Complete graph immersions in dense graphs

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Abstract

In this article we consider the relationship between vertex coloring and the immersion order. Specifically, a conjecture proposed by Abu-Khzam and Langston in 2003, which says that the complete graph with \( t \) vertices can be immersed in any \( t \)-chromatic graph, is studied.

First, we present a general result about immersions and prove that the conjecture holds for graphs whose complement does not contain any induced cycle of length four and also for graphs having the property that every set of five vertices induces a subgraph with at least six edges.

Then, we study the class of all graphs with independence number less than three. If Abu-Khzam and Langston’s conjecture is true for this class of graphs, then an easy argument shows that every graph of independence number less than 3 contains \( K_{\lceil n^2 \rceil} \) as an immersion. We show that the converse is also true. Furthermore, we show that every graph of independence number less than 3 has an immersion of \( K_{\lceil n^3 \rceil} \).

1 Introduction

Vertex coloring has been a very important topic in graph theory. The usual goal, and the one considered here, is to color every vertex of a graph such that adjacent vertices get different colors. The chromatic number of a graph \( G \), denoted \( \chi(G) \), is the minimum number of colors required to color its vertices. If \( \chi(G) = t \), then we say that \( G \) is \( t \)-chromatic.

It has been suspected for a long time that if a graph cannot be colored with \( t \) colors, then it must somehow contain the complete graph \( K_t \) with \( t \) vertices. At some point in the 40’s, Hajós [17] conjectured that the relation of containment was a topological order. This conjecture is true for \( t \leq 4 \) [10], but false for \( t \geq 7 \) [4]. It remains open for \( t \in \{5, 6\} \). In 1943 Hadwiger [16] suggested that the containment had to be the minor order, i.e. he conjectured that every \( t \)-chromatic graph contains \( K_t \) as a minor. It was shown that Hadwiger’s conjecture holds for \( t = 5 \) [26] and \( t = 6 \) [22]. But it remains uncertain until today, whether the conjecture is true for \( t \geq 7 \).

In this article we study a different order, the immersion order, which is defined by lifts of edges. A lift of two (adjacent) edges \( uv \) and \( vw \) consist of deleting \( uv \) and \( vw \), and adding the edge \( uw \). And a graph \( H \) is immersed in a graph \( G \) if \( H \) can be obtained from \( G \) by performing lifts of edges and deleting vertices and/or edges. We denote this by \( H \preceq_i G \). We also say that \( G \) contains an immersion of \( H \). It is easy to check that this definition is equivalent to the existence of an injective function \( \phi : V(H) \to V(G) \) such that:
1. For every $uv \in E(H)$, there is a path in $G$, denoted $P_{uv}$, which connects $\phi(u)$ and $\phi(v)$.

2. The paths $\{P_{uv} : uv \in E(H)\}$ are pairwise edge-disjoint.

If the paths $P_{uv}$ are internally disjoint from $\phi(V(H))$, then we say that the immersion is strong. We call the vertices in $\phi(V(H))$ the corner vertices of the immersion.

Clearly topological containment implies immersion containment (strong immersion containment, actually). However, the minor order and the immersion order are not comparable. The immersion order, although initially much less studied than the minor and topological orders, has received a large amount of attention recently [3, 11, 12, 13, 14, 18, 27]. In fact, Robertson and Seymour extended their proof of Wagner’s famous conjecture [20], to prove that the immersion order is a well-quasi-order [21].

In analogy to Hadwiger and Hajós’ conjectures, Abu-Khzam and Langston [1] conjectured the following.

Conjecture 1 (Abu-Khzam and Langston). If $\chi(G) \geq t$, then $K_t$ is immersed in $G$.

Since Hajós’ conjecture holds for $t \leq 4$, Abu-Khzam and Langston’s conjecture is true for $t \leq 4$, as topological order is just a particular case of immersion order.

Each graph $G$ with $\chi(G) = t$ must contain a $t$-critical subgraph, i.e., a graph $\tilde{G}$ such that $\chi(\tilde{G}) = t$ and $\chi(H) < t$ for every proper subgraph $H$ of $\tilde{G}$. Furthermore, it is easy to see that every $t$-critical graph must have minimum degree at least $t - 1$. Using this fact, DeVos, Kawarabayashi, Mohar and Okamura [9] resolved Abu-Khzam and Langston’s conjecture for small values of $t$.

