A CONJECTURAL GENERATING FUNCTION FOR NUMBERS OF CURVES ON SURFACES

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Abstract. I give a conjectural generating function for the numbers of $\delta$-nodal curves in a linear system of dimension $\delta$ on an algebraic surface. It reproduces the results of Vainsencher \cite{V2} for the case $\delta \leq 6$ and Kleiman-Piene \cite{K-P} for the case $\delta \leq 8$. The numbers of curves are expressed in terms of five universal power series, three of which I give explicitly as quasimodular forms. This gives in particular the numbers of curves of arbitrary genus on a K3 surface and an abelian surface in terms of quasimodular forms, generalizing the formula of Yau-Zaslow for rational curves on K3 surfaces. The coefficients of the other two power series can be determined by comparing with the recursive formulas of Caporaso-Harris for the Severi degrees in $\mathbb{P}_2$. We verify the conjecture for genus 2 curves on an abelian surface. We also discuss a link of this problem with Hilbert schemes of points.

1. Introduction

Let $L$ be a line bundle on a projective algebraic surface $S$. In the case $\delta \leq 6$ Vainsencher \cite{V2} proved formulas for the numbers $t^S_\delta(L)$ of $\delta$-nodal curves in a general $\delta$-dimensional sub-linear system of $|L|$. By a refining Vainsechers approach Kleiman-Piene \cite{K-P} extended the results to $\delta \leq 8$. The formulas hold under the assumption that $L$ is a sufficiently high power of a very ample line bundle.

In this paper we want to give a conjectural generating function for the numbers $t^S_\delta(L)$. We will have only partial success: We are able to express the $t^S_\delta(L)$ in terms of five universal generating functions in one variable $q$. Three of these are Fourier developments of (quasi-)modular forms, the other two we have not been able to identify: the formulas of \cite{C-H} for the Severi degrees on $\mathbb{P}_2$ yield an algorithm for computing their coefficients and I computed them up to degree 28. As the functions are universal, one would hope that there exists a nice closed expression for them.

If the canonical divisor of the surface $S$ is numerically trivial, only the quasimodular forms appear in the generating function. Thus we obtain (conjecturally) the numbers of curves of arbitrary genus on a K3 surface and on an abelian surface as the Fourier coefficients of quasimodular forms. The formulas generalize the calculation of \cite{Y-Z} for the numbers

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of rational curves on K3 surfaces. The fact that for K3 surfaces and abelian surfaces the numbers can be expressed solely in terms of quasimodular forms might be related to physical dualities. The numbers \( t_\delta^S(L) \) will on a general surface (including also \( \mathbb{P}_2 \)) count all curves with the prescribed numbers of nodes, including the reducible ones. It seems however that on an abelian surface one is actually counting irreducible curves, and in the case of K3 surfaces one can restrict attention to the case that \(|L|\) contains only irreducible reduced curves. Both in the case of abelian surfaces and K3 surfaces I do then expect the generating function to count the curves of given genus in sub-linear systems of \(|L|\), even if \(L\) is only assumed to be very ample and not a high multiple of an ample line bundle. The curves can then have worse singularities than nodes and a curve \(C\) should be counted with a multiplicity determined by the singularities of \(C\), as in [B], [F-G-vS]. In particular nodal (or more generally immersed) curves should count with multiplicity 1. In the case of K3 surfaces and primitive line bundle \(L\) the numbers of these curves have in the meantime been computed in [Br-Le].

The coefficients of the unknown power series are by the recursion of [C-H] the solutions to a highly overdetermined system of linear equations, the same is true for a similar recursion obtained by Vakil [Va] for rational ruled surfaces and the results of [Ch] on \(\mathbb{P}_2\) and \(\mathbb{P}_1 \times \mathbb{P}_1\). This gives an additional check of the conjecture. Finally we compute the numbers of genus 2 curves on an abelian surface.

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2. Statement of the conjecture

Let \(S\) be a projective algebraic surface and \(L\) a line bundle on \(S\). In this paper by a curve on \(S\) we mean an effective reduced divisor on \(S\). A nodal curve on \(S\) is a reduced (not necessarily irreducible) divisor on \(S\), which has only nodes as singularities. We denote by \(K_S\) the canonical bundle and by \(c_2(S)\) the degree of the second Chern class. For two line bundles \(L\) and \(M\) let \(LM\) denote the degree of \(c_1(L) \cdot c_1(M) \in H^4(S, \mathbb{Z})\).

In [V2] (for \(\delta \leq 6\)) and [K-P] (for \(\delta \leq 8\)) formulas for the number \(a_\delta^S(L)\) of \(\delta\)-nodal curves in a general \(\delta\)-dimensional linear sub-system \(V\) of \(|L|\) were proved. Here general means that \(V\) lies in an open subset of the Grassmannian of \(\delta\)-dimensional subspaces of \(|L|\). The number is expressed as a polynomial in \(c_2(S), K_S^2, LK_S\) and \(L^2\) of degree \(\delta\). The formulas are valid if \(L\) is a sufficiently high multiple of an ample line bundle. In other words, for such
L, the locally closed subset \( W^S_\delta(L) \) (with the reduced structure) of elements in \(|L|\) defining \( \delta \)-nodal curves has codimension \( \delta \), and its degree is given by a polynomial as above.

