In quantum Darwinism, the pointer observable of a system leaves redundant imprints in its environment after decoherence. Each imprint is recorded in a fraction of the environment, which amounts to a partition of the environment. An ambiguity situation may occur when another observable non-commuting to the pointer observable also leaves redundant imprints with respect to another partition of the environment. We study this problem based on a uniqueness theorem we prove. We find that within a particular subset of all possible partitions of the environment, the observable leaving redundant imprints in the environment is “unique” (where “unique” means excluding all the observables non-commuting to it). And, in a typical situation, the partitions outside this particular subset have no physical significance.
I. INTRODUCTION

The environment of macroscopic objects plays an important role in the quantum-to-classical transition problem. On one hand, the environment of a macroscopic system can destroy interference between the states of the system through the (unavoidable) system-environment interaction. This is known as decoherence or environment-induced decoherence [1,2]. On the other hand, based on the stability argument, environment “selects” a set of states (referred as preferred states or pointer states) that are stable in spite of the presence of the system-environment interaction. Here “stable” means that if the initial state is a pointer state, it remains unentangled with its environment. Which set of the states are “selected” is essentially determined by the system-environment interaction. This is known as environment-induced superselection or einselection for short [3, 4]. Typically, the pointer states are just the “classical states” we perceive in our daily lives, such as the approximate eigenstates of the center-of-mass position of a macroscopic object. In ideal situation, pointer states form a basis of the system’s Hilbert space and are eigenstates of the so-called pointer observables. For more details of decoherence theory, we refer the readers to the reviews [3, 5] and references therein.

Another important role played by environment has been recognized in quantum Darwinism [9–14]. Through decoherence process, selective information of the system spreads redundantly into the environment. According to quantum no-cloning theorem [13], the environment cannot copy the (unknown) entire quantum state of the system, so only the information of selective observables can be copied. The environment of a macroscopic object typically consists of numerous subsystems, such as photons, molecules, etc.. So after decoherence and proliferation of selective information, the information would have many copies, and each of the copies is recorded in a fraction of the environment. The importance of the existence of redundant records of selective information in the environment is that it allows many observers to observe the selected properties of the system independently without disturbing it. This is essential for the “classical world” to be objective. Here, a property of a system is objective if it is (1) simultaneously accessible to many independent observers (2) who can find out what it is without prior knowledge, and (3) who can arrive at a consensus about it [10]. Quantum Darwinism has been studied in various dynamical models [16–22]. In recent years, another idea that can explain the information redundancy and the emergence of the objectivity from quantum world, known as the spectrum broadcast structure framework, has also been proposed and studied [23–26].

Quantum Darwinism tells us that selective observables of the system can be recorded redundantly in the environment, which is actually a selection rule. In decoherence theory, pointer observables are “selected” according to the environment-induced superselection rule. The two selection rules should coincide. Indeed, it has been shown in Refs. [9, 10] that the only observables that can leave sufficiently many imprints in the environment are the pointer observables. However if the number of the imprints are not sufficiently large (the meaning of “sufficiently” shall be clear shortly), an ambiguity situation may occur. For example, consider a 4-dimensional system interacting with its environment consisting of four 2-dimensional elementary subsystems labeled as α, β, γ, δ. Suppose Λ ≡ ∑ 4 i=1 λi |φi S⟩ ⟨φi S| is the pointer observable, where |φi S⟩ (i = 1, 2, 3, 4) form an orthogonal basis in the Hilbert space of the system, and the eigenvalues λi’s are all different. If, after an ideal decoherence process, the state of the whole system (including environment) becomes

\[ |Ψ⟩ = \frac{1}{\sqrt{4}} \sum_{j=1}^{4} |φ_j S⟩ ⟨φ_j S| |φ_j B⟩ \]

\[ = \frac{1}{4} \left\{ |φ_1 S⟩ (|00⟩ + |11⟩)_A(|00⟩ + |11⟩)_B + |φ_2 S⟩ (|01⟩ + |10⟩)_A(|01⟩ + |10⟩)_B + |φ_3 S⟩ (|00⟩ − |11⟩)_A(|00⟩ − |11⟩)_B \right\}, \]  

where the subscripts A, B represent fractions of the environment: A = {α, β}, B = {γ, δ}, and we have used the notation of qubits to express the environmental subsystems. (We are considering perfect correlations here.) Then the whole information of the pointer observable Λ has been recorded in the environment with two copies in the fractions A and B, i.e., the pointer observable leaves redundant imprints in the environment. However, noticing that the environment can be partitioned in different ways, the state |Ψ⟩ can as well be expressed in another form, with respect to (w.r.t.) another partition of the environment: a = {α, γ}, b = {β, δ}, as

\[ |Ψ⟩ = \frac{1}{\sqrt{4}} \sum_{j=1}^{4} |ψ_j S⟩ ⟨ψ_j S| |ψ_j B⟩ \]

\[ = \frac{1}{4} \left\{ |ψ_1 S⟩ (|00⟩ + |11⟩)_a(|00⟩ + |11⟩)_b + |ψ_2 S⟩ (|00⟩ − |11⟩)_a(|00⟩ − |11⟩)_b + |ψ_3 S⟩ (|01⟩ + |10⟩)_a(|01⟩ + |10⟩)_b + |ψ_4 S⟩ (|01⟩ − |10⟩)_a(|01⟩ − |10⟩)_b \right\}, \]  

(1, 2)
where
\[
|\psi_i^S\rangle = \frac{1}{2} (|\phi_1^S\rangle + |\phi_2^S\rangle - |\phi_3^S\rangle + |\phi_4^S\rangle), \quad |\psi_2^S\rangle = \frac{1}{2} (|\phi_1^S\rangle - |\phi_2^S\rangle + |\phi_3^S\rangle + |\phi_4^S\rangle),
\]
\[
|\psi_3^S\rangle = \frac{1}{2} (|\phi_1^S\rangle + |\phi_2^S\rangle + |\phi_3^S\rangle - |\phi_4^S\rangle), \quad |\psi_4^S\rangle = \frac{1}{2} (|\phi_1^S\rangle - |\phi_2^S\rangle - |\phi_3^S\rangle - |\phi_4^S\rangle).
\]

$|\psi_i^S\rangle$ ($i = 1, 2, 3, 4$) form another orthonormal set, and could be eigenstates of another observable which does not commute with the pointer observable $\Lambda$. So another observable non-commuting to the pointer observable also leaves redundant imprints in the environment according to Eq. (2).

