Radiative effects and the missing energy paradox in the ideal two capacitors problem

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Abstract. Starting from the Poynting theorem, which arises from the exact Maxwell equations, we establish the balance of energy for the radiating ideal two capacitors problem. This balance of energy results in a nonlinear differential equation governing the time evolution of the voltage $V$. Boykin, Hite and Singh, following an approach not based on first principles, were the first to obtain this nonlinear differential equation and proposed an exponentially decaying voltage as a unique solution for it. We claim that the space of solutions for this differential equation is much richer. In fact, besides the exponentially decaying solution just mentioned there exist solutions with a sudden death behavior. The radiative effect introduced by Boykin, Hite and Singh, complemented with our analysis based on the exact Maxwell equations and the characterization of the more general space of solution of the nonlinear differential equation, explain the missing energy paradox in the ideal two capacitors problem.

1. Introduction
We consider a system of two capacitors in a vacuum, initially $C_1$ charged and $C_2$ uncharged, that can be connected with wires through a switch. When the switch is closed the initial charge redistributes itself and the final electrostatic energy becomes lower than the initial one. The appearance of a transient current $I$ causes radiation, in such a way that a fraction of the lacking energy within the capacitors is emitted as radiation and the complement is dissipated as heat energy by Joule effect or, for short, Joule heating. In fact, the decrease of electromagnetic energy within some volume containing the capacitors is equal to the energy radiated through the boundaries of that volume plus the energy dissipated by Joule heating.

The role of radiation for this system was first considered by Boykin, Hite and Singh [1]. They model the radiating system by a lumped element whose resistance changes with current, and such that its dissipative power equals the dissipative power of a magnetic dipole. Using Kirchhoff voltage rule without resistance $R$, nor selfinductance $L$, they obtain a third order non-linear differential equation for the voltage time evolution. Being a third order differential equation, it accepts three independent initial data. They found a particular analytical solution with the voltage decaying exponentially with time. In Boykin solution only the initial voltage is an independent datum, while its initial first and second derivatives are determined by the initial voltage itself.

We address the problem from a slightly different point of view. We consider the conservation of energy as stated in the Poynting theorem. Neglecting the system resistance $R$ and its...
selfinductance \( L \), and modeling the system radiation by a magnetic dipole, we arrive to an integral equation observing that, under certain restrictions, the third order differential equation mentioned is a differential version of this integral equation. We analyze the solutions of the third order non linear differential equation for three independent initial data \( V|_0, dV/dt|_0 \) and \( d^2V/dt^2|_0 \) on a three dimensional sector of \( R^3 \). We claim that solutions with independent initial data within this sector of \( R^3 \) present a “sudden death” behavior which is a new effect characterizing the radiation problem.

2. Conservation of energy and Maxwell equations

The balance of energy within an spherical region of radius \( r \) and volume \( v \) containing the system, for any time \( t > 0 \), is given by

\[
\int_0^t dt' \int_v \partial_u \frac{\partial u}{\partial t'} + \overrightarrow{J} \cdot \overrightarrow{E} \, d^3x + \nabla \cdot \overrightarrow{S} = 0,
\]

(1)

where \( u \) is the electromagnetic field energy per unit volume, \( \overrightarrow{J} \cdot \overrightarrow{E} \) is the rate of work done by the field on each unit volume of matter, and the magnitude of \( \overrightarrow{S} \) gives the flow of field energy per unit time across a unit area of the spherical region.

Explicit expressions for \( u \) and \( \overrightarrow{S} \) can be obtained after the Poynting theorem:

\[
u = u_e + u_m = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \varepsilon_0 c^2 B^2,
\]

(2)

\[
\overrightarrow{S} = \varepsilon_0 c^2 \overrightarrow{E} \times \overrightarrow{B},
\]

(3)

where \( \overrightarrow{S} \) is the Poynting vector. The derivation of these expressions can be found in Feynman \[2\]. The volume integral appearing in Eq. (1) is considered over the sphere of radius \( r \), and its first term gives the rate of change of the field energy \( U \) within this sphere, its second term gives the rate of work done by the field on matter inside the sphere and its third term gives the rate of the field energy flow across the sphere. The integral over time gives the total values of these physical quantities from \( t = 0 \) up to some time \( t \), while still there is current in the system. Finally we consider the integral until reaching the stationary state at \( t = \infty \).