It is clear and was noted before [K-P] that there should be a formula for all \( \delta \):

**Conjecture 2.1.** For all \( \delta \in \mathbb{Z} \) there exist universal polynomials \( T_\delta(x, y, z, t) \) of degree \( \delta \) \((T_\delta = 0 \text{ if } \delta < 0)\) with the following property. Given \( \delta \) and a pair of a surface \( S \) and a very ample line bundle \( L_0 \) there exists an \( m_0 > 0 \) such that for all \( m \geq m_0 \) and for all very ample line bundles \( M \) the line bundle \( L := L_0^{\otimes m} \otimes M \) satisfies

\[
a^S_\delta(L) = T_\delta(L^2, LKS, K^2_S, c_2(S)).
\]

**Remark 2.2.** Note that the statement is slightly stronger than that of [V2] in the case \( \delta \leq 6 \). We expect \( L \) does not have to be a high power of a very ample line bundle but that it suffices that \( L \) is sufficiently ample. In fact the results of the final two sections below suggest that there might exist a universal constant \( C > 0 \) (independent of \( S \) and \( L \)), such that, if \( L \) is \( C\delta \)-very ample (see section 5), then the conjecture holds for up to \( \delta \)-nodes.

Assuming conjecture 2.1 we will in future just write

\[
T_\delta^S(x, y) := T_\delta(x, y, K^2_S, c_2(S)), \quad t^S_\delta(L) := T^S_\delta(L^2, LKS), \quad T(S, L) := \sum_{\delta \geq 0} t^S_\delta(L)x^\delta.
\]

The aim of this note is to give a conjectural formula for the generating function \( T(S, L) \), and to give some evidence for it.

We start by noting that conjecture 2.1 imposes rather strong restrictions on the structure of \( T(S, L) \). The point is that the conjecture applies to all surfaces, including those with several connected components. In this case we will write \(|L|\) for \( \mathbb{P}(H^0(L)) \). By definition \( W^S_\delta(L) \) includes only \( f \in |L| \) which do not vanish identically on a component of \( S \).

**Proposition 2.3.** Assume conjecture 2.1. Then there exist universal power series \( A_1, A_2, A_3, A_4 \in \mathbb{Q}[[x]] \) such that

\[
T(S, L) = \exp(L^2A_1 + LKS A_2 + K^2_S A_3 + c_2(S)A_4).
\]

**Proof.** Fix \( \delta_0 \in \mathbb{Z}_{>0} \). Assume that \( S = S_1 \sqcup S_2 \) and that \( L_1 := L|_{S_1} \) and \( L_2 := L|_{S_2} \) are both sufficiently ample so that the \( W^S_{\delta_i}(L_i) \) have codimension \( \delta \) and degree \( t^S_{\delta_i}(L_i) \) in \(|L_i|\) for \( i = 1, 2 \) and \( \delta < \delta_0 \). Fix \( \delta < \delta_0 \). The application \((f + g) \mapsto (f, g)\) defines a surjective morphism \( p: U \to |L_1| \times |L_2| \), defined on the open subset \( U \subset |L| \) where neither \( f \) not \( g \) vanish identically. The fibres of \( p \) are lines in \(|L|\). Obviously

\[
W^S_\delta(L) = p^{-1}\left( \prod_{\delta_1 + \delta_2 = \delta} W^S_{\delta_1}(L_1) \times W^S_{\delta_2}(L_2) \right).
\]
In particular \( W^S_{\delta}(L) \) has codimension \( \delta \) in \( |L| \) and modulo the ideal generated by \( x^{\delta_0} \) we have \( T(S, L) = T(S_1, L_1) T(S_2, L_2) \).

Now choose \( n \in \mathbb{Z}_{>0} \) such that conjecture \([2,3]\) holds for \( Z_1 := (\mathbb{P}_2, \mathcal{O}(n)), Z_2 := (\mathbb{P}_2, \mathcal{O}(2n)), Z_3 := (\mathbb{P}_2, \mathcal{O}(3n)) \) and \( Z_3 := (\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(n, n)) \) for all \( \delta \leq \delta_0 \). Let \( S = S(a_1, a_2, a_3, a_4) \) be the disjoint union of \( a_i \) copies of each of the \( Z_i \) with \( a_i \in \mathbb{Z}_{\geq 0} \). Then by the above argument
\[
T(S, L) = \prod_i T(Z_i)^{a_i}
\]
which can be written in the form \( \exp(L^2 A_1 + L K_2 A_2 + K_3^2 A_3 + c_2(S) A_4) \), for universal power series \( A_1, A_2, A_3, A_4 \). On the other hand we note that the 4-tuple \((L^2, L K_2, K_3^2, c_2(S))\) takes on the the linearly independent values \((n^2, -3n, 9, 3), (4n^2, -6n, 9, 3), (9n^2, -9n, 9, 3), (2n^2, -4n, 8, 4)\). Thus the image of the \( S(a_1, a_2, a_3, a_4) \) is Zariski-dense in \( \mathbb{Q}^4 \), and the result follows.

