Empty-car Routing in Ridesharing Systems

Anton Braverman, J.G. Dai
Cornell University

Xin Liu, Lei Ying
Arizona State University

September 26, 2016

Abstract

This paper studies empty-car routing in modern ridesharing systems such as Didi Chuxing, Lyft, and Uber. We model a ridesharing system as a closed queueing network with three distinct features: empty-car routing, indiscrimination of passengers, and limited car supply. By routing empty cars we control car flow in the network to optimize a system-wide utility function, e.g., the availability of empty cars when a passenger arrives. Our model assumes that the destination of a passenger is not disclosed to a driver before the ride order is accepted (indiscrimination), and passenger arrival rates scale linearly with the total number of cars in the system (limited car supply). We establish the process-level and steady-state convergence of the queueing network to a fluid limit as the number of cars tends to infinity, and use this limit to study a fluid-based optimization problem. We prove that the optimal network utility obtained from the fluid-based optimization is an upper bound on the utility in the finite car system for any routing policy, both static and dynamic, under which the closed queueing network has a stationary distribution. This upper bound is achieved asymptotically under the fluid-based optimal routing policy. Simulation results with real-word data released by Didi Chuxing demonstrate the benefit of using the fluid-based optimal routing policy compared to various other policies.

1 Introduction

This paper studies the modeling and control of ridesharing systems such as Didi Chuxing, Lyft and Uber. We consider a system with \( r > 0 \) regions and \( N > 0 \) cars. The regions can be interpreted as geographic regions in a city and cars drive around between regions transporting passengers. At time \( t = 0 \), all cars start off idling empty in some region, waiting for a passenger. Passengers arrive to region \( i \) according to a Poisson process with rate \( N\lambda_i > 0 \), and arrivals to different regions are independent. When a passenger arrives to region \( i \), if there is an empty car available there, then the passenger occupies that car and travels to region \( j \) with probability \( P_{ij} \). We allow \( P_{ii} > 0 \) to represent trips within a region. Travel times from region \( i \) to \( j \) have mean \( 1/\mu_{ij} \) and are assumed to be i.i.d. exponential random variables, although we will see in Remark 9 in Section 4 that
the exponential assumption is non-essential. If no empty car is available, the passenger abandons the system and finds an alternative form of transportation to the destination. Once the passenger arrives at region \(j\), the car becomes empty. The empty car can either decide to stay in region \(j\) (and wait for a new passenger) with probability \(Q_{jj}\), or drive empty to a different region \(k\) and wait for a passenger there with probability \(Q_{jk}\). In general, the routing matrix (also called the routing policy) \(Q = (Q_{ij})\) is allowed to be state-dependent, i.e. \(Q\) may depend on the current distribution of cars in the system. We assume \(N, \lambda, \mu,\) and \(P = (P_{ij})\) are given, and seek to choose \(Q\) to optimize a system-wide utility function, e.g., to maximize system-wide availability of empty cars. To model these dynamics, we use a closed queueing network that belongs to the class of BCMP networks [4].

Because of the proliferation of ridesharing and bikesharing services, modeling and control of these systems have become important research topics over the last few years [3, 8, 13, 17, 19, 20, 21]. Our model focuses on three distinct features motivated by real-world ridesharing systems: empty-car routing, indiscrimination and limited car supply.

**Empty-car routing:** Recall that our model assumes that a driver, after dropping off a passenger at region \(i\), stays at region \(i\) with probability \(Q_{ii}\) or drives empty to region \(j\) with probability \(Q_{ij}\). The system is controlled via routing policy \(Q\). A number of existing models assume a vehicle can only be moved from one region to another when carrying a passenger [3, 20]. This is a realistic assumption for bikesharing systems, where a bike cannot autonomously move from one region to another, and only moves when a passenger rides it. In such a case, the performance of the system is largely determined by the passengers’ arrival rates and destination probabilities. Consider a simple example as shown in Figure 1, which consists of regions 1 and 2. Passengers arrive at region 1 to go to region 2 with a Poisson process with rate \(2M\) passengers/unit time, and arrive at region 2 to go to region 1 with rate \(M\) passenger/unit time for some \(M > 0\). We define the availability at region \(i\) to be the long-run fraction of time that there is at least one empty vehicle at the region available to serve passengers.

![Figure 1: A two-region example](image-url)
In a bike-sharing system, empty bikes cannot autonomously re-balance themselves. A bike taken from 1 to 2 will only return to 1 if it is brought there by a passenger. Since, on average, region 1 sees twice as many passengers as region 2, the availability of bikes at region 1 will always be approximately 50%, regardless of the number of bikes in the system. That is, region 1 will lose half of its passengers to alternative modes of transportation.

This inefficiency due to passenger imbalance has been well recognized in the literature. Proposed solutions include demand throttling via pricing [3, 20], or periodic bike rebalancing using trucks [6, 12]. Empty-car routing, however, is common in a commercial ridesharing system where drivers often wander around to find passengers.

Going back to our example, assume that there are a total of $3M$ cars in the system and that after dropping off passengers at region 2, cars drive empty to region 1 with probability $Q_{21} = 1/3$, and cars stay at 2 with probability $Q_{22} = 2/3$. After dropping off passengers at region 1, cars stay there with probability $Q_{11} = 1$. Assume also that the mean travel time in either direction is one unit. In this paper we analyze these empty-car routing decisions via fluid models. Leaving all the rigorous details to Section 2, the following is an informal illustration of how our fluid model is used. We treat the $3M$ cars as an infinitely divisible mass of fluid that circulates between regions. Let $\tilde{a}_i \in [0, 1]$ be the fluid approximation for the availability at station $i$. In the fluid model, the steady-state outflow of fluid from 1 is then $2M\tilde{a}_1$. The total inflow of fluid into 1 will be the sum of $Ma_2$ and $2M\tilde{a}_1Q_{21}$. The former represents the flow of passengers traveling from 2 to 1, and the latter represents the flow of cars that decide to drive empty from 2 to 1. In equilibrium, inflow to a region equals outflow, meaning that these rates must satisfy the flow-balance equation

$$2M\tilde{a}_1Q_{21} + Ma_2 = 2M\tilde{a}_1. \quad (1.1)$$

Furthermore, in this example flow rates also satisfy the capacity constraint

$$2M\tilde{a}_1Q_{21} + Ma_2 + 2M\tilde{a}_1 = 3M, \quad (1.2)$$

which states that the total fluid flow must equal $3M$. One can check that $\tilde{a}_1 = .75$ and $\tilde{a}_2 = 1$ is the unique solution to the above equations, suggesting that under this empty-car routing policy, regions 1 and 2 will have 75% and 100% availability, respectively. This static routing policy $Q = (Q_{ij})$ is clearly better than the no-routing policy in the bike-sharing setting, where region 1 only has 50% availability. Although $\tilde{a}_1$ and $\tilde{a}_2$ are merely fluid estimates of availability, we will see that they are asymptotically correct as $M \rightarrow \infty$. Table 1 shows simulations of the actual availabilities for the two routing strategies discussed above with $M = 400$ (a total of 1,200 cars in the system).

In the future, with autonomous cars, a ride-sharing platform will have full control of its empty cars and empty-car routing is likely to be one of primary methods the platform can use to optimize its system efficiency. This paper considers this important problem and provides structural insights on optimal empty-car routing policies. We realize that at present, companies such as Didi, Lyft and Uber do not have absolute centralized control
Empty-car routing probability | Availability
---|---
$Q_{21} = 1/3$ | Region 1: 73.21% Region 2: 97.56%
$Q_{21} = 0$ | Region 1: 49.99% Region 2: 100%

Table 1: Availabilities under two static empty-car routing policies when $M = 400$

over every single car. However, a certain degree of centralized control can be achieved using pricing schemes, or by limited sharing of passenger flow and arrival patterns with drivers to induce drivers to achieve the desired routing probabilities $Q_{ij}$. We leave it to future work to design decentralized incentive mechanisms that can achieve targeted routing probabilities. Even if they cannot fully achieve the optimal routing probabilities, our centralized policy approach provides companies with a best-case performance level for which they can strive.

**Indiscrimination:** We assume in our model that if a car decides to wait at region $i$, it will pick up the next available passenger regardless of the passenger’s destination. This assumption is motivated by the fact that many modern ridesharing platforms do not disclose the destination to a driver before the driver accepts the ride order, so that the passenger is not rejected because of the destination. As a result, controlling flow rates among origin-destination pairs (e.g. by pricing) is not feasible in our setting. Indiscrimination is also the de facto policy of the taxi industry where a passenger only reveals her destination after occupying a taxi, and the driver cannot reject the passenger because of the destination.

**Limited car supply:** In this paper, we consider the regime where passenger arrival rates grow linearly with the total number of cars $N$, i.e. the passenger arrival rate to region $i$ is $N\lambda_i$. The parameter $\lambda_i$ thus has the natural interpretation of rate of arriving passengers per car to region $i$. This allows our model to distinguish between regimes when there is an oversupply, undersupply, or critical level of supply of cars with respect to passenger demand. The latter two regimes may occur during morning or afternoon rush hours, and are precisely the regimes where an effective choice of routing matters the most. Figure 2 shows real-life ride-sharing traffic data released by Didi Chuxing. We see there the total number of passenger orders compared to the total number of accepted orders of two nearby regions during the afternoon rush hour (5PM to 6PM) over 21 consecutive days. From the figure, we can observe that there was a significant shortage of cars in region 47, pointing to a potential undersupply of cars in the system. Furthermore, there was a great disparity in car availability between the two regions, which illustrates the potential benefit of imposing an efficient car-routing policy.

This flexibility in parameter regimes is a distinguishing feature of our paper. Other cases where BCMP networks similar to ours were used to model ridesharing systems include [3, 8, 13, 21]. Although the mechanisms in those models are different from ours, a prevalent theme in those papers is posing a steady-state optimization problem to maximize some kind of system-wide performance measure. In most cases, solving the optimization problem
exactly turns out to be intractable. The normalizing constant of the stationary distribution of a BCMP network is prohibitively expensive to compute, as it grows combinatorially with the number of regions and cars in the system. As an alternative, [3, 13] consider a limiting optimization problem that arises as the number of cars in the system tends to infinity, and in [8], the authors find an approximate optimal solution that converges to the true optimal solution as the number of cars grows to infinity. In all these papers, the passenger arrival rate at each station is held fixed as the number of cars grows to infinity. This results in an asymptotic regime where some of the regions necessarily become bottleneck regions that accumulate an infinite number of empty cars, meaning that their availabilities tend to one. This regime is qualitatively different from ours, where the passenger arrival rate scales with the number of cars. In our regime, taking the number of cars to infinity does not guarantee the appearance of bottleneck regions. Hence, our model is much better suited to prescribe policies in the undersupplied regime, which is precisely the regime where a good policy is needed most. The only case where the exact optimization is efficiently solvable for a finite-sized system is in [21], but the constraints imposed there enforce equal availabilities across all regions. We will shortly demonstrate how enforcing this constraint can actually hurt overall system performance.

The papers closest to ours are [21] and [13], where the authors formulate the problem as a BCMP model, and seek to rebalance the system by routing empty cars between regions, although they use a slightly different mechanism for empty-car routing. The goal in those papers is to minimize the number of empty cars on the road while maintaining equal availabilities across all regions asymptotically. This constraint is motivated by the fact that in their scaling regime, there will always be at least one region where the availability tends to 100% as the number of cars in the system increases. Hence, maintaining equal availabilities among regions ensures that all regions will tend to 100% availability as the number of cars tends to infinity. In our model, this is achievable only when the system has an oversupply of cars. Enforcing equal availabilities can actually hurt system performance.
in the undersupplied regime. Consider again our example in Figure 1 with $3M$ cars in the system, and recall that by setting $Q_{21} = 1/3$ and $Q_{12} = 0$ we obtained availabilities of $\bar{a}_1 = .75$ and $\bar{a}_2 = 1$. One can show that to enforce equal availabilities in our fluid model, we would need to choose

$$Q_{11} = 1, Q_{12} = 0, Q_{22} = \frac{1}{2}, \text{ and } Q_{21} = \frac{1}{2},$$

which would result in $\bar{a}_1 = \bar{a}_2 = .75$, i.e. both regions now have 75% availability. Clearly this is less desirable. By allowing availabilities to vary between regions, our model allows one to consider a utility function that can be tailored to one’s need. For example, one may be interested in ensuring certain regions of a city are much better supplied by cars than others. During morning and evening rush hours, one may want to maintain 100% availability in the central business district of the city to ensure that daily commuters using the ridesharing service have a reliable means of transportation.

Finally, we wish to add that fluid models have been used in the past by $[17, 19]$ to study ridesharing networks. Those fluid models are different from the fluid model in this paper. Furthermore, they are only used as heuristics, and are not shown to be connected to an underlying stochastic system. In contrast, our fluid model is proven to be the limit of the queue length process of our BCMP network, and all fluid optimization problems considered are rigorously proven to be the limits of their stochastic counterparts.

To summarize, this paper studies ridesharing networks with empty-car routing, indiscrimination, and limited car supply. Since the stationary distribution of a finite size system is difficult to compute, we perform an asymptotic fluid analysis of the system as the number of cars $N \to \infty$. The main contributions of this paper are summarized below.

- Fixing a static empty-car routing policy $Q = (Q_{ij})$, we consider a fluid model associated with a closed queueing network composed of single and infinite server stations. We establish process level convergence of the scaled queue length process in our closed queueing network to a fluid limit. The fluid model’s equilibrium set is explicitly characterized, and we show that the fluid model converges to this equilibrium set from any initial starting condition. We then elevate the process level convergence result to convergence of steady-state distributions. See Section 4.

- To find an optimal static empty-car routing policy $Q^*$, we formulate a fluid-based optimization problem that is able to accommodate a broad class of utility functions. The latter can depend on availabilities at different regions, and fractions of both empty or occupied cars on different roads. Then $Q^*$ can be solved efficiently by solving a related problem with only linear constraints, cf. Lemma 2.

- We prove in Theorem 2 that as the number of car grows to infinity, the routing policy $Q^*$ from the fluid-based optimization is asymptotically optimal among all state dependent routing policies.
Using real-world network and passenger order data from a dataset released by Didi Chuxing, we simulated the performance of the proposed routing policy $Q^*$. We observe from simulation estimates that (a.) actual availability converges to the fluid-based solution at rate $1/\sqrt{N}$, (b.) the static routing policy $Q^*$ outperforms heuristic state-dependent routing policies.

The rest of the paper is structured as follows. In Section 2, we formulate the fluid-based optimization problem and state our main results, Theorems 1 and 2. In Section 3, we describe the numerical study performed using real-world data from Didi Chuxing, China’s largest ridesharing company. Section 4 is devoted to studying the fluid model of the ridesharing network, and establishing the machinery needed to prove our main results. Section 5 concludes.

1.1 Notation

For a function $f : \mathbb{R} \to \mathbb{R}^n$, we use $\dot{f}(t)$ to denote the derivative of $f(t)$. For any integer $n > 0$, we use $\mathbb{D}^n$ to denote the space of all cadlag functions $x : \mathbb{R}_+ \to \mathbb{R}^n$, i.e. functions that are right-continuous on $[0, \infty)$ with left limits on $(0, \infty)$. We define

$$\mathbb{D}^n_0 = \{ x \in \mathbb{D}^n : x(0) = 0 \},$$

$$\mathbb{D}^n_1 = \{ x \in \mathbb{D}^n : x(0) \in [0, 1]^n \text{ and } \sum_{i=1}^n x_i(0) = 1 \},$$

$$\mathbb{D}^n_{0+} = \{ x \in \mathbb{D}^n : x(0) \geq 0 \}.$$

For any $x \in \mathbb{D}^n$ and any $T > 0$, we define

$$\|x\|_T = \max_{1 \leq i \leq n} \sup_{0 \leq t \leq T} |x_i(t)| = \sup_{0 \leq t \leq T} \max_{1 \leq i \leq n} |x_i(t)|. \quad (1.4)$$

We let $C_0^n \subset \mathbb{D}^n$ be the subspace of continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^n$, and define $C_1^n$ analogously to $\mathbb{D}^n_1$. For any $x \in \mathbb{D}^n$, we write $\int_0^t x(s)\,ds$ to denote a vector in $\mathbb{R}^n$ whose $i$th component is $\int_0^t x_i(s)\,ds$. For a vector $x \in \mathbb{R}^n$, we use $|\cdot|$ to denote the max-norm, i.e. $|x| = \max_{1 \leq i \leq n} |x_i|$. For a set $A \subset \mathbb{Z}$, we write $|A|$ to denote the number of elements contained by this set.

2 The Ridesharing Optimization Problem

In this section we formally introduce the sequence of ridesharing networks discussed in the introduction. We then introduce the fluid-based optimization and state our main results, Theorems 1 and 2. We show that the fluid-based optimization can be solved efficiently by solving a related optimization problem with linear constraints, Lemma 2.
Recall the ridesharing network dynamics introduced in Section 1. For any time $t \geq 0$, let $E_{ij}^{(N)}(t)$ be the number of empty cars en route from region $i$ to region $j \neq i$, and let $E_{ii}^{(N)}(t)$ be the number of empty cars that are waiting in region $i$ for a new passenger. Similarly, let $F_{ij}^{(N)}(t)$ be the number of full cars driving from region $i$ to $j$. Observe that $F_{ii}^{(N)}(t)$ can be non-zero, because a passenger’s destination can be located in the same region as he was picked up. Let $Q_{ij}(t)$ and $P_{ij}(t)$ be the availability at region $i$ and the empty-car routing probability matrix at time $t$, respectively. Since we assumed that passengers arrive to regions according to independent Poisson processes, and that travel times between two regions are i.i.d. exponential random variables, the process $(\tilde{E}^{(N)}(t), \tilde{F}^{(N)}(t))$ is a continuous time Markov chain (CTMC). Go forward, we assume $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$ is irreducible under $P$ and $Q$. Since it takes values in a finite state space, it has a unique stationary distribution. Let $(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty)) \in \mathcal{T}$ be the random element having the stationary distribution of $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$. Then

$$A_i^{(N)} = \mathbb{P}(\tilde{E}_{ii}^{(N)}(\infty) > 0)$$

be the availability at region $i$, and let $A^{(N)}$ be the $r$-dimensional vector whose entries are $A_i^{(N)}$.

