The Equivalence between Uniqueness and Continuous Dependence of Solution for BSDEs with Continuous Coefficient

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Abstract

In this paper, we will prove that, if the coefficient \(g = g(t, y, z)\) of a BSDE is assumed to be continuous and linear growth in \((y, z)\), then the uniqueness of solution and continuous dependence with respect to \(g\) and the terminal value \(\xi\) are equivalent.

Keywords: Backward stochastic differential equation; Uniqueness; Continuous dependence.

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1 Introduction

We consider the following 1-dimensional backward stochastic differential equation (BSDE):

\[
y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \, dW_s, \quad t \in [0, T].
\]

(1)

where the terminal condition \(\xi\) and the coefficient \(g = g(t, y, z)\) are given. \(W\) is a \(d\)-dimensional Brownian motion. The solution \((y_t, z_t)_{t \in [0, T]}\) is a pair of square integrable processes. A foundational and interesting problem is: what is the relationship between the uniqueness of solution and continuous dependence with respect to \(g\) or \(\xi\)? In the standard situation where \(g\) satisfies linear growth condition and Lipschitz condition in \((y, z)\), it has been proved by Pardoux and Peng \([4]\) that there exists a unique solution. In this case, the continuous dependence with respect to \(g\) and \(\xi\) is is described by the following inequality (see El Karoui, Peng and Quenez \([1]\)):

\[
E \left\{ \sup_{0 \leq t \leq T} |y_t^1 - y_t^2|^2 \right\} \leq CE \left\{ |\xi^1 - \xi^2|^2 + \int_0^T |g^1(t, y_t^1, z_t^1) - g^2(t, y_t^1, z_t^1)|^2 \, dt \right\},
\]

(2)

where \((y_t^1, z_t^1)_{t \in [0, T]}\) and \((y_t^2, z_t^2)_{t \in [0, T]}\) are the unique solutions of BSDE \((g^1, \xi^1)\) and BSDE \((g^2, \xi^2)\) respectively. From this, fruitful results are derived. However in the case where \(g\) is only continuous in \((y, z)\), in place of the Lipschitz condition, Lepeltier and San Martin \([3]\) have proved that there is at least one solution. In fact, there is either one or uncountable many solutions in this situation (see Jia and Peng \([2]\)). To answer the question whether the uniqueness of solution also implies the continuous dependence with respect to \(g\) and \(\xi\) is what this paper will achieve.

In this paper we will prove that if the coefficient \(g\) satisfies the conditions given in \([3]\), then the uniqueness of solution and continuous dependence with respect to \(g\) and \(\xi\) are equivalent. This result, which can be regarded as the analog of the inequality \([2]\) in some sense, provides a useful method to study BSDEs with continuous coefficient.

This paper is organized as follows. In Section \([2]\) we formulate the problem accurately and give some preliminary results. Section \([3]\) is devoted to proving the equivalence between uniqueness and continuous dependence with respect to terminal value \(\xi\). Finally, in Section \([4]\) we will prove the equivalence of uniqueness and continuous dependence with respect to parameters \(g\) and \(\xi\).

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2 Preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((W_t)_{t \geq 0}\) be a \(d\)-dimensional standard Brownian motion in this space. Let \((\mathcal{F}_t)_{t \geq 0}\) be the filtration generated by this Brownian motion: \(\mathcal{F}_t = \sigma \{W_s, s \in [0, t]\} \cup \mathcal{N}\), \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\), where \(\mathcal{N}\) is the set of all \(P\)-null subsets.

Let \(T > 0\) be a fixed real number. In this paper, we always work in the space \((\Omega, \mathcal{F}_T, P)\). For a positive integer \(n\) and \(z \in \mathbb{R}^n\), we denote by \(\mathcal{H}_n^2 = \mathcal{H}_n^2(0, T; \mathbb{R}^n)\), the space of all \(P\)-progressively measurable \(\mathbb{R}^n\)-valued processes s.t. \(E[\int_0^T |\psi_t|^2\, dt] < \infty\), and by \(\mathcal{S}^2 = \mathcal{S}^2(0, T; \mathbb{R})\) the elements in \(\mathcal{H}_n^2(0, T; \mathbb{R})\) with continuous paths s.t. \(E[\sup_{t \in [0, T]} |\psi_t|^2] < \infty\).

