Observations on the “values” of the elliptic modular function $j(\tau)$ at real quadratics

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Abstract

We define “values” of the elliptic modular $j$-function at real quadratic irrationalities by using Hecke’s hyperbolic Fourier expansions, and present some observations based on numerical experiments.

1 Introduction

“.... von dem Studium des Verhaltens der elliptischen Modulfunktionen in der Nähe der nicht-rationalen Randpunkte noch sehr bemerkenswerte Ergebnisse erwartet werden können, die sowohl für die Funktionentheorie wie die Arithmetik wichtig sein dürften.” (Hecke, Werke S.417)

We define the “value,” written $\text{val}(w)\,^1$ of the elliptic modular function $j(\tau)$ at each real quadratic irrationality $w$ as the constant term of a hyperbolic Fourier expansion $^2$ at $w$. The map $w \mapsto \text{val}(w)$ is $\text{PSL}_2(\mathbb{Z})$-invariant and hence assigns to each $\text{PSL}_2(\mathbb{Z})$-equivalence class of real quadratic numbers a certain (real or complex) number. We conducted numerical experiments on the numbers $\text{val}(w)$ and observed the following phenomena, which we find quite remarkable, though no precise formulation (especially for (ii) and (iii)) nor proofs have yet been established:

Observations

(i) The minimum among all real values of $\text{val}(w)$ is realized at $w = (1 + \sqrt{5})/2$ (the golden ratio), with $\text{val}((1 + \sqrt{5})/2) = 706.324813540 \ldots$. Also, all real values of $\text{val}(w)$ lie in the interval $[706.324813540 \ldots, 744]$, where 744 is the constant term in the Fourier expansion of $j(\tau)$ at the cusp (which is the $\text{PSL}_2(\mathbb{Z})$-equivalence class of rational numbers and $i\infty$).

(ii) As the rational approximation of $w$ improves, $\text{val}(w)$ increases. (See the tables at the end of the paper.)

(iii) The imaginary part of any $\text{val}(w)$ lies in the interval $(-1, 1)$. Also, the distribution of the imaginary parts of $\text{val}(w)$, with the discriminants of $w$ bounded, seems to be peaked at 0 and symmetric about this peak. Furthermore, the phenomena described in (i) and (ii) also hold for the absolute value (or real part) of $\text{val}(w)$.

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$^1$Dedekind, in his seminal paper $^1$ on $j(\tau)$, used the symbol $\text{val}(\omega)$ for $j(\tau)$ (where $\omega$ is a variable in the upper half-plane) and called it the “Valenz.” We borrow his notation.

$^2$Hecke considered this type of expansion for modular forms of positive weight $^2$. 

In this paper, we give a precise definition of \( \text{val}(w) \) and then establish its basic properties, which follow almost immediately from the definition. We then describe experiments related to Markoff numbers. This also seems to support the existence of certain “Diophantine continuity” of \( \text{val} \) suggested (but not yet well-formulated) above.

2 Definition and basic properties

Let \( w \) be a real quadratic number with discriminant \( \text{disc}(w) = D > 0 \). Denote by \( \Gamma_w \) the stabilizer of \( w \) in \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) (with the action being the standard linear fractional transformation):

\[
\Gamma_w := \{ \gamma \in \Gamma \mid \gamma w = w \}.
\]

Let \( U_D \) be the group of units of norm 1 in the quadratic order \( \mathcal{O}_D \) of the discriminant \( D \) and \( \varepsilon = \varepsilon^{(1)}_D \) be a generator of the infinite cyclic part of \( U_D \). Then, if \( \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_w \), we have

\[
(cw^2 + (d - a)w - b) = 0,
\]

and thus the number \( cw \) is an algebraic integer, and

\[
(a - cw)(a - cw') = a^2 - ac(w + w') + c^2ww' = 1,
\]

that is, \( a - cw \in U_D \). Here, \( w' \) is the algebraic conjugate of \( w \). It is known that the map

\[
\Gamma_w \ni \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - cw)^2 \in U_D^2
\]

gives an isomorphism from the group \( \Gamma_w \) to \( U_D^2 \), which is an infinite cyclic group generated by \( \varepsilon^2 \). Let \( \gamma_m \) be the element in \( \Gamma_w \) that corresponds to \( \varepsilon^2 \) under this isomorphism. For \( \gamma \in \Gamma_w \), a straightforward computation shows

\[
\frac{\gamma w - \tau}{\gamma w' - \tau} = (a - cw)^2 \cdot \frac{\tau - w}{\tau - w'}
\]

and, in particular,

\[
\frac{\gamma_m w - \tau}{\gamma_m w' - \tau} = \varepsilon^2 \frac{\tau - w}{\tau - w'}.
\]

