Spinning Strings on Deformed $AdS_5 \times T^{1,1}$ with NS $B$-field

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Abstract

We study classical spinning closed string configuration on logarithmically deformed $AdS_5 \times T^{1,1}$ background with non-trivial Neveu-Schwarz $B$-field in which IIB string theory is dual to a non-conformal $\mathcal{N} = 1$ $SU(N + M) \times SU(N)$ gauge theory. The integrability on original $AdS_5 \times T^{1,1}$ background are significantly reduced by $B$-field. We find several spinning string solutions with two different ansatzes. Solutions for point-like strings and few circular strings are explicitly obtained. Folded spinning string solutions along radial direction are shown to be allowed in this background. These solutions exhibit novel properties and bring some challenges to understand them from dual quantum field theory.

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I. INTRODUCTION

Recent two years the test of AdS/CFT correspondence was received significant development due to discovery of integrability both in string theory on $AdS_5 \times S^5$ background and in $\mathcal{N} = 4$ super Yang-Mills theory (SYM). It allows us to test AdS/CFT beyond BPS sector. The integrability at the string theory side was indicated first by discovery that the IIB super-string theory is exactly solvable at pp-wave background\cite{1, 2}, and later by finding of various semiclassical spinning solutions of string sigma model on $AdS_5 \times S^5$ background\cite{3, 4, 5, 6} (see the reviews\cite{7, 8} and references therein). On the other hand, at the gauge theory side the integrability is exhibited as identification between dilatation operator of $\mathcal{N} = 4$ SYM and Hamiltonian of a integrable spin chain\cite{9, 10} (see the review\cite{11} and references therein). Following those developments the test was naturally extended into general gauge/string duality, with less supersymmetries\cite{12, 13, 14} and without conformal symmetry\cite{15, 16, 17, 18}.

The purpose of this present paper is to study the spinning closed string configurations on the Klebanov-Tseytlin (KT) background\cite{19}, which is logarithmic deformation of $AdS_5 \times T^{1,1}$ with non-trivial Neveu-Schwarz $B$-field and is conjectured to be dual to a non-conformal $\mathcal{N} = 1$ $SU(N + M) \times SU(N)$ gauge theory\cite{1}. This study is expected to enable us to precisely test general gauge/string duality beyond supergravity level, like those for AdS/CFT correspondence. One motivation to choose the KT background is that this background simple enough, but possesses some novel properties comparing with those pure Ramond-Ramond backgrounds: The NS $B$-field directly couples with world-sheet scalar fields. Hence a non-constant $B$-field enters equations of motion. The backreaction of $B$-field induces a logarithmic deformation to $AdS_5 \times T^{1,1}$ background. The resulted geometry is no longer to be simply product of $AdS_5$ and $T^{1,1}$. Rather, there are warped factors both for $AdS_5$ and $T^{1,1}$ metric. In other words, string motion in $AdS_5$ and in $T^{1,1}$ are no longer to be decoupled. The above two properties must significantly alter the integrability as well as the spinning string configurations on original $AdS_5 \times T^{1,1}$ background.

In the present paper we focus our attention on the multi-spin solutions that describes the

\[1\] It is worth to point out that various spinning string solutions have been found in $AdS_5 \times T^{1,1}$ background\cite{16}. Those solutions, however, were extracted without consideration of the NS $B$-field and logarithmic deformation of background, or are presented on $AdS_5$ part of background where the NS $B$-field plays no role.
closed strings rotating on $T^{1,1}$ part only. It according to AdS/CFT dictionary corresponds to the composite operators consisting of bi-fundamental scalar fields of a $\mathcal{N} = 1$ $SU(N + M) \times SU(N)$ gauge theory. Here the isometry of $T^{1,1}$, $SU(2) \times SU(2) \times U(1)$, is dual to a global symmetry in SYM. Hence our another motivation to study semiclassical string configurations on the KT background is that the dictionary of duality about this background is much clearer than one on other confining backgrounds[18]. In other words, even though various spinning string solutions were obtained on those confining backgrounds, it is hard to exactly construct the dual operators of the most of spinning string solutions and check gauge/string duality directly. For example, we even can not exactly know what quantum numbers in SYM correspond to conserved charges associated with strings rotating on transverse space.

The above two advantages should admit us to study the non-conformal properties of gauge theories from perspective of semiclassical solution of dual string theory. It is beyond those extracted from supergravity analysis. One typical property was known as the absence of folded string configuration along radial direction in confining background[18]. To be precise, the strings spreading along radial direction of transverse space are no longer to satisfy the periodic condition. This conclusion follows from the well-known radius/energy-scale correspondence[20, 21]. In other words, the physical parameters like coupling constants are changed with flow along radial direction. It makes that the semiclassical energy of strings as functions of conserved charges $J$ as well as ’t Hooft coupling $\lambda$ to be ambiguous. In this present paper, however, we shall show that the folded string along the radial direction is allowed by the KT background. The key point is that a combination of two coupling constants of dual SYM, $g^{-2} \sim g_{1}^{-2} + g_{2}^{-2}$, does not slide under renormalization group. Hence the semiclassical energy as function of $J$ and $g_{2}^{2}$ is well-defined.

One of the essential point to test the gauge/string correspondence is to explore the energy/charge relations for various semiclassical string configurations. This relation at large charge $J$ limit usually admits a regular expansion in positive powers of $\lambda/J^2$ with $\lambda$ the ’t Hooft coupling of dual SYM. The semiclassical expansion of string sigma model is valid at large $\lambda$ limit. Hence we actually need take double limit, $\lambda \to \infty$, $J \to \infty$, while $\lambda/J^2$ fixed. In this limit the classical energy of semiclassical string solution in any $AdS_p \times S^q$ space goes as linear function of $J$, $E = J + ...$ with appropriate normalization. This linear behavior[22] is a consequence of the constant curvature of $AdS$ space. In our case, however, we expect that the linear behavior is breakdown since no matter $AdS$ part or $T^{1,1}$ are no
longer to possess constant curvature due the logarithmic deformation. Indeed we find that the relation may be modified to \( E = E_0 + F(J) \) with \( E_0 > 0 \) when string spreads along the radial direction since the curvature of deformed \( T^{1,1} \) varies along the radius. While the linear behavior is kept for point-like (BMN-like) and circular configurations since they locate at a fixed position on radial direction. Moreover, we find that the point-like solutions is only presented at a special point on radial coordinate of deformed \( AdS_5 \). While the circular solutions are allowed only in a finite region on radial coordinate of deformed \( AdS_5 \). This phenomena should somehow exhibit non-conformal properties of dual gauge theory.

This paper is organized as follows: In section 2 we briefly review the KT background and some basic points of string sigma model on this background. In section 3 we study spinning string solutions with usual ansatz. We argue that some simpler multi-spin circular solutions are absent in this ansatz. We also show that the folded string configuration is allowed in the KT background, but it possesses infinite energy. In section 4 various spinning string solutions are constructed with another ansatz. A brief summary is devoted in section 5 and some further discussions and open questions are also included in this section.

