STATIC SPHERICALLY SYMMETRIC SOLUTIONS OF THE 
SO(5) EINSTEIN YANG-MILLS EQUATIONS

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ABSTRACT. Globally regular (i.e. asymptotically flat and regular interior), spherically symmetric and localised (“particle-like”) solutions of the coupled Einstein Yang-Mills (EYM) equations with gauge group $SU(2)$ have been known for more than 20 years, yet their properties are still not well understood. Spherically symmetric Yang-Mills fields are classified by a choice of isotropy generator and $SO(5)$ is distinguished as the simplest model with a non-Abelian residual (little) group, $SU(2) \times U(1)$, and which admits globally regular particle-like solutions. We exhibit an algebraic gauge condition which normalises the residual gauge freedom to a finite number of discrete symmetries. This generalises the well-known reduction to the real magnetic potential $w(r, t)$ in the original $SU(2)$ YM model. Reformulating using gauge invariant polynomials dramatically simplifies the system and makes numerical search techniques feasible. We find three families of embedded $SU(2)$ EYM equations within the $SO(5)$ system, one of which was first detected only within the gauge-invariant polynomial reduced system. Numerical solutions representing mixtures of the three $SU(2)$ sub-systems are found, classified by a pair of positive integers.

1. Introduction

The LHC experiments currently underway at CERN are expected to settle prominent vexing questions such as the origin of the Higgs mechanism, the existence of supersymmetry in nature, and the meaning of dark matter. However, regardless of the discoveries still to flow from the LHC, we can with confidence predict that the governing equations will incorporate the Einstein equations for the gravitational field and the Yang-Mills equations for some non-Abelian gauge group. From this perspective, the spherically symmetric solutions of the $SU(2)$-EYM equations discovered numerically in [11] are particularly important, for several reasons. First, the particle-like properties exhibited by the solutions, namely static, asymptotically flat, and globally regular with spatial topology $\mathbb{R}^3$, confound previous expectations, based on the known non-existence results for the vacuum Einstein equations and for the YM equations separately, that such solutions could not exist. On the other hand, they confirm Wheeler’s “geon” hypothesis [15], that localised semi-bound gravitational (and with hindsight, Yang–Mills) configurations might exist.

Numerical and perturbation results [17,18] show that the $SU(2)$ EYM spherically symmetric solutions may be viewed as an unstable balance between a dispersive YM and the attractive gravitational force. However, there are many aspects of these systems which have not yet been studied in depth, and it seems premature to conclude that instability is inevitable — see the review [16].
The most detailed results have generally been obtained in the static spherically symmetric setting with the simplest YM gauge group $SU(2)$, which exhibits many of the basic properties of the general non-Abelian group models. The spherically symmetric reduction of the EYM equations can be viewed as a 2D EYM-Higgs system with a residual gauge group, $G^\Lambda_1$, a residual Higgs field (to be defined below), and a “Mexican hat” potential. Most attention has focused on the case where the gauge group is $SU(n)$ and the residual group is Abelian, primarily because in this case it is possible to completely fix the gauge freedom. If the residual group is non-Abelian, then it is known [4, 5] that the issue of gauge fixing becomes much more challenging.

To better understand gauge fixing and solutions to the static spherically symmetric EYM equations we study $G = SO(5)$, which is the simplest gauge group that supports a globally regular model with non-Abelian residual group. This model was discussed in [4], but the questions of gauge fixing and the existence of solutions were not resolved. We solve this problem here by using a related system satisfied by polynomials in the gauge fields which are invariant under the action of the residual gauge group.

2. Static spherically symmetric field equations

Spherical symmetry for Yang-Mills fields is complicated to define because there are many ways to lift an $SO(3)$ action on space-time to an action on the Yang-Mills connections. For real compact semisimple gauge groups $G$, it was shown in [1, 3] that equivalent spherically symmetric Yang-Mills connections correspond to conjugacy classes of homomorphisms of the isotropy subgroup, $U(1)$, into $G$. The latter, in turn, are given by their generator $\Lambda_3$, the image of the basis vector $\tau_3$ of $su(2)$ (where $\{\tau_i, i = 1, \ldots, 3\}$ is a standard basis with $[\tau_i, \tau_j] = \epsilon^{ijk}\tau_k$), lying in an integral lattice $I$ of a Cartan subalgebra $h_0$ of the Lie algebra $g_0$ of $G$. This vector $\Lambda_3$, when nontrivial, then characterizes up to conjugacy an $su(2)$ subalgebra.

