their bootstrap percolation counterparts, the probability \(\rho_D\) of mobile droplets crucially satisfies \(\rho_D \simeq (\tau_{0}^{BP})^{-2}\) (see Proposition 3.7) and in general for FA-2f in dimension \(d\) it satisfies \(\rho_D \simeq (\tau_{0}^{BP})^{-d}\).

The extra degree of freedom in the construction of the super-good event provides a much more flexible structure that can be moved around using the FA-2f moves without going through the bottleneck corresponding to the creation of a brand new additional droplet nearby. The main consequence of this feature (see Proposition 3.9) is that the relaxation time of the FA-2f dynamics in a box of side \(L_D\) conditioned on being super-good is sub-leading w.r.t. \(\rho_D^{-1}\) as \(q \to 0\) and it contributes only to the second order term in Theorem 1.

### 3.1 Notation

For any integer \(n\), we write \([n]\) for the set \(\{1, \ldots, n\}\). We denote by \(\vec{e}_1, \vec{e}_2\) the standard basis of \(\mathbb{Z}^2\), and write \(d(x, y)\) for the Euclidean distance between \(x, y \in \mathbb{Z}^2\). Given a set \(\Lambda \subset \mathbb{Z}^2\), we set \(\partial \Lambda := \{y \in \mathbb{Z}^d \setminus \Lambda, d(y, \Lambda) = 1\}\). Given two positive integers \(a_1, a_2\), we write \(R(a_1, a_2) \subset \mathbb{Z}^d\) for the rectangle \([a_1] \times [a_2]\) and we refer to \(a_1, a_2\) as the width and height of \(R\) respectively. We also write \(\partial_x R(\partial_x R)\) for the column \([a_1 + 1] \times [a_2]\) (the column \([0] \times [a_2]\)) and \(\partial_y R(\partial_y R)\) for the the row \([a_1] \times [a_2 + 1]\) (the row \([a_1] \times [0]\)). Similarly for any rectangle of the form \(R + x, x \in \mathbb{Z}^2\).

Given \(\Lambda \subset \mathbb{Z}^2\) and \(\omega \in \Omega\), we write \(\omega_\Lambda \in \Omega_\Lambda := \{0, 1\}^\Lambda\) for the restriction of \(\omega\) to \(\Lambda\). The configuration (in \(\Omega\) or \(\Omega_\Lambda\)) identically equal to one is denoted by \(1\). Given disjoint \(\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2\), \(\omega^{(1)} \in \Omega_{\Lambda_1}\) and \(\omega^{(2)} \in \Omega_{\Lambda_2}\), we write \(\omega^{(1)} \cdot \omega^{(2)} \in \Omega_{\Lambda_1 \cup \Lambda_2}\) for the configuration equal to \(\omega^{(1)}\) in \(\Lambda_1\) and to \(\omega^{(2)}\) in \(\Lambda_2\). We write \(\mu_\Lambda\) for the marginal of \(\mu_\omega\) on
Figure 1: Black circles denote infected sites. The boundary condition $\omega$ in the figure is fully infected on $\partial_r R$ and fully healthy elsewhere. The rectangle $R$ is quasi-right-traversable under $\omega$ but neither quasi-up-, nor quasi-left-traversable. It is also down-traversable but not traversable in any other direction.

$\Omega_\Lambda$ and $\text{Var}_\Lambda(f)$ for the variance of $f$ w.r.t. $\mu_\Lambda$, given the variables $(\omega_x)_{x \notin \Lambda}$.

### 3.2 Super-good event and mobile droplets

As anticipated, mobile droplets will be defined as square regions of a certain side length in which the infection configuration satisfies a specific condition dubbed super-good. The latter requires in turn the definition of a key event for rectangles—$\omega$-traversability (see also [26])—together with a sequence of exponentially increasing length scales.

**Definition 3.1** ($\omega$-Traversability). Fix a rectangle $R = R(a_1, a_2) + x$ together with $\eta \in \Omega_R$ and a boundary configuration $\omega \in \Omega_{\partial R}$. We say that $R$ is $\omega$-right-traversable ($\omega$-left-traversable) for $\eta$ if each couple of adjacent columns of $R \cup \partial_x R$ (of $R \cup \partial_l R$) contains at least one infection. We denote the corresponding event by $T_\omega^\rightarrow(R)$ ($T_\omega^\leftarrow(R)$) and we depict it in our drawings with a dashed horizontal right (left) arrow (see Figure 1). The $\omega$-up(down)-traversability is defined similarly by looking at couples of adjacent rows of $R \cup \partial_y R$ ($R \cup \partial_d R$). The corresponding events will be denoted by $T_\omega^\uparrow(R)$ ($T_\omega^\downarrow(R)$) and they will be depicted in our drawings with a dashed up (down) arrow.

We say that $R$ is right-traversable for $\eta$ if it is $1$-right-traversable or, equivalently, if it is $\omega$-right-traversable for all $\omega$. We denote the corresponding event by $T^\rightarrow(R) \equiv T^1_\rightarrow(R)$ and we depict it in our drawings with a solid horizontal right arrow. We define analogously the left-traversable, up-traversable and down-traversable events, $T^\leftarrow(R)$, $T^\uparrow(R)$ and $T^\downarrow(R)$ respectively (see Figure 1).

**Remark 3.2.** Notice that right-traversability requires that the rightmost column contains an infection. Similarly for the other directions.
Figure 2: An example of super-good configuration in the square $\Lambda^{(6)}$. The black square, of type $\Lambda^{(2)} + x$, is completely infected and it is a super-good core for the rectangle of type $\Lambda^{(3)}$ formed by it together with the two hatched rectangles. This $\Lambda^{(3)}$-type rectangle is also super-good because of the right/left-traversability of the hatched parts (black arrows) and it is a super-good core for the square containing it and so on.

**Definition 3.3 (Length scales and nested rectangles).** For all integer $n$ we set

$$\ell_n = \begin{cases} 1 & \text{if } n = 0, \\ \lfloor \exp(n\sqrt{q}) \rfloor & \text{if } n \geq 1 \end{cases}$$

and

$$\Lambda^{(n)} = \begin{cases} R(\ell_{n/2}, \ell_{n/2}) & \text{if } n \text{ is even}, \\ R(\ell_{(n+1)/2}, \ell_{(n-1)/2}) & \text{if } n \text{ is odd}. \end{cases}$$

Notice that $(\Lambda^{(2m)})_{m \geq 0}$ is a sequence of squares, while $(\Lambda^{(2m+1)})_{m \geq 0}$ is a sequence of rectangles elongated in the horizontal direction and $\Lambda^{(n_1)} \subset \Lambda^{(n_2)}$ if $n_1 < n_2$. We also say that a rectangle $R$ is of class $n$ if there exist $w, z \in \mathbb{Z}^2$ such that $\Lambda^{(n-1)} + w \subseteq R \subseteq \Lambda^{(n)} + z$. Thus, for $n = 0$ $R$ is a single site, for $n = 2m > 0$ it is a rectangle of width $\ell_m$ and height $a_2 \in (\ell_{m-1}, \ell_m]$ and for $n = 2m + 1$, it is a rectangle of height $\ell_m$ and width $a_1 \in (\ell_m, \ell_{m+1}]$.

We are now ready to introduce the key notion of the $\omega$-super-good event on different scales. This event is defined recursively on $n$ and it has a hierarchical structure. Roughly speaking, a rectangle $R$ of the form $R = \Lambda^{(n)} + x, x \in \mathbb{Z}^2$, is $\omega$-super-good if it contains a 1-super-good rectangle $R'$ of the form $R' = \Lambda^{(n-1)} + x'$ called the core and outside the core it satisfies certain $\omega$-traversability conditions (see Figure 2).

**Definition 3.4 (\omega-Super-good rectangles).** Let us fix an integer $n \geq 0$, a rectangle $R = R(a_1, a_2) + x$ of class $n$ and $\omega \in \Omega_{\partial R}$. We say that $R$ is $\omega$-super-good and denote the corresponding event by $SG^\omega(R)$ if the following occurs.

- $n = 0$. In this case $R$ consists of a single site and $SG^\omega(R)$ is the event that this site is infected.

---

6This choice of geometrically increasing length scales is inspired by [18].
\( n = 2m \). For any \( s \in [0, \ell_m - \ell_{m-1}] \) write \( R = C_s \cup (\Lambda^{(n-1)} + x + s\vec{e}_2) \cup D_s \), where \( C_s (D_s) \) is the part of \( R \) below (above) \( \Lambda^{(n-1)} + x + s\vec{e}_2 \). With this notation we set

\[
SG_s(\omega)(R) := T^{\omega}_{\downarrow}(C_s) \cap SG_1(\Lambda^{(n-1)} + x + s\vec{e}_2) \cap T^{\omega}_{\uparrow}(D_s)
\]

and let \( SG_\omega(R) = \bigcup_{s \in [0, \ell_m - \ell_{m-1}]} SG_s(\omega)(R) \).

\( n = 2m + 1 \). In this case \( SG_\omega(R) \) requires that there is a core in \( R \) of the form \( \Lambda^{(n-1)} + x + s\vec{e}_1 \), \( s \in [0, \ell_{m+1} - \ell_m] \), which is \( 1 \)-super-good and the two remaining rectangles forming \( R \) to the left and to the right of the core are \( \omega \)-left-traversable and \( \omega \)-right-traversable respectively.

We will say that \( R \) is super-good if it is \( 1 \)-super-good and denote the corresponding event by \( SG(R) \). We have monotonicity in the boundary condition in the sense that if \( R \) is super-good then \( R \) is \( \omega \)-super-good for all \( \omega \in \Omega_{\partial R} \).

**Remark 3.5** (Irreducibility of the FA-2f chain in \( SG_\omega(R) \)). It is not difficult to verify that for all \( \eta \in SG_\omega(R) \), there exists a path of legal moves that connects \( \eta \) to the fully infected configuration. The above property implies in particular that if we consider the FA-2f chain in \( R \) restricted to \( SG_\omega(R) \), then the chain is irreducible.

Now let

\[
N := \left\lceil \frac{8 \log(1/q)}{\sqrt{q}} \right\rceil
\]

(3.3)

and observe that

\[
\ell_N = q^{17/2 + o(1)}.
\]

(3.4)

In the sequel we will refer to \( \ell_N \) as the **droplet scale**.

**Definition 3.6** (Mobile droplets). Given \( \omega \in \Omega \), a **mobile droplet** for \( \omega \) is any square \( R \) of the form \( R = \Lambda^{(2N)} + x \) for which \( \omega_R \in SG(R) \).

The two key properties of mobile droplets are the following.

**Proposition 3.7** (Probability of mobile droplets). For all \( n \leq 2N \),

\[
\mu_{\Lambda(\omega)}(SG) \geq \exp\left(- \frac{\pi^2}{9q} \left(1 + O(\sqrt{q} \log^2(1/q))\right)\right).
\]

In particular, this holds for \( \rho_D := \mu(\Lambda^{(2N)} \text{ is a mobile droplet}) \).

**Remark 3.8.** The lower bound of Proposition 3.7 is saturated on the droplet scale. Indeed, it is essentially sharp for \( n = 2N \).
The proof of Proposition 3.7 follows from standard 2n-BP techniques and it is deferred to Appendix A. The second property of mobile droplets requires a bit of preparation. Given \( R \) of class \( n \) and \( \omega \in \Omega_{\mathbb{Z}^2 \setminus R} \), let \( \gamma^\omega(R) \) be the best constant \( C \) in the Poincaré inequality
\[
\text{Var}_R(f | SG^\omega(R)) \leq C \sum_{x \in R} \mu_R(c^R,c^\omega_x(f) | SG^\omega(R)),
\]
where, for all \( \Lambda \subset \mathbb{Z}^2 \), all \( \omega \in \Omega_{\partial \Lambda} \) and \( x \in \Lambda \)
\[
c^\Lambda,\omega_x(\eta) = c_x(\eta \cdot \omega),
\]
and
\[
c_x(\omega) = \begin{cases} 1 & \text{if } \sum_{y \sim x} (1 - \omega_y) \geq 2 \\ 0 & \text{otherwise} \end{cases}
\]
with \( y \sim x \) iff \( x, y \) are nearest neighbours. The fact that FA-2f restricted to \( SG^\omega(R) \) is irreducible (see Remark 3.5) implies that \( \gamma^\omega(R) \) is finite. However, proving a good upper bound on \( \gamma^\omega(R) \) is quite hard. In the sequel we will sometimes refer to \( \gamma^\omega(R) \) as the relaxation time of \( SG^\omega(R) \).

