A Note on An Abstract Model for Branching and its Application to Mixed Integer Programming

Daniel Anderson†
Carnegie Mellon University
dlanders@cs.cmu.edu

Pierre Le Bodic
Monash University
pierre.lebodic@monash.edu

Kerri Morgan†
Deakin University
kerri.morgan@deakin.edu.au

Abstract

A key ingredient in branch and bound (B&B) solvers for mixed-integer programming (MIP) is the selection of branching variables since poor or arbitrary selection can affect the size of the resulting search trees by orders of magnitude. A recent article by Le Bodic and Nemhauser [Mathematical Programming, (2017)] investigated variable selection rules by developing a theoretical model of B&B trees from which they developed some new, effective scoring functions for MIP solvers. In their work, Le Bodic and Nemhauser left several open theoretical problems, solutions to which could guide the future design of variable selection rules. In this article, we first solve many of these open theoretical problems. We then implement an improved version of the model-based branching rules in SCIP 6.0, a modern open-source MIP solver, in which we observe an 11% geometric average time and node reduction on instances of the MIPLIB 2017 Benchmark Set that require large B&B trees.

1 Introduction

Modern mixed-integer programming (MIP) solvers and many other combinatorial optimisation technologies are driven by the branch and bound (B&B) method [16]. In MIP solvers, B&B consists in solving a linear program (LP) relaxation of the MIP and recursively splitting the domains of integer variables whose values are fractional in the solution to the relaxation. The choice of fractional variable to branch on can drastically affect the size of the resulting search trees, potentially leading to orders of magnitude variations in solving time. Modern branching rules prefer branching on variables that most improve the dual bound at the created children, and thus, for the purpose of ranking them, reduce candidate variables to couples of values corresponding to their estimated dual gap improvements. The analysis and application of these couples of improvements, or so called branching tuples for n-ary branching, is studied extensively in the context of Satisfiability by Kullman [15], where it is shown that their quality can be essentially described by a single corresponding real number called the $\tau$-value. Kullman shows that various analytic properties of the $\tau$-value can help theoretically explain why certain branching rules work better than others. Le Bodic and Nemhauser [17] study an abstract model of B&B trees, and characterise the asymptotic growth rate of the trees resulting from a single variable by the ratio value $\varphi$, which is related to the $\tau$-value of Kullman [15]. Le Bodic and Nemhauser demonstrate experimentally that branching rules incorporating the $\varphi$-value and the abstract tree model perform on average better than the default rules implemented in the state-of-the-art open-source solver SCIP 3.1.1 [2]. As part of their experimental work, they use their general variable branching (GVB) model as a simulation problem

†Work completed in part while the author was employed at Monash University
in order to tune the branching rules that they subsequently implement in SCIP. In their analysis, Le Bodic and Nemhauser leave open several problems related to the model:

- Is there a closed-form formula for computing the ratio $\varphi$?
- They conjecture that all instances of the MVB problem admit a single variable that is optimal to branch on for all sufficiently large values of the dual gap.
- They pose the question of whether GVB admits tighter hardness results, and whether it admits an approximation algorithm.

The theoretical contributions of this paper are as follows:

- We show that there exists no closed form formula for the ratio $\varphi$ in general (Section 2.1),
- We resolve the MVB conjecture, showing that it does not hold in general (Section 2.2),
- We show that GVB admits a Karp reduction from the complement of the classic KthLARGEST SUBSET problem, and that it is consequently also PP-hard under polynomial time Turing reductions (Section 2.3),
- We show that Proposition 3 of Le Bodic and Nemhauser [17] is actually false (Section 3.1),
- We give an analysis of the GVB simulation problem, showing that it can be solved in expected sub-exponential time in $n$ for scoring functions that respect dominance, which improves on the naive exponential time bound (Section 3.2).

Our practical contributions are:

- An improved implementation of the model-based branching rules in the modern state-of-the-art open-source MIP solver SCIP 6.0 [11] (Section 4.1),
- Experimental results that demonstrate an 11% geometric average speedup and tree size reduction for problems in the MIPLIB 2017 Benchmark Set [20] that required large B&B trees (Section 4.2).

These practical results were not simply performed for the sake of reproducing the results of [17]. Indeed, the experiments of [17] were performed in conditions that best verify the developed theory. For instance, measures were taken to reduce performance variability (e.g. a bound cutoff was provided), but this in turn obfuscated the extent to which the improvements found would extend to real-world use cases. By contrast, this paper uses SCIP “as is”. Furthermore, we use the most recent state-of-the-art test set, MIPLIB 2017 [20], and the latest SCIP major version, 6.0 [11], and with the help of the SCIP team, we ran the experiments in the conditions that are used for the development of SCIP. Finally, we have improved the implementation of the new branching rules. To the extent that our results are comparable to those of [17] on instances that require large trees, the new implementation significantly improves performance (from 5% in [17] to 11% improvement in geometric mean over SCIP’s default).

The practical significance of these results underlines the importance of further foundational research in the line of [17]. Hence we address some of the questions left unanswered by Le Bodic and Nemhauser [17]. In particular we prove that there is in general an unavoidable overhead to using the new branching rules, as numerical methods must be invoked for the computation of the ratio $\varphi$. We further prove that the MVB and GVB problems, essential to the comparison and calibration simulations of the branching rules, are more complex than [17] revealed. Nevertheless, we show how these simulations can be improved for GVB via an interesting analysis of the variable space. These new simulations confirm the experimental results found in [17] and in this paper.
1.1 State-of-the-art branching strategies

The state-of-the-art branching strategy employed by the open-source solver SCIP \cite{11} is a strategy called hybrid branching \cite{3}, which consists of reliability pseudocost branching augmented with several techniques from satisfiability and constraint programming. Reliability pseudocost branching consists in initially performing strong branching \cite{7}, where candidate variables are branched on in order to measure the resulting change in the dual gap. Since strong branching is computationally expensive, it is substituted with pseudocost branching \cite{9} once enough is known about the historical dual bound changes that have been exhibited by the candidates in order to estimate future changes. Strong branching is performed when the pseudocosts are unreliable, i.e., when they are uninitialized or when they are being used many times without a new exact measure being recorded. Techniques from satisfiability and constraint programming are used to further improve the estimates, yielding the state-of-the-art hybrid strategies. For a thorough description, the PhD thesis by Achterberg \cite{1} provides discussion and benchmarks of these branching rules as used in SCIP.

Given the (estimated) dual bound changes as measured by the branching rule, candidate variables are ranked based on a scoring function, and the best scoring variable is then branched on. The default scoring function used by SCIP is the product function, given by

\[
score(l, r) = \max(\epsilon, l) \times \max(\epsilon, r),
\]

where \((l, r)\) are the dual bound changes from branching left (downward) and right (upward) respectively and \(\epsilon = 10^{-6}\) is used as a tiebreaker when \(\min(l, r) = 0\). A higher product score indicates a more favourable variable to branch on. Achterberg \cite{1} emphasizes the importance of using a good scoring function by demonstrating that the product function outperforms the previously standard weighted sum scoring function by more than 10% on experimental benchmarks.

1.2 Related work

A recent trend in the design of branching heuristics has been the application of machine learning to variable selection in MIP solvers. Khalil et al. \cite{14} devise a machine learning framework for branching and show that it can produce search tree sizes in line with commercial solvers, although the running time overhead is high. Alvarez et al. \cite{5} similarly develop a machine learning approximation to strong branching, and demonstrate that it produces promising results when combined with the heuristics and separating cuts of CPLEX 12.2, although it performs less well when these are disabled. Finally, in \cite{8}, Balcan et al. devise a machine learning framework to learn a nearly-optimal linear combination of branching rules for a given distribution of instances. Although such learning-based approaches show great promise for solving problems from known distributions, they tend to lack in generality and applicability to arbitrary problem instances.

2 Le Bodic and Nemhauser’s problems

In this section, we address open problems on Le Bodic and Nemhauser’s abstract B&B model \cite{17}. The model consists of variables with known, fixed integer gains that model the dual gap changes that occur when branching on a variable. A variable \(x\) is thus represented by a pair of non-negative integers \((l, r)\) with \(1 \leq l \leq r\). A branch and bound tree (B&B tree) is a vertex-weighted, full binary tree, where each internal node is associated with a variable \((l, r)\), such that a vertex with weight \(g\) has children of weight \(g + l\) and \(g + r\). A B&B tree is said to close a gap of \(G\) if all of its leaves have weight at least \(G\), where the root node has weight \(g = 0\).
2.1 The Single Variable Branching problem

The Single Variable Branching (SVB) problem models a B&B tree consisting of a single variable \((l, r)\) branched on at every internal node. See Figure 1 for an example of an SVB tree corresponding to the variable \((2, 5)\). The size of the smallest SVB tree that closes a gap of \(G\) can be readily expressed by the \(n\)th-order linear recurrence

\[
t(G) = \begin{cases} 
1 & \text{if } G \leq 0, \\
1 + t(G - l) + t(G - r) & \text{if } G > 0.
\end{cases}
\]  

(1)

The asymptotic growth rate of the solution to the recurrence (1) is referred to as the ratio \(\varphi\), defined as

\[
\varphi = \lim_{G \to \infty} \left( \frac{t(G + l)}{t(G)} \right)^{\frac{1}{G}},
\]

whose value can be shown to be the unique root \(x \geq 1\) of the trinomial

\[
p(x) = x^r - x^{r-l} - 1.
\]  

(2)

By definition, the ratio \(\varphi\) engenders the useful approximation

\[
t(G) \approx t(\bar{G})\varphi^{G - \bar{G}}, \quad \bar{G} \leq G.
\]  

(3)

Le Bodic and Nemhauser pondered the existence of a closed-form solution to \(\varphi\), as this would be of great practical interest since \(\varphi\) has been used as an ingredient in constructing effective variable selection rules for MIP. We resolve this question in the negative.

