Critical traveling waves in a diffusive disease model

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Abstract
In this paper, the existence of a non-trivial, positive and bounded critical traveling wave solution of a diffusive disease model, whose reaction system has infinity many equilibria, is obtained for the first time. This gives an affirmative answer to an open problem left in [X. Wang, H. Wang, J. Wu, Traveling waves of diffusive predator-prey systems: disease outbreak propagation, Discrete Contin. Dyn. Syst. Ser. A 32 (2012) 3303-3324]. Our result shows that the critical traveling wave in this model is a mixed of front and pulse type.

Keywords: diffusive disease model, critical traveling wave, reaction-diffusion equation
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1. Introduction
Recently, Wang et al. [16] considered a diffusive disease model

S_i(x,t) = d_i S_{ix}(x,t) - \frac{\beta S_i(x,t) I(x,t)}{S(x,t) + I(x,t)},
I_i(x,t) = d_2 I_{ix}(x,t) + \frac{\beta S_i(x,t) I(x,t)}{S(x,t) + I(x,t)} - \gamma I(x,t),
R_i(x,t) = d_3 R_{ix}(x,t) + \gamma I(x,t),

(1.1)

where S(x,t), I(x,t) and R(x,t) are the densities of susceptible, infective and removed individuals at location x and time t, respectively. The parameters d_i > 0 (i = 1, 2, 3) denote the spatial motility of each class, \beta > 0 is the transmission coefficient and \gamma > 0 refers to the recovery rate. Since the first two equations in (1.1) form a closed system, they only studied the subsystem of

S_i(x,t) = d_i S_{ix}(x,t) - \frac{\beta S_i(x,t) I(x,t)}{S(x,t) + I(x,t)},
I_i(x,t) = d_2 I_{ix}(x,t) + \frac{\beta S_i(x,t) I(x,t)}{S(x,t) + I(x,t)} - \gamma I(x,t).

(1.2)

The traveling wave profile of (1.2) is given by the system of second-order ordinary differential equations:

c S'(\xi) = d_1 S''(\xi) - \frac{\beta S(\xi) I(\xi)}{S(\xi) + I(\xi)},
\gamma I(\xi) = d_2 I''(\xi) + \frac{\beta S(\xi) I(\xi)}{S(\xi) + I(\xi)} - \gamma I(\xi),

(1.3)

where \xi = x + ct is the wave variable and c is the wave speed. They proved that if the basic reproduction number R_0 := \beta/\gamma > 1, then there exists a critical wave speed c^* := 2 \sqrt{d_2(\beta - \gamma)} such that for each c > c^*, system (1.1) admits a non-trivial and non-negative traveling wave solution (S(\xi), I(\xi)) which satisfies the asymptotic boundary conditions

S(-\infty) = S_{-\infty}, \quad S(\infty) = S_\infty < S_{-\infty}, \quad I(\pm \infty) = 0,

(1.4)

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where $S_{-\infty} > 0$ is a given constant and $S_\infty \geq 0$ is some existing constant. They also showed that if $R_0 \leq 1$ or $c < c^*$, then system (1.1) has no non-trivial and non-negative traveling wave solutions. When $R_0 > 1$ and $c = c^*$, the existence of traveling wave solutions for (1.2) was left as an open problem in [16]. Very recently, Wang et al. [14] employed a limiting argument to solve this problem. Unfortunately, their proof of the non-triviality of the limit is problematic because they used the integrability of $I$ over $\mathbb{R}$ before they hadn't found the asymptotic boundary of the critical traveling wave (see [14], Lemma 2.1 and p. 416). In view of this, the problem on the existence of critical traveling wave solutions for (1.2) remains open.

Traveling waves, especially the ones with the minimal/critical speed, play an important role in the study of disease transmission. The minimal wave speed of a disease is a crucial threshold to predict whether the disease propagates and how fast it invades. Until recently, there has been a number of work devoting to the existence of super-critical and critical traveling waves in diffusive epidemic models [1,31]. However, all the investigation of critical traveling waves are limited to the models whose reaction systems have one disease-free equilibrium and one endemic equilibrium. Note that the reaction system of (1.2) has infinity many equilibria, then the shooting method coupled with LaSalle’s invariance principle [1] and the method of dynamic systems [3,24] are not applicable to study the existence of critical traveling wave solutions for (1.2). Also the standard limiting argument [10,11,19,20,26,27,29] fails since the $I$-component in (1.2) is not monotone.

In the present paper, motivated by [6,16], we intend to solve the problem on the existence of critical traveling wave solutions for (1.2). The main result is stated as follows.

**Theorem 1.1.** Suppose that $R_0 > 1$ and $c = c^*$. Then system (1.2) admits a non-trivial, positive and bounded traveling wave solution $(S_\ast(\xi), I_\ast(\xi))$ satisfying the asymptotic boundary conditions (1.4) and the following properties.

1. $S_\ast(\xi)$ is strictly decreasing in $\mathbb{R}$ and
   \[ 0 < S_\ast(\xi) < S_{-\infty} \quad \text{for} \quad \xi \in \mathbb{R}. \]

2. $I_\ast(\xi) = O(-\xi e^{c^*})$ as $\xi \to -\infty$ and
   \[ 0 < I_\ast(\xi) < \min \left\{ \frac{(\beta - \gamma)S_{-\infty}}{\gamma}, \frac{2c^*(S_{-\infty} - S_\infty)}{\sqrt{4c^2 + 4d\gamma + c^4}} \right\} \quad \text{for} \quad \xi \in \mathbb{R}. \]

3. \[
\int_{-\infty}^{\infty} I_\ast(\xi) d\xi = \frac{\beta}{\gamma} \int_{-\infty}^{\infty} \frac{S_\ast(\xi)I_\ast(\xi)}{S_\ast(\xi) + I_\ast(\xi)} d\xi = \frac{c^*(S_{-\infty} - S_\infty)}{\gamma}.
\]

**Remark 1.1.**

1. Theorem 1.1 shows that the critical traveling wave in (1.2) is a mixed of front and pulse type.

2. To our knowledge, Theorem 1.1 gives the first result on the existence of positive critical traveling wave solutions for diffusive disease models whose reaction systems have infinity many equilibria.

