PARTIAL EXACT CONTROLLABILITY FOR INHOMOGENEOUS MULTIDIMENSIONAL THERMOELASTIC DIFFUSION PROBLEM

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Abstract. The problem of stabilization and controllability for inhomogeneous multidimensional thermoelastic diffusion problem is considered for anisotropic material. By introducing a nonlinear feedback function on part of the boundary of the material, which is clamped along the rest of its boundary, we prove that the energy of the system decays to zero exponentially or polynomially. Both rates of decay are determined explicitly by the physical parameters. Via Russell’s “Controllability via Stabilizability” principle, we prove that the considered system is partially controllable by a boundary function determined explicitly.

1. Introduction. The research conducted in the development of high technologies after the second world war, confirmed that the field of diffusion in solids can not be ignored. Diffusion can be defined as the random walk of a set of particles from regions of high concentration to regions of lower concentration. Thermodiffusion in an elastic solid is due to coupling of the fields of strain, temperature and mass diffusion. The processes of heat and mass diffusion play a important role in many engineering applications.

The anisotropic thermoelastic diffusion materials with variables coefficients are used in several engineering applications such that metallurgical industry where these materials are processed under strong temperature gradients. An important example is the thermo-mechanical modeling of aluminium electrolytic cell [10]. Another important application of this kind of materials is the thermo-mechanical modeling of slabs during direct chill castings of alloys [9]. In all these processes and others, it is very important to take into account the mechanical heat dissipation and the non-homogeneity of the materials. The stabilization and the controllability analysis of these problems becomes very difficult due to all these aspects. For this reason the qualitative properties of solutions to thermoelastic diffusion systems becomes
a mathematical challenge, in particular when the material coefficients depend on spatial coordinate.

Since the pioneering work of Dafermos [12], it is well known that the temperature gradient always decays to zero and that, ‘usually’, the displacement also decays to zero; but no rate of decay has been given. However, he also pointed out that, when the domain is a ball, no decreasing solution can exist. Jian et al. [16] obtained the uniform rate of decay for the radial symmetric solutions in two or three dimensional space. Lebeau and Zuazua [22] proved that the decay rate is never uniform when the domain is convex. Furthermore, in the one space dimension case, it has been shown in [11] that the energy of the inhomogeneous thermoelastic diffusion system associated with various boundary conditions decays to zero exponentially and polynomially. For the multidimensional case the situation is different from that in the one-dimensional case. In fact, the degree of freedom of the displacement vector field is greater than the one of freedom of the temperature and the diffusion when the dimension is greater than $n = 1$. This means that the dissipation given by the thermal and diffusion effects are not strong enough to produce an uniform decay for the higher dimensional thermoelastic diffusion system. Thus, in order to ensure an uniform decay for thermoelastic systems, additional damping mechanisms are necessary. In this spirit, many linear and nonlinear velocity boundary feedback functions are introduced to get uniform decay in homogeneous and inhomogeneous thermoelasticity (see e.g [7, 25, 26, 28, 29, 33]).

By combining the multiplier method with other techniques, Lasiecka et al. (see e.g [18, 19, 20, 21]) have treated different dissipative systems of partial differential equations under general geometrical conditions with nonlinear feedback. Motivated by these techniques and by using a perturbed energy functional introduced by Komornik and Zuazua [17], we show that the energy in the inhomogeneous higher-dimensional thermoelastic diffusion system decays to zero at an exponential rate and also at a polynomial rate determined explicitly by the physical parameters.

While there has been extensive work on the stabilization of the linear thermoelasticity, relatively little is known about the controllability. Narukawa [31] proved the partial exact boundary controllability for a thermoelastic system in $\mathbb{R}^n$. Later, Lions [24] improved the Narukawa’s result by introducing the Hilbert Uniqueness Method. In both results, only the displacement is controlled, disregarding the values of the temperature. This is the so-called partial controllability property. Hansen [14] proved that the exact controllability of both displacement and temperature is possible for the one-dimensional thermoelastic system, by only controlling the thermal or mechanical component on the boundary. It seems that Hansen’s method is not applicable to the multi-dimensional space case. Thus, the problem of exact controllability of both the displacement and temperature is much more complicated in this case. Zuazua [35] solved this problem by introducing the concept of exact-approximate controllability and made significant progress. He proved that, if $T$ is large enough, then the thermoelastic system is exact-approximatively controllable with a control supported in a neighborhood of the boundary of $\Omega$, i.e., the displacement is shown to be exactly controllable and the temperature approximately controllable. The method of Zuazua combines multiplier techniques, compactness arguments and Holmgren’s Uniqueness Theorem among other tools. Teresa and Zuazua [13] obtained the same result for thermoelastic plates. Lebeau and Zuazua [23] proved that the system of linear thermoelasticity with periodic boundary conditions is null controllable. Liu [25, 26], via Russell’s principle, proved
that the isotropic thermoelastic system is partially controllable with boundary controls without smallness restrictions on the coupling parameters. Lasiecka in [18] provided different techniques to control several coupled systems.

The main novelties of the present paper are the following.

(i) The question of analyzing the stabilization or the controllability under the weaker feedback forces of the form (8) is an interesting open problem. This analysis has been performed in [3, 15] in elasticity context and in [25, 28] in isotropic thermoelasticity but, to our knowledge, this issue has not been addressed for inhomogeneous and anisotropic system of thermoelasticity with diffusion. The coupled system of equations considered in this study consists of a hyperbolic equation coupled with two parabolic equations. This poses new mathematical difficulties due to the nonlinear boundary conditions in stabilization or controllability issues. Under the classical polynomial growth assumption on the nonlinear boundary feedback near the origin, by using multiplier techniques and Lyapunov methods, we show that the considered system decays to zero at an exponential or polynomial rate. For the polynomial decay, some necessary modifications are imposed by the inhomogeneous and anisotropic nature of the material and the complicated coupling of our equations since they do not fall directly in the abstract frame of classical systems. Moreover, the proof of the polynomial stability has never been done previously as clearly with the weaker feedback forces given by (8) for any material. We show that the polynomial stability recover the exponential stability as a limit special case.

(ii) However, our approach, even if it uses ingredients of literature, relies essentially on the generalized Young’s inequality and it is simpler. This allows us to get more explicit expressions for the nonlinearity entering in the differential inequality governing the decay of the energy and the function of control. We show how the explicit expressions of the decay rates of the energy play an important role in the choice of material leading to faster decay and in the selection of the correct conditions imposed for stabilization or controllability.

(iii) To the best of our knowledge, there is no papers in literature discussing the controllability of the thermoelastic diffusion problem. In this paper, via Russell’s “Controllability via Stabilizability” principle we show the partial exact controllability of the considered problem with smallness restriction (see (91)) on the coupling tensors, stress-temperature and stress-diffusion. This condition shows the influence of the rate of exponential decay on the controllability. The boundary feedback function of the control will be determined explicitly. This represents a pleasant feature from the physical viewpoint.

The rest of the paper is organized as follows. In Section 2, we present the basic equations of the considered system and we give sufficient conditions needed in the sequel. In Section 3, we describe briefly the well-posedness of the corresponding system. In Section 4, we show that the considered system decays to zero at an exponential or polynomial rate depending on the assumptions on the feedback and on the system’s parameters. Finally, in Section 5, we show the partial exact controllability of the considered system.

2. Basic equations. We will study the multidimensional thermoelastic diffusion problem for inhomogeneous and anisotropic material. We denote by $\mathcal{H} = [\mathcal{H}_{i_1 \cdots i_k}]$ the tensor field of dimension $n^k$, $k = 0, 1, 2, \cdots$, defined over $\mathbb{R}^n \times (0, \infty)$, $n \in \mathbb{N}$.
We write partial derivatives with respect to the spatial variable as
\[ H_{i_1 \ldots i_k, j_1 \ldots j_s} = \frac{\partial^s H_{i_1 \ldots i_k}}{\partial x_{j_1} \ldots \partial x_{j_s}} \]
and with respect to the time derivative in the following way
\[ \dot{H}_{i_1 \ldots i_k} = \frac{\partial H_{i_1 \ldots i_k}}{\partial t}, \quad \ddot{H}_{i_1 \ldots i_k} = \frac{\partial^2 H_{i_1 \ldots i_k}}{\partial t^2}, \quad H_{i_1 \ldots i_k}^{(s)} = \frac{\partial^s H_{i_1 \ldots i_k}}{\partial t^s}, \quad s \geq 3. \]

Here we will adopt double index convention (that is double index means summation over the repeated i.e., \( H_{i_k} G_{i_k} = \sum_{i_k} H_{i_k} G_{i_k} \)).

Let us denote by \( \Omega \subset \mathbb{R}^n \) the connected bounded domain in which our material will be configurable with boundary \( \Gamma = \partial \Omega \) of class \( C^2 \). For any \( x \) of the body at a time \( t \), we will denote by \( u, \theta \) and \( P \) the displacement, the thermal and the chemical potential, respectively. We denote by \( \nu(x) \), \( x \in \Gamma = \partial \Omega \) the exterior normal vector to \( \Gamma \). We assume also that there exists a point \( x_0 \in \mathbb{R}^n \) for which we denote by \( m \) the vector field \( m(x) = x - x_0 \), such that
\[ \Gamma_0(x) = \{ x \in \Gamma, \ m(x) \cdot \nu(x) > 0 \} \quad \text{and} \quad \Gamma_1(x) = \{ x \in \Gamma, \ m(x) \cdot \nu(x) < 0 \}. \quad (1) \]
The notation \( m(x) \cdot \nu(x) \) represents the inner product in \( \mathbb{R}^n \) of the vectors \( m(x) \) and \( \nu(x) \). As example of the existence of such point \( x_0 \) we can consider the domain given by Figure 1.

