Global Behavior of Spherically Symmetric Navier-Stokes Equations with Density-Dependent Viscosity

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Abstract

In this paper, we study a free boundary problem for compressible spherically symmetric Navier-Stokes equations without a solid core. Under certain assumptions imposed on the initial data, we obtain the global existence and uniqueness of the weak solution, give some uniform bounds (with respect to time) of the solution and show that it converges to a stationary one as time tends to infinity. Moreover, we obtain the stabilization rate estimates of exponential type in $L^\infty$-norm and weighted $H^1$-norm of the solution by constructing some Lyapunov functionals. The results show that such system is stable under small perturbations, and could be applied to the astrophysics.

Keywords: Compressible Navier-Stokes equations; density-dependent viscosity; free boundary; existence; uniqueness; asymptotic behavior

1 Introduction.

We consider the compressible Navier-Stokes equations with density-dependent viscosity in $\mathbb{R}^n (n \geq 2)$, which can be written in Eulerian coordinates as

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) &= 0, \\
\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla P &= \text{div}(\mu(\nabla \vec{u} + \nabla \vec{u}^T)) + \nabla (\lambda \text{div} \vec{u}) - \rho \vec{f},
\end{aligned}
$$

(1.1)

Here $\rho$, $P$, $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{f}$ are the density, pressure, velocity and the external force, respectively; $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ are two viscosity coefficients.

In this paper, the initial conditions are

$$
\begin{aligned}
\rho(\vec{\xi}, 0) &= \rho_0(r), \ r \in [0,b], \\
\vec{u}(\vec{\xi}, 0) &= u_0(r)(\vec{\xi}, r \in (0,b], \ \vec{u}(\vec{\xi}, 0)|_{\vec{\xi}=0} = u_0(0) = 0,
\end{aligned}
$$

(1.2)

(1.3)

where $r = |\vec{\xi}| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$ and $b > 0$ is a constant, the boundary condition is

$$
\left\{(P - \lambda \text{div} \vec{u})\text{Id} - \mu(\nabla \vec{u} + \nabla \vec{u}^T)\right\} \cdot \vec{n} = P_T \vec{n}, \ \vec{\xi} \in \partial \Omega_T
$$

(1.4)
where $\partial \Omega_r = \psi(\partial \Omega_0, \tau)$ is a free boundary, $\vec{n}$ is the unit outward normal vector of $\partial \Omega_r$ and $P_\Gamma > 0$ is a external pressure. Here, $\partial \Omega_0 = \{ \vec{\xi} \in \mathbb{R}^n : |\vec{\xi}| = b \}$ is the initial boundary and $\psi$ is the flow of $\vec{u}$:

$$
\begin{align*}
\partial_\tau \psi(\vec{\xi}, \tau) &= \vec{u}(\psi(\vec{\xi}, \tau), \tau), \quad \vec{\xi} \in \mathbb{R}^n, \\
\psi(\vec{\xi}, 0) &= \vec{\xi}.
\end{align*}
$$

(1.5)

To simplify the presentation, we only consider the famous polytropic model, i.e. $P(\rho) = A \rho^\gamma$ with $\gamma > 1$ and $A > 0$ being constants. And we assume that the viscosity coefficients $\mu$ and $\lambda$ are proportional to $\rho^\theta$, i.e. $\mu(\rho) = c_1 \rho^\theta$ and $\lambda(\rho) = c_2 \rho^\theta$ where $c_1, c_2$ and $\theta$ are three constants.

For the initial-boundary value problem (1.1)-(1.4), we are looking for a spherically symmetric solution $(\rho, \vec{u})$:

$$
\rho(\vec{\xi}, \tau) = \rho(r, \tau), \quad \vec{u}(\vec{\xi}, \tau) = u(r, \tau) \frac{\vec{\xi}}{r},
$$

with the spherically symmetric external force

$$
f = f(m, r, \tau) \frac{\vec{\xi}}{r}, \quad m(\rho, r) = \int_0^r \rho(s, \tau) s^{n-1} ds, \quad r > 0
$$

and $\partial \Omega_r = \{ \vec{\xi} \in \mathbb{R}^n : |\vec{\xi}| = b(\tau), b(0) = b, b'(\tau) = u(b(\tau), \tau) \}$.

Then $(\rho, u)(r, \tau)$ is determined by

$$
\begin{align*}
\partial_\tau \rho + \partial_r (\rho u) + \frac{n-1}{r} \rho u &= 0, \\
\rho (\partial_\tau u + u \partial_r u) + \partial_r P &= (\lambda + 2 \mu) (\partial_{rr} u + \frac{n-1}{r} \partial_r u - \frac{n-1}{r^2} u) + 2 \partial_r \mu \partial_r u + \partial_r \lambda (\partial_r u + \frac{n-1}{r} u) - \rho f,
\end{align*}
$$

(1.6)

where $(r, \tau) \in (0, b(\tau)) \times (0, \infty)$, with the initial data

$$
(\rho, u)|_{\tau=0} = (\rho_0, u_0)(r), \quad 0 \leq r \leq b,
$$

(1.7)

the fixed boundary condition

$$
u|_{r=0} = 0,
$$

(1.8)

and the free boundary condition

$$
\left\{ P - 2 \mu \partial_\tau u - \lambda \left( \partial_\tau u + \frac{n-1}{r} u \right) \right\}|_{r=b(\tau)} = P_\Gamma,
$$

(1.9)

where $b(0) = b, b'(\tau) = u(b(\tau), \tau)$.

Additionally, we assume the external force $f(m, r, \tau)$ and external pressure $P_\Gamma(\tau) \in C^1(\mathbb{R}_+)$ satisfy

$$
P_\Gamma(\tau) = P_\infty + \Delta P(\tau), \quad f(m, r, \tau) = f_\infty(m, r) + \Delta f(m, r, \tau),
$$

(1.10)

for all $r \geq 0$ and $\tau \geq 0$, with

$$
f_\infty(m, r) = \frac{G m}{r^{n-1}}, \quad m(\rho, r) = \int_0^r \rho s^{n-1} ds, \quad \Delta f(m, r, \tau) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+)
$$

(1.11)

$$
\| \Delta f(\cdot, \cdot, \tau) \|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \leq f_1(\tau), \quad \| (\partial_\tau \Delta f, \partial_\tau \Delta f)(\cdot, \cdot, \tau) \|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \leq f_2(\tau),
$$

(1.12)

$$
f_1, \Delta P \in L^\infty \cap L^1(\mathbb{R}_+), \quad (\Delta P)'_1, f_2 \in L^2(\mathbb{R}_+),
$$

(1.13)

where $\mathbb{R}_+ = [0, \infty)$, $P_\infty$ and $G$ are two positive constants, perturbations $(\Delta P, \Delta f)$ tend to 0 as $\tau \to \infty$ in some weak sense. $f_\infty$ is the precise expression for its own gravitational force and $\Delta f$
expresses the influence of the outside gravitational force, in the astrophysical case (with spherical symmetry). \( P_1 \) also could express the influence of the surface tension force on the free boundary. This system can be treated as a simple model of one fluid in \( \Omega_\tau \), whose evolution is influenced by the gravitational force and the external pressure generated by the other substance in \( \mathbb{R}^n \setminus \Omega_\tau \).

We study the stabilization problem of such system, which could be applied to the astrophysics. Now, we consider the stationary problem, namely

\[
(P(\rho_\infty))_r = -\rho_\infty f_\infty(m(\rho_\infty, r), r)
\]

in an interval \( r \in (0, l_\infty) \) with the end \( l_\infty \) satisfying

\[
P(\rho_\infty(l_\infty)) = P_\infty,
\]

\[
\int_0^{l_\infty} \rho_\infty r^{n-1}dr = M := \int_0^b \rho_0 r^{n-1}dr.
\]

The unknown quantities are the stationary density \( \rho_\infty \geq 0 \) and free boundary \( l_\infty > 0 \). If \( \gamma = \frac{2n-2}{n} \) and \( Gn^{\frac{2-n}{n}}M^{\frac{2}{n}} < 2A \)

\[
(1.17)
\]

or \( \gamma > \frac{2n-2}{n} \),

\[
(1.18)
\]

from Proposition 2.5, we know that there exists a unique solution \((\rho_\infty, l_\infty)\) to the stationary system \((1.14)-(1.16)\), satisfying \( 0 < \rho \leq \rho_\infty(r) \leq \bar{\rho} < \infty \), \((\rho_\infty)_r(r) < 0 \), \( 0 < r < l_\infty \) with \( l_\infty < +\infty \).

To handle the free boundary problem \((1.6)-(1.9)\), it is convenient to reduce the problem in Eulerian coordinates \((r, \tau)\) to the problem in Lagrangian coordinates \((x, t)\), via the transformation:

\[
x = \int_0^r y^{n-1} \rho(y, \tau)dy, \quad t = \tau.
\]

Then the fixed boundary \( r = 0 \) and the free boundary \( r = b(\tau) \) become

\[
x = 0 \quad \text{and} \quad x = \int_0^{b(\tau)} y^{n-1} \rho(y, \tau)dy = \int_0^b y^{n-1} \rho_0(y)dy = M,
\]

where \( M \) is the total mass initially. So that the region \( \{(r, \tau) : 0 \leq r \leq b(\tau), \tau \geq 0 \} \) under consideration is transformed into the region \( \{(x, t) : 0 \leq x \leq M, t \geq 0 \} \).

Under the coordinate transformation \((1.19)\), the equations \((1.6)-(1.9)\) are transformed into

\[
\left\{
\begin{array}{l}
\partial_t \rho(x, t) = -\rho^2 \partial_x(r^{n-1}u), \\
\partial_t u(x, t) = r^{n-1}\{\partial_x[\rho(\lambda + 2\mu)\partial_x(r^{n-1}u) - P] - 2(n-1)\frac{\mu}{n} \partial_x \mu\} - f(x, r, t), \\
r^n(x, t) = n \int_x^M \rho^{-1}(y, t)dy,
\end{array}
\right.
\]

\[
(1.20)
\]

where \((x, t) \in (0, M) \times (0, \infty)\), with the initial data

\[
(\rho, u)|_{t=0} = (\rho_0, u_0)(x), r|_{t=0} = r_0(x) = \left(n \int_0^x \rho_0^{-1}(y)dy\right)^{\frac{1}{n}},
\]

\[
(1.21)
\]

and the boundary conditions:

\[
u(0, t) = 0,
\]

\[
(1.22)
\]
\[
\left\{ P - \rho(\lambda + 2\mu)\partial_x(r^{n-1}u) + 2(n-1)\mu u \frac{\partial u}{\partial r} \right\}
|_{x=M} = P_\Gamma, \ t > 0. \tag{1.23}
\]

It is standard that if we can solve the problem (1.20)-(1.23), then the free boundary problem (1.1)-(1.4) has a solution.

From (1.14)-(1.16), it is easy to see that \( \rho_\infty(x) \) is the solution to the stationary system,

\[
Ar^{n-1}(\rho_\infty)_x = -f_\infty(x,r_\infty), \ r_\infty^n(x) = n \int_0^x \rho_\infty^{-1}(y)dy, \ x \in (0,M), \tag{1.24}
\]

\[
\rho_\infty(M) = \left( \frac{P_\infty}{A} \right)^{\frac{1}{\gamma}}. \tag{1.25}
\]

The results in [6, 18] show that the compressible Navier-Stokes system with the constant viscosity coefficient have the singularity at the vacuum. Considering the modified Navier-Stokes system in which the viscosity coefficient depends on the density, Liu, Xin and Yang in [9] proved that such system is local well-posedness. It is motivated by the physical consideration that in the derivation of the Navier-Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order, cf. [4], the viscosity coefficient is a function of the temperature. If we consider the case of isentropic fluids, this dependence is reduced to the dependence on the density function.

Since \( n \geq 2 \) and the viscosity coefficient \( \mu \) depends on \( \rho \), the nonlinear term \( 2(n-1)\frac{1}{r}u \partial_x \mu \) in (1.20) makes the analysis significantly different from the one-dimensional case [9, 14, 17, 19, 20]. Considering the compressible spherically symmetric Navier-Stokes equations without a solid core, the techniques in the case of similar system with a solid core [1, 2, 11, 13, 21] failed to be of use in our case, so we need obtain some new \textit{a priori} estimates.

For spherically symmetric solutions of the Navier-Stokes equations with constant viscosity, in [7], the author gave an information near the origin that the solution may develop vacuum region about the origin. The difficulty of this problem is to obtain the lower bound of the density \( \rho \) and the upper bound of the term \( \frac{1}{r}u \). When the initial data are small in some sense, using some new \textit{a priori} estimates on the solution, we can obtain the lower bound of the density and the upper bound of the term \( \frac{1}{r}u \). The key ideas are using the classical continuity method and the result of Claim 1. In Claim 1, we want to prove that there is a small positive constant \( \epsilon_1 \), such that, for any \( T > 0 \), if

\[
I(t) = \|\rho(\cdot,t) - \rho_\infty\|_{L^\infty} + \left\| \frac{u}{r}(\cdot,t) \right\|_{L^\infty} \leq 2\epsilon_1, \ \forall \ t \in [0,T],
\]

then

\[
I(t) \leq \epsilon_1, \ \forall \ t \in [0,T].
\]

Let

\[
B[\rho, u, r] = \int_0^M \left[ (\rho - \rho_\infty)^2 + r^{2n-2+\alpha}(\rho - \rho_\infty)^2 + \frac{u^2}{r^2} \right. \\
+ r^{2n-2}u_x^2 + r^{2n-2+\alpha}(\rho^{1+\theta}(r^{n-1}u)_x)^2 \left. \right] dx,
\]

where \( \alpha = \frac{3}{2} - n \). In Lemmas 3.3,3.8 we get some uniform \textit{a priori} estimates (with respect to time) on the solution in the weighted Sobolev space and the upper bound of \( B[\rho, u, r] \). Using the bound of \( B[\rho, u, r] \) and Sobolev’s embedding Theorem, we can finish the proof of Claim 1. Then, we will construct a weak solution by using the finite difference approximation. Our results show that: such system does not develop vacuum states or concentration states for all
time, and the interface $\partial \Omega_t$ propagates with finite speed. Since these estimates of the solution are uniform in time, we could show that the solution converges to a stationary one as time tends to infinity. Moreover, we construct various Lyapunov functionals and obtained the stabilization rate estimates of exponential type.

We now briefly review the previous works in this direction. For the related free boundary problem of one-dimensional isentropic fluids with density-dependent viscosity (like $\mu(\rho) = c\rho^\theta$), see [9, 14, 17, 19, 20] and the references therein. For the spherically symmetric solutions of the Navier-Stokes equations with a free boundary, see [1, 2, 11, 13, 21] et al. Ducomet-Zlotnik [2, 21] studied the similar system with a solid core and without the nonlinear term $\frac{2}{n-1}u\partial_x \mu$. Also see Lions [8] and Vaigant-Kazhikhov [16] for multidimensional isentropic fluids. For the related stabilization rate estimates in the one-dimensional case, see [3, 10, 12, 15, 20] et al.

Main assumptions on $c_1$, $c_2$, $\theta$ and $\gamma$ can be stated as follows:

(A1) condition (1.17) or (1.18) holds;

(A2) $\theta \geq 0$. $c_1$ and $c_2$ satisfy that

$$c_1 > 0, \ 2c_1 + nc_2 > 0$$

and

$$[2c_1\alpha + c_2(2n - 2 + \alpha)]^2 - 4(2c_1 + c_2)[2c_1(n - 1) + c_2(n - 1)(n - 1 + \alpha)] < 0, \quad (1.26)$$

where $\alpha = \frac{3}{2} - n$.

Under the above assumptions (A1)-(A2), we will prove the existence of a global weak solution to the initial-boundary value problem (1.20)-(1.23) in the sense of the following definition.

