NORMALIZED VOLUMES OF TYPE-PQ ADJACENCY POLYTOPES 
FOR CERTAIN CLASSES OF GRAPHS

ROBERT DAVIS, JOAKIM JAKOVLESKI, AND QIZHE PAN

Abstract. The type-PQ adjacency polytope associated to a simple graph is a 0/1-polytope containing valuable information about an underlying power network. Chen and the first author have recently demonstrated that, when the underlying graph $G$ is connected, the normalized volumes of the adjacency polytopes can be computed by counting sequences of nonnegative integers satisfying restrictions determined by $G$. This article builds upon their work, namely by showing that one of their main results – the so-called “triangle recurrence” – applies in a more general setting. Formulas for the normalized volumes when $G$ is obtained by deleting a path or a cycle from a complete graph are also established.

1. Introduction

Let $N$ be a positive integer. A polytope $P \subseteq \mathbb{R}^N$ is the convex hull of finitely many points $v_1, \ldots, v_d \in \mathbb{R}^N$, that is,

$$P = \text{conv}\{v_1, \ldots, v_d\} = \left\{ \sum_{i=1}^{d} \lambda_i v_i \mid \lambda_1, \ldots, \lambda_d \geq 0, \sum_{i=1}^{d} \lambda_i = 1 \right\}.$$

To a simple, undirected graph $G$ on $[N] = \{1, \ldots, N\}$, the associated type-PQ adjacency polytope is

$$\nabla_{PQ}^G = \text{conv}\{(e_i, e_j) \in \mathbb{R}^{2N} \mid i = j \text{ or } ij \in E(G)\}$$

where $e_i$ is the $i^{th}$ standard basis vector of $\mathbb{R}^N$ and $E(G)$ is the set of edges of $G$. Note that since $G$ is undirected, an edge $ij$ can also be written $ji$, meaning that a single edge of $G$ produces two points $(e_i, e_j)$ and $(e_j, e_i)$ used in the construction of $\nabla_{PQ}^G$.

Type-PQ adjacency polytopes arise in the study of power-flow equations of electrical networks [2]. These polytopes are in contrast to type-PV adjacency polytopes, which arise in the study of systems of interconnected oscillators. The first author, together with Chen and others, have recently studied their combinatorial structures and triangulations with a view towards their use in homotopy continuation methods [3, 4, 7]. Their work has sparked a flurry of results [1, 5, 6, 10, 11] using various algebraic and combinatorial techniques, often under the name of symmetric edge polytopes, which were studied previously with a view towards Ehrhart-theoretic results [8, 9].

To state the main results of this article, we must define several additional terms. The dimension of a polytope, denoted $\text{dim}(P)$, is the dimension of the affine linear subspace spanned by $P$. One of the crucial pieces of information regarding $\nabla_{PQ}^G$ is its normalized volume, $\text{NVol}(\nabla_{PQ}^G) = \text{dim}(\nabla_{PQ}^G)! \text{vol}(\nabla_{PQ}^G)$, where $\text{vol}(P)$ is the (relative) Euclidean volume of $P$. Part of what makes this value of interest is the fact that, if $P$ is the convex hull of points in $\mathbb{Z}^N$, then $\text{NVol}(P)$ is always a positive integer. Computing the normalized volume can then be approached by attempting to find a set $S$ for which $\text{NVol}(P) = |S|$, ideally so

The authors were supported in part by NSF grant DMS-1922998 and by Colgate University.
that the structure of $S$ allows for a more feasible computation of $|S|$. This article uses such an approach, first implemented in this setting in [7], to produce formulas for $\text{NVol}(\nabla^P)$ for several classes of graphs. Our main results are the following.

**Theorem 2.9.** Let $N \geq 2$. If $\mathcal{M}$ is an $M$-element matching of $K_N$, then

$$\text{NVol}(\nabla^P_{K_N \triangle \mathcal{M}}) = 3^M \left( \frac{2(N-1)}{N-1} \right).$$

**Theorem 3.2.** Let $N \geq 4$ and $0 \leq M < N$. If $P$ is a length-$M$ path in $K_N$, then

$$\text{NVol}(\nabla^P_{K_N \setminus E(P)}) = \left( \frac{2(N-1)}{N-1} \right) - (2N-4)(M-1) + 4.$$

**Theorem 3.5.** Let $N \geq 5$ and $0 \leq M \leq N$. Denote by $E^0_M$ the set of edges of an $M$-cycle in $K_N$. For all such choices of $N$ and $M$ and any $E^0_M$,

$$\text{NVol}(\nabla^P_{K_N \setminus E^0_M}) = \begin{cases} 
\left( \frac{2(N-1)}{N-1} \right) - 2M(N-2) & \text{if } M \neq 4 \\
\left( \frac{2(N-1)}{N-1} \right) - 2(N+1)(N-2) & \text{if } M = 4.
\end{cases}$$

In Section 2, we give a brief overview of how computing normalized volumes of $\text{NVol}(\nabla^P_G)$ can be interpreted as a combinatorial problem. Within this section, we adapt an argument given in [7] to prove Theorem 2.9. Section 3 examines how $\text{NVol}(\nabla^P_G)$ relates to $\nabla^P_{K_N}$ when $G$ is obtained from $K_N$ by deleting the edges of a path or a cycle. It is in that section that we prove Theorems 3.2 and 3.5.

2. Background

One key insight provided in [7] is that $\text{NVol}(\nabla^P_G)$ can be computed by determining the number of sequences of positive integers satisfying conditions arising from $G$. To describe the sequences and the constraints placed upon them, we need to first define a number of notions.

For a positive integer $N$, let $[N] = \{1, \ldots, N\}$, and define $K_{N,N}$ to be the complete bipartite graph with partite sets $[N]$ and $[N]$. If $G$ is a graph and $v$ is a vertex of $G$, then we denote by $\mathcal{N}_G(v)$ the neighbors of $v$ in $G$. When $G \subseteq K_{N,N}$, we say a sequence $(c_1, \ldots, c_N) \in \mathbb{Z}_{\geq 0}^N$ is $G$-draconian sequence if $\sum c_i = N - 1$ and, for any choice of indices $1 \leq i_1 < \cdots < i_k \leq N$, the inequality

$$(1) \quad c_{i_1} + \cdots + c_{i_k} < \left| \bigcup_{j=1}^k \mathcal{N}_G(i_j) \right|$$

is satisfied. We will often call the inequality (1) the $G$-draconian inequality corresponding to $i_1, \ldots, i_k$, or corresponding to $c_{i_1}, \ldots, c_{i_k}$. Often, the graph $G$ and the indices $i_1, \ldots, i_k$ are understood from context, and we simply call (1) a draconian inequality. For similar reasons, if $G$ is either understood from or irrelevant to the discussion, we may call $c$ a draconian sequence.

