A mixed discontinuous Galerkin method for the time harmonic elasticity problem with reduced symmetry

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Abstract

The aim of this paper is to analyze a mixed discontinuous Galerkin discretization of the time-harmonic elasticity problem. The symmetry of the Cauchy stress tensor is imposed weakly, as in the traditional dual-mixed setting. We show that the discontinuous Galerkin scheme is well-posed and uniformly stable with respect to the mesh parameter $h$ and the Lamé coefficient $\lambda$. We also derive optimal a-priori error bounds in the energy norm. Several numerical tests are presented in order to illustrate the performance of the method and confirm the theoretical results.

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1 Introduction

In this paper we are interested in the dual-mixed formulation of the elasticity problem with weakly imposed symmetry. We introduce and analyze a mixed interior penalty discontinuous Galerkin (DG) method for the elasticity system in time-harmonic regime. The interior penalty DG method can be traced back to [1, 9] and its application for elliptic problems is now well understood; see [8] and the references cited therein for more details. The mixed interior penalty method introduced here can be viewed as a discontinuous version of the Arnold-Falk-Winther div-conforming finite element space [3]. It approximates the unknowns of the mixed formulation, given by the Cauchy stress tensor and the

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rotation, by discontinuous finite element spaces of degree $k$ and $k - 1$ respectively. This permits one to enjoy the well-known flexibility properties of DG methods for $hp$-adaptivity and to implement high-order elements by using standard shape functions. Moreover, our scheme is immune to the locking phenomenon that arises in the nearly incompressible case.

The first step in our study of the mixed DG scheme consists in providing a convergence analysis for the corresponding div-conforming Galerkin method based on the Arnold-Falk-Winther element. We point out that there are many finite element methods for the mixed formulation of the elasticity problem with reduced symmetry [3, 4, 7, 13, 16]. All of them have been analyzed in the static case, i.e., in the case $\omega = 0$ in problem (1) below. In time harmonic regime, the operator underlying the mixed formulation is not Fredholm of index zero as in the classical displacement-based formulation. The same challenge is encountered when analyzing the curl-conforming variational formulation of the Maxwell system [6, 12]. Actually, the abstract theory given in [6] can also be applied to the dual-mixed variational formulation of linear elasticity as shown (implicitly) in the analysis given in [11] for a fluid-solid interaction problem. Instead of using this approach, we take here advantage of the recent spectral analysis obtained in [15] to directly deduce the stability of the Arnold-Falk-Winther finite element approximation of the indefinite elasticity problem.

An interior penalty discontinuous Galerkin method has also been introduced in [14] for the Maxwell system. The DG formulation we are considering here is, in a certain sense, its counterpart in the $H(\text{div})$-setting. Notice that, in contrast to [14], our approach does not rely on a duality technique. We prove the convergence of the DG scheme by exploiting the stability of the corresponding div-conforming method and without requiring further regularity assumption than the one needed to write properly the right-hand side of (21) below. Moreover, if the analytic Cauchy stress tensor, its divergence and rotation belong to a Sobolev space with regularity exponent $s > 1/2$, then it is shown that the error in the DG-energy norm converges with the optimal order $O(h^{\min(s,k)})$ with respect to the mesh size $h$ and the polynomial degree $k$.

The paper is organized as follows. In Section 2, we recall the dual formulation of the linear elasticity problem with reduced symmetry and prove its well-posedness when the wave number is different from a countable set of singular values. In Section 3 we prove the convergence of the conforming Galerkin scheme based on the Arnold-Falk-Winther element. In Section 4 we introduce the mixed interior penalty discontinuous Galerkin method and its convergence analysis is carried out in Section 5. Finally, in Section 6 we present numerical results that confirm the theoretical convergence estimates.

We end this section with some of the notations that we will use below. Given any Hilbert space $V$, let $V^3$ and $V^{3 \times 3}$ denote, respectively, the space of vectors and tensors of order 3 with entries in $V$. In particular, $I$ is the identity matrix of $\mathbb{R}^{3 \times 3}$ and $0$ denotes a generic null vector or tensor. Given $\tau := (\tau_{ij})$ and $\sigma := (\sigma_{ij}) \in \mathbb{R}^{3 \times 3}$, we define as usual the transpose tensor $\tau^t := (\tau_{ji})$, the trace $\text{tr} \tau := \sum_{i=1}^{3} \tau_{ii}$, the deviatoric tensor $\tau^0 := \tau - \frac{1}{3} (\text{tr} \tau) I$, and the tensor inner product $\tau : \sigma := \sum_{i,j=1}^{3} \tau_{ij} \sigma_{ij}$.

Let $\Omega$ be a polyhedral Lipschitz bounded domain of $\mathbb{R}^3$ with boundary $\partial \Omega$. For
Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded Lipschitz polyhedron representing a solid domain. We denote by \( \mathbf{n} \) the outward unit normal vector to \( \partial \Omega \) and assume that \( \partial \Omega = \Gamma \cup \Sigma \) with \( \text{int}(\Gamma) \cap \text{int}(\Sigma) = \emptyset \). The solid is supposed to be isotropic and linearly elastic with mass density \( \rho \) and Lamé constants \( \mu \) and \( \lambda \). Under the hypothesis of small oscillations, the time-harmonic elastodynamic equations with angular frequency \( \omega > 0 \) and body force \( \mathbf{f} : \Omega \to \mathbb{R}^3 \) are given by

\[
\begin{align*}
\sigma &= C \varepsilon(u) \quad \text{in } \Omega, \\
\text{div } \sigma + \rho \omega^2 u &= \mathbf{f} \quad \text{in } \Omega, \\
\sigma \mathbf{n} &= 0 \quad \text{on } \Sigma, \\
u &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

where \( u : \Omega \to \mathbb{R}^3 \) is the displacement field, \( \varepsilon(u) := \frac{1}{2} [\nabla u + (\nabla u)^\top] \) is the linearized strain tensor and \( C \) is the elasticity operator defined by

\[C \tau := \lambda (\text{tr } \tau) \mathbf{I} + 2 \mu \tau.\]

Our aim is to introduce the Cauchy stress tensor \( \sigma : \Omega \to \mathbb{R}^{3\times 3} \) as a primary variable in the variational formulation of \((1)\). To this end, we consider the closed subspace of \( \text{H}(\text{div}, \Omega) \) given by

\[\mathcal{W} := \{ \tau \in \text{H}(\text{div}, \Omega); \quad \tau \mathbf{n} = 0 \text{ on } \Sigma \}
\]

and the space of skew symmetric tensors

\[\mathcal{Q} := \{ s \in \text{L}^2(\Omega)^{3\times 3}; \quad s = -s^\top \}.\]

Introducing the rotation \( \mathbf{r} := \frac{1}{2} [\nabla u - (\nabla u)^\top] \), the constitutive equation \((1a)\) can be rewritten as,

\[C^{-1} \sigma = \nabla u - \mathbf{r}.
\]

Testing the last identity with \( \tau \in \mathcal{W} \), integrating by parts and using the momentum equation \((1b)\) to eliminate the displacement \( u \), we end up with the following mixed variational formulation of problem \((1)\): find \( \sigma \in \mathcal{W} \) and \( \mathbf{r} \in \mathcal{Q} \) such that

\[
\int_{\Omega} \text{div } \sigma \cdot \text{div } \tau - \kappa^2 \left( \int_{\Omega} C^{-1} \sigma : \tau + \int_{\Omega} \mathbf{r} : \tau \right) = \int_{\Omega} \mathbf{f} \cdot \text{div } \tau \quad \forall \tau \in \mathcal{W} \quad (2a)
\]

\[
\int_{\Omega} \sigma : s = 0 \quad \forall s \in \mathcal{Q}, \quad (2b)
\]
where the wave number $\kappa$ is given by $\sqrt{\rho \omega}$. We notice that equation (2b) is a restriction that imposes weakly the symmetry of $\sigma$, and $r$ is the corresponding Lagrange multiplier. We also point out that the dual formulation (2) degenerates as $\omega \to 0$. The static case $\omega = 0$ is then not covered by our analysis.

