Holomorphic $L^p$-type for sub-Laplacians on connected Lie groups

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Abstract

We study the problem of determining all connected Lie groups $G$ which have the following property (hlp): every sub-Laplacian $L$ on $G$ is of holomorphic $L^p$-type for $1 \leq p < \infty$, $p \neq 2$. First we show that semi-simple non-compact Lie groups with finite center have this property, by using holomorphic families of representations in the class one principal series of $G$ and the Kunze-Stein phenomenon. We then apply an $L^p$-transference principle, essentially due to Anker, to show that every connected Lie group $G$ whose semi-simple quotient by its radical is non-compact has property (hlp). For the convenience of the reader, we give a self-contained proof of this transference principle, which generalizes the well-known Coifman-Weiss principle. One is thus reduced to studying those groups for which the semi-simple quotient is compact, i.e. to compact extensions of solvable Lie groups. In this article, we consider semi-direct extensions of exponential solvable Lie groups by connected compact Lie groups. It had been proved in [8] that every exponential solvable Lie group $S$, which has a non-regular co-adjoint orbit whose restriction to the nilradical is closed, has property (hlp), and we show here that (hlp) remains valid for compact extensions of these groups.

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1 Introduction

A comprehensive discussion of the problem studied in this article, background information and references to further literature can be found in [14]. We shall therefore content ourselves in this introduction by recalling some notation and results from [14].

Let \((X, d\mu)\) be a measure space. If \(T\) is a self-adjoint linear operator on a \(L^2\)-space \(L^2(X, d\mu)\), with spectral resolution \(T = \int_{\mathbb{R}} \lambda dE_\lambda\), and if \(F\) is a bounded Borel function on \(\mathbb{R}\), then we call \(F\) an \(L^p\)-multiplier for \(T\) \((1 \leq p < \infty)\), if \(F(T) := \int_{\mathbb{R}} F(\lambda)dE_\lambda\) extends from \(L^p \cap L^2(X, d\mu)\) to a bounded operator on \(L^p(X, d\mu)\). We shall denote by \(\mathcal{M}_p(T)\) the space of all \(L^p\)-multipliers for \(T\), and by \(\sigma_p(T)\) the \(L^p\)-spectrum of \(T\). We say that \(T\) is of holomorphic \(L^p\)-type, if there exist some non-isolated point \(\lambda_0\) in the \(L^2\)-spectrum \(\sigma_2(T)\) and an open complex neighborhood \(U\) of \(\lambda_0\) in \(\mathbb{C}\), such that every \(F \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})\) extends holomorphically to \(U\). Here, \(C_\infty(\mathbb{R})\) denotes the space of all continuous functions on \(\mathbb{R}\) vanishing at infinity.

Assume in addition that there exists a linear subspace \(\mathcal{D}\) of \(L^2(X)\) which is \(T\)-invariant and dense in \(L^p(X)\) for every \(p \in [1, \infty[\), and that \(T\) coincides with the closure of its restriction to \(\mathcal{D}\). Then, if \(T\) is of holomorphic \(L^p\)-type, the set \(U\) belongs to the \(L^p\)-spectrum of \(T\), i.e.

\[ \overline{U} \subset \sigma_p(T). \]

In particular,

\[ \sigma_2(T) \subsetneq \sigma_p(T). \]

Throughout this article, \(G\) will denote a connected Lie group.

Let \(dg\) be a left-invariant Haar measure on \(G\). If \((\pi, \mathcal{H}_\pi)\) is a unitary representation of \(G\) on the Hilbert space \(\mathcal{H} = \mathcal{H}_\pi\), then we denote the integrated representation of \(L^1(G) = L^1(G, dg)\) again by \(\pi\), i.e. \(\pi(f)\xi := \int_G f(g)\pi(g)\xi dg\) for every \(f \in L^1(G)\), \(\xi \in \mathcal{H}\). For \(X \in \mathfrak{g}\), we denote by \(d\pi(X)\) the infinitesimal generator of the one-parameter group of unitary operators \(t \mapsto \pi(\exp tX)\). Moreover, we shall often identify \(X\) with the corresponding right-invariant vector field \(X^r f(g) := \lim_{t \to 0} \frac{1}{t}[f((\exp tX)g) - f(g)]\) on \(G\) and write \(X = X^r\).
$L^2(G)$ and hypoelliptic. Denote by $\{e^{-tL}\}_{t>0}$ the heat semigroup generated by $L$. Since $L$ is right $G$-invariant, for every $t > 0$, $e^{-tL}$ admits a convolution kernel $h_t$ such that

$$e^{-tL}f = h_t * f,$$

where $*$ denotes the usual convolution product in $L^1(G)$. The function $(t, g) \mapsto h_t(g)$ is smooth on $\mathbb{R}_{>0} \times G$, since the differential operator $\frac{\partial}{\partial t} + L$ is hypoelliptic. Moreover, by [20, Theorem VIII.4.3 and Theorem V.4.2], the heat kernel $h_t$ as well as its right-invariant derivatives admit Gaussian type estimates in terms of the Carathéodory distance $\delta$ associated to the Hörmander system $X_1, \ldots, X_k$.

In particular, for every right-invariant differential operator $D$ on $G$, there exist constants $c_{D,t}, C_{D,t} > 0$, such that

$$|Dh_t(g)| \leq C_{D,t}e^{-c_{D,t}t\delta(g,e)^2}, \quad \text{for all } g \in G, t > 0. \tag{1.1}$$

Let now $F_0 \in \mathcal{M}_p(L)$. By duality, we may assume that $1 \leq p \leq 2$. With $F_0$, also the function $\lambda \mapsto F(\lambda) := e^{-\lambda}F_0(\lambda)$ lies in $\mathcal{M}_p(L)$, since $F(L) = e^{-L}F_0(L)$, where the heat operator $e^{-L}$ is bounded on every $L^p(G) (1 \leq p < \infty)$. Now by [14, Lemma 6.1], the operator $F_0(L)$ is bounded also on all the spaces $L^q(G)$, $p \leq q \leq p'$. Hence for every test function $f$ on $G$,

$$F(L)(f) = F_0(L)(e^{-L}(f)) = F_0(L)(h_1 * f) = F_0(L)(h_{1/2} * h_{1/2} * f) = (F_0(L)h_{1/2}) * h_{1/2} * f,$$

by the right invariance of the operator $F_0(L)$. Since $h_{1/2}$ is contained in every $L^q(G)$, $1 \leq q \leq \infty$, in particular in $L^1(G)$, we see that the operator $F(L)$ acts by convolution from the left with the function $(F_0(L)h_{1/2}) * h_{1/2}$ which is contained in every $L^q(G)$, $p \leq q \leq p'$, and so are all its derivatives from the right. We can thus identify the operator $F(L)$ with the $C^\infty$-function $F(L)\delta := (F_0(L)h_{1/2}) * h_{1/2}$, i.e.

$$F(L)(f) = (F(L)\delta) * f, \quad f \in \bigcup_{p \leq q \leq p'} L^q(G).$$

Recall that the modular function $\Delta_G$ on $G$ is defined by the equation

$$\int_G f(xg)dx = \Delta_G(g)^{-1} \int_G f(x)dx, \quad g \in G.$$

We put:

$$\hat{f}(g) := f(g^{-1}), \quad f^*(g) := \Delta_G^{-1}(g)f(g^{-1}).$$

Then $f \mapsto f^*$ is an isometric involution on $L^1(G)$, and for any unitary representation $\pi$ of $G$, we have:

$$\pi(f)^* = \pi(f^*). \tag{1.2}$$

The group $G$ is said to be symmetric, if the associated group algebra $L^1(G)$ is symmetric, i.e. if every element $f \in L^1(G)$ with $f^* = f$ has a real spectrum with respect to the
involutive Banach algebra $L^1(G)$.

In this paper we consider connected Lie groups for which every sub-Laplacian is of holomorphic $L^p$-type. First, in the Section 2, we consider connected semi-simple Lie groups $G$ with finite center. We construct a holomorphic family of representations $\pi(z)$ of $G$ on mixed $L^p$-spaces (see Section 2.2). Applying these representations to $h_1$, we obtain a holomorphic family of compact operators on these spaces (see Section 2.3). Using the Kunze-Stein phenomenon on semi-simple Lie groups (see Section 2.4), the eigenvectors of the operators $\pi(z)(h_1)$ allow us to construct a holomorphic family of $L^p$-functions on $G$ which are eigenvectors for $F(L)$, if $F \in \mathcal{M}_p(T) \cap C_\infty(\mathbb{R})$. From the corresponding holomorphic family of eigenvalues we can read off that $F$ admits a holomorphic extension in a neighborhood of some element in the spectrum of $L$ (see Section 2.5). This gives us:

**Theorem 1.1.** Let $G$ be a non compact connected semi-simple Lie group with finite center. Then every sub-Laplacian on $G$ is of holomorphic $L^p$-type, for $1 \leq p < \infty$, $p \neq 2$.

**Remark.** Even if at the end of the proof, we consider only ordinary $L^p$-spaces, we need representations on mixed $L^p$-spaces. They are used to get some isometry property and then to apply the Kunze-Stein phenomenon.

In Sections 3.1 and 3.2, we discuss respectively $p$-induced representations and a generalization of the Coifman-Weiss transference principle [5]. We consider a separable locally compact group $G$, and an isometric representation $\rho$ of a closed subgroup $S$ of $G$ on spaces of $L^p$-type, e.g. $L^p$-spaces $L^p(\Omega)$. Denote by $\pi_p := \text{ind}^G_{p,S} \rho$ the $p$-induced representation of $\rho$. We prove, among other results, that, for any function $f \in L^1(G)$, the operator norm of $\pi_p(f)$ is bounded by the norm of the convolution operator $\lambda_G(f)$ on $L^p(G)$, provided the group $S$ is amenable. Here, $\lambda_G$ denotes the left-regular representation. It should be noted that we do not require the group $G$ to be amenable.

As an application we obtain the $L^p$-transference of a convolution operator on $G$ to a convolution operator on the quotient group $G/S$, in the case where $S$ is an amenable closed, normal subgroup.

When preparing this article, we were not aware of J.-Ph. Anker’s article [1] which, to a large extent, contains these transference results, and which we also recommend for further references to this topic. We are indebted to N. Lohoué for informing us on Anker’s work [1] as well as on the influence of C. Herz on the development of this field (compare [9]). For the convenience of the reader, we have nevertheless decided to include our approach to these transference results, since it differs from Anker’s by the use of a suitable cross section for $G/S$, which we feel makes the arguments a bit easier.

Applying this transference principle, we obtain the following generalization of Theorem 1.1 in Section 4:

**Theorem 1.2.** Let $G = \exp \mathfrak{g}$ be a connected Lie group, and denote by $S = \exp \mathfrak{s}$ its radical. If $G/S$ is not compact, then every sub-Laplacian on $G$ is of holomorphic $L^p$-type, for any $1 \leq p < \infty$, $p \neq 2$. 
It then suffices to study connected Lie groups for which $G/S$ is compact. In Section 5, we shall consider groups $G$ which are the semi-direct product of a compact group $K$ with a non-symmetric exponential solvable group $S$ from a certain class. The exponential solvable non-symmetric Lie groups have been completely classified by Poguntke [18] (with previous contributions by Leptin, Ludwig and Boidol) in terms of a purely Lie-algebraic condition (B). Let us describe this condition, which had been first introduced by Boidol in a different context [3].

Recall that the unitary dual of $S$ is in one to one correspondence with the space of coadjoint orbits in the dual space $s^*$ of $s$ via the Kirillov map, which associates with a given point $\ell \in s^*$ an irreducible unitary representation $\pi_\ell$ (see, e.g., [8, Section 1]). If $\ell$ is an element of $s^*$, denote by

$$s(\ell) := \{X \in s \mid \ell([X,Y]) = 0, \text{ for all } Y \in s\}$$

the stabilizer of $\ell$ under the coadjoint action $\text{ad}^*$. Moreover, if $m$ is any Lie algebra, denote by

$$m = m^1 \supset m^2 \supset \ldots$$

the descending central series of $m$, i.e. $m^2 = [m,m]$, and $m^{k+1} = [m,m^k]$. Put

$$m^\infty = \bigcap_k m^k.$$ 

Then $m^\infty$ is the smallest ideal $t$ in $m$ such that $m/t$ is nilpotent. Put

$$m(\ell) := s(\ell) + [s,s].$$

Then we say that $\ell$ respectively the associated coadjoint orbit $\Omega(\ell) := \text{Ad}^*(G)\ell$ satisfies Boidol’s condition (B), if

(B) \quad \ell \mid_{m(\ell)^\infty} \neq 0.

According to [18], the group $S$ is non-symmetric if and only if there exists a coadjoint orbit satisfying Boidol’s condition.

If $\Omega$ is a coadjoint orbit, and if $n$ is the nilradical of $s$, then

$$\Omega|_n := \{\ell|_n : \ell \in \Omega\} \subset n^*$$

will denote the restriction of $\Omega$ to $n$.

We show that the methods developed in [8] can also be applied to the case of a compact extension of an exponential solvable group and thus obtain

**Theorem 1.3.** Let $G = K \ltimes S$ be a semi-direct product of a compact Lie group $K$ with an exponential solvable Lie group $S$, and assume that there exists a coadjoint orbit $\Omega(\ell) \subset s^*$ satisfying Boidol’s condition, whose restriction to the nilradical $n$ is closed in $n^*$. Then every sub-Laplacian on $G$ is of holomorphic $L^p$-type, for $1 \leq p < \infty$, $p \neq 2$.

**Remarks.**
(a) A sub-Laplacian $L$ on $G$ is of holomorphic $L^p$-type if and only if every continuous bounded multiplier $F \in \mathcal{M}_p(L)$ extends holomorphically to an open neighborhood of a non-isolated point in $\sigma_2(L)$.

(b) If the restriction of a coadjoint orbit to the nilradical is closed, then the orbit itself is closed (see [8, Thm. 2.2]).

(c) What we really use in the proof is the following property of the orbit $\Omega$:

\[ \Omega \text{ is closed, and for every real character } \nu \text{ of } \mathfrak{s} \text{ which does not vanish on } \mathfrak{s}(\ell), \text{ there exists a sequence } \{\tau_n\} \text{ of real numbers such that } \lim_{n \to \infty} (\Omega + \tau_n\nu) = \infty \text{ in the orbit space.} \]

This property is a consequence of the closedness of $\Omega|_n$. There are, however, many examples where the condition above is satisfied, so that the conclusion of the theorem still holds, even though the restriction of $\Omega$ to the nilradical is not closed (see e.g. [8, Section 7]). We do not know whether the condition above automatically holds whenever the orbit $\Omega$ is closed.

Observe that, contrary to the semisimple case, we need to consider representations on mixed $L^p$-spaces till the end of the proof.

In all the sequel, if $M$ is a topological space, $C_0(M)$ will mean the space of compactly supported continuous functions on $M$.

As usual, if $S$ is a Lie group, $\mathfrak{s}$ will denote its Lie algebra.

## 2 The semi-simple case

### 2.1 Preliminaries

If $E$ is a vector space, denote by $E^*$ its algebraic dual. If it is real, $E^*_C$ denotes its complexification. Let $F$ be a vector subspace of $E$. We identify in the sequel the restriction $\lambda|_F$ of $\lambda \in E^* \lor E^*_C$ to an element of respectively $F^*$ or $F^*_C$.

Let $G$ be a connected semisimple real Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a Cartan involution $\theta$ of $G$ and denote by $K$ the fixed point group for $\theta$. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with respect to $\theta$ is given by

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \]

where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ the $-1$-eigenspace in $\mathfrak{g}$ for the differential of $\theta$, denoted again by $\theta$.

We fix a subspace $\mathfrak{a}$ of $\mathfrak{p}$ which is maximal with respect to the condition that it is an abelian subalgebra of $\mathfrak{g}$. It is endowed with the scalar product $(\cdot, \cdot)$ given by the Killing form $B$, which is positive definite on $\mathfrak{p}$. By duality, we endow $\mathfrak{a}^*$ with the corresponding, induced scalar product, which we also denote by $(\cdot, \cdot)$. Let $| \cdot |$ be the associated norm on $\mathfrak{p}$ and $\mathfrak{a}^*$.

