Third quantization for scalar and spinor wave functions of the Universe in an extended minisuperspace

Nahomi Kan

National Institute of Technology, Gifu College, Motosu-shi, Gifu 501-0495, Japan

Takuma Aoyama, Taiga Hasegawa, and Kiyoshi Shiraishi

Graduate School of Sciences and Technology for Innovation, Yamaguchi University, Yamaguchi-shi, Yamaguchi 753–8512, Japan

(Dated: July 5, 2022)

We consider the third quantization in quantum cosmology of a minisuperspace extended by the Eisenhart–Duval lift. We study the third quantization based on both Klein–Gordon type and Dirac-type equations in the extended minisuperspace. Spontaneous creation of “universes” is investigated upon the quantization of a simple model. We find that the quantization of the Dirac-type wave function reveals that the number density of universes is expressed by the Fermi–Dirac distribution. We also calculate the entanglement entropy of the multi-universe system.

PACS numbers: 03.65.Ca, 03.65.Yz, 03.67.Mn, 03.70.+k, 04.60.-m, 04.60.Kz, 04.62.+v, 11.30.-j, 98.80.Cq, 98.80.Qc, 98.80.Jk.
I. INTRODUCTION

There are many problems with the Wheeler–DeWitt (WDW) equation in quantum cosmology [1–5]. The WDW equation is a Klein–Gordon-type second-order differential equation even in minisuperspace. The Klein–Gordon system has no positive-definite and conserved quantity, as we learned in the first class of the field theory. To obtain a conserved current and a positive-definite probability density, a square root formulation of the WDW equation was proposed originally by authors of Refs. [6, 7] and further studied by authors of Refs. [8–13].\(^1\) Incidentally, some kind of square root formulation can be found in the factorization that appears in supersymmetric quantum mechanics. The application of such elegant and mathematical structures to quantum cosmology is also being enthusiastically studied by authors of Ref. [18].

In quantum field theory, the wave equation is interpreted not regarding the wave function as a state vector, but regarding it as a second quantized field. The quantized field is expanded with creation and annihilation operators. The vacuum state is the one that becomes zero by applying the annihilation operator, and the particles are created by applying the creation operators. The conserved current is reinterpreted as the one associated with the charge, which is defined as positive for particles and negative for antiparticles.

Therefore, we can interpret the wave function of the Universe as a quantum field in minisuperspace, and the field of the universe is accompanied with the state of many universes. Such an approach has been called the third quantization [10, 19–32].

In the previous paper [33], the present authors considered a cosmology of a homogeneous and isotropic space that contains a gravitating minimally coupled scalar field, and extended the corresponding minisuperspace description with an additional degree of freedom by a method of the Eisenhart–Duval lift [34–43]. The Eisenhart–Duval lift is one of the classical methods in a Hamiltonian dynamical system, which allows for a geometric description of the system even in the presence of the potential term. We investigated the WDW equation in quantum cosmology obtained from the Hamiltonian constraint of the system, by requiring covariance in the extended minisuperspace. Using the covariance in the extended minisuperspace, we also constructed an associated Dirac equation for spinor wave function of the Universe. The introduction of the Dirac-type equation is originally motivated for obtaining

\(^1\) The Dirac-like structures naturally appear also in quantum cosmology of supergravities [14–17].
a positive-definite probability density [8–13]. We found fundamental solutions to the WDW equation \( (\nabla^2 \Phi = 0) \) and Dirac equations \( (\mathcal{D} \Psi = 0) \) in the extended minisuperspace of some simple models, of which scalar curvatures are zero. The analysis solves some factor-ordering problems [44–47] in quantum cosmology.

Since the equations obtained in the previous paper are conventional forms in use of the Laplacian operator and the Dirac operator which appear in general field theories, it is natural to bring them to the third quantization scheme [10, 19–30]. So far, the third quantization of the Dirac-type wave function of the Universe has not been studied.

In the present paper, we consider the third quantization of quantum cosmology in the extended minisuperspace. We consider the simplest case of a homogeneous and isotropic Friedmann-Lemaître–Robertson–Walker (FLRW) universe containing a single spatially constant scalar field. In Section II, we briefly review the extension of the minisuperspace by the Eisenhart–Duval lift of the WDW equation, studied in Ref. [33]. We show introduction of a fictitious additional degrees of freedom in the simplest model. The Klein–Gordon-type and the Dirac-type equations in the extended minisuperspace are constructed. In Section III, we study the third quantization in terms of two types of equations. The production of bosonic and fermionic universes from “nothing” are discussed and the distribution is calculated. We attempt to consider the extra degree of freedom as a “real coordinate” of the extended minisuperspace in Section IV. The entanglement entropy of universes is calculated in Section V. The last section will be devoted to discussions and future prospects.

II. A LIGHTNING COURSE OF EISENHART–DUVAL LIFT FOR THE WDW EQUATION

In this section, we review an application of Eisenhart–Duval lift to minisuperspace quantum cosmology. For details, please consult our previous paper [33].

As one of the simplest examples, we specify the action of the gravitating scalar field \( \phi \) with a positive cosmological constant \( V \):

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{1}{12} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V \right],
\]

where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( R \) denotes the scalar curvature constructed from \( g_{\mu\nu} \) \((\mu, \nu = 0, 1, 2, 3)\), and \( g^{\mu\nu} \) means the inverse of \( g_{\mu\nu} \). Here we use the units
\( h = c = 4\pi G/3 = 1 \), where \( G \) is Newton’s constant.

As the metric, we assume the FLRW metric with a flat space,

\[
g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)dx^2,
\]

where \( a \) represents the scale factor, and assume that the scalar field depends only on time \( t \).

The WDW equation

\[
\hat{H}\Phi = 0
\]

is the quantum Hamiltonian constraint, where \( \Phi \) denotes the wave function of the Universe. In the present system, it is known that the WDW equation in the minisuperspace presented by \( x \) and \( y \) takes the form

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2Ve^{2nx} \right) \Phi(x, y) = 0,
\]

where \( n = 3 \), \( x = \ln a \) and \( y = \phi \). In this equation, the spatial volume is interpreted to be appropriately normalized. Here we use the symbol \( n \) instead of the fixed number 3, because the WDW equations for other some simple models are known to be written in the similar form.\(^2\)

The essential idea of Eisenhart–Duval lift is to introduce the metric of the extended space with an auxiliary dimension \( z \), that is, in the present case,

\[
G_{MN}dX^M dX^N = 2Ve^{2nx}(-dx^2 + dy^2) + dz^2,
\]

where \( X^M = (x, y, z) \), and replace the WDW equation by the Laplace equation in the extended minisuperspace:

\[
\frac{1}{\sqrt{-G}}\partial_M(\sqrt{-G}G^{MN}\partial_N\Phi) = \left[ \frac{1}{2Ve^{2nx}} \left( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\partial^2}{\partial z^2} \right] \Phi = 0,
\]

Here, \( G^{MN} \) is the inverse of \( G_{MN} \), \( G = -(2V)^2e^{4nx} \) is the determinant of \( G_{MN} \), the derivatives are expressed as \( \partial_M \equiv \frac{\partial}{\partial X^M} \). If we impose an additional constraint

\[
\frac{\partial^2\Phi}{\partial z^2} = -\Phi,
\]

we can reproduce the conventional WDW equation (2.4).\(^3\)

---

\(^2\) For example, there are the models treated in Refs.\([4, 48, 49]\) (\( n = 2 \)), Ref. \([50]\) (\( n = 3 \)), and Ref. \([51]\) (\( n \) depends on values of parameters).