Denote by \( \delta(w) \) the sign of \( w - w' \). Then, if \( \tau \) is a variable in the upper half plane \( \mathcal{H} \), we have

\[
z := \delta(w) \frac{\tau - w}{\tau - w'} \in \mathcal{H},
\]

and

\[
\tau = \frac{w - \delta(w)w'z}{1 - \delta(w)z}.
\]

Let

\[
j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots \quad (q = e^{2\pi i \tau})
\]

be the classical elliptic modular function. It is \( \Gamma \)-invariant, and hence, by the relations

\[
\frac{\gamma w \tau - \tau}{\gamma w \tau' - \tau'} = \varepsilon^2 \delta(w)z
\]

2
\[
\gamma_w \tau = \frac{w - \varepsilon^2 \delta(w) w' z}{1 - \varepsilon^2 \delta(w) z},
\]
the function
\[
j(\tau) = j \left( \frac{w - \delta(w) w' z}{1 - \delta(w) z} \right) \quad (z \in \mathcal{H})
\]
is invariant under \( z \mapsto \varepsilon^2 z \). Thus, if we set \( z = e^u \), the function
\[
j \left( \frac{w - \delta(w) w' e^u}{1 - \delta(w) e^u} \right),
\]
which is holomorphic in the domain \( 0 < \text{Im}(u) < \pi \), is invariant under the translation \( u \mapsto u + 2 \log \varepsilon \). It therefore has a Fourier expansion of the form
\[
j \left( \frac{w - \delta(w) w' e^u}{1 - \delta(w) e^u} \right) = \sum_{n=-\infty}^{\infty} a_n e^{2 \pi i n \frac{u}{2 \log \varepsilon}}. \tag{1}
\]

**Definition** We define the “value,” \( \text{val}(w) \), of \( j(\tau) \) at \( w \) as the constant term of the series \( \text{(1)} \):
\[
\text{val}(w) := a_0 = \frac{1}{2 \log \varepsilon} \int_{\sigma_0}^{\sigma_0 + 2 \log \varepsilon} j \left( \frac{w - \delta(w) w' e^u}{1 - \delta(w) e^u} \right) du, \tag{2}
\]
where \( \sigma_0 \) is any complex number satisfying \( 0 < \text{Im}(\sigma_0) < \pi \).

If we set \( \sigma_0 = \pi i/2 - \log \varepsilon \) and make the change of variable \( u \mapsto u + \pi i/2 \), we have
\[
\text{val}(w) = \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{w - \delta(w) w' i e^u}{1 - \delta(w) i e^u} \right) du. \tag{3}
\]

Note that \( \text{val}(w) \) is a complex-valued function defined only on the real quadratic irrationalities.

**Proposition** The “value” function \( \text{val}(w) \) possesses the following properties.

1) If \( w \) and \( w_1 \) are \( \Gamma \)-equivalent, then \( \text{val}(w) = \text{val}(w_1) \).

2) \( \text{val}(w) = \text{val}(w') \).

3) \( \text{val}(w) = \text{val}(-w') \).

**Proof.** 1) Let \( w_1 = (aw + b)/(cw + d) \), with \( \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). Because \( j(\tau) \) is \( \Gamma \)-invariant, we have
\[
j \left( \frac{w - \delta(w) w' e^u}{1 - \delta(w) e^u} \right) = j \left( \frac{aw - \delta(w) w' e^u}{1 - \delta(w) e^u} + b \right) = j \left( \frac{aw + b - \delta(w)(aw' + b)e^u}{cw + d - \delta(w)(cw' + d)e^u} \right)
\]
\[
= j \left( \frac{aw + b - \delta(w)aw' e^u}{cw + d - \delta(w)cw' e^u} \right) = j \left( \frac{aw + b - \delta(w)(aw' + b)e^u}{cw + d - \delta(w)(cw' + d)e^u} \right)
\]
\[
= j \left( \frac{w_1 - \delta(w_1)w'_1 \text{sgn}(\eta)e^u}{1 - \delta(w_1)\text{sgn}(\eta)e^u} \right),
\]

where \( \eta = (cw' + d)/(cw + d) \) and we have used \( \delta(w) = \delta(w_1) \text{sgn}(\eta) \) [because \( w_1 - w'_1 = (w - w')/((cw + d)(cw' + d)) = (w - w')\eta/(cw + d)^2 \)]. Therefore, from (2), we obtain

\[
\text{val}(w) = \frac{1}{2 \log \varepsilon} \int_{\sigma_0}^{\sigma_0 + 2 \log \varepsilon} j \left( \frac{w_1 - \delta(w_1) \text{sgn}(\eta)e^u}{1 - \delta(w_1) \text{sgn}(\eta)e^u} \right) du.
\]

