Performance of Viterbi Decoding on Interleaved Rician Fading Channels

Lan V. Truong

Abstract

In this paper, we investigate the performance of the Viterbi decoding algorithm with/without automatic repeat request (ARQ) over a Rician flat fading channel with unlimited interleaving. We show that the decay rate of the average bit error probability with respect to the bit energy to noise ratio is of order between $d_f$ and $d_f+1$ at high bit energy to noise ratio for both cases (with ARQ and without ARQ), where $d_f$ is the free distance of the convolutional code. The Yamamoto-Itoh flag helps to reduce the average bit error probability by a factor of $4^{d_f}$ with a negligible retransmission rate. We also prove an interesting result that the average bit error probability decays exponentially fast with respect to the Rician factor for any fixed bit energy per noise ratio. In addition, the error exponent with respect to the Rician factor is equal to $d_f$ at high bit energy to noise ratio.

Index Terms

Coded ARQ, Feedback, Rician Fading, Error Exponents, Block Fading Channels.

I. INTRODUCTION

Viterbi [1] proposed a nonsequential code and derived an upper bound on error probability using random coding arguments for Discrete Memoryless (DM) and Additive White Gaussian Noise (AWGN) channels. The algorithm was thereafter shown to yield maximum likelihood decisions by Omura [2] and Forney [3]. In 1971, Viterbi proposed a method to evaluate the error probabilities of convolutional codes by using their transfer functions [4]. The performance of the convolutional codes [1] were later evaluated for time-varying channels. Using the same transfer function method, the error probability of the Viterbi decoding over the Rayleigh fading channel with unlimited interleaving was estimated by evaluating the exact pairwise error probabilities together with the union bound [5]. Vucetic evaluated the performance of punctured convolutional codes with maximum-likelihood Viterbi algorithm to enable adaptive encoding and decoding without modifying the basic structure of the encoder and the decoder for Rician fading channels [6]. Malkamaki and Leib considered union upper bound techniques for convolutional codes with limited interleaving over block fading Rician channels [7]. Although their method provides useful numerical results, the performance is intractable (hard to analyze).

Automatic Repeat reQuest (ARQ) [8] is an error-control method for data transmission that uses acknowledgments to achieve reliable data transmission over an unreliable service. As the turn-around time of the communication link increases, however, retransmission becomes expensive and a more elaborate technique with reduced retransmission is required. Coded ARQ, which combines error-correcting coding and retransmission, is then an alternative to ARQ. Fang [9] and Yamamoto-Itoh [10] studied convolutionally coded ARQ schemes with Viterbi decoding [1] and showed that a low error probability is attained by having a moderate increase in complexity for DM and AWGN channels. However, the frequency of retransmission in the Yamamoto-Itoh algorithm is much less than that of the Fang algorithm. This retransmission method is a special case of variable-length feedback coding with termination schemes (VLFT) [11]–[15]. In such schemes, the encoders with full knowledge of output sequences and receivers’ decoding algorithms can estimate the receivers’ decoded messages (if decoded) at each fixed interval and compare with the transmitted messages in order to decide whether to stop or to continue transmission after each fixed interval as in the coded-ARQ. Note that the receivers do not need to decode the messages at these fixed intervals, they only need to decode the transmitted messages at stopping times.

The Yamamoto-Itoh’s decoding flag has been implemented in the Keystone Architecture Viterbi-Decoder Coprocessor (VCP2), Texas Instruments [16]. A new decoding algorithm based on the modified Viterbi algorithm for repeat request systems was proposed in [17], which was numerically shown to achieve better error exponent than the Yamamoto-Itoh’s result [10] for the binary symmetric channel with crossover probability 0.1. The performance of the Yamamoto-Itoh algorithm [10] for different channel models was also considered in a number of papers [18]–[22]. Hashimoto [18] theoretically proved that the Yamamoto-Itoh algorithm attains better error exponent than the one shown in [10] for AWGN channels. In [21], the ROV (reliability output Viterbi algorithm) was proposed for hybrid-ARQ, and the performance was compared to the Yamamoto-Itoh algorithm. A combined scheme of the Yamamoto-Itoh algorithm and ROVA scheme was proposed in [22], and it was shown theoretically that the error exponent was improved compared to Yamamoto-Itoh algorithm for very noisy channels. Since no simulation results were shown, it is uncertain whether the combined scheme can actually attain better performance compared to the Yamamoto-Itoh algorithm for practical channels.

The author is supported by an NUS grant (R-263-000-A98-750/133) and by a Singapore Ministry of Education (MOE) Tier 2 grant (R-263-000-B61-112).
In this paper, we investigate the performance of the original Viterbi decoder [1] and the modified Viterbi decoding algorithm by Yamamoto-Itoh [10] over Rician fading channels with unlimited interleaving. The original Viterbi decoder [1] can be considered as the Yamamoto-Itoh algorithm when setting the Yamamoto-Itoh flag equal to zero \( u = 0 \) [10]. We show that the decay rate of the average bit error probability of the original Viterbi decoding scheme [1] is between \( 2d_f \) and \( 4d_f + 1 \) at high bit energy to noise ratio \( E_b/N_0 \). In addition, there exists a Yamamoto-Itoh flag such that the bit error probability of the Yamamoto-Itoh algorithm is lowered by at least a factor of \( 4^{d_f} \). When the Rician factor becomes very large, the bit error probability is shown to decay exponentially fast with the convolutional code free-distance being its exponent. To the best of the author’s knowledge, these results have not appeared in the literature before. All the existing results, thus far, were mainly evaluated for the original Viterbi decoding [1] or the Yamamoto-Itoh algorithm [10] over DM or AWGN channels. For Rayleigh or Rician fading channels, the majority of works concentrate on providing numerical results for different channel situations. In addition, the important effects of Rician factor on the error exponent for a fixed bit energy to noise ratio have not been analytically expressed in closed forms.

The rest of this paper is organized as follows: The channel model is provided in Section II. Some mathematical preliminaries are introduced in Section III. The main results are stated in Section IV. Proofs that are more technical are deferred to Sections V and VI. Tradeoffs between performance measures (bit error probability and retransmission probability) are also numerically provided in Section VI-A.

II. CHANNEL MODELS

We investigate the same channel model as [7] where the original Viterbi decoding algorithm is replaced with the Yamamoto-Itoh algorithm [10]. Note that the original Viterbi decoding is a special case of the Yamamoto-Itoh algorithm when setting the Yamamoto-Itoh flag \( u = 0 \) [10]. The model of the convolutional encoded transmission system to be analyzed is shown in Fig. 1. A block of \( Hk_e \) bits from the data source are first encoded by a rate \( R_e = k_e/n_c \) convolutional code with constraint length \( k_cK = k_c(m+1) \), where \( m \) is the memory order of the code. Before encoding, \( mk_e \) tail bits (\( m = K - 1 \) tail vectors) are added to each block of \( Hk_e \) bits to terminate the code trellis into a known state. The \( n_c(H+m) \) encoder output bits are denoted by \( x_{ij} \), where \( l = 1,2,\ldots,n_c \) indicates the code generator polynomial and \( j = 1,2,\ldots,H+m \). In the analysis, we assume that antipodal modulation, i.e., \( x_{ij} = \pm 1 \). The output symbols \( x_{ij} \) are interlaced over \( L \) subchannels. To simplify the analysis, we assume that each output of the encoder is transmitted via a different subchannel. In this paper, we consider unlimited interleaving, i.e. \( L = n_c(H+m) \).

The channel is assumed to be frequency nonselective fading Rician. In this model, the fading process is assumed to be constant over a block of \( N \) channel symbols (coherence time) and that the channel state information is perfectly estimated at the receiver. Assuming coherent detection, the received signal samples can be written as

\[
y_{ij} = \sqrt{E_c} \alpha_i x_{ij}, \quad i = 1,2,\ldots,n_c; \quad j = 1,2,\ldots,(H+m)n_c,
\]

where \( i \) indicates the subchannel, or, equivalently, the generator polynomial used, \( j \) the sample within a subchannel, \( E_c \) is the energy per transmitted code symbol, and \( n_{ij} \) are zero-mean white Gaussian noise samples with variance \( N_0/2 \). The average energy per transmitted bit can be easily shown to be equal to

\[
E_b = \left( \frac{H + m}{H} \right) \left( \frac{1}{R_c} \right) E_c.
\]

The fading envelopes \( \alpha_i \) of the \( n_c(H+m) \) subchannels involved in each decoding process are assumed to be independent of each other, identically distributed, and constant over the subchannel. Here, \( \alpha_i \) are assumed to be Rician distributed with the probability density function

\[
f(\alpha_i) = \frac{\alpha_i}{\sigma^2} e^{-((\alpha_i^2+s^2)/2\sigma^2)} I_0 \left( \frac{\alpha_i s^2}{\sigma^2} \right), \quad \alpha_i \geq 0,
\]

where \( \mathbb{E}(\alpha_i^2) = s^2 + 2\sigma^2 \). The Rician factor \( \gamma \) is defined as \( \gamma = s^2 / (2\sigma^2) \), and

\[
I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \theta} d\theta.
\]

The Viterbi decoder with repeated request proposed by Yamamoto-Itoh (Yamamoto-Itoh algorithm) [10] is used for the decoding of convolutional codes, which employs the samples \( y_{ij} \) as well as the ideal channel state information (CSI), \( \hat{\alpha}_i = \alpha_i \). The branch metrics are calculated as

\[
\lambda_j^{(r)} = \sum_{i=1}^{n_c} \alpha_i x_{ij}^{(r)} y_{ij}^{(r)}, \quad j = 1,2,\ldots,H+m,
\]

where \( \alpha_i := \alpha_{(j-1)n_c+i} \in \{\alpha_1, \alpha_2, \cdots, \alpha_{n_c}\} \), and \( x_{ij}^{(r)} \in \{x_{ij}\}_{i=1,n_c}^{j=1,(H+m)}, y_{ij}^{(r)} \in \{y_{ij}\}_{i=1,n_c}^{j=1,(H+m)} \) are obtained by reading the elements of the coded symbol matrix \( x_{ij} \) of size \( n_c \times (H+m) \) column-by-column from top-to-down.
III. SOME MATHEMATICAL PRELIMINARIES

In this paper, we use the notation \( x_+ = \max\{x, 0\} \), and \( \mathbb{R}_+ \) is the set of positive real numbers. We also use asymptotic notation such as \( O(\cdot) \) in the standard manner; \( f(x) = O(g(x)) \) holds if and only if \( \limsup_{x \to \infty} f(x)/g(x) < C \) for some positive constant \( C < \infty \). \([x]\) is the standard floor function. \( 1\{x \in A\} = 1 \) if \( x \in A \) and \( 1\{x \in A\} = 0 \) otherwise. The symbol error probability, the bit error probability, and the retransmission probability for a the Yamamoto-Itoh algorithm with \( \{x\} \) for some \( L \) are denoted by \( P_e(u), P_b(u), P_x(u) \). For \( u = 0 \), the Yamamoto-Itoh algorithm is the original Viterbi decoding algorithm, so \( P_e(0) \) and \( P_b(0) \) are the symbol error probability and bit error probability of the original Viterbi decoding [1], [4], respectively. Note that \( P_x(0) = 0 \).

**Definition 1.** For two functions \( f(x_1, x_2, \cdots, x_n) : \mathbb{R}_+^n \to \mathbb{R}_+ \) and \( g(x_1, x_2, \cdots, x_n) : \mathbb{R}_+^n \to \mathbb{R}_+ \) and a subset \( \{i_1, i_2, \cdots, i_m\} \subset \{1, 2, \cdots, n\} \), we define

\[
f(x_1, x_2, \cdots, x_n) = \omega_{x_{i_1}, x_{i_2}, \cdots, x_{i_m}} g(x_1, x_2, \cdots, x_n)
\]

if

\[
f(x_1, x_2, \cdots, x_n) \geq L g(x_1, x_2, \cdots, x_n)
\]

for some \( L \) which does not depend on the variables \( \{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\} \) but may depend on other variables \( \{x_1, x_2, \cdots, x_n\} \) \( \setminus \{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\} \).

