Universality of critical exponent in scale-free networks

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(Dated: September 28, 2021)

We study the statistical properties of observables in scale-free networks under the degree-thresholding renormalization (DTR) transformation. For BA scale-free networks at different scales, we find that their structural and dynamical observables have similar scaling behavior along the DTR flow. The finite-size scaling analysis confirms this view and reveals a scaling function with a single critical exponent that collectively captures the changes of these observables. Furthermore, by using the DTR technique, we obtain multiple downscaled subnetworks of the real scale-free network with a single resolution. An interesting result is that these observables of real scale-free networks and their subnetworks share the same critical exponent as the BA synthetic network. Our findings have important guiding significance for analyzing the critical behavior of networks. Such as in large-scale simulation scenarios with high time complexity, downscaled networks could serve as a substitute or guide for the initial network, and then quickly exploring the crucial behavior of networks.

The network widely exists in both nature and human society and is a common language for describing and modeling complex systems. Its emergence contributes to a better understanding of the structure and functional attributes of the system [1, 2]. Many real networks have a significant feature, whose degree distribution is usually approximately scale-free, which can be described by the power-law distribution $p(k) \sim k^{-\lambda}$ in the mathematical background, and the network satisfying this feature is called the scale-free network [3]. In addition, this feature effectively distinguishes scale-free networks from regular networks, ER random networks [4], and WS small-world networks [5]. As a result, scale-free networks have many peculiar properties, such as robustness under random attacks, vulnerability under target attacks [6], and faster spreading speed of virus on scale-free networks [7].

In the following years, another important feature of networks, self-similarity [8], has also been discovered by borrowing concepts related to the renormalization group [9] in statistical physics. Some major works include: Song et al. [8] proposed a box-covering renormalization (BCR) method based on shortest path lengths between network nodes to reduce the size of the network, and they found that the degree distribution of real networks such as WWW remained approximately unchanged during the BCR iteration. Serrano et al. [10] presented a simple degree-thresholding renormalization (DTR) scheme for mining the self-similar properties of real networks. Recently, García-Pérez et al. [11] offered a network geometric renormalization (GR) approach in the context of the hidden metric space model, which provides another insight for studying the structural symmetry of networks. In short, the significance of renormalization of the network is to find smaller replicas to replace the original network, so as to effectively explore the self-similarity of the real network. These approaches achieve good results for specific scenarios, but in contrast to DTR, GR and BCR all require special assumptions, such as embedding the network into space or based on the shortest path length [12].

![Degree-thresholding renormalization diagram](image_url)

**FIG. 1.** Degree-thresholding renormalization diagram. We consider a BA scale-free network with 500 nodes ($k_T = 0$) as the initial network, the parameter $m = 2$, where the parameter $m$ is the number of links increased after adding a new node. The threshold values $k_T = 2, 5, 10$ are selected to obtain three network snapshots with progressively smaller scales.

In order to avoid ad hoc assumptions, in this Letter, based on the DTR scheme [10], we conduct a series of studies on the structural and dynamical observables of scale-free networks. The specific steps of DTR are as follows: for network $G(N, E)$ with $N$ nodes and $E$ edges, given degree threshold $k_T = 0, 1, 2, \ldots$, then, nodes with degrees $k > k_T$ are extracted from $G(N, E)$ to obtain the subnetwork $G(k_T)$ (i.e., nodes with degrees less than or equal to $k_T$ are deleted), we thereby obtain a series of downscaled subnetworks, and the number of nodes contained in the subnetwork $G(k_T)$ is denoted by $N_{k_T}$. Starting from a BA scale-free network with 500 nodes, we delete those nodes with degrees $k \leq k_T = 2, 5, 10$ based on the initial network, and obtain three downscaled subnetworks (see Fig. 1). The red node in the center is the hub node, it can be found that the hub node always exists in the network during this process. Serrano et al. [10] have shown that some basic features
are self-similar, such as the complementary cumulative degree distribution, degree-dependent clustering coefficient, and degree-degree correlations (see Fig. S1 in the Supplemental Material [13]).