For any root $\alpha \in \mathfrak{a}^*$, we denote by $\mathfrak{g}_\alpha$ the corresponding root space, i.e. $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(X) \text{ for } H \in \mathfrak{a}\}$. We fix a set $R^+$ of positive roots of $\mathfrak{a}$ in $\mathfrak{g}$. Let $P$
denote the corresponding minimal parabolic subgroup of $G$, containing $A := \exp \mathfrak{a}$, and $P = MAN$ its Langlands decomposition.

Denote by $\rho$ the linear form on $\mathfrak{a}$ given by

$$\rho(X) := \frac{1}{2} \tr (\ad X|_n) \text{ for all } X \in \mathfrak{a},$$

where $n$ is the Lie algebra of $N$.

Let $\| \cdot \|$ denote the “norm” on $G$ defined in [2, §2]. Recall that, for $g \in G$, $\|g\|$ is the operator norm of $\Ad g$ considered as an operator on $\mathfrak{g}$, endowed with the real Hilbert structure, $(X,Y) \mapsto -B(X,\theta Y)$ as scalar product. This norm is $K$-biinvariant and, according to [2, Lemma 2.1], satisfies the following properties:

\begin{equation}
\|g\| = \|\theta(g)\| = \|g^{-1}\| \geq 1;
\end{equation}

\begin{align}
\|x y\| &\leq \|x\| \|y\|; \\
\text{there exists } c_1, c_2 > 0 \text{ such that, for } Y \in \mathfrak{p}, \text{ then } e^{c_1|Y|} \leq \|\exp Y\| \leq e^{c_2|Y|}; \\
\text{for all } a \in A, n \in N, \|a\| \leq \|an\|.
\end{align}

We choose a basis for $\mathfrak{a}^*$, following, for example, [6, p. 220].

Let $\alpha_1, \ldots, \alpha_r$ denote the simple roots in $\mathbb{R}^+$. By the Gram-Schmidt process, one constructs from the basis $\{\alpha_1, \ldots, \alpha_r\}$ of $\mathfrak{a}^*$ an orthonormal basis $\{\beta_1, \ldots, \beta_r\}$ of $\mathfrak{a}^*$ in such a way that, for every $j = 1, \ldots, r$, the vector space $\text{Vect}\{\beta_1, \ldots, \beta_j\}$ spanned by $\{\beta_1, \ldots, \beta_j\}$ agrees with $\text{Vect}\{\alpha_1, \ldots, \alpha_j\}$, and, for every $1 \leq k < j \leq r$, $(\beta_j, \alpha_k) = 0$.

Define $H_j$ ($j = 1, \ldots, r$) as the element of $\mathfrak{a}$ given by $\beta_k(H_j) = \delta_{jk}$ ($k = 1, \ldots, r$), and put

$$a_j := \mathbb{R}H_j \quad a^j := \sum_{k=1}^j a_k \quad R^j := R^+ \cap \text{Vect}\{\alpha_1, \ldots, \alpha_j\} \quad R_j := R^j \setminus R^{j-1}, \text{ with } R^0 := \emptyset \quad n^j := \sum_{\alpha \in R^j} \mathfrak{g}_\alpha \quad n_j := \sum_{\alpha \in R_j} \mathfrak{g}_\alpha.$$  

We define, for $j = 1, \ldots, r$, the reductive Lie subalgebra $m^j$ of $\mathfrak{g}$ by setting

$$m^j := \theta(n^j) + m + a^j + n^j.$$  

In this way, we obtain a finite sequence of reductive Lie subalgebras of $\mathfrak{g}$,

$$m := m^0 \subset m^1 \subset \cdots \subset m^r = \mathfrak{g},$$

such that

$$m^j = \theta(n_j) + m^{j-1} + a_j + n_j \quad (j = 1, \ldots, r).$$

Then $m^{j-1} \oplus a_j \oplus n_j$ is a parabolic subalgebra of $m^j$.

Observe that $G$ is a real reductive Lie group in the Harish-Chandra class (see e.g. [7, p. 58], for the definition of this class of reductive Lie groups). We can then inductively define a decreasing sequence of reductive real Lie groups $M^j$ in the Harish-Chandra class, starting from $M^r = G$, in the following way.
Let $P_j$ denote the parabolic subgroup of $M^j$ corresponding to the parabolic subalgebra $m^{j-1} \oplus a_j \oplus n_j$, and $P_j = M^{j-1}A_jN_j$ its Langlands decomposition. Here $A_j$ (resp. $N_j$) is the analytic subgroup of $M^j$ with Lie algebra $a_j$ (resp. $n_j$), $M^{j-1}A_j$ is the centralizer in $M^j$ of $A_j$, and

$$M^{j-1} := \bigcap_{\chi \in \operatorname{Hom}(M^{j-1}A_j, \mathbb{R}_{+}^*)} \ker \chi$$

(see e.g. [7, Theorem 2.3.1]). Moreover, $M^{j-1}A_j$ normalizes $N_j$ and $\theta(N_j)$, and $M^{j-1}$ is a reductive Lie subgroup of $M_j$, in the Harish-Chandra class, with Lie algebra $m^{j-1}$ (see [7, Proposition 2.1.5]).

Put $K^j := M^j \cap K$ ($j = 1, \ldots, r$). Then $K^j$ is the maximal compact subgroup of $M^j$ related to the Cartan involution $\theta_M^j$ of $M^j$ (see e.g. [7, Theorem 2.3.2, p. 68]). Hence, $M^j$ is the product

$$K^jP_j = K^jM^{j-1}A_jN_j.$$ 

Fix invariant measures $dk, dm, da, dn, dm_j, dk_j, da_j, dn_j$ for respectively $K, M, A, N, M^j, K^j, A_j, N_j$.

Choose an invariant measure $dx$ on $G$ such that

$$(2.2) \quad \int_G \varphi(x) \, dx = \int_{K \times A \times N} a^{2m} \varphi(kan) \, dkan, \quad \text{for all} \quad \varphi \in C_0(G)$$

(see, e.g., [7, Proposition 2.4.2]).

We shall next recall an integral formula. Let $S$ be a reductive Lie group in the Harish-Chandra class, and let $S = K \exp p$ be its Cartan decomposition, where $K$ is a maximal compact subgroup of $S$. Let $Q$ be a parabolic subgroup of $S$ related to the above Cartan decomposition, and let $Q = M_QA_QN_Q$ be its Langlands decomposition.

Let $K_Q := K \cap Q = K \cap M_Q$, and put, for $k \in K$, $[k] := kK_Q$ in $K/K_Q$. We extend this notation to $S$ by putting, for $s = kman$, $(k, m, a, n) \in K \times M_Q \times A_Q \times N_Q$, $[s] := kK_Q$. This is still well defined even though the representation of $s$ in $KM_QA_QN_Q$ is not unique. In fact,

$$s = kman = k'm'a'n'$$

if and only if $a' = a$, $n' = n$, and $k' = kK_Q$, $m' = k_Q^{-1}m$ for some $k_Q \in K_Q$ (see e.g. [7, Theorem 2.3.3]). From this we see that the decomposition above becomes unique, if we require $m$ to be in $\exp(m_Q \cap p)$.

Every $s \in S$ thus admits a unique decomposition $s = kman$, with $(k, m, a, n) \in K \times \exp(m_Q \cap p) \times A_Q \times N_Q$. We then write $k_Q(s) := k$, $m_Q(s) := m$, $a_Q(s) := a$ and $n_Q(s) := n$, i.e.

$$s = k_Q(s)m_Q(s)a_Q(s)n_Q(s).$$

In particular, $[s] = k_Q(s)K_Q$.

For $y \in S$ and $k \in K$, we define $y[k] \in K/K_Q$ as follows:

$$y[k] := [yk].$$

Moreover, for any $\gamma \in a^*_C$ and $Y \in a$, we put $(\exp Y)^\gamma := e^\gamma(Y)$.
Lemma 2.1. Fix an invariant measure $dk$ on $K$ and let $d[k]$ denote the corresponding left invariant measure on $K/K_Q$. For any $y \in S$, we then have
\[ d(y[k]) = a_Q(yk)^{-2\rho_Q}d[k], \]
where $\rho_Q \in \mathfrak{a}_Q^*$ is given by $\rho_Q(X) = \frac{1}{2}\text{tr}(\text{ad}X_{n_Q})$ ($X \in \mathfrak{a}_Q$); that is, for any $f \in C(K/K_Q)$,
\[ \int_{K/K_Q} f([k])d[k] = \int_{K/K_Q} a_Q(yk)^{-2\rho_Q}f(y[k])d[k]. \]

Proof. We follow the proof of [7, Proposition 2.5.4]. Let $f \in C(K/K_Q)$. Consider $f$ as a right $K_Q$-invariant function on $K$. Choose $\chi \in C_0(M_QA_QN_Q)$ such that
\[ \int_{M_Q \times A_Q \times N_Q} a^{2\rho_Q} \chi(\text{man}) d\text{mdadn} = 1 \]
and $\chi(k_Qq) = \chi(q)$, for all $k_Q \in K_Q$, $q \in Q$, and put, for $s \in S$ with $s = kman$, $(k,m,a,n) \in K \times M_Q \times A_Q \times N_Q$,
\[ h(s) := f(k)\chi(\text{man}). \]

Notice that the function $h$ is well defined, independently of the chosen decomposition $s = kman$ of $s$, since $f$ (respectively $\chi$) is right (respectively left) $K_Q$-invariant.

Let $dm$, $da$ and $dn$ be invariant measures on $M_Q$, $A_Q$ and $N_Q$, respectively. Choose an invariant measure $dx$ on $S$ such that
\[ \int_S \varphi(x) dx = \int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} \varphi(kman) d\text{dkdmdadn}, \]
for all $\varphi \in C_0(S)$ (see, e.g., [7, Proposition 2.4.3]). Then
\[ \int_K f(k)dk = \int_S h(x) dx. \]

On the other hand, by left invariance,
\[ \int_S h(x) dx = \int_S h(yx) dx = \int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} h(ykman) d\text{dkdmdadn}. \]

Write $yk = k_Q(yk)m_Q(yk)a_Q(yk)n_Q(yk)$. Since the elements of $M_Q$ and $A_Q$ commute, we get
\[ ykman = k_Q(yk)m_Q(yk)ma_Q(yk)an_Q(yk)^{(ma)^{-1}}n, \]
where $n_Q(yk)^{ma} = man_Q(yk)(ma)^{-1} \in N_Q$, since $M_QA_Q$ normalizes $N_Q$. Therefore,
\[ h(ykman) = f([yk])\chi(m_Q(yk)ma_Q(yk)an_Q(yk)^{(ma)^{-1}}n). \]

We thus obtain, by left invariance of $dm$, $da$, $dn$,
\[ \int_S h(yx) dx = \int_{K \times M_Q \times A_Q \times N_Q} a^{2\rho_Q} a_Q(yk)^{-2\rho_Q} f([yk])\chi(\text{man}) d\text{dkdmdadn} \]
\[ = \int_K a_Q(yk)^{-2\rho_Q} f(y[k]) dk. \]

The lemma follows. \qed
Remark. Let $P = MAN$ be a minimal parabolic subgroup of $G$ contained in $Q$. If we decompose $s \in S$ via the Iwasawa decomposition $S = KAN$ as

$$s = k(s)a(s)n(s),$$

where $k(s) \in K$, $a(s) \in A$ and $n(s) \in N$, we can check that $k(s) = k_Q(s)$ and $a(s) = a(m_Q(s))a_Q(s)$, where $a(m_Q(s))$ lies in fact in $\exp(\mathfrak{m}_Q \cap \mathfrak{a})$. Since this space is orthogonal to $\mathfrak{a}_Q$ with respect to the scalar product $(\cdot, \cdot)$ on $\mathfrak{p}$, for any $\lambda \in \mathfrak{a}_Q^*$ we have $\lambda |_{\mathfrak{m}_Q \cap \mathfrak{a}} = 0$, hence

$$a(s)^\lambda = a_Q(s)^\lambda.$$

With these considerations, the lemma above can also be deduced, for example, from [21, Lemma 2.4.1].

We return now to our semisimple Lie group $G$. In the sequel, we shall use another basis of $\mathfrak{a}^*$, given as follows. For $j = 1, \ldots, r$, let $\rho_j$ denote the element of $\mathfrak{a}_j^*$ defined by

$$\rho_j(X) := \frac{1}{2} \text{tr}(\text{ad}X |_{\mathfrak{n}_j}) \text{ for all } X \in \mathfrak{a}_j.$$

Notice that we can identify $\rho_j$ with the restriction $\rho |_{\mathfrak{a}_j}$ of $\rho$ to $\mathfrak{a}_j$. By [6, Lemma 4.1], $\rho_j$ and $\beta_j$ are scalar multiples of each other. In particular, the family \{\rho_j\} is an orthogonal basis of $\mathfrak{a}^*$, and therefore of $\mathfrak{a}_j^*$. For every $\nu \in \mathfrak{a}^*$ (resp. $\nu \in \mathfrak{a}^*_c$), define $\nu_j$ ($j = 1, \ldots, r$) in $\mathbb{R}$ (resp. $\mathbb{C}$) by the following:

$$\nu = \sum_{j=1}^r \nu_j \rho_j.$$

Recall that, for $j = 1, \ldots, r$, $P_j$ is a parabolic subgroup of the reductive real Lie group $M^j$, which lies in the Harish-Chandra class. Put therefore, by taking $(S, K, Q) := (M^j, K^j, P_j)$ in the discussion above, $k_j := k_{P_j}$, $m_j := m_{P_j}$, $a_j := a_{P_j}$ and $n_j := n_{P_j}$. Then, any $g \in M^j$ has a unique decomposition $g = k_j(g)m_{j-1}(g)a_j(g)n_j(g)$, with $k_j(g) \in K^j$, $m_{j-1}(g) \in \exp(m^{j-1} \cap \mathfrak{p})$, $a_j(g) \in A_j$ and $n_j(g) \in N_j$. Notice that $\mathfrak{m}_0 \cap \mathfrak{p} = \{0\}$, i.e. $m_0(\mathfrak{g}) = e$.

Lemma 2.2. Denote by $r_y$ the right multiplication with $y \in G$. Let $j \in \{1, \ldots, r\}$, $g \in M^j$ and $k_l \in K^l$ ($l = 1, \ldots, j$).

