One-loop stress-tensor renormalization in curved background: the relation between $\zeta$-function and point-splitting approaches, and an improved point-splitting procedure.

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Abstract: We conclude the rigorous analysis of the previous paper [1] concerning the relation between the (Euclidean) point-splitting approach and the local $\zeta$-function procedure to renormalize physical quantities at one-loop in (Euclidean) QFT in curved spacetime. The case of the stress tensor is now considered in general $D$-dimensional closed manifolds for positive scalar operators $-\Delta + V(x)$. Results obtained formally in previous works (in the case $D = 4$ and $V(x) = \xi R(x) + m^2$) are rigorously proven and generalized. It is also proven that, in static Euclidean manifolds, the method is compatible with Lorentzian-time analytic continuations. It is proven that, for $D > 1$, the result of the $\zeta$ function procedure is the same obtained from an improved version of the point-splitting method which uses a particular choice of the term $w_0(x, y)$ in the Hadamard expansion of the Green function, given in terms of heat-kernel coefficients. This version of the point-splitting procedure works for any value of the field mass $m$. If $D$ is even, the result is affected by an arbitrary one-parameter class of (conserved in absence of external source) symmetric tensors, dependent on the geometry locally, and it gives rise to the general correct trace expression containing the renormalized field fluctuations as well as the conformal anomaly term. Furthermore, it is proven that, in the case $D = 4$ and $V(x) = \xi R(x) + m^2$, the given procedure reduces to the Euclidean version of Wald's improved point-splitting procedure provided the arbitrary mass scale present in the $\zeta$ function is chosen opportunely. It is finally argued that the found point-splitting method should work generally, also dropping the hypothesis of a closed manifold, and not depending on the $\zeta$-function procedure. This fact is indeed checked in the Euclidean section of Minkowski spacetime for $A = -\Delta + m^2$ where the method gives rise to the correct Minkowski stress tensor for $m^2 \geq 0$ automatically.

I. Introduction.

In a previous paper [1], we have considered the relationship between the $\zeta$-function and the point-splitting procedures in renormalizing some physical quantities: effective Lagrangian, ef-
fective action and field fluctuations. The more interesting quantity, namely, the stress tensor, is the object of the present paper. The aim of this paper is hence twofold. First we want to give a rigorous mathematical foundation as well as a generalization of several propositions contained in [2] where they have been stated without rigorous proof. This is a quite untrivial task because it involves an extension of the heat kernel theory considering the derivatives of its usual “asymptotic” expansion. As we shall see shortly, this is the core of all the analyticity properties of the generalized tensorial $\zeta$ functions involved in the stress-tensor renormalization procedure. Second, we want to study the relation between our technique and the more usual point-splitting procedure in deep. This is another open issue after the appearance of [2]. We know, through practical examples, that these two approaches agree essentially in several concrete cases, but up to now, no general proof of this fact has been given. Anyhow, it was conjectured by Wald [3] that, in general, these two approaches should lead to the same results. The extension of the $\zeta$-function approach to the stress tensor has been introduced in [2] formally, this paper contains a proof of mathematical consistence of the method in a generalized case as well as a general proof of the agreement between the two approaches under our hypotheses on the manifold and the field operator.

It is a well-known fact that the point-splitting procedure faces some difficulties in the case of a field which is massless; Indeed, in such a case, one cannot make use of the Schwinger-DeWitt algorithm to fix $w_0$ in the Hadamard expansion [3, 4] and, at least in the massless conformally coupled case, the point-splitting procedure has been improved in order to get both the conformal anomaly and the conservation of the renormalized stress tensor [3]. Recently, Wald has argued that such an improved procedure, which picks out $w_0 \equiv 0$, can be generalized in more general cases [3]. Differently from the point-splitting procedure, the local $\zeta$ function approach seems to work without to distinguish between different values of mass and coupling with the curvature. This fact makes more intriguing the issue of the relation between the two procedures.

This paper is organized as follows. In the first part, we shall recall the main features of the classical theory of the stress tensor to the reader and we shall introduce the main ideas concerning the renormalization of the stress tensor via $\zeta$ function. In a second part, first we shall develop further the theory of the heat-kernel expansion in order to build up the theory of the $\zeta$ function of the stress tensor. All the work is developed in a closed $D$-dimensional manifold for a quite general Euclidean motion operator of a real scalar field. Successively, we shall state and prove several theorems concerning generalizations of several mathematical conjecture employed in [3]. The final part of this work is devoted to investigate the relation between the two considered techniques and to introduce a generalized point-splitting procedure. Indeed, within that part, we shall give a proof of the agreement of the two approaches, introducing an improved point-splitting procedure which is quite similar and generalizes that pointed out in [3, 3]. We shall see that our prescription gives all the expected result (it gives a the trivial stress tensor in Minkowski spacetime, the conformal anomaly and a conserved stress tensor in general, producing agreement with the result of the field fluctuations renormalization). A final summary ends the work. In the final appendix, the proof of some theorems and lemmata is reported.
II. Preliminaries.

Within this section, we state the general mathematical hypotheses we shall deal with and, very quickly, we review the main physical ideas concerning the classical stress tensor and its one-loop renormalization via point-splitting [4, 5, 6] and via local \( \zeta \) function [2].

We assume all the definitions and theorems given in the previous paper [1] and we shall refer to those definitions and theorems throughout all parts of this work.

A. General hypotheses and notations.

The hypotheses we shall deal with in this work are the same of the work [1]. Therefore, from now on, \( \mathcal{M} \) is a Hausdorff, connected, oriented, \( C^\infty \) Riemannian \( D \)-dimensional manifold. We suppose also that \( \mathcal{M} \) is compact without boundary (namely is “closed”). Concerning the operators, we shall consider real elliptic differential operators with the Schrödinger form "Laplace-Beltrami operator plus potential"

\[
A' = -\Delta + V : C^\infty(\mathcal{M}) \to L^2(\mathcal{M}, d\mu_g)
\]  

where, locally, \( \Delta = \nabla_a \nabla^a \), and \( \nabla \) means the covariant derivative associated to the metric connection, \( d\mu_g \) is the Borel measure induced by the metric, and \( V \) is a real function belonging to \( C^\infty(\mathcal{M}) \). We assume that \( A' \) is bounded below by some \( C \geq 0 \) (namely \( A \) is positive). (See sufficient conditions in [1]). These are the general hypotheses which we shall refer to throughout the paper.

Moreover, in the most part of the theorems, we shall use also the fact that the injectivity radius of the manifold \( r \) is strictly positive in closed manifolds (see [1] for further comments on this point).

As general remarks, we remind the reader that, as in the previous work, "holomorphic" and "analytic" are synonyms throughout this paper, natural units \( \hbar = c = 1 \) are used and the symbol \( A \) indicates the only self-adjoint extension of the essentially self-adjoint operator \( A' \). In the practice, as seen in [1], \( A \) coincides with the Friedrichs self-adjoint extension of \( A' \). \( R \) indicates the scalar curvature. Moreover, the symbol \( \sigma(x, y) \) means one half the squared geodesical distance of \( x \) from \( y \) which is continuous everywhere and \( C^\infty \) in any convex normal neighborhood.

Concerning derivative operators, we shall employ the notations in a fixed local coordinate system,

\[
D^\alpha_x := \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_D}}|_x
\]

where the multindex \( \alpha \) is defined by \( \alpha := (\alpha_1, \cdots, \alpha_D) \), \( \alpha_i \geq 0 \) is any natural number \( (i = 1, \cdots, D) \) and \( |\alpha| := \alpha_1 + \cdots + \alpha_D \). Moreover, \( n_k \) will indicate the multindex \( (0, \cdots, 0, n, 0, \cdots) \) where the only nonvanishing number is \( n \in \mathbb{N} \) which takes the \( k \)th position.

Concerning the definitions of the metric-connection symbols and curvature tensors, we shall follow the notations and conventions employed in [1] which are the same employed in [3], either for Riemannian or Lorentzian signature.
B. Physical background and classical definitions.

All quantities related to $A'$ we have considered in the previous work [1] and the averaged stress tensor we consider here, for $D = 4$, appear in (Euclidean) QFT in curved background and concern the theory of quasifree scalar fields. In several concrete cases of QFT, the form of $V(x)$ is $m^2 + \xi R(x)$ where $m^2$ is the square mass of the considered field, $R$ is the scalar curvature of the manifold, $\xi$ is a real parameter. As usual the conformal coupling is defined by

$$\xi_D := (D - 2)/[4(D - 1)].$$

(3)

Similarly to the physical quantities considered in the previous work, also the stress-tensor is formally obtained from the Euclidean functional integral

$$Z[A'] := \int D\phi e^{-\frac{1}{2} \int_M \phi A'[g] \phi d\mu_g} =: e^{-S_{\text{eff}}[A']}.$$  

(4)

$S_{\text{eff}}$ is the (Euclidean) effective action of the field. (Here, we use the opposite sign conventions in defining the effective action and thus the stress tensor, with respect the conventions employed in [2]. Our conventions are the same used in [8].)

The integral above can be considered as a partition function of a field in a particular quantum state corresponding to a canonical ensemble [8, 9]. The direct physical interpretation as a partition function should work provided the manifold has a static Lorentzian section obtained by analytically continuing some global temporal coordinate $x^0 = \tau$ of some global chart into imaginary values $\tau \to it$ and considering (assuming that they exist) the induced continuations of the metric and relevant quantities. It is required also that $\partial_\tau$ is a global Killing field of the Riemannian manifold generated by an isometry group $S_1$. Finally it is required that $\partial_\tau$ can be continued into a (generally local) time-like Killing field $\partial_t$ in the Lorentzian section (see [9] and [8]). Then one assumes that $k_B\beta$ is the inverse of the temperature of the canonical ensemble quantum state, $\beta$ being the period of the coordinate $\tau$. The limit case of vanishing temperature is also considered and in that case the manifold cannot be compact. Similar interpretations hold for the (analytic continuations of) the stress tensor.

Formally we have $Z[A', g] := \left[ \det \left( \frac{A'}{\mu} \right) \right]^{-1/2}$ where our definition of the determinant of the operator $A'$ is given by the $\zeta$ function approach [8, 10, 11] as pointed out in the previous paper [1]. The scale $\mu^2$ present in the determinant is necessary for dimensional reasons [8] and plays a central rôle in the $\zeta$–function interpretation of the determinant and in the consequent theory. Such a scale introduces an ambiguity which remains in the finite renormalization parts of the renormalized quantities and, dealing with the renormalization of the stress tensor within the semiclassical approach to the quantum gravity, it determines the presence of quadratic-curvature terms in effective Einstein’s equations [2]. Similar results are discussed in [3, 4, 5, 6] employing other renormalization procedures (point-splitting).

Coming to the (Euclidean) classical stress-tensor $T_{ab}(x)$, it is defined (e.g. see [3]) as the locally quadratic form of the field obtained by the usual functional derivative once the field $\phi$ is fixed

$$T_{ab}[\phi, g](x) := \frac{2}{\sqrt{g} \delta_{gab}(x)} \left( \frac{1}{2} \int_I \sqrt{g(x)} \phi A'[g] \phi d^D x \right).$$  

(5)
This functional derivative can be rigorously understood in terms of a Gâteaux derivative for functionals on real \( C^\infty ( I ) \) symmetric tensor fields \( g_{ab} \) and the integration above is performed in the open set \( I \) containing \( x \) where the considered coordinate system is defined. (5) means that, and this is the rigorous definition of the symmetric tensor field \( T_{ab}[\phi, g](x) \), for any \( C^\infty \) symmetric tensor field \( h_{ab} \) with compact support contained in \( I \)

\[
2 \frac{d}{d\alpha} |_{\alpha=0} S_T[g + \alpha h] = \frac{1}{2} \int_I \sqrt{g(x)} T_{ab}[\phi, g](x) h^{ab}(x) d^D x , \tag{6}
\]

where

\[
S_T[\phi, g] := \frac{1}{2} \int_I \sqrt{g(x)} \phi A'[g] \phi d^D x . \tag{7}
\]

In the case

\[
A' = -\Delta + m^2 + \xi R(x) + V'(x) , \tag{8}
\]

\( V' \) being a \( C^\infty \) function which does not depend on the metric, a direct computation of \( T_{ab}(x) \) through this procedure gives

\[
T_{ab}[\phi, g](x) = \nabla_a \phi \nabla_b \phi(x) - \frac{1}{2} g_{ab}(x) \left[ \nabla_c \phi(x) \nabla^c \phi(x) + \left( m^2 + V'(x) \right) \phi^2(x) \right] + \xi \left[ \left( R_{ab}(x) - \frac{1}{2} g_{ab}(x) R(x) \right) \phi^2(x) + g_{ab}(x) \nabla_c \nabla^c \phi^2 - \nabla_a \nabla_b \phi^2(x) \right] \tag{9}
\]

As well known, \( T_{ab} \) given in (9) and evaluated for a particular \( \phi \) is conserved \( (\nabla_a T^{ab} \equiv 0) \) provided \( \phi \) is a sufficiently smooth (customary \( C^\infty \)) solution of the Euclidean motion, namely, \( A' \phi \equiv 0 \), and \( V' \equiv 0 \). More generally for solution of Euclidean motion, in local coordinates and for any point \( x \in M \) one finds

\[
\nabla_a T^{ab}[\phi, g](x) = -\frac{1}{2} \phi^2(x) \nabla^b V'(x) . \tag{10}
\]

Another important classical property is the following one. Whenever the field \( \phi \) is massless and conformally coupled (i.e. \( V'(x) \equiv m^2 = 0 \) and \( \xi = \xi_D \)), the Euclidean action \( S_M \) is invariant under local conformal transformations and it holds also

\[
g_{ab} T^{ab}[\phi, g](x) = 0 \tag{11}
\]

everywhere, for smooth fields \( \phi \) which are solution of the (Euclidean) motion equations. (10) and (11) can be checked for the tensor in (9) directly, holding our general hypotheses.

