RATIONAL SUBSETS OF POLYCYCLIC MONOIDS AND
VALENCE AUTOMATA

Elaine Render and Mark Kambites
School of Mathematics, University of Manchester
Manchester M60 1QD, England
E.Render@maths.manchester.ac.uk
Mark.Kambites@manchester.ac.uk

Abstract. We study the classes of languages defined by valence automata
with rational target sets (or equivalently, regular valence grammars with ra-
tional target sets), where the valence monoid is drawn from the important
class of polycyclic monoids. We show that for polycyclic monoids of rank 2
or more, such automata accept exactly the context-free languages. For the
polycyclic monoid of rank 1 (that is, the bicyclic monoid), they accept a class
of languages strictly including the partially blind one-counter languages. Key
to the proof is a description of the rational subsets of polycyclic and bicyclic
monoids, other consequences of which include the decidability of the rational
subset membership problem, and the closure of the class of rational subsets
under intersection and complement.

1. Introduction

Both mathematicians and computer scientists have found applications for fi-
nite automata augmented with registers which store values from a given group or
monoid, and are modified by multiplication. These automata, variously known as
valence automata, extended finite automata or M-automata, provide an algebraic
method to characterize important language classes such as the context-free, re-
cursively enumerable and blind counter languages (see [6]). Their study provides
insight into computational problems in algebra (see, for example, [9]). These au-
tomata are also closely related to regulated rewriting systems, and in particular the
valence grammars introduced by Paun [11]: the languages accepted by M-automata
are exactly the languages generated by regular M-valence grammars [5].

Traditionally, the monoid registers are initialised to the identity element, and a
word is accepted only if it can be read by a successful computation which results
in the register being returned to the identity. Several authors have observed that the
power of these automata to describe language classes may be increased by allowing
a more general set of accepting values in the register. Fernau and Stiebe [4] began
the systematic study of the resulting valence automata with target sets, along with
the corresponding class of regulated grammars. In particular they considered the
natural restriction that the target set be a rational subset of the register monoid.

Of particular interest, when considering semigroups and monoids in relation to
automata theory, is the class of polycyclic monoids. The polycyclic monoid of
rank $n$ is the natural algebraic model of a pushdown store on an $n$-letter alphabet.
For $M$ a polycyclic monoid of rank 2 or more, it is well-known that $M$-automata are equivalent to pushdown automata, and hence accept exactly the context-free languages. The polycyclic monoid of rank 1 is called the bicyclic monoid and usually denoted $B$; we shall see below that $B$-automata accept exactly the partially blind one-counter languages defined by Greibach [7].

One of the main objectives of this paper is to consider the class of languages accepted by polycyclic monoid valence automata with rational target sets. It transpires that, for $M$ a polycyclic monoid of rank 2 or more, every language accepted by an $M$-automaton with rational target set is context-free, and hence is accepted by an $M$-automaton with target set $\{1\}$. In the rank 1 case the situation is rather different; a language accepted by a $B$-automata with rational target set need not be a partially blind one-counter language, but it is always a finite union of languages, each of which is the concatenation of two partially blind one-counter languages.

A key element of the proofs is a simple but extremely useful characterisation of the rational subsets of polycyclic monoids (Corollary 5.6 below). From this we are easily able to derive a number of other consequences which may be of independent interest. These include the facts that the rational subsets of a finitely generated polycyclic monoid form a boolean algebra (with operations effectively computable), and that membership is uniformly decidable for rational subsets of polycyclic monoids.

In addition to this introduction, the present paper is divided into four sections. Section 2 recalls some basic definitions from formal language theory and the theory of valence automata, while Section 3 establishes some foundational results about valence automata with rational target sets. In Section 4 we consider the effect of adjoining a zero to a monoid $M$ upon the classes of languages accepted by $M$-automata and by $M$-automata with rational target sets. Finally, in Section 5 we turn our attention to polycyclic and bicyclic monoids, proving our main results about both rational subsets and valence automata with rational target sets.

2. Preliminaries

Firstly, we recall some basic ideas from formal language theory. Let $\Sigma$ be a finite alphabet. Then we denote by $\Sigma^*$ the set of all words over $\Sigma$ and by $\epsilon$ the empty word. Under the operation of concatenation and with the neutral element $\epsilon$, $\Sigma^*$ forms a free monoid. A finite automaton over $\Sigma^*$ is a finite directed graph with each edge labelled with an element of $\Sigma^*$, and with a distinguished initial vertex and a set of distinguished terminal vertices. A word $w \in \Sigma^*$ is accepted by the automaton if there exists some path connecting the initial vertex with some terminal vertex, the product of whose edge labels in order is $w$. The set of all words accepted by the automaton is denoted $L$ or for an automaton $A$ sometimes $L(A)$, and is called the language accepted by $A$. A language accepted by a finite automaton is called rational or regular.

More generally, if $M$ is a monoid then a finite automaton over $M$ is a finite directed graph with each edge labelled with an element of $M$, and with a distinguished initial vertex and a set of distinguished terminal vertices. An element $m \in M$ is accepted by the automaton if there exists some path connecting the initial vertex with some terminal vertex, the product in order of whose edge labels is $m$. The subset accepted is the set of all elements accepted; a subset of $M$ which is accepted...
by some finite automaton is called a rational subset. The rational subsets of $M$ are exactly the homomorphic images in $M$ of regular languages.

We now recall the definition of a finite valence automaton, or $M$-automaton. Let $M$ be a monoid with identity $1$ and let $\Sigma$ be an alphabet. An $M$-valence automaton (or $M$-automaton for short) over $\Sigma$ is a finite automaton over the direct product $M \times \Sigma^*$. We say that it accepts a word $w \in \Sigma^*$ if it accepts $(1, w)$, that is if there exists a path connecting the initial vertex to some terminal vertex labelled $(1, w)$.

Intuitively, we visualize an $M$-automaton as a finite automaton augmented with a memory register which can store an element of $M$; the register is initialized to the identity element, is modified by right multiplication by elements of $M$, and for a word to be accepted the element present in the memory register on completion must be the identity element. We write $F_1(M)$ for the class of all languages accepted by $M$-automata, or equivalently for the class of languages generated by $M$-valence grammars [5]. More generally, an $M$-automaton with (rational) target set is an $M$-valence automaton together with a (rational) subset $X \subseteq M$. A word $w \in \Sigma^*$ is accepted by such an automaton if it accepts $(x, w)$ for some $x \in X$. We denote by $F_{\text{Rat}}(M)$ the family of languages accepted by $M$-automata with rational target sets. We recall the following result of Fernau and Stiebe [4].