Draconian sequences were first studied by Postnikov in relation to a large class of polytopes called generalized permutohedra and a generalization of Hall’s Matching Theorem [12]. He establishes formulas for normalized volumes of generalized permutohedra in terms of draconian sequences, though these formulas are generally nonalgebraic. Unfortunately, determining the number of $G$-draconian sequences for an arbitrary $G \subseteq K_{N,N}$ is computationally expensive,
and typically requires determining the set of draconian sequences themselves. In the context of finding solutions to power-flow equations, therefore, finding algebraic formulas for the number of draconian sequences, when possible, is a significant improvement.

Since draconian sequences are only associated to bipartite graphs, we must relate an arbitrary simple graph to a bipartite graph in a controlled way. For a simple graph \( G \) on \([N]\), define \( D(G) \) to be the subgraph of \( K_{N,N} \) whose edges are \( \{i,\bar{i}\} \) for each \( i \in [N] \) and \( \{i,\bar{j}\} \) and \( \{j,\bar{i}\} \) for each edge \( ij \) in \( G \). Denote by \( \mathcal{D}(G) \) the set of \( D(G) \)-draconian sequences. The following result is a crucial connection between type-PQ adjacency polytopes and draconian sequences.

**Theorem 2.1** ([7 Theorem 2.8]). For any connected graph \( G \) on \([N]\), \( \text{NVol}(\nabla^\text{PQ}_G) = |\mathcal{D}(G)| \).

We note that if \( G \) is disconnected, then the above result may not hold, but an alternate approach [7 Proposition 3.2] shows that \( \text{NVol}(\nabla^\text{PQ}_G) \) is a product of factors of the form \(|\mathcal{D}(G_i)|\) where each \( G_i \) is a connected component of \( G \). Thus, we may always assume without loss of generality that \( G \) is connected. A convenient result that we will also frequently use is the following.

**Remark 2.2** ([7 Remark 2.9]). The normalized volume of \( \nabla^\text{PQ}_G \) is invariant under permutation of vertices. Hence, \(|\mathcal{D}(G)|\) is also invariant under relabeling of the vertices of \( G \).

2.1. **Extending the triangle recurrence.** Let \( G \) be a connected graph and let \( e = uv \) be an edge of \( G \). We denote the vertex set of \( G \) by \( V(G) \). In [7], the authors defined the construction \( G \triangle e \) to be the graph on the vertex set \( V(G) \cup \{w_e\} \), where \( w_e \) is a new vertex, and the edge set \( E(G) \cup \{uw_e, vw_e\} \). Under certain conditions, we can describe \( \text{NVol}(\nabla^\text{PQ}_{G \triangle e}) \) in terms of \( \text{NVol}(\nabla^\text{PQ}_G) \).

**Theorem 2.3** ([7 Theorem 3.18]). Let \( G \) be a connected graph on \([N]\) for which \( e = uv \) is an edge with \( \deg_G(u) = 2 \). If \( \deg_G(v) = 2 \) or if the neighbors of \( u \) are neighbors of each other, then

\[
\text{NVol}(\nabla^\text{PQ}_{G \triangle e}) = 3 \text{NVol}(\nabla^\text{PQ}_G).
\]

We refer to the above recurrence as the triangle recurrence. Experimental data suggests that the conditions on the triangle recurrence can be relaxed considerably, although a complete characterization of when [2] holds remains elusive. The triangle recurrence relies on several lemmas, some of which we will need, and therefore state below.

**Lemma 2.4** ([7 Lemma 3.14]). Let \( G \) be any connected graph on \([N]\) and \( e \) any edge. If \( c \in \mathcal{D}(G) \), then \( \alpha^\triangle(c) \in \mathcal{D}(G \triangle e) \) where \( \alpha^\triangle(c) = (c,1) \). Moreover, \( \alpha^\triangle \) is injective.

**Lemma 2.5** ([7 Lemma 3.15]). Let \( G \) be a connected graph on \([N]\) and let \( e = uv \) be any edge. If \( c \in \mathcal{D}(G) \), then \( \beta^\triangle(c) \in \mathcal{D}(G \triangle e) \) where

\[
\beta^\triangle(c) = \alpha^\triangle(c) + e_u - e_{N+1}.
\]

Additionally, \( \beta^\triangle \) is injective.

We have occasional need for the notation \( \beta_G^\triangle(e) \) to denote the image of \( \gamma^\triangle(e) \). The benefit of these lemmas is their ability to apply to any connected graph. A third lemma is needed to establish the triangle recurrence, but to prove it requires additional restrictions on \( G \). The lemma we need will be an adaptation of the following.
Lemma 2.6 ([7, Lemma 3.16]). Let $G$ be a connected graph on $[N]$ and let $e = uv$ be any edge for which $\deg_G(u) = 2$. If $c \in \mathcal{D}(G)$, then $\gamma^\Delta(c) \in \mathcal{D}(G \triangle e)$ where

\[
\gamma^\Delta(c) = \begin{cases} 
\alpha^\Delta(c) + e_v - e_{N+1} & \text{if not in } B_G^\Delta(e) \\
\alpha^\Delta(c) - e_u + e_{N+1} & \text{otherwise.}
\end{cases}
\]

Additionally, $\gamma^\Delta$ is injective. \hfill $\square$

Much of the work we will need to do involves showing that the conclusion to Lemma 2.6 will still hold in our more general setting. The rest of the work will be in showing that the images of $\alpha^\Delta(G), \beta^\Delta(G)$, and $\gamma^\Delta(G)$ are disjoint, and that each appropriate $D(G)$-draconian sequence is in the image of one of these functions.

For our adaptation of Lemma 2.6, we slightly extend the definition of $G \triangle e$ as follows: If $F$ is a subset of edges of $G$, then set

\[
G \triangle F = \bigcup_{e \in F} G \triangle e.
\]

In other words, $G \triangle F$ is the graph obtained by taking each edge $e = uv$ in $F$, creating a new vertex $w_e$, and adding the edges $uw_e$ and $vw_e$.