Then, because \( \text{sgn}(\eta)\eta > 0 \), we can make the change of variable \( u \to u - \log(\text{sgn}(\eta)\eta) \), and we conclude that \( \text{val}(w) = \text{val}(w_1) \).

2) Changing \( u \) to \( -u \) in (3) and using the relation \( \delta(w') = -\delta(w) \), we have

\[
\text{val}(w) = \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{w - \delta(w)we^{-u}}{1 - \delta(w)we^{-u}} \right) du = \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{w' + \delta(w)wie^u}{1 + \delta(w)wie^u} \right) du
\]

\[
= \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{w' - \delta(w')(w')ie^u}{1 - \delta(w')ie^u} \right) du = \text{val}(w').
\]

3) By (3), we have

\[
\text{val}(w) = \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{w - \delta(w)we^{-u}}{1 - \delta(w)we^{-u}} \right) du = \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{-w' + \delta(w)wie^{-u}}{1 - \delta(w)wie^{-u}} \right) du
\]

\[
= \frac{1}{2 \log \varepsilon} \int_{-\log \varepsilon}^{\log \varepsilon} j \left( \frac{-w' - \delta(-w')(-w')ie^{-u}}{1 - \delta(-w')ie^{-u}} \right) du = \text{val}(-w').
\]

\[\Box\]

Remark. The invariance in 1) does not hold in general for other coefficients \( a_n = a_n(w) \) (\( n \neq 0 \)). The general transformation formula is similarly deduced and reads

\[
a_n \left( \frac{aw + b}{cw + d} \right) = \left| \frac{cw' + d}{cw + d} \right|^{-\frac{\pi in}{\log \varepsilon}} a_n(w).
\]

Corollary 1) Suppose \( \text{disc}(w) = D \) and let \( \varepsilon_D \) be the fundamental unit of the order \( O_D \). Then, if \( N(\varepsilon_D) := \varepsilon_D\varepsilon_D' = -1 \), we always have \( \text{val}(w) \in \mathbb{R} \).

2) If \( w \) and \( -w' \) are \( \Gamma \)-equivalent, then \( \text{val}(w) \in \mathbb{R} \).

Proof. 1) In this case, \( w \) and \( -w \) are \( \Gamma \)-equivalent, and thus, by applying 3), 2) and 1) of the Proposition in turn, we obtain

\[
\text{val}(w) = \text{val}(-w') = \text{val}(w) = \text{val}(w).
\]

2) This follows from 3) and 1) of the Proposition. \[\Box\]

We denote by \( \mathcal{A} \) the class in the narrow ideal class group \( Cl^+(D) \) to which the ideal corresponding to \( w \) belongs. By Proposition, \( \text{val}(w) \) depends only on the class \( \mathcal{A} \). (With this in mind, we may write \( \text{val}(\mathcal{A}) \).) The class corresponding to \( -w' \) is \( \mathcal{A}^{-1} \), and hence the \( \Gamma \)-equivalence of \( w \) and \( -w' \) implies \( \mathcal{A}^2 = 1 \) and vice versa. Hence, the assertion 2) in the
corollary says that the value val(A) is real if A^2 = 1.

Remark. Numerical computations reveal that not all val(w) are real.

We give three examples.

**Example 1.** The minimal discriminant for which there appears a non-real value is D = 136. The wide class number h is 2, and the narrow one h^+ is 4. A representative of the Γ-equivalence class of numbers of discriminant 136 is given by

\[ \sqrt{34}, \frac{-4 + \sqrt{34}}{18}, \frac{-1 + \sqrt{34}}{11}, \frac{1 + \sqrt{34}}{11}, \]

and these are grouped into two wide (PGL_2(Z)-equivalence) classes:

\[ \{ \sqrt{34}, \frac{-4 + \sqrt{34}}{18} \}, \{ \frac{-1 + \sqrt{34}}{11}, \frac{1 + \sqrt{34}}{11} \}. \]

The narrow class group Cl^+(136) is isomorphic to Z/4Z and is generated by the class corresponding to \((-1 + \sqrt{34})/11\). The values of val at this generator and its inverse \((1 + \sqrt{34})/11\) (this is also a generator of Cl^+(136)) are computed as

\[
\begin{align*}
\text{val} \left( \frac{-1 + \sqrt{34}}{11} \right) &= 710.600451944002489 \ldots - 0.5197938281961062 \ldots i, \\
\text{val} \left( \frac{1 + \sqrt{34}}{11} \right) &= 710.600451944002489 \ldots + 0.5197938281961062 \ldots i,
\end{align*}
\]

the two being conjugate with each other as follows from Proposition 3).