We recall some facts about quasimodular forms from \([K-Z]\). We denote by \( \mathcal{H} := \{ \tau \in \mathbb{C} \mid \Im(\tau) > 0 \} \) the complex upper half plane, and for \( \tau \in \mathcal{H} \) we write \( q := e^{2\pi i \tau} \). A modular form of weight \( k \) for \( SL(2, \mathbb{Z}) \) is a holomorphic function \( f \) on \( \mathcal{H} \) satisfying
\[
f\left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k f(\tau), \quad \tau \in \mathcal{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})
\]
and having a Fourier series \( f(\tau) = \sum_{n=0}^{\infty} a_n q^n \).

Writing \( \sigma_k(n) := \sum_{d|n} d^k \), the Eisenstein series
\[
G_k(\tau) = -\frac{B_k}{2k} + \sum_{n>0} \sigma_{k-1}(n) q^n, \quad k \geq 2, \quad B_k = k\text{th Bernoulli number}
\]
are for even \( k \geq 2 \) modular forms of weight \( k/2 \), while \( G_2(\tau) \) is only a quasimodular form. Another important modular form is the discriminant
\[
\Delta(\tau) = q \prod_{k>0} (1 - q^k)^{24} = \eta(\tau)^{24}
\]
where \( \eta(\tau) \) is the Dedekind \( \eta \) function.

For the precise definition of quasimodular forms see \([K-Z]\). They are essentially the holomorphic parts of almost holomorphic modular forms. The ring of modular forms for \( SL(2, \mathbb{Z}) \) is just \( \mathbb{Q}[G_4, G_6] \), while the ring of quasimodular forms is \( \mathbb{Q}[G_2, G_4, G_6] \). We denote by \( D \) the differential operator \( D := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq} \). Unlike the ring of modular forms the ring of quasimodular forms is closed under differentiation, i.e. for a quasimodular form \( f \) of weight \( k \) the derivative \( Df \) is a quasimodular form of weight \( k + 2 \). In fact every quasimodular form has a unique representation as a sum of derivatives of modular forms and of \( G_2 \) (see \([K-Z]\)).

**Conjecture 2.4.** There exist universal power series \( B_1, B_2 \) in \( q \) such that
\[
\sum_{\delta \in \mathbb{Z}} S^{S}_{\delta}(L)(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(L)} B_1(q)^{K_3^2} B_2(q)^{LK_2}}{\Delta(\tau)^{D^2 G_2(\tau)/q^2} \chi(\mathcal{O}_S)/2}.
\]
Remark 2.5. (1) Using the fact that $DG_2(\tau)/q$, $B_1(q)$, $B_2(q)$, $\Delta(\tau)D^2G_2(\tau)/q^2$ are power series in $q$ starting with 1, and by the standard formulas $\chi(O_S) = (K^2_S + c_2(S))/12$, $\chi(L) = (L^2 - LK_S)/2 + \chi(O_S)$ one sees that conjecture 2.4 expresses the $t_\delta^2(L)$ as polynomials of degree $\delta$ in $L^2$, $K_S L$, $K^2_S$, $c_2(S)$.

(2) I have checked that conjecture 2.4 reproduces the formulas of Vainsencher and Kleiman-Piene for $\delta \leq 8$. This determines $B_1(q)$ and $B_2(q)$ up to degree $q^8$. In remark 4.2 below we use the formulas of [C-H] for the Severi degrees in $\mathbb{P}_2$ to determine the coefficients of $B_1(q)$ and $B_2(q)$ up to degree 28 (they are given here up to degree 20).

$$B_1(q) \equiv 1 - q - 5q^2 + 39q^3 - 345q^4 + 2961q^5 - 24866q^6 + 207759q^7 - 1737670q^8$$
$$+ 14584625q^9 - 122937305q^{10} + 1040906771q^{11} - 8852158628q^{12} + 75598131215q^{13}$$
$$- 648168748072q^{14} + 5577807139921q^{15} - 48163964723088q^{16} + 417210529188188q^{17}$$
$$- 3624610235789053q^{18} + 3157529028078630q^{19} - 275758194822813754q^{20} + O(q^{21})$$

$$B_2(q) \equiv 1 + 5q + 2q^2 + 35q^3 - 140q^4 + 986q^5 - 6643q^6 + 48248q^7 - 362700q^8$$
$$+ 2802510q^9 - 22098991q^{10} + 177116726q^{11} - 1438544962q^{12} + 11814206036q^{13}$$
$$- 97940651274q^{14} + 818498739637q^{15} - 6888195294592q^{16} + 58324130994782q^{17}$$
$$- 496519067059432q^{18} + 4247266246317414q^{19} - 3648805934639524q^{20} + O(q^{21})$$

Remark 2.6. We give a reformulation of the conjecture. We define, for all $l, m, r \in \mathbb{Z}$

$$n^S_r(l, m) := T^S_{l+\chi(O_S)-1-r}(2l + m, m)$$
$$m^S_r(l, m) := n_{g-m-2+\chi(O_S)}(l, m)$$