We introduce the symmetric bilinear forms
\[ B\left((\sigma, r), (\tau, s)\right) := \int_{\Omega} \mathcal{C}^{-1} \sigma : \tau + \int_{\Omega} r : \tau + \int_{\Omega} s : \sigma \]
and
\[ A\left((\sigma, r), (\tau, s)\right) := \int_{\Omega} \nabla \sigma \cdot \nabla \tau + B\left((\sigma, r), (\tau, s)\right) \]
and denote the product norm on $H(\text{div}, \Omega) \times L^2(\Omega)$ by
\[ \| (\tau, s) \|^2 := \| \tau \|^2_{H(\text{div}, \Omega)} + \| s \|^2_{0, \Omega}. \]

**Proposition 2.1.** There exists a constant $\alpha_A > 0$, depending on $\mu$ and $\Omega$ (but not on $\lambda$), such that
\[ \sup_{(\tau, s) \in W \times Q} \frac{A\left((\sigma, r), (\tau, s)\right)}{\| (\tau, s) \|} \geq \alpha_A \| (\sigma, r) \| \quad \forall (\sigma, r) \in W \times Q. \quad (3) \]

**Proof.** It is important to notice that the bilinear form
\[ \int_{\Omega} \mathcal{C}^{-1} \sigma : \tau = \frac{1}{2\mu} \int_{\Omega} \sigma^p : \tau^p + \frac{1}{3(3\lambda + 2\mu)} \int_{\Omega} (\text{tr} \ \sigma)(\text{tr} \ \tau) \quad (4) \]
is bounded by a constant independent of $\lambda$ when $\lambda$ is too large in comparison with $\mu$. Moreover, it is shown in [15, Lemma 2.1] that there exists a constant $\alpha_0 > 0$, depending on $\mu$ and $\Omega$ (but not on $\lambda$), such that
\[ \int_{\Omega} \mathcal{C}^{-1} \tau : \tau + \int_{\Omega} \nabla \tau \cdot \nabla \tau \geq \alpha_0 \| \tau \|^2_{H(\text{div}, \Omega)} \quad \forall \tau \in W. \quad (5) \]

On the other hand, there exists a constant $\beta > 0$ depending only on $\Omega$ (see, for instance, [4]) such that
\[ \sup_{\tau \in W} \frac{\int_{\Omega} s : \tau}{\| \tau \|^2_{H(\text{div}, \Omega)}} \geq \beta \| s \|_{0, \Omega}, \quad \forall s \in Q. \]

The Babuška-Brezzi theory shows that, for any bounded linear form $L \in \mathcal{L}(W \times Q)$, the problem: find $(\sigma, r) \in W \times Q$ such that
\[ A\left((\sigma, r), (\tau, s)\right) = L\left((\tau, s)\right) \quad \forall (\tau, s) \in W \times Q, \]
is well-posed, which proves (3). \qed
We deduce from Proposition 2.1 and the symmetry of $A(\cdot, \cdot)$ that the operator $T : \mathcal{W} \times \mathcal{Q} \to \mathcal{W} \times \mathcal{Q}$ characterized by

$$A\left(T(\sigma, r), (\tau, s)\right) = B\left((\sigma, r), (\tau, s)\right) \quad \forall (\tau, s) \in \mathcal{W} \times \mathcal{Q}$$

is well-defined and bounded. It is clear that, for a given wave number $\kappa > 0$, $(\sigma, r) \neq 0$ is a solution to the homogeneous version of problem (2) if and only if $(\eta = \frac{1}{1 + \kappa^2}, (\sigma, r))$ is an eigenpair for $T$. The following characterization of the spectrum of $T$ will be useful for our analysis.

**Proposition 2.2.** The spectrum $\text{sp}(T)$ of $T$ decomposes as follows

$$\text{sp}(T) = \{0, 1\} \cup \{\eta_k\}_{k \in \mathbb{N}}$$

where $\{\eta_k\}_k$ is a real sequence of finite-multiplicity eigenvalues of $T$ which converges to 0. Moreover, $\eta = 1$ is an infinite-multiplicity eigenvalue of $T$ while $\eta = 0$ is not an eigenvalue.

**Proof.** See [15, Theorem 3.7].

**Theorem 2.1.** If $\frac{1}{1 + \kappa^2} \notin \text{sp}(T)$, then (2) is well-posed. Moreover, there exists a constant $C > 0$ independent of $\lambda$ such that, for any $f \in L^2(\Omega)^3$, the solution $(\sigma, r) \in \mathcal{W} \times \mathcal{Q}$ of (2) satisfies

$$\|(\sigma, r)\| \leq C\|f\|_{0, \Omega}. \quad (6)$$

**Proof.** Let us first recall that, given $z \in \mathbb{C} \setminus \{\text{sp}(T)\}$, the resolvent

$$(zI - T)^{-1} : \mathcal{W} \times \mathcal{Q} \to \mathcal{W} \times \mathcal{Q}$$

is bounded in $\mathcal{L}(\mathcal{W} \times \mathcal{Q})$ by a constant $C$ only depending on $\Omega$ and $|z|$.

We deduce from (3) and the symmetry of $A(\cdot, \cdot)$ that the problem: find $(\bar{\sigma}, \bar{r}) \in \mathcal{W} \times \mathcal{Q}$ solution of

$$A\left((\bar{\sigma}, \bar{r}), (\tau, s)\right) = \int_\Omega f \cdot \text{div} \tau, \quad \forall (\tau, s) \in \mathcal{W} \times \mathcal{Q},$$

is well-posed. The solution of problem (2) is then given by

$$(\sigma, r) = \frac{1}{1 + \kappa^2} \left(\frac{I}{1 + \kappa^2} - T\right)^{-1}(\bar{\sigma}, \bar{r}),$$

and (6) follows from the boundedness of $\left(\frac{I}{1 + \kappa^2} - T\right)^{-1}$. \qed
3 A continuous Galerkin discretization

We consider shape regular affine meshes $\mathcal{T}_h$ that subdivide the domain $\bar{\Omega}$ into tetrahedra $K$ of diameter $h_K$. The parameter $h := \max_{K \in \mathcal{T}_h} \{h_K\}$ represents the mesh size of $\mathcal{T}_h$. Hereafter, given an integer $m \geq 0$ and a domain $D \subset \mathbb{R}^3$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most $m$ on $D$. The space of piecewise polynomial functions of degree at most $m$ relatively to $\mathcal{T}_h$ is denoted by

$$\mathcal{P}_m(\mathcal{T}_h) := \{v \in L^2(\Omega); \ v|_K \in \mathcal{P}_m(K), \ \forall K \in \mathcal{T}_h\}.$$ 

For any $k \geq 1$, we consider the finite element spaces

$$\mathcal{W}_h := \mathcal{P}_k(\mathcal{T}_h)^{3 \times 3} \cap \mathcal{W} \quad \text{and} \quad \mathcal{Q}_h := \mathcal{P}_{k-1}(\mathcal{T}_h)^{3 \times 3} \cap \mathcal{Q}.$$ 

Let us now recall some well-known properties of the Brezzi-Douglas-Marini (BDM) mixed finite element [5]. Let $\hat{K}$ be a fixed reference tetrahedron. Given $K \in \mathcal{T}_h$, there exists an affine and bijective map $F_K : \hat{K} \to \mathbb{R}^3$ such that $F_K(\hat{K}) = K$. We consider $\tilde{B}_K := \nabla F_K$ and define

$$\mathcal{N}_k(K) := \{\mathbf{w} : \Omega \to \mathbb{R}^3; \ \mathbf{w} \circ F_K = \tilde{B}_K^{-1} \tilde{w}; \ \tilde{w} \in \mathcal{P}_{k-1}(\hat{K})^3 \oplus \mathcal{S}_k(\hat{K})\}$$

where

$$\mathcal{S}_k(\hat{K}) := \{\tilde{\mathbf{w}} \in \tilde{\mathcal{P}}_k(\hat{K})^3; \ \tilde{\mathbf{w}} \cdot \hat{x} = 0\}$$

with $\tilde{\mathcal{P}}_k(\hat{K})$ representing the space of homogeneous polynomials of total degree exactly $k$ in $\hat{x} \in \hat{K}$.

A polynomial $\mathbf{v} \in \mathcal{P}_k(K)^3$ is uniquely determined by the set of BDM degrees of freedom

$$m_\phi(\mathbf{v}) := \int_F \mathbf{v} \cdot \mathbf{n}_K \phi \quad \text{for all } \phi \in \mathcal{P}_k(F), \ \text{for all } F \in \mathcal{F}(K) \quad \text{(7)}$$

$$m_\mathbf{w}(\mathbf{v}) := \int_K \mathbf{v} \cdot \mathbf{w} \quad \text{for all } \mathbf{w} \in \mathcal{N}_{k-1}(K), \quad \text{(8)}$$

where $\mathbf{n}_K$ is the outward unit normal vector to $\partial K$. Conditions \cite{8} are avoided in the case $k = 1$.

Let us consider an arbitrary, but fixed, orientation of all internal faces $F$ of $\mathcal{T}_h$ by normal vectors $\mathbf{n}_F$. On the faces $F$ lying on $\partial \Omega$ we take $\mathbf{n}_F = \mathbf{n}|_F$. We can introduce the global BDM-interpolation operator $\Pi_h : H(\text{div}, \Omega) \cap H^s(\Omega)^3 \to \mathcal{W}_h$, characterized, for any $\mathbf{v} \in H(\text{div}, \Omega) \cap H^s(\Omega)^3$ with $s > 1/2$, by the conditions

$$\int_F \Pi_h \mathbf{v} \cdot \mathbf{n}_F \phi = \int_F \mathbf{v} \cdot \mathbf{n}_F \phi \quad \text{for all } \phi \in \mathcal{P}_k(F), \ \text{for all } F \in \mathcal{F}_h, \quad \text{(9)}$$

$$\int_K \Pi_h \mathbf{v} \cdot \mathbf{w} = \int_K \mathbf{v} \cdot \mathbf{w} \quad \text{for all } \mathbf{w} \in \mathcal{N}_{k-1}(K), \ \text{for all } K \in \mathcal{T}_h. \quad \text{(10)}$$
Moreover, the following discrete inf-sup condition is proved in [2, 4]: there exists an independent of $h$ form of $L^2$.