**Remark 1.** When $Q$ is not state-dependent, the process $(E^{(N)}, F^{(N)})$ can also be interpreted as the queue length process in a closed queueing network of $r$ single server stations and $2r^2 - r$ infinite server stations, where cars are the “customers” in the network. For
1 ≤ i ≤ r, the process \( E_{ii}^{(N)} = \{ E_{ii}^{(N)}(t), \ t ≥ 0 \} \) corresponds to a single server station with service rate \( N\lambda_i \), and

\[
E_{ij}^{(N)} = \{ E_{ij}^{(N)}(t), \ t ≥ 0 \}, \ 1 \leq i \neq j \leq r,
\]

\[
F_{ij}^{(N)} = \{ F_{ij}^{(N)}(t), \ t ≥ 0 \}, \ 1 \leq i, j \leq r,
\]
correspond to infinite server stations where the service rate of each server at station \( E_{ij}^{(N)} \) or \( F_{ij}^{(N)} \) is \( \mu_{ij} \). This network belongs to a class of closed queueing networks known as BCMP networks [4]. It is known that the queue length vector in BCMP networks has a product form stationary distribution. However, computing the normalization constant is prohibitively expensive because the state space grows combinatorially with the number of stations and customers in the network. Indeed, our case corresponds to a network with \( 2r^2 \) stations and \( N \) customers, and the state space has \( (N+2r^2-1) \) elements [9].

We are now ready to introduce the fluid-based optimization problem, and state our main results.

### 2.1 Main Results

Recall the network primitives \( \lambda, \mu, P \), and let \( q = (q_{ij}) \) be an \( r \times r \) matrix of dummy variables that represents a static empty-car routing policy \( Q \). Suppose \((\bar{e}, \bar{f}, \bar{a})\) is a point in \( \mathcal{T} \times [0,1]^r \). We consider the optimization problem

\[
\max_{q, \bar{e}, \bar{f}, \bar{a}} \sum_{i=1}^{r} c_i \lambda_i \bar{a}_i - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \bar{e}_{ij} \bar{e}_{ij} \quad \text{(2.2)}
\]

subject to

\[
\lambda_i P_{ij} \bar{a}_i = \mu_{ij} \bar{f}_{ij}, \quad 1 \leq i, j \leq r, \quad \text{(2.3)}
\]

\[
\mu_{ij} \bar{e}_{ij} = q_{ij} \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i, j \leq r, \quad j \neq i, \quad \text{(2.4)}
\]

\[
\lambda_i \bar{a}_i = \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki} + q_{ii} \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i \leq r, \quad \text{(2.5)}
\]

\[
(1 - \bar{a}_i) \bar{e}_{ii} = 0, \quad 1 \leq i \leq r, \quad \text{(2.6)}
\]

\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \bar{f}_{ij} + \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \bar{e}_{ij} + \sum_{i=1}^{r} \bar{e}_{ii} = 1, \quad \text{(2.7)}
\]

\[
0 \leq \bar{e}_{ij} \leq 1, \quad 0 \leq \bar{f}_{ij} \leq 1, \quad 0 \leq \bar{a}_i \leq 1, \quad 1 \leq i, j \leq r \quad \text{(2.8)}
\]

\[
q_{ij} \geq 0, \quad 1 \leq i, j \leq r, \quad \text{(2.9)}
\]

\[
\sum_{j=1}^{r} q_{ij} = 1, \quad 1 \leq i \leq r. \quad \text{(2.10)}
\]
In the optimization problem above, \( c_i > 0 \) are rewards for picking up a passenger at region \( i \), and \( \tilde{c}_{ij} > 0 \) are costs of sending empty cars from \( i \) to \( j \). The variables \( \bar{e}, \bar{f}, \) and \( \bar{a} \) represent \( \tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty) \) and \( \tilde{A}^{(N)} \) in the fluid model, respectively. It may not be apparent yet, but this optimization problem arises naturally from the fluid model of the ridesharing network. For this reason we refer to it as the fluid-based optimization problem. The interpretation of this is straight-forward. For region \( i \), we can interpret \( \lambda_i \bar{a}_i \) as the arrival rate of customers that successfully get a car. However, forcing cars to drive empty is wasteful, e.g. fuel is spent, and drivers do not like driving empty. So we seek to find the optimal empty-car routing policy \( q \) to maximize utility

\[
\sum_{i=1}^{r} c_i \lambda_i \bar{a}_i - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \tilde{c}_{ij} \bar{e}_{ij},
\]

which captures the tradeoff between revenue generation, and costs of empty-car routing. Constraints (2.3)–(2.5) are flow-balance constraints. Constraint (2.6) states that for region \( i \), either \( \bar{a}_i = 1 \) or \( \bar{e}_{ii} = 0 \), i.e. either availability is at 100% or the long-run fraction of empty cars at the station is zero. Additional intuition can be gained once the fluid model is introduced and its equilibrium behavior discussed in Section 4. Finally, although this optimization problem is stated for static empty-car routing policies, the connection to state-dependent policies will be made in Theorem 2.

The following are our main results. The first establishes the connection between the fluid-based optimization problem and \( (\tilde{E}^{(N)}, \tilde{F}^{(N)}) \). The second shows that asymptotically, the optimal static policy from the fluid-based optimization outperforms all state-dependent policies. Theorem 1 is proved at the end of Section 4, and Theorem 2 is proved in Appendix D.

**Theorem 1.** Let \( q, \bar{e}, \bar{f}, \bar{a} \) be a feasible solution to the optimization problem in (2.2)–(2.10). Set \( Q = q \). Assume \( P_{ij} > 0 \) for all \( 1 \leq i, j \leq r \) and \( q_{ii} > 0 \) for all \( 1 \leq i \leq r \). Then

\[
\tilde{E}^{(N)}(\infty) \Rightarrow \bar{f},
\]

\[
\tilde{E}_{ij}^{(N)}(\infty) \Rightarrow \bar{e}_{ij}, \quad 1 \leq i \neq j \leq r,
\]

\[
\tilde{E}_{ii}^{(N)}(\infty) \Rightarrow 0, \quad \text{for } i \text{ such that } \hat{a}_i < 1,
\]

\[
\sum_{i: \bar{a}_i = 1} \tilde{E}_{ii}^{(N)}(\infty) \Rightarrow \sum_{i: \hat{a}_i = 1} \hat{e}_{ii},
\]

and

\[
\mathbb{P}(\hat{E}_{ii}^{(N)}(\infty) > 0) \rightarrow \hat{a}_i, \quad 1 \leq i \leq r,
\]

as \( N \rightarrow \infty \).
Remark 2. The assumption that $P_{ij} > 0$ for all $i, j$ is made to facilitate exposition in the proof of Theorem 4, which plays a central role in establishing Theorem 1. We expect that Theorem 4, and hence Theorem 1, holds when $P$ is just irreducible. The assumption $q_{ii} > 0$ for all $i$ means that at every region, a driver has a positive probability to stay at the region after dropping off a passenger, which is a very reasonable assumption. Moreover, in our numerical study in Section 3, we have observed that all of the optimal $q_{ii}$’s were far from zero.

Theorem 2. (a) Suppose $(\tilde{E}(N), \tilde{F}(N))$ is irreducible under $P$ and $Q$, where $Q$ is a state-dependent empty-car routing policy. Let $(q^*, \tilde{e}^*, \tilde{a}^*)$ be an optimal solution of the optimization problem in (2.2)–(2.10). Then

$$\sum_{i=1}^{r} c_i \lambda_i A_i^{(N)} - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \tilde{c}_{ij} E[\tilde{E}_{ij}(\infty)] \leq \sum_{i=1}^{r} c_i \lambda_i \tilde{a}_{ii}^{*} - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \tilde{c}_{ij} \tilde{e}_{ij}^{*}, \quad N > 0.$$ 

(b) Let $(\tilde{E}(N)^*, \tilde{F}(N)^*)$ denote the CTMC under the static routing policy $q^*$. If $P_{ij} > 0$ for all $1 \leq i, j \leq r$ and $q_{ii}^* > 0$ for all $1 \leq i \leq r$, then

$$\lim_{N \to \infty} \sum_{i=1}^{r} c_i \lambda_i A_i^{(N)^*} - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \tilde{c}_{ij} E[\tilde{E}_{ij}^{(N)^*}(\infty)] = \sum_{i=1}^{r} c_i \lambda_i \tilde{a}_{ii}^{*} - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \tilde{c}_{ij} \tilde{e}_{ij}^{*}.$$ 

Remark 3. Part (a) of Theorem 2 states that the optimal value of the fluid-based optimization problem (2.2)–(2.10) is an upper bound on the expected system utility of the system with $N$ cars under any state-dependent routing policy under which the CTMC is irreducible. Part (b) states that the upper bound is asymptotically achievable under the static routing policy $q^*$ if $P_{ij} > 0$ for all $1 \leq i, j \leq r$ and $q_{ii}^* > 0$ for all $1 \leq i \leq r$.

Having established the relevance of the fluid-based optimization, we now discuss how to solve it efficiently.

2.2 Efficient Solution of the Fluid-Based Optimization

The main issue is that many of the constraints in (2.3)–(2.10) are non-linear. In this section we show that a solution to the fluid-based optimization can be found by solving a related problem with only linear constraints. Our first step is the following lemma, which removes all but one of the non-linear constraints.
Lemma 1. Consider the set of constraints

\[ \lambda_i \mu_{ij} \bar{a}_i = \mu_{ij} \bar{f}_{ij}, \quad 1 \leq i, j \leq r, \]  
(2.17)

\[ \mu_{ij} \bar{e}_{ij} \leq \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i, j \leq r, \; j \neq i, \]  
(2.18)

\[ \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki} \leq \lambda_i \bar{a}_i \leq \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki} + \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i \leq r, \]  
(2.19)

\[ \lambda_i \bar{a}_i + \sum_{j=1, j \neq i}^{r} \mu_{ij} \bar{e}_{ij} = \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki} + \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i \leq r, \]  
(2.20)

\[ \sum_{j=1}^{r} \sum_{j=1, j \neq i}^{r} \bar{f}_{ij} + \sum_{j=1, j \neq i}^{r} \bar{e}_{ij} + \sum_{i=1}^{r} \bar{e}_{ii} = 1, \]  
(2.21)

\[ 0 \leq \bar{e}_{ij} \leq 1, \quad 0 \leq \bar{f}_{ij} \leq 1, \quad 0 \leq \bar{a}_i \leq 1, \quad 1 \leq i, j \leq r, \]  
(2.22)

\[ (1 - \bar{a}_i) \bar{e}_{ii} = 0, \quad 1 \leq i \leq r. \]  
(2.23)

A point \((\bar{e}, \bar{f}, \bar{a}) \in T \times [0,1]^r\) satisfies (2.17)–(2.23) if and only if there exists an \(r \times r\) matrix \(q\) such that \((\bar{e}, \bar{f}, \bar{a})\) and \(q\) satisfy (2.3)–(2.10).

Proof. Given \((\bar{e}, \bar{f}, \bar{a})\) and \(q\) that satisfy (2.3)–(2.10), it can be easily verified that \((\bar{e}, \bar{f}, \bar{a})\) satisfies conditions (2.17)–(2.23) based on the fact that \(0 \leq q_{ij} \leq 1\), which proves the “if” part.

Now given \((\bar{e}, \bar{f}, \bar{a})\) that satisfies (2.17)–(2.23), we define

\[ q_{ij} = \frac{\mu_{ij} \bar{e}_{ij}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}} \]  
(2.24)

\[ q_{ii} = \frac{\lambda_i \bar{a}_i - \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}}. \]  
(2.25)

Then conditions (2.4) and (2.5) hold according to the definition of \(q\). Furthermore, \(q_{ij} \geq 0\) because \(\mu_{ij} \geq 0\), \(\bar{e}_{ij} \geq 0\), and \(\lambda_i \bar{a}_i \geq \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki}\) by (2.19). Finally,

\[ \sum_{j=1, j \neq i}^{r} q_{ij} = \sum_{j=1, j \neq i}^{r} \frac{\mu_{ij} \bar{e}_{ij}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}} + \frac{\lambda_i \bar{a}_i - \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}} \]  

\[ = \frac{\lambda_i \bar{a}_i + \sum_{j=1, j \neq i}^{r} \mu_{ij} \bar{e}_{ij} - \sum_{k=1, k \neq i}^{r} \mu_{ki} \bar{e}_{ki}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}} \]  

\[ = \frac{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}}{\sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}} = 1, \]  
(a)
where equality (a) is obtained based on (2.20). Therefore, (2.10) holds and we can conclude that \((\bar{e}, \bar{f}, \bar{a})\) and our newly defined \(q\) satisfy (2.3)–(2.10), which completes the proof of the “only if” part.

With the help of this lemma, the fluid-based optimization problem can be rewritten as

\[
\max_{\bar{e}, \bar{f}, \bar{a}} \sum_{i=1}^{r} c_i \lambda_i \bar{a}_i - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \tilde{c}_{ij} \bar{e}_{ij} \tag{2.26}
\]

subject to: (2.17) – (2.23). \tag{2.27}

Observe that (2.17)–(2.22) are all linear constraints, and that only (2.23) is non-linear. The following result says that we can safely ignore (2.23). It is proved in Appendix C.

**Lemma 2.** Consider the relaxed optimization problem

\[
\max_{\bar{e}, \bar{f}, \bar{a}} \sum_{i=1}^{r} c_i \lambda_i \bar{a}_i - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \tilde{c}_{ij} \bar{e}_{ij} \tag{2.28}
\]

subject to: (2.17) – (2.22). \tag{2.29}

There is always an optimal solution to (2.28)–(2.29) that also satisfies (2.23), implying it is an optimal solution to the optimization problem (2.26)–(2.27). Hence, it is also an optimal solution to the original fluid-based optimization.

The relaxed optimization problem (2.28)–(2.29) has \((4r^2 + 3r + 1)\) linear constraints, and can be solved efficiently using existing methods. In the proof of Lemma 2 we will see that given any optimal solution, finding another one that also satisfies (2.23) can be done very easily. Therefore, the problem of finding an optimal solution to (2.26)–(2.27) can be solved efficiently. Once we have an optimal solution, we can recover the optimal routing policy based on (2.24)–(2.25).

**Remark 4.** So far, we only considered the utility function defined in (2.11). It can be easily verified that both Theorem 2 and Lemma 2 hold for a general utility function \(U(\bar{e}, \bar{f}, \bar{a})\) that satisfies the following conditions:

(a) nondecreasing in \(\bar{a}_i\) for all \(i\),
(b) nondecreasing in \(\bar{f}_{ij}\) for all \(i\) and \(j\),
(c) nonincreasing in \(\bar{e}_{ij}\) for all \(i \neq j\),
(d) independent of \(\bar{e}_{ii}\) for all \(i\), and
(e) concave in \((\bar{e}, \bar{f})\).

In the next section we describe a numerical study based on real-world ridesharing data.
3 Numerical Study

In this section, we perform a numerical study to evaluate the performance of our fluid model. We use a data set released in the Di-Tech Challenge by the Didi Research Institute [http://research.xiaojukeji.com/index_en.html](http://research.xiaojukeji.com/index_en.html). Didi Chuxing is the largest ride-sharing service in China. We first describe the data and provide some statistics, and then use it to generate a realistic 9-region network on which we evaluate the performance of our fluid-based optimal routing policy.

The data set contains individual order information for trips taken between January 1, 2016 until January 21, 2016, in an unspecified city in China. The city is partitioned into a number of distinct geographical regions. The data is divided into 10-minute time slots, i.e. 144 times slots per day. Each data entry represents a single order. An order is a passenger request for a car, and may or may not be fulfilled due to lack of cars in proximity. A single data entry contains information about the origin and intended destination of the order, a time-stamp of when the order was made, whether the order was fulfilled, and in the case when the order was fulfilled, the ID of the car fulfilling the order, as well as the total trip price.

We used the data set to extract a nine-region network as shown in Figure 3. Although the data set contained more regions, we only used 9 to keep simulations, which appear later in this section, tractable. The nodes in Figure 3 represent the regions, and the edge-weights are the average trip price, in CNY per trip, between the regions. We purposely excluded edges with trip price over 18 CNY to help visualize which regions are close to each other.

![Figure 3: The nine-region network extracted from the DiDi dataset](image)

Figure 4 shows order fulfillment levels for each of the nine regions during the 5PM-6PM evening rush hour, plotted over 21 days. Each point represents the total number of orders received, and orders fulfilled during that one hour window. We can see that three of the nine regions, regions 13, 47, and 50, had significant supply shortages in most of the 21
days, while most orders in the remaining six regions were fulfilled. Unfortunately, our data did not permit us to deduce the surplus of drivers in those six regions. However, these figures clearly illustrate the significance of a good empty-car routing policy.

