Welfare-improving misreported polls

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Abstract
We introduce an electoral pollster in the canonical pivotal voting model and show that the misreporting of pre-election poll results can happen even in the absence of partisan motives, as long as reputational concerns are present. By underreporting the expected number of supporters of the most preferred candidate in society, the pollster can induce an election result more likely to be in line with its report. By doing so, not only victory chances of the most preferred candidate rise above 50%, thus breaking the unrealistic neutrality result of the pivotal voting model, but also total election costs are reduced, thus yielding welfare gains and partially offsetting the expected negative effect of polls on welfare (see Goeree and Großer in Econ Theory 31:51–68, 2007; Taylor and Yildirim in Games Econ Behav 68:353–375, 2010). Our model also allows for the simultaneous accommodation of the underdog effect (a feature of pivotal voting models) and the apparently inconsistent bandwagon effect, in the sense that the latter can actually be understood as an illusion due to the possibility of misreporting being overlooked. All of these results hold even as the electorate size grows without bound.

Keywords Costly voting · Pivotal voting model · Pre-election polls · Misreporting · Bandwagon effect

JEL Classification C70 · C72 · D72 · C46

1 Introduction
The canonical pivotal voting model with two candidates, as in Palfrey and Rosenthal (1985), Ledyard (1984), and Börgers (2004), assumes that citizens participate in an election insofar as there exists a possibility that their vote is pivotal, that is, changes
the election’s result. To compute the probability that their vote is pivotal, citizens must know the distribution of preferences for each candidate within society. In this paper, we include in the pivotal voting model an electoral pollster whose job is to provide such an information before election day.

The effects of pre-election poll results on voter behavior have been studied in Taylor and Yildirim (2010a) and Goeree and Großer (2007). Both papers, using a two-candidate costly voting model, conclude that opinion polls may be harmful to the citizens’ expected welfare by stimulating the “wrong” group of citizens (i.e., the ex-ante minority) to vote more while stimulating the “right” group to vote less. Moreover, the first effect is stronger than the second one. These polls therefore have two negative consequences: they increase the expected aggregate voting cost and decrease the probability that the most preferred candidate wins the election. In their models, however, poll results are necessarily reported truthfully.

We develop a model in which, by formally introducing an electoral pollster, the poll report is endogenously determined and is not necessarily truthful. We assume that, although the pollster knows the true distribution of citizens’ preferences between the two candidates, it may report a different distribution to improve, on average, the fit between election results and citizens’ expectations. The pollster does not have preferences among the candidates: its objective is to be highly rated, by “getting the election right.” In sum, we seek to understand how a hypothetical pollster might report pre-election poll results given that it knows these may alter voter behavior, the actual election result and, ultimately, its reputation.

We will argue that it should act as if its intent were to benefit the candidate who most likely has the support of the majority of the population. To do so, the pollster misreports the pre-election poll results in such a way that the electorate is led to believe that the political divide in society is more balanced than it really is. The real advantage to the pollster, though, is that, by misreporting in this direction, it can lower the expected turnout, and hence the variance of the election result, while still keeping these results reasonably on par with citizens’ expectations. For instance, given an electorate of size 100, a 45–30 victory (with only 25 abstentions) could raise more partisanship suspicions than a 6–4 victory (with 90 abstentions). Note that the latter result would hardly be considered a landslide, even though one candidate still got 50% more votes than the other.

Our main findings regarding the pollster’s behavior are that unless the expected number of supporters of each candidate is the same, for a large enough electorate: (i) truthful reports do not happen; (ii) the pollster underreports the probability of any given citizen favoring the majority candidate; and (iii) this misreporting is actually welfare-improving relative to reporting truthfully. All of these results hold even if the electorate size grows without bound.

To obtain (iii), we consider the limiting distribution of the vote count difference (a Skellam, or Poisson difference, distribution). This approach differs from other welfare analyses in the literature, in the sense that it considers an approximate welfare function (as the one in Taylor and Yildirim 2010a) not with the intent of merely suggesting a result about the exact welfare function but to actually prove such a result (with the aid of some other results from the theory of discrete probability distributions).
Once reputational concerns are brought into the picture, we can also revisit some implications of the canonical pivotal voting model. A classical prediction of rational voting models is the underdog effect, in which citizens in the minority group vote with a greater intensity than citizens in the majority group, as the latter have an incentive to free ride (e.g., Ledyard 1984; Palfrey and Rosenthal 1983). This result is present in and is important for the conclusions of Goeree and Großer (2007), Krasa and Polborn (2009), and Taylor and Yildirim (2010a).

An alternative prediction would be the presence of a bandwagon effect, in which constituents vote with greater intensity if they realize they belong to the majority. Grillo (2017) shows it is possible to generate a bandwagon effect in a rational voting model if citizens are assumed to be risk averse. In contrast, our work implies that, even without departing from the ubiquitous risk-neutrality assumption, a political analyst who assumes that poll reports are always truthful would see data consistent with the bandwagon effect, even though the voters’ behavior generates the underdog effect. Thus, we suggest that part of what is usually attributed to the bandwagon effect could actually be an illusion due to misreporting and not an inconsistency between the classical model and the data.

A particularly unrealistic prediction of rational voting models with fixed and homogeneous voting costs is the neutrality result, namely that both candidates should have equal chances of winning the election in equilibrium regardless of the expected number of supporters of each candidate. In other words, even in a society with a stable ideological distribution of, say, 90–10%, after 20 elections, we should supposedly see about 10 victories for each side. Taylor and Yildirim (2010b) show that, in small elections, if voting costs are not fixed but are rather independently drawn from a common distribution among citizens, the neutrality result breaks in favor of the expected majority candidate. However, for large elections (with the number of citizens tending to infinity), they show that a necessary condition for the neutrality result to break down is heterogeneity of the voting cost distribution among citizens. In our model, neutrality may disappear even if voting costs are fixed (and equal), and even in large elections. More specifically, the probability of victory of the candidate supported by the majority group is necessarily greater than, and bounded away from, 50%.

This paper also relates to the literature on prediction markets and elicitation of probabilities from experts, which goes back to Savage (1971). This literature’s main focus is to characterize proper scoring rules—those inducing the expert to reveal their true beliefs (e.g., Hossain and Okui 2013; Tarnaud 2019). For example, the well-known Brier scoring rule is proper in the simpler context in which the prediction itself does not affect the outcome upon which the expert will be evaluated (e.g., a meteorologist predicting the weather). The situation considered here, however, is more involved since the pollster’s report affects the election result through the citizens’ electoral game. In

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1 In addition to the mentioned “bandwagon abstention effect,” a “bandwagon vote choice effect” could also be considered: some citizens may switch their preference to the candidate who ranks first in the pre-election polls (see Morton and Ou 2015). In our model, however, this is not a possibility since the preference of a given citizen regarding the two candidates is held fixed.

2 Boërgers (2004), Goeree and Großer (2007), Krasa and Polborn (2009), Taylor and Yildirim (2010a,b), for instance, all implicitly assume risk neutrality.
fact, we find that the Brier score is not proper in our environment: the pollster does not reveal the true probability if it believes it will be judged by that rule.\(^3\)

Our work differs from this literature mainly in its objective: we are not looking for a proper scoring rule. Doing that would simply assume away the misreporting problem that we seek to understand from the outset. Rather, we take a positive approach and analyze the conditions under which misreporting happens for a reasonable scoring rule and the welfare consequences thereof. Perhaps surprisingly, we find that social welfare can be made higher when pre-election poll results are misrepresented. That is, at least in terms of welfare, propriety of the scoring rule is undesirable in the electoral poll market.

The rest of the paper is structured as follows. Section 2 describes the model environment, characterizes the equilibria of the citizens’ electoral game given the information released by the pollster, provides relevant comparative statics, and derives the distribution of the difference in the number of votes cast for each candidate. Section 3 presents the scoring rule to which the pollster is subjected and characterizes the solution of its optimization problem. In Sect. 4, we put several of the previous pieces together to address the welfare comparison between misreported and truthfully reported poll results. Section 5 checks for robustness of our main results in a generalized environment allowing for the existence of nonstrategic voters and alternative scoring rules. Section 6 concludes.

### 2 Model environment

The following are the model’s main elements. There are \(n \geq 2\) constituents (henceforth referred to simply as citizens), two candidates (the Blue party candidate \(B\) and the Red party candidate \(R\)), and an electoral pollster. The voting cost \(c \in \mathbb{R}_+\) is homogeneous among citizens. Besides this cost, utility also has an ideological component: a citizen gains 1 unit of utility if their preferred candidate wins the election, and loses 1 unit of utility if the other candidate is the winner. An election tie is broken by the toss of a fair coin, and voting is voluntary.

The probability that a citizen favors candidate \(B\) (or \(R\)) is \(q (1 - q)\). We assume that \(q \in [\tilde{q}, 1 - \tilde{q}]\), where \(\tilde{q} \in (0, 0.5)\).\(^4\) Only the electoral pollster knows the true probability \(q\). It reports that the probability that a citizen favors candidate \(B\) is \(p\). Since citizens do not know \(q\), they use \(p\) as an estimate for this parameter in their voting decisions.\(^5\) We say the pollster is misreporting if \(p \neq q\).

The citizens are instrumental voters. A citizen votes if \(\Pi \times \text{benefit} > c\) and does not vote if \(\Pi \times \text{benefit} < c\), where \(\Pi\) is the probability of being pivotal in the election and “benefit” represents the benefit associated with being pivotal. A voter is pivotal if

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\(^3\) Related to this point, Shi et al. (2009) characterize proper scoring rules aligned with the interests of a principal, in a setting in which the expert can affect the probabilities of events upon which they will be judged.

\(^4\) One can take \(\tilde{q}\) arbitrarily close to 0.

\(^5\) Note that \(p\) should be taken as a probability, not a frequency. Thus, citizens do not factor in their own type in their assessment of the probability of others being \(B\) or \(R\) supporters.
and only if their vote creates or breaks a tie. In both situations, the expected increase in the ideological component of utility is 1, so that benefit = 1.º

Given the probability p reported by the pollster, the citizens play among themselves a Bayesian game. We focus on type-symmetric Bayesian Nash equilibria, in which the strategies played by the citizens are homogeneous within types (B citizens or R citizens) but might differ between these types. From the point of view of a B citizen, given that all of the other B citizens are voting with probability γ and all R citizens are voting with probability δ, the probability that their vote is pivotal is:

\[
\Pi_B (n, p, \gamma, \delta) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1}{k, n-1-2k} (p \gamma)^k (1-p) \delta^k (1-p \gamma - (1-p) \delta)^{n-1-2k}
+ \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \binom{n-1}{k, k+1, n-2-2k} (p \gamma)^k (1-p) \delta^{k+1} (1-p \gamma - (1-p) \delta)^{n-2-2k},
\]

where the first summation refers to the probability of breaking a tie and the second summation refers to the probability of creating a tie. An analogous expression holds for an R citizen: \( \Pi_R (n, p, \gamma, \delta) \) simply equals \( \Pi_B (n, 1-p, \delta, \gamma) \), by the symmetry of the model.

In an interior type-symmetric electoral equilibrium \((\gamma, \delta)\), one has \( \Pi_B (n, p, \gamma, \delta) = c \) and \( \Pi_R (n, p, \gamma, \delta) = c \). Conditions characterizing other type-symmetric equilibrium possibilities can be gathered from the initial discussion in the Supplementary Appendix.

An important feature of the model is that the pollster is not ideological: by assumption, its goal is only to “get the election right.” That is, the electoral result must, in some sense, be coherent given the announced probability p. Bear in mind that this is not trivial because the voters’ equilibrium is a function of p. Since the pollster’s objective function is known only to itself (it does not play any sort of information game with the voters in our setup), its presentation and discussion will be postponed until Sect. 3.

In summary, the timing of events is as follows:

- Nature chooses the type of each citizen;
- The pollster discovers the probability q that any given citizen favors candidate B;
- The pollster reports \( p \in (0, 1) \);
- Each citizen takes \( p \) as given and chooses to vote or to abstain;
- The election takes place.

We assume that this hypothetical pollster has a technology that makes discovery of q costless. Citizens do not have access to such a technology, and are not aware whether (and how) the pollster’s possible reputational concerns are factored in in its report. Therefore, although citizens play a Bayesian game among themselves, it is not the case that they play such a game with the pollster. We make up for what could be seen

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º An implicit assumption is that citizens are risk neutral. Grillo (2017) allows for risk-averse citizens.

7 The floor function \( \lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z} \) maps each real number x into the largest integer less than or equal to x.
as a strategic naïveté on the part of individual citizens by assuming a good degree of statistical savviness on the part of society as a whole, as embodied in its institutions: after election results are in, an electoral commission is bound to check for consistency between the election results and the pre-electoral poll results. In essence, this means that the pollster may be judged as to whether it is guilty of misreporting or not, which can be achieved by means of hypothesis testing.

Involved as it may appear at first sight, when facing insufficient evidence for conviction, hypothesis-testing reasoning—even if not from a strictly mathematical point of view—is employed in real life for being the best a court or regulatory agency can do, given an observed finite sample. Not knowing the significance level (the maximum acceptable probability of a type I error) underlying society’s judgment, the pollster may find it optimal to minimize the expected statistic (“expected” because, at that time, the elections are yet to come) of the test to which it will be subjected (or, equivalently, to maximize its rating, which can also be interpreted as a scoring rule). Such a concern may result in pre-election poll results being misreported at least to a certain degree, even if the pollster has no political affiliation or preference at all.

We now discuss the equilibrium of the electoral game, which the pollster will take into consideration in its choice of the pre-election poll report \( p \) (further detailed in Sect. 3). We have two main goals for the remainder of this section. The first one is to show that, at least for a population size above a certain threshold—which does not depend on \( p \), as will prove essential in the ensuing analysis—, the Bayesian Nash equilibrium of the voting game is interior. The second goal is to provide some of the basic properties of this equilibrium, such as the neutrality result and comparative statics. The third goal is to obtain the distribution of the difference in the total vote count for each candidate, which will be taken into account by the pollster. Each of these objectives will be addressed in turn.

Proofs of this section’s results are relegated to the Supplementary Appendix. Here we present only the main upshots from that analysis.

**Lemma 1** Given \( n \geq 2 \), \( c \in \mathbb{R}^+ \) and \( p \in (0, 1) \), there is one and only one type-symmetric electoral equilibrium, \((\gamma_B(n, c, p), \gamma_R(n, c, p)) \in [0, 1]^2\).

When voting is costless \((c = 0)\), everyone votes in equilibrium \((\gamma_B(n, c, p), \gamma_R(n, c, p)) = (1, 1)\), whereas when \( c \geq 1 \), no one votes \((\gamma_B(n, c, p), \gamma_R(n, c, p)) = (0, 0)\). From now on, we focus on the \( c \in (0, 1) \) case, which brings the possibility of interior equilibria. In this case, as mentioned earlier, \( \Pi_B(n, p, \gamma_B(n, c, p), \gamma_R(n, c, p)) = c \) and \( \Pi_R(n, p, \gamma_B(n, c, p), \gamma_R(n, c, p)) = c \).

A basic yet crucial property of interior electoral equilibria is the well-known (see, e.g., Goeree and Großer 2007; Taylor and Yildirim 2010a) neutrality result of the pivotal voting model, according to which the expected turnout for each candidate \((np\gamma_B(n, c, p) \text{ and } n(1-p)\gamma_R(n, c, p)) \) should be equal. Even more relevant for the developments in the following sections is the fact that, in the case of an interior electoral equilibrium, not only are the expected turnouts for each candidate equal, but actually independent of \( p \) (despite what a first glance at those expressions could

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8 The more comprehensive version of this lemma contained in the Supplementary Appendix completely characterizes the type-symmetric electoral equilibria of the model.
suggest). In this respect, let us write $\gamma^*(n, c) := \gamma_B(n, c, 0.5)$, for all $n \geq 2$ and all $c \in (0, 1)$. We can then collect all these observations in

**Proposition 1** Given $c \in (0, 1)$, there exists an integer $n_0(c) \geq 2$ such that for all $n \geq n_0(c)$ and all $p \in [\bar{q}, 1 - \bar{q}]$, we have $(\gamma_B(n, c, p), \gamma_R(n, c, p)) \in (0, 1)^2$. Moreover, $\gamma_B(n, c, p) = \gamma^*(n, c) / (2p)$ and $\gamma_R(n, c, p) = \gamma^*(n, c) / (2(1 - p))$.

This proposition guarantees the existence of a critical population size for the interiority of the electoral equilibrium that does not depend on $p$, a choice variable of the electoral pollster.\(^9\) As an illustration, given $\bar{q} = 0.05$, for $c = 0.3$, $c = 0.5$, and $c = 0.7$, it would suffice to take $n \geq 69$, $n \geq 23$, and $n \geq 10$, respectively, to ensure interiority of the electoral equilibrium for any $p \in [\bar{q}, 1 - \bar{q}]$. Considering $n \geq n_0(c)$, this proposition also yields some basic properties of $\gamma^*(n, c)$ and of the $\gamma_B(n, c, \cdot)$ and $\gamma_R(n, c, \cdot)$ functions restricted to $[\bar{q}, 1 - \bar{q}]$: (i) $\gamma^*(n, c) \in (0, 1)$; (ii) $\gamma_B(n, c, \cdot)$ is an equilateral hyperbola; and (iii) $\gamma_R(n, c, p) = \gamma_B(n, c, 1 - p)$.