Theorem 1.1 ([9]). Let $f(k)$ be the smallest integer such that every graph of minimum degree at least $f(k)$ contains an immersion of $K_k$. Then $f(k) = k - 1$ for $k \in \{5, 6, 7\}$.

For $k \geq 8$, however, $f(k) \geq k$ [7, 8], i.e. $\delta(G) \geq k - 1$ does not guarantee an immersion of $K_k$ in $G$.

Theorem 1.1 solves Abu-Khzam and Langston’s conjecture for very small values of $t$. We are interested here in the other end of the spectrum, where $t$ is close to the number of vertices. So we restrict our attention to classes of graphs which are quite dense.

We will call a graph $(k, s)$-dense if every set of $k$ vertices induces a subgraph with at least $s$ edges. We prove the following two results.

Theorem 1.2. Every $(5,6)$-dense graph $G$ contains a strong immersion of $K_{\chi(G)}$.

Theorem 1.3. Any graph $G$ whose complement has no induced cycle of length four contains a strong immersion of $K_{\chi(G)}$.

Finally, we focus on the study of a special class of graphs, the graphs $G$ that have no independent set of size three, or equivalently, whose independence number $\alpha(G)$ is at most 2. Abu-Khzam and Langston’s conjecture restricted to that class reads as follows.

Conjecture 2. Any graph $G$ with $\alpha(G) \leq 2$ contains an immersion of $K_{\lceil \frac{n}{2} \rceil}$.

If $\alpha(G) \leq 2$, then in any vertex coloring of $G$, every color class, being an independent set, has at most two vertices, which implies that $\chi(G) \geq \frac{n}{2}$. Abu-Khzam and Langston’s conjecture would thus imply that $G$ must contain an immersion of $K_{\lceil \frac{n}{2} \rceil}$. The latter gives rise to a new conjecture.
Conjecture 3. Any graph $G$ with $\alpha(G) \leq 2$ contains an immersion of $K_{\lceil n^2 \rceil}$.

We just saw that Conjecture 2 implies Conjecture 3. However, the two conjectures are actually equivalent. Following ideas from [19] we show the next result.

Theorem 1.4. Conjectures 2 and 3 are equivalent.

A weaker version of Conjecture 3 is shown, namely the following result.

Theorem 1.5. If $G$ is a graph with $\alpha(G) \leq 2$, then $G$ contains a strong immersion of $K_{\lceil n^3 \rceil}$.

An analogous result was shown by Chudnovsky [5], namely that if $G$ is a graph with $n$ vertices and no independent set of size three, then $G$ contains a complete minor of size $\lceil n^3 \rceil$.

The technique used there is a nice use of induced paths of length two. Here we present a different technique.

This work is organized as follows. In Section 2 we present a quick review of some definitions and properties about vertex coloring that will be used through the text. In Section 3 we immerse a large complete graph into a multipartite complete graph (see Theorem 3.1), and also prove Theorems 1.2 and 1.3. And in Section 4 we prove Theorems 1.4 and 1.5 and show a series of properties that a counterexample of Conjecture 2 with minimum number of vertices should satisfy.

2 Vertex coloring

Given a coloring $c : V(G) \to \{1, \ldots, k\}$, we denote $c_i = \{u : c(u) = i\}$ and $c_{ij}$ the subgraph induced by the set of vertices $\{u : c(u) \in \{i, j\}\}$. We call a path in $c_{ij}$ a chain, and for each $u \in V(c_{ij})$, we denote $c_{ij}(u)$ the connected component of $c_{ij}$ that contains $u$. If $\{i, j\} \neq \{k, l\}$, then $c_{ij}$ and $c_{kl}$ are edge-disjoint graphs. This observation is particularly important to find immersions in graphs, considering the second definition of immersion. For this reason, the use of chains will be very helpful.

