NUMERICAL COMPUTATION OF REAL OR COMPLEX
ELLIPTIC INTEGRALS

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Abstract. Algorithms for numerical computation of symmetric elliptic integrals of all
three kinds are improved in several ways and extended to complex values of the variables
(with some restrictions in the case of the integral of the third kind). Numerical check
values, consistency checks, and relations to Legendre’s integrals and Bulirsch’s integrals
are included.

Key words. elliptic integral, algorithm, numerical computation

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COMPUTATION OF ELLIPTIC INTEGRALS
1 Introduction

Let $f(x)$ be a real function that is rational except for the square root of a cubic or quartic polynomial with at least one pair of conjugate complex zeros. Then $\int f(x)\,dx$ can be expressed in terms of standard elliptic integrals with complex variables, which are subsequently changed to real variables by using quadratic transformations \[10\]\[11\]. Since the transformations complicate the formulas, it is desirable to have algorithms for numerical computation of standard elliptic integrals with complex variables. Such integrals are met in other contexts also, for example in conformal mapping, and the complex variables might not occur in conjugate pairs.

This paper contains algorithms for numerical computation of complete and incomplete elliptic integrals of all three kinds when the variables are complex (with some restrictions for integrals of the third kind). They are similar to algorithms published earlier \[3\] for real variables, but several improvements that apply to the real case have been made in the course of extending them to complex variables. The integrals computed are the symmetric integrals of the first and third kinds and two degenerate cases, of which one is an elementary function and the other is an elliptic integral of the second kind. Other integrals can be obtained from these by using the formulas and references in Section 4. The method of computation is to iterate the duplication theorem and then sum a power series up to terms of degree five; the error ultimately decreases by a factor of $4^6 = 4096$ with each duplication. Since this method is slowest when the integrals are complete, we add a faster algorithm for computing complete integrals of the first and second kinds, including Legendre’s $K(k)$ and $E(k)$ for complex $k$, by the method of arithmetic and geometric means.

The symmetric integral of the first kind is

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty \left[ (t + x)(t + y)(t + z) \right]^{-1/2} dt ,$$  \hspace{1cm} (1)

where the square root is taken real and positive if $x, y, z$ are positive and varies continuously when $x, y, z$ become complex. The integral is well defined if $x, y, z$ lie in the complex plane cut along the nonpositive real axis (henceforth called the “cut plane”).
with the exception that at most one of \( x, y, z \) may be 0. The same requirements apply to the symmetric integral of the third kind,

\[
R_J(x, y, z, p) = \frac{3}{2} \int_0^\infty [(t + x)(t + y)(t + z)]^{-1/2}(t + p)^{-1}dt ,
\]

(2)

where \( p \neq 0 \) and the Cauchy principal value is to be taken if \( p \) is real and negative. A degenerate case of \( R_F \) that embraces the inverse circular and inverse hyperbolic functions (see Section 4) is

\[
R_C(x, y) = R_F(x, y, y) = \frac{1}{2} \int_0^\infty (t + x)^{-1/2}(t + y)^{-1}dt .
\]

(3)

It is well defined if \( x \) lies in the cut plane or is 0 and if \( y \neq 0 \); the Cauchy principal value is to be taken if \( y \) is real and negative. Professor Luigi Gatteschi pointed out to me that Fubini, while still a student at Pisa, proposed the use of a function equivalent to \( 1/R_C \) in his first published paper [12]. A degenerate case of \( R_J \) that is an elliptic integral of the second kind is

\[
R_D(x, y, z) = R_J(x, y, z, z) = \frac{3}{2} \int_0^\infty [(t + x)(t + y)]^{-1/2}(t + z)^{-3/2}dt ,
\]

(4)

which is well defined under the same conditions as \( R_F \) except that \( z \) must not be 0.

Because \( R_D \) is symmetric only in \( x \) and \( y \), it is sometimes convenient to use a completely symmetric integral of the second kind,

\[
R_G(x, y, z) = \frac{1}{4} \int_0^\infty [(t + x)(t + y)(t + z)]^{-1/2} \left( \frac{x}{t + x} + \frac{y}{t + y} + \frac{z}{t + z} \right) tdt ,
\]

(5)

where any or all of \( x, y, z \) may be 0 and those that are nonzero lie in the cut plane. If the closed convex hull of \( \{x, y, z\} \) lies in the union of 0 and the cut plane, \( R_G \) is represented by a double integral that accounts for its usefulness in problems connected with ellipsoids,

\[
R_G(x, y, z) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi (x \sin^2 \theta \cos^2 \phi + y \sin^2 \theta \sin^2 \phi + z \cos^2 \theta)^{1/2} \sin \theta d\theta d\phi .
\]