**Theorem 2.1** (Fernau and Stiebe 2001). Let $G$ be a group. Then $F_{\text{Rat}}(G) = F_1(G)$.

### 3. Automata, Transductions and Closure Properties

In this section we study the relationship between rational transductions and $M$-automata with target sets. Consider a finite automaton over the direct product $\Omega^* \times \Sigma^*$. We call an automaton of this type a rational transducer from $\Omega$ to $\Sigma$; it recognises a relation $R \subseteq \Omega^* \times \Sigma^*$ called a rational transduction. The image of a language $L \subseteq \Omega^*$ under the relation $R$ is the set of $y \in \Sigma^*$ such that $(x, y) \in R$ for some $x \in L$. We say that a language $K$ is a rational transduction of a language $L$ if $K$ is the image of $L$ under some rational transduction. The following is a straightforward generalisation of a well-known observation concerning $M$-automata (see for example [8, Proposition 2]).

**Proposition 3.1.** Let $X$ be a subset of a monoid $M$, and let $L \subseteq \Sigma^*$ be a regular language. Then the following are equivalent:

(i) $L$ is accepted by an $M$-automaton with target set $X$;

(ii) there exists a finite alphabet $\Omega$ and a morphism $\omega : \Omega^* \to M$ such that $L$ is a rational transduction of $X \omega^{-1}$.

If $M$ is finitely generated then the following condition is also equivalent to those above.

(iii) for every finite choice of generators $\omega : \Omega^* \to M$ for $M$, $L$ is a rational transduction of $X \omega^{-1}$.

**Proof.** To show that (i) implies (ii), suppose $L$ is accepted by an $M$-automaton with target set $X$. Choose a finite alphabet $\Omega$ and a map $\omega : \Omega^* \to M$ such that the image $\Omega^* \omega$ contains every element of $M$ which forms the first component of an edge-label in the automaton. We now obtain from the automaton a transducer from $\Omega$ to $\Sigma$ by replacing each edge label $(m, x)$ with $(w, x)$ where $w \in \Omega^*$ is some word such that $w \omega = m$. It is a routine exercise to verify that $L$ is the image of $X \omega^{-1}$ under the given transduction.
Conversely, suppose we are given a map \( \omega : \Omega^* \to M \) and a transducer from \( \Omega \) to \( \Sigma \). We construct from the transducer an \( M \)-automaton with target set \( X \) by replacing each edge label of the form \((w, x)\) with \((w\omega, x)\). It is readily verified that the language accepted by this \( M \)-automaton is exactly the image of \( X \) under the transduction.

Suppose now that \( M \) is finitely generated. Clearly, (iii) implies (ii). Finally, if (ii) holds then we can extend \( \omega \) arbitrarily to a finite choice of generators \( \omega' : (\Omega')^* \to M \), and check that we still have the desired property, so that (ii) holds.

\[\square\]

In particular, Proposition 3.1 gives a characterisation in terms of rational subsets and transductions of each class of languages accepted by \( M \)-automata with rational target sets.

**Proposition 3.2.** Let \( M \) be a monoid and \( L \subseteq \Sigma^* \) a language. Then the following are equivalent.

(i) \( L \in F_{\text{Rat}}(M) \);

(ii) there exists a finite alphabet \( \Omega \), a morphism \( \omega : \Omega^* \to M \) and a rational subset \( X \subseteq M \) such that \( L \) is a rational transduction of \( X\omega^{-1} \).

If \( M \) is finitely generated then the following condition is also equivalent to those above.

(iii) there exists a rational subset \( X \subseteq M \) such that for every finite choice of generators \( \omega : \Omega^* \to M \) for \( M \), \( L \) is a rational transduction of \( X\omega^{-1} \).

Recall that a rational cone (also known as a full trio) is a family of languages closed under rational transduction, or equivalently under morphism, inverse morphism, and intersection with regular languages [1, Section V.2]. Since rational transductions are closed under composition [1, Theorem III.4.4] we have the following immediate corollary.

**Corollary 3.3.** \( F_{\text{Rat}}(M) \) is a rational cone. In particular, it is closed under morphism, inverse morphism, intersection with regular languages, and (since it contains a non-empty language) union with regular languages.

### 4. Adjoining a Zero

In this section we show that adjoining a zero to a monoid \( M \) makes no difference to the families of languages accepted either by \( M \)-automata or by \( M \)-automata with rational target sets. Recall that if \( M \) is a monoid, the result of adjoining a zero to \( M \) is the monoid \( M^0 \) with set of elements \( M \cup \{0\} \) where 0 is a new symbol not in \( M \), and multiplication given by

\[
st = \begin{cases} 
   \text{the } M\text{-product } st & \text{if } s, t \in M \\
   0 & \text{otherwise}.
\end{cases}
\]

We begin with the \( M \)-automaton case, where the required result is a very simple observation.

**Proposition 4.1.** Let \( M \) be a monoid. Then \( F_1(M^0) = F_1(M) \).

\textit{Proof.} That \( F_1(M) \subseteq F_1(M^0) \) is immediate, so we need only prove the converse. Suppose \( L \in F_1(M^0) \), and let \( A \) be an \( M^0 \)-automaton accepting \( L \). Clearly any path in \( A \) containing an edge with first label component 0 will itself have first label
component 0; thus, no accepting path in $A$ can contain such an edge. It follows that by removing all edges whose label has first component 0, we obtain a new $M^0$-automaton $B$ accepting the language $L$. But now since $M$ is a submonoid of $M^0$, $B$ can be interpreted as an $M$-automaton accepting $L$, so that $L \in F_1(M)$ as required.

Next we establish the corresponding result for $M$-automata with rational target sets, which is a little more involved.

**Theorem 4.2.** Let $M$ be a monoid. Then $F_{Rat}(M^0) = F_{Rat}(M)$.

**Proof.** That $F_{Rat}(M) \subseteq F_{Rat}(M^0)$ is immediate. For the converse, suppose $L \in F_{Rat}(M^0)$. Then we may choose an $M^0$-automaton $A$ accepting $L$ with rational target set $X \subseteq M$.

Let $L_0$ be the language of words $w \subseteq \Sigma^*$ such that $(0, w)$ labels a path from the initial vertex to a terminal vertex. Let $L_1$ be the set of words $w$ such that $(m, w)$ labels a path from the initial vertex to a terminal vertex for some $m \in X \setminus \{0\}$. Clearly either $L = L_0 \cup L_1$ (in the case that $0 \in X$) or $L = L_1$ (if $0 \not\in X$). We claim that $L_0$ is regular and $L_1 \in F_{Rat}(M)$. By Proposition 3.8 this will suffice to complete the proof.

The argument to show that $L_1 \in F_{Rat}(M)$ is very similar to the proof of Proposition 4.1. We construct from the $M^0$-automaton $A$ a new $M$-automaton $B$ by simply removing each edge with label of the form $(0, m)$. The new automaton $B$ has target set $X \setminus \{0\}$. It is straightforward to show that $B$ accepts exactly the language $L_1$.