II. STRING SIGMA MODEL ON THE KLEBANOV-TSEYTLIN BACKGROUND

A. The Klebanov-Tseytlin background

The Klebanov-Tseytlin geometry is a singular background produced by placing \( N \) D3-branes and \( M \) fractional D3-branes at the apex of conifold. The angular part of conifold is a cone whose base is a coset space \( T^{1,1} = (SU(2) \times SU(2))/U(1) \), with topology \( S^2 \times S^3 \) and symmetry group \( SU(2) \times SU(2) \times U(1) \). The 10-d metric of KT geometry has the structure of a “warped product” of \( \mathbb{R}^{3,1} \) and the conifold:

\[
\begin{aligned}
ds_{10}^2 &= h^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(r)(dr^2 + r^2 g_{ij}dy^i dy^j),
\end{aligned}
\]

where \( \mu, \nu = 0, 1, 2, 3 \) and \( i, j = 4, ..., 9 \), \( g_{ij} \) denotes the Einstein metric on \( T^{1,1} \),

\[

\begin{aligned}
g_{ij}dy^i dy^j &= \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2.
\end{aligned}
\]
The insertion of fractional D3-branes involves $M$ units of the R-R 3-form flux $F_3$ as well as radial dependent NS 3-form flux $H_3 = dB_2$:

$$F_3 = \frac{M\alpha'}{2} \omega_3, \quad B_2 = \frac{3g_s M\alpha'}{2} \omega_2 \ln (r/r_0),$$

where

$$\omega_2 = \frac{1}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2),$$

$$\omega_3 = (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \wedge \omega_2.$$  \hfill (2.4)

The warped factor $h(r)$ is determined by solving the Einstein equation in presence of 3-form flux,

$$h(r) = \frac{R^4}{r^4} [1 + a \ln (r/r_0)],$$

where

$$R^4 = \frac{27}{4} \pi \alpha'^2 (g_s N + \frac{3}{8\pi} g_s^2 M^2), \quad a = \frac{12g_s M^2}{8\pi N + 3g_s M^2}. \hfill (2.6)$$

Hence this geometry is a logarithmic deformation of $AdS_5 \times T^{1,1}$. It has a naked singularity at $r = r_s$ where $h(r_s) = 0$ and approaches to $AdS_5 \times T^{1,1}$ in vicinity of the UV scale $r = r_0$. An important feature of this background is that the 5-form flux acquires a radial dependence,

$$\tilde{F}_5 = F_5 + \star F_5, \quad F_5 \propto N_{\text{eff}} = N + \frac{3}{2\pi} g_s M^2 \ln (r/r_0).$$

The reduction of 5-form flux along radius is known as RG cascade in dual gauge theory. In particular, $\tilde{F}_5$ vanishes at $r = \tilde{r}_s = r_0 \exp (\frac{1}{4} - \frac{1}{a})$ where $h(r_s) \propto 3g_s M^2/(8\pi N + 3g_s M^2) > 0$. It is well-known that the naked singularity can be removed by blow-up of $S^3$ at apex of $T^{1,1}$, it is Klebanov-Strassler solution [23]. In this present paper, however, for the sake of simpleness we focus on classical string configuration in the region $r > \tilde{r}_s$ which does not touch the singularity. Hence we still work on the KT background.

IIB superstring theory on KT background is conjectured to be dual to the $\mathcal{N} = 1$ SYM with $SU(N + M) \times SU(N)$ gauge symmetry. The matter sector of the model consists of two chiral superfields $A_1, A_2$ in the $(N + \mathbf{M}, \mathbf{N})$ representation and two fields $B_1, B_2$ in the $(\overline{\mathbf{N}}, \mathbf{M})$ representation. The model has a $SU(2) \times SU(2) \times U(1)$ global symmetry and consequently has three $U(1)$ charges for fields in matter sector. The later corresponds to.
translation invariance along $\phi_1, \phi_2$ and $\psi$ in KT geometry. The quantum numbers$^2$ of the matter under the global symmetries are given in table 1.

|         | $SU(2)$ | $SU(2)$ | $U(1)_A$ | $U(1)_B$ | $U(1)_R$ |
|---------|---------|---------|----------|----------|----------|
| $(A_1, A_2)$ | 2       | 1       | $(1, -1)$ | 0        | $\frac{1}{7}$ |
| $(B_1, B_2)$ | 1       | 2       | 0        | $(1, -1)$ | $\frac{1}{7}$ |

**TABLE I**: The quantum numbers of matter under global $SU(2) \times SU(2) \times U(1)$ symmetry.

### B. String sigma model on KT background

IIB superstrings on KT background can be described by a Green-Schwarz action whose bosonic part is a non-linear sigma model on string world-sheet,

$$I = -\frac{1}{2\pi \alpha'} \int d\tau d\sigma \sqrt{-g} \{ g^{ab} \partial_a X^M \partial_b X^N G_{MN}(X) + \epsilon^{ab} \partial_a X^M \partial_b X^N B_{MN}(X) \}, \quad (2.8)$$

where $g^{ab}$ is the world-sheet metric, $\epsilon^{ab}/\sqrt{-g}$ is the world-sheet anti-symmetric tensor, $G_{MN}$ and $B_{MN}$ are target space metric and anti-symmetric tensor respectively. The world-sheet symmetries of invariance of reparametrization and of Weyl scaling allow us to take conformal gauge, namely $g_{ab} = \eta_{ab}$, $\epsilon^{01} = 1$. The classical equations of motion that follows from the above action in conformal gauge read off

$$\partial_a \partial^a X^P + \Gamma^P_{MN} \partial_a X^M \partial^a X^N - \frac{1}{2} G^{PQ} H_{QMN} \epsilon^{ab} \partial_a X^M \partial_b X^N = 0, \quad (2.9)$$

where $\Gamma^P_{MN}$ specifies Christoffel connection of the target space and $H_{QMN}$ is NS 3-form flux. Moreover, the action (2.8) is to be supplemented with the usual conformal gauge constraints expression the vanishing of the total 2-d energy-momentum tensor

$$\left\{ \begin{array}{c}
G_{MN}(\dot{X}^M \dot{X}^N + X'^M X'^N) = 0, \\
G_{MN} \dot{X}^M X'^N = 0,
\end{array} \right. \quad (2.10)$$

where the “dot” and “prime” denote differential to $\tau$ and $\sigma$ respectively.

$^2$ It should be distinguished the $U(1)$ charges $U(1)_A$ and $U(1)_B$ in table 1 from those present in ref.~$^{23}$. The charges $U(1)_A$ and $U(1)_B$ in our paper are associated with global $SU(2) \times SU(2)$ symmetry. However, they in $^{23}$ are associated with unbroken gauge group $U(1)^N$. 
Concerning to KT background (2.1), (2.3), it is useful to explicitly write equations of motion for \( x_\mu \) and \( r \),

\[
\begin{align*}
\partial_a \partial^a x^\mu - \frac{h'}{2h} \partial_a x^\mu \partial^a r &= 0, \\
\partial_a \partial^a r + \frac{h'}{4h} \partial_a r \partial^a r + \frac{h'}{4h^2} \partial_a x_\mu \partial^a x^\mu - r (1 + \frac{r h'}{4h}) g_{ij} \partial_\alpha y^i \partial_\beta y^j - \frac{1}{2} G^{rr} H_{rij} \epsilon^{ab} \partial_\alpha y^i \partial_\beta y^j &= 0,
\end{align*}
\]

(2.11)

where \( h' = dh(r)/dr \). Due to the nontrivial \( H_3 \) and \( 1 + \frac{h'}{4h} = [4 \ln (r/r_0) + \frac{4}{a}]^{-1} \neq 0 \), the coordinates of \( T^{1,1} \), \( y_i \), in general do not decouple with radial coordinate \( r \). We will see that the presence of NS 3-form field strength in the equation of motion plays significant role to change the integrability of the simple product \( AdS_5 \times X_5 \) background. This is novel feature of string sigma model on KT background.