\(^3\) Note that the constraint (2.7) can be satisfied if we simply take \( \Phi(x, y, z) = \Phi(x, y)e^{iz} \) concretely.
Incidentally, the scalar curvature $\mathcal{R}$ constructed from $G_{MN}$, which is defined by

$$\mathcal{R} = G^{MN} (\partial_L \Gamma^L_{MN} - \partial_M \Gamma^L_{NL} + \Gamma^L_{MN} \Gamma^P_{LP} - \Gamma^L_{MP} \Gamma^P_{NL}), \quad (2.8)$$

with the Christoffel symbol $\Gamma^L_{MN}$

$$\Gamma^L_{MN} = \frac{1}{2} G^{LP} (\partial_M G_{PN} + \partial_N G_{PM} - \partial_P G_{MN}), \quad (2.9)$$

vanishes in the present case.

The two fundamental solutions of the WDW equation are known to be

$$\phi_\nu^{(1)} (x, y) \propto J_{\nu}(\sqrt{2}V e^{nx}/n)e^{iy}, \quad (2.10)$$

$$\phi_\nu^{(2)} (x, y) \propto J_{i\nu/n}(\sqrt{2}V e^{nx}/n)e^{iy}, \quad (2.11)$$

where the functions $J_\nu(Z)$ is the Bessel function of order $\nu$ [52].

The idea of square-rooting the WDW equation can be found in Refs. [8–13] and others.

The Dirac equation in the extended minisuperspace is fixed in a unique form. It is notable that the Dirac equation (without mass term) has conformal covariance.

The Dirac-like equation for a spinor wave function $\Psi$ in the extended minisuperspace can be written down as

$$\hat{D} \Psi \equiv \hat{\gamma}^M D_M \Psi \equiv \gamma^A e^M_A D_M \Psi = 0. \quad (2.12)$$

Here, the constant gamma matrices in the flat spacetime $\gamma^A (A = 1, 2, 3)$ are $\gamma^1 = \sigma^1$, $\gamma^2 = i\sigma^2$, and $\gamma^3 = i\sigma^3$, where $\sigma^1$, $\sigma^2$, $\sigma^3$ are the Pauli matrices. Note that $\{\gamma^A, \gamma^B\} = -2\eta^{AB}$, where $\eta^{AB} = \eta_{AB} = \text{diag.}(-1, 1, 1)$. The dreibein $e^A_M = \text{diag.}((2V)^{1/2}e^{nx}, (2V)^{1/2}e^{nx}, 1)$ is defined through $\eta_{AB} e^A_M e^B_N = G_{MN}$, and $e^M_A = \text{diag.}((2V)^{-1/2}e^{-nx}, (2V)^{-1/2}e^{-nx}, 1)$ is its inverse matrix. Subsequently, we find that $\{\hat{\gamma}^M, \hat{\gamma}^N\} = -2G^{MN}$. The covariant derivative $D_M$ for the spin connection $\omega_{MAB}$ is defined as $D_M \equiv \partial_M + \frac{1}{4} \omega_{MAB} \Sigma^{AB}$, where $\Sigma^{AB} \equiv -\frac{1}{2} [\gamma^A, \gamma^B]$. The spin connection $\omega_{MAB}$ is given by

$$\omega_{MAB} = \frac{1}{2} e^A_N (\partial_M e_{NB} - \Gamma^L_{MN} e_{LB}) - (A \leftrightarrow B). \quad (2.13)$$

In the present model, one can find $\omega_{y12} = -\omega_{y21} = -n$.

In the extended minisuperspace presently considered, we find that the Dirac equation (2.12) is equivalent to

$$\left[\sigma^1 \left( \frac{\partial}{\partial x} + \frac{n}{2} \right) + i\sigma^2 \frac{\partial}{\partial y} + i\sigma^3 \sqrt{2V e^{nx}} \frac{\partial}{\partial z} \right] \Psi = 0. \quad (2.14)$$
In order to reduce the equation to that of physical variables $x$ and $y$, we choose the additional constraint on $\Psi$:

$$\frac{1}{i} \frac{\partial}{\partial z} \Psi = \Psi. \tag{2.15}$$

Now, the Dirac equation reads in the matrix form,

$$\begin{pmatrix} -\sqrt{2} V e^{nx} & \frac{\partial}{\partial x} + \frac{n}{2} + \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + \frac{n}{2} - \frac{\partial}{\partial y} & \sqrt{2} V e^{nx} \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.16}$$

where $\Psi = (\Psi_+)$.

The fundamental solutions of the equation are found to be

$$\psi^{(1)}_{\pm, \nu} \propto \pm J^{-i\nu n_{\pm} + \frac{1}{2}}(\sqrt{2} Ve^{nx}/n)e^{i\nu y}, \quad \psi^{(2)}_{\pm, \nu} \propto J^{-i\nu n_{\pm} - \frac{1}{2}}(\sqrt{2} Ve^{nx}/n)e^{i\nu y}. \tag{2.17}$$

### III. THIRD QUANTIZATION AND SPONTANEOUS CREATION OF MULTIPLE UNIVERSES

The third quantization is attained by taking the wave function for an operator called as a “field” acting on the state vectors of a system of multiple universes. First we review the result of Hosoya and Morikawa [26] for the scalar wave function of the conventional WDW equation in the present model.

We start with the wave function $\Phi(x, y)$ that satisfies the WDW equation (2.4). The WDW equation is a hyperbolic second-order differential equation, and here we regard $x$ as the intrinsic time coordinate in minisuperspace. Note that when $x$ varies from $-\infty$ to $\infty$, the scale factor $a = e^x$ grows from 0 to $+\infty$.