**Proposition 3.9** (Relaxation time of mobile droplets). For all \( n \leq 2N \)
\[
\max_{\omega} \gamma^\omega(\Lambda^{(n)}) \leq \exp\left( O(\log^2(1/q)n) \right).
\]
In particular, on the droplet scale we get
\[
\max_{\omega} \gamma^\omega(\Lambda^{(2N)}) \leq e^{O(\log^3(1/q))/\sqrt{q}}.
\]

**Remark 3.10.** We stress an important difference in the definition of \( \gamma^\omega(\Lambda^{(n)}) \) w.r.t. a similar definition in \([24, (12)]\). Indeed, in (3.5) the conditioning w.r.t. the super-good event \( SG^\omega(R) \) appears in the l.h.s. and in the r.h.s. of the inequality, while in \([24, (12)]\) the conditioning was absent in the r.h.s. Keeping the conditioning also in the r.h.s. is a delicate and important point if one wants to get a Poincaré constant which is sub-leading w.r.t. \( \rho_D^{-1} \). Theorem 4.6 of \([24]\) in the context of FA-2f would give a Poincaré constant bounded from above by \( \exp(\log(1/q)^3/q) \), much bigger than \( \rho_D^{-1} \).

### 3.3 Proof of Proposition 3.9

The proof is unfortunately rather long and technical but the main idea and technical ingredients can be explained as follows.

Given the recursive definition of the super-good event \( SG^\omega(\Lambda^{(n)}) \) it is quite natural to try bounding from above its relaxation time in progressively larger and larger volumes. A high-level “dynamical intuition” here goes as follows. After every time interval
The core of $\Lambda^{(n)}$, namely a super-good rectangle of type $\Lambda^{(n-1)}$ inside $\Lambda^{(n)}$, will equilibrate under the FA-2f dynamics. Therefore, the relaxation time of $SG(\Lambda^{(n)})$ should be at most $T^{(n)}_{\text{eff}} \times \Theta(\gamma_{1}(\Lambda^{(n-1)}))$, where $T^{(n)}_{\text{eff}}$ is the time that it takes for the core to equilibrate its position inside $\Lambda^{(n)}$, assuming that at each time the infections inside it are at equilibrium. The main step necessary to transform this rather vague idea into a proof is as follows.

In order to analyse the characteristic time scale of the effective dynamics of a core, we need to improve and expand a well established mathematical technique for KCM to relate the relaxation times of two $\omega$-super-good regions on different scales. Such a technique introduces various types of auxiliary constrained block chains and a large part of our argument is devoted to proving good bounds on their relaxation times (see Section 4). The main application of this technique to our concrete problem is summarised in Lemmas 3.11 and 3.12 below which easily imply Proposition 3.9. Let

$$\Lambda^{(n, +)} = \begin{cases} R(\ell_m + 1, \ell_m) & \text{if } n = 2m, \\ R(\ell_{m+1}, \ell_m + 1) & \text{if } n = 2m + 1. \end{cases}$$

The two key steps connecting the relaxation times of super-good rectangles of increasing length scale are as follows.

**Lemma 3.11** (From $\ell_{[n/2]} + 1$ to $\ell_{[n/2]+1}$). For all $0 \leq n \leq 2N-1$

$$\max_{\omega} \gamma^\omega(\Lambda^{(n+1)}) \leq \max_{\omega} \gamma^\omega(\Lambda^{(n, +)}) \exp(O(\log^2(q))).$$

**Lemma 3.12** (From $\ell_{[n/2]}$ to $\ell_{[n/2]+1}$). For all $0 \leq n \leq 2N-1$

$$\max_{\omega} \gamma^\omega(\Lambda^{(n, +)}) \leq q^{-O(1)} \max_{\omega} \gamma^\omega(\Lambda^{(n)}).$$

The lemmas imply that

$$\max_{\omega} \gamma^\omega(\Lambda^{(n+1)}) \leq \exp(O(\log(q)^2)) \max_{\omega} \gamma^\omega(\Lambda^{(n)})$$

and Proposition 3.9 follows by induction over $n$.

**Proof of Lemma 3.11.** Given $n \leq 2N - 1$ let $K_n$ be the smallest integer $K$ such that $\left\lceil (2/3)^K (\ell_{[n/2]+1} - \ell_{[n/2]}) \right\rceil = 1$. Definition (3.1) and (3.3) imply that $\max_{n \leq 2N-1} K_n \leq O(\log(1/q))$. Consider the (exponentially increasing) sequence

$$d_k = \left\lceil (2/3)^{K_n-k} (\ell_{[n/2]+1} - \ell_{[n/2]}) \right\rceil, \quad k \leq K_n,$$

and let $s_k = d_{k+1} - d_k$ for $k \leq K_n - 1$. Next consider the collection $(R^{(k)})_{k=0}^{K_n}$ of rectangles of class $n + 1$ interpolating between $\Lambda^{(n, +)}$ and $\Lambda^{(n+1)}$ defined by

$$R^{(k)} = \begin{cases} R(\ell_m + d_k, \ell_m) & \text{if } n = 2m, \\ R(\ell_{m+1}, \ell_m + d_k) & \text{if } n = 2m + 1. \end{cases}$$
By construction, \( R^{(k)} \subset R^{(k+1)} \), \( R^{(0)} = \Lambda^{(n,+)} \) and \( R^{(K_n)} = \Lambda^{(n+1)} \). Finally, recall the events \( \mathcal{SG}^\omega(R) \) and \( \mathcal{SG}_{\vec{r}}^\omega(R) \) constructed in Definition 3.4) for any rectangle \( R \) of class \( n+1 \leq 2N \) and let

\[
a_k = \max_{\omega} \left( \mu_{R^{(k)}}(\mathcal{SG}_{d_k}^1 | \mathcal{SG}^\omega) \right)^{-2} \max_{\omega} \left( \mu_{R^{(k)}}(\mathcal{SG}_{d_k}^\omega | \mathcal{SG}^\omega) \right)^{-1}, \tag{3.8}
\]

where \( \max_\omega \) is over all \( \omega \in \Omega_{\partial R^{(k)}} \). In Corollary A.3 we prove that

\[
\mu_{R}(\mathcal{SG}_{\vec{r}}^\omega(R) | \mathcal{SG}^\omega(R)) \geq q^{O(1)}
\]

uniformly over all rectangles \( R \) of class \( n+1 \leq 2N \), all possible values of the offset \( s \) and all choices of the boundary configurations \( \omega, \omega' \in \Omega_{\partial R} \). As a consequence

\[
\max_{n \leq 2N - 1} \max_{k \in K_n} a_k \leq (1/q)^{O(1)}. \tag{3.9}
\]

With the above notation the key inequality for proving Lemma 3.11 is

\[
\max_\omega \gamma^\omega(R^{(k+1)}) \leq C a_k \times \max_\omega \gamma^\omega(R^{(k)}), \quad k \in [0, K_n - 1], \tag{3.10}
\]

for some universal constant \( C > 0 \). Recalling that \( R^{(0)} = \Lambda^{(n,+)} \) and \( R^{(K_n)} = \Lambda^{(n+1)} \), from (3.10) it follows that

\[
\max_\omega \gamma^\omega(\Lambda^{(n+1)}) \leq \left( C K_n \prod_{k=0}^{K_n-1} a_k \right) \times \max_\omega \gamma^\omega(\Lambda^{(n,+)}) \tag{3.11}
\]

which in turn implies Lemma 3.11 by (3.9) and \( K_n \leq O(\log(1/q)) \).

The proof of (3.10), which is detailed for simplicity only in the even case \( n = 2m \), relies on the Poincaré inequality for a properly chosen auxiliary block chain proved in Proposition 4.5. In order to exploit that proposition we partition \( R^{(k+1)} \) into three disjoint rectangles \( V_1, V_2, V_3 \) as follows (see Figure 3):

\[
V_1 = R(s_k, \ell_m), \quad V_2 = R^{(k)} \setminus V_1, \quad V_3 = R^{(k+1)} \setminus R^{(k)}.
\]

Then, given a boundary configuration \( \omega \in \Omega_{\partial R^{(k+1)}} \), let

\[
\mathcal{H} = \{ \eta : \eta_3 \in \mathcal{T}_{\vec{r}}^\omega(V_3) \text{ and } \eta_1 \cdot \eta_2 \in \mathcal{SG}_{\vec{r}}^\omega(V_1 \cup V_2) \}, \tag{3.12}
\]

\[
\mathcal{K} = \{ \eta : \eta_1 \in \mathcal{T}_{\vec{r}}^\omega(V_1) \text{ and } \eta_2 \cdot \eta_3 \in \mathcal{SG}_{\vec{r}}^\omega(V_2 \cup V_3) \}, \tag{3.13}
\]

\( \eta_i := \eta V_i \) and \( \eta \cdot \omega \) denotes the configuration equal to \( \omega \) on \( \partial R^{(k+1)} \) and equal to \( \eta \) on \( R^{(k+1)} \). Notice that \( \mathcal{H} \cup \mathcal{K} = \mathcal{SG}^\omega(R^{(k+1)}) \). The width of \( V_2 \) is in fact \( \ell_m + 2d_k - d_{k+1} \geq \ell_m \) and therefore any configuration in \( \mathcal{SG}^\omega(R^{(k+1)}) \) necessarily contains a super-good core in either \( V_1 \cup V_2 \) or \( V_2 \cup V_3 \).
Using (3.8) and the fact that \( V_1 \cup V_2 = R(k) \), \( V_2 \cup V_3 = R(k) + s_k \), one easily sees that
\[ T^{(1)}_{\text{aux}} \leq a_k. \]

We next introduce two additional events
\[ \mathcal{F}_{1,2} = \mathcal{S}\mathcal{G}_{s_k}^{1,\omega}(V_1 \cup V_2) \quad \text{ and } \quad \mathcal{F}_{2,3} = \mathcal{S}\mathcal{G}_0^{1,\omega}(V_2 \cup V_3), \]
where, with a slight abuse of notation, \( 1 \cdot \omega \) equals \( 1_{R(k+1)} \cdot \omega \). In words, \( \mathcal{F}_{1,2} \) (\( \mathcal{F}_{2,3} \)) consists of super-good configurations in \( V_1 \cup V_2 \) (\( V_2 \cup V_3 \)) with a super-good core of type \( \Lambda^{(n)} \) inside \( V_2 \) in the leftmost possible position. Monotonicity in the boundary condition easily implies that
\[ \{ \eta : \eta_3 \in \mathcal{T}_{\omega}(V_3) \text{ and } \eta_1 \cdot \eta_2 \in \mathcal{F}_{1,2} \} \subset \mathcal{H} \cap \mathcal{K}, \]
and similarly for \( \mathcal{F}_{2,3} \).

