Since the discovery of quantum spin Hall effect\cite{1,2,3,4}, topological insulators (TI) in both two dimensions (2D) and three dimensions (3D) have generated great interest both theoretically and experimentally\cite{5,6,7,8,9}. Motivated by such an observation, in this letter we obtain simple and explicit physical criteria for TRI topological superconductors beyond the weak pairing limit, where the pairing is only important in a small neighborhood of the Fermi surface. In 3D, the Fermi surface topological invariant (FSTI) of a TRI superconductor is determined by the sign of the pairing order parameter and the first Chern number of the Berry phase gauge field on the Fermi surfaces. In two (2D) and one (1D) dimension, the $Z_2$ topological quantum number of a TRI superconductor is determined simply by the sign of the pairing order parameter on the Fermi surfaces. We also obtain a generic and explicit expression of the $Z_2$ topological invariant in 1D and 2D.

\textbf{Fermi Surface Topological Invariants for Time Reversal Invariant Superconductors}

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A time reversal invariant (TRI) topological superconductor has a full pairing gap in the bulk and topologically protected gapless states on the surface or at the edge. In this paper, we show that in the weak pairing limit, the topological quantum number of a TRI superconductor can be completely determined by the Fermi surface properties, and is independent of the electronic structure away from the Fermi surface. In three dimensions (3D), the integer topological quantum number in a TRI superconductor is determined by the sign of the pairing order parameter and the first Chern number of the Berry phase gauge field on the Fermi surfaces. In two (2D) and one (1D) dimension, the $Z_2$ topological quantum number of a TRI superconductor is determined simply by the sign of the pairing order parameter on the Fermi surfaces. We also obtain a generic and explicit expression of the $Z_2$ topological invariant in 1D and 2D.
\[ H = \sum_k \left[ \psi_k^\dagger h_k \psi_k + \frac{1}{2} \left( \psi_k^\dagger \Delta_k \psi_k^T + H.c. \right) \right] = \sum_k \psi_k^\dagger H_k \psi_k \]
with \( \psi_k = \frac{1}{\sqrt{2}} \left( \psi_k - i T \psi_k^+ \right) \), \( H_k = \frac{1}{2} \left( h_k - i T \Delta_k^+ \right) \). (1)

In general, \( \psi_k \) is a vector with \( N \) components and \( h_k \) and \( \Delta_k \) are \( N \times N \) matrices. The matrix \( T \) is the time-reversal matrix satisfying \( T^\dagger h_k T = h_{-k}^T \), \( T^2 = -\mathbb{I} \) and \( T^\dagger T = \mathbb{I} \). We have chosen a special basis in which the BdG Hamiltonian \( H_k \) has a special off-diagonal form. It should be noted that such a choice is only possible when the system has both time-reversal symmetry and particle-hole symmetry. These two symmetries also require \( T \Delta_k \) to be Hermitian, which makes the matrix \( h_k + i T \Delta_k^+ \) generically non-Hermitian. The matrix \( h_k + i T \Delta_k^+ \) can be decomposed by singular value decomposition (SVD) as
\[ h_k + i T \Delta_k^+ = U_k^\dagger D_k V_k \]
with \( U_k, V_k \) unitary matrices and \( D_k \) a diagonal matrix with non-negative elements. One can see straightforwardly that the diagonal elements of \( D_k \) are actually the positive eigenvalues of \( H_k \).

A fully gapped superconductor, \( D_k \) is positive definite, and we can adiabatically deform it to the identity matrix \( \mathbb{I} \) without closing the superconducting gap. During this deformation the matrix \( h_k + i T \Delta_k \) is deformed to a unitary matrix \( Q_k = U_k^\dagger V_k \in U(N) \). As shown in Ref.\(^2\), the integer-valued topological invariant characterizing topological superconductors is defined as the winding number of \( Q_k \):
\[ N_W = \frac{1}{24\pi^2} \int d^3 k \epsilon^{ijk} \text{Tr} \left[ Q_k^\dagger \partial_i Q_k Q_k^\dagger \partial_j Q_k Q_k^\dagger \partial_k Q_k \right] \] (2)

Now we study \( Q_k \) in the weak pairing limit. For simplicity, in the following, we will assume the Fermi surfaces are all non-degenerate, and there are no lower dimensional zero-energy defects such as point or line nodes. All our conclusions can be easily generalized to more generic cases. When the Fermi surfaces are non-degenerate, and the weak pairing term \( \Delta_k \) is only turned on around the Fermi surfaces, the matrix elements of \( T \Delta_k^+ \) between different bands are negligible. Thus, to leading order we have
\[ h_k + i T \Delta_k^+ \simeq \sum_n \left( \epsilon_{nk} + i \delta_{nk} \right) |n,k\rangle \langle n,k| \] (3)
with \( \delta_{nk} \equiv \langle n,k| \Delta_k|n,k\rangle \in \mathbb{R} \)
where \( |n,k\rangle \) are the eigenvectors of \( h_k \). Physically, \( \delta_{nk} \) is the matrix element of \( \Delta_k \) between \( |n,k\rangle \) and its time-reversed partner \( \langle -n,-k| = T^\dagger \langle n,k| \). In this approximation, the matrix \( Q_k \) is given by
\[ Q_k = \sum_n e^{i \theta_{nk}} |n,k\rangle \langle n,k| \] (4)
with \( e^{i \theta_{nk}} = (\epsilon_{nk} + i \delta_{nk}) / |\epsilon_{nk} + i \delta_{nk}| \). In the weak pairing limit, we take \( \delta_{nk} \) to be nonzero only in a small neighborhood \(-\epsilon \leq E \leq \epsilon \) of the Fermi level. As shown in Fig.\(^1\), the phase \( \theta_{nk} \) changes from 0 to \( \pm \pi \) across the Fermi level, with the sign determined by the sign of \( \delta_{nk} \). In the limit \( \epsilon \to 0 \), such a domain wall configuration of \( \theta_{nk} \) can be expressed by the formula
\[ \nabla \theta_{nk} = -\pi v_{nk} \text{sgn}(\delta_{nk}) \delta(\epsilon_{nk}) \] (5)
in which \( v_{nk} = \nabla_k \epsilon_{nk} \) is the Fermi velocity. It should be noted that for a gapped superconductor \( \delta_{nk} \) remains nonzero for all \( k \) on the Fermi surfaces, so the sign of \( \delta_{nk} \) is fixed on each Fermi surface.

![FIG. 1](image_url)

FIG. 1: (a) The path of \( \epsilon_{nk} + i \delta_{nk} \) in the complex plane for positive (red) and negative (blue) \( \delta_{nk} \) around the Fermi surface. (b) \( \theta_{nk} \) and \( \epsilon_{nk} \) vs momentum \( k \). The change of \( \theta_{nk} \) across \( k_F \) is \( -\pi (+\pi) \) when \( \delta_{nk} \) is positive (negative), as shown by the red (blue) curve.