The values at other two points are

\[ \text{val}(\sqrt{34}) = \text{val} \left( \frac{-4 + \sqrt{34}}{18} \right) = 720.29003500445066239, \ldots \]

values being identical because \((-4 + \sqrt{34})/18\) and \(-\sqrt{34} = (\sqrt{34})'\) are PSL_2(Z)-equivalent.

**Example 2.** Consider the discriminant D = 145. In this case, we have h = h^+ = 4. As representative numbers, we may choose

\[ \frac{1 + \sqrt{145}}{2}, \frac{1 + \sqrt{145}}{6}, \frac{-5 + \sqrt{145}}{12}, \frac{7 + \sqrt{145}}{16}. \]

By Corollary 1) we know all values of val at these points are real. Numerically, they are given
as

\[
\begin{align*}
\text{val} \left( \frac{1 + \sqrt{145}}{2} \right) &= 720.484777347009813 \ldots, \\
\text{val} \left( \frac{1 + \sqrt{145}}{6} \right) &= 715.729503630174741 \ldots, \\
\text{val} \left( \frac{-5 + \sqrt{145}}{12} \right) &= 708.568357453922648 \ldots, \\
\text{val} \left( \frac{7 + \sqrt{145}}{16} \right) &= 715.729503630174741 \ldots.
\end{align*}
\]

The class group is isomorphic to \( \mathbb{Z}/4\mathbb{Z} \). This is seen from the fact that, for \( w_1 = (1 + \sqrt{145})/6 \), \(-w'_1\) is not equivalent to \( w_1 \) but equivalent to \( w_2 = (7 + \sqrt{145})/16 \). Hence \( \text{val}(w_1) = \text{val}(w_2) \).

**Example 3.** Consider \( D = 520 \). In this case, again, we have \( h = h^+ = 4 \). As representative numbers, we may choose

\[
\sqrt{130}, \quad \frac{-1 + \sqrt{130}}{3}, \quad \frac{-3 + \sqrt{130}}{11}, \quad \frac{-5 + \sqrt{130}}{15},
\]

and whose “values” are given numerically by

\[
\begin{align*}
\text{val}(\sqrt{130}) &= 721.700344576590835 \ldots, \\
\text{val} \left( \frac{-1 + \sqrt{130}}{3} \right) &= 719.032996230455907 \ldots, \\
\text{val} \left( \frac{-3 + \sqrt{130}}{11} \right) &= 713.022954982182920 \ldots, \\
\text{val} \left( \frac{-5 + \sqrt{130}}{15} \right) &= 716.888481219718920 \ldots.
\end{align*}
\]

In this case, the class group is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and all values appear to be distinct.

### 3 Experiments related to Markoff numbers

First let us recall Markoff’s theory. The classical theorem of Hurwitz asserts that, for any real irrational number \( \alpha \), there exist infinitely many rational numbers \( p/q \) that satisfy

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.
\]

The constant \( 1/\sqrt{5} \) is best possible. But if we exclude as \( \alpha \) the numbers which are \( \text{PGL}_2(\mathbb{Z}) \)-equivalent to the golden ratio \((1 + \sqrt{5})/2\), the constant \( 1/\sqrt{5} \) improves to \( 1/\sqrt{8} \). If we also exclude the numbers which are \( \text{PGL}_2(\mathbb{Z}) \)-equivalent to \( \sqrt{2} \), then we can take \( 5/\sqrt{221} \) as the
constant. In general, this continues as follows. There is an infinite sequence of integers called Markoff numbers,

\[ \{m_i\}_{i=1}^{\infty} = \{1, 2, 5, 13, 29, 34, 89, 169, 194, 233, \ldots\}, \]

and associated quadratic irrationalities \( \theta_i \) and monotonically increasing \( L_i \) whose limit is 3, with the following property: ”For any \( i \), if the number \( \alpha \) is not PGL\(_2(\mathbb{Z})\)-equivalent to any of \( \theta_1, \theta_2, \ldots, \theta_{i-1} \), then there exist infinitely many rational numbers \( p/q \) that satisfy

\[ \left| \alpha - \frac{p}{q} \right| < \frac{1}{L_i q^2}. \]

