Centre and Representations of $\mathcal{U}_q(sl(2|1))$ at Roots of Unity

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Abstract

Quantum groups at roots of unity have the property that their centre is enlarged. Polynomial equations relate the standard deformed Casimir operators and the new central elements. These relations are important from a physical point of view since they correspond to relations among quantum expectation values of observables that have to be satisfied on all physical states. In this paper, we establish these relations in the case of the quantum Lie superalgebra $\mathcal{U}_q(sl(2|1))$. In the course of the argument, we find and use a set of representations such that any relation satisfied on all the representations of the set is true in $\mathcal{U}_q(sl(2|1))$. This set is a subset of the set of all the finite dimensional irreducible representations of $\mathcal{U}_q(sl(2|1))$, that we classify and describe explicitly.

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1 Introduction

Classical and quantum Lie superalgebras and their representations play respectively an important role in the understanding and exploitation of the classical and $q$-deformed supersymmetry in physical systems. A complete classification of the finite-dimensional simple classical Lie superalgebras over $\mathbb{C}$ has been given by Kac [11, 12] and Scheunert [23]. The corresponding irreducible representations fall in two series called typical and atypical.

Irreducible representations of the quantum analogue of superalgebras are studied intensively when $q$ is not root of unity in [20, 21, 18, 25]. Partial classifications exist also in the case of $U_q(sl(3))$, in [6] for the restricted case and in [4] for periodic representations. Much progress towards a complete classification in the general case of $U_q(\mathfrak{g})$ for $\mathfrak{g}$ a simple Lie algebra was done in [3, 4].

The classification of finite dimensional irreducible representations of $U_q(\mathfrak{osp}(1|2))$ for any $q$ parallels the $U_q(sl(2))$ case [4, 16, 19, 26]. The only other fully understood case is $U_q(sl(2|1))$ [25, 27].

Our main goal in this paper is the structure of the centre of $U_q(sl(2|1))$ when $q$ is a root of unity. Complete sets of representations - to be defined below - give a convenient way to prove relations in this centre. Their construction involves a detailed knowledge of matrix elements of the finite dimension irreducible representations, whose classification is given below with emphasis on what is needed for the rest of the paper.

In Section 2, we give the definition of the quantum superalgebra $U_q(sl(2|1))$ and the expression of central elements. Generalities on the finite dimensional irreducible representations of $U_q(sl(2|1))$ are presented in section 3. In section 4, we recall some useful results on $U_q(gl(2))$ at roots of unity and we give complete sets of irreducible representations for this quantum algebra: expressions in the universal quantum enveloping algebra that vanish on such sets, vanish identically. In section 5, we classify the finite dimensional irreducible representations of $U_q(sl(2|1))$. In section 6, we present complete sets of representations corresponding to infinite subsets of the set of continuous parameters. All the representations of these complete sets have the same dimension, unlike in the classical case [1]. Finally, in section 7, we prove the relations in the centre using our complete set of irreducible representations.

2 Quantum superalgebra $U_q(sl(2|1))$ and its centre

The superalgebra $U_q(sl(2|1))$ is the associative superalgebra over $\mathbb{C}$ with generators $k_1 = q^{h_1}$, $k_1^{-1} = q^{-h_1}$, $k_2 = q^{h_2}$, $k_2^{-1} = q^{-h_2}$, $e_1$, $e_2$, $f_1$, $f_2$ and relations

$$k_1 k_2 = k_2 k_1,$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j,$$

$$k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,$$
\[
e_1 f_1 - f_1 e_1 = \frac{k_1 - k_1^{-1}}{q - q^{-1}}, \quad e_2 f_2 + f_2 e_2 = \frac{k_2 - k_2^{-1}}{q - q^{-1}}, \quad (3)
\]

\[
[e_1, f_2] = 0, \quad [e_2, f_1] = 0, \quad (4)
\]

\[
e_2^2 = f_2^2 = 0, \quad (5)
\]

\[
e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad (6)
\]

\[
f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0. \quad (7)
\]

The two last equations are called the Serre relations. The matrix \((a_{ij})\) is the distinguished Cartan matrix of \(sl(2|1)\), i.e.

\[
(a_{ij}) = \begin{pmatrix}
2 & -1 \\
-1 & 0
\end{pmatrix} \quad (8)
\]

The \(\mathbb{Z}_2\)-grading in \(\mathcal{U}_q(sl(2|1))\) is uniquely defined by the requirement that the only odd generators are \(e_2\) and \(f_2\), i.e.

\[
\text{deg} (k_1) = \text{deg} (k_2) = 0, \\
\text{deg} (k_1^{-1}) = \text{deg} (k_2^{-1}) = 0, \\
\text{deg} (e_1) = \text{deg}(f_1) = 0, \\
\text{deg} (e_2) = \text{deg}(f_2) = 1. \quad (9)
\]

We will not use the (standard) co-algebra structure in the following.

Define

\[
e_3 = e_1 e_2 - q^{-1} e_2 e_1 \quad \text{and} \quad f_3 = f_2 f_1 - q f_1 f_2. \quad (10)
\]

The quantum Serre relations become

\[
e_1 e_3 = q e_3 e_1, \\
f_3 f_1 = q^{-1} f_1 f_3. \quad (11)
\]

Furthermore,

\[
e_2 e_3 = -q e_3 e_2, \\
f_3 f_2 = -q^{-1} f_2 f_3, \quad (12)
\]

and

\[
e_3 f_3 + f_3 e_3 = \frac{k_1 k_2 - k_1^{-1} k_2^{-1}}{q - q^{-1}}, \\
e_3^2 = f_3^2 = 0. \quad (13)
\]

In the following, we will use the conventional notation

\[
[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (14)
\]
When \( q \) is not a root of unity, the centre of \( \mathcal{U}_q(sl(2|1)) \) is generated by the elements \( C_p, \ p \in \mathbb{Z} \), where

\[
C_p = k_1^{2p-1}k_2^{4p-2}(q - q^{-1})^2\left\{[h_1 + h_2 + 1][h_2] - f_1e_1 + f_2e_2([h_1 + h_2]q^{1-2p} - [h_1 + h_2 + 1]) + f_3e_3([h_2 - 2]q^{1-2p} - [h_2 - 1]) + (q - q^{-1})q^{1-p}[p]f_3e_1e_2 + (q - q^{-1})q^{-p}f_1f_2e_3k_2^{-1}[p - 1] + (q - q^{-1})^2q^{1-2p}[p][p - 1]f_2f_3e_3e_2\right\}.
\]

They satisfy the relations

\[
C_{p_1}C_{p_2} = C_{p_3}C_{p_4} \quad \text{if} \quad p_1 + p_2 = p_3 + p_4.
\]

The fact that the centre was not finitely generated in the classical case was known since [13, 24]. The explicit expression of a set of generators of the centre, together with the relations, was given in [1] in the classical case and in [3] in the quantum case.