It remains to show that $L_0$ is regular. Let $Q$ be the vertex set of the automaton $A$, and let $Q_0 = \{q_0 \mid q \in Q\}$ and $Q_1 = \{q_1 \mid q \in Q\}$ be disjoint copies of $Q$. We define from $A$ a finite automaton $C$ with

- vertex set $Q_0 \cup Q_1$;
- for each edge in $A$ from $p$ to $q$ with label of the form $(m, x)$
  - an edge from $p_0$ to $q_0$ labelled $x$ and
  - an edge from $p_1$ to $q_1$ labelled $x$;
- for each edge in $A$ from $p$ to $q$ with label of the form $(0, x)$
  - an edge from $p_0$ to $q_1$ labelled $x$ and
  - an edge from $p_1$ to $q_1$ labelled $x$;
- initial vertex $q_0$ where $q$ is the initial vertex of $A$; and
- terminal vertices $q_1$ whenever $q$ is a terminal vertex of $A$.

We shall show that $C$ accepts exactly the language $L_0$. Let $w \in L_0$. Then there exists an accepting path $\pi$ through $A$ labelled $(0, w)$. It follows from the definition of $M^0$ that no product of non-zero elements can equal 0; hence, this path must traverse at least one edge with label of the form $(0, x)$ for some $x \in \Sigma^*$. Suppose then that $\pi = \pi_1 \pi_2 \pi_3$ where $\pi_1$ is a path from the initial vertex to a vertex $p$ with label $(m_1, w_1)$, $\pi_2$ is an edge from $p$ to a vertex $q$ with label $(0, x)$, and $\pi_3$ is a path from $q$ to a terminal vertex with label $(m_3, w_3)$. It follows easily from the definition of $C$ that it has a path from the initial vertex to $p_0$ labelled $w_1$, an edge from $p_0$ to $q_1$ with label $x$, and an edge from $q_1$ to a terminal vertex with label $w_3$. Hence, $w = w_1 x w_3$ is accepted by $C$, as required.

Conversely suppose $w \in L(C)$, and let $\pi$ be an accepting path for $w$. Notice that the initial vertex of $C$ lies in $Q_0$ while all the terminal vertices lie in $Q_1$. Then $\pi = \pi_1 \pi_2 \pi_3$ where $\pi_1$ is a path from the initial vertex to some $p_0$ with label $w_1$, $\pi_2$
is an edge from \( p_0 \) to some \( q_1 \) with label \( x \), \( \pi_3 \) is a path from \( q_1 \) to a terminal vertex with label \( w_3 \) where \( w = w_1 x w_3 \). Now it follows easily from the definition of \( C \) that \( A \) has paths from the initial vertex to \( p \) with label of the form \((m_1, w_1)\), from \( p \) to \( q \) with label \((0, x)\) and from \( q \) to a terminal vertex with label of the form \((m_3, w_3)\). Thus, \( A \) accepts \((m_1 m_3, w_1 x w_3) = (0, w)\) so that \( w \in L_0 \) as required. \( \square \)

Combining Theorems 4.2 with the result of Fernau and Stiebe [4] mentioned above (Theorem 2.1) gives us the following immediate corollary.

**Corollary 4.3.** Let \( G \) be a group. Then

\[
F_{\text{Rat}}(G^0) = F_{\text{Rat}}(G) = F_1(G) = F_1(G^0).
\]

5. **Polycyclic Monoids**

In this section we study the language classes \( F_1(M) \) and \( F_{\text{Rat}}(M) \), where \( M \) is drawn from the class of polycyclic monoids, which form the natural algebraic models of pushdown stores. In the process, we obtain a number of results about rational subsets of these monoids which may be of independent interest.

Let \( X \) be a set. Recall that the polycyclic monoid on \( X \) is the monoid \( P(X) \) generated, under the operation of relational composition, by the partial bijections of the form

\[
p_x : X^* \to X^*, \quad w \mapsto wx
\]

and

\[
q_x : X^* x \to X^*, \quad wx \mapsto w.
\]

The monoid \( P(X) \) is a natural algebraic model of a pushdown store on the alphabet \( X \), with \( p_x \) and \( q_x \) corresponding to the elementary operations of *pushing* \( x \) and *popping* \( x \) (where defined) respectively, and composition to performing these operations in sequence. For a more detailed introduction see [8].

Clearly for any \( x \in X \), the composition \( p_x q_x \) is the identity map. On the other hand, if \( x \) and \( y \) are distinct letters in \( X \), then \( p_x q_y \) is the empty map which constitutes a zero element in \( P(X) \). In the case \(|X| = 1\), say \( X = \{x\} \), the monoid \( P(X) \) is called the bicyclic monoid, and is often denoted \( B \). The partial bijections \( p_x \) and \( q_x \) alone (which we shall often denote just \( p \) and \( q \)) do not generate the empty map, and so the bicyclic monoid does not have a zero element; to avoid having to treat it as a special case, it is convenient to write \( P^0(X) \) for the union of \( P(X) \) with the empty map; thus we have \( P^0(X) = P(X) \) if \(|X| \geq 2 \) but \( P^0(X) \) isomorphic to \( P(X) \) with a zero adjoined if \(|X| = 1 \).

Let \( P_X = \{p_x \mid x \in X\} \) and \( Q_X = \{q_x \mid x \in X\} \), and let \( z \) be a new symbol not in \( P_X \cup Q_X \) which will represent the zero element. Let \( \Sigma_X = P_X \cup Q_X \cup \{z\} \). Then there is an obvious surjective morphism \( \sigma : \Sigma_X \to P^0(X) \), and indeed \( P^0(X) \) admits the monoid presentation

\[
P^0(X) = \langle \Sigma_X \mid p_x q_x = 1, p_x q_y = z, z p_x = z q_x = p_x z = q_x z = z z z = z \text{ for all } x, y \in X, x \neq y \rangle.
\]

It is well-known (see for example [4, 8]) that for \(|X| \geq 2 \), a \( P(X) \)-automaton is equivalent to a pushdown automaton with stack alphabet \( X \), so that the language class \( F_1(P(X)) \) is exactly the class of context-free languages. Greibach [7] has introduced and studied the class of partially blind counter automata. The latter are non-deterministic finite automata augmented with a number of non-negative integer
counters which can be incremented and decremented but not read; attempting to
decrement a counter whose value is 0 causes the computation to fail. The counters
are initialized to 0, and a word is accepted only if some computation reading that
word places the finite state control in an accepting state and returns all counters
to 0. The following equivalence follows immediately from the definitions.

**Proposition 5.1.** For any $n > 0$, $F_1(B^n)$ is exactly the class of languages accepted
by partially blind $n$-counter automata.

We now turn our attention to the classes $F_{\text{Rat}}(P(X))$ of languages accepted by
polycyclic monoid automata with rational target sets. For $|X| \geq 2$, it transpires
that every language accepted by a $P(X)$-automaton with rational target set is
accepted by a $P(X)$-automaton, and hence that $F_{\text{Rat}}(P(X))$ is the class of context
free languages. In order to prove this, we shall need some results about rational
subsets of polycyclic monoids, which we establish using techniques from string
rewriting theory.

