On the Colin de Verdière Graph Number and Penny Graphs

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Abstract

The Colin de Verdière number of graph $G$, denoted by $\mu(G)$, is a spectral invariant of $G$ that is related to some of its topological properties. For example, $\mu(G) \leq 3$ iff $G$ is planar. A penny graph is the contact graph of equal-radii disks with disjoint interiors in the plane. In this note, we prove lower bounds on $\mu(G)$ when the complement $\overline{G}$ is a penny graph.

1 Introduction and the Main Result

A penny graph is the contact graph of equal-radii disks with disjoint interiors in the plane. To be more precise, a graph $G$ on $n$ nodes is a penny graph if there exists a one-to-one correspondence between the nodes of $G$ and a set of

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$n$ closed unit-diameter disks with disjoint interiors in the plane such that two nodes of $G$ are adjacent if and only if their corresponding disks touch. For $i = 1, \ldots, n$, let $p^i$ denote the center of the disk corresponding to node $i$. Then the set of points $\{p^1, \ldots, p^n\}$ is called a realization of $G$, and it immediately follows that the edge set of $G$ is given by

$$E(G) = \{\{i, j\} : ||p^i - p^j||^2 = 1\}, \quad (1)$$

where $||x||$ denotes the Euclidean norm of $x$, i.e., $||x|| = \sqrt{x^T x}$. Moreover, the set of missing edges of $G$, or equivalently, the edge set of the complement graph $\overline{G}$, is given by

$$E(\overline{G}) = \{\{i, j\} : ||p^i - p^j||^2 > 1\}. \quad (2)$$

As a result, penny graphs are also called minimum-distance graphs.

A graph is planar if it can be drawn in the plane with its edges intersecting at their nodes only. Evidently, penny graphs are planar. A planar graph is said to be outerplanar if all of its nodes lie on the boundary of the outer face of its drawing.

Let $A_G$ denote the generalized adjacency matrix of graph $G$, where the 1’s are replaced by arbitrary positive numbers and the diagonal entries are arbitrary. The Colin de Verdière number of $G$, denote by $\mu(G)$ [2, 3], is more or less the maximum multiplicity of the second largest eigenvalue, under a certain nondegeneracy assumption, of $A_G$. The precise definition of $\mu(G)$ is given in Subsection 2.1 below. Surprisingly, $\mu(G)$ is related to several topological properties of $G$. For example, $\mu(G)$ is minor monotone, i.e., if $H$ is a minor of $G$, then $\mu(H) \leq \mu(G)$. Also, the planarity of $G$ can be characterized in terms of $\mu(G)$ as given in the following theorem.

**Theorem 1.1** (Colin de Verdière [2]). Let $G$ be a connected graph. Then

1. $\mu(G) \leq 1$ iff $G$ is a path.

2. $\mu(G) \leq 2$ iff $G$ is outerplanar.

3. $\mu(G) \leq 3$ iff $G$ is planar.

The results on $\mu(G)$ which are most relevant to this note are given in the following two theorems.
Theorem 1.2 (Kotlov et al [10]). Let $G$ be a graph on $n$ nodes. Assume that $G$ has no twins, i.e., no two nodes with same set of neighbors. Then

1. if $\mu(G) \geq n - 3$, then $G$ is outerplanar.
2. if $\mu(G) \geq n - 4$, then $G$ is planar.

Theorem 1.3 (Kotlov et al [10]). Let $G$ be a graph on $n$ nodes. Then

1. if $G$ is a path, or a disjoint union of paths, then $\mu(G) \geq n - 3$.
2. if $G$ is outerplanar, then $\mu(G) \geq n - 4$.
3. if $G$ is planar, then $\mu(G) \geq n - 5$.

The following theorem is the main result of this note.

Theorem 1.4. Let $G$ be a penny graph on $n$ nodes, where $n \geq 5$. Then

1. if $G$ is a path, a disjoint union of paths, or a cycle, then $\mu(G) \geq n - 3$.
2. Otherwise, $\mu(G) \geq n - 4$.

It is interesting to contrast Theorems 1.3 and 1.4 since penny graphs are planar and since paths and cycles are trivially outerplanar. Furthermore, the following remark is worth pointing out. Let $K_n$ denote the complete graph on $n$ nodes and assume that $n \geq 3$. Then $\mu(K_n) = n - 1$ and $\mu(G) \leq n - 2$ for any graph $G$ on $n$ nodes that is different from $K_n$ [17].

