An approach to generalizing some impossibility theorems in social choice

Wesley H. Holliday, Eric Pacuit, and Saam Zahedian

Abstract. In social choice theory, voting methods can be classified by invariance properties: a voting method is said to be C1 if it selects the same winners for any two profiles of voter preferences that produce the same majority graph on the set of candidates; a voting method is said to be pairwise if it selects the same winners for any two preference profiles that produce the same weighted majority graph on the set of candidates; and other intermediate classifications are possible. As there are far fewer majority graphs or weighted majority graphs than there are preference profiles (for a bounded number of candidates and voters), computer-aided techniques such as satisfiability solving become practical for proving results about C1 and pairwise methods. In this paper, we develop an approach to generalizing impossibility theorems proved for C1 or pairwise voting methods to impossibility theorems covering all voting methods. We apply this approach to impossibility theorems involving “variable candidate” axioms—in particular, social choice versions of Sen’s well-known γ and α axioms for individual choice—which concern what happens when a candidate is added or removed from an election. A key tool is a construction of preference profiles from majority graphs and weighted majority graphs that differs from the classic constructions of McGarvey and Debord, especially in better commutative behavior with respect to other operations on profiles.

Contents

1. Introduction 2
2. The impossibility theorems 5
3. Preliminaries 12
4. Impossibility theorems for C1 and pairwise methods 14
5. Profile representation of (weighted) weak tournaments 14
6. C1 and pairwise projection 23
7. Preservation of axioms 24
8. Generalized impossibility theorems 28
9. Conclusion 28

Appendix A. SAT solving for C1 methods 29
Appendix B. SAT solving for pairwise methods 33

References 34

2020 Mathematics Subject Classification. 91B12, 91B14.
Key words and phrases. social choice theory, voting, impossibility theorems, SAT solving.
1. Introduction

Starting with Arrow’s seminal Impossibility Theorem, social choice theory has produced a wide range of theorems demonstrating the inconsistency of various principles for group decision making. In Arrow’s formulation of the basic problem of social choice, we are given a set of alternatives, a set of evaluators of the alternatives, and a function, called a preference profile, assigning to each evaluator a ranking of the alternatives. The goal is to output a single collective ranking of the alternatives, such that the map from preferences profiles to collective rankings, called a social welfare function, satisfies some desirable properties. This abstract setup applies to a variety of concrete problems, including ranking social states on the basis of citizens’ welfare, ranking objects of choice on the basis of multiple criteria of evaluation (e.g., ranking apartments by price, square footage, proximity to downtown, etc.), ranking scientific theories on the basis of their relative satisfaction of theoretical virtues (e.g., simplicity, explanatory power, empirical accuracy, etc.), or ranking candidates in an election on the basis of ranked ballots submitted by voters. In this paper, we are primarily interested in the electoral interpretation of Arrow’s setup. In the setting of elections, we often ask for less than a full ranking of the candidates; it suffices to choose a winning candidate or set of candidates, allowing for the possibility of ties. Thus, in lieu of a social welfare function, we ask for a voting method, understood as a function that assigns to each preference profile a set of candidates tied for the win, satisfying certain desiderata.

A well-known classification of voting methods (cf. [Fis77]), which turns out to be relevant to impossibility theorems in voting theory, groups methods according to how much information about the preference profile is needed to select the winning candidate(s). Abstractly, given an equivalence relation on the class of profiles, we can consider the class of voting methods invariant with respect to this relation, i.e., voting methods such that for any preference profiles and \( P \) and \( P' \),

\[ P \sim P' \text{ implies } F(P) = F(P'). \]

Or viewing the equivalence classes as the fibers of some function \( \phi \),

\[ \phi(P) = \phi(P') \text{ implies } F(P) = F(P'). \]

Thus, the only information \( F \) needs about \( P \) is \( \phi(P) \). Important examples include:

- **C1** methods: \( \phi(P) \) is the majority graph of \( P \);
- **Qualitative-pairwise** methods: \( \phi(P) \) is the qualitative margin graph of \( P \);
- **Pairwise** methods: \( \phi(P) \) is the margin graph of \( P \).

As shown in Figure 1, the majority graph of a profile \( P \) is the directed graph whose set of vertices is the set of candidates in \( P \) with an edge from \( x \) to \( y \) if more voters prefer \( x \) to \( y \) than prefer \( y \) to \( x \); the margin graph of \( P \) is the weighted directed graph whose underlying graph is the majority graph of \( P \), where the weight of an edge from candidate \( x \) to candidate \( y \) is the margin of \( x \) over \( y \) in \( P \), defined as the number of voters who prefer \( x \) to \( y \) minus the number of voters who prefer \( y \) to \( x \); and the qualitative margin graph of \( P \) is the majority graph of \( P \) equipped with an ordering of its edges according to the weights of the margin graph of \( P \). To illustrate these distinctions, consider the following examples:

---

1 For the subtle distinction between pairwise methods and C2 methods as in [Fis77], see [BSS18, p. 1286].
• A C1 method: the Copeland method selects as winners for \( P \) the candidates \( x \) that maximize \( \text{outdegree}(x) - \text{indegree}(x) \) in the majority graph of \( P \), i.e., the candidates who maximize the number of wins minus losses.

• A qualitative pairwise (but not C1) method: the Minimax method selects as winners for \( P \) the candidates who minimize the maximal weight of their incoming edges, i.e., whose worst loss is smallest.

• A pairwise (but not qualitative-pairwise) method: the Borda method selects as winners for \( P \) those candidates that maximize the sum of weights of their outgoing edges minus the sum of weights of their incoming edges.

The condition that a voting method lies in a given class of the classification above can be viewed as an axiom that together with other axioms may generate an impossibility theorem, e.g., of the form “There is no C1 voting method satisfying axioms \( A_1, \ldots, A_n \).” In fact, proofs of such impossibility theorem are facilitated by the fact that there are far fewer majority graphs (resp. qualitative margin graphs or margin graphs) for a fixed set of candidates than there are profiles. This makes it possible to use computer-aided methods such as satisfiability (SAT) solving—as we do in this paper—which can work on the much smaller space of majority/margin graphs (for a bounded number of candidates) rather than the space of all profiles.

Majority graphs are simply asymmetric directed graphs (cf. Theorem 3.17.1), and under the standard assumption of neutrality in voting theory (discussed below), the names of the candidates do not matter, so it is enough to consider unlabeled asymmetric directed graphs, also known as unlabeled weak tournaments. See Figure 2 for a comparison of the number of unlabeled weak tournaments vs. the number of preference profiles; even if we make the anonymity assumption that the names of voters do not matter, as in anonymous profiles (assigning to each ranking of the candidates a number, representing the number of voters casting that ballot), or that neither the names of voters nor the names of candidates matter, as in anonymous and neutral equivalence classes of profiles (or ANECs, which are equivalence classes of anonymous profiles where two anonymous profiles are equivalent when one can be obtained from the other by applying a permutation of the candidates to each ranking) [OE09, OEG13], there are still far fewer of the graph-based objects than there are of the profile-based objects. As a result, there are far fewer...
C1 voting methods than there are arbitrary voting methods, even under neutrality and anonymity. Similar points apply to pairwise methods vs. arbitrary methods.

| candidates | unlabeled weak tournaments | 6-voter ANECs | 6-voter anonymous profiles | 6-voter profiles |
|------------|---------------------------|---------------|----------------------------|-----------------|
| 3          | 7                         | 83            | 462                        | 46,656          |
| 4          | 42                        | 19,941        | 475,020                    | 191,102,976     |
| 5          | 582                       | 39,096,565    | 4,690,625,500              | 2,985,984,000,000 |

**Figure 2.** Comparison of the numbers of unlabeled weak tournaments, preference profiles, anonymous profiles, and anonymous and neutral equivalence classes of profiles (ANECs). We chose 6 voters because for each \( n \in \{3, 4, 5\} \), 6 is the least number of voters such that all \( n \)-candidate unlabeled weak tournaments are isomorphic to the majority graph of a profile with that number of voters.

In this paper, we are interested in the following problem: given an impossibility theorem proved for C1 (or pairwise) voting methods, can we generalize that impossibility theorem to hold for all voting methods? That is, can we drop the C1 (or pairwise) axiom and still prove the resulting impossibility theorem?

Our high-level strategy to do so is the following, which we will sketch for the C1 case. Where VM is the class of all voting methods and C1 is the class of C1 voting methods, we want a map

\[ \pi_{C1} : VM \rightarrow C1 \]

that preserves the axioms in the relevant impossibility theorem, i.e., if \( F \in VM \) satisfies the axioms, then so does \( \pi_{C1}(F) \). Since by the impossibility theorem, no member of C1 satisfies the axioms, it follows that no member of VM does.

Our strategy is to obtain \( \pi_{C1} \) as a composition of maps. First, we have the map \( \phi_{C1} \) that sends each profile to its majority graph. Second, we will choose an appropriate map \( \chi \) that sends each majority graph to a profile \( P \). Then we will define \( \pi_{C1} \) such that for each \( F \in VM \), the voting method \( \pi_{C1}(F) \) is defined by

\[ \pi_{C1}(F)(P) = F(\chi(\phi_{C1}(P))). \]

It is immediate that if \( \phi_{C1}(P) = \phi_{C1}(P') \), then \( \pi_{C1}(F)(P) = \pi_{C1}(F)(P') \), which shows \( \pi_{C1}(F) \in C1 \). The crucial fact to prove, however, is that \( \pi_{C1} \) preserves the axioms of the relevant impossibility theorem. This requires a careful choice of \( \chi \).

The properties required of \( \chi \) depend on the specific axioms in the impossibility theorem. As an example, let us consider the well-known axiom of neutrality. Given a profile \( P \) and candidates \( a \) and \( b \), let \( P_{a\leftrightarrow b} \) be the profile obtained from \( P \) by swapping the position of \( a \) and \( b \) in each voter’s ranking. Given a set \( Y \) of candidates, let \( Y_{a\leftrightarrow b} \) be the image of \( Y \) under the permutation that only swaps \( a \) and \( b \). Then a voting method \( F \) satisfies neutrality if for any profile \( P \),

\[ F(P_{a\leftrightarrow b}) = F(P)_{a\leftrightarrow b}. \]

Given a graph \( G \) and vertices \( a \) and \( b \), let \( G_{a\leftrightarrow b} \) be the isomorphic copy of \( G \) with \( a \) and \( b \) swapped. Then in order to show that \( \chi \) preserves neutrality, we would like
to show that for any majority graph $G$, we have
\[ \chi(G_{a \leftrightarrow b}) = \chi(G)_{a \leftrightarrow b}. \]
As we shall see, this commutativity requirement rules out a standard construction of profiles from majority graphs [McG53] as a candidate for $\chi$.

Though we have sketched the strategy above for transferring impossibility theorems for C1 voting methods to all voting methods, an analogous strategy applies to transferring impossibility theorems for pairwise voting methods to all voting methods using an appropriate map $\pi_{\text{pairwise}}$. The goal of this paper to develop these strategies in detail and to apply them to the generalization of some important impossibility theorems in voting theory.

The rest of the paper is organized as follows. In Section 2, we explain the impossibility theorems to which we will apply the strategy of generalization described above; the key axioms in these impossibility theorems apply to social choice some fundamental axioms from the theory of individual rational choice, namely versions of Sen’s $\gamma$ and $\alpha$ axioms [Sen71]. The SAT-based proofs of these initial impossibility theorems are explained in the Appendix, which includes a link to Jupyter notebooks containing the SAT code. In Section 3 we lay out the technical preliminaries needed for our approach to generalizing the initial impossibility theorems and related results. In Section 4 we state the impossibility theorems for C1 and pairwise voting methods that we wish to generalize. We then carry out the strategy of generalization sketched above in Sections 5–7. Two versions of the $\chi$ map discussed above are given in Definitions 5.2 and 5.4 of Section 5. Versions of the maps $\pi_{\text{c1}}$ and $\pi_{\text{pairwise}}$ are given in Definition 6.1 and 6.2 in Section 6. We then show that these maps preserve the axioms of the relevant impossibility theorems in Section 7. In Section 8 we put everything together to prove our generalized impossibility theorems. We conclude in Section 9 with some suggestions for future research.

2. The impossibility theorems

2.1. Origins in Individual Choice. In this paper, we consider impossibility results arising from the attempt to transfer axioms on individual choice to social choice. We briefly recall the necessary background on individual choice. Given a finite set $X$, a choice function is a map $C : \wp(X) \setminus \{\emptyset\} \to \wp(X) \setminus \{\emptyset\}$ such that for all $S \in \wp(X) \setminus \{\emptyset\}$, we have $\emptyset \neq C(S) \subseteq S$. The intuitive interpretation is that when offered a choice from the menu $S$, the agent considers all the elements in $C(S)$ choiceworthy. To force the choice of a single alternative, one can impose the following axiom on $C$:

**resoluteness**: $|C(S)| = 1$ for all $S \in \wp(X) \setminus \{\emptyset\}$.

The classic question concerning choice functions is a representation question: under what conditions can we view the agent as if she chooses from any given menu the greatest elements according to a preference relation? Where $R$ is a binary relation on $X$, we say that $C$ is represented by $R$ if for all $S \in \wp(X) \setminus \{\emptyset\}$,

$$C(S) = \{x \in S \mid \text{for all } y \in S, xRy\}.$$
Sen [Sen71] identified the following axioms on $C$ that are necessary and sufficient for representability by a binary relation:
\[
\begin{align*}
\alpha: & \text{ if } S \subseteq S', \text{ then } S \cap C(S') \subseteq C(S); \\
\gamma: & \text{ if } C(S) \cap C(S') \subseteq C(S \cup S').
\end{align*}
\]

**Theorem 2.1** (Sen). Let $C$ be a choice function on a finite set $X$.

1. $C$ is representable by a binary relation on $X$ iff $C$ satisfies $\alpha$ and $\gamma$.
2. If $C$ is resolute, then the following are equivalent:
   1. $C$ is representable by a binary relation on $X$;
   2. $C$ is representable by a linear order on $X$;
   3. $C$ satisfies $\alpha$.

