NOTES ON UNIVERSAL ALGEBRA

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Dedicated to Dennis Sullivan on the occasion of his sixtieth birthday.

Abstract. These are notes of a mini-course given at Dennisfest in June 2001. The goal of these notes is to give a self-contained survey of deformation quantization, operad theory, and graph homology. Some new results related to “String Topology” and cacti are announced in Section 27.

Introduction

Either due to the influence of string theory or just because this is what the face of mathematics was supposed to look like before the beginning of the third millennium, graphs have recently stepped forward and overwhelmed many areas of mathematics, including universal algebra. The use of graphs is similar to the Feynman diagram technique in physics; the amazing thing is that its applications to pure mathematics are extremely powerful. In these lectures we are going to discuss three topics: deformation quantization, operad theory, and graph homology, in which significant progress has been made with the help of graphs as a unifying pattern:-) If you are in any way offended with my choice of a title of this paper, please, see the disclaimer below.

Acknowledgments

I would like to thank Dennis Sullivan and the organizers of Dennisfest for inviting me to give this mini-course. These notes are based on numerous ideas of Maxim Kontsevich, which have largely influenced the subject. I am also grateful to Giovanni Felder, Kolya Ivanov, John Jones, Tom Leinster, Bob Penner, Jim Stasheff, Dennis Sullivan, and the lively Dennisfest and the University of Minnesota audiences for many helpful suggestions, incorporated in this version of the notes. I would also like to thank Kyoto University for its hospitality in July 2001, when the paper was written up.

Disclaimer. The characters and figures portrayed and the titles and notions used herein are fictitious and any resemblance to the names, character, or history of any person, except Dennis Sullivan, is coincidental and unintentional.

1. Graphs and formal algebraic quantization

One of the oldest open problems solved with the essential help of graphs is perhaps the problem of deformation quantization.

1.1. Deformations of associative algebras. Let us start with a review of deformation theory of associative algebras. Let $A$ be an associative algebra over a field $k$ of characteristic zero.

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Definition 1.1. A formal deformation of $A$ is a $k[[t]]$-bilinear multiplication law $m_t : A[[t]] \otimes k[[t]] A[[t]] \to A[[t]]$ on the space $A[[t]]$ of formal power series in a variable $t$ with coefficients in $A$, satisfying the following properties:

$$m_t(a, b) = a \cdot b + m_1(a, b)t + m_2(a, b)t^2 + \cdots$$

for $a, b \in A$, where $a \cdot b$ is the original multiplication on $A$, and $m_t$ is associative, which is equivalent to the equation

$$m_t(m_t(a, b), c) = m_t(a, m_t(b, c))$$

for $a, b, c \in A$.

Remark 1. Note that for a formal deformation $m_t$ of a commutative algebra $A$, the bracket defined by the first-order part of the commutator,

$$\{a, b\} := \frac{1}{2t}(m_t(a, b) - m_t(b, a)) \mod t,$$

defines the structure of a Poisson algebra on $A$. (All the identities of a Poisson algebra follow from the fact that $A[[t]]$ with the product $m_t(a, b)$ and the bracket $\frac{1}{2}(m_t(a, b) - m_t(b, a))$ is a noncommutative Poisson algebra in an obvious sense, e.g., see [FGV95].) In physical terms, one can regard $t$ as a quantum parameter, such as the Planck constant, the Poisson algebra given a Poisson algebra $A[[t]]$, and the algebra $A[[t]]$ as a deformation quantization of the Poisson algebra $A$. The deformation quantization problem is the inverse problem: given a Poisson algebra $A$, find a formal deformation returning the original Poisson algebra structure on $A$ in the quasi-classical limit.

The main tool in studying deformation theory is the Hochschild complex

$$0 \to C^0(A, A) \xrightarrow{d} C^1(A, A) \xrightarrow{d} C^2(A, A) \xrightarrow{d} C^3(A, A) \xrightarrow{d} \cdots,$$

where $C^n(A, A) := \text{Hom}(A^\otimes n, A)$ is the space of Hochschild $n$-cochains, i.e., the $n$-linear maps $f(a_1, \ldots, a_n)$ on $A$ with values in $A$, and the differential $d$, $d^2 = 0$, is defined as

$$(df)(a_1, \ldots, a_n) := a_1 f(a_2, \ldots, a_{n+1})$$

$$+ \sum_{i=1}^n (-1)^i f(a_1, \ldots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \ldots, a_n+1)$$

$$- (-1)^n f(a_1, \ldots, a_n)a_{n+1},$$

for $f \in C^n(A, A)$, $a_1, \ldots, a_{n+1} \in A$. The Hochschild cohomology is then the cohomology

$$H^*(A, A) := \text{Ker } d / \text{Im } d$$

of this complex. The Hochschild complex admits a bracket

$$[\cdot, \cdot] : C^m(A, A) \otimes C^n(A, A) \to C^{m+n-1}(A, A),$$

called the Gerstenhaber bracket, or the $G$-bracket. The formula for this bracket, defined by M. Gerstenhaber [Ger63] is not really inspirational to me, in spite of all those years I have spent staring at it. A conceptual definition, due to J. Stasheff, is based on the following idea. The matter is that the Hochschild complex may be identified with the space of graded derivations of the tensor coalgebra: $T^c(A[1]) := \bigoplus_{n \geq 0} A[1]^\otimes n$, and the bracket is just the commutator of derivations [Sta93]. Here
A[1] denotes the graded vector space whose only nonzero graded component is A, placed in degree −1. In general, for a graded vector space \( V = \bigoplus_n V^n \), \( V[k] \) denotes grading shift, or what is known to topologists as \( k \)-fold suspension: it is a graded vector space \( V[k] = \bigoplus_n V_{n+k} \). Note that a derivation determined by a map \( A[1] \otimes n \rightarrow A[1] \) has degree \( n-1 \), therefore, the bracket defines a differential graded (DG) Lie algebra structure on the Hochschild complex \( C^\bullet(A, A)[1] \) with a shifted grading \( \deg f = n-1 \) for \( f \in C^n(A, A) \).

Note that the (appropriately shifted) Hochschild cohomology \( H^\bullet(A, A)[1] \) inherits the structure of a graded Lie algebra.

The importance of the bracket comes from the following tautological fact, which however may be regarded as the cornerstone of deformation theory.

**Proposition 1.1.** A formal multiplication
\[
m_t(a, b) = m_0(a, b) + m_1(a, b)t + m_2(a, b)t^2 + \cdots, \quad a, b \in A,
\]
is associative, iff \([m_t, m_t] = 0\).

**Proof.** We need a little formula for the G-bracket of \( m_t \) with itself:
\[
[m_t, m_t](a, b, c) = 2(m_t(m_t(a, b), c) - m_t(a, m_t(b, c)))]
\]
which would have been obvious, if we had used Gerstenhaber’s formula to define the G-bracket. It is a good exercise to deduce it directly from the definition of the G-bracket as the commutator of derivations. The right-hand side of the formula contains the associativity equation, and we are done.

**Remark 2.** Because the original multiplication \( m_0(a, b) := a \cdot b \) is associative, the G-bracket square of it vanishes: \([m_0, m_0] = 0\). Therefore, the commutator with \( m_0 \) defines an inner differential on the Hochschild complex. This differential is in fact the Hochschild differential: another exercise is to verify that \( df = [f, m_0] \).

**Corollary 1.2.** 1. A formal multiplication
\[
m_t(a, b) = a \cdot b + m_1(a, b)t + m_2(a, b)t^2 + \cdots, \quad a, b \in A,
\]
is associative modulo \( t^2 \), iff \( dm_1 = 0 \). In this case \( m_1 \) defines a Hochschild cohomology class \( m_1 \in H^2(A, A) \).

