MONOPOLES AND LYAPUNOV EXPONENTS IN U(1) LATTICE GAUGE THEORY

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Abstract. U(1) gauge fields are decomposed into a monopole and photon part across the phase transition from the confinement to the Coulomb phase. We analyze the leading Lyapunov exponents of such gauge field configurations on the lattice which are initialized by quantum Monte Carlo simulations. We observe a strong relation between the sizes of the monopole density and the Lyapunov exponent. Evidence is found that monopole fields stay chaotic in the continuum whereas the photon fields are regular.

1. Motivation

A main question in QCD is the existence of special classes of gauge fields which are responsible for confinement. In the framework of the dual superconductor, color magnetic monopoles arise as the appropriate candidates. Their investigation necessitates a projection of the non-Abelian gauge theory on the compact U(1) theory. In lattice computations it was demonstrated that monopoles account for more than 90 percent of the string tension in the confinement and they drop toward zero across the phase transition.

The study of chaotic dynamics of classical field configurations in field theory is motivated by phenomenological applications as well as by the understanding of basic principles. The role of chaotic field dynamics for the confinement of quarks is a longstanding question. Here, we analyze the leading Lyapunov exponents of compact U(1) configurations on the lattice. The real-time evolution of the classical field equations was initialized from Euclidean equilibrium configurations created by quantum Monte Carlo simulations. The U(1) gauge field is decomposed into a monopole and photon
part. This way we expect to learn some details between the appearance of monopoles and the strength of chaotic behavior in lattice simulations.

2. Monopoles in compact quantum field theories

We have investigated 4d $U(1)$ gauge theory described by the action

$$S\{U_i\} = \sum_p (1 - \cos \theta_p) ,$$

with $U_i = U_{x,\mu} = \exp(i\theta_{x,\mu})$ and $\theta_p = \theta_{x,\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x,\nu}$ ($\nu \neq \mu$). At critical coupling $\beta_c \approx 1.01$ $U(1)$ gauge theory undergoes a phase transition between a confinement phase with mass gap and monopole excitations for $\beta < \beta_c$ and the Coulomb phase with a massless photon for $\beta > \beta_c$, see Fig. 1.

We are interested in the relationship between monopoles of $U(1)$ gauge theory and classical chaos across this phase transition. Following Refs. [1, 2, 3, 4], we have factorized our gauge configurations into monopole and photon fields. The $U(1)$ plaquette angles $\theta_{x,\mu\nu}$ are decomposed into the “physical” electromagnetic flux through the plaquette $\tilde{\theta}_{x,\mu\nu}$ and a number $m_{x,\mu\nu}$ of Dirac strings passing through the plaquette

$$\theta_{x,\mu\nu} = \tilde{\theta}_{x,\mu\nu} + 2\pi m_{x,\mu\nu} ,$$

where $\tilde{\theta}_{x,\mu\nu} \in (-\pi, +\pi]$ and $m_{x,\mu\nu} \neq 0$ is called a Dirac plaquette. Monopole and photon fields are then defined in the following way

$$\theta_{x,\mu}^{\text{mon}} = -2\pi \sum_{x'} C_{x,x'} \theta'_{\nu} m_{x',\nu\mu}$$

Figure 1. Average of the energy $\langle E \rangle$ (left) and the absolute value of the Polyakov loop squared $\langle |W|^2 \rangle$ (right) as a function of $\beta$ on a $12^3 \times 4$ lattice.
\[ \theta_{x,\mu}^{\text{phot}} = -\sum_{x'} G_{x,x'} \partial' \hat{\theta}_{x',\nu} \cdot \] (4)

Here \( \partial' \) acts on \( x' \), the quantities \( m_{x,\mu\nu} \) and \( \hat{\theta}_{x,\mu
\nu} \) are defined in Eq. (2) and \( G_{x,x'} \) is the lattice Coulomb propagator. One can show that \( \hat{\theta}_{x,\mu} \equiv \theta_{x,\mu}^{\text{mon}} + \theta_{x,\mu}^{\text{phot}} \) is up to a gauge transformation identical with the original \( \theta_{x,\mu} \) defined by \( U_{x,\mu} = \exp(i\theta_{x,\mu}) \).