**Theorem 1.** There is no closed-form formula for the positive root of the characteristic trinomial (2) in general.

We prove this theorem and provide an analysis of the algebraic characteristics of the trinomial \(f(x) = x^r - x^{r-l} - 1\). When \(r = l\), the polynomial is \(f(x) = x^r - 2\), which has the unique real root \(x = 2^{\frac{1}{r}}\). When \(l = 0\), the polynomial is \(f(x) = -1\) which has no roots. Without loss of generality, we subsequently consider the case where \(0 < l < r\).

2.1.1 Solvability by radicals

We wish to answer the question: for which values of \(r\) and \(l\), is \(f(x)\) solvable by radicals, that is by a finite number of field operations and the taking of \(n\)-th roots. Let \(r = k_1d\) and \(l = k_2d\) where \(d = \gcd(r, l)\) and \(k_1, k_2 \in \mathbb{N}\). We first note the following fact.

**Lemma 1.** If \(d > 1\), then \(f(x) = x^{k_1d} - x^{(k_1-k_2)d} - 1\) is solvable by radicals if and only if \(F(X) = X^{k_1} - X^{k_1-k_2} - 1\) is solvable by radicals.

**Proof.** We note that \(\alpha\) is a root of \(F(X)\) if and only if \(\alpha^{\frac{1}{d}}\) is a root of \(f(x)\). \(\square\)

For the remainder of the proof, it suffices to consider only cases where \(\gcd(r, l) = 1\) and \(0 < l < r\). We use the following theorem in \cite{18} which states:

**Lemma 2** (Ljunggren, 1960). If \(n = n_1d\), \(m = m_1d\), \(\gcd(n_1, m_1) = 1\), \(n \geq 2m\), then the polynomial

\[
g(x) = x^n + \epsilon x^m + \epsilon', \quad \epsilon = \pm 1, \quad \epsilon' = \pm 1,
\]

is irreducible over \(\mathbb{Q}\) apart from the following three cases where \(n_1 + m_1 \equiv 0 \pmod{3}\):
Figure 1: A minimal SVB tree closing the gap \( G = 6 \) using the variable \((2, 5)\). The edge labels represent the dual bound change and the node labels indicate the total gap closed at a particular node.

1. \( n_1, m_1 \) both odd, \( \epsilon = 1 \);
2. \( n_1 \) even, \( \epsilon' = 1 \);
3. \( m_1 \) even, \( \epsilon' = \epsilon \),

the polynomial \( g(x) \) then being the product of \( x^{2d} + (\epsilon)^m(\epsilon')^n x^d + 1 \) and a second irreducible polynomial.

We will also use the following.

**Lemma 3.** If \( 2r - l \equiv 0 \pmod{3} \) then \( r + l \equiv 0 \pmod{3} \).

**Proof.** It is clear that \( 2r - l \equiv 0 \pmod{3} \) if and only if one of the following conditions holds:

\[
\begin{align*}
& r \equiv 0 \pmod{3} \text{ and } l \equiv 0 \pmod{3}, \\
& r \equiv 1 \pmod{3} \text{ and } l \equiv 2 \pmod{3}, \text{ or} \\
& r \equiv 2 \pmod{3} \text{ and } l \equiv 1 \pmod{3}.
\end{align*}
\]

These are precisely the cases where \( r + l \equiv 0 \pmod{3} \). \( \square \)

**Lemma 4.** If \( r \) and \( l \) are co-prime and \( 0 < l < r \), then \( f(x) = x^r - x^{-l} - 1 \) is irreducible over \( \mathbb{Q} \) apart from the case where both \( r \) and \( l \) are odd and \( r + l \equiv 0 \pmod{3} \). In the latter case \( f(x) = g(x)(x^2 - x + 1) \) where \( g(x) \) is irreducible.

**Proof.** Assume \( \gcd(r, l) = 1 \). We deal with two cases, the case where \( 0 < r/2 \leq l < r \) and the case where \( r/2 < l < r \).

- **Case 1** \( (r/2 \leq l < r) \): The proof follows from Lemma 2. As the coefficients of the non-leading terms of \( f(x) \) are \(-1\), the only case when \( f(x) \) is reducible is when the third case holds. This occurs when \( 2r - l \equiv 0 \pmod{3} \) and \( r - l \) is even. By Lemma 3 when \( 2r - l \equiv 0 \pmod{3} \) then \( r + l \equiv 0 \pmod{3} \). As \( \gcd(r, 1) = 1 \) and \( r - l \) is even, it follows that \( r \) and \( l \) must be odd. Thus \( f(x) \) is irreducible except when \( r \) and \( l \) are odd, and \( r + l \equiv 0 \pmod{3} \). When \( f(x) \) is reducible, then \( f(x) = (x^2 - x - 1)g(x) \) where \( g(x) \) is an irreducible polynomial.
Lemma 6. If the symmetric group $S$ uses the following theorem in [21],

As $\gcd(a, b) = 1$, we note that at most one of $r$ and $l$ is odd and $r + l \equiv 0 \pmod{3}$. If $h(x)$ is reducible, then $h(x) = (x^2 - x + 1)g'(x)$ where $g'(x)$ is an irreducible polynomial. As $f(x) = -x^r h(1/x)$, it follows that if $f(x)$ is reducible, then

$$f(x) = -x^r \left(\frac{1}{x^2} - \frac{1}{x} + 1\right) g'(\frac{1}{x})$$

$$= -(x^r - x^{r-1} + x^{r-2}) g'(\frac{1}{x})$$

$$= (x^2 - x + 1) g(x)$$

where $g(x)$ is the irreducible polynomial $-x^{r-2}g'(\frac{1}{x})$.

If $\gcd(r, l) = 1$ and $0 < l < r$, then the polynomial $f(x) = x^r - x^{r-l} - 1$ is irreducible apart from the case where $r$ and $l$ are odd and $r + l \equiv 0 \pmod{3}$. When $f(x)$ is reducible, $f(x) = (x^2 - x + 1)g(x)$ where $g(x)$ is an irreducible polynomial.

2.1.2 Galois group of $f(x)$

We now show that when $f(x)$ is irreducible over $\mathbb{Q}$, it is not solvable by radicals for $r \geq 5$. We will use the following theorem in [21].

Lemma 5 (Osada, 1987). Let $f(X) = X^n + aX^l + b$ be a polynomial of rational integral coefficients, that is, $f(X) \in \mathbb{Z}[X]$. Let $a = a_0c^n$ and $b = b_0c^n$. Then the Galois group of $f(X)$ is isomorphic to the symmetric group $S_n$ of degree $n$ if the following conditions are satisfied:

- $f(X)$ is irreducible over $\mathbb{Q}[X]$, 
- $\gcd(a_0c(n-1), nb_0) = 1$

Lemma 6. If $f(x) = x^r - x^{r-l} - 1$ is an irreducible polynomial in $\mathbb{Q}[x]$ and $\gcd(r, l) = 1$, then $f(x)$ has Galois group $S_r$.

Proof. As $\gcd(r, l) = 1$, we note that at most one of $r$ and $l$ can be even. We consider two cases based on the parity of $l$.

- **Case 1 ($l$ is odd):** If $l$ is odd, we apply Lemma 5 with $a_0 = b_0 = -1$ and $c = 1$.

  As $f(x)$ is irreducible, by Lemma 5, $f(x)$ has Galois group $S_r$ if

  $$\gcd(-1 \times (r - r + l)(r - l), -r) = \gcd(l(l - r), -r) = \gcd(l^2, r) = 1,$$

  since $\gcd(l, r) = 1$.

- **Case 2 ($l$ is even):** If $l$ is even, then $r$ must be odd. We apply Lemma 5 with $a_0 = b_0 = 1$ and $c = -1$.

  As $f(x)$ is irreducible, by Lemma 5, $f(x)$ has Galois group $S_r$ if

  $$\gcd(-1 \times (r - r + l)(r - l), r) = \gcd(l(l - r), r) = \gcd(l^2, r) = 1,$$

  since $\gcd(l, r) = 1$.

$\square$
2.1.3 Consequences for the characteristic trinomial

The Galois group $S_{n \geq 5}$ is not solvable by radicals. This means that if $f(x)$ is irreducible over $\mathbb{Q}$:

- If $\gcd(r, l) = 1$ and $r \geq 5$ then the polynomial is not solvable by radicals,
- If $\gcd(r, l) = d$ and $r/d \geq 5$ then the polynomial is not solvable by radicals.

As any polynomial that factorises into factors of degree at most four is solvable by radicals, we have

- If $r/\gcd(r, l) \leq 4$ then the polynomial is solvable by radicals.

The open case is where $f(x)$ is reducible, that is, when both $r$ and $l$ are odd and $r + l \equiv 0 \pmod{3}$. The factor $x^2 - x - 1$ is solvable by radicals. The irreducible factor $g(x)$ has degree $r - 2$. If $r < 7$ then it is solvable by radicals. Cases where $f(x)$ is irreducible and $r \geq 7$ require further investigation. The smallest such case is $r = 7$ and $l = 5$. Here $f(x) = (x^2 - x + 1)(x^5 + x^4 - x^2 - x - 1)$ where the quintic has Galois group $S_5$ and so is not solvable by radicals.