By our choice of \( x_0 \) we easily get that hypotheses (1) holds. Note that both \( \Gamma_0 \) and \( \Gamma_1 \) are non-empty and satisfy
\[ \Gamma_0 \cap \Gamma_1 = \emptyset, \quad \Gamma = \Gamma_0 \cup \Gamma_1. \quad (2) \]

Since \( \Gamma_0 \) is a compact subset, there exists two positive constants \( a_0, \ a_1 \) for which we have
\[ 0 < a_0 \leq m(x) \cdot \nu(x) \leq a_1, \quad \forall x \in \Gamma_0. \quad (3) \]
Let us define by
\[ R_0 = \max_{\Gamma} |m(x)| \quad (4) \]
the radius of the smallest ball, with center in \( x_0 \), containing \( \Omega \).
In the absence of exterior forces, heat and chemical sources, the thermoelastic diffusion equations for inhomogeneous and anisotropic material are given by \[4\]
\[
\rho \ddot{u}_i = (C_{ijkl}u_{k,l} + \alpha_{ij}\theta + \beta_{ij}P)_j \quad \text{in} \quad \Omega \times (0, \infty),
\]
\[
\dot{\theta} + d\dot{P} = \alpha_{ij}\dot{u}_{ij} + (k_{ij}\theta_j)_i \quad \text{in} \quad \Omega \times (0, \infty),
\]
\[
d\theta + r\dot{P} = \beta_{ij}\dot{u}_{ij} + (h_{ij}P)_i \quad \text{in} \quad \Omega \times (0, \infty).
\]

We will consider the following boundary conditions
\[6\]
\[
u_i = 0, \quad \theta = 0, \quad P = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty),
\]
\[
(C_{ijkl}u_{k,l} + \alpha_{ij}\theta + \beta_{ij}P)_j = \phi_i, \quad (k_{ij}\theta_j)_i = -Q_1\theta, \quad \text{in} \quad \Gamma_0 \times (0, \infty),
\]
\[
(h_{ij}P)_i = -Q_2P \quad \text{on} \quad \Gamma_0 \times (0, \infty),
\]
with \(i, j, k, l \geq 1, \quad Q_1 > 0 \) and \( Q_2 > 0 \). In the first part of this paper, the function \( \phi \) is given by
\[8\]
\[
\phi_i(u_i) = -(m_j \cdot n_j)g_i(u_i),
\]
where \( g_i \) is at our disposal. In the second part, the function \( \phi_i \) will be determined to get the partial exact controllability of the system. We assume that the initial conditions are
\[9\]
\[
u(x, 0) = u^0(x), \quad \dot{u}(x, 0) = u^1(x), \quad \theta(x, 0) = \theta^0(x), \quad P(x, 0) = P^0(x), \quad x \in \Omega.
\]

In the above equations at a point \( x \in \Omega \), \( \rho(x) \) denotes the mass density and \( C_{ijkl}(x), \alpha_{ij}(x), \beta_{ij}(x) \) are tensor fields which represent the elasticity of the material, the stress-temperature, the stress-diffusion. By \( c(x) \), \( d(x) \) and \( r(x) \) we denote respectively the specific heat, the measure of thermal-diffusion effect and the measure of diffusive effect. \( k_{ij}(x) \) is the conductivity tensor of thermal field and \( h_{ij}(x) \) is the conductivity tensor of diffusion field.

We assume also that \( C_{ijkl}(x), \alpha_{ij}(x), \beta_{ij}(x), k_{ij}(x) \) and \( h_{ij}(x) \) are smooth over \( \overline{\Omega} \) and satisfy the following conditions
\[10\]
\[
C_{ijkl} = C_{jikl} = C_{klji}, \quad \alpha_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad k_{ij} = k_{ji}, \quad h_{ij} = h_{ji}.
\]

There exists positive constants \( \gamma_1, \gamma_2, \gamma_3, \kappa \) such that
\[11\]
\[
\int_{\Omega} \theta \dot{u}_{ij}\theta_{ij}dx \geq \gamma_1 \int_{\Omega} \theta^2 dx, \quad \int_{\Omega} P_i h_{ij} P_j dx \geq \kappa \int_{\Omega} P^2 dx.
\]

We set
\[13\]
\[
\omega_1 := \max_{x,i,j} |\alpha_{ij}(x, i, j)|, \quad \omega_2 := \max_{x,i,j} |\beta_{ij}(x, i, j)|, \quad \omega_3 := \max_{x,i,j} |\alpha_{ij,j}(x, i, j)|,
\]
\[
\omega_4 := \max_{x,i,j} |\beta_{ij,j}(x, i, j)|, \quad \omega_5 := \max_{x,i,j} |C_{ijkl}(x, i, j, k, l)|, \quad x \in \overline{\Omega}.
\]

The density \( \rho(x) \) is continuous over \( \overline{\Omega} \) and satisfies the following conditions
\[14\]
\[
\rho_j \in L^\infty(\Omega), \quad 0 < \rho_0 < \rho(x) < \rho_1, \quad \rho(x) + m_\mu(x)P_\mu(x) > \gamma_0,
\]
where \( \gamma_0 \) is a positive constant. The functions \( c(x), r(x) \) and \( d(x) \) are continuous over \( \overline{\Omega} \) and satisfy
\[15\]
\[
0 < c_0 < c(x) < c_1, \quad 0 < d_0 < d(x) < d_1, \quad 0 < r_0 < r(x) < r_1, \quad c(x)r(x) - d^2(x) > 0, \quad \forall x \in \Omega.
\]
Note that (15) implies that
\[ c(x)\theta^2(x) + 2d(x)\theta(x)P(x) + r(x)P^2(x) > 0, \quad \forall x \in \Omega. \]
This condition is needed to stabilize the thermoelastic diffusion system (see [4] for more details).

Finally, we assume that the function \( g_i \in C(\mathbb{R}^n) \), nondecreasing on \( \Gamma_0 \) and satisfies for all \( u_i \in \mathbb{R} \) the following hypothesis [28]
\[ g_i(u_i) = 0 \Leftrightarrow u_i = 0, \quad (16) \]
\[ g_i(u_i)(u_i) \geq k_3|u_i|^2, \quad \forall u_i \in \mathbb{R} \quad \text{with} \quad |u_i| \geq 1, \quad (17) \]
\[ g_i(u_i)(u_i) \geq k_3|u_i|^{p+1}, \quad \forall u_i \in \mathbb{R} \quad \text{with} \quad |u_i| \leq 1, \quad (18) \]
\[ |g_i(u_i^1) - g_i(u_i^2)| \leq k_2|u_i^1 - u_i^2|, \quad \forall u_i^1, u_i^2 \in \mathbb{R} \quad \text{with} \quad |u_i^1 - u_i^2| \leq 1, \quad (19) \]
\[ |g_i(u_i^1) - g_i(u_i^2)| \leq k_2|u_i^1 - u_i^2|, \quad \forall u_i^1, u_i^2 \in \mathbb{R} \quad \text{with} \quad |u_i^1 - u_i^2| \geq 1, \quad (20) \]
\[ (g_i(u_i^1) - g_i(u_i^2))(u_i^1 - u_i^2) \geq 0, \quad \forall u_i^1, u_i^2 \in \mathbb{R}, \quad (21) \]
for some positive constants \( k_2, k_3, p \) and \( q \) with \( 0 < q \leq 1 \).

3. Well-posedness. In this section, we shall use the theory of nonlinear semigroups to describe briefly the well-posedness of the system (5)-(9) (see [8], Chap. 3).

In what follows, \( H^s(\Omega) \) denotes the usual Sobolev space (see [1]) for any \( s \in \mathbb{R} \). For \( s \geq 0 \), \( H_0^s(\Omega) \) denotes the completion of \( C_0^\infty(\Omega) \) in \( H^s(\Omega) \), where \( C_0^\infty(\Omega) \) denotes the space of all infinitely differentiable functions on \( \Omega \) with compact support in \( \Omega \). Let \( X \) be a Banach space. We denote by \( C([0,T];X) \) the space of all \( k \) times continuously differentiable functions defined on \( [0,T] \) with values in \( X \), and write \( C([0,T];X) \) for \( C^0([0,T];X) \).

We further introduce other function space as follows
\[ H_{\Gamma_1}^1(\Omega) = \left\{ \varphi \in H^1(\Omega): \varphi = 0 \text{ on } \Gamma_1 \right\}. \]
Under assumptions (10) and (11), and \( \Gamma_1 \neq \emptyset \), one can easily show that the following norm on \( (H_{\Gamma_1}^1(\Omega))^n \)
\[ \| \varphi \|_{(H_{\Gamma_1}^1(\Omega))^n} = \frac{1}{2} \int_\Omega C_{ijkl}\varphi_{k,l}\varphi_{i,j}dx \]
is equivalent to the usual one induced by \( (H^1(\Omega))^n \) (see [25]). Since \( \Gamma_1 \neq \emptyset \), we define the function space
\[ W = (H_{\Gamma_1}^1(\Omega))^n \times (L^2(\Omega))^n. \]
Note that the norm on \( W \)
\[ \| (u,v) \|_W = \frac{1}{2} \int_\Omega \left( \rho|v|^2 + C_{ijkl}u_{k,l}u_{i,j} \right)dx \]
is equivalent to the usual one induced by \( (H^1(\Omega))^n \times (L^2(\Omega))^n. \)

Note that for function \( \varphi \in H_{\Gamma_1}^1(\Omega) \) the Poincare’s inequality remains valid, that is
\[ \| \varphi \|_{L^2(\Omega)} \leq C_p \left( \int_\Omega \varphi_j^2 \right)^{\frac{1}{2}}, \]
where \( C_p > 0 \) is the Poincare’s constant. Let \( \tau_0 \) be the smallest positive constant such that
\[ \int_{\Gamma_0} \varphi^2d\Gamma \leq \tau_0^2 \| \varphi \|_{(H_{\Gamma_1}^1(\Omega))^n}^2, \quad \forall \varphi \in (H_{\Gamma_1}^1(\Omega))^n. \quad (22) \]
Let us define $A := [A_i]$,

$$A_i \begin{pmatrix} u \\ v \\ \theta \\ P \end{pmatrix} = \begin{pmatrix} A_i u_i + \frac{1}{\rho} (\alpha_{ij}\theta + \beta_{ij}P)j \\ rC_i \theta - dD_i P + \frac{1}{\rho^2} (r\alpha_{ij} - d\beta_{ij})v_{i,j} \\ -dC_i \theta + rD_i P + \frac{c}{\rho^2} (c\beta_{ij} - d\alpha_{ij})v_{i,j} \end{pmatrix},$$

where

$$A_i u_i = \frac{1}{\rho} (C_{ijkl} u_{k,l})_j, \quad C_i \theta = \frac{1}{cr - d^2} (k_{ij}\theta)_i, \quad D_i P = \frac{1}{cr - d^2} (h_{ij}P)_i.$$

The Riesz representation theorem ensures that $A := [A_i]$ is an isometric isomorphism of $(H^1_\Gamma(\Omega))^n$ onto $[(H^1_\Gamma(\Omega))^n]'$, $C := [C_i]$ and $D := [D_i]$ are isometric isomorphisms of $H^0_\Gamma(\Omega)$ onto $[H^1_\Gamma(\Omega)]'$.

Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $(H^1_\Gamma(\Omega))^n$ and $[(H^1_\Gamma(\Omega))^n]'$ or $H^0_\Gamma(\Omega)$ and $H^{-1}(\Omega)$. Also we define a nonlinear operator $B$, by

$$\langle Bu, v \rangle = -\int_{\Omega} (m \cdot v) g(u) v d\Omega, \quad \forall u, v \in (H^1_\Gamma(\Omega))^n.$$

Note that there exist constants $c > 0$ and $b \geq 0$ such that for $u_i \in \mathbb{R}$, $g_i$ satisfies

$$|g_i(u_i)| \leq \begin{cases} c(1 + |u_i|^n), & n \geq 3, \\ c(1 + |u_i|^b), & n = 2, \end{cases}$$

then $B := [B_i]$ maps $(H^1_\Gamma(\Omega))^n$ onto $[(H^1_\Gamma(\Omega))^n]'$ (see [28] for more details).

Using the above notations we conclude that the field equation (5)-(9) is equivalent to an abstract Cauchy problem

$$(\dot{u}_i, \dot{v}_i, \dot{\theta}, \dot{P}) = A_i (u, v, \theta, P), \quad i = 1, \ldots, n$$

or in vectorial notation

$$\frac{dU(t)}{dt} = AU(t), \quad t > 0.$$

where the nonlinear operator $A$ is considered on

$$\mathcal{H} = W \times L^2(\Omega) \times L^2(\Omega),$$

endowed with the inner product

$$\langle U, Y \rangle_{\mathcal{H}} = \int_{\Omega} \left( C_{ijkl} u_{k,l} v_{i,j} + \rho v_i w_i + c\theta \phi + rP \psi + d(\theta \psi + P \phi) \right) dx,$$

where $U = (u, v, \theta, P)$ and $Y = (z, w, \phi, \psi)$ are in $\mathcal{H}$ such that $v_i = \dot{u}_i$ and $w_i = \dot{z}_i$.

Suppose that (3), (12) hold, $g_i$ satisfies (17), (18) and (24) and $Q_1 > 0, Q_2 > 0$. Then the operator $A$ is dissipative, i.e.,

$$\langle AU, U \rangle_{\mathcal{H}} = -\int_{\Omega} \left( (m \cdot v_j) g_i (v_i) v_i + Q_1 \theta^2 + Q_2 P^2 \right) d\Omega - \int_{\Omega} \theta_i k_{ij} \theta_j dx - \int_{\Omega} P_i h_{ij} P_j dx \leq 0.$$

(27)

Based on the above properties of the operators $A, B, C, D$ and suppose that $g_i$ satisfies (24), it is standard matter to show that the nonlinear operator $A$ is m-dissipative on $\mathcal{H}$ (for more details see [11, 28, 29] and references therein).
Lemma 3.1. Suppose that (2) holds and $g_i$ satisfies (16) and (24), then the domain $\mathcal{D}(A)$ is dense in $\mathcal{H}$. Further, if $g_i$ satisfies (19) and (20) we have
\[
\mathcal{D}(A) \subset \mathcal{D}_0 = \{ (u,v,\theta,P) \in \mathcal{H} : u \in (H^s(\Omega) \cap H^1_r(\Omega))^n, \ v \in (H^1_r(\Omega))^n, \\
\theta \in H^2(\Omega) \cap H^1_r(\Omega), \ P \in H^2(\Omega) \cap H^1_r(\Omega) \text{ satisfying (7) on } \Gamma_0 \},
\]
for some $s > \frac{3}{2}$.

The proof is essentially the same as that of Lemma 9.2 of [2] and Lemma 3.3 of [28]. Hence it is omitted here for the sake of brevity.

By the classical theory of nonlinear semigroups (see [8], Chap. 3), we have

Theorem 3.2. Let $\Gamma_0$ and $\Gamma_1$ are non-empty satisfying (1) and (2). Suppose that the function $g_i$ satisfies (16), (21) and (24). Then we have

(i) For every initial condition $(u^0, u^1, \theta^0, P^0) \in \mathcal{H}$, the problem (5)-(9) has a unique mild solution satisfying
\[
(u, u_t, \theta, P) \in C([0, \infty); \mathcal{H}).
\]

(ii) Further, if $g_i$ satisfies (19) and (20), then, for every initial condition $(u^0, u^1, \theta^0, P^0) \in \mathcal{D}(A)$, the problem (5)-(9) has a unique classical solution satisfying
\[
\begin{align*}
&u \in L^\infty([0, \infty); (H^s(\Omega) \cap H^1_r(\Omega))^n), \quad \dot{u} \in L^\infty([0, \infty); (H^1_r(\Omega))^n), \\
&\theta \in L^\infty([0, \infty); H^2(\Omega) \cap H^1_r(\Omega)), \quad \dot{\theta} \in L^\infty([0, \infty); L^2(\Omega)), \\
P \in L^\infty([0, \infty); H^2(\Omega) \cap H^1_r(\Omega)), \quad \dot{P} \in L^\infty([0, \infty); L^2(\Omega)).
\end{align*}
\]
for some $s > \frac{3}{2}$.

4. Decay of solutions. Firstly, we need to perform some technical lemmas. To do this end, we introduce the following energy functional
\[
E(t) = E(u_t, \theta, P, t) = \frac{1}{2} \int_\Omega \left( \rho \dot{u}_i^2 + C_{ijk} u_k^i u_l^j + c \theta^2 + 2d \theta \dot{P} + r P^2 \right) dx.
\] 
Recall that $\Gamma_0$ is a compact subset satisfying (3). The conductivity terms $k_{ij}$ and $h_{ij}$ are definite positive and satisfy (12). The function $g_i$ satisfies (17) and (18).

Finally we assume that (15) holds, $Q_1 > 0$ and $Q_2 > 0$.

Based on the above properties of the operators $A$, $B$, $C$, $D$, we multiply $(5)_1$, $(5)_2$ and $(5)_3$ respectively by $v_i$, $\theta$, $P \in H^1_r(\Omega)$ and we integrate over $\Omega$ we conclude that
\[
\frac{d}{dt} E(t) = -\int_{\Gamma_0} \left( (m_j \cdot \nu_j) g_i(\dot{u}_i) \dot{u}_i + Q_1 |\theta|^2 + Q_2 |P|^2 \right) d\Gamma \\
- \int_{\Omega} \theta_i k_{ij} \dot{\theta}_j dx - \int_{\Omega} P_i h_{ij} P_j dx < 0.
\]

First, let us introduce the following functionals
\[
F(t) = \int_{\Omega} 2 \rho (m_{i\mu} u_{\mu}) \dot{u}_i dx \quad \text{and} \quad G(t) = \int_{\Omega} \rho \dot{u}_i u_i dx,
\] 
for some $\rho > 0$.
where $\mu = 1, \cdots, n$ and $m_\mu$ is the $\mu-$component of $m$. By a direct calculation, we get

$$
\frac{d}{dt} G(t) = - \int_{\Gamma_0} (m_j \cdot \nu_j) g_i (\dot{u}_i) u_i d\Gamma + \int_\Omega \rho \ddot{u}_i \dot{u}_i dx - \int_\Omega C_{ijkl} u_k l u_{i,j} dx - \int_\Omega (\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} dx. \tag{33}
$$

**Lemma 4.1.** Let $U = (u, v, \theta, P)$ a classical solution to problem (5)-(8). Then we have

$$
\frac{d}{dt} \mathcal{F}(t) = \int_{\Gamma_0} \rho (m_\mu \cdot \nu_\mu) \dot{u}_i \dot{u}_i d\Gamma - 2 \int_{\Gamma_0} (m_j \cdot \nu_j) g_i (\dot{u}_i) m_\mu u_{i,\mu} d\Gamma - \int_{\Gamma_0} (n\rho + \rho_\mu m_\mu) \dot{u}_i \dot{u}_i dx + (n - 2) \int_\Omega C_{ijkl} u_k l u_{i,j} dx + \int_\Omega m_\mu C_{ijkl,\mu} u_k l u_{i,j} dx - 2 \int_\Omega (m_j \cdot \nu_j) (\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} d\Gamma + 2 \int_{\Gamma_0} (\alpha_{ij} \theta + \beta_{ij} P) (m_\mu u_{i,\mu})_{,j} \mu dx. \tag{34}
$$

**Proof.** Differentiating the function $\mathcal{F}(t)$ with respect to time, we get

$$
\frac{d}{dt} \mathcal{F}(t) = 2 \int_\Omega \rho m_\mu \dot{u}_i \dot{u}_i dx + 2 \int_\Omega \rho m_\mu u_{i,\mu} \dot{u}_i dx. \tag{35}
$$

By integration by parts, the first term of the right hand side of (35) can be written in the form

$$
2 \int_\Omega \rho m_\mu \dot{u}_i \dot{u}_i dx = \int_{\Gamma_0} \rho (m_\mu \cdot \nu_\mu) \dot{u}_i \dot{u}_i d\Gamma - \int_\Omega (n\rho + \rho_\mu m_\mu) \dot{u}_i \dot{u}_i dx, \tag{36}
$$

while the second term is given by

$$
\int_\Omega \rho \ddot{u}_i m_\mu u_{i,\mu} dx = - \int_{\Gamma_0} (m_j \cdot \nu_j) g_i (\dot{u}_i) m_\mu u_{i,\mu} d\Gamma + \int_{\Gamma_1} \nu_j C_{ijkl} u_k l m_\mu u_{i,\mu} d\Gamma - \int_\Omega (\alpha_{ij} \theta + \beta_{ij} P) (m_\mu u_{i,\mu})_{,j} dx. \tag{37}
$$