**Definition 1.1.** A pair of functions $(\rho, u, r)(x, t)$ is called a global weak solution to the initial boundary value problem (1.20)-(1.23), if for any $T > 0$,

$$\rho, u \in L^\infty([0, M] \times [0, T]) \cap C^1([0, T]; L^2([0, M])), \quad r \in C^1([0, T]; L^\infty([0, M])),$$

$$(r^{n-2}u)_x, (r^{n-1})_x \in L^\infty([0, T]; L^{n-\frac{1}{2}}([0, M])),$$

and

$$(r^{n-1}u)_x \in L^\infty([0, M] \times [0, T]) \cap C^{1/2}([0, T]; L^2([0, M])).$$

Furthermore, the following equations hold:

$$\rho_t + \rho^2(r^{n-1}u)_x = 0, \quad \rho(x, 0) = \rho_0(x) \ a.e.$$ \quad (2.1)

$$r_t = u, \quad r^n(x, t) = n \int_0^x \rho^{-1}(y, t)dy, \quad r(x, 0) = r_0(x) \ a.e.$$ \quad (2.2)

and

$$\int_0^\infty \int_0^M [u \psi_t + (P - \rho(\lambda + 2\mu)(r^{n-1}u)_x)(r^{n-1}\psi)_x + 2(n - 1)\mu(r^{n-2}u\psi)_x - f(x, r, t)\psi] dx dt$$

$$= \int_0^\infty P_0(r^{n-1}\psi)(M, t) dt - \int_0^M u_0(x)\psi(x, 0) dx,$$

for any test function $\psi(x, t) \in C_0^\infty(\Omega)$ with $\Omega = \{(x, t) \ : \ 0 < x \leq M, \ t \geq 0\}$. 

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In what follows, we always use $C(C_1)$ to denote a generic positive constant depending only on the initial data, independent of the given time $T$.

We now state the main theorems in this paper. Let $\bar{\rho} = \min_{x \in [0,M]} \rho(x)$ and $\rho_\infty = \max_{x \in [0,M]} \rho(x)$.

**Theorem 1.1.** Under the conditions (1.10)-(1.12) and [A1]-[A2], there exists a positive constant $\epsilon_0 > 0$, such that if

$$
\|f_1, \Delta P\|_{L_\infty \cap L_1} + \|\Delta P\|_{L_2} + \|f_2\|_{L_2} \leq \epsilon_0, \tag{1.27}
$$

$$
\|\rho_0 - \rho_\infty\|_{L_\infty}^2 + B[\rho_0, u_0, r_0] \leq \epsilon_0^2, \tag{1.28}
$$

then the system (1.20)-(1.23) has a unique global weak solution $(\rho, u, r)$ satisfying

$$
\rho(x, t) \in \left[\frac{1}{2}\rho_0, \frac{3}{2}\rho_0\right], \quad r^n(x, t) \in [C^{-1}x, Cx], \tag{1.29}
$$

$$
\left\|\frac{u}{r}(\cdot, t)\right\|_{L_\infty} \leq C\left\|\partial_x(r^{n-1}u)(\cdot, t)\right\|_{L_\infty} \leq C\epsilon_0, \tag{1.30}
$$

$$
B[\rho, u, r] \leq C\epsilon_0^2, \tag{1.31}
$$

for all $t \geq 0$ and $x \in [0, M]$. Furthermore, we have

$$
\lim_{t \to +\infty} \int_0^M \left\{x^{2n-2+\alpha} u^2 + x^{2n-2+\alpha} \left[(\rho^0)_x - (\rho^\infty)_x\right]^2\right\} dx = 0,
$$

$$
\lim_{t \to +\infty} \|u(\cdot, t)\|_{L_\infty} + \|\rho(\cdot, t) - \rho_\infty(\cdot)\|_{L_\infty} + \|r(\cdot, t) - r_\infty(\cdot)\|_{L_\infty} = 0.
$$

**Remark 1.1.** In fact, assumption (1.26) is given a restriction on $\frac{\lambda}{\mu}$, i.e.

$$
-18 + 8n + 8n^2 - 8\sqrt{3}(n-1)\sqrt{4n-3} < \frac{\lambda}{\mu} < \frac{-18 + 8n + 8n^2 + 8\sqrt{3}(n-1)\sqrt{4n-3}}{9 - 12n + 4n^2}.
$$

If $n = 3$, we can choose $\frac{\lambda}{\mu} = \frac{1}{e_1} \in \left(\frac{2}{3}(13 - 8\sqrt{3}), \frac{2}{3}(13 + 8\sqrt{3})\right)$.

**Remark 1.2.** We can choose the constant $\epsilon_0$ as in (3.75).

The proof of the uniqueness part of Theorem 1.1 also shows that the continuous dependence of the solution on the initial data holds. We may state the following result without a proof.

**Theorem 1.2.** For each $i = 1, 2$, let $(\rho_i, u_i, r_i)$ be the solution to the system (1.20)-(1.23) with the initial data $(\rho_{0i}, u_{0i}, r_{0i})$, which satisfy regularity conditions (1.29)-(1.31). Then, we have

$$
\int_0^M [(u_1 - u_2)^2 + (\rho_1 - \rho_2)^2 + x^{-\frac{2}{3}}(r_1 - r_2)^2] dx
\leq C e^{Ct} \int_0^M [(u_{01} - u_{02})^2 + (\rho_{01} - \rho_{02})^2 + x^{-\frac{2}{3}}(r_{01} - r_{02})^2] dx,
$$

for all $t \geq 0$. 

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Theorem 1.3. Under the assumptions of Theorem 1.1 and
\[ f_1(t) + f_2(t) + |\Delta P(t)| + |(\Delta P)'(t)| \leq Ce^{-a_0 t}, \quad (1.32) \]
where \(a_0\) is a positive constant, then we have
\[ \int_0^M \left\{ n^{2n-2+\alpha}(\rho - \rho_\infty)^2_x + n^{2n-2+\alpha}[\partial_x(\rho^{1+\alpha} \rho_x) + r^{n-1}u_t] \right\} dx \leq Ce^{-at}, \]
\[ \left\| \frac{u}{r}, (r^{n-1}u)_x \right\|_{L^\infty} + \|\rho(\cdot,t) - \rho_\infty(\cdot)\|_{L^\infty} + \|r(\cdot,t) - r_\infty(\cdot)\|_{L^\infty} \leq Ce^{-at}, \]
for all \(t \geq 0\), where \(a\) is a positive constant.

Remark 1.3. Considering the general case that \((\mu, \lambda)(\cdot) \in C(\mathbb{R}_+) \cap W^{1,\infty}_{loc}(\mathbb{R}_+), \) under the assumptions (1.10)-(1.12), (A1) and
\[ \mu(\rho) > 0, \ 2\mu(\rho) + n\lambda(\rho) > 0, \]
\[ [2\mu \alpha + \lambda(2n - 2 + \alpha)]^2 - 4(2\mu + \lambda)[2\mu(\rho) + \lambda(\rho - 1)(n - 1 + \alpha)] < 0, \]
for all \(\rho \in \left[ \frac{1}{2r_\infty}, \frac{3}{2} \bar{\rho} \right],\) we can obtain the same results.

Remark 1.4. In this paper, we study the case of \(\gamma > 1\) and prove the main results in this case only, since the case of \(\gamma = 1\) can be discussed through the similar process. The main different is that (2.10) is replaced by
\[ S[V] = \int_0^M \left( A \ln V_x + P_\infty V_x + \int_1^V Gx(nh) \frac{2^{2n-2}}{\alpha} dh \right) dx, \]
when \(\gamma = 1\) and \(n = 2.\)

The rest of this paper is organized as follows. First, we obtain the existence and uniqueness of the solution to the stationary problem in Section 2. In Section 3, we will prove some a priori estimates which will be used to obtain global existence of the weak solutions. In Section 4, using the finite difference approximation and a priori estimates obtained in Section 3, we prove the existence part of Theorem 1.1. In Section 5, we will prove the uniqueness of the weak solution. In Section 6, we show that the solution of the free boundary problem tends to a stationary one, as \(t \to +\infty.\) In Section 7, we will obtain the stabilization rate estimates of exponential type on the solution by constructing some Lyapunov functionals.

2 The stationary problem

We start with a proof of the existence of a positive solution to the Lagrangian stationary problem. Zlotnik-Ducomet[21] studied the stationary problem with a solid core \(r \geq r_0 > 0.\) Using similar arguments as that in [21], we can obtain the following results for the stationary problem without a solid core.

Proposition 2.1. If
\[ \gamma > \frac{2n - 2}{n}, \quad (2.1) \]
or
\[ \gamma = \frac{2n - 2}{n} \quad \text{and} \quad Gn^{\frac{2n-2}{n}} M^{\frac{2}{n}} < 2A, \quad (2.2) \]
or

\[0 < \gamma < \frac{2n - 2}{n}\]

and

\[P_\infty + \frac{G}{2} n^{\frac{2-n}{n}} M^2 \delta_3^{\frac{2n-2}{n}} \leq A\delta_3^\gamma, \quad (2.3)\]

where \(\delta_3 = \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^\frac{2}{n}}\right)^{\frac{n-2-n}{2n-2}}\), then the Lagrangian stationary problem \((1.24)-(1.25)\) has a positive solution \(\rho_\infty \in W^{1,\beta}([0,M]),\) where \(\beta \in [1, \frac{n}{n-2})\) is a constant.

Proof. We introduce the nonlinear operator

\[I : K \to W^{1,\beta}([0,M]),\]

where \(K = \left\{ f \in C([0,M]) : \min_{x \in [0,M]} f(x) \geq \left(\frac{P_\infty}{A}\right)^\gamma \right\},\) by setting

\[I(f)(x) = \left(\frac{P_\infty + \int_x^M G \frac{y}{r_{\delta_1}^{\gamma}} dy}{A}\right)^\gamma\]

with \(r_{\delta}^n(x) = n \int_0^x f^{-1}(y) dy, x \in [0,M].\) We can restate the problem \((1.24)-(1.25)\) as the fixed-point problem

\[\rho_\infty = I(\rho_\infty). \quad (2.4)\]

For all \(f \in K_\delta = \{ f \in K : f \leq \delta \}\) with \(\delta > \left(\frac{P_\infty}{A}\right)^\gamma,\) we have

\[nx\delta^{-1} \leq r_\delta^n(x)\]

and

\[P_\infty \leq A(I(f))^\gamma \leq P_\infty + G \delta^{\frac{2n-2}{n}} n^{\frac{2n-2}{n}} \int_0^M x^{\frac{2-n}{n}} dx \]

\[= P_\infty + \frac{G}{2} \delta^{\frac{2n-2}{n}} n^{\frac{2n-2}{n}} M^\frac{2}{n}.\]

If \(\gamma > \frac{2n-2}{n},\) then \(I(K_\delta_1) \subset K_\delta_1,\) where \(\delta_1\) is a positive constant satisfying \(P_\infty + \frac{G}{2} \delta_1^{\frac{2n-2}{n}} n^{\frac{2n-2}{n}} M^\frac{2}{n} \leq A\delta_1^\gamma.\) And one can immediately verify that \(I\) is a compact operator on \(K_\delta_1.\) Since \(K_\delta_1\) is a convex closed bounded non-empty subset of \(C([0,M]),\) the problem \((2.4)\) has a solution \(\rho \in K_\delta_1\) by Schauder’s fixed point theorem.

If \(\gamma = \frac{2n-2}{n}\) and \(Gn^{\frac{2-n}{n}} M^\frac{2}{n} < 2A,\) then \(I(K_\delta_2) \subset K_\delta_2,\) where \(\delta_2\) is a positive constant satisfying \(P_\infty + \frac{G}{2} \delta_2^{\frac{2n-2}{n}} n^{\frac{2n-2}{n}} M^\frac{2}{n} \leq A\delta_2^\gamma.\)

If \(\gamma < \frac{2n-2}{n}\) and

\[P_\infty + \frac{G}{2} n^{\frac{2-n}{n}} M^\frac{2}{n} \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^\frac{2}{n}}\right)^{\frac{2n-2}{n-2-n\gamma}} \leq A \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^\frac{2}{n}}\right)^{\frac{n\gamma}{2n-2-n\gamma}}\]

then \(I(K_\delta_3) \subset K_\delta_3,\) where

\[\delta_3 = \left(\frac{A\gamma n^{\frac{2n-2}{n}}}{(n-1)GM^\frac{2}{n}}\right)^{\frac{n\gamma}{2n-2-n\gamma}}.\]

We can finish the proof of the theorem immediately. 

\[\Box\]
Letting \( V_\infty = \frac{r_0}{n} \), using the equality \( \frac{1}{r_\infty} = (V_\infty)_x \), one can eliminate the function \( \rho_\infty \) from the Lagrangian stationary problem \((1.23)-(1.25)\) and obtain an equivalent boundary-value problem for a non-linear second-order ODE:

\[
(A(V_\infty)_x^{-\gamma})_x = -Gx n^{\frac{2-2n}{n}} V_\infty^{\frac{2-2n}{n}}, \quad x \in (0, M),
\]

\[
V_\infty(0) = 0, \quad (V_\infty)_x(M) = \left( \frac{A}{P_\infty} \right)^{\frac{1}{\gamma}},
\]

for a function \( V_\infty \in C^1([0, M]) \) such that \( (V_\infty)_x > 0 \).

In accordance with the method of small perturbations, we replace \( V_\infty \) by \( V = V_\infty + W \) with small \( W \) and linearized the operator in the last problem:

\[
(A(V)_x^{-\gamma})_x + Gx n^{\frac{2-2n}{n}} V_\infty^{\frac{2-2n}{n}} W = (-\gamma A(V)_x^{-\gamma-1}W)_x + (2 - 2n)Gx(nV_\infty)^{\frac{2-3n}{n}} W + \ldots, \quad x \in (0, M),
\]

\[
V(0) = 0 + W(0), \quad A(V)_x^{-\gamma}|_{x=M} - P_\infty = -\gamma A\{ (V_\infty)_x^{-\gamma-1}W_x \}|_{x=M} + \ldots,
\]

up to the terms of the second order of smallness with respect to \( W \). We define the linearized operator

\[
L[W] = (-\gamma A\rho_\infty^{-\gamma+1}W_x)_x + (2 - 2n)Gx(nV_\infty)^{\frac{2-3n}{n}} W, \quad W \in K_0,
\]

where \( K_0 = \{ W \in C^1([0, M]) : W(0) = 0, W_x(M) = 0 \} \). It is easy to get

\[
(L[W], W) = \int_0^M \left( \gamma A(r_\infty)^{1+\gamma}W_x^2 - (2n - 2)Gx(nV_\infty)^{\frac{2-3n}{n}} W^2 \right) dx, \quad W \in K_0.
\]

Let

\[
J[W] := \int_0^M \left( \gamma A(r_\infty)^{1+\gamma}W_x^2 - (2n - 2)Gx(nV_\infty)^{\frac{2-3n}{n}} W^2 \right) dx,
\]

for \( W \in K_1 = \{ f \in C^1([0, M]) : f(0) = 0 \} \).

We say a stationary solution \( V_\infty \) is statically stable if

\[
J[W] \geq \delta_3 \left( \| W_x(x) \|_{L^2(0, M)}^2 + \| x^{-1}W(x) \|_{L^2(0, M)}^2 \right),
\]

for some \( \delta_3 > 0 \) and all \( W \in K_1 \).

Now, the static potential energy takes the following form:

\[
S[V] = \int_0^M \left( \frac{A}{\gamma - 1}(V_x)^{-\gamma} + P_\infty V_x + \int_1^V Gx(nh)^{\frac{2-2n}{n}} dh \right) dx.
\]

We call \( V \in K_2 = \{ f \in C^1([0, M]) : f(0) = 0, \min(f_x) > 0 \} \) a point of local quadratic minimum of \( S \) if

\[
S[V + W] - S[V] \geq \delta_4 \left( \| W_x(x) \|_{L^2(0, M)}^2 + \| x^{-1}W(x) \|_{L^2(0, M)}^2 \right),
\]

for all \( W \in K_1 \) and \( \| W \|_{C^1([0, M])} \leq \delta_5 \), for some \( \delta_4 > 0 \) and \( \delta_5 > 0 \).

We can clarify the variational sense of the definition of statically stable as follows.