2.2 The Multiple Variable Branching problem

The Multiple Variable Branching (MVB) problem models a B&B tree consisting of a set of variables $(l_i, r_i)_{1 \leq i \leq n}$ each of which may be branched on an arbitrary number of times. The size of the smallest MVB tree that closes a gap of $G$ can be expressed as the solution to the following non-linear recurrence.

$$t(G) = \begin{cases} 
1 & \text{if } G \leq 0, \\
1 + \min_{1 \leq i \leq n} (t(G - l_i) + t(G - r_i)) & \text{if } G > 0.
\end{cases}$$

The ratio of an MVB tree is defined similarly to that of an SVB tree, such that

$$\varphi = \lim_{G \to \infty} \left( \frac{t(G + z)}{t(G)} \right)^{\frac{1}{z}},$$

where $z = \text{lcm}_{1 \leq i \leq n}(l_i, r_i)$. Le Bodic and Nemhauser showed that the ratio for an MVB tree is related to the ratios of the constituent variables such that

$$\varphi = \min_{1 \leq i \leq n} \varphi_i,$$

where $\varphi_i$ is the ratio of the single variable $(l_i, r_i)$. This interesting result spurred the following specious conjecture.

**Conjecture 1** (Le Bodic and Nemhauser, 2017 [17]). *For each instance of MVB, there exists a gap $H$ such that for all gaps greater than $H$, variable $i = \arg \min_j \varphi_j$ is always optimal to branch on at the root node.***

**Theorem 2.** *The MVB conjecture is false.*

**Proof.** Consider the instance of MVB consisting of the variables $(2, 4)$ and $(3, 3)$, whose ratios, to five significant digits are 1.27202 and 1.25992 respectively. Such an MVB tree closing a gap of $G = 8$ is depicted in Figure [2]. According to the MVB conjecture, it should be the case that $(3, 3)$
is always branched on for all $G$ above some threshold. We prove in Appendix 1 that a closed-form solution to this instance is given by

$$t(G) = \begin{cases} 
2 \cdot 4^k - 1 & \text{if } G = 6k, \\
\frac{1}{3} (2 \cdot 4^k + 1) - 1 & \text{if } G = 1 + 6k, \\
\frac{2}{3} (5 \cdot 4^k + 1) - 1 & \text{if } G = 2 + 6k, \\
4 \cdot 4^k - 1 & \text{if } G = 3 + 6k, \\
\frac{2}{3} (8 \cdot 4^k + 1) - 1 & \text{if } G = 4 + 6k, \\
\frac{4}{3} (5 \cdot 4^k + 1) - 1 & \text{if } G = 5 + 6k.
\end{cases} (4)$$

It then suffices to observe from (4) that for any gap of the form $G = 2 + 6k$, branching on $(2,4)$ yields a strictly smaller tree than branching on $(3,3)$, despite the fact that $\varphi(3,3) < \varphi(2,4)$.

![Figure 2: A minimal MVB tree for the variables (2,4) and (3,3) closing the gap $G = 8$.](image)

2.3 The General Variable Branching problem

The General Variable Branching (GVB) problem models a B&B tree consisting of a set of variables $(l_i, r_i)$, each of which may be branched on a maximum of $m_i$ times. Consequently, unlike SVB and MVB, the GVB problem need not necessarily be feasible for all possible inputs. The size of the smallest GVB tree that closes a gap $G$ is given by

$$t(G, m) = \begin{cases} 
1 & \text{if } G \leq 0, \\
\infty & \text{if } m = 0, \\
1 + \min_{1 \leq i \leq n, m_i > 0} t(G - l_i, m - I_i) + t(G - r_i, m - I_i) & \text{otherwise.}
\end{cases} (5)$$

where $m$ is the vector of multiplicities and $I_i$ is an indicator vector on the $i$th element. Le Bodic and Nemhauser showed that GVB is $\#P$-Hard under polynomial-time Turing reductions via a reduction from a variant of counting Knapsack solutions. They pose the question of whether GVB admits tighter hardness results, and whether it admits an approximation scheme. Here, we provide a Karp reduction to the complement of GVB from $K$thLARGEST SUBSET, a well-studied problem in complexity theory, and, using a recent result of Haase and Kiefer [12], we show that GVB is also PP-Hard under polynomial-time Turing reductions.
Problem: $K^{th}$ Largest Subset \[10\]
Input: Set $A$ with finite cardinal $N$, size $s(a) \in \mathbb{Z}^+$ for each $a \in A$, positive integers $K$ and $B$.
Question: Are there $K$ or more distinct subsets $A' \subseteq A$ for which the sum of the sizes of the elements in $A'$ does not exceed $B$?

Problem: General Variable Branching \[17\]
Input: $n$ variables encoded by $(l_i, r_i)$, $i = 1, \ldots, n$, an integer $G > 0$, an integer $k > 0$, and a vector of multiplicities $m \in \mathbb{Z}^n_{>0}$.
Question: Is there a B&B tree with at most $k$ nodes that closes the gap $G$, branching on each variable $i$ at most $m_i$ times on each path from the root to a leaf?

Theorem 3. There exists a Karp reduction from $K^{th}$ Largest Subset to the complement of GVB.

Proof. We will reduce $K^{th}$ Largest Subset to the complement of GVB. The proof closely resembles that of Theorem 8 in [17]. Set $n = N + 1$, and define $C = \sum_{i=0}^{N} s(a_i)$. Suppose that $B < C$, else the instance is trivial. For each $a_i \in A$, create a variable with left and right gains $(C, C + s(a_i))$ and multiplicity one. Finally, variable $n$ has gains $(C, C)$ and multiplicity one. Set the gap to $G = NC + B + 1$ and the threshold to $k = 2^{N+1} - 2 + 2K$. Observe that all variables dominate the variable $(C, C)$, and that the variables will be branched on in order of size. Each path from the root to level $N$ therefore corresponds to a subset $A' \subseteq A$, where branching left corresponds to not including some element $a_i$ in the gap, and branching right corresponds to including $a_i$. The gap closed at a particular node at level $N$, corresponding to the subset $A' \subseteq A$ (of right branches taken), is therefore

$$g = NC + \sum_{a_i \in A'} |a_i|.$$ 

If the size of $A'$ exceeds $B$, then $g \geq NC + B + 1 = K$ and hence the node is a leaf. Otherwise, the gap $g$ is strictly less than $K$ and the node will have two children, branching on the variable $(C, C)$. The gap at this level is then

$$g = NC + \sum_{a_i \in A'} |a_i| + C \geq G,$$

since $B < C$, and hence the children are leaves. Therefore, the number of subsets whose sum does not exceed $K$ is equal to half of the number of leaves at level $N + 1$, so we have,

$$\left(\# \text{ Subsets with sum } \leq B\right) = \frac{1}{2}(t(G) - (2^{N+1} - 1)).$$

Therefore, if there exists at least $K$ subsets whose sum exceeds $B$, then

$$t(G) \geq 2K + 2^{N+1} - 1 \iff \neg(t(G) \leq 2K + 2^{N+1} - 2).$$

We can conclude that the B&B tree will have at most $2^{N+1} - 2 + 2K$ nodes if and only if there are fewer than $K$ distinct subsets $A' \subseteq A$ whose sum does not exceed $B$, so the answer to $K^{th}$ Largest Subset is YES if and only if the answer to GVB is NO. Finally, observe that the reduction clearly takes polynomial time.

Corollary 1. We have the following complexity results for GVB

1. GVB is NP-Hard, $\#P$-Hard and PP-Hard under polynomial-time Turing reductions.

2. If $K^{th}$ Largest Subset is NP-Hard under Karp reductions, then GVB is coNP-Hard under Karp reductions.

Whether $K^{th}$ Largest Subset is NP-Hard under Karp reductions remains a long-standing open problem [13].
3 Analysis of the GVB simulation problem

Given a collection of variables \((l_i, r_i)\) with multiplicities \(m_i\), instead of seeking the optimal configuration of a general variable branching tree, we can instead seek to measure the size of the tree that would be built by applying a particular variable selection rule. Formally, given a variable selection rule \(f(m, G)\) returning the index of the predicted optimal variable, we can measure

\[
t(G, m) = \begin{cases} 
1 & \text{if } G \leq 0, \\
\infty & \text{if } m = 0, \\
1 + t(G - l_i, m - I_i) + t(G - r_i, m - I_i) & \text{otherwise}
\end{cases}
\]  

(6)

where \(i = f(m, G)\). In this way, the GVB problem can be used as a tool to analyse and predict the performance of a given variable selection rule. We refer to this as the \textit{GVB simulation problem}. In [17], this technique is used to evaluate and compare the \textit{product} and \textit{ratio} scoring functions on random collections of variables before testing them on real MIP problems. This situation is only tractable due to the fact that the \textit{product} and \textit{ratio} scoring functions do not depend on the dual gap, causing the candidate variables to be branched on in one specific order along any path in the tree. In this case, the recurrences (5) and (6) can be evaluated in \(O(n^2G)\) time and \(O(nG)\) space with dynamic programming after pre-computing the variable order. In general, the variables may be branched on in different orders depending on the dual gap along the path, in which case (5) and (6) naively require \(O(n^2nG)\) time and \(O(2^nG)\) space to consider all subsets of available variables, which is intractable for moderately sized test cases.

However, all scoring functions that we know of respect the notion of \textit{dominance}. For such scoring functions, the state space of the dynamic program can be restricted to those subsets of variables that are \textit{dominance free}. In this section, we provide an analysis of the expected number of dominance free subsets in a collection of random variables. This analysis implies that the expected size of the state space for such a problem is sub-exponential in \(n\).

