The Commutative Closure of Shuffle Languages over Group Languages is Regular

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Abstract. We show that the commutative closure combined with the iterated shuffle is a regularity-preserving operation on group languages. In particular, for commutative group languages, the iterated shuffle is a regularity-preserving operation. We also give bounds for the size of minimal recognizing automata. Then, we use this result to deduce that the commutative closure of any shuffle language over group languages, i.e., a language given by a shuffle expression, i.e., expressions involving shuffle, iterated shuffle, concatenation, Kleene star and union in any order, starting with the group languages, always yields a regular language.

Keywords: commutative closure · group language · permutation automaton · shuffle expression · shuffle · iterated shuffle

1 Introduction

Having applications in regular model checking [17], or arising naturally in the theory of traces [8,34], one model for parallelism, the (partial) commutative closure has been extensively studied [12,13,14,16,18,20,28,30,34].

In [16], the somewhat informal notion of a robust class was introduced, meaning roughly a class closed under some of the usual operations on languages, such as Boolean operations, product, star, shuffle, morphism, inverses of morphisms, residuals, etc. Motivated by two guiding problems formulated in [16], we formulate the following slightly altered, but related problems:

Problem 1. When is the closure of a language under [partial] commutation regular?

Problem 2. Are there any robust classes for some common operations such that the commutative closure is (effectively) regular?

By effectively regular, we mean the stipulation that an automaton of the result of the commutation operation is computable from a representational scheme for the language class at hand.

1 We relax the condition from [15] that it must be a class of regular languages. However, some mechanism to represent the languages from the class should be available.
Here, we will investigate the commutation operation on the closure of the class (or variety thereof) of group languages under union, shuffle, iterated shuffle, concatenation and Kleene star. For the class of finite languages, this closure, called the class of shuffle languages, is definable by so called shuffle expressions [0, 23, 24, 25, 26, 36]. This is also true in our case, but the atomic expressions are interpreted not as finite languages, but as group languages. In this sense, we use the term shuffle expressions, or shuffle language, in a wider sense, by allowing different atomic languages. It will turn out that the commutation operation yields a regular language on this class of languages, and it is indeed effectively regular. However, I do not know if the languages class itself consists only of regular languages.

The shuffle and iterated shuffle have been introduced and studied to understand the semantics of parallel programs. This was undertaken, as it appears to be, independently by Campbell and Habermann [4], by Mazurkiewicz [29] and by Shaw [36]. They introduced flow expressions, which allow for sequential operators (catenation and iterated catenation) as well as for parallel operators (shuffle and iterated shuffle). These operations have been studied extensively, see for example [0, 23, 24, 25].

The shuffle operation as a binary operation, but not the iterated shuffle, is regularity-preserving on all regular languages. The size of recognizing automata was investigated in [2, 3, 5, 6, 17, 19].

2 Preliminaries and Definitions

By \( \Sigma \) we denote a finite set of symbols, i.e., an alphabet. By \( \Sigma^* \) we denote the set of all words with the concatenation operation. The empty word, i.e., the word of length zero, is denoted by \( \varepsilon \). If \( u \in \Sigma \), by \( |u| \) we denote the length of \( u \), and if \( a \in \Sigma \), by \( |u|_a \) we denote the number of times the letter \( a \) appears in \( u \). A language is a subset \( L \subseteq \Sigma^* \). For a language \( L \subseteq \Sigma^* \), we set \( L^+ = \{ u_1 \cdots u_n \mid \{ u_1, \ldots, u_n \} \subseteq L, n > 0 \} \) and \( L^* = L^+ \cup \{ \varepsilon \} \). By \( \mathbb{N}_0 \), we denote the natural numbers with zero.

A finite (complete and deterministic) automaton \( \mathcal{A} = (\Sigma, Q, \delta, q_0, F) \) over \( \Sigma \) consists of a finite state set \( Q \), a totally defined transition function \( \delta : Q \times \Sigma \rightarrow Q \), start state \( q_0 \in Q \) and final state set \( F \subseteq Q \). The transition function could be extended to words in the usual way by setting, for \( u \in \Sigma^* \), \( a \in \Sigma \) and \( q \in Q \), \( \delta(q, ua) = \delta(\delta(q, u), a) \) and \( \delta(q, \varepsilon) = q \). In the following, we will drop the distinction with \( \delta \) and will denote this extension also by \( \delta : Q \times \Sigma^* \rightarrow Q \). The language recognized, or accepted, by \( \mathcal{A} \) is \( L(\mathcal{A}) = \{ u \in \Sigma^* \mid \delta(q_0, u) \in F \} \).

A permutation automaton is an automaton such that for each letter \( a \in \Sigma \), the function \( \delta_a : Q \rightarrow Q \) given by \( \delta_a(q) = \delta(q, a) \) for \( q \in Q \) is bijective. We also say that the letter \( a \) permutes the state set. For a given permutation automaton \( \mathcal{A} = (\Sigma, Q, \delta, q_0, F) \) and \( a \in \Sigma \), the order of the letter \( a \) in \( \mathcal{A} \) is the smallest number \( n > 0 \) such that \( \delta(q, a^n) = q \) for all \( q \in Q \). This equals the order of the

\footnote{Here, only complete and deterministic automata are used, hence just called automata for short.}
letter viewed as a permutation on \( Q \). The maximal order of any permutation is given by Landau’s function, which has growth rate \( O(\exp(\sqrt{n \log n})) \) [1127]. A language \( L \subseteq \Sigma^* \) is a group language, if there exists a permutation automaton \( \mathcal{A} \) such that \( L = L(\mathcal{A}) \). By \( \mathcal{G} \) we denote the class of group languages. This class could also be seen as a variety [3233].

We will also use regular expressions occasionally, for the definition of them, and also for a more detailed treatment of the above notions, we refer to any textbook on formal language theory or theoretical computer science, for example [21].

Let \( \Sigma = \{ a_1, \ldots, a_k \} \) be the alphabet. The map \( \psi : \Sigma^* \rightarrow \mathbb{N}_0^k \) given by \( \psi(w) = (|w|_{a_1}, \ldots, |w|_{a_k}) \) is called the Parikh morphism [31]. If \( L \subseteq \Sigma^* \), we set \( \psi(L) = \{ \psi(w) \mid w \in L \} \). For a given word \( w \in \Sigma^* \), we define \( \text{perm}(w) := \{ u \in \Sigma^* : \psi(u) = \psi(w) \} \). If \( L \subseteq \Sigma^* \), then the commutative (or permutational) closure is \( \text{perm}(L) := \bigcup_{w \in L} \text{perm}(w) \). A language is called commutative, if \( \text{perm}(L) = L \).

**Definition 1.** The shuffle operation, denoted by \( \uplus \), is defined by

\[
\begin{array}{l}
u \uplus v = \{ w \in \Sigma^* \mid w = x_1y_1x_2y_2\cdots x_yn, \text{ for some words } \\
x_1, \ldots, x_n, y_1, \ldots, y_n \in \Sigma^* \text{ such that } u = x_1x_2\cdots x_n \text{ and } v = y_1y_2\cdots y_n \},
\end{array}
\]

for \( u, v \in \Sigma^* \) and \( L_1 \uplus L_2 := \bigcup_{x \in L_1, y \in L_2} (x \uplus y) \) for \( L_1, L_2 \subseteq \Sigma^* \).

In writing formulas without brackets, we suppose that the shuffle operation binds stronger than the set operations, and the concatenation operator has the strongest binding.

If \( L_1, \ldots, L_n \subseteq \Sigma^* \), we set \( \biguplus_{i=1}^n L_i = L_1 \uplus \ldots \uplus L_n \). The **iterated shuffle** of \( L \subseteq \Sigma^* \) is \( L^{\uplus^n} = \bigcup_{n \geq 0} \biguplus_{i=1}^n L \).

**Theorem 2 (Fernau et al. [9]).** Let \( U, V, W \subseteq \Sigma^* \). Then,

1. \( U \uplus V = V \uplus U \) (commutative law);
2. \( (U \uplus V) \uplus W = U \uplus (V \uplus W) \) (associative law);
3. \( U \uplus (V \cup W) = (U \uplus V) \cup (U \uplus W) \) (distributive over union);
4. \( (U^{\uplus n})^{\uplus n} = U^{\uplus n} \);
5. \( (U \cup V)^{*} = U^{*} \uplus V^{*} \);
6. \( (U \uplus V)^{*} = (U \uplus (U \cup V)^{*}) \cup \{ \epsilon \} \).

The next result is taken from [9] and gives equations like \( \text{perm}(UV) = \text{perm}(U) \uplus \text{perm}(V) \) or \( \text{perm}(U^*) = \text{perm}(U)^{\uplus*} \) for \( U, V \subseteq \Sigma^* \). A semiring is an algebraic structure \( (S, +, \cdot, 0, 1) \) such that \( (S, +, 0) \) forms a commutative monoid, \( (S, \cdot, 1) \) is a monoid and we have \( a \cdot (b + c) = a \cdot b + a \cdot c, (b + c) \cdot a = b \cdot a + c \cdot a \) and \( 0 \cdot a = a \cdot 0 = 0 \).

**Theorem 3 (Fernau et al. [9]).** \( \text{perm} : \mathcal{P}(\Sigma^*) \rightarrow \mathcal{P}(\Sigma^*) \) is a semiring morphism from the semiring \( (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{ \epsilon \}) \), that also respects the iterated concatenation resp. iterated shuffle operation, to the semiring \( (\mathcal{P}(\Sigma^*), \cup, \uplus, \emptyset, \{ \epsilon \}) \).

As \( \psi(U \uplus V) = \psi(UV) \) and \( \psi(U^*) = \psi(U^{\uplus*}) \), we also find the next result.
Theorem 4. \( \text{perm} : \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*) \) is a semiring morphism from the semiring \((\mathcal{P}(\Sigma^*), \cup, \sqcup, \emptyset, \{\varepsilon\})\) to the semiring \((\mathcal{P}(\Sigma^*), \cup, \sqcup, \emptyset, \{\varepsilon\})\) that also respects the iterated shuffle operation.

In [16] it was shown that the commutative closure is regularity-preserving on \(G\) using combinatorial arguments. In [20] an automaton was constructed, yielding explicit bounds for the number of states needed in any recognizing automaton.

Theorem 5 ([20]). Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( A = (\Sigma, Q, \delta, q_0, F) \) be a permutation automaton. Then \( \text{perm}(L(A)) \) is recognizable by an automaton with at most \( \max(1, |Q| \prod_{i=1}^k |L_i|) \) states, where \( L_i \) for \( i \in \{1, \ldots, k\} \) denotes the order of \( a_i \). Furthermore, the recognizing automaton is computable.

3 Shuffle Languages over Arbitrary Language Classes

Here, we introduce shuffle languages over arbitrary language classes and prove a normal form result.

Definition 6. Let \( \mathcal{L} \) be a class of languages.

1. \( \text{SE}(\mathcal{L}) \) is the closure of \( \mathcal{L} \) under shuffle, iterated shuffle, union, concatenation and Kleene star.
2. \( \text{Shuf}(\mathcal{L}) \) is the closure of \( \mathcal{L} \) under shuffle, iterated shuffle and union.

For \( \mathcal{L}_{Alp} = \{\emptyset, \{\varepsilon\}\} \cup \{\{a\} \mid a \in \Sigma \text{ for some alphabet } \Sigma\} \) and \( \mathcal{L}_{Fin} = \{L \mid L \subseteq \Sigma^* \text{ for some alphabet and } L \text{ is finite}\} \) the resulting closures were investigated in [9,23,24,25]. Note that \( \text{SE}(\mathcal{L}_{Alp}) = \text{SE}(\mathcal{L}_{Fin}) \). By Theorem 3 we can compute a shuffle expression over \( \mathcal{L}_{Alp} \) for the commutative closure of any regular language by rewriting a regular expression and vice versa. Hence, the class \( \text{Shuf}(\mathcal{L}_{Alp}) \) equals the commutative closure of all regular languages. So, \( \text{Shuf}(\mathcal{L}_{Alp}) \neq \text{Shuf}(\mathcal{L}_{Fin}) \).

Proposition 7. Let \( L \in \text{Shuf}(\mathcal{L}) \). Then, \( L \) is a finite union of languages of the form
\[
L_1 \sqcup \ldots \sqcup L_k \sqcup L_{k+1}^{\sqcup} \sqcup \ldots \sqcup L_n^{\sqcup}
\]
with \( 1 \leq k \leq n \) and \( L_i \in \mathcal{L} \) for \( i \in \{1, \ldots, n\} \) and this expression is computable.

Proof. Theorem 2 provides an inductive proof of Proposition 7. Note that a similar statement has been shown in [23, Theorem 3.1] for \( \text{Shuf}(\mathcal{L}_{Fin}) \). However, as we do not assume that \( \mathcal{L} \) is closed under shuffle or union, we only get the form as stated.

Remark 1. By Theorem 2 we can write the languages in Proposition 7 also in the form \( L_1 \sqcup \ldots \sqcup L_k \sqcup (L_{k+1} \cup \ldots \cup L_n)^{\sqcup} \). So, if \( \mathcal{L} \) is closed under union, which is the case for languages from \( \mathcal{G} \) over a common alphabet, we can write the languages in \( \text{Shuf}(\mathcal{L}) \) as a finite union of languages of the form \( L_1 \sqcup \ldots \sqcup L_{n-1} \sqcup L_n^{\sqcup} \) with \( L_1, \ldots, L_n \in \mathcal{L} \).
Lastly, with Theorem 3 and Theorem 4, we show that up to permutational equivalence \( SE(L) \) and \( Shuf(L) \) give the same languages.

**Proposition 8.** Let \( L \) be any class of languages. Suppose \( L \in SE(L) \). Then, we can compute \( L' \in Shuf(L) \) such that \( \text{perm}(L) = \text{perm}(L') \).

**Proof.** By Theorem 3 and Theorem 4, we have, for \( U, V \subseteq \Sigma^* \), \( \text{perm}(U \oplus V) = \text{perm}(U) \oplus \text{perm}(V) = \text{perm}(U \odot V) \) and \( \text{perm}(U^\omega) = \text{perm}(U)^\omega = \text{perm}(U^\omega) \). So, inductively, for \( L \in SE(G) \), by replacing every concatenation with the shuffle and every Kleene star with the iterated shuffle, we find \( L' \in Shuf(G) \) such that \( \text{perm}(L) = \text{perm}(L') \).