Actually, the requirement of \( A' \) positive is completely unnecessary for all the definitions and results given above which hold true in any \( C^\infty \) Riemannian as well as Lorentzian manifold. In our approach, the Lorentzian stress tensor is obtained by analytic continuation of the Euclidean time as pointed out above.
Passing to the quantum averaged quantities, following Schwinger [12], the averaged one-loop stress tensor for the quantum state determined by the Feynman propagator obtained by the Green function of $A'$, can be formally defined by [4, 5, 8, 9]

$$\langle T_{ab}(x|A') \rangle := \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}(x)} S_{eff} = \int \mathcal{D}\phi e^{-\frac{1}{2} \int_M \phi A'[g] \phi} T_{ab}(\phi, g)(x) . \tag{12}$$

It is well-known that the right hand sides of (12) and the corresponding quantity in the Lorentzian section are affected by divergences whenever one tries to compute them by trivial procedures [4, 6, 8]. For instance, proceeding as usual (e.g. see [5]), interpreting the functional integral of $\phi(x)\phi(y)$ as a Green function of $A'$ (the analytic continuation of the Feynman propagator) $G(x, y)$ and then defining an off-diagonal quantum averaged stress tensor

$$\langle T_{ab}(x, y) \rangle = \int \mathcal{D}\phi e^{-\frac{1}{2} \int_M \phi A'[g] \phi} O_{ab}(x, y)\phi(x)\phi(y) = O_{ab}(x, y)G(x, y) , \tag{13}$$

where $O_{ab}(x, y)$ is an opportune bi-vectorial differential operator (see [5]), the limit of coincidence of arguments $x$ and $y$, necessary to get $\langle T_{ab}(x) \rangle$, trivially diverges. One is therefore forced to remove these divergences by hand and this is nothing but the main idea of the point-splitting procedure. Within the point-splitting procedure (10) is requested also for the quantum averaged stress tensor at least in the case $V' \equiv 0$. Conversely, the property (11) generally does not hold in the case of a conformally coupled massless field: a conformal anomaly appears [3, 4, 5, 6].

Another approach to interpret the left hand side of (12) in terms of local $\zeta$ function was introduced in [2] without rigorous mathematical discussion. Anyhow, this approach has produced correct results and agreement with point-splitting procedures in several concrete cases [2, 13] and it has pointed out a strong self-consistence and a general agreement [2] with the general axiomatic theory of the stress tensor renormalization built up by Wald [1]. (It is anyway worth stressing that Wald's axiomatic approach concerns the Lorentzian theory and thus any comparison involves an analytical continuation of the Euclidean theory. In such a way all the issues related to the locality of the theory cannot be compared directly with the general $\zeta$-function approach.) Moreover, differently from the known point-splitting techniques, no difficulty arises dealing with the case of a massless conformally coupled field.

Similarly to the cases treated in the previous work [1], the definition of the formal quantity in the left hand side of (12) given in terms of $\zeta$ function and heat kernel contains an implicit infinite renormalization procedure in the sense that the result is finally free from divergences.

C. The key idea of the $\zeta$-function regularization of the stress tensor.

The key idea of $\zeta$-function regularization of the stress tensor concerns the extension of the use of the $\zeta$ function from the effective action to the stress tensor employing some manipulations of the series involved in the $\zeta$-function technique. We remind the reader that formally one has
\[ S_{\text{eff}}[A]\mu^2 = \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left\{ - \sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \right\}. \]  \hfill (14)

\( \mu \) is the usual arbitrary mass scale necessary for dimensional reasons. Actually, the identity above holds true in the sense of the analytic continuation. Then, one can try to give some meaning to the following formal passages

\[ \langle T_{ab}(x) \rangle_{\mu^2} = \frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g_{ab}(x)} \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left\{ - \sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \right\} \]

\[ = \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left\{ - \sum_{j \in \mathbb{N}} \frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g_{ab}(x)} \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \right\} \]

\[ = \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left\{ \frac{s}{\mu^2} \sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{\mu^2} \right)^{-(s+1)} \frac{2}{\sqrt{g(x)}} \frac{\delta \lambda_j}{\delta g_{ab}(x)} \right\}. \]  \hfill (15)

The functional derivative of \( \lambda_j \) has been computed in [2], at least formally. The passages above are mathematically incorrect most likely, anyhow, in [2] it was conjectured that the series in the last line of \((15)\) converges and it can be analytically continued into a regular function \( Z_{ab}(s, x|A/\mu^2) \) in a neighborhood of \( s = 0 \). Then, one can define the renormalized averaged one-loop stress tensor as

\[ \langle T_{ab}(x) \rangle_{\mu^2} := \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} Z_{ab}(s, x|A/\mu^2). \]  \hfill (16)

The explicit form of \( Z_{ab} \) found in [2] following the route above was

\[ Z_{ab}(s, x|A/\mu^2) = \frac{2s}{\mu^2} \zeta_{ab}(s + 1, x|A/\mu^2) + sg_{ab}(x) \zeta(s, x|A/\mu^2) \]  \hfill (17)

where \( \zeta_{ab}(s, x|A/\mu^2) \) is the analytic continuation of the series

\[ \sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{\mu^2} \right)^{-s} T_{ab}[\phi_j, \phi_j^*, g](x), \]  \hfill (18)

and

\[ T_{ab}[\phi, \phi^*, g](x) := -\frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g_{ab}(x)} \frac{1}{2} \int_{\mathcal{I}} \phi A'[g] \phi^* d\mu_g \]  \hfill (19)

However, no proof of the convergence of the series above was given in [2] for the general case, but the method was checked in concrete cases finding that the series above converges really as supposed. In [2], it was showed also that, assuming reasonable mathematical properties of the involved functions, this approach in for four-dimensional operators \( A' = -\Delta + \xi R(x) + m^2 \) should
produce a stress tensor which is conserved and gives rise to the conformal anomaly. In [2], it was also (not rigorously) proven that the ambiguity arising from the presence of the arbitrary scale $\mu^2$ gives rise to conserved geometric terms added to the stress tensor, in agreement with Wald’s axioms.

We expect that the not completely rigorous procedures employed in [2] make sense provided the usual heat-kernel “asymptotic” expansion at $t \to 0$ can be derived in the variables which range in the manifold producing a similar expansion (this result is not trivial at all) and provided the series (18) can be derived under the symbol of summation (also this fact is not so obvious). Therefore, in the next parts of this work, we shall investigate also similar issues before we prove and generalize all the results found in [2].

III. The local $\zeta$-function and the one-loop stress tensor.

In this part and within our general hypotheses, we develop a rigorous theory of the $\zeta$ function of the stress tensor and give a rigorous proof of some properties of particular tensorial $\zeta$ functions introduced in [2].

The first subsection is devoted to generalize some properties of the heat-kernel concerning the smoothness of several heat-kernel expansions necessary in the second subsection.

A. The smoothness of the heat-kernel expansion and the $\zeta$ function.

A first very useful result, which we state in the form of a lemma, concerns the smoothness of the heat-kernel expansion for $t \to 0$ (Theorem 1.3 of [1]) and the possibility of deriving term by term such an “asymptotic expansion”.

Before to state the result it is worth stressing that, in the trivial case $|\alpha| = |\beta| = 0$, the statement of the lemma below and the corresponding proof include the point (a2) in Theorem 1.3 of [1] given without proof there.

**Lemma 2.1.** Let us assume our general hypotheses on $M$ and $A'$. For any $u \in M$ there is an open neighborhood $I_u$ centered on $u$ such that, for any local coordinate system defined therein, for any couple of points $x, y \in I_u$, for any couple of multindices $\alpha, \beta$ and for any integer $N > D/2 + 2|\alpha| + 2|\beta|$ ($D/2 + 2$ if $|\alpha| = |\beta| = 0$) the heat-kernel expansion (a) of Theorem 1.3 in [1] can be derived term by term obtaining ($\eta \in (0, 1)$ is fixed arbitrarily as usual),

$$D^\alpha_x D^\beta_y K(t, x, y) = \sum_{j=0}^{N} a_j(x, y|A)t^j + O_{\eta,N}(t; x, y)$$

where the derivatives are computed in the common coordinate system given above and the function $(t, x, y) \mapsto O^{(\alpha, \beta)}_{\eta,N}(t; x, y) \in C^0([0, +\infty) \times I_u \times I_u)$ at least, and for any positive
constant $K_{K_t,N}^{(\alpha,\beta)}$ and $0 \leq t < K_{K_t,N}^{(\alpha,\beta)}$, one gets
\[ |O_{K_t,N}^{(\alpha,\beta)}(t; x, y)| < M_{K_t,N}^{(\alpha,\beta)} |t| \] (21)

$M_{K_t,N}^{(\alpha,\beta)}$ being a corresponding positive constant not dependent on $x, y \in I_u$ and $t$.

**Proof.** See Appendix □.

The next lemma concerns the possibility of interchanging the operators $D_x^\alpha, D_y^\beta$ with the symbol of series in the eigenvector expansion for the heat kernel given in (b) of Theorem 1.1 in [1].

**Lemma 2.2.** Within our hypotheses on $M$ and $A'$, the eigenvector expansion of the heat kernel given in (b) of Theorem 1.1 in [1]

\[ K(t, x, y | A) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j^*(y) \] (22)

where $t \in (0, +\infty)$, $x, y \in M$ and the real numbers $\lambda_j$ ($0 \leq \lambda_0 \leq \lambda_1, \leq \lambda_2, \leq \cdots$) are the eigenvalues of $A$ with corresponding orthogonal normalized eigenvector $\phi_j$, can be derived in $x$ and $y$ passing the derivative operators under the symbol of series. Indeed, in a coordinate system defined in a sufficiently small neighborhood $I_u$ of any point $u \in M$, for $x, y \in I_u$, for $t \in (0, +\infty)$ and for any couple of multindices $\alpha, \beta$

\[ D_x^\alpha D_y^\beta K(t, x, y | A) = \sum_{j=0}^{\infty} e^{-\lambda_j t} D_x^\alpha \phi_j(x) D_y^\beta \phi_j^*(y). \] (23)

Moreover, for any $T > 0$ the following upper bounds hold

\[ |e^{-\lambda_j t} D_x^\alpha \phi_j(x) D_y^\beta \phi_j^*(y)| \leq P_T^{(\alpha,\beta)} e^{-\lambda_j (t-2T)}, \] (24)

\[ |D_x^\alpha D_y^\beta K(t, x, y | A) - D_x^\alpha D_y^\beta P_0(x, y | A)| \leq P_T^{(\alpha,\beta)} \sum_{j \in \mathbb{N}} e^{-\lambda_j (t-2T)} \] (25)

\[ \leq Q_T^{(\alpha,\beta)} e^{-\lambda (t-2T)} \] (26)

where $x, y \in I_u$ and $t \in (2T, +\infty)$, $P_T^{(\alpha,\beta)}$ and $Q_T^{(\alpha,\beta)}$ are positive constants which do not depend on $t, x, y, P_0(x, y | A)$ is the integral kernel of the projector onto the kernel of $A$, $\lambda$ is the value of the first strictly positive eigenvalue, the prime on the summation symbol indicates that the summation on the vanishing eigenvalues is not considered, and finally

\[ P_T^{(\alpha,\beta)} = \left[ \sup_{x \in I_u} \|D_x^\alpha K(T, x, . | A)\|_{L^2(M, d\mu_g)} \right] \left[ \sup_{y \in I_u} \|D_y^\beta K(T, ., y | A)\|_{L^2(M, d\mu_g)} \right]; \] (27)
Therefore, the convergence of the series in (23) is absolute in uniform sense for \((t, x, y)\) belonging in any set \([\gamma, +\infty) \times I_u \times I_u, \gamma > 0\).

**Proof.** See Appendix. ✷

**Remark.** The right hand side of (23) can be also written down as

\[
P_T^{(\alpha,\beta)} \int_M d\mu_g(z) \{K(t - 2T, z, z|A) - P_0(z, z|A)\} = P_T^{(\alpha,\beta)} Tr \left\{K(t - 2T) - P_0\right\}.
\] (28)

The two lemmata above enable us to state and prove a theorem concerning the derivability of the \(\zeta\) function. First of all let us give some definitions (in the following we shall refer to **Definition 2.1** and **Definition 2.2** in [1]).