We expand the real scalar function $\Phi(x, y)$ as

$$\Phi(x, y) = \int_{-\infty}^{\infty} d\nu \left[ a_{\nu} \phi_{\nu}(x, y) + \phi^*_{\nu}(x, y) \right], \tag{3.1}$$

where $\phi_{\nu}$ and $\phi^*_{\nu}$ are the normalized fundamental solutions of the WDW equation (2.4) and satisfy the orthonormal condition

$$(\phi_{\nu}, \phi_{\nu'}) = \delta(\nu - \nu'), \quad (\phi^*_{\nu}, \phi_{\nu'}) = 0, \tag{3.2}$$

where the inner product is defined as

$$(f, g) \equiv \int dy \sqrt{-G_{yy}} (f^* \partial_x g - \partial_x f^* g) = i \int dy (f^* \partial_x g - \partial_x f^* g). \tag{3.3}$$
Note that \((g,f) = (f,g)^* = -(f^*,g^*)\). Since this prescription is analogous to the one in quantum field theory in curved spacetime [53], \(a_\nu\) and \(a_\nu^\dagger\) are corresponding to annihilation and creation operators, respectively. We assume the following bosonic commutation relations

\[
[a_\nu, a_\nu^\dagger] = \delta(\nu - \nu'), \quad [a_\nu, a_\nu] = [a_\nu^\dagger, a_\nu^\dagger] = 0 \iff \quad [\Phi(x, y), \partial_x \Phi(x, y')] = i\delta(y - y'), \quad (3.4)
\]

which realize the canonical commutation relation of a scalar field theory. Hence the vacuum state \(|0\rangle\) is defined by

\[
\langle a_\nu |0\rangle = 0 \quad \text{(for all } \nu), \quad (3.5)
\]

and normalized as \(\langle 0|0 \rangle = 0\). The Fock space of this “bosonic” multiple universes is spanned by the states such as \(a_\nu^\dagger a_\nu^\dagger \cdots |0\rangle\), etc.

According to quantum field theory in curved space [53], the vacuum state is not unique but depends on the different region in spacetime. In the present case, we define in and out regions in minisuperspace for \(x \to -\infty\) (when the scale factor goes to zero) and \(x \to \infty\) (when the scale factor grows infinitely), respectively. When \(x \to -\infty\), \(J_{-i|\nu|/n}(\sqrt{2}V e^{nx}/n)\) is proportional to \(e^{-i|\nu| x} [52]\). Thus, the positive-frequency, in-mode function can be written as

\[
\phi_{\nu}^{\text{in}}(x, y) = \frac{1}{2} \frac{1}{\sqrt{n \sinh \frac{|\nu|}{n}}} J_{-i|\nu|/n}(\sqrt{2}V e^{nx}/n)e^{i\nu y}. \quad (3.6)
\]

On the other hand, when \(x \to +\infty\), the function \(H_{-i|\nu|/n}(\sqrt{2}V e^{nx}/n)\) is proportional to \(\exp[-i\sqrt{2}V e^{nx}/n]\), where \(H_{-i|\nu|/n}(Z)\) is the Hankel function of the second kind defined by [52]

\[
H_{-i|\nu|/n}(Z) = \frac{-i}{\sin \pi \nu}\left[e^{i\pi \nu}J_{\nu}(z) - J_{-\nu}(z)\right]. \quad (3.7)
\]

Thus, the positive-frequency, out-mode function can be written as

\[
\phi_{\nu}^{\text{out}}(x, y) = \frac{e^{-\pi|\nu|/2n}}{2\sqrt{2n}} H_{-i|\nu|/n}(\sqrt{2}V e^{nx}/n)e^{i\nu y}. \quad (3.8)
\]

Now, the field operator \(\Phi\) is expanded by two ways:

\[
\Phi(x, y) = \int_{-\infty}^{\infty} d\nu [a_\nu^{\text{in}} \phi_{\nu}^{\text{in}}(x, y) + a_\nu^{\text{in}} \phi_{\nu}^{\text{in}*}(x, y)]
\]

\[
= \int_{-\infty}^{\infty} d\nu [a_\nu^{\text{out}} \phi_{\nu}^{\text{out}}(x, y) + a_\nu^{\text{out}} \phi_{\nu}^{\text{out}*}(x, y)]. \quad (3.9)
\]

Then, we define the two vacuum states, the in-vacuum \(|0, in\rangle = \prod_\nu |0, in\rangle_\nu\) and the out-vacuum \(|0, out\rangle = \prod_\nu |0, out\rangle_\nu\). They satisfy

\[
\quad a_\nu^{\text{in}} |0, in\rangle = 0, \quad a_\nu^{\text{out}} |0, out\rangle = 0 \quad \text{(for all } \nu). \quad (3.10)
\]
Thus, the in-vacuum state represents no universe, or “nothing” in the region \( x \to -\infty \).

From Eq. (3.9), we find

\[
a^\text{out}_\nu = (\phi^\text{out}, \Phi) = \frac{1}{\sqrt{1 - e^{-2\pi|\nu|/n}}} a^\text{in}_\nu + \frac{1}{\sqrt{e^{2\pi|\nu|/n} - 1}} a^\text{in}^\dagger_\nu. \tag{3.11}
\]

From this, we can calculate the average number of “bosonic” universes \( N_{B\nu} \) for the specific value of \( \nu \) in the out region \( x \to \infty \), created from nothing

\[
N_{B\nu} \equiv \langle 0, in | a^\text{out}^\dagger_\nu a^\text{out}_\nu | 0, in \rangle = \frac{1}{e^{2\pi|\nu|/n} - 1}, \tag{3.12}
\]

which is the Planck distribution. The above results have been shown by the paper of Hosoya and Morikawa [26].

Now, we turn to the third quantization for the spinor wave function obeying the Dirac equation (2.16). The spinor wave function is decomposed as

\[
\Psi(x, y) = \int_0^\infty d\nu \left[ b_\nu u_\nu(x, y) + d^\dagger_\nu u^-_\nu(x, y) + b^-_\nu v^-_\nu(x, y) + d^\dagger^-_\nu v^+_\nu(x, y) \right]. \tag{3.13}
\]

It should be noted that the integration parameter \( \nu \) runs over positive values in this expression. The inner product of two spinors is defined by

\[
(\vartheta, \varphi) \equiv \int dy \sqrt{-G} |G^{xx}|^{1/2} \vartheta^\dagger \varphi = \sqrt{2V} e^{nx} \int dy \vartheta^\dagger \varphi. \tag{3.14}
\]

Note that \( (\varphi, \vartheta) = (\vartheta, \varphi)^* \). The mode functions in (3.13) are normalized as

\[
(u_\nu, u_\nu^\prime) = (v_\nu, v_\nu^\prime) = \delta(\nu - \nu'), \quad (u_\nu, v_\nu^\prime) = (v_\nu, u_\nu^\prime) = 0. \tag{3.15}
\]

As is known in quantum field theory, anticommutators of the fermionic operators obey\(^4\)

\[
\{b_\nu, b^\dagger_-\nu\} = \{d_\nu, d^\dagger_-\nu\} = \delta(\nu - \nu'), \quad \text{(the other anticommutators)} = 0
\]

\[
\Leftrightarrow \quad \{\Psi_\alpha(x, y), \Psi^\dagger_\beta(x, y')\} = \delta_{\alpha\beta}\delta(y - y'), \tag{3.16}
\]

where \( \alpha, \beta = +, - \).