3.1 Dominated and non-dominated variables

We recall from [17] the notion of dominance. We say that \((l_1, r_1)\) dominates \((l_2, r_2)\) if \(l_1 \geq l_2\) and \(r_1 \geq r_2\) with at least one of \(l_1 > l_2\) or \(r_1 > r_2\). We call a set of variables dominance free if it contains no dominated variables. Proposition 3 of [17] claims that for MVB and GVB, it is never strictly optimal to branch on a dominated variable before those that dominate it.

Theorem 4. Proposition 3 of [17] is false in the case of the GVB problem. That is, there exists instances of GVB such that it is strictly optimal to branch on a dominated variable.

Proof. Consider the instance of GVB consisting of the variables \((5, 6), (9, 9), (5, 10)\) all with multiplicity one. The optimal solution for this instance with a gap of \(G = 15\) contains 9 nodes as shown in Figure 3. At the root node, this tree branches on the variable \((5, 6)\), which is dominated, but we note that any tree branching on \((5, 10)\) or \((9, 9)\) that closes the same gap has at least 11 nodes.

3.2 Bounding the expected number of non-dominated subsets

The state space of the naive GVB dynamic program (all possible subsets of variables at every gap) is prohibitively large, but when considering a scoring function that only selects non-dominated variables, the only relevant states are those corresponding to a dominance-free subset of variables. For such a scoring function, we note that any subset of yet-to-be-used variables considered by
the algorithm will never contain a variable that dominates a variable that has been used, and that such a subset, of which there should be significantly fewer, can be uniquely identified by its non-dominated variables. Here, we provide an analysis of the expected number of dominance-free subsets in a collection of random variables, which shows that the GVB simulation dynamic program can in fact be solved in sub-exponential time for scoring functions that respect dominance.

**Theorem 5.** Given \( n \) variables \( (l_i, r_i)_{1 \leq i \leq n} \) such that each \( l_i \) is unique, each \( r_i \) is unique, and where each pair \( l_i, r_i \) are independent, the expected number of non-dominated subsets is given by

\[
\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k}.
\]

**Proof.** We consider all subsets of size \( k \) for some fixed constant \( k \). There are a total of \( \binom{n}{k} \) such subsets. Take an arbitrary subset \( S = \{v_1, v_2, ..., v_k\} \) and suppose we sort the pairs in descending order of \( r \), and then denote the \( i^{th} \) pair in this sorted order by \( (l_i, r_i) \), for \( 1 \leq i \leq k \). This subset is non-dominated if and only if \( S \) happens to be sorted in increasing order by \( l \), since otherwise we would have \( r_i < r_j \) and \( l_i < l_j \) for some \( i < j \). Since each pair \( l_i, r_i \) are independent, each permutation of \( l \)'s is equally likely, and since there are \( k! \) possible permutations,

\[
\Pr(S \text{ is non-dominated}) = \frac{1}{k!}.
\]

Hence the expected number of non-dominated subsets of size \( k \) is

\[
\frac{1}{k!} \binom{n}{k}.
\]

We conclude that the expected number of non-dominated subsets of all sizes is

\[
\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k}.
\]

*Observe that if \( l \) or \( r \) contain non-unique values, the chance that they are dominated only increases. Hence analysing the unique case provides an upper bound in the general case.*
Theorem 6. The expected number of non-dominated subsets is sub-exponential in \( n \). In particular, a bound on the expected number of subsets of non-dominated variables is

\[
\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} = O \left( e^{2\sqrt{n}} \right), \quad n \to \infty.
\]

From Theorem 6 it follows that the GVB simulation problem can be solved in expected sub-exponential time in \( n \), specifically, in \( O(\text{poly}(n)e^{2\sqrt{n} \cdot G}) \) time, which significantly improves upon the naive \( O(n2^n G) \) bound. We demonstrate the utility of this improved analysis in Appendix 2, where we perform simulations on the svts rule of [17] and compare it to the product and ratio rules. The proof of Theorem 6 follows from several Lemmas.

Lemma 7. The summand \((7)\) satisfies

\[
\max_{0 \leq k \leq n} \frac{1}{k!} \binom{n}{k} = O \left( \frac{1}{(k^*)!} \left( \frac{n}{k^*} \right) \right),
\]

for

\[
k^* = \sqrt{n + \frac{1}{4}} - \frac{1}{2}
\]

as \( n \to \infty \).

Proof. Consider maximising the expression

\[
\frac{1}{k!} \binom{n}{k} = \frac{n!}{(k!)^2(n-k)!},
\]

over \( 0 \leq k \leq n \). This is equivalent to minimising the denominator, \((k!)^2(n-k)!\). We consider a suitable analytic extension, valid for real \( n \) and \( k \), in terms of the gamma function \( \Gamma \), and seek the minimum over \( 0 \leq k \leq n \) of

\[
\Gamma(k+1)^2\Gamma(n-k+1).
\]

We apply the fact that the derivative of the gamma function is given by

\[
\Gamma'(x+1) = \Gamma(x+1)(H_x - \gamma),
\]

where \( H_x = \sum_{i=1}^{x} \frac{1}{i} \) is the \( i \)th Harmonic number and \( \gamma \approx 0.57721567 \) is the Euler-Mascheroni constant. Differentiating (9), we find

\[
2(\Gamma(k+1))^2(H_k - \gamma)\Gamma(n-k+1) - (\Gamma(k+1))^2\Gamma(n-k+1)(H_{n-k} - \gamma).
\]

Collecting terms and equating to zero, the minima is a solution to the equation

\[
2H_k - H_{n-k} = \gamma.
\]

Since as \( n \to \infty \), we have that \( H_x \to \log(x) + \gamma \), we seek a solution to

\[
2(\log(k) + \gamma) - (\log(n-k) + \gamma) = \gamma,
\]

and hence, using simple properties of the logarithm, we arrive at

\[
k^2 + k - n = 0.
\]

Therefore, asymptotically, the minimum (and hence the maximum of (8)) occurs at

\[
k^* = \frac{\sqrt{4n+1} - 1}{2} = \sqrt{n + \frac{1}{4}} - \frac{1}{2},
\]

as \( n \to \infty \).
Lemma 8. Let \( k = c\sqrt{n} \) for any real \( 0 < c \leq \sqrt{n} \). The summand satisfies
\[
\frac{1}{k!} \binom{n}{k} = O \left( \frac{e^{-c^2}}{c\sqrt{n} - c\sqrt{n}} \left( \frac{e}{e} \right)^{2c\sqrt{n}} \right),
\]
as \( n \to \infty \).

Proof. We have
\[
\frac{1}{k!} \binom{n}{k} = \frac{n!}{((c\sqrt{n})!)^2(n - c\sqrt{n})!}.
\]
Using Stirling’s formula, we obtain the asymptotic expansion as \( n \to \infty \),
\[
\frac{1}{k!} \binom{n}{k} \sim \frac{n!}{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \sqrt{2\pi n} \left( \frac{n}{e} \right)^n}.
\]
Canceling common terms, we find
\[
\frac{1}{k!} \binom{n}{k} \sim \frac{1}{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \sqrt{2\pi n} \left( \frac{n}{e} \right)^n}.
\]
as \( n \to \infty \). Considering the term on the right, we have
\[
\left( \frac{n}{n - c\sqrt{n}} \right)^{n-c\sqrt{n}} = \left( 1 - \frac{c}{\sqrt{n}} \right)^{\sqrt{n}(c-\sqrt{n})},
\]
and using the fact that \( (1 - c/n)^n \to e^{-c} \) as \( n \to \infty \), we deduce that
\[
\left( 1 - \frac{c}{\sqrt{n}} \right)^{\sqrt{n}(c-\sqrt{n})} = O \left( e^{c\sqrt{n}-c^2} \right),
\]
as \( n \to \infty \). The asymptotic behaviour of the summand is therefore
\[
\frac{1}{k!} \binom{n}{k} = O \left( \frac{e^{c\sqrt{n}}}{2\pi e^{2c\sqrt{n}+1} \left( n - c\sqrt{n} \right)^{n-c\sqrt{n}}} \right),
\]
\[
= O \left( \frac{e^{-c^2}}{c\sqrt{n} - c\sqrt{n}} \left( \frac{e}{e} \right)^{2c\sqrt{n}} \right),
\]
as \( n \to \infty \).