In this paper, we consider the case when \( q \) is a root of unity. Let \( l \) the smallest integer such that \( q^l = 1 \). We define:

\[
l' = \begin{cases} 
  l & \text{if } l \text{ is odd} \\
  l/2 & \text{if } l \text{ is even}
\end{cases}
\]

the elements \( z_i \equiv k_1^l, x_1 \equiv e_1^l \) and \( y_1 \equiv f_1^l \) also belong to the centre.

**Proposition 1** When \( l \) is odd, the central elements \( z_1, z_2, x_1, y_1 \) and \( C_p, \ p \in \mathbb{Z} \) satisfy the relations

\[
C_{p+l} = z_1^2z_2^4C_p,
\]

\[
C_{p+1}^l = z_1^2z_2^4C_p^l,
\]

\[
\mathcal{P}_l(C_1, \ldots, C_l) \equiv (C_1 + 1)^l - 1 + \sum_{m \geq 2, \ n > 0 \atop m + n \leq l} C_mC_1^n \frac{l}{m-1} \binom{m+n-1}{n+1} \binom{l-m}{n}
\]

\[
= \left(1 - z_1^2z_2^2\right)\left(z_2^2 - 1\right) - (q - q^{-1})^2z_1^2z_2^4y_1x_1.
\]

The first two relations follow from the expression of \( C_p \) and from (16). The third relation will be proved using complete sets of representations of \( \mathcal{U}_q(sl(2|1)) \). Furthermore, there is no other independent polynomial relation.
3 Generalities on finite dimensional irreducible representations

Let us consider a finite dimensional irreducible left module $M$ over $\mathcal{U}_q(sl(2|1))$.

- The generators $k_1$ and $k_2$ are simultaneously diagonalizable on the module $M$.
- Since $e_2^2 = 0$ and $\dim M < \infty$, there exists a subspace $V \subset M$ annihilated by $e_2$, i.e.
  \[ \forall v \in V, \quad e_2 v = 0 . \quad (19) \]
- Since $e_3^2 = 0$ and $e_2 e_3 = -q e_3 e_2$, there exists $V_0 \subset V$ annihilated by $e_3$, i.e.
  \[ \forall v \in V_0, \quad e_2 v = e_3 v = 0. \quad (20) \]
- Because of (19) the subspace $V_0$ is stable by left multiplication by $k_1$ and $k_2$.
- Because of (10) and (11) the subspace $V_0$ is stable by left multiplication by $e_1$.
- Because of $e_2 f_1 = f_1 e_2$ and $e_3 f_1 - f_1 e_3 = -e_2 k_1^{-1}$ the subspace $V_0$ is stable by left multiplication by $f_1$.

Let $\mathfrak{g}_0 \simeq gl(2)$ be the even subalgebra of $sl(2|1)$. The algebra $\mathcal{U}_q(\mathfrak{g}_0)$ is generated by $e_1$, $f_1$, $k_1$, and $k_2$.

The module $V_0$ is then a $\mathcal{U}_q(\mathfrak{g}_0)$ submodule of $M$. It is simple (as a $\mathcal{U}_q(\mathfrak{g}_0)$ module), since any submodule of $V_0$ would generate a proper submodule of $M$ by left action of $\mathcal{U}_q(sl(2|1))$. As a consequence of the simplicity of $V_0$, the element $k_1 k_2^2$ (the $U(1)$ generator) is represented by a scalar on $V_0$.

Let $\mathcal{U}_q(\mathfrak{g}_+)$ be the subalgebra of $\mathcal{U}_q(sl(2|1))$ generated by $e_1$, $f_1$, $k_1$, and $e_2$. The subspace $V_0$ is also a $\mathcal{U}_q(\mathfrak{g}_+)$-module, annihilated by $e_2$.

From $V_0$ considered as an $\mathcal{U}_q(\mathfrak{g}_+)$-module, one can construct an induced $\mathcal{U}_q(sl(2|1))$-module $M' = \mathcal{U}_q(sl(2|1)) \otimes_{\mathcal{U}_q(\mathfrak{g}_+)} V_0$. Then $M$ is equal to $M'$ if $M'$ is simple, or to the quotient of $M'$ by its maximal submodule otherwise.

Since we already know that each finite dimensional irreducible representation of $\mathcal{U}_q(sl(2|1))$ is associated to one finite dimensional irreducible representation $V_0$ of $\mathcal{U}_q(\mathfrak{g}_0)$, we will construct the classification of the former in terms of the latter. As we will see, the correspondence is one-to-one. We now need some results on $\mathcal{U}_q(gl(2))$ at roots of unity.
4 \( \mathcal{U}_q(gl(2)) \) at roots of unity

4.1 Centre of \( \mathcal{U}_q(gl(2)) \)

The elements \( k_1 k_2^2, k_1', k_2', e_1', f_1', f_1'' \) are central in \( \mathcal{U}_q(gl(2)) \). The \( q \)-deformed quadratic Casimir operator is

\[
\mathcal{C}_{\mathcal{U}_q(gl(2))} = q k_1 + q^{-1} k_1^{-1} + (q - q^{-1})^2 f_1 e_1 .
\]

When \( l \) is odd, the centre of \( \mathcal{U}_q(gl(2)) \) is actually the algebra defined by the generators \( k_1 k_2^2, k_1', e_1', f_1', \mathcal{C}_{\mathcal{U}_q(gl(2))} \) and the relation

\[
2 P_l \left( \mathcal{C}_{\mathcal{U}_q(gl(2))}/2 \right) = k_1' + k_1^{-l} + (q - q^{-1})^{2l} f_1 e_1' .
\]

The polynomial \( P_l \) is the first kind Chebychev Polynomial of degree \( l \) defined by

\[
P_l(\cos x) = \cos(lx) .
\]

4.2 Finite dimensional irreducible representations of \( \mathcal{U}_q(gl(2)) \)

All the finite dimensional simple modules over \( \mathcal{U}_q(gl(2)) \) are of course cyclic. We call type \( \mathcal{A} \) representations those that are deformation of classical representations, and type \( \mathcal{B} \) the others. Knowing the \( \mathcal{U}_q(sl(2)) \) case, we only need to add a parameter related with the value of the \( U(1) \) generator \( k_1 k_2^2 \). This parameter may be provided as \( \lambda_1 \lambda_2' \) (value of \( k_1 k_2^2 \)) and a sign, or simply by the value \( \lambda_2 \) of \( k_2 \) on a given vector. The finite dimensional irreducible representations of \( \mathcal{U}_q(sl(2)) \) are:

- **type \( \mathcal{A} \)**: Usual (nilpotent) representations, where \( k_1^{2l} = 1, e_1'' = 0, f_1'' = 0 \), characterized by their dimension \( N = 1, \cdots, l' \) and a sign \( \omega \). The highest weight \( \lambda_1 \) is \( \lambda_1 = q^{N-1} \). These representations are given explicitly in Appendix A [22]. The representation of dimension \( l' \) plays a special role. It is in fact in the intersection of this case and the following.

