The goal of this talk, and the following one by Larry Yaffe, will be to investigate the order and strength of the electroweak phase transition. I will review what can be done with standard perturbative methods and how such methods sometimes break down in cases of interest. In part 2, Larry Yaffe will discuss the application of $\epsilon$ expansion techniques to study those cases where standard perturbative methods fail.

The reason for studying the electroweak phase transition is that it plays a crucial role in scenarios of electroweak baryogenesis. Recall Sakharov’s three conditions for any scenario of baryogenesis: (1) baryon number violation, (2) $C$ and $CP$ violation, and (3) disequilibrium. As I shall briefly review, standard electroweak theory provides the required violation of baryon number. The standard model also provides $C$ and $CP$ violation, though the strength of such violation may not be sufficient to generate the observed baryon excess unless the Higgs sector is non-minimal—an issue discussed in other talks at this conference. As I shall discuss later, the role of the electroweak phase transition is to provide the required disequilibrium, and its success in this role depends on the order and strength of the transition.

In our talks, Larry and I will stick to a simple toy model: the minimal standard model with a single doublet Higgs. I call this a toy model because one probably needs to incorporate extra Higgs bosons into the theory to have sufficient $CP$ violation for electroweak baryogenesis. But multiple Higgs models have all sorts of unknown parameters in them, which makes it difficult to plot results in any simple way. It makes sense to first refine one’s techniques for studying the phase transition in the simpler one-Higgs model. With a little bit of work, everything we do should be straightforwardly extendable to more complicated models. For simplicity, we shall also generally ignore $\sin^2 \theta_w$ and focus on pure SU(2) gauge-Higgs theory.

*Talk presented at the conference Quarks ‘94: Vladimir, Russia, 1994. This work was supported by the U.S. Department of Energy, Grant No. DE-FG06-91ER40614.
1. Lightning review of electroweak B violation

I shall take a moment to quickly review baryon number (B) violation in standard electroweak theory; the formula for the B violation rate will later be relevant to motivating some of the important issues concerning the electroweak phase transition.

Baryon number violation is a bit strange in standard electroweak theory because it can’t happen perturbatively. All of the vertices in a Feynman diagram conserve quark number; whenever a quark line enters a diagram, it remains a quark line and must eventually leave again, thus conserving baryon number. However, baryon number is violated non-perturbatively due to the electroweak anomaly shown in fig. 1. This anomaly is closely analogous to the usual axial anomaly of QCD or QED. In the electroweak case, however, the axial nature of the anomaly appears in the gauge couplings rather than in the current. The formula for the anomaly is the same as in the QCD case,

$$\partial_\mu j^\mu_B \approx g_w^2 F \tilde{F},$$  

(1)

except that the field strengths $F$ are for the weak SU(2) fields rather than the gluon fields. Integrating both sides gives a formula for the amount of baryon number violation in any process:

$$\Delta B \sim g_w^2 \int d^4 x \bar{F} \tilde{F}.$$  

(2)

Note that, in order to get a $\Delta B$ of order 1, the field strengths $F$ must be of order $1/g_w$. So any process which violates baryon number involves large, non-perturbative excursions away from the vacuum $F = 0$. Also note that large field strengths imply large energies, and so any transition with $\Delta B \sim 1$ requires passing through intermediate gauge field configurations with non-negligible energy. This situation is depicted schematically in fig. 2. The horizontal axes denotes the sequence of gauge field configurations a particular process passes through when violating baryon number via (2); the vertical axis denotes the potential energy of those configurations. $E_0$ is the potential energy barrier separating the initial gauge vacuum $A = 0$ from the final gauge vacuum, which is just a gauge transform. The configuration corresponding to the minimum potential energy barrier for this process is known as the sphaleron.

* For a sample of various reviews of the subject, try ref. [1]. For a review of electroweak baryogenesis, try ref. [2].
Fig. 2. Qualitative picture of gauge configuration space for a B violating transition.