### 4 The Commutative Closure on \( SE(G) \)

By Proposition 8, the commutative closure on \( SE(L) \) for any language class \( L \) equals the commutative closure of \( Shuf(L) \). Theorem 9 of this section, stating that the commutative closure combined with the iterated shuffle is regular, is the main ingredient in our proof that the commutative closure is regularity-preserving on \( SE(G) \) and the most demanding result in this work.

Note that, in general, this combined operation does not preserves regularity, as shown by \( \text{perm}(ab)^{\omega} = \{ w \in \{a,b\}^* \mid |w|_a = |w|_b \} \).

**Theorem 9.** Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( A = (\Sigma, Q, \delta, q_0, F) \) be a permutation automaton. Then

\[
\text{perm}(L(A)^{\omega})
\]

is recognizable by an automaton with at most \( \left( |Q|^k \prod_{j=1}^k L_j \right) + 1 \) many states, where \( L_j \) for \( j \in \{1, \ldots, k\} \) denotes the order of \( a_j \), and this automaton is effectively computable.

**Proof (sketch).** The method of proof, called state label method, is an extension of the one used in [20], which also includes a detailed motivation and intuition of this method.

In what follows, we will first give an intuitive outline of the method, geared toward our intended extension, of how to use it to recognize the commutative closure of a regular language. Then, we will show how to modify it to show our statement at hand. We will only sketch the method, and will leave out some details for the sake of the bigger picture.

The method consists in labeling the points of \( N_0^{\Sigma} \) with the states of a given automaton that are reachable from the start state by all words whose Parikh image equals the point under consideration.

As it turns out, a word is in the commutative closure if and only if it ends in a state labeled by a set which contains at least one final state.

Very roughly, the resulting labeling of \( N_0^{\Sigma} \) could be thought of as a more refined version of the Parikh map for regular languages, and in some sense as a blend between the well-known powerset construction, as we label with subsets...
of states, and the Parikh map, as we not only indicate for each point if there is a word in the language or not, but additionally store all states we could reach by words whose Parikh image equals the point in question.

More specifically, let $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ be an automaton. In [20], the point $p \in \mathbb{N}_0^{[\Sigma]}$ was labeled by the set

$$S_p = \{\delta(q_0, u) \mid \psi(u) = p\}$$

and the following holds true: $v \in \text{perm}(L(\mathcal{A})) \Leftrightarrow S_{\psi(v)} \cap F \neq \emptyset$.

Then, along any line parallel to the axis, which corresponds to reading in a single fixed letter, by finiteness, the state labels are ultimately periodic. However, for each such line, the onset of the period and the period itself may change. For example, take the automaton with state set $Q = \{q_0, q_1, q_2\}$ over $\Sigma = \{a, b\}$ and transition function, for $q \in Q$ and $x \in \Sigma$,

$$\delta(q, x) = \begin{cases} q_1 & \text{if } q = q_0, x = a; \\ q_0 & \text{if } q = q_1, x = b; \\ q_2 & \text{otherwise.} \end{cases}$$

Then, $L(\mathcal{A}) = (ab)^*$ and, for $p = (p_a, p_b) \in \mathbb{N}_0^2$,

$$S_p = \begin{cases} \{q_0, q_2\} & \text{if } p_a = p_b; \\ \{q_1, q_2\} & \text{if } p_a = p_b + 1; \\ \{q_2\} & \text{otherwise.} \end{cases}$$

Let $c \in \mathbb{N}_0$. Then, along the lines $\{(p_a, p_b) \in \mathbb{N}_0^2 \mid p_a = c\}$, we have $S_{(c, c+2)} = S_{(c, c+1)}$ and the point $(c, c+1)$ is the earliest onset after which the state labeling $S_p$ gets periodic on this line.

However, if, for any line parallel to the axis, we can bound the onset of the period and the period itself uniformly, i.e., independently of the line we are considering, then the commutative closure is regular, and moreover we can construct a recognizing automaton with these uniform bounds.

This was shown in [20] and it was shown that for group languages, we have such uniform bounds.

Note that in our example, we do not have such a uniform bound, as the onset, for example, for the lines going in the direction $(0, 1)$ starting at $(c, 0)$ (i.e. reading in the letter $b$) was $c + 1$, i.e., it grows and is not uniformly bounded. In fact, $\text{perm}(ab^*) = \{u \in \{a, b\}^* : |u|_a = |u|_b\}$ is not regular.

Up to now, the method only works for the commutative closure. So, let us now describe how to modify it such that we get an automaton for the iterated shuffle of the commutative closure of a given automaton.

First, recall that, by Theorem 3, we have

$$\text{perm}(L(\mathcal{A}))^\omega,* = \text{perm}(L(\mathcal{A})*).$$

The usual construction for the Kleene star associates a final state with the start state, and this is in some sense what we are doing now. More formally, in the
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state labeling, we add the start state each time we read a final state, i.e., we have another labeling which we describe next.

Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( e_i = \psi(a_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^k \) be the vector with 1 precisely at the \( i \)-th position and zero everywhere else. If \( A = (\Sigma, Q, \delta, q_0, F) \) is an automaton, set

\[
T_{(0, \ldots, 0)} = \{q_0\} \quad \text{and} \quad T_p = \bigcup_{\exists t \in \{1, \ldots, k\}: p = q + e_t} \delta(S^+_q, a_t) \quad \text{for} \quad p \neq (0, \ldots, 0),
\]

where

\[
S^+_p = \begin{cases} 
T_p \cup \{q_0\} & \text{if} \ T_p \cap F \neq \emptyset; \\
T_p & \text{if} \ T_p \cap F = \emptyset.
\end{cases}
\]

Then, \( v \in \text{perm}(L(A)^*) \Leftrightarrow S^+_p \cap F \neq \emptyset \) or \( v = \varepsilon \).

Note the extra condition that checks for the empty word. This is a technicality, that surely could be omitted if \( q_0 \in F \), but not in the general case. Please see Figure 1 for a visual explanation in the case of a binary alphabet.

Finally, the same sufficient condition of regularity in terms of the new state labels \( S^+_p \) could be derived as in the previous case, namely if they are uniformly bounded in the axis-parallel directions, then the commutative closure is regular.

For the state label \( S_{(p_a, p_b)} \), after reading the letter \( b \), we will end up at \( S_{(p_a, p_b + 1)} \) and the state label is updated according to Equation (1) and Equation (2).

\[
S^+_{(p_a, p_b + 1)} = \begin{cases} 
T_{(p_a, p_b + 1)} \cup \{s_0\} & \text{if} \ T_{(p_a, p_b + 1)} \cap F \neq \emptyset; \\
T_{(p_a, p_b + 1)} & \text{otherwise},
\end{cases}
\]

\[
T_{(p_a, p_b + 1)} = \delta(S^+_{(p_a, p_b + 1)}, a) \cup \delta(S^+_{(p_a, p_b)}, b) \quad (1)
\]

\[
S^+_{(p_a, p_b + 1)} = \begin{cases} 
T_{(p_a, p_b + 1)} \cup \{s_0\} & \text{if} \ T_{(p_a, p_b + 1)} \cap F \neq \emptyset; \\
T_{(p_a, p_b + 1)} & \text{otherwise},
\end{cases}
\]

\[
S^+_{(p_a, p_b + 1)} = \begin{cases} 
T_{(p_a, p_b + 1)} \cup \{s_0\} & \text{if} \ T_{(p_a, p_b + 1)} \cap F \neq \emptyset; \\
T_{(p_a, p_b + 1)} & \text{otherwise},
\end{cases}
\]

\[
S^+_{(p_a, p_b + 1)} = \begin{cases} 
T_{(p_a, p_b + 1)} \cup \{s_0\} & \text{if} \ T_{(p_a, p_b + 1)} \cap F \neq \emptyset; \\
T_{(p_a, p_b + 1)} & \text{otherwise},
\end{cases}
\]

Fig. 1. Illustration of how state labels are updated for the iterated shuffle if new input symbols are read with \( \Sigma = \{a, b\} \). For the state label \( S_{(p_a, p_b)} \), after reading the letter \( b \), we will end up at \( S_{(p_a, p_b + 1)} \) and the state label is updated according to Equation (1) and Equation (2). Seen from the state label \( S_{(p_a, p_b)} \), we account for both paths given by the words \( ab \) and \( ba \) when ending at \( (p_a, p_b + 1) \), hence the union in the definition of \( T_{(p_a, p_b + 1)} \).
eventually get periodic. Also, we can show, as we only have cycles, that after a
certain number of letters have been read, we have exploited all ways that these
sets could grow, i.e., we know that after we have read a certain numbers of letters
we must end up in a period, and this period could also be bounded uniformly
(but of course, depending on \( \mathcal{A} \)).

To be a little more quantitative here, if \( L_i \) denotes the order of \( a_i \), then, for
each line going in the direction \( e_i \), we can show that after at most \( (|Q| - 1)L_j \)
many steps we must enter the period, and the smallest period has to divide \( L_j \).
This in turn could be used to derive that an automaton with at most
\[
\prod_{i=1}^{k}((|Q| - 1)L_j + L_j) = |Q|^{k} \prod_{i=1}^{k} L_j
\]
many states could recognize \( \text{perm}(L(\mathcal{A})^+) \). Note that this statement is only
valid for the state labeling \( S_\mathcal{P}^+ \), and hence only applies to \( \text{perm}(L(\mathcal{A})^+) \). So, to
recognize \( \text{perm}(L(\mathcal{A})^+) \), and incorporate the additional test for the empty word,
we have to add one more state.

Actually, a full formal treatment, especially the steps mentioned in the pre-
vious paragraphs, is quite involved and incorporates a detailed construction of
the recognizing automaton out of the state label method and a detailed analysis
of the action of the permutational letters on the state set. I refer to \cite{20} and to
the extended version of this paper, which will appear in a special issue \cite{18}, for
a treatment of these issues in the context of the mere commutative closure.

Lastly, note that the constructions are effective, as we only have to label a
bounded number of grid points of \( \mathbb{N}_0^k \), and the state labels are computable from
the transition function of \( \mathcal{A} \).

So, with Theorem \ref{thm:regularity-preserving} we can derive our next result.

**Theorem 10.** Let \( L \in \text{Shuf}(\mathcal{G}) \). Then \( \text{perm}(L) \) is effectively regular.

**Proof.** By Proposition \ref{prop:regularity-preserving}, we only need to consider languages of the form \( L_1 \uplus \ldots \uplus L_k \uplus L_{k+1}^{a_1} \ldots \uplus L_n^{a_n} \) with \( L_i \in \mathcal{G} \). By Theorem \ref{thm:regularity-preserving} \( \text{perm}(L_1 \uplus \ldots \uplus L_k \uplus L_{k+1}^{a_1} \ldots \uplus L_n^{a_n}) \) equals
\[
\text{perm}(L_1) \uplus \ldots \uplus \text{perm}(L_k) \uplus \text{perm}(L_{k+1}^{a_1}) \ldots \uplus \text{perm}(L_n^{a_n}).
\]
The shuffle is regularity-preserving \cite{35,22}, where an automaton for it is computable. So, by Theorem \ref{thm:regularity-preserving} and Theorem \ref{thm:regularity-preserving} the above language is effectively regular, where again for the commutative closure of a group language an automaton is computable similarly as outlined at the end of the proof sketch for Theorem \ref{thm:regularity-preserving}. Hence, \( \text{perm}(L) \) is effectively regular.

So, with Proposition \ref{prop:regularity-preserving} our next result follows.

**Theorem 11.** Let \( L \in \text{SE}(\mathcal{G}) \). Then \( \text{perm}(L) \) is effectively regular.
5 Commutative Group Languages

By Theorem 9, we can deduce that for commutative group languages \( L \subseteq \Sigma^* \), the iterated shuffle is a regularity-preserving operation. Also, for a commutative regular language in general, it is easy to see that for a minimal automaton \( A = (\Sigma, Q, \delta, q_0, F) \) we must have \( \delta(q, ab) = \delta(\delta(q, a), b) \) for any \( q \in Q \) and \( a, b \in \Sigma \) [10]. Furthermore, if \( A = (\Sigma, Q, \delta, q_0, F) \) is a minimal permutation automaton for a commutative language, then the order of each letter \( a \in \Sigma \) equals the minimal \( n \geq 0 \) such that \( \delta(q_0, a^n) = q_0 \). For if \( q \in Q \), then, by minimality, there exists \( u \in \Sigma^* \) such that \( \delta(q_0, u) = q \), which yields \( \delta(q, a^n) = \delta(\delta(q, u), a^n) = \delta(q_0, a^n u) = \delta(\delta(q_0, u), a^n) = \delta(q_0, u) = q \). So, combining our observations, we get the next result.

**Proposition 12.** Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( L \subseteq \Sigma^* \) be a commutative group language with minimal permutation automaton \( A = (\Sigma, Q, \delta, q_0, F) \) such that \( L = L(A) \). Then, the iterated shuffle \( L^{(i)} \subseteq \Sigma^* \) is regular and recognizable by an automaton with at most \( (|Q|^k \prod_{i=1}^k p_i) + 1 \) many states, where \( p_i > 0 \) is minimal such that \( \delta(q_0, a_i^{p_i}) = q_0 \) for \( i \in \{1, \ldots, k\} \).

6 The \( n \)-times Shuffle

We just note in passing that the method of proof of Theorem 9 could also be adapted to yield a bound for the size of a recognizing automaton of the \( n \)-times shuffle combined with the commutative closure on group languages that is better than applying the bounds from [3, 5, 20] individually.

**Proposition 13.** Let \( A_i = (\Sigma, Q_i, \delta_i, q_i, F_i) \) for \( i \in \{1, \ldots, n\} \) be \( n \) permutation automata. Then

\[
\text{sc}(\text{perm}(L(A_1)) \cup \ldots \cup \text{perm}(L(A_n))) \leq \left( \sum_{i=1}^n Q_i \right)^k \prod_{j=1}^k \text{lcm}(L_j^{(1)}, \ldots, L_j^{(n)})
\]

where \( L_j^{(i)} \) for \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, k\} \) denotes the order of the letter \( a_j \) as a permutation on \( Q_i \).

7 Conclusion

We have shown that the commutative closure of any shuffle language over group languages is regular. However, it is unknown if any shuffle language over the group languages is a regular languages itself. As a first step, the question if the iterated shuffle of a group language is regular might be investigated. I conjecture this to be true, but do not know how to proof it for general group languages. Observe that merely by noting that the commutative closure is regular, we cannot conclude that the original language is regular. For example, consider the non-regular context-free language given by the grammar \( G \) over \( \{a, b\} \) with rules

\[
S \rightarrow aTaS | \varepsilon, \quad T \rightarrow bSbT | \varepsilon.
\]
and start symbol $S$.

