Scalar Charges and Thermodynamics of Asymptotically Flat Dyonic Black Holes

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Abstract

We investigate the thermodynamics of dyonic hairy black holes in flat spacetime when the asymptotic value of the scalar field is not fixed. We use the quasilocal formalism of Brown and York and corresponding boundary terms that make the variational principle well defined to prove that the scalar charges do not contribute to the first law of thermodynamics. We also provide a unified picture of obtaining exact solutions by comparing two different methods and discuss in detail the ansatz used in each coordinate system and the relation between them.
1 Introduction

Einstein-Maxwell-dilaton theory appears naturally as a consistent truncation of low energy limit of string theory. The dilaton has a specific non-minimal coupling with the Maxwell term and the first exact hairy black hole solutions were constructed in [1], [2], [3], and [4]. Since in string theory the dilaton is a modulus whose expectation value controls the string dimensionless constant, $g_s$, it makes sense to understand how a variation of its asymptotic value affects the thermodynamics of hairy black holes. Such a study was performed in [5] with a surprising result. That is, the first law of hairy black hole thermodynamics should be supplemented with contributions from the scalar fields. This result, which is based on implementing the Arnowitt-Deser-Misner (ADM) formalism [8, 9] to hairy black hole thermodynamics, can be explicitly checked for various exact solutions. However, from a physical point of view, the modification of the first law is puzzling. The scalar fields are characterized by charges that are not conserved, particularly in all known solutions there is no integration constant associated to the classical hair and that is why it is refereed to as ‘secondary hair’. Based on some earlier suggestions on how to solve this puzzle [10, 11], Hajian and Sheikh-Jabbari have used the phase space method to show in [12] that, indeed, the asymptotic value of the dilaton, $\phi_\infty$, is generally a redundant parameter. But if the non-conserved scalar charges are not allowed to appear in the first law, what exactly is inconsistent with the original proposal of [5]? This question was completely answered in [13], where a general consistent variational principle was provided when the asymptotic value of the dilaton varies. In this case, the quasilocal energy computed by using the Brown-York formalism [17] gets a new contribution due to the boundary term associated with the scalar field. This is consistent with the interpretation from string theory where a variation of the vacuum expectation value for the dilaton is equivalent with a change of the string coupling. This new method was explicitly checked for some exact electrically charged hairy black holes in [13] and, later on, for several different examples in [18].

1 A similar result was found for asymptotically AdS black holes; see [6, 7].
2 Another approach were also presented in [14] and, for asymptotically AdS spacetime, in [15] and [16].
In this paper, we extend the work of [13] to dyonic black holes. This is particularly important because, unlike the electrically charged case, for the dyonic black holes the extremal limit is well defined (see, e.g., [19] for a general discussion of different classes of asymptotically flat hairy black hole solutions) and so the canonical ensemble is well defined. We prove again, as expected, that the scalar charges do not appear explicitly in the first law as independent contributions. We also provide a detailed discussion on how two different methods can be used to obtain exact solutions and check the consistency of changing the coordinates from the frame used in [20, 21, 22, 23, 24] to the canonical frame [1] and [2].

This paper is organised as follows. In Section 2, we describe a method used to obtain exact solutions in a non-standard radial coordinate, and we employ it to reobtain well known solutions for both electrically and dyonic hairy black holes. Also, we discuss the connection between the solutions obtained by this method and the one of the canonical frame (with the usual radial coordinate). In Section 3, we analyse the thermodynamic of these solutions when the asymptotic value of the dilaton is allowed to vary. We compute the on-shell action, the quasilocal stress tensor, the conserved energy, and verify the statistical quantum relation and first law of black hole thermodynamics.

2 Exact dyonic hairy black hole solutions

2.1 Method and setup

In this section, we start by briefly describing the method proposed in [20, 21], and also developed and used in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31], to obtain exact black hole solutions. Similar solutions were found using a different method in [32, 33]. We are going to refer to this method and its corresponding coordinate system as the ‘ $x$−frame’. We are interested in asymptotically flat hairy black hole solutions and the asymptotic value of the scalar field is not fixed in our analysis. Let us consider the Einstein-Maxwell-dilaton action

$$ I [g_{\mu \nu}, A_\mu, \phi] = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g} \left[ R - z(\phi) F^2 - \frac{1}{2} (\partial \phi)^2 \right] $$

where $F^2 \equiv F_{\mu \nu} F^{\mu \nu}$, $(\partial \phi)^2 \equiv g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$, $R$ is the Ricci scalar, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the Maxwell (gauge) field and $z(\phi)$ is the coupling function between the scalar and gauge fields. In our conventions, $\kappa = 8\pi$ and $G_N = c = 1$. The corresponding equations of motion are

$$ E_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \kappa \left( T_{\mu \nu}^{EM} + T_{\mu \nu}^\phi \right) = 0, $$

$$ \partial_\mu \left[ \sqrt{-g} z(\phi) F^{\mu \nu} \right] = 0, $$

$$ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \phi \right) = \frac{dz(\phi)}{d\phi} F^2 $$

where the energy–momentum tensors for the gauge and scalar fields have the following expressions:

$$ T_{\mu \nu}^{EM} = \frac{2z(\phi)}{\kappa} \left[ F_{\mu \alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu \nu} F^2 \right], \quad T_{\mu \nu}^\phi = \frac{1}{2\kappa} \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 \right] $$