**Proposition 2.2.** A function \( V \in K_2 \) is a point of local quadratic minimum of \( S \) if and only if \( V = V_\infty \) is a solution of the problem \((2.5)-(2.7)\) and satisfies static stability condition \((2.9)\).
Proof. Let $V \in K_2$, $W \in K_1$ and $\|W\|_{C^1([0,M])} = 1$. Using Taylor’s formula, we have

$$S[V + \epsilon W] = S[V] + \delta S[V](\epsilon W) + \frac{1}{2} \frac{d^2}{d\tau^2} S[V + \tau \epsilon W] |_{\tau = \bar{\tau}},$$

where

$$\delta S[V](\epsilon W) = \int_0^M \left( -A(V_x) - \gamma \epsilon W_x + P_{x\infty} \epsilon W_x + Gx(nV)^{2\gamma-2\alpha} \epsilon W \right) dx,$$

and

$$\frac{d^2}{d\tau^2} S[V + \tau \epsilon W] = \int_0^M \left( \gamma A(V_x + \tau \epsilon W_x)^{-1-\gamma} (\epsilon W_x)^2 \right. - (2n-2) Gx(n(V + \tau \epsilon W))^{\frac{2-3\alpha}{\gamma}} (\epsilon W)^2 \left. \right) dx,$$

for all $|\epsilon| < \frac{1}{\min V_x}$ and some $\bar{\tau} \in [0,1]$. If (2.11) holds, we have

$$\frac{d^2}{d\tau^2} S[V + \tau \epsilon W] \leq C \epsilon^2 \left( \|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])} \right)$$

and

$$C \epsilon^2 \left( \|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])} \right) + \epsilon \delta S[V](W) > 0,$$

for all $|\epsilon| \in (0, \min(\delta_5, \frac{1}{\min V_x}))$ and $\|W\|_{C^1([0,M])} = 1$. Thus, we obtain

$$\delta S[V](W) = 0,$$

i.e.

$$\int_0^M \left( -A(V_x) - \gamma \epsilon W_x + P_{x\infty} \epsilon W_x + Gx(nV)^{2\gamma-2\alpha} W \right) dx = 0,$$

for all $W \in K_1$ and $\|W\|_{C^1([0,M])} = 1$, that is, $V$ is a stationary point of $S$ and a solution of the problem (2.5)-(2.6). We can rewrite $\frac{d^2}{d\tau^2} S[V + \tau \epsilon W]$ as follows

$$\frac{d^2}{d\tau^2} S[V + \tau \epsilon W] = \delta^2 S[V](\epsilon W) + S_1,$$

where $\delta^2 S[V](\epsilon W) = \frac{d^2}{d\tau^2} S[V + \tau \epsilon W] |_{\tau = 0}$ and

$$|S_1| = \left| \frac{d^2}{d\tau^2} S[V + \tau \epsilon W] - \delta^2 S[V](\epsilon W) \right| \leq C \epsilon \left( \|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])} \right).$$

Thus, we obtain

$$\delta^2 S[V](\epsilon W) \geq (\delta_4 - C \epsilon) \left( \|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])} \right)$$

for all $\epsilon \in (0, \min(\delta_5, \frac{1}{\min V_x} + \frac{\delta_4}{2C}))$ and $\|W\|_{C^1([0,M])} = 1$. Moreover, we have

$$J[W] := \delta^2 S[V](W) \geq \frac{\delta_4}{2} \left( \|W_x(x)\|^2_{L^2([0,M])} + \|x^{-1} W(x)\|^2_{L^2([0,M])} \right), \quad (2.12)$$

for all $W \in K_1$.

If $V = V_{\infty}$ is a solution of the problem (2.5)-(2.6) and satisfies static stability condition (2.3), we can prove $V_{\infty}$ is a point of local quadratic minimum of $P$ easily.
Proposition 2.3. If \( V = V_\infty \) is a solution of the problem (2.5) - (2.6) and \( \gamma \geq \frac{2n-2}{n} \), then (2.9) and (2.11) hold.

Proof. From (A\( \rho_\infty \))\( x = -Gx(nV_\infty)^{\frac{2n-2}{n}} \), using integration by parts, we have

\[
J[W] = \int_0^M \left( \gamma A(\rho_\infty)^{1+\gamma}W_x^2 - (2n-2)Gx(nV_\infty)^{\frac{2n-2}{n}}W^2 \right) dx
\]

\[
= \int_0^M \left( \gamma A(\rho_\infty)^{1+\gamma}W_x^2 + (2n-2)A(\rho_\infty)^{\gamma}A(\rho_\infty)x(nV_\infty)^{-1}W_x^2 \right) dx
\]

\[
= \int_0^M \left( \gamma A(\rho_\infty)^{1+\gamma}W_x^2 - 2(2n-2)A(\rho_\infty)^{\gamma}A(\rho_\infty)x(nV_\infty)^{-1}WW_x \right.
\]

\[
+ \frac{2n-2}{n}A(\rho_\infty)^{\gamma}A(\rho_\infty)^{-1}V_\infty W^2 \bigg) \bigg) dx + (2n-2)P_\infty \left( \frac{W^2}{nV_\infty} \right) (M)
\]

\[
:= I_0[W] + (2n-2)P_\infty \left( \frac{W^2}{nV_\infty} \right) (M), \text{ for all } W \in K_0. \tag{2.13}
\]

If \( \gamma \geq \frac{2n-2}{n} \), we have

\[
I_0[W] \geq \int_0^M \frac{2n-2}{n}A(\rho_\infty)^{1+\gamma} \left( W_x - \frac{W}{\rho_\infty V_\infty} \right)^2 dx. \tag{2.14}
\]

If (2.9) not holds, we have for any integer \( m > 1 \), there exists \( W_m \in K_0 \) and \( \|W_m\|_{C^1([0,M])} = 1 \) such that

\[
J[W_m] < \frac{1}{m} \left( \|W_m\|_{C^2([0,M])}^2 + \|W^{-1}W_m\|_{L^2([0,M])} \right). \tag{2.15}
\]

Then, there is a subsequence \( m \to \infty \) for which

\[
W_m \to W \text{ in } C([0,M]),
\]

\[
(W_m)_x \to W_x \text{ in } L^2([0,M]).
\]

From (2.13)-(2.15), we have

\[
W_x = \frac{W}{\rho_\infty V_\infty}, \text{ in } (0,M),
\]

and \( W(0) = W(M) = 0 \). Thus, we obtain \( W \equiv 0 \). It is a contradiction.

Therefore, if \( \gamma \geq \frac{2n-2}{n} \), then (2.9) holds. From Proposition 2.1-2.2, we can obtain (2.11) immediately.

Now, we shall use the shooting method to prove the uniqueness of the solution.

Proposition 2.4. Under the assumptions (2.1)-(2.2), the Lagrangian stationary problem (1.24)- (1.25) has a unique positive solution \( \rho_\infty \).

Proof. We consider the Cauchy problem

\[
(A\rho_\infty)\rho = -Gx(nV_\infty)^{\frac{2n-2}{n}}, \quad (V_\infty)\rho = \rho_\infty^{-1}, \quad x \in (0,M), \tag{2.16}
\]

\[
\rho_\infty|_{x=0} = \sigma, \quad V_\infty|_{x=0} = 0, \tag{2.17}
\]
for the unknown functions \( \rho_\infty(\sigma, x) \) and \( V_\infty(\sigma, x) \), where \( \sigma > 0 \) is the shooting parameter. For each \( \sigma > 0 \), using similar arguments as that in Proposition 2.1, we can obtain the existence of the solution to this problem, satisfying

\[
\rho_\infty(\sigma, x) \in \left[ \frac{\sigma}{2}, \sigma \right], \quad V_\infty(\sigma, x) \in \left[ \frac{x}{\sigma}, \frac{2x}{\sigma} \right], \quad x \in [0, M_0],
\]

(2.18)

\[\rho_\infty \in W^{1, \beta}([0, M_0]), V_\infty \in C^1([0, M_0]),\]

(2.19)

where \( M_0 \) is a positive constant satisfying \( A_\gamma \sigma - \sigma \frac{2n-2}{n} \frac{2n-2}{n} M_0^2 \geq A \left( \frac{\sigma}{2} \right)^\gamma \) and \( M_0 \leq M \). If there exist two solutions \( (\rho_1, V_1) \) and \( (\rho_2, V_2) \) to this problem satisfying

\[
\rho_i \in W^{1, \beta}([0, M_i]), \quad x \in [0, M_i],
\]

(2.20)

where \( M_i \in (0, M] \), \( i = 1, 2 \). From (2.20), there exists a positive constant \( M_3 \in (0, \min\{M_1, M_2\}) \) such that

\[
\rho_i(x) \in \left[ \frac{\sigma}{2}, \sigma \right] \quad \text{and} \quad V_i(x) \in \left[ \frac{x}{\sigma}, \frac{2x}{\sigma} \right], \quad x \in [0, M_3], \quad i = 1, 2.
\]

Then, we have

\[
A\rho_1^\gamma - A\rho_2^\gamma = \int_0^x G \eta n \frac{2-n}{n} (V_2^\frac{2-n}{n} - V_1^\frac{2-n}{n}) dy \leq C \int_0^x y \frac{2-n}{n} \int_0^y |\rho_1^{-1} - \rho_2^{-1}|(z) dz dy,
\]

and

\[
\|\rho_1 - \rho_2\|_{L^\infty([0, \epsilon])} \leq C\|\rho_1 - \rho_2\|_{L^\infty([0, \epsilon])} \int_0^\epsilon y \frac{2-n}{n} dy \leq C_\sigma \epsilon^\frac{2}{n} \|\rho_1 - \rho_2\|_{L^\infty([0, \epsilon])},
\]

for all \( x, \epsilon \in (0, M_3) \). Choosing \( \epsilon < C_\sigma^{-\frac{2}{n}} \), we have

\[
\rho_1 = \rho_2, \quad \text{for all} \quad x \in [0, \epsilon].
\]

Considering the Cauchy problem

\[
(\rho_\infty^\gamma)_x = -G(\eta n V_\infty) \frac{2-n}{n}, \quad (V_\infty)_x = \rho_\infty^{-1}, \quad x \in \left( \frac{\epsilon}{2}, \epsilon \right), M_0,
\]

(2.21)

\[
\rho_\infty \big|_{x=\frac{\epsilon}{2}} = \rho_1(\sigma, \frac{\epsilon}{2}), \quad V_\infty \big|_{x=\frac{\epsilon}{2}} = \int_0^{\frac{\epsilon}{2}} \rho_1^{-1}(\sigma, y) dy,
\]

(2.22)

using the classical ODE theory, we have \( \rho_1(x) = \rho_2(x), \quad x \in \left( \frac{\epsilon}{2}, \min\{M_1, M_2\} \right) \). Thus, for each \( \sigma > 0 \), there exists a unique solution to the problem (2.16)-(2.17) satisfying \( \rho_\infty(x, \sigma) > 0 \) for \( x \in [0, M_\sigma] \), where either \( \rho_\infty \big|_{x=M_\sigma} = 0 \) and \( M_\sigma \in (0, M) \) or \( M_\sigma = M \).

Clearly, if \( \rho_\infty \) is a solution to the problem (1.21)-(1.25), then \( \rho_\infty \) satisfying (2.16)-(2.17) for some \( \sigma > 0 \). We will show that this can be possible only for one value of \( \sigma \). Using similar arguments as that in the above part and in §V.3, we obtain that \( (\partial_\sigma \rho_\infty^\gamma, \partial_\sigma V_\infty) \) is well defined and satisfies the linear Cauchy problem

\[
A(\partial_\sigma \rho_\infty^\gamma)_x = (2n - 2) G(\eta n V_\infty) \frac{2-3n}{n} \partial_\sigma V_\infty, \quad (\partial_\sigma V_\infty)_x = -\frac{1}{\gamma} \rho_\infty^{-\gamma-1} \partial_\sigma \rho_\infty^\gamma,
\]

(2.23)

where \( x \in [0, M_\sigma] \),

\[
\partial_\sigma \rho_\infty^\gamma \big|_{x=0} = 1, \quad \partial_\sigma V_\infty \big|_{x=0} = 0.
\]

(2.24)

It is easy to see that

\[
\partial_\sigma \rho_\infty^\gamma > 0, \quad (\partial_\sigma V_\infty)_x < 0, \quad \partial_\sigma V_\infty < 0
\]
hold on \([0, M_4]\), where either \(\partial_\sigma \rho^\infty |_{x=M_4} = 0\) and \(M_4 \in (0, M_\sigma)\) or \(M_4 = M_\sigma\). We claim that only \(M_4 = M_\sigma\) can occur.

Assume that \(M_4 \in (0, M_\sigma)\). Letting \(\phi = A \rho^\infty (\partial_\sigma V^\infty)_x + \frac{n}{2n-2} A \partial_\sigma \rho^\infty (V^\infty)_x\), from (2.16) and (2.23), we have

\[
\int_0^{M_4} \phi dx = \left. \left( A \rho^\infty \partial_\sigma V^\infty + \frac{n}{2n-2} A \partial_\sigma \rho^\infty V^\infty \right) \right|_0^{M_4}.
\]

By the estimates \(\rho^\infty(x, M_4) > 0\), \(\partial_\sigma \rho^\infty |_{x=M_4} = 0\), \(\partial_\sigma V^\infty |_{x=M_4} < 0\) and the initial condition (2.17) and (2.24), we get

\[
\int_0^{M_4} \phi dx < 0.
\]

On the other hand, from (2.16) and (2.23), we have

\[
\phi = A^{-1} \partial_\sigma \rho^\infty \left( \frac{n}{2n-2} - \frac{1}{\gamma} \right) \geq 0, \quad x \in (0, M_4).
\]

It is a contradiction.

Thus, we obtain

\[
\rho^\infty > 0, \quad \partial_\sigma \rho^\infty > 0, \quad x \in (0, M_\sigma),
\]

and \(M_\sigma\) is non-decreasing on \(\sigma \in (0, \infty)\). Therefore, for each fixed point \(x \in [0, \sup_{\sigma>0} M_\sigma]\), the function \(\rho^\infty(\sigma, x)\) is strictly increasing on \(\sigma > \left( \frac{P_\infty}{A} \right)^{\frac{1}{\gamma}}\), and satisfying \(A \rho^\infty |_{x=M_\sigma} = P_\infty\) for at most one value of \(\sigma\).

Using the properties of the transformation (1.19) and Propositions 2.1-2.4, we can obtain the following proposition immediately.

**Proposition 2.5.** Under the assumptions (2.1)-(2.2), the Eulerian stationary problem (1.14)-(1.16) has a unique positive solution \((\rho^\infty, l^\infty)\), satisfying

\[
0 < \rho^\infty \leq \rho^\infty(x) \leq \bar{\rho} < \infty, \quad (\rho^\infty)_r(x) < 0, \quad 0 < r < l^\infty \text{ with } l^\infty < +\infty.
\]

### 3 A priori estimates

From (1.11), (1.20) and (1.24), we could obtain the following lemma easily.

**Lemma 3.1.** Under the assumptions of Theorem 1.1, we have

\[
r_t = u, \quad (3.1)
\]

\[
A \rho^\infty(x) = P_\infty + \int_x^M \frac{G y}{r^2_{\infty}(y)} dy, \quad (3.2)
\]

\[
\left( \frac{P_\infty}{A} \right)^{\frac{1}{\gamma}} \leq \rho^\infty \leq \bar{\rho} < \infty, \quad r^\infty(x) \in [C^{-1} x, C x], \quad (3.3)
\]

\[
\frac{d}{dx} (A \rho^\infty(x)) = -G \frac{x}{r^2_{\infty}}. \quad (3.4)
\]

for all \(x \in [0, M]\).
Lemma 3.2. Under the assumptions of Theorem 1.1, we have
\[
\frac{d}{dt} \int_0^M \left( \frac{1}{2} u^2 + A \rho_i^{\gamma - 1} + \frac{P_\infty}{\rho} + \int_1^r G \frac{x}{s^{n-1}} ds \right)dx.
\]
and
\[
\begin{align*}
&+ \int_0^M \left\{ \left( \frac{2}{n} c_1 + c_2 \right) \rho_1^{1+\theta} \left[ (r^{n-1}u)_x \right]^2 + \frac{2(n-1)}{n} c_1 \rho_1^{1+\theta} \left( r^{n-1}u_x - \frac{u}{r^\theta} \right)^2 \right\}dx \\
&= - \int_0^M \Delta fudx - \Delta P(u^{n-1})(M,t). \tag{3.5}
\end{align*}
\]

Proof. Multiplying (1.20) by $u$, integrating the resulting equation over $[0, M]$, using integration by parts and the boundary conditions (1.22)-(1.23), we obtain
\[
\frac{d}{dt} \int_0^M \frac{1}{2} u^2 dx - \int_0^M A \rho_i^{\gamma - 1} \partial_x (r^{n-1}u) dx \\
+ \int_0^M \left\{ \left( \frac{2}{n} c_1 + c_2 \right) \rho_1^{1+\theta} \left[ (r^{n-1}u)_x \right]^2 - 2c_1(n-1) \rho_1^{\theta} \left( r^{n-2}u_x \right)^2 \right\} dx \\
= - P_i(u^{n-1})(M,t) - \int_0^M fudx. \tag{3.6}
\]