### 3. Classical chaotic dynamics from quantum Monte Carlo initial states

Chaotic dynamics in general is characterized by the spectrum of Lyapunov exponents. These exponents, if they are positive, reflect an exponential divergence of initially adjacent configurations. In case of symmetries inherent in the Hamiltonian of the system there are corresponding zero values of these exponents. Finally negative exponents belong to irrelevant directions in the phase space: perturbation components in these directions die out exponentially. Pure gauge fields on the lattice show a characteristic Lyapunov spectrum consisting of one third of each kind of exponents [5]. Assuming this general structure of the Lyapunov spectrum we investigate presently its magnitude only, namely the maximal value of the Lyapunov exponent, \( L_{\text{max}} \).

The general definition of the Lyapunov exponent is based on a distance measure \( d(t) \) in phase space,

\[ L := \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)}. \] (5)

In case of conservative dynamics the sum of all Lyapunov exponents is zero according to Liouville’s theorem, \( \sum L_i = 0 \). We utilize the gauge invariant distance measure consisting of the local differences of energy densities between two 3d field configurations on the lattice:

\[ d := \frac{1}{N_P} \sum_P \left| \text{tr} U_P - \text{tr} U'_P \right|. \] (6)

Here the symbol \( \sum_P \) stands for the sum over all \( N_P \) plaquettes, so this distance is bound in the interval \((0, 2N)\) for the group SU(N). \( U_P \) and \( U'_P \) are the familiar plaquette variables, constructed from the basic link variables \( U_{x,i} \),

\[ U_{x,i} = \exp \left( a A_{x,i}^c T^c \right), \] (7)

located on lattice links pointing from the position \( x = (x_1, x_2, x_3) \) to \( x + ae_i \). The generators of the group are \( T^c = -ig\tau^c/2 \) with \( \tau^c \) being the Pauli
matrices in case of SU(2) and $A^c_{x,i}$ is the vector potential. The elementary plaquette variable is constructed for a plaquette with a corner at $x$ and lying in the $ij$-plane as $U_{x,ij} = U_{x,i} U_{x+i,j} U_{x+j,i} U_{x,j}^\dagger$. It is related to the magnetic field strength $B^c_{x,k}$:

$$U_{x,ij} = \exp \left( \varepsilon_{ijk} a B^c_{x,k} T^c \right).$$

(8)

The electric field strength $E^c_{x,i}$ is related to the canonically conjugate momentum $P_{x,i} = \dot{U}_{x,i}$ via

$$E^c_{x,i} = \frac{2a}{g^3} \text{tr} \left( T^c \dot{U}_{x,i} U_{x,i}^\dagger \right).$$

(9)

The Hamiltonian of the lattice gauge field system can be casted into the form

$$H = \sum \left[ \frac{1}{2} \langle P, P \rangle + 1 - \frac{1}{4} \langle U, V \rangle \right].$$

(10)

Here the scalar product stands for $\langle A, B \rangle = \frac{1}{2} \text{tr}(AB^\dagger)$. The staple variable $V$ is a sum of triple products of elementary link variables closing a plaquette with the chosen link $U$. This way the Hamiltonian is formally written as a sum over link contributions and $V$ plays the role of the classical force acting on the link variable $U$.

Initial conditions chosen randomly with a given average magnetic energy per plaquette have been investigated in past years [6]. In the present study we prepare the initial field configurations from a standard four dimensional Euclidean Monte Carlo program on a $12^3 \times 4$ lattice varying the inverse gauge coupling $\beta$ [7]. We relate such four dimensional Euclidean lattice field configurations to Minkowskian momenta and fields for the three dimensional Hamiltonian simulation by identifying a fixed time slice of the four dimensional lattice.

4. Chaos, confinement and continuum limit

We start the presentation of our results with a characteristic example of the time evolution of the distance between initially adjacent configurations. An initial state prepared by a standard four dimensional Monte Carlo simulation is evolved according to the classical Hamiltonian dynamics in real time. Afterwards this initial state is rotated locally by group elements which are chosen randomly near to the unity. The time evolution of this slightly rotated configuration is then pursued and finally the distance between these two evolutions is calculated at the corresponding times. A typical exponential rise of this distance followed by a saturation can be inspected in Fig. 2 from an example of U(1) gauge theory in the confinement phase and in the
Figure 2. Exponentially diverging distance in real time of initially adjacent U(1) field configurations on a $12^3$ lattice prepared at $\beta = 0.9$ in the confinement phase (left) and at $\beta = 1.1$ in the Coulomb phase (right).