According to the method in the $x$−frame, let us consider the following static spherically symmetric ansatz for the metric tensor with a conformal factor

$$ ds^2 = \Omega(x) \left[ -f(x) dt^2 + \frac{r^2 dx^2}{f(x)} + d\theta^2 + \sin^2 \theta d\phi^2 \right] $$

$$ x $$
where the coordinate $x$ plays the role of a radial coordinate, $\eta$ is introduced as a parameter of the solution, and let us consider the following ansatz for the gauge field representing the dyonic sector

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = -\frac{q}{z(\phi)} dt \wedge dx + p \sin \theta d\theta \wedge d\varphi$$

(7)

where $q$ and $p$ are the electric and magnetic parameters of the solution. Note that the given ansatz satisfy the Maxwell equation (3). Now, by using the combination of Einstein’s equations $E_x^x - E_t^t$, one obtains

$$\phi'^2 = -2\Omega'' \Omega + 3 \left( \frac{\Omega'}{\Omega} \right)^2$$

(8)

which can be integrated to obtain the expression for the scalar field, provided an appropriate choice of $\Omega(x)$. Once $\phi = \phi(x)$ is known, the remaining equations of motion can be further integrated to get the metric function $f(x)$ and then the full solution.

### 2.2 Electrically charged solutions

#### 2.2.1 The solution in $x$–frame

To illustrate the method in a concrete situation, let us to consider the following coupling function

$$z(\phi) = e^{a\phi}$$

(9)

where $a$ is an arbitrary constant parametrizing the theory and let us turn off the magnetic charge. The solution we are going to reobtain for any coupling constant, $a$, is very well known \[1\], \[2\] and \[3\].

First, to conveniently decouple the equations of motion, let us separate the conformal factor as $\Omega(x) = \Omega_1(x)\tilde{\Omega}(x)$ and define a new coordinate $u \equiv \int \tilde{\Omega}(x) dx$. Then, by specifying

$$\Omega_1(u) = \frac{1}{\eta^2(u - 1)^2}$$

(10)

the ansatz (6) and (7) can be put as

$$ds^2 = \frac{1}{\eta^2(u - 1)^2} \left[ -f(u)\tilde{\Omega}(u) dt^2 + \frac{\eta^2 du^2}{f(u)\tilde{\Omega}(u)} + \tilde{\Omega}(u) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]$$

(11)

$$F = -q e^{-a\phi(u)} \frac{\tilde{\Omega}(u)}{\tilde{\Omega}(u)} dt \wedge du$$

(12)

One aspect that can be observed at this level is that, in this coordinate system, the boundary of spacetime is located at $u = 1$, where the conformal factor in the metric diverges (in the assumption that $\tilde{\Omega}$ remains finite at the boundary). The fact that $u$ can approach to 1 from the left or from the right side leads to recognize that there are two branches for this solution, corresponding to two disconnected spacetimes, one where $u$ ranges as $0 < u < 1$, which is called the negative branch, and other where $u > 1$, called the positive branch. \[3\]

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\[3\] The relation with the Schwarzschild radial coordinate is manifest in the asymptotic limit, where $\Omega(x) \approx r^2$.

\[4\] For the specific values $a = \{1, \sqrt{3}\}$, the model can be embedded in supergravity \[1\] \[2\] \[4\] \[5\] \[34\] \[35\].

\[5\] In general, these branches have actually different properties. For instance, it was recently shown that when the scalar field is provided with a non-trivial self interaction, there are stable thermodynamically black holes only in the positive branch \[36\]. Also, in the extended phase space of black hole chemistry \[37\] \[38\], where the cosmological constant is considered as a pressure term, there are non-trivial criticality phenomena in the positive branch for both canonical and grand canonical ensembles \[39\].
Note, also, that the physical charge $Q$, up to a global sign, can be obtained by the Gauss Law, i.e., by integrating the Maxwell equation on the 2-sphere at infinity.

$$Q = \frac{1}{4\pi} \oint_{s_\infty} e^{a \phi} \star F = \frac{1}{4\pi} \oint \sqrt{-g} e^{a \phi} F^{tu} d\theta \wedge d\varphi = \frac{q}{\eta}$$ (13)

Now, in order to solve the equations of motion, let us consider the combination $E_u^u - E_t^t$, which gives

$$\phi'^2 = \left( \frac{\tilde{\Omega}'}{\tilde{\Omega}} \right)^2 - \frac{2\tilde{\Omega}''}{\tilde{\Omega}}$$ (14)

where prime symbol means $d/du$. The function $\tilde{\Omega}(u)$ can be chosen in two different ways, which defines the two families of solutions, similar as in [40]. We call family 1 the family of solutions obtained by picking

$$\tilde{\Omega}(u) = \exp \left[ -a \left( \phi - \phi_\infty \right) \right],$$ (15)

and we call family 2 the family obtained by picking

$$\tilde{\Omega}(u) = \exp \left[ \frac{1}{a} \left( \phi - \phi_\infty \right) \right].$$ (16)

where $\phi_\infty$ is the asymptotic value of the scalar field.