With the vector $\Lambda_3 \in g_0$ fixed, a gauge and a coordinate system $(t, r, \theta, \phi)$ can be chosen [1, 3] so that the metric and gauge potential take the form

$$ds^2 = -S(r)^2 \left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

and

$$A = \Lambda_1(r) d\theta + (\Lambda_2(r) \sin \theta + \Lambda_3 \cos \theta) d\phi,$$

where $\Lambda_1(r), \Lambda_2(r)$ are $g_0$-valued maps satisfying

$$[\Lambda_3, \Lambda_1(r)] = \Lambda_2(r), \quad [\Lambda_2(r), \Lambda_3] = \Lambda_1(r).$$

(2.1)

It is more convenient to use the following variables,

(2.2)

$$\Lambda_0 := 2i\Lambda_3, \quad \Lambda_{\pm} := \mp\Lambda_1 - i\Lambda_2,$$

which lie in $g$, the complexification of $g_0$. The conditions (2.1) on $\Lambda_1, \Lambda_2$ then imply that the connection functions $\Lambda_{\pm}$ are valued in the residual Higgs bundles, $V_{\pm,2}$, defined by

$$\Lambda_{\pm}(r) \in V_{\pm,2} := \{X \in g | [\Lambda_0, X] = \pm 2X\}.$$  

2This ignores some interesting effects due to the fact that $SO(3)$ is not simply connected. For an analysis of $SO(3)$ actions on $SU(2)$ bundles see [2].

3Here we are assuming the ansatz, that the gauge potential is “purely magnetic.”
The residual (gauge) group $G^{\Lambda_0}$ is defined as the connected Lie subgroup with Lie algebra
$$g^{\Lambda_0} := \{ X \in g_0 \mid [\Lambda_0, X] = 0 \}.$$ We call the residual group (non-)Abelian if this is a (non-)Abelian Lie algebra. For $g^{\Lambda_0}$ to be non-Abelian it is necessary but not sufficient that $\Lambda_0$ lies on the boundary of a Weyl chamber of $h$, a Cartan subalgebra of $g$.

In the variables (2.2), the EYM equations become [6]
\begin{align*}
(2.4) & \quad m' = \frac{N}{2} |\Lambda_+'|^2 + \frac{1}{8r^2} |\Lambda_0 - [\Lambda_+, \Lambda_-]|^2, \\
(2.5) & \quad r^2 \left( S^2 N \Lambda_+ \right)' + S \left( \Lambda_+ - \frac{1}{2} [\Lambda_+, \Lambda_-], \Lambda_+ \right) = 0, \\
(2.6) & \quad [\Lambda_+', \Lambda_-] + [\Lambda_-', \Lambda_+] = 0, \\
(2.7) & \quad S^{-1} S' = \frac{1}{r} |\Lambda_+'|^2, 
\end{align*}
where $N = (1 - 2m/r)$. The $S$ parameter can be decoupled from the system, decreasing the order of the overall system by one but introducing a first order term in (2.5). Also note that the second term in (2.5) is proportional to the gradient of the second term in (2.4). Requiring that the solutions are regular and asymptotically flat gives boundary conditions
\begin{align*}
[\Lambda_+, \Lambda_-] = \Lambda_0, \quad & \text{and} \quad [[\Lambda_+, \Lambda_-], \Lambda_+] = 2\Lambda_+ 
\end{align*}
at $r = 0$, and as $r \to \infty$, respectively.

The $||\cdot||$-norm is proportional, on each irreducible component of $g$, to the real part of the Hermitean inner-product derived from the Killing form [6]. Multiplying $||\cdot||$ by a constant factor leads via a global rescaling of $m$ and $r$ to the original equations. Specifically, we have:

**Proposition 1.** If $(m(r), \Lambda_+(r))$ satisfies (2.4)-(2.6), then $(\alpha m(r/\alpha), \Lambda_+(r/\alpha))$ satisfies the equations obtained by replacing $||\cdot||$ in equations (2.4)-(2.6) with $\alpha||\cdot||$.

**Proof.** Substitution. \hfill $\square$

The EYM system (2.4)-(2.6) typically admits subfamilies of solutions which also satisfy the original $SU(2)$ but with rescaled norm $||\cdot||$ (see Section 5). We may regard such solutions as equivalent to the original $SU(2)$, since all values are consistently rescaled from the values in the original reports [7–9].