Therefore, we can use the Babuška-Brezzi theory to ensure that, for any bounded linear form $L \in \mathcal{L}(\mathcal{W} \times \mathcal{Q})$, the problem: find $(\sigma_h, r_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h$ such that

$$A(\sigma_h, r_h) = L((\tau_h, s_h)) \quad \forall (\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h$$

admits a unique solution and there exists a constant $C > 0$ independent of $h$ and $\lambda$ such that

$$\|(\sigma_h, r_h)\| \leq C\|L\|_{\mathcal{L}(\mathcal{W} \times \mathcal{Q})},$$

which gives (15).
We can now consider the discrete counterpart \( T_h : \mathcal{W}_h^c \times \mathcal{Q}_h \to \mathcal{W}_h^c \times \mathcal{Q}_h \) of \( T \) characterized, for any \((\sigma_h, r_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h\), by

\[
A(T_h(\sigma_h, r_h), (\tau_h, s_h)) = B((\sigma_h, r_h), (\tau_h, s_h)) \quad \forall (\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h.
\]

As a consequence of Proposition 3.1, \( T_h \) is well-defined and uniformly bounded with respect to \( h \) and \( \lambda \). Moreover, we deduce from [15, Theorem 5.2] that, if \( \frac{1}{1 + \kappa^2} \notin \text{sp}(T) \), there exists a mesh size \( h_0 > 0 \) such that, for \( h \leq h_0 \),

\[
\|(\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h)\| \geq C_0\|(\tau_h, s_h)\| \quad \forall (\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h,
\]

with a constant \( C_0 > 0 \) independent of \( h \) and \( \lambda \).

We introduce the bilinear form

\[
D((\sigma_h, r_h), (\tau, s)) := A((\sigma_h, r_h), (\tau, s)) - (1 + \kappa^2)B((\sigma_h, r_h), (\tau, s))
\]

and notice that there exists a constant \( M_D^c > 0 \) independent of \( h \) and \( \lambda \) such that

\[
\left|D((\sigma, r), (\tau, s))\right| \leq M_D^c\|(\sigma, r)\|\|(\tau, s)\| \quad \forall (\sigma, r), (\tau, s) \in \mathcal{W} \times \mathcal{Q}.
\]

**Proposition 3.2.** Assume that \( \frac{1}{1 + \kappa^2} \notin \text{sp}(T) \) and let \( h_0 > 0 \) be the parameter for which [16] holds true for all \( h \leq h_0 \). Then, for \( h \leq h_0 \),

\[
\inf_{(\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h} \frac{D((\sigma_h, r_h), (\tau_h, s_h))}{\|(\tau_h, s_h)\|} \geq \alpha_D^c\|(\sigma_h, r_h)\| \quad \forall (\sigma_h, r_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h,
\]

with \( \alpha_D^c > 0 \) independent of the mesh size \( h \) and \( \lambda \).

**Proof.** We deduce from Proposition 3.1 that there exists an operator \( \Theta_h : \mathcal{W}_h^c \times \mathcal{Q}_h \to \mathcal{W}_h^c \times \mathcal{Q}_h \) satisfying

\[
A(\Theta_h(\tau_h, s_h), (\tau_h, s_h)) = \|(\tau_h, s_h)\|^2 \quad \text{and} \quad \|\Theta_h(\tau_h, s_h)\| \leq \frac{1}{\alpha_A^c}\|(\tau_h, s_h)\|
\]

for all \((\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h\). It follows from [16] and (19) that

\[
D((\tau_h, s_h), \Theta_h(\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h))
\]

\[
= (1 + \kappa^2)A\left((\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h), \Theta_h(\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h)\right)
\]

\[
= (1 + \kappa^2)\|(\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h)\|^2 \geq (1 + \kappa^2)C_0\|(\tau_h, s_h)\|^2\|(\frac{I}{1 + \kappa^2} - T_h)(\tau_h, s_h)\|
\]

for all \((\sigma_h, r_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h\), with the constant \( C_0 > 0 \) from (16). The result follows now with \( \alpha_D^c = C_0(1 + \kappa^2) \). \( \square \)
Theorem 3.1. Assume that $1 + \kappa \notin \text{sp}(T)$ and let $h_0 > 0$ be the parameter for which (16) holds true for all $h \leq h_0$. Then, for $h \leq h_0$, we have the following Céa estimate,

$$
\| (\sigma, r) - (\sigma_h, r_h) \| \leq \left( 1 + \frac{M_D}{\alpha_D} \right) \inf_{(\tau_h, s_h) \in W_h \times \mathcal{Q}_h} \| (\sigma, r) - (\tau_h, s_h) \|, \quad \forall h \leq h_0. \tag{20}
$$

Moreover, if the exact solution $u$ of (1) belongs to $H^{1+s}(\Omega)$ and $\text{div} \sigma \in H^s(\Omega)$ for some $s > 1/2$ then,

$$
\| (\sigma, r) - (\sigma_h, r_h) \| \leq C h^{\min(s, k)} \| u \|_{1+s, \Omega} + \| \text{div} \sigma \|_{s, \Omega}, \quad \forall h \leq h_0,
$$

with $C > 0$ independent of $h$ and $\lambda$.

Proof. The Céa estimate (20) is a direct consequence of (17) and (18). The asymptotic error estimate follows from (11), (12) and (13). \qed

4 A discontinuous Galerkin discretization

From now on we assume that there exists $s_0 > 1/2$ such that $f|_{\Omega_j} \in H^{s_0}(\Omega_j)$ for $j = 1, \ldots, J$, where $\{\Omega_j, \ j = 1, \ldots, J\}$ is a set of polyhedral subdomains forming a disjoint partition of $\bar{\Omega}$, i.e.,

$$
\Omega_j \cap \Omega_i = \emptyset \quad \text{for all} \ 1 \leq i \neq j \leq J \quad \text{and} \quad \bar{\Omega} = \bigcup_{j=1}^{J} \bar{\Omega}_j.
$$

We deduce from this additional hypothesis on $f$ and the momentum equation (1b) that $\text{div} \sigma|_{\Omega_j}$ belongs to $H^{\min(s_0, 1)}(\Omega_j)$ for any $j = 1, \ldots, J$.

In what follows, we assume that $\mathcal{T}_h$ is compatible with the partition $\bar{\Omega} = \bigcup_{j=1}^{J} \bar{\Omega}_j$, i.e.,

$$
\bigcup_{K \in \mathcal{T}_h(\Omega_j)} K = \bar{\Omega}_j \quad \forall j = 1, \ldots, J,
$$

where $\mathcal{T}_h(\Omega_j) := \{ K \in \mathcal{T}_h, \ K \subset \bar{\Omega}_j \}$. We say that a closed subset $F \subset \bar{\Omega}$ is an interior face if $F$ has a positive 2-dimensional measure and if there are distinct elements $K$ and $K'$ such that $F = K \cap K'$. A closed subset $F \subset \bar{\Omega}$ is a boundary face if there exists $K \in \mathcal{T}_h$ such that $F$ is a face of $K$ and $F = K \cap \Gamma$. We consider the set $\mathcal{F}_h^0$ of interior faces and the set $\mathcal{F}_h^\partial$ of boundary faces. We assume that the boundary mesh $\mathcal{F}_h^\partial$ is compatible with the partition $\partial \Omega = \Gamma \cup \Sigma$, i.e.,

$$
\bigcup_{F \in \mathcal{F}_h^\partial} F = \Gamma \quad \text{and} \quad \bigcup_{F \in \mathcal{F}_h^\Sigma} F = \Sigma
$$

where $\mathcal{F}_h^\Gamma := \{ F \in \mathcal{F}_h^\partial; \ F \subset \Gamma \}$ and $\mathcal{F}_h^\Sigma := \{ F \in \mathcal{F}_h^\partial; \ F \subset \Sigma \}$. We denote

$$
\mathcal{F}_h := \mathcal{F}_h^0 \cup \mathcal{F}_h^\partial \quad \text{and} \quad \mathcal{F}_h^\star := \mathcal{F}_h^0 \cup \mathcal{F}_h^\Sigma,
$$

and for any element $K \in \mathcal{T}_h$, we introduce the set

$$
\mathcal{F}(K) := \{ F \in \mathcal{F}_h; \ F \subset \partial K \}$$
of faces composing the boundary of \( K \).