**Family 1:** By integrating the equation (14), using (15), we obtain the following expression for the scalar field

$$\phi(u) = \phi_\infty - \frac{2a}{1 + a^2} \ln(u)$$ (17)

and, by using (15) and (17), the remaining independent Einstein’s equation can be integrated to get the last unknown metric function $f(u)$

$$f(u) = (u - 1)^2 u^{-\frac{a^2 + 1}{a^2 + 1}} \eta^2 \left[ (u - 1) \left( 1 + a^2 \right) \left( q e^{-\frac{3}{2} a \phi_\infty} \right)^2 + 1 \right]$$ (18)

Notice that $\lim_{u=1} (-g_{tt}) = 1$ and $\lim_{u=1} \phi(u) = \phi_\infty$ as expected for asymptotic flatness and the boundary is located at $u = 1$, and that, as pointed out in [3], that there is only one horizon,

$$u_+ = 1 - \frac{e^{a \phi_\infty}}{(1 + a^2)q^2}$$ (19)

satisfying $h(u_+) = 0$.

**Family 2:** By integrating the equation (14), using (16), we obtain the following expression for the scalar field

$$\phi(u) = \phi_\infty + \frac{2a}{1 + a^2} \ln(u)$$ (20)

and, by using (16) and (20), the remaining independent Einstein’s equation can be integrated to get

$$f(u) = (u - 1)^2 u^{-\frac{a^2 + 1}{a^2 + 1}} \eta^2 \left[ -(u - 1) \left( 1 + a^2 \right) \left( q e^{-\frac{3}{2} a \phi_\infty} \right)^2 + u \right]$$ (21)

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6The convention is $\star F = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\alpha\beta} \partial F^{\mu\nu} dx^\alpha \wedge dx^\beta$, where $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita symbol.

7Note that the constants of integration were already assumed to be $\eta$ and $q$, therefore, any other constant appearing after integrating the equations of motion must be a suitable combination of $\eta$ and $q$. 

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with also satisfies $\lim_{u \to 1} (-g_{tt}) = 1$ and $\lim_{u \to 1} \phi = \phi_\infty$, and has only one horizon

$$u_+ = \frac{q^2 e^{-a\phi_\infty} (1 + a^2)}{q^2 e^{-a\phi_\infty} (1 + a^2) - 1}$$  \hfill (22)$$

We would like to briefly comment on these two families. Let us follow the convention in which ‘negative branch’ describes the domain $\{0 < u < 1, \phi < \phi_\infty\}$, and ‘positive branch’ describes the domain $\{u > 1, \phi > \phi_\infty\}$. From the solutions for the scalar field in both families \[17\] and \[20\], it follows that family 1 is then associated with $a \leq 0$, and in family 2 associated with $a > 0$. Therefore, it is appropriate to say that family 1 is adapted to the cases with $a \leq 0$, and family 2 to $a > 0$.

Finally, it is worth to notice that, for family 1, black hole configurations exist only in the negative branch, where the coupling function, $e^{\phi}$, can take arbitrarily large values. On the other hand, for family 2, black hole configurations exist only in the positive branch, where, again, the coupling function can take arbitrarily large values.

### 2.2.2 Comparison with the canonical frame

Now, we would like to use a suitable change of coordinate in order to rewrite the solution in the canonical frame, as in [3], by using the standard radial coordinate, $r$. For concreteness, let us consider the family 1 (negative branch). The change of coordinates is

$$u = 1 - \frac{1}{\eta r}$$  \hfill (23)$$

where $\eta$ is assumed positive definite. By noting that $g_{uu} = g_{rr} (dr/du)^2$, the solution can be easily rewritten in the canonical radial coordinate as

$$ds^2 = -a(r)^2 dt^2 + \frac{dr^2}{a(r)^2} + b(r)^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)$$  \hfill (24)$$

where

$$a^2(r) \equiv \frac{(r - r_+)(r - r_0)^{1 + a^2}}{r^{1 + a^2}}, \quad b^2(r) \equiv r^2 \left( 1 - \frac{r_0}{r} \right)^{2a^2}$$  \hfill (25)$$

It is convenient to introduce the so called ‘scalar charge’ $\Sigma$ as the component in the subleading term in the asymptotic expansion of the scalar field in the canonical coordinate, $\phi(r) = \phi_\infty + \frac{\Sigma}{r} + \mathcal{O}(r^{-2})$. In this case, the scalar field is expanded as

$$\phi(r) = \phi_\infty + \frac{2a}{(1 + a^2)\eta r} + \mathcal{O}(r^{-2})$$  \hfill (26)$$

and $\Sigma = \frac{2a}{(1 + a^2)\eta r}$. The black hole outer horizon $r_+$ and the location of the central singularity $r_0$ are then given by

$$r_+ \equiv \frac{q^2 \Sigma (1 + a^2)^2}{2a e^{a\phi_\infty}}, \quad r_0 \equiv \frac{\Sigma (1 + a^2)}{2a},$$  \hfill (27)$$

respectively. Since in this solution $\Sigma < 0$ and $a < 0$, we have that $r_0$ is positive definite. To see that $r_0$ is actually the central singularity, and not an inner horizon, observe that both the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ and the scalar field diverge at the limit $r \to r_0$

$$R = \frac{2a^2 (r - r_+)^{2a^2} - 2a^2 (a^2 + 2)}{a^2 + 1} \frac{r_0^2}{(a^2 + 1)^2 (r - r_0)^{1 + a^2}}, \quad \phi(r) = \phi_\infty - \frac{2a}{1 + a^2} \ln \left( 1 - \frac{r_0}{r} \right)$$  \hfill (28)$$