From (1.20), we have
\[
- \int_0^M A \rho_i^{\gamma - 1} \partial_x (r^{n-1}u) dx = \frac{d}{dt} \int_0^M \frac{A}{\gamma - 1} \rho^{\gamma - 1} dx, \tag{3.7}
\]
and
\[
\begin{align*}
- P_i(u^{n-1})(M,t) &= - P_\infty (r_i r^{n-1})(M,t) - \Delta P(u^{n-1})(M,t) \\
&= - \frac{d}{dt} \left\{ P_\infty \frac{r^n(M,t)}{n} \right\} - \Delta P(u^{n-1})(M,t) \\
&= - \frac{d}{dt} \int_0^M \frac{P_\infty}{\rho} dx - \Delta P(u^{n-1})(M,t), \tag{3.8}
\end{align*}
\]
and
\[
\begin{align*}
(2c_1 + c_2) \rho_1^{1+\theta} (r^{n-1}u)_x^2 - 2c_1(n-1) \rho_1^{\theta} (r^{n-2}u_x)^2 \\
= \left( \frac{2}{n} c_1 + c_2 \right) \rho_1^{1+\theta} (r^{n-1}u)_x^2 + \frac{2(n-1)}{n} c_1 \rho_1^{1+\theta} (r^{n-1}u_x - \frac{u}{r^\theta})^2. \tag{3.10}
\end{align*}
\]

From (3.6)-(3.10), we obtain (3.5) immediately. \hfill \Box

Claim 1: Under the assumptions of Theorem 1.1, there is a small positive constant $\epsilon_1$, such that, for any $T > 0$, if
\[
I(t) = \| \rho(\cdot, t) - \rho_\infty \|_{L^\infty} + \| u(\cdot, t) \|_{L^\infty} \leq 2 \epsilon_1, \ \forall \ t \in [0, T], \tag{3.11}
\]
then
\[
I(t) \leq \epsilon_1, \ \forall \ t \in [0, T].
\]

Using the results in Lemmas 3.7, 3.8, we can give the definition of $\epsilon_1$ in (3.7) and finish the proof of Claim 1.
Lemma 3.3. Under the assumptions of Theorem 1.1 and (2.11), if $\epsilon_1$ is small enough, we obtain

$$\rho(x,t) \in \left[ \frac{1}{2} \rho_\infty, \frac{3}{2} \rho_\infty \right],$$  \hspace{1cm} (3.12)$$

$$r^n(x,t) \in [C^{-1}x, Cx],$$  \hspace{1cm} (3.13)$$

$$\|u(\cdot,t)\|_{L^2} + \|\rho(\cdot,t) - \rho_\infty\|_{L^2} + \|r_\infty^{-n}(r^n - r_\infty^n)\|_{L^2} \leq C_1 \epsilon_0,$$  \hspace{1cm} (3.14)$$

$$\int_0^t \int_0^M \left( u^2 + r^{2n-2} u_x^2 + \frac{\rho^2}{r^2} \right) dx ds \leq C_1 \epsilon_0^2,$$  \hspace{1cm} (3.15)$$

for all $t \in [0, T]$ and $x \in [0, M]$.

Proof. From Lemma 3.1 and (3.11), we can easily obtain the estimate (3.12) when $2\epsilon_1 \leq \frac{1}{2} \rho$. From (1.20) and (3.12), we can obtain (3.13) immediately. From (2.10), (3.2) and (3.5), we have

$$\frac{d}{dt} \left( \int_0^M \frac{1}{2} u^2 dx + S[V] - S[V_\infty] \right)$$

$$+ \int_0^M \left\{ \left( \frac{2}{n} \epsilon_1 + c_2 \right) \rho^{1+\theta}(r^{n-1} u)_x^2 + \frac{2(n-1)}{n} c_1 \rho^{1+\theta}(r^{n-1} u_x - \frac{u}{r\rho})^2 \right\} dx$$

$$= - \int_0^M \Delta f u dx - \Delta P(u^{n-1})(M,t)$$  \hspace{1cm} (3.16)$$

where $V_\infty = \frac{r^n}{n}$ and $V = \frac{r^n}{n}$. From (1.27), (2.11), (3.12) and (3.13), and Proposition 2.3, we have

$$C^{-1} \int_0^M \left[ (\rho - \rho_\infty)^2 + \frac{(V - V_\infty)^2}{V_\infty^2} \right] dx$$

$$\leq S[V] - S[V_\infty] \leq C \int_0^M \left[ (\rho - \rho_\infty)^2 + \frac{(V - V_\infty)^2}{V_\infty^2} \right] dx,$$  \hspace{1cm} (3.17)$$

when $\|V - V_\infty\|_{C^1([0,M])} \leq C_2 \epsilon_1 \leq \delta_5$, and

$$|\Delta P(u^{n-1})(M,t)| \leq C_0 \left( \int_0^M |\partial_x (r^{n-1} u)|^2 dx \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3.18)$$

From (1.27)-(1.28), (3.12) and (3.16)-(3.18), we obtain

$$\int_0^M \left( u^2 + (\rho - \rho_\infty)^2 + r_\infty^{-2n}(r^n - r_\infty^n)^2 \right) dx$$

$$+ \int_0^t \int_0^M \left\{ (r^{n-1} u)_x^2 + (r^{n-1} u_x - \frac{u}{r\rho})^2 \right\} dx ds$$

$$\leq C_0^2 + C \int_0^t f_1(s) \|u(\cdot,s)\|_{L^2} ds,$$  \hspace{1cm} (3.19)$$

using Gronwall’s inequality and (1.27), we can obtain (3.14)-(3.15) immediately. \hfill \Box

Lemma 3.4. Under the assumptions of Lemma 3.3, if $\epsilon_0$ is small enough, we obtain

$$\int_0^t \int_0^M \left[ (\rho - \rho_\infty)^2 + r_\infty^{-2n}(r^n - r_\infty^n)^2 \right] dx ds \leq C_3 \epsilon_0^2,$$  \hspace{1cm} (3.20)$$

for all $t \in [0, T]$.  

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Proof. Multiplying (1.20) by \( r^{1-n}(\frac{r^n}{n} - \frac{r^n}{n}) \), integrating the resulting equation over \([0, M]\), using integration by parts and the boundary conditions (1.22)-(1.23), we obtain

\[
\int_0^M A(\frac{\gamma}{\rho} - \rho)(\rho^{-1} - \rho^{-1}_\infty) + Gx(r^{2-2n} - r^{2-2n}_\infty)(\frac{r^n}{n} - \frac{r^n}{n})dx \]

\[
= -\int_0^M \frac{u}{n-1}(\frac{r^n}{n} - \frac{r^n}{n})dx + \Delta P \left\{ \frac{r^n}{n} - \frac{r^n}{n} \right\} \bigg|_{x=M} \\
- \int_0^M \Delta \frac{r^{1-n}}{n}(r^n - r^n_\infty)dx + \int_0^M 2c_1(n-1)\rho^\theta \left( \frac{u}{r}(\frac{r^n}{n} - \frac{r^n}{n}) \right)_x dx \\
+ \int_0^M (2c_1 + c_2)\rho^{1+\theta} \partial_x(r^{n-1}u)(\rho^{-1}_\infty - \rho^{-1})dx \\
:= \sum_{i=1}^5 I_i. \tag{3.21}
\]

We can rewrite the left hand side of (3.21) as follows

**L.H.S of (3.21)**

\[
= \int_0^M \left[ \gamma \rho^{\gamma+1}_\infty(\rho^{-1} - \rho^{-1}_\infty)^2 - 2(n-2)Gx(r^{2-2n}_\infty)(\frac{r^n}{n} - \frac{r^n}{n})^2 \right] dx \\
+ \int_0^M g_1(\rho^{-1} - \rho^{-1}_\infty)^2 + g_2r^{2n}_\infty \left( \frac{r^n}{n} - \frac{r^n}{n}_\infty \right)^2 dx,
\]

where

\[
|g_1| = \left| \frac{A(\rho^{\gamma}_\infty - \rho^{\gamma})}{\rho^{-1} - \rho^{-1}_\infty} - \gamma A\rho^{1+\gamma}_\infty \right| \leq C_4 \epsilon_1
\]

and

\[
|g_2| = \left| Gx(2n(r^{2-2n}_\infty - r^{2-2n}_\infty))(\frac{r^n}{n} - \frac{r^n}{n}_\infty)^{-1} + (2n-2)Gx(2n(r^{2-2n}_\infty - r^{2-2n}_\infty)) \right| \leq C_4 \epsilon_1.
\]

From (2.20), we have

**L.H.S of (3.21)**

\[
\geq (2C_5 - C_4 \epsilon_1) \int_0^M \left[ (\rho^{-1} - \rho^{-1}_\infty)^2 + r^{2n}_\infty(\frac{r^n}{n} - \frac{r^n}{n}_\infty)^2 \right] dx \\
\geq C_5 \int_0^M \left[ (\rho^{-1} - \rho^{-1}_\infty)^2 + r^{2n}_\infty(\frac{r^n}{n} - \frac{r^n}{n}_\infty)^2 \right] dx, \tag{3.22}
\]

when \( C_4 \epsilon_1 \leq C_5 \).

From (3.11) and (3.12)-(3.13), using integration by parts, we can estimate \( I_i \) as follows.

\[
I_1 = -\frac{d}{dt} \int_0^M \frac{u}{r^{n-1}}(\frac{r^n}{n} - \frac{r^n}{n}_\infty)dx + \int_0^M u^2 \left( \frac{1}{n} + \frac{(n-1)r^n}{nr^n} \right)dx \\
\leq -\frac{d}{dt} \int_0^M \frac{u}{r^{n-1}}(\frac{r^n}{n} - \frac{r^n}{n}_\infty)dx + C \int_0^M u^2 dx, \tag{3.23}
\]

\[
I_2 = \Delta P \int_0^M (\rho^{-1} - \rho^{-1}_\infty)dx \leq \frac{C_5}{10} \int_0^M (\rho^{-1} - \rho^{-1}_\infty)^2 dx + C|\Delta P|^2, \tag{3.24}
\]

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\[ I_3 \leq \frac{C_5}{10} \int_0^M r^{-2n} \left( \frac{r^n}{n} - \frac{r^n}{n} \right)^2 dx + Cf_1^2, \]  
(3.25)

\[ I_4 \leq \frac{C_5}{10} \int_0^M \left[ (\rho^{-1} - \rho^{-1})^2 + r^{-2n} \left( \frac{r^n}{n} - \frac{r^n}{n} \right)^2 \right] dx 
+ C \int_0^M \left( [(r^{n-1}u)_x]^2 + \frac{u^2}{r^2} \right) dx \]  
(3.26)

and

\[ I_5 \leq \frac{C_5}{10} \int_0^M (\rho^{-1} - \rho^{-1})^2 dx + C \int_0^M (r^{n-1}u)_x^2 dx. \]  
(3.27)

From (3.21)-(3.27), we get

\[
\frac{d}{dt} \int_0^M \frac{u}{n(n-1)}(r^n - r^n) dx + C \int_0^M \left[ (\rho^{-1} - \rho^{-1})^2 + r^{-2n} \left( \frac{r^n}{n} - \frac{r^n}{n} \right)^2 \right] dx 
\leq C \int_0^M \left( r^{2n-2}u_x^2 + \frac{u^2}{r^2} \right) dx + C (|\Delta P|^2 + f_1^2). \]  
(3.28)

And from (3.12)-(3.15), we obtain (3.20) immediately.

From now on, we study the case of \( \theta > 0 \) and prove the main results in this case only, since the case of \( \theta = 0 \) can be discussed through the similar process.

**Lemma 3.5.** Under the assumptions of Lemma 3.3, if \( \epsilon_1 \) is small enough, we obtain

\[
\int_0^M [r^{2n-2}(\rho - \rho_\infty)_x^2](x,t) dx + \int_0^t \int_0^M [r^{2n-2}(\rho - \rho_\infty)_x^2](x,s) dx ds \leq C_6 \epsilon_0^2, \]  
(3.29)

for all \( t \in [0,T] \).

**Proof.** From (1.20), we have

\[
\partial_t H + \frac{A_7 \rho^{\gamma - \theta}}{2c_1 + c_2} H 
= \frac{A_7}{2c_1 + c_2} \rho^{\gamma - \theta} u + \left( \frac{2c_1 + c_2}{\theta} - 2c_1 \right) (n - 1) r^{n-2} u \rho^{\theta} - f(x,t,t) 
- \frac{(n - 1)(2c_1 + c_2)}{\theta} r^{n-2} u \rho^{\theta} - \frac{A_7 \rho^{\gamma - \theta} r^{n-1}}{\theta} (\rho_\infty)_x. \]  
(3.30)

where \( H = u + \frac{2c_1 + c_2}{\theta} r^{n-1} (\rho^{\theta} - \rho^{\theta}_\infty)_x \). Multiplying (3.30) by \( H \), integrating the resulting equation over \( [0,M] \), using the Cauchy-Schwarz inequality, we obtain

\[
\frac{d}{dt} \int_0^M H^2(x,t) dx + C_7 \int_0^M H^2(x,t) dx 
\leq C \int_0^M \left( |H\rho^{\gamma - \theta} u| + \frac{|u^2}{r^2} H^2 | + \frac{|u^2}{r^2} H + |\Delta H| \right) dx 
+ \int_0^M \left| G \frac{x}{r^{n-1}} + \frac{\rho^{\gamma - \theta} r^{n-1}}{\rho^{\theta}_\infty} (A\rho_\infty)_x \right| |H| dx + C \int_0^M |r^{n-2} u \rho^{\theta}_\infty H| dx
\]
\[ \leq C \int_0^M \left( u^2 + \frac{u^4}{r^2} + \frac{x^{2n-4}}{r_{\infty}^{2n-4}} u^2 \right) \, dx + \left( \frac{1}{4} + C_8 \epsilon_1 \right) C_7 \int_0^M H^2 \, dx \\
+ C \int_0^M \left| G \frac{x}{r^{n-1}} + \frac{\rho^{\gamma - \theta} r^{n-1}}{\rho_{\infty}^{\gamma - \theta}} (A \rho_{\infty}^{-\gamma}) \right|^2 \, dx + C f_1^2, \tag{3.31} \]

From (3.1) and (3.12), we have

\[ \int_0^M \left| G \frac{x}{r^{n-1}} + \frac{\rho^{\gamma - \theta} r^{n-1}}{\rho_{\infty}^{\gamma - \theta}} (A \rho_{\infty}^{-\gamma}) \right|^2 \, dx \]
\[ = C \int_0^M \left| G \frac{x}{r^{n-1}} - G \frac{x \rho^{\gamma - \theta} r^{n-1}}{\rho_{\infty}^{\gamma - \theta} r_{\infty}^{2n-2}} \right|^2 \, dx \]
\[ \leq C \int_0^M [(r - r_{\infty})^2 + (\rho - \rho_{\infty})^2] \, dx. \tag{3.32} \]

Then, if \( \epsilon_1 \leq 1 \) and \( C_8 \epsilon_1 \leq \frac{1}{4} \), from (3.31)-(3.32), we obtain

\[ \frac{d}{dt} \int_0^M H^2(x,t) \, dx + \frac{C_7}{2} \int_0^M H^2(x,t) \, dx \]
\[ \leq C \int_0^M (u^2 + (r - r_{\infty})^2 + (\rho - \rho_{\infty})^2) \, dx + C f_1^2. \tag{3.33} \]

From (3.12)-(3.15), (3.20) and (3.33), we obtain (3.29) immediately. \( \square \)

**Lemma 3.6.** Under the assumptions of **Lemma 3.3**, if \( \epsilon_1 \) is small enough, we obtain

\[ (\rho(M,t) - \rho_{\infty}(M))^2 + \int_0^t (\rho(M,s) - \rho_{\infty}(M))^2 \, ds \leq C_1 \epsilon_0^2, \tag{3.34} \]

\[ \int_0^M \int_0^M r^{\frac{1}{2} - m} (\rho - \rho_{\infty})^2 \, dx \, ds \leq C_{11} \epsilon_0^2, \tag{3.35} \]

\[ \int_0^M (r^{\frac{1}{2} - m} u^2)(x,t) \, dx + \int_0^t \int_0^M r^{\frac{1}{2} - m} \left( r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right) \, dx \, ds \leq C_{12} \epsilon_0^2, \tag{3.36} \]

\[ \int_0^M (r^{2n-2} - \frac{1}{2} - m) (\rho - \rho_{\infty})^2 \, dx + \int_0^t \int_0^M (r^{2n-2} - \frac{1}{2} - m) (\rho - \rho_{\infty})^2 \, dx \, ds \leq C_{15} \epsilon_0^2, \tag{3.37} \]

\[ \| \rho(\cdot,t) - \rho_{\infty}(\cdot) \|_{L^\infty} + \int_0^M \|(\rho - \rho_{\infty}) \|_{L^\infty} \, dx \leq C_{15} \epsilon_0, \tag{3.38} \]

\[ |r(x,t) - r_{\infty}(x)| \leq C_{16} \epsilon_0 x^\frac{1}{2}, \, x \in [0,M], \tag{3.39} \]

for all \( t \in [0,T] \) and \( m = 0, 1, \ldots, n - 1 \).