![Graphs showing number of orders vs. days for different regions](image)

**Figure 4:** The gaps between the number of passenger orders and the number of fulfilled orders of the nine regions

In addition to these two figures, we chose data from a single day and used it to obtain realistic parameters $N, \mu, \lambda, \text{ and } P$ for our 9-region network. To calculate $P_{ij}$, we tallied the total number of orders from $i$ to $j$, and divided by the total number of orders originating
at \( i \). The result is the matrix \( P \), which equals

\[
\begin{pmatrix}
10 & 0.2789 & 0.2585 & 0.3540 & 0.0069 & 0.0204 & 0.0543 & 0 & 0.0135 & 0.0135 \\
11 & 0.0412 & 0.6530 & 0.1617 & 0.0073 & 0.0735 & 0.0309 & 0.0103 & 0.0103 & 0.0117 \\
18 & 0.1514 & 0.2759 & 0.3161 & 0.0153 & 0.0460 & 0.1187 & 0.0192 & 0.0249 & 0.0325 \\
13 & 0 & 0.0073 & 0.0029 & 0.1798 & 0.0439 & 0.1945 & 0.1155 & 0.0936 & 0.3625 \\
19 & 0.0037 & 0.0823 & 0.0220 & 0.0366 & 0.2543 & 0.3312 & 0.2342 & 0.0119 & 0.0238 \\
27 & 0.0042 & 0.0218 & 0.0331 & 0.0880 & 0.1233 & 0.4507 & 0.1451 & 0.0444 & 0.0894 \\
45 & 0.0010 & 0.0270 & 0.0125 & 0.0694 & 0.1542 & 0.2360 & 0.4238 & 0.0202 & 0.0559 \\
47 & 0.0045 & 0.0064 & 0.0268 & 0.0771 & 0.0102 & 0.0873 & 0.0249 & 0.4092 & 0.3537 \\
50 & 0.0014 & 0.0027 & 0.0084 & 0.1102 & 0.0081 & 0.0695 & 0.0316 & 0.1837 & 0.5845
\end{pmatrix}
\]

To calculate \( \mu \), we used the average trip cost as a proxy for travel times, since travel times are not provided in the data set. Trip costs are a reasonable proxy because the price of a trip is typically a linear function of distance traveled, and time spent in car. To estimate \( \mu_{ij} \), we first calculated the average trip cost between regions \( i \) and \( j \), and then set the average travel time to equal the average trip cost. Since our time unit is a time-slot, which is 10-minute interval, we set \( 1/\mu_{ij} \) to equal the average trip cost divided by 10. For example, the average trip cost between region 47 and region 50 is 14.1 Chinese Yuan (CNY), so we assumed the average travel time is 14 minutes, and set \( 1/\mu_{47,50} = 1.41 \). The resulting matrix \( \mu \) is

\[
\begin{pmatrix}
10 & 0.791 & 1.683 & 0.982 & 3.732 & 2.717 & 2.459 & 3.958 & 2.987 & 3.803 \\
11 & 1.667 & 0.796 & 1.169 & 3.073 & 1.232 & 1.954 & 2.815 & 3.020 & 4.031 \\
18 & 0.982 & 1.183 & 0.718 & 2.723 & 1.425 & 1.393 & 2.911 & 2.138 & 2.932 \\
13 & 3.614 & 3.081 & 2.698 & 0.891 & 1.268 & 1.216 & 1.323 & 1.628 & 1.538 \\
19 & 2.625 & 1.272 & 1.424 & 1.367 & 0.771 & 0.998 & 1.322 & 2.635 & 2.703 \\
27 & 2.459 & 1.985 & 1.405 & 1.246 & 0.934 & 0.752 & 1.657 & 1.658 & 2.132 \\
45 & 3.795 & 2.831 & 2.863 & 1.254 & 1.270 & 1.605 & 0.850 & 2.936 & 2.821 \\
47 & 2.966 & 3.075 & 2.187 & 1.747 & 2.693 & 1.662 & 3.146 & 0.863 & 1.419 \\
50 & 3.769 & 4.071 & 3.008 & 1.468 & 2.634 & 2.072 & 2.766 & 1.396 & 0.957
\end{pmatrix}
\]

To determine the arrival rate to region \( i \), we counted the average number of orders to region \( i \) per time slot. This gave us an estimate \( N\lambda_i \). Our data did not provide the exact number of cars in the network \( N \). To determine a reasonable choice for \( N \), we summed up the number of fulfilled orders across all 9 regions in Figure 4. As a result, we chose \( N = 2000 \). Although not exact, this number is of the correct order of magnitude. Hence,
we set our vector \( \lambda \) to

\[
\begin{pmatrix}
\text{Region ID (i)} & \lambda_i \\
10 & 0.0122 \\
11 & 0.0567 \\
18 & 0.0435 \\
13 & 0.057 \\
19 & 0.0911 \\
27 & 0.118 \\
45 & 0.0865 \\
47 & 0.131 \\
50 & 0.248
\end{pmatrix}
\]

In the remainder of this section we evaluate our fluid-based optimization problem based on the parameters extracted from the data set. Going forward, we will be using the utility function

\[
U(\bar{e}, \bar{f}, \bar{a}) = U(\bar{a}) = \frac{\sum_{i=1}^{r} \bar{a}_i \lambda_i}{\sum_{i=1}^{r} \lambda_i}. \tag{3.1}
\]

This can be thought of as the probability that a passenger requesting a ride at any region is fulfilled. We first examine how fast the convergence in Theorem 1 occurs, and then compare our optimal fluid-based routing policies to some heuristic state-dependent policies.

### 3.1 Convergence

Let \( Q^* \) be the optimal empty-car routing matrix obtained when solving the fluid-based optimization problem with the utility function in (3.1), and let \( a^* = U(\bar{a}^*) \) be the associated optimal system-wide availability. Let \( a^{(N)} = U(A^{(N)}) \) be the system-wide availability for the finite sized system with \( N \) cars, which uses \( Q^* \) as the empty-car routing policy. Figure 5 plots \( a^* \) and \( a^{(N)} \) as \( N \) is varied from 100 to 5,000. Convergence appears to be happening at a rate of \( 1/\sqrt{N} \).

### 3.2 Performance Comparison with Dynamic Routing Policies

In Theorem 2, we showed that the expected utility under any state-dependent routing policy for the finite-sized system is upper bounded by the optimal utility of the fluid-based optimization. However, the question remains open whether a state-dependent routing policy can outperform the static routing policy \( Q^* \) for a finite-sized system. It is impossible to consider all possible dynamic routing policies, so we focus on the following intuitive heuristic motivated by the join-the-shortest-queue policy.

**Join-the-Least-Congested-Region with Threshold \( \eta \) (JLCR-\( \eta \)):** When a car drops
off a passenger at region \( i \) at time \( t \), the driver stays at region \( i \) if
\[
(1 - \eta) \frac{\sum_k E_{ki}^{(N)}(t)}{\lambda_i} \leq \min_{j=1, j \neq i} \frac{\sum_k E_{kj}^{(N)}(t)}{\lambda_j}.
\] (3.2)

Otherwise, the driver drives empty to region \( j^* \), where
\[
j^* \in \arg \min_{j=1, j \neq i} \frac{\sum_k E_{kj}^{(N)}(t)}{\lambda_j}.
\] (3.3)

Ties are broken uniformly at random.

To understand the JLCR-\( \eta \) policy, we note that \( \sum_k E_{ki}^{(N)}(t) = E_{ii}^{(N)}(t) + \sum_{k \neq j} E_{ki}^{(N)}(t) \) is the number of empty cars both currently waiting and en-route to region \( i \). Therefore, \( \frac{\sum_k E_{kj}^{(N)}(t)}{\lambda_j} \) is a measure of congestion, in terms of empty cars, at region \( i \). When \( \eta = 0 \), the policy routes empty cars to the least congested region. However, such a policy can be wasteful if congestion levels among regions are similar, because it takes time for a car to go from one region to another. We therefore introduce the threshold \( \eta \) such that a driver drives empty from \( i \) to \( j \) only if the difference in congestion levels surpasses \( \eta \frac{\sum_k E_{kj}^{(N)}(t)}{\lambda_j} \). Figure 6 illustrates the system-wide availability in the network with 2,000 cars under JLCR-\( \eta \), with \( \eta \) ranging from 0 to 1. For the system in Figure 6, the optimal threshold level is \( \eta^* = 0.5 \).
Figure 6: System-wide availability with 2,000 cars under JLCR-\(\eta\).

Figure 7 compares the static routing policy under \(Q^*\) to JLCR-\(\eta\) with different values of \(\eta\) in the nine-region network. In particular, we included JLCR policies with

- \(\eta = 0\) : Under JLCR-0, an empty car always goes to the least congested region.
- \(\eta = 1\) : Under JLCR-1, after a car drops off a passenger, it always stays at its current region.
- \(\eta = 0.5\) : JLCR-0.5 maximizes system-wide availability among all JLCR-\(\eta\) when \(N = 2,000\).

A few remarks are in order. The figure confirms that static routing with \(Q^*\) outperforms the JLCR-\(\eta\) family of policies. However, a typical quality of state-dependent policies is robustness to system parameters. In our case, computing \(Q^*\) requires knowledge of \(\lambda, \mu,\) and \(P\), whereas a JLCR-1/2 only requires knowledge of \(\lambda\). Robustness to parameters is a particularly important quality when one only has noisy observations of the true parameters, or when the true parameters change over time. The objective of this paper is not to pursue optimal state-dependent policies. Rather, it is to establish an initial, rigorous theoretical foundation for the study of ridesharing networks. Our optimal static policy can then be used as a benchmark against which one compares the performance of other routing policies. As a case in point, had Figure 7 not included the performance of static-routing under \(Q^*\), it would have been impossible to say whether JLCR-\(\eta^*\) was a good policy or not.

In the next section we introduce the fluid model together with the tools needed to prove Theorems 1 and 2.
Figure 7: Performance comparison between the static routing and JLCR in the four-region network

4 The Fluid Model

This section is devoted to understanding fluid model of the ridesharing network in Section 2. We first introduce the fluid model for static empty-car routing matrices $Q$, and establish process-level convergence of $(\bar{E}^{(N)}, \bar{F}^{(N)})$ to it as $N \to \infty$. We then characterize the fluid model’s set of equilibria, and show that the fluid model converges to this set from any starting condition. In particular, this equilibrium behavior motivates the fluid-based optimization in Section 2. We then elevate the process-level convergence result to convergence of steady-state distributions to conclude Theorem 1.

Recall the primitive parameters $\lambda, \mu, P$, and assume a static empty-car routing matrix $Q$ is given. Recall the set $T$ defined in (2.1). Let $I^{(N)}_i(t) = \int_0^t 1(E^{(N)}_{ii}(s) = 0) ds$ be the cumulative idle time of the single-server station corresponding to $E^{(N)}_{ii}$. The following is a special case of Theorem 5 in Appendix A.

Theorem 3. Assume $(\bar{E}^{(N)}(0), \bar{F}^{(N)}(0)) \Rightarrow (e(0), f(0)) \in T$ as $N \to \infty$. Let $(e, f)$:
\[
\mathbb{R}_+ \to T \text{ and } u : \mathbb{R}_+ \to \mathbb{R}_+^r \text{ be the unique solution to the dynamical system}
\]

\[
f_{ij}(t) = f_{ij}(0) + \lambda_i P_{ij}(t - u_i(t)) - \mu_{ij} \int_0^t f_{ij}(s) ds, \quad 1 \leq i, j \leq r, \quad (4.1)
\]
\[
e_{ij}(t) = e_{ij}(0) - \mu_{ij} \int_0^t e_{ij}(s) ds + Q_{ij} \sum_{k=1}^r \mu_{ki} \int_0^t f_{ki}(s) ds, \quad 1 \leq i \neq j \leq r, \quad (4.2)
\]
\[
e_{ii}(t) = e_{ii}(0) - \lambda_i (t - u_i(t)) + \sum_{j=1, j \neq i}^r \mu_{ji} \int_0^t e_{ji}(s) ds + Q_{ii} \sum_{j=1}^r \mu_{ji} \int_0^t f_{ji}(s) ds, \quad 1 \leq i \leq r, \quad (4.3)
\]

\[u(t) \text{ is non-decreasing with } u(0) = 0, \text{ and } \int_0^\infty e_{ii}(s) du_i(s) = 0 \quad \text{for all } 1 \leq i \leq r. \quad (4.4)\]

Then for all \( T \geq 0, \)
\[
\lim_{N \to \infty} \| (\tilde{E}^{(N)}, \tilde{F}^{(N)}, I^{(N)}) - (e, f, u) \|_T = 0 \quad (4.5)
\]
almost surely.

**Remark 5.** In Theorem 1, we assumed that \( P_{ij} > 0 \) for all \( i, j \) and \( Q_{ii} > 0 \) for all \( i \). However, we will see in Appendix A that Theorem 3 does not require any special assumptions on \( P \) or \( Q \).

We refer to \( (e(t), f(t), u(t)) \) as the fluid model corresponding to \( (\tilde{E}^{(N)}, \tilde{F}^{(N)}, I^{(N)}) \). Note that \( P_{ij} = 0 \) implies \( f_{ij}(t) \equiv 0 \), and \( Q_{ij} = 0 \) for \( i \neq j \) implies \( e_{ij}(t) \equiv 0 \). It will come in handy later on to know that, \( e(t), f(t), \) and \( u(t) \) are Lipschitz continuous. To see why, observe that for any \( \epsilon > 0, \) (4.5) says that we can choose \( N \) large enough such that
\[
|u(t) - u(s)| \leq |u(t) - I^{(N)}(t)| + |I^{(N)}(t) - I^{(N)}(s)| + |I^{(N)}(s) - u(s)|
\]
\[
\leq \epsilon + |t - s| + \epsilon,
\]
where in the second inequality we used the definition of \( I^{(N)} \) to get
\[
|I^{(N)}(t) - I^{(N)}(s)| \leq |t - s|, \quad 0 \leq s, t < \infty.
\]
Hence,
\[
|u(t) - u(s)| \leq |t - s|, \quad 0 \leq s, t < \infty. \quad (4.6)
\]
Combining (4.1)–(4.3) with (4.6) and the fact that \( (e(t), f(t)) \) is bounded, we deduce that both \( e(t) \) and \( f(t) \) are also Lipschitz-continuous. The following section considers the equilibrium behavior of the fluid model.
4.1 Equilibrium Points

We first characterize the set of equilibria of the fluid model. Then we show, in Theorem 4, that the fluid model converges to this equilibrium set from any initial location. We combine this result together with the process-level convergence from Theorem 3 to establish steady-state convergence in Theorem 1.

We say \((\bar{e}, \bar{f}) \in \mathcal{T}\) is an equilibrium point of \((e(t), f(t))\) if
\[
(e(0), f(0)) = (\bar{e}, \bar{f}) \implies (e(t), f(t)) = (\bar{e}, \bar{f}), \quad t \geq 0.
\]

Let
\[
\mathcal{E} = \{ (\bar{e}, \bar{f}) \in \mathcal{T} : (\bar{e}, \bar{f}) \text{ is an equilibrium point of } (e(t), f(t)) \}\]
(4.7)

We purposely excluded \(u(t)\) from the definition of an equilibrium above, because we will see that it does not always have an equilibrium (but \(\dot{u}(t)\) does). We now show that an equilibrium always exists but is not always unique, and we will give a characterization all equilibria in terms of the system primitives \((\lambda, \mu, P, Q)\). Recall from the discussion below Theorem 3 that \(\dot{e}(t), \dot{f}(t), \text{ and } \dot{u}(t)\) exist for almost all \(t \geq 0\). Differentiating (4.1)–(4.3), we get
\[
\dot{f}_{ij}(t) = \lambda_i P_{ij} (1 - \dot{u}_i(t)) - \mu_{ij} f_{ij}(t), \quad 1 \leq i, j \leq r, \tag{4.8}
\]
\[
\dot{e}_{ij}(t) = -\mu_{ij} e_{ij}(t) + Q_{ij} \sum_{k=1}^{r} \mu_{ki} f_{ki}(t), \quad 1 \leq i \neq j \leq r, \tag{4.9}
\]
\[
\dot{e}_{ii}(t) = -\lambda_i (1 - \dot{u}_i(t)) + \sum_{j=1}^{r} \mu_{ji} e_{ji}(t) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} f_{ji}(t), \quad 1 \leq i \leq r. \tag{4.10}
\]

To characterize the equilibrium points, we set the left hand sides above to zero to obtain the system of equations
\[
\lambda_i P_{ij} \bar{a}_i = \mu_{ij} \bar{f}_{ij}, \quad 1 \leq i, j \leq r, \tag{4.11}
\]
\[
\mu_{ij} \bar{e}_{ij} = Q_{ij} \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i \leq r, \quad j \neq i, \tag{4.12}
\]
\[
\lambda_i \bar{a}_i = \sum_{k=1}^{r} \sum_{j \neq i}^{r} \mu_{ki} \bar{e}_{ki} + Q_{ii} \sum_{k=1}^{r} \mu_{ki} \bar{f}_{ki}, \quad 1 \leq i \leq r, \tag{4.13}
\]
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \bar{f}_{ij} + \sum_{i=1}^{r} \sum_{j \neq i}^{r} \bar{e}_{ij} + \sum_{i=1}^{r} \bar{e}_{ii} = 1. \tag{4.14}
\]
The last equation, (4.14), follows from the fact the total amount of fluid in the system is conserved. The quantity $\bar{a}_i \in [0, 1]$ is a placeholder for $1 - \dot{u}_i(t)$, and represents the equilibrium server utilization at the station corresponding to $e_{ii}$. In addition to (4.11)–(4.14), $\bar{a}_i$ satisfies

$$
(1 - \bar{a}_i)\bar{e}_{ii} = 0, \quad 1 \leq i \leq r.
$$

To justify (4.15), we apply Lipschitz continuity of $u(t)$ to (4.4) to see that

$$
\int_0^\infty e_{ii}(s)du_i(s) = \int_0^\infty e_{ii}(s)\dot{u}_i(s)ds = 0.
$$

We know that $e_{ii}(t) \geq 0$. Furthermore, $\dot{u}_i(t) \geq 0$ because $u(t)$ is an increasing function. Therefore, for all $1 \leq i \leq r$ and all $t \geq 0$ where $\dot{u}_i(t)$ exists,

$$
e_{ii}(t)\dot{u}_i(t) = 0.
$$

We now manipulate (4.11)–(4.15) to solve for $\bar{e}$, $\bar{f}$, and $\bar{a}$, where $\bar{a}$ is the $r$-dimensional vector whose components are $\bar{a}_i$. Substituting (4.11) and (4.12) into (4.13), we obtain

$$
\lambda_i\bar{a}_i = \sum_{\ell=1}^r Q_{\ell i} \sum_{k=1}^r \mu_{\ell k} \bar{f}_{k \ell} + Q_{ii} \sum_{k=1}^r \mu_{k i} \bar{f}_{k i} = \sum_{\ell=1}^r Q_{\ell i} \sum_{k=1}^r \lambda_k P_{k \ell} \bar{a}_k
$$

$$
= \sum_{k=1}^r \lambda_k \bar{a}_k \sum_{\ell=1}^r P_{k \ell} Q_{\ell i}, \quad 1 \leq i \leq r.
$$

In matrix form, these equations can be written as

$$
(I - B)\Lambda \bar{a} = 0,
$$

where $I$ is the $r \times r$ identity matrix, and $B$ and $\Lambda$ are $r \times r$ matrices defined as

$$
B_{ij} = \sum_{\ell=1}^r P_{j \ell} Q_{\ell i}, \quad \text{and} \quad \Lambda = \text{diag}(\lambda).
$$

Observe that $B$ is a column stochastic matrix, i.e. columns sum to one. We now argue that $B$ is irreducible because the CTMC $(E^{(N)}, F^{(N)})$ is. For any $1 \leq i, j \leq r$, the entry $B_{ij}$ is the probability that a car picks up a passenger at region $i$, drives him to some region $k$, and then decides to drive empty to region $j$ to wait for a new passenger there (or stay and wait at region $j$ if $k = j$). Therefore, $B$ can be interpreted as the transition probability matrix of a discrete-time Markov chain (DTMC) that describes the motion of a single car in a network, i.e. how it serves passengers and makes routing decisions, as if travel times were zero. Irreducibility of $B$ then means that starting from any region the car in the
DTMC can visit any other region, which is clearly satisfied when \( P_{ij} > 0 \) and \( Q_{ii} > 0 \) for all \( i, j = 1, \ldots, r \).