- **type \( \mathcal{B} \)**: Coloured (nilpotent) representations, with still \( e_1'' = 0, f_1'' = 0 \), characterized by their highest weight \( \lambda_1 \), a continuous parameter. Their dimension is \( l' \). They are also described by [22].

- **type \( \mathcal{B} \)**: Periodic and semi-periodic representations, explicitly given in [48]. These representations have dimension \( l \). They depend on four complex parameters corresponding to the values of the three central elements \( k_1'', e_1'', f_1'' \) and one discrete parameter corresponding to the value of the quadratic Casimir \( \mathcal{C}_{\mathcal{U}_q(gl(2))} \) of \( \mathcal{U}_q(gl(2)) \) and related to the former through the relation [22]. The \( \mathcal{U}_q(gl(2)) \)-representation is also completely characterized by the parameters \( y = \varphi^l, \beta, \lambda_1 \) and \( \lambda_2 \) appearing in

\[
\begin{align*}
    f_1'' v_0 &= \varphi^l v_0 \\
    k_1 v_0 &= \lambda_1 v_0 = q^{\mu_1} v_0 \\
    f_1 e_1 v_0 &= \beta v_0 \\
    k_2 v_0 &= \lambda_2 v_0 = q^{\mu_2} v_0
\end{align*}
\]
The existence of periodic irreducible representations has the following consequence: the primitive ideals defined as the kernels of these representations are not the annihilator of the irreducible quotient of some Verma module, unlike in the case of classical (super)algebras 8, 17.

4.3 Complete sets of representations of $\mathcal{U}_q(sl(2))$

We prove that a set of generic (periodic) representations corresponding to an open subset of the set of parameters builds a complete set, in the following sense: if an element of $\mathcal{U}_q(sl(2))$ acts as 0 on all the representations of this set, then it is the 0 element of $\mathcal{U}_q(sl(2))$. This terminology was used in [1], where the authors found complete sets of finite dimensional irreducible representations of the classical $sl(2)$ and $sl(2|1)$. For quantum groups at roots of unity, we shall obtain rather different results.

Let $\mathcal{R} \in \mathcal{U}_q(sl(2))$ be such that it vanishes on a set $\Omega$ of representations. Let $q^{2(t-r)}$ be the $q$-grading of an element $f_1^t q^r e_1'$. We have $k_1 (f_1^t k_1^r e_1') = q^{2(t-r)} (f_1^t k_1^r e_1') k_1$. Any element of $\mathcal{U}_q(sl(2))$ is a sum of terms of given grading since the $f_1^t k_1^r e_1'$ form a basis of $\mathcal{U}_q(sl(2))$. We write $\mathcal{R} = \sum_{t=0}^{t-1} \mathcal{R}_d$, where the grading of $\mathcal{R}_d$ is $q^{2d}$. Commuting $\mathcal{R}$ with $k_1$ shows that all the $\mathcal{R}_d$ vanish separately on the representations in $\Omega$. The same is true for each $f_1^t \mathcal{R}_d$. Since $\mathcal{U}_q(sl(2))$ contains no zero divisor [14], the vanishing of $f_1^t \mathcal{R}_d$ in $\mathcal{U}_q(sl(2))$ is equivalent to that of $\mathcal{R}_d$. Hence, to prove that $\Omega$ is a complete set of representations, we only have to show that the only element of $\mathcal{U}_q(sl(2))$ commuting with $k_1$ and acting as 0 on all representations of $\Omega$ is 0.

Let $\mathcal{R}$ be an element of $\mathcal{U}_q(sl(2))$ with grading 1, and $n, n' \in \mathbb{N}$ such that $f_1^n k_1^{n'} \mathcal{R} = \sum_i a_i f_1^{r_i} k_1^{s_i} e_1^{t_i}$ has only terms with $r_i - t_i \in t' \mathbb{N}$ and $s_i \in \mathbb{N}$. Then $f_1^n k_1^{n'} \mathcal{R}$ can be written as a polynomial in $f_1^{t'}$, $k_1$, and $f_1 e_1$, which commute among themselves. The value of this polynomial on the vector $v_0$ of the representation [15] is the same polynomial evaluated on the scalars $\varphi^{t'}$, $\lambda_1$ and $\beta$. If $\Omega$ is a set of representations corresponding to an open subset of $\mathbb{C}^3$ for the values of $\varphi^{t'}$, $\lambda_1$ and $\beta$, and if $\mathcal{R}$ vanishes on all the representations of $\Omega$, then the polynomial vanishes identically in $\mathcal{U}_q(sl(2))$, and hence $\mathcal{R} = 0$ as an element of $\mathcal{U}_q(sl(2))$. We then have the following

**Proposition 2** A set of generic (periodic) representations corresponding to an open subset of the set of values for the parameters is a complete set of representations.

**Remark 1:** an element of $\mathcal{U}_q(sl(2))$ that vanishes on all type $A$ modules, or even on all nilpotent or semiperiodic modules, is not necessarily 0 in $\mathcal{U}_q(sl(2))$ (take simply $\mathcal{R} = e_1 f_1^t$). So a complete set of representations should include periodic ones.

**Remark 2:** suitably chosen infinite sets of periodic representations (not necessarily corresponding to an open set of values of the parameters) can also be complete.
5 Classification of finite dimensional irreducible representations of $\mathcal{U}(sl(2|1))$

Let $V_0$ an $N$-dimensional irreducible $\mathcal{U}_q(\mathfrak{g}_0)$-module, that we extend to a $\mathcal{U}_q(\mathfrak{g}_+)$-module by the requirement that $e_2V_0 = 0$.

