ON THE DIVERGENCE OF BIRKHOFF NORMAL FORMS

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Abstract. It is well known that a real analytic symplectic diffeomorphism of the two-dimensional annulus admitting a real analytic invariant curve with diophantine rotation number can be formally conjugated to its Birkhoff Normal Form, a formal power series defining a formal integrable symplectic diffeomorphism. We prove in this paper that this Birkhoff Normal Form is in general divergent. This solves the question of determining which of the two alternatives of Perez-Marco’s theorem [18] is true and answers a question by H. Eliasson. Our result is a consequence of the fact that the convergence of the formal object that is the BNF has strong dynamical consequences on the Lebesgue measure of the set of invariant curves in arbitrarily small neighborhoods of the original invariant curve: the measure of the complement of the set of invariant curves in these neighborhoods is much smaller than what it is in general. As a consequence, for any $d \geq 1$, the Birkhoff Normal Form of a symplectic real-analytic diffeomorphism of the $d$-dimensional annulus attached to an invariant real-analytic lagrangian torus with prescribed diophantine frequency vector is in general divergent.

1. Introduction

1.1. Birkhoff Normal Forms. Let $f : (\mathbb{R}^{2d}, 0) \to$ be a real analytic symplectic diffeomorphism preserving the canonical symplectic form $\sum dx_j \wedge dy_j$ and admitting $0$ as an elliptic fixed point: $f(0) = 0$ and the eigenvalues of $Df(0)$ are of the form $e^{\pm 2\pi i\omega_j}$, with $\omega_j \in ]0, 1/2[$, $j = 1, \ldots, d$. We can assume without loss of generality that $Df(0)$ is a symplectic rotation: for any $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_d)$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_d)$ one has $(i = \sqrt{-1})$

$$Df(0) \cdot (x, y) = (\tilde{x}, \tilde{y}) \iff \begin{cases} \tilde{x}_j + i \tilde{y}_j = e^{2\pi i \omega_j} (x_j + iy_j) \\
1 \leq j \leq d. \end{cases}$$

We shall call $\omega := (\omega_1, \ldots, \omega_d)$ the frequency vector at the origin and we say that it is nonresonant if furthermore any relation $k_0 + k_1 \omega_1 + \cdots + \omega_d \lambda_d = 0$ with $k_0, k_1, \ldots, k_d \in \mathbb{Z}$ implies that $k_0 = k_1 = \ldots = k_d = 0$. 

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More generally, we say that \( f \) is a \textit{generalized symplectic rotation} if there exist real analytic functions \( \omega_j : (\mathbb{R}^{2d}, 0) \to \mathbb{R} \) \((1 \leq j \leq d)\) such that

\[
f(x, y) = (\tilde{x}, \tilde{y}) \iff \begin{cases} 
\tilde{x}_j + i\tilde{y}_j = e^{2\pi i \omega_j(x, y)}(x_j + iy_j) \\
\forall 1 \leq j \leq d.
\end{cases}
\]

By symplecticity this implies the existence of a real analytic function \( B : (\mathbb{R}^d, 0) \to \mathbb{R}, (r_1, \ldots, r_d) \mapsto B(r_1, \ldots, r_d) \) such that

\[
\forall 1 \leq j \leq d, \quad \omega_j(x, y) = \partial_{r_j} B \left( \frac{x_1^2 + y_1^2}{2}, \ldots, \frac{x_d^2 + y_d^2}{2} \right).
\]

We then denote \( f = R_B \).

A fundamental result due to Birkhoff [2, 3], more classically proved in the context of hamiltonian systems, asserts that there exist for any \( N \in \mathbb{N}^* \), a polynomial \( B_N \in \mathbb{R}[r_1, \ldots, r_d] \) and a symplectic diffeomorphism \( Z_N : (\mathbb{R}^{2d}, 0) \cong (\text{preserving the standard symplectic form } \sum_{k=1}^d dx_k \wedge dy_k \text{ and tangent to the identity}) \) such that

\[
(1.1) \quad Z_N \circ f \circ Z_N^{-1}(x, y) = R_{B_N}(x, y) + \mathcal{O}^{2N+1}(x, y).
\]

Furthermore, the polynomials \( B_N \) and the components of \( Z_N - \text{id} \) converge as formal power series when \( N \) goes to infinity: there exists a formal \( B \in \mathbb{R}[[r_1, \ldots, r_d]] \) and a formal symplectic transformation \( Z \) such that in \( \mathbb{R}[[x, y]] \) one has

\[
(1.2) \quad Z \circ f \circ Z^{-1}(x, y) = R_B(x, y).
\]

In other words, any real analytic symplectic diffeomorphism with a non resonant elliptic fixed point is \textit{formally} integrable. The formal power series \( B \), which is unique if \( Z \) is tangent to the identity, is called the \textit{Birkhoff Normal Form}.

The Birkhoff Normal Form can also be defined for (exact) symplectic maps of the \( d \)-dimensional cylinder \( \mathbb{A}^d = \mathbb{T}^d \times \mathbb{R}^d \) \((\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d)\) which preserves the torus \( \mathcal{T}_0 = \mathbb{T}^d \times \{0\} \), but under an additional assumption on the frequency vector: let \( f : (\mathbb{T}^d \times \mathbb{R}^d, \mathcal{T}_0) \to \mathbb{R} \) be real analytic and symplectic of the form

\[
f(\theta, r) = (\theta + \omega(r), r) + \mathcal{O}^2(r).
\]

The lagrangian torus \( \mathcal{T}_0 = \mathbb{T}^d \times \{0\} \) is \( f \)-invariant and the restriction of \( f \) on \( \mathcal{T}_0 \) is a translation by the vector \( \omega(0) \). If one assumes that \( \omega(0) \) satisfies a \textit{diophantine condition}:

\[
\forall k \in \mathbb{Z}^d \setminus \{0\}, \min_{l \in \mathbb{Z}} |\langle k, \omega(0) \rangle - l| \geq \frac{\gamma}{|k|^r}
\]

one can prove the existence: (a) for any \( N \in \mathbb{N}^* \), of a polynomial \( B_N \in \mathbb{R}[r_1, \ldots, r_d] \) and of a symplectic diffeomorphism \( Z_N : (\mathbb{A}^d, \mathcal{T}_0) \cong \text{preserving the standard symplectic form } \sum_{k=1}^d d\theta_k \wedge dr_k \text{ and tangent to the identity} \) such that

\[
(1.3) \quad Z_N \circ f \circ Z_N^{-1}(\theta, r) = (\theta + \nabla B_N(r), r) + \mathcal{O}^{N+1}(r)
\]
and: (b) of a formal power series $B \in \mathbb{R}[[r_1, \ldots, r_d]]$ and of a formal symplectic transformation (tangent to the identity) such that in $C^\omega(T^d)[[r_1, \ldots, r_d]]$ one has

\begin{equation}
Z \circ f \circ Z^{-1}(\theta, r) = (\theta + \nabla B(r), r).
\end{equation}

Birkhoff Normal Forms can be defined similarly in the context of Hamiltonian flows either for nonresonant elliptic equilibria or for diophantine invariant lagrangian tori (see \cite{2}, \cite{3} and for more recent results \cite{7}, \cite{6}).

As already suggested by (1.1), (1.3) the BNF (or its approximate version $B_N$) is a precious tool in the study of the stability of an equilibrium or of a diophantine lagrangian torus, a fundamental question in Symplectic and Hamiltonian Dynamics. It is also very important when one tries to investigate the existence of quasi-periodic motions in the neighborhood of $0$ or of a diophantine lagrangian torus; indeed, mild non-degeneracy assumptions on the BNF (non-planarity) allow to prove the existence of many KAM tori accumulating the origin; in the setting of twist area-preserving diffeomorphisms of the disk or the annulus this is Moser’s famous Twist Theorem \cite{16} (for more details and references in the context of Hamiltonian systems see for example \cite{7}, \cite{6}).

The dynamical meaning of the formal integrability relation (1.2) is very limited when the map $f$ is only assumed to be smooth: equality (1.2) then only depends on the infinite jet of $H$ at the origin and cannot reflect the dynamical behavior of $f$ outside a neighborhood of the origin. The situation is less clear if one assumes, as we do, that $f$ is real-analytic – since in that case the infinite jet of $f$ determines $f$ in a neighborhood of $0$ – especially if one adds an extra arithmetic assumption on the frequency vector $\omega$ at $0$: indeed, Rüssmann \cite{19} proved (for $d = 1$) that if $f$ is real-analytic and if its BNF is trivial ($R_B = Df(0)$) then $f$ is real-analytically conjugated to its linear part provided its frequency vector at the origin satisfies a Diophantine condition: there exists a real-analytic $Z$ such that (1.2) is satisfied in the real-analytic sense. On the other hand, Siegel proved in 1954 \cite{20} (in the setting of Hamiltonian flows) that the conjugating map of a real-analytic Hamiltonian is not convergent.

These facts led H. Eliasson \cite{4}, \cite{5} (see also the references in \cite{18}) to ask:

\textit{Are there examples of real analytic symplectic diffeomorphisms or Hamiltonians admitting divergent Birkhoff Normal Form?}

R. Perez-Marco \cite{18} proved (in the setting of Hamiltonian systems having a non resonant elliptic fixed point) that for any given nonresonant quadratic part one has the following dichotomy: either the BNF is generically divergent or it always converges and Gong \cite{11} provided an example of divergent BNF with Liouville frequencies at the origin.

1 A KAM torus is an invariant lagrangian torus on which the dynamics is conjugated to a linear translation with a diophantine frequency vector.

2 In the context of CR singular points of real analytic submanifolds, analogous results of divergence of normal forms can be found in \cite{13}; cf. also \cite{17} and \cite{12}.
In the spirit of Eliasson’s question and having in mind the aforementioned Rüssmann’s Theorem, one can ask a stronger question:

*If the BNF of a real analytic symplectic diffeomorphism of the disk (resp. the annulus) having a Diophantine elliptic fixed point (resp. a Diophantine invariant circle) converges is it true that the system is integrable?*

More generally one can ask:

*What are the consequences of the convergence of a formal object like the Birkhoff Normal Form on the dynamics of a real analytic symplectic diffeomorphism?*

The main result of this paper is in some sense one answer, amongst possibly others, to the previous question, for real analytic symplectic diffeomorphisms of the two-dimensional annulus that admit an invariant diophantine circle and that have non-planar BNF: If the Birkhoff Normal Form of such a symplectic diffeomorphism converges, then the measure of the complement of the set of invariant circles accumulating the original circle is much smaller than what it should be for a general real analytic area-preserving map with twist (see the precise statement below). Combined with Pérez-Marco’s Theorem [18] (which holds in the context of symplectic diffeomorphisms but extends to our setting) we get that in any number of degrees of freedom, a generic real analytic symplectic diffeomorphism admitting an analytic diophantine invariant torus with prescribed diophantine frequency vector has a divergent Birkhoff Normal Form.

Our proof can be translated in the setting of diffeomorphisms of the disk admitting a diophantine elliptic fixed point and in the hamiltonian setting. Theorems A, B, C can thus be stated by replacing “annulus” by “disk” and “real-analytic invariant curve or torus” by “fixed point”.

**1.2. Results.** If \( f : \mathbb{T} \times [-1, 1] \to \mathbb{T} \times [-1, 1] \) is an analytic symplectic diffeomorphisms of the annulus such that \( f(\mathbb{T} \times \{0\}) = \mathbb{T} \times \{0\} \) and \( f(\theta, 0) = (\theta + \omega_0, 0) \) for some \( \omega_0 \in \mathbb{R} \), we define \( \mathcal{G}_f \) the set of real analytic graphs \( C_\gamma := \{(\theta, \gamma(\theta)), \theta \in \mathbb{T}\} \), with \( \gamma : \mathbb{T} \to [-1, 1] \) real analytic, such that \( f(C_\gamma) = C_\gamma \). For \(-1/2 < t < 1/2\), we define \( \mathcal{L}_f(t) \) the set of points of \( \mathbb{T} \times ]-t, t[ \) which are contained in an invariant graph \( C_\gamma \subset \mathbb{T} \times ]-2t, 2t[ \). We can define for \(-1/2 < t < 1/2\) the quantity \( m(t) \) as the Lebesgue measure of the complement of the set \( \mathcal{L}_f(t) \subset \mathbb{T} \times ]-1, 1[ \) in \( \mathbb{T} \times ]-1, 1[ \).

For \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) we define

\[
\tau(\omega) = \limsup_{k \to \infty} \frac{-\ln \min_{l \in \mathbb{Z}} |k\omega - l|}{\ln k}
\]

and we say that \( \omega \) is diophantine if \( \tau(\omega) < \infty \).

**Theorem A.** Let \( f : \mathbb{T} \times [-1, 1] \to \mathbb{T} \times [-1, 1] \) be a real analytic symplectic diffeomorphisms of the annulus such that \( f(\mathbb{T} \times \{0\}) = \mathbb{T} \times \{0\} \) and \( f(\theta, 0) = (\theta + \omega_0, 0) \) with \( \omega_0 \) diophantine. Assume that its formal Birkhoff Normal Form \( \Xi(r) \in \mathbb{R}[[r]] \) is non degenerate (i.e. such that \( \partial_r^2 \Xi(0) > 0 \)) and
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converges. Then,

\[ m_f(t) \leq \exp \left( -\left( \frac{1}{t} \right)^\beta(\omega_0) \right) \]

where \( \beta(\omega_0) = \frac{2}{1 + \tau(\omega_0)} \).

Remark: The same result can be proved for analytic symplectic diffeomorphisms of the disk for which the origin is a diophantine elliptic fixed point.

On the other hand one has the following quantitative version of Moser’s Twist Theorem [16]:

Theorem B. There exists a real analytic symplectic diffeomorphism \( f : \mathbb{T} \times [-1, 1] \to \mathbb{T} \times [-1, 1] \) such that \( f(\mathbb{T} \times \{0\}) = \mathbb{T} \times \{0\} \), \( f(\theta, 0) = (\theta + \omega_0, 0) \) with \( \omega_0 \) diophantine and having a non degenerate Birkhoff Normal Form for which there exists a sequence \( t_j > 0, \lim t_j = 0 \) such that

\[ m(t_j) \geq \exp \left( -\left( \frac{1}{t_j} \right)^\frac{1}{2}\beta(\omega_0) \right). \]

Using the aforementioned result by Perez-Marco this implies:

Theorem C. For any \( d \geq 1 \), the Birkhoff Normal Form of a real-analytic diffeomorphism of the annulus \( \mathbb{A}_d \) admitting an invariant real-analytic lagrangian torus with prescribed diophantine frequency vector is in general divergent.

Remarks (Added June 2019)

Theorems [A] and [B] can be extended (cf. [14]) to the case of symplectic diffeomorphisms of the disk (with twist) admitting any non-resonant fixed frequency at the origin. As a consequence, for any fixed non-resonant frequency vector, the set of real-analytic symplectic diffeomorphisms of the \( d \)-dimensional disk, \( d \geq 1 \), with a diverging BNF and with this prescribed frequency vector at the origin, is prevalent. The same result also holds in the hamiltonian setting (for 2 degrees of freedom or more).

B. Fayad [8] recently provided examples of real analytic Hamiltonian systems with 4 degrees of freedom that have a Lyapunov unstable fixed point at the origin. He can also show that for every non resonant fixed frequency vector at the origin these examples can be modified to have diverging BNF at the origin.

1.3. Sketch of the proof of Theorem [A]. A real analytic symplectic diffeomorphism of the annulus can be parametrized by using real analytic generating functions \( F : (\mathbb{R}/\mathbb{Z} \times \mathbb{R}, T_0) \to \mathbb{R} \):

\[ f_F : (\theta, r) \mapsto (\varphi, R) \iff \begin{cases} R = r - \partial_\varphi F(\varphi, r) \\ \theta = \varphi - \partial_r F(\varphi, r) \end{cases} \]
that have a complex extension to let say \(((\mathbb{R} + i) - h, h]/\mathbb{Z}) \times \mathbb{D}\). In our situation we can assume that \(f : \mathbb{A} \rightarrow \mathbb{A}\) is of the form \(f_{\Omega + F}\) where \(\Omega : r \mapsto \Omega(r)\) depends only on the \(r\) variable and \((\theta, r) \mapsto F(\theta, r)\) is small. Notice that \(f_{\Omega} : (\theta, r) \mapsto (\theta + \omega(r), r)\) with \(\omega(r) = \partial_{\theta} \Omega(r)\).

A classical procedure in this context, the KAM scheme, is to conjugate \(f := f_{\Omega + F}\) by using successive changes of coordinates to some \(f_{\Omega_n + F_n}\) where \(\Omega_n\) depends only on the \(r\) variable and where \(F_n\) is much smaller than \(F\). To do this, one has to solve linearized equations, usually called cohomological equations, which can be solved only if the frequency vector \(\omega(r)\) satisfies some (approximate) diophantine condition. These conditions cannot be satisfied for all values of \(r \in \mathbb{D}\) but instead only on complex domains \(U_n\) of the unit disk with holes (disks which have been removed). Each hole corresponds to a disk where, at some step of the KAM procedure, the frequency vector is (approximately) resonant and thus could not provide good enough estimates for the convergence of the scheme. However, the number of holes one removes at each step and their sizes can be controlled.

Parallel to the previous KAM procedure one can define, exploiting the fact that \(\omega(0)\) is diophantine, a similar iteration scheme defined on smaller and smaller domains \(T_h \times \mathbb{D}_{\rho_n}\) shrinking to \(T_h \times \{0\}\) (which contains the torus \(T_0\)) on which \(f_{\Omega + F}\) is conjugated to \(f_{\Xi_n + G_n}\), where \(G_n\) is very small and where \(\Xi_n\), which depends only on \(r\), converges (in the set of formal power series) to the BNF when \(n\) goes to infinity. This procedure is just a quantified version of the classical conjugation method producing the usual BNF.

Finally, in a neighborhood of each hole \(\Delta\) of the KAM domain \(U_n\), the presence of a resonance allows to construct another approximate normal form very similar to that of a classical pendulum that we call a Hamilton-Jacobi normal form. It gives an annulus \(\hat{\Delta} \setminus \tilde{\Delta}\) (\(\hat{\Delta} \approx \Delta \subset \tilde{\Delta}\), with \(\tilde{\Delta}\) much bigger than \(\hat{\Delta}, \Delta\)) where \(f_{\Omega + F}\) is conjugated to \(f_{\Gamma_n + H_n}\), \(\Upsilon_n\) depending only on \(r\) and \(H_n\) being very small.

So far we have thus constructed for \(f_{\Omega + F}\) three types of approximate normal forms: the KAM one \(f_{\Omega_n + F_n}\) (Section 5) on the domain with holes \(U_n\), the (approximate) BNF \(f_{\Xi_n + G_n}\) in a disk \(\mathbb{D}_{\rho_n}\) centered at 0 (Section 6) and, in each annulus \(\hat{\Delta} \setminus \tilde{\Delta}\), an approximate Hamilton-Jacobi normal form \(f_{\Gamma_n + H_n}\) (Section 7). We can compare these various normal forms (Section 8): for example since \(U_n\) and \(\hat{\Delta} \setminus \tilde{\Delta}\) have a common intersection, \(\Omega_n\) and \(\Upsilon_n\) nearly coincide on \(U_n \cap (\hat{\Delta} \setminus \tilde{\Delta})\). Similarly, since \(U_n\) and \(\mathbb{D}_{\rho_n}\) have a common intersection, \(\Omega_n\) and \(\Xi_n\) are nearly equal on \(U_n \cap \mathbb{D}_{\rho_n}\). Notice, that if one assumes that the BNF \(\Xi_{\infty}\) converges and defines a holomorphic function defined on \(\mathbb{D}\), one has that \(\Omega_n\) and \(\Xi_{\infty}\) are almost equal on \(U_n \cap \mathbb{D}_{\rho_n}\).

A natural idea in order to exploit the preceding matching conditions is to deduce by an argument from potential theory (Jensen’s formula) that \(\Omega_n\) and \(\Xi_{\infty}\) nearly coincide on (almost) the whole domain \(U_n\) since they coincide on \(U_n \cap \mathbb{D}_{\rho_n}\) (Section 5). This requires a good control on the harmonic measure.
of the domain $U_n$. Unfortunately, the number of holes and their sizes do not allow for such a control on the whole domain $U_n$. On the other hand, on a smaller domain $\hat{U}_n := U_n \cap \mathbb{D}_{\mu_n}$ that shrinks to 0 when $n$ goes to infinity but that still contains many holes, such a control is possible (Subsection 9.4) and one can prove that on this domain, $\Omega_n$ and the holomorphic function $\Xi_\omega$ (defined on $\mathbb{D}_{\mu_n}$) almost coincide. But since, $\Omega_n$ and $\Upsilon_n$ nearly coincide in any annulus $\hat{U}_n \cap (\hat{\Delta} \setminus \hat{\Delta})$ attached to a hole $\Delta$ of $\hat{U}_n$ (Subsection 9.3) one deduces that $\Upsilon_n$ and $\Xi_\omega$ coincide on this annulus. In other words, the holomorphic function $\Upsilon_n$ defined on $(\hat{\Delta} \setminus \hat{\Delta})$ which is a priori singular on $\hat{\Delta}$ coincide to some high order of accuracy with the holomorphic function $\Xi_\omega$ which is defined on the whole disk $\hat{\Delta}$. The way the function $\Upsilon_n$ is defined is more or less explicit; as a consequence, one can see that the size of its singular disk $\hat{\Delta}$ is related to the residue (defined by some Cauchy integral on the annulus $\hat{\Delta} \setminus \hat{\Delta}$) of a certain holomorphic function defined on the annulus $\hat{\Delta} \setminus \hat{\Delta}$. But the fact that $\Upsilon_n$ and $\Xi_\omega$ almost coincide show that this residue must be very small (Subsection 7.4). Consequently, one can choose $\hat{\Delta}$ much smaller than what we initially thought. This argument shows that the size of the holes of $\hat{U}_n$ is much smaller than what they should normally be, which implies the same statement for the measure of the complement of the set of invariant curves in $(\mathbb{T}_h \times \hat{U}_n) \cap (\mathbb{T} \times \mathbb{R})$.

2. Notations

If $z \in \mathbb{C}$, $\rho > 0$ we denote by $D(z, \rho) \subset \mathbb{C}$ the open disk of center $z$ and radius $\rho$; sometimes for short we shall write $\mathbb{D}_\rho$ for $D(0, \rho)$. The 1-dimensional torus is $\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z}, x \in \mathbb{R}\}$. We define its complex extensions $\mathbb{T}_h = \mathbb{R}/\{h, h(\sqrt{-1})/\mathbb{Z} \}$ for $h > 0$. If $h > 0$ and $U$ is an open set of $\mathbb{C}$ we let $\mathcal{O}(\mathbb{T}_h \times U)$ be the set of holomorphic functions on $\mathbb{T}_h \times U$. When $U \subset \mathbb{C}$ is invariant by complex conjugation we define $\mathcal{F}(\mathbb{T}_h \times U)$ (or for short $\mathcal{F}_h(U)$) the set of $F \in \mathcal{O}(\mathbb{T}_h \times \mathbb{D}_\rho)$ which are real symmetric, i.e. such that $F(\theta, r) = \bar{F}(\overline{\theta}, \overline{r})$ (the bar designs the complex conjugate), and

$$\|F\|_{h,U} = \sup_{(\theta, r) \in \mathbb{T}_h \times \mathbb{D}_\rho} |F|.$$  

When $U$ is the disk $\mathbb{D}_\rho$ we set $\|F\|_{h,\rho} = \|F\|_{h,\mathbb{D}_\rho}$. If $F \in \mathcal{F}_{h,\rho}$ we define for $k \in \mathbb{Z}$, $r \in \mathbb{D}_\rho$, its $k$-th Fourier coefficient $\hat{F}(k, r) = \int_{\mathbb{T}_h} F(\theta, r)e^{-2\pi i k \theta} d\theta$ and for $N \in \mathbb{N}$, its truncation and remainder up to order $N$

$$T_N F(\theta, r) = \sum_{|k| \leq N} \hat{F}(k, r)e^{2\pi i k \theta}, \quad R_N F(\theta, r) = \sum_{|k| > N} \hat{F}(k, r)e^{2\pi i k \theta}.$$  

For $\delta > 0$ we define $U_\delta = \{r \in U, \, \text{dist}(r, \partial U) > \delta \text{diam}(U)\}$. By Cauchy estimates we know that if $F \in \mathcal{F}_{h,\rho}$ then

$$|\hat{F}(k, r)| \lesssim e^{-2\pi |k| h} \|F\|_{h,\rho}.$$
Using this and Cauchy formula it is classical to prove for $k,l \geq 1$ and any $0 < \delta < h$

\begin{equation}
\begin{aligned}
\|c^k_0 F\|_{h-\delta,U} &\leq C_k \delta^{-(k+1)} \|F\|_{h,U} \\
\|c^l_0 F\|_{h,U_\delta} &\leq C_l \text{diam}(U)^{-l} \delta^{-(l+1)} \|F\|_{h,U}.
\end{aligned}
\end{equation}

We also have

\begin{equation}
\|R_N F\|_{h-\delta,U} \leq \delta^{-2} e^{-2pN} \|F\|_{h,U}.
\end{equation}

Notice that since $F = T_N F + R_N F$, then if $\delta^{-2} e^{-2pN} \delta \leq 1/2$ one has

\begin{equation}
\|T_N F\|_{h-\delta,U} \leq \|F\|_{h,U}
\end{equation}

and more generally without any condition on $N$

\begin{equation}
\|T_N F\|_{h-\delta,U} \leq \delta^{-1} \|F\|_{h,U}.
\end{equation}

If $F_1, \ldots, F_n$ are functions in $\mathcal{F}_{h,U}$ we denote by the generic term $O_p(F_1, \ldots, F_n)$ a function which is of “degree $p$” in the $F_1, \ldots, F_n$ and their derivatives which means that there exist $a > 0$ and a homogeneous polynomial $Q(X_1, \ldots, X_n)$ of degree $p$ in the variables $(X_1, \ldots, X_n)$ such that for any $\delta > 0$ satisfying

\begin{equation}
\text{diam}(U)^{-a} \delta^{-a} \max_{1 \leq i \leq n} \|F_i\|_{h,U} \leq 1
\end{equation}

one has

\begin{equation}
\|O_p(F_1, \ldots, F_n)\|_{h-\delta,U} \leq \text{diam}(U)^{-a} \delta^{-a} \max_{1 \leq i \leq n} \|F_i\|_{h,U}.
\end{equation}

We shall use the notation $\hat{O}_p(F_1, \ldots, F_n)$ if the polynomial $Q$ satisfies the relation $Q(X_1, 0, \ldots, 0) = 0$. If $n = 2$ and $Q(X_1, X_2) = X_1 X_2$ (resp. $Q(X_1, X_2) = X_1 X_2(X_1 + X_2)$) we shall use the notations $B_2(F_1, F_2)$ (resp. $B_3(F_1, F_2)$) instead of $O_2(F_1, F_2)$ (resp. $O_3(X_1, X_2)$).

