A stability property for coefficients in
Kronecker products of complex $S_n$ characters

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Abstract

In this note we make explicit a stability property for Kronecker coefficients
that is implicit in a theorem of Y. Dvir. Even in the simplest nontrivial case this
property was overlooked despite of the work of several authors. As an application
we give a new formula for some Kronecker coefficients.

Key Words: Kronecker product, Characters, Symmetric group, Schur func-
tions, Internal product.

1 Introduction

Let $\lambda, \mu, \nu$ be partitions of a positive integer $m$ and let $\chi^\lambda, \chi^\mu, \chi^\nu$ be their corresponding
complex irreducible characters of the symmetric group $S_m$. It is a long standing problem
to give a satisfactory method for computing the multiplicity

$$k(\lambda, \mu, \nu) := \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle$$

(1)
of $\chi^\nu$ in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of $\chi^\lambda$ and $\chi^\mu$ (here $\langle \cdot, \cdot \rangle$ denotes the inner
product of complex characters). Via the Frobenius map, $k(\lambda, \mu, \nu)$ is equal to the
multiplicity of the Schur function $s_\nu$ in the internal product of Schur functions $s\lambda * s\mu$, namely

$$k(\lambda, \mu, \nu) = \langle s\lambda * s\mu, s\nu \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of symmetric functions.

The first stability property for Kronecker coefficients was observed by F. Murnaghan
without proof in [8]. This property can be stated in the following way: Let $\overline{\lambda}, \overline{\mu}, \overline{\nu}$ be
partitions of $a, b, c$, respectively. Define $\lambda(n) := (n-a, \overline{\lambda}), \mu(n) := (n-b, \overline{\mu}), \nu(n) :=$

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\( (n-c, \nu) \). Then the coefficient \( k(\lambda(n), \mu(n), \nu(n)) \) is constant for all \( n \) bigger than some integer \( N(\lambda, \mu, \nu) \). Complete proofs of this property were given by M. Brion \[3\] using algebraic geometry and E. Vallejo \[13\] using combinatorics of Young tableaux. Both proofs give different lower bounds \( N(\lambda, \mu, \nu) \) for the stability of \( k(\lambda(n), \mu(n), \nu(n)) \), for all partitions \( \lambda, \mu, \nu \). C. Ballantine and R. Orellana \[1\] gave an improvement of one of these lower bounds for a particular case.

Here we make explicit another stability property for Kronecker coefficients that is implicit in the work of Y. Dvir (Theorem 2.4’ in \[5\]). This property can be stated as follows: Let \( p, q, r \) be positive integers such that \( p = qr \). Let \( \lambda = (\lambda_1, \ldots, \lambda_p) \), \( \mu = (\mu_1, \ldots, \mu_q) \), \( \nu = (\nu_1, \ldots, \nu_r) \) be partitions of some nonnegative integer \( m \) satisfying \( \ell(\lambda) \leq p, \ell(\mu) \leq q, \ell(\nu) \leq r \), that is, some parts of \( \lambda, \mu \) and \( \nu \) could be zero. For any positive integers \( t \) and \( n \) let \((t)_n\) denote the vector \((t, \ldots, t) \in \mathbb{N}^n \). Then we have

**Theorem 3.1.** With the above notation

\[
k(\lambda, \mu, \nu) = k(\lambda + (t)^p, \mu + (rt)^q, \nu + (qt)^r) .
\]

It should be noted that even in the simplest nontrivial case, when \( q = 2 = r \) and \( p = 4 \), this property was overlooked despite of the work of several authors \[1, 2, 9, 10\]. In this situation Remmel and Whitehead noticed (Theorems 3.1 and 3.2 in \[9\]) that the coefficient \( k(\lambda, \mu, \nu) \) has a much simpler formula if \( \lambda_3 = \lambda_4 \). The main theorem provides and explanation for that. We also obtain a new formula for \( k(\lambda, \mu, \nu) \) in this case.