At zero energy, the only way to get from one vacuum to the next, and so produce B violation through the anomaly, is by quantum tunneling. Because the barrier is non-perturbatively large, the probability for such tunneling is exponentially suppressed and turns out to be

\[ \text{rate} \sim e^{-4\pi/\alpha_w} \sim 10^{-170} = \text{zero}. \]  

(3)

Imagine instead the early universe at temperatures large compared to the barrier energy $E_0$. Such a hot thermal bath will excite states with energies large compared to $E_0$, and these can cross the barrier classically rather than tunneling beneath it. The B violation rate will not be exponentially suppressed. Now consider an intermediate situation where the universe is hot but the temperature is smaller than $E_0$. Then there’s still some chance that a random thermal fluctuation will have enough energy to cross the barrier, and this probability is naively given by a simple Boltzmann factor:

\[ \text{rate} \sim e^{-\beta E_0} \sim e^{-\beta M_w/\alpha_w}, \]  

(4)

where $\beta$ is the inverse temperature. The estimate of $E_0$ above may be understood from (1) the earlier observation that field strengths must be order $1/g_w$, which means energies are $1/g_w^2$, and (2) the fact that $E_0$ has dimensions of mass and $M_w$ is the natural mass scale of electroweak theory:

\[ E_0 \sim M_w/\alpha_w \sim \text{a few TeV}. \]  

(5)

All of this can be made more rigorous, and the numerical coefficients in these equations can be deduced, but the simple parameter dependence I am exhibiting here will be all that I’ll use for this talk. Note that my general sloppiness in writing equations extends even to exponents: the last exponent in (3) has some numerical coefficient in it which I haven’t bothered to show.
2. Disequilibrium

In GUT scenarios for baryogenesis, all the relevant physics occurs at temperatures of order $10^{16}$ GeV and the expansion of the universe directly provides the disequilibrium needed for baryogenesis. In electroweak scenarios for baryogenesis, the relevant physics occurs at temperatures of order the weak scale, when the universe is very much older, expanding very much more slowly, and so very much closer to equilibrium. Back of the envelope estimates show that the expansion is then far too slow to produce the observed number of baryons.

But there is other physics that is taking place around the same time—namely, the electroweak phase transition—and this transition can potentially supply the needed element of disequilibrium. If the transition is second order, the universe never departs significantly from equilibrium during the transition. However, if it is first order (and sufficiently strongly so), then it proceeds by the nucleation, expansion, and percolation of bubbles of the new phase inside the old—a non-equilibrium process. Each point in space will feel a non-equilibrium jolt as a bubble wall sweeps by converting it from the symmetric phase ($\phi=0$) to the asymmetric phase ($\phi \neq 0$), and so baryogenesis in these scenarios takes place on (or near) the bubble walls. Back of the envelope estimates have shown that, for some models of the Higgs sector, one has a first-order phase transition and can get in the ballpark of postdicting the observed baryon-to-photon ratio $n_B/s \sim 10^{-10}$.

To sharpen these back of the envelope estimates, there are many interesting but complicated problems one could study. How do you accurately compute the bubble wall profile? the bubble wall velocity? the amount of baryogenesis as the wall sweeps by? All of these non-equilibrium problems are complicated, and so I’m going to focus instead on a simpler problem relevant to the success or failure of electroweak baryogenesis scenarios.

3. A simpler constraint on models

After the phase transition is completed, and the universe settles down into the new, asymmetric phase with $\phi \neq 0$, it had better be the case that baryon number violation is turned off. Otherwise, the universe will simply relax back to its preferred equilibrium state of $B = 0$ and all of the baryogenesis that occurred during the phase transition will be washed away. To turn off $B$ violation, we need the rate $e^{-\beta E_0}$ to be small compared to the expansion rate of the universe, which means the exponent

$$\beta E_0 \sim M_w/\alpha_w T \sim g_w \phi/\alpha_w T$$

must be large in the asymmetric phase just after the phase transition is completed. In the minimal standard model, a leading-order calculation of this exponent (which I will review in a moment) depends qualitatively on the zero-temperature Higgs boson mass as shown in fig. [3]. I will explain why the exponent depends inversely on the Higgs mass, but for the moment let’s consider the consequences. A comparison of the $B$ violation rate to the expansion rate of the universe was made by Shaposhnikov[3]
Fig. 3. The Boltzmann exponent for baryon number violation vs. the zero-temperature Higgs mass.

and later improved by Dine et al. Using a leading-order perturbative calculation, the requirement that B violation be turned off puts a lower bound on the exponent and hence an upper bound on the Higgs mass, as depicted in the figure. For the minimal standard model, this bound on the Higgs mass is roughly 35–40 GeV, which is ruled out by the experimental lower bound of 60 GeV. So minimal standard model baryogenesis appears to fail. If one makes the Higgs sector more complicated, it is possible to evade these bounds, which is yet another reason to study multiple Higgs models.

