When $q=0$, The Forced Harmonic Oscillator Isn’t.

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Abstract

We consider the forced harmonic oscillator quantized according to infinite statistics (a special case of the ‘quon’ algebra proposed by Greenberg). We show that in order for the statistics to be consistently evolved the forcing term must be identically zero for all time. Hence only the free harmonic oscillator may be quantized according to infinite statistics.

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I. INTRODUCTION

The idea of an interpolating particle statistics to the familiar Bose and Fermi cases, although exciting, is not new. Over fifty years ago Gentile [1] proposed statistics in which at most $k$ particles could occupy a single state. Lately, much attention has been focussed onto the so-called ‘quon’ algebra proposed by Greenberg [2]

$$a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij} \quad (1)$$

where $q$ is a real parameter and the $a_j^\dagger$ ($a_i$) are creation (annihilation) operators. It can be seen that as $q$ ranges from +1 to -1, the above relation interpolates between the classical commutation and anticommutation relations appropriate for Bosons and Fermions respectively. Much work has been done with this relation: Fivel [3] and Zagier [4] showed that for $|q| < 1$ the resulting Fock space is well defined and that all states have a positive definite norm. Greenberg demonstrated that for free fields the TCP theorem and clustering remain valid.

Recently, we considered quons in curved spacetime [5]. In particular we examined Parker’s proof of the spin-statistics theorem. We found that for $|q| < 1$ a contradiction arose on attempting to consistently evolve the particle statistics through the dynamically evolving universe. (We also noted that by taking the physical limit in which the dynamical part of the evolution became constant, and demanding that the generalized commutation relations imposed remain continuous in this limit, the same contradiction should occur in flat spacetime).

In this work we examine this contradiction more closely for the special case of infinite statistics

$$a_i a_j^\dagger = \delta_{ij} \quad (2)$$

obtained by substituting $q = 0$ in relation (1). (We note that relation (2) was one of the original motivations for the study of quons [6]; it is also intrinsically interesting as it defines
a quantum group - see for example [4, 11]). Greenberg remarked that studying \( q = 0 \) is much easier than studying general \( |q| < 1 \) although the results are often qualitatively the same.

The structure of this paper is as follows. In section two, we review our previous work applying it specifically to the case of infinite statistics; in particular we demonstrate that if a system is initially quantized according to infinite statistics it remains so for all time. We then utilise this result in section three where we consider the forced harmonic oscillator in detail. In section four, we look at how this result applies in \( n \) dimensions, and possible implications for more general systems.

II. REVIEW; NO INFINITE STATISTICS FROM DYNAMICS IN CURVED SPACETIME

In this section we review our recent work and demonstrate that a contradiction arises when we attempt to consistently evolve infinite statistics through a dynamically evolving universe. The central idea is to suppose that we have a time dependent spacetime which for times \( t \leq t_1 \) and \( t \geq t_2 \) is flat, and for \( t_1 < t < t_2 \) may be dynamic. As a specific example we could consider the spatially flat Robertson-Walker spacetime

\[
ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2)
\]

where \( R(t) = R_1 \) for \( t \leq t_1 \) (which we shall call the in-region) and \( R(t) = R_2 \) for \( t \geq t_2 \) (the out-region).

We now consider a real scalar field \( \Phi(x) \) which is quantized according to infinite statistics in the in-region i.e. the field operator can be expanded in terms of the creation and annihilation operators as operators as

\[
\Phi(x) = \sum_i (F_i(x)a_i + F_i^*(x)a_i^\dagger).
\]

where
\[ a_i a_j^\dagger = \delta_{ij} \quad (5) \]

and \( \{ F_i(x) \} \) is a complete set of positive energy solutions to the Klein-Gordon equation. A similar expansion for \( \Phi(x) \) may be made in the out-region

\[ \Phi(x) = \sum_i (G_i(x)b_i + G_i^*(x)b_i^\dagger) \quad (6) \]

where

\[ b_i b_j^\dagger - qb_j^\dagger b_i = \delta_{ij} \quad (7) \]

and again \( \{ G_i(x) \} \) is a complete set of positive energy solutions to the Klein-Gordon equation. We note that we allow for the possibility of a change in particle statistics between the in and out-regions. We also assume that the quantum fields we are considering interact with a gravitational field of arbitrary strength but are otherwise free.

In general, \( a_i \neq b_i \) due to particle creation in the expanding universe. Since both sets \( \{ F_i(x) \} \) and \( \{ G_i(x) \} \) are assumed to be complete, we may expand one in terms of the other; for a spacetime of the form given in (1) this leads to the following relation:

\[ a_i = \alpha_i b_i + \beta_i^* b_i^\dagger \quad (8) \]

where \( \alpha_i \) and \( \beta_i \) are the (diagonal) Bogoliubov coefficients first used by Parker \[12\] to study particle creation in the dynamic universe. Substituting (8) and its hermitian conjugate into (5); and using (7) gives

\[ \delta_{ij} = |\alpha_i|^2 \delta_{ij} + \alpha_i \beta_j b_i b_j + \beta_i^* \alpha_j^* b_i^\dagger b_j^\dagger + \beta_i^* \beta_j b_i^\dagger b_j + q \alpha_i \beta_j^* b_i^\dagger b_j. \quad (9) \]