The proof of Theorem 1.4, which is based on the theory of Euclidean distance matrices, is presented in Section 3. The necessary background for the proof is presented in Section 2.

2 Preliminaries

We begin this section by collecting the notation used throughout this note. $E(G)$ denotes the edge set of graph $G$, while $E(G)$ denotes the edge set of the complement graph $G$. We denote by $e$ and $E$, respectively, the $n$-vector and the $n \times n$ matrix of all 1’s. $0$ denotes the zero vector or the zero matrix of appropriate dimension. $\text{null}(A)$ denotes the null space of matrix $A$ and $A_j$ denotes the $j$th column of $A$. Finally, $||x||$ denotes the Euclidean norm of $x$. 
2.1 The Colin de Verdière number of graphs

The corank of a matrix $M$ is the dimension of its null space. Let $G$ be an undirected graph on $n$ nodes. A $G$-matrix $A$ is an $n \times n$ symmetric matrix such that $a_{ij} = 0$ for all $\{i, j\} \in E(\overline{G})$. In addition, if $a_{ij} < 0$ for all $\{i, j\} \in E(G)$, then $A$ is said to be well signed. Note that there is no condition on the diagonal entries of $A$.

**Definition 2.1.** Let $G$ be a connected graph on $n$ nodes. The Colin de Verdière number of $G$, denoted by $\mu(G)$, is the maximum corank of an $n \times n$ matrix $M$ that satisfies the following conditions:

- **M1:** $M$ is a well-signed $G$-matrix.
- **M2:** $M$ has exactly one negative eigenvalue.
- **M3:** There does not exist a nonzero $\overline{G}$-matrix $X$ whose diagonal entries are all 0’s such that $MX = 0$.

Condition M3 [16] is one of several equivalent formulations of the Strong Arnold Property (SAP). Any matrix that satisfies Conditions M1, M2 and M3 is called a Colin de Verdière matrix of graph $G$. A Colin de Verdière matrix $M$ of $G$ such that $\text{corank}(M) = \mu(G)$ is called optimal.

The definition of $\mu(G)$ can be extended to disconnected graphs [17]. Let $G_1, \ldots, G_k$ be the connected components of $G$ and assume that $G$ has at least one edge. Then

$$\mu(G) = \max\{\mu(G_1), \ldots, \mu(G_k)\}.$$  

It is easy to show that $\mu(K_1) = 0$ and $\mu(K_n) = 1$ if $n \geq 2$. Also $\mu(K_n) = n - 1$ and for any graph $G$ on $n$ nodes ($n \geq 3$) such that $G \neq K_n$, we have $\mu(G) \leq n - 2$. Note that $\mu(K_2) = \mu(K_3) = 1$. For a comprehensive survey of $\mu(G)$, see the paper [17] and the recent book [12].

2.2 Penny Graphs

As was mentioned earlier, penny graphs are obviously planar. Whereas a planar graph on $n$ nodes can have at most $3n - 6$ edges, a penny graph can have at most $3n - \sqrt{12n - 3}$ edges [8]. Furthermore, unlike planar graphs which can be recognized in linear time [9], the problem of recognizing penny graphs is NP-hard [4].

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Evidently, the maximum degree of any node of a penny graph is 6 since the kissing number in the plane is 6. Also, it is easy to show that a penny graph cannot have $K_4$ or $K_{2,3}$ as subgraphs. In the remainder of this subsection, we present some properties of penny graphs which are relevant to this note.

Let $k$ be a nonnegative integer. A graph $G$ is said to be $k$-degenerate if every induced subgraph of $G$ has a node of degree at most $k$. The following are easy observations.

**Proposition 2.1** ([11]). Let $G$ be a graph on $n$ nodes. Then $G$ is $k$-degenerate if and only if the nodes of $G$ can be ordered, say $v_1, \ldots, v_n$, such that for all $i = 1, \ldots, n$, the degree of $v_i$ in the subgraph induced by nodes $\{v_i, \ldots, v_n\}$ is at most $k$. In other words, graph $G$ is $k$-degenerate iff it can be reduced to a single node by the successive removal of nodes of degree at most $k$.