For any \( s \geq 0 \), we consider the broken Sobolev space

\[
H^s(\mathcal{T}_h) := \{ v \in L^2(\Omega)^3; \; v|_K \in H^s(\Omega)^3 \ \forall K \in \mathcal{T}_h \}.
\]

For each \( v := \{ v_K \} \in H^s(\mathcal{T}_h)^3 \) and \( \tau := \{ \tau_K \} \in H^s(\mathcal{T}_h)^{3\times 3} \) the components \( v_K \) and \( \tau_K \) represent the restrictions \( v|_K \) and \( \tau|_K \). When no confusion arises, the restrictions of these functions will be written without any subscript. We will also need the space given on the skeletons of the triangulations \( \mathcal{T}_h \) by

\[
L^2(\mathcal{F}_h) := \prod_{F \in \mathcal{F}_h} L^2(F).
\]

Similarly, the components \( \mu_F \) of \( \mu := \{ \mu_F \} \in L^2(\mathcal{F}_h)^3 \) coincide with the restrictions \( \mu|_F \) and we denote

\[
\int_{\mathcal{F}_h} \mu := \sum_{F \in \mathcal{F}_h} \int_F \mu_F \quad \text{and} \quad \|\mu\|_{0,\mathcal{F}_h}^2 := \int_{\mathcal{F}_h} \mu^2, \quad \forall \mu \in L^2(\mathcal{F}_h).
\]

From now on, \( h_F \in L^2(\mathcal{F}_h)^3 \) is the piecewise constant function defined by \( h_F|_F := h_F \) for all \( F \in \mathcal{F}_h \) with \( h_F \) denoting the diameter of face \( F \).

Given a vector valued function \( v \in H^t(\mathcal{T}_h)^3 \), with \( t > 1/2 \), we define averages \( \{ v \} \in L^2(\mathcal{F}_h)^3 \) and jumps \( [ v ] \in L^2(\mathcal{F}_h)^3 \) by

\[
\{ v \}_F := (v_K + v_{K'})/2 \quad \text{and} \quad [ v ]_F := v_K \cdot n_K + v_{K'} \cdot n_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K'),
\]

where \( n_K \) is the outward unit normal vector to \( \partial K \). On the boundary of \( \Omega \) we use the following conventions for averages and jumps:

\[
\{ v \}_F := v_K \quad \text{and} \quad [ v ]_F := v_K \cdot n \quad \forall F \in \mathcal{F}(K) \cap \partial \Omega.
\]

Similarly, for matrix valued functions \( \tau \in H^t(\mathcal{T}_h)^{3\times 3} \), we define \( \{ \tau \} \in L^2(\mathcal{F}_h)^{3\times 3} \) and \( [ \tau ] \in L^2(\mathcal{F}_h)^3 \) by

\[
\{ \tau \}_F := (\tau_K + \tau_{K'})/2 \quad \text{and} \quad [ \tau ]_F := \tau_K n_K + \tau_{K'} n_{K'} \quad \forall F \in \mathcal{F}(K) \cap \mathcal{F}(K')
\]

and on the boundary of \( \Omega \) we set

\[
\{ \tau \}_F := \tau_K \quad \text{and} \quad [ \tau ]_F := \tau_K n \quad \forall F \in \mathcal{F}(K) \cap \partial \Omega.
\]

For any \( k \geq 1 \) we introduce the finite dimensional space \( \mathcal{W}_h := \mathcal{P}_k(\mathcal{T}_h)^{3\times 3} \) and consider \( \mathcal{W}(h) := \mathcal{W} + \mathcal{W}_h \). Given \( \tau \in \mathcal{W}_h \) we define \( \text{div}_h \tau \in L^2(\Omega)^3 \) by \( \text{div}_h \tau|_K = \text{div}(\tau|_K) \) for all \( K \in \mathcal{T}_h \) and endow \( \mathcal{W}(h) \) with the seminorm

\[
|\tau|_{\mathcal{W}(h)}^2 := \|\text{div}_h \tau\|_{0,\Omega}^2 + \|h_F^{-1/2}[\tau]\|_{0,\mathcal{F}_h}^2
\]
and the norm
\[ \| \tau \|^2_{W_h} := |\tau|_{W(h)}^2 + \| \tau \|^2_{0, \Omega}. \]
For the sake of simplicity, we will also use the notation
\[ \| (\tau, s) \|^2_{DG} := \| \tau \|^2_{W(h)} + \| s \|^2_{0, \Omega}. \]

Given a parameter \( a > 0 \), we introduce the symmetric bilinear form
\[ D_h^a((\sigma, r), (\tau, s)) := \int_\Omega \text{div}_h \sigma : \text{div}_h \tau - \kappa^2 B((\sigma, r), (\tau, s)) + \int_{T_h} a_{\text{ref}}^{-1} [\sigma] : [\tau] - \int_{T_h} (\{\text{div}_h \sigma\} : [\tau] + \{\text{div}_h \tau\} : [\sigma]) \forall (\sigma, r), (\tau, s) \in W_h \times Q_h \]
and the linear form
\[ L_h((\tau, s)) := \int_\Omega f : \text{div}_h \tau - \int_{T_h} \{ f \} : [\tau] \quad \forall (\tau, s) \in W_h \times Q_h, \]
and consider the DG method: find \((\sigma_h, r_h) \in W_h \times Q_h\) such that
\[ D_h^a((\sigma_h, r_h), (\tau, s)) = L_h((\tau, s)) \quad \forall (\tau, s) \in W_h \times Q_h. \tag{21} \]
We notice that, as it is usually the case for DG methods, the essential boundary condition is directly incorporated within the scheme. We need the following technical result to show that the bilinear form \( D_h^a(\cdot, \cdot) \) is uniformly bounded on \( W_h \).

**Proposition 4.1.** There exists a constant \( C > 0 \) independent of \( h \) such that
\[ \| h_F^{1/2} \{ v \} \|_{0, F_h} \leq C \| v \|_{0, \Omega} \quad \forall v \in P_k(T_h). \tag{22} \]

**Proof.** It is straightforward that
\[ \| h_F^{1/2} \{ v \} \|^2_{0, F_h} = \sum_{F \in T_h} \| h_F^{1/2} \{ v \} \|^2_{0, F} \leq \sum_{K \in T_h} \sum_{F \in T(K)} h_F \| v \|_{0, F}^2 \leq \sum_{K \in T_h} h_K \| v \|^2_{0, \partial K}. \]
The result follows now from the following discrete trace inequality (cf. [3]):
\[ h_K \| v \|^2_{0, \partial K} \leq C_0 \| v \|^2_{0, K} \quad \forall v \in P_k(K), \quad \forall K \in T_h, \]
where \( C_0 > 0 \) is independent of \( K \).

With the aid of the Cauchy-Schwarz inequality and Proposition 4.1, we can easily prove that there exists constants \( M_D^a > 0 \) independent of \( h \) and \( \lambda \) such that
\[ |D_h^a((\sigma, r), (\tau, s))| \leq M_D^a \left( \| \sigma \|^2_{W(h)} + \| h_F^{1/2} \{ \text{div} \sigma \} \|^2_{0, F_h} + \| r \|^2_{0, \Omega} \right)^{1/2} \| (\tau, s) \|_{DG} \]
for all \((\sigma, r) \in W(h) \times Q\) with \( \text{div}_h \sigma \in H^s(T_h)^3 \) for a given \( s > 1/2 \) and for all \((\tau, s_h) \in W_h \times Q_h\).

We end this section by showing that the DG scheme (21) is consistent.
Proposition 4.2. Let \( u \) be the solution of (1) and let \( \sigma := C \varepsilon(u) \) and \( r := \frac{1}{2}(\nabla u - (\nabla u)^t) \). Then,

\[
D_h^a \left( (\sigma - \sigma_h, r - r_h), (\tau_h, s_h) \right) = 0 \quad \forall (\tau_h, s_h) \in \mathcal{W}_h \times \mathcal{Q}_h.
\]

Proof. By definition,

\[
D_h^a \left( (\sigma, r), (\tau_h, s_h) \right) = \int_\Omega \text{div } \sigma \cdot \text{div } \tau_h - \kappa^2 \left( \int_\Omega C^{-1} \sigma : \tau_h + \int_\Omega r : \tau_h \right)
- \int_{\partial\Omega} \{\text{div } \sigma\} \cdot [\tau_h].
\]

The identity \( \text{div } \sigma = f - \kappa^2 u \) and integration by parts yield

\[
\int_\Omega \text{div } \sigma \cdot \text{div } \tau_h = \int_\Omega f \cdot \text{div } \tau_h - \kappa^2 \sum_{K \in \mathcal{T}_h} \int_K u \cdot \text{div } \tau_h + \kappa^2 \sum_{K \in \mathcal{T}_h} \int_K \nabla u : \tau_h - \kappa^2 \sum_{K \in \mathcal{T}_h} \int_{\partial K} u \cdot \tau_h n_K.
\]

Substituting back into (25) by taking into account that \( \nabla u = C^{-1} \sigma - r \) and

\[
\sum_{K \in \mathcal{T}_h} \int_{\partial K} u \cdot \tau_h n_K = \int_{\partial \Omega} \{u\} \cdot [\tau_h]
\]

we obtain

\[
D_h^a \left( (\sigma, r), (\tau_h, s_h) \right) = \int_\Omega f \cdot \text{div } \tau_h - \int_{\partial\Omega} \{\text{div } \sigma + \kappa^2 u\} \cdot [\tau_h]
= \int_\Omega f \cdot \text{div } \tau_h - \int_{\partial\Omega} \{f\} \cdot [\tau_h]
\]

and the result follows.

