On the $1/N$–expansion in chiral perturbation theory

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In the first part of the talk, I presented a review of the results for the light quark masses obtained on the basis of chiral perturbation theory. As this material is described in ref. [1], the following notes only concern the second part, which dealt with the behaviour of the effective theory in the large–$N_c$ limit.

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1. Ward identity

The low energy properties of QCD are governed by an approximate, spontaneously broken symmetry, which originates in the fact that three of the quarks happen to be light. If $m_u, m_d, m_s$ are turned off, the symmetry becomes exact: The QCD hamiltonian is then invariant under independent rotations of the right- and lefthanded quark fields. For such a theory to describe the strong interactions observed in nature, it is crucial that this symmetry is spontaneously broken, the ground state being invariant only under the subgroup generated by the charges of the vector currents. There are theoretical arguments indicating that chromodynamics indeed leads to the formation of a quark condensate, which is invariant under the subgroup generated by the vector charges, but correlates the right- and lefthanded fields and thus breaks chiral invariance [2]. The available lattice results also support the hypothesis. In the following I take this generally accepted picture for granted.

It is easy to see why a nonzero quark condensate implies that the spectrum of the theory contains massless particles. Consider the two–point–function formed with an axial vector current and a pseudoscalar density,

$$ A^a_{\mu} = \bar{q} \gamma_{\mu} \gamma_5 \lambda^a q, \quad P^a = \bar{q} i \gamma_5 \lambda^a q, $$

where $\lambda^0, \ldots, \lambda^8$ is a complete set of hermitean $3 \times 3$ matrices in flavour space, normalized by $\text{tr}(\lambda^a \lambda^b) = 2 \delta^{ab}$. If the quark masses are dropped, this function obeys the Ward identity

$$ \partial^\mu \langle 0 | T A^a_{\mu}(x) P^b(0) | 0 \rangle = $$

$$ -2 \delta(x) \langle 0 \bar{q} (\lambda^a, \lambda^b) q | 0 \rangle $$

$$ + \text{tr}(\lambda^a) \langle 0 | T \omega(x) P^b(0) | 0 \rangle. $$

The first term on the r.h.s. is proportional to the quark condensate, which in the chiral limit is flavour independent, $\langle 0 \bar{q} \gamma^{\mu} q | 0 \rangle = \delta_{ff} \langle 0 | \bar{u} u | 0 \rangle$. The second term arises from the anomaly and involves the winding number density $\omega = \text{tr}_e(G_{\mu\nu} \tilde{G}^{\mu\nu})/(16\pi^2)$.

On account of Lorentz invariance, the Fourier transform of $\langle 0 | T A^a_{\mu}(x) P^b(0) | 0 \rangle$ is of the form $q_{\mu} F^{ab}(q^2)$. For the octet components, where the anomaly term does not contribute, the Ward identity states that $q^2 F^{ab}(q^2)$ is a constant. A nonzero quark condensate thus implies that $F^{ab}(q^2)$ contains a pole at $q^2 = 0$. The explicit solution of the Ward identity reads ($a = 1, \ldots, 8$):

$$ \int d^4 x e^{ix\phi} \langle 0 | T A^a_{\mu}(x) P^b(0) | 0 \rangle = $$

$$ 2 \delta^{ab} \frac{q_{\mu}}{q^2} \langle 0 | \bar{u} u | 0 \rangle. $$

A pole term of this form arises if and only if the sum over intermediate states on the left exclusively receives contributions from massless one–particle states: The spectrum of QCD must contain an octet of pseudoscalar particles, which become massless in the chiral limit.

In the case of the singlet current, the anomaly spoils current conservation. The Ward identity does not require the Fourier transform to contain a pole, but merely implies that the inte-
2. Large $N_c$

