Linear perturbations of a Schwarzschild black hole by thin disc

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Abstract. The article written by C.M. Will (1974) describes a method how to obtain a perturbation of an (originally) Schwarzschild metric by a slowly rotating light ring. This scheme is revisited in order to adapt it to the case of a thin disc. It turns out for some class of mass distribution on the disc bounded between some radii the solution (or at least its decomposition into spherical harmonics) can be found explicitly.

1. Introduction
Many of the very compact active astrophysical objects (for example active galactic nuclei and X-ray binaries) are believed to contain disc-like structures. However these very compact objects generates strong and inhomogenous gravitational field so Newton’s theory is not sufficient to describe them. On the other hand general relativity is non-linear and the simple superposition cannot be used. Recently these and even more complicated (for example time dependant) fields are successfully treated numerically. However so far the only known “analytical” solutions of this problem contains a very high degree of symmetry, usually they are axisymmetric and stationary. Even in this “symmetric” model the equations are quite complicated so it is a bit surprising in the static case the superposition can be used and the equations solved explicitly (Weyl class of solutions).

The fundamental treatment of stationary axisymmetric thin discs can be found in the paper by Bardeen [1]. It was succeeded by the article by Will [2], which remains a major reference in this subject even now.

2. Basic concepts
Considering axisymmetric stationary case we can write metric in form

\[ ds^2 = -e^{2\nu} dt^2 + r^2 B^2 e^{-2\nu} \sin^2 \theta (d\varphi - \omega dt)^2 + e^{2\zeta-2\nu} \left( dr^2 + r^2 d\theta^2 \right), \]  

(1)

where \( \nu, \omega, B \) and \( \zeta \) stands for an unknown metric functions of coordinates \( r \) and \( \theta \).

The energy-momentum tensor of thin dust disc can expressed

\[ T^\alpha_\beta = \sigma e^{2\nu} u^\alpha u^\beta r^{-1} \delta(\cos \theta), \]  

(2)
where $\sigma$ describes surface mass density and $u^\alpha$ is four-velocity of the fluid. It is convenient to write four-velocity in the form $u^\alpha = \frac{e^{-\nu}}{1-\nu^2}(1, 0, 0, \Omega)$, where $\Omega = \frac{d\phi}{dt}$ is coordinate angular velocity and $\nu = r \sin \theta e^{-2\nu}(\Omega - \omega)$ linear velocity\(^1\) of the fluid.

It can be shown function $B$ does not contain physically relevant information. As long as it satisfies the equation $\nabla \cdot (r \sin \theta \nabla B) = 0$ (where the divergence $(\nabla \cdot)$ and gradient $(\nabla)$ operators are the same as in the virtual Euclidean space-time with spherical coordinates $r$, $\theta$ and $\phi$) it can be chosen arbitrary\(^2\) and the solutions for different $B$ will only differ in the choice of coordinates.

The important pair Einstein equations is

$$\nabla \cdot (B \nabla \nu) - \frac{1}{2} r^2 \sin^2 \theta e^{-4\nu} \nabla \omega \cdot \nabla \omega = 4\pi B \sigma \frac{1 + \nu^2}{1 - \nu^2} \frac{1}{r} \delta(\cos \theta),$$

(3)

$$\nabla \cdot \left(r^2 \sin^2 \theta B^3 e^{-4\nu} \nabla \omega \right) = -16\pi B^2 \sigma e^{-2\nu} \frac{v}{1 - \nu^2} \delta(\cos \theta)$$

(4)

and the last two independent equations can be used to express $\zeta$ using known $\nu$ and $\omega$.

Now let us assume the background metric is perturbed by a thin dust disc. The metric function can be written in form $\nu = \nu_0 + \nu_1 + \cdots$ where the lower index corresponds with the power of the disc mass it contains (and analogously for the functions $\omega$ and $\zeta$). In the subsequent text we will work only with the linear perturbations.