5 Well-posedness and stability of the DG method

By using the transformation rules

\[
\phi \circ F_K = \hat{\phi}, \quad v \circ F_K = \frac{B_K \hat{v}}{|\det B_K|} \quad \text{and} \quad w \circ F_K = B_K^{-1} \hat{w},
\]

we can easily show that

\[
\int_F v \cdot n_K \phi = \int_{\hat{F}} \hat{v} \cdot n_{\hat{K}} \hat{\phi} \quad \text{and} \quad \int_K v \cdot w = \int_{\hat{K}} \hat{v} \cdot \hat{w},
\]

where \( F \) is the image of the face \( \hat{F} \) under the affine map \( F_K : \hat{K} \to \mathbb{R}^3 \) defined in Section 3.
Proposition 5.1. There exists a constant $C > 0$ independent of $h$ such that

$$
\left( \| \nabla v \|_{0,K}^2 + h_K^{-2} \| v \|_{0,K}^2 \right)^{1/2} \leq C \left( h_K^{-1} \sup_{w \in \mathcal{N}_{k-1}(K)} \frac{\int_K v \cdot w}{\| w \|_{0,K}} + \sum_{F \in F(K)} h_F^{-1/2} \sup_{\phi \in \mathcal{P}_k(F)} \frac{\int_F v \cdot n_K \phi}{\| \phi \|_{0,F}} \right) \quad (28)
$$

for all $v \in \mathcal{P}_k(K)^3$.

Proof. We will use here the notation $a \lesssim b$ to express that there exists $C > 0$ independent of $h$ such that $a \leq C b$ for all $h$. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$ simultaneously. We first notice that, thanks to the unisolvency of conditions (7)-(8), the norms

$$
\left( \| \nabla \tilde{v} \|_{0,K}^2 + \| \nabla \hat{v} \|_{0,K}^2 \right)^{1/2} \quad \text{and} \quad \sup_{w \in \mathcal{N}_{k-1}(K)} \frac{\int_K \tilde{v} \cdot \tilde{w}}{\| \tilde{w} \|_{0,K}} + \sum_{F \in F(K)} \sup_{\hat{\phi} \in \mathcal{P}_k(F)} \frac{\int_F \tilde{v} \cdot n_K \hat{\phi}}{\| \hat{\phi} \|_{0,F}}
$$

are equivalent on the finite dimensional space $\mathcal{P}_k(\hat{K})^3$. Standard scaling arguments show that

$$
h_K \| v \|_{0,K}^2 \simeq \| \tilde{v} \|_{0,K}^2, \quad h_K^3 \| \nabla v \|_{0,K}^2 \simeq \| \nabla \tilde{v} \|_{0,K}^2
$$

and

$$
\| \phi \|_{0,F} \simeq h_F \| \hat{\phi} \|_{0,F}, \quad \| w \|_{0,K} \simeq h_K \| \hat{w} \|_{0,K}.
$$

Hence, we deduce from (27) that

$$
\left( h_K \| v \|_{0,K}^2 + h_K^3 \| \nabla v \|_{0,K}^2 \right)^{1/2} \lesssim \left( \| \tilde{v} \|_{0,K}^2 + \| \nabla \hat{v} \|_{0,K}^2 \right)^{1/2} \lesssim \sup_{w \in \mathcal{N}_{k-1}(K)} \frac{\int_K \tilde{v} \cdot \tilde{w}}{\| \tilde{w} \|_{0,K}} + \sum_{F \in F(\hat{K})} \sup_{\hat{\phi} \in \mathcal{P}_k(F)} \frac{\int_F \tilde{v} \cdot n_K \hat{\phi}}{\| \hat{\phi} \|_{0,F}} \lesssim \frac{h_K^{1/2}}{\sup_{w \in \mathcal{N}_{k-1}(K)} \frac{\int_K v \cdot w}{\| w \|_{0,K}}} \sum_{F \in F(K)} h_F \sup_{\phi \in \mathcal{P}_k(F)} \frac{\int_F v \cdot n_K \phi}{\| \phi \|_{0,F}},
$$

and the result follows. 

We introduce the projection $\mathcal{P}_h : \mathcal{W}_h \to \mathcal{W}_h$ uniquely characterized, for any $\tau \in \mathcal{W}_h$, by the conditions

$$
\int_F (\mathcal{P}_h \tau) n_F \cdot \varphi = \int_F \tau n_F \cdot \varphi \quad \forall \varphi \in \mathcal{P}_k(F)^3, \forall F \in \mathcal{F}_h^\Gamma, \quad (29)
$$

$$
\int_F (\mathcal{P}_h \tau) n_F \cdot \varphi = 0 \quad \forall \varphi \in \mathcal{P}_k(F)^3, \forall F \in \mathcal{F}_h^\Sigma, \quad (30)
$$

$$
\int_F (\mathcal{P}_h \tau) n_F \cdot \varphi = \int_F \{ \tau \}_F n_F \cdot \varphi \quad \forall \varphi \in \mathcal{P}_k(F)^3, \forall F \in \mathcal{F}_h^0, \quad (31)
$$

$$
\int_K \mathcal{P}_h \tau : W = \int_K \tau : W \quad \forall W \text{ with rows in } \mathcal{N}_{k-1}(K), \forall K \in \mathcal{T}_h. \quad (32)
$$
We point out that the projection $\mathcal{P}_h$ may be viewed as the div-conforming counterpart of the projection with curl-conforming range introduced in [14].

**Proposition 5.2.** The norm equivalence

$$\|\tau\|_{\mathcal{W}(h)} \leq \left(\|\mathcal{P}_h \tau\|_{H(\text{div},\Omega)} + \|h_F^{-1/2}\|_{0,F_h}^{2}\|\tau\|_{0,F_h}^{2}\right)^{1/2} \leq \bar{C}\|\tau\|_{\mathcal{W}(h)} \quad (33)$$

holds true on $\mathcal{W}_h$ with constants $\underline{C} > 0$ and $\bar{C} > 0$ independent of $h$.

**Proof.** Using Proposition 5.1 row-wise we deduce that there exists $C_0 > 0$ independent of $h$ such that

$$\|\text{div}(\tau - \mathcal{P}_h \tau)\|_{0,K}^2 + h_K^2 \|\tau - \mathcal{P}_h \tau\|_{0,K}^2 \leq C_0 \sum_{F \in \mathcal{F}(K)} h_F^{-1} \left(\sup_{\phi \in \mathcal{P}_h(F)^3} \int_F (\tau - \mathcal{P}_h \tau) n_K \cdot \phi\right)^2.$$

It is easy to obtain, from the definition of $\mathcal{P}_h$, the identity

$$\int_F (\tau - \mathcal{P}_h \tau) n_K \cdot \phi = \begin{cases} \frac{1}{2} \int_F [\tau]_F \cdot \phi & \text{if } F \in \mathcal{F}_h^0 \\ \int_F [\tau]_F \cdot \phi & \text{if } F \in \mathcal{F}_h^\Sigma \\ 0 & \text{if } F \in \mathcal{F}_h^\Gamma \end{cases}.$$

Hence, using the Cauchy-Schwarz inequality and summing up over $K \in \mathcal{T}_h$ we deduce that

$$\|\text{div}_h(\tau - \mathcal{P}_h \tau)\|_{0,\Omega}^2 + \sum_{K \in \mathcal{T}_h} h_K^{-2} \|\tau - \mathcal{P}_h \tau\|_{0,K}^2 \leq C_0 \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\tau\|_{0,F}^2, \quad (34)$$

which proves that

$$\|\tau - \mathcal{P}_h \tau\|_{\mathcal{W}(h)}^2 \leq (1 + C_0) \|h_F^{-1/2}\|_{0,F_h}^{2}\|\tau\|_{0,F_h}^{2} \quad \forall \tau \in \mathcal{W}_h. \quad (35)$$

The lower bound of (33) is then a consequence of the uniform boundedness of $\mathcal{P}_h$ on $\mathcal{W}_h$,

$$\|\mathcal{P}_h \tau\|_{H(\text{div},\Omega)}^2 = \|\mathcal{P}_h \tau\|_{\mathcal{W}(h)}^2 \leq 2\|\tau\|_{\mathcal{W}(h)}^2 + 2\|\tau - \mathcal{P}_h \tau\|_{\mathcal{W}(h)}^2 \leq 2(2 + C_0)\|\tau\|_{\mathcal{W}(h)}^2. \quad (36)$$