Let $M'$ be the induced module $\mathcal{U}_q(sl(2|1)) \otimes_{\mathcal{U}_q(\mathfrak{g}_+)} V_0$. Then

$$M' = V_0 \oplus f_2V_0 \oplus f_3V_0 \oplus f_2f_3V_0.$$  \hspace{1cm} (25)

The subspaces $(f_2V_0 \oplus f_3V_0)$ and $f_2f_3V_0$ are representations of $\mathcal{U}_q(\mathfrak{g}_0)$ with the same value for central elements $k_1^l$, $k_2$, $e_1$, $f_1$ as for $V_0$. If we write the value of quadratic Casimir $C_{\mathcal{U}_q(gl(2))}$ of $\mathcal{U}_q(gl(2))$ as $\xi + \xi^{-1}$, then its eigenvalues on the different subspaces are

| Subspace | $C_{\mathcal{U}_q(gl(2))}$ |
|----------|-----------------|
| $V_0$ :  | $\xi + \xi^{-1}$ |
| $(f_2V_0 \oplus f_3V_0)$ : | $q\xi + q^{-1}\xi^{-1}$, $q^{-1}\xi + q\xi^{-1}$ |
| $f_2f_3V_0$ : | $\xi + \xi^{-1}$ |

The elements $f_2^p f_3^\rho f_1^\sigma$, for $p \in \mathbb{N}$, $\rho, \sigma \in \{0, 1\}$ build a Poincaré–Birkhoff–Witt basis of the subalgebra $\mathcal{U}^-$ generated by $f_1$ and $f_2$. The elements $e_1^{p'} e_3^{\rho'} e_2^{\sigma'}$, for $p' \in \mathbb{N}$, $\rho', \sigma' \in \{0, 1\}$ build a Poincaré–Birkhoff–Witt basis of the subalgebra $\mathcal{U}^+$ generated by $e_1$ and $e_2$. Together with the basis $k_1^{s_1} k_2^{s_2}$ (with $s_i \in \mathbb{Z}$) for the Cartan subalgebra, this provide a basis for $\mathcal{U}_q(sl(2|1))$.

Let $w_{0,0,0}$, $w_{0,0,1}$, $\cdots$, $w_{0,0,N-1}$, be a basis of $V_0$. Then it follows from the definition of $V_0$ and of the Poincaré–Birkhoff–Witt basis of $\mathcal{U}_q(sl(2|1))$ given above that the vectors $f_2^p f_3^\rho w_{0,0,\rho}$, $\rho, \sigma \in \{0, 1\}$, $p \in \{0, N-1\}$ build a basis of $M'$. In particular

$$\dim M' = 4N,$$  \hspace{1cm} (27)

i.e. four times the dimension of $V_0$.

Since the dimension $N$ of $V_0$ is bounded by $l$, we already know that the dimension of a simple $\mathcal{U}_q(sl(2|1))$-module is bounded by $4l$. Since nilpotent representations of $\mathcal{U}_q(\mathfrak{g}_0)$ have dimension less or equal to $l'$, the dimension of nilpotent representations of $\mathcal{U}_q(sl(2|1))$ is bounded by $4l'$.

5.1 Usual (type A) representations

We now start from a $\mathcal{U}_q(\mathfrak{g}_0)$-module $V_0$ which is the $q$-deformation of a classical module. Let $N$ be its dimension $(1 \leq N \leq l')$.  


The module $M'$ is then a highest weight module with highest weight vector $w_{0,0,0}$ on which
\begin{align*}
e_1 w_{0,0,0} &= 0 , \quad e_2 w_{0,0,0} = 0 , \\
k_1 w_{0,0,0} &= \lambda_1 w_{0,0,0} , \quad k_2 w_{0,0,0} = \lambda_2 w_{0,0,0} \tag{28}
\end{align*}
with $\lambda_1 = \omega q^{N-1}$, $\omega = \pm 1$.

The Casimir operators $C_p$ have the following scalar value on $M'$:
\begin{align*}
C_p &= (q - q^{-1})^2 \lambda_1^{2p-1} \lambda_2^{4p-2} [\mu_2][\mu_1 + \mu_2 + 1] \tag{29}
\end{align*}
where, again, $q^{\mu_i} \equiv \lambda_i$.

A basis of $M'$ is given by
\begin{align*}
w_{\rho,\sigma,p} &= f_2^p f_3^\sigma f_1^\rho w_{0,0,0} , \quad \text{with} \quad \begin{cases} \rho, \sigma \in \{0, 1\} \\
 p \in \{0, \ldots, N - 1\} \end{cases} \tag{30}
\end{align*}
By convention, we set
\begin{align*}
w_{\rho,\sigma,N} &\equiv 0 . \tag{31}
\end{align*}

A non zero vector in a representation is called singular if it is annihilated by $e_1$ and $e_2$ and is contained in a proper subrepresentation. Any submodule of $M'$ contains a singular vector for $M'$. Indeed, any submodule of $M'$ has its own $U_q(g_0)$-submodule annihilated by $e_2$. This last module is also of type $A$ because this property is determined by the scalar value of the central elements, which are determined by $V_0$. The module $M'$ is simple if, and only if it contains no singular vector $v_s \neq 0$.

**Lemma 1** The non-vanishing of the Casimir operators $C_p$ is a sufficient condition for $M'$ to be simple.

The comparison of the values of the Casimir operators on the highest weight vector and on the singular vector indeed shows that
\begin{align*}
[\mu_2][N + \mu_2] &= 0 \tag{32}
\end{align*}
is a necessary condition for the existence of a singular vector (which cannot be in $V_0$ since $V_0$ is a simple $U_q(g_0)$-module). This condition amounts to the vanishing of all the Casimir operators $C_p$. We shall see that this is actually a necessary and sufficient condition for the simplicity of $M'$.

### 5.1.1 Typical type $A$ representations

**Proposition 3** If (32) is not satisfied, the module $M'$ is simple. It has dimension $4N$. Its explicit expression is given in (50). It is called typical.

**Proof:** If (32) is not satisfied, $M'$ contains no singular vector.

For $N = 1, \ldots, l' - 1$, the subspace $f_2 V_0 \oplus f_3 V_0$ is the direct sum of $U_q(g_0)$-modules characterized by the dimensions $N \pm 1$ and sign $\omega$. For $N = l'$, $f_2 V_0 \oplus f_3 V_0$ is an indecomposable $U_q(g_0)$-module which is isomorphic to the tensor product of $V_{l'}$ with the spin $1/2$ representation, and which contains the dim = $l' - 1$ (sign = $\omega$) simple sub-$U_q(g_0)$-module.
5.1.2 Atypical type A representations

We now consider the case $[\mu_2][N + \mu_2] = 0$ (i.e. $(\lambda_2^2 - 1)(\lambda_2^2 - q^{-2N}) = 0$). We will prove the following:

**Proposition 4** If the Casimir operators $C_p$ vanish on $M'$, there exists a maximal submodule $M''$ of $M'$. The quotient space $M = M'/M''$ is a simple module, called atypical. We can consider three cases:

- If $[\mu_2] = 0$ and $[N + \mu_2] \neq 0$, then $\dim M = 2N - 1$.
- If $[\mu_2] \neq 0$ and $[N + \mu_2] = 0$ then $\dim M = 2N + 1$.
- If $[\mu_2] = 0$ and $[N + \mu_2] = 0$ (and hence $N = l'$) then $\dim M = 2l' - 1$.