However, the situation is more complicated than I have just made it out to be. As I shall discuss, the leading-order calculation used to derive these constraints may be inadequate and higher-order corrections may be crucial. But first, let me outline how the leading-order calculation is made.

4. The leading-order calculation

Consider the classical Higgs potential:

\[ V_0 \sim -\mu^2 \phi^2 + \lambda \phi^4. \]  

\( V_0 \) basically tells us the “vacuum” energy as a function of \( \phi \), and at zero temperature the ground-state is determined by minimizing it. At finite temperature, the ground state is determined by minimizing the free energy. At finite temperature, the system is not in vacuum—there is a plasma of real, on-shell particles such as W’s, quarks, and leptons, and all contribute to the free energy. For the sake of pedagogy, let me just focus on the contribution of W’s in the plasma, and for the time being let’s ignore interactions. The free energy of an ideal Bose gas is something you can easily look up

\( \dagger \) “Leading” order here means leading order after improvement by resumming hard thermal loops (daisies).

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\( \beta \) E(sphaleron) 35 GeV

\( m(\text{Higgs}) \)
in a graduate textbook:

$$\Delta F \sim T \int d^3 k \ln \left(1 - e^{-\beta E_k}\right),$$  \hspace{1cm} (8)

where the relativistic energy is just

$$E_k = \sqrt{\vec{k}^2 + M^2_w} \sim \sqrt{\vec{k}^2 + g^2 \phi^2}.$$  \hspace{1cm} (9)

Note that the W mass is proportional to $g\phi$ in a background Higgs field $\phi$, and so the W gas contribution $\Delta F$ to the free energy is a function of $\phi$. At high temperature ($T >> M_w$), $\Delta F$ can be expanded in powers of $1/T$ to give

$$\Delta F \sim \# T^4 + \# M^2_w T^2 - \# M^3_w T + \cdots,$$  \hspace{1cm} (10)

where I haven’t bothered showing the numerical coefficients #. Henceforth I won’t bother writing in the # signs either.

To get the total free energy, we just add the “vacuum” contribution (7) and the W gas contribution (10):

$$F \sim V_0 + \Delta F \sim \text{const.} + (-\mu^2 + g^2 T^2)\phi^2 - g^3 \phi^3 T + \lambda \phi^4 \cdots.$$  \hspace{1cm} (11)

The $\phi$-independent term “const.” doesn’t affect the properties of the transition and will be ignored. The $g^2 T^2 \phi^2$ term comes from the $M^2_w T^2$ term of (10) and is responsible for driving the phase transition: it turns the curvature of the free energy at $\phi = 0$ from concave-down at $T = 0$ to concave-up at sufficiently large $T$. The cubic term $-g^3 \phi^3 T$ comes from the $-M^3_w T$ term and is responsible for making the phase transition first-order, as depicted in fig. 4, rather than second-order.

We are now in a position to estimate the order of magnitude (or more specifically, the parameter dependence) of quantities related to the phase transition. Examine fig. 4.
and consider the critical temperature at which the two ground states are degenerate, and consider the region of $\phi$ which, roughly, encompasses the maximum and asymmetric minimum of $F(\phi)$. The only way for the free energy to have that shape is if the quadratic, cubic, and quartic terms of (11) all have the same order of magnitude in that region of $\phi$. If the quadratic term were negligible, it wouldn’t curve up at the origin; if the cubic term were negligible, it wouldn’t curve down later; and if the quartic term were negligible, it wouldn’t be turning up again. So

$$(-\mu^2 + g^2 T^2)\phi^2 \sim g^3 \phi^3 T \sim \lambda \phi^4. \quad (12)$$

The last relation, in particular, then easily yields

$$\phi \sim \frac{g^3}{\lambda} T. \quad (13)$$

Now let’s return to the rate of B violation in the asymmetric phase. The Boltzmann exponent $\beta E_0$ is then

$$\frac{M_w}{g^2 T} \sim \frac{\phi}{g^2} \sim \frac{g^2}{\lambda} \sim \frac{M^2(W)_{T=0}}{m^2(\text{Higgs})_{T=0}}. \quad (14)$$

This is how one finds the inverse dependence on the Higgs mass that was depicted in fig. 3.