On the other hand,

$$\|\tau\|_{\mathcal{W}(h)}^2 \leq 2\|\mathcal{P}_h \tau\|_{H(\text{div},\Omega)}^2 + 2\|\tau - \mathcal{P}_h \tau\|_{\mathcal{W}(h)}^2 \leq 2(1 + C_0) \left(\|\mathcal{P}_h \tau\|_{H(\text{div},\Omega)}^2 + \|h_F^{-1/2}\|_{0,F_h}^2\|\tau\|_{0,F_h}^2\right)$$

for all $\tau \in \mathcal{W}_h$ which gives the upper bound of (33). \hfill \Box

**Proposition 5.3.** Assume that $\frac{1}{1 + \lambda^2} \notin \text{sp}(T)$. There exist parameters $h^* > 0$ and $a > 0$ such that, for $h \leq h^*$ and $a \geq a^*$,

$$\sup_{(\tau_h, s_h) \in \mathcal{W}_h \times \mathcal{Q}_h} \frac{D_h^a\left((\sigma_h, r_h), (\tau_h, s_h)\right)}{\|\tau_h, s_h\|_{DG}} \geq a_D^{d}(\sigma_h, r_h)_{DG}, \forall (\sigma_h, r_h) \in \mathcal{W}_h \times \mathcal{Q}_h \quad (37)$$

with $a_D^{d} > 0$ independent of the mesh size $h$ and $\lambda$. 
Proof. We deduce from (38) that there exists an operator \( \Xi_h : \mathcal{W}_h^c \times \mathcal{Q}_h \to \mathcal{W}_h^c \times \mathcal{Q}_h \) such that, for \( h \leq h_0 \),

\[
D_h \left( (\tau_h, s_h), \Xi_h(\tau_h, s_h) \right) = \| (\tau_h, s_h) \|^2 \quad \text{and} \quad \| \Xi_h(\tau_h, s_h) \| \leq \frac{1}{\alpha_D} \| (\tau_h, s_h) \| \tag{38}
\]

for all \( (\tau_h, s_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h \). Given \( (\tau_h, s_h) \in \mathcal{W}_h \times \mathcal{Q}_h \), the decomposition \( \tau_h = \tau_h^c + \tilde{\tau}_h \), with \( \tau_h^c := P_h \tau_h \) and \( \tilde{\tau}_h := \tau_h - P_h \tau_h \), and (38) yield

\[
D_h^a \left( (\tau_h, s_h), \Xi_h(\tau_h^c, s_h) + (\tilde{\tau}_h, 0) \right) = \| (\tau_h^c, s_h) \|^2 + D_h^a \left( (\tau_h^c, s_h), (\tilde{\tau}_h, 0) \right) + \]

\[
D_h^a \left( (\tilde{\tau}_h, 0), \Xi_h(\tau_h^c, s_h) \right) + D_h^a \left( (\tilde{\tau}_h, 0), (\tilde{\tau}_h, 0) \right). \tag{39}
\]

Using the Cauchy-Schwarz inequality we have that

\[
D_h^a \left( (\tilde{\tau}_h, 0), (\tilde{\tau}_h, 0) \right) = \| \text{div}_h \tilde{\tau}_h \|^2_{0, \Omega} + a \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2 - \kappa^2 \int_{\Omega} \left( C^{-1} \tilde{\tau}_h : \tilde{\tau}_h \right)
\]

\[
- 2 \int_{F_h^c} \{ \text{div}_h \tilde{\tau}_h \} \cdot [\tilde{\tau}_h] \geq a \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2 - C_1 \| \tilde{\tau}_h \|^2_{0, \Omega}
\]

\[
- 2 \| h^{1/2} \{ \text{div}_h \tilde{\tau}_h \} \|_{0, F_h^c} \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}.
\]

It follows from (22) and (35) that

\[
D_h^a \left( (\tilde{\tau}_h, 0), (\tilde{\tau}_h, 0) \right) \geq (a - C_2) \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2
\]

with a constant \( C_2 > 0 \) independent of \( h \) and \( \lambda \).

The remaining two terms of (39) are bounded from below by using (23), (22) and (35). Indeed, it is straightforward that

\[
D_h^a \left( (\tau_h^c, s_h), (\tilde{\tau}_h, 0) \right) \geq - M_D^d \| (\tau_h^c, s_h) \| \| \tilde{\tau}_h \|_{\mathcal{W}(h)} \geq - \frac{1}{4} \| (\tau_h^c, s_h) \|^2 - C_3 \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2
\]

and

\[
D_h^a \left( (\tilde{\tau}_h, 0), \Xi_h(\tau_h^c, s_h) \right) \geq - M_D^d \| \Xi_h(\tau_h^c, s_h) \| \| \tilde{\tau}_h \|_{\mathcal{W}(h)} \geq - \frac{M_D^d}{\alpha_D} \| (\tau_h^c, s_h) \| \| \tilde{\tau}_h \|_{\mathcal{W}(h)} \geq - \frac{1}{4} \| (\tau_h^c, s_h) \|^2 - C_4 \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2,
\]

with \( C_3 > 0 \) and \( C_4 > 0 \) independent of \( h \) and \( \lambda \). Summing up, we have that

\[
D_h^a \left( (\tau_h, s_h), \Xi_h(\tau_h^c, s_h) + (\tilde{\tau}_h, 0) \right) \geq \frac{1}{2} \| (\tau_h^c, s_h) \|^2 + (a - C^*) \| h^{-1/2} [\tilde{\tau}_h] \|_{0, F_h^c}^2,
\]
with \( C^* := C_2 + C_3 + C_4 \). Hence, if \( a > C^* + 1/2 \), by virtue of (33) we have that

\[
D_h^a \left( (\tau_h, s_h), \Xi_h(\tau_h^e, s_h) + (\tilde{\tau}_h, 0) \right) \geq \frac{1}{2} \left( \| (\tau_h^e, s_h) \|^2 + \| \theta^{\frac{1}{2}} \|_{0,F_h}^2 \right) \geq \frac{C^2}{2} \| (\tau_h, s_h) \|^2_{\text{DG}}.
\]

Finally, using (35) and (38) we deduce that there exists \( \alpha_D^e > 0 \) such that,

\[
D_h^a \left( (\tau_h, s_h), \Xi_h(\tau_h^e, s_h) + (\tilde{\tau}_h, 0) \right) \geq \alpha_D^e \| (\tau_h, s_h) \|_{\text{DG}} \left( \| \Xi_h(\tau_h^e, s_h) + (\tilde{\tau}_h, 0) \|_{\text{DG}}^2 \right),
\]

provided that \( h \) is sufficiently small and \( a \) is sufficiently large, which gives (37).

The first consequence of the inf-sup condition (37) is that the DG problem (21) admits a unique solution. Moreover, we have the following Céa estimate.

**Theorem 5.1.** Assume that \( \frac{1}{1 + \kappa^2} \notin \text{sp}(T) \) and let \((\sigma, r) \in W \times Q\) be the solution of (2a)–(2b). There exist parameters \( h^* > 0 \) and \( a^* > 0 \) such that, for \( h \leq h^* \) and \( a \geq a^* \),

\[
\| (\sigma, r) - (\sigma_h, r_h) \|_{\text{DG}} \leq (1 + \frac{M_D^a}{\alpha_D^a}) \inf_{(\tau_h, s_h) \in W \times Q_h} \left( \| \sigma - \tau_h \|_{W(h)} \right. \\
+ \left. \| h_{F}^{\frac{1}{2}} \{ \text{div}(\sigma - \tau_h) \} \|_{0,F_h} + \| r - s_h \|_{0,\Omega} \right).
\]

Moreover, if the exact solution \( u \) of (1) belongs to \( H^{1+s}(\Omega)^3 \) for some \( s > 1/2 \) and if \( \text{div} \sigma \in H\times(\Omega)^3 \), then the error estimate

\[
\| (\sigma, r) - (\sigma_h, r_h) \|_{\text{DG}} \leq C h^{\min(s,k)} \left( \| u \|_{1+s,\bar{K}} + \| \text{div} \sigma \|_{s,\bar{K}} \right), \quad \forall h \leq h^*,
\]

holds true with a constant \( C > 0 \) independent of \( h \) and \( \lambda \).