3. An SO(5) Model

We now specialize to the gauge group $G = SO(5)$. The complexified Lie algebra $g = so(5, \mathbb{C})$ has the Cartan decomposition
$$g = h \oplus \bigoplus_{\alpha \in R} \mathbb{C} e_\alpha,$$where $h = \text{span}_\mathbb{C}[h_{a_1}, h_{a_2}]$ is the Cartan subalgebra, $R = \{ \alpha_{\pm i}, i = 1, \ldots, 4 \}$ is a root system in $h^*$ with root diagram as in Figure 1.

The reduced spherically symmetric EYM model depends strongly on the choice of isotropy generator $\Lambda_0$. For $G = SO(5)$, we choose [4]
$$\Lambda_0 = 2H_1.$$
With this choice, the residual group is $SU(2) \times U(1)$, while (see (2.3))

$$V_2 = \text{span}_\mathbb{C}[e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}],$$

and hence

\begin{align}
\Lambda_+(r) &= w_1(r)e_{\alpha_1} + w_2(r)e_{\alpha_2} + w_3(r)e_{\alpha_3}, \\
\Lambda_-(r) &= \overline{w}_1(r)e_{\alpha_{-1}} + \overline{w}_2(r)e_{\alpha_{-2}} + \overline{w}_3(r)e_{\alpha_{-3}}.
\end{align}

Substituting (3.1) and (3.2) into (2.4)-(2.6), we find that the static spherically symmetric EYM equations in terms of the $w_i$ are

\begin{align}
(3.3) 
& m' = \frac{N}{2}(|w_1'|^2 + |w_2'|^2 + |w_3'|^2) + \frac{1}{2} \mathcal{P}, \\
(3.4) 
& r^2 N w_1'' + (2m - \frac{1}{r} \mathcal{P}) w_1' + w_1(1 - |w_1|^2) + w_2 \left(\frac{w_2 w_3}{2} - w_1 \overline{w}_2\right) = 0, \\
(3.5) 
& r^2 N w_2'' + (2m - \frac{1}{r} \mathcal{P}) w_2' + w_2(1 - \frac{|w_2|^2}{2} - |w_1|^2 - |w_3|^2) + \overline{w}_2 w_1 w_3 = 0, \\
(3.6) 
& r^2 N w_3'' + (2m - \frac{1}{r} \mathcal{P}) w_3' + w_3(1 - |w_3|^2) + w_2 \left(\frac{w_2 \overline{w}_3}{2} - \overline{w}_2 w_3\right) = 0, \\
(3.7) 
& w_1' \overline{w}_1 - w_1 \overline{w}_1' + w_2' \overline{w}_2 - w_2 \overline{w}_2' + w_3' \overline{w}_3 - w_3 \overline{w}_3' = 0, \\
(3.8) 
& w_1' \overline{w}_1 - w_1 \overline{w}_1' = (w_1 \overline{w}_1 - w_3 \overline{w}_3) = 0, \\
(3.9) 
& w_2' \overline{w}_2 - w_1 \overline{w}_1' = (w_2 \overline{w}_2 - w_2 \overline{w}_2) = 0,
\end{align}

where

\begin{align}
\mathcal{P} := & \frac{(1 - |w_1|^2)^2}{2} + \frac{(1 - |w_3|^2)^2}{2} + \frac{|w_2|^2}{2}(|w_1|^2 + |w_3|^2 + \frac{|w_2|^2}{4} - 1) - \Re[w_2 \overline{w}_1 w_3] \\
= & \frac{1}{4} (|w_1|^2 + |w_2|^2 + |w_3|^2 - 2)^2 + \frac{1}{4} (|w_1|^2 + |w_3|^2)^2 + \frac{1}{2} |w_1 \overline{w}_2 - w_2 \overline{w}_3|^2,
\end{align}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{root_diagram.png}
\caption{Root Diagram for $\mathfrak{so}(5, \mathbb{C})$}
\end{figure}
The boundary conditions are

\[ |w_1|^2 + |w_2|^2 + |w_3|^2 = 2, \]  
\[ |w_1| - |w_3| = 0, \]  
\[ w_1\overline{w}_2 - w_2\overline{w}_3 = 0, \]

at \( r = 0 \), and

\[ w_1(1 - |w_1|^2) + w_2(\frac{|w_2|}{2} - w_1\overline{w}_2) = 0, \]  
\[ w_2(1 - \frac{|w_2|^2}{2}) - w_2(|w_1|^2 + |w_3|^2) + \overline{w}_2w_1w_3 = 0, \]  
\[ w_3(1 - |w_3|^2) + w_2(\frac{\overline{w}_1}{2} - w_3\overline{w}_3) = 0, \]

as \( r \to \infty \).