Lemma 2.7. Let $\mathcal{M}$ be an $M$-element matching of $K_N$ and $e = uv$ an edge of $K_N$ such that $\mathcal{M}' = \mathcal{M} \cup \{e\}$ is also a matching of $K_N$. If $c \in \mathcal{D}(K_N \triangle \mathcal{M})$, then $\gamma^\Delta(c) \in \mathcal{D}(K_N \triangle \mathcal{M}')$ where

\[
\gamma^\Delta(c) = \begin{cases} 
\alpha^\Delta(c) + e_v - e_{w_e} & \text{if not in } B_{K_N \triangle \mathcal{M}}^\Delta(e) \\
\alpha^\Delta(c) - e_u + e_{w_e} & \text{otherwise.}
\end{cases}
\]

Additionally, $\gamma^\Delta$ is injective.

Proof. Let $\mathcal{M}$ be an $M$-element, non-maximal matching of $K_N$. By Remark 2.2 we may assume, without loss of generality, that $V(K_N) = [N]$, $V(K_N \triangle \mathcal{M}) = [N + M]$, $e = \{N + M - 1, N + M\}$, and the new vertex added to $K_N \triangle \mathcal{M}$ is $N + M + 1$.

That $\gamma^\Delta$ is injective is clear from the definition. We wish to show, then, that if $c = (c_1, \ldots, c_{N+M}) \in \mathcal{D}(G \triangle \mathcal{M})$, then $\gamma^\Delta(c) \in \mathcal{D}(G \triangle \mathcal{M}')$, where

\[
\gamma^\Delta(c) = \begin{cases} 
(c_1, \ldots, c_{N+M-1}, c_{N+M} + 1, 0) & \text{if not in } B_{K_N \triangle \mathcal{M}}^\Delta(e) \\
(c_1, \ldots, c_{N+M-1} - 1, c_{N+M}, 2) & \text{otherwise.}
\end{cases}
\]

Note that in each case, the sum of all entries is $N + M$, as needed.

First suppose that $\gamma^\Delta(c) = (c_1, \ldots, c_{N+M-1}, c_{N+M} + 1, 0)$. For notational convenience, let the entries of $\gamma^\Delta(c)$ be denoted by $(\gamma^\Delta_1, \ldots, \gamma^\Delta_{N+M+1})$. We will verify that the required $D(K_N \triangle \mathcal{M}')$-draconian inequalities hold.

Let $1 \leq i_1 < \cdots < i_k \leq N + M + 1$. If $i_k < N + M - 1$, then $\gamma^\Delta_{i_j} = c_{i_j}$ for each $j$, and, since $c \in \mathcal{D}(K_N \triangle \mathcal{M})$, we have

\[
\gamma^\Delta_{i_1} + \cdots + \gamma^\Delta_{i_k} = c_{i_1} + \cdots + c_{i_k} < \bigcup_{j=1}^k \mathcal{N}_{D(K_N \triangle \mathcal{M})}(i_j) = \bigcup_{j=1}^k \mathcal{N}_{D(K_N \triangle \mathcal{M}')}(i_j).
\]
If $i_k \in \{N + M - 1, N + M\}$, then
\[
\gamma_{i_1}^\triangle + \cdots + \gamma_{i_k}^\triangle \leq c_{i_1} + \cdots + c_{i_k} + 1
\]
\[
< \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M})}(i_j) \right| + |\{N + M + 1\}|
\]
\[
= \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M'})}(i_j) \right|.
\]
Lastly, if $i_k = N + M + 1$, then, by the previous cases,
\[
\gamma_{i_1}^\triangle + \cdots + \gamma_{i_k}^\triangle = c_{i_1} + \cdots + c_{i_{k-1}}
\]
\[
< \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(KN_{N+M'})}(i_j) \right| \leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M'})}(i_j) \right|.
\]

Thus, $\gamma^\triangle(c) \in \mathcal{D}(KN_{N,M'})$ if it is not already a member of $B^\triangle_{KN_{N,M'}}(e)$.

Now consider the case in which $\gamma^\triangle(c) = (c_1, \ldots, c_{N+M-1} - 1, c_{N+M}, 2)$. Here, we know $(c_1, \ldots, c_{N+M-1}, c_{N+M} + 1, 0) \in B^\triangle_{KN_{N,M'}}(e)$, so that $c_{N+M-1} \geq 1$. This time, denote the entries of $\gamma^\triangle(c)$ by $(\gamma_{i_1}^\triangle, \ldots, \gamma_{N+M+1}^\triangle)$ and let $1 \leq i_1 < \cdots < i_k \leq N + M + 1$. If $i_k < N + M + 1$, then
\[
\gamma_{i_1}^\triangle + \cdots + \gamma_{i_k}^\triangle \leq c_{i_1} + \cdots + c_{i_k} < \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(KN_{N+M})}(i_j) \right| \leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M})}(i_j) \right|.
\]

If $i_k = N + M + 1$, then there are two cases to consider. If $i_j > N$ for each $j < k$, then no two $i_j$ and $i_{j'}$ share a neighbor in $D(KN_{N,M'})$. Therefore,
\[
\gamma_{i_1}^\triangle + \cdots + \gamma_{i_k}^\triangle \leq 2 + \cdots + 2 < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M})}(i_j) \right|.
\]

Otherwise, suppose $i_j \leq N$ for some $j < k$. If $i_\ell = N + M - 1$, for some $\ell$, then
\[
\gamma_{i_1}^\triangle + \cdots + \gamma_{i_k}^\triangle = c_{i_1} + \cdots + c_{i_{k-1}} - 1 + 2
\]
\[
< \left| \bigcup_{j=1}^{k-1} \mathcal{N}_{D(KN_{N+M})}(i_j) \right| + \{N + M + 1\}
\]
\[
\leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(KN_{N+M'})}(i_j) \right|.
\]

If $i_j \neq N + M - 1$ for all $j$, then we use the fact that
\[
\mathcal{N}_{KN_{N,M}}(N + M - 1) \subseteq \bigcup_{j=1}^{k-1} \mathcal{N}_{D(KN_{N+M})}(i_j)
\]
Lemma 2.8. Let \( \gamma \) since all draconian inequalities hold, we have \( \gamma \) we assume that the conclusion holds for any non-maximal, Lemmas 2.4, 2.5, 2.7, and 2.8, we have we will proceed by induction. The case of \( i \) and such that the unique vertex of \( K \) to obtain \( 6 \) ROBERT DAVIS, JOAKIM JAKOVLESKI, AND QIZHE PAN

\[
\gamma_{i_1} + \cdots + \gamma_{i_k} \leq c_{i_1} + \cdots + c_{i_{k-1}} + c_{N+M-1} - 1 + 2
\]

\[
< \left( \bigcup_{j=1}^{k-1} N_{D(K_N \triangle M)}(i_j) \right) \cup N_{K_N \triangle M}(N + M - 1) \bigg| + |N + M + 1|)
\]

\[
\leq \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M')}(i_j) \right|
\]

Since all draconian inequalities hold, we have \( \gamma^{\Delta}(c) \in \mathcal{D}(K_N \triangle M') \).