2. Suppose that a formal multiplication as above is associative modulo \( t^2 \). Then the existence of \( m_2 \) such that \( m_t \) is associative modulo \( t^3 \) is equivalent to the vanishing \([m_1, m_1] = 0\) in Hochschild cohomology \( H^3(A, A) \).

**Proof.** Expand the equation \( \frac{1}{2}[m_t, m_t] = 0 \) in powers of \( t \) and collect terms by \( t^n \) for \( n = 0, 1, \) and 2. We will get the following.
\[
t^0 : \quad \frac{1}{2}[m_0, m_0] = 0,
\]
\[
t^1 : \quad dm_1 = 0,
\]
\[
t^2 : \quad dm_2 + \frac{1}{2}[m_1, m_1] = 0.
\]
The first equation is always satisfied, because the original multiplication is associative, see Remark 2. The other two equations explain both statements of the corollary.
1.2. Deformation quantization. Deformation quantization usually refers to a specific deformation quantization problem in a geometric/physical setting. The following theorem, which solves the deformation quantization problem posed by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [BFF+78], is a remarkable breakthrough in pure mathematics achieved by applying ideas motivated by Feynman diagrams.

**Theorem 1.3** (M. Kontsevich [Kon97]). Every Poisson manifold \((M, \{\cdot, \cdot\})\) may be deformation quantized, i.e., there exists a formal deformation quantization, see Remark [\[\]_1],

\[ f \star g := m_1(f, g) = fg + m_1(f, g)t + m_2(f, g)t^2 + \cdots, \quad f, g \in C^\infty(M), \]
of the Poisson algebra \(A = C^\infty(M)\) of smooth functions, so that all the \(m_i\)'s are local, that is, bidifferential operators on \(M\). According to our definition of deformation quantization, the star product must be associative and also recover the Poisson algebra of functions in the quasi-classical limit, i.e., \((m_1(f, g) - m_1(g, f))/2 = \{f, g\}\).

**Proof.** We will only consider the case of \(M = \mathbb{R}^d\) with an arbitrary Poisson structure, where the situation is already highly nontrivial. Globalization, which is done using a Fedosov-type connection, see [Kon97, CFT00], lies outside the main theme of these notes: no pattern in it has to do with graphs.

First, we will sketch Kontsevich’s original proof, giving an explicit formula for the star product \(f \star g\):

\[ f \star g := \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_{n,2}} W_\Gamma B_\Gamma(f, g). \]

which is explained in the this paragraph. The interior summation runs over the set \(G_{n,2}\) of directed graphs \(\Gamma\) of a certain type with vertices labeled 1, 2, \ldots, \(n\), 1, 2. The set \(G_{n,2}\) of graphs consists of the graphs satisfying the following conditions. Each vertex of first type, i.e., labeled 1, 2, \ldots, or \(n\), has exactly two outgoing edges, labeled the first one and the second, and there are no other edges whatsoever. No edge may form a loop, i.e., start and end at one and the same vertex. \(B_\Gamma(f, g)\) is a bidifferential operator defined by an explicit formula [Kon97], which we will describe using an example.

For \[\Gamma = \begin{array}{ccc}
1 & j & 2 \\
i & k & 1 \\
\bar{1} & 2
\end{array}\]

the corresponding bidifferential operator will be

\[ B_\Gamma(f, g) := \alpha^{ij}\partial_j(\alpha^{kl})\partial_k\partial_i(f)\partial_l(g), \]

where \(\alpha^{ij}\) denotes the corresponding component of the Poisson tensor in a fixed coordinate system \((x_1, \ldots, x_d)\) in \(\mathbb{R}^d\) and we assume summation over the repeating indices. Finally, the coefficient \(W_\Gamma\) is given by another formula:

\[ W_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int_{C^+_{n,2}} \prod_{r \rightarrow s} d\phi(z_r, z_s), \]
where $C_{n,2}^+$ is the configuration space of $n$ distinct points $z_1, \ldots, z_n$ in the upper half-plane and two fixed points $z_1 = 0$ and $z_2 = 1$ on the real line; $\phi(z_r, z_s)$, $r$ and $s$ running over $\{1, 2, \ldots, n, 1, 2\}$, is the directed angle at $z_r$ between the hyperbolic line through $z_r$ and $\infty$ and the hyperbolic line through $z_r$ and $z_s$. The order in the wedge product is given by the lexicographic order of the vertices $\{1, \ldots, n\}$ and the orders of the set of edges going out of the vertices. Kontsevich proves that this improper integral is absolutely convergent.

The associativity of the star product may be verified explicitly, see [Kont97]. However, I will sacrifice the rigor for the moral and give a more conceptual, physical explanation of the associativity, following A. Cattaneo and G. Felder [CF00].

Cattaneo and Felder define the star product as

$$(2) \quad (f \star g)(x) := \int_{\mathcal{P}_x} f(X(0))g(X(1))e^{iS(X, \eta)/t} DXD\eta.$$ 

This is a Feynman integral over the infinite dimensional “path” space, which is the following space of fields $X$ and $\eta$ on the upper half-plane $\mathcal{H}$:

$$\mathcal{P}_x := \{ X : \tilde{\mathcal{H}} \to \mathbb{R}^d, \eta \in \Omega^1(\tilde{\mathcal{H}}) \otimes \mathbb{R}^d \mid X(\infty) = x \quad \text{and} \quad \eta \text{ vanishes on tangent vectors to the boundary} \}.$$ 

The function $S(X, \eta)$ is a certain action functional defining a Poisson sigma model on $\mathcal{H}$, see [CF00]:

$$S = \int_{\mathcal{H}} (\eta_{\mu i} \partial_{\nu} X^i + \frac{1}{2} \alpha^{ij}(X) \eta_{\mu i} \eta_{\nu j}) du^\mu du^\nu.$$ 

A rigorous definition of the Feynman integral would be the very formula (1). However, physics takes the opposite viewpoint and treats (1) as the saddle-point expansion of the integral (2) in parameter $t$ obtained by formally applying the rules by analogy with the finite-dimensional case. The advantage of this approach is that the mystery of Kontsevich’s formula (1) is now replaced by the mystery of Equation (2), which is not so mysterious to a physicist, for whom it represents a standard integral quantization formula. Another advantage is that it offers the following explanation of the associativity.

Consider the integral

$$\langle f, g, h \rangle_p(x) := \int_{\mathcal{P}_x} f(X(0))g(X(1))h(X(p))e^{iS(X, \eta)/t} DXD\eta,$$ 

where $p \in (1, \infty) \subset \mathbb{R} \subset \tilde{\mathcal{H}}$ is a fixed point on the real line between 1 and $\infty$ and $\mathcal{P}_x$ is as above in (2). This integral is independent of the choice of this point $p$, because the action $S$ is diffeomorphism invariant and, roughly speaking, by integrating over all fields $X$ and $\eta$, we take an average over all possible positions of $p$. Thus, the limits of $\langle f, g, h \rangle_p$ as $p \to 1$ and $p \to \infty$ will be the same. On the other hand, in the moduli space of configurations of four points 0, 1, $p$, and $\infty$ on the boundary of $\mathcal{H}$, these configurations will degenerate as follows:
which means that

\[
\lim_{p \to 1} (f, g, h)_p = f \ast (g \ast h),
\]
\[
\lim_{p \to \infty} (f, g, h)_p = (f \ast g) \ast h,
\]
yielding the associativity.

