Difference between three quantities

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May 2, 2014

Abstract

The notion of difference between three and more quantities is introduced. The method is based on one of the remarkable properties of the Vandermonde’s determinant.

Keywords: Distance, Vandermonde determinant, geometry, matrix, polynomial.

Introduction.

The notion difference between two quantities $a$ and $b$ given by $(a - b)$ plays a basic role in mathematics, consequently in all branches of human activity where the mathematics is applied. However the long stand question is: what is the difference between three (or more) quantities?

This question frequently arises, for example, in the physics of systems consisting of many particles, in economics etc.

The binary operation $[a, b] = (a - b)$ possesses the following principal feature: with respect to the third quantity $c$ this operation is decomposed into a sum of the same operations between $a$ and $c$, and $c$ and $b$, i.e.,

$[a, b] = [a, c] + [c, b].$

There were several problems of mathematics and physics where investigators needed in the notion of the difference between three quantities. Y. Nambu [1] quantizing the generalized Poisson structure on three dimensional phase space met the problem of extension of the notion of commutator. In fact, the notion of commutator can be defined only for the pair of operators $A$ and $B$ as a difference $AB$ and $BA$. Thus, if one wants to extend this notion for triple operators he will need on the notion of the difference between three quantities. In Refs.[2],[3], the following approach has been developed. By noting that in formula $(a - b) = (a + \theta b$ the value $\theta = -1$ is a primitive root of quadratic polynomial $x^2 - 1$, the author suggested the following definition of the difference between three quantities

$[a, b, c] = a + b\theta + c\theta^2,$

where $\theta$ is a primitive root of polynomial $x^3 - 1$. In Refs.[5],[4], this formula of difference have been used in order to formulate a notion of ternary commutator. Apparently, this definition belong to the field of complex numbers and possesses with the following feature

$[a, b, c] = [a, d, f] + [d, a, f] + [d, f, c].$

In the present paper we suggest a definition of the notion of the difference between three and more quantities making use of a feature of the Vandermonde determinant. Denote by $[a, b, c]$
difference between three quantities $a, b, c$. With respect to additional quantity $d$ this definition of the difference is decomposed as follows

$$[a, b, c] = [d, b, c] + [a, d, c] + [a, b, d].$$

All quantities belong to the field of real numbers.

1 Difference between three quantities

Let us start with the fraction of type

$$\frac{1}{x^3 - 3p_1x^2 + 2p_2x - p^2} = \frac{1}{(x - x_3)(x - x_2)(x - x_1)},$$

(1.1)

where $x_i, i = 1, 2, 3$ are roots of the cubic polynomial

$$P_3(x) := x^3 - 3p_1x^2 + 2p_2x - p^2.$$  

(1.2)

The following expansion for that fraction holds true

$$\frac{1}{x^3 - 3p_1x^2 + 2p_2x - p^2} = \frac{(x_3 - x_2)}{V} \frac{1}{x - x_1} + \frac{(x_1 - x_3)}{V} \frac{1}{x - x_2} + \frac{(x_2 - x_1)}{V} \frac{1}{x - x_3},$$

(1.3)

where by $V$ we denoted the Vandermonde’s determinant of matrix of order $(3 \times 3)$

$$V = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = Det\begin{pmatrix}1&1&1 \\
x_1&x_2&x_3 \\
x_1^2&x_2^2&x_3^2\end{pmatrix}.\quad (1.4)$$

Collecting the terms in the right-hand side of (1.3) we obtain

$$\frac{(x_3 - x_2)}{V} \frac{1}{x - x_1} + \frac{(x_1 - x_3)}{V} \frac{1}{x - x_2} + \frac{(x_2 - x_1)}{V} \frac{1}{x - x_3} =$$

$$\frac{1}{V (x - x_3)(x - x_2)(x - x_1)} (x_3 - x_2)(x - x_2)(x - x_3) + (x - x_2)(x - x_1)(x_1 - x_2) + (x - x_3)(x - x_1)(x_3 - x_1) \right). \quad (1.5)$$

From this equality we come to the conclusion, that

$$(x_3 - x_2)(x - x_2)(x - x_3) + (x - x_2)(x - x_1)(x_1 - x_2) + (x - x_3)(x - x_1)(x_3 - x_1) = V. \quad (1.6)$$

This equation displays an interesting feature of Vandermonde’s determinant, which to our knowledge still has not been revealed [6].

