Attractivity and stability in the competitive systems of PDEs of Kolmogorov type

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Abstract

In this paper we consider $N$-species nonautonomous competitive systems of partial differential equations of Kolmogorov type. Under the Neumann boundary conditions we give a sufficient condition for the system to be uniformly stable and globally attractive.

Keywords: Reaction – diffusion system, upper average, lower average, persistence, permanence, Lyapunov functional, global attractivity, stability.

1. Introduction

In this paper we consider Kolmogorov systems of reaction-diffusion parabolic partial differential equations (PDEs)

$$\frac{\partial u_i}{\partial t} = \Delta u_i + f_i(t, x, u_1, \ldots, u_N)u_i \quad t > 0, \quad x \in \Omega, \quad 1 \leq i \leq N.$$ 

From the biological viewpoint this problem represents a model of population growth where $u_i(t, x)$ is the density of the $i$-th species at time $t$ and spatial location $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded habitat, $f_i(t, x, u_1, \ldots, u_N)$ is the local per capita growth rate of the $i$-th species. The system is usually considered with appropriate boundary conditions, either Dirichlet or Neumann, or Robin boundary conditions. In this paper we investigate the system under the Neumann boundary conditions. We deal with competitive systems. It means that the derivatives $\frac{\partial f_i}{\partial u_j}$, $1 \leq i, j \leq N$ are nonpositive.

One of the major problems in models of population growth are permanence and stability. The authors of many papers focus on finding necessary or sufficient conditions for permanence or stability in models which are described by ordinary or partial differential equations. Permanence means that any positive solution of the model has all its components, for large $t$, bounded away from zero and the lower bound is independent of the solution. Global attractivity states that the
difference of any two positive solutions tends to zero as \( t \) goes to infinity. In \([1]\) S. Ahmad and A.C. Lazer, using the upper and lower averages of a function, found sufficient conditions for permanence and global attractivity of the competitive \( N \) species Lotka-Volterra system of ODEs. In \([6]\) we extended their result for the \( N \) species competitive Kolmogorov system of ODEs. In terms of the upper and lower averages of a function we found sufficient conditions for such a system to be permanent and globally attractive.

In \([5]\) the authors considered two species Lotka–Volterra system of PDEs. They investigated behavior of this system not only as \( t \) goes to infinity, but in the "pullback" sense. It means that we are starting with a fixed initial condition further and further back in time. They introduced a notion "pullback permanence" which has an interesting biological interpretation. When two species have already been competing for a long time then neither species will have died out. Their densities are bounded below in a uniform way, no matter how long this population has been running.

In \([2]\) we investigate the \( N \) species competitive Kolmogorov system of PDEs. Using the methods of supersolutions and subsolutions for parabolic PDEs we found sufficient conditions for permanence in such systems.

To establish the global attractivity many authors use appropriate Lyapunov function in the case of ODE or Lyapunov functional in the case of PDE (see for example \([1]\), \([6]\), \([7]\), \([8]\), \([9]\)). In \([3, 4]\) K. Gopalsamy dealt with, respectively, periodic and almost periodic solution of a Lotka-Volterra system of ODEs. A. Tineo \([9]\) considered the nonautonomous Kolmogorov system of ODEs

\[
\begin{aligned}
u_i' &= f_i(t, u)u_i, \quad 1 \leq i \leq N, \\
\end{aligned}
\]

(K)

where \( f : \mathbb{R} \times \mathbb{R}_+^N \to \mathbb{R}^N \) is a continuous function and \( \mathbb{R}_+^N = \{ u \in \mathbb{R}^N : u_i \geq 0, 1 \leq i \leq N \} \). He found a sufficient condition for the system (K) to be globally attractive. In \([6]\) we considered system (K) with slightly weaker assumptions for the function \( f \) than those in \([9]\). In \([8]\) Sze-Bi Hsu examined Lyapunov functions (or functionals) for various ecological models which take form of ODE systems (or reaction-diffusion PDE systems). First the author constructed Lyapunov functions for the predator prey system and then constructed Lyapunov functionals for the corresponding reaction-diffusion PDE system. Sze-Bi Hsu considered the models in which the reaction term is independent on \( t \) and \( x \).

