Boolean algebras and Lubell functions

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Abstract

Let $2^{[n]}$ denote the power set of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. A collection $\mathcal{B} \subseteq 2^{[n]}$ forms a $d$-dimensional Boolean algebra if there exist pairwise disjoint sets $X_0, X_1, \ldots, X_d \subseteq [n]$, all non-empty with perhaps the exception of $X_0$, so that $\mathcal{B} = \left\{ X_0 \cup \bigcup_{i \in I} X_i : I \subseteq [d] \right\}$. Let $b(n, d)$ be the maximum cardinality of a family $\mathcal{F} \subseteq 2^{[n]}$ that does not contain a $d$-dimensional Boolean algebra. Gunderson, Rödl, and Sidorenko proved that $b(n, d) \leq c_d n^{1 - 1/2^d} \cdot 2^n$ where $c_d = 10^{d^2 - 2^d - d}$.

In this paper, we use the Lubell function as a new measurement for large families instead of cardinality. The Lubell value of a family of sets $\mathcal{F}$ with $\mathcal{F} \subseteq 2^{[n]}$ is defined by $h_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$. We prove the following Turán type theorem. If $\mathcal{F} \subseteq 2^{[n]}$ contains no $d$-dimensional Boolean algebra, then $h_n(\mathcal{F}) \leq 2(n + 1)^{1 - 2^{1-d}}$ for sufficiently large $n$. This result implies $b(n, d) \leq C n^{-1/2^d} \cdot 2^n$, where $C$ is an absolute constant independent of $n$ and $d$. With some modification, the ideas in Gunderson, Rödl, and Sidorenko’s proof can be used to obtain this result. We apply the new bound on $b(n, d)$ to improve several Ramsey-type bounds on Boolean algebras. We also prove a canonical Ramsey theorem for Boolean algebras.

1 History

Given a ground set $[n]$ with $[n] = \{1, 2, \ldots, n\}$, let $2^{[n]}$ denote the power set of $[n]$.

Definition 1. A collection $\mathcal{B} \subseteq 2^{[n]}$ forms a $d$-dimensional Boolean algebra if there exist pairwise disjoint sets $X_0, X_1, \ldots, X_d \subseteq [n]$, all non-empty with perhaps the exception of $X_0$, so that

$$\mathcal{B} = \left\{ X_0 \cup \bigcup_{i \in I} X_i : I \subseteq [d] \right\}.$$ 

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We view all $d$-dimensional Boolean algebras as copies of a single structure $B_d$. Thus, a $d$-dimensional Boolean algebra forms a copy of $B_d$, and a family $F \subseteq 2^{[n]}$ is $B_d$-free if it does not contain a copy of $B_d$.

The starting point of this paper is to explore the question of how large a family of sets can be if it does not contain a $d$-dimensional Boolean algebra. The first result in this area is due to Sperner. The simplest example of a non-trivial Boolean algebra, $B_1$, is a pair of sets, one properly contained in the other. Sperner’s theorem can be restated as follows. If $F \subseteq 2^{[n]}$ is $B_1$-free, then $|F| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Erdős and Kleitman [4] considered the problem of determining the maximum size of a $B_2$-free family in $2^{[n]}$. General extremal problems on Boolean algebras of sets were most recently studied by Gunderson, Rödl, and Sidorenko in [12].

Given an $n$-element set $X$ and a positive integer $d$, define $b(n,d)$ to be the maximum cardinality of a $B_d$-free family contained in $2^{[n]}$. In [12], the following bounds on $b(n,d)$ are proved:

$$n^{-\frac{1+o(1)}{d+1}} \cdot 2^n \leq b(n,d) \leq 10^d 2^{-2^{1-d}} d^{d-2^d} n^{-1/2^d} \cdot 2^n. \tag{1}$$