**Definition 2.1.** Let us assume our general hypotheses on \(M\) and \(A'\). Fixing a sufficiently small neighborhood \(I_u\) of any point \(u \in M\), considering a coordinate system defined in \(I_u\) and choosing a couple of multindices \(\alpha,\beta\), the \textbf{off-diagonal derived local \(\zeta\) function} of the operator \(A\) is defined for \(x, y \in I_u, \text{Re } s > D/2 + |\alpha| + |\beta|\) as

\[
\zeta^{(\alpha,\beta)}(s, x, y|A/\mu^2) := D_x^\alpha D_y^\beta \zeta(s, x, y|A/\mu^2)
\] (29)

provided the right hand side exists, where both derivatives are computed in the coordinate system defined above.

**Definition 2.2.** Let us assume our general hypotheses on \(M\) and \(A'\). Fixing a sufficiently small neighborhood \(I_u\) of any point \(u \in M\), considering a coordinate system defined in \(I_u\) and choosing a couple of multindices \(\alpha,\beta\), the \textbf{derived local \(\zeta\) function} of the operator \(A\) is defined for \(x \in I_u, \text{Re } s > D/2 + |\alpha| + |\beta|\) as

\[
\zeta^{(\alpha,\beta)}(s, x|A/\mu^2) := \left\{D_x^\alpha D_y^\beta \zeta(s, x, y|A/\mu^2)\right\}_{x=y}
\] (30)

provided the right hand side exists, where both derivatives are computed in the coordinate system defined above.

**Remark.** The use of a common coordinate system either for \(x\) and \(y\) is essential in these definitions.

The following theorem proves that the given definitions make sense.

**Theorem 2.1.** Let us assume our general hypotheses on \(M\) and \(A'\). The local off-diagonal \(\zeta\) function of the operator \(A\) defined for \(x, y \in M, \text{Re } s > D/2, \mu > 0\) (\(\mu\) being a constant with the dimension of a mass)

\[
\zeta(s, x, y|A/\mu^2) = \frac{1}{\Gamma(s)} \int_0^{+\infty} d(\mu^2 t) (\mu^2 t)^{s-1} \{K(t, x, y|A) - P_0(x, y|A)\}
\] (31)
can be derived in $x$ and $y$ under the symbol of integration in a common coordinate system defined in a sufficiently small neighborhood $I_u$ of any point $u \in \mathcal{M}$, provided $\text{Re } s$ is sufficiently large. In particular, for any choice of multindices $\alpha, \beta$ and $x, y \in I_u$ it holds

(a) for $\text{Re } s > D/2 + |\alpha| + |\beta|$ the derived local $\zeta$ functions are well-defined holding

$$D_x^\alpha D_y^\beta \zeta(s, x, y|A/\mu^2) = \frac{1}{\Gamma(s)} \int_0^{+\infty} d(\mu^2 t) (\mu^2 t)^{s-1} D_x^\alpha D_y^\beta \{ K(t, x, y|A) - P_0(x, y|A) \} . \quad (32)$$

Moreover, the right hand side of $(32)$ defines a s-analytic function which belongs to $C^0(\{ s \in \mathcal{C} | \text{Re } s > D/2 + |\alpha| + |\beta| \} \times I_u \times I_u)$ together with all its s-derivatives.

(b) Whenever $x \neq y$ are fixed in $I_u$,

(1) the right hand side of $(32)$ can be analytically continued in the variable $s$ in the whole complex plane.

(2) Varying $s \in \mathcal{C}$ and $(x, y) \in (I_u \times I_u) - \mathcal{D}_{I_u}$, the s-continued function in $(31)$ defines an everywhere s-analytic function which belongs to $C^\infty(\mathcal{C} \times \{(I_u \times I_u) - \mathcal{D}_{I_u}\})$ (where $\mathcal{D}_{I_u} := \{(x, y) \in I_u \times I_u | x = y \}$) and it holds, $\mathcal{C} \times \{(I_u \times I_u) - \mathcal{D}_{I_u}\}$,

$$D_x^\alpha D_y^\beta \zeta(s, x, y|A/\mu^2) = \zeta^{(\alpha, \beta)}(s, x, y|A/\mu^2) \quad (33)$$

where the function $\zeta$ in the left hand side and the function $\zeta^{(\alpha, \beta)}$ in the right hand side are the respective s-analytic continuations of the initially defined $\zeta$ function $(31)$ and the right hand side of $(32)$.

(3) Eq. $(32)$ holds also when the left hand side is replaced by the s-continued function $\zeta^{(\alpha, \beta)}$ for $\text{Re } s > 0$, or everywhere provided $D_x^\alpha D_y^\beta P_0(x, y|A) = 0$ in the considered point $(x, y)$.

(c) Whenever $x = y$ is fixed in $I_u$,

(1) the right hand side of $(30)$ can be analytically continued in the variable $s$ in the complex plane obtaining a meromorphic function with possible poles, which are simple poles only, situated in the points

$$s_j^{(\alpha, \beta)} = D/2 + |\alpha| + |\beta| - j, \quad j = 0, 1, 2, \cdots \quad \text{if } D \text{ is odd};$$

$$s_j^{(\alpha, \beta)} = D/2 + |\alpha| + |\beta| - j, \quad j = 0, 1, 2, \cdots D/2 - 1 + |\alpha| + |\beta| \quad \text{if } D \text{ even}. \quad \text{These poles and the corresponding residues are the same of the set of analytic functions, labeled by the integer } N > D/2 + 2|\alpha| + 2|\beta|, \quad (N > D/2 + 2 \text{ if } |\alpha| = |\beta| = 0)$$

$$R_N(s, x)_{\mu_0}^{-2} := \frac{\mu^{2s}}{(4\pi)^{D/2} \Gamma(s)} \sum_{j=0}^{N} \int_0^{\mu_0^{-2}} dt \times$$

$$\times \left\{ D_x^\alpha D_y^\beta e^{-\sigma(x, y)/2t} a_j(x, y|A) \right\}_{x=y} t^{s-1-D/2+j} , \quad (34)$$

defined for $x \in I_u$ and $\text{Re } s > D/2 + |\alpha| + |\beta|$ and then continued in the s-complex plane. $\mu_0$ is an arbitrary strictly positive mass scale which does not appear in the residues.

(2) Varying $x \in I_u$, the s-continued function belongs to $C^0(\mathcal{C} - \mathcal{P}^{(\alpha, \beta)} \times \mathcal{M})$ together with all its s-derivatives, $\mathcal{P}^{(\alpha, \beta)}$ being the set of the actual poles (each for some $x$) among the points listed
well-defined in a neighborhood of \( (s, x) \in \mathcal{C} \). Moreover, for any coordinate \( x^k \) and \( (s, x) \in \{ \mathcal{C} - (\mathcal{P}^{(\alpha, \beta)} \cup \mathcal{P}^{(\alpha + 1_k, \beta)} \cup \mathcal{P}^{(\alpha, \beta + 1_k)}) \} \times I_\mu \), the \( \frac{\partial}{\partial x^k} \zeta^{(\alpha, \beta)}(s, x; A/\mu^2) \) exists, is continuous in \((s, x)\) with all of its \( s \) derivatives, analytic in the variable \( s \) and

\[
\frac{\partial}{\partial x^k} \zeta^{(\alpha, \beta)}(s, x; A/\mu^2) = \zeta^{(\alpha + 1_k, \beta)}(s, x; A/\mu^2) + \zeta^{(\alpha, \beta + 1_k)}(s, x; A/\mu^2)
\]

(35)

where \( \zeta^{(\alpha, \beta)} \) is the analytic continuation of the initially defined function (30).

(d) For \( x, y \in I_\mu \), the analytic continuations of the right hand sides of (29) and (30) are well-defined in a neighborhood of \( s = 0 \) and it holds, and the result does not depend on the values of \( \mu_0 > 0 \) and \( \mu > 0 \),

\[
\left[D_x^\alpha D_y^\beta \zeta(s, x, y; A/\mu^2)\right]_{\alpha=0} + D_x^\alpha D_y^\beta \rho(x, y) = \delta_D \delta_{x, y} \lim_{\mu_0 \to 0} R_N(s, x; A/\mu_0^{-2})
\]

(36)

where \( |s=0| \) means the analytic continuation from \( \text{Re} \, s > D/2 + |\alpha| + |\beta| \) to \( s = 0 \) of the considered function and \( N \) is any integer \( > D/2 + 2|\alpha| + 2|\beta| \) \( (D/2 \) whenever \( |\alpha| = |\beta| = 0 \)). Finally, we have defined \( \delta_D = 0 \) if \( D \) is odd and \( 1 \) otherwise, \( \delta_{x, y} = 0 \) if \( x \neq y \) or \( 1 \) otherwise.

**Sketch of Proof.** The proof of this theorem is a straightforward generalization of the proof of Theorem 2.2 in \([1]\), so we just sketch this proof. As in the proof of Theorem 2.2 in \([1]\), the main idea is to break off the integration in (31) for \( \text{Re} \, s > D/2 \) as

\[
\zeta(s, x, y; A/\mu^2) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^{+\infty} dt \, t^{s-1} \left[ K(t, x, y; A) - P_0(x, y; A) \right]
\]

(37)

\[
= \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} \{ \ldots \} + \frac{\mu^{2s}}{\Gamma(s)} \int_{\mu_0^{-2}}^{+\infty} \{ \ldots \}
\]

(38)

where \( \mu_0 > 0 \) is an arbitrary mass cutoff. Then one studies the possibility of computing the derivative passing \( D_x^\alpha \) and \( D_y^\beta \) under the symbol of integration in both integrals in the right hand side above. This is possible provided the absolute values of the derived integrand are \( x, y \) uniformly bounded by integrable functions dependent on \( \alpha, \beta \) in general, for any choice of \( \alpha \) and \( \beta \). This assures also the continuity of the derivatives because the derivatives of the integrands are continuous functions. The analyticity in \( s \) can be proved by checking the Cauchy-Riemann conditions passing the derivative under the symbol of integration once again. The \( s \)-derivatives of the integrand at any order can be still proven to be bounded with the same procedure. Then the proof deal with similarly to the proof of Theorem 2.2 of \([1]\). One uses Lemma 2.2 and (26) (choosing \( 2T < \mu_0^{-2} \)) in place of the corresponding formula (99) of \([1]\), to prove that the latter integral in the right hand side of (38) can be derived under the symbol of integration obtaining a \( s \)-analytic function continuous with all of its \( s \) derivatives, for \( s \in \mathcal{C} \) and \( x, y \in I_\mu \). The former integral can be studied employing Lemma 1.1 and, in particular, (20). The requirement \( N > D/2 + 2 \) in the expansion in \([1]\) has to be changed \( N > D/2 + 2|\alpha| + 2|\beta| \) in the present case. The requirement in the point (a) \( \text{Re} \, s > D/2 + |\alpha| + |\beta| \) arises by the term with \( j = 0 \) in the heat kernel expansion when all the derivatives either in \( x \) and in \( y \) act on the exponential producing a factor \( t^{-|\alpha| - |\beta|} \) and posing \( x = y \) in the end. Eq. (101) and the successive ones
of \( O_\eta \) have to be changed employing \( O_\eta^{(\alpha,\beta)} \) in place of \( O_\eta \) and \( t^{s-1+N-D/2-|\alpha|-|\beta|} \) in place of \( t^{s-1+N-D/2} \).

The requirement \( D_x^\alpha D_y^\beta P_0(x,y|A) = 0 \) in (b3) is simply due to the divergence of the integral \( \int_0^{\mu_0^2} dt t^{s-1} \) for \( s \leq 0 \). (b3) is essentially due to the presence of the factor \( 1/\Gamma(s) \) in all considered integrals, which vanishes with a simple zero as \( s \to 0 \). \( \square \)

Comments
(1) The right hand side of (b3), for \( x = y \) and when \( D \) is even has the form

\[
\frac{D_x^\alpha D_y^\beta a_{D/2}(x,y|A)}{(4\pi)^{D/2}} + \cdots
\]

(39)

Where the dots indicate a finite number of further terms consisting of derivatives of product of heat kernel coefficients and powers of \( \sigma(x,y) \), computed in the coincidence limit of the arguments. In the case \( |\alpha| = |\beta| = 0 \) this agrees with the found result for the simple local \( \zeta \) function given in [1].

(2) It is worth noticing that the right hand side of (b3) proves that the procedures of \( s \)-continuing \( D_x^\alpha D_y^\beta \zeta(s,x,y|A/\mu^2) \) and that of taking the coincidence limit of arguments \( x,y \) generally do not commute. This means that, understanding both sides in the sense of the analytic continuation, in general

\[
\zeta^{(\alpha,\beta)}(s,x,y|A/\mu^2)|_{x=y} \neq \zeta^{(\alpha,\beta)}(s,x|A/\mu^2).
\]

Above the coincidence limit is taken after the analytic continuation. Obviously, whenever \( \Re s > D/2 + |\alpha| + |\beta| \)

\[
\zeta^{(\alpha,\beta)}(s,x,y|A/\mu^2)|_{x=y} = \zeta^{(\alpha,\beta)}(s,x|A/\mu^2).
\]

(3) The point (b2) proves that, for \( x \neq y \), the Green function of any operator \( A^n, n = 0,1,2,\ldots \)
defined in [1] via local \( \zeta \) function, is \( C^\infty \) as one could have to expect.