We define recursively the element $g_l$ of $M^l$, $l = 1, \ldots, j$, starting from $l = j$, by putting $g_j := g$ and $g_{l-1} := m_{l-1}(g_l k_l)$, i.e.

$$g_l = m_l \circ (r_{k_{l+1}} \circ m_{l+1}) \circ \cdots \circ (r_{k_j} \circ m_j)(g), \quad 1 \leq l \leq j - 1.$$

Then, the following estimate holds:

$$\|\prod_{l=j}^1 a_l(g_l k^l)\| \leq \|g\|.$$

Proof. We first show that, for $1 \leq p \leq j$,

$$\|g\| = \|\prod_{l=j}^p a_l(g_l k_l) \cdot m_{p-1}(g_p k_p) \cdot \Pi_{l=p}^j m_l(g_l k_l) k^{-1}_l\|.$$ (2.3)
(Here the products are non-commutative products, in which the order of the factors is indicated by the order of indices.) We use an induction, starting from $p = j$. If $p = j$ and $g \in M^j$, then
\[
\|g\| = \|gk_j k_j^{-1}\| = \|k_j(gk_j)m_{j-1}(gk_j)a_j(gk_j)n_j(gk_j)k_j^{-1}\|.
\]
Using the left $K$-invariance of the norm and the fact that $a_j(gk_j) \in A_j$ and $m_{j-1}(gk_j) \in M^{j-1}$ commute, we find that
\[
\|g\| = \|a_j(gk_j)m_{j-1}(gk_j)n_j(gk_j)k_j^{-1}\|,
\]
so that (2.3) holds for $p = j$. Assume now, by induction, that (2.3) is true for $p + 1$ in place of $p$, i.e.
\[
\|g\| = \|\Pi_{i=j}^{p+1} a_i(gk_i) \cdot m_p(g_{p+1}k_{p+1}) \cdot \Pi_{i=p+1}^j n_i(gk_i)k_i^{-1}\|.
\]
We then decompose
\[
m_p(g_{p+1}k_{p+1})k_p = g_pk_p = k_p(g_pk_p)m_{p-1}(g_pk_p)\alpha_p(g_pk_p)n_p(g_pk_p).
\]
Since $k_p(g_pk_p) \in K^p \subset M^l$, for $p \leq l \leq j$, it commutes with $\alpha_l(g_l k_l)$, for $l = p+1, \ldots, j$, and therefore, because of the $K$-invariance of $\| \cdot \|$, we have
\[
\|g\| = \|\Pi_{i=j}^{p+1} a_i(gk_i) \cdot m_{p-1}(g_pk_p)\alpha_p(g_pk_p)n_p(g_pk_p)k_p^{-1} \cdot \Pi_{i=p+1}^j n_i(gk_i)k_i^{-1}\|.
\]
Moreover, $\alpha_p(g_pk_p)$ commutes with $m_{p-1}(g_pk_p)$, and so (2.3) follows. Applying now (2.3) for $p = 1$, we obtain
\[
(2.4) \quad \|\Pi_{i=j}^1 a_i(gk_i)\Pi_{i=1}^j n_i(gk_i)k_i^{-1}\| = \|g\|.
\]
By right $K$-invariance of the norm, the left hand side of this equation is equal to
\[
\|\Pi_{i=j}^1 a_i(gk_i)\Pi_{i=1}^j n_i(gk_i)k_i^{-1}\Pi_{i'=j}^1 k_{l'}\|.
\]
Notice that we can write $\Pi_{i=1}^j n_i(gk_i)k_i^{-1}\Pi_{i'=j}^1 k_{l'}$ as follows:
\[
n_1(gk_1) (k_1^{-1}n_2(gk_2)k_1) ((k_2k_1)^{-1}n_3(gk_3)k_2k_1) \cdots ((\Pi_{i=j-1}^1 k_i)^{-1}n_j(gjk_j)\Pi_{i'=j-1}^1 k_{l'}).
\]
Since $(\Pi_{i'=p-1}^1 k_{l'})^{-1}$, $2 \leq p \leq j$, lies in $K^{p-1} \subset M^{p-1}$ and thus normalizes $N_p$, we get that
\[
\Pi_{i=1}^j n_i(gk_i)k_i^{-1}\Pi_{i'=j}^1 k_{l'} \in N.
\]
Using the last property of the norm given in (2.1), the left hand side of (2.4) is then greater or equal to $\|\Pi_{i=j}^1 a_i(gk_i)\|$, which proves the lemma. \[\square\]
2.2 A holomorphic family of representations of $G$ on mixed $L^p$-spaces

For $\nu \in \mathfrak{a}_G$, let $\mathcal{M}(G, P, \nu)$ denote the space of complex valued measurable functions $f$ on $G$ satisfying the following covariance property:

$$f(g m a n) = a^{-(\nu + \rho)} f(g) \text{ for all } g \in G, \ m \in M, \ a \in A, \ n \in N.$$  

The space $\mathcal{M}(G, P, \nu)$ is endowed with the left regular action of $G$, denoted by $\tilde{\pi}_\nu$, i.e.,

$$[\tilde{\pi}_\nu(g)f](g') = f(g^{-1}g').$$

The representations $\tilde{\pi}_\nu$ form the class-one principal series.

Let $\mathcal{M}(K/M)$ be the space of right $M$-invariant measurable functions on $K$.

The restriction to $K$ of functions on $G$ gives us a linear isomorphism from $\mathcal{M}(G, P, \nu)$ onto $\mathcal{M}(K/M)$. Denote by $I_\nu : f \mapsto f_\nu$ the inverse mapping. Then $f_\nu \in \mathcal{M}(G, P, \nu)$ is given by

$$f_\nu(k a n) := a^{-(\nu + \rho)} f(k) \text{ for all } k \in K, \ a \in A, \ n \in N,$$

if $G = K A N$ denotes the Iwasawa decomposition of $G$.

If we intertwine the representation $\tilde{\pi}_\nu$ with $I_\nu$, we obtain a representation $\pi_\nu$ of $G$ on $\mathcal{M}(K/M)$, given by

$$(\pi_\nu(g)f)_\nu = \tilde{\pi}_\nu(g)f_\nu, \text{ if } f \in \mathcal{M}(K/M), \ g \in G.$$  

Denote by $d \hat{k}_j$, for $j = 1, \ldots, r$, the quotient measure on $K^j/K^{j-1}$ coming from $dk_j$. It is invariant by left translations. Notice that $K^{j-1} = K^j \cap M^{j-1}$.

We choose a left invariant measure $d k$ on $K/M$ such that, for any $f \in C(K/M)$,

$$(2.5) \quad \int_{K/M} f(k) d\hat{k} = \int_{K/K^{r-1}} \cdots \int_{K^1/M} f(k_r \cdots k_1) d\hat{k}_1 \cdots d\hat{k}_r.$$  

Let $\underline{p} = (p_1, \ldots, p_r) \in [1, +\infty]^r$.

One can easily see that, for every $f \in \mathcal{M}(K/M)$, $k' \in K$, the function on $K^j$ given by

$$k' \mapsto \left(\int_{K^{j-1}/K^{j-2}} \cdots \left(\int_{K^1/M} \left| f(k' k k_{j-1} \cdots k_1)\right|^{p_1} d\hat{k}_1 \right)^{p_2/p_1} \cdots \left(\int_{K^1/M} \right)^{1/p_j},$$

is right $K^{j-1}$-invariant.

We can thus define the mixed $L^p$-space $L^p(K/M)$, as the space of all (equivalent classes of) functions $f$ in $\mathcal{M}(K/M)$ whose mixed $L^p$-norm

$$\|f\|_{\underline{p}} := \left(\int_{K^{r-1}/K^{r-2}} \cdots \left(\int_{K^1/M} \left| f(k_r \cdots k_1)\right|^{p_1} d\hat{k}_1 \right)^{p_2/p_1} \cdots \left(\int_{K^1/M} \right)^{1/p_r}$$

is finite, endowed with this norm. This definition extends to the case where some of the $p_j$ are infinite, by the usual modifications.

Let $d$ denote the right $G$- and left $K$-invariant metric on $G$, given by

$$d(g, g') := \frac{1}{c_1} \log \|g' g^{-1}\| \quad (g, g' \in G),$$

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where $c_1$ is the positive constant appearing in (2.1). Notice that $d(g,e) = 0$ if and only if $g$ lies in the center of $G$. In particular, $d$ is not separating. 

Then, for $a = \exp Y$, with $Y \in \mathfrak{a} \subset \mathfrak{p}$, and $\gamma \in \mathfrak{a}^*$, we have, in view of the fourth property of $\| \cdot \|$ in (2.1), that

\begin{equation}
(2.6) \quad \|a\gamma| = |e^{\gamma(Y)}| \leq e^{\|\gamma\|/c_1} \leq \|a\|^{\|\gamma\|/c_1} = e^{d(a,e)}.
\end{equation}

**Proposition 2.1.** For every $f \in L^2(K/M)$ and $g \in G$, we have

$$
\|\pi_\nu(g)f\|_p \leq e^{\sum_j (\frac{2}{p_j} - \text{Re } \nu_j - 1)\rho_j d(g,e)} \|f\|_p.
$$

Thus $\pi_\nu$ defines a representation $\pi_\nu^p$ of $G$ on $L^2(K/M)$. Furthermore, this gives us an analytic family $\{\pi_\nu^p\}_{\nu \in \mathfrak{a}^*}$ of representations of $G$ on $L^2(K/M)$.

Before giving the proof, we show the following statement. We keep the same notations as in Lemma 2.2.

**Lemma 2.3.** Let $g \in M^j, k \in K$ and $f_\nu \in \mathcal{M}(G,P,\nu)$. Then

$$
\left( \int_{K^j/K^{j-1}} \cdots \left( \int_{K^1/M} \left| f_\nu(k g k_j \cdots k_1) \right|^{p_1} dk_1 \right)^{p_2/p_1} \cdots dk_j \right)^{1/p_j}
$$

$$
= \left( \int_{K^j/K^{j-1}} \cdots \left( \int_{K^1/M} \left| \Pi_{l=1}^j a_l(g k_l) \right|^{(\text{Re } \nu_l + 1)p_i} f_\nu(k \Pi_{l=1}^j k_l(g k_l)) \right|^{p_1} dk_1 \right)^{p_2/p_1} \cdots dk_j \right)^{1/p_j}.
$$

**Proof.** We use induction on $j$. For $j = 0$, one has, by right $M$-invariance of $f$ and since $g \in M^0 = M$, that

$$
|f_\nu(k g)| = |f_\nu(k)|.
$$

Assume that the statement is true for $j - 1$. Observe that $a_j(g k_j)$ commutes with $k_{j-1} \cdots k_1 \in M^{j-1}$, and that $(k_{j-1} \cdots k_1)^{-1} n_j(g k_j) k_{j-1} \cdots k_1 \in N$. Therefore, the covariance property of $f_\nu$ applied to the integration over $K^j/K^{j-1}$, implies

$$
\left( \int_{K^j/K^{j-1}} \cdots \left( \int_{K^1/M} \left| f_\nu(k g k_j \cdots k_1) \right|^{p_1} dk_1 \right)^{p_2/p_1} \cdots dk_j \right)^{1/p_j}
$$

$$
= \left( \int_{K^j/K^{j-1}} \cdots \left( \int_{K^1/M} \left| a_j(g k_j)^{-\text{Re } \nu_j} f_\nu(k k_j(g k_j))^m_{j-1} k_{j-1} \cdots k_1 \right|^{p_1} dk_1 \right)^{p_2/p_1} \cdots dk_j \right)^{1/p_j}.
$$

The statement holds, using the induction hypothesis, since $g = g_j$ and $m_{j-1}(g k_j) = g_{j-1} \in M^{j-1}$.

**Proof of Proposition 2.1.** If we apply (2.6) to $\gamma := \sum_{j=1}^r (\frac{2}{p_j} - \text{Re } \nu_j - 1)\rho_j$ and notice that the $a_j$'s are pairwise orthogonal with respect to $(\cdot, \cdot)$, we get, in view of Lemma 2.2:

$$
\sup_{k_j \in K^j, j = 1, \ldots, r} \left( \prod_{j=1}^r a_j(g_j k_j)^{\left(\frac{2}{p_j} - \text{Re } \nu_j - 1\right)\rho_j} \right) \leq \|g\|^{\|\gamma\|/c_1} = e^{d(g,e)}.
$$

On the other hand, according to the above lemma and Lemma 2.1, applied successively to the integrations over $K^j/K^{j-1}, j = 1, \ldots, r$, we have

$$
\|\pi_\nu(g^{-1})f\|_p \leq \sup_{k_j \in K^j, j = 1, \ldots, r} \left( \Pi_{j=1}^r a_j(g_j k_j)^{\left(\frac{2}{p_j} - \text{Re } \nu_j - 1\right)\rho_j} \right) \|f\|_p.
$$
In order to prove the analyticity of the family of representations $\pi_p^p$, choose $p = (p_1, \ldots, p_r) \in [1, \infty]^r$ and denote by $p' = (p'_1, \ldots, p'_r) \in [1, \infty]^r$ the tuple of conjugate exponents, i.e., $1/p_j + 1/p'_j = 1$. Then, for $f \in L^2(K/M)$, $u \in L^2(K/M) = (L^2(K/M))'$ and $g \in G$, we have

$$\langle \pi_p^p(g)f, u \rangle = \int_{K/M} \langle \pi_p^p(g)f(k)\overline{u(k)}dk = \int_{K/M} a(g^{-1}k)^{-(\nu+p)}f(\kappa(g^{-1}k))\overline{u(k)}dk.$$  

Here, the functions $\kappa(\cdot), a(\cdot), n(\cdot)$ on $G$ are given by the unique factorization $g = \kappa(g)a(g)n(g)$ of $g$, according to the Iwasawa decomposition $G = KAN$.

Obviously, the expression above is analytic in $\nu \in \mathfrak{a}_c^*$, which finishes the proof. 

For $t = (t_1, \ldots, t_r) \in [0, +\infty[^r$, let $\Omega_t := \{\nu \in \mathfrak{a}_c^* \mid |\Re \nu_j| < t_j$ for all $j = 1, \ldots, r\}$. Moreover, for $p \geq 0$, let $\overline{p} := (p, \ldots, p) \in \mathbb{R}^r$.

**Proposition 2.2.**

(i) For all $p \in [1, +\infty[^r$, $f \in L^2(K/M)$, $\nu \in \Omega_t$, $g \in G$, we have

$$\|\pi_{\overline{p}}^p(g)f\|_p \leq e^{\sum_j (t_j + 1)|\rho_j|d(g,e)}\|f\|_{\overline{p}}.$$

(ii) Let $\nu \in \mathfrak{a}_c^*$, and let $q$ be an element of $[1, +\infty[^r$ satisfying

$$\Re \nu_j = \frac{2}{q_j} - 1, \quad j = 1, \ldots, r.$$

Then, for all $g \in G$, $f \in L^2(K/M)$,

$$\|\pi_{\overline{p}}^p(g)f\|_q = \|f\|_q.$$

Furthermore, for $\nu \in i\mathfrak{a}^*$, $\pi_{\overline{p}}^p$ is a unitary representation of $G$.

**Proof.** (i) results immediately from the estimate given in Proposition 2.1 and (ii) from Lemmas 2.3 and 2.1, since, for such $q$, we have $a^{-((\Re \nu_j + 1)\rho_j)} = a^{-2\rho_j/q_j}$, if $a \in A_t$.  

### 2.3 A holomorphic family of compact operators

Let $L = -\sum_1^k X_j^2$ be a fixed sub-Laplacian on $G$. The estimate (1.1), in combination with the estimate in Proposition 2.2 (i), easily implies that the operator

$$\pi_{\overline{p}}^p(h_1)f := \int_G h_1(x)\pi_{\overline{p}}^p(x)f dx, \quad f \in L^2(K/M),$$

is well defined and bounded on $L^2(K/M)$. In fact these operators are even compact. To see this, let us put, for $\nu \in \Omega_1$ and $k_1, k_2 \in K$,

$$K_\nu(k_1, k_2) := c_G \int_{M \times A \times N} a^{-\nu+p}h_1(k_1(ma)^{-1}k_2^{-1})dmdadn,$$

where $c_G$ is the positive constant given by $d(x^{-1}) = c_Gdx$ (which exists, since $G$ is unimodular).
**Lemma 2.4.** The integral in (2.7) is absolutely convergent and defines a continuous, right $M$-invariant kernel function on $K \times K$, i.e. $K_\nu(k_1m', k_2m') = K_\nu(k_1, k_2)$ for every $m' \in M$.

**Proof.** In order to prove that the integral in (2.7) is absolutely convergent, we put

$$I := \int_{M \times A \times N} |a^{-\nu+\rho}h_1(k_1(man)^{-1}k_2^{-1})|dmdadn.$$ 

Then, in view of (1.1), we have

$$I \leq C \int_{M \times A \times N} a^{-\Re \nu+\rho}e^{-cd(k_1(man)^{-1}k_2^{-1}, e)^2}dmdadn.$$ 

Using the $K$-bi-invariance of the norm $\| \cdot \|$ on $G$ and the inclusion $M \subset K$, we get that

$$d(k_1(man)^{-1}k_2^{-1}, e) = d(kan, e),$$ 

for any $k \in K$.

Moreover, by (2.6) and (2.1),

$$a^{-2\rho}a^{-\Re \nu+\rho} = a^{-\Re \nu-\rho} \leq e^{\Re \nu + \rho(d(kan, e))}.$$ 

We thus get, since $|\Re \nu + \rho| \leq 2 \sum_j |\rho_j|$ for $\nu \in \Omega_1$,

$$I \leq C \int_{K \times A \times N} a^{2\rho}e^{2\sum_j |\rho_j|d(kan, e)}e^{-cd(kan, e)^2}dkdmdadn,$$ 

for every $k_1, k_2 \in K$, which is in fact equal to

$$C \int_G e^{2\sum_j |\rho_j|d(x, e)}e^{-cd(x, e)^2}dx.$$ 

Since $G$ is unimodular and has exponential volume growth, it is easy to see that this integral is finite. Moreover, since the integrand in (2.7) depends continuously on $k_1$ and $k_2$, we see that $K_\nu$ is continuous.