In the lower bound of (1), the $o(1)$ term represents a function that tends to 0 as $n$ grows for each fixed $d$. A rough overview of the proof of the upper bound on $b(n,d)$ in [12] follows. Given $F \subseteq 2^{[n]}$, the set $[n]$ is partitioned into $d$ parts $X_1, \ldots, X_d$ whose sizes differ by at most 1. Next, probabilistic techniques are used to select a family of chains $C_1, \ldots, C_d$, where each $C_i$ is a chain in $2^{X_i}$ of length at least $2^{\lfloor \sqrt{n/d} \rfloor}$. Let $F_0$ be the set of all $A \in F$ such that $A \cap X_i \in C_i$ for each $i$. The chains $C_1, \ldots, C_d$ are chosen so that $F_0$ is large. Let $H$ be the $d$-partite $d$-uniform hypergraph with parts $C_1, \ldots, C_d$ where $E(H) = \{\{A \cap X_1, \ldots, A \cap X_d\}: A \in F_0\}$. If $H$ contains a copy of the complete $d$-partite $d$-uniform hypergraph with 2 vertices in each part (denoted by $K^{(d)}(2,\ldots,2)$), then $F_0$ contains a $d$-dimensional Boolean algebra. A result of Erdős [3] implies that for $n$ sufficiently large in terms of $d$, each $n$-vertex $d$-uniform hypergraph with at least $n^{d-2^{1-d}}$ edges contains a copy of $K^{(d)}(2,\ldots,2)$. Since $|E(H)| = |F_0|$ and $|F_0|$ is large enough that Erdős’s result applies, it follows that $F$ contains a $d$-dimensional Boolean algebra.

With some work, the argument in [12] can be modified to eliminate the large multiplicative factor in inequality (1) that is asymptotic to $(10d)^d$. The most important modification is to exploit that $H$ is $d$-partite, and in this case fewer edges force a copy of $K^{(d)}(2,\ldots,2)$. A second, more technical modification is also necessary: the chains should be chosen to have length at least $2^{\lfloor \sqrt{n/d} \rfloor}$. In this paper, we obtain this improvement directly, by extending a well-known result on affine cubes to Boolean algebras.

**Definition 2.** A set $H$ of integers is called a $d$-dimensional affine cube or an affine $d$-cube if there exist $d+1$ integers $x_0 \geq 0$, and $x_1, \ldots, x_d \geq 1$, such that

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [d] \right\}.$$
A set of non-negative integers is $B_d$-free if it contains no affine $d$-cube.

In one of the first Ramsey-type results, Hilbert [13] showed that for all $d$ and $k$, there exists an integer $n$ such that every $k$-coloring of $[n]$ contains a monochromatic $d$-dimensional affine cube. Nearly 80 years later, Szemerédi [18] strengthened Hilbert’s result by proving a density version: for each positive $\varepsilon$ and for each integer $d$, there exists an integer $n$ such that every $\varepsilon$-coloring of $[n]$ contains a monochromatic $d$-dimensional affine cube. Nearly 80 years later, Szemerédi [18] strengthened Hilbert's result by proving a density version: for each positive $\varepsilon$ and for each integer $d$, there exists an integer $n$ such that every $\varepsilon$-coloring of $[n]$ contains a monochromatic $d$-dimensional affine cube. Nearly 80 years later, Szemerédi [18] strengthened Hilbert's result by proving a density version: for each positive $\varepsilon$ and for each integer $d$, there exists an integer $n$ such that every $\varepsilon$-coloring of $[n]$ contains a monochromatic $d$-dimensional affine cube. Nearly 80 years later, Szemerédi [18] strengthened Hilbert's result by proving a density version: for each positive $\varepsilon$ and for each integer $d$, there exists an integer $n$ such that every $\varepsilon$-coloring of $[n]$ contains a monochromatic $d$-dimensional affine cube.

