Expanded solutions of force-free electrodynamics on general Kerr black holes

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Abstract

In this work, expanded solutions of force-free magnetospheres on general Kerr black holes are derived through a radial distance expansion method. From the regular conditions both at the horizon and at spatial infinity, two previously known asymptotical solutions (one of them is actually an exact solution) are identified as the only solutions that satisfy the same conditions at the two boundaries. Taking them as initial conditions at the boundaries, expanded solutions up to the first few orders are derived by solving the stream equation order by order. It is shown that our extension of the exact solution can (partially) cure the problems of the solution: it leads to magnetic domination and a mostly timelike current for restricted parameters.

1 Introduction

Black hole magnetospheres are believed to play essential roles in many high-energy astrophysical objects. In the popular Blandford-Znajek model [1, 2], energy can be extracted from a rotating black hole via a stationary and force-free magnetosphere on it to eject dipole relativistic jets, which may account for most of the high-energy phenomena in active galactic nuclei, gamma-ray bursts and microquasars.

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In the simplest configuration, the force-free magnetosphere is well described by the clean and precise electrodynamics on a Kerr black hole. However, the present understanding of such a system strongly relies on numerical simulations. We have very few options in analytical approaches to date.

One of the approaches is the perturbation method first given in the original work of Blandford and Znajek [1]. Based on the split monopole and paraboloidal solutions on a non-rotating black hole, analytical solutions on a slow-rotating black hole are derived by expanding the functions and stream equation to leading orders of the spin parameter. So the solutions apply to slowly rotating black holes. To get analytical properties of magnetospheres on rapidly rotating black holes, which may be more interesting to us, we need to calculate higher order corrections. But it seems uneasy to do so [3]. Recently, the solution up to the fourth order has been derived [4].

Solutions that go beyond the slow-rotation limit in the perturbation approach can be obtained in some limited regions. In the work [5, 6] of Menon and Dermer (MD), asymptotic solutions (and their generalisation [7]) were derived in regions far away from the horizon. The solutions can apply to black holes with general angular momentum. But these solutions are radial distance independent. In particular, one of the solutions is even the only known exact solution so far that can solve the full stream equation, which leaves it very interesting. However, the current for this solution is along the infalling principle null geodesic. This means that charged particles must move at the speed of light, which is not allowed. Besides, the electromagnetic fields from the solution are also null. A relieving method is given by the authors in [8, 9]. The lightlike current is artificially decomposed into a linear combination of two timelike currents with opposite charges.

On the other hand, in past years, exact solutions on extremely fast rotating black holes were obtained by focusing on the near-horizon region [10, 11, 12, 13, 14]. But, a smooth connection between these near- and far-region solutions is lacking.

In this work, we consider a different expansion method other than the one in the traditional perturbation approach. In terms of the boundary conditions of a magnetosphere, we expand the functions and stream equation in series of the radial distance, instead of the spin parameter. Analytical solutions that depend on both poloidal coordinates can be derived order by order following a precise procedure. So this approach hopefully can help us extend solutions in far region to the ones in the near region. Moreover, this provides a method to generalise the MD exact solution and relax its problems of null current and electromagnetic fields.

The paper is organised as follows. In Section 2, the stream equation of a force-free magnetosphere is instructed and presented. In Section 3, we show the boundary condi-
tions at the horizons and at infinity, which can be determined from the stream equation. Two special cases of the boundary conditions lead to the previously known asymptotic solutions. In Section 4, the expansion forms and solving procedure of the stream equation are introduced in terms of the boundary conditions. Examples of solutions are derived and analysed in Section 5. Then we summarise in the last section.

2 The stream equation

On the Boyer-Lindquist coordinates, a Kerr black hole is depicted by the metric:

\[ ds^2 = -\Lambda^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \varpi^2 (d\phi - \omega dt)^2, \]  

(1)

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad a = \frac{J}{M}, \]

\[ \Lambda^2 = \frac{\rho^2 \Delta}{A}, \quad \varpi^2 = \frac{A \sin^2 \theta}{\rho^2}, \quad \omega = \frac{2Mar}{A}, \]

\[ \Delta = (r - r_+)(r - r_-), \quad r_{\pm} = M \pm \sqrt{M^2 - a^2}, \]

\[ A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta = 2Mr(r^2 + a^2) + \Delta \rho^2. \]

The spin parameter \( a \) measures the angular momentum \( J \) per unit mass \( M \) of the black hole. The inner and outer horizons are located at \( r = r_- \) and \( r = r_+ \) respectively. From the above relations, the velocities of the black hole at the horizons are:

\[ \omega_{\pm} \equiv \frac{a}{r_{\pm}^2 + a^2}. \]  

(2)

In the \( 3 + 1 \) split formulation [15], the four-dimensional spacetime (1) is replaced by a three-dimensional absolute space and a universal time coordinate. The electrodynamics on a Kerr black hole can be equivalently dealt with on the following absolute space:

\[ ds^2_A = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \varpi^2 d\phi^2. \]  

(3)

The four-dimensional quantities and equations are split accordingly. The quantities we deal with on the absolute space are measured by the so-called zero-angular momentum observers (ZAMOs). From the inverse metric, the unit basis vectors are given by

\[ e_r = \sqrt{\frac{\Delta}{\rho^2}} \partial_r, \quad e_\theta = \frac{1}{\sqrt{\rho^2}} \partial_\theta, \quad e_\phi = \frac{\sqrt{\rho^2}}{\sqrt{A \sin \theta}} \partial_\phi. \]  

(4)
The Kerr spacetime has Killing vectors along the time and along the toroidal directions. For simplicity, we consider the stationary and axisymmetric case of electrodynamics on the spacetime. The relevant inhomogeneous Maxwell’s equations relate the electromagnetic fields to the electric charge and current densities ($\rho_e, j$):

$$\nabla \cdot E = 4\pi \rho_e, \quad (5)$$

$$\nabla \times (\Lambda B) = 4\pi \Lambda j - \omega (E \cdot \nabla \omega) e^\phi. \quad (6)$$