(6)

Except that at most one of \( x, y, z \) may be 0, \( R_F \) has the same representation with exponent \(-1/2\) instead of \( 1/2 \). With the same exception and with \( x, y, z \) permuted so that \( z \neq 0 \), \( R_G \) can be obtained from \( R_F \) and \( R_D \) by using the relation

\[
2R_G(x, y, z) = zR_F(x, y, z) - \frac{1}{3}(x - z)(y - z)R_D(x, y, z) + \sqrt{\frac{xy}{z}} .
\]

(7)
Algorithms for direct computation of $R_G$ and $R_F$ with real variables by successive Landen transformations are given in [3]. In Section 2 of the present paper, one of those algorithms is extended to complex variables, but only in the complete case. For each of the functions defined above, the complete case is the case in which one of $x, y, z$ is 0.

The functions $R_F$ and $R_C$ are homogeneous of degree $-1/2$ in their variables, $R_J$ and $R_D$ of degree $-3/2$, and $R_G$ of degree $+1/2$. Their homogeneity and symmetry replace a set of linear transformations of Legendre’s integrals, simplify their quadratic transformations and other properties, and make it possible (see [7]-[11]) to unify many of the formulas in customary integral tables.

The earlier versions of the algorithms in Section 2 (except the last one) were published in [3], modified in [6, (4),(5)] to avoid underflows, and implemented by Fortran codes in several major software libraries, in the Supplements to [7] and [8], and in [13] and [14, §6.11]. Codes in C can be found in [15, §6.11]. The algorithms in the present paper are preferable to those in [3] in the following four respects (in addition to incorporating the modification in [3]):

1. All complex values of the variables are admissible for which $R_F, R_D,$ and $R_C$ were defined above, with the exception that computing the Cauchy principal value of $R_C(x, y)$ when $y < 0$ requires the preliminary transformation (21). The variables of $R_J$ are restricted by conditions that are sufficient but not necessary to keep the fourth variable from being transformed to 0 by the duplication theorem.

2. The algorithms have been rearranged to reduce substantially the number of arithmetic operations.

3. A bound on the fractional error of the result can be specified directly.

4. Because $R_C$ is used repeatedly in the algorithm for $R_J$, its computation has been speeded up a little by including terms up to degree seven (instead of five) in the truncated power series.

Section 3 contains information for checking codes based on the algorithms of Section 2, and Section 4 relates other integrals to the integrals used here.
2 Algorithms

We shall summarize briefly the method of computation; details are given in [5] for the case of real variables, and a full discussion of the complex case with proof of error bounds will appear eventually in a book now in preparation. Validity of algorithms can be tested in the complex domain by the consistency checks in Section 3 and sometimes by comparison of the complete case with the last algorithm in the present Section.

If we obtain \( x_{m+1}, y_{m+1}, z_{m+1} \) from \( x_m, y_m, z_m \) as prescribed by (11) below, the duplication theorem implies

\[
R_F(x_0, y_0, z_0) = R_F(x_m, y_m, z_m), \quad m = 1, 2, 3, \ldots.
\]

Although \( x_m, y_m, z_m \) converge to a common limit in the cut plane as \( m \to \infty \), differences like \( x_m - y_m \) decrease by a factor of only 4 when \( m \) increases by 1. When the fractional differences between \( x_m, y_m, z_m \) and their arithmetic average become smaller than an amount determined by the desired accuracy of the final result, \( R_F \) is expanded in a multiple series of powers of these fractional differences, denoted by \( X, Y, Z \). Because of the symmetry of \( R_F \) the series can be rewritten in terms of the elementary symmetric functions \( E_1, E_2, E_3 \) of \( X, Y, Z \); and because \( E_1 = X + Y + Z = 0 \) the terms up to degree 5 are very simple. The truncation error, being of degree 6, ultimately decreases by a factor of \( 4^6 \) with each duplication. An estimate of truncation error is provided by [5, (A.10)].

**Algorithm for \( R_F \).** We suppose that at most one of \( x, y, z \) is 0 and those that are nonzero have phase less in magnitude than \( \pi \). The function \( R_F(x, y, z) \) defined by (4) is to be computed with relative error less in magnitude than \( r \). (We assume \( r < 3 \times 10^{-4} \).)

Let \( x_0 = x, y_0 = y, z_0 = z \), and

\[
A_0 = \frac{x + y + z}{3}, \quad Q = (3r)^{-1/6} \max\{|A_0 - x|, |A_0 - y|, |A_0 - z|\}.
\]

For \( m = 0, 1, 2, \ldots \), define

\[
\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}, \quad A_{m+1} = \frac{A_m + \lambda_m}{4},
\]

\[
(10)
\]
where each square root has nonnegative real part.

