ON COQUASITRIANGULAR BIALGEBRAS

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1. Introduction

Coquasitriangularity is one of the most fundamental concepts in quantum group theory. Some early papers on this notion are [LT], [S1], [H], [Doi], [Mj]. A bialgebra or a Hopf algebra is called coquasitriangular if it is equipped with a universal r-form. The latter concept is the main object of study in this paper.

Definition 1.1 Let A be a bialgebra. A universal r-form on A is a linear functional on A ⊗ A which is invertible with respect to the convolution multiplication and satisfies the following three conditions for arbitrary a, b, c ∈ A:

(CQT.1) r(c ⊗ ab) = r(c(1) ⊗ b)r(c(2) ⊗ a),
(CQT.2) r(ab ⊗ c) = r(a ⊗ c(1))r(b ⊗ c(2)),
(CQT.3) r(a(1) ⊗ b(1))a(2)b(2) = r(a(2) ⊗ b(2))b(1)a(1).

The present paper deals with three topics on coquasitriangular bialgebras. In Section 2 we give a characterization of a universal r-form in terms of (certain) Yetter-Drinfeld modules. In Section 3 we study the uniqueness of universal r-forms for the coordinate Hopf algebras of the quantum groups GLq(N), SLq(N), Oq(N) and Spq(N). Section 4 is concerned with some linear functionals on a coquasitriangular Hopf algebra A with universal r-form r which describe the square of the antipode of A. To be more precise, it is known that S^2 = f_r * id * f_r, where f_r is the linear functional on A defined by f_r(a) = r(a(1), S(a(2))), a ∈ A, and f_r is the convolution inverse of f_r. Among others we show that F_r := f_r * f_r^{-1} is a character on A which coincides with Woronowicz's modular character f_{-2} when A is the coordinate Hopf *-Algebra for the standard compact quantum groups U_q(N), SU_q(N), O_q(N; ℜ), and USp_{pq}(N). Dual versions for quasitriangular Hopf algebras of some results in Section 4 have been proved by V.G. Drinfeld [D].

Let us fix some notation. We use the Sweedler notations Δ(a) = a(1) ⊗ a(2) for the comultiplication Δ and δ(m) = m(0) ⊗ m(1) for a right coaction δ. The multiplication map of an algebra A is denoted by m_A. We write * for the convolution multiplication. The convolution inverse of a functional f is denoted by f̄.
2. Universal $r$-forms and Yetter-Drinfeld modules

If not stated otherwise, $A$ denotes a bialgebra in this section. First we recall a well-known definition (see [Y], [S2] or [Mo]).

**Definition 2.1.** A (right) **Yetter-Drinfeld module** of $A$ is a right $A$-module and a right $A$-comodule $M$ satisfying the compatibility condition

$$m(0) \triangleleft a(1) \otimes m(1)a(2) = (m \triangleleft a(2))(0) \otimes a(1)(m \triangleleft a(2))(1)$$

(1)

for all $m \in M$ and $a \in A$. Here the right action of $a \in A$ on $m \in M$ is denoted by $m \triangleleft a$ and the right coaction $\delta: M \to M \otimes A$ is expressed by the Sweedler notation $\delta(m) = m(0) \otimes m(1)$, $m \in M$.

For a right $A$-comodule $M$, let $C(M) = \text{Lin}\{m'(m(0)) \ m(1) \ m \in M, m' \in M'\}$ denote the coefficient coalgebra of $M$ (see, for instance, [KS], p.399). For the following considerations we assume that $r$ is a convolution invertible linear functional on the bialgebra $A \otimes A$. Recall that the convolution inverse of $r$ is denoted by $\bar{r}$.

Let $M$ be a right $A$-comodule with right coaction $\delta(m) = m(0) \otimes m(1)$. For $m \in M$ and $a \in A$, we define

$$m \triangleleft_1 a = r(m(1) \otimes a)m(0) \quad \text{and} \quad m \triangleleft_2 a = \bar{r}(a \otimes m(1)m(0)).$$

Let $M_i$, $i = 1, 2$, denote the right $A$-comodule $M$ equipped with the mapping $M \times A \ni (m, a) \mapsto m \triangleleft_i a \in M$. If $r$ is a universal $r$-form, then it is known (see [Mo], Example 10.6.14) that $M_1$ is a Yetter-Drinfeld module. We shall strengthen this fact and give a characterization of universal $r$-forms in this manner.