This note is organized as follows. Section 2 contains the definitions and notation about partitions needed in this paper. In Section 3 we give the proof of the main theorem. Section 4 deals with the Kronecker coefficient \( k(\lambda, \mu, \nu) \) when \( \ell(\lambda) = \ell(\mu) \ell(\nu) \). In particular, we give, in this case, a new vanishing condition. Finally, in Section 5 we give an application of the main theorem.

## 2 Partitions

In this section we recall the notation about partitions needed in this paper. See for example \[6, 7, 11, 12\].

For any nonnegative integer \( n \) let \([n] := \{1, \ldots, n\} \). A *partition* is a vector \( \lambda = (\lambda_1, \ldots, \lambda_p) \) of nonnegative integers arranged in decreasing order \( \lambda_1 \geq \cdots \geq \lambda_p \). We consider two partitions equal if they differ by a string of zeros at the end. For example \((3, 2, 1)\) and \((3, 2, 1, 0, 0)\) represent the same partition. The *length* of \( \lambda \), denoted by \( \ell(\lambda) \), is the number of positive parts of \( \lambda \). The *size* of \( \lambda \), denoted by \(|\lambda|\) is the sum of its parts; if \(|\lambda| = m \), we say that \( \lambda \) is a partition of \( m \) and denote it by \( \lambda \vdash m \). The partition conjugate to \( \lambda \) is denoted by \( \lambda' \). A *composition* of \( m \) is a vector \( \pi = (\pi_1, \ldots, \pi_r) \) of positive integers such that \( \sum_{i=1}^{r} \pi_i = m \).
The diagram of $\lambda$, also denoted by $\lambda$, is the set of pairs of positive integers
\[
\lambda = \{(i, j) \mid i \in [p], \ j \in [\lambda_i]\}.
\]
The identification of $\lambda$ with its diagram permits us to use set theoretic notation for partitions. If $\delta$ is another partition and $\delta \subseteq \lambda$, we denote by $\lambda/\delta$ the skew diagram consisting of the pairs in $\lambda$ that are not in $\delta$, and by $|\lambda/\delta|$ its cardinality. If $\mu$ is another partition, then $\lambda \cap \mu$ denotes the set theoretic intersection of $\lambda$ and $\mu$.

3 Main theorem

3.1 Theorem. Let $\lambda, \mu, \nu$ be partitions of some integer $m$. Let $p, q, r$ be integers such that $p \geq \ell(\lambda)$, $q \geq \ell(\mu)$, $r \geq \ell(\nu)$ and $p = qr$. Then for any positive integer $t$ we have
\[
k(\lambda, \mu, \nu) = k(\lambda + (t)^p, \mu + (rt)^q, \nu + (qt)^r).
\]

The proof of the main theorem will follow from Dvir’s theorem

3.2 Theorem. Let $\lambda, \mu, \nu$ be partitions of $n$ such that $\ell(\nu) = |\lambda \cap \mu'|$. Let $l = \ell(\nu)$ and $\rho = \nu - (1^l)$. Then
\[
k(\lambda, \mu, \nu) = \langle \chi^{\lambda/\lambda \cap \mu'} \otimes \chi^{\mu/\lambda \cap \mu}, \chi^\rho \rangle.
\]

Proof of theorem. It is enough to prove the theorem for $t = 1$. The general case follows by repeated application of the particular case. Let $\alpha = \lambda + (1)^p$, $\beta = \mu + (r)^q$ and $\gamma = \nu + (q)^r$. Then $\beta \cap \gamma' = (r^q)$. In particular, $|\beta \cap \gamma'| = p = \ell(\alpha)$. So, we have $\beta/\beta \cap \gamma' = \mu$ and $\gamma/\beta' \cap \gamma = \nu$. Thus, by Dvir’s theorem, we have
\[
k(\beta, \gamma, \alpha) = k(\mu, \nu, \lambda).
\]
The claim follows from the symmetry $k(\lambda, \mu, \nu) = k(\mu, \nu, \lambda)$ of Kronecker coefficients.