Finally, notice that the limit $a = 0$ corresponds to the Reissner-Nordström limit. In that particular case, the scalar field becomes a constant, the Ricci scalar automatically vanishes and, therefore, $r_0 = 1/\eta$ becomes the inner horizon.
2.3 The dyonic solution, \( a = -1 \)

2.3.1 The solution obtained in the \( x \)-frame

In this subsection, we consider the case \( z(\phi) = e^{\pm \phi} \) with both electric and magnetic charges turned on. We are going to reobtain the exact solution in [10], by using the method shown before. Let us, again, focus on the family 1 (\( a = -1 \))\(^8\), and consider the ansatz

\[
ds^2 = \Omega(u) \left[ -f(u)dt^2 + \frac{\eta^2 du^2}{w^2 f(u)} + d\theta^2 + \sin^2 \theta d\varphi^2 \right]
\]

\[
F = -\frac{qe^{\phi}}{u} dt \wedge du + p \sin \theta d\theta \wedge d\varphi
\]

where \( \eta, q \) and \( p \) are the three constants of integration (the parameters) of the solution. They are related with the physical electric and magnetic charges, \( P \) and \( Q \), which are

\[
Q = \frac{1}{4\pi} \int_{s^2} e^{-\phi} F = \frac{q}{\eta}, \quad P = \frac{1}{4\pi} \int_{s^2} F = p,
\]

respectively. The combination of the Einstein’s equations \( E^t_t - E^u_u \) leads to the equation

\[
\phi'^2 = 3 \left( \frac{\Omega'}{\Omega} \right)^2 - \frac{2}{u} \left( \frac{\Omega''}{\Omega} + \frac{\Omega'}{\Omega} \right)
\]

which, provided the conformal factor

\[
\Omega(u) = \frac{u}{\eta^2(u-1)^2},
\]

can be integrated to obtain the following expression for the scalar field

\[
\phi(u) = \phi_{\infty} + \ln(u)
\]

Now, the remaining independent equation of motion is solved by the following metric function

\[
f(u) = \frac{\eta^2(u-1)^2}{u^2} \left[ u + 2u(u-1) \left( qe^{\frac{1}{2} \phi_{\infty}} \right)^2 - 2\eta^2(u-1) \left( pe^{-\frac{1}{2} \phi_{\infty}} \right)^2 \right]
\]

Unlike the purely electrically charged solution, the horizon equation \( f(u) = 0 \), in this case, implies the existence of two horizons,

\[
u_{\pm} = \frac{1}{2} + \frac{2\eta^2 e^{-\phi_{\infty}} p^2 - 1}{4q^2 e^{\phi_{\infty}}} \pm \left( \mp \right) \sqrt{\frac{4e^{-2\phi_{\infty}} p^4 q^4 + 4e^{2\phi_{\infty}} q^4 - 2(2qp\eta)^2 - 4\eta^2 e^{-\phi_{\infty}} p^2 - 4q^2 e^{\phi_{\infty}} + 1}{4q^2 e^{\phi_{\infty}}}}
\]

The choice \( \pm \) or \( \mp \) depends on branch being considered. In negative branch \( (0 < u < 1) \), one must ensure that \( u_- < u_+ \) and, in positive branch \( (1 < u < \infty) \), \( u_+ < u_- \).

\(^8\)The analysis is identical for the family 2 \( (a = +1) \).
2.3.2 Comparison with the canonical frame

Let us consider the same change of coordinate than for the purely electrically charged solution, (23). In the same manner, the solution can be rewritten as

\[ ds^2 = -a^2(r) dt^2 + \frac{dr^2}{a^2(r)} + b^2(r) (d\theta^2 + \sin^2 \theta d\varphi^2) \]  

where

\[ a^2(r) = \frac{(r - r_+)(r - r_-)}{b^2(r)}, \quad b^2(r) = r (r + \Sigma) \]  

By asymptotically expanding the scalar field in \( r \) coordinates, one gets \( \Sigma = -\frac{1}{\eta} \), and the inner and outer horizon are given by \( r_{\pm} = 1 - \frac{1}{\eta u_\pm} \), or, explicitly,

\[ r_{\pm} = \frac{\Sigma q^2}{e^{\phi_{\infty}}} - \frac{p^2 e^\phi_{\infty}}{\Sigma} - \frac{\Sigma}{2} \pm \frac{1}{2\Sigma} \sqrt{\frac{4(q\Sigma)^4}{e^{2\phi_{\infty}}} - \frac{8(qp\Sigma)^2}{\Sigma e^{2\phi_{\infty}}} + 4e^{\frac{1}{2}\phi_{\infty} p} \Sigma^4 - 4(\Sigma e^{\frac{1}{2}\phi_{\infty} p})^2 + \Sigma^4} \]  

Note that, since \( \Sigma \) is negative, both the Ricci scalar and the scalar field diverge in the limit \( r = -\Sigma \), which is the location of the central singularity,

\[ R = \frac{\Sigma^2 (r - r_+) (r - r_-)}{2r^3 (r + \Sigma)^3}, \quad \phi(r) = \phi_{\infty} + \ln \left( 1 + \frac{\Sigma}{r} \right) \]  