For example, we have

\[
x_1^2 x_2 + x_2^2 = \omega_{x_1} (5x_1^2 + x_2),
\]

since \( x_1^2 x_2 + x_2^2 \geq (1/5)x_2(5x_1^2 + x_2) \) for any \( x_1, x_2 \in \mathbb{R}_+ \).

The following fact holds.

**Lemma 1.** For two functions \( f: \mathbb{R}_+^n \to \mathbb{R}_+ \) and \( g: \mathbb{R}_+^n \to \mathbb{R}_+ \). If \( f(x_1, x_2, \cdots, x_n) = \omega_{x_{i_1}, x_{i_2}, \cdots, x_{i_m}} (g(x_1, x_2, \cdots, x_n)) \) then \( f(x_1, x_2, \cdots, x_n) = \omega_{S} (g(x_1, x_2, \cdots, x_n)) \) for any \( S \subset \{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\} \).

Next, we prove two preliminary lemmas which will be used later in upper bounding and approximating reliability performances.

**Lemma 2.** For any variables \( \Phi_1 > 0, \Phi_2 \in \mathbb{R}, z > 0 \), where \( \Phi_1 \) and \( \Phi_2 \) can be dependent on each other but \( z \) is independent of \( \Phi_1 \) and \( \Phi_2 \), define

\[
g(\Phi_1, \Phi_2, z):= \int_0^z \alpha \exp \left(-\Phi_1 \alpha^2 - \Phi_2 \alpha \right) d\alpha.
\]

Then, the following expressions hold:

\[
g(\Phi_1, \Phi_2, z) = \frac{1}{2\Phi_1} \left[ 1 - \exp(-\Phi_1 z^2 - \Phi_2 z) - \Phi_2 \sqrt{2\pi} \exp \left( \frac{\Phi_2}{4\Phi_1} \right) \left[ Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \sqrt{2\Phi_1} \left[ \frac{z + \Phi_2}{2\Phi_1} \right] \right) \right] \right], \tag{10}
\]

\[
g(\Phi_1, \Phi_2, \infty) = \frac{1}{2\Phi_1} \left[ 1 - \frac{\Phi_2 \sqrt{2\pi}}{\sqrt{2\Phi_1}} \exp \left( \frac{\Phi_2}{4\Phi_1} \right) \right] Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right), \tag{11}
\]

1 Here, two variables are said to be dependent if one variable is a function of the other variable. Two variables are said to be independent if no variable is a function of the other.
where \( Q(x) \) is defined as
\[
Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt, \quad \forall t \in \mathbb{R}.
\] (12)

In addition, for any \( \Phi_2 \leq 0 \) we have
\[
g(\Phi_1, \Phi_2, z) = \omega_{\Phi_1, \Phi_2} \left( \frac{1}{2\Phi_1} \left[ 1 - \exp(-\Phi_1 z^2 - \Phi_2 z) - \frac{\Phi_2 \sqrt{2\pi}}{2\Phi_1} \exp\left( \frac{\Phi_2^2}{4\Phi_1} \right) \right] \right) \times \frac{z\sqrt{2\Phi_1}}{\sqrt{2\pi}} \min \left\{ \exp \left[ -\frac{1}{2} \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right)^2 \right], \exp \left[ -\frac{1}{2} \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} + z\sqrt{2\Phi_1} \right)^2 \right] \right\}.
\] (13)

**Lemma 3.** Consider a convolutional code with transfer function \( T(D, N)^2 \). Assuming that \( \{a_k\}_{k=1}^\infty \) and \( \{c_k\}_{k=1}^\infty \) are coefficients in the expansions of the transfer function and the derivative of the transfer function, respectively, i.e.,
\[
T(D, N) = \sum_{k=d_f}^{\infty} a_k D^k,
\] (14)
\[
\frac{dT(D, N)}{dN} \bigg|_{N=1} = \sum_{k=d_f}^{\infty} c_k D^k.
\] (15)

Then, the following bounds hold:
\[
a_k \leq \sum_{L=1}^{\lfloor \frac{L}{d_f} \rfloor} \sum_{l=0}^{L-1} \binom{L}{l} \binom{k-1}{L-l-1} 1\{L-l-1 \leq k-1\},
\] (16)
\[
c_k \leq k k_c a_k.
\] (17)

**IV. MAIN RESULTS**

Our main contribution in this paper is the following theorem.

**Theorem 1.** The decay rate of the bit error probability \( P_b(0) \) of the original Viterbi decoding scheme satisfies the following constraints
\[
h(\gamma, \sigma, H, k_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-d_f} \geq P_b(0) \geq f(\gamma, \sigma, H, k_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-(d_f+1)}
\] (18)
for some finite but high \( E_b/N_0 \) and some constants \( h(\gamma, \sigma, H, k_c, d_f, R_c, m), f(\gamma, \sigma, H, k_c, d_f, R_c, m) \) which depend on convolutional code parameters \( H, k_c, d_f, R_c, m \) and channel parameters \( \gamma, \sigma \).

If the channel allows ARQ, then there exists a Yamamoto-Itoh flag \( u_0 > 0 \) such that the retransmission probability \( P_x(u_0) \) and the decay rate of the bit error probability \( P_b(u_0) \) for the Yamamoto-Itoh algorithm satisfy
\[
P_x(u_0) = O \left( \left( \frac{E_b}{N_0} \right)^{-d_f} \right),
\] (19)
\[
P_b(u_0) \leq 4^{-d_f} P_b(0).
\] (20)

In addition, for any fixed transmission bit energy to noise ratio \( (E_b/N_0) \) and Yamamoto-Itoh flag \( u \geq 0 \) such that
\[
\frac{1}{\sigma^2} \left[ \left( \sqrt{\frac{2E_c}{N_0}} - \frac{u}{d_f} \right)^2 + \frac{1}{\sigma^2} \right]^{-1} < 1,
\] (21)
the following holds
\[
d_f \geq \limsup_{\gamma \to \infty} \frac{-\ln P_b(u)}{\gamma} \geq \liminf_{\gamma \to \infty} \frac{-\ln P_b(u)}{\gamma}
\] (22)
\[
\geq d_f \left( 1 - \frac{1}{\sigma^2} \left[ \left( \sqrt{\frac{2E_c}{N_0}} + \frac{u}{d_f} \right)^2 + \frac{1}{\sigma^2} \right]^{-1} \right),
\] (23)
for any convolutional code using the Yamamoto-Itoh algorithm for decoding.

\(^2\)The transfer function for convolutional codes is defined by Viterbi [4].
The lower bounds in this Theorem are provided in Proposition 2. The upper bounds in this Theorem are provided in Proposition 3.

**Remark 1.** Some remarks are in order.

- The original Viterbi decoding [1] is a special case of Yamamoto-Itoh algorithm when setting the flag \( u = 0 \) [10]. The upper bounds work the same for two cases: with ARQ \(( u > 0)\) and the original Viterbi decoding [1].
- Although parameter \( \sigma \) plays an important role in characterizing the coding performance, we can scale the \( E_c \) and set \( \sigma = 1 \). Therefore, we are not interested in finding the effect of this parameter in this paper. In other words, this parameter will behave as the \( E_b/N_0 \) from this viewpoint.
- This theorem shows that in a fading environment with unlimited interleaving, the decay order of average error probability is between \( d_f \) and \( d_f + 1 \) in \( E_b/N_0 \). This is a new contribution.
- The average error exponent with respect to the Rician factor is shown to be positive and approximately equal to \( d_f \) at high \( E_b/N_0 \). It shows a very interesting result that the average bit error probability decays exponentially fast with respect to the Rician factor.
- Yamamoto-Itoh flag is well-known to double the reliability function for the DM channel [10]. For the fading channel, this paper shows that it can help to reduce the error probability by a factor of \( 4^{d_f} \) with the same resources \((E_b/N_0, H, m, d_f, R_c, k_c)\) and negligible retransmission probability. This result is surprising.

**V. LOWER BOUNDS ON PERFORMANCE**

**A. General Lower Bounds at finite \( H \) and \((E_b/N_0)\)**

**Proposition 1.** For any convolutional code with transfer function \( T(D, N) \) which uses the Yamamoto-Itoh algorithm with flag \( u \) for decoding, the following inequalities hold

\[
P_e(u) \leq \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, u/d_f, \sigma, s)]^k,
\]

\[
P_x(u) \leq \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, -u/d_f, \sigma, s)]^k,
\]

\[
P_b(u) \leq \sum_{k=d_f}^{n_c(H+m)} c_k [D(E_c/N_0, u/d_f, \sigma, s)]^k,
\]

where

\[
D(E_c/N_0, u/d_f, \sigma, s) = \frac{1}{\pi} \exp(-\gamma) \int_0^{\pi} \frac{1}{2A\sigma^2} \left[ 1 - \frac{B_{\theta}\sqrt{2\pi}}{2\sqrt{2A}} \exp\left(\frac{B_{\theta}^2}{4A}\right) \right] Q\left(\frac{B_{\theta}}{\sqrt{2A}}\right) d\theta,
\]

and

\[
A = \frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0}} + \frac{u}{d_f} \right)^2 + \frac{1}{2\sigma^2},
\]

\[
B_{\theta} = -\frac{s \cos \theta}{\sigma^2}.
\]

Here, \( \{a_k\}_{k=1}^{\infty} \) and \( \{c_k\}_{k=1}^{\infty} \) are coefficients in expansion of transfer function and derivative of transfer function, respectively, i.e.,

\[
T(D, N) = \sum_{k=d_f}^{\infty} a_k D^k,
\]

\[
\left. \frac{dT(D, N)}{dN} \right|_{N=1} = \sum_{k=d_f}^{\infty} c_k D^k.
\]
Corollary 1. For any convolutional code with transfer function $T(D, N)$ which uses the Yamamoto-Itoh algorithm with flag $u$ for decoding, the following inequalities hold

$$P_e(u) \leq T(D, N) \bigg|_{D = D(E_c/N_0, u/d_f \sigma^2),}^{N_0=1}$$

$$P_x(u) \leq T(D, N) \bigg|_{D = D(E_c/N_0, u/d_f \sigma^2),}^{N_0=1}$$

$$P_b(u) \leq \frac{dT(D, N)}{dN} \bigg|_{D = D(E_c/N_0, u/d_f \sigma^2),}^{N_0=1}$$

Proof: Assume that $x_{ij}^{(r)}$ is the true transmitted code sequence and $(x_{ij}^{(r)})'$ is another code sequence in the Viterbi trellis diagram. For this case, we modify the Yamamoto-Itoh decoding scheme [10] as follows. Since we assume perfect CSI at the receiver, the error event happens if

$$\sum_{j} \sum_{i=1}^{n_c} \alpha_{ij} \left( \left| (x_{ij}^{(r)})' - x_{ij}^{(r)} \right| y_{ij} \geq \frac{u}{d_f} \sqrt{N_0/2} \sum_{j} \sum_{i=1}^{n_c} \left| (x_{ij}^{(r)})' - x_{ij}^{(r)} \right| \alpha_{ij}^2 \right).$$

