Reduction of balance laws in (3+1)–dimensions to autonomous conservation laws by means of equivalence transformations

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**Abstract**

A class of partial differential equations (a conservation law and four balance laws), with four independent variables and involving sixteen arbitrary continuously differentiable functions, is considered in the framework of equivalence transformations. These are point transformations of differential equations involving arbitrary elements and live in an augmented space of independent, dependent and additional variables representing values taken by the arbitrary elements. Projecting the admitted symmetries into the space of independent and dependent variables, we determine some finite transformations mapping the system of balance laws to an equivalent one with the same differential structure but involving different arbitrary elements; in particular, the target system we want to recover is an autonomous system of conservation laws. An application to a physical problem is considered.

**Keywords.** Systems of balance laws; Equivalence transformations; Derivation of autonomous and homogeneous conservation laws.
1 Introduction

Physical laws are often expressed mathematically by systems of partial differential equations (PDEs) in the form of balance laws \[1\] [2],

\[
\sum_{i=1}^{n} \frac{\partial F^i(u)}{\partial x_i} = G(u),
\]

where \(u \in \mathbb{R}^m\) denotes the set of unknown fields, \(x \in \mathbb{R}^n\) the set of independent variables, \(F^i(u)\) the components of a flux, and \(G(u)\) the production term; when \(G(u) \equiv 0\), we have a system of conservation laws. In this paper, the first component \(x_1\) of the independent variables is the time, and the components of \(F^1\) are the densities of some physical quantities. The presence of the source terms in systems in divergence form implies additional mathematical difficulties in solving various problems. For instance, from a numerical point of view, the presence of source terms may require fractional step splitting methods where one alternates between solving a homogeneous system of conservation laws and an ordinary differential system obtained from the system of balance laws by dropping the terms involving space derivatives. It is known [3] that for some type of problems fractional step splitting methods perform quite poorly. Systems like (1) fall in the more general class of nonhomogeneous quasilinear first order systems of PDEs. Further mathematical difficulties may arise in those problems where the coefficients involved in the differential equations depend also on the independent variables \(x\), accounting for material inhomogeneities, or special geometric assumptions, or external actions.

In dealing with differential equations, Lie group theory [4] [5] [6] [7] [8] [9] [10] [11] yields general algorithmic methods either for the determination of special (invariant) solutions [12] [13] [14] [15] [16] of initial and boundary value problems, or the derivation of conserved quantities, or the construction of relations between different differential equations that turn out to be equivalent [11] [17] [18] [19] [20] [21] [22] [23] [24].

In this paper, in the context of equivalence transformations of differential equations [4] [25] [26] [27] [28] [29] [30] [31], we consider a \((3 + 1)\)–dimensional system of first order PDEs consisting of a linear conservation law and four general balance laws involving some arbitrary functions. The aim is to identify classes of systems that can be mapped through an invertible point transformation to a system of autonomous conservation laws. Recently [24], it has been shown that the transformation of a general nonautonomous and/or nonhomogeneous first order
quasilinear system of PDEs (which every system of first order balance laws reduces to) into autonomous and homogeneous quasilinear form is possible if and only if a suitable algebra of point symmetries is admitted. The theorem proved in Ref. [24] generalizes a theorem established in Ref. [22] for $2 \times 2$ quasilinear first order systems. Both theorems may be applied when we consider a given system of PDEs and the required hypotheses are fulfilled. If one is interested to identify the systems of balance laws (possibly nonautonomous) that can be transformed by an invertible point transformation to an autonomous system of conservation laws, a convenient approach consists in using equivalence transformations. A similar approach has been used recently in [32] for a $2 \times 2$ first order quasilinear system of PDEs, and in [33] for a system of three balance laws in three independent variables.

The plan of the paper is the following. In Section 2, we recall the very basic elements concerning the Lie symmetries of differential equations; also, the main ideas about equivalence transformations of differential equations (and the way they are used) are introduced. In Section 3 we investigate a class of differential equations, involving four independent and five dependent variables, expressed under the form of a linear conservation law and four nonlinear balance laws; the considered system involves sixteen arbitrary functions of the independent and dependent variables. The equivalence transformations are determined and the finite transformations generated by the admitted generators are constructed. As a consequence, the equivalent conservation laws are characterized. An example of physical interest (3D Euler equations for an ideal gas in a non–inertial frame and subject to gravity) is also considered in Section 4.

2 Equivalence transformations

In this Section, to fix the notation and to render the paper self–contained, we briefly recall the main elements of Lie group analysis of differential equations, and the results, especially concerned with equivalence transformations, that will be used throughout this paper.