It is well-known that there is residual symmetry on world-sheet sigma model under transformation \( \sigma^+ = \tau + \sigma \rightarrow \tilde{\sigma}^+(\sigma^+) \), \( \sigma^- = \tau - \sigma \rightarrow \tilde{\sigma}^-(\sigma^-) \). For flat target space this invariance allows us to transfer arbitrary non-trivial solutions of \( x_0(\tau) \) or \( x^+ = x_0 + x_1 = x^+(\tau) \) to the static gauge \( x^0 = \kappa \tau \) or the light-cone gauge \( x^+ = \tau^+ \tau \). In the curved background, however, it is in general impossible to achieve those gauge. For example, in the KT background following from the first equation in (2.11) it is possible to take static gauge \( x^0 = \kappa \tau \) only for those special solutions with \( r \) constant. Furthermore, \( r \) can not be arbitrary constant due to the constraint from the second equation of (2.11). In simple product backgrounds such as \( AdS_5 \times S^5 \) or \( AdS_5 \times T^{1,1} \) it is determined by \( h'/h^2 = 0 \rightarrow r = 0 \) unless \( R \rightarrow \infty \) with \( R \) the radius of \( AdS_5 \). In the KT background, however, the value of \( r \) depends on motion of string inside \( T^{1,1} \). More generally, we may try to find a class of special solutions in which \( r \) depends on \( \sigma \) only and satisfies the period boundary condition, i.e., \( r(\sigma + 2\pi) = r(\sigma) \). A special solution for \( x^0 \) following from this ansatz is

\[
x^0 = \kappa \tau + \beta(\sigma), \quad \beta'(\sigma) = c_0 \sqrt{h(r(\sigma))},
\]

(2.12)

with \( c_0 \) constant and boundary condition \( \beta(\sigma + 2\pi) = \beta(\sigma) \).

In this present paper, we focus our attention on those configurations describing closed strings rotating on \( T^{1,1} \). Hence we simply set \( x^i = \text{constant}, \ (i = 1, 2, 3) \).
C. Conserved charges

The symmetry of $T^{1,1}$, $SU(2) \times SU(2) \times U(1)$, admits three conserved charges. They are angular momenta corresponding to strings rotating along $\phi_1, \phi_2$ and $\psi$ directions:

$$J_A = P_{\phi_1} = \frac{1}{6\pi \alpha'} \int_0^{2\pi} d\sigma r^2 \sqrt{h(r)} \left\{ \left(1 - \frac{1}{3} \cos^2 \theta_1 \right) \dot{\phi}_1 + \frac{2}{3} \cos \theta_1 \left( \psi + \cos \theta_2 \dot{\phi}_2 \right) \right\} - \frac{3g_s M}{8\pi} \int_0^{2\pi} d\sigma \theta'_1 \ln (r/r_0) \sin \theta_1,$$

$$J_B = P_{\phi_2} = \frac{1}{6\pi \alpha'} \int_0^{2\pi} d\sigma r^2 \sqrt{h(r)} \left\{ \left(1 - \frac{1}{3} \cos^2 \theta_2 \right) \dot{\phi}_2 + \frac{2}{3} \cos \theta_2 \left( \psi + \cos \theta_1 \dot{\phi}_1 \right) \right\} + \frac{3g_s M}{8\pi} \int_0^{2\pi} d\sigma \theta'_2 \ln (r/r_0) \sin \theta_2,$$

$$J_R = P_{\psi} = \frac{1}{9\pi \alpha'} \int_0^{2\pi} d\sigma r^2 \sqrt{h(r)} \left( \psi + \cos \theta_1 \dot{\phi}_1 + \cos \theta_2 \dot{\phi}_2 \right). \quad (2.13)$$

In addition, the classical energy is associated with translation invariance along $x^0$. For special solutions like (2.12), its explicit expression is

$$E = P_0 = \frac{\kappa}{\pi \alpha'} \int_0^{2\pi} d\sigma h^{-1/2}(r). \quad (2.14)$$

III. TYPE A SOLUTIONS

Inserting KT metric (2.1) and NS 3-form field (2.2) into equations of motion (2.9) we obtain some highly involved non-linear equations. To solve them some ansatz are necessary. Here the word “type A” just denotes that in this section we study the various special solutions with ansatz $\dot{\theta}_i = \dot{\phi}_i = \dot{\psi} = 0$, ($i = 1, 2$). Then equations of motion of $\phi_i$ and $\psi$ reduce to

$$\ddot{\phi}_i = \ddot{\psi} = 0 \Rightarrow \begin{cases} \phi_i = \omega_i \tau, & i = 1, 2 \\ \psi = \omega \tau. \end{cases} \quad (3.1)$$

Furthermore, the second constraint in (2.10) implies $\beta(\sigma) \sim \text{constant}$. While another conformal constraint reduces to

$$\frac{k^2}{r^2 h(r)} = \frac{r'^2}{r^2} + \frac{1}{6} \sum_{i=1}^{2} \left( \theta_i'^2 + \omega_i^2 \sin^2 \theta_i \right) + \frac{1}{9} (\omega + \omega_1 \cos \theta_1 + \omega_2 \cos \theta_2)^2. \quad (3.2)$$
Using this constraint in equations of \( r \) and \( \theta_i \) we have

\[
\frac{r''}{r} - \frac{r'^2}{r^2} - \frac{k^2}{r^2h}(1 - \frac{a/2}{1 + a \ln (r/r_0)}) - \frac{a(\theta_1'^2 + \theta_2'^2)}{12(1 + a \ln (r/r_0))} + \frac{R^2}{3r^2\sqrt{a}}(\omega_1 \sin \theta_1 \theta_1' - \omega_2 \sin \theta_2 \theta_2') = 0,
\]

\[
\theta_1'' + \frac{a \theta_1'}{1 + a \ln (r/r_0)} \frac{r'}{2r} - \frac{\omega_1 \sin \theta_1}{3}(2\omega + 2\omega_2 \cos \theta_2 - \omega_1 \cos \theta_1 + \frac{3R^2}{r^2\sqrt{2h}}) = 0,
\]

\[
\theta_2'' + \frac{a \theta_2'}{1 + a \ln (r/r_0)} \frac{r'}{2r} - \frac{\omega_2 \sin \theta_2}{3}(2\omega + 2\omega_1 \cos \theta_1 - \omega_2 \cos \theta_2 - \frac{3R^2}{r^3\sqrt{2h}}) = 0. \quad (3.3)
\]

For \( a = 0 \) the above equations of motion just describe strings rotating on \( AdS_5 \times T^{11,1} \) background where \( r \) and \( \theta_i \) decouple. Moreover, to describe closed string configuration, the following boundary conditions should be supplemented,

\[
\theta_i(\sigma + 2\pi) = \theta_i(\sigma) + 2m_i\pi, \quad r(\sigma + 2\pi) = r(\sigma), \quad (3.4)
\]

with \( m_i \) integers.

A. Point-like solution

The point-like string configurations indicates that none of coordinates acquires the \( \sigma \)-dependence. Hence equations of motion (3.3) reduce

\[
\begin{cases}
    r_* = r_0e^{\frac{1}{2} - \frac{a}{4}} \Rightarrow N_{\text{eff}} = \frac{3}{8\pi} g_s M^2, \\
    \omega_1 = 0, & \text{or} \quad 2\omega + 2\omega_2 \cos \theta_2 - \omega_1 \cos \theta_1 = 0, \\
    \omega_2 = 0, & \text{or} \quad 2\omega + 2\omega_1 \cos \theta_1 - \omega_2 \cos \theta_2 = 0.
\end{cases} \quad (3.5)
\]

Here we used a fact that \( \sin \theta_i = 0 \) means that the point-like string locates at north- or south-pole of \( S^2 \). Classically it does not make sense to claim the point-like string rotating along \( \phi_i \). Therefore, \( \sin \theta_i = 0 \) implies \( \omega_i = 0 \). From (3.5) we see that value of \( r_* \) is uniquely determined by the number of the background fluxes and string coupling.

- Single-spin solution: \( \omega_1 = \omega_2 = 0 \).