**Proof.** From (1.20) and the boundary condition (1.23), we have

\[ A \rho^\gamma (M,t) - P_t + \frac{(2c_1 + c_2)}{\theta} \partial_t (\rho^\theta)(M,t) = -2c_1(n-1) \left( \frac{\rho^\theta u}{r} \right)(M,t). \]

Multiplying the above equality by \( \rho^\theta(M,t) - \rho_{\infty}^\theta(M) \), we obtain

\[ \frac{2c_1 + c_2}{2} d \frac{d}{dt} (\rho^\theta - \rho_{\infty}^\theta)^2 \bigg|_{x=M} + (\rho^\theta(M,t) - \rho_{\infty}^\theta(M))(A \rho^\gamma(M,t) - P_{\infty}) \]

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by parts and the boundary conditions (1.22)-(1.23), we obtain
\[ m = \frac{2}{r} \frac{\partial}{\partial x}(\rho^\theta - \rho^\theta_\infty) \left|_{x=M} \right. + \Delta P(\rho^\theta(M,t) - \rho^\theta_\infty(M)). \]

Combining (3.12)-(3.13), using the Cauchy-Schwarz inequality, we get
\[ \frac{d}{dt}(\rho^\theta(M,t) - \rho^\theta_\infty(M))^2 + C^{-1}(\rho^\theta(M,t) - \rho^\theta_\infty(M))^2 \]
\[ \leq C|\Delta P|^2 + C(u^2 r^n)(M,t) = C|\Delta P|^2 + C \int_0^M \partial_x (u^2 r^n) dx \]
\[ \leq C|\Delta P|^2 + C \int_0^M (r^{2n-2} u_x^2 + \frac{u^2}{r^2}) dx. \quad (3.40) \]

Integrating the above inequality over \([0,t]\), using the estimates (3.12) and (3.15), we can obtain (3.34). From (3.14)-(3.15), (3.20) and (3.29), we know that the estimates (3.35)-(3.37) hold with \( m = 0 \).

**Claim 2:** If that (3.35)-(3.37) hold with \( m \leq k, k \in [0,n-2] \), then the estimates (3.35)-(3.37) hold with \( m = k + 1 \).

We could prove Claim 2 as follows. Let \( \alpha_k = \frac{1}{2} - k - 1 \). Using Hölder’s inequality, we have
\[ \int_0^M r^{\alpha_k}(\rho - \rho_\infty)^2 dx \]
\[ = \int_0^M r^{\alpha_k} \left( \rho(M,s) - \rho_\infty(M) \right) - \int_x^M \partial_x (\rho - \rho_\infty) dy \right)^2 dx \]
\[ \leq C(\rho(M,s) - \rho_\infty(M))^2 \]
\[ + C \int_0^M r^{\alpha_k} \int_x^M r^{2n-2+\alpha_k-1}(\rho - \rho_\infty)^2 dy \int_x^M r^{2n-2+\alpha_k-1} dy dx \]
\[ \leq C(\rho(M,s) - \rho_\infty(M))^2 + C \int_0^M r^{2n-2+\alpha_k-1}(\rho - \rho_\infty)^2 dx. \quad (3.41) \]

From (3.34), (3.37) \((m = k)\) and (3.41), we obtain (3.35) \((m = k + 1)\).

Multiplying (3.20) by \( ur^{\alpha_k} \), integrating the resulting equation over \([0,M]\), using integration by parts and the boundary conditions (1.22)-(1.23), we obtain
\[ \frac{d}{dt} \int_0^M \frac{1}{2} r^{\alpha_k} u^2 dx - \int_0^M \frac{\alpha_k}{2} r^{\alpha_k-1} u^3 dx \]
\[ = - \int_0^M [(2c_1 + c_2) \rho^{1+\theta} (r^{n-1} u) (r^{n-1+\alpha_k} u) x - 2c_1(n-1) \rho^{\theta} (r^{n-2+\alpha_k} u^2)_x] dx \]
\[ + \int_0^M A(\rho^n - \rho^n_\infty)(r^{n-1+\alpha_k} u)_x dx - \int_0^M \Delta f r^{\alpha_k} dx \]
\[ - \Delta P(ur^{n-1+\alpha_k})(M,t) + \int_0^M (r^{n-1+\alpha_k} u)_x \int_x^M \left( \frac{Gy}{r^{2n-2}} - \frac{Gy}{r^{2n-2}} \right) dy dx \]
\[ = \sum_{i=1}^5 L_i. \quad (3.42) \]

We can estimate \( L_1 \) as follows.
\[ - L_1 = \int_0^M \left\{ (2c_1 + c_2) r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 \right\} \]
+ [2\alpha_k c_1 + c_2(2n - 2 + \alpha_k)] \rho^{\theta - r_{\alpha_k u u_x}}
+ [c_2(n - 1) + c_2(n - 1)(n - 1 + \alpha_k)] \rho^{\theta - r_{\alpha_k u^2}} dx.

(3.43)

Since

\[ [2\alpha c_1 + c_2(2n - 2 + \alpha)]^2 - 4(2c_1 + c_2)[2c_1(n - 1) + c_2(n - 1)(n - 1 + \alpha)] < 0, \]

where \( \alpha = \frac{\gamma}{2} - n \) and

\[ [c_2(2n - 2)]^2 - 4(2c_1 + c_2)[2c_1(n - 1) + c_2(n - 1)^2] < 0, \]

we have

\[ [2\alpha_k c_1 + c_2(2n - 2 + \alpha)]^2 - 4(2c_1 + c_2)[2c_1(n - 1) + c_2(n - 1)(n - 1 + \alpha_k)] < 0. \]

Then there exists a positive constant \( C_{13} \) such that

\[ -L_1 \geq C_{13} \int_0^M \left( r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta - r_{\alpha_k u^2}} u^2 \right) dx. \]

(3.44)

From (3.12)-(3.13), using the Cauchy-Schwarz inequality, we obtain

\[
L_2 = \int_0^M A(\rho^\gamma - \rho^\gamma_\infty) r^{n+\alpha_k} u_x dx + \int_0^M A(n - 1 + \alpha_k) r^{\alpha_k - 1} \rho^\gamma - \rho^\gamma_\infty \rho^\gamma dx
\leq \frac{C_{13}}{8} \int_0^M \left( r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta - r_{\alpha_k u^2}} u^2 \right) dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty) dx,
\]

(3.45)

\[
L_3 \leq \frac{1}{8} C_{13} \int_0^M \rho^{\theta - r_{\alpha_k u^2}} u^2 dx + C f_1^2,
\]

(3.46)

\[
L_4 = -\int_0^M \Delta P \partial_x (r^{n+\alpha_k} u) dx
\leq \frac{1}{8} C_{13} \int_0^M \left( r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta - r_{\alpha_k u^2}} u^2 \right) dx + C |\Delta P|^2
\]

(3.47)

and

\[
L_5 \leq \frac{1}{8} C_{13} \int_0^M \left( r^{2n-2+\alpha_k} \rho^{1+\theta} u_x^2 + \rho^{\theta - r_{\alpha_k u^2}} u^2 \right) dx
\leq \frac{1}{8} C_{13} \int_0^M \left( \frac{Gy}{r^{2n-2_\infty}} + \frac{Gy}{r^{2n-2}} \right) dy \int_0^M \left( r^{\alpha_k} u^2 \right) dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty) dx.
\]

(3.48)

From (3.11), (3.12)-(3.13) and (3.42)-(3.48), we obtain

\[
\frac{d}{dt} \int_0^M (r^{\alpha_k} u_x^2)(x,t) dx + 2C_{14}^{-1} \int_0^M r^{\alpha_k} \left( r^{2n-2} u_x^2 + u^2 \right) dx
\leq C (f_1^2 + |\Delta P|^2) + C_{14} \int_0^M r^{\alpha_k} \left( \frac{u_x^2}{r^2} \right) dx + C \int_0^M r^{\alpha_k} (\rho - \rho_\infty) dx.
\]

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When $C_{14}^{2} \epsilon_{1} \leq \frac{1}{4}$, using the estimate (3.35) $(m = k + 1)$, we can get

\[
\frac{d}{dt} \int_{0}^{M} (r^{\alpha_k}u^2)(x,t)dx + C_{14}^{-1} \int_{0}^{M} r^{\alpha_k} \left( r^{n-2}u_x^2 + \frac{u^2}{r^2} \right) dx \leq C (f_1^2 + |\Delta P|^2) + C \int_{0}^{M} r^{\alpha_k} (\rho - \rho_{\infty})^2 dx
\]

(3.49)

and (3.36) $(m = k + 1)$ holds.

From (1.20), we have

\[
H_1 = \frac{A_{\gamma}}{2c_1 + c_2} \rho^{-\theta} H_1
\]

(3.50)

where $H_1 = r^{\frac{\alpha_k}{2}} \rho + \frac{2c_1 + c_2}{\theta} r^{\frac{n-1}{2}} (\rho - \rho_{\infty})$. Multiplying (3.50) by $H_1$, integrating the resulting equation over $[0,M]$, using the Cauchy-Schwarz inequality, we obtain

\[
\frac{d}{dt} \int_{0}^{M} H_1^2(x,t)dx + C_{18} \int_{0}^{M} H_1^2(x,t)dx \leq C \int_{0}^{M} \left( |H_1\rho^{-\theta} r^{\frac{\alpha_k}{2}} u| + \left| \frac{u}{r} \right| H_1^2 + r^{\frac{\alpha_k}{2}} \rho^{-\theta} H_1^2 \rho^{-n+1} \rho^{-\theta} |\Delta f H_1| \right) dx
\]

\[
+ C \int_{0}^{M} \left( r^{n-2}u_x^2 + \rho^{-n} (A_\gamma \rho_{\infty}) \right) |H_1| dx
\]

\[
\leq C \int_{0}^{M} \left( r^{\alpha_k} u^2 + r^{\alpha_k} u^4 + \frac{2^{n-4} + \alpha_k}{r^{4n-4}} u^2 \right) dx + C f_1^2
\]

\[
+ \left( \frac{1}{4} + C_{17} \epsilon_1 \right) C_{18} \int_{0}^{M} H_1^2 dx + C \int_{0}^{M} r^{\alpha_k} \left( \frac{G \rho^{-n+1} + \rho^{-\theta} \rho_{\infty}^{-2}}{\rho_{\infty}} \right)^2 dx.
\]

(3.51)

If $\epsilon_1 \leq 1$, $C_{17} \epsilon_1 \leq \frac{1}{4}$, using the estimates (3.11) and (3.12)-(3.13), we have

\[
\frac{d}{dt} \int_{0}^{M} H_1^2(x,t)dx + C_{18} \int_{0}^{M} H_1^2(x,t)dx \leq C \int_{0}^{M} \left[ r^{\alpha_k} u^2 + r^{\alpha_k} (\rho - \rho_{\infty})^2 + r^{\alpha_k} (\rho - \rho_{\infty})^2 \right] dx + C f_1^2
\]

(3.52)

And from (3.35)-(3.36) $(m = k + 1)$, we have

\[
\int_{0}^{M} H_1^2(x,t)dx + \int_{t}^{\infty} \int_{0}^{M} H_1^2(x,s)dxds \leq C t_0^2.
\]

From (3.12) and (3.35)-(3.36) $(m = k + 1)$, we obtain (3.37) $(m = k + 1)$ immediately and finish the proof of Claim 2.
From Claim 2, we obtain that the estimates (3.35)–(3.37) \((m = 0, \ldots, n - 1)\) hold. From (3.13) and (3.37), using Hölder’s inequality, we obtain

\[
\int_0^M |(\rho - \rho_\infty)_x| dx \leq \left( \int_0^M r^{2n-2+\alpha}(\rho - \rho_\infty)_x^2 dx \right)^{\frac{1}{2}} \left( \int_0^M r^{-2n+2-\alpha} dx \right)^{\frac{1}{2}} \leq C\epsilon_0. \tag{3.53}
\]

From (3.13)–(3.14) and (3.53), using Sobolev’s embedding Theorem, we could obtain (3.38)–(3.39) immediately.

**Lemma 3.7.** Under the assumptions of Lemma 3.3, if \(\epsilon_1\) is small enough, we obtain

\[
\int_0^M \left( \frac{u^2}{r^2} + r^{2n-2}u_x^2 \right)(x,t) dx + \int_0^t \int_0^M u_t^2(x,s) dx ds \leq C_9\epsilon_0^2 (1 + \|(r^{n-1}u)_x\|_{L^\infty_t}), \tag{3.54}
\]

for all \(t \in [0,T]\).

**Proof.** Multiplying (1.20) by \(u_t\), integrating the resulting equation over \([0,M]\), using integration by parts and the boundary conditions (1.22)–(1.23), we obtain

\[
\int_0^M u_t^2 dx + \int_0^M (2c_1 + c_2)\rho^{1+\theta}(r^{n-1}u)_x(r^{n-1}u_t)_x dx
\]

\[
= \int_0^M A\rho^\gamma(r^{n-1}u)_x dx - P_t(r^{n-1}u_t)(M,t)
\]

\[
+ \int_0^M 2c_1(n-1)\rho^\theta(r^{n-2}uu)_x dx - \int_0^M f u_t dx
\]

\[
= \sum_{i=1}^4 N_i. \tag{3.55}
\]

From (3.11), (3.12) and (3.15), using the Cauchy-Schwarz inequality, we obtain

\[
\int_0^M (2c_1 + c_2)\rho^{1+\theta}(r^{n-1}u)_x(r^{n-1}u_t)_x dx
\]

\[
= \frac{d}{dt} \int_0^M \frac{2c_1 + c_2}{2} \rho^{1+\theta}[(r^{n-1}u)_x]^2 dx
\]

\[
- \int_0^M (2c_1 + c_2)(n-1)\rho^{1+\theta}(r^{n-1}u)_x(r^{n-2}u)_x dx
\]

\[
+ \int_0^M \frac{2c_1 + c_2}{2} (1+\theta)\rho^{2+\theta}[(r^{n-1}u)_x]^3 dx
\]

\[
\geq \frac{d}{dt} \int_0^M \frac{2c_1 + c_2}{2} \rho^{1+\theta}[(r^{n-1}u)_x]^2 dx
\]

\[
- C(\|(r^{n-1}u)_x\|_{L^2_x}^2 + \left\| \frac{u}{r} \right\|^2_{L^2_x} ) (1 + \|(r^{n-1}u)_x\|_{L^\infty_t}), \tag{3.56}
\]

\[
N_1 = \frac{d}{dt} \int_0^M A\rho^\gamma(r^{n-1}u)_x dx + \int_0^M A\rho^{\gamma+1}[(r^{n-1}u)_x]^2 dx
\]

\[
- \int_0^M 2A(n-1)\rho^\gamma \frac{M}{r}(r^{n-1}u)_x dx + \int_0^M An(n-1)\rho^{\gamma-1} \frac{u^2}{r^2} dx
\]

\[
\geq \frac{d}{dt} \int_0^M A\rho^\gamma(r^{n-1}u)_x dx + \int_0^M A\rho^{\gamma+1}[(r^{n-1}u)_x]^2 dx
\]

\[
- \int_0^M 2A(n-1)\rho^\gamma \frac{M}{r}(r^{n-1}u)_x dx + \int_0^M An(n-1)\rho^{\gamma-1} \frac{u^2}{r^2} dx
\]
\begin{align}
\frac{d}{dt} \int_0^M A \rho^\gamma (r^{n-1} u)_x dx + C(\| (r^{n-1} u)_x \|^2_{L^2_x} + \| u \|^2_t + L^2_x), \quad (3.57)
\end{align}

\begin{align}
N_2 &= -\frac{d}{dt} \int_0^M (P_\infty + \Delta P(t))(r^{n-1} u)_x dx \\
&\quad + \int_0^M (n-1)(P_\infty + \Delta P)(r^{n-2} u^2)_x dx + (\Delta P)' \int_0^M (r^{n-1} u)_x dy ds \\
&\leq -\frac{d}{dt} \int_0^M (P_\infty + \Delta P(t))(r^{n-1} u)_x dx \\
&\quad + C \left( \| (r^{n-1} u)_x \|^2_{L^2_x} + \| u \|^2_t + \| (\Delta P)' \|^2 \right), \quad (3.58)
\end{align}

\begin{align}
N_3 &= \frac{d}{dt} \int_0^M c_1(n-1)\rho^\theta (r^{n-2} u^2)_x dx + \int_0^M 2\theta c_1(n-1)\rho^{\theta+1} u \int (r^{n-1} u)_x^2 dx \\
&\quad - \int_0^M \theta c_1 n(n-1)\rho^\theta \frac{u^2}{r^2} (r^{n-1} u)_x dx + \int_0^M 2n c_1(n-1)(n-2)\rho^{\theta-1} u^3 \frac{3}{r^3} dx \\
&\quad - \int_0^M 3c_1(n-1)(n-2)\rho^\theta \frac{u^2}{r^2} (r^{n-1} u)_x dx \\
&\leq \frac{d}{dt} \int_0^M c_1(n-1)\rho^\theta (r^{n-2} u^2)_x dx + C(\| (r^{n-1} u)_x \|^2_{L^2_x} + \| u \|^2_t) \quad (3.59)
\end{align}

and
\begin{align}
N_4 &\leq -\frac{d}{dt} \int_0^M G \frac{x u}{r^{n-1}} dx + \int_0^M (1-n)G x r^{-n} u^2 dx + \frac{1}{2} \int_0^M u_t^2 dx + C f_1^2. \quad (3.60)
\end{align}