Let $c : V(G) \to \{1, \ldots, k\}$ be a vertex coloring of $G$ and let $i \in \{1, \ldots, k\}$. We say that $u \in V(G)$ is a dominating vertex for color $i$, if $c(u) = i$ and if for each color $j \neq i$, there is a vertex $v$ such that $c(v) = j$ and $uv \in E(G)$. If $c : V(G) \to \{1, \ldots, \chi(G)\}$ is a minimum vertex coloring of $G$, then every $i \in \{1, \ldots, \chi(G)\}$ has a dominating vertex. Indeed, if $i$ is such that for each $u \in c_i$, there exists $j_u \neq i$, such that for all $v \in c_{j_u}$, $uv \notin E(G)$ then we can assign color $j_u$ to each $u$, thus eliminating color $i$, and get a smaller coloring.

3 Complete Graph Immersions

Let us see first, that in a complete multipartite graph we can find an immersion of a complete graph of relatively large size.

Theorem 3.1. Let $G$ be a complete multipartite graph of $k \geq 2$ classes with $s$ vertices each.
Then $G$ has a strong immersion of $H$, where,

$$
H = \begin{cases} 
K_{(k-1)s+1} & \text{if } s \text{ is even} \\
K_{(k-1)s} & \text{if } s \neq 1 \text{ and } s \text{ is odd} \\
K_2 & \text{if } s = 1
\end{cases}
$$

**Proof.** The $s = 1$ case is trivial, so we can assume $s > 1$. We choose the vertices of $k - 1$ classes as corner vertices (in the case that $s$ is even, we will add an additional corner vertex later), and the vertices of the remaining class, let us call it $U$, will be used for the edge-disjoint paths. The paths between two vertices from different classes already exist (they are the edges between them), so we only need to worry about those vertices that are in the same class. We know that $\chi'(K_s) = s - 1$ if $s$ is even, and $\chi'(K_s) = s$ if $s$ is odd ([24, p.133]).

For each class of $s$ corner vertices, consider a $\chi'(K_s)$-edge-coloring of the edges that are missing (all of them). As $|U| \geq \chi'(K_s)$, we can assign each of the used colors on the edges of $K_s$ to some vertex in $U$. Say vertex $u_i \in U$ gets color $i$. Then, for two corner vertices $v$ and $w$ in the same class, we assign $P_{vw} = vu_iw$ where $vw$ is colored with color $i$.

Observe that these paths are edge-disjoint. Indeed, if two paths $P_{vw}$ and $P_{xy}$ share an edge, then $vw$ would have to be adjacent to $xy$. In addition, we would have $P_{vw} = vu_iw$, $P_{xy} = xu_iy$ for some $i \leq \chi'(G)$. That is, both $vw$ and $xy$ would have assigned color $i$, which is a contradiction.

Note that if $s$ is even, then in $U$ there is a vertex that is not being used in the edge-disjoint paths, so we can add it as a corner vertex of the immersion, as it is adjacent to all other corner vertices. Thus, we find the desired immersion, which is strong because no corner vertex is used as an internal vertex of some path.

**Observation.** Actually, a more general result follows directly from the proof of the theorem. If $G$ is a complete multipartite graph with $k \geq 2$ classes of sizes $s_1, s_2, \ldots, s_k$, with $s_k \geq s_i$, for $i \leq k - 1$, then $G$ contains a strong immersion of $K_{s_1+s_2+\ldots+s_{k-1}}$.

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** Let us suppose first that $G$ has fewer than five vertices. The cases $\chi(G) \in \{1, 2\}$ are trivial. If $\chi(G) = 3$, $G$ must contain a triangle, so $K_3 \subseteq G$. And if $\chi(G) = 4$, it is easy to check that the only option is $G = K_4$. So, we can assume $|V(G)| \geq 5$.

Let $c$ be a minimum coloring of $V(G)$ and let $k = \chi(G)$. Note that $|c_i| \leq 3$, for $1 \leq i \leq k$, since there cannot be independent sets of size four. This, because if there were any, then, adding any other vertex, we would have a set of five vertices inducing less than six edges.

Observe that if $c_i = \{u, x\}$ and $c_j = \{v, y\}$ are such that $c_{ij}$ is not connected, then the vertices in $c_i \cup c_j$ are adjacent to all other vertices (thus, they are dominant). Indeed, if $c_{ij}$ is not connected, it has exactly two edges. Then, any other vertex must be adjacent to $u$, $v$, $x$ and $y$, because of the $(5,6)$-density of $G$.

