On the Bound of Inverse Images of a Polynomial Map *

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Abstract

Let \( f_1(x), \ldots, f_n(x) \) be some polynomials. The upper bound on the number of \( x \in \mathbb{F}_p \) such that \( f_1(x), \ldots, f_n(x) \) are roots of unit of order \( t \) is obtained. This bound generalize the bound of the paper \([1]\) to the case of polynomials of degrees greater than one. The bound is obtained over fields of positive characteristic and over the complex field.

1 Introduction

Consider the field \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) where \( p \) is a prime number. Let \( \mathbb{F}_p^* \) be the multiplicative group of the field \( \mathbb{F}_p \) and let \( \Gamma \) be a subgroup of \( \mathbb{F}_p^* \) of an order \( t = |\Gamma| \). Garcia and Voloch have proved that for any subgroup \( \Gamma \subseteq \mathbb{F}_p^* \), such that \( |\Gamma| < (p - 1)/((p - 1)^{1/2} + 1) \) and for any \( \mu \in \mathbb{F}_p^* \) the inequality

\[
|\Gamma \cap (\Gamma + \mu)| \leq 4|\Gamma|^{2/3}
\]

holds. Heath-Brown and Konyagin have re-proved this result (see \([5]\)). They also have improved it for the case of a set of equations. Shkredov and Vyugin have generalized the bound to the case of several additive shifts (see \([1]\)).

**Theorem 1** (Shkredov and V. \([1]\)). Let \( \Gamma \subseteq \mathbb{F}_p^* \) be a subgroup and let \( \mu_1, \ldots, \mu_n \in \mathbb{F}_p^* \) be pairwise distinct non-zero elements of \( \mathbb{F}_p \), \( n \geq 2 \). Suppose that

\[
32n2^{20n \log(n+1)} \leq |\Gamma|, \quad 4n|\Gamma|(|\Gamma|^{1/(2n+1)} + 1) \leq p.
\]

Then we have

\[
|\Gamma \cap (\Gamma + \mu_1) \cap \ldots \cap (\Gamma + \mu_n)| \leq 4(n + 1)(|\Gamma|^{1/(2n+1)} + 1)^{n+1}.
\]

In other words this theorem gives us that

\[
|\Gamma \cap (\Gamma + \mu_1) \cap \ldots \cap (\Gamma + \mu_n)| \ll_n |\Gamma|^{1/2 + \alpha_n},
\]

where \( 1 \ll_n |\Gamma| \ll_n p^{1-\beta_n} \), and \( \{\alpha_n\}, \{\beta_n\} \) are real sequences, such that \( \alpha_n, \beta_n \to 0, n \to \infty \).

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Consider the map
\[ x \mapsto (x, x - \mu_1, \ldots, x - \mu_n). \tag{2} \]

It is easy to see that
\[ \Gamma \cap (\Gamma + \mu_1) \cap \ldots \cap (\Gamma + \mu_n) = \varphi(\Gamma^{n+1}), \]
where \( \Gamma^{n+1} = \Gamma \times \ldots \times \Gamma \) \((n + 1)\)-times. We generalize Theorem 1 to the case of polynomial map.

Consider a subgroup \( \Gamma \subset F_p^* \), cosets \( \Gamma_1, \ldots, \Gamma_n \) by subgroup \( \Gamma \) \((\Gamma_i = g_i \Gamma, g_i \in F_p^*)\) and a map
\[ f : x \mapsto (f_1(x), \ldots, f_n(x)), \quad n \geq 2 \tag{3} \]
with polynomials \( f_1(x), \ldots, f_n(x) \in F_p[x] \).

Let us call the set of polynomials
\[ f_1(x), \ldots, f_n(x) \tag{4} \]
the admissible set of polynomials, if there exist such \( x_1, \ldots, x_n \) that:
\[ f_i(x_i) = 0, \quad f_j(x_i) \neq 0, \quad j \neq i, \quad 0 \leq i, j \leq n, \]
and \( f_i(0) \neq 0, i = 1, \ldots, n \).