In this paper, we shall see that there are difficulties transferring even weaker versions of the above axioms from individual choice to social choice. First, consider the following weakening of $\gamma$:

**Binary $\gamma$**: if $x \in C(A)$ and $C(\{x, y\}) = \{x\}$, then $x \in C(A \cup \{y\})$.

As Bordes [Bor83] puts it (in the context of choice between candidates in voting, to which we turn in Section 2.2), “if $x$ is a winner in $A$ . . . one cannot turn $x$ into a loser by introducing new alternatives to which $x$ does not lose in duels” (p. 125).

In fact, Bordes’ formulation suggests replacing $C(\{x, y\}) = \{x\}$ by $x \in C(\{x, y\})$, resulting in a slightly stronger axiom, but we will work with the weaker version, as impossibility results with weaker axioms are stronger results. Binary $\gamma$ is an obvious consequence of $\gamma$, setting $S = A$ and $S' = \{x, y\}$.

We will also consider a weakening of $\alpha$ with the same antecedent as binary $\gamma$:

**Binary $\alpha$**: if $x \in C(A)$ and $C(\{x, y\}) = \{x\}$, then $C(A \cup \{y\}) \subseteq C(A)$.

In other words, adding to the menu an alternative that loses to an initially chosen alternative does not lead to any new choices from the menu. That binary $\alpha$ follows from $\alpha$ is not quite as immediate as the analogous implication for $\gamma$—but almost.

**Proposition 2.2.** Any choice function satisfying $\alpha$ also satisfies binary $\alpha$.

**Proof.** Suppose $x \in C(A)$ and $C(\{x, y\}) = \{x\}$. Since $\{x, y\} \subseteq A \cup \{y\}$, we have $\{x, y\} \cap C(A \cup \{y\}) \subseteq C(\{x, y\})$ by $\alpha$. The since $y \notin C(\{x, y\})$, we conclude $y \notin C(A \cup \{y\})$. Hence $C(A \cup \{y\}) \subseteq A$. By $\alpha$ again, $A \cap C(A \cup \{y\}) \subseteq C(A)$, which with the previous sentence implies $C(A \cup \{y\}) \subseteq C(A)$. \qed

Finally, we will consider a weakening of resoluteness. This weakening says that while some non-singleton choice sets may be allowed, we should never enlarge the alternative that loses to some initially chosen alternative:

**$\alpha$-resoluteness**: if $x \in C(A)$ and $C(\{x, y\}) = \{x\}$, then $|C(A \cup \{y\})| \leq |C(A)|$.

Clearly $\alpha$-resoluteness is implied by binary $\alpha$ and by resoluteness.

**2.2. From Individual Choice to Social Choice.** We now explain how the axioms on individual choice from the previous section can be applied to social choice. Here we proceed informally, deferring formal definitions to later sections.
A profile $P$ assigns to each voter in an election a ranking of the set $X(P)$ of candidates in the election, which we will assume is a linear order without ties. Given two candidates $x, y$, we say that $x$ is majority preferred to $y$ in $P$ if more voters rank $x$ above $y$ than rank $y$ above $x$ (i.e., $x$ has a positive margin over $y$ as in Section 4). Given a set $A$ of candidates, the profile $P|_A$ assigns to each voter their ranking of the candidates in $P$ restricted to $A$, and for a particular candidate $y$, we write $P_{-y} = P|_{X(P)\{y\}}$. A voting method $F$ assigns to each profile $P$ a nonempty set $F(P)$ of candidates who are tied for winning the election.

All of the axioms on individual choice functions from the previous section correspond to axioms on voting methods as follows:

1. **Resoluteness**: $|F(P)| = 1$ for all profiles $P$;
2. **Binary $\gamma$**: if $x \in F(P_{-y})$ and $F(P|_{\{x,y\}}) = \{x\}$, then $x \in F(P)$;
3. **Binary $\alpha$**: if $x \in F(P_{-y})$ and $F(P|_{\{x,y\}}) = \{x\}$, then $F(P) \subseteq F(P_{-y})$;
4. **$\alpha$-Resoluteness**: if $x \in F(P_{-y})$ and $F(P|_{\{x,y\}}) = \{x\}$, then $|F(P)| \leq |F(P_{-y})|$.

We will be interested in the interaction of the above axioms borrowed from individual choice with some standard axioms on voting methods that are distinctively social-choice theoretic:

1. **Anonymity**: whenever $P$ and $P'$ differ only in swapping the rankings assigned to two voters, we have $F(P) = F(P')$;
2. **Neutrality**: whenever $P$ and $P'$ differ only in swapping the position of two candidates in each voter’s ranking, then $F(P)$ and $F(P')$ differ only in swapping those two candidates.

One of the most basic facts in voting theory is that there is no voting method satisfying anonymity, neutrality, and resoluteness. In fact, the problem already arises with two candidates:

1. **Binary Resoluteness**: $|F(P)| = 1$ for all profiles $P$ such that $|X(P)| = 2$.

Consider a profile $P$ with only two candidates $x, y$ and an even number of voters, half of whom rank $x$ above $y$ and half of whom rank $y$ above $x$. Then anonymity and neutrality imply $F(P) = \{x, y\}$, violating resoluteness.

---

2. This linearity assumption strengthens our impossibility results. That is, our impossibility results for voting methods defined only on profiles of linear orders immediately imply corresponding impossibility results for voting methods defined on profiles of weak orders; for if there were a voting method satisfying the relevant axioms defined on profiles of weak orders, then restricting the voting method to profiles of linear orders would yield the kind of voting method we will prove does not exist. By contrast, proving that no voting method defined on profiles of weak orders satisfies certain axioms does not immediately imply that no voting method defined only on linear orders satisfies the axioms. Selecting winners in profiles of weak orders may be more difficult.

3. This statement builds the axiom of “universal domain” into the definition of a voting method, by requiring that the voting method is defined on all profiles. In fact, the domain of $F$ contains profiles with different sets of candidates, so we are in a variable-candidate setting.

4. Note that binary $\gamma$ for choice functions can be written equivalently as: if $x \in C(B \setminus \{y\})$ and $C(\{x,y\}) = \{x\}$, then $x \in C(B)$. 
Proposition 2.3. There is no voting method satisfying anonymity, neutrality, and binary resoluteness.

A natural response is to weaken binary resoluteness to the following:

**binary quasi-resoluteness**: $|F(P)| = 1$ for all profiles $P$ such that $|X(P)| = 2$ and for $x, y \in X(P)$ with $x \neq y$, the number of voters ranking $x$ above $y$ is not equal to the number of voters ranking $y$ above $x$.

Unlike binary resoluteness, binary quasi-resoluteness is consistent with anonymity and neutrality. Indeed, almost all standard voting methods satisfy binary quasi-resoluteness, including, e.g., the Plurality voting method that selects in a given profile the candidates with the most first-place votes.

In this paper, we are interested in impossibility results that arise by adding axioms to the initial package of anonymity, neutrality, and binary quasi-resoluteness. First, we add binary $\gamma$. As argued in [HP20], binary $\gamma$ can be viewed as an axiom that mitigates spoiler effects in elections.

Consider one of the most famous U.S. elections involving a spoiler effect: the 2000 Presidential Election in Florida. Though the election did not collect ranked ballots from voters, polling suggests that Gore would have beaten Bush in the two-candidate election $P \{(Bush, Gore)\}$, and Gore would have beaten Nader in the two-candidate election $P \{(Gore, Nader)\}$, yet Gore was not a winner in the three-candidate election $P \{(Bush, Gore, Nader)\}$ according to Plurality voting. Thus, Plurality voting violates binary $\gamma$.

In the example above, Gore is the *Condorcet winner*, a candidate who is majority preferred to every other candidate. Thus, the example also shows that Plurality voting violates the property of *Condorcet consistency*, which requires that a method selects the Condorcet winner whenever one exists. However, binary $\gamma$ and Condorcet consistency are independent properties. For example, the Condorcet consistent methods Beat Path [Sch11, Sch13], Minimax [Sim69, Kra77], and Ranked Pairs [Tid87, ZT89] all violate binary $\gamma$. Conversely, the trivial voting method that always declares all candidates tied for the win satisfies binary $\gamma$ but not Condorcet consistency. However, there is a connection between binary $\gamma$ and the concept of the Condorcet winner: if a voting method satisfies binary $\gamma$ and reduces to majority voting in two-candidate elections, then the Condorcet winner, if one exists, is always among the winners of the election. In fact, the examples of voting methods in the literature satisfying binary $\gamma$, including Banks [Ban85], Top Cycle [Smi73, Sch86], Uncovered Set [Mil80, Dug13], and Split Cycle [HP20, HP21], are all Condorcet consistent. Thus, Condorcet consistency and binary $\gamma$ are cross-cutting but related properties, as shown by the list of voting methods in Figure 3.

Though we can consistently add binary $\gamma$ to our initial package of axioms, adding binary $\gamma$ together with even the weakest consequence of $\alpha$ that we have discussed—$\alpha$-resoluteness—leads to an impossibility theorem. We will prove a first

---

5The axiom called *stability for winners* in [HP20] is almost the same as binary $\gamma$, but with $'F(P \{|x,y\}) = \{x\}'$ in the statement of binary $\gamma$ replaced by ‘the margin of $x$ over $y$ in $P$ is positive’. Assuming $F$ reduces to majority voting in two-candidate profiles, $F$ satisfying binary $\gamma$ is equivalent to $F$ satisfying stability for winners.

6Of course there were more than three candidates in this election, but this simplified example contains the essential points.
GENERALIZING SOME IMPOSSIBILITY THEOREMS IN SOCIAL CHOICE

| methods violating binary $\gamma$ | methods satisfying binary $\gamma$ |
|-----------------------------------|-----------------------------------|
| Beat Path*                        | Banks*                            |
| Borda                             | Top Cycle*                         |
| Instant Runoff                    | Uncovered Set*                     |
| Minimax*                          | Split Cycle*                       |
| Plurality                         | Ranked Pairs*                      |

**Figure 3.** Examples of methods violating or satisfying binary $\gamma$. Condorcet consistent voting methods are marked with a $\star$.

A version of the theorem here and then explain how the theorem can be strengthened with the help of SAT solving.

**Theorem 2.4.** There is no voting method satisfying anonymity, neutrality, binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness.

**Proof.** Suppose for contradiction that there is a voting method $F$ satisfying the axioms. By anonymity, neutrality, and binary quasi-resoluteness, for every possible margin $m > 0$ and number $k$ of voters, either (i) for every two-candidate profile with $k$ voters in which one candidate has a margin of $m$ over the other, the unique winner is the majority winner or (ii) for every two-candidate profile with $k$ voters in which one candidate has a margin of $m$ over the other, the unique winner is the minority winner. For $m = 6$ and $k = 18$, suppose that (i) holds; the proof in case (ii) is analogous. Now consider the profile $P$ in Figure 4 whose margin graph is shown in Figure 5 (all arrows in/out of a rectangle go in/out of every node in the rectangle). By (i), if there is an arrow from $x$ to $y$ in Fig. 5, then $F(P|\{x,y\}) = \{x\}$.

**Figure 4.** We will repeatedly use the fact that in a perfectly symmetrical three-candidate profile realizing a majority cycle, such as $P_{\{a,b,c\}}$, $P_{\{b,c,d\}}$, and $P_{\{a,b,c\}}$, anonymity and neutrality imply that every candidate is chosen. Thus, $c \in F(P_{\{a,b,c\}})$, so $c \in F(P_{\{a,a',a'',b,c\}})$ by two applications of binary $\gamma$. By anonymity and neutrality we also have that

$$\{a,a',a''\} \cap F(P_{\{a,a',a'',b,c\}}) \neq \emptyset$$

if and only if $\{a,a',a''\} \subseteq F(P_{\{a,a',a'',b,c\}})$. Thus, one of the following three cases holds.
and hence contradict \( \alpha \). Moreover, by anonymity and neutrality, \( d \in F(\mathbf{P}_{\{b, c, d\}}) \), so \( d \in F(\mathbf{P}_{\{a, a', a'', b, c, d\}}) \) by three applications of binary \( \gamma \). But together \( F(\mathbf{P}_{\{a, a', a'', b, c\}}) = \{c\}, F(\mathbf{P}_{\{c, d\}}) = \{c\}, \) and \( c, d \in F(\mathbf{P}_{\{a, a', a'', b, c, d\}}) \) contradict \( \alpha \)-resoluteness.

Case 2: \( F(\mathbf{P}_{\{a, a', a'', b, c\}}) = \{b, c\} \). Hence \( b \in F(\mathbf{P}_{\{a, a', a'', b, c, e\}}) \) by binary \( \gamma \). By anonymity and neutrality,

\[
\{c, c'\} \cap F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \neq \emptyset \quad \text{if and only if} \quad \{c, c'\} \subseteq F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}),
\]

and

\[
\{a, a', a''\} \cap F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \neq \emptyset \quad \text{if and only if} \quad \{a, a', a''\} \subseteq F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}).
\]

Then given \( F(\mathbf{P}_{\{a, a', a'', b, c\}}) = \{b, c\} \) and \( F(\mathbf{P}_{\{b, c'\}}) = \{b\} \), \( \alpha \)-resoluteness implies \( F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) = \{b\} \). Next, by anonymity and neutrality, \( e \in F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \) and hence \( e \in F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \) by four applications of binary \( \gamma \). Moreover, given \( F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) = \{b\} \), we have \( b \in F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \) by binary \( \gamma \). But together \( F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) = \{b\}, F(\mathbf{P}_{\{b, c\}}) = \{b\}, \) and \( b, e \in F(\mathbf{P}_{\{a, a', a'', b, c, c'\}}) \) contradict \( \alpha \)-resoluteness.

