LOW DIMENSIONAL LINEAR REPRESENTATIONS OF \text{SAut}(F_n)

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Abstract. We prove that \text{SAut}(F_n), the unique subgroup of index two in the automorphism group of a free group of rank \(n\), admits no non-trivial linear representation of degree \(d < n\) for any field of characteristic not equal to two.

1. Introduction

In this work we study low dimensional linear representations of the group \text{SAut}(F_n) and we prove strong rigidity results for these. To be precise, let \(\mathbb{Z}^n\) be the free abelian group and \(F_n\) the free group of rank \(n\). The abelianization map \(F_n \rightarrow \mathbb{Z}^n\) gives a natural epimorphism \(\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})\). The group \text{SAut}(F_n) is defined as the preimage of \(\text{SL}_n(\mathbb{Z})\) under this map. The group \(\text{SL}_n(\mathbb{Z})\) acts faithfully by linear transformations on \(\mathbb{R}^n\). We shall prove that the linear representation: \(\text{SAut}(F_n) \rightarrow \text{SL}_n(\mathbb{Z}) \hookrightarrow \text{SL}_n(\mathbb{R})\) is minimal in the following sense:

Theorem. Let \(n \geq 3\) and let \(\rho : \text{SAut}(F_n) \rightarrow \text{SL}_d(K)\) be a linear representation of degree \(d\) over a field \(K\) with \(\text{char}(K) \neq 2\). If \(d < n\), then \(\rho\) is trivial.

We note that the group \text{SAut}(F_n) is perfect, therefore the image of a linear representation of \text{SAut}(F_n) is a subgroup of \(\text{SL}_d(K)\).

Indeed, BRIDSON and VOGTMANN proved in [BV11] that if \(n \geq 3\) and \(d < n\), then \text{SAut}(F_n) cannot act non-trivially by homeomorphisms on any contractible manifold of dimension \(d\). Using their techniques we proved in a purely group theoretical way the above theorem.

2. Linear representations

2.1. The automorphism group of a free group. As the main protagonist in this work is the group \text{SAut}(F_n), we start with the definition of this group, and establish some notation to be used throughout.

We begin with the definition of the automorphism group of the free group of rank \(n\). Let \(F_n\) be the free group of rank \(n\) with a fixed basis \(X := \{x_1, \ldots, x_n\}\). We denote by \(\text{Aut}(F_n)\) the automorphism group of \(F_n\) and by \text{SAut}(F_n) the unique subgroup of index two in \(\text{Aut}(F_n)\).

Let us first introduce a notations for some elements of \(\text{Aut}(F_n)\). We define involutions \((x_i, x_j)\) and \(e_i\) for \(i, j = 1, \ldots, n, i \neq j\) as follows:

\[
(x_i, x_j)(x_k) := \begin{cases} 
  x_j & \text{if } k = i, \\
  x_i & \text{if } k = j, \\
  x_k & \text{if } k \neq i, j.
\end{cases}
\]

\[
e_i(x_k) := \begin{cases} 
  x_i^{-1} & \text{if } k = i, \\
  x_k & \text{if } k \neq i.
\end{cases}
\]
2.2. Some finite subgroups of $\text{Aut}(F_n)$. We begin this subsection by describing some finite subgroups of $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$:

\[
\Sigma_n := \{ \alpha \in \text{Aut}(F_n) \mid \alpha|_X \in \text{Sym}(X) \} \cong \text{Sym}(n),
\]

\[
N_n := \{ (e_i) \mid i = 1, \ldots, n \} \cong \mathbb{Z}_2^n,
\]

\[
\text{SN}_n := N_n \cap \text{SAut}(F_n) = \{ (e_1 e_i) \mid i = 2, \ldots, n \} \cong \mathbb{Z}_2^{n-1},
\]

\[
W_n := \langle N_n \cup \Sigma_n \rangle = N_n \times \Sigma_n,
\]

\[
\text{SW}_n := W_n \cap \text{SAut}(F_n).
\]

The following variant of a result by Bridson and Vogtmann [BV11, 3.1] will be used here to prove that certain actions on spaces of $\text{SAut}(F_n)$ are indeed trivial. For a detailed proof the reader is referred to [Var10, 1.13].

**Proposition 2.1.** Let $n \geq 3$, $G$ be a group and $\phi : \text{SAut}(F_n) \to G$ a group homomorphism.

(i) If there exists $\alpha \in \text{SW}_n - \{ \text{id}_{F_n}, e_1 e_2 \ldots e_n \}$ with $\phi(\alpha) = 1$, then $\phi$ factors through $\text{SL}_n(\mathbb{Z}_2)$.

(ii) If $n$ is even and $\phi(e_1 e_2 \ldots e_n) = 1$, then $\phi$ factors through $\text{PSL}_n(\mathbb{Z})$.