Recall that a monadic rewriting system $\Lambda$ over an alphabet $\Sigma$ is a subset of
$\Sigma^* \times (\Sigma \cup \{\epsilon\})$. We normally write an element $(w, x) \in \Lambda$ as $w \to x$. Then we
write $u \Rightarrow v$ if $u = rws \in \Sigma^*$ and $v = rxs \in \Sigma^*$ with $w \to x$. Denote by $\Rightarrow^*$ the
transitive, reflexive closure of the relation $\Rightarrow$. If $u \Rightarrow^* v$ we say that $u$ is an ancestor
of $v$ under $\Lambda$ and $v$ is a descendant of $u$ under $\Lambda$; we write $L_{\Lambda}$ for the set of all
descendants of words in $L$. It is well-known that if $L$ is regular then $L_{\Lambda}$ is again
a regular language; if moreover the rewriting system $\Lambda$ is finite, a finite automaton
recognising $L_{\Lambda}$ can be effectively computed from a finite automaton recognising $L$.
For more information on such systems see [2, 3].

**Theorem 5.2.** Let $X$ be a finite alphabet and $R$ a rational subset of $P^0(X)$. Then
there exists a regular language

$$L \subseteq Q_X^* P_X^* \cup \{z\}$$

such that $L\sigma = R$. Moreover, there is an algorithm which, given an automaton
recognizing a regular language $G \subseteq \Sigma_X^*$, constructs an automaton recognising a
language $L \subseteq Q_X^* P_X^* \cup \{z\}$ with $L\sigma = G\sigma$.

**Proof.** Since $R$ is rational, there exists a regular language $K \subseteq \Sigma_X^*$ such that
$K\sigma = R$. We define a monadic rewriting system $\Lambda$ on $\Sigma_X^*$ with the following rules:

- $p_xq_x \rightarrow \epsilon$,  \hspace{1cm}  $p_xq_y \rightarrow z$
- $p_xz \rightarrow z$,  \hspace{1cm}  $q_xz \rightarrow z$
- $zz \rightarrow z$

for all $x, y \in X$ with $x \neq y$. Notice that the language of $\Lambda$-irreducible words is
exactly $Q_X^* P_X^* \cup \{z\}$. With this in mind, we define

$$L = K\Lambda \cap (Q_X^* P_X^* \cup \{z\})$$

Certainly $L$ is regular, and moreover an automaton for $L$ can be effectively com-
puted from an automaton for $K$. Thus, it will suffice to show that $L\sigma = R$.

By definition $L\sigma \subseteq (K\Lambda)\sigma$, and since the rewriting rules are all relations satisfied
in $P^0(X)$,

$$(K\Lambda)\sigma \subseteq K\sigma = R.$$
Conversely, if \( s \in R \) then \( s = w\sigma \) for some \( w \in K \). Now the rules of \( \Lambda \) are all length-reducing, so \( w \) must clearly have an irreducible descendant, say \( w' \). But now \( w' \in L \) and \( w'\sigma = w\sigma = s \) so that \( s \in L\sigma \). Thus, \( L\sigma = R \) as required. \( \square \)

As an immediate corollary, we obtain a corresponding result for bicyclic monoids

**Corollary 5.3.** Let \( R \) be a rational subset of a bicyclic monoid \( B \), and \( \sigma : \{ p, q \}^* \rightarrow B \) the natural morphism. Then there exists a regular language \( L \subseteq q^*p^* \) such that \( L\sigma = R \). Moreover, there is an algorithm which, given an automaton recognizing a regular language \( G \subseteq \{ p, q \}^* \), constructs an automaton recognizing a language \( L \subseteq q^*p^* \) with \( L\sigma = G\sigma \).

Before proceeding to apply the theorem to polycyclic monoid automata with target sets, we note some general consequences of Theorem 5.2 for rational subsets of polycyclic monoids. Recall that a collection of subsets of a given base set is called a boolean algebra if it is closed under union, intersection and complement within the base set.

**Corollary 5.4.** The rational subsets of any finitely generated polycyclic monoid form a boolean algebra. Moreover, the operations of union, intersection and complement are effectively computable.

*Proof.* The set of rational subsets of a monoid is always (effectively) closed under union, as a simple consequence of non-determinism. Since intersection can be described in terms of union and complement, it suffices to show that the rational subsets of polycyclic monoids are closed (effectively) under complement. To this end, suppose first that \( R \) is a rational subset of a finitely generated polycyclic monoid \( P(X) \) with \(|X| \geq 2\). Then by Theorem 5.2 there is a regular language \( L \subseteq (Q^*_X P^*_X \cup \{ z \}) \) such that \( L\sigma = R \). Let \( K = (Q^*_X P^*_X \cup \{ z \}) \setminus L \). Then \( K \) is regular and, since \( Q^*_X P^*_X \cup \{ z \} \) contains a unique representative for every element of \( P(X) \), it is readily verified that \( K\sigma = P(X) \setminus (L\sigma) \). Thus, \( P(X) \setminus (L\sigma) \) is a rational subset of \( P(X) \), as required.

For effective computation of complements, observe that given an automaton recognizing a language \( R = \Sigma^*_X \), we can by Theorem 5.2 construct an automaton recognizing a regular language \( L \subseteq (Q^*_X P^*_X \cup \{ z \}) \) with \( L\sigma = R\sigma \). Clearly we can then compute the complement \( K = (Q^*_X P^*_X \cup \{ z \}) \setminus L \) of \( L \) in \( (Q^*_X P^*_X \cup \{ z \}) \), and since \( K\sigma = P(X) \setminus (L\sigma) \), this suffices.

In the case that \(|X| = 1\), the statement can be proved in a similar way but using Corollary 5.3 in place of Theorem 5.2. \( \square \)

Recall that the rational subset problem for a monoid \( M \) is the algorithmic problem of deciding, given a rational subset of \( M \) (specified using an automaton over a fixed generating alphabet) and an element of \( M \) (specified as a word over the same generating alphabet), whether the given element belongs to the given subset. The decidability of this problem is well-known to be independent of the chosen generating set [9, Corollary 3.4]. As another corollary, we obtain the decidability of this problem for finitely generated polycyclic monoids.

**Corollary 5.5.** Finitely generated polycyclic monoids have decidable rational subset problem.

*Proof.* Let \(|X| \geq 2\) [respectively, \(|X| = 1\)]. Suppose we are given a rational subset \( R \) of \( P(X) \) (specified as an automaton over \( \Sigma^*_X \) [respectively \( \{ p, q \}^* \)]) and an element
there exists an integer

Corollary 5.6.