3. Wang et al. [16] obtained that the super-critical traveling wave solutions of (1.2) are non-negative and the $S$-component is monotonically decreasing in $\mathbb{R}$. While Theorem 1.1 indicates that the critical traveling wave solution of (1.2) is positive and the $S$-component is strictly decreasing in $\mathbb{R}$. In fact, one can apply the method adopted by us to improve the corresponding results in [16].

The remainder of this paper is organized as follows. Section 2–Section 4 are devoted to the proof of Theorem 1.1. In Section 2, the existence of a critical traveling wave solution $(S_\ast(\xi), I_\ast(\xi))$ of (1.2) is established by Schauder’s fixed point theorem. In Section 3, the asymptotic boundary of $(S_\ast(\xi), I_\ast(\xi))$ at minus infinity is found with the aid of squeeze theorem. While the asymptotic boundary of $S_\ast(\xi)$ at plus infinity is derived from the monotonicity and boundedness of $S_\ast(\xi)$ in $\mathbb{R}$ and the asymptotic boundary of $I_\ast(\xi)$ at plus infinity is shown by the integrability of $I_\ast(\xi)$ and the boundedness of $I_\ast(\xi)$ in $\mathbb{R}$. In Section 4, the positivity and upper bound of $(S_\ast(\xi), I_\ast(\xi))$ are obtained via the strong maximum principle and auxiliary function method.
2. Existence of critical traveling wave solution

In this section, we will establish the existence of a critical traveling wave solution for (1.2) by Schauder’s fixed point theorem.

Linearizing the second equation in (1.3) at $(S_{-\infty}, 0)$, we have

$$d_2 I''(\xi) - c I'(\xi) + (\beta - \gamma) I(\xi) = 0.$$  

Letting $I(\xi) = e^{\xi \lambda}$, we then get the characteristic equation

$$\Delta(\lambda, c) := d_2 \lambda^2 - c \lambda + \beta - \gamma = 0.$$  

Obviously, when $c = c^* = 2 \sqrt{d_2 (\beta - \gamma)}$, equation (2.1) has a unique real root

$$\lambda^* = \frac{c^*}{2d_2} = \sqrt{\frac{\beta - \gamma}{d_2}}.$$

Now we define four non-negative, continuous and bounded functions in $\mathbb{R}$.

$$\overline{S}(\xi) := S_{-\infty},$$

$$\overline{T}(\xi) := \min \left\{ - L_1 \xi e^{\xi \epsilon}, M \right\} = \begin{cases} - L_1 \xi e^{\xi \epsilon}, & \xi \leq \xi_1, \\ M, & \xi > \xi_1, \end{cases}$$

$$\underline{S}(\xi) := \max \left\{ S_{-\infty} \left( 1 - \frac{1}{\epsilon} e^{\xi \epsilon} \right), 0 \right\} = \begin{cases} S_{-\infty} \left( 1 - \frac{1}{\epsilon} e^{\xi \epsilon} \right), & \xi \leq \xi_2, \\ 0, & \xi > \xi_2, \end{cases}$$

$$\underline{T}(\xi) := \max \left\{ - L_1 \xi - L_2 (-\xi)^\gamma e^{\xi \epsilon}, 0 \right\} = \begin{cases} - L_1 \xi - L_2 (-\xi)^\gamma e^{\xi \epsilon}, & \xi \leq \xi_3, \\ 0, & \xi > \xi_3, \end{cases}$$

where

$$M := \frac{(\beta - \gamma) S_{-\infty}}{\gamma}, \quad L_1 := e M \lambda^*, \quad \xi_1 := -\frac{1}{\lambda^*}, \quad \xi_2 := \frac{1}{\epsilon} \ln \epsilon, \quad \xi_3 := \left( \frac{L_2}{L_1} \right)^2,$$

and $\epsilon$ and $L_2$ are two positive constants to be determined.

**Lemma 2.1.** There exist a sufficiently small constant $\epsilon > 0$ and a large enough constant $L_2 > 0$, such that the functions $\overline{S}(\xi), \overline{T}(\xi), \underline{S}(\xi)$ and $\underline{T}(\xi)$ satisfy

$$c \overline{T}(\xi) \geq d_2 \overline{T}'(\xi) + \frac{\beta \overline{S}(\xi) \overline{T}(\xi)}{\overline{S}(\xi) + \overline{T}(\xi)} - \gamma \overline{T}(\xi) \quad \text{for} \quad \xi \neq \xi_1,$$

$$-\beta \overline{T}(\xi) \geq -d_1 \overline{T}''(\xi) + c \overline{T}'(\xi) \quad \text{for} \quad \xi < \xi_2,$$

and

$$\frac{\beta \underline{S}(\xi) \underline{T}(\xi)}{\underline{S}(\xi) + \underline{T}(\xi)} - \gamma \underline{T}(\xi) \geq -d_2 \underline{T}''(\xi) + c \underline{T}'(\xi) \quad \text{for} \quad \xi < \xi_3.$$