**Lemma 2.2** (i) $M_1$ is a Yetter-Drinfeld module of $A$ with right action $\triangleleft_1$ if and only if (CQT.1) holds for all $b, c \in C(M)$ and (CQT.3) holds for all $a \in C(M)$ and $b \in A$.

(ii) $M_2$ is a Yetter-Drinfeld module of $A$ with right action $\triangleleft_2$ if and only if (CQT.2) is fulfilled for all $b, c \in C(M)$ and (CQT.3) holds for all $a \in A$ and $b \in C(M)$.

**Proof.** We prove the assertion for $M_1$. It is obvious that the condition $(m \triangleleft_1 a) \triangleleft_1 b = m \triangleleft_1 ab$ is equivalent to equation (CQT.1) for $c \in C(M)$ and $a, b \in A$. We show that the latter implies that $m \triangleleft_1 1 = m, m \in M$. Indeed, using the convolution inverse $\bar{r}$ of $r$ and condition (CQT.1) for $a = b = 1$, we get

$$\bar{r}(c \otimes 1) = \bar{r}(c(1) \otimes 1)\bar{r}(c(2) \otimes 1)\bar{r}(c(3) \otimes 1) = \bar{r}(c(1) \otimes 1)c(2) \otimes 1 = \varepsilon(c).$$

for $c \in C(M)$, so that $m \triangleleft_1 1 = r(m(1) \otimes 1)m(0) = m$. The left hand and right hand sides of the Yetter-Drinfeld condition (1) are equal to

$$r(m(1) \otimes a(1))m(0) \otimes m(2)a(2) \quad \text{and} \quad r(m(2) \otimes a(2))m(0) \otimes a(1)m(1),$$

respectively. Thus, $\square$ is equivalent to (CQT.3) for $a \in C(M)$ and $b \in A$.

The assertion for $M_2$ follows by some slight modifications of the preceding reasoning. The
relation \((m \circ_2 a) \circ_2 b = m \circ_2 ab\), where \(a, b \in \mathcal{A}\) and \(m \in M\), is fulfilled iff \(\bar{r}(a \otimes c_2(b \otimes c_1)) = \bar{r}(a \otimes c_1(b \otimes c_2))\) for \(a, b \in \mathcal{A}\) and \(c \in \mathbb{C}(M)\). The latter is obviously equivalent to the fact that \(r(a \otimes c_1(b \otimes c_2)) = r(a \otimes c_2(b \otimes c_1))\) for \(a, b \in \mathcal{A}\) and \(c \in \mathcal{A}\). The Yetter-Drinfeld condition \((\mathcal{I})\) is satisfied iff \(\bar{r}(a(1) \otimes b(1))b(2)a(2) = \bar{r}(a(1) \otimes b(2))a(1)b(2)\) for all \(a \in \mathcal{A}\) and \(b \in \mathbb{C}(M)\) which in turn is equivalent to equation (CQT.3) for \(a \in \mathcal{A}\) and \(b \in \mathbb{C}(M)\).

An immediate consequence of Lemma 2.2 is

**Proposition 2.3** Let \(\mathcal{R}\) be a family of right comodules of a bialgebra \(\mathcal{A}\) such that the linear span of the coefficient coalgebras \(\mathbb{C}(M), M \in \mathcal{R}\), coincides with \(\mathcal{A}\). Let \(r\) be a convolution invertible linear functional on \(\mathcal{A} \otimes \mathcal{A}\). Then \(r\) is a universal \(r\)-form of \(\mathcal{A}\) if and only if \(M_1\) and \(M_2\) are Yetter-Drinfeld modules for any comodule \(M \in \mathcal{R}\).

**Corollary 2.4** A convolution invertible linear functional \(r\) on \(\mathcal{A} \otimes \mathcal{A}\) is a universal \(r\)-form of a bialgebra \(\mathcal{A}\) if and only if \(\mathcal{A}\) becomes a Yetter-Drinfeld module with the comultiplication as right coaction and with the mappings \(\circ_1\) and \(\circ_2\) as right actions.

**Remark 2.5** As shown by P. Schauenburg [S2], the category of right Yetter-Drinfeld modules is equivalent to the category of Hopf bimodules over a Hopf algebra \(\mathcal{A}\). Using this correspondence the preceding results can be reformulated in terms of Hopf bimodules of \(\mathcal{A}\). In this setting they were crucial for the construction of bicovariant differential calculi on general coquasitriangular Hopf algebras (see [SS], last line on p.189, and [KS], Section 14.5).