Let us now consider the limit $N_c \to \infty$, at a fixed value of the renormalization group invariant scale $\Lambda_{\text{QCD}}$. In this limit, the running coupling constant $g$ disappears: $N_c g^2$ tends to a constant. The leading contributions to the connected correlation functions of the quark currents stem from those graphs that contain a single quark loop (contributions from graphs with $\ell$ quark loops are at most of order $N_c^{2-\ell}$). In particular, the two-point function considered above is dominated by graphs with a single quark loop, so that the left-hand side of the Ward identity is of order $N_c$. On the right, the contribution from the quark condensate is also of this order. The graphs relevant for the anomalous term $\langle 0 | T \omega P^0 | 0 \rangle$, however, are at most of order $(N_c)^0$. For large values of $N_c$, the term that invalidates the conservation law for the singlet axial current is thus suppressed by one power of $1/N_c$, so that the argument given in the preceding section then also applies to the singlet current: In the limit $N_c \to \infty$, the spectrum of QCD must contain a ninth Goldstone boson. In other words, if the quark masses $m_u, m_d, m_s$ are turned off and if the number of colours is sent to infinity, the mass of the lightest bound state with the quantum numbers of $P^0 | 0 \rangle$, the $\eta'$, tends to zero.

To my knowledge, the large-$N_c$ limit represents the only coherent theoretical explanation of the Okubo-Zweig-Iizuka rule, whose approximate validity is documented by many examples. It is clear, however, that a world containing nine massless strongly interacting particles resembles the one we live in only vaguely. In particular, the limit strongly distorts the mass spectrum of the pseudoscalars. We need to account for the terms generated by the anomaly, even if they tend to zero when $N_c$ becomes large. This can be done by replacing the limit through an expansion in powers of $1/N_c$. The method is extensively discussed in the literature and the leading terms of the effective lagrangian are known since a long time. More recently, the expansion in powers of $1/N_c$, momenta and quark masses was extended to first non-leading leading order. I wish to discuss some new results obtained on the basis of this approach.

One particular reason for studying the large-$N_c$ limit in the framework of chiral perturbation theory is an ambiguity that affects phenomenological determinations of the quark mass ratios. As pointed out by Kaplan and Manohar, the standard framework only exploits the symmetry properties of the quark mass term and these remain the same if the quark mass matrix is subject to the transformation $m' = m + \alpha (m^1)^{-1} \det m$, where $\alpha$ is an arbitrary parameter. Indeed, the standard effective lagrangian is invariant under the operation, provided the effective coupling constants are transformed accordingly. This implies that, beyond leading order, the quark mass ratios cannot be determined on purely phenomenological grounds. As noted already in ref. [8], however, the ambiguity disappears in the large-$N_c$ limit, because the above transformation of the quark mass matrix violates the OZI rule: The transformation law for the effective coupling constants shows that the parameter $\alpha$ is a quantity of order $1/N_c$. Indeed, in the expansion of the effective lagrangian introduced below, the ambiguity only shows up at next-to-next-to leading order.

The limit $N_c \to \infty$ enlarges the symmetry group of the massless hamiltonian from $G = SU(3)_R \times SU(3)_A \times U(1)_Y$ to $G = U(3)_R \times U(3)_A$, while the subgroup that leaves the ground state invariant remains the same, $H = U(3)_A$. The Goldstone bosons live on the coset space $G/H$. For a finite number of colours, $G/H = SU(3)$, while at $N_c = \infty$, $G/H = U(3)$. The occurrence of an extra Goldstone boson implies that the standard chiral lagrangian does not cover the large-$N_c$ limit: A coherent effective field theory only results if all of the Goldstone bosons are treated as dynamical variables. The standard framework, where the effective field $U(x)$ represents an element of $SU(3)$ must be replaced by one with $U(x) \in U(3)$. The unimodular part of the field $U(x)$ contains the degrees of freedom of the
pseudoscalar octet, while the phase
\[ \det U(x) = e^{i\phi(x)} \]
describes the \( \eta' \).

The effective Lagrangian is formed with the field \( U(x) \) and its derivatives,
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots) . \]

The expression may be expanded in powers of \( 1/N_c \), powers of momenta and powers of the quark mass matrix \( m \). It is convenient to order this triple series by counting the three expansion parameters as small quantities of order \( 1/N_c = O(\delta) \), \( p = O(\sqrt{\delta}) \) and \( m = O(\delta) \), respectively. The first two are familiar from the standard effective Lagrangian. They involve the quark mass matrix \( \tau \), related to the quark condensate. The coefficient \( \kappa \) is expressed through the phase of the determinant in terms of the trace of the field, \( \psi = \langle \phi \rangle \). For the off–diagonal components of \( \phi \), this yields the standard mass formulae of current algebra:
\[ M_{\pi}^2 = (m_u + m_d) B , \]
\[ M_{K^0}^2 = (m_u + m_s) B , \]
\[ M_{K^+}^2 = (m_d + m_s) B . \]

