Scully-Lamb quantum laser model for parity-time-symmetric whispering-gallery microcavities: Gain saturation effects and non-reciprocity

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We use a non-Lindbladian master equation of the Scully-Lamb laser model for the analysis of light propagation in parity-time symmetric photonic systems. Performing the semiclassical approximation, we obtain a set of two nonlinear coupled differential equations describing the time evolution of intracavity fields. These coupled equations are able to explain the experimentally-observed light non-reciprocity [Peng et al., Nature Physics 10, 394 (2014), Chang et al., Nature Photonics 8, 524 (2014)]. We show that this effect results from the interplay between gain saturation in the active microcavity and parity-symmetry breaking. Additionally, using this approach, we study the effect of the gain saturation on exceptional points, i.e., exotic degeneracies in non-Hermitian systems. Namely, we demonstrate that the inclusion of gain saturation leads to a modification of the exceptional points in the presence of intense intracavity fields. The Scully-Lamb master equation for systems of coupled optical structures, as proposed and applied here, constitutes a promising tool for the study of quantum optical effects in coupled systems with losses, gain, and gain saturation.

I. INTRODUCTION

Recent years has witnessed an increasingly intense research effort to explore a class of non-Hermitian systems described by parity-time (PT) symmetric Hamiltonians [1], (for reviews see [2, 3]). A system described by the Hamiltonian $H$ is $PT$-symmetric if it is invariant under the combined action of the parity $P$ and the time-reversal $T$ operators (i.e., $H$ commutes with the $PT$ operator: $[H, PT] = 0$) but not necessarily with the $P$ or $T$ operator alone. An important consequence of this is the necessary, but not sufficient, condition for $PT$-symmetry: The complex potential $V(x) = V_r(x) + iV_i(x)$ of the Hamiltonian should satisfy $V(x) = V^*(−x)$, where the superscript * denotes complex conjugation. In other words, the real part of the potential should be an even function of $x$, while its imaginary part should be an odd function of $x$, i.e., $V_r(x) = V_r(−x)$ and $V_i(x) = −V_i(−x)$. A $PT$-symmetric system exhibits two very distinct phases: an unbroken $PT$ phase (also known as the exact-$PT$ regime), where the Hamiltonian supports real eigenvalues despite being non-Hermitian, and a broken $PT$ phase where some eigenvalues form complex conjugate pairs. The transition between these two phases takes place spontaneously as a result of parametric variation of the Hamiltonian. This real-to-complex spectral phase transition or the $PT$-phase transition point exhibits all properties of an exceptional point (EP), which is defined as a singularity in the parameter space of a non-Hermitian system at which two or more eigenvalues and their associated eigenvectors coalesce.

A decade after the ground-breaking work of Bender and Boettcher in 1998 which initiated the mathematical framework and fundamental understanding of $PT$-symmetric systems [1], it was realized that $PT$-symmetry and its breaking at an EP can be observed in photonics by imposing the necessary condition for $PT$-symmetry on a complex optical potential, that is on the complex refractive index, $n(x) = n_r(x) + in_i(x)$, which leads to $n_r(x) = n_r(−x)$ and $n_i(x) = −n_i(−x)$ [4, 5]. Thus, an optical system with the $PT$-symmetric potential has a symmetric index profile but an asymmetric gain/loss profile. Such a refractive index profile can be obtained in two coupled optical structures, such as waveguides or resonators: one having loss and the other having gain compensating the loss of the other. This discovery opened a very fertile research direction, where the interplay between gain, loss, and the strength of the coupling between them provides entirely new features and device functionalities [6, 9]. In these non-Hermitian systems, with coupled loss and gain components, EPs emerge as the coupling strength or the amplification (gain) to dissipation (loss) ratio being controlled or tuned. Thus, EPs can drastically alter the overall response of the system. This leads to a plethora of nontrivial phenomena [6, 8], such as enhanced light-matter interactions [10, 11], unidirectional invisibility [13, 14], lasers with enhanced mode selectivity [15, 16], low-power budget nonreciprocal light transmission [17, 18], loss-induced lasing [19, 20], thresholdless phonon lasers [21, 22] to name a few. In parallel to
these efforts in photonics, the concepts of $\mathcal{PT}$-symmetry have been put into use in electronics, optomechanics, acoustics, plasmonics, and metamaterials. More recently, there is a trend in investigating the features of $\mathcal{PT}$-symmetric quantum systems and the effect of $\mathcal{PT}$-symmetry and its breaking on the quantum states of light and the properties of quantum information.

Although $\mathcal{PT}$-symmetry and related concepts have their roots in quantum-field theories, the experimental demonstrations and the majority of theoretical works are focused on classical systems, as in the example of two coupled optical microresonators: where the energy loss in one of them is compensated by the gain in the other, and the system is probed with light from a laser. In experiments, gain in such systems can be provided optically via parametric gain or Raman gain of the amplifying material, while the resonator is made from emitters or rare-earth ions embedded in it. The theoretical framework to analyze such systems relies on linearly-coupled rate equations of classical fields, where the gain and loss correspond to a different sign of the imaginary part of the complex frequencies (e.g., minus for gain and plus for loss, or vice versa) without a reference to how the gain and loss are generated. For example, the theoretical model, developed in Ref. [17], was linear without any nonlinearity, although the experimental results reported in that paper showed the presence of nonlinearity leading to nonreciprocal light transmission. Thus, the theoretical framework, applied there, failed to describe the observed non-reciprocity. On the other hand, in Ref. [18], followed closely Ref. [17], but a term was added phenomenologically to the linear rate equations to include the effect of gain saturation, which provides the required nonlinearity for the nonreciprocal light transmission, without explicitly describing where this term comes from. On a different note, as the field is shifting from the classical to the quantum realm, it is important that the rate equations for the field operators are studied and the quantum-mechanical origins of gain and loss are properly described and incorporated into the models. Thus, a theoretical framework that addresses these concerns is highly desirable. In order to address this goal, we revisit the system, studied in Ref. [17], and later on in Ref. [18], of two coupled optical structures with loss, gain, and gain saturation by using a non-Lindbladian master equation originally derived for the Scully-Lamb laser model. We apply this master equation for the density operators of the optical fields in optical structures. Our approach explains (at a fundamental level) the results obtained in Refs. [17, 18] and explicitly describes the interplay of loss, gain, and gain saturation.

The Hamiltonian of the coupled microresonators with one driving coherent field, which drives the active cavity, can be written as

$$\hat{H} = \sum_{k=1}^{2} \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k + i \hbar \left[ \kappa \hat{a}_1^\dagger \hat{a}_2^\dagger + \epsilon \hat{a}_1 e^{i \omega t} - \text{h.c.} \right],$$

where $\hat{a}_k$ ($\hat{a}_k^\dagger$) is the boson annihilation (creation) operator of the mode $k = 1, 2$, with frequency $\omega_k$; and h.c. denotes Hermitian conjugate. The parameters $\kappa$ and $\epsilon$ are coupling constants between the microresonators as well as between the active cavity field $\hat{a}_1$ and the driving coherent field with frequency $\omega_1$, respectively. The coupling constant $\epsilon$ can be expressed by the power $P$ of the driving field as $\epsilon \equiv \sqrt{\gamma_1} P / (\hbar \omega_1)$.

