Gaussian-Hermite Moment Invariants of General Vector Functions to Rotation-Affine Transform

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Abstract—With the development of data acquisition technology, multi-channel data is collected and widely used in many fields. Most of them can be expressed as various types of vector functions. Feature extraction of vector functions for identifying certain patterns of interest is a critical but challenging task. In this paper, we focus on constructing moment invariants of general vector functions. Specifically, we define rotation-affine transform to describe real deformations of general vector functions, and then design a structural frame to systematically generate Gaussian-Hermite moment invariants to this transform model. This is the first time that a uniform frame has been proposed in the literature to construct orthogonal moment invariants of general vector functions. Given a certain type of multi-channel data, we demonstrate how to utilize the new method to derive all possible invariants and to eliminate various dependences among them. For RGB images, 2D and 3D flow fields, we obtain the complete and independent sets of the invariants with low orders and low degrees. Based on synthetic and popular datasets of vector-valued data, the experiments are carried out to evaluate the stability and discriminability of these invariants, and also their robustness to noise. The results clearly show that the moment invariants proposed in our paper have better performance than other previously used moment invariants of vector functions in RGB image classification, vortex detection in 2D vector fields and template matching for 3D flow fields.

Index Terms—General vector functions, rotation-affine transform, orthogonal moments, Gaussian-Hermite moment invariants, RGB image classification, vortex detection, 3D flow fields, template matching

1 INTRODUCTION

How to extract effective features of various types of data is one of the most fundamental problems in the field of pattern recognition. In order to obtain the ability to capture intrinsic information of the given data, a desirable feature should be invariant to the deformations caused by the sensor’s setup, the influence of environment and many other factors. For decades, researchers have designed numerous invariant features for different data and practical applications. Among them, moments and moment invariants play a very important role.

From the mathematical point of view, moments are “projections” of a function onto a polynomial basis, and classical moment invariants are special homogeneous polynomials of moments, which have invariance to certain transforms. For example, the standard power basis leads to geometric moments, and they are widely used in the image analysis community. In 1962, based on the theory of algebraic invariants, Hu first derived seven geometric moment invariants of grayscale images to 2D rotations [1]. These invariants are referred to as Hu moment invariants. Thirty years later, Reiss and Flusser and Suk independently published several affine moment invariants [2], [3], [4]. Since then, an effort has been put into designing simpler and more transparent ways to systematically generate rotation or affine moment invariants of grayscale images [5], [6]. In 2018, Li et al. further proved the existence of image projective invariants using finite combinations of weighted geometric moments [7]. Unfortunately, geometric moments and the moment invariants from them are very sensitive to additive noise, and there is a high level of information redundancy. To solve these problems, a series of papers focused on defining image moments by using a variety of orthogonal polynomial bases. The experimental results in these papers clearly showed that image orthogonal moments have better numerical stability and recognition ability than geometric moments and complex moments. The existing orthogonal moments can be divided into two groups, namely moments orthogonal on a circle and moments orthogonal on a square. The former group includes Zernike moments [8], [9], Pseudo Zernike moments [10], Fourier-Mellin moments [11], [12], Jacobi-Fourier moments [13] and Chebyshev-Fourier [14]. The values of these moments are complex numbers, and the magnitudes of them are naturally invariant to 2D rotations. Before calculating them, we have to first map a grayscale image into a unit circle. The interpolation not only leads to the loss of precision but also increases the computation time. Hence, the latter one, moments orthogonal on a square, are preferred by many researchers, such as Legendre moments [15], [16], Chebyshev moments [17], [18], Krawtchouk moments [19], [20], Gegenbauer moments [21] and Gaussian-Hermite moments [22]. However, the construction of rotation invariants from these moments is very difficult. The only exception is the Gaussian-Hermite moments. Yang et al. proved that any geometric moment invariants to 2D rotations, such as Hu moment invariants, still keep their invariance when replacing geometric mo-
ments by the corresponding Gaussian–Hermite moments. Based on this discovery, they derived a set of Gaussian-Hermite moment invariants to 2D rotations [23], [24] and further achieved their invariance to 2D scale transform [25].

All of the above-mentioned moment invariants are calculated from grayscale images. Some construction methods of them can also be generalized to derive 3D moment invariants [26], [27]. Note that both grayscale images and 3D shapes are single-channel data, which can be expressed as scalar functions. When constructing moment invariants of scalar functions, researchers just discussed and achieved the invariance of them to geometric transforms acting on the spatial coordinates. With the development of simulation and measurement techniques, multi-channel data sets are rapidly growing in size and becoming prevalent in many fields, including pattern recognition, computer vision, visualization and computer graphics. Most of multi-channel data, such as RGB images, 2D vector fields, 3D flow fields and color volume data (see in Fig. 1), can be expressed as different types of vector functions. However, in most cases, traditional geometric transforms do not have the ability to describe the realistic deformations of vector functions, because these deformations usually simultaneously act on spatial domain and vector domain. To address this issue, researchers designed several more complicated and effective transform models for various vector functions, such as total rotation transform (TR) and total affine transform (TA). In the last decade, moment invariants of vector functions to these models have attracted considerable attention. According to data types, we can classify the relevant moment invariants into the following three categories

- **RGB images**: There are a lot of papers about constructing moment invariants of color images, but most of them just focused on the invariance under geometric transforms [28], [29], [30], [31], [32], [33]. In 2017, using the geometric primitive introduced by Xu and Li [26], Gong et al. first derived geometric moment invariants of color images to TA [34]. These features are invariant to both 2D affine transform of spatial coordinates and 3D affine transform of color space. In fact, TA is the best linear model to simulate color image deformations caused by imaging geometry and the changes of illumination condition [2], [3], [35], [36], [37]. Before that, Mindru et al. also constructed similar invariants under restricted TA, in which the color affine transform degenerated into 3D diagonal transform [38], [39].

- **2D vector fields**: In 2017, Schlemmer et al. first generated complex moment invariants of 2D vector fields to TR [40], [41]. Then, Bujack et al. obtained invariant complex moments to TR by using the normalization approach [42], [43], [44], [45]. Specifically, they estimated the parameters of TR by means of several complex moments of low orders. However, using this approach, we can not accurately determine the angle of rotation, which is the most important parameter of TR, and have to calculate the normalized complex moments for a set of candidate angles. In addition, the potential errors of transform parameters will affect the stability of all invariant moments, whereas this issue has not been studied and experimentally tested in their papers [46]. In 2018, Yang et al. constructed orthogonal moment invariants to TR from Zernike moments and Gaussian-Hermite moments [47], [48]. Recently, Kostková et al. also designed a procedure to systematically generate geometric moment invariants to TA [49], [50], which is similar to Gong’s approach [34].

- **3D vector fields and color volume data**: Since most of fluid flow data are intrinsically three-dimensional, researchers are more desirable to get moment invariants of 3D vector fields than those of 2D ones. Unfortunately, there is little work on this topic. The main reason is that many methods of deriving moment invariants of 2D vector fields can not be extended to 3D. For 3D vector fields and even more general tensor fields, Hagen and Langbein designed the tensor-valued functions of geometric moments, which are invariant to TR [51]. However, these moment tensors are not classical moment invariants, and the authors neither provided explicit formulas of them nor carried out numerical experiments to test their performance. Based on their previous work, Bujack et al. proposed a more complicated normalization approach to calculate invariant geometric moments of 3D vector fields to TR [52]. Recently, Hao et al. generalized Gong’s method [34] to generate geometric moment invariants of general vector functions to TA [53], and first showed the expansions of classical moment invariants of 3D vector fields and color volume data.
or 5D vector fields in a similar way. Meanwhile, as shown in [27], it is not very easy to prove that the isomorphism relationship still holds for higher-dimensional data. That is why Yang et al. did not get orthogonal moment invariants of 3D vector fields to TR using this approach. In addition, given a vector function, some methods demand that the dimension of the vector domain must be equal to the dimension of the spatial domain [40], [42], [48], [52]. However, many types of multi-channel data do not meet this condition, such as RGB images.