Once the behavior of \( \theta_{nk} \) in the Brillouin zone is simplified to Eq.\(^3\) in the weak pairing limit, the winding number \( N_W \) can be simplified to the following simple FSTI:
\[ N_W = \frac{1}{2} \sum_s \text{sgn}(\delta_s) C_{1s} \] (6)
where \( s \) is summed over all disconnected Fermi surfaces, and \( \text{sgn}(\delta_s) \) denotes the sign of \( \delta_{nk} \) on the \( s \)-th Fermi surface. \( C_{1s} \) is the first Chern number of the \( s \)-th Fermi surface (denoted by \( \text{FS}_s \)):
\[ C_{1s} = \frac{1}{2\pi} \int_{\text{FS}_s} d\Omega^{ij} (\partial_i a_{sj}(k) - \partial_j a_{si}(k)) \] (7)
with \( a_{si} = -i \langle sk| \partial / \partial k_i |sk \rangle \) the adiabatic connection defined for the band \( |sk \rangle \) which crosses the Fermi surface,
and $d\Omega^J$ the surface element 2-form of the Fermi surface. More details of the derivation of Eq. (8) are included in the supplementary material.\textsuperscript{29}

As an example, consider a two-band Hamiltonian $h_k = k^2/2m - \mu + \alpha k \cdot \sigma$ for $\mu > 0$, the system has two Fermi surfaces which are concentric spheres around $k = 0$. (The two-band model should be regularized on the lattice, but the lattice regularization is unimportant as long as no other Fermi surfaces are introduced.) Denoting the electron states at the inner (outer) Fermi surface by $|k, +(-)\rangle$, we have $\sigma \cdot k |k, +\rangle = \pm |k| |k, \pm\rangle$. It is easy to check that the two Fermi surfaces carry opposite Chern number $C_\pm = \pm 1$. Thus, according to (6), we can obtain a topological superconductor if the two Fermi surfaces have opposite signs of pairing. The time-reversal matrix is $T = i\sigma_y$ in this system. If we have $\Delta_k = i\Delta_0 \sigma_y$, then $iT\Delta_k^\dagger = \Delta_0^\dagger i\sigma_y$ which has the same sign on the two Fermi surfaces and leads to $N_W = 0$. On the other hand, if we have $\Delta_k = i\Delta_0 \sigma_y \sigma \cdot k$, then $iT\Delta_k^\dagger = \Delta_0^\dagger \sigma_y \sigma \cdot k$ has opposite sign on the two Fermi surfaces, so that $N_W = 1$ if $\Delta_0 > 0$. If we take the limit $\alpha \to 0$, we obtain a topological superconductor with quadratic kinetic energy term and pairing $\Delta_k = i\Delta_0 \sigma_y \sigma \cdot k$, which is exactly the BdG Hamiltonian of the $^3$HeB phase. This example also illustrates how the FSTI (8) can be generalized to systems with degeneracies on the Fermi surface: One can always add a small perturbation proportional to $T\Delta_k^\dagger$ to the Hamiltonian to lift the degeneracy, while preserving the topological properties of the superconductor.

**Dimensional reduction to 2D.** The FSTI can be generalized to lower dimensions i.e. 2D and 1D. The TRI topological superconductors in 2D and 1D are related to the one in 3D by dimensional reduction, similar to the identity I:

$$Q_{k,\theta} = \begin{cases} 
1, & \theta = 0 \\
Q_k, & \theta = \pi
\end{cases} \quad (8)$$

It should be noted that $Q_k$ satisfies $T^\dagger Q_k T = Q_k^T$ due to time-reversal symmetry. Thus, if we define $Q_{k,-\theta} = T^\dagger Q_{k,\theta}^T T$ for $\theta \in [0, \pi]$, we obtain $Q_{k,\theta}$ for $\theta \in [-\pi, \pi]$ which is continuous and periodic in $\theta \to \theta + 2\pi$. Considering $\theta$ as a momentum in an additional dimension, $Q_{k,\theta}$ describes a 3D TRI superconductor, which is characterized by the winding number (2). If there are two different interpolations $Q_{k,\theta}$ and $Q_{k,\theta}'$ which both interpolate between $Q_k$ and 1, it can be shown that time-reversal symmetry requires their winding numbers to be different by an even number: $N_W(Q) - N_W(Q') = 0 \mod 2$. Thus the parity $(-1)^{N_W(Q)}$ is independent of the choice of interpolation path, and is a $\mathbb{Z}_2$ topological invariant uniquely determined by $Q_k$.

Now we study the expression of such a $\mathbb{Z}_2$ invariant in the weak pairing limit. In this limit, the interpolation of $Q_k$ to $Q_{k,\theta}$ is equivalent to interpolating the 1D Fermi circles of the 2D normal state Hamiltonian $h_k$ to Fermi surfaces in a 3D Brillouin zone parameterized by $(k_x, k_y, \theta)$. We can simply extrapolate the pairing on the Fermi circles to the Fermi surfaces, as illustrated in Fig. 2. If the Fermi surfaces remain nondegenerate during the interpolation, we obtain the $2\mathbb{Z}_2$ FSTI as the parity of the winding number given by Eq. (8):

$$N_{2D} = (-1)^{N_W} = (-1)^{\frac{1}{2} \sum \text{sgn(\delta_s)} C_{1s}} = \prod_s \left(\text{sgn(\delta_s)}\right)^{C_{1s}}. \quad (9)$$

Such a formula can be further simplified by noticing the following two properties of the Chern number $C_{1s}$ carried by the Fermi surfaces: i) The Chern number of each Fermi surface satisfies $(-1)^{C_{1s}} = (-1)^{m_s}$, where $m_s$ is the number of TRI points enclosed by the $s$-th Fermi surface. ii) The net Chern number of all Fermi surfaces vanishes, $\sum_s C_{1s} = 0$. We will leave a more detailed demonstration of these two conclusions to the supplementary material\textsuperscript{29}, and only sketch the physical reasons for them here. The conclusion i) comes from the fact that a Fermi surface which only encloses one TRI point, such as Fermi surface 1 in Fig. 2, always enclose a singularity at the TRI point due to Kramers’ degeneracy. One can prove that the Chern number is always odd by making use of time-reversal symmetry. The Fermi surfaces enclosing multiple TRI points can be adiabatically deformed into several Fermi surfaces, each enclosing a single TRI point. The conclusion ii) is a consequence of the Nielsen-Ninomiya theorem\textsuperscript{29} which states that the total chirality of a 3D lattice system must be zero.