Throughout the paper, the operator $\nabla$ denotes the covariant derivative associated with the 3-dimensional spatial dimensions (3). The homogeneous Maxwell’s equations tell us that the electromagnetic fields can be expressed as the gauge potentials ($A_0, A$):

$$E = \frac{1}{\Lambda} (\nabla A_0 + \omega \nabla A^\phi), \quad (7)$$

$$B = \nabla \times A. \quad (8)$$

We ignore the dynamics of plasma and impose the force-free condition

$$\rho_e E + j \times B = 0, \quad (9)$$

which automatically satisfies

$$j \cdot E = 0, \quad E \cdot B = 0. \quad (10)$$

Under the conditions, the electrodynamics is described by three correlated functions: the flux $\psi = 2\pi A^\phi$ and the total electric current $I(\psi)$ flowing through the area enclosed by an axisymmetric loop, and the angular velocity of the electromagnetic field lines $\Omega(\psi) = -dA_0/dA^\phi$ on the loop.

The electromagnetic fields read

$$E = -\frac{\Omega - \omega}{2\pi \Lambda \sqrt{\rho^2}} \left( \sqrt{\Delta} \partial_\psi \psi e_r + \partial_\theta \psi e_\theta \right), \quad (11)$$

$$B = \frac{1}{2\pi \sqrt{A} \sin \theta} \left( \partial_\psi \psi e_r - \sqrt{\Delta} \partial_\psi \psi e_\theta + \frac{4\pi I \sqrt{\rho^2}}{\Lambda} e^\phi \right). \quad (12)$$

The charge and current densities are respectively

$$\rho_e = -\frac{1}{8\pi^2} \nabla \cdot \left( \frac{\Omega - \omega}{\Lambda} \nabla \psi \right), \quad (13)$$

$$j = \frac{1}{\Lambda} \rho_e \omega (\Omega - \omega) e^\phi + I|B|. \quad (14)$$
where the prime stands for the derivative with respect to $\psi$. Note that the total current $I$ is defined to flow upwards, with opposite sign to that defined in the original paper [15].

From the above equations and expressions, we will find that the force-free electrodynamics on a Kerr spacetime can be described by the following unique stream equation [15]

$$
\nabla \cdot \left\{ \frac{\Lambda}{\omega^2} \left[ 1 - \frac{(\Omega - \omega)^2}{\Lambda^2} \right] \nabla \psi \right\} + \frac{\Omega - \omega}{\Lambda} \Omega' (\nabla \psi)^2 + \frac{16\pi^2}{\Lambda \omega^2} II' = 0. \tag{15}
$$

For the electromagnetic system, the poloidal components of the energy and angular momentum flux densities from the hole are given by:

$$
\mathcal{E}^r = \Omega \mathcal{L}^r = -\Omega \frac{I}{2\pi \sin \theta \rho^2} \partial_\psi, \tag{16}
$$

$$
\mathcal{E}^\theta = \Omega \mathcal{L}^\theta = \Omega \frac{I}{2\pi \sin \theta \rho^2} \partial_\psi. \tag{17}
$$

### 3 Boundary behaviours

In regions that are accessible to us, the differential equation (15) has two boundaries: one at the horizon and the other at spatial infinity (if the force-free region extends far away from the outer horizon). In some sense, the two boundaries have similar behaviours and features, e.g., they both attain the following radiation condition [16]:

$$
E^\theta = \pm B^\phi, \tag{18}
$$

for the electromagnetic fields (11) and (12) as they are approached. The condition can be obtained directly from the stream equation (15). To make this clear, we re-express the stream equation as the following form

$$
\frac{\Delta}{A} \left\{ \frac{A^2 \sin^2 \theta (\Omega - \omega)^2}{\rho^4} - \Delta \right\} \partial_r^2 \psi + \frac{A^2 \sin^2 \theta (\Omega - \omega)}{\rho^4} \partial_r \Omega \partial_r \psi
$$

$$
+ 2 \frac{A \sin^2 \theta}{\rho^2} \left[ r \Omega^2 - \frac{Ma^2 \sin^2 \theta (r^2 - a^2 \cos^2 \theta) (\Omega - \Omega_N)^2}{\rho^4} \right] \partial_r \psi
$$

$$
- \partial_\theta^2 \psi + \frac{1}{2} \left[ \frac{\Delta}{A} + \frac{A \sin^2 \theta (\Omega^2 - \Omega_N^2)}{\rho^4} \right] \partial_\theta^2 \partial_\theta \psi \right\}
$$

$$
+ \sqrt{A} \sin \theta (\Omega - \omega) \frac{1}{\rho^2} \partial_\theta \sqrt{A} \sin \theta (\Omega - \omega) \partial_\theta \psi - 16\pi^2 II' = 0,
$$

where $\Omega_N = 1/(a \sin^2 \theta)$. 

5
3.1 Conditions at horizons

From the equation (19), we can see that only the last line of the equation remains at the horizons \( r = r_\pm \) or \( \Delta = 0 \):

\[
16\pi^2 I \frac{\partial}{\partial \psi} I = \frac{\sqrt{A} \sin \theta (\Omega - \omega)}{\rho^2} \sqrt{A} \sin \theta (\Omega - \omega) \frac{\partial \psi}{\rho^2} \frac{\partial}{\partial \psi} \sqrt{A} \sin \theta (\Omega - \omega) \frac{\partial \psi}{\rho^2}.
\]

Approaching the event horizon, \( \psi \) is a function only dependent on \( \theta \) [15]. This structure in the stream equation is also found in the near-horizon treatments of magnetospheres on near-extreme Kerr black holes [12].