As mentioned in the introductory section, the neutrality result implied by the above proposition is at odds with electoral data. In the next section we argue that the apparent discrepancy between theory and data can be better understood if one recognizes that the pollster may have incentives to misreport; that is, honesty is not assumed from the outset.

The comparative statics of the type-symmetric electoral equilibrium with respect to parameters $c$ and $p$ are described in the following proposition. A more detailed version of this result covering the issues of strict monotonicity, continuity, and differentiability (needed for our propositions) is stated and proved in the Supplementary Appendix.

**Proposition 2** Given $n \geq 2$, the functions $\gamma_B(n, \cdot, p)$, $\gamma_B(n, c, \cdot)$ and $\gamma_R(n, \cdot, p)$ are decreasing, while $\gamma_R(n, c, \cdot)$ is increasing.

The intuition is as follows. First, as voting becomes more costly, the net benefit of voting decreases, and thus we should expect a smaller turnout. Second, the probability according to which citizens vote is decreasing in the perceived proportion of citizens of their type, for if citizens believe there are many others who support their preferred candidate, then they have a larger incentive to free ride, thus avoiding the cost $c$.

The following lemma collects asymptotic results regarding $\gamma^*(n, c)$ (and, consequently, $\gamma_B(n, c, p)$ and $\gamma_R(n, c, p)$).

**Lemma 2** Given $c \in (0, 1)$, we have $\lim_{n \to \infty} \gamma^*(n, c) = 0$ and $\lim_{n \to \infty} n \gamma^*(n, c) > 0$.

The convergence of $\gamma^*(n, c)$ (and, by Proposition 1, of $\gamma_B(n, c, p)$ and $\gamma_R(n, c, p)$ for any fixed $p \in [\bar{q}, 1 - \bar{q}]$) to 0 originates from Palfrey and Rosenthal (1985), while the convergence of $n \gamma^*(n, c)$ (whence that of $n \gamma_B(n, c, p)$ and $n \gamma_R(n, c, p)$ as well) stems from Taylor and Yildirim (2010a). We will refer to $\lim_{n \to \infty} n \gamma^*(n, c)$ as $m(c)$.

\(^9\) This is where the assumption $q \in [\bar{q}, 1 - \bar{q}]$ comes into play. With this assumption in hand, the pollster’s choice of $p$ is naturally constrained to the $[\bar{q}, 1 - \bar{q}]$ interval as well. If we had let $q \in (0, 1)$, there would not exist a uniform critical population size capable of guaranteeing that, regardless of the $p \in (0, 1)$ chosen by the pollster, the electoral equilibrium is interior.
The results mentioned so far hold regardless of the announced probability \( p \) being equal to the true probability \( q \) or not, and will be useful for the analysis and intuition regarding endogenization of \( p \) in the following sections. There, we will also see that it makes sense, in analyzing both the pollster’s behavior and its welfare implications, to consider the distribution of the difference in the number of votes cast for the two candidates. Since such analysis is not standard in this literature, it is presented below.

The election result is a random vector \((b, r, a)\), where \( b \), \( r \), and \( a \) denote the vote count for \( B \), the vote count for \( R \), and the number of abstentions, respectively. The true probability distribution (known only to the pollster) of \((b, r, a)\) is

\[
\text{Multinomial}(n, q \gamma_B(n, c, p), (1 - q) \gamma_R(n, c, p), 1 - q \gamma_B(n, c, p)) .
\]

\[ (1) \]

To simplify the notation momentarily, write the distribution of \((b, r, a)\) in \((1)\) as Multinomial \((n, \beta, \rho, 1 - \beta - \rho)\). Let us call the distribution of \( b - r \) the multinomial difference distribution, \( \text{MultiDiff}(n, \beta, \rho) \). Its characteristic function \( \varphi_{\text{MultiDiff}(n, \beta, \rho)}(t) \) can be obtained as follows.

First, note that \( b - r = d \) if and only if \( d + n = b + (n - r) = b + (b + a) = 2b + a \). Now, on the one hand, we have, for any \( t \in \mathbb{R} \),

\[
\varphi_{\text{MultiDiff}(n, \beta, \rho)}(t) = E\left( e^{itd} \right) = e^{-itn} E\left( e^{it(2b+a)} \right) .
\]

On the other hand, \( E\left( e^{it(2b+a)} \right) \) can be thought of as \( \varphi_{\text{Multinomial}(n, \beta, \rho, 1 - \beta - \rho)}(2t, 0, t) \), which equals \( (\beta e^{i2t} + \rho e^{i0} + (1 - \beta - \rho) e^{it})^n \), or still, \( e^{itn} (1 + \beta (e^{it} - 1) + \rho (e^{-it} - 1))^n \). Therefore,

\[
\varphi_{\text{MultiDiff}(n, \beta, \rho)}(t) = \left( 1 + \beta \left( e^{it} - 1 \right) + \rho \left( e^{-it} - 1 \right) \right)^n .
\]

\[ (2) \]

Once we have studied the determination of \( p \) by the pollster, the characteristic function given in \((2)\), in conjunction with Proposition 1, will also allow us to find (through Levy’s Continuity Theorem) the asymptotic distribution of the difference of votes. This will be key in establishing our welfare results, which will hold not only in the limit but also for any sufficiently large \( n \).

### 3 Electoral pollster’s behavior

The main concern of the pollster is to be considered of good quality, by “getting the election right.” The election result, a realization of the random vector \((b, r, a)\), is known to all citizens right after the election. Although \((b, r, a)\) is distributed according to \((1)\), the citizens, based on the information reported by the pollster, \( p \), believe that it

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\[ ^{10} \text{The characteristic function of the multinomial distribution can be checked in Johnson et al. (1997, p. 37).} \]
is distributed as

\[
\text{Multinomial} \left( n, p \gamma_B(n, c, p), (1 - p) \gamma_R(n, c, p), 1 - p \gamma_B(n, c, p) \right)
- (1 - p) \gamma_R(n, c, p)) .
\]

(3)

The pollster’s “rating” depends on how the actual election result compares to the result implied by the report. We assume that given an election result \((b, r, a)\), this rating is given by

\[
- \left[ (b - np \gamma_B(n, c, p))^2 + (r - n (1 - p) \gamma_R(n, c, p))^2 \right],
\]

which is a measure of how far the election result was from the result expected by society. Defining \(e_B(b, n, c, p) := b - np \gamma_B(n, c, p)\) and \(e_R(r, n, c, p) := r - n (1 - p) \gamma_R(n, c, p)\) as the pollster’s prediction errors, from the perspective of society, regarding \(B\) votes and \(R\) votes, respectively, the rating is

\[
- \left[ e_B(b, n, c, p)^2 + e_R(r, n, c, p)^2 \right] .
\]

(4)

This resembles, in absolute value,

\[
\chi^2 := \frac{e_B(b, n, c, p)^2}{np \gamma_B(n, c, p)} + \frac{e_R(r, n, c, p)^2}{n (1 - p) \gamma_R(n, c, p)}
+ \frac{(a - n (1 - p) \gamma_B(n, c, p) - (1 - p) \gamma_R(n, c, p))^2}{n (1 - p) \gamma_B(n, c, p) - (1 - p) \gamma_R(n, c, p)},
\]

the value of the statistic of Pearson’s chi-squared test of goodness of fit for a multinomial distribution, with a null hypothesis that the true multinomial distribution of \((b, r, a)\) has parameters \((p \gamma_B(n, c, p), (1 - p) \gamma_R(n, c, p), 1 - p \gamma_B(n, c, p) - (1 - p) \gamma_R(n, c, p))\); that is, that the pollster is being truthful. In fact, by Proposition 1, for a sufficiently large \(n\), both the first and the second denominators in this formula equal \(n \gamma^*(n, c) / 2\), so that, by Lemma 2, they both converge as \(n \to \infty\) to the same positive number \(m(c) / 2\), whereas the third denominator, equal to \(n - n \gamma^*(n, c)\), diverges. Therefore, it is only logical for a zero weight to be attached to the third term in Pearson’s formula (the one concerning abstentions) and for equal positive weights to be given to its first two terms, as in (4).

Not only is this zero weight advantageous from an analytical perspective, but it also ensures that we are not forcing the turnout of the election (or, equivalently, its abstentions) to be a driving force behind our results. In fact, including an \((a - n (1 - p) \gamma_B(n, c, p) - (1 - p) \gamma_R(n, c, p))^2\) term in the pollster’s objective function would only intensify its misreporting behavior (although the resulting solution to the pollster’s problem would retain the same characteristics as the solution to be presented).\(^{11}\)

Nevertheless, as we will see shortly, for large elections it so hap-

\(\text{11} \) We thank an anonymous referee for raising this point.
pens that the pollster ultimately chooses to misreport pre-election polls in a way that mimics an interest to minimize the election turnout.

Since the pollster has no clue as to the significance level to be employed in society’s hypothesis-testing-based judgment, it is in its best interest to generate the lowest value of \( \chi^2 \)—that is, the highest rating—possible. As the actual election result is random, we pose that the pollster is interested in maximizing the expected value of its rating.

\[
T (n, c, p, q) := -E \left[ e_B (b, n, c, p)^2 + e_R (r, n, c, p)^2 \right],
\]

where \( E \) is taken with respect to the true probability distribution (1), which depends on \( q \). Thus, the pollster’s optimization problem is

\[
\max_{p \in [\bar{q}, 1-\bar{q}]} T (n, c, p, q).
\]

However, it should be stressed that specifics of the pollster’s decision process regarding the reporting of pre-election poll results, such as whether it has an objective function at all or what that would be, are unknown to the public.

Function \( T \) can be thought of as a scoring rule used to measure the accuracy of the pollster’s prediction (i.e., the probabilistic prediction that poll results should follow distribution (3)). In this decision-theoretic terminology, what we are about to show in this section is that, typically, \( T \) is not proper—in other words, it usually pays off to misreport pre-election polls. An exception would be a situation in which everybody votes regardless of the pollster’s actions, as will be verified in Proposition 3 (which is more akin to the context of reporting a weather forecast, to which air masses cannot react).\(^{12}\) In the next section, we argue that this nonpropriety is actually a good thing.

Section 5 explores these issues further, both by employing some other standard scoring rules, and by allowing for the existence of nonstrategic voters, to test the robustness of our findings.

We denote a solution to problem (5) by \( p^*_n (c, q) \). Because it is commonly known that \( q \in [\bar{q}, 1-\bar{q}] \), all citizens would know for sure that the pollster was not being truthful if it released \( q < \bar{q} \) or \( q > 1-\bar{q} \). That is why the constraint in that problem is set as \( p \in [\bar{q}, 1-\bar{q}] \) which, along with the continuity of \( T \) in \( p \), ensures existence of a solution.

To better grasp the incentives the pollster faces, we can rewrite its objective function as

\[
T (n, c, p, q) = - \left[ E^2 (e_B (b, n, c, p)) + \Var (e_B (b, n, c, p)) 
+ E^2 (e_R (r, n, c, p)) + \Var (e_R (r, n, c, p)) \right],
\]

where expected values and variances are computed using distribution (1). Thus, the pollster faces a tradeoff between minimizing the expected value and the variance of its prediction errors.

\(^{12}\) We are indebted to an anonymous associate editor for the observation that our model could be framed in this way.
On the one hand, since

\[ E(e_B(b, n, c, p)) = n(q - p)\gamma_B(n, c, p) \]  

(7)

and

\[ E(e_R(r, n, c, p)) = n(p - q)\gamma_R(n, c, p), \]  

(8)

to minimize

\[ E^2(e_B(b, n, c, p)) + E^2(e_R(r, n, c, p)) = n^2(p - q)^2\left(\gamma_B^2(n, c, p) + \gamma_R^2(n, c, p)\right) \]

the pollster should release a truthful report \((p=q)\). This would make the election a tie in expected terms since, by Proposition 1, the pollster’s estimated number of votes for \(B\) and \(R\) would coincide: \(n(1-q)\gamma_R(n, c, p) = n(1-p)\gamma_R(n, c, p) = np\gamma_B(n, c, p) = nq\gamma_B(n, c, p)\). Any other choice of \(p\) would favor the electability of one of the candidates (candidate \(B\) if \(p < q\) and candidate \(R\) if \(p > q\), potentially undermining the pollster’s credibility.

On the other hand, since

\[ \text{Var}(e_B(b, n, c, p)) = nq\gamma_B(n, c, p)(1 - q\gamma_B(n, c, p)) \]  

(9)

and

\[ \text{Var}(e_R(r, n, c, p)) = n(1 - q)\gamma_R(n, c, p)(1 - (1 - q)\gamma_R(n, c, p)), \]  

(10)

if \(c \in (0, 1)\) and \(n \geq n_0(c)\), Proposition 1 applies and we obtain

\[ \text{Var}(e_B(b, n, c, p)) + \text{Var}(e_R(r, n, c, p)) = n \left(\frac{q \gamma^*}{p} \left(1 - \frac{q \gamma^*}{p^2} \right) + \frac{1 - q \gamma^*}{1 - p} \left(1 - \frac{1 - q \gamma^*}{1 - p^2} \right)\right), \]  

(11)

where \(\gamma^*\) is short for \(\gamma^*(n, c)\). This is not minimized by taking \(p=q\). In fact, the derivative of this expression with respect to \(p\) is

\[ \frac{n\gamma^*}{2p^3(1-p)^3} \left( (2q - 1)p^4 - (2\gamma^*q^2 + 2(2 - \gamma^*)q + \gamma^* - 1) p^3 + 3q(\gamma^*q + 1)p^2 - q(3\gamma^*q + 1)p + \gamma^*q^2 \right), \]

which, at \(p=q\), equals \(n\gamma^*(1-\gamma^*)(2q-1)/(2q(1-q))\), which is only 0 if \(q=0.5\) (in which case the pollster will indeed report truthfully, as shown in Proposition 4 ahead).

Given this tradeoff, to avoid putting its reputation at risk, the pollster is typically willing to accept nonzero expected errors, as long as their variance is sufficiently small. For instance, if \(q > 0.5\), since this expression for the partial derivative of
Var \((e_B (b, n, c, p)) + Var \((e_R (r, n, c, p))\) with respect to \(p\) at \(p = q\) becomes positive, reporting \(p < q\) may be beneficial to the pollster—even in the absence of any partisan interest whatsoever. That this is in fact the case will be proved in Proposition 4, the central result of this section.

An alternative but related way of understanding this sum of variances is to note, from the covariance formula of a multinomial distribution, that

\[
\text{Cov} (b, r) = -nq\gamma_B (n, c, p) (1 - q) \gamma_R (n, c, p) = -n \left( \frac{\gamma^*}{2} \right)^2 \frac{q}{p} \frac{1 - q}{1 - p},
\]

so that, since \(\text{Var} (b - r) = \text{Var} (b) + \text{Var} (r) - 2 \text{Cov} (b, r)\), (11) gives

\[
\text{Var} (b - r) = \frac{n\gamma^*}{2} \left( \frac{q}{p} \left(1 - q \gamma^* \right) + \frac{1 - q}{1 - p} \left(1 - \frac{1 - q \gamma^*}{2} \right) \right) + \frac{n\gamma^* q}{2} \frac{1 - q}{p} \gamma^*.
\]

Lemma 2 thus yields that, for large \(n\), \(\text{Var} (b - r)\) should approach

\[
\frac{m(c)}{2} \left( \frac{q}{p} + \frac{1 - q}{1 - p} \right), \tag{12}
\]

just as \(\text{Var} (e_B (b, n, c, p)) + \text{Var} (e_R (r, n, c, p))\) does (from (11)). One can readily note that expression \(q / p + (1 - q) / (1 - p)\) equals 2 both if \(p = q\) and if \(p = 0.5\), is lower than 2 for all \(p\) lying between 0.5 and \(q\), and greater than 2 otherwise. This indicates that, unlike the average channel of the pollster’s problem (which always points to \(p = q\)), the variance channel points to a \(p\) between 0.5 and \(q\).

Expression (12) can also be interpreted as the asymptotic expected turnout of the election:

\[
\lim_{n \to \infty} \left( nq\gamma_B (n, c, p) + n (1 - q) \gamma_R (n, c, p) \right) = \lim_{n \to \infty} \frac{n\gamma^* (n, c)}{2} \left( \frac{q}{p} + \frac{1 - q}{1 - p} \right).
\]

In this sense, for large \(n\), to minimize \(\text{Var} (b) + \text{Var} (r)\), the pollster may want to choose a \(p\) close to the one that minimizes the asymptotic turnout, (12), instead—that is, \(p = \phi (q) := \left(1 + \sqrt{1/q - 1}\right)^{-1}\).

In fact, as a step in showing the comparative statics results in Proposition 5, we will establish that the \(p\) reported by the pollster necessarily lies between \(q\) and \(\phi (q)\) (see Lemma 3).

This makes sense since, although the only way of matching the \(B\) and the \(R\) turnouts on average (in an interior equilibrium setting, \(nq\gamma_B (n, c, p) = n (q / p) \gamma^* / 2\) and \(n (1 - q) \gamma_R (n, c, p) = n ((1 - q) / (1 - p)) \gamma^* / 2\)) is by reporting \(p = q\), the pollster’s rating is based not on averages of elections, but on this particular election. In

13 The expression within parentheses in the expression for the asymptotic turnout, \(q / p + (1 - q) / (1 - p)\), is also what should be minimized by a pollster concerned solely with the average of a Brier or logarithmic scoring rule. This is duly addressed in Sect. 5. We are grateful to an associate editor for suggesting this line of investigation to us.