By following the analysis of Yamamoto and Imamoğlu, we consider our system as an ideal laser system in the Scully-Lamb laser model, which can be described by a non-Lindbladian master equation. This equation for the density operator $\hat{\rho}$ of the optical fields and for the Hamiltonian $\hat{H}$ reads as follows (see...
the master equation, given in Eq. (2), can be simplified in the active microcavity, i.e., we consider only the weakly saturated regime. When the laser operates not far above from the threshold waveguide. The loss due to the coupling of the $i$th cavity, and $\gamma_i$ stands for the decay rate of the atoms, and $r$ accounts for the pump rate of the gain medium. In Eq. (2), the decaying rates for both cavities are denoted by $(i = 1, 2)$

$$\Gamma_i = C_i + \gamma_i, \quad \text{where} \quad C_i = \frac{\omega_i}{Q_i} \tag{4}$$

is the intrinsic loss of the $i$th cavity, and $\gamma_i$ stands for the loss due to the coupling of the $i$th cavity to the $i$th waveguide.

It is important to mention that the master equation, given in Eq. (2), remains valid and its use is justified when the laser operates not far above from the threshold in the active microcavity, i.e., we consider only the weakly saturated regime.

We also note that under the considered approximation, the master equation, given in Eq. (2), can be simplified to the Lindbladian form as [37]:

$$\frac{d}{dt} \hat{\rho} = \frac{1}{i\hbar} \left[ \hat{H}, \hat{\rho} \right] - \frac{1}{2} \sum_{i=1}^{4} \left( \hat{L}_i \hat{L}_i \hat{\rho} + \hat{\rho} \hat{L}_i \hat{L}_i - 2 \hat{L}_i \hat{\rho} \hat{L}_i \right), \tag{5}$$

where the Lindblad operators $\hat{L}_i$ (for $i = 1, \ldots, 4$) are defined as:

$$\hat{L}_1 = \sqrt{A} \hat{a}_1 \left( 1 - \frac{\hat{B}}{2} \hat{a}_1 \hat{a}_1^\dagger \right), \quad \hat{L}_2 = \frac{1}{2} \sqrt{3B} \hat{a}_1 \hat{a}_1^\dagger, \quad \hat{L}_3 = \sqrt{\Gamma_1} \hat{a}_1, \quad \hat{L}_4 = \sqrt{\Gamma_2} \hat{a}_2. \tag{6}$$

The Lindblad form in Eq. (5) is equivalent to the master equation (2) if the terms of second order in $B\hat{a}_1 \hat{a}_1^\dagger/(2A)$ are neglected for the weak gain saturation regime.

III. RESULTS

A. Rate equations

The master equation for the field operators, given in Eq. (2), and the Hamiltonian in Eq. (1), yield the rate equations for the averaged boson operators $\hat{a}_1$ and $\hat{a}_2$. Namely, by using the formula

$$\frac{d}{dt} \langle \hat{a}_j \rangle = \text{Tr} \left[ \hat{a}_j \frac{d}{dt} \hat{\rho} \right], \quad j = 1, 2, \tag{7}$$

and utilizing the cyclic property of the trace operation, after substituting Eq. (2) in Eq. (7), we obtain

$$\frac{d}{dt} \langle \hat{a}_1 \rangle = -i\omega_1 \langle \hat{a}_1 \rangle + \frac{G_1}{2} \langle \hat{a}_1 \rangle - \kappa \langle \hat{a}_2 \rangle - \frac{B}{2} \langle \hat{a}_1 \hat{a}_1^\dagger \rangle, \tag{8}$$

where $G_1 = A - \Gamma_1 - \frac{r}{3}B$. As one can see, the rate equations, given in Eq. (8) for the averaged quantum amplitudes, are nonlinear due to the presence of the gain saturation in the active cavity (i.e., the term $B/2 \langle \hat{a}_1 \hat{a}_1 \rangle$ represents the nonlinearity).

B. Classical Limit

In the classical limit, i.e., in the case of large intensities of the fields, the quantum field operators can be represented by $c$-number amplitudes as $\hat{a}_i \rightarrow \langle \hat{a}_i \rangle \equiv a_i$. Then the rate equations in Eq. (8) can be rewritten in the classical limit as

$$\frac{d}{dt} a_1 = -i\omega_1 a_1 + \frac{G_1}{2} a_1 - \kappa a_2 - \frac{B}{2} |a_1|^2 a_1 - \epsilon \exp (-i\omega t), \tag{9}$$

$$\frac{d}{dt} a_2 = -i\omega_2 a_2 - \frac{\Gamma_2}{2} a_2 + \kappa a_1,$$

where the term $B/2 |a_1|^2 a_1$ represents the nonlinearity due to gain saturation. By substituting $a_k = A_k(t) \exp(-i\omega t)$ into Eq. (9), one arrives at

$$\frac{d}{dt} A_1 = i\Delta A_1 + \frac{G_1}{2} A_1 - \kappa A_2 - \frac{B}{2} |A_1|^2 A_1 - \epsilon,$$

$$\frac{d}{dt} A_2 = i\Delta A_2 - \frac{\Gamma_2}{2} A_2 + \kappa A_1, \tag{10}$$
where $\Delta = \omega - \omega_c$ is the frequency-detuning parameter, and we assumed that the frequency of the driving coherent field is $\omega_f = \omega$, and the frequencies of the cavities are the same, $\omega_1 = \omega_2 = \omega_c$.

It is clearly seen from Eq. (10) that the rate equations simplify to a linear form if the gain saturation in the active laser cavity are neglected.

C. Eigenfrequencies in the steady-state and exceptional point

In the steady state, by considering $\gamma_1 = \gamma_2 = \gamma$ for symmetry reasons, we find from Eq. (10) as:

$$\left(i\Delta + \frac{G_1'}{2}\right) A_1 - \kappa A_2 = \epsilon,$$

$$\left(i\Delta - \frac{\Gamma_2}{2}\right) A_2 + \kappa A_1 = 0,$$

where $G_1' = G_1 - B|A_1|^2$. Setting the driving field to zero ($\epsilon = 0$), we obtain the eigenvalue equation of the system from Eq. (11) as

$$\left(\Delta - i\frac{G_1'}{2}\right) \left(\Delta + i\frac{\Gamma_2}{2}\right) - \kappa^2 = 0,$$

from which we find the eigenfrequencies to be

$$\Delta_{\pm} = \frac{iG_1'}{4} \pm \sqrt{\frac{4\kappa^2 - 1}{4} \left(G_1' + \Gamma_2\right)^2}.$$

By recalling that $\Delta_{\pm} = \omega_{\pm} - \omega_c$, we obtain

$$\omega_{\pm} = \omega_c + i \frac{1}{4} \left(A - C_1 - C_2 - BI_1 - 2\gamma\right)$$

$$\pm \sqrt{\frac{4\kappa^2 - 1}{4} \left(A - C_1 + C_2 - BI_1\right)^2},$$

where $I_1 = |A_1|^2$ is the dimensionless intensity of the field in the active cavity in the steady-state.