The coefficient \(g\) of BSDE is a function \(g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) satisfying the following assumptions:

(H1): linear growth: there exists a nonnegative constant \(A\), such that \(|g(\omega, t, y, z)| \leq A(1 + |y| + |z|), \forall t, \omega, y, z\)

(H2): \((g(t, y, z))_{t \in [0, T]} \in \mathcal{H}_2^2\), for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\)

(H3): \(g(\omega, t, y, z)\) is continuous for fixed \((t, \omega)\).

Given by Lepeltier and San Martin [3 Th. 1], under (H1)—(H3) and for each given \(\xi \in L^2(\Omega, \mathcal{F}_T, P)\), there exists at least one solution \((y, z)_{t \in [0, T]} \in \mathcal{S}^2 \times \mathcal{H}_2^2\) of BSDE. [3] also gives the existence of the maximal solution \((y, z)_{t \in [0, T]}\) and the minimal solution \((\tilde{y}, \tilde{z})_{t \in [0, T]}\) of BSDE in the sense that any solution \((y, z)_{t \in [0, T]} \in \mathcal{S}^2 \times \mathcal{H}_2^2\) of BSDE must satisfy \(\tilde{y} \leq y \leq \bar{y}\), a.s., for all \(t \in [0, T]\).

It is well known that under the standard assumptions where \(g\) is Lipschitz continuous in \((y, z)\), for any random variable \(\xi\) in \(L^2(\mathcal{F}_T)\), the BSDE has a unique adapted solution, say \((y, z)_{t \in [0, T]}\) such that \(z \in \mathcal{H}_2^2\) and \(y \in \mathcal{S}^2\) (see [3]). And we have the following estimate for solution of BSDEs with Lipschitz continuous generator \(g\) coming from [1].

Lemma 1 If \(\xi^1, \xi^2 \in L^2(\mathcal{F}_T)\) and \(g\) is Lipschitz continuous in \((y, z)\). Then, for the solutions \((y^1, z^1)_{t \in [0, T]}\) and \((y^2, z^2)_{t \in [0, T]}\) of the BSDEs \((\text{g}, T, \xi^1)\) and \((\text{g}, T, \xi^2)\) respectively, we have

\[ E[\sup_{0 \leq t \leq T} |y^1_t - y^2_t|^2] \leq C E[|\xi^1 - \xi^2|^2] \]

where \(C\) is a positive constant only depending on Lipschitz constant of \(g\).

Now, we will recall some properties and associated approximation about BSDEs with \(g\) satisfying Assumptions (H1)—(H3)(see [3] for details).

Lemma 2 If \(g\) satisfies Assumptions (H1)—(H3), and we set

\[ \underline{g}_m(t, y, z) := \inf_{(u, v) \in \mathbb{R}^{1+d}} \{g(t, u, v) + m(|y - u| + |z - v|)\}, \]

and

\[ \bar{g}_m(t, y, z) := \sup_{(u, v) \in \mathbb{R}^{1+d}} \{g(t, u, v) - m(|y - u| + |z - v|)\}, \]

then for any \(m \geq A\), we have

1. For any \(y \in \mathbb{R}, z \in \mathbb{R}^d\) and \(t \in [0, T]\), \(\underline{g}_m(t, y, z) \leq A(|y| + |z| + 1)\), and \(\bar{g}_m(t, y, z) \leq A(|y| + |z| + 1)\).

2. For any \(y \in \mathbb{R}, z \in \mathbb{R}^d\) and \(t \in [0, T]\), \(\underline{g}_m(t, y, z)\) is non-decreasing in \(m\) and \(\bar{g}_m(t, y, z)\) is non-increasing in \(m\).

3. \(\underline{g}_m\) and \(\bar{g}_m\) are Lipschitz functions, i.e., for any \(y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d\) and \(t \in [0, T]\),

\[ |\underline{g}_m(t, y_1, z_1) - \underline{g}_m(t, y_2, z_2)| \leq m(|y_1 - y_2| + |z_1 - z_2|) \]

and

\[ |\bar{g}_m(t, y_1, z_1) - \bar{g}_m(t, y_2, z_2)| \leq m(|y_1 - y_2| + |z_1 - z_2|). \]

4. If \((y_m, z_m) \to (y, z)\) as \(m \to \infty\), then \(\underline{g}_m(t, y_m, z_m) \to g(t, y, z)\) and \(\bar{g}_m(t, y_m, z_m) \to g(t, y, z)\) as \(m \to \infty\).