Since \( B \) is column stochastic and irreducible, (4.17) has a unique solution up to a multiplicative constant. That is, any solution to (4.17) must be of the form \( \bar{a}a^* \geq 0 \), where \( c > 0 \), and \( a^* \geq 0 \) is a unique vector in \( \mathbb{R}^r_+ \). To determine \( \bar{a} \), we choose the constant \( c > 0 \) so that (4.14) is satisfied and set \( \bar{a} = ca^* \). The following Lemma confirms that solutions to (4.11)\textendash(4.15) are indeed satisfy our definition of an equilibrium point. It is proved at the end of Appendix A.1.

**Lemma 3.** Assume \((\bar{e}, \bar{f}) \in \mathcal{T} \) and \( \bar{a} \in [0,1]^r \) satisfy (4.11)\textendash(4.15), and \( P_{ij} > 0 \) and \( Q_{ii} > 0 \) for all \( i, j = 1, \ldots, r \). Then \((\bar{e}, \bar{f})\) is an equilibrium of \((e(t), f(t))\).

**Remark 6.** From (4.11)\textendash(4.15), it is clear that \( \bar{a} \) uniquely determines \( \bar{f}, \bar{e}_{ij} \) for \( i \neq j \), and \( \bar{e}_{ii} \) for \( i \) such that \( \bar{a}_i < 1 \). Define the quantity \( \bar{m} \geq 0 \) to be

\[
\bar{m} = \sum_{i: \bar{a}_i = 1} \bar{e}_{ii} = 1 - \sum_{i=1}^{r} \sum_{j=1}^{r} \bar{f}_{ij} - \sum_{i=1}^{r} \sum_{j=1}^{r} \bar{e}_{ij} \tag{4.19}
\]

is unique. If \( \bar{m} > 0 \) and more than one element of \( \bar{a} \) equals one, then the equilibrium is not unique. To see why, suppose \( \bar{a}_i = 1 \) and \( \bar{a}_j = 1 \) for \( i \neq j \). Then provided \( \bar{m} > 0 \), there is an uncountable number of choices for \( \bar{e}_{ii} \) and \( \bar{e}_{jj} \) that satisfy (4.15) and (4.19). However, \( \bar{f}, \bar{e}_{ij} \) for \( i \neq j \), \( \bar{e}_{ii} \) for \( i \) such that \( \bar{a}_i < 1 \), and \( \bar{m} \) are always identical for any equilibrium point \((\bar{e}, \bar{f}) \in \mathcal{E}\).

The following theorem establishes that the fluid model approaches the equilibrium set from any starting position. The proof is delayed to Appendix B.

**Theorem 4.** Let \((e(t), f(t), u(t))\) be the unique solution to (4.1)\textendash(4.4) with initial condition \((e(0), f(0)) \in \mathcal{T}\). Assume \( P_{ij} > 0 \) and \( Q_{ii} > 0 \) for all \( i, j = 1, \ldots, r \). Then for any \( \epsilon > 0 \), there exists a \( T > 0 \) such that

\[
\inf_{x \in \mathcal{E}} |(e(t), f(t)) - x| < \epsilon, \quad t \geq T.
\]

**Remark 7.** Suppose \( \mathcal{E} \) contains multiple points, and let \( i \) and \( j \) be regions for which \( \bar{a}_i = \bar{a}_j = 1 \). As it is currently stated, Theorem 4 does not exclude the possibility that mass oscillates indefinitely between \( e_{ii}(t) \) and \( e_{jj}(t) \). It only guarantees that \( \sum_{i: \bar{a}_i = 1} e_{ii}(t) \) converges to \( \bar{m} \) as \( t \to \infty \). We do not pursue a proof for excluding such oscillations because the current statement of Theorem 4 is sufficient for our needs.

**Remark 8.** In Lemma 3 and Theorem 4, we make use of the assumption that \( P_{ij} > 0 \) for all \( i, j \). We expect that the result holds if \( P \) is only assumed to be irreducible, but our extra assumption greatly facilitates exposition in the proof of Theorem 4, which is already rather cumbersome.
We now have proved the process-level convergence of the queuing network and the convergence of the fluid-model to its equilibrium point. The proof of (2.12)–(2.15) follows from Theorem 5.1 of [1]. We repeat the argument below for completeness. We know that the sequence $\{(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))\}_N$ is tight, because for any $N > 0$, the support of $(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))$ is the compact set $\mathcal{T}$. It follows by Prohorov’s Theorem [5] that the sequence is also relatively compact. We will now show that any subsequence of $\{(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))\}_N$ has a further subsequence that converges weakly to a probability measure under which the measure of $\mathcal{E}$ is one, thereby proving (2.12)–(2.15).

Fix $N > 0$ and initialize $(\tilde{E}^{(N)}(0), \tilde{F}^{(N)}(0))$ according to $(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))$. We know that for any subsequence

\[\{(\tilde{E}^{(N')}(0), \tilde{F}^{(N')}(0))\}_{N'} \subset \{(\tilde{E}^{(N)}(0), \tilde{F}^{(N)}(0))\}_N,\]

there exists a further subsequence

\[\{(\tilde{E}^{(N'')}(0), \tilde{F}^{(N'')}(0))\}_{N''} \subset \{(\tilde{E}^{(N')}(0), \tilde{F}^{(N')}(0))\}_{N'},\]

that converges weakly to some probability measure $(e_0, f_0)$ whose support is $\mathcal{T}$. Now for any $t \geq 0$,

\[\tilde{E}^{(N'')}(0), \tilde{F}^{(N'')}(0) \overset{d}{=} (\tilde{E}^{(N'')})(t), \tilde{F}^{(N'')})(t) \Rightarrow (e(t), f(t)),\]

as $N'' \to \infty$, where $(e(t), f(t))$ is the fluid model with initial condition $(e(0), f(0)) = (e_0, f_0)$, and the weak convergence follows from Theorem 3. Since $(e(t), f(t))$ converges to the set $\mathcal{E}$ as $t \to \infty$, it must be the case that $(e_0, f_0) \in \mathcal{E}$ with probability one.

To prove (2.16), we need to use the generator of $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$, which we call $G^{(N)}$. Since $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$ takes values in a bounded set, [10, Proposition 3] tells us that any function $g : \mathcal{T} \to \mathbb{R}$ satisfies

\[\mathbb{E}[G^{(N)} g(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))] = 0. \tag{4.20}\]

In particular, fix $i, j$ between 1, \ldots, $r$ and choose $g(e, f) = f_{ij}$. Then

\[G^{(N)} g(e, f) = N \lambda_i \alpha_{ij} 1(e_{ii} > 0)((f_{ij} + 1/N) - f_{ij}) + \mu_{ij} N f_{ij}((f_{ij} - 1/N) - f_{ij}),\]

which implies

\[\mathbb{E}[G^{(N)} g(\tilde{E}^{(N)}(\infty), \tilde{F}^{(N)}(\infty))] = \lambda_i \alpha_{ij} \mathbb{P}(\tilde{E}^{(N)}_{ii}(\infty) > 0) - \mu_{ij} \mathbb{E}\tilde{F}^{(N)}_{ij}(\infty) = 0. \tag{4.21}\]

Hence,

\[\lim_{N \to \infty} \mathbb{P}(\tilde{E}^{(N)}_{ii}(\infty) > 0) = \lim_{N \to \infty} \frac{\mu_{ij}}{\lambda_i \alpha_{ij}} \mathbb{E}\tilde{F}^{(N)}_{ij}(\infty) = \frac{\mu_{ij}}{\lambda_i P_{ij}} f_{ij} = \tilde{a}_i,\]

where in the second equality we used (2.12) and the fact that $\tilde{F}^{(N)}_{ij}(\infty) \in [0, 1]$ to conclude that the sequence of expected values $\mathbb{E}[\tilde{F}^{(N)}_{ij}(\infty)]$ converges to $\tilde{f}_{ij}$, and in the last equality we used (4.11).
Remark 9. Recall that \((E^{(N)}, F^{(N)})\) is a BCMP network. In full generality, BCMP networks only require that the service time distributions in the infinite server stations have rational Laplace transforms. This class of distributions is dense in the set of all probability distributions on \((0, \infty)\) [2]. Furthermore, stationary distribution of BCMP networks is known to depend only on the service rates at the stations, and not on the entire distribution. Hence, the results of Theorem 1 hold for travel time distributions with rational Laplace transforms.

5 Conclusions

This paper considered a ridesharing model with three distinct features (empty-car routing, indiscrimination and limited car supply) and provided a comprehensive analysis of the design of an optimal empty-car routing policy based on asymptotic fluid analysis. Simulation results using real-world ridesharing data confirmed the effectiveness of our solution. Our paper is only a first step to understand ridesharing networks, and poses some interesting problems that are for future research directions:

Decentralized routing: Our routing policy is a centralized solution that assumes the ridesharing platform has full control over its empty cars, which provides a best-case benchmark. One of our future research topics is the design of decentralized incentive mechanisms for achieving the routing probabilities of the centralized solution. Potential mechanisms include surge pricing and targeted information sharing. Using the centralized routing policy as a benchmark, we will be able to quantify the efficiency of decentralized mechanisms under a variety of practical constraints.

Time-varying parameters: This paper relied on steady-state analysis to attain its results. In particular, we assumed that primitive parameters such as passenger arrival rates and travel times do not change over time. Suppose instead that the daily evolution of passenger arrival rates was given to us. Could this model, or perhaps a similar one, be applied to derive effective empty-car routing policies in the case of time-varying parameters? How does one route cars to preemptively prepare for rush hours, or other foreseen changes in passenger flow patterns.

Robust routing policies: We witnessed in Figure 7 that certain state-dependent policies can attain performance levels close to those of the optimal static policy, yet have the benefit of not requiring explicit knowledge of system parameters. Now that we have established a benchmark, can we find state-dependent policies that are provably asymptotically optimal, yet rely on as little as possible on primitive system parameters?

Acknowledgments: The authors thank Siddhartha Banerjee and Daniel Freund for feedback and stimulating discussion on this work. They also thank Peter Frazier for arranging a visit to Uber headquarters, where they received invaluable feedback. This research is supported in part by NSF Grants CNS-1248117, CMMI-1335724, and CMMI-1537795.
A Process Level Convergence for Closed Networks of Single and Infinite Server Stations

In this section we consider general closed queueing networks with exponential service times that consist of both single-server and infinite-server stations. We prove that as the number of customers in the network increases, the appropriately scaled queue length process converges to a fluid limit. The model in Section 2 is a special type of such networks, meaning that Theorem 3 will be a special case of the results here. We remark that process-level convergence for the class of networks considered here is a consequence of the results in [15]. However, in that paper the limiting process is defined as the solution to a differential inclusion, and our proof technique is sufficiently different to merit a separate write-up.

We consider a closed queueing network with \( N > 0 \) customers and \( J \) stations, consisting of both single-server and infinite-server stations. We let the stations be indexed by the set \( \mathcal{J} = \{1, \ldots, J\} \). Let \( \mathcal{I} \subset \mathcal{J} \) and \( \mathcal{S} \subset \mathcal{J} \) be the non-empty index sets corresponding to the infinite server stations, and single server stations, respectively. To describe the network dynamics, we introduce the following primitives. Let \( Q^{(N)}(0) \in \mathbb{Z}_+^J \) with \( \sum_{i \in \mathcal{J}} Q^{(N)}_i(0) = N \) be the vector representing the initial customer distribution in the network. To keep track of service completions at each stations, we let

\[
S_i = \{S_i(t), t \geq 0\}, \quad i \in \mathcal{J}
\]

be a collection of unit-rate Poisson processes with \( S_i \) independent of \( S_j \) for \( i \neq j \). Let \( \lambda, \mu \in \mathbb{R}_+^J \) be two vectors with \( \lambda_i = 0 \) for \( i \in \mathcal{I} \) and \( \mu_i = 0 \) for \( i \in \mathcal{S} \), which we will use to represent the service rates at different stations. We assume that the service rate of each server at station \( i \in \mathcal{J} \) is

\[
N \lambda_i, \quad i \in \mathcal{S} \quad \mu_i, \quad i \in \mathcal{I}.
\]

Let

\[
\{(\Phi_{i1}(n), \ldots, \Phi_{iJ}(n)) \in \mathbb{Z}_+^J, \ n \in \mathbb{Z}_+\}, \quad i \in \mathcal{J}
\]

be a collection of routing processes defined as follows. For each \( n \in \mathbb{Z}_+ \) and each \( i \in \mathcal{J} \), the vector

\[
(\Phi_{i1}(n), \ldots, \Phi_{iJ}(n)) = \sum_{m=1}^n \phi_i(m),
\]

where \( \{\phi_i(m) \in \{0,1\}^J\}_{m=1}^{\infty} \) is a sequence of i.i.d. random variables with

\[
\mathbb{P}(\phi_i(1) = e^{(j)}) = R_{ij}, \quad i, j \in \mathcal{J}.
\]
Furthermore, the sequences \( \{ \phi_i(m) \}_{m=1}^{\infty} \) and \( \{ \phi_j(m) \}_{m=1}^{\infty} \) are assumed to be independent for \( i \neq j \). Let \( R \) be the routing probability matrix whose \( i,j \)-th entry is \( R_{ij} \), and observe that it is a column stochastic matrix.

By a sample-path construction involving the primitives above, one can argue that there exists a unique process

\[
Q^{(N)} = \{ Q^{(N)}(t) = (Q^{(N)}_1(t), \ldots, Q^{(N)}_J(t)), t \geq 0 \}
\]

satisfying

\[
Q^{(N)}_i(t) = Q^{(N)}_i(0) - S_i(N\lambda_i T^{(N)}_i(t)) + \sum_{j \in S} \Phi_{ji} \left( S_j(N\lambda_j T^{(N)}_j(t)) \right)
+ \sum_{k \in I} \Phi_{ki} \left( S_k \left( \mu_k \int_0^t Q^{(N)}_k(s) \, ds \right) \right), \quad i \in S,
\]

\[
Q^{(N)}_i(t) = Q^{(N)}_i(0) - S_i \left( \mu_i \int_0^t Q^{(N)}_i(s) \, ds \right) + \sum_{j \in S} \Phi_{ji} \left( S_j(N\lambda_j T^{(N)}_j(t)) \right)
+ \sum_{k \in I} \Phi_{ki} \left( S_k \left( \mu_k \int_0^t Q^{(N)}_k(s) \, ds \right) \right), \quad i \in I,
\]

where

\[
T^{(N)}_i = \left\{ T^{(N)}_i(t) = \int_0^t 1(Q^{(N)}_i(s) > 0) \, ds \right\}, \quad i \in S,
\]

is the cumulative busy time process of the server at each single server station. At any time \( t \geq 0 \), \( Q^{(N)}_i(t) \) is the customer count at station \( i \in J \). We omit the details of the sample-path construction due to their bulkiness. It is a straightforward exercise to verify that \( Q^{(N)} \) satisfies the Markov property and is therefore a CTMC. Furthermore, one can see that the CTMC \( (E^{(N)}, F^{(N)}) \) introduced in Section 2 is a special case of \( Q^{(N)} \).

To write (A.1)–(A.2) in a form that is more convenient for analysis, for any \( t \geq 0 \) let us define

\[
\widehat{S}_i(t) = S_i(t) - t, \quad i \in J,
\]

\[
\widehat{\Phi}_{ij}(n) = \Phi_{ij}(n) - R_{ij} n, \quad i, j \in J, \quad n \in \mathbb{Z}_+,
\]
and
\[
\bar{M}_i^{(N)}(t) = -\bar{S}_i(N\lambda_i T_i^{(N)}(t)) + \sum_{j \in S} \left[ \hat{\Phi}_{ji} \left( S_j(N\lambda_j T_j^{(N)}(t)) \right) + R_{ji} \hat{S}_j(N\lambda_j T_j^{(N)}(t)) \right] \\
+ \sum_{k \in I} \left[ \hat{\Phi}_{ki} \left( S_k(\mu_k \int_0^t Q_k^{(N)}(s)ds) \right) + R_{ki} \hat{S}_k(\mu_k \int_0^t Q_k^{(N)}(s)ds) \right], \quad i \in S, \tag{A.3}
\]
\[
\bar{M}_i^{(N)}(t) = -\bar{S}_i(\mu_0 \int_0^t Q_i^{(N)}(s)ds) + \sum_{j \in S} \left[ \hat{\Phi}_{ji} \left( S_j(N\lambda_j T_j^{(N)}(t)) \right) + R_{ji} \hat{S}_j(N\lambda_j T_j^{(N)}(t)) \right] \\
+ \sum_{k \in I} \left[ \hat{\Phi}_{ki} \left( S_k(\mu_k \int_0^t Q_k^{(N)}(s)ds) \right) + R_{ki} \hat{S}_k(\mu_k \int_0^t Q_k^{(N)}(s)ds) \right], \quad i \in I, \tag{A.4}
\]
and let \( \hat{M}^{(N)}(t) \) be the vector whose components are \( \hat{M}_i^{(N)}(t) \). For \( t \geq 0 \), we also define
\[
I_i^{(N)}(t) = 0 \text{ for } i \in I \quad \text{and} \quad I_i^{(N)}(t) = t - T_i^{(N)}(t) \text{ for } i \in S, \tag{A.5}
\]
and let \( I^{(N)} = \{I^{(N)}(t) \in \mathbb{R}_+, t \geq 0 \} \). Then for \( i \in S \), \( I_i^{(N)}(t) \) represents the cumulative idle time up to time \( t \). Setting
\[
Q^{(N)}(t) = \frac{1}{N} Q^{(N)}(t) \quad \text{and} \quad \bar{M}^{(N)}(t) = \frac{1}{N} \hat{M}^{(N)}(t),
\]
we from (A.1)–(A.4) that
\[
\bar{Q}_i^{(N)}(t) = \bar{Q}_i^{(N)}(0) + \bar{M}_i(t) + \left( \sum_{j \in S} R_{ji} \lambda_j - \lambda_i \right) t + \sum_{k \in I} R_{ki} \mu_k \int_0^t Q_k^{(N)}(s)ds \\
+ \lambda_i I_i^{(N)}(t) - \sum_{j \in S} R_{ji} \lambda_j I_j^{(N)}(t), \quad i \in S, \tag{A.6}
\]
\[
\tilde{Q}_i^{(N)}(t) = \tilde{Q}_i^{(N)}(0) + \tilde{M}_i(t) + \left( \sum_{j \in S} R_{ji} \lambda_j \right) t - \mu_i \int_0^t \tilde{Q}_i^{(N)}(s)ds \\
+ \sum_{k \in I} P_{ki} \mu_k \int_0^t \tilde{Q}_k^{(N)}(s)ds - \sum_{j \in S} R_{ji} \lambda_j I_j^{(N)}(t), \quad i \in I. \tag{A.7}
\]
In the next section, we describe the fluid model to which the process \( \tilde{Q}^{(N)} \) will converge to as \( N \to \infty \).
A.1 The Fluid Model

Recalling that \(\mu_i = 0\) for \(i \in S\) and \(\lambda_i = 0\) for \(i \in I\), we set
\[
M = \text{diag}(\mu) \quad \text{and} \quad \Lambda = \text{diag}(\lambda).
\] (A.8)

Recall the routing matrix \(R\) and define the \(J \times J\) matrix \(\tilde{R}\) by setting
\[
\tilde{R}_{ij} = R_{ij}, \quad i \in S,
\]
\[
\tilde{R}_{ij} = 0, \quad i \in I.
\]

That is \(\tilde{R}\) is the matrix \(R\) with all rows corresponding to infinite server stations being set to zero. Since \(I \neq \emptyset\), the matrix \(\tilde{R}^T\) is sub-stochastic. The following lemma is proved in Section A.2.