\( \square \)

Lemma 9. For
\[
k^* = \sqrt{n + 1} - \frac{1}{2},
\]
we have
\[
\frac{1}{(k^*)!} \binom{n}{k^*} = O \left( \frac{1}{(\sqrt{n})!} \left( \frac{n}{\sqrt{n}} \right) \right).
\]
Proof. We make use of the fact that \((n + \Delta)! \geq n! \Delta\). This property holds for integer \(\Delta\), but can also be shown, using properties of the Gamma function to be valid for non-integral \(\Delta\). We will abuse notation and proceed by writing factorials anyway. Let \(\hat{k} = \sqrt{n + \frac{1}{4}}\), and write, using the aforementioned fact

\[
\frac{1}{(k^*)!} \binom{n}{k^*} = \frac{n!}{((\hat{k} - \frac{1}{2})!)^2 (n - \hat{k} + \frac{1}{2})!} \leq \frac{n!}{(\hat{k}!)^2 \hat{k}^{-1} (n - \hat{k})!(n - \hat{k})^{\frac{1}{2}}}
\]

Now we can write

\[
\frac{1}{(k^*)!} \binom{n}{k^*} \leq \frac{n!}{(k!)^2 (n-k)!} \cdot \frac{\hat{k}}{\sqrt{n-k}} = O\left(\frac{n!}{(k!)^2 (n-k)!}\right),
\]

which follows from the fact that

\[
\frac{\sqrt{n + \frac{1}{4}}}{\sqrt{n - \sqrt{n + \frac{1}{4}}}} \rightarrow 1,
\]

as \(n \rightarrow \infty\). Finally, since \(\hat{k} \rightarrow \sqrt{n}\) as \(n \rightarrow \infty\), we have

\[
\frac{1}{(k^*)!} \binom{n}{k^*} = O\left(\frac{n!}{(\sqrt{n})^2 (n - \sqrt{n})!}\right) = O\left(\frac{1}{(\sqrt{n})!} \left(\frac{n}{\sqrt{n}}\right)\right).
\]

\[
\Box
\]

Proof of Theorem 6. By Lemma 7, Lemma 8 with \(c = 1\), and Lemma 9, we bound the maximum term of the sum as \(n \rightarrow \infty\),

\[
\frac{1}{(\sqrt{n})!} \binom{n}{\sqrt{n}} = O\left(\frac{e^{2\sqrt{n-1}}}{2\pi \sqrt{n - \sqrt{n}}}\right).
\]

Then, observe that when \(c > e\), (10) is a decaying exponential, hence by Lemma 8 there are at most \(|e\sqrt{n} + 1|\) terms that contribute to the asymptotic behaviour of the sum, and therefore,

\[
\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} = O\left((e\sqrt{n} + 1)\frac{e^{2\sqrt{n-1}}}{\sqrt{n - \sqrt{n}}}\right),
\]

\[
= O\left(e^{2\sqrt{n}}\right),
\]

as \(n \rightarrow \infty\). \(\Box\)
4 Implementation and Experiments

From their abstract B&B model, Le Bodic and Nemhauser derive two scoring functions for MIP, the *ratio* rule, and the *svts* rule. The *ratio* rule first estimates the height of the branch and bound tree as \( \lfloor G/l \rfloor \), and if this is greater than 10, score the variables based on their ratio. If the estimated tree height is at most 10, then the *product* score is used instead. The *svts* rule scores variables based on their single variable tree size, either computed exactly, or approximated using (3) if the gap is very large.

4.1 Efficient computation of the ratio

Since the ratio \( \varphi \) of all non-dominated candidate variables is computed at every node of height greater than 10 for the *ratio* rule, and for any node at which the dual gap is high for *svts*, computing it efficiently is very important. Since we showed that there exists no closed-form formula for \( \varphi \), we describe here an improved numerical algorithm for computing it. Similarly to [17], we first scale the gains of a variable \((l, r)\) to \((1, \frac{r}{l})\), so that the resulting value computed is precisely \( \varphi' \). If \( r/l \leq 200 \), then we use Laguerre’s method [4] to find the root of the scaled trinomial, \( x^r - x^{r-1} - 1 \). This method typically converges within two or three iterations. Otherwise, if \( r/l > 200 \), then we use the fixed-point method given by the following recurrence:

\[
\begin{align*}
  f(x) &= (1 - \frac{1}{x})^{-\frac{1}{r}}. \\
\end{align*}
\]

We make an additional optimisation by caching the value of the ratio for each variable. Since in practice, the gains of a variable, and hence also its ratio do not change significantly between two nodes (as they are often given by pseudocosts), we initialise the method at the variable’s most recently computed ratio. Experiments show that this improved fixed-point method converges in roughly half as many iterations as the original fixed-point method given in [17].

4.2 Performance tests

We implemented the *ratio* and *svts* scoring functions in SCIP 6.0 [11], a state-of-the-art academic MIP solver. No additional modifications besides those required to perform branching decisions were made. We test the performance of our implementation on the MIPLIB 2017 Benchmark Set, the current state-of-the-art performance benchmarking suite for MIP solvers [20]. The set contains 240 problem instances representing a diverse range of real-world problems. All experiments were run on a cluster with 48 nodes equipped with Intel Xeon Gold 5122 at 3.60GHz and 96GB RAM. Jobs were run exclusively on a node. Each problem is given a time limit of two hours. We ignore in our results any problem that is solved in less than one second, solved at the root node without any branching (e.g. via presolving), or that is not solved by any of the scoring functions at all. To reduce variability, each instance is solved three times, with different random seeds used to permute the input.

One noteworthy difference between our experiments and those of Le Bodic and Nemhauser [17] is that other than the change in branching rules, we allow SCIP to use its default settings. This contrasts with the experiments of Le Bodic and Nemhauser, which provided problems with their primal bound, disabled primal heuristics, disabled cuts after the root node, and disabled the connected components presolver. Although these changes may lead to reduced variability [19], such setups often do not reflect the true performance of the solver on real world instances with real
settings. Additionally, the MIPLIB 2017 Benchmark Set has been designed to include problems that exhibit strong numerical stability, and hence less variability, for this reason.

The summary results are depicted in Table 1. The results are divided into three major columns, representing respectively the product, ratio, and svts scoring functions. The three subcolumns of product show the number of instances that were solved, the time taken, and the search tree size respectively. For ratio and svts, these are measured relative to product. We report summaries in terms of the totals, the geometric means, and shifted geometric means (with shifts of 10 and 100 for time and nodes respectively, as is standard for MIP benchmarking). The best performing rule for each measurement is shown in bold. A table with per-instance statistics can be found in Appendix 3.

| Instance | product | ratio | svts |
|----------|---------|-------|------|
|          | # Time  | Nodes | # Time  | Nodes | # Time  | Nodes |
| Total    | 305     | 296.61k | 296.61k | 135.63m | 135.63m | 0.88  |
| Geo. mean| 319.07  | 2.75k   | 0.99    | 1.04   | 1.08   | 0.99  |
| Sh. geo. mean | 351.50 | 4.84k   | 0.99    | 1.02   | 1.00   | 1.00  |

Table 1: Performance results on all instances of the MIPLIB 2017 Benchmark Set

Table 1 shows that both ratio and svts solve more instances than product, and require fewer nodes in total. However, the speedups are relatively small, with only a 1% geometric average speedup for svts, and a 1% - 2% slowdown for ratio. The larger improvements in arithmetic averages (i.e. totals) than geometric averages allude to the fact that the methods perform particularly well on instances requiring large B&B trees. Since these scoring functions are indeed designed to work well on instances requiring large trees, we restrict our attention to the subset of instances for which at least one setting required at least 10k, 50k, and 100k nodes. The summary results for these subsets of instances are shown in Tables 2, 3, and 4 respectively.

| Instance | product | ratio | svts |
|----------|---------|-------|------|
|          | # Time  | Nodes | # Time  | Nodes | # Time  | Nodes |
| Total    | 141     | 216.35k | 216.35k | 135.49m | 135.49m | 0.88  |
| Geo. mean| 858.30  | 92.19k  | 0.96    | 1.02   | 0.99   | 1.00  |
| Sh. geo. mean | 882.15 | 92.93k  | 0.96    | 1.02   | 0.99   | 1.00  |

Table 2: Performance results on instances of the MIPLIB 2017 Benchmark Set that required at least 10k nodes for some setting

| Instance | product | ratio | svts |
|----------|---------|-------|------|
|          | # Time  | Nodes | # Time  | Nodes | # Time  | Nodes |
| Total    | 93      | 144.30k | 144.30k | 134.87m | 134.87m | 0.87  |
| Geo. mean| 1.02k   | 326.96k | 0.96    | 1.00   | 0.99   | 1.00  |
| Sh. geo. mean | 1.03k | 327.19k  | 0.96    | 1.02   | 0.99   | 1.00  |

Table 3: Performance results on instances of the MIPLIB 2017 Benchmark Set that required at least 50k nodes for some setting
On instances requiring at least 10k nodes, svts achieves a 7% and 6% geometric average speedup and tree size reduction respectively, and solves more instances than product. On instances requiring 50k nodes, this improves to 11%, remaining consistent at 100k nodes. The ratio scoring function performs less well, only outperforming product by 4% in geometric average time, and not yielding smaller trees on average. These results confirm that svts is significantly better than product for MIPs that require very large B&B trees.

Finally, to assess whether our branching rules might consistently perform better for some problem instances, we compute the performance of an auto-opt setting, i.e. the results that would be achieved if each problem instance could select its best branching rule ahead of time. The results for the full benchmark set and the subset of instances in which some setting required at least 50k nodes are shown in Tables 5 and 6 respectively.

### Table 4: Performance results on instances of the MIPLIB 2017 Benchmark Set that required at least 100k nodes for some setting

| Instance        | product | ratio | svts |
|-----------------|---------|-------|------|
| #               | Time    | Nodes | #    | Time  | Nodes |
| Total           | 83      | 125.22k | +3  | 1.03 | 0.87 |
| Geo. mean       | 1.08k   | 441.63k | 0.96 | 1.01 | 0.89 |
| Sh. geo. mean   | 1.09k   | 441.84k | 0.96 | 1.01 | 0.89 |

### Table 5: Performance of an auto-opt setting on all instances of the MIPLIB 2017 Benchmark Set

| Instance        | product | ratio | svts | opt |
|-----------------|---------|-------|------|-----|
| #               | Time    | Nodes | #    | Time  | Nodes | #    | Time  | Nodes |
| Total           | 305     | 296.61k | +1  | 1.04 | 0.88 | +3  | 0.96 | 0.82 |
| Geo. mean       | 319.07  | 2.75k  | 1.01 | 1.08 | 0.99 | 1.02 | 0.83 | 0.80 |
| Sh. geo. mean   | 351.50  | 4.84k  | 1.02 | 1.05 | 0.99 | 1.00 | 0.83 | 0.81 |

### Table 6: Performance of an auto-opt setting on instances of the MIPLIB 2017 Benchmark Set that required at least 50k nodes for some setting

| Instance        | product | ratio | svts | opt |
|-----------------|---------|-------|------|-----|
| #               | Time    | Nodes | #    | Time  | Nodes | #    | Time  | Nodes |
| Total           | 93      | 144.30k | +4  | 1.00 | 0.87 | +5  | 0.86 | 0.81 |
| Geo. mean       | 1.02k   | 326.96k | 0.96 | 1.02 | 0.89 | 0.89 | 0.73 | 0.72 |
| Sh. geo. mean   | 1.03k   | 327.19k | 0.96 | 1.02 | 0.89 | 0.89 | 0.73 | 0.72 |

Tables 5 and 6 clearly show that there is an advantage to using the model-based branching rules for some problems, with the auto-opt setting achieving a 17% speedup across all instances, and a 27% speedup for instances requiring large B&B trees.