A second and last theorem concerns the possibility to compute the derived local \( \zeta \) functions through a series instead of an integral.

**Theorem 2.2** Within our hypotheses on \( \mathcal{M} \) and \( A' \) and \( \mu > 0 \), the (off-diagonal and not) derived local \( \zeta \) function can be computed as the sum of a series. Indeed, choosing a couple of multindices \( \alpha, \beta \), in a common coordinate system defined in a sufficiently small neighborhood of any point \( u \in \mathcal{M} \) one has, in the sense of the punctual convergence

\[
\zeta^{(\alpha,\beta)}(s,x,y|A/\mu^2) = \sum_{j \in \mathbb{N}}' \left( \frac{\lambda}{\mu^2} \right)^{-s} D_x^\alpha \phi_j(x) D_y^\beta \phi_j^*(y)
\]

(42)

\[
\zeta^{(\alpha,\beta)}(s,x|A/\mu^2) = \sum_{j \in \mathbb{N}}' \left( \frac{\lambda}{\mu^2} \right)^{-s} D_x^\alpha \phi_j(x) D_x^\beta \phi_j^*(x),
\]

(43)
provided \( \text{Re } s > 3D/2 + |\alpha| + |\beta| \) and \((x, y) \in I_u \times I_u\).

**Proof.** First of all it is worth stressing that, in the considered domain for \( s \), the functions are continuous in all variables and

\[
\zeta^{(\alpha,\beta)}(s, x, y| A/\mu^2)|_{x=y} = \zeta^{(\alpha,\beta)}(s, x| A/\mu^2).
\]

So we perform our proof in the general case \( x \neq y \) and then consider the coincidence limit of arguments. Therefore, from Theorem 2.1., for \( \text{Re } s > D/2 + |\alpha| + |\beta| \), one has

\[
\zeta^{(\alpha,\beta)}(s, x, y| A/\mu^2) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt \ t^{s-1}D^\alpha_x D^\beta_y \{ K(t, x, y| A) - P_0(x, y| A) \} + \frac{\mu^{2s}}{\Gamma(s)} \int_{\mu_0^{-2}}^{+\infty} dt \ t^{s-1}D^\alpha_x D^\beta_y \{ K(t, x, y| A) - P_0(x, y| A) \}.
\]

\( \mu_0 > 0 \) arbitrarily. Let us focus attention on the second integral. It can be written also

\[
\frac{\mu^{2s}}{\Gamma(s)} \int_{\mu_0^{-2}}^{+\infty} dt \ \sum_{j \in \mathbb{N}} t^{s-1}D^\alpha_x \phi_j(x)D^\beta_y \phi_j^*(y)e^{-\lambda_j t},
\]

where we have used Lemma 2.2. We want to show that it is possible to interchange the symbol of series with that of integration. We shall prove a similar fact for the other integral in (45), then the well-known formula \( a > 0 \)

\[
a^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt \ t^{s-1}e^{-at}
\]

will complete the proof of the theorem.

To prove the possibility of interchanging the integration with the summation in the integral (46) it is sufficient to show that the absolute value of the function after the summation symbol is integrable in the measure \( \int dt \times \sum_j \), then Fubini’s theorem allows one to interchange the integrations. From Lemma 2.2, we know that for \( t > 2T > 0 \)

\[
\sum_{j \in \mathbb{N}} t^{s-1}D^\alpha_x \phi_j(x)D^\beta_y \phi_j^*(y)e^{-\lambda_j t} \leq P^{(\alpha,\beta)}_T \sum_{j \in \mathbb{N}} t^{Re s-1}e^{-\lambda_j (2t-2T)} \leq Q^{(\alpha,\beta)}_T t^{Re s-1}e^{-\lambda(t-2T)},
\]

where \( \lambda \) is the first strictly positive eigenvalue of \( A \). We choose the constant \( T < \mu_0^{-2}/2 \). The \( t \)-integration in \([\mu_0^{-2}, +\infty)\) of the last line above is finite for any \( s \in \mathcal{C} \). Thus, a part of Fubini’s theorem prove that the function after the summation symbol in (46) is integrable in the product measure.

Let us perform a similar proof for the first integral in the right hand side of (45). It can be written down

\[
\frac{\mu^{2s}}{\Gamma(s)} \int_0^{\mu_0^{-2}} dt \ \sum_{j \in \mathbb{N}} t^{s-1}D^\alpha_x \phi_j(x)D^\beta_y \phi_j^*(y)e^{-\lambda_j t}.
\]
We want to show that it is possible to interchange the symbol of series with that of integration.

Posing $T = t/4$ we have, for $t \in (0, \mu^{-2})$

$$
\sum_{j \in \mathcal{N}} |t^{s-1}D_x^\alpha \phi_j(x)D_y^\beta \phi^*_j(y)e^{-\lambda_j t}| \leq P^{(\alpha,\beta)}_{t/4} t^{Re s - 1} Tr \left\{ K_{t/2} - P_0 \right\} .
$$

where for (27)

$$
P^{(\alpha,\beta)}_T := \left[ \sup_{x \in \bar{I}_u} ||D_x^\alpha K(T, x, . | A)|| \right] \left[ \sup_{y \in \bar{I}_u} ||D_y^\beta K(T, . , y| A)|| \right].
$$

Employing (20) of Lemma 2.1 and taking account of the finite volume of the manifold one finds that there is a positive constant $A$ such that, for $t \in (0, \mu^{-2})$

$$
P^{(\alpha,\beta)}_T \leq AT^{-D-|\alpha|-|\beta|}.
$$

This is due to the leading order for $t \to 0$ of the heat-kernel expansion (20). This upper bound, inserted in (50) with $T = t/4$, together with the $x$-integral of the heat-kernel expansion (19) in Theorem 1.3 of [1], entails

$$
\sum_{j \in \mathcal{N}} |t^{s-1}D_x^\alpha \phi_j(x)D_y^\beta \phi^*_j(y)e^{-\lambda_j t}| \leq Bt^{Re s - 1 - 3D/2 - |\alpha|-|\beta|} .
$$

where $B$ is a positive constant.

Dealing with as in previously considered case, for $Re s > 3D/2 + |\alpha| + |\beta|$, we can interchange the symbol of integral with that of series also in the second integral of (45), then (47) entails the thesis. ✷

Notice that, in the case $|\alpha| = |\beta| = 0$, the convergence of the series (43) arises for $Re s > D/2$ and it is uniform as well-known [1]. Actually, our theorem uses a quite rough hypothesis. Nevertheless, this is enough for the use we shall make of the theorem above.

Following the way traced out in 1.3, we can give a precise definition concerning the $\zeta$ function of the stress tensor. We shall assume, more generally than in [2], $A' := -\Delta + V$ where $V(x) := m^2 + \xi R + V'(x)$ and $V'$ is real and $\in C^\infty(M)$ does not depend on the metric. Moreover, in this paper we consider a general $D$ dimensional manifold rather than the more physical case $D = 4$ studied in [2]. Also, as required by our general hypotheses, $A'$ must be positive. It is worth stressing that this does not entails necessarily $m^2 + \xi R(x) > 0$ everywhere also when $V' \equiv 0$, and neither $V(x) > 0$ everywhere in the general case (see [1]).

B. The $\zeta$-regularized stress tensor and its properties.

For future convenience, let us define the symmetric tensorial field in a local coordinate system,

$$
T_{ab}[\phi, \phi^*, g](x) := \frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g_{ab}(x)} \frac{1}{2} \int_{\mathcal{I}} \phi A'[g] \phi^* d\mu_g ,
$$

15
where $\phi \in C^\infty(M)$ and the functional derivative has been defined in 1.2. The precise form of $T_{ab}[\phi, \phi^*, g](x)$ reads in our case

$$
T_{ab}[\phi, \phi^*, g](x) = \frac{1}{2} (\nabla_a \phi(x) \nabla_b \phi^*(x) + \nabla_a \phi^*(x) \nabla_b \phi(x))
- \frac{1}{2} g_{ab}(x) \left[ \nabla_c \phi(x) \nabla^c \phi^*(x) + (m^2 + V'(x)) |\phi|^2(x) \right]
+ \xi \left[ \left( R_{ab}(x) - \frac{1}{2} g_{ab}(x) R(x) \right) |\phi|^2(x) + g_{ab}(x) \nabla_c \nabla^c |\phi|^2(x) 
- \nabla_a \nabla_b |\phi|^2(x) \right].
$$

(54)

A few trivial manipulations which make use of $A' \phi_j = \lambda_j \phi_j$ lead us to a simpler form for $T_{ab}[\phi_j, \phi_j^*, g](x)$, namely

$$
T_{ab}[\phi_j, \phi_j^*, g](x) = \frac{1}{2} (\nabla_a \phi_j(x) \nabla_b \phi_j^*(x) + \nabla_a \phi_j^*(x) \nabla_b \phi_j(x))
- \xi \nabla_a \nabla_b |\phi_j|^2 + \left( \xi - \frac{1}{4} \right) g_{ab}(x) \Delta |\phi_j|^2(x)
+ \xi R_{ab}(x) |\phi_j|^2 - \frac{g_{ab}(x)}{2} \lambda_j |\phi_j|^2(x).
$$

(55)

Following the insights given in 1.3 as well as [3], we can give the following definition.

**Definition 2.3.** Within our hypotheses on $M$ and $A' := -\Delta + m^2 + \xi R + V'(x)$ defined above ($m, \xi \in \mathbb{R}$), the local $\zeta$ function of the stress tensor is the symmetric tensorial field defined in local coordinates as

$$
Z_{ab}(s, x|A/\mu^2) := 2 \frac{s}{\mu^2} \zeta_{ab}(s + 1, x|A/\mu^2) + s g_{ab}(x) \zeta(s, x|A/\mu^2)
$$

(56)

where $\zeta_{ab}(s, x|A/\mu^2)$ is defined as the sum of the series below, in a sufficiently small neighborhood $I_u$ of any point $u \in M$ and for $Re \ s > 3D/2 + 2$,

$$
\sum_{j \in \mathbb{N}} \left( \frac{\lambda_j}{\mu^2} \right)^s T_{ab}[\phi_j, \phi_j^*, g](x),
$$

(57)

and $T_{ab}[\phi_j, \phi_j^*, g](x)$ is defined in (3) and (8) with respect to a base of smooth orthogonal normalized eigenvector of $A$.

**Comments**

(1) The definition given above makes sense since the relevant series converges for $Re \ s > 3D/2+2$ because of Theorem 2.2. Notice that the given definition does not depend on the base of smooth orthogonal normalized eigenvectors of $A$ (take account that each eigenspace has finite dimension as follows from Theorem 1.1 in [4]).

(2) The fact that the coefficients $Z_{ab}(s, x|A/\mu^2)$ do define a tensor is a direct consequence of
Theorem 2.3. In our general hypotheses on $\mathcal{M}$ and $A'$ (a) each component of $Z_{ab}(s, x|A/\mu^2)$ can be analytically continued into a meromorphic function of $s$ whenever $x$ is fixed. In particular, in the sense of the analytic continuation, it holds, for $x$ belonging to a sufficiently small neighborhood of any point $u \in \mathcal{M}$

\[
Z_{ab}(s, x|A/\mu^2) = \frac{s}{\mu^2} \left[ \zeta^{(1a,1b)}(s+1, x|A/\mu^2) + \zeta^{(1b,1a)}(s+1, x|A/\mu^2) \right] + \frac{2s}{\mu^2} \left[ (\xi - \frac{1}{4}) g_{ab}(x) \Delta + \xi R_{ab}(x) - \xi \nabla_a \nabla_b \right] \zeta(s+1, x|A/\mu^2)
\]
(b) The possible poles of each component of \( Z_{ab}(s,x|A/\mu^2) \), which are simple poles only, are situated in the points

\[
\begin{align*}
    s_j &= D/2 - j + 1, & j = 0, 1, 2, \ldots & \text{if } D \text{ is odd;} \\
    s_j &= D/2 - j + 1, & j = 0, 1, 2, \ldots D/2 & \text{if } D \text{ even.}
\end{align*}
\]

(c) Varying \( x \in I_u \) and \( s \in C \) the \( s \)-analytically continued symmetric tensorial field \( (s,x) \mapsto Z_{ab}(s,x|A/\mu^2) \) defines a \( s \)-analytic tensorial field of \( C^0((C - \mathcal{P}) \times I_u) \) together with all its \( s \) derivatives, where \( \mathcal{P} \) is the set of the actual poles (each for some \( x \)) among the points listed above.

