Skyrme and Faddeev models in the low-energy limit of 4d Yang-Mills-Higgs theories

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Abstract

Firstly, we consider Yang–Mills theory on $\mathbb{R}^{3,1}$ with an adjoint Higgs field spontaneously breaking a compact gauge group $G$ to a subgroup $H$, so that the Higgs vacuum manifold forms the coset $G/H$. It is shown that in the low-energy limit, when the Higgs vacuum value is large, the 4d Yang–Mills–Higgs theory reduces to the Faddeev sigma model on $\mathbb{R}^{3,1}$ with $G/H$ as target. Its action contains the standard two-derivative sigma-model term as well as the four-derivative Skyrme-type term, which stabilizes solutions against scaling. Secondly, we put the Higgs field in the bi-fundamental representation of $G = U_+(N) \times U_-(N)$, realizing the simplest $A_2$-type quiver gauge theory. Breaking $G$ to $H = \text{diag}(G)$, the vacuum manifold $G/H \cong U(N)$ is a group. In this case, when the Higgs vacuum value is large, the 4d $A_2$-quiver gauge theory reduces to the Skyrme sigma model on $\mathbb{R}^{3,1}$ with $U(N)$ as target. Thus, both the Skyrme and the Faddeev model arise as effective field theories in the infrared of Yang–Mills–Higgs models.
1 Introduction and summary

In 1975, Faddeev introduced a (3+1)-dimensional SU(2)/U(1) coset sigma model that includes a term quartic in derivatives to stabilize classical solutions [1]. This model is similar to the Skyrme model [2], which features maps from \( \mathbb{R}^{3,1} \) into SU(2). Despite their similarity, these models are quite different from one another. Solitons of the Skyrme model have a point-like core and are supposed to describe baryons (see e.g. [3] for a review and references). On the other hand, solitons in the Faddeev model take the form of stable knotted strings characterized by the Hopf charge (homotopy class of maps \( S^3 \to S^2 \)) [4]-[7]. It was proposed by Faddeev and Niemi that the latter describe glueballs [8, 9], and applications in condensed matter systems have also been suggested [10, 11]. Exact solutions to the Faddeev model are not known, but numerical approximations with topological charge up to 16 have been found and studied (see e.g. [5, 6, 7, 12] and references therein). For more recent investigations of the Faddeev model and possible applications, see e.g. [13]-[17] and references therein.

There have been persistent attempts of different authors (see e.g. [8, 9, 18]) to show that 4d Yang–Mills theory can be reduced to the Faddeev model in a low-energy limit with the help of a generalized Cho ansatz [18] and arguments from quantum Yang–Mills theory, but various obstacles remain. Here we will show that the Faddeev model indeed appears in the low-energy limit if one adds an adjoint Higgs field to the 4d Yang–Mills theory for proper symmetry breaking. Our method is the adiabatic approach, which was used in field theory for the first time by Manton [19]. For a review of this approach see [20]; brief discussions can be found e.g. in [21]-[24].

The standard Skyrme model [2] supposedly describes pion degrees of freedom and does capture many properties of nuclei, which are identified with the topological solitons of the model (see e.g. [25, 26, 27] and references therein). Other mesons can be incorporated into an extended 4d Skyrme model, which is obtained from 5d Yang–Mills theory on an AdS-type manifold \( M^5 \) with boundary \( \partial M^5 = \mathbb{R}^{3,1} \) as derived from D-brane configurations in string theory and the holographic approach [28] (see e.g. [29, 30, 31] for reviews and references). This extended Skyrme model also arises in the adiabatic limit of the 5d Yang–Mills system on \( \mathbb{R}^{3,1} \times I \), where \( I \) is a short interval [32]. Here we will demonstrate that the standard (unextended) U(\( N \)) Skyrme model also emerges in the low-energy limit of the 4d Yang–Mills–Higgs model for the choice of a gauge group U(\( N \)) \times U(\( N \)) broken to its diagonal subgroup with a bi-fundamental Higgs field. To summarize, we demonstrate that both the Skyrme model and the Faddeev model occupy an infrared corner of four-dimensional Yang–Mills–Higgs theory.