We suggest to use formula (1.6) as a formula of difference between three amounts. The interval of this formal distance is bounded by three points $x_1, x_2, x_3$ and it is divided into three parts by using only one point, $x$, among them. Thus, the following formula

$$V = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \text{ is an analogue of the interval } (x_1 - x_2),$$

and the property given by formula (1.6) is an analogue of the following property of the interval between two points

$$x_1 - x_2 = (x_1 - x) + (x - x_2). \quad (1.7)$$
The Proof of formula (1.6).

Formula (1.6) is a consequence of one of the properties of Vandermonde’s determinant. We have to prove that

\[
\det \begin{pmatrix}
    1 & 1 & 1 \\
    x_1 & x_2 & x_3 \\
    x_1^2 & x_2^2 & x_3^2
\end{pmatrix}
= \det \begin{pmatrix}
    1 & 1 & 1 \\
    x & x_2 & x_3 \\
    x^2 & x_2^2 & x_3^2
\end{pmatrix} + \det \begin{pmatrix}
    1 & 1 & 1 \\
    x & x & x_3 \\
    x^2 & x^2 & x_3^2
\end{pmatrix} + \det \begin{pmatrix}
    1 & 1 & 1 \\
    x_1 & x_2 & x \\
    x_1^2 & x_2^2 & x^2
\end{pmatrix}.
\] (1.8)

Consider the following matrix

\[
AV := \begin{pmatrix}
    1+1 & 1+1 & 1+1 \\
    x_1+x & x_2+x & x_3+x \\
    x_1^2+x^2 & x_2^2+x^2 & x_3^2+x^2
\end{pmatrix},
\] (1.9)

and calculate determinant of this matrix in two ways.

Firstly, let us calculate the determinant on making use of the method of expansion with respect to lines of the matrix. In this way we find that

\[
\det(AV) = \det \begin{pmatrix}
    1+1 & 1+1 & 1+1 \\
    x_1+x & x_2+x & x_3+x \\
    x_1^2+x^2 & x_2^2+x^2 & x_3^2+x^2
\end{pmatrix}
= \det \begin{pmatrix}
    1+1 & 1+1 & 1+1 \\
    x_1 & x_2 & x_3 \\
    x_1^2+x^2 & x_2^2+x^2 & x_3^2+x^2
\end{pmatrix} + \det \begin{pmatrix}
    1+1 & 1+1 & 1+1 \\
    x & x & x \\
    x_1^2+x^2 & x_2^2+x^2 & x_3^2+x^2
\end{pmatrix}
= \det \begin{pmatrix}
    2 & 2 & 2 \\
    x_1 & x_2 & x_3 \\
    x_1^2+x^2 & x_2^2+x^2 & x_3^2+x^2
\end{pmatrix} + \det \begin{pmatrix}
    2 & 2 & 2 \\
    x_1^2 & x_2^2 & x_3^2 \\
    x^2 & x^2 & x^2
\end{pmatrix}.
\]

The last determinant is equal to zero. In this way we get

\[
\det(AV) = 2V.
\] (1.20)

Secondly, let us expand the determinant with respect to columns.