2.4 The dyonic solution, \( a = -\sqrt{3} \)

Since this solution is algebraically more complicated, let us consider the well known exact solution [34] as written in [10]. The metric is

\[ ds^2 = -a(r)^2 dt^2 + \frac{dr^2}{a(r)^2} + b(r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \]  

\[ a(r)^2 = \frac{(r - r_+)(r - r_-)}{\sqrt{AB}}, \quad b(r)^2 = \sqrt{AB}, \quad \phi(r) = \phi_{\infty} + \frac{\sqrt{3}}{2} \ln \left( \frac{A}{B} \right) \]  

where

\[ r_{\pm} = M \pm c, \quad c = \frac{1}{2} \sqrt{4M^2 + \Sigma^2 - 4q^2 e^{\sqrt{3}\phi_{\infty}} - 4p^2 e^{-\sqrt{3}\phi_{\infty}}} \]  

and

\[ A = (r - r_{A_+}) (r - r_{A_-}), \quad B = (r - r_{B_+}) (r - r_{B_-}) \]  

with

\[ r_{A_\pm} = -\frac{\Sigma}{2\sqrt{3}} \pm pe^{-\frac{1}{2}} e^{\sqrt{3}\phi_{\infty}} \sqrt{\frac{2\Sigma}{\Sigma + 2\sqrt{3}M}}, \quad r_{B_\pm} = \frac{\Sigma}{2\sqrt{3}} \pm qe^{\frac{1}{2}} e^{\sqrt{3}\phi_{\infty}} \sqrt{\frac{2\Sigma}{\Sigma - 2\sqrt{3}M}} \]  

and the constraint

\[ \frac{1}{6} \Sigma = \frac{p^2 e^{-\sqrt{3}\phi_{\infty}}}{\Sigma + 2\sqrt{3}M} + \frac{q^2 e^{\sqrt{3}\phi_{\infty}}}{\Sigma - 2\sqrt{3}M} \]  

The gauge field, on the other hand, is

\[ F = \frac{qe^{\sqrt{3}\phi}}{b^2} dt \wedge dr + p \sin \theta d\theta \wedge d\varphi \]
Now, let us schematically show the connection between the solution in canonical frame, as given above, with the $x$–frame. The change of coordinates that connects them is

$$u = \left(\frac{A}{B}\right)^{\pm 1}$$

(48)

so that the scalar field can be rewritten as

$$\phi(u) = \phi_\infty \pm \frac{\sqrt{3}}{2} \ln(u)$$

(49)

The choice of $\pm$ sign defines the two families of the solution in $x$–frame, according to the previous discussions. If, by considering the change of coordinates (48), we define $\Omega(u) = u^{\pm \frac{1}{2}} B(u)$, then, the metric can be rewritten as

$$ds^2 = \Omega(u) \left(-h(u)dt^2 + \frac{\eta^2 du^2}{f(u)} + d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(50)

where $h(u)$ and $f(u)$ must be determined in order to complete the relation between the $x$–frame and the canonical frame.

3 Thermodynamics and scalar charges

In this section, we are going to use the counterterm method and the quasilocal formalism for the asymptotically flat black hole solutions presented before, in order to compute the conserved energy and obtain the regularized on-shell action. This will allow to verify the quantum statistical relation and the first law of black hole thermodynamics, under the consideration that $\phi_\infty$ is allowed to vary.

According with the variational principle when the asymptotic value of the scalar field is not fixed, the gravitational action for Einstein-Maxwell-dilaton theories, which consists of the bulk part of the action, the Gibbons-Hawking boundary term $I_{GH}$ and the gravitational counterterm $I_{ct}$, should be supplemented with a boundary term for the scalar field $I_\phi$, as

$$I = I_{\text{bulk}} + I_{GH} + I_{ct} + I_\phi$$

(51)

The gravitational counterterm that cancels infrared divergences in the theory, regularizing the action, is [41, 42, 43, 44]

$$I_{ct} = -\frac{1}{\kappa} \int_{\partial M} d^3y \sqrt{-h} \sqrt{2R(3)}$$

(52)

where $y^a = (t, \theta, \varphi)$ are the coordinates on the boundary $\partial M$, which is the hypersurface $u = \text{const}$. The boundary term for scalar field, $I_\phi$, is [45]

$$I_\phi = -\frac{1}{2\kappa} \int_{\partial M} d^3y \sqrt{-h} \left[ \frac{(\phi - \phi_\infty)^2}{\Sigma^2} W(\phi_\infty) \right]$$

(53)

where the function $W$ is defined by means of the general boundary condition $\Sigma \equiv \frac{dW}{d\phi_\infty}$, which is similar with the one proposed in [45] for AdS black holes. The action (54) corresponds with the grand canonical ensemble, where $\delta A_\mu|_{\partial M} = 0$ is the boundary condition for the gauge potential. This corresponds to fixing the electric potential $\Phi \equiv A_\mu|_{\partial M} - A_\mu|_{\text{horizon}}$. The canonical ensemble, given by

\[\text{A concrete relation } \Sigma = \Sigma(\phi_\infty) \text{ in not required and we can work in the general situation, } W = \int \Sigma(\phi_\infty) d\phi_\infty.\]
the boundary condition \( \delta (z(\phi) \ast F)|_{\partial M} = 0 \), which fixes the electric charge \( Q \), is obtained by added to the action a new boundary term\(^{10}\)

\[
I_A = -\frac{2}{\kappa} \int_{\partial M} d^3 y \sqrt{-h} \delta(\phi) n_\mu F^{\mu \nu} A_\nu
\]

where \( n_\mu \) is the normal unit to the boundary.

