GENERATING THE MONOID OF $2 \times 2$ MATRICES OVER MAX-PLUS AND MIN-PLUS SEMIRINGS

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Abstract. In this short note, we describe generating sets for the monoids of consisting of all $2 \times 2$ matrices over certain finite tropical semirings.

A semigroup is a set $S$ with an associative binary operation $\ast$. A semigroup $S$ has an identity if there exists $e \in S$ such that $e \ast s = s \ast e = s$ for all $s \in S$. A semigroup with identity is called a monoid. The semigroup $S$ has a zero if there exists $z \in S$ such that $z \ast s = s \ast z = z$ for all $s \in S$. A semigroup $S$ is commutative if $s \ast t = t \ast s$ for all $s, t \in S$.

A semiring is a set $K$ equipped with two binary operations, which we will denote by $\oplus$ and $\otimes$, and two distinguished elements $0, 1 \in K$, such that the following conditions hold:

(i) $(K, \oplus)$ is a commutative monoid with identity 0 (called the zero of $K$);
(ii) $(K, \otimes)$ is a monoid with identity 1 (called the one of $K$);
(iii) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ and $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ for all $x, y, z \in K$;
(iv) $0 \otimes x = x \otimes 0 = 0$ for all $x \in K$.

The simplest examples of semirings are rings themselves, such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ with the usual addition and multiplication. An example of semiring that is not a ring is the boolean semiring $\mathbb{B} = \{0, 1\}$ with operations $\oplus$ and $\otimes$ defined by:

$$
\begin{array}{c|cc}
\oplus & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1
\end{array}
\quad
\begin{array}{c|cc}
\otimes & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
$$

The min-plus semiring is defined to be the set $\mathbb{N} \cup \{\infty\}$ with the operations $\oplus = \min$ and $\otimes = +$, where $\otimes$ extends the usual addition on $\mathbb{N}$ in a natural way, so that

$$x \otimes \infty = \infty \otimes x = \infty \quad \text{for all} \quad x \in \mathbb{N} \cup \{\infty\}.$$ 

We will denote the min-plus semiring by $K^\infty$. The one of $K^\infty$ is 0 and the zero is $\infty$. The min-plus semiring was first introduced by Simon [8] in the context of automata theory; see also Pin [6].

The max-plus semiring is defined to be the set $\mathbb{N} \cup \{-\infty\}$ with the operations $\oplus = \max$ and $\otimes = +$, where $\otimes$ extends the usual addition on $\mathbb{N}$ in a natural way, so that

$$x \otimes -\infty = -\infty \otimes x = -\infty \quad \text{for all} \quad x \in \mathbb{N} \cup \{-\infty\}.$$ 

We will denote the max-plus semiring by $K^{-\infty}$. The one of $K^{-\infty}$ is 0 and the zero is $-\infty$. A variant of the max-plus semiring was first introduced by Mascle [4]

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(Mascle’s semiring included both $\infty$ and $-\infty$). The monoid of $2 \times 2$ max-plus matrices with entries in $\mathbb{R} \cup \{-\infty\}$ has also been studied in detail; see [3].

Both the min-plus and max-plus semirings, and several others which we will not be concerned with here, are referred to as tropical semirings. A congruence on a semiring $K$ is just an equivalence relation $\rho \subseteq K \times K$ that is compatible with $\oplus$ and $\otimes$. More specifically, if $(a, b), (c, d) \in \rho$, then

$$(a \oplus c, b \oplus d), (a \otimes c, b \otimes d) \in \rho.$$ 

We are also concerned with the following finite quotients of the semirings $K_\infty$ and $K_{-\infty}$:

$$K^\infty_t = K_\infty / (t = t + 1) \quad \text{and} \quad K^{-\infty}_t = K^{-\infty} / (t = t + 1)$$

where $t \in \mathbb{N}$ and $(t = t + 1)$ denotes the least congruence such that $t$ and $t + 1$ are equivalent.

More explicitly, if $t \in \mathbb{N}$, then

$$K^\infty_t = \{0, \ldots, t, \infty\}$$

and the operations on $K^\infty_t$ are defined by

$$a \oplus b = \min\{a, b\} \quad \text{and} \quad a \otimes b = \begin{cases} \min\{a + b, t\} & \text{if } a, b \neq \infty \\ \infty & \text{if } a \text{ or } b = \infty. \end{cases}$$

Similarly,

$$K^{-\infty}_t = \{0, \ldots, t, -\infty\}$$

and the operations on $K^{-\infty}_t$ are defined by

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \otimes b = \min(a + b, t).$$

We denote by $M_n(K)$ the semigroup of $n \times n$ matrices with entries in a semiring $(K, \oplus, \otimes)$, and the usual multiplication of matrices with respect to $\oplus$ and $\otimes$. Since every semiring $K$ has a one and a zero, $M_n(S)$ is a monoid whose identity element is just the usual identity matrix (with ones of the semiring on the diagonal and zeros elsewhere).