**Proposition 14.** The language $L \subseteq \{a,b\}^*$ generated by the above grammar $G$ is not regular, but its commutative closure is regular.

**Proof.** 1. $L \cap (ab)^*(ba)^* = \{(ab)^n(ba)^n \mid n \geq 0\}$.

It is easy to see that $\{(ab)^n(ba)^n \mid n \geq 0\} \subseteq L \cap (ab)^*(ba)^*$. For the other inclusion, we will first show that if

$$S \rightarrow u$$

with $u \in (ab)^*(ba)^*$, then $u = \varepsilon$ or $S \rightarrow abSba \rightarrow u$ with $u = abvba$, which implies $v \in (ab)^*(ba)^*$. So assume $S \rightarrow u$ with $u \neq \varepsilon$. Then, we must have

$$S \rightarrow aTaS \rightarrow u,$$

As, by assumption $u \notin \Sigma^*aa\Sigma^*$, we must apply $S \rightarrow \varepsilon$ and could not apply $T \rightarrow \varepsilon$. So, the following steps are necessary

$$S \rightarrow aTaS \rightarrow aT \rightarrow abSba \rightarrow u. \quad (3)$$

Assume we expand $T$ into a non-empty word, then

$$abSba \rightarrow abSbbSba.$$

As the factor $bb$ occurs at most once in any word from $(ab)^*(ba)^*$, the above must expand to $abSbbSba$. This, in turn, implies that the first $S$ must expand into a word from $(ab)^*a$. However, such a word always contains either an odd number of $a$’s or an odd number of $b$’s, and by the production rules, as these letters are always introduced in pairs, this is not possible. Hence, we cannot expand $T$ in Equation (3) into a non-empty word and we must have $T \rightarrow \varepsilon$.

Then,

$$S \rightarrow aTaS \rightarrow aT \rightarrow abSba \rightarrow u.$$

So, we can write $u = abvba$ with $v \in (ab)^*(ba)^*$.

Finally, we reason inductively. If $u = \varepsilon$, then $u \in \{(ab)^n(ba)^n \mid n \geq 0\}$.

Otherwise, by the previously shown statement, we have $u = abvba$ with $S \rightarrow v$ and $v \in (ab)^*(ba)^*$. Hence, inductively, we can assume $v = (ab)^n(ba)^n$ for some $n \geq 0$, which implies $u = (ab)^{n+1}(ba)^{n+1}$.

2. The generated language is not regular.

Assume $L$ is regular. Then, with the above result, also $\{(ab)^n(ba)^n \mid n \geq 0\}$ would be regular. However, for the homomorphism $\varphi : \{c,d\}^* \rightarrow \{a,b\}^*$ given by $\varphi(c) = ab$, $\varphi(d) = ba$ we have $\{c^n d^n \mid n \geq 0\} = \varphi^{-1}(\{(ab)^n(ba)^n \mid n \geq 0\})$. As the last language is well-known to be not regular, and as regular languages are closed under inverse homomorphic mappings, the language $\{(ab)^n(ba)^n \mid n \geq 0\}$ could not be regular.
3. The commutative closure of $L$ is $\{u \in \{a, b\}^* : |w|_a \equiv 0 \pmod{2}, |w|_b \equiv 0 \pmod{2}, |w|_a \geq \min(1, |w|_b)\}$, which is a regular language.

We have, for any $n \geq 0$ and $m \geq 0$, that $a(bb)^m(aa)^n \in L$ and $\varepsilon \in L$. Also, as each rule introduces the letters $a$ or $b$ in pairs, any word in $L$ has an even number of $a$ and $b$'s and as we can only introduce the letter $b$ with the non-terminal $T$, which we only can apply after producing at least one $a$, we see that if we have at least one $b$, then we need to have at least one $a$. Combining these observations yields that the commutative closure equals the language written above and the defining conditions of this language could be realized by automata.

So, we have shown the claims made in the proposition. \(\square\)

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A Proofs for Section 4 (The Commutative Closure on $SE(G)$)

First, we collect some results that we will need later. Thereafter, we introduce the state label method in a more general formulation. Then, we apply it and give the full proof of Theorem 1

The method is technical and non-trivial, and the reader might notice, what I find quite remarkable, that the main difficulty in the results presented in this work poses Theorem 3 for which the state label method was developed. All the other results follow less or more readily.

In the following, we call a semi-automaton a tuple $A = (\Sigma, Q, \delta)$ with $\Sigma$ the input alphabet, $Q$ the finite state set and transition function $\delta : Q \times \Sigma \rightarrow Q$. This is an automaton without a start state and a final state set, and all notions that do not explicitly use the start state or the final state set carry over from automata to semi-automata.

For a natural number $n \in \mathbb{N}_0$, we set $[n] = \{0, \ldots, n-1\}$. Also, let $M \subseteq \mathbb{N}_0$ be some finite set. By max $M$ we denote the maximal element in $M$ with respect to the usual order, and we set $\max \emptyset = 0$. Also for finite $M \subseteq \mathbb{N}_0 \setminus \{0\}$, i.e., $M$ is finite without zero in it, by lcm $M$ we denote the least common multiple of the numbers in $M$, and set lcm $\emptyset = 0$.

A.1 Other Results Needed

Here, we state the following result about unary languages, which we will need later in this subsection.

**Unary Languages** Let $\Sigma = \{a\}$ be a unary alphabet. In this section we collect some results about unary languages. Suppose $L \subseteq \Sigma^*$ is regular with an accepting complete deterministic automaton $A = (\Sigma, S, \delta, q_0, F)$. Then by considering the sequence of states $\delta(q_0, a^1), \delta(q_0, a^2), \delta(q_0, a^3), \ldots$ we find numbers $i \geq 0$, $p > 0$ with $i$ and $p$ minimal such that $\delta(q_0, a^i) = \delta(q_0, a^{i+p})$. We call these numbers the index $i$ and the period $p$ of the automaton $A$. Suppose $A$ is initially connected, i.e., $\delta(q_0, \Sigma^*) = Q$. Then $i + p = |S|$ and the states $\{q_0, \delta(q_0, a), \ldots, \delta(q_0, a^{i-1})\}$ constitute the tail and the states $\{\delta(q_0, a^i), \delta(q_0, a^{i+1}), \ldots, \delta(q_0, a^{i+p-1})\}$ constitute the unique cycle of the automaton. When we speak of the cycle, tail, index or period of an arbitrary unary automaton we nevertheless mean the above sets, even if the automaton is not initially connected and the automaton graph might have more than one cycle or more than one straight path.

**Lemma 15.** Let $A = (\Sigma, Q, \delta, q_0, F)$ be some unary automaton. If $\delta(s, a^k) = s$ for some state $s \in Q$ and number $k > 0$, then $k$ is divided by the period of $A$.

**Proof.** Let $i$ be the index, and $p$ the period of $A$. We write $k = np + r$ with $0 \leq r < p$. First note that $s$ is on the cycle of $A$, i.e.,

$$s \in \{\delta(q_0, a^i), \delta(q_0, a^{i+1}), \ldots, \delta(q_0, a^{i+p-1})\}$$

3 Recall, the automaton is assumed to be complete.
as otherwise \( i \) would not be minimal. Then if \( s = \delta(q_0, a^{i+j}) \) for some \( 0 \leq j < p \) we have
\[
\delta(q_0, a^{i+k}) = \delta(q_0, a^{i+p+k}) = \delta(q_0, a^{i+j+k+(p-j)}) = \delta(q_0, a^r).
\]
So \( \delta(q_0, a^i) = \delta(q_0, a^{i+k}) = \delta(q_0, a^{i+n+p+r}) = \delta(q_0, a^{i+r}) \) which gives \( r = 0 \) by minimality of \( p \).

\[ \square \]

A.2 Overview of the State Label Method

The state label method was implicitly used in [Hof20] to give a state complexity bound for the commutative closure of a group language, see [Hof20] for an intuitive explanation and examples in this special case.

Here, we extract the method of proof from [Hof20] in a more abstract setting and formulate it independently of any automata. Intuitively, we want to describe a commutative language by labeling points from \( \mathbb{N}_0^k \) with subsets. We call these subsets state labels, as in our applications they arise from the states of given automata. Intuitively and very roughly, the method could be thought of as both a refined Parikh map for regular languages and a power set construction for automata that incorporates the commutativity condition. The connection to languages is stated in Theorem 22. In the framework of the state label method we construct unary automata, see Definition 19, that are used to decompose the state label map, see Proposition 20.

Please also see the proof sketch of Theorem 9 supplied in the main text to get a bird’s-eye view of the method applied to our situation.

Outline of the Method and How to Apply It

Before giving the formal definitions, let us give a rough outline of the method and how to apply it. First, the actual labeling is computed by using another function \( f \) that operates on the subsets of a set \( Q \), which is, in our applications, related to the states of one or more automata. This other function gives us flexibility in the way the other state labels are formed. Usually, the labels are formed by the transitions function(s) of one or more automata and additional operations, like adding a start state when a condition is meet. The state label function itself, which will be called \( \sigma \), uses \( f \), and computes the state labels out of the predecessor state labels, where a predecessor of a state label at a point is a state label that corresponds to a point strictly smaller, in the componentwise order, than the point in question. For a given automaton \( A \), we will introduce functions \( f \) that are compatible with \( A \), a term made precise later, and we will show that for permutation automata and arbitrary state labels \( \sigma \) induced by function compatible with the automaton, the commutative closure is regular. In summary, the application of the state label method, to show regularity of the commutative closure for operations on permutation automata, consists in the following steps:

1. Define a state labeling \( \sigma \) with the help of a function \( f \) that arises out of the operation and automata in question.
2. Link the state labeling to the Parikh image of the operation. More precisely, determine the resulting state labeling more precisely, and show that the inverse image of a suitable chosen subset of state labels equals the Parikh image of the operation in question applied to the language of the permutation automaton. This yields that the state label map could be used to describe the commutative closure.
3. Apply regularity conditions, i.e., apply Proposition 24 and Theorem 22 to deduce regularity and a bound for the size of an automaton.
A.3 The State Label Method

Our first definition in this section will be the notion of a state label map.

**Definition 16.** Let $\Sigma = \{a_1, \ldots, a_k\}$ and $Q$ be a finite set. A state label function is a function $\sigma : \mathbb{N}_0^k \to \mathcal{P}(Q)$ given by another function $f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ so that

$$\sigma(p) = \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma} f(\sigma(q), b)$$

for $p \neq (0, \ldots, 0)$ and $\sigma(0, \ldots, 0) \in \mathcal{P}(Q)$ is arbitrary.

In this context, we call the elements from $Q$ states, even if they do not correspond to an automaton. The function $f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ could be extended to words by setting $f(S, \varepsilon) = S$ and $f(S, ux) = f(f(S, u), x)$. With this extension the next equation could be derived.

**Lemma 17.** Let $\sigma : \mathbb{N}_0^k \to \mathcal{P}(Q)$ be a state label function given by $f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ and $p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k$. If $1 \leq n \leq p_1 + \ldots + p_k$, then

$$\sigma(p) = \bigcup_{(q,u) \in \mathbb{N}_0^k \times \Sigma^n} f(\sigma(q), u).$$

**Proof.** For $n = 1$ this is simply Definition [16] where $p \neq (0, \ldots, 0)$ by the assumptions. For $n > 1$, by Definition [16] we get

$$\sigma(p) = \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma} f(\sigma(q), b).$$

If $b \in \Sigma$, then $p = q + \psi(b)$ implies $q_1 + \ldots + q_k = p_1 + \ldots + p_k - 1$. Hence $1 \leq n - 1 \leq q_1 + \ldots + q_k$ and, as inductively

$$\sigma(q) = \bigcup_{(q', u) \in \mathbb{N}_0^k \times \Sigma^{n-1}} f(\sigma(q'), u),$$

we get

$$\sigma(p) = \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma} f\left(\bigcup_{(q', u) \in \mathbb{N}_0^k \times \Sigma^{n-1}} f(\sigma(q'), u), b\right)$$

$$= \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma \{q', u\} \in \mathbb{N}_0^k \times \Sigma^{n-1}} f(f(\sigma(q'), u), b)$$

$$= \bigcup_{(q,u) \in \mathbb{N}_0^k \times \Sigma^{n-1}, u \in \Sigma} f(f(\sigma(q), u), b)$$

$$= \bigcup_{(q,u) \in \mathbb{N}_0^k \times \Sigma^{n-1}, u \in \Sigma} f(f(\sigma(q), u), b)$$

$$= \bigcup_{(q,u) \in \mathbb{N}_0^k \times \Sigma^n} f(\sigma(q), u).$$
Assume \( b \) \( (\text{state label map decomposition}) \) Suppose \( p \). They are meant to read inputs in the direction \( \psi(a_j) \), which is orthogonal to \( H_j \).

This will be stated more precisely in Proposition 20.

Definition 19. (unary automata along letter \( a_j \) \( (\Sigma \subseteq \{a_1, \ldots, a_k\} \)) Let \( \Sigma \subseteq \{a_1, \ldots, a_k\} \) \( \sigma : \mathbb{N}_0^k \rightarrow \mathfrak{P}(Q) \) be a state label function, with defining function \( f : \mathfrak{P}(Q) \times \Sigma \rightarrow \mathfrak{P}(Q) \) and finite set \( Q \). Fix \( j \in \{1, \ldots, k\} \) and \( p \in H_j \). We define a unary automaton \( \mathcal{A}_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p^{(j)}, s_p^{(0,j)}, F_p^{(j)}) \). But suppose for points \( q \in \mathbb{N}_0^k \) with \( p = q + \psi(b) \) for some \( b \in \Sigma \) the unary automata \( \mathcal{A}_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p^{(j)}, s_p^{(0,j)}, F_p^{(j)}) \) are already defined. Set\(^4\)

\[
\mathcal{P} = \{A_q^{(j)} \mid p = q + \psi(b) \text{ for some } b \in \Sigma\}.
\]

Let \( I \) be the maximal index and \( P \) the least common multiple\(^5\) of the periods of the unary automata in \( \mathcal{P} \). Then set

\[
Q_p^{(j)} = \mathfrak{P}(Q) \times [I + P],
\]

\[
s_p^{(0,j)} = (\sigma(p), 0),
\]

\[
\delta_p^{(j)}((S,i),a_j) = \begin{cases} (T, i + 1) & \text{if } i + 1 < I + P; \\ (T, I) & \text{if } i + 1 = I + P; \end{cases}
\]

where \( S \subseteq Q \), \( i \in [I + P] \), \( j \in \{1, \ldots, k\} \),

\[
T = f(S,a_j) \cup \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma} f\left(\pi_1(\delta_q^{(j)}(s_q^{(0,j)}, a_j^{i+1})), b\right)
\]

and \( F_p^{(j)} = \{(S,i) \mid S \cap F = \emptyset\} \). For a state \((S,i) \in Q_p^{(j)}\) the set \( S \subseteq Q \) will be called the state (set) label, or the state set associated with it.