Suppose that polynomials \( f_1(x), \ldots, f_n(x) \) have degrees \( m_i \) \((\deg f_i(x) = m_i, i = 1, \ldots, n)\). Let define the set
\[ M = \{ x \mid f_i(x) \in \Gamma_i, \ i = 1, \ldots, n \}. \]

Theorem 2 gives us the upper bound of cardinality of \( M \).

**Theorem 2.** Let \( \Gamma \) be a subgroup of \( F_p^* \) \((p \text{ is prime})\), let \( \Gamma_1, \ldots, \Gamma_n \) be cosets by subgroup \( \Gamma \), \( n \geq 2 \) and let \( f_1(x), \ldots, f_n(x) \) be an admissible set of polynomials of degrees \( m_1, \ldots, m_n \). Let us suppose that
\[ C_1(m, n) < |\Gamma| < C_2(m, n)p^{1 - \frac{1}{n+1}}, \]
where \( C_1(m, n), C_2(m, n) \) are constants depending only on \( n \) and \( m = (m_1, \ldots, m_n) \). Then we have the bound
\[ |M| \leq C_3(m, n)|\Gamma|^{\frac{1}{2} + \frac{1}{2n}}, \]
where \( C_3(m, n) \) is a constant which depends on \( n \) and \( m \).

**Remark 1.** Constants \( C_1(m, n), C_2(m, n), C_3(m, n) \) can be given as follow:
\[ C_1(m, n) = 2^{2n}m_n^{4n}, \quad C_2(m, n) = (n + 1)^{-\frac{2n}{2n+1}}(m_1 \ldots m_n)^{-\frac{2}{2n+1}}, \]
\[ C_3(m, n) = 4(n + 1)M_n(m_1 \ldots m_n)^{\frac{1}{n}}. \]
2 Proof of Theorem \[2\]

We use the idea of proof of papers \[1, 2\]. Let us describe Stepanov method (see \[5\]) applied to a polynomial map. Let us denote by \(M'\) the set:

\[
M' = \{ x \mid x \in M, x f_1(x) \ldots f_n(x) \neq 0 \}.
\]

Let us construct such polynomial \(\Psi(x)\) that satisfy to the following conditions:

1) if \(x \in M'\) then \(x\) is a root of the polynomial \(\Psi(x)\) of an order not less than \(D\);
2) \(\Psi(x) \neq 0\).

If such polynomial \(\Psi(x)\) exists, than we have the bound:

\[
|M| \leq 1 + \sum_{i=1}^{n} m_i + \frac{\deg \Psi(x)}{D}. \tag{5}
\]

Actually, condition 2) gives us that \(\Psi(x)\) is a non-zero polynomial. Condition 1) gives us that \(|M'| \leq \deg \Psi(x)\) because any element of \(M'\) is a root of polynomial \(\Psi(x)\) having the order not less than \(D\) by condition 1). A cardinality of the set \(M \setminus M'\) is not greater than \(1 + \sum_{i=1}^{n} m_i\) by definition of \(M'\).

Let us construct the polynomial \(\Psi(x)\). Let \(b\) be a vector \((b_1, \ldots, b_n)\), and \(t = |\Gamma|\). Consider the polynomial

\[
\Psi(x) = \sum_{a, b} \lambda_{a, b} x^a f_1^{b_1 t}(x) \ldots f_n^{b_n t}(x), \quad a < A, \ b_i < B_i, \ i = 1, \ldots, n \tag{6}
\]

with coefficients \(\lambda_{a, b}\). Let us define coefficients \(\lambda_{a, b}\) which satisfy to conditions:

\[
\left. \frac{d^k}{dx^k} \Psi(x) \right|_{x \in M'} = 0, \quad k = 0, \ldots, D - 1. \tag{7}
\]

Let us suppose that \(x \in M'\), then the condition (7) is equivalent to

\[
\left. \left( f_1(x) \ldots f_n(x) \right)^k \frac{d^k}{dx^k} \Psi(x) \right|_{x \in M'} = 0, \quad k = 0, \ldots, D - 1. \tag{8}
\]