### 3. Uniqueness of universal \(r\)-forms for quantized matrix groups

In this section let \(G_q\) denote one of the quantum groups \(GL_q(N), SL_q(N), O_q(N)\) or \(Sp_q(N)\) and \(O(G_q)\) its coordinate Hopf algebra as defined in [FRT], [T1] or [KS], Chapter 9. It is known (see, for instance, [KS], Theorem 10.9) that the Hopf algebra \(O(G_q)\) is coquasitriangular and there exists a universal \(r\)-form \(r_z\) of \(O(G_q)\) such that

\[
r_z(u^i_j \otimes u^m_n) = zR^{in}_{jm}, \quad i, j, n, m = 1, \ldots, N. \tag{2}
\]

Here \(u = (u^i_j)_{i,j=1,\ldots,N}\) is the fundamental matrix of \(O(G_q)\), \(R\) is the corresponding \(R\)-matrix as given in [FRT], (1.5) and (1.9), or in [KS], (9.13) and (9.30), and \(z\) is a fixed complex number such that \(z \neq 0\) for \(G_q = GL_q(N), z^N = q^{-1}\) for \(G_q = SL_q(N)\) and \(z^2 = 1\) for \(G_q = O_q(N), Sp_q(N)\).

Throughout this section we assume that \(q\) is a complex number which is not a root of unity. (A closer look at the proof given below shows that it suffices to exclude only very few roots of unity.) We shall show that the above functionals \(r_z\) exhaust all universal \(r\)-forms of \(O(G_q)\). In order to place this result in a more general context, we need some preliminaries.

**Definition 3.1** A **central bicharacter** of a bialgebra \(\mathcal{A}\) is a convolution invertible linear functional \(c\) on \(\mathcal{A} \otimes \mathcal{A}\) such that for arbitrary \(a, b, c \in \mathcal{A}\):

(CB.1) \(c(ab \otimes c) = c(a \otimes c_1(b \otimes c_2))\) and \(c(c \otimes ab) = c(c_1 \otimes b)c(c_2 \otimes a)\),

(CB.2) \(c(a \otimes b_1)b_2 = c(a \otimes b_2)b_1\) and \(c(a_1 \otimes b)a_2 = c(a_2 \otimes b)a_1\).
Condition (CB.1) means that \( c(\cdot \otimes \cdot) \) is a dual pairing of the bialgebras \( A \) and \( A^{op} \), where \( A^{op} \) is the bialgebra with the same comultiplication and the opposite multiplication as \( A \).

Condition (CB.2) is equivalent to the requirement that for any \( a \in A \) the linear functionals \( c(a \otimes \cdot) \) and \( c(\cdot \otimes a) \) on \( A \) are central in the dual algebra \( A' \). A trivial example of central bicharacter is the counit of \( A \otimes A \).

Suppose that \( c \) is a central bicharacter and \( r \) is a universal \( r \)-form of \( A \). Using the condition (CQT.1)–(CQT.3) and (CB.1)–(CB.2) it is a straightforward matter to verify that the convolution product \( c * r \) is again a universal \( r \)-form of \( A \). Recall ([KS], Proposition 10.2(iv)) that \( r_{21} \) is another universal \( r \)-form of \( A \), where \( r_{21}(a \otimes b) := r(b \otimes a) \), \( a, b \in A \). Thus \( c * r_{21} \) is also a universal \( r \)-form of \( A \).

**Definition 3.2** A coquasitriangular bialgebra \( A \) is said to have an **essentially unique universal** \( r \)-**form** if there exists a universal \( r \)-form \( r \) of \( A \) such that for any universal \( r \)-form \( s \) of \( A \) there exists a central bicharacter \( c \) such that \( s = c * r \) or \( s = c * r_{21} \).

**Example 3.3** Let \( A = CG \) be the group Hopf algebra of an abelian group \( G \). Then the universal \( r \)-forms of \( A \) are in one-to-one correspondence to the bicharacters of the group \( G \).

Since \( CG \) is cocommutative and \( G \) is abelian, the universal \( r \)-forms are precisely the central bicharacters of \( A \) and hence \( A = CG \) has obviously an essentially unique universal \( r \)-form.