**Lemma 4.** For each \(x \in \mathbb{D}_1^J\), there exists a unique \((q, v) \in \mathbb{D}_2^J\), with \(q(t) \in \mathbb{R}_+^J\) and \(v(t) \in \mathbb{R}_+^J\) for all \(t \geq 0\), such that
\[
q(t) = x(t) - (I - R^T)M \int_0^t q(s)ds + (I - \tilde{R}^T)v(t) \tag{A.9}
\]
\[
q(t) \geq 0, \quad t \geq 0, \tag{A.10}
\]
\[
v(\cdot) \text{ is non-decreasing with } v(0) = 0, \tag{A.11}
\]
\[
\int_0^\infty q_i(s)dv_i(s) = 0, \quad i \in J. \tag{A.12}
\]

Furthermore, the map \(\Upsilon : \mathbb{D}_1^J \to \mathbb{D}_2^J\) given by \(\Upsilon(x) = (q, v)\) is well-defined and is Lipschitz-continuous, in the sense that for any \(x, \tilde{x} \in \mathbb{D}_1^J\), and any \(T > 0\), there exists a constant \(c_T\) such that
\[
\|\Upsilon(x) - \Upsilon(\tilde{x})\|_T \leq c_T\|x - \tilde{x}\|_T, \tag{A.13}
\]

**Theorem 5.** Assume \(\hat{Q}^{(N)}(0) \to a\) as \(N \to \infty\) for some \(a \in [0, 1]^J\) with \(\sum_{i=1}^J a_i = 1\). Let \(e \in \mathbb{R}^J\) be the vector of ones, and let \(\gamma : \mathbb{R}_+ \to \mathbb{R}_+\) be the identity map defined by \(\gamma(t) = t\). Set
\[
(q, v) = \Upsilon\left(a + \gamma(R^T - I)\Lambda e\right). \tag{A.14}
\]

Then for any \(T > 0\),
\[
\lim_{n \to \infty} \|\hat{Q}^{(N)}(t) - q(t)\|_T = 0,
\]
\[
\lim_{n \to \infty} \|I^{(N)}(t) - v(t)\|_T = 0.
\]
Proof of Theorem 5. From (A.6)–(A.7) we see that
\[
(\bar{Q}^{(N)}, I^{(N)}) = \Upsilon \left( \bar{Q}^{(N)}(0) + \bar{M}^{(N)} + \gamma(R^T - I)\Lambda e \right),
\]
where \( \bar{M}^{(N)} = \{ \bar{M}^{(N)}(t) \in \mathbb{R}^J, t \geq 0 \} \). Suppose we knew that for every \( T \geq 0 \),
\[
\lim_{N \to \infty} \| \bar{M}^{(N)} \|_T = 0 \quad \text{almost surely.}
\]
(A.15)

Then the continuous mapping theorem [5], together with (A.13) would imply Theorem 5. The proof of (A.15) involves a standard argument using the functional strong law of large numbers (FSLLN), and is therefore omitted. For an example of such an argument, see the proof of (5.6) in [7]. \( \square \)

Remark 10. Since \( I_i^{(N)}(t) = 0 \) for \( i \in \mathcal{I} \) and \( t \geq 0 \), Theorem 5 implies
\[
v_i(t) = 0, \quad i \in \mathcal{I}.
\]
(A.16)

Establishing (A.16) by relying on convergence of \( (\bar{Q}^{(N)}, I^{(N)}) \) to \((q, v)\) may seem strange, because (A.16) should be a standalone property of \( \Upsilon (a + \gamma(R^T - I)\Lambda e) \). Indeed, it is possible to establish (A.16) using a direct argument that relies on Proposition 1 of [18]. However, we avoid using said argument because it is significantly longer.

We immediately see that Theorem 3 is a special case of Theorem 5. In particular, it means that \((e(t), f(t), u(t))\) of Theorem 3 can be represented as in (A.14), which facilitates the proof of Lemma 3.

Proof of Lemma 3. Let \((e(t), f(t), u(t))\) be a solution of (4.1)–(4.4) with \((e(0), f(0)) = (\bar{e}, \bar{f})\). Since Theorem 3 is just a special case of Theorem 5, from (A.14) we see that \((e(t), f(t), u(t))\) is uniquely determined by \((\bar{e}, \bar{f})\). That is, if two solutions to (4.1)–(4.4) share the same initial condition, then they must coincide for all \( t \geq 0 \). To conclude Lemma 3, observe that \((e(t), f(t)) \equiv (\bar{e}, \bar{f})\), and \( u_i(t) = (1 - \bar{a}_i)t \) solves (4.1)–(4.4) with initial condition \((e(0), f(0)) = (\bar{e}, \bar{f})\). Hence, \((\bar{e}, \bar{f})\) is an equilibrium of \((e(t), f(t))\). \( \square \)

The rest of this section is devoted to proving Lemma 4.

A.2 Proof of Lemma 4

In order to prove Lemma 4, we first need to introduce the Skorohod problem. Let \( \tilde{Q} \) be a \( J \times J \) column sub-stochastic matrix with non-negative entries. For any \( x \in \mathbb{D}_{\tilde{a}+} \), let

\[\]
(\(z, y\)) ∈ \(D^{2J}\) with \(z(t) \in \mathbb{R}^J_+\) and \(y(t) \in \mathbb{R}^J_+\) for all \(t ≥ 0\) be the solution to

\[
\begin{align*}
z &= x + (I - \tilde{Q}^T)y, & (A.17) \\
z &≥ 0, & (A.18) \\
y(\cdot) \text{ is non-decreasing and } y(0) = 0, & (A.19) \\
\int_0^\infty z_i(s)dy_i(s) = 0, & 1 ≤ i ≤ J. & (A.20)
\end{align*}
\]

Existence and uniqueness of \((z, y)\) was proved in [11] when \(x\) is continuous, but the arguments there hold for \(x \in \mathbb{D}_0^J\) as well. We refer to \((A.17)-(A.20)\) as the Skorohod problem associated with \((x, \tilde{Q})\), and write \(SP(x, \tilde{Q})\) for short. Furthermore, for any \(x \in \mathbb{D}_0^J\) we define the Skorohod map \(\Psi_{\tilde{Q}} : \mathbb{D}_0^J \to \mathbb{D}^{2J}\) by

\[
\Psi_{\tilde{Q}}(x) = (\Psi_{\tilde{Q}}^z(x), \Psi_{\tilde{Q}}^y(x)) = (z, y),
\]

where \((z, y)\) is the solution of \(SP(x, \tilde{Q})\). From \((A.17)\) it is clear that

\[
\Psi_{\tilde{Q}}^z(x) = x + (I - \tilde{Q}^T)\Psi_{\tilde{Q}}^y(x), \quad x \in \mathbb{D}^J, \; x(0) ≥ 0.
\]

We claim that both \(\Psi_{\tilde{Q}}^z\) and \(\Psi_{\tilde{Q}}^y\) are Lipschitz-continuous, in the sense that for any \(T > 0\), there exists constants \(c^T_z, c^T_y > 0\), which depend on \(\tilde{Q}\), such that for any \(x, \tilde{x} \in \mathbb{D}_0^J\),

\[
\begin{align*}
||\Psi_{\tilde{Q}}^z(x) - \Psi_{\tilde{Q}}^z(\tilde{x})||_T &≤ c^T_z ||x - \tilde{x}||_T, \\
||\Psi_{\tilde{Q}}^y(x) - \Psi_{\tilde{Q}}^y(\tilde{x})||_T &≤ c^T_y ||x - \tilde{x}||_T,
\end{align*}
\]

where \(||·||_T\) is defined in \((1.4)\). This fact was established in [11, p. 305] when both \(x\) and \(\tilde{x}\) are continuous, but the argument used there holds for \(x, \tilde{x} \in \mathbb{D}_0^J\) as well. We are now ready to prove Lemma 4.

\textbf{Proof of Lemma 4.} Fix \(x \in \mathbb{D}_0^J\) and omitting the superscript \(\tilde{R}\), let \((\Psi_z, \Psi_y)\) be the Skorohod map associated with \(SP(x, \tilde{R})\). Define the integral operator \(\alpha : \mathbb{D}_0^J \times \mathbb{D}^J \to \mathbb{D}^J\) by

\[
\alpha(x, w)(t) = x(t) - (I - R^T)M \int_0^t w(s)ds,
\]

and consider the integral equation

\[
\begin{align*}
q &= \Psi_z(\alpha(x, q)) = \alpha(x, q) + (I - \tilde{R}^T)\Psi_y(\alpha(x, q)) & (A.23) \\
q &\in \mathbb{D}^J, & (A.24)
\end{align*}
\]
where the second equality in \((A.23)\) follows from \((A.21)\). Provided \((A.23)-(A.24)\) has a unique solution \(q\), we can set \(v = \Psi_y(\alpha(x,q))\) and observe that by definition of \(\alpha\) and \((\Psi_z,\Psi_y)\),

\[
(q,v) = \Upsilon(x).
\]

Hence, we now establish the existence and uniqueness of a solution \(q\) to \((A.23)-(A.24)\).

Construct a sequence \(\{q^n \in \mathbb{D}^J\}_{n=0}^{\infty}\) by letting

\[
q^0(t) \equiv x(0), \quad t \geq 0,
\]

\[
q^{n+1}(t) = \Psi_z(\alpha(x,q^n))(t), \quad t \geq 0.
\]

Observe that \(q^n(0) = x(0)\) for all \(n \geq 0\). We first show that for any \(T > 0\), this sequence is a Cauchy sequence in the Hilbert space \((\mathbb{D}^J[0,T], \| \cdot \|_T)\). Let \(\bar{\mu} = \max_{i \in I} \{ \mu_j \}\) (remembering that \(\mu_i = 0\) for \(i \in S\)), and observe that

\[
\|q^{n+1} - q^n\|_T \leq c^T \|\alpha(\cdot)\|_{L^1(0,T)} \leq c^T \max_{i \in I} \sup_{0 \leq t \leq T} |\alpha(x,q^n) - \alpha(x,q^{n-1})| \leq c^T \bar{\mu}J \max_{i \in I} \left\{ \int_0^T |q^n_i(s) - q^{n-1}_i(s)| \, ds \right\} \leq c^T \bar{\mu}J \int_0^T \max_{i \in I} |q^n_i(s) - q^{n-1}_i(s)| \, ds \leq \left( \frac{c^T \bar{\mu}JT}{n!} \right) \|q^1 - q^0\|_T.
\]

The first inequality follows from the Lipschitz property of \(\Psi_z\), the second inequality is from the form of \(\alpha\), and the last inequality follows by recursion. From this point it is not hard to conclude (see for instance (11.22) of [16]) that \(\{q^n\}_{n=0}^{\infty}\) is a Cauchy sequence in \((\mathbb{D}^J[0,T], \| \cdot \|_T)\) for each \(T > 0\). Therefore, \(q^n\) converges to some limit \(q \in \mathbb{D}^J[0,T]\) that satisfies \((A.23)\). Since the choice of \(T > 0\) was arbitrary, we have proved existence of a solution to \((A.23)-(A.24)\). Uniqueness can be argued by taking two potential solutions \(q\) and \(\tilde{q}\), and applying the chain of arguments in \((A.25)\) with \(\|q - \tilde{q}\|_T\) on the left hand side there.

It remains to prove the Lipschitz-continuity of \(\Upsilon\). Fix any \(x, \tilde{x} \in \mathbb{D}^J\), and set

\[
(q,v) = \Upsilon(x) \quad \text{and} \quad (\tilde{q},\tilde{v}) = \Upsilon(\tilde{x}).
\]
Repeating the logic used to obtain (A.25), we see that for any \( n \geq 1 \),
\[
\|q - \bar{q}\|_T = \|\Psi_z(\alpha(x, q)) - \Psi_z(\alpha(\bar{x}, \bar{q}))\|_T \\
\leq c_z^T \|x - \bar{x}\|_T + c_z^T \mu J \int_0^T \max_{i \in I} |q_i(s) - \bar{q}_i(s)| \, ds \\
\leq c_z^T \|x - \bar{x}\|_T \sum_{k=1}^n \frac{(c_z^T \mu J T)^k}{k!} + \frac{(c_z^T \mu J T)^n}{n!} \|q - \bar{q}\|_T,
\]
where the last inequality follows by recursion. Choosing \( n \) large enough so that \( \frac{(c_z^T \mu J T)^n}{n!} < 1 \), we conclude the existence of a constant \( c_{T,q} > 0 \) such that
\[
\|q - \bar{q}\|_T \leq c_{T,q}^T \|x - \bar{x}\|_T.
\]
Similarly, we see that
\[
\|v - \bar{v}\|_T = \|\Psi_y(\alpha(x, q)) - \Psi_y(\alpha(\bar{x}, \bar{q}))\|_T \\
\leq c_y^T \|x - \bar{x}\|_T + c_y^T \mu J \int_0^T \max_{i \in I} |q_i(s) - \bar{q}_i(s)| \, ds \\
\leq c_y^T \|x - \bar{x}\|_T + c_y^T \mu J T \|q - \bar{q}\|_T,
\]
and by (A.26), there exists a constant \( c_{T,v} > 0 \) satisfying
\[
\|v - \bar{v}\|_T \leq c_{T,v}^T \|x - \bar{x}\|_T.
\]
This establishes (A.13) and concludes the proof the lemma.

B Proof of Theorem 4

This section is devoted to proving Theorem 4. For the remainder of this section, we fix \((\bar{e}, \bar{f}) \in \mathcal{E}\) and let \( \bar{a} \) and \( \bar{m} \) be defined by (4.11) and (4.19), respectively. For \((x, y) \in T\), define the Lipschitz-continuous function \( V : T \to \mathbb{R}_+ \) by
\[
V(x, y) = \sum_{i=1}^r \sum_{j=1}^r |y_{ij} - \bar{f}_{ij}| + \sum_{i=1}^r \sum_{j=1}^r |x_{ij} - \bar{e}_{ij}| + \sum_{i: \bar{a}_i < 1} x_{ii} + \left| \bar{m} - \sum_{i: \bar{a}_i = 1} x_{ii} \right|.
\]

To prove Theorem 4, we will show that \( V(e(t), f(t)) \) is a Lyapunov function. We know that \( V(e(t), f(t)) \) is a Lipschitz-continuous function from \( \mathbb{R}_+ \to \mathbb{R}_+ \) because \( V(\cdot), e(\cdot), \) and \( f(\cdot) \) are all Lipschitz-continuous.

We say \( t > 0 \) is a regular point of \( V(e(t), f(t)) \) if for all \( i, j = 1, \ldots, r \), the functions \( e_i(t), |f_{ij}(t) - \bar{f}_{ij}|, |e_{ij}(t) - \bar{e}_{ij}| \) for \( i \neq j \), and \( |\bar{m} - \sum_{i: \bar{a}_i = 1} e_{ii}(t)| \) are differentiable at
Since these functions are all Lipschitz-continuous, then almost every point is a regular point. Furthermore, if \( t \) is a regular point then

\[
\begin{align*}
    f_{ij}(t) &= \bar{f}_{ij} \quad \Rightarrow \quad \dot{f}_{ij}(t) = 0, \quad 1 \leq i, j \leq r \\
    e_{ii}(t) &= 0 \quad \Rightarrow \quad \dot{e}_{ii}(t) = 0, \quad 1 \leq i \leq r, \\
    e_{ij}(t) &= \bar{e}_{ij} \quad \Rightarrow \quad \dot{e}_{ij}(t) = 0, \quad 1 \leq i \neq j \leq r \\
    \sum_{i:a_i=1} e_{ii} &= \bar{m} \quad \Rightarrow \quad \sum_{i:a_i=1} \dot{e}_{ii} = 0, \\
    \dot{u}_i(t) &\text{ exists for all } 1 \leq i \leq r.
\end{align*}
\]

Most of these properties are simple consequence of the definition of a regular point. For (B.3) we also need to recall that \( e_{ii}(s) \geq 0 \) for all \( s \geq 0 \), and (B.6) is true because \( \dot{u}_i(t) \) must exist in order for \( \dot{e}_{ii}(t) \) to exist (see (4.10)).

**Lemma 5.** Let \( t > 0 \) be a regular point of \( V(e(t), f(t)) \). If \( (e(t), f(t)) \notin E \), then

\[
\dot{V}(e(t), f(t)) < 0.
\]

It is clear that if \( (e(t), f(t)) \in E \) for some \( t \geq 0 \), then \( V(e(s), f(s)) = 0 \) and \( \dot{V}(e(s), f(s)) = 0 \) for all \( s \geq t \). We postpone the proof of Lemma 5 to the end of this section, and first demonstrate how it is used to prove Theorem 4. We first need to introduce the notion of an invariant set. A set \( B \subset T \) is said to be invariant if for any \( s \geq 0 \),

\[
    (e(s), f(s)) \in B \quad \Rightarrow \quad (e(t), f(t)) \in B, \quad \forall t \geq 0.
\]

Let

\[
    \mathcal{N} = \{(e(t), f(t)) \in T : t > 0 \text{ is a regular point and } \dot{V}(e(t), f(t)) = 0\},
\]

and let \( \mathcal{M} \subset \mathcal{N} \) be the largest invariant set contained in \( \mathcal{N} \). The following result is known as LaSalle’s Invariance Principle.