In this Letter, our results show that some basic and important topological observables of BA scale-free networks and real scale-free networks follow general scaling laws under the DTR flow, such as largest node degree, average degree (edge density), average clustering coefficient, and the ith moment of node degree. Detailed results are provided in the Supplemental Material [13]. However, because the actual system is affected by the finite size effect, even if we consider a scale-free network with sufficient initial size, excessive DTR iterations will eventually destroy the structural symmetry (or scale invariance) of the original network, resulting in some observables deviating from thermodynamic limit behavior. Here, we show that the potential scale invariance of network observables is often masked by finite-size effects. We employ the finite-size scaling (FSS) [14] analytical tool to prove this view and reveal a scaling function with a single critical exponent that jointly captures the critical behavior of these observables. Using \( Z \) represents a particular observable, we find that under the FSS hypothesis,

\[
Z = f \left( n_{k_T} N_0^{1/\alpha} \right),
\]

where \( N_0 \) is the size of the initial network, \( n_{k_T} \) \( (n_{k_T} = N_{k_T}/N_0) \) is the relative size of the subnetwork \( G(k_T) \), and the behavior of the observable \( Z \) is completely determined by the exponent \( \alpha \) and the scaling function \( f \).

For BA scale-free networks, Fig. 2(a), (c), and (e) show the dependence of observables \( \bar{k}_{k_T, \text{max}} \), \( \langle k \rangle_{k_T} \), \( \langle c \rangle_{k_T} \) of the subnetwork \( G(k_T) \) on the relative size \( n_{k_T} \) respectively. Our results show that these observables are all scaling functions of the variable \( n_{k_T} \). By using the exponent \( \alpha \) to rescale the horizontal axis, the observables curves for different network sizes could collapse onto a master curve [see Fig. 2(b), (d), and (f)]. Notably, these observables share the same critical exponent, \( \alpha \approx 1 \). Here, the optimal exponent \( \alpha \) is obtained by measuring the quality \( S \) of the collapse plot, where the \( S \) is used to measure the mean square distance between the collapse data and the master curve, and its detailed definition is shown in Ref. [15]. Taking the normalized largest degree as an example, Fig. S8 [13] shows the dependence of the \( S \) on exponent \( \alpha \), when \( \alpha = 1.01 \), \( S \) achieves the minimum value, which means that different curves have the best collapse.

As a supplement, we also consider some global observables that depend on the underlying structure of the network. Such as the normalized smallest nonzero eigenvalue of the Laplace matrix, \( \lambda_{k_T, 2}(L) = [\Lambda_{k_T, 2}(L)]/N_{k_T} \),
As shown in Fig. S9 [13], we observed that the scaling behavior of these dynamic observables is similar to that in Fig. 2. Under the rescaling of Eq. (1), these curves of BA networks with different sizes could collapse onto the same master curve. In particular, an same critical exponent, \( \alpha = 1 \), is produced here. Our results have important guiding significance for studying the dynamic behavior of large-scale networks. For instance, in practice, one possible outcome is to predict the synchronization stability of the initial large-scale network by taking the subnetwork as the study object and combining this critical exponent.

In fact, empirical studies have shown that the degree distribution of a large number of real-world networks approximately satisfies \( p(k) \sim k^{-\lambda} \), in general, the exponent \( \lambda \in (2, 3) \). Therefore, the BA scale-free network is only a particular case among many scale-free networks. Next, we employ a more general scale-free network model, Chung-Lu (CL) scale-free network model [23–25], as an extension model, where the degree distribution exponent \( \lambda \) is used to control the heterogeneity of the network. We perform the same study as Fig. 2 on CL networks with different heterogeneity exponent \( \lambda \), examining the dependence of critical exponent \( \alpha \) on the heterogeneity exponent \( \lambda \), as shown in Fig. 3 (●, ■, and ▼). Our results show that, for each observable, the critical exponent \( \alpha \) decreases gradually with the increase of the \( \lambda \), which means that the critical exponent \( \alpha \) strongly depends on the heterogeneity of the network. Moreover, for different observables, the critical exponent \( \alpha \) has a large difference. Particularly, when \( \lambda = 3 \), the difference is almost negligible, that is, each observable corresponds to an approximately same critical exponent, i.e., \( \alpha \approx 1 \), which is consistent with the results of BA scale-free networks (see Fig. 2).