The sets of boundary conditions (3.10)-(3.12) and (3.13)-(3.15) define algebraic varieties on \( \mathbb{C}^3 \) and as such they do not restrict the initial and final values to one particular point in \( \mathbb{C}^3 \) (or indeed a discrete set of such points). Given a point lying on one of these varieties there will in general be a set of gauge-equivalent points but it is not immediately obvious what these equivalence classes of points are from the defining equations. This is a problematic feature of all models with a non-Abelian residual gauge group and will be discussed elsewhere.

### 4. Gauge fixing and the reduced equations

For the \( SU(5) \) model, a direct computation shows that the ansatz

\[ \Lambda_\pm(r) = u(r)e_{\alpha_1} + 0e_{\alpha_2} + v(r)e_{\alpha_3}, \quad u(r), v(r) \in \mathbb{R} \]

automatically satisfies the constraint equations (2.4) (equivalently (3.7)-(3.9)), while the equations (2.5) (equivalently (3.8)-(3.10)) reduce to

\[ r^2Nu'' + \left( 2m - \frac{(1 - u^2)^2 + (1 - v^2)^2}{2r} \right) u' + u(1 - u^2) = 0, \]  
\[ r^2Nv'' + \left( 2m - \frac{(1 - u^2)^2 + (1 - v^2)^2}{2r} \right) v' + v(1 - v^2) = 0, \]  
\[ m' = \frac{N}{2}((u')^2 + (v')^2) + \frac{1}{4r^2}((1 - u^2)^2 + (1 - v^2)^2), \]

with boundary conditions

\[ u^2 + v^2 = 2, \quad |u| = |v| \]

at \( r = 0 \) and

\[ u(1 - u^2) = 0, \quad v(1 - v^2) = 0 \]

as \( r \to \infty \).

The above results show that (4.1) is a consistent ansatz for the static spherically symmetric EYM equations. In fact the equations that result could have been obtained from a model with gauge group \( SU(2) \times SU(2) \). However, it is not obvious
that the choice (4.1) is equivalent to fixing a section of the gauge fields. To see this we use the fact that the following two polynomials

\[ K_1 := 2||\Lambda_+||^2 = 2\left(|w_1|^2 + |w_2|^2 + |w_3|^2\right), \]
\[ K_2 := 2||\Lambda_-, \Lambda_+||^2 = 4|w_1|^4 + 2|w_2|^4 + 4|w_3|^4 + 8|w_1|^2|w_2|^2 + 8|w_3|^2|w_2|^2 - 8\text{Re}[w_2^2w_1w_3], \]

are a complete set of generators for the ring of residual-group-invariant polynomials in \( \Lambda_+ \). When \( (w_1, w_2, w_3) = (u, 0, v) \) for real \( u, v \), these become

\[ K_1 = 2(u^2 + v^2), \quad \text{and} \quad K_2 = 4(u^4 + v^4). \]

These equations can be inverted to give

\[ u = \pm_1 \left( \frac{K_1}{4} \pm \sqrt{\frac{2K_2 - K_1^2}{4}} \right)^{\frac{1}{2}}, \quad \text{and} \quad v = \pm_2 \left( \frac{K_1}{4} \mp \sqrt{\frac{2K_2 - K_1^2}{4}} \right)^{\frac{1}{2}} \]

which are well-defined for all \( K_1, K_2 \) since the inequalities

\[ 2K_2 - K_1^2 \geq 0, \quad \text{and} \quad K_2 - K_1^2 \leq 0 \]

follow directly from the definition of \( K_1 \) and \( K_2 \). The \( \pm \)-signs amount to choosing which function is \( u \) and which is \( v \) and deciding on an arbitrary convention for their initial values.

5. Numerical Results

From the form of the reduced equations (4.2)-(4.4), it is clear that we can make two distinct simplifying assumptions \( u = v \) or \( v = \pm_1 \) (equivalently \( u = \pm_1 \)).