**Proof.** The first estimate follows from (28), (24) and (37) as shown in [8, Theorem 1.35]. On the other hand, under the regularity hypotheses on \( u \) and \( \sigma \),

\[
\| (\sigma, r) - (\sigma_h, r_h) \|_{\text{DG}} \leq (1 + \frac{M_D^a}{\alpha_D^a}) \left( \| \sigma - \Pi_h \sigma \|_{H(\text{div},\Omega)} + \| h_{F}^{\frac{1}{2}} \{ \text{div}(\sigma - \Pi_h \sigma) \} \|_{0,F_h} \right. \\
+ \left. \| r - \mathcal{S}_h r \|_{0,\Omega} \right)
\]

and we notice that

\[
\| h_{F}^{\frac{1}{2}} \{ \text{div}(\sigma - \Pi_h \sigma) \} \|_{0,F_h} \leq \sum_{K \in \mathcal{T}_h} \sum_{F \in \mathcal{F}(K)} h_{F} \| \text{div}(\sigma - \Pi_h \sigma) \|_{0,F}^2.
\]

Using the commuting diagram property satisfied by \( \Pi_h \), the trace theorem and standard scaling arguments we obtain that

\[
\| h_{F}^{\frac{1}{2}} \| \text{div}(\sigma - \Pi_h \sigma) \|_{0,F} = h_{F}^{\frac{1}{2}} \| \text{div} \sigma - \mathcal{R}_K \| \text{div} \sigma \|_{0,F} \leq C h_{K}^{\min(k,\bar{K})} \| \text{div} \sigma \|_{s,\bar{K}}
\]

for all \( F \in \mathcal{F}(K) \), where the \( L^2(K) \)-orthogonal projection \( \mathcal{R}_K := \mathcal{R}_h|_K \) onto \( P_{k-1}(K) \) is applied componentwise. Consequently, by virtue of the error estimates (11), (12) and (13),

\[
\| (\sigma, r) - (\sigma_h, r_h) \|_{\text{DG}} \leq C_3 h^{\min(k,\bar{K})} \left( \| u \|_{1+s,\bar{K}} + \| \text{div} \sigma \|_{s,\bar{K}} \right)
\]

and the result follows. \( \square \)
6 Numerical results

We present a series of numerical experiments confirming the good performance of the continuous Galerkin scheme (14) and the discontinuous Galerkin scheme (21). For simplicity we consider our model problem in two dimensions. The corresponding theory and results from three dimensions apply with trivial modifications.

All the numerical results have been obtained by using the FEniCS Problem Solving Environment [10]. We choose $\Omega = (0,1) \times (0,1)$, $\lambda = \mu = 1$ and select the data $f$ so that the exact solution is given by

$$u(x_1, x_2) = \begin{pmatrix} -x_2 \sin(\kappa \pi x_1) \\ 0.5 \pi x_2 \cos(\kappa \pi x_1) \end{pmatrix}.$$ 

We also assume that the body is fixed on the whole $\partial \Omega$ and the non-homogeneous Dirichlet boundary condition is imposed by adding an adequate boundary term to the right-hand side of (21). The numerical results obtained below for the continuous and discontinuous Galerkin schemes have been obtained by considering nested sequences of uniform triangular meshes $T_h$ of the unit square $\Omega$. The individual relative errors produced by the continuous Galerkin method are given by

$$e^c_\kappa(\sigma) := \frac{\|\sigma - \sigma_h\|_{H(\text{div}, \Omega)}}{\|\sigma\|_{H(\text{div}, \Omega)}}, \quad e^c_\kappa(r) := \frac{\|r - r_h\|_{0, \Omega}}{\|r\|_{0, \Omega}} ,$$

where $(\sigma, r) \in \mathcal{W} \times \mathcal{Q}$ and $(\sigma_h, r_h) \in \mathcal{W}_h^c \times \mathcal{Q}_h$ are the solutions of (2) and (14) respectively. We introduce the experimental rates of convergence

$$r^c_\kappa(\sigma) := \frac{\log(e^c_\kappa(\sigma)/\hat{e}^c_\kappa(\sigma))}{\log(h/\hat{h})}, \quad r^c_\kappa(r) := \frac{\log(e^c_\kappa(r)/\hat{e}^c_\kappa(r))}{\log(h/\hat{h})},$$

where $e^c_\kappa$ and $\hat{e}^c_\kappa$ are the errors corresponding to two consecutive triangulations with mesh sizes $h$ and $\hat{h}$, respectively.

Similarly, we denote the individual relative errors of the discontinuous Galerkin scheme

$$e^d_\kappa(\sigma) := \frac{\|\sigma - \sigma_h\|_{H(\text{div}, \Omega)}}{\|\sigma\|_{H(\text{div}, \Omega)}}, \quad e^d_\kappa(r) := \frac{\|r - r_h\|_{0, \Omega}}{\|r\|_{0, \Omega}} ,$$

where, in this case, $(\sigma_h, r_h) \in \mathcal{W}_h \times \mathcal{Q}_h$ is the solution of (21). Accordingly, the experimental rates of convergence of the DG scheme are given by

$$r^d_\kappa(\sigma) := \frac{\log(e^d_\kappa(\sigma)/\hat{e}^d_\kappa(\sigma))}{\log(h/\hat{h})}, \quad r^d_\kappa(r) := \frac{\log(e^d_\kappa(r)/\hat{e}^d_\kappa(r))}{\log(h/\hat{h})}.$$ 

We begin by testing the convergence order of the continuous Galerkin method (14) for the range of values $\kappa = 4, 8, 16, 32$ of the wave number. We report in Tables [1 2 3 4] the relative errors (40) and the convergence orders (41) obtained in the cases $k = 2$ and...
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Table 1: Convergence of the CG method in $\sigma$ for different wave numbers ($k = 2$).

| $1/h$ | $e^4_\sigma$ | $e^8_\sigma$ | $e^{16}_\sigma$ | $e^{32}_\sigma$ |
|-------|---------------|---------------|-----------------|-----------------|
| 8     | 6.90e-02     | 2.96e-01     | 1.64e+00       | 2.43e+00       |
| 16    | 1.79e-02     | 1.94e-02     | 2.12e-01       | 2.41e+00       |
| 32    | 5.53e-03     | 1.77e-02     | 1.94e-01       | 2.19e+00       |
| 64    | 1.13e-03     | 2.00e-02     | 3.99e-01       | 1.94e+00       |
| 128   | 2.84e-04     | 2.00e-02     | 3.99e-01       | 1.94e+00       |
| 256   | 7.10e-05     | 2.00e-02     | 3.99e-01       | 1.94e+00       |

Table 2: Convergence of the CG method in $r$ for different wave numbers ($k = 2$).

| $1/h$ | $e^4_r$ | $e^8_r$ | $e^{16}_r$ | $e^{32}_r$ |
|-------|--------|--------|------------|------------|
| 8     | 1.96e-01 | 7.64e-01 | 9.97e+00 | 1.59e+01 |
| 16    | 5.32e-02 | 1.88e-01 | 9.32e-01 | 6.80e+00 |
| 32    | 1.36e-02 | 1.97e-02 | 1.95e-01 | 7.80e-01 |
| 64    | 3.34e-03 | 1.99e-02 | 5.27e-02 | 1.95e-01 |
| 128   | 8.60e-04 | 2.00e-02 | 1.34e-02 | 5.27e-02 |
| 256   | 2.15e-04 | 2.00e-02 | 3.38e-03 | 1.34e-02 |

Table 3: Convergence of the CG method in $\sigma$ for different wave numbers ($k = 4$).

| $1/h$ | $e^4_\sigma$ | $e^8_\sigma$ | $e^{16}_\sigma$ | $e^{32}_\sigma$ |
|-------|---------------|---------------|-----------------|-----------------|
| 8     | 9.33e-04     | 1.67e-02     | 2.34e-01       | 1.50e+00       |
| 16    | 5.95e-05     | 3.97e-02     | 1.70e-02       | 1.16e-01       |
| 32    | 3.74e-06     | 3.99e-02     | 8.81e-04       | 1.69e-02       |
| 64    | 2.34e-07     | 4.00e-02     | 5.61e-05       | 8.78e-04       |
| 128   | 1.46e-08     | 4.00e-02     | 3.53e-06       | 5.60e-05       |

Table 4: Convergence of the CG method in $r$ for different wave numbers ($k = 4$).

| $1/h$ | $e^4_r$ | $e^8_r$ | $e^{16}_r$ | $e^{32}_r$ |
|-------|--------|--------|------------|------------|
| 8     | 1.88e-03 | 3.31e-02 | 1.52e+00 | 8.98e+00 |
| 16    | 1.23e-04 | 3.93e-02 | 3.64e-02 | 5.24e-01 |
| 32    | 3.80e-06 | 3.98e-02 | 1.75e-03 | 3.41e-02 |
| 64    | 4.90e-07 | 3.99e-02 | 1.13e-04 | 1.74e-03 |
| 128   | 3.27e-08 | 3.91e-02 | 7.15e-06 | 1.13e-04 |
parameter \( \kappa \) notice that the higher the value of the wave number \( \kappa \) with \( \lambda \) needed to reduce the error below a given tolerance. The subsequent numerical tests are for the discontinuous Galerkin scheme (21). We take a stabilization parameter \( a = 100 \), respectively. It is clear that the correct quadratic and quartic convergence rates of the errors are attained in each variable and for each fixed wave number \( \kappa \).