In \([10]\) A. Tineo and J. Riviero dealt with the two species nonautonomous competitive Lotka-Volterra system with diffusion. They resigned from the Lyapunov function (functional) to establish of attractivity of this system. They used a powerful tool which is the iteration monotone methods.

Usually the same conditions which guarantee permanence in the ODE model of population growth ensure that the model is globally attractive (see for example \([1]\), \([6]\)). But when we try to prove that a system of Kolmogorov type of PDEs is globally attractive then some difficulties occur. Firstly, we can prove that the system is globally attractive only under the Neumann boundary conditions. Secondly, besides the conditions which guarantee the permanence we
need additional inequalities which contain the parameters which occur in the
definition of permanence. Even in the case of two species problem it is a diffi-
cult to obtain sufficient conditions which guarantee both permanence and global
attractivity of Kolmogorov system of PDEs. One can try to introduce a partial
ordering. The idea is that we can compare solutions of our system of PDE with
the functions which are independent on $x$ in a way that when we apply partial
ordering our solutions of PDE are between the functions independent on $x$. A
difficulty occurs when we let the parameter which is a spatial location to be
arbitrary.

In this paper we find sufficient conditions for $N$ – species system of PDEs
of Kolmogorov type under the Neumann boundary conditions to be globally
attractive. Conditions which we obtained are stronger than those guaranteeing
permanence. Assumptions for a function $f$ in this paper is the same as in [2].

This paper is organized as follows.

In Section 2 we present basic assumptions and definitions.

In Section 3 we formulate the main theorem of this paper. By Lyapunov
functional some sufficient conditions are obtained to guarantee attractivity and
uniform stability Kolmogorov system of PDEs.

In Section 4 we give some remarks.

2. Preliminaries

We consider a nonautonomous system of PDEs of Kolmogorov type

$$
\begin{align*}
\frac{\partial u_i}{\partial t} &= \Delta u_i + f_i(t, x, u_1, \ldots, u_N)u_i, & t > 0, & x \in \Omega, & i = 1, \ldots, N \\
B u_i &= 0, & t > 0, & x \in \partial \Omega, & i = 1, \ldots, N,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial \Omega$, $\Delta$
is the Laplace operator on $\Omega$,

$$
\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},
$$

and $B$ is the boundary operator of the Neumann type

$$
Bu_i = \frac{\partial u_i}{\partial \nu} \text{ on } \partial \Omega,
$$

where $\nu$ denotes the unit normal vector on $\partial \Omega$ pointing out of $\Omega$.

Now we make the following assumptions

(A1) (see [2]) $f_i: [0, \infty) \times \Omega \times [0, \infty)^N \to \mathbb{R}$ $(1 \leq i \leq N)$, as well as their
first derivatives $\partial f_i/\partial t$ $(1 \leq i \leq N)$, $\partial f_i/\partial u_j$ $(1 \leq i, j \leq N)$, and $\partial f_i/\partial x_k$
$(1 \leq i \leq N, 1 \leq k \leq n)$, are continuous.

(A1) guarantees that for any regular initial function $u_0(x) = (u_{01}, \ldots, u_{0N})$, $x \in \Omega$ where $u_{0i} \geq 0$ $(1 \leq i \leq N, x \in \Omega)$ there exists a unique maximally
defined solution \( u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) of system \((R)\), \( (t, x) \in [0, \tau_{\text{max}}) \times \bar{\Omega} \), where \( \tau_{\text{max}} > 0 \), satisfying the initial condition \( u(0, x) = u_0(x) \). Moreover, the solution of system \((R)\) is classical.

\[(A2)\text{(see [2]) Functions }[0, \infty) \times \bar{\Omega} \ni (t, x) \mapsto f_i(t, x, 0, \ldots, 0) \in \mathbb{R} \text{, } 1 \leq i \leq N \text{, are bounded.}\]

**Definition 1.** A solution \( u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) of system \((R)\) is positive if \( u_i(t, x) > 0 \) for all \( i = 1, \ldots, N \), \( x \in \Omega \), \( t > 0 \).

**Definition 2.** System \((R)\) is permanent if there exist positive constants \( \delta \), \( \bar{\delta} \) such that for any positive solution \( u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) there exists \( T = T(u) \geq 0 \) with the property

\[ \delta \leq u_i(t, x) \leq \bar{\delta} \]

for all \( t \geq T \), \( x \in \bar{\Omega} \), \( i = 1, \ldots, N \).

