ON THE PULLBACK OF AN ARITHMETIC THETA FUNCTION

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1. Introduction

In this paper, we consider the relation between the simplest types of arithmetic theta series, those associated to the cycles on the moduli space $C$ of elliptic curves with CM by the ring of integers $O_k$ in an imaginary quadratic field $k$, on the one hand, and those associated to cycles on the arithmetic surface $M$ parametrizing 2-dimensional abelian varieties with an action of the maximal order $O_B$ in an indefinite quaternion algebra $B$ over $\mathbb{Q}$, on the other.

To be more precise, let $C$ be the moduli stack of elliptic curves $(E, \iota)$ with $O_k$ action, so that $C$ is an arithmetic curve over $\text{Spec}(O_k)$, [13]. Let $L(E, \iota)$ be the space of special endomorphisms, i.e., endomorphisms $j$ of $E$ such that $j \circ \iota(\alpha) = \iota(\alpha^\sigma) \circ j$ for all $\alpha \in O_k$, where $\sigma$ is the nontrivial Galois automorphism of $k/\mathbb{Q}$. Fix a fractional ideal $a$ and elements $\lambda \in \partial^{-1}a/a$ and $r \in \partial^{-1}/O_k$, where $\partial$ is the different of $k/\mathbb{Q}$. For a positive integer $m$, let $Z_C(m) = Z_C(m; a, \lambda, r)$ be the locus of triples $(E, \iota, \beta)$, where

1. $(E, \iota)$ is an object in $C(S)$
2. $\beta \in L(E, \iota)a^{-1}$ is a special quasi-endomorphism such that
3. $r + \beta \lambda \in O_E := \text{End}(E/S)$, and
4. $\deg(\beta) = m \Delta(\lambda)$.

Note that $L(E, \iota)a^{-1}$ is a lattice in $V(E, \iota) = L(E, \iota) \otimes \mathbb{Z} \mathbb{Q}$ and, for $x \in V(E, \iota)$, $-x^2 = \deg(x) 1_E$. The special cycle $Z_C(m)$ is either empty or is a 0-cycle on $C$ supported in characteristic $p$ for a prime $p$ determined by $k$ and $m$. There is a corresponding generating function (see (2.10) for the precise definition)

$$\hat{\phi}_C(\tau; a, \lambda, r) = \sum_m \hat{Z}_C(m, v) q^m, \quad m = m/\Delta(\lambda)$$

for the images under the arithmetic degree map $\hat{\text{CH}}^1(C) \rightarrow \mathbb{R}$, of the classes defined by these 0-cycles in the first arithmetic Chow group $\hat{\text{CH}}^1(C)$ of $C$. Here, $q = e(\tau)$, $\tau = u + iv \in \mathcal{H}$, the upper half, and the divisor $\Delta(\lambda) \mid \Delta$ is given in (1.4). For positive $m$, $\hat{Z}_C(m, v) = \hat{Z}_C(m)$ is independent of $v$. For $m \leq 0$, additional terms, depending on $v$, are defined in section 2.

These cycles and generating series are generalizations of those defined in [13]. Indeed, when $k$ has prime discriminant, the series $\hat{\phi}_C(\tau; O_k, 0, 0)$ coincides, up to a constant factor, with that of [13] and was shown there to be a (non-holomorphic) modular form of weight 1.

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Also associated to the data \((a, \lambda, r)\) is a normalized incoherent Eisenstein series \(E^*(\tau, s; a, \lambda, r)\) of weight 1 and character \(\chi\) for \(\Gamma_0(|\Delta|)\), where \(\chi\) is the quadratic character associated to \(k/Q\). The analytic continuation of such a series vanishes at \(s = 0\), and our first main result, which generalizes that of [13], describes the leading term there.

**Main Theorem A.** Assume that \(2 \nmid \Delta\). Then

\[
E^*(\tau, 0; a, \lambda, r) = -2 \hat{\phi}_C(\tau; a, \lambda, r).
\]

In particular, \(\hat{\phi}(\tau; a, \lambda, r)\) is a non-holomorphic modular form of weight 1.

The second type of arithmetic theta series are associated to the arithmetic surfaces whose generic fibers are Shimura curves over \(Q\). For an indefinite quaternion algebra \(B\) over \(Q\) with a fixed maximal order \(O_B\), the moduli stack \(\mathcal{M}\) of abelian surfaces \((A, \iota)\) with \(O_B\)-action is an arithmetic surface over \(\text{Spec}(Z)\). This surface has a rich supply of divisors \(Z(t), t \in \mathbb{Z}_{>0}\), defined as the locus of triples \((A, \iota, x)\), where \(x\) is a ‘special’ endomorphism of \(A\) with square \(x^2 = -t\). Recall that such an endomorphism commutes with the given action of \(O_B\) and has trace zero. In [15], an extensive study was made of the classes defined by the cycles \(Z(t)\) in the arithmetic Chow group of \(\widehat{CH}^1(\mathcal{M})\). More precisely, for a positive real number \(v\), there is a Green function \(\Xi(t, v)\) for \(Z(t)\), and a resulting class \(\widehat{Z}(t, v) = (Z(t), \Xi(t, v)) \in \widehat{CH}^1(\mathcal{M})\). We refer the reader to [15] for more details. One of the main results of [15] is that, for \(\tau = u + iv\) in the upper half plane, the generating series

\[
(1.2) \quad \hat{\phi}(\tau) = \sum_t \widehat{Z}(t, v) q^t,
\]

the arithmetic theta function of our title, is a (non-holomorphic) modular form of weight \(\frac{3}{2}\) and level \(4D(B)_o\) valued in \(\widehat{CH}^1(\mathcal{M})\), where \(D(B)_o\) is the product of the odd primes at which \(B\) is ramified. Here, classes for \(t \leq 0\) are also included in the series\(^1\).

In the present paper, we suppose that embeddings

\[
k \rightarrow B \rightarrow M_2(k)
\]

are given, with \(i(O_k) \subset O_B\). For an \(O_k\)-lattice \(\Lambda \subset k^2\), let \(O_\Lambda\) be the maximal order in \(M_2(k)\) which stabilizes \(\Lambda\). For each \(O_k\)-lattice \(\Lambda\) such that \(O_B = O_\Lambda \cap B\), we define a morphism of moduli stacks

\[
\jmath_\Lambda : \mathcal{C} \rightarrow \mathcal{M}' = \mathcal{M} \times_{\text{Spec}(Z)} \text{Spec}(O_k),
\]

corresponding to the functorial construction of an \(O_B\)-module \((A, \iota)\) from a elliptic curve \((E, \iota)\) with CM by \(O_k\) given by the Serre construction, \(A = \Lambda \otimes_{O_k} E\). We assume that \(\Delta\) and \(D(B)\) are relatively prime so that the base change \(\mathcal{M}'\) of \(\mathcal{M}\) to \(\text{Spec}(O_k)\) is again regular. Then there is a natural map

\[
\overline{CH}^1(\mathcal{M}) \rightarrow \overline{CH}^1(\mathcal{M}'),
\]

and we abuse notation and also write \(\hat{\phi}(\tau)\) for the image of the generating series (1.2) under this map. The morphism \(\jmath_\Lambda\) determines a map

\[
\jmath_\Lambda : \overline{CH}^1(\mathcal{M}') \rightarrow \overline{CH}^1(\mathcal{C}),
\]

of arithmetic Chow groups, and our main goal is to determine the arithmetic degree of the pullback \(\jmath_\Lambda^* \hat{\phi}(\tau)\) of the arithmetic theta series. The result is the following.

\(^1\)This series was denoted in [15] by \(\hat{\phi}_1(\tau)\); here we omit the subscript, since the genus two generating function \(\hat{\phi}_2(\tau)\) of [15] will play no role in the present paper.
Main Theorem B. Assume that $2 \nmid \Delta$ and that $\Delta$ and $D(B)$ are relatively prime. Then
\[ \widehat{\deg} j_A^*(\widehat{\phi}(\tau)) = \sum_{r \in \partial^{-1}/O_k \atop \text{tr}(r)=0} \theta(\tau; r) \phi_c(D(B)\tau; \bar{a}, \lambda', r), \]
where,
\[ \theta(\tau; r) = \sum_{\alpha \in \partial^{-1} \atop \text{tr}(r)=0} q^{N(\alpha)} \]
is a theta series of weight $1/2$ depending on $r$. Here $a$ is a fractional $O_k$-ideal and $\lambda$ is a generator for the cyclic $O_k$-module $\partial^{-1}a/a$ determined by the embedding of $O_k$ into $O_B$, cf. Proposition 7.1, and $\lambda'$ is a twist of $\lambda$, cf. Proposition 9.3.

Such a relation is analogous to the following simple identity for classical theta series. Suppose that $L$ is an integral lattice in a quadratic space $(V, Q)$ and that an orthogonal decomposition $V = V_0 + V_1$ is given. Then, the classical theta series
\[ \theta(\tau, L) = \sum_{x \in L} q^{Q(x)} \]
has a factorization
\[ \theta(\tau, L) = \sum_{r \in L^\vee/L} \theta(\tau, L_0, r_0) \theta(\tau, L_1, r_1), \tag{1.3} \]
where $r = r_0 + r_1$ runs over the cosets of $L$ in the dual lattice $L^\vee$, and $L_i = L \cap V_i$. We expect that such relations will hold for the pullbacks of other arithmetic theta series and that they will be useful in applications to special values of derivatives of $L$-functions, just as the factorization formula (1.3) plays an important role in the study of special values of $L$-function.

For example, as an application of our results, we can determine the pullback of the classes
\[ \widehat{\theta}(f) = \langle f, \widehat{\phi} \rangle_{\text{Pet}} \in \widehat{\mathbb{C}H}^1(M) \]
associated to a newform $f$ of weight $\frac{3}{2}$ on $\Gamma_0(4D(B)\alpha)$ via the arithmetic theta lift, [15], Chapter IX. We compute
\[
\widehat{\deg} j_A^*(\widehat{\theta}(f)) = \langle f, \widehat{\deg} j_A^*(\widehat{\phi}) \rangle_{\text{Pet}} \n = \sum_{r \in \partial^{-1}/O_k \atop \text{tr}(r)=0} \langle f, \theta(\tau; r) \phi_c(D(B)\tau; \bar{a}, \lambda', r) \rangle_{\text{Pet}} \n = -\frac{1}{2} \frac{\partial}{\partial s} \left( \sum_{r \in \partial^{-1}/O_k \atop \text{tr}(r)=0} \langle f, \theta(\tau; r) E(D(B)\tau, s; \bar{a}, \lambda', r) \rangle_{\text{Pet}} \right) \bigg|_{s=0}.
\]
Here the second line follows from the first by Main Theorem B while the third line follows from the second by Main Theorem A. The inner integrals in the last line are the Rankin-Selberg integrals studied by Shimura in his seminal paper [19] on modular forms of half integral weight, and they represent the Hecke $L$-function of the corresponding newform $F$ of weight 2. In this way, we find that $\deg j_A^*(\widehat{\theta}(f))$ is proportional to $L'(1, F) \cdot a(|\Delta|, f)$, where $a(m, f)$ is the $m$-th Fourier coefficient of $f$. We hope to give the details of this computation elsewhere.
There are still two restrictions in the present paper. First, we assume that \( \Delta \) is odd. This is due to a certain lack of information about the local Whittaker functions, section 5, and could be removed with more calculation. The second restriction, that \( D(B) \) and \( \Delta \) be relatively prime, arises from the fact that the regularity of the base change \( \mathcal{M}' \) of the arithmetic Shimura surface \( \mathcal{M} \) to \( \text{Spec}(O_k) \) is lost if there are primes dividing \( D(B) \) that are ramified in \( k \). This will result in a slight shift in the contribution of the arithmetic Hodge bundle that remains to be determined.

This paper has been in progress for a long time. Hidden just below the surface are elaborate relations involving the genus theory of the field \( k \) and its interaction with the arithmetic of the quaternion algebra \( B \). Earlier versions were disfigured by complicated explicit computations with the genera. Thanks to the nice idea of Ben Howard about how to use the Serre construction in this situation, an idea he introduced in his lectures [7] at the Morningside Center in Beijing in the summer of 2009, we were able to eliminate these calculations in the present version.

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1.1. Notation. We write \( A \) for the ad\'ele ring of \( \mathbb{Q} \). We fix an imaginary quadratic field \( k = \mathbb{Q}(\sqrt{\Delta}) \) with discriminant \( \Delta < 0 \), and let \( O_k \) be its ring of integers. Let \( \chi = \chi_{k/\mathbb{Q}} = (\Delta, )_\lambda \) be the quadratic Dirichlet character associated to \( k/\mathbb{Q} \), and let \( \partial = \sqrt{\Delta}O_k \) be the different. Let \( \text{Cl}(k), h_k, \) and \( w_k \) be the ideal class group, the class number and the number of root of unity in \( k \) respectively.

For a fractional ideal \( a \) and an element \( \lambda \in \partial^{-1}a/a \), let \( \partial_{\lambda} \mid \partial \) be the divisor of \( \partial \) such that \( \lambda \) generates \( \partial^{-1}a/a \). We write

\[
\Delta(\lambda) = N(\partial_{\lambda}).
\]

Let \( \psi = \prod \psi_p \) be the ‘canonical’ unramified additive character of \( \mathbb{Q}\backslash A \) such that \( \psi_{\infty}(x) = e(x) = e^{2\pi ix} \).

**Part I**

2. The moduli problem and the incoherent Eisenstein series

2.1. The moduli problem and special cycles. For our fixed imaginary quadratic field \( k \) with ring of integers \( O_k \), let \( \mathcal{C} \) be the moduli stack over \( O_k \) of CM elliptic curves \( (E, \iota) \) as in [13], to which we refer the reader for more details. For an \( O_k \)-scheme \( S \) and a CM elliptic curve \( (E, \iota) \in \mathcal{C}(S) \), let

\[
L(E, \iota) = \{ j \in \text{End}(E/S) \mid j \circ \iota(a) = \iota(\bar{a}) \circ j, \quad a \in O_k \}
\]

be the lattice of special endomorphisms, and let \( V(E, \iota) = L(E, \iota) \otimes \mathbb{Q} \). This \( \mathbb{Q} \)-vector space is equipped with a canonical quadratic form \( Q(j) = \deg j \).

We now introduce special cycles that are a slight generalization of those defined in [13].

**Definition 2.1.** Fix a rational number \( m \geq 1 \), a fractional ideal \( a \), an element \( \lambda \in \partial^{-1}a/a \), and an element \( r \in \partial^{-1}/O_k \). Let \( Z(m) = Z(m; a, \lambda, r) \) be the fibered category over \( \text{Sch}/O_k \) that associates to an \( O_k \)-scheme \( S \) the category \( Z(m; a, \lambda, r)(S) \) whose objects are triples \( (E, \iota, \beta) \), where
(1) \((E, \iota)\) is an object in \(\mathcal{C}(S)\)
(2) \(\beta \in L(E, \iota)a^{\lambda -1}\),
(3) \(r + \beta \lambda \in O_E := \text{End}(E/S)\),
(4) \(\deg(\beta) = \frac{m}{N(\alpha)}\).

The morphisms in the category are \(O_k\)-linear isomorphisms \(\phi : (E, \iota) \to (E', \iota')\) of elliptic schemes over \(S\) such that \(\phi^* \beta' = \beta\).

**Remark 2.2.** (i) Here \(\beta\) is an element of \(\text{End}^0(E/S) = \text{End}(E/S) \otimes_{\mathbb{Z}} \mathbb{Q}\) and, in condition (3), we choose any representative of \(r\) in \(\partial^{-1}\) and of \(\lambda\) in \(\partial^{-1}a\).

(ii) The cycle \(\mathcal{Z}(m; a, \lambda, r)\) coincides with that defined in [13] in the case \(a = O_k\) with \(r = \lambda = 0\).

(iii) This particular generalization of the definition in [13] is motivated by the results about the pullback for cycles on Shimura curves that will be obtained in Part II of this paper.

(iv) For any \((E, \iota)\) in \(\mathcal{C}(S)\), the Serre construction gives rise to an elliptic scheme \(E_a := a \otimes_{O_k} E\) over \(S\) with CM by \(O_k\). Then, the element \(\beta\) can be viewed as an \(O_k\)-anti-linear homomorphism
\[\beta \in \text{Hom}((E_a, \iota), (E, \iota)),\]
where \((E, \iota)\) is the elliptic scheme \(E\) with \(O_k\)-action given by \(\iota(a) = \iota(\alpha)\).

The same argument as in [13, Section 5] shows that this moduli problem is represented by a stack, still denoted by \(\mathcal{Z}(m)\), whose coarse moduli scheme \(\mathcal{Z}(m)\) is a finite Artinian \(O_k\)-scheme which is only supported on primes non-split in \(k\). It is clear that \(\mathcal{Z}(m)\) is empty unless \(m\) is a positive integer and \(\partial_r \subseteq \partial_\lambda\). Here \(\partial_r\) (resp. \(\partial_\lambda\)) is the divisor of \(\partial\) such that \(r\) (resp. \(\lambda\)) generates \(\partial_r^{-1}/O_k\) (resp. \(\partial_\lambda^{-1}a/\alpha\)).

The forgetful functor defines a morphism of stacks
\[\text{pr} : \mathcal{Z}(m) \to \mathcal{C}, \quad (E, \iota, \beta) \mapsto (E, \iota)\).