5. Review of finite-temperature formalism

Our next goal will be to discuss the validity of the above leading-order treatment, where the W bosons were treated as an ideal gas. Discussing corrections due to interactions requires a little more careful treatment of finite temperature calculations, and so in this section I shall very briefly review the formalism of finite-temperature field theory. Recall that the basic tool for studying equilibrium questions at finite temperature is the partition function:

$$Z = \text{tr} e^{-\beta H}. \quad (15)$$

A path integral expression for the partition function may be easily found by noting that $e^{-\beta H}$ is nothing but the time-evolution operator for an imaginary amount of time $t = i\beta$. So, in the exact same way one derives the path integral for infinite-time evolution to attack zero-temperature problems, one may derive the completely analogous result

$$Z = \int [\mathcal{D}\phi] \exp \left[ -\int_0^\beta dt \int d^3x L_E(\phi) \right], \quad (16)$$

where $L_E$ is the Euclidean action density. The only difference is that the integral in the exponent is over Euclidean time $\beta$ rather than over infinite real time. Also, the trace in the partition function is implemented by requiring the “initial” state be the same as the “final” state, which amounts to requiring the boundary condition

$$\phi(\tau=0, \vec{x}) = \phi(\tau=\beta, \vec{x}) \quad (17)$$
on the path integral.

Since the only difference between finite temperature and zero temperature is the extent of Euclidean time, the only difference in Feynman rules will be that, when integrating over internal momenta \( k \), only the frequencies \( k_0 = 2\pi n T \) which are periodic in time \( \beta \) are relevant. So Fourier integrals get replaced by Fourier sums, and the only difference in the Euclidean Feynman rules is the replacement

\[
\int d^4 k \rightarrow T \sum_{k_0} \int d^3 k, \quad k_0 = 2\pi n T, \tag{18}
\]

for integrations over internal momenta. Note for later the factor of \( T \) in front of the Fourier series sum, which makes the dimensions the same as \( d^4 k \).

Now consider what happens in the large temperature limit \( \beta \rightarrow 0 \). In this case, the extent of the (Euclidean) temporal dimension shrinks to zero. So the four-dimensional theory reduces to an effective three-dimensional theory of the static \( (k_0 = 0) \) modes. (More precisely, this reduction takes place if we are studying equilibrium quantities at distance scales large compared to \( \beta \).) I should note that fermions turn out to have anti-periodic boundary conditions and so cannot have any static modes. As a result, fermions completely decouple from long-distance, equilibrium physics in the high-temperature limit.

The introduction of temperatures into Feynman rules via (18) may seem a bit formal. However, it turns out to have a fairly simple physical interpretation when one does actual calculations. If one carries out the Euclidean frequency sum for a simple one-loop integral, one finds

\[
\begin{align*}
\begin{array}{c}
\text{\( k \)} \\
\end{array}
= \begin{array}{c}
\text{\( T=0 \)} \\
\end{array} + \int \frac{d^3 k}{(2\pi)^3 2E_k} \frac{1}{e^{\beta E_k} - 1} \begin{array}{c}
\text{\( k \)} \\
\end{array} \begin{array}{c}
\text{\( k \)} \\
\end{array}.
\end{align*}
\tag{19}
\]

The first term on the right-hand side denotes the zero-temperature result. The second term—which contains all the temperature dependence—is nothing more than the amplitude for the external particle to forward scatter off of a real, physical particle present in the thermal bath: the \( 1/(e^{\beta E_k} - 1) \) is the Bose probability for finding such a particle, and the \( d^3 k/(2\pi)^3 2E_k \) is just the usual measure for phase space.

6. Loop Expansion Parameter

We’re now in a position to discuss when leading-order calculations are adequate at finite temperature. The basic cost of adding a loop to a diagram at high temperature

\[\text{\footnote{For a review, try ref. 5.}}\]
The $g^2$ is just the usual cost of extra coupling constants. The factor of $T$ is the explicit factor of $T$ from the Fourier sum (18) associated with the additional loop momentum. But the cost of adding a loop should be dimensionless, so $g^2T$ must be divided by whatever gives the mass scale of the problem—in this case, $M_w$. Because of the factor of $T/M_w$, the loop expansion is not necessarily small at high temperature! The criteria that the loop expansion parameter be small, and that therefore a perturbative expansion around mean field theory be useful, is known to condensed matter physicists as the Ginzburg criteria.