Case 3: \( \{c, a, a', a''\} \subseteq F(\mathbf{P}_{\{a, a', a'', b, c\}}) \). By anonymity and neutrality, we have \( F(\mathbf{P}_{\{a, b, c\}}) = \{a, b, c\} \). Then since \( F(\mathbf{P}_{\{a'\}}) = \{c\} \), it follows by binary \( \gamma \) and \( \alpha \)-resoluteness that \( c \in F(\mathbf{P}_{\{a, a', b, c\}}) \) and \( |F(\mathbf{P}_{\{a, a', b, c\}})| \leq 3 \), which with \( F(\mathbf{P}_{\{a, a', a''\}}) = \{c\} \) and \( \alpha \)-resoluteness implies \( |F(\mathbf{P}_{\{a, a', a'', b, c\}})| \leq 3 \), which contradicts the assumption of the case.

\[\square\]

**Remark 2.5.**

1. If we drop either anonymity or neutrality, then there are voting methods satisfying the remaining axioms from Theorem 2.4. For an anonymous (resp. neutral) voting method \( F \) satisfying the other axioms, fix a linear order \( L \) of the set of all possible candidates (resp. for each possible set \( V \) of voters, fix a voter \( i_\mathbf{v} \in V \), and for any profile \( \mathbf{P} \) (whose set of voters is \( V(\mathbf{P}) \)), let \( F(\mathbf{P}) = \{x\} \) where \( x \) is the maximal element in the linear order \( L \) restricted to \( X(\mathbf{P}) \) (resp. in \( i_\mathbf{v}(\mathbf{P}) \)’s linear order of \( X(\mathbf{P}) \)) such that no candidate ranked below \( x \) in this linear order is majority preferred to \( x \).
Observe that $F$ satisfies binary $\gamma$. Then since $F$ always selects a unique winner, it also satisfies $\alpha$-resoluteness.

(2) If we drop binary quasi-resoluteness, then there are voting methods satisfying the remaining axioms from Theorem 2.4, e.g., the trivial voting method that always selects all candidates as winners (note that this method trivially satisfies binary $\gamma$ and $\alpha$-resoluteness, because the condition $F(P\{(x,y)\}) = \{x\}$ is always false).

(3) It is open whether if we drop binary $\gamma$, there are voting methods satisfying the remaining axioms from Theorem 2.4. However, we do know from SAT solving (see the Appendix) that there are voting methods satisfying the remaining axioms at least for profiles having up to 6 candidates.

(4) If we drop $\alpha$-resoluteness, then all the voting methods in the right column of Figure 3 satisfy the remaining axioms from Theorem 2.4.

Because axioms like binary $\gamma$ and $\alpha$-resoluteness are variable candidate axioms, involving choosing winners from elections involving different sets of candidates, we are especially interested in exactly how many candidates are needed to trigger impossibility results. The proof of Theorem 2.4 only uses the assumption that the voting method satisfies the relevant axioms for profiles having up to seven candidates, rather than for all profiles. Thus, it proves the stronger statement that there is no voting method satisfying anonymity, neutrality, binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness for profiles of up to seven candidates. In fact, using the methods to be introduced in this paper, we can prove the following.

**Theorem 2.6.** There is no voting method satisfying anonymity, neutrality, binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness for profiles up to 6 candidates.

We prove Theorem 2.6 as follows: first, we use SAT solving to show that there is no C1 voting method satisfying the relevant axioms; then, using the strategy outlined Section 1, we generalize the SAT-based C1 impossibility theorem to Theorem 2.6. This result is optimal in terms of the number of candidates, since we know via SAT solving (see the Appendix) that there are voting methods satisfying all of the axioms for profiles up to 5 candidates.

While binary $\gamma$ and $\alpha$-resoluteness for individual choice functions are easily satisfied—in particular by any choice function representable by a binary relation—Theorems 2.4 and 2.6 show that binary $\gamma$ and $\alpha$-resoluteness for voting methods are together inconsistent with some of the most uncontroversial axioms for voting, namely anonymity, neutrality, and binary quasi-resoluteness.

A natural response to Proposition 2.3 and Theorem 2.6 is that asking for any kind of resoluteness property with respect to all profiles, including highly symmetric profiles, is asking for too much. This motivates a move to more qualified resoluteness properties that only apply to profiles with sufficiently broken symmetries. For example, call a profile $P$ uniquely weighted if for any two distinct pairs $a, b$ and $x, y$ of distinct candidates, the margin of $a$ over $b$ in $P$ (recall Section 1) is not equal to the margin of $x$ over $y$ in $P$. Thus, the profile in Figure 3 is not uniquely weighted.

Optimizing these impossibility results for the number of voters is an interesting subject for future research.

By “satisfying the axioms for profiles up to seven candidates,” we mean that all the quantification over profiles in the statements of the axioms should be restricted to profiles with up to seven candidates.
By contrast, in elections with many voters, one would expect with high probability to obtain a uniquely weighted profile. Now consider restricting resoluteness only to uniquely weighted profiles:

**quasi-resoluteness**: $|F(P)| = 1$ for all uniquely weighted profiles $P$.

There are many voting methods satisfying the trio of anonymity, neutrality, and quasi-resoluteness, including Minimax, Ranked Pairs, and Beat Path, as well as any voting method that uses a quasi-resolute voting method as a tiebreaker, e.g., Plurality with Minimax tiebreaking (in the event of a tie for the most first place votes). However, we will show that even this weakening of resoluteness is incompatible with basic principles of individual rational choice applied to voting—in particular, with the binary $\gamma$ axiom alone.

**Theorem 2.7.** There is no voting method satisfying anonymity, neutrality, binary $\gamma$, and quasi-resoluteness.

Theorem 2.8 explains the observation from the literature that voting methods satisfying binary $\gamma$, listed in Figure 3, all violate quasi-resoluteness, while the quasi-resolute methods mentioned above all violate binary $\gamma$.

Unlike in the case of Theorem 2.4 to prove Theorem 2.7 we rely essentially on SAT solving to show there is no pairwise voting method satisfying the relevant axioms and then on our strategy, sketched in Section 1, for generalizing impossibility theorems for pairwise methods to all voting methods in order to obtain Theorem 2.7. At the same time, we obtain the following strengthening of Theorem 2.7 that is optimized in terms of the number of candidates required.

**Theorem 2.8.** There is no voting method satisfying anonymity, neutrality, binary $\gamma$, and quasi-resoluteness for profiles of up to 4 candidates.

By contrast, all of the voting methods in the right column of Figure 3 satisfy the relevant axioms for profiles up to 3 candidates.

In the rest of the paper, we will build up to proofs of Theorems 2.6 and Theorem 2.8. As we have seen, these impossibility theorems show that core principles of individual rational choice (viz. weakenings of $\gamma$ and $\alpha$) are inconsistent with natural principles of social choice. Though these impossibility results raise important conceptual questions about which axioms ought to be weakened in response, in this paper we focus only on the methodology for proving the results themselves. We expect that this methodology will be applicable to other significant impossibility results in social choice.

### 3. Preliminaries

**3.1. Profiles.** Fix an infinite set $X$ of candidates. For later use, we fix a distinguished strict linear order $L_\star$ on $X$. For $X \subseteq X$, let $L(X)$ be the set of all strict linear orders on $X$. Given $L \in L(X)$, we write $aLb$ for $(a, b) \in L$. We may display a linear order as, e.g., $abcd$ to indicate that $aLb$, $bLc$, and $cLd$, etc.

---

9Note that Minimax, Ranked Pairs, and Beat Path are pairwise voting methods, while Plurality with Minimax tiebreaking is not pairwise.

10For a human-readable proof that binary $\gamma$ and quasi-resoluteness are incompatible with an axiom satisfied by many qualitative-pairwise voting methods, see [HP20, Theorem 6.20] (which phrases the result in terms of *stability for winners* instead of binary $\gamma$).
Definition 3.1. An anonymous profile—henceforth profile—is a function $P$ such that for some finite $X(P) \subset \mathcal{X}$, we have $P : \mathcal{L}(X(P)) \rightarrow \mathbb{N}$.

It will be convenient for certain intermediate calculations to be able to work with a generalization of anonymous profiles allowing the number of voters who submit a certain ballot to be negative (cf. [Saa08, p. 135]).

Definition 3.2. A generalized anonymous profile—henceforth generalized profile—is a function $P$ such that for some finite $X(P) \subset \mathcal{X}$, we have $P : \mathcal{L}(X(P)) \rightarrow \mathbb{Z}$.

We define the margin of $a$ over $b$ in $P$, denoted $\mu_{a,b}(P)$, as in Section 1.

Definition 3.3. Given a generalized profile $P$ and $a,b \in X(P)$, we define

$$\mu_{a,b}(P) = \left( \sum_{L \in \mathcal{L}(X(P)) : a \triangleright b} P(L) \right) - \left( \sum_{L \in \mathcal{L}(X(P)) : b \triangleright a} P(L) \right).$$

Addition, subtraction, and multiplication of profiles are defined as expected.

Definition 3.4. Given generalized profiles $P$ and $P'$ with $X(P) = X(P')$, we define $P + P'$ with $X(P + P') = X(P)$ (resp. $P - P'$ with $X(P - P') = X(P)$) such that for all $L \in \mathcal{L}(X(P))$,

$$(P + P')(L) = P(L) + P'(L)$$
and $$(P - P')(L) = P(L) - P'(L).$$

Lemma 3.5. For any generalized profiles $P, P'$ with $X(P) = X(P')$ and $a,b \in X(P)$:

$$\mu_{a,b}(P + P') = \mu_{a,b}(P) + \mu_{a,b}(P')$$
$$\mu_{a,b}(P - P') = \mu_{a,b}(P) - \mu_{a,b}(P').$$

Definition 3.6. Given a profile $P$ and $n \in \mathbb{Z}$, we define the profile $nP$ with $X(nP) = X(P)$ such that for all $L \in \mathcal{L}(X(P))$,

$$(nP)(L) = n \times P(L).$$

For the next operation of transposition, given a linear order $L$ on $X \subseteq \mathcal{X}$ and $a,b \in X$, let $L_{a \leftrightarrow b}$ be the linear order obtained from $L$ by switching the positions of $a$ and $b$. Transposition switches the positions of $a$ and $b$ on each voter’s ballot.

Definition 3.7. Given a generalized profile $P$ and $a,b \in X(P)$, we define the profile $P_{a \leftrightarrow b}$ such that for any linear order $L$ on $X(P)$,

$$P_{a \leftrightarrow b}(L) = P(L_{a \leftrightarrow b}).$$

Finally, for each finite set $X$ of candidates, we define two distinguished profiles: a profile in which each linear order on $X$ is submitted by exactly one voter and a profile with only one voter who submits the restriction to $X$ of the distinguished linear order $L_*$ on $\mathcal{X}$ fixed above.

Definition 3.8. Given a finite $X \subset \mathcal{X}$, define the profile $L_X$ such that for all $L \in \mathcal{L}(X)$, $L_X(L) = 1$; define the profile $P_{*,X}$ such that $P_{*,X}(L) = 1$ if $L = L_* \cap X^2$ and $P_{*,X}(L) = 0$ otherwise.
3.2. Tournaments, majority graphs, weighted tournaments, and margin graphs. In this section, we go over more formally the relation between profiles and majority graphs (resp. margin graphs) sketched in Section 1. Recall that we fixed an infinite set $\mathcal{X}$ of candidates.

**Definition 3.9.** A weak tournament is a directed graph $T = (X(T), \to)$ where $X(T)$ is a finite subset of $\mathcal{X}$ and the edge relation $\to$ is asymmetric: for all $x, y \in X(T)$, if $x \to y$, then not $y \to x$. A tournament is a weak tournament whose edge relation is connected: for all $x, y \in X(T)$, if $x \neq y$, then $x \to y$ or $y \to x$. Let $\mathcal{W}_T$ be the set of weak tournaments and $\mathcal{T}$ the set of tournaments.

**Definition 3.10.** A weighted weak tournament (resp. weighted tournament) is a pair $T = (X(T), w)$ where $X(T)$ is a finite subset of $\mathcal{X}$ and $w : X(T)^2 \to \mathbb{Z}$ (resp. $w : X(T)^2 \to \mathbb{Z} \setminus \{0\}$) is such that for all $a, b \in X(T)$, $w(a, b) = -w(b, a)$. Let $\mathcal{W}_T^w$ be the set of weighted weak tournaments and $\mathcal{T}^w$ the set of weighted tournaments.

We say that a weighted tournament $T$ is uniquely weighted if for all $(a, b), (c, d) \in X(T)^2$ with $a \neq b$, $c \neq d$, and $(a, b) \neq (c, d)$, we have $w(a, b) \neq w(c, d)$.

Subtraction and multiplication for weighted weak tournaments works as expected.

**Definition 3.11.** Given weighted weak tournaments $T_1 = (X(T_1), w_1)$ and $T_2 = (X(T_2), w_2)$ with $X(T_1) = X(T_2)$, we define $T_1 - T_2 = (X(T_1), w_{1-2})$ by $w_{1-2}(a, b) = w_1(a, b) - w_2(a, b)$.

Given a weighted weak tournament $T = (X(T), w)$ and $n \in \mathbb{Z}^+$, we define $nT = (X(T), w_n)$ by $w_n(a, b) = n \times w(a, b)$.

The transposition operation on profiles (Definition 3.7) has the following analogue for weak tournaments and weighted weak tournaments.