3. Poisson-Jensen formula on domains with holes

Let $D$ be a disk (that we can assume to be the unit disk) and $U$ be an open subset of $D$ of the form $U = D \setminus (\bigcup_{1 \leq j \leq N} D_j)$ where $(D_j)_{j=1}^N$ is a collection of open sub-disks. We can define the Green function of $U$, $g_U : U \times U \to \mathbb{R}$ as follows: for any $z \in U$, $-g(z, \cdot)$ is the function equal to 0 on the boundary $\partial U$ of $U$, which is subharmonic on $U$, harmonic on $U \setminus \{w\}$ and which behaves like $\log|z - w|$ when $z \to w$. We denote by $\omega_U : U \times \text{Bor}(\partial U) \to [0,1]$ the harmonic measure of the disk $D$. One can see this harmonic measure the following way: if $z \in U$ and $I \in \text{Bor}(\partial U)$ (one can assume $I$ is an arc for example) then

\begin{equation}
\omega_U(z, I) = E(1_I (W_z(T_{z,I})))
\end{equation}

where $W_z(t)$ is the value at time $t$ of a brownian motion issued from the point $z$ (at time 0) and $T_{z,I}$ is the stopping time adapted to the filtration $\mathcal{F}_z$ of hitting $I$ before $\partial U \setminus I$.

Poisson-Jensen formula asserts that for any subharmonic function $u$

\begin{equation}
u(z) = \int_{\partial U} u(w) d\omega_U(z, w) - \int_U g(z, w) \Delta u(w).
\end{equation}

\begin{equation}
u \omega_U(z, I) = E(1_I (W_z(T_{z,I})))
\end{equation}

where $W_z(t)$ is the value at time $t$ of a brownian motion issued from the point $z$ (at time 0) and $T_{z,I}$ is the stopping time adapted to the filtration $\mathcal{F}_z$ of hitting $I$ before $\partial U \setminus I$.

Poisson-Jensen formula asserts that for any subharmonic function $u$

\begin{equation}
\int_{\partial U} u(w) d\omega_U(z, w) - \int_U g(z, w) \Delta u(w).
\end{equation}
In particular, if $f$ is a holomorphic function on $U$, the application of this formula to $u(z) = \ln |f(z)|$ gives

$$\ln |f(z_0)| = \int_{\partial U} \ln |f(w)| d\omega_U(z, w) - \sum_{w: f(w) = 0} g_U(z_0, w)$$

and thus

$$\sum_{w: f(w) = 0} g_U(z_0, w) \leq \int_{\partial U} \ln |f(w)| d\omega_U(z, w) - \ln |f(z_0)|.$$

If one has a good estimates on $\omega_U$ and $g_U$ then one can give a bound from above on the zeros of $f$ located in $U$.

**Lemma 3.1.** Let $U$ be a domain $U = D(0, \rho) \smallsetminus \bigcup_{1 \leq j \leq N} D(z_j, \varepsilon_j)$ where $D$ is a disk centered at 0, and let $B \subset U$, $B = D(0, \sigma)$. Assume that $f \in \mathcal{O}(U)$ satisfies

$$\|f\|_U \leq 1$$

and

$$\|f\|_{\partial B} \leq m.$$  

Then for any point $z \in \hat{U} := D(0, \rho) \smallsetminus \bigcup_{1 \leq j \leq N} D(z_j, d_j))$

$$\ln |f(z)| \leq \left( \frac{\ln(|z|/\rho)}{\ln(\sigma/\rho)} - \sum_{j=1}^N \frac{\ln(d_j/\rho)}{\ln(\varepsilon_j/\rho)} \right) \ln m.$$

**Proof.** Replacing $z$ by $z/\rho$, $\sigma$ by $\sigma/\rho$, and each $\varepsilon_j$ by $\varepsilon_j/\rho$ we can reduce to the case $D = (0, 1)$.

By Poisson-Jensen formula

$$\ln |f(z)| \leq \int_{\partial (U \setminus B)} \ln |f(w)| d\omega_U(z, w)$$

$$\leq \omega_{U \setminus B}(z, \partial B) \ln m.$$  

We now compare $\omega_{U \setminus B}(z, \partial B)$ with $\omega_{D \setminus B}(z, \partial B)$. The function $z \mapsto \omega_{U \setminus B}(z, \partial B)$ is the unique harmonic function which is 1 on $\partial B$ and 0 on $\partial D \cup \bigcup_{1 \leq j \leq N} \partial D_j$ while the function $z \mapsto \omega_{D \setminus B}(z, \partial B)$ is the unique harmonic function which is 1 on $\partial B$ and 0 on $\partial D$. So $\omega_{U \setminus B}(z, \partial B) - \omega_{D \setminus B}(z, \partial B)$ is a harmonic function $v$ defined on $U \setminus B$ which is 0 on $\partial B \cup \partial D$ and $\|v\|_{U \setminus B} \leq 1$. Let $v_j$ be the harmonic function defined on $D \setminus B$ which is 0 on $\partial (D \setminus B)$ and 1 on $\partial D_j$. Then by the maximum principle on $D \setminus B$

$$v \leq \sum_{j=1}^N v_j$$

and since by the maximum principle

$$v_j(z) \leq \frac{\ln |z - z_j|}{\ln \varepsilon_j}$$
one has
\begin{equation}
(3.28) \quad v(z) \leq \sum_{j=1}^{N} \frac{\ln d_j}{\ln \varepsilon_j}.
\end{equation}

The same argument shows that
\begin{equation}
(3.29) \quad -v(z) \geq -\sum_{j=1}^{N} \frac{\ln d_j}{\ln \varepsilon_j}.
\end{equation}

On the other hand
\begin{equation}
(3.30) \quad \omega_{D-B}(z,B) = \frac{|z|}{\ln \sigma}.
\end{equation}

Finally from (3.25)
\begin{equation}
(3.31) \quad \ln |f(z)| \leq \left( \frac{|z|}{\ln \sigma} - \sum_{j=1}^{N} \frac{\ln d_j}{\ln \varepsilon_j} \right) m.
\end{equation}

\[\square\]

4. Symplectic diffeomorphisms of the annulus

4.1. Parametrization. If $F : \mathbb{T}_h \times \mathbb{D} \to \mathbb{C}$ is a real symmetric holomorphic function the notation $O_2(F)$ denotes a function that is quadratic in $F$ and its first two derivatives.

4.1.1. Symplectic vector fields. If $F : \mathbb{T}_h \times \mathbb{D} \to \mathbb{C}$ is a real symmetric holomorphic function we denote by $\phi_{J \nabla F}$ the time 1 map of the symplectic vector field $J \nabla F$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and we set $\Phi_F = \phi_{J \nabla F}$. If $G$ is another real symmetric holomorphic function we define the Poisson bracket
\begin{equation}
(4.32) \quad \{F,G\} := \langle J \nabla F, \nabla G \rangle = \partial_q F \partial_r G - \partial_r F \partial_q G.
\end{equation}

If $f$ is a symplectic diffeomorphism one has
\begin{equation}
(4.33) \quad f \circ \Phi_F \circ f^{-1} = \Phi_{f^*F}
\end{equation}
where $f^*F = F \circ f^{-1}$. In fact, $f \circ \phi_{J \nabla F} \circ f^{-1} = \phi_{f^*_{\ast}(J \nabla F)}$ and we compute using the fact that $f^{-1}$ is symplectic $(Df^{-1} \circ J \circ Df^{-1}) = J$ \begin{equation}
(4.34) \quad f^*_{\ast}(J \nabla F) = (Df \cdot J \nabla F) \circ f^{-1}
\end{equation}
\begin{equation}
(4.35) \quad = (D(f^{-1}))^{-1} J (\nabla F) \circ f^{-1}
\end{equation}
\begin{equation}
(4.36) \quad = J (D(f^{-1})) (\nabla F) \circ f^{-1}
\end{equation}
\begin{equation}
(4.37) \quad = J \nabla (F \circ f^{-1}).
\end{equation}

We have the following formula
\begin{equation}
(4.38) \quad \frac{d}{dt}(G \circ \Phi_F^t)_{|t=0} = L_{J \nabla F} G = \{F,G\}
\end{equation}
\begin{equation}
(4.39) \quad \Phi_F \circ \Phi_G = \Phi_{F \circ G + O_2(F,G)}
\end{equation}
Using (4.33) we have if $G \in \mathcal{F}_{h,\rho}$, $\|G\|_{h,\rho} \leq 1$

(4.41) $\Phi_Y \circ \Phi_G \circ \Phi_Y^{-1} = \Phi_{G+\{Y\}G}$

(4.42)

where we denote

(4.43) $[Y] \cdot G = G \circ \Phi_Y - G.$

4.1.2. Generating functions. Let $F : \mathbb{T}_h \times \mathbb{D} \to \mathbb{C}$ be real symmetric ($\Omega(\overline{z}) = \Omega(z)$) holomorphic functions. We assume that $\|F\|_{T_h \times \mathbb{D}}$ is small enough so that the symplectic diffeomorphism $f_{-F} : (\theta, r) \mapsto (\varphi, R)$ implicitly defined by

(4.44) $R = r + \partial_\varphi F(\varphi, r), \quad \theta = \varphi + \partial_r F(\varphi, r)$

is close to the identity. This diffeomorphism is exact symplectic which means that differential form $Rd\varphi - r d\theta$ is exact. Indeed (we work first on the universal cover $(\mathbb{R} + i \mathbb{R} - h, \mathbb{R}) \times \mathbb{D}_\rho$),

(4.45) $Rd\varphi - r d\theta = Rd\varphi + \theta dr - d(\theta r)$

(4.46) $= (r + \partial_\varphi F(\varphi, r)) d\varphi + (\varphi + \partial_r F(\varphi, r)) dr - d(\theta r)$

(4.47) $= dF + d(\varphi r) - d(\theta r)$

(4.48) $= d(F + (\varphi - \theta) r)$. 

Conversely, if a diffeomorphism $(\theta, r) \mapsto (\varphi, R)$ is exact symplectic and close enough to the identity, it admits this type of parametrization.

We make the following remark: by the Fixed Point Theorem, if $\|\partial_r F\|_{h, U} \leq c_1 < h$, $\|\partial_{\varphi r} F\|_{h, U} \leq c_2 < 1$, then for any $\theta \in \mathbb{T}_{h-c_1}$ and $r \in U$ there exists a unique $\varphi \in \mathbb{T}_h$ such that $\theta = \varphi + \partial_r F(\varphi, r)$. If $r \in U_{c_1/diam(U)}$ and if $\|\partial_\theta F\|_{h, U} \leq c_1 < \text{diam}(U)$ then $R = r + \partial_\theta F(\varphi, r) \in U$. More precisely:

**Lemma 4.1.** Any exact symplectic real (symmetric) analytic symplectic diffeomorphism $f : \mathbb{T} \times \mathbb{D} \to \mathbb{T} \times \mathbb{D}$ close enough to the identity is of the form $f_F$ where

(4.49) $F = O_1(f - id)$.

Conversely, given $F \in \mathcal{F}_{h, U}$ one has

(4.50) $f_F = id + J \nabla F + O_2(F)$.

**Proof.** See the Appendix.

The composition of two exact symplectic maps is again exact symplectic and more precisely

**Lemma 4.2.** Let $F, G \in \mathcal{F}_{h, U}$ and $\delta$ satisfy then

(4.51) $f_G \circ f_F = f_{F+G+B_2(F,G)}$. 
If $F$ does not depend on the variable $r$ then

\[(4.52)\quad f_G \circ f_F = f_{F+G}\]

**Proof.** See the Appendix. \qed

The main result of this section is the following:

**Proposition 4.3.** If $F, Y \in \mathcal{F}_{h,U}$ and $\Omega \in \mathcal{F}_U$ depends only on $r$ ($\Omega : r \mapsto \Omega(r)$) and is such that $\|\Omega\|_U \leq 1$, then

\[(4.53)\quad f_Y \circ f_{\Omega+F} \circ f_Y^{-1} = f_{\Omega+F+[\Omega]}Y + \dot{\Omega}_2(Y,F)\]

where,

\[(4.54)\quad [\Omega] \cdot Y = Y - Y \circ f_Y^{-1}.\]

*If $\Omega = 0$ then*

\[(4.55)\quad f_Y \circ f_F \circ f_Y^{-1} = f_{F+(F,Y) \cdot B_3(F,Y)}\]

**Proof.**

If $\Omega = 0$ we write $f_Y = id + v$ and $f_F = id + u$. From Lemma A.4 one has

\[(4.56)\quad f_F^{-1} \circ f_Y \circ f_F \circ f_Y^{-1} = id + J\nabla\{F, Y\} + O_3(F,Y).\]

Hence,

\[(4.57)\quad f_Y \circ f_F \circ f_Y^{-1} = f_F \circ f_{(F,Y) \cdot B_3(F,Y)}\]

\[(4.58)\quad = f_{F+(F,Y) \cdot B_3(F,Y)}.\]

In the general case we observe that from Lemma A.6

\[(4.59)\quad f_Y \circ f_{\Omega} \circ f_Y^{-1} = f_{\Omega+[\Omega]}Y + O_2(Y)\]

so

\[(4.60)\quad f_Y \circ f_{\Omega+F} \circ f_Y^{-1} = f_Y \circ f_F \circ f_Y^{-1} \circ f_Y \circ f_{\Omega+F} \circ f_Y^{-1}\]

\[(4.61)\quad = f_{F+(F,Y) \cdot B_3(F,Y)} \circ f_{\Omega+[\Omega]}Y + O_2(Y)\]

\[(4.62)\quad = f_{F+(F,Y) \cdot B_3(F,Y)} \circ f_{[\Omega]}Y + O_2(Y) \circ f_{\Omega}\]

\[(4.63)\quad = f_{F+(F,Y) \cdot [\Omega]}Y + \dot{\Omega}_2(Y,F) \circ f_{\Omega}\]

\[(4.64)\quad = f_{\Omega+F+(F,Y)+[\Omega]}Y + \dot{\Omega}_2(Y,F).\]

In the following we shall parametrize exact symplectic diffeomorphisms of the annulus as maps $f_{\Omega+F}$ where $\Omega$ depends only on the $r$ variable and $F$ is small.
5. KAM scheme on domains with holes

5.1. One step of non-resonant KAM. Let $\Omega_1 \in \mathcal{F}(\mathbb{T}_{h_1})$, $F_1 \in \mathcal{F}(\mathbb{T}_{h_1} \times U_1)$, diam$(U_1) \geq 1$. We now try to conjugate the symplectic diffeomorphism $f_{\Omega_1} + F_1$ to a symplectic diffeomorphism $f_{\Omega_2} + F_2$ with $F_2$ much smaller than $F_1$. To do that we look for a change of coordinates of the form $f_{Y_1}$:

\[(5.65)\]

\[f_{Y_1} \circ f_{\Omega_1} + F_1 \circ f_{Y_1}^{-1} = f_{\Omega_1} + F_1 + \{\Omega_1\}_{Y_1} + O_2(F_1, Y_1).\]

**Lemma 5.1.** Let $F \in \mathcal{F}(\mathbb{T}_h \times U)$, $\Omega \in \mathcal{F}(U)$, $N \in \mathbb{N} \cup \{\infty\}$ and $K > 0$ such that for any $r \in U$

\[(5.66)\]

\[\forall k \in \mathbb{Z}, \ 0 < |k| \leq N, \forall l \in \mathbb{Z}, \ |k \nabla \Omega(r) - l| \geq \frac{K^{-1}}{|k|^\tau}.\]

Then, for any $0 < \delta < h$, there exists $Y \in \mathcal{F}(\mathbb{T}_{h-\delta} \times U)$ such that

\[(5.67)\]

\[\|Y\|_{h-\delta, U} \leq K\delta^{-(1+\tau)}\|F\|_{h, U}.\]

**Proof.** Equation (5.67) is equivalent to

\[(5.68)\]

\[Y(\theta, r) - Y(\theta - \nabla \Omega(r), \theta) = F(\theta, r) - \int_\mathbb{T} F(\theta, \cdot) d\theta\]

which in Fourier is

\[(5.69)\]

\[\forall k \in \mathbb{Z}, \hat{Y}(k, r)(1 - e^{-2\pi ik \nabla \Omega(r)}) = \hat{F}(k, r).\]

We define

\[(5.70)\]

\[Y(\theta, r) = \sum_{0 < |k| \leq N} \frac{\hat{F}(k, r)}{1 - e^{-2\pi ik \nabla \Omega(r)}}.\]

This converges since

\[(5.71)\]

\[\|Y\|_{h-\delta, U} \leq K\|F\|_{h, U} \sum_{0 < |k| \leq N} |k|^\tau e^{-2\pi |k| \delta} \leq C_\tau \delta^{-\tau-1} K\|F\|_{h, U}.\]

For $N, K$ fixed let

\[(5.72)\]

\[U_{2,N,K} = U_1 \setminus \{r \in U_1, \forall (k, l) \in \mathbb{Z}^2, \ 0 < |k| \leq N, \ |k \nabla \Omega_1(r) - l| > |k|^{-\tau} K^{-1}\}\]

and

\[(5.73)\]

\[U_{2,\delta}^{N,K} = \{r \in U_2^{N,K}, \ \text{dist}(r, \partial U_2^{N,K}) > \delta\}.\]

Let us assume that $U_{2,\delta}^{N,K}$ is not empty.
Proposition 5.2. There exists $Y_1$ and $F_2$ defined on $U_{2}^{N,K}$ such that
\begin{equation}
(5.76) \quad f_{Y_1} \circ f_{\Omega_1 + F_1} \circ f_{Y_1}^{-1} = f_{\Omega_2 + F_2}
\end{equation}
with
\begin{equation}
(5.77) \quad \Omega_2 = \Omega_1 + \int_T F_1(\theta, \cdot) \, d\theta
\end{equation}
and
\begin{equation}
(5.78) \quad \|Y_1\|_{h_1 - \delta, U_{2,\delta}^{N,K}} \leq K \delta^{-(1+\tau)} \|F_1\|_{h_1, U_1}
\end{equation}
\begin{equation}
(5.79) \quad \|F_2\|_{h_1 - \delta, U_{2,\delta}^{N,K}} \leq K^2 \delta^{-a_1} \|F\|_{h_1, U_1}^2 + \delta^{-2} e^{-2\pi \delta N} \|F\|_{h_1, U_1}
\end{equation}
provided
\begin{equation}
(5.80) \quad K^2 \delta^{-a_1} \|F\|_{h_1} \leq 1.
\end{equation}

Proof. Using Lemma 5.1 we choose $Y_1$ such that $[\Omega_1] \cdot Y_1 = -T_N F_1 + \int_T F_1(\theta, \cdot) \, d\theta$. Equation (5.65) gives us
\begin{equation}
(5.81) \quad f_{Y_1} \circ f_{\Omega_1 + F_1} \circ f_{Y_1}^{-1} = f_{\Omega_2 + R_N F_1 + O_2(F_1, Y_1)}.
\end{equation}
From (5.68) and (2.12) we have for some $a > 0$
\begin{equation}
(5.82) \quad \|R_N F_1 + D_2(F_1, Y_1)\|_{h_1 - \delta, U_{2,\delta}^{N,K}} \leq \delta^{-2} e^{-2\pi \delta N} \|F\|_{h_1, U_1} + \text{diam}(U_1)^{-a} \delta^{-a} O_2(\|F\|_{h_1, U_1}, \|Y\|_{h_1, U_1})
\end{equation}
\begin{equation}
(5.83) \quad \leq \delta^{-2} e^{-2\pi \delta N} \|F\|_{h_1, U_1} + \text{diam}(U_1)^{-a} \delta^{-a} K^2 \delta^{-2(\tau + 1)} \|F\|_{h_1, U_1}^2.
\end{equation}
This gives the conclusion with $a_1 = 2(\tau + 1) + a$. \hfill \Box

5.2. Description of $U_2$, location of resonances.

Proposition 5.3. Assume that $\min_{[-1,1] \cap U_1} \Omega'' \geq c > 0$, $\max_{\mathbb{D}} \|D^3 \Omega\| \leq C$ and that $\Omega'$ is increasing on $[-1, 1] \cap U_1$. Also we assume that $\text{diam}(U_1) \leq c/C$. Then, the set $U_{2,\delta}^{N,K}$ contains a set $\hat{U}_{2,\delta}^{N,K} = U_1 \setminus \bigcup_{j=1}^n D(c_j, \rho_j)$ where $n \leq N^2$, $N^{-(1+\tau)} K^{-1} \leq \rho_j \leq K^{-1}$, $c_j \in \mathbb{R}$. Furthermore, if $c_j \notin U_1 \cap \mathbb{R}$ then dist$(c_j, \partial U_1) \leq K^{-1}$ and for any $1 \leq j, j' \leq n$,
\begin{equation}
(5.86) \quad N^{-2} \leq |c_j - c_{j'}|.
\end{equation}

Proof. Indeed, for each $(k, l) \in \mathbb{Z}^2$, $0 < |k| \leq N$, the equation $|k \nabla \Omega_1(t + \sqrt{-1}s) - l| \leq |k|^{-\tau} K^{-1}$ is equivalent to
\begin{equation}
(5.87) \quad |\nabla \Omega_1(t + \sqrt{-1}s) - \frac{l}{k}| \leq |k|^{-(\tau + 1)} K^{-1}.
\end{equation}
Writing $|\nabla \Omega_1(t + \sqrt{-1}s) - \nabla \Omega_1(t) + is D^2 \Omega(t)| \leq C s^2$ and taking the imaginary parts we have
\begin{equation}
(5.88) \quad cs \leq Cs^2 + |k|^{-(\tau + 1)} K^{-1}
\end{equation}
and so
\begin{equation}
\begin{aligned}
\left\{ \begin{array}{l}
s \leq 2|k|^{-(\tau+1)K^{-1}} \\
|\nabla \Omega_1(t) - \frac{t}{k}| \leq 2|k|^{-(\tau+1)K^{-1}}.
\end{array} \right.
\end{aligned}
\end{equation}

Since \( \Omega'_1 \) is increasing on \((-1,1) \cap U_1\) with a derivative bounded by below by \( c \), the set of points \( t \in U_1 \cap \mathbb{R} \) for which \( |\nabla \Omega_1(t) - \frac{t}{k}| \leq 2|k|^{-(\tau+1)K^{-1}} \) has at most one connected components provided
\begin{equation}
2(C/c)|k|^{-(\tau+1)K^{-1}} < g(U_1)
\end{equation}
where \( g(U_1) \) is the largest diameter of the connected components of \((-1,1) \cap U_1^c\).

\[ \square \]

5.3. Iterative estimates. Now the preceding conjugation can be repeated let's say \( p \) times with the same \( N,K,\delta \) on \( U_2 \) as long as (5.80) and (5.89) are satisfied:
\begin{equation}
\varepsilon_{k+1} \leq K^2N^2\varepsilon_k^2 + N^\frac{e^{-N\delta}}{\delta}\varepsilon_k
\end{equation}
with \( \varepsilon_k = \|F_k\|_{h_k,\rho_k} \). We notice that as long as \( K^2N^2\varepsilon_k^2 \geq N^\frac{e^{-N\delta}}{\delta}\varepsilon_k \) one has
\begin{equation}
\varepsilon_{k+1} \leq 2K^2N^2\varepsilon_k^2.
\end{equation}
We now see that with the choice
\begin{equation}
\begin{aligned}
N_k &= e^{k\alpha}N, \\
\bar{\varepsilon}_k &= e^{-N_k/(\ln N_k)^\alpha}, \\
K_k^{-1} &= \varepsilon_k^{(1/2)-(\alpha/2)}, \\
\delta_k &= (\ln N_k)^{-\mu}, \\
h_k &= h - \delta_k, \quad \rho_k = e^{-\delta_k}\rho
\end{aligned}
\end{equation}
one has
\begin{equation}
\|F_k\|_{h_k,\rho_k} \leq \bar{\varepsilon}_k = e^{-N_k/(\ln N_k)^\alpha}.
\end{equation}

**Proposition 5.4.** There exist changes of coordinates \( Y_1, \ldots, Y_k \) and \( \Omega_k, F_k \) holomorphic and real symmetric on \( U_k \) such that if \( Z_k = f_{Y_{k-1}} \circ \cdots \circ f_{Y_1} \)
\begin{equation}
Z_k \circ f_{\Omega+F} \circ Z_k^{-1} = f_{\Omega+F_k}
\end{equation}
with
\begin{equation}
\|F_k\|_{h_k,\rho_k} \leq e^{-N_k/(\ln N_k)^\alpha}
\end{equation}
\begin{equation}
\Omega_k(r) = \Omega_{k-1}(r) + \int_{T^d} F_{k-1}(r,\theta) d\theta
\end{equation}
and \( U_k \) is of the form \( \mathbb{D} \setminus \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} D(c_{i,j},\rho_{i,j}) \) where \( K_{i-1}^{-1}N_{i-1}^{-(1+\tau)} \leq \rho_{i,j} \leq K_{i-1}^{-1} \) and \( n_i \leq N_{i-1}^2 \).
6. Birkhoff Normal Form

6.1. Reminders on BNF. An element of $\mathcal{F}[[r]]$ (where $\mathcal{F} = \cup_{h>0}\mathcal{F}_h$) is a formal power series

$$f = f(\varphi, r) = \sum_{n \in \mathbb{N}} f_n(\varphi)r^n. \quad \text{(6.98)}$$

**Proposition 6.1.** Let $H = \omega_0r + O^2(r)$. There exist $f(\varphi, r) \in \mathcal{F}[[r]] \cap O^2(r)$ and $\Xi(r) \in \mathbb{R}[[r]]$ such that

$$f_H(\varphi - \partial_r F(\varphi, r), r) = (\varphi + \partial_r \Xi(r), r - \partial_r F(\varphi, r)). \quad \text{(6.99)}$$

Moreover, $\Xi(r)$ is unique and $f$ is uniquely determined by fixing its mean value. The series $\Xi(r)$ is called the Birkhoff Normal Form of $H$.