The goal of our paper is to address those limitations, and the main contributions can be summarized as follows

- We define rotation-affine transform (RA) of general vector functions, clarify the relationship between TR, TA and RA, and explain why RA is more suitable to model the deformations of multi-channel data.
- Using two fundamental differential operators and one fundamental primitive, we develop a novel and intuitive method to derive orthogonal Gaussian-Hermite moment invariants of general vector functions to RA (RAGHMs). We believe that no previous research has yielded similar results.
- For RGB images, 2D and 3D flow fields, all possible RAGHMs with low orders and low degrees are generated. We identify various types of dependencies among them, and derive the complete and independent sets of them. As far as we are aware, this is the first time that orthogonal moment invariants of 3D vector fields have been proposed in the literature.
- Numerical experiments are carried out on synthetic and widely used vector-valued datasets. We demonstrate the stability and discriminability of RAGHMs, and also test their robustness to common additive noise. Most of existing moment invariants of vector functions are chosen for comparison. The experimental results show that RAGHMs have better performance in RGB image classification, vortex detection in 2D vector fields and template matching of 3D flow fields.

The rest of our paper is organized as follows. Section 2 formulates the definitions and concepts used throughout the paper. Sections 3 and 4 are the main contributions of this paper. We introduce the structural framework of RAGHMs, and then derive some instances with low degrees and low orders for widely used multi-channel data. In Section 5, we conduct experiments to validate the performance of RAGHMs in various recognition tasks. Finally, Section 6 presents our conclusions.

## 2 Basic Definitions and Notations

In this section, we will introduce some basic concepts and definitions used in the following sections.

### 2.1 Vector Functions

A general vector function \( F(X) : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n \) can be defined as

\[
X = (x_1, x_2, \ldots, x_m)^T \\
F(X) = (f_1(X), f_2(X), \ldots, f_n(X))^T
\]  

where

- Both \( m \) and \( n \) are positive integers.
- For any \( \beta \in \{1, 2, \ldots, n\} \), the scalar function \( f_\beta(X) \) acts as mapping from the domain \( \Omega \) to \( \mathbb{R} \). When \( n = 1 \), \( F(X) \) degenerates into a scalar function.

We can instantiate (1) by setting specific \( m \) and \( n \). Some instances are listed in Table 1.

### 2.2 Rotation-Affine Transform

Suppose that the vector function \( F(X) \) defined by (1) is changed to \( G(Y) : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n \) using a rotation-affine transformation \( RA = (R; A; T) \), we have

\[
Y = R \cdot X, \quad G(Y) = A \cdot F(X) + T
\]  

where

- \( Y = (y_1, \ldots, y_m)^T \) and \( G(Y) = (g_1(Y), \ldots, g_n(Y))^T \).
- The inner rotation transformation \( IR = R \in \mathbb{R}^{m \times m} \) is an orthogonal matrix, meaning that \( R^{-1} = R^T \) and \( |R| = 1 \).
- In the outer affine transformation \( OA = (A; T) \), \( A \in \mathbb{R}^{n \times n} \) is a non-singular matrix, and \( T \in \mathbb{R}^n \) represents the outer translation.

When constructing invariant features of vector functions to \( OA = (A; T) \), researchers usually omit the outer transformation \( T \), because its parameters can be easily removed by subtracting the average of \( F(X) \) over the domain \( \Omega \). Thus, for \( OA \), our paper will just achieve the invariance to the \( m \times m \) matrix \( A \). It should be noted that we also do not discuss the inner translation \( S \in \mathbb{R}^n \) in \( IR \), namely do not utilize more complicated inner transform model \( Y = R \cdot X + S \). In fact, in many practical applications, such as template matching and interest point detection, a local coordinate system with a given point as the origin is first established. Then, based on this coordinate system, the selected features are calculated from the local region around the point. This makes these features naturally invariant to \( S \).

Since \( RA \) acts simultaneously on the domain and the codomain of \( F(X) \), it is more suitable to describe realistic deformations of multi-channel data than those geometric transformations only acting on spatial coordinates. For example, previous studies have proved that 3D affine transform is the best linear model of the photometric changes for RGB images [35], [36], [37]. Two color images \( (R(x, y), G(x, y), B(x, y)) \) and \( (R'(u, v), G'(u, v), B'(u, v)) \) of the same planar object taken from different angles and lighting conditions can be related by (3), where \( \theta \in (0, 2\pi] \)

| (m,n) | Instances |
|-------|-----------|
| (2,2) | 2D Vector Fields: \( F(X) = (U(x, y), V(x, y)) \) |
| (2,3) | RGB Images: \( F(X) = (R(x, y), G(x, y), B(x, y)) \) |
| (3,3) | Color Volume Data: \( F(X) = (R(x, y, z), G(x, y, z), B(x, y, z)) \) 3D Flow Fields: \( F(X) = (U(x, y, z), V(x, y, z), W(x, y, z)) \) |

### TABLE 1: Some specific instances of (1).
represents the angle of rotation, the other parameters are real numbers and the 3D matrix is non-singular. Clearly, we can make use of (2) to represent this transform model via setting \( m = 2 \) and \( n = 3 \). In Fig. 2a, we show four \( RA \) versions of the same RGB image. Similarly, when \( m = 2 \) and \( n = 2 \), \( RA \) is also able to model realistic local deformations of 2D vector fields. Fig. 2b shows several \( RA \) versions of an original vector field.

Additionally, it is necessary to clear the relationship between \( RA \) and widely used transform models for vector functions. As stated previously, most of existing geometric or complex moment invariants of vector fields are invariant to \( TR \) or \( TA \). In \( TR \), the general \( m \times m \) matrix \( A \) in \( OA \) degenerates into an orthogonal matrix and is equal to the inner rotation \( R \). Thus, we have \( TR \subset RA \). It shows that moment invariants to \( RA \) are also invariant to \( TR \). In \( TA \), the orthogonal matrix \( R \) in \( IR \) is generalized to a general \( n \times n \) matrix, which means \( RA \subset TA \). Note that the linearity assumption does not seriously restrict the capacity of \( TA \), because any nonlinear transformations acting on the small-enough neighborhood around each point can be approximated by linear ones. Theoretically, compared with \( RA, TA \) is able to model more complicated transformations of multi-channel data. However, in most cases, it is difficult to determine an affine-equivariant region for calculating invariant features around a given point, when we do not know the parameters of the inner affine transformation \( (IA) \). Kostković et al. also found this problem. For detecting vortices in 2D vector fields, they calculated moment invariants on a circular region with a fixed radius around each of the points [49], [50]. However, to comply with the definition of \( TA \), the circular region should be transformed into an ellipse using \( IA \). In fact, Kostković et al. have implicitly assumed that \( TA \) degenerates into \( RA \). That is the reason why our paper mainly focuses on constructing moment invariants to \( RA \) rather than to \( TA \).