Besides, when using the Yamamoto-Itoh flag $u$ with ARQ, the retransmission event is a subset of the following event

$$\frac{u}{d_f} \sqrt{N_0/2} \sum_{j} \sum_{i=1}^{n_c} \left| (x_{ij}^{(r)})' - x_{ij}^{(r)} \right| \alpha_{ij}^2 \geq -\frac{u}{d_f} \sqrt{N_0/2} \sum_{j} \sum_{i=1}^{n_c} \left| (x_{ij}^{(r)})' - x_{ij}^{(r)} \right| \alpha_{ij}^2.$$

Then if the Hamming distance between the two sequences is $k$, the pairwise error probability is bounded by

$$P_{k,e}(u|\alpha) = \mathbb{P} \left( \sum_{j} \sum_{i=1}^{n_c} \alpha_{ij} \left( (x_{ij}^{(r)})' - x_{ij}^{(r)} \right) y_{ij} \geq \frac{u}{d_f} \sqrt{N_0/2} \sum_{j} \sum_{i=1}^{n_c} \left| (x_{ij}^{(r)})' - x_{ij}^{(r)} \right| \alpha_{ij}^2 \right)$$

$$= \mathbb{P} \left( -2 \sum_{r=1}^{k} \alpha_{r} y_{r} \geq \frac{u}{d_f} \sqrt{N_0/2} \sum_{r=1}^{k} \alpha_{r}^2 \right)$$

$$= \mathbb{P} \left( \sum_{r=1}^{k} \alpha_{r} y_{r} \leq -\frac{u}{d_f} \sqrt{N_0/2} \sum_{r=1}^{k} \alpha_{r}^2 \right).$$

Here,

$$\sum_{r=1}^{k} \alpha_{r} y_{r} \sim \mathcal{N} \left( \sum_{r=1}^{k} \sqrt{E_c \alpha_{k}} \frac{N_0}{2} \sum_{r=1}^{k} \alpha_{r}^2 \right).$$

It follows that

$$P_{k,e}(u|\alpha) = Q \left( \frac{\sqrt{E_c/\alpha_{k}}} \frac{N_0}{2} \sum_{r=1}^{k} \alpha_{r}^2 + \frac{u}{d_f} \sqrt{\sum_{r=1}^{k} \alpha_{r}^2} \right)$$

$$= Q \left( \sqrt{\frac{E_c}{N_0} + \frac{u}{d_f} \sqrt{\sum_{r=1}^{k} \alpha_{r}^2}} \right).$$

Similarly, the probability that a path is surviving with $\lambda$ in the Yamamoto-Itoh sense [10] can be shown to be upper bounded by

$$P_{k,\lambda}(0) = Q \left( \sqrt{\frac{2E_c}{N_0} - \frac{u}{d_f} \sqrt{\sum_{r=1}^{k} \alpha_{r}^2}} \right) - Q \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f} \sqrt{\sum_{r=1}^{k} \alpha_{r}^2}} \right).$$

For the case $u = 0$ (Viterbi decoding without Yamamoto-Itoh flag), this probability is equal to zero. This means that

$$P_{k,\lambda}(0) = \mathbb{E} \left[ P_{k,\lambda}(0|\alpha) \right] = 0.$$
Now, we consider the case $u > 0$. Observe that $Q(x) \geq 0, \ \forall x$, hence

$$P_{k,n}(u|\alpha) \leq Q \left( \sqrt{\frac{2E_c}{N_0} - \frac{u}{d_f}} \right) \sqrt{\sum_{r=1}^{k} \alpha_r^2} \tag{45}$$

Hence, for a realization of fast fading, i.e. $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, by Viterbi [4] the expected results for each branch are as follows:

$$P_e(u|\alpha) \leq \sum_{k=d_f}^{n_c(H+m)} a_k Q \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right) \sqrt{\sum_{r=1}^{k} \alpha_r^2} \tag{46}$$

$$P_x(u|\alpha) \leq \sum_{k=d_f}^{n_c(H+m)} a_k Q \left( \sqrt{\frac{2E_c}{N_0} - \frac{u}{d_f}} \right) \sqrt{\sum_{r=1}^{k} \alpha_r^2} \tag{47}$$

$$P_b(u|\alpha) \leq \sum_{k=d_f}^{n_c(H+m)} c_k Q \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right) \sqrt{\sum_{r=1}^{k} \alpha_r^2} \tag{48}$$

It follows that

$$P_e(u) = \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P_e(u|\alpha) \prod_{r=1}^{L} \frac{\alpha_r e^{-\left((\alpha_r^2 + s^2) / 2\sigma^2\right)}}{\sqrt{\sigma^2}} d\alpha_1 d\alpha_2 \cdots d\alpha_L \tag{49}$$

$$\leq \sum_{k=d_f}^{B(n_c+m)} a_k \int_0^{\infty} \cdots \int_0^{\infty} Q \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right) \sqrt{\sum_{r=1}^{k} \alpha_r^2} \prod_{r=1}^{k} \frac{\alpha_r e^{-\left((\alpha_r^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_r^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1 d\alpha_2 \cdots d\alpha_k \tag{50}$$

$$
\leq \sum_{k=d_f}^{B(n_c+m)} a_k \int_0^{\infty} \cdots \int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \alpha_1^2 \right) \prod_{r=1}^{k} \frac{\alpha_r e^{-\left((\alpha_r^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_r^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1 d\alpha_2 \cdots d\alpha_k \tag{51}
$$

$$= \sum_{k=d_f}^{B(n_c+m)} a_k \int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \alpha_1^2 \right) e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)} d\alpha_1 \tag{52}$$

Observe that

$$\int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1$$

$$\int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1$$

$$\int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1 d\theta$$

$$= \int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1$$

Now, from Lemma 2 we have

$$\int_0^{\infty} \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 \right) \frac{\alpha_1 e^{-\left((\alpha_1^2 + s^2) / 2\sigma^2\right) I_0 \left( (\alpha_1^2 / \sigma^2) \right)}}{\sqrt{\sigma^2}} d\alpha_1$$

$$= \exp(-\gamma) \left( \frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}} \right)^2 + \frac{1 - s^2}{\sigma^2} \right)$$

$$= \frac{\exp(-\gamma)}{2A\sigma^2} \left[ 1 - \frac{B_\theta \sqrt{2\pi}}{\sqrt{2A}} \exp \left( \frac{B_\theta^2}{4A} \right) \right] Q \left( \frac{B_\theta}{\sqrt{2A}} \right) \tag{55}$$
Hence, we obtain

\[
\int_0^\infty \exp \left( -\frac{1}{2} \left( \sqrt{\frac{2E_c}{N_0}} + \frac{u}{d_f} \right)^2 \sigma^2 \right) \alpha_1 \left( \frac{\sigma^2}{\sigma^2} \right) e^{-((\alpha_1^2 + s^2)/2\sigma^2)} I_0 \left( \frac{\alpha_1 s}{\sigma^2} \right) \, d\alpha_1
\]

\[
= \frac{1}{2\pi} \exp(-\gamma) \int_{-\pi}^\pi \frac{1}{2} \left[ 1 - \frac{B_0 \sqrt{2\pi}}{\sqrt{2A}} \exp \left( \frac{B_0^2}{4A} \right) \right] Q \left( \frac{B_0}{\sqrt{2A}} \right) \, d\theta
\]

\[
= \frac{1}{\pi} \exp(-\gamma) \int_0^\pi \frac{1}{2} \left[ 1 - \frac{B_0 \sqrt{2\pi}}{\sqrt{2A}} \exp \left( \frac{B_0^2}{4A} \right) \right] Q \left( \frac{B_0}{\sqrt{2A}} \right) \, d\theta
\]

\[
= D(E_c/N_0, u/d_f, \sigma, s).
\]

Here, (a) follows from the fact that \( B_0 = -s \cos \theta/\sigma^2 \) is an even function in \( \theta \).

From (52) and (55) we obtain

\[
P_e(u) \leq \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, u/d_f, \sigma, s)]^k.
\]

Similarly, for \( u < d_f \sqrt{\frac{2E_c}{N_0}} \) we also obtain

\[
P_x(u) \leq \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, -u/d_f, \sigma, s)]^k
\]

In addition, we also have

\[
P_b(u) \leq \sum_{k=d_f}^{n_c(H+m)} c_k [D(E_c/N_0, u/d_f, \sigma, s)]^k
\]

Further upper bound (that does not depend on \( B \)) can be obtained by taking the limit \( B \to \infty \), i.e.

\[
P_e(u) \leq \sum_{k=d_f}^{\infty} a_k [D(E_c/N_0, u/d_f, \sigma, s)]^k
\]

\[
= T(D, N) \bigg|_{D = D(E_c/N_0, u/d_f, \sigma, s)}^{N = \infty}
\]

\[
P_x(u) \leq \sum_{k=d_f}^{\infty} a_k [D(E_c/N_0, -u/d_f, \sigma, s)]^k
\]

\[
= T(D, N) \bigg|_{D = D(E_c/N_0, u/d_f, \sigma, s)}^{N = \infty}
\]

\[
P_b(u) \leq \sum_{k=d_f}^{\infty} a_k [D(E_c/N_0, u/d_f, \sigma, s)]^k
\]

\[
= T(D, N) \bigg|_{D = D(E_c/N_0, u/d_f, \sigma, s)}^{N = \infty}
\]

Finally, by using (2) we obtain the final results of this lemma. That concludes our proof.

B. Reliability Evaluations

**Proposition 2.** The decay rate of the bit error probability \( P_b(0) \) of the original Viterbi decoding scheme satisfies

\[
P_b(0) \leq h(\gamma, H, k_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-d_f}
\]

for some finite but sufficiently large \( E_b/N_0 \). Here, \( h(\gamma, H, k_c, d_f, R_c, m) \) is a constant which depends on convolutional code parameters.
There exists a Yamamoto-Itoh flag $u_0 > 0$ such that the retransmission probability $P_x(u_0)$ and the decay rate of the bit error probability $P_b(u_0)$ for the Yamamoto-Itoh algorithm satisfy

$$P_x(u_0) = O\left(\left(\frac{E_b}{N_0}\right)^{-d_f}\right), \quad (70)$$

$$P_b(u_0) \leq 4^{-d_f} P_b(0). \quad (71)$$

In addition, for any fixed transmission bit energy to noise ratio $(E_b/N_0)$ and Yamamoto-Itoh flag $u \geq 0$ such that

$$P_x(u_0) = O\left(\left(\frac{E_b}{N_0}\right)^{-d_f}\right), \quad (70)$$

and

$$P_b(u_0) \leq 4^{-d_f} P_b(0). \quad (71)$$

the following holds

$$\liminf_{\gamma \to \infty} - \frac{\ln P_b(u)}{\gamma} \geq d_f \left(1 - \frac{1}{\sigma^2} \left[\sqrt{\frac{2E_c}{N_0} - u d_f} + \frac{1}{\sigma^2}\right]^{-1}\right), \quad (73)$$

for any convolutional code using the Yamamoto-Itoh algorithm for decoding.

**Remark 2.** There are some modifications in the decoding algorithm compared to the Yamamoto-Itoh algorithm which compensate for the Rician fading channels. However, these changes only affect the decoding performances when we use the Yamamoto-Itoh flag $u > 0$. Our results show that the Yamamoto-Itoh flag can contribute in reducing the BER by a factor of $4^{d_f}$ with negligible retransmission.

