Detecting entanglement by symplectic uncertainty relations

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A hierarchy of multimode uncertainty relations on the second moments of \( n \) pairs of canonical operators is derived in terms of quantities invariant under linear canonical (i.e. ‘symplectic’) transformations. Conditions for the separability of multimode continuous variable states are derived from the uncertainty relations, generalizing the inequalities obtained in [Phys. Rev. Lett. 96, 110402 (2006)] to states with some transposed symplectic eigenvalues equal to 1. Finally, to illustrate the methodology proposed for the detection of continuous variable entanglement, the separability of multimode noisy GHZ-like states is analysed in detail with the presented techniques, deriving a necessary and sufficient condition for the separability of such states under an ‘even’ bipartition of the modes.

I. INTRODUCTION

The effective detection of entanglement in continuous variable systems has become a task of crucial importance, notably in view of the rising relevance of such systems for quantum information and communication protocols, especially in quantum optical settings. To this aim, several methods have been proposed lately, variously emphasising theoretical and practical aspects of the matter.

In first seminal contributions to this aim, the positivity of the partial transposition has been shown to be necessary and sufficient for the separability of (1 + \( n \))-mode Gaussian states (i.e. of states with Gaussian Wigner function shared by two subsystems of 1 and \( n \) modes respectively) \[2, 3\]. Furthermore, a general – ‘computationally efficient’, but not analytical – criterion has been found for the separability of any Gaussian state under any bipartition, based on the iteration of a non-linear map \[4\]. More recently, a full set of conditions has been developed to detect genuine multipartite entanglement in multimode states \[5\] and a complete theory of optimised linear and ‘curved’ entanglement witnesses for second moments (capable of detecting even bound entangled states) has been developed \[6\]. Also, a hierarchy of inequalities for higher moments (generally sufficient for entanglement) has been obtained \[7\], allowing to significantly improve the entanglement qualification of non-Gaussian states as well.

This paper focuses on the separability of multimode continuous variable states under any bipartition. The approach of Ref. \[8\] , introducing a class of (generally) sufficient conditions for entanglement on second statistical moments, will be recovered and discussed in detail. Moreover, such an approach is extended to a more general class of inequalities, capable of dealing with ‘pathological’ instances (entangled states which were not detected by the original inequalities derived in Ref. \[8\] ). Finally, a relevant class of multimode continuous variable states (“GHZ-like”) will be explicitly considered and analysed with the presented methodology, yielding a compact necessary and sufficient condition for the separability of such states of 2\( n \)-mode systems under \((n + n)\)-mode bipartitions. It is worth remarking that, as is clearly the case for all conditions on second moments alone, the experimental test of the inequalities we will derive does not require a complete tomography of the state under examination – generally a daunting task – but only the measurement of the second moments, which might turn out to be delicate for multimode systems but seems in the reach of present techniques \[9\].

The inequalities we will present are based on a class of uncertainty relations on second moments, which may be compactly expressed in terms of symplectic invariants (i.e. of quantities invariant under linear canonical transformations). Besides retaining a major fundamental interest \(\text{per s`e}\), uncertainty relations have also lately been acknowledged as a main ingredient in the detection and qualification of entanglement, whose general scope goes well beyond the scenario of continuous variable systems \[10, 11, 12, 13\]. In fact, for any state \(\varrho\) of a bipartite quantum system, the positivity of the partially transposed density matrix \(\tilde{\varrho}\) (obtained from \(\varrho\) by transposing the Hilbert space of only one of the two subsystems) is a necessary condition for the state to be separable \[14, 15\]. In other words, the violation of the positivity of \(\tilde{\varrho}\) is a proof of the presence of quantum entanglement in the state \(\varrho\), which can in such a case be exploited to various quantum informational aims. Now, uncertainty relations for quantum observables derive only from the commutation relations and from the positivity of the generic density matrix \(\varrho\). Therefore, any relation derived from an uncertainty relation by replacing the state \(\varrho\) with the partially transposed state \(\tilde{\varrho}\) provides a way of testing the positivity of \(\tilde{\varrho}\) and constitutes thus a sufficient condition for the state \(\varrho\) to be entangled. In the present instance, since the original uncertainty relations come in terms of second moments alone, the entanglement conditions are cast in terms of second moments as well, and are therefore
of great experimental relevance. Furthermore, as it will be shown, these conditions turn out to be also necessary for the presence of entanglement whenever the positivity of the partial transpose is also sufficient for separability, namely in \((1+n)\)-mode Gaussian states and \((m+n)\)-mode bisymmetric Gaussian states.