From (3.55)-(3.60), using the fact that
\begin{align}
\int_0^M \left\{ \frac{1}{2}(2c_1 + c_2)\rho^{1+\theta} [(r^{n-1} u)_x]^2 - c_1(n-1)\rho^\theta (r^{n-2} u^2)_x \right\} dx \\
= \int_0^M \left\{ \frac{1}{2} \left( \frac{2}{n} c_1 + c_2 \right) \rho^{1+\theta} (r^{n-1} u)_x^2 + \left( \frac{n-1}{n} - c_1 \rho \right) (r^{n-1} u_x - \frac{u}{r \rho})^2 \right\} dx,
\end{align}
we have
\begin{align}
\int_0^M \frac{1}{2} u_t^2 dx \\
+ \frac{d}{dt} \int_0^M \left\{ \frac{1}{2} \left( \frac{2}{n} c_1 + c_2 \right) \rho^{1+\theta} (r^{n-1} u)_x^2 + \left( \frac{n-1}{n} - c_1 \rho \right) (r^{n-1} u_x - \frac{u}{r \rho})^2 \right\} dx \\
\leq \frac{d}{dt} \left\{ \int_0^M \left[ (A \rho^\gamma - A \rho_\infty - \Delta P)(r^{n-1} u)_x + (r^{n-1} u)_x \int_x^M \left( \frac{G y}{r_{2n-2}^{2n-2}} \frac{G y}{r_{2n-2}^{2n-2}} \right) dy \right] dx \right\} \\
+ C(1 + \| (r^{n-1} u)_x \|_{L^\infty_x}) (\| (r^{n-1} u)_x \|_{L^2_x} + \| u \|^2_t + f_1^2 + \| (\Delta P)' \|^2). \quad (3.61)
\end{align}

Integrating (3.61) over [0, t], using the estimates (3.13)-(3.15) and the Cauchy-Schwarz inequality, we can obtain (3.54). \qed
Lemma 3.8. Under the assumptions of Lemma 3.3, we obtain
\[
\int_0^M (r^\alpha u_t^2)(x,t)dx + \int_0^t \int_0^M (r^{2n-2+\alpha} u_{xt}^2 + r^{\alpha-2} u_t^2) dxds \leq C_{19} \epsilon_0^2, \tag{3.62}
\]
\[
\left\| \frac{u}{t} \right\|_{L^\infty} + \left\| (r^{n-1} u)_x \right\|_{L^\infty} \leq C_{20} \epsilon_0, \tag{3.63}
\]
\[
\int_0^M \left( \frac{u^2}{r^2} + r^{2n-2} u_x^2 \right)(x,t)dx + \int_0^t \int_0^M u_t^2(x,s)dxds \leq C_{21} \epsilon_0^2, \tag{3.64}
\]
for all \( t \in [0,T] \), where \( \alpha = \frac{3}{2} - n \).

Proof. We differentiate the equation (1.20)\_2 with respect to \( t \), multiply it by \( u_t r^\alpha \) and integrate it over \([0,M]\), using the boundary conditions (1.22)-(1.23), then derive
\[
\frac{d}{dt} \int_0^M \frac{1}{2} r^\alpha u_t^2 dx - \frac{\alpha}{2} \int_0^M r^{\alpha-1} u u_t dx = - \int_0^M \left[ (2c_1 + c_2)\rho^{1+\theta}(r^{n-1} u)_x - A \rho^{\gamma} + P_\infty - 2c_1(n-1)\rho^{\theta} \frac{u}{r} \right]
\times((n-1)r^{n+2+\alpha} u u_t)_x dx - \int_0^M \partial_t \left[(2c_1 + c_2)\rho^{1+\theta}(r^{n-1} u)_x - A \rho^{\gamma} + A \rho^{\gamma} - 2c_1(n-1)\rho^{\theta} \frac{u}{r} \right] (r^{n+\alpha} u_t)_x dx
+ \int_0^M 2c_1(n-1)\rho^{\theta} \partial_t (r^{n+\alpha} u_t) xdxds - \int_0^M f r^\alpha u_t dx
\quad - \left[ (n-1)\Delta P(r^{n+2+\alpha} u u_t)(M,t) + (\Delta P)'(r^{n+\alpha} u_t)(M,t) \right]_0^M.
\]
\[
=: J_1 + J_2 + J_3 + J_4 + J_5. \tag{3.65}
\]
From (3.12), (3.35) and (3.36), using the Cauchy-Schwarz inequality, we obtain
\[
J_1 \leq \epsilon \int_0^M \left[ r^{\alpha-2} u_t^2 + r^{2n-2+\alpha} u_{xt}^2 \right] dx
+ C_{\epsilon} (1 + \|(r^{n-1} u)_x\|_{L^\infty}^2) \int_0^M \left[ r^{2n-2+\alpha} u_x^2 + r^{\alpha-2} u^2 + r^{\alpha}(\rho - \rho_\infty)^2 \right] dx. \tag{3.66}
\]
From (3.12)-(3.13), using the same argument in the proof of (3.44) and the Cauchy-Schwarz inequality, we get
\[
J_2 + J_3
\]
\[
= - \int_0^M \left[ (2c_1 + c_2)\rho^{1+\theta}(r^{n-1} u)_x (r^{n+\alpha} u_t)_x + 2c_1(n-1)\rho^{\theta}(r^{n+\alpha} u_t^2)_x \right] dx
+ \int_0^M \left\{ (2c_1 + c_2)(1 + \theta)\rho^{\theta+2}[(r^{n-1} u)_x]^2 - (n-1)(2c_1 + c_2)\rho^{1+\theta}(r^{n-2} u^2)_x \right.
\left. - \gamma \rho^{\gamma+1}(r^{n-1} u)_x - 2c_1(n-1)\rho^{\theta+1}(r^{n-1} u)_x \frac{u}{r} - 2c_1(n-1)\rho^{\theta} \frac{u^2}{r^2} \right\}
\times \left[ (n-1 + \alpha) r^{\alpha} u_t + r^{n-1+\alpha} u_{xt} \right] dx
+ 2c_1(n-1) \int_0^M \left\{ (n-1)r^{n+\alpha} u \rho^{\theta} \left( \frac{u}{r} \right)_x \right. u_t
\]
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Using the estimates (3.11), (3.12)-(3.13), (3.35)-(3.37) and (3.71), conclude that

\[-\theta r^{n-1+\alpha}\rho^{\theta+1}(r^{n-1}u)_{x} \left( \frac{u}{r} \right)_{x} u_{t} - r^{n-1+\alpha}\rho^{\theta} \left( \frac{u^{2}}{r^{2}} \right)_{x} u_{t} \] 

\[ \leq - (C_{22} - \epsilon) \int_{0}^{M} (r^{2n-2+\alpha} u_{x t x} + r^{\alpha-2} u_{t}^{2}) dx \]

\[ + C_{\epsilon} (1 + \|(r^{n-1} u)_{x} \|_{L_{\infty}^{2}}) \int_{0}^{M} \left[ r^{2n-2+\alpha} u_{x}^{2} + r^{\alpha-2} u_{x}^{2} + r^{\alpha}(\rho - \rho_{\infty})^{2} \right] dx, \quad (3.67) \]

\[ J_{4} \leq \epsilon \int_{0}^{M} r^{\alpha-2} u_{t}^{2} dx + C_{\epsilon} \int_{0}^{M} ((1-n)G x r^{-n} u + \partial_{i} \Delta f u + \partial_{x} \Delta f) r^{2+\alpha} dx \]

\[ \leq \epsilon \int_{0}^{M} r^{\alpha-2} u_{t}^{2} dx + C_{\epsilon} (f_{2}^{2} + \int_{0}^{M} r^{2+\alpha} u_{x}^{2} dx) \quad (3.68) \]

and

\[ J_{5} = - \int_{0}^{M} [(\Delta P)'(r^{n-1+\alpha} u_{t})_{x} + (n-1)\Delta P(r^{n-2+\alpha} u u_{t})_{x}] dx \]

\[ \leq \epsilon \int_{0}^{M} (r^{\alpha-2} u_{t}^{2} + r^{2n-2+\alpha} u_{x t}^{2}) dx + C_{\epsilon} (|\Delta P|^{2} + |(\Delta P)'|^{2}). \quad (3.69) \]

Let \( \epsilon = \frac{1}{8} C_{22} \), from (3.65)-(3.69), we have

\[ \frac{d}{dt} \int_{0}^{M} r^{\alpha} u_{t}^{2} dx + \frac{C_{22}}{4} \int_{0}^{M} (r^{2n-2+\alpha} u_{x t}^{2} + r^{\alpha-2} u_{t}^{2}) dx \]

\[ \leq C(1 + \|(r^{n-1} u)_{x} \|_{L_{\infty}^{2}}) \int_{0}^{M} \left[ r^{2n-2+\alpha} u_{x}^{2} + r^{\alpha-2} u_{x}^{2} + r^{\alpha}(\rho - \rho_{\infty})^{2} \right] dx \]

\[ + C (f_{2}^{2} + |\Delta P|^{2} + |(\Delta P)'|^{2}) \]

\[ (3.70) \]

and

\[ \int_{0}^{M} (r^{\alpha} u_{t}^{2})_{x} (x, t) dx + \int_{0}^{t} \int_{0}^{M} (r^{2n-2+\alpha} u_{x t}^{2} + r^{\alpha-2} u_{t}^{2}) (x, s) dx ds \]

\[ \leq C_{\epsilon}^{2} (1 + \|(r^{n-1} u)_{x} \|_{L_{\infty}^{2}}). \quad (3.71) \]

From the equation (1.20), we have

\[(2c_{1} + c_{2}) r^{n-1} \partial_{x} (\rho^{1+\theta} \partial_{x} (r^{n-1} u)) = u_{t} + Ar^{n-1}(\rho^{\gamma})_{x} + 2c_{1} (n-1) r^{n-2} u (\rho^{\theta})_{x} + f, \]

and using the estimates (3.11), (3.12)-(3.13), (3.35)-(3.37) and (3.71), conclude that

\[ \int_{0}^{M} r^{2n-2+\alpha} \partial_{x} (\rho^{1+\theta} \partial_{x} (r^{n-1} u))^{2} dx \leq C_{\epsilon}^{2} (1 + \|(r^{n-1} u)_{x} \|_{L_{\infty}^{2}}), \]

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and
\[ \int_0^M |\partial_x(\rho^{1+\theta}\partial_x (r^{n-1}u))|dx \leq C\epsilon_0(1 + \|(r^{n-1}u)_x\|_{L^\infty_x}), \quad (3.72) \]
for all \( t \in [0,T] \). From (3.12), (3.54) and (3.72), using Sobolev’s embedding theorem \( W^{1,1} \to L^\infty \), we can obtain
\[ \|\partial_x(r^{n-1}u)\|_{L^\infty_x} \leq C_{24}\epsilon_0(1 + \|(r^{n-1}u)_x\|_{L^\infty_x}). \quad (3.73) \]
If \( C_{24}\epsilon_0 \leq \frac{\epsilon}{2} \), from (3.54), (3.71) and (3.73), we can get (3.62)- (3.64) immediately.

Now, we can let
\[ \epsilon_1 = (C_{16} + C_{20})\epsilon_0. \quad (3.74) \]
If \( (1 + \frac{4}{\rho} + \frac{C_2}{\delta_5} + \frac{C_4}{C_5} + 4C_8 + 2C_{14} + 4C_{17} + 2C_{23})\epsilon_1 + 2C_{24}\epsilon_0 \leq 1 \), using the results in Lemmas 3.3-3.8, we finish the proof of the Claim 1. Thus, we can let \( \epsilon_0 \) be a positive constant satisfying
\[ (1 + \frac{4}{\rho} + \frac{C_2}{\delta_5} + \frac{C_4}{C_5} + 4C_8 + 2C_{14} + 4C_{17} + 2C_{23})(C_{16} + C_{20})\epsilon_0 + 2C_{24}\epsilon_0 = 1. \quad (3.75) \]

Using the classical continuity method, we can obtain the following lemma.

**Lemma 3.9.** Under the assumptions in Theorem 1.1, the solution \((\rho, u)\) satisfies the estimates (3.12)-(3.15), (3.30), (3.32), (3.34)-(3.39), (3.62)-(3.64) for all \( t \geq 0 \).

**Remark 3.1.** In this section, we just give some ideas. In deed, we use the continuity method to obtain the bound of \( \frac{u_i(t)}{r_i(t)} \), \( i = 0, \ldots, N + 1 \), in Section 1. The basic theory of ordinary differential equations guarantees \( \rho_i(t), u_i(t), r_i(t) \in C([0,T^*]), i = 0, \ldots, N \). Since \( r_j^n(t) \geq h, j = 0, \ldots, N \), we have \( \frac{u_i(t)}{r_i(t)} \in C([0,T^*]), i = 0, \ldots, N \). Thus, we can use the continuity method.

From Lemma 3.9, we can obtain the following lemma easily.

**Lemma 3.10.** Under the assumptions in Theorem 1.1, if \( \epsilon_0 \) is small enough, we have
\[
\begin{align*}
\|\rho(\cdot, t_1) - \rho(\cdot, t_2)\|_{L^2} & \leq C|t_1 - t_2|, \\
\|u(\cdot, t_1) - u(\cdot, t_2)\|_{L^2} & \leq C|t_1 - t_2|, \\
\|r(\cdot, t_1) - r(\cdot, t_2)\|_{L^\infty} & \leq C|t_1 - t_2|, \\
\|\partial_x(r^{n-1}u)(\cdot, t_1) - \partial_x(r^{n-1}u)(\cdot, t_2)\|_{L^2} & \leq C|t_1 - t_2|^\frac{1}{2}, \\
\|((r^{n-2}u)_x, (r^{n-1})_x)(\cdot, t)\|_{L^{n-\frac{1}{2}}} & \leq C,
\end{align*}
\]
for all \( t_1, t_2, t \geq 0 \).

4 Difference scheme and approximate solutions.

In this section, applying a discrete difference scheme as in [1], we construct approximate solutions to the initial boundary value problem (1.20)-(1.23).