In reality, things are more complicated than I have made them appear: the Feynman diagram expansion involves gauge fixing and renormalization, which is achieved by introducing ghosts and antighosts (and what not) and using the BV formalism, see \cite{CF00}. The behavior of the Feynman integral with respect to the compactification of the configuration spaces is another issue suppressed in the above. I hope to learn these things before the next Dennisfest:-)

2. Trees and Operads

The combinatorics of trees essentially describes the combinatorics of operads, while more general graphs would be related to PROP’s and modular operads. The main idea is that the trees form a free operad (More precisely, the linear span of the set of labeled trees with a certain differential forms the $A_\infty$ operad), and vice versa, any free operad may be described via decorated trees: the tree remembers how to get to its root by applying the operations put at its vertices, given no identities, such as the associativity, whatsoever. Here we will review these notions and related concepts of homotopy algebra.

Operads in general are spaces of operations with certain rules on how to compose operations to get new ones. In this sense operads are directly related to Lawvere’s algebraic theories and represent true objects of universal algebra. However, operads as such appeared in topology in the works of J. P. May, J. M. Boardman and R. M. Vogt as a recognition tool for based multiple loop spaces. Stasheff earlier described the first example of an operad, the associahedra, which recognized based loop spaces, while about the same time, Gerstenhaber, studying the algebra of the Hochschild complex, introduced the notion of a composition algebra, which is nothing but the notion of an operad of graded vector spaces.

2.1. PROP’s. Unfortunately, the formal definition of an operad resembles in a way the definition of a Deligne-Mumford stack — it is very easy to drown in the ocean of sheaves of groupoids over the site of schemes on your first trip to the beach.
It is much easier to define a PROP (=PROducts and Permutations) and think of an operad as certain part of a PROP.

Definition 2.1. A PRO is a symmetric monoidal (i.e., tensor) category whose set of objects is identified with the set \( \mathbb{Z}_+ \) of nonnegative integers, where the tensor law on \( \mathbb{Z}_+ \) is given by addition. A PROP is a PRO along with a right action of the permutation group \( S_m \) and a left action of \( S_n \) on \( \text{Mor}(m, n) \) for each pair \( m, n \geq 0 \). These actions should be in a natural way compatible with the tensor structure and the composition of morphisms in the category that constitutes the PROP, see, for example, the founding fathers’ sources, such as, J. F. Adams’ book [Ada78] or S. Mac Lane’s paper [ML65], or a postmodern view [http://www.math.umn.edu/~voronov/8390/lec2.pdf](http://www.math.umn.edu/~voronov/8390/lec2.pdf) for more detail.

Usually, PROP’s are enriched over another symmetric monoidal category, that is, the morphisms in the PROP are taken as objects of the other symmetric monoidal category. This gives the notions of a PROP of sets, vector spaces, complexes, topological spaces, manifolds, etc. Examples of PROP’s include the following. We will only specify the morphisms, because the objects are already given by the definition.

Example 2.1. The endomorphism PROP of a vector space \( V \) has the space of morphisms \( \text{Mor}(m, n) = \text{Hom}(V^\otimes m, V^\otimes n) \). This is a PROP of vector spaces. The composition and tensor product of morphisms is given by same of linear maps.

Example 2.2. The Segal PROP is a PROP of infinite dimensional complex manifolds. The space of morphisms is defined as the moduli space \( P_{m,n} \) of complex Riemann surfaces bounding \( m + n \) labeled nonoverlapping holomorphic holes. The surfaces should be understood as compact smooth complex curves, not necessarily connected, along with \( m + n \) biholomorphic maps of the closed unit disk to the surface. The more exact nonoverlapping condition is that the closed disks in the inputs do not intersect pairwise and the closed disks in the outputs do not intersect pairwise, however, an input and an output disk may have common boundary, but are still not allowed to intersect at an interior point. This technicality brings in the identity morphisms to the PROP, but does not create singular Riemann surfaces by composition. The moduli space means that we consider isomorphism classes of such objects. The composition of morphisms in this PROP is given by sewing the Riemann surfaces along the boundaries, using the equation \( zw = 1 \) in the holomorphic parameters coming from the standard one on the unit disk. The tensor product of morphisms is the disjoint union. This PROP plays a crucial role in Conformal Field Theory, as we will see now.

2.2. Algebras over a PROP. We need to define another important notion before we proceed.

Definition 2.2. We say that a vector space \( V \) is an algebra over a PROP \( P \), if a morphism of PROP’s from \( P \) to the endomorphism PROP of \( V \) is given. A morphism of PROP’s should be a functor respecting the symmetric monoidal structure and the symmetric group actions, and also equal to the identity on the objects.

An algebra over a PROP could have been called a representation, but since algebras over operads, which are similar objects, are nothing but familiar types of algebras, it is more common to use the term “algebra”.
Example 2.3. An example of an algebra over a PROP is a Conformal Field Theory (CFT), which may be defined as an algebra over the Segal PROP. The fact that the functor respects compositions of morphisms translates into the sewing axiom of CFT in the sense of G. Segal. Usually, one also asks for the functor to depend smoothly on the point in the moduli space $P_{m,n}$. This definition of a CFT describes only theories with a vanishing central charge. One needs to extend the Segal PROP by a line bundle to cover the case of an arbitrary charge, see [Hua97].

Example 2.4 (Sullivan). Another example of an algebra over a PROP is a Lie bialgebra. Sullivan has shared with me a nice graph description of the corresponding PROP, see http://www.math.umn.edu/~voronov/8390/lec4.pdf.

2.3. Operads. Now we are ready to deal with operads. Informally, an operad is the part $\text{Mor}(n,1)$, $n \geq 0$, of a PROP. Of course, given only the collection of morphisms $\text{Mor}(n,1)$, it is not clear how to compose them. The idea is to take the union of a $m$ elements from $\text{Mor}(n,1)$ to be able to compose them with an element of $\text{Mor}(m,1)$. This leads to cumbersome notation and ugly axioms, compared to those of a PROP. However operads are in a sense more basic than the corresponding PROP's: the difference is similar to the difference between Lie algebras and the universal enveloping algebras.

Definition 2.3. An operad $O$ is a collection of sets (vector spaces, complexes, topological spaces, manifolds, . . . , objects of a symmetric monoidal category) $O(n)$, $n \geq 0$, with

1. A composition law:

$$\gamma : O(m) \otimes O(n_1) \otimes \cdots \otimes O(n_m) \to O(n_1 + \cdots + n_m).$$

2. A right action of the symmetric group $S_n$ on $O(n)$.

3. A unit $e \in O(1)$.

such that the following properties are satisfied:

1. The composition is associative, i.e., the following diagram is commutative:

$$\begin{array}{c}
\begin{array}{c}
\{ O(l) \otimes O(m_1) \otimes \cdots \otimes O(m_l) \} \\
\otimes O(n_{i1}) \otimes \cdots \otimes O(n_{il,n_l})
\end{array}
\xrightarrow{id \otimes \gamma_{l}}
\begin{array}{c}
O(l) \otimes O(n_{11}) \otimes \cdots \otimes O(n_{11,n_l})
\end{array}
\xrightarrow{\gamma_{l}}
\begin{array}{c}
O(n_{11}) \otimes \cdots \otimes O(n_{m,n_m})
\end{array}
\xrightarrow{\gamma}
\begin{array}{c}
O(n)
\end{array}
\end{array},$$

where $m = \sum_i m_i$, $n_i = \sum_j n_{ij}$, and $n = \sum_i n_i$.

2. The composition is equivariant with respect to the symmetric group actions: the groups $S_m, S_{n_1}, \ldots, S_{n_m}$ act on the left-hand side and map naturally to $S_{n_1 + \cdots + n_m}$, acting on the right-hand side.

3. The unit $e$ satisfies natural properties with respect to the composition: $\gamma(e; f) = f$ and $\gamma(f; e, \ldots, e) = f$ for each $f \in O(k)$.

The notion of a morphism of operads is introduced naturally.
Remark 3. One can consider non-$\Sigma$ operads, not assuming the action of the symmetric groups. Not requiring the existence of a unit $e$, we arrive at nonunital operads. Do not mix this up with operads with no $O(0)$, algebras over which (see next section) have no unit. There are also good examples of operads having only $n \geq 2$ components $O(n)$.