We choose a dominant vertex of each color for the corner vertices of the immersion. In the case that $c_{ij}$ is not connected, for two colors classes of size two, then choose the pair of dominant vertices that are adjacent as corner vertices. Note that this choice is possible, because of the above observation. Let us call $u_i$ the dominant vertex of color $i$ chosen as corner vertex. Let $i, j$ be any two colors and let us see that its respective corner vertices are connected by a chain.
If $|c_i| = 1$, then $u_i u_j \in E(G)$, as $u_j$ is dominant. The edge $u_i u_j$ is the chain we want.

- If $|c_i| = 2, |c_j| = 3$, then $u_i u_j \in E(G)$, due to the $(5, 6)$-density of the graph.

- If $|c_i| = 3, |c_j| = 3$, then considering the set of $c_i$’s vertices plus $u_j$ and a second vertex from $c_j$, it holds that the induced subgraph must necessarily be a complete bipartite graph, because of the $(5, 6)$-density of $G$. Then, $u_i u_j \in E(G)$.

- If $|c_i| = 2, |c_j| = 2$, there are two options. If $c_{ij}$ is connected, we can always find a chain between $u_i$ and $u_j$. And if $c_{ij}$ is not connected, then by the choice of the corner vertices, it holds that $u_i u_j \in E(G)$.

By symmetry, the above are all possible cases, and so, between each pair of corner vertices there is a chain that connects them, and therefore, we have found an immersion of $K_{\chi(G)}$. The immersion is strong as the paths being chains between corner vertices of different colors, none of them uses another corner vertex as an internal vertex.

Let us prove now Theorem 1.3.

**Proof of Theorem 1.3.** Let $c$ be a minimum vertex coloring of $G$ and choose a dominant vertex of each color as the set of corner vertices. Consider two corner vertices, $u$ and $v$, with $c(u) = i, c(v) = j$ and let us see that there is a chain that joins them (so we ensure that paths will be edge-disjoint).

If $uv \in E(G)$, then the edge $uv$ is the chain we want. If $uv \notin E(G)$, there are vertices $x \in c_j, y \in c_i$, such that $ux, vy \in E(G)$, because $u$ and $v$ are dominant. Also, as $C_4 \not\subseteq G$, necessarily $xy \in E(G)$. Thus $uxyv$ is the chain we want. Then we have an immersion of $K_{\chi(G)}$, which is strong since the paths being chains, they do not use another corner vertex as an internal vertex.

**Observation.** At first, the condition that there are no induced cycles of length four in the complement of the graph might seem too restrictive, however, unlike in $(5, 6)$-dense graphs, color classes can be relatively big. Indeed, the authors of [13] proved that if $G$ is a graph such that $C_4 \not\subseteq G$, with $n$ vertices and average degree $a$, then $\omega(G) \geq 0.1 a^2 n^{-1}$. That means if $C_4 \not\subseteq G$, then $\alpha(G) \geq (n - 1 - a)^2 n^{-1}/10$. I.e. there might be colorings with classes of sizes $(n - 1 - a)^2 n^{-1}/10$.

**4 Graphs with small independence number**

The class of graphs with no independent set of size three has been extensively studied in an attempt to solve Hadwiger’s conjecture (see [2, 5, 6, 13]). It is for this reason that we are interested in Abu-Khzam and Langston’s conjecture restricted to these graphs, as stated in Conjecture 2.

It is easy to check that the non-neighbourhood of any vertex of a graph $G$ with $\alpha(G) \leq 2$ induces a complete graph.

We shall now see that if we replaced $K_{\lceil \frac{n}{3} \rceil}$ with $K_{\lceil \frac{n}{2} \rceil}$ in Conjecture 3, then the claim is true, as stated in Theorem 1.5. Moreover, either $G$ contains $K_{\lceil \frac{n}{2} \rceil}$ as a subgraph, or any set of $\lceil \frac{n}{3} \rceil$ vertices can be a set of corner vertices. Also, the immersion is strong.
Proof of Theorem 1.4 Let us define, for a vertex \( v \in V(G) \) and a set \( U \subseteq V(G) \),

\[
N_U(v) = N(v) \cap U.
\]

We can assume \( \delta(G) \geq \lceil \frac{2n}{3} \rceil \), since otherwise there is a vertex \( v \) with \( d(v) < \lceil \frac{2n}{3} \rceil \). Then the non-neighborhood of \( v \) has size at least \( \lceil \frac{2n}{3} \rceil \), and as it induces a complete graph, we would have \( G \) containing \( K_{\lceil \frac{2n}{3} \rceil} \) as a subgraph.