**Proposition 2.2** ([5]). Let $G$ be a penny graph. Then $G$ is 3-degenerate.

Proposition 2.2 follows since each vertex of the convex hull of any subgraph of a penny graph has degree at most 3. The following technical lemmas easily follow from the geometry of penny graphs.

**Lemma 2.1.** Let $\{p^1, \ldots, p^n\}$ be a realization of a penny graph $G$; and assume that $p^1, \ldots, p^n$ lie on a circle of radius $\rho$. If $n \geq 5$, then $\rho^2 > \frac{1}{2}$.

*Proof.* Let $\theta$ be the central angle formed by two touching disks. Then obviously, $\theta \leq 2\pi/5$. Hence,

$$\rho^2 \geq \frac{1}{2(1 - \cos \theta)} \geq \frac{1}{2(1 - \cos(2\pi/5))} > \frac{1}{2}. \quad \square$$

**Lemma 2.2.** Let $G$ be a penny graph on $n$ nodes, where $n \geq 5$. If the complement graph $\overline{G}$ is not connected, then $n = 5, 6$ or 7; and $G$ is realized by one disk touching, respectively, 4, 5 or 6 other disks.

*Proof.* Assume that $\overline{G}$ is not connected. If $\overline{G}$ has 4 or more connected components, then it is easy to see that $G$ has a $K_4$ as a subgraph, a contradiction. Similarly, if $\overline{G}$ has 3 connected components, then $G$ has a $K_{2,3}$ as a subgraph, also a contradiction. Hence, $\overline{G}$ has 2 connected components. Now if one of these components has 2 or more nodes and the other component has 3 or more nodes, then $G$ has a $K_{2,3}$ as a subgraph, again a contradiction. Therefore, $\overline{G}$ must have one isolated node, i.e., we must have one disk, corresponding to this isolated node, touching the disks corresponding to all other nodes. The result follows since the kissing number in the plane is 6, and since $n \geq 5$. \quad \square
2.3 Euclidean Distance Matrices

As was mentioned earlier, the proof of Theorem 1.4 is based on the theory of Euclidean distance matrices (EDMs). In this subsection, we present the results of EDMs that are most relevant to this note. For a comprehensive treatment see the monograph [1].

An \( n \times n \) matrix \( D \) is called a Euclidean distance matrix (EDM) if there exist points \( p^1, \ldots, p^n \) in some Euclidean space such that

\[
d_{ij} = ||p^i - p^j||^2 \quad \text{for all} \quad i, j = 1, \ldots, n.
\]

The points \( p^1, \ldots, p^n \) are called the generating points of \( D \) and the dimension of their affine span is called the embedding dimension of \( D \). Obviously, an EDM \( D \) is symmetric with zero diagonal and nonnegative offdiagonal entries.

Let \( e \) denote the vector of all 1’s in \( \mathbb{R}^n \), and let \( V \) be an \( n \times (n - 1) \) matrix such that \( Q = [e/\sqrt{n} \ V] \) is an \( n \times n \) orthogonal matrix. Then we have the following well-known characterization of EDMs.

**Theorem 2.1.** [7, 14, 18] Let \( D \) be an \( n \times n \) symmetric matrix of zero diagonal. Then \( D \) is an EDM if and only if \( V^T(-D)V \) is positive semidefinite; in which case, the embedding dimension of \( D \) is equal to the rank of \( V^T D V \).

That is, a symmetric matrix \( D \) with zero diagonal is an EDM iff \( D \) is negative semidefinite on \( e^\perp \), the orthogonal complement of \( e \) in \( \mathbb{R}^n \). EDMs have the nice property that \( e \) lies in the column space of every nonzero EDM \( D \) [7], i.e., for any EDM \( D \neq 0 \), there exists a vector \( w \) such that \( Dw = e \).

An EDM \( D \) is said to be spherical if its generating points lie on a sphere. Otherwise, it is said to be nonspherical. If the generating points of \( D \) lie on a sphere of radius \( \rho \), we will refer to \( \rho \) as the radius of \( D \). Spherical and nonspherical EDMs have many different characterizations. The most relevant for our purposes are those given in the following two theorems.