Recall from section 5 of [13] that the Arakelov degree of \(\mathcal{Z}(m)\) is defined to be
\begin{equation}
\hat{\deg} \mathcal{Z}(m) = \sum_p \log N(p) \sum_{x \in \mathcal{Z}(m)(\kappa(p))} \frac{1}{|\text{Aut}_C(\text{pr}(x))|} \log(x)
= \frac{1}{w_k} \sum_p \log N(p) \sum_{x \in \mathcal{Z}(m)(\kappa(p))} \log(x).
\end{equation}

Here \(p\) runs over the primes of \(k\), \(\kappa(p)\) is the residue field of \(k\) at \(p\), and \(\log(x)\) is the length of the local ring \(O_{\mathcal{Z}(m), x}\):
\begin{equation}
\log(x) = \text{length of } O_{\mathcal{Z}(m), x} = \text{length of } \hat{O}_{\mathcal{Z}(m), x}.
\end{equation}

**Remark 2.3.** We choose a preimage \(\tilde{\lambda}\) of \(\lambda\) in \(\partial^{-1}a\) and \(\tilde{r}\) of \(r\) in \(\partial^{-1}\). For \(\ell \mid \Delta\), denote by \(\lambda_\ell\) (resp. \(r_\ell\)) the image of \(\tilde{\lambda}\) (resp. \(\tilde{r}\)) in \(\partial^{-1}a \otimes \mathbb{Z}_\ell\) (resp. \(\partial^{-1} \otimes \mathbb{Z}_\ell\)). Then the condition \(r + \beta \lambda \in O_E := \text{End}(E)\) is the same as
\[r_\ell + \beta_\ell \lambda_\ell \in O_{E, \ell} = O_E \otimes \mathbb{Z}_\ell, \quad \text{for all } \ell \mid \Delta.
\]
We write \(\lambda\) (resp. \(r\)) for the adele with components \(\lambda_\ell\) (resp. \(r_\ell\)) for \(\ell \mid \Delta\) and 0 elsewhere.

The following scaling relation is immediate.

**Lemma 2.4.** For \(\alpha \in k^x\), there is an isomorphism
\[\mathcal{Z}(m; a, \lambda, r) \xrightarrow{\sim} \mathcal{Z}(m; a\alpha, a\lambda, r), \quad (E, \iota, \beta) \mapsto (E, \iota, \beta \iota(\alpha)^{-1}).\]
For a negative integer $m < 0$, we define an ‘arithmetic cycle for $C$ supported at $\infty$’ as follows\(^2\). For a CM elliptic curve $(E, \iota)$ over $\mathbb{C}$, let $E^{\text{top}}$ be the underlying real torus. As in (2.1), we define the lattice of special endomorphisms

$$L(E^{\text{top}}, \iota) = \{ j \in \text{End}(E^{\text{top}}) : j \circ \iota(a) = \iota(a) \circ j, \ a \in O_k \},$$

equipped with a quadratic form $Q(j) = -j^2$. It is easy to check that $L(E^{\text{top}}, \iota)$ is a projective $O_k$-module of rank 1 and that the quadratic form $Q$ is negative definite.

**Definition 2.5.** Let $Z^{\text{top}}(m)$ be the category of triples $(E, \iota, \beta)$ where

1. $(E, \iota) \in C(\mathbb{C})$
2. $\beta \in L(E^{\text{top}}, \iota) a^{-1}$,
3. $r + \beta \lambda \in O_{E^{\text{top}}} : = \text{End}(E^{\text{top}})$,
4. $Q(\beta) = \frac{m}{N(a)}$.

The forgetful functor defines a map $\text{pr} : Z^{\text{top}}(m) \to C(\mathbb{C})$ with finite fibers. We denote the set of isomorphism classes of objects in $Z^{\text{top}}(m)$ by $\underline{Z}^{\text{top}}(m)$.

For a negative integer $m < 0$ and a parameter $v \in \mathbb{R}_+^\times$, we define a real valued function on $C(\mathbb{C})$ by

\[(2.4) \quad Z(m, v)(E, \iota) = Z(m, v; a, \lambda, r)(E, \iota) = \sum_{x \in \underline{Z}^{\text{top}}(m)} \beta_1(4\pi |m|v) \text{lg}(x, v),\]

where

\[(2.5) \quad \beta_1(a) = \int_1^\infty u^{-1} e^{-ua} du\]

is ‘length’ of the $x$. Here, $\Delta(\lambda)$ is given by (1.4) and

$$\beta_1(a) = \int_1^\infty u^{-1} e^{-ua} du$$

is the partial Gamma function or exponential integral.

As in section 6 of [13], we view $Z(m, v)$ as an Arakelov divisor on $C$. Its Arakelov degree is

\[(2.6) \quad \wh{\deg} Z(m, v) = \sum_{x \in \underline{Z}^{\text{top}}(m)} \frac{1}{|\text{Aut}_C(\text{pr}(x))|} \text{lg}(x, v) = \frac{1}{w_\varepsilon} \beta_1(4\pi |m|v) |\underline{Z}^{\text{top}}(m)|.\]

Note that, for $m < 0$, the coefficient of $q^m$ in the generating function in [13] is $\beta_1(4\pi |m|) w_\varepsilon \rho(-m)$ where $w_\varepsilon = 2$. For comparison with this case, we note the following.

**Lemma 2.6.** Suppose that $\Delta$ is a prime. Then, for $a = O_k$ and $r = \lambda = 0$,

$$|\underline{Z}^{\text{top}}(m)| = w_\varepsilon \rho(-m),$$

where $\rho(n)$ is the number of ideal classes of $k$ of norm $n$.

**Proof.** Suppose that $E = \mathbb{C}/b$, and let $j_\sigma : \mathbb{C}/b \overset{\sim}{\to} \mathbb{C}/\bar{b}$ be the topological isomorphism given by complex conjugation. Then, for $\beta \in L(E^{\text{top}}, \iota)$ with $Q(\beta) = m$, map $j_\sigma \circ \beta : \mathbb{C}/b \to \mathbb{C}/\bar{b}$ is $O_k$-linear and holomorphic and thus is given by multiplication by some $\beta \in \bar{b}^{-1}$ with $N(\beta) = -Q(\beta) = -m$. The integral ideal $\beta \bar{b}^{-1} \subset O_k$ has norm $-m$ and lies in the ideal class $[\bar{b}^{-1}] = [b]^2$. As $b$ varies

\(^2\)This construction is based on Gross’s observation [13, Page 383] that $E^{\text{top}}$ should be viewed as the archimedean analogue of a supersingular elliptic curve, since its endomorphism algebra $\text{End}(E^{\text{top}}) \simeq M_2(\mathbb{Z})$ is a maximal order in the quaternion algebra $\mathbb{H}$ ramified at $\infty$ and $\infty$, i.e., everywhere unramified!
over representatives for the ideal classes, so does \([6]^{2}\), since the genus group is trivial for a prime discriminant, hence the claim.

\[\square\]

Finally, we define \(Z(0, v)\) to be the Arakelov divisor supported at \(\infty\) with degree
\[
\deg \tilde{Z}(0, v) = \begin{cases} 
-\Lambda'(1, \chi) - \frac{1}{2} \Lambda(1, \chi) \log(v) & \text{if } r \in O_k, \\
0 & \text{if } r \not\in O_k,
\end{cases}
\]

where \(^3\)
\[
\Lambda(s, \chi) = |\Delta|^{\frac{s}{2}} L_{\infty}(s, \chi)L(s, \chi)
\]

is the complete \(L\)-function of \(\chi\). Here
\[
L_{\infty}(s, \chi) = \pi^{-\frac{\chi(1)}{2}} \Gamma(s + \frac{1}{2}).
\]

Note that the constant term of the generating function in \([13]\) is
\[
-w_k \left( \Lambda'(1, \chi) + \frac{1}{2} \Lambda(1, \chi) \log(v) \right).
\]

With these definitions and for \(\tau = u + iv \in \mathfrak{H}\), we define the generating function
\[
\tilde{\phi}(\tau; a, \lambda, r) = \deg \tilde{Z}(0, v) + \sum_{m \in \mathbb{Z}_{<0}} \deg \tilde{Z}(m, v) q^m + \sum_{m \in \mathbb{Z}_{>0}} \deg \tilde{Z}(m) q^m,
\]

where we have suppressed the dependence on \((a, \lambda, r)\) on the right side.

The purpose of Part I is to prove that the generating function \(\tilde{\phi}(\tau; a, \lambda, r)\) is a (non-holomorphic) modular form of weight 1 for some congruence subgroup of \(SL_2(\mathbb{Z})\), and to identify it with the central derivative of an incoherent Eisenstein series. This is a generalization of the main result in \([13]\). Indeed, in the case \(|\Delta| = q, a = O_k, \lambda = r = 0, \) our \(\tilde{\phi}(\tau, O_k, 0, 0)\) is just \(1/w_k\) times the generating function in \([13]\). The new idea is to use the Siegel-Weil formula to avoid some lengthy explicit calculations.

### 2.2. Quadratic spaces and Eisenstein series

We briefly review some basic facts about the Weil representation and Eisenstein series for use in the rest of the paper, referring to \([16]\) for more information. Let \(\chi\) be a quadratic character of \(\mathbb{A}^\times/\mathbb{Q}^\times\).

For a quadratic space \((V, Q)\) of dimension \(n\) over \(\mathbb{Q}\) with \(^4\) \(\chi_V = \chi\), we obtain a collection \(V = \{V_p\}_{p<\infty}\) of local quadratic spaces, \(V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p\), and \(V(A) = \prod_{p \leq \infty} V_p\) is the restricted product with respect to the collection of compact open subgroups \(L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p\), for any lattice \(L\) in \(V\).

More generally, we can consider a collection \(V = \{V_p\}\) of local quadratic spaces with \(\chi_{V_p} = \chi_p\) for all \(p\) and agreeing with a coherent collection at almost all places\(^5\). The restricted product \(V(A) = \prod_{p \leq \infty} V_p\) is then defined, and the collection \(V\) is called coherent (resp. incoherent) if the global Hasse invariant
\[
\epsilon(V) = \prod_{p \leq \infty} \epsilon(V_p)
\]

\(^3\)Note that we have added the factor \(|\Delta|^{\frac{1}{2}}\) here as compared with the convention used in \([13]\).

\(^4\)Recall that \(\chi_V(\sigma) = ((-1)^{n+1} \det V, \sigma)\).

\(^5\)This means that there exists a global quadratic space \(V\) and isomorphisms \(\phi_p : V_p \tilde{\to} V_p\) for almost all \(p\). If \((V', (\phi'_p))\) is another such collection, we require that \(\phi_p^{-1} \circ \phi'_p : V'_p \tilde{\to} V_p\) carry \(L'_p\) to \(L_p\) for almost all \(p\) for some lattices \(L\) in \(V\) and \(L'\) in \(V'\).
is +1 (resp. −1). In the coherent case, the collection \( \mathcal{V} \) arises by localization from a global quadratic space, unique up to isomorphism. In the incoherent case, there are infinitely many such global spaces at ‘distance one’ from \( \mathcal{V} \).

For a collection \( \mathcal{V} \) and our fixed additive character \( \psi \), there is a Weil representation \( \omega_{\mathcal{V}, \psi} \) of \( \text{SL}_2(\mathbb{A}) \) on \( S(\mathcal{V}(\mathbb{A})) = \bigotimes_p S(\mathcal{V}_p) \). It is given by

\[
\omega_{\mathcal{V}, \psi}(n(b))\varphi(x) = \psi(b Q_\mathbb{A}(x))\varphi(x), \quad b \in \mathbb{A},
\]

\[
\omega_{\mathcal{V}, \psi}(m(a))\varphi(x) = \chi(a)|a|^m\varphi(xa), \quad a \in \mathbb{A}^{	imes},
\]

\[
\omega_{\mathcal{V}, \psi}(w^{-1})\varphi(x) = \gamma(\mathcal{V}) \int_{\mathcal{V}(\mathbb{A})} \varphi(y)\psi(-(x, y))dy.
\]

Here \( \gamma(\mathcal{V}) = \prod_p \gamma(\mathcal{V}_p) \), where \( \gamma(\mathcal{V}_p) \) is the local Weil index [17, 9], an 8-th root of unity, and we write

\[
n(b) = \left( \begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right), \quad m(a) = \left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right), \quad w = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right).
\]

Let \( I(s, \chi) \) be the representation of \( \text{SL}_2(\mathbb{A}) \) induced from the character of the Borel subgroup \( B \) given by \( n(b)m(a) \mapsto \chi(a)|a|^{s+1} \). The induction is normalized in the standard way so that \( \text{Re}(s) = 0 \) is the unitary axis. There is an \( \text{SL}_2(\mathbb{A}) \)-equivariant map

\[
\Phi : S(\mathcal{V}(\mathbb{A})) \to I(s_0, \chi), \quad \Phi_{\varphi}(g) = \omega_{\mathcal{V}, \psi}(g)\varphi(0),
\]

where \( s_0 = \frac{n-2}{2} \). We denote the image of this map by \( R(\mathcal{V}) \), when \( \mathcal{V} \) is incoherent, and by \( R(V) \), when \( \mathcal{V} \) is coherent with associated global quadratic space \( V \). Notice that for \( n = 2 \), the representation \( I(0, \chi) \) is unitarizable, the \( R(\mathcal{V}) \)'s and \( R(V) \)'s are irreducible, and there is a decomposition

\[
I(0, \chi) = (\oplus_{\epsilon(\mathcal{V})=+1} R(\mathcal{V})) \oplus (\oplus_{\epsilon(\mathcal{V})=-1} R(\mathcal{V})).
\]

For \( \varphi \in S(\mathcal{V}(\mathbb{A})) \), let \( \Phi(s) = \Phi_{\varphi}(s) \in I(s, \chi) \) be the associated standard\(^6\) section with \( \Phi(g, s_0) = \Phi_{\varphi}(g) \), and let

\[
E(g, s, \varphi) = \sum_{\gamma \in B(\mathbb{Q}) \setminus \text{SL}_2(\mathbb{Q})} \Phi(g, s).
\]

be the associated Eisenstein series. It is absolutely convergent for \( \text{Re}s > 1 \), has meromorphic analytic continuation in \( s \), and is holomorphic at \( s = s_0 \). The Eisenstein series attached to \( \varphi \in S(\mathcal{V}(\mathbb{A})) \) will be called coherent (resp. incoherent) if \( \mathcal{V} \) is coherent (resp. incoherent). When \( \mathcal{V}_\infty \) is positive definite, and \( \varphi_\infty(x) = e^{-\pi Q_\infty(x)} \),

\[
E(\tau, s, \varphi) = v^{-\frac{n}{2}} E(g_\tau, s, \varphi)
\]

is a (non-holomorphic) modular form of weight \( \frac{n}{2} \) for some congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), where, for \( \tau = u + iv \in \mathbb{H} \), the upper half-plane, \( g_\tau = n(u)m(\sqrt{v}) \). The Eisenstein series has Fourier expansion

\[
E(g, s, \varphi) = \sum_{m \in \mathbb{Q}} E_m(g, s, \varphi),
\]

and, for \( m \neq 0 \),

\[
E_m(g, s, \varphi) = \prod_p W_{m,p}(g_p, s, \varphi_p).
\]

Here, for any \( m \in \mathbb{Q} \),

\[
W_{m,p}(g_p, s, \varphi_p) = \int_{\mathbb{Q}_p} \Phi_p(w^{-1}n(b)g_p, s)\psi_p(-mb)\,db
\]

\(^6\)This means that the restriction of \( \Phi(s) \) to the maximal compact subgroup \( K = K_\infty \prod_p \text{SL}_2(\mathbb{Z}_p) \) is independent of \( s \).
is the local Whittaker function. The constant term is given by
\[ E_0(g, s, \varphi) = \Phi(g, s) + \prod_p W_{0,p}(g_p, s, \varphi_p). \]

Finally, when \( \dim V = 2 \) and \( \chi = \chi_{k/Q} \), we define the normalized Eisenstein series
\[ E^*(\tau, s, \varphi) = \Lambda(s + 1, \chi) E(\tau, s, \varphi), \]
and normalized local Whittaker functions
\[ W^*_{m,p}(g_p, s, \varphi_p) = |\Delta|^{\frac{s+1}{2}} L_p(s + 1, \chi) W_{m,p}(g_p, s, \varphi_p), \]
for \( p < \infty \), and
\[ W^*_{m,\infty}(\tau, s, \varphi_\infty) = v^{-\frac{s}{2}} L_\infty(s + 1, \chi) W_{m,\infty}(g, s, \varphi_\infty). \]

For a finite prime \( p \), we will frequently write
\[ W^*_{m,p}(s, \varphi_p) = W^*_{m,p}(1, s, \varphi_p), \]
when \( g_p = 1 \).

Recall that the values at \( s = 0 \) of coherent Eisenstein series are given in terms of binary theta series, a classical version of the Siegel-Weil formula, while all incoherent Eisenstein series vanish at this point.

### 2.3. The central derivative of an incoherent Eisenstein series.

We fix data \( a, \lambda \in \mathcal{O}_k \) and \( r \in \mathcal{O}_k \). Let \( V = k \) with quadratic form \( Q(x) = -N(\partial^{-1} a) N(x) \). Let \( V \) be the incoherent collection with \( V_p = V_p \) for all finite primes and with \( V_\infty = \mathbb{R}_\ell, Q_\infty(x) = N(\partial^{-1} a) N(x) \). Recall that \( N(\partial_{\lambda}) = \Delta(\lambda), (1.4) \).

Following [10], for a non-zero rational number \( m \), let \( \text{Diff}(V, m) \) be the set of primes \( p \) where \( V_p \) does not represent \( m \). It is clear that \( p \in \text{Diff}(V, m) \) if and only if
\[ \chi_p(-m) = \begin{cases} -1 & \text{if } p < \infty, \\ 1 & \text{if } p = \infty, \end{cases} \]
where \( \chi_p(x) = (\Delta, x)_p \). In particular, \( |\text{Diff}(V, m)| \) is odd.

Let \( \varphi = \varphi_{a,\lambda,r} = \otimes_\ell \varphi_\ell \in S(V(A)) \), where
\[ \varphi_\ell(x) = \begin{cases} \text{char}(a_\ell^{-1})(x) & \text{if } \ell \nmid \Delta_\infty, \\ \text{char}(a_\ell)(x) \cdot \text{char}(-r_\ell + O_{k,\ell})(x \lambda_\ell) & \text{if } \ell | \Delta, \\ e^{-2\pi Q_\infty(x)} & \text{if } \ell = \infty. \end{cases} \]

Here \( r_\ell \) is the image of \( r \) in \( (\mathcal{O}_k) \otimes \mathbb{Z}_\ell \). Let \( E^*(\tau, s; \varphi) = E^*(\tau, s; a, \lambda, r) \) be the associated normalized incoherent Eisenstein series.