**Proof:** From Lemma 3 we know that

$$a_k \leq \sum_{L=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{L-1} \binom{L}{l} \binom{k-1}{L-l-1} 1\{L-l-1 \leq k-1\} \quad (74)$$

$$(a) \sum_{L=1}^{k} \sum_{l=0}^{L-1} \binom{L}{l} \binom{k-1}{L-l-1} 1\{L-l-1 \leq k-1\} \quad (75)$$

$$(b) \sum_{L=1}^{k} \sum_{l=0}^{L-1} \binom{L}{l} 2^{k-1} \quad (76)$$

$$(c) \sum_{L=0}^{k-1} 2^L 2^{k-1} \quad (77)$$

$$< 2^{k-1} \sum_{L=0}^{k-1} 2^L \quad (78)$$

$$= 4^k. \quad (80)$$

Here, (a) follows from the fact that $n_c \geq 1$, (b) follows from the fact that

$$\binom{k-1}{L-l-1} 1\{L-l-1 \leq k-1\} < \sum_{t=0}^{k-1} \binom{k-1}{t} = 2^{k-1}, \quad (81)$$

and (c) follows from the fact that $\sum_{t=0}^{L-1} \binom{L}{t} < \sum_{t=0}^{L} \binom{L}{t} = 2^L$. Observe from (27) that

$$D(E_c/N_0, u/d_f, \sigma, s) = \frac{1}{\pi} \exp(-\gamma) \int_0^\pi \frac{1}{2A\sigma^2} \left[1 - \frac{B_\theta \sqrt{2\pi}}{\sqrt{2A}} \exp\left(\frac{B_\theta^2}{4A}\right)\right] Q\left(\frac{B_\theta}{\sqrt{2A}}\right) d\theta, \quad (82)$$

and

$$A = \frac{1}{2} \left(\sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}}\right)^2 + \frac{1}{2\sigma^2}, \quad (83)$$

$$B_\theta = -\frac{s \cos \theta}{\sigma^2}. \quad (84)$$
It follows that

\[
D(E_c/N_0, u/d_f, \sigma, s) = \frac{1}{\pi} \exp(-\gamma) \int_0^{\pi/2} \frac{1}{2A\sigma^2} \left[ 1 - B_\theta \sqrt{2\pi} \exp \left( \frac{B_\theta^2}{4A} \right) \right] \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta \\
+ \frac{1}{\pi} \exp(-\gamma) \int_{\pi/2}^{\pi} \frac{1}{2A\sigma^2} \left[ 1 - B_\theta \sqrt{2\pi} \exp \left( \frac{B_\theta^2}{4A} \right) \right] \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta
\]  

(85)

\[
(\text{a)} \quad \frac{1}{\pi} \exp(-\gamma) \int_0^{\pi/2} \frac{1}{2A\sigma^2} \
\left[ 1 - B_\theta \sqrt{2\pi} \exp \left( \frac{B_\theta^2}{4A} \right) \right] \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta \leq \frac{1}{\pi} \exp(-\gamma) \int_{\pi/2}^{\pi} \frac{1}{2A\sigma^2} Q(0) d\theta
\]  

(86)

\[
\leq \frac{1}{\pi} \exp(-\gamma) \int_0^{\pi/2} \frac{1}{2A\sigma^2} \left[ 1 - B_\theta \sqrt{2\pi} \exp \left( \frac{B_\theta^2}{4A} \right) \right] \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta + \exp(-\gamma) \frac{\exp(-\gamma)}{8A\sigma^2}.
\]  

(87)

Here, (a) follows from the fact that \( B_\theta \geq 0 \) for all \( \pi/2 \leq \theta \leq \pi \).

Now, we note that

\[
\int_0^{\pi/2} \frac{1}{2A} \left[ 1 - B_\theta \sqrt{2\pi} \exp \left( \frac{B_\theta^2}{4A} \right) \right] \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta \\
= \frac{1}{2A} \int_0^{\pi/2} Q \left( \frac{B_\theta}{\sqrt{2A}} \right) d\theta - \frac{\sqrt{2\pi}}{2A\sqrt{2A}} \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta
\]  

(88)

\[
\leq \frac{1}{2A} \int_0^{\pi/2} d\theta - \frac{\sqrt{2\pi}}{2A\sqrt{2A}} \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta
\]  

(89)

\[
= \frac{\pi}{4A} - \frac{\sqrt{2\pi}}{2A\sqrt{2A}} \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) Q \left( \frac{B_\theta \sqrt{2A}}{\sqrt{2A}} \right) d\theta.
\]  

(90)

It follows that

\[
D(E_c/N_0, u/d_f, \sigma, s) \leq \frac{3}{8A\sigma^2} \exp(-\gamma) - \frac{1}{\pi} \exp(-\gamma) \left[ \frac{\sqrt{2\pi}}{2A\sigma^2 \sqrt{2A}} \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta \right]
\]  

(91)

Now, for fixed \( \gamma, \sigma \), observe that

\[
\eta(A) := -\exp(-\gamma) \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) \frac{Q \left( B_\theta \sqrt{2A} \right)}{\sqrt{2A}} d\theta
\]  

(92)

\[
(\text{a)} \quad \leq -\exp(-\gamma) \int_0^{\pi/2} B_\theta \exp \left( \frac{B_\theta^2}{4A} \right) d\theta
\]  

(93)

\[
(\text{b)} \quad \leq \exp(-\gamma) \int_0^{\pi/2} \frac{s}{\sigma^2} \exp \left( \frac{s^2}{4A\sigma^4} \right) d\theta,
\]  

(94)

\[
= \frac{\pi}{2} \exp(-\gamma) \frac{s}{\sigma^2} \exp \left( \frac{s^2}{4A\sigma^4} \right).
\]  

(95)

Here, (a) follows from the fact that \( B_\theta = -s \cos \theta/\sigma^2 \leq 0 \) for all \( 0 \leq \theta \leq \pi/2 \) and that \( Q(x) \leq 1, \forall x \in \mathbb{R} \), (b) follows from the fact that \( |B_\theta| = s \cos \theta/\sigma^2 \leq s/\sigma^2 = B \).

It follows that

\[
D(E_c/N_0, u/d_f, \sigma, s) \leq \frac{3}{8A\sigma^2} \exp(-\gamma) + \frac{1}{\pi} \frac{\sqrt{2\pi}}{2A\sigma^2 \sqrt{2A}} \frac{\pi}{2} \exp(-\gamma) \frac{s}{\sigma^2} \exp \left( \frac{s^2}{4A\sigma^4} \right)
\]  

(96)

\[
= \exp(-\gamma) \left[ \frac{3}{8A\sigma^2} + \frac{\sqrt{4\pi\gamma}}{4A\sigma^3 \sqrt{2A}} \exp \left( \frac{\gamma}{2A\sigma^2} \right) \right]
\]  

(97)

\[
= O \left( \frac{1}{A} \right) \quad \text{as} \quad A \to \infty.
\]  

(98)
Therefore, we have
\[
P_c(u) = \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, -u/d_f, \sigma, s)]^k
\]
(99)
\[
\leq \sum_{k=d_f}^{n_c(H+m)} 4^k [D(E_c/N_0, -u/d_f, \sigma, s)]^k
\]
(100)
\[
\leq \sum_{k=d_f}^{\infty} 4^k [D(E_c/N_0, -u/d_f, \sigma, s)]^k
\]
(101)
\[
= \frac{4^d_f [D(E_c/N_0, -u/d_f, \sigma, s)]^{d_f}}{1 - 4D(E_c/N_0, -u/d_f, \sigma, s)}
\]
(102)
\[
= O \left( \left( \frac{2E_c}{N_0} - \frac{u}{d_f} \right)^{-2d_f} \right)
\]
(103)

Here, (a) holds if
\[
4D(E_c/N_0, -u/d_f, \sigma, s) \leq 4 \exp(-\gamma) \left[ \frac{3}{8A\sigma^2} + \frac{\sqrt{4\pi\gamma}}{4A\sigma^3\sqrt{2A}} \exp \left( \frac{\gamma}{2A\sigma^2} \right) \right] < 1.
\]
(105)

(i.e. \(A\) is large enough) and (b) holds if we choose \(u\) such that
\[
\sqrt{\frac{2E_b}{N_0}} \left( \frac{H}{H+m} \right) R_c - \frac{u}{d_f} > 0.
\]
(106)

Now, we choose \(u\) such that
\[
\sqrt{\frac{2E_b}{N_0}} \left( \frac{H}{H+m} \right) R_c - \frac{u}{d_f} = \delta \sqrt{\frac{2E_b}{N_0}} \left( \frac{H}{H+m} \right) R_c,
\]
(107)
for some \(\delta > 0\). This means that we choose
\[
u_0 = d_f(1-\delta) \sqrt{\frac{2E_b}{N_0}} \left( \frac{H}{H+m} \right) R_c.
\]
(108)

Then from (103) we achieve
\[
P_x(u_0) = O \left( \left( \frac{E_b}{N_0} \left( \frac{H}{H+m} \right) R_c \right)^{-d_f} \right),
\]
(109)
so \(P_x(u_0) \to 0\) as \((E_b/N_0) \to \infty\).

With this choice of \(u_0\) we have
\[
P_b(u_0) \leq \sum_{k=d_f}^{n_c(H+m)} c_k [D(E_c/N_0, u_0/d_f, \sigma, s)]^k
\]
(110)
\[
\leq \sum_{k=d_f}^{n_c(H+m)} k a_k k_c [D(E_c/N_0, u_0/d_f, \sigma, s)]^k
\]
(111)
\[
\leq \sum_{k=d_f}^{n_c(H+m)} k 4^k k_c [D(E_c/N_0, u_0/d_f, \sigma, s)]^k
\]
(112)
\[
\leq \sum_{k=d_f}^{n_c(H+m)} 5^k k_c [D(E_c/N_0, u_0/d_f, \sigma, s)]^k
\]
(113)
\[
= \frac{5^{d_f} [D(E_c/N_0, u_0/d_f, \sigma, s)]^{d_f}}{1 - 5D(E_c/N_0, u_0/d_f, \sigma, s)}
\]
(114)
\[
= (2 - \delta)^{-2d_f} O \left( \left( \frac{E_b}{N_0} \left( \frac{H}{H+m} \right) R_c \right)^{-d_f} \right).
\]
(115)
Here, (a) follows from Lemma 3, (b) holds if
\[
5D(E_c/N_0, u/d_f, \sigma, s) = 5 \exp(-\gamma) \left[ \frac{3}{8A\gamma^2} + \frac{\sqrt{3\pi\gamma}}{4A\sigma^2\sqrt{2A}} \exp\left(\frac{\gamma}{2A\sigma^2}\right) \right] < 1, \tag{116}
\]
(i.e. A sufficiently large but finite and the threshold can be estimated even tighter), (c) follows from (2). Hence, we obtain
\[
P_b(u) \leq h(\gamma, \sigma, H, k_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-d_f}, \tag{117}
\]
and
\[
P_x(u) = O \left( \left( \frac{E_b}{N_0} \right)^{-d_f} \right), \tag{118}
\]
In addition, by noting that the Viterbi decoding without Yamamoto-Itoh flag corresponds to the case \( \delta = 1 \), from (115) we see that
\[
P_b(u_0) = (2 - \delta)^{-2d_f} P_b(0). \tag{119}
\]
Since \( \delta > 0 \) can be arbitrarily chosen, we have
\[
P_b(u_0) \approx 4^{-d_f} P_b(0). \tag{120}
\]
Now, for each fixed \( A, \sigma \) we consider the following function
\[
\mu(\gamma) = -\exp(-\gamma) \int_0^{\pi/2} B_\theta \exp\left(\frac{B_\theta^2}{2A}\right) Q\left(\frac{B_\theta}{\sqrt{2A}}\right) d\theta
\]
\[
= \int_0^{\pi/2} \exp(-\gamma) \frac{s \cos \theta}{\sigma^2} \exp\left(\frac{s^2 \cos^2 \theta}{4A\sigma^4}\right) Q\left(\frac{-s \cos \theta}{\sigma\sqrt{2A}}\right) d\theta
\]
\[
= \int_0^{\pi/2} \exp(-\gamma) \frac{\sqrt{2}\gamma \cos \theta}{\sigma} \exp\left(\frac{\gamma \cos^2 \theta}{2A\sigma^2}\right) Q\left(\frac{-\sqrt{2\gamma} \cos \theta}{\sigma\sqrt{2A}}\right) d\theta.
\]
\[
= \sqrt{2\gamma} \exp\left(-\gamma \left(1 - \frac{1}{2A\sigma^2}\right)\right) \int_0^{\pi/2} \frac{\cos \theta}{\sigma} \exp\left(-\frac{\gamma \sin^2 \theta}{2A\sigma^2}\right) Q\left(\frac{-\sqrt{2\gamma} \cos \theta}{\sigma\sqrt{2A}}\right) d\theta. \tag{121}
\]
Define
\[
\mu(\gamma, \theta) := \frac{\cos \theta}{\sigma} \exp\left(-\frac{\gamma \sin^2 \theta}{2A\sigma^2}\right) Q\left(\frac{-\sqrt{2\gamma} \cos \theta}{\sigma\sqrt{2A}}\right) d\theta. \tag{125}
\]
Note that \( \mu(\gamma, \theta) \) is non-negative non-increasing for \( \theta \in [0, \pi] \), integrable for \( \gamma = 1 \), and
\[
\lim_{\gamma \to \infty} \mu(\gamma, \theta) = 0. \tag{126}
\]
Hence, by Dominated Convergence Theorem [23], we have
\[
\lim_{\gamma \to \infty} \int_0^{\pi/2} \frac{\cos \theta}{\sigma} \exp\left(-\frac{\gamma \sin^2 \theta}{2A\sigma^2}\right) Q\left(\frac{-\sqrt{2\gamma} \cos \theta}{\sigma\sqrt{2A}}\right) d\theta
\]
\[
= \int_0^{\pi/2} \lim_{\gamma \to \infty} h(A, \gamma, \theta) d\theta \tag{127}
\]
\[
= 0. \tag{128}
\]
This means that
\[
\mu(\gamma) = \sqrt{2\gamma} \exp\left(-\gamma \left(1 - \frac{1}{2A\sigma^2}\right)\right) o(1), \tag{129}
\]
as $\gamma \to \infty$. It follows that