The ambiguity of this type has been studied in Ref. [10]. The authors of that paper proved that as long as the redundancies of two observables $R_0(X_1)$ and $R_0(X_2)$ (where, $R_0(X)$ is, roughly speaking, the number of the copies recording the information about $X$ in the environment) satisfy $R_0(X_1) R_0(X_2) > N$ where $N$ is the number of the environmental subsystems, the ambiguity will not occur. So, they set a requirement that the redundancy should be sufficiently large, i.e., $R_0(X) > \sqrt{N}$. The example presented here doesn’t satisfy this requirement.

It is also argued that the partition of the environment (equivalently, the decomposition of the Hilbert space into tensor products) should be fixed [11]. In Ref. [27], a proposition has been proposed that the decomposition of the Hilbert space should be induced by measurable observables. This proposition essentially implies that the decomposition is determined by what measurements people can perform. So it is not suitable to apply in quantum Darwinism (which is aiming at explaining the emergence of classical properties) context.

In this work, we reconsider this ambiguity problem from a different perspective by studying the entanglement structure of the state of the whole system. We find a uniqueness theorem (named as the extended tridecomposition uniqueness theorem) which applies to a particular subset of all possible partitions of the whole system. Within this subset, the observable leaving redundant imprints in the environment is “unique”, where “unique” means excluding all the observables non-commuting to it. By “redundant”, we mean that the redundancy is larger or equal to 2. In a special case, when every fraction of the environment is accessible to one local observer for the partitions within the subset that the uniqueness theorem applies, for partitions outside this subset, every fraction of the environment is only accessible to a super/nonlocal observer (the meaning of which will be explained later), which has no physical significance. This helps us to gain further insight of the emergence of the objectivity problem. In addition, based on the theorem, we can reproduce the relevant results given in Ref. [10]. The benefit of our method is that the results are free from dynamical details or model details. Our results presented here can also be applied to the spectrum broadcast structure framework.

The remaining part of this paper is organized as follows. In section II, we set up conventions and introduce the extended tridecomposition uniqueness theorem. In section III based on this theorem, we study the problem whether and on what condition only the pointer observable can leave redundant imprints in the environment. The proof of the theorem is presented in section IV and section V is devoted to concluding remarks.

II. THE EXTENDED TRIDECOMPOSITION UNIQUENESS THEOREM

A. Definitions and Conventions

Consider a macroscopic object bathed in its environment. We shall call the object the system of interest or simply the system, but refer the system+environment as the whole system. Both the system of interest and its environment are composed of numerous elementary systems, such as atoms, photons, etc.. So the Hilbert space of the whole system is constructed by the direct product of the Hilbert spaces of all the elementary systems $H = \bigotimes_i H^{i_{ES}}$. Although the system of interest is usually described by a density matrix, the whole system can be described by a state vector in the whole Hilbert space. Now let’s consider the general situation without making the distinction between the system and the environment. The whole system can be partitioned into subsystems/fractions in different ways, correspondingly, the Hilbert space can be decomposed into subspaces in different ways. For a given whole system, any two possible partitions can be clarified by writing the decompositions of the Hilbert space

\[ A : H = H^{(1)}_A \otimes H^{(2)}_A \otimes \cdots \otimes H^{(n)}_A, \]
\[ B : H = H^{(1)}_B \otimes H^{(2)}_B \otimes \cdots \otimes H^{(n)}_B, \]

and specifying each fractions $H^{(i)}_{A/B}$ (we use the same notation for a fraction and its corresponding Hilbert subspace). For a given fraction in Partition A (PAT.A), say $H^{(i)}_A$, its elementary systems must be distributed in one or several
fractions of Partition B (PAT.B); \( H_B^{(j_1)}, H_B^{(j_2)}, \ldots, H_B^{(j_k)} \). In other words, elementary systems of \( H_B^{(j_i)} \) belong to (and no more than) \( H_B^{(j_1)}, H_B^{(j_2)}, \ldots, H_B^{(j_k)} \) when looked in PAT.B. We denote the set of fractions \( H_B^{(j_1)}, H_B^{(j_2)}, \ldots, H_B^{(j_k)} \) as \( \xi_{A \rightarrow B} = \{ H_B^{(j_1)}, H_B^{(j_2)}, \ldots, H_B^{(j_k)} \} \).

For example, a whole system consisting of six elementary systems can be partitioned in two ways as \( A : \{ 1, 2, 3, 4, 5, 6 \} \) and \( B : \{ 1, 2, 4, 3, 5, 6 \} \), where the numbers represent the elementary systems, and the fractions are separated with the vertical lines. Correspondingly, the whole Hilbert space is decomposed in two ways as

\[
A : H = H_A^{(1)} \otimes H_A^{(2)} \otimes H_A^{(3)},
\]

\[
B : H = H_B^{(1)} \otimes H_B^{(2)} \otimes H_B^{(3)},
\]

where \( H_A^{(1)} = H_{2S}^E \otimes H_{3S}^E \otimes H_{4S}^E, H_A^{(2)} = H_{5S}^E \otimes H_{6S}^E; H_B^{(1)} = H_{2S}^E \otimes H_{4S}^E, H_B^{(2)} = H_{3S}^E \otimes H_{5S}^E \otimes H_{6S}^E; \) and \( H_A^{(1)} = H_B^{(1)} = H_{ES} \). The elementary systems of fraction \( H_A^{(2)} \), i.e., \( \{ 2, 3, 4 \} \) are distributed in \( H_B^{(2)} \) and \( H_B^{(3)} \) when shown in partition B. So \( \xi^{A \rightarrow B}_2 \) denotes the set \( \{ H_B^{(2)}, H_B^{(3)} \} \).

**Definitions:** We call PAT.B comparable to PAT.A if there exists at least two fractions in PAT.B, say \( H_B^{(j_1)} \) and \( H_B^{(j_2)} \) (\( j_1 \neq j_2 \)), such that the corresponding sets \( \xi_{B \rightarrow A}^{j_1} \) and \( \xi_{B \rightarrow A}^{j_2} \) satisfy (a) \( \xi_{B \rightarrow A}^{j_1} \cap \xi_{B \rightarrow A}^{j_2} = \emptyset \) and (b) \( \xi_{B \rightarrow A}^{j_1} \cup \xi_{B \rightarrow A}^{j_2} \subseteq \{ A \} \), where \( \{ A \} \equiv \{ H_A^{(1)}, H_A^{(2)} \ldots H_A^{(n)} \} \) denotes the set of all fractions of PAT.A. If PAT.A is also comparable to PAT.B, they are mutually comparable. For a given partition, say PAT.A, all the partitions that are mutually comparable to PAT.A and PAT.A itself form a set, which is called the comparable set of PAT.A.

According to the definition, the partitions inside a comparable set must be m-partitions with \( m \geq 3 \). In the example illustrated by Eqs. (5, 6), the two partitions are mutually comparable.