This paper is organised as follows. Canonical systems of many modes (like discrete bosonic fields in second quantization or motional degrees of freedom of material particles in first quantization), notation and uncertainty relations are introduced in Sec. IV. In Sec. V quantities invariant under symplectic operations on the field modes are constructed as functions of the second moments of the field operators. In terms of such invariant quantities, simple uncertainty relations for the second moments of any \(n\)-mode system are derived in Sec. VI. The partial transposition of such a relation will promptly lead to conditions for entanglement in terms of the second moments for any bipartition of the modes in Sec. VII. Finally, specific instances of multimode states are considered in Sec. VIII and conclusions are drawn in Sec. IX.

II. UNCERTAINTY RELATIONS FOR CANONICAL SYSTEMS

Let us consider a quantum mechanical system described by \(n\) pairs of canonically conjugated operators \(\{\hat{x}_j, \hat{p}_j\}\), each of them satisfying the canonical commutation relations (CCR). As well known, by virtue of the Stone-von Neumann theorem the CCR admit, for any finite \(n\), only a unique, infinite dimensional representation, thus allowing for the occurrence of continuous spectra for the canonical operators. Therefore the system in case, whose variables could be motional degrees of freedom of particles in first quantization or quadratures of a bosonic field in second quantization, is commonly referred to as a “continuous variable” (CV) system. Grouping the canonical operators together in the vector \(\hat{R} = (x_1, p_1, \ldots, x_n, p_n)^T\) allows to concisely express the CCR as \(\{\hat{R}_j, \hat{R}_k\} = 2i\Omega_{jk}\), where the symplectic form \(\Omega\) is defined by

\[
\Omega \equiv \oplus^n_1 \omega, \quad \omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

A major role in the description of the system is played by the transformations which, acting in Heisenberg picture on the vector of operators \(\hat{R}\), preserve the fundamental commutation relations set out in Eq. (1). All the conceivable dynamical evolutions of the system have to comply with such a prescription. In particular, Eq. (1) shows that allowed transformations acting linearly on \(\hat{R}\) must preserve the symplectic form \(\Omega\) under congruence: such linear transformations form the real symplectic group \(Sp_{2n, R}\) (reality being necessary to preserve the hermiticity of the canonical operators):

\[
S \in SL(2n, \mathbb{R}) : S \in Sp_{2n, R} \Leftrightarrow S^T \Omega S = \Omega.
\]

At the Hilbert space level, these transformations correspond to unitary operations generated by second order polynomials in the canonical operators. This fact, rigorously justified by the construction of the infinite dimensional ‘metaplectic’ representation of the symplectic group (allowed by the Stone-von Neumann theorem), can be heuristically understood by considering the Heisenberg equation of motion for \(\hat{R}\) under the action of a quadratic Hamiltonian and by recalling Eq. (1).

Clearly, any state of a \(n\)-mode CV system is described by a hermitian, positive, trace-class operator \(\rho\). Let us define the matrix of second moments, or “covariance matrix” \((\text{CM})\), \(\sigma\) (with entries \(\sigma_{jk}\)) of the state \(\rho\) as

\[
\sigma_{jk} \equiv \text{Tr}[\{\hat{R}_j, \hat{R}_k\}\rho]/(2) - \text{Tr}[\hat{R}_j\rho]\text{Tr}[\hat{R}_k\rho].
\]

The CM is a symmetric \(2n \times 2n\) matrix. The positivity of \(\rho\) and the CCR imply the following semidefinite constraint to be satisfied by any \textit{bona fide} CM \(\sigma\):

\[
\sigma + i\Omega \succeq 0.
\]

This well known inequality (whose proof is, for the ease of the reader and to provide a self-contained exposition, reported in Appendix A) is the only constraint a symmetric \(2n \times 2n\) matrix has to satisfy to qualify as the CM of a physical state. For future convenience, let us write down the CM of an \(n\)-mode system in terms of \(2 \times 2\) submatrices \(\gamma_{jk}\)

\[
\sigma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}.
\]

Symmetry implies \(\gamma_{jk} = \gamma_{kj}^\dagger\). Let us also notice that, because of the skew-symmetry of \(\Omega\), Inequality (7) ensures the strict positivity of \(\sigma\):

\[
\sigma > 0.
\]

For a single mode system, the uncertainty principle can be more explicitly recast as

\[
\text{Det}\sigma \geq 1,
\]

which corresponds to the well known Robertson-Schrödinger uncertainty relation. Inequality (9) together with the condition (10) are equivalent to the uncertainty principle for single-mode systems. For a two-mode system, the uncertainty principle reads

\[
\text{Det}\sigma + 1 \geq \Delta_1^2,
\]

where \(\Delta_1^2 \equiv \sum_{j,k=1}^2 \text{Det}\gamma_{jk}\).
Note that the quantities $\text{Det}\sigma$ and $\Delta_j^2$, entering into Inequalities 4 and 8, are invariant under symplectic transformations $\Xi$. Clearly, one could provide a complete set of conditions determining the physicality of the CM by just verifying the positivity of the matrix $\sigma + i\Omega$, which can be checked by considering the determinants of the principal submatrices (see, e.g., 24). Still, the invariant nature of the uncertainty principle implies that the explicit expression of the uncertainty relation for a general $n$-mode CM $\sigma$ must be possible in terms of symplectic invariants constructed from the entries of $\sigma$. In the following section we shall single out such invariants, by which an explicit, general expression for a multimode uncertainty relation will be subsequently derived.