### 2.3 Gaussian-Hermite moments of vector functions

Hermite polynomials were defined by Pierre-Simon Laplace and studied in detail by Pafnuty Chebyshev. However, Chebyshev's work was overlooked, and they were named later after Charles Hermite. The \( p \)-order Hermite polynomial is defined as

\[
H_p(x) = (-1)^p \exp(x^2) \frac{d^p}{dx^p} \exp(-x^2)
\]

where \( p \) is a non-negative integer, and \( H_p(x) \) satisfies the following recurrence relation

\[
H_p(x) = 2xH_{p-1}(x) - 2(p - 1)H_{p-2}(x)
\]

where \( H_0(x) = 1 \) and \( H_1(x) = 2x \). The Hermite polynomials are orthogonal on \((-\infty, +\infty)\) with the weight \( \omega(x) = \exp(-x^2) \)

\[
\int_{-\infty}^{+\infty} \exp(-x^2)H_{p_1}(x)H_{p_2}(x)dx = p_1!2^{p_1}\sqrt{\pi}\delta_{p_1p_2}
\]

where \( p_1 \) and \( p_2 \) are non-negative integers, and \( \delta_{p_1p_2} \) is the Kronecker delta.

Due to a high range of values and poor localization, original Hermite polynomials are difficult to be employed directly without any normalization. To overcome this, some researchers modulated Hermite polynomials with a Gaussian function. Specifically, the \( p \)-order Gaussian–Hermite polynomial is defined as

\[
\hat{H}_p(x; \sigma) = \exp \left( -\frac{x^2}{2\sigma^2} \right) H_p \left( \frac{x}{\sigma} \right) = (-\sigma)^p \exp \left( \frac{x^2}{2\sigma^2} \right) \frac{d^p}{dx^p} \exp \left( -\frac{x^2}{\sigma^2} \right)
\]

where the parameter \( \sigma \) is the user-defined standard deviation of the Gaussian function, which controls the attenuation of the polynomial. When setting \( \sigma = 1 \), we plot Gaussian-Hermite polynomials up to the sixth order in Fig. 3. Note that in order to show them in a similar range, we normalize \( \hat{H}_p(x; \sigma) \) by \( 1/\sqrt{p!2^p\sigma\sqrt{\pi}} \).

Given a general vector function (1), we can define its \((p_1 + p_2 + \cdots + p_m)\)-order Gaussian-Hermite moment as

\[
\eta_{p_1p_2\cdots p_m} = \int_{\Omega} \cdots \int_{\Omega} \prod_{a=1}^{m} \hat{H}_{p_a}(x_a; \sigma) f_\beta(X)dx_1dx_2\cdots dx_m
\]

(8)
where \( p_1, p_2, \ldots, p_m \) are non-negative integers, and \( \beta \in \{1,2,\ldots,n\} \). The scale parameter \( \sigma \) influences the performance of \( \eta_{p_1,p_2,\ldots,p_m} \), but there is no exact rule for setting its value. Finding appropriate \( \sigma \) is a heuristic which depends on the size and content of the data \([46]\). In addition, to ensure better numerical stability, previous research proposed that \( \eta_{p_1,p_2,\ldots,p_m} \) should be normalized by \( 1/\sqrt{\gamma 12^\gamma \sigma^7 \pi^3} \), where \( \gamma = p_1 + p_2 + \cdots + p_m \), and also proved that this normalization did not influence the invariance of classical moment invariants from \( \eta_{p_1,p_2,\ldots,p_m} \) \([32],[46],[48]\).

In \([23],[27],[32],[47],[48]\), Yang et al. defined Gaussian-Hermite moments of grayscale images, 3D shapes, RGB images and 2D vector fields. In fact, all of these moments are special instances of \((8)\).

### 3 The structural framework of RAGHMI

This section demonstrates how to systematically generate RAGHMI is general vector functions by means of two fundamental differential operators and one fundamental primitive.

#### 3.1 Invariant differential operators to IR

**Definition 1.** Let the symbol \( \nabla_i \) denote the gradient operator with respect to \( X_i = (x_1, x_2, \ldots, x_m)^T \), namely

\[
\nabla_i = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m} \right)^T
\]

Then, for \( m \) arbitrary points \( X_1, X_2, \ldots, X_m \), two fundamental differential operators \( \phi_{12} \) and \( \psi_{12} \ldots m \) can be defined as

\[
\phi_{12} = \nabla_1^T \cdot \nabla_2 = \sum_{\alpha=1}^{m} \frac{\partial^2}{\partial x_1^\alpha \partial x_2^\alpha}
\]

and

\[
\psi_{12} \ldots m = \left| (\nabla_1, \nabla_2, \ldots, \nabla_m) \right|
\]

where

- The symbol \( \left| \cdot \right| \) denotes the determinant of the given matrix.
- The function \( \varepsilon \) is the permutation of the set \( \{1,2,\ldots,m\} \), and the value in the \( i \)-th position after the reordering \( \varepsilon \) is denoted by \( \varepsilon_i \). The symbol \( S_m \) denotes the set of all such permutations.
- The signature of \( \varepsilon \), namely \( sgn(\varepsilon) \), is defined to be \(+1\) whenever the reordering given by \( \varepsilon \) can be achieved by successively interchanging two entries an even number of times, and \(-1\) whenever it can be achieved by an odd number of such interchanges.

**Lemma 1.** Suppose that \( X_i \) is transformed into \( Y_i = (y_1, y_2, \ldots, y_m)^T \) using \( IR = R \), where \( i = 1,2,\ldots,m \). Let differential operators \( \phi_{12} \) and \( \psi_{12} \ldots m \) be defined as

\[
\phi_{12} = \nabla_1^T \cdot \nabla_2
\]

\[
\psi_{12} \ldots m = \left| \begin{array}{cccc}
\nabla_1 & \nabla_2 & \cdots & \nabla_m \\
\end{array} \right|
\]

where

\[
\nabla_i = \left( \frac{\partial}{\partial y_{i1}}, \frac{\partial}{\partial y_{i2}}, \ldots, \frac{\partial}{\partial y_{im}} \right)^T
\]

Then, we have \( \phi_{12} = \phi_{12} \) and \( \psi_{12} \ldots m = \psi_{12} \ldots m \), where \( \phi_{12} \) and \( \psi_{12} \ldots m \) are defined by \((10)\) and \((11)\), respectively.

**Proof:** Since \( Y = R \cdot X \) and \( R^{-1} = R^T \), according to the chain’s rule of composite functions, the gradient operators \( \nabla’_i \) and \( \nabla_i \) are related by

\[
\nabla’_i = \left( \frac{\partial}{\partial y_{i1}}, \frac{\partial}{\partial y_{i2}}, \ldots, \frac{\partial}{\partial y_{im}} \right)^T
\]

\[
= \left( \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \ldots, \frac{\partial}{\partial x_{1m}} \right)^T
\]

\[
= R^{-1} \nabla_i = R^T \nabla_i
\]

Substituting the relations \( R^{-1} = R^T \), \( |R| = 1 \) and \( \nabla’_i = R^T \nabla_i \) into \((12)\), we can get

\[
\phi_{12} = \left( R^T \nabla_1 \right)^T R^T \nabla_2 = \nabla_1^T R R^T \nabla_2 = \nabla_1^T \nabla_2 = \phi_{12}
\]

\[
\psi_{12} \ldots m = \left| \left( R^T \nabla_1, \ldots, R^T \nabla_m \right) \right| = \left| R^T \left| (\nabla_1, \ldots, \nabla_m) \right| \right|
\]

\[
= \psi_{12} \ldots m
\]

The proof is completed. \( \square \)

Lemma 1 indicates that two fundamental differential operators \( \phi_{12} \) and \( \psi_{12} \ldots m \) are invariant to \( IR \).