### B. The Theorem

The physical situation we are interested in is that the whole state of system-environment has become entangled after decoherence. Since entanglement is only meaningful w.r.t. particular partitions of the whole system, a state entangled w.r.t. one partition may be unentangled or entangled in a different manner w.r.t. another partition. The following theorem clarifies some aspects of this problem.

**Definition:** For a partition of the whole system denoted as PAT.A: \( H = H_A^{(1)} \otimes H_A^{(2)} \otimes \cdots \otimes H_A^{(n)} \) (\( n \geq 3 \)), if a state can be written as \( |\Psi\rangle = \sum_{i=1}^{n} \alpha_i |\psi_i^{(1)}\rangle |\psi_i^{(2)}\rangle \cdots |\psi_i^{(n)}\rangle \), where each set of \( |\psi_i^{(j)}\rangle \) is a linearly independent and normalized set in the subspace \( H_A^{(j)} \), every \( \alpha_i \) is nonzero and \( i \geq 2 \), then |\Psi\rangle is called a semi-GHZ state w.r.t. PAT.A.

**Theorem:** If a state is a semi-GHZ state w.r.t. PAT.A: \( |\Psi\rangle = \sum_{i=1}^{n} \alpha_i |\psi_i^{(1)}\rangle |\psi_i^{(2)}\rangle \cdots |\psi_i^{(n)}\rangle \), then with the comparable set of PAT.A, the state cannot be a semi-GHZ state w.r.t. a different partition, say PAT.B: \( \sum_{i=1}^{n} \kappa_i |\phi_i^{(1)}\rangle |\phi_i^{(2)}\rangle \cdots |\phi_i^{(m)}\rangle \), unless \( |\psi_i^{(1)}\rangle |\psi_i^{(2)}\rangle \cdots |\psi_i^{(n)}\rangle = |\phi_i^{(1)}\rangle |\phi_i^{(2)}\rangle \cdots |\phi_i^{(m)}\rangle \) for every \( i \) (up to a phase), with a proper ordering of the expanding bases.

This theorem is an extension to the tridecomposition uniqueness theorem which works for a fixed partition \( \mathbb{2} \), so we shall call this theorem as the extended tridecomposition uniqueness theorem.

The theorem actually says that it is impossible to expand a state in two different sets of correlation bases w.r.t. two mutually comparable partitions. To check whether the extended tridecomposition uniqueness theorem applies or not, one needs to check whether two partitions are mutually comparable or not. There exists a sufficient condition for two partitions to be mutually comparable. Now, for a given partition C, denote the number of elementary systems in one fraction (\( j \)) as \( L_C^{(j)} \), and the largest(smallest) \( L_C^{(j)} \) among all fractions as \( L_C^{\text{max}}(L_C^{\text{min}}) \). Two partitions PAT.A and PAT.B, where PAT.A is an n-partition and PAT.B is an m-partition, are mutually comparable if

\[
\begin{align*}
n > L_A^{\text{min}}(L_B^{\text{max}} - 1) + 1, \quad & n > L_B^{\text{min}} + L_B^{\text{max}}, \\
m > L_B^{\text{min}}(L_A^{\text{max}} - 1) + 1, \quad & m > L_A^{\text{min}} + L_A^{\text{max}}.
\end{align*}
\]
III. PHYSICAL IMPLICATIONS OF THE THEOREM

A. Perfect Correlation

Now consider a system of interest has decohered by its environment. The pointer bases of the system are represented by \(|\psi_i^{(S)}\rangle \ (i = 1, 2, \ldots, d_S)\), where \(d_S\) is the dimensionality of the system. Suppose a perfect correlation among the system and the fractions of the environment has established, and the state has a GHZ-like form w.r.t. a particular partition:

\[
|\Psi\rangle = \sum_{i=1}^{d} \alpha_i |\psi_i^{(S)}\rangle |\psi_i^{(1)}\rangle \cdots |\psi_i^{(n)}\rangle,
\]

where \(|\psi_i^{(j)}\rangle\)'s form a set of orthogonal bases of the Hilbert subspace of the environmental fraction \(j\). In Eq. (9) every \(\alpha_i\) is nonzero and \(i\) is no greater than the minimum dimensionality among the system and every environmental fraction. Denote this partition as \(\text{PAT.A}\) which is clarified by \(H = H_A^{(S)} \otimes H_A^{(1)} \otimes \cdots \otimes H_A^{(n)}\), where \(H_A^{(S)}\) represents the Hilbert space of the system and \(H_A^{(j)}\) represents the Hilbert space of the \(j\)th environmental fraction. (If we consider the density matrix \(\rho|\Psi\rangle\langle\Psi|\) and trace out one of the fractions of the environment, we obtain a so-called spectrum broadcast structure [22].)

The structure of the state shows that the information of the pointer observable has proliferated in the environment, which can be seen by evaluating the classical mutual information between the system and each fraction of the environment. The classical mutual information of an observable \(\hat{\Lambda}\) in the Hilbert space of the system and an observable \(\hat{F}\) in the Hilbert space of an environmental fraction is defined as

\[
I(\hat{S} : \hat{F}) = H(\hat{S}) - H(\hat{S}|\hat{F}),
\]

where \(H(\hat{S}) = -\sum_i p(S_i) \log p(S_i)\) is the Shannon entropy, and \(p(S_i)\) is the probability associated with the measurement outcome \(S_i\). \(H(\hat{S}|\hat{F}) = -\sum_{ij} p(F_j)p(S_i|F_j) \log p(S_i|F_j)\), where \(p(S_i|F_j)\) is the conditional probability of \(S_i\) given \(F_j\). The maximum \(I(\hat{S} : \hat{F}) = H(\hat{S})\) occurs when \(H(\hat{S}|\hat{F}) = 0\), which means we can gain the complete information about \(\hat{S}\) by measuring \(\hat{F}\). Now introduce the pointer observable \(\hat{\Lambda} = \sum_i \lambda_i |\psi_i^{(S)}\rangle \langle \psi_i^{(S)}|\), where \(\lambda_i\)'s are all real and nonequal, and an observable of the environmental fraction \(j\): \(\hat{\Omega} = \sum_i \omega_i |\psi_i^{(j)}\rangle \langle \psi_i^{(j)}|\), where \(\omega_i\)'s are all real and nonequal. (We only consider the non-degenerate case because \(\hat{\Lambda}\) and \(\hat{\Omega}\) could represent a complete set of commuting observables in their corresponding subspaces. There is a one-to-one correspondence between \(\lambda_i/\omega_i\) and the set of eigenvalues of the commuting observables.) From Eq. (9), it is easy to see that \(H(\hat{\Lambda}|\hat{\Omega}) = 0\), so the pointer observable leaves a perfect imprint in the fraction \(j\), which is also true for any other environmental fraction in \(\text{PAT.A}\). On the other hand, if \(H(\hat{\Lambda}|\hat{F}^{(j)}) = 0\) for some observable \(\hat{F}^{(j)}\) acting on the \(j\)th fraction and for every \(j\), then the state \(|\Psi\rangle\) must be able to be expressed in a GHZ-like form as Eq. (9). (If the eigenstates of \(\hat{F}^{(j)}\) are not correlated to \(|\psi_i^{(S)}\rangle\)'s, we can always construct some other observable \(\hat{F}^{(j)}\) whose eigenstates are one-to-one correlated to \(|\psi_i^{(S)}\rangle\)'s.)