2 Yang–Mills–Higgs model

**Notation.** On Minkowski space \( \mathbb{R}^{3,1} \ni x^\mu \) with the metric \((\eta_{\mu\nu}) = \text{diag}(-1,1,1,1)\), we consider a real scalar (Higgs) field \( \phi \), a gauge potential \( A = A_\mu \, dx^\mu \) and the Yang–Mills field \( F = dA + A \wedge A \) with components \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \), where \( \partial_\mu := \partial / \partial x^\mu \) and \( \mu, \nu = 0, 1, 2, 3 \). For the generators \( I_i \) of a compact connected gauge group \( G \) we use the standard normalization \( \text{tr}(I_i I_j) = -2\delta_{ij} \) with \( i, j = 1, \ldots, \dim G \). All fields \( A \), \( F \) and \( \phi \) live in the adjoint representation of the Lie algebra \( g = \text{Lie}G \).

**Lagrangian.** The standard Yang–Mills–Higgs (YMH) action functional reads

\[
S = -\frac{1}{8} \int_{\mathbb{R}^{3,1}} d^4 x \left\{ \text{tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2 D_\mu \phi D^\mu \phi \right) + \lambda V(\phi) \right\},
\] (2.1)
where \( e \) is the gauge coupling constant, \( D_\mu = \partial_\mu + [A_\mu, \cdot] \), \( V(\phi) \geq 0 \) is a Higgs potential and \( \lambda \geq 0 \) is a real constant. The energy density \( H \) of YMH configurations described by (2.1) is
\[
H = -\frac{1}{8} \text{tr} \left( \frac{e^2}{2} F_{\alpha a} F_{\alpha a} + 2 D_\alpha \phi D_\alpha \phi + \frac{e^2}{2} F_{ab} F_{ab} + 2 D_a \phi D_a \phi \right) + \frac{\lambda}{8} V(\phi) ,
\]
(2.2)

where \( a, b = 1, 2, 3 \). Here both \( V(\phi) \) and \( H \) are positive-semidefinite and gauge-invariant functions.

**Vacua.** A YMH vacuum configuration \((\hat{A}, \hat{F}, \hat{\phi})\) is defined by the vanishing of the energy density (2.2). This is achieved by
\[
\hat{F}_{\mu \nu} = 0 , \quad \hat{D}_\mu \hat{\phi} = 0 \quad \text{and} \quad V(\hat{\phi}) = 0 ,
\]
(2.3)
where the last equation defines the (gauge-invariant) Higgs vacuum manifold. The explicit form of \( V(\phi) \) is not essential; its function is to spontaneously break the gauge group \( G \) to a subgroup \( H \), which stabilizes any chosen reference vacuum \( \hat{\phi} \). Hence, only the coset \( G/H \) acts effectively on \( \hat{\phi} \), so its adjoint orbits identify the Higgs vacuum manifold with \( G/H \),
\[
\{ \phi \mid V(\phi) = 0 \} = \{ \phi = g \hat{\phi} g^{-1} \forall g \in G \} \cong G/H .
\]
(2.4)

### 3 Higgs vacuum manifold

**Geometry of \( G/H \).** The space \( G/H \) of adjoint orbits consists of the left cosets \( gH \) of \( G \) with the natural projection
\[
\pi : \quad G \overset{H}{\rightarrow} G/H \quad \text{with} \quad g \mapsto gH .
\]
(3.1)
Accordingly, we have the splitting
\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{g} = \text{Lie } G \quad \text{and} \quad \mathfrak{h} = \text{Lie } H ,
\]
(3.2)
and \( \mathfrak{m} \) is chosen to be orthogonal to \( \mathfrak{h} \) with respect to the Cartan–Killing form. At any point in \( G/H \) the tangent space is isomorphic to \( \mathfrak{m} \). We choose a basis \( \{ I_i \} \) for \( \mathfrak{g} \) in such a way that
\[
I_{\bar{i}} \quad \text{for} \quad \bar{i} = 1, \ldots, \dim \mathfrak{m} \quad \text{and} \quad I_i \quad \text{for} \quad i = \dim \mathfrak{m} + 1, \ldots, \dim \mathfrak{g}
\]
(3.3)
form bases for \( \mathfrak{m} \) and \( \mathfrak{h} \), respectively, with \( \text{tr}(I_i I_j) = 0 \).