The diagonal components undergo mixing – the levels \( \pi^0, \eta \) and \( \eta' \) repel. In particular, the neutral pion is pushed down and winds up at a mass that is slightly lower than the one of the charged pion (the effect is proportional to \( (m_d - m_u)^2 \) and is tiny – the observed mass difference is due almost entirely to the e.m. interaction). The prediction for \( M_\eta \) and \( M_{\eta'} \) depends on the relative size of the quark masses and the topological susceptibility, which it is convenient to parametrize through the ratio
\[ \kappa \equiv \frac{F^2 B m_s}{9 \tau} . \]

For \( \kappa \ll 1 \), the singlet component of the field effectively decouples, so that the mass of the \( \eta \) is approximately given by the Gell-Mann-Okubo formula, \( M_\eta = \frac{1}{3}(4M_K^2 - M_{\pi}^2) \), while \( M_{\eta'} \simeq 6 \tau/F^2 \).

In the opposite limit, \( \kappa \gg 1 \), where the susceptibility term becomes irrelevant, the \( \eta \) is degenerate with the \( \pi \) and \( M_{\eta'}^2 = 2M_K^2 - M_\pi^2 \).

The standard framework with \( U(x) \in SU(3) \) results if the above effective lagrangian is expanded in powers of \( \kappa \), or equivalently, in inverse powers of \( \tau \). The \( \eta' \) then counts among the massive states, which do not show up as dynamical degrees of freedom, but only manifest themselves indirectly, in the effective coupling constants. The singlet component \( \psi \) may then be integrated out, generating a correction of the form \( L_7 \langle \chi^\dagger U - U^\dagger \chi \rangle^2 \), with \( \kappa \)
\[ L_7 = -\frac{F^4}{288 \tau} \left\{ 1 + O \left( \frac{1}{N_c} \right) \right\} . \]

This result gives rise to a paradox \[ \Box \]: If the limit \( N_c \to \infty \) is taken at fixed quark masses, the “correction term” proportional to \( L_7 \) grows with \( N_c \) and thus dominates over the “leading part” of the effective lagrangian, which is of order \( N_c \).
The shift is proportional to $\kappa$ and thus lowers the value of $\eta$. It is approximately given by $\eta = \kappa$. Nevertheless, it is perfectly meaningful to consider the large-$N_c$ behaviour of the effective coupling constants occurring in the standard framework. By definition, these are independent of the quark masses. They may be worked out by treating $m$ as an infinitesimal quantity, so that the condition $\kappa \ll 1$ is obeyed. The resulting low energy theorem is not in conflict with general properties of QCD, but merely shows that, if the limit $N_c$ is taken at fixed $m$, we are necessarily leaving the region covered by standard ChPT.

We may determine the topological susceptibility with the mass of the $\eta'$. The mass formula for the $\eta$ then takes the form

$$M_\eta^2 = m_0^2 - \frac{8 (M_K^2 - M_\eta^2)^2}{9 (M_\eta^2 - m_0^2)},$$

where $m_0^2 \equiv \frac{1}{4} (4 M_K^2 - M_\eta^2)$ is the value of $M_\eta^2$ that follows from the Gell-Mann-Okubo formula. The remainder originates in the repulsion between $\eta$ and $\eta'$ and represents an SU(3)–breaking effect of second order. Inserting the observed masses, we obtain $M_\eta = 494$ MeV, to be compared with the experimental value, $M_\eta = 547$ MeV and with the Gell-Mann-Okubo formula, which predicts $M_\eta = 566$ MeV. The repulsion between the two levels thus lowers the value of $M_\eta$ by about 70 MeV. The shift is proportional to $\kappa$: The mass of the $\eta$ is approximately given by $M_\eta^2 \approx m_0^2 (1 - \kappa)$. Although the shift is about four times too large, it does make sense to treat it as a correction – the value $\kappa \approx \frac{2}{3} M_\eta^2 / M_\eta^2$ is in the range where standard ChPT applies.