At this stage we can apply Proposition 4.5 with parameters \( \Omega_i = \Omega_{V_i} \) for \( i \in \{1, 2, 3\} \),
\( \mathcal{A}_1 = \mathcal{T}_{\omega}(V_1), \mathcal{A}_3 = \mathcal{T}_{\omega}(V_3), \mathcal{B}_{1,2} = \mathcal{S}\mathcal{G}^{\eta,\omega}(V_1 \cup V_2) \) and \( \mathcal{B}_{2,3} = \mathcal{S}\mathcal{G}^{\eta,\omega}(V_2 \cup V_3) \) to get
\[ \text{Var}_{R(k+1)} \left( f \mid \mathcal{S}\mathcal{G}^{\omega}(R(k+1)) \right) = \text{Var}_{R(k+1)} \left( f \mid \mathcal{H} \cup \mathcal{K} \right) \leq c T^{(1)}_{\text{aux}} \times \frac{\mu_{R(k+1)}(1_{\mathcal{H}} \text{Var}(f \mid \mathcal{H}, \eta_3) + 1_{\mathcal{K}} \text{Var}(f \mid \mathcal{K}, \eta_1) \mid \mathcal{H} \cup \mathcal{K})}{\mu_{R(k+1)}(\mathcal{F}_{1,2})} \times \frac{\mu_{R(k+1)}(\mathcal{S}\mathcal{G}^{\eta,\omega}(V_1 \cup V_2))}{\mu_{R(k+1)}(\mathcal{F}_{2,3})}, \]
for some universal constant \( c > 0 \), where
\[ T^{(1)}_{\text{aux}} = \max_{\eta \in T_{\omega}(V_1) \cap \Omega_{V_1}} \left( \frac{\mu_{R(k+1)}(\mathcal{S}\mathcal{G}^{\omega}(V_1 \cup V_2))}{\mu_{R(k+1)}(\mathcal{F}_{1,2})} \right)^2 \times \frac{\mu_{R(k+1)}(\mathcal{S}\mathcal{G}^{\eta,\omega}(V_2 \cup V_3))}{\mu_{R(k+1)}(\mathcal{F}_{2,3})}. \]

Using (3.8) and the fact that \( V_1 \cup V_2 = R(k), V_2 \cup V_3 = R(k) + s_k \), one easily sees that
\[ T^{(1)}_{\text{aux}} \leq a_k. \]
In order to conclude the proof of (3.10) we are left with the analysis of the average w.r.t. \( \mu_{R(k+1)}(\cdot | \mathcal{H} \cup \mathcal{K}) \) in the r.h.s. of (3.14). Recalling (3.5), for any \( \eta \in \Omega_{R(k+1)} \) such that \( \eta_3 \in \mathcal{T}^\omega_\to(V_3) \) we get
\[
\text{Var}(f | \mathcal{H}, \eta_3) \leq \max_{\omega' \in \Omega_{R(k)}} \gamma^{\omega'}(R(k)) \times \sum_{y \in R(k)} \mu_{R(k)}(c_y^{R(k),\eta_3\omega} \text{Var}_y(f) | SG^{\eta_\omega}(R(k))) \tag{3.15}
\]
An analogous inequality holds for \( \text{Var}(f | \mathcal{K}, \eta_1) \) when \( \eta_1 \in \mathcal{T}^\omega_\leftarrow(V_1) \). Finally,
\[
\mu_{R(k+1)}(\mathbb{1}_{\mathcal{H}} \mu_{R(k)}(c_y^{R(k),\eta_3\omega} \text{Var}_y(f) | SG^{\eta_\omega}(R(k))) | \text{SG}^{\omega}(R(k+1)))) = \mu_{R(k+1)}(\mathbb{1}_{\mathcal{H}} c_y^{R(k+1),\omega} \text{Var}_y(f) | \text{SG}^{\omega}(R(k+1)))
\]
and the similarly for \( \mathcal{K} \). Inserting the above into (3.14), we get
\[
\text{Var}_{R(k+1)}(f | \text{SG}^{\omega}(R(k+1))) \leq O(a_k) \times \max_{\omega'} \gamma^{\omega'}(R(k)) \times \sum_{x \in R(k+1)} \mu_{R(k+1)}(c_x^{R(k+1),\omega} \text{Var}_x(f) | \text{SG}^{\omega}(R(k+1))),
\]
which proves (3.10).

**Proof of Lemma 3.12.** The proof is similar to the proof of Lemma 3.11, but in this case we plan to use Proposition 4.7. Again we provide the details only in the case \( n = 2m \).

The result for the case \( m = 0 \) follows immediately since \( \Lambda^{(0,+)} \) contains only two sites. If \( m \geq 1 \) we begin by writing \( \Lambda^{(n,+)} = R(\ell_m + 1, \ell_m) = V_1 \cup V_2 \cup V_3 \), where \( V_1 \) denotes the leftmost column, \( V_3 \) the rightmost column and \( V_2 \) all the remaining columns (see Fig 4). By construction \( V_1 \cup V_2 \) and \( V_2 \cup V_3 \) are translates of \( \Lambda^{(n)} \). Then, for any given \( \omega \in \Omega_{\partial \Lambda^{(n,+)}} \), we introduce the events
\[
\mathcal{M} = \mathcal{T}^\omega_\to(V_3) \cap \text{SG}(V_1 \cup V_2) \quad \mathcal{N} = \mathcal{T}^\omega_\leftarrow(V_1) \cap \text{SG}(V_2 \cup V_3)
\]
and observe that \( \text{SG}^{\omega}(\Lambda^{(n,+)}) = \mathcal{M} \cup \mathcal{N} \). In order to be able to use Proposition 4.7 we need some further events. The first one is the event \( \overline{\text{SG}}(V_2) \) which is best explained by Figure 4. It corresponds to requiring that inside the rectangle \( V_2 = R(\ell_m - 1, \ell_m) + \vec{e}_1 \) there exists a \( 1 \)-super-good square \( R(\ell_m-1, \ell_m-1) + x \) and the remaining rectangles in \( V_2 \setminus R(\ell_m-1, \ell_m-1) + x \) which sandwich \( R(\ell_m-1, \ell_m-1) + x \) are \( 1 \)-traversable. The formal Definition A.4 is left to Appendix A.

It is immediate to verify that for any \( \eta_2 \in \overline{\text{SG}}(V_2) \) there exist two vertical intervals \( I_1 = I_1(\eta_2) \subset V_1 \) and \( I_3 = I_3(\eta_2) \subset V_3 \) such that \( \eta_1 \neq 1 \) implies that \( \eta_1 \cdot \eta_2 \in \text{SG}(V_1 \cup V_2) \) and similarly if \( \eta_{I_3} \neq 1 \). Here, as before, \( \eta_i := \eta_{V_i} \). We then set
\[
\hat{C}_{1,2} := \{ \eta : \eta_2 \in \overline{\text{SG}}(V_2), \eta_1(\eta_2) \neq 1 \} \tag{3.16}
\]
Figure 4: The partition of $\Lambda^{(n,+)}$ into the rectangle $V_2$ and the two columns $V_1$ and $V_3$. Here we illustrate the event $SG(V_2)$: the grey region is a super-good rectangle of the type $\Lambda^{(n-2)}$, while the patterned rectangles are 1-traversable in the arrow directions. If there is at least one infection in $I_3$ then the rectangle $V_2 \cup V_3$ is super-good. Analogously for $I_1$.

and for $\eta \in \hat{C}_{1,2}$ we let
$$A^{\eta_1,\eta_3}_{3} = \{ \eta_{I_3(\eta_2)} \neq 1 \}.$$

By construction
$$\{ \eta : \eta_1 \cdot \eta_2 \in \hat{C}_{1,2} \text{ and } \eta_3 \in A^{\eta_1,\eta_3}_{3} \} \subset \mathcal{M} \cap \mathcal{N}.$$

We can finally apply Proposition 4.7 with parameters $C_{1,2} = SG(V_1 \cup V_2), C_{2,3} = SG(V_2 \cup V_3), A_1 = T_\omega(V_1)$ and $A_3 = T_\omega(V_3)$ to get that
$$\text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{M} \cup \mathcal{N}) \leq c T_{\text{aux}}^{(2)} \times \mu_{\Lambda^{(n,+)}}(\mathcal{A}_{3}^{\eta_1,\eta_3}) \times \mu_{\Lambda^{(n,+)}}(\mathcal{C}_{1,2}) \times \mu_{\Lambda^{(n,+)}}(\hat{C}_{1,2}).$$

for some universal constant $c > 0$, with
$$T_{\text{aux}}^{(2)} = \max_{\eta \in \hat{C}_{1,2}} \frac{\mu_{\Lambda^{(n,+)}}(\mathcal{A}_{3}^{\eta_1,\eta_3})}{\mu_{\Lambda^{(n,+)}}(\mathcal{A}_{3})} \times \frac{\mu_{\Lambda^{(n,+)}}(\mathcal{C}_{1,2})}{\mu_{\Lambda^{(n,+)}}(\hat{C}_{1,2})}.$$

Clearly, $\min_{\eta \in \hat{C}_{1,2}} \mu_{\Lambda^{(n,+)}}(A^{\eta_1,\eta_3}_{3}) \geq q$. Furthermore, in Lemma A.5 we will establish that $\mu_{V_1 \cup V_2}(\hat{C}_{1,2} \mid C_{1,2}) \geq q^{O(1)}$. All together
$$T_{\text{aux}}^{(2)} \leq q^{-O(1)}.$$

We now turn to examine the four averages w.r.t. $\mu_{\Lambda^{(n,+)}}(\cdot \mid \mathcal{M} \cup \mathcal{N})$ appearing in the r.h.s. of (3.17). Recall that $\mathcal{M} \cup \mathcal{N} = SG^\omega(\Lambda^{(n,+)})$. For the first and second average we can mimic what we did for (3.15) and get that they are both bounded from above by
$$\max_{\omega'} \gamma^{\omega'}(\Lambda^{(n)}) \sum_{x \in V_1 \cup V_2} \mu_{\Lambda^{(n,+)}}(c^{(n,+)}_{x}, \omega) \text{Var}_{x}(f \mid SG^\omega(\Lambda^{(n,+)})).$$

(3.19)
We will now explain how to upper bound the third average,

$$\mu_{A(n,+)} \left( \mathbb{1}_M \operatorname{Var}(f \mid A_3, \eta_1, \eta_2) \mid M \cup N \right),$$

the forth one being similar. We need to distinguish two cases, according to whether the boundary condition \( \omega \) has an infection on the column \( V_3 + \vec{e}_1 \) or not.