**Definition 3.12.** Given a weak tournament $T = (X(T), \to)$ and $a, b \in X(T)$, we define the weak tournament $T_{a \leftrightarrow b} = (X(T), \to_{a \leftrightarrow b})$ such that for all $x, y \in X(T)$, we have $x \to_{a \leftrightarrow b} y$ in $T_{a \leftrightarrow b}$ if $\pi_{a \leftrightarrow b}(x) \to \pi_{a \leftrightarrow b}(y)$ in $T$, where $\pi_{a \leftrightarrow b}$ is the bijection on $X(T)$ with $\pi_{a \leftrightarrow b}(a) = b$, $\pi_{a \leftrightarrow b}(b) = a$, and $\pi_{a \leftrightarrow b}(x) = x$ for $x \in X(T) \setminus \{a, b\}$. Similarly, for a weighted weak tournament $T = (X(T), w)$, we define $T_{a \leftrightarrow b} = (X(T), w_{a \leftrightarrow b})$ where $w_{a \leftrightarrow b}(x, y) = w(\pi_{a \leftrightarrow b}(x), \pi_{a \leftrightarrow b}(y))$.

Each profile gives rise to a weak tournament and weighted weak tournament as follows.

**Definition 3.13.** Given a profile $P$, we define $M(P)$, the majority graph of $P$ (resp. $M(P)$, the margin graph of $P$) to be the directed graph (resp. weighted directed graph) whose set of vertices is $X(P)$ with an edge from $a$ to $b$ (resp. weighted by $\mu_{a,b}(P)$) if $\mu_{a,b}(P) > 0$.

The following result is standard and straightforward to prove.

---

11Here we use the term ‘weak tournament’ as in [BF07].
Proposition 3.14. For any profile \( P \), \( M(P) \) is a weak tournament (resp. \( M(P) \) is a weighted weak tournament in which all weights have the same parity). If the number of voters in \( P \) is odd, then \( M(P) \) is a tournament (resp. \( M(P) \) is a weighted tournament).

Later we will use the obvious fact that transposition commutes with taking the majority or margin graph of a profile and with multiplication.

Lemma 3.15. For any profile \( P \) and \( a, b \in X(P) \), \( M(P_{a=b}) = M(P)_{a=b} \) and \( M(P_{a=b}) = M(P)_{a=b} \), and for any weighted weak tournament \( T \) and \( n \in \mathbb{Z}^+ \), \( n(T_{a=b}) = (nT)_{a=b} \).

As a converse to Proposition 3.14, McGarvey [McG53] showed that every weak tournament can be represented as the majority graph of a profile, and Debord [Deb87] generalized the result to weighted weak tournaments. We review the construction for comparison with our alternative representation in Section 5.

Definition 3.16. For any weighted weak tournament \( T \) such that all weights are even numbers, we define the profile \( \text{Deb}(T) \) as follows. For each \( a, b \in X(T) \) such that \( a \rightarrow b \) in \( T \) with weight \( n \), where \( x_1, \ldots, x_k \) enumerates \( X(T) \setminus \{a,b\} \) according to the order \( L_a \) (recall Section 3.1), we add \( \frac{n}{2} \) voters with the linear order \( abx_1 \ldots x_k \) and \( \frac{n}{2} \) voters with the linear order \( x_k \ldots x_1ab \).

For a weighted weak tournament \( T \) with odd weights, we define the profile \( \text{Deb}(T) \) as follows, using \( P_{\ast,X(T)} \) from Definition 3.3. As \( T - M(P_{\ast,X(T)}) \) has even number weights, we set

\[
\text{Deb}(T) = \text{Deb}(T - M(P_{\ast,X(T)})) + P_{\ast,X(T)}.
\]

Finally, for a weak tournament \( T \), let \( T_2 \) be the weighted weak tournament based on \( T \) in which each edge has weight 2, and define \( \text{Mc}(T) = \text{Deb}(T_2) \).

It is easy to see that

\[ M(\text{Deb}(T)) = T, \]

which yields the following.

Theorem 3.17.

1. [McG53] Every weak tournament is the majority graph of a profile.
2. [Deb87] If \( T \) is a weighted weak tournament in which all weights have the same parity, which is even if \( T \) is not a weighted tournament, then \( T \) is the margin graph of a profile.

3.3. Restriction to a set of candidates. Our variable candidate axioms from Section 2.2 such as binary \( \gamma \) and \( \alpha \)-resoluteness, involve the restriction of a profile or tournament to a set of candidates. Given a binary relation \( R \) on a set \( X \) and \( Y \subseteq X \), the restriction of \( R \) to \( Y \) is the binary relation \( R_{\mid Y} \) on \( Y \) such that for all \( a, b \in Y \), \( (a, b) \in R_{\mid Y} \) if and only if \( (a, b) \in R \).

Definition 3.18. Given a profile \( P \) and \( Y \subseteq X(P) \), the restriction of \( P \) to \( Y \) is the profile \( P_{\mid Y} \) with \( X(P_{\mid Y}) = Y \) such that for all \( L \in \mathcal{L}(Y) \),

\[
P_{\mid Y}(L) = \sum_{L' \in \mathcal{L}(X(P) \setminus \{x\}) : L = L'_{\mid Y}} P(L')
\]

For \( x \in X(P) \), we write \( P_{-x} \) for \( P_{\mid X(P) \setminus \{x\}} \).
profiles are in the domain of election. methods, which assign to each profile a set of candidates tied for winning the ∅ and for any a,b weak tournament is defined analogously, with the weight function restricted to ∈ X(T), and for any a,b ∈ Z, (P_{T|Z})_{a\leftrightarrow b} = (P_{a\leftrightarrow b})_{|Z}. For any weighted weak tournament T, ∅ ≠ Z ⊆ X(T), and n ∈ Z^+, n(T|Z) = (nT)|Z, and for any a,b ∈ Z, (T_{|Z})_{a\leftrightarrow b} = (T_{a\leftrightarrow b})_{|Z}.

3.4. Voting methods. Our main objects of interest in this paper are voting methods, which assign to each profile a set of candidates tied for winning the election.

Definition 3.21. An anonymous voting method (henceforth voting method) is a function F whose domain is a set of profiles such that for any P ∈ dom(F), we have ∅ ≠ F(P) ⊆ X(P). A voting method F satisfies universal domain if all profiles are in the domain of F.

We recall the key invariance axioms on voting methods that we will consider.

Definition 3.22. Let F be a voting method.
(1) F is C1 if for all P, P' ∈ dom(F), M(P) = M(P') implies F(P) = F(P').
(2) F is pairwise if for all P, P' ∈ dom(F), M(P) = M(P') implies F(P) = F(P').

Finally, we officially define the axioms on voting methods from Section 2.2.

Definition 3.23. Let F be a voting method.
(1) F is neutral if for all P ∈ dom(F) and a, b ∈ X(P), if P_{a\leftrightarrow b} ∈ dom(F), then F(P_{a\leftrightarrow b}) = F(P)_{a\leftrightarrow b}, where for Y ⊆ X, Y_{a\leftrightarrow b} comes from Y by swapping in/out a and b, i.e., Y_{a\leftrightarrow b} is Y if Y \cap \{a, b\} = ∅ or \{a, b\} ⊆ Y, (Y \setminus \{b\}) \cup \{a\} if Y \cap \{a, b\} = \{b\}, and (Y \setminus \{a\}) \cup \{b\} if Y \cap \{a, b\} = \{a\}.
(2) F satisfies quasi-resoluteness (resp. binary quasi-resoluteness) if for any uniquely weighted P ∈ dom(F) (resp. |X(P)| = 2), we have |F(P)| = 1.
(3) F satisfies binary γ if for any P ∈ dom(F) and x, y ∈ X(P), if x ∈ F(P_{-y}) and F(P_{\{x,y\}}) = \{x\}, then x ∈ F(P).
(4) F satisfies α-resoluteness if for any P ∈ dom(F) and x, y ∈ X(P), if x ∈ F(P_{-y}) and F(P_{\{x,y\}}) = \{x\}, then |F(P)| ≤ |F(P_{-y})|.

Note that the definition of neutrality using transpositions of two candidates is equivalent to the other standard definition using permutations of candidates, as every permutation can be obtained from transpositions.
4. Impossibility theorems for C1 and pairwise methods

Let us now state the impossibility theorems proved with the help of SAT solving that we will eventually strengthen to Theorems 2.6 and 2.8 from Section 2.2. For the proofs of the following theorems, see Corollaries A.13 and B.4 in the Appendix.

**Theorem 4.1.** For any finite \( Y \subset X \) with \( |Y| \geq 6 \), there is no anonymous, neutral, and C1 voting method satisfying binary quasi-resoluteness, binary \( \gamma \), and \( \alpha \)-resoluteness on the domain \( \{ P \mid X(P) \subseteq Y \} \).

By contrast, there is such a voting method on the domain \( \{ P \mid |X(P)| \leq 5 \} \).

**Theorem 4.2.** For any \( Y \subset X \) with \( |Y| \geq 4 \), there is no anonymous, neutral, and pairwise voting method satisfying binary \( \gamma \) and quasi-resoluteness on the domain

\[
\{ P \mid X(P) \subseteq Y, M(P) \text{ is uniquely weighted, and all positive weights belong to } \{2, 4, 6, 8, 10, 12\} \}.
\]

By contrast, there are such voting methods on the domain \( \{ P \mid |X(P)| \leq 3 \} \), including all of the voting methods in the right column of Figure 3.

5. Profile representation of (weighted) weak tournaments

The key to extending our results for C1 methods to all voting methods is an alternative to McGarvey’s representation of weak tournaments by profiles (recall Definition 3.16). For this alternative representation, we use the following notation and terminology.

**Definition 5.1.** Given a finite \( Y \subset X \) and \( a, b \in Y \), let \( L_{ab-Y} \) be the profile such that \( L_{ab-Y}(L) = 1 \) if \( L \) is a linear order on \( Y \) of the form \( aby_1 \ldots y_n \) for \( y_1, \ldots, y_n \in Y \setminus \{a, b\} \) and \( L_{ab-Y}(L) = 0 \) otherwise. We call \( L_{ab-Y} \) an \( ab-Y \)-block.

Note that given a finite \( Y \subset X \) and \( a, b \in Y \), the number of ballots in an \( ab-Y \)-block is \( (|Y| - 2)! \). Later it will be useful to express margins as multiples of

\[
\psi_Y = 2 \times (|Y| - 2)!,
\]

in which case the number of ballots in an \( ab-Y \)-block is \( \psi_Y/2 \). When \( Y \) is clear from context, we will write \( \psi \) for \( \psi_Y \).

**Definition 5.2.** Given a finite \( Y \subset X \) and weak tournament \( T \) with \( X(T) \subseteq Y \), we define the profile representation of \( T \) relative to \( Y \), \( \mathcal{L}_Y(T) \), as follows. Start with the profile \( L_Y \) (recall Definition 3.8). For each pair \( a, b \in X(T) \) with \( a \rightarrow b \) in \( T \), flip the first two candidates of the \( ba-Y \)-block to obtain an additional \( ab-Y \)-block, resulting in a profile \( \mathcal{L}_Y(T) \). More formally, let

\[
(5.1) \quad \mathcal{L}_Y(T) = L_Y + \sum_{a \rightarrow b \text{ in } T} (L_{ab-Y} - L_{ba-Y}).
\]

Then let

\[
(5.2) \quad \mathcal{P}_Y(T) = \mathcal{L}_Y(T)|_{X(T)}.
\]

It is easy to see that like McGarvey’s construction, Definition 5.2 allows us to represent any weak tournament as the majority graph of a profile.
Proposition 5.3. For any finite $Y \subset \mathcal{X}$, $T \in \mathcal{W}_Y$, we have $T = M(\mathcal{P}_Y(T))$.

This proposition follows from Propositions 5.5 and 5.7 proved below.

In order to extend results for pairwise methods to all voting methods, we define a similar construction for certain weighted weak tournaments as follows. Recall the definition of $P_{*,X(T)}$ from Definition 5.2.

Definition 5.4. Given a finite $Y \subset \mathcal{X}$, $m \in \mathbb{Z}^+$, and weighted weak tournament $T$ with $X(T) \subseteq Y$ each of whose weights is $k \psi_Y$ for $k \in \mathbb{Z}^+$ and less than or equal to $m \psi_Y$, we define the profile representation of $T$ relative to $Y$ and $m$, $\mathcal{P}_{Y,m}(T)$, as follows. Start with $mL_Y$ (Definition 5.3), so for each $a, b \in X(T)$, there are $m$ $ab$-blocks in $mL_Y$. For each pair $a, b$ with $a \rightarrow b$ in $T$ with weight $k \psi_Y$, in $k$ $ba$-$Y$-blocks in $mL_Y$, flip the first two candidates to obtain an additional $k$ $ab$-$Y$-blocks, resulting in a profile $\mathcal{L}_{Y,m}(T)$. More formally,

$$\mathcal{L}_{Y,m}(T) = mL_Y + \sum_{a \rightarrow b \in T} (kL_{ab,Y} - kL_{ba,Y}).$$

Then let

$$\mathcal{P}_{Y,m}(T) = \mathcal{L}_{Y,m}(T)|_{X(T)}.$$

Finally, for a weighted weak tournament $T$ each of whose weights is $k \psi_Y + 1$ for some $k \in \mathbb{Z}^+$ and less than or equal to $m \psi_Y + 1$, as the weights in $T - M(P_{*,X(T)})$ are each of the form $k \psi_Y$ for some $k \in \mathbb{Z}^+$, we define

$$\mathcal{P}_{Y,m}(T) = \mathcal{P}_{Y,m}(T - M(P_{*,X(T)})) + P_{*,X(T)}.$$

We can regard the construction in Definition 5.2 as a special case of that in Definition 5.4.

Proposition 5.5. Given a finite $Y \subset \mathcal{X}$ and weak tournament $T$ with $X(T) \subseteq Y$, let $T_\psi$ be the weighted tournament based on $T$ with all weights being $\psi_Y$. Then $\mathcal{P}_Y(T) = \mathcal{P}_{Y,1}(T_\psi)$.

Proof. Since all weights in $T_\psi$ are $\psi_Y$, the $k$ in Definition 5.4 is 1. Then since $m = 1$, (5.3) reduces to (5.1).