(iii) If there exists $\alpha \in \text{SW}_n - \text{SN}_n$ with $\phi(\alpha) = 1$, then $\phi$ is trivial.

We divide the proof of our theorem into two steps. In the first step we show that for $d < n$ any homomorphism $\overline{\rho} : \text{SL}_n(\mathbb{Z}_2) \to \text{SL}_d(K)$ is trivial. In the second step we prove by induction on $n$ that for $d < n$ any homomorphism $\rho : \text{SAut}(F_n) \to \text{SL}_d(K)$ factors through $\text{SL}_n(\mathbb{Z}_2)$.

\[
\begin{array}{ccc}
\text{SAut}(F_n) & \xrightarrow{\rho} & \text{SL}_d(K) \\
\pi & & \overline{\rho}
\end{array}
\]

A key observation is the following proposition.

**Proposition 2.2.** Let $K$ be a field with $\text{char}(K) \neq 2$ and let $\phi : \mathbb{Z}_2^m \to \text{SL}_d(K)$ be a group homomorphism. If $d \leq m$, then $\phi$ is not injective.

**Proof.** First, we recall an important characterization of diagonalizable linear maps: a linear map is diagonalizable over the field $K$ if and only if its minimal polynomial is a product of distinct linear factors over $K$, see [Ho90] Thm. 6, p. 204. The minimal polynomial of an involution is a divider of $x^2 - 1 = (x + 1)(x - 1)$ and satisfies this characterization if $\text{char}(K) \neq 2$. We observe that all elements in the image of $\phi$ have order less or equal to two, therefore the elements in the image of $\phi$ are diagonalizable.

Next, we note that all elements in the image of $\phi$ commute, therefore these are simultaneously diagonalizable, see [Ho90] p. 51-53.

For $A$ in the image of $\phi$ we have $\text{Eig}(A,1) \oplus \text{Eig}(A,-1) = K^d$, where we denote by $\text{Eig}(A,1)$ resp. $\text{Eig}(A,-1)$ the eigenspace for the eigenvalue $1$ resp. $-1$. We note that $\det(A) = 1$. The number of diagonal matrices in $\text{SL}_d(K)$ with an even number of entries equal to $-1$ is given by

\[
\sum_{i=0}^{d} \binom{d}{2i} = 2^{d-1}.
\]

Therefore the image of $\phi$ contains at most $2^{d-1}$ elements. But we have

\[
\text{ord}(\mathbb{Z}_2^m) = 2^m > 2^{d-1} = \text{ord}(\text{im}(\phi(\mathbb{Z}_2^m)))
\]

and therefore $\phi$ is not injective. □

Using Proposition 2.2 we obtain the following result.
**Proposition 2.3.** Let $n \geq 3$ and $\overline{\rho} : \text{SL}_n(\mathbb{Z}_2) \to \text{SL}_d(K)$ be a linear representation of degree $d$ over a field $K$ with $\text{char}(K) \neq 2$. If $d < n$, then $\overline{\rho}$ is trivial.

*Proof.* By simplicity of $\text{SL}_n(\mathbb{Z}_2)$ the kernel of the map $\overline{\rho}$ is either trivial or all of $\text{SL}_n(\mathbb{Z}_2)$. The group $\text{SL}_n(\mathbb{Z}_2)$ contains a subgroup $U$ which is isomorphic to $\mathbb{Z}_2^{n-1}$. $U = \{E_{12}, \ldots, E_{1n}\}$, where we denote by $E_{ij}$ the matrix which has ones on the main diagonal and in the entry $(1, i)$ and zeros elsewhere. By Proposition 2.2 the restriction of $\overline{\rho}$ to this group is not injective, therefore the kernel of the map $\overline{\rho}$ is equal to $\text{SL}_n(\mathbb{Z}_2)$.

First, we show the main theorem for $n = 3$ and $n = 4$ and then proceed by induction on $n$.

**Lemma 2.4.** Let $\rho : \text{SAut}(F_3) \to \text{SL}_d(K)$ be a linear representation of degree $d$ over a field $K$ with $\text{char}(K) \neq 2$. If $d < 3$, then $\rho$ is trivial.

*Proof.* The group $\text{SN}_3$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and by Proposition 2.2 it follows that the restriction of $\rho$ to this group is not injective. Therefore there exists an element $\alpha \in \text{SN}_3 - \{\text{id}_3\}$ with $\rho(\alpha) = I_d$, where $I_d$ is the unit matrix. If $\alpha$ is not equal to $e_1e_2e_3e_4$, then by Proposition 2.2 it follows that the restriction of $\rho$ to this group is not injective. Therefore there exists an element $\beta \in \text{SN}_3 - \{\text{id}_3\}$ with $\rho(\beta) = I_d$. By Proposition 2.2 it follows that $\rho$ factors through $\text{SL}_d(\mathbb{Z}_2)$ and the triviality of $\rho$ follows by Proposition 2.1. If $\alpha$ is equal to $e_1e_2e_3e_4$ then by Proposition 2.2 the map $\rho$ factors through $\text{PSL}_4(\mathbb{Z})$.