Let us introduce the polynomial \(P_{k, a, b}(x)\) such that

\[
\left( f_1(x) \ldots f_n(x) \right)^k \frac{d^k}{dx^k} \left( x^a f_1^{b_1 t}(x) \ldots f_n^{b_n t}(x) \right) = f_1^{b_1 t}(x) \ldots f_n^{b_n t}(x) P_{k, a, b}(x). \tag{9}
\]

It is easy to see that polynomials \(P_{k, a, b}(x)\) are identity zeros or

\[
\deg P_{k, a, b}(x) \leq A + \mathcal{M}_n k - k, \tag{10}
\]

where \(\mathcal{M}_k = \sum_{i=1}^{k} m_i\). The degree of polynomial in left hand side of equality (9) is equal to \(\mathcal{M}_n k + a + \sum_{i=1}^{n} b_i m_i t - k\), consequently, (10) holds.
Let us substitute \( f_i^t(x) \) by \( g_i^t \) in formulas (6) and (8). Actually, \( f_i^t(x) = g_i^t \), \( i = 1, \ldots, n \) if \( x \in M \), because if \( x \in M \) than \( f_i(x) \in \Gamma_i = g_i \Gamma \). It means that \( f_i^t(x) = g_i^t \) as \( t \) — is the order of subgroup \( \Gamma \). Consequently, we have the equality:

\[
\left[ f_1(x) \ldots f_n(x) \right]^k \frac{d^k}{dx^k} \left( x^a f_1^{b_1 t} \ldots f_n^{b_n t} \right) \bigg|_{x \in M} = (g_1^{b_1 t} \ldots g_n^{b_n t}) P_{k,a,b}(x),
\]

and

\[
\left[ f_1(x) \ldots f_n(x) \right]^k \frac{d^k}{dx^k} \Psi(x) \bigg|_{x \in M} = \sum_{a,b} \lambda_{a,b} P_{k,a,b}(x) = P_k(x). \tag{11}
\]

Formulas (10) and (11) gives us that degrees of polynomials \( P_k(x) \) are bounded as follow:

\[
\deg P_k(x) \leq A + \mathcal{M}_n k - k, \quad k = 0, 1, \ldots, D - 1. \tag{12}
\]

For implicity of the condition (7), it is sufficient to find such \( \lambda_{a,b} \) that they are not vanish simultaneously and such that the following condition

\[
\forall k = 0, \ldots, D - 1 \quad P_k(x) \equiv 0 \tag{13}
\]

holds. Let us obtain coefficients \( \lambda_{a,b} \). Coefficients of polynomials \( P_k(x) \) are homogeneous linear combinations of coefficients \( \lambda_{a,b} \), it follows from (11). The condition (13) is equivalent to the system of homogeneous linear equations. The system of linear homogeneous equations has a non-zero solution if the number of variables \( \lambda_{a,b} \) is grater than the number of equations. Note that the number of \( \lambda_{a,b} \) is equal to \( AB_1 \ldots B_n \), but the number of equations is equal to the number of coefficients fo all polynomials \( P_k(x) \), \( k = 0, 1, \ldots, D - 1 \). It does not exceed of \( AD + \mathcal{M}_n \frac{D^2}{2} \), because there exist bounds (12). Consequently, we have the sufficient condition:

\[
AD + \mathcal{M}_n \frac{D^2}{2} < AB_1 \ldots B_n, \tag{14}
\]

of existence of non-zero set \( \lambda_{a,b} \) such that the condition 1) holds.