The conserved charges corresponding to angular momenta following from Eq. (2.13) read off

\[
J_A = J_B = J_R = J = \frac{\sqrt{2a}}{9\alpha'} R^2 \omega. \quad (3.6)
\]
Then using conformal constraint (3.2) we obtain the classical energy of this configuration

\[ E = \frac{2}{3\alpha'} r_* \omega \propto J. \]  

(3.7)

According to the table 1, we can see that its dual operators is:

\[ \text{Tr}(A_1B_1)^J. \]  

(3.8)

- Two-spin solution: \( \omega_1 = 0, \omega_2 \neq 0 \).

From Eq. (3.5) it is straightforward to obtain \( \cos \theta_2 = 2\omega/\omega_2 \Rightarrow |\omega| \leq |\omega_2|/2 \) and conserved charges as follows:

\[ J_B = \frac{\sqrt{2a}}{6\alpha'} R^2 \omega_2, \quad J_R = \frac{\sqrt{2a}}{3\alpha'} R^2 \omega \leq J_B, \quad J_A = J_R \cos \theta_1. \]  

(3.9)

Because there is no other constraint to fix the value of \( \cos \theta_1 \), the dual operators of this solution is rather complicated. In particular, the \( U(1) \) in dual SYM is anomalous if \( \cos \theta_1 \neq n/J_R \) with \( n, J_R \) integers. Finally, the classical energy reads off

\[ E = \frac{\sqrt{6}}{3 \sqrt{aR^2}} r_* \sqrt{J_B^2 + 2J_R^2}. \]  

(3.10)

- Three-spin solution: \( \omega_1 \neq 0, \omega_2 \neq 0 \).

It is straightforward to obtain the conserved charges from the solution \( \cos \theta_i = -2\omega/\omega_i \),

\[ J_A = \frac{\sqrt{2a}}{6\alpha'} R^2 \omega_1, \quad J_B = \frac{\sqrt{2a}}{6\alpha'} R^2 \omega_2, \quad J_R = -\frac{\sqrt{2a}}{3\alpha'} R^2 \omega. \]  

(3.11)

They should be supplemented by the condition

\[ |J_R| \leq \min(|J_A|, |J_B|). \]  

(3.12)

The classical energy following from conformal constraint is again expressed by the above charges,

\[ E = \frac{\sqrt{6}}{\sqrt{aR^2}} r_* \sqrt{2(\frac{J_A^2}{J_B^2} + J_B^2)} - J_R^2. \]  

(3.13)

Its dual operators have to be out of holomorphic (or anti-holomorphic) sector. To see this fact, let us consider an holomorphic operators consisting of \( n_1 (A_1B_1), n_2 (A_1B_2), n_3 (A_2B_1) \) and \( n_4 (A_2B_2) \). If there are no global \( U(1) \) anomaly, we have

\[ J_R = \sum_{i=1}^{4} n_i, \quad J_A = n_1 + n_2 - n_3 - n_4, \quad J_B = n_1 - n_2 + n_3 - n_4. \]  

(3.14)
The above conclusion, however, conflicts with condition (3.12) unless \( n_1 = n_2 = n_3 = 0 \) which corresponds to \( |\omega_1| = |\omega_2| = 2|\omega| \Rightarrow \sin \theta_i = 0 \). According to previous discussion, this result conflicts with assumption \( \omega_i \neq 0 \). Therefore, we conclude that the dual operator of three-spin solution must mixing operators \( A_iB_j \) with their Hermitian conjugated partners.

B. Circular solution with \( r' = 0 \)

The circular solutions describe those string configurations in which at least one of \( \theta_i \) acquires \( \sigma \)-dependence and satisfies the boundary condition (3.4) with nonzero winding number \( m_i \). Here we focus our attention on the circular solutions with \( r' = 0 \). The solutions with \( r' \neq 0 \) will be discussed in the next subsection.

- single-spin solution: \( \omega_1 = \omega_2 = 0 \).

From equations of \( \theta_i \) we have \( \theta_i = m_i \sigma \) with \( m_i \) integers. Then the consistency between conformal constraint \( 3.2 \) and the first equation in \( 3.3 \) implies

\[
r_s = r_0 \exp \left( \frac{\omega^2}{3(m_1^2 + m_2^2) + 2\omega^2} - \frac{1}{a} \right).
\]

While the condition \( N_{\text{eff}} > 0 \) imposes a constraint \( 2\omega^2 > 3(m_1^2 + m_2^2) \).

The angular momenta and classical energy are as follows:

\[
J_A = J_B = 0, \quad J_R = \frac{R^2}{9\alpha'} \sqrt{\frac{2a}{3}} \mathcal{J} = \frac{2R^2}{9\alpha' \omega} \sqrt{\frac{a\omega^2}{3(m_1^2 + m_2^2) + 2\omega^2}},
\]

\[
E = \frac{2r_0}{\alpha'} e^{-1/a} \sqrt{m_1^2 + m_2^2 + \frac{2}{3} \omega^2} \exp \left( \frac{\omega^2}{3(m_1^2 + m_2^2) + 2\omega^2} \right)
\rightarrow \frac{2\sqrt{2}r_0}{\alpha'} e^{\frac{1}{2} - \frac{1}{4}} \mathcal{J} (1 + \frac{m_1^2 + m_2^2}{\mathcal{J}^2} + \ldots).
\]

Hence we again have the regular expansion to classical energy at large angular momentum limit. Its dual operators have the form like

\[
\text{Tr}(A_1B_1)^{J_R/2}(A_2B_2)^{J_R/2}.
\]

Otherwise they should be out of holomorphic sector.

Other single-spin solutions, e.g., \( \omega = \omega_1 = 0, \omega_2 \neq 0 \) will be studied in the following discussions.
Two-spin solution: $\omega_1 \neq 0$, $\omega_2 = 0$.

The solutions of Eq. (3.3) describing the circular string configuration is

$$\theta_2 = m\sigma, \quad \cos \theta_1 = 2\omega/\omega_1 = \text{constant.} \quad (3.17)$$

Then

$$r_* = r_0 \exp \left( \frac{\omega_1^2 + 2\omega^2}{2m^2 + 2\omega_1^2 + 4\omega^2} - \frac{1}{a} \right)$$

follows from the consistency between conformal constraint (3.2) and the first equation in (3.3). The conversed charges and classical energy reads off:

$$J_A = \frac{2R^2}{6\alpha'} \omega_1 \sqrt{\frac{2}{2m^2 + 2\omega_1^2 + 4\omega^2}}, \quad J_B = 0,$$

$$J_R = \frac{2R^2}{3\alpha'} \omega_1 \sqrt{\frac{2}{2m^2 + 2\omega_1^2 + 4\omega^2}} \leq J_A,$$

$$E = \frac{2r_0}{\alpha'} e^{-1/a} \sqrt{m^2 + \omega_1^2 + 2\omega^2} \exp \left( \frac{\omega_1^2 + 2\omega^2}{2m^2 + 2\omega_1^2 + 4\omega^2} \right)$$

$$\xrightarrow{\mathcal{J} \to \infty} 2\sqrt{2} r_0 \mathcal{J}^{1/2} \left( 1 + \frac{m^2}{\mathcal{J}^2} + \ldots \right), \quad (3.18)$$

where $\mathcal{J}^2 \propto \sqrt{2}J_A^2 + J_R^2$. Its dual operators in (anti-)holomorphic sector have the form like

$$\text{Tr}(A_1 B_1)^{J_R/2}(A_1 B_2)^{J_R/2}.$$ 

It exists only for $J_R = J_A$ and even $J_R$. For other choices of charges they are no longer to be (anti-)holomorphic.

If we set $\omega = 0$, the above solution reduces to the single-spin solution with $\omega = \omega_1 = 0$, $\omega_2 \neq 0$. Moreover, It should be pointed out that NS $B$-field plays trivial role in all of the above solutions. In the following we will see that some simpler multi-spin solutions (such as those found in $AdS_5 \times X^5$ background) are absent when NS $B$-field plays nontrivial role.