So, every GHZ-like expression of a given state corresponds to an information (perfect) multiplication of a particular observable.

Given Eq. (9), for \(\text{PAT.A}\), there is no alternative expansion of \(|\Psi\rangle\) having a GHZ-like structure according to the tridecomposition uniqueness theorem [22], so only the pointer observable \(\hat{\Lambda}\) has redundant imprints in the environment w.r.t. \(\text{PAT.A}\). But if different partitions of environment are possible, then the state may have a different GHZ-like expression (as shown by the example displayed in Introduction), and a different observable of the system may leave multiple imprints in the environment as well. Regarding this problem, the extended tridecomposition uniqueness theorem tells that, as long as the partitions are restricted within the comparable set of \(\text{PAT.A}\), there is no alternative expansion of \(|\Psi\rangle\) having a GHZ-like structure. Thus, within the comparable set of \(\text{PAT.A}\), only the pointer observable \(\hat{\Lambda}\) has redundant imprints in the environment.

Then, how about the partitions outside the comparable set of \(\text{PAT.A}\)? We argue that, if for \(\text{PAT.A}\), every fraction of the environment is accessible to a local observer, then for any partition (with one subsystem being the system of interest) outside the comparable set of \(\text{PAT.A}\), every fraction of the environment is only accessible to a super/nonlocal observer, where a super/nonlocal observer is an observer that is able to gain the information about the whole environment. To explain this, we first show what mutually comparable implies in the present case. Consider another partition denoted as \(\text{PAT.B}: H = H_B^{(S)} \otimes H_B^{(1)} \otimes \cdots \otimes H_B^{(n)}\). Since we are considering the partitions with the system of interest fixed, we have \(H_A^{(S)} = H_B^{(S)}\) and \(\xi^{B\rightarrow A} = \{H_A^{(S)}\}\) and vice versa. And it is obvious that \(\xi^{B\rightarrow A} \cap \xi^{B\rightarrow A} = \emptyset\) for any fraction \(j\) of the environment w.r.t. \(\text{PAT.B}\). So, if \(\xi_{j}^{B\rightarrow A} \subset \{H_A^{(1)}, H_A^{(2)}, \ldots, H_A^{(n)}\}\) for
at least one \( j \), then \( \text{PAT}.B \) is comparable to \( \text{PAT}.A \); on the other hand, if \( \xi_{j}^{B \rightarrow A} = \{ H_{A}^{(1)}, H_{A}^{(2)}, \ldots, H_{A}^{(n_{e})} \} \) for every \( j \) then \( \text{PAT}.B \) is \textit{not} comparable to \( \text{PAT}.A \). In addition, one can prove that if \( \xi_{j}^{B \rightarrow A} = \{ H_{A}^{(1)}, H_{A}^{(2)}, \ldots, H_{A}^{(n_{e})} \} \) for every \( j \), then \( \xi_{j}^{A \rightarrow B} = \{ H_{B}^{(1)}, H_{B}^{(2)}, \ldots, H_{B}^{(n_{e})} \} \) for every \( j \). So, \( \text{PAT}.B \) is not comparable to \( \text{PAT}.A \) if and only if

\[
\xi_{j}^{B \rightarrow A} = \{ H_{A}^{(1)}, H_{A}^{(2)}, \ldots, H_{A}^{(n_{e})} \} \text{ for every } j.
\]

The physical meaning of the condition \( \xi_{j}^{B \rightarrow A} \) must be distributed in all the fractions of the environment in \( \text{PAT}.A \). When the environment is divided into fractions each accessible to a local observer, an observer that is able to receive information from all these fractions should be a nonlocal observer, because in this situation there is no reason to forbid the observer to gain information of the whole environment. So, if every environmental fraction of \( \text{PAT}.A \) is accessible to a local observer, then every environmental fraction of \( \text{PAT}.B \) can only be “watched” by a nonlocal observer. To get an intuitive picture, consider a macroscopic object decohered by its environmental photons. The photons spread into a wide open space so no local observer can receive whole of them, but a small fraction of them which occupies a location with finite extension could be received by a local observer. Dividing the whole environment into such fractions, then an observer who receives photons from all the fractions must occupy the whole space that the environment spreads into, so this observer is able to receive the whole environmental photons and is a nonlocal observer. Recalling that for decoherence theory and quantum Darwinism to work, nonlocal observers should be excluded, because for a super observer “watching” the whole state, decoherence had never happened. Since in the present case, every environmental fraction of \( \text{PAT}.B \) is only accessible to a nonlocal observer, this partition has no physical significance. Although we use the word “observer” in our arguments, it doesn’t necessarily refer to a live being. A local observer is a physical system with finite extension which can only communicate with part of the environment; on the contrary, a nonlocal/super observer is the physical system that can communicate with the whole system.