If $L$ is sufficiently ample with respect to $\delta = \chi(L) - 1 - r$ and $S$ (and thus in particular $\chi(L) = H^0(S, L)$ and $r \geq 0$), then $n^S_r((L^2 - LK_S)/2, LK_S)$ counts the $\delta$-nodal curves in a general $r$-codimensional sub-linear system of $|L|$. Then

$$\sum_{l \in \mathbb{Z}} n^S_r(l, m)q^l = B_1(q)^{K^2_S}B_2(q)^m(DG_2(\tau))^r\frac{D^2G_2(\tau)}{(\Delta(\tau)D^2G_2(\tau))^{\chi(O_S)/2}}.$$
Proof. If \( f(q) \) and \( g(q) \) are power series in \( q \) and \( g(q) \) starts with \( q \), then we can develop \( f(q) \) as a power series in \( g(q) \) and

\[
\text{Coeff}_{g(q)^k} f(q) = \text{Res}_{g(q)=0} \frac{f(q) dg(q)}{g(q)^{k+1}} = \text{Coeff}_{q^0} \frac{f(q) D g(q)}{g(q)^{k+1}}.
\]

We apply this with \( g(q) = DG_2(\tau) \).

\[\square\]

3. COUNTING CURVES ON K3 SURFACES AND ABELIAN SURFACES

Let now \( S \) be a surface with numerically trivial canonical divisor. We denote \( n^S_r(l) := n^S_r(l,0) \), i.e. for \( L \) sufficiently ample \( n^S_r(L^2/2) \) is the number of \( \chi(L) - r - 1 \)-nodal curves in an \( r \)-codimensional sub-linear system of \( L \). \( n^S_r(l) \) can be expressed in terms of quasimodular forms.

For \( S \) a K3 surface, \( A \) an abelian surface and \( F \) an Enriques or bielliptic surface we get

\[
\sum_{l \in \mathbb{Z}} n^S_r(l) q^l = (DG_2(\tau))^r/\Delta(\tau) \tag{3.0.1}
\]

\[
\sum_{l \in \mathbb{Z}} n^A_r(l) q^l = (DG_2(\tau))^r D^2 G_2(\tau), \tag{3.0.2}
\]

\[
\sum_{l \in \mathbb{Z}, r \geq 0} n^A_r(l) q^l \frac{z^r}{r!} = \frac{1}{z} D(\exp(DG_2(\tau) z))
\]

\[
\sum_{l \in \mathbb{Z}, r \geq 0} n^F_r(l) q^l = (DG_2(\tau))^r \left( \left( (D^2 G_2(\tau)) / \Delta(\tau) \right)^{1/2} \right) \tag{3.0.3}
\]

Note that \( m^S_g(l) = n^S_g(l) \), \( m^F_g(l) = n^F_g(l) \), \( m^A_g(l) = n^A_g(l) \).

Remark 3.1. In the case of an abelian surface or a K3 surface we expect that the numbers \( n^A_r(l) \) and \( n^S_r(l) \) have a more interesting geometric significance.

(1) Let \((S, L)\) be a polarized K3 surface with \( \text{Pic}(S) = \mathbb{Z} L \). Then the linear system \(|L|\) contains only irreducible curves. The number \( n^S_r(L^2/2) \) is a count of curves \( C \in |L| \) of geometric genus \( r \) passing through \( r \) general points on \( S \). In this case the numbers of rational curves have been calculated in [Y-Z] and [B] and (3.0.1) is a generalization to arbitrary genus. The rational curves that are counted are not necessarily nodal. In the count a rational curve \( C \) is assigned the Euler number \( e(\bar{JC}) \) of its compactified Jacobian (which is 1 if \( C \) is immersed) as multiplicity.

Let \( \overline{M}_{g,n}(S, \beta) \) be the moduli space of \( n \)-pointed genus \( g \) stable maps of homology class \( \beta \) (see [K-N]). It comes equipped with an evaluation map \( \mu \) to \( S^n \). In [P-G-vS] it is shown that for a rational curve \( C \) on a K3 surface \( e(\bar{JC}) \) is just the multiplicity of \( \overline{M}_{0,0}(S, |L|) \) at the point defined by the normalization of \( C \). Here \( |L| \) denotes the homology class Poincaré dual to \( c_1(L) \). In other words \( n^S_r(L^2/2) \) is just the length of the \( 0 \)-dimensional scheme \( \overline{M}_{0,0}(S, |L|) \). I expect that for curves of arbitrary genus the corresponding result should hold:
If $S$ and $L$ are general and $x$ is a general point in $S^r$ then the fiber $μ^{-1}(x) ⊂ \overline{M}_{r,2}(S, [L])$ should be a finite scheme and $n^S_r(L^2/2)$ should just be its length. More generally $n^S_r(L^2/2)$ should be a generalized Gromov-Witten invariant as defined and studied in [Br-L] in the symplectic setting and in [Be-F2] in the algebraic geometric setting. In the meantime this invariant has been computed in [Br-L] for curves of arbitrary genus on K3 surfaces confirming the conjecture.