We now return to our main task of proving that $F_{\text{Rat}}(M) = F_1(M)$ for $M$ a polycyclic monoid of rank 2 or more, that is, that polycyclic monoid automata with target sets accept only context-free languages. We shall need some preliminary results.

Corollary 5.6. Let $R$ be a rational subset of $P^0(X)$ and suppose that $0 \notin R$. Then there exists an integer $n$ and regular languages $Q_1, \ldots, Q_n \subseteq Q_X^*$ and $P_1, \ldots, P_n \subseteq P_X^*$ such that

$$R = \bigcup_{i=1}^{n} \left( Q_iP_i \right) \sigma.$$ 

Proof. By Theorem 5.2 there is a regular language $L \subseteq Q_X^*P_X^*$ such that $L\sigma = R$. Let $A$ be a finite automaton accepting $L$, with vertices numbered $1, \ldots, n$. Suppose without loss of generality that the edges in $A$ are labelled by single letters from $Q_X \cup P_X$. For each $i$ let $Q_i$ be the set of all words in $Q_X^*$ which label paths from the initial vertex to vertex $i$. Similarly, let $P_i$ be the set of all words in $P_X^*$ which label words from vertex $i$ to a terminal vertex.

Now if $w \in Q_iP_i$ then $w = uv$ where $u \in Q_X^*$ labels a path from the initial vertex to vertex $i$, and $v \in P_X^*$ labels a path from vertex $i$ to a terminal vertex. Hence $uv = w$ labels a path from the initial vertex to a terminal vertex, and so $w \in L$. Conversely, if $w \in L \subseteq Q_X^*P_X^*$ then $w$ admits a factorisation $w = uv$ where $u \in Q_X^*$ and $v \in P_X^*$. Since the edge labels in $A$ are single letters, an accepting path for $w$ must consist of a path from the initial vertex to some vertex $i$ labelled $u$, followed by a path from $i$ to a terminal vertex labelled $v$. It follows that $u \in Q_i$ and $v \in P_i$, so that $w \in Q_iP_i$. Thus we have

$$L = \bigcup_{i=1}^{n} Q_iP_i$$

and so

$$R = L\sigma = \left( \bigcup_{i=1}^{n} Q_iP_i \right) \sigma = \bigcup_{i=1}^{n} \left( Q_iP_i \right) \sigma$$

as required. □

For the next proposition, we shall need some notation. For a word $q = q_{x_1}q_{x_2} \cdots q_{x_n} \in Q_X^*$, we let $q' = p_{x_n} \cdots p_{x_2}p_{x_1} \in P_X^*$. Similarly for a word $p = p_{x_n}p_{x_2} \cdots p_{x_2} \in Q_X^*$, we let $p' = q_{x_n} \cdots q_{x_2}q_{x_1} \in Q_X^*$. Note that $p'' = p$ and $q'' = q$. Note also that $p'\sigma$ is the unique right inverse of $p\sigma$, and $q'\sigma$ is the unique left inverse of $q\sigma$.

Proposition 5.7. Let $u \in \Sigma_X^*$, and let $q \in Q_X^*$ and $p \in P_X^*$. Then $u\sigma = (qp)\sigma$ if and only if there exists a factorisation $u = u_1u_2$ such that $(q'u_1)\sigma = 1 = (u_2p')\sigma$.

Proof. Suppose first that $u\sigma = (qp)\sigma$. Let $\Lambda$ be the monadic rewriting system defined in the proof of Theorem 5.2. Then $u$ is reduced by $\Lambda$ to $qp$. Notice that the only rules in $\Lambda$ which can be applied to words not representing zero remove factors

$w$ (specified as a word in the appropriate alphabet). Clearly, we can compute $\{w\}$ as a regular language. Now by Corollary 5.4 we can compute a regular language $K \subseteq \Sigma_X^*$ [respectively, $(p,q)^*\sigma$] such that $K\sigma = R \cap \{w\}\sigma$. So $w\sigma \in R$ if and only if $R \cap \{w\}$ is non-empty, that is, if and only if $K$ is non-empty. Since emptiness of regular languages is testable, this completes the proof. □
representing the identity; it follows easily that \( u \) admits a factorisation \( u = u_1u_2 \)
where \( u_1\sigma = q\sigma \) and \( u_2\sigma = p\sigma \). Now we have
\[
(q'u_1)\sigma = (q'\sigma)(u_1\sigma) = (q'\sigma)(q\sigma) = 1
\]
and symmetrically \( (u_2p')\sigma = 1 \) as required.

Conversely, \( q\sigma \) is the unique right inverse of \( q'\sigma \), so if \( (q'u_1)\sigma = (q'\sigma)(u_1\sigma) = 1 \)
then we must have \( u_1\sigma = q\sigma \). Similarly, if \( (u_2p')\sigma = 1 \) then \( u_2\sigma = p\sigma \), and so we
deduce that \( u\sigma = (u_1u_2)\sigma = (qp)\sigma \) as required. \( \square \)

We are now ready to prove our main theorem about \( M \)-automata with rational
target sets where \( M \) is a polycyclic monoid.

**Theorem 5.8.** Suppose \( L \in Frat(P^0(X)) \). Then \( L \) is a finite union of languages,
each of which is the concatenation of one or two languages in \( F_1(P^0(X)) \).

**Proof.** Let \( M = P^0(X) \) and let \( A \) be an \( M \)-automaton with rational target set \( R \)
accepting the language \( L \). By Corollary 5.6 there exists an integer \( n \) and regular
languages \( Q_1, \ldots, Q_n \subseteq Q^*_X \) and \( P_1, \ldots, P_n \subseteq P^*_X \) such that
\[
R = R_0 \cup \bigcup_{i=1}^{n} (Q_iP_i)\sigma,
\]
where either \( R_0 = \emptyset \) or \( R_0 = \{0\} \) depending on whether \( 0 \in R \). For \( 1 \leq i \leq n \), we
let \( R_i = (Q_iP_i)\sigma \). It follows easily that we can write
\[
L = L_0 \cup L_1 \cup \cdots \cup L_n
\]
where each \( L_i \) is accepted by a \( M \)-automaton with target set \( R_i \). Clearly it suffices
to show that each \( L_i \) is a finite union of languages, each of which is the concatenation
of at most two languages in \( F_1(M) \).