The situation discussed before is common in real world, but in principle, there could be more sophisticated situations such as when not all of the environmental fractions in \( \text{PAT}.A \) are accessible to local observers. In such situations, we still need the requirement already indicated in Ref. [10] that the redundancy should be sufficiently large to guarantee no occurrence of ambiguity. To give a quantitative criterium, introduce the redundancy of an operator \( \hat{X} \) (acting on the Hilbert space of the system of interest): \( R_{\delta}(\hat{X}) \), defined as the number of the disjoint fractions of the environment containing all but a fraction \( \delta \) of the information about \( \hat{X} \) present in the entire environment. In the perfect correlation case, we have \( \delta = 0 \) and the redundancy of the pointer observable is \( R_{0}(\hat{A}) \). Implied by Eq. (8), the redundancy \( R_{0}(\hat{A}) \) is equal or greater than \( n_{e} \). When \( R_{0}(\hat{A}) \) is greater than \( n_{e} \), we could fine-grain the partition of the environment and eventually, we could write the state in a form such that each fraction can no longer be finer-grained while still contain the whole information of \( \hat{A} \). Suppose Eq. (8) is the finest-grained in this sense, we have \( R_{0}(\hat{A}) = n_{e} \). If there is another partition, say \( \text{PAT}.B \), with respect to which the state \( \{ \Psi \} \) can have a (finest-grained) GHZ-like form, then there is another observable \( \hat{A} \) whose redundancy is \( R_{0}(\hat{A}) = m_{e} \) where \( m_{e} \) is the number of environmental fractions. Suppose the total number of elementary systems of the environment is \( N_{e} \), in order for these two partitions to satisfy the requirements to apply the extended tridecomposition uniqueness theorem, we only need \( n_{e} > L_{B_{\text{min}}} \) or \( m_{e} > L_{A_{\text{min}}} \), where \( L_{A_{\text{min}}} (L_{B_{\text{min}}}) \) represents the minimum number of elementary systems in one fraction of \( \text{PAT}.A \) (\( \text{PAT}.B \)). Noticing that \( L_{A_{\text{min}}} \leq \frac{N_{e}}{n_{e}} \) and \( L_{B_{\text{min}}} \leq \frac{N_{e}}{m_{e}} \), so if \( n_{e}m_{e} > N_{e} \) or equivalently \( R_{0}(\hat{A})R_{0}(\hat{A}') > N_{e} \), then the requirements would be satisfied and the extended tridecomposition uniqueness theorem applies, which means that the two GHZ-like expressions of \( \{ \Psi \} \) w.r.t. \( \text{PAT}.A \) and \( \text{PAT}.B \) must be identical and \( [\hat{A}, \hat{A}'] = 0 \). It follows immediately that if we only admit redundancies satisfying \( R_{0} > \sqrt{N_{e}} \), then the observable leaving redundant imprints in the environment is “unique”. This is exactly the result obtained in Ref. [10]. We obtain it using a different method.

### B. Imperfect Correlation

In most realistic situations, decoherence processes only induce a fast damping of the coherent phases, and the perfect decoherence never occur in finite time. So it is important to study the case when the correlation among the system and the environmental fragments is imperfect. In this case, we can express the state as

\[
\{ \Psi \} = \sum_{i=1}^{i_{\text{f}}} \alpha_{i}|\psi_{i}^{(S)}\rangle|\psi_{i}^{(1)}\rangle \ldots |\psi_{i}^{(n_{e})}\rangle + \sum_{\{ i \}} \beta_{\{ i \}}|\psi_{i}^{(S)}\rangle|\psi_{i}^{(1)}\rangle \ldots |\psi_{i}^{(n_{e})}\rangle,
\]

(11)

where \( \{ i \} \) denotes \( \{ i_{s}, i_{1}, \ldots, i_{n_{e}} \} \) excluding the cases when \( i_{s} = i_{1} = \cdots = i_{n_{e}} \). Such an expansion for any state is always possible because all the states \( |\psi_{i}^{(S)}\rangle|\psi_{i}^{(1)}\rangle \ldots |\psi_{i}^{(n_{e})}\rangle \) form a complete set of orthonormal bases in the whole Hilbert space. We call each of the bases a product basis w.r.t. \( \text{PAT}.A \). Only when \( \sum_{\{ i \}} |\beta_{\{ i \}}|^{2} \) is finite and
\[ \sum_{(i)} |\beta_{(i)}|^2 \ll \sum_j |\alpha_j|^2, \] the state is said to have an approximate GHZ-like form w.r.t. PAT.A, and the observable \( \hat{A} \) leaves redundant but imperfect imprints with redundancy \( R_{\hat{A}}(\hat{A}) \geq n_e \) in the environment.

Before we can draw any conclusion, let’s first consider the following situation. Suppose a state \( |\Psi_0\rangle \) has a perfect correlation among the system and the environmental fractions w.r.t. PAT.A. This state can also be expanded in another complete set of product bases w.r.t. PAT.B as

\[ |\Psi_0\rangle = \sum_{i=1}^{i'} \alpha_i |\psi_i^{(S)}\rangle |\psi_i^{(1)}\rangle \ldots |\psi_i^{(n_e)}\rangle = \sum_{i=1}^{i'} \alpha_i' |\phi_i^{(S)}\rangle |\phi_i^{(1)}\rangle \ldots |\phi_i^{(m_e)}\rangle + \sum_{(i)} \beta_{(i)}' |\phi_{(i)}^{(S)}\rangle |\phi_{(i)}^{(1)}\rangle \ldots |\phi_{(i)}^{(m_e)}\rangle, \]

where for each \( x \), \( |\phi_x^{(x)}\rangle \)'s form a complete set of orthogonal bases in the Hilbert space of the subsystem \( x \). For PAT.B, there are infinitely many complete sets of product bases. For the same bases, we serve different ordering as forming different sets. (For instance, for a bipartite whole system, if one of its complete sets of product bases is \( \{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle, |\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle, |\phi_2^{(1)}\rangle, |\phi_1^{(2)}\rangle\} \), then after reordering \( |\phi_2^{(2)}\rangle \), i.e. \( |\phi_2^{(2)}\rangle \), the new set is considered as a different one.) When varying the product bases, the coefficients \( \alpha_i' \) and \( \beta_{(i)}' \) vary accordingly.

In complete set, the \( k \)th product basis can be described by its wave function, i.e., \( C^k(i) \), in a reference representation. Then \( \delta_2 \equiv \sum_{(i)} |\beta_{(i)}'|^2 \) is an analytic function of \( C^k(i) \)'s. For PAT.B within the comparable set of PAT.A, the extended tridecomposition uniqueness theorem tells that \( \delta_2 = 0 \) only when the set of \( |\psi_1^{(S)}\rangle |\psi_1^{(1)}\rangle \ldots |\psi_1^{(m_e)}\rangle \) and the set of \( |\phi_1^{(S)}\rangle |\phi_1^{(1)}\rangle \ldots |\phi_1^{(m_e)}\rangle \) coincide (up to a phase). So for sufficiently small \( \delta_2 \), the set of \( |\phi_1^{(S)}\rangle |\phi_1^{(1)}\rangle \ldots |\phi_1^{(m_e)}\rangle \) must be approximately coincide with the set of \( |\psi_1^{(S)}\rangle |\psi_1^{(1)}\rangle \ldots |\psi_1^{(m_e)}\rangle \). To be precise, writing

\[ \langle \psi_1^{(n_e)}, \ldots, \psi_1^{(1)} | \psi_1^{(1)} \rangle | \psi_2^{(1)} \rangle \ldots | \psi_2^{(m_e)} \rangle = e^{i\theta_2} (\delta_{ij} + \epsilon_{ij}), \]

then for sufficiently small \( \delta_2 \), we must have \( |\epsilon_{ij}| \ll 1 \) for every legal \( i, j \) under a proper ordering.