Let us return to the coquasitriangular Hopf algebra \( O(G_q) \). We shall say that a complex number \( \zeta \) is **admissible** if \( \zeta \neq 0 \) for \( G_q = GL_q(N), \zeta^N = 1 \) for \( G_q = SL_q(N) \) and \( \zeta^2 = 1 \) for \( G_q = O_q(N), Sp_q(N) \). For any admissible number \( \zeta \) there exists a unique central bicharacter \( c_{\zeta} \) of \( O(G_q) \) such that

\[
c_{\zeta}(u_j^i \otimes u_m^n) = \delta_{ij}\delta_{nm}\zeta, \quad i, j, n, m = 1, \ldots, N. \tag{3}
\]

In order to prove this, one first extends \([3]\) to a linear functional on \( \mathbb{C}(u_j^i) \otimes \mathbb{C}(u_j^i) \) such that (CB.1) is satisfied, where \( \mathbb{C}(u_j^i) \) denotes the free bialgebra with generators \( u_j^i, i, j = 1, \ldots, N \).

Because \( \zeta \) is assumed to be admissible, it follows from the defining relations for the algebra \( O(G_q) \) that this functional passes to a functional \( c_{\zeta} \) of \( O(G_q) \otimes O(G_q) \). It is easily seen that \( c_{\zeta} \) is a central bicharacter of \( O(G_q) \).

We now fix a universal \( r \)-form \( r_{z_0} \) of \( O(G_q) \) with parameter \( z_0 \) as above and denote it by \( r \).

Then it is clear that any universal \( r \)-form \( r_z \) of \( O(G_q) \) given by \([3]\) is of the form \( r_z = c_{\zeta} * r \) for some admissible number \( \zeta \).

**Proposition 3.4** Suppose that \( q \) is not a root of unity. For any universal \( r \)-form \( s \) of \( O(G_q) \) there exists an admissible complex number \( \zeta \) such that \( s = c_{\zeta} * r \) or \( s = c_{\zeta} * r_{21} \). The coquasitriangular Hopf algebra \( O(G_q) \) has an essentially unique universal \( r \)-form.

**Proof.** Suppose that \( s \) is a universal \( r \)-form of \( O(G_q) \). We define \( N^2 \times N^2 \)-matrices \( T = (T_{jm}^{ni}) \) and \( \hat{T} = (\hat{T}_{jm}^{ni}) \) by

\[
\hat{T}_{jm}^{ni} = T_{jm}^{ni} := s(u_j^i \otimes u_m^n), \quad i, j, n, m = 1, \ldots, N.
\]

From the general theory of coquasitriangular Hopf algebras (see, for instance, [KS], 10.1) we conclude that \( \hat{T} \) belongs to the centralizer algebra Mor(\( u \otimes u \)) of the tensor product.
corepresentation \( u \otimes u \) and that \( T \) satisfies the quantum Yang-Baxter equation, so \( \hat{T} \) fulfills the braid relation

\[
\hat{T}_{12}\hat{T}_{23}\hat{T}_{12} = \hat{T}_{23}\hat{T}_{12}\hat{T}_{23}
\]

on the space of the tensor product corepresentation \( u \otimes u \otimes u \).

We shall carry out the proof in the cases \( O_q(N) \) and \( Sp_q(N) \). Then it is well-known (see [Re] or [KS], Proposition 8.40) that there exists a homomorphism \( \pi \) of the Birman-Wenzl-Murakami algebra \( \mathrm{BWM}_3(q,q^N) \) to the centralizer algebra \( \text{Mor}(u \otimes u \otimes u) \) such that \( \pi(G_i) = R_{i,i+1}, i = 1,2 \), where \( \epsilon = 1 \) for \( O_q(N) \) and \( \epsilon = -1 \) for \( Sp_q(N) \). The generators of the BWM-algebra \( \mathrm{BWM}_3(q,q^N) \) are denoted by \( G_1, G_2, E_1, E_2 \) and their images under the homomorphism \( \pi \) by \( g_1, g_2, e_1, e_2 \). The matrices \( \{ \hat{R}, \hat{R}^{-1}, I \} \) form a basis of the vector space \( \text{Mor}(u \otimes u) \). Since \( \hat{T} \in \text{Mor}(u \otimes u) \), there are complex numbers \( \alpha, \beta, \gamma \) such that

\[
\hat{T} = \alpha \hat{R} + \beta \hat{R}^{-1} + \gamma \cdot I.
\]

The crucial step of the proof is to show that the braid relation for \( \hat{T} \) implies that \( \hat{T} \) is a complex multiple of either \( \hat{R}, \hat{R}^{-1} \) or \( I \). In order to prove this, we essentially use the relations of the BWM-algebra \( \mathrm{BWM}_3(q,q^N) \) (see [BW], p. 225).