**Definition 2.** Given a positive integer \( K \) and a scale parameter \( \sigma \), we can define the cumulative product of two fundamental differential operators with respect to \( X_i = (x_1, x_2, \ldots, x_m)^T \), where \( i = 1,2,\ldots,K \)

\[
D = \prod_{b_1,\ldots,b_m=1}^{K} (-\sigma \psi_{b_1 b_2 \cdots b_m}) q_{b_1 \cdots b_m} \prod_{a_1, a_2=1}^{K} (-\sigma \phi_{a_1 a_2}) p_{a_1 a_2}
\]

where \( p_{a_1 a_2} \) and \( q_{b_1 b_2 \cdots b_m} \) are non-negative integers, and each \( X_i (\nabla_i) \) must be involved at least once in \( D \). We just consider \( a_1 \leq a_2 \) because \( \phi_{a_1 a_2} = \phi_{a_2 a_1} \). If all elements of one column are identical with the elements of some other columns, the determinant is zero. And the interchange of any two columns of the determinant only changes its sign. Thus, for the determinant \( \psi_{b_1 b_2 \cdots b_m} \), it is meaningful to consider only \( b_1 < b_2 < \cdots < b_m \). As shown in \((10)\) and \((11)\), both \( \phi_{a_1 a_2} \) and \( \psi_{b_1 b_2 \cdots b_m} \) can be expressed as the homogeneous polynomials in partial derivative operators to \( X_i \). Obviously, the cumulative product of them also holds this property. In fact, the differential operator \( D \) takes the form of the following homogeneous polynomial of \( K \) degree

\[
D = \sum_{j=1}^{M} s_j \prod_{i=1}^{K} \left( \left( \sigma \right)^{p_{i1}^j + p_{i2}^j + \cdots + p_{im}^j} \left( \frac{\partial}{\partial x_{i1}^j} \right)^{p_{i1}^j} \left( \frac{\partial}{\partial x_{i2}^j} \right)^{p_{i2}^j} \cdots \left( \frac{\partial}{\partial x_{im}^j} \right)^{p_{im}^j} \right)
\]
where $M$ denotes the number of the product terms, the coefficients $s_j$ are non-zero integers, and for any $\alpha \in \{1, 2, ..., m\}$, the non-negative integer $p_i^l(\alpha)$ represents the order of the partial derivative operator to $x_i^l$. As mentioned above, we demand that each $X_i$, namely each gradient operator $\nabla_i$, must be involved at least once. This assures $\min \{\sum_{i=1}^{m} p_i^l(\alpha)\} \geq 1$. In fact, $T^l_i = \sum_{i=1}^{m} p_i^l(\alpha)$ is equal to the number of times $X_i(\nabla_i)$ appears in $D$. This implies that for any $j \in \{1, 2, ..., M\}$, we have $T^l_i = T^l_j$. In our paper, $\max \{T^l_i\} = \max \{T^l_i\}$ is called the order of $D$.

Lemma 2. Using the differential operator $D$ defined by (16), we can generate a homogeneous polynomial in terms of Gaussian-Hermite polynomials

$$W = G_2 \cdot D(G_1) = \sum_{j=1}^{M} s_j \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \tilde{H}_{p_i^l(\alpha)}(x_i^l; \sigma) (18)$$

where

$$G_1 = \prod_{i=1}^{K} \exp \left( -\frac{X_i^T X_i}{\sigma^2} \right) = \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \exp \left( -\frac{(x_i^l)^2}{2\sigma^2} \right)$$

$$G_2 = \prod_{i=1}^{K} \exp \left( \frac{X_i^T X_i}{2\sigma^2} \right) = \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \exp \left( \frac{(x_i^l)^2}{2\sigma^2} \right) (19)$$

Proof: According to (7) and (17), we have (20). The proof is completed.

3.2 Relative invariant primitive to RA

Definition 3. Given a general vector function $F(X)$ defined by (1) and $n$ arbitrary points $X_1, X_2, ..., X_n \in \Omega$, we define the fundamental primitive as

$$\Gamma_{12 \cdots n} = |(F(X_1), F(X_2), \cdots, F(X_n))|$$

$$= \sum_{\varepsilon \in S_n} \text{sgn}(\varepsilon)f_{\varepsilon_1}(X_1)f_{\varepsilon_2}(X_2)\cdots f_{\varepsilon_n}(X_n) (21)$$

Since the definition of the permutation $\varepsilon \in S_n$ is similar to $\varepsilon \in S_{m}$ used in (11), we skip it here.

Lemma 3. Let the vector function $F(X)$ be transformed using (2) into $G(Y)$. Suppose that $Y_i = (y_{i1}, y_{i2}, ..., y_{im})^T \in \Omega$ is the corresponding point of $X_i \in \Omega$, where $i = 1, 2, ..., n$. Then, we have

$$\Gamma'_{12 \cdots n} = |A| |\Gamma_{12 \cdots n}|$$

(22)

where

$$\Gamma'_{12 \cdots m} = |(G(Y_1), G(Y_2), \cdots, G(Y_n))|$$

(23)

Proof: According to (2), we have $G(Y_i) = A \cdot F(X_i)$ for any $i \in \{1, 2, ..., n\}$. Thus,

$$\Gamma'_{12 \cdots n} = |A| (F(X_1), F(X_2), \cdots, F(X_n))|$$

$$= |A| |(F(X_1), F(X_2), \cdots, F(X_n))|$$

$$= |A| |\Gamma_{12 \cdots n}|$$

(24)

The proof is completed.

Lemma 3 demonstrates that the fundamental primitive $\Gamma_{12 \cdots n}$ is invariant, more exactly, is relatively invariant to RA.

Definition 4. Given a general vector function $F(X)$ defined by (1) and an integer $K \geq n$, we can construct the cumulative product of the fundamental primitive with respect to $K$ arbitrary points $X_1, X_2, ..., X_K \in \Omega$

$$\prod_{c_1, c_2, \cdots, c_n = 1}^{K} (\Gamma_{c_1, c_2, \cdots, c_n})^{q_{c_1, c_2, \cdots, c_n}} (25)$$

where $q_{c_1, c_2, \cdots, c_n}$ are non-negative integers. Similar to (16), the properties of determinants allow us to consider only $c_1 < c_2 < \cdots < c_n$. Clearly, $P$ is also able to be expressed as a homogeneous polynomials in $f_{\beta}(X_i)$, where $\beta \in \{1, 2, ..., n\}$ and $i \in \{1, 2, ..., K\}$. We further request that $F(X_i)$ must be used once and only once in $P$. Hence, $q_{c_1, c_2, \cdots, c_n}$ can only equal 0 or 1. This constraint makes $P$ take the following form

$$P = \sum_{i=1}^{N} t_i \prod_{i=1}^{K} f_{\beta_i}(X_i) (26)$$

where $N$ is the number of product terms, the coefficients $t_i$ are non-zero integers and the subscript $\beta_i \in \{1, 2, ..., n\}$.

3.3 The construction of RAGHMIs

Let us now design a general structural frame of RAGHMIs based on the definitions and the lemmas in Section 3.1 and 3.2.