Our first main result is the following.

**Theorem 2.7.** Assume \( 2 \nmid \Delta \). Then
\[ E^*'(\tau, 0; a, \lambda, r) = -2 \hat{\phi}(\tau; a, \lambda, r). \]

**Remark 2.8.** The assumption \( 2 \nmid \Delta \) is technical and is only made to simplify Proposition 5.5 below.
We will prove the theorem by comparing Fourier coefficients, i.e., by proving that, for each integer \( m = m/\Delta(\lambda) \),
\[
E^s_{m}(\tau, 0; a, \lambda, r) = -2\tilde{\deg} Z(m, v) q^m.
\]

For \( m < 0 \) we will sketch a proof very similar to the case \( m > 0 \) in Section 6. The case \( m = 0 \) follows from the definition of \( Z(0, v) \) and the computation of the constant term of the Eisenstein series, and is left to the reader. The proof of the case \( m > 0 \) can be divided into three parts. It is easy to show that \( Z(m) \) can only be supported at primes \( p \) non-split in \( k \). First, in Section 3, we compute \( Z(m)(\mathbb{F}_p) \), for such a \( p \). Using the fact that the class group \( \text{Cl}(k) \) acts transitively on \( C(\mathbb{F}_p) \), we can write this quantity as a theta integral, thereby avoiding a lengthy calculation involving genus theory that disfigured an earlier draft of this paper. This device is inspired by Ben Howard’s lectures at the Morningside Center of Mathematics in Beijing, [7], in summer 2009. Using the Siegel-Weil formula, we see that \( Z(m)(\mathbb{F}_p) \) is equal to the \( m \)-th Fourier coefficient of a coherent Eisenstein series \( E^s(\tau, 0, \varphi(p)) \) (Theorem 3.6), which comes from a coherent collection \( V^{(p)} \) differing from \( V \) exactly at \( p \). In Section 4, we use Gross’s canonical lifting to compute the length \( \log(x) \) for a point \( x = (E, \iota, \beta) \in Z(m)(\mathbb{F}_p) \), and proved that it depends only on \( m \). Therefore, one has (Theorem 4.2 )
\[
\tilde{\deg} Z(m) q^m = \frac{1}{4} c_p(m) \log p \cdot E^s_{m}(\tau, 0; \varphi(p))
\]
for some number \( c_p(m) \) (basically \( \log(x) \)) depending only on \( m \). In Section 5, we first observe that the incoherent Eisenstein series \( E(\tau, 0; \varphi) \) and the coherent Eisenstein series \( E(\tau, 0; \varphi(p)) \) are closely related—a general phenomenon (Proposition 5.1):
\[
W^s_{m,p}(0, \varphi_p) E^s_{m}(\tau, 0; \varphi) = W^s_{m,p}(0, \varphi_p) E^s_{m}(\tau, 0; \varphi(p)).
\]

Now all we need is to prove
\[
\frac{W^s_{m,p}(0, \varphi_p)}{W^s_{m,p}(0, \varphi_p)} = -\frac{1}{2} c_p(m) \log p,
\]
which we do in Section 5 by explicit calculation.

2.4. Variants. Many variations are possible. For example, on the geometric side, Bruinier and the second author, [2], consider a slight different moduli problem with a different motivation as follows. Let \( b \) be a fractional ideal of \( k \), and let \( L = b \) be equipped with the integral quadratic form \( Q(x) = -\frac{N(x)}{N(b)} \), so that its dual lattice is \( L' = \partial^{-1}b \). For \( \mu \in L'/L \) and \( m \in \mathbb{Q}_{>0} \), we consider the moduli stack \( Z_\mu \) representing triples \( (E, \iota, \beta) \) such that

1. \( (E, \iota) \) is a CM elliptic curve as above,
2. \( \beta \in L(E, \iota)\partial^{-1}b = L(E, \iota) \otimes \mathbb{Z} L' \),
3. \( \mu + \beta \in O_E(b) \).
4. \( \deg \beta = mN(b) \).

For \( 2 \nmid \Delta \), choose a generator \( \lambda \) of \( b^{-1}/b^{-1} \partial \). Then it is easy to see that
\[
Z(m, \mu; b) = Z(m|\Delta|; \partial b^{-1}, \lambda, \lambda \mu).
\]

So Theorem 2.7 immediately gives the following result that was used in [2], Theorem 6.4.

**Corollary 2.9.** Let \( V = k \) with quadratic form \( Q(x) = -N(b)^{-1}N(x) \) as above, and assume that \( 2 \nmid \Delta \). Let \( \mathcal{V} b \) be the incoherent collection with \( \mathcal{V} b = V_\ell \) at all finite primes \( \ell \) and with \( \mathcal{V} b_\infty = k_\mathbb{R} \).
with $Q_\infty(x) = N(b)^{-1}N(x)$. Let $L = b$ and view $\hat{L}$ as a lattice in $V^b(\mathbb{A}_f)$. For $\mu \in L'/L$, let

$$\phi^\mu = \text{char}(\mu + \hat{L}) e^{-2\pi \frac{\cdot}{\Delta |b|}} \in S(V^b).$$

Then

$$\varphi_\mu(\tau) := \sum_{m \in \frac{1}{|\Delta|} \mathbb{Z}} Z(m, \mu, v; b) q^m = \frac{1}{2} E_{m}\tau,0,\phi^\mu.$$

Here $Z(m, \mu, v; b) = Z(m, \mu; b)$ for $m > 0$ as above, and is defined to be $Z(m|\Delta, v; \partial b^{-1}, \lambda, \lambda \mu)$ for $m \leq 0$.

In [2], this theorem is derived from a result in an early version of this paper. This theorem implies that

$$\varphi_L(\tau) = \sum_{\mu \in L'/L} \varphi_\mu(\tau) \text{char}(\mu + \hat{L})$$

is a vector-valued modular form for $SL_2(\mathbb{Z})$ with respect to the Weil representation [2].

3. Counting

We fix data $a, \lambda \in \partial^{-1}a/a$, $r \in \partial^{-1}/O_\mathfrak{k}$ and $m \in \mathbb{Q}_{\geq 0}$ as before, and we recall that the cycle $Z(m) = Z(m; a, \lambda, r)$ has support in the non-split primes of $O_\mathfrak{k}$. In this section, we express the quantity $|Z(m)(\mathbb{F}_p)|$ as a Fourier coefficient of a coherent Eisenstein series.

We fix a non-split prime $p$ and choose a supersingular CM elliptic curve $(E, \iota)$ over $\overline{\mathbb{F}}_p$. Then the space of special endomorphisms $V^E = V(E, \iota)$ is a one dimensional $\mathbb{k}$-vector space with a positive definite $\mathbb{Q}$-valued quadratic form $Q(j) = \deg j$. Define $\varphi^E = \bigotimes_\ell \varphi_\ell^E \in S(V^E(\mathbb{A}))$ with local components

$$\varphi_\ell^E(x) = \begin{cases} \text{char}(L(E, \iota)a_\ell^{-1})(x) & \text{if } \ell \nmid \Delta \infty, \\ \text{char}(L(E, \iota)a_\ell^{-1})(x) \cdot \text{char}(O_{E, \ell})(r_\ell + x \lambda_\ell) & \text{if } \ell | \Delta, \\ e^{-2\pi \text{deg} x} & \text{if } \ell = \infty. \end{cases}$$

Here $r_\ell$ and $\lambda_\ell$ are defined in Remark 2.3. Thus, writing $\varphi_\ell^E$ for the finite part of $\varphi^E$, we have $\varphi_\ell^E(\beta) \neq 0$ for $\beta \in V^E(\mathbb{Q}) = V(E, \iota)$ precisely when $\beta$ satisfies conditions (2) and (3) in the definition of $Z(m)$.

**Proposition 3.1.**

$$|Z(m)(\mathbb{F}_p)| q^m \frac{w_k}{N_a} (\tau, 0; \varphi^E).$$

**Proof.** Let $H = \text{SO}(V^E)$, so that $H(\mathbb{Q}) \cong \mathbb{k}_1$, and let $dh$ be the Tamagawa measure on $H(\mathbb{A})$, so that $\text{vol}(H(\mathbb{Q}) \backslash H(\mathbb{A})) = 2$. Let

$$\theta(\tau, h, \varphi^E) = v^{-\frac{1}{2}} \sum_{x \in V^E(\mathbb{Q})} \omega_{V^E, \psi}(g_\tau) \varphi^E(h^{-1}x) \varphi^E(h^{-1}_* x)$$

be the theta kernel, and let

$$I(\tau, \varphi^E) = \int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \theta(\tau, h, \varphi^E) \, dh$$

be the associated theta integral. Then the Siegel-Weil formula asserts [11, Theorem 4.1]

$$I(\tau, \varphi^E) = E(\tau, 0; \varphi^E).$$
On the other hand, a simple calculation gives

\[ \theta(\tau, h, \varphi^E) = \sum_{x \in V(E(Q))} \varphi^E(h^{-1}x) q^{\deg x}, \]

so that

\[ \theta_m(\tau, h, \varphi^E) = q^m \sum_{x \in V(E(Q)), \deg x = m} \varphi^E(h^{-1}x), \]

and

\[ I_m(\tau, \varphi^E) = \int_{H(Q) \backslash H(\mathbb{A})} \theta_m(\tau, h, \varphi^E) \, dh \]
\[ = \text{vol}(H(\mathbb{R})) \int_{H(Q) \backslash H(\mathbb{A})} \sum_{x \in V(E(Q)), \deg x = m} \varphi^E(h^{-1}x) \, dh \]

Let \( \pi(t) = tt^{-1} \) be the map from \( k^\times \) to \( H(Q) \). It induces an isomorphism

\[ \text{Cl}(k) = k^\times \backslash k^\times_1 / \tilde{O}_k^\times \cong H(Q) \backslash H(\mathbb{A}_f) / \pi(\tilde{O}^\times_1). \]

Recall [13, Section 5] that \( \text{Cl}(k) \) acts simply transitively on \( C(F_p) \) via the Serre construction. For a finite idèle \( t \in k^\times_1 \), we write \( t.E = (t) \otimes O_k E \), where \((t)\) is the ideal ‘generated by’ \( t \). For any \( t \in \tilde{O}_k^\times \), one can check that

\[ \varphi^E_f(\pi(t)x) = \varphi^E_f(x), \]

and thus we have

\[ I_m(\tau, \varphi^E) q^{-m} \]
\[ = \text{vol}(H(\mathbb{R})) \frac{\text{vol}(\pi(\tilde{O}_k^\times))}{|k^1 \cap \pi(\tilde{O}_k^\times)|} \sum_{x \in V(E(Q)), \deg x = m} \sum_{h \in H(Q) \backslash H(\mathbb{A}_f) / \pi(\tilde{O}_k^\times)} \varphi^E_f(h^{-1}x) \]
\[ = C \sum_{x \in V(E(Q)), \deg x = m} \sum_{[t] \in \text{Cl}(k)} \varphi^E_f(\pi(t^{-1})x) \]
for some constant \( C \). Now \( \varphi^E_f(\pi(t^{-1})x) = 1 \) or 0, and is 1 if and only if

\[ \pi(t^{-1})x \in L(E, \iota) a^{-1}, \quad r + \pi(t^{-1})x \in O_E, \]

or, equivalently,

\[ x \in \pi(t)L(E, \iota) a^{-1} = L(t.E, \iota) a^{-1}, \quad r + \beta \lambda \in tO_E t^{-1} = O_{t.E}. \]

Thus we have

\[ I_{m,\alpha}(\tau, \varphi^E) q^{-\frac{m}{\alpha(m)}} = C \sum_{[t] \in \text{Cl}(k)} \sum_{x \in L(t.E, \iota) a^{-1}} ^{\deg x = m} 1 \]
\[ = C \left| \mathcal{Z}(m)(\mathbb{F}_p) \right|. \]

To determine \( C \), replacing the theta function in the formula involving defining \( C \) by 1, one sees that

\[ 2 = \int_{H(Q) \backslash H(\mathbb{A})} dh = C \sum_{t \in \text{Cl}(k)} 1 = C h_k \]

and

\[ \Lambda(1, \chi) = |\Delta|^{\frac{1}{2}} \pi^{-1} L(1, \chi) = \frac{2h_k}{w_k}. \]
So $C^{-1} = \frac{w_\mathfrak{e}}{E} \Lambda(1, \chi)$, and

$$|Z(m)(\overline{\mathcal{O}_p})| q^{\frac{m}{N}} = C^{-1} I_{\frac{m}{N}}(\tau; \varphi^E) = \frac{w_\mathfrak{e}}{E} E_{\frac{m}{N}}(\tau; 0; \varphi^E),$$

as claimed.

The next step is to rewrite the Eisenstein series here in terms of data that does not involve the choice of $(E, \iota)$. Recall that $O_E = \text{End}(E)$ is a maximal order in the quaternion algebra $\mathbb{B}$ ramified at $p$ and $\infty$, and that, $\mathbb{B}$, pp. 376–378, we can write $L(E, \iota) = b \mathfrak{b}^{-1} \mathcal{P}_0^{-1} \delta$ for a fractional ideal $\mathfrak{b}$ and a certain auxiliary prime ideal $\mathcal{P}_0 | p_0$ which is split in $\mathbb{B}$. Here $\delta \in \mathbb{B}$ with $\iota(\delta) = \delta \iota(\bar{\delta})$, and $\delta^2 = \kappa$, with $\kappa = p_0 p$ if $p$ is inert in $\mathbb{k}$ and $\kappa = -p_0$ if $p$ is ramified in $\mathbb{k}$. Moreover $\chi_{E}(\kappa) = 1$ for $\ell \neq p$ and $\chi_{\ell}^{(E)}(\kappa) = -1$. The following result is an easy consequence of this.

**Lemma 3.2.** (Howard [7]) Write $L(E, \iota) \otimes \mathbb{Z}_\ell = \delta \bar{\ell} O_{\mathbb{Z}_\ell}$ with $\delta \bar{\ell}^2 = \kappa_{\ell} \in \mathbb{Z}_\ell$.

1. If $\ell \neq p$, $\delta \bar{\ell}$ can be chosen so that $\kappa_{\ell} = 1$.
2. For $\ell = p$, $(\Delta, \kappa_{\ell}) = -1$ and $\text{ord}_p \kappa_{\ell} = 1$ or 0 depending on whether $p$ is inert or ramified in $\mathbb{k}$.

The maximal order $O_E$ can then be described locally as follows, cf. [4], and Proposition 7.1 below.

**Lemma 3.3.** Let the notation be as in Lemma 3.2.

1. When $\ell$ is ramified in $\mathbb{k}$ and $\ell \neq p$, $\delta \bar{\ell}$, satisfying the conditions in Lemma 3.2, can be chosen so that

$$O_{E, \ell} = \{ \alpha + \beta \delta \bar{\ell} : \alpha \in \partial_{\ell}^{-1}, \alpha + \beta \in O_{\ell, \kappa} \}.$$

2. When $\ell = p$ is ramified in $\mathbb{k}$, there is a element $\mu_p \in O_{p, \kappa}$ with

$$\mu_p \bar{\mu}_p - \kappa_p \in p^{-1} \Delta \mathbb{Z}_p,$$

and such that

$$O_{E, \ell} = \{ \alpha + \beta \delta \bar{\ell} : \alpha, \beta \in \mathbb{w}_{p, \partial_{p, \kappa}^{-1}, \alpha + \bar{\mu}_p \bar{\beta} \in O_{p, \kappa} \}.$$

Here $\mathbb{w}_p$ is a uniformizer in $\mathbb{k}_p$.

These two lemmas allow us to identify the coherent collection associated to $V^E$ as $\mathbb{V}^{(p)} = \{ \mathbb{V}_{(p, \ell)} \}$, for $\mathbb{V}_{(p, \ell)} = \mathbb{k}_p$ with $Q_{E}(x) = \partial_{\ell} \mathbb{N}(x)$ where

$$\mathbb{\partial}_{\ell} = \begin{cases} -1 & \text{if } \ell \nmid p \infty, \\ -\kappa_p & \text{if } \ell = p, \\ 1 & \text{if } \ell = \infty. \end{cases}$$

When $\ell < \infty$, the local isomorphism is given by $\delta_{E} x \rightarrow x$ since $L(E, \iota) \otimes \mathbb{Z}_\ell = \delta \bar{\ell} O_{\mathbb{Z}_\ell}$. Under this identification, $\varphi^E$ becomes $\mathbb{\varphi}^{(p)} = \otimes \mathbb{\varphi}^{(p)}_{(p, \ell)}$ with $\varphi^{(p)}_{(p, \ell)}(x)$ given by

$$\varphi^{(p)} = \begin{cases} \text{char}((a_{\ell}^{-1})^{(x)}(x) & \text{if } \ell \nmid \Delta \infty, \\ \text{char}((a_{\ell}^{-1})^{(x)}(x) \cdot \text{char}(\bar{\mathfrak{e}}_{\ell} + O_{\kappa_{\ell}}(x) \lambda_{\ell}) & \text{if } \ell \nmid \Delta, \ell \neq p, \\ \text{char}((a_{\ell}^{-1})^{(x)}(x) \cdot \text{char}(\mathbb{w}_{p, \partial_{p, \kappa}^{-1}}(x) \lambda_{p}) \cdot \text{char}(\bar{\mathfrak{e}}_{p} + O_{\kappa_{p}}(\mu_{p} x) \lambda_{p}) & \text{if } \ell \Delta, \ell = p \\ e^{-2\pi x x} & \text{if } \ell = \infty. \end{cases}$$

Notice that $\varphi^{(p)}$ does not depend explicitly on the choice of $(E, \iota)$. The above argument gives

$$E(\tau, s; \varphi^E) = E(\tau, s; \varphi^{(p)}).$$

---

7The only linkage is through the element $\mu_p$. 
To obtain our result in final form, we rescale slightly.