In the framework of Lie group analysis, given a system of differential equations, say

$$\Delta \left( \mathbf{x}, \mathbf{u}, \mathbf{u}^{(r)} \right) = 0, \quad (2)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the set of the independent variables, $\mathbf{u} \in \mathbb{R}^m$ the set of the dependent variables, and $\mathbf{u}^{(r)}$ the set of all partial derivatives of the $\mathbf{u}$’s with respect
to the x's up to the order r, one is interested to find the admitted group of Lie symmetries. Lie group analysis provides a powerful and unified approach to differential equations, both ordinary and partial; for ordinary differential equations it algorithmically leads to lowering the order (or reducing to quadrature), for partial differential equations they allow for the construction of group–invariant solutions or for introducing invertible point transformations mapping the differential equations in equivalent forms.

In many situations we have differential equations involving arbitrary elements (constants or functions), so that one has a class of differential equations. Here we shall consider a class $\mathcal{E}(p)$ of first order PDEs involving some arbitrary continuously differentiable functions $p_k(x,u)\ (k = 1,\ldots,\ell)$,

$$\Delta(x,u,u^{(1)};p,p^{(1)}) = 0,$$

whose elements are given once we fix the functions $p_k$ ($p^{(1)}$ denotes the set of first order partial derivatives of the p's with respect to their arguments). To face this problem, it is convenient to consider equivalence transformations, i.e., transformations that preserve the differential structure of the equations in the system but may change the form of the constitutive functions and/or parameters \[4, 25, 26, 27, 28, 30, 31, 32, 33\].

**Definition 1** (Equivalence transformations \[4\]). A one–parameter Lie group of equivalence transformations of a family $\mathcal{E}(p)$ of PDEs is a one–parameter Lie group of transformations given by

$$X = X(x,u,p;a), \quad U = U(x,u,p;a), \quad P = P(x,u,p;a),$$

a being the parameter, which is locally a $C^\infty$ diffeomorphism and maps a class $\mathcal{E}(p)$ of differential equations into itself; thus, it may change the differential equations (the form of the arbitrary elements therein involved) but preserves the differential structure.

In the following we shall assume that the transformations of the independent and dependent variables do not involve the arbitrary elements $p$.

In an augmented space $\mathcal{A} \equiv \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \ [4, 26]$, where the independent variables, the dependent variables and the arbitrary functions live, respectively, the generator of the equivalence transformation,

$$Z = \sum_{i=1}^n \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{A=1}^m \eta^A(x,u) \frac{\partial}{\partial u_A} + \sum_{k=1}^\ell \mu^k(x,u,p) \frac{\partial}{\partial p_k},$$

4
involves also the infinitesimals \( \mu^k(x,u,p) \) accounting for the arbitrary functions \( p_k \). The search for continuous equivalence transformations can be exploited by using the Lie infinitesimal criterion \[4\].

The first prolongation of \( \Xi \) writes as

\[
\Xi^{(1)} = \Xi + \sum_{A=1}^{m} \sum_{i=1}^{n} \eta^A_{[i]} \frac{\partial}{\partial u_{A,i}} + \sum_{k=1}^{\ell} \sum_{\alpha=1}^{n+m} \mu^k_{[\alpha]} \frac{\partial}{\partial p_{k,\alpha}},
\]

with

\[
\eta^A_{[i]} = \frac{D\eta^A}{Dx_i} - \sum_{j=1}^{n} u_{A,j} \frac{D\xi_j}{Dx_i}, \quad \mu^k_{[\alpha]} = \frac{D\mu^k}{Dz_\alpha} - \sum_{\beta=1}^{n+m} p_{k,\beta} \frac{D\zeta_\beta}{Dz_\alpha},
\]

\( (u_{A,j} = \frac{\partial u^A}{\partial x_j}, p_{k,\alpha} = \frac{\partial p_k}{\partial z_\alpha}, z = (x,u), \xi = (\xi,\eta)) \), where the Lie derivatives are

\[
\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{A=1}^{m} u_{A,i} \frac{\partial}{\partial u^A}, \quad \frac{D}{Dz_\alpha} = \frac{\partial}{\partial z_\alpha} + \sum_{k=1}^{\ell} \frac{\partial}{\partial p_k}.
\]

### 2.1 Finite transformations in the projected space

In the augmented space \( \mathcal{A} \), the arbitrary functions determining the class of differential equations are assumed as dependent variables, and we require the invariance of the class in this augmented space. If we project the symmetries on the space \( \mathcal{Z} \equiv \mathbb{R}^n \times \mathbb{R}^m \) of the independent and dependent variables (this is possible because the infinitesimals of independent and dependent variables are assumed to be independent of \( p \)), we obtain a transformation changing an element of the class of differential equations to another element in the same class (same differential structure but in general different arbitrary elements). Such projected transformations map solutions of a system in the class to solutions of a transformed system in the same class.