Figure 6 illustrates our construction from Definition 5.4, contrasting it with Debord’s from Definition 5.10. This construction allows us to represent any weighted weak tournament in the following class as the margin graph of a profile.

Definition 5.6. Given a finite $Y \subset \mathcal{X}$ and $m \in \mathbb{Z}^+$, let $\mathcal{W}_{Y,m}$ be the set of all weighted weak tournaments $T$ with $X(T) \subseteq Y$ such that either each weight in $T$ is of the form $k \psi_Y$ for some $k \in \mathbb{Z}^+$ and less than or equal to $m \psi_Y$ or each weight in $T$ is of the form $k \psi_Y + 1$ for some $k \in \mathbb{Z}^+$ and less than or equal to $m \psi_Y + 1$.

Proposition 5.7. For any finite $Y \subset \mathcal{X}$, $m \in \mathbb{Z}^+$, and $T \in \mathcal{W}_{Y,m}$, we have $T = M(\mathcal{P}_{Y,m}(T))$.

Proof. Case 1: each weight in $T$ is $k \psi_Y$ for some $k \in \mathbb{Z}^+$ and less than or equal to $m \psi_Y$. Hence $\mathcal{P}_{Y,m}$ is defined by (5.3)–(5.4). Concerning (5.3), observe that for any $a, b \in X(T)$:

(i) $\mu_{a,b}(mL_Y) = 0$;

(ii) if $a \rightarrow b$ in $T$, then $\mu_{a,b}(kL_{ab,Y} - kL_{ba,Y}) = k \psi_Y$;
(iii) for any \((c, d) \neq (a, b)\) and \((d, c) \neq (a, b)\), if \(c \rightarrow d\), then \(\mu_{a,b}(kL_{cdY} - kL_{dcY}) = 0\).

Part (i) is immediate from the definition of \(L_Y\). For (ii), we have:
\[
\mu_{a,b}(kL_{abY} - kL_{baY}) = \mu_{a,b}(kL_{abY}) - \mu_{a,b}(kL_{baY}) = 
\]
\[
= k\mu_{a,b}(L_{abY}) - k\mu_{a,b}(L_{baY}) = 
\]
\[
= k\frac{\psi_Y}{2} - \left(-k\frac{\psi_Y}{2}\right) = 
\]
\[
= k\psi_Y. 
\]

For (iii), we have:
\[
\mu_{a,b}(kL_{cdY} - kL_{dcY}) = \mu_{a,b}(L_{cdY}) - \mu_{a,b}(L_{dcY}). 
\]

Recall that \((c, d) \neq (a, b)\) and \((d, c) \neq (a, b)\). If \(\{c, d\} \cap \{a, b\} = \emptyset\), so \(a, b \in Y\), then \(\mu_{a,b}(L_{cdY}) = \mu_{a,b}(L_{dcY}) = 0\), so \(\mu_{a,b}(kL_{cdY} - kL_{dcY}) = 0\). Now suppose
\(|\{c, d\} \cap \{a, b\}| = 1\). Then without loss of generality suppose \(a \in \{c, d\}\) and \(b \in Y\). Then \(\mu_{a,b}(L_{cd-Y}) = \mu_{a,b}(L_{dc-Y}) = \psi_Y/2\), so again \(\mu_{a,b}(kL_{cd-Y} - kL_{dc-Y}) = 0\).

It follows from (i)–(iii) that for each pair \(a, b\) with \(a \rightarrow b\) in \(T\) with weight \(k\psi_Y\), we have \(\mu_{a,b}(\Sigma_{Y,m}(T)) = k\psi_Y\); moreover, for each pair \(a, b\) with neither \(a \rightarrow b\) nor \(b \rightarrow a\) in \(T\), we have \(\mu_{a,b}(\Sigma_{Y,m}(T)) = 0\). Thus, \(T = \mathcal{M}(\mathfrak{P}_{Y,m}(T))\).

Case 2: each weight in \(T\) is of the form \(k\psi_Y + 1\) for some \(k \in \mathbb{Z}_+\) and less than or equal to \(m\psi_Y + 1\). Hence \(\mathfrak{P}_{Y,m}\) is defined by (5.5). Without loss of generality, suppose \(L_{*,X}(T)\) has \(a\) over \(b\). Then where \(k\psi_Y + 1\) is the weight of \(a, b\) in \(T\), the weight of \(a, b\) in \(T - \mathcal{M}(\mathfrak{P}_{*,X}(T))\) is \(k\psi_Y\). Then by the proof for Case 1, the margin of \(a, b\) in \(\mathfrak{P}_{Y,m}(T - \mathcal{M}(\mathfrak{P}_{*,X}(T))) + \mathfrak{P}_{*,X}(T)\) is \(k\psi_Y + 1\), as desired. \(\Box\)

\[\begin{array}{c}
\text{Figure 7. Representation commutes with restriction (above), but the McGarvey/Debord construction does not commute with restriction (below).}
\end{array}\]
Figure 8. Representation commutes with transposition (above), but the McGarvey/Debord construction does not commute with transposition (below).
There are two key differences between our construction and those of McGarvey and Debord. First, as illustrated in Figure 7, unlike their constructions, representation commutes with restriction.

**Proposition 5.8.** For any finite $Y \subset X$, $T \in \mathcal{WT}_Y$, $m \in \mathbb{Z}^+$, and $\mathbb{T} \in \mathcal{WT}_Y^w$,:

1. if $\emptyset \neq Z \subseteq X(T)$, then $\mathcal{P}_Y(T|_Z) = \mathcal{P}_Y(T)|_Z$;
2. if $\emptyset \neq Z \subseteq X(T)$, then $\mathcal{P}_{Y,m}(\mathbb{T}|_Z) = \mathcal{P}_{Y,m}(\mathbb{T})|_Z$.

**Proof.** We give only the proof for (1), as the proof for (2) is essentially the same but with more notation. Define $\rho = Y \to Y$ by $\rho(a) = b$, $\rho(b) = a$, and $\rho(c) = c$ for all $c \notin \{a, b\}$. The key observations are that $(L_Y)_{a=b} = L_Y$ and for any $c, d \in X(T)$, $(L_{cd-Y})_{a=b} = L_{\rho(c)\rho(d)-Y}$. Then we have

\[
\mathcal{P}_Y(T)|_Z = (\mathcal{Q}_Y(T|_{X(T)}))Z \quad \text{by (5.2)}
\]

\[
= \mathcal{L}_Y(T)|_Z \text{ since } Z \subseteq X(T)
\]

\[
= (L_Y + \sum_{a \nrightarrow b} (L_{ab-Y} - L_{ba-Y}))|_Z
\]

\[
= (L_Y + \sum_{a \nrightarrow b} \sum_{(a,b) \in Z} (L_{ab-Y} - L_{ba-Y}))|_Z
\]

\[
= (L_Y + \sum_{a \nrightarrow b} \sum_{(a,b) \subseteq Z} (L_{ab-Y} - L_{ba-Y}))|_Z
\]

Second, as illustrated in Figure 8, unlike the constructions of McGarvey and Debord, representation commutes with transposition.

**Proposition 5.9.** For any finite $Y \subset X$, $T \in \mathcal{WT}_Y$, $m \in \mathbb{Z}^+$, and $\mathbb{T} \in \mathcal{WT}_Y^w$:

1. if $\emptyset \neq Z \subseteq X(T)$, then $\mathcal{P}_Y(T_{a=b}) = \mathcal{P}_Y(T)_{a=b}$;
2. if $\emptyset \neq Z \subseteq X(T)$, then $\mathcal{P}_{Y,m}(\mathbb{T}_{a=b}) = \mathcal{P}_{Y,m}(\mathbb{T})_{a=b}$.

**Proof.** We give only the proof for (1), as the proof for (2) is essentially the same but with more notation. Define $\rho : Y \to Y$ by $\rho(a) = b$, $\rho(b) = a$, and $\rho(c) = c$ for all $c \notin \{a, b\}$. The key observations are that $(L_Y)_{a=b} = L_Y$ and for any $c, d \in X(T)$, $(L_{cd-Y})_{a=b} = L_{\rho(c)\rho(d)-Y}$. Then we have
indeed a C1 (resp. pairwise) voting method. Moreover, C1 and pairwise projection
select winners according to $F$ not represent $M$ where $\psi$.$P$
representation by Definition 5.4. Though the resulting profile
pairwise projection of $F$ step is to associate with each voting method
we call its C1 projection $C1$ projection
It is easy to see that the C1 (resp. pairwise) projection of a voting method is
Note that multiplying 6.2 Definition
We define an analogous notion of pairwise projection using Definition 5.4.
Using the representation of weak tournaments in Definition 5.2, the next key
and finite $Y \subset X$ a method that we call its C1 projection (relative to $Y$).

DEFINITION 6.1. Given a voting method $F$ and finite $Y \subset X$, define the C1 projection of $F$ relative to $Y$, $F_{C1,Y}$, as the voting method with
dom($F_{C1,Y}$) = \{$P$ | $X(P) \subseteq Y$ and $\Psi_Y(M(P)) \in \text{dom}(F)$\}
and

$$F_{C1,Y}(P) = F(\Psi_Y(M(P))).$$

We define an analogous notion of pairwise projection using Definition 5.4

DEFINITION 6.2. Given a voting method $F$, finite $Y \subset X$, and $m \in \mathbb{Z}^+$, define the pairwise projection of $F$ relative to $Y$ and $m$, $F_{p,Y,m}$, as the voting method with
dom($F_{p,Y,m}$) = \{$P$ | $X(P) \subseteq Y$, all weights in $M(P)$ are $\leq m$, and
and

$$F_{p,Y,m}(P) = F(\Psi_{Y,m}(\psi M(P))).$$

where $\psi = 2 \times (|Y| - 2)!$.

Note that multiplying $M(P)$ by $\psi$ gives us a weighted tournament suitable for representation by Definition 5.3. Though the resulting profile $\Psi_{Y,m}(\psi M(P))$ does not represent $M(P)$, this does not matter, since we only use $\Psi_{Y,m}(\psi M(P))$ to select winners according to $F$.

It is easy to see that the C1 (resp. pairwise) projection of a voting method is indeed a C1 (resp. pairwise) voting method. Moreover, C1 and pairwise projection
preserve neutrality thanks to the fact in Proposition 5.9 that representation commutes with transposition (in contrast to the McGarvey/Debord construction) plus Lemma 3.15. Thus, we have the following.

**Lemma 6.3.** Given a voting method $F$, finite $Y \subset X$, and $m \in \mathbb{Z}^+$:

1. $F_{C1,Y}$ is C1;
2. $F_{p,Y,m}$ is pairwise;
3. if $F$ is neutral, then $F_{C1,Y}$ and $F_{p,Y,m}$ are neutral.

**Proof.** Parts 1 and 2 are immediate from Definitions 6.1 and 6.2, respectively. For part 3, assuming $F$ is neutral, we have

$$F_{C1,Y}(\mathbf{P}_{a\leftrightarrow b}) = F(\mathbf{P}_Y(M(\mathbf{P}_{a\leftrightarrow b})))$$

by Definition 6.1

$$= F(\mathbf{P}_Y(M(\mathbf{P}_{a\leftrightarrow b})))$$

by Lemma 3.15

$$= F(\mathbf{P}_Y(M(\mathbf{P}_{a\leftrightarrow b})))$$

by Proposition 5.9

$$= F_{C1,Y}(\mathbf{P}_{a\leftrightarrow b})$$

by Definition 6.1

so $F_{C1,Y}$ is neutral. The proof for $F_{p,Y,m}$ is analogous. □

### 7. Preservation of axioms

We now turn to the key question of which axioms on voting methods are preserved by $C_1$ and pairwise projection, besides neutrality (Lemma 6.3).

#### 7.1. Axioms as sets.

To prove general results about the preservation of axioms by $C_1$ and pairwise projections, we will make a move that is unusual in social choice theory but standard in mathematical logic, by treating axioms on voting methods as mathematical objects. Thinking syntactically, we could introduce a formal language and define a *universal axiom* as a formula of the form

$$\forall \mathbf{P}_1 \ldots \forall \mathbf{P}_n \varphi(\mathbf{P}_1, \ldots, \mathbf{P}_n)$$

where $\varphi$ contains no profile quantifiers (but possibly quantifiers over voters and candidates). Instead, we will view universal axioms semantically as follows.\(^{\text{12}}\)

Let $\mathcal{S}$ be the set of all pairs $\langle \mathbf{P}, Y \rangle$ where $\mathbf{P}$ is a profile and $Y \subseteq X(\mathbf{P})$.

**Definition 7.1.** An $n$-ary universal axiom is a subset $Ax \subseteq \mathcal{S}^n$. A voting method $F$ satisfies $Ax$ if $\{\langle \mathbf{P}, F(\mathbf{P}) \rangle | \mathbf{P} \text{ a profile}\}^n \subseteq Ax$.

We illustrate this concept by formalizing some of the axioms we have discussed.