6.2. Speed of convergence of a BNF scheme. We recall that Lemma 4.3 says that there exists $a > 0$, $C > 0$ such that for any $G, Y \in \mathcal{F}_{h, \rho}$, $\Omega \in \mathcal{F}_\rho$ with $\|\Omega\|_{\rho} \leq 1$, satisfying $\rho^{-a}\delta^{-a}\max(\|G\|_{h, \rho}, \|Y\|_{h, \rho}) \leq 1$ one has

$$f_Y \circ f_{\Omega+G} \circ f_Y^{-1} = f_{\Omega+G+[1:Y]+O_2(G,Y)}. \quad \text{(6.100)}$$

with

$$\|O_2(G,Y)\|_{h-\delta,e^{-4}\rho} \leq C\rho^{-a}\delta^{-a}\|G\|_{h,\rho}\|Y\|_{h,\rho}. \quad \text{(6.101)}$$

We can assume $a$ to be in $\mathbb{N}$ and $a \geq 5$. We shall use the notation $p = a$.

**Proposition 6.2.** Let $\omega_0$ satisfy a diophantine condition

$$\forall k \in \mathbb{Z}^*, \min_{l \in \mathbb{Z}} |k\omega_0 - l| \geq \gamma^{-1} \frac{1}{|k|^2} \quad \text{(6.102)}$$

and let $\Omega_0(r) = \omega_0r$ and $F \in \mathcal{F}_{h_0,\rho_0}$ with $F(\theta,r) = O(r^2)$. Then, there exists $\overline{p} > 0$ such that such for any $0 < \rho \leq \overline{p}$ and any

$$n \leq \left(\frac{\overline{p}}{\rho}\right)^{\frac{1}{a}} \quad \text{(6.103)}$$

one can define changes of coordinates $W_n \in \mathcal{F}_{h/4,\rho_1/2}$, $f_{W_n} = id + O(r)$, $n \geq 1$ and $\Xi_n \in \mathcal{F}_{\rho_1/2}$ such that

$$f_{W_{n-1}} \circ f_{\Omega_0} \circ f_{W_n}^{-1} = f_{\Xi_n+G_n} \quad \text{(6.104)}$$

where $G_n(r, \theta) = O(r^{n+1})$ and where $\Xi_0(r) = \Omega_0(r)$, and $\Xi_n(r) - \Xi_{n-1}(r) = O(r^{n+1})$. One has for $0 < \rho \leq \rho_0(\tau)$ the estimates

$$\|\Xi_n - \Omega_0\|_{\rho} \leq \sum_{k=2}^{n-1} \rho^kk_l^b \quad \text{(6.105)}$$

$$\|W_n\|_{h/2,\rho} \leq \sum_{k=1}^{n-1} \rho^kk_l^b \quad \text{(6.106)}$$

$$\|G_n\|_{h,\rho} \leq (2\rho)^{n+1}(n-1)!a_2 \quad \text{(6.107)}$$
(6.108) \[ \|G_k\|_{h_{n},\rho} \leq (\rho/\overline{\rho})^{n+p+1}(n-1)!^{1/(\tau+a)}. \]

**Proof.** Performing \( p = a \) steps of the classical Birkhoff Normal Form, we can by some \( f_{W_0} \) conjugate \( \Omega_0(r) + F(\theta, r) \) to \( \Xi_1(r) + G_1(\theta, r) \), with \( G_1(\theta, r) = O(r^{p+2}) \) and where \( \Xi_1 \in \mathcal{F}_{\overline{P}_1} \), \( G_1 \in \mathcal{F}_{h_1,\overline{P}_1} \) for some \( h_1 = h/2 \) and some \( \overline{P}_1 > 0 \). For \( l \geq 1 \) let \( \delta_l = c_1(l+\varepsilon) \) such that \( \sum_{l=1}^{\varepsilon} \delta_l = h/4 \), let \( \Delta_l = \delta_1 \cdots \delta_l \) (with \( \Delta_0 = 1 \)) and for \( 0 < \rho < \overline{P}_1 \) define inductively for \( l \geq 1 \), \( h_{l+1} = h_l - \delta_l, h_1 := h/4, \rho_{l+1}(\rho) = e^{-\delta_l}\rho_l(\rho), \rho_1(\rho) = \rho \). We observe that for \( l \geq 1 \), \( \rho_l/2 < \rho_l(\rho) < \rho \).

For \( 1 \leq k \leq k_*(\rho) \), we construct inductively sequences, \( \Xi_{k,\rho} \in \mathcal{F}_{h_k,\rho_k(\rho)} \), \( G_{k,\rho} \in \mathcal{F}_{h_k,\rho_k(\rho)} \), \( Y_{k,\rho} \in \mathcal{F}_{h_k-h_k,\rho_k(\rho)} \), such that for all \( 1 \leq k \leq k_*(\rho) \)

(6.109) \[ f_{Y_{k,\rho}} \circ f_{\Xi_{k,\rho}} + G_{k,\rho} \circ f_{Y_{k,\rho}}^{-1} = f_{\Xi_{k+1,\rho}} + G_{k+1,\rho} \]

with \( \|G_{k,\rho}\|_{h_{k},\rho_k} \leq (\rho_k/\overline{\rho})^{1/(\tau+a)} \). These sequences will satisfy the following restriction property: for any \( 0 < \rho' < \rho < \rho_* \) and any \( 1 \leq k \leq k_*(\rho) \) the restrictions of the functions \( G_{k,\rho} \) (resp. \( \Xi_{k,\rho} \), resp. \( Y_{k,\rho} \)) to \( \mathbb{T}_h \times D(0, \rho_k(\rho')) \) (resp. \( D(0, \rho_k(\rho')) \)) coincide with the functions \( G_{k,\rho'} \) (resp. \( \Xi_{k,\rho'} \), resp. \( Y_{k,\rho'} \)). The proof of the proposition will follow since \( f_{W_0} = id + O(r) \).

In the sequel we remove the dependence of \( \rho \) on the sequences \( \rho_k(\rho), G_{k,\rho} \) etc.

We define \( Y_k \) as the unique solution in \( \mathcal{F}_{h_k-h_k,\rho_k} \) of

(6.110) \[ -[\Omega_0] \cdot Y_k = G_k(\theta, r) - \int_{\mathbb{T}_h} G_k(\theta, r) d\theta. \]

We have \( Y_k(r, \theta) = O(r^{k+p+1}) \) and from Lemma 5.1

(6.111) \[ \|Y_k\|_{h_k-h_k,\rho_k} \leq \delta_k^{-1}(1+\tau) \|G_k\|_{h_k,\rho_k}. \]

We then have from Lemma 4.3

(6.112) \[ f_{Y_k} \circ f_{\Xi_k} + G_k \circ f_{Y_k}^{-1} = f_{\Xi_k + [\Omega_0] Y_k + G_k + O_2(Y_k, G_k)} \]

(6.113) \[ = f_{\Xi_k + [\Omega_0] Y_k + ([\Xi_k] - [\Omega_0]) \cdot Y_k + G_k + O_2(Y_k, G_k)} \]

(6.114) \[ = f_{\Xi_{k+1} + G_{k+1}} \]

with

(6.115) \[ \Xi_{k+1}(r) = \Xi_k(r) + \int_{\mathbb{T}_h} G_k(\theta, r) d\theta \]

and

(6.116) \[ G_{k+1} = ([\Xi_k] - [\Omega_0]) \cdot Y_k + O_2(Y_k, G_k). \]

We thus have \( G_{k+1} = O(r^{k+p+2}) \) and if

(6.117) \[ \rho_k^{-a} \delta_k^{-a} \max(\|Y_k\|_{h_k,\rho_k}, \|G_k\|_{h_k,\rho_k}) \leq 1 \]
then

\begin{align}
\|G_{k+1}\|_{h_k, \delta, e^{-\delta} \rho_k} & \leq \rho_k^{-a} \delta_k^{-\rho_k^{-a} \delta_k^{-\delta} \rho_k} \|G_k\|_{h_k, \delta_k}^2 + \rho_k \|\delta_k Y_k\|_{h_k, \delta_k, e^{-\delta} \rho_k} \\
& \leq \rho_k^{-a} \delta_k^{-(a+2(1+\tau))} \|G_k\|_{h_k, \delta_k}^2 + \rho_k \delta_k^{-\delta_k} \|Y_k\|_{h_k, \delta_k, e^{-\delta} \rho_k} \\
& \leq \rho_k^{-a} \delta_k^{-(a+2(1+\tau))} \|G_k\|_{h_k, \delta_k}^2 + \rho_k \delta_k^{-\delta_k} \|Y_k\|_{h_k, \delta_k, e^{-\delta} \rho_k} \\
& \leq C_1 \rho_k \delta_k^{-(a+1)} \|G_k\|_{h_k, \rho_k} (1 + C_1 \rho_k^{-1} \delta_k^{-(a+\tau)}} \|G_k\|_{h_k, \rho_k}) \\
& \leq 2C_1 \rho_k \delta_k^{-(a+\tau)} \|G_k\|_{h_k, \rho_k} (1 + C_1 \rho_k^{-1} \delta_k^{-(a+\tau)}} \|G_k\|_{h_k, \rho_k})
\end{align}

for some universal constant $C_1$. Now define $C_1 \geq \max(1, C_1, \rho_1^{-p+2}) \|G_1\|_{h_1, \rho_1}$.

By the maximum principle

\begin{align}
\|G_1\|_{h_1, \rho} \leq (4C_1 \rho)^{p+2}.
\end{align}

Now assume that for $1 \leq l \leq k$

\begin{align}
\|G_l\|_{h_l, \rho_l} \leq (4C_1 \rho)^{k+l+1} \Delta_k^{-(a+\tau)}
\end{align}

Then as long as \((6.117)\) holds together with

\begin{align}
C_1 \rho_k^{-(a+1)} \delta_k^{-(a+1)} \|G_k\|_{h_k, \rho_k} \leq 1
\end{align}

one has

\begin{align}
\|G_{k+1}\|_{h_{k+1}, \rho_{k+1}} \leq (4C_1 \rho)^{k+l+1} \Delta_k^{-(a+\tau)}
\end{align}

Now conditions \((6.117)\) and \((6.125)\) are satisfied provided

\begin{align}
(8C_1)^{a+1} (4C_1 \rho)^{k+l-a} \Delta_k^{-(a+1)} \leq 1.
\end{align}

Since we have chosen $p = a$, the preceding condition is equivalent to

\begin{align}
(8C_1)^{a+1} (4C_1 \rho)^{k} \Delta_k^{-(a+1)} \leq 1
\end{align}

With our choice $\delta_k = c k^{-1+\varepsilon}$ we see that this is implied by

\begin{align}
(8C_1)^{a+1} (4C_1 c^{-a+1}) \rho_k^{k} \Delta_k^{-(a+1)} \leq 1.
\end{align}

The quantity $k \mapsto g^k k!^b$ $(g, k > 0)$ is decreasing on $[1, k_*]$, $k_* := g^{-1/(b+1)}$, so the last condition is satisfied if

\begin{align}
\rho \leq \rho_* := \min((8C_1)^{-(a+1)} c^{a+1}, \rho_1/2) \\
1 \leq k \leq k_*(\rho) := \left( \frac{\rho_*}{\rho} \right)^{1/a}.
\end{align}

Since $\ln(a^k k!^b)$ attains its maximum $(-\ln a/(b+1)) a^{-1/(b+1)}$ at $k_* := a^{-1/(b+1)}$ we hence get the
Corollary 6.3. There exists a constant $b$, such that for any $0 < \rho << 1$ there exists $\Xi_\rho \in \mathcal{F}_\rho$, $G_\rho, W_\rho \in \mathcal{F}_{\rho/2}$ such that on $\mathbb{T}_{\rho/2} \times \mathbb{D}_{\rho}$ one has

$$W^{-1} \circ f_{\Theta g} \circ W_\rho = f_{\Xi_\rho + G_\rho}$$

$$\|G_\rho\|_{\rho/2} \lesssim \exp(-\rho^{-3}).$$

One has

$$G_\rho(\theta, r) = O(r^{-\rho^{-3}})$$

and thus $\Xi - \Xi_\rho$ (viewed as a formal series) satisfies $\Xi - \Xi_\rho = O(r^{-\rho^{-3}})$.

By Proposition 6.1

Corollary 6.4. The Birkhoff Normal Form of $f_{\Theta g}$ is the limit $\Xi$ in $\mathbb{R}[\![r]\!]$ of the sequence $(\Xi_n)_{n \in \mathbb{N}}$ (which converges in $\mathbb{R}[\![r]\!]$ since $\Xi_n(r) - \Xi_{n-1}(r) = O(r^{n+1})$).

7. Hamilton-Jacobi Normal Forms

7.1. Resonant normal form. Let $\varepsilon > 0$, $N$, $K$ such that

$$\varepsilon = e^{-N/(\ln N)^{\kappa}}$$

$$K^{-1} = \varepsilon^{(1/2)-\kappa/2}.$$ 

We assume that

$$\|F\|_{h, \rho} \leq \varepsilon$$

and that there exists $r_0 \in [-1, 1]$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$,

$$0 < q < N, \quad \gcd(p, q) = 1$$

such that

$$\nabla \Omega(r_0) = p/q.$$ 

Denoting $2c = D^2 \Omega(r_0) > 0$ we have the following Taylor expansion

$$\Omega(r) = \Omega(r_0) + \frac{p}{q}(r - r_0) + c(r - r_0)^2 + O((r - r_0)^3).$$

Let $\hat{D}$ be the disk centered at $r_0$ and of radius $\hat{K}^{-1}$, with

$$K^{-1} \leq \hat{K}^{-1} \leq N^{-2}.$$ 

A typical choice we shall make is

$$\hat{K} = N^{50P}$$

where $P$ is large (another choice could have been $\hat{K} = e^{N^{1/2}}$). Let $\hat{N}$ be such that $\hat{N}^2 \hat{K}^{-1} \leq 1$, for example

$$\hat{N} = \hat{K}^{1/2-\nu/4}.$$ 

Lemma 7.1. For any $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, $|k| \leq \hat{N}$, either there exists $m \in \mathbb{Z}$ such that $(k, l) = m \cdot (q, p)$ or

$$\forall r \in D(r_0, \hat{K}^{-1}), \quad |k \nabla \Omega(r) - l| \geq \hat{K}^{-1}.$$
Notice that if \((k,l) \in \mathbb{Z}^2\), \(0 < |k| \leq \hat{N}\) satisfies for some \(r \in D\), \(|k \nabla \Omega(r) - l| \leq \hat{K}^{-1}\) then since
\[
|k \nabla \Omega(r_0) - l| \leq |k \nabla \Omega(r) - l| + |k \nabla \Omega(r) - \nabla \Omega(r_0)|
\]
\[
\leq \hat{K}^{-1} + k\|D^2 \Omega\|_D \hat{K}^{-1}
\]
\[
\leq \hat{N} \hat{K}^{-1}
\]
one has
\[
\left(\begin{array}{cc}
q & -p \\
k & -l
\end{array}\right)
\begin{pmatrix}
\nabla \Omega(r_0)
\end{pmatrix}
= O(\hat{N} \hat{K}^{-1}).
\]
In particular, if \((q,p)\) and \((k,l)\) are not co-linear, \(|\det \left(\begin{array}{cc}
q & -p \\
k & -l
\end{array}\right)\| \geq 1\) and so
\[
|\nabla \Omega(r_0)| \leq \hat{N}^2 \hat{K}^{-1} < 1
\]
which is a contradiction. Consequently, \(ql = kp\) and since \(\gcd(p,q) = 1\) we deduce that there exists \(m \in \mathbb{Z}\) such that \((q,p) = m \cdot (k,l)\).

We now have the following proposition.

**Proposition 7.2.** There exists \(Z : \mathbb{T}_{h/2} \times D(r_0, \hat{K}^{-1}/2) \to \mathbb{C}\) holomorphic such that
\[
\tilde{f}_Z \circ f_{\Omega + F} \circ f_Z^{-1} = f_{\Omega + \hat{F}}
\]
where \(\hat{F}(\theta, r) = O(F)\) can be written \(F = \tilde{F}_{\text{per}} + \tilde{F}_{\text{aper}}\) with \(\tilde{F}_{\text{per}}\) \(1/q\)-periodic
\[
\tilde{F}_{\text{per}}(\theta + q^{-1}, r) = \tilde{F}_{\text{per}}(\theta, r)
\]
and
\[
\tilde{F}_{\text{aper}} = O(e^{-\hat{K}^{(1/2)-\nu}}).
\]

**Proof.** The proof follows a classical linearization procedure. Observe that here we can eliminate non-resonant terms up to order \(\hat{N}\). We construct inductively sequences \(Y_k, F_k = F_k^{\text{res}} + F_k^{\text{nr}}\) such that
\[
\tilde{f}_{Y_k} \circ f_{\Omega + F_k^{\text{nr}} + F_k^{\text{res}}} \circ f_{Y_k}^{-1} = f_{\Omega + F_{k+1}^{\text{nr}} + F_{k+1}^{\text{res}}}
\]
where \(F_k^{\text{res}}\) is \(1/q\)-periodic and \(\|F_k^{\text{nr}}\|_{h_{k,\rho_k}}\) will decrease rapidly.

When \(g : \mathbb{R}/\mathbb{Z} \times D \to \mathbb{R}\) we define \(\hat{g}(n, r) = \int_{\mathbb{T}} g(\theta, r) e^{-2\pi i n \theta} d\theta\), \(T_N g(\theta, r) = \sum_{|n| \leq N} e^{2\pi i n \theta} \hat{g}(n, r)\), \(T_N^{\text{res}} g(\theta, r) = \sum_{|n| \leq N} e^{2\pi i n \theta} \hat{g}(n, r)\), \(T_N^{\text{nr}} = T_N g - T_N^{\text{res}} g\). We thus define
\[
F_k^{\text{res}} = T_N^{\infty} F_k, \quad F_k^{\text{nr}} = F_k - F_k^{\text{res}}.
\]

If \(Y_k\) is the solution of
\[
[\Omega] \cdot Y_k = -T_N F_k^{\text{nr}}
\]
we have from Proposition 4.3
\[ f_{Y_k} \circ f_{\Omega + F_k} \circ f_{\bar{Y}_k}^{-1} = f_{\Omega + F_{k+1}} \]
with
\[ F_{k+1} = F_k + [\Omega] \cdot Y_k + D_2(Y_k, F_k) \]
\[ = F_{nr}^k + F_{res}^k + [\Omega] \cdot Y_k + D_2(Y_k, F_k) \]
\[ = R_{\bar{N}} F_{nr}^k + F_{res}^k + D_2(Y_k, F_{nr}^k + F_{res}^k) \]
where \( D_2(Y_k, F_k) = \hat{\Omega}_2(Y_k, F_k) \). We now define
\[ F_{res}^k = F_{nr}^k + T^\infty D_2(Y_k, F_k) \]
and
\[ F_{nr}^k = R_{\bar{N}} F_{nr}^k + T^\infty D_2(Y_k, F_k) \]
One has
\[ \|Y_k\|_{h_k, e^{-\delta} \rho_k} \leq \hat{K} \bar{N} \|F_{nr}^k\|_{h, \rho} \]
and
\[ \|R_{\bar{N}} F_{nr}^k\|_{h-\delta, \rho} \leq \delta^{-2} e^{-2\pi \bar{N} \delta} \|F_{nr}^k\|_{h, \rho} \]
On the other hand for \(* = nr, res\)
\[ \|T^a D_2(Y_k, F_k)\|_{h-\delta, \rho} \leq \delta^{-1} \|D_2(Y_k, F_k)\|_{h-\delta/2, \rho} \]
\[ \leq \delta^{-1} \delta^{-\alpha} \rho^{-\alpha} (\|Y_k\|_{h, \rho}^2 + \|Y_k\|_{h, \rho} \|F_k\|_{h, \rho}) \]
so that
\[ \|F_{nr}^k\|_{h_{k+1}, \rho_{k+1}} \leq \delta_k^{-2} e^{-2\pi \bar{N} \delta_k} \|F_{nr}^k\|_{h_k, \rho_k} \]
We choose \( \delta_k = c/(k \ln(k)^2) \) so that \( \sum_{k=1}^{\infty} \delta_k = 1/2 \). If \( \sigma_k = \|F_{nr}^k\|_{h_k, \rho_k} \) we then have as long as
\[ \|F_{res}^k\|_{h_k, \delta_k} \leq 2\varepsilon, \quad \rho_k \geq \hat{K}^{-1/2} \]
the inequalities
\[ \sigma_{k+1} \leq A_k \sigma_k, \quad A_k = \delta_k^{-b} (e^{-2\pi \bar{N} \delta_k} + \varepsilon \hat{K}^{-b+4}). \]
As long as
\[ e^{-2\pi \bar{N} \delta_k} \leq \varepsilon \hat{K}^{-b+4} \]
we then have
\[ \sigma_k \leq k!^{b+1} (\varepsilon \hat{K}^{-b+4})^k \]
Notice that (7.165) is satisfied for \( 0 \leq k \leq k_{max} = [(2\pi \bar{N})/\ln(\varepsilon \hat{K}^{-b+4})]^{1-\nu/2} \)
so
\[ \sigma_{k_{max}} \leq k_{max}!^{b+1} (\varepsilon \hat{K}^{-b+4})^k \]
\[ \leq \exp(- \bar{N}^{1-\nu/2} \ln(\varepsilon \hat{K}^{-b+4})^{\nu/2}) \].
To conclude, we define
\begin{equation}
\tilde{F}_{\text{per}} = F_{k_{\text{max}}}^{\text{res}}, \quad \tilde{F}_{\text{nper}} = F_{k_{\text{max}}}^{\text{nres}}.
\end{equation}

**Remark 7.1.** We notice that
\begin{equation}
\tilde{F}_{\text{per}} = T_{\omega}^{\text{res}} F + O_2(F).
\end{equation}

### 7.2. Approximation by a Hamiltonian System.

**Proposition 7.3.** There exists $\Pi$ such that on on $\mathbb{T}_{\frac{1}{2}} \times D(r_0, \hat{K}^{-1/4})$
\begin{equation}
f_{\Omega + \hat{F}} = f_{\hat{F}_2} \circ \Phi_{\Pi} \circ R_{p/q}
\end{equation}
where $\Pi(\theta, r_0 + r) = cr^2 + O(r^3) + r^2 \varphi(\theta, r)$ with $\varphi$ $(1/q)$-periodic in $\theta$ (or equivalently $\Pi \circ R_{p/q} = \Pi$) satisfying $\varphi = O(F)$ and $\hat{F}_2 = O(e^{-\hat{K}^{1/20}})$.