## 5 Conclusions

In this paper, we resolved many of the open problems of Le Bodic and Nemhauser's B&B model. We showed that there is no closed-form formula for the ratio value $\varphi$, and that the MVB conjecture
is false. Additionally, we showed tighter hardness results for the GVB problem, and showed that the GVB simulation problem can be solved in expected sub-exponential time in \( n \) for scoring functions that respect dominance. We then implemented improved branching rules for the open-source MIP solver SCIP 6.0, which yielded an 11% geometric average speedup and tree size reduction for problems in the MIPLIB 2017 Benchmark Set that required large B&B trees.

Since these rules perform well on MIP instances that lead to large B&B trees, an interesting line of future work would be to incorporate the methods of Anderson et al. \([6]\) for predicting B&B tree sizes to select branching rules at run time, or to reconsider the choice of branching rule after performing a restart.

**Acknowledgments** This research was funded by a Monash Faculty of IT grant. We would like to thank Professor Graham Farr for introducing the authors to one another, hence without whom this research may have never happened. We also thank Gregor Hendel and the SCIP team for assistance with the performance tests.

**References**

[1] T. Achterberg. *Constraint Integer Programming*. PhD thesis, Technische Universität Berlin, 2007.

[2] T. Achterberg. Scip: solving constraint integer programs. *Mathematical Programming Computation*, 1(1):1–41, 2009.

[3] T. Achterberg and T. Berthold. Hybrid branching. In *International Conference on AI and OR techniques in constraint programming for combinatorial optimization problems (CPAIOR)*, 2009.

[4] F. S. Acton. *Numerical methods that work*. Mathematical Association of America, 1990.

[5] A. M. Alvarez, Q. Louveaux, and L. Wehenkel. A machine learning-based approximation of strong branching. *INFORMS Journal on Computing*, 29(1):185–195, 2017.

[6] D. Anderson, G. Hendel, P. Le Bodic, and M. Viernickel. Clairvoyant restarts in branch-and-bound search using online tree-size estimation. In *AAAI Conference on Artificial Intelligence*, 2019.

[7] D. Applegate, R. Bixby, V. Chvatal, and B. Cook. Finding cuts in the tsp (a preliminary report). Technical report, Center for Discrete Mathematics & Theoretical Computer Science, 1995.

[8] M.-F. Balcan, T. Dick, T. Sandholm, and E. Vitercik. Learning to branch. In *International Conference on Machine Learning (ICML)*, 2018.

[9] M. Bénichou, J.-M. Gauthier, P. Girodet, G. Hentges, G. Ribiére, and O. Vincent. Experiments in mixed-integer linear programming. *Mathematical Programming*, 1(1):76–94, 1971.

[10] M. R. Garey and D. S. Johnson. *Computers and intractability*. W. H. Freeman & Co., 2002.

[11] A. Gleixner, M. Bastubbe, L. Eifler, T. Gally, G. Gamrath, R. L. Gottwald, G. Hendel, C. Hojný, T. Koch, M. E. Lübbecke, S. J. Maher, M. Miltenberger, B. Müller, M. E. Pfetsch, C. Puchert, D. Rehfeldt, F. Schlösser, C. Schubert, F. Serrano, Y. Shinano, J. M. Viernickel,
M. Walter, F. Wegscheider, J. T. Witt, and J. Witzig. The SCIP Optimization Suite 6.0. Technical report, Optimization Online, July 2018.

[12] C. Haase and S. Kiefer. The complexity of the Kth largest subset problem and related problems. *Information Processing Letters*, 116(2):111–115, 2016.

[13] S. Homer and A. L. Selman. *Computability and complexity theory*. Springer, 2011.

[14] E. B. Khalil, P. Le Bodic, L. Song, G. L. Nemhauser, and B. N. Dilkina. Learning to branch in mixed integer programming. In *AAAI Conference on Artificial Intelligence*, 2016.

[15] O. Kullmann. Fundaments of branching heuristics. *Handbook of Satisfiability*, 185:205–244, 2009.

[16] A. H. Land and A. G. Doig. An automatic method of solving discrete programming problems. *Econometrica: Journal of the Econometric Society*, pages 497–520, 1960.

[17] P. Le Bodic and G. Nemhauser. An abstract model for branching and its application to mixed integer programming. *Mathematical Programming*, 166(1-2):369–405, 2017.

[18] W. Ljunggren. On the irreducibility of certain trinomials and quadrinomials. *Mathematica Scandinavica*, 8(1):65–70, 1960.

[19] A. Lodi and A. Tramontani. *Performance Variability in Mixed-Integer Programming*, chapter Chapter 1, pages 1–12. INFORMS, 2013.

[20] MIPLIB 2017, 2018.

[21] H. Osada. The Galois groups of the polynomials $X^n + aX^l + b$. *Journal of number theory*, 25(2):230–238, 1987.
Appendices

Appendix 1: Solution to the MVB conjecture counterexample

We prove the correctness of (1) by induction. For $G \leq 5$, we can confirm exhaustively that $t(0) = 1$, $t(1) = 3$, $t(2) = 3$, $t(3) = 3$, $t(4) = 5$, $t(5) = 7$. Then, for $G \geq 6$, suppose that (1) is a solution to the instance. We have

\[
t(0 + 6k) = 1 + \min \begin{cases} t(0 + 6k - 2) + t(0 + 6k - 4), \\ t(0 + 6k - 3) + t(0 + 6k - 3), \end{cases}
\]

\[= 1 + \min \begin{cases} t(4 + 6(k - 1)) + t(2 + 6(k - 1)), \\ t(3 + 6(k - 1)) + t(3 + 6(k - 1)), \end{cases}
\]

\[= 1 + \min \begin{cases} \frac{2}{3} (2 \cdot 4^k + 1) - 1 + \frac{1}{6} (5 \cdot 4^k + 4), \\ 4^k - 1 + 4^k - 1, \end{cases}
\]

\[= \min \begin{cases} \frac{1}{6} (13 \cdot 4^k + 8) - 1, \\ 2 \cdot 4^k - 1, \end{cases}
\]

\[= 2 \cdot 4^k - 1.
\]

\[
t(1 + 6k) = 1 + \min \begin{cases} t(1 + 6k - 2) + t(1 + 6k - 4), \\ t(1 + 6k - 3) + t(1 + 6k - 3), \end{cases}
\]

\[= 1 + \min \begin{cases} t(5 + 6(k - 1)) + t(3 + 6(k - 1)), \\ t(4 + 6(k - 1)) + t(4 + 6(k - 1)), \end{cases}
\]

\[= 1 + \min \begin{cases} \frac{1}{3} (5 \cdot 4^k + 4) - 1 + 4^k - 1, \\ \frac{2}{3} (2 \cdot 4^k + 1) - 1 + \frac{2}{3} (2 \cdot 4^k + 1), \end{cases}
\]

\[= \min \begin{cases} \frac{4}{3} \left(2 \cdot 4^k + 1\right) - 1, \\ \frac{4}{3} \left(2 \cdot 4^k + 1\right) - 1, \end{cases}
\]

\[= \frac{4}{3} \left(2 \cdot 4^k + 1\right) - 1.
\]

\[
t(2 + 6k) = 1 + \min \begin{cases} t(2 + 6k - 2) + t(2 + 6k - 4), \\ t(2 + 6k - 3) + t(2 + 6k - 3), \end{cases}
\]

\[= 1 + \min \begin{cases} t(0 + 6k) + t(4 + 6(k - 1)), \\ t(5 + 6(k - 1)) + t(5 + 6(k - 1)), \end{cases}
\]

\[= 1 + \min \begin{cases} 2 \cdot 4^k - 1 + \frac{2}{3} (2 \cdot 4^k + 1), \\ \frac{1}{3} (5 \cdot 4^k + 4) - 1 + \frac{1}{3} (5 \cdot 4^k + 4), \end{cases}
\]

\[= \min \begin{cases} \frac{2}{3} (5 \cdot 4^k + 1) - 1, \\ \frac{2}{3} (5 \cdot 4^k + 4) - 1, \end{cases}
\]

\[= \frac{2}{3} \left(5 \cdot 4^k + 1\right) - 1.
\]
Therefore by induction on $G$, we can conclude that $P$ is a solution to the instance.

**Appendix 2: Evaluating scoring functions using the GVB simulation problem**

We demonstrate the utility of the improved analysis of the GVB simulation problem as a tool for calibrating and predicting the performance of proposed variable selection rules.
A practical algorithm for the GVB simulation problem

Using the fact that the expected number of non-dominated subsets of a set of random variables is sub-exponential, a dynamic programming algorithm that, at each step, filtered out the dominated variables and only generated states implicitly could solve (6) in expected sub-exponential time and space. We present here some techniques that lead to an even more practical algorithm.