**Example 7.2.**

1. Given a profile $\mathbf{P}$, a *Condorcet winner* is a candidate $a \in X(\mathbf{P})$ such that for all $b \in X(\mathbf{P}) \setminus \{a\}$, the margin of $a$ over $b$ in $\mathbf{P}$ is positive. A voting method $F$ is *Condorcet consistent* if for all $\mathbf{P} \in \text{dom}(F)$, if $\mathbf{P}$ has

\(^{12}\)For those familiar with mathematical logic (see, e.g., [CK90]), consider a two-sorted model $\mathfrak{A}$ whose domains contains all profiles and all nonempty finite sets of candidates. Where $P_i$ is a variable for a profile and $X_i$ is a variable for a set of candidates, a formula $\varphi(P_1, \ldots, P_n, X_1, \ldots, X_n)$ defines the set of all tuples $\langle P_1, \ldots, P_n, X_1, \ldots, X_n \rangle$ such that $\mathfrak{A} \models \varphi(P_1, \ldots, P_n, X_1, \ldots, X_n)[P_1, \ldots, P_n, X_1, \ldots, X_n]$. We repackage $\langle P_1, \ldots, P_n, X_1, \ldots, X_n \rangle$ as $\langle P_1, X_1 \rangle, \ldots, (P_n, X_n) \rangle$ in Definition 7.1.
a Condorcet winner, then \( F(P) \) contains only that candidate. Condorcet consistency corresponds to the unary axiom
\[
\text{CC} = \{ \langle P, X \rangle \mid \text{if } P \text{ has a Condorcet winner, then } X \text{ contains only that candidate} \}.
\]

(2) C1 (Definition \[3.22.1\]) corresponds to the binary axiom
\[
\text{C1} = \{ \langle \langle P, X \rangle, \langle P', X' \rangle \rangle \mid \text{if } M(P) = M(P'), \text{ then } X = X' \}.
\]

(3) neutrality (Definition \[3.22.1\]) corresponds to the binary axiom
\[
\text{N} = \{ \langle \langle P, X \rangle, \langle P', X' \rangle \rangle \mid \text{if } P' = P_{a\equiv b}, \text{ then } X' = X_{a\equiv b} \}.
\]

(4) binary quasi-resoluteness (Definition \[3.22.2\]) corresponds to the axiom
\[
\text{BQR} = \{ \langle P, X \rangle \mid \text{if } M(P) \text{ is a tournament with } 2 \text{ nodes, then } |X| = 1 \}.
\]

(5) quasi-resoluteness (Definition \[3.22.2\]) corresponds to the axiom
\[
\text{QR} = \{ \langle P, X \rangle \mid \text{if } M(P) \text{ is uniquely weighted, then } |X| = 1 \}.
\]

### 7.2. C1 and homogeneously pairwise axioms.

Inspired by the notions of C1/pairwise voting methods, we will define the notions of C1/pairwise axioms, inspired by the notion of a voting method \( F \) being homogeneous \[SmI73\], which means that for all \( P \in \text{dom}(F) \) and \( n \in \mathbb{Z}^+ \), if \( nP \in \text{dom}(F) \), then \( F(P) = F(nP) \).

**Definition 7.3.** Let \( Ax \) be an \( n \)-ary universal axiom. We say that \( Ax \) is C1 (resp. homogeneously pairwise) if for all profiles \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_n \) such that for \( 1 \leq i \leq n \), \( M(P_i) = M(Q_i) \) (resp. for which there is an \( n \in \mathbb{Z}^+ \) such that for \( 1 \leq i \leq n \), \( nM(P_i) = M(Q_i) \)), we have that for all \( X_1 \subseteq X(P_1), \ldots, X_n \subseteq X(P_n) \),
\[
\langle \langle P_1, X_1 \rangle, \ldots, \langle P_n, X_n \rangle \rangle \in Ax \text{ if and only if } \langle \langle Q_1, X_1 \rangle, \ldots, \langle Q_n, X_n \rangle \rangle \in Ax.
\]

Note that being C1 implies being homogeneously pairwise.

The axioms from Example 7.2 can be classified using Definition 7.3 as follows.

**Example 7.4.**

1. Condorcet consistency is C1 because if \( \langle P, X \rangle \in \text{CC} \) and \( M(P) = M(Q) \), then clearly \( \langle Q, X \rangle \in \text{CC} \).
2. The C1 axiom is obviously C1.
3. Neutrality is not homogeneously pairwise, because there are profiles \( P, P', Q, Q' \) such that \( P' = P_{a\equiv b}, M(P) = M(Q), \) and \( M(P') = M(Q') \), but \( Q' \neq Q_{a\equiv b} \). Then for \( X, X' \subseteq X(P) \) such that \( X' \neq X_{a\equiv b} \), we have \( \langle \langle P, X \rangle, \langle P', X' \rangle \rangle \not\in N \) but trivially \( \langle \langle Q, X \rangle, \langle Q', X' \rangle \rangle \in N \).
4. BQR is obviously C1.
5. QR is homogeneously pairwise (for \( M(P) \) is uniquely weighted iff \( nM(P) \) is), but it is not C1, because there are profiles \( P, P' \) such that \( M(P) = M(P'), M(P) \) is uniquely weighted, but \( M(P') \) is not. Then for \( X \subseteq X(P) \) with \( |X| > 1 \), \( \langle P, X \rangle \not\in QR \) but trivially \( \langle P', X \rangle \in QR \).

We now verify that all C1 (resp. homogeneously pairwise) universal axioms are preserved by the operation of C1 (resp. pairwise) projection from Section 6.

\[\text{\textsuperscript{13}Other examples of C1 axioms include the Condorcet loser criterion [Nur87] and Smith criterion [SmI73].}\]
Lemma 7.5. For any C1 (resp. pairwise) universal axiom Ax, if F satisfies Ax, then \( F_{C1,Y} \) (resp. \( F_{p,Y,m} \)) satisfies Ax.

Proof. Suppose Ax is a C1 n-ary universal axiom. To show that \( F_{C1,Y} \) satisfies Ax, consider any \( P_1, \ldots, P_n \in \text{dom}(F_{C1,Y}) \). We must show that
\[
\langle \langle P_1, F_{C1,Y}(P_1) \rangle, \ldots, \langle P_n, F_{C1,Y}(P_n) \rangle \rangle \in Ax.
\]
By definition of \( F_{C1,Y} \) (Definition 6.1), this is equivalent to
\[
\langle \langle P_1, F(\mathcal{P}_Y(M(P_1))) \rangle, \ldots, \langle P_n, F(\mathcal{P}_Y(M(P_n))) \rangle \rangle \in Ax.
\]
By Lemma 5.3 for all \( i, M(P_i) = M(\mathcal{P}_Y(M(P_i))) \). Then since Ax is C1, setting \( Q_i = \mathcal{P}_Y(M(P_i)) \) in Definition 7.3, the previous membership statement is equivalent to
\[
\langle \langle \mathcal{P}_Y(M(P_1)), F(\mathcal{P}_Y(M(P_1))) \rangle, \ldots, \langle \mathcal{P}_Y(M(P_n)), F(\mathcal{P}_Y(M(P_n))) \rangle \rangle \in Ax,
\]
which holds by our assumption that F satisfies Ax. The proof for the pairwise case is also analogous, using Lemma 5.7.

Now suppose Ax is a pairwise n-ary universal axiom. To show that \( F_{p,Y,m} \) satisfies Ax, consider any \( P_1, \ldots, P_n \in \text{dom}(F_{p,Y,m}) \). We must show that
\[
\langle \langle P_1, F_{p,Y,m}(P_1) \rangle, \ldots, \langle P_n, F_{p,Y,m}(P_n) \rangle \rangle \in Ax.
\]
By definition of \( F_{p,Y,m} \) (Definition 6.2), this is equivalent to
\[
\langle \langle P_1, F(\mathcal{P}_{Y,m}(\mathcal{P}(P_1))) \rangle, \ldots, \langle P_n, F(\mathcal{P}_{Y,m}(\mathcal{P}(P_n))) \rangle \rangle \in Ax.
\]
By Lemma 5.7 for all \( i, \mathcal{P}(P_i) = \mathcal{P}_{Y,m}(\mathcal{P}(P_i)) \). Then since Ax is homogeneously pairwise, setting \( Q_i = \mathcal{P}_{Y,m}(\mathcal{P}(P_i)) \) in Definition 7.3, the previous membership statement is equivalent to
\[
\langle \langle \mathcal{P}_{Y,m}(\mathcal{P}(P_1)), F(\mathcal{P}_{Y,m}(\mathcal{P}(P_1))) \rangle, \ldots, \langle \mathcal{P}_{Y,m}(\mathcal{P}(P_n)), F(\mathcal{P}_{Y,m}(\mathcal{P}(P_n))) \rangle \rangle \in Ax,
\]
which holds by our assumption that F satisfies Ax.

Lemma 7.5 provides a method of extending impossibility theorems proved for C1 or pairwise methods to all voting methods. However, it does not go far enough for us, because the variable candidate axioms of binary \( \gamma \) and \( \alpha \)-resoluteness are not pairwise axiom (see Lemma 7.8).

### 7.3. Variable candidate axioms

Next we extend Lemma 7.5 to handle variable candidate axioms—axioms that concern adding or removing a candidate from each voter’s ballot.

Definition 7.6. Let Ax be an n-ary universal axiom. We say that Ax is a variable candidate axiom if for all profiles \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_n \) such that

1. for \( 1 \leq i \leq n \), \( X(P_i) = X(Q_i) \), and
2. for \( 1 \leq i, j \leq n \), if \( P_i = (P_j)|_{X(P_i)} \), then \( Q_i = (Q_j)|_{X(Q_i)} \),

we have that for all \( X_1 \subseteq X(P_1), \ldots, X_n \subseteq X(P_n) \),

3. if \( \langle \langle Q_1, X_1 \rangle \rangle, \ldots, \langle Q_n, X_n \rangle \rangle \in \text{Ax} \), then \( \langle \langle P_1, X_1 \rangle \rangle, \ldots, \langle P_n, X_n \rangle \rangle \in \text{Ax} \).

The intuition is that if the \( Q_i \)'s are related to each other in terms of restriction in all of the ways that the \( P_i \)'s are, and \( X_1, \ldots, X_n \) are acceptable choices for the \( Q_i \)'s according to the relevant variable candidate axiom (e.g., binary \( \gamma \) or \( \alpha \)-resoluteness), then they are also acceptable choices for the \( P_i \)'s.
Example 7.7. Binary γ and α-resoluteness correspond to the following axioms:

\[ \text{BG} = \{ \langle \langle P, X \rangle, \langle P', X' \rangle, \langle P'', X'' \rangle \rangle \mid P' = P_{-b} \text{ for some } b \in X(P), \ P'' = P_{\{a, b\}} \text{ for } a \in X(P'), \ a \in X', \text{ and } X'' = \{a\}, \text{ then } a \in X \}; \]

\[ \text{AR} = \{ \langle \langle P, X \rangle, \langle P', X' \rangle, \langle P'', X'' \rangle \rangle \mid P' = P_{-b} \text{ for some } b \in X(P), \ P'' = P_{\{a, b\}} \text{ for } a \in X(P'), \ a \in X', \text{ and } X'' = \{a\}, \text{ then } |X| \leq |X'|. \]

Lemma 7.8. BG and AR are variable candidate axioms but neither C1 nor homogeneously pairwise axioms.

Proof. We give only the proof for BG, as the proof for AR is almost the same. Suppose (i) that \( P, P', P'' \) and \( Q, Q', Q'' \) are related as in Definition 7.6.1. Further suppose (ii) that \( \langle \langle Q, X \rangle, \langle Q', X' \rangle, \langle Q'', X'' \rangle \rangle \in \text{BG} \). Finally, suppose (iii) that \( P' = P_{-b} \text{ for some } b \in X(P), \ P'' = P_{\{a, b\}} \text{ for } a \in X(P'), \ a \in X', \text{ and } X'' = \{a\}. \) By (i) and (iii), we have that \( Q' = Q_{-b} \text{ for } b \in X(Q) = X(P), \ Q'' = Q_{\{a, b\}} \text{ for } a \in X(Q') = X(P'), \ a \in X', \text{ and } X'' = \{a\}. \) Then by (ii), we have \( a \in X. \) Thus, we conclude that \( \langle \langle P, X \rangle, \langle P', X' \rangle, \langle P'', X'' \rangle \rangle \in \text{BG} \). To see that BG is not pairwise, it is easy to find \( \langle \langle P_1, X_1 \rangle, \langle P_2, X_2 \rangle, \langle P_3, X_3 \rangle \rangle \) and \( \langle \langle Q_1, X_1 \rangle, \langle Q_2, X_2 \rangle, \langle Q_3, X_3 \rangle \rangle \) such that while \( M(P_1) = M(Q_1) \) and \( P_2 = (P_1)_b \), we have \( Q_2 \neq (Q_1)_{X(Q_2)} \); then even if \( \langle \langle P_1, X_1 \rangle, \langle P_2, X_2 \rangle, \langle P_3, X_3 \rangle \rangle \notin \text{BG} \), from \( Q_2 \neq (Q_1)_{X(Q_2)} \) we trivially have \( \langle \langle Q_1, X_1 \rangle, \langle Q_2, X_2 \rangle, \langle Q_3, X_3 \rangle \rangle \in \text{BG}. \)

Fortunately, not only C1 (resp. pairwise) axioms but also variable candidate axioms are preserved by our operation of C1 (resp. pairwise) projection, thanks to the fact that our representation of weak (resp. weighted weak) tournaments commutes with restriction, in contrast to McGarvey and Debord’s constructions.

Proposition 7.9. Let \( F \) be a voting method satisfying a variable candidate universal axiom \( A_x \). Then for any finite \( Y \subset X \), \( F_{C_1,Y} \) (resp. \( F_{f,Y,m} \)) satisfies \( A_x \).

Proof. Suppose \( A_x \) is a C1 n-ary universal axiom. To show that \( F_{C_1,Y} \) satisfies \( A_x \), consider any \( P_1, \ldots, P_n \in \text{dom}(F_{C_1,Y}) \). We must show that

\[ \langle \langle P_1, F(M(P_1)), \ldots, P_n, F(M(P_n)) \rangle \rangle \in A_x. \]

By definition of \( F_{C_1,Y} \), this is equivalent to

\[ \langle \langle P_1, F(M(P_1)), \ldots, P_n, F(M(P_n)) \rangle \rangle \in A_x. \]

By Proposition 6.3 and Lemma 3.20 we have

\[ Y(M(P_j)|_{X(Y(M(P_j)))}) = Y(M(P_j)|_{X(Y(M(P_i)))}) = Y(M(P_j)|_{X(P_i)}). \]

Thus, for \( 1 \leq i, j \leq n \), we have that

\[ \text{if } P_i = (P_j)|_{X(P_i)}, \text{ then } \varphi_Y(M(P_i)) = \varphi_Y(M(P_j)|_{X(P_i)}). \]
Then since $Ax$ is a variable candidate axiom, setting $Q_i = \mathcal{P}_Y(M(P_i))$ in Definition 7.6 the membership statement displayed above is a consequence of

$$\langle \mathcal{P}_Y(M(P_1)), F(\mathcal{P}_Y(M(P_1))), \ldots, \mathcal{P}_Y(M(P_n)), F(\mathcal{P}_Y(M(P_n))) \rangle \in Ax,$$

which holds by our assumption that $F$ satisfies $Ax$. The proof for the pairwise case is analogous.