From the above relation at the horizons, we have

\[
r = r_\pm : \quad I^2 = C_\pm + \left[ \frac{\sqrt{A} \sin \theta (\Omega - \omega)}{4\pi \rho^2} \partial_\theta \right]^2,
\]

where \( C_\pm \) are constants. So we can conclude that any solution \( \psi \) (with any given correlated functions \( \Omega_F(\psi) \) and \( I(\psi) \)) satisfying the stream equation (19) will always satisfy the condition (21), only if the sum of the terms within the brace in Eq. (19) are non-singular compared with the rest terms at the horizons.

It is easy to find that the Znajek boundary condition can be obtained by setting the special value:

\[
C_\pm = 0.
\]

When the positive sign is chosen, the conditions (21) at the horizons read

\[
I_+ = \frac{Mr_+ \sin \theta (\Omega_+ - \omega_+)}{2\pi \rho_{+,+}^2} \partial_\theta \psi_+,
\]

\[
I_- = \frac{Mr_- \sin \theta (\Omega_- - \omega_-)}{2\pi \rho_{-,+}^2} \partial_\theta \psi_-,
\]

where \( \psi_\pm = \psi(r_\pm) \), \( \Omega_\pm = \Omega(\psi(r_\pm)) \), \( I_\pm = I(\psi(r_\pm)) \) and \( \rho_{\pm,\pm}^2 = r_\pm^2 + a^2 \cos^2 \theta \). The former is exactly the Znajek regularity condition at the outer horizon [2], which corresponds to the positive sign case of the radiation condition (18): \( E^\theta = B^\phi \). The positive sign is chosen because this means current flow is directed outwards for \( 0 < \Omega_+ < \omega_+ \) [15, 17], which leads to energy and angular momentum extraction from the hole across the event horizon, as implied by Eq. (16).
3.2 Condition at spatial infinity

Let us now turn to the behaviours at spatial infinity. As shown in Eq. (16) and (17), the energy and momentum extraction rates are different by the angular velocity \( \Omega \). Since the energy and momentum extracted from the hole must be finite at spatial infinity, \( \Omega \) should be independent of \( r \) at infinity:

\[
\Omega(r, \theta) \to \Omega_0(\theta) \quad \text{as} \quad r \to \infty,
\]

(25)
as noticed in [5, 6]. Further, since \( I(\Omega) \) and \( \psi(\Omega) \) are functions of \( \Omega \), then the associated functions \( I_0 \) and \( \psi_0 \) at infinity should be functions of \( \Omega_0 \) as well:

\[
r \to \infty : \quad \psi(\Omega) \to \psi_0(\Omega_0(\theta)), \quad I(\Omega) \to I_0(\Omega_0(\theta)).
\]

(26)

That is, all three correlated functions should be independent of \( r \) at infinity if one is. This is quite similar to the situation at the outer horizon, where the functions are also only dependent on \( \theta \).

In the present work, we consider the case that \( \Omega_0 \) is not a constant or other trivial functions of \( \theta \) (so do \( \psi_0 \) and \( I_0 \)). With these asymptotic conditions (25) and (26), we can find that the stream equation (19) becomes the following simple form at infinity:

\[
16\pi^2 I_0 \frac{\partial I_0}{\partial \psi_0} = \sin \theta \Omega_0 \partial_\theta (\sin \theta \Omega_0 \partial_\theta \psi_0).
\]

(27)

Similarly, we have

\[
I_0^2 = C_0 + \frac{1}{16\pi^2} (\sin \theta \Omega_0 \partial_\theta \psi_0)^2,
\]

(28)

where \( C_0 \) is a constant. So any solutions satisfying the boundary conditions (25) and (26) must satisfy this relation. Here, we also choose the special case \( C_0 = 0 \), for which the above relation reads

\[
I_0 = -\frac{1}{4\pi} \sin \theta \Omega_0 \partial_\theta \psi_0.
\]

(29)

Here, the negative sign is chosen when \( \Omega_+ \leq \omega_+ \), which guarantees outflow of energy by inserting the current (29) into Eq. (19). This also corresponds to the positive sign case of the radiation condition (18). Note that, when \( \Omega_+ > \omega_+ \), we need to choose positive sign in the equation (29) (corresponding to the minus sign case of the radiation condition: \( E^\theta = -B^\phi \)), which leads to influx of energy at spatial infinity. The reason is that the direction of energy can not reverse on a field line [1]. If the energy inflows across the event horizon for \( \Omega_+ > \omega_+ \), we should also have influx at infinity.
3.3 The cases with identical boundary conditions

As in usual second-order differential equations, a set of solutions can be defined by con-
straining appropriate conditions on the two boundaries. On the other hand, as we stated
above, the behaviours are similar at the two boundaries: the functions are purely \( \theta \)-
dependent and satisfy the radiation condition (18). So it is natural to consider the special
cases that the conditions on the two boundaries are identical.

Generalising the condition (29) to include the positive sign case, we can express the
boundary condition at infinity as

\[
\pm \frac{4\pi}{\sin \theta} = \Omega_0 \frac{\partial_\theta \psi_0}{I_0}. \tag{30}
\]

On the other hand, the Znajek boundary condition (23) can be re-expressed as

\[
4\pi \left( \frac{1}{\sin \theta} - a \sin \theta \omega_+ \right) = (\Omega_+ - \omega_+) \frac{\partial_\theta \psi_+}{I_+}. \tag{31}
\]

Now we consider the special case that the functions satisfy the same boundary conditions
at the horizon and at infinity\(^2\):

\[
\Omega_0 = \Omega_+, \quad \psi_0 = \psi_+, \quad I_0 = I_+. \tag{32}
\]

(1) If we choose the positive sign in Eq. (30), we can have \( \Omega_+ \partial_\theta \psi_+/I_+ = 4\pi/\sin \theta \) and
\( \partial_\theta \psi_+/I_+ = 4\pi a \sin \theta \) by comparing the equations (30) and (31). From the factor difference
between them, we have:

\[
\Omega_+ (\theta) = \Omega_0 (\theta) = \Omega_N \equiv \frac{a}{2 M r_+ - \rho_+^2} = \frac{1}{a \sin^2 \theta}. \tag{33}
\]

As expected, the angular velocity is larger than the one of the black hole. That is why
we have chosen positive sign in the condition (30), as stated in the previous subsection.