14 This also happens to be the value that the pollster would choose if it wished its report to reflect the expected turnout for \(B\) divided by the expected turnout for \(R\) in the election (i.e., so that \(p / (1 - p) = (q\gamma_B (n, c, p)) / ((1 - q) \gamma_R (n, c, p)) = (q / p) / ((1 - q) / (1 - p)))\).
this way, it is willing to report a $p$ that does not imply an expected election tie as long as the vote count difference between $B$ and $R$ is still small and has less dispersion. The simplest way to achieve this effect is by making less people show up on election day. This can be achieved by picking a $p$ strictly between $0.5$ and $q$, so that (say, in the $q > 0.5$ case), although the expected turnout for $B$ rises from $n\gamma^*(n, c)/2$ to $(n\gamma^*(n, c)/2)(q/p)$, this increase is more than offset by the expected decrease in the turnout for $R$, from $n\gamma^*(n, c)/2$ to $(n\gamma^*(n, c)/2)((1 - q)/(1 - p))$. In fact, if the turnout given $p \in (0.5, q)$ were not smaller than that given $p = q > 0.5$, one would have

$$\frac{n\gamma^*(n, c)}{2} \left(\frac{q}{p} + \frac{1 - q}{1 - p}\right) \geq \frac{n\gamma^*(n, c)}{2} \left(\frac{q}{q} + \frac{1 - q}{1 - q}\right),$$

thus implying (for an interior equilibrium) $q(1 - p) + (1 - q)p \geq 2p(1 - p)$, or $q(1 - 2p) = q - 2pq \geq 2p(1 - p) - p = p(1 - 2p)$, so that $q \leq p$, a contradiction.

Note that if voting were costless, then it would never be in the pollster’s best interest to misreport, as the following proposition shows (its proof is available in the Appendix, along with the proofs of all the remaining results of the paper).

**Proposition 3** Given $n \geq 2$ and $q \in [\bar{q}, 1 - \bar{q}]$, if $c = 0$, then $p^*_n(c, q) = q$ is the unique solution to the pollster’s problem.

Thus, it is most natural to analyze the issue of misreporting of pre-election poll results within a costly voting framework, in which the pollster’s choice of $p$ affects turnouts (and as a result, the probability that either party will win the election), as presented in Proposition 2.

We are now ready to state and prove the main result of this section, which confirms the misreporting behavior suggested in the discussion above. According to it, misreporting is the norm, rather than the exception. This proposition is also at the heart of the welfare results in the next section.

**Proposition 4** Given $q \in [\bar{q}, 1 - \bar{q}]$, $c \in (0, 1)$ and $n \geq n_0(c)$, the solution of the pollster’s optimization problem, $p^*_n(c, q)$, is unique and such that

i. if $q = 0.5$, then $p^*_n(c, q) = q$;

ii. if $q > 0.5$, then $p^*_n(c, q) \in (0.5, q)$; and

iii. if $q < 0.5$, then $p^*_n(c, q) \in (q, 0.5)$.

Thus, if there exists an expected majority group in society ($q \neq 0.5$) and $n$ is sufficiently large so that the electoral equilibrium is interior, then a rational pollster driven purely by reputational motives will always misreport information. Moreover, it will do so in such a way that society will be led to believe that the majority group is not as large as it actually is, thus tilting upward the expected vote count for this group’s candidate and downward the expected vote count for the other candidate.

This proposition also establishes that fixed $n \geq 2$, $c \in (0, 1)$ and $n \geq n_0(c)$, $p^*_n(c, \cdot) : [\bar{q}, 1 - \bar{q}] \to [\bar{q}, 1 - \bar{q}]$ is a well-defined, single-valued function. It is plotted in Fig. 1 for $n = 100$ and $c \in \{0.3, 0.5, 0.7\}$, together with the 45° line (the graph of $p^*_n(0, \cdot)$, by Proposition 3).
Also present in the figure is the aforementioned bound $\phi(q)$, which lies between 0.5 and $q$. This hints to the following strengthening of Proposition 4.

**Lemma 3** Given $c \in (0, 1)$ and $n \geq n_0(c)$, if $q \in (0.5, 1 - \bar{q})$, then $p^*_n(c, q) \in (\phi(q), q)$, and if $q \in [\bar{q}, 0.5)$, then $p^*_n(c, q) \in (q, \phi(q))$, where $\phi(q) := (1 + \sqrt{1/q - 1})^{-1}$.

This more precise bound helps in the addressing of the comparative statics issues regarding $p^*_n$, stated in the following proposition.

**Proposition 5** Given $q \in (\bar{q}, 1 - \bar{q})$, $c \in (0, 1)$ and $n \geq n_0(c)$, the function $p^*_n$ is continuously differentiable. Furthermore,

i. $\partial p^*_n(c, q) / \partial q > 0$; and

ii. $\partial p^*_n(c, q) / \partial c \preceq 0$ if $q \succeq 0.5$.

A couple of comments about the implications of our model are in order. First, given the possibility of misreporting, the election will not be a tie in expected terms. Rather, if $q > 0.5$, then the expected result of the election is a win for candidate $B$, as could be anticipated by the fact that the expected vote count for $B$ ($nq\gamma^*/(2p)$) is greater than the expected vote count for $R$ ($n(1-q)\gamma^*/(2(1-p))$). Thus, our model suggests a mechanism to explain why, given $q \neq 0.5$, the candidate of the minority is not expected to win every other election, as would be implied by the canonical pivotal voting model with fixed and equal voting costs and truthful pre-election poll reports.\(^{15}\)

This is formalized in

\(^{15}\) Alternatively, Taylor and Yildirim (2010b) show that for a finite population size, the neutrality result may also break in favor of the majority as long as voting costs are drawn from a common nondegenerate distribution.
Lemma 4: Given $c \in (0, 1)$, $n \geq n_0(c)$ and $q \in [\bar{q}, 1 - \bar{q}]$, we have $\Pr(B \text{ wins } | n, c, q, \bar{q}) = 0.5$, and

i. if $q = 0.5$, then $\Pr(B \text{ wins } | n, c, p_n^*(c, q)) = 0.5$;

ii. if $q > 0.5$, then $\Pr(B \text{ wins } | n, c, p_n^*(c, q)) > 0.5$; and

iii. if $q < 0.5$, then $\Pr(B \text{ wins } | n, c, p_n^*(c, q)) < 0.5$.

Second, we provide a new interpretation for the emergence of the so-called bandwagon effect—the phenomenon according to which supporters of a specific candidate are more likely to cast their votes if their candidate ranks first in pre-election polls. Our model implies that part of this effect could actually be an illusion. In an interior equilibrium, citizens expect that, on average, the election will be a tie—or, in other words, the ratio of the expected vote count for each candidate should be

$$\frac{n p_n^*(c, q) \gamma_B(n, c, p_n^*(c, q))}{n(1 - p_n^*(c, q)) \gamma_R(n, c, p_n^*(c, q))} = 1,$$

by Proposition 1. However, in actuality, this ratio would be

$$\frac{n q \gamma_B(n, c, p_n^*(c, q))}{n(1 - q) \gamma_R(n, c, p_n^*(c, q))} = \frac{q}{p_n^*(c, q)},$$

which is greater than 1, by Proposition 4.

For example, if $q = 0.80$, $c = 0.7$, and $n = 100$, the pollster would report $p = p_{100}^*(0.7, 0.80) \approx 0.71$, and the above ratio would be approximately 1.62. Thus, candidate $B$ would receive, on average, 62% more votes than candidate $R$. By observing such a discrepancy in a given election, observers who believe that poll results are truthfully reported could erroneously be led into thinking that, rather than the underdog effect predicted by Proposition 2, the bandwagon effect was in place. In particular, it could occur to them that the electoral equilibrium $(\gamma_B(n, c, p), \gamma_R(n, c, p))$ generated by the pivotal voting model was incorrect—namely, that the first (second) argument of this equilibrium vector was higher (lower) than predicted—without realizing that candidate $B$’s ($R$’s) vote count should not be compared to $np \gamma_B(n, c, p)$ $(n(1 - p) \gamma_R(n, c, p))$ in the first place, but to $n q \gamma_B(n, c, p)$ $(n(1 - q) \gamma_R(n, c, p))$.

In this way, our model implies that part of what is usually attributed to the bandwagon effect can actually be a direct consequence of misreporting.

Having studied the pollster’s rational reporting behavior for a fixed (and sufficiently large) population size, one could ask whether such behavior is qualitatively different as the electorate size grows without bound. The following proposition shows that this is not the case.

Proposition 6: Given $c \in (0, 1)$ and $q \in [\bar{q}, 1 - \bar{q}]$, $\lim_{n \to \infty} p_n^*(c, q)$ exists and, if denoted by $p^*_\infty(c, q)$, is such that

i. if $q = 0.5$, then $p^*_\infty(c, q) = q$;

16 There is a slight abuse of notation here, in the sense that $n, c, p$ and $q$ are not events but parameters.
ii. if $q > 0.5$, then $p^*_\infty (c, q) \in (0.5, q)$; and

iii. if $q < 0.5$, then $p^*_\infty (c, q) \in (q, 0.5)$.

Along with Proposition 1 and the characteristic function of the multinomial difference distribution in (2), this result makes the derivation of the asymptotic distribution of the vote count difference possible, as stated in the following lemma. As anticipated in Sect. 2, this is important for the welfare discussion to be carried out in the next section.

**Lemma 5** Given $c \in (0, 1)$ and $q \in [\bar{q}, 1 - \bar{q}]$, the number of $B$ votes minus the number of $R$ votes converges in distribution to a random variable $Z$ distributed as Skellam $\left( (q / p^*_\infty (c, q)) m (c) / 2, \left( (1 - q) / \left( 1 - p^*_\infty (c, q) \right) \right) m (c) / 2 \right)$.

Given the possibility of misreporting (for $q \neq 0.5$), $p^*_\infty (c, q)$ exists and is not equal to $q$, by Proposition 6. Lemma 5 says that even the asymptotic distribution of the difference of votes will be skewed. In fact, if we use $q > 0.5$ to fix ideas, it will be skewed in favor of candidate $B$ since, by Proposition 6, $(q / p^*_\infty (c, q)) m (c) / 2 > m (c) / 2 > \left( (1 - q) / \left( 1 - p^*_\infty (c, q) \right) \right) m (c) / 2$.

### 4 Welfare analysis

Now that we have characterized the electoral equilibrium and the solution to the pollster’s problem, we can analyze the welfare implications of misreporting. Let $\mathcal{I} (n, c, p, q)$ correspond to the expected (in the eyes of an outside observer, who knows not only $p$ but also $q$) aggregate ideological component of utility within society (i.e., citizens favoring the winning candidate earn +1, and all others earn −1). The expected cost $\mathcal{C} (n, c, p, q)$ of the election is given by

$$\mathcal{C} (n, c, p, q) = nq \gamma_B (n, c, p) c + n (1 - q) \gamma_R (n, c, p) c,$$

whereas the welfare function $\mathcal{W}$ is given by $\mathcal{W} (n, c, p, q) := \mathcal{I} (n, c, p, q) - \mathcal{C} (n, c, p, q)$.

To proceed with the computation of $\mathcal{I} (n, c, p, q)$, the random vector $(b, r, a)$ is insufficient, since we do not know how many of those $a$ abstentions correspond to $B$ supporters and how many correspond to $R$ supporters. If we denote by $n_B$ the number of $B$ citizens, the expected aggregate ideological component of utility within society, given $n, c, p$ and $q$, is

$$\mathcal{I} (n, c, p, q) = \mathbb{E}_{n_B} \left[ \operatorname{Pr} (B \text{ wins} | n, c, p, q, n_B) (n_B - (n - n_B)) + \operatorname{Pr} (R \text{ wins} | n, c, p, q, n_B) ((n - n_B) - n_B) \right]$$

$$= \mathbb{E}_{n_B} \left[ (2 \operatorname{Pr} (B \text{ wins} | n, c, p, q, n_B) - 1) (2n_B - n) \right].$$

---

17 Given $m_B, m_R > 0$, Skellam $(m_B, m_R)$ is the distribution of $X - Y$, where $X \sim \text{Poisson} (m_B)$ and $Y \sim \text{Poisson} (m_R)$ are independent. It is also called the Poisson difference distribution.

18 We are only interested in the utility of the citizens; that is, the utility function of the pollster does not enter the welfare function.
since $\Pr (R \text{ wins } | n, c, p, q, n_B) = 1 - \Pr (B \text{ wins } | n, c, p, q, n_B)$.

An exact expression for this ideological component of utility would be rather difficult to work with.\(^{19}\) However, a tremendous shortcut would be made possible if we were to disregard the dependence between $n_B$ and the probability that candidate $B$ wins the election. In that case, we could use the following approximation for $I (n, c, p, q)$:

$$I (n, c, p, q) = (2 \Pr (B \text{ wins } | n, c, p, q) - 1) E_{n_B} (2n_B - n)$$

$$= (2 \Pr (B \text{ wins } | n, c, p, q) - 1) (2n_B - n), \quad (14)$$

where $E_{n_B} (n_B)$ equals $nq$ since $n_B$ is binomially distributed with parameters $(n, q)$.

Let us argue that this is a reasonable approximation when $n$ is large, in the sense that $\lim_{n \to \infty} (( I (n, c, p, q) - I (n, c, p, q)) /n) = 0$ (i.e., the difference in the per capita ideological component of utility when measured through $I$ and $I$ is negligible). This approach differs from other welfare analyses present in the literature in the sense that it considers an approximate welfare function ($I - C$, as in Taylor and Yildirim, 2010a) not as a means of merely suggesting a result about the exact welfare function ($I - C$) but to actually prove such a result (with the aid of properties of the Skellam distribution).

Call $x_{n,n_B} := \Pr (B \text{ wins } | n, c, p, q, n_B)$ and $y_n := \Pr (B \text{ wins } | n, c, p, q)$, just to save a bit on notation. Note that

$$I (n, c, p, q) - I (n, c, p, q) = \frac{E_{n_B} (2x_{n,n_B} - 1) (2n_B - n) - (2y_n - 1) (2q - 1)}{n}$$

$$= E_{n_B} (2x_{n,n_B} - 1) \left( \frac{2n_B}{n} - 1 \right) - (2y_n - 1) (2q - 1),$$

$$= 2 E_{n_B} (2x_{n,n_B} - 1) \left( \frac{n_B}{n} - q \right),$$

where we have used the Law of Iterated Expectations to get $E_{n_B} [(2x_{n,n_B} - 1) (2q - 1)] = 2 (2q - 1) E_{n_B} (x_{n,n_B} - y_n) = 0$.

Note that both $2x_{n,n_B} - 1$ and $n_B/n - q$ have absolute values bounded by 1 and that, by the Strong Law of Large Numbers, $n_B/n - q$ converges almost surely to 0.

\(^{19}\) If abstentions $a$ are decomposed as $\tilde{b} + \tilde{r}$, where $\tilde{b}$ and $\tilde{r}$ represent $B$ and $R$ supporters who choose not to vote, then we could write $I (n, c, p, q)$ as

$$\sum_{b,r,\tilde{b},\tilde{r} \geq 0 \atop b + \tilde{r} = n} (b + \tilde{b} - r - \tilde{r}) \left( \sum_{\tilde{b},r,\tilde{b},\tilde{r}}^{n} (q_{Y_B}) (q_{1 - q} - Y_B)^r (q_{1 - Y_B})^{\tilde{b}} (1 - q) (1 - Y_B) \right)^{\tilde{r}}$$

$$+ \sum_{b,r,\tilde{b},\tilde{r} \geq 0 \atop b + \tilde{r} = n} (r + \tilde{b} - b - \tilde{r}) \left( \sum_{\tilde{b},r,\tilde{b},\tilde{r}}^{n} (q_{Y_B}) (q_{1 - q} - Y_B)^r (q_{1 - Y_B})^{\tilde{b}} (1 - q) (1 - Y_B) \right)^{\tilde{r}},$$

where $\gamma_B$ and $\gamma_R$ stand for $\gamma_B (n, c, p)$ and $\gamma_R (n, c, p)$, respectively.
Therefore, the problem is reduced to one of showing that if $X_n$ and $Y_n$ are random variables such that $|X_n|, |Y_n| \leq 1, \forall n \in \mathbb{N}$, and $Y_n \xrightarrow{a.s.} 0$, then $\lim_{n \to \infty} E(X_nY_n) = 0$.\footnote{We thank Elisabeti Kira for sharing this insight with us.} That this is true can be shown as follows: from $Y_n \xrightarrow{a.s.} 0$ we have $|Y_n| \xrightarrow{a.s.} |0| = 0$, and since $|Y_n| \leq 1$ and $E \left( 1 \right) = 1 < \infty$, the Dominated Convergence Theorem ensures that $\lim_{n \to \infty} E(|Y_n|) = E(0) = 0$. Since $E$ is an increasing operator and $-|Y_n| \leq X_nY_n \leq |Y_n|$, $\forall n \in \mathbb{N}$, we must have $E(X_nY_n)$ lying between $-E(|Y_n|)$ and $E(|Y_n|)$ so that, by the Squeeze Theorem, $\lim_{n \to \infty} E(X_nY_n) = 0$.