**Theorem 6.** Suppose \( \dot{V}(e(t), f(t)) \leq 0 \) for almost every \( t > 0 \). Then

\[
\lim_{t \to \infty} \inf_{x \in \mathcal{M}} \|(e(t), f(t)) - x\| = 0.
\]

Since the set of all equilibria \( E \) is an invariant set, Lemma 5 and Theorem 6 immediately imply Theorem 4.

**Proof of Theorem 6.** Theorem 6 is simply a restatement of Theorem 4.4 of [14] with two differences. The first difference is that [14] requires the function \( V(x, y) \) to be continuously differentiable in \( x \) and \( y \), but in our \( V(x, y) \) is only Lipschitz-continuous. The continuous
differentiability assumption is in fact unnecessary, and the reader can check that the proof of Theorem 4.4 in [14] goes through with the weaker assumption that $V(x, y)$ is only Lipschitz-continuous.

The second difference is that in [14], the author states Theorem 4.4 for functions $x : \mathbb{R} \to \mathbb{R}^n$ (with $n \in \mathbb{Z}_+$) that satisfy

$$\dot{x} = g(x),$$  \hspace{1cm} (B.7)

for some locally Lipschitz $g : \mathbb{R}^n \to \mathbb{R}^n$. The solution to (B.7) is guaranteed to exist, be unique, and be continuous with respect to its initial condition. Stated more precisely, the latter property means that if $x$ and $\tilde{x}$ both satisfy (B.7) with $x(0) \neq \tilde{x}(0)$, then for every $t \geq 0$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x(0) - \tilde{x}(0)| < \delta \Rightarrow |x(t) - \tilde{x}(t)| < \epsilon. \hspace{1cm} (B.8)$$

These three properties of $x$ are needed to establish Lemma 4.1 in [14], which is a key tool in the proof of Theorem 4.4 there. In our case, $(e, f)$ are defined as the solution to (4.1)–(4.4) and cannot be written in the form of (B.7). In fact, $(\dot{e}(t), \dot{f}(t))$ is not even guaranteed to exist for all $t \geq 0$. Nevertheless, we know that $(e, f)$ is unique by Lemma 4, and that it satisfies the analogue of (B.8) by (A.13) of Lemma 4. Hence, Lemma 4.1 of [14] holds for $(e, f)$ as well.

The rest of this section is devoted to proving Lemma 5. Fix a regular point $t > 0$. For notational simplicity, we omit the time index $t$ when referring to $V(e(t), f(t))$, $e_{ij}(t)$, $f_{ij}(t)$, $u_i(t)$, or their derivatives. From (4.1)–(4.3) we can see that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \dot{e}_{ij} + \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{f}_{ij} = 0. \hspace{1cm} (B.9)$$

Set

$$\hat{f}_{ij} = f_{ij} - \bar{f}_{ij}, \hspace{1cm} 1 \leq i, j \leq r,$$

$$\hat{e}_{ij} = e_{ij} - \bar{e}_{ij}, \hspace{1cm} 1 \leq i \neq j \leq r.$$
Recall that \( t \) is a regular point, meaning (B.2)–(B.6) hold. Therefore,

\[
\frac{1}{2} \dot{V}(e, f) = \frac{1}{2} \sum_{i : \bar{a}_i < 1} r \dot{e}_{ii} + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{e}_{ij} \left( e_{ij} > \bar{e}_{ij} \right) - \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{e}_{ij} \left( e_{ij} < \bar{e}_{ij} \right) \\
+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{f}_{ij} \left( f_{ij} > \bar{f}_{ij} \right) - \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{f}_{ij} \left( f_{ij} < \bar{f}_{ij} \right) \\
+ 1 \left( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \right) \frac{1}{2} \sum_{i : \bar{a}_i = 1} \dot{e}_{ii} - 1 \left( \sum_{i : \bar{a}_i = 1} e_{ii} < \bar{m} \right) \frac{1}{2} \sum_{i : \bar{a}_i = 1} \dot{e}_{ii} \\
= \sum_{i : \bar{a}_i < 1} \dot{e}_{ii} + \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{e}_{ij} \left( e_{ij} > \bar{e}_{ij} \right) + \sum_{i=1}^{r} \sum_{j=1}^{r} \dot{f}_{ij} \left( f_{ij} > \bar{f}_{ij} \right) \\
+ 1 \left( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \right) \sum_{i : \bar{a}_i = 1} \dot{e}_{ii},
\]  

(B.10)

where the second equality follows from (B.9). The expression in (B.10) will soon become very bulky to work with, so before moving forward we first present a toy example to help the reader gain some intuition.

**B.1 An Illustrative Example**

Suppose \( r = 3 \) and that \((\lambda, \mu, P, Q)\) are such that \( \bar{a}_1 < 1, \bar{a}_i = 1 \) for \( i = 2, 3 \), and \( \bar{m} > 0 \). For simplicity, we also assume that all entries of \( P \) and \( Q \) are strictly positive. We now compute \( \frac{1}{2} \dot{V}(e, f) \) for several configurations of \((e, f)\), and show that it is always negative. The list of cases we consider is by no means exhaustive, but is nevertheless helpful to develop intuition.

**Case 1:** Suppose \( e_{ii} > 0 \) for \( i = 1, 2, 3 \) and \( e_{22} + e_{33} > \bar{m} \). Furthermore, suppose \( \dot{f}_{ij} < 0 \) for all \( i, j \), and \( \dot{e}_{ij} < 0 \) for all \( i \neq j \). Using (B.10) and (4.10), we see that

\[
\frac{1}{2} \dot{V}(e, f) = \dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33} = \sum_{i=1}^{3} \left[ - \lambda_i (1 - \bar{u}_i) + \sum_{j=1}^{3} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{3} \mu_{ji} \dot{f}_{ji} \right].
\]

By (4.16), \( \dot{u}_i = 0 \) for \( i = 1, 2, 3 \) because \( e_{ii} > 0 \). Furthermore, using (4.13) we see that

\[
\sum_{j=1}^{3} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{3} \mu_{ji} \dot{f}_{ji} = \sum_{j=1}^{3} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{3} \mu_{ji} \dot{f}_{ji} = \lambda_i \bar{a}_i + \sum_{j=1}^{3} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{3} \mu_{ji} \dot{f}_{ji}, \quad 1 \leq i \leq 3.
\]
Recalling that $\bar{a}_2 = \bar{a}_3 = 1$, we arrive at

$$\frac{1}{2} \dot{V}(e, f) = \lambda_1 (1 - \bar{a}_1) + \sum_{i=1}^{3} \left[ \sum_{j=1}^{3} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{3} \mu_{ji} \dot{f}_{ji} \right] < 0.$$ 

**Case 2:** Suppose $e_{11} = e_{22} = 0, e_{33} > \bar{m}, \dot{f}_{ij} = 0$ for all $i, j, \dot{e}_{12} < 0$ and $\dot{e}_{ij} = 0$ for all other $i, j$ with $i \neq j$. In such a case, (B.10) and (4.10) tell us that

$$\frac{1}{2} \dot{V}(e, f) = \dot{e}_{33} = -\lambda_3 (1 - \dot{u}_3) + \sum_{j=1}^{3} \mu_{j3} \dot{e}_{j3} + Q_{33} \sum_{j=1}^{3} \mu_{j3} \dot{f}_{j3}.$$ 

Now $e_{33} > 0$ and (4.16) implies that $\dot{u}_3 = 0$. Furthermore, we know $\dot{e}_{j3} = 0$ for $j = 1, 2$ and $\dot{f}_{j3} = 0$ for $j = 1, 2, 3$. Therefore, we can use (4.13) to see that

$$\frac{1}{2} \dot{V}(e, f) = -\lambda_3 + \sum_{j=1}^{3} \sum_{j \neq 3} \mu_{j3} \dot{e}_{j3} + Q_{33} \sum_{j=1}^{3} \mu_{j3} \dot{f}_{j3} = 0,$$

which appears to contradict what we set out to prove. However, it turns out that the time $t$ corresponding to this configuration of $(e, f)$ is not a regular point. If $t$ were a regular point, then by (B.2) we would have $\dot{f}_{2k} = 0$ for all $k = 1, 2, 3$, because $\dot{f}_{2k} = 0$. However, by (4.8), the fact that $\dot{f}_{2k} = 0$, and (4.11) we see that

$$\dot{f}_{2k} = \lambda_2 P_{2k} (1 - \dot{u}_2) - \mu_{2k} \dot{f}_{2k} = \lambda_2 (1 - \dot{u}_2) - \lambda_2 P_{2k} \bar{a}_2.$$

Since $e_{22} = 0$, (B.3) forces $\dot{e}_{22} = 0$, which together with (4.10) implies that

$$\lambda_2 P_{2k} (1 - \dot{u}_2) = \sum_{j=1}^{3} \mu_{j2} \dot{e}_{j2} + Q_{22} \sum_{j=1}^{3} \mu_{j2} \dot{f}_{j2} < \sum_{j=1}^{3} \mu_{j2} \dot{e}_{j2} + Q_{22} \sum_{j=1}^{3} \mu_{j2} \dot{f}_{j2} = \lambda_2 P_{2k} \bar{a}_2,$$

where in the inequality we used that $\dot{f}_{j2} = \dot{e}_{32} = 0$, and $\dot{e}_{12} < 0$, and in the last equality we used (4.13). Therefore, we just showed that $\dot{f}_{2k} < 0$, which is a contradiction to the assumption that $t$ is a regular point.

Having worked through the toy example, we now present a general algebraic expansion of $\frac{1}{2} \dot{V}(e, f)$ in Lemma 6. A line by line inspection of (B.11) and (B.12) confirms $\dot{V}(e, f) \leq 0$. We prove Lemma 6 in Section B.3, right after proving Lemma 5.
Lemma 6. If \( \{ i : \bar{a}_i = 1 \} \neq \emptyset \), \( m > 0 \), and \( \sum_{i: \bar{a}_i = 1} e_{ii} < m \), then

\[
\frac{1}{2} \dot{V}(e, f) = \sum_{i: \bar{a}_i < 1} \lambda_i (\bar{a}_i - 1) \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) 1(\dot{u}_i = 0)
\]

\[- \sum_{i=1}^{r} \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0) \right) 1(\dot{u}_i > 0) \]

\[+ \sum_{i=1}^{r} \left( \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \leq 0) \right) 1(\dot{u}_i > 0) \]

\[+ \sum_{i: \bar{a}_i < 1} 1(\dot{u}_i = 0) \left[ \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \leq 0) \right] \]

\[+ \sum_{i=1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} 1(\hat{f}_{ki} \leq 0) \]

\[- \sum_{i: \bar{a}_i < 1} \left( 1 - Q_{ii} - \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \right) \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0) \]

\[- \sum_{i: \bar{a}_i = 1} 1(\dot{u}_i = 0) \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) \]

\[- \sum_{i: \bar{a}_i = 1} \left( 1 - Q_{ii} 1(\dot{u}_i > 0) - \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \right) \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0). \]  (B.11)
and if \{i : \bar{a}_i = 1\} = \emptyset, or \{i : \bar{a}_i = 1\} \neq \emptyset and \sum_{i, a_i = 1} e_{ii} > \bar{m}, then

\[
\frac{1}{2} \dot{V}(e, f) = \sum_{i : \bar{a}_i < 1} \lambda_i (\bar{a}_i - 1) \left( 1 - \sum_{j=1}^{r} P_{ij}1(\dot{f}_{ij} > 0) \right) 1(\dot{u}_i = 0) \\
- \sum_{i=1}^{r} \left( 1 - \sum_{j=1}^{r} P_{ij}1(\dot{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji}1(\dot{e}_{ji} \geq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji}1(\dot{f}_{ji} \geq 0) \right) 1(\dot{u}_i > 0) \\
+ \sum_{i=1}^{r} \left( \sum_{j=1}^{r} P_{ij}1(\dot{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji}1(\dot{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji}1(\dot{f}_{ji} \leq 0) \right) 1(\dot{u}_i) \\
+ \sum_{i=1}^{r} 1(\dot{u}_i = 0) \left[ \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji}1(\dot{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji}1(\dot{f}_{ji} \leq 0) \right] \\
+ \sum_{i=1}^{r} \sum_{j=1}^{r} Q_{ij}1(\dot{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki}1(\dot{f}_{ki} \leq 0) \\
- \sum_{i=1}^{r} \left( 1 - Q_{ii} - \sum_{j=1}^{r} Q_{ij}1(\dot{e}_{ij} > 0) \right) \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji}1(\dot{f}_{ji} \geq 0).
\] (B.12)

**B.2 Proof of Lemma 5**

*Proof of Lemma 5.* For notational simplicity, we omit the argument \(t\) when referring to \(V(e(\cdot), f(\cdot)), e(\cdot), f(\cdot), u(\cdot)\) and their derivatives. We first work in the case when \(\{i : \bar{a}_i = 1\} \neq \emptyset, \bar{m} > 0, \) and \(\sum_{i, a_i = 1} e_{ii} < \bar{m},\) meaning that \(\frac{1}{2} \dot{V}(e, f)\) is given by (B.11). The following is a list of conditions that are necessary for \(\dot{V}(e, f) = 0.\) To obtain them, we use the fact that since the term in (B.11) is non-positive, each line there must equal zero for \(\dot{V}(e, f)\) to equal zero. Any condition on \(\dot{e}_{ij}\) for some \(i, j\) with \(i \neq j\) assumes \(Q_{ij} > 0,\) because if \(Q_{ij} = 0\) then \(e_{ij}\) always equals zero.

1. For all \(i\) such that \(\bar{a}_i < 1\) and \(\dot{u}_i = 0,\) we need

\[
\hat{f}_{ij} > 0, \quad \hat{f}_{ji} \geq 0, \quad 1 \leq j \leq r \\
\hat{e}_{ij} > 0, \quad \hat{e}_{ji} \geq 0, \quad 1 \leq j \leq r, \quad j \neq i.
\]

2. For all \(i\) such that \(\bar{a}_i < 1\) and \(\dot{u}_i > 0,\) one of the following three mutually exclusive conditions must hold:
a. 
\[ \hat{f}_{ij} > 0, \quad \hat{f}_{ji} \geq 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ij} > 0, \quad \hat{e}_{ji} \geq 0, \quad 1 \leq j \leq r, \ j \neq i, \]

b. 
\[ \hat{f}_{ij} \leq 0, \quad \hat{f}_{ji} \leq 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ji} \leq 0, \quad 1 \leq j \leq r, \ j \neq i, \]

and if \( \hat{f}_{ki} < 0 \) for some \( k \), then \( \hat{e}_{ij} \leq 0 \) for all \( j \neq i \).

c. \( f_{ij} > 0 \) and \( f_{ik} \leq 0 \) for some \( j, k = 1, \ldots, r \), and
\[ \hat{f}_{ji} = 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ji} = 0, \quad 1 \leq j \leq r, \ j \neq i, \]

3. For all \( i \) such that \( \bar{a}_i = 1 \) and \( \dot{u}_i = 0 \), we need
\[ \hat{f}_{ij} \leq 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ji} \leq 0, \quad 1 \leq j \leq r, \ j \neq i, \]

and if \( \hat{f}_{ki} < 0 \) for some \( k \), then \( \hat{e}_{ij} \leq 0 \) for all \( j \neq i \).

4. For all \( i \) such that \( \bar{a}_i = 1 \) and \( \dot{u}_i > 0 \), one of the following three mutually exclusive conditions must hold:

a. 
\[ \hat{f}_{ij} > 0, \quad \hat{f}_{ji} \geq 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ij} > 0, \quad \hat{e}_{ji} \geq 0, \quad 1 \leq j \leq r, \ j \neq i, \]

b. 
\[ \hat{f}_{ij} \leq 0, \quad \hat{f}_{ji} \leq 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ji} \leq 0, \quad 1 \leq j \leq r, \ j \neq i, \]

and if \( \hat{f}_{ki} < 0 \) for some \( k \), then \( \hat{e}_{ij} \leq 0 \) for all \( j \neq i \).

c. \( f_{ij} > 0 \) and \( f_{ik} \leq 0 \) for some \( j \) and \( k \), and
\[ \hat{f}_{ji} = 0, \quad 1 \leq j \leq r \]
\[ \hat{e}_{ji} = 0, \quad 1 \leq j \leq r, \ j \neq i, \]
We now argue that there does not exist a configuration of \(e_{ij}\)'s and \(f_{ij}\)'s that satisfies conditions 1–4. If a region satisfies one of the conditions above, we refer to it as a region of type corresponding to the condition it satisfies. For example, if region \(i\) satisfies condition 2b, we refer to it as a type 2b region.

First observe that a region \(i\) can never be of type 4a or 4c. For such a region, the fact that \(\dot{u}_i > 0\), together with (4.16) and (B.3) will imply that \(\dot{e}_{ii} = 0\), or

\[
\lambda_i (1 - \dot{u}_i) = \sum_{j=1}^{r} \mu_{ji} e_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} f_{ji}.
\]

However, the conditions in 4a and 4c also imply that

\[
\sum_{j=1}^{r} \mu_{ji} e_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} f_{ji} \geq \sum_{j=1}^{r} \mu_{ji} \bar{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \bar{f}_{ji} = \lambda_i,
\]

where the equality above follows from (4.13) and the fact that \(\bar{a}_i = 1\). This leads to a contradiction because conditions 4a and 4c require that \(\dot{u}_i > 0\).

Second, by our assumption that \(\{i : \bar{a}_i = 1\} \neq \emptyset\), there must always be a region of either type 3 or type 4b. This implies that there are no type 1 or type 2a regions. If \(i\) were a type 1 or 2a region and \(j\) were a type 3 or 4b region, then the conditions in 1 and 2a would require \(\bar{f}_{ij} > 0\), but the conditions in 3 and 4b would require that \(\bar{f}_{ij} \leq 0\), which is a contradiction.

Third, we argue that there cannot be a type 2c region. Suppose \(i\) is a type 2c region. Then for some region \(j\), we would have \(\bar{f}_{ij} > 0\). However, this region \(j\) could not belong to any of types 2b, 2c, 3, or 4b, causing a contradiction.