**Definition 3.** System \((R)\) is globally attractive if any two positive solutions \( u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) and \( v(t, x) = (v_1(t, x), \ldots, v_N(t, x)) \) of system \((R)\) satisfy

\[ \lim_{t \to \infty} (u_i(t, x) - v_i(t, x)) = 0 \]

for \( 1 \leq i \leq N \), uniformly in \( x \in \bar{\Omega} \).

**Definition 4.** System \((R)\) is uniformly stable on \([t_1, \infty)\), \( t_1 > 0 \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for any two positive solutions \( u(t, x) = (u_1(t, x), \ldots, u_N(t, x)) \) and \( v(t, x) = (v_1(t, x), \ldots, v_N(t, x)) \) if \( t_2 \in [t_1, \infty) \) and \( \|u(t_2, x) - v(t_2, x)\| < \delta \) then \( \|u(t, x) - v(t, x)\| < \varepsilon \) for \( t \geq t_2 \).

The next assumption is

\[(A3)\text{(see [2]) There exist } b_{ii} > 0 \text{ such that } \frac{\partial f_i}{\partial u_i}(t, x, u) \leq -b_{ii} \forall t \geq 0, x \in \bar{\Omega}, u \in [0, \infty)^N, 1 \leq i \leq N.\]

Define

\[ B(\varepsilon) := \left[ 0, \frac{\tilde{a}_1}{b_{11}} + \varepsilon \right] \times \cdots \times \left[ 0, \frac{\tilde{a}_N}{b_{NN}} + \varepsilon \right], \quad \varepsilon \geq 0. \]

\[(A4)\text{(see [2]) } \partial f_i / \partial u_j, 1 \leq i, j \leq N \text{, are bounded and uniformly continuous on each set } [0, \infty) \times \bar{\Omega} \times B.\]

For \( 1 \leq i, j \leq N \) define

\[ \tilde{b}_{ij}(\varepsilon) := \sup \left\{ -\frac{\partial f_i}{\partial u_j}(t, x, u) : t \geq 0, x \in \bar{\Omega}, u \in B(\varepsilon) \right\}. \]

and \( \tilde{b}_{ij}(0) := \bar{b}_{ij} \).
Assumptions (A3) and (A4) imply that $b_{ij}(\epsilon) \geq 0$, $1 \leq i, j \leq N$, and $\tilde{b}_{ii}(\epsilon) > 0$, $1 \leq i \leq N$, whereas it follows from (A5) that $\tilde{b}_{ij}(\epsilon) < \infty$, and $\lim_{\epsilon \to 0^+} \tilde{b}_{ij}(\epsilon) = \tilde{b}_{ij}$, for $1 \leq i, j \leq N$.

Recall that in [2] we found average conditions for system (R) to be permanent. The average conditions for system (R) under the Neumann boundary conditions are

$$m[f_i] > \frac{\sum_{j=1, j \neq i}^{N} \tilde{b}_{ij} M[f_j]}{\tilde{b}_{ii}}, \quad \text{(AC)}$$

where

$$m[f_i] := \liminf_{t-s \to \infty} \frac{1}{t-s} \int_{s}^{t} \min_{x \in \Omega} f_i(\tau, x, 0, \ldots, 0) d\tau,$$

and

$$M[f_i] := \limsup_{t-s \to \infty} \frac{1}{t-s} \int_{s}^{t} \max_{x \in \Omega} f_i(\tau, x, 0, \ldots, 0) d\tau.$$

3. Main Theorem

In this section we state the main theorem of this paper and with the help of a Lyapunov functional we prove attractivity and stability of system (R).

**Theorem 3.1.** Assume (A1) – (A5) and (AC). Let $\delta \geq 0$ be such that for each $1 \leq i \leq N$

$$\delta \tilde{b}_{ii} > \sum_{j=1, j \neq i}^{N} \tilde{b}_{jj}. \quad \text{(2.1)}$$

Suppose that $u(t, x) = (u_1(t, x), \ldots, u_N(t, x))$ and $v(t, x) = (v_1(t, x), \ldots, v_N(t, x))$ be two positive solutions of system (R) such that $\tilde{\delta} \leq u_i(t, x), v_i(t, x) \leq \tilde{\delta}$ for sufficiently large $t$ and all $x \in \Omega$. Then there exist positive constants $Z, \gamma$ and $\tilde{T} \geq 0$ such that for $i = 1, \ldots, N$ and $t \geq \tilde{T}$ one has

$$\sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(t, x) - v_i(t, x)| \leq Z \sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(\tilde{T}, x) - v_i(\tilde{T}, x)| \cdot \exp\left(-\gamma(t-\tilde{T})\right),$$

In particular system (R) is globally attractive and uniformly stable.