Thus, in the augmented space \( \mathcal{A} \), given the equivalence generator (5) the integration of Lie’s equations

\[
\frac{dX}{da} = \xi(X,U), \quad \frac{dU}{da} = \eta(X,U), \quad \frac{dP}{da} = \mu(X,U,P),
\]

\( X(0) = x, \quad U(0) = u, \quad P(0) = p \)

provides the finite transformation which maps the class into itself. On the contrary, the integration of the Lie’s equations (9) in the projected space \( \mathcal{Z} \) gives an equivalence transformation mapping a system in the class into another system in the same class.
3 The model equations

Consider the class $E(p)$ with $p = (p_1, \ldots, p_{16})$ of systems
\begin{align*}
\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 + \partial_{x_4} u_4 &= 0, \\
\partial_{x_1} u_2 + \partial_{x_2} p_1 + \partial_{x_3} p_2 + \partial_{x_4} p_3 &= p_{13}, \\
\partial_{x_1} u_3 + \partial_{x_2} p_4 + \partial_{x_3} p_5 + \partial_{x_4} p_6 &= p_{14}, \\
\partial_{x_1} u_4 + \partial_{x_2} p_7 + \partial_{x_3} p_8 + \partial_{x_4} p_9 &= p_{15}, \\
\partial_{x_1} u_5 + \partial_{x_2} p_{10} + \partial_{x_3} p_{11} + \partial_{x_4} p_{12} &= p_{16}, \\
\end{align*}
(10)
where $x \equiv (x_1, x_2, x_3, x_4)$ are the independent variables, $u \equiv (u_1, u_2, u_3, u_4, u_5)$ the dependent variables, whereas $p \equiv (p_1, \ldots, p_{16})$ stand for arbitrary continuously differentiable functions of $x$ and $u$. For instance, three-dimensional Euler equations of ideal gas–dynamics fall into the class (10).

By requiring the invariance of the class $E(p)$ in the augmented space $\mathcal{A} \equiv \mathbb{R}^4 \times \mathbb{R}^5 \times \mathbb{R}^{16}$ through the Lie’s infinitesimal criterion \([4]\), we determine 24 symmetry operators, whose expression is too long to be written here. In view of the results we want to achieve, we report the projections of the admitted operators on the space $\mathcal{Z} \equiv \mathbb{R}^4 \times \mathbb{R}^5$:
\begin{align*}
\Xi_1 &= f_1(x_1) \partial_{x_1} - f'_1(x_1)(u_2 \partial_{u_2} + u_3 \partial_{u_3} + u_4 \partial_{u_4}), \\
\Xi_i &= f_i(x) \partial_{x_i} + \sum_{k=1}^{4} (u_k \partial_{x_k} f_i(x) \partial_{u_i} - u_k \partial_{x_i} f_i(x) \partial_{u_k}), \quad (i = 2, 3, 4), \\
\Xi_{4+i} &= u_i f_{4+i}(x) \partial_{u_5}, \quad (i = 1, \ldots, 5), \\
\Xi_{10} &= f_{10}(x) \partial_{u_5}, \quad \Xi_{11} = \sum_{k=1}^{4} f_{10+k}(x) \partial_{u_k}, \quad \Xi_{12} = \sum_{k=1}^{4} u_k \partial_{u_k}, \\
\end{align*}
(11)
where the functions $f_i (i = 1, \ldots, 14)$ are arbitrary functions of the indicated variables, along with the condition
\begin{equation*}
\sum_{k=1}^{4} \partial_{x_k} f_{10+k}(x) = 0, \\
(12)
\end{equation*}
and the prime $'$ denotes the differentiation with respect to the argument.