Since over $\Gamma_1$ we have $u_i = 0$, then we obtain

$$
\nu_j u_{i,\mu} = \nu_\mu u_{i,j} \tag{38}
$$

so the second term of r. h. s of (37) is given by

$$
\int_{\Gamma_1} \nu_j C_{ijkl} u_k l m_\mu u_{i,\mu} d\Gamma = \int_{\Gamma_1} (m_\mu \cdot \nu_j) C_{ijkl} u_k l u_{i,j} d\Gamma. \tag{39}
$$

Using the symmetry on the coefficient which implies

$$
(C_{ijkl} u_k l u_{i,j})_{,\mu} = C_{ijkl,\mu} u_k l u_{i,j} + 2 C_{ijkl} u_k l u_{i,j}. \mu,
$$
the last term of (37) becomes
\[
\int_{\Omega} C_{ijkl} u_{k,i} (m_{\mu} u_{i,\mu})_j dx
- \frac{n}{2} \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) C_{ijkl} u_{k,i} u_{i,j} d\Gamma + \frac{1}{2} \int_{\Gamma_1} (m_{\mu} \cdot \nu_{\mu}) C_{ijkl} u_{k,i} u_{i,j} d\Gamma
- \frac{1}{2} \int_{\Omega} m_{\mu} C_{ijkl,\mu} u_{k,i} u_{i,j} dx.
\] (40)

Then using (39) and (40), (37) becomes
\[
\int_{\Omega} \rho \dot{u}_i u_{i,\mu} dx = - \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) g_i(u_i) m_{\mu} u_{i,\mu} d\Gamma + \frac{1}{2} \int_{\Gamma_1} (m_{\mu} \cdot \nu_{\mu}) C_{ijkl} u_{k,i} u_{i,j} d\Gamma
- \frac{1}{2} \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) C_{ijkl} u_{k,i} u_{i,j} d\Gamma + \frac{n}{2} \int_{\Omega} (m_{\mu} \cdot \nu_{\mu}) C_{ijkl} u_{k,i} u_{i,j} dx
+ \frac{1}{2} \int_{\Omega} m_{\mu} C_{ijkl,\mu} u_{k,i} u_{i,j} dx - \int_{\Omega} (\alpha_{ij} \theta + \beta_{ij} P) (m_{\mu} u_{i,\mu})_j dx.
\] (41)

Substituting Eqs. (36) and (41) into (35) we obtain (34). \qed

Lemma 4.2. Let \( K(t) = F(t) + (n-1)G(t) \), then
\[
\frac{d}{dt} K(t) \leq - \frac{\varrho_2}{\varrho_1} E(t) + \Lambda_1 \int_{\Gamma_0} \left( |g_i(u_i)|^2 + |\dot{u}_i|^2 + |\theta|^2 + |P|^2 \right) d\Gamma + \Lambda_2 \int_{\Omega} \left( |\theta_j|^2 + |P_j|^2 \right) dx,
\] (42)

where
\[
\varrho_1 = \max \left\{ \rho_1, \omega_5, c_1 + d_1, \rho_1 + d_1 \right\}, \quad \varrho_2 = \min \left\{ \gamma_0, \frac{\gamma_2}{2} \right\}, \quad \Lambda_1 = M_1 + \varrho_2 C_p, \quad \Lambda_2 = M_1 + \varrho_2 C_p,
\]

\[
M_1 = M_2 + M_3,
\]

\[
M_2 = \max \left\{ \frac{2(n-1)^2 \omega_2^2 C_p}{\gamma_1}, \frac{2(n-1) \gamma_2 \omega_2^2 C_p}{\gamma_1} \right\},
\]

\[
M_3 = \max \left\{ \frac{4R_1^2}{\gamma_1} (\omega_2^2 C_p + \omega_2^2), \frac{4R_1^2}{\gamma_1} (\omega_2^2 C_p + \omega_2^2) \right\}.
\] (43)

Proof. After differentiating the function \( K(t) \) with respect to time and using (33), (34) and (1) we get
\[
\frac{d}{dt} K(t) \leq \int_{\Gamma_0} \rho (m_{\mu} \cdot \nu_{\mu}) u_i \dot{u}_i d\Gamma - 2 \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) g_i(u_i) m_{\mu} u_{i,\mu} d\Gamma
- (n - 1) \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) g_i(u_i) u_i d\Gamma - (n - 1) \int_{\Omega} (\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} dx
- \int_{\Omega} (\rho + \rho_{\mu} m_{\mu}) \dot{u}_i \ddot{u}_i dx - \int_{\Omega} (C_{ijkl} - m_{\mu} C_{ijkl,\mu}) u_{k,i} u_{i,j} dx
- 2 \int_{\Gamma_0} (m_{\mu} \cdot \nu_{\mu}) (\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} d\Gamma + 2 \int_{\Omega} (\alpha_{ij} \theta + \beta_{ij} P) (m_{\mu} u_{i,\mu})_j dx.
\] (44)
where \( R_0 \) is given by (4). The third term of the r. h. s of (44) can be estimated as follows

\[
- \int_{\Gamma_0} (m_j \cdot \nu_j) g_i(\hat{u}_i) u_i d\Gamma
\]

Using (13), the fourth term of the r. h. s of (44) becomes

\[
- (n - 1) \int_{\Omega} (\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} dx \leq M_2 \int_{\Omega} (|\theta_{,j}|^2 + |P_{,j}|^2) dx + \frac{\gamma_1}{4} \int_{\Omega} |u_{i,j}|^2 dx,
\]

where \( M_2 \) is given by (43). From (14)_3 and (11)_1 the fifth and sixth terms of the r. h. s of (44) are given by

\[
- \int_{\Omega} \rho \mu_i u_i \hat{u}_i dx - \int_{\Omega} \left( C_{ijkl} - m_\mu C_{ijkl,\mu} \right) u_{k,i} u_{i,j} dx
\]

\[
\leq - \gamma_0 \int_{\Omega} |\hat{u}_i|^2 dx - \gamma_1 \int_{\Omega} |u_{i,j}|^2 dx.
\]

The two last terms of the r. h. s of (44) can be estimated as follows

\[
- 2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu)(\alpha_{ij} \theta + \beta_{ij} P) u_{i,j} d\Gamma
\]

\[
\leq \omega_1^2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |\theta|^2 d\Gamma + \omega_2^2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |P|^2 d\Gamma + \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |u_{i,j}|^2 d\Gamma
\]

and

\[
2 \int_{\Omega} (\alpha_{ij} \theta + \beta_{ij} P)_{,\mu} (m_j u_{i,j}) dx \leq M_3 \int_{\Omega} (|\theta_{,\mu}|^2 + |P_{,\mu}|^2) dx + \frac{\gamma_1}{4} \int_{\Omega} |u_{i,j}|^2 dx,
\]

where \( M_3 \) is given by (43). By substituting Eqs. (45)-(51) into (44), we get

\[
\frac{d}{dt} \mathcal{K}(t) \leq -\varphi_2 \int_{\Omega} (|\hat{u}_i|^2 + |u_{i,j}|^2) dx + \frac{\epsilon}{\tau_0} \int_{\Gamma_0} |\hat{u}_i|^2 d\Gamma + 2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |u_{i,j}|^2 d\Gamma
\]

\[
+ \rho_1 a_1 \int_{\Gamma_0} |\hat{u}_i|^2 d\Gamma + \omega_1^2 \int_{\Gamma_0} (m_j \cdot \nu_j) |\theta|^2 d\Gamma + \omega_2^2 \int_{\Gamma_0} (m_j \cdot \nu_j) |P|^2 d\Gamma
\]

\[
+ \left( R_0^2 + \frac{(n - 1)^2 a_1 \tau_0^2}{4\epsilon} \right) \int_{\Gamma_0} (m_j \cdot \nu_j) |g_i(\hat{u}_i)|^2 d\Gamma + M_1 \int_{\Omega} (|\theta_{,\mu}|^2 + |P_{,\mu}|^2) dx,
\]

where \( \mathcal{K}(t) \) is defined by (42).
where $q_2$ and $M_1$ are given by (43). It is easy to see that there exist a constant $q_1 > 0$ such that

$$2E(t) \leq q_1 \int_\Omega \left( |\dot{u}_i|^2 + |u_{i,j}|^2 + |\theta|^2 + |P|^2 \right) dx,$$

where $q_1$ is given by (43). Then we infer from (53) that

$$-q_2 \int_\Omega \left( |\dot{u}_i|^2 + |u_{i,j}|^2 \right) dx$$

$$= -q_2 \left( \int_\Omega \left( |\dot{u}_i|^2 + |u_{i,j}|^2 + |\theta|^2 + |P|^2 \right) dx + \int_\Omega (|\theta|^2 + |P|^2) dx \right)$$

$$\leq -\frac{2q_2}{q_1} E(t) + q_2 \int_\Omega (|\theta|^2 + |P|^2) dx.$$  

From (22) and by taking $\epsilon = \frac{q_2}{q_1}$ in (52), the second term of the r. h. s of (52) can be estimated as follows

$$\frac{q_2}{2} \int_\Gamma_0 |u_i|^2 d\Gamma \leq \frac{q_2}{q_1} E(t).$$  

From (11)$_2$, the third term of the r. h. s of (52) can be estimated as follows

$$2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |u_{i,j}|^2 d\Gamma \leq \frac{2}{\alpha} \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) C_{ijkl} u_{kl} u_{ij} d\Gamma.$$  

Using (7)$_1$ and Young’s inequality, we get

$$2 \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |u_{i,j}|^2 d\Gamma \leq \frac{4R_0^2}{\alpha^2} \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |g_i(\dot{u}_i)|^2 d\Gamma$$

$$+ \frac{4\omega_1^2}{\alpha^2} \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |\theta|^2 d\Gamma + \frac{4\omega_2^2}{\alpha^2} \int_{\Gamma_0} (m_\mu \cdot \nu_\mu) |P|^2 d\Gamma.$$  

Substituting (54)-(56) into (52), we get (42).
we have
\[ E(t) \left( 1 - \delta \varrho_0 E^\sigma(t) \right) \leq \mathcal{L}(t) \leq E(t) \left( 1 + \delta \varrho_0 E^\sigma(t) \right), \] (59)
where \( \varrho_0 = \max \left\{ \frac{2B_0}{\rho}, \frac{(n-1)}{a}, \frac{B_0 a^2}{2g} + (n-1) \frac{a e C_a}{4} \right\} \).