\[
D(E_c/N_0, u/d_f, \sigma, s) \leq \frac{3}{8A} \exp(-\gamma) + \sqrt{2\gamma} \exp\left(-\gamma \left(1 - \frac{1}{2A\sigma^2}\right)\right) o(1),
\]

\[
= \frac{3}{8A} \sqrt{2\gamma} \exp\left(-\gamma \left(1 - \frac{1}{2A\sigma^2}\right)\right) \left[\exp\left(-\frac{\gamma}{2A\sigma^2}\right) + o(1)\right],
\]

\[
\leq \frac{3}{2} \sqrt{\frac{\gamma}{2}} \left[\left(\sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}}\right)^2 + \frac{1}{\sigma^2}\right]^{-1} \exp\left(-\gamma \left(1 - \frac{1}{\sigma^2}\right) \left[\left(\sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}}\right)^2 + \frac{1}{\sigma^2}\right]^{-1}\right),
\]

\[
= O \left(\sqrt{\frac{\gamma}{2}} \exp\left(-\gamma \left(1 - \frac{1}{\sigma^2}\right) \left[\left(\sqrt{\frac{2E_c}{N_0} + \frac{u}{d_f}}\right)^2 + \frac{1}{\sigma^2}\right]^{-1}\right)\right),
\]

as $\gamma \to \infty$.

Now, from (25) we have

\[
\mathbb{P}_x(u) \leq \sum_{k=d_f}^{n_c(H+m)} a_k [D(E_c/N_0, -u/d_f, \sigma^2, s)]^k
\]

\[
\leq \sum_{k=d_f}^{\infty} 4^k [D(E_c/N_0, -u/d_f, \sigma^2, s)]^k
\]

\[
= 4^{d_f} [D(E_c/N_0, -u/d_f, \sigma^2, s)]^{d_f}
\]

\[
1 - 8D(E_c/N_0, -u/d_f, \sigma^2, s)
\]

for $\gamma$ sufficiently large, and

\[
\frac{1}{\sigma^2} \left[\left(\frac{2E_c}{N_0} - \frac{u}{d_f}\right)^2 + \frac{1}{\sigma^2}\right]^{-1} < 1.
\]

Similarly, we have

\[
\mathbb{P}_b(u) \leq \sum_{k=d_f}^{n_c(H+m)} c_k [D(E_c/N_0, u/d_f, \sigma^2, s)]^k
\]

\[
\leq \sum_{k=d_f}^{\infty} k \mathbb{A}_k d_f [D(E_c/N_0, u/d_f, \sigma^2, s)]^k
\]

\[
\leq \sum_{k=d_f}^{n_c(H+m)} k \mathbb{A}_k d_f [D(E_c/N_0, u/d_f, \sigma^2, s)]^k
\]

\[
\leq k_c \sum_{k=d_f}^{\infty} 8^k [D(E_c/N_0, u/d_f, \sigma^2, s)]^k
\]

\[
= k_c \mathbb{A}_k d_f [D(E_c/N_0, u/d_f, \sigma^2, s)]^{d_f}
\]

\[
1 - 8D(E_c/N_0, u/d_f, \sigma^2, s)
\]

for $\gamma$ sufficiently large, and the following holds

\[
\frac{1}{\sigma^2} \left[\left(\frac{2E_c}{N_0} - \frac{u}{d_f}\right)^2 + \frac{1}{\sigma^2}\right]^{-1} < 1.
\]

It follows that if

\[
\frac{1}{\sigma^2} \left[\left(\frac{2E_c}{N_0} - \frac{u}{d_f}\right)^2 + \frac{1}{\sigma^2}\right]^{-1} < 1,
\]
we will obtain both (136) and (143).

It follows that

$$\lim_{\gamma \to \infty} -\frac{\ln P_b(u)}{\gamma} \geq d_f \left( 1 - \frac{1}{\sigma^2} \left[ \left( \frac{2E_c}{N_0} + \frac{u}{d_f} \right)^2 + \frac{1}{\sigma^2} \right]^{-1} \right),$$

(146)

if the condition (145) holds.

VI. UPPER BOUNDS ON PERFORMANCE

**Proposition 3.** The decay rate of the bit error probability $P_b(u)$ of the Yamamoto-Itoh algorithm satisfies

$$P_b(u) \geq f(\gamma, \sigma, H, K_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-(d_f+1)}, \quad \forall u \geq 0$$

(147)

for some constant $f(\gamma, \sigma, H, K_c, d_f, R_c, m) > 0$ for high $E_b/N_0$, which depends on the convolutional code parameters and channel parameters $\gamma, \sigma$.

In addition, for a fixed transmission bit energy to noise ratio $E_b/N_0$ we have

$$\limsup_{\gamma \to \infty} \frac{-\ln P_b(u)}{\gamma} \leq d_f, \quad \forall u \geq 0.$$  

(148)

**Remark 3.** To prove (148), we can use exactly the same arguments as for the case $u = 0$. Therefore, for simplicity of notations, we provide the following proof for $u = 0$. Besides, the following proof applies to both with ARQ and without ARQ channel and for any chosen value of $u$.

**Proof:** For $u = 0$, the Viterbi decoding algorithm is MLSD (Maximum Likelihood Sequence Decoding), i.e., by comparing the correlation metrics between paths. This means that the error event happens if

$$\sum_i \sum_j \alpha_{ij} \left( \left[ (x_{ij}^{(r)})' - x_{ij}^{(r)} \right] y_{ij} \right) \geq 0,$$

(149)

where $x_{ij}, 1 \leq i \leq n_c, 1 \leq j \leq n_c(H + m)$ is the true path (the path that corresponds to the transmitted sequence).

Since the convolutional code is linear, for error analysis, we can assume that the transmitted sequence is all zeros. The error event in (149) can be lower bounded by an error event that at some decoding steps the decoding algorithm chooses a non-all-zero path which merges with the all-zero path and has higher correlation metrics with received sequences. This means that given a realization of fading vector, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n_c(H + m)})$, we have

$$P_e(0|\alpha) \geq \max_{k \geq 1} P_{k,e}(\alpha),$$

(150)

where $P_{k}(\alpha)$ is the probability of the error event that the incorrect path has higher correlation metric than the all-zero path and also has the Hamming distance $k$ from all zeros sequences given a realization of fading vector $\alpha$. Using the same arguments as Viterbi [4], we have for the given realization of fading $\alpha$, that

$$P_{k,e}(\alpha) = \mathbb{P} \left( \sum_{r=1}^{k} \alpha_r y_r \leq 0 \right),$$

(151)

where

$$y_r = \sqrt{E_c} \alpha_r + N_r,$$

(152)

$$N_r \sim (0, N_0/2).$$

(153)

It follows that

$$\sum_{r=1}^{k} \alpha_r y_r \sim \mathcal{N} \left( \sqrt{E_c} \sum_{r=1}^{k} \alpha_r^2, \frac{N_0}{2} \sum_{r=1}^{k} \alpha_r^2 \right).$$

(154)

This means that (see also (41) with $u = 0$)

$$P_{k,e}(\alpha) = Q \left( \sqrt{\frac{2E_c}{N_0}} \sqrt{\sum_{r=1}^{k} \alpha_r^2} \right),$$

(155)
Now, it is easy to see that for each fixed realization vector $\alpha$ we have

$$P_e(0|\alpha) \geq \max_{k \geq d_f} P_{k,e}(\alpha)$$  \hspace{1cm} (156)

$$= \max_{k \geq d_f} Q \left[ \left( \frac{2E_c}{N_0} \right)^{\alpha_r^2} \right] \hspace{1cm} (157)$$

$$= Q \left[ \left( \frac{2E_c}{N_0} \right)^{\alpha_r^2} \right] \hspace{1cm} (158)$$

It follows that

$$P_e(0) = \mathbb{E} [P_e(0|\alpha)]$$  \hspace{1cm} (159)

$$= \mathbb{E} \left( Q \left[ \left( \frac{2E_c}{N_0} \right)^{\alpha_r^2} \right] \right) \hspace{1cm} (160)$$

$$\geq \mathbb{E} \left( Q \left[ \left( \frac{2E_c d_f}{N_0} \right)^{\max_{1 \leq r \leq d_f} \alpha_r} \right] \right) \hspace{1cm} (161)$$

$$\geq \mathbb{E} \left( Q \left[ \left( \frac{2E_c d_f}{N_0} \right)^{\alpha_r} \right] \right) \hspace{1cm} (162)$$

Now, we find the PDF of $\hat{\alpha} := \sqrt{\frac{2E_c d_f}{N_0}} \max_{1 \leq r \leq d_f} \alpha_r$. Observe that

$$P(\hat{\alpha} \leq \beta) = P \left( \sqrt{\frac{2E_c d_f}{N_0}} \max_{1 \leq r \leq k} \alpha_r \leq \beta \right)$$  \hspace{1cm} (163)

$$= \prod_{r=1}^{d_f} P \left( \alpha_r \leq \beta \sqrt{\frac{N_0}{2E_c d_f}} \right) \hspace{1cm} (164)$$

$$= P(\alpha_1 \leq \beta \sqrt{\frac{N_0}{2E_c d_f}})^{d_f} \hspace{1cm} (165)$$

$$= \int_0^\beta \left[ \int_0^{\frac{\sqrt{N_0}}{\sqrt{2E_c d_f}}} \frac{\alpha}{\sigma^2} e^{-\left(\alpha^2+\beta^2\right)\frac{\alpha^2}{\sigma^2}} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha \right]^{d_f} \frac{\beta N_0}{2\sigma^2 E_c d_f} e^{-\left(\frac{\beta^2 N_0}{2E_c d_f}+\frac{s^2}{2}\right)\frac{\beta^2 N_0}{2E_c d_f}} I_0 \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) d\beta. \hspace{1cm} (166)$$