If \( \Psi(x) \) is not identity vanish, than

\[
|M'| \leq \frac{\deg \Psi(x)}{D}. \tag{15}
\]

We prove that \( \Psi(x) \not\equiv 0 \) if we prove that products

\[
x^a f_1^{b_1 t} \ldots f_n^{b_n t}(x) \tag{16}
\]

where \( a < A, b_i < B_i, i = 1, \ldots, n \) are linearly independent over \( \mathbb{F}_p \), because \( \Psi(x) \) is a linear combination of products (16) with coefficients \( \lambda_{a,b} \) (\( \lambda_{a,b} \) do not vanish simultaneously).
Lemma 1. Products

\[ x^af_1^{b_1t}(x) \cdots f_n^{b_nt}(x), \quad a < A, \ b_i < B_i, \ i = 1, \ldots, n \]  \hspace{1cm} (17)

are linearly independent over the field \( \mathbb{F}_p \), if

\[ A - 1 + (B_1m_1 + \ldots + B_{n-1}m_{n-1} - \mathcal{M}_{n-1})t < p, \]  \hspace{1cm} (18)

and

\[ t > AB_1 \ldots B_{n-1} + \mathcal{M}_n \frac{(B_1 \ldots B_{n-1})^2}{2}. \]  \hspace{1cm} (19)

Proof. Let us prove Lemma 1 by induction on \( n \). In the case \( n = 0 \) Lemma 1 is obvious. Actually, the statement of Lemma 1 is equivalent to the condition that the monomials

\[ 1, x, \ldots, x^{A-1} \]  

are linearly independent over \( \mathbb{F}_p \). Let us prove a step of induction. Suppose that the products

\[ x^af_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x), \quad a < A, \ b_i < B_i, \ i = 1, \ldots, n - 1, \]  \hspace{1cm} (20)

are linearly independent. We will prove the step of induction from the case \( n - 1 \) to the case \( n \) by contradiction.

Let us suppose that products (17) are linearly dependent. Then there exists a non-trivial set of coefficients \( \tilde{\lambda}_{a,b} \), such that

\[ \tilde{\Psi}(x) = \sum_{a,b} \tilde{\lambda}_{a,b} x^af_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x) \equiv 0. \]

Without loss of generality let us suppose that \( \min_{a,b} b_n = 0 \). If \( \min_{a,b} b_n \neq 0 \) than consider the polynomial \( \tilde{\Psi}(x)/f_n^{t\min_{a,b} b_n}(x) \) instead of \( \tilde{\Psi}(x) \). Let us present the polynomial \( \tilde{\Psi}(x) \) in the following form:

\[ \tilde{\Psi}(x) = f_n^{t}(x) \sum_{a,b,b_n \neq 0} \tilde{\lambda}_{a,b} x^af_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x) + \sum_{a,b,b_n = 0} \tilde{\lambda}_{a,b} x^af_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x) \equiv 0. \]  \hspace{1cm} (21)

Consider the polynomial

\[ \Phi(x) = \sum_{a,b,b_n = 0} \tilde{\lambda}_{a,b} x^af_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x). \]

Polynomial \( \Phi(x) \) is divided by \( f_n^{t}(x) \), because the equality (21) holds. The first term in equality (21) is divided by \( f_n^{t}(x) \), and all sum is divided by \( f_n^{t}(x) \) too, consequently, the second term is divided by \( f_n^{t}(x) \). By the proposition of induction \( \Phi \neq 0 \). Consequently, to obtain the contradiction to proposition of induction just to prove that if \( f_n^{t}(x) \mid \Phi(x) \) then \( \Phi(x) \equiv 0 \).

Rewrite \( \Phi(x) \) in the following form:

\[ \Phi(x) = \sum_{b: b_n = 0} H_b(x)f_1^{b_1t}(x) \cdots f_{n-1}^{b_{n-1}t}(x), \]
where \( H_b(x) = \sum a \tilde{\lambda}_{a,b} x^a \), and all \( b \) are pairwise distinct, \( b_i \in \{0, \ldots, B_i - 1\} \), \( i = 1, \ldots, n - 1 \). Note that for all \( b \): \( \deg H_b(x) < A \).

Let us introduce the polynomials \( Q_b(x) = H_b(x) f_1^{b_1 t}(x) \cdots f_{n-1}^{b_{n-1} t}(x) \), where \( \tilde{b} \) is the vector \((b_1, \ldots, b_{n-1})\). Polynomials \( Q_b(x) \) are linearly independent, because by the proposition of induction the polynomials \( (20) \) are linearly independent, and \( Q_b(x) \) is the linear combination of polynomials \( (20) \). Let us define: \( B_n := \prod_{i=1}^n B_i \).