The above approximation plays a major role in the proof (see the Appendix) of this section’s main result, Proposition 7. Using the limiting distribution given in Lemma 5, we can establish an asymptotic version of Lemma 4, so that even as $n \to \infty$, one should expect the expected majority candidate to win the election more often than not.

**Lemma 6** Given $c \in (0, 1)$ and $q \in [\overline{q}, 1 - \overline{q}]$, we have 
\begin{align*}
\text{i.} & \quad \text{if } q = 0.5, \text{ then } \lim_{n \to \infty} \Pr (B \text{ wins } | n, c, p^*_n (c, q), q) = 0.5; \\
\text{ii.} & \quad \text{if } q > 0.5, \text{ then } \lim_{n \to \infty} \Pr (B \text{ wins } | n, c, p^*_n (c, q), q) > 0.5; \text{ and} \\
\text{iii.} & \quad \text{if } q < 0.5, \text{ then } \lim_{n \to \infty} \Pr (B \text{ wins } | n, c, p^*_n (c, q), q) < 0.5.
\end{align*}

Thus, even if the electorate size grows without bound, the probability of a victory of the “right” candidate—namely, the candidate who most likely has the support of the majority—is greater than $50\%$. This contrasts with the conclusions of Campbell (1999) and Taylor and Yildirim (2010b), where, given a truthful report of poll results, departure from the neutrality result in the limit can only occur if the voting cost (or, equivalently, the ideological component of utility) of $B$ supporters is different from that of $R$ supporters.\footnote{More precisely, Taylor and Yildirim (2010b) show that even if voting costs are stochastic, as long as they are drawn from a common distribution, asymptotically we should expect an election tie even if one of the candidates has an expected majority of supporters. Both Campbell (1999) and Taylor and Yildirim (2010b) show a converse result: if the voting costs faced by $R$ supporters are first-order stochastically dominated by the voting costs faced by $B$ supporters, then, no matter how large an expected majority $B$ has, one should expect a victory of $R$.}

Once we relax the assumption of truthful poll reports, even in a model with fixed and homogeneous voting costs, and even as $n$ tends to infinity, the unrealistic neutrality prediction that both candidates have equal chances of winning the election disappears in favor of the candidate of the expected majority group.

The next proposition shows that for sufficiently large elections, misreporting of poll results actually generates a welfare improvement. Properties of the Skellam distribution are key in arriving at this central result of this work, alongside the basic property proved in Proposition 4 that $p^*_n (c, q)$ lies between $0.5$ and $q$ (see the Appendix).

**Proposition 7** Given $c \in (0, 1)$ and $q \in [\overline{q}, 1 - \overline{q}] \setminus \{0.5\}$, for sufficiently large $n$, we have 
\begin{align*}
\text{i.} & \quad C \left( n, c, p^*_n (c, q), q \right) < C \left( n, c, q, q \right); \\
\text{ii.} & \quad \mathcal{I} \left( n, c, p^*_n (c, q), q \right) > \mathcal{I} \left( n, c, q, q \right); \text{ and} \\
\text{iii.} & \quad \mathcal{W} \left( n, c, p^*_n (c, q), q \right) > \mathcal{W} \left( n, c, q, q \right).
\end{align*}

The intuition behind this proposition is as follows. Suppose that $q > 0.5$. In this case, it would be best for society if candidate $B$ were the winner of the election as,
more often than not, the number of $B$ citizens should be larger than the number of $R$ citizens. Recall that given a truthful report ($p = q$), the expected result of the election is a tie, so that the probability of candidate $B$ winning is 50%. However, due to the misreporting behavior explained in Proposition 4, $p_n^*(c, q) \in (0.5, q)$, and by Proposition 2, $B$ citizens ($R$ citizens) vote with a higher (lower) probability relative to the truthful report, so that the probability of $B$ winning is greater than 50%. Besides that, as explained in the previous section, in absolute value, the probability associated with a $B$ vote varies less than the probability associated with an $R$ vote, so that the expected voting cost decreases. Therefore, for purely self-centered reasons, the pollster acts in a way that unambiguously improves welfare.

Finally, with Lemma 6 at our disposal, we can also check the robustness of the conclusion that misreporting polls is welfare-improving by showing that it holds even as the size of the electorate grows without bound. This is facilitated by our proof of Proposition 7, which, given the asymptotic nature of the approximation of $I$ via $I$, naturally calls for the taking of limits as $n \to \infty$ (in contrast to the fixed $n$ approach in Proposition 4 of Goeree and Großer, 2007).

**Proposition 8** Given $c \in (0, 1)$ and $q \in [\bar{q}, 1 - \bar{q}] \setminus \{0.5\}$,

i. $\lim_{n \to \infty} (C(n, c, p_n^*(c, q), q) / n) = \lim_{n \to \infty} (C(n, c, q, q) / n) = 0$;

ii. $\lim_{n \to \infty} (I(n, c, p_n^*(c, q), q) / n) > \lim_{n \to \infty} (I(n, c, q, q) / n)$; and

iii. $\lim_{n \to \infty} (W(n, c, p_n^*(c, q), q) / n) > \lim_{n \to \infty} (W(n, c, q, q) / n)$.

Therefore, the misreporting of pre-elections poll results enhances the welfare of society relative to a truthful report not only for a finite population size but also asymptotically.

### 5 Robustness

This section suggests a few variations on our model as to check for the robustness of our results. One possible concern could be related to the well-known fact that in the pivotal voting model, the expected turnout rate vanishes if the electorate were allowed to increase without bound. An equally important concern would involve the robustness of our results to the choice of alternative scoring rules, such as those surveyed in Savage (1971) and Selten (1998). We address each of these concerns in turn.

First, we consider an extension in which some voters may be nonstrategic, in the sense that they vote regardless of their probability of being pivotal. This yields a positive asymptotic turnout rate, in contrast to the previous, more basic model. More precisely, there is a probability $\varepsilon \in [0, 1)$ that any given voter will be nonstrategic. This yields a new electoral game, where $\varepsilon p, (1 - \varepsilon) p, \varepsilon (1 - p)$ and $(1 - \varepsilon) (1 - p)$ are, in turn, the probabilities that a citizen is nonstrategic and supports $B$, strategic and supports $B$, nonstrategic and supports $R$, and strategic and supports $R$. The probability

---

22 However, as explained in the introduction, the search for a proper scoring rule in this context in which predictions affect outcomes (as in Shi et al. 2009, for instance) would fall outside the scope of the present work.

23 We are indebted to Marcos Nakaguma for discussions that led to the consideration of this extension.
of being pivotal needs to be adjusted accordingly. Also, in this setting, the election result distributions (1) and (3) become

\[
\text{Multinomial} \left( n, q \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right), (1 - q) \left( (1 - \varepsilon) \gamma_R(n, c, p, \varepsilon) + \varepsilon \right), 1 - q \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right) - (1 - q) \left( (1 - \varepsilon) \gamma_R(n, c, p, \varepsilon) + \varepsilon \right) \right),
\]

and

\[
\text{Multinomial} \left( n, p \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right), (1 - p) \left( (1 - \varepsilon) \gamma_R(n, c, p, \varepsilon) + \varepsilon \right), 1 - p \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right) - (1 - p) \left( (1 - \varepsilon) \gamma_R(n, c, p, \varepsilon) + \varepsilon \right) \right),
\]

respectively.

Let us illustrate how this would affect the probability of each strategic citizen being pivotal in an election by considering the simplest case possible, \( n = 2 \). The probability of a \( B \) citizen being pivotal is \( (1 - p) + (1 - \varepsilon) p (1 - \gamma_B) \), where the first term refers to the probability of the other citizen favoring \( R \) and the second term represents the probability of the other citizen favoring \( B \) but not voting. Similarly, the probability of an \( R \) citizen being pivotal is \( p + (1 - \varepsilon) (1 - p) (1 - \gamma_R) \). In an interior equilibrium, both of these probabilities must equal \( c \), thus yielding

\[
\gamma_B(2, c, p, \varepsilon) = \frac{1 - c - p \varepsilon}{p (1 - \varepsilon)}
\]

and

\[
\gamma_R(2, c, p, \varepsilon) = \frac{1 - c - (1 - p) \varepsilon}{(1 - p) (1 - \varepsilon)}.
\]

As can be seen from the expressions above, and is true for any \( n \) (supposing an interior equilibrium), a modified neutrality result still applies:

\[
n p \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right) = n \left( 1 - p \right) \left( (1 - \varepsilon) \gamma_R(n, c, p, \varepsilon) + \varepsilon \right),
\]

that is, citizens expect an equal turnout for \( B \) and \( R \). This allows us to address the misreporting problem as done in Sect. 3, since we can define \( \gamma^*(n, c, \varepsilon) := \gamma_B(n, c, 0.5, \varepsilon) \) and modify the argument in the proof of Lemma S2 in the Supplementary Appendix to obtain

\[
p \left( (1 - \varepsilon) \gamma_B(n, c, p, \varepsilon) + \varepsilon \right) = 0.5 \left( (1 - \varepsilon) \gamma^*(n, c, \varepsilon) + \varepsilon \right),
\]

so that both \( \gamma_B(n, c, p, \varepsilon) \) and \( \gamma_R(n, c, p, \varepsilon) \) could be expressed in terms of \( \gamma^*(n, c, \varepsilon) \) in an interior equilibrium:

\[
\gamma_B(n, c, p, \varepsilon) = \frac{\gamma^*(n, c, \varepsilon)}{2p} - \frac{\varepsilon}{2p - 1},
\]

and

\[
\gamma_R(n, c, p, \varepsilon) = \frac{\gamma^*(n, c, \varepsilon)}{2(1 - p)} - \frac{\varepsilon}{1 - \varepsilon 2(1 - p)}.
\]
By plugging these expressions into distribution (15), the pollster’s objective function $T$ in (6) can be recalculated and reoptimized, only to give a very similar $p^*_n (c, q, \epsilon)$ to the one previously obtained, $p^*_n (c, q)$ (i.e., $p^*_n (c, q, 0)$). This can be noted in Fig. 2, which uses $n = 10$, $\epsilon = 0.2$, and $c_0 \in (0, 1)$ chosen so that the expected turnout rate without misreporting $(1 - \epsilon) \gamma^* (n, c_0, \epsilon) + \epsilon$ adds up to 40% (in other words, $\gamma^* (n, c_0, \epsilon) = 0.25$).\(^{24}\)

As for the possibility of a welfare improvement induced by the pollster’s misreporting behavior, it would follow similar reasoning as that of Sect. 4. Without loss of generality, let us assume that $q > 0.5$. On the one hand, we already know that the expected aggregate ideological component of utility can be approximated by $I$ given in (14), which in this case is increasing in $\Pr (B \text{ wins } | n, c, p, q, \epsilon)$. As argued above, misreporting gives candidate $B$ an edge in the election by raising $\Pr (B \text{ wins } | n, c, p, q, \epsilon)$ above the 50% mark, thus increasing welfare.

On the other hand, the aggregate cost is lower with misreporting than with a truthful report due to a reduction in the expected turnout. In fact, the expected aggregate cost $C$ of the election is now given by

$$C (n, c, p, q, \epsilon) = nc \left[ q ((1 - \epsilon) \gamma_B (n, c, p, \epsilon) + \epsilon) + (1 - q) ((1 - \epsilon) \gamma_R (n, c, p, \epsilon) + \epsilon) \right].$$

\(^{24}\) If we were to use $n = 100$ as in Fig. 1, these functions would be indistinguishable to the naked eye, differing among themselves starting at the third decimal place only.
Assuming an interior electoral equilibrium, we can plug in the electoral equilibrium expressions (16) and (17) to obtain

\[
\mathcal{C}(n, c, p, q, \varepsilon) = nc \left[ q \left(1 - \varepsilon\right) \left(\frac{\gamma^*(n,c,\varepsilon)}{2p} - \frac{\varepsilon}{1 - \varepsilon} \frac{2p-1}{2p}\right) + \varepsilon \right] \\
+ n \left(1 - q\right) \left(1 - \varepsilon\right) \left(\frac{\gamma^*(n,c,\varepsilon)}{2(1-p)} - \frac{\varepsilon}{1 - \varepsilon} \frac{1-2p}{2(1-p)}\right) + \varepsilon \right]
\]

\[
= \frac{nc}{2} \left(1 - \varepsilon\right) \gamma^*(n, c, \varepsilon) + \varepsilon \left(\frac{q}{p} + \frac{1 - q}{1 - p}\right).
\]

This is lower than \(\mathcal{C}(n, c, q, q, \varepsilon)\) if and only if \(q/p + (1 - q)/(1 - p)\) is lower than 2 or, equivalently, \((2p - 1)(q - p) > 0\), which is clearly the case for the solution \(p^*\) illustrated in Fig. 2. In sum, both of our main results still hold if the electorate is not composed only of citizens who make their electoral decisions based on the probability of being pivotal.

Let us now address a second possible variation of our model and see that it would lead to this same general conclusion. As mentioned earlier, other scoring rules on the part of society could be considered. A well-known alternative would be the Brier score (Brier 1950), which we briefly present below. As before, given the report \(p\), citizens expect that a \(B\) vote will occur with probability \(p \gamma_B(n, c, p)\) and an \(R\) vote with probability \((1 - p) \gamma_R(n, c, p)\) (the derivations below are done with \(\varepsilon = 0\) in mind, but could also be easily extended). For each \((b, r, a)\) realization of (1), the Brier score will be given by

\[
\psi(b, r, a, p) = \frac{1}{n} \left[ b \left(\left(p \gamma_B - 1\right)^2 + (1 - p) \gamma_R - 0\right)^2 + (1 - p) \gamma_B - (1 - p) \gamma_R - 0\right)^2 \right] \\
+ r \left(\left(p \gamma_B - 0\right)^2 + (1 - p) \gamma_R - 1\right)^2 + (1 - p) \gamma_B - (1 - p) \gamma_R - 0\right)^2 \right] \\
+ a \left(\left(p \gamma_B - 0\right)^2 + (1 - p) \gamma_R - 0\right)^2 + (1 - p) \gamma_B - (1 - p) \gamma_R - 1\right)^2 \right]
\]

where the \((n, c, p)\) arguments of \(\gamma_B\) and \(\gamma_R\) are omitted for the sake of brevity.

In the expression above, we use the fact that society is able to deduce from the report variable \(p\) the implied probabilities of each of the three categories \((B\) voter, \(R\) voter, nonvoter) for each of the \(n\) citizens. Since the realization \((b, r, a)\) is not known ex-ante, one could assume that the pollster wishes to minimize its expected Brier score \(E(\psi(b, r, a, p))\), where this expectation is taken with respect to (1). This can be obtained by simply replacing \(b, r\) and \(a\) in the expression for \(\psi(b, r, a, p)\) by \(E(b) = nq \gamma_B\), \(E(r) = n(1 - q) \gamma_R\) and \(E(a) = n(1 - q) \gamma_B - (1 - q) \gamma_R\).

Proposition 1 then implies that for a sufficiently large \(n\), we can get rid of each \(\gamma_B\) and \(\gamma_R\) in the resulting expression to obtain

\[
\frac{q}{p} \frac{\gamma^*}{2} \left(\left(1 - \frac{\gamma^*}{2}\right)^2 + \left(\frac{\gamma^*}{2}\right)^2 + \left(1 - \gamma^*\right)^2\right) \\
+ \frac{1 - q}{1 - p} \frac{\gamma^*}{2} \left(\left(\frac{\gamma^*}{2}\right)^2 + \left(1 - \frac{\gamma^*}{2}\right)^2 + \left(1 - \gamma^*\right)^2\right)
\]
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\[
+ \left( 1 - \frac{q \gamma^*}{p} \right) - \frac{1 - q \gamma^*}{1 - p} \right) \left( 2 \left( \frac{\gamma^*}{2} \right)^2 + \gamma^* \right),
\]

where \( \gamma^* \) stands for \( \gamma^* (n, c) \). After a few algebraic manipulations, this can be seen to equal

\[
\gamma^* \left( 1 - \frac{3}{2} \gamma^* \right) z + \frac{3}{2} \gamma^* \gamma^2,
\]

where \( z := q/p + (1 - q) / (1 - p) \). By Lemma 2, we must have, for sufficiently large \( n \), \( \gamma^* (n, c) < 2/3 \). Therefore, minimizing the Brier score entails nothing but minimizing \( z \), thus obtaining the solution \( p = \phi (q) = \left( 1 + \sqrt{1/q - 1} \right)^{-1} \)—quite similar to \( p_n^* (c, q) \), as noted already in the discussion of Fig. 1. Another way to state this result is that, in large elections, the pollster once again acts as if to minimize the asymptotic turnout, as explained in Sect. 3. The welfare discussion would follow the exact same logic as that of Sect. 4.