By considering the corresponding Lie’s equations we will be able to compute the finite corresponding transformations, say
\begin{equation*}
X = X(x, u; a), \quad U = U(x, u; a) \\
(13)
\end{equation*}
allowing us to map the original system \( (10) \) to a different system with the same differential structure; in particular, we are interested to the case where the target system is an autonomous system of conservation laws:

\[
\begin{align*}
\partial_{X_1} U_1 + \partial_{X_2} U_2 + \partial_{X_3} U_3 + \partial_{X_4} U_4 &= 0, \\
\partial_{X_1} U_2 + \partial_{X_2} P_1 + \partial_{X_3} P_2 + \partial_{X_4} P_3 &= 0, \\
\partial_{X_1} U_3 + \partial_{X_2} P_4 + \partial_{X_3} P_5 + \partial_{X_4} P_6 &= 0, \\
\partial_{X_1} U_4 + \partial_{X_2} P_7 + \partial_{X_3} P_8 + \partial_{X_4} P_9 &= 0, \\
\partial_{X_1} U_5 + \partial_{X_2} P_{10} + \partial_{X_3} P_{11} + \partial_{X_4} P_{12} &= 0, \\
\end{align*}
\]

where \( P_i \equiv P_i(U_1, U_2, U_3, U_4, U_5) \), \( i = 1, \ldots, 12 \). Of course, a given system falling in the class \( (10) \) can be mapped by an equivalence transformation to a system having the form \( (14) \) provided that the functions \( p_i(x, u) \) \( (i = 1, \ldots, 16) \) have special functional forms. To simplify the computation, we exchange the source and target system; in fact, taking the inverse transformation of \( (13) \) (which is obtained by exchanging lower and capital letters and replacing \( a \) with \(-a\)), and starting from the autonomous system \( (14) \) of conservation laws, we are able to obtain the equivalent nonautonomous system of balance laws. In such a way, we are able to identify, for a given equivalence transformation, the elements of the class \( (10) \) that can be mapped to a system of autonomous conservation laws.

Now, since we start from an autonomous system of conservation laws to arrive to a nonautonomous system of balance laws, let us write the operators \( (11) \) in terms of the capital letters; then, we build the corresponding finite transformations.

The most general finite transformation can be recovered by composition of the finite transformations induced by each generator.

### 3.1 Finite transformations generated by \( \Xi_1 \)

By considering the generator \( \Xi_1 \),

\[
\Xi_1 = f(X_1) \partial_{X_1} - f'(X_1) \left( U_2 \partial_{U_2} + U_3 \partial_{U_3} + U_4 \partial_{U_4} \right),
\]

where we set \( f(X_1) = f_1(X_1) \), we get the finite transformation

\[
\begin{align*}
x_1 &= \tilde{x}_1(X_1; a), \quad x_2 = X_2, \quad x_3 = X_3, \quad x_4 = X_4, \\
u_1 &= U_1, \quad u_2 = U_2 \frac{f(X_1)}{f(x_1)}, \quad u_3 = U_3 \frac{f(X_1)}{f(x_1)}, \quad u_4 = U_4 \frac{f(X_1)}{f(x_1)}, \quad u_5 = U_5,
\end{align*}
\]
\( \tilde{x}_1(X_1; a) \) being such that \( \partial_{X_1} \tilde{x}_1 = \frac{f(x_1)}{f(X_1)} \), whereupon we may write

\[
U_1 = u_1, \quad U_2 = u_2 \partial_{X_1} \tilde{x}_1, \quad U_3 = u_3 \partial_{X_1} \tilde{x}_1, \quad U_4 = u_4 \partial_{X_1} \tilde{x}_1, \quad U_5 = u_5, \tag{17}
\]

and system (10) is equivalent to (14) with

\[
p_k = \frac{P_k}{(\partial_{X_1} \tilde{x}_1)^2}, \quad k = 1, \ldots, 12,
\]

\[
p_{13} = -u_2 \frac{\partial^2_{X_1 X_1} \tilde{x}_1}{(\partial_{X_1} \tilde{x}_1)^2}, \quad p_{14} = -u_3 \frac{\partial^2_{X_1 X_1} \tilde{x}_1}{(\partial_{X_1} \tilde{x}_1)^2}, \quad p_{15} = -u_4 \frac{\partial^2_{X_1 X_1} \tilde{x}_1}{(\partial_{X_1} \tilde{x}_1)^2}, \tag{18}
\]

where \( p_k = P_k(u_1, u_2 \partial_{X_1} \tilde{x}_1, u_3 \partial_{X_1} \tilde{x}_1, u_4 \partial_{X_1} \tilde{x}_1, u_5) \), \( k = 1, \ldots, 12 \).