Let us consider the Wronskian

\[
W(x) = \begin{vmatrix}
Q_{(0,\ldots,0)}(x) & \ldots & Q_{(B_1-1,\ldots,B_{n-1}-1)}(x) \\
Q_{(0,\ldots,0)}'(x) & \ldots & Q_{(B_1-1,\ldots,B_{n-1}-1)}'(x) \\
\vdots & \ddots & \vdots \\
Q_{(B_{n-1}-1)}(x) & \ldots & Q_{(B_1-1,\ldots,B_{n-1}-1)}(x)
\end{vmatrix}.
\]

It is constructed by functions \( Q_b(x) \), \( b = (b_1, \ldots, b_n) \), \( b_i = 0, 1, \ldots, B_i \), \( i = 1, \ldots, n \). Wronskian \( W(x) \) is not equal to zero identity. Actually, the theorem of F.K. Shmidt (see [6],[3]) states that if Wronskian \( W(x) \) is identity vanish then polynomials \( Q_b(x) \) are linearly independent over the ring \( \mathbb{F}_p[[x^p]] \) of formal power series of the variable \( x^p \). The inequality \( (15) \) gives us that degrees of polynomials \( Q_b(x) \) are less than \( p \). It means that the linear independence of \( Q_b(x) \) over the ring \( \mathbb{F}_p[[x^p]] \) follows from linear independence of \( Q_b(x) \) over the field \( \mathbb{F}_p \).

Wronskian \( W(x) \) is devided by

\[
R(x) = \prod_b f_1^{b_1 t-B_{n-1}+1}(x) \cdots f_{n-1}^{b_{n-1} t-B_{n-1}+1}(x),
\]

because all elements of column with index \( b \) are devided by \( f_1^{b_1 t-B_{n-1}+1}(x) \cdots f_{n-1}^{b_{n-1} t-B_{n-1}+1}(x) \) for each \( b \).

Easy to obtain that

\[
\deg(W(x)/R(x)) \leq AB_{n-1} + MB_{n-1}^2 \frac{B_{n-1}}{2}.
\]

By means of elementary tranformations (adding one column to other with some coefficient) Wronskii matrix can be transformed to the form such that elements of one column are function \( \Phi(x) \) and its derivatives of orders \( 1, \ldots, B_{n-1} - 1 \).

We have that \((x-x_n)^t \mid \Phi(x)\), because \( f_1^n(x) \mid \Phi(x)\), consequentily, we have that \((x-x_n)^{t-(B_{n-1}-1)}(x) \mid W(x)\). It means that the degree of \((x-x_n)^{t-(B_{n-1}-1)}(x)\) must be greater than or equal to the degree of \( W(x)/R(x) \). It is equivalent to:

\[
t - (B_{n-1} - 1) \leq AB_{n-1} + MB_{n-1} \frac{B_{n-1}^2}{2}.
\]

Consequently, if

\[
t \geq AB_{n-1} + MB_{n-1} \frac{B_{n-1}^2}{2} + B_{n-1},
\]

then polynomials \( (17) \) are linearly independent. We have proved the step of induction. \( \square \)
2.1 Setting of Parameters

To prove Theorem 2 we have to set the parameters $A, B_1, \ldots, B_n, D$, and proved that they are satisfy to the necessary conditions (14), (18), (19). The bound can be obtained by substituting of parameters to formulas (5), (6).

Without loss of generality, let us set the following

\[ m_1 \leq \ldots \leq m_n, \quad (23) \]

where $m_i = \deg f_i(x)$, $i = 1, \ldots, n$. A permutation of polynomials do not change conditions of Theorem 2.