**Proof:**

Atypical type A representations with $[\mu_2] = 0$ and $[N + \mu_2] \neq 0$. In this case, the vector $f_2 w_{0,0,0} = w_{1,0,0}$ is a singular vector. The action of $U_q(sl(2|1))$ on it generates a $2N + 1$-dimensional submodule $M''$ spanned by

$$
\begin{align*}
  f_1^p w_{1,0,0} &= q^{-p} w_{1,0,p} - q^{-1}[p] w_{0,1,p-1}, & p = 0, \cdots, N \\
  f_1^p f_3 w_{1,0,0} &= -q^{-1} w_{1,1,p}, & p = 0, \cdots, N - 1
\end{align*}
$$

(33)

This submodule is maximal. Quotienting $M'$ by $M''$ provides a $2N - 1$-dimensional simple module $M$, the expression of which is given in (31).

Atypical type A representations with $[\mu_2] \neq 0$ and $[N + \mu_2] = 0$. Looking by direct computation for a singular vector, we see that $N = 1$, $[1 + \mu_2] = 0$ is a particular case: it is the only case of existence of a singular vector in $f_2 f_3 V_0$ (one-dimensional in this case). The singular vector is $w_{1,1,0} = f_2 f_3 w_{0,0,0}$. It generates only $f_2 f_3 V_0$ as $U_q(sl(2|1))-submodule$. The quotient $M'/f_2 f_3 V_0$ is three dimensional. It is actually the $q$-deformed three dimensional atypical fundamental representation.

If $N \in \{2, \cdots, l' - 1\}$, there is a singular vector given by

$$
v_s = \lambda_2 q^1 w_{1,0,1} + [\mu_1] w_{0,1,0}.
$$

(34)

It generates the $2N - 1$-dimensional maximal submodule $M''$ spanned by

$$
\begin{align*}
  f_1^p v_s &= \lambda_2 q^{1-p} w_{1,0,p+1} + [\mu_1 - p] w_{0,1,p}, & p = 0, \cdots, N - 2 \\
  f_1^p f_2 v_s &= [\mu_1] w_{1,1,p}, & p = 0, \cdots, N - 1
\end{align*}
$$

(35)

The quotient $M = M'/M''$ is a $2N + 1$-dimensional simple module, explicitly given in (32).
Atypical type $A$ representations with $N = l'$ If $[\mu_2] = 0$ and $N = l'$, the vector $w_{1,0,0}$ is singular. The submodule it generates is similar to (33), except that now $f_1^{l'} w_{1,0,0} = 0$. However, the vector $u_{0,1,0,1}$ is subsingular, i.e. its image by $e_1$ and $e_2$ is contained in the submodule generated by $w_{1,0,0}$. It belongs to the maximal submodule $M''$ of $M'$. Note that $f_1 w_{1,0,0} \in M''$ is also singular. The submodule $M''$ has dimension $2l' + 1$ and $M = M'/M''$ has dimension $2l' - 1$. It is also described by (51).

5.2 Nilpotent type $B$ representations

We now consider the case where $V_0$ is a type $B$ nilpotent $U_q(\mathfrak{g}_0)$-module, of dimension $N = l'$, with two parameters $\lambda_1$ and $\lambda_2$. We assume $[\mu_1 + 1] \neq 0$ since this case was treated as type $A$. As in the type $A$ case we consider the induced module $M'$, on which (28) applies. A basis for $M'$ is also given by (30) with $N = l'$. We also have

**Proposition 5** Nilpotent type $B$ representations fall into two classes

- If $[\mu_2][\mu_1 + \mu_2 + 1] \neq 0$, i.e. $C_p \neq 0$, then $M'$ is simple. Its dimension is $4l'$ and the parameters are $\lambda_1$ and $\lambda_2$. Its explicit expression is given in (50) (typical case).

- If $[\mu_2][\mu_1 + \mu_2 + 1] = 0$, i.e. $C_p = 0$, then $M'$ has a maximal submodule $M''$ of dimension $2l'$. Then $M = M'/M''$ has dimension $2l'$ (atypical case).

**Proof:** As in the type $A$ case, there is no singular vector if the $C_p$ do not vanish. Suppose now that $[\mu_2][\mu_1 + \mu_2 + 1] = 0$. We can separate this case into two subcases, according to which term of the product vanishes (both terms cannot vanish simultaneously, since $[\mu_1 - p + 1] \neq 0$ for any integer $p$ in type $B$ $U_q(\mathfrak{g}_0)$-modules).

- If $[\mu_2] = 0$, the vector $w_{1,0,0}$ is singular. It generates the submodule $M''$ given as in (33) with $N = l'$, except that now $f_1^{l'} w_{1,0,0} = 0$. Then dim $M'' = 2l'$. The quotient module hence has dimension $2l'$. It is described by (51).

- If $[\mu_1 + \mu_2 + 1] = 0$, then there is a singular vector given by (34). It generates the submodule $M''$ given as in (35) with $N = l'$, except that now $f_1^{l-1} v_s \neq 0$ also belongs to $M''$, so that dim $M'' = 2l'$. Again, dim $M = 2l'$ and $M$ is described by (52).

5.3 Periodic and semi-periodic type $B$ representations

Let us now consider the case when $V_0$ is a periodic or semi-periodic $U_q(\mathfrak{g}_0)$-module, i.e. with non vanishing (scalar) value of the central element $f_1^l$.

$$f_1^l = \varphi^l \text{id}$$  (36)
In $\mathcal{U}_q(sl(2|1))$, $f^l_1$ is central too, so (36) also holds in $M'$.