\[
\begin{array}{ccc}
\text{SAut}(F_3) & \overset{\rho}{\longrightarrow} & \text{SL}_d(K) \\
\pi & \searrow & \nearrow \overline{\rho} \\
& \text{PSL}_4(\mathbb{Z}) & \\
\end{array}
\]

The group $\text{PSL}_4(\mathbb{Z})$ contains a subgroup $U$ which is isomorphic to $\mathbb{Z}_2^3$, namely

$U = \{([E_{-1}E_{-2}], [E_{-2}E_{-3}], [P_{12}P_{34}], [P_{13}P_{24}]\}$,

where we denote by $E_{ij}$ the matrix which has ones on the main diagonal except the entry $(i, i)$ and zeros everywhere and the entry $(i, i)$ is equal to $-1$. The matrix $P_{ij}$ is a permutation matrix.

By Proposition 2.2 it follows that there exists an element $[A] \in U - \{[I_4]\}$ with $\overline{\rho}([A]) = I_d$. We consider a preimage of $[A]$ under $\pi$ and note that there exists an element $\beta \in \text{SW}_n - \{\text{id}_n, e_1e_2e_3e_4\}$ with $\rho(\beta) = \overline{\rho} \circ \pi(\beta) = \overline{\rho}([A]) = I_d$. By Proposition 2.1 it follows that $\rho$ factors through $\text{SL}_4(\mathbb{Z}_2)$ and by Proposition 2.3 we obtain triviality of $\rho$.

*Proof of main theorem.* We proceed by induction on $n$. We assume that $n > 4$. If $\rho(e_1e_2) = I_d$, then by Proposition 2.1 the map $\rho$ factors through $\text{SL}_n(\mathbb{Z}_2)$ and by Proposition 2.3 it follows that $\rho$ is trivial. Otherwise $\rho(e_1e_2)$ is a non-trivial involution. We note that the dimension of the eigenspace $\text{Eig}(\rho(e_1e_2), 1)$ is equal to $d - 2 \cdot l$ for some $l \in \mathbb{N}_{>0}$. The centralizer of $e_1e_2$, which we will denote by $C(e_1e_2)$, contains a subgroup $U$ isomorphic to $\text{SAut}(F_{n-2})$. More precisely: let $\{x_1, \ldots, x_n\}$ be a basis of $F_n$ and let $\{x_3, \ldots, x_n\}$ be a basis of $F_{n-2}$. We define

$U := \{f \in C(e_1e_2) \mid f(x_1) = x_1, f(x_2) = x_2 \text{ and } f|_{\{x_3, \ldots, x_n\}} \in \text{SAut}(F_{n-2})\}$.

Then $U$ is isomorphic to $\text{SAut}(F_{n-2})$. By the induction assumption any homomorphism

$\rho' : \text{SAut}(F_{n-2}) \to \text{SL}_d(K)$
for \( d' < n - 2 \) is trivial. The restriction of \( \rho \) to \( U \) acts on \( \text{Eig}(\rho(e_1e_2), 1) \cong K^{d-2} \) and therefore
\[
\rho_U : U \to \text{SL}_{d-2}(K)
\]
is trivial. In particular, the element \( e_3e_4 \) is in \( U \) and acts trivially on \( \text{Eig}(\rho(e_1e_2), 1) \). It follows that \( \text{Eig}(\rho(e_1e_2), 1) \subseteq \text{Eig}(\rho(e_3e_4), 1) \). This argument is symmetric and we obtain
\[
\text{Eig}(\rho(e_1e_2), 1) = \text{Eig}(\rho(e_3e_4), 1).
\]
But then we have two commuting involutions with equal eigenspaces of eigenvalue 1, therefore \( \rho(e_1e_2) \) and \( \rho(e_3e_4) \) are equal and the element \( e_1e_2e_3e_4 \) acts trivially. We have \( n > 4 \) and by Proposition 2.1 the map \( \rho \) factors through \( \text{SL}_n(\mathbb{Z}_2) \). By Proposition 2.3 it follows that \( \rho \) is trivial. \( \square \)

We finish this work with the following remark.

**Remark 2.6.** The key ingredient in the proof of main theorem is that there exist only finitely many pairwise commuting involutions in \( \text{SL}_d(K) \) when \( \text{char}(K) \neq 2 \), see Proposition 2.2. This is no longer true for infinite fields of characteristic 2, as we can for example consider:

\[
\left\{ \begin{array}{ccc}
1 & a & 0 \\
0 & 1 & 0 \\
0 & 0 & I_{d-2}
\end{array} \right\} \quad a \in K^*, \quad \subseteq \text{SL}_d(K).
\]

**References**

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