### 3.2 Finite transformations generated by \( \Xi_2, \Xi_3 \) and \( \Xi_4 \)

By taking the generators \( \Xi_i, i = 2, 3, 4 \),

\[
\Xi_i = f(X) \partial_{X_i} + \sum_{k=1}^{4} (U_k \partial_{X_i} f(X) \partial U_i - U_k \partial_{X_i} f(X) \partial U_k), \quad (i = 2, 3, 4), \tag{19}
\]

where we set \( f(X) = f_i(X) \), we may write the general finite transformation arising from the integration of Lie’s equations in the three cases in a unified form:

\[
x_k = \begin{cases} 
X_k, & k = 1, \ldots, 4, k \neq i \\
\tilde{x}_k(X; a), & k = i 
\end{cases}
\]

\[
u_k = \begin{cases} 
U_k, & k = 1, \ldots, 4, k \neq i \\
U_k + f(X) \sum_{j=1, j \neq i}^{4} U_j \int_{0}^{a} \frac{\partial_{X_j} f(x)}{f(x)} da, & k = i \\
U_k, & k = 5 
\end{cases} \tag{20}
\]

where \( \tilde{x}_i(X; a) \) is such that

\[
\partial_{X_1} \tilde{x}_i = \begin{cases} 
\frac{f(x)}{f(X)} \int_{0}^{a} \frac{\partial_{X_i} f(x)}{f(x)} da, & k \neq i \\
\frac{f(x)}{f(X)}, & k = i. \tag{21}
\end{cases}
\]
By introducing the matrix $J$ with the $(j,k)$–entry equal to $\partial X_i \tilde{x}_j$ $(j,k = 1, \ldots, 4)$, the $(5,5)$–entry equal to $\partial X_i \tilde{x}_i$ and all remaining entries vanishing, we may write

$$u = AU, \quad A = \frac{J}{\partial X_i \tilde{x}_i};$$

(22)

moreover, by defining the matrices

$$q = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & 0 \\ u_2 & p_1 & p_2 & p_3 & 0 \\ u_3 & p_4 & p_5 & p_6 & 0 \\ u_4 & p_7 & p_8 & p_9 & 0 \\ u_5 & p_{10} & p_{11} & p_{12} & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & 0 \\ U_2 & P_1 & P_2 & P_3 & 0 \\ U_3 & P_4 & P_5 & P_6 & 0 \\ U_4 & P_7 & P_8 & P_9 & 0 \\ U_5 & P_{10} & P_{11} & P_{12} & 0 \end{bmatrix},$$

system (14) is mapped to system (10) with

$$q = AQJ^T,$$

$$p_{11+m} = \sum_{j=1}^{5} A_{mj} \sum_{\ell=1}^{5} \left( 4 \sum_{k=1}^{4} u_k \frac{\partial^2 R_{\ell j}}{\partial U_\ell \partial X_k} - \sum_{k=1}^{5} u_k \frac{\partial^2 R_{\ell j}}{\partial U_k \partial X_\ell} \right), \quad m = 2, \ldots, 5,$$

(23)

where $R_{\ell j}$ is the generic entry of the matrix $JP^T$, and it is $P_k = P_k(U)$, $(k = 1, \ldots, 12)$, with $U$ defined by (22); note that the right hand side of (23) is vanishing for $m = 1$.

### 3.3 Finite transformations generated by $\Xi_5, \Xi_6, \Xi_7, \Xi_8$

By considering the generator $\Xi_{4+i}$ $(i = 1, \ldots, 4),$

$$\Xi_{4+i} = U_i f(X) \partial U_5,$$

(24)

where we set $f(X) = f_{4+i}(X)$, we get from Lie’s equations the finite transformation

$$x_k = X_k, \quad u_k = U_k, \quad k = 1, \ldots, 4, \quad u_5 = U_5 - aU_i f(X).$$

(25)
System (14) is equivalent to system (10) if

\[ p_k = P_k, \quad k = 1, \ldots, 9, \]

\[ p_{9+k} = \begin{cases} 
  P_{9+k} + au_{k+1}f(x), & i = 1, \\
  P_{9+k} + aP_{3i+k-6}f(x), & i = 2, 3, 4, \\
  k = 1, 2, 3, 
\end{cases} \]

\[ p_{12+k} = au_i \sum_{j=2}^{4} \partial X_j f(x) \partial U_5 P_{3k+j-4} \quad k = 1, 2, 3, \]

\[ p_{16} = \begin{cases} 
  au_i \left( \partial X_i f(x) + \sum_{j=2}^{4} \partial X_j f(x) \partial U_5 P_{8+j} \right), & i = 1 \\
  au_i \left( \partial X_i f(x) + \sum_{j=2}^{4} \partial X_j f(x) \left( \partial U_5 P_{8+j} + af(x) \partial U_5 P_{3i+j-7} \right) \right), & i = 2, 3, 4, \end{cases} \]

where \( P_k = P_k(u_1, u_2, u_3, u_4, u_5 + au_i f(x)), k = 1, \ldots, 12. \)