Explicitly, the Markoff numbers \( m_i \) appear as solutions of the diophantine equation

\[ x^2 + y^2 + z^2 = 3xyz, \] (4)

and

\[ L_i = \sqrt{9 - 4/m_i^2}, \quad \theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i^2 - 4}}{2m_i}, \] (5)

where \( k_i \) is an integer that satisfies \( a_i k_i \equiv b_i \pmod{m_i} \) and here \( (a_i, b_i, m_i) \) is a solution of equation (4) with \( m_i \) maximal. If \( (p, q, r) \) is a solution of (4), then \( (p, q, 3pq - r) \) and \( (p, r, 3pr - q) \) are also solutions. This gives to all solutions a structure of tree, and we can arrange Markoff numbers like the picture below.

**Figure 1: The tree of Markoff numbers**

![Tree of Markoff numbers](image)

We computed several values of \( \text{val}(\theta_i) \), and observed the following.

**Observation (iv) Only real values are**

\[ \text{val}(\theta_1) = \text{val}\left(\frac{-1 + \sqrt{5}}{2}\right) = 706.32481354\ldots \]

and

\[ \text{val}(\theta_2) = \text{val}(-1 + \sqrt{2}) = 709.89289091\ldots. \]
No other values $\text{val}(\theta_i) \ (i \geq 3)$ seem to be real.

Note that in Markoff’s theory only $\text{PGL}_2(\mathbb{Z})$-equivalence class is relevant, but we need $\text{PSL}_2(\mathbb{Z})$-equivalence to distinguish non-real $\text{val}(\theta_i)$ and its conjugate. Here, the order of $(a_i, b_i)$ in the definition of $\theta_i$ in (5) becomes relevant. We introduce the following refinement.

Let $(a, b, m_i)$ be the Markoff triple associated to the $i$th Markoff number $m_i$ and assume the order of $a$ and $b$ is so chosen that their positions in the tree is like

\[
\begin{array}{c}
 a \\
 m_i \\
 b
\end{array}
\]

(b is on the right of $a$, so, $(13, 5, 194)$ for 194, $(5, 29, 433)$ for 433 etc.) Define two numbers $\theta_{i,1}$ and $\theta_{i,2}$ by

\[
\theta_{i,1} = \frac{-3m_i + 2k_{i,1} + \sqrt{9m_i^2 - 4}}{2m_i} \quad \text{and} \quad \theta_{i,2} = \frac{-3m_i + 2k_{i,2} + \sqrt{9m_i^2 - 4}}{2m_i}
\]

with $k_{i,1}$ and $k_{i,2}$ being integers that satisfy

\[
 ak_{i,1} \equiv b \pmod{m_i} \quad \text{and} \quad bk_{i,2} \equiv a \pmod{m_i}
\]

respectively.

**Observation (v)** The imaginary part of $\text{val}(\theta_{i,1})$ (resp. $\text{val}(\theta_{i,2})$) is always positive (resp. negative).

**Observation (vi)** Suppose three Markoff numbers $m, m', m''$ are in the position like

\[
\begin{array}{c}
 m \\
 m' \\
 m''
\end{array}
\]

in the Markoff tree, and let $\theta_1, \theta_2, \theta'_1, \theta'_2, \theta''_1, \theta''_2$ be the associated (refined) quadratic numbers. Then, for $j = 1, 2$, both the real and the imaginary parts of $\theta''_j$ lie between those of $\theta_j$ and $\theta'_j$ (the case of $m = 1, m' = 2, m'' = 5$ is exceptional, where the imaginary parts of $\text{val}(\theta_j)$ and $\text{val}(\theta''_j)$ are both 0, while the real part of $\text{val}(\theta''_j)$ is indeed in between those of $\text{val}(\theta_j)$ and $\text{val}(\theta'_j)$).

