P_Z(S)-METRICS AND P_Z(S)-METRIC SPACES

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Abstract. In this paper, we define notions of P_Z(S)-metric and P_Z(S)-metric space and we show that every P_Z(S)-metric space, analogous to an ordinary metric space and generally, a Λ-metric space, is a topological space, and in continuance, we show that from a topological point of view, some properties of P_Z(S)-metric spaces, and Λ-metric spaces, have coordination.

1. Introduction

A metric on a nonempty set X, is defined as d: X × X → R which satisfies three following condition:
1) ∀x, y ∈ X, 0 ≤ d(x, y), d(x, y) = 0 ⇔ x = y,
2) ∀x, y ∈ X, d(x, y) = d(y, x),
3) ∀x, y, z ∈ X, d(x, y) ≤ d(x, z) + d(z, y).

The second condition, is known as symmetry and the third, as triangle inequality. Under this conditions, (X, d) is said to be a metric space.

Also, if Λ is an abelian totally ordered group with a total order <, a Λ-metric on a nonempty set X, is defined as d: X × X → Λ which satisfies three above conditions, with this changing that 0, < and +, are interpreted to the identity element of Λ, the total order on Λ, and the operation of group Λ, respectively.

Under this conditions, (X, d) is named a Λ-metric space. Since (R, <), where < is ordinary order on R, is an additive abelian totally ordered group, each ordinary metric space is a Λ-metric space. Of course, all notions and properties of the metric spaces is not generalizable to Λ-metric spaces (see [2]).

In a Λ-metric space X, for every ε(> 0) ∈ Λ and every x ∈ X, the open ε-ball with center x, shown by B(x, ε), is defined as {y ∈ X | d(x, y) < ε}, and the definition of limit point, interior point, open and closed sets, and some another relevent notions, are counterparts of the definitions of same notions in ordinary metric spaces. The collection of all open sets is formed the metric topology, T_d, on X, in each metric space (X, d), and the topological space (X, T_d) is Hausdorff and normal. The generalization of these, is established for any Λ-metric space (X, d) (See [2]).

We replace P(S) (the power set of S), with Λ in definition of Λ-metric, where S is a nonempty set, of course with consideration a nonempty subset Z of S, and define a notion named P_Z(S)-metric and from that, P_Z(S)-metric space, and then we make counterparts of some existent notions in Λ-metric spaces for P_Z(S)-metric

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spaces, and in continuance, we proceed to make counterparts of results, for $P_Z(S)$-metric spaces. Specially, we prove that the set of all open sets in every $P_Z(S)$-metric space $X$ is a topology for $X$, and that all open $\varepsilon$-balls in $X$ form a bases for this topology.

The $\Lambda$-metrics, have fined plenty of applications in group theory, especially in action of a group on a set (See [1], [2], [3], [5]). Therefore it is possible that $P_Z(S)$-metrics will be helpful in group theory as well general topology.

2. Basic definitions and examples

In the sequel we use [4], for standard terminology and notation on topology.

Before all, We give definition of $P_Z(S)$-metric.

2.1. Definition. Let $P(S)$ be the power set of a nonempty set $S$ and $\emptyset \neq Z \subseteq S$, and let $X$ be a nonempty set. $(d, Z)$ is said to be a $P_Z(S)$-metric on $X$, if $d: X \times X \rightarrow P(S)$, be a function with three following properties:

1) $\forall x, y \in X, \ d(x, y) = \emptyset \iff x = y,$
2) $\forall x, y \in X, \ d(x, y) = d(y, x),$
3) $\forall x, y, z \in X, \ d(x, y) \subseteq d(x, z) \cup d(z, y).$

We name the second property, symmetry, and the third, triangle inequality.

If there is no chance for ambiguity, we mention merely $P_Z(S)$-metric $d$, instead of $P_Z(S)$-metric $(d, Z)$.

The role of the set $Z$ will be specified in continuance.