\( G/H \) supports an orthonormal frame of one-forms \( \{ e^i \} \) locally giving the \( G \)-invariant metric as
\[
ds_{G/H}^2 = \delta_{\bar{i}j} e^\bar{i} e^j = \delta_{\bar{i}j} e_\alpha^\bar{i} e_\beta^j \text{d}X^\alpha \text{d}X^\beta =: g_{\alpha \beta} \text{d}X^\alpha \text{d}X^\beta \quad \text{for} \quad \alpha, \beta = 1, \ldots, \dim G/H ,
\]
(3.4)
where \( \{ X^\alpha \} \) is a set of local coordinates of a point \( X \in G/H \), and \( \partial_\alpha = \partial/\partial X^\alpha \) will denote derivatives with respect to them.

**Canonical connection.** On the principal \( H \)-bundle (3.1) there exists a unique \( G \)-equivariant connection, the so-called canonical connection (see e.g. [33]-[35]),
\[
A_{G/H} = e^i I_i = e^i_\alpha I_i \text{d}X^\alpha .
\]
(3.5)
The one-forms $e^i = (e^i, e^j)$ obey the Maurer–Cartan equations,
\[ \text{de}^i = - f^i_{jk} e^j \wedge e^k - \frac{1}{2} f^i_{jk} e^j \wedge e^k \quad \text{and} \quad \text{de}^i = - \frac{1}{2} f^i_{jk} e^j \wedge e^k, \] (3.6)
where we assumed the reductivity property $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ for $G/H$. The curvature of the canonical connection (3.5) follows as
\[ F_{G/H} = - \frac{1}{2} f^i_{jk} I_I e^j \wedge e^k = - \frac{1}{2} f^i_{jk} I_I e^j e^k \wedge dX^\alpha \wedge dX^\beta. \] (3.7)

**Variation of $\phi$.** By letting $X^\alpha$ run over $G/H$ we obtain a local section $\phi(X^\alpha)$ of the bundle (3.1). We need to look at the infinitesimal change of this section induced by a small variation of the coordinates $X^\alpha$. According to the split (3.2) of $T_X(G) \cong \mathfrak{g}$, the corresponding tangent vector is the sum of two components,
\[ \partial_\alpha \phi = \delta_\alpha \phi + [\phi, \epsilon_\alpha] \quad \text{with} \quad \delta_\alpha \phi \in \pi^* T_X(G/H) \cong \mathfrak{m} \quad \text{and} \quad [\phi, \epsilon_\alpha] = \delta_\epsilon_\alpha \phi \in T_X(H) \cong \mathfrak{h}, \] (3.8)
where $\epsilon_\alpha$ are $\mathfrak{h}$-valued gauge parameters generating the (compensating) infinitesimal gauge transformations which after the $\partial_\alpha$-shift bring $\partial_\alpha \phi$ back to $\pi^* T_X(G/H) \cong \mathfrak{m}$.

### 4 Faddeev model in the infrared limit of 4d YMH

**Dependence on $x^\mu$.** Now we return to Yang–Mills–Higgs theory on $\mathbb{R}^{3,1}$. The spontaneous symmetry breaking introduces a mass scale $M$ because the Higgs vacuum value $\hat{\phi}$ has the dimension of mass. For small excitations around $G/H$ with energies much lower than $M$, the Higgs field $\phi(x)$ can be considered as a map
\[ \phi: \mathbb{R}^{3,1} \to G/H \subset \mathfrak{g}. \] (4.1)

The moduli-space approximation then postulates that all fields depend on the spacetime coordinates $x = \{x^\mu\}$ only via coordinates $X^\alpha = X^\alpha(x)$ on $G/H$ (see e.g. [19]–[24] and references therein). By substituting $\phi(X^\alpha(x))$ and $A(X^\alpha(x))$ into the initial action (2.1), we obtain an effective field theory describing small fluctuations around the vacuum manifold.