Lastly, we show why it cannot be that \(\dot{V}(e, f) = 0\). We have shown that all regions must be of types 2b, 3, or 4b. Together with our assumption that \(\bar{m} > 0\) and \(\sum_{i: \bar{a}_i = 1} e_{ii} < \bar{m}\), this implies that

\[
\sum_{i=1}^{r} \sum_{j=1}^{r} f_{ji} + \sum_{i=1}^{r} \sum_{j=1}^{r} e_{ji} + \sum_{i=1}^{r} e_{ii} = \sum_{i=1}^{r} f_{ji} + \sum_{i=1}^{r} e_{ji} + \sum_{i: \bar{a}_i = 1} e_{ii} < \sum_{i=1}^{r} \sum_{j=1}^{r} \bar{f}_{ji} + \sum_{i=1}^{r} \bar{e}_{ji} + \bar{m} = 1,
\]

where in the first equality we used the fact that any region \(i\) with \(\bar{a}_i < 1\) must satisfy \(\dot{u}_i > 0\), which together with by (4.16) implies \(e_{ii} = 0\). The result above implies that the total mass in the system is strictly less than one, which is impossible because the total mass in the system must always equal one. Hence, we have just shown that at any regular point, \(\dot{V}(e, f) < 0\) in the case when \(\{i : \bar{a}_i = 1\} \neq \emptyset\), \(\bar{m} > 0\), and \(\sum_{i: \bar{a}_i = 1} e_{ii} < \bar{m}\).
We now assume that \( \{ i : \bar{a}_i = 1 \} = \emptyset \), or \( \{ i : \bar{a}_i = 1 \} \neq \emptyset \) and \( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \). Just as before, we assume that \( \hat{V}(e, f) = 0 \) and use (B.12) to list the necessary conditions required for that to happen. As before, conditions 1, 2, and 4 must hold, and a region cannot be of type 4a or 4c. Only condition 3 changes slightly to become

3’ For all \( i \) such that \( \bar{a}_i = 1 \) and \( \hat{u}_i = 0 \),

\[
\begin{align*}
\hat{f}_{ji} & \geq 0, & 1 \leq j \leq r \\
\hat{e}_{ji} & \geq 0, & 1 \leq j \leq r, \ j \neq i,
\end{align*}
\]

and if \( \hat{f}_{ki} > 0 \) for some \( k \), then \( \hat{e}_{ij} > 0 \) for all \( j \neq i \).

Again, any condition on \( \hat{e}_{ij} \) assumes \( Q_{ij} > 0 \), because otherwise \( e_{ij} \) always equals zero.

First, we show that there must be a region of type 2b, 2c or 4b. If that were not the case, then all regions would be of type 1, 2a or 3’. Together with our assumption that \( (e, f) \notin \mathcal{E}, \{ i : \bar{a}_i = 1 \} = \emptyset \), or \( \{ i : \bar{a}_i = 1 \} \neq \emptyset \) and \( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \), this would imply that

\[
\sum_{i=1}^{r} \sum_{j=1}^{r} f_{ij} + \sum_{i=1}^{r} \sum_{j=1}^{r} e_{ij} + \sum_{i=1}^{r} e_{ii} > 1,
\]

which cannot happen. The fact that there is always a region of type 2b, 2c or 4b implies that there cannot be any type 1 or 2a regions.

Second, we show that there must always be a region of type 3’. If there is a region of type 4b then it must be that \( \{ i : \bar{a}_i = 1 \} \neq \emptyset \) and \( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \), meaning that there is also a region of type 3’. Now suppose there is no region of type 4b. Then it cannot be the case that all regions are exclusively of type 2b or exclusively of type 2c. The former case would imply that the total mass in the system is strictly less than one, and the latter case cannot happen by definition of 2c (i.e. if \( i \) were of type 2c then there would be some \( j \) with \( \hat{f}_{ij} > 0 \), but this \( j \) could not be of type 2b or 2c). The same reasoning implies that the regions cannot be a mixture of type 2b and 2c regions exclusively. Therefore, there must always exist a region of type 3’.

Lastly, we show why it cannot be that \( \hat{V}(e, f) = 0 \). Let \( i \) be a type 3’ region with \( e_{ii} > 0 \) (such a region must always exist because \( \sum_{i : \bar{a}_i = 1} e_{ii} > \bar{m} \)). There must exist some \( k, \ell \), such that either \( \hat{f}_{k\ell} < 0 \), or \( k \neq \ell \) and \( \hat{e}_{k\ell} < 0 \). Observe that \( \ell \) cannot be a type 2c or 3’ region, so it must be a type 2b or 4b region. Furthermore, since \( i \) is of type 3’ and \( \ell \) is of type 2b or 4b, it must be true that \( \hat{f}_{i\ell} = 0 \), and by (B.2) this would imply that \( \hat{f}_{\ell i} = 0 \). However, we will now show that \( \hat{f}_{\ell i} \) must also be strictly less than zero, leading to a contradiction. By \( f_{\ell i} = \hat{f}_{\ell i} \) and (4.11), it follows that

\[
0 = \hat{f}_{\ell i} = \lambda_{\ell} P_{\ell i}(1 - \hat{u}_{\ell}) - \mu_{\ell i} f_{\ell i} = \lambda_{\ell} P_{\ell i}(1 - \hat{u}_{\ell}) - \lambda_{\ell} P_{\ell i} \bar{a}_{\ell}.
\]
We know that $\dot{u}_\ell > 0$ by definition of a type 2b and 4b region. Hence, $\dot{e}_{\ell\ell} = 0$ by (4.16), which in turn means that $\dot{e}_{\ell\ell} = 0$ by (B.3), and therefore

$$
\lambda_\ell (1 - \dot{u}_\ell) 1(\dot{u}_\ell > 0) = \sum_{j=1}^{r} \mu_{j\ell} e_{j\ell} + Q_{\ell\ell} \sum_{j=1}^{r} \mu_{j\ell} f_{j\ell}
$$

$$
= \lambda_\ell \bar{a}_\ell + \sum_{j=1}^{r} \mu_{j\ell} \dot{e}_{j\ell} + Q_{\ell\ell} \sum_{j=1}^{r} \mu_{j\ell} \dot{f}_{j\ell},
$$

where the second equality follows from (4.13). We combine (B.13) with the form of $\dot{f}_{\ell i}$ and the fact that either $\dot{f}_{k\ell} < 0$, or $k \neq \ell$ and $\dot{e}_{k\ell} < 0$, we see that

$$
\dot{f}_{\ell i} = P_{\ell i} \left( \lambda_\ell \bar{a}_\ell + \sum_{j=1}^{r} \mu_{j\ell} \dot{e}_{j\ell} + Q_{\ell\ell} \sum_{j=1}^{r} \mu_{j\ell} \dot{f}_{j\ell} \right) - \lambda_\ell P_{\ell i} \bar{a}_\ell < 0,
$$

which is a contradiction because $\dot{f}_{\ell i} = 0$ at regular points. This concludes the proof of Lemma 5.

B.3 Proof of Lemma 6

Proof of Lemma 6. We begin with the case that $\{ i : \bar{a}_i = 1 \} \neq \emptyset$, $\bar{m} > 0$, and $\sum_{i: \bar{a}_i = 1} e_{ii} < \bar{m}$. From (B.10) and (4.1)–(4.3), we know that

$$
\frac{1}{2} \dot{V}(e, f) = \sum_{i: \bar{a}_i < 1} \left[ -\lambda_i (1 - \dot{u}_i) + \sum_{j=1}^{r} \mu_{ji} e_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} f_{ji} \right]
$$

$$
+ \sum_{i=1}^{r} \sum_{j=1}^{r} \left[ -\mu_{ij} e_{ij} + Q_{ij} \sum_{k=1}^{r} \mu_{ki} f_{ki} \right] 1(\dot{e}_{ij} > 0)
$$

$$
+ \sum_{i=1}^{r} \sum_{j=1}^{r} \left[ \lambda_i P_{ij} (1 - \dot{u}_i) - \mu_{ij} f_{ij} \right] 1(\dot{f}_{ij} > 0),
$$

44
Similarly we use (4.12) to get the first equality. Using (4.13), we see that the sum in (B.15) equals
\[
\sum_{i:a_i<1} \left[ \lambda_i(1 - \dot{u}_i) + \sum_{j=1 \atop j \neq i}^r \mu_{ji}\dot{e}_{ji} + Q_{ii} \sum_{j=1}^r \mu_{ji}\dot{f}_{ji} \right] + \sum_{i:a_i<1} \left[ \sum_{j=1 \atop j \neq i}^r \mu_{ji}\dot{e}_{ji} + Q_{ii} \sum_{j=1}^r \mu_{ji}\dot{f}_{ji} \right]
\]
\[
= \sum_{i:a_i<1} \lambda_i(\dot{u}_i - 1)1(\dot{u}_i = 0) + \sum_{i:a_i<1} \left( -\lambda_i(1 - \dot{u}_i) + \lambda_i\dot{a}_i \right) \lambda_i(\dot{u}_i > 0)
\+ \sum_{i:a_i<1} \left[ \sum_{j=1 \atop j \neq i}^r \mu_{ji}\dot{e}_{ji} + Q_{ii} \sum_{j=1}^r \mu_{ji}\dot{f}_{ji} \right]. \tag{B.17}
\]

Above, we used (4.13) to get the first equality. Using (4.12), we see that the sum in (B.15) equals
\[
\sum_{i=1 \atop i:a_i<1} \sum_{j=1 \atop j \neq i}^r \left[ -\mu_{ij}\dot{e}_{ij} + Q_{ij} \sum_{k=1}^r \mu_{ki}\hat{f}_{ki} \right] 1(\dot{e}_{ij} > 0) + \sum_{i=1 \atop i:a_i<1} \sum_{j=1 \atop j \neq i}^r \left[ -\mu_{ij}\dot{e}_{ij} + Q_{ij} \sum_{k=1}^r \mu_{ki}\hat{f}_{ki} \right] 1(\dot{e}_{ij} > 0)
\]
\[
= -\sum_{i=1 \atop i:a_i<1} \sum_{j=1 \atop j \neq i}^r \mu_{ji}\dot{e}_{ji} 1(\dot{e}_{ji} > 0) + \sum_{i=1 \atop i:a_i<1} \sum_{j=1 \atop j \neq i}^r Q_{ij} 1(\dot{e}_{ij} > 0) \sum_{k=1}^r \mu_{ki}\hat{f}_{ki}. \tag{B.18}
\]

Similarly we use (4.11) to see that the sum in (B.16) equals
\[
\sum_{i=1 \atop i:a_i<1} \sum_{j=1}^r \left[ \lambda_i\dot{P}_{ij}(1 - \dot{u}_i) - \mu_{ij}\dot{f}_{ij} \right] 1(\dot{f}_{ij} > 0) - \sum_{i=1 \atop i:a_i<1} \sum_{j=1}^r \mu_{ij}\dot{f}_{ij} 1(\dot{f}_{ij} > 0)
\]
\[
= \sum_{i=1}^r 1(\dot{u}_i = 0)\lambda_i(1 - \dot{u}_i) \sum_{j=1}^r \dot{P}_{ij} 1(\dot{f}_{ij} > 0)
\+ \sum_{i=1}^r 1(\dot{u}_i > 0) \sum_{j=1}^r \dot{P}_{ij} \left[ \lambda_i(1 - \dot{u}_i) - \lambda_i\dot{a}_i \right] 1(\dot{f}_{ij} > 0) - \sum_{i=1 \atop i:a_i<1} \sum_{j=1}^r \mu_{ji}\dot{f}_{ji} 1(\dot{f}_{ji} > 0) \tag{B.19}
\]
Making use of (B.13), we can combine (B.17)–(B.19) to see that

\[
\frac{1}{2} \dot{V}(e, f) = \sum_{i: 0 < a_i < 1} \lambda_i (a_i - 1) \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) 1(\hat{u}_i = 0) \tag{B.20}
\]

\[
- \sum_{i: 0 < a_i < 1} \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} \right) 1(\hat{u}_i > 0) \tag{B.21}
\]

\[
+ \sum_{i: 0 < a_i < 1} \left( \sum_{k=1}^{r} P_{ik} 1(\hat{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} \right) 1(\hat{u}_i > 0) \tag{B.22}
\]

\[
+ \sum_{i: 0 < a_i < 1} \left[ \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} \right] \tag{B.23}
\]

\[
- \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} > 0) + \sum_{i=1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} \tag{B.24}
\]

\[
- \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} > 0). \tag{B.25}
\]

Since the term above is very bulky, we manipulate one line at a time to help exposition. We start with (B.21), which we decompose into

\[
- \sum_{i: 0 < a_i < 1} \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} \right) 1(\hat{u}_i > 0)
\]

\[
- \sum_{i: 0 < a_i < 1} \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\hat{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} \right) 1(\hat{u}_i > 0).
\]

The term in (B.23) equals

\[
\sum_{i: 0 < a_i < 1} \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} > 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} > 0)
\]

\[
+ \sum_{i: 0 < a_i < 1} \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} > 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} > 0).
\]
The term in (B.24) equals

\[ - \sum_{i: \hat{a}_i < 1} r \sum_{j: j \neq i}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) + \sum_{i: \hat{a}_i < 1} r \sum_{j: j \neq i}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} 1(\hat{f}_{ki} \leq 0) \]

\[ + \sum_{i: \hat{a}_i < 1} r \sum_{j: j \neq i}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} 1(\hat{f}_{ki} \geq 0) \]

\[ - \sum_{i: \hat{a}_i = 1} r \sum_{j: j \neq i}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) + \sum_{i: \hat{a}_i = 1} r \sum_{j: j \neq i}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki}, \]

and the term in (B.25) equals

\[ - \sum_{i: \hat{a}_i < 1} r \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0) - \sum_{i: \hat{a}_i = 1} r \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0). \]
Putting all of these expansions back into (B.20)–(B.25), we see that

\[
\frac{1}{2} \dot{V}(e, f) = \sum_{i: a_i < 1} \lambda_i (\bar{a}_i - 1) \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\dot{f}_{ij} > 0) \right) 1(\dot{u}_i = 0)
\]

\[
- \sum_{i: a_i < 1} \left( 1 - \sum_{j=1}^{r} P_{ij} 1(\dot{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \geq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0) \right) 1(\dot{u}_i = 0)
\]

\[
+ \sum_{i: a_i < 1} \left( \sum_{j=1}^{r} P_{ij} 1(\dot{f}_{ij} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \leq 0) \right) 1(\dot{u}_i > 0)
\]

\[
+ \sum_{i: a_i < 1} 1(\dot{u}_i = 0) \left[ \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \leq 0) \right]
\]

\[
+ \sum_{i: a_i < 1} \sum_{j=1}^{r} \sum_{k=1, j \neq i} Q_{ij} 1(\dot{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki} 1(\dot{f}_{ki} \leq 0)
\]

\[
- \sum_{i: a_i < 1} \left( 1 - Q_{ii} - \sum_{j=1}^{r} Q_{ij} 1(\dot{e}_{ij} > 0) \right) \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0)
\]

\[
+ \sum_{i: a_i = 1} \left( \sum_{k=1}^{r} P_{ik} 1(\dot{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} \right) 1(\dot{u}_i > 0)
\]

\[
- \sum_{i: a_i = 1} \sum_{j=1, j \neq i} \sum_{k=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \geq 0) + \sum_{i: a_i = 1} \sum_{j=1, j \neq i} Q_{ij} 1(\dot{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki}
\]

\[
- \sum_{i: a_i = 1} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0).
\]
It remains to manipulate the terms in (B.32)–(B.34) to get them into the form we need. We begin with (B.32), which equals

$$
\sum_{i: \hat{a}_i = 1}^{r} \left( \sum_{k=1}^{r} P_{ik} 1(\hat{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \leq 0) \right) 1(\hat{u}_i > 0)
$$

$$
\sum_{i: \hat{a}_i = 1}^{r} \left( \sum_{k=1}^{r} P_{ik} 1(\hat{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0) \right) 1(\hat{u}_i > 0).
$$

The term in (B.33) equals

$$
- \sum_{i: \hat{a}_i = 1}^{r} \sum_{j=1}^{r} \mu_{ji} \hat{e}_{ji} 1(\hat{e}_{ji} \geq 0) + \sum_{i: \hat{a}_i = 1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} 1(\hat{f}_{ki} \leq 0)
$$

$$
+ \sum_{i: \hat{a}_i = 1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\hat{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \hat{f}_{ki} 1(\hat{f}_{ki} \geq 0),
$$

and the term in (B.34) equals

$$
- \sum_{i: \hat{a}_i = 1}^{r} \left( 1 - Q_{ii} 1(\hat{u}_i > 0) \right) \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0) - \sum_{i: \hat{a}_i = 1}^{r} Q_{ii} 1(\hat{u}_i > 0) \sum_{j=1}^{r} \mu_{ji} \hat{f}_{ji} 1(\hat{f}_{ji} \geq 0).
$$
Combining all of these expansions together, we arrive at
\[
\sum_{i:a_i=1}^{r} \left( \sum_{k=1}^{r} P_{ik}(\dot{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{jj} \dot{e}_{ji} + Q_{ij} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} \right) 1(\dot{u}_i > 0) \\
- \sum_{i:a_i=1}^{r} \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \geq 0) + \sum_{i:a_i=1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\dot{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki} \\
- \sum_{i:a_i=1}^{r} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0) \\
= \sum_{i:a_i=1}^{r} \left( \sum_{k=1}^{r} P_{ik}(\dot{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{jj} \dot{e}_{ji} 1(\dot{e}_{ji} \leq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \leq 0) \right) 1(\dot{u}_i > 0) \\
- \sum_{i:a_i=1}^{r} \left( 1 - \sum_{k=1}^{r} P_{ik}(\dot{f}_{ik} > 0) \right) \left( \sum_{j=1}^{r} \mu_{jj} \dot{e}_{ji} 1(\dot{e}_{ji} \geq 0) + Q_{ii} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0) \right) 1(\dot{u}_i > 0) \\
- \sum_{i:a_i=1}^{r} 1(\dot{u}_i = 0) \sum_{j=1}^{r} \mu_{ji} \dot{e}_{ji} 1(\dot{e}_{ji} \geq 0) + \sum_{i:a_i=1}^{r} \sum_{j=1}^{r} Q_{ij} 1(\dot{e}_{ij} > 0) \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki} 1(\dot{f}_{ki} \leq 0) \\
- \sum_{i:a_i=1}^{r} \left( 1 - Q_{ii} 1(\dot{u}_i > 0) - \sum_{j=1}^{r} Q_{ij} 1(\dot{e}_{ij} > 0) \right) \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} 1(\dot{f}_{ji} \geq 0).
\]

The form of $\frac{1}{2} \dot{V}(e, f)$ we just derived can be compared with (B.11) to see that it matches.