- Fate of other multi-spin circular solutions with $r' = 0$.

It should be noticed that the conformal constraint (3.2) is consistent with the equations of motion of $\theta_1$. So that there are three independent of equations.

We first consider solutions with $\omega_1 \neq 0$, $\omega_2 \neq 0$ (or $\omega_1 = 0$, $\omega_2 \neq 0$, $\omega_2 \neq 0$). When $\theta_2' = 0$, the equation of motion of $\theta_2$ implies that $\omega_2 \sin \theta_2 = 0$. According to the
previous argument, classically it always indicates $\omega_2 = 0$, i.e., $\theta_2' = 0 \Rightarrow \omega_2 = 0$. Those solutions, however, are forbidden by inconsistency between the conformal constraint (3.2) and the first equation in (3.3). In particular, the NS field plays crucial role in this assertion, as we have mentioned.

Hence we should consider the multi-spin solutions with $\omega_1 \neq 0$, $\omega_2 \neq 0$. It is convenient to rewrite equations of motion (3.3) in terms of introducing $\eta = \omega_1 \cos \theta_1$, $\xi = \omega_2 \cos \theta_2$ and $z = \cos \sigma$,

$$
(1 - z^2)(\eta' - \xi')^2 = b^2[\omega_1^2 + \omega_2^2 - \eta^2 - \xi^2 + \frac{2}{3}(\omega + \eta + \xi)^2 + c]^2, \\
(1 - z^2)\eta'^2 = \frac{1}{3}(\omega_1^2 - \eta^2)(\eta^2 - 4\omega \eta - 4\eta \xi + 4F(z) + d_1), \\
(1 - z^2)\xi'^2 = \frac{1}{3}(\omega_2^2 - \xi^2)(\xi^2 - 4\omega \xi - 4F(z) + d_2),
$$

(3.19)

where the “prime” denotes the differential to $z$, $F(z) = \int \eta(z)\xi'(z)dz$, $d_1$, $d_2$ are integral constants and $b$, $c$ are independent constants since they depend on $r_*, \kappa, a$ and $R$. The appearance of the first equation in (3.19), which forbids some simpler solution such as $\eta = \xi$, is entirely due to coupling between $r$ and $\theta$ and presence of NS B-field. Since $\eta$ and $\xi$ are required to be periodic functions in region $\sigma \in [0, 2\pi]$, the general non-singular periodic solutions of $\eta$ and $\xi$ can be expressed by polynomial of $z$,

$$
\eta = \sum_{n=0}^{\infty} a_n z^n, \quad \xi = \sum_{n=0}^{\infty} b_n z^n. \quad (3.20)
$$

Unfortunately, it is hard to answer whether the solution like (3.20) is allowed or not for highly non-linear equations (3.19) since the above polynomial can not be truncated at finite order.

C. Folded solutions with $r' \neq 0$

The equations of motion (3.3) in general is hard to solve when $r' \neq 0$ due to coupling among $r$ and $\theta_i$. Fortunately, these differential equations can be performed the first integral if we set $\omega_1 = \omega_2 = 0$. The solution reads off

$$
\theta_i' = \frac{c_i}{\sqrt{6(a \ln (r/r_0) + 1)}}, \\
\frac{r'^2}{r^2} = \frac{\kappa^2 r^2}{a \ln (r/r_0) + 1} - \frac{c_1^2 + c_2^2}{a \ln (r/r_0) + 1} - \frac{\omega^2 9}{9}. \quad (3.21)
$$
It is not hard to check that the above equations are consistent with conformal constraints \((3.2)\). The general solutions of \((3.21)\) can not be expressed by elementary functions, so to find the folded string solutions following from \((3.21)\) are nontrivial task. For the simplest case with \(a = 0\), i.e., the \(AdS_5 \times T^{1,1}\) background. It is straightforward to obtain the following solution

\[
\theta_i = n_i \sigma, \quad r^2 = r_0^2/ \cos^2 \Omega \sigma, \quad \kappa^2 r_0^2 = \Omega^2 = \frac{1}{6}(n_1^2 + n_2^2) + \frac{\omega^2}{9}. \tag{3.22}
\]

The above solution satisfies periodic boundary condition \((3.4)\) with \(n_i\) integers and \(\Omega\) half-integer. It is a circular string for nonzero \(n_i\) and folded one when \(n_1 = n_2 = 0\). The latter describes a closed string spreading from \(r_0\) to infinity and folded at \(r(\sigma = 0) = r(\sigma = \pi) = r(\sigma = 2\pi) = r_0\) and \(r(\sigma = \pi/2) = r(\sigma = 3\pi/2) = \infty\). These solutions, however, are ill-defined since they following from \((2.14)\) yield infinite energy.

In general we may consider a first order differential equation with the following form

\[
r^2 = F(r)^2. \tag{3.23}
\]

If \(F(r)\) is positive in region \(0 < r_1 < r < r_2\), the function \(v(r)\) defined by integral

\[
v(r) = \int_{r_1}^{r} \frac{dr'}{F(r')} \tag{3.24}
\]

is single-valued in region \(r_1 < r < r_2\), \(0 < v < v(r_2)\), so one can unambiguously define inverse function of \(v(r)\), \(R(v) = v^{-1}(r)\) in this region. Then we can construct a special solution of Eq. \((3.23)\):

\[
R(\sigma - \frac{2m\pi}{n}), \quad \text{if} \quad \sigma \in \left(\frac{2m\pi}{n}, \frac{(2m+1)\pi}{n}\right),
\]

\[
R\left(\frac{2(m+1)\pi}{n} - \sigma\right), \quad \text{if} \quad \sigma \in \left(\frac{(2m+1)\pi}{n}, \frac{2(m+1)\pi}{n}\right), \tag{3.25}
\]

with \(m, \ n\) integers and \(m = 0, 1, ..., n - 1\). The above solution satisfies the boundary condition \(r(\sigma + 2\pi) = r(\sigma)\) and has properties

\[
r(\frac{2m\pi}{n} + \sigma) = r(\sigma), \quad r\left(\frac{2(m+1)\pi}{n} - \sigma\right) = r(\sigma - \frac{2m\pi}{n}). \tag{3.26}
\]

The crucial point to admit this kind of solutions is because Eq. \((3.23)\) allows us to choose \(r' = F(r)\) in the region \(2m\pi/n \leq \sigma \leq (2m+1)\pi/n\) and \(r' = -F(r)\) in the region \((2m+1)\pi/n \leq \sigma \leq 2(m+1)\pi/n\). Moreover, in order that the solution \((3.25)\) correctly describes
the folded closed string configuration, the turning points at $\sigma = k\pi/n$ must be fixed points, i.e., $r'|_{\sigma=k\pi/n} = 0$ or $F(r(\sigma = k\pi/n)) = 0$. Without loss of generality, we may assume $r_1, r_2$ are fixed points of Eq. (3.23). Then an additional consistent condition,

$$v(r_2) = \frac{\pi}{n},$$

has to be imposed.