**Two-derivative part of effective action.** Multiplying (3.8) by $\partial_\mu X^\alpha$, we obtain
\[ \partial_\mu \phi = (\partial_\mu X^\alpha) \delta_\alpha \phi + [\phi, \epsilon_\mu], \] (4.2)
where $\epsilon_\mu = (\partial_\mu X^\alpha) \epsilon_\alpha$ is the pull-back of $\epsilon_\alpha$ from $G/H$ to $\mathbb{R}^{3,1}$. It immediately follows that
\[ D_\mu \phi \equiv \partial_\mu \phi + [A_\mu, \phi] = (\partial_\mu X^\alpha) \delta_\alpha \phi + [A_\mu - \epsilon_\mu, \phi]. \] (4.3)

We see that $D_\mu \phi$ are tangent\(^1\) to $C^\infty(\mathbb{R}^{3,1}, G/H)$ if
\[ A_\mu = \epsilon_\mu. \] (4.4)

\(^1\)This is a key requirement of the adiabatic approach. It is necessary for the description of small fluctuations around the initial moduli space when the dynamical fields are collective coordinates.
Substituting (4.3) with (4.4) into the (2.1), we obtain

$$S_{\text{kin}} = -\frac{1}{4} \int_{\mathbb{R}^{3,1}} d^4x \, \eta^{\mu
u} \text{tr}(D_{\mu}\phi D_{\nu}\phi) = \frac{M^2}{2} \int_{\mathbb{R}^{3,1}} d^4x \, \eta^{\mu
u} g_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta$$

(4.5)

where

$$g_{\alpha\beta} = -\frac{1}{2M^2} \text{tr}(\delta_\alpha \phi \delta_\beta \phi) = \delta_{ij} e_i^\alpha e_j^\beta$$

(4.6)

are the components of the metric (3.4) on $G/H$ pulled back to $\mathbb{R}^{3,1}$, so $g_{\alpha\beta}(X^\gamma(x))$ now depend on $x$. We introduced the mass scale $M$ to render $g_{\alpha\beta}$ dimensionless. Thus, this part of the action (2.1) reduces to the standard non-linear sigma model on $\mathbb{R}^{3,1}$ with the coset $G/H$ as its target.

**Four-derivative part of effective action.** As discussed earlier, the potential term in (2.1) vanishes since $\phi(x)$ takes values in the vacuum manifold $G/H$. For calculating the first term in (2.1), we identify $\epsilon_\alpha$ with the canonical connection (3.5) in bundle (3.1) (cf. [21, 22]),

$$\epsilon_\alpha = A_\alpha = e_i^\alpha I_i \Rightarrow A_\mu = \epsilon_\mu = (\partial_\mu X^\alpha) e_i^\alpha I_i \ .$$

(4.7)

Then for the curvature of $A$ we obtain

$$F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu = -\frac{1}{2} f^i_{jk} I_i e_j^\alpha e_k^\beta \partial_\mu X^\alpha \partial_\nu X^\beta \, dx^\mu \wedge dx^\nu ,$$

(4.8)

allowing one to extract the components $F_{\mu\nu}$. Substituting (4.8) into (2.1) we obtain

$$S_{\text{Fad}} = -\frac{1}{8\epsilon^2} \int_{\mathbb{R}^{3,1}} d^4x \, \text{tr}(F_{\mu\nu} F^{\mu\nu}) = \frac{1}{4\epsilon^2} \int_{\mathbb{R}^{3,1}} d^4x \, \delta_{ij} f^i_{jk} f^j_{lm} e_l^\alpha e_k^\beta e_m^\gamma e_n^\delta \partial_\mu X^\alpha \partial_\nu X^\beta \partial_\rho X^\gamma \partial_\sigma X^\delta ,$$

(4.9)

where $\partial^\mu := \eta^{\mu\nu} \partial_\nu$. Thus, in the infrared limit the Yang–Mills–Higgs action (2.1) is reduced to the Faddeev action,

$$S_{\text{eff}} = \int_{\mathbb{R}^{3,1}} d^4x \left\{ \frac{M^2}{2} g_{\alpha\beta} \partial_\mu X^\alpha \partial_\mu X^\beta + \frac{1}{4\epsilon^2} \delta_{ij} f^i_{jk} f^j_{lm} e_l^\alpha e_k^\beta e_m^\gamma e_n^\delta \partial_\mu X^\alpha \partial_\nu X^\beta \partial_\rho X^\gamma \partial_\sigma X^\delta \right\} ,$$