3.4 Finite transformations generated by \( \Xi_9 \)

By considering the generator \( \Xi_9 \),

\[ \Xi_9 = U_5 f(X) \partial U_5, \]

where we set \( f(X) = f_9(X) \), we get from Lie’s equations the finite transformation

\[ x_k = X_k, \quad u_k = U_k, \quad k = 1, \ldots, 4, \quad u_5 = U_5 \exp(af(X)). \]  

The system (14) is equivalent to system (10) provided that:

\[ p_k = P_k, \quad k = 1, \ldots, 12, \]

\[ p_{12+k} = a \exp(-af(x)) \left( \sum_{j=2}^{4} \partial X_j f(x) \partial U_5 P_{3k+j-4} \right) u_5, \quad k = 1, 2, 3, \]

\[ p_{16} = a \left( \partial X_i f(x) + \sum_{j=2}^{4} \partial X_j f(x) \partial U_5 P_{8+j} \right) u_5, \]

where \( P_k = P_k(u_1, u_2, u_3, u_4, u_5 \exp(-af(x))), i = 1, \ldots, 12. \)
3.5 Finite transformations generated by $\Xi_{10}$

By considering the generator $\Xi_{10}$,

$$\Xi_{10} = f(X) \partial_{U_5}, \quad (30)$$

where we set $f(X) = f_{10}(X)$, we get from Lie’s equations the finite transformation

$$x_k = X_k, \quad u_k = U_k, \quad k = 1, \ldots, 4, \quad u_5 = U_5 + af(X), \quad (31)$$

and the equivalence between (10) and (14) is recovered provided that

$$p_k = P_k, \quad k = 1, \ldots, 12,$$

$$p_{12+k} = a \sum_{j=2}^{4} \partial X_j f(x) \partial_{U_5} P_{3k+j-4}, \quad k = 1, 2, 3, \quad (32)$$

$$p_{16} = a \left( \partial X_1 f(x) + \sum_{j=2}^{4} \partial X_j f(x) \partial_{U_5} P_{8+j} \right), \quad (33)$$

where $P_k = P_k(u_1, u_2, u_3, u_4, u_5 - af(x)), k = 1, \ldots, 12$.

3.6 Finite transformations generated by $\Xi_{11}$

By considering the generator $\Xi_{11}$,

$$\sum_{k=1}^{4} g_k(X) \partial_{U_k}, \quad (34)$$

where we set $g_k(X) = f_{10+k}(X)$, along with the constraint $\partial X_k g_k(x) = 0$, and integrating the Lie’s equations, the following finite transformation arises:

$$x_k = X_k, \quad u_k = U_k + ag_k(x), \quad k = 1, \ldots, 4, \quad u_5 = U_5. \quad (35)$$

System (14) is equivalent to system (10) provided that:

$$p_k = P_k, \quad k = 1, \ldots, 12,$$

$$p_{12+k} = a \left( \partial X_1 g_{k+1}(x) + \sum_{i=1}^{4} \sum_{j=2}^{4} \partial X_j g_i(x) \partial_{U_i} P_{3k+j-4} \right), \quad k = 1, 2, 3, \quad (36)$$

$$p_{16} = a \left( \sum_{i=1}^{4} \sum_{j=2}^{4} \partial X_j g_i(x) \partial_{U_i} P_{8+j} \right), \quad (37)$$

where $P_k = P_k(u_1 - ag_1(x), u_2 - ag_2(x), u_3 - ag_3(x), u_4 - ag_4(x), u_5)$. 
3.7 Equivalence transformations generated by $\Xi_{12}$

In this case the finite transformation consists of a uniform scaling of the dependent variables,

\[ x = X, \quad u = \exp(a)U, \]

and for such a transformation there are no balance laws equivalent to conservation laws.

4 Physical application

In this Section, we make some assumptions on the form of the functions involved in the generators of equivalence transformations in order to deal with physically relevant systems of differential equations. In particular, we construct the finite transformations corresponding to the infinitesimal generator $\sum_{i=1}^{4} \Xi_{i}$, where we assume

\[ f_2 = n_1(X_1)X_2 + n_2(X_1)X_3, \quad f_3 = -n_2(X_1)X_2 + n_1(X_1)X_3, \quad f_4 = n_3(X_1), \]

with $n_i(X_1), i = 1, \ldots, 3,$ arbitrary functions of $X_1$.