Differentiating \( \mathcal{L}(t) \) with respect to time, using (58) and (31), we get
\[ \frac{d}{dt} \mathcal{L}(t) \leq \frac{d}{dt} E(t) - \delta \sigma \varrho_0 \left( \frac{d}{dt} E(t) E^\sigma(t) + \delta E^\sigma(t) \frac{d}{dt} K(t) \right). \]
From (31) and (42) of Lemma 4.2, we infer that
\[ \frac{d}{dt} \mathcal{L}(t) \leq -\frac{\delta \varrho_2}{\varrho_1} E^{\sigma+1}(t) + \left( \delta E^\sigma(t) (\sigma \varrho_0 + \varrho_3) - 1 \right) \int_\Omega (\theta_i k_{ij} \theta_j + P_i h_{ij} P_j) dx \]
\[ + \left( \delta E^\sigma(t) (\Lambda_1 + \sigma \varrho_4) - \varrho_5 \right) \int_{\Gamma_0} (|\theta|^2 + |P|^2) d\Gamma \]
\[ - \left( 1 - \delta \sigma \varrho_0 E^\sigma(t) \right) \int_{\Gamma_0} (m_j \cdot \nu_j) g_i(\dot{u}_i) \dot{u}_i d\Gamma \]
\[ + \delta \Lambda_1 E^\sigma(t) \int_{\Gamma_0} (|g_i(\dot{u}_i)|^2 + |\dot{u}_i|^2) d\Gamma, \]
(60)
where \( \varrho_3 = \Lambda_2 \max \{ \frac{1}{2}, \frac{1}{q} \} \), \( \varrho_4 = \varrho_0 \max \{ Q_1, Q_2 \} \) and \( \varrho_5 = \min \{ Q_1, Q_2 \} \).

We now distinguish the cases \( p = q = 1 \) and \( p + 1 > 2q \).

4.1. Exponential decay.

**Theorem 4.3.** Let \( p = q = 1 \). Under the assumptions on the physical coefficient given by (10)-(15) and under the hypothesis (16)-(20), for any initial data \((u^0, u^1, \theta^0, P^0) \in H\), there exists two constants \( \Lambda > 1 \) and \( \omega > 0 \) such that
\[ E(t) \leq \Lambda E(0) e^{-\omega t}, \] (61)
where \( E(t) \) is defined by (30) and
\[ \delta_1 = \min \left\{ \frac{1}{\varrho_3}, \frac{\varrho_5}{\Lambda_1}, \frac{a_0 k_3}{(1 + k_2^2) \Lambda_1}, \frac{1}{2 \varrho_0} \right\}, \quad \Lambda = \frac{1 + \delta_1 \varrho_0}{1 - \delta_1 \varrho_0} \quad \text{and} \quad \omega = \frac{\delta_1 \varrho_2}{\varrho_1 (1 + \delta_1 \varrho_0)}. \] (62)

**Proof.** By taking \( p = q = 1 \), we have \( \sigma = 0 \). Substituting \( \sigma = 0 \) into (60) and using (3) and (17)-(20), we get
\[ \frac{d}{dt} \mathcal{L}(t) \leq -\frac{\delta \varrho_2}{\varrho_1} E(t) + \left( \delta \varrho_3 - 1 \right) \int_\Omega (\theta_i k_{ij} \theta_j + P_i h_{ij} P_j) dx \]
\[ + \left( \delta \Lambda_1 - \varrho_5 \right) \int_{\Gamma_0} (\theta^2 + P^2) d\Gamma - a_0 \int_{\Gamma_0} g_i(\dot{u}_i) \dot{u}_i d\Gamma \]
\[ + \delta \Lambda_1 \int_{\Gamma_0} (|g_i(\dot{u}_i)|^2 + |\dot{u}_i|^2) d\Gamma. \]
Choosing \( \delta = \delta_1 \) (as in (62)1) such that we have \( \delta \varrho_3 - 1 < 0 \) and \( \delta \Lambda_1 - \varrho_5 < 0 \), we infer from (16)-(20) that
\[ \frac{d}{dt} \mathcal{L}(t) \leq -\frac{\delta_1 \varrho_2}{\varrho_1} E(t) + \left( \delta_1 \Lambda_1 (1 + k_2^2) - a_0 k_3 \right) \int_{\Gamma_0} |\dot{u}_i|^2 d\Gamma \]
\[ \leq -\frac{\delta_1 \varrho_2}{\varrho_1} E(t) \quad \text{(use (59))} \]
By Gronwall’s inequality and using again (59), we get (61). From (62), we have $\delta_1 \leq \frac{1}{\tau_0}$. This implies that $1 - \delta_1 \theta_0 > 0$ and $\Lambda > 1$.

4.2. Polynomial decay.

**Theorem 4.4.** Let $p + 1 > 2q$. Under the assumptions on the physical coefficient given by (10)-(15) and under the hypothesis (16)-(20), for any initial data $(u^0, \mathbf{u}^1, \theta^0, P^0) \in \mathcal{H}$, there exists a positive constant $\lambda$ depending on initial data $(u^0, \mathbf{u}^1, \theta^0, P^0) \in \mathcal{H}$, such that

$$E(t) \leq gE(0)\left(1 + \sigma \lambda t\right)^{-\frac{1}{p}},$$

(63)

where

$$g = \frac{1 + \delta_2 \theta_0 E^\sigma(0)}{1 - \delta_2 \theta_0 E^\sigma(0)}, \quad \tau_1 = \frac{\Lambda_1 (1 + k_2^2)}{2}, \quad \lambda = \frac{\theta_0^2 \delta_2 E^\sigma(0)}{2\Lambda_1 \left(1 + \delta_2 \theta_0 E^\sigma(0)\right)},$$

$$r = \frac{p + 1}{p + 1 - 2q}, \quad \tau_2 = (2\tau_1)^\frac{\theta_1}{\delta_2},$$

$$\delta_2 = \min \left\{ \frac{1}{E^\sigma(0)(\sigma \theta_0 + \rho_3)}, \frac{\theta_5}{E^\sigma(0)(\Lambda_1 + \sigma \rho_4)}, \frac{\sigma \theta_0 \alpha_1 E^\sigma(0)}{E^\sigma(0) + \frac{\tau_2}{k_3} E^\sigma(0)}, \frac{\sigma \theta_0 \alpha_1 E^\sigma(0)}{\sigma \theta_0 \alpha_1 E^\sigma(0) + \frac{\tau_2}{k_3} E^\sigma(0)} \right\}.$$

(64)

**Proof.** Using (16)-(20) for the boundary integrals, (60) becomes

$$\frac{d}{dt} E^\sigma(t) \leq -\delta \frac{\delta_2}{\theta_1} E^\sigma(t) + \left(\delta E^\sigma(t)(\sigma \theta_0 + \rho_3) - 1\right) \int_{\Omega} \left(\theta_i \partial_{ij} \theta_{ij} + P_{ij} h_{ij} P_{ij}\right) dx$$

$$+ \left(\delta E^\sigma(t)(\Lambda_1 + \sigma \rho_4) - \theta_5\right) \int_{\Gamma_0} (|\theta|^2 + |P|^2) d\Gamma$$

$$+ \left(\delta \sigma \theta_0 \alpha_1 E^\sigma(t) - \theta_5\right) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} g_i(\mathbf{u}_i) \mathbf{u}_i d\Gamma$$

$$+ \delta \Lambda_1 E^\sigma(t) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} (|g_i(\mathbf{u}_i)|^2 + |\mathbf{u}_i|^2) d\Gamma$$

(65)

Using (31), (16), (17) and (20), the last integral of (65) becomes

$$\delta \Lambda_1 E^\sigma(t) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} (|g_i(\mathbf{u}_i)|^2 + |\mathbf{u}_i|^2) d\Gamma \leq \frac{\delta \tau_1}{k_3} E^\sigma(0) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} g_i(\mathbf{u}_i) \mathbf{u}_i d\Gamma,$$

(66)

where $\tau_1$ is given by (64). Now we estimate the fifth term of r. h. s of (65) by using (16) and (19)

$$\delta \Lambda_1 E^\sigma(t) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} (|g_i(\mathbf{u}_i)|^2 + |\mathbf{u}_i|^2) d\Gamma \leq \delta \tau_1 E^\sigma(t) \int_{\Gamma_0 \cap \{|\mathbf{u}| \leq 1\}} |\mathbf{u}_i|^{2q} d\Gamma.$$