Now, go back to the general situation Eq. (11) when the state has an approximate GHZ-like form. Expanding it in another complete set of product bases w.r.t. PAT.B:

\[ |\Psi\rangle = \sum_{i=1}^{i'} \alpha_i |\psi_i^{(S)}\rangle |\psi_i^{(1)}\rangle \ldots |\psi_i^{(n_e)}\rangle + \sum_{(i)} \beta_{(i)} |\psi_{(i)}^{(S)}\rangle |\psi_{(i)}^{(1)}\rangle \ldots |\psi_{(i)}^{(n_e)}\rangle = \sum_{i=1}^{i'} \alpha_i' |\phi_i^{(S)}\rangle |\phi_i^{(1)}\rangle \ldots |\phi_i^{(m_e)}\rangle + \sum_{(i)} \beta_{(i)}' |\phi_{(i)}^{(S)}\rangle |\phi_{(i)}^{(1)}\rangle \ldots |\phi_{(i)}^{(m_e)}\rangle. \]

Moving the term \( \sum_{(i)} \beta_{(i)} |\psi_{(i)}^{(S)}\rangle |\psi_{(i)}^{(1)}\rangle \ldots |\psi_{(i)}^{(n_e)}\rangle \) to the right hand side of the second equality sign, and expanding it in the \( |\phi_{(i)}^{(S)}\rangle |\phi_{(i)}^{(1)}\rangle \ldots |\phi_{(i)}^{(m_e)}\rangle \) bases, we can make the same argument as we did for Eq. (12). It turns out that for sufficiently small \( \sum_{(i)} |\beta_{(i)}|^2 + \sum_{(i)} |\beta_{(i)}'|^2 \geq |\psi_{(i)}^{(S)}\rangle |\psi_{(i)}^{(1)}\rangle \ldots |\psi_{(i)}^{(n_e)}\rangle \) and \( |\phi_{(i)}^{(S)}\rangle |\phi_{(i)}^{(1)}\rangle \ldots |\phi_{(i)}^{(m_e)}\rangle \) approximately coincide, i.e., \( |\epsilon_{ij}| \ll 1 \) for every legal \( i, j \), under a proper ordering. Writing \( \langle \psi_i^{(S)} | \phi_i^{(S)} \rangle = e^{|\theta_1|} (\delta_{ij} + \epsilon_{ij}), \langle \psi_i^{(1)} | \phi_i^{(1)} \rangle = e^{|\theta_2|} (\delta_{ij} + \epsilon_{ij}), \ldots \), we find \( e^{|\theta_1|} (\delta_{ij} + \epsilon_{ij}) = e^{|\theta_2|} (\delta_{ij} + \epsilon_{ij}) e^{|\theta_3|} (\delta_{ij} + \epsilon_{ij}) \cdots = e^{|\theta_s|+|\theta_1|+\ldots} (\delta_{ij} + \delta_{ij} \epsilon_{ij} + \delta_{ij} \epsilon_{ij} + \cdots) \) using Eq. (13). For \( i = j, \epsilon_{ii} \) is (at most) the same order as \( \epsilon_{ii} \), i.e., \( |\epsilon_{ii}| \ll 1 \). So, within the comparable set of PAT.A, the states having redundant imperfect imprints in the environment are approximately the pointer states.

IV. PROOF OF THE THEOREM

In this section, we prove the extended tridecomposition uniqueness theorem.

Lemma: Consider a Hilbert space \( \mathcal{H} \) and its two decompositions PAT.A having \( n \) subspaces and PAT.B having \( m \) subspaces (see Eqs. (4) (5)). A factorized state \( |\Psi\rangle = |\psi^{(1)}\rangle |\psi^{(2)}\rangle \ldots |\psi^{(n)}\rangle \) w.r.t. PAT.A can not be written as \( |\Psi\rangle = \sum_{i=1}^{n'} v_i |\phi_i^{(1)}\rangle |\phi_i^{(2)}\rangle \ldots |\phi_i^{(m)}\rangle (v_i \neq 0) \) w.r.t. PAT.B where \( n' \geq 2 \), if the following two conditions are satisfied:
(a) There exists at least two fractions of PAT.B, say \( H_B^{(j_1)} \) and \( H_B^{(j_2)} \) \((j_1 \neq j_2)\), such that the corresponding sets \( \xi_1^{B \rightarrow A} \) and \( \xi_2^{B \rightarrow A} \) satisfy \( \xi_1^{B \rightarrow A} \cap \xi_2^{B \rightarrow A} = \emptyset \). (b) The states \( |\phi_i^{(1)}\rangle \) \((i=1,2,\ldots)\) are independent, and the states \( |\phi_i^{(2)}\rangle \) \((i=1,2,\ldots)\) are non-collinear, or vice versa.
Proof: Without losing generality, we take \( j_1 = 1 \) and \( j_2 = 2 \), and \( \xi^{B \rightarrow A} (\xi^{B \rightarrow A}) \) as the set of subsystems \( \{ H_A^{(1)}, H_A^{(2)}, \ldots, H_A^{(l)} \} \) \( \{ (H_A^{(i+1)}, H_A^{(i+2)}, \ldots, H_A^{(l+1)}) \} \) of PAT.A. Suppose the set of states \( |\phi_i^{(1)}\rangle \) (i = 1, 2, \ldots) is an independent set in subspace \( H_B^{(1)} \). One can always find a state orthogonal to all states but \( |\phi_1^{(1)}\rangle \) in this subspace, and denote this state as \( |\phi_{\perp 1}\rangle \). Define the projector \( \hat{P}_1 \equiv |\phi_{\perp 1}\rangle \langle \phi_{\perp 1}| \) and

\[
|\Phi\rangle \equiv \mathcal{N}\hat{P}_1|\Psi\rangle = |\phi_{\perp 1}\rangle|\phi_1^{(2)}\rangle \ldots |\phi_1^{(m)}\rangle,
\]

where \( \mathcal{N} \) is a proper normalization factor so that \( \langle \Phi|\Phi\rangle = 1 \). Partial trace \( |\Phi\rangle \langle \Phi| \) over the subspace \( H_B^{(2)} \), then calculate the von Neumann entropy: \( S(\rho) \equiv -\text{Tr} \rho \log \rho \). Since

\[
\text{Tr}^{(2)}|\Phi\rangle \langle \Phi| = |\phi_{\perp 1}\rangle|\phi_1^{(3)}\rangle \ldots |\phi_1^{(m)}\rangle|\phi_1^{(3)}\rangle \ldots |\phi_1^{(3)}\rangle \langle \phi_{\perp 1}|,
\]

we obtain

\[
S(\text{Tr}^{(2)}|\Phi\rangle \langle \Phi|) = 0.
\]

This entropy is just the entanglement entropy of \( |\Phi\rangle \) w.r.t. the bipartition \( H_B^{(2)} \otimes \overline{H_B^{(2)}} \), where \( \overline{H_B^{(2)}} \) represents the complementary subspace of \( H_B^{(2)} \). It is obvious \( |\Phi\rangle \) is non-entangled w.r.t. this bipartition.