(2) If we choose the negative sign in Eq. (30), we have: \( \Omega_+ \partial_\theta \psi_+/I_+ = -4\pi/\sin \theta \) and
\( \partial_\theta \psi_+/I_+ = -4\pi(2 M r_+ + \rho_+^2)/(a \sin \theta) \), which leads to

\[
\Omega_+ (\theta) = \Omega_0 (\theta) = \Omega_P \equiv \frac{a}{2 M r_+ + \rho_+^2}. \tag{34}
\]

The two solutions (33) and (34) at boundaries are exactly the same as the asymptotical
solutions found in [5, 6] (the MD solutions). The first solution is even the only known

\(^2\)Actually, the latter two stringent conditions can be simply replaced by the unique one \( \partial_\theta \psi_0/I_0 = \partial_\theta \psi_+/I_+ \) when the relations between \( I, \psi \) and \( \Omega \) are not necessarily the same at the two boundaries.
exact solution to date that can solve the full stream equation. In deriving the above solutions, the functions $I$ and $\psi$ are identical but not specified at the boundaries.

In what follows, we only take them as initial values at the two identical boundaries, instead of asymptotical solutions, to explore analytical solutions that are $(r, \theta)$-dependent in between the boundaries.

4 The expansion method

In terms of the boundary properties, we may derive solutions to the stream equation by expanding the functions in series of the radial distance $r$, as done in Appendix for Schwarzschild black hole case. If $\Omega_0$, $\psi_0$ and $I_0$ are all nontrivial functions of $\theta$ (i.e., not zero or constant), we can take the three correlated functions as the following general expanded forms:

$$
\Omega = \sum_{n=0}^{\infty} \Omega_n(\theta) r^{-n},
$$

$$
\psi = \sum_{n=0}^{\infty} \psi_n(\theta) r^{-n},
$$

$$
I = \sum_{n=0}^{\infty} I_n(\theta) r^{-n}.
$$

We assume these expanded forms to be valid in all force-free regions outside the event horizon of the Kerr spacetime. These forms saturate the conditions (25) and (26) at infinity. Inserting the expanded forms into the stream equation, solutions can be derived order by order. The solving procedure is as follows.

First, we need to choose the right zero-th order functions $\Omega_0$, $\psi_0$ and $I_0$, i.e., the conditions at infinity. But we only need to know two of them, because the third one can be determined via Eq. (29) (or (30) more generally) when the two are given.

Second, we need to know the function forms $\Omega(\psi)$ and $I(\psi)$ of $\psi$ (we can also take $\psi(\Omega)$ and $I(\Omega)$ as functions of $\Omega$). The function relations can be simply determined by the zero-th order ones:

$$
\psi, \Omega(\psi), I(\psi) \leftrightarrow \psi_0, \Omega_0(\psi_0), I_0(\psi_0),
$$

since the former will always lead to the latter as $r \to \infty$. With the specific forms of the functions $\Omega(\psi)$ and $I(\psi)$, we can determine the values $\psi_+, \Omega_+$ and $I_+$ at the horizon by inserting the functions into the Znajek regularity condition (23). This is how the
conditions at the two boundaries are correlated. So the zero-th order functions can be
adjusted if the conditions at the horizon are found to be inappropriate.

Finally, with all the functions and expanded forms inserted into the stream equation,
we can solve the equation order by order. The obtained solutions should apply for rotating
black holes with general $a$.

In summary, the derived solutions in this method completely rely on the choices of the
conditions at the two boundaries. Given any two of $\psi_0$, $\Omega_0$ and $I_0$, the general function
relations among $\psi$, $\Omega$ and $I$ can be determined by the zero-th order ones. This further
leads to the determinant of the condition at the horizon. So, with appropriate conditions
at both boundaries, a set of solutions are defined.

Besides, there is an extra problem that needs to be classified: the convergency at the
horizon in the extreme limit. The coefficient of the $n$-th order term of the derived solution
should be order of

$$\psi_{-n} \sim \mathcal{O}(m^n), \quad (n \geq 1) \tag{37}$$

where

$$m^n = \prod_{i=1}^{n} m_i, \quad m_i = (a, M). \tag{38}$$

Thus, in the extreme limit $r_+ = M = a$, each term of the expanded forms (35) is order
$\mathcal{O}(1)$ at the coincident horizon. So every term is important close to the horizon in the
extreme case. We must check this convergency of the solution, which is hard to do because
we usually can not derive the full solution of all orders. Fortunately, its convergency
should be guaranteed by the Znajek regularity condition at the horizon, since it applies
for arbitrary $a$.

## 5 Solutions

As examples, we shall adopt the special boundary conditions obtained in Section. (3.3)
to make solutions in what follows.

### 5.1 $\Omega_0 = \Omega_P$

As shown in [5], this case may correspond to the split monopole because it is expanded
to leading order of $a$ as $\Omega_P = a/(8M^2) + \cdots$ in the slow-rotating limit. Thus, we take the
zero-th order flux $\psi_0$ as that in the split monopole solution (on the upper half hemisphere
$0 \leq \theta < \pi/2$)

$$\psi_0 = \alpha(1 - \cos \theta), \tag{39}$$
with $\alpha$ a constant. Then we have from Eq. (29)

$$I_0 = -\frac{\alpha}{4\pi} \sin^2 \theta \Omega_P. \quad (40)$$

In terms of the relations between $\psi_0$ and $I_0(\psi_0)$, $\Omega_0(\psi_0)$, we can generalise them by assuming that the relations apply for any $r$:

$$\Omega(\psi) = \frac{\alpha^2 a}{B(\psi)}, \quad I(\psi) = -\frac{\alpha a \psi(2\alpha - \psi)}{4\pi B(\psi)}, \quad (41)$$

where $B(\psi) = \alpha^2 (r_+^2 + 2Mr_+) + a^2 (\alpha - \psi)^2$. Thus, we have

$$II' = \frac{Mr_+ \alpha^4 a^2 \psi(\alpha - \psi)(2\alpha - \psi)}{2\pi^2 B^3}. \quad (42)$$