4. Wess-Zumino-Witten term

The anomalies of the fermion determinant not only equip the $\eta'$ with a mass. They also explain the lifetime of the $\pi^0$: At leading order of the low energy expansion, the transitions $\pi^0 \to \gamma\gamma$, $\eta \to \gamma\gamma$ and $\eta' \to \gamma\gamma$ are described by the Wess-Zumino-Witten term, which accounts for the anomalies in the framework of the effective theory and is proportional to the number of colours. The piece relevant for the above transitions is given by

$$\mathcal{L}_{\text{WZW}} = -\frac{N_c \alpha}{4 \pi} F_{\mu\nu} \tilde{F}^{\mu\nu} (Q_\alpha^2 \phi),$$

where $\alpha$, $F_{\mu\nu}$ and $Q_\alpha$ denote the fine structure constant, the e.m. field strength and the quark charge matrix, respectively. I normalize the corresponding decay rates with $F_\pi$,

$$\Gamma_{\rho \to \gamma \gamma} = \frac{\alpha^2 M_\rho^2 N_\rho^2}{576 \pi^3 F_\pi^2} C_\rho^2.$$

The experimental values given by the Particle Data Group correspond to $c_{\pi^0} = 1.001 \pm 0.036$, $c_\eta = 0.944 \pm 0.040$, $c_{\eta'} = 1.242 \pm 0.027$.

At leading order, the standard framework with 8 Goldstone bosons leads to $c_{\pi^0} = 1$, $c_\eta = 3 (Q^2 \lambda^8) = 1 / \sqrt{3}$. The prediction for the lifetime of the $\pi^0$ is in perfect agreement with the data, but the result $c_\eta \approx 0.58$ is too low. Also, this framework does not shed any light on the value of $c_{\eta'}$.

In the extended effective theory, the octet interferes with the singlet. At leading order, the corresponding mixing angle is determined by

$$\tan \vartheta = -\frac{\sqrt{3} (M_\rho^2 - M_\pi^2)}{3 (M_\eta^2 - m_0^2)}.$$

The result for $c_{\pi^0}$ remains the same, but the one for $c_\eta$ is modified by mixing. Moreover, we now also obtain a prediction for the $\eta'$:

$$c_\eta = \frac{1}{\sqrt{3}} (\cos \vartheta - \sqrt{3} \sin \vartheta),$$

$$c_{\eta'} = \frac{1}{\sqrt{3}} (\sqrt{3} \cos \vartheta + \sin \vartheta).$$

Numerically, these relations lead to $\vartheta = -20^\circ$, $c_\eta = 1.09$, $c_{\eta'} = 1.34$. The extended lagrangian thus yields a decent approximation for all three photonic transitions – the observed amplitudes differ from the prediction by less than 15%.

5. Georgi’s inequality and higher orders

Above, I have fixed the quark mass ratios with the mass formulae that follow from the effective lagrangian. We may drop this input and work out the masses $M_\eta$, $M_{\eta'}$, leaving the ratio

$$S = \frac{m_s}{m_u} = \frac{2 m_s}{m_u + m_d}.$$
As pointed out by Georgi, the result for $M_{c}/M_{q}$ is smaller than what is observed, quite irrespective of the value of $S$. The number which results for $S = (2M_{K}^{2} - M_{p}^{2})/M_{p}^{2} = 25.9$ is very close to the upper bound, which is too low by about 10%.

Another evident deficiency of the effective lagrangian is that it leads to $F_{K} = F_{π}$, while the observed values are in the ratio $F_{K} = 1.22 F_{π}$. Clearly, the higher order terms cannot be neglected at the 10% level. In fact, it was noted already in ref. that these generate a shift in the mass of the $η$ that counteracts the repulsion from the $η'$.

The explicit expression for the terms of order $δ$ reads

$$L^{(1)} = L_{5} \langle \partial_{µ} U^{†} U (\chi^{1} U + U^{†} \chi) \rangle + L_{6} \langle \chi^{1} U \chi U + h.c. \rangle + \frac{1}{12} \Lambda_{1} F^{2} \partial_{µ} \psi \partial_{µ} \psi$$

$$+ \frac{1}{12} F_{π} L_{1}^{2} i \psi (\chi U - U^{†} \chi) + \mathcal{L}_{WZW}.$$