Compute $A_m$ for $m = 0, 1, \ldots, n$, where $4^{-n}Q < |A_n|$. Define

$$X = \frac{A_0 - x}{4^n A_n}, \quad Y = \frac{A_0 - y}{4^n A_n}, \quad Z = -X - Y,$$

$$E_2 = XY - Z^2, \quad E_3 = XYZ. \quad (12)$$

Then

$$R_F(x, y, z) \approx A_n^{-1/2} \left( 1 - \frac{1}{10} E_2 + \frac{1}{14} E_3 + \frac{1}{24} E_2^2 - \frac{3}{44} E_2 E_3 \right) \quad (14)$$

with relative error less in magnitude than $r$.

The statement that the relative error is less in magnitude than $r$ means that the true value of the function lies inside a circle in the complex plane with center at the computed value and radius equal to $r$ times the distance of the center from the origin.

We assume that $r$ is large compared to the machine precision, so that roundoff error is negligible compared to the error produced by approximations made in the algorithm. These remarks apply to all the algorithms in this paper.

We note the relations

$$A_m = \frac{x_m + y_m + z_m}{3}, \quad A_m - x_m = \frac{A_0 - x}{4^m}, \quad X = 1 - \frac{x_n}{A_n}, \quad (15)$$

and similar relations obtained by permuting $(x, X), (y, Y), (z, Z)$. Although $A_0$ can be 0 in the complex case, $A_n \neq 0$ because $Q \geq 0$. (It can be shown that $A_m \neq 0$ if $m \geq 1$.) If $x_m, y_m, z_m$ are real and nonnegative, the inequality of arithmetic and geometric means implies $\lambda_m \leq 3A_m$, whence $A_{m+1} \leq A_m$.

If $y = z$ then $R_F$ reduces to $R_C$. Because $E_2$ and $E_3$ are no longer independent, the series in (14) becomes a series in one variable. By including terms up to degree seven (instead of five), we can usually save one duplication, which seems worthwhile because computation of $R_J$ requires one computation of $R_C$ in each cycle of iteration.
ALGORITHM FOR $R_C$. Let $x$ and $y$ be nonzero and have phase less in magnitude than $\pi$, with the exception that $x$ may be 0. The function $R_C(x, y)$ defined by (3) is to be computed with relative error less in magnitude than $r$. (We assume $r < 2 \times 10^{-4}$.)

Let $x_0 = x$, $y_0 = y$, and

$$A_0 = \frac{x + 2y}{3}, \quad Q = (3r)^{-1/8} |A_0 - x|.$$ \hspace{1cm} (16)

For $m = 0, 1, 2, \ldots$, define

$$\lambda_m = 2 \sqrt{x_m} \sqrt{y_m} + y_m, \quad A_{m+1} = \frac{A_m + \lambda_m}{4},$$ \hspace{1cm} (17)

$$x_{m+1} = \frac{x_m + \lambda_m}{4}, \quad y_{m+1} = \frac{y_m + \lambda_m}{4},$$ \hspace{1cm} (18)

where each square root has nonnegative real part.

Compute $A_m$ for $m = 0, 1, \ldots, n$, where $4^{-n} Q < |A_n|$, and define

$$s = \frac{y - A_0}{4^n A_n}.$$ \hspace{1cm} (19)

Then

$$R_C(x, y) \approx A_n^{-1/2} \left(1 + \frac{3}{10} s^2 + \frac{1}{7} s^3 + \frac{3}{8} s^4 + \frac{9}{22} s^5 + \frac{159}{208} s^6 + \frac{9}{8} s^7\right),$$ \hspace{1cm} (20)

with relative error less in magnitude than $r$.

If the second variable of $R_C$ is real and negative, the Cauchy principal value is

$$R_C(x, -y) = \left(\frac{x}{x + y}\right)^{1/2} R_C(x + y, y), \quad y > 0,$$ \hspace{1cm} (21)

by [16, (4.8)] and (73). This vanishes if $x = 0$.