The value of the central element $e^l_1$ will be a free parameter (possibly zero for semi-periodic representations). One would get the representations with a vanishing value for $f^l_1$ and a non vanishing value for $e^l_1$, using the automorphism of $\mathcal{U}_q(sl(2|1))$ given by

$$\psi(e_i) = f_i, \quad \psi(f_i) = e_i$$

$$\psi(k_1) = k_1^{-1}, \quad \psi(k_2) = -k_2^{-1}$$

(37)

The module $M'$ is actually characterized by the following actions on a vector $w_{0,0,0}$ of $V_0$:

$$f^l_1 w_{0,0,0} = \varphi^l_1 w_{0,0,0}$$

$$k_1 w_{0,0,0} = \lambda_1 w_{0,0,0} = q^{\mu_1} w_{0,0,0}$$

$$f_1 e_1 w_{0,0,0} = \beta w_{0,0,0}$$

$$k_2 w_{0,0,0} = \lambda_2 w_{0,0,0} = q^{\mu_2} w_{0,0,0}$$

(38)

Those values determine the values of $e^l_1$ (using (22)) and of $C_p$

$$C_p = (q - q^{-1})^2 \lambda_1^{2p-1} \lambda_2^{2p-2} ([\mu_2][\mu_1 + \mu_2 + 1] - \beta)$$

(39)

A basis of $M'$ is given by

$$w_{\rho,\sigma,p} \equiv \varphi^{-\rho} f_2^p f_3^\sigma f^l_1 w_{0,0,0}$$

with $\rho, \sigma \in \{0, 1\}$ and $p \in \{0, \ldots, l - 1\}$.

(40)

**Proposition 6** For periodic and semi-periodic representations, the following alternative holds:

- (i). If $[\mu_2][\mu_1 + \mu_2 + 1] - \beta \neq 0$, the module $M'$ is irreducible and its dimension is equal to $4l$. It is described explicitly in equation (53).

- (ii). If $[\mu_2][\mu_1 + \mu_2 + 1] - \beta = 0$, the module $M'$ is not simple. It has a submodule $M''$ of dimension $2l$ and the factor space $M'/M''$ is an irreducible module of dimension $2l$, explicitly given by equation (54).

The cases (ii) corresponds to atypical periodic representations and $[\mu_2][\mu_1 + \mu_2 + 1] = \beta$ is the condition for the vanishing of the Casimir operators $C_p$ on $M'$.

**Proof.** By direct computation, we check that $[\mu_2][\mu_1 + \mu_2 + 1] - \beta = 0$ is the necessary and sufficient condition for the existence of a vector (not belonging to $V_0$), annihilated by both $e_2$ and $e_3$. This vector then belongs to $f_2 V_0 \oplus f_3 V_0$ and it generates a $2l$-dimensional subspace spanned by the vectors $w_{1,1,p}$ and $[\mu_2 + p + 1] w_{0,1,p} - \lambda_2^{-1} q^{-p} w_{1,0,p+1}$ for $p \in \{0, \ldots, l - 1\}$. The quotient of $M'$ by this submodule is simple.
6 Complete sets of representations of $\mathcal{U}_q(sl(2|1))$

Proposition 7 A set of typical periodic representations corresponding to an open subset of the set of values of the parameters is a complete set of representations.

Proof: Let $\Omega$ be a set of representations, and $\mathcal{R} \in \mathcal{U}_q(sl(2|1))$ such that $\mathcal{R}$ vanishes on all the representations of $\Omega$. As for $\mathcal{U}_q(sl(2))$, we can restrict ourselves to the case where $k_i \mathcal{R} k_i^{-1} = q^{d_i} \mathcal{R}$ for given gradings $d_i (i = 1, 2)$.

We have in fact to consider five cases, according to the possible gradings with respect to $k_1 k_2^2$. All the possible values for $d_1 + 2d_2$ are actually $-2, -1, 0, 1, 2$ (This is due to the fact that the squares of fermionic generators vanish, and it can also be read from the Poincaré–Birkhoff–Witt basis).

\[
d_1 + 2d_2 = -2 \quad \mathcal{R}^{(-2)} = \mathcal{R}_1 e_3 e_2
\]
\[
d_1 + 2d_2 = -1 \quad \mathcal{R}^{(-1)} = \mathcal{R}_2 e_2 + \mathcal{R}_3 e_3 + \mathcal{R}_4 f_2 e_3 e_2 + \mathcal{R}_5 f_3 e_3 e_2
\]
\[
d_1 + 2d_2 = 0 \quad \mathcal{R}^{(0)} = \mathcal{R}_6 + \mathcal{R}_7 f_2 e_2 + \mathcal{R}_8 f_3 e_2 + \mathcal{R}_9 f_2 e_3 + \mathcal{R}_{10} f_3 e_3 + \mathcal{R}_{11} f_2 f_3 e_3 e_2
\]
\[
d_1 + 2d_2 = 1 \quad \mathcal{R}^{(1)} = \mathcal{R}_{12} f_2 + \mathcal{R}_{13} f_3 + \mathcal{R}_{14} f_2 f_3 e_2 + \mathcal{R}_{15} f_2 f_3 e_3
\]
\[
d_1 + 2d_2 = 2 \quad \mathcal{R}^{(2)} = \mathcal{R}_{16} f_2 f_3
\]

where the $\mathcal{R}_i$ are elements of $\mathcal{U}_q(g_0)$. We have to prove that all of them vanish. Since $\Omega$ is a set of representations corresponding to an open subset of the set of values of the parameters, the representations of $\mathcal{U}_q(g_0)$ given by the corresponding $V_0$ is a complete set. If we identify $V_0$ and $f_2 f_3 V_0$ (as $\mathcal{U}_q(g_0)$-modules), we see that the vanishing of $\mathcal{R}_1$ and $\mathcal{R}_{16}$ results from this. Let us now consider $\mathcal{R}^{(0)}$, the cases of $\mathcal{R}^{(-1)}$ and $\mathcal{R}^{(1)}$ being simpler. Since $\mathcal{R}^{(0)} e_3 e_2 = \mathcal{R}_6 e_3 e_2$ act as zero on all the representations of $\Omega$, then $\mathcal{R}_6 = 0$. Now, $\mathcal{R}^{(0)} e_2 = (\mathcal{R}_0 f_2 + \mathcal{R}_{10} f_3) e_3 e_2$. This operator sends $f_2 f_3 V_0$ into $f_2 V_0 \oplus f_3 V_0$ and is supposed to act as zero. Looking at the explicit action of this operator on the vector $v_{1,1,p}$ and using the fact that $f_2 V_0 \oplus f_3 V_0$ is generically a direct sum of two inequivalent $\mathcal{U}_q(g_0)$-modules, we learn that $\mathcal{R}_9 = \mathcal{R}_{10} = 0$. Multiplying $\mathcal{R}^{(0)}$ on the right by $e_3$, we then prove in a similar way that $\mathcal{R}_7 = \mathcal{R}_8 = 0$. Finally, the proof that $\mathcal{R}_{11} = 0$ mimics the proof of proposition 4.