The reader might consult [Hof20] for examples. The next statement makes precise what we mean by decomposing the state label map along the hyperplanes into the automata \( \mathcal{A}_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p^{(j)}, s_p^{(0,j)}, F_p^{(j)}) \). Moreover, it justifies calling the first component of any state \((S,i) \in Q_p^{(j)}\) also the state set label.

Proposition 20. (state label map decomposition) Suppose \( \Sigma = \{a_1, \ldots, a_k\} \) \( Q \) is a finite set. Let \( \sigma : \mathbb{N}_0^k \rightarrow \mathfrak{P}(Q) \) be a state label map, \( 1 \leq j \leq k \) and \( p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k \). Assume \( \mathcal{P} \in H_j \) is the projection of \( p \) onto \( H_j \), i.e., \( \mathcal{P} = (p_1, \ldots, p_j-1, 0, p_{j+1}, \ldots, p_k) \). Then

\[
\sigma(p) = \pi_1(\hat{\psi}_{\mathcal{P}}(s_{\mathcal{P}}^{\psi_{\mathcal{P}}(0,j)}, a_j^{p_j})),
\]

for the automata \( \mathcal{A}_{\mathcal{P}}^{(j)} = (\{a_j\}, Q_{\mathcal{P}}^{(j)}, \delta_{\mathcal{P}}^{(j)}, s_{\mathcal{P}}^{(0,j)}, F_{\mathcal{P}}^{(j)}) \) from Definition 19.

\(^4\) Note that in the definition of \( \mathcal{P} \), as \( p \in H_j \), we have \( b \neq a_j \) and \( q \in H_j \). In general, points \( q \in \mathbb{N}_0^k \) with \( p = q + \psi(b) \) for some \( b \in \Sigma \) are predecessor points in the grid \( \mathbb{N}_0^k \).

\(^5\) Note max \( \emptyset = 0 \) and lcm \( \emptyset = 1 \).
Proof. Notation as in the statement. Also, let \( f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q) \) be the defining function for the state label map. For \( p = (0, \ldots, 0) \) this is clear. If \( p_j = 0 \), then \( p = \overline{p} \), and, by Equation 3,

\[
\pi_1(\delta_{\overline{p}}^{(j)}(s_{\overline{p}}^{(0,j)}, \varepsilon)) = \pi_1(s_{\overline{p}}^{(0,j)}) = \sigma(p).
\]

Suppose \( p_j > 0 \) from now on. Then, the set \( \{ (q, b) \in \mathbb{N}_0^b \times \Sigma \mid p = q + \psi(b) \} \) is non-empty and we can use Equation 4 and, inductively, that \( \sigma(q) = \pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j})) \), which gives

\[
\sigma(p) = \bigcup_{(q, b) \in \mathbb{N}_0^b \times \Sigma} f(\sigma(q), b)
= \bigcup_{(q, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j})), b)
\tag{8}
\]

where \( q = (q_1, \ldots, q_k) \) and \( \overline{q} = (q_1, \ldots, q_{j-1}, 0, q_{j+1}, \ldots, q_k) \) \( \in H_j \). As \( p_j > 0 \) we have \( p = q + \psi(a_j) \) for some unique point \( q = (p_1, \ldots, p_{j-1}, p_j - 1, p_{j+1}, \ldots, p_k) \). For all other points \( r = (r_1, \ldots, r_k) \) with \( p = r + \psi(b) \) for some \( b \in \Sigma \), the condition \( r \neq q \) implies \( b \neq a_j \) and \( r_j = p_j \) for \( r = (r_1, \ldots, r_k) \). Also, if \( \overline{q} \in H_j \) denotes the projection to \( H_j \), we have \( \overline{q} = \overline{p} \) for our chosen \( q \) with \( p = q + \psi(a_j) \). Hence, taken all this together, we can write Equation (8) in the form

\[
\sigma_A(p) = \bigcup_{(r, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j})), b) \cup \bigcup_{(q, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j-1})), a_j).
\]

Let \( b \in \Sigma \). As for \( a_j \neq b \), we have that \( p = r + \psi(b) \) if and only if \( \overline{p} = \overline{r} + \psi(b) \), with the notation as above for \( p, r; \overline{p} \) and \( \overline{r} = (r_1, \ldots, r_{j-1}, 0, r_{j+1}, \ldots, r_k) \), we can simplify further and write

\[
\sigma_A(p) = \bigcup_{(r, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j})), b) \cup \bigcup_{(q, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j-1})), a_j). \tag{9}
\]

Set \( S = \pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j-1})) \), \( T = \sigma(p) \) and\(^6\)

\[
\mathcal{P} = \{ A_{\overline{q}}^{(j)} \mid \overline{p} = r + \psi(b) \text{ for some } b \in \Sigma \}.
\]

Let \( I \) be the maximal index, and \( P \) the least common multiple of all the periods, of the unary automata in \( \mathcal{P} \). We distinguish two cases for the value of \( p_j > 0 \).

(i) \( 0 < p_j \leq I \).

By Equation 13, \( \delta_{\overline{p}}^{(j)}(s_{\overline{p}}^{(0,j)}, a_j^{p_j-1}) = (S, p_j - 1) \). In this case Equation 14 equals Equation 17, if the state \( (S, p_j - 1) \) is used in Equation 16, i.e.,

\[
T = f(S, a_j) \cup \bigcup_{(r, b) \in \mathbb{N}_0^b \times \Sigma} f(\pi_1(\delta_{\overline{q}}^{(j)}(s_{\overline{q}}^{(0,j)}, a_j^{p_j})), b).
\]

\(^6\) Note that for \( \overline{p} \in H_j \), the condition \( \overline{p} = q + \psi(b) \), for some \( b \in \Sigma \), implies \( q \in H_j \) and \( b \neq a_j \).
This gives
\[ \delta^j_p((S, p_j - 1), a_j) = (T, p_i). \]
Hence \( \pi_1(\delta^j_p((S, p_j - 1), a_j)) = T = \sigma(p)\).

(ii) \( I < p_j\).

Set \( y = I + (p_j - 1 - I) \mod P\). Then \( I \leq y < I + P\). By Equation \(9\),
\[ \delta^j_{\overline{p}}((\overline{p}, a_j^{p_j-1}) = (S, y). \]
So, also by Equation \(9\),

\[ \delta^j_{\overline{p}}((\overline{p}, a_j^{p_j-1}) = \delta^j_{\overline{p}}((S, y), a_j) = \begin{cases} (R, y + 1) & \text{if } I \leq y < I + P - 1 \\ (R, I) & \text{if } y = I + P - 1, \end{cases} \]

where, by Equation \(10\),

\[ R = f(S, a_j) \cup \bigcup_{p \in \overline{p} \psi(b)} f(\pi_1(\delta^j_{\overline{p}}((\overline{p}, a_j^{p_j+1}), b)). \]

Let \( \overline{p} \in H_j \) with \( \overline{p} = \overline{p} + \psi(b) \) for some \( b \in \Sigma \), and \( \overline{p} \in H_j \) the point from the statement of this Proposition. Then, as the period of \( A^j_{\overline{p}} \) divides \( P \), and \( y \) greater than or equal to the index of \( A^j_{\overline{p}} \), we have

\[ \delta^j_{\overline{p}}((\overline{p}, a_j^{p_j-1}) = \delta^j_{\overline{p}}((\overline{p}, a_j^y)). \]

So \( \delta^j_{\overline{p}}((\overline{p}, a_j^{p_j}) = \delta^j_{\overline{p}}((\overline{p}, a_j^{p_j+1})). \) Hence, comparing Equation \(10\) with Equation \(11\), we find that they are equal, and so \( R = T \).

By Proposition \(20\) the state label sets of the axis-parallel rays in \( \mathbb{N}_0 \) correspond to the state set labels of unary automata. Hence, the next is implied.

**Corollary 21.** A state label map is ultimately periodic along each ray. More formally, if \( \sigma : \mathbb{N}_0 \rightarrow \mathcal{P}(\mathcal{Q}) \) is a state label function, \( p \in \mathbb{N}_0 \) and \( j \in \{1, \ldots, k\} \), then the sequence of state sets \( \sigma(p + i \cdot \psi(a_j)) \) for \( i = 0, 1, 2\ldots \) is ultimately periodic.

Our final result in this section is the mentioned regularity condition. It says that if the automata from Definition \(19\) underlying the state set labels, as stated in Proposition \(20\) do not grow, i.e., have a bounded number of states, then we can deduce that the languages we get if we look at the inverse images of the state label map and the Parikh map are regular. This is equivalent with the condition that the state set labels all get periodic behind specific points, i.e., outside of some bounded rectangle in \( \mathbb{N}_0 \).

**Theorem 22.** Let \( \sigma : \mathbb{N}_0 \rightarrow \mathcal{P}(\mathcal{Q}) \) be a state label map and \( \psi : \Sigma^* \rightarrow \mathbb{N}_0 \) be the Parikh map. Suppose for every \( j \in \{1, \ldots, k\} \) and \( p \in H_j \) the automata \( A^j_{\psi(p)} = ([a_j], Q^j_{\psi(p)}, \delta^j_{\psi(p)}, \gamma^j_{\psi(p)}, F^j_{\psi(p)}) \) from Definition \(12\) have a bounded number of states \(4\), i.e., \( |Q^j_{\psi(p)}| \leq N \) for some \( N \geq 0 \) independent of \( p \) and \( j \). Then for \( F \subseteq \mathcal{P}(\mathcal{Q}) \) the commutative language

\[ \psi^{-1}(\sigma^{-1}(F)) \]

\[ \hat{\psi}^{-1}(\sigma^{-1}(F)) \]

\[ \text{Equivalently, the index and period is bounded, which is equivalent with just a finite number of distinct automata, up to semi-automaton isomorphism. We call two semi-automata isomorphic if one semi-automaton can be obtained from the other one by renaming states and alphabet symbols.} \]
is regular and could be accepted by an automaton of size \( \prod_{j=1}^{k} (I_j + P_j) \), where \( I_j \) denotes the largest index among the unary automata \( \{ A_w^{(j)} \mid p \in H_j \} \) and \( P_j \) the least common multiple of all the periods of these automata. In particular, by the relations of the index and period to the states from Section 3.4.1, the state complexity of \( \psi^{-1}(\sigma^{-1}(F)) \) is bounded by \( N^k \).

**Proof.** We use the same notation as introduced in the statement of the theorem. Let \( p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k \) and \( j \in \{1, \ldots, k\} \). Denote by \( \sigma : \mathbb{N}_0 \to \mathcal{P}(Q) \) the state label function from Definition 16. By Proposition 20, if \( p_j \geq I_j \), we have

\[
\sigma(p_1, \ldots, p_{j-1}, p_j + P_j, p_{j+1}, \ldots, p_k) = \sigma(p_1, \ldots, p_k). \tag{11}
\]

Construct the unary semi-automaton \( A_j = (\{ a_j \}, Q_j, \delta_j) \) with

\[
Q_j = \{ s_0^{(j)} \cdot s_1^{(j)} \cdots s_{p_j - 1}^{(j)} \},
\]

\[
\delta_j(s_i^{(j)}, a_j) = \begin{cases} 
    s_{i+1}^{(j)} & \text{if } i < I_j \\
    s_{I_j + (i - I_j + 1) \mod P_j}^{(j)} & \text{if } i \geq I_j.
\end{cases}
\]

Then build \( C = (\Sigma, Q_1 \times \cdots \times Q_k, \mu, s_0, E) \) with

\[
s_0 = (s_0^{(1)}, \ldots, s_0^{(k)}), \\
\mu((t_1, \ldots, t_k), a_j) = (t_1, \ldots, t_{j-1}, \delta_j(t_j, a_j), t_{j+1}, \ldots, t_k) \quad \text{for all } 1 \leq j \leq k,
\]

\[
E = \{ \mu(s_0, u) : \sigma(\psi(u)) \in F \}.
\]

By construction, for words \( u, v \in \Sigma \) with \( u \in \text{perm}(v) \) we have \( \mu((t_1, \ldots, t_k), u) = \mu((t_1, \ldots, t_k), v) \) for any state \( (t_1, \ldots, t_k) \in Q_1 \times \cdots \times Q_k \). Hence, the language accepted by \( C \) is commutative. We will show that \( L(C) = \{ u \in \Sigma^* \mid \sigma(\psi(u)) \in F \} \subseteq L(C) \). Conversely, suppose \( w \in L(C) \). Then \( \mu(s_0, w) = \mu(s_0, u) \) for some \( u \in \Sigma^* \) with \( \sigma(\psi(u)) \in F \). Next, we will argue that we can find \( u' \in L(C) \) and \( \psi(u') \in F \) with \( \sigma(\psi(u')) \in F \), \( \mu(s_0, w') = \mu(s_0, w) = \mu(s_0, u) = \mu(s_0, u') \) and \( \max\{|w'|_{a_j}, |u'|_{a_j}\} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \).

(i) By construction of \( C \), if \( |w|_{a_j} \geq I_j + P_j \), we can find \( u' \) with \( |u'|_{a_j} = |w|_{a_j} - P_j \) such that \( \mu(s_0, w') = \mu(s_0, w) \). So, applying this procedure repeatedly, we can find \( u' \in \Sigma^* \) with \( |u'|_{a_j} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \) and \( \mu(s_0, w') = \mu(s_0, w) \).

(ii) If \( |w|_{a_j} \geq I_j + P_j \), by Equation 11, we can find \( u' \) with \( |u'|_{a_j} = |w|_{a_j} - P_j \) and \( \sigma(\psi(u')) \in F \). By construction of \( C \), we have \( \mu(s_0, u) = \mu(s_0, u') \). So, after repeatedly applying the above steps, we find \( u' \in \Sigma^* \) with \( \sigma(\psi(u')) \in F \), \( \mu(s_0, u) = \mu(s_0, u') \) and \( u'|_{a_j} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \).

