Research Article

Second and Secondary Lattice Modules

Fethi Çallıalp,1 Ünsal Tekir,2 Emel Aslankarayıgit Uğurulu,2 and Kürsat Hakan Oral3

1 Department of Mathematics, Beykent University, Ayazaga-Maslak, 34396 Istanbul, Turkey
2 Department of Mathematics, Marmara University, Ziverbey, Göztepe, 34722 Istanbul, Turkey
3 Department of Mathematics, Yıldız Technical University, 34210 Istanbul, Turkey

Correspondence should be addressed to Kürsat Hakan Oral; khoral@yildiz.edu.tr

Received 22 April 2014; Accepted 7 September 2014; Published 14 October 2014

Academic Editor: Aldo Humberto Romero

Copyright © 2014 Fethi Çallıalp et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $M$ be a lattice module over the multiplicative lattice $L$. A nonzero $L$-lattice module $M$ is called second if for each $a \in L, a1_M = 1_M$ or $a1_M = 0_M$. A nonzero $L$-lattice module $M$ is called secondary if for each $a \in L, a1_M = 1_M$ or $a^n1_M = 0_M$ for some $n > 0$. Our objective is to investigate properties of second and secondary lattice modules.

A multiplicative lattice $L$ is a complete lattice in which there is defined as a commutative, associative multiplication which distributes over arbitrary joins and has the compact greatest element $1_L$ (least element $0_L$) as a multiplicative identity (zero). An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in $L$ is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If $0_L$ is prime, then $L$ is said to be a domain. For $a \in L$, we define $\sqrt{a} = \{x \in L : x^\alpha \leq a$ for some integer $n\}$. An element $p < 1_L$ in $L$ is said to be primary if $ab \leq p$ implies either $a \leq p$ or $b \leq p$.

If $a, b$ belong to $L, (a; b)$ is the join of all $c \in L$ such that $cb \leq a$. An element $e$ of $L$ is called meet principal if $a \land b = (a; b) \land b$ for all $a, b \in L$. An element $e$ of $L$ is called join principal if $(a \lor b; e) = a \lor (b; e)$ for all $a, b \in L$. An element $e$ in $L$ is said to be principal if $e$ is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \land e = e(a; b)$ (a $\lor (0; e) = (a; e))$ for all $a \in L$. An element $a$ of a multiplicative lattice $L$ is called compact if $a \leq \bigvee b$ implies $a \leq b_1 \lor b_2 \lor \cdots \lor b_n$ for some subsets $\{a_1, a_2, \ldots, a_n\}$. If each element of $L$ is a join of principal (compact) elements, then $L$ is called a $PG$-lattice (CG-lattice). If $L$ is a CG-lattice and $p$ is a primary element, then $\sqrt{p}$ is prime [1, Lemma 2.1].

Let $M$ be a complete lattice. Recall that $M$ is a lattice module over the multiplicative lattice $L$ or simply an $L$-module in case there is a multiplication between elements of $L$ and $M$, denoted by $IB$ for $I \in L$ and $B \in M$, which satisfies the following properties:

1. $(b)B = I(bB)$;
2. $(\bigvee a_k)(\bigvee B_k) = \bigvee a_k B_k$;
3. $1_B B = B$;
4. $0_B B = 0_M$.

for all $I, I_a, B$ in $L$ and for all $B, B_2$ in $M$.

Let $M$ be an $L$-module. If $N, K$ belong to $M, (N; K)$ is the join of all $a \in L$ such that $aK \leq N$. Particularly, $(0_M; 1_M)$ is denoted by $ann(M)$. If $a \in L$ and $N \in M$, then $(N; M)$ is the join of all $H \in M$ such that $aH \leq N$. An element $N$ is called meet principal if $(b \land (B; N)) = bN \land B$ for all $b \in L$ and for all $B \in M$. An element $N$ of $M$ is called join principal if $b \lor (B; N) = (b \lor B; N)$ for all $b \in L$ and for all $B \in M$. $N$ is said to be principal if it is both meet principal and join principal. In special cases, an element $N$ of $M$ is called weak meet principal (weak join principal) if $(B; N) = B \land N \lor (B; N) = b \lor (0_M; N)$ for all $b \in M$ (for all $b \in L$). $N$ is said to be weak principal if $N$ is both weak meet principal and weak join principal.