Proof. Sketched above. \( \square \)

Remark. Eq. (59) could be used as an independent definition of the \( \zeta \) function of the stress tensor. The important point is that it does not refer to any series of eigenvectors. It could be considered as the starting point for the generalization of this theory in the case the spectrum of the operator \( A \) is continuous provided the functions in the right hand side of (59) are defined in terms of \( t \) integrations of derivatives of the heat kernel.

**Definition 2.4.** In our general hypotheses on \( M \) and \( A' \) and for \( x \in I_u \) where \( I_u \) is a sufficiently small neighborhood of \( u \in M \), the one-loop renormalized stress tensor is defined in a local coordinate system in \( I_u \), by the set of functions \( (a,b = 1, \ldots, D) \)

\[
(T_{ab}(x|A))_{\mu^2} := \frac{1}{2} \frac{d}{ds}|_{s=0} Z_{ab}(s,x|A/\mu^2),
\]

where the tensorial field \( Z_{ab} \) which appears in the right hand side is the \( s \)-analytic continuation of that defined above and \( \mu^2 > 0 \) is any fixed constant with the dimensions of a squared mass.

We can state and prove the most important properties of \( \langle T_{ab}(x|A) \rangle_{\mu^2} \) in the following theorem. These results generalize previously obtained results \( [2, 13] \) for a more general operator \( A \) and for any dimension \( D > 0 \).

**Theorem 2.4.** In our general hypotheses on \( M \) and \( A' \), the functions \( x \mapsto \langle T_{ab}(x|A) \rangle_{\mu^2} \) defined above satisfy the following properties.

(a) The functions \( x \mapsto \langle T_{ab}(x|A) \rangle_{\mu^2} \) \( (a,b = 1, 2, \ldots, D) \) define a \( C^\infty \) symmetric tensorial field on \( M \).

(b) This tensor is conserved for \( V' \equiv 0 \), and more generally

\[
\nabla^a \langle T_{ab}(x|A) \rangle_{\mu^2} = -\frac{1}{2} \langle \phi^2(x|A) \rangle_{\mu^2} \nabla_b V'(x)
\]

everywhere in \( M \).
(c) For any rescaling $\mu^2 \to \alpha \mu^2$, where $\alpha > 0$ is a pure number, one has

$$\langle T_{ab}(x|A)\rangle_{\mu^2} \to \langle T_{ab}(x|A)\rangle_{\alpha \mu^2} = \langle T_{ab}(x|A)\rangle_{\mu^2} + (\ln \alpha) t_{ab}(x|A) \tag{62}$$

where $t_{ab}(x|A) = Z_{ab}(0, x|A)/2$, which coincides also with the residue of the pole of $\zeta_{ab}(s+1, x|A)$ at $s = 0$, is a, conserved for $V' \equiv 0$, symmetric tensor not dependent on $\mu$ built up by a linear combination of product of the metric, curvature tensors, $V'(x)$ and their covariant derivatives evaluated at the point $x$. In general it satisfies

$$\nabla^a t_{ab}(x|A) = -\delta_D \frac{a_D/2-1(x,x|A)}{2(4\pi)^D/2} \nabla_b V'(x), \tag{63}$$

where $\delta_D = 0$ when $D$ is odd and $\delta_D = 1$ otherwise. In terms of heat-kernel coefficients one has also

$$t_{ab}(x|A) = \frac{\delta_D}{(4\pi)^{D/2}} \left\{ a_{D/2-1,(ab)}(x,x|A) + \frac{g_{ab}(x)}{2} a_{D/2}(x,x|A) + \left[ \left( \xi - \frac{1}{4} \right) g_{ab}(x) \Delta + \xi R_{ab}(x) - \xi \nabla_a \nabla_b \right] a_{D/2-1}(x,x|A) \right\}, \tag{64}$$

where we have employed the notations (using the same coordinate system both for $x$ and $y$)

$$a_{j,(ab)}(x,x|A) := \frac{1}{2} \left[ \left( \nabla_{(x)a} \nabla_{(y)b} + \nabla_{(y)a} \nabla_{(x)b} \right) a_j(x,y|A) \right] |_{x=y} \tag{65}$$

(d) Concerning the trace of $\langle T_{ab}(x|A)\rangle_{\mu^2}$ one has

$$g^{\alpha\beta}(x) \langle T_{ab}(x|A)\rangle_{\mu^2} = \left( \frac{\xi_D - \xi}{4\xi_D - 1} \Delta - m^2 - V'(x) \right) \langle \phi^2(x|A)\rangle_{\mu^2} + \delta_D \frac{a_{D/2}(x,x|A)}{(4\pi)^{D/2}} - P_0(x,x|A) \tag{66}.$$

Above, $\langle \phi^2(x|A)\rangle_{\mu^2}$ is the value of the averaged quadratic fluctuations of the field computed by the $\zeta$-function approach [13, 13].

(The coefficient $(4\xi_D - 1)^{-1}$ above is missprinted in [13] where $(2\xi_D)^{-1}$ appears in place of it.)

**Sketch of Proof.** Barring the issue concerning the smoothness, the property (a) is a trivial consequence of the corresponding fact for $Z_{ab}(s, x|A/\mu^2)$ discussed in Comment (2) after **Definition 2.3**. The tensorial field belongs to $C^\infty$ because of the $C^\infty$ smoothness of the functions $(s,x) \mapsto Z_{ab}(s, x|A/\mu^2)$ for $(s,x) \in J_0 \times \mathcal{U}$, where $J_0$ and $\mathcal{U}$ are respectively neighborhood of $s = 0$ in $\mathcal{O}$ and $u \in \mathcal{M}$. Indeed, first of all, no pole at $s = 0$ arises in the functions $(s,x) \mapsto Z_{ab}(s, x|A/\mu^2)$ and in their $x$ derivatives. This is because, considering (56) and (c2) of **Theorem 2.1**, one notices that if any pole appears in the various $\zeta^{(\alpha,\beta)}$ functions used building up $Z_{ab}$ it has to be a simple pole. Anyhow, the factor $s$ makes the global functions $Z_{ab}$ regular at $s = 0$. Using recursively (c2) of **Theorem 2.1** one has that each function $x \mapsto Z_{ab}(s, x|A/\mu^2)$
is $C^\infty$ in a neighborhood of $u$ for any fixed $u \in \mathcal{M}$ and $s = 0$. More generally, this results holds for $s$ fixed in neighborhood of $0$ because, by (c1) of Theorem 2.1, one has that no pole can arise in a open disk centered in $s = 0$ with radius $\rho = 1/2$. The functions $Z_{ab}$ and all their $x$ derivatives are also $s$-analytic for $x$ fixed in a neighborhood of $0$. Then, we can conclude that any function $(s, x) \mapsto Z_{ab}(s, x|A/\mu^2)$ is $C^\infty$ in a neighborhood of $(0, u)$ for any fixed $u \in \mathcal{M}$. The $C^\infty$ smoothness of the stress tensor then follows trivially from (66) directly.

The property (b) can be proved as follows. From the point (c2) of Theorem 2.1 and taking account of (a) of Theorem 2.3 and the definition (60), we have that (b) holds true if

$$\nabla^a Z_{ab}(s, x|A/\mu^2) = -Z(s, x|A/\mu^2)\nabla_b V'(x)$$

for the considered point $x$ and $s \in \mathcal{C}$ away from the poles, the function $Z$ in the right hand side that of the field fluctuations (see Definition 2.7 in [1]). By the theorem of the uniqueness of the analytic continuation, if one is able to prove such an identity for $Re\ s$ sufficiently large this assures also the validity of $\nabla^a Z_{ab}(s, x|A/\mu^2) = -Z(s, x|A/\mu^2)\nabla_b V'(x)$ everywhere in the variable $s$. Therefore, let us prove that there is a $M > 0$ such that $\nabla^a Z_{ab}(s, x|A/\mu^2) = -Z(s, x|A/\mu^2)\nabla_b V'(x)$ for $Re\ s > M$ and this will be enough to prove the point (b). To get this goal we represent $\nabla^a Z_{ab}(s, x|A/\mu^2)$ employing (69) for each function $Z_{ab}$. Then we make recursive use of (65) of Theorem 2.1 and obtain $\nabla^a Z_{ab}(s, x|A/\mu^2)$ written as a linear combination of functions $\zeta^{(a, b)}(s + 1, x|A/\mu^2)$. Finally we can expand all these functions in series of the form (63) of Theorem 2.2, provided $Re\ s > M$ for an opportune $M > 0$. Taking account of the comment (4) after Definition 2.3, the explicit expression of the final series of $\nabla^a Z_{ab}(s, x|A/\mu^2)$ reads, for $Re\ s > M$

$$\nabla^a Z_{ab}(s, x|A/\mu^2) = s \sum_{j \in \mathbb{N}} \frac{2}{\lambda_j} \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \nabla^a \left\{ T_{ab}[^{\phi_j, ^\phi_j^*} g](x) + \frac{\lambda_j g_{ab}(x)}{2} \phi_j(x)\phi_j^*(x) \right\}. \quad (67)$$

Finally, using the form of the $\zeta$ function of the field fluctuatius given in [1], one has to prove that for any $x \in \mathcal{M}$

$$\nabla^a \left\{ T_{ab}[^{\phi_j, ^\phi_j^*} g](x) + \frac{\lambda_j g_{ab}(x)}{2} \phi_j(x)\phi_j^*(x) \right\} = -\frac{1}{2} \phi_j(x)\phi_j^*(x)\nabla_b V'(x). \quad (68)$$

This is nothing but the generalized “conservation law” of the stress tensor for the action

$$S_j[^{\phi, ^\phi^*}] = \frac{1}{2} \int_{\mathcal{M}} \left[ \phi A'[g]\phi^* - \lambda_j \phi\phi^* \right] d\mu_g \quad (69)$$

Indeed, (68) holds when the field $\phi$ satisfies the motion equations for the action above $A'[\phi] = \lambda_j \phi$. This is satisfied by the $C^\infty$ eigenfunctions of $A\phi$ with eigenvalue $\lambda_j$. Therefore (68) holds true and (b) is proven.

Concerning the point (c), (62) with $t_{ab}(x|A) = Z_{ab}(0, x|A)/2$ arises as a direct consequence of the definition (60), noticing that, from Theorem 2.3, $Z(s, x|A/\mu^2)$ is analytic at $s = 0$ and, from (66), (65) can be written down also

$$\langle T_{ab}(x|A) \rangle_{\mu^2} = \frac{1}{2} \frac{d}{ds}\big|_{s=0} Z_{ab}(s, x|A) + \frac{1}{2} Z_{ab}(0, x|A) \ln \mu^2, \quad (70)$$

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Similarly, from Definition 2.3, one sees also that \( Z(s, x|A/\mu^2)/2 \) evaluated at \( s = 0 \) takes contribution only from the possible pole at \( s = 1 \) of the function \( s \mapsto \zeta_{ab}(s, x|A/\mu^2) \) (the remaining simple \( \zeta \) function is regular for \( s = 0 \)) and coincides with the value of residue of the pole of this function at \( s = 0 \). When \( D \) is odd, no pole of \( s \mapsto \zeta_{ab}(s, x|A/\mu^2) \) arises at \( s = 1 \) because of (c) of Theorem 2.1, this is the reason for the \( \delta_D \) in the right hand side of (54). The form (54) of \( t_{ab} \) assures that it is built up as a liner combination of product of the metric, curvature tensors, \( V'(x) \) and their covariant derivatives, everything evaluated at the same point \( x \). The property (63) is consequence of the heat-kernel expansion which, once integrated in \( t \) (taking account of the factor \( t^s \) in the

\[
\lim_{s \to 0} s \zeta_{ab}(s + 1, x|A) = \left. \frac{\delta_D}{(4\pi)^{D/2}} \right\{ a_{D/2-1, (ab)}(x, x|A) + \frac{g_{ab}(x)}{2}a_{D/2}(x, x|A) \right\}.
\]

From (a) of Theorem 2.3 this is equivalent to

\[
\lim_{s \to 0} \frac{1}{2} \left[ \zeta^{(1a,1b)}(s + 1, x|A) + \zeta^{(1b,1a)}(s + 1, x|A) \right] + \lim_{s \to 0} \left[ \left( \xi - \frac{1}{4} \right) g_{ab}(x) \Delta + \xi R_{ab}(x) - \xi \nabla_a \nabla_b \right] \zeta(s + 1, x|A/\mu^2)
\]

\[
= \left. \frac{\delta_D}{(4\pi)^{D/2}} \right\{ a_{D/2-1, (ab)}(x, x|A) + \frac{g_{ab}(x)}{2}a_{D/2}(x, x|A) \right\} + \left. \left[ \left( \xi - \frac{1}{4} \right) g_{ab}(x) \Delta + \xi R_{ab}(x) - \xi \nabla_a \nabla_b \right] a_{D/2-1}(x, x|A) \right\}.
\]

The proof of this identity is very straightforward so we sketch its way only. Using Theorem 2.1, it is sufficient to consider the decomposition for large \( Re \) \( s \) (and a similar decomposition interchanging \( a \) with \( b \))

\[
s\zeta^{(1a,1b)}(s + 1, x, y|A/\mu^2)|_{x=y} = \frac{s}{\Gamma(s+1)} \int_0^{+\infty} dt \, t^s D^1_x D^1_y \left[ K(t, x, y|A) - P_0(x, y|A) \right] |_{x=y}
\]

\[
= \frac{s}{\Gamma(s+1)} \int_{\mu_0}^{+\infty} D^1_x D^1_y \{ \ldots \} |_{x=y} + \frac{s}{\Gamma(s+1)} \int_{\mu_0}^{+\infty} D^1_x D^1_y \{ \ldots \} |_{x=y}.
\]

Then, we can expand the integrand the first integral in the second line of (72) using (20) of Lemma 2.1 and we can continue both integral in the second line of (72) as far as \( s = 0 \). A direct computation proves that, because of the presence of the factor \( s \), only the first integral in (72), expanded as said above, gives contribution. The contribution arises from the terms of the heat-kernel expansion which, once integrated in \( t \) (taking account of the factor \( t^s \) in the
integrands), have a pole for \( s = 0 \). This pole is canceled out by the factor \( s \) giving a finite result. The terms which have no pole at \( s = 0 \) vanish due to the factor \( s \), in the limit \( s \rightarrow 0 \). A similar procedure can be employed concerning the second limit in (74). In performing calculations, it is worth to remind that \( \nabla_{(x)} \sigma(x, y) \) and \( \nabla_{(y)} \sigma(x, y) \) vanish in the limit \( x \rightarrow y \), and furthermore \( \nabla_{(x)} \nabla_{(y)} \sigma(x, y) \big|_{x=y} = -g_{ab}(y) \). Summing all contributions one obtains (54).