When evaluating the critical exponent \( \alpha \) of synthetic networks, we can obtain this result by generating a series of networks with different initial sizes. For real networks, however, there is usually a single initial size. A natural question, then, is how to calculate the critical exponent for a particular network. To address this issue, we carried out further research on scale-free networks with a single initial size. First, taking CL scale-free models as an example, we generate an initial network \( G_0 \) with \( N_0 = 10^5 \), set the threshold \( K_T \), and obtain a subnetwork \( G(K_T) \) of \( G_0 \) by using DTR technology. Notice that the \( K_T \) is used here to distinguish \( k_T \). Then, \( G_0 \) and its subnetworks are taken as research objects to study the FSS behavior of their observables under the DTR flow. Thus, under the FSS hypothesis, the observable \( Z \) satisfies,

\[
Z = f \{ n_{k_T} [N(K_T)]^{1/\alpha} \},
\]

where \( N(K_T) \) is the size of the subnetwork \( G(K_T) \). An interesting result is that the critical exponent implied by these observables of the initial network and its subnetwork is independent of the heterogeneity exponent \( \lambda \) and

\[
\text{FIG. 3. Dependence of critical exponent } \alpha \text{ on heterogeneity exponent } \lambda. \text{ Solid symbols (●, ■, and ▼) are the results obtained by FSS analysis of scale-free networks with different initial sizes. Here we consider four initial sizes, } N_0 = 10000, 20000, 50000, \text{ and } 100000. \text{ Hollow symbols (○, □, and ▲) represent the results of a single specific scale-free network and its subnetworks. The gray horizontal straight line shows the position of } \alpha = 1. \text{ Where } Z_1, Z_2, \text{ and } Z_3 \text{ represent } k_{\text{max}}, \langle k \rangle_{k_T}, \text{ and } \langle c \rangle_{k_T}, \text{ respectively, and all results are averaged over 100 independent realizations.}
\]
shares the same critical exponent with the BA network, namely, \( \alpha = 1 \), as shown in Fig. 3 (\( \Box \), \( \square \), and \( \nabla \)). In this case, different observables’ critical exponents are approximately equal. Here we give only the results of three topological observables, we confirm the universality of \( \alpha = 1 \) from another aspect and combine this exponent to predict the largest eigenvalue of the adjacency matrix of large-scale heterogeneous networks (see Fig. S10 [13]), which usually determines the epidemic spreading threshold of the network.

Finally, we perform further studies on some real scale-free networks, and details of these real network datasets are shown in Table S1 [13]. Figure 4 shows the observables results of the Internet and its three subnetworks \( G(K_T) \), where \( K_T = 1, 2, 4 \). To ensure the subnetwork \( G(K_T) \) has approximately the same degree distribution as the original network (see Fig. S11 [13]), the value of \( K_T \) should not be selected too large. Taking Internet and these three subnetworks as research objects, we perform DTR transform on them, respectively. By rescaling the horizontal axis according to Eq. (2), these observables curves of subnetworks and the original Internet network can collapse onto the same master curve, where the exponent \( \alpha \approx 1 \). For other real scale-free networks, an approximately equal exponent \( \alpha \) is obtained, as shown in the last column of Table S1 [13]. The result further confirm that scale-free networks and their subnetworks share the same universality class critical exponent.

The scaling function used in this Letter is similar in spirit to the Refs. [26–28], by comparison, our work is carried out on DTR renormalization techniques with fewer external constraints. Specifically, we study the statistical properties of some representative observables of scale-free networks under the DTR transformation. In the context of FSS analysis, we find a universal scaling method that always yields observed data collapse at different network sizes. Most importantly, our results show that for a particular scale-free network, the observables of both itself and its subnetworks share the same universal class critical exponent, \( \alpha = 1 \), which implies that downscaled networks could also be applied to perform finite-size scaling and the critical exponent can be determined from a single snapshot of the topology. From the perspective of the application, the critical exponent obtained here can be used as the foundation for predicting the epidemic spreading threshold of the large-scale network, which has important guiding significance for reducing the time complexity of the large-scale numerical simulation.

This work was supported by the National Natural Science Foundation of China under Grants No. 61991412 and 61873318, the Frontier Research Funds of Applied Foundation of Wuhan under Grant No. 2019010701011421, and the Program for HUST Academic Frontier Youth Team under Grant.
No. 2018QYTD07.

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Supplemental Material for “Universality of critical exponent in scale-free networks”

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I. NETWORK MODELS

As one of the most classic scale-free network models, BA model successfully explains the most basic evolution characteristics of the real-world network. Its generating mechanisms are dominated by two factors: growth and preferential attachment [1]. Under this background, the network eventually develops into a scale-free network with a degree distribution exponent $\lambda = 3$. The exact form of the degree distribution of the BA model is $p(k) = [2m(m + 1)]/[k(k + 1)(k + 2)]$. For large $k$, approximately satisfies $p(k) \sim k^{-3}$, where the parameter $m$ is the number of links increased after adding a new node. The paper uses the Python-based NetworkX library [2] to generate a BA scale-free network.