On the other hand, since \( |\Psi\rangle = |\psi^{(1)}\rangle|\psi^{(2)}\rangle \ldots |\psi^{(n)}\rangle \) and the projector \( \hat{P}_1 \) only acts on \( |\psi^{(1)}\rangle \ldots |\psi^{(l)}\rangle \), we can write

\[
|\Phi\rangle = \mathcal{N}\hat{P}_1|\Psi\rangle = \langle \mathcal{N}\hat{P}_1|\psi^{(1)}\rangle \ldots |\psi^{(l)}\rangle |\psi^{(l+1)}\rangle \ldots |\psi^{(n)}\rangle.
\]

Then

\[
\text{Tr}^{(2)}|\Phi\rangle \langle \Phi| = \rho_1 \otimes \rho_2,
\]

where

\[
\rho_1 = \mathcal{N}\hat{P}_1|\psi^{(1)}\rangle \ldots |\psi^{(l)}\rangle \langle \psi^{(l)}| \ldots \langle \psi^{(l)}| \hat{P}_1 \mathcal{N},
\]

and

\[
\rho_2 = \text{Tr}^{(2)}|\psi^{(l+1)}\rangle \ldots |\psi^{(n)}\rangle \langle \psi^{(n)}| \ldots \langle \psi^{(l+1)}|.
\]

Since

\[
S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2),
\]

and \( S(\rho_1) = 0 \), and we already know that \( S(\rho_1 \otimes \rho_2) = 0 \) from Eq. (17), we find \( S(\rho_2) = 0 \).

Now consider the state without the projection. On one hand, we have

\[
\text{Tr}^{(2)}|\Psi\rangle \langle \Psi| = |\psi^{(1)}\rangle \ldots |\psi^{(l)}\rangle \langle \psi^{(l)}| \ldots \langle \psi^{(l)}| \text{Tr}^{(2)}|\psi^{(l+1)}\rangle \ldots |\psi^{(n)}\rangle \langle \psi^{(n)}| \ldots \langle \psi^{(l+1)}|,
\]

\[
S(\text{Tr}^{(2)}|\Psi\rangle \langle \Psi|) = S(|\psi^{(1)}\rangle \ldots |\psi^{(l)}\rangle \langle \psi^{(l)}| \ldots \langle \psi^{(l)}|) + S(\rho_2) = S(\rho_2) = 0.
\]

On the other hand, \( S(\text{Tr}^{(2)}|\Psi\rangle \langle \Psi|) \) is just the entanglement entropy of \( |\Psi\rangle \) w.r.t. the bipartition \( H_B^{(2)} \otimes \overline{H_B^{(2)}} \). If \( |\Psi\rangle = \sum \kappa_i |\phi^{(1)}_i\rangle |\phi^{(2)}_i\rangle \ldots |\phi^{(m)}_i\rangle \), then \( |\Psi\rangle \) must be entangled, and we must have

\[
S(\text{Tr}^{(2)}|\Psi\rangle \langle \Psi|) > 0,
\]

which contradicts Eq. (24).

To be more precise, given the expression \( |\Psi\rangle = \sum \kappa_i |\phi^{(1)}_i\rangle |\phi^{(2)}_i\rangle \ldots |\phi^{(m)}_i\rangle \) and the conditions required in the Lemma, \( |\Psi\rangle \) cannot be a factorized state w.r.t. the bipartition \( H_B^{(2)} \otimes \overline{H_B^{(2)}} \). So when written into its Schmidt decomposition form \( |\Psi\rangle = \sum \lambda_i |A_i\rangle |B_i\rangle \), \( |\Psi\rangle \) must have the Schmidt rank larger than 1 with \( \lambda_i < 1 \), so we must have

\[
S(\text{Tr}^{(2)}|\Psi\rangle \langle \Psi|) = -\sum \lambda_i^2 \log \lambda_i^2 > 0.
\]

The extended tridecomposition uniqueness theorem:

If a state is a semi-GHZ state w.r.t. PAT.A: \( |\Psi\rangle = \sum \alpha_i |\psi^{(1)}_i\rangle |\psi^{(2)}_i\rangle \ldots |\psi^{(n)}_i\rangle \), then within the comparable set of
PAT A, the state \( \text{can not be} \) a semi-GHZ state w.r.t. a different partition, say PAT B: \( \sum_{j=1}^{\tilde{\gamma}} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \), unless \( |\psi_i^{(1)}\rangle|\psi_i^{(2)}\rangle \ldots |\psi_i^{(n)}\rangle = |\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle \ldots |\phi_i^{(m)}\rangle \) for every \( i \) (up to a phase), with a proper ordering of the expanding bases.

**Proof:** Since there are two fractions of PAT B satisfy \( \xi_{1i}^{B \to A} \cap \xi_{2i}^{B \to A} = 0 \) and \( \xi_{1i}^{B \to A} \cup \xi_{2i}^{B \to A} \subset \{ A \} \), without losing generality, we can take \( j_1 = 1 \) and \( j_2 = 2 \), and \( \xi_{1i}^{B \to A} = \{ H_A^{(1)} \}, H_A^{(2)}, \ldots H_A^{(l)} \} \) and \( \xi_{2i}^{B \to A} = \{ H_A^{(l+1)} \}, H_A^{(l+2)}, \ldots H_A^{(l+r)} \} \). Then we have \( \xi_{i}^{A \to B} \subseteq \{ H_B^{(3)} \ldots H_B^{(m)} \} \).

Since \( |\psi_i^{(l \to l') + 1}\rangle \) is independent, we may introduce more independent states \( |\psi_i^{(l \to l') + 1}\rangle \) with \( i \) runs beyond \( \tilde{i} \), such that the states \( |\psi_i^{(l \to l') + 1}\rangle \) \((i = 1, 2, \ldots, d_{A^{(l \to l') + 1}})\) forms an independent and complete set, where \( d_{A^{(l \to l') + 1}} \) is the dimension of the subspace \( A^{(l \to l') + 1} \). So if \( |\Psi\rangle = \sum_{j=1}^{\tilde{\gamma}} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \), then each state \( |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \) \( (j = 1, \ldots, \tilde{j}) \) can be written as \( \sum_{k=1}^{d_{A^{(l \to l') + 1}}} \gamma_k^j \langle \bar{\psi}_k^{(l \to l') + 1}| \bar{\psi}_k^{(l \to l') + 1} \rangle |\bar{\psi}_k^{(l \to l') + 1}\rangle. \) |\bar{\psi}_k^{(l \to l') + 1}\rangle \) may not be orthogonal or independent. Then we have

\[
|\Psi\rangle = \sum_{i=1}^{\tilde{i}} \alpha_i |\psi_i^{(1)}\rangle|\psi_i^{(2)}\rangle \ldots |\psi_i^{(n)}\rangle = \sum_{j} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \left( \sum_k \gamma_k^j \langle \bar{\psi}_k^{(l \to l') + 1}| \bar{\psi}_k^{(l \to l') + 1} \rangle |\bar{\psi}_k^{(l \to l') + 1}\rangle \right)
\]