We note again that, in reality, the loss rates $\gamma_1$, arising from the coupling with the input-output channels, are not true losses, because these describe the energy transfer from the system to the output (or from the input to the system). Hence, the concept of the $\mathcal{PT}$-symmetry in our system, with balanced gain and loss, can be expressed as

$$A - C_1 - C_2 - BI_1 = 0.$$

This is valid because the system is $\mathcal{PT}$-symmetric regardless of how it is probed (i.e., waveguides in the coupled resonator systems are used only to probe the system). When the condition, given in Eq. (15), is satisfied we find from Eq. (14) that:

$$\omega_{\pm}^{\mathcal{PT}} = \omega_c - i \frac{\gamma}{2} \pm \sqrt{\frac{4\kappa^2 - C_2^2}{2}}.$$

Note that, when changing the input signal $\epsilon$ (and hence the resulting steady-state intensity $I_1$), one has to adjust

the losses correspondingly in order to satisfy the $\mathcal{PT}$-symmetry condition.

The analysis of the frequency spectrum, given in Eqs. (14) and (16), provides two different regimes depending on the sign of the expression under the square-root sign. When the expression under the root is positive (that is, $\kappa > C_2/2$), there are always two supermodes with non-degenerate real frequencies $\omega_{\pm} = \omega_c$ that propagate in the system. Note that this is true for the system itself, where the coupling loss $\gamma$ (which is not inherent to the system of coupled resonators) is zero (i.e., $\gamma = 0$). When that expression is negative (that is, $\kappa < C_2/2$), the real spectrum becomes degenerate, indicating that the system displays two modes with the same resonance frequency but with different decay rates. The transition between these two regimes takes place at an EP given by

$$\kappa_{\text{EP}} = \frac{C_2}{2}.$$

In Fig. 2 we show the real and imaginary parts of the eigenfrequencies of the supermodes. As expected by the inspection of Eq. (17), the $\kappa_{\text{EP}}$ does not depend on the field intensity. For $\kappa > \kappa_{\text{EP}}$, we observe that the imaginary part is different from zero. This is due to the contributions of the loss rates $\gamma_1$ arising from the coupling with the input-output channels. The gray dash-dotted curve in Fig. 2 describes the imaginary parts of the complex eigenfrequencies, when the input-output coupling losses are neglected (i.e., $\gamma = 0$).

We now consider the case, where the $\mathcal{PT}$-symmetry condition is achieved at low input rates, so the gain saturation effects are negligible ($BI_1 \simeq 0$). In this case, as for the $\mathcal{PT}$-symmetry condition, we can use Eq. (15).
FIG. 3. Real part, Re[\omega], (red solid curve) and imaginary part, Im[\omega], (blue dashed curve) of the eigenfrequencies of the supermodes as a function of the coupling \( \kappa \) under the \(\mathcal{PT}\)-symmetry condition \( A - C_1 - C_2 = 0 \) (i.e., excluding the gain saturation term \( B I_1 \)) for different values of the coupling coefficient \( \epsilon \) of the driving field to a microresonator (see Fig. 1). It is seen that that exceptional point depends on gain saturation. (a) The linear regime with \( B = 0.05 \text{ Hz} \) and \( \epsilon = 1 \text{ MHz} \); and (b, c, d) the nonlinear regime according to Eq. (20) with \( B = 0.05 \text{ Hz} \) with (b) \( \epsilon = 2 \text{ GHz} \), (c) \( \epsilon = 20.5 \text{ GHz} \), and (d) \( \epsilon = 25 \text{ GHz} \). The observed inclination of the imaginary part of the eigenfrequencies (blue dashed curve) towards negative values near the EP indicates an additional loss due to gain saturation caused by stronger driving fields. We assumed a passive cavity loss of \( C_2 = A - C_1 = 1 \text{ MHz} \), an active cavity gain of \( A = 301 \text{ MHz} \), and a waveguide-microresonator coupling strength of \( \gamma = 1.15 \text{ MHz} \). The vertical black dashed line denotes the EP \( \kappa_{\text{EP}} \) for the linear system, when the gain saturation term \( B I_1 \) is either exactly zero or negligible compared to the system parameters \( A \) and \( C_i \); thus, the behavior of the EP is identical to that presented in Fig. 2 [see also Figs. 3(a) and 4(a)].

Eq. (18). The condition for the exceptional point becomes [see Eq. (14)]:

\[
\kappa_{\text{EP}} = \frac{1}{4} |A - C_1 + C_2 - B I_1| = \frac{1}{4} |2 C_2 - B I_1| .
\] (19)

From inspection of Eq. (19), we observe that, in this case, the EP changes when the steady-state field intensity increases in the active cavity (see Fig. 2). In particular, as an example, in Fig. 4 we plot the real and imaginary parts of the eigenfrequencies of the supermodes for the linear [Fig. 4(a)] and nonlinear [Figs. 4(b)–(d)] regimes for different values of the driving-field coupling \( \epsilon \) when driving the system at a resonant frequency \( \omega_c \).

In the linear regime, the gain saturation term \( B I_1 \) is either exactly zero or negligible compared to the system parameters \( A \) and \( C_i \); thus, the behavior of the EP is identical to that presented in Fig. 2 [see also Figs. 3(a) and 4(a)]. The same conclusion applies when the driving field is far away from the resonance \( \omega_c \), as in that case, the steady-state intensity also tends to zero.

An interesting situation arises when the gain saturation term \( B I_1 \) becomes comparable with the passive cavity loss \( C_2 \). However, at the same time, it is much less than the gain coefficient \( A \) in the active cavity, i.e.,
FIG. 5. Dimensionless steady-state intensities $I_1$ (green solid curve) and $I_2$ (yellow dashed curve) in the active and passive cavities, respectively, as a function of the intercavity coupling $\kappa$ for the driving field propagating in the direction $1 \rightarrow 4$ (see Fig. 1). The panels (a) – (d) are obtained for different values of the driving field coupling $\epsilon$ and correspond to the panels in Fig. 3.