An equivalent definition of an operad may be given in terms of operations $f \circ_i g = \gamma(f; \text{id}, \ldots, \text{id}, g, \text{id}, \ldots, \text{id})$, $i = 1, \ldots, m$, for $f \in O(m), g \in O(n)$. Then the associativity condition translates as $f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h$ plus a natural symmetry condition for $(f \circ_i g) \circ_j h$, when $g$ and $h$ “fall into separate slots” in $f$, see e.g., [KSV96].

Example 2.5 (The Riemann surface and the endomorphism operads). $\mathcal{P}(n)$ is the space of Riemann spheres with $n+1$ boundary components, i.e., $n$ inputs and 1 output. Another example is the endomorphism operad of a vector space $V$: $\mathcal{E}nd_V(n) = \text{Hom}(V^\otimes n, V)$, the space of $n$-linear mappings from $V$ to $V$.

2.4. Algebras over an operad.

Definition 2.4. An algebra over an operad $O$ (in other terminology, a representation of an operad) is a morphism of operads $O \rightarrow \mathcal{E}nd_V$, that is, a collection of maps $O(n) \rightarrow \mathcal{E}nd_V(n)$ for $n \geq 0$ compatible with the symmetric group action, the unit elements, and the compositions. If the operad $O$ is an operad of vector spaces, then we would usually require the morphism $O \rightarrow \mathcal{E}nd_V$ to be a morphism of operads of vector spaces. Otherwise, we would think of this morphism as a morphism of operads of sets. Sometimes, we may also need a morphism to be continuous or respect differentials, or have other compatibility conditions.

2.4.1. The commutative operad. The commutative operad is the operad of $k$-vector spaces with the $n$th component $\text{Comm}(n) = k$ for all $n \geq 0$. We assume that the symmetric group acts trivially on $k$ and the compositions are just the multiplication of elements in the ground field $k$. An algebra over the commutative operad is nothing but a commutative associative algebra with a unit, as we see from the following exercise.

Another version of the commutative operad is $\text{Comm}(n) = \text{point}$ for all $n \geq 0$. This is an operad of sets. An algebra over it is also the same as a commutative associative unital algebra.

Exercise 1. Show that the operad $\mathcal{T}op(n) = \{\text{the set of diffeomorphism classes of Riemann spheres with } n \text{ input holes and 1 output hole}\}$ is isomorphic to the commutative operad of sets.

Exercise 2. Prove that the structure of an algebra over the commutative operad $\text{Comm}$ on a vector space is equivalent to the structure of a commutative associative algebra with a unit.

2.4.2. The associative operad. The associative operad $\text{Assoc}$ can be considered as a one-dimensional analogue of the commutative operad $\mathcal{T}op$. $\text{Assoc}(n)$ is the set of equivalence classes of connected planar binary (each vertex being of valence 3) trees that have a root edge and $n$ leaves labeled by integers 1 through $n$: 

\[
\text{Example 2.5 (The Riemann surface and the endomorphism operads). } \mathcal{P}(n) \text{ is the space of Riemann spheres with } n+1 \text{ boundary components, i.e., } n \text{ inputs and 1 output. Another example is the endomorphism operad of a vector space } V: \mathcal{E}nd_V(n) = \text{Hom}(V^\otimes n, V), \text{ the space of } n\text{-linear mappings from } V \text{ to } V.\]

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\[
O(n) \rightarrow \mathcal{E}nd_V(n) \quad \text{for } n \geq 0\]

\[
\text{compatible with the symmetric group action, the unit elements, and the compositions. If the operad } O \text{ is an operad of vector spaces, then we would usually require the morphism } O \rightarrow \mathcal{E}nd_V \text{ to be a morphism of operads of vector spaces. Otherwise, we would think of this morphism as a morphism of operads of sets. Sometimes, we may also need a morphism to be continuous or respect differentials, or have other compatibility conditions.}\]

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\text{Another version of the commutative operad is } \text{Comm}(n) = \text{point} \text{ for all } n \geq 0. \text{ This is an operad of sets. An algebra over it is also the same as a commutative associative unital algebra.}\]

\[
\text{Exercise 1. Show that the operad } \mathcal{T}op(n) = \{\text{the set of diffeomorphism classes of Riemann spheres with } n \text{ input holes and 1 output hole}\} \text{ is isomorphic to the commutative operad of sets.}\]

\[
\text{Exercise 2. Prove that the structure of an algebra over the commutative operad } \text{Comm} \text{ on a vector space is equivalent to the structure of a commutative associative algebra with a unit.}\]

\[
2.4.2. \text{ The associative operad. The associative operad } \text{Assoc} \text{ can be considered as a one-dimensional analogue of the commutative operad } \mathcal{T}op. \text{ Assoc}(n) \text{ is the set of equivalence classes of connected planar binary (each vertex being of valence 3) trees that have a root edge and } n \text{ leaves labeled by integers 1 through } n:\]
\]
If \( n = 1 \), there is only one tree — it has no vertices and only one edge connecting a leaf and a root. If \( n = 0 \), the only tree is the one with no vertices and no leaves — it only has a root. Unfortunately, I have a problem sketching it: it probably exists only in the quantum world.

Two trees are equivalent if they are related by a sequence of moves of the kind

\[ \begin{array}{c}
\text{1 2 3} \\
\text{1 2 3}
\end{array} \]

performed over pairs of two adjacent vertices of a tree. The symmetric group acts by relabeling the leaves, as usual. The composition is obtained by grafting the roots of \( m \) trees to the leaves of an \( m \)-tree, no new vertices being created at the grafting points. Note that this is similar to sewing Riemann surfaces and erasing the seam, just as we did to define operad composition in that case. By definition, grafting a 0-tree to a leaf just removes the leaf and, if this operation creates a vertex of valence 2, we should erase the vertex.

**Exercise 3.** Prove that the structure of an algebra over the associative operad \( \text{Assoc} \) on a vector space is equivalent to the structure of an associative algebra with a unit.

2.4.3. *The Lie operad.* The Lie operad \( \text{Lie} \) is another variation on the theme of a tree operad. Consider the same planar binary trees as for the associative operad, except that we do not include a 0-tree, i.e., the operad has only positive components \( \text{Lie}(n), n \geq 1 \), and there are now two kinds of equivalence relations:

\[ \begin{array}{c}
\text{1 2 2 1} \\
\text{1 2 3 2 3 1 3 2 1}
\end{array} \]

Skew Symmetry

and

\[ \begin{array}{c}
\text{1 2 3} \\
\text{1 2 3} \\
\text{1 3 2}
\end{array} \]

Jacobi Identity

Now that we have arithmetic operations in the equivalence relations, we consider the Lie operad as an operad of vector spaces. We also assume that the ground field
is of a characteristic other than 2, because otherwise we will arrive at the wrong
definition of a Lie algebra.

**Exercise 4.** Prove that the structure of an algebra over the Lie operad \( \mathcal{L}_{\text{ie}} \) on a
vector space over a field of a characteristic other than 2 is equivalent to the structure
of a Lie algebra.

**Exercise 5.** Describe algebraically an algebra over the operad \( \mathcal{L}_{\text{ie}} \), if we modify it
by including a 0-tree, whose composition with any other tree is defined as (a) zero,
(b) the one for the associative operad.