In order to prove the right $M$-invariance of $K_\nu$, let $m' \in M$. One has, for any $(m, a, n) \in M \times A \times N$,

$$(man)m' = mm'anm'.$$ 

According to the invariance of $dm$, we then have, for any $k_1, k_2 \in K$,

$$K_\nu(k_1m', k_2m') = c_G \int_{M \times A \times N} a^{-\nu+\rho}h_1(k_1(man)^{-1}k_2^{-1})dmdadn.$$ 

Furthermore, it is easy to check that, for any $\varphi \in C_0(N)$,

$$\int_N \varphi(nm') \, dn = \int_N \varphi(n) \, dn.$$ 

Indeed, since $G = KAN$, there exists $\phi \in C_0(G)$ such that

$$\varphi(n) = \int_{K \times A} a^{2\rho} \phi(kan) \, dkda.$$ 

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According to our choice of the Haar measure $dx$ on $G$ (c.f. (2.2)), we have
\[
\int_G \phi(x) \, dx = \int_{K \times A \times N} a^{2\rho\phi(kan)} \, dk \, dn = \int_N \varphi(n) \, dn,
\]
and using the invariance of $dx$ and $dk$, in combination with the commutation and normalization properties of $m' \in M$, we see that
\[
\int_N \varphi(n) \, dn = \int_G \phi(x^{m'}) \, dx = \int_{K \times A \times N} a^{2\rho\phi(kan^{m'})} \, dk \, dn = \int_N \varphi(n^{m'}) \, dn.
\]
We thus conclude that
\[
K_\nu(k_1m', k_2m') = K_\nu(k_1, k_2).
\]

Put, for $\nu \in \Omega_1$,
\[
T(\nu) := \pi_\nu(h_1).
\]

**Proposition 2.3.** The operator $T(\nu)$ is represented by the integral kernel $K_\nu$, i.e.
\[
(T(\nu)f)(k_1M) = \int_{K/M} K_\nu(k_1, k) f(k) \, dk, \quad f \in L^1(K/M).
\]

In particular, $T(\nu) = \pi^p_\nu(h_1)$ is a compact operator on every mixed $L^p$-space $L^p(K/M)$, $p \in [1, +\infty]$\(^{\star}\), which we then shall also denote by $T_p(\nu)$, in order to indicate the space on which it acts. Moreover, the family of compact operators $\nu \mapsto T_p(\nu)$ is analytic (in the sense of Kato [10]) on $\Omega_1$.

Furthermore, for $\nu \in i\mathbb{A}^\ast$, the operator $T_\mathcal{F}(\nu)$ is self-adjoint on $L^2(K/M)$.

**Proof.** We have, by definition, for any $k_1 \in K$, that
\[
(T(\nu)f)(k_1) = \int_G h_1(x)(\pi_\nu(x)f)(k_1) \, dx.
\]

By invariance of $dx$, this is equal to
\[
c_G \int_G h_1(k_1x^{-1})f_\nu(x) \, dx.
\]

Now, according to our choice of $dx$ and using the covariance property of $f_\nu$, we obtain
\[
(T(\nu)f)(k_1) = c_G \int_{K \times A \times N} a^{2\rho a^{-\nu}(\nu+\rho)h_1(k_1(an)^{-1}k^{-1})} f_\nu(k) \, dk \, dn \, dn.
\]

Since $dk$ is invariant and $M \subset K$, this can be written, using also the right $M$-invariance of $f_\nu$, as follows:
\[
(T(\nu)f)(k_1) = c_G \int_{K \times M \times A \times N} a^{-\nu+\rho} h_1(k_1(mn)^{-1}k^{-1}) f_\nu(k) \, dk \, dn \, dn.
\]
But, $f_{\nu} = f$ on $K$, and thus (2.8) follows, by Fubini’s theorem.

Since the kernel $K_{\nu}$ is continuous on the compact space $K \times K$, by Lemma 2.4, it follows that $T_{\nu}(\nu)$ is a compact operator on $L_{p}(K/M)$, and the analytic dependence of $K_{\nu}$, which is evident from (2.8), implies that, for every $p [1, +\infty \rangle$, the family of operators $T_{\nu}(\nu)$ is analytic on $\Omega_{1}$.

Finally, if $\nu \in i\mathfrak{a}^{*}$, then $\pi_{\nu}^{\mathcal{T}}$ is unitary, and since $h_{1}(x) = \overline{h_{1}(x^{-1})}$, we see (from (1.2)) that the operator $\pi_{\nu}^{\mathcal{T}}(h_{1})$ is self-adjoint. □

2.4 Some consequences of the Kunze-Stein phenomenon

Observe that, by Hölder’s inequality, for any $p \in [1, 2]$, and any $q \in [p, p']$, we have

$$\|f\|_{q} \leq \|f\|_{p'}, \quad \text{for all } f \in L_{p'}(K/M),$$

since the compact space $K/M$ has normalized measure 1. Therefore, $L_{p'}(K/M)$ is a subspace of $L_{q}(K/M)$.

Notice also that $L_{p}(K/M) = L_{q}(K/M)$ and $\|\cdot\|_{p} = \|\cdot\|_{q}$, by our choice of measure on $K/M$ (c.f. (2.5)).

As a consequence of the Kunze-Stein phenomenon (see [12] and [6]), we shall prove:

**Proposition 2.4.** Let $1 < p_{0} < 2$ and $\nu_{0} \in \mathfrak{a}^{*} \setminus \{0\}$. There exist $\varepsilon > 0$ and $C > 0$, such that, for any $\xi, \eta \in L^{p_{0}}(K/M)$ and $z \in \mathbb{C}$ with $\text{Re } z < \varepsilon$,

$$\|\langle \pi_{\nu_{0}}(\cdot)\xi, \eta \rangle\|_{L_{p_{0}}}(G) \leq C\|\xi\|_{p_{0}}\|\eta\|_{p_{0}},$$

(2.10)

**Proof.** Observe that, for every $\nu \in i\mathfrak{a}^{*}$, the representation $\pi_{\nu}$ is unitarily equivalent to $\tilde{\pi}_{\nu}$. Therefore, given $\delta > 0$, we obtain from [6], that there is a constant $C_{\delta} > 0$, such that, for any $2 + \delta \leq r' \leq \infty$ and $\xi, \eta \in L^{2}(K/M)$, we have:

$$\|\langle \pi_{\nu}(\cdot)\xi, \eta \rangle\|_{L'_{p}(G)} \leq C_{\delta}\|\xi\|_{2}\|\eta\|_{2}, \quad \text{provided } \text{Re } \nu = 0.$$

(2.11)

Indeed, in [6], this is only stated for $\nu = 0$, but the proof easily extends to arbitrary $\nu \in i\mathfrak{a}^{*}$.

On the other hand, we have the estimate:

$$\|\langle \pi_{\nu}(\cdot)\xi, \eta \rangle\|_{L_{r'}(G)} \leq \|\xi\|_{\overline{p}_{j}}\|\eta\|_{q'}, \quad q' \in [1, +\infty \rangle,$$

(2.12)

for any $\xi \in L_{2}(K/M), \eta \in L_{p'}(K/M)$, provided that:

$$\text{Re } \nu_{j} = \frac{1}{q_{j}} - 1, \quad j = 1, \ldots, r.$$

(2.13)

This is an immediate consequence of Proposition 2.2 (ii), since $\pi_{\nu}^{\mathcal{T}}$ is isometric, if (2.13) is satisfied.

Let $\theta_{0} \in [0, 1]$ be given by $\frac{2}{p_{0}} = 1 + \theta_{0}$. Since $2 \in [p_{0}, p']$ and since $q \in [\overline{p}_{0}, \overline{p}_{0}]$, if $q$ satisfies (2.13) and $|\text{Re } \nu_{j}| \leq \theta_{0}$, $j = 1, \ldots, r$, we can unify (2.11) and (2.12), using
Moreover, according to \([8, Lemma 6.1]\), it suffices to consider the case where \(2 < p \leq \infty\), as follows.

As in the introduction, we can replace \(L_p\) by \(\mathcal{L}_p\) and \(L_{\infty}(G)\) by \(\mathcal{L}_{\infty}(G)\). Then, for \(\nu \in \mathbb{C}\), we have that

\[
\|\langle \pi_\nu(\cdot)\xi, \eta \rangle\|_{L_{p'}(G)} \leq C_\delta \|\xi\|_{T_\delta_p} \|\eta\|_{T_\delta_p}, \quad \text{for all } \nu \in \mathbb{C},
\]

and

\[
\|\langle \pi_\nu(\cdot)\xi, \eta \rangle\|_{L_{\infty}(G)} \leq \|\xi\|_{T_\delta_p} \|\eta\|_{T_\delta_p}, \quad \text{if } |\Re \nu_j| \leq \theta_0, \quad j = 1, \ldots, r.
\]

If we choose \(\nu = z \nu_0\), and put \(\Psi_z := \langle \pi_{z \nu_0}(\cdot)\xi, \eta \rangle\), for \(\xi, \eta \in L_{\infty}(K/M)\) fixed, this implies that

\[
\|\Psi_{iz}\|_{L_{p'}(G)} \leq C_\delta \|\xi\|_{T_\delta_p} \|\eta\|_{T_\delta_p}, \quad \text{for all } p' \geq 2 + \delta, \quad z \in \mathbb{C}.
\]

But, since \(p_0' > 2\), we can choose \(\delta > 0\) and \(\varepsilon > 0\) so small that \((1 - \frac{\varepsilon}{\delta_1})p_0' \geq 2 + \delta\).

Then, for \(|\Re z| \leq \varepsilon\), if we choose \(r' := p_0' (1 - \frac{|\Re z|}{\delta_1})\) in (2.14), we have \(r' \geq 2 + \delta\), and hence:

\[
\|\Psi_{iz}\|_{L_{p_0'}(G)} \leq C_\delta \|\xi\|_{T_\delta_p} \|\eta\|_{T_\delta_p}.
\]

\[\square\]

### 2.5 Proof of Theorem 1.1

Let \(p \in [1, \infty[, \ p \neq 2\). The aim is to find a non-isolated point \(\lambda_0\) in the \(L^2\)-spectrum \(\sigma_2(L)\) of \(L\) and an open neighbourhood \(\mathcal{U}\) of \(\lambda_0\) in \(\mathbb{C}\) such that, if \(F_0 \in C_{\infty}(\mathbb{R})\) is an \(L^p\)-multiplier for \(L\), then \(F_0\) extends holomorphically to \(\mathcal{U}\). Recall that \(C_{\infty}(\mathbb{R})\) denotes the space of continuous functions on \(\mathbb{R}\) vanishing at infinity.

Since the \(L^2\)-spectrum of \(L\) is contained in \([0, +\infty[\), we may assume that \(F_0 \in C_{\infty}([0, +\infty[\). Moreover, according to \([8, Lemma 6.1]\), it suffices to consider the case where \(2 < p' < \infty\).

As in the introduction, we can replace \(F_0\) by the function \(F = F_0 e^{-z}\), so that \(F(L)\) acts on the spaces \(L^q(G), \ q \in [p, p']\) by convolution with the function \(F(L)\delta \in \bigcap_{q=p}^{p'} L^q(G)\). The Kunze-Stein phenomenon implies now that every \(L^p\) function defines a bounded operator on \(L^2(G)\) and also on every Hilbert space \(\mathcal{H}\) of any unitary representation \(\pi\) of \(G\), which is weakly contained in the left regular representation. Indeed, we know that for any coefficient \(c_{\xi, \eta}(x) := (\pi(x)\xi, \eta)\) of \(\pi\), we have that

\[
\|c_{\xi, \eta}\|_{L^p(G)} \leq C_p \|\xi\| \|\eta\|, \quad \xi, \eta \in \mathcal{H},
\]

for some constant \(C_p > 0\). Hence for \(f \in L^p(G)\),

\[
\left| \int_G f(x)c_{\xi, \eta}(x)dx \right| \leq \|f\|_p \|c_{\xi, \eta}\|_{L^p} \leq C_p \|f\|_p \|\xi\| \|\eta\|.
\]
Hence there exists a unique bounded operator $\pi(f)$ on $\mathcal{H}$, such that $\|\pi(f)\|_{\text{op}} \leq C_p\|f\|_p$ and

$$\langle \pi(f)\xi,\eta \rangle = \int_G f(x)c_{\xi,\eta}^\pi(x)dx, \quad \xi, \eta \in \mathcal{H}.$$  

Choosing now a sequence $(f_\nu)_\nu$ of continuous functions with compact support, which converges in the $L^p$-norm to $F(L)\delta$, we see that the operators $\lambda(f_\nu)$ converge in the operator norm to $\lambda(F(L)) = F(\lambda(L))$, and so for every unitary representation $(\pi, \mathcal{H})$ of $G$ which is weakly contained in the left regular representation $\lambda$, we have that:

$$\int_G (F(L)\delta)(x)c_{\xi,\eta}^\pi(x)dx = \lim_{\nu \to \infty} \int_G f_\nu(x)c_{\xi,\eta}^\pi(x)dx$$

$$= \lim_{\nu \to \infty} \langle \pi(f_\nu)\xi,\eta \rangle = \langle \pi(F(L))\xi,\eta \rangle = \langle F(\pi(L))\xi,\eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$  

In particular,

$$(F(L)\delta) * (c_{\xi,\eta}^\pi) = \int_G (F(L)\delta)(y)\langle \pi(y)\xi,\pi(x)\eta \rangle dy$$

$$= \langle F(\pi(L))\xi,\pi(x)\eta \rangle, \quad x \in G, \xi, \eta \in \mathcal{H}. \quad (2.15)$$

In a first step in order to find $\lambda_0 \in \mathbb{R}$ and its neighborhood $\mathcal{U}$, we choose a suitable direction $\nu_0$ in $\mathfrak{a}^*$. To this end, let $\omega$ be the Casimir operator of $G$, and let $\nu \in i\mathfrak{a}^*$. Then $\pi_\nu$ is a unitary representation, and we can define the operator $d\pi_\nu(\omega)$ on the space of smooth vectors in $L^2(K/M)$ with respect to $\pi_\nu$. Moreover, $\pi_\nu$ is irreducible (see [11, Theorem 1]), and therefore

$$d\pi_\nu(\omega) = \chi(\nu)\text{Id},$$

where $\chi$ is a polynomial function on $\mathfrak{a}^*$, given by the Harish-Chandra isomorphism. Thus, $p$ is in fact a quadratic form.

Choose $\nu_0 \in \mathfrak{a}^*$, $\nu_0 \neq 0$, such that $p(\nu_0) \neq 0$. Then, clearly,

$$|\chi(iy\nu_0)| \to +\infty \text{ as } y \to +\infty \text{ in } \mathbb{R}. \quad (2.16)$$

Put $p_0 := p'$. According to Proposition 2.4, there is an $\varepsilon > 0$ and a constant $C > 0$, such that (2.10) holds, for every $z \in U_1 := \{z \in \mathbb{C} | |\text{Re } z| < \varepsilon\}$. Put

$$\pi(z) := \pi_{z\nu_0} \text{ and } \widetilde{T}(z) := T(z\nu_0).$$

Then $(\widetilde{T}(z))_{z \in U_1}$ is an analytic family of compact operators on $L^{p_0}(K/M)$ (see Proposition 2.3).

And, by an obvious analogue to [8, Proposition 5.4], there exists an open connected neighbourhood $U_{y_0}$ of some point $iy_0$ in $U_1$, with $y_0 \in \mathbb{R}$, and two holomorphic mappings

$$\lambda : U_{y_0} \to \mathbb{C} \text{ and } \xi : U_{y_0} \to L^{p_0}(K/M)$$

such that, for all $z \in U_{y_0}$ and some constant $C > 0$,

$$\widetilde{T}(z)\xi(z) = \lambda(z)\xi(z); \quad \xi(z) \neq 0 \text{ and } \|\xi(z)\|_{p_0} \leq C. \quad (2.17)$$
Since $\pi_{(iy)}$ is unitary for every $y \in \mathbb{R}$, $\lambda$ is real-valued on $U_{y_0} \cap i\mathbb{R}$.

Fix a non-trivial function $\eta$ in $C^\infty(K/M)$.