**Proof.** We assume to alleviate the notations $r_0 = 0$. Let $\Omega(r) = \Omega(r) - \langle \nabla \Omega(0), r \rangle = \Omega(r) - (p/q)r$. We write
\begin{equation}
f_{\Omega + \hat{F}} = f_{\hat{F}_{nper} + \hat{F}_{\text{per}}} \circ f_{\Omega} = f_{\Omega}(\hat{F}_{nper}) \circ f_{\hat{F}_{\text{per}}} \circ f_{\hat{F}_2} \circ R_{p/q}.
\end{equation}

Since
\begin{equation}
\|\hat{F}_{\text{per}}\|_{h, \rho} \leq \hat{K}^{-10}
\end{equation}
applying Lemma B.2 we see that there exists $\Pi$ which is $1/q$-periodic and $F_2$ such that on $\mathbb{T}_{\frac{1}{2}} \times D(r_0, \hat{K}^{-1/4})$
\begin{equation}
\Pi = \Omega + \hat{F}_{\text{per}} + O(\rho \hat{F}_{\text{per}}),
\end{equation}
\begin{equation}
f_{\hat{F}_{\text{per}}} \circ \Phi_{\Pi} = f_{F_2} \circ \Phi_{\Pi}, \quad \|F_2\|_{h/8, \hat{K}^{-1/8}} = O(e^{-\hat{K}^{1/20}}).
\end{equation}

We then have
\begin{align*}
f_{\Omega + F} &= f_{\Omega}(\hat{F}_{nper}) \circ f_{F_2} \circ \Phi_{\Pi} \circ R_{p/q} \\
&= f_{\hat{F}_2} \circ \Phi_{\Pi} \circ R_{p/q}.
\end{align*}

**Corollary 7.4.** If $c \leq D^2 \Omega(\cdot)$ then for all $\theta \in \mathbb{T}_h$, $c \leq D^2 \Pi(\theta, r_0)$.

**Proof.** This is a consequence of the last statement of Lemma B.2 of the Appendix: $D^2 \Pi(\theta, r_0) = D^2 F(\theta, r) + D^2 \Omega(r_0) + O(\|\nabla F\|_{h} \|\nabla \Omega\|_{\rho})$. \qed
7.3. Further normal form: Hamilton–Jacobi. Let $\Pi : (\theta, r) : (\mathbb{C}/q \mathbb{Z}) \times D(r_0, K^{-1}/4) \to \mathbb{C}$ as before and define $\Pi_q : (\mathbb{C}/q \mathbb{Z}) \times qD(0, K^{-1}/4) \to \mathbb{C}$ by $\Pi_q(\tilde{\theta}, \tilde{r}) = q^2 \Pi(\theta, r_0 + r)$ where $\tilde{\theta} = q\theta, \tilde{r} = qr$. The Hamilton–Jacobi equations associated to $\Pi_q$ with respect to the canonical symplectic structure $d\tilde{\theta} \wedge d\tilde{r}$ on $\Pi_q, \mathbb{T}_q \times qD(0, K^{-1}/4)$ are equivalent to the ones associated to $\Pi$ and $d\theta \wedge dr$:

$$
\begin{align*}
\dot{\tilde{\theta}} &= \tilde{\chi}_r \Pi_q(\tilde{\theta}, \tilde{r}) \\
\dot{\tilde{r}} &= -\tilde{\chi}_\theta \Pi_q(\tilde{\theta}, \tilde{r}) \\
\iff \quad \dot{\theta} &= \chi_r \Pi(\theta, r) \\
\dot{r} &= -\chi_\theta \Pi(\theta, r).
\end{align*}
$$

From Corollary 7.4 we can assume that $c := D^2\Pi(0, \cdot)$ satisfies $c \approx 1$. We have

$$
\Pi(\theta, r) = cr^2 + O(r^3) + \tilde{f}_0(q\theta) + \tilde{f}_1(\theta)r + r^2 \tilde{f}(\theta, r)
$$

where $\tilde{f}_0, \tilde{f}_1, \tilde{f}$ are in $\mathcal{F}_{h/4, K^{-1}/4}$ and $1/q$-periodic in the $\theta$-variable.

$$
\max_{C_{h/4} \times D(0, K^{-1}/4)} (|\tilde{f}_0|, |\tilde{f}_1|, |\tilde{f}|) \lesssim \hat{K}^a \varepsilon
$$

so that

$$
\Pi_q(\tilde{\theta}, \tilde{r}) = q^2(c(\tilde{r}/q)^2 + O((\tilde{r}/q)^3) + \tilde{f}_0(\tilde{\theta}/q) + \tilde{f}_1(\tilde{\theta}/q)(\tilde{r}/q) + (\tilde{r}/q)^2 \tilde{f}(\tilde{\theta}/q, \tilde{r}/q))
$$

$$
= cr^2 + q^{-1}O(\tilde{r}^3) + q^2 \tilde{f}_0(\tilde{\theta}/q) + q \tilde{f}_1(\tilde{\theta}/q) \tilde{r} + \tilde{r}^2 \tilde{f}(\tilde{\theta}/q, \tilde{r}/q)
$$

$$
= cr^2 + q^{-1}O(\tilde{r}^3) + f_0(\tilde{\theta}) + f_1(\tilde{\theta}) \tilde{r} + \tilde{r}^2 \tilde{f}(\tilde{\theta}, \tilde{r})
$$

with $f_0, f_1, \tilde{f} \in \mathcal{F}_{h/4, \rho_q}$ such that

$$
\max_{\mathbb{T}_{qh/4} \times D(0, \rho_q)} (|f_0|, |f_1|, |\tilde{f}|) \lesssim \hat{K}^a \varepsilon
$$

where

$$
\rho_q = q\hat{K}^{-1}/4.
$$

To simplify the notations we assume that $c = 1$. We then write

$$
\Pi_q(\theta, r) = r^2 + O(r^3) + f_0(\theta) + f_1(\theta)r + r^2 \tilde{f}(\theta, r)
$$

$$
= (1 + f_2(\theta))r^2 + f_0(\theta) + f_1(\theta)r + O(r^3) + r^3 \tilde{f}(\theta, r)
$$

$$
= (1 + f_2(\theta))r^2 + f_0(\theta) + f_1(\theta)r + r^2 f(\theta, r)
$$

so that

$$
\Pi_q(\theta, r) = (1 + f_2(\theta))(r - c_0(\theta))^2 - c_1(\theta) + r^2 f(\theta, r)
$$
with
\[
e_0(\theta) = \frac{1}{2} \frac{f_1(\theta)}{1 + f_2(\theta)}, \quad e_1(\theta) = -f_0(\theta) + \frac{1}{4} \frac{f_1(\theta)^2}{1 + f_2(\theta)}
\]
with
\[
\varepsilon := \max(\|f_0\|, \|f_1\|, \|f\|_{\lambda, \rho_q}) < q \hat{K}^{-1}/10.
\]
Since \( \Pi_q \) is defined up to an additive constant we can assume
\[
\int_T (1 + f_2(\theta))^{-1/2} f_0(\theta) d\theta = \int_T (1 + f_2(\theta))^{-1/2} \frac{f_1(\theta)^2}{4(1 + f_2(\theta))} d\theta
\]
hence
\[
\int_T (1 + f_2)^{-1/2} e_1 = 0.
\]
We denote by
\[
\varepsilon_0 = \|e_0\|_{C^0}, \quad \varepsilon_1 = \|e_1\|_{C^0},
\]
and we introduce
\[
\lambda = L \varepsilon_1^{1/2}, \quad L >> 1.
\]

**Notation:** For \( 0 < a_1 < a_2 \) and \( z \in \mathbb{C} \) we denote by \( \Lambda(z; a_1, a_2) \) the annulus centered at \( z \) with inner and outer radii of sizes respectively \( \sim a_1 \) and \( \sim a_2 \).
When \( z = 0 \) we simply denote this annulus by \( \Lambda(a_1, a_2) \)

**Lemma 7.5.** There exists a holomorphic function \( g \) defined on \( T_{\hat{q}h} \times \Lambda(\lambda, \rho_q) \) such that one has
\[
\Pi_q(\theta, g(\theta, z)) = z^2.
\]
and
\[
(9/10)|z| \leq |g(\theta, z)| \leq (11/10)|z|.
\]

**Proof.**
We refer to the Appendix for the definition and properties of \( z \mapsto (z^2 + a)^{1/2} \) on \( \mathbb{C} \setminus \{|z| > |a|^{1/2}\} \).
We assume \( \varepsilon_0 \) and \( \varepsilon_1 \) small enough so that \( \lambda < 1/10 \). The classical fixed point theorem (with parameter \( z \)) shows that for \( z \) in the annulus
\[
\Lambda(\lambda, \rho_q) = \{z \in \mathbb{C}, \lambda/8 \leq |z| \leq \rho_q\}
\]
the map
\[
g \mapsto e_0 + \frac{1}{(1 + f_2)^{1/2}} (z^2 + e_1 - g^3 f(\cdot, g))^{1/2}
\]
is contracting on the unit ball of the set \( F(T_{\hat{q}h}/4 \times D(0, \rho_q)) \) and consequently admits a unique fixed point \( g \) which depends holomorphically on \( z \). This \( g \) solves
\[
z^2 = (1 + f_2(\theta)) g(\theta, z)^2 + f_0(\theta) + f_1(\theta) g(\theta, z) + O(g(\theta, z)^3)
\]
Lemma 7.6. There exists a solution $\mathbf{z}$ provided for all $A$ (7.197) or $p$ (7.200) $\Gamma$ (7.196)

which we write

$$z^2 = (1 + f_2(\theta))(g(\theta, r) - e_0(\theta))^2 - e_1(\theta) + g(\theta, r)^3 f(\theta, g(\theta, r))$$

or

$$g(\theta, r) = e_0(\theta) + \frac{1}{(1 + f_2(\theta))^{1/2}}(z^2 + e_1(\theta) + g(\theta, r)^3 f(\theta, g(\theta, r)))^{1/2}.$$

In other words

$$\Pi_p(\theta, g(\theta, z)) = z^2.$$

We notice that for $\mathbf{z} \in A_{\lambda, \rho_q}$, $\varphi \in \mathbb{T}_{qh}$

$$|g(\varphi, z)| \leq (11/10)|z|.$$

We now define the function $\Gamma \in \mathcal{F}(A(\lambda, \rho_q))$ by $\Gamma : A(\lambda, \rho_q) \to \mathbb{C}$

$$\Gamma(u) = \int_0^1 g(\varphi, u)d\varphi.$$

Lemma 7.6. There exists a solution $h \in \mathcal{F}(A(2\lambda, \rho_q/2))$ of the equation

$$\Gamma(h(z)) = z.$$

Proof. For $\mathbf{t} \in \mathbb{D}$ let $g_t$ be the unique solution of

$$z^2 = (1 + f_2(\theta))g_t(\theta, z)^2 + tf_0(\theta) + tf_1(\theta)g_t(\theta, z) + tO(g_t(\theta, z)^3)$$

$$z^2 = (1 + f_2(\theta))(g_t(\theta, z) - e_0(\theta))^2 - te_1(\theta) + t g_t(\theta, z)^3 f(\theta, g_t(\theta, z))$$

$$g_t = e_0 + \frac{1}{(1 + f_2)^{1/2}}(z^2 + te_1 - tg_t^3 f(\cdot, g_t))^{1/2}.$$

This $g_t$ depends holomorphically on $\mathbf{t} \in \mathbb{D}$ and the same is true for $\Gamma_t := \int_0^1 g_t(\varphi, \cdot)d\varphi$. For $\mathbf{z} \in \mathbb{C}$, the map $\mathbb{C} \ni t \mapsto \deg(\Gamma_t, \mathbb{A}(\lambda, \rho_q), z)$ is constant provided for all $\mathbf{t} \in \mathbb{D}$, $\mathbf{z} \notin \Gamma_t(\mathbb{A}(\lambda, \rho_q))$, a condition that is satisfied if $\mathbf{z} \in \mathbb{A}(2\lambda, \rho_q/2)$ (since $\Gamma_t(\mathbf{r})$ as well as $g_t(\mathbf{r})$ compares with $\mathbf{r}$ for $\mathbf{r} \in \mathbb{A}(2\lambda, \rho_q/2)$). But for $\mathbf{t} = 0$ and $\mathbf{z} \in \mathbb{A}(2\lambda, \rho_q/2)$ this degree is equal to 1. This implies that $\Gamma$ is injective on $\mathbb{A}(2\lambda, \rho_q/2)$ and contains $\mathbb{A}(2\lambda, \rho_q/2)$ in its image.

Proposition 7.7. There exists a symplectic change of coordinates $W_q$ defined on $\mathbb{A}(2\lambda, \rho_q/2)$ such that

$$W_q \circ \Phi_{\Pi_q} \circ W_q^{-1} = \Phi_R.$$

Proof. Let $h$ be the function defined by the previous lemma and define for $\mathbf{z} \in \mathbb{A}(2\lambda, \rho_q/2)$ and $\theta \in \mathbb{T}_{qh}$

$$S(\theta, z) = \int_0^\theta g(\varphi, h(\mathbf{z}))d\varphi.$$

We have

$$\hat{c}_\theta S(\theta, z) = g(\theta, h(\mathbf{z}))$$
and so $S$ is a solution of the Hamilton-Jacobi equation

$$\Pi_q(\theta, \frac{\partial S}{\partial \theta}(\theta, z)) = \Pi_q(\theta, g(\theta, h(z)))$$

(by (7.191)).

Since by definition of $h$ (cf. lemma 7.3)

$$\int_0^1 g(\varphi, h(z))d\varphi = \Gamma(h(z)) = z$$

we see that

$$\partial_z S(\theta + 1, z) - \partial_z S(\theta, z) = \partial_z (S(\theta + 1, z) - S(\theta, z))$$

$$= \partial_z \int_\theta^{\theta+1} g(\varphi, h(z))dz$$

$$= \partial_z \int_0^1 g(\varphi, h(z))dz$$

$$= \partial_z \Gamma(h(z))$$

and so $\partial_z S(\theta, z) - \theta$ is 1-periodic in the variable $\theta$. If the change of variable

$$W_q : (\theta, w) \mapsto (\varphi, z) \iff \begin{cases} w = \frac{\partial S}{\partial \theta} \\ \varphi = \frac{\partial S}{\partial z} \end{cases}$$

is well defined, then it is symplectic and conjugates $\Phi_{\Pi(\theta, w)}$ to $\Phi_{h(z)^2}$. The fact that it is well defined follows from the following. Indeed, (7.217) amounts to

$$W_q : (\theta, w) \mapsto (\varphi, z) \iff \begin{cases} w = g(\theta, h(z)) \\ \varphi = \frac{\partial S}{\partial z}(\theta, z). \end{cases}$$

So given $(\theta, w)$ we can find by the first equation, $u$ solution of $w = g(z, u)$, then set $z = \Gamma(u)$ and $\varphi$ is determined by the second equation. Conversely, given $(\varphi, z)$ we can determine first $\theta$ (if $L$ in (7.190) is chosen big enough) and then $w$.

**Corollary 7.8.** There exists a symplectic change of coordinates $W_{in}$ such that on $A(r_0; 2q\lambda, q\rho, q/2)$

$$W_{in} \circ \Phi_{\Pi(\cdot, r_0+\cdot)} \circ W_{in}^{-1} = \Phi_{h^2(q)}$$

and which satisfies

$$W_{in} \circ R_{p/q} = R_{p/q} \circ W_{in}.$$ 

**Proof.** The symplectic diffeomorphism $W$ on $(C_h/[\frac{1}{q}Z]) \times D(r_0, \tilde{K}^{-1}/4)$ defined by $W = \Lambda \circ W_q \circ \Lambda^{-1}$ where $\Lambda(\theta, r) = (q\theta, qr)$ is $(1/q)$-periodic in $\theta \in C_h$ and thus commutes with $R_{p/q}$. 

\[ \square \]
Corollary 7.9. Let \( T(r_0 + r) = h^2(qr) + (p/q)r \). One has on the annulus \( A(r_0; 2q\lambda, q\rho q/2) \)
\[
W_{in} \circ f^{-1} \circ \Phi \circ W^{-1} = f_{F_2} \circ \Phi_T
\]
with \( F_2 = O(e^{-K^{1/20}}) \).

7.4. Obstruction to extending the linearizing map inside the hole. In general the maps \( g, \Gamma, h \) are not holomorphic on the whole disk \( qD \). In this sub-section we quantify to which extent the domains of holomorphy of these maps can be extended.

We notice that for \( z \in A(\lambda/2, \lambda/4) \), \(|g(\theta, z)|\) compares to \( \lambda \) and that
\[
z^2 = (1 + f_2)(g - e_0)^2 - e_1 + O(\bar{z}g^3)
\]
so that
\[
g = e_0 + \frac{1}{(1 + f_2)^2}(z^2 + e_1 + O(\bar{z}g^3))^{1/2}
\]
\[
= e_0 + \frac{1}{(1 + f_2)^2}(z^2 + e_1)^{1/2} + O(\bar{z}\lambda^2).
\]

Let’s introduce
\[
\tilde{g} = e_0 + \frac{1}{(1 + f_2)^2}(z^2 + e_1)^{1/2}
\]
\[
\tilde{\Gamma}(\cdot) = \int_\mathbb{Z} \tilde{g}(\theta, \cdot) d\theta, \quad \tilde{h} = \tilde{\Gamma}^{-1}
\]
where the inverse is with respect to composition. The functions \( \tilde{\Gamma} \) and \( \tilde{h} \) are defined on \( \{ z \in \mathbb{C}, L_\epsilon^{1/2} < |z| \} \) for some fixed \( L >> 1 \).

Notation: In the following we denote by \( C(0, t) \) the circle of center 0 and radius \( s > 0 \).

Proposition 7.10. If there exists a holomorphic function \( \tilde{\Xi} \) defined on \( D(0, \rho_q) \) such that
\[
\| \tilde{\Xi} - h^2 \|_{C(0, \rho_q/2)} \leq \nu
\]
then
\[
\| e_1 \|_{C^0(\mathbb{T})} \leq \nu^{1/6} L^{-1} + h^{-1} e^{-h/(2L^2\nu)}.
\]

7.4.1. Computation of a residue.

Lemma 7.11. For any circle \( C(0, t) \) centered at 0 with \( t \geq \lambda \) one has
\[
\frac{1}{2\pi i} \int_{C(0, t)} z^2 \tilde{h}(z)^2 dz = (1/8) \left( \int_\mathbb{T} (1 + f_2)^{-1/2} \right)^2 \int_\mathbb{T} (1 + f_2)^{-1/2} e_1^2.
\]
Proof. We compute the expansion of $\tilde{g}$ into Laurent series:

$$\tilde{g} = e_0 + \frac{1}{(1 + f_2)^{1/2}} z(1 + z^{-2}e_1)^{1/2}$$

(7.229)

$$= e_0 + \frac{1}{(1 + f_2)^{1/2}} z(1 + \frac{1}{2} \frac{e_1}{z^2} - \frac{1}{8} \frac{e_1^2}{z^4} + \left( \frac{1}{2} \right) \frac{e_1^3}{3} \frac{1}{z^6} + \left( \frac{1}{2} \right) \frac{e_1^4}{4} \frac{1}{z^8} + O(z^{-10}))$$

(7.230)

$$= e_0 + \frac{1}{(1 + f_2)^{1/2}} (z + \frac{1}{2} \frac{e_1}{z} - \frac{1}{8} \frac{e_1^2}{z^3} + O(z^{-5}))$$

(7.231)

As a consequence since $\tilde{\Gamma}(z) = \int_0^1 \tilde{g}(\theta, z) d\theta$ we have with the notation $\gamma = \int_\mathbb{T} (1 + f_2)^{-1/2}$ the identity

$$\tilde{\Gamma}(z) = \gamma (z + \sum_{k=-4}^{0} a_k z^{-k}) + O(z^{-5})$$

(7.232)

where

$$a_0 = \gamma^{-1} \int_\mathbb{T} e_0, \quad a_{-1} = \gamma^{-1} (1/2) \int_\mathbb{T} (1 + f_2)^{-1/2} e_1$$

(7.233)

$$a_{-2} = 0, \quad a_{-3} = \gamma^{-1} (-1/8) \int_\mathbb{T} (1 + f_2)^{-1/2} e_1^2, \quad a_{-4} = 0.$$  

(7.234)

We can thus write

$$\tilde{\Gamma} = \Lambda_\gamma \circ (id + u)$$

where $\Lambda_\gamma z = \gamma z$ and

$$u(z) = \sum_{k=-3}^{0} a_k z^{-k} + O(z^{-5}).$$

(7.235)

By our choice [(**)] we have

$$a_{-1} = \int_\mathbb{T} (1 + f_2)^{-1/2} e_1 = 0.$$  

(7.236)

We now find the expansion into Laurent series of $v = \sum_{k=-3}^{0} b_k z^{-k}$ defined by $id + v := (id + u)^{-1}$. Since $u^{(k)}(z) = O(z^{-(k+1)})$, we have

$$v = - u \circ (id + v)$$

$$= - u(z) - u'(z)v(z) - (1/2)u''(z)v(z)^2 - (1/6)u'''(z)v(z)^3 + O(z^{-5})$$

$$= - u(z) - u'(z)(-u(z) - u'(z)v(z + O(z^{-3})) - (1/2)u''(z)(-u(z) + O(z^{-2}))^2$$

$$- (1/6)u'''(z)v(z)^3 + O(z^{-5})$$

$$= - u(z) - u'(z)(-u(z) - u'(z)u(z)) - (1/2)u''(z)(-u(z))^2 + O(z^{-5})$$

$$= - u(z) + u'(z)u(z) + u'(z)^2 u(z) - (1/2)u''(z)u(z)^2 + O(z^{-5})$$

(7.237)
so that
\[(7.238) \quad \sum_{k=-3}^{0} b_k z^{-k} = -(a_0 + a_{-1}z^{-1} + a_{-3}z^{-3} + a_{-4}z^{-4})
+ (a_0 + a_{-1}z^{-1} + a_{-3}z^{-3})(-a_{-1}z^{-2} - 3a_{-3}z^{-4}) + a_0a_{-1}^2z^{-4}
- (1/2)(a_0 + a_{-1}z^{-1} + a_{-3}z^{-3})^2(2a_{-1}z^{-3}) + O(z^{-5})\]
\[(7.239) \quad \sum_{k=-3}^{0} b_k z^{-k} = (-a_0) + (-a_{-1})z^{-1} + (-a_0a_{-1})z^{-2} + (-a_{-3} - a_{-1}^2 - a_0^2a_{-1})z^{-3}
- a_0a_{-1}^2z^{-4} + O(z^{-5})\]
so
\[(7.240) \quad b_0 = -a_0, \quad b_{-1} = -a_{-1}, \quad b_{-2} = -a_0a_{-1}, \quad b_{-3} = -a_{-3} - a_{-1}^2 - a_0^2a_{-1}\]
\[(7.241) \quad b_{-4} = -a_0a_{-1}^2.\]