**Proof.** Fix two positive solutions $u(t, x) = (u_1(t, x), \ldots, u_N(t, x))$ and $v(t, x) = (v_1(t, x), \ldots, v_N(t, x))$ of system (R). By assumption (A5) we can take $\epsilon > 0$ such that

$$\tilde{\delta}, \tilde{\delta}_i > \sum_{j=1, j \neq i}^{N} \tilde{b}_{jj}(\epsilon) \quad \text{for all} \quad i = 1, \ldots, N.$$
Let $t_0 > 0$ be such that $u(t, x), v(t, x) \in B(\varepsilon)$ and $\delta \leq u_i(t, x), v_i(t, x) \leq \overline{\delta}$ for all $t \geq t_0, x \in \Omega, 1 \leq i \leq N$. By a result in matrix theory (see \[7, Section 5\]) we know that there exist $\alpha_1, \ldots, \alpha_N > 0$ such that

$$\alpha_i \delta \overline{b}_{ij} \leq \sum_{j=1}^{N} \alpha_j b_{ij}(\varepsilon), \quad 1 \leq i \leq N. \quad (3.1)$$

Denote

$$\Theta(t) = \sum_{i=1}^{N} \alpha_i \Theta_i(t), \text{ where } \Theta_i(t) := \sup_{x \in \Omega} |\ln \frac{u_i(t, x)}{v_i(t, x)}|. \quad (3.2)$$

Fix $1 \leq i \leq N$. We prove that there exists $\bar{T} \geq t_0$ such that

$$D^+ \Theta_i(t) \leq -\delta b_i \Theta_i(t) + \delta \sum_{j=1}^{N} \overline{b}_{ij}(\varepsilon) \Theta_j(t), \quad t \geq \bar{T}. \quad (3.3)$$

where $D^+ \Theta_i$ denotes the upper derivative of $\Theta_i$.

We take for $\bar{T}$ any number $\geq t_0$. For $t \geq \bar{T}$ denote $\overline{\Omega}^{(i)}(t)$ to be the set of the points $x^{(i)} \in \Omega$ such that the supremum of the function $\ln \frac{u_i(t, x)}{v_i(t, x)}$ is realized.

Then $\overline{\Omega}^{(i)}(t) = \overline{\Omega}^{(i)}_+(t) \cup \overline{\Omega}^{(i)}_-(t)$, $\overline{\Omega}^{(i)}_+(t) \cap \overline{\Omega}^{(i)}_-(t) = \emptyset$, where

$$\overline{\Omega}^{(i)}_+(t) = \{x^{(i)} \in \overline{\Omega}^{(i)}(t) : u_i(t, x^{(i)}) > v_i(t, x^{(i)})\}$$

and

$$\overline{\Omega}^{(i)}_-(t) = \{x^{(i)} \in \overline{\Omega}^{(i)}(t) : u_i(t, x^{(i)}) < v_i(t, x^{(i)})\}$$

Therefore by (3.2)

$$D^+ \Theta_i(t) \leq \max \left\{ \sup_{x \in \overline{\Omega}^{(i)}_+(t)} \left\{ \frac{\partial}{\partial t} \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} : x^{(i)} \in \overline{\Omega}^{(i)}_+(t) \right\}, \right.$$

$$\sup_{x \in \overline{\Omega}^{(i)}_-(t)} \left\{ \frac{\partial}{\partial t} \ln \frac{v_i(t, x^{(i)})}{u_i(t, x^{(i)})} : x^{(i)} \in \overline{\Omega}^{(i)}_-(t) \right\} \right\}. \quad (3.4)$$

Denote $\kappa_i := \Theta_i(t)$. Now we consider four (not mutually exclusive) cases.