The Schwarzschild black hole in the isotropic coordinates takes form

$$ds^2 = -\left(\frac{2r - M}{2r + M}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(d\nu^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right),$$

(5)

so the background metric functions are

$$B = 1 - \frac{M^2}{4r^2}, \quad \nu_0 = \ln \left(\frac{2r - M}{2r + M}\right), \quad \omega_0 = 0, \quad \zeta_0 = \ln \left(1 - \frac{M^2}{4r^2}\right).$$

(6)

The linear perturbation of this black hole by disc will lead us to the equations

$$\frac{\partial}{\partial x} \left[(x^2 - 1) \frac{\partial \nu_1}{\partial x}\right] + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial \nu_1}{\partial \theta}\right] = 4\pi r \sigma \frac{1 + \nu^2}{1 - \nu^2} \delta(\cos \theta),$$

(7)

$$\frac{\partial}{\partial x} \left[(x + 1)^4 \frac{\partial \omega_1}{\partial x}\right] + \frac{(x + 1)^3}{x - 1} \frac{1}{\sin^3 \theta} \frac{\partial}{\partial \theta} \left[\sin^3 \theta \frac{\partial \omega_1}{\partial \theta}\right] = -\frac{\pi}{4M^2} \frac{(2r + M)^3}{2r - M} \frac{\nu}{\sigma} \frac{1}{1 - \nu^2} \delta(\cos \theta).$$

(8)

where $x = \frac{r}{M} \left[1 + M^2/(4r^2)\right]$ is new radial coordinate. Considering the particles are moving on geodesics their velocity can be written\(^3\) as $v = (x - 1)^{-1/2}$.

To solve these equations we will use expansion into the Legendre (for the function $\nu_1$) and Gegenbauer ($\omega_1$) polynomials\(^4\). So let us define

$$\nu_1 = \sum_{n=0}^{\infty} \nu_1 n(x) P_n(\cos \theta), \quad \omega_1 = \sum_{n=0}^{\infty} \omega_1 n(x) T_n^{3/2}(\cos \theta).$$

(9)

\(^1\) With respect to the zero angular momentum observer.

\(^2\) With the exception $B = -\frac{1}{\ln \nu_0}$, which represents plane waves solutions instead of stationary axisymmetric ones.

\(^3\) In the zeroth order of expansion. We need not to consider first order since the velocity is used in the source term of the equations, which is clearly proportional to the disc mass solely because of $\sigma$.

\(^4\) It can be shown the series converges absolutely anywhere besides the axis. On the axis $\nu_1$ converges absolutely, however in the case of $\omega_1$ it is still not clear. The conjecture is $\omega_1$ converges, but not absolutely.
Moreover we will write the r.h.s. of Einstein equations using these polynomials

\[ 4\pi r \sigma \frac{1 + v^2}{1 - v^2} \delta(\cos \theta) = \sum_{n=0}^{\infty} R_n P_n(\cos \theta), \]

where \( R_n = (4n + 2)\pi P_n(0)x\chi(x) \) and

\[ -\pi \frac{(2r + M)^3}{4M^2 - 2r - M} \frac{\sigma v}{1 - v^2} \delta(\cos \theta) = \sum_{n=0}^{\infty} S_n T_n^{3/2}(\cos \theta), \]

where \( S_n = \frac{\pi T_n^{3/2}(0)(2n + 3)(x + 1)^{3/2}}{-2M(n + 1)(n + 2)} \). \hspace{1cm} (10)

In both expressions \( \chi(x) = r \sigma/(x - 2) \) stands for the new function used to describe mass density. Also we can see \( R_n = S_n = 0 \) when \( n \) is odd.

After substitution into linearized Einstein equations and their separation we will obtain

\[ \frac{d}{dx} \left[ (x^2 - 1) \frac{d \omega_1}{dx} \right] - n(n + 1) \omega_1 = R_n, \hspace{1cm} (12) \]

\[ \frac{d}{dx} \left[ (x + 1)^3 \frac{d \omega_1}{dx} \right] - n(n + 3) \frac{(x + 1)^3}{x - 1} \omega_1 = S_n. \hspace{1cm} (13) \]