Fig. S1(a), (b) and (c) show the complementary cumulative degree distribution, degree-dependent clustering coefficient, and degree-degree correlations of a BA scale-free network and its corresponding subnetworks. The average degree of the initial network satisfies $\langle k \rangle \approx 2m = 10$. We find that each feature almost collapses on the same main curve in the process of DTR, which indicates that the main structural properties of the network are self-similar along the DTR flow.

II. SCALING PROPERTIES OF SCALE-FREE NETWORKS ALONG THE DTR FLOW

In this section, we study the statistical properties of observables of BA scale-free networks and real scale-free networks, and the results show that these observables obey the general scaling law under the DTR flow. Specifically, we investigate the most basic and important topological characteristics of the network, such as the largest node degree, average degree (edge density), average clustering coefficient, and the $i$th moment of node degree.

Taking the largest degree as an example, in general, the probability that a node’s degree value is equal to or higher than $k_{\text{max}}$ can be found in the network is $1/N$ [3]. For power-law degree distribution $p(k) \sim k^{-\lambda}$, since $\int_{k_{\text{max}}}^{\infty} k^{-\lambda} dk = 1/N$, we have $k_{\text{max}} \sim N^{1/(\lambda-1)}$. Based on this, for a network $G_0$ with $N_0$ nodes, the largest degree satisfies $k_{0,\text{max}} \sim N_0^{1/(\lambda-1)}$, for subnetwork $G(k_T)$, the largest degree denotes $k_{k_T,\text{max}}$. The results of Ref. [4] show that, within the range of finite DTR iterations, the subnetwork $G(k_T)$ and the original network approximately obey the
power-law distribution of the same degree distribution exponent, therefore, $k_{kT,max} \sim N_{kT}^{1/(\lambda-1)}$. For a BA scale-free network, Fig. S2(a) shows that the largest degree of subnetwork $G(k_T)$ is a monotonically decreasing function as the threshold value $k_T$. The inset shows the dependence of largest degree on subnetwork size, showing that $k_{kT,max} \sim N_{kT}^{\beta}$, where $\beta = 0.5228$. Here, if using the maximum likelihood method [5] to estimate the degree distribution exponent of the BA scale-free network, we can get $\lambda \approx 2.89$, obviously, $\beta \approx 1/(\lambda - 1)$. This result implies that the DTR transform can maintain the scaling characteristics of the largest node degree of the original BA scale-free network.

FIG. S2. Observables of the BA scale-free network as a function of the degree threshold, the inset shows the dependence of each observable on the size of the subnetwork $G(k_T)$. (a) The largest degree $k_{kT,max}$ of the subnetwork $G(k_T)$ as a function of the degree threshold $k_T$, where the inset shows $k_{kT,max} \sim N_{kT}^{\beta}$. (b) The average degree $\langle k \rangle_{k_T}$ of the subnetwork $G(k_T)$ as a function of the degree threshold $k_T$. The inset shows the dependence of the normalized average degree (i.e., the link density) $\langle k \rangle_{k_T}/N_{kT}$ on $N_{kT}$, approximately satisfies $\langle k \rangle_{k_T} \sim N_{kT}^{-\alpha}$. (c) The average clustering coefficient $\langle c \rangle_{k_T}$ of the subnetwork $G(k_T)$ as a function of the degree threshold $k_T$, the inset shows $\langle c \rangle_{k_T} \sim N_{kT}^{-\mu}$, the red square is the simulation result, $\mu = 0.6966$, and the green circle is the theoretical result, $\mu = 0.6733$. The size of the initial network is $N = 10^4$, the parameter $m = 5$, error bars show the standard deviations for different realizations, and all results are averaged over 100 independent realizations.