Inserting (3) into (2), both sides of (2) are sums of 27 summands. From the braid relation

\[
G_1G_2G_1 = G_2G_1G_2
\]

in the BWM-algebra we get

\[
g_1^2g_2g_1 = g_2g_1^2g_2, \quad g_1^{-1}g_2^{-1}g_1^{-1} = g_2^{-1}g_1^{-1}g_2^{-1},
\]

\[
g_1^{-1}g_2g_1 = g_2g_1^2g_2^{-1}, \quad g_1^2g_1^{-1} = g_2^{-1}g_1^{-1}g_2, \quad g_1^{-1}g_2^{-1}g_1 = g_2g_1^2g_1^{-2} \quad \text{and} \quad g_1g_2g_1^{-1} = g_2^2g_1.
\]

Inserting these relations and cancelling equal terms on both sides of (2) we finally obtain

\[
\alpha^2\beta g_1g_2^{-1}g_1 + \alpha^2g_2g_1^{-1}g_1^{-1} + \alpha^2g_2^{-1}g_1^{-1} + \beta^2g_1^{-2} = \alpha^2\beta g_2g_1^{-1}g_2 + \alpha^2g_2^{-1}g_1^{-1}g_1^{-1} + \alpha^2g_2^{-1}g_2^{-1} + \beta^2g_2^{-2}.
\]

Now we recall the following relations in the BWM-algebra (see [BW], p. 255):

\[
G_i^{-1} - G_i = \lambda E_i - \lambda \cdot 1, \quad G_2E_1G_2 = G_2^{-1}E_2G_2^{-1}, \quad G_2^{-1}E_1G_2^{-1} = G_1E_2G_1,
\]

\[
E_1E_2G_1 = E_1G_2^{-1}, \quad G_1E_2E_1 = G_2^{-1}E_1, \quad E_1E_2E_1 = E_1,
\]

where \( \lambda := q - q^{-1} \). Applying the images of these relations under the homomorphism \( \pi \), a straightforward computation shows that (2) reduces to the equation

\[
\alpha^2(\gamma - \beta \lambda)(g_1^2 - g_2^2) + \beta^2(\alpha \lambda + \gamma)(g_1^{-2} - g_2^{-2}) + \alpha\beta(\alpha + \beta)\lambda^2\{e_2g_1 + e_1e_2 - e_1g_2^{-1} - g_2^{-1}e_1 + \lambda(e_1e_2 + e_2e_1 + e_1 + e_2)\} = 0.
\]

The BWM-algebra \( \mathrm{BWM}_3(q,q^N) \) is 15-dimensional and the elements

\[
1, G_1, G_2, G_1G_2, G_2G_1, G_1G_2G_1, E_1, E_2, E_1E_2, E_2E_1, G_1E_2, E_2G_1, G_2^{-1}E_1, E_1G_2^{-1}, G_1E_2G_1
\]

form a vector space basis (see [BW] or [KS], Proposition 8.39). From the representation theory of quantum groups it is well-known (see, for instance, [KS], 8.6.2) how to decompose
the tensor product $u \otimes u \otimes u$ into irreducible components. From these decompositions it follows that the homomorphism $\pi$ is injective for $O_q(N), N \geq 3$, and for $Sp_q(N), N \geq 6$. In these cases the images of the elements (8) under $\pi$ are also linearly independent. Therefore, the coefficient of, say, $e_2g_1$ in (7) is zero, so that

$$\alpha\beta(\alpha + \beta) = 0. \quad (9)$$

Hence the second line of (7) vanishes identically and the coefficients of $g_1$ and $e_1$ in (7) have to be zero as well. Using the relation $G_1 E_1 = e_q^{q^{-N}} E_1$ in the algebra $\text{BWM}_3(q, e_q^{N-\epsilon})$, these coefficients are computed as

$$-\alpha^2(\gamma \beta \lambda)\lambda + \beta^2(\alpha \lambda + \gamma)\lambda = 0, \quad (10)$$

$$\alpha^2(\gamma - \beta \lambda)\lambda e_q^{q^{-N}} + \beta^2(\alpha \lambda + \gamma)\lambda(e_q^{N-\epsilon} - \lambda) = 0, \quad (11)$$

respectively. In the case of the quantum group $Sp_q(4)$ the corepresentation corresponding to the Young tableaux of a column with 3 boxes does not occur in the decomposition of tensor product $u \otimes u \otimes u$ (see [KS], p.289) and we have $\dim \text{Mor}(u \otimes u \otimes u) = 14$. An explicit computation shows that the images of the basis elements (8) satisfy the linear relation

$$1 - q^{-1} g_1 - q^{-1} g_2 + q^{-2} g_1 g_2 + q^{-3} g_1 g_2 g_1 - q^{-4} g_1 - q^{-6} e_1 - q^{-2} e_1 - q^{-4} e_1 e_2$$

$$- q^{-4} e_2 e_1 + q^{-3} g_1 g_2 + q^{-3} g_2 g_1 + g^{-5} e_1 g_2 + q^{-1} e_1 g_2 - q^{-4} e_1 g_2 g_1 = 0. \quad (12)$$