Let $\Phi_z$, $z \in U_{y_0}$, denote the function on $G$ given by

$$
\Phi_z(g) := \langle \pi(z)(g^{-1})\xi(z), \eta \rangle.
$$

Then $\Phi_z(g)$ depends continuously on $z$ and $g$. Moreover, by (2.10) and (2.17), there exists a constant $C_0 > 0$, such that:

$$
(2.18) \quad \|\Phi_z\|_{L^p_0(G)} \leq C_0, \quad \text{for all } z \in U_{y_0}.
$$

Thus, for any $z \in U_{y_0}$, $\Phi_z \in L^p_0(G)$, and consequently $F(L)\Phi_z \in L^p_0(G)$ is well-defined, since $F$ is an $L^p_0$-multiplier for $L$.

Put $\mu(z) := - \log \lambda(z)$ ($z \in U_{y_0}$), where log denotes the principal branch of the logarithm. Since, for $z \in U_{y_0}$, $\xi(z)$ is an eigenvector of $\overline{T}(z) = \pi(z)(h_1)$ associated to the eigenvalue $\lambda(z)$, where $h_1$ is the convolution kernel of $e^{-L}$, one has by (2.15), for all $z \in U_{y_0} \cap i\mathbb{R}$, $g \in G$:

$$
(2.19) \quad (F(L)\Phi_z)(g) = \langle F(\pi(z)(L))\xi(z), \pi(z)\eta \rangle = F(\mu(z))\langle \pi(z)(g^{-1})\xi(z), \eta \rangle.
$$

Let $\psi$ be a fixed element of $C_0(G)$ such that:

$$
\int_G \Phi_{iy_0}(x)\psi(x)\,dx \neq 0.
$$

By shrinking $U_{y_0}$, if necessary, we may assume that $\int_G \Phi_z(x)\psi(x)\,dx \neq 0$ for all $z \in U_{y_0}$.

Then, (2.19) implies that:

$$
(2.20) \quad (F \circ \mu)(z) = \frac{\int_G (F(L)\Phi_z)(x)\psi(x)\,dx}{\int_G \Phi_z(x)\psi(x)\,dx}, \quad \text{for } z \in U_{y_0} \cap i\mathbb{R}.
$$

Observe that the enumerator and the denominator in the right-hand side of (2.20) are holomorphic functions of $z \in U_{y_0}$. Indeed, $\langle F(L)\Phi_z, \psi \rangle = \langle \Phi_z, F(L)\psi \rangle$, where $F(L)^*\psi \in L^p_0$ and $\|\Phi_z\|_{L'_{p_0}} \leq C$, by (2.18). This implies that the mapping $z \mapsto \langle F(L)\Phi_z, \psi \rangle$ is continuous, and the holomorphy of this mapping then follows easily from Fubini’s and Morera’s theorems.

Therefore, $F \circ \mu$ has a holomorphic extension to $U_{y_0}$.

Moreover, since $\omega h_1 \in L^1(G)$, in view of Proposition 2.2, the norm

$$
\|\pi_{(iy)}(\omega h_1)\|_{\text{op}} \leq \|\omega h_1\|_{L^1(G)}
$$

is uniformly bounded, for $y \in \mathbb{R}$. On the other hand, $\pi_{(iy)}(\omega h_1) = d\pi_{(iy)}(\omega)\pi_{(iy)}(h_1) = \chi(i\nu h_0)\pi_{(iy)}(h_1)$, and so (2.16) implies that

$$
\lim_{y \to +\infty} \|\overline{T}(iy)\| = \lim_{y \to +\infty} \|\pi_{(iy)}(h_1)\| = 0.
$$

This shows that $\lambda$ is not constant, and hence, varying $y_0$ slightly, if necessary, we may assume that $\mu'(iy_0) \neq 0$. Then $\mu$ is a local bi-holomorphism near $iy_0$, which implies, in combination with (2.20), that $F$ has a holomorphic extension to a complex neighbourhood of $\lambda_0 := \mu(iy_0) \in \mathbb{R}$. 

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3 Transference for \( p \)-induced representations

3.1 \( p \)-induced representations

Let \( G \) be a separable locally compact group and \( S \subset G \) a closed subgroup. By [15], there exists a Borel measurable cross-section \( \sigma : G/S \to G \) for the homogeneous space \( H := G/S \) (i.e. \( \sigma(x) \in x \) for every \( x \in G/S \)) such that \( \sigma(K) \) is relatively compact for any compact subset \( K \) of \( H \). Then, every \( g \in G \) can be uniquely decomposed as

\[
g = \sigma(x)s, \quad \text{with } x \in H, s \in S.
\]

We put \( \Phi : H \times S \to G, \Phi(x,s) := \sigma(x)s \).

Then \( \Phi \) is a Borel isomorphism, and we write \( \Phi^{-1}(g) =: (\eta(g), \tau(g)) \).

Then

\[
g = \sigma \circ \eta(g) \tau(g), \quad g \in G.
\]

For later use, we also define

\[
\tau(g,x) := \tau(g^{-1}\sigma(x)), \quad \eta(g,x) := \eta(g^{-1}\sigma(x)), \quad g \in G, x \in H.
\]

Let \( dg \) denote the left-invariant Haar measure on \( G \), and \( \Delta_G \) the modular function on \( G \), i.e.

\[
\int_G f(gh)dg = \Delta_G(h)^{-1} \int_G f(g)dg, \quad h \in G.
\]

Similarly, \( ds \) denotes the left-invariant Haar measure on \( S \), and \( \Delta_S \) its modular function.

On a locally compact measure space \( Z \), we denote by \( \mathcal{M}_b(Z) \) the space of all essentially bounded measurable functions from \( M \) to \( \mathbb{C} \), and by \( \mathcal{M}_0(Z) \) the subspace of all functions which have compact support, in the sense that they vanish a.e. outside a compact subset of \( Z \). For \( f \in \mathcal{M}_0(G) \), let \( \tilde{f} \) be the function on \( G \) given by

\[
\tilde{f}(g) := \int_S f(gs) \Delta_G,S(s) ds, \quad g \in G,
\]

where we have put \( \Delta_G,S(s) = \Delta_G(s)/\Delta_S(s), s \in S \). Then \( \tilde{f} \) lies in the space

\[
\mathcal{E}(G,S) := \{ \tilde{h} \in \mathcal{M}_b(G) : \tilde{h} \text{ has compact support modulo } S, \text{ and } \tilde{h}(gs) = (\Delta_G,S(s))^{-1} \tilde{h}(g) \text{ for all } g \in G, s \in S \}
\]

In fact, one can show that \( \mathcal{E}(G,S) = \{ \tilde{f} : f \in \mathcal{M}_0(G) \} \). Moreover, one checks easily, by means of the use of a Bruhat function, that \( \tilde{f} = 0 \) implies \( \int_G f(g)dg = 0 \).

From here it follows that there exists a unique positive linear functional, denoted by \( \int_{G/S} d\tilde{g} \), on the space \( \mathcal{E}(G,S) \), which is left-invariant under \( G \), such that

\[
(3.1) \quad \int_G f(g)dg = \int_{G/S} \tilde{f}(g)d\tilde{g} = \int_{G/S} \int_S f(gs) \Delta_G,S(s) ds \ d\tilde{g}.
\]
By means of the cross-section $\sigma$, we can next identify the function $\tilde{h} \in \mathcal{E}(G,S)$ with the measurable function $h \in \mathcal{M}_0(H)$, given by

$$h(x) := R\tilde{h}(x) := \tilde{h}(\sigma(x)), \quad x \in H.$$  

Notice that, given $h \in \mathcal{M}_0(H)$, the corresponding function $\tilde{h} := R^{-1}h \in \mathcal{E}(G,S)$ is given by

$$\tilde{h}(\sigma(x)s) = h(x)\Delta_{G,S}(s)^{-1}.$$  

The mapping $h \mapsto \int_{G/S} \tilde{h}(g)d\hat{g}$ is then a positive Radon measure on $C_0(H)$, so that there exists a unique regular Borel measure $dx$ on $H = G/S$, such that

$$(3.2) \quad \int_{G/S} \tilde{h}(g)d\hat{g} = \int_H h(x)dx, \quad h \in C_0(H).$$  

Formula (3.1) can then be re-written as

$$(3.3) \quad \int_G f(g)d\hat{g} = \int_H \int_S f(\sigma(x)s)\Delta_{G,S}(s)ds \, dx.$$  

Notice that the left-invariance of $\int_{G/S} d\hat{g}$ then translates into the following quasi-invariance property of the measure $dx$ on $H$:

$$(3.4) \quad \int_H \tilde{h}(g(x))\Delta_{G,S}(\tau(g,x))^{-1}dx = \int_H h(x)dx \quad \text{for every } g \in G.$$  

Formula (3.3) remains valid for all $f \in L^1(G)$. Next, let $\rho$ be a strongly continuous isometric representation of $S$ on a complex Banach space $(X, \| \cdot \|_X)$, so that in particular

$$\|\rho(s)v\|_X = \|v\|_X \quad \text{for every } s \in S, v \in X.$$  

Fix $1 \leq p < \infty$, and let $L^p(G,X; \rho)$ denote the Banach space of all Borel measurable functions $\tilde{\xi} : G \to X$, which satisfy the covariance condition

$$\tilde{\xi}(gs) = \Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\tilde{\xi}(g)], \quad \text{for all } g \in G, s \in S,$$

and have finite $L^p$-norm $\|\tilde{\xi}\|_p := \left(\int_{G/S} \|\tilde{\xi}(g)\|_{X}^{p}d\hat{g}\right)^{1/p}$. Notice that the function $g \mapsto \|\tilde{\xi}(g)\|_{X}^{p}$ satisfies the covariance property of functions in $\mathcal{E}(G,S)$, so that the integral $\int_{G/S} \|\tilde{\xi}(g)\|_{X}^{p}d\hat{g}$ is well-defined.

The $p$-induced representation $\pi_p = \text{ind}_p^G \rho$ is then the left-regular representation $\lambda_G = \lambda$ of $G$ acting on $L^p(G,X; \rho)$, i.e.

$$\left[\pi_p(g)\tilde{\xi}\right](g') := \tilde{\xi}(g^{-1}g'), \quad g, g' \in G, \tilde{\xi} \in L^p(G,X; \rho).$$

By means of the cross-section $\sigma$, one can realize $\pi_p$ on the $L^p$-space $L^p(H,X)$. To this end, given $\tilde{\xi} \in L^p(G,X; \rho)$, we define $\xi \in L^p(H,X)$ by

$$\xi(x) := T\tilde{\xi}(x) := \tilde{\xi}(\sigma(x)), \quad x \in H.$$  

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Because of (3.2), \( T : L^p(G, X; \rho) \to L^p(H, X) \) is a linear isometry, with inverse
\[
T^{-1}\xi(\sigma(x)s) := \tilde{\xi}(\sigma(x)s) = \Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\xi(x)].
\]
Since, for \( g \in G, y \in H \) and \( \tilde{\xi} \in L^p(G, X; \rho) \),
\[
\tilde{\xi}(g^{-1}\sigma(y)) = \tilde{\xi}(\sigma \circ \eta(g^{-1}\sigma(y))\tau(g^{-1}\sigma(y))) = \tilde{\xi}(\sigma(\eta(g,y))\tau(g,y)) = \Delta_{G,S}(\tau(g,y))^{-1/p}\rho(\tau(g,y)^{-1})\left[\tilde{\xi}(\sigma(\eta(g,y)))\right],
\]
we see that the induced representation \( \pi_p \) can also be realized on \( L^p(H, X) \), by
\[
(3.5) \quad [\pi_p(g)\xi](y) = \Delta_{G,S}(\tau(g,y))^{-1/p}\rho(\tau(g,y)^{-1})\left[\xi(\eta(g,y))\right],
\]
for \( g \in G, y \in H, \xi \in L^p(H, X) \).
Observe that \( \pi_p(g) \) acts isometrically on \( L^p(H, X) \), for every \( g \in G \). This is immediate from the original realization of \( \pi_p \) on \( L^p(G, X; \rho) \), but follows also from (3.4), in the second realization given by (3.5).

**Examples 3.1.**

(a) If \( S \triangleleft G \) is a closed, normal subgroup, then \( H = G/S \) is again a group, and one finds that, for a suitable normalization of the left-invariant Haar measure \( dx \) on \( H \), we have
\[
\int_G f(g)dg = \int_H \int_S f(\sigma(x)s)dsdx, \quad f \in L^1(G).
\]
In particular, \( \Delta_{G|S} = \Delta_S \), so that \( \Delta_{G,S} = 1 \) and \( dx \) in (3.3) agrees with the left-invariant Haar measure on \( H \).
Furthermore, there exists a measurable mapping \( q : H \times H \to S \), such that
\[
\sigma(x)^{-1}\sigma(y) = \sigma(x^{-1}y)q(x,y), \quad x, y \in H,
\]
since \( \sigma(x)^{-1}\sigma(y) \equiv \sigma(x^{-1}y) \) modulo \( S \). Thus, if \( g = \sigma(x)s \), then
\[
g^{-1}\sigma(y) = s^{-1}\sigma(x)^{-1}\sigma(y) = s^{-1}\sigma(x^{-1}y)q(x,y) = \sigma(x^{-1}y)((s^{-1})^{\sigma(x^{-1}y)^{-1}}q(x,y)).
\]
(Here we use the notation \( s^g := gs^{-1}, \ s \in S, g \in G \).)
This shows that \( \tau(g,y) = (s^{-1})^{\sigma(x^{-1}y)^{-1}}q(x,y) \) and \( \eta(g,y) = x^{-1}y \). Hence \( \pi_p \) is given as follows:
\[
(3.6) \quad [\pi_p(\sigma(x)s)\xi](y) = \rho(q(x,y)^{-1}s^{\sigma(x^{-1}y)^{-1}})[\xi(x^{-1}y)],
\]
for \( (x,s) \in H \times S, y \in H, \xi \in L^p(H, X) \).
We remark that it is easy to check that:
\[
q(x,y)^{-1}s^{\sigma(x^{-1}y)^{-1}} = s^{\sigma(y)^{-1}\sigma(x)}q(x,y)^{-1}.
\]
Notice that (3.6) does not depend on \( p \).
In the special case where $\rho = 1$ and $S$ is normal, the induced representation $\iota = \text{ind}_G^S 1$ is given by

$$[\iota(\sigma(x)s)\xi](y) = \xi(x^{-1}y).$$

For the integrated representation, we then have

$$[\iota(f)\xi](y) = \int_H \int_S f(\sigma(x)s)\xi(x^{-1}y)ds\,dx$$

$$= \int_H \tilde{f}(x)\xi(x^{-1}y)dx$$

$$= \left[\lambda_H(\tilde{f})\xi\right](y),$$

i.e.

$$\iota(f) = \lambda_H(\tilde{f}),$$

where

$$\tilde{f}(x) := \int_S f(\sigma(x)s)ds,$$

i.e. $\tilde{f}$ is the image of $f$ under the quotient map from $G$ onto $G/S$.

### 3.2 A transference principle

If $\xi \in L^p(H, X)$, and if $\phi : S \to \mathbb{C}$, we define the $\rho$-twisted tensor product

$$\xi \otimes^\rho \phi : G \to X \quad \text{by}$$

$$[\xi \otimes^\rho \phi] \cdot (\sigma(x)s) := \phi(s)\Delta_{G,S}(s)^{-1/p}\rho(s^{-1})[\xi(x)], \quad (x, s) \in H \times S.$$ 

Let us denote by $X^*$ the dual space of $X$. For any complex vector space $Y$, we denote by $\overline{Y}$ its complex conjugate, which, as an additive group, is the space $Y$, but with scalar multiplication given by $\lambda y$, for $\lambda \in \mathbb{C}$ and $y \in Y$. In the sequel, we assume that $X$ contains a dense, $\rho$-invariant subspace $X_0$, which embeds via an anti-linear mapping $i : X_0 \hookrightarrow \overline{X^*}$ into the complex conjugate of the dual space of $X$, in such a way that

$$||x|| = \sup_{\{v \in X_0 : ||v||_{X^*} = 1\}} |\langle x, v \rangle|$$

for every $x \in X$.

Here, we have put

$$\langle x, v \rangle := i(v)(x), \quad v \in X_0, x \in X.$$ 

Moreover, we assume that

$$||i(\rho(s)v)||_{X^*} = ||i(v)||_{X^*}$$

for every $v \in X_0, s \in S$, and

$$\langle \rho(s)x, \rho(s)v \rangle = \langle x, v \rangle$$

for every $x \in X, v \in X_0, s \in S$.