We separate then \( V(G) \) into two disjoint sets \( U \) and \( W \), such that \( |U| = \lceil \frac{n}{3} \rceil \) and \( |W| = \lceil \frac{2n}{3} \rceil \). The vertices from the set \( U \) will be the corner vertices and, as always, \( P_{uv} \) is the path between \( u \) and \( v \) in the immersion. We arrange the pairs \( \{u, v\} \) with \( u, v \in U \) arbitrarily and we assign the paths of the immersion as follows. If \( uv \in E(G) \), then \( P_{uv} = uv \). If \( uv \notin E(G) \), then \( P_{uv} = uzv \), with \( z \in N_W(u) \cap N_W(v) \) and such that \( z \) has not been used at some \( P_{ax} \), with \( x \in U \) or some \( P_{vx} \), with \( x \in U \). Note that given the latter condition, the paths will be edge-disjoint. Furthermore, no corner vertex is used as an internal vertex of a path, so the immersion is indeed strong. Let us see that this assignment is possible (we only need to verify this for the case \( uv \notin E(G) \)). Note that:

\[
|N_W(u) \cup N_W(v)| = |N_W(u)| + |N_W(v)| − |N_W(u) \cap N_W(v)| \\
\geq \left\lceil \frac{2n}{3} \right\rceil − |N_U(u)| + \left\lceil \frac{2n}{3} \right\rceil − |N_U(v)| − |N_W(u) \cap N_W(v)|.
\]

Since \( uv \notin E(G) \) and \( \alpha(G) \leq 2 \), we have that for each \( w \in W, uw \in E(G) \) or \( vw \in E(G) \). This implies that \( N_W(u) \cup N_W(v) = W \). Then,

\[
|N_U(u)| + |N_U(v)| \geq \frac{2n}{3} − |N_W(u) \cap N_W(v)|.
\]

Observe that the vertices in \( U \) that need a path to \( u \) of length greater than 1 are its non-neighbors, i.e. \( \lceil \frac{n}{3} \rceil − 1 − |N_U(u)| \) vertices. Similarly, the vertices in \( U \) that need a path to \( v \) of length greater than 1 are \( \lceil \frac{n}{3} \rceil − 1 − |N_U(v)| \). Thus, the total number of paths with length greater than 1 that need to use a vertex from \( N_W(u) \cap N_W(v) \) are

\[
2 \left\lceil \frac{n}{3} \right\rceil − 2 − (|N_U(u)| + |N_U(v)|) < |N_W(u) \cap N_W(v)|,
\]

by (1).

That is, less vertices are needed than those that are available, to construct the edge-disjoint paths of the immersion. Therefore, there exists \( z \in N_W(u) \cap N_W(v) \) that has not been used in other paths \( P_{ax} \) or \( P_{vx} \), and then we can assign \( P_{uv} = uzv \). Thus, we have obtained a strong immersion of \( K_{\lceil \frac{2n}{3} \rceil} \) in \( G \).

\[\square\]

4.1 Equivalence of Conjectures 2 and 3

The proof of Theorem 1.4 is strongly inspired from [19]. We will need to use some preliminary results. Suppose Conjecture 2 fails, and let \( G \) be a counterexample that minimizes the number of vertices. Observe that the number of vertices is bounded by the product of the independence number and the chromatic number. So we have the following inequality:
\[ |V(G)| \leq 2\chi(G). \]  \hspace{1cm} (2)

We will prove some properties that \( G \) satisfies.

**Definition 4.1.** A graph \( G \) is \( k \)-color-critical if \( \chi(G) = k \) and \( \chi(G - v) < k \), for each \( v \in V(G) \).