**Proof.** Proof of (2.2). When $\xi < \xi_1$, $\overline{T}(\xi) = - L_1 \xi e^{\xi \epsilon}$. It follows that

$$d_2 \overline{T}'(\xi) - c \overline{T}'(\xi) + \frac{\beta \overline{S}(\xi) \overline{T}(\xi)}{\overline{S}(\xi) + \overline{T}(\xi)} - \gamma \overline{T}(\xi) \leq d_2 \overline{T}'(\xi) - c \overline{T}'(\xi) + (\beta - \gamma) \overline{T}(\xi)$$

$$= (d_2 \lambda^*)^2 - c \lambda^* + \beta - \gamma (- L_1 \xi e^{\xi \epsilon})$$

$$= 0 \quad \text{for} \quad \xi < \xi_1.$$
When \( \xi > \xi_1 \), \( \overline{T}(\xi) = M = (\beta - \gamma)S_{-\infty}/\gamma \). We have

\[
d_1 \overline{T}''(\xi) - c' \overline{T}(\xi) + \frac{\beta S(\xi) \overline{T}(\xi)}{S(\xi) + \overline{T}(\xi)} - \gamma \overline{T}(\xi) \leq \frac{\beta S_{-\infty} M}{S_{-\infty} + M} - \gamma M
\]

\[
= 0 \quad \text{for} \quad \xi > \xi_1.
\]

**Proof of (2.4).** Let \( \epsilon > 0 \) be sufficiently small such that \( \epsilon^{-1} \ln \epsilon < -(\lambda^*)^{-1} \), i.e., \( \xi_2 < \xi_1 \). When \( \xi < \xi_2 \),

\[
\overline{T}(\xi) = -L_1 \epsilon \overline{e}^{\gamma \xi}
\]

and

\[
S(\xi) = S_{-\infty}\left(1 - \frac{1}{\epsilon} \overline{e}^{\gamma \xi}\right).
\]

Then inequality (2.3) is equivalent to

\[
\beta L_1 \epsilon \overline{e}^{\gamma \xi} \geq d_1 S_{-\infty} \epsilon \overline{e}^{\gamma \xi} - c' S_{-\infty} \epsilon \overline{e}^{\gamma \xi}
\]

\[
= S_{-\infty}(d_1 \epsilon - c') \epsilon \overline{e}^{\gamma \xi} \quad \text{for} \quad \xi < \xi_2,
\]

that is,

\[
S_{-\infty}(c' - d_1 \epsilon) \geq -\beta L_1 \epsilon \overline{e}^{(\lambda^*)^{-1} \xi} \quad \text{for} \quad \xi < \xi_2,
\]

which holds for sufficiently small \( \epsilon \in (0, \min\{c' / d_1, \lambda^*\}) \).

**Proof of (2.4).** Let \( L_2 > 0 \) be large enough such that \( L_2 > L_1 \sqrt{-\epsilon^{-1} \ln \epsilon} \). When \( \xi < \xi_3 \),

\[
S(\xi) = S_{-\infty}\left(1 - \frac{1}{\epsilon} \overline{e}^{\gamma \xi}\right)
\]

and

\[
\overline{I}(\xi) = \left[-L_1 \epsilon - L_2 (-\xi)^{3/2} \right] \overline{e}^{\gamma \xi}.
\]

Then to prove (2.4) is to show

\[
-\frac{\beta \overline{I}''(\xi)}{S(\xi) + \overline{I}(\xi)} \geq -d_2 \overline{I}''(\xi) + c' \overline{I}'(\xi) - (\beta - \gamma) \overline{I}(\xi) \quad \text{for} \quad \xi < \xi_3,
\]

that is,

\[
\frac{-\beta(-L_1 \epsilon - L_2 (-\xi)^{3/2})^2 \overline{e}^{3 \gamma \xi}}{S_{-\infty}\left(1 - \frac{1}{\epsilon} \overline{e}^{\gamma \xi}\right) + (-L_1 \epsilon - L_2 (-\xi)^{3/2}) \epsilon \overline{e}^{\gamma \xi}} \geq -\frac{d_2 L_2}{4} (-\xi)^{3/2} \epsilon \overline{e}^{\gamma \xi} \quad \text{for} \quad \xi < \xi_3.
\]

It suffices to verify that

\[
d_2 L_2 S_{-\infty}(1 - \frac{1}{\epsilon} \overline{e}^{\gamma \xi}) \geq 4\beta(-L_1 \epsilon)^2 (-\xi)^{3/2} \epsilon \overline{e}^{\gamma \xi}
\]

\[
= 4\beta L_1^2 (-\xi)^{3/2} \epsilon \overline{e}^{\gamma \xi} \quad \text{for} \quad \xi < \xi_3.
\]

Let \( g(\xi) := \beta L_1^2 (-\xi)^{3/2} \epsilon \overline{e}^{\gamma \xi} \) for \( \xi \leq 0 \). Then \( g(\xi) \in C(-\infty, 0] \) and \( g(-\infty) = f(0) = 0 \) and there exists a constant \( \tilde{C} > 0 \) independent of \( L_2 \), such that

\[
g(\xi) \leq \tilde{C} \quad \text{for} \quad \xi \leq 0.
\]

Therefore, inequality (2.5) holds for large enough \( L_2 > 0 \). This ends the proof. \( \square \)

We rewrite system (1.3) in an equivalent form:

\[
\begin{cases}
(d_1 S''(\xi) - c' S'(\xi) - \beta_1 S(\xi)) + \left(\frac{\beta S(\xi) H(\xi)}{S(\xi) + H(\xi)}\right) = 0, \\
(d_2 I''(\xi) - c' I'(\xi) - \beta_2 I(\xi)) + \left((\beta_2 - \gamma) I(\xi) + \frac{\beta S(\xi) H(\xi)}{S(\xi) + H(\xi)}\right) = 0.
\end{cases}
\]
Here we choose $\beta_1 > \beta$ and $\beta_2 > \gamma$, such that

$$a_1(S, I) := \beta_1 S - \frac{\beta S I}{S + I}$$

is non-decreasing in $S$ and non-increasing in $I$, while

$$a_2(S, I) := (\beta_2 - \gamma) I + \frac{\beta S I}{S + I}$$

is non-decreasing in $S$ and $I$.