3. The \( RC \) model with radiation

We assume that the current \( I \) is a function of the time only and not of the spatial coordinates. This assumption seems to be reasonable in order to give a qualitative analysis of the system energy. The field energy \( U \) has electric and magnetic components: \( U = U_e + U_m \), where

\[
U_e = \int_v u_e d^3x = \int_v \frac{1}{2} \varepsilon_0 E^2 \, d^3x,
\]

(4)

\[
U_m = \int_v u_m d^3x = \int_v \frac{1}{2} \varepsilon_0 c^2 B^2 \, d^3x.
\]

(5)

Also, for this physical system

\[
\int_v \overrightarrow{J} \cdot \overrightarrow{E} \, d^3x = I^2 R.
\]

(6)
3.1. The magnetic dipole model for the energy radiated in the two capacitors problem

The system radiated power \( P_{\text{rad}}(t) \equiv \frac{dU_{\text{rad}}(t)}{dt} \) is the outgoing flux of the Poynting vector \( \vec{S} \), and is calculated through the spherical surface \( \Sigma \) of radius \( r \),

\[
\frac{dU_{\text{rad}}(t)}{dt} = \int_\Sigma \vec{S} \cdot d\vec{a} = \int_v \vec{\nabla} \cdot \vec{S} \ d^3x,
\]

where the last equality is Gauss’s theorem. For this purpose the system can be modeled by a loop of radius \( b \) and current \( I \), which is a magnetic dipole. The radiated power due to this magnetic dipole has been calculated in the paper of Boykin [1], its result being

\[
\frac{dU_{\text{rad}}(t)}{dt} = \pi b^4 \epsilon_0 c^5 \frac{1}{6} \left( \frac{d^2 I}{dt^2} (t - r/c) \right)^2 \quad \text{if} \quad t \geq r/c,
\]

and \( \frac{dU_{\text{rad}}(t)}{dt} = 0 \quad \text{if} \quad t < r/c. \) In Eq. (8) the quantity \( t - r/c \) is the retarded time. For other models see the paper of Choy [3]. Within this model the field energy radiated up to time \( t \) becomes

\[
U_{\text{rad}}(t) = \int_{r/c}^t K \left( \frac{d^2 I}{dt^2} (t' - r/c) \right)^2 \ dt',
\]

where \( K = \pi b^4 / 6 \epsilon_0 c^5 \). After the change of variable \( \tau = t' - r/c \), we get

\[
U_{\text{rad}}(t) = \int_0^{t-r/c} K \left( \frac{d^2 I}{d\tau^2} \right)^2 d\tau - \int_{t-r/c}^t K \left( \frac{d^2 I}{d\tau^2} \right)^2 d\tau.
\]

After replacing Eqs. (11), (6), (5) and (4) back into Eq. (1), we obtain

\[
\int_0^t dt' \left( \frac{d}{dt'} \left( U_e + U_m \right) + I^2 R + K \left( \frac{d^2 I}{dt^2} \right)^2 \right) - \int_{t-r/c}^t dt' K \left( \frac{d^2 I}{dt^2} \right)^2 = 0.
\]

We can split the electric energy as \( U_e = U_{e}^{\text{in}} + U_{e}^{\text{out}} \), where \( U_{e}^{\text{in}} \) is the energy of the electric field in the capacitors including border effects, and \( U_{e}^{\text{out}} \) is the energy of the electric field in the complementary space. If we define a physical quantity \( V \) as

\[
V = \frac{Q_1}{C_1} - \frac{Q_2}{C_2},
\]

we get the relationship

\[
I = -C_{eq} \frac{dV}{dt},
\]

where

\[
\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}.
\]

Plugging Eq. (14) back in Eq. (12) we obtain

\[
\int_0^t \left( K \left( \frac{d^2 V}{dt^2} \right)^2 + R \left( \frac{dV}{dt} \right)^2 + \frac{1}{C_{eq}} V \frac{dV}{dt'} \right) dt' = \]
This equation is an expression of the conservation of the energy for any time $t$ during the process. Taking the time derivative on both sides we have

$$
K \left( \frac{d^3V}{dt^3} \right)^2 + R \left( \frac{dV}{dt} \right)^2 + \frac{1}{C_{eq}} V \frac{dV}{dt} = \nonumber
$$

$$- \frac{1}{C_{eq}^2} \frac{d}{dt} \left[ U_e^{\text{out}}(t) + U_m(t) \right] |t|^t + K \left( \frac{d^3V}{dt^3} \right)^2 \bigg|_{t-r/c}. \quad (17)
$$

For the case where the time $t$ is much larger than $r/c$ and $(-1/C_{eq}^2) d \left[ U_e^{\text{out}}(t) + U_m(t) \right] /dt \approx 0$, we can neglect the right hand side of Eq. (17) and obtain

$$
K \left( \frac{d^3V}{dt^3} \right)^2 + R \left( \frac{dV}{dt} \right)^2 + \frac{1}{C_{eq}} V \frac{dV}{dt} = 0. \quad (18)
$$