Also,
\[(7.242) \quad (z + v(z))^2 = (z + b_0 + b_{-1}z^{-1} + b_{-2}z^{-2} + b_{-3}z^{-3} + b_{-4}z^{-4})^2 + O(z^{-4})\]
\[(7.243) \quad = z^2 + 2b_0z + (b_0^2 + 2b_{-1}) + 2(b_{-2} + b_0b_{-1})z^{-1}
+ (b_{-1}^2 + 2b_{-3} + 2b_0b_{-2})z^{-2} + 2(b_0b_{-3} + b_{-1}b_{-2} + b_{-4})z^{-3} + O(z^{-4}).\]
We have
\[(7.245) \quad 2(b_{-2} + b_0b_{-1}) = 2(-a_0a_{-1} + (-a_0(-a_{-1}))
= 0\]
\[(7.246) \quad (b_{-1}^2 + 2b_{-3} + 2b_0b_{-2}) = a_{-1}^2 + 2(-a_{-3} - a_{-1}^2 - a_0^2a_{-1}) + 2a_0^2a_{-1}\]
\[(7.247) \quad = -a_{-1}^2 - 2a_{-3} - 2a_0^2a_{-1} + 2a_0^2a_{-1}\]
\[(7.248) \quad = -2(a_{-1}^2 + a_{-3})\]
\[(7.249) \quad 2(b_0b_{-3} + b_{-1}b_{-2} + b_{-4}) = 2(-a_0(-a_{-3} - a_{-1}^2 - a_0^2a_{-1}) + a_{-1}^2a_0 - a_0a_{-1}^2)
= 2a_0(a_{-3} + a_{-1}^2 + a_0^2a_{-1}).\]
Finally,
\[(7.250) \quad (z + v(z))^2 = z^2 - 2a_0z + a_0^2 - a_{-3}z^{-3} + O(z^{-4}).\]
Now if $\tilde{h}$ is the inverse of $\tilde{G}$, $z = \tilde{G} \circ \tilde{h}$ we have $\tilde{h} = (id + u)^{-1} \circ \Lambda^{-1}$ and we get

$$\tilde{h}(z)^2 = \gamma^{-2}z^2 - 2a_0\gamma^{-1} + a_0^2 - a_{-3}\gamma^3 z^{-3} + O(z^{-4}).$$

(7.253)

We observe that for any circle $C(0, t)$ centered at 0, $t \geq \lambda$:

$$\frac{1}{2\pi i} \int_{C(0, t)} z^2 \tilde{h}(z)^2 dz = -a_{-3}\gamma^3$$

(7.254)

$$= (\gamma^2/8) \int_{\mathbb{T}} (1 + f_2)^{-1/2} e_1^2.$$

(7.255)

7.4.2. Proof of Proposition 7.10

Lemma 7.12. Let $\lambda \geq L\varepsilon_1^{1/2}$. One has for $z \in A(\lambda/4, \lambda/2)$

$$|h(z)^2 - \tilde{h}(z)^2| \lesssim \lambda^3.$$  

(7.256)

Proof. For $z \in A(\lambda/4, \lambda/2)$, $\theta \in \mathbb{T}$ one has

$$|g(\theta, z) - \tilde{g}(\theta, z)| \lesssim \varepsilon\lambda^2$$

(7.257)

so

$$|\Gamma(z) - \tilde{\Gamma}(z)| \lesssim \varepsilon\lambda^2$$

(7.258)

On the other hand, from Lemma C.1 hence

$$e^{-L} \leq \left| \frac{\tilde{g}(\theta, z) - \tilde{g}(\theta, z')} {z - z'} \right| \leq e^L$$

(7.259)

and

$$e^{-L} \leq \left| \frac{\tilde{\Gamma}(z) - \tilde{\Gamma}(z')} {z - z'} \right| \leq e^L.$$

(7.260)

Since $z = \Gamma(h(z)) = \tilde{\Gamma}(\tilde{h}(z))$ one has

$$|\tilde{\Gamma}(h(z)) - \tilde{\Gamma}(\tilde{h}(z))| \lesssim \varepsilon\lambda^2$$

(7.261)

and so

$$|\tilde{h}(z) - h(z)| \lesssim \varepsilon\lambda^2.$$  

(7.262)

We conclude

$$|\tilde{h}(z)^2 - h(z)^2| \lesssim \varepsilon\lambda^3.$$  

(7.263)

\[ \square \]

We recall that $\varepsilon_1 = \|e_1\|_{C^0(\mathbb{T})}$. The function $\tilde{\Xi} - h^2$ satisfies

$$\|\tilde{\Xi} - h^2\|_{C(0,\rho/2)} \leq \nu,$$

(7.264)

$$\|\tilde{\Xi} - h^2\|_{C(0, L\varepsilon_1^{1/2})} \leq 1.$$
Let $M > 1$ and $\lambda_M := (\rho_0/2)^{1/M}(L\varepsilon_1^{1/2})^{1-1/M} \leq (L\varepsilon_1^{1/2})^{1-1/M}$ (we can assume $\rho_0 \leq 1$). By the three circles theorem,
\begin{equation}
\|\tilde{\Xi} - h^2\|_{C(0, \lambda_M)} \leq \nu^{1/M}
\end{equation}
Lemma 7.12 tells us that
\begin{equation}
\|\tilde{\Xi} - \tilde{h}^2\|_{C(0, \lambda_M)} \leq \nu^{1/M} + \varepsilon \lambda_M^3
\end{equation}
hence for any $z$ in the circle $C(0, \lambda_M)$
\begin{equation}
|z^2\tilde{\Xi}(z) - z^2\tilde{h}^2(z)| \leq (\nu^{1/M} + \varepsilon \lambda_M^3)\lambda_M^2
\end{equation}
and
\begin{equation}
\left|\frac{1}{2\pi i} \int_{C(0, \lambda_M)} z^2\tilde{\Xi}(z) - z^2\tilde{h}^2(z)dz\right| \leq \lambda_M^3(\nu^{1/M} + \varepsilon \lambda_M^3).
\end{equation}
Since $z^2\tilde{\Xi}$ is holomorphic on $D(0, 2\lambda_M)$, $\int_{C(0, \lambda_M)} z^2\tilde{\Xi}(z)dz = 0$ and by Lemma 7.11 we get
\begin{equation}
\int_T (1 + f_2)^{-1/2}e_1^2 \leq \lambda_M^3(\nu^{1/M} + \varepsilon \lambda_M^3).
\end{equation}
This gives
\begin{equation}
(1 - \varepsilon_2)\int_T e_1^2 \leq \lambda_M^3(\nu^{1/M} + \varepsilon \lambda_M^3)
\end{equation}
hence
\begin{equation}
\|e_1\|_{L^2(T)} \leq \lambda_M^{3/2}\nu^{1/2} + \varepsilon^{1/2}\lambda_M^3
\end{equation}
\begin{equation}
\leq L^{(3/2)(1-1/M)}\varepsilon_1^{(3/4)(1-1/M)}\nu^{1/2} + \varepsilon^{1/2}L^{3(1-1/M)}\varepsilon_1^{3/2}(1-1/M)\nu^{1/2}L^{3(1-1/M)}\varepsilon_1^{(3/2)(1-1/M)}
\end{equation}
If we choose $M = 3$ we get (recall that $\varepsilon_1 \leq \varepsilon$)
\begin{equation}
\|e_1\|_{L^2(T)} \leq L\varepsilon^{1/2}\nu^{1/6} + L^2\varepsilon^{1/2}\|e_1\|_{C^0(T)}
\end{equation}
and using Lemma C.32
\begin{equation}
\|e_1\|_{C^0(T)} \leq \nu^{1/6}L^{-1} + h^{-1}e^{-h/(2L\varepsilon)}.
\end{equation}
This completes the proof of Proposition 7.10.

\section{Comparing the Various Normal Forms}

\textbf{Proposition 8.1.} Let $0 < \rho_1 < \rho_2$ and denote by $A$ or $A(\rho_1, \rho_2)$ some annulus $\mathbb{A}(w; \rho_1, \rho_2)$. Assume that for $j = 1, 2$ there exist $\hat{\Omega}, \hat{\Omega}_j, \in \hat{\mathcal{F}}_A$, $W_j, F, F_j \in \hat{\mathcal{F}}_{h,A},$ such that
\begin{enumerate}
\item[(1)] on $\mathbb{T}_h \times A$ one has
\begin{equation}
f_{W_j} \circ f_{\hat{\Omega} + F} \circ f_{W_j}^{-1} = f_{\hat{\Omega} + F_j}, \quad j = 1, 2
\end{equation}
with
\begin{equation}
\max_{j=1,2}(\|F_j\|_{h,A}) = \nu
\end{equation}
\end{enumerate}
Then there exists $a \geq 1$ such that if $0 < \rho_1 + \nu^{1/a} < \rho < \rho_2 - \nu^{1/a}$
(8.277) \[ \| \bar{\mathcal{C}}\Omega_1 - \bar{\mathcal{C}}\Omega_2 \|_{A(\bar{\rho}_1, \bar{\rho}_2)} \leq \rho^{-a} \nu \]
with $\bar{\rho}_1 = \rho_1 + \nu^{1/a}$, $\bar{\rho}_2 = \rho_1 - \nu^{1/a}$ and there exists a twisted rotation \( R_\alpha : (\theta, r) = (\theta + \alpha(r), r) \) \( (\alpha \in F_\mathcal{A}(\bar{\rho}_1, \bar{\rho}_2)) \) of the annulus such that
(8.278) \[ f_{W_2} = f_G \circ f_{W_1} \circ R_\alpha, \quad \| G \|_{h-\delta,A(\bar{\rho}_1, \bar{\rho}_2)} \leq \rho^{-a} \nu. \]

Proof. We denote by $A_\delta$ the annulus $\mathcal{A}(w; e^\delta \rho, e^{-\delta} \rho_2)$
We can write $f_{W_1}^{-1} \circ f_{W_2} = f_Z$ so that If we denote by $F_3$ the exact symplectic diffeomorphism
\[
f_{F_3} = f_{Z}^{-1} \circ f_{F_1}^{-1} \circ f_Z \circ f_{F_2}
\]
one has
(8.279) \[ f_{F_3} \circ \Phi_{\Omega_2} \circ f_{Z}^{-1} = f_{Z}^{-1} \circ \Phi_{\Omega_1} \]
with
(8.280) \[ \| F_3 \|_{h-\delta,A_\delta} \leq (\rho \delta)^{-b} \nu \]
Let us introduce the notations $\omega_i = \nabla \Omega_i$, $i = 1, 2$ and $f_{Z}^{-1}(\theta, r) = (\theta + u(\theta, r), r + v(\theta, r)).$ We have
(8.281) \[ f_{Z}^{-1} \circ \Phi_{\Omega_1} \circ f_{Z}^{-1} = (\theta + \omega_1(r) + u(\theta + \omega_1(r), r), r + v(\theta + \omega_1(r), r)) \]
and
(8.282) \[ \Phi_{\Omega_2} \circ f_{Z}^{-1} = (\theta + u(\theta, r) + \omega_2(r + v(\theta, r)), r + v(\theta, r)). \]
We thus have
(8.283) \[
\begin{cases} 
\omega_2(r + v(\theta, r)) - \omega_1(r) = A + u(\theta + \omega_1(r), r) - u(\theta, r) \\
v(\theta + \omega_1(r), r) - v(\theta, r) = B 
\end{cases}
\]
with $\max(\| A \|_{h-\delta,A_\delta}, \| B \|_{h-\delta,A_\delta}) = O((\rho \delta)^{-b} \nu)$. We observe that since $\bar{\mathcal{C}}\Omega_1 > 1/2$, there exists a set $R \subset A_\delta$ of Lebesgue measure less than $\rho^3$, which is a union of disks centered on the real axis, such that one has for any $r \in A_\delta \setminus R$
and any $k \in \mathbb{Z}^*$
(8.284) \[ \min_{i \in \mathbb{Z}} | \omega_1(r) - \frac{1}{k} | \geq \frac{\rho^3}{k^3} \]
so that the second identity in (8.283) gives for any $r \in A_\delta \setminus R$
(8.285) \[ \| v(\cdot, r) - \int_T v(\theta, r)d\theta \|_{h^{-2\delta}} \leq \delta^{-2} \rho^{-3}(\rho \delta)^{-b} \nu. \]

We now notice that (provided $\rho \leq \delta$) there exists $1 < t < 2$ such that $R \cap \bar{\mathcal{C}}A_\delta = \emptyset$. The maximum principle applied to the holomorphic function $v(\cdot, r) - \int_T v(\theta, r)d\theta$ for any $\varphi \in T$ shows that (8.285) holds for any $r \in A_{2\delta}$. We thus have and so $\| \bar{\mathcal{C}}v \|_{h^{-2\delta},A_{2\delta}} = O((\rho \delta)^{-(4+b)} \nu)$. Now by definition
\[ f_Z^{-1}(\theta, r) = (\varphi, R) \text{ if and only if } r = R + \varphi Z(\theta, R), \text{ and } \varphi = \theta + \varphi Z(\theta, R) \]

so
\[ 0 = v(\theta, r) + \varphi Z(\theta, r + v(\theta, r)). \]

Together with (8.285) this shows that (the value of \( b \) may change in the following)
\[ \| \varphi Z(\theta, r + v(\theta, r)) \|_{\mathcal{H}_{-2\delta,A_{2\delta}}} \lesssim O((\rho \delta)^{-b} \nu) \]
and hence
\[ \| Z(\theta, r + v(\theta, r)) - Z(0, r + v(0, r)) \|_{\mathcal{H}_{-2\delta,A_{2\delta}}} \lesssim O((\rho \delta)^{-b} \nu). \]

Since \( u(\theta, r) = \varphi RZ(\theta, r + v(\theta, r)) \) this shows that
\[ \| u(\theta, r) - u(0, r) \|_{\mathcal{H}_{-2\delta,A_{2\delta}}} \lesssim O((\rho \delta)^{-b} \nu)). \]
Notice that by (8.286) and (8.288) one has (integrate with respect to the \( \theta \) variable)
\[ \| v(\theta, r) \|_{\mathcal{H}_{-2\delta,A_{2\delta}}} \lesssim O((\rho \delta)^{-b} \nu)). \]
Finally, the second equation of (8.283) (and the fact that the first derivative of \( \omega_2 \) is bounded by 1) shows that
\[ |\omega_2(r) - \omega_1(r)| = O((\rho \delta)^{-b} \nu)). \]

The assertion on the conjugations follows from (8.289) and (8.290) with \( \alpha(r) = u(0, r) \).

**Remark:** The last assertion of the previous proposition tells us that the domains where the linearizations hold do match to some very good approximation.

9. **Consequences of the Convergence of the BNF**

Let \( \Omega \in \mathcal{F}_{\rho_0} = \omega_0 r + r^2/2 + O(r^3) \) with \( \omega_0 \) diophantine and \( F \in \mathcal{F}_{h,\rho_0}, F = O(r^3) \). From Proposition 5.4 we know that we can construct domains \( U_i \in \mathcal{U}, \) conjugating maps \( \Omega_i \in \mathcal{F}_U, F_i \in \mathcal{F}_{h_i} \) such that
\[ f_{Y_i} \circ f_{\Omega_i} \circ f_{\Phi_i} = f_{\Omega_{i+1}} \circ f_{\Phi_{i+1}} \]

On the other hand Corollary 7.9 applied to each \( F_i \) shows that for every \( D := D(z_i, K_i^{-1}) \in D(U_i) \setminus D(U_{i+1}) \) one can find a change of coordinates \( \hat{W}_{i,D} \) defined on the annulus \( A_{D,D} := D(z_i, \hat{K}_i^{-1}) \setminus D(z_i, K_i^{-1}) \) such that
\[ \hat{W}_{i,D} \circ f_{\Omega_i} \circ f_{\Phi_i} = f_{Y_{i+1},D,D} \]
with
\[ \| f_{Y_{i+1},D,D} \|_{A(D,D)} = O(\exp(-\hat{K}_i^{-1})). \]

From now on we fix \( \rho \in [0, \rho_0/2] \). We define
\[ i_-(\rho) = \max\{i \geq 0, \ D(0, 2\rho) \cap U_i = D(0, 2\rho)\}. \]
Since \( \delta^2 \Omega \approx 1 \) we notice that Dirichlet approximation theorem implies that
\[
\rho \lesssim (N_{i-}(\rho))^{-2}
\]
and the fact that \( \omega_0 \in DC(\kappa, \tau) \)
\[
(N_{i-}(\rho))^{-1(1+\tau)} \lesssim \rho.
\]
There is thus \( t := t(\rho) \) such that
\[
(9.298) \quad \rho = (N_{i-}(\rho))^{-t(\rho)}, \quad 2 \leq t(\rho) \leq 1 + \tau
\]
For \( 1 < \mu < 2 \) fixed we define \( i_+(\rho) \in \mathbb{N} \) by
\[
(9.299) \quad (N_{i-}(\rho))^\mu \leq N_{i+}(\rho) < e^{\mu \tau} (N_{i-}(\rho))^\mu
\]
which implies
\[
(9.300) \quad \rho^{-\mu/i(\rho)} \leq N_{i+}(\rho) \leq e^{\mu \tau} \rho^{-\mu/i(\rho)}.
\]
We thus have
\[
\| F_{i_+}(\rho) \|_{h/2, U_{i_+}(\rho)} \lesssim \exp\left(-N_{i+}(\rho)/(\ln(N_{i+}(\rho)))^\mu \right)
\]
\[
(9.302) \quad \lesssim \exp(-\rho^{-\mu/i(\rho)}).
\]
We notice that since \( \omega_0 \) is diophantine and \( 2 \geq \delta^2 \Omega \geq 1/2, U_{i+}(\rho) \) contains a disk \( D(0, N_{i+}(\rho)) \) (with \( \tau = \tau(\omega_0) \)) hence
\[
(9.303) \quad D(0, \rho^{(\tau+2)/i(\rho)}) \subset U_{i+}(\rho).
\]
We introduce one more notation:
\[
\mathcal{D}_\rho = \{ D \in \mathcal{D}(U_{i_+}(\rho)), D \cap D(0, \rho) \neq \emptyset \}
\]
and if \( D \) is a disk of the form \( D(c, K_i^{-1}) \) we denote \( \hat{D} = D(c, \hat{K}_i^{-1}) \).

Our main proposition is the following

**Proposition 9.1.** For every \( i_-(\rho) \leq i \leq i_+(\rho) - 1 \), any disk \( D \in D(c, K_i^{-1}) \) \( D(U_{i_+}) \setminus \mathcal{D}(U_i) \) such that \( D \subset D(0, \rho) \), the maps \( Y_{i+1, D, D}, \tilde{W}_{i, D, D}, \hat{F}_{i+1, D, D} \) can be extended to analytic functions defined on \( A_\rho(D) := A(\exp(-\rho^{(\mu/i(\rho))^-}), \hat{K}_i^{-1}) \) and we still have
\[
(9.304) \quad \delta^2 \hat{Y}_{i+1, D, D} \approx 1
\]
\[
(9.305) \quad \| \hat{F}_{i+1, D, D} \|_{h/4, A_\rho(D)} = O(\exp(-\hat{K}_i^{(1/20)})) = O(\exp(-\rho^{5/i(\rho)})).
\]
We prove this proposition in the next subsections.
9.1. Comparing $\Xi$ and $\Xi_{\rho}$. We refer to Corollary 6.3 for the definition of $\Xi_{\rho}$.

**Lemma 9.2.** Assume that the Birkhoff Normal Form $\Xi$ has a positive radius of convergence (let say $\rho_\Xi$). Then for $\rho$ small enough one has on $D(0, \rho^b)$

$$\|\Xi - \Xi_{\rho}\|_{\rho^b} \leq \exp(-\rho^{-3}).$$

**Proof.** From Corollary 6.3 we know that the first $n_\rho := [\rho^{-3}]$ coefficients in the formal series defining $\Xi$ and $\Xi_{\rho}$ coincide. We thus have

$$\Xi_{\rho} = \sum_{k=1}^{n_\rho} \Xi(k)r^k$$

and so

$$| (\Xi - \Xi_{\rho})(r) | \leq \sum_{k=n_\rho+1}^{\infty} \Xi(k)r^k$$

$$\leq \|\Xi\|_{\rho_\Xi} \sum_{k=n_\rho+1}^{\infty} (r/\rho_\Xi)^k$$

$$\leq \|\Xi\|_{\rho_\Xi} \frac{1}{1 - (r/\rho_\Xi)^{n_\rho}}.$$

\[\square\]

9.2. Comparing $\Xi_{\rho}$ and $\Omega_{i_+(\rho)}$.

**Lemma 9.3.** One has

$$\|\Omega_{i_+(\rho)} - \Xi_{\rho}\|_{A(2\rho^b, (1/2)\rho^b/2)} \leq \exp(-\rho^{-(\mu/\rho)})^{-}).$$

**Proof.** We can assume that the exponent $b$ of Corollary 6.3 is larger than $2\mu(\tau + 2)/\iota(\rho)$ so

$$D(0, \rho^b) \subset D(0, \rho^{b/2}) \subset U_{i_+(\rho)}.$$ We know that on $D(0, \rho^{b/2})$

$$f_{\Gamma_{i_+(\rho)}-1} \circ f_{\Omega + F} \circ f_{\Gamma_{i_+(\rho)}-1}^{-1} = f_{\Omega_{i_+(\rho)} + F_{i_+(\rho)}}$$

and on $D(0, \rho^b)$

$$W_{\rho} \circ f_{\Omega + F} \circ W_{\rho}^{-1} = f_{\Xi_{\rho} + G_{\rho}}$$

with

$$\|F_{i_+(\rho)}\|_{\rho^b, D(0, \rho^b)} \leq \exp(-\rho^{-\mu/\rho}), \quad \|G_{\rho}\|_{\rho^b} \leq \exp(-\rho^{-3}).$$

The assumptions of Lemma 8.1 are satisfied and we hence get (observe that $\Omega_{i_+(0)} = \Xi_{\rho}(0) = 0$)

$$\|\Omega_{i_+(\rho)} - \Xi_{\rho}\|_{A(2\rho^b, (1/2)\rho^b/2)} \leq \rho^{-\alpha} \exp(-\rho^{-\mu/\rho})$$

$$\leq \exp(-\rho^{-\mu(1/\rho)})^{-}.$$
9.3. **Comparing** $\Omega_{i_-(\rho)}$ and $\Upsilon_{i+1,D,\hat{D}}$. Let $i_-(\rho) \leq i \leq i_+(\rho) - 1$. We notice that for any $D \in \mathcal{D}(U_{i+1}) \setminus \mathcal{D}(U_i)$ such that $D \subset D(0,\rho)$ one has $\hat{D} \subset D(0,\rho)$. Indeed, $\hat{K}_i^{-1} \leq \hat{K}_{i-1}^{-1} = N_{i_-(\rho)}^{-20\rho}$ if $\rho \geq 10$. Notice that
\begin{equation}
\hat{K}_{i-(\rho)}^{-1} \approx \rho^{20\rho/i(\rho)}.
\end{equation}

**Lemma 9.4.** For every $i_-(\rho) \leq i \leq i_+(\rho) - 1$, any disk $D = D(c, K_i^{-1}) \in \mathcal{D}(U_{i+1}) \setminus \mathcal{D}(U_i)$, such that $D(c, K_i^{-1}) \subset D(0,\rho)$ and any $0 < \alpha < \mu/i(\rho)$ one has
\begin{equation}
\|\Upsilon_{i+1,D,\hat{D}} - \Omega_{i-(\rho)} - \text{cst}\|_{h/2,A_{\alpha,\rho}} \lesssim \exp(-\rho^{-(\mu/i(\rho))^{-}})
\end{equation}
where $A_{\alpha,\rho}$ is the annulus $A_{\alpha,\rho}(D) := \{z \in \mathbb{C} : e^{-\rho^{-\alpha}} < |z - c| < \rho^{20\rho/i(\rho)}\}$.

**Proof.** We have
\begin{equation}
\hat{W}_{i, D, \hat{D}} \circ f_{\Omega_i} \circ f_i = \hat{W}_{i, D, \hat{D}}^{-1} = f_{\Upsilon_{i+1,D,\hat{D}} - \hat{F}_{i+1,D,\hat{D}}},
\end{equation}
with
\begin{equation}
\|\hat{F}_{i+1,D,\hat{D}}\|_{h/2,A(D,\hat{D})} = O(\exp(\hat{K}_i^{1/20})) = O(\exp(-\rho^{-P/(3h(\rho))}))
\end{equation}
and for $Z := Z_{i,i+1} = f_{\Upsilon_{i+1,D,\hat{D}} - 1} \circ \cdots \circ f_{\Upsilon_i}$
\begin{equation}
Z \circ f_{\Upsilon_i} \circ f_i \circ Z^{-1} = f_{\Upsilon_i} + F_{i+1(\rho)}
\end{equation}
with
\begin{equation}
\|F_{i+1(\rho)}\|_{h/2,A(D,\hat{D})} = O(\exp(-\rho^{-\mu/i(\rho)})).
\end{equation}
We can apply Lemma 8.1 on $A_{\alpha,\rho}$:
\begin{equation}
\|\Omega_{i+1(\rho)} - \Upsilon_{i+1,D,\hat{D}}\|_{h/2,A_{\alpha,\rho}} \lesssim (\exp(-\rho^{-\alpha}))^{-a} \exp(-\rho^{-\mu/i(\rho)})
\end{equation}
\begin{equation}
\lesssim \exp(-\rho^{-(\mu/i(\rho))^{-}}).
\end{equation}

\[ \square \]

9.4. **Comparing** $\Xi$ and $\Upsilon_{i+1,D,\hat{D}}$.