Now let us turn to folded strings in deformed $AdS_5 \times T^{1,1}$. We may introduce new variable $\rho = r_0 e^{-1/a}/r$ and rewrite the second equation in (3.21) as follows

$$\rho^2 = \frac{\kappa^2 r_0^2 e^{-2/a} \tilde{c}^2 \rho^2 - 1 - \tilde{\omega}^2 \rho^2 \ln \rho}{\ln \rho},$$

where

$$\tilde{c}^2 = \frac{c_1^2 + c_2^2}{\kappa^2 r_0^2 e^{-2/a}}, \quad \tilde{\omega}^2 = \frac{\alpha \omega^2}{9\kappa^2 r_0^2 e^{-2/a}}.$$

In the physical region, $\rho \in [0, e^{-1/4})$, the equation (3.28) can only have two possible fixed points: $\rho = 0$ and $\rho = \rho_c$ with $\rho_c$ solution of algebraic equation

$$\tilde{c}^2 \rho^2 - 1 - \tilde{\omega}^2 \rho^2 \ln \rho = 0.$$ 

In the region $\rho \in (0, \rho_c)$ it is possible to choose $\tilde{c}$ and $\tilde{\omega}$ to ensure r.h.s. of (3.28) to be positive. Then if condition following from (3.29),

$$\int_0^{\rho_c} d\rho \sqrt{\frac{\ln \rho}{\tilde{c}^2 \rho^2 - 1 - \tilde{\omega}^2 \rho^2 \ln \rho}} = \frac{\kappa r_0 e^{-1/a} \pi}{\sqrt{a} n},$$

is satisfied, we may construct a solution like (3.25). This solution describes a folded closed string spreading from $\rho = 0$ to $\rho_c$ (or from $r_c$ to infinity). Indeed it is not hard to prove that the integral in (3.31) is convergent. For example, if we choose $\tilde{c} = 1$, $\tilde{\omega} = 5$ and $n = 4$, we obtain $\rho_c = 0.1416$ and $\kappa r_0 e^{-1/a}/\sqrt{a} = 0.49$.

It is interesting that the equation (3.21) admits static configuration ($\omega = 0$), which is not admitted for $AdS_5 \times T^{1,1}$. Explicitly, for $\omega = 0$ we have

$$r = \kappa^{-1} \sqrt{\frac{c_1^2 + c_2^2}{\cos \sqrt{\theta_1^2 + \theta_2^2}}}.$$ 

By means of the boundary conditions listed in (3.4) it is easy to check that the configuration describes a circular closed string when

$$m_1 \theta_1 = m_2 \theta_2, \quad \frac{m_1}{m_2} \sqrt{m_1^2 + m_2^2} \in \mathbb{Z},$$

(3.33)
with winding number \(m_1, m_2\) integers. For other cases we have to set \(m_1 = m_2 = 0\) so that it describes a folded string spreading from \(r_c = \sqrt{c_1^2 + c_2^2}/\kappa\) to infinity. However, it is tilted folded configuration comparing with one on \(AdS_5 \times T^{1,1}\). Namely both of \(\theta_i\) and \(r\) acquire \(\sigma\)-dependence. Those static string configurations yield infinite energy

\[
E_s = \frac{1}{\epsilon} + ..., \tag{3.34}
\]

with \(\epsilon\) a cut-off. Moreover, the spinning folded string configurations found previously yield infinity energy too,

\[
E_{\text{spin}} = E_s + E_0 + F(J), \quad F(J) > 0, \tag{3.35}
\]

with \(E_0\) finite constant. Although the infinite energy implies that those configurations are not well-defined, it may make sense to ask whether energy/charge relation \(E = E_{\text{spin}} - E_s = E_0 + F(J)\) can be reproduced from dual SYM. If true, the usual relation \(E \propto J\) is modified, as we expected\(^3\).

IV. TYPE B SOLUTION

Here the word “type B” means that in this section we shall use a ansatz in which

\[
\psi' = \dot{\theta}_1 = \dot{\theta}_2 = 0, \tag{4.1}
\]

but \(\phi_1\) and \(\phi_2\) acquire both of \(\tau\)- and \(\sigma\)-dependence. Then following from Eq. (2.9) the equation of motion of \(\psi\) reads off

\[
3\ddot{\psi} = - \sum_{i=1}^{2} \frac{3 + \cos^2 \theta_i}{\sin \theta_i} \theta_i' \phi_i' + 2 \cot \theta_1 \cot \theta_2 (\sin \theta_2 \theta_1' \phi_2' + \sin \theta_1 \theta_2' \phi_1'). \quad (4.2)
\]

The \(\sigma\)-independence of \(\psi\) requires the r.h.s of the above equation is independent of \(\sigma\) too. The simplest solution is then

\[
\theta_1' \phi_1' = \theta_2' \phi_2' = \theta_1' \phi_2' = \theta_2' \phi_1' = 0, \quad \psi = \omega \tau. \quad (4.3)
\]

\[^3\] We do not know whether \(F(J) \propto J\) at large \(J\) limit is still kept. It need get the explicit expression of the solutions.
Since we hope that at least one of $\phi'_i$ does not vanish, we have $\theta'_1 = \theta'_2 = 0$. The conformal constraints in (2.10) reduce to

$$\frac{k^2}{r^2 h(r)} = \frac{r'^2}{r^2} + \frac{1}{6} \sum_{i=1}^{2} \sin^2 \theta_i (\dot{\phi}'_i^2 + \phi'^2_i) + \frac{1}{9} (\omega + \dot{\phi}_1 \cos \theta_1 + \dot{\phi}_2 \cos \theta_2)^2$$

$$+ \frac{1}{9} (\dot{\phi}'_1 \cos \theta_1 + \dot{\phi}'_2 \cos \theta_2)^2,$$

$$\sum_{i=1}^{2} (1 - \frac{1}{3} \cos^2 \theta_i) \dot{\phi}_i \phi'_i + \frac{2}{3} \cos \theta_1 \cos \theta_2 (\dot{\phi}_1 \phi'_2 + \dot{\phi}_2 \phi'_1) = 0.$$  (4.4)

Using those constraints and the above ansatz in equations of motion, one has

$$\frac{r''}{r} - \frac{r'^2}{r^2} - \frac{k^2}{r^2 h(r)} (1 - \frac{a/2}{1 + a \ln (r/r_0)})$$

$$- \frac{a/12}{1 + a \ln (r/r_0)} \left\{ \sum_{i=1}^{2} \phi'^2_i \sin^2 \theta_i + \frac{2}{3} (\phi'_1 \cos \theta_1 + \phi'_2 \cos \theta_2)^2 \right\} = 0,$$

$$2 \omega \dot{\phi}_1 - (\phi'^2_1 - \phi'^2_2) \cos \theta_1 + 2 (\dot{\phi}_1 \phi'_2 - \phi'_1 \phi'_2) \cos \theta_2 + \frac{3 R^2 \sqrt{a}}{r^3 \sqrt{2h}} r' \phi_1 = 0,$$

$$2 \omega \dot{\phi}_2 - (\phi'^2_2 - \phi'^2_1) \cos \theta_2 + 2 (\dot{\phi}_1 \phi'_2 - \phi'_1 \phi'_2) \cos \theta_1 - \frac{3 R^2 \sqrt{a}}{r^3 \sqrt{2h}} r' \phi_2 = 0,$$

$$- \ddot{\phi}_i + \phi''_i + \frac{a \phi'_i}{1 + a \ln (r/r_0)} \frac{r'}{2r} = 0, \quad (i = 1, 2).$$  (4.5)

Here we used the fact that $\sin \theta_i = 0$ corresponds to string shrink to a point at south or north-pole of $S^2$. It is uninterested by us, i.e., we shall always assume that $\sin \theta_i \neq 0$ in rests part of this paper. It should be noticed that, in equations of $r$ and $\phi_i$, the terms involving NS field vanish. Moreover, the equation (4.5) should be supplemented by boundary condition

$$\phi_i(\sigma + 2\pi) = \phi_i(\sigma) + 2m_i \pi$$

with $m_i$ integers.