### III. CONSTRUCTION OF THE SYMPLECTIC INVARIANTS

To begin with let us recall that, because of the positivity condition of the Williamson normal form of the CM $\sigma$ (originally concerning the classification of quadratic Hamiltonians) to the positive quadratic form $\sigma$ to infer the following result, which constitutes a basic tool of symplectic analysis: for any CM $\sigma$ there exists a (non-unique) symplectic transformation $S \in Sp_{2n,\mathbb{R}}$ such that

$$S^T \sigma S = \nu,$$

where

$$\nu = \oplus_{j=1}^n \text{diag} (\nu_j, \nu_j).$$

The quantities $\{\nu_j\}$ are referred to as *symplectic eigenvalues*, while the transformation $S$ is said to perform a *symplectic diagonalisation* of the CM $\sigma$. The matrix $\nu$ is the so called Williamson normal form of the CM $\sigma$. The uncertainty principle 11 can be succinctly recast in terms of the symplectic eigenvalues $\{\nu_j\}$ as

$$\nu_j \geq 1$$

for $j = 1, \ldots, n$. Inequality 9 is fully equivalent to the uncertainty relation 11. Its proof is immediate, since it derives from the straightforward computation of the eigenvalues of the matrix $\nu + i\Omega$, where $\nu$ is the normal form associated to the CM $\sigma$.

We can proceed now to the construction of the symplectic invariants for an $n$-mode state. The symplectic eigenvalues are of course symplectic invariants but, for an $n$-mode system, their analytical expression in terms of the second moments turns out to be rather cumbersome (when possible at all). As a matter of fact, the symplectic eigenvalues can be computed by diagonalising the matrix $\Omega\sigma$, whose eigenvalues turn out to be $\{\pm i\nu_j\}$ for $j = 1, \ldots, n$. The latter statement is easily proved by checking it on the normal form $\nu$ and by considering that

$$\Omega\nu = \Omega S^T \sigma S = S^{-1} \Omega\sigma S$$

for some $S \in Sp_{2n,\mathbb{R}}$. Acting by similarity, the transformation $S$ preserves the spectrum of $\Omega\sigma$ which thus coincides with the one of $\Omega\nu$. Now, a natural choice of symplectic invariants, dictated by Eq. 14, is given by the principal minors of the matrix $\Omega\sigma$, which are invariant under symplectic transformations acting by congruence on $\sigma$. As we will see shortly, a compact and elegant uncertainty principle can be expressed in terms of such invariants for any number of modes. Let $M_k(\alpha)$ be the principal minor of order $k$ of the matrix $\alpha$ 24. Then, we define the symplectic invariants of a $n$-mode state $\{\Delta_j^n\}$ for $j = 1, \ldots, n$ as

$$\Delta_j^n \equiv M_{2j}(\Omega\sigma).$$

The principal minors of odd order vanish because of the alternate sign in the spectrum of $\Omega\sigma$, leaving us with $n$ independent symplectic invariants $\{\Delta_j^n\}$ (as one should have expected, since the number of symplectic eigenvalues is $n$). The quantities $\{\Delta_j^n\}$ are also known as “quantum universal invariants” 30. We are now interested in the expression of the invariants $\{\Delta_j^n\}$ in terms of the symplectic eigenvalues $\{\nu_j\}$, which can be straightforwardly retrieved by considering the normal form $\nu$ and turns out to be the following

$$\Delta_j^n = \sum_{S^j_n} \prod_{k \in S^j_n} \nu_k^2,$$

where the sum runs over all the possible $j$-subsets $S^j_n$ of the first $n$ natural integers (i.e. over all the possible combinations of $j$ integers smaller or equal than $n$). In the following, $\Delta_j^n(\nu_1, \ldots, \nu_n)$ will stand for the functions relating the symplectic eigenvalues to the symplectic invariants, according to Eq. 12. Clearly, one has $\Delta_0^n = \text{Det}\sigma$ while, for two-mode states, it is immediate to verify that the invariant $\Delta_2^n$ coincides with the quantity appearing in Inequality 8 (this correspondence is clarified and generalised in Appendix B, where an alternative way to compute the invariants $\{\Delta_j^n\}$ is described).