In the notation of (26) to (28), the duplication theorem for $R_J$ is

$$R_J(x_m, y_m, z_m, p_m) = \frac{1}{4} R_J(x_{m+1}, y_{m+1}, z_{m+1}, p_{m+1}) + \frac{6}{d_m} R_C(1, 1 + e_m).$$ \hspace{1cm} (22)

The duplication theorem for $R_C$ has been applied to [3, (5.1)] to allow a wider range of complex phase for the variables. Iteration of (22) yields

$$R_J(x_0, y_0, z_0, p_0) = 4^{-n} R_J(x_n, y_n, z_n, p_n) + 6 \sum_{m=0}^{n-1} \frac{4^{-m}}{d_m} R_C(1, 1 + e_m).$$ \hspace{1cm} (23)
The first term on the right side can be treated as a symmetric function of the five variables \(x_n, y_n, z_n, p_n, p_n\) by writing \((t + p)^{-1}\) in (3) as \((t + p)^{-1/2}(t + p)^{-1/2}\). When the fractional differences \(X, Y, Z, P, P\) between these five variables and their arithmetic average become small enough, \(R_J\) is expanded in powers of the elementary symmetric functions \(E_2, E_3, E_4, E_5\) of \(X, Y, Z, P, P\) (since \(E_1 = 0\)).

**Algorithm for \(R_J\).** Let \(x, y, z\) have nonnegative real part and at most one of them be 0, while \(\text{Re} p > 0\). Alternatively, if \(p \neq 0\) and \(|\text{ph} p| < \pi\), either let \(x, y, z\) be real and nonnegative and at most one of them be 0, or else let two of the variables \(x, y, z\) be nonzero and conjugate complex with phase less in magnitude than \(\pi\) and the third variable be real and nonnegative. The function \(R_J(x, y, z, p)\) defined by (3) is to be computed with relative error less in magnitude than \(r\). (We assume \(r < 10^{-4}\).)

Let \((x_0, y_0, z_0, p_0) = (x, y, z, p)\) and

\[
A_0 = \frac{x + y + z + 2p}{5}, \quad \delta = (p - x)(p - y)(p - z),
\]

\[
Q = (r/4)^{-1/6}\max\{|A_0 - x|, |A_0 - y|, |A_0 - z|, |A_0 - p|\}.
\]

For \(m = 0, 1, 2, \ldots\), define

\[
\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{z_m} \sqrt{y_m} + \sqrt{z_m} \sqrt{z_m}, \quad A_{m+1} = \frac{A_m + \lambda_m}{4},
\]

\[
x_{m+1} = \frac{x_m + \lambda_m}{4}, \quad y_{m+1} = \frac{y_m + \lambda_m}{4}, \quad z_{m+1} = \frac{z_m + \lambda_m}{4}, \quad p_{m+1} = \frac{p_m + \lambda_m}{4},
\]

\[
d_m = (\sqrt{p_m} + \sqrt{x_m})(\sqrt{p_m} + \sqrt{y_m})(\sqrt{p_m} + \sqrt{z_m}), \quad e_m = \frac{4^{-3m} \delta}{d_m^2},
\]

where each square root has nonnegative real part.

Compute \(A_m\) for \(m = 0, 1, \ldots, n\), where \(4^{-n}Q < |A_n|\). Compute also \(R_C(1, 1 + e_m)\) with relative error less in magnitude than \(r\) for \(m = 0, 1, \ldots, n - 1\). Define

\[
X = \frac{A_0 - x}{4^n A_n}, \quad Y = \frac{A_0 - y}{4^n A_n}, \quad Z = \frac{A_0 - z}{4^n A_n}, \quad P = (-X - Y - Z)/2,
\]

\[
E_2 = XY + XZ + YZ - 3P^2, \quad E_3 = XYZ + 3E_2P + 4P^3, \quad E_4 = (2XYZ + E_2P + 3P^3)P, \quad E_5 = XYZP^2.
\]
Then
\[ R_J(x, y, z, p) \approx 4^{-n} A_n^{-3/2} \left( 1 - \frac{3}{14} E_2 + \frac{1}{6} E_3 + \frac{9}{88} E_2^2 - \frac{3}{22} E_4 - \frac{9}{52} E_2 E_3 + \frac{3}{26} E_5 \right) \]
\[ + 6 \sum_{m=0}^{n-1} \frac{4^{-m}}{d_m} R_C(1, 1 + e_m) \]  \hspace{1cm} \text{(32)}
with relative error less in magnitude than \( r \).

When the variables are complex, the bound on relative error is not rigorous because of the possibility of some cancellation between terms that individually have error less than \( r \). In practice, however, the error is usually much smaller than the bound.

If \( x, y, z \) are real and nonnegative, at most one of them is 0, and the fourth variable of \( R_J \) is negative, the Cauchy principal value is given by \([16, (4.6)]\) and \((21)\):
\[ (y + q)R_J(x, y, z, -q) = (p - y)R_J(x, y, z, p) - 3R_F(x, y, z) \]
\[ + 3 \left( \frac{xyz}{xz + pq} \right)^{1/2} R_C(xz + pq, pq), \quad q > 0, \]  \hspace{1cm} \text{(33)}
where \( p - y = (z - y)(y - x)/(y + q) \). If we permute the values of \( x, y, z \) so that \((z - y)(y - x) \geq 0\), then \( p \geq y > 0 \) and all terms on the right side can be computed by the preceding algorithms. If \( x, y, z \) are complex, conditions of validity have not been established for this method of computing the Cauchy principal value.