Therefore, the derivation of equations (4)–(11) from (7) is also valid in the case $Sp_q(4)$. Since $(q - e_q^{q^{-N}}(q + e_q^{q^{-N}}) \neq 0$ by assumption, (11) and (11) imply that

$$\alpha^2(\gamma - \beta \lambda) = \beta^2(\alpha \lambda + \gamma) = 0. \quad (13)$$

The solutions of equations (4) and (11) are $\beta = \gamma = 0, \alpha = \gamma = 0$ and $\alpha = \beta = 0$ which means that $\hat{T}$ is a multiple of either $\hat{R}, \hat{R}^{-1}$ or $I$.

For the quantum groups $GL_q(N)$ and $SL_q(N)$ the proof is similar and much simpler. Then there is a homomorphism of the Hecke algebra $H_3(q)$ on the centralizer algebra $\text{Mor}(u \otimes u \otimes u)$ which is injective for $N \geq 3$. In the case $N=2$ we have $\dim H_3(q) = 1 + \dim \text{Mor}(u \otimes u \otimes u) = 6$ and the corresponding linear relation is obtained from (12) by setting $e_1 = e_2 = 0$ therein. Since the Hopf algebras $\mathcal{O}(Sp_q(2))$ and $\mathcal{O}(SL_q^2(2))$ are isomorphic, we also cover the case $Sp_q(2)$ in this manner which was excluded during preceding considerations.

Summarizing, we have shown that $\hat{T} = z \cdot \hat{R}$ or $\hat{T} = z \cdot \hat{R}^{-1}$ or $\hat{T} = z \cdot I$ for some complex number $z$. The rest of the proof is more or less routine. Using the fact that $\mathfrak{s}$ is a dual pairing of $\mathcal{O}(G_q)$ and $\mathcal{O}(G_q)^{op}$ it follows that the case $\hat{T} = z \cdot I$ is impossible (because it is not compatible with the defining relation $\hat{R}u_1 u_2 = u_1 u_2 \hat{R}$) and that the number $z$ must be as described at the beginning of this section. Thus we have $\mathfrak{s} = \mathfrak{r}_z$ or $\mathfrak{s} = (\mathfrak{r}_z)_{21}$. Fixing a universal $r$-form $\mathfrak{r}_{\zeta_0}$ and reformulating the latter in terms of a central bicharacter $c_{\zeta}$, the proof will be completed.

**Remark 3.5** As I have learned from the referee, the universal $r$-forms of the bialgebra $\mathcal{O}(M_q(N))$ have been described recently in the paper [2]. This result implies the assertion of Proposition 3.4 in the case $G_q = GL_q(N)$. 


4. On functionals describing the square of the antipode

Throughout this section we assume that \( \mathcal{A} \) is a coquasitriangular Hopf algebra and \( \mathbf{r} \) is a universal \( r \)-form of \( \mathcal{A} \).

Let \( f_\mathbf{r} \) and \( \tilde{f}_\mathbf{r} \) denote the linear functionals on \( \mathcal{A} \) defined by

\[
f_\mathbf{r}(a) = \mathbf{r}(a(1) \otimes S(a(2))) \quad \text{and} \quad \tilde{f}_\mathbf{r}(a) = \mathbf{r}(S(a(1)) \otimes a(2)), \quad a \in \mathcal{A}.
\]

Then it is well-known (see, for instance, [KS, Proposition 10.3]) that \( \bar{\mathbf{r}} \) using the properties (CQT.1)–(CQT.3) of the universal \( r \)-form \( \mathbf{r} \) and that the square of the antipode \( S \) is given by

\[
S^2 = \tilde{f}_\mathbf{r} \ast \mathbf{id} \ast f_\mathbf{r}, \quad \text{that is,} \quad S^2(a) = \tilde{f}_\mathbf{r}(a(1))a(2)f_\mathbf{r}(a(3)), a \in \mathcal{A}.
\]

**Lemma 4.1** The functional \( f_\mathbf{r} \) satisfies the equations

\[
r_{21} \ast \mathbf{r} \ast (f_\mathbf{r} \circ m_\mathcal{A}) = (f_\mathbf{r} \circ m_\mathcal{A}) \ast r_{21} \ast \mathbf{r} = f_\mathbf{r} \otimes f_\mathbf{r}.
\]