1. We first note that dominance is clearly transitive by definition, and that if $v_1$ dominates $v_2$, then $v_2$ can not dominate $v_1$. The dominance relation therefore defines a directed acyclic graph (DAG) on the variables. Note that the order that a non-dominating variable selection rule chooses to branch on a set of variables must therefore be a topological sort of the given DAG. For random data, the dominance DAG can be very dense, so, for efficiency, our algorithm computes the transitive reduction of the dominance DAG. The transitive reduction of the dominance DAG contains for each vertex, edges to those visible on the upper convex hull from the point of view of that vertex, hence the expected outdegree will be $O(\log(n))$. See Figure 4.

2. The dynamic programming states are indexed by the current subset of available non-dominated variables and the current gap. In order to speed up indexing the states, the algorithm first pre-computes the set of all non-dominated subsets using heuristics (1) and (3) and a depth-first search through the state space.

3. When a variable is used, the algorithm examines its successors in the reduced dominance DAG. For each successor, if the selected variable was its last dominator, that variable is no longer dominated and is added to the subset of available non-dominated variables. This can be maintained in constant time by keeping track of the current indegree for each variable (à la Kahn’s topological sorting algorithm). Alternatively, if the multiplicities are all one and we use fewer than $w$ variables, where $w$ is the size of a machine word, we can pre-compute the adjacency matrix of the reverse dominance DAG and store the current subset as a bitmask. Checking whether a variable is the last remaining dominator is then achieved in one operation with a bitwise AND.

Figure 4: The transitive reduction of the dominance graph of 15 random variables in $[0,1000]^2$. 
Test parameters

For each test case, the algorithm generates 60 random variables \((l, r)\). The test cases each fall into one of four categories:

- **Balanced Instances**: The gains \(l\) and \(r\) are generated uniformly from the range \([1,1000]\). If \(l > r\), then the two are switched to ensure \(l \leq r\).

- **Unbalanced Instances**: The gains \(l\) and \(r\) are generated uniformly from the ranges \([1,500]\) and \([501,1000]\) respectively.

- **Very Unbalanced Instances**: The gains \(l\) and \(r\) are generated uniformly from the ranges \([1,250]\) and \([251,1000]\) respectively.

- **Extremely Unbalanced Instances**: The gains \(l\) and \(r\) are generated uniformly from the ranges \([1,125]\) and \([126,1000]\) respectively.

For simplicity, all variables have multiplicity 1, so each leaf node of a resulting B&B tree will have a depth at most 60. We generate 3000 test cases for each category, using three different values of \(G\) to measure the effectiveness of the rules as the trees become larger. For the four categories above, we use the gap values \(G\) depicted in Table 7. Note that the gap decreases as the variables become less balanced since the trees will require an intractable number of nodes otherwise.

| Category                | Small | Medium | Large |
|-------------------------|-------|--------|-------|
| Balanced                | 5000  | 9000   | 12000 |
| Unbalanced              | 4000  | 7000   | 9000  |
| Very Unbalanced         | 3000  | 5000   | 6000  |
| Extremely Unbalanced    | 2000  | 3000   | 3500  |

Table 7: The gap sizes \(G\) used in each of the computational tests.

The gaps in Table 7 are chosen slightly differently to Le Bodic and Nemhauser [17] in order to ensure that the simulations remain computationally feasible.

GVB simulation experiments

Le Bodic and Nemhauser [17] used the GVB simulation problem to tune the hybrid ratio rule, which selects from either the ratio or the product score depending on the estimated tree height. Due to the intractability of the GVB problem, they were unable to perform experiments on the svts rule. In this section, we use our improved GVB simulation algorithm to compare the product rule, the ratio rule and svts.

Table 8 shows for each category and gap combination, for each scoring function, the estimated geometric average tree size relative to product (hence product shows all zeros).

The results depicted in Table 8 clearly predict that the svts scoring function will outperform the hybrid ratio function, yielding tree size reductions between 5% and 20%. This backs up the results of our MIP benchmarks, in which svts does indeed outperform ratio on average.
Table 8: Results of the GVB simulation problem. The best performing rule for each category and gap combination is shown in bold.

### Appendix 3: Full performance test results

Here we present complete data on the performance tests of the MIPLIB 2017 Benchmark Set. For each row (i.e. problem instance), we report the geometric average time and tree size of the best $N$ permutations of each scoring function, where $N$ is the minimum number of permutations solved by any scoring function on that problem. This gives scoring functions that solve more permutations a fair advantage. When a rule fails to solve any permutation of an instances, time and node comparisons for that row are omitted, but number of instances solved is still depicted. Note that for fairness, the totals and averages in Section 4.2 are still computed based on all of the solved instances, not on the averages presented on each row here. As in the summary results, measurements for *ratio* and *svts* are given relative to *product*.

| Instance           | product | ratio | svts |
|--------------------|---------|-------|------|
|                    | # Time  | Nodes | # Time | Nodes | # Time | Nodes |
| 30n20b8            | 3 281.40 | 138.00 | +0 0.92 | 1.26 | +0 0.70 | 0.56 |
| CMS750_4           | 3 857.93 | 11.91k | +0 1.25 | 1.03 | +0 2.15 | 2.06 |
| air05              | 3 30.10  | 333.67 | +0 1.20 | 1.91 | +0 1.17 | 1.19 |
| app1-1             | 3 5.97   | 4.00   | +0 1.01 | 1.00 | +0 0.99 | 1.00 |
| app1-2             | 3 1.04k  | 68.33  | +0 0.83 | 0.62 | +0 1.28 | 0.85 |
| assign1-5-8        | 3 3.35k  | 5.94m  | +0 0.96 | 0.93 | +0 0.85 | 0.83 |
| beasleyC3          | 3 22.57  | 2.00   | +0 1.00 | 1.00 | +0 1.00 | 1.00 |
| binkar10_1         | 3 28.47  | 2.58k  | +0 0.79 | 0.88 | +0 0.83 | 0.86 |
| bnatt400           | 3 1.18k  | 6.85k  | +0 0.96 | 1.07 | +0 0.96 | 1.07 |
| bnatt500           | 3 5.02k  | 30.07k | +0 0.97 | 0.96 | +0 0.97 | 0.96 |
| bppc4-08           | 0 - -    | +2 - - | +2 - - | - - |
| brazil3            | 3 4.01k  | 1.69k  | +0 1.00 | 1.47 | +0 1.04 | 1.39 |
| chromaticindex512-7| 3 2.21k  | 5.89k  | +0 1.01 | 1.24 | +0 1.01 | 1.24 |

continues on next page...
| Instance         | product | ratio | svts |
|------------------|---------|-------|------|
|                  | # Time  | Nodes | # Time  | Nodes | # Time  | Nodes |
| co-100           | 2 4.53k | 6.11k | +0 0.93 | 1.64 | +0 1.09 | 2.01 |
| cod105           | 3 315.37 | 105.00 | +0 1.16 | 2.05 | +0 1.18 | 2.04 |
| cost266-UUE      | 3 3.18k | 234.69k | +0 1.34 | 1.46 | +0 0.96 | 0.95 |
| csched007        | 3 3.11k | 266.24k | +0 0.53 | 0.51 | +0 0.71 | 0.57 |
| csched008        | 3 1.03k | 94.19k | +0 1.27 | 1.24 | +0 1.36 | 1.42 |
| dano3_3          | 3 106.70 | 12.33 | +0 1.10 | 1.68 | +0 1.02 | 1.41 |
| dano3_5          | 3 307.80 | 165.00 | +0 0.86 | 0.89 | +0 0.89 | 1.04 |
| drayage-100-23   | 3 16.83 | 33.33 | +0 0.95 | 1.03 | +0 0.89 | 0.90 |
| drayage-25-23    | 3 129.80 | 106.45k | -2 4.99 | 18.14 | -2 0.97 | 2.67 |
| eil33-2          | 3 70.93 | 797.00 | +0 0.89 | 0.91 | +0 0.97 | 0.97 |
| fast057          | 3 217.13 | 840.00 | +0 1.44 | 1.54 | +0 1.20 | 1.15 |
| fastxgemm-n2r6s0t2 | 3 605.93 | 113.70k | +0 1.35 | 1.42 | +0 0.76 | 0.73 |
| fiball           | 1 1.47k | 3.56k | +1 0.89 | 0.80 | +1 1.17 | 1.16 |
| gen-ip002        | 3 1.55k | 5.59m | +0 0.91 | 0.93 | +0 0.96 | 1.00 |
| gen-ip054        | 3 3.04k | 12.77m | +0 0.63 | 0.62 | +0 0.63 | 0.63 |
| glass-sc         | 3 3.16k | 278.80k | +0 0.93 | 0.96 | +0 0.95 | 0.98 |
| glass4           | 3 2.17k | 1.73m | +0 1.54 | 2.37 | +0 1.14 | 1.63 |
| graph20-20-1rand| 2 5.17k | 38.68k | +0 0.34 | 0.37 | +1 0.49 | 0.49 |
| graphdraw-domain | 3 1.39k | 2.52m | +0 1.15 | 1.11 | +0 1.03 | 1.01 |
| h80x6320d        | 3 3.80k | 72.00k | +0 0.89 | 0.80 | +1 1.17 | 1.16 |
| icir97-tension   | 3 12.90 | 5.33 | +0 1.01 | 1.00 | +0 1.00 | 1.00 |
| irp              | 3 179.57 | 302.33 | +0 0.95 | 1.03 | +0 1.02 | 0.99 |
| istanbul-no-cutoff | 3 775.90 | 1.15k | +0 1.01 | 1.32 | +0 1.11 | 1.43 |
| map10            | 3 775.90 | 1.15k | +0 1.01 | 1.32 | +0 1.11 | 1.43 |
| map16715-04      | 3 1.75k | 1.73k | +0 1.17 | 1.35 | +0 1.21 | 1.31 |
| markshare-140    | 3 320.37 | 2.51m | +0 0.78 | 0.79 | +0 0.87 | 0.88 |
| mas74            | 3 2.29k | 7.07m | +0 0.89 | 0.84 | +0 0.77 | 0.78 |
| mas76            | 3 132.87 | 301.16k | +0 0.98 | 1.02 | +0 0.90 | 0.88 |
| mc11             | 3 1.12k | 2.26k | +0 1.01 | 0.95 | +0 1.55 | 2.12 |
| mcsched          | 3 240.10 | 11.39k | +0 0.99 | 1.31 | +0 1.00 | 1.08 |
| mik-250-20-75-4  | 3 37.23 | 16.05k | +0 0.74 | 0.73 | +0 0.79 | 0.76 |
| mzzv11           | 3 336.90 | 1.57k | +0 0.90 | 0.98 | +0 0.94 | 0.80 |
| mzzv42z          | 3 168.13 | 259.67 | +0 1.26 | 1.22 | +0 1.19 | 1.34 |
| n2seq36q         | 3 921.67 | 4.12k | +0 0.94 | 0.52 | +0 0.95 | 0.86 |
| n5-3             | 3 30.50 | 785.33 | +0 1.07 | 1.17 | +0 1.06 | 1.22 |
| neos-1445765     | 3 59.53 | 72.33 | +0 1.03 | 1.29 | +0 1.01 | 1.00 |
| neos-1456979     | 1 - | - | -1 - | -1 - | 1 - | - |
| neos-1582420     | 3 37.30 | 664.33 | +0 0.99 | 1.02 | +0 1.21 | 1.67 |
| neos-2657525-crna | 0 - | - | -1 - | -1 - | 1 - | - |
| neos-3004026-kanka | 3 68.13 | 5.25k | +0 0.82 | 0.66 | +0 0.82 | 0.66 |
| neos-3024952-love | 2 3.12k | 135.87k | +1 1.25 | 0.92 | +1 0.99 | 0.74 |
| neos-3083819-mbu | 3 14.93 | 2.33k | +0 0.86 | 1.01 | +0 0.89 | 1.00 |
| neos-3216931-puriri | 1 - | - | -1 - | -1 - | 1 - | - |
| neos-3402294-bobin | 3 1.84k | 12.76k | +0 1.02 | 1.44 | +0 0.68 | 0.53 |
| neos-3627168-kasai | 0 - | - | +1 - | +1 - | 1 - | - |
| neos-4413714-turia | 3 387.20 | 2.00 | +0 0.99 | 1.00 | +0 1.00 | 1.00 |
| neos-4722843-widden | 3 1.60k | 2.62k | +0 1.12 | 1.15 | +0 1.01 | 1.01 |