We will find generating sets for $M_2(K^\infty)$ and $M_2(K^{-\infty})$, and hence also for their finite quotients $M_2(K^\infty_t)$ and $M_2(K^{-\infty}_t)$ for all $t \in \mathbb{N}$.

It is possible to compute subsemigroups of the monoids $M_n(K^\infty)$ and $M_n(K^{-\infty})$ defined by a generating set, for some relatively small values of $n$ and $t$, using the C++ library libsemigroups [5] or Semigroupe [7] (a C program by Jean-Eric Pin). The motivation for finding generating sets for the entire monoids $M_n(K^\infty_t)$ and $M_n(K^{-\infty}_t)$ stems from the second and third authors experiments when they reimplemented the Froidure-Pin Algorithm [1] in libsemigroups [5].

1. Min-plus matrices

In this section, we are concerned with the monoid $M_2(K^\infty)$ of $2 \times 2$ matrices over the min-plus semiring $K^\infty$ and its finite quotients $K^\infty_t$ for all $t \in \mathbb{N}$. The identity of $M_2(K^\infty)$ is

$$\begin{pmatrix} 0 & \infty \\ \infty & 0 \end{pmatrix}.$$
Theorem 1.1. The monoid \( M_2(K^\infty) \) of \( 2 \times 2 \) min-plus matrices is generated by the matrices:

\[
A_i = \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \infty \\ \infty & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} \infty & \infty \\ \infty & 0 \end{pmatrix}
\]

where \( i \in \mathbb{N} \cup \{\infty\} \).

Proof. Pre-multiplying any matrix \( X \) in \( M_2(K^\infty) \) by \( A_\infty \) swaps the rows of \( X \), and post-multiplying \( X \) by \( A_\infty \) swaps the columns of \( X \). Hence it suffices to express one representative of every matrix in \( M_2(K^\infty) \) up to rearranging the rows and the columns.

It is routine to verify the following equalities:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} C = \begin{pmatrix} \infty & b \\ \infty & d \end{pmatrix} \quad \text{and} \quad C \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \infty & \infty \\ c & d \end{pmatrix}
\]

and

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} B = \begin{pmatrix} a \otimes 1 & b \\ c \otimes 1 & d \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c & d \end{pmatrix}.
\]

If \( i \in K^\infty \) is arbitrary, then

\[
\begin{pmatrix} \infty & \infty \\ \infty & i \end{pmatrix} = C \begin{pmatrix} \infty & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad C = C A_\infty A_i A_\infty C,
\]

and in this way it is possible to generate every matrix in \( M_2(K^\infty) \) containing three or four occurrences of \( \infty \), using the given matrices.

If \( i, j \in K^\infty \setminus \{\infty\} \) and \( k \in K^\infty \) are arbitrary, then

\[
\begin{pmatrix} \infty & i \\ \infty & j \end{pmatrix} = B^i \begin{pmatrix} \infty & 0 \\ 0 & j \end{pmatrix} = B^i \begin{pmatrix} \infty & 0 \\ 0 & j \end{pmatrix} \quad \text{and} \quad C = C A_\infty A_j A_\infty C,
\]

\[
\begin{pmatrix} \infty & \infty \\ i & j \end{pmatrix} = \begin{pmatrix} \infty & \infty \\ 0 & j \end{pmatrix} \quad \text{and} \quad B^i \begin{pmatrix} \infty & 0 \\ 0 & j \end{pmatrix} B^j = C A_\infty A_j A_\infty B^i,
\]

\[
\begin{pmatrix} \infty & i \\ j & k \end{pmatrix} = B^i \begin{pmatrix} \infty & 0 \\ 0 & k \end{pmatrix} B^j = B^i A_\infty A_k A_\infty B^j.
\]

(Note that the third equation gives \( \begin{pmatrix} \infty & i \\ j & \infty \end{pmatrix} = B^i A_\infty B^j \). Hence every matrix in \( M_2(K^\infty) \) with at least one occurrence of \( \infty \) can be expressed as a product of the given matrices.