Lemma 3 If the processes \((\bar{y}^m, \bar{z}^m)_{t \in [0, T]}\) and \((\tilde{y}^m, \tilde{z}^m)_{t \in [0, T]}\) are the unique solutions of the BSDEs \((\underline{g}_m, T, \xi)\) and \((\bar{g}_m, T, \xi)\) respectively, then

\[ (\bar{y}^m, \bar{z}^m)_{t \in [0, T]} \to (\bar{y}, \bar{z})_{t \in [0, T]}, \quad \text{and} \quad (\tilde{y}^m, \tilde{z}^m)_{t \in [0, T]} \to (\tilde{y}, \tilde{z})_{t \in [0, T]}, \quad (m \to \infty) \]

in \(\mathcal{S}^2 \times \mathcal{H}_2^2\), where \((\bar{y}, \bar{z})_{t \in [0, T]}\) and \((\tilde{y}, \tilde{z})_{t \in [0, T]}\) are the minimal solution and maximal solution of BSDE.
3 Main Results

In this section, we will prove the equivalence of uniqueness of solution and continuous dependence with respect to terminal value $\xi$.

**Theorem 4** If Assumptions (H1)–(H3) hold for $g$, then the following two statements are equivalent.

(i). Uniqueness: The equation (11) has a unique solution.

(ii). Continuous dependence with respect to $\xi$: For any $\{\xi_n\}_{n=1}^\infty$, $\xi \in L^2(\mathcal{F}_T)$, if $\xi_n \to \xi$ in $L^2(\mathcal{F}_T)$ as $n \to \infty$, then

$$\lim_{n \to \infty} E\left[ \sup_{t \in [0,T]} \left| y_t^{\xi_n} - y_t^\xi \right|^2 \right] = 0 \tag{3}$$

where $(y_t^{\xi_n}, z_t^{\xi_n})_{t \in [0,T]}$ is any solution of BSDE (11) and $(\bar{y}_t^{\xi}, z_t^{\xi})_{t \in [0,T]}$ are any solutions of the BSDEs $(g, T, \xi^n)$.

**Proof.** Firstly, we will prove that (i) implies (ii). Given $n$, we note that for any solution $(y_t^{\xi_n}, z_t^{\xi_n})_{t \in [0,T]}$ of BSDE $(g, T, \xi^n)$, we have

$$y_t^{\xi_n} \leq \bar{y}_t^{\xi}, \quad P - a.s. \quad t \in [0,T], \tag{4}$$

Now, we consider the following equations:

$$y_t^{m,\xi_n} = \xi_n + \int_t^T \bar{g}_s(y_s^{m,\xi_n}, z_s^{m,\xi_n}) \, ds - \int_t^T z_s^{m,\xi_n} \, dW_s \tag{5}$$

and

$$\bar{y}_t^{m,\xi_n} = \xi_n + \int_t^T \bar{g}_s(\bar{y}_s^{m,\xi_n}, \bar{z}_s^{m,\xi_n}) \, ds - \int_t^T \bar{z}_s^{m,\xi_n} \, dW_s \tag{6}$$

where $(y_t^{m,\xi_n}, z_t^{m,\xi_n})_{t \in [0,T]}$ and $(\bar{y}_t^{m,\xi_n}, \bar{z}_t^{m,\xi_n})_{t \in [0,T]}$ are unique solutions of (5) and (6) respectively. Thanks to Lemma 3, we know that

$$(y_t^{m,\xi_n}, z_t^{m,\xi_n}) \to (y_t^{\xi_n}, z_t^{\xi_n}), \text{ and } (\bar{y}_t^{m,\xi_n}, \bar{z}_t^{m,\xi_n}) \to (\bar{y}_t^{\xi}, \bar{z}_t^{\xi}), \quad t \in [0,T].$$

in $\mathcal{S}^2 \times \mathcal{H}_d^2$ as $m \to \infty$, and get the following inequalities

$$y_t^{m,\xi_n} \leq y_t^{\xi_n} \leq \bar{y}_t^{\xi}, \quad \text{for any } n, t \in [0,T] \text{ and } m \geq A \tag{7}$$