Let us put

\[ B = (m_1 \ldots m_n)^{\frac{1}{n} t^{\frac{1}{2n}}}, \]

\[ B_i = \left\lfloor \frac{B}{m_i} \right\rfloor, \quad i = 1, \ldots, n, \quad A = B_1 \ldots B_n, \quad D = \left\lfloor \frac{A}{M_n} \right\rfloor, \]

where $\left\lfloor \cdot \right\rfloor$ is the integer part of the number. Let us check conditions (14), (18), (19).

The condition (14) has the form:

\[ AD + M_n D^2 < \frac{1}{M_n} A^2 + \frac{1}{2M_n} A^2 < A^2 = AB_1 \ldots B_n. \]

It is true, because $M_n \geq n \geq 2$.

The condition (18) has the form:

\[ \deg \Psi(x) \leq A + \sum_{i=1}^{n} (B_i - 1)m_i t < A + nBt \leq \]

\[ \leq t^{1/2} + n(m_1 \ldots m_n)^{\frac{1}{n}} t^{1 + \frac{3}{2n}} \leq (n + 1)(m_1 \ldots m_n)^{\frac{1}{n}} t^{\frac{2n+1}{2n}} < p. \quad (24) \]

Actually, $\left\lfloor \frac{B}{m_i} \right\rfloor \leq \frac{B}{m_i}$ and, consequently,

\[ A = \prod_{i=1}^{n} \left\lfloor \frac{B}{m_i} \right\rfloor \leq \frac{B^n}{m_1 \ldots m_n} = t^{1/2}. \]

The last inequality in (24) follows from

\[ t < (n + 1)^{-\frac{2n}{2n+1}} (m_1 \ldots m_n)^{-\frac{2}{2n+1}} p^{1-\frac{1}{2n+1}}. \quad (25) \]

The inequality (18) is proved.

Let us show that the condition (19) is also holds. Let us consider the right hand side of the inequality (19):

\[ AB_1 \ldots B_{n-1} + \frac{M_n}{2} (B_1 \ldots B_{n-1})^2 = (B_1 \ldots B_{n-1})^2 \left( B_n + \frac{M_n}{2} \right) \]
(Use now that \( B_i = \left[ \frac{B}{m_i} \right] \geq \gamma \frac{B}{m_i}, \quad i = 1, \ldots, n \), where \( \gamma \) is the following

\[
\gamma = \frac{B - \max_{1 \leq i \leq n} m_i}{B} = \frac{B - m_n}{B},
\]

using the condition \((23)\))

\[
< \gamma^{n-1} \frac{(m_1 \cdots m_n)^{2-2/n}}{(m_1 \cdots m_{n-1})^2} t^{1-1/n} \left( \frac{(m_1 \cdots m_n)^{1/n}}{m_n} t^{\frac{1}{2n}} + \frac{\mathcal{M}_n}{2} \right).
\]

(Use that \( (m_1 \cdots m_n)^{1/n} \leq 1 \), and that \( t > (\mathcal{M}_n/2)^{2n} \).)

\[
< 2\gamma^{n-1} m_n^2 t^{1-\frac{1}{2n}} < t.
\]

The last inequality follows from

\[
t^{\frac{1}{2n}} > 2\gamma^{n-1} m_n^2 = 2 \left( \frac{B - m_n}{B} \right)^{2n-1} m_n^2,
\]

this follows from \( (\frac{B-m_n}{B}) < 1 \), and

\[
t > 2^{2n} m_n^{4n}.\tag{26}
\]

Use \((5)\) for estimation of \(|M|\):

\[
|M| \leq 1 + \mathcal{M}_n + \frac{A + \sum_{i=1}^n (B_i - 1)m_it}{D} <
\]

(Use now that \( t^{1/2} > 1 + \mathcal{M}_n \). It follows from \((26)\).)

\[
< \frac{A + nBt}{D} = \frac{A + nBt}{[A/\mathcal{M}_n]} <
\]

(Obtain the inequality by means \((24)\).)

\[
\leq \frac{(n + 1)(m_1 \cdots m_n)^{\frac{1}{n}} t^{\frac{2n+1}{2n}}}{\gamma n^{1/2}/\mathcal{M}_n} \leq
\]