Suppose that \( a, b, c, d \in K^\infty \setminus \{\infty\} \). We may further suppose without loss of generality that \( a = \min\{a, b, c, d\} \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d - b & c - a \end{pmatrix} B^a A_\infty B^a \quad \text{if} \ b \leq d
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b - d & c - a \end{pmatrix} B^d A_\infty B^a \quad \text{if} \ b > d.
\]

It therefore suffices to note that if \( i, j \in K^\infty \setminus \{\infty\} \) and \( i \geq j \), then

\[
\begin{pmatrix} 0 & i \\ j & 0 \end{pmatrix} = A_i A_j \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ i & j \end{pmatrix} = A_\infty B^j A_\infty \begin{pmatrix} 0 & 0 \\ i - j & 0 \end{pmatrix} = A_\infty B^j A_\infty A_0 A_{i-j}.
\]

□
Corollary 1.2. Let \( t \in \mathbb{N} \) be arbitrary. Then the finite monoid \( M_2(K^\infty_t) \) of \( 2 \times 2 \) min-plus matrices is generated by the \( t + 4 \) matrices:

\[
A_i = \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \infty \\ \infty & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \infty & \infty \\ \infty & 0 \end{pmatrix}
\]

where \( i \in \{0, 1, \ldots, t, \infty\} \).

We do not know if the generating set in Corollary 1.2 has minimal size. However, it has been verified computationally for small values of \( t \), that the generating set in Corollary 1.2 is irredundant, in the sense none of the generators belongs to the semigroup generated by the other generators.

While knowing a generating set for the \( 2 \times 2 \) matrices is good, it would, of course, be much better to have a generating set for the \( n \times n \) matrices for all \( n \in \mathbb{N} \), \( n > 2 \). It seems unlikely that a case-by-case approach (such as that performed in the proof of Theorem 1.1) will be successful for values of \( n \) greater than 2. However, the following conjecture is suggested by our computational experiments.

Conjecture 1.3. Let \( t \in \mathbb{N} \) be arbitrary. Then the monoid \( M_3(K^\infty_t) \) of \( 3 \times 3 \) min-plus matrices is generated by the \((2t^3 + 9t^2 + 19t + 36)/6\) matrices:

\[
\begin{pmatrix} \infty & \infty & 0 \\ 0 & \infty & \infty \\ 0 & 0 & \infty \end{pmatrix}, \quad \begin{pmatrix} \infty & \infty & 0 \\ 0 & 0 & \infty \\ 0 & i & \infty \end{pmatrix}, \quad \begin{pmatrix} \infty & \infty & \infty \\ 0 & \infty & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \infty & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{pmatrix},
\]

where \( i \in \{0, 1, \ldots, t\} \).

2. Max-plus matrices

In this section, we are concerned with the monoid \( M_2(K^{-\infty}) \) of \( 2 \times 2 \) matrices over the max-plus semiring \( K^{-\infty} \) and its finite quotients \( K^{-\infty}_t \) for all \( t \in \mathbb{N} \).

The identity of \( M_2(K^{-\infty}) \) is

\[
\begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.
\]

Theorem 2.1. The monoid \( M_2(K^{-\infty}) \) of \( 2 \times 2 \) max-plus matrices is generated by the matrices

\[
X_i = \begin{pmatrix} i & 0 \\ 0 & -\infty \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \quad W_{jk} = \begin{pmatrix} 0 & j \\ k & 0 \end{pmatrix},
\]

where \( i \in \mathbb{N} \cup \{-\infty\} \) and \( j, k \in \mathbb{N} \) such that \( 0 < j \leq k \).
Proof. Pre-multiplying any matrix \( A \) in \( M_2(K^{-\infty}) \) by \( X_{-\infty} \) swaps the rows of \( A \), and post-multiplying \( A \) by \( X_{-\infty} \) swaps the columns of \( A \). Hence it suffices to express one representative of every matrix in \( M_2(K^{-\infty}) \) up to rearranging the rows and the columns, as a product of the given generators.

It is routine to verify the following equalities:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} Z = \begin{pmatrix} -\infty & b \\ -\infty & d \end{pmatrix} \quad \text{and} \quad Z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\infty & -\infty \\ c & d \end{pmatrix}
\]

and

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} Y = \begin{pmatrix} a \otimes 1 & b \\ c \otimes 1 & d \end{pmatrix} \quad \text{and} \quad Y \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c & d \end{pmatrix}.
\]

If \( i \in K^{-\infty} \) is arbitrary, then

\[
\begin{pmatrix} -\infty & -\infty \\ -\infty & i \end{pmatrix} = Z \begin{pmatrix} -\infty & 0 \\ 0 & i \end{pmatrix} Z = Z X_{-\infty} X_i X_{-\infty} Z,
\]

and in this way it is possible to generate every matrix in \( M_2(K^{-\infty}) \) containing three or four occurrences of \( -\infty \), using the given matrices.