From inequality (7), we have

$$y_t^{\xi_n} - \bar{y}_t^{\xi} = y_t^{\xi_n} - \bar{y}_t^{\xi_n} + \bar{y}_t^{\xi_n} - \bar{y}_t^{\xi} + \bar{y}_t^{\xi} - y_t^{\xi_n}$$

$$\leq (\bar{y}_t^{\xi_n} - \bar{y}_t^{\xi}) + (\bar{y}_t^{\xi} - y_t^{\xi_n})$$

and

$$y_t^{\xi_n} - y_t^{\xi} = y_t^{\xi_n} - y_t^{\xi_n} + (y_t^{\xi_n} - y_t^{\xi}) + (y_t^{\xi} - y_t^{\xi_n})$$

$$\geq (y_t^{\xi_n} - y_t^{\xi}) + (y_t^{\xi} - y_t^{\xi_n})$$

Thus

$$E\left[ \sup_{t \in [0,T]} \left| y_t^{\xi_n} - y_t^{\xi} \right|^2 \right] \leq 2E\left[ \sup_{t \in [0,T]} \left| y_t^{\xi_n} - y_t^{\xi} \right|^2 \right] + 2E\left[ \sup_{t \in [0,T]} \left| y_t^{\xi} - y_t^{\xi_n} \right|^2 \right]$$

$$+ 2E\left[ \sup_{t \in [0,T]} \left| \bar{y}_t^{\xi} - \bar{y}_t^{\xi_n} \right|^2 \right] + 2E\left[ \sup_{t \in [0,T]} \left| \bar{y}_t^{\xi_n} - \bar{y}_t^{\xi} \right|^2 \right]$$

where $(y_t^{m,\xi}, z_t^{m,\xi})_{t \in [0,T]}$ and $(\bar{y}_t^{m,\xi}, \bar{z}_t^{m,\xi})_{t \in [0,T]}$ are solutions of BSDEs $(u_n, T, \xi)$ and $(\bar{u}_m, T, \xi)$ respectively.

By Lemma 1 and Lemma 2 as $n \to \infty$, we have

$$E\left[ \sup_{t \in [0,T]} \left| y_t^{m,\xi_n} - y_t^{m,\xi} \right|^2 \right] \to 0, \text{ and } E\left[ \sup_{t \in [0,T]} \left| y_t^{m,\xi} - y_t^{m,\xi_n} \right|^2 \right] \to 0, \quad \text{for any } m.$$
By Lemma 3 and the uniqueness of solution for BSDE (1), we get
\[
E \left[ \sup_{t \in [0,T]} \left| y^m_t - y^n_t \right|^2 \right] \to 0, \text{ and } E \left[ \sup_{t \in [0,T]} \left| \tilde{y}^m_t - y^n_t \right|^2 \right] \to 0
\]
as \(m \to \infty\). That is (ii).

Now, we will prove that (ii) implies (i). We take \(\xi_n = \xi\). For equations \((g, T, \xi^n)\), we set \(\tilde{y}^n_t = \tilde{y}^n_T\). For the equation \((\xi,\lambda, \mathbb{1})\), we set \(\tilde{y}^n_t = \tilde{y}^n_T\). The proof is complete. \(\blacksquare\)

**Remark 5** In fact, when the solution of \((\xi,\lambda, \mathbb{1})\) is not unique, the continuous dependence may not hold true in general. For example, we take \(g(t, y, z) = 3y^{2/3}, \xi = 0\). It is easy to know that \((y, z_t)_{t \in [0,T]} = (0,0)_{t \in [0,T]}\) and \((Y_t, Z_t)_{t \in [0,T]} = ((T-t)^{3}, 0)_{t \in [0,T]}\) both are solutions of BSDE
\[
y_t = \int_t^T 3y^2_s ds - \int_t^T z_s dW_s; \quad 0 \leq t \leq T.
\]
Set \(\xi_n = 1/n\), the BSDEs
\[
y_t = \frac{1}{n} + \int_t^T 3y^2_s ds - \int_t^T z_s dW_s; \quad 0 \leq t \leq T, \quad n = 1, 2, \ldots \]
have unique solutions \((\tilde{y}^n_t, z^n_t) = ((T-t + \frac{1}{\sqrt{n}})^{3}, 0)\) for \(n = 1, 2, \ldots\). But
\[
\lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| \tilde{y}^n_t - y_t \right|^2 \right] = T^6 \neq 0 = \lim_{n \to \infty} E \left[ \sup_{t \in [0,T]} \left| y^0_t - Y_t \right|^2 \right]
\]