Graham [6] strengthened Szemerédi’s cube lemma by reducing the bound on $|A|$ which suffices to force a $d$-dimensional affine cube (see also [7]). Let $b'(n,d)$ be the maximum size of a $B_d$-free subset of $\{0,\ldots,n\}$. Using similar methods as in [6] and [7], problem 14.12 in [15] contains a proof that $b'(n,d) < (4(n+1))^{1-2^{-d}}$ when $n$ is sufficiently large in terms of $d$. Gunderson and Rödl [11] improved the coefficient, showing that the following holds for sufficiently large $n$:

$$b'(n,d) \leq 2(n+1)^{1-2^{-d}}. \quad (2)$$

If $F \subseteq \{0,\ldots,n\}$ and $F = \{ A \in 2^{[n]} : |A| \in F \}$, then $F$ contains an affine $d$-cube if and only if $F$ contains a $d$-dimensional Boolean algebra. Hence, constructions that yield lower bounds on $b'(n,d)$ also yield lower bounds on $b(n,d)$. Similarly, upper bounds on $b(d,n)$ translate to upper bounds on $b'(d,n)$. The connection between large $B_d$-free families in $2^{[n]}$ and large $B_d$-free families in $\{0,\ldots,n\}$ is simplified by using the Lubell function.

**Definition 3.** Given a family $F \subseteq 2^{[n]}$, we define the Lubell function $h_n(F)$ as follows:

$$h_n(F) = \sum_{F \subseteq [n]} \frac{1}{\binom{n}{|F|}}.$$

With this definition in mind, we see that

$$b'(n,d) \leq \max \{ h_n(F) : F \text{ is } B_d\text{-free} \}. \quad (3)$$

The Lubell function has been widely used in the study of extremal families of sets forbidding given subposets (see [1, 8, 9, 10, 14, 17]) and in Turán problems on Non-uniform hypergraphs [16]. The advantage of using the Lubell function is its convenient probabilistic interpretation. Suppose that $C$ is full-chain in $2^{[n]}$ chosen uniformly at random, i.e. $C = \{ \emptyset, \{ i_1 \}, \{ i_1, i_2 \}, \ldots, [n] \}$. Let $X$ be the random variable $X = |C \cap F|$. Then we have that $E(X) = h_n(F)$. This interpretation allows us to use tools from the probability theory (such as conditional expectation and convexity) and simplify many counting arguments.

**Theorem 1.** There is a positive constant $C$, independent of $d$, such that for every $d$ and all sufficiently large $n$, the following is true.

$$b(n,d) \leq Cn^{-1/2^d} \cdot 2^n. \quad (4)$$

Our next theorem extends well-known ideas in Graham’s proof of Szemerédi’s cube lemma from integers to set families. The Lubell function plays a critical role and replaces cardinality as our metric for the size of a set family.
Theorem 2. For \( d \geq 1 \), define \( \alpha_d(n) \) recursively as follows. Let \( \alpha_1(n) = 1 \) and \( \alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{3}{4}} \) for \( d \geq 2 \). For \( n \geq d \geq 1 \) if a family \( \mathcal{F} \subseteq 2^{[n]} \) satisfies \( h_n(\mathcal{F}) > \alpha_d(n) \), then \( \mathcal{F} \) contains a \( d \)-dimensional Boolean algebra.

The rest of the paper is organized as follows. In section 2, we prove Theorem 1 and Theorem 2. In section 3, we prove several Ramsey-type results.

## 2 Proofs of Theorems 1 and 2

Note that the sequence \( \{\alpha_d(n)\}_{d \geq 1} \) satisfies

\[
\left( \frac{\alpha_{d+1}(n)}{2} \right) = n\alpha_d(n) \quad \text{for} \quad d \geq 1.
\] (5)

The function \( \alpha_d(n) \) is used in [11] implicitly. Note that for any fixed \( d \geq 2 \), \( \alpha_d(n) \) is an increasing function of \( n \). We have \( \alpha_1(n) = 1 \), \( \alpha_2(n) = \frac{1}{2} + \sqrt{2n + \frac{3}{4}} \). For \( d \geq 3 \), it was implicitly shown in [11] that

\[ \alpha_d(n) \leq 2^{1-2^{-d}} \left( \sqrt{n+1} + 1 \right) 2^{-2^{-d}} \quad \text{for} \quad n + 1 \geq 2^{d^2-1}/(2^{d-1}-1) \]

and

\[ \alpha_d(n) \leq 2(n+1)^{1-2^{-d}} \quad \text{for} \quad n + 1 \geq (2^d - 2/\ln 2)^2. \]