(i) Setting \( u = v \), the reduced equations (4.2)-(4.4) become

\[ r^2Nu'' + \left( 2m - \frac{(1-u^2)^2}{r} \right)u' + u(1-u^2) = 0, \]
\[ m' = N(u')^2 + \frac{1}{2r^2}(1-u^2)^2. \]

which is the \( SU(2) \) equation and has the well-known Bartnik-McKinnon family of solutions [7,11]. We note that these correspond to the embedded \( SU(2) \) solutions as they satisfy \( \Lambda_\pm(r) = u(r)\Omega_\pm \) where \( \Omega_\pm \) are fixed vectors satisfying \( [\Lambda_0, \Omega_\pm] = 2\Omega_\pm \) and \( [\Omega_+, \Omega_-] = \Lambda_0 \).

(ii) Setting \( v = 1 \), the reduced equations (4.2)-(4.4) become

\[ r^2Nu'' + \left( 2m - \frac{(1-u^2)^2}{2r} \right)u' + u(1-u^2) = 0, \]
\[ m' = \frac{N}{2}(u')^2 + \frac{1}{4r^2}(1-u^2)^2. \]

As in Proposition 4 rescaling these equations by \( \sqrt{2} \) in \( m \) and \( r \) results in the \( SU(2) \)-equations above, giving another family of solutions where the \( u \) field is a radially-scaled Bartnik-McKinnon solution with smaller mass and the \( v \) field remains constant.

\[ \text{This can be established by computing the Molien function to determine the appropriate orders of the polynomials and then verifying } K_1 \text{ and } K_2 \text{ are algebraically independent. See [10] for some useful formulae.} \]
Figure 2. A plot of the solutions in the space of the invariant polynomials. All solutions start at \((4,8)\). The type (i) SU(2) solutions lie on the \(K_2 = \frac{1}{2}K_1^2\) curve, while the type (ii) SU(2) solutions lie on the \(K_2 = (K_1 - 2)^2 + 4\) curve. The type (iii) solutions are curves in the combined interior of the three parabolas touching the boundaries tangentially (an illustration of the \((2,1)\) type (iii) solution is shown).

For the above two special situations, existence of a countably infinite number of solutions is guaranteed by the existence theorems in [7, 12, 13]. Integrating the reduced equations (4.2)-(4.4) numerically, we also find evidence for solutions that are not of the type (i) or (ii) above. We will call these solutions type (iii). Figure 2 indicates where these solutions lie in the space of invariant polynomials. All three types of solutions can be characterized by the number of nodes of the functions \(u\) and \(v\). Figure 3 enumerates all of the solutions that were found numerically or that are known to exist analytically. The type (i) solutions lie on the diagonal \(n_u = n_v\), while the type (ii) solutions lie on either the horizontal \(n_v = 0\) or vertical axis \(n_u = 0\). The remaining solutions are of type (iii) and are indicated on the node diagram by the larger circles. We conjecture, based on the diagram, that for each \((n_u, n_v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\) there exists a solution to the reduced equations (4.2)-(4.4) satisfying the boundary conditions (4.5)-(4.6).

For the numerical analysis we use the “shooting to a fitting point” method [14], using the Matlab routine ode45 which is a fourth-order Runge-Kutta solver. Using
Figure 3. Node Diagram for solutions to the reduced equation

the power series expansions

\[ u = 1 + a_1 r^2 + \left( \frac{2}{5} a_1^3 + \frac{2}{5} a_1 a_2^2 + \frac{3}{10} a_1^2 a_2^2 \right) r^4 + O(r^6), \]

\[ v = 1 + a_2 r^2 + \left( \frac{2}{5} a_2 a_1^2 + \frac{2}{5} a_2^3 + \frac{3}{10} a_2^2 r^4 \right) + O(r^6), \]

\[ m = (a_1^2 + a_2^2) r^3 + \left( \frac{4}{5} a_1^3 + \frac{4}{5} a_2^3 \right) r^5 + O(r^7), \]

near \( r = 0 \), and

\[ u = \pm 1 + \frac{b_1}{r} + O\left( \frac{1}{r^2} \right), \]

\[ v = \pm 1 + \frac{b_2}{r} + O\left( \frac{1}{r^2} \right), \]

\[ m = m_\infty + O\left( \frac{1}{r^2} \right), \]

near \( r = \infty \). We shoot from the origin out trying to maximise (by searching in the parameters \( a_i \) evaluated at \( r = 0.01 \)) the \( r \)-value that is attained before the solution violates one of the necessary conditions for a global smooth solution. Suitable large \( r \)-values will then give an indication what the \( b_i \) and \( m_\infty \) parameters should approximately be for the shooting back from some large \( r \)-value (we use \( r = 10000 \)). We then use these parameter values as an initial guess in the “shooting to a fitting point” search, where the fitting was done at \( r = 10 \).