Let $\lambda_i^+$ be the roots of equation $d_i \lambda^2 - c^* \lambda - \beta_i = 0$ ($i = 1, 2$), then we obtain

$$\lambda_i^+ = \frac{c^* \pm \sqrt{(c^*)^2 + 4d_i \beta_i}}{2d_i} \quad \text{for } i = 1, 2.$$ 

Now we introduce a subset of $C(\mathbb{R}, \mathbb{R}^2)$

$$\Gamma := \{(S(.), I(.)) \in B_{\mu}(\mathbb{R}, \mathbb{R}^2) : S(\xi) \leq S(\xi^-) \leq S(\xi) \leq S(\xi^-), \text{ } I(\xi) \leq I(\xi^-) \leq \overline{I}(\xi), \text{ } \overline{I}(\xi) \}
$$

where

$$B_{\mu}(\mathbb{R}, \mathbb{R}^2) = \max \left\{ \Psi = (\phi(\cdot), \varphi(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) \left\vert \sup_{\xi \in \mathbb{R}} \|\phi(\xi)\|e^{-\mu|\xi|} < \infty, \sup_{\xi \in \mathbb{R}} \|\varphi(\xi)\|e^{-\mu|\xi|} < \infty \right\} \right\}$$

equipped with the norm

$$\|\Psi\|_{\mu} := \max \left\{ \sup_{\xi \in \mathbb{R}} \|\phi(\xi)\|e^{-\mu|\xi|}, \sup_{\xi \in \mathbb{R}} \|\varphi(\xi)\|e^{-\mu|\xi|} \right\}.$$ 

Here the constant $\mu$ satisfies $0 < \mu < \min\{-\lambda_1^-, -\lambda_2^-, -\lambda_2^+\}$. Then $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$ is a Banach space with the norm $\| \cdot \|_{\mu}$ and $\Gamma$ is uniformly bounded with respect to the norm $\| \cdot \|_{\mu}$.

Define a function $f : \Gamma \rightarrow C(\mathbb{R})$

$$f(S, I)(\xi) := \begin{cases} \frac{\beta S(\xi)I(\xi)}{S(\xi^-) + I(\xi)^-}, & S(\xi)I(\xi) \neq 0, \\ 0, & S(\xi)I(\xi) = 0. \end{cases}$$

and an operator

$$F = (F_1(S, I)(\xi), F_2(S, I)(\xi)) : \Gamma \rightarrow C(\mathbb{R}, \mathbb{R}^2),$$

where

$$F_i(S, I)(\xi) = \frac{1}{\Lambda_i} \left( \int_{-\infty}^{\xi} e^{c(\xi-y)} h_i(S, I)(y) dy + \int_{\xi}^{\infty} e^{c(\xi-y)} h_i(S, I)(y) dy \right) \quad \text{for } i = 1, 2,$$

$$\Lambda_i = d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{(c^*)^2 + 4d_i \beta_i},$$

$$h_1(S, I)(\xi) = \beta_1 S(\xi) - f(S, I)(\xi),$$

and

$$h_2(S, I)(\xi) = (\beta_2 - \gamma) I(\xi) + f(S, I)(\xi).$$

Next we will prove that the operator $F$ satisfies the conditions of Schauder’s fixed point theorem.

**Lemma 2.2.** The operator $F : \Gamma \rightarrow \Gamma$ is completely continuous with respect to $\| \cdot \|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$. 

Proof. We will divide the proof into the following three steps.

Step 1. We prove that $F$ maps $\Gamma$ into $\Gamma$.
For any $(S(\xi), I(\xi)) \in \Gamma$, we only need to show

$$S(\xi) \leq F_1(S, I)(\xi) \leq S_{-\infty} \text{ and } I(\xi) \leq F_2(S, I)(\xi) \leq T(\xi) \text{ for } \xi \in \mathbb{R}.$$ 

Let $(S(\xi), I(\xi)) \in \Gamma$, then we obtain

$$F_1(S, I)(\xi) = \left\{ \begin{array}{l}
\frac{1}{\lambda_1}\left[ \int_{-\infty}^{\xi} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy + \int_{\xi}^{\infty} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy \right] \\
\geq \frac{1}{\lambda_1}\left[ \int_{-\infty}^{\xi} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy + \int_{\xi}^{\infty} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy \right]
\end{array} \right.$$ 

When $\xi < \xi_2$, we deduce from Lemma 2.1 and the definition of $S(\xi)$ that

$$F_1(S, I)(\xi) = \frac{1}{\lambda_1}\left[ \int_{-\infty}^{\xi} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy + \int_{\xi}^{\infty} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy \right]$$

$$\geq \frac{1}{\lambda_1}\left[ \frac{1}{\lambda_1}\left[ \int_{-\infty}^{\xi} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy + \int_{\xi}^{\infty} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy \right] \right]$$

When $\xi > \xi_2$, we get

$$F_1(S, I)(\xi) \geq \frac{1}{\lambda_1}\left[ \frac{1}{\lambda_1}\left[ \int_{-\infty}^{\xi} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy + \int_{\xi}^{\infty} e^{\xi^2(\xi^2-\gamma)}(\beta S - f(S, I)(y))dy \right] \right]$$