The following lemma is straightforward from the definitions of \( \alpha^{\Delta} \), \( \beta^{\Delta} \), and \( \gamma^{\Delta} \), so we omit its proof. In it, we use \( A_{K_N \triangle M}(e) \) to denote the image of \( \alpha^{\Delta} \) and \( C_{K_N \triangle M}(e) \) to denote the image of \( \gamma^{\Delta} \).

**Lemma 2.8.** Let \( M \) be matching of \( K_N \) and \( e = uv \) an edge of \( K_N \) such that \( M' = M \cup \{e\} \) is also a matching of \( K_N \). The sets \( A_{K_N \triangle M}(e), B_{K_N \triangle M}(e), \) and \( C_{K_N \triangle M}(e) \) are pairwise disjoint.

**Theorem 2.9.** Let \( N \geq 2 \). If \( M \) is an \( M \)-element matching of \( K_N \), then

\[
N\text{Vol}(\nabla_{K_N \triangle M}^{\text{PQ}}) = 3^M \left( \frac{2(N - 1)}{N - 1} \right).
\]

**Proof.** We will proceed by induction. The case of \( M = \emptyset \) is exactly [7, Proposition 2.10]. We assume that the conclusion holds for any non-maximal, \( M \)-element matching of \( K_N \).

Let \( M \) be an arbitrary non-maximal, \( M \)-element matching of \( K_N \). Since \( M \) is non-maximal, there is an edge \( e \) of \( K_N \) such that \( M' = M \cup \{e\} \) is a matching of \( K_N \) as well. By Remark 2.2, we may freely relabel the vertices of \( K_N \triangle M \) such that \( e = \{N + M - 1, N + M\} \) and such that the unique vertex of \( K_N \triangle M' \) not appearing in \( K_N \triangle M \) is \( N + M + 1 \). Applying Lemmas 2.4, 2.5, 2.7, and 2.8 we have

\[
A_{K_N \triangle M}(e) \cup B_{K_N \triangle M}(e) \cup C_{K_N \triangle M}(e) \subseteq \mathcal{D}(K_N \triangle M').
\]

Hence, we only need to establish the reverse inclusion.

Let \( d = (d_1, \ldots, d_{N+M+1}) \in \mathcal{D}(K_N \triangle M') \). We will show that \( d \) is in one of \( A_{K_N \triangle M}(e), B_{K_N \triangle M}(e), C_{K_N \triangle M}(e) \) by proving each of the following claims:

1. if \( d_{N+M+1} = 1 \), then \( (d_1, \ldots, d_{N+M}) \in \mathcal{D}(K_N \triangle M) \);
2. if \( d_{N+M+1} = 0 \), then one of \( (d_1, \ldots, d_{N+M-1}, d_{N+M}) \) and \( (d_1, \ldots, d_{N+M-1}, d_{N+M-1}) \) is in \( \mathcal{D}(K_N \triangle M) \);
3. if \( d_{N+M+1} = 2 \), then \( (d_1, \ldots, d_{N+M-1}, d_{N+M+1}) \) and \( (d_1, \ldots, d_{N+M-1}+1, d_{N+M}) \) are in \( \mathcal{D}(K_N \triangle M) \).

If \( d_{N+M+1} = 1 \), then we claim \( (d_1, \ldots, d_{N+M}) \in \mathcal{D}(G) \). Let \( 1 \leq i_1 < \cdots < i_k \leq N + M \). If \( i_k < N + M - 1 \), then

\[
d_{i_1} + \cdots + d_{i_k} < \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M')}(i_j) \right| = \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M)}(i_j) \right|
\]
If $i_k \geq N + M - 1$, then
\[ d_1' + \cdots + d_{i_k}' = d_1 + \cdots + d_{i_k} + 1 - 1 \]
\[ < \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M')} \left( i_j \right) \right| - 1 \]
\[ = \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M)} \left( i_j \right) \right| \]

Next suppose that $d_{N+M+1} = 0$. If we also have $d_{N+M} = d_{N+M-1} = 0$, then we would have
\[ N + M = d_1 + \cdots + d_{N+M+1} = d_1 + \cdots + d_{N+M-2} < \left| \bigcup_{j=1}^{N+M-2} N_{D(K_N \triangle M')} \left( i_j \right) \right| = N + M, \]
which is a contradiction. Thus, at least one of $d_{N+M}, d_{N+M-1}$ is positive. Without loss of generality we assume $d_{N+M} > 0$ and will show that $(d_1, \ldots, d_{N+M-1}, d_{N+M} - 1)$ is in $\mathcal{D}(K_N \triangle M)$.

For notational convenience, let
\[ d' = (d'_1, \ldots, d'_{N+M}) = (d_1, \ldots, d_{N+M-1}, d_{N+M} - 1). \]

Consider a sum of the form $d'_1 + \cdots + d'_{i_k}$. If $i_k < N + M - 1$, then the neighbors of $i_j$ are the same in $D(K_N \triangle M')$ and $D(K_N \triangle M)$, so the corresponding draconian inequality immediately holds. If $i_k = N + M - 1$, then
\[ d'_1 + \cdots + d'_{i_k} - 1 = d_1 + \cdots + d_{i_k-1} + d_{N+M} - 1 \]
\[ < \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M')} \left( i_j \right) \right| - |N + M + 1| \]
\[ = \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M)} \left( i_j \right) \right| \]

A nearly-identical argument holds if $i_k = N + M$ and $i_{k-1} = N + M - 1$, so we omit it. Lastly, if $i_k = N + M$ and $i_{k-1} < N + M - 1$, then
\[ d'_1 + \cdots + d'_{i_k} \leq d_1 + \cdots + d_{i_k} + d_{N+M-1} - 1 \]
\[ < \left| \left( \bigcup_{j=1}^{k} N_{D(K_N \triangle M')} \left( i_j \right) \right) \cup N_{D(K_N \triangle M')} \left( N + M - 1 \right) \right| - |N + M + 1| \]
\[ = \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M') \left( i_j \right)} \right| - |N + M + 1| \]
\[ = \left| \bigcup_{j=1}^{k} N_{D(K_N \triangle M)} \left( i_j \right) \right| \]