Using these properties of $C_{1s}$, we finally obtain the following expression for the $2\mathbb{Z}_2$ FSTI which is independent of the interpolation to 3D:

$$N_{2D} = \prod_s \left(\text{sgn(\delta_s)}\right)^{m_s}. \quad (9)$$

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**FIG. 2:** Dimensional reduction from a 3D TRI superconductor to a 2D TRI superconductor. The 2D TRI superconductor corresponds to the $\theta = \pi$ section of a 3D superconductor. The Fermi surfaces with blue (red) color are those with positive (negative) pairing amplitude $\delta_s$. Due to the same symmetry reason as the 3D case, the BdG Hamiltonian $H_k$ of a 2D TRI superconductor can also be written in the form of (1), so that one can also define a matrix $Q_{k,\theta}$, $\theta \in [0, \pi]$ which interpolates between $Q_k$ and the...
The criterion shown in Eq. (9) is quite simple: a 2D TRI superconductor is nontrivial (trivial) if there are an odd (even) number of Fermi surfaces each of which encloses one TRI point in the Brillouin zone and has negative pairing.

**Dimensional reduction to 1D and generic expression of the \( Z_2 \) invariant.** Following the same logic, the dimensional reduction can be carried out again to obtain the \( Z_2 \) FSTI in 1D. This results in an identical formula to Eq. (9). Since in 1D each Fermi “surface” (which consists of two points at \( k_F \) and \(-k_F\)) always encloses one TRI invariant point, the FSTI is simply

\[
N_{1D} = \prod_s (\text{sgn}(\delta_s))
\]

(10)

where \( s \) is summed over all the Fermi points between 0 and \( \pi \). In other words, a 1D TRI superconductor is nontrivial (trivial) if there are an odd number of Fermi points between 0 and \( \pi \) with negative pairing. Two examples with

![Graph showing nontrivial and trivial pairing](image)

**FIG. 3:** Simple examples of (a) nontrivial and (b) trivial pairing in a 1D system. The red and blue dots are Fermi points with negative and positive pairing, respectively. The phase \( \theta = \theta_{nk} \) is taken to be the same at \( k = 0 \) for the two bands. At \( k = \pi \), \( \theta \) of the two bands are differ by \( 2\pi \) for nontrivial pairing (a) and by \( 0 \) for trivial pairing (b).

Interestingly, from Fig. 3 we can get an alternative understanding of the 1D topological superconductor, which can apply to a generic 1D TRI superconductor beyond the weak pairing limit. As discussed earlier in Fig. 1, the sign of the pairing \( \delta_s \) determines the winding of the phase \( \theta_{nk} \) across the Fermi point. On the other hand, we have shown that time-reversal symmetry requires \( T^\dagger Q_k T = Q_{-k}^T \), from which we can find that \( \theta_{nk} = \theta_{n-k} \) if \( |n, k\rangle \) and \( |n, -k\rangle \) label a Kramers’ pair. Thus, along the path from \( k = 0 \) to \( k = \pi \), the change of \( \theta_{nk} \) and \( \theta_{sk} \) must be the same modulo \( 2\pi \):

\[
\int_0^\pi dk (\partial_k \theta_{nk} - \partial_k \theta_{sk}) = 2\pi n, \quad n \in \mathbb{Z}
\]

In the examples shown in Fig. 3 we have \( n = 1 \) for the nontrivial pairing and \( n = 0 \) for the trivial pairing. Such a parity difference of the winding number of \( \theta_{nk} \) turns out to be generic, and can be captured by the following \( Z_2 \) FSTI:

\[
N_{1D} = \frac{\text{Pf} (T^\dagger Q_{k=\pi})}{\text{Pf} (T^\dagger Q_{k=0})} \exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial_k Q_k \right] \right)
\]

(11)

where we have used \( T^\dagger Q_k T = Q_{-k}^T \Rightarrow T^\dagger Q_k = - (T^\dagger Q_{-k})^T \), so that \( T^\dagger Q_k \) is anti-symmetric for \( k = 0, \pi \), and the Pfaffian is well-defined. It is straightforward to show that \( N_{1D} = \pm 1 \) is a \( Z_2 \) quantity, and also a topological invariant. More details on the properties of the \( Z_2 \) FSTI (D1) and its relation to the FSTI (10) are given in the supplementary materials. Eq. (11) is the topological superconductor analog of Kane and Mele’s \( Z_2 \) invariant in quantum spin Hall insulators. Following the same approach as Refs. 22, 23, one can obtain three \( Z_2 \) invariants in 2D, one of which is the “strong topological invariant” \( N_{2D} = N_{1D}(k_y = 0) N_{1D}(k_y = \pi) \) with \( N_{1D}(k_y = 0(\pi)) \) the 1D topological invariant defined for the \( k_y = 0(\pi) \) system, respectively. This topological invariant is robust to disorder, and is equivalent to the one described by Eq. (9).

In summary, we have presented the criteria for TRI topological superconductivity in the physical dimensions one, two and three. When the Fermi surfaces are nondegenerate, the criteria are very simple. In three dimensions, the winding number is an integer which is determined by the sign of pairing order parameter and the Chern number of the Fermi surfaces. In one and two dimensions, a pairing around the Fermi surface is nontrivial if there are an odd number of Fermi surfaces with a negative pairing order parameter. We also obtained an explicit formula for the \( Z_2 \) invariants applicable to generic 1D and 2D TRI superconductors. Our results provide simple and physical criteria that can be used in the search of topological superconductors. Our FSTI’s suggest to search for topological superconductors in the nonconventional superconductors with strong inversion symmetry breaking and strong correlation. The strong inversion symmetry breaking is necessary to generate spin-split Fermi surfaces, and strong electron-electron Coulomb interactions prefer the pairing to have a nonuniform sign in the Brillouin zone.

