Characterizations of the Generators of Positive Semigroups on $C^*$- and von Neumann Algebras†

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Abstract

Generators of positive $C_0$-semigroups on $C^*$-algebras and $C^*_0$-semigroups on von Neumann algebras are examined. A characterization due to Bratteli and Robinson in the $C_0$-case is proven in the $C^*_0$-case. Under the additional assumptions of unitality and contractivity of the semigroup another characterization of the generator is given. This result is restated for the dual and predual semigroup.

Keywords: $C^*$-algebra, von Neumann algebra, positivity, $C_0$-semigroup, $C^*_0$-semigroup, infinitesimal generator

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1 Introduction

In this note we examine infinitesimal generators of positive one parameter semigroups and of positive, unital and contractive one-parameter semigroups $T_t$ on $C^*$- and von Neumann algebras. Unitality of $T_t$ means $T_t(1) = 1$, where $1$ is the unit of the algebra.

Uniformly continuous, positive semigroups were studied by Evans and Hanche-Olsen [1]. They characterized the generators $L$ of such semigroups by the inequalities

\[ L(a^2) + aL(1)a \geq L(a)a + aL(a), \text{ for all self-adjoint } a \in \mathcal{A}, \]
\[ L(1) + u^*L(1)u \geq L(u^*)u + u^*L(u), \text{ for all unitary } u \in \mathcal{A}. \]

†Part of this work is contained in the author’s thesis [2].
This result was generalized by Bratteli and Robinson [2] to positive $C_0$-semi-
groups on $C^*$-algebras. They derived the above inequalities with the generator $L$ replaced by the resolvent $(\lambda - L)^{-1}$. In section 2 this result is restated for $C_0^*$-semigroups on von Neumann algebras. The proof of this characterization is mainly based on the proof of Bratteli and Robinson in the $C_0$-case. The modification of the proof holds for both $C_0$-semigroups on $C^*$-algebras and $C_0^*$-semigroups acting on von Neumann algebras.

In section 3 it is additionally assumed that the semigroups under consideration are contractive and unital. A characterization of the generators themselves can be given, which is not only a characterization in terms of their resolvents. This result for positive, unital semigroups of contractions admits a reformulation for the related dual and predual semigroups. This leads to a generalization of a result due to Kossakowski [14].

With the help of the results in section 3 it is proven in [13] that the generators of uniformly continuous, unital and positive semigroups of contractions on the $C^*$-algebra $\mathcal{M}(n, \mathbb{D})$ of $n \times n$ matrices are essentially the difference of two generators of completely positive semigroups.

Semigroups which satisfy this stronger notion of positivity are the natural candidates for a description of a Markovian irreversible time evolution in quantum statistical mechanics, in particular for so called reduced dynamics. These dynamics arise from a Hamiltonian dynamic through an averaging procedure over a subsystem or some internal degrees of freedom, i.e. one is interested in the family of maps $N \circ \alpha_t$, where $\alpha_t$ is a one-parameter group of $*$-automorphisms of the algebra $\mathcal{A}$ and $N : \mathcal{A} \to \mathcal{B}$ is the conditional expectation from $\mathcal{A}$ into a $C^*$-subalgebra $\mathcal{B}$. If $N$ and $\alpha_t$ are completely positive, so is the composition. Of course, $N \circ \alpha_t$ is in general not a one-parameter semigroup because of memory effects. In order to eliminate these memory effects, one usually carries out a limiting procedure, for example a weak or singular coupling limit [8]. For a detailed discussion see [10].

When the completely positive, unital semigroup is uniformly continuous, the infinitesimal generator can be written in the so-called Lindblad form [12], [16], [7], see also [6] p. 222-226. In the case of non-uniformly continuous semigroups, a canonical form of the generator is not yet known, but there are interesting results in this direction, see [13], [6] and references therein.

The description of irreversible dynamics by completely positive semigroups is not without criticism, see e.g. [1], [17] and [19].

For a detailed discussion on positive and completely positive semigroups see [13].

2 Positive semigroups

In this section we discuss positive semigroups and their generators. We state a characterization of the generators of both positive $C_0$-semigroups on a $C^*$-
algebra and positive $C^*_0$-semigroups on a von Neumann algebra $A$, in the $C_0$-case due to Bratteli and Robinson [2].