### 4 The General Case

In this section, we will deal with the more general case, that is, the relationship between uniqueness of solution and continuous dependence with respect not only to \(\xi\) but also to \(g\). Now, we consider the following BSDEs:
\[
y^\lambda_t = \xi^\lambda + \int_t^T g^\lambda(s, y^\lambda_s, z^\lambda_s) ds - \int_t^T z^\lambda_s dW_s,
\]
where \(\lambda\) belongs to a nonempty set \(D \subset \mathbb{R}\). The coefficient \(g^\lambda\) is a function \(g(\omega, t, y, z) : D \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) satisfying the following assumptions:

(H1'): linear growth: there exists a nonnegative constant \(A\), such that \(|g^\lambda(\omega, t, y, z)| \leq A(1 + |y| + |z|), \forall \lambda, t, \omega, y, z\).

(H2'): \(g(t, y, z)_{t \in [0,T]} \in \mathcal{H}^2\), for each \((y, z) \in \mathbb{R} \times \mathbb{R}^d\) and \(\lambda \in D\).

(H3'): \(g(\omega, t, \cdot, \cdot, \cdot)\) is continuous for fixed \((t, \omega, \lambda)\).

(H4'): uniform continuity: \(g^\lambda\) is continuous in \(\lambda = \lambda_0\) uniformly with respect to \((y, z)\).

When (H1') and (H3') are replaced by Lipschitz condition (L), i.e., there exists a nonnegative constant \(K\), such that \(|g^\lambda(\omega, t, y_1, z_1) - g^\lambda(\omega, t, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|), \forall \lambda, t, \omega, y_1, z_1 \text{ and } y_2, z_2\), the BSDE (8) has a unique adapted solution for any \(\lambda \in D\). And we have the following property:

**Lemma 6** If \(\xi^\lambda \to \xi^\lambda_0\) in \(L^2(\mathcal{F}_T)\) as \(\lambda \to \lambda_0\), Assumption (H2'), (L) and (H4') hold for \(g^\lambda\). Moreover \((y^\lambda_t, z^\lambda_t)_{t \in [0,T]}\) and \((y^\lambda_0, z^\lambda_0)_{t \in [0,T]}\) are the solutions of the BSDEs \((g^\lambda, T, \xi^\lambda)\) and \((g^\lambda_0, T, \xi^\lambda_0)\) respectively, then
\[
E \left[ \sup_{t \in [0,T]} \left| y^\lambda_t - y^{\lambda_0}_t \right|^2 \right] \leq CE \left| \xi^\lambda - \xi^{\lambda_0} \right|^2 + CE \int_0^T \left| g^\lambda(t, y^\lambda_t, z^\lambda_t) - g^{\lambda_0}(t, y^{\lambda_0}_t, z^{\lambda_0}_t) \right|^2 ds
\]
where \(C\) is a positive constant only depending on Lipschitz constant \(K\). Moreover, we have
\[
\lim_{\lambda \to \lambda_0} E \left[ \sup_{t \in [0,T]} \left| y^\lambda_t - y^{\lambda_0}_t \right|^2 \right] = 0.
\]
Proof. With the usual techniques of BSDE we can get inequality (9) (see [3] for detail). Because of the continuity of \(g^\lambda\) in \(\lambda = \lambda_0\) and Lebesgue dominated convergence theorem we take limit to both sides of (9) and get equation (10). The proof is complete. ■

Now, we introduce the approximation sequences of \(g^\lambda\) as follows:

\[
g_m^\lambda(t, y, z) = \inf_{(u, v) \in \mathbb{R}^{1+d}} \left\{ g^\lambda(t, u, v) + m(|y - u| + |z - v|) \right\}, \quad (11)
\]

and

\[
g_m^\lambda(t, y, z) = \sup_{(u, v) \in \mathbb{R}^{1+d}} \left\{ g^\lambda(t, u, v) - m(|y - u| + |z - v|) \right\}. \quad (12)
\]

**Lemma 7** If \(g^\lambda\) satisfies (H1')—(H4'), then for any \(m \geq A\), we have

(1) \(|g_m^\lambda(t, y, z)| \leq A(|y| + |z| + 1)\), and \(|g_m^\lambda(t, y, z)| \leq A(|y| + |z| + 1)\), for any \(y \in \mathbb{R}, z \in \mathbb{R}^d, \lambda \in D\) and \(t \in [0, T]\).