The following section is devoted to understanding how the uncertainty relation constrains the values allowed for the symplectic invariants of a $n$-mode state, giving rise to ‘symplectic’ uncertainty relations.

### IV. SYMPLECTIC UNCERTAINTY RELATIONS

Let us first introduce some further notation. Let us consider an $n$-mode CV system and define the quantity $\Sigma_n$ as

$$\Sigma_n = \sum_{j=0}^n (-1)^{n+j} \Delta_j^n,$$

where we assume $\Delta_0^n \equiv 1$. Likewise, $\Sigma_n(\nu_1, \ldots, \nu_n)$ will stand for the function relating the symplectic eigenvalues $\{\nu_j\}$ to $\Sigma_n$, according to Eqs. 12 and 13.

**Symplectic uncertainty relation:** Let $\sigma$ be the covariance matrix of a $n$-mode continuous variable state.
The symplectic invariant \( \Sigma_n \), determined according to Eqs. (17) and (18), fulfills the inequality

\[ \Sigma_n \geq 0 . \quad (14) \]

**Proof.** The statement (obviously true for \( n = 1 \), for which it reduces to \( \nu_1^2 \geq 1 \)) will be proven by induction. Let us then assume that, for a generic \( n, \Sigma_{n-1} \geq 0 \). One has

\[ \partial_{\nu_j} \Sigma_n (\nu_1, \ldots, \nu_n) = \Sigma_{n-1} (\nu_1, \ldots, \nu_{k-1}, \nu_{k+1}, \ldots, \nu_n) \geq 0 \]

(because of the inductive hypothesis) for any \( k \leq n \) and for any value of the \( \{ \nu_j \} \). Therefore, because of the bound (9), the minimum of \( \Sigma_n \) is attained for \( \nu_j = 1 \) \( \forall j \leq n \). In such a case one has

\[ \Sigma_n = (-1)^n \sum_{j=1}^{n} (-1)^j C^n_j = 0 , \]

where \( C^n_j \) is the binomial coefficient. Inequality (14) is thus established. \( \square \)

Alternately it may be shown, exploiting Eqs. (12) and (13), that

\[ \Sigma_n = \prod_{j=1}^{n} (\nu_j^2 - 1) . \quad (15) \]

Inequality (14) reduces to the well known relations (1) and (2) for, respectively, \( n = 1 \) and \( n = 2 \), provides a general and elegant way of expressing a necessary uncertainty relation constraining the symplectic invariants. Rather remarkably, such a relation can be analytically checked for any (finite) number of bosonic degrees of freedom. However Eq. (17) shows that, actually, Inequality (14) is only necessary and not sufficient for the full uncertainty relation (9) to be satisfied as it is not able to detect unphysical CMs for which an even number of symplectic eigenvalues are not necessarily fully saturated, as the other symplectic eigenvalues could be greater than one. This situation is described as partial saturation of the uncertainty relation. ‘Partially saturating’ states have been proven to be endowed with peculiar properties. For instance, concerning two-mode Gaussian states, partially saturating states turn out to coincide with the states with minimal entanglement for given purities (31, 32), which in turn prove to be relevant in characterizing the multipartite entanglement structure of three-mode Gaussian states (33).

Actually, even for states featuring partial saturation a simple necessary uncertainty relation may always be found in terms of the quantities \( \{ \Delta^n_j \} \). In fact, let us suppose that \( p < n \) symplectic eigenvalues \( \nu_j \) are equal to 1. In this case, one can iteratively exploit Eq. (17), for \( \nu = 1 \), to express all the invariants of lower order \( \{ \Delta_j^{n-p} \} \) for \( j = 0, \ldots, n-p \) in terms of the original invariants \( \{ \Delta^n_j \} \), obtaining

\[ \Delta_j^{n-p} = (-1)^p \sum_{k_1=j+1}^{n-p+1} \sum_{k_2=k_1+1}^{n-p+2} \ldots \sum_{k_p=k_{p-1}+1}^{n} (-1)^{j+k_p} \Delta^n_{k_p} , \quad (18) \]

where \( \{ k_j \} \) is a set of \( p \) summation indexes. Finally, one can recast the uncertainty relation of lower order

\[ \sum_{j=0}^{n-p} (-1)^{n-j} \Delta_j^{n-p} \geq 0 \text{ simply as} \]

\[ \sum_{j=0}^{n-p} \sum_{k_1=j+1}^{n-p+1} \ldots \sum_{k_p=k_{p-1}+1}^{n} (-1)^{n+k_p} \Delta^n_{k_p} = \sum_{j=p}^{n} [(-1)^{n+j} p! C^n_j \Delta^n_j] \geq 0 , \quad (19) \]

where \( C^n_j \) is the binomial coefficient defined as customary by \( C^n_j = j! / [(j-p)! p!] \). Eq. (19) cannot be saturated if, as per hypothesis, only \( p \) symplectic eigenvalues are equal to 1. Therefore, in case of saturation of Inequality (19), one may proceed to test Inequality (19) for increasing values of \( p \), until the saturation disappears: the first
nonzero value will determine the legitimacy of the CM. Note that, in general, Inequality (14) hold for partially saturating states, when at least \( p \) symplectic eigenvalues are equal to 1 (in fact, the inequality has been derived under such an assumption).