We begin with \( L_0 \). Let \( Z = \{ u \in \Sigma^*_X \mid u\sigma = 0 \} \) and \( W = \{ w \in \Sigma^*_X \mid w\sigma = 1 \} \).
It is easily seen (for example, by considering the rewriting system \( A \) from the proof
of Theorem 5.2) that \( u \in Z \) if and only if either \( u \) contains the letter \( z \), or \( u \)
factorizes as \( u_1p_xu_2q_yu_3 \) where \( x, y \in X \), \( x \neq y \) and \( u_1, u_2, u_3 \in \Sigma_X^* \) are such that
\( u_2 \) represents the identity, that is, such that \( u_2 \in W \). Thus,
\[
Z = \Sigma_X^* \{ z \} \Sigma_X^* \cup \bigcup_{x,y \in X, x \neq y} \Sigma_X^* \{ p_x \} W \{ q_y \} \Sigma_X^*.
\]
From this expression it is a routine matter to show that \( Z \) is a rational transduction
of \( W \). By Proposition 3.1 \( L_0 \) is a rational transduction of the language \( Z \). Since
the class of rational transductions is closed under composition, it follows that \( L \)

We now turn our attention to the languages \( L_i \) for \( i \geq 1 \). Recall that \( L_i \) is
accepted by a \( M \)-automaton with target set \( R_i = (Q_iP_i)\sigma \). Let
\[
P'_i = \{ (p', \epsilon) \mid p \in P_i \} \subseteq Q^*_X \times \Sigma^*
\]
and similarly
\[
Q'_i = \{ (q', \epsilon) \mid q \in Q_i \} \subseteq P^*_X \times \Sigma^*.
\]
It is readily verified that \( P'_i \) and \( Q'_i \) are rational subsets of \( \Sigma^*_X \times \Sigma^* \); let \( A_P \) and
\( A_Q \) be finite automata accepting \( P'_i \) and \( Q'_i \) respectively, and assume without loss of
generality that the first component of every edge label is either a single letter in
\( \Sigma_X \) or the empty word \( \epsilon \).
By Proposition 3.1 there is a rational transduction $\rho \subseteq \Sigma^* \times \Sigma^*$ such that $w \in L_i$ if and only if $(u, w) \in \rho$ for some $u \in \Sigma^*_X$ such that $w\sigma \in R_i$. Let $A$ be an automaton recognizing $\rho$, again with the property that the first component of every edge label is either a single letter in $\Sigma_X$ or the empty word $\epsilon$. We construct a new automaton $B$ with

- vertex set the disjoint union of the state sets of $A_Q$, $A$, and $A_P$;
- all the edges of $A_Q$, $A$ and $A_P$;
- initial vertex the initial vertex of $A_Q$;
- terminal vertices the terminal vertices of $A_P$;
- an extra edge, labelled $(\epsilon, \epsilon)$, from each terminal vertex of $A_Q$ to the initial vertex of $A$; and
- an extra edge labelled $(\epsilon, \epsilon)$, from each terminal vertex of $A$ to the initial vertex of $A_P$.

It is immediate that $B$ recognizes the relation

$$\tau = Q_i'P_i' = \{(q'xp', w) \mid q \in Q_i, p \in P_i, (x, w) \in \rho\} \subseteq \Sigma^*_X \times \Sigma^*$$

and again has the property that the first component of every edge label is either a single letter or the empty word.

Let $Q$ be the vertex set of $A$, viewed as a subset of the vertex set of $B$. For each vertex $y \in Q$, we let $K_y$ be the language of all words $w$ such that $(u, w)$ labels a path in $B$ from the initial vertex of $B$ to $y$ for some $u$ with $w\sigma = 1$. By considering $B$ as a transducer but with terminal vertex $y$, we see that $K_y$ is a rational transduction of the word problem of $P(X)$, and hence by Proposition 3.1 lies in the class $F_1(P(X))$.

Dually, we let $L_y$ be the language of all words $w$ such that $(u, w)$ labels a path in $B$ from $y$ to a terminal vertex for some $u$ with $w\sigma = 1$. This time by considering $B$ as a transducer but with initial vertex $y$, we see that $L_y$ is also a rational transduction of the word problem of $P(X)$, and hence also lies in $F_1(P(X))$.

We claim that

$$L_i = \bigcup_{y \in Q} K_y L_y,$$

which will clearly suffice to complete the proof.

Suppose first that $w \in L_i$. Then there exists a word $u \in \Sigma^*_X$ such that $w\sigma \in R_i$ and that $(u, w) \in \rho$. Since $R_i = (Q_iP_i)\sigma$ we have $u\sigma = (q\rho)p\sigma$ for some $q \in Q_i$ and $p \in P_i$. Note that $(q'up', w) \in \tau$ is accepted by $B$. By Proposition 5.7, $u$ admits a factorization $u = u_1u_2$ such that $(q'u_1)\sigma = 1$ and $(u_2p')\sigma = 1$. Now in view of our assumption on the edge labels of $B$, $w$ must admit a factorization $w = w_1w_2$ such that $B$ has a path from the initial vertex to some vertex $y$ labelled $(q'u_1, w_1)$ and a path from $y$ to a terminal vertex labelled $(u_2p', w_2)$; moreover, the vertex $y$ can clearly be assumed to lie in $Q$. Since $(q'u_1)\sigma = 1 = (u_2p')\sigma$, it follows that $w_1 \in K_y$ and $w_2 \in L_y$ so that $w = w_1w_2 \in K_yL_y$, as required.

Conversely, suppose $y \in Q$ and that $w = w_1w_2$ where $w_1 \in K_y$ and $w_2 \in L_y$. Then $B$ has a path from the initial vertex to vertex $y$ labelled $(u_1, w_1)$ and a path from the vertex $y$ to a terminal vertex labelled $(u_2, w_2)$ for some $u_1$ and $u_2$ with $u_1\sigma = u_2\sigma = 1$. Since $y \in Q$, it follows from the definition of $B$ that $u_1 = q'v_1$ and $u_2 = v_2p'$ for some $q \in Q_i$ and $p \in P_i$ and $v_1$ and $v_2$ such that $(v_1v_2, w) \in \rho$. But now $(q'v_1)\sigma = u_1\sigma = 1$ and $(v_2p')\sigma = u_2\sigma = 1$, so we deduce by Proposition 5.7 that
Let \( \rho \) be a \( B \)-automaton.

Proof. \( v_1 \sigma = q \sigma \) and \( v_2 \sigma = p \sigma \). But then \((v_1v_2)\sigma = (qp)\sigma \in R_i \subseteq R \) and \((v_1v_2, w) \in \rho \), from which it follows that \( w \in L_i \) as required.

Thus, we have written \( L \) as a finite union of languages \( L_i \), where each \( L_i \) either lies in \( F_1(M) \) (in the case \( i = 0 \)) or is a finite union of concatenations of two languages in \( F_1(M) \). This completes the proof. \( \square \)

In the case that \( |X| = 2 \), we have \( P^0(X) = P(X) \) and \( F_1(P(X)) \) is the class of context-free languages, which is closed under both finite union and concatenation. Hence, we obtain the following easy consequence.

**Theorem 5.9.** If \( |X| \geq 2 \) then \( F_{Rat}(P(X)) \) is the class of context-free languages.

In the case \( |X| = 1 \), we have that \( P^0(X) \) is isomorphic to the bicyclic monoid \( B = P(X) \) with a zero adjoined. Combining Theorem 5.8 with Proposition 4.11 and Theorem 4.12 we thus obtain.