Let $M$ be an $L$-module. An element $N$ in $M$ is called compact if $N \leq \bigvee a_k$ implies $N \leq b_1 \lor b_2 \lor \cdots \lor b_n$ for some subsets $\{a_1, a_2, \ldots, a_n\}$. The greatest element of $M$ will be denoted by $1_M$. If each element of $M$ is a join of principal (compact) elements of $M$, then $M$ is called a $PG$-lattice module (CG-lattice module).

Let $M$ be an $L$-module. An element $N \in M$ is said to be proper if $N < 1_M$. For all elements $N$ of $M, [N, 1_M]$ is a set of
all \( K \in M \) such that \( N \leq K \leq 1_M \) and \([N,1_M]\) is an \( L \)-lattice module with \( a \cdot K = aK \lor N \) for all \( a \in L \) and \( K \in M \) such that \( N \leq K \).

For various characterizations of lattice modules, the reader is referred to [2–9].

**Definition 1.** A nonzero \( L \)-lattice module \( M \) is called second if for each \( a \in L \), \( a1_M = 1_M \) or \( a1_M = 0_M \).

**Definition 2.** A nonzero \( L \)-lattice module \( M \) is called secondary if for each \( a \in L \), \( a1_M = 1_M \) or \( a^*1_M = 0_M \) for some \( n > 0 \).

**Example 3.** Let \( Z \) be the integers, let \( Q \) be the rational numbers, and let \( Q \) be \( Z \)-module. Suppose \( L = L(Z) \) is the set of all ideals of \( Z \) and \( M = L(Q) \) is the set of all submodules of \( Q \). Thus, \( M \) as \( L \)-lattice module is a second module, since for every integer \( n \in Z \), \( (nZ)Q = Q \) or \( (nZ)Q = 0 \).

**Remark 4.** Every second lattice module is a secondary lattice. But the converse is not true. For this, we can give the following example.

**Example 5.** Let \( Z \) be the integers and let \( Z_4 \) be \( Z \)-module. Suppose that \( L = L(Z) \) is the set of all ideals of \( Z \) and \( M = L(Z_4) \) is the set of all submodules of \( Z_4 \). Thus, \( M \) as \( L \)-lattice module is a secondary lattice module, which is not a second lattice module.

**Example 6.** Let \( Z \) be the integers and \( L = L(Z) \) be the set of all ideals of \( Z \). Thus, \( L \) as \( L \)-lattice module is neither a second lattice module nor a secondary lattice module.

**Proposition 7.** Let \( L \) be a \( CG \)-lattice and let \( M \) be a nonzero \( L \)-lattice module. If for each compact \( a \in L \), \( a1_M = 1_M \) or \( a1_M = 0_M \), then \( M \) is a second \( L \)-lattice module.

**Proof.** Let \( r \in L \). Since \( L \) is a \( CG \)-lattice, then we have \( r = \bigvee c_i \) such that \( c_i \)'s are compact elements of \( L \). Then, we obtain \( r1_M = (\bigvee c_i)1_M = \bigvee c_i 1_M \). We have two cases.

Case 1. If \( c_i 1_M = 0_M \) for each compact \( c_i \in L \), then we have \( r1_M = (\bigvee c_i)1_M = \bigvee c_i 1_M = 0_M \).

Case 2. If \( c_i 1_M = 1_M \) for some compact \( c_i \in L \), then we have \( r1_M = (\bigvee c_i)1_M = \bigvee c_i 1_M = 1_M \).

Hence, \( r1_M = 1_M \) or \( r1_M = 0_M \) for each \( r \in L \). Consequently, \( M \) is second.

**Proposition 8.** If \( M \) is a second \( L \)-lattice module, then \( \text{ann}(M) = (0_M \lor 1_M) = p \) is a prime element of \( L \). In this case, \( M \) is called \( p \)-second lattice module.