2.2. Definition. We say $(X, (d, Z))$ to be a $P_Z(S)$-metric space, if $(d, Z)$ be a $P_Z(S)$-metric on $X$.

Again, where there is no chance for ambiguity, we mention only $P_Z(S)$-metric space $X$ instead of $P_Z(S)$-metric space $(X, (d, Z))$.

2.3. Examples. a) Let $S$ be a nonempty set and let $Z, L$ be nonempty subsets of $S$, and let $X$ be an arbitrary nonempty set. If we define:

$$d: X \times X \rightarrow P(S)$$

$$d(x, y) = \begin{cases} \emptyset & \text{if } x = y, \\ L & \text{otherwise}, \end{cases}$$

then $(d, Z)$ is a $P_Z(S)$-metric on $X$ and therefore $(X, (d, Z))$ is a $P_Z(S)$-metric space.

b) Let $X = \mathbb{R}, S = \mathbb{R}, Z = (a, b)$, where $a, b \in \mathbb{R}$. If

$$d: X \times X \rightarrow P(S)$$

$$d(x, y) = \begin{cases} \emptyset & \text{if } x = y, \\ \{x, y\} & \text{otherwise}, \end{cases}$$

then $(d, Z)$ is a $P_Z(S)$-metric on $X$. 

2.4. Definition. Let \((X, d)\) be a \(P_Z(S)\)-metric space. If \(\varepsilon \in P(S)\) contains \(Z\), we say that \(\varepsilon\) is positive under \(Z\), and we write \(\varepsilon >_Z 0\) or \(0 <_Z \varepsilon\). If there is no chance for ambiguity, we are satisfied with writing \(\varepsilon > 0\) or \(0 < \varepsilon\), respectively.

It is clear that in each \(P_Z(S)\)-metric space, \(\emptyset\) is never positive, and \(S, Z\) are constantly positive. Also, if \(\varepsilon > 0\) and \(\varepsilon \subseteq \beta\), then \(\beta > 0\), where \(\beta \in P(S)\).

2.5. Definition. Let \((X, d)\) be a \(P_Z(S)\)-metric space and let \(\varepsilon(\subseteq S) > 0\), and \(x \in X\). The open \(\varepsilon\)-ball with center \(x\), is:
\[
\{ y \in X \mid d(x, y) \subset \varepsilon \},
\]
and is denoted by \(B(x, \varepsilon)\).

Under above conditions, \(B(x, \varepsilon) - \{x\}\) is said to be deleted open \(\varepsilon\)-ball with center \(x\) and is denoted by \(B^*(x, \varepsilon)\).

2.6. Examples. In the Example (2.3) in part (a) for all \(x \in X\),
\[
B(x, \varepsilon) = \begin{cases} X & \text{if } L \subset \varepsilon, \\ \{x\} & \text{otherwise}, \end{cases}
\]
and therefore if \(L \subset Z\), then for all \(\varepsilon > 0\) and all \(x \in X\), \(B(x, \varepsilon) = X\).

In part (b), for all \(x \in X\),
\[
B(x, \varepsilon) = \begin{cases} \varepsilon & \text{if } x \in \varepsilon, \\ \{x\} & \text{otherwise}, \end{cases}
\]
As it is observed in the above examples, in each \(P_Z(S)\)-metric space, \(x \in B(x, \varepsilon)\), for all \(\varepsilon > 0\), the same that its similar is established in \(\Lambda\)-metric spaces.

2.7. Definition. Let \(X\) be a \(P_Z(S)\)-metric space,
1) \(x \in X\) is said to be an interior point of a subset \(A\) of \(X\), if,
\[
\exists \varepsilon(\subseteq S) > 0, \ B(x, \varepsilon) \subseteq A.
\]
We denote by \(A^0\) or \(Int(A)\), the set of all interior points of \(A\) and name it inter of \(A\).

2) \(x \in X\) is said to be a limit point of a subset \(A\) of \(X\), if,
\[
\forall \varepsilon(\subseteq S) > 0, \ B^*(x, \varepsilon) \cap A \neq \emptyset.
\]
We denote by \(A'\), the set of all limit points of \(A\).