(2) Let $A$ be an abelian surface with a very ample line bundle $L$. We claim that in general all the curves counted in $n^A_r(L^2/2)$ will be irreducible and reduced: The set of $δ$-nodal curves in $|L|$ has expected dimension $χ(L) − δ − 1 = L^2/2 − δ − 1$. On the other hand the set of reducible $δ$-nodal curves $C_1 + \ldots + C_n ∈ |L|$ with $C_i ∈ |L_i|$ has expected dimension

$$\sum_{i=1}^n (L_i^2/2 − 1) − δ + \sum_{1 \leq i < j \leq n} L_iL_j = L^2/2 − δ − n.$$ 

Therefore in general $n^A_r(L^2/2)$ should count the irreducible curves $C ∈ |L|$ of geometric genus $r + 2$ passing through $r$ general points. Again we expect that this result holds in a modified form also if $L$ is not required to be sufficiently ample, and if not all the curves in $|L|$ are immersed. The moduli space $\overline{M}_{g,n}(A, [L])$ is naturally fibered over $Pic^L(A)$. The fibers are the spaces $\overline{M}_{g,n}(A, |M|)$ of stable maps $ϕ : W → A$ with $ϕ_∗(W)$ a divisor in $|M|$, where $c_1(M) = c_1(L)$. Again for $A$ and $L$ general and $x$ a general point in $A^r$ the number $n^A_r(L^2/2)$ should be the length of the fiber $μ^{-1}(x) ⊂ \overline{M}_{r+2,r}(A, |L|)$.

We want to show conjecture [2.3] and the expectations of remark [3.1] for abelian surfaces in a special case.

**Theorem 3.2.** Let $A$ be an abelian surface with an ample line bundle $L$ such that $c_1(L)$ is a polarization of type $(1, n)$. Assume that $A$ does not contain elliptic curves. Write again $σ_1(n) = \sum_{d|n} d$. Then the number of genus 2 curves in $|L|$ is $n^2σ_1(n)$. Moreover all these curves are irreducible and immersed, and the moduli space $\overline{M}_{2,0}(A, |L|)$ consists of $n^2σ_1(n)$ points corresponding to the their normalizations.

**Proof.** We will denote by $[C]$ the homology class of a curve $C$ and by $[L]$ the Poincaré dual of $c_1(L)$. For a divisor $D$ we write $c_1(D)$ for $c_1(\mathcal{O}(D))$. Let $ϕ : C → A$ be a morphism from a connected nodal curve of arithmetic genus 2 to $A$, with $ϕ_∗([C]) = [L]$. Put $D := ϕ(C)$. As $A$ does not contain curves of genus 0 or 1, $C$ must be irreducible and smooth, and $ϕ$ must be generically injective. In particular $[D] = [L]$. Let $J(C)$ be the Jacobian of $C$. We freely use standard results about Jacobians of curves, see ([L-E] chap. 11) for reference. For each $c ∈ C$ the Abel-Jacobi map $α_c : C → J(C)$ is an embedding with $α_c(c) = 0$. We write $C_c := α_c(C)$ and $θ_C := c_1(α_c(C))$. By the Torelli theorem the isomorphism class of $C$ is
determined uniquely by the pair $(J(C), \theta_C)$. For all $a \in A$ we denote by $t_a$ the translation by $a$. By the universal property of the Jacobian there is a unique isogeny $\tilde{\varphi} : J(C) \to A$ such that $\varphi = t_{\varphi(c)} \circ \tilde{\varphi} \circ \alpha_c$ for all $c \in C$. As $\tilde{\varphi}$ is étale $\varphi$ is an immersion and $\varphi : C \to D$ is the normalization map. We also see that $\tilde{\varphi}^*(c_1(L)) = n\theta_C$.

On the other hand, let $(B, \gamma)$ be a principally polarized abelian surface and $\psi : B \to A$ an isogeny with $\psi^*(c_1(L)) = n\gamma$. By the criterion of Matsusaka-Ran and the assumption that $A$ does not contain elliptic curves we obtain that $(B, \gamma) = (J(C), \theta_C)$ for $C$ a smooth curve of genus 2 and $\psi = \tilde{\varphi}$ for a morphism $\varphi : C \to A$ with $\varphi^*([C]) = [L]$. $\tilde{\varphi}$ depends only on $\varphi$ up to composition with a translation in $A$ and $\varphi = t_{\varphi(c)} \circ \tilde{\varphi} \circ \alpha_c$ is determined by $\tilde{\varphi}$ up to translation in $A$.

By the universal property of $J(C)$ applied to the embedding $\alpha_c$, an automorphism $\psi$ of $C$ induces an automorphism $\hat{\psi}$ of $J(C)$. If $\epsilon$ is an automorphism of $J(C)$ with $\epsilon^*(\theta_C) = \theta_C$, then it is $\hat{\psi}$ for some automorphism $\psi$ of $C$. $(H^0(J(C), \mathcal{O}(C_c)) = 1$, and $a \mapsto t_a^\epsilon(\mathcal{O}(C_c))$ defines an isomorphism $J(C) \to \text{Pic}^\theta C(J(C))$.

Therefore we see that the set $M_1$ of morphisms $\varphi : C \to A$ from curves of genus 2 with $\varphi^*([C]) = [L]$ modulo composition with automorphisms of $C$ and with translations of $A$ can be identified with the set $M_2$ of morphisms $\psi : B \to A$ from a principally polarized abelian surface $(B, \gamma)$, such that $\psi^*(c_1(L)) = n\gamma$ modulo composition with automorphisms $\eta : B \to B$ with $\eta^*(\gamma) = \gamma$.