**Lemma 4.2.** \( G \) is \( \chi(G) \)-color-critical.

**Proof.** Indeed, if there is a vertex \( v \in V(G) \), such that \( \chi(G - v) = \chi(G) \), then as \( G - v \) has less vertices than \( G \) and \( \alpha(G - v) \leq 2 \), we would have that, \( K_{\chi(G)} = K_{\chi(G-v)} \preceq_i G - v \preceq_i G \), which contradicts the fact that \( G \) is a counterexample for Conjecture 2.

**Lemma 4.3.** \( \overline{G} \) is connected.

**Proof.** If not, \( G \) consists of two disjoint subgraphs \( G_1 \) and \( G_2 \), such that for all \( u \in V(G_1) \) and for all \( v \in V(G_2) \), \( uv \in E(G) \). Then, as both \( G_1 \) and \( G_2 \) have less vertices than \( G \), it holds that \( K_{\chi(G_1)} \preceq_i G_1 \) and \( K_{\chi(G_2)} \preceq_i G_2 \), and then, \( K_{\chi(G)} = K_{\chi(G_1)+\chi(G_2)} \preceq_i G \), which leads to a contradiction.

For the next property, we will use a well-known theorem of Gallai.

**Theorem 4.4** (Gallai, see [25]). Every \( k \)-color-critical graph \( G \), such that \( G \) is connected, has at least \( 2k - 1 \) vertices.

**Lemma 4.5.** \( |V(G)| = 2\chi(G) - 1 \).

**Proof.** By (2) we know that \( |V(G)| \leq 2\chi(G) \). If \( |V(G)| \) is equal to \( 2\chi(G) \), then for each \( v \in V(G) \), we have
\[
2\chi(G) - 1 = |V(G)| - 1 = |V(G - v)| \leq 2\chi(G - v) \leq 2\chi(G).
\]
That is,
\[
\chi(G) - \frac{1}{2} \leq \chi(G - v) \leq \chi(G).
\]
Thus, \( \chi(G - v) = \chi(G) \), which contradicts Lemma 4.2. Therefore, \( |V(G)| \leq 2\chi(G) - 1 \).

For the other inequality, simply use Lemmas 4.2 and 4.3 together with Theorem 4.4 to obtain that \( |V(G)| \geq 2\chi(G) - 1 \), so \( |V(G)| = 2\chi(G) - 1 \).

We are now able to prove Theorem 1.4.

**Proof of Theorem 1.4.** By Lemma 4.5 we have that \( \left\lceil \frac{|V(G)|}{2} \right\rceil = \frac{|V(G)| + 1}{2} = \chi(G) \). Then, \( K_{\left\lfloor \frac{\chi(G)}{2} \right\rfloor} \not\preceq_i G \) and therefore, \( G \) is also a counterexample for Conjecture 3.

Observe that \( G \) turns out to be a counterexample with minimum number of vertices for Conjecture 3 as well.
4.2 Counterexample of Conjecture 2

In this subsection we will prove a series of properties that a counterexample of Conjecture 2 with minimum number of vertices satisfies besides those mentioned by Lemmas 4.2, 4.3 and 4.5. We will need the next result.

**Theorem 4.6** ([19]). Suppose $G$ is a connected graph with $\alpha(G) \leq 2$ and suppose that $G$ is $\alpha$-critical. Then if $x$ and $y$ are any two vertices in $G$, $d(x, y) \leq 2$.

**Theorem 4.7.** Let $G$ be a counterexample to Conjecture 2 which minimizes the number of vertices. Then the following hold:

1. $G$ is a counterexample to Conjecture 2 which minimizes the chromatic number.
2. For every $v \in V(G)$, $\overline{G} - v$ has a perfect matching.
3. For every pair $x, y \in V(G)$, such that $xy \notin E(G)$, it holds that $|N(x) \cap N(y)| \leq \frac{n-1}{2}$.
4. $\omega(G) \geq \frac{n+1}{2}$.
5. $G$ is connected.
6. $\delta(G) \geq \lceil \frac{n}{2} \rceil$.
7. $G$ is Hamiltonian.
8. For every $v \in V(G)$, $G - v$ has a perfect matching.
9. $\chi(G) \geq 8$.