In other words, in a context in which forecasts affect outcomes, one should not expect strict propriety, or even propriety, of the Brier score. All of this can be seen to apply to other scoring rules such as the logarithmic (Good 1952) and the spherical (Roby 1964) ones, by following the exact same lines above. Just as an illustration of this point, the former scoring rule would imply

\[
E (\psi (b, r, a, p)) = \frac{1}{n} \left[ b \log (p \gamma_B) + r \log ((1 - p) \gamma_R) + a \log (1 - p \gamma_B - (1 - p) \gamma_R) \right] .
\]

As before, taking expectations and using Proposition 1, the pollster’s objective function becomes

\[
\frac{q \gamma^*}{p} \log \left( \frac{\gamma^*}{2} \right) + \frac{1 - q \gamma^*}{1 - p} \log \left( \frac{\gamma^*}{2} \right) + \left( 1 - \frac{q \gamma^*}{p} \right) \frac{1 - q \gamma^*}{1 - p} \log (1 - \gamma^*)
\]

\[
= \gamma^* \log \left( \frac{\gamma^*}{2 (1 - \gamma^*)} \right) z + \log (1 - \gamma^*),
\]

once again affine in \( z \), so that it is maximized at \( p = \phi (q) \).

6 Concluding remarks

In this paper, we formally introduce an electoral pollster in a two-candidate costly voting model and conclude that, driven by reputational concerns only, it underreports the probability of any given citizen favoring the candidate of the expected majority group in society. This implies that, relative to a truthful report, citizens in this group will vote more intensely, with the opposite being true for citizens in the expected minority group. As a consequence, not only are the chances of the most preferred candidate in society raised, but total election costs are also reduced, thus yielding a welfare gain.
Thus, by acknowledging the possibility of nontruthful reports of pre-election poll results, we show that even in a model with fixed and homogeneous voting costs, the candidate who most likely is supported by the majority of citizens wins the election with probability greater than 50%, in contrast to the canonical pivotal voting model and its troubling neutrality result. This also holds asymptotically, that is, for an electorate size growing without bound. Our analysis is robust also to variations of the model employed, such as different scoring rules or the inclusion of nonstrategic citizens.

This work also suggests that part of what is usually attributed to the bandwagon effect could actually be an illusion due to misreporting. That is, even if citizens’ behavior generates the underdog effect, an observer who disregards the possibility of nontruthful poll reports could incorrectly conclude that the bandwagon effect was in place.

A possible extension of this work would be to analyze the effects of different incentive schemes regarding the release of poll results, especially when there is competition among different pollsters. Also of interest would be an analysis of the behavioral and welfare effects in the case of multiple simultaneous state elections (for governors or legislative representatives), with or without state-dependent $q$ variables.

Our analysis can also be linked to the issue of voluntary versus compulsory voting, studied in Börgers (2004), Krasa and Polborn (2009) and Faravelli and Man (2021). This could be achieved by letting a social planner aware of the pollster’s misreporting behavior choose $c$ so as to maximize welfare.

Finally, Campbell (1999) provides a model that favors the electability of the minority group, due to an asymmetry in the ideological component of both groups’ payoffs. Whether the pollster’s report bias under such circumstances, as well as the resulting welfare implications, would be strengthened or weakened, is left for future research.

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Declarations

Conflict of interest The authors declare no conflict of interest.

Appendix

Here we present proofs for all of the results stated in Sects. 3 and 4.
Before getting into the proofs of the propositions and lemmas of Sect. 3, we present a computation that will be used in several of them. Given \( n \geq 2, c \in \mathbb{R}_+ \) and \( p \in [\bar{q}, 1 - \bar{q}] \), by plugging (7), (8), (9) and (10) into (6) we get

\[
T(n, c, p, q) = \left[ n^2 (p - q)^2 \left( \gamma_B^2(n, c, p) + \gamma_R^2(n, c, p) \right) + n (q \gamma_B(n, c, p)) (1 - q \gamma_B(n, c, p)) \right]. \tag{A.1}
\]

If \( c \in (0, 1) \) and \( n \geq n_0(c) \), Proposition 1 yields \( \gamma_B(n, c, p) = \gamma^*/(2p) \) and \( \gamma_R(n, c, p) = \gamma^*/(2 (1 - p)) \), where \( \gamma^* = \gamma^*(n, c) \in (0, 1) \). In this case, (A.1) can be rewritten as

\[
T(n, c, p, q) = -n^2 (p - q)^2 \left( \frac{\gamma^*}{4p^2} + \frac{\gamma^*}{4 (1 - p)^2} \right)
- \frac{n}{2} \frac{\gamma^*}{p} \left( 1 - \frac{q}{p} \right) - n \frac{1 - q}{p} \frac{\gamma^*}{2} \left( 1 - \frac{1 - q}{1 - p} \right)
= \left[ -\frac{n \gamma^* q}{2} \frac{p}{p} + \frac{n \gamma^* q}{2} \frac{4}{p^2} - \frac{n^2 \gamma^2}{4} (p - q)^2 \right]
+ \left[ -\frac{n \gamma^* (1 - q)}{2} \frac{1}{1 - p} + \frac{n \gamma^* (1 - q)^2}{4} \frac{1 - p}{(1 - p)^2} - \frac{n^2 \gamma^2 (p - q)^2}{4} \right]. \tag{A.2}
\]

Several proofs below will involve differentiation of \( T \) with respect to \( p \), the pollster’s choice variable (viewed as a function of \( p \), the above expression obviously belongs to \( C^\infty ([\bar{q}, 1 - \bar{q}]) \)). As seen in (A.2), in doing so, we need not worry about \( p \) entering \( T(n, c, p, q) \) implicitly, perhaps inside a \( \gamma_B(n, c, p) \) or \( \gamma_R(n, c, p) \) expression as in (A.1).

**Proof of Proposition 3** For any \( p \in [\bar{q}, 1 - \bar{q}] \), since \( c = 0 \), the electoral equilibrium is \( (\gamma_B(n, c, p), \gamma_R(n, c, p)) = (1, 1) \) (see Proposition 1 in the Supplementary Appendix). Therefore, by (A.1),

\[
T(n, c, p, q) = -\left[ n^2 (p - q)^2 2 + nq (1 - q) + n (1 - q) (1 - (1 - q)) \right]
= -2n^2 (p - q)^2 - 2nq (1 - q).
\]

It is clear that the maximum value of \( T(n, c, p, q) \) occurs only at \( p = q \). \( \square \)

The following claim enables us to focus our attention on the \( q \geq 0.5 \) case in the proofs below.

**Claim A1** Given \( q \in [\bar{q}, 1 - \bar{q}] \), \( c \in (0, 1) \) and \( n \geq n_0(c) \), if \( p_n^*(c, q) \) is a solution to \( \max_{p \in [\bar{q}, 1 - \bar{q}]} T(n, c, p, q) \), then \( 1 - p_n^* (c, q) \) is a solution to \( \max_{p \in [\bar{q}, 1 - \bar{q}]} T(n, c, p, 1 - q) \).

**Proof** Since \( c \in (0, 1) \) and \( n \geq n_0(c) \), (A.2) holds. Note from that expression that \( T(n, c, p, 1 - q) = T(n, c, 1 - p, q) \).
Now, if \( r \in [\bar{q}, 1 - \bar{q}] \) were such that \( T(n, c, r, 1 - q) > T(n, c, 1 - p^*_n(c, q), 1 - q) \), then we would have \( T(n, c, 1 - r, q) > T(n, c, p^*_n(c, q), q) \). Since \( 1 - r \in [\bar{q}, 1 - \bar{q}] \), this would contradict the fact that \( p^*_n(c, q) \) is a solution of \( \max_{p \in [\bar{q}, 1 - \bar{q}]} T(n, c, p, q) \).

Before proving Proposition 4, we state a couple of lemmas about the pollster’s objective function. Since we assume that the reader is more interested in following the development of proof strategies, rather than big calculations, in the proofs below we unashamedly make use of mathematical software to help with the computation of derivatives and algebraic manipulations; this is indicated by the “\( \ast \)” symbol. The symbol “\( \sim \)” stands for “shares its sign with” (whenever it is not being used to inform the distribution of a random variable; the correct interpretation will always be clear from context).

**Lemma A1** Given \( q \in [\bar{q}, 1 - \bar{q}] \), \( c \in (0, 1) \) and \( n \geq n_0(c) \), if \( \gamma^*(n, c) \leq 1/n \), then \( \partial^2 T(n, c, p, q) / \partial p^2 < 0 \) for all \( p \in (\bar{q}, 1 - \bar{q}) \).

**Proof** Since \( c \in (0, 1) \) and \( n \geq n_0(c) \), (A.2) holds, regardless of the value of \( p \in (\bar{q}, 1 - \bar{q}) \). We now apply the \( \partial^2 / \partial p^2 \) operator to each term in square brackets in (A.2). First, note that

\[
\frac{\partial^2}{\partial p^2} \left( -\frac{n\gamma^*}{2} q + \frac{n\gamma^*}{4} q^2 - \frac{n^2\gamma^*}{2} (p - q)^2 \right) = -\frac{1}{2} n\gamma^* q (2p - 3q\gamma^* - 2np\gamma^* + 3nq\gamma^*) \sim (3q + 2np - 3nq) \gamma^* - 2p.
\]

This is an affine function of \( \gamma^* \) which, when evaluated at \( \gamma^* = 0 \), equals \(-2p < 0\), and when evaluated at \( \gamma^* = 1/n \), equals \(3q/n + 2p - 3q - 2p = 3q(1 - n)/n < 0\). Therefore, it is negative for all \( \gamma^* \in (0, 1/n) \).

Similarly, we have

\[
\frac{\partial^2}{\partial p^2} \left( -\frac{n\gamma^*}{2} \frac{1 - q}{1 - p} + \frac{n\gamma^*}{4} \frac{(1 - q)^2}{(1 - p)^2} - \frac{n^2\gamma^*}{4} \frac{(p - q)^2}{(1 - p)^2} \right) = -\frac{1}{2} \frac{n\gamma^*}{(1 - p)^4} (1 - q) (n\gamma^* - 3\gamma^* - 2p + 3q\gamma^* + 2np\gamma^* - 3nq\gamma^* + 2) \\
\sim (3nq - 3q - 2np - n + 3) \gamma^* + 2p - 2,
\]

and the same reasoning applies: it is an affine function of \( \gamma^* \) which, when evaluated at \( \gamma^* = 0 \), equals \(2p - 2 < 0\), and when evaluated at \( \gamma^* = 1/n \), equals \(3q - 3q/n - 2p - 1 + 3/n + 2p - 2 = 3q - 3q/n + 3/n - 3 = -3(1 - q)(n - 1)/n < 0\). Therefore, it is negative for all \( \gamma^* \in (0, 1/n) \).

Thus, \( \partial^2 T(n, c, p, q) / \partial p^2 < 0 \). \( \square \)
To analyze the $\gamma^* (n, c) > 1/n$ case as well (for which, unfortunately, $T$ will not necessarily be concave in $p$), it is useful to establish the notation

$$f_n(p, q, \gamma) := \left[ (2nqy - ny - 2q + 1) p^4 + (4q + y - 2qy + 2q^2 y - 2nqy - 2nq^2 y - 1) p^3 + (3nqy - 3q^2 y - 3q + 3nq^2 y) p^2 + \left( q + 3q^2 y - nqy - 3nq^2 y \right) p + nq^2 y - q^2 y \right].$$

(A.3)

since differentiation of (A.2) yields

$$\frac{\partial}{\partial p} T(n, c, p, q) = \frac{1}{2} \frac{n\gamma^*}{p^3 (1 - p)^3} f_n(p, q, \gamma^*),$$

(A.4)

where, as usual, $\gamma^*$ stands for $\gamma^* (n, c)$.

**Lemma A2** Given $q \in [\bar{q}, 1 - \bar{q}]$, $c \in (0, 1)$ and $n \geq n_0 (c)$, if $\gamma^* (n, c) > 1/n$, then $f_n(p, q, \gamma^* (n, c))$ is strictly decreasing in $p$ for $p \in [\bar{q}, 1 - \bar{q}]$.

**Proof** Since $c \in (0, 1)$ and $n \geq n_0 (c)$, (A.2) holds, regardless of the value of $p \in [\bar{q}, 1 - \bar{q}]$. From the hypothesis on $\gamma^* (= \gamma^* (n, c))$ and Proposition 1, we have $\gamma^* \in (1/n, 1)$.

Since $f_n(p, q, \gamma^*)$ is an affine function of $\gamma^*$, so is, for any $p \in (\bar{q}, 1 - \bar{q})$, its partial derivative

$$\frac{\partial}{\partial \gamma} f_n(p, q, \gamma^*) \doteq \left( 6p^2 q^2 - 3nq^2 - 6pq^2 - 6p^2 q - 4np^3 - nq + 3p^2 \right) + \left( + 3q^2 + 6npq - 6np^2 q + 8np^3 q - 6np^2 q^2 + 6npq \right) \gamma^*$$

$$+ q + 12p^2 q - 8p^3 q - 6pq - 3p^2 + 4p^3.$$  

At $\gamma^* = 1/n$, this expression simply amounts to

$$\frac{\partial}{\partial p} f_n(p, q, 1/n) \doteq -3 \left( -2pq^2 - 2p^2 q + 2p^2 q^2 + p^2 + q^2 \right) \frac{n - 1}{n}$$

$$= -3 \left( p^2 (1 - q)^2 + q^2 (1 - p)^2 \right) \frac{n - 1}{n} < 0,$$

whereas at $\gamma^* = 1$, it is equal to

$$\frac{\partial}{\partial p} f_n(p, q, 1) \doteq \left( \left( 6p - 6p^2 - 3 \right) q^2 + \left( 8p^3 - 6p^2 + 6p - 1 \right) q - 4p^3 \right) (n - 1)$$

$$\sim (8q - 4) p^3 + (6q^2 - 6q) p^2 + (6q^2 + 6q) p + (-3q^2 - q)$$

$$= : g(p, q).$$

If $q = 0.5$, then this becomes the quadratic $(-18p^2 + 18p - 5)/4$, the maximum value of which, obtained at the vertex $p = 0.5$, is $-1/8$. If $q \neq 0.5$, then $g(p, q)$ is a cubic in $p$, with discriminant

$$(-6q^2 - 6q)^2 (6q^2 + 6q)^2 - 4 (8q - 4) (6q^2 + 6q)^3 - 4 (-6q^2 - 6q)^3 (-3q^2 - q)$$

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\[-27(8q - 4)^2(-3q^2 - q)^2 + 18(8q - 4)(-6q^2 - 6q)(6q^2 + 6q)(-3q^2 - q)\]
\[\equiv -432q^2(1 - q)^2(3q^4 - 6q^3 + q^2 + 2q + 1) \sim 16(-3q^4 + 6q^3 - q^2 - 2q - 1)\].

By letting \(s := (2q - 1)^2\), it may be noted that \(-3s^2 + 14s - 27\) is identical to the above expression.

Since the discriminant of this concave quadratic in \(s\) is negative, it is negative itself, and so is the discriminant above. Thus, \(g(\cdot, q)\) has exactly one real root \(p^*\). This root cannot lie in the \((0, 1)\) interval. In fact, in case \(q > 0.5\), \(\lim_{p \to +\infty} g(p, q) = +\infty\) and \(g(1, q) = -3q^2 + 7q - 4 = (1 - q)(3q - 4) < 0\) yield \(p^* > 1\), whereas in case \(q < 0.5\), \(\lim_{p \to -\infty} g(p, q) = +\infty\) and \(g(0, q) = -3q^2 - q < 0\) yield \(p^* < 0\). Therefore, in either case, \(g(p, q) < 0\) for all \(p \in (\bar{q}, 1 - \bar{q})\).

Just as the thesis of Lemma A1 may not hold if \(\gamma^*(n, c) > 1/n\), the thesis of Lemma A2 may not hold if \(\gamma^*(n, c) \leq 1/n\) (these claims can be checked numerically). That is why we need both lemmas in the following proof.

**Proof of Proposition 4** Since \(c \in (0, 1)\) and \(n \geq n_0(c)\), \((A.2)\) holds, regardless of the value of \(p \in [\bar{q}, 1 - \bar{q}]\), and \(\partial T(n, c, p, q) / \partial p\) shares its sign with \(f_n(p, q, \gamma^*)\), where \(\gamma^* = \gamma^*(n, c)\).

(i) Note that

\[fn(p, 0.5, \gamma^*) = \frac{1}{4}(2p - 1)(\gamma^* - 2p - n\gamma^* - p\gamma^* + p^2\gamma^* + 2p^2 + 3np\gamma^* - 3np^2\gamma^*)\]

so that \(fn(0.5, 0.5, \gamma^*) = 0\), and 0.5 is a critical point of \(T(n, c, \cdot, 0.5)\). If \(\gamma^* \leq 1/n\), then Lemma A1 ensures 0.5 is the unique solution to (5). If \(\gamma^* > 1/n\), then Lemma A2 implies that, for all \(p \in [\bar{q}, 0.5]\), \(fn(p, 0.5, \gamma^*) > fn(0.5, 0.5, \gamma^*) = 0\), and, for all \(p \in (0.5, 1 - \bar{q})\), \(fn(p, 0.5, \gamma^*) < fn(0.5, 0.5, \gamma^*) = 0\). Since \(\partial T(n, c, p, q) / \partial p\) shares its sign with \(f_n(p, q, \gamma^*)\), \(T(n, c, \cdot, 0.5)|_{[\bar{q}, 0.5]}\) is strictly increasing and \(T(n, c, \cdot, 0.5)|_{[0.5, 1 - \bar{q}]}\) is strictly decreasing, so that 0.5 is once again the unique solution to (5).