We write $A = \mathbb{C}^2/\Gamma$ and $B = \mathbb{C}^2/\Lambda$. Then $c_1(L)$ is given by an alternating form $a : \Gamma \times \Gamma \to \mathbb{Z}$ such that there is a basis $x_1, x_2, y_1, y_2$ of $\Gamma$ with $a(x_1, y_1) = 1$, $a(x_2, y_2) = n$, $a(x_1, x_2) = a(y_1, y_2) = 0$. A homomorphism $\psi : B \to A$ is given by a linear map $\tilde{\psi} : \mathbb{C}^2 \to \mathbb{C}^2$ with $\psi(\Lambda) \subset \Gamma$. We see that $M_2$ can be identified with the set $M_3$ of sublattices $\Lambda \subset \Gamma$ of index $n$ with $a(\Lambda, \Lambda) \subset n\mathbb{Z}$. We claim that $M_3$ has $\sigma_1(n)$ elements.

First we want to see that this claim implies the theorem. Let $\text{Pic}^L(A)$ be the group of line bundles on $A$ with first Chern class $c_1(L)$. By proposition 4.9 the morphism $\varphi_L : A \to \text{Pic}^L(A); a \mapsto t_a^*L$ is étale of degree $n^2$. By the claim this means that for each $L_1 \in \text{Pic}^L(A)$ the linear system $|L_1|$ contains precisely $n^2\sigma_1(n)$ curves of genus 2.

Finally we show the claim. Via the basis $x_1, y_1, x_2, y_2$ (in that order) we identify $\Gamma$ with $\mathbb{Z}^2 \times \mathbb{Z}^2$. We see that $\Lambda$ must be of the form $\Lambda' \times \mathbb{Z}^2$, where $\Lambda'$ is a sublattice of index $n$ in $\mathbb{Z}^2$, satisfying $b(\Lambda', \Lambda') \subset n\mathbb{Z}$, for the alternating form $b$ defined by $a(x_1, y_1) = 1$. Let $\Lambda'$ be a sublattice of $\mathbb{Z}^2$ of index $n$. We claim that $b(\Lambda', \Lambda') \subset n\mathbb{Z}$. Then the result follows by the well-known fact that the number of sublattices of index $n$ in a rank two lattice is $\sigma_1(n)$. Let $L_1 := p_2(\Lambda')$ for the second projection $\mathbb{Z}^2 \to \mathbb{Z}$ and put $L_2 := \ker(p_2|_{\Lambda'})$. Then $L_1 = d_1\mathbb{Z}$ and $L_2 = d_2\mathbb{Z}$ for $d_1, d_2 \in \mathbb{Z}$ with $d_1d_2 = n$. Choose $x = (k, d_1) \in \Lambda' \cap p_2^{-1}(d_1)$. Then $\Lambda'$ is generated by $L_2$ and $x$, and in particular $b(\Lambda', \Lambda') \subset n\mathbb{Z}$.
4. Severi degrees on $\mathbb{P}_2$ and rational ruled surfaces

The Severi degree $N^{d,\delta}$ is the number of plane curves of degree $d$ with $\delta$ nodes passing through $(d^2 + 3d)/2 - \delta$ general points. In [R2] a recursive procedure for determining the $N^{d,\delta}$ is shown, and in [C-H] a different recursion formula is proven. In the number $N^{d,\delta}$ also reducible curves are included, they however occur only if $d \leq \delta + 1$, furthermore the numbers of irreducible curves can be determined from them ([C-H] see also [Ge]). For simplicity I will write $t_{\delta}(d) := t_{\mathbb{P}_2}(\mathcal{O}(d))$. If $d$ is sufficiently large with respect to $\delta$, then $N^{d,\delta}$ should be equal to $t_{\delta}(d)$:

**Conjecture 4.1.** If $\delta \leq 2d - 2$, then $N^{d,\delta} = t_{\delta}(d)$.

**Remark 4.2.** (1) Conjecture 4.1 and [C-H] provide an effective method of determining the coefficients of the two unknown power series $B_1(q)$ and $B_2(q)$. Using a suitable program I computed the $N^{d,\delta}$ via the recursive formula from [C-H] for $d \leq 16$ and $\delta \leq 30$. We write $x = DG_2(\tau)$. By conjectures 2.4 and 4.1 one has for all $d > 0$ moduli the identity

$$\sum_{\delta \in \mathbb{Z}} N^{d,\delta} x^\delta \equiv \exp(d^2C_1(x) + dC_2(x) + C_3(x))$$

(4.2.1)

Here $C_1(x)$ is known by conjecture 2.4 and the first $k$ coefficients of $C_2(x)$ and $C_3(x)$ determine the first $k$ coefficients of $B_1(q)$ and $B_2(q)$. Taking logarithms on both sides gives, for any two degrees $d_1 < d_2$ and $\delta \leq 2d_1 - 2$, a system of two linear equations for the coefficients of $x^\delta$ in $C_2(x)$ and $C_3(x).$ Note that this also gives a test of the conjecture. It is a nontrivial fact that the generating function has the special shape (4.2.1). In particular each pair $d_1 < d_2$ with $\delta \leq 2d_1 - 2$ already determines the coefficients of $x^\delta$.