It follows that the PDF of $\hat{\alpha}$ is as follows

$$f_{\hat{\alpha}}(\beta) = d_f \int_0^\beta \int_0^{\frac{\sqrt{N_0}}{\sqrt{2E_c d_f}}} \frac{\alpha}{\sigma^2} e^{-\left(\alpha^2+\beta^2\right)\frac{\alpha^2}{\sigma^2}} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha \frac{\beta N_0}{2\sigma^2 E_c d_f} e^{-\left(\frac{\beta^2 N_0}{2E_c d_f}+\frac{s^2}{2}\right)\frac{\beta^2 N_0}{2E_c d_f}} I_0 \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) d\beta. \hspace{1cm} (167)$$

Hence, we obtain

$$P_e(0) \geq \int_0^\infty Q(\beta) d\beta \int_0^\beta \int_0^{\frac{\sqrt{N_0}}{\sqrt{2E_c d_f}}} \frac{\alpha}{\sigma^2} e^{-\left(\alpha^2+\beta^2\right)\frac{\alpha^2}{\sigma^2}} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha \frac{\beta N_0}{2\sigma^2 E_c d_f} e^{-\left(\frac{\beta^2 N_0}{2E_c d_f}+\frac{s^2}{2}\right)\frac{\beta^2 N_0}{2E_c d_f}} I_0 \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) d\beta \hspace{1cm} (168)$$

$$= \int_{-\pi}^\pi \frac{1}{2\pi} \int_0^\infty Q(\beta) d\beta \int_0^\beta \int_0^{\frac{\sqrt{N_0}}{\sqrt{2E_c d_f}}} \frac{\alpha}{\sigma^2} e^{-\left(\alpha^2+\beta^2\right)\frac{\alpha^2}{\sigma^2}} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha \frac{\beta N_0}{2\sigma^2 E_c d_f} e^{-\left(\frac{\beta^2 N_0}{2E_c d_f}+\frac{s^2}{2}\right)\frac{\beta^2 N_0}{2E_c d_f}} I_0 \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) d\beta d\theta \hspace{1cm} (169)$$

$$= \int_{-\pi}^\pi \frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) d\beta \int_0^\beta \int_0^{\frac{\sqrt{N_0}}{\sqrt{2E_c d_f}}} \frac{\alpha}{\sigma^2} e^{-\left(\alpha^2+\beta^2\right)\frac{\alpha^2}{\sigma^2}} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha d\beta$$
Now, we observe that for any \( \beta \geq 0 \)

\[
\int_{0}^{\beta} \frac{\alpha}{\sigma^2} e^{-(\alpha^2 + s^2)/2\sigma^2} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha = \int_{0}^{\beta} \frac{\alpha}{\sigma^2} e^{-(\alpha^2 + s^2)/2\sigma^2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{\alpha s \cos \theta}{\sigma^2} \right) d\theta d\alpha
\]

(171)

\[
= \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{0}^{\beta} \frac{\alpha}{\sigma^2} e^{-(\alpha^2 + s^2)/2\sigma^2} \exp \left( \frac{\alpha s \cos \theta}{\sigma^2} \right) d\alpha d\theta
\]

(172)

\[
\geq \lim_{\beta \to \infty} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{0}^{\beta} \frac{\alpha}{\sigma^2} e^{-(\alpha^2 + s^2)/2\sigma^2} \exp \left( \frac{\alpha s \cos \theta}{\sigma^2} \right) d\alpha d\theta
\]

(173)

\[
= \exp(-\gamma) \int_{0}^{\pi/3} g \left( \frac{1}{2\sigma^2}, -\frac{s \cos \theta}{\sigma^2}, \sqrt{\frac{N_0}{2E_c d_f}} \right) d\theta.
\]

(174)

(175)

For \( \theta \in [0, \pi/3] \) we have \(-s \cos \theta/\sigma^2 \leq 0\). Hence, by Lemma 2 we know that

\[
g \left( \frac{1}{2\sigma^2}, -\frac{s \cos \theta}{\sigma^2}, \sqrt{\frac{N_0}{2E_c d_f}} \right)
\]

(176)

\[
= \omega_{E_c/N_0, \beta, s, \sigma, \gamma} \left\{ \sigma \beta \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( -\frac{s^2 \cos^2 \theta}{2\sigma^2} \right) \right] \right\}
\]

(177)

From (175) and (176) we obtain

\[
\int_{0}^{\beta} \frac{\alpha}{\sigma^2} e^{-(\alpha^2 + s^2)/2\sigma^2} I_0 \left( \frac{\alpha s}{\sigma^2} \right) d\alpha
\]

(178)

\[
\geq \lim_{\beta \to \infty} \int_{0}^{\pi/3} \omega_{E_c/N_0, \beta, s, \sigma, \gamma} \left\{ \sigma \beta \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( -\frac{s^2 \cos^2 \theta}{2\sigma^2} \right) \right] \right\} d\theta
\]

(179)

\[
= \exp(-\gamma) \int_{0}^{\pi/3} \omega_{E_c/N_0, \beta, s, \sigma, \gamma} \left\{ \sigma \beta \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( -\frac{s^2 \cos^2 \theta}{2\sigma^2} \right) \right] \right\} d\theta,
\]

(179)
\[
\times \min \left\{ 1, \exp \left( \frac{s \cos \theta}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta N_0}{2E_c d_f \sigma^2} \right) \right\} \} d\theta, \tag{180}
\]

\[
\exp(-\gamma) \sqrt{\frac{\pi}{\pi \sigma^2}} \int_0^{\pi/3} \omega_{E_c/N_0, \beta, s, \gamma} \left\{ \sigma^2 \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( \frac{s}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) + \sqrt{2\pi} \frac{s \cos \theta}{\sigma} \right] \right\} d\theta \tag{181}
\]

\[
\min \left\{ 1, \exp \left( \frac{-s \sigma^2}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta^2 N_0}{2E_c d_f \sigma^2} \right) \right\} \} d\theta \tag{182}
\]

\[
\exp(-\gamma) \sqrt{\frac{\pi}{\pi \sigma^2}} \omega_{E_c/N_0, \beta, s, \gamma} \left\{ \sigma^2 \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( \frac{s}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) + \sqrt{2\pi} \frac{s \cos \theta}{\sigma} \right] \right\} d\theta \tag{183}
\]

\[
= \exp(-\gamma) \sqrt{\frac{\pi}{\pi \sigma^2}} \omega_{E_c/N_0, \beta, s, \gamma} \left\{ \sigma^2 \sqrt{\frac{N_0}{2E_c d_f}} \left[ 1 - \exp \left( \frac{s}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} \right) + \sqrt{2\pi} \frac{s \cos \theta}{\sigma} \right] \right\} d\theta \tag{184}
\]

Here, (a) follows from the fact that \( \cos \theta \geq 1/2 \) for \( 0 \leq \theta \leq \pi/3 \).

From (170) and (184) and the fact that the function \( \cos \theta \) is even, we have

\[
P_e(0) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \exp \left[ -\frac{1}{2\sigma^2} \left( \frac{\beta N_0}{2E_c d_f} + s^2 \right) \right] \frac{\beta N_0}{2\sigma^2 E_c} \times \exp \left( \frac{-s \sigma^2}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta^2 N_0}{2E_c d_f \sigma^2} \right) \right\} d^2 \beta d\theta \tag{185}
\]

\[
= \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \exp \left[ -\frac{1}{2\sigma^2} \left( \frac{\beta N_0}{2E_c d_f} + s^2 \right) \right] \frac{\beta N_0}{2\sigma^2 E_c} \times \exp \left( \frac{-s \sigma^2}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta^2 N_0}{2E_c d_f \sigma^2} \right) \right\} d^2 \beta d\theta \tag{186}
\]

\[
\geq \int_{0}^{\pi/2} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \exp \left[ -\frac{1}{2\sigma^2} \left( \frac{\beta N_0}{2E_c d_f} + s^2 \right) \right] \frac{\beta N_0}{2\sigma^2 E_c} \times \exp \left( \frac{-s \sigma^2}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta^2 N_0}{2E_c d_f \sigma^2} \right) \right\} d^2 \beta d\theta \tag{187}
\]

\[
(a) \int_{0}^{\pi/2} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) \exp \left[ -\frac{1}{2\sigma^2} \left( \frac{\beta N_0}{2E_c d_f} + s^2 \right) \right] \frac{\beta N_0}{2\sigma^2 E_c} \times \exp \left( \frac{-s \sigma^2}{\sigma^2} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\beta^2 N_0}{2E_c d_f \sigma^2} \right) \right\} d^2 \beta d\theta \tag{188}
\]
Here, (a) follows from the fact that \( \cos \theta \geq 0, 0 \leq \theta \leq \pi/2 \). The last equality holds for \( \beta \geq 0 \).

It follows from (187) that for some \( J > 0 \) which does not depend on \( \beta, s, \sigma, \gamma \) we have

\[
P_c(0) \geq \int_0^{\pi/2} \frac{1}{2 \pi} \frac{N_0}{2 \sigma^2 E_c} \left( \exp(-\gamma) \left( \frac{N_0}{2 E_c d_f} \right)^{d_f-1} \right) dt d\beta d\theta.
\]

(188)

Now, we see that

\[
\int_0^{\pi/2} \frac{1}{2 \pi} \frac{N_0}{2 \sigma^2 E_c} \left( \exp(-\gamma) \left( \frac{N_0}{2 E_c d_f} \right)^{d_f-1} \right) dt d\beta d\theta.
\]

(189)

Now, we see that

\[
\int_0^{\pi/2} \frac{1}{2 \pi} \frac{N_0}{2 \sigma^2 E_c} \left( \exp(-\gamma) \left( \frac{N_0}{2 E_c d_f} \right)^{d_f-1} \right) dt d\beta d\theta.
\]

(190)

If \( \sqrt{N_0/E_c} \leq t \), then

\[
I(d_f, t) := \int_0^t \beta \left[ 1 - \exp \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2 E_c d_f}} \right) + \left( \sqrt{2 \pi} \frac{s \beta}{\sigma^2} \right) \right]^{d_f-1} \exp \left( - \frac{N_0 \beta^2}{4 \sigma^2 E_c d_f} \right) d\beta.
\]

(191)

\[
I(d_f, t) := \int_0^t \beta \left[ 1 - \exp \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2 E_c d_f}} \right) + \left( \sqrt{2 \pi} \frac{s \beta}{\sigma^2} \right) \right]^{d_f-1} \exp \left( - \frac{N_0 \beta^2}{4 \sigma^2 E_c d_f} \right) d\beta.
\]

(192)

\[
I(d_f, t) := \int_0^t \beta \left[ 1 - \exp \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2 E_c d_f}} \right) + \left( \sqrt{2 \pi} \frac{s \beta}{\sigma^2} \right) \right]^{d_f-1} \exp \left( - \frac{N_0 \beta^2}{4 \sigma^2 E_c d_f} \right) d\beta.
\]

(193)

\[
I(d_f, t) := \int_0^t \beta \left[ 1 - \exp \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2 E_c d_f}} \right) + \left( \sqrt{2 \pi} \frac{s \beta}{\sigma^2} \right) \right]^{d_f-1} \exp \left( - \frac{N_0 \beta^2}{4 \sigma^2 E_c d_f} \right) d\beta.
\]

(194)

\[
I(d_f, t) := \int_0^t \beta \left[ 1 - \exp \left( \frac{s \beta}{\sigma^2} \sqrt{\frac{N_0}{2 E_c d_f}} \right) + \left( \sqrt{2 \pi} \frac{s \beta}{\sigma^2} \right) \right]^{d_f-1} \exp \left( - \frac{N_0 \beta^2}{4 \sigma^2 E_c d_f} \right) d\beta.
\]

(195)
Here, we replaced $s = \sigma \sqrt{2 \gamma}$ in (a).