It would be desirable to give the algorithm less restrictive conditions on \( x, y, z, p \) that do not rule out so many cases in which \( p = z \) and \( R_J \) reduces to \( R_D \), for in such cases the algorithm gives correct results under the much weaker conditions stated below in the algorithm for \( R_D \). We note that \( p = z \) implies \( e_m = 0 \) and \( d_m = 2\sqrt{z_m}(z_m + \lambda_m) \) for all \( m \), whence \((23)\) becomes
\[ R_D(x, y, z) = 4^{-n} R_D(x_n, y_n, z_n) + 3 \sum_{m=0}^{n-1} \frac{4^{-m}}{\sqrt{z_m}(z_m + \lambda_m)}. \]  \hspace{1cm} \text{(34)}
The function \( R_D(x, y, z) \) is treated as a symmetric function of \( x, y, z, z \).

**Algorithm for \( R_D \).** Let \( x, y, z \) be nonzero and have phase less in magnitude than \( \pi \), with the exception that at most one of \( x \) and \( y \) may be 0. The function \( R_D(x, y, z) \)
defined by (4) is to be computed with relative error less in magnitude than \( r \). (We assume \( r < 10^{-4} \).)

Let \( x_0 = x, \ y_0 = y, \ z_0 = z \), and

\[
A_0 = \frac{x + y + 3z}{5}, \quad Q = (r/4)^{-1/6} \max\{|A_0 - x|, |A_0 - y|, |A_0 - z|\}. \tag{35}
\]

For \( m = 0, 1, 2, \ldots \), define

\[
\lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}, \quad A_{m+1} = \frac{A_m + \lambda_m}{4}, \tag{36}
\]

\[
x_{m+1} = \frac{x_m + \lambda_m}{4}, \quad y_{m+1} = \frac{y_m + \lambda_m}{4}, \quad z_{m+1} = \frac{z_m + \lambda_m}{4}, \tag{37}
\]

where each square root has nonnegative real part.

Compute \( A_m \) for \( m = 0, 1, \ldots, n \), where \( 4^{-n}Q < |A_n| \). Define

\[
X = \frac{A_0 - x}{4^n A_n}, \quad Y = \frac{A_0 - y}{4^n A_n}, \quad Z = -(X + Y)/3, \quad E_2 = XY - 6Z^2, \quad E_3 = (3XY - 8Z^2)Z, \quad E_4 = 3(XY - Z^2)Z^2, \quad E_5 = XYZ^3. \tag{38}
\]

Then

\[
R_D(x, y, z) \approx 4^{-n}A_n^{-3/2}\left(1 - \frac{3}{14}E_2 + \frac{1}{6}E_3 + \frac{9}{88}E_2^2 - \frac{3}{22}E_4 - \frac{9}{52}E_2E_3 + \frac{3}{26}E_5\right) + 3 \sum_{m=0}^{n-1} \frac{4^{-m}}{\sqrt{z_m} (z_m + \lambda_m)}, \tag{41}
\]

with relative error less in magnitude than \( r \).

The remark about relative error immediately following the algorithm for \( R_J \) applies again here. The function \( R_G \) can be computed from \( R_F \) and \( R_D \) by using (7) if at most one of \( x, y, z \) is 0. Neither (7) nor the next algorithm can be used to compute \( R_G(0, 0, z) = \sqrt{z}/2 \).

A faster and simpler way of computing the complete case of \( R_F \) and \( R_G \) with complex variables is useful because Legendre’s complete integrals of the first and second kinds are

\[
K(k) = R_F(1 - k^2, 1, 0), \quad E(k) = 2R_G(1 - k^2, 1, 0). \tag{42}
\]
The method of successive arithmetic and geometric means (a special case of successive Landen transformations) serves this purpose for complex $k^2 \notin [1, +\infty)$. (Algorithms using Landen transformations for incomplete $R_F$ and $R_G$ are given in [3] but only for real variables; conditions of validity for complex variables are not obvious.)

**ALGORITHM FOR $R_F(x, y, 0)$ AND $R_G(x, y, 0)$**. Let $x$ and $y$ be nonzero and have phase less in magnitude than $\pi$. The complete elliptic integrals $R_F(x, y, 0)$ and $R_G(x, y, 0)$ are to be computed with relative error less in magnitude than $r$.