As in the scalar case, the spinor field can be expressed as

\[
\Psi(x, y) = \int_0^\infty d\nu [b^\text{in}_\nu u^\text{in}_\nu(x, y) + d^\text{in}^\dagger_\nu u^-_\nu(x, y) + b^-_\nu v^-_\nu(x, y) + d^\text{in}^\dagger^-_\nu v^+_\nu(x, y)]
\]

\[
= \int_0^\infty d\nu [b^\text{out}_\nu u^\text{out}_\nu(x, y) + d^\text{out}^\dagger_\nu u^-_\nu(x, y) + b^-_\nu v^-_\nu(x, y) + d^\text{out}^\dagger^-_\nu v^+_\nu(x, y)]. \tag{3.17}
\]

\(^4\)In the third quantization, we consider the description borrowing the state and creation and annihilation operators from the quantum field theory. Note that the signature of the minisuperspace is Lorentzian. It is very interesting that the spin-statistics theorem does not hold, but it is expected that difficulties will appear when introducing interactions among multiple universes.
Here the in-mode spinor functions are obtained from
\[
\begin{align*}
    u^\text{in}_\nu(x, y) &= \frac{1}{2\sqrt{n \cosh \frac{\pi \nu}{n}}} \left( J_{-i\frac{\nu}{n} + \frac{1}{2}}(\sqrt{2V e^{nx}/n}) - J_{-i\frac{\nu}{n} - \frac{1}{2}}(\sqrt{2V e^{nx}/n}) \right) e^{i\nu y}, \\
v^\text{in}_\nu(x, y) &= \frac{1}{2\sqrt{n \cosh \frac{\pi \nu}{n}}} \left( J_{i\frac{\nu}{n} + \frac{1}{2}}(\sqrt{2V e^{nx}/n}) - J_{i\frac{\nu}{n} - \frac{1}{2}}(\sqrt{2V e^{nx}/n}) \right) e^{i\nu y},
\end{align*}
\] (3.18)
while the out-mode spinor functions are given by
\[
\begin{align*}
    u^\text{out}_\nu(x, y) &= U_\nu(x, y), \quad u^\text{out}_{\nu}^*(x, y) = -iV_{\nu}(x, y), \quad (\nu > 0) \quad (3.19) \\
v^\text{out}_\nu(x, y) &= -iU_{\nu}(x, y), \quad v^\text{out}_\nu(x, y) = V_\nu(x, y), \quad (\nu > 0) \quad (3.20)
\end{align*}
\]
where
\[
\begin{align*}
    U_\nu(x, y) &= \frac{e^{-\frac{\pi \nu}{n}}}{2\sqrt{2\pi}} \left( \frac{H^{(2)}_{-i\frac{\nu}{n} + \frac{1}{2}}(\sqrt{2V e^{nx}/n})}{H^{(2)}_{-i\frac{\nu}{n} - \frac{1}{2}}(\sqrt{2V e^{nx}/n})} \right) e^{i\nu y} = \frac{u^\text{in}_\nu(x, y)}{\sqrt{1 + e^{-2\pi \nu/n}}} + i \frac{v^\text{in}_\nu(x, y)}{\sqrt{e^{2\pi \nu/n} + 1}}, \quad (3.21) \\
    V_\nu(x, y) &= \frac{ie^{\frac{\pi \nu}{n}}}{2\sqrt{2\pi}} \left( \frac{H^{(1)}_{-i\frac{\nu}{n} + \frac{1}{2}}(\sqrt{2V e^{nx}/n})}{H^{(1)}_{-i\frac{\nu}{n} - \frac{1}{2}}(\sqrt{2V e^{nx}/n})} \right) e^{i\nu y} = i \frac{u^\text{in}_\nu(x, y)}{\sqrt{e^{2\pi \nu/n} + 1}} + \frac{v^\text{in}_\nu(x, y)}{\sqrt{1 + e^{-2\pi \nu/n}}}. \quad (3.22)
\end{align*}
\]
Then, one can find the relation among the in- and out-operators as
\[
\begin{align*}
b^\text{out}_\nu &= \frac{1}{\sqrt{1 + e^{-2\pi |\nu|/n}}} b^\text{in}_\nu - i \frac{1}{\sqrt{e^{2\pi |\nu|/n} + 1}} d^\text{in}_\nu, \quad (\nu > 0) \quad (3.23) \\
b^\text{out}_{\nu}^* &= \frac{1}{\sqrt{1 + e^{-2\pi |\nu|/n}}} b^\text{in}_\nu + i \frac{1}{\sqrt{e^{2\pi |\nu|/n} + 1}} d^\text{in}_\nu, \quad (\nu > 0) \quad (3.24) \\
d^\text{out}_\nu^\dagger &= \frac{1}{\sqrt{1 + e^{-2\pi |\nu|/n}}} d^\text{in}_\nu^\dagger + i \frac{1}{\sqrt{e^{2\pi |\nu|/n} + 1}} b^\text{in}_\nu, \quad (\nu > 0) \quad (3.25) \\
d^\text{out}_{\nu}^\dagger &= \frac{1}{\sqrt{1 + e^{-2\pi |\nu|/n}}} d^\text{in}_\nu^\dagger - i \frac{1}{\sqrt{e^{2\pi |\nu|/n} + 1}} b^\text{in}_\nu. \quad (\nu > 0) \quad (3.26)
\end{align*}
\]
Accordingly, with the aid of
\[
|b^\text{in}_\nu| = |d^\text{in}_\nu| = 0 \quad (\nu > 0)
\]
and so on, we find the average number of “fermionic” universes in the form
\[
N_{F\nu} \equiv \langle 0, in|b^\text{out}_\nu b^\text{out}_{\nu}^*|0, in \rangle = \frac{1}{e^{2\pi |\nu|/n} + 1}, \quad (\infty < \nu < \infty) \quad (3.27)
\]
and that of “anti-universes” as
\[
\tilde{N}_{F\nu} \equiv \langle 0, in|d^\text{out}_\nu d^\text{out}_{\nu}^*|0, in \rangle = \frac{1}{e^{2\pi |\nu|/n} + 1} = N_{F\nu}, \quad (\infty < \nu < \infty) \quad (3.28)
\]

We conclude that the finite number density of universes is the Fermi–Dirac distribution in the context of spontaneous creation of fermionic universes derived from the third quantization for the Dirac-type equation in quantum cosmology.
IV. IS THERE DEPENDENCE ON THE AUXILIARY DIMENSION OF THE EXTENDED MINISUPERSPACE?