Hence, all real parts of $\text{val}(\theta_{i,j})$ ($j = 1, 2$), conjecturally, lie in the interval

$$[706.3248135 \ldots, 709.8928909 \ldots]$$

and imaginary parts in

$$[-0.2670397 \ldots, 0.2670397 \ldots],$$
where 0.2670397... is the imaginary part of
\[
\text{val}(\theta_{3,1}) = \text{val}((-1 + \sqrt{221})/10) = 708.90991972... + 0.267039735...i.
\]

Choose any Markoff number \(m\). This determines a connected unbounded region \(R\) in the tree. If we trace the edges of \(R\) downward, we obtain the sequence of Markoff numbers associated to the neighboring region with respect to those edges. Let

\[
n_1^L, n_2^L, n_3^L, \ldots \quad \text{and} \quad n_1^R, n_2^R, n_3^R, \ldots
\]

be those sequences corresponding to the left and the right edges respectively. (When \(m = 1\) (resp. \(m = 2\)), only the sequence \(\{n_k^L\}\) (resp. \(\{n_k^R\}\) occur.)

**Observation (vii)** Let \(\theta_1^{(m)}\) and \(\theta_2^{(m)}\) be the Markoff irrationalities associated to \(m\) as explained above (by fixing the order of \(a\) and \(b\) in the triple \((a, b, m)\)), and similarly \(\theta_{k,j}^L\) (\(j = 1, 2\)) (resp. \(\theta_{k,j}^R\) (\(j = 1, 2\))) the irrationalities associated to \(n_k^L\) (resp. \(n_k^R\)). Then, we surmise

\[
\lim_{k \to \infty} \text{val}(\theta_{k,1}^R) = \text{val}(\theta_1^{(m)}) \quad \text{and} \quad \lim_{k \to \infty} \text{val}(\theta_{k,2}^L) = \text{val}(\theta_2^{(m)}).
\]

Below, we repeat the observations made at the beginning of the paper, in the form of several questions:

- Is \(\text{val}((1 + \sqrt{5})/2) = 706.32481354081...\) minimal (in absolute value) among all the values of \(j(\tau)\) at real quadratics? Do all real values of \(\text{val}(w)\), or all absolute values or real parts of \(\text{val}(w)\), lie in the interval \([706.32481354081... , 744]\)? If this is the case, is 744 the best possible upper bound?

- Does \(\text{val}(w)\) possess some information concerning the Diophantine approximation of \(w\)? For instance, does \(\text{val}(w)\) increase as the rational approximation of \(w\) improves?

- Does the imaginary part of \(\text{val}(w)\) always lie in the interval \((-1, 1)\)? What is the distribution of the imaginary parts?

**Problem** Formulate rigorous statements and find proofs of them that answer all of these questions and, above all, find an arithmetic meaning of \(\text{val}(w)\).

**Remark.** 1) Concerning the nature of the value \(\text{val}(w)\), numerical experiments suggest that it is very unlikely that \(\text{val}(w)\) is itself an algebraic number. The author has spent a fair amount of time, using “lindep” or “algdep” facilities of Pari-GP, or “Plouffe’s inverter” website, to see if any multiplicative combination of \(\text{val}(w)\), \(\log \varepsilon, \pi\) etc. becomes algebraic, but all in vain so far.

2) Recent work of Duke, Imamoğlu and Tóth [2] reveals that the “trace” of \(\text{val}(w)\) appears as the Fourier coefficient of a weakly harmonic modular forms of weight 1/2. It would be an important problem to understand our observations in light of their results.

At the end of the paper, we present some tables of values of \(\text{val}(w)\). The computations were carried out using Mathematica.
We denote by \([b_1, b_2, \ldots, b_n]\) a purely periodic (ordinary) continued fraction of period length \(n\). For example, we have \([1] = (1 + \sqrt{5})/2\), \([2, 1] = 1 + \sqrt{3}\), etc. The fundamental unit of norm 1 (a generator of \(U_D\) in §2) of the order \(O_D\) of discriminant \(D\) is denoted by \(\varepsilon\).

Table 1: Values of \(\text{val}(w)\) at \(w = [n]\).

| \(w\) | \(D\) | \(\text{val}(w)\) | \(\log \varepsilon\) |
|------|------|-----------------|------------------|
| 1    | 5    | 706.3248135408125820559603 \ldots | 0.9624236501192 \ldots |
| 2    | 8    | 709.8928909199123368059253 \ldots | 1.7627471740390 \ldots |
| 3    | 13   | 713.2227192129106375260272 \ldots | 2.3895264345742 \ldots |
| 4    | 20   | 715.8658310509644536878287 \ldots | 2.887209503576 \ldots |
| 5    | 29   | 717.9165510885627097946754 \ldots | 3.294462927421 \ldots |
| 6    | 40   | 719.5292195149241565812037 \ldots | 3.6368929184641 \ldots |
| 7    | 53   | 720.824753829016929089184 \ldots | 3.9314409432993 \ldots |
| 8    | 68   | 721.887832620286958905005 \ldots | 4.189425094522 \ldots |
| 9    | 85   | 722.7768914565219262830724 \ldots | 4.4186954172306 \ldots |
| 10   | 104  | 723.5327700907464960378584 \ldots | 4.6248766825455 \ldots |
| 20   | 404  | 727.6296000047325464824629 \ldots | 5.996445905959 \ldots |
| 30   | 904  | 729.4314438625732480951697 \ldots | 6.804632909611 \ldots |
| 50   | 2504 | 731.2426027524741005593885 \ldots | 7.824855312825 \ldots |
| 100  | 10004| 733.1113065597372736130899 \ldots | 9.210543419828 \ldots |
Table 2: Values of $\text{val}(w)$ at $w = [n, 1]$.