The results in Fig. S2(b) show the average degree $\langle k \rangle_{k_T}$ of the subnetwork decreases gradually during renormalization. The inset shows the dependence of the normalized average degree $\langle k \rangle_{k_T}/N_{kT} - 1$ on the size of the subnetwork $N_{kT}$, which approximately satisfies the power-law behavior, $\langle k \rangle_{k_T} \sim N_{kT}^{-\gamma}$. Figure S2(c) shows the average clustering coefficient $\langle c \rangle_{k_T}$ of the network gradually increases along the direction of DTR flow, and the results of the inset show that $\langle c \rangle_{k_T} \sim N_{kT}^{-\mu}$, as shown in the red box, and the dashed line is the fitting result. Klemm et al. [6] have proved that the average clustering coefficient of the BA network meets $\langle c \rangle \sim (\ln N)^2/N$, and the average clustering coefficient of subnetwork $G(k_T)$ can be calculated based on this conclusion, as shown in the green circle. It can be seen that this result is roughly equal to the actual simulation result (i.e., the green box and the red circle roughly overlap), which implies that DTR transformation will not change the statistical law of the average clustering coefficient for the initial BA network. It also reflects from the side that the DTR downsampling process of the BA network can be approximately used as the inverse process of evolutionary growth. Therefore, DTR is of great significance for predicting the structural characteristics of large-scale networks.

In fact, empirical studies show that the degree distribution of a large number of real-world networks approximately satisfies $p(k) \sim k^{-\lambda}$, and in general, the exponent $\lambda \in (2, 3)$. Therefore, BA scale-free network is just a special case
among many scale-free networks. Next, we consider eight real scale-free networks, which belong to four different categories: Biological, Technological, Text, and Affiliation [7]. The detailed topological information is shown in Table S1. After examining the statistical properties of these networks and their subnetworks, respectively, a conclusion similar to Fig. S2(a) is obtained, that is, the largest degree of the subnetwork satisfies \( k_{kT, \text{max}} \sim n_{kT}^\beta = [N_{kT}/N_0]^{\beta} \sim N_{kT}^\beta \), as shown in Fig. S3. We have roughly plotted the relationship between exponent \( \beta \) and \( \lambda \), the detailed results as shown in Fig. S4, the results show that \( \beta \approx 1/(\lambda - 1) \).

![Figure S3](image_url)

**FIG. S3.** For real scale-free networks, shows the dependence of the largest degree \( k_{kT, \text{max}} \) of subnetwork \( G(k_T) \) on the relative size \( n_{kT} \), where \( k_{kT, \text{max}} \sim N_{kT}^\beta \), and the value of exponent \( \beta \) is shown in Table 1.

Furthermore, we also studied the dependence of the normalized average degree of subnetwork \( G(k_T) \) on \( n_{kT} \) (detailed results as shown in Fig. S5). The results further show that \( \langle k \rangle_{kT} \sim n_{kT}^{-\gamma} = [N_{kT}/N_0]^{-\gamma} \sim N_{kT}^{-\gamma} \), where the value of \( \gamma \) is shown in Table 1. However, our results also show that excessive DTR iteration can cause network’s \( \langle k \rangle_{kT} \) to deviate seriously from the power-law behavior, which is obvious on Proteome, Words, Frenchbookinter, Japanesebookinter, and YouTube. It can be expected that even if a scale-free network with a sufficiently large initial size is considered, the normalized average degree of the subnetwork obtained after DTR transformation will eventually deviate from the power-law behavior, and self-similar properties (scale invariance) of the network will cease to exist.

Recently, Serafino et al. [8] have demonstrated through finite-size scaling analysis and moment ratio tests that the scale-free properties of real-world networks are usually hidden by finite-size effects. At the end of this section, we use the moment ratio test to further verify the scale invariance of scale-free networks under DTR transformation from another aspect. Specifically, for a scale-free network, the ratio of the \( i \)th moment to the \((i - 1)\)th moment of node degree is independent of \( i \), satisfying \( \langle k^i \rangle / \langle k^{i-1} \rangle \propto N^d \), where \( d > 0 \) [8]. As shown in Fig. S6(a), the moment ratio for different \( i \) is a group of parallel lines. We consider the BA scale-free network with the initial size of \( N = 10^5 \), and perform DTR downscaling transformation on it to obtain a series of subnetworks. Then, applying the moment ratio experiment to these subnetworks, we get a similar result [see Fig. S6(b)], namely \( \langle k^i \rangle / \langle k^{i-1} \rangle \propto N_{kT}^{d_{kT}} \), where \( N_{kT} \) is the size of the subnetwork \( G(k_T) \). Notably, \( d \approx d_{kT} \), which implies that the DTR transform can accurately reproduce the scale invariance of the original BA network. For above real networks, this conclusion is still true, i.e., \( \langle k^i \rangle / \langle k^{i-1} \rangle \propto N_{kT}^{d_{kT}} \), detailed results as shown in Fig. S7.
FIG. S4. The positions of the BA scale-free network and real scale-free networks on the phase diagram, the vertical axis represents the exponent \( \beta \) of the largest degree of the subnetwork, and the horizontal axis represents the degree distribution exponent \( \lambda \) of the original network. The blue solid circle is the result of the BA scale-free network, the corresponding initial network size is \( N = 10^4 \) and the parameter \( m = 5 \). The remaining shapes show the positions of the eight real networks. The solid red line satisfies \( \beta = 1/(\lambda - 1) \).