### Proof of Theorem 2:

The proof is by induction on \( d \). For the initial case \( d = 1 \), we have \( h_n(\mathcal{F}) > \alpha_1(n) = 1 \). Let \( X \) be the number of sets in both \( \mathcal{F} \) and a random full chain. Then \( E(X) = h_n(\mathcal{F}) > 1 \). There is an instance of \( X \) satisfying \( X \geq 2 \). Let \( A \) and \( B \) be two sets in both \( \mathcal{F} \) and a full chain. Clearly, the pair \( \{A, B\} \) forms a copy of \( B_1 \).

Assume that the statement is true for \( d \). For \( d+1 \), suppose \( \mathcal{F} \subseteq 2^{[n]} \) satisfies \( h_n(\mathcal{F}) > \alpha_{d+1}(n) \). Let \( X \) be the number of sets in both \( \mathcal{F} \) and a random full chain. By the convex inequality, we have

\[
E \left( \frac{X}{2} \right) \geq \left( \frac{EX}{2} \right) > \left( \frac{\alpha_{d+1}(n)}{2} \right) = n\alpha_d(n).
\]

For each subset \( S \) of \([n]\), let \( \mathcal{F}_S = \{A \in \mathcal{F}: A \cap S = \emptyset \text{ and } A \cup S \in \mathcal{F}\} \). We show that for some non-empty set \( S \), the Lubell function of \( \mathcal{F}_S \) in \( 2^{[n]} \setminus S \) exceeds \( \alpha_d(n - |S|) \). It follows by induction that \( \mathcal{F}_S \) contains a copy of \( B_d \) generated by some sets \( S_0, S_1, \ldots, S_d \), and with \( S \) these sets generate a copy of \( B_{d+1} \) in \( \mathcal{F} \). Let \( Z = \{ (A, B) \in \mathcal{F} \times \mathcal{F}: A \subseteq B \} \). For each \( (A, B) \in Z \), the probability that...
a random full-chain in $2^n$ contains both $A$ and $B$ is $1/(|A|,|B|−|A|,n−|B|)$. We compute

$$E\left(\binom{X}{2}\right) = \sum_{(A,B)\in Z} \frac{1}{\binom{|A|,|B|−|A|,n−|B|}}$$

$$= \sum_{0\leq S\subseteq [n]} \sum_{A\in\mathcal{F}_S} \frac{1}{\binom{|A|,|S|,n−|A|−|S|}}$$

$$= \sum_{0\leq S\subseteq [n]} \frac{1}{\binom{|S|}} \sum_{A\in\mathcal{F}_S} \binom{n−|S|}{|A|}$$

$$= \sum_{0\leq S\subseteq [n]} \frac{1}{\binom{|S|}} h_{n−|S|}(\mathcal{F}_S)$$

$$= \sum_{k=1}^{n} \frac{1}{\binom{k}} \sum_{S\in\binom{[n]}{k}} h_{n−k}(\mathcal{F}_S).$$

Since $E\left(\binom{X}{2}\right) > n\alpha_d(n)$, it follows that $\frac{1}{\binom{k}} \sum_{S\in\binom{[n]}{k}} h_{n−k}(\mathcal{F}_S) > \alpha_d(n)$ holds for some $k$. In turn, $h_{n−k}(\mathcal{F}_S) > \alpha_d(n) \geq \alpha_d(n−k)$ for some set $S$ of size $k$. □

The following is a corollary which can be viewed as the generalization of inequality (2) and (3).

**Corollary 1.** For $d \geq 3$ and $n \geq \left(2^d−2/\ln 2\right)^2$, every family $\mathcal{F} \subseteq 2^n$ containing no $d$-dimensional Boolean algebra satisfies $h_n(\mathcal{F}) \leq 2(n+1)^{1−2^{−d}}$.

Before proving Theorem 1, we need bounds on ratios of binomial coefficients.