(67)
To estimate the last term of (67), we use Young’s inequality
\[ AB \leq \frac{A^s}{s} + \frac{B^r}{r}, \]  
(68)
where
\[ A = 2\tau_1 \left( \frac{\rho_1}{\rho_2} \right)^{\frac{s}{r}} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma, \]
\[ B = \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right)^{\frac{s}{r}} E^{\sigma} (t), \quad s = \frac{p+1}{2q}, \quad r = \frac{p+1}{p+1-2q}. \]
Since \( p+1 > 2q \), we get
\[ \delta \tau_1 E^{\sigma} (t) \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \]
\[ \leq \frac{\delta}{s} \left( 2\tau_1 \left( \frac{\rho_1}{\rho_2} \right)^{\frac{s}{r}} \right)^{\frac{s}{q}} \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \right)^{\frac{s}{r}}\]
\[ + \frac{\delta}{r} \left( \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right)^{\frac{s}{r}} E^{\sigma} (t) \right)^{\frac{s}{r}} \quad \text{(use } \frac{1}{s} \leq 1 \text{ and } \frac{1}{r} \leq 1) \]
\[ \leq \delta \tau_2^{\frac{s}{r}} \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \right)^{\frac{s}{r}} + \frac{\delta \rho_2}{2\rho_1} \left( E^{\sigma} (t) \right)^{\frac{s}{r}}, \]  
(69)
where \( \tau_2 \) is given by (64)_5. Now we estimate the first term in the r. h. s. of (69) by Hölder’s inequality
\[ \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \right)^{\frac{s}{r}} \leq \left\{ \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} d\Gamma \right)^{\frac{s}{r}} \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \right)^{\frac{s}{q}} \right\}^{\frac{s}{r}} \]
\[ \leq \left( \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} d\Gamma \right)^{\frac{s}{r}} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{p+1} d\Gamma \]
\[ \leq \left( \text{mes } \Gamma_0 \right)^{\frac{s}{r}} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} g_1 (\hat{u}_i) |\hat{u}_i|^{p+1} d\Gamma \quad \text{(use } (18)) \]
\[ \leq \frac{\left( \text{mes } \Gamma_0 \right)^{\frac{s}{r}}}{k_3} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} g_1 (\hat{u}_i) |\hat{u}_i|^{p+1} d\Gamma. \]  
(70)
By (70), the last term of (67) (given by (69)) becomes
\[ \delta \tau_1 E^{\sigma} (t) \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} |\hat{u}_i|^{2q} d\Gamma \]
\[ \leq \delta \tau_2^{\frac{s}{r}} \frac{\left( \text{mes } \Gamma_0 \right)^{\frac{s}{r}}}{k_3} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} g_1 (\hat{u}_i) |\hat{u}_i|^{p+1} d\Gamma + \frac{\delta \rho_2}{2\rho_1} \left( E^{\sigma} (t) \right)^{\frac{s}{r}}. \]  
(71)
Finally by substituting (71) into (67), we get
\[ \delta \Lambda_1 E^{\sigma} (t) \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} (|g_1 (\hat{u}_i)|^2 + |\hat{u}_i|^2) d\Gamma \]
\[ \leq \delta \frac{\tau_2 \text{mes } \Gamma_0}{k_3} \int_{\Gamma_0 \cap \{|\hat{u}_i| \leq 1\}} g_1 (\hat{u}_i) |\hat{u}_i|^{p+1} d\Gamma + \frac{\delta \rho_2}{2\rho_1} \left( E^{\sigma} (t) \right)^{\frac{s}{r}}. \]  
(72)
In light of (72) and (66), (65) takes the following form
\[ \frac{d}{dt} L(t) \leq - \frac{\delta \rho_2}{\rho_1} E^{\sigma+1} (t) + \frac{\delta \rho_2}{2\rho_1} \left( E^{\sigma} (t) \right)^{\frac{s}{r}} \]
\[ + \left( \delta E^{\sigma} (0) (\sigma \rho_0 + \rho_0) - 1 \right) \int_{\Omega} (\theta_i k_{ij} \theta_j + P_i h_{ij} P_j) dx \]
5. Controllability. We now use Russell’s “Controllability via Stabilizability” principle [34] to get the controllability of the problem (5)-(7) with the following homogeneous initial conditions

\[ u(x, 0) = 0, \quad \dot{u}(x, 0) = 0, \quad \theta(x, 0) = 0, \quad P(x, 0) = 0 \quad x \in \Omega, \quad (74) \]

and \( \phi = (\phi_1, \ldots, \phi_n) \) in (7) is a nonlinear control function (that will be determined later) acting on the boundary \( \Gamma_0 \). Given \( T > 0 \), let \( \Upsilon = \Omega \times (0, T), \quad \Sigma = \Gamma \times (0, T), \quad \Sigma_0 = \Gamma_0 \times (0, T) \) and \( \Sigma_1 = \Gamma_1 \times (0, T) \).

The partial exact controllability problem can be stated as follows [6, 24, 27, 30]:

**Definition 5.1.** (Partial Exact Controllability) We say that the system (5) is partially exactly controllable if there exist \( T > 0 \), such that for any given initial data \((u^0, u^1, \theta^0, P^0)\) in a suitable space, there exist a control function such that the corresponding solution to system (5) satisfies

\[ u(x, T) = u^0(x), \quad \dot{u}(x, T) = u^1(x), \quad x \in \Omega, \quad (75) \]

disregarding the values of the temperature and the chemical potential.

As stated in [24], this is equivalent to steering every initial state \((u^0, u^1)\) of the displacement in the function space to the state \((u(T), \dot{u}(T)) = (0, 0)\), disregarding the values of the temperature and the chemical potential.

Let \( \bar{\sigma} \) be the smallest positive constant such that [25, 26]

\[ \| \text{div} \, u \|_{-1} \leq \bar{\sigma} \| u \|_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega). \quad (76) \]
For \((z^0, z^1) \in W\), we now consider the Lamé system
\[
\begin{align*}
\rho \ddot{z}_i &= (C_{ijkl} z_{k,l})_{,j} & \text{in } & \mathcal{Y}, \\
z_i &= 0 & \text{on } & \Sigma_1, \\
C_{ijkl} z_{k,l} \nu_j &= (m_j \cdot \nu_j) g_i(\dot{z}_i) & \text{on } & \Sigma_0, \\
z_i(T) &= z^0_i, & \dot{z}_i(T) &= z^1_i & \text{in } & \Omega.
\end{align*}
\]  
\tag{77}

From the classical theory of nonlinear semigroup (see [8], chapter 3) one can easily prove that (77) has an unique solution
\[(z(t), \dot{z}(t)) \in C([0,T]; W).\]
Moreover, by Theorem 4.3 (since (77) is a special case of (5)-(9)) there exists a positive constant \(\omega\) such that (see Liu [25, 26] and references therein),
\[E_L(z,t) \leq E_L(z,T) e^{1-\omega(T-t)}, \quad \forall t \in [0,T] \tag{78}\]
where
\[E_L(z,t) = \frac{1}{2} \int_\Omega (\rho \dddot{z}_i^2 + C_{ijkl} z_{k,l} z_{i,j}) dx. \tag{79}\]

In (77) and in the sequel, \(g_i\) is an unknown boundary function that satisfies the hypothesis (16)-(20). Using the solution to (77), we consider the following thermo-diffusion problem
\[
\begin{align*}
\alpha \ddot{\chi} + r \dot{\chi} &= -\alpha_{ij} \text{div} \dot{z} + (k_{ij} \xi_{,j,i})_i & \text{in } & \mathcal{Y}, \\
\beta \ddot{\chi} + r \dot{\chi} &= -\beta_{ij} \text{div} \dot{z} + (h_{ij} \chi_{,j,i})_i & \text{in } & \mathcal{Y}, \\
\dot{\chi} + \chi &= 0 & \text{on } & \Sigma_1, \\
(k_{ij} \xi_{,j}) \nu_i &= -Q_1 \xi & \text{on } & \Sigma_0, \\
(h_{ij} \chi_{,j}) \nu_i &= -Q_2 \chi & \text{on } & \Sigma_0, \\
\xi(0) &= \chi(0) = 0 & \text{in } & \Omega.
\end{align*}
\]  
\tag{80}

Since \(\text{div} \dot{z} \in \left(L^2(0,T; H^{-1}(\Omega))\right)^n\), it follows from the semigroup theory that the problem (80) has a unique solution
\[(\xi, \chi) \in \left(C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))\right)^2.\]

**Lemma 5.2.** Let (15) holds and \((\xi, \chi)\) be solution to (80), then we have
\[
\int_0^T \| (\alpha_{ij} \xi + \beta_{ij} \chi)_i \|^2_{L^2(\Omega)} dt \leq \frac{\theta^2 \eta^2}{\omega} E_L(z,T), \tag{81}
\]
where
\[
\theta^2 = 2 \left( \frac{(\omega_3 C_p + \omega_1)^2}{\zeta} + \frac{(\omega_4 C_p + \omega_2)^2}{\kappa} \right) \quad \text{and} \quad \eta^2 = \max \left( \frac{\omega_1^2 + \omega_3^2 C_p}{\zeta}, \frac{\omega_2^2 + \omega_4^2 C_p}{\kappa} \right). \tag{82}
\]

**Proof.** Multiplying (80)1 by \(\xi\) and (80)2 by \(\chi\), we get after integrating and summing up
\[
\begin{align*}
\frac{1}{2} \int_\Omega \left( \alpha_{ij} \xi_{,j} + \beta_{ij} \chi_{,j} \right) \text{div} \dot{z} dx + \int_\Sigma_0 (Q_1 \xi^2 + Q_2 \chi^2) d\Sigma &= - \int_0^T \int_\Omega (\alpha_{ij} \xi_{,j} + \beta_{ij} \chi_{,j}) \text{div} \dot{z} dx dt \\
- \int_\Sigma_0 (Q_1 \xi^2 + Q_2 \chi^2) d\Sigma - \int_0^T \int_\Omega k_{ij} \xi_{,j} \xi_{,i} dx dt - \int_0^T \int_\Omega h_{ij} \chi_{,j} \chi_{,i} dx dt.
\end{align*}
\]
Using the positivity of $Q_1$ and $Q_2$, Young’s inequality, (12) and (76), the above identity can be estimated as

$$
\int_{\Omega} \left( c\xi^2(T)dx + r\chi^2(T) + 2d\xi(T)\chi(T) \right)dx + \int_0^T \left( \zeta||\xi||^2_{L^2(\Omega)} + \kappa||\chi||^2_{L^2(\Omega)} \right)dt \\
\leq \frac{\vartheta^2\sigma^2}{2} \int_0^T ||\hat{\dot{z}}||dt \text{ (use (79) and (78))} \\
\leq \frac{\vartheta^2\sigma^2}{2} \int_0^T E_L(z, T)e^{1-\omega(T-t)}dt \\
\leq \frac{\vartheta^2\sigma^2}{\omega} E_L(z, T),
$$

where $\vartheta$ is given by (82)₁. Since (15) holds, hence the first term in (83) is positive and it follows

$$
\int_0^T \left( \zeta||\xi||^2_{L^2(\Omega)} + \kappa||\chi||^2_{L^2(\Omega)} \right)dt \leq \frac{\vartheta^2\sigma^2}{\omega} E_L(z, T). \tag{83}
$$