**Lemma 9.5.** Let $0 < \alpha < 1/i(\rho)$. Then, one has (we recall that $A_{\alpha,\rho}(D) := \{z \in \mathbb{C} : \exp(-\rho^{-\alpha}) < |z - c| < \rho^{20\rho/i(\rho)}\}$)
\begin{equation}
\|\Xi - \Omega_{i_+(\rho)} - \text{cst}\|_{h/2,A_{\alpha,\rho}} \lesssim \exp(-\rho^{-(\mu/i(\rho))^{-}}).
\end{equation}

**Proof.** From Lemmas 9.2 and 9.3 we have
\begin{equation}
\|\Xi - \Omega_{i_+(\rho)} - \text{cst}\|_{h/2,C(0,\rho)} \lesssim \exp(-\rho^{-(\mu/i(\rho))^{-}}).
\end{equation}
Let us write
\begin{equation}
U_{i_+(\rho)} \cap D(0,\rho) = D(0,\rho) \setminus \left( D \cup \bigcup_{D' \in \mathcal{D}_{\rho} \setminus D} D' \right)
\end{equation}
\begin{equation}
= D(0,\rho) \setminus \left( D \cup \bigcup_{l=i_-(\rho)}^{i_+(\rho)} \bigcup_{D' \in \mathcal{D}_{\rho} \setminus \mathcal{D}(U_l),D' \neq D} D' \right)
\end{equation}
and denote \( n_t = \#(\mathcal{D}_\rho \cap \mathcal{D}(U_l)) \). For \( z \in A_{\alpha,\rho}(D) \) one has
\[
\text{dist}(z, D) = e^{-\rho^{-\alpha}}
\]
and for \( D' \in \mathcal{D}_\rho \cap \mathcal{D}(U_l) \)
\[
\text{dist}(z, D') \geq N_{\alpha,\rho}^{-3} \geq \rho^{-3\mu/\iota(\rho)}
\]
while
\[
n_t \leq \rho N_{l-1}^2.
\]
From Lemma 3.1 we deduce that for \( z \in A_{\alpha,\rho} \)
\[
\ln |(\Xi - \Omega_{\alpha,\rho})(z)| \leq -\rho^{(\mu/\iota(\rho))} \left( \frac{\ln(1/(2\rho))}{\ln(\rho^\beta/\rho)} - \frac{\ln(\text{dist}(z, D)/\rho)}{\ln(K_{l-1}/\rho)} - \frac{\sum_{l=i-(\rho)+1}^{i+(\rho)} \sum_{D' \in \mathcal{D}_\rho \cap \mathcal{D}(U_l), D' \neq D} \ln(\text{dist}(z, D'))/\rho}{\ln(K_{l-1}/\rho)} \right).
\]
If
\[
(I) = \frac{\ln(\text{dist}(z, D)/\rho)}{\ln(K_{l-1}/\rho)} + \frac{\sum_{l=i-(\rho)+1}^{i+(\rho)} \sum_{D' \in \mathcal{D}_\rho \cap \mathcal{D}(U_l), D' \neq D} \ln(\text{dist}(z, D'))/\rho}{\ln(K_{l-1}/\rho)}
\]
we see that
\[
(I) \leq \frac{\ln(\rho^{-1}e^{-\rho^{-\alpha}})}{\ln(\rho^{-1}e^{-N_{l-1}/(\ln N_{l-1})^\beta})} + \frac{\sum_{l=i-(\rho)+1}^{i+(\rho)} \ln(\rho^{-10})}{\ln(K_{l-1}/\rho)}
\]
\[
\leq \rho^{(1/\iota(\rho)) - \alpha} + \rho \sum_{l=i-(\rho)+1}^{i+(\rho)} \rho N_{l-1}^2 \frac{\ln(\rho^{-10})}{(b \ln N_{l-1}) - \ln \rho}
\]
\[
\leq \rho^{(1/\iota(\rho)) - \alpha} + \rho \sum_{l=i-(\rho)+1}^{i+(\rho)} \rho N_{l-1}^2
\]
\[
\leq \rho^{(1/\iota(\rho)) - \alpha} + \rho N_{i+1}(\rho)
\]
\[
\leq \rho^{(1/\iota(\rho)) - \alpha} + \rho^{1-\mu/\iota(\rho)}.
\]
From (9.333) we see that
\[
\ln |(\Xi - \Omega_{\alpha,\rho})(z)| \leq -\rho^{(\mu/\iota(\rho))^-}.
\]
\(\square\)
9.5. **Consequences on the size of the holes.** We can now apply Proposition 7.10.

**Lemma 9.6.** For every \( i_-(\rho) \leq i \leq i_+(\rho) - 1 \), any disk \( D = D(c, K_i^{-1}) \in \mathcal{D}(U_{i+1}) \backslash \mathcal{D}(U_i) \), such that \( D(c, K_i^{-1}) \subset D(0, \rho) \), the maps \( \mathcal{V}_{i+1,D,D}, \hat{W}_{i,D,D}, \hat{F}_{i+1,D,D} \) can be extended to analytic functions defined on \( A_\rho(D) := A(\exp(-\rho^{(\mu/i(\rho))^-}), \hat{K}_i^{-1}) \) and we still have

\[
\|\hat{F}_{i+1,D,D}\|_{h/4,A_\rho(D)} = O(\exp(-\hat{K}_i^{1/20})).
\]

10. **Estimates on the measure of the linearization domain**

10.1. **Classical KAM measure estimates.** We recall first a classical statement in KAM theory

**Proposition 10.1.** Let \( U \subset \mathbb{C} \) real symmetric, \( \Omega \in \mathcal{F}_U \), \( F \in \mathcal{F}_{h,U} \) such that \( \partial^2 \Omega \geq 1/2 \). Let \( B \) be the complement in \( \mathbb{T} \times (U \cap \mathbb{R}) \) of the union of all curves, not homotopic to the identity and invariant by \( f_{\Omega+\hat{F}} \). Then, one has

\[
m(B) \leq \|F\|_{h,U}^{(1/2)}.
\]

10.2. **Proof of Theorem A.** The Remark in Section 5 and Proposition 9.5 show that the collection of sets \( f_{\Omega_{i+}\hat{F}}(\mathbb{T} \times (U_{i+}(\rho) \cap \mathbb{R})) \) and \( W_{i,D,D}(\mathbb{T} \times (A_\rho(D) \cap \mathbb{R})) \), \( D \in \mathcal{D}_\rho \) is modulo sets of Lebesgue measure less than

\[
O(\exp(-\rho^{-(\mu/i(\rho))^-}))
\]
a partition of \( \mathcal{F}\mathcal{T}(f_{\Omega_{i+}(\rho)}(\mathbb{T} \times (U_{i+}(\rho) \cap \mathbb{R}))) \) (\( \mathcal{F}\mathcal{T}(E) \) denotes the filled-in set of the open set \( E \), which is by definition the union of \( E \) and all its contractible connected components). Now Proposition 10.1 applied to \( f_{\Omega_{i+}(\rho)+F_{i+}(\rho)} \) shows that

\[
\text{Leb}_2(f_{\Omega_{i+}(\rho)}(\mathbb{T} \times (U_{i+}(\rho) \cap \mathbb{R})) \cap G_\rho) \leq \exp(-1/4)N_{i+}(\rho)^{-1} \leq \exp(-\rho^{-(\mu/i(\rho))^-})
\]

and applied to each \( f_{\hat{\Omega}_{i+1,D,D}+\hat{F}_{i+1,D,D}} \), \( i_- \leq i \leq i_+ - 1 \), \( D \in cD_\rho \), shows that

\[
\text{Leb}_2(W_{i,D,D}(\mathbb{T} \times (A_\rho(D) \cap \mathbb{R})) \cap G_\rho) \leq \exp(-\hat{K}_i^{1/20}) \leq \exp(-\rho^{-(5/\ell(\rho))^-}).
\]

Finally

\[
m(\rho) \leq \exp(-\rho^{-(\mu/i(\rho))^-})
\]

and since \( 2 \leq \ell(\rho) \leq 1 + \tau \) and \( \mu \) can be taken arbitrarily close to 2, this concludes the proof of Theorem A. \( \square \)
11. Construction of a counterexample

Proof of Theorem B

11.1. Creating hyperbolic points.

Proposition 11.1. Let \(N_{n-1} < q \leq N_n\) and
\[
G_q(\theta, r) = \tau_n e^{-2\pi q h/10} r^2 \cos(2\pi q \theta)
\]
which satisfies
\[
\|G_n\|_{h/10,1/2} = O(e^{-N_n^2}).
\]
There exist a point \(c = \left|\omega_0 - \frac{p}{q}\right|\) and an interval \(I_n \subset \mathbb{R}\) of length \(\geq \exp(-N_n^{1+})\) such that the diffeomorphism \(\Phi_{G_q} \circ f_{\Omega + F}\) has no piece of invariant (horizontal) circle in the region \(\mathbb{T} \times I_n\).

Proof.

11.1.1. Preliminary reduction. We consider the situation where after some steps of the KAM procedure described in Section 5 we have conjugated the initial symplectic diffeomorphism to a diffeomorphism \(f_{\Omega + F}\) defined on a domain \(\mathbb{T}_{h_n} \times U_n\) (see Proposition 5.4):
\[
Z_n = f_{\Omega + F} \circ G_n \circ f_{\Omega + F}^{-1}.
\]
Hence \(Z_n\) is defined on the domain with holes \(\mathbb{T}_{h_n} \times U_n\). The conjugation \(Z_n\) is in fact obtained as a composition
\[
Z_n = f_{Y_{n-1}} \circ \cdots \circ f_{Y_1}
\]
where each \(Y_i\), \(1 \leq i \leq n - 1\) is defined on a domain \(\mathbb{T}_{h_i} \times U_i\) and satisfies
\[
\|Y_i\|_{h_i, U_i} \lesssim N_i^2 \tau_i.
\]
Furthermore, since \(\mathbb{T} \times \{r = 0\}\) is invariant by \(f_{Y_i}\), each \(Y_i\) can be written on a domain \(\mathbb{T}_{h_i} \times D(0, N_i^{-2}/10) \subset \mathbb{T}_{h_i} \times U_i\) as
\[
Y_i(\theta, r) = r^2 \tilde{Y}_i(\theta, r)
\]
where \((b = a + 2)\)
\[
\|\tilde{Y}_i\|_{h_i, D(0, N_i^{-2}/10)} \lesssim N_i^2 \tau_i.
\]
We recall that by Proposition 5.3 we have

Lemma 11.2. Let \(N_{n-1} \leq q \leq N_n\) and assume that for some \(p \in \mathbb{Z}\), \(|\omega_0 - p/q| \leq q^{-2}\). There exists \(c \in \mathbb{R}\) which belongs to the connected component of \(U_n \cap \mathbb{R}\) containing \(\theta\) such that
\[
0 < c = |\omega_0 - \frac{p}{q}|, \quad \nabla_{\Omega_n}(c) = \frac{p}{q}, \quad |q| \leq N_n \quad \text{and} \quad \text{dist}(c, \partial U_n) \gtrsim N_n^{-2}.
\]
From Propositions 11.2 and Lemma A.8 there exist \( W_n, F_{n,\text{per}} \) and \( F_{n,\text{aper}} \) such that on \( \mathbb{T}_{h/2} \times D(c, N_n^{-2}/2) \)

\[
(11.353) \quad f_{W_n} \circ f_{\Omega_n} + f_{n} \circ f_{W_n}^{-1} = f_{\Omega_n} + \tilde{F}_{n,\text{per}} + \tilde{F}_{n,\text{aper}}
\]

with \( \tilde{F}_{n,\text{aper}} \) 1/q-periodic

\[
(11.354) \quad \tilde{F}_{n,\text{per}}(\theta + q^{-1} r, r) = \tilde{F}_{n,\text{per}}(\theta, r) = O(F_n)
\]

and

\[
(11.355) \quad \tilde{F}_{n,\text{aper}} = O(e^{-\tilde{K}(1/2)^{-\nu}}).
\]

We notice that since \( \tilde{F}_{n,\text{aper}} \) is 1/q periodic one has

\[
(11.356) \quad \|\tilde{F}_{n,\text{aper}}\|_{h/4, D(0, N_n^{-2}/5)} \lesssim e^{-2\pi q h/2 \xi_n}.
\]

This implies that \( \tilde{F}_n := \tilde{F}_{n,\text{per}} + \tilde{F}_{n,\text{aper}} \) satisfies

\[
(11.357) \quad \|\tilde{F}_n\|_{h/4, D(0, N_n^{-2}/5)} \lesssim e^{-2\pi q h/2 \xi_n}.
\]

11.1.2. Effect of the perturbation on \( \Omega_n + F_n \). We recall that

\[
G_q(\theta, r) = \xi_n e^{-2\pi q h/10} r^2 \cos(2\pi q \theta)
\]

satisfies

\[
\|G_q\|_{\mathbb{T}_{h/10}\times \mathbb{D}} \lesssim \xi_n
\]

and we consider the following perturbation of \( f_{\Omega + F} \):

\[
(11.358) \quad \Phi_{G_q} \circ f_{\Omega + F}
\]

and we define \( \tilde{G}_q \) by

\[
\Phi_{G_q} := (f_{W_n} \circ f_{Y_{n-1}} \circ \cdots \circ f_{Y_1}) \circ \Phi_{G_q} \circ (f_{W_n} \circ f_{Y_{n-1}} \circ \cdots \circ f_{Y_1})^{-1}
\]

or equivalently

\[
(11.359) \quad \tilde{G}_q = G \circ f_{Y_1}^{-1} \circ \cdots \circ f_{Y_{n-1}}^{-1} \circ f_{W_n}^{-1}.
\]

Notice that

\[
(11.360) \quad (f_{W_n} \circ Z_n) \circ \Phi_{G_q} \circ f_{\Omega + F} \circ (f_{W_n} \circ Z_n)^{-1} = \Phi_{\tilde{G}_q} \circ f_{\Omega_n + \tilde{F}_n}.
\]

**Lemma 11.3.** One can write

\[
\tilde{f}_{Y_1}^{-1} \circ \cdots \circ \tilde{f}_{Y_n}^{-1} \circ \tilde{f}_{W_n}^{-1}(\theta, r) = (\theta + ra(\theta, r), r + r^2b(\theta, r))
\]

with

\[
\max(\|a\|_{h/2, D(c, \tilde{K}_n^{-1})}, \|b\|_{h/2, D(c, \tilde{K}_n^{-1})}) \leq 1.
\]

**Proof.** Since \( f_{Y_k} \) leaves invariant \( \mathbb{T} \times \{r = 0\} \), \( Y_k \) can be written \( Y_k(\theta, r) = r^2 \tilde{Y}_k(\theta, r) \) with

\[
\|\tilde{Y}_k\|_{V_k} \lesssim N_k^h \|Y_k\|_{V_k}.
\]

Let \( f_{Z_k} = (f_{Y_1}^{-1} \circ \cdots \circ f_{Y_k}^{-1})^{-1} \) for \( 1 \leq k \leq n - 1 \), \( f_{Z_n} = f_{W_n} \circ f_{Z_{n-1}} \) and for \( 1 \leq k \leq n \), \( \tilde{Z}_k \) be such that \( \tilde{Z}_k(\theta, r) = r^2 \tilde{Z}_k(\theta, r) \). Since

\[
f_{Z_{k+1}} = f_{Y_{k+1}} \circ f_{Z_k}
\]
we have from Remark A.1
\[ Z_{k+1} = Z_k + B_2(\nabla Z_k, \nabla Y_{k+1}) \]
which implies
\[ \tilde{Z}_{k+1} = \tilde{Z}_k + B_2(\nabla \tilde{Z}_k, \nabla \tilde{Y}_{k+1}) \]
and
\[ \|\nabla \tilde{Z}_{k+1}\|_{\nu_{k+1}} = \|\nabla \tilde{Z}_k + B_2(\nabla \tilde{Z}_k, \nabla \tilde{Y}_{k+1})\| \leq \|\nabla \tilde{Z}_k\|_{\nu_k} (1 + N_k^b \varepsilon) \]
for some \( b > 0 \). In particular, \( \|\nabla \tilde{Z}_{n-1}\|_{\nu_n} \leq 1 \) and \( \|\nabla \tilde{Z}_n\|_{\nu_n} \leq 1 \). If \( f_{\tilde{Z}_n} = f_{\tilde{Z}_1} \) we clearly have \( Z_n^* (\theta, r) = r^2 \tilde{Z}_n^* (\theta, r) \) and \( \|\nabla \tilde{Z}_n\|_{\nu_n} \leq 1 \) which is the conclusion.

\[ \Box \]

**Lemma 11.4.** One has
\[ \tilde{G}_q (\theta, r) = \varepsilon_n^a e^{-2\pi q h/10} r^2 \cos (2\pi q \theta) + r^2 S (\theta, r) \]
where
\[ \|S\|_{h/10,D(c,\hat{K}^{-1})} \lesssim \varepsilon_n q^{-1} \lesssim \varepsilon_n N_n^{-1}. \]

**Proof.** From the previous Lemma 11.3 and (11.359) we can write
\[ \tilde{G}_q (\theta, r) = \varepsilon_n^a e^{-2\pi q h/10} (r + r^2 b(\theta, r))^2 \cos (2\pi q (\theta + ra(\theta, r))) \]
with
\[ \max (\|a\|_{h/2,D(c,\hat{K}^{-1})}, \|b\|_{h/2,D(c,\hat{K}^{-1})}) \lesssim 1. \]
Using Taylor expansion
\[ \cos (2\pi q (\theta + ra(\theta, r))) = \cos (2\pi q \theta + 2\pi q r a(\theta, r)) \]
\[ = \cos (2\pi q \theta) + \sum_{l=1}^{\infty} \frac{\cos (l) (2\pi q \theta)}{l!} (2\pi q r a(\theta, r))^l \]
\[ = \cos (2\pi q \theta) + (q r) \tilde{a}(\theta, r) \]
where
\[ \|\tilde{a}\|_{h/10,r^*} \leq e^{2\pi q h/10} \|a\|_{h/10,r^*}. \]
So
\[ \tilde{G}_q (\theta, r) = \varepsilon_n^a e^{-2\pi q h/10} r^2 (1 + 2 r b(\theta, r) + r^2 b(\theta, r)^2) \cos (2\pi q \theta) + (q r) \tilde{a}(\theta, r) \]
\[ = \varepsilon_n^a e^{-2\pi q h/10} r^2 \cos (2\pi q \theta) + r^2 S (\theta, r) \]
and if \( \theta \in T_{h/10} \) and \( |r| \leq r^* \) satisfies \( q r^* \leq q^{-1} \) one has from (11.362)
\[ |S (\theta, r)| \lesssim \varepsilon_n q^{-1} \max (\|a\|_{h/10,r^*}, \|b\|_{h/10,r^*}). \]
Since we assume from (11.352) that \( c \lesssim q^{-2} \)
one has from (11.361)
\[ \|S\|_{h/10,D(c,\hat{K}^{-1})} \lesssim \varepsilon_n q^{-1}. \]

We now come back to the perturbation (11.358):
\[
(W_n \circ Z_n) \circ \Phi_{G,q} \circ f_{\Omega + F} \circ (W_n \circ Z_n)^{-1} = \Phi_{\tilde{G}_q} \circ f_{\hat{\Omega}_n + \tilde{F}_n}
\]
and
\[
\Phi_{\tilde{G}_q} \circ f_{\hat{\Omega}_n + \tilde{F}_n} = f_{\tilde{G}_q + \mathcal{O}^2(G_q)} \circ f_{\tilde{F}_n} \circ f_{\hat{\Omega}_n} = f_{\tilde{G}_q + \tilde{F}_n + \mathcal{O}_2(G_q,F_n)} \circ f_{\hat{\Omega}_n} = f_{\tilde{\Omega}_n + \tilde{G}_q + \mathcal{O}_2(\tilde{G}_q,\tilde{F}_n)} = f_{\tilde{\Omega}_n + \tilde{G}_q^*}
\]
where
\[(11.364) \quad \tilde{G}_q^* = \tilde{G}_q + \tilde{F}_n + \mathcal{O}_2(\tilde{G}_q,\tilde{F}_n).\]

We have

**Lemma 11.5.** There exists \( \tilde{W}_n \) such that on \( \mathbb{T}_{h/20} \times D(c,\hat{K}^{-1}/2) \)
\[(11.365) \quad f_{\tilde{W}_n} \circ f_{\tilde{\Omega}_n + \tilde{G}_q^*} \circ f_{\tilde{W}_n} = f_{\tilde{\Omega}_n + \tilde{G}_q^*},
\]
where
\[
\tilde{G}_q = \tilde{G}_q^* + \tilde{G}_q^{per}
\]
with \( \tilde{G}_q^{per} = O(\tilde{G}_q^*) \) 1/q-periodic and
\[
\tilde{G}_q^{per} = O(\exp(-K_n^{-1/2})).
\]

**Proof.** Apply Proposition 7.2 to \( \tilde{\Omega}_n + \tilde{G}_q^* \).

**Lemma 11.6.** One has on \( (\theta,r) \in \mathbb{T}_{h/21} \times D(c,\hat{K}^{-1}) \)
\[
\tilde{G}_q^{per}(\theta,r) = \varepsilon_n e^{-2\pi q/10}(1 - O(N_n^{-1}) r^2 \cos(2\pi q \theta) + O(\varepsilon_n e^{-2\pi q h/5}).
\]

**Proof.** From Remark 7.4, (11.364) and Lemma 11.4
\[
\tilde{G}_q^{per}(\theta,r) = T^{res}\tilde{G}_q^*(\theta,r) + \mathcal{O}_2(\tilde{G}_q^*)
\]
\[
= \varepsilon_n e^{-2\pi q/10} r^2 \cos(2\pi q \theta) + r^2 T^{res} S(\theta,r) + O(\tilde{F}_n).
\]
If \( S(\theta,r) = \sum_{k \in \mathbb{Z}} \hat{S}_k(r)e^{2\pi i k \theta} \) one has
\[
T^{res} S(\theta,r) = \hat{S}_q(r)e^{2\pi q \theta} + \sum_{|l| \geq 2} \hat{S}_q(l)e^{2\pi q l \theta}
\]
and by Lemma 11.4
\[
|\hat{S}_q(r)| \lesssim e^{-2\pi q h/10} \varepsilon_n N_n^{-1}
\]
and by Cauchy estimates for \((\theta, r) \in T_{h/21} \times D(c, \hat{K}_{n}^{-1})\)

\[
| \sum_{|l| \geq 2} \hat{S}_{q}(r)e^{2\pi i q \theta} | \lesssim e^{-2\pi (2q)/10} \xi_{n} N_{n}^{-1}
\]

\[
\lesssim e^{-2\pi q h/5} \xi_{n} N_{n}^{-1}.
\]

The conclusion then follows from (11.357). 

11.1.3. Perturbation of a pendulum. Coming back to 11.5 we write

\[
f_{\hat{\Theta}_{q}} + \hat{G}_{q} = f_{\hat{\Theta}_{q}} \circ f_{\hat{\Theta}_{q}}
\]

\[
= \tilde{f}_{\hat{\Theta}_{q},2} \circ \tilde{f}_{\hat{G}_{q}^{\text{per}}} \circ \hat{\Theta}_{q} - \langle \nabla \hat{\Theta}_{q}(\hat{c}), \cdot \rangle \circ R_{p/q}
\]

\[
= \tilde{f}_{\hat{\Theta}_{q},2} \circ \hat{H}_{q} \circ \hat{G}_{q}^{\text{per}} \circ R_{p/q}
\]

(11.366)

where

(11.367) \[ \| \bar{G}_{q,2} \|_{h/20, D(c, \hat{K}_{n}^{-1})} = O(\exp(-\hat{K}_{n}^{-1/20})) \]

\(\hat{c}\) is a point \(O(\exp(-\hat{K}_{n}^{-1/20})\) close to \(c\) where \(\nabla \hat{\Theta}_{q}(\hat{c}) = 0\) and

\(H_{q} = \hat{\Theta}_{q} - \langle \nabla \hat{\Theta}_{q}(\hat{c}), \cdot \rangle + \hat{G}_{q}^{\text{per}}\)

is a \(1/q\)-periodic (in \(\theta\)) function.