Until the previous section, we have treated the extra degree of freedom, or the extra auxiliary dimension $z$ of the extended minisuperspace, as a fictitious one. That is, owing to the additional constraint, we have considered that wave functions and also quantized fields are defined in two dimensional minisuperspace as concerned with the present model.

In this section, we consider the third quantized field theory without the additional constraint in full extended dimensions, i.e., three dimensional extended minisuperspace in our model.

Again, we restart with the scalar case. The equation of motion of the scalar field in the extended minisuperspace of our model is

$$\frac{1}{2Vc^{2nx}} \left( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, y, z) = 0. \quad \text{(4.1)}$$

The expansion of the scalar field in terms of mode functions is given by

$$\Phi(x, y, z) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\nu \left[ a_{\nu,k}^{\text{in}} \phi_{\nu,k}^{\text{in}}(x, y, z) + a_{\nu,k}^{\text{in} \dagger} \phi_{\nu,k}^{\text{in}*}(x, y, z) \right],$$

(4.2)

where

$$\phi_{\nu,k}^{\text{in}}(x, y, z) = \frac{1}{2} \sqrt{\frac{1}{1 - e^{2\pi|\nu|/n}}} J_{-i|\nu|/n}(|k|\sqrt{2Ve^{nx}/n}) e^{i\nu y + ikz}, \quad \text{(4.3)}$$

and

$$\phi_{\nu,k}^{\text{out}}(x, y, z) = \frac{e^{-\pi|\nu|/4n}}{\pi\sqrt{2n}} H^{(2)}_{-i|\nu|/n}(|k|\sqrt{2Ve^{nx}/n}) e^{i\nu y + ikz}. \quad \text{(4.4)}$$

We presume the commutation relations $[a_{\nu,k}^{\text{in}}, a_{\nu',k'}^{\text{in} \dagger}] = [a_{\nu,k}^{\text{out} \dagger}, a_{\nu',k'}^{\text{out}}] = \delta(\nu - \nu')\delta(k - k')$, etc. Similar analysis to one in the previous section leads to the relation between in- and out-operators:

$$a_{\nu,k}^{\text{out}} = \frac{1}{\sqrt{1 - e^{-2\pi|\nu|/n}}} a_{\nu,k}^{\text{in}} + \frac{1}{\sqrt{e^{2\pi|\nu|/n} - 1}} a_{-\nu,k}^{\text{in} \dagger}, \quad \text{(4.5)}$$

whose form is apparently independent of the index $k$. Then, the number density is found to be independent of $k$ also in this case:

$$N_{B_{\nu,k}} \equiv \langle 0, \text{in} | a_{\nu,k}^{\text{out} \dagger} a_{\nu,k}^{\text{out}} | 0, \text{in} \rangle = \frac{1}{e^{2\pi|\nu|/n} - 1}. \quad \text{(4.6)}$$
Therefore we conclude, at least in the present subject to study, the variable $z$ can be considered as fictitious and unphysical.\footnote{This may be trivial if we have observed that the relation of in- and out-operators are independent of the constant $\sqrt{2V}$.} As discussed in our previous paper [33], the eigenvalue $|k|$ can be absorbed in redefinition in the original dynamical system, and thus this is a natural result.

Just in case, let us consider the case of the spinor fermionic field. The solution of the Dirac equation (2.12)

\[
\left[ \sigma^1 \left( \frac{\partial}{\partial x} + \frac{n}{2} \right) + i \sigma^2 \frac{\partial}{\partial y} + i \sigma^3 \sqrt{2V} e^{n_x} \frac{\partial}{\partial z} \right] \Psi = 0 ,
\]  

(4.7)

can be expanded in modes with a new index $k$ as

\[
\Psi(x, y, z) = \int_0^\infty dk \int_0^\infty dv \left[ b_{\nu, k}^{in} u_{\nu, k}^{in}(x, y, z) + b_{\nu, -k}^{in} u_{\nu, -k}^{in}(x, y, z) 
+ d_{\nu, k}^{in\dagger} u_{\nu, k}^{in\dagger}(x, y, z) + d_{\nu, -k}^{in\dagger} u_{\nu, -k}^{in\dagger}(x, y, z) 
+ b_{\nu, k}^{in\dagger} v_{\nu, k}^{in\dagger}(x, y, z) + b_{\nu, -k}^{in\dagger} v_{\nu, -k}^{in\dagger}(x, y, z) 
+ d_{\nu, k}^{in\dagger} v_{\nu, k}^{in\dagger}(x, y, z) + d_{\nu, -k}^{in\dagger} v_{\nu, -k}^{in\dagger}(x, y, z) \right] ,
\]  

(4.8)

and similar expansion in terms of out-modes.

The construction of mode functions are performed as in the previous section and here we use in-mode functions

\[
u_{\nu, k}^{in}(x, y) = \frac{1}{2\sqrt{2\pi kn \cosh \frac{2\pi n}{v}}} \left( J_{\frac{\nu - k}{2}}(k\sqrt{2V}e^{n_x}/n) \right) e^{\imath \nu y + \imath k z} ,
\]  

(4.9)

\[
u_{\nu, k}^{in\dagger}(x, y) = \frac{1}{2\sqrt{2\pi kn \cosh \frac{2\pi n}{v}}} \left( J_{\frac{k - \nu}{2}}(k\sqrt{2V}e^{n_x}/n) \right) e^{\imath \nu y + \imath k z} ,
\]  

(4.10)

\[
u_{\nu, -k}^{in}(x, y) = \frac{1}{2\sqrt{2\pi kn \cosh \frac{2\pi n}{v}}} \left( J_{\frac{-\nu - k}{2}}(k\sqrt{2V}e^{n_x}/n) \right) e^{\imath \nu y - \imath k z} ,
\]  

(4.11)

\[
u_{\nu, -k}^{in\dagger}(x, y) = \frac{1}{2\sqrt{2\pi kn \cosh \frac{2\pi n}{v}}} \left( J_{\frac{k - \nu}{2}}(k\sqrt{2V}e^{n_x}/n) \right) e^{\imath \nu y - \imath k z} ,
\]  

(4.12)

and out-mode functions are constructed from

\[
u_{\nu, \pm k}(x, y, z) = \frac{u_{\nu, \pm k}^{in}(x, y, z)}{\sqrt{1 + e^{-2\pi \nu/n}}} + \frac{v_{\nu, \pm k}^{in\dagger}(x, y, z)}{\sqrt{1 + e^{-2\pi \nu/n}}} ,
\]  

(4.13)

\[
u_{\nu, \pm k}(x, y, z) = \frac{i u_{\nu, \pm k}^{in}(x, y, z)}{\sqrt{1 + e^{2\pi \nu/n}}} + \frac{i v_{\nu, \pm k}^{in\dagger}(x, y, z)}{\sqrt{1 + e^{2\pi \nu/n}}} .
\]  

(4.14)
Finally, we obtain the result
\[
\langle 0, \text{in} | b_{\nu,k} \dagger b_{\nu,k}^\dagger | 0, \text{in} \rangle = \langle 0, \text{in} | a_{\nu,k} \dagger a_{\nu,k}^\dagger | 0, \text{in} \rangle = \frac{1}{e^{2\pi |\nu|/n} + 1}. \quad (-\infty < \nu, k < \infty)
\] (4.15)

We thus confirmed the \(k\)-independence in the third quantization method for the case with the fermionic field.