continues on next page...
| Instance                     | product | ratio | svts |
|------------------------------|---------|-------|------|
|                              | #       | Time  | Nodes | #       | Time  | Nodes |
| neos-4738912-atrato          | 3       | 1.02k | 103.05k | +0     | 0.71  | 0.60  |
| neos-5107597-kakapo          | 3       | 2.10k | 682.60k | -2     | 2.67  | 3.52  |
| neos-5188808-nattai          | 3       | 2.22k | 27.40k  | +0     | 0.83  | 0.69  |
| neos-5195221-niemur          | 3       | 2.68k | 21.97k  | +0     | 1.10  | 1.09  |
| neos-848589                 | 1       | -     | -      | +0     | -     | -     |
| neos-860300                 | 3       | 16.73 | 2.00    | +0     | 1.00  | 1.00  |
| neos-911970                 | 2       | 1.12k | 623.61k | -1     | 2.55  | 2.37  |
| neos-933966                 | 3       | 1.87k | 582.60  | +0     | 1.60  | 0.91  |
| neos-950242                 | 3       | 283.60| 73.33   | +0     | 0.82  | 0.72  |
| neos-960392                 | 3       | 1.22k | 65.67   | +0     | 1.09  | 1.04  |
| neos17                     | 3       | 22.07 | 16.59k  | +0     | 1.17  | 1.06  |
| neos5                      | 3       | 104.37| 255.77k | +0     | 0.92  | 0.96  |
| net12                      | 3       | 1.34k | 2.07k   | +0     | 0.71  | 0.85  |
| netdiversion                | 3       | 951.75| 13.00   | -1     | 1.66  | 2.08  |
| nexp-150-20-8-5             | 2       | 2.35k | 3.60k   | +0     | 1.34  | 2.18  |
| ns1208400                  | 3       | 513.60| 1.26k   | +0     | 0.79  | 1.21  |
| ns1644855                  | 2       | 2.09k | 8.00    | -1     | 1.01  | 1.00  |
| ns1830653                  | 3       | 115.40| 6.23k   | +0     | 1.24  | 1.20  |
| ns1952667                  | 3       | 1.56k | 3.19k   | +0     | 0.53  | 1.16  |
| mu25-pr12                  | 3       | 6.60  | 97.00   | +0     | 0.99  | 0.88  |
| nursesched-sprint02        | 3       | 39.87 | 13.00   | +0     | 1.02  | 0.97  |
| nw04                      | 3       | 23.87 | 7.33    | +0     | 1.01  | 1.00  |
| p200x1188c                 | 3       | 2.90  | 3.00    | +0     | 0.99  | 1.00  |
| peg-solitaire-a3           | 1       | 4.09k | 1.92k   | +0     | 1.53  | 2.02  |
| pg                        | 3       | 20.47 | 552.67  | +0     | 0.99  | 0.94  |
| pg5_34                     | 3       | 2.09k | 191.13k | +0     | 0.84  | 0.98  |
| physiciansched6-2          | 1       | 165.20| 151.00  | +0     | 2.15  | 12.62 |
| piperout-08                | 3       | 858.30| 604.00  | +0     | 0.93  | 0.81  |
| piperout-27                | 3       | 320.93| 174.00  | +0     | 1.65  | 2.13  |
| pk1                       | 3       | 149.17| 386.87k | +0     | 1.03  | 1.06  |
| qap10                      | 3       | 108.17| 3.33    | +0     | 0.99  | 1.00  |
| rail507                    | 3       | 255.57| 1.32k   | +0     | 1.03  | 0.80  |
| rannée14x18-disj-8         | 3       | 1.52k | 446.14k | +0     | 0.67  | 0.55  |
| rd-rphsc-21                | 0       | -     | -      | +1     | -     | -     |
| reblock115                | 1       | 5.77k | 802.39k | +1     | 0.87  | 1.04  |
| rmatr100-p10              | 3       | 174.13| 857.67  | +0     | 1.04  | 1.11  |
| rocl-4-11                 | 3       | 727.85| 13.22k  | -1     | 5.34  | 8.54  |
| roccocoC10-001000          | 3       | 985.13| 74.65k  | +0     | 0.89  | 1.30  |
| ro2-alpha3n4             | 3       | 844.00| 5.64k   | +0     | 1.51  | 1.97  |
| roll3000                   | 3       | 46.90 | 2.59k   | +0     | 0.75  | 0.61  |
| s250r10                   | 2       | 3.00k | 28.38k  | +1     | 1.41  | 1.58  |
| seymour1                   | 3       | 60.80 | 1.49k   | +0     | 0.91  | 0.74  |
| sp98ar                    | 0       | -     | -      | +1     | -     | -     |
| supportcase18              | 1       | -     | -      | -1     | -     | -     |
| supportcase26              | 1       | 6.16k | 7.60m   | +2     | 0.20  | 0.22  |
| supportcase33              | 3       | 1.67k | 15.08k  | +0     | 1.23  | 1.20  |
| supportcase40              | 3       | 1.11k | 12.08k  | +0     | 1.05  | 1.05  |
| Instance       | product | ratio | svts |
|----------------|---------|-------|------|
|                | #       | Time  | Nodes| #   | Time | Nodes| #   | Time | Nodes |
| supportcase7   | 3       | 177.13| 28.33| +0  | 1.00 | 1.05| +0  | 1.00 | 1.06 |
| swath1         | 3       | 14.93 | 387.00| +0  | 1.03 | 1.05| +0  | 0.99 | 0.94 |
| swath3         | 3       | 425.53| 72.00k| +0  | 0.33 | 0.32| +0  | 0.38 | 0.30 |
| tbfp-network   | 3       | 1.02k | 51.00 | -1  | 3.36 | 6.81| +0  | 1.84 | 3.28 |
| timtab1        | 3       | 52.40 | 39.40k| +0  | 1.22 | 1.33| +0  | 1.25 | 1.33 |
| tr12-30        | 3       | 832.57| 501.01k| +0  | 0.73 | 0.72| +0  | 1.04 | 1.00 |
| uct-subprob    | 3       | 2.50k | 94.91k| +0  | 0.78 | 0.81| +0  | 0.76 | 0.73 |
| unitcal_7      | 3       | 280.30| 180.33| +0  | 1.02 | 0.96| +0  | 1.00 | 0.60 |
| var-smallemer-y-m6j6 | 0     | -     | -     | +0  | -    | -    | +1  | -    | -    |
| wachplan       | 3       | 857.33| 45.68k| +0  | 1.14 | 1.16| +0  | 1.13 | 1.10 |

Table 9: Detailed performance results on all instances of the MIPLIB 2017 Benchmark Set. The best performing rule for each problem is shown in bold.