4 The case $\ell(\lambda) = \ell(\mu)\ell(\nu)$

In this section we give a general result for the Kronecker coefficient $k(\lambda, \mu, \nu)$ when $\ell(\lambda) = \ell(\mu)\ell(\nu)$. On the one hand it gives a new vanishing condition. On the other hand, when this vanishing condition does not hold, it reduces the computation of $k(\lambda, \mu, \nu)$ to the computation of a simpler Kronecker coefficient.

Let $m$ be a positive integer, $\lambda, \mu$ be partitions of $m$ and $\pi = (\pi_1, \ldots, \pi_r)$ be a composition of $m$. Let $\rho(i) \vdash \pi_i$ for $i \in [r]$. A sequence $T = (T_1, \ldots, T_r)$ of tableaux
is called a Littlewood-Richardson multitableau of shape \( \lambda \), content \((\rho(1), \ldots, \rho(r))\) and type \( \pi \) if

1. there exists a sequence of partitions
   \[ \emptyset = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda \]
such that \(|\lambda(i)/\lambda(i - 1)| = \pi_i\) for all \( i \in [r] \), and
2. \( T_i \) is Littlewood-Richardson tableau of shape \( \lambda(i)/\lambda(i - 1) \) and content \( \rho(i) \), for all \( i \in [r] \).

Let \( LR(\lambda, \mu; \pi) \) denote the set of pairs \((S, T)\) of Littlewood-Richardson multitableaux of shape \((\lambda, \mu)\), same content and type \( \pi \). This means that \( S = (S_1, \ldots, S_r) \) is a Littlewood-Richardson multitableau of shape \( \lambda \), \( T = (T_1, \ldots, T_r) \) is a Littlewood-Richardson multitableau of shape \( \mu \) and both \( S_i \) and \( T_i \) have the same content \( \rho(i) \) for some partition \( \rho(i) \) of \( \pi_i \), for all \( i \in [r] \). Let \( c_{(\rho(1), \ldots, \rho(r))}^{\lambda, \mu} \) denote the number of Littlewood-Richardson multitableaux of shape \( \lambda \) and content \((\rho(1), \ldots, \rho(r))\) and let \( lr(\lambda, \mu; \pi) \) denote the cardinality of \( LR(\lambda, \mu; \pi) \). Then

\[
lr(\lambda, \mu; \pi) = \sum_{\rho(1) = \pi_1, \ldots, \rho(r) = \pi_r} c_{(\rho(1), \ldots, \rho(r))}^{\lambda, \mu} c_{(\rho(1), \ldots, \rho(r))}^{\mu, \nu}.
\]

Similar numbers have already proved to be useful in the study of minimal components, in the dominance order of partitions, of Kronecker products [14].

The number \( lr(\lambda, \mu; \pi) \) can be described as an inner product of characters. For this description we need the permutation character \( \phi^\pi := \text{Ind}^S_{\pi_\lambda}(1_{\pi}) \), namely, the induced character from the trivial character of \( S_\pi = S_{\pi_1} \times \cdots \times S_{\pi_r} \). It follows from Frobenius reciprocity and the Littlewood-Richardson rule that (see also [6 2.9.17])

4.1 Lemma. Let \( \lambda, \mu, \pi \) be as above. Then

\[
lr(\lambda, \mu; \pi) = \langle \chi^\lambda \otimes \chi^\mu, \phi^\pi \rangle.
\]

Since Young’s rule and Lemma 4.1 imply that \( lr(\lambda, \mu; \nu) \geq k(\lambda, \mu, \nu) \), then we have

4.2 Corollary. Let \( \lambda, \mu, \nu \) be partitions of \( m \). If \( lr(\lambda, \mu; \nu) = 0 \), then \( k(\lambda, \mu, \nu) = 0 \).

4.3 Lemma. Let \( \lambda, \mu, \nu \) be partitions of \( m \) of lengths \( p, q, r \), respectively. If \( p = qr \), and \( \mu_q < r \lambda_p \) or \( \nu_r < q \lambda_p \), then \( lr(\lambda, \mu; \nu) = 0 \).