Combining (2.6) and (2.7) and the continuities of $S(\xi)$ and $F_1(S, I)(\xi)$ in $\mathbb{R}$, we conclude that

$$F_1(S, I)(\xi) \geq S(\xi) \text{ for } \xi \in \mathbb{R}.$$
When $\xi < \xi_1$, in view of Lemma 2.1 and the definition of $I(\xi)$, we infer that

$$F_2(S, I)(\xi) = \frac{1}{\Lambda_2} \left\{ \int_{-\infty}^{\xi} e^{\xi_{2}(\xi-y)} ((\beta_2 - \gamma) H(y) + f(S, I)(y)) dy + \int_{\xi}^{\infty} e^{\xi_{2}(\xi-y)} ((\beta_2 - \gamma) H(y) + f(S, I)(y)) dy \right\}$$

$$\geq \frac{1}{\Lambda_2} \left\{ \int_{-\infty}^{\xi} e^{\xi_{2}(\xi-y)} \left[ (\beta_2 - \gamma) I + \frac{\beta S_1}{S + I} \right] (y) dy + \int_{\xi}^{\infty} e^{\xi_{2}(\xi-y)} \left[ (\beta_2 - \gamma) I + \frac{\beta S_1}{S + I} \right] (y) dy \right\}$$

$$\geq \frac{1}{\Lambda_2} \left\{ \int_{-\infty}^{\xi} e^{\xi_{2}(\xi-y)} ((\beta_2 - \gamma) L - dL^\prime + c^* L^\prime)(y) dy + \int_{\xi}^{\infty} e^{\xi_{2}(\xi-y)} ((\beta_2 - \gamma) L - dL^\prime + c^* L^\prime)(y) dy \right\}$$

$$= \frac{1}{\Lambda_2} \left\{ \int_{-\infty}^{\xi} e^{\xi_{2}(\xi-y)} \left[ (d_2(\lambda^\prime)^2 - c^* \lambda^\prime - \beta_2) L_{1y} e^{\xi y} + L_2 (d_2(\lambda^\prime)^2 - c^* \lambda^\prime - \beta_2)(-y)^2 e^{\xi y} \right. \right.$$

$$- \frac{1}{4} d_2 L_2 (-y)^{-2} e^{\xi y} \left. ) dy \right. + \int_{\xi}^{\infty} e^{\xi_{2}(\xi-y)} \left[ (d_2(\lambda^\prime)^2 - c^* \lambda^\prime - \beta_2) L_{1y} e^{\xi y} + L_2 (d_2(\lambda^\prime)^2 - c^* \lambda^\prime - \beta_2)(-y)^2 e^{\xi y} \right. \right.$$

$$- \frac{1}{4} d_2 L_2 (-y)^{-2} e^{\xi y} \left. ) dy \right)$$

$$\geq \left[-L_1 \xi - L_2 (-\xi)\right] e^{\xi \xi} + \frac{L_1 d_2}{2 \Lambda_2} e^{\xi \xi}, e^{\xi_{2}(\xi-y)}$$

(2.8)

When $\xi > \xi_3$, we obtain

$$F_2(S, I)(\xi) \geq \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\xi_{2}(\xi-y)} \left[ \beta S_1 \frac{1}{S + I} + (\beta_2 - \gamma) I \right] (y) dy$$

$$\geq 0 \text{ for } \xi > \xi_3,$$

which together with (2.3) and the continuities of $I(\xi)$ and $F_2(S, I)(\xi)$ in $\mathbb{R}$ yields that

$$F_2(S, I)(\xi) \geq I(\xi) \text{ for } \xi \in \mathbb{R}.$$

When $\xi < \xi_1$, we have

$$\mathcal{T}(\xi) = -L_1 \xi e^{\xi \xi}.$$
When $\xi > \xi_1$, $\tilde{T}(\xi) = M$. It follows from Lemma 2.1 and the definition of $\tilde{T}(\xi)$ that

$$
F_2(S, I)(\xi) \leq \frac{1}{\Lambda_2} \left\{ \int_{-\infty}^{\xi} e^{\xi[\beta_-(\gamma)]} \left[ (\beta_2 - \gamma) \tilde{T} + \frac{\beta S}{S + I} \right](y) dy + \int_{\xi}^\infty e^{\xi[\beta_-(\gamma)]} \left[ (\beta_2 - \gamma) \tilde{T} + \frac{\beta S}{S + I} \right](y) dy \right\}
$$

analogously.

which implies that

By (2.9), (2.10) and the continuities of $F_1$, we have

$$
F_1(S, I)(\xi) \leq \frac{1}{\Lambda_1} \left\{ \int_{-\infty}^{\xi} e^{\xi[\beta_+(\gamma)]} \left[ (\beta_1 + \beta) \tilde{T} + \frac{\beta S}{S + I} \right](y) dy + \int_{\xi}^\infty e^{\xi[\beta_+(\gamma)]} \left[ (\beta_1 + \beta) \tilde{T} + \frac{\beta S}{S + I} \right](y) dy \right\}
$$

Step 2. We prove that $F$ is continuous respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$.

For any $(S_1, I_1), (S_2, I_2) \in \Gamma$, we have

$$
|F_1(S_1, I_1)(\xi) - F_1(S_2, I_2)(\xi)| e^{-\rho|\xi|} \leq \frac{1}{\Lambda_1} \left\{ \int_{-\infty}^{\xi} e^{\xi[\beta_+(\gamma)]} |(S_1 - S_2)(y)| e^{-\rho|\xi|} dy + \int_{\xi}^\infty e^{\xi[\beta_+(\gamma)]} |S_1(y) - S_2(y)| e^{-\rho|\xi|} dy \right\}
$$

which implies that $F_1$ is continuous with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$. The continuity of $F_2$ can be shown analogously.