From (190), (192), and (198) we obtain

\[
\begin{align*}
&\times \exp \left( -\frac{\sigma \sqrt{2 \gamma}}{\sigma^2} \min \{t, t/\sqrt{\gamma}\} \sqrt{\frac{N_0}{2E_c d_f}} - \frac{\min \{t, t/\sqrt{\gamma}\}^2 N_0}{4E_c d_f \sigma^2} \right) \bigg|_{d_f}^{d_f-1} \\
&\times \exp \left( -\frac{N_0 \min \{t, t/\sqrt{\gamma}\}^2}{4\sigma^2 E_c d_f} \right) \int_0^{\min \{t, t/\sqrt{\gamma}\}} \beta^{d_f} d\beta \\
&= \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}}{\sigma} \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp \left( -\frac{N_0 \min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \min \{t, \sqrt{\gamma}, t\}^{(d_f+1)/2} \\
&\times \frac{1}{(d_f+1) \gamma^{(d_f+1)/2}} \left( \frac{N_0}{E_c} \right)^{d_f+1}.
\end{align*}
\]

From (190), (192), and (198) we obtain

\[
\begin{align*}
P_e(0) \geq & \int_0^{\pi/2} \frac{1}{\pi} \frac{N_0}{2\sigma^2 E_c} \left( \frac{\exp(-\gamma)}{\sigma^2} \right) \frac{1}{2E_c d_f} \sqrt{N_0}^{d_f-1} \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}}{\sigma} \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp \left( -\frac{N_0 \min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \min \{t, \sqrt{\gamma}, t\}^{(d_f+1)/2} \\
&\times \frac{1}{\sqrt{2\pi} d_f+1} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2} \\
&\times \int_{\sqrt{N_0/E_c}}^{\infty} \exp \left( \frac{-t^2}{2} \right) \frac{1}{\sqrt{2\pi}} \sqrt{N_0} \left[ \frac{1}{\sqrt{2E_c d_f}} \right] \gamma^{-(d_f+1)/2} \\
&\times \int_{\sqrt{N_0/E_c}}^{\infty} \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}}{\sigma} \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \left( \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp(-\gamma) \\
&\geq \int_0^{\pi/2} \frac{1}{\pi} \frac{N_0}{2\sigma^2 E_c} \left( \frac{\exp(-\gamma)}{\sigma^2} \right) \frac{1}{2E_c d_f} \sqrt{N_0}^{d_f-1} \exp(-\gamma) \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}}{\sigma} \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp(-\gamma) \\
&\times \frac{1}{\sqrt{2\pi} d_f+1} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2} \\
&\times \int_{\sqrt{N_0/E_c}}^{\infty} \exp \left( \frac{-t^2}{2} \right) \frac{1}{\sqrt{2\pi}} \sqrt{N_0} \left[ \frac{1}{\sqrt{2E_c d_f}} \right] \gamma^{-(d_f+1)/2} \\
&\times \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \left( \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \left( \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp(-\gamma) \\
&\geq \int_0^{\pi/2} \frac{1}{\pi} \frac{N_0}{2\sigma^2 E_c} \left( \frac{\exp(-\gamma)}{\sigma^2} \right) \frac{1}{2E_c d_f} \sqrt{N_0}^{d_f-1} \exp(-\gamma) \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}}{\sigma} \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp(-\gamma) \\
&\times \frac{1}{\sqrt{2\pi} d_f+1} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2} \\
&\times \int_{\sqrt{N_0/E_c}}^{\infty} \exp \left( \frac{-t^2}{2} \right) \frac{1}{\sqrt{2\pi}} \sqrt{N_0} \left[ \frac{1}{\sqrt{2E_c d_f}} \right] \gamma^{-(d_f+1)/2} \\
&\times \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \left( \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \\
&\times \left[ 1 - \exp \left( \frac{\min \{t, \sqrt{\gamma}, t\}^2}{4\sigma^2 E_c d_f} \right) \left( \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma} \right) \right]^{d_f-1} \exp(-\gamma) \\
&\geq \int_0^{\pi/2} \frac{1}{\pi} \frac{N_0}{2\sigma^2 E_c} \left( \frac{\exp(-\gamma)}{\sigma^2} \right) \frac{1}{2E_c d_f} \sqrt{N_0}^{d_f-1} \exp(-\gamma).
\end{align*}
\]
The coding scheme has the transfer function

\[
T(D, N) = \frac{D^5 N}{1 - 2 D N} = \sum_{k=5}^{\infty} 2^{k-5} D^k N^{k-1},
\]

and

\[
\frac{dT(D, N)}{dN} = \sum_{k=5}^{\infty} 2^{k-5} (k - 1) D^k N^{k-2}.
\]

It follows from (59)–(61) that

\[
P_e(u) \leq \sum_{k=5}^{4H} 2^{k-5} \left[ D(E_c/N_0, u/d_f, \sigma^2, s) \right]^k.
\]

A. Numerical Evaluations

Let us consider the Viterbi coding scheme with \(m = 2(K = 3)\), \(n_c = 2, k_c = 1, d_f = 5\) in [4, Section II]. This convolution coding scheme has the transfer function

\[
\times \frac{1}{\sqrt{2\pi d_f + 1}} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2}
\]

\[
\times \sqrt{\frac{\max\{1, \sqrt{N_0/E_c}\}}{N_0/E_c}} \exp\left(-\max\{1, \sqrt{N_0/E_c}\} \sigma d_f \right)
\]

\[
\times \left[ 1 - \exp\left(\max\{1, \sqrt{N_0/E_c}\} \min\{1, \sqrt{\gamma}\} \right) \sqrt{\frac{N_0}{E_c d_f}} - \frac{N_0 \max\{1, \sqrt{N_0/E_c}\} 2 \min\{1, \gamma\}}{4 \sigma^2 \gamma E_c d_f} \right]^{d_f-1} \exp(-\gamma)
\]

\[
= \frac{1}{4\sigma^2} \left( \frac{\exp(-\gamma)}{\sigma^2} \right)^{d_f-1} \exp(-\gamma)
\]

\[
\times \frac{1}{\sqrt{2\pi d_f + 1}} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2}
\]

\[
\times \exp\left(-\max\{1, \sqrt{N_0/E_c}\}^2 \right) \exp\left(-\frac{N_0 \max\{1, \sqrt{N_0/E_c}\} 2 \min\{1, \gamma\}}{4 \sigma^2 \gamma E_c d_f} \right)
\]

\[
\times \left[ 1 - \exp\left(\max\{1, \sqrt{N_0/E_c}\} \min\{1, \sqrt{\gamma}\} \right) \sqrt{\frac{N_0}{E_c d_f}} + \sqrt{\pi \gamma}
\]

\[
\times \exp\left(-\max\{1, \sqrt{N_0/E_c}\} \min\{1, \sqrt{\gamma}\} \right) \sqrt{\frac{N_0}{E_c d_f}} - \frac{N_0 \max\{1, \sqrt{N_0/E_c}\} 2 \min\{1, \gamma\}}{4 \sigma^2 \gamma E_c d_f} \right]^{d_f-1} \max\left\{ 1, \frac{N_0}{E_c} \right\} - \sqrt{\frac{N_0}{E_c}}
\]

Hence, we obtain

\[
P_b(0) \geq f(\gamma, \sigma, H, k_c, d_f, R_c, m) \left( \frac{E_b}{N_0} \right)^{-(d_f+1)}
\]

\[
\gamma, \sigma \text{ fixed},
\]

\[
\gamma, \sigma \text{ fixed},
\]

for some constant \(f\) depending on convolutional code parameters and \(\gamma, \sigma\), and for \(E_b/N_0\) sufficiently large. Combining right above result and (2) we have

\[
\limsup_{\gamma \to \infty} \frac{-\ln P_b(0)}{\gamma} \leq d_f, \quad \text{as } \gamma \to \infty, \quad (E_b/N_0), \sigma \text{ fixed.}
\]

That concludes our proof.

\[
A. \textbf{Numerical Evaluations}
\]

Let us consider the Viterbi coding scheme with \(m = 2(K = 3)\), \(n_c = 2, k_c = 1, d_f = 5\) in [4, Section II]. This convolution coding scheme has the transfer function

\[
\times \frac{1}{\sqrt{2\pi d_f + 1}} \left( \frac{N_0}{E_c} \right)^{d_f+1} \gamma^{-(d_f+1)/2}
\]
Similarly, for $u < 5\sqrt{\frac{2E_b}{N_0}}$ we also obtain

$$P_x(u) \leq \sum_{k=5}^{4H} 2^{k-5} \left[D\left(\frac{E_c}{N_0}, -u/d_f, \sigma^2, s\right)\right]^k.$$  \hspace{1cm} (211)

In addition, we also have

$$P_b(u) \leq \sum_{k=5}^{4H} (k-1)2^{k-5} \left[D\left(\frac{E_c}{N_0}, u/d_f, \sigma^2, s\right)\right]^k.$$  \hspace{1cm} (212)

Here, $D\left(\frac{E_c}{N_0}, u/d_f, \sigma^2, s\right)$ is defined as (27) of Proposition 1. Figures 2, 4 show a tradeoff between the bit error probability $P_e(u)$ and retransmission probability $P_x(u)$ as a function of Yamamoto-Itoh flag $u$. The Figure 2 shows $P_e(u)$ and $P_x(u)$ as functions of bit energy to noise ratio $E_b/N_0$ for a fixed Rician factor while the Figure 4 shows $P_e(u)$ and $P_x(u)$ as functions of Rician factor for a fixed bit energy to noise ratio $E_b/N_0$. These figures indicate that, as we increase the Yamamoto-Itoh flag $u$, the bit error probability $P_e(u)$ decreases but the retransmission probability $P_x(u)$ increases. Figures 3 and 5 show that the increasing of Yamamoto-Itoh flag $u$ leads to the decreasing of the bit error probability $P_b(u)$. The Viterbi decoding scheme using Yamamoto-Itoh flag helps to reduce the bit error probability $P_e(u)$ compared with the Viterbi’s decoding scheme without using this flag (i.e., $u = 0$).