**Assume** \( \omega_{V_3 + \vec{e}_1} = 1 \)  
In this case \( A_3 = T_\rightarrow(V_3) = \Omega_{V_3 \setminus \{1\}} \) and Proposition 4.2 (1), gives that

$$\operatorname{Var}(f \mid A_3, \eta_1, \eta_2) = \operatorname{Var}_{V_3}(f \mid T_\rightarrow(V_3)) \leq q^{-O(1)} \sum_{x \in V_3} \mu_{V_3}(\tilde{c}_x \operatorname{Var}_x(f) \mid T_\rightarrow(V_3)), \quad (3.20)$$

with \( \tilde{c}_x(\eta) = 1 \) if \( x \) has at least one infected neighbour inside \( V_3 \) and \( \tilde{c}_x(\eta) = 0 \) otherwise. For \( x \in V_3 \) let

$$A_x = \mu_{A(n,+)} \left( \mathbb{1}_M \mu_{V_3}(\tilde{c}_x \operatorname{Var}_x(f) \mid T_\rightarrow(V_3)) \mid M \cup N \right).$$

Using \( M \subset \mathcal{SG}^\omega(\Lambda(n,\omega)) \), \( \mu_{A(n,+)}(T_\rightarrow(V_3) \mid \mathcal{SG}^\omega(\Lambda(n,\omega))) = 1 \), \( V_1 \cup V_2 = \Lambda(n) \) and the fact that the average of a conditional variance is not more than the total variance, we get

$$A_x = \mu_{A(n,+)} \left( \mathbb{1}_M \tilde{c}_x \operatorname{Var}_x(f) \mid \mathcal{SG}^\omega(\Lambda(n,\omega)) \right) \leq \mu_{A(n,+)} \left( \mathbb{1}_M \tilde{c}_x \operatorname{Var}_x(f) \mid \mathcal{SG}(\Lambda(n)) \right).$$

Next, we use Proposition 4.3 to write

$$\operatorname{Var}_{\{x \cup A\}}(f \mid \mathcal{SG}(\Lambda(n))) \leq \frac{2}{q} \left( \operatorname{Var}_{\Lambda(n)}(f \mid \mathcal{SG}(\Lambda(n))) + \mathbb{1}_{\eta_x = 0} \operatorname{Var}_x(f) \right).$$

Recalling (3.5), we get

$$\operatorname{Var}_{\Lambda(n)}(f \mid \mathcal{SG}(\Lambda(n))) \leq \gamma(\Lambda(n)) \sum_{y \in \Lambda(n)} \mu_{\Lambda(n)} \left( c_y^{A(n)} \mathbb{1} \operatorname{Var}_y(f) \mid \mathcal{SG}(\Lambda(n)) \right) \leq \gamma(\Lambda(n)) \sum_{y \in \Lambda(n)} \mu_{\Lambda(n)} \left( c_y^{A(n,\omega)} \operatorname{Var}_y(f) \mid \mathcal{SG}(\Lambda(n)) \right),$$

because \( c_y^{A(n),1} \leq c_y^{A(n,\omega)} \). Finally, observe that \( \mathbb{1}_{\eta_x = 0} \tilde{c}_x \leq c_x^{A(n,\omega)} \), because if \( x \in V_3 \) has an infected neighbour in \( V_3 \) (the constraint \( \tilde{c}_x \)) and \( x - \vec{e}_1 \in V_2 \) is also infected then \( x \) has two infected neighbours in \( \Lambda(n,+) \). Putting all together, we conclude that

$$A_x \leq \frac{2}{q} \gamma(\Lambda(n)) \mu_{A(n,+)} \left( c_x^{A(n,\omega)} \operatorname{Var}_x(f) \right) + \sum_{y \in \Lambda(n)} c_y^{A(n,\omega)} \operatorname{Var}_y(f) \mid \mathcal{SG}(\Lambda(n,\omega)).$$
In conclusion, using $|V_3| = \ell_m = q^{-O(1)}$ we get that when $\omega_{V_3 + \vec{e}_1} = 1$ the third average in the r.h.s. of (3.17) satisfies

$$\mu_{\Lambda^{(n,+)}} \left( \mathbb{1}_{\mathcal{M}} \text{Var}(f \mid A_3, \eta_1, \eta_2) \mid \mathcal{M} \cup \mathcal{N} \right) \leq \sum_{x \in V_3} A_x \leq \frac{\gamma_1^{\Lambda^{(n)}}}{q^{-O(1)}} \sum_{y \in \Lambda^{(n,+)}} \mu_{\Lambda^{(n,+)}} \left( c_{\omega y}^{\Lambda^{(n,+)}} \text{Var}_{\omega y}(f) \mid \mathcal{M} \cup \mathcal{N} \right). \quad (3.21)$$

Combining (3.17), (3.18), (3.19) and (3.21) (and its analogue for the forth average in the r.h.s. of (3.17)), we conclude the proof of Lemma 3.12 in this case.

Assume $\omega_{V_3 + \vec{e}_1} \neq 1$ In this case $T^\omega_{V_3}(V_3) = \Omega_{V_3}$, so $\text{Var}_{V_3}(f \mid T^\omega_{V_3}(V_3)) = \text{Var}_{V_3}(f)$. The proof is then identical to the previous one except for inequality (3.20) which now follows from Proposition 4.2 (2) with any site in $V_3$ neighbouring an infection of $\omega_{V_3 + \vec{e}_1}$ being unconstrained.

## 4 Constrained Poincaré inequalities

In this section we state and prove various Poincaré inequalities for the auxiliary chains that were instrumental for the proofs of Lemmas 3.11 and 3.12.

### 4.1 FA-1f-type Poincaré inequalities

Fix $\Lambda \subset \mathbb{Z}^2$ a connected set and let $\tilde{\Omega}_\Lambda = \Omega_\Lambda \setminus 1$. Given $x \in \Lambda$ let $N^\Lambda_x$ be the set of neighbours of $x$ in $\Lambda$ and let $N^\Lambda_x$ be the event that $N^\Lambda_x$ contains at least one infection. For any $z \in \Lambda$ consider the two Dirichlet forms

$$D^{FA-1f}_\Lambda(f) = \mu_\Lambda \left( \sum_{x \in \Lambda} \mathbb{1}_{N^\Lambda_x} \text{Var}_x(f) \mid \tilde{\Omega}_\Lambda \right), \quad f : \tilde{\Omega}_\Lambda \to \mathbb{R},$$

$$D^{FA-1f,z}_\Lambda(f) = \mu_\Lambda \left( \sum_{x \in \Lambda} \mathbb{1}_{N^\Lambda_x} \text{Var}_x(f) + \text{Var}_z(f) \right), \quad f : \Omega_\Lambda \to \mathbb{R}.$$

Remark 4.1. The alert reader will recognise the above expressions as the Dirichlet forms of the FA-1f process on $\tilde{\Omega}_\Lambda$ or on $\Omega_\Lambda$ with the site $z$ unconstrained.

Our first tool is a Poincaré inequality for these Dirichlet forms.

**Proposition 4.2.** Let $\Lambda$ be a connected subset of $\mathbb{Z}^2$ and let $z \in \Lambda$ be an arbitrary site. Then:

1. for any $f : \tilde{\Omega}_\Lambda \to \mathbb{R}$,

$$\text{Var}_\Lambda(f \mid \tilde{\Omega}_\Lambda) \leq \frac{1}{q^{O(1)}} D^{FA-1f}_\Lambda(f); \quad (4.1)$$
(2) for any \( f : \Omega_\Lambda \to \mathbb{R} \),
\[
\text{Var}_\Lambda(f) \leq \frac{1}{qO(1)} \mathcal{D}^{\text{FA}-1f,z}_\Lambda(f),
\]
where the constants in the \( O(1) \) do not depend on \( z \) or \( \Lambda \).

**Proof.** Inequality (4.1) is proved in [7, Theorem 6.1]. In order to prove (4.2), consider the auxiliary Dirichlet form
\[
\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\mathbb{1}_{\tilde{\Omega}_\Lambda} \text{Var}_\Lambda(f | \tilde{\Omega}_\Lambda)).
\]

The corresponding ergodic, continuous time Markov chain on \( \Omega_\Lambda \), reversible w.r.t. \( \mu_\Lambda \), updates the state of \( z \) at rate 1 and, if \( \omega \in \tilde{\Omega}_\Lambda \), it updates the entire configuration w.r.t. \( \pi(\cdot | \tilde{\Omega}_\Lambda) \). Since the chain enters \( \tilde{\Omega}_\Lambda \) at rate \( q \) (by flipping \( \omega_z \)), a simple coupling argument shows that its relaxation time is \( O(1/q) \). Hence,
\[
\text{Var}_\Lambda(f) \leq O(1)/q \left( \mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\mathbb{1}_{\tilde{\Omega}_\Lambda} \text{Var}_\Lambda(f | \tilde{\Omega}_\Lambda)) \right)
\leq \frac{1}{qO(1)} \left( \mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\tilde{\Omega}_\Lambda) \mathcal{D}^{\text{FA}-1f}_\Lambda(f) \right),
\]
where the second inequality follows from (4.1). We may then conclude by observing that
\[
\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\tilde{\Omega}_\Lambda) \mathcal{D}^{\text{FA}-1f}_\Lambda(f) \leq 2 \mathcal{D}^{\text{FA}-1f,z}_\Lambda(f). \quad \square
\]

Our second tool is a general constrained Poincaré inequality for two independent random variables.

**Proposition 4.3** (See [24, Lemma 3.10]). Let \( X_1, X_2 \) be two independent random variables taking values in two finite sets \( X_1, X_2 \) respectively. Let also \( \mathcal{H} \subset X_1 \) with \( \mathbb{P}(X_1 \in \mathcal{H}) > 0 \). Then for any \( f : X_1 \times X_2 \to \mathbb{R} \) it holds
\[
\text{Var}(f) \leq 2 \mathbb{P}(X_1 \in \mathcal{H})^{-1} \mathbb{E}(\text{Var}_1(f) + 1_{\{X_1 \in \mathcal{H}\}} \text{Var}_2(f)).
\]
with \( \text{Var}_i(f) = \text{Var}(f(X_1, X_2) | X_i) \).

### 4.2 Constrained block chains

In this section we define two auxiliary constrained reversible Markov chains and give an upper bound for the corresponding Poincaré constants (Lemmas 4.5 and 4.7).

Let \((\Omega_i, \pi_i)_{i=1}^3 \) be finite probability spaces and let \((\Omega, \pi)\) denote the associated product space. For \( \omega \in \Omega \) we write \( \omega_i \in \Omega_i \) for its \( i \)-th coordinate and we assume for simplicity that \( \pi_i(\omega_i) > 0 \) for each \( \omega_i \). Fix \( \mathcal{A}_3 \subset \Omega_3 \) and for each \( \omega_3 \in \mathcal{A}_3 \) consider an event \( B^\omega_{1,2} \subset \Omega_1 \times \Omega_2 \). Analogously, fix \( \mathcal{A}_1 \subset \Omega_1 \) and for each \( \omega_1 \in \mathcal{A}_1 \) consider an event \( B^\omega_{2,3} \subset \Omega_2 \times \Omega_3 \). We then set
\[
\mathcal{H} = \{ \omega : \omega_3 \in \mathcal{A}_3 \text{ and } (\omega_1, \omega_2) \in B^\omega_{1,2} \}, \quad \mathcal{K} = \{ \omega : \omega_1 \in \mathcal{A}_1 \text{ and } (\omega_2, \omega_3) \in B^\omega_{2,3} \}
\]
and let for any \( f : \mathcal{H} \cup \mathcal{K} \rightarrow \mathbb{R} \)
\[
D^{(1)}_{\text{aux}}(f) = \pi(1_{\mathcal{H}} \text{Var}_{\pi}(f \mid \mathcal{H}, \omega_3) + 1_{\mathcal{K}} \text{Var}_{\pi}(f \mid \mathcal{K}, \omega_1) \mid \mathcal{H} \cup \mathcal{K}).
\]

**Remark 4.4.** It is easy to check that \( D^{(1)}_{\text{aux}}(f) \) is the Dirichlet form of the continuous time Markov chain on \( \mathcal{H} \cup \mathcal{K} \) in which if \( \omega \in \mathcal{H} \) the pair \((\omega_1, \omega_2)\) is resampled with rate one from \( \pi_1 \otimes \pi_2(\cdot \mid B^{\omega_1}_{1,2}) \) and if \( \omega \in \mathcal{K} \) the pair \((\omega_2, \omega_3)\) is resampled with rate one from \( \pi_2 \otimes \pi_3(\cdot \mid B^{\omega_3}_{2,3}) \). This chain is reversible w.r.t. \( \pi(\cdot \mid \mathcal{H} \cup \mathcal{K}) \) and its constraints, contrary to what happens for general KCM, depend on the to-be-updated variables.