7 Proof of the relation in the centre

We now use a complete set of representation to prove the relation

\[
\mathcal{P}_l(\mathcal{C}_1, \ldots, \mathcal{C}_t) \equiv (\mathcal{C}_1 + 1)^l - 1 + \sum_{m \geq 2, n \geq 0}^{m+n \leq t} \mathcal{C}_m \mathcal{C}_n^l \frac{l}{m-1} \binom{m+n-1}{n+1} \binom{l-m}{n} \\
= (1 - z_1^2 z_2^2) (z_2^2 - 1) - (q^{-1} z_1^{-2} z_2^4 y_1 x_1) \cdot (q - q^{-1})^{2l} z_1^2 z_2^4 y_1 x_1. \tag{42}
\]
On a typical type B periodic representation characterized by the parameters $\lambda_1$, $\lambda_2$, $\varphi^1$ and $\beta$, the value of $C_p$ is

$$ C_p = (\lambda_1 \lambda_2^2)^{2p-2} C_1 $$

$$ = \lambda_1^{2p-1} \lambda_2^{4p-2} \left((q \lambda_1 \lambda_2 - q^{-1} \lambda_1^{-1} \lambda_2^{-1})(\lambda_2 - \lambda_2^{-1}) - (q - q^{-1})^2 \beta\right) $$

$$ = \lambda_1^{2p-1} \lambda_2^{4p-2} \left(q \lambda_1 \lambda_2^2 + q^{-1} \lambda_1^{-1} \lambda_2^{-2} - (\xi + \xi^{-1})\right) $$

$$ = \lambda_1^{2p-1} \lambda_2^{4p-2} \left(q^{1/2} \lambda_1^{1/2} \lambda_2 \xi^{1/2} - q^{-1/2} \lambda_1^{-1/2} \lambda_2^{-1} \xi^{-1/2}\right) $$

$$ \cdot \left(q^{1/2} \lambda_1^{1/2} \lambda_2 \xi^{1/2} - q^{-1/2} \lambda_1^{-1/2} \lambda_2^{-1} \xi^{-1/2}\right) $$

(43)

where $(q - q^{-1})^{-2}(\xi + \xi^{-1}) \equiv (q - q^{-1})^{-2}(q \lambda_1 + q^{-1} \lambda_1^{-1}) + \beta$ is the value of the $U_q(\mathfrak{g}_0)$ quadratic Casimir operator on the subspace $V_0$.

The polynomial $P_l$ in (48) is such that, if we set

$$ C_1 = \lambda_1 \lambda_2^2 (x_1 - x_1^{-1})(x_2 - x_2^{-1}) $$

$$ \frac{x_2}{x_1} = \lambda_1 \lambda_2^2 $$

(44)

then

$$ \mathcal{P}(C_1, \ldots, C_l) = \lambda_1^l \lambda_2^{2l} \left(x_1^l - x_1^{-l}\right) \left(x_2^l - x_2^{-l}\right) $$

(45)

so that

$$ \mathcal{P}(C_1, \ldots, C_l) = \lambda_1^l \lambda_2^{2l} \left(\lambda_1^l \lambda_2^{2l} + \lambda_1^{-l} \lambda_2^{-2l} - (\xi^l + \xi^{-l})\right). $$

(46)

Using the polynomial relation (22) in $U_q(\mathfrak{g}_0)$, we identify $(\xi^l + \xi^{-l})$ with the value of $(q - q^{-1})^{2l} f_1 e_1^l + (k^l + k^{-l})$ and we get the evaluation of the right hand side of (48) on the representation. Since this is true for any typical periodic representations, and since the set of those representations is complete, the relation is true in the enveloping algebra.

The existence of any other independent polynomial relation in the centre would imply more relations among the parameters of the periodic representations, so we also conclude that there is no other independent relation.

**Appendix A  Finite dimensional irreducible representations of $U_q(gl(2))$**

Nilpotent modules of $U_q(gl(2))$

$$ k_1 v_p = \lambda_1 q^{-2p} v_p, \quad \text{for } p \in \{0, \ldots, N - 1\}, $$

(47)

$$ f_1 v_p = v_{p+1}, \quad \text{for } p \in \{0, \ldots, N - 2\}, \quad \text{and } f_1 v_{N-1} = 0, $$

$$ e_1 v_p = [p][\mu_1 - p + 1] v_{p-1}, \quad q^{\mu_1} \equiv \lambda_1, $$

$$ k_2 v_p = \lambda_2 q^p v_p, \quad \text{for } p \in \{0, \ldots, N - 1\}. $$
The dimension $N$ is the smallest non-negative integer satisfying $[N][\mu_1 - N + 1] = 0$. For usual type $A$ representations, $N \in \{1, \cdots, l\}$ and the highest weight is related to $N$ by $\lambda_1 = \omega q^{N-1}$, with $\omega = \pm 1$.

For nilpotent type $B$ representations $N = l'$ and $\lambda_1$ is a free parameter.

If $N = l'$ and $\lambda_1 = \pm q^{-1}$, the representation is still the $q$-deformation of a classical one, but it has $q$-dimension $[N] = 0$. This case plays a special role.

**Appendix B  Finite dimensional irreducible representations of $U_q(gl(2))$**

The following relations are used to determine the action of the generators on the representations:

\[
\begin{align*}
k_1 v_p &= \lambda_1 q^{-2p} v_p, \\
f_1 v_p &= \varphi v_{p+1}, \\
e_1 v_p &= \varphi^{-1}(p)[\mu_1 - p + 1] + \beta) v_{p-1}, \\
k_2 v_p &= \lambda_2 q^p v_p, \\
f_2 f_3 f_1^p f_1^{\rho+1} &= q^{\sigma} f_2^p f_3 f_1^{\rho+1} f_1^p, \\
f_2 f_3 f_1^p &= (1-\rho) f_2^p f_3 f_1^p, \\
[ e_1, f_2^p f_3 f_1^p ] &= \sigma(1-\rho)(-1)^\sigma f_2^p f_3 f_1^p q^{h_1-2p+1} + [p] f_2^p f_3 f_1^{p-1}[h_1 - p + 1], \\
e_2 f_2^p f_3 f_1^p - (-1)^{\rho+\sigma} f_2^p f_3 f_1^p e_2 &= \rho f_2^p f_3 f_1^p [h_2 + p + \sigma] + \sigma(-1)^\rho f_2^p f_3 f_1^{p+1} q^{h_2 - p},
\end{align*}
\]

where $(p, \rho, \sigma) \in \mathbb{N} \times \{0, 1\} \times \{0, 1\}$.