(4.10)

generalized from the target SU($n+1$)/U(1)$^n$ [1, 4, 36] to arbitrary cosets $G/H$. For the maximal adjoint breaking $H = T$, where $T$ denotes the maximal torus in $G$, the effective action tolerates residual Abelian gauge fields $A_\mu$ since

$$A_\mu = \epsilon_\mu + gA_\mu g^{-1} \Rightarrow [A_\mu - \epsilon_\mu, \phi] = g [A_\mu, \hat{\phi}] g^{-1} = 0 .$$

(4.11)

However, a consideration of this more general case is beyond the scope of our paper.
5 A₂-quiver gauge theory

Fields. Is it possible to obtain not only the Faddeev model but also the original Skyrme model from Yang–Mills–Higgs theory in four dimensions? To achieve this, we should like to realize a group manifold, say U(N), as a coset G/H. The simplest way to do this is to choose

\[ G = U_+(N) \times U_-(N) \quad \text{and} \quad H = \text{diag}(G) \quad \text{with} \quad N \geq 2. \]  \hspace{1cm} (5.1)

This breaking is achieved by a bi-fundamental complex Higgs field. Let \( \{ I_i \} \) be a basis of the Lie algebra \( \mathfrak{g} = \text{Lie}G = u_+(N) \oplus u_-(N) \) realized as \( 2N \times 2N \) block-diagonal matrices. We use the normalization \( \text{tr}(I_i I_j) = -\frac{1}{2} \delta_{ij} \) for \( i = 1, \ldots, 2N^2 \). Introducing indices \( i_+ = 1, \ldots, N^2 \) and \( i_- = N^2+i_+ \), we split \( \{ I_i \} = \{ I_{i_+}, I_{i_-} \} \) where \( \{ I_{i_\pm} \} \) generates \( U_\pm(N) \). Then a gauge potential \( A \) and the gauge field \( F \) can be written as

\[ A = A^+ \oplus A^- = A^{i_+} I_{i_+} \oplus A^{i_-} I_{i_-}, \quad \text{and} \quad F = F^+ \oplus F^- = F^{i_+} I_{i_+} \oplus F^{i_-} I_{i_-}. \]  \hspace{1cm} (5.2)

Accordingly, the covariant derivative of the bi-fundamental \( N \times N \) Higgs field \( \phi \) reads (see e.g. [37])

\[ D_\mu \phi = \partial_\mu \phi + A^+_{\mu} \phi - \phi A^-_{\mu}. \]  \hspace{1cm} (5.3)

Gauge transformations. We denote by \( \mathcal{G} \) the infinite-dimensional group \( C^\infty(\mathbb{R}^{3,1}, G) \) of gauge transformations which are parametrized by \( g = (g_+, g_-) \) with \( g_\pm \in C^\infty(\mathbb{R}^{3,1}, U_\pm(N)) \). Then \( A^\pm \) and \( \phi \) are transformed as

\[ A^\pm \mapsto g(A^\pm) = g_\pm A^\pm g_\pm^{-1} + g_\pm \text{d}g_\pm^{-1} \quad \text{and} \quad \phi \mapsto g \phi = g_+ \phi g_-^{-1}. \]  \hspace{1cm} (5.4)

For the infinitesimal action of \( \mathcal{G} \) we have

\[ A^\pm \mapsto \delta_\epsilon A^\pm = \text{d}\epsilon^\pm + [A^\pm, \epsilon^\pm] \quad \text{and} \quad \delta_\epsilon \phi = \epsilon^+ \phi - \phi \epsilon^-, \]  \hspace{1cm} (5.5)

where \( \epsilon = \epsilon^+ \oplus \epsilon^- \in \text{LieG} \), and \( \epsilon^\pm \) are \( u_\pm(N) \)-valued gauge parameters.

Lagrangian. We consider the YMH action functional

\[ S = -\frac{1}{2} \int_{\mathbb{R}^{3,1}} \text{d}^4 x \text{tr}\left\{ \frac{1}{e^2} F_{\mu \nu}^+, F_{\mu \nu}^+ + \frac{1}{e^2} F_{\mu \nu}^-, F_{\mu \nu}^- \right\} + 2 \left( D_\mu \phi \right)^\dagger D^\mu \phi + (M^2 \mathbf{1}_N - \phi^\dagger \phi)^2 \right\}, \]  \hspace{1cm} (5.6)

where the last term in (5.6) is the Higgs potential including a mass scale \( M \). This action describes the simplest quiver gauge theory (see e.g. [37] and references therein) corresponding to the quiver

\[ A_2 : \quad \mathbb{C}^N_- \xrightarrow{\phi} \mathbb{C}^N_+ \]  \hspace{1cm} (5.7)

where \( \mathbb{C}^N_\pm \) at the two vertices are the fundamental representation spaces of \( U_\pm(N) \), and the arrow \( \phi \in \text{Hom}(\mathbb{C}^N_-, \mathbb{C}^N_+) = \text{Mat}(N, \mathbb{C}) \) denotes the map between them. For more details and references see [37].