(ii) The proof follows the exact same lines as part (i), once we are able to pin down a zero of \(f_n(\cdot, q, \gamma^*)\) between 0.5 and \(q\). It should be noted that

\[fn(0.5, q, \gamma^*) = \frac{1}{16}((n - 2)\gamma^* + 1)(2q - 1) > 0\]

and

\[fn(q, q, \gamma^*) = - (1 - \gamma^*)q^2(2q - 1)(1 - q)^2 < 0\].

Because \(f_n(p, q, \gamma^*)\) is a polynomial in \(p\) (hence, continuous), by the Intermediate Value Theorem, there exists \(p^* \in (0.5, q)\) such that \(f_n(p^*, q, \gamma^*) = 0\). That is,
\( p^* \) is a critical point of \( T(n, c, \cdot, q) \) in the interval \([\bar{q}, 1 - \bar{q}]\). Now both cases \( \gamma^* \leq 1/n \) and \( \gamma^* > 1/n \) must be considered, in the exact same fashion as in part (i).

(iii) This follows directly from part (ii) and Claim A1.

The following claim follows directly from the proof of Proposition 4.

**Claim A2** Given \( q \in [\bar{q}, 1 - \bar{q}] \), \( c \in (0, 1) \) and \( n \geq n_0(c) \), \( p_n^*(c, q) \) is the only \( p \in [\bar{q}, 1 - \bar{q}] \) such that \( f_n(p, q, \gamma^*(n, c)) = 0 \).

As a preliminary result for Proposition 5, we have

**Lemma A3** Given \( c \in (0, 1) \) and \( n \geq n_0(c) \), the function \( p_n^*(c, \cdot) \) is one to one.

**Proof** In fact, let \( q_1, q_2 \in [\bar{q}, 1 - \bar{q}] \) be such that \( p_n^*(c, q_1) = p_n^*(c, q_2) \) and \( q_1 \neq q_2 \). Without loss of generality, assume \( q_1 < q_2 \). By Proposition 4, it must then be the case that either \( q_1, q_2 \in [\bar{q}, 0.5) \) or \( q_1, q_2 \in (0.5, 1 - \bar{q}] \). To fix ideas, we assume the latter for the time being.

Claim A2 guarantees that \( f_n(p, q, \gamma^*(n, c)) = 0 \), together with the constraint \( p \in (0, 1) \), defines \( p_n^*(c, q) \). In order to conclude that this function is smooth, we must first check for the smoothness of \( \gamma^* \) with respect to its second argument.

Since \( n \geq n_0(c) \), Proposition 1 gives \( \gamma^*(n, c) = \gamma_B(n, c, 0.5) = \gamma_R(n, c, 0.5) \) \( (\in (0, 1)) \). If we define \( P_n : [0, 0.5] \rightarrow \mathbb{R} \) by

\[
P_n(\alpha) = \sum_{k=0}^{[\frac{n-1}{2}]} (k, k, n-1-2k)\alpha^{2k}(1-2\alpha)^{n-1-2k} + \sum_{k=0}^{[\frac{n-2}{2}]} (k, k+1, n-2-2k)\alpha^{2k+1}(1-2\alpha)^{n-2-2k},
\]

the interior equilibrium conditions \( \Pi_B(n, 0.5, \gamma^*(n, c), \gamma^*(n, c)) = c \) and \( \Pi_R(n, 0.5, \gamma^*(n, c), \gamma^*(n, c)) = c \) both become \( P_n(0.5\gamma^*(n, c)) = c \). Because \( P_n \) is a \( C^1 \) function and \( P_n'(0.5\gamma^*(n, c)) < 0 \) (see Lemma S1 in the Supplementary Appendix), the Implicit Function Theorem guarantees that \( \gamma^*(n, c) \) is also continuously differentiable in \( c \in (0, 1) \).

It must also be noted that \( \partial f_n(p^*, q, \gamma^*) / \partial p \), where \( p^* \) is short for \( p_n^*(c, q) \), cannot be zero. In fact, the \( \gamma^* > 1/n \) case is covered in Lemma A2, while if \( \gamma^* \leq 1/n \), then Lemma A1 and (A.4) give

\[
0 > \frac{\partial^2}{\partial p^2} T(n, c, p^*, q) = \frac{ny^*}{2} \left( \frac{1}{p^{3*} (1-p^{*})^3} \right) \frac{\partial}{\partial p} f_n(p^*, q, \gamma^*) + \alpha \frac{\partial}{\partial p} \left( \frac{1}{p^{3*} (1-p^{*})^3} f_n(p^*, q, \gamma^*) \right) = \frac{ny^*}{2} \left( \frac{1}{p^{3*} (1-p^{*})^3} \right) \frac{\partial}{\partial p} \left( f_n(p^*, q, \gamma^*) + \alpha f_n(p^*, q, \gamma^*) \right) \sim \frac{\partial}{\partial p} f_n(p^*, q, \gamma^*).
\]

Now, since \( f_n \) is \( C^1 \) as well (it is polynomial), a new application of the Implicit Function Theorem, but now to the identity provided in Claim A2, guarantees that \( p_n^* \) itself is \( C^1 \) and.
\[
\frac{\partial}{\partial q} p_n^*(c, q) = -\frac{\partial}{\partial q} f_n \left( p_n^*(c, q), q, \gamma^*(n, c) \right) - \frac{\partial}{\partial p} f_n \left( p_n^*(c, q), q, \gamma^*(n, c) \right)
\]

(A.5)

at any \( q \). The most important feature of this identity is that the derivative in the numerator is only partial (with respect to the second argument).

By Rolle’s Theorem applied to \( p_n^*(c, \cdot) \), there exists \( \tilde{q} \in (q_1, q_2) \) such that \( \partial p_n^*(c, \tilde{q}) / \partial q = 0 \). Expression (A.5) then yields \( \partial f_n \left( p_n^*(c, \tilde{q}), \tilde{q}, \gamma^* \right) / \partial q = 0 \), where \( \gamma^* = \gamma^*(n, c) \). In other words, \( f_n \left( p_n^*(c, \tilde{q}), \cdot, \gamma^* \right) \) is a quadratic in its second argument, as can be seen in (A.3), with \( \tilde{q} \) as a root, and at which its slope is zero. This implies that this quadratic has a zero discriminant. However, writing \( \tilde{p} \) for \( p_n^*(c, \tilde{q}) \), the discriminant can be computed as

\[
\begin{align*}
\left( (2ny^* - 2) \tilde{p}^3 + (4 - 2ny^* - 2\gamma^*) \tilde{p}^3 + (3ny^* - 3) \tilde{p}^2 + (1 - ny^*) \tilde{p} \right)^2 \\
-4 \left( (2\gamma^* - 2ny^*) \tilde{p}^3 + (3ny^* - 3\gamma^*) \tilde{p}^2 + (3\gamma^* - 3ny^*) \tilde{p} - \gamma^* + ny^* \right) \\
\times \left( \tilde{p}^3 (\gamma^* - 1) - \tilde{p} (ny^* - 1) \right) \\
\pm \tilde{p}^2 (1 - \tilde{p})^2 \times \left[ 4 \left( ny^* - 1 \right)^2 \tilde{p}^4 - 8 (ny^* - 1)^2 \tilde{p}^3 + 8 (ny^* - 1)^2 - 4 \gamma^* (1 - \gamma^*) (n - 1) \right] \tilde{p}^2 \\
= \tilde{p}^2 (1 - \tilde{p})^2 \left[ (2\tilde{p} + 2\tilde{p}^2 + 1)^2 (ny^* - 1)^2 + 4\tilde{p} (1 - \tilde{p}) \gamma^* (1 - \gamma^*) (n - 1) \right] > 0.
\end{align*}
\]

a contradiction.

Also, if it were the case that \( q_1, q_2 \in [\tilde{q}, 0.5] \), then, using Claim A1, \( p_n^*(c, q_1) = p_n^*(c, q_2) \) would imply \( p_n^*(c, 1 - q_1) = 1 - p_n^*(c, q_1) = 1 - p_n^*(c, q_2) = p_n^*(c, 1 - q_2) \), where \( 1 - q_1 \neq 1 - q_2 \). Thus, by the argument given above, we again arrive at a contradiction.

Therefore, \( p_n^*(c, \cdot) \) is indeed one to one. \( \square \)

**Proof of Lemma 3** Suppose \( q \in (0.5, 1 - \tilde{q}) \). First note that, in this case, \( \phi(q) < q \), so that \( (\phi(q), q) \) is an nondegenerate interval indeed. In fact, since \( 0 < 1/q - 1 < 1 \), we have \( \sqrt{1/q - 1} > 1/q - 1 \), so that \( \phi(q) = (1 + \sqrt{1/q - 1})^{-1} < (1 + 1/q - 1)^{-1} = q \). Also, note that \( \phi(q) = (1 + \sqrt{1/q - 1})^{-1} > (1 + \sqrt{1/0.5 - 1})^{-1} = 0.5 \) (if that were not the case, the present proof could actually be done by a simple call to Proposition 4).

Now, as already noted in the proof of Proposition 4, we have

\[
f_n \left( q, q, \gamma^* \right) \approx - (1 - \gamma^*) q^2 (2q - 1) (1 - q)^2 \sim 1 - 2q < 0,
\]

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Using the notation established in Lemma A3, we have, from the Implicit Function Theorem, 

\[
\phi(q, \gamma^*) = (n-1) \gamma^* \frac{q^2 (1-q) (2q-1) \sqrt{1-q}}{1+4q (1-q) + 4q \sqrt{1-q}} > 0.
\]

Thus, by the Intermediate Value Theorem, \( f_n (\cdot, q, \gamma^*) \) has a root between \( \phi(q) \) and \( q \) which, by Claim A2, is necessarily \( p^*_n (c, q) \).

If \( q \in [\bar{q}, 0.5) \), then \( 1-q \in (0.5, 1-\bar{q}) \), so that we have already proved that \( p^*_n (c, 1-q) \in (\phi(1-q), 1-q) \). Now, \( p^*_n (c, 1-q) = 1 - p^*_n (c, q) \) by Claim A1, and, from \( \sqrt{1/(1-q)} - 1 \sqrt{1/q} - 1 = \sqrt{q}/(1-q)(1-q)/q = 1 \), we see that \( \phi(1-q) \) equals

\[
\left(1 + \sqrt{\frac{1}{1-q} - 1}\right)^{-1} = \left(1 + \frac{1}{\sqrt{q} - 1}\right)^{-1} = \frac{\sqrt{q} - 1}{1 + \sqrt{q} - 1} = 1 - \frac{1}{1 + \sqrt{q} - 1} = 1 - \phi(q).
\]

Therefore, \( 1 - p^*_n (c, q) \in (1 - \phi(q), 1-q) \), that is, \( p^*_n (c, q) \in (q, \phi(q)) \).

\begin{proof}
(i) Under conditions \( c \in (0, 1) \) and \( n \geq n_0 (c) \), as argued in the proof of Lemma A3, \( p^*_n (c, \cdot) \) is a smooth, and hence continuous, function (this in itself could also be obtained through Berge’s Theorem). Since it also injective by that lemma, it is strictly monotone (the real analysis lemma we hereby employ follows immediately from the Intermediate Value Function). Finally, as to whether it is strictly increasing or strictly decreasing: since, by Proposition 4, \( p^*_n (c, \bar{q}) < 0.5 = p^*_n (c, 0.5) \), it is strictly increasing.

(ii) Using the notation established in Lemma A3, we have, from the Implicit Function Theorem and the Chain Rule,

\[
\frac{\partial}{\partial c} p^*_n (c, q) = -\frac{\partial}{\partial y} f_n \left( p^*_n (c, q), q, \gamma^* (n, c) \right) \frac{\partial}{\partial c} \gamma^* (n, c).
\]

By Proposition 1 and Proposition 2, \( \partial \gamma^* (n, c) / \partial c = \partial \gamma_B (n, c, 0.5) / \partial c < 0 \). Also, as explained in the proof of Lemma A3, \( \partial f_n (p^*, q, \gamma^*) / \partial p < 0 \), where \( p^* \) and \( \gamma^* \) stand for \( p^*_n (c, q) \) and \( \gamma^* (n, c) \). Therefore, \( \partial p^*_n (c, q) / \partial c \sim -\partial f_n (p^*, q, \gamma^*) / \partial \gamma \).

Now, if we write expression (A.3) as \( f_n (p, q, \gamma) = \kappa_n (p, q) + \lambda_n (p, q) \gamma \), since \( f_n (p^*, q, \gamma^*) = 0 \) by Claim A2, we have

\[
\frac{\partial}{\partial \gamma} f_n (p^*, q, \gamma^*) = \lambda_n (p, q) = -\frac{\kappa_n (p^*, q)}{\gamma^*} \sim -\kappa_n (p^*, q) = (1 - 2q) p^{*4} + (1 - 4q) p^{*3} + 3qp^{*2} - qp^*.
\]
\end{proof}
\[ p^* (1 - p^*) \left( (1 - 2q) p^{*2} + 2qp^* - q \right) \sim (1 - 2q) p^{*2} + 2qp^* - q = Q(p^*), \]

where \( Q(x) := (1 - 2q)x^2 + 2q x - q. \) Thus, \( \partial p^*_n(c, q) / \partial c \sim -Q(p^*), \) and, in order to conclude that \( \partial p^*_n(c, q) / \partial c \leq 0 \) if \( q \leq 0.5, \) it suffices to argue that \( Q(p^*) \sim 2q - 1. \)

If \( q = 0.5, \) then \( Q(p^*) = p^* - 0.5, \) which is 0 by Proposition 4.

If \( q \neq 0.5, \) then \( Q(x) \) is a quadratic in \( x, \) and \( \phi(q) \) happens to be one of its roots:

\[ Q(\phi(q)) = (1 - 2q)(\phi(q))^2 + 2q\phi(q) - q = \frac{1 - 2q}{1 + \sqrt{\frac{1}{q} - 1}} \leq + \frac{2q}{1 + \sqrt{rac{1}{q} - 1}} - q \]

\[ = \frac{1 - 2q + 2q \left( 1 + \sqrt{\frac{1}{q} - 1} \right) - q \left( 1 + \sqrt{\frac{1}{q} - 1} \right)^2}{\left( 1 + \sqrt{\frac{1}{q} - 1} \right)^2} = 0. \]

Call its second root \( \phi_2(q). \)

In the \( q > 0.5 \) case, \( Q \) is concave and \( \phi_2(q) > 1 > \phi(q), \) since \( Q(1) = 1 - 2q + 2q - q > 0. \) Thus, \( Q(x) > 0, \forall x \in (\phi(q), 1) \leq (\phi(q), 1), \) so that, by Lemma 3, \( Q(p^*) > 0. \)

In the \( q < 0.5 \) case, \( Q \) is convex and \( \phi_2(q) < 0 < \phi(q), \) since \( Q(0) = -q < 0. \) Thus, \( Q(x) < 0, \forall x \in (q, \phi(q)) \leq (0, \phi(q)), \) so that, by Lemma 3, \( Q(p^*) < 0 \) and we are done.

The following fact will be used in the proof of Lemma 4 ahead.

Claim A3 A coin that lands heads up with probability \( s \in [0, 1] \) is tossed \( l \geq 1 \) times. Given any \( k \in \{1, \ldots, l\}, \) the probability of obtaining at least \( k \) heads out of those \( l \) tosses is strictly increasing in \( s. \)

Proof Let \( F_{l,s} \) denote the cumulative distribution function of Binomial \( (l, s). \) Then for any \( k \in \{1, \ldots, l\}, \) the probability of obtaining at least \( k \) heads is

\[ 1 - F_{l,s}(k - 1) = \frac{\int_0^s y^{k-1} (1 - y)^{l-k} dy}{\int_0^1 y^{k-1} (1 - y)^{l-k} dy}, \]

obviously strictly increasing in \( s.\)

Proof of Lemma 4 Given these hypotheses, Proposition 1 is applicable. If the pollster is truthful and reports \( q, \) the distribution of \( (b, r, a) \) in (1) becomes

\[
\text{Multinomial} \left( n, q \gamma_R(n, c, q), (1-q) \gamma_R(n, c, q), 1 - q \gamma_R(n, c, q) - (1-q) \gamma_R(n, c, q) \right)
\]

\[
= \text{Multinomial} \left( n, \frac{\gamma^*(n, c)}{2}, \frac{\gamma^*(n, c)}{2}, 1 - \gamma^*(n, c) \right).
\]

---

25 The preceding formula is provided in Wadsworth and Bryan (1974, p. 51).
symmetric in \((b, r)\). Therefore, \(\Pr(B \text{ wins} \mid n, c, q, q) = \Pr(R \text{ wins} \mid n, c, q, q) = 0.5\).

(i) If \(q = 0.5\), then Proposition 4 gives \(p^*_n(c, q) = q\). Therefore, as shown above, \(\Pr(B \text{ wins} \mid n, c, p^*_n(c, q), q) = \Pr(B \text{ wins} \mid n, c, q, q) = 0.5\).