Thus, $d' \in \mathcal{D}(K_N \triangle M)$, as desired. Because a symmetric argument holds if $d_{N+M-1} > 0$, this completes the case in which $d_{N+M+1} = 0$. 
Finally, suppose $d_{N+M+1} = 2$. Showing that $(d_1, \ldots, d_{N+M-1}+1, d_{N+M}+1, 0) \in \mathcal{B}_{K_N \Delta M}(e)$ is equivalent to showing that $(d_1, \ldots, d_{N+M-1}, d_{N+M}+1) \in \mathcal{D}(K_N \Delta M)$, and it is this latter statement that we will establish. For notational convenience, set

$$d'' = (d''_1, \ldots, d''_{N+M}) = (d_1, \ldots, d_{N+M-1}, d_{N+M}+1).$$

Consider a sum of the form $d''_{i_1} + \cdots + d''_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq N + M$. As before, if $i_k < N + M - 1$, then the neighbors of $i_j$ are the same in $D(K_N \Delta M')$ and $D(K_N \Delta M)$, so the corresponding draconian inequality immediately holds. If $i_k = N + M - 1$, then, using the fact that $\mathcal{N}_{D(K_N \Delta M')}(N + M + 1) \subseteq \mathcal{D}(K_N \Delta M)(N + M - 1)$,

$$d''_{i_1} + \cdots + d''_{i_k} < d_{i_1} + \cdots + d_{i_{k-1}} + d_{N+M-1} + 1(1 - 1)$$

$$\leq \left| \left( \bigcup_{j=1}^k \mathcal{N}_{D(K_N \Delta M')}(i_j) \right) \cup \mathcal{N}_{D}(K_N \Delta M')(N + M + 1) \right| - |\{N + M + 1\}|$$

$$= \left| \bigcup_{j=1}^k \mathcal{N}_{D}(K_N \Delta M')(i_j) \right| - |\{N + M + 1\}|$$

$$= \left| \bigcup_{j=1}^k \mathcal{N}_{D}(K_N \Delta M')(i_j) \right|.$$ 

If $i_k = N + M$, then the same reasoning as in the case $i_k = N + M - 1$ applies, with the only change being that the first strict inequality becomes a weak inequality. In all instances, the corresponding $D(K_N \Delta M)$-inequalities hold, showing that $d'' \in \mathcal{D}(K_N \Delta M)$. A symmetric argument also shows that $(d_1, \ldots, d_{N+M-1}+1, d_{N+M}) \in \mathcal{D}(K_N \Delta M)$. This completes the case for $d_{N+M+1} = 2$.

The completion of the above three cases shows that

$$\mathcal{A}_{K_N \Delta M}(e) \cup \mathcal{B}_{K_N \Delta M}(e) \cup \mathcal{C}_{K_N \Delta M}(e) = \mathcal{D}(K_N \Delta M').$$

Therefore, by our inductive assumption and the fact that $\alpha^\Delta, \beta^\Delta, \gamma^\Delta$ are all injections,

$$|\mathcal{D}(K_N \Delta M')| = |\mathcal{A}_{K_N \Delta M}(e)| + |\mathcal{B}_{K_N \Delta M}(e)| + |\mathcal{C}_{K_N \Delta M}(e)|$$

$$= 3|\mathcal{D}(K_N \Delta M)|$$

$$= 3^{M+1} \binom{2(N-1)}{N-1},$$

as desired. \[\square\]

3. Deleting edges from $K_N$

In this section, we will obtain formulas for $\text{NVol}((\nabla^0_G)^\gamma)$ where $G$ is constructed in a way different than that of Section 2.1. Rather than introduce new new vertices and edges, we will instead remove edges.

3.1. Deleting a path. In this section, we will examine what happens when we delete a path from the complete graph $K_N$. It will help to introduce some new notation. If $a \leq b$ are integers, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$. \[\]
Lemma 3.1. Let $N \geq 4$ and let $P$ be the length-$M$ path on $[N - M, N]$ where $uv$ is an edge if and only $|u - v| = 1$. Set
\[
\mathcal{P}_1 = \{e_i + (N - 2)e_j \in \mathbb{R}^N \mid 1 \leq i \leq N, N - M + 1 \leq j \leq N - 1\}
\]
and
\[
\mathcal{P}_2 = \{re_i + (N - 1 - r)e_j \in \mathbb{R}^N \mid 0 \leq r \leq N - 1, N - M \leq i \leq N - 2\}.
\]
Then $\mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E(P)) = \mathcal{P}_1 \cup \mathcal{P}_2$.

Proof. First, it is clear that $\mathcal{P}_1 \cup \mathcal{P}_2 \subseteq \mathcal{D}(K_N)$ since each sequence in $\mathcal{P}_1 \cup \mathcal{P}_2$ is a weak composition of $N - 1$ into $N$ parts, and $|\mathcal{N}_{D(K_N)}(i)| = N$ for each $i \in [N]$. For ease of notation as we continue, let $G = K_N \setminus E(P)$.

For each vertex $i \in [N - M + 1, N - 1]$, $|\mathcal{N}_{D(G)}(i)| = N - 2$, since neither of $i - 1$ nor $i + 1$ are a neighbor of $i$ in $D(G)$. Hence, any sequence in $\mathcal{D}(K_N)$ in which the $i$th coordinate is at least $N - 2$, where $N - M + 1 \leq i \leq N - 1$, cannot be a sequence in $\mathcal{D}(G)$; these are the sequences in $\mathcal{P}_1$. Moreover, we further know that for each $i \in [N - M + 1, N - 1]$, $|\mathcal{N}_{D(G)}(i - 1) \cup \mathcal{N}_{D(G)}(i + 1)| = N - 1$. It follows that any sequence in $\mathcal{D}(K_N)$ in which the $(i - 1)$th and $(i + 1)$th coordinates sum to $N - 1$, where $N - M + 1 \leq i \leq N - 1$, cannot be a sequence in $\mathcal{D}(G)$; these are the sequences in $\mathcal{P}_2$. Therefore, $(\mathcal{P}_1 \cup \mathcal{P}_2) \cap \mathcal{D}(G) = \emptyset$, and $\mathcal{P}_1 \cup \mathcal{P}_2 \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E(P))$.

We now need to show that if $c = (c_1, \ldots, c_N) \in \mathcal{D}(K_N) \setminus \mathcal{D}(G)$, then $c \in \mathcal{P}_1 \cup \mathcal{P}_2$. There are several cases to consider.