Vacua. The energy density of YMH configurations described by the action (5.6) has the form (2.2), and vacuum configurations are described by (2.3), which implies

\[ \phi^\dagger \phi = M^2 \mathbf{1}_N. \]  \hspace{1cm} (5.8)
Taking as a reference Higgs vacuum the solution $\hat{\phi} = M 1_N$, the gauge transformations (5.4) generate the Higgs vacuum configurations

$$\phi = g_+ \hat{\phi} g_-^{-1} = M g_+ g_-^{-1}.$$  

(5.9)

Since each of the latter is clearly inert under a right action of $\text{diag}(G)$,

$$(g_+, g_-) \mapsto (g_+ h, g_- h) \quad \text{for} \quad (h, h) \in \text{diag}(G) \cong U_{\text{diag}}(N) = H,$$  

(5.10)

it follows that the Higgs vacua are parametrized by the group manifold

$$G/H = U_+(N) \times U_-(N)/U_{\text{diag}}(N) \cong U(N),$$  

(5.11)

and we have the decomposition

$$\mathfrak{g} = u_+(N) \oplus u_-(N) = \mathfrak{h} \oplus \mathfrak{m} = u(N)_{\text{diag}} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{h} = \{(h, h) \mid h \in u(N)\}. \quad (5.12)$$

**Geometry of $G/H$.** The geometry of a group manifold considered as a homogeneous space has some characteristic features (see e.g. [38, 39, 40]) which we briefly describe here. In the split (5.12), $\mathfrak{m}$ is not necessarily orthogonal to $\mathfrak{h}$ with respect to the Cartan–Killing form. In fact, there are three natural reductive decompositions of $\mathfrak{g}$ in (5.12) with the following versions of $\mathfrak{m}$:

$$\mathfrak{m}_0 = \{(m, -m)\}, \quad \mathfrak{m}_+ = \{(m, 0)\}, \quad \mathfrak{m}_- = \{(0, -m)\}, \quad \text{with} \quad m \in u(N). \quad (5.13)$$

The first case yields $G/H$ as a symmetric space with $\mathfrak{m}_0$ orthogonal to $\mathfrak{h}$. With the choice $\mathfrak{m}_+$ or $\mathfrak{m}_-$ the coset (5.11) becomes a nonsymmetric homogeneous manifold. Obviously, $\mathfrak{m} \cong u(N)$ in all three cases. The choices of $\mathfrak{m}_0$, $\mathfrak{m}_+$ and $\mathfrak{m}_-$ correspond to the gauges $g_- = g_+^{-1}$, $g_-=1$ and $g_+=1$, respectively, which determine different coset representatives, i.e. sections of the bundle

$$\pi : \quad G = U_+(N) \times U_-(N) \longrightarrow U_+(N) \times U_-(N)/U_{\text{diag}}(N) = G/H.$$  

(5.14)

As in Section 3, we introduce a basis $\{I_i\}$ for $\mathfrak{m}$ and $\{I_{\hat{i}}\}$ for $\mathfrak{h}$, where $i = 1, \ldots, N^2$ and $\hat{i} = N^2+1, \ldots, 2N^2$. We have an orthonormal frame of one-forms $\{e^i\}$ on $G/H$, the metric (3.4) and the canonical connection (3.5) for all three cases $\mathfrak{m}_0$, $\mathfrak{m}_+$ and $\mathfrak{m}_-$. However, the Maurer–Cartan equations (3.6) now take the form

$$\mathfrak{m}_0 : \quad de^\hat{i} = -f_{\hat{i}jk} e^j \wedge e^k \quad \text{and} \quad de^\hat{i} = -\frac{1}{2} f_{\hat{i}jk} e^j \wedge e^k - \frac{1}{2} f_{\hat{i}jk} e^j \wedge e^k,$$  