In the next section, criteria for the separability and entanglement of quantum states will be obtained from the uncertainty relation previously derived.

V. SYMPLECTIC SEPARABILITY CRITERIA

The positivity of the partially transposed state ("PPT criterion") is a necessary condition for the separability of any bipartite quantum state (i.e. for the possibility of creating the state by local operations and classical communication alone). Conversely, the violation of such positivity is a sufficient condition for a quantum state to be entangled, in which case quantum correlations are at disposal and may be exploited for quantum informational tasks. Moreover, as far as the CV systems here addressed are concerned, the PPT criterion turns out to be sufficient as well for the separability of \((1+n)\)-mode Gaussian states (i.e. of \((1+n)\)-mode states with Gaussian Wigner and characteristic functions) and of bisymmetric \((m+n)\)-mode Gaussian states (here and in what follows, we refer to a bipartite (\((m+n)\)-mode) CV state as to a state separated into a subsystem \( A \) of \( m \) modes, owned by party \( A \), and a subsystem \( B \) of \( n \) modes, owned by party \( B \).)

These facts come in especially handy for CV systems, as the action of partial transposition on covariance matrices is easily described. Let \( \varrho \) be a \((m+n)\)-mode bipartite CV state with \( 2(m+n) \)-dimensional CM \( \sigma \). Then the CM \( \tilde{\sigma} \) of the partially transposed state \( \tilde{\varrho} \) with respect to, say, subsystem \( A \), is obtained by switching the signs of the \( m \) momenta \( \{ p_j \} \) belonging to subsystem \( A \). In formulae:

\[
\tilde{\sigma} = T \sigma T, \quad \text{with} \quad T \equiv \bigoplus_m^1 \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \oplus \mathbb{I}_{2n}, \quad (20)
\]

where \( \mathbb{I}_{2n} \) stands for the \( 2n \)-dimensional identity matrix. Now, in analogy with Inequality (14) derived from the positivity of the density matrix \( \varrho \), a (generally) sufficient condition for separability derived by the PPT criterion is given by (21):

\[
\tilde{\sigma} + i \Omega \geq 0 \quad (21)
\]

or, in terms of the symplectic eigenvalues \( \{ \tilde{\nu}_j \} \) of the partially transposed CM \( \tilde{\sigma} \) (whose normal form will be henceforth denoted by \( \tilde{\nu} \)), as

\[
\tilde{\nu}_j \geq 1. \quad (22)
\]

The results of the previous section allow one to recast such separability criteria for \((m+n)\)-mode states in terms of partially transposed symplectic invariants \( \{ \tilde{\Delta}_j^{m+n} \} \), defined by \( \tilde{\Delta}_j^{m+n} \equiv M_2(\Omega \tilde{\sigma}) \).

This simple accessory result will be precious in what follows:

**Little lemma.** Let \( \sigma \) be the physical CM of a state of a \((m+n)\)-mode CV system, with \( m \leq n \). Let \( \tilde{\sigma} \) be the partial transposition of \( \sigma \) with respect to any of the two subsystems. Then, at most \( m \) symplectic eigenvalues \( \tilde{\nu}_j \) of \( \tilde{\sigma} \) can violate Inequality (23).

**Proof.** Suppose to perform the transposition in the \( m \)-mode subsystem: \( \tilde{\sigma} = T_m \sigma T_m \), where \( T_m \) is the partial transposition matrix in the \( m \)-mode subsystem (equal to the matrix \( T \) of Eq. (20)). Let \( D(\alpha) \) be the dimension of the subspace upon which the generic matrix \( \alpha \) is negative definite. Since \( T_m \) reduces to the identity on a \((2m+n)\)-dimensional subspace, Inequality (21) reduces to the (definitely satisfied) Inequality (14) on such a subspace, thus implying \( D(\tilde{\sigma} + i \Omega) \leq m \). One has then \( D(\tilde{\nu} + i \Omega) = D(\tilde{\sigma} + i \Omega) \leq m \), where the inequality holds because the signature is preserved under congruence transformations ("Sylvester’s inertia law") and \( \tilde{\nu} + i \Omega = S^T (\tilde{\sigma} + i \Omega) S \) for some \( S \in S_{2(2m+n)} \). Straightforward computation shows that the eigenvalues of \( \tilde{\nu} + i \Omega \) are given by \( \{ \tilde{\nu}_j + 1 \} \), thus proving the result as the \( \{ \tilde{\nu}_j \} \) have to be positive (\( \tilde{\sigma} = T_m \sigma T_m > 0 \) because \( \sigma > 0 \)). The choice of the transposed subsystem is not relevant, since \( (T_n T_m)(\tilde{\sigma} + i \Omega)(T_n T_m) = \tilde{\sigma}' - i \Omega \) (where \( T_n \) and \( \sigma' \) are, respectively, the partial transposition matrix and the partially transposed CM with respect to the \( n \)-mode subsystem), and \( \tilde{\sigma}' - i \Omega \geq 0 \) is equivalent to \( \tilde{\sigma} + i \Omega \geq 0 \).