The quasilocal formalism of Brown and York\(^ {17}\) provides a powerful method to obtain conserved quantities in general relativity. According to this, the conserved energy \( E \) is the conserved quantity associated with the time-translational symmetry of the metric tensor, given by the Killing vector \( \xi = \partial/\partial t \). If one consider a quasilocal surface with a stress tensor defined as

\[
\tau_{ab} \equiv 2 \sqrt{-h} \delta I \delta h_{ab}
\]

where \( I \) is the total action given by \(^{51}\), then, the conserved energy is

\[
E = \oint_{\Sigma} d^2 \sigma \sqrt{\sigma} n^a \tau_{ab} \xi^b
\]

where \( d^2 \sigma = d\theta d\varphi \) (for the spherical cross section) and \( n_a = (-g^{tt})^{-1/2} \delta_a^t \) is the normal unit to the hypersurface \( t = const \) at the asymptotic limit. The concrete expression for the quasilocal stress tensor in this case is \(^{46}\) is given by

\[
\tau_{ab} = \frac{1}{\kappa} \left[ K_{ab} - h_{ab} K - \Psi \left( R_{ab}^{(3)} - R_{ab}^{(3)} h_{ab} \right) - h_{ab} \Box \Psi + \Psi \tau_{ab} \right] + \frac{h_{ab} (\phi - \phi_\infty)^2 W}{2\kappa} \Sigma^2
\]

where \( \Psi \equiv \left( \frac{1}{2\kappa} R^{(3)} \right)^{-1/2} \).

### 3.1 Electrically charged solutions

In this subsection, we are going to consider the electrically charged solution given in subsection 2.2 (family 1, negative branch) to compute the conserved energy, obtain the regularized euclidean on-shell action and verify the quantum statistical relation.

From the trace of Einstein’s equation, \(^{(2)}\), it follows that \( R = \frac{1}{2} (\partial \phi)^2 \) and, therefore,

\[
I_{bulk}^{\text{E}} = -\frac{q^2 \beta}{2\eta} \int_{u_{t+}}^{u_b} \frac{du}{\Omega(u)e^{a\phi}} = \frac{\beta q^2 e^{-a\phi_\infty}}{2\eta} (u_+ - 1)
\]

where \( u_b \) is the coordinate at the boundary, which at the end should be pushed to \( u_b \to 1 \). \( \beta \) is the periodicity in the imaginary time, \( \tau^E = -it \), which removes the conical singularities in the euclidean section, then \( \beta = T^{-1} \), where \( T \) is the Hawking temperature of the black hole. The Gibbons-Hawking boundary term is

\[
I_{GH}^{\text{E}} = \frac{\beta}{4\eta} \left[ 3(a^2 + 1)q^2 e^{-a\phi_\infty} + \frac{a^2 + 3}{a^2 - 1} \right] + \frac{\beta}{\eta(u_b - 1)} + O(u_b - 1)
\]

By noting that the Ricci scalar of the foliation \( u = \text{const} \) is \( R^{(3)} = 2\eta^2 (u - 1)^2/\Omega \), we obtain the gravitational counterterm,

\[
I_{ct}^{\text{E}} = -\frac{\beta}{2\eta} \left[ 1 + (a^2 + 1)q^2 e^{-a\phi_\infty} \right] - \frac{\beta}{\eta(u_b - 1)} + O(u_b - 1)
\]

\(^{10}\)The thermodynamic potentials of grand canonical ensemble and canonical ensemble are related by a Legendre transform in \( Q-\Phi \).

\(^{11}\)Concrete applications of this method for asymptotically flat black holes can be found in \(^ {11, 27, 35, 39} \)
and, finally, the boundary term of the scalar field has a finite contribution at the boundary

\[ I^E_\phi = -\frac{\beta}{4\eta} \Omega(u_b) \left[ f(u_b)\tilde{\Omega}(u_b)\right]^{1/2} \left( \frac{\ln u_b}{u_b - 1} \right)^2 W = \frac{1}{4} \beta W \]  

(61)

In consequence, the total on-shell action at the limit \( u_b = 1 \) is

\[ I^E = I^E_{\text{bulk}} + I^E_{\text{GH}} + I^E_{\text{cl}} + I^E_\phi = \beta \left[ \frac{(a^2 + 2u_+ - 1)}{4\eta} q^2 e^{-a\phi_\infty} - \frac{1}{4\eta} \left( \frac{a^2 - 1}{a^2 + 1} \right) + \frac{1}{4} W \right] \]  

(62)