**Lemma 1.** If $k \leq n$, then $\frac{\binom{2n}{k}}{\binom{n}{k}} \leq e^{-\frac{k}{2}(n−k)}$.

**Proof.** Note that $\frac{\binom{2n}{k}}{\binom{n}{k}} = \frac{\prod_{i=k+1}^{2n} i}{\prod_{i=0}^{n-k} i} \leq \prod_{j=0}^{n-k-1} \frac{n−j}{n+j+1} \leq e^{-2x}$ for $x \geq 0$ with $x = j/n$ to find $\frac{\binom{2n}{k}}{\binom{n}{k}} \leq e^{-\frac{k}{2}(n−k)}$. □

**Proof of Theorem 1:** Let $\mathcal{F} \subseteq 2^n$ be a $\mathcal{B}_d$-free family. For $0 \leq a \leq b \leq n$, let $\mathcal{F}(a,b) = \{A \in \mathcal{F} : a \leq |A| \leq b\}$. For two sets $A$ and $B$ with $A \subseteq B$, the interval $[A,B]$ is the set $\{X \in 2^n : A \subseteq X \subseteq B\}$. Let $Z = \{(A,B) : A \subseteq B, |A| = a, |B| = b\}$. Since $\mathcal{F}$ is $\mathcal{B}_d$-free and $[A,B]$ is a copy of the $(b−a)$-dimensional Boolean algebra, Theorem 2 implies that $h_{b−a}(\mathcal{F} \cap [A,B]) \leq \alpha_d(b−a)$ for each $(A,B) \in Z$. Since a random chain is equally likely to intersect levels $a$ and $b$ at all pairs in $Z$, it follows that $h_n(\mathcal{F}(a,b))$ is the average, over all $(A,B) \in Z$, of $h_{b−a}(\mathcal{F} \cap [A,B])$. Therefore $h_n(\mathcal{F}(a,b)) \leq \alpha_d(b−a)$.

We may assume without loss of generality that $n$ is an even integer $2m$, and let $\ell = \lfloor \sqrt{m} \rfloor$. We first bound the number of sets in $\mathcal{F}$ whose size is at most $m$; to do this, we partition $\{A \in \mathcal{F} : |A| \leq m\}$ into subsets of the form $\mathcal{F}(a,b)$ where
$b-a$ is at most $\ell$. Let $t$ be the largest integer such that $m-t\ell-1 \geq 0$. We define $x_0, \ldots, x_{t+1}$ by setting $x_0 = m$, $x_j = m - j\ell - 1$ for $1 \leq j \leq t$, and $x_{t+1} = -1$. For $0 \leq j \leq t$, we define $F_j = F(x_{j+1} + 1, x_j)$, and note that $x_j - (x_{j+1} + 1) \leq \ell$ for all $j$. Hence $h_n(F_j) \leq \alpha_d(\ell)$ for all $j$. Since $h_n(F_j) \geq |F_j|/(2m)$, it follows that $|F_j| \leq \alpha_d(\ell)(2m)$.

We compute

$$
\sum_{j=0}^{t} |F_j| \leq \alpha_d(\ell) \sum_{j=0}^{t} \binom{2m}{x_j} \\
\leq \alpha_d(\ell) \binom{2m}{m} \sum_{j=0}^{t} e^{-\frac{2}{m}(m-x_j)} \\
\leq \alpha_d(\ell) \binom{2m}{m} \sum_{j=0}^{t} e^{-\frac{j}{m}(\ell)^2} \\
\leq \alpha_d(\ell) \binom{2m}{m} \sum_{j \geq 0} e^{-\frac{e^2}{m}j} \\
\leq \alpha_d(\ell) \binom{2m}{m} \frac{1}{1 - e^{-\ell^2/m}},
$$

where we have applied Lemma 1. Since $\ell \geq \sqrt{m}$, the series is bounded by the absolute constant $1/(1 - e^{-1})$. Using that $(2m/m) \leq \sqrt{\frac{2e}{\pi}} \sqrt{\frac{22m}{m}}$ for all $m$ and applying our bound $\alpha_d(\ell) \leq (4\ell)^{1-2^{-d}} \leq (4(\sqrt{m} + 1))^{1-2^{-d}} \leq 8(\sqrt{m})^{1-2^{-d}}$ yields

$$
\sum_{j=0}^{t} |F_j| \leq 8\sqrt{\frac{2e^2}{\pi(e-1)}} \cdot m^{-1/2^d} \cdot 2^{2m}.
$$