I have discarded contributions of the type $⟨∂U⟩^{4}$, because they do not contribute to the quantities under discussion. The terms with $L_{5}, L_{6}$ are familiar from the standard effective lagrangian. Since $L_{4}$ and $L_{6}$ are of order $N_{c}$, they only show up in $L^{(2)}$ (for a detailed discussion of the large-$N_{c}$ counting rules in the framework of the effective theory, see ref. ). Concerning $L_{7}$, we need to distinguish the coupling constant $L_{7}$ occurring in the $U(3)$–lagrangian from the one appearing in the standard framework. While $L_{7}$ is of order $N_{c}$ and does therefore not show up in $L^{(1)}$, the integration over the singlet component of the effective field gives rise to the additional contribution discussed above, so that the effective coupling constant relevant for the SU(3)–lagrangian is given by

$$L_{7} = - \frac{F^{4} (1 + \Lambda_{2})^{2}}{288 \pi} + L_{7}.$$

The coupling constants $Λ_{1}$ and $Λ_{2}$ are of order $1/N_{c}$ (I have extracted a factor of $F^{2}$, so that these constants are dimensionless). They describe differences in the dynamics of the octet and singlet components of the effective field, which arise from violations of the Okubo-Zweig-Iizuka rule – in the large–$N_{c}$ limit, this rule becomes exact. The term with $Λ_{1}$ modifies the kinetic energy of the singlet field; the normalization factor is chosen such that this modification amounts to $∂_{µ} \psi \partial_{µ} \psi \rightarrow (1 + Λ_{1})∂_{µ} \psi \partial_{µ} \psi$. The constant $Λ_{2}$ affects the interference term between octet and singlet, $ψ(χ)$, $Λ_{2}$) $ψ(χ)$, and also modifies the mass term of the singlet.

### 6. Anomalous dimensions

As is well-known, the dimension of the singlet axial current is anomalous. This implies that the singlet components of the matrix elements $⟨0| T \gamma_{µ} \overline{χ}^{A} q |P⟩ = i P_{ν} F_{ν}^{A}$ depend on the running scale:

$$μ \frac{d F_{ν}^{A}}{d μ} = \gamma_\nu F_{ν}^{A}, \quad \gamma_\nu = - \left( \frac{g}{2π} \right)^{4} O(g^{5}).$$

A change in $μ$ leads to a multiplicative renormalization: $F_{ν}^{A} → Z_{A} F_{ν}^{A}$. The problem does not manifest itself at leading order, where $F_{ν}^{A} = - \sin θ F_{ν}$, $Z_{A}^{0} = \cos θ F_{ν}$, because the anomalous dimension is of order $1/N_{c}$. Indeed, the scale dependence is very weak: $F_{ν}^{0} \propto \exp(4/β_{0} L)$, with $L = ℓ(μ/Λ_{QCD})$, $β_{0} = 11 - 2/3 N_{f}$.

For the generating functional of QCD to remain invariant under a change of the running scale, it does therefore not suffice to only renormalize the external scalar and pseudoscalar fields. In addition, we also need to renormalize the singlet axial field and the vacuum angle. The symmetry properties of the generating functional ensure that the renormalization factors of $a^{µ}(x)$ and $θ(x)$ are the same. To simplify the renormalization group behaviour of the scalar and pseudoscalar external fields, it is convenient to replace $s(x)$ and $p(x)$ by the combination $m = e^{iθ/3}(s + ip)$, which is invariant under the transformations generated by the singlet axial charge. The generating functional of QCD does remain invariant under a change of the running scale if the external fields are subject to the transformation

$$m(x) \rightarrow Z_{m}^{-1} m(x),$$

$$a^{µ}(x) \rightarrow Z_{A}^{-1} a^{µ}(x),$$

$$θ(x) \rightarrow Z_{θ}^{-1} θ(x),$$

and the generating functional $Z_{A} Z_{θ} Z_{m}$ remains invariant under a change of the external fields.
where the term $Z^{-1}_M$ represents the familiar factor that describes the scale dependence of the quark masses.

This property of the generating functional is readily translated into the language of the effective theory. To maintain the transformation law \( U \rightarrow V_0 U V_0^\dagger \), we need to subject the singlet field \( \psi \) to the same multiplicative renormalization as the vacuum angle, \( \psi \rightarrow Z^{-1}_x \psi \), while the octet components of \( \phi(x) \) are scale independent. A change in scale thus modifies the effective coupling constants according to \( Z \rightarrow Z^\dagger \tau, 1+\Lambda_1 \rightarrow Z^\dagger_1 (1+\Lambda_1), 1+\Lambda_2 \rightarrow Z_2 (1+\Lambda_2) \), while \( F, \chi, L_5 \) and \( L_8 \) are scale independent.