For any given positive integer \( N \), let \( h = \frac{1}{N} \) be an increment in \( x \) and \( x_j = jh \) for \( j \in \{0, \ldots, N\} \). For each integer \( N \), we construct the following time-dependent functions:
\[
(\rho_j(t), u_j(t), r_j(t)), \quad j = 0, \ldots, N,
\]
that form a discrete approximation to \((\rho, u, r)(x_j, t)\) for \( j = 0, \ldots, N \).
First, $\rho_i(t)$, $u_j(t)$ and $r_{i+1}(t)$, $i = 0, \ldots, N$, $j = 1, \ldots, N$, are determined by the following system of $3N + 2$ differential equations:

\[
\frac{d}{dt}\rho_i = -\rho_i^2 \delta(r_i^{n-1} u_i),
\]

\[
\frac{d}{dt}u_j = r_j^{n-1} \delta \sigma_j - 2(n-1)r_j^{n-2}u_j \delta(\mu_{j-1}) - f_j,
\]

\[
\frac{d}{dt}r_{i+1} = u_{i+1},
\]

with the boundary conditions:

\[
u_0(t) = 0, \quad r_0^0(t) = h,
\]

\[P_N - \rho_N(\lambda_N + 2\mu_N)\delta(r_N^{n-1} u_N) + 2(n \mu_{N+1})\mu_N = P_T,
\]

and initial data

\[(\rho_j, u_j)(0) = \left(\frac{1}{h} \int_{(j-1)h}^{jh} \rho_0(y)dy, \frac{1}{h} \int_{(j-1)h}^{jh} u_0(y)dy\right), \quad j = 1, \ldots, N,
\]

\[\rho_0(0) = \rho_1(0), u_0(0) = 0, \quad r_0^0(0) = h,
\]

\[r_i^n(0) = h + n \sum_{l=0}^{i-1} \frac{r_i^l(0)}{\rho_i(0)}, \quad i = 1, \ldots, N + 1,
\]

and $u_{N+1}(0)$ satisfies

\[P_N(0) - \rho_N(0)(\lambda_N(0) + 2\mu_N(0))\delta(r_N^{n-1}(0) u_N(0)) + 2(n \mu_{N+1}(0))\mu_N(0) = P_T(0),
\]

where $\delta$ is the operator defined by $\delta w_j = (w_{j+1} - w_j)/h$, and

\[
\sigma_j(t) = \rho_{j-1}(\lambda_{j-1} + 2\mu_{j-1})\delta(r_{j-1}^{n-1} u_{j-1}) - P_{j-1},
\]

\[
\lambda_j = \lambda(\rho_j), \mu_j = \mu(\rho_j), P_j = P(\rho_j),
\]

\[
f_j(t) = f(jh, r_j, t).
\]

The boundary conditions (4.4)-(4.5) are consistent with the initial data. The condition (4.5) determines $u_{N+1}(t)$.

Let $(\rho_{\infty,i}, r_{\infty,i}) = (\rho_{\infty}(ih), h + r_{\infty}(ih))$, $i = 0, \ldots, N$, we have

\[r_{\infty,j}^{n-1} \delta(A \rho_{\infty,j-1}) = -\frac{G_{jh}}{r_{\infty,j}^{n-1}} + Q_{1j},
\]

\[r_{\infty,j}^n = h + \sum_{k=0}^{j-1} \frac{h}{\rho_{\infty,k}} + Q_{2j},
\]

and

\[|Q_{1j}| \leq C(jh)^{\frac{1-n}{2}} h, \quad |Q_{2j}| \leq C(jh)^{\frac{3}{2}} h, \quad j = 1, \ldots, N.
\]
Then, for any small $h$, the initial data $(\rho_i, u_i, r_i)(0)$ and the external force $f_i$, $i = 0, \ldots, N$, satisfies
\[
\max_{i \in \{0, \ldots, N\}} |\rho_i(0) - \rho_{\infty,i}|^2 + \sum_{j=0}^{N-1} \left[ r_j^{2n-2+\alpha}(\delta \rho_j - \delta \rho_{\infty,j})^2 \right] (0) h \leq C\epsilon_0^2, \quad (4.10)
\]
\[
C^{-1}(i + 1)h \leq r_i^n(0) \leq C(i + 1)h, \quad \sum_{j=0}^{N} \left[ r_j^{-2}u_j^2 + r_j^{2n-2}(\delta u_j)^2 \right] (0) h \leq C\epsilon_0^2, \quad (4.11)
\]
\[
\sum_{j=1}^{N} \left( r_j^{2n-2\alpha} \left[ \delta \left( \rho_j^{1+\alpha}(r_j^{n-1}u_{j-1}) \right) \right] \right)^2 (0) h \leq C\epsilon_0^2, \quad (4.12)
\]
where $C > 0$ are independent of $h$.

The basic theory of differential equations guarantees the local existence of smooth solutions $(\rho_i, u_i, r_i)$ ($i = 0, \ldots, N$) to the Cauchy problem (4.1)-(4.9) on an interval $[0, T^h)$, such that
\[
0 < \rho_i(t) < \infty, |u_i(t)|, |r_i(t)| < \infty, i = 0, \ldots, N,
\]
with the aid of (4.10)-(4.12).

For any fixed $T > 0$, by virtue of Lemma 3.1-3.10 and using similar arguments as in [1, 7], we can obtain the following lemma and prove that the Cauchy problem (4.1)-(4.9) has a unique solution for $t \in [0, T]$ when $h \leq h_{T, \epsilon_0}$, where $h_{T, \epsilon_0} > 0$ is a constant dependent on $T$ and $\epsilon_0$.

**Lemma 4.1.** For any $h \in (0, h_{T, \epsilon_0}]$, there exist a positive constant $C$ independent of $h$ such that
\[
\rho_i(t) \leq \left[ \frac{1}{4 \rho_0} \frac{3}{2} \right], \quad \|\rho_i(t) - \rho_{\infty,i}\|_{L^\infty} \leq C\epsilon_0,
\]
\[
r_i^n(t) \in [C^{-1}(i + 1)h, C(i + 1)h], \quad \sum_{j=0}^{N} (u_j^2(t)) + |\rho_j(t) - \rho_{\infty,j}|^2 h \leq C\epsilon_0^2,
\]
\[
\frac{|u_i(t)|}{r_i(t)} \leq C\epsilon_0,
\]
\[
\int_{0}^{t} \sum_{j=0}^{N} \left( u_j^2 + r_j^{2n-2}(\delta u_j)^2 + \frac{u_j^2}{r_j^2} \right) (s) h ds \leq C\epsilon_0^2,
\]
\[
\sum_{j=0}^{N} \left( \frac{u_j^2}{r_j^2} + r_j^{2n-2}(\delta u_j)^2 \right) (t) h + \int_{0}^{t} \sum_{j=0}^{N} \left( \frac{d}{dt} u_j \right)^2 (s) h ds \leq C\epsilon_0^2,
\]
\[
\int_{0}^{t} \sum_{j=0}^{N} \left[ r_j^\alpha (\rho_j - \rho_{\infty,j})^2 + (r_j - r_{\infty,j})^2 \right] h ds \leq C\epsilon_0^2,
\]
\[
\sum_{j=0}^{N} (r_j^\alpha u_j^2(t)) dx + \int_{0}^{t} \sum_{j=0}^{N} r_j^\alpha \left( r_j^{2n-2}(\delta u_j)^2 + \frac{u_j^2}{r_j^2} \right) h ds \leq C\epsilon_0^2,
\]
\[
\sum_{j=0}^{N-1} \left( r_j^{2n-2+\alpha}(\delta \rho_j - \delta \rho_{\infty,j})^2 \right) (t) h + \int_{0}^{t} \sum_{j=0}^{N-1} \left( r_j^{2n-2+\alpha}(\delta \rho_j - \delta \rho_{\infty,j})^2 \right) (s) h ds \leq C\epsilon_0^2,
\]

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for each fixed \( N \) by using the Friedrichs mollifier. Since the following estimates are valid for the solutions with the regularity indicated in Theorem 1.1,

\[
\sum_{j=0}^{N-1} |\delta \rho_j - \delta \rho_{\infty,j}|(t) h \leq C \epsilon_0,
\]
\[
\sum_{j=0}^{N} \left[ r_j^0 \left( \frac{d}{dt} u_j \right)^2 \right] (t) h \leq C \epsilon_0^2,
\]
\[
|\delta (r_i^{n-1} u_i)(t)| \leq C \epsilon_0,
\]
\[
\sum_{j=0}^{N} \left( |\delta (r_j^{n-1} u_j)(t) - \delta (r_j^{n-1} u_j)(t_2)|^2 \right) h \leq C |t_1 - t_2|,
\]
\[
\sum_{j=0}^{N} \left( |\delta (r_j^{n-2} u_j)|^{n-1 \over 2} + |\delta (r_j^{n-1})|^{n-3 \over 2} \right) h \leq C,
\]

for all \( t_1, t_2, t \in [0, T] \), \( i \in \{0, \ldots, N\} \) and \( l \in \{1, \ldots, N + 1\} \).

Now, we can define our approximate solutions \((\rho^N, u^N, r^N)(x,t)\) for the Cauchy problem (1.20)-(1.23). For each fixed \( N \) and \( t \in [0, T] \), we define piecewise linear continuous functions \((\rho^N, u^N, r^N)(x,t)\) with respect to \( x \) as follows: when \( x \in ([xN], [xN] + 1) \)

\[
\rho^N(x,t) = \rho_{[xN]}(t) + (xN - [xN])(\rho_{[xN]+1}(t) - \rho_{[xN]}(t)),
\]
\[
u^N(x,t) = u_{[xN]}(t) + (xN - [xN])(u_{[xN]+1}(t) - u_{[xN]}(t)),
\]
\[
r^N(x,t) = \left( r_{[xN]}^n(t) + (xN - [xN])(r_{[xN]+1}^n(t) - r_{[xN]}^n(t)) \right)^{1/n}.
\]

From Lemma 4.1 using similar arguments as in [7], we can obtain the compactness of approximate solutions \((\rho^N, u^N, r^N)\) and prove the existence part of Theorem 1.1. Since the constant \( C \) in Lemma 4.1 is independent of \( T \), we can obtain the regularity estimates (1.20)-(1.31) easily.

5 Uniqueness.

In this section, applying energy method, we will prove the uniqueness of the solution in Theorem 1.1. Let \((\rho_1, u_1, r_1)(x,t)\) and \((\rho_2, u_2, r_2)(x,t)\) be two solutions in Theorem 1.1. Then, we have

\[
\rho_i(x,t) \in \left[ \frac{1}{2} \rho_0, \frac{3}{2} \rho_0 \right], \quad C^{-1} x^{1 \over \pi} \leq r_i(x,t) \leq C x^{1 \over \pi},
\]

\[
| x^{1 \over \pi} u_i(x,t)| + | x^{n-1 \over \alpha} \partial_x u_i(x,t)| \leq C.
\]

For simplicity, we may assume that \((\rho_1, u_1, r_1)(x,t)\) and \((\rho_2, u_2, r_2)(x,t)\) are suitably smooth since the following estimates are valid for the solutions with the regularity indicated in Theorem 1.1 by using the Friedrichs mollifier.

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Let
\[ \varrho = \rho_1 - \rho_2, \ w = u_1 - u_2, \ R = r_1 - r_2. \]

From (3.1), we have
\[
\frac{d}{dt} \int_0^M x^{-2} R^2(x,t) \, dt = 2 \int_0^M x^{-\frac{2}{n}} R \varrho \, dx
\leq \epsilon \int_0^M x^{-\frac{2}{n}} w^2 \, dx + C \epsilon \int_0^M x^{-\frac{2}{n}} R^2 \, dx.
\]
\hspace{8cm} (5.3)

From (1.20) and (5.1)-(5.2), we have
\[
\frac{d}{dt} \int_0^M \varrho^2(x,t) \, dt = 2 \int_0^M \varrho \partial_t (\rho_1 - \rho_2) \, dx
= 2 \int_0^M \varrho \left( -\rho_1 r_1^{n-1} \partial_x u_1 + \rho_2 r_2^{n-1} \partial_x u_2 - (n-1) \frac{\rho_1 u_1}{r_1} + (n-1) \frac{\rho_2 u_2}{r_2} \right) \, dx
\leq \epsilon \int_0^M \left( x^{\frac{2n-2}{n}} w_2^2 + x^{-\frac{2}{n}} w^2 \right) \, dx + C \epsilon \int_0^M (\varrho^2 + x^{-\frac{2}{n}} R^2) \, dx.
\]
\hspace{8cm} (5.4)

From the equation (1.20)2 and boundary conditions (1.22)-(1.23), we get
\[
\frac{d}{dt} \int_0^M \frac{1}{2} w^2(x,t) \, dx
+ \int_0^M \left\{ \left( \frac{2}{n} c_1 + c_2 \right) \rho_1^{1+\theta} (r_1^{n-1} w)_x + \frac{2(n-1)}{n} c_1 \rho_1^{1+\theta} (r_1^{n-1} w_x - \frac{w}{r_1 \rho_1})^2 \right\} \, dx
= -\int_0^M \partial_x (r_1^{n-1} w) \left[ \left( 2c_1 + c_2 \right) \left( \rho_1^{1+\theta} - \rho_2^{1+\theta} \right) \partial_x (r_1^{n-1} u_2)
+ (2c_1 + c_2) \rho_2^{1+\theta} \partial_x ((r_1^{n-1} - r_2^{n-1}) u_2) - (\rho_1^{1+\theta} - \rho_2^{1+\theta}) \right] \, dx
+ \int_0^M 2c_1 (n-1) \partial_x \left[ r_1^{n-1} w u_2 \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \right] \rho_1^\theta \, dx
+ \int_0^M 2c_1 (n-1) \partial_x \left[ r_1^{n-1} w u_2 \frac{u_2}{r_2} \right] \rho_1^\theta \, dx
+ \int_0^M 2c_1 (n-1) \partial_x \left[ r_1^{n-1} w u_2 \frac{u_2}{r_2} \right] \rho_1^\theta \, dx
- \int_0^M \partial_x (r_1^{n-1} - r_2^{n-2} w) \left[ \left( 2c_1 + c_2 \right) \rho_2^{1+\theta} \partial_x (r_2^{n-1} u_2) - \rho_2^\theta \right] \, dx
+ \int_0^M w G x (r_2^{1-n} - r_1^{1-n}) \, dx + \int_0^M w (\Delta f(x,r_2,t) - \Delta f(x,r_1,t)) \, dx.
\]
\hspace{8cm} (5.5)

From (5.1)-(5.2) and (5.3), we have
\[
\frac{d}{dt} \int_0^M \frac{1}{2} w^2(x,t) \, dx + C_{22} \int_0^M \left\{ x^{\frac{2n-2}{n}} w_2^2 + x^{-\frac{2}{n}} w^2 \right\} \, dx
\leq C \int_0^M \left( x^{-\frac{2}{n}} R^2 + \varrho^2 + w^2 \right) \, dx.
\]
\hspace{8cm} (5.6)

From (5.3)-(5.4) and (5.6), letting \( \epsilon = \frac{1}{4} C_{22} \), we obtain
\[
\frac{d}{dt} \int_0^M \left[ w^2 + \varrho^2 + x^{\frac{2n}{n}} R^2 \right] \, dx \leq C \int_0^M \left( x^{-\frac{2}{n}} R^2 + \varrho^2 + w^2 \right) \, dx.
\]
Using Gronwall’s inequality, we have for any \( t \in [0, T] \),
\[
\int_0^M \left[ w^2 + \varrho^2 + x^{-\frac{\varrho}{2}} R^2 \right] dx = 0.
\]

This proves the uniqueness of solution in Theorem 1.1.

6 Asymptotic behavior

In this section, we consider the asymptotic behavior of the solution to the free boundary problem (1.20)-(1.23). We will show that the solution to the free boundary problem tends to the stationary solution as \( t \to +\infty \).

The following lemma is proved in [15].

Lemma 6.1. Suppose that \( y \in W^{1,1}_{loc}(\mathbb{R}^+) \) satisfies
\[
y = y_1' + y_2,
\]
and
\[
|y_2| \leq \sum_{i=1}^n \alpha_i, \quad |y'| \leq \sum_{i=1}^n \beta_i, \quad \text{on } \mathbb{R}^+
\]
where \( y_1 \in W^{1,1}_{loc}(\mathbb{R}^+) \), and \( \lim_{s \to +\infty} y_1(s) = 0 \) and \( \alpha_i, \beta_i \in L_p(\mathbb{R}^+) \) for some \( p_i \in [1, \infty) \), \( i = 1, \ldots, m \). Then \( \lim_{s \to +\infty} y(s) = 0 \).

Proposition 6.1. Under the assumptions of Theorem 1.1, the total kinetic energy
\[
E(t) := \int_0^M \frac{1}{2} u^2(x,t) dx \to 0 \quad \text{as} \quad t \to +\infty.
\]

Proof. From (3.15) and Lemma 3.9, we have \( E(t) \in L^1(\mathbb{R}^+) \). Using the Cauchy-Schwarz inequality, we obtain
\[
|E'(t)| \leq E(t) + \int_0^M u^2 dx.
\]
Taking into account the estimate (3.64) and Lemma 3.9, applying Lemma 6.1, we finish the proof.

Proposition 6.2. Under the assumptions of Theorem 1.1, we have
\[
\int_0^M (r - r_\infty)^2(x,t) dx \to 0 \quad \text{as} \quad t \to +\infty.
\]

Proof. From (3.20) and Lemma 3.9, we have \( \int_0^M (r - r_\infty)^2(x,t) dx \in L^1(\mathbb{R}^+) \). Using the Cauchy-Schwarz inequality, we obtain
\[
\frac{d}{dt} \int_0^M (r - r_\infty)^2 dx = 2 \int_0^M (r - r_\infty) u dx \leq 2E(t) + \int_0^M (r - r_\infty)^2 dx.
\]
Taking into account the estimate \( E(t) \in L^1(\mathbb{R}^+) \), applying Lemma 6.1, we finish the proof.
Proposition 6.3. Under the assumptions of Theorem 1.1, we have
\[
\int_0^M (\rho - \rho_\infty)^2(x,t)dx \to 0,
\] (6.1)
and
\[
\|(\rho - \rho_\infty)(\cdot,t)\|_{L^q} \to 0, \quad q \in [1, \infty),
\] (6.2)
as \(t \to +\infty\).