Let us recall some definitions. If $X$ is a Banach space then a one parameter family $T_t, t \geq 0,$ of bounded linear operators on $X$ is defined to be a strongly continuous or $C_0$-semigroup if it satisfies

1. $T_{t+s} = T_t T_s, \forall t, s \geq 0,$
2. $T_0 = \text{id}_X,$
3. $\lim_{t \downarrow 0} \|T_t(x) - x\| = 0, \forall x \in X.$

Similarly, if $X$ is a Banach space with predual space $X_*$ a one parameter family $T_t, t \geq 0,$ is defined to be a $C^*_0$-semigroup if it satisfies 1 and 2 from above and the weak-$^*$-continuity conditions

4. $t \mapsto \eta(T_t(x))$ is continuous $\forall x \in X, \forall \eta \in X_*,$
5. $x \mapsto \eta(T_t(x))$ is continuous $\forall \eta \in X_*, \forall t \geq 0.$

If one replaces the predual space $X_*$ in the above definition by the dual space $X^*$ one gets a weakly continuous semigroup. It can be proven that each weakly continuous semigroup is a $C_0$-semigroup, see [2] p. 233. The generator of a semigroup $T_t$ is defined as the linear operator $L$ whose domain $D(L)$ consists of those $x \in X$ for which there exists a $y \in X$ such that

$$\lim_{t \downarrow 0} \frac{1}{t}(T_t(x) - x) - y = 0$$

and the action of $L$ is then defined by $L(x) = y.$ The limit is taken in the weak topology if $T_t$ is a $C_0$-semigroup and in the weak-$^*$-topology in the $C^*_0$-case.

We are interested in $C_0$-semigroups on $C^*$-algebras and $C^*_0$-semigroups on von Neumann algebras with further properties. First we look at positive semigroups $T_t$, i.e. semigroups which leave the positive cone $A^+$ of the algebra invariant: $T_t(A^+) \subseteq A^+.$ In Section 3 we assume additionally the unitality $T_t(1) = 1$ and the contractivity of the semigroups. We always assume that the $C^*$-algebras possess a unit $1$.

In order to handle $C_0$- and $C^*_0$-semigroups simultaneously, we use the notation $\sigma(A,F)$-continuous semigroup, where $\sigma(A,F)$ is either the locally-convex topology on the $C^*$-algebra $A$ induced by the functionals in the dual $F = A^*$ corresponding to $C_0$-semigroups. On the other hand $A$ can be viewed as a von Neumann algebra with predual space $F = A_*$. In this case $\sigma(A,F)$ is the weak-$^*$-topology corresponding to $C^*_0$-semigroups, see [2] Section 2.5.3 and Section 3.1.2. For a discussion of the different continuity properties on $C^*$- and von Neumann algebras, see [3] Section 3.2.2.
A linear mapping $T : D(T) \to A$ on a $C^*$- and von Neumann algebra $A$ respectively with domain $D(T) \subseteq A$ is called symmetric, if $T(x^*) = T(x)^*$ for all $x \in D(T)$.

For the theory of $C^*$- and von Neumann algebras the reader is referred to [3] and [22]. References for the theory of one-parameter semigroups are e.g. [3], [9], [18]. In [3] the theories of $C_0$- and $C_0^*$-semigroups are developed together.

**Theorem 1** Let $L$ be the generator of a $\sigma(A,F)$-continuous semigroup $T_t$ of symmetric operators on a unital algebra $A$. Then the following conditions are equivalent:

1. $T_t$ is positive for all $t \geq 0$.
2. $(\lambda - L)^{-1}$ is positive for all large $\lambda > 0$.
3. $(\lambda - L)^{-1}(a^2) + a(\lambda - L)^{-1}(1)a \geq (\lambda - L)^{-1}(a)a + a(\lambda - L)^{-1}(a)$, for all self-adjoint $a \in A$ and all large $\lambda > 0$.
4. $(\lambda - L)^{-1}(1) + u^*(\lambda - L)^{-1}(1)u \geq (\lambda - L)^{-1}(u^*)u + u^*(\lambda - L)^{-1}(u)$, for all unitary $u \in A$ and all large $\lambda > 0$.
5. $T_t(a^2) + aT_t(1)a \geq T_t(a)a + aT_t(a)$, for all self-adjoint $a \in A$ and $t \geq 0$.
6. $T_t(1) + u^*T_t(1)u \geq T_t(u^*)u + u^*T_t(u)$, for all unitary $u \in A$ and $t \geq 0$.