The Wess-Zumino-Witten term (6) is invariant under a change of scale only up to contributions of order \( 1/N_c \). To arrive at a scale invariant effective lagrangian, we need to add the term

\[
\mathcal{L}^{(2)} = - \frac{N_c \alpha_\Lambda}{18 \pi} F_{\mu\nu} \tilde{F}^{\mu\nu} \psi ,
\]

which belongs to \( \mathcal{L}^{(2)} \). The coupling constant \( \alpha_\Lambda \) describes the OZI-violations in the transitions \( P \rightarrow \gamma \gamma \) and is of order \( 1/N_c \). It transforms with \( Z^\dagger \)

At order \( \delta_5^2 \), the effective lagrangian contains further contributions. In particular, it contains terms that contribute to the symmetry breaking in the photonic transition matrix elements \( \delta_5 \).

In the following, I ignore this complication, assuming that the symmetry breaking is dominated by the interference between the octet and the singlet, which the framework described here does account for to first nonleading order.

The discussion below is based in the diploma work of R. Kaiser [12]. It relies on the assumption that the expression \( \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} \) represents a decent approximation to the full effective lagrangian. As mentioned earlier, an ambiguity of the Kaplan-Manohar type does not show up in this framework: As the corresponding transformation of the quark mass matrix violates the OZI rule, it represents a change of the type \( m \rightarrow m(1+O(\delta_5^2)) \), which is beyond the accuracy of our analysis.

7. Results

Since loops start contributing only at order \( \delta_5^2 \), it suffices to evaluate the tree graphs. For the masses and decay constants, we may again restrict ourselves to the terms quadratic in \( \phi = \sum_a \lambda^a \phi_a \). While the masses are the square roots of the eigenvalues of this quadratic form, the decay constants are the entries of the matrix that diagonalizes it: The eigenstates are given by \( \varphi = \sum_a F^a_\varphi \phi_a \). For the photonic transition matrix elements, the above lagrangian implies

\[
\sqrt{3} (F_\eta c_\eta + F_\pi c_\pi) = F_\pi , \quad (9)
\]

\[
\sqrt{3} (F_\eta c_\eta + F_\pi c_\pi) = \sqrt{3} F_\pi (1 + \Lambda_3) . \quad (10)
\]

Both of these relations are manifestly scale invariant.

Next, I exploit the fact that the coupling constants \( \Lambda_1 \) and \( \Lambda_2 \) do not affect the part of the lagrangian that is quadratic in the component \( \phi_8 \) of the effective field. If the quark mass ratio \( S = m_8/m_\pi \) is taken as known, we may determine the constants \( L_5 \) and \( L_8 \) with \( F_\pi \), \( F_\eta \), \( M_5 \) and \( M_K \), so that we arrive at a parameter free representation of this part of the lagrangian. Comparing the result with the expression that follows from the representation \( \phi = \sum a F^a \phi_a + \ldots \) for the eigenstates, we obtain two analogues of the Gell-Mann-Okubo formula:

\[
3 \{(F_\eta^8)^2 + (F_\pi^8)^2\} = 4F_\pi^2 - F_\eta^2 , \quad (11)
\]

\[
3 \{(F_\eta^8)^2 M_5^2 + (F_\pi^8)^2 M_\eta^2\} = 4F_\pi^2 M_5^2 - \frac{2S}{(S+1)} - F_\eta^2 M_\eta^2 (2S-1) . \quad (12)
\]

Both of these relations are valid to first nonleading order. The first one implies

\[
F_\eta^8 = \cos \vartheta_8 F_8 , \quad F_\pi^8 = \sin \vartheta_8 F_8 ,
\]

with \( F_8 = 1.28 F_\pi \). The mixing angle may be determined with eq. (13). The observed values of the transition matrix elements \( c_\eta, c_\pi \) require \( \vartheta_8 = -20.5^\circ \). Inserting this in (12), we may then determine the quark mass ratio. The result, \( S = 26.6 \), is remarkably close to the current algebra prediction, \( S = (2M_5^2 - M_\eta^2)/M_\eta^2 = 25.9 \).