In order to identity \( I^E \) with the thermodynamic potential, let us now obtain the expressions for the thermodynamic quantities of this solutions. The Hawking temperature \( T \), the Bekenstein-Hawking entropy \( S \), the electric charge \( Q \) and the conjugate potential \( \Phi \) are

\[ T = \frac{\tilde{\Omega}(u_+)}{4\pi\eta} \left. \frac{d\tilde{h}(u)}{du} \right|_{u=u_+} = \frac{\eta(u_+ - 1)}{4\pi u_+} \left[ -1 + \frac{2(a^2 + 2u_+ - 1)}{(a^2 + 1)(u_+ - 1)} + (3a^2 + 4u_+ - 1)q^2 e^{-a\phi_\infty} \right] \]  

(63)

\[ S = \frac{\pi \tilde{\Omega}(u_+)}{\eta^2(u_+ - 1)^2}, \quad Q = \frac{q}{\eta}, \quad \Phi = -qe^{-a\phi_\infty}(u_+ - 1), \]  

(64)

while the conserved energy is obtained by computing the quasilocal stress tensor, which, in this case, has the following form

\[ \tau_{tt} = \frac{1}{\kappa} \left\{ \frac{a^2 - 1}{a^2 + 1} - (a^2 + 1)q^2 e^{-a\phi_\infty} \right\} \eta - \frac{1}{2} \eta^2 W \]  

(65)

With this result, the conserved energy is

\[ E = \frac{1}{2\eta} \left[ (a^2 + 1)q^2 e^{-a\phi_\infty} - \frac{a^2 - 1}{a^2 + 1} \right] + \frac{1}{4} W \]  

(66)

which matches with the ADM mass \( M \), obtained by expanding \( g_{tt} \), when \( W = 0 \), that is, when \( \phi_\infty \) is considered fixed from the beginning. We conclude that, in this scenario, the conserved energy is \( E = M + \frac{1}{4} W \), where

\[ M = \frac{1}{2\eta} \left[ (a^2 + 1)q^2 e^{-a\phi_\infty} - \frac{a^2 - 1}{a^2 + 1} \right] \]  

(67)

It is easy to verify that the both the quantum statistical relation and the first law of black hole thermodynamics

\[ I^E = \beta (E - TS - Q\Phi) \equiv \beta \mathcal{G}, \quad dE = TdS + \Phi dQ \]  

(68)

where \( \mathcal{G} \) is the thermodynamic potential for grand canonical ensemble, hold without any extra term.

In canonical ensemble, where the electric charge is fixed, the boundary term \([54]\) is \( I^E_A = \beta Q\Phi \) and the quantum statistical relation reads \( \bar{I}^E = \beta (E - TS) \), where \( \bar{I}^E = I^E + I^E_A \).

Despite for the purely electrically charged solution the extremal limit is not well defined for any value of the coupling constant \( a \), for the dyonic solution, one can use the entropy function formalism \([50, 51]\) to show that the extremal limit is allowed. \([10]\).

\[ ^{12} \text{Notice that the factor } 1/4, \text{ appearing in this expression, does not appear in } [13]. \text{ This is due to the convention used in the action for the scalar field kinetic term.} \]
3.2 Thermodynamic in the dyonic sector, $a = -1$

Now, we would like to perform a similar analysis in the presence of the magnetic charge, for the particular case $a = -1$. We proceed in the same manner that the previous subsection.

The total on-shell action is

$$I^E = I^E_{bulk} + I^E_{GH} + I^E_{ct} + I^E_{\phi} = \beta \left[ \frac{q^2 e^{fi}}{2\eta} u_+ - \eta p^2 e^{-fi} \left( 2u_+ - 1 \right) + \frac{1}{4} W \right] + O \left[(u_b - 1)\right]$$

(69)

and, therefore, one can compute the relevant component of the quasilocal stress tensor

$$\tau_{tt} = \frac{1}{\kappa} \left[ 2\eta \left( -q^2 e^{fi} + \eta p^2 e^{-fi} \right) - \frac{1}{2} \eta^2 W \right] (u_b - 1)^2 + O \left[(u_b - 1)^3\right]$$

(70)

which allows to obtain the conserved energy, giving

$$E = \frac{q^2 e^{fi}}{\eta} - \eta p^2 e^{-fi} + \frac{1}{4} W = M + \frac{1}{4} W$$

(71)

where $M$ is the ADM mass.

To verify the quantum statistical relation, let us obtain the thermodynamic quantities. The temperature and entropy can be written as

$$T = \frac{u_+}{4\pi \eta} \frac{d\eta(u)}{du} \bigg|_{u_+} = \frac{\eta (u_+ - 1)^2}{2\pi u_+^2} \left[ u_+ (2u_+ + 1) q^2 e^{fi} - (u_+ + 2) \eta p^2 e^{-fi} - \frac{u_+ (u_+ + 1)}{2(u_+ - 1)} \right]$$

(72)

$$S = \pi \Omega(u_+)$$

(73)

The electric and magnetic charges are given in (31) and the conjugate potentials, the electric $\Phi$ and magnetic one $\Psi^p$, are

$$\Phi = - (u_+ - 1) q e^{fi}, \quad \Psi^p = A^p(u = 1) - A^p(u_+) = - \frac{\eta (u_+ - 1)}{u_+} p e^{fi}$$

(74)

where, in the language of differential forms, $dA^p = F^p \equiv e^{-\phi} \wedge F$ or, equivalently, $\Psi^p = \int_{u=1}^{u+} F^p u^d u$. One can see that, again, both the the quantum statistical relation and the first law,

$$I^E = \beta \left(E - T S - Q\Phi\right), \quad dE = T dS + \Phi dQ + \Psi^p dP$$

(75)

hold, as expected, and no extra term is required.