The most important example for us will be an $L^p$-space $X = L^p(\Omega)$, $1 \leq p < \infty$, on a measure space $(\Omega, d\omega)$, and a representation $\rho$ of $G$ which acts isometrically on $L^p(\Omega)$.
as well as on its dual space $L'\,\Omega$ (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). In this case, by interpolation, we have $||\rho(g)||_r \leq ||\xi||_r$, for $|\frac{1}{p} - \frac{1}{2}| \leq |\frac{1}{p'} - \frac{1}{2}|$, $g \in G$, which implies that indeed $\rho(g)$ acts isometrically on $L'\,\Omega$, for $|\frac{1}{p} - \frac{1}{2}| \leq |\frac{1}{p'} - \frac{1}{2}|$. In particular, $\rho$ is a unitary representation on $L^2(\Omega)$. We can then choose $X_0 := L^p(\Omega) \cap L^p(\Omega) \subset L^2(\Omega)$, and put

$$i(\eta)(\xi) := \int_{\Omega} \xi(\omega)\overline{\eta(\omega)}\,d\omega, \quad \eta \in L^p(\Omega) \cap L^p(\Omega), \xi \in L^p(\Omega).$$

Notice that (3.8) and (3.9) are always satisfied, if $\rho$ is a unitary character.

**Lemma 3.1.** Let $\phi \in L^p(S), \psi \in L^p(S), \xi \in L^p(H, X_0)$ and $\eta \in L^p(H, X_0)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for every $g \in G$,

$$\langle \lambda_G(g)(\xi \otimes^p \phi), \eta \otimes^p \psi \rangle = \int_H \phi^* \psi(\tau(g, x))([\pi_p(g)\xi](x), \eta(x))\,dx. \tag{3.10}$$

**Proof.** By (3.3), we have

$$\langle \lambda_G(g)(\xi \otimes^p \phi), \eta \otimes^p \psi \rangle = \int_H \int_S \langle \xi \otimes^p \phi(g^{-1}\sigma(x)s), \eta \otimes^p \psi(\sigma(x)s) \rangle \Delta_G,S(s)\,ds\,dx \tag{3.4}$$

$$= \int_H \int_S \langle \xi \otimes^p \phi(\sigma(\eta(g,x))\tau(g,x)s), \eta \otimes^p \psi(\sigma(x)s) \rangle \Delta_G,S(s)\,ds\,dx \tag{3.5}$$

$$= \int_H \int_S \Delta_G,S(\tau(g,x)s)^{-\frac{1}{p'}} \Delta_G,S(s)^{-\frac{1}{p}} \phi(\tau(g,x)s)\overline{\psi(s)} \langle \rho(s^{-1}\tau(g,x)^{-1})[\xi(\eta(g,x))], \rho(s^{-1})[\eta(x)] \rangle \Delta_G,S(s)\,ds\,dx \tag{3.6}$$

$$= \int_H \int_S \Delta_G,H(\tau(g,x))^{-\frac{1}{p'}} \rho(\tau(g,x)^{-1})[\xi(\eta(g,x))], \eta(x)] \rangle \Delta_G,H(\tau(g,x))^{-\frac{1}{p}} \phi(\tau(g,x)s)\overline{\psi(s)} \,ds\,dx \tag{3.7}$$

Here, we have used that, by (3.9), $\langle \rho(s^{-1})v_1, \rho(s^{-1})v_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in X_0$. But,

$$\int_S \phi(\tau(g,x)s)\overline{\psi(s)}\,ds = \int_S \phi(s)\psi(\tau(g,x)^{-1}s)\,ds = \phi^* \psi(\tau(g,x)),$$

and

$$\Delta_G,H(\tau(g,x))^{-\frac{1}{p'}} \rho(\tau(g,x)^{-1})[\xi(\eta(g,x))], \eta(x)] \rangle \Delta_G,H(\tau(g,x))^{-\frac{1}{p}} \phi(\tau(g,x)s)\overline{\psi(s)} \,ds\,dx \tag{3.8}$$

and thus (3.10) follows. \hfill \Box

From now on, we shall assume that the group $S$ is amenable.

Since $G$ is separable, we can then choose an increasing sequence $\{A_j\}$ of compacta in $S$ such that $A_j^{-1} = A_j$ and $S = \bigcup_j A_j$, and put

$$\phi_j = \phi^p_j := \frac{\chi_{A_j}}{|A_j|^{1/p}}, \quad \psi_j = \psi^{p'}_j := \frac{\chi_{A_j}}{|A_j|^{1/p'}}.$$
where \( \chi_A \) denotes the characteristic function of the subset \( A \). Then \( \bar{\psi}_j = \psi_j, \ ||\phi_j||_p = ||\psi_j||_{p'} = 1 \), and, because of the amenability of \( S \) (see [16]), we have

\[
(3.11) \quad \chi_j := \phi_j \ast \psi_j \text{ tends to } 1, \text{ uniformly on compacta in } S.
\]

**Proposition 3.1.** Let \( \pi_p = \text{ind}^G_{p,S} \rho \) be as before, where \( S \) is amenable, and let \( \xi, \eta, \in C_0(H, X_0) \).

Then

\[
\langle \pi_p(g) \xi, \eta \rangle = \lim_{j \to \infty} \langle \lambda_G(g)(\xi \otimes^p \phi_j), \eta \otimes^p \psi_j \rangle,
\]

uniformly on compacta in \( G \).

**Proof.** By Lemma 3.1,

\[
\langle \lambda_G(g)(\xi \otimes^p \phi_j), \eta \otimes^p \psi_j \rangle = \int_H \chi_j(\tau(g, x)) \langle [\pi_p(g)](x), \eta(x) \rangle \, dx.
\]

Fix a compact set \( K = K^{-1} \subset H \) containing the supports of \( \xi \) and \( \eta \), and let \( Q \subset G \) be any compact set. We want to prove that \( \{ \tau(g, x) | g \in Q, x \in K \} \) is relatively compact, for then, by (3.11), we immediately see that

\[
\lim_{j \to \infty} \langle \lambda_G(g)(\xi \otimes^p \phi_j), \eta \otimes^p \psi_j \rangle = \int_H \langle [\pi_p(g)](x), \eta(x) \rangle \, dx = \langle \pi_p(g) \xi, \eta \rangle,
\]

uniformly for \( g \in Q \).

Recall that \( \tau(g, x) = \tau(g^{-1} \sigma(x)) \). Therefore, since \( \sigma(K) \) is relatively compact, it suffices to prove that \( \tau \) maps compact subsets of \( G \) into relatively compact sets in \( S \). So, let again \( Q \) denote a compact subset of \( G \), and put \( M := Q \mod S < g H = G/S \). Then \( M \) is compact, so that \( \sigma(M) \) is compact in \( S \). And, since \( \tau(\sigma(x) s) = s \) for every \( x \in H, s \in S \), we have

\[
\tau(Q) = \{ s \in S | \sigma(x) s \in Q \text{ for some } x \in M \} = \sigma(M)^{-1} Q,
\]

which shows that \( \tau(Q) \) is indeed relatively compact. \( \square \)

**Theorem 3.1.** For every bounded measure \( \mu \in M^1(G) \), we have

\[
|| \pi_p(\mu) ||_{L^p(H, X)} \to L^p(H, X) \leq || \lambda_G(\mu) ||_{L^p(G, X)} \to L^p(G, X).
\]

**Proof.** Let \( \xi, \eta \in C_0(H, X_0) \). Observe first that, for \( g \in G \),

\[
| \langle \lambda_G(g)(\xi \otimes^p \phi_j), \eta \otimes^p \psi_j \rangle | \\
\leq || \lambda_G(g)(\xi \otimes^p \phi_j) ||_{L^p(G, X)} || i \circ (\eta \otimes^p \psi_j) ||_{L^p(G, X^*)} \\
= || \xi \otimes^p \phi_j ||_{L^p(G, X)} || i \circ (\eta \otimes^p \psi_j) ||_{L^p(G, X^*)},
\]

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Proof. If \( \rho(s^{-1}) \) is isometric on \( X \), so that
\[
(3.12) \quad \| \xi \otimes_{\rho}^p \phi_j \|_{L^p(G,X)} = \| \xi \|_{L^p(H,X)},
\]
and similarly, because of (3.8),
\[
(3.13) \quad \| i \circ (\eta \otimes_{\rho'}^p \psi_j) \|_{L^{p'}(G,X^*)} = \| i \circ \eta \|_{L^{p'}(H,X^*)}.
\]
This implies
\[
\| \langle \lambda_G(g)(\xi \otimes_{\rho}^p \phi_j), \eta \otimes_{\rho'}^p \psi_j \rangle \| \leq \| \xi \|_{L^p(H,X)} \| i \circ \eta \|_{L^{p'}(H,X^*)}.
\]
Therefore, if \( \mu \in M^1(G) \), Proposition 3.1 implies, by the dominated convergence theorem, that
\[
(3.14) \quad \langle \pi_p(\mu) \xi, \eta \rangle = \lim_{j \to \infty} \langle \lambda_G(\mu)(\xi \otimes_{\rho}^p \phi_j), \eta \otimes_{\rho'}^p \psi_j \rangle.
\]
Moreover, by (3.12) and (3.13),
\[
\| \langle \lambda_G(\mu)(\xi \otimes_{\rho}^p \phi_j), \eta \otimes_{\rho'}^p \psi_j \rangle \|
= \| i \circ (\eta \otimes_{\rho'}^p \psi_j) \|_{L^{p'}(H,X^*)}.
\]
By (3.14), we therefore obtain
\[
(3.15) \quad \| \langle \pi_p(\mu) \xi, \eta \rangle \| \leq \| \lambda_G(\mu) \|_{L^p(G,X) \to L^p(G,X)} \| \xi \|_{L^p(H,X)} \| i \circ \eta \|_{L^{p'}(H,X^*)}.
\]
In view of (3.7), this implies the theorem, since \( C_0(H,X_0) \) lies dense in \( L^p(H,X) \). \( \Box \)

**Corollary of Theorem 3.1 (Transference).** Let \( X = L^p(\Omega) \). Then, for every \( \mu \in M^1(G) \), we have
\[
\| \pi_p(\mu) \|_{L^p(H,L^p(\Omega)) \to L^p(H,L^p(\Omega))} \leq \| \lambda_G(\mu) \|_{L^p(G) \to L^p(G)}.
\]

**Proof.** If \( X = L^p(\Omega) \) and \( h \in L^p(G,X) \), then, by Fubini’s theorem,
\[
\| \lambda_G(\mu) h \|_{L^p(G,X)}^p = \int_{\Omega} \| \mu \ast h(\cdot, \omega) \|_{L^p(G)}^p \, d\omega \leq \| \lambda_G(\mu) \|_{L^p(G) \to L^p(G)}^p \| h \|_{L^p(G,X)},
\]
hence
\[
\| \lambda_G(\mu) \|_{L^p(G,X) \to L^p(G,X)} \leq \| \lambda_G(\mu) \|_{L^p(G) \to L^p(G)}.
\]
In combination with (3.15), we obtain the desired estimate. \( \Box \)
Remark. We call a Banach space $X$ to be of $L^p$-type, $1 \leq p < \infty$, if there exists an embedding $\iota : X \hookrightarrow L^p(\Omega)$ into an $L^p$-space such that

$$\frac{1}{C} ||x||_X \leq ||\iota(x)||_{L^p(\Omega)} \leq C ||x||_X \quad \text{for every } x \in X,$$

for some constant $C \geq 1$.

For instance, any separable Hilbert space $\mathcal{H}$ is of $L^p$-type, for $1 \leq p < \infty$, or, more generally, any space $L^p(Y, \mathcal{H})$. This follows easily from Khintchin’s inequality. Corollary 3.2 remains valid for spaces $X$ of $L^p$-type, by an obvious modification of the proof.

Denote by $C_r^*(G)$ the reduced $C^*$-algebra of $G$. If $p = 2$, we can extend (3.14) to $C_r^*(G)$.

**Proposition 3.2.** If $p = 2$ and $X = L^2(\Omega)$, then the unitary representation $\pi_2$ is weakly contained in the left-regular representation $\lambda_G$. In particular, for any $K \in C_r^*(G)$, the operator $\pi_2(K) \in \mathcal{B}(L^2(H, L^2(\Omega)))$ is well-defined.

Moreover, for all $\xi, \eta \in C_0(H, L^2(\Omega))$, we have

(3.16) $\langle \pi_2(K)\xi, \eta \rangle = \lim_{j \to \infty} \langle \lambda_G(K)(\xi \otimes^2 \phi_j), \eta \otimes^2 \phi_j \rangle$.

**Proof.** If $K \in C_r^*(G)$, then we can find a sequence $\{f_k\}_k$ in $L^1(G)$, such that $\lambda_G(K) = \lim_{k \to \infty} \lambda_G(f_k)$ in the operator norm $|| \cdot ||$ on $L^2(G)$. But, (3.15) implies that

(3.17) $||\pi_2(f)|| \leq ||\lambda_G(f)||$, for all $f \in L^1(G)$,

where $|| \cdot ||$ denotes the operator norm on $\mathcal{B}(L^2(H, L^2(\Omega)))$ and $\mathcal{B}(L^2(G))$, respectively. Therefore, the $\{\pi_2(f_k)\}_k$ form a Cauchy sequence in $\mathcal{B}(L^2(H, L^2(\Omega)))$, whose limit we denote by $\pi_2(K)$.

It does not depend on the approximating sequence $\{f_k\}_k$. Moreover, from (3.17) we then deduce that

(3.18) $||\pi_2(K)|| \leq ||\lambda_G(K)|| = ||K||_{C_r^*(G)}$, for all $K \in C_r^*(G)$.

In particular, we see that $\pi_2$ is weakly contained in $\lambda_G$. It remains to show (3.16).

Given $\varepsilon > 0$, we choose $f \in C_0(G)$ such that $||K - f||_{C_r^*(G)} < \varepsilon/4$. Next, by (3.15), we can find $j_0$ such that

$$|\langle \pi_2(f)\xi, \eta \rangle - \langle \lambda_G(f)(\xi \otimes^2 \phi_j), \eta \otimes^2 \phi_j \rangle| < \varepsilon/4 \quad \text{for all } j \geq j_0.$$

Assume without loss of generality that $||\xi||_2 = ||\eta||_2 = 1$. Then, by (3.18),

$$|\langle \pi_2(K)\xi, \eta \rangle - \langle \pi_2(f)\xi, \eta \rangle| \leq ||K - f||_{C_r^*(G)} ||\xi||_2 ||\eta||_2 < \varepsilon/4,$$

and furthermore

$$|\langle \lambda_G(K)(\xi \otimes^2 \phi_j), \eta \otimes^2 \phi_j \rangle - \langle \lambda_G(f)(\xi \otimes^2 \phi_j), \eta \otimes^2 \phi_j \rangle| \leq ||K - f||_{C_r^*(G)} ||\xi \otimes^2 \phi_j||_2 ||\eta \otimes^2 \phi_j||_2 < \frac{\varepsilon}{4} ||\xi||_2 ||\eta||_2 = \varepsilon/4.$$

Combining these estimates, we find that

$$|\langle \pi_2(K)\xi, \eta \rangle - \langle \lambda_G(K)(\xi \otimes^2 \phi_j), \eta \otimes^2 \phi_j \rangle| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon \quad \text{for all } j \geq j_0.$$
Corollary of Proposition 3.2. Assume that $\rho$ is a unitary representation on a separable Hilbert space $X$, for instance a unitary character of $S$, and that $\Delta_{G,S} = 1$. Let $K \in C^*_r(G)$, and assume that $\lambda_G(K)$ extends from $L^2(G) \cap L^p(G)$ to a bounded linear operator on $L^p(G)$, where $1 \leq p < \infty$.

Then $\pi_2(K)$ extends from $L^2(G/S) \cap L^p(G/S)$ to a bounded linear operator on $L^p(G/S)$, and

$$\|\pi_2(K)\|_{L^p(G/S) \rightarrow L^p(G/S)} \leq \|\lambda_G(K)\|_{L^p(G) \rightarrow L^p(G)}.$$  

Moreover, for $f \in L^1(G)$, we have $\pi_p(f) = \pi_2(f)$ on $C_0(G/S)$.