Under these restrictions, the solutions of Eq. (18) will be a good approximation to the behavior of the system. When we compare Eq. (18) with the well known equation of the $RC$ model without radiation which is

$$
R \left( \frac{dV}{dt} \right) + \frac{1}{C_{eq}} V = 0, \quad (19)
$$

we see that, on the one hand, Eq. (19) is a first order linear differential equation for $V$, whose solution depends just upon one initial datum, which is $V|_0$. From its solution one gets that $dV/dt|_0 = -(Q_0^0/C_1) (1/RC_{eq})$ which, in the limit when $R$ tends to zero, diverges to minus infinity. On the other hand, Eq. (18) is a third order non-linear differential equation for $V$, whose solution depends on three initial data, which are $V|_0$, $dV/dt|_0$ and $d^2V/dt^2|_0$. In this case $dV/dt|_0$ is a finite initial datum which may be chosen to be independent of the value of $R$ and, therefore, in the limit when $R$ tends to zero, it remains unaffected. Actually, when $R$ tends to zero, $dV/dt$ remains bounded for all $t$, and the second term in Eq. (18) can be neglected.

4. The model with just capacitance and radiation

In what follows we restrict the discussion of Eq. (18) to the case $R = 0$. Then the equation for $V$ becomes

$$
\left( \frac{d^3V}{dt^3} \right)^2 + \frac{1}{KC_{eq}} V \frac{dV}{dt} = 0. \quad (20)
$$

This third order non-linear differential equation for $V$ was first derived by Boykin, Hite and Singh [1]. We notice that Eq. (20) arises from the more general Eq. (17), after neglecting the terms we have already mentioned. Equation (17) is a consequence of first principles, and it is a proof of the validity of the differential equation in reference [1].

A particular explicit solution for Eq. (20) was found by Boykin, Hite and Singh [1], given by

$$
V(t) = V(0) \exp(-\alpha^{-1/5} t), \quad (21)
$$

where $\alpha = KC_{eq} > 0$. (Just for the sake of clarity, in Boykin et al. is used $V(t) \equiv Q_2(t)/C_2 - Q_1(t)/C_1 \Rightarrow V(0) = -Q_0^0/C_1$; whereas in Eq. (13) we have defined $V(t) \equiv Q_1(t)/C_1 - Q_2(t)/C_2 \Rightarrow V(0) = Q_0^0/C_1$.)
4.1. The space of solutions for $V$

The solution given in Eq. (21) is obtained by taking the initial data of Eq. (20) to depend just upon one parameter: $V(0)$, which we also denote as $V|_0$. In fact, for this solution $dV/dt|_0 = \alpha^{-1/5} V|_0$ and $d^2V/dt^2|_0 = \alpha^{-2/5} V|_0$. However Eq. (20) has a much larger space of solutions because its initial data in general depend upon three independent parameters: $V|_0$, $dV/dt|_0$ and $d^2V/dt^2|_0$.

In the problem that we consider here $V|_0 > 0$, and Eq. (20) implies $dV/dt|_0 < 0$, while $d^2V/dt^2|_0$ can take negative or positive values. The sector of $R^3$ we are going to consider is the following: $V|_0 > 0$, $dV/dt|_0 < 0$ and $d^2V/dt^2|_0 < 0$. We claim that any solution $V$ with these initial conditions reaches the value zero at a finite time $T$, and that its behavior near $T$ is given by

$$V(t) \approx G(t - T),$$

where $G$ has a negative value. We say that these solutions have a sudden death behavior. The model given by Eq. (20) needs to be upgraded for $t$ near $T$ by taking into account the selfinductance of the system. The effect of the selfinductance is to smooth out the behavior of $V$ when it approaches zero. We will give an explicit proof of these claims elsewhere.

5. Conclusions

We have analyzed the ideal two-capacitor problem. Starting from Maxwell’s equations and using the conservation of energy as given by the Poynting theorem, we derived a third order non-linear differential equation for the physical quantity $V$. After neglecting some terms in the equation we arrived at Boykin et al. equation which was obtained using a different approach.

There exist real solutions for this differential equation depending on the initial data $V|_0$, $dV/dt|_0$, and $d^2V/dt^2|_0$. Without loss of generality, we have taken $V|_0$ strictly positive, i.e. $V|_0 > 0$. Real solutions exist if and only if $dV/dt|_0 \leq 0$. We claim that for any $d^2V/dt^2|_0 \leq 0$ there exists a unique solution satisfying the initial data at $t = 0$. The solution satisfying $dV/dt|_0 = 0$, $d^2V/dt^2|_0 = 0$ is the constant $V$ solution which corresponds to the uninteresting case when the switch remains open. In all other cases we expect that the solution will present the “sudden death” behavior. That is, it will decay to $V = 0$ in a finite time $T$.

It is a very interesting property of the nonlinear differential equation that it admits two qualitatively different kinds of solutions: the exponential decay and the “sudden death” decay solutions. The appearance of the “sudden death” solutions in the radiated system has not been noticed before.

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