Proposition 7.9 is the key to extending impossibility theorems involving variable candidate axioms proved for C1 or pairwise methods to all voting methods, as we show next.

### 8. Generalized impossibility theorems

We can now put everything together to extend the impossibility results in Section 4 for C1 methods and pairwise methods, respectively, to all voting methods.

**Theorem 8.1.** For any $Y \subseteq X$ with $|Y| \geq 6$, there is no anonymous and neutral voting method satisfying binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness on the domain $\{P | X(P) \subseteq Y\}$.

**Proof.** Suppose for contradiction that there is such a method $F$. Then $F_{C1,Y}$ is an anonymous voting method on the domain $\{P | X(P) \subseteq Y\}$ that is C1 and neutral by Lemma 6.3, satisfies binary quasi-resoluteness by Lemma 7.5 (for C1 axioms), and satisfies binary $\gamma$ and $\alpha$-resoluteness by Lemma 7.8 and Proposition 7.9. But this contradicts Theorem 4.1.

**Theorem 8.2.** For any $Y \subseteq X$ with $|Y| \geq 4$, there is no anonymous and neutral voting method satisfying binary $\gamma$ and quasi-resoluteness on the domain $\{P | X(P) \subseteq Y\}$.

**Proof.** Suppose for contradiction that there is such a method $F$. Then $F_{p,Y,12}$ is an anonymous voting method on the domain

$$\{P | X(P) \subseteq Y, 12 \text{ is the maximum weight in } \mathcal{M}(P)\}$$

that is pairwise and neutral by Lemma 6.3, quasi-resolute by Lemma 7.5 (for pairwise axioms), and satisfies binary $\gamma$ by Lemma 7.8 and Proposition 7.9. But this contradicts Theorem 4.2.

Thus, starting from the initial impossibility results in Theorems 4.1 and 4.2 provided by SAT, our theories of C1 and pairwise projection allow us to obtain the more general impossibility results in Theorems 8.1 and 8.2.

### 9. Conclusion

In this paper, we developed a methodology for extending impossibility theorems originally proved for C1 or pairwise voting methods with the help of SAT solving to impossibility theorems for all voting methods. Using this methodology, we showed there is no voting method satisfying anonymity, neutrality, binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness for profiles up to 6 candidates (but there is up to 5 candidates); and there is no voting method satisfying anonymity, neutrality, binary $\gamma$, and quasi-resoluteness for profiles up to 4 candidates (but there is up to 3 candidates). Rather than extending the SAT formalization to encode the preference profiles used by non-C1 or non-pairwise voting methods, which is infeasible in light of Figure 2, we developed a theory of “C1 projection” and “pairwise...
projection" for automatically generalizing the initial C1 or pairwise impossibility theorems. A natural question for future research concerns optimizing the number of voters needed to obtain our impossibility results, in the spirit of \cite{BGP17, BMS21}. We also believe our methodology should be applicable to other impossibility results in voting theory involving variable candidate axioms, which we hope to explore.

### Appendix A. SAT solving for C1 methods

In this Appendix and the next, we explain how the SAT solving approach of Brandt and Geist \cite{BG16} can be applied to prove our Theorems 4.1 and 4.2 for C1 and pairwise voting methods, respectively.

Jupyter notebooks with our SAT formalization are available at https://github.com/szahedian/generalizing-sat.

#### A.1. Results.

Instead of working with profiles and C1 methods, our first SAT approach works with tournaments and tournament solutions.

**Definition A.1.** A weak tournament solution (resp. tournament solution) is a function $F$ on a set of weak tournaments (resp. tournaments) such that for all $T \in \text{dom}(F)$, we have $\emptyset \neq F(T) \subseteq X(T)$.

The axioms of neutrality, binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness from Section 2.2 apply in the obvious way to weak tournament solutions.

**Definition A.2.** Let $F$ be a weak tournament solution.

1. $F$ is neutral if for all $T, T' \in \text{dom}(F)$, $F(T') = h[F(T)]$ for any isomorphism $h : T \rightarrow T'$.
2. $F$ satisfies binary quasi-resoluteness if for any tournament $T \in \text{dom}(F)$ with $|X(T)| = 2$, we have $|F(T)| = 1$.
3. $F$ satisfies binary $\gamma$ if for every $T \in \text{dom}(F)$ and $a, b \in X(T)$, if $a \in F(T - b)$ and $F(T_{\{a,b\}}) = \{a\}$, then $a \in F(T)$.
4. $F$ satisfies $\alpha$-resoluteness if for any $T \in \text{dom}(F)$ and $a, b \in X(T)$, if $a \in F(T - b)$ and $F(T_{\{a,b\}}) = \{a\}$, then $|F(T)| \leq |F(T - b)|$.

We used SAT to find the dividing line in terms of number of candidates between the consistency and inconsistency of these axioms.

**Theorem A.3.**

1. There is a neutral weak tournament solution satisfying binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness on the domain $\{T \in WT \mid |X(T)| \leq 5\}$.
2. For any finite $Y \subset X$ with $|Y| \geq 6$, there is no neutral tournament solution satisfying binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness on the domain $\{T \in T \mid X(T) \subseteq Y\}$.

This theorem is a consequence of Theorem A.12 and Lemma A.10 below.

#### A.2. SAT encoding: canonical tournaments and solutions.

For computational feasibility, following Brandt and Geist \cite{BG16} we work with tournament solutions on a smaller domain of canonical tournaments up to a certain number of candidates.
For each canonical representative and pick a tournaments \( T \) from \( WT \) that \( T \) is a function tournament solution is the most straightforward.

Definition A.4. Given \( T, T' \in WT \), let \( T \cong T' \) if \( T \) and \( T' \) are isomorphic. For each \( T \in WT \), define the equivalence class \( [T] = \{ T' \in WT \mid T \cong T' \} \) and pick a canonical representative \( T_C \in [T] \). Let \( WT_C = \{ T_C \mid T \in WT \} \) and \( T_C = \{ T_C \mid T \in T \} \) be the sets of canonical weak tournaments and canonical tournaments, respectively. A canonical weak tournament solution (resp. canonical tournament solution) is a function \( F \) on a subset of \( WT_C \) (resp. \( T_C \)) such that for all \( T \in \text{dom}(F) \), we have \( \emptyset \neq F(T) \subseteq X(T) \).

Next we consider analogues for canonical weak tournament solutions of the axioms on weak tournament solutions from Definition A.2. Binary quasi-resoluteness is the most straightforward.

Definition A.5. A canonical weak tournament solution \( F \) satisfies binary quasi-resoluteness if for every \( T \in T_C \) with \( |X(T)| = 2 \), we have \( |F(T)| = 1 \).

As for neutrality, there is a one-to-one correspondence between neutral weak tournament solutions and canonical weak tournament solutions satisfying what Brandt and Geist call the orbit condition. Given \( T \in WT_C \) and \( a, b \in X(T) \), let \( a \sim_T b \) if there exists an automorphism \( h \) of \( T \) such that \( h(a) = b \). The orbit of \( a \) in \( T \) is \( \{ b \in T \mid a \sim_T b \} \). Let \( O_T \) be the set of all orbits of elements of \( T \).

Definition A.6. Given a canonical weak tournament \( T \) and \( Y \subseteq X(T) \), we say that \( (T, Y) \) satisfies the orbit condition if for all \( O \in O_T \), we have \( O \subseteq Y \) or \( O \cap Y = \emptyset \). A canonical weak tournament solution \( F \) satisfies the orbit condition if for all \( T \in \text{dom}(F) \), \( (T, F(T)) \) satisfies the orbit condition. Let

\[ O(T) = \{ Y \subseteq X(T) \mid Y \neq \emptyset \text{ and } (T, Y) \text{ satisfies the orbit condition} \}. \]

To state the one-to-one correspondence, we need the following preliminary lemma.

Lemma A.7. Let \( F \) be a canonical weak tournament solution satisfying the orbit condition. Then for any \( T \in WT \) and isomorphisms \( g : T_C \to T \) and \( h : T_C \to T \), we have \( g[F(T_C)] = h[F(T_C)] \).

Proof. Suppose \( a \in g[F(T_C)] \), so \( g^{-1}(a) \in F(T_C) \). Since \( h^{-1} \circ g \) is an automorphism of \( T_C \) such that \( h^{-1}(g^{-1}(a)) = h^{-1}(a) \), we have \( g^{-1}(a) \sim_{T_C} h^{-1}(a) \). Then since \( F \) satisfies the orbit condition, \( g^{-1}(a) \in F(T_C) \) implies \( h^{-1}(a) \in F(T_C) \), so \( a \in h[F(T_C)] \). The other direction is analogous. \( \square \)

Lemma A.7 ensures that the function \( F^* \) in the following is well defined.

Lemma A.8.

1. Given a canonical weak tournament solution \( F \) satisfying the orbit condition, the function \( F^* \) on \( \{ T \in WT \mid T_C \in \text{dom}(F) \} \) defined by \( F^*(T) = h_T[F(T_C)] \) for an isomorphism \( h_T : T_C \to T \) is a neutral weak tournament solution.

2. Given a neutral weak tournament solution \( F \), the function \( F_* \) on \( \{ T_C \mid T \in \text{dom}(F) \} \) defined by \( F_*(T_C) = F(T_C) \) is a canonical weak tournament solution satisfying the orbit condition.

3. For any canonical weak tournament solution \( F \) satisfying the orbit condition, \( (F^*)_* = F \); and for any neutral weak tournament solution \( F \), \( (F_*)^* = F \).
Next we state the analogues of binary $\gamma$ and $\alpha$-resoluteness for canonical weak tournament solutions. Given weak tournaments $T$ and $T'$, an embedding of $T'$ into $T$ is an injective function $e : X(T') \rightarrow X(T)$ such that for all $a,b \in X(T')$, we have $a \rightarrow b$ in $T'$ if and only if $e(a) \rightarrow e(b)$ in $T$, in which case we write $e : T' \hookrightarrow T$.

**Definition A.9.** Let $F$ be a canonical weak tournament solution.

1. $F$ satisfies canonical binary $\gamma$ if for any $T,T',T'' \in \text{dom}(F)$, if $|X(T)| = |X(T')| + 1$, $|X(T'')| = 2$, and there are embeddings $e : T' \hookrightarrow T$ and $f : T'' \hookrightarrow T$ such that $X(T) \setminus e[X(T')] \subseteq f[X(T'')]$, then for $a \in e[X(T')] \cap f[X(T'')]$, if $e^{-1}(a) \in F(T')$ and $F(T'') = \{f^{-1}(a)\}$, then $a \in F(T)$.

2. $F$ satisfies canonical $\alpha$-resoluteness if for any $T,T',T'' \in \text{dom}(F)$, if $|X(T)| = |X(T')| + 1$, $|X(T'')| = 2$, and there are embeddings $e : T' \hookrightarrow T$ and $f : T'' \hookrightarrow T$ such that $X(T) \setminus e[X(T')] \subseteq f[X(T'')]$, then for $a \in e[X(T')] \cap f[X(T'')]$, if $e^{-1}(a) \in F(T')$ and $F(T'') = \{f^{-1}(a)\}$, then $|F(T)| \leq |F(T')|$. Then we state the analogues of binary $\alpha$-resoluteness, where $F$ is a neutral weak tournament solution satisfying binary $\gamma$ (resp. $\alpha$-resoluteness, binary quasi-resoluteness).

**Lemma A.10.**

1. If $F$ is a canonical weak tournament solution satisfying the orbit condition and canonical binary $\gamma$ (resp. canonical $\alpha$-resoluteness, binary quasi-resoluteness), then $F^*$ defined as in Lemma A.8.1 is a neutral weak tournament solution satisfying binary $\gamma$ (resp. $\alpha$-resoluteness, binary quasi-resoluteness).

2. If $F$ is a neutral weak tournament solution satisfying binary $\gamma$ (resp. $\alpha$-resoluteness, binary quasi-resoluteness), then $F_*$ defined as in Lemma A.8.2 is a canonical weak tournament solution satisfying the orbit condition and canonical binary $\gamma$ (resp. canonical $\alpha$-resoluteness, binary quasi-resoluteness).

**A.3. SAT encoding: formulas.** We now explain how the axioms above can be encoded as formulas of propositional logic for the purposes of SAT solving. Following [BG16], our propositional variables are of the form $A_{T_C,Y}$, where $T_C$ is a canonical weak tournament, $\emptyset \neq Y \subseteq X(T_C)$, and $Y \in \mathcal{O}(T_C)$ (Definition A.6). The intended interpretation is that $A_{T_C,Y}$ evaluates to true if and only if $Y$ is assigned as the set of winners of $T_C$. Next we introduce propositional formulas, written in conjunctive normal form for processing by the SAT solver, which formalize the relevant axioms.