$BI_1 \ll A$, so the weakly saturated regime still holds and the validity of the master equation in Eq. (2) remains. This also implies that $C_1 \gg C_2$. In this case, the system starts exhibiting some nonlinear features in its eigenspectrum. In short, the described condition can be written as

$$BI_1 \approx C_2 \ll A \approx C_1.$$  \hfill (20)

In what follows, we will always call the system to be in the nonlinear $\mathcal{PT}$-symmetric regime, whenever both conditions, given in Eqs. (18) and (20), are satisfied. In that case, one can observe that the critical value of $\kappa_{EP}$ significantly changes depending on the gain saturation term $BI_1$. Moreover, the steady-state intensity $I_1$ in the active cavity by itself becomes dependent on the direction of the propagation of the driving field, i.e., on whether the driving field is coupled to the active (through port 1) or the passive cavity (through port 3) [see Fig. 1]. Consequently, the gain saturation term $BI_1$ also depends on the driving-field direction. For instance, in the case when the driving field is coupled to the active cavity, by increasing the strength of the driving field, at first, one observes a shift of the critical values of $\kappa_{EP}$ towards lower values, i.e., the value of the modulus in the r.h.s. of Eq. (19) decreases as gain saturation term $BI_1$ approaches $C_2$ [see also Fig. 3(b)]. At the same time, the imaginary part

FIG. 6. Dimensionless steady-state intensities $I_1$ (green solid curve) and $I_2$ (yellow dashed curve) in the active and passive cavities, respectively, as a function of the intercavity coupling $\kappa$ for the driving field propagating in the direction $4 \rightarrow 1$ (see Fig. 1). The panels (a) – (d) are obtained for different values of the driving-field coupling $\epsilon$ and correspond to the panels in Fig. 4.

FIG. 7. Dimensionless steady-state intensity $I_1^{(4 \rightarrow 1)}$ (green solid curve) in the active cavity for the driving field propagating in the direction $4 \rightarrow 1$ from Fig. 6(d), and the steady-state intensity $I_2^{(1 \rightarrow 4)}$ (yellow dashed curve) in the passive cavity for the driving field propagating in the direction $1 \rightarrow 4$ from Fig. 5(d) for lower values of $\kappa$. For the broken-$\mathcal{PT}$-symmetric phase in the nonlinear regime for small values of $\kappa < \kappa_{EP}$, the steady-state intensity $I_1^{(4 \rightarrow 1)}$ is two orders of magnitude larger than $I_2^{(1 \rightarrow 4)}$, implies nonreciprocal light propagation.
of the eigenfrequencies demonstrates a slight inclination towards negative values near the critical $\kappa_{\text{EP}}$ because the imaginary part assumes additional negative values related to the increased values of gain saturation term $B_1 I$ (see Fig. 6). For $\kappa < \kappa_{\text{EP}}$ and for strong driving fields both eigenvalues are negative, regardless of whether the losses $\gamma$ are included or not in the $\mathcal{PT}$-symmetric condition. With further increase of $\epsilon$, the critical value of $\kappa_{\text{EP}}$ then moves back to the starting point, because now the gain saturation term $B_1 I$ becomes larger than $C_2$ in Eq. (19) [see Fig. 6(c)], and then towards larger values for stronger driving fields [see Fig. 6(d)]. Moreover, the observed inclination of the imaginary part of the eigenfrequencies, in the proximity of $\kappa_{\text{EP}}$, also increases [Figs. 3(b) and 3(c)]. The latter is responsible for an additional rising loss and, thus, for an extra line broadening of the eigenmodes near $\kappa_{\text{EP}}$. As such, that broadening, due to gain saturation, makes the two supermodes more difficult to resolve for larger $\kappa > \kappa_{\text{EP}}$.

In the case when the driving field is coupled to the passive cavity, the behavior of the EP shows only the shift towards lower values for the considered values of $\epsilon$ (see Fig. 5), because the steady-state intensity $I_1$ acquires lower values compared to the case when the driving field is coupled to the active resonator in the proximity of $\kappa_{\text{EP}}$ (Fig. 6). As a result, the gain saturation term $B_1 I$ does not exceed $C_2$ in Eq. (19). Moreover, the imaginary part of the eigenfrequencies attains a minimum for $\kappa > \kappa_{\text{EP}}$ for larger $\epsilon$ [Figs. 4(c) and 4(d)]. Therefore, when driving $\kappa$ from this minimum towards the EP, the negative values of the imaginary eigenfrequencies decrease: hence, the losses also decrease compared to the case when one instead drives the active cavity.

The same analysis applies when one fixes the driving field and varies the strength of the gain saturation coefficient $B$ instead.

Note that, with an increasing strength of the driving field and, therefore, with an increasing nonlinearity in the system, in the broken-$\mathcal{PT}$-symmetric phase for small values of $\kappa < \kappa_{\text{EP}}$, the steady-state intensity $I_1$ (for the direction $4 \rightarrow 1$) can attain values which are of two orders larger than the steady-state intensity $I_2$ when the driving field propagates in the opposite direction (see Fig. 7). The latter property can lead to the observation of the non-reciprocal light behavior for small $\kappa$.

In summary, the critical value of $\kappa$, at which the real eigenfrequencies become degenerate, can be shifted towards lower or larger values depending on gain saturation term $B_1 I$. As a result, this shift also depends on the propagation direction of the driving field in the nonlinear regime, in which the gain saturation losses are comparable to the passive cavity losses (see Figs. 6 and 4). In addition, we also observe that both real and imaginary parts of the eigenfrequencies of the system are modified in the presence of intense resonant intracavity fields in the nonlinear regime and that modification is asymmetric in the driving field propagation. In particular, the imaginary part (blue dashed line in Figs. 6 and 4) assumes additional negative values due to the gain saturation term $B_1 I$ (see Fig. 5), and for sufficiently large intensities when $\kappa \approx \kappa_{\text{EP}}$ one can observe either negative or positive inclination of the imaginary part of the eigenmodes depending on whether the driving field is coupled to the active or passive cavity, respectively. The negative values of the imaginary part of the spectrum are responsible for the losses imposed on eigenmodes by the gain saturation in the active cavity, and, as such, near the EP $\kappa_{\text{EP}}$ these losses can become larger or lower depending on the propagation of the driving resonant fields.

D. Transmission spectra

Here we focus on the spectral properties of the driving fields that propagate through the system.

By rewriting the complex amplitudes $A_k$ of the fields [as in Eq. (10)] as $A_k = |A_k| \exp(i\phi_k)$, one arrives at a cubic equation for the field intensity $I_1$ in the active steady state (see Appendix B, for details):

$$\lambda_1 I_1^3 + \lambda_2 I_1^2 + \lambda_3 I_1 + \lambda_4 = 0,$$

(21)

with coefficients $\lambda_i$ defined as:

$$\lambda_1 = \frac{B^2}{4}, \quad \lambda_2 = BF, \quad \lambda_3 = -\epsilon^2, \quad \lambda_4 = F^2 + \Delta^2 \left(\frac{4\epsilon^2}{C_2^2 + 4\Delta^2} - 1\right)^2, \quad F = \left(\frac{2\epsilon^2 C_2^2 - \frac{\epsilon^2}{2}}{C_2^2 + 4\Delta^2}\right).$$

(22)

Equation (21) has only one real solution, when its discriminant is negative, which is always the case when, e.g., $A \approx \Gamma_1$ regardless of $\Gamma_2$ and $\kappa$). This real root is found as

$$I_1 = \frac{1}{6\lambda_1} \left[ x^3 - \frac{4(3\lambda_1 \lambda_4 - \lambda_3^2)}{x^2} - 2\lambda_2^2 \right],$$