**Proof.** Suppose that \( M \) is a second \( L \)-lattice module. Clearly, \( \text{ann}(M) = p \) is a proper element of \( L \). Let \( ab \leq p \) and assume that \( b \not\leq p \); that is, \( b1_M \neq 0_M \). But \( M \) is a second \( L \)-lattice module; then \( b1_M = 1_M \). Since \( b1_M = 1_M \) and \( ab1_M = 0_M \), then \( a1_M = 0_M \), which implies that \( a \leq p \).

**Proposition 9.** If \( M \) is a secondary \( L \)-lattice module, then \( \sqrt{\text{ann}(M)} \) is a primary element of \( L \).

**Proof.** Suppose that \( M \) is a secondary \( L \)-lattice module. Let \( ab \leq \sqrt{\text{ann}(M)} \) and \( b \neq \sqrt{\text{ann}(M)} \); we prove that \( a \leq \sqrt{\text{ann}(M)} \).

Since \( ab \leq \sqrt{\text{ann}(M)} \) and \( b \not\leq \sqrt{\text{ann}(M)} \), we have \( ab1_M = 0_M \) and \( (b^*)1_M \neq 0_M \) for each \( n > 0 \). Since \( M \) is secondary, we have \( b1_M = 1_M \). Then \( ab1_M = a1_M = 0_M \), which implies \( a \leq \sqrt{\text{ann}(M)} \).
contradiction. Therefore, \( r \leq \sqrt{ann(M)} \). Consequently, we obtain \( \sqrt{ann([N,1_M])} \leq \sqrt{ann(M)} \).

Let \( a \in L \). Since \( N \) is pure, we have \( aN = a1_M \cap N \). As \( M \) is a secondary lattice module, then either \( a1_M = 1_M \) or there exists a positive integer \( n \) such that \( a^n1_M = 0_M \). This implies that either \( aN = N \) or \( a^2N = a1_M \cap N = 0_M \). Therefore, we have \( a \cdot 1_{[0,a,N]} = a \cdot N = aN \cap 0_M = aN = N = 1_{[0,a,N]} \) or \( a^n \cdot 1_{[0,a,N]} = a^n \cdot N = a^nN \cap 0_M = a^nN = 0_M = 0_{[0,a,N]} \). Hence, \([0_M,N] \) is a secondary lattice module.

Now we show that \( \sqrt{ann(M)} = \sqrt{ann([0_M,N])} \). Clearly, \( \sqrt{ann(M)} \leq \sqrt{ann([0_M,N])} \). Let \( c \) be compact and \( c \leq \sqrt{ann([0_M,N])} \). Since \( c \) is compact, there exists a positive integer \( k \) such that \( c^k \cdot N = c^kN = 0_{[0,a,N]} = 0_M \). Since \( N \) is pure, we have \( c^kN = c^k1_M \cap N = 0_M \). If \( c \notin \sqrt{ann(M)} \), then \( c1_M = 1_M \). This implies that \( 0_M = c^kN = N \cap c^k1_M = N \cap 1_M = N \), a contradiction. Therefore, \( c \leq \sqrt{ann(M)} \).

Proposition 14. Let \( M \) be a non-zero \( L \)-lattice module and let \( N \) be a pure element of \( M \). Then \( M \) is a \( p \)-second lattice module if and only if \([0_M,N] \) and \([N,1_M] \) are both \( p \)-second lattice modules.

Proof. \( \Rightarrow \): Suppose that \( M \) is a \( p \)-second lattice module. Let \( a \in L \). Since \( N \) is pure, we have \( aN = N \cap a1_M \). As \( M \) is a second lattice module, then either \( a1_M = 0_M \) or \( a1_M = 1_M \). This implies that either \(aN = 0_M \) or \( aN = N \). Hence, \([0_M,N] \) is a second lattice module. Now, we show that \( \sqrt{ann(M)} = \sqrt{ann([0_M,N])} \). Clearly, \( \sqrt{ann(M)} \leq \sqrt{ann([0_M,N])} \). Let \( r \leq \sqrt{ann([0_M,N])} \). Thus, we have \( rN = 0_M \). Now we assume that \( r \notin \sqrt{ann(M)} \). Then we obtain \( r1_M = 1_M \), since \( M \) is a second lattice module. This implies that \( 0_M = rN = r1_M \cap N = 1_M \cap N = N \), a contradiction. Therefore, \( r \leq \sqrt{ann(M)} \).