**Proof.** If $\xi, \eta \in C_o(H)$, then, since $\Delta_{G,S} = 1$,

$$\langle \pi_2(K)\xi, \eta \rangle = \lim_{j \rightarrow \infty} \langle \lambda_G(K)(\xi \otimes_R \phi^p_j), \eta \otimes_R \psi^p_j \rangle = \lim_{j \rightarrow \infty} \langle \lambda_G(K)(\xi \otimes_R \phi^p_j), \eta \otimes_R \psi^p_j \rangle.$$

And,

$$\|\langle \lambda_G(K)(\xi \otimes_R \phi^p_j), \eta \otimes_R \psi^p_j \rangle\| \leq \|\lambda_G(K)\|_{L^p(G) \rightarrow L^p(G)} \|\xi \otimes_R \phi^p_j\|_{L^p(G)} \|\eta \otimes_R \psi^p_j\|_{L^p(G)}$$

Estimate (3.19) follows.

That $\pi_p(f) = \pi_2(f)$ on $C_0(H)$, if $f \in L^1(G)$, is evident, since $\Delta_{G,S} = 1$. \qed

4 The case of a non-compact semi-simple factor

In this section, we shall give our proof of Theorem 1.2.

Let us first notice the following consequence of Corollary 3.2.

Assume that $S$ is a closed, normal and amenable subgroup of $G$, and let $L = -\sum_j X_j^2$ be a sub-Laplacian on $G$. Denote by $\iota_2 := \text{ind}_S^G 1$ the representation of $G$ induced by the trivial character of $S$ (compare Example 3.1), and let $\tilde{L} = -\sum_j (X_j \mod s)^2 = d_2(L)$ be the corresponding sub-Laplacian on the quotient group $H := G/S$. Then

$$\mathcal{M}_p(L) \cap C_{\infty}(\mathbb{R}) \subset \mathcal{M}_p(\tilde{L}) \cap C_{\infty}(\mathbb{R}).$$  

In particular, if $\tilde{L}$ is of holomorphic $L^p$-type, then so is $L$.

In order to prove (4.1), assume that $F$ is an $L^p$-multiplier for $L$ contained in $C_{\infty}(\mathbb{R})$. Then $F(L)$ lies in $C^*_r(G)$, and by Corollary 3.2 the operator $\iota_2(F(L)) = F(d_2(L)) = F(\tilde{L})$ extends from $L^2(H) \cap L^p(H)$ to a bounded operator on $L^p(H)$, so that $F \in \mathcal{M}_p(\tilde{L}) \cap C_{\infty}(\mathbb{R})$.

Let now $G$ be a connected Lie group, with radical $S = \text{exp} \mathfrak{s}$. Then there exists a connected, simply connected semi-simple Lie group $H$ such that $G$ is the semi-direct product of $H$ and $S$, and this Levi factor $H$ has a discrete center $Z$ (see [4]). Let $L$ be a sub-Laplacian on $G$, and denote by $\tilde{L}$ the corresponding sub-Laplacian on $G/S \simeq H$.
and by $\tilde{L}$ the sub-Laplacian on $H/Z$ corresponding to $\tilde{L}$ on $H$. We have that $Z$ and $S$ are amenable groups, and $H/Z$ has finite center. From Theorem 1.1, we thus find that $\tilde{L}$ is of holomorphic $L^p$-type for every $p \neq 2$, if we assume that $H$ is non-compact, and (4.1) then allows us to conclude that the same is true of $\tilde{L}$, and then also of $L$.

5 Compact extensions of exponential solvable Lie groups

5.1 Compact operators arizing in induced representations

Let now $K = \exp \mathfrak{k}$ be a connected compact Lie group acting continuously on an exponential solvable Lie group $S = \exp \mathfrak{s}$ by automorphisms $\sigma(k) \in \text{Aut}(S), k \in K$. We form the semi-direct product $G = K \ltimes S$ with the multiplication given by:

$$(k, s) \cdot (k', s') = (kk', \sigma(k'^{-1})ss'), \quad k, k' \in K, s, s' \in S.$$ 

The left Haar measure $dg$ is the product of the Haar measure of $K$ and the left Haar measure of $S$. Let us choose a $K$-invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{s}$ of $S$. Denote by $\mathfrak{n}$ the nil-radical of $\mathfrak{s}$. Since every derivation $d$ of $\mathfrak{s}$ maps the vector space $\mathfrak{s}$ into the nil-radical, it follows that the orthogonal complement $\mathfrak{b}$ of $\mathfrak{n}$ in $\mathfrak{s}$ is in the kernel of $d\sigma(X)$ for every $X \in \mathfrak{k}$. The following decomposition of the solvable Lie algebra $\mathfrak{s}$ has been given in [3]. Choose an element $X \in \mathfrak{b}$, which is in general position for the roots of $\mathfrak{s}$, i.e., for which $\lambda(X) \neq \mu(X)$ for all roots $\mu \neq \lambda$ of $\mathfrak{s}$. Let $\mathfrak{s}_0 = \{Y \in \mathfrak{s}; \text{ad}^l(X)Y = 0 \text{ for some } l \in \mathbb{N}^\ast\}$. Then $\mathfrak{s}_0$ is a nilpotent subalgebra of $\mathfrak{s}$, which is $K$-invariant (since $[X, \mathfrak{t}] = \{0\}$) and $\mathfrak{s} = \mathfrak{s}_0 + \mathfrak{n}$. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n} \cap \mathfrak{s}_0$ in $\mathfrak{s}_0$. Then $\mathfrak{a}$ is also a $K$-invariant subspace of $\mathfrak{s}$ (but not in general a subalgebra) and $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. Let $N = \exp \mathfrak{n} \subset S$ be the nil-radical of the group $S$. Then $S$ is the topological product of $A = \exp \mathfrak{a}$ and $N$. Finally our group $G$ is the topological product of $K, A$ and $N$. Hence every element $g$ of $G$ has the unique decomposition:

$$g = k_g \cdot a_g \cdot n_g, \text{ where } k_g \in K, a_g \in A \text{ and } n_g \in N.$$ 

We shall use the notations and constructions of [8] in the following but we have to replace there the symbol $G$ with the letter $S$.

Let $h : G \to \mathbb{C}$ be a function. For every $x \in G$, we denote by $\tilde{h}(x)$ the function on $S$ defined by:

$$\tilde{h}(x)(s) = h(xs), \quad s \in S.$$ 

Also for a function $r : S \to \mathbb{C}$ and for $x \in G$, we let $x_r : S \to \mathbb{C}$ be defined by:

$$x_r(s) := r(xsx^{-1}).$$ 

We say that a Borel measurable function $\omega : G \to \mathbb{R}^\ast_+$ is a weight, if $1 \leq \omega(x) = \omega(x^{-1})$ and $\omega(xy) \leq \omega(x)\omega(y)$ for every $x, y \in G$. Then the space

$$L^p(G, \omega) = \{f \in L^p(G) \mid \|f\|_{\omega,p} := \int_G |f(g)|^p \omega(g) \, dg < \infty\},$$ 

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for \(1 \leq p \leq \infty\), is a subspace of \(L^p(G)\) and for \(p = 1\) it is even a Banach algebra for the norm \(\| \cdot \|_{\omega,1}\).

**Proposition 5.1.** Let \(G\) be a locally compact group and let \(S\) be a closed normal subgroup of \(G\). Let \(\omega\) be a continuous weight on \(G\) such that the inverse of its restriction to \(S\) is integrable with respect to the Haar measure on \(S\). Let \(f, g : G \to \mathbb{C}\) be two continuous functions on \(G\), such that \(\omega \cdot g\) is uniformly bounded and such that \(f \in L^1(G, \omega)\). Let \(h := f \ast g \in L^1(G, \omega)\). Then for every \(t \in G\), the function \(\tilde{h}(t)\) is in \(L^1(S)\) and the mapping \(G \times G \to L^1(S); (t, u) \mapsto u \tilde{h}(t)\) is continuous.

**Proof.** Since \(\omega\) is a weight, we have that \(\omega(s) \leq \omega(u)\omega(u^{-1}s)\), i.e. \(\frac{1}{\omega(u^{-1}s)} \leq \frac{\omega(u)}{\omega(s)}\), \(s, u \in G\). Hence, for \(t \in G, s \in S\),

\[
|\tilde{h}(t)(s)| = | \int_G f(u)g(u^{-1}ts) du | = | \int_G f(tu)g(u^{-1}s) du |
\]

\[
\leq \int_G |f(tu)||g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(u^{-1}s)} du \leq \int_G |f(tu)||\omega(u)||g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(s)} du
\]

and so

\[
\|\tilde{h}(t)\|_1 \leq \int_S \int_G |f(tu)||\omega(u)||g(u^{-1}s)|\frac{\omega(u^{-1}s)}{\omega(s)} duds
\]

\[
\leq \int_S \int_G |f(tu)||\omega(u)||g||_{\omega,\infty}\frac{\omega(u^{-1}s)}{\omega(s)} duds = \omega(t)|f||\omega,1||g||_{\omega,\infty}\|\frac{1}{\omega}\|_1 duds
\]

(5.1)

So for every \(t \in G\), the function \(\tilde{h}(t)\) is in \(L^1(S)\). Furthermore, for \(t, t' \in G\), by (5.1),

\[
\|h(t) - h(t')\|_1 \leq \int_S \int_G |f(tu) - f(t'u)||\omega(u)||g||_{\omega,\infty}\frac{\omega(u^{-1}s)}{\omega(s)} duds
\]

\[
\leq \| (\lambda(t^{-1})f - \lambda(t'^{-1})f) \|_{\omega,1}||g||_{\omega,\infty}\|\frac{1}{\omega}\|_1 duds,
\]

where \(\lambda\) denotes left translation by elements of \(G\). Since left translation in \(L^1(G, \omega)\) and conjugation in \(L^1(S)\) are continuous, it follows that the mapping \((t, u) \mapsto u \tilde{h}(t)\) from \(G \times G\) to \(L^1(S)\) is continuous too. \(\square\)

Let as in (1.1) \(\delta\) denote the Carathéodory distance associated to our sub-Laplacian \(L\) on \(G\) and \((h_t)_{t>0}\) its heat kernel. Then the function \(\omega_d(g) := e^{d\delta(x,e)}, g \in G, d \in \mathbb{R}_+\), defines a weight on \(G\). Since we have the Gaussian estimate

\[
|h_t(g)| \leq C_t e^{-C_t \delta(g,e)^2}, \text{ for all } g \in G, t > 0,
\]

it follows that:

(5.2)

\[h_t \in L^1(G, \omega_d) \cap L^\infty(G, \omega_d)\] for every \(t > 0\) and \(d > 0\).
Proposition 5.2. Let $G$ be the semidirect product of a connected compact Lie group $K$ acting on an exponential solvable Lie group $S$. Then there exists a constant $d > 0$, such that $\frac{1}{\omega_d} |S|$ is in $L^1(S)$.

Proof. Let $U$ be a compact symmetric neighborhood of $e$ in $G$ containing $K$. Since $S$ is connected, we know that $G = \cup_{k \in \mathbb{N}} U^k$. This allows us to define $\tau_U = \tau : G \to \mathbb{N}$ by:

$$\tau(x) = \min \{ k \in \mathbb{N} | x \in U^k \}.$$  

Then $\tau$ is sub-additive and defines thus a distance on $G$, which is bounded on compact sets. Since $\tau$ is clearly connected in the sense of [20], it follows that $\tau$ and the Carathéodory distance $\delta$ are equivalent at infinity, i.e.

$$1 + \tau(x) \leq D(1 + \delta(x)) \leq D'(1 + \tau(x)), \quad x \in G.$$  

We choose now a special compact neighborhood of $e$ in the following way. We take our $K$-invariant scalar-product on $s$, the unit-ball $B_a$ in $a$ and the unit-ball $B_n \subset n$. Both balls are $K$-invariant. Let $U_a = \exp B_a$ and $U_n = \exp B_n$. Then $U = KU_a U_n \cap U_n U_a K$ is a compact symmetric neighborhood of $e$. Let us give a rough estimate of the radii of the ”balls” $U^l$, $l \in \mathbb{N}$. For simplicity of notation, we shall denote all the positive constants which will appear in the following arguments (and which will be assumed to be integers, if necessary) by $C$.

Let $k_i a_i n_i \in KU_a U_n$, $i = 1, \cdots, l$ and $g := \Pi_{i=1}^l k_i a_i n_i$. We have

$$g = \Pi_{i=1}^l k_i a_i n_i = (\Pi_{i=1}^l k_i a_i) ((k_2 a_2 \cdots k_l a_l)^{-1} n_1(k_2 a_2 \cdots k_l a_l) \cdots (k_l a_l)n_{l-1}(k_l a_l)n_l).$$  

Since $U_a$ is $K$-invariant, it follows that:

$$g = \Pi_{i=1}^l k_i a_i s_i = k' a' \Pi_{i=1}^l (a'' k'') n_i (a' k')^{-1},$$

where $k', k_1'' \cdots k_l'' \in K$, $a' \in U^l_a$, $a''_1 \in U^{-1}_a$, ..., $a''_{l-1} \in U_a$. Hence there exists $X_1, \cdots, X_l \in B_a$, such that

$$a' = \exp X_1 \cdots \exp X_l = \exp (X_1 + \cdots + X_l) \exp q_l(X_1, \cdots, X_l)$$

for some element $q_l(X_1, \cdots, X_l) \in n \cap s_0$. Since $s_0$ is a nilpotent Lie algebra we have that $\|q_l(X_1, \cdots, X_l)\| \leq C(1 + l)^l$, $l \in \mathbb{N}$. Hence

$$a' \in \exp (lB_a) \exp [C(1 + l)^l B_n] \subset \exp (lB_a) U_n^{C(1 + l)^l}.$$  

Furthermore, because $U_a$ is compact, $\sup_{a \in U_a} \| \text{Ad}(a) \|_\infty \leq C < \infty$ and so $(a'' k'') n_i (a'' k'')^{-1} \in \exp C(l-i) B_n \subset U_n^{C(l-i)} (i = 1, \cdots, l)$. Finally for some integer constants $C$,

$$g = k' a' \Pi_{i=1}^l (a'' k'') n_i (a'' k'')^{-1} \in K \exp l U_a U_n^{C(1 + l)^l} \left( \Pi_{i=1}^{l-1} U_n^{C(l-i)} \right) U_n \subset K \exp l U_a U_n^{C(1 + l)^l + \sum_{i=1}^{l-1} C(l-i + 1)}$$

$$\subset K \exp l U_a U_n^{C l}$$

(5.3) 

$$\subset K \exp l U_a \exp C^d B_n$$
Hence for any $g \in G$, for $\tau_U(g) = l$, we have that $g \in (KU_a U_a)'$ and so, denoting by $\log : S \to s$ the inverse map of $\exp : s \to S$, $g = k g_a n_g$, with $k_g \in K$, $a_g \in \exp a$, $\|\log (a_g)\| \leq l = \tau_U(g)$ and $n_g \in N$ with $\|\log (n_g)\| \leq C'$, i.e. $\log (1 + \|\log (n_g)\|) \leq Cl = C \tau_G (g)$. Whence for our weight $\omega_d$, $(d \in \mathbb{R}^+)$, we have that:

$$\omega_d (g) = e^{\delta (g)} \geq C e^{d \tau_U (g)} \geq C e^{d C (\|\log (a_g)\| + \log (1 + \|\log (n_g)\|))} = C e^{d C \|\log (a_g)\| \cdot (1 + \|\log (n_g)\|)^d C}.$$  

Therefore, for $d$ big enough,

$$\int_S \frac{1}{\omega_d (s)} ds = \int_S \frac{1}{\omega_d (\exp \times \exp Y)} dY dX \leq C \int_S \frac{1}{e^{-d C \|X\|} (1 + \|Y\|)^d C} dY dX < \infty.$$

\[\square\]

**Proposition 5.3.** Let $T$ be a compact topological space and let $k : T \times T \to \mathcal{K}(\mathcal{H})$ be a continuous mapping into the space of compact operators on a Hilbert space $\mathcal{H}$. Let $\mu$ be a Borel probability measure on $T$. Then the linear mapping

$K : L^2 (T, \mathcal{H}) \to L^2 (T, \mathcal{H})$,

$K \xi (t) := \int_T k(t, u) \xi (u) du, \ t \in T, \ \xi \in L^2 (T, \mathcal{H})$,

is compact too.