By using (13) and Poincaré’s inequality, we get

$$
\int_{\Omega} |(\alpha_{ij}\xi + \beta_{ij}\chi)_{,j}|^2 dx \leq \eta^2 \int_{\Omega} (\zeta||\xi||^2 + \kappa||\chi||^2)dx, \tag{84}
$$

where $\eta$ is given by (82)₂. Integrating (84) from 0 to $T$ and combining with (83) our conclusion follows. \hfill \square

Now, by using the solution $(\xi, \chi)$ to (80) and the initial data of (77), we consider the following problem

\begin{align*}
\rho\dddot{w}_i &= (C_{ijkl}w_{k,l} + \alpha_{ij}\varphi + \beta_{ij}\psi)_{,j} + (\alpha_{ij}\xi + \beta_{ij}\chi)_{,j} \quad \text{in } \Upsilon, \\
c\dot{\psi} + d\dot{\psi} &= \alpha_{ij} \text{ div } \dot{w} + (k_{ij}\varphi)_{,j}, \quad \text{in } \Upsilon, \\
d\dddot{\psi} + r\dddot{\psi} &= \beta_{ij} \text{ div } \dot{w} + (h_{ij}\psi)_{,j}, \quad \text{in } \Upsilon, \\
\dot{w}_i = \varphi &= \psi = 0 \quad \text{on } \Sigma_1, \\
(C_{ijkl}w_{k,l} + \alpha_{ij}\varphi + \beta_{ij}\psi + \alpha_{ij}\xi + \beta_{ij}\chi)\nu_j &= -((m_j \cdot \nu_j))g_i(\dot{w}_i) \quad \text{on } \Sigma_2, \\
(k_{ij}\varphi)_{,j}\nu_i &= -Q_1\varphi \quad \text{on } \Sigma_0, \\
(h_{ij}\psi)_{,j}\nu_i &= -Q_2\psi \quad \text{on } \Sigma_0, \\
\dot{w}_i(x, 0) &= z_i(0), \quad \dot{w}_i(x, 0) = \dot{z}_i(0), \quad \varphi(x, 0) = 0, \quad \psi(x, 0) = 0 \quad x \in \Omega. \tag{85}
\end{align*}

Since $(\xi, \chi) \in \left(L^2(0, T; H_{x,t}^1(\Omega)) \right)^2$, from (84) we infer that $(0, -(\alpha_{ij}\xi + \beta_{ij}\chi)_{,j}, 0, 0) \in L^1(0, T; \mathcal{H})$. Since $(\dot{w}(0), \ddot{w}(0), \varphi(0), \psi(0)) \in \mathcal{H}$, by the classical theory of non-linear semigroups, the problem (85) has a unique solution

$$
(w, \ddot{w}, \varphi, \psi) \in C([0, T]; \mathcal{H}),
$$

which can be expressed as

$$
(w, \dot{w}, \varphi, \psi) = S(t) \left( w_0, \dot{w}_0, \varphi_0, \psi_0 \right) + \int_0^t S(t - \tau)(0, -(\alpha_{ij}\xi + \beta_{ij}\chi)_{,j}, 0, 0)d\tau, \tag{86}
$$

where $S(t)$ is the semigroup generated by the linear part of the operator.
where $S(t)$ denotes the strongly continuous semigroup of contractions generated by the system (5)-(9). From Theorem 4.3, we have

$$\| S(t) \| \leq \Lambda \frac{t}{2} e^{-\frac{\omega t}{2}} \leq \Lambda \frac{t}{2} e^{-\frac{\omega t}{2}} \quad \forall t \geq 0.$$  

(87)

Therefore, combining (86) and (87), we deduce that the energy $\mathcal{E}(t)$ of system (85) satisfies

$$\mathcal{E}(t) \leq 2\| S(t) \|^{2}\mathcal{E}(0) + 2 \left( \int_{0}^{t} \| S(\tau) \| \| (\alpha_{ij}\xi + \beta_{ij}\chi)_{,j} \|_{L^{2}(\Omega)} d\tau \right)^{2} \leq 2\Lambda e^{-\omega t}\mathcal{E}(0) + 2\Lambda \int_{0}^{t} e^{1-\omega(t-\tau)} d\tau \int_{0}^{t} \| (\alpha_{ij}\xi + \beta_{ij}\chi)_{,j} \|_{L^{2}(\Omega)} d\tau \quad \text{(use (81))}$$

$$\leq 2\Lambda e^{-\omega t}\mathcal{E}(0) + 4 \frac{\Lambda \omega^{2} \eta \overline{\gamma}^{2} \bar{e}^{2}}{\omega^{2}} E_{L}(z, T),$$

(88)

where $\mathcal{E}(t)$ is also defined by (30).

**Lemma 5.3.** Let $(z, \dot{z})$ solution to the Lamé system (77) and let $(w, \dot{w}, \varphi, \psi)$ mild solution to system (85). Then, there exists a positive constant $c_{1}$ such that

$$\mathcal{E}(T) + \int_{0}^{T} \left( \varsigma \| \varphi, i \|_{L^{2}(\Omega)} + \kappa \| \psi, i \|_{L^{2}(\Omega)} \right) dt + a_{0} \int_{\Sigma_{0}} g_{i}(\dot{w}_{i})\dot{w}_{i} d\Sigma \leq c_{1} E_{L}(z, T),$$

(89)

where $c_{1} = 1 + 2\overline{\sigma}^{2} \Lambda \frac{\xi_{0}}{2} + \frac{\bar{\eta}^{2}\bar{\gamma}^{2}}{2\omega} + \frac{4\overline{\Lambda} \bar{\tau}^{2} \bar{e}^{2}}{\omega^{2}}$ and $a_{0}$ is defined by (3).

**Proof.** Multiplying (85)$_{1}$ by $\dot{w}_{i}$, (85)$_{2}$ by $\varphi$, (85)$_{3}$ by $\psi$, integrating and summing up, we get after using (12) and (3)

$$\mathcal{E}(T) + \int_{0}^{T} \left( \varsigma \| \varphi, i \|_{L^{2}(\Omega)} + \kappa \| \psi, i \|_{L^{2}(\Omega)} \right) dt + a_{0} \int_{\Sigma_{0}} g_{i}(\dot{w}_{i})\dot{w}_{i} d\Sigma \leq \mathcal{E}(0) - \int_{0}^{T} \int_{\Omega} (\alpha_{ij}\xi + \beta_{ij}\chi) \text{div } \dot{w} \, dx \, dt.$$

(90)

Using Young’s inequality and (30), the last term of (90) can be estimated as follows

$$- \int_{0}^{T} \int_{\Omega} (\alpha_{ij}\xi + \beta_{ij}\chi) \text{div } \dot{w} \, dx \, dt$$

$$\leq \int_{0}^{T} \| (\alpha_{ij}\xi + \beta_{ij}\chi)_{,j} \|_{L^{2}(\Omega)} \| \text{div } \dot{w} \|_{-1} \, dt \quad \text{(use (76))}$$

$$\leq \frac{1}{2} \int_{0}^{T} \| (\alpha_{ij}\xi + \beta_{ij}\chi)_{,j} \|_{L^{2}(\Omega)}^{2} \, dt + \frac{\bar{\sigma}^{2}}{2} \int_{0}^{T} \| \dot{w} \|^{2} \, dt \quad \text{(use (81))}$$

$$\leq \frac{\bar{\eta}^{2}\bar{\gamma}^{2}}{2\omega} E_{L}(z, T) + \sigma^{2} \int_{0}^{T} \mathcal{E}(t) \, dt \quad \text{(use (88))}$$

$$\leq \left( \frac{\bar{\eta}^{2}\bar{\gamma}^{2}}{2\omega} + 4\overline{\Lambda} \bar{\tau}^{2} \bar{e}^{2} \frac{\bar{\Lambda} e}{\omega} \right) E_{L}(z, T) + 2\overline{\sigma}^{2} \Lambda \frac{\xi_{0}}{2} \mathcal{E}(0).$$

Substituting this inequality into (90) and using $\mathcal{E}(0) = E_{L}(z, 0) < E_{L}(z, T)$, our conclusion follows.

We now state the main result of this section.
Theorem 5.4. Suppose that  
\[ \vartheta \eta < \frac{\omega}{2\sqrt{\Lambda} \sigma e}, \]  
(91)
where \( \omega \) and \( \Lambda \) are the constants in Theorem 4.3, \( \eta \) and \( \vartheta \) are given by (82) and \( \bar{\sigma} \) is defined by (76). Let \( T_0 \) be large enough such that  
\[ 2\Lambda e^{(1-\omega T_0)} < 1 - \frac{4\Lambda \delta^2 \eta^2 \sigma^2 e^2}{\omega^2} \]  
(92)
and \( T \geq T_0 \). Then for any \((u^0, u^1) \in W\), there exists a boundary control function \( \phi(x, t) \) defined by  
\[ \phi = -(m \cdot \nu) \left( g(\hat{w}) + g(\hat{z}) \right), \]  
(93)
such that the solution to the problem (5)-(7) with the initial conditions (74), satisfies (75).

Moreover, \( g \) satisfies  
\[ \| g(\hat{w}) + g(\hat{z}) \|_{(L^2(S_\omega))^n} \leq C \| (u^0, u^1) \|_W \]  
(94)
where \( C \) is a positive constant independent of \((u^0, u^1)\).