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APPENDIX A: BACKGROUND AND DERIVATION OF THE BDG HAMILTONIAN

We first list some basic properties of time-reversal invariant superconductors in generic dimensions and then go on to derive the form of Eq. (1) from the Letter. Consider a general TRI superconductor with the Hamiltonian

\[ H = \sum_k \left[ \psi_k^\dagger h_k \psi_k + \frac{1}{2} \left( \psi_k^\dagger \Delta_k \psi_k^\dagger + H.c. \right) \right] = \sum_k \left( \psi_k^\dagger, \psi_k^\dagger \right) H(k) \left( \begin{array}{c} \psi_k \\ \psi_k^\dagger \end{array} \right) \]  

(A1)

with

\[ H(k) = \begin{pmatrix} h(k) & \Delta(k) \\ \Delta^\dagger \Delta & -h^T(-k) \end{pmatrix} \]  

(A2)

The normal state Hamiltonian \( h(k) \) is time-reversal invariant, which means there is a matrix \( T \) satisfying

\[ T^{-1} \psi_k T = \psi_{-k}^\dagger, \quad T^\dagger h_{-k} T = h_{-k}, \quad T = -T^T, \quad T^\dagger T = I \]  

(A3)

From the transformation property of \( \psi_k \) we can obtain

\[ T^{-1} \psi_k^\dagger T = \psi_{-k}^\dagger \]  

(A4)

so that

\[ T^{-1} \left( \begin{array}{c} \psi_k \\ \psi_{-k}^\dagger \end{array} \right) T = \left( \begin{array}{cc} T^\dagger & -T \\ T & -T^\dagger \end{array} \right) \left( \begin{array}{c} \psi_k \\ \psi_{-k}^\dagger \end{array} \right) = \]  

(A5)

and the time-reversal symmetry of the Hamiltonian requires

\[ T^\dagger H(k) T = H(-k)^T, \text{ with } T = \left( \begin{array}{cc} T & -T^\dagger \end{array} \right) \]  

(A6)

On the other hand, the following identity

\[ \left( \begin{array}{c} \psi_k \\ \psi_{-k}^\dagger \end{array} \right)^\dagger \left( \begin{array}{cc} I & \ I \\ \ I & -I \end{array} \right) \left( \begin{array}{c} \psi_{-k} \\ \psi_k^\dagger \end{array} \right) = \]  

(A7)

requires the particle-hole symmetry of the BdG Hamiltonian:

\[ C^\dagger H(k) C = -H(-k)^T, \text{ with } C = \left( \begin{array}{cc} I & \ I \\ \ I & -I \end{array} \right) \]  

(A8)

The two symmetries (A6) and (A8) require the pairing matrix \( \Delta(k) \) to satisfy

\[ \Delta(k) = -\Delta^\dagger(-k), \quad \left( T \Delta^\dagger(k) \right)^\dagger = T \Delta^\dagger(k) \]  

(A9)

If we define

\[ \chi = iTC^\dagger = \left( \begin{array}{c} i^T \\ -i^T \end{array} \right), \]  

(A10)

then we have

\[ \chi^\dagger H(k) \chi = CT^\dagger H(k) TC^\dagger = CH(-k)^T C^\dagger = -H(k) \]  

(A11)

The “chirality operator” \( \chi \) can be diagonalized by

\[ \chi = V^\dagger \left( \begin{array}{cc} I & \ I \\ \ I & -I \end{array} \right) V, \text{ with } V = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} I & \ I \\ \ I & I \end{array} \right) \left( \begin{array}{cc} I & -iI \end{array} \right) \]  

(A12)

To derive Eq. 1 from the main letter we transform the basis to the eigenbasis of \( \chi \) to get the Hamiltonian form

\[ \tilde{H}(k) = VH(k)V^\dagger = \left( \begin{array}{cc} h_k & iT \Delta_k^\dagger \\ -iT \Delta_k & h_k \end{array} \right) \]  

(A13)
As is mentioned in main text, the matrix \( A_k = h_k + i \mathcal{T} \Delta_k^\dagger \) can be decomposed by singular value decomposition (SVD):

\[
A_k \equiv h_k + i \mathcal{T} \Delta_k^\dagger = U_k^\dagger D_k V_k
\]

(A14)
in which \( D_k \) is a diagonal matrix with nonnegative real diagonal components, and \( U_k \) and \( V_k \) are unitary. The Hamiltonian \( \tilde{H}(k) \) can be diagonalized as

\[
\tilde{H}(k) = \begin{pmatrix} U_k^\dagger D_k V_k \end{pmatrix}
\]

\[
\tilde{H}(k) = \begin{pmatrix} U_k^\dagger & D_k & U_k \\ V_k & D_k & V_k \end{pmatrix} \left( \begin{array}{c} U_k \\ D_k \\ V_k \end{array} \right) = \begin{pmatrix} 1 \sqrt{2} & 0 & 1 \sqrt{2} \\ 0 & -1 \sqrt{2} & 0 \end{pmatrix} \left( \begin{array}{c} U_k^\dagger \\ -V_k^\dagger \\ V_k \end{array} \right)
\]

(A15)

Thus we see that the eigenvalues of the Hamiltonian are given by the eigenvalues of \( D_k \) and \( -D_k \). For a gapped Hamiltonian, all the eigenvalues of \( D_k \) are positive, so that we can adiabatically deform \( D_k \) to \( I \), which deforms the Hamiltonian to the form

\[
\tilde{H}(k) = \begin{pmatrix} Q_k^\dagger & Q_k \end{pmatrix}, \quad Q_k \equiv U_k^\dagger V_k \in U(N)
\]

(A16)

It can be seen from the derivation above that \( Q_k \) is uniquely determined by the BdG Hamiltonian \( H_k \), up to a \( k \)-independent \( U(N) \times U(N) \) rotation

\[
Q_k \rightarrow gQ_k h, \quad g, h \in U(N)
\]

(A17)

All physical information carried by \( Q_k \), such as the topological invariants, is insensitive to this global \( U(N) \times U(N) \) rotation.

**APPENDIX B: DETAILED DERIVATION OF THE 3D FERMI SURFACE FORMULA**

In this section we will show the detailed calculation of the 3d Fermi-surface formula. Beginning with the generic form of the winding number in 3d we will show how to derive Eq. 6 from the main letter. The general formula for the integer valued topological number is

\[
N_W = \frac{1}{24\pi^2} \int d^3k \varepsilon^{ijk} \text{Tr} \left[ Q_k^\dagger \partial_i Q_k Q_k^\dagger \partial_j Q_k Q_k^\dagger \partial_k Q_k \right].
\]

(B1)

First of all, if \( \Delta_k = 0 \) for some region of \( k \), the winding number density vanishes in that region. To see that, notice that for \( \Delta_k = 0 \), \( Q_k \) is an adiabatic deformation of \( A_k = h_k \), so that \( Q_k \) is Hermitian, and \( Q_k^\dagger Q_k = \tau_i Q_k = \tau_i \). Consequently the winding number density is given by

\[
\rho_W = \frac{1}{24\pi^2} \varepsilon^{ijk} \text{Tr} \left[ Q_k^\dagger \partial_i Q_k Q_k^\dagger \partial_j Q_k Q_k^\dagger \partial_k Q_k \right].
\]

(B2)

By making use of \( \partial_i (Q_k^2) = Q_k^\dagger \partial_i Q_k + \partial_i Q_k Q_k^\dagger = 0 \), i.e., \( \{Q_k, \partial_i Q_k \} = 0 \), one can prove that \( \rho_W = 0 \). This confirms our statement that in the weak pairing limit, when only the pairing around Fermi surfaces is considered, the topological invariant \( N_W \) is completely determined by the physics in the neighborhood of the Fermi surfaces.