Inserting the functions $\Omega(\psi)$ and $I(\psi)$ into the equation (19), we get an equation of $\psi$. The resulting equation can be solved order by order by using the expanded forms (35), in analogy to the Schwarzschild case shown in the Appendix. Similarly, let us define

$$L_\theta^2 \equiv \partial_\theta^2 + (2a\Omega_P \sin 2\theta + \cot \theta) \partial_\theta + 6 - 8a\Omega_P \cos^2 \theta - \frac{4}{\sin^2 \theta}. \quad (43)$$

The vanishing of the coefficients of $r^0$ gives rise to the equation about $\psi_0$, which is automatically saturated because it is just the condition chosen at infinity. Comparing all the terms at order $r^{-1}$ leads to the following equation about $\psi_{-1}$:

$$L_\theta^2 \psi_{-1} = 0. \quad (44)$$

This equation can be solved by

$$\psi_{-1} = \beta a \sin^2 \theta, \quad (45)$$

with $\beta$ being an arbitrary dimensionless constant.

In order to make higher order calculations simpler, we may set the free parameter $\beta$ to be the special value 0. Then the equation about $\psi_{-2}$ can be obtained and simplified:

$$(L_\theta^2 + 2)\psi_{-2} = -8Mr_+ \alpha a \cos \theta \sin^2 \theta \Omega_P. \quad (46)$$

A solution to the equation is

$$\psi_{-2} = \frac{1}{2} \alpha a^2 \cos \theta \sin^2 \theta. \quad (47)$$

Accurate to this order, the solution is somehow similar to the slow-rotating solution in the large $r$ limit: $A_\phi = C(1 - \cos \theta) + \mathcal{O}(a^2/M)(C \cos \theta \sin^2 \theta)r^{-1} + \cdots$, obtained in the
perturbation approach [1]. The difference is in that the next-to-leading order is at $r^{-2}$ for
the former and it is at $r^{-1}$ for the latter.

The equation of $\psi_{-3}$ is

$$(L_0^2 + 6)\psi_{-3} = 2\alpha M \cos \theta \left( 3a\Omega_P^{-1} + 4Mr_+ - 8Mr_+a \sin^2 \theta \Omega_P \right).$$

(48)

No analytical solution is found for the equation and so the calculation procedure can not
proceed.

Inserting the relations in Eq. (41) into the the Znajek condition (23 ), we can find
that the condition of $\psi$ at the horizon is the same as the one at infinity: $\psi_+ = \psi_0$, as we mentioned in the previous section. This means that all higher order terms $\psi_{-n}$
($1 \leq n < \infty$) of a legal solution $\psi$ must cancel out on the horizon, which is a constraint
of the Znajek regularity condition.

At boundaries, the solution satisfies $\Omega_0 = \Omega_+ \geq \omega_+/2$, where the equality occurs at
$\theta = 0$. Generally, the solution up to the second-order also satisfies

$$\Omega(r, \theta) = \frac{a}{2Mr_+ + r_+^2 + a^2 \cos^2 \theta (1 - \frac{1}{2}a^2 \sin^2 \theta r^{-2})^2} > \frac{1}{2}\omega_+. $$

(49)

This means that the magnetosphere in the valid regions is stable [18, 4] against the screw
instability [19]. Since the obtained solution is quite similar to the split monopole pertur-
bation solution at large $r$, other properties about the solution will be not reconsidered
here.

5.2 $\Omega_0 = \Omega_N$

5.2.1 The expanded solution

As stated previously, $\Omega = \Omega_N$ is the MD exact solution of the force-free magnetosphere
on general rotating black holes. But this $r$-independent solution is unrealistic. It is
interesting to investigate the situation by extending the solution to the $r$-dependent case
through the expansion approach given above.

We take $\Omega_N$ as the initial value at the boundary to derive the $r$-dependent solution.
Obviously, $\Omega_0 = \Omega_N$ is singular at poles $\theta = 0, \pi/2$. Thus, we demand $\psi_0$ to be non-
singular by taking the simple form:

$$\psi_0 = c\Omega_0^{-k} + d, \quad (c > 0, \quad k > 0)$$

(50)

where $c, d$ and $k$ are constants. By choosing positive sign in Eq. (30) instead, we have

$$I_0 = \frac{kc}{2\pi} \cos \theta \Omega_0^{1-k}.$$
since $\Omega_0 = 1/(a\sin^2\theta)$ is already faster than $\omega_+$.

In terms of the relations among the zero-th order functions, we can get the functional relations at general $r$:

$$\psi(\Omega) = c\Omega^{-k} + d, \quad (52)$$

$$I(\Omega) = \frac{kc}{2\pi}\sqrt{1 - (a\Omega)^{-1}\Omega^{1-k}}, \quad (53)$$

which lead to

$$\frac{16\pi^2}{kc}I'I\Omega^{2+k} = 4(k - 1)\Omega^4 - \frac{2(2k - 1)}{a}\Omega^3. \quad (54)$$

It is convenient for later calculations to redefine

$$\tilde{\Omega} \equiv \frac{\Omega}{\Omega_N} \quad \text{with} \quad \tilde{\Omega}_{-n}(\theta) \equiv \frac{\Omega_{-n}(\theta)}{\Omega_N}, \quad (55)$$