(23)

where

$$x = 12\sqrt{3} \left(27 \lambda_3^2 \lambda_4^2 - 18 \lambda_1 \lambda_2 \lambda_3 \lambda_4 + 4 \lambda_1 \lambda_3^3 + 4 \lambda_4^3 \lambda_3 - \lambda_2^3 \lambda_4 \right)^{\frac{1}{3}} + 36 \lambda_1 \lambda_2 \lambda_3 - 108 \lambda_2^3 \lambda_4 - 8 \lambda_3^2.$$

(24)

The transmission spectrum can be calculated as follows

$$T(\omega) = \left| \frac{A_{\text{out}}}{A_{\text{in}}} \right|^2.$$

(25)

To obtain the transmission spectrum $T_{1 \rightarrow 4}(\omega)$ at port 4 when sending the signal from port 1, one needs to know the expressions for the corresponding input-output fields. The output field at port 4 is found as $A_{\text{out}} = \sqrt{\gamma_4} A_2$. The input driving field sent from port 1 can be expressed as $A_{\text{in}} = \epsilon / \sqrt{\gamma_1}$, where $\epsilon$ is the coupling constant. By rewriting the field $A_2$ via $A_1$ and using Eq. (10), one finally obtains

$$T_{1 \rightarrow 4}(\Delta) = \frac{4\kappa^2 \gamma_1 \gamma_2}{\epsilon^2 (C_2^2 + 4\Delta^2)} I_1.$$

(26)
C3

the passive cavity loss
and $\epsilon$

ties is $A = 301 \text{ MHz}$, and the waveguides coupling with both cavities
is observed in the transmission spectrum $\tilde{T}_{1 \rightarrow 4}$ (see also Fig. 1, for details), for
different values of the intercavity coupling $\kappa$: (a) $\kappa = 0.5 \text{ MHz}$ and (b) $\kappa = 1.8 \text{ MHz}$ with the $\mathcal{PT}$-symmetry condition $A - C_1 - C_2 = 0$ (excluding the gain saturation term $B I_1$). The linear regime with $B = 0.05 \text{ Hz}$ and $\epsilon = 1 \text{ MHz}$ (red solid curve); the nonlinear regime, according to Eq. (20),
with $B = 0.05 \text{ Hz}$ and $\epsilon = 2 \text{ GHz}$ (green dashed curve) and $\epsilon = 20.5 \text{ GHz}$ (blue dash-dotted curve). Assuming
the passive cavity loss $C_2 = 1 \text{ MHz}$, the active cavity gain $A = 301 \text{ MHz}$, and the waveguides coupling with both cavities
is $\gamma = 1.15 \text{ MHz}$.

The same analysis can be carried out for the case when the driving coherent field is sent from port 4, and the signal is detected at port 1. In that case, one obtains the same cubic equation for the field intensity $I_1$, as in
Eq. (21), but with different $\lambda_k$; $k = 1, \ldots, 4$. The expression for the transmission spectra $T_{1 \rightarrow 1}(\omega)$ then attains the following form

$$T_{1 \rightarrow 1}(\omega) = \frac{\gamma_1 \gamma_2}{\epsilon^2} I_1. \tag{27}$$

Similarly, the transmission spectrum $T_{1 \rightarrow 2}$ can be found using the input-output relation: $A_{\text{out}} = A_{\text{in}} + \sqrt{\gamma_1 A_1}$.

As an example, in Fig. 8 we plot the normalized transmission spectrum $\tilde{T}_{1 \rightarrow 4} = T_{1 \rightarrow 4}(\Delta)/T_{1 \rightarrow 4}(0)$ for the $\mathcal{PT}$-symmetric condition $A - C_1 - C_2 = 0$, for different values of the intercavity coupling $\kappa$, and for different intensities of the driving field (i.e., by varying $\epsilon$) in both linear ($B I_1/C_2 \ll 1$) and nonlinear ($B I_1/C_2 \approx 1$) regimes [for details regarding the nonlinearity condition, see Eq. (20)]. One can observe an increasing spectral line broadening of the transmitted light for stronger driving fields $\epsilon$ in the nonlinear regime, indicating rising losses
due to the gain saturation [green dashed and blue dash-dotted curves in Fig. 8(a)]. Also, in the nonlinear case, the splitting of the supermodes occurs for larger $\kappa > \kappa_{\text{EP}}$
and more intensive fields [Fig. 8(b)]. A similar behavior is observed in the transmission spectrum $\tilde{T}_{4 \rightarrow 1}$.

IV. A COMPARISON WITH EXPERIMENTAL RESULTS OF REF. [17]

In this section, we discuss possible applications of the semiclassical Scully-Lamb laser theory in the prediction of some nontrivial light behavior that was experimentally observed in Refs. [17, 18]. In that paper, the authors experimentally studied a system of coupled $\mathcal{PT}$-symmetric whispering-gallery microcavities, i.e., a system that is identical to that presented in Fig. 1 and which is the focus of the theoretical study of this manuscript. Below, we theoretically reproduce some of the experimental graphs of Ref. [17] in a rather qualitative than quantitative way. Meaning that we make some additional assumptions regarding the system parameters, used in constructing the graphs here. Nevertheless, as was just mentioned, a qualitative comparison can be made, and somewhat positive conclusions can be inferred regarding the applicability of the Scully-Lamb laser model, in its semiclassical limit, to explain some of the results of Ref. [17]. We note that in the construction of the graphs shown here, which presented in Figs. 9, 12 we do not invoke $\mathcal{PT}$-symmetry in the system (see also the text below).

For clarity, in the captions of some of our figures we indicate the corresponding experimental plots of Ref. [17] that we try to theoretically reproduce. Also, in order to stress the similarity between the figures reproduced here and the original experimental graphs of Ref. [17], we keep the axis scales of the plots to be the same as those given in Ref. [17].

For simplicity, when plotting the graphs in this section,
we assume that the losses in both cavities are comprised mainly by the losses due to the coupling of the cavities to the waveguides \( \Gamma_i = C_i + \gamma_i = \gamma_i, \quad i = 1, 2, \) i.e., we set the intrinsic losses \( C_i \) to zero. The latter assumption also implies that the system considered has a broken \( \mathcal{PT} \)-symmetry, i.e., \( A - C_1 - C_2 \neq 0 \), according to Eq. (18). At the same time, the total losses in the system are expected to be larger than the gain, i.e., \( \gamma_1 - \gamma_2 < 0 \). Also, the active microcavity is assumed to operate near the threshold \( A \approx \gamma_1 > \gamma_2 \).

For example, in Fig. 9, which corresponds to the experimentally-obtained Fig. 1(f) in Ref. [17], the transmission spectrum \( T_{1\rightarrow4} = T_{4\rightarrow1} \) in the linear regime when: (a) \( A = 21 \text{ MHz}, \kappa = 1 \text{ MHz}, \) and \( P = 100 \text{ nW}; \) (b) \( A = 21 \text{ MHz}, \kappa = 20 \text{ MHz}, \) and \( P = 100 \text{ nW}. \) This figure qualitatively reproduces the experimentally-obtained Fig. 3 in Ref. [17].