Doubling this, we have that $|F| \leq \frac{8\sqrt{2e^2}}{\pi(e-1)} \cdot m^{-1/2^d} \cdot 2^{2m}$, and substituting $m = n/2$ gives $|F| \leq \frac{16e^2}{\pi(e-1)} \cdot n^{-1/2^d} \cdot 2^{2n} < 2^{n-1/2^d} 2^{n}$. \(\square\)

We note that our constant 22 can be reduced by sharpening the analysis in the proof of Theorem 1 in several places; we make no attempt to further reduce the constant.

### 3 Ramsey-type results

#### 3.1 Multi-color Ramsey results

Given positive integers $n$ and $d$, define $r(d, n)$ to be the largest integer $r$ so that every $r$-coloring of $2^{[n]}$ contains a monochromatic copy of $B_d$. Gunderson, Rödl, and Sidorenko [12] proved for $d > 2$,

$$
cn^{1/2^d} \leq r(d, n) \leq n^{\frac{d}{2^d-2}}(1+o(1)).
$$

Using Theorem 2, we improve the lower bound.
Theorem 3. For $d > 2$, we have

$$r(d, n) \geq \left\lfloor \frac{1}{2} n^{2/2^d} \right\rfloor.$$ 

Proof of Theorem 3: Let $r = \left\lfloor \frac{1}{2} n^{2/2^d} \right\rfloor$. For every $r$-coloring of $2^n$ and $1 \leq i \leq r$, let $\mathcal{F}_i$ be the family of sets in color $i$. By linearity, we have

$$\sum_{i=1}^{r} h_n(\mathcal{F}_i) = h_n(2^n) = n + 1.$$ 

By the pigeonhole principle, there is a color $i$ with $h_n(\mathcal{F}_i) \geq \frac{n+1}{r} > 2(n + 1)^{1-2^{1-d}}$. For all $r, d \geq 2$, we have $n + 1 \geq (2^d - 2/\ln 2)^2$. Thus,

$$h_n(\mathcal{F}_i) \geq \frac{n+1}{r} > 2(n+1)^{1-2^{1-d}} \geq \alpha_d(n).$$

By Theorem 2, $\mathcal{F}_i$ contains a copy of $\mathcal{B}_d$. 

For positive integers $t_1, t_2, \ldots, t_r$, let $R(\mathcal{B}_{t_1}, \ldots, \mathcal{B}_{t_r})$ be the least integer $N$ such that for any $n \geq N$ and any $r$-coloring of $2^n$ there exists an $i$ such that $\mathcal{B}_n$ contains a monochromatic copy of $\mathcal{B}_{t_i}$ in color $i$. In this language, Theorem 3 states that

$$R(\mathcal{B}_{t_1}, \ldots, \mathcal{B}_{t_r}) \leq (2r)^{2^{t-1}} - 1.$$ 

Next, we establish an exact result for $R(\mathcal{B}_s, \mathcal{B}_1)$. Our lower bound on $R(\mathcal{B}_s, \mathcal{B}_1)$ requires a numerical result. A sequence of positive integers is complete if every positive integer is the sum of a subsequence. In 1961, Brown [2] showed that a non-decreasing sequence $x_1, x_2, \ldots$ of positive integers with $x_1 = 1$ is complete if and only if $\sum_{i=1}^{k} x_i \leq 1 + x_{k+1}$ for each $k$. We adapt Brown’s argument to obtain a sufficient condition for a finite variant; we include the proof for completeness.

Lemma 2 (Brown [2]). Let $x_1, \ldots, x_s$ be a list of positive integers with sum at most $2s - 1$. For each $k$ with $0 \leq k \leq s$, there is a sublist with sum $k$.