Proof. Set  
\[ u = w - z, \quad \theta = \varphi + \xi, \quad P = \psi + \chi, \]  
(95)
where \((w, \varphi, \psi)\) solution to (85), \( z \) solution to (77) and \((\xi, \chi)\) solution to (80). Then \((u, \theta, P)\) satisfies  
\[ \rho u_i = (C_{ijkl}u_{k,l} + \alpha_{ij}\theta + \beta_{ij}P),_j \]  
in \( \Upsilon \),
\[ c\theta + dP = \alpha_{ij} \text{div}\ u + (k_{ij}\theta)_j,i \]  
in \( \Upsilon \),
\[ d\theta + rP = \beta_{ij} \text{div}\ u + (h_{ij}P)_j,i \]  
in \( \Upsilon \),
\[ w = \theta = P = 0 \]  
on \( \Sigma_1 \),
\[ (C_{ijkl}u_{k,l} + \alpha_{ij}\theta + \beta_{ij}P)\nu_j = \phi_i \]  
on \( \Sigma_0 \),
\[ (k_{ij}\theta)_j\nu_i = -Q_1\theta, \quad (h_{ij}P)_j\nu_i = -Q_2P \]  
on \( \Sigma_0 \),
\[ u_i(0) = \dot{u}_i(0) = 0, \quad \theta(0) = 0, \quad P(0) = 0 \]  
\[ u_i(T) = w_i(T) - z_0^i, \quad \dot{u}_i(T) = \dot{w}_i(T) - z_1^i \]  
\[ u_i(T) = w_i(T) - z_0^i, \quad \dot{u}_i(T) = \dot{w}_i(T) - z_1^i \]  
in \( \Omega \),
\[ \| (w^0, z^1) \|_{W} \]  
(96)
where \( \phi \) is given by (93) and \( g \) satisfies (16)-(20). Now, we define the linear operator \( N \) from \( W \) into \( W \) by  
\[ N(z^0, z^1) = (w(T), \dot{w}(T)), \]  
(97)
where \((z^0, z^1) \in W\) are defined by (77)\(_4\). Moreover, we have  
\[ \| N(z^0, z^1) \|_{W}^2 \]  
\[ = \| (w(T), \dot{w}(T)) \|_{W}^2 \]  
\[ \leq 2\Lambda e^{(1-\omega T)} E(z, T) \]  
(98)
(\text{use (88))}
\[ \leq \left( 2\Lambda e^{(1-\omega T)} + \frac{4\Lambda \delta^2 \eta^2 \sigma^2 e^2}{\omega^2} \right) E_L(z, T) \]  
\[ \leq \left( 2\Lambda e^{(1-\omega T)} + \frac{4\Lambda \delta^2 \eta^2 \sigma^2 e^2}{\omega^2} \right) E_L(z, T) \]  
\[ = \left( 2\Lambda e^{(1-\omega T)} + \frac{4\Lambda \delta^2 \eta^2 \sigma^2 e^2}{\omega^2} \right) \| (z^0, z^1) \|_{W}^2, \]  
(99)
which ready implies that

\[ \| \mathcal{N} \|^2 \leq 2\Delta e^{(1 - \omega T)} + \frac{4A\delta^2 \eta^2 \sigma^2 \epsilon^2}{\omega^2}. \]  

(98)

Let (91) holds and let \( T \geq T_0 \) and \( T_0 \) be large enough such that (92) holds. Then \( \| \mathcal{N} \|^2 < 1 \) and \( \mathcal{N} - I \) is an isomorphism from \( W \) onto \( W \). Thus, for any \((u^0, u^1) \in W\), there exists a unique \((z^0, z^1) \in W\) such that

\[ (u^0, u^1) = \mathcal{N}(z^0, z^1) - (z^0, z^1) \]  

(use (97))

\[ = (w(T), w(T)) - (z^0, z^1) \]  

(use (95) and (77)),

\[ = (u(T), \dot{u}(T)). \]  

(99)

Consequently, we have constructed a control function given by (93) solving the partial exact controllability problem (5)-(7) with the initial conditions (74).

Using (93) and (1), we have

\[ \frac{1}{2} \int_{\Sigma_0} (m_j \cdot \nu_j) |g_i(\dot{w_i}) + g_i(\dot{z_i})|^2 d\Sigma \]

\[ \leq \int_{\Sigma_0 \cap \{|\dot{w_i}| \geq 1\}} (m_j \cdot \nu_j) |g_i(\dot{w_i})|^2 d\Sigma + \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} (m_j \cdot \nu_j) |g_i(\dot{w_i})|^2 d\Sigma \]

\[ + \int_{\Sigma_0 \cap \{|\dot{z_i}| \geq 1\}} (m_j \cdot \nu_j) |g_i(\dot{z_i})|^2 d\Sigma + \int_{\Sigma_0 \cap \{|\dot{z_i}| \leq 1\}} (m_j \cdot \nu_j) |g_i(\dot{z_i})|^2 d\Sigma \]  

(use (19) and (20))

\[ \leq k_1 \int_{\Sigma_0 \cap \{|\dot{w_i}| \geq 1\}} (m_j \cdot \nu_j) |\dot{w_i}|^2 d\Sigma + k_2 \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} (m_j \cdot \nu_j) |\dot{w_i}|^2 d\Sigma \]

\[ + k_3 \int_{\Sigma_0 \cap \{|\dot{z_i}| \geq 1\}} (m_j \cdot \nu_j) |\dot{z_i}|^2 d\Sigma + k_4 \int_{\Sigma_0 \cap \{|\dot{z_i}| \leq 1\}} (m_j \cdot \nu_j) |\dot{z_i}|^2 d\Sigma \]  

(use (17))

\[ \leq \frac{a_1k_1^2}{a_0k_3} c_1 E_L(z, T) + 2 \frac{k_2^2}{k_3} E_L(z, T) + a_1k_4^2 \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} |\dot{w_i}|^2 d\Sigma \]

\[ + a_1k_2^2 \int_{\Sigma_0 \cap \{|\dot{z_i}| \leq 1\}} |\dot{z_i}|^2 d\Sigma. \]  

(100)

Using (68) with

\[ A = a_1k_1^2 (E_L(z, 0))^\frac{1}{q}, \quad B = (E_L(z, 0))^{-\frac{1}{q}} \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} |\dot{w_i}|^2 d\Sigma \]

and \( r = \frac{p + 1}{2q}, \quad s = \frac{p + 1}{p + 1 - 2q}, \)

the third term of (100) becomes

\[ a_1k_2^2 \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} |\dot{w_i}|^2 d\Sigma \]

\[ \leq \frac{1}{s} \left( a_1k_2^2 (E_L(z, 0))^\frac{1}{q} \right)^s + \frac{1}{r} (E_L(z, 0))^{-\frac{1}{q}} \left( \int_{\Sigma_0 \cap \{|\dot{w_i}| \leq 1\}} |\dot{w_i}|^2 d\Sigma \right)^r. \].
Using the same argument, the last term of \((100)\) can be estimated as

\[
\int_{\Sigma_0 \cap \{|\zeta_i| \leq 1\}} |\dot{z}_i|^2 d\Sigma \leq \left( \frac{1}{s} \left( a_1 k_2^2 \right)^s + \frac{c_1}{r k_3 a_0} \left( \mes \Sigma_0 \right) \right) E_L(z,T).
\]

Substituting the two last inequalities in \((100)\), we get

\[
\int_{\Sigma_0} (m_j \cdot \nu_j) |g_i(\dot{w}_i) + g_i(\dot{z}_i)|^2 d\Sigma \leq C E_L(z,T) \leq C \| (z_0^1, z_1^1) \|^2.
\]

Since \(N - I\) is an isomorphism from \(W\) onto itself and from \((92)\) and \((98)\) we obtain that \(0 < \| N - I \| \leq 2\). Thus it follows that

\[
\int_{\Sigma_0} (m_j \cdot \nu_j) |g_i(\dot{w}_i) + g_i(\dot{z}_i)|^2 d\Sigma \leq C \| (N - I)^{-1}(u_0^1, u_1^1) \|^2 \quad (\text{use } (99))
\]

for a positive constant \(C\). Using \((3)\), our conclusion follows.

6. **Conclusion.** (i) In this paper the partial exact controllability is proved for the multidimensional thermoelastic diffusion problem for inhomogeneous and anisotropic material with smallness restriction \((91)\) on the coupling tensors, stress-temperature and stress-diffusion. This condition states that more the magnitudes of the coupling tensors are small or more the energy of the system decreases exponentially faster to zero \((\omega \text{ large enough})\), more the system is controllable. From the numerical values of explicit exponential decay rate, Aouadi and Moulahi found in [5] that the decay rate is best for copper, than for aluminum alloy, silicon, diamond, steel, indium arsenide, for germanium and finally for gallium arsenide, sequentially. The choice of materials in applications should be based on this. Also the boundary feedback function of the control has been determined explicitly. These results represent a pleasant feature from the physical viewpoint.

(ii) The explicit expressions of the decay rates \(\omega\) and \(\lambda\) play an important role in stabilization and controllability issues. They describe the dependence of the decay rate on the assumptions on the feedback and the physical parameters leading to the best choice of materials realizing the fastest decay of the energy. This is useful to select the correct setting of the conditions imposed for stabilization or controllability. For example, if we take a linear boundary feedback \(g(s) = k_3 s\)
where $k_3$ is the parameter appearing in conditions (17) and (18) and let $k_3 \to 0$, then it follows from (62), (1) that $\omega \to 0$ and from (64), (2) that $\lambda \to 0$. Thus we lose the stability and consequently the controllability of the problem (see (91)).

(iii) In addition, by these expressions, we can analyze the limit of the polynomial decay rate $\left(1 + \sigma \lambda t\right)^{-\frac{1}{2}}$ as $p, q$ tends to 1 and recover the exponential decay of the case $p = q = 1$. Indeed, it is easy to see that

$$\lim_{p,q \to 1} \delta_2(p,q) = \delta_1 \quad \text{and} \quad \lim_{p,q \to 1} \lambda(p,q) = \frac{\omega}{2}.$$ 

Consequently we have

$$\lim_{p,q \to 1} \left(1 + \sigma \lambda t\right)^{-\frac{1}{2}} = e^{-\frac{\omega}{2}}. \quad (102)$$

Despite the exponential stability is a result more important than the polynomial stability in our study, it turns out that it is a special case of the latter.

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