In analogy with Eq. (18), let us now define

\[
\tilde{\Sigma}_{m+n} = \sum_{j=0}^{m+n} (-1)^{m+n+j} \tilde{\Delta}_j^{m+n}, \quad (23)
\]

The inequality

\[
\tilde{\Sigma}_{m+n} \geq 0, \quad (24)
\]

being necessary for Inequality (21) to be satisfied, is a necessary condition for separability under \((m+n)\)-mode bipartitions and is thus a sufficient condition to detect entanglement in such a multimode system, irrespective of the nature of the state under examination.

Let us remark that, while for Gaussian states Inequality (21) (which ultimately reduces to a condition on the second moments) is necessary and sufficient for the positivity of the partial transpose, this is not the case for non-Gaussian states. For such states conditions resorting to higher moments lead to a sharper detection of negative partial transposition, and thus of entanglement.

Even for Gaussian states, Inequality (24) cannot detect the negativity of the partial transpose if an even number of symplectic eigenvalues violate condition (23). However, let us focus on Gaussian states under \((1+n)\)-mode bipartitions (for which the PPT criterion is necessary and sufficient for separability). Because of the previous lemma, for these states at most one partially
transposed symplectic eigenvalue can violate Inequality \(22\). Inequality \(24\) is then capable of detecting such a violation.

The same argument applies to ‘bisymmetric’ Gaussian states, defined as the \((m+n)\)-mode Gaussian states which are invariant under mode permutations internal to the \(m\)-mode and \(n\)-mode subsystems. A bisymmetric Gaussian state with CM \(\sigma\) can be reduced, by local symplectic operations (on the \(m\)-mode and \(n\)-mode subsystems), to the tensor product of a two-mode Gaussian state and of uncorrelated thermal states \(12\), with global CM \(\sigma_2: \sigma_2 = S_i^T \sigma S_i\) for some \(S_i \in Sp_{2m,R} \oplus Sp_{2n,R}\). The lemma above can be applied to obtain \(D(T \sigma T + i\Omega) \leq 1\) from which, observing that

\[(T S_i^T T)(T \sigma T + i\Omega)(T S_i T) = T \sigma_2 T + i\Omega,\]

one infers that at most one partially transposed symplectic eigenvalue of the CM \(\sigma\) can violate Inequality \(22\). Notice that the locality of the operation \(S_i\) is crucial in establishing this result, since it implies \((T S_i^T T)\Omega(T S_i T) = \Omega\) (which would not generally hold for a \(S_i\) with nondiagonal terms relating the \(m\)-mode to the \(n\)-mode subsystem).

Inequality \(24\) is thus necessary and sufficient for the separability of all \((1+n)\)-mode and bisymmetric \((m+n)\)-mode Gaussian states, except for the set of ‘null measure’ for which any \(\nu_j\) is identical to 1 (that could be entangled but have \(\sum_{j=0}^J = 0\)). However, as in the case of the uncertainty relation, a necessary and sufficient condition for separability in terms of the quantities \(\{\Delta_j\}\) may be found for such states as well. If \(p\) symplectic eigenvalues \(\nu_j\) (for \(p \leq n\)) are equal to 1, such a relation reads, in analogy with Inequality \(19\),

\[\sum_{j=p}^{m+n} (-1)^{m+n+j} p! C_p^j \Delta_j^{m+n} \geq 0.\]  \(25\)

Eq. \(26\) cannot be saturated if, as per hypothesis, only \(p\) eigenvalues were equal to 1. Therefore, in case of saturation of Inequality \(21\), one may proceed to test Inequality \(24\) for increasing values of \(p\), until the saturation disappears: the first nonzero value will determine the positivity of the partial transposition of the state at the level of second moments. Notice that the case for which \(\nu_j = 1\) for some \(j\) is indeed very relevant in designing and detecting entangled resources for quantum information as, typically, it corresponds to pure CV states. For \(p = 0\), Inequality \(25\) correctly reduces to Inequality \(24\), which has thus been generalised. Summing up, Inequality \(21\) – together with Inequalities \(24\) in case of partial saturation – are necessary and sufficient for the separability of \((1+n)\)-mode and of bisymmetric \((m+n)\)-mode Gaussian states.