Table 5: Convergence of the DG method in \( \sigma \) for different wave numbers (\( k = 4, a = 100 \)).

| 1/h | \( e^4_d(\sigma) \) | \( x^4_d(\sigma) \) | \( e^8_d(\sigma) \) | \( x^8_d(\sigma) \) | \( e^{16}_d(\sigma) \) | \( x^{16}_d(\sigma) \) | \( e^{32}_d(\sigma) \) | \( x^{32}_d(\sigma) \) |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 8   | 9.41e-04      | -              | 1.68e-02      | -              | 2.35e-01      | -              | 1.49e+00      | -              |
| 16  | 6.01e-05      | 3.97           | 9.00e-04      | 4.22           | 1.70e-02      | 3.79           | 1.17e-01      | 3.68           |
| 32  | 3.78e-06      | 3.99           | 5.75e-05      | 3.97           | 8.89e-04      | 4.26           | 1.70e-02      | 2.78           |
| 64  | 2.36e-07      | 4.00           | 3.61e-06      | 3.99           | 5.68e-05      | 3.97           | 8.87e-04      | 4.26           |
| 128 | 1.48e-08      | 4.00           | 2.26e-07      | 4.00           | 3.57e-06      | 3.99           | 5.67e-05      | 3.97           |

Table 6: Convergence of the DG method in \( r \) for different wave numbers (\( k = 4, a = 100 \)).

| 1/h | \( e^{16}_d(r) \) | \( x^{16}_d(r) \) | \( e^{28}_d(r) \) | \( x^{28}_d(r) \) | \( e^{32}_d(r) \) | \( x^{32}_d(r) \) | \( e^{40}_d(r) \) | \( x^{40}_d(r) \) |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 8   | 1.88e-03      | -              | 3.30e-02      | -              | 1.52e+00      | -              | 8.84e+00      | -              |
| 16  | 1.23e-04      | 3.93           | 1.77e-03      | 4.22           | 3.63e-02      | 5.38           | 5.22e-01      | 4.08           |
| 32  | 7.81e-06      | 3.98           | 1.15e-04      | 3.94           | 1.75e-03      | 4.38           | 3.40e-02      | 3.94           |
| 64  | 4.90e-07      | 3.99           | 7.27e-06      | 3.98           | 1.13e-04      | 3.95           | 1.74e-03      | 4.28           |
| 128 | 3.14e-08      | 3.97           | 4.56e-07      | 4.00           | 7.15e-06      | 3.99           | 1.13e-04      | 3.95           |

Table 7: Convergence of the DG method in \( \sigma \) for different wave numbers (\( k = 6, a = 100 \)).

| 1/h | \( e^{16}_d(\sigma) \) | \( x^{16}_d(\sigma) \) | \( e^{28}_d(\sigma) \) | \( x^{28}_d(\sigma) \) | \( e^{32}_d(\sigma) \) | \( x^{32}_d(\sigma) \) | \( e^{40}_d(\sigma) \) | \( x^{40}_d(\sigma) \) |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 8   | 1.46e-02      | -              | 4.41e-01      | -              | 4.89e-01      | -              | 1.24e+00      | -              |
| 16  | 3.69e-04      | 5.30           | 8.02e-03      | 5.78           | 9.42e-03      | 5.70           | 5.25e-02      | 4.56           |
| 32  | 4.80e-06      | 6.27           | 1.30e-04      | 5.95           | 3.70e-04      | 4.67           | 1.05e-03      | 5.65           |
| 40  | 4.29e-07      | 5.96           | 1.19e-05      | 5.89           | 2.62e-05      | 6.53           | 9.74e-05      | 5.86           |
| 56  | 1.71e-07      | 5.97           | 4.77e-06      | 5.93           | 1.05e-05      | 5.91           | 3.93e-05      | 5.88           |
| 64  | 7.68e-08      | 5.98           | 2.16e-06      | 5.94           | 4.77e-06      | 5.93           | 1.79e-05      | 5.90           |

\( k = 4 \), respectively. To test the locking-free character of the method in the nearly incompressible case, we consider now Lamé coefficients \( \lambda \) and \( \mu \) corresponding to a Poisson ratio \( \nu = 0.499 \) and a Young modulus \( E = 10 \). We fix the polynomial degree to \( k = 2 \), take a stabilization parameter \( a = 50 \) and report in Tables 9 and 10 the experimental rates of convergence for \( \kappa = 4, 8, 16, 32 \). We observe that the method is thoroughly robust for nearly incompressible.
Table 8: Convergence of the DG method in $r$ for different wave numbers ($k = 6$, $a = 100$).

| $1/h$ | $e_d^{16}(r)$ | $r_d^{16}(r)$ | $e_d^{28}(r)$ | $r_d^{28}(r)$ | $e_d^{32}(r)$ | $r_d^{32}(r)$ | $e_d^{40}(r)$ | $r_d^{40}(r)$ |
|-------|----------------|---------------|----------------|----------------|----------------|---------------|----------------|---------------|
| 8     | 8.13e-03      | 2.08e+00     | 2.65e+00       | 3.93e+00       |                |               |                |               |
| 16    | 8.08e-04      | 6.65          | 6.38           | 6.42           | 1.67e-01       | 4.56          |                |               |
| 32    | 9.71e-06      | 6.38          | 6.58           | 5.30           | 2.09e-03       | 6.32          |                |               |
| 40    | 8.68e-07      | 5.96          | 5.89           | 6.65           | 1.96e-04       | 5.84          |                |               |
| 56    | 3.45e-07      | 5.97          | 5.93           | 5.91           | 7.94e-05       | 5.87          |                |               |
| 64    | 1.55e-07      | 5.99          | 5.94           | 5.93           | 9.64e-06       | 5.90          |                |               |

Table 9: Convergence of the DG method in $\sigma$ for different wave numbers ($k = 2$, $a = 50$, $\nu = 0.499$).

| $1/h$ | $e_d^4(\sigma)$ | $r_d^4(\sigma)$ | $e_d^8(\sigma)$ | $r_d^8(\sigma)$ | $e_d^{16}(\sigma)$ | $r_d^{16}(\sigma)$ | $e_d^{32}(\sigma)$ | $r_d^{32}(\sigma)$ |
|-------|-----------------|-----------------|-----------------|-----------------|-------------------|-------------------|-------------------|-------------------|
| 8     | 6.90e-02       | 3.27e-01       | 3.58e-02       | 1.80e-02       |                   |                   |                   |                   |
| 16    | 1.80e-02       | 1.94           | 6.83e-02       | 2.26           | 3.27e-01         | 1.83e-02         |                   |                   |
| 32    | 4.54e-03       | 1.99           | 1.78e-02       | 1.94           | 6.81e-02         | 2.27             | 3.27e-01         |                   |
| 64    | 1.14e-03       | 2.00           | 4.49e-03       | 1.99           | 1.77e-02         | 1.94             | 6.80e-02         | 2.27             |
| 128   | 2.84e-04       | 2.00           | 1.13e-03       | 2.00           | 4.48e-03         | 1.99             | 1.77e-02         | 1.94             |

Table 10: Convergence of the DG method in $r$ for different wave numbers ($k = 2$, $a = 50$, $\nu = 0.499$).

| $1/h$ | $e_d^4(r)$ | $r_d^4(r)$ | $e_d^8(r)$ | $r_d^8(r)$ | $e_d^{16}(r)$ | $r_d^{16}(r)$ | $e_d^{32}(r)$ | $r_d^{32}(r)$ |
|-------|------------|------------|------------|------------|----------------|---------------|----------------|---------------|
| 8     | 1.00e+00  | 1.15e+01  | 5.19e+00  | 3.60e+00  |                |               |                |               |
| 16    | 1.38e-01  | 2.87       | 1.08e-00  | 3.42       | 1.26e+01       | 2.92e+00       |                |               |
| 32    | 2.08e-02  | 2.73       | 1.37e-01  | 2.98       | 1.13e+00       | 3.48           | 1.32e+01       |                |
| 64    | 3.94e-03  | 2.40       | 2.04e-02  | 2.75       | 1.38e-01       | 3.03           | 1.16e+00       | 3.50           |
| 128   | 8.91e-04  | 2.15       | 3.88e-03  | 2.40       | 2.04e-02       | 2.76           | 1.39e-01       | 3.06           |

possible materials. However, it seems that the pre-asymptotic region increases in this case for big values of $\kappa$.

We now study the influence of $\kappa$ on the choice of the stabilization parameter $a$ of the discontinuous Galerkin scheme (21). To this end, we present in Figure 4 different approximations corresponding to $\kappa = 2, 4, 8, 16, 32$, obtained with the mesh $h = 1/32$ and a polynomial degree $k = 3$. In each case, we represent in a double logarithmic scale the errors versus the parameter $a$. Clearly, $a$ is not sensible to the variations of $\kappa$. However, higher polynomial degrees $k$ require higher values for the stabilization parameter $a$. This is made clear in Figure 2 where the polynomial degrees $k = 1, \cdots, 7$ are considered on a fixed mesh $h = 1/32$, with a fixed wave number $\kappa = 16$. In each case, the errors are depicted versus $a$. 
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Figure 1: Errors of the DG method versus \( a \) with \( h = 1/32 \) and \( k = 3 \).

Figure 2: Errors of the DG method versus \( a \) with \( h = 1/32 \) and \( \kappa = 16 \).

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