**Lemma 3.4.** Let $V_1 = V_2 = V$ be a vector space over $\mathbb{Q}_\ell$ of even dimension $n$, with quadratic forms $Q_1(x) = aQ_2(x)$ for some $a \in \mathbb{Q}_\ell^{\times}$. Let $\varphi \in S(V) = S(V_1)$, and let $\Phi_i \in I(s, \chi)$ be the associated standard sections where $\chi = \chi_{V_1} = \chi_{V_2} = (\frac{(-1)^{n(n-1)/2}}{2} \det V_i).$ Then the corresponding Whittaker functions have the following relation

$$W_{m, \ell}(s_0, \Phi_1) = \frac{\gamma(V_1)}{\gamma(V_2)} |a|_{\ell}^{\frac{n}{2}} W_{m, \ell}(s_0, \Phi_2).$$

Here $s_0 = \frac{n-2}{2}$ and the convention of (2.12) is used.

**Proof.** This follows from simple calculation using (2.11). Write $w = \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right)$. Then

$$W_{m, \ell}(s_0, \Phi_1) = \int_{\mathbb{Q}_\ell} \omega V_1 \left( w^{-1} n(b) \right) \varphi(0) \psi(-mb) \, db$$

$$= \frac{\gamma(V_1)}{\gamma(V_2)} \int_{\mathbb{Q}_\ell} \omega V_1 \frac{\varphi(x) dV_1 x}{V_1 x} \psi(-mb) \, db$$

$$= \frac{\gamma(V_1)}{\gamma(V_2)} |a|_{\ell}^{\frac{n}{2}} \int_{\mathbb{Q}_\ell} \varphi(bQ_1(x)) \psi(-mb) \, db$$

$$= \frac{\gamma(V_1)}{\gamma(V_2)} |a|_{\ell}^{\frac{n}{2}} \int_{\mathbb{Q}_\ell} \varphi(bQ_2(x)) \psi(-mb) \, db$$

$$= \frac{\gamma(V_1)}{\gamma(V_2)} |a|_{\ell}^{\frac{n}{2}} W_{m, \ell}(s_0, \Phi_2),$$

as claimed. Here $dV_1$ is the Haar measure on $V_1$ self-dual with respect to quadratic forms $Q_i$. \qed

**Proposition 3.5.** Let $V_1 = (V, Q_1)$ and $V_2 = (V, Q_2)$ be positive definite quadratic spaces over $\mathbb{Q}$ of even dimension $n$ with $Q_1 = aQ_2$ for $a \in \mathbb{Q}_\ell^{\times}$. Let $\varphi \in S(V_1(\mathbb{A})) = S(V_2(\mathbb{A}))$. Let $\varphi_i, \infty \in S(V_i(\mathbb{R}))$, $\varphi_i, \infty(x) = e^{-2\pi Q_i(x)}$ be the Gaussian, and let $\Phi_i \in I(s, \chi)$ be the standard section associated to $\varphi_i, \infty \otimes \varphi \in S(V_i(\mathbb{A}))$, where $\chi = \chi_{V_i}$. Let $s_0 = \frac{n-2}{2}$. Then

$$E_{am}(\tau, s_0; \Phi_1) q^{-am} = a_{\frac{n-2}{2}} E_{m}(\tau, s_0; \Phi_2) q^{-m},$$

i.e., the $(am)$-th Fourier coefficient of $E(\tau, 0; \Phi_1)$ is the same as the $m$-th Fourier coefficient of $E(\tau, 0; \Phi_2)$, up to a constant multiple.

**Proof.** When $m \neq 0$, one has by Lemma 3.4

$$E_{am}(\tau, s_0; \Phi_1) = W_{am, \infty}(\tau, s_0; \Phi_{1, \infty}) \prod_{\ell \leq \infty} W_{am, \ell}(s_0; \Phi_{1, \ell})$$

$$= W_{am, \infty}(\tau, s_0; \Phi_{1, \infty}) \prod_{\ell \leq \infty} \frac{\gamma(V_{1, \ell})}{\gamma(V_{2, \ell})} |a|_{\ell}^{\frac{n-2}{2}} \prod_{\ell \leq \infty} W_{m, \ell}(s_0, \Phi_{2, \ell}).$$

Recall from [13, Proposition 2.6] that $\Phi_{1, \infty}(s) = \Phi_{2, \infty}(s)$ is the normalized eigenfunction of weight $\frac{n}{2}$ and

$$W_{am, \infty}(\tau, s_0; \Phi_{1, \infty}) = W_{am, \infty}(\tau, s_0; \Phi_{2, \infty}) = W_{m, \infty}(\tau, s_0; \Phi_{2, \infty}) q^{am} q^{-m}.$$ 

Recall also [17] that

$$\prod_{\ell \leq \infty} \gamma(V_{1, \ell}) = \prod_{\ell \leq \infty} \gamma(V_{2, \ell}) = 1.$$

Since $\gamma(V_{1, \infty}) = \gamma(V_{2, \infty})$, we have proved the proposition for $m \neq 0$. The case $m = 0$ is similar and is left to the reader. \qed
For $a \in \mathbb{Q}_{>0}$, let $\tilde{V}^{(p,a)}$ be the coherent collection with $\tilde{V}^{(p,a)}_\ell = k_\ell$ and $Q_\ell(x) = \beta^{(p,a)}_\ell N(x)$, where

\begin{equation}
\beta^{(p,a)}_\ell = \begin{cases} 
-a & \text{if } \ell \nmid p\infty, \\
a & \text{if } \ell = \infty, \\
-k_p a & \text{if } \ell = p.
\end{cases}
\end{equation}

Here $k_p \in \mathbb{Z}$ with $(\Delta, k_p)_p = -1$ and $\text{ord}_p k_p = 1$ or 0, depending on whether $p$ is inert or ramified in $k$.

Let $\tilde{\varphi}^{(p,a)} = \otimes \tilde{\varphi}^{(p,a)}_\ell \in S(V^{(p,a)}(\mathbb{A}))$ be given by

\begin{equation}
\varphi^{(p,a)}_\ell(x) = \begin{cases} 
\text{char}(\mathbb{F}_p^{-1})(x) & \text{if } \ell \nmid \Delta \infty, \\
\text{char}(\mathbb{F}_p^{-1})(x) \cdot \text{char}(-\bar{\ell} + O_{k\ell})(x\ell) & \text{if } \ell | \Delta, \ell \neq p, \\
\text{char}(\mathbb{F}_p^{-1})(x) \cdot \text{char}(\alpha_p^{-1})(x\ell) \cdot \text{char}(-\bar{\ell} + O_{k\ell})(\mu_p x\lambda_p) & \text{if } \ell | \Delta, \ell = p, \\
e^{-2\pi q\infty(x)} & \text{if } \ell = \infty.
\end{cases}
\end{equation}

In the special case where $a = N(\partial_\lambda^{-1} a)$, we write $V^{(p)} = \tilde{V}^{(p,a)}$ and $\varphi^{(p)} = \tilde{\varphi}^{(p,a)}$. The main result of this section is the following.

**Theorem 3.6.** Assume that $p$ is non-split in $k$. For any $a \in \mathbb{Q}_{>0}$,

$$|Z(m)(\mathbb{F}_p)| q^{m/\Delta(a)} = \frac{w_k}{4} E_{\frac{m}{N(a)}}^1(\tau, 0; \varphi^{(p,a)}).$$

In particular, for $a = N(\partial_\lambda^{-1} a)$, and $m = \frac{m}{\Delta(a)}$,

$$|Z(m)(\mathbb{F}_p)| q^{m} = \frac{w_k}{4} E_{\frac{m}{N(a)}}(\tau, 0; \varphi^{(p)}).$$

**Proof.** Noting that $\tilde{\varphi}^{(p)} = \varphi^{(p),1}$, we have, by Proposition 3.5, (3.3), and Proposition 3.1,

$$E_{\frac{m}{N(a)}}^1(\tau, 0; \varphi^{(p,a)}) q^{-\frac{m}{N(a)}} = E_{\frac{m}{N(a)}}^1(\tau, 0; \varphi^{(p)}) q^{-\frac{m}{N(a)}} = E_{\frac{m}{N(a)}}(\tau, 0; \varphi^{(p)}) q^{-\frac{m}{N(a)}}.$$
Proposition 4.1. Let the notation be as above. Then
\[ \hat{\mathcal{O}}_{\mathcal{Z}(m),x} = W_{O_p}(\overline{\mathbb{F}}_p)/(\pi^\nu), \]
where
\[ \nu = \nu_p(m) = \begin{cases} \frac{1}{2} (\text{ord}_p(m) + 1) & \text{if } p \nmid \Delta, \\ \text{ord}_p(mN(\partial \lambda^{-1})) & \text{if } p \mid N(\partial). \end{cases} \]
In particular, \( \log(x) = \nu_p(m) \) depends only on \( \text{ord}_p(m) \) and \( \lambda \).

Proof. The point \( x \) corresponds to a collection \( (E, \iota, \beta) \) over \( \overline{\mathbb{F}}_p \) for a special quasi-endomorphism \( \beta \in L(E, \iota) \) with \( N(a)N(\beta) = m \), and \( r + \beta \lambda \in \text{End}(E) \).

Let \( X = E[\infty] \) be the \( p \)-divisible group of \( E \). Then \( r + \beta \lambda \) determines an endomorphism \( \xi_p = (r + \beta \lambda)_p \) of \( X \). Gross’s result, [5, Proposition 4.3], determines the deformation locus of \( (X, \iota, \xi_p) \) inside that of \( (X, \iota) \) as \( \text{Spf} W_{O_p}(\overline{\mathbb{F}}_p)/(\pi^\nu) \), where
\[ \nu = \left\{ \begin{array}{ll} \frac{1}{2} (\text{ord}_p(m) + 1) & \text{if } p \nmid \Delta, \\ \text{ord}_p(m \Delta \lambda^{-1}) & \text{if } p \mid \Delta, \end{array} \right. \]
as claimed. \( \square \)

Combining this proposition with Theorem 3.6, we obtain the following intermediate result.

Theorem 4.2. For a positive integer \( m > 0 \) with \( \text{Diff}(\mathcal{V}, m) = \{p\} \), and any positive rational number \( a > 0 \), then
\[ \hat{\deg} \mathcal{Z}(m) q_{\text{am}(\alpha)} = \frac{1}{4} c_p(m) \log p \cdot E_{\text{am}(\alpha)}^*(\tau, 0, \varphi^{(p,a)}). \]
Here
\[ c_p(m) = \left\{ \begin{array}{ll} \text{ord}_p(m) + 1 & \text{if } p \nmid \Delta, \\ \text{ord}_p(m \Delta \lambda^{-1}) & \text{if } p \mid \Delta. \end{array} \right. \]

5. Whittaker functions and their derivatives

Proposition 5.1. Suppose \( p \in \text{Diff}(\mathcal{V}, \frac{m}{\Delta(\lambda)}) \). Then
\[ W_{m,p}^{*}(0, \varphi^{(p)}) E_{m}^{*}(\tau, 0, \varphi) = W_{m,p}^{*}(0, \varphi^{(p)}) E_{m}^{*}(\tau, 0, \varphi^{(p)}). \]
Here \( \varphi \) is defined in (2.14) and \( \varphi^{(p)} \) is defined in (3.5).
Proof. Notice first that \( \mathcal{V}_\ell = V^{(p)}_\ell \) as quadratic spaces and \( \varphi_\ell = \varphi^{(p)}_\ell \) for all \( \ell \neq p \). Since \( p \in \text{Diff}(\mathcal{V}, \mathfrak{m}) \), \( W^*_m(0, \varphi_p) = 0 \), and so

\[
W^*_m(0, \varphi_p) E^*_m(\tau, 0, \varphi) = W^*_m(0, \varphi_p) \prod_{\ell \neq p} W^*_m(0, \varphi_\ell) = W^*_m(0, \varphi_p) \prod_{\ell} W^*_m(0, \varphi_\ell) = W^*_m(0, \varphi_p) E^*_m(\tau, 0, \varphi^{(p)}).
\]

\[\square\]

To compute the central derivative of the Whittaker functions, we need the following few lemmas, whose proofs can be found in [8]. Special cases can be found in [13], [21], and [16].

Let \( K \) be a \( p \)-adic local field, and let \( L \) be a quadratic extension (including \( K \oplus K \)) with associated quadratic character \( \chi \). Let \( \psi_K \) be a fixed unramified character of \( K \). For \( t \in O_K \), \( t \neq 0 \), let \( V_t = (L, t\overline{x}) \) be the corresponding binary quadratic space, and, for \( \mu \in \partial^{-1}_L \), let \( \phi^\mu = \text{char}(\mu + O_L) \in S(V_t) \). Let \( \tau \) be a uniformizer of \( K \) and \( q = |O_K/\tau| \). Let

\[
W^*_m(s, \phi^\mu) = |d_{L/K}|^{s+\frac{1}{2}} L(s + 1, \chi) W_m(s, \phi^\mu)
\]

be the normalized Whittaker function.

The following is well known, see for example [13], [21].

**Lemma 5.2.** Suppose that \( L/K \) is unramified and that \( t \in O_K^\times \). If \( m \notin O_K \), then \( W^*_m(s, \phi^0) = 0 \). If \( m \in O_K \), then

\[
\gamma(V_t)^{-1} W^*_m(s, \phi^0) = \sum_{r=0}^{\text{ord}_K m} (\chi(\tau)q^{-s})^r.
\]

In particular, \( W^*_m(0, \phi^0) = 0 \) if and only if \( \chi(m) = -1 \) and \( \text{ord}_K m \) is even, and, in this case,

\[
\gamma(V_t)^{-1} W^*_m(0, \phi^0) = \frac{1}{2}(1 + \text{ord}_K m) \log q.
\]

**Lemma 5.3.** ([8, Lemma 4.6.3]) Suppose that \( L/K \) is ramified, and let \( N = \text{ord}_K m, c = \text{ord}_K t, \) and \( X = q^{-s} \).

(1) If \( m \notin O_K \), then \( W^*_m(s, \Phi_t^0) = 0 \). If \( m \in O_K \), then

\[
\gamma(V_t)^{-1} W^*_m(s, \phi^0) = \begin{cases} 
\frac{|t|}{(1 - X) \sum_{n=0}^N (qX)^n} & \text{if } N < c, \\
\frac{|t|}{(1 - X) \sum_{n=0}^{c-1} (qX)^n + (X^c + \chi(tm)X^{f+N})} & \text{if } N \geq c.
\end{cases}
\]

(2) Suppose that \( m \in O_K, m \neq 0 \), then

\[
\gamma(V_t)^{-1} W^*_m(0, \phi^0) = \begin{cases} 
0 & \text{if } N < c, \\
1 + \chi(mt) & \text{if } N \geq c.
\end{cases}
\]

In particular, \( W^*_m(0, \phi^0) = 0 \) if and only if \( N < c \) or \( \chi(tm) = -1 \). In this case

\[
\gamma(V_t)^{-1} W^*_m(0, \phi^0) = \log q \cdot \begin{cases} 
q^{-c} \sum_{n=0}^N q^n & \text{if } N < c, \\
\frac{1 - q^{-c}}{q(1 - q^{-1})} + (N + f - c) & \text{if } N \geq c.
\end{cases}
\]
Lemma 5.4. ([8, Lemma 4.6.4]) Let the notation be as in Lemma 5.3 and assume that \( p \neq 2 \) and that \( \mu \notin O_K \). If \( m \notin \ell \mu + O_K \), then \( W^*_m(s, \phi_{\ell}^\mu) = 0 \). If \( m \in \ell \mu + O_K \), write \( c(m, \mu) = \ord_p(m - \ell \mu) \). Then

\[
\gamma(V_{\ell})^{-1} W^*_m(s, \phi_{\ell}^\mu) = \begin{cases} 
[t] (1 - X) \sum_{0 \leq n < c(m, \mu)} (qX)^n, & \text{if } c(m, \mu) < c, \\
[t] (1 - X) \sum_{0 \leq n < c} (qX)^n + X^c, & \text{if } c(m, \mu) \geq c.
\end{cases}
\]

In particular, \( W^*_m(0, \phi_{\ell}^\mu) = 0 \) if and only if \( c(m, \mu) < c \). In this case,

\[
\gamma(V_{\ell})^{-1} W'_m(0, \phi_{\ell}^\mu) = |t| \log q \sum_{0 \leq n < c(m, \mu)} q^n.
\]

Proposition 5.5. Let \( m > 0 \) be a rational number, and assume that \( p \in \Diff(\mathcal{V}, m) \).

1. Suppose that \( p \) is inert in \( \kappa \). If \( m \notin \mathbb{Z}_p \), then \( W^*_{m,p}(s, \varphi^{(p)}) = 0 \). If \( m \in \mathbb{Z}_p \), then

\[
\frac{W^*_{m,p}(0, \varphi_p)}{W^*_{m,p}(0, \varphi_p^{(p)})} = -\frac{1}{2} (1 + \ord_p m) \log p.
\]

2. If \( p \neq 2 \) is ramified in \( \kappa \) and \( r_p \notin O_{K,p} \),

\[
W^*_{m,p}(s, \varphi^{(p)}) = 0, \quad \text{and} \quad W^*_{m,p}(s, \varphi_p) = 0.
\]

3. Suppose that \( p \neq 2 \) is ramified in \( \kappa \) and that \( r_p \in O_{K,p} \). If \( m \notin \mathbb{Z}_p \), then \( W^*_{m,p}(s, \varphi^{(p)}) = 0 \). If \( m \in \mathbb{Z}_p \), then

\[
\frac{W^*_{m,p}(0, \varphi_p)}{W^*_{m,p}(0, \varphi_p^{(p)})} = -\frac{1}{2} \ord_p (mN(\partial \partial_{\lambda}^{-1})) \log p.
\]

4. In all cases,

\[
W^*_{m,p}(0, \varphi_p) = -\frac{1}{2} c_p(m) \log p \cdot W^*_{m,p}(0, \varphi_p^{(p)}),
\]

where \( c_p(m) \) is the number given in Theorem 4.2.