As discussed in Eq. (4) of the letter, in the weak pairing limit \( Q_k \) can be written as

\[
Q_k = \sum_n e^{i\theta_n k} |n, k \rangle \langle n, k|
\]

(B3)

with \( e^{i\theta_n k} = (\epsilon_{nk} + i\delta_{nk}) / |\epsilon_{nk} + i\delta_{nk}| \) and \( \delta_{nk} = |n, k, T \Delta_k^\dagger |n, k \rangle \). To the leading order, near the Fermi surface we have

\[
e^{i\theta_n k} \simeq \frac{\nu_F (k - k_F) + i\delta_{nkF}}{\sqrt{\nu_F^2 (k - k_F)^2 + \delta_{nkF}^2}}
\]

(B4)
In the limit $\delta_{nkF} \to 0$, we have

$$\lim_{\delta_{nkF} \to 0} \theta_{nk} \to \pi \text{sgn} (\delta_{nkF}) \eta (k_F - k_{\perp})$$

(B5)

with $\eta(x)$ the step function satisfying $\eta(x) = 1$, $x \geq 0$ and $\eta(x) = 0$, $x < 0$. Thus we obtain

$$\partial_{k_{\perp}} \theta_{nk} = -\pi \text{sgn} (\delta_{nk}) \delta(k_{\perp} - k_F)$$

(B6)

In the vector form, this equation can be written as Eq. (5) of the letter:

$$\nabla \theta_{nk} = -\pi \nu_{nk} \text{sgn} (\delta_{nk}) \delta (\epsilon_{nk})$$

(B7)

Now we simplify the winding number formula by using Eq. (B3). After some algebra we obtain

$$N_W = \frac{i}{2\pi^2} \int_{FS} d^2k \, \sum_{n,s} \delta_{nk} \cos \frac{\theta_{ns}}{2} \left( a_{nk}^s \sin \frac{\theta_{ns}}{2} \right) - \frac{2i}{3} \sum_p \left( a_p^s \sin \frac{\theta_{np}}{2} \right) \left( a_p^s \sin \frac{\theta_{np}}{2} \right) \left( a_p^s \sin \frac{\theta_{np}}{2} \right)$$

(B8)

where $\theta_{ns} = \theta_n - \theta_s$ and $a_{nk}^i = -i \langle n, k | \partial_i | s, k \rangle$ is the non-Abelian adiabatic connection. When we restrict the pairing to an energy shell $-\epsilon < \epsilon_{nk} < \epsilon$ and take the $\epsilon \to 0$ limit, the only nonvanishing term is the one with $\partial_i \theta_n$ which has a $\delta$ function on the Fermi surface. This leads to

$$N_W = -\frac{i}{2\pi^2} \int_{FS} d^2k \, \sum_{n,s} \partial_{nk} \sin \frac{\theta_{ns}}{2} \left( a_{nk}^s \sin \frac{\theta_{ns}}{2} \right)$$

$$= -\frac{i}{2\pi^2} \int_{FS} d^2k \, \sum_{n,s} \left[ \int d\theta \sin \frac{\theta_n - \theta_s}{2} \right] \left( a_{nk}^s \sin \frac{\theta_{ns}}{2} \right)$$

$$= -\frac{i}{2\pi^2} \int_{FS} d^2k \, \sum_{n,s} \frac{\theta_n - \theta_s}{2} \left( a_{nk}^s \sin \frac{\theta_{ns}}{2} \right)$$

(B9)

where $\theta_n^\pm$ are the values of $\theta_n$ right outside and inside the Fermi surface, respectively. When there is only one band that crosses the Fermi surface, $\theta_n^+ = \theta_n^-$ for all other bands. Labelling the single band crossing the Fermi surface with $n = 0$, we have

$$N_W = -\frac{i}{2\pi^2} \int_{FS} d^2k \, \sum_{s \neq 0} \frac{\theta_0 - \sin \theta_s}{2} \left( a_{0k}^s \sin \frac{\theta_{0s}}{2} \right) \left( a_{0k}^s \sin \frac{\theta_{0s}}{2} \right).$$

(B10)

Since $\theta_0^\pm$ and $\theta_s$ all have the values 0 or $\pi$, the second term $\sin(\theta_0 - \theta_s)$ vanishes, and we have

$$N_W = -\frac{i}{4\pi^2} \int_{FS} d^2k \, \sum_{s \neq 0} \Delta \theta_s \left( a_{0k}^s a_{0k}^{s0} - a_{0k}^{s0} a_{0k}^s \right)$$

$$= \frac{1}{4\pi} \sum_{s \neq 0} \text{sgn} \langle \delta_{nk} \rangle \int_{FS} d^2k \, \partial_1 a_{1k}^{s0} - \partial_2 a_{1k}^{s0} \rangle$$

$$= \frac{1}{2} \sum_{s \neq 0} \text{sgn} (\delta_s) C_{1s}. \quad \text{(B11)}$$

It should be noted that, (i) The superconducting gap on the Fermi surface is given by $|\delta_{nk}|$, so that $\text{sgn}(\delta_{nk})$ is the same for all the $k$ on the same Fermi surface, otherwise the superconducting gap would vanish for some $k$. (ii) The Chern number of the $s$-th Fermi surface $C_{1s}$ is defined with the normal vector along the direction of $v_F$, which is opposite for an electron pocket and a hole pocket.
APPENDIX C: PROOF OF CHERN-NUMBER PROPERTIES USED IN THE DIMENSIONAL REDUCTION TO 2D SECTION OF MAIN LETTER

In the main letter, we have used the following two properties of the Fermi surface Chern number to obtain the 2d $Z_2$ formula:

1. The Chern number of each Fermi surface satisfies $(-1)^{C_{1s}} = (-1)^{m_s}$, where $m_s$ is the number of TRI points enclosed by the $s$th Fermi surface.

2. The net Chern number of all Fermi surfaces vanishes, $\sum_s C_{1s} = 0$.

In this section, we will prove both properties.