**Proposition 4.5.** There exists a universal constant \( c \) such that the following holds. Suppose that there exist two events \( \mathcal{F}_{1,2}, \mathcal{F}_{2,3} \) such that
\[
\{ \omega : \omega_3 \in \mathcal{A}_3 \text{ and } (\omega_1, \omega_2) \in \mathcal{F}_{1,2} \} \subset \mathcal{H} \cap \mathcal{K},
\]
\[
\{ \omega : \omega_1 \in \mathcal{A}_1 \text{ and } (\omega_2, \omega_3) \in \mathcal{F}_{2,3} \} \subset \mathcal{H} \cap \mathcal{K}\]  
and let
\[
T^{(1)}_{\text{aux}} = \max_{\omega_3 \in \mathcal{A}_3} \left( \frac{\pi(B^{\omega_3}_{1,2})}{\pi(\mathcal{F}_{1,2})} \right)^2 \max_{\omega_1 \in \mathcal{A}_1} \frac{\pi(B^{\omega_1}_{2,3})}{\pi(\mathcal{F}_{2,3})}.
\]
Then, for all \( f : \mathcal{H} \cup \mathcal{K} \rightarrow \mathbb{R} \),
\[
\text{Var}_{\pi}(f \mid \mathcal{H} \cup \mathcal{K}) \leq cT^{(1)}_{\text{aux}}D^{(1)}_{\text{aux}}(f).
\]

**Proof.** Consider the Markov chain \((\omega(t))_{t \geq 0}\) determined by the Dirichlet form \( D^{(1)}_{\text{aux}} \) as described in Remark 4.4. Given two arbitrary initial conditions \( \omega(0) \) and \( \omega'(0) \) we will construct a coupling of the two chains such that with probability \( \Omega(1) \) we have \( \omega(t) = \omega'(t) \) for any \( t > T^{(1)}_{\text{aux}} \). Standard arguments [27] then prove that the mixing time of the chain is \( O(T^{(1)}_{\text{aux}}) \) and the conclusion of the proposition follows. To construct our coupling we use the following representation of the Markov chain.

We are given two independent Poisson clocks with rate one and the chain transitions occur only at the clock rings. Suppose that the first clock rings. If the current configuration \( \omega \) does not belong to \( \mathcal{H} \) the ring is ignored. Otherwise, a Bernoulli variable \( \xi \) with probability of success \( \pi(F_{1,2} \mid B^{\omega_1}_{1,2}) \) is sampled. If \( \xi = 1 \), then the pair \((\omega_1, \omega_2)\) is resampled w.r.t. the measure \( \pi(\cdot \mid F_{1,2}, B^{\omega_1}_{1,2}) \), while if \( \xi = 0 \), then \((\omega_1, \omega_2)\) is resampled w.r.t. the measure \( \pi(\cdot \mid F_{1,2}, B^{\omega_1}_{1,2}) \). Clearly, in doing so the couple \((\omega_1, \omega_2)\) is resampled w.r.t. \( \pi(\cdot \mid F^{\omega_1}_{1,2}) \). Similarly if the second clock rings but with \( \mathcal{H} \), \((\omega_1, \omega_2)\), \( F_{1,2} \) and \( B^{\omega_1}_{1,2} \) replaced by \( \mathcal{K} \), \((\omega_2, \omega_3)\), \( F_{2,3} \) and \( B^{\omega_3}_{2,3} \) respectively. It is important to notice that \( \pi(\cdot \mid F_{1,2}, B^{\omega_1}_{1,2}) = \pi(\cdot \mid F_{1,2}) \) for all \( \omega_3 \in \mathcal{A}_3 \), as, by assumption, \( F_{1,2} \subset \bigcap_{\omega_3 \in \mathcal{A}_3} B^{\omega_3}_{1,2} \). Similarly, \( \pi(\cdot \mid F_{2,3}, B^{\omega_3}_{2,3}) = \pi(\cdot \mid F_{2,3}) \) for all \( \omega_1 \in \mathcal{A}_1 \).

In our coupling both chains use the same clocks. Suppose that the first clock rings and that the current pair of configurations is \( (\omega, \omega') \). Assume also that at least one of them, say \( \omega \), is in \( \mathcal{H} \) (otherwise, both remain unchanged). In order to construct the coupling update we proceed as follows.
• If $ω′ ∉ H$ then $ω$ is updated as described above, while $ω′$ stays still.

• If $ω′ ∈ H$ we first maximally couple the two Bernoulli variables $ξ, ξ′$ corresponding to $ω, ω′$ respectively. Then:
  
  – if $ξ = ξ′ = 1$, we update both $(ω_1, ω_2)$ and $(ω′_1, ω′_2)$ to the same couple $(η_1, η_2) ∈ F_{1,2}$ with probability $π((η_1, η_2) | F_{1,2})$;
  
  – otherwise we resample $(ω_1, ω_2)$ and $(ω′_1, ω′_2)$ independently from their respective law given $ξ, ξ′$.

Similarly if the ring comes from the second clock. The final coupling is then equal to the Markov chain on $Ω × Ω$ with the transition rates described above. Suppose now that there are three consecutive rings occurring at times $t_1 < t_2 < t_3$ such that:

• the first and last ring come from the first clock while the second ring comes from the second clock, and

• the sampling of the Bernoulli variables (if any) at times $t_1, t_2$ and $t_3$ all produce the value one.

Then we claim that at time $t_3$ the two copies are coupled.

To prove the claim, we begin by observing that after the first update at $t_1$ both copies of the coupled chain belong to $K$. Here we use (4.3). Indeed, if the first update is successful for $ω$ (i.e. $ω ∈ H$) then the updated configuration belongs to $F_{1,2} × {ω_3} ⊂ K$, because of our assumption $ξ = 1$. If, on the contrary, the first update fails (i.e. $ω ∉ H$) then $ω ∈ K \setminus H$ before and after the update. The same applies to $ω′$.

Next, using again the assumption on the Bernoulli variables together with the previous observation, we get that after the second ring the new pair of current configurations agree on the second and third coordinate. Moreover both copies belong to $H$ thanks to (4.4). Finally, after the third ring the two copies couple on the first and second coordinates using again the assumption on the outcome for the Bernoulli variables.

In order to conclude the proof of the proposition it is enough to observe that for any given time interval $Δ$ of length one the probability that there exist $t_1 < t_2 < t_3$ in $Δ$ satisfying the requirements of the claim is bounded from below by

$$c \min_{ω_3 ∈ A_3} \pi(F_{1,2} | B_{1,2}^{ω_3})^2 \min_{ω_1 ∈ A_1} \pi(F_{2,3} | B_{2,3}^{ω_1}),$$

for some constant $c > 0$. □

In the same setting consider two other events $C_{1,2} ⊂ Ω_1 ⊗ Ω_2, C_{2,3} ⊂ Ω_2 ⊗ Ω_3$ and let

$$M = A_3 ∩ C_{1,2}, \quad N = A_1 ∩ C_{2,3},$$
The Dirichlet form of our second Markov chain on $\mathcal{M} \cup \mathcal{N}$ is then
\[
\mathcal{D}_{\text{aux}}^{(2)}(f) = \pi \left( 1_{\mathcal{M}} \text{Var}(f \mid \mathcal{C}_{1,2}, \omega_3) + 1_{\mathcal{M}} \text{Var}(f \mid \mathcal{A}_3, \omega_1, \omega_2) \right) \\
+ 1_{\mathcal{N}} \text{Var}(f \mid \mathcal{C}_{2,3}, \omega_1) + 1_{\mathcal{N}} \text{Var}(f \mid \mathcal{A}_1, \omega_2, \omega_3) \mid \mathcal{M} \cup \mathcal{N}. \tag{4.5}
\]

**Remark 4.6.** Similarly to the first case, the continuous time chain defined by (4.5) is reversible w.r.t. $\pi(\cdot \mid \mathcal{M} \cup \mathcal{N})$ and it can be described as follows. If $\omega \in \mathcal{M}$ then with rate one $(\omega_1, \omega_2)$ is resampled w.r.t. $\pi_1 \otimes \pi_2(\cdot \mid \mathcal{C}_{1,2})$ and, independently at unit rate, $\omega_3$ is resampled w.r.t. $\pi_3(\cdot \mid \mathcal{A}_3)$. Similarly, independently from the previous updates at rate one, if $\omega \in \mathcal{N}$ then $(\omega_2, \omega_3)$ is resampled w.r.t. $\pi_2 \otimes \pi_3(\cdot \mid \mathcal{C}_{2,3})$ and, independently, $\omega_1$ is resampled from $\pi_1(\cdot \mid \mathcal{A}_1)$.

**Proposition 4.7.** There exists a universal constant $c$ such that the following holds. Suppose that there exist an event $\hat{\mathcal{C}}_{1,2} \subset \mathcal{C}_{1,2}$ and a collection $(\mathcal{A}_3^{\omega_1, \omega_2})_{(\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2}}$ of subsets of $\mathcal{A}_3$ such that
\[
\{ \omega : (\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2} \text{ and } \omega_3 \in \mathcal{A}_3^{\omega_1, \omega_2} \} \subset \mathcal{M} \cap \mathcal{N}, \tag{4.6}
\]
and let
\[
T_{\text{aux}}^{(2)} = \max_{(\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2}} \frac{\pi(\mathcal{A}_3)}{\pi(\mathcal{A}_3^{\omega_1, \omega_2})} \times \frac{\pi(\mathcal{C}_{1,2})}{\pi(\hat{\mathcal{C}}_{1,2})}.
\]
Then there exists $c > 0$ such that for all $f : \mathcal{M} \cup \mathcal{N} \to \mathbb{R}$,
\[
\text{Var}(f \mid \mathcal{M} \cup \mathcal{N}) \leq c T_{\text{aux}}^{(2)} \mathcal{D}_{\text{aux}}^{(2)}(f).
\]

**Proof.** We proceed as in the proof of Proposition 4.5 with the following representation for the Markov chain. We are given four independent Poisson clocks of rate one and each clock comes equipped with a collection of i.i.d. random variables. The four independent collections, the first being for the first clock etc, are
\[
\left( (\omega_1^{(i)}, \omega_2^{(i)}) \right)_{i=1}^{\infty}, \quad \left( \eta_3^{(i)} \right)_{i=1}^{\infty}, \quad \left( (\omega_2^{(i)}, \omega_3^{(i)}) \right)_{i=1}^{\infty}, \quad \left( \eta_1^{(i)} \right)_{i=1}^{\infty},
\]
where the laws of the collections are $\pi_1 \otimes \pi_2(\cdot \mid \mathcal{C}_{1,2})$, $\pi_3(\cdot \mid \mathcal{A}_3)$, $\pi_2 \otimes \pi_3(\cdot \mid \mathcal{C}_{2,3})$ and $\pi_1(\cdot \mid \mathcal{A}_1)$ respectively.

At each ring of the first and second clocks the configuration is updated with the variables from the corresponding collection iff $\omega \in \mathcal{M}$. Similarly for the third and fourth clocks with $\mathcal{N}$. In order to couple different initial conditions, we use the same collections of clock rings and update configurations.

Suppose now that there are four consecutive rings $t_1 < t_2 < t_3 < t_4$, coming from the first, second, third and forth clocks in that order, such that:

- at $t_1$ the proposed update $(\eta_1, \eta_2)$ of the first two coordinates belongs to $\hat{\mathcal{C}}_{1,2}$, and
at $t_2$ the proposed update $\eta_3$ of the third coordinate belongs to $A_{3}(\eta_1, \eta_2)$.

We then claim that after $t_4$ all initial conditions $\omega$ are coupled. To prove this, we first observe that after the second ring each chain belongs to $N$. Indeed, if $\omega \notin M$, then the first two proposed updates are ignored and the configuration $\omega \in N \setminus M$. If, on the contrary, $\omega \in M$, then both updates are successful and the configuration is updated to $(\eta_1, \eta_2, \eta_3) \in \hat{C}_{1, 2} \times A_{3}(\eta_1, \eta_2) \subset M \cap N$ by (4.6).

Since after $t_2$ the state of the chain is necessarily in $N$, the third and fourth updates to states $(\eta'_2, \eta'_3)$ and $\eta'_1$ respectively are both successful and thus any initial condition leads to the state $(\eta'_1, \eta'_2, \eta'_3)$ after $t_4$, which proves the claim. The proof is then completed as in Proposition 4.5.