First, the following formula $\text{func}$ ensures that the assignment of sets of winners to canonical weak tournaments yields a function:

$$\bigwedge_{T_C} \left( \bigvee_{Y \in \mathcal{O}(T_C)} A_{T_C,Y} \right) \land \bigwedge_{T_C} \left( \bigwedge_{Y,Z \in \mathcal{O}(T_C), Y \neq Z} (\neg A_{T_C,Y} \lor \neg A_{T_C,Z}) \right).$$

To formalize binary quasi-resoluteness, where $T_{••}$ is the canonical tournament with two nodes, we define $\text{bqr}$ by:

$$\bigvee_{Y \in \mathcal{O}(T_{••}), |Y|=1} A_{T_C,Y}.$$
Then let $bg$ be the formula:

$$\bigwedge_{T_C} \bigwedge_{(T_C', e) \in E(T_C)} \bigwedge_{(T_C', f) \in F(T_C)} \varphi$$

with $\varphi$ defined as follows, where $e[X(T_C')] \cap f[X(T_C')] = \{a\}$:

$$\varphi = \bigwedge_{Y' \in \Theta(T_C')} (e^{-1}(a) \in Y') \bigwedge_{Y'' \in \Theta(T_C')} (f^{-1}(a) \in Y'') \bigvee_{Y \in \Theta(T_C)} (A_{T_C, Y}).$$

The formula $ar$ is defined in the same way but with the last disjunction replaced by

$$\bigvee_{Y \in \Theta(T_C)} (|Y| \leq |Y'|) A_{T_C, Y}.$$

Propositional valuations $v : \{A_{T_C, Y} \mid Y \in \Theta(T_C)\} \rightarrow \{0, 1\}$ satisfying the formula $func \land bqr \land bg \land ar$ correspond to canonical weak tournament solutions satisfying the relevant axioms.

**Lemma A.11.**

1. If a valuation $v$ satisfies $func \land bqr \land bg \land ar$, then $\{A_{T_C, Y} \mid Y \in \Theta(T_C)\}$ is a canonical weak tournament solution satisfying the orbit condition, binary quasi-resoluteness, canonical binary $\gamma$, and canonical $\alpha$-resoluteness.

2. Given a canonical weak tournament solution $F$ satisfying the orbit condition, binary quasi-resoluteness, canonical binary $\gamma$, and canonical $\alpha$-resoluteness, the valuation $v$ defined by $v(A_{T_C, Y}) = 1$ if and only if $F(T_C) = Y$ satisfies $func \land bqr \land bg \land ar$.

Using a SAT solver and Lemma A.11, we obtain the following.

**Theorem A.12.**

1. There is a canonical weak tournament solution satisfying the orbit condition, binary quasi-resoluteness, canonical binary $\gamma$, and canonical $\alpha$-resoluteness on the domain $\{T \in WT_C \mid |T| \leq 5\}$.

2. There is no canonical tournament solution satisfying the orbit condition, binary quasi-resoluteness, canonical binary $\gamma$, and canonical $\alpha$-resoluteness on the domain $\{T \in T_C \mid |T| \leq 6\}$.

Finally, combining Theorem A.12 and Lemma A.10 establishes Theorem A.3 from Section A.1.

**A.4. From tournament solutions to C1 voting methods.** Having obtained Theorem A.3 for (weak) tournament solutions, it is straightforward using McGarvey’s theorem (Theorem 3.17) to obtain an analogous statement for C1 voting methods.

**Corollary A.13.**

1. There is an anonymous, neutral, and C1 voting method satisfying binary quasi-resoluteness, binary $\gamma$, and $\alpha$-resoluteness on the domain $\{P \mid |X(P)| \leq 5\}$. 
(2) For any finite \( Y \subset X \) with \(|Y| \geq 6\), there is no anonymous, neutral, and \( C_1 \) voting method satisfying binary quasi-resoluteness, binary \( \gamma \), and \( \alpha \)-resoluteness on the domain \( \{P \mid X(P) \subseteq Y\} \).

**Proof.** For part 1, define the method \( F \) as follows: where \( G \) is the weak tournament solution from Theorem A.3.1, given a profile \( P \), let \( F(P) = G(M(P)) \).

For part 2, suppose for contradiction there is such a method \( F \). We define a canonical tournament solution \( G \) as follows: given a canonical tournament \( T \) (we may assume \( X(T) \subseteq Y \)), let \( G(T) = F(Mc(T)) \). Since \( F \) is \( C_1 \) and satisfies the relevant axioms, it is not difficult to show that \( G \) satisfies the tournament solution versions of the relevant axioms, contradicting Theorem A.3.2. \( \square \)

Theorem 8.1 in the main text drops the assumption of \( C_1 \) from Corollary A.13.2.

**Appendix B. SAT solving for pairwise methods**

**B.1. Results.** Using SAT we also obtained an impossibility theorem for the conjunction of binary \( \gamma \), quasi-resoluteness, and pairwiseness. However, the relevant SAT approach works not with profiles and pairwise voting methods but with weighted tournaments and weighted tournament solutions.

**Definition B.1.** A weighted tournament solution is a function \( F \) on a set of weighted tournaments such that for all \( T \in \text{dom}(F) \), we have \( \emptyset \neq F(T) \subseteq X(T) \).

The binary \( \gamma \) and quasi-resoluteness axioms from Section 2.2 apply in the obvious ways to weighted tournament solutions.

**Definition B.2.** Let \( F \) be a weighted tournament solution.

1. \( F \) satisfies binary \( \gamma \) if for every \( T \in \text{dom}(F) \) and \( a, b \in X(T) \), if \( a \in F(T_{-b}) \) and \( F(T_{\{a,b\}}) = \{a\} \), then \( a \in F(T) \).
2. \( F \) satisfies quasi-resoluteness if for every uniquely weighted \( T \in \text{dom}(F) \), we have \( |F(T)| = 1 \).

The following result explains the phenomenon noted in Section 2.2 that no known (anonymous and neutral) pairwise voting method satisfies binary \( \gamma \) and quasi-resoluteness. Note that since the SAT formalization requires a finite domain, we fix a finite set of weights for our weighted tournaments.

**Theorem B.3.** There is no neutral weighted tournament solution satisfying quasi-resoluteness and binary \( \gamma \) on the domain

\[ \{T \in T^w \mid |X(T)| \leq 4, \, T \text{ uniquely weighted}, \text{ and all positive weights belong to } \{2, 4, 6, 8, 10, 12\} \}. \]

**B.2. SAT encoding.** The SAT encoding strategy for weighted tournament solutions follows the strategy for tournament solutions in Sections A.2 and A.3 for computational feasibility, we work with weighted tournament solutions defined on a smaller domain of canonical weighted tournaments. Moreover, we restrict attention to those in which all weights are distinct. Automorphisms (for the orbit condition) and embeddings (for binary \( \gamma \)) are now automorphisms and embeddings of weighted tournaments in the obvious way. In the setting in which all weights in

14 In fact, Theorem A.32 shows that there is no such voting method on the domain \( \{P \mid X(P) \subseteq Y; |V(P)| \text{ odd}\} \) or on the domain \( \{P \mid X(P) \subseteq Y; |V(P)| \text{ even}\} \).
a weighted tournament are assumed to be distinct, the quasi-resoluteness qr axiom can be formalized simply as resoluteness:

\[ \bigwedge_{(T_C,Y): |Y| > 1} \neg A_{T_C,Y}. \]

Binary \( \gamma \) is formalized as a formula \( bg \) in the same way as binary \( \gamma \) in Section A.3 but using canonical weighted tournaments \( T_C \) instead of tournaments \( T_C \). The unsatisfiability of \( bg \wedge qr \) according to the SAT solver yields Theorem B.3 by reasoning analogous to that in Section A.2.

B.3. From weighted tournament solutions to pairwise voting methods. It is a short step to transfer the impossibility result in Theorem B.3 from weighted tournament solutions to pairwise voting methods. The proof of the following corollary is analogous to that of Corollary A.13 except that we use Debord’s representation of weighted tournaments instead of McGarvey’s representation of tournaments, and we do not need SAT solving for part 1, as Minimax, Ranked Pairs, Beat Path, and Split Cycle are all examples (indeed, these methods agree on all profiles up to 3 candidates).

Corollary B.4.

1. There is an anonymous, neutral, and pairwise voting method satisfying binary \( \gamma \) and quasi-resoluteness on the domain \( \{P : |X(P)| \leq 3\} \).
2. For any \( Y \subset X \) with \( |Y| \geq 4 \), there is no anonymous, neutral, and pairwise voting method that satisfies binary \( \gamma \) and quasi-resoluteness on the domain

\[ \{P : X(P) \subseteq Y, M(P) \text{ is uniquely weighted, and all positive weights belong to } \{2, 4, 6, 8, 10, 12\}\}. \]

Theorem 8.2 in the main text drops the pairwise assumption from Corollary B.4.2.

References

[AR86] Kenneth J. Arrow and Hervé Raïnaud, Social choice and multicriterior decision-making, MIT Press, Cambridge, Mass., 1986.
[Arr63] Kenneth J. Arrow, Social choice and individual values, 2nd ed., John Wiley & Sons, Inc., New York, 1963.
[Ban85] J. S. Banks, Sophisticated voting outcomes and agenda control, Social Choice and Welfare 1 (1985), no. 4, 295–306.
[BF07] Felix Brandt and Felix Fischer, PageRank as a weak tournament solution, Internet and Network Economics: Third International Workshop, WINE 2007 (Berlin) (Xiaotie Deng and Fan Chung Graham, eds.), Springer-Verlag, 2007, pp. 300–305.
[BG16] Felix Brandt and Christian Geist, Finding strategyproof social choice functions via SAT solving, Journal of Artificial Intelligence Research 55 (2016), 565–602.
[BGP17] Felix Brandt, Christian Geist, and Dominik Peters, Optimal bounds for the no-show paradox via SAT solving, Mathematical Social Sciences 90 (2017), 18–27.
[BMS21] Felix Brandt, Marie Matthäus, and Christian Saile, Minimal voting paradoxes, Working paper, 2021.
[Bor83] Georges Bordes, On the possibility of reasonable consistent majoritarian choice: Some positive results, Journal of Economic Theory 31 (1983), no. 1, 122–132.
[BSS18] Felix Brandt, Christian Saile, and Christian Stricker, Voting with ties: Strong impossibilities via SAT solving, Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018) (M. Dastani, G. Sukthankar, E. André, and S. Koenig, eds.), International Foundation for Autonomous Agents and Multiagent Systems, 2018, pp. 1285–1293.
[CK90] C. C. Chang and H. Jerome Keisler, *Model theory*, North-Holland, Amsterdam, 1990.

[CK02] Donald E. Campbell and Jerry S. Kelly, *Impossibility theorems in the Arrovian framework*, Handbook of Social Choice and Welfare (Kenneth J. Arrow, Amartya K. Sen, and Kōtarō Suzumura, eds.), vol. 1, North-Holland, Amsterdam, 2002, pp. 35–94.

[Deb87] Bernard Debon, *Caractérisation des matrices des préférences nettes et méthodes d’agrégation associées*, Mathématiques et sciences humaines 97 (1987), 5–17.

[Dug13] John Duggan, *Uncovered sets*, Social Choice and Welfare 41 (2013), no. 3, 489–535.

[Fis77] Peter C. Fishburn, *Condorcet social choice functions*, SIAM Journal on Applied Mathematics 33 (1977), no. 3, 469–489.

[HP20] Wesley H. Holliday and Eric Pacuit, *Split Cycle: A new Condorcet consistent voting method independent of clones and immune to spoilers*, arXiv:2004.02350, 2020.

[HP21] __________, *Axioms for defeat in democratic elections*, Journal of Theoretical Politics 33 (2021), no. 4, 475–524, arXiv:2008.08451.

[Kra77] Gerald H. Kramer, *A dynamical model of political equilibrium*, Journal of Economic Theory 16 (1977), no. 2, 310–334.

[McG53] David C. McGarvey, *A theorem on the construction of voting paradoxes*, Econometrica 21 (1953), no. 4, 608–610.

[Mil80] Nicholas R. Miller, *A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting*, American Journal of Political Science 24 (1980), no. 1, 68–96.

[Nur87] Hannu Nurmi, *Comparing voting systems*, D. Reidel, Dordrecht, 1987.

[OEG13] Ömer Egecioglu and Ayça E. Giritligil, *The impartial, anonymous, and neutral culture model: A probability model for sampling public preference structures*, Journal of Mathematical Sociology 37 (2013), 203–222.

[Oka11] Samir Okasha, *Theory choice and social choice: Kuhn versus Arrow*, Mind 120 (2011), no. 477, 83–115.

[Saa08] Donald G. Saari, *Disposing dictators, demystifying voting paradoxes*, Cambridge University Press, Cambridge, 2008.

[Sch86] Thomas Schwartz, *The logic of collective choice*, Columbia University Press, New York, 1986.

[Sch11] Markus Schulze, *A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method*, Social Choice and Welfare 36 (2011), 267–303.

[Sch18] __________, *The Schulze method of voting*, arXiv:1804.02973, 2018.

[Sen71] Amartya Sen, *Choice functions and revealed preference*, The Review of Economic Studies 38 (1971), no. 3, 307–317.

[Sen17] __________, *Collective choice and social welfare: An expanded edition*, Harvard University Press, Cambridge, Mass., 2017.

[Sim69] Paul B. Simpson, *On defining areas of voter choice: Professor Tullock on stable voting*, The Quarterly Journal of Economics 83 (1969), no. 3, 478–490.

[Smi73] John H. Smith, *Aggregation of preferences with variable electorate*, Econometrica 41 (1973), no. 6, 1027–1041.

[Tid87] T. Nicolaus Tideman, *Independence of clones as a criterion for voting rules*, Social Choice and Welfare 4 (1987), 185–206.

[ZT89] T. M. Zavist and T. Nicolaus Tideman, *Complete independence of clones in the ranked pairs rule*, Social Choice and Welfare 6 (1989), 167–173.

DEPARTMENT OF PHILOSOPHY AND GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE, UNIVERSITY OF CALIFORNIA, BERKELEY

Email address: wesholliday@berkeley.edu

DEPARTMENT OF PHILOSOPHY, UNIVERSITY OF MARYLAND

Email address: epacuit@umd.edu

STANFORD INSTITUTE FOR ECONOMIC POLICY RESEARCH, STANFORD UNIVERSITY

Email address: zahedian@stanford.edu