### A. Circular solutions

Let us first study the circular solutions by imposing $r' = 0$. Here equations of motion of $\phi_i$ reduce to one of free fields on world-sheet. It has the general solution

$$\phi_i = \omega_i \tau + m_i \sigma + \sum_n \{a_{in} \cos n(\tau + \sigma) + b_{in} \cos n(\tau - \sigma)\},$$  (4.6)

with $m, n$ integers. Meanwhile, we notice that the last term in the first equation of (4.5) has to be constant. Substituting the solution (4.6) into this equation one has $a_{in} = b_{in} = 0$. 

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Furthermore, the consistence between the first equation of motion in (4.5) and the first conformal constraint in (4.4) implies
\[ r = r_0 \exp \left( \frac{\Omega^2}{2(\Omega^2 + M^2)} - \frac{1}{a} \right), \] (4.7)

where
\[ \Omega^2 = \sum_{i=1}^{2} \omega_i^2 \sin^2 \theta_i + \frac{2}{3}(\omega_1 \cos \theta_1 + \omega_2 \cos \theta_2)^2, \]
\[ M^2 = \sum_{i=1}^{2} m_i^2 \sin^2 \theta_i + \frac{2}{3}(m_1 \cos \theta_1 + m_2 \cos \theta_2)^2. \] (4.8)

The “physical” condition \( N_{\text{eff}} > 0 \) further imposes that \( \Omega^2 > M^2 \). The above results have been enough to determine the classical energy following from eqs. (2.14) and (4.4):
\[ E = \frac{\sqrt{2} r_0}{\sqrt{3} \alpha'} e^{-1/a} \sqrt{\Omega^2 + M^2} \exp \left( \frac{\Omega^2}{2(\Omega^2 + M^2)} - \frac{1}{a} \right). \] (4.9)

It exhibits an interesting exponential factor, like one for circular string in “type A” solutions.

The task to explore the relation between \( E \) and various spins then reduces to find the relation between \( \Omega, M \) and spins. For all of the following special solutions we shall show that we can always define a convenient “angular momentum” satisfying
\[ J^2 = \frac{\Omega^4}{2(\Omega^2 + M^2)}. \] (4.10)

Then at large charge limit \( J \to \infty \), the classical energy can be expand regularly
\[ E = \frac{2r_0}{\sqrt{3} \alpha'} e^{\frac{1}{2} - \frac{1}{4}J} \left( 1 + \frac{M^2}{J^2} + ... \right). \] (4.11)

In order to find the explicit solutions, we shall solve the remainder equations of motion that now reduces to the linear algebraic equations on \( \cos \theta_i \),
\[ \begin{cases} (\omega_1^2 - m_1^2) \cos \theta_1 - 2(\omega_1 \omega_2 - m_1 m_2) \cos \theta_2 = 2 \omega \omega_1, \\ (\omega_2^2 - m_2^2) \cos \theta_2 - 2(\omega_1 \omega_2 - m_1 m_2) \cos \theta_1 = 2 \omega \omega_2, \end{cases} \] (4.12)
supplemented by the second conformal constraint in (4.4). The equations in (4.12) has several different solutions:

- \( \omega \omega_1 = \omega \omega_2 = 0 \).
The equation (4.12) in this case has trivial solution \( \cos \theta_1 = \cos \theta_2 = 0 \) unless
\[
\omega_1 \omega_2 = m_1 m_2, \quad \omega_1 m_2 = \omega_2 m_1, \quad \Rightarrow \quad \omega_1^2 = \omega_2^2 = m_1^2 m_2^2. \tag{4.13}
\]
Using the above equation to the second conformal constraint in (4.4) one has \( \omega_1 = \omega_2 \).
If \( \omega = 0 \), therefore, the nontrivial spinning configuration is presented only for the trivial solution \( \cos \theta_1 = \cos \theta_2 = 0 \).

1. Two-spin solution: \( \omega = 0, \cos \theta_1 = \cos \theta_2 = 0 \).

The second conformal constraint in (4.4) indicates \( \omega_1 m_1 + \omega_2 m_2 = 0 \), while equation (4.8) reduces to
\[
\Omega^2 = \omega_1^2 + \omega_2^2, \quad M^2 = m_1^2 + m_2^2.
\]
The conserved charges following from eqs. (2.13), (2.14) and (4.4) read off
\[
J_A = \frac{R^2}{3\alpha'} \omega_1 \sqrt{\frac{a \Omega^2}{2(\Omega^2 + M^2)}}, \quad J_B = J_A (\omega_1 \leftrightarrow \omega_2), \quad J_R = 0,
\]
\[
J = \frac{3\alpha'}{R^2 \sqrt{a}} \sqrt{J_A^2 + J_B^2}. \tag{4.14}
\]
Its dual operators in SYM should be like
\[
\text{Tr}(A_1 A_2^\dagger)^{J/2}(B_1 B_2^\dagger)^{J/2}. \tag{4.15}
\]
It is out of holomorphic sector.

2. Single-spin solution: \( \omega \neq 0, \omega_1 = \omega_2 = 0 \).

In this case the equation (4.12) admits two solutions that correspond to different operators in dual field theory. One of them is \( \cos \theta_1 = \cos \theta_2 = 0 \) which leads the conserved charges
\[
J_A = J_B = 0, \quad J_R = \frac{R^2}{9\alpha'} \sqrt{\frac{2 a J}{3}} \frac{2R^2}{9\alpha'} \omega \sqrt{\frac{a \omega^2}{2 \omega^2 + 3 M^2}}. \tag{4.16}
\]
For even \( J \), its dual operators in the holomorphic sector have the form
\[
\text{Tr}(A_1 B_1)^{J/2}(A_2 B_2)^{J/2}. \tag{4.17}
\]
If \( J \) is odd, they should be no longer to be holomorphic operators. Meanwhile, another solution \( m_1 = 0, \cos \theta_2 = 0, \cos \theta_1 = \text{arbitrary constant} \) (or \( m_2 = 0, \cos \theta_1 = 0, \cos \theta_2 = \text{arbitrary} \)) leads to

\[
J_R = \frac{R^2}{9\alpha'} \sqrt{\frac{2a}{3}} \mathcal{J} = \frac{2R^2}{9\alpha' \omega} \sqrt{\frac{a\omega^2}{2\omega^2 + 3M^2}}, \\
J_A = J \cos \theta_1, \quad J_B = 0. \tag{4.18}
\]

Its dual operators must be out of holomorphic sector. In addition, if \( \cos \theta_1 \neq n/J_R \) with \( n, J_R \) integers, the \( U(1)_A \) is anomalous.

- \( \omega \omega_1 \neq 0 \) and/or \( \omega \omega_2 \neq 0 \).

For this assumption the solutions of equation (4.12) together with the second conformal constraint in (4.4) in general lead to tedious expressions of angular momenta and classical energy. For the sake of convenience, we consider an example of two-spin solution with \( \omega_2 = 0 \) that leads the solution of (4.12) as follows

\[
\cos \theta_1 = \frac{2\omega \omega_1}{\omega_1^2 + 3m_1}, \quad \cos \theta_2 = \frac{4\omega \omega_1 m_1}{m_2(\omega_1^2 + 3m_1)}. \tag{4.19}
\]

Substituting the above solution into the second conformal constraint in (4.4) one has

\[
m_1 = 0, \quad \rightarrow \quad \cos \theta_1 = \frac{2\omega}{\omega_1}, \quad \cos \theta_2 = 0. \tag{4.20}
\]

It is then straightforward to write the explicit expressions for angular momenta,

\[
J_R = \frac{2R^2}{3\alpha' \omega} \sqrt{\frac{a\Omega^2}{2(\Omega^2 + M^2)}}, \quad J_A = \frac{\omega_1}{2\omega} J_R, \quad J_B = 0, \\
\mathcal{J} = \frac{3\alpha'}{R^2 \sqrt{a}} \sqrt{J_A^2 + J_R^2/2}. \tag{4.21}
\]

Its dual operators again must be out of holomorphic sector.