(ii) If \(q > 0.5\), let us first argue that, conditional on the number of abstentions \(a\), \(\Pr(b > r \mid n, c, p^*_n(c, q), q, a) > \Pr(b < r \mid n, c, p^*_n(c, q), q, a)\) (unless \(a = n\), in which case both probabilities vanish). In fact, given \(a \in \{0, \ldots, n - 1\}\), \(b\) will be distributed as Binomial \((n - a, s)\), where

\[
\begin{align*}
\frac{q \gamma_B(n, c, p^*_n(c, q))}{q \gamma_B(n, c, p^*_n(c, q)) + (1 - q) \gamma_R(n, c, p^*_n(c, q))} &= \frac{q \gamma^*_{(n,c)} + (1 - q) \gamma^*_{(n,c)} 2p^*_n(c, q)}{2p^*_n(c, q)} \\
&= \frac{1}{1 + \frac{1}{p^*_n(c, q)} - 1} > \frac{1}{1 + 1} = 0.5,
\end{align*}
\]

where we first used Proposition 1 to get rid of the \(\gamma_B\) and \(\gamma_R\) terms, and then applied Proposition 4 to conclude that, since \(0 < p^*_n(c, q) < q < 1\), \(0 < (1/q - 1) / (1/p^*_n(c, q) - 1) < 1\). Note that this probability parameter of the distribution of \(b\) would be exactly 0.5 if pre-election poll results could not be misreported:

\[
\frac{q \gamma_B(n, c, q)}{q \gamma_B(n, c, q) + (1 - q) \gamma_R(n, c, q)} = \frac{q \gamma^*_{(n,c)} / 2q + (1 - q) \gamma^*_{(n,c)} / 2(1-q)}{1} = \frac{1}{1 + 1} = 0.5.
\]

Since, conditional on \(a\), the event \(b > r\) could also be written as \(b \geq \lfloor (n - a) / 2 \rfloor + 1\) (which is at least 1), Claim A3 then yields that the probability of this event under misreporting is larger that it would be under truthful reporting of pre-election poll results: \(\Pr(b > r \mid n, c, p^*_n(c, q), q, a) > \Pr(b > r \mid n, c, q, q, a)\).

Similarly, conditional on \(a\), \(r \sim\) Binomial \((n - a, 1 - s)\), where \(1 - s < 0.5\), and the event \(b < r\) is the same as \(r \geq \lfloor (n - a) / 2 \rfloor + 1\), so Claim A3 gives \(\Pr(b < r \mid n, c, p^*_n(c, q), q, a) < \Pr(b < r \mid n, c, q, q, a)\).

Finally, note that \(\Pr(b > r \mid n, c, q, q, a) = \Pr(b < r \mid n, c, q, q, a)\) (if we toss an unbiased coin \(n - a\) times, the probability of obtaining more heads than tails equals the probability of obtaining more tails than heads). Therefore, for all \(\tilde{a} \in \{0, \ldots, n - 1\}\),

\[
\Pr(b > r \mid n, c, p^*_n(c, q), q, \tilde{a}) > \Pr(b > r \mid n, c, q, q, \tilde{a}) \]

\[
= \Pr(b < r \mid n, c, q, q, \tilde{a}) > \Pr(b < r \mid n, c, p^*_n(c, q), q, \tilde{a}),
\]
while \( \Pr \left(b > r \mid n, c, p_n^* (c, q), q, n \right) = \Pr \left(b < r \mid n, c, p_n^* (c, q), q, n \right) = 0 \). By the Law of Iterated Expectations,

\[
\Pr \left(b > r \mid n, c, p_n^* (c, q), q \right) = E \left( \Pr \left(b > r \mid n, c, p_n^* (c, q), q, a \right) \right) > E \left( \Pr \left(b < r \mid n, c, p_n^* (c, q), q, a \right) \right) = \Pr \left(b < r \mid n, c, p_n^* (c, q), q \right),
\]

where expectations are taken with respect to the distribution of \( a \) (it is a marginal distribution of \( (1, a \sim \text{Binomial} \left(n, 1 - q \gamma_B \left(n, c, p_n^* (c, q)\right) - (1 - q) \gamma_R \left(n, c, p_n^* (c, q)\right)\right)\)).

Thus,

\[
\Pr \left(B \text{ wins } \mid n, c, p_n^* (c, q), q \right) = \Pr \left(b > r \mid n, c, p_n^* (c, q), q \right) + \frac{\Pr \left(b = r \mid n, c, p_n^* (c, q), q \right)}{2} > \Pr \left(b < r \mid n, c, p_n^* (c, q), q \right) + \frac{\Pr \left(b = r \mid n, c, p_n^* (c, q), q \right)}{2} = \Pr \left(R \text{ wins } \mid n, c, p_n^* (c, q), q \right) = 1 - \Pr \left(B \text{ wins } \mid n, c, p_n^* (c, q), q \right),
\]

that is, \( \Pr \left(B \text{ wins } \mid n, c, p_n^* (c, q), q \right) > 0.5 \).

(iii) If \( q < 0.5 \), then the proof is entirely analogous, the difference being that now Proposition 4 implies \( s < 0.5 \), since \( 0 < q < p_n^* (c, q) < 1 \). \( \square \)

Before going on to prove Proposition 6, we must establish a bit more notation. Given \( q \in \left[\overline{q}, 1 - \overline{q}\right] \), we can define, for all \( p \in \left[\overline{q}, 1 - \overline{q}\right] \),

\[
\hat{T} (c, p, q) = \left[\frac{-m (c) q}{2} - \frac{(m (c))^2 (p - q)^2}{4 p^2}\right] + \left[\frac{-m (c) 1 - q}{2} - \frac{(m (c))^2 (p - q)^2}{4 (1 - p)^2}\right]. \tag{A.6}
\]

Due to the limits given in Lemma 2, it is just a matter of comparing (A.6) to (A.2) to note that \( \hat{T} (c, p, q) \) is simply the pointwise limit of \( (T (n, c, p, q))_{n \in \mathbb{N}} \), viewed as a sequence of functions of \( p \).

It will be important in the arguments below to note that this convergence is actually uniform. Since \( p \) takes its values on the compact set \( \left[\overline{q}, 1 - \overline{q}\right] \), which is bounded away from 0 and 1, not only the coefficients of the terms \( p^{-1}, p^{-2}, (1 - p)^{-1}, (1 - p)^2 \) appearing in (A.2) are bounded, but also these very terms. Since the coefficients converge uniformly simply by being constant in \( p \) and pointwise convergent, and the terms converge uniformly to themselves since they have no dependence on \( n \), we have that \( (T (n, c, p, q))_{n \in \mathbb{N}} \) converges uniformly to \( \hat{T} (c, p, q). \) \( ^{26} \)

\[ ^{26} \] This is just a matter of repeatedly using exercise 2 in Rudin (1976, ch. 7).
Proof of Proposition 6 (i) Assume $q = 0.5$. By Proposition 4, the sequence 
$(p_n^*(c, 0.5))_{n \in \mathbb{N}}$ is eventually constant at 0.5. Therefore, 
$$\lim_{n \to \infty} p_n^*(c, q) = 0.5 = q.$$ 

(ii) Assume $q > 0.5$, and let $n \geq n_0(c)$. Let us first solve an auxiliary problem, 
that of maximizing $\tilde{T}(c, p, q)$ for $p \in [\bar{q}, 1 - \bar{q}]$. It may be checked, just as in (A.4), 
that, for all $p \in (\bar{q}, 1 - \bar{q})$,
\[
\frac{\partial}{\partial p} \tilde{T}(c, p, q) = \frac{1}{2} \frac{m(c)}{p^3 (1 - p)^3} \hat{f}(c, p, q),
\]
where
\[
\hat{f}(c, p, q) := (2m(c)q - m(c) - 2q + 1)p^4 + \left(4q - 2m(c)q - 2m(c)q^2 - 1\right)p^3 \\
+ \left(3m(c)q - 3q + 3m(c)q^2\right)p^2 \\
+ \left(q - m(c)q - 3m(c)q^2\right)p + m(c)q^2.
\]

Since \(\hat{f}(c, 0.5, q) \equiv (2q - 1)(m(c) + 1)/16 > 0\) and \(\hat{f}(c, q, q) \equiv -q^2(2q - 1)(1 - q)^2 < 0\), by the Intermediate Value Theorem, there exists a root of \(\hat{f}(c, \cdot, q)\) in \((0.5, q)\). For the time being, let us call it \(p^*_\infty(c, q)\).

Inspired by the roles of Lemmas A1 and A2 in the proof of Proposition 4, let us split 
our analysis in two. On one hand, if \(m(c) \leq 1\), then we have 
\(\frac{\partial^2}{\partial p^2} \tilde{T}(c, p, q) / \partial p^2 < 0, \forall p \in (\bar{q}, 1 - \bar{q})\). In order to see this, it is just a matter of applying the \(\partial^2 / \partial p^2\) operator to each term in square brackets in (A.6) in turn. The first one gives
\[
\frac{\partial^2}{\partial p^2} \left(\frac{-m(c)q}{2} - \frac{(m(c))^2}{4} \frac{p - q)^2}{p^2}\right) \\
\equiv - \frac{1}{2} \frac{m(c)}{p^4}q(2p - 2m(c)p + 3m(c)q) \sim (2p - 3q)m(c) - 2p.
\]

This is negative because it is affine in \(m(c)\), it is equal to \(-2p < 0\) when \(m(c) = 0\), 
and equal to \(-3q < 0\) when \(m(c) = 1\). Now for the second one:
\[
\frac{\partial^2}{\partial p^2} \left(\frac{-m(c)}{2} \frac{1 - q}{1 - p} - \frac{(m(c))^2}{4} \frac{p - q)^2}{(1 - p)^2}\right) \\
\equiv - \frac{1}{2} \frac{m(c)}{(1 - p)^4} (3q - 2p - 1)m(c) + 2p - 2.
\]

Again, this is negative because it is affine in \(m(c)\), it is equal to \(2p - 2 < 0\) when 
\(m(c) = 0\), and equal to \(3q - 3 < 0\) when \(m(c) = 1\).
On the other hand, if \( m(c) > 1 \), then \( \hat{f}(c, p, q) \) is strictly decreasing in \( p \) for \( p \in [\tilde{q}, 1 - \tilde{q}] \). In fact, given any \( p \in (\tilde{q}, 1 - \tilde{q}) \), we have

\[
\frac{\partial}{\partial p} \hat{f}(c, p, q) = \left( -3q^2 - 4p^3 - q + 6pq^2 - 6p^2q + 8p^3q - 6p^2q^2 + 6pq \right) m(c) + q + 12p^2q - 8p^3q - 6pq - 3p^2 + 4p^3.
\]

once more an affine expression in \( m(c) \). At \( m(c) = 1 \), this equals

\[
-3 \left( -2pq^2 - 2p^2q + 2p^2q^2 + p^2 + q^2 \right) = -3 \left( p^2 (1 - q)^2 + q^2 (1 - p)^2 \right) < 0.
\]

As for the behavior of \( \partial \hat{f}(c, p, q)/\partial p \) when \( m(c) \to +\infty \), note that \( \partial \hat{f}(c, p, q)/\partial p = g(p, q) m(c) + q + 12p^2q - 8p^3q - 6pq - 3p^2 + 4p^3 \), where we have borrowed the \( g \) notation from the proof of Lemma A2. Since we show there that \( g(p, q) < 0 \), we have \( \lim_{m(c) \to +\infty} \partial \hat{f}(c, p, q)/\partial p = -\infty \), so that \( \partial \hat{f}(c, p, q)/\partial p < 0 \), \( \forall p \in (\tilde{q}, 1 - \tilde{q}) \).

Thus, in either case, \( \hat{p}(c, q) \) is the one and only solution to \( \max_{p \in [\tilde{q}, 1 - \tilde{q}]} \hat{T}(c, p, q) \).

Finally, since \( T(n, c, p, q) \) converges uniformly to \( \hat{T}(c, p, q) \) and we have shown that \( \arg \max_{p \in [\tilde{q}, 1 - \tilde{q}]} \hat{T}(c, p, q) \) is a singleton \( \{p^*_n(c, q)\} \), it follows from Theorem 2.2 of Schochetman (1990) that \( \lim_{n \to \infty} p^*_n(c, q) \) exists and equals \( p^*_n(c, q) \), which, as we have already shown, belongs to the \((0.5, q)\) interval.

(iii) Assume \( q < 0.5 \). Then, by the argument above, \( 0.5 < \lim_{n \to \infty} p^*_n(c, 1 - q) < 1 - q \). By Claim A1, \( \lim_{n \to \infty} p^*_n(c, 1 - q) = \lim_{n \to \infty} (1 - p^*_n(c, q)) = 1 - \lim_{n \to \infty} p^*_n(c, q) \). Therefore, \( 0.5 < 1 - \lim_{n \to \infty} p^*_n(c, q) < 1 - q \), or yet \( 0.5 > \lim_{n \to \infty} p^*_n(c, q) > q \), as wished. \( \square \)

**Proof of Lemma 5**

By Levy’s Continuity Theorem, it suffices to show that, for each \( t \in \mathbb{R} \), \( \varphi_{\text{MultiDiff}}(n, q \gamma_B(n, c, p^*_n(c, q)), 1 - q) \gamma_R(n, c, p^*_n(c, q)) \) \((t)\) converges, as \( n \to \infty \), to \( \varphi_{\text{Skellam}}((q/p^*_n(c, q)) m(c)/2, ((1 - q)/(1 - p^*_n(c, q))) m(c)/2) \) \((t)\). In turn, the characteristic function of Skellam \((m_B, m_R)\) is given by \( \varphi_{\text{Skellam}}(m_B, m_R) \) \((t)\) = \( \exp(m_B (e^{it} - 1) + m_R (e^{-it} - 1)) \).\(^{27}\)

Let \( n \geq n_0(c) \), so that, by Proposition 1, \( \gamma_B(n, c, p^*_n(c, q)) = \gamma^*(n, c) / (2p^*_n(c, q)) \) and \( \gamma_R(n, c, p^*_n(c, q)) = \gamma^*(n, c) / (2(1 - p^*_n(c, q))) \). For any \( t \in \mathbb{R} \), (2) then yields

\[
\varphi_{\text{MultiDiff}}(n, q \gamma_B(n, c, p^*_n(c, q)), 1 - q) \gamma_R(n, c, p^*_n(c, q)) \) \((t)\)

\[
= \left( 1 + q \gamma_B(n, c, p^*_n(c, q)) (e^{it} - 1) + (1 - q) \gamma_R(n, c, p^*_n(c, q)) (e^{-it} - 1) \right)^n.
\]

\(^{27}\) This can be verified in Skellam (1946), where the probability generating function of this distribution is provided: \( G(t) = \exp(m_B t + m_R t^{-1} - m_B - m_R) \). Then it is just a matter of writing \( \varphi_{\text{Skellam}}(m_B, m_R) \) \((t)\) = \( G(e^{it}) \).
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variables, we may write

\[ \Pr \left( e^{it} - 1 \right) = \left( 1 + \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \right) \frac{1}{2} \left( e^{it} - 1 \right) + \frac{1 - q \gamma^* (n, c)}{1 - p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \]

\[ = \left( 1 + \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \right) \frac{1}{2} \left( e^{it} - 1 \right) + \frac{1 - q \gamma^* (n, c)}{1 - p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \]

At the same time, note that

\[ \lim_{n \to \infty} \left( \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \right) \frac{1}{2} \left( e^{it} - 1 \right) = \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \]

\[ = \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) + \frac{1 - q \gamma^* (n, c)}{1 - p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \]

by Proposition 6. Therefore,

\[ \lim_{n \to \infty} \varphi_{\text{MultiDiff}} (n, q y_B (n, c, p^*_n (c, q)), (1 - q) y_B (n, c, p^*_n (c, q))) (t) \]

\[ = \exp \left( \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) + \frac{1 - q \gamma^* (n, c)}{1 - p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \right) \]

\[ = \varphi_{\text{Skellam}} \left( \frac{q \gamma^* (n, c)}{p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) + \frac{1 - q \gamma^* (n, c)}{1 - p^*_n (c, q)} \frac{1}{2} \left( e^{it} - 1 \right) \right) \cdot \]

The following lemma will be used in the proof of Lemma 6.

Lemma A4 Given \( q \in (0.5, 1 - \overline{q}) \) and \( \mu > 0 \), let \( Z_p \sim \text{Skellam} \left( \left( (q/p) \mu, \left( (1 - q)/(1 - p) \mu \right) \right) \right) \), for all \( p \in (0.5, q] \). Then \( \Pr (Z_p > 0) + \Pr (Z_p = 0) / 2 \) is strictly decreasing in \( p \).