By construction of \( C \), for words \( u, v \in \Sigma^* \) with \( \max\{|w|_{a_j}, |v|_{a_j}\} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \), we have

\[
\mu(s_0, u) = \mu(s_0, v) \iff u \in \text{perm}(v) \iff \psi(u) = \psi(v). \tag{12}
\]

Hence, using Equation 12 for the words \( w' \) and \( u' \) from (i) and (ii) above, as \( \mu(s_0, u') = \mu(s_0, w') \), we find \( \psi(u') = \psi(w') \). So \( \sigma(\psi(w')) = \sigma(\psi(u')) \in F \). Now, again using Equation 11, this gives \( \sigma(\psi(w)) \in F \). 

I refer to [Hof20] for examples and more explanations.

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8 The term semi-automaton is used for automata without a designated initial state, nor a set of final states.
A.4 Automata Induced State Label Maps

We call a state label map \( \sigma : \mathbb{N}_0^3 \rightarrow \mathcal{P}(Q) \) given by a function \( f : \mathcal{P}(Q) \times \Sigma \rightarrow \mathcal{P}(Q) \) an automaton induced state label map, if there exists some semi-automaton \( A = (\Sigma, Q, \delta) \) such that \( \delta(S,a) \subseteq f(S,a) \) for each \( a \in \Sigma \). We also say that such an (semi-)automaton is compatible with the state label map. This gives inductively that \( \delta(S,w) \subseteq f(S,w) \) for each word \( w \in \Sigma^* \) and set \( S \subseteq Q \).

**Lemma 23.** Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton and suppose the state label map \( \sigma : \mathbb{N}_0^3 \rightarrow \mathcal{P}(Q) \) is compatible with \( A \). Let \( p,q \in \mathbb{N}_0^3 \) with \( q < p \), then \( \delta(\sigma(q),w) \subseteq \sigma(p) \) for each \( w \in \Sigma^* \) with \( p = \psi(w) + q \).

**Proof.** Let \( w \in \Sigma^* \) with \( p = \psi(w) + q \). Set \( n = |w| \). As \( q < p \) we have \( 1 \leq n \leq p_1 + \ldots + p_k \). Hence, by Lemma 17 \( f(\sigma(q),w) \subseteq \sigma(p) \). As the state label map is compatible with \( A \), we have \( \delta(\sigma(q),w) \subseteq f(\sigma(q),w) \).

Our most important result, which generalizes a corresponding result from [Hof20] to automata induced state label maps, is stated next.

**Proposition 24.** Let \( A = (\Sigma, Q, \delta, s_0, F) \) be a permutation automaton and \( \sigma : \mathbb{N}_0^3 \rightarrow \mathcal{P}(Q) \) a state map compatible with \( A \). Then for every automaton \( A^{(j)}_\psi \) from Definition 19 its index equals at most \( (|Q| - 1)L_j \) and its period is divided by \( L_j \), where \( L_j \) denotes the order of the letter \( a_j \) viewed as a permutation of \( Q \), i.e., \( \delta(q,a_j^{|Q|}) = q \) for any \( q \in Q \) and \( L_j \) is minimal with this property.

**Proof.** It might be helpful for the reader to have some idea of how the symmetric group (or any permutation group) acts on subsets of its permutation domain, see for example [Cam99] for further information. We also say that the letter \( a_j \) acts (or operates) on a subset \( S \subseteq Q \), the action being given by the transition function \( \delta : Q \times \Sigma \rightarrow Q \), where \( \delta(S,a_j) \) is the result of the action of \( a_j \) on \( S \). Set

\[
\mathcal{P} = \{ A^{(j)}_\psi \mid p = q + \psi(b) \text{ for some } b \in \Sigma \}.
\]

Denote by \( I \) the maximal index and by \( P \) the least common multiple of the periods of the unary automata in \( \mathcal{P} \).

First the case \( \mathcal{P} = \emptyset \), which is equivalent with \( p = (0, \ldots, 0) \). In this case, \( I = 0, P = 1, Q^{(j)}_\psi = \mathcal{P}(Q) \times \{0\} \) and Equation (6) reduces to

\[
A^{(j)}_\psi((S,0),a_j) = (f(S,a_j),0)
\]

for \( S \subseteq Q \). As the state label map is compatible with \( A \), we have \( \delta(S,a_j) \subseteq f(S,a_j) \).

So, as \( a_j \) permutes the states \( Q \), if \( |S| = |f(S,a_j^n)| \) for \( n \geq 0 \), then \( f(S,a_j^n) = \delta(S,a_j^n) \).

As for each \( S \subseteq Q \) we have \( \delta(S,a_j^{|Q|}) = S \), if \( |f(S,a_j^n)| = |S| \), which gives \( f(S,a_j^n) = \delta(S,a_j^n) \), we find \( 0 \leq m < L_j \) with \( f(S,a_j^m) = f(S,a_j^n) \). Let

\[
R = \{ n > 0 : |f(\sigma(p),a_j^{n-1})| < |f(\sigma(p),a_j^n)| \}.
\]

If \( R = \emptyset \), then \( f \) does not add any states as symbols are read, and the automaton \( A^{(j)}_\psi \) is essentially the action of \( a_j \) starting on the set \( \sigma(p) \), i.e., the orbit \( \{ \sigma(p), \delta(\sigma(p),a_j), \delta(\sigma(p),a_j^2), \ldots \} \).

\footnote{For every automaton \( A = (\Sigma, Q, \delta, s_0, F) \) we can consider the corresponding semi-automaton \( A = (\Sigma, Q, \delta) \) and we will do so without special mentioning.}
Hence we have index zero and some period dividing \( L_j \), as the letter \( a_j \) is a permutation of order \( L_j \) on \( Q \). If \( R \neq \emptyset \), then \( R \) is finite, as the sets could not grow indefinitely. Let \( m = |R| \) and write \( R = \{ n_i \mid i \in \{1, \ldots, m \} \} \) with \( n_i < n_{i+1} \) for \( i \in \{1, \ldots, m - 1\} \), i.e., the sequence orders the elements from \( R \). We have \( n_{i+1} - n_i \leq L_j \) and \( n_1 \leq L_j \), for if \( n_i \leq k < n_{i+1} \) (or \( k < n_1 \)), then with \( S = f(\sigma(p), a_j^0) \) (or \( S = \sigma(p) \)), as argued previously, we find \( f(S, a_j^1) = \delta(S, a_j^1) \). Assuming \( n_{i+1} - n_i = L_j \) (or similarly \( n_1 > L_j \)) would then yield \( f(S, a_j^{L_j}) = S \), and so for every \( k \geq n_i \), writing \( k = qL_j + r \), we have \( f(S, a_j^k) = f(S, a_j^r) \) and the cardinalities could not grow anymore, i.e., we would be stuck in a cycle. So by definition of \( R \), \( |\sigma(p)| < |f(\sigma(p), a_j^{n_1}| < \ldots < |f(\sigma(p), a_j^{n_m}| \leq |Q| \). This gives \( m \leq |Q| - |\sigma(p)| \). By choice, for \( n \geq n_m \) we have \( |f(\sigma(p), a_j^{n_m})| = |\sigma(p), a_j^{n_m})| \). Hence it is again just the action of \( a_j \) starting on the subset \( f(\sigma(p), a_j^{n_m}) \). So we are in the cycle, and the period of \( A_p^{(j)} \) divides \( L_j \), as the operation of \( A_p^{(j)} \) could be identified with the function \( f : P(Q) \times \Sigma \to P(Q) \) for \( p = (0, \ldots, 0) \). Note that \( n_m \) is precisely the index of \( A_p^{(j)} \), and by the previous considerations \( n_m \leq (|Q| - |\sigma(p)|)L_j \).

So, now suppose \( P \neq \emptyset \). We split the proof into several steps. Note that the statements (ii), (iii), (iv), (v) written below are also proven by the above considerations for the case \( P = \emptyset \). Hence, we can argue inductively in their proofs. Let \( S, T \subseteq Q \).

(i) **Claim:** If \((T, y) = \delta_p^{(j)}((S, x), a_j^r)\) for some \( r \geq 0 \), then \(|T| \geq |S|\). In particular, the state labels of cycle states all have the same cardinality.

**Proof of Claim (i):** By Equations (7), \( f(S, a_j^r) \subseteq \pi_1(\delta_p^{(j)}((S, x), a_j^r)) \). As the state map is compatible with \( A \), we have \( \delta(S, a_j^r) \subseteq f(S, a_j^r) \), and as \( a_j \) is a permutation on the states, we have \(|S| = |\delta(S, a_j^r)| \). Hence \(|S| \leq |\pi_1(\delta_p^{(j)}((S, x), a_j^r))| \), which gives the claim inductively. As states on the cycle could be mapped to each other, the state labels from cycle states all have the same cardinality.

(ii) **Claim:** Let \( L_S = \text{lcm}\{\{|\delta(s, a_j^r)| : s \in S\} : r \geq 0\} \), i.e. the least common multiple of the orbit length\(^{10}\) of all elements in \( S \). For \( x \geq I \) and \((T, y) = \delta_p^{(j)}((S, x), a_j^{\text{lcm}(P, L_S)})) \), if \(|T| = |S|\), then \((T, y) = (S, x)\). So, by Lemma 19 the period of \( A_p^{(j)} \) divides \( \text{lcm}(P, L_S) \).

**Proof of Claim (ii):** From Equation 9 of Definition 10 and the fact that the state map is compatible with \( A \), we get inductively \( \delta(S, a_j^i) \subseteq f(S, a_j^i) \subseteq \pi_1(\delta_p^{(j)}((S, x), a_j^i)) \) for all \( i \geq 0 \). So, as \( \delta(s, a_j^{i-r}) = s \) for all \( s \in S \), this gives \( S \subseteq T \). Hence, as \(|S| = |T|\), we get \( S = T \). Furthermore, as \( x \geq I_i \), by Equation 9 of Definition 10 as \( P \) divides \( \text{lcm}(P, L_S) \), we have \( x = y \). By Lemma 19 this implies that the period of \( A_p^{(j)} \) divides \( \text{lcm}(P, L_S) \).

(iii) **Claim:** With the notation from (ii), the number \( \text{lcm}(P, L_S) \) divides \( L_j \) and the period of \( A_p^{(j)} \) divides \( L_j \).

**Proof of Claim (iii):** With the notation from (ii), as \( L_j = \text{lcm}\{\{|\delta(q, a_j^r)| : r \geq 0\} : q \in Q\} \), \( L_S \) divides \( L_j \). Inductively, the periods of all unary automata in \( P \) divide

---

\(^{10}\)For a permutation \( \pi : [n] \to [n] \) on a finite set \([n]\) and \( m \in [n] \), the orbit length of \( m \) under the permutation \( \pi \) is \(|\pi^i(m) : i \geq 0|\). In [Hof20], the orbit length of an element is also called the cycle length of that element, as it is precisely the size of the unique cycle in which the element \( m \) appears with respect to the permutation.
For \( P \) the least common multiple of these periods, also \( P \) divides \( L_j \). Hence \( \text{lcm}(P, L_j) \) divides \( L_j \). So, with Claim (ii), the period of \( \mathcal{A}_p^{(j)} \) divides \( L_j \).

(iv) **Claim:** For \( x \geq I \) and \( (T, y) = \delta_p^{(j)}((S, x), a_j^{L_j}) \), if \( |T| = |S| \), then \( (T, y) = (S, x) \).

**Proof of Claim (iv):** With the notation from (ii) and Claim (iii), we can write \( L_j = m \cdot \text{lcm}(P, L_j) \) for some natural number \( m \geq 1 \). Set \( (R, z) = \delta_p^{(j)}((S, x), a_j^{\text{lcm}(P, L_j)}) \).

By (i), we have \( |S| \leq |R| \leq |T| \). By assumption \( |S| = |T| \), hence \( |S| = |R| \). So, we can apply (ii), which yields \( (R, z) = (S, x) \). Applying this repeatedly \( m \) times gives \( (T, y) = (S, x) \).

(v) **Claim:** If \( T \) is the state label of any cycle state of \( \mathcal{A}_p^{(j)} \), then the index of \( \mathcal{A}_p^{(j)} \) is bounded by \( (|T| - 1)L_j \).

**Proof of Claim (v):** We define a sequence \( (T_n, y_n) \in Q_p^{(j)} \) of states for \( n \in \mathbb{N}_0 \). Set \( (T_0, y_0) = \delta_p^{(j)}((S, 0), a_j^{L_j}) \), which implies \( y_0 = I \) by Equation (3), and

\[
(T_n, y_n) = \delta_p^{(j)}((T_{n-1}, y_{n-1}), a_j^{L_j})
\]

for \( n > 0 \). Note that, as \( P \) divides \( L_j \), by Equation (6), we have \( y_n = I \) for all \( n \geq 0 \).

**Claim 1:** Let \( (T, x) \in Q_p^{(j)} \) be some state from the cycle of \( \mathcal{A}_p^{(j)} \). Then the state \( (T|\tau|\pi_\tau y_0|\pi_\tau y_0|\pi_\tau y_0) = \delta_p^{(j)}((s_p^{(0, j)}, a_j^{L_j})) = \delta_p^{(j)}((s_p^{(0, j)}, a_j^{L_j})) \) is also from the cycle of \( \mathcal{A}_p^{(j)} \).

By construction, and Equation (10) from Definition (19) we have \( y_n \geq I \) for all \( n \). If \( T_{n+1} \neq T_n \), then, by (iv) and (i), we have \( |T_{n+1}| > |T_n| \) (remember \( y_n = y_{n+1} = I \)). Hence\(^{11} \) by finiteness, we must have a smallest \( m \) such that \( T_{n+1} = T_m \). As also \( y_{n+1} = y_{n+1} \), we are on the cycle of \( \mathcal{A}_p^{(j)} \), and the period of this automaton divides \( L_j \) by (iv). This yields \( (T_n, y_n) = (T_m, y_m) \) for all \( n \geq m \). By (i), the size of the state label sets on the cycle stays constant, and just grows before we enter the cycle. As we could add at most \( |T_m| - |T_0| \) elements, and for \( T_0, T_1, \ldots, T_m \) each time at least one element is added, we have, as \( m \) was chosen minimal, that \( m \leq |T| - |T_0| \), where \( T \) is any state label on the cycle, all which have the same cardinality \( |T| = |T_0| \) by (i). This means we could read at most \( |T| - |T_0| \) times the sequence \( a_j^{L_j} \), starting from \( (T_0, I) \), before we enter the cycle of \( \mathcal{A}_p^{(j)} \).