The simple condition \(24\) on the second moments, which can be promptly analytically verified, may be very helpful in the detection of interesting CV entangled states (in the same spirit as in Refs. \(36\)). This aspect is especially relevant in view of the recent experimental developments in the implementation of multiparticle CV quantum information protocols, usually relying on symmetric resources (discriminated by the previous condition \(33\)).

The next section will better illustrate the efficacy of Inequality \(21\) for the detection of CV entanglement, by addressing the separability of a relevant example of multimode states.

VI. EXAMPLE: NOISY GHZ-LIKE STATES

The CV “GHZ-type” states \(22\) are a class of fully symmetric multimode Gaussian states introduced in Ref. \(36\) as the prototypical resource for the implementation of a CV teleportation network. Experimentally, they can be generated by inserting squeezed vacua into an array of beam splitters (for a detailed description of the generating scheme, see \(32\) and \(37\)). Moreover, they turn out to be the symmetric Gaussian states maximising both the couplewise (between any pair of modes) and the genuine multipartite entanglement \(33\) and their multipartite entanglement proves to be especially resilient to decoherence \(33\). We will study here the separability of GHZ-type states determining the partially transposed symplectic invariants and employing Inequality \(21\).

To fix ideas, we will consider a GHZ-type state with an even number of total modes \(2n\) under the \(n+n\) bipartition. The CM \(\sigma\) of such a state, generated from impinging squeezed vacua with squeezing parameter \(r\), is described by Eq. \(\delta\) with \(2 \times 2\) blocks given by

\[\gamma_{jj} = q \left( \frac{e^{2r} + (2n-1)e^{-2r}}{2n} \right), \quad 1 \leq j \leq 2n,\]

\[\gamma_{jk} = q \left( \frac{0}{\frac{n}{\sinh(2r)}} - \frac{\sinh(2r)}{\sinh(2r)} \right), \quad j \neq k.\]  \(26\)

Realistically, we have assumed a thermal noise with mean photon number \((q - 1) \geq 0\) to affect the creation of the states (this amounts to multiplying the CM by \(q\)).

After some basic algebra (see also Appendix B), the partially transposed minors \(\Delta_{j+n}\) can be explicitly worked out and found to be

\[\Delta_{j+n} = q^{2j} \left[ (C_j^n - g_j^n) + g_j^n \cosh(4r) \right],\]  \(27\)

where the coefficients \(g_j^n\) are determined by the following sums

\[g_j^n = \left( \sum_{k=0}^{j-1} (-1)^{j+k-1} 2^{j-k} C_k^{2n} \right) - \left( \sum_{k=0}^{j-3} (-1)^{j+k-1} 2^{j-k-2} C_k^{2n} \right),\]  \(28\)

and satisfy a recursive relation which will shortly be useful

\[g_j^n = 2(C_j^{2n} - g_{j-1}^n) - g_{j-2}^n, \quad \text{for } 2 \leq j \leq 2n.\]  \(29\)
Also, one has $g_0^n = 0$ and $g_1^n = 2$. Inequality (24) then reads
\[
\sum_{j=0}^{2n} (-1)^{j+2n} q^{2j} \left[ (C_j^{2n} - g_j^n) + g_j^n \cosh(4r) \right] \geq 0. \tag{30}
\]

Now, the polynomial in $q$ in the LHS of Inequality (30) admits $4n - 4$ degenerate roots for $q = \mp 1$ and $4$ more roots for $q = \mp e^{\mp 2r}$ [as can be shown by employing Eq. (24)]. Because the polynomial is obviously diverging for infinite $q$ (the leading order in $q$ has a positive sign) and accounting for the physical condition $q \geq 1$, we find that Inequality (30) is equivalent to
\[
q \geq e^{2r}. \tag{31}
\]

Such a simple criterion, derived from Inequality (24), is necessary and sufficient for the separability of $2n$-mode noisy GHZ-like states under $(n + n)$-mode bipartitions, regardless of the total number of modes (let us recall that, in the absence of noise, such states are already known to be always inseparable under any bipartition). Note also that the physical significance of the derived condition is immediately evident: in order to maintain the entanglement in the final state, the thermal noise has to be counterbalanced with a corresponding level of squeezing in the initially uncorrelated input modes.

Notably, an analogous analysis – based on Inequality (24) – may be applied for any number of modes under any bipartition and for more general states, getting more and more useful as the number of modes increases.