Define $x_0 = \sqrt{x}$, $y_0 = \sqrt{y}$, and

\[ x_{m+1} = \frac{x_m + y_m}{2}, \quad y_{m+1} = \sqrt{x_my_m}, \quad m = 0, 1, 2, \ldots, \]  

where each square root has positive real part. Compute $x_m$ and $y_m$ for $m = 0, 1, \ldots, n$, where

\[ |x_n - y_n| < 2.7 \sqrt{r} |x_n|. \]  

Then

\[ R_F(x, y, 0) \approx \pi \frac{x_n + y_n}{x_n + y_n} \]  

with relative error less in magnitude than $r$. Also,

\[ 2R_G(x, y, 0) \approx \left( \frac{x_0 + y_0}{2} \right)^2 - \sum_{m=1}^{n} 2^{m-2}(x_m - y_m)^2 \]  

with relative error less in magnitude than $r$ if we neglect terms of order $r^2$. The summation is empty if $n = 0$.

This algorithm can be used to compute also

\[ R_D(0, y, z) = \frac{3}{z(y - z)} \left[ 2R_G(y, z, 0) - z R_F(y, z, 0) \right], \quad 0 \neq y \neq z \neq 0. \]  

The exceptional case with $y = z \neq 0$ is

\[ R_D(0, y, y) = \frac{3\pi}{4} y^{-3/2}. \]
3 Numerical checks

Codes based on the algorithms of Section 2 can be checked against the following assortment of numerical values for complete and incomplete integrals with real, conjugate complex, or nonconjugate complex variables.

\[ R_F(1, 2, 0) = 1.3110287771461 \]
\[ R_F(i, -i, 0) = R_F(0.5, 1, 0) = 1.8540746773014 \]
\[ R_F(i - 1, i, 0) = 0.79612586584234 - i 1.2138566698365 \]
\[ R_F(2, 3, 4) = 0.58408284167715 \]
\[ R_F(i, -i, 2) = 1.0441445654064 \]
\[ R_F(i - 1, i, 1 - i) = 0.93912050218619 - i 0.53296252018635 \]

\[ R_C(0, 1/4) = \pi = 3.1415926535898 \]
\[ R_C(9/4, 2) = \ln 2 = 0.69314718055995 \]
\[ R_C(0, i) = (1 - i) 1.1107207345396 \]
\[ R_C(-i, i) = 1.2260849569072 - i 0.3447136988768 \]
\[ R_C(1/4, -2) = \frac{\ln 2}{3} = 0.23104906018665 \]
\[ R_C(i, -1) = 0.77778596920447 + i 0.19832484993429 \]

\[ R_J(0, 1, 2, 3) = 0.77688623778582 \]
\[ R_J(2, 3, 4, 5) = 0.14297579667157 \]
\[ R_J(2, 3, 4, -1 + i) = 0.13613945827771 - i 0.38207561624427 \]
\[ R_J(i, -i, 0, 2) = 1.6490011662711 \]
\[ R_J(-1 + i, -1 - i, 1, 2) = 0.94148358841220 \]
\[ R_J(i, -i, 0, 1 - i) = 1.8260115229009 + i 1.2290661908643 \]
\[ R_J(-1 + i, -1 - i, 1, -3 + i) = -0.61127970812028 - i 1.0684038390007 \]
\[ R_J(-1 + i, -2 - i, -i, -1 + i) = 1.8249027393704 - i 1.2218475784827 \]
The last case does not fit the assumptions, but the algorithm yields a value agreeing with \( R_D(-2 - i, -i, -1 + i) \) below. If \( x, y, z \) are strictly positive, the Cauchy principal value of \( R_J \) changes sign at a negative value of the fourth variable, as illustrated by

\[
R_J(2, 3, 4, -0.5) = 0.24723 \, 81970 \, 3052 \\
R_J(2, 3, 4, -5) = -0.12711 \, 23004 \, 2964
\]

In this example \( R_J \) vanishes when the fourth variable is approximately \(-1.2552\), and cancellation between terms on the right side of (33) will lead to loss of significant figures.