Remark: The proof of the $C^*_0$-case requires a modification of the proof of the $C_0$-case in [3]. There one uses a convergence argument to prove the equivalence of 5 and 6 to 1–4. This argument does not work in the case of $C^*_0$-semigroups. In order to overcome this difficulty a Laplace transform argument is used in the following proof. For the sake of convenience, the complete proof is given.

Proof: 1. $\Rightarrow$ 2.: This follows from the Laplace transformation:

$$(\lambda - L)^{-1}(a) = \int_0^\infty e^{-\lambda t}T_t(a)dt, \quad \text{for all } \lambda > 0, \text{ for all } a \in A.$$

The usual relation for $C_0$-semigroups between the resolvent and the Laplace transform holds also for $C^*_0$-semigroups, see [3] Proposition 3.1.6.

2. $\Rightarrow$ 1.:

$$T_t(a) = \lim_{n \to \infty} \left( \frac{T_t}{T_t(L)^{-1}} \right)^n(a), \quad \text{for all } t > 0, \text{ for all } a \in A,$$

where the limit is taken in the $\sigma(A,F)$-topology. Thus, the semigroup is positive whenever the resolvent is positive for $\lambda$ sufficiently large.
1. ⇒ 3. (2. ⇒ 4.): In the case of uniformly continuous semigroups and bounded generators respectively conditions 1. and 2. are equivalent to one of the following two conditions [11]:

\[
L(a^2) + aL(1)a \geq L(a)a + aL(a), \quad \text{for all self-adjoint } a \in \mathcal{A}, \quad (1)
\]
\[
L(1) + u^*L(1)u \geq L(u^*)u + u^*L(u), \quad \text{for all unitary } u \in \mathcal{A}. \quad (2)
\]

With this result in mind, it is sufficient to show the equivalence of 1. and 2. to the following condition.

7. The uniformly continuous semigroup \( S_t \) with generator \((\lambda - L)^{-1}\) is positive for all \( t \geq 0 \) and large \( \lambda > 0 \).

2. ⇒ 7.: Follows from

\[
S_t = e^{t(\lambda - L)^{-1}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\lambda - L)^{-n}.
\]

7. ⇒ 1.: Let \( U^\lambda_t \) be the uniformly continuous semigroup whose generator \( L_\lambda \) is given by the Yosida approximation of \( L \), i.e. \( L_\lambda := \lambda L(\lambda - L)^{-1} = \lambda^2(\lambda - L)^{-1} - \lambda \). \( U^\lambda_t \) is positive if \( S_t \) is positive, since

\[
U^\lambda_t = e^{t(\lambda^2(\lambda - L)^{-1} - \lambda)} = e^{-t\lambda}e^{\lambda^2t(\lambda - L)^{-1}} = e^{-t\lambda}S_{\lambda^2t}.
\]

\( T_t \) is the \( \sigma(\mathcal{A}, \mathcal{F}) \)-limit of \( U^\lambda_t \) for \( \lambda \to \infty \), see [8] Theorem 1.5.5. for the result in the \( C_0 \)-case the \( C_0^* \)-result follows by duality. So \( T_t \) is positive.

5. ⇒ 3. (6. ⇒ 4.): We have

\[
T_t(a^2) + aT_t(1)a - (T_t(a)a + aT_t(a)) \geq 0,
\]

for all \( t \geq 0 \) and all self-adjoint (unitary \( a \in \mathcal{A} \)). So

\[
(\lambda - L)^{-1}(a^2) + a(\lambda - L)^{-1}(1)a - ((\lambda - L)^{-1}(a)a + a(\lambda - L)^{-1}(a))
\]

\[
= \int_0^\infty e^{-\lambda t}(T_t(a^2) + aT_t(1)a - (T_t(a)a + aT_t(a)) \geq 0
\]

follows by Laplace transformation.