1. Proof of Property 1

We first study a simple Fermi surface enclosing one TRI point, e.g., the $\Gamma$ point, as shown in Fig. 4 (a). Denote the states at the Fermi level as $|s, k\rangle$. The Berry phase gauge potential is defined by $a_{ij}^{ss} = -i \langle s, k | \partial_i | s, k \rangle$. In the following, we will denote $a_i = a_{ii}^{ss}$ for simplicity. The time-reversal invariance of the normal state Hamiltonian $h_k$ requires the time-reversed state $T |(s, k)\rangle$ to also be on the Fermi surface. When the bands are non-degenerate on the Fermi surface, in general we have

$$T |(s, k)\rangle = e^{i\varphi_k} |s, -k\rangle$$

$$a_i(-k) = -i \langle s, -k | \frac{\partial}{\partial (-k_i)} | s, -k \rangle = i T (|s, k\rangle e^{i\varphi_k} \partial_i [e^{-i\varphi_k} T |s, k\rangle])$$

$$= \partial_i \varphi_k + i (\langle s, k | \partial_i | s, k \rangle)^*$$

$$= \partial_i \varphi_k + a_i(k).$$

Thus the gauge curvature is

$$f_{ij}(k) = \partial_i a_j(k) - \partial_j a_i(k) = -f_{ij}(-k).$$

We denote the upper half of the Fermi surface with $k_z \geq 0$ as $FS_+$ and the lower half as $FS_-$. Thus

$$C_{1s} = \frac{1}{2\pi} \int_{FS} d\Omega^{ij} f_{ij}(k) = \frac{1}{2\pi} \int_{FS_+} d\Omega^{ij} f_{ij}(k) + \frac{1}{2\pi} \int_{FS_-} d\Omega^{ij} f_{ij}(k)$$

(C4)

Since the two form $d\Omega^{ij}$ denoting the normal direction of the Fermi surface is also odd in $k$, the contributions of $FS_+$ and $FS_-$ to the Chern number are equal, so that

$$C_{1s} = \frac{1}{\pi} \int_{FS_+} d\Omega^{ij} f_{ij}(k).$$

(C5)

Since $FS_+$ is a manifold with boundary, the Chern form is equivalent to a boundary integral:

$$C_{1s} = \frac{1}{\pi} \oint_{\partial FS_+} dl^i a_i(k)$$

(C6)

in which $\partial FS_+$ is the boundary of $FS_+$, i.e., the $k_z = 0$ section of the Fermi surface, and $dl^i$ is the tangent vector to $\partial FS_+$. However, it should be noted that Eq. (C6) holds only if $a_i(k)$ is continuous in the whole $FS_+$. In a generic gauge transformation $a_i(k) \rightarrow a_i(k) + \partial_i \phi_k$ on the boundary $\partial FS_+$, the right hand side of Eq. (C6) can change by an even number:

$$\frac{1}{\pi} \oint_{\partial FS_+} dl^i a_i(k) \rightarrow \frac{1}{\pi} \oint_{\partial FS_+} dl^i a_i(k) + \frac{1}{\pi} \oint_{\partial FS_+} dl^i \partial_i \phi_k = \frac{1}{\pi} \oint_{\partial FS_+} dl^i a_i(k) + 2n, \; n \in \mathbb{N}$$

(C7)

Thus we have

$$C_{1s} = \frac{1}{\pi} \oint_{\partial FS_+} dl^i a_i(k) \text{ mod } 2$$

(C8)
in a generic gauge choice.

Since the section \( k_z = 0 \) of the Fermi surface is also symmetric under time-reversal, we can always reduce the Chern number to a \( \pi \) integral over the upper half of the Fermi surface.

\( \oint_{\partial\Sigma} d\ell^i a_i(k) = -\oint_{\partial\Sigma} d\ell^i a_i(k) - \oint_{\partial\Sigma} d\ell^i \partial_i \varphi_k \)

\( \Rightarrow C_{1s} = \frac{1}{\pi} \oint_{\partial\Sigma} d\ell^i a_i(k) = \frac{1}{\pi} \left( \oint_{L_1} + \oint_{L_2} \right) d\ell^i a_i(k) = -\frac{1}{\pi} \oint_{L_2} d\ell^i \partial_i \varphi_k. \)  \( (C9) \)

One can always split the boundary so that there are only two points \( A \) and \( B \) on the interface between \( L_1 \) and \( L_2 \). Due to time-reversal symmetry, the two points must be the time-reversed partners of each other, and the formula above becomes

\( C_{1s} = -\frac{1}{\pi} (\varphi_A - \varphi_B) \mod 2 \)  \( (C10) \)

Denote the momentum of \( A \) and \( B \) as \( k_A \) and \( k_B = -k_A \), according to the definition Eq. \( (C1) \) we have

\[ T \left( |s, k_A \rangle \right) = e^{i\varphi_A} |s, k_B \rangle, \quad T \left( |s, k_B \rangle \right) = e^{i\varphi_B} |s, k_A \rangle \]

\( \Rightarrow T \left( T \left( |s, k_A \rangle \right) \right) = T \left( e^{i\varphi_A} |s, k_B \rangle \right) = e^{-i\varphi_A} e^{i\varphi_B} |s, k_A \rangle \)  \( (C11) \)

On the other hand, we have \( T^2 = -1 \) for each state, so that

\[ e^{i(\varphi_A - \varphi_B)} = -1 \Rightarrow C_{1s} = -1 \mod 2 \]  \( (C12) \)

Thus we have proved t

FIG. 4: (a) Schematic picture of a Fermi surface enclosing one TRI point \((0, 0, 0)\). The Fermi surface is separated to two parts \( FS_+ \) and \( FS_- \) by \( k_z = 0 \) plane. The interface between the two parts is further split into curves \( L_1 \) (red curve) and \( L_2 \) (blue curve) which are time-reversal partner of each other. The interface of \( L_1 \) and \( L_2 \) are given by points \( A \) and \( B \). (b) Schematic picture of a Fermi surface enclosing two TRI points \((0, 0, 0)\) and \((0, 0, \pi)\). Similar to (a), the Fermi surface is separated to \( FS_+ \) and \( FS_- \), and the interface between \( FS_+ \) and \( FS_- \) are split into \( L_1 \) (red curve) and \( L_2 \) (blue curve), which intersect at two pairs of points \( A_1, B_1 \) and \( A_2, B_2 \).