With the above relations, then the stream equation can be expressed as

$$\rho^2\tilde{\Omega}[A\tilde{\Omega}(\tilde{\Omega} - 2a\sin^2\theta\omega) - a^2\sin^2\theta(\rho^2 - 2Mr)](\Delta\partial_r^2\tilde{\Omega} + \partial_\theta^2\tilde{\Omega})$$

$$-\rho^2[A\tilde{\Omega}(k\tilde{\Omega} - (2k + 1)a\sin^2\theta\omega) - (k + 1)a^2\sin^2\theta(\rho^2 - 2Mr)][\Delta(\partial_r\tilde{\Omega})^2 + (\partial_\theta\tilde{\Omega})^2]$$

$$+ 2\Delta\tilde{\Omega}r^2\tilde{\Omega}^2 - Ma^2\sin^2\theta(r^2 - a^2\cos^2\theta)(\tilde{\Omega} - 1)^2\partial_r\tilde{\Omega}$$

$$+ \cot\theta\tilde{\Omega}\{(4k - 1)^2\rho^2[A\tilde{\Omega}(\tilde{\Omega} - 2a\sin^2\theta\omega) - a^2\sin^2\theta(\rho^2 - 2Mr)]$$

$$+ 2a^2\sin^2\theta A[\tilde{\Omega}(\tilde{\Omega} - 2a\sin^2\theta\omega) + a\sin^2\theta\omega] - 2\rho^2(r^2 + a^2)^2\tilde{\Omega}^2\partial_\theta\tilde{\Omega} \quad (56)$$

$$+ 2[2(k - 1)\rho^2(A - a^2(\rho^2 + 2Mr)) + 2Mr(r^2 + a^2)(r^2 - a^2\cos^2\theta) + \rho^4\Delta]\tilde{\Omega}^4$$

$$+ 2[(2k - 1)\rho^2(4Mr a^2 \cos^2\theta - \rho^4) - 4Mr a^2 \sin^2\theta(r^2 - a^2\cos^2\theta)]\tilde{\Omega}^3$$

$$+ 2[2(k - 1)\rho^2 a^2 \cos^2\theta(\rho^2 - 2Mr) - \rho^2 a^2 \sin^2\theta(\rho^4 - 2Mr(r^2 - a^2\cos^2\theta))]\tilde{\Omega}^2 = 0.$$
In what follows we shall consider more general \((r\text{-dependent})\) solutions other than this trivial case.

With \(\Omega_0 = \Omega_N\), the vanishing of the terms at order \(r^6\) is automatically satisfied, as expected. For the order \(r^5\), the obtained equation is

\[
L_\theta^2 \tilde{\Omega}_1 = 0.
\]  

(60)

The equation has the simple solution

\[
\tilde{\Omega}_1 = -2\alpha a \cos \theta,
\]  

(61)

where \(\alpha\) is an arbitrary dimensionless constant. We assume that the solution applies to the upper hemisphere \(\theta \in [0, \pi/2]\) since it is asymmetric about the equatorial plane.

The equation (60) has the second kind of solution, which is symmetric. The solution generally can be expressed in terms of the hypergeometric functions. But we can have their explicit forms when \(4k - 3\) is an odd number. For example, the solution for \(k = 3/2\) is

\[
\tilde{\Omega}_1 = -\beta a \left[ 1 - \frac{1}{2} \cot^2 \theta + \frac{3}{4} \cos \theta \ln \frac{1 - \cos \theta}{1 + \cos \theta} \right],
\]  

(62)

where \(\beta\) is a dimensionless constant. This solution is symmetric under \(\cos \theta \rightarrow -\cos \theta\). But the solution forms closed magnetic field lines, which is excluded for a force-free magnetosphere [15, 20]. So this solution is abandoned.

At order \(r^4\), the resulting equation about \(\Omega_{-2}\) can be simply reduced by inserting the solution (61)

\[
(L_\theta^2 + 2) \tilde{\Omega}_{-2} = 4k\alpha^2 a^2.
\]  

(63)

A simple solution to this equation is

\[
\tilde{\Omega}_{-2} = \alpha^2 a^2.
\]  

(64)

The equation of \(\tilde{\Omega}_{-2}\) by inserting the second solution (62) is hard to be solved and is not considered.

The vanishing of the coefficients of \(r^3\) leads to the equation about \(\Omega_{-3}\), which can be reduced to

\[
(L_\theta^2 + 6) \tilde{\Omega}_{-3} = 4\alpha^2 Ma^2 (1 - 2k \cos^2 \theta) + 4\alpha a^3 \cos \theta [3 + 2(1 - 2k) \cos^2 \theta].
\]  

(65)

When \(k \neq 3/2\), a solution to this equation is

\[
\tilde{\Omega}_{-3} = 2\alpha a^2 \cos^3 \theta + \frac{\alpha^2 Ma^2}{2k - 3} \left( 4k \cos^2 \theta - \frac{3}{k + 1} \right).
\]  

(66)
| $\psi$ | $\mathcal{L}^\theta$ | $I$, $\mathcal{L}^r$ | $E^\theta$ | $E^r$ |
|--------|----------------------|---------------------|----------|------|
| $k \geq 0$ | $k \geq \frac{3}{4}$ | $k \geq 1$ | $k \geq \frac{5}{4}$ | $k \geq \frac{3}{2}$ |

Table 1: The conditions of $k$ for the corresponding quantities to be non-singular on the rotation axis.

The solution at the critical value $k = 3/2$ is not found.

The solutions at higher orders are hard to derive due to the involvement of many more terms. But we can make some simple analysis based on the equation (56) and the derived solutions above. For higher orders, we can find that the function $F_n$ for each $n$ in Eq. (57) should be some polynomial of $\cos \theta$:

$$F_n(\theta) = \sum_{i} f_i(\alpha, k) \cos^i \theta,$$

where $f_i$ are coefficients and are of order $m^n$, as pointed out in Eq. (37). So the equation (57) with the form of $F_n$ should be solvable except that abnormal situations emerge. This implies that an exact solution may be eventually obtained or guessed by following the procedure if we could successfully handle all the terms to higher enough orders.