**Case one.** $x^{(i)} \in \Omega \cap \overline{\Omega}^{(i)}_+(t)$.

By the definition of $\kappa$ it follows that

$$u_i(t, x) \leq e^{\kappa_i} v_i(t, x) \quad \text{for all } x \in \Omega \quad \text{and} \quad u_i(t, x^{(i)}) = e^{\kappa_i} v_i(t, x^{(i)}), \quad (3.5)$$

and

$$\frac{d}{dt} \left( \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} \right) = \frac{\Delta u_i(t, x^{(i)})}{u_i(t, x^{(i)})} - \frac{\Delta v_i(t, x^{(i)})}{v_i(t, x^{(i)})} + f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)})).$$
By (3.5),
\[
\frac{\Delta u_i(t, x^{(i)})}{u_i(t, x^{(i)})} - \frac{\Delta v_i(t, x^{(i)})}{v_i(t, x^{(i)})} = \frac{\Delta(u_i(t, x^{(i)}) - e^{\kappa}v_i(t, x^{(i)}))}{u_i(t, x^{(i)})} \leq 0.
\]

Hence
\[
\frac{d}{dt} \left( \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} \right) \leq f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)})).
\]

**Case two.** \(x^{(i)} \in \Omega \cap \bar{\Omega}_-^{(i)}(t)\).

In a similar manner we prove that
\[
\frac{d}{dt} \left( \ln \frac{v_i(t, x^{(i)})}{u_i(t, x^{(i)})} \right) \leq f_i(t, x^{(i)}, v(t, x^{(i)})) - f_i(t, x^{(i)}, u(t, x^{(i)})).
\]

**Case three.** \(x^{(i)} \in \partial \Omega \cap \bar{\Omega}_+^{(i)}(t)\)

This implies that \(u_i(t, x) \leq e^{\kappa}v_i(t, x^{(i)})\) for all \(x \in \Omega\) and \(u_i(t, x^{(i)}) = e^{\kappa}v_i(t, x^{(i)})\). Denote \(g(x) := u_i(t, x) - e^{\kappa}v_i(t, x)\). The restriction \(g|_{\partial \Omega}\) of \(g\) to \(\partial \Omega\) has largest value at the point \(x^{(i)}\), consequently \(\nabla g|_{\partial \Omega}(x^{(i)}) = 0\). By the definition of \(g\) it follows that \(\frac{\partial g}{\partial v} = \frac{\partial u_i}{\partial v} - e^{\kappa} \frac{\partial v_i}{\partial v} = 0\). Hence \(\nabla g(x^{(i)}) = 0\).

As \(g\) attains its maximum at \(x^{(i)}\), this implies that \(\Delta g(x^{(i)}) \leq 0\). Therefore
\[
\frac{d}{dt} \left( \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} \right) = \frac{\Delta u_i(t, x^{(i)})}{u_i(t, x^{(i)})} - \frac{\Delta v_i(t, x^{(i)})}{v_i(t, x^{(i)})} + f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)}))
\]
\[
\leq f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)})).
\]

**Case four.** \(x^{(i)} \in \Omega \cup \bar{\Omega}_-^{(i)}(t)\)

In a similar manner we prove that
\[
\frac{d}{dt} \left( \ln \frac{v_i(t, x^{(i)})}{u_i(t, x^{(i)})} \right) \leq f_i(t, x^{(i)}, v(t, x^{(i)})) - f_i(t, x^{(i)}, u(t, x^{(i)})).
\]

The cases 1 - 4 with (3.4) give
\[
D^+ \Theta_i(t) \leq \sum_{i=1}^{N} \max\{ \sup\{f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)})) : x^{(i)} \in \bar{\Omega}_+^{(i)} \}, \sup\{f_i(t, x^{(i)}, u(t, x^{(i)})) - f_i(t, x^{(i)}, v(t, x^{(i)})) : x^{(i)} \in \bar{\Omega}_-^{(i)} \} \}.
\]

By assumptions (A3), (A4) and (A5), it follows that
\[
f_i(t, x^{(i)}, u_1(t, x^{(i)}), \ldots, u_N(t, x^{(i)})) - f_i(t, v_1(t, x^{(i)}), \ldots, v_N(t, x^{(i)}))
\]
\[
\leq -p_i(u_i(t, x^{(i)}) - v_i(t, x^{(i)})) + \sum_{j=1}^{N} b_{ij}(\varepsilon) |u_j(t, x^{(i)}) - v_j(t, x^{(i)})|
\]

for each \( x^{(i)} \in \bar{\Omega}_+^{(i)} \).