V. ENTANGLEMENT ENTROPY OF MULTIPLE UNIVERSES

As in a quantum system, the von Neumann entropy in an expanding universe has been calculated in many works including Ref. [54, 55]. The von Neumann entropy is an index for quantitatively evaluating entanglement [56, 57]. It has been demonstrated that the von Neumann entropy can be calculated in third-quantized quantum cosmology [58], and we will show both bosonic and fermionic system originated from the usual WDW equation and the Dirac-type equation in our present model. We return to the constrained version, or we discard the index \(k\), as in Section III.

First, we examine “bosonic” universes originated from the wave function of the usual WDW equation. From Eq. (3.9), we find
\[
a_{\nu}^\text{in} = \frac{1}{\sqrt{1 - e^{-2\pi |\nu|/n}}} a_{\nu}^\text{out} - \frac{1}{\sqrt{e^{2\pi |\nu|/n} - 1}} a_{-\nu}^\text{out} \dagger.
\] (5.1)

For simplicity, we focus on the fixed mode of \(\nu\). If \(|0, \text{in}\rangle_\nu = \sum_{p=0}^{\infty} A_p |p_\nu, p, \text{out}\rangle\), where \(|p_\nu, p', \text{out}\rangle = |p, \text{out}\rangle_\nu |p', \text{out}\rangle_{-\nu} = \frac{1}{\sqrt{p}} (a_{\nu}^\text{out} |p, \text{out}\rangle_\nu a_{-\nu}^\text{out} |p', \text{out}\rangle_{-\nu})\), is assumed, \(a_{\nu}^\text{in} |0, \text{in}\rangle_\nu = 0\) with the use of Eq. (5.1) yields
\[
|0, \text{in}\rangle_\nu = \sqrt{1 - e^{-2\pi |\nu|/n}} \sum_{p=0}^{\infty} e^{-p\pi |\nu|/n} |p_\nu, p, \text{out}\rangle.
\] (5.2)

After the modes with the order \(-\nu\) are traced out, we obtain the reduced density matrix
\[
\rho_\nu \equiv \text{Tr}_{-\nu} \left[ |0, \text{in}\rangle_\nu \langle 0, \text{in}| \right] = (1 - e^{-2\pi |\nu|/n}) \sum_{p=0}^{\infty} e^{-2\pi |\nu|p/n} |p, \text{out}\rangle_\nu \langle p, \text{out}|.
\] (5.3)

Using this, the von Neumann entanglement entropy defined by
\[
S \equiv -\text{Tr} \left[ \rho_\nu \log_2 \rho_\nu \right].
\] (5.4)

becomes
\[
S_B = \log_2 \left[ \frac{\exp\left[\frac{2\pi |\nu|/n}{e^{2\pi |\nu|/n} - 1}\right]}{1 - e^{-2\pi |\nu|/n}} \right],
\] (5.5)
where the subscript “B” is assigned because universes are bosonic in this case.

Incidentally, the Rényi entropy, which is another index of the entanglement, is defined as

\[ S(q) \equiv \frac{1}{1 - q} \log_2 \left[ \text{Tr} \rho_\nu^q \right]. \]  

(5.6)

Note that the limit of \( q \to 1 \) reduces the Rényi entropy to the von Neumann entropy exactly. In our bosonic case, one can find

\[ S_B(q) = \frac{1}{1 - q} \log_2 \left[ \frac{1 - e^{-2\pi|\nu|/n}q}{1 - e^{-2\pi|\nu|q/n}} \right]. \]  

(5.7)

Next, we consider the fermionic universes originated from the Dirac field. We consider an in-vacuum on a specific \( \nu \), \( |0, \text{in}\rangle_\nu |\bar{0}, \text{in}\rangle_{-\nu} \), where \( b^\dagger_\nu |0, \text{in}\rangle_\nu = d^\dagger_{-\nu} |\bar{0}, \text{in}\rangle_{-\nu} \). Similarly to the bosonic case, the inverse relation of Eq. (3.23–3.26) leads to

\[ |0, \text{in}\rangle_\nu |\bar{0}, \text{in}\rangle_{-\nu} = \frac{1}{\sqrt{1 + e^{-2\pi|\nu|/n}}} (|0, \text{out}\rangle_\nu |\bar{0}, \text{out}\rangle_{-\nu} - i \text{sgn}(\nu) e^{-\pi|\nu|/n} |1, \text{out}\rangle_\nu |\bar{1}, \text{out}\rangle_{-\nu} \), \]  

(5.8)

where \( |1, \text{out}\rangle_\nu \equiv b^\dagger_\nu |0, \text{out}\rangle_\nu \) and \( |\bar{1}, \text{out}\rangle_{-\nu} \equiv d^\dagger_{-\nu} |\bar{0}, \text{out}\rangle_{-\nu} \), of course with the relations

\[ b^\dagger_\nu |0, \text{out}\rangle_\nu = d^\dagger_{-\nu} |\bar{0}, \text{out}\rangle_{-\nu} = 0 \]  

for the out-vacuum.

Tracing over the anti-universe modes with the index \( -\nu \), we obtain

\[
\rho_\nu \equiv \text{Tr}_{-\nu} \left[ |0, \text{in}\rangle_\nu |\bar{0}, \text{in}\rangle_{-\nu} \langle \bar{0}, \text{in}\rangle_{-\nu} |0, \text{in}\rangle_\nu \right] \\
= \frac{1}{1 + e^{-2\pi|\nu|/n}} |0, \text{out}\rangle_\nu \langle 0, \text{out}\rangle_\nu + \frac{e^{-2\pi|\nu|/n}}{1 + e^{-2\pi|\nu|/n}} |1, \text{out}\rangle_\nu \langle 1, \text{out}\rangle_\nu.
\]  

(5.9)

From this reduced matrix, we find the von Neumann entropy

\[ S_F = \log_2 \left( \frac{1 + e^{-2\pi|\nu|/n}}{\exp[-\frac{2\pi|\nu|/n}{e^{2\pi|\nu|/n} + 1}]} \right), \]  

(5.10)

and the Rényi entropy

\[ S_F(q) = \frac{1}{1 - q} \log_2 \left[ \frac{1 + e^{-2\pi|\nu|q/n}}{1 + e^{-2\pi|\nu|q/n}} \right]. \]  

(5.11)

In Fig. 1, the dependence of the entanglement entropy on \( \nu \) for different values of the parameter \( q \). The entanglement monotonically decreases with \( \nu \) for both cases, while it diverges when \( \nu \to 0 \) only in the bosonic case.