If \( i, j \in K^{-\infty} \setminus \{-\infty\} \) and \( k \in K^{-\infty} \) are arbitrary, then

\[
\begin{pmatrix} -\infty & -\infty \\ i & j \end{pmatrix} = Z X_{-\infty} X_j X_{-\infty} Y^i,
\]

\[
\begin{pmatrix} -\infty & i \\ -\infty & j \end{pmatrix} = Y^i X_{-\infty} X_j X_{-\infty} Z,
\]

\[
\begin{pmatrix} -\infty & i \\ j & k \end{pmatrix} = Y^i X_{-\infty} X_k X_{-\infty} Y^j.
\]

Hence every matrix in \( M_2(K^{-\infty}) \) with at least one occurrence of \( -\infty \) can be expressed as a product of the given matrices.

Suppose that \( a, b, c, d \in K^{-\infty} \setminus \{-\infty\} \). We may further suppose without loss of generality that \( a = \min \{a, b, c, d\} \). Then

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 0 \\ d - b & c - a \end{pmatrix} Y^b X_{-\infty} Y^a & \text{if} \ b \leq d \\
\begin{pmatrix} b - d & 0 \\ 0 & c - a \end{pmatrix} Y^d X_{-\infty} Y^a & \text{if} \ b > d.
\end{cases}
\]

It therefore suffices to note that if \( i, j \in K^{-\infty} \setminus \{-\infty\} \) and \( i \geq j \), then

\[
\begin{pmatrix} 0 & i \\ j & 0 \end{pmatrix} = W_{ij}
\]

and

\[
\begin{pmatrix} 0 & 0 \\ i & j \end{pmatrix} = X_{-\infty} Y^j \begin{pmatrix} i - j & 0 \\ 0 & 0 \end{pmatrix} X_{-\infty} Y^j \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix} \begin{pmatrix} i - j & -\infty \\ 0 & 0 \end{pmatrix},
\]

with both \( \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix} \) and \( \begin{pmatrix} i - j & -\infty \\ 0 & 0 \end{pmatrix} \) being a product of the given matrices, since they each have an entry equal to \( -\infty \).

\[\square\]

**Corollary 2.2.** Let \( t \in \mathbb{N} \) be arbitrary. Then the finite monoid \( M_2(K_t^{-\infty}) \) of \( 2 \times 2 \) max-plus matrices is generated by the \( (t^2 + 3t + 8)/2 \) matrices

\[
X_i = \begin{pmatrix} i & 0 \\ 0 & -\infty \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \quad W_{jk} = \begin{pmatrix} 0 & j \\ k & 0 \end{pmatrix},
\]
where $i \in \{-\infty, 0, \ldots, t\}$ and $j, k \in \{1, \ldots, t\}$ with $j \leq k$.

Although we are able to produce generating sets for $M_3(K_i^{-\infty})$, for some small values of $t$, using GAP [2], we have not been able to recognise a pattern in these matrices due to their large number and complexity. GAP [2] produces irredundant generating sets for $M_3(K_i^{-\infty})$ with 19, and 78 generators, respectively, when $t = 1$ and 2.

References

[1] Véronique Froidure and Jean-Eric Pin. Algorithms for computing finite semigroups. In *Foundations of computational mathematics (Rio de Janeiro, 1997)*, pages 112–126. Springer, Berlin, 1997.

[2] The GAP Group. *GAP – Groups, Algorithms, and Programming. Version 4.11.0*, 2020.

[3] Marianne Johnson and Mark Kambites. Multiplicative structure of $2 \times 2$ tropical matrices. *Linear Algebra Appl.*, 435(7):1612–1625, 2011.

[4] Jean-Paul Mascle. *Automata, Languages and Programming: 13th International Colloquium Rennes, France, July 15–19, 1986 Proceedings*, chapter Torsion matrix semigroups and recognizable transductions, pages 244–253. Springer Berlin Heidelberg, Berlin, Heidelberg, 1986.

[5] J. D. Mitchell et al. *libsemigroups - C++ library for semigroups and monoids, Version 1.3.1*, August 2020.

[6] Jean-Eric Pin. Tropical semirings. *http://www.liafa.jussieu.fr/~jep/PDF/Tropical.pdf*.

[7] Jean-Eric Pin. *Semigroupe 2.01: a software for computing finite semigroups*. Laboratoire d’Informatique Algorithmique : Fondements et Applications (LIAFA), CNRS et Université Paris 7, 2.01 edition, 2009.

[8] Imre Simon. Limited subsets of a free monoid. *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, 0:143–150, 1978.