Definition 5. Given a general vector function $F(X)$ defined by (1), suppose that $K$ arbitrary points $X_1, X_2, ..., X_K \in \Omega$ and $K \geq n$. Then, we can define RAGHMI as

$$RAGHMI = \int_{\Omega^K} (W \cdot P) \prod_{i=1}^{m} \prod_{\alpha=1}^{m} dx_i^\alpha (27)$$

where $W$ and $P$ are given by (18) and (25), respectively. According to (18) and (26), $(W \cdot P)$ is able to be expressed as

$$W \cdot P = \sum_{i=1}^{N} t_i \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \hat{H}_{p_i^l(\alpha)}(x_i^l; \sigma)f_{\beta_i}(X_i) (28)$$

Thus, we have (29), which means that RAGHMI is a homogeneous polynomial of $K$ degree in terms of Gaussian-Hermite moments of $F(X)$. The order of the RAGHMI is the highest order of Gaussian-Hermite moments it depends upon, namely $\max \{\sum_{\alpha=1}^{m} p_i^l(\alpha)\}$. Obviously, it is the same as the order of the differential operator $D$ defined by (16), and is equal to the maximum number of times that $X_i(\nabla_i)$ appears in $D$.

Theorem 1. Let $F(X)$ be transformed using (2) into $G(Y)$, and $Y_i \in \Omega'$ is the corresponding point of $X_i \in \Omega$, where $i = 1, 2, ..., K$ and the integer $K \geq n$. Then, we have

$$RAGHMI' = |A| \sum_{c_1, c_2, \cdots, c_n}^{q_{c_1, c_2, \cdots, c_n}} RAGHMI (30)$$
\[ W = G_2 \cdot D(G_1) = G_2 \cdot \left\{ \sum_{j=1}^{M} s_j \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \left( -\sigma \right)^{p_{i,\alpha}} \exp \left( -\frac{x_{i,\alpha}^2}{\sigma^2} \right) \right\} \]
\[ = \sum_{j=1}^{M} s_j \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \left( -\sigma \right)^{p_{i,\alpha}} \exp \left( -\frac{x_{i,\alpha}^2}{2\sigma^2} \right) \]
\[ \frac{d^{p_{i,\alpha}}}{d(x_{i,\alpha})^{p_{i,\alpha}}} \exp \left( -\frac{x_{i,\alpha}^2}{\sigma^2} \right) = \sum_{j=1}^{M} s_j \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \hat{H}_{p_{i,\alpha}}(x_{i,\alpha}; \sigma) \]
\[ (20) \]
\[ \text{RAGHMI} = \int_{\Omega^K} \cdots \int_{\Omega^K} \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \hat{H}_{p_{i,\alpha}}(x_{i,\alpha}; \sigma) f_{\beta i}(X_i) \]
\[ = \sum_{l=1}^{N} \sum_{j=1}^{M} \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \hat{H}_{p_{i,\alpha}}(x_{i,\alpha}; \sigma) f_{\beta i}(X_i) \prod_{\alpha=1}^{m} dx_{\alpha} \]
\[ (29) \]

where the RAGHMI is defined by (27) and
\[ \text{RAGHMI}' = \int_{\Omega^K} \cdots \int_{\Omega^K} \left( W \cdot P' \right) \prod_{i=1}^{K} \prod_{\alpha=1}^{m} dy_{\alpha} \]
\[ (31) \]

The definitions of all functions used in the RAGHMI are similar to the corresponding ones in the RAGHMIs. Specifically,
\[ W' = G_2' \cdot D'(G_1') \]
\[ P' = \prod_{c_1, c_2, \ldots, c_n=1}^{K} \left( \Gamma'_{c_1 c_2 \cdots c_n} \right) q_{c_1 c_2 \cdots c_n} \]
\[ (32) \]

where
\[ D' = \prod_{b_1, \ldots, b_m=1}^{K} \left( -\sigma \psi_{b_1 \cdots b_m}^{c_{b_1 \cdots b_m}} \right) \prod_{a_1, a_2=1}^{K} \left( -\sigma \phi_{a_1 a_2}^{c_{a_1 a_2}} \right) \]
\[ (33) \]

and
\[ G_1' = \prod_{i=1}^{K} \exp \left( -\frac{Y_i^T Y_i}{\sigma^2} \right) = \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \exp \left( -\frac{\left( y_{i,\alpha}^2 \right)}{\sigma^2} \right) \]
\[ G_2' = \prod_{i=1}^{K} \exp \left( \frac{Y_i^T Y_i}{2\sigma^2} \right) = \prod_{i=1}^{K} \prod_{\alpha=1}^{m} \exp \left( \frac{\left( y_{i,\alpha}^2 \right)}{2\sigma^2} \right) \]
\[ (34) \]

Note that \( \psi_{b_1 \cdots b_m}^{c_{b_1 \cdots b_m}} \) and \( \Gamma'_{c_1 c_2 \cdots c_n} \) are defined by (12) and (23), respectively.

**Proof:** First, by substituting (15) into (33), we can get \( D' = D \). Meanwhile, since \( Y_i = R \cdot X_i \) and \( R^{-1} = R^T \), we have
\[ Y_i^T Y_i = X_i R^T R X_i = X_i^T R^{-1} R X_i = X_i^T X_i \]
\[ (35) \]
meaning that \( G_1' = G_1 \) and \( G_2' = G_2 \). Hence, the following relation can be obtained
\[ W' = G_2' \cdot D'(G_1') = G_2 \cdot D(G_1) = W \]
\[ (36) \]

The proof is completed. \( \Box \)

Theorem 1 shows that all of RAGHMI's generated by (27) are relatively invariant to arbitrary RA defined by (2). To eliminate \( |A|^{q_{c_1 c_2 \cdots c_n}} \) and obtain an absolute invariant,
we can normalize a relative invariant by other relative invariants or by proper sum/power of them so that Jacobians get canceled. In fact, this kind of normalization approach has been used in many previous papers [34], [49], [50], [53].

As far as we know, it is the first structure frame for deriving orthogonal moment invariants of general vector functions. In [53], Hao et al. introduced the method to construct geometric moment invariants of general vector functions to $TA$. In theory, all polynomial bases are equivalent, because they generate the same space of functions. Thus, any geometric moments can be expressed in terms of orthogonal moments and vice versa [46]. This seems to indicate that we can directly derive RAGHMIs based on Hao’s method. However, this property does not apply to Gaussian-Hermite moments. In fact, due to the Gaussian function, any Gaussian-Hermite moments can not be expressed as the finite combinations of geometric moments.

4 The instances of RAGHMIs

Based on Definition 5, we can generate all possible RAGHMIs of a certain type of $F(X)$ up to the degree $K$ and the order $O$. This procedure can be described as following four steps. Note that some symbolic computation software can help us to achieve each of them.

- **Step 1**: Since we demand the degree of each RAGHMI is less than or equal to $K$, there are at most $K$ points which can be utilized to construct $W$ and $P$. We first generate all possible fundamental differential operators $\phi_{a_1 a_2}$ and $\psi_{b_1 b_2 \ldots b_n}$ with respect to $X_1, X_2, \ldots, X_K$, and the set of them is denoted as $U$. Similarly, we can also derive the set $V$ of all possible fundamental primitives $\Gamma_{c_1 c_2 \ldots c_n}$.