Proof. \( \Leftarrow \): Suppose that \( M \) and \( M \) are two second lattice modules with \( \sqrt{ann([0_M,N])} = \sqrt{ann([N,1_M])} = p \). Let \( r \in L \). We have two cases.

Case 1. If \( r \leq \sqrt{ann([0_M,N])} = \sqrt{ann([N,1_M])} \), then \( rN = 0_M \) and \( r \cdot 1_{[N,1_M]} = r \cdot 1_M = r1_M \cap N = N \), which implies \( r1_M \leq N \). Thus, \( 0_M = rN = N \cap r1_M = r1_M \).

Case 2. If \( r \notin \sqrt{ann([0_M,N])} = \sqrt{ann([N,1_M])} \), then \( rN = N \) since \([0_M,N] \) is a second lattice module. Hence, we have \( N = rN = N \cap r1_M \), which is \( N \leq r1_M \), since \( N \) is pure. Because \([N,1_M] \) is a second lattice module and \( r \notin \sqrt{ann([N,1_M])} \), we obtain \( r \cdot 1_{[N,1_M]} = r1_M \cap N = 1_{[N,1_M]} = 1_M \). Therefore, we obtain that \( r1_M = 1_M \). Consequently, \( M \) is a second lattice module.

Now we show that \( \sqrt{ann(M)} = p \). Clearly \( \sqrt{ann(M)} \leq \sqrt{ann([N,1_M])} \). Let \( s \leq \sqrt{ann([N,1_M])} \). Then we have \( s \cdot 1_{[N,1_M]} = 0_{[N,1_M]} \), that is, \( s \cdot 1_M = N \). Thus, \( s1_M \cap N = N \), and so \( s1_M \leq N \). Now, we assume that \( s \notin \sqrt{ann(M)} \). Then, we have \( s1_M = 1_M \), since \( M \) is second. Hence, \( 1_M = s1_M \leq N \), a contradiction. Consequently, we have \( s \leq \sqrt{ann(M)} \).

Definition 15. An \( L \)-module \( M \) is called a multiplication lattice module if for every element \( N \in M \), there exists an element \( a \in L \), such that \( N = a1_M \).

Definition 16. A element \( N \) of an \( L \)-module \( M \) is called prime element if \( N \neq 1_M \) and whenever \( r \in L \) and \( X \in M \) with \( rX \leq N \), then \( X \leq N \) or \( r \leq (N;1_M) \).

Definition 17. A element \( N \) of an \( L \)-module \( M \) is called semiprime element if \( N \neq 1_M \) and whenever \( r \in L \) and \( X \in M \) with \( r^2X \leq N \), then \( rX \leq N \).

Remark 18. Let \( N \) be a proper element of an \( L \)-module \( M \). Then \( N \) is a semiprime element if and only if whenever \( r \in L \), \( X \in M \) and \( k \) is a positive integer with \( r^kX \leq N \), then \( rX \leq N \).

We know that a prime element is semiprime, but the converse is not true in general. The following proposition shows that the converse is true when the module is secondary and multiplication.

Proposition 19. Let \( M \) be a multiplication and secondary \( L \)-lattice module. For all element \( N \) of \( M \) such that \( 1_M \neq N \in M \), \( N \) is a semiprime element of \( M \) if and only if \( N \) is a prime element of \( M \).

Proof. \( \Rightarrow \): Suppose that \( N \) is a semiprime element of \( M \) and let \( rX \leq N \), where \( r \in L \). Since \( M \) is a secondary lattice module, then either \( r^n1_M = 0_M \) for some positive integer \( n \) or \( r1_M = 1_M \).

Case 1. If \( r^n1_M = 0_M \), then \( r^n1_M \leq N \). Since \( N \) is a semiprime element, we have \( r1_M \leq N \).

Case 2. If \( r1_M = 1_M \), then we have \( X = rX \), since \( M \) is a multiplication lattice module. Then we have \( rX \leq X \).

Therefore, \( N \) is a prime element of \( M \).

\( \Leftarrow \): It is obvious.