Proof. We assume that \( lr(\lambda, \mu; \nu) > 0 \) and show that \( \mu_q \geq r \lambda_p \) and \( \nu_r \geq q \lambda_p \). Let \((S, T)\) be an element in \( LR(\lambda, \mu; \nu) \) having content \((\rho(1), \ldots, \rho(r))\). Since \( T_i \) is contained in \( \mu \), one has, by elementary properties of Littlewood-Richardson tableaux, that \( \ell(\rho(i)) \leq \ell(\mu) \leq q \). For any \( i \), let \( n_i \) be the number of squares of \( S_i \) that are in column \( \lambda_p \) of \( \lambda \), then \( n_i \leq q \). We conclude that \( p = n_1 + \cdots + n_r \leq rq = p \). Therefore \( n_i = q = \ell(\rho(i)) \) for all
i. This forces that each $S_i$ contains a $j$ in the squares $(j+(i-1)q, 1), \ldots, (j+(i-1)q, \lambda_p)$ of $\lambda$, for all $j \in [q]$. So, $\rho(i)_j \geq \lambda_p$ for all $j$. In particular, for $i = r$, since $S_r$ has $\nu_r$ squares, one has $\nu_r \geq q \lambda_p$. Now, since $\ell(\mu) = q$, all entries of $T_i$ equal to $q$ must be in row $q$ of $\mu$. Then $\mu_q \geq \rho(1)_q + \cdots + \rho(r)_q \geq r \lambda_p$. The claim follows.

4.4 Corollary. Let $\lambda, \mu, \nu$ be partitions of $m$ of length $p, q, r$, respectively. If $p = qr$, and $\mu_q < r \lambda_p$ or $\nu_r < q \lambda_p$, then $k(\lambda, \mu, \nu) = 0$.

Proof. This follows from Lemma 4.3 and Corollary 4.2.

4.5 Theorem. Let $\lambda, \mu, \nu$ be partitions of $m$ of length $p, q, r$, respectively. Let $t = \lambda_p$ and assume $p = qr$, then we have

1. If $\mu_q < rt$ or $\nu_r < qt$, then $k(\lambda, \mu, \nu) = 0$.

2. If $\mu_q \geq rt$ and $\nu_r \geq qt$, let $\tilde{\lambda} = \lambda - (t)^p$, $\tilde{\mu} = \mu - (rt)^q$, and $\tilde{\nu} = \nu - (qt)^r$. Then, $k(\lambda, \mu, \nu) = k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$.

5 Applications

We conclude this paper with an application to the expansion of $\chi^{\mu} \otimes \chi^{\nu}$ when $\ell(\mu) = 2 = \ell(\nu)$. It is well known that any component of $\chi^{\mu} \otimes \chi^{\nu}$ corresponds to a partition of length at most $|\mu \cap \nu'| \leq 4$, see Satz 1 in [4], Theorem 1.6 in [5] or Theorem 2.1 in [9]. Even in this simple case a nice closed formula seems unlikely to exist. J. Remmel and T. Whitehead (Theorem 2.1 in [9]) gave a close, though intricate, formula for $k(\lambda, \mu, \nu)$ valid for any $\lambda$ of length at most 4; M. Rosas (Theorem 1 in [10]) gave a formula of combinatorial nature for $k(\lambda, \mu, \nu)$, which requires taking subtractions, also valid for any $\lambda$ of length at most 4; C. Ballantine and R. Orellana (Proposition 4.12 in [2]) gave a simpler formula for $k(\lambda, \mu, \nu)$, at the cost of assuming an extra condition on $\lambda$.

Note that when $\ell(\lambda) = 1$ the coefficient $k(\lambda, \mu, \nu)$ is trivial to compute. For $\ell(\lambda) = 2$ Remmel-Whitehead formula for $k(\lambda, \mu, \nu)$ reduces to a simpler one (Theorem 3.3 in [9]). This formula was recovered by Rosas in a different way (Corollary 1 in [10]). So, the nontrivial cases are those for which $\ell(\lambda) = 3, 4$. Corollary 4.1 deals with the case of length 4. On the one hand it gives a new vanishing condition. On the other hand, when this vanishing condition does not hold, it reduces the case of length 4 to the case of length 3. Thus, this reduction would help to simplify the proofs of the formulas given by Remmel-Whitehead and Rosas.