Step 3. We prove that $F$ is compact with respect to the norm $|\cdot|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^2)$.

The proof can be carried out by the similar argument as that in [16, Lemma 3.5] or [21, Lemma 3.7]. For completeness, we give the details here. For any $(S, I) \in \Gamma$, we deduce that

$$
|F_1(S, I)(\xi)| = \left| \frac{\lambda_1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\xi[\beta_+(\gamma)]} h_1(S, I)(y) dy + \frac{\lambda_1}{\Lambda_1} \int_{\xi}^\infty e^{\xi[\beta_+(\gamma)]} h_1(S, I)(y) dy \right|
$$

analogously.

$$
|F_2(S, I)(\xi)| = \left| \frac{\lambda_1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\xi[\beta_+(\gamma)]} \beta S(y) dy + \frac{\lambda_1}{\Lambda_1} \int_{\xi}^\infty e^{\xi[\beta_+(\gamma)]} \beta S(y) dy \right|
$$

(2.11)
and
\[
|F'_2(S, I)(\xi)| = \left| \frac{\lambda_1}{\lambda_2} \int_{-\infty}^{\xi} e^{\xi_2(\xi^{-\gamma})} h_3(S, I(y))dy + \frac{\lambda_3}{\lambda_2} \int_{\xi}^{\infty} e^{\xi_2(\xi^{-\gamma})} h_3(S, I(y))dy \right|
\]
\[
\leq -\frac{\lambda_1}{\lambda_2} \int_{-\infty}^{\xi} e^{\xi_2(\xi^{-\gamma})}(\beta + \beta_2 - \gamma)I(y)dy + \frac{\lambda_3}{\lambda_2} \int_{\xi}^{\infty} e^{\xi_2(\xi^{-\gamma})}(\beta + \beta_2 - \gamma)I(y)dy
\]
\[
\leq -\frac{(\beta + \beta_2 - \gamma)M_1\lambda_2}{\lambda_2} \int_{-\infty}^{\xi} e^{\xi_2(\xi^{-\gamma})}dy + \frac{(\beta + \beta_2 - \gamma)M_1\lambda_2}{\lambda_2} \int_{\xi}^{\infty} e^{\xi_2(\xi^{-\gamma})}dy
\]
\[
= \frac{2(\beta + \beta_2 - \gamma)M_1}{\lambda_2}.
\]

Hence \( F = (F_1, F_2) \) is equi-continuous on any compact interval in \( \mathbb{R} \). Moreover, one can infer from the argument in Step 2 that
\[
|F'_1(S, I)(\xi)| \leq S_{-\infty} \quad \text{and} \quad |F'_2(S, I)(\xi)| \leq M \quad \text{for} \quad \xi \in \mathbb{R}.
\]

Hence, for any \( \varepsilon > 0 \), there exists a positive integer \( N \) such that
\[
(|F'_1(S, I)(\xi)| + |F'_2(S, I)(\xi)|)e^{-\varepsilon|\xi|} \leq (S_{-\infty} + M)e^{-\varepsilon|\xi|} < \varepsilon \quad \text{for} \quad |\xi| > N.
\]

Applying (2.11)-(2.13) and the Arzelà-Ascoli theorem, one can find finite elements in \( F(\xi) \) such that there exist a finite \( \varepsilon \)-net of \( F(\xi) \) in sense of supremum norm if they are restricted on \([-N, N]\), which is also a finite \( \varepsilon \)-net of \( F(\xi) \) on \((-\infty, \infty)\) in sense of the norm \( |\cdot|_{\mu} \) (by (2.14)). The compactness of \( F \) with respect to the norm \( |\cdot|_{\mu} \) then follows. This completes the proof. \[\square\]

It is easy to see that \( \Gamma \) is nonempty, bounded, closed and convex. Then from Lemma 2.2 and Schauder’s fixed point theorem we conclude that \( F \) admits a fixed point \((S(\xi), I(\xi)) \in \Gamma \), which is a solution of the system
\[
\begin{align*}
  c'S'_1(\xi) &= d_1S'_1(\xi) - \frac{\beta S_1(\xi)I_1(\xi)}{S_1(\xi) + I_1(\xi)}, \\
  c'I'_1(\xi) &= d_2I'_1(\xi) + \frac{\beta S_1(\xi)I_1(\xi)}{S_1(\xi) + I_1(\xi)} - \gamma I_1(\xi).
\end{align*}
\]

Therefore, we have the following existence result for (1.2).

**Proposition 2.1.** Suppose that \( R_0 > 1 \) and \( c = c^* \). Then (1.2) has a non-negative traveling wave solution \((S(\xi), I(\xi)) \) satisfying
\[
S(\xi) \leq S_*(\xi) \leq S_{-\infty} \quad \text{and} \quad I(\xi) \leq I_*(\xi) \leq \bar{I}(\xi)
\]
for \( \xi \in \mathbb{R} \).

3. **Asymptotic boundary of the critical traveling wave solution**

In this section, we will derive the asymptotic boundary of the critical traveling wave solution for (1.2), whose existence is established in Section 2.