2.4.4. *The Poisson operad.* Recall that a *Poisson algebra* is a vector space \( V \) (over
a field of characteristic zero) with a unit element \( e \), a dot product \( ab \), and a bracket \( [a, b] \) defined, so that the dot product defines the structure of commutative associ-ative unital algebra, the bracket defines the structure of a Lie algebra, and the
bracket is a derivation of the dot product:

\[
[a, bc] = [a, b]c + b[a, c] \quad \text{for all } a, b, \text{ and } c \in V.
\]

**Exercise 6.** Define the *Poisson operad*, using a tree model similar to the previous
examples. Show that an algebra over it is nothing but a Poisson algebra. [*Hint:*
Use two kinds of vertices, one for the dot product and the other one for the bracket.]

2.4.5. *The Riemann surface operad and vertex operator algebras.* Just for a change,
let us return to the operad \( \mathcal{P} \) of Riemann surfaces, more exactly, isomorphism
classes of Riemann spheres with holomorphic holes. What is an algebra over it?
Since there are infinitely many nonisomorphic pairs of pants, there are infinitely
many (at least) binary operations. In fact, we have an infinite dimensional family
of binary operations parameterized by classes of pairs of pants. However modulo
the unary operations, those which correspond to cylinders, we have only one funda-mental binary operation corresponding to a fixed pair of pants. An algebra over
this operad \( \mathcal{P} \) is part of a CFT data at the tree level, the central charge \( c = 0 \). If we
consider a holomorphic algebra over this operad, that is, require that the defining
mappings \( \mathcal{P}(n) \to \mathcal{E}nd_{V}(n) \), where \( V \) is a complex vector space, be holomorphic,
then we get part of a chiral CFT, or an object which may be called a *vertex op-
erator algebra* (VOA). This kind of object is not equivalent to what people used
to call a VOA, but according to Y.-Z. Huang’s Theorem, a true VOA is a holo-morphic algebra over a “partial pseudo-operad of Riemann spheres with rescaling”,
which is a version of \( \mathcal{P} \), where the disks are allowed to overlap. The fundamental binary operation \( Y(a, z)b \) for \( a, b \in V \) of a VOA is commonly chosen to be the
one corresponding to a pair of pants which is the Riemann sphere with a standard
holomorphic coordinate and three unit disks around the points 0, \( z \), and \( \infty \) (No
doubt, these disks overlap badly, but we shrink them on the figure to look better):

![Diagram](image)

The famous associativity identity

\[
Y(a, z - w)Y(b, -w)c = Y(Y(a, z)b, -w)c
\]
for vertex operator algebras comes from the following natural isomorphism of the Riemann surfaces:

\[
\begin{align*}
\begin{array}{ccc}
\text{a} & \text{c} & \text{b} \\
\text{w} & \text{z} & \text{0} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{ccc}
\text{c} & \text{0} & \text{b} \\
\text{w} & \text{z} & \text{0} \\
\end{array}
\end{align*}
\]

2.5. **Operads via generators and relations.** The tree operads that we looked at above, such as the associative and the Lie operads, are actually operads defined by generators and relations. Here is a way to define such operads in general. To fix notation, assume throughout this section that we work with operads \(O(n), n \geq 1\), of vector spaces.

**Definition 2.5.** An *ideal* in an operad \(O\) is a collection \(I\) of \(S_n\)-invariant subspaces \(I(n) \subset O(n)\), for each \(n \geq 1\), such that whenever \(i \in I\), \(\gamma((\ldots,i,\ldots)) \in I\).

The intersection of an arbitrary number of ideals in an operad is also an ideal, and one can define the ideal generated by a subset in \(O\) as the minimal ideal containing the subset.

**Definition 2.6.** For an operad ideal \(I \subset O\), the *quotient operad* \(O/I\) is the collection \(O(n)/I(n), n \geq 1\), with the structure of operad induced by that on \(O\).

The free operad \(F(S)\) generated by a collection \(S = \{S(n) \mid n \geq 1\}\) of sets, is defined as follows.

\[ F(S)(n) = \bigoplus_{n\text{-trees } T} k \cdot S(T), \]

where the summation runs over all planar rooted trees \(T\) with \(n\) labeled leaves and

\[ S(T) = \text{Map}(v(T), S), \]

the set of maps from the set \(v(T)\) of vertices of the tree \(T\) to the collection \(S\) assigning to a vertex \(v\) with \(\text{In}(v)\) incoming edges an element of \(S(\text{In}(v))\) (the edges are directed toward the root). In other words, an element of \(F(S)(n)\) is a linear combination of planar \(n\)-trees whose vertices are decorated with elements of \(S\). There is a special tree with no vertices:

\[
\begin{array}{c}
\vdots
\end{array}
\]

The component \(F(S)(1)\) contains, apart from \(S(1)\), the one-dimensional subspace spanned by this tree.

The following data defines an operad structure on \(F(S)\).

1. The identity element is the special tree in \(F(S)(1)\) with no vertices.
2. The symmetric group \(S_n\) acts on \(F(S)(n)\) by relabeling the inputs.
3. The operad composition is given by grafting the roots of trees to the leaves of another tree. No new vertices are created.

**Definition 2.7.** Now let \(R\) be a subset of \(F(S)\), i.e., a collection of subsets \(R(n) \subset F(S)(n)\). Let \((R)\) be the ideal in \(F(S)\) generated by \(R\). The quotient operad \(F(S)/(R)\) is called the *operad with generators \(S\) and defining relations \(R\).*
Example 2.6. The associative operad $\mathcal{A}\mathsf{ssoc}$ is the operad generated by a point $S = S(2) = \{•\}$ with a defining relation given by the associativity condition, see Section 2.4.2, expressed in terms of trees.

Example 2.7. The Lie operad $\mathcal{L}\mathsf{ie}$ is the operad also generated by a point $S = S(2) = \{•\}$ with defining relations given by the skew symmetry and the Jacobi identity, see Section 2.4.3.

Example 2.8. The Poisson operad is the operad also generated by a two-point set $S = S(2) = \{•, ◦\}$ with defining relations given by the commutativity and the associativity for simple trees decorated only with $•$’s, the skew symmetry and the Jacobi identity for simple trees decorated with $◦$’s, and the Leibniz identity for binary 3-trees with mixed decorations, see Section 2.4.4.

2.6. Homotopy algebra. The idea of a homotopy “something” algebra is to relax the axioms of the “something” algebra, so that the usual identities are satisfied up to homotopy. For example in a homotopy associative algebra, the associativity identity looks like

$$(ab)c - a(bc)$$

is homotopic to zero.

Or in a homotopy Gerstenhaber (G-) algebra, the Leibniz rule is

$$[a, bc] - [a, b]c \mp b[a, c]$$

is homotopic to zero.

Usually, a homotopy something algebra arises when one wants to lift the structure of a something algebra a priori defined on cohomology to the level of cochains.

This kind of relaxation seems to be too lax for many, practical and categorical, purposes, and one usually requires that the null-homotopies, regarded as new operations, satisfy their own identities, up to their own homotopy. These homotopies should also satisfy certain identities up to homotopy and so on. This resembles Hilbert’s chains of syzygies in early homological algebra. Algebras with such chains of homotopies are called strongly homotopy “something” algebras or “something” $\mathcal{A}_\infty$-algebras.

Operads are especially helpful when one needs to work with something$_\infty$-algebras. We already know that defining the class of something algebras is equivalent to defining the something operad. Thus, if we have an operad $\mathcal{O}$, what is $\mathcal{O}_\infty$, the corresponding strongly homotopy operad? M. Markl’s paper [Mar96] provides a satisfactory answer to this question: the operad $\mathcal{O}_\infty$ is a minimal model of the operad $\mathcal{O}$. A minimal model is unique up to isomorphism. The idea is borrowed from Sullivan’s rational homotopy theory: a minimal model is, first of all, a free resolution of $\mathcal{O}$ in the category of operads of complexes, i.e., an operad of complexes free as an operad of graded vector spaces, whose cohomology is $\mathcal{O}[0]$, the operad $\mathcal{O}$ sitting in degree zero, if it is an operad of vector spaces, or the operad $\mathcal{O}$ sitting in the original degrees, if it is already an operad of graded vector spaces. Second of all, a minimal model must satisfy a minimality condition: its differential must be decomposable.