First suppose $c_\ell > 0$ for exactly one $\ell \in [N]$, and consider a sum of the form $c_{i_1} + \cdots + c_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq N$. If $i_j \neq \ell$ for all $j$, then the inequality
\[
ci_1 + \cdots + c_{i_k} < 0 < \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(K_N \setminus E(G))}(i_j) \right| \leq \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(K_N)}(i_j) \right|
\]
always holds. On the other hand, if $i_j = \ell$ for some $j$, then the inequalities of the form
\[
ci_1 + \cdots + c_{i_k} = c_\ell = N - 1 < \left| \mathcal{N}_{D(K_N \setminus E(G))}(\ell) \right| = \left| \bigcup_{j=1}^{k} \mathcal{N}_{D(K_N \setminus E(G))}(i_j) \right|
\]
all hold if and only if $\ell < N - M$. Therefore, the sequences $(N - 1)e_{N-M}, \ldots, (N - 1)e_N \in \mathcal{D}(K_N) \setminus \mathcal{D}(G)$. The sequences $(N - 1)e_{N-M}$ and $(N - 1)e_N$ are in $\mathcal{P}_2$, and the rest are in $\mathcal{P}_1$.

Next suppose that there are exactly two indices $1 \leq \ell < m \leq N - 1$ for which $c_\ell, c_m > 0$. There are three ways in which a $D(G)$-draconian inequality can fail to hold:

1. if $|\mathcal{N}_{D(G)}(\ell)| \leq c_\ell$;
2. if $|\mathcal{N}_{D(G)}(m)| \leq c_m$; and
3. if $|\mathcal{N}_{D(G)}(\ell) \cup \mathcal{N}_{D(G)}(m)| \leq N - 1$.

We first examine when $|\mathcal{N}_{D(G)}(\ell)| \leq c_\ell$. By construction, we know that for any $i \in [N]$, $N - 2 \leq |\mathcal{N}_{D(G)}(i)|$. If $|\mathcal{N}_{D(G)}(\ell)| = N - 1$, then this would mean $c_\ell = N - 1$. This would require $c_m = 0$, which is a contradiction to the case we are in. So, the only way to have $|\mathcal{N}_{D(G)}(\ell)| \leq c_\ell$ would be if $c_\ell = N - 2$, in which case $c_m = 1$. This is only possible when $\ell \in [N - M + 1, N - 1]$; each of the resulting sequences are in $\mathcal{P}_1$. A symmetric argument holds if $|\mathcal{N}_{D(G)}(m)| \leq c_m$. 
Now suppose \(|\mathcal{N}_{D(G)}(\ell) \cup \mathcal{N}_{D(G)}(m)| \leq N-1|; we may freely assume \(\ell < m\). It is immediate that for this to happen, we have \(\ell, m \in [N-M,N]\). Moreover, it is straightforward to see that if \(m-\ell \neq 2\), then \(|\mathcal{N}_{D(G)}(\ell) \cup \mathcal{N}_{D(G)}(m)| = N\), hence, \(m = \ell + 2\). These two coordinates in \(c\) must sum to \(N - 1\); all such sequences satisfying these conditions are in \(\mathcal{P}_2\). Therefore, if \(c \in \mathcal{D}(K_N) \setminus \mathcal{D}(G)\) has exactly two nonzero coordinates, then \(c \in \mathcal{P}_1 \cup \mathcal{P}_2\).

Lastly, suppose \(c \in \mathcal{D}(K_N)\) has at least three nonzero coordinates. This time, if \(c_\ell, c_m, c_n > 0\) with \(1 \leq \ell < m < n \leq N\), then each is at most \(N - 3\). Also, the sum of any two will be at most \(N - 2\), and the union of the two corresponding sets of neighbors in \(D(G)\) will have at least \(N - 1\) elements. Continuing via now-routine arguments, one sees that \(c \in \mathcal{D}(G)\). Therefore, we have established the reverse inclusion \(\mathcal{D}(K_N) \setminus \mathcal{D}(G) \subseteq \mathcal{P}_1 \cup \mathcal{P}_2\), completing our proof.

Lemma 3.1 is enough to determine what happens when a path is deleted from a complete graph.

**Theorem 3.2.** Let \(N \geq 4\) and \(0 \leq M < N\). If \(P\) is a length-\(M\) path in \(K_N\), then

\[
\text{Vol}_{\mathcal{D}}(\mathcal{P}Q_{K_N \setminus E(P)}) = \left(\frac{2(N-1)}{N-1}\right) - (2N - 4)(M - 1) + 4.
\]

**Proof.** Since \(N \geq 4\), \(K_N \setminus E(P)\) is connected, and we may apply Theorem 2.1. By Remark 2.2, we assume that the edges of \(P\) can be written as \(ij\) where \(j = i + 1\) and \(N - M \leq i \leq N - 1\). With the help of Lemma 3.1, we know that this is

\[
|\mathcal{D}(K_N)| - |\mathcal{P}_1 \cup \mathcal{P}_2| = \left(\frac{2(N-1)}{N-1}\right) - |\mathcal{P}_1| - |\mathcal{P}_2| + |\mathcal{P}_1 \cap \mathcal{P}_2|.
\]

It is immediate to see that \(|\mathcal{P}_1| = N(M-1)|. In \(\mathcal{P}_2\) there is some redundancy, since, each vector of the form \(0e_i + (N-1)e_{i+2}\) with \(i \in [N-M,N-4]\) is also obtained as \((N-1)e_{i+2} + 0e_i+4\). Thus, \(|\mathcal{P}_2| = N(M-1) - (M-3)\). Computing \(|\mathcal{P}_1 \cap \mathcal{P}_2| = 2(M-1) + (M-2)\) is similarly straightforward, so we omit the details. Therefore,

\[
|\mathcal{D}(K_N \setminus E(P))| = \left(\frac{2(N-1)}{N-1}\right) - (N(M-1) + N(M-1) - (M-3) + (2(M-2) + (M-1)))
\]

\[
= \left(\frac{2(N-1)}{N-1}\right) - (2N - 4)(M - 1) + 4,
\]

as desired. \(\square\)

### 3.2. Deleting a cycle.

In this section we examine what happens when the edges of a cycle are deleted from \(K_N\). To prove our main result, we establish more notation. Throughout this section, let \(N \geq 5\) and let \(C_M^o = (V_M^o, E_M^o)\) be an \(M\)-element cycle subgraph of \(K_N\). Label the elements of \(V_M^o\) by \(v_0, \ldots, v_{M-1}\) so that \(v_i, v_j\) form an edge if and only if \(j - i = \pm 1 \pmod M\). We then denote by \(\mathcal{X}(C_M^o)\) the set of sequences

\[
\mathcal{X}(C_M^o) = \{(N-2)e_{v_i} + e_j \in \mathbb{R}^N \mid 0 \leq i \leq M - 1, j \in [N]\}.
\]