## 5 Proof of Theorem 1: upper bound

As already announced we will only discuss the two dimensional case. The starting point is as in [24, Section 5]. Let $\kappa$ be a large enough constant, let

$$t_* = \exp \left( \frac{\pi^2}{9q} (1 + \kappa \sqrt{q} \log^{3}(1/q)) \right)$$

and let $T = \lfloor \exp(\log^4(1/q)/q) \rfloor$. Then

$$E_{\mu}(\tau_0) = \int_0^{+\infty} ds \, P_{\mu}(\tau_0 > s)$$

$$= \int_0^{t_*} ds \, P_{\mu}(\tau_0 > s) + \int_{t_*}^{T} ds \, P_{\mu}(\tau_0 > s) + \int_{T}^{+\infty} ds \, P_{\mu}(\tau_0 > s)$$

$$\leq t_* + T P_{\mu}(\tau_0 > t_*) + \int_{T}^{+\infty} ds \, P_{\mu}(\tau_0 > s).$$

The term $t_*$ has exactly the form required in (1.5). The last term in the r.h.s. above tends to zero as $q \to 0$ if $c$ is large enough. Indeed, using [30, Theorem 2] (see also [24, 30]) we have that for any $s \geq 0$ $P_{\mu}(\tau_0 > s) \leq e^{-s \lambda_0}$ with $\lambda_0 \geq e^{-\Omega((\log q)^3/q)}$. In conclusion, the proof of (1.5) boils down to proving

$$\lim_{q \to 0} T P_{\mu}(\tau_0 > t_*) = 0. \quad (5.2)$$

The key ingredients to prove (5.2) are Propositions 3.7 and 3.9 and Proposition 5.1 below. The latter is a Poincaré inequality for an auxiliary process, the generalised coalescing and branching symmetric exclusion process (g-CBSEP), preliminarily studied in [22]. Once we have these key ingredients, the strategy to prove (5.2) is similar to the one in [24, Section 5]. In particular, for the first part of the proof (Section 5.2) we will omit most of the details and refer to [24, Section 5] for a more detailed explanation.
5.1 The $g$-CBSEP process

Given a finite connected graph $G = (V, E)$ and a finite probability space $(S, \pi)$, assign a variable $\sigma_x \in S$ to each vertex $x \in V$ and write $\sigma = (\sigma_x)_{x \in V}$ and $\pi_G(\sigma) = \prod_x \pi(\sigma_x)$. Fix also a bipartition $S_1 \sqcup S_0 = S$ such that $\pi(S_1) > 0$ and define the projection $\varphi : S^V \to \{0, 1\}^V$ by $\varphi(\sigma) = (1_{\{\sigma_x \in S_1\}})_{x \in V}$. We will say that a vertex $x$ is occupied by a particle if $\sigma_x \in S_1$ and we will write $\Omega^+_G \subset \Omega_G = S^V$ for the set of configurations $\sigma$ with at least one particle. Finally, for any edge $e = \{x, y\} \in E$ let $E_e$ be the event that there exists a particle at $x$ or at $y$.

The $g$-CBSEP continuous time Markov chain on $\Omega^+_G$ with parameters $(S, S_1, \pi)$ runs as follows. The state $\{\sigma_x, \sigma_y\}$ of every edge $e = \{x, y\}$ for which $E_e$ holds is resampled with rate one (independently of all the other edges) w.r.t. $\pi_x \otimes \pi_y (\cdot \mid E_e)$. Thus, an edge containing exactly one particle can swap the position of the particle between its endpoints or can create a new particle at the empty endpoint (a branching transition). An edge with two particles can kill one of them (a coalescing transition) with equal probability or keep them untouched. It is immediate to check that $g$-CBSEP is ergodic on $\Omega^+_G$ with reversible stationary measure $\pi^+_G := \pi_G (\cdot \mid \Omega^+_G)$ and that its Dirichlet form $\mathcal{D}^{g\text{-CBSEP}}(f)$ for $f : \Omega^+_G \to \mathbb{R}$, takes the form

$$\mathcal{D}^{g\text{-CBSEP}}(f) = \sum_{e \in E} \pi^+_G (1_{E_e} \operatorname{Var}_e(f) \mid E_e)),$$

where $\operatorname{Var}_e(f \mid E_e)$ is the variance w.r.t. $\sigma_x, \sigma_y$ conditioned on $E_e$ if $e = \{x, y\}$. Let now $T_{rel}^{g\text{-CBSEP}}$ be the relaxation time of $g$-CBSEP on $\Omega^+_G$ defined as the best constant $C$ in the Poincaré inequality

$$\operatorname{Var}_{\pi^+_G}(f) \leq C \mathcal{D}^{g\text{-CBSEP}}(f).$$

In the above setting the main result needed to prove (5.2) is as follows. For any positive integers $d$ and $L$ set $n = L^d$ and let $\mathbb{Z}_L = \{0, 1, \ldots, L - 1\}$ be the set of remainders modulo $L$. The $d$-dimensional discrete torus with $n$ vertices, $\mathbb{T}^d_n$ in the sequel, is the set $\mathbb{Z}_L^d$ endowed with the graph structure inherited from $\mathbb{Z}_L^d$.

**Proposition 5.1.** Let $G = \mathbb{T}^d_n$ and assume that $S$, $S_1$ and $\pi$ depend on $n$ in such a way that $\lim_{n \to \infty} \pi(S_1) = 0$ and $\lim_{n \to \infty} n \pi(S_1) = +\infty$. Then, as $n \to \infty$,

$$T_{rel}^{g\text{-CBSEP}} \leq O(\pi(S_1)^{-1} \log (\pi(S_1)^{-1})).$$

5.2 Transforming (5.2) into a Poincaré inequality

Using standard “finite speed of propagation” bounds (see [24, Section 5.2.1]) it is enough to prove (5.2) for FA-2f on the discrete torus $\mathbb{T}^d_n$ with linear size $\sqrt{n} = 2T$. Next we fix a small positive constant $\delta < 1/2$ and choose $N_\delta = N - \lfloor \log(1/\delta)/\sqrt{q} \rfloor$ where
\[ N = \lceil \frac{8 \log(1/q)}{\sqrt{q}} \rceil \] is the final scale in the droplet construction (see (3.3)). With this choice \( \ell_N = \delta_N = \delta/q^{17/2+o(1)} \) (cf. (3.1)) and w.l.o.g. we assume that \( \ell_N \) divides \( 2T \).

We partition the torus \( \mathbb{T}_n^2 \) into \( M = n/\ell_{N_S}^2 \) equal mesoscopic disjoint boxes \( (Q_j)_{j=1}^M \), where each \( Q_j \) is a suitable lattice translation by a vector in \( \mathbb{T}_n^2 \) of the box \( Q = [\ell_N]^2 = \Lambda^{(2N_S)} \) (see (3.2)). The labels of the boxes can be thought of as belonging to the new torus \( \mathbb{T}_M^2 \) and we assume that \( Q_i, Q_j \) are neighbouring boxes in \( \mathbb{T}_n^2 \) iff \( i, j \) are neighbouring sites in \( \mathbb{T}_M^2 \).

In \( \Omega_{\mathbb{T}_n^2} \) we consider the event
\[ E = \bigcup_{j \in \mathbb{T}_M^2} SG_j \cap \bigcap_{i \in \mathbb{T}_M^2} G_i \]
where \( SG_i \) is the event that \( Q_i \) is super-good (see Definition 3.6) and \( G_i \) is the event that any row and any column (of lattice sites) of \( Q_i \) contains an infected site.

In order to apply the same strategy as [25, Section 5] it is crucial to have that the “environment” characterised by \( E \) is so likely that (cf. [25, (28)])
\[ \lim_{q \to 0} \mu(E^c)T^3t_* = 0. \quad (5.3) \]
It is such a requirement that guided us in the choice of the side length \( \ell_{N_S} \) of the mesoscopic boxes. Using Proposition 3.7 together with trivial bounds on the probability that a row/column of a box \( Q_i \) does not contain an infection it is immediate to verify (5.3). An easy consequence of (5.3) (cf. [24, Section 5.2.3]) is that it is sufficient to prove (5.2) for the stationary FA-2f in \( \mathbb{T}_n^2 \) restricted to \( E \). For the latter process we follow the standard “variational” approach (see [24, Section 5.2.4]) and the overall conclusion is that
\[ T \mathbb{P}_\mu(\tau_0 \geq t_*) \leq T e^{-t_*\lambda_F} + o(1), \]
with
\[ \lambda_F \geq \inf_f q \frac{\mathcal{D}_{\mathbb{T}_n^2}(f)}{\operatorname{Var}_{\mathbb{T}_n^2}(f|E)}, \]
where \( \mathcal{D}_{\mathbb{T}_n^2}(f) \) is the Dirichlet form of FA-2f on the torus \( \mathbb{T}_n^2 \) and the supremum is taken over all \( f : E \to \mathbb{R} \).

### 5.3 Bounding \( \lambda_F \) from below

The last and most important step is to prove that
\[ \lambda_F \geq e^{-O(\log^3(1/q)/\sqrt{q})} \rho_D, \quad (5.4) \]
where \( \rho_D \geq \exp(-\frac{\pi^2}{36q}(1 + O(\sqrt{q}\log^2(1/q)))) \) is the probability that a box \([\ell_N]^2 \) is super-good (cf. Proposition 3.7). Once (5.4) is established, the proof of (5.2) is complete because \( t_*\lambda_F \) diverges rapidly enough as \( q \to 0 \) if the constant \( \kappa \) in the definition (5.1) of \( t_* \) is chosen large enough.

The proof of (5.4) is crucially based on Propositions 3.9 and 5.1 and is divided into two parts.
We next compare the sum appearing in the r.h.s. of (5.1) to the FA-2f Dirichlet form \( D_{\mathbb{T}_n^2} (f | \mathcal{E}) \) as \( f \) ranges over the functions in \( \mathcal{S}_G \). Using Proposition 5.1 we conclude that

\[
\var_{\pi_G^+} (f_G) = \var_{\mathbb{T}_n^2} (f | \mathcal{E}), \\
D^{g \text{-CBSEP}} (f_G) = \sum_{i \sim j} \mu_{\mathbb{T}_n^2} (1_{S_{G_{i,j}}} \var_{Q_i \cup Q_j} (f | S_{G_{i,j}}) | \mathcal{E}),
\]

where \( S_{G_{i,j}} \) is a shorthand notation for the event \( (S_{G_i} \cup S_{G_j}) \cap S_i \cap S_j \) and \( \sum_{i \sim j} \) denotes the sum over pairs, each counted once, of adjacent boxes. Using Proposition 5.1 we conclude that

\[
\var_{\mathbb{T}_n^2} (f | \mathcal{E}) = \var_{\pi_G^+} (f_G) \leq O(\varpi (S_1)^{-1} \log (1/\varpi (S_1))) \var D^{g \text{-CBSEP}} (f_G)
\]

\[
= \exp \left( \frac{\pi^2}{9q} (1 + O(\sqrt{q} \log^2 (1/q))) \right) \times \sum_{i \sim j} \mu_{\mathbb{T}_n^2} (1_{S_{G_{i,j}}} \var_{Q_i \cup Q_j} (f | S_{G_{i,j}}) | \mathcal{E})
\]

(5.5)

Application of Proposition 3.9 We next compare the sum appearing in the r.h.s. of (5.5) to the FA-2f Dirichlet form \( D_{\mathbb{T}_n^2} (f | \mathcal{E}) \) and prove that the “comparison cost” is at most \( \exp \left( O(\log^3 (1/q)/\sqrt{q}) \right) \), so sub-leading w.r.t. the main term \( \exp (\frac{\pi^2}{9q}) \) above.

Lemma 5.2.

\[
\sum_{i \sim j} \mu_{\mathbb{T}_n^2} (1_{S_{G_{i,j}}} \var_{Q_i \cup Q_j} (f | S_{G_{i,j}}) | \mathcal{E}) \leq e^{O(\log^3 (1/q)/\sqrt{q})} \sum_{x \in \mathbb{T}_n^2} \mu_{\mathbb{T}_n^2} (c_{x}^{\mathbb{T}_n^2} \var_{x} (f)),
\]

where \( c_{x}^{\mathbb{T}_n^2} \) is the FA-2f constraint at \( x \) for the torus \( \mathbb{T}_n^2 \) (see (3.6)).