If we now take the expectation value of (9) with the out-region vacuum state \( |0, \text{out} > \) defined by

\[ b_i |0, \text{out} > \equiv 0. \quad (10) \]

then we discover
\[ 1 = |\alpha_i|^2 \] 

If we now substitute this into relation (9) we obtain

\[ 0 = \alpha_i \beta_j b_i b_j + \beta_i^* \alpha_j^* b_i^\dagger b_j^\dagger + \beta_i^* \beta_j b_i^\dagger b_j^\dagger + q \alpha_i \alpha_j^* b_i^\dagger b_j^\dagger \] 

Because the operators which appear on the RHS of this equation are independent of each other, the coefficients must vanish separately. In particular

\[ \beta_i^* \beta_i = 0 \] 

\[ q \alpha_i \alpha_j^* = 0 \]

These must hold when \( i = j \). Then we obtain

\[ \beta_i = 0 \] 

\[ q = 0 \]

We conclude that if a real scalar field \( \Phi(x) \) is quantized according to infinite statistics in the in-region, then it remains quantized according to infinite statistics in the out-region. However, we find that \( |\alpha_i| = 1 \) and \( \beta_i = 0 \) implying that positive energy solutions in the in-region remain positive energy solutions in the out-region - this corresponds to an absence of particle creation in the dynamic universe. This is in contradiction to Parker’s argument [13] that, for a spacetime of the form given in (3), in general \( \beta_i \neq 0 \). (A number of spacetimes leading to \( \beta_i \neq 0 \) are known; for example see [14,15].)

III. TOY MODEL: THE FORCED HARMONIC OSCILLATOR

In order to investigate further, we construct a toy model of the situation described in section two. We examine a single oscillator quantized according to infinite statistics with a forcing term \( f(t) \) which acts only for \( t_1 < t < t_2 \). We take our (time dependent) Hamiltonian to be
\[ H(t) = N(t)\omega + f(t)a(t) + f^*(t)a^\dagger(t) \]  

(15)

where we have set \( \hbar = 1 \) for convenience. \( N(t) \) is the ‘physical’ number operator satisfying

\[ [N(t), a(t)] = -a(t) \]  

(16)

( Greenberg gives an explicit formula for \( N(t) \) but we will not require that here ). Our quantization scheme is

\[ a(t)a^\dagger(t) = 1 \]  

(17)

which we assume to hold true for all times \( t \) ( this is consistent with the propagation of infinite statistics as shown in section two ). For times \( t \leq t_1 \) and \( t \geq t_2 \), \( H(t) \) becomes the free Hamiltonian.

We now construct the Heisenberg equation of motion for the annihilation operator. This yields

\[ i\frac{da(t)}{dt} = [a(t), H(t)] = \omega a(t) + f^*(t)(1 - a^\dagger(t)a(t)) \]  

(18)

This relation differs considerably from the usual forced harmonic oscillator relation ( see for example Merzbacher [16] ) due to the presence of the \( f^*(t)a^\dagger(t)a(t) \) term. However, this is easily dealt with; we act with \( a(t) \) from the left on both sides of relation (18); then using (17) we obtain

\[ a(t) \left( i\frac{da(t)}{dt} - \omega a(t) \right) = 0 \]  

(19)

Since \( a(t) \) is non-zero, we find that the solution to the above relation is given by

\[ a(t) = a \exp(-i\omega t) \]  

(20)

If we now substitute this relation and its hermitian conjugate back into our original equation of motion for \( a(t) \), we discover
\[ 0 = f^*(t)(1 - a^\dagger a) \quad (21) \]

We now take the expectation value of this relation with the vacuum state \(|0\rangle\) defined by

\[ a|0\rangle \equiv 0 \quad (22) \]

and find that

\[ f(t) = 0 \quad (23) \]

Thus we conclude the harmonic oscillator quantized according to infinite statistics makes sense only as a free system.

**IV. DISCUSSION**

The result we obtained in section three has been arrived at as follows; first we showed that if a real scalar field \(\Phi\) was originally quantized according to infinite statistics then it would remains so for all time. The only assumption we made was that the field interacted with the gravitational field. We then used this result to show that \(\Phi\) had to be the free field i.e. it could not interact with the gravitational field - this is the source then of our contradiction. Hence we must originally assume that \(\Phi\) does not interact with the gravitational field. A similar argument could be presented for a scalar field interacting with an electromagnetic field which can also lead to particle creation.

We also note that since the number of spacetime dimensions does not enter into the calculation in section two, we would expect the same results to be obtained in any number of spacetime dimensions. ( This is a special case of Chen’s recent result. ) The result obtained at the end of section three should also hold in any number of spacetime dimensions.

It is also possible to wonder if the results of section three apply to general \(|q| < 1\). Since the same problem described in section two arises for \(|q| < 1\), it may have the same root cause. If it is impossible to define the forced harmonic oscillator for general \(|q| < 1\), then the future of quantum field theories based on quons is extremely bleak.
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