The loop expansion parameter (20) is very important, so let’s understand it in several different ways. First, consider adding a loop with internal momentum $k$ to any diagram, denoted by a shaded circle below. As discussed earlier, the effect of finite temperature on loops is to incorporate the physics of particles forward-scattering off of real particles in the plasma:

\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{E_k e^{\beta E_k} - 1}
\]

The origin of the large $T/M_w$ factor in the loop expansion parameter is simply the divergent behavior of the Bose factor $1/(e^{\beta E_k} - 1)$ as $E_k \to 0$; there are a large number of low-energy bosons present in a high-temperature plasma.

Let’s understand the loop expansion parameter in yet another way. As mentioned earlier, the high-temperature limit $\beta \to 0$ reduces the four-dimensional Euclidean theory to an effective three-dimensional theory of the static ($k_0 = 0$) modes. Restricting

\[
\int \frac{d^3k}{(2\pi)^3} \frac{T}{E_k}
\]
attention to the static modes, the integrand of the path integral then has the form
\[ e^{-S_E} = \exp \left[ -\frac{1}{g^2} \int_0^\beta d\tau \int d^3x \mathcal{L}_E \right] \rightarrow \exp \left[ -\frac{1}{g^2T} \int d^3x \mathcal{L}_E \right]. \] (22)

In the first equality, I have normalized the fields so that the coupling constant appears explicitly out in front of the action as \(1/g^2\). When I specialize to static field configurations, the Euclidean time integration becomes trivial, giving a factor of \(1/T\). But now we see that \(g^2\) always appears in the combination \(g^2T\). Then dimensional analysis gives us the loop expansion parameter \(\lambda\) as before.

Note that for pure, unbroken gauge theory (where there is no Higgs \(\phi\) to give a mass to the W), (22) shows that the only scale in the theory would be \(g^2T\) itself. If we were to study the physics at that scale, then the loop expansion parameter given by the left-hand side of (24) would be order one; the physics is strongly-coupled even though \(g^2\) is small. This is known as the infrared (or “magnetic mass”) problem of high-temperature non-Abelian gauge theory.

7. Why life is not simple

We’re now finally in a position to discuss under what conditions perturbation theory might be adequate to study the electroweak phase transition. Our loop expansion parameter \(\lambda\) is nothing other than the inverse of the B violation rate exponent \(\epsilon\), and so we may borrow the earlier analysis of the parameter dependence:\footnote{For a more detailed discussion of the loop expansion, see ref. [1].}
\[ \frac{g^2T}{M_w} \sim \frac{\lambda}{g^2} \sim \frac{m^2(Higgs)_{T=0}}{M^2(W)_{T=0}}. \] (23)

A basic result of this review, which should be remembered for Larry Yaffe’s talk on the \(\epsilon\) expansion, is then:

THE LOOP EXPANSION WORKS WHEN \(\lambda \ll g^2\).

Now once can wonder how well perturbation theory is doing at the upper bound \(m(Higgs) = 35\) GeV that we earlier discussed for electroweak baryogenesis in the minimal standard model. Is 35 GeV small compared to the W mass of 80 GeV? The answer, of course, depends on all the factors of 2 and \(\pi\) that left out of the simple minded “\(\sim\)” equalities presented in this review. But there’s a way to check it. One can (1) formally assume \(\lambda \ll g^2\), (2) explicitly compute the next-to-leading order (i.e. two-loop) correction to the free-energy \(F(\phi)\), and then (3) see if the correction is numerically large for a 35 GeV Higgs.

I should note in passing that from (23) one sees that my constant use of the high-temperature limit \(T \gg M_w\) is justified provided \(g^4 \ll \lambda\). So the calculation actually formally assumes
\[ g^4 \ll \lambda \ll g^2, \] (24)
where the first inequality is for the high-temperature expansion and the second for the loop expansion.
The effective potential at the critical temperature for $m_h(0) = 35$ GeV and $m_t(0) = 110$ GeV. The dashed and solid lines are the one-loop and two-loop results respectively. [Why a 110 GeV top mass? Because this is an old graph. But the results aren’t particularly sensitive to $m_t$.]

The details may be found in refs. 6 and 7, but the result of the calculation is shown in fig. 5. Both the one-loop and 2-loop results for $F(\phi)$ are shown at the temperature that makes their minima degenerate. The value of $\phi$ in the asymmetric ground state has only shifted by about 20%. On the other hand, the height of the hump has shifted by almost a factor of three! The moral is that for some quantities, $(35 \text{ GeV})/(80 \text{ GeV})$ is not small, and perturbation theory cannot necessarily be trusted. In particular, we have no way to know a priori whether the B violation rate might not be a quantity which gets substantial corrections. (Though the B violation rate exponent is proportional to $\phi$ at leading order, this simple relation does not hold beyond leading order.)