For certain specific classes of operads, one manages to describe a minimal model explicitly. For example, V. Ginzburg and M. Kapranov [GK94] do it (even earlier than the notion of a minimal model for an operad surfaced) for the so-called Koszul operads. Below we describe an example of such kind, giving rise to the notion of an $A_\infty$-algebra and the $A_\infty$ operad.
2.6.1. $A_{\infty}$-algebras.

**Definition 2.8.** An $A_{\infty}$-algebra, or a strongly homotopy associative algebra, is a complex $V = \bigoplus_{i \geq 0} V^i$ with a differential $d$, $d^2 = 0$, of degree 1 and a collection of $n$-ary operations, called products:

$$M_n(v_1, \ldots, v_n) \in V, \quad v_1, \ldots, v_n \in V, \quad n \geq 2,$$

which are homogeneous of degree $2 - n$ and satisfy the relations

$$dM_n(v_1, \ldots, v_n) - (-1)^n \sum_{i=1}^{n} \epsilon(i) M_n(v_1, \ldots, dv_i, \ldots, v_n)$$

$$= \sum_{k+l=i+1}^{k,l \geq 2} (-1)^{i+l(n-i-l)} \sigma(i) M_k(v_1, \ldots, v_l, M_l(v_{i+1}, \ldots, v_{i+l}), \ldots, v_n),$$

where $\epsilon(i) = (-1)^{|v_1|+\cdots+|v_{i-1}|}$ is the sign picked up by taking $d$ through $v_1, \ldots, v_{i-1}$, $|v|$ denoting the degree of $v \in V$, and $\sigma(i)$ is the sign picked up by taking $M_l$ through $v_1, \ldots, v_l$.

It is remarkable to look at these relations for $n = 2$ and 3:

$$dM_2(v_1, v_2) - M_2(dv_1, v_2) - (-1)^{|v_1|} M_2(v_1, dv_2) = 0,$$

$$dM_3(v_1, v_2, v_3) + M_3(dv_1, v_2, v_3) + (-1)^{|v_1|} M_3(v_1, dv_2, v_3)$$

$$+ (-1)^{|v_1|+|v_2|} M_3(v_1, v_2, dv_3)$$

$$= M_2(M_2(v_1, v_2), v_3) - M_2(v_1, M_2(v_2, v_3)),$$

which mean that the differential $d$ is a derivation of the bilinear product $M_2$ and the trilinear product $M_3$ is a homotopy for the associativity of $M_2$, respectively.

$A_{\infty}$-algebras can be described as algebras over a certain tree operad. This operad is the tree part of the graph complex, which will be the topic of the following sections.

2.6.2. The $A_{\infty}$ operad. Let $A_{\infty}(n)$ be the linear span of the set of equivalence classes of connected planar trees that have a root edge and $n$ leaves labeled by integers 1 through $n$, with vertices of a valence at least 3, $n \geq 2$. For $n = 1$ take one tree with a unique edge connecting a leaf and a root. Let us not include anything for $n = 0$, although one could do that similar to the associative operad case, so that the corresponding notion of an $A_{\infty}$-algebra would have a unit.

We grade each vector space $A_{\infty}(n)$ by defining the degree $|T|$ of a tree $T \in A_{\infty}(n)$ via

$$|T| := v(T) + 1 - n = e(T) + 1 - 2n,$$

where $v(T)$ is the number of vertices and $e(T)$ the number of edges of $T$. Notice that $2 - n \leq |T| \leq 0$ for $n \geq 1$.

Let us define an operad structure on these spaces of trees. The symmetric group acts by relabeling the leaves, and the operad composition is obtained by grafting, as in the examples above, except one needs to take a sign into account. When we graft a tree $T_2$ to the $i$th leaf of a tree $T_1$, the result must be the grafted tree multiplied by a sign, which is $(-1)$ to the power $(e(T_2) - 1)(n)$ (the number of edges to the right of the $i$th leaf in $T_1$), where the edges to the right of a leaf are the edges
which are strictly on the right-hand side of a unique path from the leaf to the root. The reason for the sign above is that grafting must respect the differential, which is introduced below.

**Exercise 7.** Show that this operad is a free operad of vector spaces generated by the following trees for \( n \geq 2 \), which are sometimes called *corollas*.

\[
\delta_n := \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & \cdots \\
& & &  n \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{array}
\]

**Remark 4.** There is no need to mark directions on the edges of a tree: from now on we will assume the edges are directed from top to bottom.

2.6.3. *The tree complex.* The above operad of trees is not yet the \( A_\infty \)-operad, but only its underlying operad of graded vector spaces. The \( A - \infty \)-operad is a DG operad, i.e., an operad of complexes. The DG structure, or a differential, is defined as follows.

Before defining it, we will define the operation of internal-edge contraction on the set of trees.

**Definition 2.9.** We use the notation \( T/e \) to denote the tree obtained from a tree \( T \) by contracting an internal edge \( e \):

\[
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
e & & & \\
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{array}
\]

We can now define a *differential* \( d : A_\infty(n) \to A_\infty(n) \) by the formula

\[
dT := \sum_{T': T = T'/e} \epsilon T',
\]

where \( \epsilon \) is the sign given by counting the number of edges below and to the left of the edge \( e \) in the tree \( T' \), not counting the root.
In particular,

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\vdots \\
n
\end{array}
\]

\begin{align*}
(4) \\
\cdots
\end{align*}

\[
= \sum_{k + \ell = n + 1} \sum_{i=0}^{k-1} (-1)^i 
\]

**Proposition 2.1.** 1. The operator \(d\) satisfies \(d^2 = 0\) and \(\deg d = 1\).
2. The operad structure on \(A_\infty = \{ A_\infty(n) \mid n \geq 1 \}\) is compatible with the differential \(d\):

\[
d(T_1 \circ_i T_2) =dT_1 \circ_i T_2 + (-1)^{|T_1|} T_1 \circ_i dT_2,
\]

i.e., \(A_\infty\) is a DG operad.

**Definition 2.10.** We will call the DG operad \(A_\infty\) the \(A_\infty\) operad.

**Remark 5.** The complex \(A_\infty(n)\) is part of the (cochain) graph complex, see Section 3. A similar operad \(L_\infty\), based on abstract, i.e., nonplanar trees, was introduced by V. Hinich and V. Schechtman [HS93]. The operad \(A_\infty\) is the dual cobar operad in the sense of Ginzburg and Kapranov [GK94] of the associative operad \(\text{Assoc}\). They also show that the cohomology of the operad \(A_\infty\) is the associative operad \(\text{Assoc}\) of Section 2.4.2, implying that \(A_\infty\) is a free, and in fact, minimal, resolution of \(\text{Assoc}\).

The following theorem shows that the \(A_\infty\) operad describes the class of \(A_\infty\)-algebras.

**Theorem 2.2** ([GK94]). An algebra over the \(A_\infty\) operad is an \(A_\infty\)-algebra. Each \(A_\infty\)-algebra admits a natural structure of an algebra over the \(A_\infty\) operad.

**Proof.** For a complex \(V\) of vector spaces with a differential \(d\) of degree 1, \(d^2 = 0\), the structure of an algebra over the operad \(A_\infty\) on \(V\) is a morphism of DG operads:

\[
\phi : A_\infty(n) \to \text{End}_V(n), \quad n \geq 1,
\]

where \(\text{End}_V(n) := \text{Hom}(V^\otimes n, V)\) is the endomorphism operad, which is also a DG operad (with the usual internal differential determined by \(d\)). Given such a morphism \(\phi\), we define the \(n\)-ary product on \(V\):

\[
M_n(v_1, \ldots, v_n) := \phi(\delta_n)(v_1 \otimes \cdots \otimes v_n).
\]

Note that the degree of the product is equal to that of the corolla \(\delta_n\), which is \(2 - n\). Since \(\phi\) is a morphism of DG operads, \(d\phi = \phi d\), and in view of (4), this is equivalent to the identity (3).