Proof. First assume that \( p \nmid \Delta \) is unramified in \( \kappa \). Then we can choose \( \alpha \in \mathfrak{a}_p \) with \( N_{\mathbb{Q}_p/\kappa} \alpha = N(\partial \partial_{\lambda}^{-1}) \) in \( \mathbb{Q}_p^\times \). So \( x \mapsto x \alpha \) gives an isomorphism of quadratic space \( V^{(p)}(\kappa_p, -N(\partial \partial_{\lambda}^{-1})x \alpha) \) to \( \hat{V}_p = (\kappa_p, -x \alpha) \), under which, \( \varphi_p = \text{char}(\mathfrak{a}_p^{-1}) \) becomes \( \tilde{\varphi}_p = \text{char}(O_{K,p}) \). Then, by Lemma 5.2, noting that \( \ord_p \partial_{\lambda} = 0 \),

\[
\gamma(V^{(p)}_p)^{-1} W^*_{m,p}(s, \varphi_p) = \gamma(\hat{V}_p)^{-1} W^*_{m,p}(s, \tilde{\varphi}_p) = \sum_{0 \leq r \leq \ord_p m} (\chi_p(p)^{-s})^r.
\]

For the same reason, \( x \mapsto x \alpha \) gives an isomorphism of quadratic space \( V^{(p)}_p = (\kappa_p, -\kappa_p N(\partial \partial_{\lambda}^{-1})x \alpha) \) to \( \hat{V}_p = (\kappa_p, -\kappa_p x \alpha) \), under which, \( \varphi^{(p)}_p = \text{char}(\mathfrak{a}_p^{-1}) \) becomes \( \tilde{\varphi}^{(p)}_p = \text{char}(O_{K,p}) \). So Lemma 5.2 gives

\[
\gamma(V^{(p)}_p)^{-1} W^*_{m,p}(0, \varphi^{(p)}_p) = \gamma(\hat{V}_p)^{-1} W^*_{m,p}(0, \tilde{\varphi}^{(p)}_p) = \text{char}(\mathbb{Z}_p)(m).
\]

Here we have used the fact that \( p \in \Diff(\mathcal{V}, m) \), which implies that \( \chi_p(m) = -1 \). So \( W^*_{m,p}(0, \varphi^{(p)}_p) \neq 0 \) if and only if \( m \in \mathbb{Z}_p \), and, in this case, we have

\[
\frac{W^*_{m,p}(0, \varphi_p)}{W^*_{m,p}(0, \varphi_p^{(p)})} = -\frac{1}{2} (1 + \ord_p m) \log p.
\]
where the negative sign comes from the fact

\[ \gamma(V_p^{(p)}) = -\gamma(V_p). \]

Next, we assume that \( p \neq 2 \) is ramified. Then \( \partial_p = \varpi O_{k,p} \), and so

\[ \varphi_p^{(p)}(x) = \text{char}(a_p^{-1})(x) \cdot \text{char}(\varpi \partial^{-1})(x\lambda_p) \cdot \text{char}(-\bar{r}_p + O_{k,p})(\mu_p x\lambda_p) \]

\[ = \begin{cases} 0 & \text{if } \text{ord}_p r_p < 0, \\ \text{char}(O_{k,p})(x\lambda_p) & \text{if } \text{ord}_p r_p = 0. \end{cases} \]

On the other hand,

\[ \varphi_p = \text{char}(a_p^{-1})(x) \cdot \text{char}(-\bar{r}_p + O_{k,p})(x\lambda_p) \]

\[ = \begin{cases} \text{char}(a_p^{-1})(x) \cdot \text{char}(-\bar{r}_p + O_{k,p})(x\lambda_p) & \text{if } \text{ord}_p r_p < 0, \\ \text{char}(O_{k,p})(x\lambda_p) & \text{if } \text{ord}_p r_p = 0. \end{cases} \]

To prove (2), we first assume that \( r_p \notin O_{k,p} \). This implies that \( \lambda_p \) generates \( \partial_p^{-1}a_p \), i.e., that \( p|\Delta(\lambda) \). In this case, \( x \mapsto x\lambda_p \) gives an isomorphism from \( V_p \) to \( (k, \frac{N(a\partial^{-1})}{\lambda_p\lambda_p} x \bar{x}) \), under which \( \varphi_p \)

becomes \( \bar{\varphi}_p = \text{char}(-\bar{r}_p + O_{k,p}) \). Set \( t = -\frac{N(a\partial^{-1})}{\lambda_p\lambda_p} \), and note that \( \text{ord}_p t = 0 \). Lemma 5.4 shows that \( W_{m,p}^*(s, \varphi_p) = 0 \) unless

\[ m - t(r_p \bar{r}_p) \in \mathbb{Z}_p, \]

which is impossible since \( p \in \text{Diff}(V, m) \).

To prove (3), we next assume that \( r_p \in O_{k,p} \). In this case, the map \( x \mapsto x\lambda_p \) gives an isomorphism between \( V_p \) and \( (k, \frac{N(a\partial^{-1})}{\lambda_p\lambda_p} x \bar{x}) \), under which \( \varphi_p \) becomes \( \text{char}(O_{k,p}) \). Set \( t = \frac{N(a\partial^{-1})}{\lambda_p\lambda_p} \in \mathbb{Z}_p^\times \). Then Lemma 5.3 shows that \( W_{m,p}^*(s, \varphi_p) = 0 \) unless \( m \in \mathbb{Z}_p \). In that case, the same lemma shows that

\[ \gamma(V_p)^{-1} W_{m,p}^*(0, \varphi_p) = \text{ord}_p(m \Delta) \log p, \]

and

\[ \gamma(V_p^{(p)})^{-1} W_{m,p}^*(0, \varphi_p^{(p)}) = 2. \]

Since \( \gamma(V_p) = -\gamma(V_p^{(p)}) \), the proposition is proved. \( \Box \)

**Proof of Theorem 2.7 for \( m > 0 \).** We may assume that \( m \) is an integer and \( \text{Diff}(V, m) = \{ p \} \), since otherwise both sides are zero. Furthermore, we assume \( r_p \in O_{k,p} \) and \( m \in \mathbb{Z}_p \), since otherwise both sides are zero by Proposition 5.5. Under these conditions, \( W_{m,p}^*(0, \varphi_p^{(p)}) \neq 0 \) by Proposition 5.5. Now Theorem 4.2 with \( a = \frac{N(a\partial^{-1})}{\lambda_p\lambda_p} \)

gives

\[ \widehat{\text{deg}} Z(m) q^m = -\frac{1}{4} c_p(m) E_m^*(\tau, 0, \varphi^{(p)}). \]

On the other hand, Propositions 5.1 and 5.5 give

\[ E_{m^{'}}^*(\tau, 0, \varphi) = \frac{W_{m,p}^*(0, \varphi_p^{(p)})}{W_{m,p}^*(0, \varphi_p^{(p)})} E_m^*(\tau, 0, \varphi^{(p)}) \]

\[ = -\frac{1}{2} c_p(m) E_m^*(\tau, 0, \varphi) \]

\[ = -2 \widehat{\text{deg}} Z(m) q^m. \]

as claimed.
Remark 5.6. Instead of \( V \) and \( \varphi \), we can use \( V^{(a)} \) and \( \varphi^{(a)} \) for any rational number \( a > 0 \). The only problem is that the identity

\[
\frac{W_{m,p}^{*(a)}(0, \varphi^{(a)}_p)}{W_{m,p}^{*}(0, \varphi^{(p,a)}_p)} = \frac{1}{2} c_p(m) \log p
\]

is not true in general. One can fix this by using a modified Eisenstein series, as was done in [15]. Let \( S \) be the set of primes dividing either the numerator or denominator of \( \frac{a\Delta(x)}{\Delta(x)} \), then there are holomorphic functions \( c_p(s) \) with \( c_p(0) = 0 \) and coherent Eisenstein series \( E^*(\tau, s, \varphi^{(p,a)}) \) such that the modified Eisenstein series

\[
\tilde{E}(\tau, s, \varphi^{(a)}) = E^*(\tau, s, \varphi^{(a)}) + \sum_{p \in S} c_p(s) E^*(\tau, s, \varphi^{(p,a)})
\]

satisfies

\[
\text{deg} Z(m; \alpha, \lambda, r) q^{\frac{am}{N(\alpha)}} = \tilde{E}_{N(\alpha)}^{*}(\tau, 0, \varphi^{(a)}).
\]

6. The case \( m < 0 \)

In this section, we prove Theorem 2.7 in the case \( m < 0 \). This case is simpler than the case \( m > 0 \) since the ‘length’ is artificially defined to be independent of the ‘points’ \( (E, t, \beta) \). The calculation of \( |Z_C(m)| \) can be dealt exactly the same way as in Section 3, so we only give sketch of the proof.

Let \( V^{(\infty)} = (k, -N(a\Omega^{-1})\tau \bar{x}) \) be the coherent quadratic space over \( A \). Notice that \( V^{(\infty)} \) differ from \( V \) at exact the prime \( \infty \).

We define \( \varphi^{(\infty)} = \prod \varphi^{(\infty)}_\ell \in S(V^{(\infty)}) \) as follows.

\[
(6.1) \quad \varphi^{(\infty)}_\ell(x) = \begin{cases} \text{char}(a_\ell^{-1})(x) & \text{if } \ell \nmid \Delta_{\infty}, \\ \text{char}(a_\ell^{-1})(x) \cdot \text{char}(-\bar{r}_\ell + O_{k, \ell})(x\lambda_\ell) & \text{if } \ell | \Delta, \\ e^{-2\pi N(a\Omega^{-1})\tau \bar{x}} & \text{if } \ell = \infty. \end{cases}
\]

Then we have

**Proposition 6.1.**

\[
|Z_C(m)| q^m = \frac{u_k}{4} E^*_m(\tau, 0, \varphi^{(\infty)}).
\]

**Proof.** (sketch) Fix \( (E, t) = (k \otimes_{Q} \mathbb{R})/O_{k, t} \in C(\mathbb{C}) \) with the fixed embedding of \( k \) into \( \mathbb{C} \) giving the complex structure on \( k \otimes_{Q} \mathbb{R} \). Let \( j_0(\tau \otimes x) = \bar{\tau} \otimes x \). Then \( j_0 \in L(E^{\text{top}}, t) \), and \( j_0^0 = 1 \). Moreover, \( L(E^{\text{top}}, t) = j_0 O_{k} \). Since \( \text{Cl}(k) \) acts on \( C(\mathbb{C}) \) simply transitively, the rest is exact the same argument as in Section 3. \( \square \)

**Proposition 6.2.** (1) For \( m < 0 \), one has

\[
W^*_m,\infty(\tau, 0, \varphi^{(\infty)}_\infty) E^*_m(\tau, 0, \varphi) = W^*_m,\infty,\prime(\tau, 0, \varphi^{(\infty)}_\infty) E^*_m(\tau, 0, \varphi^{(\infty)}).
\]

(2) For \( m < 0 \), \( W^*_m,\infty(\tau, 0, \varphi^{(\infty)}_\infty) \neq 0 \), and

\[
\frac{W^*_m,\infty,\prime(\tau, 0, \varphi^{(\infty)}_\infty)}{W^*_m,\infty(\tau, 0, \varphi^{(\infty)}_\infty)} = -\frac{1}{2} \beta_l(4\pi |m| v).
\]
Proof. (1) is exactly the same as Proposition 5.1.

(2) Notice that \( V^{(\infty)} = -V_\infty \). Thus, as in Lemma 3.4 and [13, Proposition 2.6]
\[
\gamma(V^{(\infty)})^{-1} W^{m}_{\frac{m+1}{2}}(\tau, 0, \varphi^{(\infty)}) q^{-\frac{m}{2\Delta(\lambda)}} = \gamma(V_\infty)^{-1} W^{m}_{\frac{m}{2\Delta(\lambda)}}(\tau, 0, \varphi_\infty) q^\frac{m}{2\Delta(\lambda)} = 2.
\]
By the same proposition, one has
\[
\gamma(V^{(\infty)})^{-1} W^{m}_{\frac{m+1}{2}}(\tau, 0, \varphi^{(\infty)}) q^{-m} = \beta_1(4\pi |m| v)
\]
This proves (2) since \( \gamma(V_\infty) = -\gamma(V^{(\infty)}) \).

Proof of Theorem 2.7 for \( m < 0 \). Now the proof for the case \( m < 0 \) is the same as that of \( m > 0 \). We leave the details to the reader. \( \square \)

Part II

7. Maximal orders and optimal embeddings

In this section, we summarize the results we need concerning maximal orders in a quaternion algebra \( B \) over \( \mathbb{Q} \) with an optimal embedding of \( O_k \). Of course, some of this material is rather classical and well known, [6], [3], [4], but we need certain detailed information for which we found no good reference. In this section, we do not assume that \( B \) is indefinite.

Let \( \Delta \) be the discriminant of \( O_k \), and let \( \chi \) be the Dirichlet character associated to \( k \). Let \( D = D(B) \) be the product of the primes that ramify in \( B \) and write \( D(B) = D_0D_1 \), where \( D_1 \) is the product of the primes which ramify in both \( k \) and \( B \). The primes dividing \( D_0 \) are all inert in \( k \). Let \( \partial \) be the different of \( k/\mathbb{Q} \) and let \( \partial_1 | \partial \) be the factor with \( N(\partial_1) = D_1 \). Write \( \partial = \partial_1\partial_2 \), and note that, if 2 is ramified in both \( B \) and \( k \), then \( \partial_1 \) and \( \partial_2 \) are not relatively prime. Let \( \Delta_2 = N(\partial_2) \).

Fix an embedding \( i : k \rightarrow B \) and write
\[
B = k + k\delta, \quad \text{with } \delta^2 = \kappa \in \mathbb{Q}^\times,
\]
where \( \delta\alpha = \bar{\alpha}\delta \) for \( \alpha \in k \). The element \( \delta \) is unique up to scaling by an element of \( k^\times \). Note that
\[
(\kappa, \Delta)_p = \inv_p(B). \quad \text{We write } [\alpha, \beta] = \alpha + \beta\delta \text{ and obtain an embedding}
\]
\[
(7.2) \quad B \hookrightarrow M_2(k), \quad [\alpha, \beta] \mapsto \begin{pmatrix} \alpha & \beta \\ \kappa\beta & \bar{\alpha} \end{pmatrix}.
\]
Note that we have an isomorphism
\[
(7.3) \quad B \overset{\sim}{\rightarrow} k^2, \quad [\alpha, \beta] \mapsto \begin{pmatrix} \beta \\ \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \kappa\beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
equivariant for the left action of \( B \) and such that
\[
[\alpha, \beta][\alpha_0, 0] \mapsto \bar{\alpha}_0 \begin{pmatrix} \beta \\ \bar{\alpha} \end{pmatrix},
\]
i.e., conjugate linear for the right action of \( k \). Note that this action is the restriction to \( B \otimes k \) of the action of \( B \otimes B \) on \( B \) defined by \( b_1 \otimes b_2 : x \mapsto b_1 \cdot x b_2 \).

Let \( \Max_B(O_k) \) be the set of maximal orders \( O_B \) in \( B \) with \( i^{-1}(O_B) = O_k \). Such orders can be described as follows. Consider pairs \( (a, \lambda) \) where \( a \) is a fractional ideal with
\[
(7.4) \quad N(a) = \kappa |\Delta| D,
\]
and \( \lambda \) is a generator of the cyclic \( O_\kappa \)-module \( \partial_2^{-1}a/a \) such that

\[
N(\lambda) \equiv \kappa \text{ in } \Delta_2^{-1}N(a)/N(a).
\]

For a given pair \((a, \lambda)\), let

\[
O_{a,\lambda,B} = \{ [\alpha, \beta] \mid \alpha \in \partial_2^{-1}, \beta \in a^{-1}, \alpha + \lambda\beta \equiv 0 \text{ in } \partial_2^{-1}/O_\kappa \}.
\]

The following result is a slight generalization of the description of maximal orders given [3] and [4].

**Proposition 7.1.** (i) \( O_{a,\lambda,B} \) is a maximal order in \( \text{Max}_B(O_\kappa) \).

(ii) Every maximal order in \( \text{Max}_B(O_\kappa) \) has the form \( O_B = O_{a,\lambda,B} \) for some \( a \) and \( \lambda \).

(iii) The finite idèles\(^8 \mathbb{A}_k^\times \) act on the set \( \text{Max}_B(O_\kappa) \) by conjugation,

\[
O_B \mapsto bO_Bb^{-1} = b\hat{O}_Bb^{-1} \cap B,
\]

where \( b \in \mathbb{A}_k^\times \) and \( \hat{O}_B = O_B \otimes \mathbb{Z} \hat{\mathbb{Z}} \). Explicitly,

\[
bO_{a,\lambda,B} b^{-1} = O_{a_b,\lambda_b,B},
\]

where

\[
a_b = bb^{-1}a, \quad \text{and} \quad \lambda_b = bb^{-1}\lambda,
\]

and \( b \) is the fractional ideal determined by \( b \). This action is transitive.

(iv) Suppose that \( \delta \) is replaced by \( \delta' = \beta_0\delta \), for \( \beta_0 \in \kappa^\times \). Then,

\[
O_B = O_{a,\lambda,B} = O_{a',\lambda',B}
\]

where \( a' = \beta_0a \), and \( \lambda' = \beta_0\lambda \).