The following corollary is a particular case of Theorem 4.5

5.1 Corollary. Let $\lambda, \mu, \nu$ be a partitions of $m$ of length 4, 2, 2, respectively. Let $t = \lambda_4$, then we have
(1) If \( \mu_2 < 2t \) or \( \nu_2 < 2t \), then \( k(\lambda, \mu, \nu) = 0 \).

(2) If \( \mu_2 \geq 2t \) and \( \nu_2 \geq 2t \), let \( \lambda = (\lambda_1 - t, \lambda_2 - t, \lambda_3 - t) \), \( \tilde{\mu} = (\mu_1 - 2t, \mu_2 - 2t) \) and \( \tilde{\nu} = (\nu_1 - 2t, \nu_2 - 2t) \). Then, \( k(\lambda, \mu, \nu) = k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) \).

Another observation of Remmel and Whitehead (Theorems 3.1 and 3.2 in [9]) is that their formula simplifies considerably in the case \( \lambda_3 = \lambda_4 \). Corollary 5.1 explains this phenomenon since, in this case, the computation of \( k(\lambda, \mu, \nu) \) reduces to the computation of a Kronecker coefficient involving only three partitions of length at most 2, which have a simple nice formula (Theorem 3.3 in [9]). In fact, combining our re-
computation of a Kronecker coefficient involving only three part itions of length at most 5.2 Theorem.

From Corollary 5.1 and Theorem 5.2 we obtain

5.3 Theorem. Let \( \lambda, \mu, \nu \) be partitions of \( m \) of length 2. Let \( x = \max \left(0, \left\lceil \frac{\nu_2+\mu_2-\lambda_2}{2} - \lambda_3 \right\rceil \right) \) and \( y = \left\lceil \frac{\nu_2+\mu_2-\lambda_2+1}{2} \right\rceil \). Assume \( \nu_2 \leq \mu_2 \leq \lambda_2 \). Then

\[ k(\lambda, \mu, \nu) = (y - x)(y \geq x). \]

Proof. Let \( \tilde{\lambda} = (\lambda_1 - \lambda_3, \lambda_2 - \lambda_3) \), \( \tilde{\mu} = (\mu_1 - 2\lambda_3, \mu_2 - 2\lambda_3) \) and \( \tilde{\nu} = (\nu_1 - 2\lambda_3, \nu_2 - 2\lambda_3) \). These are partitions of \( m - 4\lambda_3 \). Then, by Corollary 5.1 \( k(\lambda, \mu, \nu) = k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) \). Since \( \ell(\tilde{\lambda}) = \ell(\tilde{\mu}) = \ell(\tilde{\nu}) = 2 \), we can apply Theorem 5.2. Due to the symmetry of the Kronecker coefficients we are assuming \( \nu_2 \leq \mu_2 \). We have to consider three cases: (a) \( \lambda_2 - \lambda_3 \leq \nu_2 - 2\lambda_3 \), (b) \( \nu_2 - 2\lambda_3 < \lambda_2 - \lambda_3 \leq \mu_2 - 2\lambda_3 \) and (c) \( \mu_2 - 2\lambda_3 < \lambda_2 - \lambda_3 \). In the first two cases Remmel-Whitehead formula yields the same formula for \( k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) \). So, we have only two cases to consider: (1) \( \lambda_2 + \lambda_3 \leq \mu_2 \) and (2) \( \mu_2 < \lambda_2 + \lambda_3 \). In the first case Theorem 5.2 yields

\[ k(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}) = (y' - x')(y' \geq x') \]

where \( x' = \max \left(0, \left\lceil \frac{\nu_2-2\lambda_3+\lambda_2-\lambda_3+\mu_2-2\lambda_3-(m-4\lambda_3)}{2} \right\rceil \right) \) and \( y' = \left\lceil \frac{\nu_2-2\lambda_3+\lambda_2-\lambda_3+(m-2\lambda_3)+1}{2} \right\rceil \). It is straightforward to check that \( x' = x \) and \( y' = y \), so the first claim follows.

The second case is similar. \( \square \)
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