Remark 5.3. As it will be clear from the proof, we actually prove a stronger statement, namely the constraint \( c_{x}^{\mathbb{T}_n^2} \) above will appear multiplied by the indicator that \( x \) belongs to a droplet. Whileos many choices of \( f \) the presence of this additional constraint may completely change the average \( \mu_{\mathbb{T}_n^2} (c_{x}^{\mathbb{T}_n^2} \var_{x} (f)) \), it is possible to exhibit choices of \( f \), for which \( \mathbb{I}_{\{x \text{ belongs to a "droplet"}\}} c_{x}^{\mathbb{T}_n^2} \var_{x} (f) \simeq c_{x}^{\mathbb{T}_n^2} \var_{x} (f) \).
Figure 5: Illustration of Observation 5.5. The shaded square of shape $\Lambda^{(2N_\delta)}$ is $SG$ and the arrows indicate the presence of an infection in each row/column, as guaranteed by $G(\Lambda_{i,j})$ with $\Lambda_{i,j}$ being the larger square of shape $\Lambda^{(2N)}$. Observation 5.5 asserts that these events combined imply $SG(\Lambda_{i,j})$ (see Figure 2).

**Proof of Lemma 5.2.** The lemma follows by summing the bound from Claim 5.4. □

**Claim 5.4.** Fix two adjacent boxes $Q_i, Q_j$ and let $\Lambda_{i,j} \supset Q_i \cup Q_j$ be a translate of the box $\Lambda^{(2N)}$. Then

$$
\mu_{T_n^q}(\mathbb{1}_{SG_{i,j}} \text{Var}_{Q_i\cup Q_j}(f \mid SG_{i,j}) \mid \mathcal{E}) \leq e^{O\left(\log^2(1/q)/\sqrt{n}\right)} \sum_{x \in \Lambda_{i,j}} \mu_{T_n^q}(\mathbb{1}_{SG(\Lambda_{i,j})} c_{T_n^q} \text{Var}_x(f)).
$$

**Proof Claim 5.4.** Let $\mathcal{G} = \bigcap_{k \in T_n^q} \mathcal{G}_k \supset \mathcal{E}$ and recall that $\mu(\mathcal{E}) = 1 - o(1)$. Next, let $\mathcal{G}(\Lambda_{i,j})$ be the event that any $\ell_{N_\delta}$ lattice sites contained in $\Lambda_{i,j}$ forming either a row or a column of some $Q_k$ contain an infection.

**Observation 5.5.** The event $SG_{i,j} \cap \mathcal{G}(\Lambda_{i,j})$ implies the event $SG(\Lambda_{i,j})$.

The formal proof of this observation, as illustrated in Figure 5 is left to the reader.

Write $\rho_{i,j}$ for $\mu(SG_{i,j} \mid \mathcal{G})$ and observe that the term $\text{Var}_{Q_i\cup Q_j}(\cdot)$ does not depend on the variables $\omega_{Q_i}, \omega_{Q_j}$. This fact together with the Cauchy-Schwarz inequality allows us to write

$$
\mu_{T_n^q}(\mathbb{1}_{SG_{i,j}} \text{Var}_{Q_i\cup Q_j}(f \mid SG_{i,j}) \mid \mathcal{E})
= (1 + o(1)) \mu_{T_n^q}(\mathbb{1}_{SG_{i,j}} \text{Var}_{Q_i\cup Q_j}(f \mid SG_{i,j}) \mid \mathcal{G})
= (1 + o(1)) \rho_{i,j} \mu_{T_n^q}(\text{Var}_{\Lambda_{i,j}}(f \mid SG_{i,j}) \mid \mathcal{G}(\Lambda_{i,j})) \mid \mathcal{G})
\leq (1 + o(1)) \rho_{i,j} \mu_{T_n^q}(\text{Var}_{\Lambda_{i,j}}(f \mid SG_{i,j} \cap \mathcal{G}(\Lambda_{i,j})) \mid \mathcal{G})
\leq (1 + o(1)) \rho_{i,j} \frac{\mu_{T_n^q}(SG(\Lambda_{i,j}) \mid \mathcal{G})}{\mu_{T_n^q}(SG_{i,j} \cap \mathcal{G}(\Lambda_{i,j}) \mid \mathcal{G})} \mu_{T_n^q}(\text{Var}_{\Lambda_{i,j}}(f \mid SG(\Lambda_{i,j})) \mid \mathcal{G})
= (1 + o(1)) \mu_{T_n^q}(SG(\Lambda_{i,j}) \mid \mathcal{G}) \mu_{T_n^q}(\text{Var}_{\Lambda_{i,j}}(f \mid SG(\Lambda_{i,j})) \mid \mathcal{G}).
$$

In the last inequality we used Observation 5.5 together with the inequality $\text{Var}(X \mid A) \leq \mathbb{P}(B)/\mathbb{P}(A) \text{Var}(X \mid B)$ valid for any (finite) random variable $X$ and any two events $A \subset...
By applying now Proposition 3.9 to the term $\text{Var}_{\Lambda_{i,j}}(f | S\mathcal{G}(\Lambda_{i,j}))$ and, using $c_{x}^{\Lambda_{i,j}} \lesssim c_{x}^{\mathbb{N}}$, we conclude that

$$
\mu_{T_{2}^{n}}(\mathbb{I}_{S\mathcal{G}_{i,j}} \text{Var}_{Q \cup Q_{j}}(f | S\mathcal{G}_{i,j}) | \mathcal{E}) \\
\leq e^{O(\log^{3}(1/q)/\sqrt{q})} \mu_{T_{2}^{n}}(S\mathcal{G}(\Lambda_{i,j}) | \mathcal{G}) \times \sum_{x \in \Lambda_{i,j}} \mu_{T_{2}^{n}}(c_{x}^{\Lambda_{i,j}} \text{Var}_{x}(f) | S\mathcal{G}(\Lambda_{i,j}) | \mathcal{G}) \\
= e^{O(\log^{3}(1/q)/\sqrt{q})} \sum_{x \in \Lambda_{i,j}} \mu_{T_{2}^{n}}(\mathbb{I}_{S\mathcal{G}(\Lambda_{i,j})} c_{x}^{\mathbb{N}} \text{Var}_{x}(f) | \mathcal{G}).
$$

The proof is complete because $\mu_{T_{2}^{n}}(\mathcal{G}) = 1 - o(1)$.

## A Probability of super-good events

In this appendix we prove Proposition 3.7 and we gather several more technical and relatively standard bootstrap percolation estimates on the probability of super-good events used in Section 3.

For $z > 0$ we define

$$
g(z) = -\log \left( \beta(1 - e^{-z}) \right),
$$

where $\beta(u) = (u + \sqrt{u(4 - 3u)})/2$. It is known [26, Proposition 5(ii)] that $\int_{0}^{\infty} g(z) \, dz = \pi^{2}/18$. We next recall some straightforward properties of $g$.

**Fact A.1.** The function $g$ is positive, decreasing, differentiable and convex on $(0, \infty)$.

Moreover, the following asymptotic behaviour holds:

$$
g(z) \sim \frac{1}{2} \log(1/z), \quad g'(z) \sim -\frac{1}{2z}, \quad \text{as } z \to 0,
$$

$$
g(z) \sim e^{-2z}, \quad g'(z) \sim -2e^{-2z}, \quad \text{as } z \to \infty,
$$

where $x \sim y$ stands for $x = (1 + o(1))y$.

The relevance of this function comes from its link to the probability of traversability. Recalling Definition 3.1, for any positive integers $a$ and $b$ we set

$$
T^{1}(a, b) = \mu(T^{1}_{\uparrow}(R(a, b))), \quad T^{0}(a, b) = \mu(T^{0}_{\uparrow}(R(a, b)));
$$

where 0 stands for the fully infected configuration. Note that these probabilities are the same for left-traversability, while for up or down-traversability $a$ and $b$ are inverted in the r.h.s. The next lemma follows easily from [26, Lemma 8]. Let $q' = -\log(1 - q) = q + O(q^{2})$.

**Lemma A.2.** For any positive integers $a$ and $b$ and $\omega \in \{0, 1\}$ we have

$$
T^{\omega}(a, b) = q^{O(1)} e^{-a\omega(bq')}.
$$
**Corollary A.3.** For any positive integers $a$ and $b$ we have

$$\max_{0 \leq s, s' \leq b} \frac{T^0(a, s)T^0(a, b - s)}{T^1(a, s')T^1(a, b - s')} \leq q^{-O(1)}. \tag{A.1}$$

In particular, for any boundary conditions $\omega, \omega'$ and rectangle $R$ of class $1 \leq n \leq 2N$ with $n$ odd, we have

$$\mu_R(\mathcal{SG}^\omega_s(R) \mid \mathcal{SG}^\omega_{s'}(R)) \geq q^{O(1)} \tag{A.2}$$

uniformly over all possible values of $s$ and boundary conditions $\omega, \omega'$ and similarly for even $n$.

**Proof.** (A.1) follows immediately from Lemma A.2. To obtain (A.2), recall that

$$\mathcal{SG}^\omega_{s'}(R) = \bigcup_{s} \mathcal{SG}^\omega_{s'}(R);$$

there are $q^{-O(1)}$ possible values of $s'$; by (A.1), for all $s, s', \omega$ and $\omega'$,

$$\mu_R(\mathcal{SG}^\omega_s) / \mu_R(\mathcal{SG}^\omega_{s'}) \geq q^{O(1)}. \quad \square$$

We are now ready for the main result of this appendix.

**Proof of Proposition 3.7.** We will prove the same bound for the super-good event occurring with all $s = 0$ in Definition 3.4 on all scales, i.e. the initial infection $\Lambda^{(0)}$ being in the bottom-left corner of $\Lambda^{(n)}$. Once the offsets are fixed, it suffices to prove the bound on this probability for $n = 2N$, in which case it reads

$$q \prod_{m=1}^{N} T^1(\ell_m - \ell_{m-1}, \ell_m)T^1(\ell_m - \ell_{m-1}, \ell_{m-1})$$

$$= q^{O(N)} \exp\left(- \sum_{m=1}^{N} (\ell_m - \ell_{m-1})(g(q'\ell_m) + g(q'\ell_{m-1}))\right),$$

by Lemma A.2 and symmetry. Since $g$ is decreasing, the last sum is at most

$$2 \sum_{m=1}^{\infty} (\ell_m - \ell_{m-1})g(q'\ell_{m-1}).$$

The term for $m = 1$ is $O(\log(1/q)/\sqrt{q})$ by Fact A.1. For the other terms we use that by convexity for any $0 < a < b$

$$(b - a)g(a) \leq \int_{a}^{b} g(z) \, dz - O((b - a)^2 g'(a)).$$

Using Fact A.1, we get

$$-(b - a)^2 g'(a) \leq O((b - a)^2) \times \begin{cases} 1/a & \text{if } a = O(1) \\ e^{-a} & \text{if } a = \Omega(1). \end{cases}$$
Finally, we have \( \ell_m - \ell_{m-1} \leq 2\sqrt{q}\ell_{m-1} \) by (3.1), so

\[
q' \sum_{m=2}^{m_0} \left( \frac{\ell_m - \ell_{m-1}}{\ell_{m-1}} \right)^2 \leq O(q'\sqrt{q}\ell_{m_0}) = O(q)
\]

\[
(q')^2 \sum_{m=m_0}^{\infty} (\ell_m - \ell_{m-1})^2 e^{-q\ell_{m-1}} = O(q),
\]

setting \( m_0 = \max\{m, \ell_m \leq 1/q\} = O(\log(1/q))/\sqrt{q} \). Putting these bounds together and recalling (3.3), we conclude the proof of Proposition 3.7. \( \square \)

We next turn to defining the event \( \overline{SG}(V_2) \) required in the proof of Lemma 3.12, so we fix \( n = 2m \in [2, 2N] \) and \( R = R(\ell_m - 1, \ell_m) = V_2 - \tilde{e}_1 \).