FIG. S5. For real scale-free networks, shows the dependence of the normalized average degree \( \langle k \rangle_{kT} \) of subnetwork \( G(k_T) \) on the relative size \( n_{kT} \), where \( \langle k \rangle_{kT} \sim N_{kT}^{-\gamma} \), and the value of exponent \( \gamma \) is shown in Table 1.
FIG. S6. Moment ratio test of BA scale-free networks. (a) Generate a series of BA networks with sizes between $[500, 10^5]$, examines the dependence of the moment ratio of node degrees on $N$, and the exponent $d$ is the slope obtained by fitting these results. (b) Starting from the BA scale-free network with the initial size of $N = 10^5$, a series of subnetworks are obtained by DTR downscaling transformation, shows the dependence of the moment ratio of the subnetwork on the size $N_{k_T}$, and the exponent $d_{k_T}$ is the slope obtained by fitting. The parameter $m = 5$, and all results are averaged over 100 independent realizations.

FIG. S7. Moment ratio test of real scale-free networks. For each real scale-free network, a series of subnetworks are obtained by DTR downscaling transformation, shows the dependence of the moment ratio $\langle k^i \rangle / \langle k^{i-1} \rangle$ of the subnetwork on the size $N_{k_T}$, satisfying $\langle k^i \rangle / \langle k^{i-1} \rangle \propto N_{k_T}^{d_{k_T}}$. 
III. FSS ANALYSIS OF SCALE-FREE NETWORKS

FIG. S8. Taking the normalized largest degree as an example, we show the dependence of the quality $S$ of the collapse plot for different curves on exponent $\alpha$, and the minimum point of $S$ gives an estimate of the optimal exponent $\alpha$.

FIG. S9. Scaling analysis of dynamical observables for BA scale-free networks with different initial sizes. (a), (c), and (e) show the dependence of each observable on the relative size of the subnetwork, respectively. (a) The normalized smallest nonzero eigenvalue $\lambda_{kT,2}(L)$ of the Laplace matrix, (c) the $Q_{kT}$ of the subnetwork, and (e) the normalized largest eigenvalue of the adjacency matrix $\lambda_{kT,n}(A)$. (b), (d), and (f) show the collapse diagram after rescaling the horizontal axis using the exponent $\alpha$, respectively. The parameter $m = 5$, and all results are averaged over 100 independent realizations.
FIG. S10. Scaling analysis of normalized largest eigenvalues of the adjacency matrix for a scale-free network and its subnetwork. First, taking CL scale-free models as an example, we generate an initial network $G_0$ with $N_0 = 50000$, set the threshold $K_T$, and obtain its subnetwork $G(K_T)$. We applied the DTR transformation to $G_0$ and $G(K_T)$, respectively, and calculate their observable curves, as shown in solid lines. The horizontal axis of the observable curve (green diamond) of $G(K_T)$ is rescaled as $n_{k_T} [N(K_T)]^{1/\alpha} / N_0^{1/\alpha}$, and the result (black dashed line) can accurately reproduce the observable result of the initial network $G_0$ (the red solid line and the black dashed line almost collapse on the same master curve), where the exponent $\alpha = 1$, which further confirmed the conclusion of the main text.
IV. THE COMPLEMENTARY CUMULATIVE DEGREE DISTRIBUTION OF SCALE-FREE NETWORKS AND THEIR SUBNETWORKS

FIG. S11. Self-similarity of real network’s complementary cumulative degree distribution along the DTR flow. Where $K_T$ is the degree threshold set for the initial network, which is different from $k_T$. To ensure that the obtained subnetwork $G(K_T)$ has approximately the same degree distribution as the original network, the value of $K_T$ should not be selected too large.

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