- **Step 2**: Based on the sets $U$ and $V$, we can further generate $\bar{U} = \bigcup_{r=1}^{R_1} U^r$ and $\bar{V} = \bigcup_{r=1}^{R_2} V^r$, where $U^r$ ($V^r$) represents the cumulative Cartesian product of $U$ ($V$). Each of the elements in $U^r$ ($V^r$) contains $r$ fundamental differential operators (fundamental primitives), which can be used for constructing the differential operator $D$ in Definition 2 ($P$ in Definition 4). Note that two constrains can be employed to determine the values of $R_1$ and $R_2$, respectively. First, the order of each RAGHMI must be less than or equal to $O$. Thus, for each of the elements in the set $U^{R_1+1}$, we demand that at least one point $X_i$ ($\nabla_i$) appears more than $O$ times, where $i \in \{1, 2, \ldots, K\}$. In addition, as stated in Definition 4, each of $F(X_i)$ must be used only once in $P$. Therefore, we also demand that all elements in $V^{R_2+1}$ do not satisfy this condition.

- **Step 3**: Obviously, each of the elements in $\bar{U} \times \bar{V}$ represents a construction formula of RAGHMIs, namely (27). Since a pair of $W$ and $P$ must be constructed by using the same points, those elements that do not meet this requirement can also be discarded. Then, we can derive the expansions of RAGHMIs by means of (17), (20), (26), (28) and (29).

- **Step 4**: The expansions of some construction formulas are identically zero, and one invariant can also be generated repeatedly. We further eliminate these meaningless RAGHMIs.

However, the above procedure does not guarantee that there are no dependent invariants in the generated set, which means some of RAGHMIs are the functions of the others. In many practical applications, a single real-valued RAGHMI does not provide enough discriminative ability, and all of RAGHMIs in the set must be used simultaneously as a feature vector. Previous research has found that dependent moment invariants not only contribute nothing to the discrimination power of the vector, but also increase the dimensionality of feature space and the computational time [46]. Thus, identifying and discarding dependent RAGHMIs in the set is highly desirable. In [54], we have designed various approaches to eliminate linear dependencies, polynomial dependencies and functional dependencies among the elements in a set. Based on them, we get the complete and functionally independent sets of the RAGHMIs up to the degree $K$ and the order $O$ for RGB images ($K = 3, O = 3$), 2D vector fields ($K = 2, O = 3$) and 3D flow fields ($K = 3, O = 3$). The structural information of the invariants in these sets, including $K, O, D$ and $P$, and their explicit expansions can be found in the Supplementary Materials. As an instance, Table 2 shows the expansions of seven RAGHMIs in the set of 2D vector fields. According to Theorem 1, for any invariants in these sets, we have $RAGHMI = |A| RAGHMI$, because all of them make use of only one fundamental primitive ($\Gamma_{12}$ or $\Gamma_{123}$) to construct $P$. Obviously, the sum of these invariants also holds this property. Hence, we can normalize each of RAGHMI by the sum of all invariants in the same set, which eliminates the constant $|A|$ and makes RAGHMIs absolutely invariant to $RA = (R; A, T)$.

5 Experiments and Discussion

In this section, we conduct experiments on various types of multi-channel data, including RGB images, 2D vector fields and 3D flow fields, to verify the stability and discriminability of RAGHMIs generated in Section 4, and to evaluate the robustness of them to additive noises and their
recognition ability in RGB image classification, vortex detection and template matching. Also, most of existing moment invariants of vector functions are chosen for comparison.

5.1 The classification of RGB images

As shown in the Supplementary Materials, when setting \( K \leq 3 \) and \( O \leq 3 \), we derive thirteen \( \text{RAGHMI} \)s of RGB images, which form a complete and independent set. In this subsection, we test the performance of them in RGB image classification. Ten color images were randomly selected from the USC-SIPI image dataset (https://sipi.usc.edu/database/), which were denoted as \( \text{Img}_i \), where \( i = 1, 2, ..., 10 \). They were used as training images, each scaled to \( 257 \times 257 \) pixels. Also, for each image, we changed the origin of image coordinate system to the center \((129, 129)\), and only the content located in the area of the inscribed circle \((x - 129)^2 + (y - 129)^2 \leq 128\) was used for feature extraction (see Fig.4a).

As stated previously, \( RA \) can simulate realistic deformations of color images caused by viewpoint changes and illumination changes. Hence, using the formula (3), we synthetically generated sixty \( RA \) versions \( \text{Img}_i^j \) of \( \text{Img}_i \), where \( j = 1, 2, ..., 60 \). Specifically, each training image was first rotated around its origin by angles from 0 to \( 2\pi /60 \), and then sixty rotated versions were further transformed using sixty color affine transformations, respectively. These 3D affine transformations of color space, namely \( OA = (A, T) \), were randomly generated. In order to insure that \((R(x, y), G(x, y), B(x, y))\) was not mapped outside of the RGB cube \([0, 1] \times [0, 1] \times [0, 1]\) as far as possible, we added some constraints to the parameters of \( OA \). In fact, according to QR decomposition, the \( 3 \times 3 \) matrix \( A \) can be expressed as \( R_x \cdot R_y \cdot R_z \cdot U \), where the first three ones represent the rotations about \( x \)-axis, \( y \)-axis and \( z \)-axis, respectively, and \( U \) is defined as

\[
U = \begin{pmatrix}
S_z & M_1 & M_2 \\
0 & S_y & M_3 \\
0 & 0 & S_z
\end{pmatrix}
\]  

(40)

When generating these matrices, we damended that \( \theta_x, \theta_y, \theta_z \in [-\pi/10, \pi/10] \), which are the angles of \( R_x, R_y \) and \( R_z \), respectively; \( S_x, S_y, S_z \in [0.7, 0.9] \); \( M_1, M_2, M_3 \in [-0.1, 0.1] \). For the outer translation \( T \), we had \( t_1, t_2, t_3 \in [-0.1, 0.1] \). Note that some \( RA \) versions \( \text{Img}_2 \) have been shown in Fig.2a.

The above process yielded \( 10 \times 60 = 600 \) testing images. Using the formula (8), the numerical values of thirteen \( \text{RAGHMI} \)s were calculated from each image (including training images and testing ones). Before that, we had subtracted per-channel mean from the image, which eliminates three parameters of the outer translation \( T \). In order to observe how the scale parameter \( \sigma \) influences the performance of \( \text{RAGHMI} \)s, we set \( \sigma = 30, 40, ..., 80 \). When setting \( \sigma < 30 \approx 257/9 \), the quantity of \( \hat{H}_p(x; \sigma) \) vanishes so rapidly that the polynomial basis cannot cover the whole domain \([-128, 128]\), which leads to image information loss. On the other hand, if \( \sigma > 80 \approx 257/3 \), \( \hat{H}_p(x; \sigma) \) has a large number of non-zero values outside \([-128, 128]\), which also decreases the recognition ability of \( \text{RAGHMI} \). As stated in Section 4, the values of these invariants were normalized by the sum of them. Note that thirteen normalized \( \text{RAGHMI} \)s are linearly dependent, because the sum of them is equal to 1. Thus, we discard the last one, and the first twelve of them were used as the final feature vector of RGB images.

First, the mean relative error \( (MRE) \) is used to evaluate the numerical stability of each normalized \( \text{RAGHMI} \), which defined by

\[
MRE = \left( \frac{1}{600} \sum_{i=1}^{10} \sum_{j=1}^{60} E_{ij} \right) \times 100\%
\]

(41)

\[
E_{ij} = \left| \frac{\text{RAGHMI}(\text{Img}_i^j) - \text{RAGHMI}(\text{Img}_i)}{\text{RAGHMI}(\text{Img}_i)} \right|
\]

In Fig.5a, we visualized the \( MRE \) of twelve normalized invariants with different \( \sigma \). It can be observed that most of them are less than 1%, and even the highest one is only 1.47%. These small errors are caused by the image resampling. This further illustrates that \( \text{RAGHMI} \)s of RGB images derived by our method are actually invariant to \( RA \).