**Theorem 2.2** [6, 7, 13, 15]. Let \( D \) be a nonzero \( n \times n \) EDM of embedding dimension \( r \) and let \( Dw = e \). If \( r = n-1 \), then \( D \) is spherical; and if \( r \leq n-2 \), then the following statements are equivalent:

1. \( D \) is spherical of radius \( \rho \).
2. \( \text{rank}(D) = r + 1 \).
3. \( e^T w > 0 \) and \( \rho^2 = 1/(2e^T w) \).
4. There exists a scalar $\beta$ such that $\beta E - D$ is positive semidefinite; moreover, $\beta = 2\rho^2$ is the smallest such scalar.

**Theorem 2.3** (Gower [6, 7]). Let $D$ be a nonzero EDM of embedding dimension $r$ and let $Dw = e$. Then the following statements are equivalent:

1. $D$ is nonspherical.
2. $\text{rank}(D) = r + 2$.
3. $e^T w = 0$.

The following lemmas will be needed when dealing with the Strong Arnold Property.

**Lemma 2.3.** Let $D$ be an EDM and let $M = E - D$. Then $\text{null}(D) \subseteq \text{null}(M)$.

*Proof.* Let $x \in \text{null}(D)$. Then $Mx = e^T x e$. But $e^T x = 0$ since $e$ lies in the column space of $D$. Therefore $x \in \text{null}(M)$ and thus $\text{null}(D) \subseteq \text{null}(M)$. \qed

**Lemma 2.4.** Let $D$ be an EDM and let $M = E - D$. Assume that $D$ is nonspherical or spherical with radius $\rho \neq 1/\sqrt{2}$. Then $\text{null}(M) \subseteq \text{null}(D)$.

*Proof.* Let $x \in \text{null}(M)$. Then $Dx = e^T x e$. Let $Dw = e$. Then $w^T Dx = e^T x e^T w$. Hence, $e^T x (1 - e^T w) = 0$. Now if $D$ is nonspherical, then, it follows from Theorem 2.3 that $e^T w = 0$ and hence $e^T x = 0$. On the other hand, if $D$ is spherical with $\rho^2 \neq 1/2$, then it follows from part 3 of Theorem 2.2 that $e^T w \neq 1$ and hence again $e^T x = 0$. Consequently, $Dx = 0$ and thus $\text{null}(M) \subseteq \text{null}(D)$. \qed

The following corollary immediately follows from Lemmas 2.3 and 2.4 and Lemma 2.1.

**Corollary 2.1.** Let $D$ be an $n \times n$ EDM, where $n \geq 5$, and let $M = E - D$. Then $\text{null}(M) = \text{null}(D)$.

**Lemma 2.5.** Let $D$ be an $n \times n$ EDM, where $n \geq 3$, and assume that all of its offdiagonal entries are positive. Then any 3 columns of $D$ are linearly independent.
Proof. By way of contradiction, assume that the columns $D_i, D_{i_2}, D_{i_3}$ of $D$ are linearly dependent, and let $	ilde{D}$ denote the $3 \times 3$ principal submatrix of $D$ induced by the indices $\{i_1, i_2, i_3\}$. Then the EDM $\tilde{D}$ is singular. Let $p^{i_1}, p^{i_2}, p^{i_3}$ denote the generating points of $\tilde{D}$. Then, by our assumption, $p^{i_1}, p^{i_2}, p^{i_3}$ are distinct.

Now if $p^{i_1}, p^{i_2}, p^{i_3}$ are collinear, then $\tilde{D}$ is nonspherical of embedding dimension 1, and hence, by Theorem 2.3, $\text{rank}(\tilde{D}) = 3$, a contradiction. On the other hand, if $p^{i_1}, p^{i_2}, p^{i_3}$ are not collinear, then $\tilde{D}$ has embedding dimension 2, and hence, $\tilde{D}$ is spherical. Therefore, by Theorem 2.2, $\text{rank}(\tilde{D}) = 3$, also a contradiction. Hence, the result follows.