In Fig. 10, which corresponds to the experimentally-obtained Fig. 3 in Ref. [17], we show the transmission spectra for the light propagating in the directions \( 1 \rightarrow 4 \) and \( 4 \rightarrow 1 \) in the linear regime, i.e., the laser cavity is assumed to be below the lasing threshold \( A < \gamma_1 \).

Non-reciprocity of light propagation

Another important result of our work is the theoretical microscopic prediction of non-reciprocity of the propagating light in the considered coupled active-passive microcavities system (see Fig. 11 which corresponds to the experimentally-obtained Fig. 4 in Ref. [17]). Namely, there is an enhancement in the transmitted light from port 4 to port 1, and tending to zero transmission \( T_{1\rightarrow4} \) in the opposite direction for small values of \( \kappa \). Meaning that the system starts behaving nonreciprocally. The latter nonlinear effect was observed in Ref. [17], but, naturally, could not be explained based on the used linear rate equations there. Utilizing the semiclassical laser theory, one can attain the needed nonlinear term arising from the laser gain saturation in the active microcavity. Moreover, this non-reciprocity is observed without invoking \( \mathcal{PT} \)-symmetry, because it can be already observed when the gain and loss are unbalanced in the system (see Fig. 11). Importantly, as the inset of Fig. 11(iii) indicates, there is no unidirectional behavior of light if there is no input signal, ensuring that the observed nonlinearity is not caused by the lasing initiated by spontaneous emission in the active cavity.

In Fig. 12, which corresponds to the experimentally-obtained Fig. S6 in Ref. [17], displays the transmitted spectrum \( T_{1\rightarrow2} \) versus the detuning \( \Delta = \omega - \omega_c \), for various values of the gain \( A \) and the intercavity coupling \( \kappa \). One can see the appearance of the supermodes splitting in the system with increasing values of \( \kappa \). Conversely, for smaller values of \( \kappa \), the two supermodes coalesce resulting in only one peak in the spectrum.

V. CONCLUSIONS

We have applied the quantum Scully-Lamb laser theory to a pair of \( \mathcal{PT} \)-symmetric coupled whispering-galley microcavities, i.e., a system, which consists of both active and passive microring cavities, such that gain and losses are balanced in the system. It has been shown that, in the nonlinear regime, or more precisely, under the condition in which the gain saturation in the active cavity is comparable to the losses in the passive cavity, the intense intracavity fields of the steady state lead to the modification of the eigenmodes and of the EPs of the \( \mathcal{PT} \)-symmetric system. Namely, the imaginary part of the eigenspectrum acquires an extra negative term due to the gain saturation effects. This effect leads to the shift of the EP either to lower or larger values depending on the gain saturation \( B \) and the propagation direction of the driving fields. Starting from the master equation for this coupled system, including dissipation, gain, and gain saturation, and applying the semiclassical approximation, we are able to describe the experimental results obtained in Refs. [17, 18]. In particular, this approach is able to reproduce the observed non-reciprocal light propagation in the coupled system of whispering-gallery microcavities. We have also shown that the gain saturation mechanism in the active cavity is crucial for the observation of light non-reciprocity. Moreover, we have found that the unidirectional light propagation can be observed.
FIG. 11. Transmission spectrum $T_{1 \rightarrow 4}$ versus detuning $\Delta = \omega - \omega_c$ for: a (i) no amplification $A = 0$ MHz, $\kappa = 1$ MHz; a (ii) $A = 25.7$ MHz, $\kappa = 20$ MHz; a (iii) $A = 25.7$ MHz, $\kappa = 0.15$ MHz. The power of the input signal is $P = 1$ µW. Graphs a (iii) and b (iii) clearly show non-reciprocity of light propagation. The insets of b (ii) and b (iii) show the spectra assuming no input signal. This figure qualitatively reproduces the experimentally-obtained Fig. 4 in Ref. [17].

FIG. 12. Transmission spectrum $T_{1 \rightarrow 2}$ when there is a coupling between the microresonators $R_1$ and $R_2$ for: (a) $A = 10$ MHz, $\kappa = 5.8$ MHz; (b) $A = 18$ MHz, $\kappa = 10.4$ MHz; and (c) $A = 22$ MHz, $\kappa = 16.5$ MHz. The power of the input signal is $P = 100$ nW. This figure qualitatively reproduces the experimentally-obtained Fig. S6 in Ref. [17].

even when the $\mathcal{PT}$-symmetry condition is not fulfilled. It should be stressed that neither $\mathcal{PT}$-symmetry nor its
breaking is required for nonreciprocity. The nonreciprocity observed in our system is a result of a nonlinearity and, in the broken PT-regime, such nonlinearity can be observed at much lower input intensity.

In summary, we proposed, applied, and validated the Scully-Lamb laser model, with the non-Lindbladian master equation, for coupled resonators with losses, gain, and gain saturation. Although we studied non-reciprocity and exceptional points applying the semiclassical approximation, this master equation allows for a quantum description of the cavity fields. This approach constitutes a promising tool for the study of quantum optical effects in coupled resonators with balanced (or unbalanced) gain and losses.

Appendix A: Derivation of the master equation in Eq. (2)

To make this article self-consistent, in this Appendix, we present a derivation of the quantum laser master equation, given in Eq. (2), for the Scully-Lamb laser model based on the derivation of Yamamoto and Imamoglu in Ref. [35]. This derivation bears a phenomenological character and, as such, naturally allows to incorporate all the terms of the interaction Hamiltonian of the fields without the need to solve the Schrödinger equation directly. Another derivation of the master equation [2] can be found in Ref. [36].

The active cavity is represented by a general four-level laser system in which the two intermediate energy levels are coupled by the laser mode (see Fig. 13). In this limit, the uppermost level of the atom may be adiabatically eliminated to give an effective three-level system. The latter assumption is valid as long as the decay rate from the uppermost state |1⟩ to the upper laser level |e⟩ is much faster than all other rates in the atom-field system. In this limit, one has an effective incoherent pumping rate r from the atomic ground state |0⟩ into |e⟩. Additionally, the laser mode in the active cavity is coupled to the passive microresonator.

The interaction Hamiltonian \( \hat{H}_I \), describing such two coupled active-passive microresonators system, is given by

\[
\hat{H}_I = i g \left( \hat{\sigma}_+ \hat{a}_1 - \hat{\sigma}_- \hat{a}_1^\dagger \right) + i \kappa \left( \hat{a}_1 \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_1^\dagger \right). \tag{A1}
\]

Specifically, the second term in Eq. (A1) describes the linear interaction between the modes in the active (\( a_1 \)) and passive (\( a_2 \)) resonators; while the first term describes the interaction between the mode \( a_1 \) with only two levels (|g⟩ and |e⟩) within the standard Jaynes-Cummings model. The operators \( \hat{\sigma}_k, k = +, - \), are spin-raising and spin-lowering operators of the atom in the active medium, respectively. The constants \( g, \kappa \) denote the coupling strength between the atom and the field in the laser cavity, and between the two fields propagating in two different cavities, respectively. We also assumed that the atomic and active field resonances coincide.