If the principle of maximum entropy can be applied to the initial state of the multiple universes, it may be said that the universe starts with a small value of the mode number
ν of the scalar field. In other words, the probability distribution of the value of the scalar field will be nearly uniform.

Before closing this section, we must mention the physical meaning of the entanglement entropy of the multiple-universe system. The problem of observers already existed in quantum cosmology (who exists outside the Universe?). There is no observer in quantum cosmology, where no measurement was carried out. Thus, the definition of information or entropy deduced from measurement of a system should be ambiguous.

Nevertheless, Hosoya and Morikawa [26] considered “detector” of universes, as an ambitious attempt. Unfortunately, their approach may bring about more ambiguities in the model, because it should be additionally assumed the interaction with multiple universes. The inclusion of interactions, or “wormholes” making bridges between (baby) universes, is still an interesting idea [22], since the interaction may fix the physical constants in our (specific) universe.

VI. DISCUSSION AND OUTLOOK

In conclusion, we have investigated the third quantization of quantum cosmology in a simple model by using the extended minisuperspace. As the equation of motion of the quantized field, we considered both Klein–Gordon-type and Dirac-type equations, which describe annihilation and creation of bosonic and fermionic universes, respectively. By utilizing the technique of quantum field theory in curved space, we obtain the exact results...
on the average number of bosonic and fermionic universes which are spontaneously created. It is found that the average number is expressed by the Planck distribution for bosonic universes and the Fermi–Dirac distribution for fermionic universe, respectively. We also show the calculable entanglement entropies for both systems. Although we have studied a simple model in this paper, exact results may play some important role in understanding the crude concepts of the third quantization in future study. The analyses in various cosmological models are required as a straightforward extension of the present study.

We have chosen the in-vacuum state as the initial state, but condensate/degenerate states are equally considerable as a hypothesis. On the other hand, for operators, interaction terms may fix the constant in nature [22] and provide significance of entanglement. Moreover, the dependence of physical quantities on the auxiliary dimension is quite possible if we consider general interaction of fields. We can postulate supersymmetric third-quantized field theory as well. The study in the above mentioned directions will be advanced in future publication.

NOTE ADDED

After completion of the present manuscript, Prof. S. J. Robles-Pérez informed us another approach to the Wheeler–DeWitt equation [59]. The massive Klein–Gordon field in curved superspace considered in Ref. [59] seems to be equivalent to the dimensional reduction (rather than the Kaluza–Klein compactification) of the field in the extended superspace.

ACKNOWLEDGMENTS

We would like to thank Prof. S. J. Robles-Pérez and Prof. P. V. Moniz for providing information about their remarkable works.

[1] J. B. Hartle and S. W. Hawking, “Wave function of the universe”, Phys. Rev. D28 (1983) 2960.
[2] S. W. Hawking, “The quantum state of the universe”, Nucl. Phys. B239 (1984) 257.
[3] J. J. Halliwell, “Introductory lectures on quantum cosmology”, in: Proceedings of 7th Jerusalem Winter School for Theoretical Physics: Quantum Cosmology and Baby Universes,
edited by S. Coleman, J. B. Hartle, T. Piran and S. Weinberg (World Scientific, Singapore, 1991), pp. 159–243. arXiv:0909.2566 [gr-qc].

[4] C. Kiefer, *Quantum Gravity*, Int. Ser. Monogr. Phys. Vol. 155 (Clarendon Press, Oxford, 2012).

[5] C. Kiefer, “Conceptual problems on quantum gravity and quantum cosmology”, ISRN Mathematical Physics **2013** (2013) 509316. arXiv:1401.3578 [gr-qc].

[6] M. P. Ryan and L. C. Shepley, “Homogeneous Relativistic Cosmologies”, (Princeton Univ. Press, Princeton, 1975).

[7] M. Ryan, “Hamiltonian Cosmology”, (Springer, Berlin Heidelberg, 1972).

[8] P. D. D’Eath, S. W. Hawking and O. Obregón, “Supersymmetric Bianchi models and the square root of the Wheeler–DeWitt equation”, Phys. Lett. **B300** (1993) 44.

[9] C.-M. Kim and S.-K. Oh, “Dirac-square-root formulation of some types of minisuperspace quantum cosmology”, J. Korean Phys. Soc. **29** (1996) 549.

[10] Y.-G. Shen and Z.-Y. Cheng, “Quantization of the Bianchi type universe”, Chin. Astron. Astrophys. **21** (1997) 389.

[11] H. Yamazaki and T. Hara, “Dirac decomposition of Wheeler–DeWitt equation in the Bianchi class A models”, Prog. Theor. Phys. **106** (2001) 323. gr-qc/0101066.

[12] S. A. Hojman and F. A. Asenjo, “Supersymmetric Majorana quantum cosmologies”, Phys. Rev. **D92** (2015) 083518. arXiv:1506.02939 [gr-qc].

[13] C. A. Rubio, F. A. Asenjo and S. A. Hojman, “Quantum cosmologies under geometrical unification of gravity and dark energy”, Symmetry **11** (2019) 860.

[14] P. V. Moniz, “Quantum cosmology — the supersymmetric perspective”, vols. 1 and 2, Lect. Notes Phys. **803** and **804** (Springer, Berlin Heidelberg 2010).

[15] P. V. Moniz, “Supersymmetric quantum cosmology shaken, not stirred”, Int. J. Mod. Phys. **A11** (1996) 4321. gr-qc/9604025.

[16] P. V. Moniz, “Coserved currents in supersymmetric quantum cosmology?”, Int. J. Mod. Phys. **D6** (1997) 625. gr-qc/9606047.

[17] P. V. Moniz, “Supersymmetric quantum cosmology: a ‘Socratic’ guide”, Gen. Rel. Grav. **46** (2014) 1618.

[18] S. Jalalzadeh, S. M. M. Rasouli and P. Moniz, “Shape invariant potentials in supersymmetric quantum cosmology”, Universe **8** (2022) 316. arXiv:2206.00083 [gr-qc].
[19] N. Caderni and M. Martellini, “Third quantization formalism for Hamiltonian cosmologies”, Int. J. Theor. Phys. 23 (1984) 233.

[20] M. McGuigan, “Third quantization and the Wheeler–DeWitt equation”, Phys. Rev. D38 (1988) 3031.