Denote by \(\mathcal{Y}(C_M^o)\) the set of sequences

\[
\mathcal{Y}(C_M^o) = \{re_{v_i} + (N-1 - r)e_{v_{i+2}} \in \mathbb{R}^N \mid v_i \in V_M^o, 2 \leq r \leq N - 3\}.
\]

where the subscripts on the indices are taken modulo \(M\). When \(M = 4\), we will need the following third set of sequences:

\[
\mathcal{Z}(C_M^o) = \{re_{v_i} + (N-2 - r)e_{v_{i+2}} + e_s \mid v_i \in V_M^o, 1 \leq r \leq N - 3, s \in [N] \setminus \{v_i, v_{i+2}\}\}.
\]
Lemma 3.3. The sets $\mathcal{X}(C_M^o)$, $\mathcal{Y}(C_M^o)$ and $\mathcal{Z}(C_M^o)$ are pairwise disjoint.

Proof. Note that each sequence in $\mathcal{X}(C_M^o)$ has a nonzero entry of $N - 2$, while each entry of each sequence in $\mathcal{Y}(C_M^o) \cup \mathcal{Y}(C_M^o)$ is at most $N - 3$. Thus, $\mathcal{X}(C_M^o) \cap \mathcal{Y}(C_M^o) = \emptyset$ and $\mathcal{X}(C_M^o) \cap \mathcal{Z}(C_M^o) = \emptyset$. To see that $\mathcal{Y}(C_M^o) \cap \mathcal{Z}(C_M^o) = \emptyset$, note that each sequence in $\mathcal{Y}(C_M^o)$ has exactly two nonzero entries while each sequence in $\mathcal{Z}(C_M^o)$ has exactly three nonzero coordinates. Therefore, the three sets are pairwise disjoint. □

Lemma 3.4. For each $N \geq 5$ and $3 \leq M \leq N$,

$$\mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M) = \begin{cases} \mathcal{X}(C_M^o) \cup \mathcal{Y}(C_M^o) & \text{if } M \neq 4, \\ \mathcal{X}(C_M^o) \cup \mathcal{Z}(C_M^o) & \text{if } M = 4. \end{cases}$$

Proof. We will first show that $\mathcal{X}(C_M^o) \cup \mathcal{Y}(C_M^o) \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$ for all $M$, and that $\mathcal{Z}(C_M^o) \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$. By Remark 2.2, we may assume that $V_M = [N - M + 1, N]$ and $ij \in E_M$ if and only if $j - i = \pm 1 \pmod{M}$. Since each sequence $c \in \mathcal{X}(C_M^o) \cup \mathcal{Y}(C_M^o) \cup \mathcal{Z}(C_M^o)$ is a weak composition of $N - 1$ into $N$ parts, we know $c \in \mathcal{D}(K_N)$.

For any sequence $c = (c_1, \ldots, c_N) \in \mathcal{X}(C_M^o)$, there exists some $N - M + 1 \leq i \leq N$ such that $c_i \geq N - 2$. However, since we are removing the edges in $E_M$, every vertex $i \in [N - M + 1, N]$ satisfies $\deg_{K_N \setminus E_M}(i) = N - 3$, meaning that $c_i \geq |\mathcal{N}_{K_N \setminus E_M}(i)|$ for some $i$. This violates one of the draconian inequalities, so $c \notin \mathcal{D}(K_N \setminus E_M)$, and it follows that $\mathcal{X}(C_M^o) \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$.

Next, for any $c \in \mathcal{Y}(C_M^o)$, there are some $i, j \in [N - M + 1, N]$ such that $i \leq N - 1$ and $j - i = \pm 2 \pmod{M}$. However, in $K_N \setminus E_M$, neither of these vertices will have $i + 1$ (mod $M$) as a neighbor. Therefore,

$$|\mathcal{N}_{K_N \setminus E_M}(i) \cup \mathcal{N}_{K_N \setminus E_M}(j)| \leq N - 1 = c_i + c_j,$$

implying $c \notin \mathcal{D}(K_N \setminus E_M)$. It again follows that $\mathcal{Y}(C_M^o) \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$.

When $M = 4$, deleting the edges in $C_M^o$ means $i$ and $j$ will have neither $i + 1$ nor $j + 1$ as common neighbors when $i, j \in [N - 3, N]$ and $j - i = \pm 2 \pmod{M}$. Using analogous reasoning as in the previous paragraph, we obtain $\mathcal{Z}(C_M^o) \subseteq \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$. It now remains to establish the reverse inclusions.

Let $c \in \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$. First, if $c$ only has one nonzero entry, then the value of that entry is $N - 1$. Since $|\mathcal{N}(K_N \setminus E_M)| \leq N - 1$ if and only if $i \in [N - M + 1, N]$, then $c = (N - 1)e_i$ for some $i \in [N - M + 1, N]$. Thus, $c \in \mathcal{X}(C_M^o)$.

Now suppose $c$ has two nonzero entries, $c_i$ and $c_j$ with $c_i \leq c_j$. If $c_j = N - 2$, then, by an argument similar to that of the previous paragraph, $c \in \mathcal{X}(C_M^o)$. Otherwise, $2 \leq c_i \leq c_j \leq N - 3$. We already know that $\deg_{K_N \setminus E_M}(\ell) \in \{N - 2, N\}$ for each $\ell \in [N]$. Because $N \geq 5$, this means

$$2 = c_i \leq c_j = N - 3 < N - 2 \leq \deg_{K_N \setminus E_M}(\ell)$$

for all $\ell$, namely, when $\ell \in \{i, j\}$. Hence, since $c \in \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M)$, we know that the following inequality holds:

$$N - 1 = c_i + c_j \geq |\mathcal{N}_{K_N \setminus E_M}(i) \cup \mathcal{N}_{K_N \setminus E_M}(j)|.$$