Now suppose that \{i : \tilde{a}_i = 1\} = \emptyset, or \{i : \tilde{a}_i = 1\} \neq \emptyset and \sum_{i:a_i=1}^{r} e_{ii} > \bar{m}. We have already done all the work to justify (B.12) does not require any extra work. Indeed, by the same logic used to derive (B.14)–(B.16), we know that
\[
\frac{1}{2} \dot{V}(e, f) = \sum_{i=1}^{r} \left[ -\lambda_i(1 - \tilde{u}_i) + \sum_{j=1}^{r} \mu_{jj} e_{ji} + Q_{ij} \sum_{j=1}^{r} \mu_{ji} \dot{f}_{ji} \right] \\
+ \sum_{i=1}^{r} \sum_{j=1}^{r} \left[ -\mu_{ij} e_{ij} + Q_{ij} \sum_{k=1}^{r} \mu_{ki} \dot{f}_{ki} \right] 1(\dot{e}_{ij} > 0) \\
+ \sum_{i=1}^{r} \sum_{j=1}^{r} \left[ \lambda_i P_{ij} (1 - \tilde{u}_i) - \mu_{ij} \dot{f}_{ij} \right] 1(\dot{f}_{ij} > 0).
\]

The difference between the form of $\frac{1}{2} \dot{V}(e, f)$ above and that of (B.14)–(B.16) is that the summation in the first line is over all $i = 1, \ldots, r$, as opposed to only those $i$ for which
\( \bar{a}_i < 1 \). Therefore, \( \frac{1}{2} \bar{V}(e,f) \) equals (B.26)--(B.31) with summations over all \( i = 1, \ldots, r \), and not just those \( i \) such that \( \bar{a}_i < 1 \). This verifies (B.12) and concludes the proof of this lemma. \[ \square \]

C Proof of Lemma 2

Proof of Lemma 2. Let \((\bar{e}^*, \bar{f}^*, \bar{a}^*)\) be one optimal solution to the relaxed optimization problem (2.28)--(2.29).

Case 1: Suppose \( \bar{a}_i^* < 1 \), for all \( 1 \leq i \leq r \). We argue by contradiction that this implies \( \bar{e}_{ii}^* = 0 \), for all \( i \). Suppose there exists region \( i' \) such that \( \bar{e}_{i'i'}^* > 0 \). We now construct another solution \((\tilde{e}, \tilde{f}, \tilde{a})\) that is better than \((\bar{e}^*, \bar{f}^*, \bar{a}^*)\). Assume for now that there exists an \( r \times r \) matrix with non-negative entries \( \pi_{ij} \), such that

\[
P_{ij} \sum_k \pi_{ki} \mu_{ki} = \mu_{ij} \pi_{ij}, \quad 1 \leq i, j \leq r. \tag{C.1}
\]

Fix \( \epsilon > 0 \) to be specified later, and let

\[
\begin{align*}
\tilde{e}_{ij} &= \bar{e}_{ij}^* - \epsilon, \\
\tilde{f}_{ij} &= \bar{f}_{ij}^* + \epsilon \pi_{ij}, \quad 1 \leq i, j \leq r, \\
\tilde{a}_i &= \bar{a}_i^* + \epsilon \sum_{k=1}^{r} \pi_{ki} \mu_{ki}, \quad 1 \leq i \leq r.
\end{align*}
\]

Since \( \tilde{a}_i \geq \bar{a}_i^* \) and \( \bar{f}_{ij} \geq \bar{f}_{ij}^* \) for all \( i, j \), and \( \tilde{e}_{ij} = \tilde{e}_{ij}^* \) for \( i \neq j \), it follows that

\[
\sum_{i=1}^{r} c_i \lambda_i \tilde{a}_i^* - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \tilde{e}_{ij} \bar{e}_{ij}^* \leq \sum_{i=1}^{r} c_i \lambda_i \bar{a}_i^* - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \tilde{e}_{ij} \bar{e}_{ij}, \tag{C.2}
\]

We now check that \((\tilde{e}, \tilde{f}, \tilde{a})\) satisfies (2.17)--(2.22), and is therefore a feasible solution. Since \((\bar{e}^*, \bar{f}^*, \bar{a}^*)\) is a feasible solution and satisfies (2.17), it follows that

\[
\lambda_i P_{ij} \tilde{a}_i = \lambda_i P_{ij} \left( \bar{a}_i^* + \epsilon \sum_{k=1}^{r} \pi_{ki} \mu_{ki} / \lambda_i \right) = \mu_{ij} \bar{f}_{ij}^* + \epsilon P_{ij} \sum_{k=1}^{r} \pi_{ki} \mu_{ki} \]

\[
= \mu_{ij} \tilde{f}_{ij}^* + \epsilon \mu_{ij} \pi_{ij} \]

\[
= \mu_{ij} \tilde{f}_{ij}, \quad 1 \leq i, j \leq r,
\]

meaning \((\tilde{e}, \tilde{f}, \tilde{a})\) satisfies (2.17). Next, we see that

\[
\mu_{ij} \tilde{e}_{ij} = \mu_{ij} \tilde{e}_{ij}^* \leq \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki}^* \leq \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki}, \quad 1 \leq i \neq j \leq r,
\]

51
Therefore $(\tilde{e}, \tilde{f}, \tilde{a})$ satisfies (2.18). To verify (2.20), observe that
\[
\lambda_i \tilde{a}_i = \lambda_i \left( \tilde{a}_i^* + \epsilon \frac{\sum_{k=1}^{r} \pi_{ki} \mu_{ki}}{\lambda_i} \right) \leq \sum_{k=1, k \neq i}^{r} \mu_{ki} \tilde{e}_{ki} + \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki} + \epsilon \sum_{k=1}^{r} \pi_{ki} \mu_{ki} + \epsilon \sum_{k=1}^{r} \pi_{ki} \mu_{ki} - \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki} + \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki} + \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki}
\]
\[
= \sum_{k=1, k \neq i}^{r} \mu_{ki} \tilde{e}_{ki} + \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki}, \quad 1 \leq i \leq r.
\]
Therefore $(\tilde{e}, \tilde{f}, \tilde{a})$ satisfies (2.19). To verify (2.20), observe that
\[
\lambda_i \tilde{a}_i + \sum_{j=1, j \neq i}^{r} \mu_{ij} \tilde{e}_{ij} - \sum_{k=1, k \neq i}^{r} \mu_{ki} \tilde{e}_{ki} - \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki}
\]
\[
= \lambda_i \tilde{a}_i^* + \sum_{j=1, j \neq i}^{r} \mu_{ij} \tilde{e}_{ij} - \sum_{k=1, k \neq i}^{r} \mu_{ki} \tilde{e}_{ki} - \sum_{k=1}^{r} \mu_{ki} \tilde{f}_{ki} + \epsilon \sum_{k=1}^{r} \pi_{ki} \mu_{ki} - \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki} - \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki} + \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki} + \epsilon \sum_{k=1}^{r} \mu_{ki} \pi_{ki}
\]
\[
= 0,
\]
and therefore (2.20) holds under $(\tilde{e}, \tilde{f}, \tilde{a})$. Lastly, (2.21) holds under because $\sum_{ij} \pi_{ij} = 1$, and $\epsilon$ can always be chosen small enough to ensure (2.22) holds. We conclude that $(\tilde{e}, \tilde{f}, \tilde{a})$ is a feasible solution to (2.28)–(2.29) that is better than $(\tilde{e}^*, \tilde{f}^*, \tilde{a}^*)$, which contradicts the fact that $(\tilde{e}^*, \tilde{f}^*, \tilde{a}^*)$ is an optimal solution.

It remains to verify that we can choose non-negative $\pi_{ij}$’s to satisfy $\sum_{ij} \pi_{ij} = 1$ and (C.1). Consider a CTMC defined on the space $\{1, \cdots, r\}^2$. For all $1 \leq i, j, k \leq r$, the transition rate from $(k, i)$ to $(i, k)$ is $\mu_{ki} P_{ik}$. No other transitions are possible. Since the CTMC is defined on a finite state space, it has a stationary distribution. Furthermore, any stationary distribution $\nu$ must satisfy the flow-balance equations
\[
\sum_{k=1}^{r} \nu_{ki} \mu_{ki} P_{ij} = \sum_{\ell=1}^{r} \nu_{ij} \mu_{ij} P_{ij}, \quad 1 \leq i, j \leq r,
\]
or
\[
P_{ij} \sum_{k=1}^{r} \nu_{ki} \mu_{ki} = \nu_{ij} \mu_{ij}, \quad 1 \leq i, j \leq r,
\]

52
which are precisely the same as (C.1). Therefore we can take the $\pi_{ij}$’s to be any of the stationary distributions of such a CTMC.

**Case 2:** Now consider the case that there exists $i'$ such that $\bar{a}^*_{i'} = 1$. Without loss of generality, assume $i' = 1$. We then consider the following solution:

\[
\begin{align*}
\tilde{e}_{ij} &= e_{ij}^*, \quad 1 \leq i \neq j \leq r, \\
\tilde{e}_{11} &= \sum_{i=1}^{r} e_{ii}^*, \\
\tilde{e}_{ii} &= 0, \quad i \neq 1, \\
\tilde{f}_{ij} &= f_{ij}^*, \quad 1 \leq i, j \leq r, \\
\tilde{a}_i &= a_i^*, \quad 1 \leq i \leq r.
\end{align*}
\]

It is straightforward to verify that $(\tilde{e}, \tilde{f}, \tilde{a})$ is a feasible solution and yields the same objective value as $(\bar{e}^*, \bar{f}^*, \bar{a}^*)$. Furthermore, constraint (2.23) holds under $(\tilde{e}, \tilde{f}, \tilde{a})$. This completes the proof.

**D Proof of Theorem 2**

*Proof of Theorem 2.* We will show that $(\mathbb{E}[\bar{E}^{(N)}(\infty)], \mathbb{E}[\bar{F}^{(N)}(\infty)], A^{(N)})$ is a feasible solution to the optimization problem (2.28)–(2.29). Lemma 2 then implies part (a) of the theorem holds. Part (b) is an immediate consequence of Theorem 1 by setting $Q = q^*$. Recall from (4.20) that any function $g : T \to \mathbb{R}$ satisfies

\[
\mathbb{E}[G^{(N)}(\bar{E}^{(N)}(\infty), \bar{F}^{(N)}(\infty))] = 0.
\]

We now show that $(\mathbb{E}[\bar{E}^{(N)}(\infty)], \mathbb{E}[\bar{F}^{(N)}(\infty)], A^{(N)})$ satisfies (2.17)–(2.22).

- Condition (2.17) was already verified in (4.21).

- To check condition (2.18), we fix $i \neq j$, and use the test function $g(e, f) = e_{ij}$. Then

\[
G^{(N)}(e, f) = Q_{ij}(e, f) \sum_{k=1}^{r} \mu_{ki} N f_{ki} ((e_{ij} + 1/N) - e_{ij}) + \mu_{ij} N e_{ij} ((e_{ij} - 1/N) - e_{ij}),
\]

where $Q_{ij}(e, f)$ is the probability that upon dropping a passenger off at region $i$, a car drives empty to region $j$ given the current state of the system is $(e, f)$. Using (D.1) and the fact that $Q_{ij}(e, f) \in [0, 1]$, we see that
\[0 = \mathbb{E}\left[ Q_{ij}\left( \tilde{E}(N)(\infty), \tilde{F}(N)(\infty) \right) \sum_{k=1}^{r} \mu_{ki} \tilde{F}_{ki}(N)(\infty) - \mu_{ij} \tilde{E}_{ij}(N)(\infty) \right] \] (D.2)

\[
\leq \sum_{k=1}^{r} \mu_{ki} \mathbb{E}[\tilde{F}_{ki}(N)(\infty)] - \mu_{ij} \mathbb{E}[\tilde{E}_{ij}(N)(\infty)].
\]

• For condition (2.19), we fix \(i\) and use the test function \(g(e, f) = e_{ii}\). Then

\[G^{(N)} g(e, f) = N\lambda_{i} 1(e_{ii} > 0)((e_{ii} - 1/N) - e_{ii})
\]

\[+ \left( Q_{ii}(e, f) \sum_{k=1}^{r} \mu_{ki} N f_{ki} + \sum_{k=1, k \neq i}^{r} \mu_{ki} N e_{ki} \right) ((e_{ii} + 1/N) - e_{ii}).\]

Taking the expected value and using (D.1), we see that

\[\lambda_{i} \mathbb{P}(\tilde{E}_{ii}(N)(\infty) > 0) = \mathbb{E}\left[ Q_{ii}(\tilde{E}(N)(\infty), \tilde{F}(N)(\infty)) \sum_{k=1}^{r} \mu_{ki} \tilde{F}_{ki}(N)(\infty) + \sum_{k=1, k \neq i}^{r} \mu_{ki} \tilde{E}_{ki}(N)(\infty) \right] \] (D.3)

Using the fact that \(Q_{ii}(e, f) \in [0, 1]\), we conclude that

\[0 \leq \sum_{k=1}^{r} \mu_{ki} \mathbb{E}[\tilde{F}_{ki}(N)(\infty)] + \sum_{k=1, k \neq i}^{r} \mu_{ki} \mathbb{E}[\tilde{E}_{ki}(N)(\infty)] - \lambda_{i} \mathbb{P}(\tilde{E}_{ii}(N)(\infty) > 0),\]

and

\[0 \geq \sum_{k=1, k \neq i}^{r} \mu_{ki} \mathbb{E}[\tilde{E}_{ki}(N)(\infty)] - \lambda_{i} \mathbb{P}(\tilde{E}_{ii}(N)(\infty) > 0).\]

• Fix \(i\). To check condition (2.20), we could use the test function \(g(e, f) = \sum_{j=1}^{r} e_{ij}\). Alternatively, it is easier to just add up (D.2) for all \(j \neq i\) together with (D.3) to arrive at

\[\lambda_{i} \mathbb{P}(\tilde{E}_{ii}(N)(\infty) > 0) + \sum_{j=1, j \neq i}^{r} \mu_{ij} \mathbb{E}[\tilde{E}_{ij}(N)(\infty)]
\]

\[= \sum_{k=1}^{r} \mu_{ki} \mathbb{E}[\tilde{F}_{ki}(N)(\infty)] + \sum_{k=1, k \neq i}^{r} \mu_{ki} \mathbb{E}[\tilde{E}_{ki}(N)(\infty)].\]

• Conditions (2.21) and (2.22) hold trivially.

\[\square\]
References

[1] Anantharam, V. and Benchekroun, M. (1993). A technique for computing sojourn times in large networks of interacting queues. Probability in the Engineering and Informational Sciences, 7 441–464. URL http://journals.cambridge.org/article_S0269964800003065.

[2] Asmussen, S. (2003). Applied probability and queues, vol. 51 of Applications of Mathematics (New York). 2nd ed. Springer-Verlag, New York. Stochastic Modelling and Applied Probability.

[3] Banerjee, S., Freund, D. and Lykouris, T. (2016). Multi-objective pricing for shared vehicle systems. Preprint, URL http://arxiv.org/abs/1608.06819.

[4] Baskett, F., Chandy, K. M., Muntz, R. R. and Palacios, F. G. (1975). Open, closed and mixed networks of queues with different classes of customers. Journal of the Association for Computing Machinery, 22 248–260.

[5] Billingsley, P. (1999). Convergence of probability measures. 2nd ed. Wiley, New York.

[6] Chemla, D., Meunier, F. and Calvo, R. W. (2013). Bike sharing systems: Solving the static rebalancing problem. Discrete Optimization, 10 120 – 146. URL http://www.sciencedirect.com/science/article/pii/S1572528612000771.

[7] Dai, J. G., He, S. and Tezcan, T. (2010). Many-server diffusion limits for G/Ph/n + GI queues. Annals of Applied Probability, 20 1854–1890.

[8] George, D. K. and Xia, C. H. (2011). Fleet-sizing and service availability for a vehicle rental system via closed queueing networks. European Journal of Operational Research, 211 198 – 207. URL http://www.sciencedirect.com/science/article/pii/S0377221710008817.

[9] George, D. K., Xia, C. H. and Squillante, M. S. (2012). Exact-order asymptotic analysis for closed queueing networks. J. Appl. Probab., 49 503–520. URL http://dx.doi.org/10.1239/jap/1339878801.

[10] Glynn, P. W. and Zeevi, A. (2008). Bounding stationary expectations of Markov processes. In Markov processes and related topics: a Festschrift for Thomas G. Kurtz, vol. 4 of Inst. Math. Stat. Collect. Inst. Math. Statist., Beachwood, OH, 195–214. URL http://dx.doi.org/10.1214/074921708000000381.

[11] Harrison, J. M. and Reiman, M. I. (1981). Reflected Brownian motion on an orthant. Annals of Probability, 9 302–308.
[12] Henderson, S. G., O’Mahony, E. and Shmoys, D. B. (2016). (Citi)Bike sharing. Submitted for publication.

[13] Iglesias, R., Rossi, F., Zhang, R. and Pavone, M. (2016). A bcmp network approach to modeling and controlling autonomous mobility-on-demand systems. Preprint, URL http://arxiv.org/abs/1607.04357.

[14] Khalil, H. (2002). Nonlinear Systems. 3rd ed. Pearson Education, Prentice Hall.

[15] Krichagina, E. V. (1992). Asymptotic analysis of queueing networks. Stochastics and Stochastic Reports, 40 43–76. http://dx.doi.org/10.1080/17442509208833781, URL http://dx.doi.org/10.1080/17442509208833781.

[16] Mandelbaum, A., Massey, W. A. and Reiman, M. I. (1998). Strong approximations for Markovian service networks. Queueing Systems, 30 149–201.

[17] Pavone, M., Smith, S. L., Frazzoli, E. and Rus, D. (2012). Robotic load balancing for mobility-on-demand systems. The International Journal of Robotics Research, 31 839–854. http://ijr.sagepub.com/content/31/7/839.full.pdf+html, URL http://ijr.sagepub.com/content/31/7/839.abstract.

[18] Reiman, M. I. (1984). Open queueing networks in heavy traffic. Mathematics of Operations Research, 9 441–458.

[19] Waserhole, A. and Jost, V. (2013). Vehicle Sharing System Pricing Regulation: A Fluid Approximation. Working paper or preprint, URL https://hal.archives-ouvertes.fr/hal-00727041.

[20] Waserhole, A. and Jost, V. (2016). Pricing in vehicle sharing systems: optimization in queueing networks with product forms. EURO Journal on Transportation and Logistics, 5 293–320. URL http://dx.doi.org/10.1007/s13676-014-0054-4.

[21] Zhang, R. and Pavone, M. (2016). Control of robotic mobility-on-demand systems. Int. J. Rob. Res., 35 186–203. URL http://dx.doi.org/10.1177/0278364915581863.