(2) For any given \(y \in \mathbb{R}, z \in \mathbb{R}^d, \lambda \in D\) and \(t \in [0, T]\), \(g_m^\lambda(t, y, z)\) is nondecreasing in \(m\) and \(g_m^\lambda(t, y, z)\) is non-increasing in \(m\).

(3) \(g_m^\lambda\) and \(g_m^\lambda\) are Lipschitz continuous in \((y, z)\), that is, for any \(y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d\) and \(\lambda \in D\), we have \(\left| g_m^\lambda(t, y_1, z_1) - g_m^\lambda(t, y_2, z_2) \right| \leq m(|y_1 - y_2| + |z_1 - z_2|)\), and \(\left| g_m^\lambda(t, y_1, z_1) - g_m^\lambda(t, y_2, z_2) \right| \leq m(|y_1 - y_2| + |z_1 - z_2|)\).

(4) If \((y_m, z_m) \to (y, z)\) as \(m \to \infty\), then \(g_m^\lambda(t, y_m, z_m) \to g^\lambda(t, y, z)\), and \(g_m^\lambda(t, y_m, z_m) \to g^\lambda(t, y, z)\) as \(m \to \infty\).

(5) Both \(g_m^\lambda\) and \(g_m^\lambda\) are continuous in \(\lambda = \lambda_0\).

Proof. It is easy to check (1) — (4) (see [3]). Now, we will prove (5). For any \(\varepsilon > 0\), by the definition of \(g_m^\lambda\), there exist \((y^\varepsilon, z^\varepsilon, \lambda)\) and \((y^\varepsilon, \lambda_0, z^\varepsilon, \lambda_0)\) such that

\[
g^\lambda(t, y^\varepsilon, z^\varepsilon, \lambda) + m |y - y^\varepsilon, \lambda| + |z - z^\varepsilon, \lambda| - \varepsilon \leq g_m^\lambda(t, y, z)
\]

\[\leq g^\lambda(t, y^\varepsilon, \lambda_0, z^\varepsilon, \lambda_0) + m |y - y^\varepsilon, \lambda_0| + |z - z^\varepsilon, \lambda_0|\]

and

\[
g^\lambda(t, y^\varepsilon, \lambda_0, z^\varepsilon, \lambda_0) + m |y - y^\varepsilon, \lambda_0| + |z - z^\varepsilon, \lambda_0| - \varepsilon \leq g_m^\lambda(t, y, z)
\]

\[\leq g^\lambda(t, y^\varepsilon, z^\varepsilon, \lambda) + m |y - y^\varepsilon, \lambda| + |z - z^\varepsilon, \lambda|\]

thus

\[
g^\lambda(t, y^\varepsilon, z^\varepsilon, \lambda) - g^\lambda(t, y^\varepsilon, \lambda, z^\varepsilon, \lambda) - \varepsilon
\]

\[\leq g_m^\lambda(t, y, z) - g_m^\lambda(t, y, z)
\]

\[\leq g_m^\lambda(t, y^\varepsilon, z^\varepsilon, \lambda) - g_m^\lambda(t, y^\varepsilon, \lambda_0, z^\varepsilon, \lambda_0) + \varepsilon\]

Because \(g^\lambda\) is continuous when \(\lambda = \lambda_0\) uniformly with respect to \((y, z)\), we obtain the continuity of \(g_m^\lambda\) and \(g_m^\lambda\) in \(\lambda = \lambda_0\). The proof is complete. ■