**Definition A.4.** We say that \( \overline{SG}(R) \) occurs if there exist integers \( 0 \leq s_1 \leq \ell_m - \ell_{m-1} - 1 \) and \( 0 \leq s_2 \leq \ell_m - \ell_{m-1} \) such that the intersection of the following events, in the sequel \( \overline{SG}_{s_1,s_2}(R) \), occurs (see Figure 4)

\[
SG(\Lambda^{(n-2)} + s_1\tilde{e}_1 + s_2\tilde{e}_2);
\]

\[
\mathcal{T}_- (R(s_1, \ell_{m-1}) + s_2\tilde{e}_2);
\]

\[
\mathcal{T}_+ (R(\ell_m - \ell_{m-1} - 1 - s_1, \ell_{m-1}) + (s_1 + \ell_{m-1})\tilde{e}_1 + s_2\tilde{e}_2);
\]

\[
\mathcal{T}_i (R(\ell_m - 1, s_2));
\]

\[
\mathcal{T}_i (R(\ell_m - 1, \ell_m - \ell_{m-1} - s_2) + (s_2 + \ell_{m-1})\tilde{e}_2).
\]

The event \( \overline{SG}(V_2) \) is defined by translation of \( \overline{SG}(R) \). Then for any \( \omega_2 \in \overline{SG}_{s_1,s_2}(V_2) \), the segments \( I_1 \) and \( I_3 \) are given by

\[
I_1(\eta_2) = R(1, \ell_{m-1}) + s_2(\omega_2)\tilde{e}_2 \subset V_1 = R(1, \ell_m),
\]

\[
I_3(\eta_2) = R(1, \ell_{m-1}) + s_2(\omega_2)\tilde{e}_2 + \ell_m\tilde{e}_1 \subset V_3 = V_1 + \ell_m\tilde{e}_1.
\]

**Lemma A.5.** Recalling (3.16), we have

\[
\mu_{V_1 \cup V_2} (\hat{C}_{1,2} | SG) \geq q^{-O(1)}.
\]

**Proof.** Recall that \( V_1 \cup V_2 = \Lambda^{(n)} \) and assume \( SG(\Lambda^{(n)}) \) occurs. For any \( 0 \leq s_1, s_2 \leq \ell_m - \ell_{m-1} \) we write

\[
SG_{s_1,s_2}(\Lambda^{(n)}) = SG_{s_2}(\Lambda^{(n)}) \cap SG_{s_1}(\Lambda^{(n-1)} + s_2\tilde{e}_2).
\]

Then by Corollary A.3 for any such \( s_1, s_2 \) we have

\[
\mu_{\Lambda^{(n)}} (SG_{s_1,s_2}(\Lambda^{(n)})) = \mu_{\Lambda^{(n)}} (SG(\Lambda^{(n)})) q^{O(1)},
\]
so it suffices to show that

\[ \mu_{V^2}(\overline{SG}_{0,0}(V^2)) \geq \mu_{\Lambda(n)}(\overline{SG}_{1,0}(\Lambda(n)))q^{O(1)}, \]

since \( \mu(T_n(I_1(\eta_{V^2}))) \geq q \) for any \( \omega_{V^2} \in \overline{SG}(V^2) \).

However, by Definitions 3.4 and A.4 and symmetry we have

\[
\frac{\mu_{V^2}(\overline{SG}_{0,0}(V^2))}{\mu_{\Lambda(n)}(\overline{SG}_{1,0}(\Lambda(n)))} \geq \frac{T^1(\ell_m - \ell_{m-1} - 1, \ell_{m-1})}{T^1(\ell_m - \ell_{m-1}, \ell_m)T^1(1, \ell_{m-1})} = q^{O(1)}e^{-O(\ell_m - \ell_{m-1})/\ell_m} \]

the last equality following from Lemma A.2.

By convexity of \( g \) we get

\[ g((\ell_m - 1)q') - g(\ell_m q') \leq -q'g'((\ell_m - 1)q'). \quad (A.3) \]

By Fact A.1 we have that the r.h.s. of (A.3) is \( O(1/\ell_m) \). Putting this together we obtain

\[
\frac{\mu_{V^2}(\overline{SG}_{0,0}(V^2))}{\mu_{\Lambda(n)}(\overline{SG}_{1,0}(\Lambda(n)))} \geq q^{O(1)}e^{-O(\ell_m - \ell_{m-1})/\ell_m} = q^{O(1)}, \quad (A.4)
\]
as desired, the last equality coming from (3.1).

**B Proof of Proposition 5.1**

Let \((\mathcal{S}, \mathcal{S}_1, \pi)\) be the parameters of the \( g \)-CBSEP on \( \mathbb{T}_n^d \) and let \( \ell = [\pi(S_1)]^{-1/d} \). For simplicity we assume that \( n^{1/d}/\ell \in \mathbb{N} \) and we partition the torus \( \mathbb{T}_n^d \) into \( M = (n/\ell)^d \) equal boxes \((B_j)_{j=1}^M\), where each \( B_j \) is a suitable lattice translation by a vector in \( \mathbb{T}_n^d \) of the box \( B = [\ell]^d \). The labels of the boxes can be thought of as belonging to \( \mathbb{T}_M^d \) and we assume that \( B_i, B_j \) are neighbouring boxes in \( \mathbb{T}_n^d \) iff \( i \sim j \) in \( \mathbb{T}_M^d \).

We then set \( \hat{\mathcal{S}} = B, \hat{\pi}((\sigma_x)_{x \in B}) = \bigotimes_{x \in B} \pi(\sigma_x), \hat{\mathcal{S}}_1 = \bigcup_{x \in B} \{\sigma_x \in \mathcal{S}_1\} \) and we consider the auxiliary renormalized \( g \)-CBSEP (in the sequel \( \hat{g} \)-CBSEP) on the graph \( \hat{G} = \mathbb{T}_M^d \) with parameters \((\hat{\mathcal{S}}, \hat{\mathcal{S}}_1, \hat{\pi})\). Using the assumption \( \lim_{n \to \infty} \pi(S_1) = 0 \), we have that \( \hat{\pi}(\hat{\mathcal{S}}_1) = (1 - \pi(S_1))^{\ell_d} \to 1/e \) as \( n \to +\infty \).

**Lemma B.1.** Let \( T_{\text{rel}}^{\hat{g} \text{CBSEP}} \) be the relaxation time of \( \hat{g} \)-CBSEP on \( \hat{G} \). Then there exists a constant \( C > 0 \) depending on \( d \) such that \( T_{\text{rel}}^{\hat{g} \text{CBSEP}} \leq C \).

**Proof.** This follows by comparison between \( g \)-CBSEP and generalised FA-1f analogous to [22, (4)] combined with [30, Proposition 3.5]. \( \square \)
Proof of Proposition 5.1. For any pair of neighbouring boxes \( B_i \) and \( B_j \) write \( \hat{E}_{i,j} \) for the event \( \bigcup_{x \in B_i \cup B_j} \{ \sigma_x \in S^1 \} \). Using Lemma B.1 and the definition of \( T_{\text{rel}}^{\hat{g}^{\text{CBSEP}}} \) we get that
\[
\text{Var}_{\tau_{\text{rel}}}^+(f) \leq C \sum_{i \sim j} \pi_{\text{T}_{\text{n}}}^+(\mathbb{1}_{\hat{E}_{i,j}} \text{Var}_{B_i \cup B_j}(f | \hat{E}_{i,j}))
\]
where the sum in the r.h.s. is an equivalent way to express the Dirichlet form of \( \hat{g} \)-CBSEP. Now fix a pair of adjacent boxes \( B_i, B_j \) and let \( T_{\text{rel}}^{\hat{g}^{\text{CBSEP}}} (i, j) \) be the relaxation time of our original \( g \)-CBSEP with parameters \( (S, S_1, \pi) \) on \( B_i \cup B_j \). By symmetry \( T_{\text{rel}}^{\hat{g}^{\text{CBSEP}}} (i, j) \) does not depend on \( i, j \) and the common value will be denoted by \( \tilde{T}_{\text{rel}} \). If we plug the Poincaré inequality for \( g \)-CBSEP on \( B_i \cup B_j \)
\[
\text{Var}_{B_i \cup B_j}(f | \hat{E}_{i,j}) \leq \tilde{T}_{\text{rel}} \sum_{x \sim y \in B_i \cup B_j} \pi_{\text{T}_{\text{n}}}^+(\| \hat{E}_{x,y} \| \text{Var}_{x,y}(f | \hat{E}_{x,y})))
\]
into the r.h.s. above of we get
\[
\text{Var}_{\tau_{\text{rel}}}^+(f) \leq C \tilde{T}_{\text{rel}} \sum_{i \sim j} \sum_{x \sim y \in B_i \cup B_j} \pi_{\text{T}_{\text{n}}}^+(\| \hat{E}_{x,y} \| \text{Var}_{x,y}(f | \hat{E}_{x,y})))
\]
\[
\leq C d \tilde{T}_{\text{rel}} \sum_{x \sim y \in \text{T}_{\text{n}}} \pi_{\text{T}_{\text{n}}}^+(\| \hat{E}_{x,y} \| \text{Var}_{x,y}(f | \hat{E}_{x,y})))
\]
\[
= C d \tilde{T}_{\text{rel}} T_{\text{mix}}^{g^{\text{CBSEP}}}(f),
\]
i.e. \( T_{\text{rel}}^{g^{\text{CBSEP}}} \leq O(\tilde{T}_{\text{rel}}) \). It remains to bound \( \tilde{T}_{\text{rel}} \) from above.

Let \( T_{\text{mix}}^{g^{\text{CBSEP}}} \) be the mixing time of \( g \)-CBSEP on \( G_{i,j} \) with parameters \( S' = \{0, 1\}, S'_1 = \{1\} \) and \( \pi'(1) = \pi(S_1) = 1 - \pi'(0) \). Let \( T_{\text{cov}}^{g^{\text{CBSEP}}} \) be the cover time of the continuous-time random walk on \( G_{i,j} \). Theorem 2 of [22] implies that \( \tilde{T}_{\text{rel}} \leq O(T_{\text{mix}}^{g^{\text{CBSEP}}} + T_{\text{cov}}^{g^{\text{CBSEP}}}) \). It is well known (see e.g. [27]) that \( T_{\text{cov}}^{g^{\text{CBSEP}}} \leq O(\ell d \log(\ell)) = O(\pi(S_1)^{-1} \log(1/\pi(S_1))) \) and [22, Corollary 3.1] proves\(^7\) the same bound for \( T_{\text{mix}}^{g^{\text{CBSEP}}} \). In conclusion,
\[
\tilde{T}_{\text{rel}} \leq O(\pi(S_1)^{-1} \log(1/\pi(S_1))).
\]

Acknowledgments. This work is supported by ERC Starting Grant 680275 “MALIG”, ANR-15-CE40-0020-01 and PRIN 2015PAWZB “Large Scale Random Structures”. We acknowledge enlightening discussions with P. Balister, B. Bollobás, J. Balogh, H. Duminil-Copin, R. Morris and P. Smith and the hospitality of IHES during the informal workshop in 2017 “Kinetically constrained spin models and bootstrap percolation”. In that occasion P. Balister suggested a flexible structure for the droplets, featuring freedom in the position of the internal core at all scales, which he conjectured would remove the spurious log-corrections in the bound (1.8) available at that time.

\(^7\)Strictly speaking [22, Corollary 3.1] deals with the torus of cardinality \( \pi(S_1)^{-1} \) but the same proof extends to our case of the graph \( G_{i,j} \).
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