Integration of Lie’s equations provides:

\[
\begin{align*}
  x_1 &= \ddot{x}_1(X_1;a), \quad x_4 = \ddot{x}_4(X_1,X_4;a) = X_4 + m_3(X_1;a), \\
  x_2 &= \ddot{x}_2(X_1,X_2,X_3;a) = \exp(m_1(X_1;a))(X_2 \cos(m_2(X_1;a)) + X_3 \sin(m_2(X_1;a))), \\
  x_3 &= \ddot{x}_3(X_1,X_2,X_3;a) = \exp(m_1(X_1;a))(-X_2 \sin(m_2(X_1;a)) + X_3 \cos(m_2(X_1;a))), \\
  U_1 &= \exp(2m_1(X_1;a))u_1, \\
  U_2 &= \exp(m_1(X_1;a)) [(u_2 \cos(m_2(X_1;a)) - u_3 \sin(m_2(X_1;a)))] \frac{\partial}{\partial x_1} \ddot{x}_1 \\
&\quad -u_1 \left( \frac{\partial}{\partial x_1} \ddot{x}_2 \cos(m_2(X_1;a)) - \frac{\partial}{\partial x_2} \ddot{x}_3 \sin(m_2(X_1;a)) \right), \\
  U_3 &= \exp(m_1(X_1;a)) [(u_2 \sin(m_2(X_1;a)) + u_3 \cos(m_2(X_1;a)))] \frac{\partial}{\partial x_1} \ddot{x}_1 \\
&\quad -u_1 \left( \frac{\partial}{\partial x_1} \ddot{x}_2 \sin(m_2(X_1;a)) + \frac{\partial}{\partial x_2} \ddot{x}_3 \cos(m_2(X_1;a)) \right), \\
  U_4 &= \exp(2m_1(X_1;a))(u_4 \frac{\partial}{\partial x_1} \ddot{x}_1 - u_1 \frac{\partial}{\partial x_1} \ddot{x}_4), \quad U_5 = u_5,
\end{align*}
\]

(37)
where

\[ m_i(X_1; a) = \int_{x_1}^{x_i} \frac{n_i(s)}{f_1(s)} ds, \]
\[ \tilde{n}_i(X_1; a) = n_i(x_1) - n_i(X_1), \quad i = 1, 2, 3, \]
\[ \partial_{X_1} \tilde{x}_1 = f_1(x_1), \]
\[ \partial_{X_1} \tilde{x}_2 = \frac{\tilde{n}_1(X_1; a)x_2 + \tilde{n}_2(X_1; a)x_3}{f_1(X_1)}, \]
\[ \partial_{X_1} \tilde{x}_3 = \frac{-\tilde{n}_2(X_1; a)x_2 + \tilde{n}_1(X_1; a)x_3}{f_1(X_1)}, \]
\[ \partial_{X_1} \tilde{x}_4 = \frac{\tilde{n}_3(X_1; a)}{f_1(X_1)}. \]

System (14) describes the 3D unsteady flow of an ideal fluid subject to no extra-
neous force along with the choices

\[ U_1 = \rho, \quad U_2 = \rho u, \quad U_3 = \rho v, \quad U_4 = \rho w, \quad U_5 = \rho S, \]
\[ P_1 = \frac{U_2^2}{U_1} + p(U_1, U_5), \quad P_2 = P_4 = \frac{U_2U_3}{U_1}, \quad P_3 = P_7 = \frac{U_2U_4}{U_1}, \]
\[ P_5 = \frac{U_3^2}{U_1} + p(U_1, U_5), \quad P_6 = P_8 = \frac{U_3U_4}{U_1}, \quad P_9 = \frac{U_4^2}{U_1} + p(U_1, U_5), \]
\[ P_{10} = \frac{U_2U_5}{U_1}, \quad P_{11} = \frac{U_3U_5}{U_1}, \quad P_{12} = \frac{U_4U_5}{U_1}. \]