The base of fundamental system of the first equation is the \( n \)-th Legendre polynomial \( P_n(x) \) and the Legendre function of the second kind \( Q_n(x) \). Analogously we can obtain fundamental system of the second equation. It consists of the polynomial \( F_n(x) \) and the “function of the second kind” \( G_n(x) \) defined as

\[ F_n(x) = 2F_1(-n, n + 3; (x + 1)/2), \]

\[ G_n(x) = F_n(x) \int \frac{d\xi}{F_n(\xi)} (\xi + 1)^4, \]

where \( 2F_1(a, b; c; z) \) denotes hypergeometric function. Assuming asymptotic flatness and regularity of the perturbation one can construct corresponding Green functions \( G_n^\nu(x, x') = -P_n[\min(x, x')]Q_n[\max(x, x')] \) and \( G_n^\nu(x, x') = -F_n[\min(x, x')]G_n[\max(x, x')] \). \hspace{1cm} (15)

These can be used to calculate first perturbation, however the integrals are quite complicated. In special case, where \( \chi(x) \) is an even polynomial, particular solution can be found explicitly.

\[ \nu_{1P}^{\text{Part}} \] is exactly

\[ x\chi(x) = \sum_{l=0}^{m} c_{2l+1} P_{2l+1}(x) \Rightarrow \nu_{1P}^{\text{Part}} = \sum_{l=0}^{m} \frac{c_{2l+1}}{2(l + 1)(2l + 1) - n(n + 1)} P_{2l+1}(x), \]

where \( m \) and \( c_j \) are some constants. The case of \( \omega_{1P}^{\text{Part}} \) is slightly more difficult, but it takes form

\[ \omega_{1P}^{\text{Part}} = \frac{A_n(x)}{\sqrt{x + 1}} + C_n F_n(x) \log \left( \frac{\sqrt{x + 1} + \sqrt{2}}{\sqrt{x + 1} - \sqrt{2}} \right), \]

where \( A_n(x) \) stands for a suitable polynomial and \( C_n \) is some constant.

However we would like to obtain solution with disc spanned between some radii \( r_{min} \) and \( r_{max} \) and we want the solution to be regular on the horizon and asymptotically flat. Its uniqueness is clear from the uniqueness of the Green functions. It can be also described by

\[ \nu_1 = \begin{cases} 
A<P_n(x) + B_{<} Q_n(x) + \nu_{1P}^{\text{Part}}(x) & \text{for } \ x < x_{\text{min}} \\
A=_x P_n(x) + \nu_{1P}^{\text{Part}}(x) & \text{for } \ x_{\text{min}} \leq x < x_{\text{max}} \\
B> Q_n(x) & \text{for } \ x \geq x_{\text{max}} 
\end{cases}, \]

where \( A_>, A_=_x, B_< \) and \( B_> \) are constants obtained by the the requirement of continuity of metric functions and their first derivatives at the radii of the rims of the disc (and analogously we can express \( \omega_1 \)).

### 3. Numerical examples

Let us present some numerical examples of the procedure described above. In both cases we take \( M = 1 \), \( r_{min} = 5 \) and \( r_{max} = 6 \) and we consider only first 30 orders in the “harmonic” expansion (i.e. \( P_n(\cos \theta) \) and \( T_n^{3/2}(\cos \theta) \)).
The first one (Fig. 1 and 2) deals with $\chi = (40 - x^2) \cdot 10^5$ and the second one (Fig. 3 and 4) with $\chi = 10^{-7}$.

**Figure 1.** Gravitational potential $\nu = \nu_0 + \nu_1$. The lines corresponds with the potential values $-1.5, -1.4, \ldots, -0.1$.

**Figure 2.** Dragging $\omega = \omega_0 + \omega_1$. The lines corresponds with the values $8 \cdot 10^{-6}, 7 \cdot 10^{-6}, \ldots, 1 \cdot 10^{-6}$.

**Figure 3.** Gravitational potential $\nu = \nu_0 + \nu_1$. The lines corresponds with the potential values $-0.20, -0.18, \ldots, -0.04$.

**Figure 4.** Dragging $\omega = \omega_0 + \omega_1$. The lines corresponds with the values $10 \cdot 10^{-9}, 9 \cdot 10^{-9}, \ldots, 1 \cdot 10^{-9}$.

4. Conclusion

Analogously to the paper by Will [2] we were able to obtain procedure calculating the first order perturbations of Schwarzschild black hole by a thin dust disc for some class of matter distributions. However mentioned procedure is not suited well for second or higher order expansion in the mass because of the problems associated with multiplication of spherical harmonics. Moreover it is not well suited for the numerical calculations (as we can see in the numerical examples).

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References

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