\[
R_D(0, 2, 1) = 1.7972 \, 10352 \, 1034 \\
R_D(2, 3, 4) = 0.16510 \, 52729 \, 4261 \\
R_D(i, -i, 2) = 0.65933 \, 85415 \, 4220 \\
R_D(0, i, -i) = 1.2708 \, 19627 \, 1910 + i \, 2.7811 \, 12015 \, 9521 \\
R_D(0, i - 1, i) = -1.8577 \, 23543 \, 9239 - i \, 0.96193 \, 45088 \, 8839 \\
R_D(-2 - i, -i, -1 + i) = 1.8249 \, 02739 \, 3704 - i \, 1.2218 \, 47578 \, 4827
\]

\[
R_G(0, 16, 16) = 2E(0) = \pi = 3.1415 \, 92653 \, 5898 \\
R_G(2, 3, 4) = 1.7255 \, 03028 \, 0692 \\
R_G(0, i, -i) = 0.42360 \, 65423 \, 9699 \\
R_G(i - 1, i, 0) = 0.44660 \, 59167 \, 7018 + i \, 0.70768 \, 35235 \, 7515 \\
R_G(-i, i - 1, i) = 0.36023 \, 39218 \, 4473 + i \, 0.40348 \, 62340 \, 1722 \\
R_G(0, 0.0796, 4) = E(0.99) = 1.0284 \, 75809 \, 0288
\]

Consistency checks that do not use external information are furnished by the following equations, in which \( x, y, p \) are positive, \( \lambda \mu = xy \), and \( |\text{ph} \lambda| < \pi \):

\[
R_F(x + \lambda, y + \lambda, \lambda) + R_F(x + \mu, y + \mu, \mu) = R_F(x, y, 0); \quad (49)
\]
\[ R_C(\lambda, x + \lambda) + R_C(\mu, x + \mu) = R_C(0, x); \quad (50) \]

\[ R_J(x + \lambda, y + \lambda, \lambda, p + \lambda) + R_J(x + \mu, y + \mu, \mu, p + \mu) = R_J(x, y, 0, p) - 3 R_C(a, b), \quad (51) \]

where

\[ a = p^2(\lambda + \mu + x + y), \quad b = p(p + \lambda)(p + \mu), \quad b - a = p(p - x)(p - y); \quad (52) \]

\[ R_D(\lambda, x + \lambda, y + \lambda) + R_D(\mu, x + \mu, y + \mu) = R_D(0, x, y) - \frac{3}{y \sqrt{x + y + \lambda + \mu}}. \quad (53) \]

These equations are special cases of addition theorems. Another consistency check is furnished by

\[ R_D(x, y, z) + R_D(y, z, x) + R_D(z, x, y) = 3 \sqrt{x \sqrt{y \sqrt{z}}}, \quad (54) \]

where \( x, y, z \) lie in the cut plane and each square root has positive real part.

### 4 Other integrals

Legendre’s complete elliptic integrals \( K \) and \( E \) are given by

\[ K(k) = R_F(0, 1 - k^2, 1), \quad (55) \]
\[ E(k) = 2 R_G(0, 1 - k^2, 1) = \frac{1 - k^2}{3} [R_D(0, 1 - k^2, 1) + R_D(0, 1, 1 - k^2)], \quad (56) \]

\[ K(k) - E(k) = \frac{k^2}{3} R_D(0, 1 - k^2, 1), \quad (57) \]
\[ E(k) - (1 - k^2)K(k) = \frac{k^2(1 - k^2)}{3} R_D(0, 1, 1 - k^2). \quad (58) \]

In Legendre’s incomplete integrals we shall use the abbreviation \( c = \csc^2 \phi = 1/\sin^2 \phi \):

\[ F(\phi, k) = (\sin \phi) R_F(\cos^2 \phi, 1 - k^2 \sin^2 \phi, 1) = R_F(c - 1, c - k^2, c), \quad (59) \]
\[ E(\phi, k) = R_F(c - 1, c - k^2, c) - \frac{k^2}{3} R_D(c - 1, c - k^2, c), \quad (60) \]
\[ \Pi(\phi, k, n) = \int_0^\phi (1 + n \sin^2 \theta)^{-1}(1 - k^2 \sin^2 \theta)^{-1/2} d\theta \]
\[ = R_F(c - 1, c - k^2, c) - \frac{n}{3} R_J(c - 1, c - k^2, c, c + n). \quad (61) \]
Some related integrals are
\[
D(\phi, k) = \int_0^\phi \sin^2 \theta (1 - k^2 \sin^2 \theta)^{-1/2} d\theta = \frac{1}{3} R_D(c - 1, c - k^2, c), \tag{62}
\]
\[
K(k)Z(\beta, k) = \frac{k^2}{3} \sin \beta \cos \beta \sqrt{1 - k^2 \sin^2 \beta} R_F(1, 1, 1 - k^2, 1 - k^2 \sin^2 \beta), \tag{63}
\]
\[
\Lambda_0(\beta, k) = \frac{2}{\pi} \frac{(1 - k^2) \sin \beta \cos \beta}{\Delta} \left[ R_F(0, 1 - k^2, 1) + \frac{k^2}{3\Delta^2} R_J\left(0, 1 - k^2, 1, 1 - \frac{k^2}{\Delta^2}\right)\right], \tag{64}
\]
where \(\Delta = \sqrt{1 - (1 - k^2) \sin^2 \beta}\). The function \(Z(\beta, k)\) is Jacobi’s zeta function \[2, 140.03\], and \(\Lambda_0(\beta, k)\) is Heuman’s lambda function \[2, 150.01\].