Conversely, given a collection of \(n\)-ary brackets on \(V\), \(n \geq 2\), we define a morphism \(\phi\) on the generators \(\delta_n\) by the above formula. The \(A_\infty\) operad is freely
generated by the corollas $\delta_n$, with a differential defined by $\mathcal{L}$, so the mappings $\phi$ define a morphism of DG operads, if the relations $\mathcal{O}$ are satisfied by the $\phi(\delta_n)$'s. Equations $\mathcal{P}$ show that this is the case.

2.7. **Cacti.** One of the very recent applications of operads moves somewhat southwest and goes through cacti rather than trees. However, the word “homology” will mean rational homology throughout this section.

The cacti operad may be used to explain the structure of a BV-algebra on the homology of a free loop space on a compact oriented manifold discovered by M. Chas and D. Sullivan [CS99]. We will be brief and not describe what exactly a BV-algebra structure is. Even if you are new to the subject, you may bear with me by taking a tautological approach: think of BV as the algebraic structure induced on homology from the structure of an algebra over the cacti operad on a topological space. Some of the missing details are also captured by Ralph Cohen’s lecture and his paper with John Jones [CJ01], where they develop a more civilized version of the cacti operad action in the category of spectra and show that Chas-Sullivan’s BV-structure is the same as the BV-structure coming naturally after identifying the homology of the free loop space with the Hochschild homology of the cochain algebra of the target space.

The cacti operad is an operad $c = \{C(n) \mid n \geq 1\}$ of topological spaces. Its $n$th component $C(n)$ for $n \geq 1$ may be described as follows.

$C(n)$ is the set of ordered tree-like configurations of parameterized circles (lobes) of varying (positive) radii, along with a cyclic order of components at the intersection points and the choice of a point “0” on the whole configuration along with the choice of one of the circles on which this point 0 lies. The latter is, of course essential when 0 happens to be an intersection point. The topology on the set of configurations before choosing cyclic orders and marking a point 0 is induced from a natural embedding into $(S^1)^{(\frac{n}{2})}$. The choice of cyclic orders defines a finite covering. After these choices are made, we can define a continuous flow on the cactus which goes along the parameters on the circles and jumps from one component to the next in the cyclic order at the intersection points. Two choices of a zero point are considered close to each other, if they are close in the sense of this flow. Thus the space of cacti with a marked point 0 is an $S^1$-bundle over the configuration space. In short, before marking 0 the topology we are talking about is Gromov’s topology: “two pictures are close to each other, if they appear so”. This principle is quite subtle and does not always work the same way: a classical example of its violation is perhaps the right side mirror on your car, cf. [Spi93].

![Diagram of cacti structure](image-url)
The operad structure on the cacti comes from the following observation. The choice of a point 0 and a component on a cactus gives a natural map from $S^1$ to the cactus. First rescale the radius of $S^1$ to match the sum of the radii of the lobes forming the cactus. Then wind this $S^1$ around the cactus and follow the flow along the lobes, starting with the chosen lobe at 0. Topologically, the constructed map will identify a few groups of points on $S^1$ and therefore will have a degree one on each lobe. Given two cacti and the $i$th lobe in the first one, the operad composition $\circ_i$ will be given by further collapsing the $i$th circle according to the map given by the second cactus.

The following theorem describes both the cacti operad $C$ and how it produces the BV-structure on the homology of a free loop space $LM$ in a compact oriented manifold $M$ of dimension $d$.

**Theorem 2.3.**

1. The cacti operad is homotopy equivalent to the framed little disks operad.
2. The cacti operad $C$ “acts” on $LM$ in the following sense. The diagram

$$C(n) \times (LM)^n \xleftarrow{i} L^{(n)} M \xrightarrow{e} LM,$$

where $L^{(n)} M$ is the space of continuous maps of $n$-component cacti to $M$, induces a composite map

$$H_* (C(n)) \otimes H_* (LM)^n \xrightarrow{i^*} (H_* (L^{(n)} M))[(1-n)d] \xrightarrow{e_*} H_* (LM)[(1-n)d],$$

where $i^*$ denotes the pullback in homology. The collection of such maps for $n \geq 1$ is compatible with the operad structure on $C$.
3. The composite map $e_* i^*$ produces the structure of an algebra over the homology cacti operad $H_* (C)$ on the space $H_* (LM)[d]$.

Combining Statements 1 and 3 of the theorem with a theorem of E. Getzler [Get94] which says that the homology framed little disks operad is the operad describing BV-algebras and checking what the basic operations (the dot product and the BV operator) really are, we obtain the following result.

**Corollary 2.4.** The space $H_* (LM)$ (after an appropriate degree shift) has the natural structure of a BV-algebra, coinciding with the one constructed by Chas and Sullivan.

Cacti of higher dimensions are much subtler and are the topic of an upcoming paper with Sullivan. Here we will just mention one of the applications, which uses a generalization of Getzler’s theorem to higher dimensions by P. Salvatore and N. Wahl [SW01].

**Theorem 2.5** (Sullivan-A.V. [SV01]). Let $M$ be a compact oriented manifold and $S^n M = \text{Map}(S^n, M)$ the sphere space. Then the space $H_* (S^n M)$ (after an appropriate degree shift) is naturally an algebra over the homology framed little $(n+1)$-disks operad. In particular, the homology of the sphere space has the following algebraic structure. For $n$ odd, it is a $BV_{n+1}$-algebra, which is the same as a usual $BV$-algebra, except that the $BV$-operator has degree $n$. For $n$ even, this structure is the same as that of an $(n+1)$-algebra with a differential of degree $2n-1$. 

3. **Graph homology**

Even more interesting things start happening when you pass from trees to graphs. On of these things is that such a classical, analytic and algebraic geometric object as the moduli space of Riemann surfaces miraculously emerges in the horizon the very moment you say the word “graphs”. Out of the numerous versions of the graph complex, we have chosen the ribbon one, which is most closely related to the moduli spaces of Riemann surfaces. The notion of a ribbon graph and the graph complex are due to R. Penner [Pen86, Pen88], who used the term “fatgraph”. The ribbon graph complex is a generalization of the planar tree complex we considered in the previous section, except that we no longer allow any free legs. Other versions of the graph complex include those which do not require cyclic orders at vertices, see M. Culler and K. Vogtmann [CV86], or, on the contrary, have more complicated decorations at vertices, see Getzler-Kapranov’s Feynman transform [GK98]. There is also a dual version of the graph complex, producing graph cohomology. Kontsevich [Kon93], who noticed a common pattern in Penner’s and Culler-Vogtmann’s work and related different versions of the graph complex to different fundamental types of algebras (or operads that describe them), had an enormous influence on the subject.