8. More on the reduction to 3 dimensions

When discussing the loop expansion parameter back in section 3, I assumed that the relevant “physics scale” for any diagram was determined by particle masses. This is true if diagrams are ultraviolet (UV) convergent and so dominated by their infrared...
behavior. However, here’s a quadratically divergent diagram, for which it’s not true:

\[ k \vdash (T=0 \text{stuff}) + g^2 T^2 \]  \hspace{1cm} (25)

[Note: as usual, I’m leaving out numerical coefficients and just showing the order of magnitude of terms.] As in (19), the finite-temperature piece of this diagram comes from interactions with real particles of momentum \( k \) in the plasma. The quadratic divergence is then cut-off by \( T \) for this piece because there are no such particles with \( k \gg T \). This gives the result of order \( g^2 T^2 \) indicated above, and the diagram is dominated by loop momenta \( k \) of order \( T \).

Because the important momentum scale for this diagram is \( T \) (and not the particle masses), this diagram is sensitive to the Euclidean temporal direction. That is, \( k_0 \neq 0 \) modes are \textit{not} suppressed in UV divergent diagrams. But this is the usual story for the decoupling of heavy degrees of freedom in field theory. At large distances compared to \( \beta = 1/T \), the Euclidean time dimension decouples \textit{except} for renormalizations of the masses and couplings of the theory. The \( g^2 T^2 \) contribution to (25) is just the renormalization of the scalar mass in matching the original four-dimensional theory to the effective three-dimensional theory at long distances.

The way to systematically construct the effective three-dimensional theory, and to relate its parameters to those of the more fundamental four-dimensional theory, is to compute the effective interactions among the \( k_0=0 \) modes generated by integrating out the \( k_0 \neq 0 \) modes. For instance, diagrams like

\[ \text{give } m_3^2 = m_4^2 + g^2 T^2 + \text{higher-order}, \]  \hspace{1cm} (26)

where the double-lines indicate the non-static \( k_0 \neq 0 \) modes, \( m_4 \) is the scalar mass in the fundamental four-dimensional theory, and \( m_3 \) is the scalar mass in the effective three-dimensional theory. A similar thing happens for the temporal polarization of the photon. Diagrams like

\[ \text{give } M_3^2(A_0) = g^2 T^2 + \cdots. \]  \hspace{1cm} (27)

This is the Debye screening mass for static electric fields in a hot plasma. Coupling

\[ ** \text{For more details of the reduction from four to three dimensions in the particular context of the electroweak phase transition, see ref. 8. } \]
constants also get contributions from the $k_0 \neq 0$ modes, such as

\[
giving \quad \lambda_3 = \lambda_4(T) + \cdots, \tag{28}
\]

where $\lambda_4(T)$ is the four-dimensional coupling evaluated at renormalization scale $T$. There is also an effective $\phi^6$ interaction, which is a marginal operator in three dimensions:

\[
\text{marginal.} \tag{29}
\]

Other interactions generated by integrating out the heavy modes are irrelevant—that is they decouple as powers of the physics scale over $T$:

\[
\text{irrelevant operators.} \tag{30}
\]

The most important point to be made about this matching of the four-dimensional to an effective three-dimensional theory is that the matching is perturbative in $g^2$ (and $\lambda$). To integrate out the $k_0 = 0$ modes is to account for physics whose scale is set by $T$ (the inverse size of the temporal dimensions) and not by particle masses $M$. The loop expansion parameter for this integration is therefore

\[
\frac{g^2 T}{\text{scale}} \sim g^2, \tag{31}
\]

and everything is under the control. It is only when one goes to solve the effective three-dimensional theory of the $k_0 = 0$ modes that one encounters the infrared enhancements that gave the potentially large loop expansion parameter of section 6.