Proof. From (3.20) and Lemma 3.9 we have \(\int_0^M (\rho - \rho_\infty)^2(x,t)dx \in L^1(\mathbb{R}^+)\). From (1.29), using the Cauchy-Schwarz inequality, we obtain
\[
\left| \frac{d}{dt} \int_0^M (\rho - \rho_\infty)^2dx \right| \leq \int_0^M (\rho - \rho_\infty)^2dx + C \int_0^M (r^{n-1}u)^2dx.
\]
Taking into account the estimates (3.15) and Lemma 3.9 applying Lemma 6.1, we obtain (6.1). From (1.29), (3.3) and (6.1), we can obtain (6.2) easily.

Proposition 6.4. Under the assumptions of Theorem 1.1, we have
\[
\int_0^M x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x)^2(x,t)dx \to 0, \quad as \ t \to +\infty.
\]

Proof. From (1.29), (3.35), (3.37) and Lemma 3.9 we have
\[
\int_0^M x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x)^2(x,t)dx \in L^1(\mathbb{R}^+).
\]
From (1.20), (1.29)-(1.30) and (3.3), using the Cauchy-Schwarz inequality, we have
\[
\left| \frac{d}{dt} \int_0^M x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x)^2dx \right| = 2\theta \left| \int_0^M x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x)(\rho^\theta_\infty)\partial_x(r^{n-1}u)dx \right|
\]
\[
= \frac{2\theta}{2c_1 + c_2} \left| \int_0^M x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x) \left( \frac{u_t}{r^{n-1}} + A(\rho^\gamma)_x \right) 
+ 2c_1(n-1)\frac{u(\rho^\theta)_x}{r} + f(x,r,t) \right| dx
\]
\[
\leq C \int_0^M \left[ x^{\frac{2n-2+a}{n}}((\rho^\theta)_x - (\rho^\theta_\infty)_x)^2 + r^\alpha u^2 + r^\alpha (r - r_\infty)^2 
+ r^\alpha (\rho - \rho_\infty)^2 + r^\alpha u^2 \right] dx + f_1.
\]
Taking into account the estimates (1.29), (3.35)-(3.37), (3.62) and Lemma 3.9 applying Lemma 6.1 we end the proof.

From Proposition 6.3, 6.4 using Sobolev’s embedding Theorem, we can obtain the following corollary immediately.

Corollary 6.1. Under the assumptions of Theorem 1.1, we have
\[
\|\rho(\cdot,t) - \rho_\infty(\cdot)\|_{L^\infty} + \|r(\cdot,t) - r_\infty(\cdot)\|_{L^\infty} \to 0, \quad as \ t \to +\infty.
\]
Proposition 6.5. Under the assumptions of Theorem 1.1, we have
\[ \int_0^M x^{2n-2+\alpha} u_x^2(x,t)dx \to 0, \text{ as } t \to +\infty. \]

Proof. From the estimates (1.29), (3.36) and Lemma 3.9, we have
\[ \int_0^M x^{2n-2+\alpha} u_x^2(x,t)dx \in L^1(\mathbb{R}^+). \]

Using the Cauchy-Schwarz inequality, we have
\[ \left| \frac{d}{dt} \int_0^M x^{2n-2+\alpha} u_x^2 dx \right| \leq \int_0^M x^{2n-2+\alpha} u_x^2 dx + \int_0^M x^{2n-2+\alpha} u_{xx}^2 dx. \]

Taking into account the estimates (1.29), (3.36), (3.62) and Lemma 3.9 applying Lemma 6.1, we end the proof.

From Proposition 6.1 and 6.5 using Sobolev’s embedding Theorem, we can obtain the following corollary immediately.

Corollary 6.2. Under the assumptions of Theorem 1.1, we have
\[ \|u(\cdot, t)\|_{L^\infty} \to 0, \text{ as } t \to +\infty. \]

Thus, we finish the proof of Theorem 1.1.

7 Stabilization rate estimates

Now we are in position to estimate the stabilization rate. We first state the following proposition which gives the stabilization rate estimates in \(L^2([0,M])\)-norm of the solution.

Proposition 7.1. Under the assumptions of Theorem 1.3, we have
\[ \int_0^M \left( u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r_\infty^n)^2 \right) dx \leq Ce^{-a_1 t} \quad (7.1) \]

and
\[ |\rho(M,t) - \rho_\infty(M)| + \left( \int_0^M r^{2n-2}(\rho - \rho_\infty)^2 dx \right)^{\frac{1}{2}} + \|r(\cdot, t) - r_\infty(x)\|_{L^2} \leq Ce^{-a_1 t}, \quad (7.2) \]

for all \( t \geq 0 \), where \( a_1 \) is a positive constant.

Proof. Let
\[ V_1 = \int_0^M \frac{1}{2} u^2 dx + S[V] - S[V_\infty], \]
\[ W_1 = \int_0^M \left\{ \left( r^{n-1} u \right)_x^2 + r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right\} dx. \]
From (1.32), (3.16)-(3.18), we have
\[ V'_1 + 2C_{31}W_1 \leq C f_1 V_1^\frac{1}{2} + C |\Delta P|^2 \leq C e^{-a_0 t} V_1^\frac{1}{2} + C e^{-2a_0 t}, \]  
(7.3)

\[ C_{32}^{-1} \int_0^M (u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2) \, dx \]
\[ \leq V_1 \leq C_{32} \int_0^M (u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2) \, dx, \]  
(7.4)

and
\[ C_{33} \|u(\cdot, t)\|_{L^2} \leq W_1. \]  
(7.5)

From (3.28), we have
\[ \int_0^M \left[ (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2 \right] \, dx \]
\[ \leq -C_{38} \frac{d}{dt} \int_0^M \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n_\infty}{n} \right) \, dx + C_{38} W_1 + C e^{-2a_0 t}. \]  
(7.6)

From (1.29), we obtain
\[ \left| C_{38} \int_0^M \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n_\infty}{n} \right) \, dx \right| \leq C_{39} \int_0^M (u^2 + |\rho - \rho_\infty|^2) \, dx. \]  
(7.7)

Let
\[ V_2 = V_1 + \epsilon C_{38} \int_0^M \frac{u}{r^{n-1}} \left( \frac{r^n}{n} - \frac{r^n_\infty}{n} \right) \, dx, \]
\[ W_2 = C_{31} W_1 + \epsilon \int_0^M \left[ (\rho^n - \rho^n_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2 \right] \, dx, \]
where \( \epsilon = \min \{ \frac{C_{31}}{C_{39}}, \frac{1}{2C_{32}C_{39}} \}. \) From (1.32) and (1.41)-(1.74), we have
\[ V'_2 + W_2 \leq C e^{-2a_0 t}, \]  
(7.8)

\[ C_{39}^{-1} \int_0^M (u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2) \, dx \]
\[ \leq V_2 \leq C_{39} \int_0^M (u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2) \, dx, \]  
(7.9)

and
\[ C_{40} \int_0^M (u^2 + (\rho - \rho_\infty)^2 + x^{-2}(r^n - r^n_\infty)^2) \, dx \leq W_2. \]  
(7.10)

Thus \( V_2 \) is a Lyapunov functional. From (1.32), we obtain the estimate (7.1). From (1.29), (3.33), (3.40) and (7.1), we can get (7.2) easily.

**Proposition 7.2.** Under the assumptions of Theorem 1.3, we obtain
\[ \int_0^M \left( \frac{u^2}{r^2} + r^{2n-2}u_x^2 \right) (x, t) \, dx \leq C e^{-a_3 t}, \]  
(7.11)

for all \( t \geq 0 \), where \( a_3 \) is a positive constant.
Proof. Let
\[ V_3 = \int_0^M \left\{ \frac{1}{2}c_1 + c_2 \right\} \rho^{1+\theta} [(r^{n-1}u)_x]^2 + \frac{(n-1)}{n} c_1 \rho^\theta (r^{n-1}u_x - \frac{u}{r^\rho})^2 \]
\[ + (A\rho_\infty^\gamma - A\rho^\gamma + \Delta P)(r^{n-1}u)_x + u \left( G(x) - G(x) \right) \} dx. \]

From \((3.30)\) and \((3.61)\), we have
\[ V'_3 + \int_0^M \frac{1}{2} u^2 x dx \leq C_{41} \left( f^2 + (|\Delta P'|^2 + W_2) \right). \]

From \((3.13)-(3.14)\), we have
\[ V_3 \geq \int_0^M \left\{ C_{42} [(r^{n-1}u)_x^2 + \frac{u^2}{r^2} + r^{2n-2}u_x^2] - C_{43} ((\rho - \rho_\infty)^2 + |\Delta P|^2) \right\} dx, \]
\[ V_3 \leq \int_0^M \left\{ C_{44} [(r^{n-1}u)_x^2 + \frac{u^2}{r^2} + r^{2n-2}u_x^2] + C_{43} ((\rho - \rho_\infty)^2 + |\Delta P|^2) \right\} dx. \]

Letting \( V_4 = V_2 + \eta V_3 + \eta C_{43} |\Delta P|^2 \), where \( \eta = \min\{ \frac{1}{2}, \frac{1}{2 \cdot 3 \cdot 35}, \frac{1}{2 \cdot C_{41}} \} \). From \((7.8)-(7.10)\), we have
\[ CW_2 \geq V_4 \geq C^{-1} W_1, \]
and
\[ V'_4 + C^{-1} W_2 \leq C \left( f^2 + (|\Delta P'|^2 + (\Delta P')^2) \right). \]

Thus \( V_4 \) is a Lyapunov functional. From \((1.32)\), we can obtain the estimate \((7.11)\). \( \square \)

**Proposition 7.3.** Under the assumptions of Theorem 1.3, we obtain
\[ \int_0^M r^{\frac{1}{2} - m} (\rho - \rho_\infty)^2 dx ds \leq Ce^{-at}, \]
\[ \int_0^M r^{\frac{1}{2} - m} (r - r_\infty)^2 dx ds \leq Ce^{-at}, \]
\[ \int_0^M (r^{\frac{1}{2} - m} u^2)(x,t) dx \leq Ce^{-at}, \]
\[ \int_0^M (r^{2n-2 + \frac{1}{2} - m} (\rho - \rho_\infty)^2)(x,t) dx \leq Ce^{-at}, \]
for all \( t \geq 0 \) and \( m = 0, 1, \ldots, n - 1 \), where \( a \) is a positive constant.

**Proof.** From \((7.11)-(7.13)\), we know that the estimates \((7.12)-(7.15)\) hold with \( m = 0 \).

**Claim 3:** If that \((7.12)-(7.15)\) hold with \( m \leq k \), \( k \in [0, n-2] \), then the estimates \((7.12)-(7.15)\) hold with \( m = k + 1 \).

We could prove Claim 3 as follows. Let \( \alpha_k = \frac{1}{2} - k - 1 \). From \((3.41)\), \((7.1)\) and \((7.15)(m = k)\), we have
\[ \int_0^M r^{\alpha_k} (\rho - \rho_\infty)^2 dx \leq Ce^{-at}, \]
and \((7.12)(m = k + 1)\) holds. From \((1.29)\) and \((7.12)\), we can obtain \((7.13)(m = k + 1)\) easily.
From (3.49), we obtain
\[
\frac{d}{dt} \int_0^M r^{\alpha_k} u_x^2 \, dx + C_{42} \int_0^M r^{\alpha_k} \left( r^{2n-2} u_x^2 + \frac{u^2}{r^2} \right) \, dx \leq C \left( f_1^2 + |\Delta P|^2 \right) + C e^{-at}. \tag{7.16}
\]
Thus \( \int_0^M (r^{\alpha_k} u_x^2)(x,t) \, dx \) is a Lyapunov functional, and we obtain (7.14) \((m = k + 1)\) immediately. From (3.52) and (7.12)-(7.14), we have
\[
\frac{d}{dt} \int_0^M H_1^2 \, dx + \frac{C_{18}}{2} \int_0^M H_1^2 \, dx \leq C \int_0^M \left( r^{\alpha_k} u_x^2 + r^{\alpha_k} (\rho - \rho_\infty)^2 + r^{\alpha_k} (r - r_\infty)^2 \right) + C f_1^2 \leq C e^{-at}.
\]
Thus \( \int_0^M H_1^2 \, dx \) is a Lyapunov functional. Using the estimates (1.29) and (7.12)-(7.14), we obtain (7.15) \((m = k + 1)\), finish the proof of Claim 3 and Proposition 7.3 immediately. \( \square \)

From (1.29), (7.12) and (7.15), using Hölder’s inequality and Sobolev’s embedding Theorem, we could obtain the following proposition.

**Proposition 7.4.** Under the assumptions of Theorem 1.3 we obtain
\[
\|\rho(\cdot, t) - \rho_\infty(\cdot)\|_{L^\infty} + \|r(\cdot, t) - r_\infty(\cdot)\|_{L^\infty} \leq C e^{-at},
\]
for all \( t \geq 0 \), where \( a \) is a positive constant.

**Proposition 7.5.** Under the assumptions of Theorem 1.3 we obtain
\[
\int_0^M (r^{\alpha} u_t^2)(x,t) \, dx \leq C e^{-at}, \tag{7.17}
\]
for all \( t \geq 0 \), where \( \alpha = \frac{3}{2} - n \) and \( a \) is a positive constant.

**Proof.** From (3.30) and (3.70), we have
\[
\frac{d}{dt} \int_0^M r^\alpha u_t^2 \, dx + C_{45} \int_0^M \left( r^{2n-2+\alpha} u_{xt}^2 + r^{\alpha-2} u_t^2 \right) \, dx \leq C_{46} \int_0^M \left( r^{2n-2+\alpha} u_x^2 + r^{\alpha-2} u_t^2 + r^{\alpha} (\rho - \rho_\infty)^2 \right) \, dx + C \left( f_1^2 + |\Delta P|^2 + |(\Delta P)|^2 \right). \tag{7.18}
\]
Let \( V_4 = \int_0^M (r^{\alpha} u_t^2)(x,t) \, dx + \frac{C_{42}}{2 C_{46}} \int_0^M (r^{\alpha} u_t^2)(x,t) \, dx \). From (1.32), (7.12), (7.16) \((k = n - 2)\) and (7.18), we have
\[
V_4 + C^{-1} V_4 \leq C e^{-a t} + C e^{-a t}.
\]
Thus \( V_4 \) is a Lyapunov functional, and we obtain (7.17) immediately. \( \square \)

**Proposition 7.6.** Under the assumptions of Theorem 1.3 we obtain
\[
\int_0^M r^{2n-2+\alpha \partial_x(\rho^{1+\theta} \partial_x(r^{n-1} u))} \, dx + \left\| \left( \frac{u}{r}, (r^{n-1} u)_x \right)(\cdot, t) \right\|_{L^\infty} \leq C e^{-at},
\]
for all \( t \geq 0 \), where \( \alpha = \frac{3}{2} - n \) and \( a \) is a positive constant.
Proof. From the equation \((1.20)_2\), we have
\[
(2c_1 + c_2)r^{n-1}\partial_x(r_1t^\gamma \partial_x(r^{n-1}u)) = ut + Ar^{n-1}(\rho_1^\gamma)_x + 2c_1(n-1)r^{n-2}u(\rho^\gamma)_x + f,
\]
and using the estimates \((1.29)-(1.30)\), \((1.32)\), \((7.12)-(7.15)\) and \((7.17)\), conclude that
\[
\int_0^M r^{2n-2+\alpha} [\partial_x(r^{n-1}u)]^2 dx \leq Ce^{-at},
\]
and
\[
\int_0^M [\partial_x(r^{n-1}u)] dx \leq Ce^{-at}. \tag{7.19}
\]
From \((1.29)\), \((7.11)\) and \((7.19)\), using Sobolev’s embedding Theorem $W^{1,1} \hookrightarrow L^\infty$, we can obtain
\[
\|\partial_x(r^{n-1}u)(\cdot, t)\|_{L^\infty} \leq Ce^{-at},
\]
and
\[
\left\| \frac{u}{r}(\cdot, t) \right\|_{L^\infty} \leq Ce^{-at}.
\]

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