First, for every \( |\psi_i^{(l \to l') + 1}\rangle \) with \( i > \tilde{i} \), one can find at least one state orthogonal to all the states \( |\psi_k^{(l \to l') + 1}\rangle \) with \( k \neq i \) but \( |\psi_i^{(l \to l') + 1}\rangle \) itself. Denoting this state as \( |\psi_i^{(1)}\rangle \), we have

\[
\langle \psi_i^{(1)}|\Psi\rangle = 0 = \sum_{j} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \langle \psi_i^{(1)}| \psi_i^{(l \to l') + 1} \rangle |\bar{\psi}_k^{(l \to l') + 1}\rangle.
\]

Since \( |\phi_j^{(1)}\rangle \) are independent and all the \( \kappa_j \)'s are non-zero, so we have \( \gamma_j^i = 0 \) for \( i > \tilde{i} \) and any \( j \).

Now for \( i \leq \tilde{i} \), we have

\[
\langle \psi_i^{(1)}|\Psi\rangle = \sum_{j} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \left( \sum_k \gamma_k^j \langle \bar{\psi}_k^{(l \to l') + 1}| \bar{\psi}_k^{(l \to l') + 1} \rangle \bar{\psi}_k^{(l \to l') + 1}\rangle \right).
\]

According to the Lemma, the only possibility is that only one of the \( \gamma_j^i \)'s (for a given \( i \)) is non-zero. Suppose \( \gamma_j^i \)'s are non-zero, then we have

\[
\langle \psi_i^{(1)}|\Psi\rangle = \sum_{j} \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle \left( \sum_k \gamma_k^j \langle \bar{\psi}_k^{(l \to l') + 1}| \bar{\psi}_k^{(l \to l') + 1} \rangle \bar{\psi}_k^{(l \to l') + 1}\rangle \right).
\]

for every \( i \leq \tilde{i} \). It is important to notice that \( \gamma_j^i \delta_{jk} = 0 \). For a given \( i \leq \tilde{i} \), there exists only one \( j = j_i \) such that \( \gamma_j^i \neq 0 \). In general, for a given \( j \) there could be more than one \( i \) such that \( \gamma_j^i \neq 0 \) while for all the other \( i \)'s \( \gamma_j^i = 0 \). This is happening when some \( j_i \)'s are equal, say \( j_{i_a} = j_{i_b} = \cdots = j \), then we could write (denoting \( W \equiv \{ i_a, i_b, \ldots \} \))

\[
\sum_{i \in W} \frac{|\psi_i^{(l \to l') + 1}\rangle|\psi_i^{(l \to l') + 1}\rangle\langle \Psi\rangle}{\langle \psi_i^{(1)}| \psi_i^{(l \to l') + 1} \rangle} = \sum_{i \in W} \alpha_i |\psi_i^{(1)}\rangle \ldots |\psi_i^{(n)}\rangle = \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \cdots |\phi_j^{(m)}\rangle \left( \sum_k \gamma_k^j \langle \bar{\psi}_k^{(l \to l') + 1}| \bar{\psi}_k^{(l \to l') + 1} \rangle \bar{\psi}_k^{(l \to l') + 1}\rangle \right).
\]

Since for this \( \tilde{j} \), all the \( \gamma_j^i = 0 \) for \( i \notin W \), \( i \) can run through all legal values, so

\[
\sum_{i \in W} \alpha_i |\psi_i^{(1)}\rangle \ldots |\psi_i^{(n)}\rangle = \kappa_j |\phi_j^{(1)}\rangle|\phi_j^{(2)}\rangle \ldots |\phi_j^{(m)}\rangle.
\]

Since PAT A is also comparable to PAT B, according to the Lemma, this equation cannot hold unless \( W \) has only one element. Hence, the only possibility is that for a given \( j \) there is only one \( i \) such that \( \gamma_j^i \neq 0 \).

To sum up, for every \( i \leq \tilde{i} \) there is only one \( j \) such that \( \gamma_j^i \neq 0 \), and for every \( j \) there is only one \( i \) such that \( \gamma_j^i \neq 0 \). So by rearrange the order we can write \( \gamma_j^i \propto \delta_{ij} \) and \( \tilde{i} = \tilde{j} \), then we have

\[
\alpha_i |\psi_i^{(1)}\rangle \ldots |\psi_i^{(n)}\rangle = \kappa_i |\phi_i^{(1)}\rangle \ldots |\phi_i^{(m)}\rangle,
\]
and \( |\alpha_i| = |\kappa_i| \) for every \( i \leq \tilde{i} \). This ends the proof.
V. CONCLUDING REMARKS

In real world, not all the partitions of an object’s environment have physical meanings, although mathematically any partition is possible (as long as the Hilbert space of the environment is a tensor product of the elementary system’s Hilbert space). Then among all the possible mathematical partitions of the environment, which partitions are physically meaningful? We need objective criteria to settle this problem. In the case when the system has proliferated its classical properties into its environment, the extended tridecomposition uniqueness theorem combined with the distinguishing of local and nonlocal observers provides one such criterion for a particular situation. Although, there are other complicated situations in general, our arguments may help to gain some physical insights.

The ambiguity we presented in Introduction is not common in dynamical decoherence models. However, for quantum Darwinism to be a fundamental mechanism to explain the emergence of the objectivity of classical properties, it should resolve this ambiguity even if it may occur only in rare conditions, which calls for the present study.

We didn’t mention how to deal with identical particles, while the constituents of the environment may be identical particles of a few types. To deal with identical particles, we need to turn to quantum field theories. The present form of the theorem cannot be applied to continuum quantum field theories, but can be applied to lattice quantum field theories. We may consider each elementary system to be a state at each spatial point, such as using $|0\rangle_x$, $|1\rangle_x$ and $|2\rangle_x$ to represent there are 0, 1 and 2 particles (of a particular type) at point $x$ respectively. Then the arguments in the main context are still valid. Currently, it is not clear whether the extension to continuum space-time is straightforward or not, and this problem deserves further efforts in the future.

At last, we expect the uniqueness theorem we proved here is also useful in quantum multipartite entanglement studies.

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