Suppose now that $G$, among all counterexamples of Conjecture 2 minimizing the number of vertices, is one that minimizes the number of edges. Then the next additional properties hold:

10. For every edge $e \in E(G)$, it holds that $\alpha(G - e) > \alpha(G)$, i.e. $G$ is $\alpha$-critical.
11. For every $x, y \in V(G)$, it holds that $d(x, y) \leq 2$.

**Proof.** Let $\overline{G}$ be any counterexample to Conjecture 2 with minimum chromatic number. Then,

$$2\chi(G) - 1 = |V(G)| \leq |V(\overline{G})| \leq 2\chi(\overline{G}) \leq 2\chi(G).$$

Therefore, $\chi(G) = \chi(\overline{G})$.

Indeed, we know by Lemma 4.2 that $\chi(G - v) \leq \chi(G) - 1$, and the inequality $\chi(G - v) \geq \chi(G) - 1$ holds everytime. Then,

$$2\chi(G - v) = 2\chi(G) - 2 = |V(G)| - 1 = |V(G - v)|.$$

That is, $G - v$ has a $(\chi(G) - 1)$-coloring, in which every color class has exactly two vertices. This corresponds to a perfect matching in $\overline{G} - v$.

If this is not so, let $x$ and $y$ be two vertices such that $xy \notin E(G)$ and $|N(x) \cap N(y)| \geq \frac{n-1}{2} + 1$. As $G$ is a minimal counterexample for Conjecture 3 we know that

$$K_{\frac{n-1}{2}} = K_{\lfloor \frac{n-1}{2} \rfloor} \approx_i G - \{x, y\}.$$
Let $U$ be the set of corner vertices of such an immersion and let $W = V(G - \{x, y\}) \setminus U$. As $\alpha(G) \leq 2$ and $xy \notin E(G)$, we have that for every $u \in U$, $ux \in E(G)$ or $uy \in E(G)$. Without loss of generality, assume that $x$ is adjacent to at least half of the vertices in $U$ (and that $x$ has more neighbors than $y$ in $U$). Note that every non-neighbor of $x$ has to be adjacent to $y$.

Let us see that $x$ is connected to every vertex $u$ in $U$, by edge-disjoint paths $P_{xu}$. If $xu \in E(G)$, then $P_{xu} = xu$. If $xu \notin E(G)$, then $P_{xu} = xzyu$, with $z \in N_W(x) \cap N_W(y)$. Observe that for this to work, it needs to hold that $|N_W(x) \cap N_W(y)| \geq |N_U(x)|$.

We know that $|N_U(x)| \leq \frac{n-1}{4}$, so,

$$|N_U(x)| = \left\lfloor \frac{n-1}{4} \right\rfloor - i \text{ with } i \in \{0, ..., \left\lfloor \frac{n-1}{4} \right\rfloor \}.$$  

Besides,

$$|N_W(x) \cap N_W(y)| = |N(x) \cap N(y)| - |N_U(x) \cap N_U(y)| \geq |N(x) \cap N(y)| - (2i + 1).$$

The last inequality is obtained by assuming that $|N_U(x)| \geq |N_U(y)|$, so the number of neighbors that $x$ and $y$ share in $U$ is bounded. Indeed, $N_U(y) = N_U(x) \cup (N_U(x) \cap N_U(y))$ and as we assumed $|N_U(y)| \leq |N_U(x)|$, we have that

$$|N_U(x)| + |N_U(x) \cap N_U(y)| \leq |N_U(x)|.$$

Then,

$$|N_U(x) \cap N_U(y)| \leq |N_U(x)| - |N_U(x)|$$

$$= \left(\left\lfloor \frac{n-1}{4} \right\rfloor + i\right) - \left(\left\lfloor \frac{n-1}{4} \right\rfloor - i\right)$$

$$\leq 2i + 1.$$

And as $|N(x) \cap N(y)| \geq \frac{n-1}{4} + 1$, we have that

$$|N_U(x)| = \left\lfloor \frac{n-1}{4} \right\rfloor - i \leq |N(x) \cap N(y)| - (2i + 1) \leq |N_W(x) \cap N_W(y)|.$$

It is important to notice that the paths $P_{xu}$, from $x$ to $u \in U$, do not interfere with the already existing paths between corner vertices in $U$. This is so, because the new paths only use edges which are incident to $x$ and $y$. Therefore, we get an immersion of $K_{\left\lceil \frac{n-2}{2} \rightceil + 1} = K_{\left\lceil \frac{n}{4} \right\rceil}$ in $G$, which is a contradiction.