(2) The conjecture implies in particular that for $\delta \leq 2d - 2$ the numbers $N^{d,\delta}$ are given by a polynomial $P_{\delta}(d)$ of degree $2\delta$ in $d$. This was already conjectured in [D-I]. Denote by $p_\mu(\delta)$ the coefficient of $d^{2\delta - \mu}$ in $P_{\delta}$. In [D-I] a conjectural formula for the leading coefficients $p_\mu(\delta)$ for $\mu \leq 6$ is given. Kleiman and Piene have determined $p_7(\delta)$ and $p_8(\delta)$ [K-P]. In [C-H] the $N^{d,\delta}$ for $\delta \leq 4$ are computed using the recursive method of [R2], and as an application $p_0(\delta)$ and $p_1(\delta)$ are determined. Using conjectures 2.4 and 4.1 there is an algorithm to determine the $p_\mu(\delta)$. Again we use the formula (4.2.1), and collect terms. From knowing the coefficients of $C_2(x)$ and $C_3(x)$ up to degree 28 we get the $p_\mu(\delta)$ for $\mu \leq 28$. For $\mu \leq 8$ they coincide with those from [D-I], [C-H] and [K-P]. Let $[\ ]$ denote the integer part. For $\mu \leq 28$ we observe that $p_\mu(\delta)$ is of the form

$$p_\mu(\delta) = \frac{3^{\delta-[\mu/2]}(\delta-[\mu/2])!}{(\delta-[\mu/2])!} Q_\mu(\delta),$$
where \(Q_\mu(\delta)\) is a polynomial of degree \([\mu/2]\) in \(\delta\) with integer coefficients, which have only products of powers of 2 and 3 as common factors. In particular

\[
Q_8(\delta) = -2^4(282855 \delta^4 - 931146 \delta^3 + 417490 \delta^2 + 425202 \delta + 1141616), \\
Q_9(\delta) = -2^33^2(128676 \delta^4 + 268644 \delta^3 - 1011772 \delta^2 - 1488377 \delta - 1724779), \\
Q_{10}(\delta) = 2^43^2(4345998 \delta^5 - 15710500 \delta^4 - 3710865 \delta^3 + 7300210 \delta^2 \\
+ 57779307 \delta + 98802690).
\]

Note that for \(d > \delta + 1\) all the curves are irreducible, so that in this case we get a conjectural formula for the stable Gromov-Witten invariants of \(\mathbb{P}_2\), i.e. the numbers of irreducible curves \(C\) of degree \(d\) with \(g(C) \geq \binom{d-2}{2}\).

**Remark 4.3.** Let \(\Sigma_e\) be a rational ruled surface, \(E\) the curve with \(E^2 = -e\) and \(F\) a fiber of the ruling. For simplicity denote \(t^\Sigma_e_6(n,m) := t_6^((\Sigma_e_6)(O(nF + mE))). \) [CH] determined the \(N^{2,m}_{0,\delta}\) for \(\delta \leq 9\) as polynomials of degree \(\delta\) in \(m\) and several numbers \(N^{3,m}_{0,\delta}\). In [Va] a recursion formula very similar to that of [CH] is proved for the generalized Severi degrees \(N^{n,m}_{e,\delta}\), i.e. the number of \(\delta\)-nodal curves in \(|nF + mE|\) which do not contain \(E\) as a component. Using a suitable program I computed the \(N^{n,m}_{e,\delta}\) for \(e \leq 4\), \(\delta \leq 10\), \(n \leq 11\) and \(m \leq 8\). These results are compatible with conjecture 4.1 if one conjectures that for \((n,m) \neq (1,0)\) one has \(N^{n,m}_{e,\delta} = t_6^((\Sigma_e_6)(n,m))\) if and only if \(\delta \leq \min(2m,n - em)\) or \(\delta \leq \min(2m,2n)\) in case \(e = 0\).

**Remark 4.4.** A slightly sharpened version of conjecture 4.1 can be reformulated as saying that \(N^{d,\delta} = t_\delta(d)\) if and only if \(H^0(\mathbb{P}_2, O(d)) > \delta\) and the locus of nonreduced curves in \(|O(d)|\) has codimension bigger then \(\delta\). In a similar way the conjecture of remark 4.3 can be reformulated as saying that \(N^{n,m}_{e,\delta} = t_6^((\Sigma_e_6)(n,m))\) if and only if \(H^0(\Sigma_e, O(nF + mE)) > \delta\) and the locus of curves in \(|O(nF + mE)|\) which are nonreduced or contain \(E\) as a component has codimension bigger then \(\delta\). One would expect that nonreduced curves contribute to the count of nodal curves, and the recursion formula of [Va] only counts curves not containing \(E\). Therefore it seems that, at least in the case of \(\mathbb{P}_2\) and of rational ruled surfaces, \(t_\delta^\Sigma(L)\) is the actual number of \(\delta\)-nodal curves in a general \(\delta\)-codimensional linear system, unless this cannot be expected for obvious geometrical reasons.

5. Connection with Hilbert schemes of points

Let again \(S\) be an algebraic surface and let \(L\) be a line bundle on \(S\). Let \(S^{[n]}\) be the Hilbert scheme of finite subschemes of length \(n\) on \(S\), and let \(Z_n(S) \subset S \times S^{[n]}\) denote the universal family with projections \(p_n : Z_n(S) \rightarrow S\), \(q_n : Z_n(S) \rightarrow S^{[n]}\). Then \(L_n := (q_n)_*(p_n)^*(L)\) is a locally free sheaf of rank \(n\) on \(S^{[n]}\).
Definition 5.1. Let $S^δ_2 \subset S^{[3][\delta]}$ be the closure (with the reduced induced structure) of the locally closed subset $S^δ_{2,0}$ which parametrizes subschemes of the form $\bigsqcup_{i=1}^δ \text{Spec} (O_{S,x_i}/m_{S,x_i}^2)$, where $x_1, \ldots, x_δ$ are distinct points in $S$. It is easy to see that $S^δ_2$ is birational to $S^{[\delta]}$. We put $d_n(L) := \int_{S^δ_2} c_{2\delta}(L^\delta)$.