### 3.1. The graph complex

The right analogue of a planar tree is a **ribbon graph**, which is a nonempty connected finite cell complex \( \Gamma \) of dimension one with the choice of a cyclic order on the set of half-edges around each vertex. We also require that the valences of vertices must be greater than or equal to 3. A **boundary component** of a ribbon graph \( \Gamma \) is a cyclic sequence \( \vec{e}_0, \vec{e}_1, \ldots, \vec{e}_q = \vec{e}_0 \) of directed edges (i.e., edges with directions chosen on each of them) of \( \Gamma \), so that for each pair \((\vec{e}_i, \vec{e}_{i+1})\) of two subsequent edges, the tail of \( \vec{e}_{i+1} \) is the half-edge that follows the head of \( \vec{e}_i \) in the cyclic order at a common vertex of \( \vec{e}_i \) and \( \vec{e}_{i+1} \). To have a more direct connection with moduli spaces of Riemann surfaces with labeled boundary components, we will assume that graphs under consideration will have their boundary components labeled. We identify ribbon graphs which differ by an isomorphism, which is just a cellular homeomorphism preserving the cyclic orders at vertices and fixing each of the boundary components (but not necessarily each edge forming a boundary component). The **group of automorphisms** of a ribbon graph is denoted by \( \text{Aut}_\partial(\Gamma) \). This group is infinite, but has a finite number of connected components, \( \pi_0(\text{Aut}_\partial(\Gamma)) \), which is what would usually be called the group of automorphisms of a graph. An **orientation** on a graph \( \Gamma \) is an orientation on the vector space \( \mathbb{R}^{e(\Gamma)} \), where \( e(\Gamma) \) is the set of edges of \( \Gamma \).

For each \( m \geq 2 \), let \( G_m \) be the free abelian group generated by the set of isomorphism classes of oriented ribbon graphs \( \Gamma \) as above with \( m \) edges, modded out by the defining relations \( \Gamma + (-\Gamma) = 0 \), where \(-\Gamma\) is the same graph as \( \Gamma \), but with the opposite orientation.

Define a differential \( d : G_m \to G_{m-1} \), so that \( d^2 = 0 \), as follows:

\[
d\Gamma := \sum_{\substack{\text{edges } e \in e(\Gamma) \\
\text{which are not loops}}} \Gamma/e,
\]

where \( \Gamma/e \) is the ribbon graph obtained from \( \Gamma \) by contracting edge \( e \) to a point. The cyclic order at the new vertex created by merging the two ends of \( e \) is obtained...
by a natural insertion. The orientation is defined by the natural isomorphism

\[ \Lambda^{\max}_{\mathbb{R}e} (\Gamma/e) \rightarrow \Lambda^{\max}_{\mathbb{R}e} (\Gamma), \]

\[ \omega \mapsto \omega \wedge e. \]

The complex \( G_\bullet \) with the differential \( d \) is called the graph (chain) complex and its homology is called graph homology.

Obviously, the differential preserves the number \( n \) of boundary components, as well as the Euler characteristic \( \chi(\Gamma) := v(\Gamma) - e(\Gamma) \). It also preserves the genus \( g \) defined by the equation \( \chi = 2 - 2g - n \). The solution \( g = 1 - (\chi + n)/2 \) is a nonnegative integer, because if we glue in \( n \) disks into the boundary components, we will get a compact orientable topological surface whose Euler characteristic equals \( \chi + n = 2 - 2g \) for some nonnegative integer \( g \). Thus, the graph complex splits into the direct sum \( G_\bullet = G^{g,n}_\bullet \) of subcomplexes \( G^{g,n}_\bullet \) with a fixed genus \( g \) and a number \( n \) of boundary components, \( g \geq 0, n \geq 1 \).

3.2. Metric ribbon graphs and moduli spaces. The graph complex above is in fact the chain complex of a certain cell complex, that of metric ribbon graphs. A metric on a ribbon graph is an assignment of a positive real number, a length, to each edge of the graph. Obviously, the space of isomorphism classes of metric ribbon graphs with an underlying graph \( \Gamma \) is the space \( \mathbb{R}e^\bullet_{\bullet}(\Gamma)/\text{Aut}_\partial(\Gamma) = \mathbb{R}e^\bullet_{\bullet}(\Gamma)/\pi_0(\text{Aut}_\partial(\Gamma)) \), which is, in fact, an orbifold. For different ribbon graphs \( \Gamma \) of a fixed genus \( g \) and a number \( n \) of boundary components, these orbifolds glue together by identifying metric graphs some of whose lengths degenerate to zero with the metric graphs obtained by contracting the zero length edges. Thus, these orbifolds become what is called the rational cells of an orbifold, the space \( G^{\text{met}}_{g,n} \) of metric ribbon graphs.

If we forget the orbifold structure, these rational cells are just cells of a topological space, but not a cell complex, because only part of the boundary of each cell, the part with lengths approaching zero, is glued up to other cells. To get an honest cell complex, we may take the one-point compactification of the space of metric ribbon graphs, thus, gluing in a single point to all cells as lengths tend to positive infinity. This gives a nonorbifold-type singularity, and we get a compact smooth orbifold \( \overline{G}_{g,n} \) with one singular point. It may also be viewed as an ordinary cell complex with a base point. This space was designed with the following obvious proposition in mind.

**Proposition 3.1.** The chain complex computing the rational reduced homology of the orbifold \( \overline{G}^{\text{met}}_{g,n} \) is isomorphic to the ribbon graph complex \( G^{g,n}_\bullet \otimes \mathbb{Q} \).

**Remark 6.** We had to take the rational coefficients to make sure that we were computing the orbifold homology. Over the integers, it would be the computation of the ordinary homology of the underlying space, which would be somewhat misleading.

The relevance of the graph complex and the metric ribbon graph space is explained by the following theorem, which is one of the deepest results in mathematics in the past twenty years.

**Theorem 3.2** (Harer-Mumford-Thurston, Penner, see a remark below). For \( g, n > 0 \) (or \( g = 0 \) and \( n > 2 \)), there is an orbifold isomorphism

\[ \mathcal{M}_{g,n} \times \mathbb{R}^n_+ \cong G^{\text{met}}_{g,n}, \]
where $\mathcal{M}_{g,n}$ is the moduli space of compact smooth Riemann surfaces of genus $g$ with $n$ labeled punctures.

**Remark 7.** The idea of such combinatorial description of the moduli space belongs to D. Mumford and W. Thurston. Mumford’s approach, which was realized by J. Harer [Har86], uses the theory of Strebel differentials on a Riemann surface, an analytic result producing a unique meromorphic quadratic differential $q$ in such a way that a metrized ribbon emerges as the set of critical horizontal trajectories of $q$ and the residues of $\sqrt{q}$ at punctures give elements in $\mathbb{R}^n_+$, see more detail in E. Looijenga [Loo95], R. Hain-Looijenga [HL97], and M. Mulase-M. Penkava [MP98]. Penner [Pen87] came up with a quite different way to obtain the above result, in which the points of $\mathbb{R}^n_+$ on the left-hand side were interpreted as the hyperbolic lengths of horocycles attached to the punctures in a hyperbolic model of the Riemann surface with the punctures removed. Kontsevich [Kon92] considered a natural compactification of the space of metric ribbon graphs and compared it with Deligne-Mumford’s one for $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$.

Combining this theorem with Proposition 3.1, one gets the following purely combinatorial way of computing the rational homology of the moduli spaces.

**Corollary 3.3.** The homology of the ribbon graph complex $G^{g,n}_\otimes \mathbb{Q}$ is isomorphic to the rational reduced homology of the one-point compactification of $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$.

Applying Poincaré-Lefschetz duality, we see that

$$H_k(G^{g,n}_\otimes \mathbb{Q}) = H^{6g-6+3n-k}(\mathcal{M}_{g,n}; \mathbb{Q}).$$

The above corollary has been used to prove Harer’s stability theorem, estimate the homological dimension of $\mathcal{M}_{g,n}$, and compute its virtual (i.e., orbifold) Euler characteristic, see Hain-Looijenga [HL97]. Unfortunately, the combinatorics of the graph complex is complicated enough not to produce more general, explicit computations of its homology as of yet. However, this combinatorial model was one of the key elements in Kontsevich’s proof [Kon92] of the Witten conjecture [Wit91] on the intersection theory on moduli spaces and the KP hierarchy.

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