The coupling constant \( L_5 \) generates an off-diagonal part in the kinetic term. Comparing
the coefficient of $\partial_\mu \phi_8 \partial^\mu \phi_0$ with the one that follows from the representation in terms of the eigenstates, we obtain

$$3 \left\{ F_\eta F^0_\eta + F^8_\eta F^8_\eta' \right\} = -2\sqrt{2} (F^2_\eta - F^2_8) (1 + O(\delta)) \ .$$

The relation shows that, at this order of the low energy expansion, the vectors $(F^8_\eta, F^\prime_\eta)$ are not orthogonal to one another: We need to distinguish the mixing angle of the singlet components,

$$F_\eta = - \sin \vartheta_0 F_0 \ , \ F^\prime_\eta = \cos \vartheta_0 F_0$$

from the one introduced above. The relation determines the difference:

$$\sin(\vartheta_0 - \vartheta_8) = \frac{2\sqrt{2} (F^2_\eta - F^2_8)}{3 F^2_8} (1 + O(\delta)) \ .$$

I have written the relation in scale invariant form, making use of the fact that the difference between $F_0$ and $F_8$ is of order $\delta$. Numerically, the formula yields $\vartheta_0 \simeq -4^\circ$. The term responsible for the difference between $F_K$ and $F_\pi$ thus also generates a substantial difference in the two mixing angles. Since $\vartheta_0$ turns out to be remarkably small, the state $|\eta\rangle$ is nearly orthogonal to $A^0_8 \langle 0 |$ – in this sense, the $\eta$ is nearly pure octet. To my knowledge, this is a new result (see the review by Bijnens [12]; for more recent discussions of related phenomena, I refer to [13],[19]).

The relation implies $F_0 = 1.25 F_\eta (1 + \Lambda_3)$. The numerical value of $F_0$ cannot be determined phenomenologically, because it depends on the renormalization scale – all of the coupling constants $\tau, \Lambda_1, \Lambda_2, \Lambda_3$ are scale dependent. We may express the result for these in terms $\Lambda_3$. The numerical values of the scale invariant combinations are $\tau/(1 + \Lambda_3)^2 = (195 \text{ MeV})^4$, $\Lambda_1 - 2\Lambda_3 = 0.25$, $\Lambda_2 - \Lambda_3 = 0.28$. For the remaining coupling constants, we obtain $F = 90.6 \text{ MeV}$, $L_5 = 2.2 \cdot 10^{-3}$, $L_7 = -0.3 \cdot 10^{-3}$, $L_8 = 1.0 \cdot 10^{-3}$.

8. Discussion

The sensitivity of the result for the quark mass ratio to the input used for the decay rates is shown in fig.1. The tilted lines represent constant values of $S$. The rectangle corresponds to the experimental errors quoted above.

The calculation described in the preceding section gives rise to another paradox: According to ref. [9], the effective lagrangian we are using here leads an upper bound for $S$. The argument does not invoke the experimental information about the decay rates and implies that the current algebra value $S = 25.9$ represents an upper limit. Although this is within the uncertainties of the above result, it is instructive to identify the origin of the difference.

In ref. [4], the matrix elements of the $2 \times 2$ matrices that occur in the diagonalization procedure are expanded in powers of $1/N_c$ and only the first two terms are retained. In particular, $\Delta_N \equiv 24 L_5 \tau F^{-4} + \frac{1}{2} \Lambda_1 - \Lambda_2$ is treated as a small parameter, because it represents a term of order $1/N_c$. The contributions from the OZI-violating coupling constants $\Lambda_1, \Lambda_2$ are indeed small, but the first term is not: While the susceptibility is related to the mass of the $\eta'$, $\tau \simeq \frac{1}{6} M^2_\eta F^2$, the coupling constant $L_5$ is dominated by the exchange of scalar resonances, $L_5 \simeq \frac{1}{4} F^2/M^2_\eta$, so that $\Delta_N \simeq M^2_\eta/M^2_\eta$. Since the two masses are nearly the same, this estimate yields $\Delta_N \simeq 1$ (indeed, the explicit evaluation of the coupling
constants with the input specified above yields $\Delta_N = 1.0$). Despite the fact that $\Delta_N$ is of order $1/N_c$, it is not numerically small for $N_c = 3$.