Let $x, y$ be any two vertices such that $xy \notin E(G)$ and divide the rest of the vertices into $A = N(x) \setminus N(y)$, $B = N(x) \cap N(y)$ and $C = N(y) \setminus N(x)$. Observe that both $A$ and $C$ induce a complete graph. Besides, by property $\mathbb{K}$ it holds that $|B| \leq \frac{n-1}{4}$. Therefore, at least one of the other two sets, say $A$, satisfies that $|A \cup \{x\}| \geq \frac{n+1}{4}$. And as $\omega(G) \geq |A \cup \{x\}|$, we are done.

Indeed, if not so, $G$ would have at least two connected components. In fact, since $\alpha(G) \leq 2$, it would have exactly two connected components and every component would be a complete graph. Then, $K_{\chi(G)} \subseteq G$, which contradicts that $G$ is a counterexample for Conjecture $\mathbb{K}$. 

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6. Observe first that it is straightforward to prove that $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$, since the non-neighborhood of any vertex induces a complete graph. Indeed, if $\delta(G) < \lfloor \frac{n}{2} \rfloor$, $K_{\lfloor \frac{n}{2} \rfloor}$ would be a subgraph of $G$, a contradiction.

So suppose that $\delta(G) = \lfloor \frac{n}{2} \rfloor$ and let $v$ be such that $d(v) = \delta(G)$. Divide $V(G) - v$ into the neighbors and the non-neighbors of $v$. We know, by property 2, that $G - v$ has a perfect matching. And given that $\overline{N}(v)$ is a complete graph, every vertex in $\overline{N}(v)$ is matched with a vertex in $N(v)$. Besides, as $|N(v)| = \lfloor \frac{n}{2} \rfloor$, then $|\overline{N}(v)| = \lceil \frac{n}{2} \rceil$.

This matching represents a coloring of $G$, in which all color classes have exactly two vertices. We claim that $K_{\chi(G)} \preceq_i G$, where the corner vertices are $\{v\} \cup N(v)$.

Indeed, $vu \in E(G)$, for every $u \in N(v)$. Then, we can assign $P_{vu} = vu$. Consider now $u, w \in N(v)$. If $uw \in E(G)$, then $P_{uw} = uw$. If $uw \notin E(G)$, then, as $\alpha(G) \leq 2$, it holds that $ux_w, wx_u \in E(G)$, where $x_u, x_w$ are the vertices that are matched with $w$ and with $u$, respectively. Also, $x_u x_w \in E(G)$, since $x_u, x_w \in \overline{N}(v)$, which is a complete graph. Therefore, we can assign $P_{uw} = ux_w x_u w$. The paths are edge-disjoint, because by seeing the matching in $G - v$ as a coloring in $G - v$, the chosen paths are precisely chains between corner vertices of different colors.

7. It follows from property 6 along with Dirac’s well known Theorem.

8. It is implied by property 7 and Lemma 4.5

9. It follows directly from Theorem 1.1

10. If there were an edge $e \in E(G)$, such that $\alpha(G - e) \leq \alpha(G) = 2$, then,

$$K_{\chi(G-e)} \preceq_i G - e \preceq_i G.$$ 

So, $\chi(G - e) \leq \chi(G) - 1$. Therefore, $G - e$ has $|V(G)| = 2\chi(G) - 1$ vertices and can be colored with $\chi(G) - 1$ colors. Necessarily one color class has at least 3 vertices, which is a contradiction.

11. It is implied by properties 5 and 10 along with Theorem 4.6

5 Conclusion

The question of whether Abu-Khzam and Langston’s conjecture is true still remains open, even in the special case of $\alpha(G) \leq 2$. A possible way would be to continue studying a counterexample of Conjecture 2 minimizing the number of vertices. More structural properties can be found in [23].

After seeing the proofs of Theorems 1.2 and 1.3 it is tempting to try to look for an immersion of a complete graph with a vertex of every color as the set of corner vertices and chains as paths between them. However there are examples of graphs with colorings in which it is impossible to find this type of immersion (the reader is referred also to [23]).
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