Then, to verify the discriminability of these normalized \( \text{RAGHMI} \)s, we utilized the Nearest Neighbor classifier for image classification. The Chi-Square distance was used to measure the similarity of two images in the space of features. In fact, the magnitudes of different moment invariants are not the same. As a result, the Euclidean distance between two feature vectors will be dominated by certain invariants with large magnitudes instead of all of them. Previous studies have proved that the Chi-Square distance can solve this problem naturally [7], [34], [55]. In addition to \( \text{RAGHMI} \)s, two moment invariants were chosen for benchmarking.

1) \( \text{GPDs} \): Twenty one geometric moment invariants of color images proposed in [39]. They have invariance to both 2D affine transform of spatial coordinates and 3D diagonal transform of color space.
As mentioned previously, this transform model is a kind of restricted \( TA \).

2) \( SCAMIs \): Twenty four geometric moment invariants of color images to \( TA \), which were proposed in [34].

In fact, from what we know, \( GPDs \) and \( SCAMIs \) are the only two kinds of image moment invariants in previous literature, which are invariant simultaneously to viewpoint changes and illumination changes. The classification accuracy rates from using these invariants are listed in the first column of Table 3. For any \( \sigma \in \{30, 40, ..., 80\} \), twelve normalized \( RAGHMIs \) achieved 100% classification accuracy. Using \( SCAMIs \), we derived the same result. This is because both of them are invariant to \( RA \) in theory. The rate 95.83% from using twenty one \( GPDs \) is slightly lower than \( RAGHMIs \) and \( SCAMIs \), because they are just invariant to some special \( OA \).

Further, we tested the robustness of \( RAGHMIs \) of RGB images to additive noises. Specifically, we added a zero-mean Gaussian noise to testing images, and then repeated the classification experiment in a similar fashion. For the zero-mean Gaussian noise, we set the standard deviation \( \sigma_G = 0.0005 \times K \), where \( K = 1, 2, ..., 6 \). With the increase of \( \sigma_G \), the signal-to-noise ratio (SNR) gradually changed from 22.24 dB to 8.34 dB. Fig. 4b (top row) shows some degraded instances on different noise levels of the same testing image. The classification results are listed in Table 3. It is obvious that twelve normalized \( RAGHMIs \) significantly outperform \( SCAMIs \) and \( GPDs \) in terms of accuracy. We can observe that \( SCAMIs \) and \( GPDs \) are extremely sensitive with respect to noise. Even if testing images are disturbed by low Gaussian noise \( (\sigma_G = 0.005, SNR=22.24 \text{ dB}) \), the success rate from using \( SCAMIs \) drops from 100% to 64.83%. As stated earlier, this is mainly because they are constructed based on geometric moments. In contrast, \( RAGHMIs \) are much more robust with respect to Gaussian noise. Even for heavy noise \( (\sigma_G = 0.030, SNR=8.34 \text{ dB}) \), they still achieved 95.83% accuracy rate \( (GPDs: 15.00\%, SCAMIs: 19.33\%) \). Note that the robustness of \( RAGHMIs \) calculated using small \( \sigma \), such as 10 and 20, is worse than those with large \( \sigma \). In fact, the Gaussian function involved in \( \tilde{H}_p(x; \sigma) \) naturally smooths the images and reduces the level of noise, and as the scale \( \sigma \) increases, the degree of smoothing increases. As shown in Table 4, we also conducted the classification on testing images disturbed by a Salt & Pepper noise (see the bottom row of Fig.4b), where the noise density \( r = 0.02 \times K \) and \( K = 1, 2, ..., 6 \). The performance of three features slightly rises while \( RAGHMIs \) is also the best one. For example, when setting \( r = 0.12 \) (SNR=12.80), the accuracy rates from using \( SCAMIs \), \( GPDs \) and \( RAGHMIs (\sigma = 50) \) are 40.33%, 54.67% and 98.67%, respectively.

### 5.2 Vortex detection in 2D vector fields

For seven \( RAGHMIs \) of 2D vector fields listed in Table 2, this subsection mainly demonstrates their applicability in vortex detection, which is an important task in the field of fluid dynamics engineering. For this purpose, we downloaded a 1501-frame video of 2D cylinder flow from https://cgl.ethz.ch/Downloads/Data/ScientificData/cylinder2d_vti.zip, which shows the time-development of a von Kármán vortex street. In each of the frames, there are many similar patterns of swirling vortices with different orientations. For example, Fig.6 shows the line integral convolution (LIC) [56] of the frame 500. The resolution of all frames is 80 × 640.

Using the procedure stated in Section 5.1, we first evaluated the stability of six normalized \( RAGHMIs \) of 2D vector fields. As shown in Fig.7a, ten \( 41 \times 41 \) templates were randomly selected from the frame 500, which were denoted as \( 2VF_i \), where \( i = 1, 2, ..., 10 \). Then, for each \( 2VF_i \), sixty \( RA \) versions were generated. Four transformed examples of \( 2VF_2 \) are shown in Fig.7b. Unlike generating \( RA \) of RGB images, we did not add any constraints to the parameters of \( RA \) of 2D vector fields. When setting the scale parameter \( \sigma = 4, 6, ..., 14 \) (4 \( \approx \) 41/9 and 14 \( \approx \) 41/3), the MRE of six normalized \( RAGHMIs \) can be seen in Fig.5b. We can find that all of them are less than 1.5%, which confirms the invariance of these \( RAGHMIs \) to \( RA \) of 2D vector fields.

Then, our task is to find all vortices with different orientations in the frames 500 ~ 1500. We selected a \( 65 \times 65 \) template with a typical vortex from the frame 500 (see Fig.6), and calculated six normalized \( RAGHMIs \) from the
TABLE 3: The classification accuracy from various moment invariants on RGB images disturbed by Gaussian noise.

| SNR | GPDs [39] | SCAMIs [34] | RAGHMIs(σ = 30) | RAGHMIs(σ = 40) | RAGHMIs(σ = 50) | RAGHMIs(σ = 60) | RAGHMIs(σ = 70) | RAGHMIs(σ = 80) |
|-----|-----------|--------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.00 | 95.83     | 100.00       | 100.00          | 100.00          | 100.00          | 100.00          | 100.00          | 100.00          |
| 0.02 | 94.17     | 94.67        | 98.67           | 98.67           | 98.67           | 98.67           | 98.67           | 98.67           |
| 0.04 | 51.67     | 91.67        | 98.00           | 97.00           | 97.17           | 96.50           | 95.00           | 93.50           |
| 0.06 | 37.00     | 89.00        | 89.33           | 85.00           | 94.00           | 97.67           | 96.00           | 94.00           |
| 0.08 | 28.17     | 83.00        | 89.33           | 85.00           | 94.00           | 97.67           | 96.00           | 94.00           |
| 0.10 | 19.33     | 54.67        | 50.00           | 50.00           | 50.00           | 50.00           | 50.00           | 50.00           |
| 0.12 | 15.00     | 19.33        | 19.33           | 19.33           | 19.33           | 19.33           | 19.33           | 19.33           |

TABLE 4: The classification accuracy from various moment invariants on RGB images disturbed by Salt & Pepper noise.

| r   | SNR |    |    |    |    |    |    |    |
|-----|-----|----|----|----|----|----|----|----|
|     | 0.00| 62.99 dB | 17.50 dB | 15.79 dB | 14.55 dB | 13.60 dB | 12.80 dB |    |
| 0.02| 95.83 | 94.67 | 90.33 | 80.17 | 72.33 | 61.83 | 54.67 |    |

![Images](image1.png)

Fig. 7: Some examples of 2D vector fields used for testing the stability of six normalized RAGHMIs.

template by setting $\sigma_G = 17$ (65/9 < 17 < 65/3). In a similar fashion, the values of these RAGHMIs were calculated from a 65 × 65 neighborhood of each location in 1000 frames. We made use of the Chi-Square distance to measure the similarity between the neighborhood and the template in the space of features. In previous papers [47], [48], [49], [50], all local minima of the distance below a user-defined threshold were regarded as the locations of vortices in a frame. However, the approach discards those locations that have smaller distances (namely are more similar to the template) but do not meet the local minimum condition. This makes the corresponding results unable to fully reflect the stability and discriminability of moment invariants. To resolve this issue, in this paper, we sorted all points of a frame (80 × 640 = 51200) in ascending order by the Chi-Square distance, and then marked the first two thousand ones on the frame. Also, three existing moment invariants of 2D vector fields were selected for comparison.