**B. Folded solution with \( r' \neq 0 \)**

From eq. (4.5) the most general solution of \( \phi_i \) in this case has the following form:

\[
\phi_i(\tau, \sigma) = \eta_i \tau^2 + \omega_i \tau + T_i(\tau) S_i(\sigma). \tag{4.22}
\]
Recalling that $r$ is independent of $\tau$, the second and the third equations in (4.5) indicates that $\eta_i = 0$, while the first equation in (4.5) implies $T_i$ to be constants. Hence it is straightforward to obtain

$$S'_i = \frac{c_i}{\sqrt{1 - a \ln (r/r_0)}}.$$  \hfill (4.23)

with $c_i$ constants. Substituting Eqs. (4.22) and (4.23) into the second and the third equations in (4.5) we can easily to check that they are inconsistent with the first conformal constraint in (4.4) unless $\omega_1 = \omega_2 = 0$. In other words, there is only single-spin “type B” solution when $r' \neq 0$. It is different from one in “type A” solutions where we can not claim the same conclusion although it is hard to solve corresponding equations of motion with two or three nonzero spins.

In terms of the above results, there is a remainder constraint, $c_1 \cos \theta_1 = c_2 \cos \theta_2 = 0$, following from Eq. (4.5). Finally to solve the equations of motion of $r$ we get nothing but just Eq. (3.21). Hence following the same process in section (III C) we can show that there are folded string solutions with infinite energy.

V. SUMMARY AND DISCUSSIONS

We have studied various spinning closed string configurations on Klebanov-Tseytlin background. This background is described by a logarithmically deformed $AdS_5 \times T^{1,1}$ geometry induced by nontrivial Neveu-Schwarz flux and is dual to a nonconformal $SU(N+M) \times SU(N)$ $\mathcal{N} = 1$ SYM. We studied spinning string solutions with two types of different ansatzs. They describe the classical closed strings rotating on $T^{1,1}$ part of KT background. Explicit expressions for point-like solutions and some circular solutions were found and the energy/charge relation for these solutions were obtained. The folded string solutions along the radial direction was discussed. These folded solutions will in general break usual energy/charge relation $E \propto J$ at large $J$ limit. The dual operators in gauge theory were constructed for diverse solutions.

A. Comparing with solutions in $AdS_5 \times T^{1,1}$ background

To manifest how presence of NS field and deformation of geometry change the integrability of original $AdS_5 \times T^{1,1}$ background, it is worth to compare the solutions obtained in this
present paper with those obtained in $AdS_5 \times T^{1,1}$ geometry. It is obvious that the KT geometry reduces to $AdS_5 \times T^{1,1}$ at the limit $a \to 0$, where the $AdS$ metric is given by Poincaré coordinate. If none of Poincaré coordinates of $AdS_5$ acquiring $\sigma$-dependence, there is no finite energy the spinning string solutions $T^{1,1}$ with static gauge $x^0 = \tau$ unless the radius of $AdS_5$ goes to infinity. This conclusion follows from Eqs. (2.14) and (3.2) that indicates that $E \propto r$ while Eq. (3.3) requires $r^2/R^4 \to 0$ when $r' = 0, \; a \to 0$. The KT background, in contrast, allows the finite energy spinning string configurations in $T^{1,1}$ without requirement to take the limit $R \to \infty$. These solutions, however, locate at special region along the radius of $AdS_5$: The point-like strings locate at $r = r_* = r_0 \exp (1/2 - 1/a)$. While the circular strings are presented in the region $\tilde{r}_s < r \leq r_*$. From perspective of world-sheet sigma model, the KT background bring some interesting features in contrast to $AdS_5 \times T^{1,1}$ background: 1) The equation of motion of radial coordinate $r$ does not decouple with those of $T^{1,1}$ coordinates. This feature induces extra constraints to usual spinning string solutions on $AdS_5 \times T^{1,1}$ background. So that some solutions are forbidden. 2) The NS $B$-field is involved in equations of motion. In fact, the NS field plays trivial role in all of solutions presented in this present paper. So far all attempts to find the solutions in which the NS field plays nontrivial role are failed. In particular, we have shown that those simpler circular solutions are forbidden by NS field role. For solutions with $r' \neq 0$, it is easy to check that the NS field plays nontrivial role only if $\omega_i \neq 0$ and $\theta_i' \neq 0, \; \theta_1 \neq \theta_2$ following from Eqs. (3.2) and (3.3). Seemingly it is impossible to find analytic solutions satisfying all of these requirements. In other words, the presence of NS field should break the integrability of $AdS_5 \times T^{1,1}$ background.

B. Challenges in dual gauge theory

We expect that the various solutions found in the present paper enable us to precisely test general gauge/string duality, like those for AdS/CFT correspondence. Many interesting

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[4] The assertion is not right for global coordinate of $AdS_5$. It is known that there are the spinning string solutions in $S^5$ or $T^{1,1}$ at the center of $AdS_5$, without requirement of the limit $R \to \infty$. The key point is that the static gauge in global coordinate is no longer to be static one in Poincaré coordinate. In other words, both of $x^0$ and $r$ in Poincaré coordinate acquire not only $\tau$-dependence, but also $\sigma$-dependence. In this present paper we insist on using the Poincaré coordinate because the radial coordinate has manifest physical meaning in dual quantum field theory.
features of these solutions, however, bring new challenges to understand them from perspective of dual quantum field theory. For example, how to obtain an integrable spin chain from SYM, and how to construct the dual operators of folded strings?

We have mentioned that in our solutions, the point-like strings locate at \( r = r_s = r_0 \exp \left( \frac{1}{2} - \frac{A}{b} \right) \) and the circular strings are presented in the region \( \tilde{r}_s < r \leq r_s \). It implies that \( r = r_s \) plays special roles in these solutions. Since the flow along radial direction of background in string theory is known to be dual to renormalization group flow in quantum field theory, it indicates that there is a special energy scale in quantum field theory where a sort of phase transition occurs. It is not necessary that this scale is at IR since the value of \( r_* \) depends on \( a \sim g_s M^2/N \). Although in KT background one has \( M/N \ll 1 \), it is possible to impose a large \( g_s M \) such that \( a \) has finite value. Moreover, the phase transition does not correspond to the breaking of chiral symmetry associating with gauge group \( SU(N + M) \times SU(N) \) revealed in [23]. Here it should be associated with the global symmetry \( SU(2) \times SU(2) \).

In all of our solutions, the charges \( J \propto R^2 \sqrt{\alpha \omega / \alpha'} \sim g_s M \omega \). Hence the real parameter controlling regular expansion at double limit is \( \lambda_M^2 / J^2 \) where \( \lambda_m = g_s M \) is ‘t Hooft couplings corresponding to \( SU(M) \) group. While from the point of view of dual SYM, there are two ‘t Hooft couplings \( \lambda_1 = g_1^2 (N + M) \) and \( \lambda_2 = g_2^2 N \) in which \( g_1^2 + g_2^2 \sim g_s^{-1} \sim \text{constant} \), but \( g_1^2 - g_2^2 \) flows logarithmically under renormalization group. These known results yield

\[
\lambda_M \sim \frac{\lambda_1 \lambda_2}{\lambda_1 N/M + \lambda_2 (1 + N/M)}.
\]

It is puzzled since seemingly it is out of one-loop corrections of quantum field theory.

In several solutions we have obtained an implication that the \( U(1)_A \) and/or \( U(1)_B \) presented in table 1 should be anomalous. Therefore another question is how to explore this phenomena from dual SYM. In addition, in ref. [25] authors studied the hadron dynamics in Witten model in terms of semiclassical solutions in dual string theory. The similar studies are expected to be performed for the KT background and its dual gauge theory.

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