The leading contribution to $\Delta_N$ represents a ratio of two terms in the effective lagrangian: Compare the term $2L_5\langle \partial_\mu \phi \partial^\mu \phi \chi \rangle$ from $\mathcal{L}^{(1)}$ to the term $\frac{1}{4}F^2\phi^2\chi$ from $\mathcal{L}^{(0)}$. For the $\eta'$ matrix elements, the square of the momentum is equal to $M_{\eta'}^2$, so that the ratio of the two terms is given by $8L_5M_{\eta'}^2/F^2 \simeq 48L_5\pi/F^4 \simeq 2\Delta_N$. Hence the matrix element of one of the terms in $\mathcal{L}^{(1)}$ is about twice as large as the corresponding matrix element of one of those contained in $\mathcal{L}^{(0)}$. At first sight, this looks like a disaster for the expansion we are using here, which treats $\mathcal{L}^{(1)}$ as a perturbation. It is not the expansion as such which fails, however. The ratio is large because the term we picked out from $\mathcal{L}^{(0)}$ does not represent the main contribution, which arises from $\frac{1}{2}\tau\psi^2$. Compared to this term, the one from $\mathcal{L}^{(1)}$ does indeed represent a small correction. In fact, the results obtained for the masses and decay constants explicitly show that, throughout, the first order corrections are reasonably small compared to the leading terms. In particular, the OZI-violating coupling constants are small. In this sense, the $1/N_c$–expansion is a coherent scheme, also in the pseudoscalar channel. As witnessed by the mass ratio $M_p^2/M_{\eta'}^2$, it is not true, however, that all dimensionless quantities of physical interest that vanish for $N_c \rightarrow \infty$ are numerically small for $N_c = 3$. The formal expansion used in ref. [4] in effect treats each one of the terms in $\mathcal{L}^{(1)}$ as small compared to each one of those in $\mathcal{L}^{(0)}$ – this is not the case.

9. Conclusion

The simplest form of the large–$N_c$ hypothesis is the assumption that all dimensionless quantities of physical interest that disappear when $N_c \rightarrow \infty$ are numerically small for $N_c = 3$. In this form, the hypothesis evidently fails: In the world we live in, the mass ratio $M_{\eta'}^2/M_{\eta'}^2$ is about equal to one, despite the fact that it represents a quantity of order $1/N_c$. If we wish, we may blame this on the fact that the topological susceptibility of Gluodynamics happens to be rather large.

The analysis described here relies on a weaker form of the large–$N_c$ hypothesis, namely the assumption that the terms occurring in the effective lagrangian approximately obey the Okubo-Zweig-Iizuka rule. It implies that those effective coupling constants which do not receive contributions from graphs with a single quark loop are suppressed. At order $p^2$, this hypothesis requires the terms

$$\Lambda_1 F^2(\partial\psi)^2 \quad \text{and} \quad \Lambda_2 F^2i\psi(\chi U - U^\dagger \chi)$$

to be small compared to

$$F^2(\partial U^\dagger \partial U) \quad \text{and} \quad F^2(\chi U + U^\dagger \chi),$$

respectively. At order $p^4$, the same hypothesis implies that the terms proportional to $L_4, L_6$ and $L_7$ are small compared to those with $L_5, L_8$. In this form, the $1/N_c$ expansion indeed leads to a coherent picture for the masses, decay constants and photonic transitions of the pseudoscalar nonet.

The effective lagrangian used collects all terms of first nonleading order in the simultaneous expansion in powers of $1/N_c$, momenta and quark masses. The main limitation of the calculation reported here is that this lagrangian does not account for the symmetry breaking corrections to the Wess-Zumino-Witten term: I am assuming that the symmetry breaking in the photonic transitions is dominated by the one due to the interference between the octet and the singlet.

The observed rates for $\eta \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ indicate that the OZI-violations in these matrix elements are small. The result obtained for the quark mass ratio $m_u/m_d$ is very close to the current algebra value. The main effect generated by the corrections to the well-known leading order lagrangian concerns $\eta - \eta'$ mixing: At the order of the low energy expansion considered here, we need to distinguish two mixing angles. The analysis leads to a low energy theorem, which states that the difference between the two is determined by $F_K - F_{\pi}$. The mixing angle seen in the singlet components of the decay constants turns out to be much smaller than the one in the octet components: $\vartheta_8 \simeq -20^\circ, \vartheta_0 \simeq -4^\circ$.

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