Concerning the point (d), the proof is dealt with as follows. Starting from (58) one finds

\[
g^{ab} T_{ab}[\phi_j, \phi_j^*, g] = \nabla_c \phi_j \nabla^c \phi_j^* + \left\{ \xi R + \left[ \xi (D - 1) - \frac{D}{4} \right] \Delta \right\} |\phi_j|^2 - \frac{D}{2} \lambda_j |\phi_j|.
\]

Then, employing the identities \( 2\nabla_c \phi^* \nabla^c \phi = \Delta |\phi|^2 - \phi \Delta \phi^* - \phi^* \Delta \phi \) and \( (-\Delta + \xi R + m^2 + V') \phi_j = \lambda_j \phi_j \) we have also

\[
g^{ab} T_{ab}[\phi_j, \phi_j^*, g] = \left[ \xi (D - 1) - \frac{D-2}{4} \right] \Delta |\phi_j|^2 - (m^2 + V') |\phi|^2 + \frac{2-D}{2} \lambda_j |\phi_j|.
\]

Since \( \xi_D = (D-2)/[4(D-1)] \), we have finally

\[
g^{ab} T_{ab}[\phi_j, \phi_j^*, g] = \left[ \frac{\xi_D - \xi}{\xi_D D - 1} \Delta - m^2 - V' \right] s |\phi_j|^2 + \frac{2-D}{2} \lambda_j |\phi_j|.
\]

From Definition 2.3, this entails that, for \( Re \ s \) sufficiently large

\[
g^{ab} Z_{ab}(s, x|\mu^2) = \frac{2}{\mu^2} \left[ \frac{\xi_D - \xi}{\xi_D D - 1} \Delta - m^2 - V' \right] s \zeta(s + 1, x|A/\mu^2) + 2s \zeta(s, x|A/\mu^2).
\]

The function \( (s, x) \mapsto s \zeta(s + 1, x|A/\mu^2) \) is \( C^\infty \) in a neighborhood of \((0, u)\) for any \( u \in \mathcal{M} \) the proof is similar to that given in (b) above for \((s, x) \mapsto Z_{ab}(s, x|\mu^2)\). Finally, employing Definition 2.4 taking also account of Definition 2.7 in [4] and (34) in Theorem 2.2 in [4] (i.e. (74) below), one finds (58). \( \square \)

Comments.

(1) Concerning the point (b) which generalizes the classical law [10], we stress that this result is strongly untrivial. We have not put this result somewhere “by hand” in the definitions and hypotheses we have employed. Notice also that, in the case \( V' \equiv 0 \), the tensor \( T_{ab}[\phi_j, \phi_j^*, g] \) we have used in the definitions is not conserved. Nevertheless, the final stress tensor is conserved. This should means that the local \( \zeta \) function approach is a quite deep approach.

(2) Concerning the point (c), we notice that this result is in agreement with Wald’s axioms [3] and, on a purely mathematical ground, it reduces the ambiguity allowed by Wald’s theorem. Indeed, Wald’s theorem involves at least two arbitrary terms dependent on two free parameters. Recently it has been proven that in the case of massive field which are not conformally coupled such an ambiguity should be much larger [14]. The point (d) proves that the corresponding ambiguity related to the field fluctuations is consistent with that which arises from the stress tensor. Assuming the \( \zeta \) function procedure the only ambiguity remaining is just that related to the initial arbitrary mass scale \( \mu \). On the other hand there is no physical evidence that the \( \zeta \)-function procedure is the physically correct one and thus one cannot conclude that this
method gets rid of the ambiguity pointed out by Wald et al.

(3) Concerning the point (d), we notice that, in the case $ξ = ξ_D$ and $V' \equiv m^2 = 0$ the usual conformal anomaly arises provided $D$ is even and the kernel of $A$ is trivial. Anyhow, in the case $\text{Ker } A$ is untrivial, the trace anomaly takes a contribution from the null modes also when $D$ is odd. In any cases, for the anomalous term, it holds ((34) in Theorem 2.2 in [1])

$$δ_D \frac{a_{D/2}(x, x|A)}{(4\pi)^{D/2}} - P_0(x, x|A) = ζ(0, x|A/μ^2)$$

(74)
also for $ξ \neq ξ_D$.

Let us consider some issues related to the physical interpretations of the theory. Suppose $S_1$ acts as a globally one-parameter isometry group on the Riemannian manifold $M$ giving rise to closed orbits with period $β > 0$. Suppose also that there exist a $D - 1$ embedded submanifold $Σ$ which intersects each orbit just once and is orthogonal to the Killing vector field of the isometry group $K$ (notice that any submanifold $Σ_τ$, obtained by the action on $Σ$ of the isometry group on the points of $Σ$, remains orthogonal to the Killing vector field). In this case the Riemannian metric is said static, the parameter of the group $τ$ is said the Euclidean time of the manifold with period $β$ and the submanifold $Σ$ is said the Euclidean space of the manifold.

As is well known, $β$ is interpreted as the “statistical mechanics” inverse temperature of the quantum state, anyway, it has no direct physical meaning because it can be changed by rescaling the normalization of the Killing vector $K$ everywhere by a constant factor. The physical temperature which, in principle, may be measured by a thermometer is the local rescaling-invariant Tolman temperature $T_T := 1/[\sqrt{(K, K)}β]$.

Whenever $M$ is static and $Σ$ is endowed with a global coordinate system $(x^1, \cdots, x^{D-1}) \equiv \vec{x}$, $M$ is endowed with a natural coordinate system $(τ, \vec{x})$, $τ \in (0, β)$ $\vec{x} \in Σ - F$, where $F$ is the set of the fix points of the group (which, anyhow, may be empty). This coordinate system is obtained by the evolution of the coordinates on $Σ$ along the orbits of the isometry group and is almost global in the sense that is defined everywhere in $M$ except for the set of the (coincident) endpoints of each orbit at $\vec{x}$ constant including the fix points of the group. This set has anyway negligible measure. Coordinates $(τ, \vec{x})$ given above are said static coordinates. Notice that, in these coordinates, $\partial_τ g_{ab} = 0$ and $g_{τa} = 0$ for $a = 1, \cdots, D - 1$ everywhere. Local static coordinates are defined similarly.

The important result is that, under our general hypotheses, supposing also that $M$ is static and admits static coordinates $(τ, \vec{x})$ and $V'$ does not depend on $τ$ one has that the stress tensor depends on $\vec{x}$ only and satisfies everywhere

$$\langle T_{τa}(\vec{x}|A) \rangle_\mu^2 = \langle T_{aτ}(\vec{x}|A) \rangle_\mu^2 = 0$$

(75)
for $a = 1, \cdots, D - 1$. The remarkable point as far as the physical ground is concerned, is that this result allows one to look for analytic continuations towards Lorentzian metrics performing the analytical continuation $τ \rightarrow it$ and without encountering imaginary components of the continued stress tensor. Notice that also $\langle φ^2(x|A) \rangle_\mu^2$ and the effective Lagrangian $L_{\text{eff}}(x|A)_\mu^2$
(see Definition 2.5 in [1]) do not depend on the temporal coordinate and, moreover, all results contained in Theorem 2.4 hold true in the Lorentzian section of the manifold, considering the trivial analytic continuations of all the terms which appear in the thesis. One has:

**Theorem 2.5.** Within our hypotheses on $\mathcal{M}$ and $A$, suppose $\mathcal{M}$ is static with Euclidean time $\tau \in (0, \beta)$ ($\beta > 0$) and $V'$ is invariant under Euclidean time displacements. In this case, for any $\mu^2 > 0$ and any point $x \in \mathcal{M}$,

$$
\langle T_{ab}(x|A) \rangle_{\mu^2} K^a(x)\sigma^b(x) = 0 ,
$$

where $K$ is the Killing vector field associated to the time $\tau$ and $\sigma(x)$ is any vector orthogonal to $K$ at $x$. Furthermore, denoting the Lie derivative along $K$ by $\mathcal{L}(K)$, it holds everywhere on $\mathcal{M}$

$$
\mathcal{L}(K)c \langle T^{ab}(x|A) \rangle_{\mu^2} = 0 ,
$$

$$
\mathcal{L}(K)\langle \phi^2(x|A) \rangle_{\mu^2} = 0 ,
$$

$$
\mathcal{L}(K)\mathcal{L}_{\text{eff}}(x|A)_{\mu^2} = 0 .
$$

Finally, all the results of Theorem 2.4 hold in the Lorentzian section provided one considers the Lorentzian-time-continued quantities in place of the corresponding Riemannian ones everywhere.

**Sketch of Proof.** In the given hypotheses and fixed $x \in \mathcal{M}$ ($x$ different from any fixed point in such a case the thesis being trivial since $K(x) = 0$), let us consider a generally local coordinate system $\tilde{x}$ on the Euclidean space $\Sigma$ around the intersection of the orbit passing from $x \in \mathcal{M}$, this induces a natural local coordinate system on $\mathcal{M}$, $(\tau, \tilde{x})$ (where $\tau \in (0, \beta)$) which includes the same point $x$. In our hypotheses (76) is trivially equivalent to (75) in the considered coordinate system.

Concerning the form of the $\zeta$ function of the stress tensor given in Definition 2.3 taking account of (54), since $g^{\tau\alpha}(x) = 0$ and $\partial_\tau g_{ab}(x) = 0$, only the first line of (54) and the last term in the last line may produce the considered components of the stress tensor. Actually, the dependence from $\tau$ of the eigenfunctions $\phi_j$ of the operator $A$, can be taken of the form $e^{i\omega \tau}$ with $\omega \in \mathbb{R}$ just because $\partial_\tau = K$ is a Killing field as we shall prove shortly. Then, the last term in the last line of (54) immediately vanishes concerning the considered components because the argument of the covariant derivatives (which commute on scalar fields) does not depend on $\tau$; furthermore, taking account that $A$ is real and thus $\phi_j$ and $\phi_j^*$ correspond to the same eigenvalue, one sees that the contribution coming from the first line of (54) computed for $b \neq a = \tau$ and $a \neq b = \tau$ vanishes when one sum over $j$ to get the stress-tensor $\zeta$ function. The validity of (77), (78), (79) is also obvious working in local static coordinates where the Lie derivative reduces to the ordinary $\tau$ derivative and taking account of the imaginary exponential dependence form $\tau$ of the modes. In fact, this dependence is canceled out directly in the various $\zeta$ functions due to the product of $\phi_j$ and $\phi_j^*$ (or corresponding derivatives) which appear in their definitions.