| $w$  | $D$ | $\text{val}(w)$                  | $\log \varepsilon$ |
|------|-----|----------------------------------|---------------------|
| [2, 1] | 12  | 709.7923590080320102702826\ldots | 1.3169578969248\ldots |
| [3, 1] | 21  | 713.246137271926341372589\ldots   | 1.5667992369724\ldots |
| [4, 1] | 32  | 715.8764861800141880351424\ldots | 1.7627471740390\ldots |
| [5, 1] | 45  | 717.8834096374473486546884\ldots | 1.9248473002384\ldots |
| [6, 1] | 60  | 719.4559616552358003854302\ldots | 2.0634370688955\ldots |
| [7, 1] | 77  | 720.72156829624895450810\ldots   | 2.1846437916051\ldots |
| [8, 1] | 96  | 721.7640368038035489169855\ldots | 2.2924316695611\ldots |
| [9, 1] | 117 | 722.639624217652465181309\ldots | 2.389526435742\ldots |
| [10, 1] | 140 | 723.387187954432922287542\ldots | 2.477887302884\ldots |
| [20, 1] | 480 | 727.4935574326730521838984\ldots | 3.089699048446\ldots |
| [30, 1] | 1020 | 729.32406313730436667693\ldots | 3.4647579066758\ldots |
| [50, 1] | 2700 | 731.1703417153765088105933\ldots | 3.9508736907744\ldots |
| [100, 1] | 10400 | 733.0728964687665155522285\ldots | 4.6248766825455\ldots |

Table 3: Values of $\text{val}(w)$ at $w = [n, 2]$.

| $w$  | $D$ | $\text{val}(w)$                  | $\log \varepsilon$ |
|------|-----|----------------------------------|---------------------|
| [3, 2] | 60  | 711.9275163995819056553017\ldots | 2.0634370688955\ldots |
| [4, 2] | 24  | 713.8258642873420364918902\ldots | 2.2924316695611\ldots |
| [5, 2] | 140 | 715.4007874465895012696492\ldots | 2.477887302884\ldots |
| [6, 2] | 48  | 716.6952844238825705424260\ldots | 2.6339157938496\ldots |
| [7, 2] | 252 | 717.7711201642989402376217\ldots | 2.7686593833135\ldots |
| [8, 2] | 80  | 718.6786015779022038417819\ldots | 2.8872709503576\ldots |
| [9, 2] | 396 | 719.4552346952050033894397\ldots | 2.9932228461263\ldots |
| [10, 2] | 120 | 720.1286213941960093536607\ldots | 3.0889699048446\ldots |
Table 4: Values of val(w) at w = [2, 1, ... , 1].

| w          | D | val(w)                              | log ε          |
|------------|---|-------------------------------------|----------------|
| [2]        | 8 | 709.8928909199123368059253...       | 1.7627471740390... |
| [2, 1]     | 12| 709.79235900080302102702826...     | 1.316958969248... |
| [2, 1, 1]  | 40| 708.5134481348921906198907...      | 3.636892918464... |
| [2, 1, 1, 1]| 96| 708.1560508416661547689422...      | 2.292431669561... |
| [2, 1, 1, 1, 1]| 260| 707.8064656210238322953785...     | 5.5529445614474... |
| [2, 1, 1, 1, 1, 1]| 672| 707.5978542380262638805993...     | 3.2566139548000... |
| [2, 1, 1, 1, 1, 1, 1]| 1768| 707.4305612244349322611838... | 7.4764720605230... |

Table 5: Values of val(w) at w = [3, 1, ... , 1].