Definition 20. Let \( M \) be an \( L \)-lattice module and let \( N \) be a proper element of \( M \). \( N \) is called a primary element of \( M \), if whenever \( a \in L \), \( X \in M \) such that \( aX \leq N \), then \( N = a \cdot 1_M \) or \( a \leq (N;1_M) \). Particularly, if \( M \) is nonzero and \( 0_M \) is primary, then \( M \) is said to be primary lattice module.

Definition 21. An \( L \)-lattice module \( M \) is said to be simple lattice module if \( M = [0_M,1_M] \).
Proposition 22. Every multiplication secondary lattice module is a primary lattice module.

Proof. Let $M$ be a multiplication secondary module and $rX = 0_M$ for some $r \in L, X \in M$. Now, we assume that $r\sqrt{ann}(M)$. Since $M$ is a secondary module, then we have $r1_M = 1_M$. Because $M$ is a multiplication, then we have $rX = X$. Consequently, we obtain $X = 0_M$.

Proposition 23. Every multiplication second lattice module is a simple lattice module.

Proof. Let $M$ be a multiplication and second module. Since $M$ is a multiplication, for every $N \in M$, there exists $a \in L$ such that $N = a1_M$. Then we obtain $a1_M = 1_M$ or $a1_M = 0_M$, since $M$ is second. Thus, we have $N = 1_M$ or $N = 0_M$ for every $N \in M$; that is, $M$ is simple.

Definition 24. Let $L$ be a domain and let $M$ be a nonzero $L$-lattice module. If $r1_M = 1_M$ for every $0_L \neq r \in L$, then $M$ is said to be divisible.

Definition 25. A nonzero $L$-lattice module $M$ is said to be torsion if there exists $0_L \neq r \in L$ such that $r1_M = 0_M$.

Proposition 26. Let $L$ be a domain. Let $M$ be a secondary $L$-lattice module. Then either $M$ is a divisible module or $M$ is a torsion module.

Proof. Suppose that $M$ is a secondary module over a domain $L$. If $M$ is not divisible, then there exists $0_L \neq r \in L$ such that $r1_M \neq 1_M$. Since $M$ is a secondary lattice module, then there exists a positive integer $n$ such that $r^n1_M = 0_M$. Since $0_L \neq r$ and $L$ is a domain, then we have $r^n \neq 0_L$. Consequently, there exists $0_L \neq r^n = s \in L$ such that $s1_M = 0_M$. Therefore, $M$ is a torsion lattice module.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. A. Johnson, "The structure of a class of $r$-lattices," Commentarii Mathematici Universitatis Sancti Pauli, vol. 32, no. 2, pp. 89–94, 1983.

[2] F. Callialp and U. Tekir, "Multiplication lattice modules," Iranian Journal of Science and Technology, vol. 35, no. 4, pp. 309–313, 2011.

[3] E. W. Johnson and J. A. Johnson, "Lattice modules over principal element domains," Communications in Algebra, vol. 31, no. 7, pp. 3505–3518, 2003.

[4] D. S. Culhan, Associated Primes and Primal Decomposition in Modules and Lattice Modules, and Their Duals, University of California Riverside, 2005.

[5] E. A. Al-Khouja, "Maximal elements and prime elements in lattice modules," Damascus University for Basic Sciences, vol. 19, pp. 9–20, 2003.

[6] H. M. Nakkar, "Localization in multiplicative lattice modules," Istoriko-Matematicheskie Issledovaniya, vol. 2, no. 32, pp. 88–108, 1974 (Russian).

[7] H. M. Nakkar and I. A. Al-Khouja, "Multiplication elements and distributive and supporting elements in lattice modules," Research Journal of Aleppo University, vol. 11, pp. 91–110, 1989.

[8] H. M. Nakkar and I. A. Al-Khouja, "Nakayama’s Lemma and the principal elements in Lattice Modules over multiplicative lattices," Research Journal of Aleppo University, vol. 7, pp. 1–16, 1985.

[9] H. M. Nakkar and D. D. Anderson, "Associated and weakly associated prime elements and primary decomposition in lattice modules," Algebra Universalis, vol. 25, no. 2, pp. 196–209, 1988.