**Proposition 3.1.** Suppose that \((S_*(\xi), I_*(\xi)) \) is a critical traveling wave solution of (1.2), then it has the following asymptotic boundary

1. \( S_*(\xi) \to S_{-\infty}, \quad I_*(\xi) \to 0, \quad I_*(\xi) = O(-\xi e^{\xi}) \quad \text{as} \quad \xi \to -\infty, \)
2. \( \lim_{\xi \to \infty} S_*(\xi) := S_{\infty} \) exists, \( S_{\infty} < S_*(-\infty) = S_{-\infty} \) and \( I_*(\xi) \to 0 \) as \( \xi \to \infty, \)

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Proof. By squeeze theorem and (2.16), we obtain
\[ S_s(\xi) \to S_{-\infty}, \quad I_s(\xi) \to 0 \quad \text{and} \quad L_s(\xi) = O(-e^{l_1 \xi}) \quad \text{as} \quad \xi \to -\infty. \]  
(3.1)

Moreover, it follows from (2.15), (2.16) and L’Hospital’s rule that
\[ S_s'(\xi) \to 0, \quad I_s'(\xi) \to 0 \quad \text{as} \quad \xi \to \pm \infty. \]  
(3.2)

By the first equation in (2.15) one can derive
\[ \left( e^{-\frac{2}{\gamma} S_s'(\xi)} \right)' = e^{-\frac{2}{\gamma} S_s'} \frac{\beta S_s(\xi) I_s(\xi)}{d_1(S_s(\xi) + L_s(\xi))}. \]  
(3.3)

Integrating (3.3) over $(\xi, \infty)$, using (3.2) and noting that $S_s(\xi)$ and $I_s(\xi)$ are non-negative, we then get
\[ S_s'(\xi) = -e^{-\frac{2}{\gamma} S_s} \int_{ \xi }^{ \infty } e^{-\frac{2}{\gamma} v} \frac{\beta S_s(v) I_s(v)}{d_1(S_s(v) + L_s(v))} dv \leq 0 \quad \text{for} \quad \xi \in \mathbb{R}, \]  
(3.4)

which implies that $S_s(\xi)$ is monotonically decreasing in $\mathbb{R}$. Additionally, $S_s(\xi)$ is bounded in $\mathbb{R}$ (by (2.16)). Then it follows that
\[ \lim_{\xi \to \infty} S_s(\xi) := S_{\infty} \quad \text{exists}. \]  
(3.5)

Furthermore, since $S_s(\xi) \geq S(\xi) > 0$ and $I_s(\xi) \geq I(\xi) > 0$ for $\xi < \min(\xi_2, \xi_3)$, we have from (3.4) that
\[ S_s'(\xi) < 0 \quad \text{for} \quad \xi < \min(\xi_2, \xi_3). \]  
(3.6)

Then one can infer that
\[ S_s < S_s(-\infty) = S_{-\infty}. \]

Now integrating the first equation in (2.15) over $\mathbb{R}$ and using (3.1), (3.2) and (3.5) give that
\[ \int_{-\infty}^{\infty} \frac{S_s(v) I_s(v)}{S_s(v) + L_s(v)} dv = \frac{e^\gamma (S_{-\infty} - S_\infty)}{\beta}. \]  
(3.7)

Another integration of the second equation in (2.15) from $-\infty$ to $\xi$ leads to
\[ \int_{-\infty}^{\xi} I_s(v) dv = \frac{d_2}{\gamma} I_s'(\xi) = \frac{e^\gamma}{\gamma} I_s(\xi) + \frac{\beta}{\gamma} \int_{-\infty}^{\xi} \frac{S_s(v) I_s(v)}{S_s(v) + L_s(v)} dv, \]  
(3.8)

where we have used (3.1) and (3.2). In view of (2.16), (3.2), (3.7) and (3.8), we have
\[ \int_{-\infty}^{\infty} I_s(v) dv < \infty. \]  
(3.9)

Since $I_s'(\xi)$ is bounded in $\mathbb{R}$ (by (3.2)), one can obtain that
\[ I_s(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \]  
(3.10)

Moreover, from (3.2), (3.7), (3.8) and (3.10) we deduce that
\[ \int_{-\infty}^{\infty} I_s(v) dv = \frac{\beta}{\gamma} \int_{-\infty}^{\infty} \frac{S_s(v) I_s(v)}{S_s(v) + L_s(v)} dv = \frac{e^\gamma (S_{-\infty} - S_\infty)}{\gamma}. \]  
(3.11)

The proof is completed. \( \square \)
4. Positivity and upper bound of the critical traveling wave solution

In the section, we will derive the positivity and upper bound of the critical traveling wave solution and finish the proof of Theorem 1.1.

Proposition 4.1. Let \( (S_1(\xi), I_1(\xi)) \) be a critical traveling wave solution of (1.2), then we have

\[
0 < S_1(\xi) < S_{-\infty} \quad \text{and} \quad 0 < I_1(\xi) < \min\left\{ \frac{(\beta - \gamma)S_{-\infty}}{\gamma}, \frac{2c^* (S_{-\infty} - S_\infty)}{\sqrt{(c^*)^2 + 4d_2^2 + c^*}} \right\}
\]

for \( \xi \in \mathbb{R} \).

Proof. Firstly, it is easy to see from (3.1), (3.4) and (3.6) that

\[
S_1(\xi) < S_{-\infty} \quad \text{for} \quad \xi \in \mathbb{R}.
\]

(4.1)

Secondly, we claim that \( S_1(\xi) > 0 \) for \( \xi \in \mathbb{R} \). By contradiction, we assume that \( S_1(\xi) = 0 \) for some \( \xi \in \mathbb{R} \). Then there exist two constants \( l_1, l_2 \in \mathbb{R} \) such that \( l_1 < \xi_2 \leq l_2 \) and \( \xi \in (l_1, l_2) \). This implies that \( S(\xi) \) attains its minimum in \((l_1, l_2)\). From the first equation in (2.15) we get

\[
-d_1S''_1(\xi) + c^*S'_1(\xi) + \beta S_1(\xi) \geq 0 \quad \text{for} \quad \xi \in [l_1, l_2].
\]