1. ⇒ 5. (6.): Provided \( T_t \geq 0 \) for all \( t \geq 0 \),

\[
e^{sT_t} = \sum_{n=0}^{\infty} \frac{(sT_t)^n}{n!}
\]

is positive for all \( s \geq 0 \) and \( t \geq 0 \). Thus, the estimates (1) and (2) hold for the generator \( T_t \) of the semigroup \( e^{sT_t} \).
3 Positive, unital semigroups of contractions

In the previous section we discussed $\sigma(A, F)$-continuous, positive semigroups. Now we assume in addition that the semigroup is unital and contractive. These properties allow a characterization of the generator which is different to the characterizations in Theorem 1. There the characterizations are in terms of the resolvent of the generator, in the following we provide a characterization in terms of the generator only.

**Theorem 2** Let $L$ be the generator with domain $D(L)$ of a $\sigma(A, F)$-continuous semigroup $T_t$ of contractions on $A$. The following two conditions are equivalent:

1. $T_t$ is positive and unital.
2. $1 \in D(L)$, $L(1) = 0$ and $L$ is symmetric.

**Proof:**

1. $\Rightarrow$ 2.: $1 \in D(L)$ and $L(1) = 0$ follow from $\lim_{h \to 0} \frac{1}{h}(T_h(1) - 1) = 0$, where the limit is taken in the $\sigma(A, F)$-topology.

Every element $x \in A$ can be decomposed into a linear combination of four positive elements $a, b, c, d \in A^+$ such that

$$x = a - b + i(c - d).$$

Since $T_t$ is positive and linear we get with the help of this decomposition

$$T_t(x^*) = T_t(x)^*,$$

thus $T_t$ is symmetric. Let $x \in D(L)$, then by using the uniform-continuity of the $*$-map

$$L(x)^* = \lim_{h \to \infty} \frac{1}{h}(T_h(x) - x)^*$$

$$= \lim_{h \to \infty} \frac{1}{h}(T_h(x^*) - x^*),$$

the limit is performed in the $\sigma(A, F)$-topology. Thus $x^* \in D(L)$ and $L(x^*) = L(x)^*$.

2. $\Rightarrow$ 1.: Property 2. reads in terms of the resolvent $(\lambda - L)^{-1}(1) = \frac{1}{A}1$. By approximation in the $\sigma(A, F)$-topology we get

$$T_t(1) = \lim_{n \to \infty} \left(\frac{n}{T} \left(\frac{n}{T} - L\right)^{-1}\right)^n(1) = \lim_{n \to \infty} \left(\frac{n}{T} \left(\frac{n}{T} - L\right)^{-1}\right)^n(1) = 1,$$

which proves the unitality of $T_t$.

By assumption $L$ is the generator of a semigroup of contractions. Therefore the range of $\lambda - L$ is the whole algebra $A$ for $\lambda > 0$. For $x \in A$ exists a unique
element \( y \in D(L) \), such that \((\lambda - L)^{-1}(x) = y\). Clearly, \(\lambda - L\) is symmetric for \(\lambda > 0\). Therefore
\[
((\lambda - L)^{-1}(x))^* = ((\lambda - L)^{-1}(\lambda - L))(y^*) = (\lambda - L)^{-1}(x^*).
\]
The semigroup \(T_t\) can be defined by \(T_t(x) = \lim_{n \to \infty} (\frac{t}{n}(\frac{t}{n} - L))^{-1}(x)\). In the \(C^*_0\)-case the limit is taken in the weak-\*-topology on the von Neumann algebra \(\mathcal{A}\). The weak-\* topology on \(\mathcal{A}\) is the relative \(\sigma\)-weak topology on \(\mathcal{A}\) and the involution \(^*\) is continuous in this topology. So one can interchange the involution and the limit, thus \(T_t\) is symmetric. In the \(C_0\)-case the limit is taken in the weak topology on the \(C^*\)-algebra \(\mathcal{A}\). Every weakly continuous semigroup is also strongly continuous, see [23] p.233, and the involution \(^*\) is continuous with respect to the uniform topology on the \(C^*\)-algebra \(\mathcal{A}\). As in the von Neumann case we can interchange the limit with the involution and the symmetry of \(T_t\) follows.