Let us finally prove that one can define the normalized orthogonal eigenvectors of $A'$ (and thus $A$) in order to have the dependence from $\tau$ said above. Reminding that each eigenfunction
of $A$ is a $C^\infty(\mathcal{M})$ function, and working in the local coordinate system around the orbit of $x$ considered above where $g_{ab}$ does not depend on $\tau$, one trivially has that

$$A' \partial_\tau \phi_{jk} = \partial_\tau A' \phi_{jk} = \lambda_j \partial_\tau \phi_{jk}, \quad (80)$$

where $\phi_{jk}$ is an eigenvector of $A$ with eigenvalue $\lambda_j$. This holds in the considered coordinates and therefore, choosing different local coordinate systems in $\Sigma$ and reasoning similarly, the above identity can be proven to hold almost everywhere in $\mathcal{M}$ provided $\partial_\tau \phi_{jk}$ is interpreted as the $C^\infty(\mathcal{M})$ scalar field $(K, \nabla \phi_{jk})$. Reminding that the dimension of each eigenspace $d_j$ is finite (Theorem 1.1. in [1]), it must be

$$\partial_\tau \phi_{jk}(x) = \sum_{l_j=1}^{d_j} c_{k_jl_j} \phi_{jl_j}(x), \quad (81)$$

almost everywhere. Remind that locally $\partial_\tau g_{ab} = 0$ and, since $g_{\tau\alpha} = 0$, $g = (K,K)h$ where $h$ is the determinant of the metric induced in $\Sigma$, one finds from (81)

$$c_{k_jh_j} + c_{h_kj} = \int_\mathcal{M} \partial_\tau \left\{ \phi_{k_j}^*(x) \phi_{h_j}(x) \right\} d\mu_g(x)$$

where $p$ is any point of the submanifold $\Sigma$ and $\nu$ is its (finite) Riemannian measure induced there from the metric. We have passed the derivative through the symbol of integration employing Fubini’s theorem and Lebesgue’s dominate convergence theorem. The right hand side of the identity above vanishes taking account that, for any fixed $p$

$$\lim_{\tau \to 0^+} \phi(\tau, p)_{jk} = \lim_{\tau \to \beta^-} \phi(\tau, p)_{jk} \quad (82)$$

because the orbits of the coordinate $\tau$ are closed and the functions are continuous in the whole manifold. Therefore, the matrix of the coefficients $c_{pq}$ is anti-hermitian. Finally, in the considered eigenspace, we can choose an orthogonal base of smooth normalized eigenfunctions where the matrix above is represented by a diagonal matrix, the eigenvalues being $i\omega_{l_j}$, $\omega_{lj} \in \mathbb{R}$ and $l_j = 1, 2, \cdots, d_j$. In the new base one re-writes (81), in local coordinates,

$$\partial_\tau \phi_{jk}(x) = i\omega_{l_j} \phi_{jk}(x), \quad (83)$$

and this entails trivially, with $\omega_{k_j} = 2\pi n_{jk}/\beta$, $n_{jk} \in \mathbb{Z}$ by (82),

$$\phi_{jk}(x) = e^{i\omega_{k_j}^\tau} \phi_{jk}(x^1, \cdots, x^{D-1}) \quad (84)$$

We leave to the reader the simple proof of the last statement of our theorem which can be carried out in local coordinates. $\square$

As a final remark notice that changes in the period $\beta$ of the the manifold which correspond to actual increases of the proper length of the orbits (and not to a simple rescaling of the normalization of the Killing vector), in general produce conical singularities in the fix points of the Lie group provided they exist. In such a case the manifold fails to be smooth and, in general, the theorems proven in this work and in [1] may not hold.
IV. The relation between the $\zeta$ function and the point-splitting to renormalize the stress tensor. An improved formula for the point-splitting procedure.

Similarly to the previous work, we prove here that, in our general hypotheses, a particular (improved) form of the point-splitting procedure can be considered as a consequence of the $\zeta$ function technique.

A. The point-splitting renormalization.

Let us summarize the point-splitting approach to renormalize the one-loop stress tensor \[4, 5, 6\] in the Euclidean case. First of all, we want to rewrite (9) into a more convenient form. Employing the motion equations $A^\prime \phi \equiv 0$ one can rewrite the right hand side of (9) as

$$
T_{ab}(\phi, g)(x) = (1 - 2\xi) \left[ \nabla_a \phi(x) \nabla_b \phi(x) + \phi(x) \nabla_a \nabla_b \phi(x) \right]
+ \left( 2\xi - \frac{1}{2} \right) g_{ab}(x) \left[ \nabla_c \phi(x) \nabla^c \phi(x) + \phi(x) \Delta \phi(x) \right]
+ \left[ \frac{g_{ab}(x)}{D} \phi(x) \Delta \phi(x) - 2\xi \phi_a \nabla_b \phi(x) \right]
- \xi \left[ R_{ab}(x) - \frac{g_{ab}(x)}{D} R(x) \right] \phi^2(x)
- \frac{V'(x) + m^2}{D} g_{ab}(x) \phi^2(x).
$$

Notice that the first two lines in the right hand side of (85) produce a vanishing trace in the case of $\xi = \xi_D (:= (D - 2)/(4(D - 1)))$, the third and the fourth line have separately a vanishing trace not depending on $\xi$. Finally, the trace of the last line is $-\left[ V'(x) + m^2 \right] \phi^2(x)$ trivially. It is obvious that, in the case of conformal coupling ($\xi = \xi_D, V' \equiv m^2 = 0$), the trace of the stress tensor vanishes. Conversely, for $\xi \neq \xi_D$ one get also

$$
g^{ab}(x) T_{ab}(x) = \left( \frac{\xi_D - \xi}{4\xi_D - 1} \Delta - m^2 - V'(x) \right) \phi^2(x).
$$

This is nothing but the classical version of (85). The most important difference is the lack of the trace anomaly term which is related to the last two terms in the right hand side of (66).

The point-splitting procedure can be carried out employing the expression above for the stress tensor (actually one expects that the same final result should arise starting from different but equivalent expressions of the stress tensor). The basic idea is very simple \[4, 5, 6, 15, 16\]. One defines the $a, b$ component of the one-loop renormalized stress tensor in the point $y$ as the result of the following limit

$$
\left< T_{ab}(y) \right> := \lim_{x \to y} D_{ab}(x, y) \left\{ \langle \phi(x) \phi(y) \rangle - H(x, y) \right\},
$$

where the quantum average of the couple of fields is interpreted as the Green function of the field equation corresponding to the quantum state one is considering, $H(x, y)$ is a Hadamard
on this symmetrization procedure), after an opportune symmetrization of the arguments (again, the final result should not depend on the quantum state. The operator $D_{ab}(x,y)$ “splits” the point $y$ and it is written down following (83), after an opportune symmetrization of the arguments (again, the final result should not depend on this symmetrization procedure).

\[
D_{ab}(x,y) = \frac{1}{2} [I_a^b \partial_{(x)b} \partial_{(y)a} + I_b^a \partial_{(x)a} \partial_{(y)b}]
\]

Above $I_a^b = I_{(y)a}^b (y,x)$ is a generic component of the bitensor of parallel displacement from $y$ to $x$, so the (co)tangent space at the point $x$ is identified with the fixed (co)tangent space at the point $y$.

What one has to fix, in order to use (87) for a particular quantum state, is the Hadamard solution $H$. It is known that, in the case $D$ is even, this solution is not unique [5, 6] and is determined once one has fixed the term $w_0(x,y)$ (see Comment (2) of Theorem 2.6 in [1]). This term, differently from the case of the renormalization of the field fluctuations, is not completely arbitrary. Indeed, it is possible to show that there are terms $w_0$ producing a left hand side of (87) which is not conserved [5]. Moreover, the massless conformally coupled case, and more generally, the case $m = 0$ and $V' \equiv 0$, involves some difficulties for the choice of $w_0$. For $m \neq 0$, it is possible to fix $w_0$ through the Schwinger-deWitt algorithm [4] obtaining a conserved renormalized stress tensor [4, 6]. This is not possible for $m = 0$ because Schwinger-deWitt’s algorithm becomes singular in that case. Anyhow, there is a further prescription due to Adler, Lieberman, and Ng [18] (see also [3, 5]) which seems to overcome this drawback: this is the simplest choice $w_0(x,y) \equiv 0$. However, in the case of a massless conformally coupled field at least, as pointed out by Wald [3], another drawback arises: the above prescription cannot produce a conserved stress tensor. Nevertheless, as proven in [3], in the case of a (analytic in the cited reference) either Lorentzian or Riemannian manifold, it is still possible to add a finite term in the right hand side of (87) which takes account of the failure of the conservation law in order to have a conserved final left hand side. This further term carries also a contribution to the trace of the final tensor which then fails to vanish and coincides with the well-known conformal anomaly. In [3], it has been argued that such an improved procedure can be generalized to any value of $m$ and $\xi$ getting

\[
\langle T_{ab}(y) \rangle := g_{ab}(y)Q(y) + \lim_{x \to y} D_{ab}(x,y) \left\{ \langle \phi(x) \phi(y) \rangle - H^{(0)}(x,y) \right\},
\]
where $H^{(0)}$ is the Hadamard solution determined by the choice $w_0 \equiv 0$ and $Q$ is a term fixed by imposing both the conservation of the left hand side of (88) and the request that the renormalized stress tensor vanishes in the Minkowski vacuum. Employing the local $\zeta$-function approach, we shall find out a point-splitting procedure which, in the case of a compact manifold, generalizes Wald’s one for a general operator $-\Delta + V$ in $D > 1$ dimensions in a Riemannian, not necessarily analytic, manifold and gives an explicit expression for $Q$ automatically.

B. Local $\zeta$ function and point-splitting procedure. An improved point-splitting prescription.

In this part of the work we shall state a theorem concerning the relation between the two considered techniques proving their substantial equivalence within our general hypotheses.

**Theorem 3.1** Let us assume our general hypotheses on $\mathcal{M}$ and $A'$ and suppose also $D > 1$.

(a) The renormalized stress tensor $(T_{ab}(y|A))_{\mu^2}$ defined in Definition 2.4 can be also computed as the result of a point-splitting procedure. Indeed one has, for any $\mu^2 > 0$

$$
(T_{ab}(y|A))_{\mu^2} = \lim_{x \to y} D_{ab}(x,y) \left\{ G(x,y|A) - H_{\mu^2}(x,y) \right\}
+ \frac{g_{ab}(y)}{D} \left( \delta_D a_D/2(y,y|A) \right)
- \frac{a_D/2(y,y|A)}{(4\pi)^{D/2}} - P_0(y,y|A)
$$

where $D_{ab}$ is defined in (88), $G(x,y|A) := \mu^{-2}\zeta(1,x,y|A/\mu^2)$ is the $\mu^2$ independent “Green function” of $A$ defined in [4], $P_0(y,y|A)$ is the $C^\infty$ integral kernel of the projector on the kernel of $A$ and $H_{\mu^2}(x,y)$ is defined as (the summation appears for $D \geq 4$ only)

$$
H_{\mu^2}(x,y) = \sum_{j=0}^{\lfloor D/2 \rfloor - 2} (D/2 - j - 2)! \left( \frac{2}{\sigma} \right)^{D/2-j-1} a_j(x,y|A) \frac{a_D/2-1(x,y|A)}{(4\pi)^{D/2}} - \ln \left( \frac{\sigma}{2} \right)
$$

if $D$ is even, and (the summation appears for $D \geq 5$ only)

$$
H_{\mu^2}(x,y) = \sum_{j=0}^{(D-5)/2} \frac{(D - 2j - 4)!!}{2(D-3)/2-j} \left( \frac{2}{\sigma} \right)^{D/2-j-1} a_j(x,y|A) \frac{a_D/2-1(x,y|A)}{(4\pi)^{D/2}}
+ \frac{a_{D-3/2}(x,y|A)}{(4\pi)^{D/2}} \sqrt{\frac{2\pi}{\sigma}} - \frac{a_{D-1/2}(x,y|A)}{(4\pi)^{D/2}} \sqrt{2\pi \sigma}
$$

if $D$ is odd.

(b) $H_{\mu^2}$ is a particular Hadamard local solution of the operator $A'$ truncated at the orders $L, M, N$, indeed one has

$$
H_{\mu^2}(x,y) = \frac{\Theta_D}{(4\pi)^{D/2}(\sigma/2)^{D/2-1}} \sum_{j=0}^{L} u_j(x,y) \sigma^j(x,y) + \delta_D \left( \sum_{j=0}^{M} v_j(x,y) \sigma^j \right) \ln \left( \frac{\sigma}{2} \right)
$$
\[ \delta_D \sum_{j=0}^{N} w_j(x, y) \sigma^j \]  

where \( \delta_D = 0 \) if \( D \) is odd and \( \delta_D = 1 \) if \( D \) is even; \( \Theta_D = 0 \) for \( D = 2 \) and \( \Theta_D = 1 \) otherwise; and furthermore:

1. \( L = D/2 - 2, M = 1 \) and \( N = 0 \) for \( D \) even, and \( L = (D - 1)/2 \) when \( D \) is odd.
2. The coefficients \( u_j \) and \( v_j \) of the above Hadamard expansion are completely determined by fixing the value as \( x \to y \) of the coefficient of the leading divergent term in order that this expansion for \( L, M, N \to +\infty \) defines a Green function formally. Using our conventions, this means:

\[ u_0(y, y) = \frac{4\pi^{D/2}}{D(D - 2)\omega_D}, \]  

for \( D \geq 3 \), \( \omega_D \) being the volume of the unitary \( D \)-dimensional disk, and

\[ v_0(y, y) = \frac{1}{4\pi} \]  

for \( D = 2 \).