We now define \(\tilde{H}\) on \(T_{qh/20} \times D(0, q\hat{K}_{n}^{-1/20})\) by

(11.368) \[ \tilde{H}(q\theta, qr) = q^{2} H(\theta, \hat{c} + r) \]

so that (we assume to simplify the notations that \(D^{2} \hat{\Theta}_{q}(\hat{c}) = 1\))

\[
\tilde{H}(\theta, r) = r^{2}/2 + q^{2} \xi_{n} e^{-2\pi q h/10} (1 - O(N_{n}^{-1})(\hat{c} + q^{-1}r)^{2} \cos(2\pi \theta) + O(\xi_{n} e^{-2\pi q h/5})
\]

\[
= r^{2}/2 + \nu \cos(2\pi \theta) + \nu \delta f(\theta, r)
\]

where, denoting

(11.369) \[ r_{*} = \hat{K}_{n}^{-1/20} \]

one has

(11.370) \[ \nu = (q \hat{c})^{2} \xi_{n} e^{-2\pi q h/10} (1 - O(N_{n}^{-1}), \quad \delta = \max(\hat{c} q r_{*}, r_{*}^{2}, O(\xi_{n} e^{-2\pi q h/5})) \]

and where \(f\) has bounded \(C^{3}\)-norm:

\[
\| f \|_{C^{3}(\mathbb{T} \times I(\hat{c}, r_{*}))} < 1.
\]

Lemma 11.7. The symplectic diffeomorphism \( f_{\tilde{H}} \) has a hyperbolic fixed point \( \hat{p} \in \mathbb{T} \times I(\hat{c}, r_{*}) \) with eigenvalues \( \lambda_{\pm} \), in \( \lambda_{\pm} = \pm \nu^{1/2} \) and eigendirections \( \begin{pmatrix} 1 \\ \hat{m}_{\pm} \end{pmatrix} \) with \( \hat{m}_{\pm} = \pm \nu^{1/2} \). The stable and unstable manifolds of \( \hat{p} \) are graphs \( \theta \mapsto \hat{w}_{\pm}(\theta) \) defined on a domain of size \( r_{*} \nu^{1/2} \) and on this domain

\[
3\hat{m}_{-} \theta < \hat{w}_{-}(\theta) < (1/3)\hat{m}_{-} \theta < (1/3)\hat{m}_{+} \theta < \hat{w}_{+}(\theta) < 3\hat{m}_{+} \theta
\]
Proof. The fixed point of $f_{\bar{H}}$ satisfy
\[
\begin{cases}
  r + \delta \nu \hat{c}_r f(\theta, r) = 0 \\
  -\nu \sin \theta + \delta \nu \hat{c}_\theta f(\theta, r) = 0 \\
  r + \delta \nu \hat{c}_r f(\theta, r) = 0 \\
  -\sin \theta + \delta \hat{c}_\theta f(\theta, r) = 0
\end{cases}
\]
and by the Implicit Function Theorem there exists a solution
\[
\begin{cases}
  \theta_\delta = \delta \hat{c}_\theta f(0,0) + O(\delta^2) \\
  r_\delta = -\delta \nu (\hat{c}_r f(0,0) + O(\delta))
\end{cases}
\]
At this point
\[
D^2 H_\delta(\theta_\delta, r_\delta) = \begin{pmatrix}
  -\nu + O(\delta \nu) & O(\delta \nu) \\
  O(\delta \nu) & 1 + O(\delta \nu)
\end{pmatrix}
\]
which has determinant $-\nu + O(\delta \nu) = -\nu(1 + O(\delta))$.

The remaining part is a consequence of classical theorems on hyperbolic fixed point.

Corollary 11.8. The symplectic diffeomorphism $f_{\bar{H}} \circ R_{p/q}$ has a hyperbolic periodic point $p \in \mathbb{T} \times I(\hat{c}, r_*)$ with eigenvalues $\lambda_\pm$, $\ln \lambda_\pm \approx \pm \nu^{1/2}$ and eigendirections $\left(\frac{1}{m_\pm}\right)$ with $m_\pm \approx \pm \nu^{1/2}$. The stable and unstable manifolds of $p$ are graphs $\theta \mapsto w_\pm(\theta)$ defined on a domain of size $\approx q^{-2} r_* \nu^{1/2}$ and on this domain
\[
3m_\theta < w_-(\theta) < (1/3)m_\theta < (1/3)m_\theta < w_+(\theta) < 3m_\theta.
\]
Proof. This is immediate from the definition (11.368) of $\bar{H}$. \qed

Corollary 11.9. The symplectic diffeomorphism $f_{\hat{\Omega}_q + \hat{c}_q}$ has a hyperbolic periodic point $p_q \in \mathbb{T} \times I(\hat{c}, r_*)$ with eigenvalues $\lambda_\pm$, $\ln \lambda_\pm \approx \pm \nu^{1/2}$ and eigendirections $\left(\frac{1}{m_\pm}\right)$ with $m_\pm \approx \pm \nu^{1/2}$. The stable and unstable manifolds of $p$ are graphs $\theta \mapsto w_\pm(\theta)$ defined on a domain of size $\approx q^{-2} r_* \nu^{1/2}$ and on this domain
\[
2m_\theta < w_-(\theta) < (1/2)m_\theta < (1/2)m_\theta < w_+(\theta) < 2m_\theta.
\]
Proof. Using classical result on perturbations of hyperbolic compact sets (in our case a periodic orbit) the claim is a consequence of the preceding Corollary, equality (11.366) and the estimates (11.367) (notice that $\exp(-\hat{R}_n^{-1/20})$ is much smaller than $\nu$). \qed

Corollary 11.10. There exists an open set of area $\geq \exp(-N_n^{-1+})$ in a neighborhood of $p_q$ that has an empty intersection with any invariant curve of the symplectic diffeomorphism $f_{\hat{\Omega}_q + \hat{c}_q}$. 
Proof. By a theorem of Birkhoff [3] (cf. also [10]), the invariant curves of the twist diffeomorphism \( f_{\hat{\Omega}_n + \hat{G}_q} \) are graphs; if they intersect the stable or unstable manifold of \( p_q \) they must be included in the union of these stable and unstable manifolds. As a consequence, the open set
\[
\begin{align*}
    \{ (\theta, r) &\in [-q^{-2}r^2\nu^{1.2}, q^{-2}r^2\nu^{1.2}], |r| < (1/2) \min(w_+(\theta), |w_-(\theta)|) \}
\end{align*}
\]
has an empty intersection with any invariant curve. The estimates of Corollary \[11.9, (11.370), (11.369)\] and the fact that \( q \approx N^{-1} \) show that this open set has an area \( \geq (q^{-2}r^2\nu^{1/2})^3 \geq \exp(-N^{-1+}) \).

11.1.4. Proof of Proposition 11.1. From \[11.360, (11.363), (11.365)\] we have
\[
(f_{\hat{\Omega}_n} \circ f_{W_n} \circ f_{Z_n}) \circ (\Phi_{G_q} \circ f_{\Omega+F}) \circ (f_{\hat{\Omega}_n} \circ f_{W_n} \circ f_{Z_n})^{-1} = f_{\hat{\Omega}_n + \hat{G}_q}.
\]
The proof of Proposition 11.1 now follows from Corollary 11.10.

11.2. Construction of a sequence of hyperbolic periodic points accumulating the origin. We now use inductively Proposition 11.1 to construct a sequence of hyperbolic periodic points accumulating the origin: if \( G_1, \ldots, G_n \) have already been constructed, we define \( G_{n+1}, c_{n+1}, I_{n+1} \) by applying Proposition 11.1 to \( \Omega + F + G_1 + \cdots + G_n \). The argument given at the end of the proof of Proposition 11.1 shows that the hyperbolic periodic points created at step 1, \ldots, \( n \) are not destroyed by adding \( G_{n+1} \).

We have thus proved

**Proposition 11.11.** Let \( n_k \) be a sequence of integers going to infinity, \( q_{n_k} \) such that \( N_{n_k}^{-1} < q_{n_k} \leq N_{n_k} \). Then there exist \( G \in \mathcal{F}(\mathbb{T}_h \times \mathbb{D}), \|G\|_{h/2} = O(\|F\|_{h, p}) \) and \( p_{n_k} \in \mathbb{Z}, c_{n_k} \in \mathbb{R} \) such that \( c_{n_k} \approx |\omega_0 - p_{n_k} / q_{n_k}| \), and interval \( I_{n_k} \subset \mathbb{R} \) of length \( \geq \exp(-N_{n_k}^{1+}) \) such that each region \( \mathbb{T} \times I_{n_k} \) does not contain any piece of invariant circle for the diffeomorphism \( f_{\Omega+F+G} \).

11.3. Proof of Theorem 3. Let \( f := f_{\Omega+F+G} \). Let \( \sigma > 0 \). From \[11.5\] there are sequences of integers going to infinity \( p_k, q_k \) such that
\[
\begin{align*}
    \frac{1}{q_k^{1+\sigma}} &\leq \frac{1}{q_k^{1+\sigma}} \\
    \frac{1}{q_k^{1+\sigma}} &\leq \frac{1}{q_k^{1+\sigma}} \\
    \frac{1}{q_k^{1+\sigma}} &\leq \frac{1}{q_k^{1+\sigma}} \\
    \frac{1}{q_k^{1+\sigma}} &\leq \frac{1}{q_k^{1+\sigma}}
\end{align*}
\]
and from the previous estimates
\[
m(t_k) \geq \exp(-N_{n_k}^{1+})
\]
and
\[
m(t_k) \geq \exp(-(1/t_k)^{(1/(1+\sigma))^{1+}}).
\]
Appendix A. Estimates on composition and inversion

A.1. General estimates.

Lemma A.1. Let \( u \in \mathcal{O}(\mathbb{T}_h \times U) \) satisfy
\[
\| u \|_{h,U} \text{diam}(U)^{-1} \delta^{-2} \lesssim 1/2
\]
Then \((id + u)\) is invertible and one can define \((id + u)^{-1}\) on \( \mathbb{T}_{h-\delta/2} \times U_{\delta/2} \) and
\[
(id + u)^{-1} = id - u + O_2(u)
\]
with
\[
\| O_2(u) \|_{h-\delta,U_{\delta}} = \text{diam}(U)^{-1} \delta^{-2} \| u \|_{h,U}^2.
\]

Proof. The map \((\theta,r) \mapsto (\varphi, R) - u(\theta,r)\) is \(1/2\)-contracting on \( \mathbb{T}_{h-\delta/2} \times U_{\delta/2} \) if \( \text{diam}(U)^{-1} (\delta/2)^{-4} \| u \|_{h,U} \leq 1/2 \) and for \((\varphi, R) \in \mathbb{T}_{h-\delta/2} \times U_{\delta/2}\) it sends \( \mathbb{T}_{h-\delta/2} \times U_{\delta/2} \) to itself. It thus admits a unique fixed point: \((\theta, r) + u(\theta, r) = (\varphi, R)\) which depends continuously and even holomorphically on \((\varphi, R)\). Now if \((id + u)^{-1} = id + v\) one has \((id + u) \circ (id + v) = id\) and so \(v = -u \circ (id + v) = -u + (u - u(id + v))\). The first equality and condition \([A.374]\) imply that \(\| v \|_{h-\delta,U_{\delta}} \leq \| u \|_{h,U}\) and the second
\[
\| v + u \|_{h-3\delta/4,U_{3\delta/4}} \leq \| Du \|_{h-\delta/2,U_{\delta/2}} \| v \|_{h-\delta/2,U_{\delta/2}}
\]
\[
\| Du \|_{h-\delta/2,U_{\delta/2}} \| u \|_{h-\delta/2,U_{\delta/2}} \leq \text{diam}(U)^{-1} \delta^{-2} \| u \|_{h-\delta/2,U_{\delta/2}}^2.
\]

Lemma A.2. If \( u, v \in \mathcal{O}(\mathbb{T}_h \times U) \) satisfy
\[
\max(\| u \|_{h,U}, \| v \|_{h,U}) \text{diam}(U)^{-1} \delta^{-2} \lesssim 1/2
\]
then
\[
(id + u) \circ (id + v) = id + u + v + B_2(u, v)
\]
with
\[
\| B_2(u, v) \|_{h-\delta,U_{\delta}} = \text{diam}(U)^{-1} \delta^{-2} \| u \|_{h,U} \| v \|_{h,U}.
\]

Proof. Indeed, \((id + u) \circ (id + v) = id + v + u \circ (id + v) = id + u + v + (u \circ (id + v) - u)\) and
\[
\| u \circ (id + v) - u \|_{h-\delta/2,U_{\delta/2}} \leq \| Du \|_{h-\delta/4,U_{\delta/4}} \| v \|_{h,U}
\]
\[
\| Du \|_{h-\delta/2,U_{\delta/2}} \| v \|_{h,U} \leq \text{diam}(U)^{-1} \delta^{-2} \| u \|_{h,U} \| v \|_{h,U}.
\]

Lemma A.3. If \( u, v \in \mathcal{O}(\mathbb{T}_h \times U) \) satisfy
\[
\max(\| u \|_{h,U}, \| v \|_{h,U}) \text{diam}(U)^{-1} \delta^{-2} \lesssim 1/2
\]
then

\[(A.386) \quad (id + u) \circ (id + v) \circ (id + u)^{-1} \circ (id + v)^{-1} = id + [u, v] + B_3(u, v)\]

with \([u, v] = Dv \cdot u - Du \cdot v\) and \n\[(A.387) \quad \|B_3(u, v)\|_{h^{-\delta}, U_{\delta/2}} = \text{diam}(U)^{-2}\delta^{-4}(\|u\|_{h, U}^2 \|v\|_{h, U} + \|u\|_{h, U} \|v\|_{h, U}^2).\]

**Proof.** Let \(w\) be defined by \((id + u) \circ (id + v) = (id + w) \circ (id + v) \circ (id + u)\) one has

\[(A.388) \quad v + u \circ (id + v) = u + v \circ (id + u) + w(id + u + v \circ (id + u))\]

and so

\[(A.389) \quad w(id + v + u + v \circ (id + v)) = Du \cdot v - Dv \cdot u + A\]

\[(A.390) \quad = [u, v] + A\]

where

\[(A.391) \quad \|A\|_{h^{-\delta/2}, U_{\delta/2}} \leq (\|D^2 u\|_{h^{-\delta/4}, U_{\delta/4}} \|v\|_{h^{-\delta/4}, U_{\delta/4}} + \|D^2 v\|_{h^{-\delta/4}, U_{\delta/4}} \|u\|_{h^{-\delta/4}, U_{\delta/4}})^2\]

\[(A.392) \quad \leq \text{diam}(U)^{-2}\delta^{-4}\|u\|_{h, U} \|v\|_{h, U}(\|u\|_{h, U} + \|v\|_{h, U})\]

and

\[(A.393) \quad w = ([u, v] + A) \circ (id + v + u + v \circ (id + u))^{-1}\]

\[(A.394) \quad = [u, v] + C_3(u, v)\]

with

\[(A.395) \quad \|C_3(u, v)\|_{h^{-\delta}, U_{\delta}} \leq \|A\|_{h^{-\delta/2}, U_{\delta/2}} + \|D([u, v])\|_{h^{-\delta/4}, U_{\delta/4}}(\|u\|_{h, U} + \|v\|_{h, U})\]

\[(A.396) \quad \leq \text{diam}(U)^{-2}\delta^{-4}\|u\|_{h, U} \|v\|_{h, U}(\|u\|_{h, U} + \|v\|_{h, U}).\]

\[\square\]

**Lemma A.4.** If \(u, v \in \mathcal{O}(T_h \times U)\) satisfy

\[(A.397) \quad \max(\|u\|_{h, U}, \|v\|_{h, U})\text{diam}(U)^{-1}\delta^{-3} \leq 1\]

then

\[(A.398) \quad (id + v) \circ (id + u) \circ (id + v)^{-1} = id + u + [u, v] + D_3(u, v)\]

with \([u, v] = Dv \cdot u - Du \cdot v\) and \n\[(A.399) \quad \|B_3(u, v)\|_{h^{-\delta}, U_{\delta}} = \text{diam}(U)^{-2}\delta^{-4}(\|u\|_{h, U}^2 \|v\|_{h, U} + \|u\|_{h, U} \|v\|_{h, U}^2).\]

**Proof.** If \((id + v) \circ (id + u) = (id + \tilde{u}) \circ (id + v)\) one has

\[(A.400) \quad u + v \circ (id + u) = v + \tilde{u} \circ (id + v)\]

and so

\[(A.401) \quad \tilde{u} = u + v \circ (id + u) - v - (\tilde{u} \circ (id + v) - \tilde{u})\]

\[(A.402) \quad = u + Dv \cdot u - D\tilde{u} \cdot v + I_1\]

\[(A.403) \quad = u + Dv \cdot u - Du \cdot v + I_2\]
Proof.

Indeed, if \( (id + v) \circ (id + u) = (id + u) \circ (id + v) \) one has
\[
\begin{equation}
A.410 
\begin{aligned}
& u + v \circ (id + u) = v + u \circ (id + v) + w(id + v + u \circ (id + v))
\end{aligned}
\end{equation}
\]
and so
\[
\begin{equation}
A.411 
\begin{aligned}
& w(id + u + v + u \circ (id + u)) = -Dv \cdot u + Du \cdot v + B_3(u, v)
\end{aligned}
\end{equation}
\]
\[
\begin{equation}
A.412 
\begin{aligned}
& = [u, v] + O_3(u, v)
\end{aligned}
\end{equation}
\]
\[
\begin{equation}
A.413 
\begin{aligned}
& w = [u, v] \circ (id + u + v + u \circ (id + v))^{-1} + B_3(u, v)
\end{aligned}
\end{equation}
\]
\[
\begin{equation}
A.414 
\begin{aligned}
& = [u, v] + B_3(u, v).
\end{aligned}
\end{equation}
\]

**Lemma A.5.** If \( g - id \in \mathcal{O}(U) \) and satisfies \( \| g - id \|_U \leq 1, \ v \in \mathcal{O}(T_h \times U) \)
\[
\begin{equation}
A.415 
\begin{aligned}
& (id + v) \circ g \circ (id + v)^{-1} = (id + [g] \cdot v) \circ g
\end{aligned}
\end{equation}
\]
where
\[
\begin{equation}
A.416 
\begin{aligned}
& [g] \cdot v = v - (Dg \cdot v) \circ g^{-1}.
\end{aligned}
\end{equation}
\]

**Proof.**

\[
\begin{equation}
A.417 
\begin{aligned}
& (id + v) \circ g \circ (id + v)^{-1} = (id - v + O_2(u)) + v \circ g \circ (id - v + O_2(u))
\end{aligned}
\end{equation}
\]
\[
\begin{equation}
A.418 
\begin{aligned}
& = g - Dg \cdot v + v \circ g + O_2(v)
\end{aligned}
\end{equation}
\]
\[
\begin{equation}
A.419 
\begin{aligned}
& = (id + v - Dg \circ g^{-1} \circ v \circ g^{-1} + O_2(g)) \circ g.
\end{aligned}
\end{equation}
\]

Lemma A.6. If $\Omega \in \mathcal{F}_U$, $Y \in \mathcal{F}_{h,U}$ then

(A.420) \[ f_Y \circ f_\Omega \circ f_Y^{-1} = f_{\Omega + [\Omega]Y + O_2(Y)}. \]

Proof. We just have to prove that if $g = f_\Omega = id + J\nabla \Omega$ and $id + v = f_Y$ then

(A.421) \[ [g] \cdot v = J\nabla ([\Omega] \cdot Y) \]

We have

(A.422) \[ [g] \cdot v = v - (Dg \cdot v) \circ g^{-1} \]

(A.423)

where

(A.424) \[ [\Omega] \cdot Y = Y - Y \circ f_\Omega^{-1} \]

We have $g(\theta, r) = (\theta + J\nabla \Omega(r), r)$ and $g^{-1}(\theta, r) = (\theta - J\nabla \Omega(r), r)$ so $Dg^{-1} = I - JD^2 \Omega$ and

(A.425) \[ J\nabla (Y \circ g^{-1}) = J^t Dg^{-1} \cdot \nabla Y \circ g^{-1} \]

(A.426) \[ = J(I + D^2 \Omega J) \nabla Y \circ g^{-1} \]

(A.427) \[ = J\nabla Y \circ g^{-1} + JD^2 \Omega J\nabla Y \circ g^{-1} \]

(A.428) \[ = (Dg \cdot v) \circ g^{-1}. \]

So

(A.429) \[ J\nabla (Y - Y \circ f_\Omega^{-1}) = v - (Dg \cdot v) \circ g^{-1}. \]

\[ \Box \]

A.2. Proof of Lemma 4.1

Lemma A.7. Any exact symplectic real (symmetric) analytic symplectic diffeomorphism $f : T \times D \rightarrow T \times D$ close enough to the identity is of the form $f_F$ where

(A.430) \[ F = O_1(f - id). \]

Conversely, given $F \in \mathcal{F}_{h,U}$ one has

(A.431) \[ f_F = id + J\nabla F + O_2(F). \]

Proof. For the first part we proceed as follows: if $f : T_h \times U$ is a symplectic diffeomorphism on its image (preserving $T \times \{0\}$) close enough to the identity, the 1-form $Rd\varphi - rd\theta$ is exact of the form $dS$: $Rd\varphi - rd\theta = dS$. So $Rd\varphi +
\[ \theta dr = d(S + r\theta). \]

Since \( dS = \partial_\varphi S d\varphi + \partial_\theta S d\theta = \partial_\varphi S((\partial_\varphi / \partial \theta)d\theta + (\partial_\varphi / \partial r) dr) + \partial_\theta S d\theta \)

\[
(A.432) \quad S(\varphi(t, r), \theta) = \int_0^1 \left[ \partial_\varphi S(\varphi(t, r), \theta) \left( \frac{\partial_\varphi}{\partial \theta}(t, r)\theta + \frac{\partial_\varphi(t, r)}{\partial r} r \right) + \partial_\theta S(\varphi(t, r), \theta) \right] dt
\]

\[
(A.433) \quad = \int_0^1 \left[ R(t, t, r, \theta) \left( \frac{\partial_\varphi}{\partial \theta}(t, r)\theta + \frac{\partial_\varphi(t, r)}{\partial r} r \right) + \theta r \right] dt
\]

\[
(A.434) \quad = -\theta r + \int_0^1 \left[ R(t, t, r, \theta) \left( \frac{\partial_\varphi}{\partial \theta}(t, r)\theta + \frac{\partial_\varphi(t, r)}{\partial r} r \right) \right] dt.
\]

From this we see that

\[
(A.436) \quad \|S + \theta r\|_{h-\delta, U_\delta} \leq \|D(f - id)\|_{h-\delta, U_\delta}
\]

\[
(A.437) \quad \leq \text{diam}(U)^{-1} \delta^{-2} \|f - id\|_{h, U}.
\]

For the second part we write

\[
(A.438) \quad \begin{cases}
R = r - \partial_\varphi F(\varphi, r) \\
\varphi = \theta + \partial_\varphi F(\varphi, r)
\end{cases}
\]

and we observe that

\[
(A.439) \quad \begin{cases}
\|\partial_\varphi F(\varphi, r) - \partial_\varphi F(\theta, r)\|_{h-\delta, U_\delta} \leq \|\partial_\varphi F\|_{h, U} \|\partial_\varphi F\|_{h, U} \\
\|\partial_\varphi F(\varphi, r) - \partial_\varphi F(\theta, r)\|_{h-\delta, U_\delta} \leq \|\partial_\varphi F\|_{h, U} \|\partial_\varphi F\|_{h, U}
\end{cases}
\]

and use (2.11).

\[\square\]

A.3. **Proof of Lemma 4.2** The composition of two exact symplectic maps is again exact symplectic and more precisely

**Lemma A.8.** Let \( F, G \in \mathcal{F}_{h, U} \) and \( \delta \) satisfy then

\[
(A.440) \quad f_G \circ f_F = f_{F + G + B_2(F, G)}.
\]

If \( F \) does not depend on the variable \( r \) then

\[
(A.441) \quad f_G \circ f_F = f_{F + G}.
\]