3.3 Thermodynamics in the dyonic sector, $a = -\sqrt{3}$

To obtain the Euclidean action, we proceed as before. Note that, from Einstein equations we have $R = \frac{1}{2} (\partial \phi)^2$ and, from Klein-Gordon equation, (31), we have $\sqrt{3} e^{-\sqrt{3} \phi} F^2 = - \frac{1}{\sqrt{-g}} \left(a^2 b^2 \sin^2 \theta \phi' \right)'$. Therefore, the bulk part of the action is

$$I^E_{bulk} = - \frac{\beta}{4\sqrt{3}} \left(a^2 b^2 \phi' \right) \bigg|_{r_+}^r = \frac{1}{4\sqrt{3}} \beta \Sigma + O(r_b^{-1})$$

(76)

The Gibbons-Hawking boundary term, the gravitational counterterm and the boundary term for the scalar field are

$$I^E_{GH} = \frac{3}{2} \beta M - \beta r_b + O(r_b^{-1}), \quad I^E_{ct} = \beta r_b - \beta M + O(r_b^{-1}), \quad I^E_{\phi} = \frac{1}{4} \beta W + O(r_b^{-1})$$

(77)
and the total action is then

\[ I^E = I^E_{\text{bulk}} + I^E_{\text{GH}} + I^E_{\text{ct}} + I^E_{\phi} = \beta \left( \frac{\sum}{4\sqrt{3}} + \frac{1}{2}M + \frac{1}{4}W \right) \]  

(78)

One can show that, by computing the time component of the quasilocal stress tensor, the conserved energy is

\[ E = -a^3b \left[ ab' - 1 - \frac{1}{4\Sigma^2} b (\phi - \phi_\infty)^2 W \right] = M + \frac{1}{4} W \]  

(79)

as expected.

To verify the quantum statistical relation, let us obtain the thermodynamic quantities. The temperature and entropy, in the compact form, can be written as

\[ T = \frac{r_+ - r_-}{4\pi \sqrt{A(r_+)B(r_+)}}, \quad S = \pi \sqrt{A(r_+)B(r_+)} \]  

(80)

The electric and magnetic charges are \( Q = q \) and \( P = p \), and the concrete expressions for electric and magnetic potentials can be written as

\[ \Phi = \frac{36MQe^{\sqrt{3}\phi_\infty}}{\sqrt{3\Sigma - 6r_+} \sqrt{3\Sigma - 6M}}, \quad \Psi^p = \frac{36MP e^{-\sqrt{3}\phi_\infty}}{\sqrt{3\Sigma + 6r_+} \sqrt{3\Sigma + 6M}} \]  

(81)

respectively. One can now easily verify that, by using the expression (43), (45) and (46) for this solution, both the quantum statistical relation and the first law

\[ I^E = \beta(E - TS - Q\Phi), \quad dE = TdS + \Phi dQ + \Psi^p dP \]  

(82)

hold, once again.

4 Conclusion

In this work, we use the method developed in [21] to re-obtain some known exact hairy black hole solutions of Einstein-Maxwell-dilaton asymptotically flat theories [3]. However, we have generalized the method to obtain solutions when the asymptotic value of the scalar field is not fixed, which provides an interesting tool for constructing this type of exact solutions. We have obtained two different families of solutions and each of them with two different branches or domains of the radial coordinate. We have also analyzed some specific aspects of black hole thermodynamics, using the quasilocal formalism, supplemented with the boundary terms that make the action principle well defined when the asymptotic value of the asymptotic scalar field varies.

With respect to thermodynamics, we first analyzed hairy electrically charged black holes for more slightly general theories than considered previously in [13] for asymptotically flat spacetimes, concretely, for arbitrary value of the constant in the coupling function \( e^{a\phi} \) between the scalar and Maxwell fields. We checked that the conserved energy has received a new contribution coming from the asymptotic value of the scalar field, \( E = M + \frac{1}{4} W(\phi_\infty) \), where \( W \) is defined by means of the relation between the quantities appearing in the leading and subleading term in the asymptotic expansion of the scalar field. Once the correct total energy is computed, there is no need of including the scalar charges in the first law of thermodynamics.

One observation, not new in this context [3], is that hairy (non self-interacting scalar field) electrically charged asymptotically flat exact black hole solutions in four dimensions do not have a well defined extremal limit. In this limit, the solution becomes a naked singularity. This is particularly
problematic in the scenario when we do thermodynamics with $Q$ fixed, since, as it was pointed out in [52], a vacuum solution with a fixed $Q \neq 0$ is not a regular solution of Einstein’s equations and, therefore, cannot be chosen as the ground state of the theory. A natural candidate that can be used as a background for the canonical ensemble is, then, the extremal black hole and so the importance of constructing solutions with such a limit becomes evident. However, by turning on the magnetic charge, we obtain a regular extremal hairy dyonic black hole solutions. Consequently, the canonical ensemble can be appropriately defined. In this context, we verified that the first law is indeed satisfied without including the extra term due to the non-conserved scalar charges. Therefore, we have generalized the results presented in [13] to the dyonic sector.

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