Proof Because \( Z_p \) is the difference of two independent Poisson-distributed random variables, we may write

\[ \Pr (Z_p > 0) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} e^{-\frac{q}{p} \mu} \frac{\left( \frac{q}{p} \mu \right)^i}{i!} e^{-\frac{1 - q}{1 - p} \mu} \frac{(1 - q)/(1 - p) \mu)^j}{j!} \]

and

\[ \Pr (Z_p = 0) = 1 - \Pr (Z_p > 0) - \Pr (Z_p < 0) = 1 - \Pr (Z_p > 0) - \Pr (-Z_p > 0) \]

Let us call the first and second parameters of the Skellam distribution \( m_B \) and \( m_R \), respectively. We now analyze the partial derivatives of the above probabilities with respect to \( m_B \) and \( m_R \).

\[ \frac{\partial}{\partial m_B} \Pr (Z_p > 0) = -\sum_{i=1}^{\infty} \sum_{j=0}^{i-1} e^{-m_B} (m_B)^i \frac{(m_B)^i}{i!} e^{-m_R} (m_R)^j \frac{(m_R)^j}{j!} \]

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\[ + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} e^{-m_R} \frac{(m_B)^{i-1}}{(i-1)!} e^{-m_R} \frac{(m_R)^j}{j!} \]

\[ = -\Pr(Z_p > 0) + \sum_{i'=0}^{\infty} \sum_{j=0}^{i'} e^{-m_B} \frac{(m_B)^{i'}}{(i')!} e^{-m_R} \frac{(m_R)^j}{j!} \]

\[ = -\Pr(Z_p > 0) + \Pr(Z_p \geq 0) = \Pr(Z_p = 0). \]

Since we can also write

\[ \Pr(Z_p > 0) = \sum_{i=1}^{\infty} e^{-\frac{q}{p} \mu} \frac{(\frac{q}{p} \mu)^i}{i!} e^{-\frac{1-q}{1-p} \mu} \left( 1 + \sum_{j=1}^{i-1} \frac{(\frac{1-q}{1-p} \mu)^j}{j!} \right), \]

we have

\[ \frac{\partial}{\partial m_R} \Pr(Z_p > 0) = -\sum_{i=1}^{\infty} e^{-\frac{q}{p} \mu} \frac{(\frac{q}{p} \mu)^i}{i!} e^{-\frac{1-q}{1-p} \mu} \left( 1 + \sum_{j=1}^{i-1} \frac{(\frac{1-q}{1-p} \mu)^j}{j!} \right) \]
\[ + \sum_{i=1}^{\infty} e^{-\frac{q}{p} \mu} \frac{(\frac{q}{p} \mu)^i}{i!} e^{-\frac{1-q}{1-p} \mu} \sum_{j=1}^{i-1} \frac{(\frac{1-q}{1-p} \mu)^j}{j!} \]

\[ = -\Pr(Z_p > 0) + \sum_{i=1}^{\infty} e^{-\frac{q}{p} \mu} \frac{(\frac{q}{p} \mu)^i}{i!} e^{-\frac{1-q}{1-p} \mu} \sum_{j'=0}^{i-2} \frac{(\frac{1-q}{1-p} \mu)^{j'}}{j'!} \]
\[ = -\Pr(Z_p > 0) + \Pr(Z_p \geq 2) = -\Pr(Z_p = 1). \]

By the symmetry property of the Skellam distribution \((Z \sim \text{Skellam}(m_B, m_R) \Leftrightarrow -Z \sim \text{Skellam}(m_R, m_B))\), we have

\[ \frac{\partial}{\partial m_B} \Pr(-Z_p > 0) = -\Pr(-Z_p = 1) = -\Pr(Z_p = -1) \]

and

\[ \frac{\partial}{\partial m_R} \Pr(-Z_p > 0) = \Pr(-Z_p = 0) = \Pr(Z_p = 0). \]

Therefore,

\[ \frac{\partial}{\partial p} \Pr(Z_p > 0) = \frac{\partial \Pr(Z_p > 0)}{\partial m_B} \frac{\partial m_B}{\partial p} + \frac{\partial \Pr(Z_p > 0)}{\partial m_R} \frac{\partial m_R}{\partial p} \]
\[ = \Pr(Z_p = 0) \left( -\frac{1}{2} \frac{m}{p^2} q \right) - \Pr(Z_p = 1) \frac{1}{2} \frac{m}{(1-p)^2} (1-q) \]
\[
= -\frac{m}{2} \left( \frac{q}{p^2} \Pr(Z_p = 0) + \frac{1-q}{(1-p)^2} \Pr(Z_p = 1) \right)
\]

and
\[
\frac{\partial}{\partial p} \Pr(Z_p < 0) = \frac{\partial}{\partial p} \Pr(Z_p < 0) \frac{\partial m_B}{\partial p} + \frac{\partial}{\partial m_R} \Pr(Z_p < 0) \frac{\partial m_R}{\partial p} \\
= -\Pr(Z_p = -1) \left( -\frac{1}{2} \frac{m}{p^2} q \right) + \Pr(Z_p = 0) \frac{1}{2} \frac{m}{(1-p)^2} (1-q)
\]
\[
= \frac{m}{2} \left( \frac{q}{p^2} \Pr(Z_p = -1) + \frac{1-q}{(1-p)^2} \Pr(Z_p = 0) \right).
\]

Now note that
\[
\Pr(Z_p > 0) + \frac{1}{2} \Pr(Z_p = 0) = \Pr(Z_p > 0) + \frac{1}{2} \left( 1 - \Pr(Z_p > 0) - \Pr(Z_p < 0) \right) \\
\quad = \frac{1}{2} \left( 1 + \Pr(Z_p > 0) - \Pr(Z_p < 0) \right).
\]

Thus,
\[
\frac{d}{dp} \left( \Pr(Z_p > 0) + \frac{1}{2} \Pr(Z_p = 0) \right) \\
\quad = \frac{1}{2} \left( \frac{\partial}{\partial p} \Pr(Z_p > 0) - \frac{\partial}{\partial p} \Pr(Z_p < 0) \right) \\
\quad = \frac{1}{2} \left( -\frac{m}{2} \left( \frac{q}{p^2} \Pr(Z_p = 0) + \frac{1-q}{(1-p)^2} \Pr(Z_p = 1) \right) \\
\quad \quad - \frac{m}{2} \left( \frac{q}{p^2} \Pr(Z_p = -1) + \frac{1-q}{(1-p)^2} \Pr(Z_p = 0) \right) \right) \\
\quad = -\frac{m}{4} \left( \frac{q}{p^2} + \frac{1-q}{(1-p)^2} \right) \Pr(Z_p = 0) + \frac{1-q}{(1-p)^2} \Pr(Z_p = 1) \\
\quad \quad + \frac{q}{p^2} \Pr(Z_p = -1) < 0.
\]

\[\square\]

**Proof of Lemma 6** Since \(\Pr(B \text{ wins } | n, c, p_n^* (c, q), q) = \Pr(b - r > 0 | n, c, p_n^* (c, q), q) + \Pr(b - r = 0 | n, c, p_n^* (c, q), q) / 2\), Lemma 5 yields \(\lim_{n \to \infty} \Pr(B \text{ wins } | n, c, p_n^* (c, q), q) = \Pr(Z > 0) + \Pr(Z = 0) / 2\), where \(Z \sim \text{Skellam } ((q/p_{\infty}^* (c, q)) m (c) / 2, ((1-q) / (1-p_{\infty}^* (c, q))) m (c) / 2)\). Thus, at least the existence of the limit is ensured. Let us see how it compares to 0.5.

(i) If \(q = 0.5\), then the thesis follows directly from Lemma 4, since \(\Pr(B \text{ wins } | n, c, p_n^* (c, q), q) = 0.5, \forall n \geq n_0 (c)\). Alternatively (and this second argument will be useful below), it could also be seen to follow from the fact above, since
in this case, by Lemma 6, \( p^*_\infty(c, q) = q \), and \( Z \) will be distributed as the symmetric Skellam \((m(c)/2, m(c)/2)\), so that \( \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 = 1 - \text{Pr}(Z < 0) - \text{Pr}(Z = 0)/2 = 1 - \text{Pr}(Z > 0) - \text{Pr}(Z = 0)/2 \). Thus, \( \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) = \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 = 0.5 \).

(ii) If \( q > 0.5 \), then, by Lemma 6, \( p^*_\infty(c, q) \in (0.5, q) \). Therefore, by Lemma A4, \( \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 \) is strictly larger than \( \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 \), where \( Z_q \sim \text{Skellam} ((q/m)/2, (1 - q)/2) \). As explained in part (i), \( \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) = \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 = 0.5 \), so that \( \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) = \text{Pr}(Z > 0) + \text{Pr}(Z = 0)/2 = 0.5 \).

(iii) If \( q < 0.5 \), then \( \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) = \text{Pr}(R \text{ wins } | n, c, 1 - p^*_n(c, q), 1 - q) = 1 - \text{Pr}(B \text{ wins } | n, c, p^*_n(c, 1 - q), 1 - q) \), where we have used Claim A1. Since \( 1 - q > 0.5 \), part (ii) yields \( \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) = 1 - \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, 1 - q), 1 - q) < 1 - 0.5 = 0.5 \).

Proof of Proposition 7

(i) Fix any \( n \geq n_0(c) \), so that, by (13) and Proposition 1,

\[
\mathcal{C}(n, c, q) = \mathcal{C}(n, c, p^*_n(c, q), q) = nc\left(\frac{q\gamma^*(n, c)}{2q} + (1 - q)\frac{\gamma^*(n, c)}{2(1 - q)}\right)
- ncp^*_n(c, q)\left(\frac{q}{2p^*_n(c, q)} + \frac{1 - q}{p^*_n(c, q)}\right)
= nc\gamma^*(n, c)\left(1 - \frac{1}{2}\left(\frac{q}{p^*_n(c, q)} + \frac{1 - q}{p^*_n(c, q)}\right)\right)
\sim 2p^*_n(c, q)(1 - p^*_n(c, q)) - ((1 - p^*_n(c, q))q + p^*_n(c, q)(1 - q))
= (2p^*_n(c, q) - 1)(q - p^*_n(c, q)).
\]

If \( q \geq 0.5 \), then \( p^*_n(c, q) \geq 0.5 \) and \( p^*_n(c, q) \leq q \) due to Proposition 4. In either case, \( 2p^*_n(c, q) - 1 \) and \( q - p^*_n(c, q) \) have the same sign, so that \( \mathcal{C}(n, c, q, q) - \mathcal{C}(n, c, p^*_n(c, q), q) \) is indeed positive.

(ii) Let \( J(c, p, q) := (2q - 1)\left(2 \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) - 1\right) \). As done in the proof of Lemma 6, note that (14) and Lemma 5 yield

\[
\lim_{n \to \infty} \frac{I(n, c, p^*_n(c, q), q)}{n} = \left(2q - 1\right)\left(2 \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) - 1\right)
= (2q - 1)\left(2 \lim_{n \to \infty} \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q) - 1\right)
= J(c, p^*_\infty(c, q), q).
\]

This implies that also \( \lim_{n \to \infty} \left(I(n, c, p^*_n(c, q), q) / n\right) = J(c, p^*_\infty(c, q), q) \). In fact, a simple variation of the argument given in Sect. 4 would show that \( \lim_{n \to \infty} \left(I(n, c, p^*_n(c, q), q) - I(n, c, p^*_n(c, q), q) / n\right) = 0 \) (just redefine \( x_{n,B} \) as \( \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q, n_B) \) and \( y_n \) as \( \text{Pr}(B \text{ wins } | n, c, p^*_n(c, q), q, n_B) \).
welfare-improving misreported polls). Then, given \( \varepsilon > 0 \), it is only a matter of choosing \( n_1 \in \mathbb{N} \) large enough so that \( n \geq n_1 \) implies both \(|I(n, c, p_n^*(c, q), q) - I(n, c, p_n(c, q), q)| / n| < \varepsilon / 2\) and \(|I(n, c, p_n^*(c, q), q) - J(c, p_n^*(c, q), q) - J(c, p_n(c, q), q)| < \varepsilon / 2\), and then applying the triangle inequality.

Since \( I(n, c, q, q) = (2q - 1) (2 \Pr B(n, c, q, q) - 1) = 0 \) for all \( n \geq n_0(c) \) by Lemma 4, we have \( \lim_{n \to \infty} (I(n, c, q, q) / n) = 0 \) and \( \lim_{n \to \infty} (I(n, c, q, q) / n) = 0 \) (take \( x_{n,n_B}^* \) as \( \Pr (B(n, c, q, q, n_B) \) and \( y_n \) as \( \Pr (B(n, c, q, q, 0)) \). Because

\[
J(c, q, q) = (2q - 1) (2 \Pr (Z_{q,q} > 0) + \Pr (Z_{q,q} = 0) - 1)
\]

\[
= (2q - 1) (\Pr (Z_{q,q} > 0) + \Pr (Z_{q,q} < 0) + \Pr (Z_{q,q} = 0) - 1) = (2q - 1)(1 - 1) = 0
\]

by the symmetry of Skellam \( (m(c) / 2, m(c) / 2) \), we can also write \( \lim_{n \to \infty} (I(n, c, q, q) / n) = J(c, q, q) \). Therefore,

\[
\lim_{n \to \infty} \frac{I(n, c, p_n^*(c, q), q) - I(n, c, q, q)}{n} = J(c, p_n^*(c, q), q) - J(c, q, q).
\]

If \( q > 0.5 \), then \( p_n^*(c, q) \in (0.5, q) \) by Proposition 6, so that

\[
J(c, p_n^*(c, q), q) - J(c, q, q)
\]

\[
= 2 (2q - 1) \left( \Pr (Z_{p_n^*(c, q), q} > 0) + \frac{\Pr (Z_{p_n^*(c, q), q} = 0)}{2} \right)
\]

\[
- \left( \Pr (Z_{q,q} > 0) + \frac{\Pr (Z_{q,q} = 0)}{2} \right)
\]

> 0,

by Lemma A4.

To see that this same inequality holds for the \( q < 0.5 \) case, first note that \( J(c, p, q) \equiv J(c, 1 - p, 1 - q) \). In fact,

\[
J(c, 1 - p, 1 - q) = (2 (1 - q) - 1) (2 \Pr (Z_{1-p,1-q} > 0) + \Pr (Z_{1-p,1-q} = 0) - 1),
\]

where \( Z_{1-p,1-q} \sim \) Skellam \( \left( (1 - q) / (1 - p) \right) m(c) / 2, (q/p) m(c) / 2 \). By the symmetry property of the Skellam distribution, we then have \( -Z_{1-p,1-q} \sim \) Skellam \( \left( q/p \right) m(c) / 2, ((1 - q) / (1 - p)) m(c) / 2 \), the exact same distribution of \( Z_{p,q} \). Thus,

\[
J(c, 1 - p, 1 - q) = (1 - 2q) \left( 2 \Pr (-Z_{1-p,1-q} < 0) + \Pr (-Z_{1-p,1-q} = 0) - 1 \right)
\]

\[
= (1 - 2q) \left( 2 \Pr (Z_{p,q} < 0) + \Pr (Z_{p,q} = 0) - 1 \right)
\]

\[
= (1 - 2q) \left( 2 \Pr (Z_{p,q} > 0) - \Pr (Z_{p,q} = 0) \right)
\]
\[
\begin{align*}
&+ \Pr \left( Z_{p,q} = 0 \right) - 1 \\
&= (1 - 2q) \left[ -2 \Pr \left( Z_{p,q} > 0 \right) - \Pr \left( Z_{p,q} = 0 \right) + 1 \right] \\
&= (2q - 1) \left[ 2 \Pr \left( Z_{p,q} > 0 \right) + \Pr \left( Z_{p,q} = 0 \right) - 1 \right] \\
&= J(c, p, q).
\end{align*}
\]

Having noted this, since \( 1 - p_{\infty}^*(c, q) = p_{\infty}^*(c, 1 - q) \) by Claim A1, we once more have

\[
J(c, p_{\infty}^*(c, q), q) - J(c, q, q) = J(c, p_{\infty}^*(c, q), q) = J(c, 1 - p_{\infty}^*(c, q), 1 - q) \\
= J(c, p_{\infty}^*(c, 1 - q), 1 - q) = J(c, p_{\infty}^*(c, 1 - q), 1 - q) - J(c, 1 - q, 1 - q) > 0,
\]

since \( 1 - q > 0.5 \).

Thus, in either case, we have \( \lim_{n \to \infty} \left( \left( I(n, c, p_{\infty}^*(c, q), q) - I(n, c, q, q) \right) /n \right) > 0 \), so that \( I(n, c, p_{\infty}^*(c, q), q) > I(n, c, q, q) \) for sufficiently large \( n \).

(iii) It follows immediately from parts (i) and (ii) that, for sufficiently large \( n \),

\[
W(n, c, p_{\infty}^*(c, q), q) > W(n, c, q, q).
\]

\textbf{Proof of Proposition 8} (i) Note from (13) and Proposition 1 that, for all \( n \geq n_0(c) \),

\[
\frac{C(n, c, q, q)}{n} = n \frac{\gamma^*(n,c)}{2} c + n \frac{\gamma^*(n,c)}{2} c = \gamma^*(n,c)c.
\]

Hence, Lemma 2 yields \( \lim_{n \to \infty} \left( C(n, c, q, q) /n \right) = 0 \).

Since \( C(n, c, p_{\infty}^*(c, q), q) < C(n, c, q, q) \) for sufficiently large \( n \) (Proposition 7) and \( C(n, c, p_{\infty}^*(c, q), q) \geq 0 \), the Squeeze Theorem gives \( \lim_{n \to \infty} \left( C(n, c, p_{\infty}^*(c, q), q) /n \right) = 0 \) too.

(ii) It was already shown in the proof of part (ii) of Proposition 7 that, both for the \( q > 0.5 \) and the \( q < 0.5 \) cases, \( \lim_{n \to \infty} \left( I(n, c, p_{\infty}^*(c, q), q) /n \right) = J(c, p_{\infty}^*(c, q), q) = J(c, q, q) = \lim_{n \to \infty} \left( I(n, c, q, q) /n \right) \).

(iii) This follows immediately from parts (i) and (ii).

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