**Corollary 5.10.** Every language in \( F_{Rat}(B) \) is a finite union of languages, each of which is the concatenation of one or two blind one-counter languages.

Since the class \( F_1(B) \) of partially blind one-counter languages is not closed under concatenation, however, we cannot here conclude that \( F_{Rat}(B) = F_1(B) \). Indeed, the following result shows that this is not the case.

**Theorem 5.11.** The language

\[ \{a^i b^i | i, j \geq 0\} \]

lies in \( F_{Rat}(B) \) but not in \( F_1(B) \).

Proof. Let \( L = \{a^i b^i | i, j \geq 0\} \). First, we claim that the \( B \)-automaton with rational target set shown in Figure 1 accepts the language \( L \). Indeed, it is easily seen to accept exactly pairs of the form

\[ p^{i_0} q^{i_1} q p^{i_2} q^{i_3} a^{i_0} b^{i_1} a^{i_2} b^{i_3} = p^{i_0} q^{i_1+1} q^{i_2+1} q^{i_3}, a^{i_0} b^{i_1} a^{i_2} b^{i_3} \]

for \( i_0, i_1, i_2, i_3 \in \mathbb{N} \). A straightforward argument shows that \( p^{i_0} q^{i_1+1} p^{i_2+1} q^{j_3} = qp \)

if and only if \( i_0 = i_1 \) and \( i_2 = i_3 \), which suffices to establish the claim and proof that \( L \in F_{Rat}(B) \).

Assume now for a contradiction that \( L \in F_1(B) \). Then there exists a \( B \)-automaton \( A \) accepting \( L \), with \( N \) vertices say. For \( i \geq 0 \) let \( \pi_i \) be an accepting path for \( a^i b^i a^i b^i \). Suppose without loss of generality that the right-hand sides of edge labels in \( A \) are all \( a \), \( b \) or \( \epsilon \). Then we can write \( \pi_i = \alpha_i \beta_i \gamma_i \delta_i \) and where \( \alpha_i \)
has label \((s_i,a^i)\), \(\beta_i\) has label \((t_i,b^i)\), \(\gamma_i\) has label \((u_i,a^i)\) and \(\delta_i\) has label \((v_i,b^i)\) for some \(s_i,t_i,u_i,v_i \in B\).

The proof will proceed by considering loops (that is, closed paths) in the automaton \(A\); we begin by introducing some terminology to describe particular types of loops. A loop with label \((q^kp^i,x)\) is called an increment loop if \(j > k\), a stable loop if \(k = j\) and a decrement loop if \(k > j\). We call the loop an epsilon loop if \(x = \epsilon\) and a non-epsilon loop otherwise. A path which does not traverse any loops is called a simple path.

First, notice that since there are only finitely many simple paths, there exists a constant \(K\) such that every simple path in \(A\) has label of the form \((q^kp^h, x)\) with \(g + h < K\).

Now let us consider paths of the form \(\alpha_i\). We claim that for all but at most \(KN\) values of \(i\), the path \(\alpha_i\) contains a non-epsilon increment loop. For all \(i \geq N\), we can write \(\alpha_i = \alpha_i^{(1)}\alpha_i^{(2)}\) where \(\alpha_i^{(1)}\) has label \((s_i^{(1)},a^{i-N})\) and \(\alpha_i^{(2)}\) has label \((s_i^{(2)},a^N)\).

First note that the only elements of \(B\) which generate a right ideal [left ideal] including the identity element, are those of the form \(p^k\) [respectively \(q^k\)] for some \(k \geq 0\). Thus, we must have that both \(s_i\) and \(s_i^{(1)}\) are powers of \(p\), and that \(v_i\) is a power of \(q\). In particular, we can let \(f_i \geq 0\) be such that \(s_i^{(1)} = p^{f_i}\).

First suppose \(i\) is such that \(\alpha_i^{(1)}\) does not traverse an increment loop. Let \(\alpha_i'\) be the path obtained from \(\alpha_i^{(1)}\) by removing all loops, and suppose \(\alpha_i'\) has label \((q^kp^h,a^i)\). Since none of the loops removed were increment loops, it follows easily that

\[ f_i \leq h - g \leq h + g \leq K. \]

Suppose now for a contradiction than more than \(KN\) values of \(i \geq N\) are such that \(\alpha_i'\) contains no increment loop. Then by the pigeonhole principle, there exist \(i \neq j\) with \(i \geq N\) and \(j \geq N\) such that \(f_i = f_j\) and the paths \(\alpha_i^{(1)}\) and \(\alpha_j^{(1)}\) end at the same state. But now the composition \(\alpha_i^{(1)}\alpha_j^{(2)}\beta_j\delta_j\) is an accepting path with label

\[
(s_i^{(1)}t_ju_jv_j,a^{i-N}a^N b^ja^3b^j) = (p^{f_i^{(1)}}, s_j^{(2)}t_ju_jv_j,a^ib^ja^3b^j) \\
= (s_j^{(1)}s_j^{(2)}t_ju_jv_j,a^ib^ja^3b^j) \\
= (s_jt_ju_jv_j,a^ib^ja^3b^j) \\
= (1,a^ib^ja^3b^j)
\]

so that \(a^ib^ja^3b^j\) is accepted by \(A\), giving a contradiction. Thus, we have established that for all but \(KN\) values of \(i \geq N\), the path \(\alpha_i^{(1)}\) must traverse an increment loop. Hence, for all but \(KN + N = (K + 1)N\) values of \(i \geq 0\), the path \(\alpha_i^{(1)}\) must traverse an increment loop.

Now let \(i\) be such that \(\alpha_i^{(1)}\) traverses an increment loop and suppose for a contradiction that \(\alpha_i\) does not traverse a non-epsilon increment loop. Consider the path \(\alpha_i^{(2)}\). Clearly, since this path has label with right-hand-side \(a^N\), and the right-hand-sides of edge labels in the automaton are single letters or \(\epsilon\), this path must traverse a non-epsilon loop. Since \(\alpha_i\) does not traverse a non-epsilon increment loop, \(\alpha_i^{(2)}\) must traverse a non-epsilon stable or decrement loop, say with label \((q^kp^h,a^k)\) where \(0 \leq h \leq g\) and \(0 < k\). We also know that \(\alpha_i^{(1)}\) traverses an epsilon increment loop, say with label \((q^{x}p^h,\epsilon)\) where \(0 \leq x < y\). Clearly, by traversing the
latter loop an additional \((g - h)\) times and the former loop an additional \((y - x)\) times, we obtaining an accepting path for the word \(a^{i+(y-x)k}b^ia^ib^i\), which gives the required contradiction.