For the Fermi surfaces enclosing more TRI points, as shown in Fig. 4 (b), the proof is similar. Due to the time-reversal symmetry, we can always reduce the Chern number to an integral over the upper half of the Fermi surface \( FS_+ \) as in Eq. \( (C5) \) and \( (C6) \). Generically, \( FS_+ \) has two boundaries at \( k_z = 0 \) and \( k_z = \pi \), so that

\[ C_{1s} = \frac{1}{\pi} \oint_{FS_+} d\ell^i f_{ij} \langle k \rangle = \frac{1}{\pi} \left( \oint_{\partial_0 FS_+} - \oint_{\partial_\pi FS_+} \right) d\ell^i a_i(k) \]  \( (C13) \)

where \( \partial_0 FS_+ \) stands for the boundary of \( FS_+ \) at \( k_z = 0 \) and \( k_z = \pi \) respectively. In the same way as above, the boundary at \( k_z = 0 \) can each be separated into two parts \( L_1 \) and \( L_2 \), with several pairs of interface points \( A_i, B_i, i = 1, 2, ..., p_0 \). By the same derivation as above one can prove \( \varphi_{A_i} - \varphi_{B_i} = \pi \mod 2\pi \), and

\[ \frac{1}{\pi} \oint_{\partial_\pi FS_+} d\ell^i a_i(k) = -\frac{1}{\pi} \sum_{i=1}^{p_0} (\varphi_{A_i} - \varphi_{B_i}) = p_0 \mod 2 \]  \( (C14) \)
The same argument works for the \( k_z = \pi \) boundary. Denoting the number of interface points at the \( k_z = \pi \) boundary by \( p_\pi \), we have

\[
C_{1s} = p_\pi - p_0 \mod 2. \tag{C15}
\]

If the boundary \( \partial_{0,\pi} \text{FS}_+ \) encloses \( m_{0,\pi} \) number of TRI points, respectively, we have \( p_0 = m_0 \mod 2 \), \( p_\pi = m_\pi \mod 2 \). Thus

\[
(-1)^{C_{1s}} = (-1)^{p_\pi - p_0} = (-1)^{m_\pi + m_0} = (-1)^{m_s} \tag{C16}
\]

with \( m_s = m_\pi + m_0 \) the total number of TRI points enclosed by the Fermi surface. Thus we have proved the property 1.

### 2. Proof of Property 2

To prove property 2, we take a simple \( s \)-wave pairing

\[
\Delta_k = \Delta_0 T \tag{C17}
\]

with \( \Delta_0 \) a real number. For such a pairing the matrix \( A_k = h_k + i T \Delta_k^\dagger = h_k + i \Delta_0 i \), so that the pairing on all the Fermi surfaces has the same sign:

\[
\delta_s = \langle s, k | T \Delta_k^\dagger | s, k \rangle = \Delta_0, \quad \forall s. \tag{C18}
\]

Consequently, the topological invariant in the weak pairing limit is given by

\[
N_W(\Delta_0) = \frac{1}{2} \sum_s \text{sgn}(\delta_s) C_{1s} = \frac{\text{sgn}(\Delta_0)}{2} \sum_s C_{1s}. \tag{C19}
\]

On the other hand, the BdG Hamiltonian (A13) for this simple pairing can be diagonalized easily to obtain the eigenvalues

\[
E_{nk}^\pm = \pm \sqrt{\epsilon_{nk}^2 + \Delta_0^2}. \tag{C20}
\]

Thus for finite \( \Delta_0 \), the spectrum of the BdG Hamiltonian is always gapped, so that the winding number \( N_W(\Delta_0) \) remains invariant for all \( \Delta_0 > 0 \). Thus we can compute \( N_W \) in the limit \( \Delta_0 \to +\infty \). The unitary matrix \( Q_k \) is given by

\[
Q_k = \sum_n |n, k\rangle \frac{\epsilon_{nk} + i \Delta_0}{\sqrt{\epsilon_{nk}^2 + \Delta_0^2}} |n, k\rangle, \quad \Rightarrow \lim_{\Delta_0 \to +\infty} Q_k = i \sum_n |n, k\rangle \langle n, k| = i \mathbb{I} \tag{C21}
\]

Obviously, the winding number \( N_W(\Delta_0 \to +\infty) = 0 \), so that \( N_W(\Delta_0) = 0 \) for any \( \Delta_0 \). According to Eq. (C19) we have proven property 2:

\[
\sum_s C_{1s} = 0. \tag{C22}
\]

## APPENDIX D: PROPERTIES OF THE 1D \( Z_2 \) TOPOLOGICAL INVARIANT (11)

In this section, we will study some basic properties of the \( Z_2 \) topological invariant defined in Eq. (11) of the letter, and show how it is reduced to the Fermi surface formula (10) in the weak pairing limit.

We start from Eq. (10) of the letter:

\[
N_{1d} = \frac{\text{Pf}(T_k^{\dagger} Q_{k=\pi})}{\text{Pf}(T_k^{\dagger} Q_{k=0})} \exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_k^{\dagger} \partial_k Q_k \right] \right) \tag{D1}
\]
First of all, $\mathcal{T}^\dagger Q_k$ is antisymmetric since

$$\mathcal{T}^\dagger h_k \mathcal{T} = h_k^\dagger, \quad \mathcal{T}^\dagger \left( \mathcal{T} \Delta_k^\dagger \right) \mathcal{T} = \Delta_k^\dagger \mathcal{T} = -\Delta_k^\dagger \mathcal{T} = \left( \mathcal{T} \Delta_k^\dagger \right)^T$$

$$\Rightarrow \mathcal{T}^\dagger Q_k \mathcal{T} = Q_k^\dagger \mathcal{T} \Rightarrow \mathcal{T}^\dagger Q_k = Q_k^\dagger \mathcal{T}^\dagger = - (\mathcal{T}^\dagger Q_k)^T \tag{D2}$$

Thus the Pfaffian is well-defined at $k = 0$ and $k = \pi$.

Since $Q_k \in U(N)$, we have $\det Q_k = e^{i\varphi_k}$ which is a U(1) phase. Since $\text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] = \text{Tr} \left[ \log Q_k \right] = \log \det Q_k = i\varphi_k$, we have $\int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] = i(\varphi(\pi) - \varphi(0)) \mod 2\pi$, so that

$$\exp \left( -\int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] \right) = e^{-i(\varphi(\pi) - \varphi(0))} = \frac{\det (\mathcal{T}^\dagger Q_{k=0})}{\det (\mathcal{T}^\dagger Q_{k=\pi})} \tag{D3}$$

Thus

$$N_{1d}^2 = \frac{\det (\mathcal{T}^\dagger Q_{k=\pi})}{\det (\mathcal{T}^\dagger Q_{k=0})} \exp \left( -\int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] \right) \equiv 1 \tag{D4}$$

so that $N_{1d}$ always takes the value of $\pm 1$.

Now we show that $N_{1d}$ is a topological invariant. For an infinitesimal deformation $Q_k > Q_k' = Q_k + \delta Q_k$, the phase factor $\exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] \right)$ only depends on the deformation of $Q_k$ at $k = 0$ and $\pi$:

$$\exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] \right) = \exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_k^\dagger \partial h Q_k \right] \right) e^{-\frac{1}{2} \delta \varphi(\pi) - \delta \varphi(0)} \tag{D5}$$

On the other hand, the change of Pfaffian is given by

$$\text{Pf} \left( \mathcal{T}^\dagger Q_{k=0} \right) = e^{\frac{1}{2} \delta \varphi_{k=0,\pi}} \text{Pf} \left( \mathcal{T}^\dagger Q_{k=\pi} \right) \tag{D6}$$

Consequently, we see that $\delta N_{1d} = 0$ in any smooth deformation of the unitary matrix $Q_k$ as long as time-reversal symmetry is preserved.