(It is easy to see that \( \gamma^{n} = (1 - \frac{m_n}{B})^n > 1/4 \), because \( \frac{m_n}{B} < 1/n \) follows from \((26)\).)

\[
\leq 4(n + 1)\mathcal{M}_n (m_1 \cdots m_n)^{\frac{1}{n}} t^{\frac{1}{2} + \frac{1}{2n}}.
\]

It is easy to see that constants \( C_1(m, n), C_2(m, n) \quad C_3(m, n) \) can be setted as follows: from the inequality \((26)\) let us obtain:

\[
C_1(m, n) = 2^{2n} m_n^{4n},
\]

let us remind that \( m_n = \max_{1 \leq i \leq n} m_i \); from the inequality \((25)\) obtain:

\[
C_2(m, n) = (n + 1)^{-\frac{3n}{2n+1}} (m_1 \cdots m_n)^{-\frac{2}{2n+1}};
\]

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and from the final bound obtain the value of the last constant:

$$C_3(m, n) = 4(n + 1)M_n(m_1 \ldots m_n)^{\frac{1}{2}}.$$  

These constants prove Remark 1. \hfill □

Let us consider the linear map (2). Let us obtain Corollary 1 which is the result of the paper [1].

**Corollary 1.** Let $\Gamma$ be a subgroup of $\mathbb{F}_p^*$ ($p$ is a prime number), $n \geq 1$. Let the following inequality

$$C_1(n) < |\Gamma| < C_2(n)p^{1 - \frac{1}{2m_1}},$$

where $C_1(n), C_2(n)$ are constant depending only on $n$, holds. Then we have the following bound:

$$|\Gamma \cap (\Gamma + \mu_1) \cap \ldots \cap (\Gamma + \mu_{m_1})| \leq C_3(n)|\Gamma|^{\frac{1}{2} + \frac{1}{2n}},$$

where $C_3(n)$ is some constant depending only on $n$, holds.

### 3 Polynomial Maps over $\mathbb{C}$

Let us consider the analog Theorem 2 for the complex field. Let $G = \{x \mid x^t = 1\}$ be a subgroup of roots of orders $t$ of unity of the group $\mathbb{C}^*$. Let us denote cosets of the subgroup $G$ by $G_1, \ldots, G_n$. Consider the map

$$f : x \mapsto (f_1(x), \ldots, f_n(x)), \quad n \geq 2, \tag{27}$$

where $f_1(x), \ldots, f_n(x) \in \mathbb{C}[x]$ are polynomials. The definition of admissibility of polynomials is analogous to the definition for polynomials over $\mathbb{F}_p$.

For the cardinality of the set

$$M = \{x \mid f_i(x) \in G_i, \; i = 1, \ldots, n\}$$

the following theorem holds.

**Theorem 3.** Let $G$ be a subgroup of $\mathbb{C}^*$ of roots of unity of some order, $G_1, \ldots, G_n$ are cosets of $G$, $n \geq 2$, $f_1(x), \ldots, f_n(x)$ is an admissible set of polynomials of degrees $m_1, \ldots, m_n$. Let us suppose that:

$$|G| > \tilde{C}_1(m, n),$$

where $\tilde{C}_1(m, n)$ is a constant depending only on $n$ and $m$. Then we have the following bound:

$$|M| \leq \tilde{C}_2(m, n)|\Gamma|^{\frac{1}{2} + \frac{1}{2n}},$$

where $\tilde{C}_2(m, n)$ is a constant depending only on $n$ and $m$.

**Remark 2.** Constants can be setted as follows: $\tilde{C}_1(m, n) = C_1(m, n)$, $\tilde{C}_2(m, n) = C_3(m, n)$.

The proofs of Theorem and Remark 2 almost completely repeat the proofs of Theorem 2 and Remark 1. We will not repeat these proofs. We only describe two small changes. We do not require that degree of polynomial $\Psi(x)$ is the less than characteristic of the field. It gives us that the restriction (24) is not actual. Also instead of theorem of F.K. Shmidt we use the theorem on linear dependence of a set of functions and vanishing of Wronskian.
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