Figure 3. Monopole density (left) and Lyapunov exponent (right) of the decomposed U(1) fields as a function of coupling. The Lyapunov exponent being the only observable with visible error bars.

Coulomb phase. While the saturation is an artifact of the compact distance measure of the lattice, the exponential rise (the linear rise of the logarithm) can be used for the determination of the leading Lyapunov exponent. The left plot exhibits that in the confinement phase the original field and its monopole part have similar Lyapunov exponents whereas the photon part has a smaller $L_{max}$. The right plot in the Coulomb phase suggests that all slopes and consequently the Lyapunov exponents of all fields decrease substantially.

We now turn to a comparison with the monopole density in U(1) quan-
Figure 4. Comparison of average maximal Lyapunov exponents as a function of the scaled average energy per plaquette $ag^2E$. The full U(1) theory shows an approximately quadratic behavior in the weak coupling regime which is more pronounced for the photon field. The monopole field indicates a linear relation.

In Euclidean field configurations, 

$$Q_{\text{mon}} = \frac{1}{4V_4} \sum_{x,\mu\nu} |m_{x,\mu\nu}|. \quad (11)$$

The left plot of Fig. 3 exhibits for a statistics of 100 independent configurations that $Q_{\text{mon}}$ decreases sharply between the strong and the weak coupling regime. It can be seen that the photon part from the decomposition carries also a few monopoles.

The main result of the present study is the dependence of the leading Lyapunov exponent $L_{\text{max}}$ on the inverse coupling strength $\beta$, displayed in the right plot of Fig. 3. As expected the strong coupling phase, where confinement of static sources has been established many years ago by proving the area law behavior for large Wilson loops, is more chaotic. The transition reflects the critical coupling to the Coulomb phase. The right plot of Fig. 3 shows that the monopole fields carry Lyapunov exponents of nearly the same size as the full U(1) fields. The photon fields yield a non-vanishing value in the confinement ascending toward $\beta = 0$ for randomized fields which indicates that the decomposition (1) works perfectly for ideal configurations only.

An interesting result concerning the continuum limit can be viewed from Fig. 4 which shows the energy dependence of the Lyapunov exponents for the U(1) theory and its components. One observes an approximately linear relation for the monopole part while a quadratic relation is suggested for the photon part in the weak coupling regime. From scaling arguments one
expects a functional relationship between the Lyapunov exponent and the energy \[5, 8\]

\[ L(a) \propto a^{k-1}E^k(a), \tag{12} \]

with the exponent \(k\) being crucial for the continuum limit of the classical field theory. A value of \(k < 1\) leads to a divergent Lyapunov exponent, while \(k > 1\) yields a vanishing \(L\) in the continuum. The case \(k = 1\) is special leading to a finite non-zero Lyapunov exponent. Our analysis of the scaling relation (12) gives evidence, that the classical compact U(1) lattice gauge theory and especially the photon field have \(k \approx 2\) and with \(L(a) \to 0\) a regular continuum theory. The monopole field signals \(k \approx 1\) and stays chaotic approaching the continuum.

5. Conclusions

We investigated the classical chaotic dynamics of U(1) lattice gauge field configurations prepared by quantum Monte Carlo simulation. The fields were decomposed into a photon and monopole part. The maximal Lyapunov exponent shows a pronounced transition as a function of the coupling strength indicating that on a finite lattice configurations in the strong coupling phase are substantially more chaotic than in the weak coupling regime. The computations give evidence that the Lyapunov exponents in the original U(1) field and in its monopole part are very similar. The situation for the monopole density is analogous and serves as a consistency check of the decomposition. We conclude that classical chaos in field configurations and the existence of monopoles are intrinsically connected to the confinement of a theory.

We found evidence that the monopole fields stay chaotic in the continuum while the photon fields and the full U(1) fields possess a regular continuum theory. So far the monopoles are extracted from the Euclidean gauge fields. It will be interesting to compute their counterparts in Minkowski space and perform an analysis concerning monopole annihilation.

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