**Proof.** We notice that if

\[
(A.442) \quad f_F(\varphi, r) = (\varphi, R), \quad \begin{cases}
R = r - \partial_\varphi F(\varphi, r) \\
\theta = \varphi - \partial_\varphi F(\varphi, r)
\end{cases}
\]

\[
(A.443) \quad f_G(\varphi, R) = (\psi, Q), \quad \begin{cases}
Q = R - \partial_\psi G(\psi, R) \\
\varphi = \psi - \partial_\psi G(\psi, R)
\end{cases}
\]

then

\[
(A.444) \quad Q d\psi - r d\theta = Q d\psi - R d\varphi + R d\varphi - r d\theta
\]

\[
(A.445) \quad = d(F + G + (\varphi - \theta)r + (\psi - \varphi)R).
\]
If \( f_G \circ f_F = f_H \) then one has \( Qd\psi - r d\theta = d(-H + r(\psi - \theta)) \) and then
\[
\begin{align*}
0 &= d(-H + F + G + r(\psi - \theta) - r(\varphi - \theta) - R(\psi - \varphi)) \\
&= d(-H + F + G - (R - r)(\psi - \varphi))
\end{align*}
\]
and so
\[
H(\psi, r) = \text{cst} + F(\varphi, r) + G(\psi, R) - (R - r)(\psi - \varphi).
\]
Let us write \( H(\psi, r) = F(\psi, r) + G(\psi, r) + A(\psi, r) \) where
\[
A = F(\varphi, r) - F(\psi, r) + G(\psi, R) - G(\psi, r) + (R - r)(\psi - \varphi).
\]
We can now estimate
\[
\begin{align*}
\|A\|_{h-\delta,U_\delta} &\leq \|\partial_\varphi F\|_{h,U} \|\varphi - \psi\|_{h,U} + \|\partial_R G\|_{h,U} \|R - r\|_{h,U} \\
&\quad + \|\partial_\varphi F(\varphi, r)\|_{h,U} \|\partial_R G(\psi, R)\|_{h,U} \\
&\quad \leq \|\partial_\varphi F\|_{h,U} \|\partial_R G\|_{h,U} + \|\partial_R G\|_{h,U} \|\partial_\varphi F\| \\
&\quad + \|\partial_\varphi F(\varphi, r)\|_{h,U} \|\partial_R G(\psi, R)\|_{h,U}.
\end{align*}
\]
\[
\Box
\]

**Remark A.1.** The preceding proof shows in fact that
\[
f_G \circ f_F = f_{F+G+B_\delta(\nabla F, \nabla G)}.
\]

**Lemma A.9.** If \( F, G \in \mathcal{F}_{h,U} \), with \( \text{diam}(U)^{-1}\delta^{-2} \max(\|F\|_{h,U}, \|G\|_{h,U}) \leq 1 \)
\[
f_{F+G} = f_{A_1(F,G)} \circ f_G
\]
where
\[
\|A_1(F, G)\|_{h-\delta,U_\delta} \leq \text{diam}(U)^{-1}\delta^{-2}\|F\|_{h,U}.
\]

**Proof.** We write
\[
\begin{align*}
\begin{cases}
R = r - \partial_\varphi (G + F)(\varphi, r) \\
\theta = \varphi - \partial_r (G + F)(\varphi, r)
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\tilde{R} = r - \partial_\varphi G(\tilde{\varphi}, r) \\
\tilde{\theta} = \tilde{\varphi} - \partial_r G(\tilde{\varphi}, r)
\end{cases}
\end{align*}
\]
We have
\[
\begin{align*}
\begin{cases}
\tilde{R} - R = \partial_\varphi G(\varphi, r) - \partial_\varphi G(\tilde{\varphi}, r) + \partial_\varphi F(\varphi, r) \\
\tilde{\varphi} - \varphi = \partial_r G(\tilde{\varphi}, r) - \partial_r G(\varphi, r) - \partial_r F(\varphi, r)
\end{cases}
\end{align*}
\]
From the second equation we see that as long as \((\varphi, r), (\tilde{\varphi}, r)\) are in \( \mathbb{T}_h \times U \)
\[
|\tilde{\varphi} - \varphi| \leq \|\partial_r G\|_{h,U}|\tilde{\varphi} - \varphi| + \|\partial_r F\|_{h,U}
\]
and so if \( \|\partial_r G\|_{h,U} \leq 1/2 \)
\[
|\tilde{\varphi} - \varphi| \leq 2\|\partial_r F\|_{h,U}.
\]
Now the first inequality of (A.460) gives
\[ |\bar{R} - R| \leq \| \partial_x G \|_{h,U} |\tilde{\varphi} - \varphi| + \| \partial_x F \|_{h,U} \]
\[ \leq 2\| \partial_x F \|_{h,U} \| \partial_x G \|_{h,U} + \| \partial_x F \|_{h,U}. \]

\[ \square \]

**Appendix B. Approximations by vector fields**

**Lemma B.1.** Let \( \Omega(r) = O(r^2) \), \( F, G \in \mathcal{F}_{h,\rho} \) such that \( \max(\|D^2 F\|_{h,\rho}, \|D^2 G\|_{h,\rho}) \leq 1 \). There exists \( A(F,G) \) such that
\[ f_G \circ \Phi_{\Omega + F} = f_A(F,G) \circ \Phi_{\Omega + F + G - \Phi_{\Omega/2}} \]
with
\[ A(F,G) \|_{h-\delta,e-\delta,\rho} \leq \rho^5 \| F \|_{h,\rho} + \rho^{-8} \delta^{-10} (\| F \|_{h,\rho} + \| G \|_{h,\rho}) |\mathcal{G}|_{h,\rho}. \]

**Proof.** We denote by \( \tilde{h} = h - \delta, \tilde{\rho} = e^{-\delta} \rho \) and we assume that \( F \) and \( G \) are small enough so that the images of the domain \( \mathcal{D}_h \times D(0,\tilde{\rho}) \) by \( \Phi_{\Omega}, \Phi_{\Omega + F}, \Phi_{\Omega + F + G} \) are contained in \( \mathcal{T}_h \times D(0,\rho) \).

Let \( \omega(r) = \nabla \Omega(r) \).

Let \( r, \theta \) be the observables \((\theta, r) \mapsto r \) and \((\theta, r) \mapsto \theta \). We have
\[
\begin{align*}
\frac{d}{dr} r(\Phi_{\Omega + F + G}^t) &= \{ \Omega + F + G, r \}(\Phi_{\Omega + F + G}^t) = \{ F + G, r \}(\Phi_{\Omega + F + G}^t) \\
\frac{d}{d\theta} (\Phi_{\Omega + F + G}^t) &= \{ \Omega + F + G, \theta \}(\Phi_{\Omega + F + G}^t) = (\omega(r) + \{ F + G, \theta \}) (\Phi_{\Omega + F + G}^t)
\end{align*}
\]
and so for \( t \in [0,1] \)
\[
\begin{align*}
\frac{d}{dt} r(\Phi_{\Omega + F + G}^t) &= O(\varepsilon) \\
\theta(\Phi_{\Omega + F + G}^t) &= \theta + t \omega(r) + O(\varepsilon)
\end{align*}
\]
hence
\[ \Phi_{\Omega + F + G}^t(\theta, r) = \Phi_{\Omega}^t(\theta + t \omega(r), r) + O(\varepsilon) \]
where \( \varepsilon = \max(\| \nabla F \|_{\tilde{h},\tilde{\rho}}, \| \nabla G \|_{h,\rho}) \). The same estimates hold for \( \Phi_{\Omega + F}^t \). We can also compute
\[ \frac{d}{dt} r(\Phi_{\Omega + F + G}^t) - \frac{d}{dt} r(\Phi_{\Omega + F}^t) = \{ F + G, r \} (\Phi_{\Omega + F + G}^t) - \{ F, r \} (\Phi_{\Omega + F}^t) \]
\[ = \{ F, r \} (\Phi_{\Omega + F + G}^t) - \{ F, r \} (\Phi_{\Omega + F}^t) + \{ G, r \} (\Phi_{\Omega + F + G}^t) \]
\[ \frac{d}{dt} \theta(\Phi_{\Omega + F + G}^t) - \frac{d}{dt} \theta(\Phi_{\Omega + F}^t) = \{ F + G, \theta \} (\Phi_{\Omega + F + G}^t) - \{ F, \theta \} (\Phi_{\Omega + F}^t) \]
\[ + r(\Phi_{\Omega + F + G}^t) - r(\Phi_{\Omega + F}^t) \]
\[ = \{ F, \theta \} (\Phi_{\Omega + F + G}^t) - \{ F, \theta \} (\Phi_{\Omega + F}^t) + \{ G, \theta \} (\Phi_{\Omega + F + G}^t) \]
\[ + r(\Phi_{\Omega + F + G}^t) - r(\Phi_{\Omega + F}^t). \]

Let us denote for \( x = (\theta, r) \),
\[ \Delta_r(t, x) = r(\Phi_{\Omega + F + G}^t)(x) - r(\Phi_{\Omega + F}^t)(x) \]
\[
\Delta_{\theta}(t, x) = \theta(\Phi_{\Omega + F, + G}^t)(x) - \theta(\Phi_{\Omega + F}^t)(x).
\]

Since \( \{G, r\} = \partial_{\theta} G \) and \( \{G, \theta\} = -\partial_{r} G \), we get using Taylor formula and the fact that \( \omega(r) = r + O(r^2) \)

\[
\{G, r\}(\Phi_{\Omega + F, + G}^t(x)) = \{G, r\}(\theta + t\sigma(r) + O(\varepsilon), r + O(\varepsilon))
\]

\[
= \partial_{\theta} G(\theta + t\omega(r), r) + O(\|D^2 G\|_{\tilde{h}, \tilde{\rho}})\varepsilon
\]

\[
= (\partial_{\theta} G(\theta, r) + t\omega(r)\partial_{\theta}^2 G(\theta, r) + O(\rho^2 \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \|D^2 G\|_{\tilde{h}, \tilde{\rho}}))
\]

and

\[
\{G, \theta\}(\Phi_{\Omega + F, + G}^t(x)) = \{G, \theta\}(\theta + t\omega(r) + O(\varepsilon), r + O(\varepsilon))
\]

\[
= -\partial_{\theta} G(\theta + t\omega(r), r) + O(\|D^2 G\|_{\tilde{h}, \tilde{\rho}})\varepsilon
\]

\[
= -\partial_{\theta} G(\theta, r) - t\omega(r)\partial_{\theta}^2 G(\theta, r) + O(\rho^2 \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \|D^2 G\|_{\tilde{h}, \tilde{\rho}})).
\]

On the other hand we have

\[
|\{F, r\}(\Phi_{\Omega + F, + G}^t(x)) - \{F, r\}(\Phi_{\Omega + F}^t(x))| = |\partial_{\theta} F(\Phi_{\Omega + F}^t(x)) + (\Delta_{\theta}(t, x), \Delta_{r}(t, x))
\]

\[
- \partial_{\theta} F(\Phi_{\Omega + F}^t(x))| \leq \|D\partial_{\theta} F\|_{\tilde{h}, \tilde{\rho}}(|\Delta_{\theta}(t, x)| + |\Delta_{r}(t, x)|)
\]

and similarly

\[
|\{F, \theta\}(\Phi_{\Omega + F, + G}^t(x)) - \{F, \theta\}(\Phi_{\Omega + F}^t(x))| \leq \|D\partial_{\theta} F\|_{\tilde{h}, \tilde{\rho}}(|\Delta_{\theta}(t, x)| + |\Delta_{r}(t, x)|).
\]

We introduce

(B.471) \[
\Delta(t, x) = |\Delta_{\theta}(t, x)| + |\Delta_{r}(t, x)|.
\]

Consequently, from (B.467)

\[
\Delta_{r}(t, x) = t\partial_{\theta} G(\theta, r) + (t^2/2)\omega(r)\partial_{\theta}^2 G(\theta, r) + O(\|D^2 F\|_{\tilde{h}, \tilde{\rho}} \int_0^t \Delta(s, x) ds)
\]

\[
+ O(\rho^2 \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \|D^2 G\|_{\tilde{h}, \tilde{\rho}}).\]

The same way,

(B.472) \[
\Delta_{\theta}(t, x) = -t\partial_{\theta} G(\theta, r) - (t^2/2)\omega(r)\partial_{\theta}^2 G(\theta, r) + \int_0^t \Delta_{r}(s, x) ds
\]

\[
+ O(\|D^2 F\|_{\tilde{h}, \tilde{\rho}} \int_0^t \Delta(s, x) ds) + O(\rho^2 \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \|D^2 G\|_{\tilde{h}, \tilde{\rho}}).
\]

and

(B.473) \[
\Delta_{\theta}(t, x) = -t\partial_{r} G(\theta, r) - (t^2/2)\omega(r)\partial_{r}^2 G(\theta, r)
\]

\[
+ (t^3/6)\omega(r)\partial_{r}^2 G(\theta, r) + O(\|D^2 F\|_{\tilde{h}, \tilde{\rho}} \int_0^t \Delta(s, x) ds)
\]

\[
+ O(\rho^2 \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \max_{j=1, 2} \|\partial^{2j}_{\theta} \partial_{r} G\|_{\tilde{h}, \tilde{\rho}} + \|D^2 G\|_{\tilde{h}, \tilde{\rho}}).\]
From (B.472) and (B.473) we obtain
\[
\Delta(t,x) \leq C \|D^2F\|_{h,\rho} \int_0^t \Delta(s,x)ds + B
\]
where
\[
B = O(\|DG\|_{\tilde{h},\tilde{\rho}} + \rho \|D^2G\|_{\tilde{h},\tilde{\rho}}) + O((\max_{j=1,2} \|\partial^j_\rho \partial_\theta G\|_{\tilde{h},\tilde{\rho}} + \max_{j=1,2} \|\partial^j_\rho \partial_\theta G\|_{\tilde{h},\tilde{\rho}}) + \|D^2G\|_{\tilde{h},\tilde{\rho}}).
\]
By Gronwall inequality
\[
\Delta(t,x) \leq te^{C \|D^2F\|_{\tilde{h},\tilde{\rho}}} B \leq B
\]
and thus
\[
O(\|D^2F\|_{\tilde{h},\tilde{\rho}} \int_0^t \Delta(s,x)ds) = O(\|D^2F\|_{\tilde{h},\tilde{\rho}} B) \leq O(A_\delta(\tilde{h},\tilde{\rho}))
\]
where
\[
A_\delta(h,\rho) = \rho^2(\max_{j=1,2} \|\partial^j_\rho \partial_\theta G\|_{h-\delta,\rho} + \max_{j=1,2} \|\partial^j_\rho \partial_\theta G\|_{h-\delta,\rho}) +
(\|D^2F\|_{h-\delta,\rho} + \|D^2G\|_{h-\delta,\rho}) \|D^2G\|_{h-\delta,\rho} + \rho \max_{j=1,2} \|\partial^j_\rho \partial_\theta G\|_{\tilde{h},\tilde{\rho}}
\]
and for short \( A = A_\delta(h,\rho). \)

For \( t = 1 \) the preceding equations become
\[
\Delta_r(1,x) = \partial_\theta G(\theta, r) + (\omega(r)/2) \partial^2_\theta G(\theta, r) + O(A)
\]
\[
\Delta_\rho(1,x) = -\partial_r G(\theta, r) - (\omega(r)/2) \partial^2_\theta G(\theta, r)
+ (1/2) \partial_\theta G(\theta, r) + (\omega(r)/6) \partial^3_\theta G(\theta, r) + O(A)
\]
and using Taylor formula again
\[
\Delta_r(1,x) = \partial_\theta G(\theta + \omega(r)/2, r) + O(A)
\]
\[
\Delta_\rho(t,x) = -\partial_r G(\theta + \omega(r)/2, r) + (1/2) \partial_\theta G(\theta + \omega(r)/2, r)
+ O(\rho \|\partial^3_\theta G\|_{\tilde{h},\tilde{\rho}}) + O(A)
\]
We can rewrite this last formula
\[
\Delta_\rho(t,x) = -\partial_r G(\theta + \omega(r)/2, r) + (1/2) \partial_\theta G(\theta + \omega(r)/2, r)
+ O(\rho \|\partial^3_\theta G\|_{\tilde{h},\tilde{\rho}}) + O(A)
\]
Now let's introduce
(B.474) \( \tilde{G}(\theta, r) = G(\theta - \omega(r)/2, r) \)
then
(B.475) \( \partial_\theta \tilde{G}(\theta, r) = \partial_\theta G(\theta - \omega(r)/2, r) \)
and since $\omega'(r) = 1 + O(r)$

\begin{equation}
\partial_t \tilde{G}(\theta, r) = -(1/2)\omega'(r)\partial_\theta \theta G(\theta - \omega(r)/2, r) + \partial_r G(\theta - \omega(r)/2, r)
\end{equation}

so that

\begin{equation}
\begin{aligned}
\Delta_t(t, x) &= \partial_\theta \tilde{G}(\theta + \omega(r), r) + O(A) \\
\Delta_\theta(t, x) &= -\partial_r \tilde{G}(\theta + \omega(r), r) + O(A)
\end{aligned}
\end{equation}

Finally,

\begin{equation}
\Phi_{\Omega+F+G} = \Phi_{\Omega+F} + J\nabla \tilde{G} \circ \Phi_{\Omega} + O(A)
\end{equation}

we get

\begin{equation}
\Phi_{\Omega+F+G} = \Phi_{\Omega+G} + O(A)
\end{equation}

or since $\tilde{G} = G \circ \Phi_{-\Omega/2}$

\begin{equation}
\Phi_{\Omega+F+G} \circ \Phi_{\Omega/2} = \Phi_{G_2} \circ \Phi_{G} \circ \Phi_{\Omega+F}
\end{equation}

with

\begin{equation}
\|G_2\| = O(A).
\end{equation}

We now estimate $A_\delta(h, \rho)$:

\begin{equation}
A_\delta(h, \rho) \lesssim |h|^{5-5} \rho^{-8} \delta^{-10}(\|F\|_{h, \rho} + \|G\|_{h, \rho}) \|F\|_{h, \rho}
\end{equation}

\[ \square \]

**Lemma B.2.** Let $F, S \in C^\omega_{h, \rho}$ such that $\|F\|_{h, \rho} \lesssim \rho^9$, and $\Omega, (r \mapsto \Omega(r))$. Then, there exists $\Pi \in C^\omega_{h/2, \rho/2}$ such that

\begin{equation}
f_F \circ \Phi_\Omega = f_{F_2} \circ \Phi_\Pi
\end{equation}

with

\[ \Pi = \Omega + F + O_1(\rho F) \]

and

\begin{equation}
\|F_2\|_{h/2, \rho/2} \lesssim \exp(-\frac{1}{\rho^{1/20}}).
\end{equation}
Proof. Let $\delta_n = c/(n + 1)^2$, $h_n = h - \delta_n$, $\rho_n = e^{-\delta_n} \rho_n$ and $c$ chosen such that $h_n \geq h/2$, $\rho_n \geq \rho/2$ for all $n$. Using Lemma B.1 we construct sequences $S_n, G_n$ such that $S_0 = 0$, $G_0 = F$

\begin{equation}
 f_{G_n} \circ \Phi_{\Omega + S_n} = f_{G_{n+1}} \circ \Phi_{\Omega + S_{n+1}}
\end{equation}

\begin{equation}
 \begin{cases}
 S_{n+1} = S_n + G_n \circ \Phi_{\Omega/2} \\
 G_{n+1} = A(S_n, G_n)
\end{cases}
\end{equation}

with
\[ \|S_{n+1}\|_{h_n+1, \rho_{n+1}} \lesssim \|S_n\|_{h_n, \rho_n} + \|G_n\|_{h_n, \rho_n} \]

and
\begin{equation}
 \|A(S_n, G_n)\|_{h_n, \rho_n} \lesssim \rho \delta_n^{-5} \|G_n\|_{h_n, \rho_n} + \rho^{-8} \delta_n^{-10} (\|S_n\|_{h_n, \rho_n} + \|G_n\|_{h_n, \rho_n}) \|G_n\|_{h_n, \rho_n}.
\end{equation}

With $\varepsilon_n = \|G_n\|_{h_n, \rho_n}$ and $\sigma_n := \|S_n\|_{h, \rho}$ we have
\[ s_{n+1} \leq s_n + \varepsilon_n \]

and
\begin{equation}
 \varepsilon_{n+1} \leq \rho \delta_n^{-10} (\delta_n^5 + \rho^{-9} (\sigma_0 + \varepsilon_0 + \cdots + \varepsilon_n)) \varepsilon_n
\end{equation}

and if for all $0 \leq n \leq k$
\begin{equation}
 \rho^{-9} (\sigma_0 + \varepsilon_0 + \cdots + \varepsilon_n) \leq 1
\end{equation}

we have
\begin{equation}
 \varepsilon_{n+1} \leq (2\rho) \delta_n^{-10} \varepsilon_n
\end{equation}

that is
\begin{equation}
 \varepsilon_{n+1} \leq (2c^{-1}) \rho n^{120} \varepsilon_0.
\end{equation}

Since $a^n n!^b \leq \exp(n \ln a + bn \ln n)$ we see that for $n \leq (1/(ae))^{1/6}$ one has $a^n n!^b \leq e^{-n}$ and so if $\rho^{-9}(\sigma_0 + \varepsilon_0) \leq 1$ condition B.493 is satisfied. We then have for $k = (1/(C\rho))^{1/20}$,
\[ \varepsilon_k \leq \exp(- (1/(C\rho))^{1/20}). \]

To conclude we notice that we notice that
\[ (G_0 + G_1 + \cdots + G_n) \circ \Phi_{\Omega/2} = F + \mathcal{O}_1(\rho F). \]

\[ \square \]
Appendix C. Some other lemmas

Lemma C.1. Let $a \in \mathbb{C}^*$. There exist a unique function $m_a$ univalent on $\mathbb{C} \setminus D(0, |a|^{1/2})$ such that

\begin{equation}
\label{C.496}
m_a^2(z) = z^2 + a.
\end{equation}

It satisfies for $z, z' \in \{w \in \mathbb{C}, |w| > t^{-1}|a|^{1/2}\}$

\begin{equation}
\label{C.497}
(1 - t) \leq \left| \frac{m_a(z) - m_a(z')}{z - z'} \right| \leq \frac{1}{1 - t}.
\end{equation}

We denote it by $m_a(z) = (z^2 + a)^{1/2}$. One has for $a, a' \in \mathbb{C}^*$

\begin{equation}
\label{C.498}
|m_a(z) - m_{a'}(z)| \leq \frac{|a - a'|}{|m_a(z) + m_{a'}(z)|}.
\end{equation}

**Proof.** The existence of $m_a$ follows from a classical monodromy argument. The map $z \mapsto 1/m_a(|a|^{1/2}z^{-1})$ can be extended to a holomorphic function defined on the unit disk $\mathbb{D}$ and sends the circle $\{|z| = t\}$ in $\{(t/(1 - t) \leq |z|\}$.

We then get the conclusion by Schwarz Lemma.

Lemma C.2. Let $f \in C_0^0(\mathbb{T})$ be such that

\begin{equation}
\label{C.499}
\|f\|_{L^2(\mathbb{T})} \leq \delta \|f\|_{C_0^0(\mathbb{T})} + \mu.
\end{equation}

Then

\begin{equation}
\label{C.500}
\|f\|_{C_0^0(\mathbb{T})} \leq \delta^{-1} \mu + \frac{1}{h} e^{-\frac{2\pi h}{10}} \|f\|_h.
\end{equation}

**Proof.** If

\begin{equation}
\label{C.501}
f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k \theta}
\end{equation}

is the Fourier expansion of $f$, one has for $N \in \mathbb{N}$

\begin{equation}
\label{C.502}
\|f\|_{C_0^0} \leq \sum_{|k| \leq N} |\hat{f}(k)| + \frac{1}{h} e^{-2\pi h N} \|f\|_h
\end{equation}

\begin{equation}
\label{C.503}
\leq (2N + 1)^{1/2} \|f\|_{L^2(\mathbb{T})} + \frac{1}{h} e^{-2\pi h N} \|f\|_h
\end{equation}

\begin{equation}
\label{C.504}
\leq (2N + 1)^{1/2} (\delta \|f\|_{C_0^0(\mathbb{T})} + \mu) + \frac{1}{h} e^{-2\pi h N} \|f\|_h.
\end{equation}

If we choose $N = \delta^{-2}/10$ we have $(2N + 1)^{1/2} \delta < 1/2$ and

\begin{equation}
\label{C.505}
\|f\|_{C_0^0(\mathbb{T})} \leq \delta^{-1} \mu + \frac{2}{h} e^{-h/(2\delta^2)} \|f\|_h.
\end{equation}

By the three circles theorem

\begin{equation}
\label{C.506}
\|f\|_h \leq \|f\|_{C_0^0(\mathbb{T})} \|f\|_h^{1-t}
\end{equation}

\begin{equation}
\label{C.507}
\leq (\delta^{-1} \mu + \frac{2}{h} e^{-h/(2\delta^2)} \|f\|_h) (1-t) \|f\|_h^{1-t}.
\end{equation}
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