Let

\[
\Lambda(a, \kappa, \Delta_2) = \{ \lambda \in \partial_2^{-1}a/a \mid \lambda \text{ a generator, } N(\lambda) \equiv \kappa \text{ in } \Delta_2^{-1}N(a)/N(a) \}
\]

be the set of possible choices of \( \lambda \). For a place \( w \mid \partial_2 \), let \( \lambda_w \) be the image of \( \lambda \) in \( (\partial_2^{-1}a/a)_w \).

**Lemma 7.2.** (i) For each \( w \mid \partial \), with \( w \nmid \partial_1 \), there are two choices \( \lambda_w \) and \( \varpi_w\varpi^{-1}w \lambda_w \) of the local component \( \lambda_w \in (\partial_2^{-1}a/a)_w \) of \( \lambda \). Here \( \varpi_w \) is a local uniformizer at \( w \).

(ii) If \( w \mid \partial_2 \) and \( w \mid \partial_1 \) (i.e., \( w \mid 2 \) is ramified and \( w \mid D_1 \)), then there is a unique choice of \( \lambda_w \).

(iii) In particular,

\[
|\Lambda(a, \kappa, \Delta_2)| = 2^{o(\Delta)-o(D_1)}
\]

where \( o(\Delta) \) (resp. \( o(D_1) \)) is the number of prime factors of \( \Delta \) (resp. \( D_1 \)).

Note that for a odd place \( w \), the possible local components are just \( \pm \lambda_w \). For \( w \mid 2 \), an elementary calculation gives the result.

Given a maximal order \( O_B \in \text{Max}_B(O_\kappa) \), we will be interested in \( O_\kappa \)-lattices \( \Lambda \) in \( \kappa^2 \) that are stable under the action of \( O_B \subset M_2(\kappa) \). Note that, for such a lattice \( \Lambda \), we have \( O_B = B \cap O_\Lambda \), where \( O_\Lambda \subset M_2(\kappa) \) is the maximal order stabilizing \( \Lambda \). Under the isomorphism (7.3), such lattices correspond to left \( O_B \)-ideals that are stable under the right action of \( O_\kappa \). The finite idèles \( \mathbb{A}_k^\times \) act on the set of such ideals by right homotheties,

\[
I \mapsto I \cdot b = \tilde{I} \cdot b \cap B.
\]

\(^8\text{imbedded in } (B \otimes \mathbb{A}_Q,f)^\times \text{ via } i\)
Since the right order of such an ideal is again an element of \( \text{Max}_B(O_k) \), part (iii) of Proposition 7.1 implies that every \( \mathbb{A}_{k,f}^\times \)-orbit contains a two-sided \( O_B \) ideal. For a subset \( S \) of the set of primes dividing \( D_0 \), let 
\[
I_S = \prod_{p \in S} \mathcal{P},
\]
where \( \mathcal{P} \) is the two-sided \( O_B \)-ideal with \( \mathcal{P}^2 = pO_B \). Then the \( I_S \)'s are a set of representatives for the \( \mathbb{A}_{k,f}^\times \)-homothety classes. In particular, there are \( 2^{\text{ord}(D_0)} \) such classes. Let \( \Lambda_S \) be the image of \( I_S \) under the isomorphism (7.3). Then \( \Lambda_{O_B'} := \Lambda_{0} \) is the image of \( O_B \).

Any \( O_k \)-lattice in \( k^2 \) has the form \( gA_0 \), where \( A_0 = O_k^2 \) and \( g \in \text{GL}_2(\mathbb{A}_{k,f}) \). Choose a finite idèle \( a \) with \( \text{ord}_w(a) = \text{ord}_w(a) \) for all \( w \), and an adèle \( \lambda \) with components \( \lambda_w \) having image \( \lambda_w \) in \( (\mathbb{Z}_2^{-1}a/a)_w \) for \( w \mid \mathbb{Z}_2 \) and \( \lambda_w = 0 \) otherwise. We will sometimes write \( O_{a,\lambda,B} \) in place of \( O_{a,\lambda,B} \).

Then a short calculation shows that, for \( O_B = O_{a,\lambda,B} \),
\[
(7.8) \quad \Lambda_{O_B} = gA_0, \quad \text{where} \quad g = a^{-1} \left( \begin{array}{c} 1 \\ \lambda \\ a \end{array} \right). \]

8. Morphisms \( j_\Lambda : \mathcal{C} \to \mathcal{M} \)

In this section, we suppose that an indefinite quaternion algebra \( B \) with a fixed maximal order \( O_B \) and an imaginary quadratic field \( k = \mathbb{Q}(\sqrt{\Delta}) \), with an embedding \( i : O_k \to O_B \) are given. If \( \Lambda \subseteq B \) is a left \( O_B \)-ideal which is stable for the right action of \( O_k \), we view \( \Lambda \) as an \( O_B \otimes_{\mathbb{Z}} O_k \)-module by the rule\(^9\)
\[
(8.1) \quad (b \otimes \alpha) : x \mapsto bx\bar{\alpha}.
\]

Then there is a functor, given by Serre’s construction, [18], [20], \( E \to A = \Lambda \otimes_{O_k} E \), from elliptic curves \( (E, \iota) \) with \( O_k \) action over an \( O_k \)-scheme \( S \) to \( O_B \)-modules \( (A, \iota_B) \) over \( S \).

Fix an element \( \delta \) as in section 7 and hence an identification \( B \cong k^2 \) as in (7.3). Let \( A_0 \subseteq B \) be the free \( O_k \)-module corresponding to \( O_k^2 \subseteq k^2 \). Then there is a natural quasi-isogeny
\[
(8.2) \quad A_0 = \Lambda \otimes_{O_k} E \longrightarrow \Lambda \otimes_{O_k} E = A,
\]
and \( A_0 = E \times S E \).

**Lemma 8.1.** The \( O_B \)-module \( A = \Lambda \otimes_{O_k} E \) satisfies the Drinfeld special condition or Kottwitz condition (3.1.2) in [15].

**Proof.** The action of \( O_k \) on \( \text{Lie}(E/S) \) satisfies
\[
\text{char}((\iota(\alpha)|\text{Lie}(E))(T) = T - i(\alpha) \in \mathcal{O}_S[T],
\]
for \( \alpha \in O_k \to \mathcal{O}_S \). Thus, for \( \xi \in M_2(O_k) \),
\[
\text{char}((\iota(\xi)|\text{Lie}(E \times E))(T) = T^2 - \text{tr}(i(\xi))T + \text{det}(i(\xi)) \in \mathcal{O}_S[T],
\]
and, hence, for \( b \in O_B \),
\[
\text{char}((\iota(b)|\text{Lie}(A))(T) = T^2 - \text{tr}(b)T + \nu(b) \in \mathcal{O}_S[T],
\]
as required, since this formula is unchanged under the quasi-isogeny from \( A_0 \) to \( A \). \( \square \)

\(^9\)The \( \bar{\alpha} \) occurs here due to the conventions of section 7 above.
Thus there is a morphism
\[ j_\Lambda : \mathcal{C} \rightarrow \mathcal{M}, \quad (E, \iota) \mapsto (A, \iota_B), \]
of moduli stacks.

Let \( O_\Lambda = \text{End}_{O_\mathbf{k}}(\Lambda) \) be the stabilizer of \( \Lambda \subset \mathbf{k}^2 \) in \( M_2(\mathbf{k}) \).

**Lemma 8.2.** For \( A = \Lambda \otimes_{O_\mathbf{k}} E \), \( \text{End}_{O_\mathbf{k}}(A/S) = O_\Lambda \).

**Proof.** By functorial properties of the Serre construction, there is a natural map
\[ O_\Lambda \rightarrow \text{End}_{O_\mathbf{k}}(A/S), \quad \phi \mapsto \phi \otimes 1_E, \]
and
\[ \text{End}_{O_\mathbf{k}}(A/S) = \text{End}_{O_\mathbf{k}}((\Lambda \otimes_{O_\mathbf{k}} E)/S) \cong O_\Lambda \otimes_{O_\mathbf{k}} \text{End}_{O_\mathbf{k}}(E/S) = O_\Lambda. \]
since \( \text{End}_{O_\mathbf{k}}(E/S) = O_\mathbf{k} \). \( \square \)

Note that for any \( a \in A_{k,f} \) there is a functor \( F_a : \mathcal{C} \rightarrow \mathcal{C} \) given by \( E \mapsto (\mathbf{a}) \otimes_{O_\mathbf{k}} E \). Then, keeping in mind the conjugation in (8.1), we have the relation \( j_{\Lambda a} = j_\Lambda \circ F_a \). Thus we need only consider the morphisms \( j_\Lambda \) for \( \Lambda \) in the set of representatives for the \( A_{k,f} \)-homothety classes of lattices described in the previous section.

Next suppose that \( I \) is a two-sided \( O_B \)-ideal. By a (slightly noncommutative) analogue of the Serre construction, there is a functor
\[ F_I : \mathcal{M} \rightarrow \mathcal{M}, \quad A \mapsto I \otimes_{O_B} A, \]
where the action of \( O_B \) on \( I \otimes_{O_B} A \) arises from its left action on \( I \). Then, for any \( \Lambda \) as above, we have \( j_{IA} = F_I \circ j_\Lambda \), since \( IA \cong I \otimes_{O_B} \Lambda \).

As a consequence of the previous observations, we can and do assume from now on that \( \Lambda \) is the given order \( O_B \), which we will frequently identify with the corresponding \( O_\mathbf{k} \)-lattice in \( \mathbf{k}^2 \) via (7.3).

### 9. Special endomorphisms

In this section, we determine the endomorphism rings \( \text{End}(A) \) and \( \text{End}(A, \iota_B) \) for \( (A, \iota_B) \) coming from \( (E, \iota) \) by the construction of the previous section in various cases. We then determine the space of special endomorphisms
\[ L(A, \iota_B) = \{ x \in \text{End}(A, \iota_B) \mid \text{tr}(x) = 0 \}. \]
We write \( V(A, \iota_B) = L(A, \iota_B) \otimes_{\mathbb{Z}} \mathbb{Q} \).

For \( (E, \iota)/S \), let \( O_E = \text{End}(E/S) \) and let \( \mathbb{B} = \text{End}^0(E/S) \), so that \( O_E \) is an order in \( \mathbb{B} \), with a given embedding \( \iota' = \iota : O_\mathbf{k} \rightarrow O_E \). First we consider rational endomorphisms. As explained above, we have homomorphisms\(^{10}\)
\[ \mathbf{k} \xrightarrow{\iota} B \xrightarrow{(7.3)} M_2(\mathbf{k}) \xrightarrow{\iota'} M_2(\mathbb{B}) = \text{End}^0(E \times E) \xrightarrow{(8.2)} \text{End}^0(A). \]
and \( C := \text{End}^0(A, \iota_B) \), is the centralizer of \( \iota'(B) \) in \( M_2(\mathbb{B}) \). Explicitly, a simple computation shows that
\[ C = \{ [\alpha, \beta] = \left( \begin{array}{cc} \alpha & \beta \\ \kappa & \beta \end{array} \right) \in M_2(\mathbb{B}) \mid \alpha \in \iota'(\mathbf{k}), \beta \in V(E, \iota) \}. \]
\(^{10}\)Here (7.3) depends on the choice of \( \delta \) and (8.2) depends on the choice of the free lattice \( \Lambda_0 \). Both of these choices are fixed at the outset.
where \( V(E, \iota) = L(E, \iota) \otimes_{\mathbb{Z}} \mathbb{Q} \) for

\[
V(E, \iota) = \{ x \in \text{End}(E/S) \mid x \iota(\alpha) = \iota(\bar{\alpha}) x \}
\]

the lattice of special endomorphisms of \((E, \iota)\). Thus

\[
(9.3) V(A, \iota_B) = \{ [\alpha, \beta] \in C \mid \text{tr}(\alpha) = 0 \},
\]

and \( L(A, \iota_B) \) is the space of trace zero elements in the order \( O_C = \text{End}(A, \iota_B) \) in \( C \). Note that, for \( x = [\alpha, \beta] \in V(A, \iota_B) \),

\[
(9.4) -x^2 = N(\alpha) + \kappa N(\beta).
\]

From now on, we will suppress \( \iota' \) from the notation.

Remark 9.1. The construction just explained, based on (9.1), is functorial in \( E/S \). In particular, it provides us with idempotents

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

and \( e_2 = 1_2 - e_1 \) in \( \text{End}^0(A/S) \). If a special endomorphism \( x \in L(A, \iota_B) \) is given, then the corresponding ‘components’ are \( \beta = e_1 x e_2 \) and \( \alpha = e_1 x e_1 \) in \( \text{End}^0(E/S) \), and their construction is functorial, hence, for example, commutes with any base change.

Lemma 9.2. Suppose that \((E, \iota)/S\) is ordinary, i.e., that \( \text{End}(E/S) = O_k \). Then

(i) \( \text{End}(A/S) = O_\Lambda \).
(ii) \( \text{End}^0(A, \iota_B) = k \) and \( \text{End}(A, \iota_B) = O_k \).
(iii) \( L(A, \iota_B) = \{ \alpha \in O_k \mid \text{tr}(\alpha) = 0 \} \).

Proof. In this case, \( \text{End}(A/S) \) is an order in \( M_2(k) \) containing the maximal order \( O_\Lambda \). This proves (i). Since \( V(E, \iota) = 0 \), parts (ii) and (iii) are then immediate from (i) and (9.3). \( \square \)

In general, for a CM elliptic curve \( E/S \), we write \( \text{End}^0(E) = \mathbb{B} \), so that \( \text{End}(E) \) is an order in \( \mathbb{B} \). Notice that the matrix \( g \) of (7.8) depends only on the lattices \( \Lambda \) and \( \Lambda_0 \) and not on \( E \). We view \( \text{End}(A) \) and \( \text{End}(A_0) \) as orders in \( \text{End}^0(A) = \text{End}^0(A_0) = M_2(\mathbb{B}) \). Then

\[
\text{End}(A) = g M_2(O_E) g^{-1},
\]

and, similarly,

\[
O_C = \text{End}(A, \iota_B) = C \cap g M_2(O_E) g^{-1}.
\]

We now describe \( O_C \) and the space of special endomorphisms more explicitly.

**Proposition 9.3.** Let \( O_B = O_{a, \lambda, B} \) for \( a \) and \( \lambda \) as in section 7, and let \( g \) be given by (7.8). In particular, \( a = (a) \).

(i) The order \( O_C = \text{End}(A, \iota_B) \) in \( C \) is the set of all

\[
[\alpha, \beta] = \begin{pmatrix} \alpha & \beta \\ \kappa \beta & \alpha \end{pmatrix} \in M_2(\mathbb{B})
\]

such that

\[
\beta \in L(E, \iota) \bar{a}^{-1}, \quad \alpha \in \partial_2^{-1}, \quad \text{and} \quad \alpha + \beta \lambda' \in O_E.
\]

Here \( \lambda' \) is an element of \( \partial_2^{-1} \bar{a} \) whose image in \( \partial_2^{-1} \bar{a} / \bar{a} \) coincides with that of \( \lambda' = \bar{a} a^{-1} \).
Proof. By (7.8),
\[ g = a^{-1} \begin{pmatrix} 1 & a \\ \lambda & \alpha \end{pmatrix}, \]
so that
\[ \gamma = \begin{pmatrix} \alpha & \beta \\ \kappa \beta & \alpha \end{pmatrix} \in C \subset M_2(\mathbb{B}), \]
lies in \( O_C \) if and only if \( g^{-1} \gamma g \in M_2(\mathcal{O}_E) \), i.e.,
\[ (9.5) \quad \begin{pmatrix} \alpha + \lambda \beta & a \beta \\ a^{-1}(\kappa - \lambda \lambda) \beta & \alpha - \lambda a^{-1}a \beta \end{pmatrix} \in M_2(\mathcal{O}_E), \]
with \( \tilde{\beta} = a^{-1}a \beta. \)

The conditions that the off diagonal entries lie in \( \mathcal{O}_E \) are then
\[ \tilde{\beta} a \in \mathcal{O}_E, \quad \tilde{\beta} a^{-1}(\kappa - \lambda \lambda) \in \mathcal{O}_E. \]

A little case by case calculation gives
\[ (9.6) \quad \min(\text{ord}_w(\alpha), \text{ord}_w(a^{-1}(\kappa - \lambda \lambda))) = \text{ord}_w(a), \]
for each finite place \( w \) of \( k \). Thus, the off diagonal condition is simply
\[ (9.7) \quad \tilde{\beta} \in \mathcal{O}_E a^{-1}, \quad \text{i.e.} \quad \beta \in O_E a^{-1}a \beta = O_E \bar{a}^{-1}. \]

The diagonal entries in (9.5) lie in \( \mathcal{O}_E \) if and only if
\[ \alpha + \lambda \beta \in \mathcal{O}_E, \quad \text{and} \quad \alpha - \lambda a^{-1}a \beta \in \mathcal{O}_E. \]
But the second condition is a consequence of the first condition and (9.7). Indeed,
\[ \alpha - \lambda a^{-1}a \beta = \alpha + \lambda \beta - \text{tr}(\lambda a^{-1})\beta a. \]

When \( w \mid \partial_2 \), so that \( \lambda_w = 0 \), there is nothing to check. When \( w \not| \partial_2 \),
\[ \text{ord}_w(\lambda a^{-1}) = \text{ord}_w(\kappa) - \text{ord}_w(a) = -\text{ord}_w(\partial_2), \]
Thus, \( \text{tr}(\lambda a^{-1}) \) is integral, while, by (9.7), \( (\beta a)_w \in (O_E)_w. \)

\[ \square \]

10. The pullback on arithmetic Chow groups

In this section, we determine the cycle \( j^*_A(Z(t)) \) on \( \mathcal{C} \) where, for \( t \in \mathbb{Z}_{\geq 0}, Z(t) \) is the special cycle on \( \mathcal{M} \) defined in [14], [15]. This pullback is defined as the fiber product
\[ j^*_A(Z(t)) \longrightarrow Z(t) \]
\[ i \downarrow \quad \downarrow \]
\[ \mathcal{C} \quad j^*_A \quad \mathcal{M}. \]
Thus, \( j^*_A(Z(t)) \) is the stack over \( \text{Sch}/\mathcal{O}_k \) which associates to a base scheme \( S \) the category of collections \( (E, i, A, \imath_B, x) \) with \( (E, i) \) in \( \mathcal{C}(S) \), \( (A, \imath_B) \) in \( \mathcal{M}(S) \) with \( (A, \imath_B) \simeq j_A(E, i) \) and \( x \in V(A, \imath_B) \) a special endomorphism with \( Q(x) = -x^2 = t. \)

We assume that \( \mathbb{Q}(\sqrt{-t}) \) and \( k \) are distinct, so that the images of \( \mathcal{C} \) and \( Z(t) \) are disjoint on the generic fiber \( \mathcal{M}_0 \). It follows that the generic fiber of \( j^*_A(Z(t)) \) is empty. Since \( Z(t) \) is relatively representable over \( \mathcal{M} \) by an unramified morphism, [15], (3.4.3), the same is true for \( j^*_A(Z(t)) \) over \( \mathcal{C} \). In particular, the morphism \( i \) is finite and unramified, and the coarse moduli scheme corresponding to \( j^*_A(Z(t)) \) is an artinian scheme.
On the other hand, recall that in Definition 2.1, we introduced the moduli stacks \( \mathcal{Z}_C(m, a, \lambda, r) \) over \( C \), where we now add the subscript \( C \) to distinguish them from the cycles \( \mathcal{Z}(t) \) on \( \mathcal{M} \). The main result of this section is the following.