**Typical nilpotent modules**

\[
\begin{align*}
k_1 w_{\rho,\sigma,p} &= \lambda_1 q^{\rho-\sigma-2p} w_{\rho,\sigma,p}, \\
k_2 w_{\rho,\sigma,p} &= \lambda_2 q^{\sigma+p} w_{\rho,\sigma,p}, \\
f_1 w_{\rho,\sigma,p} &= q^{\rho-\sigma} w_{\rho,\sigma,p+1} - (1-\rho)q^{-p} w_{\rho-1,\sigma+1,p}, \\
f_2 w_{\rho,\sigma,p} &= (1-\rho) w_{p+1,\sigma,p}, \\
e_1 w_{\rho,\sigma,p} &= -\sigma(1-\rho)\lambda_1 q^{2p+1} w_{\rho+1,\sigma-1,p} + [p][\mu_1 - p + 1] w_{\rho,\sigma,p-1}, \\
e_2 w_{\rho,\sigma,p} &= \rho[\mu_2 + p + \sigma] w_{p-1,\rho,\sigma} + \sigma(-1)^\rho \lambda_2^{-1} q^{-p} w_{\rho,\sigma-1,p+1}.
\end{align*}
\]

[14]
with \((p, \rho, \sigma) \in \{0, \cdots, N - 1\} \times \{0, 1\} \times \{0, 1\}\) in the left hand side and, by convention, \(w_{p,\sigma,N} = 0\) in the right hand side. For type \(A\) modules, \(q^{\mu_1} \equiv \lambda_1 = \omega q^{N-1}\). For type \(B\) nilpotent modules, \(N = l'\) and \(q^{\mu_1} \equiv \lambda_1\) is free.

**Atypical nilpotent modules; case** [\(\mu_2 = 0\)]

\[
\begin{align*}
k_1 w_{\sigma,p} &= \lambda_1 q^{-\sigma - 2p} w_{\sigma,p}, \\
k_2 w_{\sigma,p} &= \varepsilon q^{\sigma + p} w_{\sigma,p}, \\
f_1 w_{\sigma,p} &= q^2 w_{\sigma,p+1}, \\
f_2 w_{\sigma,p} &= (1 - \sigma)q^{p-2}[\mu_1 - p + 1] w_{\sigma+1,p-1}, \\
e_1 w_{\sigma,p} &= q^{-\sigma} [\mu_1 + 1 - p - \sigma] w_{\sigma,p-1}, \\
e_2 w_{\sigma,p} &= \sigma q^{-p} w_{\sigma-1,p+1}.
\end{align*}
\]

(51)

where \(\sigma \in \{0, 1\}\). For type \(A\) representations, \(p \in \{0, \cdots, N - 1 - \sigma\}\) and the dimension is \(2N - 1\). For type \(B\) representations, \(p \in \{0, \cdots, l' - 1\}\) and the dimension is \(2l'\).

**Atypical nilpotent modules; case** [\(\mu_1 + \mu_2 + 1 = 0\)]

\[
\begin{align*}
k_1 w_{\sigma,p} &= \lambda_1 q^{-\sigma - 2p} w_{\sigma,p}, \\
k_2 w_{\sigma,p} &= \varepsilon\lambda_1^{-1} q^{\sigma + p - 1} w_{\sigma,p}, \\
f_1 w_{\sigma,p} &= q^2 w_{\sigma,p+1}, \\
f_2 w_{\sigma,p} &= -(1 - \sigma)\lambda_1^{-1} q^{p-2}[\mu_1 - p + 1] w_{\sigma+1,p-1}, \\
e_1 w_{\sigma,p} &= q^{-\sigma} [p + \sigma] [\mu_1 + 1 - p] w_{\sigma,p-1}, \\
e_2 w_{\sigma,p} &= \sigma q^{-p+1} w_{\sigma-1,p+1}.
\end{align*}
\]

(52)

where \(\sigma \in \{0, 1\}\). For type \(A\) representations, \(p \in \{-\sigma, \cdots, N - 1\}\) and the dimension is \(2N + 1\). For type \(B\) representations, \(p \in \{0, \cdots, l' - 1\}\) and the dimension is \(2l'\).

**Typical periodic modules** The actions of the generators \(e_1, e_2, f_1\) and \(f_2\) on a typical periodic \(M\) module are given by

\[
\begin{align*}
k_1 w_{p,\sigma,p} &= \lambda_1 q^{\sigma - 2p} w_{p,\sigma,p}, \\
k_2 w_{p,\sigma,p} &= \lambda_2 q^{\sigma + p} w_{p,\sigma,p}, \\
f_1 w_{p,\sigma,p} &= \varphi q^{2p} w_{p,\sigma,p+1} - \varphi q^{p} w_{p-1,\sigma-1,p}, \\
f_2 w_{p,\sigma,p} &= (1 - p) w_{p+1,\sigma,p}, \\
e_1 w_{p,\sigma,p} &= -\varphi^{-1}(1 - p)\lambda_1 q^{2p+1} w_{p+1,\sigma-1,p} + \varphi^{-1}([p][\mu_1 - p + 1] + \beta) w_{p,\sigma,p-1}, \\
e_2 w_{p,\sigma,p} &= p[\mu_2 + p + \sigma] w_{p-1,\sigma,p} + \sigma(1 - p)\lambda_2 q^{p-1} w_{p,\sigma-1,p+1},
\end{align*}
\]

(53)

with \((\rho, \sigma) \in \{0, 1\}^2\) and \(p \in \{0, \cdots, l - 1\}\).
Atypical periodic modules

\[ k_1 w_{\sigma,p} = \lambda_1 q^{-\sigma-2p} \tilde{w}_{\sigma,p}, \]
\[ k_2 w_{\sigma,p} = \lambda_2 q^{\sigma+p} \tilde{w}_{\sigma,p}, \]
\[ f_1 w_{\sigma,p} = \varphi q^\sigma w_{\sigma,p+1}, \]
\[ f_2 w_{\sigma,p} = (1 - \sigma) \lambda_2 q^{p-1} [\mu_2 + p] \tilde{w}_{\sigma+1,p-1}, \]
\[ e_1 w_{\sigma,p} = \varphi^{-1} q^{-\sigma} [p + \mu_2] [\mu_1 + \mu_2 - p + 1 - \sigma] w_{\sigma,p-1}, \]
\[ e_2 w_{\sigma,p} = \sigma \lambda_2^{-1} q^{-p} w_{\sigma-1,p+1}, \]

(54)

with \( \sigma \in \{0, 1\} \) and \( p \in \{0, \ldots, l-1\} \).

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