The quantum Liouville equation for the density operator \( \hat{\rho} \) in the interaction picture, which describes the atom-field-field dynamics is:

\[
\frac{d}{dt} \hat{\rho} = \frac{1}{i\hbar} \left[ \hat{H}_I, \hat{\rho} \right]. \tag{A2}
\]

In the active cavity, the optical gain is provided by the excited atoms, which are pumped by an external field. The interaction between the atom and an external pumping field, that provides an inverse population in our effective three-level atom laser (depicted in Fig. 13), can be described by the following equation

\[
\frac{d}{dt} \hat{\rho} = - \frac{r}{2} \left( \hat{\sigma}_{00} \hat{\rho} - \hat{\rho} \hat{\sigma}_{00} - 2 \hat{\sigma}_{e0} \hat{\rho} \hat{\sigma}_{0e} \right), \tag{A3}
\]

where \( \hat{\sigma}_{00} = \hat{\sigma}_{0e} \hat{\sigma}_{e0} \), and \( \hat{\sigma}_{0e} \) (\( \hat{\sigma}_{e0} \)) is the spin operator for the atomic transition from |0⟩ (|e⟩) to |e⟩ (|0⟩). The coefficient \( r \) accounts for the pumping rate of the atom.

To include spontaneous emission and the emission caused by external dephasing processes into an external reservoir field, we need to add in Eq. (A2) the following terms:

\[
\frac{d}{dt} \hat{\rho} = - \gamma_{sp} \left( \hat{\sigma}_+ \hat{\rho} \hat{\sigma}_- + \hat{\rho} \hat{\sigma}_- \hat{\sigma}_+ - 2 \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+ \right) - \gamma_d \left( \hat{\sigma}_{ee} \hat{\rho} \hat{\sigma}_{gg} + \hat{\sigma}_{gg} \hat{\rho} \hat{\sigma}_{ee} \right), \tag{A4}
\]

where \( \gamma_{sp} \) and \( \gamma_d \), are, respectively, the rates of spontaneous emission, and the emission imposed by additional dephasing processes.

Now, collecting together Eqs. (A2)–(A4), one arrives at the master equation for the density operator \( \hat{\rho} \)

\[
\frac{d}{dt} \hat{\rho} = \frac{1}{i\hbar} \left[ \hat{H}_I, \hat{\rho} \right] - \frac{1}{2} \sum_{i=1}^2 C_i \hat{L}_i^\dagger(\hat{\rho}) - \frac{r}{2} \left( \hat{\sigma}_{00} \hat{\rho} - \hat{\rho} \hat{\sigma}_{00} - 2 \hat{\sigma}_{e0} \hat{\rho} \hat{\sigma}_{0e} \right) - \gamma_{sp} \left( \hat{\sigma}_+ \hat{\rho} \hat{\sigma}_- + \hat{\rho} \hat{\sigma}_- \hat{\sigma}_+ - 2 \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+ \right) - \gamma_d \left( \hat{\sigma}_{ee} \hat{\rho} \hat{\sigma}_{gg} + \hat{\sigma}_{gg} \hat{\rho} \hat{\sigma}_{ee} \right). \tag{A5}
\]
where we introduced a Lindbladian damping super operator as $\hat{L}_i^d(\hat{O}) = \hat{a}_i^{\dagger} \hat{a} \hat{O} + \hat{O} \hat{a}_i^{\dagger} \hat{a}_i - 2 \hat{a}_i \hat{O} \hat{a}_i^{\dagger}$.

To obtain the master equation for the reduced field density operator $\hat{\rho}_f$, which describes the dynamics of the optical fields in the cavities, one has to trace out the density operator $\hat{\rho}$ in Eq. (A5) over the atom states. Namely,

$$\frac{d}{dt}\hat{\rho}_f = \text{Tr}_\alpha \left[ \frac{d}{dt}\hat{\rho} \right] = \sum_{i=0, e, g} \left< i | \frac{d}{dt}\hat{\rho} | i \right>.$$  \hspace{1cm} (A6)

$$\frac{d}{dt}\hat{\rho}_f = \frac{1}{\hbar} \left[ \hat{H}_f, \hat{\rho}_f \right] - \frac{1}{2} \sum_{i=1}^2 C_i \hat{L}_i^d(\hat{\rho}_f) + g(\hat{a}_1 \hat{\rho}_{eg} - \hat{a}_1^{\dagger} \hat{\rho}_{ge} - \hat{\rho}_{ge} \hat{a}_1 + \hat{\rho}_{eg} \hat{a}_1^{\dagger}),$$  \hspace{1cm} (A7)

where the Hamiltonian $\hat{H}_f$ accounts for the field interaction between active and passive cavities, i.e., it is the second term in Eq. (A1). The operator $\hat{\rho}_{ge} = \left< g | \hat{\rho} | e \right>$, from Eq. (A5), obeys the following equation

$$\frac{d}{dt}\hat{\rho}_{ge} = g(\hat{\rho}_{gg} \hat{a}_1^{\dagger} - \hat{a}_1^{\dagger} \hat{\rho}_{ee}) - \gamma_T \frac{r}{2} \hat{\rho}_{ge},$$  \hspace{1cm} (A8)

where $\gamma_T = \gamma_{sp} + \gamma_d$ is the total decay rate of the atom. The same relation, given in Eq. (A8), also holds true for the operator $\hat{\rho}_{eg}$.

Assuming $\gamma_T \gg C_1, C_2, \kappa$, one eliminates $\hat{\rho}_{ge}$ and $\hat{\rho}_{eg}$ in the adiabatic approximation, i.e., $\frac{d}{dt}\hat{\rho}_{ge} = 0$. Thus, one obtains

$$\hat{\rho}_{ge} = \frac{2g}{\gamma_T} (\hat{\rho}_{gg} \hat{a}_1^{\dagger} - \hat{a}_1^{\dagger} \hat{\rho}_{ee}).$$  \hspace{1cm} (A9)

Substituting $\hat{\rho}_{ge}$ and $\hat{\rho}_{eg}$ in Eq. (A9) into Eq. (A7), one obtains

$$\frac{d}{dt}\hat{\rho}_f = \frac{1}{\hbar} \left[ \hat{H}_f, \hat{\rho}_f \right] - \frac{1}{2} \sum_{i=1}^2 C_i \hat{L}_i^d(\hat{\rho}_f) - \frac{2g^2}{\gamma_T} \left( \hat{L}_1^d(\hat{\rho}_{gg}) + \hat{L}_2^d(\hat{\rho}_{ee}) \right).$$  \hspace{1cm} (A10)

where $L_i^d(\hat{O}) = \hat{a}_i \hat{O} + \hat{O} \hat{a}_i^{\dagger} - 2 \hat{a}_i \hat{O} \hat{a}_i^{\dagger}$ is the Lindbladian amplification super operator.