Bulirsch’s elliptic integrals \[1\] are
\[
el 1(x, k_c) = x R_F(1, 1 + k_c^2 x^2, 1 + x^2), \tag{65}
\]
\[
el 2(x, k_c, a, b) = ax R_F(1, 1 + k_c^2 x^2, 1 + x^2) + \frac{1}{3} (b - a) x^3 R_D(1, 1 + k_c^2 x^2, 1 + x^2), \tag{66}
\]
\[
el 3(x, k_c, p) = x R_F(1, 1 + k_c^2 x^2, 1 + x^2) + \frac{1}{3} (1 - p) x^3 R_J(1, 1 + k_c^2 x^2, 1 + x^2, 1 + px^2), \tag{67}
\]
\[
cel(k_c, p, a, b) = a R_F(0, k_c^2, 1) + \frac{1}{3} (b - pa) R_J(0, k_c^2, 1, p). \tag{68}
\]

In the real domain, inverse circular and inverse hyperbolic functions are expressed in terms of \(R_C\) by
\[
\ln\left(\frac{x}{y}\right) = (x - y) R_C\left(\left(\frac{x + y}{2}\right)^2, xy\right), \quad x > 0, \tag{69}
\]
\[
\arctan(x/y) = x R_C(y^2, y^2 + x^2), \quad -\infty < x < \infty, \tag{70}
\]
\[
\arctanh(x/y) = x R_C(y^2, y^2 - x^2), \quad -y < x < y, \tag{71}
\]
\[
\arcsin(x/y) = x R_C(y^2 - x^2, y^2), \quad -y \leq x \leq y, \tag{72}
\]
\[
\arcsinh(x/y) = x R_C(y^2 + x^2, y^2), \quad -\infty < x < \infty, \tag{73}
\]
\[
\arccos(x/y) = (y^2 - x^2)^{1/2} R_C(x^2, y^2), \quad 0 \leq x \leq y, \tag{74}
\]
\[
\arccosh(x/y) = (x^2 - y^2)^{1/2} R_C(x^2, y^2), \quad x \geq y, \tag{75}
\]
where \(y > 0\) in each case. These equations remain valid in the complex domain provided
that the variables of $R_C$ satisfy the conditions accompanying (3). If $y = 1$ the function multiplying $R_C$ shows in each case the asymptotic behavior as the left side tends to 0.

Many elliptic integrals of the form

$$\int_y^x \prod_{i=1}^n (a_i + b_it)^{p_i/2} \, dt,$$

(76)

where $p_1, \ldots, p_n$ are integers and the integrand is real, are reduced in [7]-[11] to the integrals in Section 2. Use of the algorithm for $R_F$ is illustrated by numerical computation of the integral

$$I = \int_y^x \frac{dt}{\sqrt{(f_1 + 2g_1t + h_1t^2)(f_2 + 2g_2t + h_2t^2)}},$$

(77)

where all quantities are real, $x > y$, the two quadratic (or, if $h_i = 0$, linear) polynomials are positive on the open interval of integration, and their product has at most simple zeros on the closed interval. Let

\[
q_i(t) = f_i + 2g_it + h_it^2, \quad i = 1, 2,
\]

(78)

\[(x - y)U = \sqrt{q_1(x)q_2(y)} + \sqrt{q_1(y)q_2(x)},\]

(79)

\[T = 2g_1g_2 - f_1h_2 - f_2h_1,\]

(80)

\[V = 2\sqrt{(g_1^2 - f_1h_1)(g_2^2 - f_2h_2)}.\]

(81)

Then

$$I = 2 \, R_F(U^2 + T + V, U^2 + T - V, U^2).$$

(82)

Except for notation this is the same as [4, (34)]. If the interval of integration is infinite, $U$ is obtained by taking a limit; for example, if $x = +\infty$, then $U = \sqrt{h_1q_2(y)} + \sqrt{q_1(y)h_2}$. If exactly one of $q_1$ and $q_2$ has conjugate complex zeros, then $V$ is pure imaginary; otherwise the variables of $R_F$ are real. The algorithm for $R_F$ in this paper can be used in both cases. However, if both polynomials have conjugate complex zeros, the quadrilateral with the zeros as vertices has diagonals that must not intersect at an interior point of the interval of integration. If they do, the integral must be split into two parts at the point of intersection. This restriction is discussed in [4, §4] and removed by a Landen transformation in [11].
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