By assumptions (A3), (A4) and (A5), it follows that

\[
f_i(t, x^{(i)}), u_1(t, x^{(i)}), \ldots, u_N(t, x^{(i)}) - f_i(t, v_1(t, x^{(i)})), \ldots, v_N(t, x^{(i)}) \\
\leq -b_i(v_i(t, x^{(i)}) - u_i(t, x^{(i)})) + \sum_{j=1, j \neq i}^N b_{ij}(\varepsilon)|v_j(t, x^{(i)}) - u_j(t, x^{(i)})|
\]

for each \( x^{(i)} \in \bar{\Omega}_+^{(i)} \).

Hence

\[
D^+\Theta_i(t) \leq \max_{x^{(i)} \in \bar{\Omega}_+^{(i)}} \{ -\hat{b}_i(u_i(t, x^{(i)}) - v_i(t, x^{(i)})) + \sum_{j=1, j \neq i}^N \hat{b}_{ij}(\varepsilon)|u_j(t, x^{(i)}) - v_j(t, x^{(i)})| \}
\]

Since \( \delta_i \leq u_i(t, x), v_i(t, x) \leq R_i \) for all \( i = 1, \ldots, N \) and for sufficiently large \( t \), using the mean value theorem we have that

\[
\frac{1}{\delta} |u_i(t, x) - v_i(t, x)| \leq |\ln \frac{u_i(t, x)}{v_i(t, x)}| \leq \frac{1}{\delta} |u_j(t, x) - v_j(t, x)| \quad (3.6)
\]

for all \( t \geq \hat{T}, x \in \Omega, 1 \leq j \leq N \).

Therefore

\[
D^+\Theta_i(t) \leq \max \left\{ \sup_{x \in \Omega} \left\{ -\hat{b}_i \left| \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} \right| + \delta \sum_{j=1, j \neq i}^N \hat{b}_{ij}(\varepsilon) \left| \ln \frac{u_j(t, x^{(i)})}{v_j(t, x^{(i)})} \right| : x^{(i)} \in \bar{\Omega}_+^{(i)} \right\} , \right.
\]

\[
\left. \sup_{x \in \Omega} \left\{ -\hat{b}_i \left| \ln \frac{u_i(t, x^{(i)})}{v_i(t, x^{(i)})} \right| + \delta \sum_{j=1, j \neq i}^N \hat{b}_{ij}(\varepsilon) \left| \ln \frac{u_j(t, x^{(i)})}{v_j(t, x^{(i)})} \right| : x^{(i)} \in \bar{\Omega}_-^{(i)} \right\} \right\}
\]

\[
\leq \left( -\hat{b}_i \sup_{x \in \Omega} \left| \ln \frac{u_i(t, x)}{v_i(t, x)} \right| + \delta \sum_{j=1, j \neq i}^N \hat{b}_{ij}(\varepsilon) \sup_{x \in \Omega} \left| \ln \frac{u_j(t, x)}{v_j(t, x)} \right| \right), \quad t \geq \hat{T}.
\]

Hence we proved (3.3).
By (3.1), it follows that
\[
\sum_{i=1}^{N} \alpha_i \left( -\bar{b}_i \theta_i(t) + \bar{\bar{b}} \sum_{j=1 \atop j \neq i}^{N} \alpha_j b_{ji}(\varepsilon) \theta_i(t) \right)
\]
\[
= -\sum_{i=1}^{N} \alpha_i b_i \theta_i(t) + \bar{\bar{b}} \sum_{j=1 \atop j \neq i}^{N} \alpha_j b_{ji}(\varepsilon) \theta_i(t)
\]
\[
\leq -\epsilon \sum_{i=1}^{N} \theta_i(t), \quad t \geq \tilde{T},
\]
where
\[
\epsilon = \min_{1 \leq i \leq N} \left\{ \alpha_i b_i - \sum_{j=1 \atop j \neq i}^{N} \alpha_j b_{ji}(\varepsilon) \right\}.
\]
Hence by (3.3) and (3.7),
\[
D^+ \theta(t) \leq \sum_{i=1}^{N} \alpha_i D^+ \theta_i(t) \leq -\epsilon \sum_{i=1}^{N} \theta_i(t) \leq -\frac{\epsilon}{\alpha^*} \theta(t), \quad t \geq \tilde{T}
\]
(3.8)
where \(\alpha^* = \max \{\alpha_i : 1 \leq i \leq N\}\). Therefore
\[
\theta(t) \leq \theta(\tilde{T}) \exp \left( -\frac{\epsilon}{\alpha^*} (t - \tilde{T}) \right)
\]
(3.9)
for all \(t \geq \tilde{T}\).