Proof. We use induction on $s$. Since the empty list of numbers has sum 0 which is larger than $2 \cdot 0 - 1$, the lemma holds vacuously when $s = 0$. For $s \geq 1$, index the integers so that $1 \leq x_1 \leq \cdots \leq x_s$. If $x_s = 1$, then $x_j = 1$ for each $j$ and the lemma holds. Otherwise, $x_s \geq 2$ and $x_1, \ldots, x_{s-1}$ has sum at most $2(s - 1) - 1$. By induction, for each $k$ with $0 \leq k \leq s - 1$, some sublist of $x_1, \ldots, x_{s-1}$ has sum $k$. Note that $x_s \leq s$, or else $x_s \geq s + 1$ and $x_j \geq 1$ for $1 \leq j \leq s - 1$ would contradict that the list $x_1, \ldots, x_s$ has sum at most $2s - 1$. Since $s - x_s$ is in the range $\{0, \ldots, s - 1\}$, we obtain a sublist with sum $s$ by adding $x_s$ to a sublist of $x_1, \ldots, x_{s-1}$ with sum $s - x_s$. 

Theorem 4. For all $s \geq 1$, we have $R(\mathcal{B}_s, \mathcal{B}_1) = 2s$. 
Proof: First we show \( R(B_s, B_1) \leq 2s \). Let \( n = 2s \), let \( c \) be a red-blue coloring of \( 2^{[n]} \), and suppose for a contradiction that \( c \) contains neither a red copy of \( B_s \) nor a blue copy of \( B_1 \). We claim that every blue set has size \( s \). If \( A \) is blue, then all points in the up-set of \( A \) and all points in the down-set of \( A \) are red, or else the coloring has a blue copy of \( B_1 \). If \( |A| < s \), then the up-set of \( A \) contains red copies of \( B_s \), and if \( |A| > s \), then the down-set of \( A \) contains red copies of \( B_s \). Therefore \( |A| = s \) as claimed. Consider the copy of \( B_s \) generated via setting \( X_0 = \emptyset \), \( X_j = \{ j \} \) for \( 1 \leq j \leq s - 1 \), and \( X_s = \{ s, s + 1, \ldots, 2s \} \). None of the sets in this copy of \( B_s \) have size \( s \), and therefore this is a red copy of \( B_s \), a contradiction.

Now we show that \( R(B_s, B_1) > 2s - 1 \). Let \( n = 2s - 1 \). We construct a 2-coloring of \( 2^{[n]} \) that contains no red copy of \( B_s \) and no blue copy of \( B_1 \) as follows. Color all sets of size \( s \) blue and all other sets red. The blue sets form an antichain, so the coloring avoids blue copies of \( B_1 \). It suffices to show that there is no red copy of \( B_s \). Suppose for a contradiction that a red copy of \( B_s \) is generated by sets \( X_0, X_1, \ldots, X_s \), and let \( x_j = |X_j| \). Since \( X_1, \ldots, X_s \) are disjoint in \( [2s - 1] \) and non-empty, it follows that \( x_1, \ldots, x_s \) is a list of positive integers with sum at most \( 2s - 1 \). Since \( 0 \leq x_0 \leq s - 1 \), we apply Lemma 2 to obtain \( I \subseteq [s] \) such that \( \sum_{i \in I} x_i = s - x_0 \). It follows that \( X_0 \cup \bigcup_{i \in I} X_i \) has size \( s \), contradicting that the coloring contains a red copy of \( B_s \). \( \square \)

In 1950, Erdős and Rado \([5]\) proved the Canonical Ramsey Theorem, which lists structures that arise in every edge-coloring of the complete graph on countably many vertices. It states that each edge-coloring of the complete graph on the natural numbers contains an infinite subgraph \( H \) such that either all edges in \( H \) have the same color, or the edges in \( H \) have distinct colors, or the edges in \( H \) are colored lexicographically by their minimum or maximum endpoint. By a standard compactness argument, the Canonical Ramsey Theorem implies a finite version, stating that for each \( r \), there is a sufficiently large \( n \) such that every edge-coloring of the complete graph on vertex set \([n]\) contains a subgraph on \( r \) vertices that is colored as in the infinitary version.