1) **TAGMIs**: Eight geometric moment invariants to TA, which were proposed in [50]. They create a complete and independent set of the second and third-order TAGMI.

2) **TRGHI Ms**: Nine Gaussian-Hermite moment invariants up to the third order, which are invariant to TR [48].

3) **TRCMIs**: Five complex moment invariants to TR proposed in [40].

It is difficult to evaluate the detection results quantitatively, because we do not know the ground-truth locations of vortices on each frame. Fortunately, previous researchers have found that visual inspection of the resulting videos provides a good insight into the performance of the features [47], [48], [49], [50]. The videos showing the detected locations from using various moment invariants can be downloaded from the link https://drive.google.com/drive/folders/1JYyi0jMibEHsKrr9ZOAq-u9GebOnH73q?usp=sharing. As an instance, the detected results obtained on the frame 1000 are shown in Fig.8. It can be seen that RAGHMIs and TAGMIs correctly located most of vortices and outperform TRGHI Ms and TRCMIs in this task. This indicates that RA and TA are more suitable than TR to model real deformations of 2D vector fields. There are several undetected vortices which can be regarded as the spatial scaling versions of the template. As mentioned previously, the main reason for that is all moment invariants were calculated on a circular region with a fixed radius (65) around each of the locations in a frame. To test the robustness with respect to noise, we added a zero-mean Gaussian noise to each of the frames, where $\sigma_G = 0.050$ and SNR=8.947 dB, and then repeated the detection experiment again. The resulting videos can also be found at the above link. The visual results on the frame 1000 are shown in Fig.9. By comparing Fig.8 and 9, we can clearly observe that the performance of TAGMIs, TRGHI Ms and TRCMIs decreased significantly while RAGHMIs still achieve an excellent detection result.

Note: The images are not included in the text.
that both $\textit{RAGHMI}s$ and $\textit{TRGHMI}s$ are constructed based on Gaussian-Hermite moments up to the third-order. In theory, $\textit{TRGHMI}s$ should also be insensitive to noise. However, this was not reflected in our experiment. A possible explanation is that they are just invariant to simple $\textit{TR}$ model, which limits their numerical stability to noise when complicated transformations ($\textit{RA}$ or $\textit{TA}$) and noise appear simultaneously on the data.

5.3 Template matching for 3D flow fields

In Section 4, we also derive fourteen orthogonal moment invariants of 3D flow fields (see the Supplementary Materials). Thus, in the last experiment, 3D flow fields were employed to evaluate the performance of these $\textit{RAGHMI}s$ in template matching. We downloaded the numerical simulations of a 3D flow around a half cylinder at 151 times ($t$) from https://cgl.ethz.ch/Downloads/Data/ScientificData/halfcylinder3d-Re320.vti.zip. When $t = 11.90$s, Fig.10a shows the LIC of three basic reference planes of the 3D flow field. Note that all the experiments below were carried out on this flow. Similar to Section 5.1 and 5.2, we first verified the stability of $\textit{RAGHMI}s$ of 3D vector fields to $\textit{RA}$. As shown in Fig.10b, we randomly selected ten $41 \times 41 \times 41$ templates from the 3D flow and generated sixty $\textit{RA}$ versions of each of them. By setting the scale parameter $\sigma = 4, 6, ..., 14$, the $\textit{MRE}$ of thirteen normalized $\textit{RAGHMI}s$ were calculated from ten templates and their $\textit{RA}$ versions (see Fig.5c). We can see that the error of each invariant is less than 3% regardless of $\sigma$ was chosen to calculate its value. Clearly, the stability of these invariants is slightly worse than $\textit{RAGHMI}s$ of 2D vector fields and RGB images. This is because the expansions of them are more complicated.

Then, we further selected one thousand templates from the 3D flow, which were denoted as $3VF_i$, where $i = 1, 2, ..., 1000$. For each $3VF_i$, we randomly generated a $\textit{RA}$ version $3VF'_i$. The template matching is to establish correct matches between one thousand $3VF_i$ and one thousand $3VF'_i$ based on feature vectors. We also used the Chi-Square distance to measure the similarity between $3VF_i$ and $3VF'_i$ in the feature space. Besides thirteen normalized $\textit{RAGHMI}s$, eighteen geometric moment invariants of 3D vector fields proposed in [53] were chosen for a comparison, which are invariant to $\textit{TA}$. They were denoted as $\textit{TAGMI}s$. As mentioned earlier, to our knowledge, $\textit{TAGMI}s$ are the only classical moment invariants of 3D flows in the previous literature. To increase the difficulty of this task and show robustness, each $3VF'_i$ was further corrupted by additive zero-mean Gaussian noise. Note that the standard deviation $\sigma_G$ of the Gaussian noise changed from 0.005 to 0.030 while SNR declined from 37.61 dB to 22.03 dB. For each noise level, we ran the experiment again. The number of correct matches from using $\textit{RAGHMI}s$ and $\textit{TAGMI}s$ are listed in Table 5. It is obvious that $\textit{RAGHMI}s$ provide a higher robustness to the white Gaussian noise. For example, when setting $\sigma_G = 0.005$ (SNR=37.61 dB), we mismatched nearly half of the templates by means of $\textit{TAGMI}s$, while using $\textit{RAGHMI}s(\sigma = 10)$, only 98 ones were mismatched. Also, we can find that $\textit{RAGHMI}s$ calculated using larger $\sigma$ are
more insensitive to noise. These results are consistent with those in Sections 5.1 and 5.2.

6 Conclusions

In this paper, we propose a unified framework for deriving Gaussian-Hermite moment invariants of general vector functions to rotation-affine transform, and introduce how to utilize the frame to generate all possible instances of RAGHMI\(s\) of a certain type of vector function. For commonly used vector-valued data, including RGB images, 2D and 3D flow fields, the complete and independent sets of RAGHMI\(s\) up to low order and low degree are generated. To illustrate the stability and discriminability of RAGHMI\(s\), we conducted a series of numerical experiments on synthetic and popular vector-valued datasets, and also verified their sensitivity to additive noise. For a comparison, we selected nearly all of moment invariants of vector functions published in previous papers. Our results clearly show that RAGHMI\(s\) have better performance in RGB image classification, vortex detection in 2D vector fields and template matching of 3D flow fields.

In the future, we plan to design some approaches to enhance the invariance of RAGHMI\(s\) to spatial scaling, and further explore the application of them in the fields of computer vision, pattern recognition and visualization.

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