**Proposition 10.1.** For \( t > 0 \) with \( Q(\sqrt{-t}) \neq k \),

\[
(10.2) \quad j_\Lambda^*(\mathcal{Z}(t)) = \prod_{\alpha \in \partial_2^{-1} \text{tr}(\alpha) = 0 \land N(\alpha) < t} \mathcal{Z}_C(\frac{|\Delta|}{D}(t - N(\alpha)), \tilde{a}, \lambda', r_\alpha),
\]

where \( r_\alpha \) is the image of \( \alpha \) in \( \partial^{-1}/O_k \) and \( \lambda' \in \partial_2^{-1} \tilde{a}/\tilde{a} \subset \partial^{-1} \tilde{a}/\tilde{a} \).

Here we view the union on the right side of (10.2) as a disjoint union of stacks indexed by \( \alpha \). No terms with \( N(\alpha) = t \) can occur, due to our assumption that \( k \neq Q(\sqrt{-t}) \).

**Proof.** First we define a functor from \( j_\Lambda^*(\mathcal{Z}(t)) \) to union on the right side. Given \( (A, \iota_B, E, \iota, x) \) over \( S \), we have components \( \alpha = e_1 e_1 \in \partial_2^{-1} \) and \( \beta = e_1 e_2 \in L(E, \iota) \tilde{a}^{-1} \) with \( \text{tr}(\alpha) = 0 \) and \( \alpha + \beta \lambda' \in O_E \). Moreover, \( t = N(\alpha) + \kappa N(\beta) \). The collection \( (E, \iota, \beta) \) is then an object of \( \mathcal{Z}_C(m, \tilde{a}, \lambda', r_\alpha) \) over \( S \), where \( r_\alpha \) is the image of \( \alpha \) in \( \partial^{-1}/O_k \) and

\[
m = N(\alpha) N(\beta) = \frac{|\Delta|}{D} (t - N(\alpha)).
\]

Note that \( N(\beta) \geq 0 \), so that \( N(\alpha) \leq t \). This construction is functorial. Conversely, over a base \( S \), an object of the right side of (10.2) in the term with index \( \alpha \) is a collection \( (E, \iota, \beta) \) in \( \mathcal{Z}_C(m, \tilde{a}, \lambda', r_\alpha) \) where an \( \alpha \in \partial_2^{-1} \) is given by the index, \( r = r_\alpha \), and \( m = \frac{|\Delta|(t-N(\alpha))}{D} \). Then we obtain \( (A, \iota_B, E, \iota, x) \) by taking \( A = j_\Lambda^*(E) \) and \( x = [\alpha, \beta] \). These constructions are inverses of each other. \( \Box \)

11. **Pullbacks of Green functions**

In this section, we will compute the pullback \( j_\Lambda^* \Xi(t, v) \) of the Green function associated to the cycle \( \mathcal{Z}(t) \) and its contribution to the arithmetic degree \( \deg(j_\Lambda^* \Xi(t, v)) \). We continue to assume that the cycles \( j_\Lambda^*(C) \) and \( \mathcal{Z}(t) \) are disjoint on the generic fiber.

We begin by giving a more intrinsic description of the Green function \( \Xi(t, v) \) defined in [10] and sections 3.2 and 3.4 of [15] for any \( t \in \mathbb{Q}^\times \) and \( v \in \mathbb{R}_+^\times \). We view \( \Xi(t, v) \) as a real valued function on \( \mathcal{M}(\mathbb{C}) \); its value on an object \( (A, \iota_B) \) of \( \mathcal{M}(\mathbb{C}) \) is defined as follows. Let \( T_e(A) \) be the tangent space at the identity and let \( T_e(A^{\text{top}}) \) be the underlying real vector space, where \( A^{\text{top}} \) is the underlying real torus. For \( x \in \text{End}(T_e(A^{\text{top}})) \), let \( |N(x)| = |\det(x)|^{\frac{1}{2}} \). Write

\[
(11.1) \quad \text{End}(T_e(A^{\text{top}}), \iota_B) = U^+ + U^-,
\]

where \( U^+ \) (resp. \( U^- \)) is the space of complex linear (resp. anti-linear) endomorphisms of \( T_e(A^{\text{top}}) \) commuting with the action of \( O_B \). If \( x \in V(A^{\text{top}}, \iota_B) = \{ x \in \text{End}(A^{\text{top}}, \iota_B) \mid \text{tr}(x) = 0 \} \) is a special endomorphism of \( A^{\text{top}} \) and \( x \in \text{End}(T_e(A^{\text{top}}), \iota_B) \) is the induced endomorphism of \( T_e(A^{\text{top}}) \), let \( \text{pr}_-(x) \) be its \( U^- \)-component. Then we let

\[
(11.2) \quad \Xi(t, v)(A, \iota_B) = \sum_{x \in L(A^{\text{top}}, \iota_B), Q(x) = t} \beta_1(4\pi v |N(\text{pr}_-(x))|),
\]
where $\beta_1(r) = -\text{Ei}(-r)$ is the exponential integral, [15], (3.5.2). Here recall that the quadratic form $Q$ on $V(A^{\text{top}}, \iota_B)$ is defined by $-x^2 = Q(x)\text{id}_A$. This sum converges if there is no element $x \in V(A^{\text{top}}, \iota_B)$ with $Q(x) = t$ and $\text{pr}_-(x) = 0$. Note that the image in $\mathcal{M}(\mathbb{C})$ of the cycle $\mathcal{Z}(t)(\mathbb{C})$ consists precisely of those $(A, \iota_B)$'s for which such a holomorphic special endomorphism $x$ exists, so that the function $\Xi(t, v)$ is well defined outside of this cycle. In particular, if $t < 0$, then there are no such holomorphic endomorphisms and $\Xi(t, v)$ is a smooth function on all of $\mathcal{M}(\mathbb{C})$.

The relation between this description and that given in [15] arises as follows. Writing $A(\mathbb{C}) \simeq T_e(A)/L$ where $L$ is a lattice, we can choose an isomorphism

$$B_\mathbb{R} = B \otimes_\mathbb{Q} \mathbb{R} \sim T_e(A), \quad O_B \sim L.$$  

This is unique up to right multiplication by an element of $O_B^\times$. Note that

$$B_\mathbb{R} \sim \text{End}(T_e(A^{\text{top}}), \iota_B), \quad b \mapsto (x \mapsto xb'),$$

and, under this isomorphism,

$$O_B \sim \text{End}(A^{\text{top}}, \iota_B).$$

Also, $|N(x)|$ is the absolute value of the reduced norm of $x$. The complex structure on $T_e(A)$ is then given by right multiplication by an element $J \in B_\mathbb{R}^\times$, and the decomposition (11.1) becomes $B_\mathbb{R} = U^+ + U^-$ where $U^\pm$ is the $\pm 1$-eigenspace of $\text{Ad}(J)$. If $V$ is the space of trace zero elements in $B$ with inner product $(x, y) = \text{tr}(xy^*)$, then $U^- \subset V(\mathbb{R})$ is a negative 2-plane, oriented by the action of $J$, i.e., an element $z$ of the space $D$. The $U^-$-component of $x \in O_B \cap V$ is denoted by $\text{pr}_z(x)$ in [15], and $|\text{pr}_z(x), \text{pr}_z(x)| = 2|N(\text{pr}_z(x))|$. We then have

$$\Xi(t, v)(A, \iota_B) = \Xi(t, v)(z) = \sum_{\substack{x \in O_B \cap V \\ N(x) = t \\ \text{exists}}} \beta_1(2\pi v |\text{pr}_z(x), \text{pr}_z(x)|),$$

as an $O_B^x$-invariant function on $D$, with singularities at the points $z$ for which $\text{pr}_z(x) = 0$ for some $x$ in $O_B \cap V$ with $Q(x) = t$.

We now determine the pullback of this function to $\mathcal{C}(\mathbb{C})$ under $j_\Lambda : \mathcal{C}(\mathbb{C}) \rightarrow \mathcal{M}(\mathbb{C})$. The analysis of special endomorphism made in section 9 is based on the functorial properties of the Serre construction. Hence it applies without change to the real tori $E^{\text{top}}$ and $A^{\text{top}}$ underlying $(E, \iota)/\mathbb{C}$ and $A = j_\Lambda(E)$, and yields the following result.

**Proposition 11.1.** Let $\mathbb{B} = \text{End}^0(E^{\text{top}})$, $O_{E^{\text{top}}} = \text{End}(E^{\text{top}})$ and

$$L(E^{\text{top}}, \iota) = \{ x \in \text{End}(E^{\text{top}}) \mid x\iota(\alpha) = \iota(\check{\alpha})x \}.$$  

Then

$$\text{End}(A^{\text{top}}, \iota_B) = \{ [\alpha, \beta] \in M_2(\mathbb{B}) \mid \beta \in L(E^{\text{top}}, \iota)\check{\alpha}^{-1}, \ \alpha \in \delta_2^{-1}, \ \alpha + \beta \lambda' \in O_{E^{\text{top}}} \}.$$  

If $x = [\alpha, \beta]$ is a special endomorphism, i.e., if $\text{tr}(\alpha) = 0$, then $Q(x) = N(\alpha) + \kappa N(\beta)$, where $-\beta^2 = N(\beta)\text{id}_E$.

Using this in (11.2), we obtain the following formula for the pullback.

**Proposition 11.2.** Assume that $t \neq 0$ and that $Q(\sqrt{-t}) \neq k$. Then

$$\Xi(t, v)(j_\Lambda(E, \iota)) = \sum_{\substack{\alpha \in \delta_2^{-1} \\ \text{tr}(\alpha) = 0 \\ N(\alpha) > t}} \mathcal{Z}_C\left(\frac{|\lambda|}{D}(t - N(\alpha)), D_{2v}; \check{\alpha}, \lambda', r_\alpha\right)(E, \iota),$$

where $\mathcal{Z}_C(m, v; \alpha, \lambda, r)$ is the function on $\mathcal{C}(\mathbb{C})$ defined in (2.4).
Proof. Note that, if \(x = [\alpha, \beta] \in V(A^{\top}, \iota_B)\), then \(N(x_\perp) = \kappa N(\beta)\). Also note that \(N(\beta) \leq 0\), and that \(\beta \neq 0\) due to our assumption that \(\mathbb{Q}(\sqrt{-t}) \neq \kappa\). Recall that \(\kappa = N(a)D/|\Delta|\). Thus

\[
\Xi(t, v)(A, \iota_B) = \sum_{x = [\alpha, \beta] \in \text{End}(A^{\top}, \iota_B)} \beta_1(4\pi v N(a)D/|\Delta| N(\beta))
\]

\[
= \sum_{\alpha \in O_k^{-1}} \sum_{\beta \in \mathcal{L}(E^{\top}, \iota)\bar{a}^{-1}} \beta_1(4\pi v mD_2 N(\beta))
\]

\[
= \sum_{\alpha \in O_k^{-1}} \mathcal{Z}_C(|\Delta| D(t - N(\alpha)), D_2v; \bar{a}, \lambda', r_0)(E, \iota),
\]

where, in the second line, \(N(\beta) = \kappa^{-1}(t - N(\alpha)) = m N(\alpha)^{-1}\), so that

\[
m = |\Delta| D(t - N(\alpha)).
\]

Here recall that \(\Delta(\lambda') = N(\partial_2)\). 

\[
\square
\]

12. The main formula for \(\widehat{\deg} j_A^*(\widehat{\phi}(\tau))\).

We now give a formula for the pullback

\[
\widehat{\deg} j_A^*(\widehat{\phi}(\tau)) = \sum_t \widehat{\deg} j_A^*(\widehat{Z}(t, v)) q^t,
\]

which, by the results of [15], is a modular form of weight \(\frac{3}{2}\) and level \(4D(B)o\), where \(D(B)o\) is the odd part of \(D(B)\). To express the result, we introduce the theta functions of weight \(\frac{1}{2}\), defined by

\[
\theta(\tau; r) = \sum_{\alpha \in O_k^{-1}} q^{N(\alpha)} \quad \text{for } r \in \partial_1/O_k.
\]

Theorem 12.1. Suppose that \(t \neq 0\) and that \(\mathbb{Q}(\sqrt{-t}) \neq \kappa\). Then the quantity \(\widehat{\deg} j_A^*(\widehat{Z}(t, v))\) is the coefficient of \(q^t\) in the modular form

\[
(12.1) \sum_{r \in \partial_2^{-1}/O_k} \theta(\tau; r) \widehat{\phi}_C(D_2\tau; \bar{a}, \lambda', r),
\]

where \(\widehat{\phi}_C(\tau; a, \lambda, r)\) is the generating function defined in (2.10) and \(D = D(B) = D_1D_2\), as in section 7.

If we assume disjoint ramification, then we have the following stronger result, whose proof will be completed in the next section.
Theorem 12.2. Assume that ∆ and \( D(B) \) are relatively prime. Then
\[
j^*_\Lambda(\hat{\phi}(\tau)) = \sum_{r \in \mathcal{O}_k^{1}/\mathcal{O}_k \text{tr}(r) = 0} \theta(\tau; r) \hat{\phi}_C(D(B)\tau; \bar{a}, \lambda', r),
\]
where \( \hat{\phi}_C(\tau; a, \lambda, r) \) is the generating function defined in (2.10).

Remark 12.3. Note that when, in addition, \( 2 \nmid \Delta \), then by Theorem 2.7, we have
\[
\sum_{r \in \mathcal{O}_k^{1}/\mathcal{O}_k \text{tr}(r) = 0} \theta(\tau; r) \hat{\phi}_C(D_2\tau; \bar{a}, \lambda', r) = -\frac{1}{2} \frac{\partial}{\partial s} \left( \sum_{r \in \mathcal{O}_k^{1}/\mathcal{O}_k \text{tr}(r) = 0} \theta(\tau; r) E^*(D_2\tau, s; \bar{a}, \lambda', r) \right) \bigg|_{s=0}.
\]

Proof of Theorem 12.1. By the results of the previous sections, if \( t \neq 0 \) and \( \mathbb{Q}(\sqrt{-t}) \neq \mathbb{k} \), the quantity \( \deg j^*_\Lambda(\tilde{Z}(t, v)) \) is the sum of the terms
\[
(12.3) \quad \sum_{\alpha \in \mathcal{O}_k^{1}/\mathcal{O}_k \text{tr}(\alpha) = 0 \atop N(\alpha) < t} \deg Z_C(\frac{|\Delta|}{D}(t - N(\alpha)), \bar{a}, \lambda', r_\alpha),
\]
and
\[
(12.4) \quad \sum_{\alpha \in \mathcal{O}_k^{1}/\mathcal{O}_k \text{tr}(\alpha) = 0 \atop N(\alpha) > t} \deg Z_C(\frac{|\Delta|}{D}(t - N(\alpha)), D_2v; \bar{a}, \lambda', r_\alpha).
\]
The term here for a fixed \( \alpha \) is the coefficient of \( q^{\frac{D_2m}{\Delta(\lambda')}} \) in \( \hat{\phi}_C(D_2\tau; \bar{a}, \lambda', r) \), where
\[
\frac{D_2m}{\Delta(\lambda')} = \frac{D_2|\Delta|}{\Delta(\lambda')D}(t - N(\alpha)) = t - N(\alpha),
\]
since \( \Delta(\lambda') = N(\partial_2) \). Recall that \( \lambda \) is a generator for the cyclic module \( \partial_2^{-1}a/a \). This gives the claimed identity. \( \square \)

For later use, we compute the remaining Fourier coefficients of the modular form (12.1).

Proposition 12.4. (i) The constant term of (12.1) is
\[
\deg Z_C(0, D_2v) = -\mathcal{N}(1, \chi) - \frac{1}{2} \Lambda(1, \chi) \log(D_2v).
\]

Note that \( D(B) = D_2 \frac{|\Delta|}{\Delta(\lambda)} \), and that, when \( D(B) \) and \( \Delta \) are relatively prime, \( D(B) = D_2 \).