An analogous result holds for colorings of \( 2^{[n]} \). A coloring of \( 2^{[n]} \) is **rainbow** if all sets receive distinct colors. Let \( \text{CR}(r, s) \) be the minimum \( n \) such that every coloring of \( 2^{[n]} \) contains a rainbow copy of \( B_r \) or a monochromatic copy of \( B_s \). Although it is not immediately obvious that \( \text{CR}(r, s) \) is finite, our next theorem provides an upper bound.

**Theorem 5.** \( \text{CR}(r, s) \leq r2^{(2r+1)2^{r-1}-2} \) for positive \( r \) and \( s \).

Proof. Set \( t = 2^{(2r+1)2^{r-1}-2} \) and \( n = tr \), and consider a coloring of \( 2^{[n]} \) that does not contain a monochromatic copy of \( B_s \). We obtain a rainbow copy of \( B_s \) with the probabilistic method. Partition the ground set \([n]\) into \( r \) sets \( U_1, \ldots, U_r \) each of size \( t \). Independently for each \( i \) in \([r]\), choose a subset \( X_i \) from \( U_i \) so that sets are chosen proportionally to their Lubell mass in the Boolean algebra on \( U_i \). That is, each \( k \)-set in \( U_i \) has probability \( \frac{1}{t+1} \binom{t}{k}^{-1} \) of being selected for \( X_i \). For each pair \( \{ I, J \} \) with \( I, J \subseteq [r] \), let \( A_{I,J} \) be the event that both \( \bigcup_{i \in I} X_i \) and \( \bigcup_{j \in J} X_j \) receive the same color.
We obtain an upper bound on the probability that $A_{I,J}$ occurs. Since $I$ and $J$ are distinct sets, we may assume without loss of generality that there exists $m \in I - J$. Fix the selection of all sets $X_1, \ldots, X_r$ except $X_m$. This determines the color $c$ of $\bigcup_{j \in J} X_j$, and the probability that $A_{I,J}$ occurs is at most the probability that $\bigcup_{i \in I} X_i$ has color $c$. Let $L$ be the $t$-dimensional Boolean sublattice with ground set $U_m$, and color $B \in L$ with the same color as $B \cup \bigcup_{i \in I - \{m\}} X_i$. Let $F$ be the elements in $L$ with color $c$. Since $F$ does not contain a monochromatic copy of $B_s$, Theorem 2 implies that $h_t(F) \leq (4t)^{1-2^{1-s}}$. Since $h_t(F) = \sum_{B \in F} \left( \frac{t}{|B|} \right)^{-1} = (t+1) \sum_{B \in F} \Pr[X_m = B] \geq (t+1) \Pr[A_{I,J}]$, we have that $\Pr[A_{I,J}] \leq (4t)^{1-2^{1-s}}/(t+1) < 4/(4t)^{2^{1-s}}$. Using the union bound, we have that the probability that at least one of the events $A_{I,J}$ occurs is less than $\left( \frac{t}{3} \right) \cdot 4/(4t)^{2^{1-s}}$, which is at most 1. It follows that for some selection of the sets $X_1, \ldots, X_r$, none of the events $A_{I,J}$ occur. These sets generate a rainbow copy of $B_r$.

Note that Equation (6) implies that if $k > n^{\frac{s}{2}}(1+o(1))$, then there is a $k$-coloring of $2^{[n]}$ that does not contain a monochromatic copy of $B_s$. Of course, with $k = 2^r - 1$, there is also no rainbow copy of $B_r$. It follows that $2^r - 1 > n^{\frac{s}{2}}(1+o(1))$ implies that $CR(r,s) > n$, and hence $CR(r,s) \geq 2^{\frac{r^s}{2} - 1}(1-o(1))$ where the $o(1)$ term tends to 0 as $r$ increases.

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