(5.15)

$$\mathfrak{m}_+ : \quad de^i = -f_{ijk} e^j \wedge e^k - \frac{1}{2} f_{ijk} e^j \wedge e^k \quad \text{and} \quad de^i = -\frac{1}{2} f_{ijk} e^j \wedge e^k,$$  

(5.16)

$$\mathfrak{m}_- : \quad de^\hat{i} = -f_{\hat{i}jk} e^j \wedge e^k + \frac{1}{2} f_{\hat{i}jk} e^j \wedge e^k \quad \text{and} \quad de^\hat{i} = -\frac{1}{2} f_{\hat{i}jk} e^j \wedge e^k.$$  

(5.17)

Furthermore, on the group manifold (5.11) one can introduce a family of connections

$$A_{\mathcal{G}/H}^\kappa = \kappa e^\hat{i} I_i \quad \text{with} \quad \kappa \in \mathbb{R}$$  

(5.18)

and curvature

$$\mathcal{F}_{\mathcal{G}/H}^\kappa = \frac{1}{2} \kappa(\kappa-1)f_{\hat{i}jk} I_i e^j \wedge e^k - \frac{1}{2} \kappa f_{\hat{i}jk} I_i e^j \wedge e^k.$$  

(5.19)
For the cases $m_{\pm}$ the last term in (5.19) vanishes. The connection (5.18) is the unique $G$-equivariant connection on the bundle (5.14) [38, 39].

**Variation of $\phi$.** In the following we adopt the gauge $g_-=1$ fixing $m=m_+$, so $f^i_{jk}=0$ in (5.19). Then, we can embed the (dimensionless) bi-fundamental Higgs into $U_+(N) \times G/H$,

$$
\frac{1}{M} \phi =: \varphi \iff (\varphi, 1_N) \implies g_{\varphi}(\varphi, 1_N) \equiv (g_+, g_-)(\varphi, 1_N) = (g_+ \varphi g_-^{-1}, 1_N)(g_-, g_-),
$$

(5.20)

where the last equality restores the gauge. Hence, the $G$-action

$$
\varphi \mapsto g_{\varphi} = g_+ \varphi g_-^{-1}
$$

(5.21)

maps one coset representative into another one. The infinitesimal version of this map yields, similarly to (3.8),

$$
\partial_\alpha \phi - \phi \epsilon_\alpha = \delta_\alpha \phi \quad \text{with } \phi^{-1} \delta_\alpha \phi \in \pi^*T_\phi(G/H) \cong m_+ \quad \text{and } \epsilon_\alpha \in T_\phi(H) \cong u(N). \quad (5.22)
$$

We note that $\partial_\alpha \varphi - \varphi \epsilon_\alpha$ with gauge parameters $\epsilon_\alpha$ are the covariant derivatives of the section $\varphi$ of the bundle (5.14), and $\epsilon_\alpha dX^\alpha$ can be identified with the connection (5.18) on this bundle (cf. [21, 22]),

$$
\epsilon_\alpha dX^\alpha = \epsilon_\mu dx^\mu = \kappa \epsilon_\mu^I I_i = \kappa \varphi^{-1} d\varphi = \kappa \varphi^{-1} d\phi.
$$

(5.23)

### 6 Skyrme model in the infrared limit of 4d YMH theory

The derivation of the Skyrme model as an effective theory for 4d YMH theory (5.6) is similar to the derivation of the Faddeev model from the YMH action (2.1). The main difference is that now the Higgs vacuum manifold $\mathcal{M} = G/H$ is a group manifold, whose geometry was described in Section 5.

According to the philosophy of the adiabatic method, we assume that the gauge potentials $A_{\pm}$ and the complex Higgs field depend on the $\mathbb{R}^{3,1}$ coordinates $x$ only via coordinates $X^\alpha = X^\alpha(x^\mu)$ on $U(N)$ and substitute $A_{\pm}(X^\alpha(x))$ and $\phi(X^\alpha(x))$ into the action (5.6) by using results of Section 5.