3) A subset \(U\) of \(X\), is said to be open, if each its point be an interior point.

4) A subset \(A\) of \(X\), is said to be closed, if each its limit point, be an element of it.

It is observed that these definitions, are counterparts of the definitions of similar notions in \(\Lambda\)-metric spaces.

2.8. Remark. Clearly in each \(P_Z(S)\)-metric space, \(\emptyset\) is open, by negation of antecedent and \(X\) is open by definition.
2.9. Examples for open sets. In part (a) of (2.3), upon (2.6) if \( A \subset X \), then \( x \in A \) will be an interior point of \( A \), iff there exist some \( \varepsilon > 0 \) so that \( L \) be not a real subset of \( \varepsilon \) and also, \( X \) is open clearly. Therefore if \( L \subset Z \), then all \( \gamma (> 0) \), satisfy \( L \subset \gamma \), and therefore excluding \( X \), \( \emptyset \), there exist no another open sets, and the interior of each another set, is \( \emptyset \). But if there exist some member of \( L \), as \( l \), so that it is not belong to \( Z \), or \( Z \subseteq L \), then each subset \( A \) of \( X \), is open, because in the first case, if \( \varepsilon = Z \), and in the second case, if \( \varepsilon = L \), then \( L \) is not a real subset of \( \varepsilon \). Therefore in that example, each set is open, or the open sets are restricted to \( \emptyset \) and \( X \).

In part (b) of (2.3), for each subset \( A \) of \( X \), upon (2.6), \( x \in A \) is an interior point of \( A \), iff \( x \notin (a,b) \) or \( x \in (a,b) \subseteq A \) (In the first case, take \( \varepsilon = (a,b) \), and in the second, \( \varepsilon = A \)). Therefore, the open sets are only \( X \), \( \emptyset \) and each set \( A \) that \( A \cap (a,b) = \emptyset \) or \( (a,b) \subseteq A \).

3. Statement of results

In each metric space, every open \( \varepsilon \)-ball is an open set. The following Lemma, is the counterpart of this case, for \( P_Z(S) \)-metric spaces.

3.1. Lemma. In every \( P_Z(S) \)-metric space, each open \( \varepsilon \)-ball is an open set.

Proof: Let \( B(x, \varepsilon) = U \), be an open \( \varepsilon \)-ball in \( P_Z(S) \)-metric space \( X \), and let \( y \in U \). Set:

\[
\delta = (\varepsilon - d(x, y)) \cup Z.
\]

Then \( \delta > 0 \) and \( B(y, \delta) \subseteq U \), since if \( a \in B(y, \delta) \), then,

\[
d(a, x) \leq d(a, y) + d(y, x) \subset \delta \cup d(x, y) = \varepsilon. \quad \Delta
\]

3.2. Theorem. The collection of all open sets in a \( P_Z(S) \)-metric space \( X \), is a topology on \( X \).

Proof: Let:

\[
\mathcal{T} = \{ U \subseteq X \mid U \text{ is open in } X \}.
\]

We have:

1) Upon what we said above, \( \emptyset, X \in \mathcal{T} \).

2) If \( \{ U_i \}_{i \in I} \), be a collection of elements of \( \mathcal{T} \), then \( \bigcup_{i \in I} U_i \) is an open set in \( X \), and therefore is an element of \( \mathcal{T} \), because if \( x \in \bigcup_{i \in I} U_i \), then:

\[
\exists i_o \in I, \ x \in U_{i_o}, \quad \exists \varepsilon_{i_o} (\subseteq X) > 0, \ B(x, \varepsilon_{i_o}) \subseteq U_{i_o} \subseteq \bigcup_{i \in I} U_i.
\]

and since \( U_{i_o} \) is open, then:

\[
\exists i_o \in I, \ \exists \varepsilon_{i_o} (\subseteq X) > 0, \ B(x, \varepsilon_{i_o}) \subseteq U_{i_o} \subseteq \bigcup_{i \in I} U_i.
\]