[21] M. McGuigan, “Universe creation from the third-quantized vacuum”, Phys. Rev. D39 (1989) 2229.

[22] A. Strominger, “Third quantization”, Phil. Trans. R. Soc. Lond. A329 (1989) 395.

[23] Y. Ohkuwa, Y. Ezawa and M. Faizal, “Constraints on operator ordering from third quantization”, Ann. Phys. (NY) 365 (2016) 54. arXiv:1505.02754 [physics.gen-ph].

[24] Y. Ohkuwa, Y. Ezawa and M. Faizal, “Operator ordering ambiguity and third quantization”, Ann. Phys. (NY) 414 (2020) 168072. arXiv:1907.02042 [hep-th].

[25] S. J. Robles-Pérez, “Quantum cosmology in the light of quantum mechanics”, Galaxies 7 (2019) 50. arXiv:1812.10657 [gr-qc].

[26] A. Hosoya and M. Morikawa, “Quantum field theory of the Universe”, Phys. Rev. D39 (1989) 1123.

[27] Y. Ohkuwa, “Third quantization of Kaluza–Klein cosmology and compactification”, Int. J. Mod. Phys. A13 (1998) 4091. gr-qc/9712035.

[28] L. O. Pimentel and C. Mora, “Third quantization of Brans–Diche cosmology”, Phys. Lett. A280 (2001) 191. gr-qc/0009016.

[29] S.-P. Kim, “Third quantization and quantum universes”, Nucl. Phys. B (Proc. Suppl.) 246-247 (2014) 68. arXiv:1212.5355 [gr-qc].

[30] L. Campanelli, “Creation of universes from the third-quantized vacuum”, Phys. Rev. D102 (2020) 043514. arXiv:2007.01732 [gr-qc].

[31] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, “Expanding and contracting universes in third quantized string cosmology”, Class. Quant. Grav. 14 (1997) L97. hep-th/9701146.

[32] M. Gasperini, “Quantum string cosmology”, Universe 7 (2021) 14. arXiv:2101.01070 [gr-qc].

[33] N. Kan, T. Aoyama, T. Hasegawa and K. Shiraishi, “Eisenhart–Duval lift for minisuperspace quantum cosmology”, Phys. Rev. D104 (2021) 086001. arXiv:2105.09514 [gr-qc].

[34] L. P. Eisenhart, “Dynamical trajectories and geodesics”, Ann. Math. 30 (1928) 591.
[35] C. Duval, G. Burdet, H. P. Künzle and M. Perrin, “Bargmann structures and Newton–Cartan theory”, Phys. Rev. D31 (1985) 1841.

[36] C. Duval, G. Gibbons and P. Horváthy, “Celestial mechanics, conformal structures, and gravitational waves”, Phys. Rev. D43 (1991) 3907. hep-th/0512188.

[37] L. Casetti, M. Pettini and E. G. O. Cohen, “Geometric approach to Hamiltonian dynamics and statistical mechanics”, Phys. Rep. 337 (2000) 237. cond-mat/9912092.

[38] M. Pettini, *Geometry and topology in Hamiltonian dynamics and statistical mechanics* (Springer, New York, 2007)

[39] M. Cariglia, “Hidden symmetries of dynamics in classical and quantum physics”, Rev. Mod. Phys. 86 (2014) 1283. arXiv:1411.1262 [math-ph].

[40] M. Cariglia, A. Galajinsky, G. W. Gibbons and P. A. Horvathy, “Cosmological aspects of the Eisenhart–Duval lift”, Eur. Phys. J. C78 (2018) 314. arXiv:1802.03370 [gr-qc].

[41] K. Finn and S. Karamitsos, “Finite measure for the initial conditions of inflation”, Phys. Rev. D99 (2019) 063515. arXiv:1812.07095 [gr-qc].

[42] K. Finn, “Initial conditions of inflation in a Bianchi I universe”, Phys. Rev. D101 (2020) 063512. arXiv:1912.04306 [gr-qc].

[43] S. Dhasmana, A. Sen and Z. K. Silagadze, “Equivalence of a harmonic oscillator to a free particle and Eisenhart lift”, Ann. Phys. (NY) 434 (2021) 168623. arXiv:2106.09523 [quant-ph].

[44] S. W. Hawking and D. N. Page, “Operator ordering and the flatness of the universe”, Nucl. Phys. B264 (1986) 185.

[45] I. Moss, “Quantum cosmology and the self observing universe”, Ann. Inst. Henri Poincaré 49 (1988) 341.

[46] J. J. Halliwell, “Derivation of the Wheeler–DeWitt equation from a path integral for minisuperspace models”, Phys. Rev. D38 (1988) 2468.

[47] N. Kontoleon and D. L. Wiltshire, “Operator ordering and consistency of the wave function of the Universe”, Phys. Rev. D59 (1999) 063513. gr-qc/9807075.

[48] S. W. Hawking and D. N. Page, “Spectrum of wormholes”, Phys. Rev. D42 (1990) 2655.

[49] C. Kiefer, “Wave packets in minisuperspace”, Phys. Rev. D38 (1988) 1761.

[50] A. A. Andrianov, C. Lan, O. O. Novikov and Y.-F. Wang, “Integrable cosmological models with Field: energy density self-adjointness and semiclassical Wave Packets”, Eur. J. Phys.
C78 (2018) 786. arXiv:1802.06720 [hep-th].

[51] A. Paliathanasis, “Dust fluid component from Lie symmetries in scalar field cosmology”, Mod. Phys. Lett. A32 (2017) 1750206. arXiv:1710.08666 [gr-qc].

[52] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, vol. 2 (McGraw-Hill, New York, 1953).

[53] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space, (Cambridge Univ. Press, Cambridge, 1982).

[54] J. L. Ball, I. Fuentes-Schuller and F. P. Schuller, “Entanglement in an expanding spacetime”, Phys. Lett. A359 (2006) 550. quant-ph/0506113.

[55] I. Fuentes, R. B. Mann, E. Martín-Martínez and S. Moradi, “Entanglement of Dirac fields in an expanding spacetime”, Phys. Rev. D82 (2010) 045030. arXiv:1007.1569 [quant-ph].

[56] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, “Quantum entanglement”, Rev. Mod. Phys. 81 (2009) 865. quant-ph/0702225.

[57] T. Nishioka, “Entanglement entropy: holography and renormalization group”, Rev. Mod. Phys. 90 (2018) 035007. arXiv:1801.10352 [hep-th].

[58] A. Balcerzak and K. Marosek, “Doubleverse entanglement in third quantized non-minimally coupled varying constants cosmologies”, Eur. Phys. J. 80 (2020) 709. arXiv:2003.06380 [gr-qc].

[59] S. J. Robles-Pérez, “Quantum cosmology with third quantization”, Universe 7 (2021) 404. arXiv:2110.05785 [gr-qc].