Then it follows from the strong maximum principle that

\[
S_1(\xi) \equiv 0 \quad \text{for} \quad \xi \in [l_1, l_2],
\]

which contradicts the fact that \( S_1(\xi) \geq S(\xi) > 0 \) for \( \xi \in [l_1, \xi_2] \). Hence we have

\[
S_1(\xi) > 0 \quad \text{for} \quad \xi \in \mathbb{R}.
\]

(4.2)

In an analogous manner, one can get that

\[
I_1(\xi) > 0 \quad \text{for} \quad \xi \in \mathbb{R}.
\]

(4.3)

Thirdly, we assert that \( I_1(\xi) = M = (\beta - \gamma)S_{-\infty}/\gamma \) for \( \xi \in \mathbb{R} \). Assume by way of contradiction that \( I_1(\xi) = M \) for some \( \xi \in \mathbb{R} \). Then we have \( I_1'(\xi) = 0 \) and \( I_1''(\xi) \leq 0 \). Evaluating the second equation in (2.15) at the point \( \xi \) and using (4.1), we deduce that

\[
0 = c^*I_1'(\xi)
= d_2I_1''(\xi) + \frac{\beta S_1(\xi)I_1(\xi)}{S_1(\xi) + I_1(\xi)} - \gamma I_1(\xi)
< \frac{\beta S_{-\infty}M}{S_{-\infty} + M} - \gamma M
= 0,
\]

which leads to a contradiction. Therefore we have

\[
I_1(\xi) < \frac{(\beta - \gamma)S_{-\infty}}{\gamma} \quad \text{for} \quad \xi \in \mathbb{R}.
\]

(4.4)

Finally, we prove that

\[
I_1(\xi) < \frac{2c^* (S_{-\infty} - S_\infty)}{\sqrt{(c^*)^2 + 4d_2^2 + c^*}} \quad \text{for} \quad \xi \in \mathbb{R}.
\]

To this end, we introduce an auxiliary function

\[
P(\xi) = I_1(\xi) + \frac{2\gamma}{\sqrt{(c^*)^2 + 4d_2^2 + c^*}} \int_{-\infty}^{\xi} I_1(\nu)\,d\nu.
\]

(4.5)
Then by (2.16) and (3.9), one can see that $P(\xi)$ is well-defined in $\mathbb{R}$. Since $I_1(\xi) \geq I(\xi) > 0$ on $(-\infty, \xi_1)$, we have $P(\xi) > 0$ and $I_1(\xi) < P(\xi)$ for $\xi \in \mathbb{R}$. Note that $I_1(\pm \infty) = 0$, then we get from (4.5) and (3.11) that

$$P(-\infty) = 0 \quad \text{and} \quad P(\infty) = \frac{2c^*(S_{-\infty} - S_\infty)}{(c^*)^2 + 4d_2\gamma + c^*}. \quad (4.6)$$

Differentiating (4.5) with respect to $\xi$ yields

$$P'(\xi) = I_1'(\xi) + \frac{2\gamma}{\sqrt{(c^*)^2 + 4d_2\gamma + c^*}} I_1(\xi). \quad (4.7)$$

Then from $I_1(\pm \infty) = I_1'(\pm \infty) = 0$ we obtain that $P'(\pm \infty) = 0$. Differentiating (4.7) with respect to $\xi$ and using the second equation in (2.15), we get

$$-d_2P''(\xi) + bP'(\xi) = -d_2I_1''(\xi) + c^*I_1'(\xi) + \gamma I_1(\xi)$$

$$= \beta S_\ast(\xi)I_1(\xi) - \beta S_\ast(\xi)I_1(\xi), \quad (4.8)$$

where

$$b = \frac{c^* + \sqrt{(c^*)^2 + 4d_2\gamma}}{2}.$$

From (4.8) and using $P'(\infty) = 0$ and (2.16), we infer that

$$P'(\xi) = \frac{\beta}{d_2} \left[ \int_\xi^\infty e^{\frac{\sqrt{\beta}}{2}(\xi - v)} \frac{S_\ast(v)I_1(v)}{S_\ast(v) + I_1(v)} dv \right] \geq 0,$$

which together with (4.6) yields

$$P(\xi) \leq P(\infty) = \frac{2c^*(S_{-\infty} - S_\infty)}{(c^*)^2 + 4d_2\gamma + c^*} \quad \text{for} \quad \xi \in \mathbb{R}.$$ 

Then from (4.5) and (4.3), we obtain that

$$I_1(\xi) < \frac{2c^*(S_{-\infty} - S_\infty)}{(c^*)^2 + 4d_2\gamma + c^*} \quad \text{for} \quad \xi \in \mathbb{R}.$$ 

All the claims of this proposition are shown. 

Remark 4.1.  

(1) In fact, the inequalities (3.4), (4.2) and (4.3) imply that $S_\ast(\xi)$ is strictly decreasing in $\mathbb{R}$.

(2) When $R_0 > 1$ and $c > c^*$, Wang et al. [10] introduced a function

$$I(\xi) := I(\xi) + \frac{\gamma}{c} \int_{-\infty}^\xi I(y)dy + \frac{\gamma}{c} \int_{\xi}^\infty e^{-\frac{\sqrt{\beta}}{2}(\xi - y)} I(y)dy, \quad \xi \in \mathbb{R}$$

to obtain an upper bound of I-component in system (1.2) with $c > c^*$, that is,

$$I(\xi) \leq S_{-\infty} - S_\infty \quad \text{for} \quad \xi \in \mathbb{R}.$$ 

While using the function $P(\xi)$ in (4.5) by replacing $c^*$ by $c$, one can get a better estimate of I-component in (1.2) with $c > c^*$, i.e.,

$$I(\xi) < \frac{2c(S_{-\infty} - S_\infty)}{c^* + 4d_2\gamma + c} \quad \text{for} \quad \xi \in \mathbb{R}.$$
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