3) Let \( \{ U_i \}_{i=1}^n \) is a finite collection of members of \( \mathcal{T} \). Then \( \bigcap_{i=1}^n U_i \) is an open set in \( X \) and therefore is an element of \( \mathcal{T} \), because if \( x \in \bigcap_{i=1}^n U_i \), then since each \( U_i \) is open,

\[
\forall i \in I, \ \exists \varepsilon_i, \ B(x, \varepsilon_i) \subseteq U_i.
\]

Now set:

\[
\varepsilon = \bigcap_{i=1}^n \varepsilon_i.
\]

Since for each \( i \), \( \varepsilon_i > o \), then \( \varepsilon > o \). Now,

\[
\forall 1 \leq i \leq n, \ B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq U_i.
\]
Then $B(x, \varepsilon) \subseteq \bigcap_{i=1}^{n} U_i$. \(\triangle\)

Similar to $\Lambda$-metric spaces, the topology obtained from open sets of $P_Z(S)$-metric space $(X, d)$, is said to be the $P_Z(S)$-metric topology obtained from $P_Z(S)$-metric $d$ on $X$, and is denoted by $T_d$.

3.3. **Example.** In the Example (a) of (2.3), if $L$ is not a real subset of $Z$, the $P_Z(S)$-metric topology obtained from $P_Z(S)$-metric $d$, is the discrete topology and else, that is the indiscrete topology.

3.4. **Theorem.** The collection of all open balls in each $P_Z(S)$-metric space $(X, d)$, forms a bases for the $P_Z(S)$-metric topology $T_d$ on $X$.

**Proof:** Set:
$$B = \{ B(x, \varepsilon) \mid \varepsilon > 0, x \in X \}.$$ Firstly, according to (3.1), the members of $B$ are open. Secondly, if $U$ is an open set in $T_d$, then for each $x \in U$, we have:
$$\exists \varepsilon_x > 0, \quad B(x, \varepsilon_x) \subseteq U,$$
and since for each $x \in U$, $x \in B(x, \varepsilon_x)$, then,
$$U = \bigcup_{x \in U} B(x, \varepsilon_x),$$
and therefore $B$ is a bases for the topology $T_d$ on $X$. \(\triangle\)

Upon above theorem, to be an element of $X$ as an interior or limit point for a set, and therefore to be a set, closed in a $P_Z(S)$-metric space $(X, T_d)$, with meaning which we said in (2.7), is equivalent with the analogous cases in topological space $(x, T_d)$.

With the upper descriptions, if $(X, d)$, be a $P_Z(S)$-metric space, then $(X, T_d)$ is a topological space and from this, all contexts and corollaries, concerned with topological spaces, are current in it.

If $(X, d)$ is a $P_Z(S)$-metric space and $(x_n)$ is a sequence in $X$, and if $x \in X$, then upon (3.4), $\lim_{x \to \infty} x_n = x$ (in topological mining), iff,
$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad d(x_n, x) \subset \varepsilon.$$ 

3.5. **Definition.** Let $(X, d)$ be a $P_Z(S)$-metric space. A sequence $(x_n)$ in $X$ is said to be a Cauchy sequence, if,
$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall i, j \geq N, \quad d(x_i, x_j) \subset \varepsilon.$$ 

3.6. **Theorem.** Let $(X, d)$ be a $P_Z(S)$-metric space and $(x_n)$ be a convergent sequence in $X$. Then $(x_n)$ is a Cauchy sequence in $X$.