| w          | D | val(w)                              | log ε          |
|------------|---|-------------------------------------|----------------|
| [3]        | 13| 713.2227192129106375260272...       | 2.3895264345742... |
| [3, 1]     | 21| 713.2461372719263413372589...       | 1.5667992369724... |
| [3, 1, 1]  | 17| 711.046084409650318879502...        | 4.1894250452222... |
| [3, 1, 1, 1]| 165| 710.3366093961225252087583...      | 2.5589789770286... |
| [3, 1, 1, 1, 1]| 445| 709.647535497912849968007...       | 6.0935646743888... |
| [3, 1, 1, 1, 1, 1]| 288| 709.2118541585357042188756...      | 3.5254943480781... |
| [3, 1, 1, 1, 1, 1, 1]| 3029| 708.8593091672155721790085... | 8.0153271998839... |
Table 6: First several non-real values.

| $w$       | $D$ | $\text{val}(w)$ |
|-----------|-----|-----------------|
| $(12 + \sqrt{34})/11$ | 136 | $710.60045194400248945 \ldots + 0.51979382819610620 \ldots i$ |
| $(10 + \sqrt{34})/11$ | 136 | $710.60045194400248945 \ldots - 0.51979382819610620 \ldots i$ |
| $(33 + \sqrt{205})/34$ | 205 | $714.16034018225715592 \ldots + 0.75363913959038068 \ldots i$ |
| $(25 + \sqrt{205})/30$ | 205 | $714.16034018225715592 \ldots - 0.75363913959038068 \ldots i$ |
| $(21 + \sqrt{221})/22$ | 221 | $708.90991972070874730 \ldots + 0.26703973546028996 \ldots i$ |
| $(23 + \sqrt{221})/22$ | 221 | $708.90991972070874730 \ldots - 0.26703973546028996 \ldots i$ |
| $(47 + \sqrt{305})/56$ | 305 | $716.13898693848579303 \ldots + 0.82184193359696810 \ldots i$ |
| $(35 + \sqrt{305})/46$ | 305 | $716.13898693848579303 \ldots - 0.82184193359696810 \ldots i$ |
| $(23 + \sqrt{79})/25$ | 316 | $712.65948582687702503 \ldots + 0.3254553768732463 \ldots i$ |
| $(13 + \sqrt{79})/15$ | 316 | $712.65948582687702503 \ldots - 0.3254553768732463 \ldots i$ |
| $(17 + \sqrt{79})/15$ | 316 | $712.65948582687702503 \ldots + 0.3254553768732463 \ldots i$ |
| $(17 + \sqrt{79})/21$ | 316 | $712.65948582687702503 \ldots - 0.3254553768732463 \ldots i$ |

Table 7: First several values at Markoff irrationalities.

| $i$ | $m_i$ | $\theta_{i,1}$ | $\text{val}(\theta_{i,1})$ |
|-----|-------|----------------|-----------------------------|
| 1   | 1     | $(-3 + \sqrt{5})/2$ | $706.32481354081258205 \ldots$ |
| 2   | 2     | $-1 + \sqrt{2}$ | $709.89289091991233680 \ldots$ |
| 3   | 5     | $(-11 + \sqrt{221})/10$ | $708.909919720708747 \ldots + 0.267039735460289 \ldots i$ |
| 4   | 13    | $(-29 + \sqrt{1517})/26$ | $708.257588242846779 \ldots + 0.228635826664936 \ldots i$ |
| 5   | 29    | $(-63 + \sqrt{7565})/58$ | $709.302611667387656 \ldots + 0.165196473942199 \ldots i$ |
| 6   | 34    | $(-19 + 5\sqrt{26})/17$ | $707.858372382696744 \ldots + 0.184765335383999 \ldots i$ |
| 7   | 89    | $(-199 + \sqrt{71285})/178$ | $707.594565998876317 \ldots + 0.153386774906169 \ldots i$ |
| 8   | 169   | $(-367 + \sqrt{257045})/338$ | $709.469768024657232 \ldots + 0.118518079083046 \ldots i$ |
| 9   | 194   | $(-108 + \sqrt{21170})/97$ | $708.534665666479421 \ldots + 0.245013213468323 \ldots i$ |
| 10  | 233   | $(-521 + \sqrt{488597})/466$ | $707.408028846873175 \ldots + 0.130903420887032 \ldots i$ |
Figure 2: Values in the Markoff tree

706.32481...  709.89289...

708.90991... ±0.26703...i

708.2575... ±0.2286...i
708.534... ±0.2450...i

707.8583... ±0.1847...i
709.534... ±0.2036...i

709.3026... ±0.1651...i
709.4697... ±0.1185...i

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