Moreover, $\hat{\rho}_{ee}$ and $\hat{\rho}_{gg}$ satisfy the following equations

$$\frac{d}{dt}\hat{\rho}_{ee} = \frac{2g^2}{\gamma_T} \hat{\rho}_{gg} - \gamma_{sp} \hat{\rho}_{ee} + r \hat{\rho}_{00},$$  \hspace{1cm} \hspace{1cm} (A11)

$$\frac{d}{dt}\hat{\rho}_{gg} = -\frac{2g^2}{\gamma_T} \hat{L}_1^d(\hat{\rho}_{gg}) + \gamma_{sp} \hat{\rho}_{ee}.$$

If the population of the lower energy level $|g\rangle$ is very low, i.e., the energy quickly decays into the ground state $|0\rangle$, and if the gain saturation is weak, then one may write $\hat{\rho}_{gg} \simeq 0$, and $\hat{\rho}_{00} \simeq \hat{\rho}_f$, \hspace{1cm} (A12)

Applying standard perturbation techniques to Eq. (A11), one arrives at

$$\hat{\rho}_{ee} \simeq \frac{r}{\gamma_{sp}} \hat{\rho}_f - \frac{2g^2 r}{\gamma_T \gamma_{sp}} \left( \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f + \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \right).$$  \hspace{1cm} (A13)

Combining now Eqs. (A12), (A13), and Eq. (A10), one attains the master equation for the two field operator in the interaction picture

$$\frac{d}{dt}\hat{\rho}_f = \frac{1}{\hbar} \left[ \hat{H}_f, \hat{\rho}_f \right] - \frac{A}{2} \hat{L}_1^d(\hat{\rho}_f) - \frac{1}{2} \sum_{i=1}^2 C_i \hat{L}_i^d(\hat{\rho}_f) + \frac{B}{2} \left( \hat{a}_1 \hat{\rho}_f \hat{a}_1^{\dagger} + 2 \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f \hat{a}_1 + \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \right)^2 - 2 \hat{a}_1 \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f \hat{a}_1^{\dagger},$$  \hspace{1cm} (A14)

where

$$A = \frac{4g^2 r}{\gamma_T \gamma_{sp}}, \quad B = \frac{2g^2}{\gamma_T \gamma_{sp}} A.$$

The coefficient $A$ stands for the linear gain, and $B$ is the gain saturation coefficient.

For the case of the ideal laser system with $\Gamma_e = \Gamma_g = \Gamma$ and $\gamma_{dep} = \gamma_{eg} = 0$, one attains the quantum laser master equation within the Scully-Lamb laser theory. In the weakly saturated regime, the spontaneous emission on the laser transition can be discarded, and one obtains

$$\hat{\rho}_{gg} \simeq \frac{2g^2}{\gamma_T} \hat{a}_1 \hat{\rho}_{ee} \hat{a}_1 = \frac{2g^2}{\gamma_T} \hat{a}_1 \hat{\rho}_f \hat{a}_1,$$

$$\hat{\rho}_{ee} \simeq \frac{r}{\Gamma} \hat{\rho}_f - \frac{r}{\gamma_T} \left[ \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f + \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \right].$$  \hspace{1cm} (A15)

Substituting equations in Eqs. (A15) into Eq. (A10), we obtain

$$\frac{d}{dt}\hat{\rho}_f = \frac{1}{\hbar} \left[ \hat{H}_f, \hat{\rho}_f \right] - \frac{A}{2} \hat{L}_1^d(\hat{\rho}_f) - \frac{1}{2} \sum_{i=1}^2 C_i \hat{L}_i^d(\hat{\rho}_f) + \frac{B}{8} \left( \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} + 3 \hat{a}_1 \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} - 4 \hat{a}_1 \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \hat{\rho}_f \hat{a}_1 \hat{a}_1^{\dagger} \right) + \text{h.c.},$$  \hspace{1cm} (A16)

where the gain and gain saturation coefficients are now expressed as

$$A = \frac{4g^2 r}{\gamma_T^2}, \quad B = \frac{4g^2}{\gamma_T^2} A.$$  \hspace{1cm} (A17)

Note a typo in the prefactor of the gain saturation coefficient $B$ in Eq. (A17) in Ref. [35]. Namely, there is a prefactor 1, instead of 4.

We may also add the Hamiltonian term related to the coupling between the cavity field $\hat{a}_1$ and the external driving coherent classical field into the master equation. Such a Hamiltonian can be given by

$$\hat{H}_{\text{drv}} = i \epsilon \left( \hat{a}_1 e^{i \omega t} - \hat{a}_1^{\dagger} e^{-i \omega t} \right),$$  \hspace{1cm} (A18)

where the coupling constant $\epsilon = \sqrt{\gamma_P/\hbar \omega}$ accounts for the coupling between the driving external coherent field with power $P$ and the cavity field $\hat{a}_1$.

By rewriting the field Hamiltonian $\hat{H}_f$ in the Schrödinger picture, we finally arrive at Eq. (2).
Appendix B: Derivation of Eq. (21)

Working in the reference frame where the phase of the driving field is zero, and by expressing the complex amplitudes in the rate equations in Eq. (10) as $A_k = |A_k|e^{i\phi_k}$, one obtains the following equations for the steady-state:

\[ i\Delta |A_1| + \frac{G}{2}|A_1| - \kappa |A_2|e^{i(\phi_2 - \phi_1)} - B/2 |A_1|^3 - |A_1|\epsilon e^{-i\phi_1} = 0, \]
\[ i\Delta |A_2| - \frac{G}{2}|A_2| + \kappa |A_1|e^{-i(\phi_2 - \phi_1)} = 0. \]  
(B1)

Replacing now $|A_2|\exp(i(\phi_2 - \phi_1))$ by $|A_1|$ in the second equation of Eq. (B1) and inserting it into the first equation, we attain

\[ \left( i\Delta + \frac{G}{2} - \frac{2\kappa^2}{C_2 - 2i\Delta} \right) |A_1| - \frac{B}{2} |A_1|^3 - \epsilon \exp(-i\phi_1) = 0. \]  
(B2)

Separating the real and imaginary parts in Eq. (B2) and equalizing them to zero, one arrives at

\[ \cos \phi_1 = \frac{|A_1|}{\epsilon} \left( \frac{G}{2} - \frac{2\kappa^2 C_2}{C_2^2 + 4\Delta^2} - \frac{B}{2} |A_1|^2 \right), \]
\[ \sin \phi_1 = \frac{|A_1|\Delta}{\epsilon} \left( \frac{4\kappa^2}{C_2^2 + 4\Delta^2} - 1 \right). \]  
(B3)

Utilizing now the standard trigonometric relation and collecting together the coefficients at each order of the real amplitude, we finally obtain the cubic equation for the field intensity $I_1 = |A_1|^2$, given in Eq. (21).

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