**Claim 2:** We have \( I \leq (|T_0| - 1)L_j \).

Remember, the case \( P = \emptyset \) was already handled, for then \( p = (0, \ldots, 0) \) and \( I = 0 \). Otherwise, let \( \mathcal{A}_q^{(j)} \in \mathcal{P} \) with \( p = q + \psi(b) \) for \( b \in \Sigma \). Let \( (S, x) = \delta_q^{(j)}((s_q^{(0, j)}, a_j^n), a_j^{L_j}) \) with \( n \geq 0 \). If \( n > 0 \), by Equation (11) from Definition (19) we have, with \( (R, z) = \delta_q^{(j)}((s_q^{(0, j)}, a_j^{L_j})) \),

\[
\pi_1(\delta_q^{(j)}((s_q^{(0, j)}, a_j^n))) = \pi_1(\delta_q^{(j)}((R, z), a_j^n)) = f(R, a_j^n) \cup \bigcup_{(r, a) \in (s_q^{(0, j)}, a_j^{L_j}), \pi_1(\pi_1(\delta_q^{(j)}(s_q^{(0, j)}, a_j^n)), a_j^{L_j}))} f(\pi_1(\delta_q^{(j)}((s_q^{(0, j)}, a_j^n)), a_j^{L_j})).
\]

If \( n = 0 \), we have

\[
\pi_1(\delta_q^{(j)}((s_q^{(0, j)}, a_j^n))) = \pi_1((\psi(p), 0)) = \psi(p).
\]

\(^{11} \) Also, as \( T_0 = \delta(T_0, a_j^{L_j}) \leq f(T_0, a_j^{L_j}) \leq \pi_1(\delta_p^{(j)}((T_0, I), a_j^{L_j})), \) we find \( T_0 \subseteq T_{n+1} \).
In the latter case, also \( S = \pi_1(\delta_{i,j}^{(0)}(s_q^{(0,j)}, a_j^n)) = \sigma(q) \) and, as \( p \neq (0, \ldots, 0) \) (which is equivalent to \( P \neq \varnothing \)), by Equation (4) and as the state label map is compatible with \( \mathcal{A} \), we have \( \delta(S,b) \subseteq f(S,b) \subseteq \sigma(p) \). In the former case \( n > 0 \),

\[
\delta(S,b) \subseteq f(S,b) \subseteq \bigcup_{(r,a) \in \mathbb{N}_0^n \times \Sigma} f(\pi_1(\delta_{i,j}^{(0)}(s_r^{(0,j)}, a_j^n)), a)
\]

So, in any case, \( \delta(S,b) \subseteq \pi_1(\delta_{i,j}^{(0)}(s_q^{(0,j)}, a_j^n)) \). In particular for \( n = I \) we get \( \delta(S,b) \subseteq T_0 \), and as \( b \) induces a permutation on the states, this gives \( |S| \leq |T_0| \). Also for \( n \geq I \), we are on the cycle of \( A_i^{(1)} \). Hence, inductively, the index of \( A_i^{(1)} \) is at most \((|S| - 1)L_j \leq (|S| - 1)L_j \). As \( A_i^{(1)} \in \mathcal{P} \) was chosen arbitrary, we get \( I \leq (|T_0| - 1)L_j \).

With Claim (2) above, we can derive the upper bound \((|T| - 1)L_j \) for the length of the word \( a_j^{I+((T_0|-T_0))} \) from Claim (1), as

\[
I + (|T| - |T_0|)L_j \leq (|T_0| - 1)L_j + (|T| - |T_0|)L_j = (|T| - 1)L_j.
\]

And as Claim (1) essentially says that the index of \( A_i^{(1)} \) is smaller than \( I + (|T| + |T_0|)L_j \), this gives Claim (v). Also, as \( |T| \leq |Q| \), the claim about the index of the statement in Proposition 24 is proven. So, in total, (iii) and (v) give Proposition 24.

\[\square\]

A.5 The State Label Method for Iterated Shuffle

So, after we have introduced the state label method, state label maps that are compatible with an automaton and a general regularity-criterion for permutation automata, we are ready to give the proof of Theorem 9.

First, we need to define our state label map.

**Definition 25.** Let \( \mathcal{A} = (\Sigma, Q, \delta, s_0, F) \) be a finite automaton. Denote by \( \sigma_{\mathcal{A},+} : \mathbb{N}_0^I \rightarrow \mathcal{P}(Q) \) the state label function given by \( f : \mathcal{P}(Q) \times \Sigma \rightarrow \mathcal{P}(Q) \), where

\[
f(Q,a) = \begin{cases} 
\delta(Q,a) \cup \{s_0\} & \text{if } \delta(Q,a) \cap F \neq \varnothing; \\
\delta(Q,a) & \text{otherwise};
\end{cases}
\]

and \( \sigma_{\mathcal{A},+}(0, \ldots, 0) = \{s_0\} \).

To derive our results, we need the following formula for the image of the state label map at a given point.

**Proposition 26.** Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( \mathcal{A} = (\Sigma, Q, \delta, s_0, F) \) be a finite automaton. For the state-label function from Definition 25 we have

\[
\sigma_{\mathcal{A},+}(p) = \begin{cases}
A_p \cup B_p & \text{if } (A_p \cup B_p) \cap F = \varnothing; \\
A_p \cup B_p \cup \{s_0\} & \text{otherwise};
\end{cases}
\]

where \( A_p = \{\delta(s_0, w) \mid \psi(w) = p\} \) and \( B_p = \{\delta(s_0, w) \mid \exists q \in \mathbb{N}_0^I : q < p \text{ and } q + \psi(w) = p \text{ and } \sigma_{\mathcal{A},+}(q) \cap F \neq \varnothing\} \).
Proof. For $p = (0, \ldots, 0)$ the statement is clear. If $p \neq (0, \ldots, 0)$, then by definition

$$\sigma_{A^+}(p) = \bigcup_{(q,b) \in \mathbb{N}_0^k} f(\sigma_{A^+}(q), b).$$

For $q$ with $p = q + \psi(b)$ for some $b \in \Sigma$ set

$$A_q = \{\delta(s_0, w) \mid \psi(w) = q\}$$

$$B_q = \{\delta(s_0, w) \mid \exists r \in \mathbb{N}_0^k : r < q \text{ and } r + \psi(w) = q \text{ and } \sigma_{A^+}(r) \cap F \neq \emptyset\}.$$

Inductively,

$$\sigma_{A^+}(q) = \begin{cases} A_q \cup B_q & \text{if } (A_q \cap B_q) \cap F = \emptyset, \\ A_q \cup B_q \cup \{s_0\} & \text{otherwise.} \end{cases}$$

Hence, using Definition 25, $f(\sigma_{A^+}(q), b)$ equals

$$\begin{cases} \delta(A_q \cup B_q, b) & \text{if } (A_q \cup B_q) \cap F = \emptyset, \delta(A_q \cup B_q, b) \cap F = \emptyset, \\ \delta(A_q \cup B_q, b) \cup \{s_0\} & \text{if } (A_q \cup B_q) \cap F = \emptyset, \delta(A_q \cup B_q, b) \cap F \neq \emptyset, \\ \delta(A_q \cup B_q \cup \{s_0\}, b) & \text{if } (A_q \cup B_q) \cap F \neq \emptyset, \delta(A_q \cup B_q \cup \{s_0\}, b) \cap F = \emptyset, \\ \delta(A_q \cup B_q \cup \{s_0\}, b) \cup \{s_0\} & \text{if } (A_q \cup B_q) \cap F \neq \emptyset, \delta(A_q \cup B_q \cup \{s_0\}, b) \cap F \neq \emptyset. \end{cases}$$

(15)

Under the induction hypothesis, i.e., that the formula holds true for $q < p$, in particular if $p = q + \psi(b)$ for some $b \in \Sigma$, we prove various claims that we use to derive our final formula.

Claim 1: For $q \in \mathbb{N}_0^k$ with $p = q + \psi(b)$ for some $b \in \Sigma$ we have

$$(A_q \cup B_q) \cap F \neq \emptyset \Leftrightarrow \sigma_{A^+}(q) \cap F \neq \emptyset.$$  

Proof of the claim. If $(A_q \cup B_q) \cap F \neq \emptyset$, then $\sigma_{A^+}(q) \cap F \neq \emptyset$ by induction hypothesis. If $\sigma_{A^+}(q) \cap F \neq \emptyset$, assume $(A_q \cup B_q) \cap F = \emptyset$. Then, using inductively that the formula holds true for $q$, this gives $\sigma_{A^+}(q) = A_q \cup B_q$, which implies $(A_q \cup B_q) \cap F \neq \emptyset$. Hence, this is not possible and we must have $(A_q \cup B_q) \cap F \neq \emptyset$. \hfill \Box

Claim 2: We have

$$A_p = \bigcup_{(q,b) \in \mathbb{N}_0^k} \delta(A_q, b),$$

$$B_p = \bigcup_{(q,b) \in \mathbb{N}_0^k} \delta(B_q, b) \cup \bigcup_{(q,b) \in \mathbb{N}_0^k} \delta(\{s_0\}, b).$$

Proof of the claim. The first equation is obvious. For the other, first let

$$\delta(s_0, w) \in B_p$$

for some $w \in \Sigma^*$. Then, we have $r \in \mathbb{N}_0^k$ such that

$$r < p, \quad r + \psi(w) = p, \quad \text{and } \sigma_{A^+}(r) \cap F \neq \emptyset.$$

Write $w = ub$ with $b \in \Sigma$ (note that by definition of the sets $B_q$ we have $|w| > 0$ here). If $r < p - \psi(b)$, then $\delta(s_0, u) \in B_{p-\psi(b)}$ and so $\delta(s_0, w) \in \delta(B_{p-\psi(b)}, b)$.

Otherwise $r = p - \psi(b)$, which implies $u = \varepsilon$ and $w = b$. In this case,

$$\delta(s_0, b) \in \bigcup_{(q,b) \in \mathbb{N}_0^k} \delta(\{s_0\}, b).$$
Hence, $B_p$ is included in the set on the right hand side. The inclusion of the other two sets in $B_p$ is obvious. □

Claim 3: We have $(A_p \cup B_p) \cap F \neq \emptyset$ if and only if there exists $q \in \mathbb{N}_0^\ast$ and $b \in \Sigma^\ast$ with $p = q + \psi(b)$ such that at least one of the conditions is fulfilled:

(1) $(A_q \cup B_q) \cap F = \emptyset$ and $\delta(A_q \cup B_q, b) \cap F \neq \emptyset$.
(2) $(A_q \cup B_q) \cap F \neq \emptyset$ and $\delta(A_q \cup B_q \cup \{s_0\}, b) \cap F \neq \emptyset$.

Proof of the claim. Assume $(A_p \cup B_p) \cap F \neq \emptyset$. We distinguish the two cases $A_p \cap F \neq \emptyset$ or $B_p \cap F \neq \emptyset$. First, suppose $A_p \cap F \neq \emptyset$. By Claim (2) then $\delta(A_q,b \cap F \neq \emptyset$ for some $q \in \mathbb{N}_0^\ast$ and $b \in \Sigma$ with $p = q + \psi(b)$. As $\delta(A_q, b) \subseteq \delta(A_q \cup B_q, b) \subseteq \delta(A_q \cup B_q \cup \{s_0\}, b)$, both conditions (1) and (2) are fulfilled. Now, suppose $B_p \cap F \neq \emptyset$. Using Claim (2), we have two cases.

1. It is $\delta(B_q, b) \cap F \neq \emptyset$ for some $q \in \mathbb{N}_0^\ast$ and $b \in \Sigma$ with $p = q + \psi(b)$.

As $\delta(B_q, b) \subseteq \delta(A_q \cup B_q, b) \subseteq \delta(A_q \cup B_q \cup \{s_0\}, b)$, both conditions (1) and (2) are fulfilled.

2. We find, also using Claim (1), some $q \in \mathbb{N}_0^\ast$ and $b \in \Sigma$ with $p = q + \psi(b)$ and $(A_q \cup B_q) \cap F \neq \emptyset$ such that $\delta(s_0, b) \in F$.

Then condition (2) is fulfilled.

Conversely, assume condition (1) is fulfilled. Then, by Claim (2), we have $A_p \cap F \neq \emptyset$ or $B_p \cap F \neq \emptyset$. Otherwise, assume condition (2) is fulfilled. If $\delta(A_q \cup B_q, b) \cap F \neq \emptyset$, we have $(A_p \cup B_p) \cap F \neq \emptyset$ as before. So, assume $\delta(A_q \cup B_q, b) \cap F = \emptyset$. But then, we must have $\delta(s_0, b) \in F$, using Claim (2), which gives, as $(A_q \cup B_q) \cap F \neq \emptyset$ and using Claim (1) and Claim (2), that $\delta(s_0, b) \in B_p$, hence $B_p \cap F \neq \emptyset$. □

First, assume $(A_p \cup B_p) \cap F = \emptyset$. Then, by Equation (15) together with Claim (3) and Equation (14),

$$
\sigma_{A,+}(p) = \bigcup_{(q,b)} f(\sigma_{A,+}(q), b)
= \bigcup_{(q,b)} \delta(A_q \cup B_q, b) \cup \bigcup_{(q,b)} \delta(\{s_0\}, b).
$$

By Claim (1) and Claim (2), we get $\sigma_{A,+}(p) = A_p \cup B_p$. Otherwise, if $(A_p \cup B_p) \cap F \neq \emptyset$, by Equation (15) together with Claim (3) and Equation (14),

$$
\sigma_{A,+}(p) = \bigcup_{(q,b)} f(\sigma_{A,+}(q), b)
= \{s_0\} \cup \bigcup_{(q,b)} \delta(A_q \cup B_q, b) \cup \bigcup_{(q,b)} \delta(\{s_0\}, b).
$$

As above, this equals $\{s_0\} \cup A_p \cup B_p$. □

The next statement gives a connection between the Parikh image of $L(A)^\ast$ and $\sigma_{A,+} : \mathbb{N}_0^\ast \rightarrow \mathcal{P}(Q)$. 