For that purpose it is convenient to use the following notation of the Vandermonde determinant:

\[
V = \det \begin{pmatrix}
    1 & 1 & 1 \\
    x_1 & x_2 & x_3 \\
    x_1^2 & x_2^2 & x_3^2
\end{pmatrix} = [x_1, x_2, x_3].
\]

In this notation \(\det(AV) = [x_1 + x, x_2 + x, x_3 + x]\). Then, the following expansion holds true

\[
[x_1 + x, x_2 + x, x_3 + x] = [x_1, x_2 + x, x_3 + x] + [x, x_2 + x, x_3 + x] = [x_1, x_2, x_3] + [x_1, x_2, x] + [x_1, x, x_3] + [x, x_2, x_3].
\] (1.21)

Notice, the first term is the Vandermonde’s determinant. Therefore,

\[
\det(AV) = V + [x_1, x_2, x] + [x_1, x, x_3] + [x, x_2, x_3] = 2V.
\]

Hence,

\[
V = [x_1, x_2, x] + [x_1, x, x_3] + [x, x_2, x_3].
\] (1.22)

End of proof.

\[f\]
2 Difference between $n \geq 2$ quantities

Since we have found the concept of difference between three quantities this generalization to the case of $n \geq 3$ quantities is straightforward.

Consider $n$-th order Vandermonde’s matrix

$$V_{ik} := \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
x_1 & x_2 & x_3 & \ldots & x_n \\
x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1}
\end{pmatrix}. \quad (2.1)$$

The determinant of this matrix is given by well-known Vandermonde’s formula:

$$V = Det(V_{ij}) = \prod_{i > k} (x_i - x_k). \quad (2.2)$$

Consider the following auxiliary matrix

$$AV(x) := \begin{pmatrix}
1 + 1 & \ldots & 1 + 1 & \ldots & 1 + 1 \\
x_1 + x & \ldots & x_k + x & \ldots & x_n + x \\
x_1^2 + x^2 & \ldots & x_k^2 + x^2 & \ldots & x_n^2 + x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^l + x^l & \ldots & x_k^l + x^l & \ldots & x_n^l + x^l \\
x_{n-1} + x^{n-1} & \ldots & x_k^{n-1} + x^{n-1} & \ldots & x_n^{n-1} + x^{n-1}
\end{pmatrix}. \quad (2.3)$$

Now, let us prove that the determinant of this matrix equal to $2V$. Firstly, expand this determinant with respect to $l$-th line. The result is given by sum of two determinants

$$Det(AV(x)) = Det(AV_1(x, x_k^l)) + Det(AV_2(x, x^l)), \quad (2.4)$$

where we denoted

$$AV_1(x, x^l) = \begin{pmatrix}
1 + 1 & \ldots & 1 + 1 & \ldots & 1 + 1 \\
x_1 + x & \ldots & x_k + x & \ldots & x_n + x \\
x_1^2 + x^2 & \ldots & x_k^2 + x^2 & \ldots & x_n^2 + x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^l & \ldots & x_k^l & \ldots & x_n^l \\
x_1^{n-1} + x^{n-1} & \ldots & x_k^{n-1} + x^{n-1} & \ldots & x_n^{n-1} + x^{n-1}
\end{pmatrix},$$

and,

$$AV_2(x, x^l) = \begin{pmatrix}
1 + 1 & \ldots & 1 + 1 & \ldots & 1 + 1 \\
x_1 + x & \ldots & x_k + x & \ldots & x_n + x \\
x_1^2 + x^2 & \ldots & x_k^2 + x^2 & \ldots & x_n^2 + x^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^l & \ldots & x_k^l & \ldots & x_n^l \\
x_1^{n-1} + x^{n-1} & \ldots & x_k^{n-1} + x^{n-1} & \ldots & x_n^{n-1} + x^{n-1}
\end{pmatrix}.$$
The determinant of the second matrix is trivial because there \( l \)-th line is proportional to the first one. Continue to expand the first determinant \( AV_1(x, x^l) \) with respect to other lines. At the final step of this process the determinant of the auxiliary matrix is reduced to the following form

\[
Det(AV(x)) = Det \begin{pmatrix}
1 + 1 & 1 + 1 & 1 + 1 \\
x_1 & x_k & x_n \\
x_1^2 & x_k^2 & x_n^2 \\
\vdots & \vdots & \vdots \\
x_1^{l-1} & x_k^{l-1} & x_n^{l-1}
\end{pmatrix},
\]

which obviously equal to \( 2V \),

\[
Det(AV(x)) = 2V. \tag{2.6}
\]