In order for the union to have fewer than $N$ elements, we need $i, j \in [N - M - 1, N]$ and $j - i = \pm 2 \pmod{M}$. Therefore, $c \in \mathcal{Y}(C_M^o)$. 
Now suppose that \( c \) has three nonzero entries, \( c_i, c_j, c_k \). In this case,
\[
c_\ell \leq N - 3 < N - 2 \leq |\mathcal{N}_{D(K_N \setminus E_M^o)}(\ell)|
\]
for each \( \ell = i, j, k \), and
\[
c_i + c_j + c_k = N - 1 < N = \left| \bigcup_{\ell = i, j, k} \mathcal{N}_{D(K_N \setminus E_M^o)}(\ell) \right|.
\]
Thus, since we know \( c \in \mathcal{D}(K_N) \setminus \mathcal{D}(K_N \setminus E_M^o) \), an inequality of the following form must hold:
\[
(3) \quad c_i + c_j \geq \left| \bigcup_{\ell = i, j} \mathcal{N}_{D(K_N \setminus E_M^o)}(\ell) \right|
\]
Note, though, that we always have
\[
(4) \quad c_i + c_j \leq N - 2 \leq \left| \bigcup_{\ell = i, j} \mathcal{N}_{D(K_N \setminus E_M^o)}(\ell) \right|
\]
so that the only way for both (3) and (4) to hold is if all of the inequalities are equalities.
This occurs if and only if both \( i, j \in [N - M + 1, N] \) and \( i \) and \( j \) have the same neighbors in \( K_N \setminus E_M^o \). These conditions hold if and only if \( M = 4 \) and \( j - i = \pm 2 \pmod{M} \), and subsequently \( c_k = 1 \); hence, \( c \in \mathcal{Z}(C_4^2) \). This establishes the desired reverse inclusions. By Lemma 3.3 we are done. \( \square \)

We are now ready to prove our main result of the section.

**Theorem 3.5.** For all \( N \geq 5 \) and \( 0 \leq M \leq N \),
\[
\text{NVol}(\nabla_{K_N \setminus E_M^o}^{\text{PQ}}) = \begin{cases} 
\binom{2(N-1)}{N-1} - 2M(N-2) & \text{if } M \neq 4 \\
\binom{2(N-1)}{N-1} - 2(N+1)(N-2) & \text{if } M = 4.
\end{cases}
\]

**Proof.** Since \( N \geq 5 \), \( K_N \setminus E_M^o \) is connected, and we may apply Theorem 2.1. By Lemma 3.4 the desired formula comes from computing
\[
|\mathcal{D}(K_N \setminus E_M^o)| = \begin{cases} 
|\mathcal{D}(K_N)| - |\mathcal{X}(C_M^o)| - |\mathcal{Y}(C_M^o)| & \text{if } M \neq 4, \\
|\mathcal{D}(K_N)| - |\mathcal{X}(C_M^o)| - |\mathcal{Y}(C_M^o)| - |\mathcal{Z}(C_4^2)| & \text{if } M = 4.
\end{cases}
\]
We already know that \( |\mathcal{D}(K_N)| = \binom{2(N-1)}{N-1} \). Elementary counting arguments and algebra will verify that
\[
|\mathcal{X}(C_M^o)| = MN \quad \text{and} \quad |\mathcal{Y}(C_M^o)| = M(N-4)
\]
for all \( M \). Thus, when \( M \neq 4 \), we get
\[
|\mathcal{D}(K_N)| - |\mathcal{X}(C_M^o)| - |\mathcal{Y}(C_M^o)| = \binom{2(N-1)}{N-1} - MN - M(N-4) = \binom{2(N-1)}{N-1} - 2M(N-2),
\]
and when \( M = 4 \), we get
\[
|\mathcal{D}(K_N)| - |\mathcal{X}(C_M^o)| - |\mathcal{Y}(C_M^o)| - |\mathcal{Z}(C_4^2)| = \binom{2(N-1)}{N-1} - 2(N+1)(N-2) - 2M(N-2),
\]
and the desired formula follows.
as claimed. When \( M = 4 \), it is also not difficult to directly show that 
\[
|Z(C_4^\circ)| = 2(N - 3)(N - 2),
\]
so that
\[
\begin{align*}
\bar{\mathcal{D}}(K_N \setminus E_4^\circ) &= |\mathcal{D}(K_N)| - |\mathcal{X}(C_4^\circ)| - |\mathcal{Y}(C_4^\circ)| - |\mathcal{Z}(C_4^\circ)| \\
&= \binom{2(N - 1)}{N - 1} - (4N + 4(N - 4) + 2(N - 3)(N - 2)) \\
&= \binom{2(N - 1)}{N - 1} - 2(N + 1)(N - 2),
\end{align*}
\]
as claimed.

\[\square\]

**References**

[1] Benjamin Braun and Kaitlin Bruegge. Facets of symmetric edge polytopes for graphs with few edges. 2022. arXiv:2201.13303.

[2] T. Chen and D. Mehta. On the network topology dependent solution count of the algebraic load flow equations. IEEE Transactions on Power Systems, 33(2):1451–1460, 2018.

[3] Tianran Chen, Robert Davis, and Evgeniia Korchevskaia. Facets of symmetric edge polytopes for graphs with few edges. 2021. arXiv:2107.12315.

[4] Tianran Chen, Robert Davis, and Dhagash Mehta. Counting equilibria of the Kuramoto model using birationally invariant intersection index. SIAM J. Appl. Algebra Geom., 2(4):489–507, 2018.

[5] Alessio D’Ali, Emanuele Delucchi, and Mateusz Michałek. Many faces of symmetric edge polytopes. 2019. arXiv:1910.05193.

[6] Alessio D’Ali, Martina Juhnke-Kubitzke, Daniel Köhne, and Lorenzo Venturello. On the gamma-vector of symmetric edge polytopes. 2022. arXiv:2201.09835.

[7] Robert Davis and Tianran Chen. Computing Volumes of Adjacency Polytopes via Draconian Sequences. 2020. arXiv:2007.11051.

[8] Akihiro Higashitani, Katharina Jochemko, and Mateusz Michałek. Arithmetic aspects of symmetric edge polytopes. Mathematika, 65(3):763–784, 2019.

[9] Tetsushi Matsui, Akihiro Higashitani, Yuuki Nagazawa, Hidefumi Ohsugi, and Takayuki Hibi. Roots of Ehrhart polynomials arising from graphs. J. Algebraic Comb., 34(4):721–749, 2011.

[10] Hidefumi Ohsugi and Akiyoshi Tsuchiya. Symmetric edge polytopes and matching generating polynomials. 2020. arXiv:2008.08621.

[11] Hidefumi Ohsugi and Akiyoshi Tsuchiya. PQ-type adjacency polytopes of join graphs. 2021. arXiv:2103.15045.

[12] Alexander Postnikov. Permutohedra, associahedra, and beyond. Int. Math. Res. Not. IMRN, (6):1026–1106, 2009.