(ii) Suppose that \( t \in \mathbb{Z}_{>0} \) with \( \mathbb{Q}(\sqrt{-t}) = \mathbb{k} \), and write \( 4t = n^2|\Delta| \). Then the \( t \)-th Fourier coefficient of (12.1) is the sum of the following terms:
\begin{enumerate}
\item[(a)] The contribution where \( \alpha = \pm \sqrt{t} \),
\[
2 \deg Z_C(0, D_2v).
\]
\item[(b)] Terms given by (12.3), where \( N(\alpha) < t \), and (12.4), where \( N(\alpha) > t \).
\end{enumerate}
Proof. Part (ii) is immediate. To prove (i), observe that the constant term of (12.1) is \( \hat{\deg} Z_C(0, D_2v) \), the contribution of the \( \alpha = 0 \) term, together with the sum

\[
(12.5) \quad \sum_{\alpha \in \beta^{-1} \setminus \text{tr}(\alpha) = 0} \hat{\deg} Z(m, D_2v; \hat{a}, \lambda', r_{\alpha}),
\]

where \(-N(\alpha) = D_2m\). Note that \( \text{End}^0(E^{\top}) = M_2(\mathbb{Q}) = k + kj \), where \( j \in L(E^{\top}, \iota) \), with \((\Delta, j^2)_p = 1 \) for all primes \( p \leq \infty \). Then, in Definition 2.5, \( \hat{\beta} = \beta j \), with \( \beta \in k \), and \( Q(\beta) = -N(\beta)j^2 = m/N(\alpha) \). Recalling that \( N(\alpha) = k|\Delta|/D(B) \), we have

\[
N(\alpha) = \frac{a^2}{|\Delta|} = \frac{D(B)}{|\Delta|} N(\beta) j^2 \kappa = \frac{D(B)}{|\Delta|} (\Delta, \kappa). \]

But then, for any \( p \leq \infty \), the Hilbert symbol has value

\[
1 = (\Delta, N(\alpha))_p = (\Delta, N(\beta) j^2 \kappa)_p = (\Delta, \kappa)_p.
\]

Since \((\Delta, \kappa)_p = -1 \) for \( p \mid D(B) \), this contradicts the fact that \( B \) is a division algebra. Thus, no such \( \beta \) can exist and the sum (12.5) is empty.

\[
\hfill \Box
\]

13. Arithmetic adjunction and \( j^*_\Lambda(\hat{Z}(0, v)) \)

In this section, we determine \( \hat{\deg} j^*_\Lambda \hat{Z}(t, v) \) in the case where \( k = \mathbb{Q}(\sqrt{-1}) \) so that \( j^*_\Lambda(C) \) and \( Z(t) \) have common components, using the arithmetic adjunction formula, discussed in section 2.7 of [15]. We also determine the arithmetic degree \( \hat{\deg} j^*_\Lambda \hat{Z}(0, v) \) of the pullback of the constant term \( \hat{Z}(0, v) \) of the generating function.

First consider the constant term.

**Proposition 13.1.**

\[
\hat{\deg} j^*_\Lambda \hat{Z}(0, v) = -N'(1, \chi) - \frac{1}{2} \Lambda(1, \chi) \log(vD(B)).
\]

**Proof.** Recall that, [15], (3.5.7),

\[
\hat{Z}(0, v) = -\hat{\omega} - (0, \log(vD(B))) \in \hat{\text{CH}}^1(\mathcal{M}),
\]

where \( \hat{\omega} \) is the Hodge bundle, metrized as in [14], p. 987. Thus,

\[
\hat{\deg} j^*_\Lambda \hat{Z}(0, v) = -2 \frac{2h_k}{w_k} h_{\text{Fal}}(E) - \deg_Q(C) \cdot \log(vD(B)).
\]

Here the initial factor of 2 comes in because the pullback of the Hodge line bundle \( \omega \) of \( \mathcal{M} \), cf. section 3.3 of [15], is the square of the Hodge line bundle for the moduli space \( C \) of CM elliptic curves. The factor \( 2h_k \) arises from the fact that the Faltings height \( h_{\text{Fal}}(E) \) is given by the arithmetic degree of the Hodge line bundle divided by the degree of the Hilbert class field. Finally, the factor \( w_k \) in the denominator arises from the stack. Also note that \( \deg_Q(C) = h_k/w_k \). Using the fact that, [14], (10.82),

\[
(13.1) \quad 2 h_{\text{Fal}}(E) = \frac{1}{2} \log |\Delta| + \frac{L'(1, \chi)}{L(1, \chi)} - \frac{1}{2} \log(\pi) - \frac{1}{2} \gamma = \frac{\Lambda'(1, \chi)}{\Lambda(1, \chi)},
\]

and \( \Lambda(1, \chi) = 2h_k/w_k \), where \( \Lambda(s, \chi) \) is given by (2.8), we obtain the claimed expression. \( \hfill \Box \)
Now we turn to the case where $t > 0$ with $\mathbb{Q}(\sqrt{-t}) = k$, so that $Z(t)$ and $j_\Lambda(C)$ are not disjoint on the generic fiber, and we write $4t = n^2|\Delta|$. Then, there is a decomposition, [15], (7.4.7),

$$Z(t) = \sum_{c|n, (c, D(B)) = 1} Z^{\text{hor}}(t : c) + Z^{\text{ver}}(t),$$

of divisors on $\mathcal{M}$. The idea is that, in the definition of $Z(t)$, we are imposing an action of the order $\mathbb{Z}[\sqrt{-t}]$ of conductor $n$, while, along the component $Z^{\text{hor}}(t : c)$ there is an action of the order $O_{\mathbb{Z}[\sqrt{-t}]}$ of conductor $c$. The divisor $Z^{\text{ver}}(t)$ consists of vertical components in the fibers of bad reduction for $p \mid D(B)$.

**Lemma 13.2.** (i) If the $O_k$-lattices $\Lambda$ and $\Lambda'$ in $k^2$ are associated to two-sided $O_B$-ideals that are inequivalent for the right translation action of $\mathbb{A}_k^\times \cap N(O_B)$, then the cycles $j_\Lambda(C)$ and $j_{\Lambda'}(C)$ are disjoint on the generic fiber. The number of such inequivalent $\Lambda$'s is $2^{o(D_2)}$, where $o(D_2)$ is the number of divisors of $D(B)$ that are inert in $k$.

(ii) As a divisor on $\mathcal{M}$,

$$Z^{\text{hor}}(t : 1) = 2 \sum_{\Lambda} j_{\Lambda}(C),$$

where $\Lambda$ runs representatives for the equivalence classes of $O_k$-lattices in $k^2$ described in (i).

**Proof.** It suffices to check this on the generic fiber, since the cycles in question are flat over Spec $(O_k)$. The generic fiber of the divisor on the right side is contained in that of the one on the left. Note that, by a slight variant of (3.4.5) of [15] with the same proof,

$$\deg_Q Z^{\text{hor}}(t : c) = 2^{o(D_2)+1} \frac{h(c^2|\Delta|)}{w(c^2|\Delta|)}.$$ 

Recalling that $\deg Q C = h_k/w_k$, we see that the degrees of the two sided coincide.  

**Remark 13.3.** There is an error at this point in Lemma 7.4.2 of [15], where the irreducibility of $Z^{\text{hor}}(t : c)$ is incorrectly claimed, whereas its actual decomposition into irreducible components can be obtained by the construction of Remark 3.4.7.

Let

$$Z(t)^o = Z(t) - 2 j_\Lambda(C),$$

and, for convenience, write $Z_o = j_\Lambda(C)$. Let $\hat{Z}(t, v)^o$ and $\hat{Z}_o(tv)$ denote the corresponding classes in $\hat{CH}^1(\mathcal{M})$, where the Green functions are defined as in [15], section 3.5. Recall that these Green functions depend on the auxilliary parameter $v \in \mathbb{R}_{>0}$. Then

$$\hat{Z}(t, v) = \hat{Z}(t, v)^o + 2 \hat{Z}_o(tv),$$

and

$$\hat{\deg} j^* \hat{Z}(t, v) = \hat{\deg} j^* \hat{Z}(t, v)^o + 2 \hat{\deg} j^* \hat{Z}_o(tv).$$

The quantity $\hat{\deg} j^* \hat{Z}_o(tv)$ is given by the arithmetic adjunction formula, (i) of Theorem 2.7.2 in [15]. More precisely,

$$\hat{\deg} j^* \hat{Z}_o(tv) = -\hat{\deg} j^* \hat{\omega} + \hat{\omega}_{Z_o} + \frac{1}{2} \sum_{P, P' \in \mathcal{Z}_o(C), P \neq P'} e_{P}^{-1} e_{P'}^{-1} \sum_{\gamma \in \Gamma} \gamma g^0_{tv}(z, \gamma z'),$$

where, for $z$ and $z' \in D$, the function $g^0_{tv}(z, \gamma z')$ is defined in Proposition 7.3.1 and (7.3.42) of [15], and where $\hat{\omega}_{Z_o}$ is the discriminant term. In the sum, $z$ (resp. $z'$) is a preimage of $P$ (resp. $P'$) in $D$. 


Note that, as explained in the proof of Proposition 7.5.1, p.226 of [15], $\hat{\omega}$ is the relative dualizing sheaf on $\mathcal{M}$ with metric determined by $-g_{tv}^0$. Thus, by Lemma 7.5.2,
\[
\hat{\omega} = \omega + (0, \log(4tvD(B)),
\]
and
\[
-\deg j^*\hat{\omega} = -\deg j^*\omega - \deg_{\mathbb{Q}}(C) \log(4tvD(B))
\]
\[
= -2 \frac{2h_k}{w_k} h_{Fal}^*(E) - \deg_{\mathbb{Q}}(C) \log(4tvD(B))
\]
\[
= -\Lambda'(1, \chi) - \frac{1}{2} \Lambda(1, \chi) \log(vD(B)) - \frac{1}{2} \Lambda(1, \chi) \log(n^2|\Delta|).
\]
Also, the discriminant term is simply
\[
d_{Z_0} = \frac{h_k}{w_k} \log |\Delta| = \frac{1}{2} \Lambda(1, \chi) \log |\Delta|.
\]

**Lemma 13.4.** When $D(B) = D_2$,
\[
-2\deg j^*\hat{\omega} + 2d_{Z_0} = 2\deg Z_C(0, D_2v) - \Lambda(1, \chi) \log(n^2).
\]

Next we consider the finite part of $\deg j^*_\Lambda Z(t, v)^\omega$.

**Proposition 13.5.** Assume that $\Delta$ and $D(B)$ are relatively prime. Then the finite part of $\deg j^*_\Lambda Z(t, v)^\omega$ is the sum of (12.3) and the quantity $2 \deg_{\mathbb{Q}}(C) \log(n^2)$.

**Proof.** Here we have to take into account the following rather subtle phenomenon. First, note that, for an object $(E, t)$ in $\mathcal{C}(S)$, the abelian scheme $A = \Lambda \otimes_{\mathcal{O}_k} E$ has a ‘distinguished’ special endomorphism $x = 1 \otimes t(\sqrt{-t})$, where $2\sqrt{-t} = n\sqrt{\Delta}$. We utilize the notation of Chapter 7 of [15] and refer the reader to that chapter for further details.

In the case where $p \nmid D(B)$, the formal branches of $i_*Z(t)$ through a point $x \in \mathcal{M}_p$ corresponding to $(A, \iota_B)$ are described in Proposition 7.7.4 of [15] as follows:
\[
(i_*Z(t))_x^\Delta = \sum_{y \in V(A, \iota_B)} \sum_{Q(y)=t}^{\text{ord}_p(n)} W_s(\psi_y).
\]

Here $i : Z(t) \rightarrow \mathcal{M}$ is the unramified morphism, and $W_s(\psi)$ is the quasi-canonical divisor of level $s$ associated to $\psi$, [15], p.240. If $x$ lies in $j_\Lambda(C)$, then $(j_\Lambda(C))_x^\Lambda = W_0(\psi_{y_0})$, where $y_0$ is the distinguished special endomorphism. Thus,
\[
(13.4) \quad (i_*Z(t)^\omega)_x^\Lambda = \sum_{y \in V(A, \iota_B)} \sum_{Q(y)=t}^{\text{ord}_p(n)} W_s(\psi_y) + 2 \sum_{s=1}^{\text{ord}_p(n)} W_s(\psi_{y_0}),
\]
where the factor of 2 in the second summand is due to the fact that $W_s(\psi_{y_0}) = W_s(\psi_{-y_0})$. In particular, the component $W_0(\psi_{y_0})$ has been removed. Recall that the local intersection number is given by, [15], Proposition 7.7.7, $(W_0(\psi), W_s(\psi)) = m_0(p)$ where $m_0(p) = 2$ if $p$ is ramified in $k$. 


and $m_0(p) = 1$ otherwise. Thus, the contribution of the terms in the second summand in (13.4), summed over the points of $C(\mathbb{F}_p)$, is

$$\frac{1}{w_k} 2 \text{ord}_p(n) m_0(p) \sum_{x \in C(\mathbb{F}_p)} \log |\kappa(x)| = 2 \frac{h_k}{w_k} \text{ord}_p(n^2) \log(p).$$

It follows that the log part of $\deg j^* \hat{Z}(t, v)^0$ is given by the log part of (12.3) together with the additional term

$$(13.5) \quad 2 \text{deg}_Q(C) \text{ord}_p(n^2) \log p.$$

Next suppose that $p \mid D(B)$. Again, we begin with a calculation on $\mathcal{M}$ and consider both inert and ramified $p$. Now the formal branches of $i_*Z(t)$ through a point $x \in \mathcal{M}_p$ corresponding to $(A, \iota_B)$ can be described using the results about the $p$-adic uniformization of the special cycles given section 8 of [12]. We have

$$(13.6) \quad (i_*Z(t))_x^\wedge = \sum_{y \in V(A, \iota_B) \cap Q(y) = t} \mathcal{N}(\psi_y)_x,$$

where the notation, which differs slightly from that of [12], is as follows. Let $(X, \iota_B)$ be the $p$-divisible group of $A$ with its $O_B$-action and let $\mathcal{N}$ be the Rapoport-Zink space parametrizing special formal $O_B$ modules $(X, \rho)$, cf. [12], p.154, where we require the quasi-isogeny $\rho$ to have height 0. A special endomorphism $y$ of $(A, \iota_B)$ gives rise to a special endomorphism $\psi_y$ of $(X, \iota_B)$, and we denote by $\mathcal{N}(\psi_y)$ the locus in $\mathcal{N}$ where it deforms. This locus was denoted by $Z(j)$ in [12]. Finally, we denote by $\mathcal{N}(\psi_y)_x$ the branches of $\mathcal{N}(\psi_y)$ at the point $x$. The decomposition (13.6) then follows from the description of the $p$-adic uniformization of the special cycle given in (8.17) and (8.20) of [12].

Now suppose that $x$ lies in $j_\Lambda(C)$, and let $y_0$ be the distinguished special endomorphism. Then

$$(13.7) \quad (i_*Z(t)^0)_x^\wedge = \sum_{y \in V(A, \iota_B) \cap Q(y) = t, y \neq y_0} \mathcal{N}(\psi_y)_x + 2 \mathcal{N}(\psi_{y_0})_x^{\text{ver}},$$

where $\mathcal{N}(\psi_{y_0})_x^{\text{ver}}$ is the vertical part of $\mathcal{N}(\psi_{y_0})_x$. Note that the divisors $\mathcal{N}(\psi_{y_0})_x^{\text{hor}}$ have been omitted on the right side of (13.6).

First suppose that $p \neq 2$, and write $4t = n^2|\Delta|$. Then $\text{ord}_p(t) = 2 \text{ord}_p(n)$ (resp. $2 \text{ord}_p(n) + 1$) if $p$ is inert (resp. ramified) in $k$. The structure of $\mathcal{N}(\psi_{y_0})$ is then shown in the pictures on p.161 of [12] and described in Proposition 4.5. In the inert case, $\mathcal{N}(\psi_{y_0})_x^{\text{ver}}$ consists of a single component of the special fiber $\mathcal{M}_p$ with multiplicity $\text{ord}_p(n)$, and the intersection multiplicity at $x$ of this component with $j_\Lambda(C)$ is 1. In the ramified case, $\mathcal{N}(\psi_{y_0})_x^{\text{ver}}$ consists of 2 components of $\mathcal{M}_p$ meeting at $x$, each with multiplicity $\text{ord}_p(n)$, and the intersection multiplicity at $x$ of each of them with $j_\Lambda(C)$ is 1, cf. Lemma 4.9. Again, it follows that the log part of $\deg j^* \hat{Z}(t, v)^0$ is given by the log part of (12.3) together with the additional term (13.5).

Finally, suppose that $p = 2$. In this case the structure of $\mathcal{N}(\psi_{y_0})$ is given in the appendix to section 11 of [14] and in section 6A.2 of [15]. Since the field $k = \mathbb{Q}(\sqrt{-t})$ splits $B$, we need only consider the cases in which $2$ is not split in $k$, using the descriptions on p.187 of [15]. Again we write $4t = n^2|\Delta|$ and $t = \varepsilon p^\alpha \in \mathbb{Z}_2$. In the inert case (2), there is one ‘central’ component of $\mathcal{N}(\psi_{y_0})_x^{\text{ver}}$. It has multiplicity $\mu = \frac{n}{2} + 1$ and its intersection number with the divisor $j_\Lambda(C)_x^{\text{hor}}$ is 1. Note that $\text{ord}_2(\Delta) = 0$ in this case, so that $\mu = \text{ord}_2(n)$. In the ramified case (3), there are a
pair of ‘central’ components of $N(\psi y_0)_x^{\text{ver}}$ meeting in a unique superspecial point and each having multiplicity $\mu = \frac{\alpha}{2}$. The intersection number of $j_\Lambda(C) \wedge x$ with each of these central components is 1, and, since $\text{ord}_2(\Delta) = 2$, $\mu = \text{ord}_2(n)$. Finally, the configuration in case (3) is the same as in case (2), except that now the multiplicity of the central components is $\mu = \frac{\alpha - 1}{2}$. Since $\text{ord}_2(\Delta) = 3$, we again find that $\mu = \text{ord}_2(n)$. Thus, we find the same contribution as in the other cases. □

Proof of Theorem 12.2. We simply observe that the Fourier coefficients of the pullback that are not covered by Theorem 12.1 have now been computed and match those described in Proposition 12.4. Note that the archimedean contributions include the archimedean term in the adjunction formula. □

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