By (3.6) it follows that
\[
\theta(t) \geq \alpha \sum_{i=1}^{N} \theta_i(t) \geq \alpha \bar{\bar{b}} \sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(t,x) - v_i(t,x)|, \quad t \geq \tilde{T},
\]
(3.10)
and
\[
\theta(\tilde{T}) \leq \alpha^* \sum_{i=1}^{N} \theta_i(t) \leq \alpha^* \bar{\bar{b}} \sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(\tilde{T},x) - v_i(\tilde{T},x)|, \quad t \geq \tilde{T}
\]
(3.11)
where \(\alpha^* = \min \{\alpha_i : 1 \leq i \leq N\}\).

From (3.9), (3.10) and (3.11) we have that
\[
\sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(t,x) - v_i(t,x)| \leq Z \sum_{i=1}^{N} \sup_{x \in \Omega} |u_i(\tilde{T},x) - v_i(\tilde{T},x)| \cdot \exp (-\gamma (t - \tilde{T})), \quad t \geq \tilde{T},
\]
(3.12)
where $Z = \frac{a}{2a}$ and $\gamma = \frac{\delta}{\alpha}$ ($Z$ and $\gamma$ we can take independently on the solutions $u(t, x)$ and $v(t, x)$).

By (3.12) it follows that the system $\text{(R)}$ is globally attractive and uniformly stable.

4. Remarks

Remark 1. Note that conditions (3.1) contain $\delta$ and $\bar{\delta}$ which appear in definition of permanence of system $\text{(R)}$. In [2] we showed that if we change conditions (AC) to slightly stronger that is

$$m[f_i] > \sum_{j=1, j\neq i}^N \frac{b_{ij}M_{ij}}{b_{ij}} \quad \text{(AC')},$$

then we can estimate $\delta$ and $\bar{\delta}$ in terms of parameters of system $\text{(R)}$. By dissipativity of system $\text{(R)}$ it follows that we can take $\max_{1 \leq i \leq N} \left\{ \frac{M_{ii}}{b_{ii}} \right\}$ as $\delta$. Concerning the lower bound (see [2, Theorem 4.1]) as $\delta$ we can take

$$\min_{1 \leq i \leq N} \left\{ \frac{1}{b_{ii}} \left( \alpha_i - \sum_{j=1, j\neq i}^N \frac{b_{ij}M_{ij}}{b_{ij}} \right) \right\}$$

when

$$\alpha_i > \sum_{j=1, j\neq i}^N \frac{b_{ij}M_{ij}}{b_{ij}}$$

for all $1 \leq i \leq N$.

If we define $\delta$ and $\bar{\delta}$ as in the above condition (2.1), take a form

$$\alpha_i m[f_i] \geq \sum_{j=1}^N \left( \alpha_i \frac{b_{ij}M_{ij}}{b_{ij}} + \alpha_j \frac{b_{ij}M_{ij}}{b_{ij}} \right).$$

Remark 2.

Definition 5. System $\text{(R)}$ is uniformly asymptotically stable on $[t_1, \infty)$ if it is uniformly stable and if there is a $r > 0$ so that for every $\varepsilon > 0$ there is a $T(\varepsilon) > 0$ so that for any two positive solutions $u(t, x) = (u_1(t, x), \ldots, u_N(t, x))$ and $v(t, x) = (v_1(t, x), \ldots, v_N(t, x))$ if $t_2 \geq t_1$ and $\|u(t_2, x) - v(t_2, x)\| < r$ then $\|u(t, x) - v(t, x)\| < \varepsilon$ for $t \geq t_2 + T(\varepsilon)$.

Note that from attractivity and uniform stability of system $\text{(R)}$ it follows that system $\text{(R)}$ is uniformly asymptotically stable.
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