I should also make a few remarks about choice of scale in the regularization of the three-dimensional theory. For simplicity, imagine a simple UV momentum cut-off $\Lambda$. To get the effective three-dimensional theory, we now integrate out not only the $k_0 \neq 0$ modes but also all $|\vec{k}| > \Lambda$ for the $k_0 = 0$ modes. For the latter integration, the loop expansion parameter is

\[
\frac{g^2 T}{\text{scale}} \sim \frac{g^2 T}{\Lambda}. \tag{32}
\]

The natural choice for $\Lambda$ is $T$—the scale at which we’re doing the matching. If we picked $\Lambda$ to be very small, the matching would no longer be well-behaved perturbatively. The
Fig. 6. Hierarchy of scales and effective theories. (I have shown the phase transition between $gT$ and $g^2T$. If the Higgs mass is light enough so that $\lambda \ll g^3$, then it would instead be above the $gT$ line.)
Fig. 7. The uncertainties in $F(\phi)$ in different regions of $\phi$. Perturbation theory is controlled by $\lambda/g^2$ in the region of the hump and the asymmetric minimum, designated qualitatively by the solid line, but it breaks down close to the origin. The size of the problem region is small when $\lambda/g^2$ is small.

moral is that the matching is simple and perturbatively calculable provided one chooses a renormalization scale of order $T$.

Fig. 6 shows the hierarchy of important energy scales present at finite temperature. At scale $T$, we integrate out the fourth dimension to get an effective 3-dimensional theory. In the process, $A_0$ picks up a Debye screening mass of order $gT$. So, to study physics below $gT$, one should also integrate out $A_0$ as well. The final effective theory at large distances is then a 3-dimensional gauge theory of $\vec{A}$ and the Higgs field $\phi$. At the scale $g^2T$, the loop expansion parameter becomes strong and the theory is no longer perturbatively solvable. If $\lambda\ll g^2$, the scale associated with the phase transition will be larger than $g^2T$ and one can apply perturbative techniques.

9. Breakdown at $\phi \sim 0$

Assume for the moment that $\lambda/g^2$ is arbitrarily small. When I made the original estimate (23) that the loop expansion parameter $g^2T/M_w$ is of order $\lambda/g^2$, I assumed that $\phi$ was the same order of magnitude as its value in the asymmetric ground state. However, since $M_w \sim g\phi$, the loop expansion parameter $g^2T/M_w$ must eventually get large as I approach the symmetric ground-state $\phi=0$, no matter how small $\lambda/g^2$ is. This situation is depicted in fig. 4. For small $\lambda/g^2$, there will be a small region around $\phi=0$ where perturbation theory breaks down and our calculation of the free energy is uncertain. How big is this uncertainty? In particular, this uncertainty introduces an uncertainty in computing the critical temperature $T_c$ at which the two ground states become degenerate, and $T_c$ affects every other property of the transition we might compute.

†† This section contains material I didn’t have time to cover in my talk in Vladimir.
There’s a simple way to estimate the magnitude of our ignorance of the free energy $F$ in the symmetric phase. In that phase, we have an unbroken 3-dimensional gauge theory. As discussed in section 6, the only parameter of the theory is then $g^2 T$. So, by dimensional analysis, the 3-dimensional action density is then

$$\frac{1}{V} \ln Z \sim (g^2 T)^3,$$  \hspace{1cm} (33)

and so

uncertainty in $F(0) \sim g^6 T^4$. \hspace{1cm} (34)

Now compare this to the accuracy of a perturbative calculation in the asymmetric phase, where the loop expansion parameter is small. The uncertainty in $F(0)$ turns out to be comparable to the accuracy of a four-loop calculation of $F$ in the asymmetric phase. (For details on the power-counting, see sec. II of ref. 6.) So, in general, a perturbative treatment of the phase transition is useful when $\lambda/g^2$ is small, but it is useful only up to a certain order in $\lambda/g^2$.

10. Summary

The following are the elements of this talk that you need to remember for Larry Yaffe’s talk on the $\epsilon$ expansion.

- The hard part of studying the phase transition is solving a three-dimensional theory of $\vec{A}$ and $\phi$.

- That theory has a simple relationship to the original $d=4$ couplings if it is defined at a renormalization scale $\Lambda \sim T$:

$$m_3^2 \sim m_4^2 + g^2 T^2,$$ \hspace{1cm} (35)

$$g_3 \sim g_4(T),$$ \hspace{1cm} (36)

$$\lambda_3 \sim \lambda_4(T).$$ \hspace{1cm} (37)

- The theory can be studied with straightforward perturbation theory (at least to some order) when

$$\lambda \ll g^2.$$ \hspace{1cm} (38)

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‡‡ For more on this topic, try ref. 9.
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