**Proof.** Using the properties (CQT.1)–(CQT.3) of the universal \( r \)-form \( \mathbf{r} \) we compute

\[
r_{21}(a(1) \otimes b(1))\mathbf{r}(a(2) \otimes b(2))f_\mathbf{r}(a(3)b(3))
\]

\[
= \mathbf{r}(b(1) \otimes a(1))\mathbf{r}(a(2) \otimes b(2))\mathbf{r}(a(3)b(3) \otimes S(a(4)b(4)))
\]

\[
= \mathbf{r}(b(1) \otimes a(1))\mathbf{r}(a(3) \otimes b(3))\mathbf{r}(b(2)a(2) \otimes S(b(4))S(a(4)))
\]

\[
= \mathbf{r}(a(4) \otimes b(4))\mathbf{r}(b(1) \otimes a(1))\mathbf{r}(b(2)a(2) \otimes S(a(3)))\mathbf{r}(b(4)a(4) \otimes S(b(5)))
\]

\[
= \mathbf{r}(a(4) \otimes b(4))\mathbf{r}(b(2) \otimes a(2))\mathbf{r}(a(1)b(1) \otimes S(a(3)))\mathbf{r}(b(3) \otimes S(b(5)))\mathbf{r}(a(3) \otimes S(b(5)))
\]

\[
= \mathbf{r}(b(2) \otimes a(2))\mathbf{r}(a(1)b(1) \otimes S(a(3)))\mathbf{r}(b(3) \otimes S(b(4)))
\]

\[
= \mathbf{r}(b(2) \otimes a(2))\mathbf{r}(a(1) \otimes S(a(4)))\mathbf{r}(b(1) \otimes S(a(3)))f_\mathbf{r}(b(3))
\]

\[
= f_\mathbf{r}(a)f_\mathbf{r}(b)
\]

for \( a, b \in \mathcal{A} \). This proves the first equality \( r_{21} \ast \mathbf{r} \ast (f_\mathbf{r} \circ m_\mathcal{A}) = f_\mathbf{r} \otimes f_\mathbf{r} \). Applying condition (CQT.3) twice to this relation we get the second equality \( (f_\mathbf{r} \circ m_\mathcal{A}) \ast r_{21} \ast \mathbf{r} = f_\mathbf{r} \otimes f_\mathbf{r} \). ■

In some sense the functional \( r_{21} \ast \mathbf{r} \) on \( \mathcal{A} \otimes \mathcal{A} \) measures the distance of \( f_\mathbf{r} \) from being a character. In particular, Lemma 3.1 implies

**Corollary 4.2.** The functional \( f_\mathbf{r} \) is a character of \( \mathcal{A} \) (that is, \( f_\mathbf{r}(ab) = f_\mathbf{r}(a)f_\mathbf{r}(b) \) for \( a, b \in \mathcal{A} \)) if and only if \( \mathcal{A} \) is cotriangular (that is, \( \bar{\mathbf{r}} = r_{21} \)).

Recall that any coquasitriangular Hopf algebra \( \mathcal{A} \) has a second universal \( r \)-form \( \mathbf{s} := r_{21} \) given by \( \mathbf{s}(a \otimes b) = \mathbf{r}(b \otimes a), a, b \in \mathcal{A} \).

**Proposition 4.3** The functionals \( f_\mathbf{r}, \tilde{f}_\mathbf{r}, f_\mathbf{s}, \tilde{f}_\mathbf{s} \) pairwise commute in the algebra \( \mathcal{A}' \), \( z := f_\mathbf{r} \circ f_\mathbf{s} \) belongs to the center of \( \mathcal{A}' \) and \( g := f_\mathbf{s} \circ f_\mathbf{r} \) is a character of \( \mathcal{A} \) such that \( S^4 = \bar{g} \ast \mathbf{id} \ast g \).
Proof. The antipode $S$ is bijective and we have $S \ast \tilde{f}_s = \tilde{f}_s \ast S^{-1}$ (\cite{KS}, Proposition 10.3). Using this fact, the relation $\mathbf{r}(S(a) \otimes S(b)) = \mathbf{r}(a \otimes b)$ and (\ref{eq:3}), we obtain

$$f_r \ast \tilde{f}_s(a) = \mathbf{r}(a_{(1)} \otimes S(a_{(2)})) \tilde{f}_s(a_{(3)}) = \mathbf{r}(a_{(1)} \otimes S^{-1}(a_{(3)})) \tilde{f}_s(a_{(2)})$$

$$= \mathbf{r}(S^2(a_{(1)}) \otimes S(a_{(3)})) \tilde{f}_s(a_{(2)}) = \mathbf{r}(a_{(2)} \otimes S(a_{(3)})) \tilde{f}_s(a_{(1)}) = \tilde{f}_s \ast f_r(a).$$

Hence the functionals $f_r, \tilde{f}_r, f_s, \tilde{f}_s$ pairwise commute.