3 Proof of Theorem 1.4

To prove Theorem 1.4, it suffices to exhibit a Colin de Verdière matrix of the complement graph $\overline{G}$ of the appropriate corank. To this end, every realization $\{p^1, \ldots, p^n\}$ of a penny graph $G$ defines an EDM $D = (d_{ij} = ||p^i - p^j||^2)$. Hence, by Equations (1) and (2), it immediately follows that all the offdiagonal entries of $D$ are $\geq 1$. To be more precise, let

$$ M = E - D, \quad (3) $$

then

$$ M_{ij} \begin{cases} = 1 & \text{if } i = j, \\ = 0 & \text{for all } \{i,j\} \in E(G), \\ < 0 & \text{for all } \{i,j\} \in E(\overline{G}). \end{cases} \quad (4) $$

Therefore, $M$ is a well-signed $\overline{G}$-matrix, and thus $M$ satisfies Condition M1 of Definition 2.1 for the complement graph $\overline{G}$. To prove that $M$ is the desired Colin de Verdière matrix for $\overline{G}$, we need to show that matrix $M$ also satisfies Conditions M2 and M3 for $\overline{G}$. This we do next.

3.1 Proof that $M$ Satisfies Condition M2

It follows from Theorem 2.1 that $V^T(-D)V = V^T MV$ is an $(n-1) \times (n-1)$ positive semidefinite matrix of rank 2. Let $Q = [e/\sqrt{n}, V]$. Then matrices $M$ and

$$ Q^T MQ = \begin{bmatrix} e^T Me/n & e^T MV/\sqrt{n} \\ V^T Me/\sqrt{n} & V^T MV \end{bmatrix} $$

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are similar. Let $\lambda_1$ and $\lambda_2$ denote, respectively, the smallest and the second smallest eigenvalues of $M$. Thus, it follows from the eigenvalue interlacing theorem that $\lambda_2 = 0$ and $\lambda_1 \leq 0$ since $V^TMV$ is positive semidefinite of rank $2$. Recall that we assume that $n \geq 5$.

Now if $\lambda_1 = 0$, then $M$ is positive semidefinite, then it follows from Theorem 2.1 and part 4 of Theorem 2.2 that $D$ is a spherical EDM with radius $\rho \leq 1/\sqrt{2}$. This contradicts Lemma 2.1 since we assume that $n \geq 5$. Therefore, $\lambda_1 < 0$ and thus $M$ satisfies Condition M2.

### 3.2 Proof that $M$ Satisfies Condition M3

Let $X$ be a $G$-matrix whose diagonal entries are all 0's and let $MX = 0$. Then it follows from Corollary 2.1 that $DX = 0$. Therefore,

$$\sum_{j: j \in N(i)} x_{ij}D_j = 0 \text{ for all } i = 1, \ldots, n, \quad (5)$$

where $N(i)$ denotes the set of nodes of $G$ that are adjacent to node $i$, and $D_j$ denotes the $j$th column of $D$. Now Propositions 2.1 and 2.2 imply that $G$ can be reduced to a single node by the successive removal of nodes of degree at most 3. Therefore, by solving the $n$ systems of equations of (5) in the same order as that of removing these nodes, we obtain that $X = 0$ since, by Lemma 2.5 any 3 columns of $D$ are linearly independent. Consequently, $M$ satisfies Condition M3 and as a result, $M$ is a Colin de Verdière matrix of $G$.

### 3.3 Establishing the Corank of $M$

Assume that $G$ is a path, a disjoint union of paths, or a cycle. Then $G$ has a realization whose corresponding EDM $D$ is spherical. Then, by part 2 of Theorem 2.2 and Lemmas 2.3 and 2.4, $\text{rank}(M) = \text{rank}(D) = 3$. Hence, $\text{corank}(M) = n - 3$ and thus $\mu(G) \geq n - 3$. Note that, by Lemma 2.2, $G$ is connected.

Now assume that $G$ is not a path, a disjoint union of paths, or a cycle. Then $G$ has a realization whose corresponding EDM $D$ is nonspherical. Assume that $\overline{G}$ is connected. Then, by part 2 of Theorem 2.3, $\text{rank}(M) = 4$. Hence, $\text{corank}(M) = n - 4$ and thus $\mu(\overline{G}) \geq n - 4$. On the other hand, if $\overline{G}$ is not connected, then by Lemma 2.2, $G$ consists of one isolated node and one connected component with $n - 1$ nodes. Again, by Lemma 2.2 this connected
component has a realization whose corresponding \((n - 1) \times (n - 1)\) EDM \(D'\) is spherical. Thus, matrix \(M' = E' - D'\) has rank 3, where \(E'\) is the matrix of all 1’s of order \(n - 1\). Hence, \(\text{corank}(M') = n - 1 - 3 = n - 4\). Again \(\mu(G) \geq n - 4\).

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