\section*{VII. DISCUSSION AND OUTLOOK}

The separability criteria here presented, compactly cast in terms of symplectic invariants, have been proven to be necessary and sufficient for two relevant classes of Gaussian states and have led to remarkably simple conditions for the separability of GHZ-like states. Besides this immediate usefulness, the presented analysis, deriving from a recasting of the uncertainty relations in terms of symplectic invariants, touches the very core of the symplectic structure, encompassing the full description of Gaussian states of light and discrete bosonic systems in general. More specifically, it further clarified the specific constraints imposed by quantum mechanics on the second moments of canonical operators, in a naturally canonically invariant framework.

This fundamental analysis might yield further interesting results concerning the entanglement characterization of CV states. In particular, the parametrization of Gaussian states through symplectic invariants has provided remarkable insight into the entanglement properties of two-mode states \cite{21, 22} and could be, employing the techniques here presented, carried over to the analysis of multipartite continuous variable entanglement, which has been lately drawing considerable attention \cite{32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44} mostly in view of the remarkable experimental perspectives uprising in quantum optical systems.

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\section*{APPENDIX A: PROOF OF INEQUALITY (24)}

Let us consider the operator $\hat{y} = (\hat{R} - \text{Tr}[\varrho \hat{R}])^T Y$, where $Y \in \mathbb{C}^{2n}$ is an arbitrary complex vector. The positivity of $\varrho$ implies $\text{Tr}[\varrho \hat{y}^\dagger \hat{y}] \geq 0$, from which
\[
\text{Tr}[\varrho \hat{y}^\dagger \hat{y}] = Y^\dagger \tau Y = Y^\dagger (\sigma + i\Omega) Y \geq 0, \tag{A1}
\]
where the matrix $\tau$ has entries $\tau_{jk} \equiv \text{Tr}[\varrho \hat{R}_j \hat{R}_k] - \text{Tr}[\varrho \hat{R}_j] \text{Tr}[\varrho \hat{R}_k]$ and the CCR \cite{11} have been employed. Due to the arbitrariness of $Y \in \mathbb{C}^{2n}$, Inequality (A1) corresponds to Inequality (24), which is thus proven.

Notice how the uncertainty relation has been derived assuming solely the CCR \cite{11} and the positivity of the density matrix $\varrho$.

\section*{APPENDIX B: COMPUTATION OF THE SYMPLECTIC INVARIANTS}

An alternative way of computing the symplectic invariants defined by Eq. (12) is here outlined, providing some deeper insight into the symplectic architecture of covariances of bosonic systems.

Let $\hat{m} = \{m_1, \ldots, m_k\}$ stand for a ordered $k$-subset of natural integers smaller or equal than $n$, such that $m_j \in \mathbb{N}$ and $m_j \geq m_{j-1}$ for $j = 1, \ldots, k$, and let $\mathcal{N}_n^k$ be the set of all such $k$-subsets. Now, for $\hat{l}, \hat{m} \in \mathcal{N}_n^k$, let us define the $2k \times 2k$ submatrix $\sigma_{l,\hat{m}}^k$ of the CM $\sigma$ as
\[
\sigma_{l,\hat{m}}^k = \begin{pmatrix}
\gamma_{l_1 m_1} & \cdots & \gamma_{l_1 m_k} \\
\vdots & \ddots & \vdots \\
\gamma_{l_k m_1} & \cdots & \gamma_{l_k m_k}
\end{pmatrix}, \tag{B1}
\]
where the $2 \times 2$ submatrices $\gamma_{jk}$ are defined as in Eq. (B6).

The symplectic invariants $\Delta_n^k$ are then given by
\[
\Delta_n^k = \sum_{l,\hat{m} \in \mathcal{N}_n^k} \text{Det} \sigma_{l,\hat{m}}^k, \tag{B2}
\]
where the sum runs over all the possible ordered $k$-subset. This definition amounts to considering the sums of the determinants of all the possible (also non principal) $2k \times 2k$ submatrices of $\sigma$ obtained by selecting all the combinations of $2 \times 2$ blocks (each block describing one mode or the correlations between a pair of modes). For instance, for any $n$, one has $\Delta_1^n = \sum_{j,k=1}^{n} \text{Det} \gamma_{jk}$.
that is just the sum of the determinants of the $2 \times 2$ blocks themselves (this is the multimode generalisation of the invariant entering in the uncertainty relation \[\mathcal{S}\] for two-mode states).

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[18] The CCR could be, more rigorously, introduced in terms of the (bounded) displacement operators $D_\xi = \exp (i R \Omega \xi)$, for $\xi \in \mathbb{R}^{2n}$, as $D_\xi^\dagger D_\xi = D_\xi D_\xi^\dagger \exp (4i \xi^\dagger \Omega \xi^\prime)$). However, to our aims, the simpler, straightforward definition given by Eq. (4) will suffice. Note that, through all the paper, it is understood that the second moments of the field operators \{\hat{R}_j\} are finite for the considered states.
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