**Kinetic term.** Multiplying (5.22) by $\partial_\mu X^\alpha$, we obtain

$$
D_\mu \phi \equiv \partial_\mu \phi + A^+_\mu \phi - \phi A^-_\mu = (\partial_\mu X^\alpha) \delta_\alpha \phi + A^+_\mu \phi - \phi (A^-_\mu - \epsilon_\mu), \quad (6.1)
$$

where $\epsilon_\mu = (\partial_\mu X^\alpha) \epsilon_\alpha$ is the pull-back of $\epsilon_\alpha$ from $U(N)$ to $\mathbb{R}^{3,1}$. To render $\phi^{-1} D_\mu \phi$ tangent to $C^\infty(\mathbb{R}^{3,1}, G/H)$ we choose the simplest option

$$
A^+_\mu = 0 \quad \text{and} \quad A^-_\mu = \epsilon_\mu. \quad (6.2)
$$

Furthermore, (5.22) and (5.23) imply that

$$
\epsilon_\alpha = \kappa \varphi^{-1} \partial_\alpha \phi \quad \Rightarrow \quad \delta_\alpha \phi = (1-\kappa) \partial_\alpha \phi \quad \Rightarrow \quad D_\mu \phi = (1-\kappa) \partial_\mu \phi. \quad (6.3)
$$

Substituting (6.1)–(6.3) into (5.6), we obtain

$$
S_{\text{kin}} = -\int_{\mathbb{R}^{3,1}} d^4 x \, \eta^{\mu \nu} \text{tr}\left\{ (D_\mu \phi)^\dagger D_\nu \phi \right\} = -\frac{1}{2} f_\pi^2 \int_{\mathbb{R}^{3,1}} d^4 x \, \eta^{\mu \nu} \text{tr}(L_\mu L_\nu), \quad (6.4)
$$

7
with
\[ L_\mu := \varphi^{-1} \partial_\mu \varphi \quad \text{and} \quad \frac{1}{4} f_\pi^2 = (\varphi - 1)^2 M^2, \]  
where \( f_\pi \) may be interpreted as the pion decay constant. Thus, this part of the action (5.6) reduces to the standard non-linear sigma model on \( \mathbb{R}^{3,1} \) with the group manifold \( U(N) \) as its target space.

**Skyrme term.** For calculating the \( F^2 \)-terms in (5.6) we use (6.2) and (5.23). For the curvature \( F^- \) of \( A^- \) we obtain
\[ F^- = dA^- + A^- \wedge A^- = \varphi(\varphi - 1) \varphi^{-1} d\varphi \wedge \varphi^{-1} d\varphi = \frac{1}{2} \varphi(\varphi - 1) [L_\mu, L_\nu] dx^\mu \wedge dx^\nu \]  
(6.6)
since \( A^-_\mu = \epsilon_\mu = \varphi^{-1} \partial_\mu \varphi \) after the pull-back to \( \mathbb{R}^{3,1} \). Substituting \( F^+_{\mu\nu} = 0 \) and (6.6) into (5.6), we obtain
\[ S_{\text{Sky}} = -\frac{1}{2e^2} \int_{\mathbb{R}^{3,1}} d^4 x \, \text{tr} \{ F^-_{\mu\nu} F^-^{\mu\nu} \} = -\frac{1}{32\zeta^2} \int_{\mathbb{R}^{3,1}} d^4 x \eta^{\mu\lambda} \eta^{\nu\sigma} \text{tr} ([L_\mu, L_\nu][L_\lambda, L_\sigma]), \]  
(6.7)
where
\[ \frac{1}{32\zeta^2} = \frac{\varepsilon^2(\varphi - 1)^2}{8e^2}, \]  
(6.8)
and \( \zeta \) is the dimensionless Skyrme parameter. Hence, in the infrared limit the Yang–Mills–Higgs action (5.6) is reduced to the action of the Skyrme model,
\[ S_{\text{eff}} = -\int_{\mathbb{R}^{3,1}} d^4 x \left\{ \frac{f_\pi^2}{4} \eta^{\mu\nu} \text{tr}(L_\mu L_\nu) + \frac{1}{32\zeta^2} \eta^{\mu\lambda} \eta^{\nu\sigma} \text{tr} ([L_\mu, L_\nu][L_\lambda, L_\sigma]) \right\}. \]  
(6.9)
Thus, both Skyrme and Faddeev models appear as effective field theories in the infrared of Yang–Mills–Higgs models.

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