At the moment, the above solution up to the first few orders should be valid for asymptotical regions far away from the horizon. The solution is consistent with our near-horizon solution for near-extreme black holes [12]. It can be checked that the solution forms open magnetic field lines, which may be separated by a current sheet on the equatorial plane, just like the split monopole solution.

### 5.2.2 Analysis of the solution

Our solution generalises the MD exact solution $\Omega = \Omega_N$ [5, 6] to the $(r, \theta)$-dependent case. The MD exact solution is taken as an initial condition at both boundaries and is recovered from the generalised solution as the parameter $\alpha = 0$. The exact solution has difficulties to describe a realistic magnetosphere since its four-current and the electromagnetic field are both null. Here we examine the situation for our generalised solution.

Before doing that, we first determine the conditions that the quantities from the solution are not singular on the poles $\theta = 0$, which are summarised in Table 1. For $k \geq 3/2$, all the quantities on the table (as well as the electromagnetical fields) are non-singular on the poles.

The existence of magnetosphere in all frames requires it should be magnetically dom-
Figure 1: Illustrations of the invariant $F^2$ at different radial distances $r$ and different poloidal angles $\theta = q\pi/2$. The parameter $c = 1$ and the spin parameter $a = 0.8M$. The line groups: (1) $\alpha = 1$ and $k = 1.3; (2) \alpha = 0.1$ and $k = 1.49; (3) \alpha = 1$ and $k = 2$.

inated, i.e., the following invariant to be

$$F^2 = 2(B^2 - E^2) > 0.$$  \hspace{1cm} (68)

Inserting the solution into the expressions, we can get the invariant. We find that its sign is strongly affected by the parameter $k$ but not sensitive to other parameters like $\alpha$ and $a$. As shown in Fig. 1, the invariant $F^2$ is positive for $k < 3/2$ while negative for $k > 3/2$ (the case $k = 3/2$ can not be judged since the solution is not available here). This indicates that the magnetic fields can get dominated only when (part of) the quantities are singular on the poles. As expected, the values all asymptotically approach 0 at large $r$ as the $(r, \theta)$-dependent solution recovers the MD exact solution at the far boundary.

Whether the four-current $J^\mu$ is timelike, lightlike or spacelike can be determined by judging $J^2 = J_\mu J^\mu$ (contracted by the four-dimensional metric (1)) to be negative, null, or positive, respectively. The contracted current is related to the charge and current densities measured in ZAMOs via

$$J^2 = -\rho_e^2 + \mathbf{j} \cdot \mathbf{j},$$  \hspace{1cm} (69)

with their components satisfying

$$\frac{1}{\Lambda} \rho_e = J^0, \quad j^r = J^r,$$  \hspace{1cm} (70)
The parameter are chosen to be: $c = 1$ and $a = 0.8M$. Left panel: (1) $\alpha = 1$ and $k = 1.3$; (2) $\alpha = 1$ and $k = 3$. Right panel: $\alpha = 0.1$ and $k = 1.49$.

\[ j^\theta = J^\theta, \quad j^\phi = -\omega J^\theta + J^\phi. \]  

(71)

The three-current $j$ is contracted by the metric (3) of the absolute space.

The sign of $J^2$ is also sensitive to $k$, as shown in Fig. 2. For $k > 3/2$, the values of $J^2$ are almost all positive at all angles $\theta$. For $k < 3/2$, they are not always positive and are negative for larger $\theta$, i.e., near the equatorial plane. The only case that its values are mostly negative happens when $k \to 3/2$ from the $k < 3/2$ side. The case $k = 1.49$ (to regularise $\tilde{\Omega}_3$ to be not too large, we adopt a small $\alpha = 0.1$) is shown on the right panel of Fig. 2. It can be seen that the values of $J^2$ grow with $\theta$ increasing from negative values at the small angle $\theta = 0.01\pi/2$, and turn to be slightly positive at around $\theta = \pi/4$. Then they turn back to be negative again for larger angles. It can be checked that the values of $J^2$ are also negative for angles smaller than $\theta = 0.01\pi/2$. But they all will tend to be null: $J^2 = 0$, at exactly $\theta = 0$.

6 Summary

In this work, we adopt a new expansion method to explore analytical solutions of force-free magnetospheres on black holes with arbitrary spin parameter. The functions and stream equation are expanded in series of the radial distance in terms of the boundary conditions at the event horizon and at spatial infinity. With the conditions at the two boundaries chosen, a set of solutions can be defined and solved order by order.

In terms of the regular conditions at both boundaries, the two asymptotical solutions found by MD in [5, 6] are identified as the solutions that have the same conditions at the
two boundaries. By taking them as initial conditions at the boundaries, we derived the corresponding expanded solutions to higher orders. The first one corresponds to the split monopole solution obtained in the perturbation approach when we take the $a \to 0$ limit. It is found to have similar asymptotical profile to the latter at large $r$, though not with the same $r$ dependence.

The second solution can be viewed as an extension of the $r$-independent MD exact solution to the $(r, \theta)$-dependent case. With an appropriate choice of the relation between $\psi$ and $\Omega$ at the far boundary, we find that the expanded stream equation should be solvable at each order. So an exact solution (probably with a closed form) is hopefully derived or guessed if we could calculate to all or higher enough orders, though we only get the expanded solution up to first few orders in this work.

Based on the obtained solution, we show that the extended solution can (partially) avoid the problems of the $r$-independent MD solution: the four-current and the electromagnetic field are both null. When the parameter $k$ tends to the critical value $3/2$ from the $k < 3/2$ side, our solution leads to a force-free magnetosphere which is magnetically dominated with timelike current in most directions $\theta$. The current gets slightly spacelike at around $\theta = \pi/4$ and lightlike at exactly $\theta = 0$. A difficulty for the solution with $k$ less than and close to $3/2$ is that the energy extraction (integration of $E^r$) highly converges along the rotation axis in a singular way. Similar singular behaviours also exist in the relieving method [8, 9]. But, in our case, the singular mode is very slight for $k \to 3/2$. Nevertheless, we may still have to exclude the $\theta = 0$ direction or assume that the force-free condition is violated by dense plasma in this region.