Proposition 27. Let \( A = (\Sigma, Q, \delta, s_0, F) \) be a finite automaton. Then \( \psi(L(A)^*) = \sigma_{A,+}^{-1}\left(\{S \subseteq Q \mid S \cap F \neq \emptyset\}\right) \cup \{(0,\ldots,0)\} \).

Proof. First suppose \( p \in \psi(L(A)^*) \). Then either \( p = (0,\ldots,0) \) or we find \( p_1,\ldots,p_n \) with \( n > 0 \), \( p = p_1 + \ldots + p_n \) and words \( w_1,\ldots,w_n \) with \( p_i = \psi(w_i) \) and \( w_i \in L(A) \) for \( i \in \{1,\ldots,n\} \). If \( p \neq (0,\ldots,0) \), then we can assume \( w_i \neq \varepsilon \) for \( i \in \{1,\ldots,n\} \), which is equivalent with \( p_i \neq (0,\ldots,0) \). Note that, with the notation from Proposition 26, for any \( p \in \mathbb{N}_0^n \)

\[
\sigma_{A,+}(p) \cap F \neq \emptyset \iff (A_p \cup B_p) \cap F \neq \emptyset. \tag{16}
\]

For if \( \sigma_{A,+}(p) \cap F = \emptyset \) then obviously \( (A_p \cup B_p) \cap F = \emptyset \). And if \( (A_p \cup B_p) \cap F = \emptyset \) holds true, then \( \sigma_{A,+}(p) \neq \emptyset \) implies \( s_0 \in \sigma_{A,+}(p) \).

Claim: For \( i \in \{1,\ldots,n\} \) we have \( \sigma_{A,+}(p_1 + \ldots + p_{i-1}) \cap F \neq \emptyset \).

Proof of the claim. As \( w_1 \in L(A) \) and for every \( w \in \Sigma^* \) by Definition 25 and Lemma 17 we have \( \delta(s_0, w) \in f(\{s_0\}, w) \subseteq \sigma_{A,+}(\psi(w)) \), we get \( \delta(s_0, w_1) \in \sigma_{A,+}(p_1) \). Hence \( \sigma_{A,+}(p_1) \cap F \neq \emptyset \) as \( \delta(s_0, w_1) \in F \). Now, suppose inductively that for \( i \in \{1,\ldots,n-1\} \) we have

\[\sigma_{A,+}(p_1 + \ldots + p_i) \cap F \neq \emptyset.\]

By Equation 16 and the remarks thereafter, \( s_0 \in \sigma_{A,+}(p_1 + \ldots + p_i) \). By Definition 25 and Lemma 17 then, as \( p_1 + \ldots + p_i + p_{i+1} = p_1 + \ldots + p_i + \psi(w_{i+1}) \),

\[
\delta(s_0, w_{i+1}) \in \delta(\sigma_{A,+}(p_1 + \ldots + p_i), w_{i+1})
\]

\[
\subseteq f(\sigma_{A,+}(p_1 + \ldots + p_i), w_{i+1}) \quad \text{[Definition 25]}
\]

\[
\subseteq \sigma_{A,+}(p_1 + \ldots + p_i + p_{i+1}). \quad \text{[Lemma 17]}
\]

As \( \delta(s_0, w_{i+1}) \in F \) we find \( \sigma_{A,+}(p_1 + \ldots + p_i + p_{i+1}) \cap F \neq \emptyset. \]

With the above claim, for \( i = n \), we find \( \sigma_{A,+}(p) \cap F \neq \emptyset \).

Conversely, assume \( \sigma_{A,+}(p) \cap F \neq \emptyset \) or \( p = (0,\ldots,0) \). In the latter case we have \( p \in \psi(L(A)^*) \) by definition of the star operation. Hence, assume the former holds true. If \( p \in \psi(L(A)) \subseteq \psi(L(A)^*) \) we have nothing to prove. So, assume \( p \notin \psi(L(A)) \). Then, we claim the next.

Claim: There exists \( q \in \mathbb{N}_0^n \) with \( q < p \) such that \( p = q + \psi(w) \) for some \( w \in L(A) \) and \( \sigma_{A,+}(q) \cap F \neq \emptyset \).

Proof of the claim. As \( p \notin \psi(L(A)) \), we have

\[
\{\delta(s_0, w) \mid \psi(w) = p\} \cap F = \emptyset.
\]

Set \( B_p = \{\delta(s_0, w) \mid \exists q < p, \psi(w) + q = p, \sigma_{A,+}(q) \cap F \neq \emptyset\} \). Assume \( B_p \cap F = \emptyset \), then by Proposition 26 this implies \( \sigma_{A,+}(p) = \{\delta(s_0, w) \mid \psi(w) = p\} \cup B_p \). But then, as \( \sigma_{A,+}(p) \cap F \neq \emptyset \), this is not possible and we must have \( B_p \cap F \neq \emptyset \), which gives the claim.

By the above claim, choose \( q \in \mathbb{N}_0^n \) with \( q < p \) and \( p = q + \psi(w) \) for some \( w \in L(A) \) and \( \sigma_{A,+}(q) \cap F \neq \emptyset \). By induction hypothesis, we find \( u \in L(A)^* \) with \( \psi(u) = q \).

Then \( p = \psi(u) + \psi(w) = \psi(uw) \) and we have \( uw \in L(A)^* \), i.e. \( p \in \psi(L(A)^*) \).

We will also need the next lemma.
Lemma 28. Let $\Sigma = \{a_1, \ldots, a_k\}$ and $L \subseteq \Sigma^*$ be a regular language with $sc(L) = n$. Then $sc(L \cup \{\varepsilon\}) \leq n + 1$ and this bound is sharp. If $L$ is commutative with index vector $(i_1, \ldots, i_k)$, then the index vector of $L \cup \{\varepsilon\}$ is at most $(i_1 + 1, \ldots, i_k + 1)$ and both languages have the same period.

Proof. Let $A = (\Sigma, Q, \delta, s_0, F)$ be an automaton for $L$. Choose $s_0' \notin Q$ and construct $A' = (\Sigma, Q \cup \{s_0'\}, \delta', s_0', F \cup \{s_0'\})$ with

$$\delta'(s_0', x) = \delta(s_0, x)$$

for $x \in \Sigma$, and $\delta'(q, x) = \delta(q, x)$ for $q \in Q$, $x \in \Sigma$.

1. $L(A') \subseteq L \cup \{\varepsilon\}$.
   Let $w \in \Sigma^*$ be a word with $\delta'(s_0', w) \in F \cup \{s_0'\}$. By construction, if $\delta'(s_0', w) = s_0'$, then $w = \varepsilon$. Otherwise, if $\delta'(s_0', w) \neq s_0'$, then $|w| > 0$ and $\delta'(s_0', w) = \delta(s_0, w)$. So $w \in L$.

2. $L \cup \{\varepsilon\} \subseteq L(A')$.
   As $s_0'$ is a final state, the empty word is accepted. Now suppose that $w \in L \setminus \{\varepsilon\}$. Hence $\delta(s_0, w) \in F$. Then, as $\delta'(s_0, w) = \delta(s_0, w)$, we have $w \in L(A')$.

Let $m > 0$. That the bound is sharp is demonstrated by the (unary group) language $L = a^{m-1}(a^n)^*$. We have $sc(L) = m$ and $sc(L \cup \{\varepsilon\}) = m + 1$. Note that $L \cup \{\varepsilon\}$ is in general not a group language anymore. If $A$ is the minimal commutative automaton from [Hof19, Hof, GA08] it is easy to see that the above construction increases the index for each letter by one, but leaves the period untouched.

With this, we can derive our state complexity bound for the combined operation of the commutative closure and of the shuffle closure on group languages.

Theorem 9. Let $\Sigma = \{a_1, \ldots, a_k\}$ and $A = (\Sigma, Q, \delta, q_0, F)$ be a permutation automaton. Then

$$perm(L(A)^{\mu,*})$$

is recognizable by an automaton with at most $\left(\left|Q\right| \prod_{j=1}^{k-1} L_j\right) + 1$ many states, where $L_j$ for $j \in \{1, \ldots, k\}$ denotes the order of $a_j$, and this automaton is effectively computable.

Proof. Let $\Sigma = \{a_1, \ldots, a_k\}$ and $A = (\Sigma, Q, \delta, s_0, F)$ be a permutation automaton. Denote by $\sigma_{A,+} : \mathbb{N}_0^{\Sigma} \rightarrow \mathcal{P}(Q)$ the state label map from Definition 28 and by $\psi : \Sigma^* \rightarrow \mathbb{N}_0$ the Parikh map. By Proposition 26 we have

$$perm(L(A))^{\mu,*} = \psi^{-1}(\sigma_{A,+}^{-1}(F)) \cup \{\varepsilon\}$$

with $F = \{S \subseteq Q \mid S \cap F \neq \emptyset \} \subseteq \mathcal{P}(Q)$. Inspecting Definition 28 we see that the state label map is compatible with $A$. So, by Proposition 24 the indices of the automata $A^{(i)}_p$ from Definition 19 are universally bounded by $\left(|Q| - 1\right) L_j$ and the periods divide $L_j$. Hence, applying Theorem 22 give

$$sc(\psi^{-1}(\sigma_{A,+}^{-1}(F))) \leq |Q|^k \prod_{j=1}^{k} L_j.$$

Finally, using Lemma 28 gives the result for the iterated shuffle. 

12 The set $\psi^{-1}(\sigma_{A,+}^{-1}(F))$ equals $perm(L(A))^{\mu,+}$. This is not explicitly stated but could be extracted from the proof of Proposition 26.
B  Proofs for Section 6 (The n-times Shuffle)

For notational simplicity, we only do the case $n = 2$, the general case works the same way. In fact, the general case is only a notational complication, but nothing more.

We use the general scheme for the application of the state label method as outlined at the start of this appendix. First, we have to define a state label map.

**Definition 29.** Let $A = (\Sigma, Q_A, \delta_A, s_A, F_A)$ and $B = (\Sigma, Q_B, \delta_B, s_B, F_B)$ be finite automata with disjoint state sets, i.e., $Q_A \cap Q_B = \emptyset$. Denote by $\sigma_{A,B} : \mathbb{N}_0 \rightarrow \mathcal{P}(Q_A \cup Q_B)$ the state label function given by $f : \mathcal{P}(Q_A \cup Q_B) \times \Sigma \rightarrow \mathcal{P}(Q_A \cup Q_B)$, where

$$f(S, a) = \begin{cases} \delta_A(S \cap Q_A, a) \cup \delta_B(S \cap Q_B, a) \cup \{s_B\} & \text{if } \delta_A(S \cap Q_A, a) \cap F_A \neq \emptyset; \\ \text{otherwise}; \end{cases}$$

for $S \subseteq Q_A \cup Q_B$, $a \in \Sigma$, and $\sigma_{A,B}(p) = \begin{cases} \{s_A, s_B\} & \text{if } s_A \in F_A; \\ \{s_A\} & \text{otherwise}. \end{cases}$

The requirement $Q_A \cap Q_B = \emptyset$ in most statements of this section is not a limitation, as we could always construct an isomorphic copy of any one of the involved automata if this is not fulfilled. It is more a technical requirement of the constructions, to not mix up what is read up to some point.

**Lemma 30.** Let $p \in \mathbb{N}_0^k$ and $A = (\Sigma, Q_A, \delta_A, s_A, F_A)$, $B = (\Sigma, Q_B, \delta_B, s_B, F_B)$ be finite automata with disjoint state sets. Denote by $\sigma_{A,B} : \mathbb{N}_0 \rightarrow \mathcal{P}(Q_A \cup Q_B)$ the state label map from Definition 29. If for all $q \in \mathbb{N}_0^k$ with $q \leq p$ we have $\sigma_{A,B}(q) \cap F_A = \emptyset$, then $\sigma_{A,B}(p) \cap Q_B = \emptyset$.

**Proof.** For $p = (0, \ldots, 0)$ the claim follows by Definition 29. Suppose $p \neq (0, \ldots, 0)$. Then

$$\sigma_{A,B}(p) = \bigcup_{p-q+\psi(b)} f(\sigma_{A,B}(q), b).$$

By assumption $\sigma_{A,B}(p) \cap F_A = \emptyset$. Hence, for $q \in \mathbb{N}_0^k$ and $b \in \Sigma$ with $p = q + \psi(b)$, we have $f(\sigma_{A,B}(q), b) \cap F_A = \emptyset$. By Definition 29 $\delta_A(\sigma_{A,B}(q) \cap Q_A, b) \subseteq f(\sigma_{A,B}(q), b)$, so that $\delta_A(\sigma_{A,B}(q) \cap Q_A, b) \cap F_A = \emptyset$. Again, by Definition 29 then

$$f(\sigma_{A,B}(q), b) = \delta_A(\sigma_{A,B}(q) \cap Q_A, b) \cup \delta_B(\sigma_{A,B}(q) \cap Q_B, b)$$

Inductively, we can assume $\sigma_{A,B}(q) \cap Q_B = \emptyset$. So the above set equals

$$\delta_A(\sigma_{A,B}(q) \cap Q_A, b),$$

which is contained in $Q_A$. Hence, as this holds for any $q \in \mathbb{N}_0^k$ and $b \in \Sigma$ with $p = q + \psi(b)$, we have

$$\sigma_{A,B}(p) = \bigcup_{p-q+\psi(b)} f(\sigma_{A,B}(q), b) \subseteq Q_A$$

which is equivalent with $\sigma_{A,B}(p) \cap Q_B = \emptyset$.

We will also need the following stronger version of Lemma 30.