Following [B-S] we call $L$ $k$-very ample if for all subschemes $Z \subset S$ of length $k + 1$ the natural map $H^0(S,L) \rightarrow H^0(L \otimes O_Z)$ is surjective. If $L$ and $M$ are very ample, then $L^\otimes k \otimes M^\otimes l$ is $(k + l)$-very ample.

Proposition 5.2. Assume $L$ is $(3\delta - 1)$-very ample, then a general $\delta$-dimensional linear subsystem $V \subset |L|$ contains only finitely many curves $C_1, \ldots, C_s$ with $\geq \delta$ singularities. There exist positive integers $n_1, \ldots, n_s$ such that $\sum_i n_i = d_\delta(L)$. If furthermore $L$ is $(5\delta - 1)$-very ample (5-very ample if $\delta = 1$), then the $C_i$ have precisely $\delta$ nodes as singularities.

Proof. Assume first that $L$ is $(3\delta - 1)$-very ample. We apply the Thom-Porteous formula to the restrictions of the evaluation map $H^0(S,L) \otimes O_S^{[3][\delta]} \rightarrow L_{3\delta}$ to $S^\delta_2$ and to $S^\delta_2 \setminus S^\delta_{2,0}$. As $L$ is $(3\delta - 1)$-very ample the evaluation map is surjective. Then (Fu. ex. 14.3.2) applied to $S^\delta_2$ gives that for a general $\delta$-dimensional sublinear system $V \subset |L|$ the class $d_n(L)$ is represented by the class of the finite scheme $W$ of $Z \subset S^\delta_2$ with $Z \subset D$ for $D \in V$. The scheme structure of $W$ might be nonreduced. The application of (Fu. ex. 14.3.2) to $S^\delta_2 \setminus S^\delta_{2,0}$ and a dimension count give that $W$ lies entirely in $S^\delta_{2,0}$.

Now assume that $L$ is $(5\delta - 1)$-very ample. Let $V \subset |L|$ again be general $\delta$-dimensional subsystem of $|L|$. The Porteous formula applied to the restriction of $L_{3\delta+3}$ to $S^\delta_{2,0}$ and a dimension count shows that there will be no curves in $V$ with more than $\delta$ singularities.

Let $S^\delta_{3,0} \subset S^{[5\delta]}$ be the locus of schemes of the form $Z_1 \sqcup Z_2 \ldots \sqcup Z_\delta$, where each $Z_i$ is of the form $\text{Spec} (O_{S,x_i}/(m^3 + xy))$ with $x, y$ local parameters at $x_i$ and let $S^\delta_3$ be the closure. If a curve $C$ with precisely $\delta$ singularities does not contain a subscheme corresponding to a point in $S^\delta_3 \setminus S^\delta_{3,0}$, then it has $\delta$ nodes as only singularities. It is easy to see that $S^\delta_{3,0}$ is smooth of dimension $4\delta$. Applying the Porteous formula to the restriction of $L_{5n}$ to $S^\delta_3 \setminus S^\delta_{3,0}$ and a dimension count we see that all the curves in $V$ with $\delta$ singularities have precisely $\delta$ nodes. \hfill \Box

Conjecture 5.3. $d_\delta(L) = T_\delta(L^2, LK_S, K^2_S, c_2(S))$.

Conjecture 5.3 gives the hope of proving conjecture 2.4 via the study of the cohomology of Hilbert schemes of points.

Remark 5.4. Note that one can generalize the above to singular points of arbitrary order: Let $\mu = (m_1, \ldots, m_{t(\mu)})$ where $m_i \in \mathbb{Z}_{\geq 2}$. Let $N(\mu) := \sum_{i=1}^s \binom{m_i + 1}{2}$ and let $S_\mu$ be the
closure in $S^{[N(\mu)]}$ of the subset of schemes of the form $\coprod_{i=1}^{l(\mu)} \text{Spec}(O_{S,x_i}/m_{S,x_i}^\mu)$. Denote $d_\mu(L) := \int_{S_\mu} c_2(\mu)(L_{N(\mu)})$. We call a curve $D \in |L|$ of type $\mu$ if there are distinct points $x_1, \ldots, x_{l(\mu)}$ in $S$ such that the ideal $I_{D,x_i}$ is contained in $m_{S,x_i}^\mu$. A straightforward generalization of the proof of proposition 5.2 shows that, for $V$ a general $(N(\mu) - 2l(\mu))$-dimensional linear subsystem of an $N(\mu)$-very ample line bundle $|L|$, $d_\mu(L)$ counts the finite number of curves of type $\mu$ in $V$ with positive multiplicities. Again I expect $d_\mu(L)$ to be a polynomial of degree $l(\mu)$ in $L^2$, $LK_S$, $K^2_S$ and $c_2(S)$.

In a similar way one can also deal with cusps instead of nodes.

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