Applying (\ref{eq:4}) to both $r$ and $s$, we get

$$z \ast \mathbf{id} = f_r \ast \tilde{f}_s \ast \mathbf{id} = f_r \ast S^2 \ast \tilde{f}_s = \mathbf{id} \ast f_r \ast \tilde{f}_s = \mathbf{id} \ast z,$$

so $z$ is in the center of the algebra $A'$.

Since $\mathbf{r}_{21} \ast \mathbf{r} \ast \mathbf{s}_{21} \ast s = \varepsilon_{A \otimes A}$, the equations (\ref{eq:9}) easily imply that $g(ab) = g(a)g(b)$ for $a, b \in A$, that is, $g$ is a character on $A$. By (\ref{eq:5}), we have $S^4 = \bar{g} \ast \mathbf{id} \ast g$.

We illustrate the preceding by a simple example.

Example 4.4 Let $A$ be the Hopf algebra $\mathbb{C}Z$ of the group of integers. Then any universal $r$-form of $A$ is of the form $\mathbf{r}(n \otimes m) = \lambda^{nm}, n, m \in \mathbb{Z}$, for some fixed $\lambda \in \mathbb{C}, \lambda \neq 0$ (see also Example 3.3). Thus we have $f_r(n) = \lambda^n$, $\tilde{f}_s(n) = \lambda^{-n}$ and $F_r(n) = 1$ for $n \in \mathbb{Z}$, so that $F_r = \varepsilon$. Since $F_r(n + m) = f_r(n)f_r(m)\lambda^{2nm}$ for $n, m \in \mathbb{Z}$, the functional $f_r$ on $A$ is far from being a character in general.

Let us suppose now that $A$ is a coquasitriangular Hopf algebra equipped with a universal $r$-form $r$ and that $A$ is also a CQG-algebra (see \cite{DR} or \cite{KS}, 11.3.1, for this notion). Let $f_z, z \in \mathbb{C}$, denote Woronowicz’s modular characters on $A$ (see \cite{W} or \cite{KS}, 11.3.4). Recall that $f_z \ast f_{z'} = f_{z+z'}$, for $z, z' \in \mathbb{C}$ and $S^2(a) = f_2 \ast \mathbf{id} \ast f_{-2}$ for $a \in A$. Then the functionals $F_r$ and $f_{-2}$ are both characters on $A$ which implement $S^4$, that is,

$$S^4(a) = \bar{F}_r(a_{(1)}a_{(2)}F_r(a_{(3)}) = f_2(a_{(1)})a_{(2)}f_{-2}(a_{(3)}), a \in A.$$ Hence $F_r f_2$ is a character of $A$ which is central in $A'$. This suggests the following

**PROBLEM:** Do the characters $F_r$ and $f_{-2}$ on $A$ coincide?

If the Hopf algebra $A$ is cocommutative (in particular, if $A$ is the group algebra $\mathbb{C}G$ of an abelian group $G$), then we have $F_r = f_{-2} = \varepsilon$ and so the answer is affirmative.

A more interesting case is the Hopf $*$-algebra $O(G_q)$, where $G_q$ is one of the compact forms $U_q(N)$, $SU_q(N)$, $O_q(N, \mathbb{R})$, $USp_q(N)$ of the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$, $Sp_q(N)$, respectively, and $q$ is real. Then $O(G_q)$ is a CQG-algebra (\cite{KS}, Example 11.7) and a coquasitriangular Hopf algebra with universal $r$-form $r$ determined by (\ref{eq:2}).

**Proposition 4.5** Then we have $F_r = f_{-2}$.

**Proof.** From the explicit formulas for $f_r(u_i^j), f_s(u_i^j)$ and $f_{-2}(u_i^j)$ listed in \cite{KS}, p.341 and p.425, respectively, we see that $F_r(u_i^j) = f_{-2}(u_i^j)$ for $i, j = 1, \ldots, N$. Therefore, since $F_r$ and $f_{-2}$ are both characters, they coincide on the whole algebra $O(G_q)$. 

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