**Proof.** We show that $K$ is the norm-limit of a sequence of operators of finite rank. Let $\varepsilon > 0$. Since $T$ is compact and $k$ is continuous, there exists a finite partition of unity of $T \times T$ consisting of continuous non-negative functions $(\varphi_i)_{i = 1}^N$, such that $\|k(t, t') - k(u, u')\|_{op} < \frac{\varepsilon}{2}$ for every $(t, t')$, $(u, u')$ contained in the support $\varphi_i$. Choose for $i = 1, \ldots, N$ an element $(t_i, t_i')$ in supp $\varphi_i$. Since $k(t_i, t_i')$ is a compact operator, we can find a bounded endomorphism $F_i$ of $\mathcal{H}$ of finite rank, such that $\|k(t_i, t_i') - F_i\|_{op} < \frac{\varepsilon}{2}$, hence $\|k(t, t') - F_i\|_{op} < \varepsilon$ for every $(t, t') \in$ supp $\varphi_i$, $i = 1, \ldots, N$. The finite rank operator $F_i$ has the expression $F_i = \sum_{k=1}^{N_i} P_{\eta, \eta'}^{\varphi_i \otimes \eta, \eta}$, where for $\eta, \eta' \in \mathcal{H}$, $P_{\eta, \eta'}$ denotes the rank one operator $P_{\eta, \eta'} (\eta'') = \langle \eta'' \rangle \eta$, $\eta'' \in \mathcal{H}$. We approximate the continuous functions $\varphi_i$ uniformly on $T \times T$ up to an error of at most $\frac{\varepsilon}{2} R$ by tensors $\psi_i = \sum_{j=1}^{M_i} \varphi_{i, j} \otimes \varphi_{i, j}' \in C (T, \mathbb{R}_+) \otimes C (T, \mathbb{R}_+)$ for some $R > 0$ to be determined later on. Let $K_\varepsilon$ be the finite rank operator

$$K_\varepsilon = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \sum_{k=1}^{N_i} \psi_{i, j} \otimes \psi_{i, j}' \otimes \varphi_{i, j} \otimes \eta_{i, j} \otimes \eta_{i, j}' \otimes \eta_{i, j}.$$  

In order to estimate the difference $K - K_\varepsilon$, we let first $K_{\varepsilon, 1}$ be the kernel operator with kernel $k_{\varepsilon, 1} (s, t) = \sum_{i=1}^{N} \varphi_i (s, t) F_i$. Then for $\xi \in L^2 (T, \mathcal{H})$

$$\|K_{\varepsilon, 1} \xi - K \xi\|_2^2 = \int_T \| \sum_{i=1}^{N} \int_T \varphi_i (s, t) (k(s, t) - F_i) \xi (t) dt \|_2^2 dB.$$  

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\[ \leq \int_T \left( \sum_{i=1}^N \int_T \varphi_i(s, t) \| \xi(t) \| dt \right)^2 ds = \int_T \left( \int_T \| \xi(t) \| dt \right)^2 ds \leq \varepsilon^2 \| \xi \|^2; \]

hence \( \| K - K_{\varepsilon, 1} \|_{op} \leq \varepsilon \). Moreover

\[ \| (K_{\varepsilon, 1} - K_{\varepsilon}) \xi \|^2 = \int_T \left( \int_T \sum_{i=1}^N (\varphi_i(s, t) - \sum_{j=1}^{M_i} \varphi_{i,j}(s) \varphi'_{i,k}(t)) F_i \xi(t) dt \right)^2 ds \]

\[ \leq \int_T \left( \int_T \sum_{i=1}^N \frac{\varepsilon}{R} \| F_i \|_{op} \| \xi(t) \| dt \right)^2 ds \leq \frac{\varepsilon^2}{R^2} \left( \sum_{i=1}^N \| F_i \|_{op} \right)^2 \| \xi \|^2. \]

So, if we let \( R = \frac{1}{1 + \sum_{i=1}^N \| F_i \|_{op}} \), then

\[ \| K - K_{\varepsilon} \|_{op} \leq \| K - K_{\varepsilon, 1} \|_{op} + \| K_{\varepsilon, 1} - K_{\varepsilon} \|_{op} \leq 2\varepsilon. \]

Let now \( \pi \) be an isometric representation of the group \( S \) on a Banach space \( X \) and denote by \( \rho^\pi := \text{ind}_G^S \pi \) the corresponding induced representation of \( G \) on \( L^p(G, X; \pi) \). Here we follow notation of Section 3.1.

Let \( h \) be in \( L^1(G) \) and assume furthermore that \( \tilde{h}(g) \in L^1(S) \) for all \( g \in G \) and that the mapping \( \tilde{h} : G \to L^1(S) \) is continuous. Then the operator \( \rho^\pi(h) \) is a kernel operator, whose kernel \( k(t, u) \), \( t, u \in G \), is given by:

\begin{equation}
(5.4) \quad k(t, u) = \Delta_G(u^{-1}) \pi(u \tilde{h}(tu^{-1}))
\end{equation}

(in the notations of Proposition 5.1). Indeed, for \( \xi \in L^p(G, X; \pi) \), \( t \in G \),

\[ [\rho^\pi(h)\xi](t) = \int_G h(g)\xi(g^{-1}t) \, dg = \int_G \Delta_G(g^{-1})h(tg^{-1})\xi(g) \, dg \]

\[ = \int_{G/S} \int_S \Delta_G(s^{-1}g^{-1})h(ts^{-1}g^{-1})\xi(gs) \, ds \, dg \]

\[ = \int_{G/S} \int_S \Delta(g^{-1})\Delta_S(s^{-1})h(ts^{-1}g^{-1})\xi(gs) \, ds \, dg \]

\[ = \int_{G/S} \int_S \Delta(g^{-1})h(tg^{-1}(gsg^{-1}))\pi(s)\xi(g) \, ds \, dg \]

\[ = \int_{G/S} \Delta(g^{-1})\pi(q \tilde{h}(tg^{-1}))\xi(g) \, dg. \]

Moreover the kernel \( k \) satisfies the following covariance property under \( S \):

\begin{equation}
(5.5) \quad k(ts, us') = \pi(s^{-1})k(t, u)\pi(s'), \quad t, u \in G, s, s' \in S.
\end{equation}

**Proposition 5.4.** Let \( G \) be the semidirect product of a connected compact Lie group \( K \) acting on an exponential solvable Lie group \( S \). Let \( (\pi, \mathcal{H}) \) be an irreducible unitary representation of the normal closed subgroup \( S \) of \( G \) whose Kirillov-orbit \( \Omega_\pi = \Omega \subset s^* \) is closed. Let \( \rho = \text{ind}_G^S \pi \). Then the operator \( \rho(h_t) \) is compact for every \( t > 0 \).
Furthermore we have that $h_t = h_{t/2} * h_{t/2}$. Hence by the Propositions 5.2 and 5.1 the mapping $G \times G \to L^1(S)$, $(s, u) \mapsto h_t(u^{-1})$, is continuous and so the operator valued kernel function $k(s, u) := \Delta_{G}(u^{-1})\pi^*(h_t(su^{-1}))$ is continuous too. It follows from the preceding discussion that the $k$ is just the integral kernel of the operator $\rho(h_t)$. The fact that the Kirillov orbit of $\pi \in \hat{S}$ is closed in $\mathfrak{s}^*$ implies that for every $\varphi \in L^1(S)$, the operator $\pi(\varphi) = \int_S f(s)\pi(s)ds$ is compact (see [13] and [8]). Hence $k(s, u)$ is compact for every $(s, u) \in G \times G$ and in particular for every $(s, u) \in K \times K$. We apply Proposition 5.3 to the restriction of $k$ to $K \times K$. The related kernel operator on $L^2(K, \mathcal{H})$ is then compact. Now, since $\pi$ is unitary, the restriction map to $K$ is an isometric isomorphism from $L^2(G, \mathcal{H}; \pi)$ onto $L^2(K, \mathcal{H})$, and we thus see that $\rho(h_t)$ is compact too. 

5.2 Proof of Theorem 1.3

We now turn to the proof of Theorem 1.3, which follows closely the notation and argumentation in [8]. In the sequel, we always make the following

Assumption. $\ell \in \mathfrak{s}^*$ satisfies Boidol’s condition (B), and $\Omega(\ell)|_n$ is closed.

Moreover, we assume that $p \in [1, \infty[, \ p \neq 2$, is fixed.

Since $\ell$ satisfies (B), the stabilizer $\mathfrak{s}(\ell)$ is not contained in $\mathfrak{n}$. Let $\nu$ be the real character of $\mathfrak{s}$, which has been defined in [8, Section 5], trivial on $\mathfrak{n}$ and different from 0 on $\mathfrak{s}(\ell)$. We denote by $\pi_\ell = \text{ind}_P^S \chi_\ell$ the irreducible unitary representation of $S$ associated to $\ell$ by the Kirillov map; here $P = P(\ell)$ denotes a suitable polarizing subgroup for $\ell$, and $\chi_\ell$ the character $\chi_\ell(p) := e^{i\ell(\log p)}$ of $P$.

For any complex number $z$ in the strip

$$
\Sigma := \{ \zeta \in \mathbb{C} : |\text{Im} \ \zeta| < 1/2 \},
$$

let $\Delta_z$ be the complex character of $S$ given by

$$
\Delta_z(\exp X) := e^{-iz\nu(X)}, \quad X \in \mathfrak{s},
$$

and $\chi_z$ the unitary character

$$
\chi_z(\exp X) := e^{-i\text{Re} \ z\nu(X)}, \quad X \in \mathfrak{s}.
$$

If we define $p(z) \in ]1, \infty[$ by the equation

$$
(5.1) \quad \text{Im} \ z = 1/2 - 1/p(z),
$$

it is shown in [8] that the representation $\pi^{z}_\ell$, given by

$$
(5.2) \quad \pi^{z}_\ell(x) := \Delta_z(x)\pi_\ell(x) = \chi_z(x)\pi^{p(z)}_\ell(x), \quad x \in G,
$$

is an isometric representation on the mixed $L^p$-space $L^{p(z)}(S/P, \ell)$. Here, $\pi^{p(z)}_\ell$ denotes the $p(z)$-induced representation of $S$ on $L^{p(z)}(S/P, \ell)$ defined in [8], where $p(z)$ is a
multi-index of the form \((p(z), \ldots, p(z), 2, \ldots, 2)\).

Observe that for \(\tau \in \mathbb{R}\), we have \(p(\tau) = 2\), and \(\pi^\tau_\ell = \chi_\tau \otimes \pi_\ell\) is a unitary representation on \(L^2(S/P, \ell)\). Moreover,

\begin{equation}
\pi^\tau_\ell \simeq \pi_{\ell - \tau \nu},
\end{equation}

since the mapping \(f \mapsto \chi_\tau f\) intertwines the representations \(\chi_\tau \otimes \pi_\ell\) and \(\pi_{\ell - \tau \nu}\).

We take now for \(z \in \Sigma\) the \(p(z)\)-induced representation \(\rho^z := \text{ind}_{G}^{p(z), S} \pi^z_\ell\) of \(G\) which acts on the space \(L^p(G/P, \ell) := L^p(S/P, \ell; \pi^z_\ell)\).

Let us shortly write

\(L^p := L^p(G/P, \ell), \quad 1 \leq p < \infty\),

for the space of \(\rho^z_\ell\).

We can extend the character \(\Delta_z, z \in \Sigma\), of \(S\) to a function on \(G\) by letting

\(\Delta_z(\text{kan}) := \Delta_z(\text{an}) = e^{-iz\nu(\text{Log}(a))}, \quad k \in K, a \in A, n \in N.\)

Since \(\nu\) is trivial on \(n\) and since \(kak^{-1} \in aN\) for all \(k \in K, a \in A,\) we have that

\(\Delta_z(kank') = \Delta_z(\text{an}), \quad k, k' \in K, a \in A, n \in N,\)

and in particular \(\Delta_z\) is a character of \(G\).

Define the operator \(T(z), z \in \Sigma,\) by:

\(T(z) := \rho^z_\ell(h_1).\)

Then by the relations (5.4) and (5.2), for \(z \in \Sigma\) and \(\xi \in L^p\), (since \(\Delta_z\) is \(K\)-invariant)

\[
T(z)\xi(k) = \int_K \pi^z_\ell(k' h_1(\bar{k}k'^{-1})k(k')\xi(k')dk'.
\]

\[
= \int_K \pi_\ell((\Delta_z|_S)k' h_1(\bar{k}k'^{-1})k(k')\xi(k')dk'.
\]

\[
= \int_K \pi_\ell(k' (\Delta_z h_1)(\bar{k}k'^{-1})k(k')\xi(k')dk'.
\]

\[
= |\rho_\ell(\Delta_z h_1)|(\xi(k)).
\]

Hence

\begin{equation}
T(z) = \rho^z_\ell(h_1) = \rho_\ell(\Delta_z h_1), \quad z \in \Sigma.
\end{equation}

Since by (5.2), for every continuous character \(\chi\) of \(G\) which is trivial on \(N\) the function \(\chi h_1\) is in \(L^1(G)\), it follows from [8, Corollary 5.2 and Proposition 3.1] that the operator \(T(z)\) leaves \(L^q\) invariant for every \(1 \leq q < \infty\), and is bounded on all these spaces. Moreover, by Proposition 5.4, \(T(\tau)\) is compact for \(\tau \in \mathbb{R}\). From here on we can proceed...
exactly as in the proof of [8, Theorem 1], provided that we can prove a \textquotedblleft Riemann-Lebesgue\textquotedblright
type lemma like [8, Theorem 2.2] in our present setting, since \( G = K \ltimes S \) is amenable.

We must show that \( T(\tau) \) tends to 0 in the operator norm if \( \tau \) tends to \( \infty \) in \( \mathbb{R} \). The condition we have imposed on the coadjoint orbit \( \Omega \) of \( \ell \), namely that the restriction of \( \Omega \) to \( n \) is closed, tells us that \( \lim_{\tau \to \infty} \Omega + \tau \nu = \infty \) in the orbit space, which means that \( \lim_{\tau \to \infty} \| \pi_{\ell+\tau \nu}(f) \|_{op} = 0 \) for every \( f \in L^1(S) \). Now, by (5.4) the operator \( T(\tau) = \rho_{\ell}^\tau(h_1) \) is a kernel operator whose kernel \( K_\tau \) has values in the bounded operators on \( \mathcal{H}_\ell \). The kernel \( K_\tau \) is given by:

\[
K_\tau(k, k') = \int_S \Delta_\tau(s) h_1(k^{-1}sk'^{-1}) \pi_\ell(s) ds = \pi_\ell^\tau(h_1(k, k')) ,
\]

where \( h_1(k, k') \) is the function on \( S \) defined by \( h_1(k, k')(s) \equiv h_1(ksk'^{-1}) \). Hence

\[
\lim_{\tau \to \infty} \| \pi_\ell^\tau(h_1(k, k')) \|_{op} = 0
\]

for every \( k, k' \in K \). Moreover for \( k, k' \in K \),

\[
\| \pi_\ell^\tau(h_1(k, k')) \|_{op} \leq \| h_1(k, k') \|_1 \leq \sup_{k'' \in K} \| \tilde{h}_1(k'') \|_1.
\]

We know from Proposition 5.1 that, for every \( k'' \in K \),

\[
\| h_1(k'') \|_1 \leq \| \omega_p |k| \|_\infty \| h_{1/2} \|_{\omega_{d,1}} \| h_{1/2} \|_{\omega_{d,\infty}} \| (\frac{1}{\omega_d})_S \|_1 ,
\]

which is finite by Proposition 5.2 and relation (5.2) (if \( d \) is big enough). Hence, by Lebesgue\textquoteright s dominated convergence theorem, we see that:

\[
\lim_{\tau \to \infty} \int_K \int_K \| \pi_\ell^\tau(h_1(k, k')) \|_{op}^2 dk dk' = 0.
\]

This shows that:

\[
\lim_{\tau \to \infty} \| \rho_\ell^\tau(h_1) \|_{op} = 0.
\]

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