**Lemma 31.** Let $p \in \mathbb{N}_0^k$ and $A = (\Sigma, Q_A, \delta_A, s_A, F_A)$, $B = (\Sigma, Q_B, \delta_B, s_B, F_B)$ be finite automata with disjoint state sets. Denote by $\sigma_{A,B} : \mathbb{N}_0 \rightarrow \mathcal{P}(Q_A \cup Q_B)$ the state label map from Definition 29. Then

$$\sigma_{A,B}(p) \cap Q_B = \bigcup_{p-q+\psi(b)} \delta_B(\{s_B\}, u).$$
Proof. If $p = (0, \ldots, 0)$, then

$$\sigma_{A,B}(p) = \begin{cases} \{s_A, s_B\} & \text{if } s_A \in F_A; \\ \{s_A\} & \text{otherwise.} \end{cases}$$

Hence, as then $p = q + \psi(u)$ implies $q = p = (0, \ldots, 0)$ and $u = \varepsilon$, so that $\delta(\{s_B\}, u) = \{s_B\}$, we have

$$\sigma_{A,B}(p) \cap Q_B = \begin{cases} \delta(\{s_B\}, u) & \text{if } \sigma_{A,B}(p) \cap F_A \neq \emptyset; \\ \emptyset & \text{otherwise.} \end{cases}$$

So, the equation holds. If $p \neq (0, \ldots, 0)$, then we can reason inductively,

$$\sigma_{A,B}(p) \cap Q_B = \bigcup_{(q,b)} f(\sigma_{A,B}(q), b) \cap Q_B = \bigcup_{(q,b)} (f(\sigma_{A,B}(q), b) \cap Q_B)$$

By Equation (17), the set $f(\sigma_{A,B}(q), b) \cap Q_B$ equals

$$\begin{cases} \delta_B(\sigma_{A,B}(q) \cap Q_B, b) \cup \{s_B\} & \text{if } \delta_A(\sigma_{A,B}(q) \cap Q_A, b) \cap F_A \neq \emptyset; \\ \delta_B(\sigma_{A,B}(q) \cap Q_B, b) & \text{otherwise.} \end{cases}$$

By induction hypothesis, we can assume

$$\sigma_{A,B}(q) \cap Q_B = \bigcup_{q = r + \psi(u)} \delta_B(\{s_B\}, u).$$

Hence

$$\delta_B(\sigma_{A,B}(q) \cap Q_B, b) = \bigcup_{q = r + \psi(u)} \delta_B(\delta_B(\{s_B\}, u), b)$$

If for all $b \in \Sigma$ and $q \in \mathbb{N}_0^\Sigma$ with $p = q + \psi(b)$ we have $\delta_A(\sigma_{A,B}(q) \cap Q_A, b) \cap F_A = \emptyset$, then by combining the above equations

$$\sigma_{A,B}(p) = \bigcup_{(q,b)} f(\sigma_{A,B}(q), b) \cap Q_B$$

$$\begin{aligned} & = \bigcup_{(q,b)} \delta_B(\{s_B\}, ub) \\ & = \bigcup_{(r,w)} \delta_B(\{s_B\}, w). \end{aligned}$$
Otherwise
\[
\sigma_{A,B}(p) = \bigcup_{(q,b)} \left( \bigcup_{p=r+\psi(b)} \delta_B(\delta_B(\sigma_B(\varepsilon), u), b) \right) \cup \{s_B\}
\]
\[
= \bigcup_{p=r+\psi(u)+\psi(b)} \delta_B(\sigma_B(\varepsilon), u) \cup \{s_B\}
\]
\[
= \bigcup_{(r,u)} \delta_B(\sigma_B(\varepsilon), u)
\]
where the last equation holds, as \{s_B\} = \delta(\{s_B\}, \varepsilon) and \sigma_{A,B}(p) \cap F_A \neq \emptyset, so that \(p, \varepsilon\) is part of the union. So, by induction, the equation from the lemma holds true.

With Lemma [30] we can derive a connection between the Parikh image of \(L(A)L(B)\) and the state label map.

**Proposition 32.** Suppose we have finite automata \(A = (\Sigma, Q_A, \delta_A, s_A, F_A)\) and \(B = (\Sigma, Q_B, \delta_B, s_B, F_B)\) with \(Q_A \cap Q_B = \emptyset\). Then
\[
\psi(L(A)L(B)) = \sigma_{A,B}^{-1}([S \subseteq Q_A \cup Q_B \mid S \cap F_B \neq \emptyset]).
\]

**Proof.** By assumption \(Q_A \cap Q_B = \emptyset\). Set \(Q = Q_A \cup Q_B\). Construct the semi-automaton \(\mathcal{C} = (\Sigma, Q, \delta)\) with
\[
\delta(q, x) = \begin{cases} 
\delta_A(q, x) & \text{if } q \in Q_A; \\
\delta_B(q, x) & \text{if } q \in Q_B.
\end{cases}
\]
Then \(\delta(S, a) = \delta_A(S \cap Q_A, a) \cup \delta_B(S \cap Q_B, a)\) for each \(S \subseteq Q\). Let \(f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)\) be the function from Definition [29] and \(\sigma_{A,B} : \mathbb{N}_0^* \to \mathcal{P}(Q)\) the corresponding state label map. Then, for each \(S \subseteq Q_A \cup Q_B\) and \(a \in \Sigma\), we have \(\delta(S, a) \subseteq f(S, a)\) by Equation [17], i.e., the semi-automaton \(\mathcal{C}\) is compatible with the state label map.

(i) First, let \(p \in \psi(L(A)L(B))\). Then \(p = \psi(u) + \psi(v)\) with \(u \in L(A)\) and \(v \in L(B)\). By Lemma [29] \(\delta(\sigma_{A,B}(0, \ldots, 0), u) \subseteq \sigma_{A,B}(\psi(u))\). By Definition [29] \(\{s_A\} \subseteq \sigma_{A,B}(0, \ldots, 0)\). As \(u \in L(A)\), we have \(\delta_A(s_A, u) \in F_A\). We will show that this implies \(\{s_B\} \subseteq \sigma_{A,B}(\psi(u))\).

**Claim:** \(\{s_B\} \subseteq \sigma_{A,B}(\psi(u))\) for \(u \in L(A)\).

**Proof of the claim.** If \(|u| = 0\), then \(s_A \in F_A\). Hence, by Definition [29] \(\{s_A, s_B\} = \sigma_{A,B}(0, \ldots, 0) = \sigma_{A,B}(\psi(u))\). Otherwise, write \(u = wa\) for some \(a \in \Sigma, w \in \Sigma^*\) and set \(S = f(\sigma_{A,B}(0, \ldots, 0), w)\). So,
\[
f(\sigma_{A,B}(0, \ldots, 0), u) = f(S, a)
\]
by the extension of \(f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)\) to words. As \(\mathcal{C}\) is compatible with \(\sigma_{A,B}\), we find
\[
\delta(\sigma_{A,B}(0, \ldots, 0), w) \subseteq S.
\]
As \(\{s_A\} \subseteq \sigma_{A,B}(0, \ldots, 0)\), this gives, by construction of \(\mathcal{C}\), then \(\delta_A(\{s_A\}, w) \subseteq S\). Hence, \(\delta_A(S \cap Q_A, a) \cap F_A \neq \emptyset\). But then, by Equation [17],
\[
f(S, a) = \delta(S, a) \cup \{s_B\}.
\]
By Lemma 17 and as \( \psi(u) = q + \psi(w) \) for \( |u| = |w| \) implies \( q = (0, \ldots, 0) \),

\[
\sigma_{A,B}(\psi(u)) = \bigcup_{(q,w) \in \mathbb{N}_0^k \times \\psi(u)=q+\psi(w)} f(\sigma_{A,B}(0, \ldots, 0), w).
\]

Hence \( f(S,a) = f(\sigma_{A,B}(0, \ldots, 0), u) \subseteq \sigma_{A,B}(\psi(u)) \) and we can deduce \( \{s_B\} \subseteq \sigma_{A,B}(\psi(u)) \). □

Using Lemma 17 we find

\[
f(\sigma_{A,B}(\psi(u)), v) \subseteq \sigma_{A,B}(\psi(u) + \psi(v)).
\]

As

\[
\delta_B(\{s_B\}, v) \subseteq \delta_B(\sigma_{A,B}(\psi(u)) \cap Q_B, v) \subseteq \delta(\sigma_{A,B}(\psi(u)), v) \subseteq f(\sigma_{A,B}(\psi(u)), v)
\]

and \( \delta_B(s_B, v) \in F_B \), we find \( \sigma_{A,B}(p) \cap F_B \neq \emptyset \). This shows \( \psi(L(A)L(B)) = \sigma_{A,B}^{-1}(\{S \subseteq Q_A \cup Q_B \mid S \cap F_B \neq \emptyset\}) \).

(ii) Conversely, assume \( F_B \cap \sigma_{A,B}(p) \neq \emptyset \).

**Claim**: For each \( S \subseteq Q \) and \( w \in \Sigma^* \)

\[
f(S, w) \cap Q_A = \delta_A(S \cap Q_A, w).
\]

**Proof of the claim.** If \( |w| = 0 \), then \( f(S, w) \cap Q_A = S \cap Q_A = \delta(S \cap Q_A, w) \) by definition of the extension of \( f \) and the transition function to words. Otherwise, write \( w = w'a \) with \( w' \in \Sigma^* \) and \( a \in \Sigma \). Then \( f(S, w'a) = f(f(S, w'), a) \). By Equation (17), in either case \( \delta_A(f(S, w') \cap Q_A, a) \cap F_A \neq \emptyset \) or \( \delta_A(f(S, w') \cap Q_A, a) \cap F_A = \emptyset \), we have

\[
f(f(S, w'), a) \cap Q_A = \delta_A(f(S, w') \cap Q_A, a).
\]

Inductively, \( f(S, w') \cap Q_A = \delta_A(S \cap Q_A, w') \), so that \( f(S, w) = \delta_A(f(S, w') \cap Q_A, a) = \delta_A(S \cap Q_A, w') \cap Q_A, a) = \delta_A(S \cap Q_A, w). \)

□

Then Lemma 17 and Equation (18) give, for any \( q \in \mathbb{N}_0^k \),

\[
\sigma_{A,B}(q) \cap Q_A = \left( \bigcup_{\psi(w)=q} f(\sigma_{A,B}(0, \ldots, 0), w) \right) \cap Q_A
\]

\[
= \bigcup_{\psi(w)=q} \left( f(\sigma_{A,B}(0, \ldots, 0), w) \cap Q_A \right)
\]

\[
= \bigcup_{\psi(w)=q} \delta_A(\sigma_{A,B}(0, \ldots, 0) \cap Q_A, w).
\]

By Lemma 17 as \( \sigma_{A,B}(p) \cap F_B \neq \emptyset \), we have some \( v \in L(B) \) and \( q \in \mathbb{N}_0^k \) with \( p = q + \psi(v) \) and \( \sigma_{A,B}(q) \cap F_A \neq \emptyset \). By the above equations, we find \( w \in \Sigma^* \) with \( \psi(w) = q \) and \( \delta_A(\sigma_{A,B}(0, \ldots, 0) \cap Q_A, w) \cap F_A \neq \emptyset \). As, by Equation (17), \( \sigma_{A,B}(0, \ldots, 0) \cap Q_A = \{s_A\} \), this gives \( w \in L(A) \). So, we have \( p = \psi(w) + \psi(v) \) with \( w \in L(A) \) and \( v \in L(B) \). This yields \( p = \psi(L(A)L(B)) \).

□

Hence, as \( \text{perm}(L) = \psi^{-1}(\psi(L)) \) for any \( L \subseteq \Sigma^* \), we can conclude that this state labeling could be used to describe the commutative closure of the concatenation, which, by Theorem 3 equals \( \text{perm}(L(A)) \cup \text{perm}(L(B)) \).
Corollary 33. Suppose we have finite automata $A = (\Sigma, Q_A, \delta_A, s_A, F_A)$ and $B = (\Sigma, Q_B, \delta_B, s_B, F_B)$ with $Q_A \cap Q_B = \emptyset$. Then
\[
\text{perm}(L(A) \cup L(B)) = \psi^{-1}(\sigma_{A,B}^{-1}(\{S \subseteq Q_A \cup Q_B \mid S \cap F_B \neq \emptyset\})).
\]

Constructing an appropriate automaton over $Q_A \cup Q_B$ and applying Theorem 22 then gives the next result.

Proposition 13. Let $A_i = (\Sigma, Q_i, \delta_i, q_i, F_i)$ for $i \in \{1, \ldots, n\}$ be $n$ permutation automata. Then
\[
\text{sc}(\text{perm}(L(A_1)) \sqcup \ldots \sqcup \text{perm}(L(A_n))) \leq \left( \sum_{i=1}^{n} Q_i \right)^k \prod_{j=1}^{k} \text{lcm}(L_j^{(1)}, \ldots, L_j^{(n)})
\]
where $L_j^{(i)}$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$ denotes the order of the letter $a_j$ as a permutation on $Q_i$.

Proof. As said, we only do the case $n = 2$. Let $A = (\Sigma, Q_A, \delta_A, s_A, F_A)$ and $B = (\Sigma, Q_B, \delta_B, s_B, F_B)$ be finite permutation automata. Suppose $L_j$ and $K_j$ denote the order of the letter $a_j$ viewed as a permutation on $Q_A$ and $Q_B$ respectively. We show that
\[
\text{sc}(\text{perm}(L(A)) \sqcup \text{perm}(L(B))) \leq (Q_A + Q_B)^k \prod_{j=1}^{k} \text{lcm}(L_j, K_j).
\]

We can assume $Q_A \cap Q_B = \emptyset$. Set $Q = Q_A \cup Q_B$. Construct the semi-automaton $C = (\Sigma, Q, \delta)$ with
\[
\delta(q, x) = \begin{cases} 
\delta_A(q, x) & \text{if } q \in Q_A; \\
\delta_B(q, x) & \text{if } q \in Q_B.
\end{cases}
\]
Let $f : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ be the function from Definition 29 and $\sigma_{A,B} : \mathbb{N}_0^k \to \mathcal{P}(Q)$ the corresponding state label map. Then, for each $S \subseteq Q_A \cup Q_B$ and $a \in \Sigma$, we have $\delta(S, a) \subseteq f(S, a)$ by Equation 29, i.e., the semi-automaton $C$ is compatible with the state label map. The automaton $C$ is a permutation semi-automaton, and each letter $a_j \in \Sigma$ has order $\text{lcm}(L_j, K_j)$, viewed as a permutation on $Q_A \cup Q_B$. By Proposition 24 the automata $A_p^{(j)}$ from Definition 19 have index at most $|Q_A \cup Q_B| \cdot \text{lcm}(L_j, K_j)$ and period at most $\text{lcm}(L_j, K_j)$. Hence, using Theorem 22 the language $\psi^{-1}(\sigma_{A,B}(F))$ with $F = \{S \subseteq Q_A \cup Q_B \mid S \cap F_B \neq \emptyset\}$ is accepted by an automaton of size at most
\[
\prod_{j=1}^{k} \left( (|Q_A \cup Q_B| - 1) \text{lcm}(L_j, K_j) + \text{lcm}(L_j, K_j) \right).
\]
By Corollary 33 the result follows. $\square$

References for the Appendix

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