Thus, we have shown that for all but at most \((K + 1)N\) values of \(i\), the path \(\alpha_i\) traverses a non-epsilon increment loop. A left-right symmetric argument can be used to establish firstly that each \(v_i = q^{g_i}\) for some \(g_i \geq 0\), and then that for \(i\) sufficiently large, \(\delta_i\) must traverse a non-epsilon decrement loop. Thus, for all but at most \(2(K + 1)N\) values of \(i\), the paths \(\alpha_i\) and \(\delta_i\) traverse respectively a non-epsilon increment loop and a non-epsilon decrement loop.

Now choose \(i\) such that this holds, and let \((q^ip^k, a^m)\) label a non-epsilon increment loop in \(\alpha_i\) and let \((q^{i'}p^{k'}, b^{m'})\) label a non-epsilon decrement loop in \(\delta_i\) where \(k > j\), \(k' > j'\) and \(m, m' > 0\). Let \(\pi_i'\) be the path obtained from \(\pi_i\) by traversing the given increment loop an additional \(j' - k\) times, and the given decrement loop an additional \(k - j\) times. Then \(\pi_i\) has label of the form

\[
\left( t(q^ip^k)(j' - k)^{+1}u(q^{i'}p^{k'})(k-j)^{+1}v, a^{i+m(j'-k')}b^ia^ib^{m'(k-j)} \right)
\]

where \(t, u\) and \(v\) are such that \(\pi\) has label

\[
\left( tq^ip^kuq^i p^{k'}v, a^ib^ia^b^i \right)
\]

so that in particular \(tq^ip^kuq^i p^{k'}v = 1\). Now by our argument above regarding right and left ideals, the element \(tq^i \in B\) must be a power of \(p\), while \(q^{i'}v \in B\) must be a power of \(q\). Noting that powers of \(p\) commute with each other, and powers of \(q\) commute with each other, we get

\[
t(q^ip^k)(j' - k)^{+1}u(q^{i'}p^{k'})(k-j)^{+1}v = tq^ip^{(k-j)(j'-k)}p^kuq^{(k-j)(j'-k)}p^{k'}v
\]

\[
= p^{(k-j)(j'-k)}tq^ip^kuq^{i'}p^{k'}vq^{(k-j)(j'-k)}
\]

\[
= p^{(k-j)(j'-k)}\cdot 1q^{(k-j)(j'-k)}
\]

\[
= 1.
\]

Therefore \(\pi_i'\) is an accepting path. Thus, the automaton accepts the word

\[a^{i+m(j'-k')}b^ia^ib^{m'(k-j)}\]

which is not in the language \(L\), giving the required contradiction. This completes the proof that \(L \notin F_1(B)\). \(\square\)

It is possible, however, to describe concatenations of partially blind one-counter languages using partially blind two-counter automata. Indeed more generally we have the following proposition.

**Proposition 5.12.** Let \(M_1\) and \(M_2\) be monoids and \(L_1\) and \(L_2\) languages over the same alphabet. If \(L_1 \in F_1(M_1)\) and \(L_2 \in F_1(M_2)\) then \(L_1L_2 \in F_1(M_1 \times M_2)\).

**Proof.** By Proposition \([\ddagger]\) for \(i = 1, 2\) there are alphabets \(\Omega_i\), morphisms \(\omega_i : \Omega_i^* \to M_i\) and rational transductions \(\rho_i \subseteq \Omega_i^* \times \Sigma^*\) such that \(L_i = \{1\}^{\omega_i^{-1}}\rho_i\). Assume without loss of generality that \(\Omega_1\) and \(\Omega_2\) are disjoint, and let \(\Omega = \Omega_1 \cup \Omega_2\). Then there is a natural morphism \(\omega : \Omega^* \to M_1 \times M_2\) extending \(\omega_1, \omega_2\). Now let \(\rho\) be the product of \(\rho_1\) and \(\rho_2\):

\[
\rho = \{(u_1u_2, w_1w_2, | (u_1, w_1) \in \rho_1, (u_2, w_2) \in \rho_2)\} \subseteq \Omega^* \times \Sigma^*.
\]
Then \( \rho \) is a rational transduction from \( \Omega^* \) to \( \Sigma^* \). Clearly, if \( u_1 \in \Omega_1^* \) and \( u_2 \in \Omega_2^* \) then \( u_1u_2 \) represents the identity element in \( M_1 \times M_2 \) if and only if \( u_1 \) and \( u_2 \) represent the identity elements in \( M_1 \) and \( M_2 \) respectively. It follows that \( w \) is in the image under \( \rho \) of the identity language of \( M_1 \times M_2 \) if and only if \( w = u_1u_2 \) where \( w_1 \in L_1 \) and \( w_2 \in L_2 \), so that \( w \in L_1L_2 \). Thus, \( L_1L_2 \) is a rational transduction of the identity language of \( M_1 \times M_2 \), so applying Proposition 3.1 again we see that \( L_1L_2 \in F_1(M_1 \times M_2) \) as required.

Since classes of the form \( F_1(M) \) are closed under union, Proposition 5.1, Theorem 5.8, and Proposition 5.12 combine to give the following inclusion.

**Corollary 5.13.**

\[
F_{\text{Rat}}(B) \subseteq F_1(B^2). 
\]

**Acknowledgements**

The research of the second author was supported by an RCUK Academic Fellowship.

**References**

[1] J. Berstel. *Transductions and Context-Free Languages*. Teubner StudienbM-(ucher, Stuttgart, 1979.
[2] R.V. Book, M. Jantzen, and C. Wrathall. Monadic Thue systems. *Theoretical Computer Science*, 19:231–251, 1982.
[3] R.V. Book and F. Otto. *String rewriting systems*. Springer Verlag, New York, 1993.
[4] H. Fernau and R. Stiebe. Valence grammars with target sets. In S. Yu M. Ito, Gh. Paun, editor, *Words, Semigroups and Transductions*, pages 129–140. World Scientific, Singapore, 2001.
[5] H. Fernau and R. Stiebe. Sequential grammars and automata with valences. *Theoretical Computer Science*, 276:377–405, 2002.
[6] R.H. Gilman. Formal languages and infinite groups. In *Geometric and Computational Perspectives on Infinite Groups (Minneapolis, MN and New Brunswick, NJ, 1994)*, DIMACS Series, volume 25 of *Discrete Mathematics and Theoretical Computer Science*, Providence RI, 1996. American Mathematical Society.
[7] S.A. Greibach. Remarks on blind and partially blind one-way multicontroller machines. *Theoretical Computer Science*, 7(3):311–324, 1978.
[8] M. Kambites. Formal languages and groups as memory. *Communications in Algebra* (to appear).
[9] M. Kambites, P.Y. Silva, and B. Steinberg. On the rational subset problem for groups. *J. Algebra*, 309(2):622–639, 2007.
[10] Gh. Paun. A new generative device: valence grammars. *Rev. Roumaine Math. Pures Appl.*, XXV(6):911–924, 1980.