Now let us calculate the determinant \( Det(AV(x)) \) by expanding with respect to columns. For the sake of convenience denote the Vandermonde’s determinant (2.1) as follows

\[
V = Det(V[ij]) = [x_1...x_k...x_n]. \tag{2.7}
\]

Correspondingly, the determinant of the auxiliary matrix will be written in the form

\[
Det(AV(x)) = [x_1 + x...x_k + x...x_n + x]. \tag{2.8}
\]

The expansion process with respect to columns of this determinant is worked out as follows.

\[
[x_1 + x...x_k + x...x_n + x] = [x_1, x_2 + x...x_k + x...x_n + x] + [x, x_2 + x...x_k + x...x_n + x]. \tag{2.9}
\]

The second term in right-hand side is equal to

\[
[x, x_2 + x...x_k + x...x_n + x] = [x, x_2...x_k...x_n].
\]

Continue to expand the first of the sum

\[
[x_1, x_2 + x...x_k + x...x_n + x] = [x_1, x_2, x_3 + x...x_k + x...x_n + x] + [x, x_2, x...x_k + x...x_n + x].
\]

The last term is equal to zero. The first one is represented as follows

\[
[x_1, x_2, x, x_4 + x...x_k + x...x_n + x] = [x, x_2, x_3, x_4 + x...x_k...x_n].
\]

At the final step of this process we come to the following equation

\[
Det(AV(x)) = 2V = \sum_{k=1}^{n} [x_1, x_2, ...x_{k-1}, x, x_{k+1}...x_n] + V, \tag{2.10}
\]

On the other hand, according to (2.6) \( Det(AV(x)) = 2V \). Hence,

\[
V = \sum_{k=1}^{n} [x_1, x_2, ...x_{k-1}, x, x_{k+1}...x_n]. \tag{2.11}
\]

This formula implies one of the important features of the Vandermonde’s determinant. That is the formula which we suggest to use as a definition of the difference between \( n \) quantities.

Concluding remarks.
Ternary algebraic operations and cubic relations have been considered, although quite sporadically, by several authors already in the XIX-th century, e.g. by A. Cayley ([7]) and J.J. Sylvester ([8]. The development of Cayley’s ideas, which contained a cubic generalization of matrices and their determinants, can be found in a recent book by M. Kapranov, I.M. Gelfand and A. Zelevinskii ([9]). A discussion of the next step in generality, the so called \(n - ary\) algebras, can be found in ([10]).

The difference between two quantities has direct geometrical interpretation as a distance between two points on a straight line. Let \(O, A, B\) be a set of points on the line and let point \(O\) be a point on the left-hand side of the points \(A\) and \(B\). Let the values \(d(OA), d(OB)\) mean distances between points \(A\) and \(B\) of the point \(O\), correspondingly. Then the difference \([d(OA), d(OB)]\) does not depend of the motion of the point \(O\) and means the distance between points \(A\) and \(B\).

In the similar way, let \(O\) be a point on the straight line on the left-hand side of three points \(A, B, C\) installed on the same line. Let \(d(OA), d(OB), d(OC)\) be distances from \(O\) till points \(A, B, C\), correspondingly. Then the difference \([d(OA), d(OB), d(OC)]\) does not depend of the motion of the point \(O\) along the straight line. This definition of the difference we suggest use in geometry in order to found a concept of ternary distance between three points. A generalization to the case of \(n \geq 3\) points is straightforward.

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