**APPENDIX A**

**PROOF OF LEMMA 2**

We have

$$\int_0^z \exp\left(-\Phi_1\alpha^2 - \Phi_2\alpha\right) d\alpha = \exp\left(\frac{\Phi_2}{4\Phi_1}\right) \int_0^z \exp\left[-\Phi_1\left(\alpha + \frac{\Phi_2}{2\Phi_1}\right)^2\right] d\alpha$$  \hspace{1cm} (213)

$$= \exp\left(\frac{\Phi_2}{4\Phi_1}\right) \int_{\Phi_2/(2\Phi_1)}^{z+\Phi_2/(2\Phi_1)} \exp(-\Phi_1\alpha^2)d\alpha$$  \hspace{1cm} (214)
\[
\frac{1}{\sqrt{2\Phi_1}} \exp \left( \frac{\Phi_2^2}{4\Phi_1} \right) \int_{\sqrt{2\Phi_1} \left[ \Phi_2/(2\Phi_1) \right]}^{\infty} \exp(-t^2/2) dt
\]
\[
= \frac{\sqrt{2\pi}}{\sqrt{2\Phi_1}} \exp \left( \frac{\Phi_2^2}{4\Phi_1} \right) \left[ Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \sqrt{2\Phi_1} \left[ \gamma + \frac{\Phi_2}{2\Phi_1} \right] \right) \right] .
\]

(215)

It follows for \( \Phi_2 \leq 0 \) that
\[
g(\Phi_1, \Phi_2, \gamma) = \int_0^\gamma \alpha \exp \left( -\Phi_1 \alpha^2 - \Phi_2 \alpha \right) d\alpha
\]
\[
= -\frac{1}{2\Phi_1} \int_0^\gamma (-2\Phi_1 \alpha - \Phi_2) \exp \left( -\Phi_1 \alpha^2 - \Phi_2 \alpha \right) d\alpha - \frac{\Phi_2}{2\Phi_1} \int_0^\gamma \exp \left( -\Phi_1 \alpha^2 - \Phi_2 \alpha \right) d\alpha
\]
\[
= -\frac{1}{2\Phi_1} \exp \left( -\Phi_1 \alpha^2 - \Phi_2 \alpha \right) \bigg|_0^\gamma - \frac{\Phi_2}{2\Phi_1} \int_0^\gamma \exp \left( -\Phi_1 \alpha^2 - \Phi_2 \alpha \right) d\alpha
\]
\[
= \frac{1}{2\Phi_1} \left[ 1 - \exp(-\Phi_1 \gamma^2 - \Phi_2 \gamma) \right] - \frac{\Phi_2}{2\Phi_1} \sqrt{2\pi} \exp \left( \frac{\Phi_2^2}{4\Phi_1} \right) \left[ Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \sqrt{2\Phi_1} \left[ \gamma + \frac{\Phi_2}{2\Phi_1} \right] \right) \right]
\]
\[
= \frac{1}{2\Phi_1} \left[ 1 - \exp(-\Phi_1 \gamma^2 - \Phi_2 \gamma) \right] - \frac{\Phi_2}{2\Phi_1} \sqrt{2\pi} \exp \left( \frac{\Phi_2^2}{4\Phi_1} \right) \left[ Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \sqrt{2\Phi_1} \left[ \gamma + \frac{\Phi_2}{2\Phi_1} \right] \right) \right] .
\]

(220)

(221)

To show (13), observe that the following holds
\[
Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \gamma \sqrt{2\Phi_1} + \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) = \int_{\gamma \sqrt{2\Phi_1} + \frac{\Phi_2}{\sqrt{2\Phi_1}}}^{\infty} \exp \left( -\frac{t^2}{2} \right) dt
\]
\[
= \sqrt{2\pi} \Phi_1 \min \left\{ \exp \left[ -\frac{1}{2} \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right)^2 \right] , \exp \left[ -\frac{1}{2} \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} + \gamma \sqrt{2\Phi_1} \right)^2 \right] \right\} .
\]

(222)

(223)
It follows that for any $\Phi_2 \leq 0$

$$g(\Phi_1, \Phi_2, z) = \frac{1}{2\Phi_1} \left\{ 1 - \exp(-\Phi_1 z^2 - \Phi_2 z) \right\} - \frac{\Phi_2 \sqrt{2\pi}}{2\Phi_1 \sqrt{2\Phi_1}} \exp \left( \frac{\Phi_2}{4\Phi_1} \right) \left[ Q \left( \frac{\Phi_2}{\sqrt{2\Phi_1}} \right) - Q \left( \sqrt{2\Phi_1} \left[ z + \frac{\Phi_2}{2\Phi_1} \right] \right) \right],$$

(a)

$$\mathbf{(224)}$$

$$= \left. \frac{\Phi_2 \sqrt{2\pi}}{\sqrt{2\Phi_1}} \right| \min \left\{ \exp \left[ -\frac{1}{2} \frac{(\Phi_2)}{(\sqrt{2\Phi_1})^2} \right], \exp \left[ -\frac{1}{2} \frac{(\Phi_2 + z\sqrt{2\Phi_1})^2}{(\sqrt{2\Phi_1})^2} \right] \right\}. \mathbf{(225)}$$

Here, (a) follows from (223) and the assumption that $\Phi_2 \leq 0$. Combining with the fact that $g(\Phi_1, \Phi_2, z) \geq 0, \forall \Phi_1, \Phi_2, z > 0$, we achieve (13).

**APPENDIX B**

PROOF OF LEMMA 3

Observe that for each $k = 1, 2, \ldots, a_k$ is equal to the number of paths of the Hamming distance $k$ from the all zeros and $c_k$ is equal to the number of bits in error caused by an incorrect choice of the surviving path of free distance $k$ from the correct one [4]. Since convolutional codes are linear, each path of the Hamming distance $k$ from the correct one can be bijectively mapped to a path of the Hamming distance $k$ from all zeros. Therefore, it is obvious that

$$c_k \leq k k_c a_k. \mathbf{(226)}$$

Now, we find the number of paths $a_k(L)$ of the Hamming distance $k$ and length $L$, which merge with the all zeros at a given node in a Viterbi trellis diagram. Observe that for each branch the Hamming distance between two paths is at most $L$, hence it is easy to see that

$$a_k(L) = 0, \quad \forall k > Ln_c. \mathbf{(227)}$$

Now, for each $k \leq L/n_c$, we will show that

$$a_k(L) \leq \sum_{l=0}^{L-1} \binom{L}{l} \left( \binom{k-1}{L-l-1} \right) 1\{L - l - 1 \leq k - 1\}. \mathbf{(228)}$$

Indeed, for a given path of length $L$, denote by $b_j$ the weight of this path at branch $j$ for all $j = 1, 2, \ldots, L$. Then, each path of length $L$ with the Hamming distance $k$ from the all zeros corresponds to a tuple $(b_1, b_2, \ldots, b_L)$ such that $\sum_{j=1}^{L} b_j = k$ where $0 \leq b_j \leq n_c$. Note that the number of tuples $(b_1, b_2, \ldots, b_L)$ such that $\sum_{j=1}^{L} b_j = k$ is at most $\sum_{l=0}^{L-1} \binom{L}{l} \binom{k-1}{L-l-1}$.

This fact can be easily proved as follows

- Choose $l$ out of $L$ numbers $(b_1, b_2, \ldots, b_L)$ such that $b_1 = 0$. There are $\binom{L}{l}$ ways of choices.
- Find $L - l$ positive integers which sum up to $k$.

To find $L - l$ positive integers which sum up to $k$, we put $k$ ones in a row, and then find $L - l - 1$ positions to divide this row of 1’s into $L$ parts where $b_j$ is equal to the number of 1 in the part $j$ for each $j = 1, 2, \ldots, L$. There are $\binom{k-1}{L-l-1} 1\{L - l - 1 \leq k - 1\}$ ways of dividing this row of ones into $L$ parts where each part is non-empty set. Hence, without constraining all the numbers less than or equal to $n_c$, the number of tuples of positive integers which sum up to $k$ is $\binom{k-1}{L-1} 1\{L - l - 1 \leq k - 1\}$. Note that since we have a constraint $0 \leq b_j \leq n_c$, hence the number of positive integer tuples which sum up to $k$ is at most $\binom{k-1}{L-1} 1\{L - l - 1 \leq k - 1\}$.

It follows that

$$a_k \leq \sum_{L=1}^{\left\lfloor \frac{k}{n_c} \right\rfloor} a_k(L) \mathbf{(229)}$$

$$\leq \sum_{L=1}^{\left\lfloor \frac{k}{n_c} \right\rfloor} \sum_{l=0}^{L-1} \binom{L}{l} \binom{k-1}{L-l-1} 1\{L - l - 1 \leq k - 1\}. \mathbf{(230)}$$

That concludes our proof.

**Acknowledgements**

The author would also like to thank Prof. Vincent Y. F. Tan, Prof. Teng J. Lim (National University of Singapore), and Prof. Hirosuke Yamamoto (The University of Tokyo) for constructive suggestions to improve the manuscript.
REFERENCES

[1] A. J. Viterbi. Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. *IEEE Trans. on Inform. Th.*, 13(2):260–269, 1967.

[2] J. K. Omura. On the viterbi decoding algorithm. *IEEE Trans. on Inform. Th.*, 15(1):177–179, Jan 1969.

[3] J. G. D. Fomey. Convolutional codes ii: Maximum likelihood decoding. *IEEE Trans. on Inform. Th.*, 25(2):222–226, Jul 1974.

[4] A. J. Viterbi. Convolutional codes and their performance in communication systems. *IEEE Transactions on Communication Technology*, 19(5):751–772, 1971.

[5] John G. Proakis. *Digital Communications*. McGraw-Hill, 4th edition, 2001.

[6] B. Vucetic. An adaptive coding scheme for time-varying channels. *IEEE Transactions on Communications*, 39(5):653–663, 1991.

[7] E. Malkamaki and H. Leib. Evaluating the performance of convolutional codes over block fading channels. *IEEE Trans. on Inform. Th.*, 45(5):1643–1646, 1999.

[8] S. Lin and Jr. D. J. Costello. *Error Control Coding*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[9] R. J. F. Fang. Lower bounds on reliability functions of variable-length nonsystematic convolutional codes for channels with noiseless feedback. *IEEE Trans. on Inform. Th.*, 17(2):161–171, Sep 1971.

[10] H. Yamamoto and K. Itoh. Viterbi decoding algorithm for convolutional codes with repeat request. *IEEE Trans. on Inform. Th.*, 26(5):540–547, Sep 1980.

[11] M. V. Burnashe. Data transmission over a discrete channel with feedback. Random transmission time. *Problems of Information Transmission*, 12(4):10–30, 1976.

[12] H. Yamamoto and K. Itoh. Asymptotic performance of a modified Schalkwijk-Barron scheme for channels with noiseless feedback. *IEEE Trans. on Inform. Th.*, 25(6):729–733, 1979.

[13] Y. Polyanskiy, H. V. Poor, and S. Verdú. Feedback in the non-asymptotic regime. *IEEE Trans. on Inform. Th.*, 57(8):4903–4925, 2011.

[14] L. V. Truong and V. Y. F. Tan. On awgn channels and gaussian macs with variable-length feedback. *IEEE Trans. on Inform. Th.*, 2016. arXiv:1609.00594 [cs.IT].

[15] L. V. Truong and V. Y. F. Tan. On the reliability function of the common-message broadcast channel with variable-length feedback. *IEEE Trans. on Inform. Th.*, 2017. arXiv:1701.01530 [cs.IT].

[16] Texas Instruments. Keystone architecture viterbi coprocessor (vcp2). Literature Number: SPRUGV6A, Rev. A:1–59, June 2011.

[17] B. D. Kudryashov. Error probability for repeat request systems with convolutional codes. *IEEE Trans. on Inform. Th.*, 39(5):1680–1684, 1993.

[18] T. Hashimoto. On the error exponent of convoluntionally coded ARQ. *IEEE Trans. on Inform. Th.*, 40(2):567–575, Mar 1994.

[19] H. Fujiwara, H.Yamamoto, and J. Ren. A convolutional coded ARQ scheme with retransmission criterion based on an estimated decoding error rate. *IEICE Trans. on Fundamentals*, E78-A(1):100–110, Jan 1995.

[20] H. Fujiwara and H.Yamamoto. The performance of the new convolutional coded ARQ scheme for moderately time-varying channels. *IEICE Trans. on Fundamentals*, E78-A(3):403–411, Mar 1995.

[21] A. R. Raghavan and C. W. Baum. A reliability output Viterbi algorithm with applications to hybrid ARQ. *IEEE Trans. on Information Theory*, 44(3):1214–1216, May 1998.

[22] T. Hashimoto. Composite scheme LR+Th for decoding with erasure and its effective equivalence to Foney’s rule. *IEEE Trans. on Information Theory*, 45(1):78–93, Jan. 1999.

[23] P. Billingsley. *Probability and Measure*. Wiley-Interscience, 3rd edition, 1995.