6. Discussion

We have found three solutions using the numerical technique from the previous section. When combined with the two families of \( SU(2) \) type solutions, they suggest that there is a solution for each point \( (n_u, n_v) \) on the nonnegative integer lattice, where \( n_u, n_v \) denote the number of nodes of \( u \) and \( v \) respectively. The numerical values are given in Table 1 and some type (iii) solutions are displayed in Figures 4 and 5.
It is only with the reduced variables we have presented that these solutions could be found. As an indication of the difficulties otherwise, consider the particulars of the shooting method technique. In the original $w$-variables the ODE system is given in terms of four complex variables and the mass, $m$. This leads to eight real shooting parameters near $r = 0$ and nine shooting parameters near $r = \infty$. Furthermore, after finding a promising approximate solution by shooting outward from $r = 0$ and another by shooting inward from $r = \infty$, they will generally need to be gauge rotated to match at the fitting point (if indeed they are gauge equivalent-solutions). This all amounts to a difficult optimization problem in 17 variables. This problem was discussed in [4] where some approximate solutions were obtained by shooting but a global solution was not found.

In contrast, shooting for the reduced variables has only five parameters to search in, two parameters near $r = 0$ and three near $r = \infty$ with no need to gauge rotate possible solutions at the fitting point. The removal of degeneracy from the numerical problem and minimization of the number of shooting parameters is what has made it tractable to find the solutions presented here.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$m_{\infty}$ & $u_{\infty}$ & $v_{\infty}$ & $a_1$ & $a_2$ & $b_1$ & $b_2$ & $n_u$ & $n_v$ & type \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (i) \\
0.58595 & -1 & 1 & -0.9074325 & 0 & 0.631715 & 0 & 1 & 0 & (ii) \\
0.68685 & 1 & 1 & -1.303451 & 0 & -6.26770 & 0 & 2 & 0 & (ii) \\
0.70380 & -1 & 1 & -1.394080 & 0 & 41.6713 & 0 & 3 & 0 & (ii) \\
0.70655 & 1 & 1 & -1.409757 & 0 & -259.038 & 0 & 4 & 0 & (ii) \\
0.70700 & -1 & 1 & -1.412337 & 0 & 1592.32 & 0 & 5 & 0 & (ii) \\
0.70710 & 1 & 1 & -1.412759 & 0 & -9770.35 & 0 & 6 & 0 & (ii) \\
0.82865 & -1 & -1 & -0.45372 & -0.45372 & 0.8934 & 0.8934 & 1 & 1 & (i) \\
0.92377 & 1 & -1 & -1.187117 & -0.117170 & -6.32743 & 1.68255 & 2 & 1 & (iii) \\
0.93982 & -1 & -1 & -1.371735 & -0.022338 & 37.63922 & 1.67940 & 3 & 1 & (iii) \\
0.97135 & 1 & 1 & -0.65173 & -0.65173 & -8.8639 & -8.8639 & 2 & 2 & (i) \\
0.98729 & -1 & 1 & -1.246544 & -0.155699 & 42.52411 & -13.88445 & 3 & 2 & (iii) \\
0.99532 & -1 & -1 & -0.69704 & -0.69704 & 58.9326 & 58.9326 & 3 & 3 & (i) \\
0.99924 & 1 & 1 & -0.70488 & -0.70488 & -366.335 & -366.335 & 4 & 4 & (i) \\
0.99988 & -1 & -1 & -0.70617 & -0.70617 & 2251.89 & 2251.89 & 5 & 5 & (i) \\
0.99998 & 1 & 1 & -0.70638 & -0.70638 & -13817.4 & -13817.4 & 6 & 6 & (i) \\
\hline
\end{tabular}
\caption{Table of Numerical Parameters}
\end{table}
Figure 4. The (2,1) type (iii) solution. The mass, $m$, increases monotonically from 0 to 0.924. $u$ starts and ends at 1 with two nodes and $v$ goes from +1 to -1 with one node.

Figure 5. The (3,2) type (iii) solution. The mass, $m$, increases monotonically from 0 to 0.987, $u$ goes from 1 to -1 with three nodes and $v$ goes from +1 to +1 with two nodes.

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