**Proof:** Suppose that $\varepsilon > 0$ and let $x_n \to x$, where $x \in X$. We have,
$$\exists N \in \mathbb{N}, \quad \forall n \geq N, \quad d(x_n, x) \subset \varepsilon.$$ 

Now, if $i, j \geq N$, then,
$$d(x_i, x) \subset \varepsilon, \quad d(x_j, x) \subset \varepsilon.$$ 

then,
$$d(x_i, x_j) \subseteq d(x_i, x) \cup d(x, x_j) \subset \varepsilon \cup \varepsilon = \varepsilon. \quad \triangle$$
3.7. **Theorem.** If a Cauchy sequence \((x_n)\) in \(P_Z(S)\)-metric space \((X, d)\), has a subsequence convergent to \(x\), then \((x_n)\) is convergent to \(x\).

**Proof:** Let \((x_{n_k})\) be a subsequence of \((x_n)\) such that \(x_{n_k} \to x\). With hypothesis for \(\varepsilon > 0\), we have,
\[
\exists \, N_1 \in \mathbb{N}, \forall k \geq N_1, \, d(x_{n_k}, x) < \varepsilon,
\]
\[
\exists \, N_2 \in \mathbb{N}, \forall i, j \geq N_2, \, d(x_i, x_j) < \varepsilon.
\]
Now if we take \(N \in \mathbb{N}\) such that \(N \geq N_1\) and \(n_N \geq N_2\), we have,
\[
d(x_i, x) \subseteq d(x_i, x_{n_k}) \cup d(x_{n_k}, x) \subseteq \varepsilon \cup \varepsilon = \varepsilon,
\]
for every \(i \geq N\). \(\triangle\)

3.8. **Theorem.** Let \((X_1, d_1)\) is a \(P_Z(S_1)\)-metric space and \((X_2, d_2)\) is a \(P_Z(S_2)\)-metric space. Then \(f : X_1 \to X_2\) is continuous (in topological mining), iff for each \(a \in X_1\), we have,
\[
\forall \varepsilon > Z_2, 0, \exists \delta > Z_1, 0, \forall x \in X_1, \, d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon. \quad (*)
\]

**Proof:** Let \(f : X_1 \to X_2\) is continuous and let \(a \in X_1\) and \(\varepsilon > Z_2, 0\). Then upon definition of the continuity, \(f^{-1}(B(f(a), \varepsilon))\) is open in \(X\). Therefore since \(a \in f^{-1}(B(f(a), \varepsilon))\),
\[
\exists \delta > Z_1, 0, B(a, \delta) \subseteq f^{-1}(B(f(a), \varepsilon)).
\]
Now, if \(x \in X_1\) and \(d_1(x, a) < \delta\), then \(x \in f^{-1}(B(f(a), \varepsilon))\) and therefore
\[
d_2(f(x), f(a)) < \varepsilon.
\]
Conversely, let \((*)\) is established. We prove that \(f\) is continuous. Upon (3.4), it is enough that we prove that the inverse image of each open \(\varepsilon\)-ball in \(X_2\), is open in \(X_1\). Let \(B(y, \varepsilon)\) be an open \(\varepsilon\)-ball in \(X_2\), and let \(a \in f^{-1}(B(y, \varepsilon))\). Then upon \((*)\),
\[
\exists \delta > Z_2, 0, \forall x \in X_1, \, d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon. \quad (I)
\]
We have \(B(a, \delta) \subseteq f^{-1}(B(y, \varepsilon))\), because if \(x \in B(a, \delta)\), then \(d_1(x, a) < \delta\) and therefore upon \((I)\), \(d_2(f(x), f(a)) < \varepsilon\). Then,
\[
d_2(f(x), y) \subseteq d_2(f(x), f(a)) \cup d_2(f(a), y) \subseteq \varepsilon \cup \varepsilon = \varepsilon. \quad \triangle
\]

At the end, we notice that for a nonempty set \(S\), \(P(S)\) with the operation \(\cup\), is not a group. Also, if \(S \neq \emptyset\) is not of one element, \((P(S), \subseteq)\) is not a totally ordered set.

We hope this paper give a motivation for characterization compact and Hausdorff \(P_Z(S)\)-metric spaces as well to prove a version of fixed point theorem for these spaces.

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