In the weak pairing limit, the general formula (D1) can be reduced to the Fermi surface formula given by Eq. (10) of the letter. In the weak pairing limit, assume there are $M$ Fermi points $k_s$, $s = 1, 2, ..., M$ between 0 and $\pi$. As discussed in the main text, we require that the Fermi level does not cross any band at $k = 0$ or $\pi$. As discussed in Fig. 1 of the letter, each Fermi point leads to a domain wall of $\theta_{sk}$ for the corresponding band $s$ crossing the Fermi level. According to Eq. (B3) in the weak pairing limit we have

$$\det Q_k = \exp \left( i \sum_n \theta_{nk} \right) \tag{D7}$$

Across each Fermi point $k_{Fs}$, the phase $\theta_{sk}$ will jump by $-\pi \text{sgn} (v_{Fs} \delta_{sk_s})$ and the $\theta_{nk}$ for other bands remain invariant. It should be noted that the sign of $v_{Fs}$ enters the expression since the winding of $\theta_{sk}$ is given by $-\pi \text{sgn} (\delta_{sk})$ along the direction of the Fermi velocity $v_{Fs}$. Consequently, the phase log det $Q_k = i \sum_n \theta_{nk}$ is changed by $-i \pi \text{sgn} (v_{Fs} \delta_{sk_s})$ across the $s$-th Fermi point, and the net change of log det $Q_k$ from 0 to $\pi$ is given by

$$\int_0^\pi dk \partial k \log \det Q_k = -i \pi \sum_{s=1}^{M} \text{sgn} (v_{Fs} \delta_{sk_s})$$

$$\Rightarrow \exp \left( -\frac{1}{2} \int_0^\pi dk \text{Tr} \left[ Q_{k_s}^\dagger \partial h Q_{k_s} \right] \right) = \prod_s e^{-\frac{1}{2} \text{sgn} (v_{Fs} \delta_{sk_s})} \equiv \prod_s (-\text{sgn} (v_{Fs} \delta_{sk_s}))$$

$$\Rightarrow \prod_s (-\text{sgn} (v_{Fs})) = (-i)^m i^n = e^{\frac{1}{2} \pi (n-m)} = (-1)^{\frac{N_2(\pi) - N_2(0)}{2}} \tag{D8}$$

When there are $m$ Fermi points with positive $v_{Fs}$ and $n$ Fermi points with negative $v_{Fs}$, $n - m$ gives the number of bands which are above the Fermi level at $k = 0$, but below the Fermi level at $k = \pi$. If we denote $N_2(0)$ and $N_2(\pi)$ as the number of bands occupied at $k = 0$ and $k = \pi$, respectively, then $n - m = N_2(\pi) - N_2(0)$. Since all bands are paired in Kramers pairs at $k = 0, \pi$, $N_2(0)$ and $N_2(\pi)$ must be even. Thus we have

$$\prod_s (-\text{sgn} (v_{Fs})) = (-i)^m i^n = e^{\frac{1}{2} \pi (n-m)} = (-1)^{\frac{N_2(\pi) - N_2(0)}{2}} \tag{D9}$$
Now we study the Pfaffian \( \text{Pf} (\mathcal{T}^\dagger Q_{k=0,\pi}) \). Since we have assumed the Fermi level does not cross the bands at \( k = 0,\pi \), in the weak pairing limit we have \( \Delta_{k=0,\pi} = 0 \). If the normal state Hamiltonian \( h_k \) is diagonalized to

\[
h_k = U_k^\dagger \begin{pmatrix} \epsilon_1(k) & \cdots & \epsilon_N(k) \end{pmatrix} U_k \tag{D10}
\]

\( Q_k \) can be obtained by

\[
Q_k = U_k^\dagger \begin{pmatrix} I_{N_1 \times N_1} - I_{N_2 \times N_2} \end{pmatrix} U_k \tag{D11}
\]

in which \( N_1 \) and \( N_2 \) are the number of unoccupied and occupied bands, respectively. The Pfaffian \( \text{Pf} (\mathcal{T}^\dagger Q_k) \) can be obtained as

\[
\text{Pf} (\mathcal{T}^\dagger Q_k) = \text{Pf} \left( \mathcal{T}^\dagger U_k^\dagger \begin{pmatrix} I_{N_1 \times N_1} - I_{N_2 \times N_2} \end{pmatrix} U_k \right) = \text{Pf} \left( U_k^\dagger T^\dagger U_k^\dagger \begin{pmatrix} I_{N_1 \times N_1} & -I_{N_2 \times N_2} \end{pmatrix} \right) \cdot \det U_k \tag{D12}
\]

By making use of the time-reversal invariance condition \( T^\dagger Q_k T = Q_{-k}^\dagger \), one can prove that

\[
\left[ U_k^\dagger T^\dagger U_k^\dagger, \begin{pmatrix} I_{N_1 \times N_1} & -I_{N_2 \times N_2} \end{pmatrix} \right] = 0 \tag{D13}
\]

for \( k = 0,\pi \), so that the matrix \( U_k^\dagger T^\dagger U_k^\dagger \) is block diagonal. Consequently, we have

\[
\text{Pf} \left( U_k^\dagger T^\dagger U_k^\dagger \begin{pmatrix} I_{N_1 \times N_1} & -I_{N_2 \times N_2} \end{pmatrix} \right) = (-1)^{N_2/2} \text{Pf} \left( U_k^\dagger T^\dagger U_k^\dagger \right) = (-1)^{N_2/2} \text{Pf} (T^\dagger) \cdot \det U_k^\dagger \tag{D14}
\]

Thus

\[
\text{Pf} (\mathcal{T}^\dagger Q_k) = (-1)^{N_2/2} \text{Pf} (T^\dagger) \tag{D15}
\]

for \( k = 0,\pi \). It should be noted that the number of occupied bands \( N_2 \) is always even for \( k = 0,\pi \) due to Kramers degeneracy.

Combining Eq. (D8), (D9) and (D15) we obtain

\[
N_{1d} = \prod_s (\text{sgn} (\delta_{s_{k_s}})) \tag{D16}
\]

Thus we have proved that the general \( Z_2 \) invariant (11) in the main text is equivalent to Eq. (10) in the main text in the weak pairing limit.

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