As we can see, the solution with $k = 3/2$ is an interesting case, but is not found in this work, which is left for further study. We also need to derive the expanded solution to higher orders and to check whether the results from the present solution still hold (or even improve). Moreover, more varieties of the relation between $\psi_0$ and $\Omega_0$ other than (50) are under consideration to find saturate results.

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Appendix

A Solutions on a Schwarzschild black hole

A detailed discussion of exact solutions of magnetospheres on Schwarzschild black holes have been made in [21]. Here, we use the expansion method in the text to rederive the solutions. The Schwarzschild metric is

\[ ds^2 = - \left(1 - \frac{r_0}{r}\right) dt^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \]

where the horizon is located at \( r_0 = 2M \).

In the non-rotating case, the stream equation reduces to

\[ x^2 \partial_x \left( (1 - x^{-1}) \partial_x \psi \right) + L_\theta^2 \psi = 0, \]

where

\[ x \equiv \frac{r}{r_0}, \]
\[ L_\theta^2 = \sin \theta \partial_\theta \left( \sin^{-1} \theta \partial_\theta \right) = \partial_\theta^2 - \cot \theta \partial_\theta. \]

Here, we take \( \psi \) to be dimensionless to simplify the notation. This equation is essentially the Maxwell’s equation on Schwarzschild in the absence of sources, i.e., \( \rho_e = j = 0 \).

So the force-free condition (9) is trivial here and does not really provide any extra constraint. We need to find alternative boundary conditions. Let us adopt the ansatz of a general solution:

\[ \psi(x, \theta) = \psi_1(\theta)x + \psi_\ast \ln x + \psi_0(\theta) + \psi_{-1}(\theta)x^{-1} + \psi_{-2}(\theta)x^{-2} + \cdots. \]

We choose this ansatz to guarantee that the electromagnetic fields vanish at \( x \to \infty \).

Inserting the expanded form into the equation and comparing the coefficients of each order of \( x \), we get the following equations:

\[ L_\theta^2 \psi_1 = 0, \]
\[ L_\theta^2 \psi_\ast = 0, \]
\[ L_\theta^2 \psi_0 = \psi_\ast - \psi_1, \]
\[ (L_\theta^2 + 2)\psi_{-1} = -2\psi_\ast, \]
\[ [L_\theta^2 + n(n + 1)]\psi_{-n} = (n^2 - 1)\psi_{-(n-1)}. \quad (n \geq 2) \]

(I) Non-separable solutions
The first four equations (6)-(9) are closed and complete to give exact solutions. We first set the coefficients $\psi_{-n} = 0$ $(n \geq 1)$ so that $\psi_* = 0$ since they are always solutions. Then from Eq. (6), a solution of $\psi_1$ can be written as the form

$$\psi_1 = 1 - \cos \theta,$$  (11)

With it, the equation (8) can be expressed as

$$\partial_y^2 \psi_0 = -\frac{1}{y},$$  (12)

where $y = 1 + \cos \theta$. A general solution to this is

$$\psi_0 = \alpha + \beta \cos \theta - (1 + \cos \theta) \ln(1 + \cos \theta),$$  (13)

where $\alpha$ and $\beta$ are constants. When $-\alpha = \beta = 1$, the full solution is

$$\psi = (x - 1)(1 - \cos \theta) - (1 + \cos \theta) \ln(1 + \cos \theta),$$  (14)

which is exactly the non-separable solution [21].

(II) Separable solutions

(1) Zeroth-order solution

From Eqs. (6)-(8), we can impose the general solution

$$\psi_1 = \psi_* = b + c \cos \theta, \quad \psi_0 = d\psi_1 + \alpha \cos \theta + \beta,$$  (15)

where $b$, $c$, $d$ and $e$ are constants. So we have

$$\psi_{-n} = -\frac{1}{n} \psi_1. \quad (n \geq 1)$$  (16)

Adopting the relation $\ln(1 - z) = -\sum_{n=1}^{\infty} z^n / n$, we can express the solution as

$$\psi = \alpha \cos \theta + \beta + (b + c \cos \theta)[d + x + \ln(x - 1)].$$  (17)

This is the lowest order separable solution with $m = 0$ given in [21]. The case $b = c = 0$ is the (split) monopole solution.

(2) First-order solution

From the first three equations (6)-(8), we consider the case:

$$\psi_1 = \psi_* = 0,$$  (18)

and

$$\psi_0 = \alpha \cos \theta + \beta.$$  (19)
Then Eq. (9) becomes $L_\theta^2 \psi_{-1} = -2 \psi_{-1}$. So the general solution of $\psi_{-1}$ can be

$$\psi_{-1} = g \sin^2 \theta,$$

(20)

where $g$ is an arbitrary constant. Thus, we can have generically from Eq. (10):

$$\psi_{-n} = \frac{3g}{n + 2} \sin^2 \theta. \quad (n \geq 2)$$

(21)

By using the expansion expression of $\ln(1 - z)$, we can express the full solution as

$$\psi = \alpha \cos \theta + \beta - 3g\sin^2 \theta \left[ \frac{1}{2} + x + x^2 \ln \left( 1 - \frac{1}{x} \right) \right].$$

(22)

The solution with $\alpha = \beta = 0$ is clearly the separable solution at the order $m = 1$ given in [21].

(3) Higher order solutions

If we consider the case $\psi_1 = \psi_* = \psi_0 = \psi_{-1} = 0$, then Eq. (10) becomes $L_\theta^2 \psi_{-2} = -6 \psi_{-2}$. It solves as $\psi_{-2} = 3h \cos \theta \sin^2 \theta$. We can then insert the solution into the general $\psi_{-n}$. Following the same approach above, we can derive the separable solution at the $m = 3$ order. Similarly, we can derive all higher order separable solutions.

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