Now let \(a \in \mathcal{A}^+\) and \(a_1 := \frac{a}{\|a\|}\). \(\mathcal{A}^+\) is a convex cone, thus \(a_1 \in \mathcal{A}^+\) and \(\|a_1\| = 1\). So the spectrum, \(\sigma(a_1)\), of \(a_1\) is contained in \([0, 1]\). Therefore
\[
1 - \sigma(a_1) = \sigma(1 - a_1) \subseteq [0, 1]
\]
and
\[
\|1 - a_1\| \leq 1.
\]
Thus
\[
\|1 - T_t(a_1)\| = \|T_t(1) - T_t(a_1)\| \leq \|T_t\||1 - a_1\| \leq 1.
\]
The semigroup \(T_t\) is symmetric for all \(t \geq 0\), hence \(T_t(a_1) \in \mathcal{A}^{sa}\). With this
\[
1 - \sigma(T_t(a_1)) = \sigma(1 - T_t(a_1)) \subseteq [-1, 1].
\]
It follows that \(\sigma(T_t(a_1)) \subseteq [0, 2]\), therefore \(T_t(a_1) \in \mathcal{A}^+\). One concludes
\[
T_t(a) = \|a\|T_t(a_1) \in \mathcal{A}^+,
\]
thus \(T_t\) is positive, which proves the theorem.

Remark: The direction 2. \(\Rightarrow\) 1. can be sharpened using the fact, that a unital, contractive map between two unital \(C^*\)-algebras and two von Neumann algebras respectively is positive, see [21], [3] Corollary 3.2.6.

**Theorem 3** Let \(L\) be the generator of a \(\sigma(\mathcal{A}, F)\)-continuous semigroup \(T_t\) of contractions on \(\mathcal{A}\).
\[
1 \in D(L), \ L(1) = 0 \quad \Rightarrow \quad T_t \text{ is positive.}
\]
Proof: The unitality is proven as in Theorem 2. Since $T_t$ is a contraction, i.e. $\|T_t\| \leq 1$ for all $t \geq 0$, we have $\|T_t\| = 1$. The conclusion now follows from the above remark.

**Corollary 1** A unital $\sigma(A, F)$-continuous semigroup of contractions on an operator algebra is positive.

In the following, the above characterization is restated for the dual and the predual semigroup respectively.

Let $T_t$ be a $C_0$-semigroup of contractions on a $C^*$-algebra $A$. Then, the adjoint operators $T_t^* : A^* \to A^*$ form a $C_0^*$-semigroup of contractions on the dual space $A^*$ of $A$, see [3]. The generator of $T_t^*$ is given by the adjoint operator $L^*$ of the generator $L$ of the semigroup $T_t$. In $A^*$, the set

$$A^{+*} := \{ \eta \in A^* : \eta(a) \geq 0 \ \forall a \in A^+ \}$$

is a positive cone [4]. Let $\eta \in A^{+*}$ and $T_t$ be a positive $C_0$-semigroup. Then $\eta(T_t(a)) \geq 0$ for all $a \in A^+$. So $T_t^*$ is also positive since $(T_t^*(\eta))(a) = \eta(T_t(a))$. For a unital semigroup $T_t$, one obtains $(T_t^*(\eta))(1) = \eta(1)$ for all $\eta \in A^*$. $L(1) = 0$ implies $L^*(\eta)(1) = 0$ for all $\eta \in A$.

In the case that $T_t$ is a $C_0^*$-semigroup of contractions on a von Neumann algebra $A$, there exists a $C_0$-semigroup $T_t^*$ of contractions on the predual $A_*$ which is adjoint to $T_t$. The unitality and positivity of the semigroup $T_t$ and $L(1) = 0$ can be transferred to $T_t^*$ and $L^*$ as in the case of a $C^*$-algebra. Finally, one gets

**Corollary 2** Let $T_t^*$ be a $C_0^*$-semigroup of contractions on $A^*$ the dual of a $C^*$-algebra $A$ or a $C_0$-semigroup of contractions on the predual $A_*$ of a von Neumann algebra $A$. Denote the generator of $T_t^*$ by $L^*$. The following two conditions are equivalent:

1. $T_t^*$ is positive and $T_t^*(\eta)(1) = \eta(1)$ for all $\eta \in A^*$ or for all $\eta \in A_*$.
2. $L^*(\eta)(1) = 0$ for all $\eta \in D(L^*)$.

If one applies the last result to $A_* = L^1(\mathcal{H})$ the Banach space of trace class operators and restricts the semigroup $T_t^*$ to the real Banach space $L^1(\mathcal{H})^{sa}$ of all selfadjoint operators in $L^1(\mathcal{H})$, one sees that the result due to Kossakowski [14] is contained in Corollary 2.

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