\( \rho \) being the fluid mass density, \((u, v, w)\) the components of its velocity, \(S\) the en-
tropy, and \(p(\rho, S)\) the pressure. Thorough the transformation (37) we get the system
(10) with

\[ p_1 = \frac{u_2^2}{u_1} + p(\exp(-2m_1)u_1, u_5), \quad p_2 = p_4 = \frac{u_2 u_3}{u_1}, \quad p_3 = p_7 = \frac{u_2 u_4}{u_1}, \]

\[ p_5 = \frac{u_2^3}{u_1} + p(\exp(-2m_1)u_1, u_5), \quad p_6 = p_8 = \frac{u_3 u_4}{u_1}, \quad p_9 = \frac{u_4^2}{u_1} + p(\exp(-2m_1)u_1, u_5), \]

\[ p_{10} = \frac{u_2 u_5}{u_1}, \quad p_{11} = \frac{u_3 u_5}{u_1}, \quad p_{12} = \frac{u_4 u_5}{u_1}, \]

\[ p_{13} = 2 \frac{\partial x_1 m_1 u_2 + \partial x_1 m_2 u_3}{\partial x_1 1} - \frac{\partial^2 x_1 x_1 1}{(\partial x_1 1)^2} u_2 \]

\[ + \frac{x_2(\partial^2 x_1 x_1 m_1 - (\partial x_1 m_1)^2 + (\partial x_1 m_2)^2)}{(\partial x_1 1)^2} + x_3(\partial^2 x_1 x_1 m_2 - 2\partial x_1 m_1 \partial x_1 m_2) \]

\[ p_{14} = 2 \frac{\partial x_1 m_1 u_3 - \partial x_1 m_2 u_1}{\partial x_1 1} - \frac{\partial^2 x_1 x_1 1}{(\partial x_1 1)^2} u_3 \]

\[ + \frac{x_2(\partial^2 x_1 x_1 m_2 - 2\partial x_1 m_1 \partial x_1 m_2)}{(\partial x_1 1)^2} + x_3(\partial^2 x_1 x_1 m_1 - (\partial x_1 m_1)^2 + (\partial x_1 m_2)^2) \]

\[ p_{15} = -\frac{\partial^2 x_1 x_1 1}{(\partial x_1 1)^2} u_4 + \frac{\partial^2 x_1 x_1 1}{(\partial x_1 1)^2} u_4, \quad p_{16} = 2 \frac{\partial x_1 m_1}{\partial x_1 1} u_5. \]

By choosing \( \partial x_1 1 = 1 \) (whereupon \( x_1 = X_1 + a \), \( m_1 = 0 \), \( m_2 = \omega X_1 + X_{10} \), \( m_3 = \frac{u_5^2}{u_1} + a_1 X_1 + a_0 \), where \( \omega \), \( a_0 \), \( a_1 \), \( a_2 \), \( g \) and \( X_{10} \) are constants, we recover the system

\[ \partial x_1 u_1 + \partial x_2 u_2 + \partial x_3 u_3 + \partial x_4 u_4 = 0, \]

\[ \partial x_1 u_2 + \partial x_2 \left( \frac{u_2^2}{u_1} + p(u_1, u_5) \right) + \partial x_3 \left( \frac{u_2 u_3}{u_1} \right) + \partial x_4 \left( \frac{u_2 u_4}{u_1} \right) = 2\omega u_3 - \omega^2 x_2 u_1, \]

\[ \partial x_1 u_3 + \partial x_2 \left( \frac{u_2 u_3}{u_1} \right) + \partial x_3 \left( \frac{u_3^2}{u_1} + p(u_1, u_5) \right) + \partial x_4 \left( \frac{u_3 u_4}{u_1} \right) = -2\omega u_2 + \omega^2 x_3 u_1, \]

\[ \partial x_1 u_4 + \partial x_2 \left( \frac{u_2 u_4}{u_1} \right) + \partial x_3 \left( \frac{u_3 u_4}{u_1} \right) + \partial x_4 \left( \frac{u_4^2}{u_1} + p(u_1, u_5) \right) = gu_1, \]

\[ \partial x_1 u_5 + \partial x_2 \left( \frac{u_2 u_5}{u_1} \right) + \partial x_3 \left( \frac{u_3 u_5}{u_1} \right) + \partial x_4 \left( \frac{u_4 u_5}{u_1} \right) = 0. \]

With obvious identifications, we recognize the equations of an ideal gas in a non-inertial frame rotating with constant angular velocity \( \omega \) around the vertical \( x_4 \)-axis.
and subject to gravity. This implies that the Euler equations for an ideal gas in a non–inertial frame rotating with constant angular velocity around a vertical axis and subject to gravity can be transformed in a form where the gravity and apparent forces disappear.

5 Conclusions

In this paper we have characterized classes of PDEs in four independent variables expressed under the form of a linear conservation law and four nonautonomous nonlinear balance laws that can be transformed by an invertible point transformation into an autonomous system of conservation laws. This has been accomplished through the use of equivalence transformations. A physical application has been provided: it has been shown the equivalence of the 3D unsteady Euler equations of an ideal gas subject to gravity and Coriolis forces with the corresponding system where forces are absent.

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