**Lemma 8** If \(g^\lambda\) satisfies (H1')—(H4'), and the processes \((y^\lambda, z^\lambda)_{t \in [0, T]}\) and \((\tilde{y}^\lambda, \tilde{z}^\lambda)_{t \in [0, T]}\) are the unique solutions of the BSDEs \((g_m^\lambda, T, \xi^\lambda)\) and \((\tilde{g}_m^\lambda, T, \xi^\lambda)\) respectively, then, for any \(\lambda \in D\), we have

\[
(y^\lambda, z^\lambda)_{t \in [0, T]} \to (y^\lambda, z^\lambda)_{t \in [0, T]}, \quad \text{and} \quad (\tilde{y}^\lambda, \tilde{z}^\lambda)_{t \in [0, T]} \to (\tilde{y}^\lambda, \tilde{z}^\lambda)_{t \in [0, T]},
\]

in \(S^2 \times \mathcal{H}_2^c\) as \(m \to \infty\), where \((y^\lambda, z^\lambda)_{t \in [0, T]}\) and \((\tilde{y}^\lambda, \tilde{z}^\lambda)_{t \in [0, T]}\) are the minimal solution and maximal solution of BSDE (3).

Now, we give our result for the general case.
Theorem 9 If $g^\lambda$ satisfies (H1')—(H4'), then the following statements are equivalent:

(iii). Uniqueness: there exists a unique solution of BSDE (8) when $\lambda = \lambda_0$, that is, the solution of $(g^{\lambda_0}, T, \xi^{\lambda_0})$ is unique.

(iv). Continuous dependence with respect to $g$ and $\xi$: for any $\xi^\lambda$, $\xi^{\lambda_0} \in L^2(\mathcal{F}_T)$, if $\xi^\lambda \to \xi^{\lambda_0}$ in $L^2(\mathcal{F}_T)$ as $\lambda \to \lambda_0$, $(y_t^\lambda, z_t^\lambda)_{t \in [0, T]}$ are any solutions of BSDEs (8), $(y_t^{\lambda_0}, z_t^{\lambda_0})_{t \in [0, T]}$ is any solution of BSDE (8) when $\lambda = \lambda_0$, then

$$\lim_{\lambda \to \lambda_0} E[\sup_{t \in [0, T]} |y_t^\lambda - y_t^{\lambda_0}|^2] = 0.$$ 

Proof. This proof is similar to that of Theorem 4. For the sake of completeness, we give the sketch of proof. Firstly, we prove (iii) implies (iv). We can get the inequalities similarly to (7), that is, $y_t^{\lambda, m, \lambda} \leq y_t^\lambda \leq y_t^m \leq \bar{y}_t = \bar{\xi}$, for any $t \in [0, T]$ and $m \geq A$. So,

$$E[\sup_{t \in [0, T]} |y_t^\lambda - y_t^{\lambda_0}|^2] \leq 2E[\sup_{t \in [0, T]} |y_t^{\lambda, m} - y_t^{\lambda_0, m}|^2] + 2E[\sup_{t \in [0, T]} |y_t^{\lambda_0, m} - y_t^{\lambda_0}|^2] + 2E[\sup_{t \in [0, T]} |\bar{y}_t^{\lambda_0, m} - y_t^{\lambda_0}|^2].$$

Fixed $m$, with the help of Lemma 6 and Lemma 7 and the continuity of $g^\lambda$ and $\bar{g}_m$ when $\lambda = \lambda_0$, we have,

$$E[\sup_{t \in [0, T]} |y_t^{\lambda, m} - y_t^{\lambda_0, m}|^2] \to 0, \text{ and } E[\sup_{t \in [0, T]} |y_t^{\lambda_0, m} - y_t^{\lambda_0}|^2] \to 0$$

as $\lambda \to \lambda_0$, for any $m \geq A$. By Lemma 8 and the uniqueness of solution for $(g^{\lambda_0}, T, \xi^{\lambda_0})$ (Condition (iii)), we obtain, as $m \to \infty$,

$$E[\sup_{t \in [0, T]} |y_t^{\lambda_0, m} - y_t^{\lambda_0}|^2] \to 0, \text{ and } E[\sup_{t \in [0, T]} |\bar{y}_t^{\lambda_0, m} - y_t^{\lambda_0}|^2] \to 0.$$ 

This implies (iv).

Now we will prove that (iv) implies (iii). Take $\xi^\lambda = \xi^{\lambda_0}$, $g^\lambda = g^{\lambda_0}$. For equation (8), set $y_t^\lambda := \bar{y}_t = \bar{\xi}$. For equation $(g^{\lambda_0}, T, \xi^{\lambda_0})$, take $y_t^{\lambda_0} = \bar{y}_t$. By (iv), we have $\bar{y}_t = \bar{\xi}$. The proof is complete. 

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