3. The coefficients \( w_j \), when \( D \) is even, are completely determined by posing

\[ w_0(x, y) := -\frac{a_{D/2-1}(x, y|A)}{(4\pi)^{D/2}} \left[ 2\gamma + \ln \mu^2 \right]. \]  

Proof. See Appendix. \( \square \)

Comments

1. Whenever \( D \) is even, the logarithm in (93) contains a dimensional quantity. At first sight, this may look like a mistake. Actually, this apparent drawback means that the third summation in (93) has to contain terms proportional to \( \ln \mu^2 \) which can be reabsorbed in the second summation transforming the argument of the logarithm from \( \sigma/2 \) into the nondimensional one \( \sigma\mu^2/2 \). Indeed, the term in the right hand side of (96) makes this job concerning the term \( v_0 \) in (93). Since (93) is computed up to \( M = 1 \), one may expect the presence of a corresponding term \( w_1 \) in the last summation in (93). Actually, this term gives no contribution to the stress tensor employing (88) and (88) as one can check directly, taking account that in any coordinate system around any \( x \in M \) (with an obvious meaning of the notations)

\[ I_b^a(x, y)|_{x=y} = \delta_b^a \]  

and

\[ \nabla_{(x)}a \nabla_{(x)}b \sigma(x, y)|_{x=y} = -\nabla_{(x)}a \nabla_{(y)}b \sigma(x, y)|_{x=y} \]

\[ = \nabla_{(y)}a \nabla_{(y)}b \sigma(x, y)|_{x=y} = -\nabla_{(y)}a \nabla_{(x)}b \sigma(x, x')|_{x=y} = g_{ab}(y) \]  

In particular, one can check that each line of the right hand side of (88) vanishes separately when it is evaluated for the considered terms of the Hadamard expansion and \( x \to y \). This is
the reason because we have put $N = 0$ in (93) and we have omitted the corresponding term in (91). Notice that, conversely, in the usual version of point-splitting procedure [4, 3, 5] the term $w_1(x, y)$ is necessary. Similarly, the terms of order $\sigma^n \ln \sigma$ with $n > 1$ give no contribution to the stress tensor and thus we have omitted them in (93).

(2) In the case $D$ is odd, the expansions (92) and (93) do not consider terms corresponding to $\sigma^{k + (1/2)}$ with $k = 1, 2, \ldots$. In fact these terms give no contribution to the stress tensor via (90) and (88). Also in this case, each line of right hand side of (88) gives a contribution which vanishes separately for $x \to y$. Since (88) and (91) involve that the result does not depend from the coordinate system, one can check this fact working in Riemannian normal coordinates centered in $y$.

(3) The point-splitting procedure suggested in [5] for $D = 4$, differently from our procedure, requires $w_0 \equiv 0$ rather than (96). Actually, the function $\sigma$ which appears in [5] is defined as two times our function $\sigma$. Therefore, taking account that the argument of the logarithm in the second line of (77) is a quarter of Wald’s one, Wald’s prescription corresponds to take $w_0(x, y) - v_0(x, y) \ln 4 = 0$ in our case. Actually, as clarified in [3], the logarithm argument which appear in Wald’s prescription has to be understood as $\ln (\sigma/u^2)$, where $u$ is the unit of length employed. In our formalism, this correspond, in particular, to perform the changes $\ln(r/2) \to \ln(\sigma/(2u^2))$ and $\ln \mu^2 \to \ln(\mu^2u^2)$ in (91). (77) with the changes above entails that Wald’s prescription, namely $w_0(x, y) - v_0(x, y) \ln 4 = 0$, is satisfied provided one fixes a $\mu^2$ such that $2\gamma + \ln(\mu^2u^2/4) = 0$ namely $\mu = 2e^{-\gamma}/u$.

This proves that, under our hypotheses, our prescription generalize Wald’s one when the latter is understood in the Euclidean section of the manifold. Moreover, our prescription, differently from [5], gives explicitly the form of the Hadamard local function to subtract to the Green function in the general case as well as an explicit expression for the term $Q$ in (99), in terms of heat-kernel coefficients and, trivially, for any choice of the value of $\mu^2$.

(4) Rescaling the parameter $\mu^2$, the expression of the final stress tensor changes by taking a term $(\ln \alpha) t_{ab}(y)$. We know the explicit form of such a term, indeed, it must be that given in the point (c) of Theorem 2.4. Notice also that the obtained point-splitting method, also concerning the rescaling of $\mu^2$ agrees with the corresponding point-splitting procedure for computing the field fluctuations given in [1]. For example, the point (d) of Theorem 2.4 holds, provided both sides are renormalized with the point-splitting procedures above and the same value of $\mu^2$ is fixed.

(5) The point-splitting procedure we have found out uses the heat-kernel expansion in Theorem 1.3 of [1] and nothing further. This expansion can be built up also either in noncompact manifolds or manifolds containing boundary, essentially because it is based upon local considerations (see discussion in [3] concerning Schwinger-DeWitt’s expansion). Therefore, it is natural to expect that the obtained procedure, not depending on the $\zeta$-function approach, may work in the general case (namely, it should produce a symmetric conserved stress tensor with the known properties of the trace also in noncompact or containing boundary, manifolds), provided the Green function of the considered quantum state has the Hadamard behaviour.

(6) As a final comment, let us check on the found point-splitting method in the Euclidean section of Minkowski spacetime which is out of our general hypotheses, without referring to the
ζ-function approach. In this case, for $A = -\Delta + m^2$, one has that the heat kernel referred to globally flat coordinates reads

$$K(t, x, y|A) = e^{-\sigma/2t} \frac{e^{-m^2t}}{(4\pi)^2t^2}$$

(99)

and thus, supposing $m^2 > 0$,

$$a_j(x, y|A) = \frac{(-1)^j m^2 j}{j!}$$

(100)

As is well known, the (Euclidean) Green function of Minkowski vacuum can be computed directly

$$G(x, y|A) = \int_0^{+\infty} K(t, x, y) dt = \frac{2m}{(4\pi)^2 \sqrt{\sigma/2}} K_1 \left( 2 \sqrt{\frac{\sigma m^2}{2}} \right).$$

(101)

Expanding $K_1$ in powers and logarithms of $\sigma$ one get

$$G(x, y|A) = \frac{2}{(4\pi)^2 \sigma} + \frac{1}{(4\pi)^2} \left\{ m^2 + \frac{m^4}{4} \sigma + \sigma^2 f(\sigma) \right\} \ln \left( \frac{\sigma}{2} \right)$$

$$+ \frac{m^2}{(4\pi)^2} \left( 2\gamma - 1 + \ln m^2 \right) + \sigma g(\sigma),$$

(102)

where $f$ and $g$ are smooth bounded functions. Then, employing (99), (100) and (100), it is a trivial task to prove that, provided the choice $\mu = me^{-3/4}$ is taken in (96), one gets $< T_{ab}(y) > \equiv 0$ as it is expected. In particular one finds also $Q(y) \equiv m^4/(128\pi^2)$ for the coefficient of $g_{ab}(y)$ in the last line of (90). For a general value of $\mu^2$, the computation of the stress-tensor trace via the formula in (d) in Theorem 2.4 reproduces the correct Coleman-Weinberg results [19] for the field fluctuations still obtained by the local ζ function approach [13] as well as by using the point-splitting formula given in Theorem 2.6 in [1].

The case $m = 0$ is much more trivial. In this case the heat kernel is given by (99) with $m = 0$, and thus only $a_0(x, y) \equiv 1$ survives in the heat-kernel expansion. In this case, $A$ is not positive defined but positive only, the manifold is not compact and the Minkowski vacuum Green function can be still computed integrating the heat kernel despite the local ζ function does not exist. Moreover Green function coincides with the Hadamard local solution $2/[(4\pi)^2 \sigma]$, furthermore $Q(y) \equiv 0$, and thus our procedure gives a vanishing stress tensor as well.

V. Summary and outlooks.

In this paper we have concluded the rigorous analysis started in [1], concerning the mathematical foundation of the theory of the local ζ-function renormalization of the one-loop stress tensor introduced in [2]. The other important point developed herein has been the relation between the local ζ-function approach and the (Euclidean) point-splitting procedure.
Concerning the first point, we have proven that the \( \zeta \)-function theory of the stress tensor can be rigorously defined at least in closed manifold giving results which agree with and generalize previous results concerning the \( \zeta \)-function renormalization of the field fluctuations. On the mathematical ground, we have also proven a few of new theorems about the smoothness of the heat-kernel expansions.

Concerning the second proposed goal, we have found out that the two methods (\( \zeta \)-function and point-splitting) agree essentially, provided a particular form of point-splitting procedure is employed. Within the hypotheses of a Riemannian compact \( C^\infty \) manifold, this point-splitting procedure is a natural generalization (in any \( D > 1 \) and for a larger class of Euclidean motion operators) of Wald’s improved procedure presented in \([3]\) and also discussed in \([1]\) defined in a Lorentzian manifold (but the same arguments employed can by trivially extended to Riemannian manifolds). Our procedure gives also explicitly the form of the various terms which are employed in the point-splitting procedure in terms of the heat-kernel expansion.

In our opinion, the found point-splitting procedure should work also without the employed hypotheses and independently from the \( \zeta \)-function procedure. We have anyhow checked this conjecture in the Euclidean Minkowski spacetime proving that it holds true as expected either in the case \( m = 0 \) or \( m > 0 \). Moreover, the obtained results concerning the point splitting procedure should be trivially generalized for static Lorentzian manifolds at least.

Appendix A. Proof of some lemmata and theorems.

Proof of Lemma 2.1. Let us consider the form of the heat kernel as it was built up in \([2]\) Section 4 Chapter VI. This construction holds also in the case of an operator \( A' := -\Delta + V \) and not only \( A' := -\Delta \) as pointed out in the previous work \([1]\). In our notations, one has by (45) in Section 4 of Chapter VI of \([2]\)

\[
F_N(t, x, y) = e^{-\sigma(x,y)/2t} \chi(\sigma(x,y)) \sum_{j=0}^{N} a_j(x, y|A)t^j,
\]

\( N > D/2 + 2 \) is a fixed integer. (Actually, the equation (45) in Section 4 of Chapter VI of \([2]\) is missprinted in \([2]\) because of the unnecessary presence of the operator \( L_x \) in the right hand side of the first line of (45) in Section 4 of Chapter VI of \([2]\). Since the absence of this operator in the correct formula, we cannot get the second line of (45) in a direct way. In fact, in \([1]\) we have used a different [but equivalent in the practice] form of the remaining of the heat-kernel expansion with respect to that which appears in (45). Some other parts of Section 4 of Chapter VI in \([2]\) contain several other missprints like the requirement \( F \in C^0(M \times M \times [0+\infty)) \) in Lemma 2 which has to be corrected into \( F \in C^1(M \times M \times [0+\infty)) \).

Some other parts of Section 4 of Chapter VI in \([2]\) contain several other missprints like the requirement \( F \in C^0(M \times M \times [0+\infty)) \) in Lemma 2 which has to be corrected into \( F \in C^1(M \times M \times [0+\infty)) \).

\[
K(t, x, y|A) = F_N(t, x, y) + (F_N * F)(t, x, y)
\]

where \( F_N \) is the \( C^\infty((0, +\infty) \times M \times M) \) parametrix defined in \([1]\).

The remaining proportional to \( O_{\eta,N} \) in (20) of Lemma 2.1 of \([1]\) is therefore \( (F_N * F)(t, x, y) \).
We remind the reader that $\sigma(x, y)$ is one half the squared geodesical distance $(d(x, y))$ from $x$ to $y$ and defines an everywhere continuous function in $\mathcal{M} \times \mathcal{M}$ which is also $C^\infty$ in the set of the points $x, y$ such that $d(x, y) < r$. $\chi(u)$ is a nonnegative $C^\infty([0, +\infty))$ function which takes the constant value $1$ for $|u| < r^2 / 16$ and vanishes for $|u| \geq r^2 / 4$, $r$ being the injectivity radius of the manifold. The convolution $*$ has been defined in Section 4 of Chapter VI of [20].

\[(G * H)(t, x, y) := \int_0^t d\tau \int_\mathcal{M} d\mu_g(z) G(\tau, x, z) H(t - \tau, z, y), \quad (105)\]

whenever the right hand side makes sense. Finally, the function $F$ which appears in [104] is defined by a uniformly convergent series in $[0, T] \times \mathcal{M} \times \mathcal{M}$ for any $T > 0$ (see (43) in Section 4 Chapter VI of [20]).

\[F(t, x, y) := \sum_{l=1}^\infty \left[(A'_{\alpha} - \partial/\partial t)F_N\right]^l(t, x, y). \quad (106)\]

($B^l$ means $B * B * \cdots * B$ $l$ times.). This function belongs to $C^L([0, +\infty) \times \mathcal{M} \times \mathcal{M}$) provided $M > D/2 + 2L$ (see [20]). [104] satisfies the heat-kernel equation provided $F$ is $C^1$ in all variables, namely $N > D/2 + 2$.

Let us consider [104]. The remaining of the “asymptotic” expansion of the heat kernel computed up to the coefficient $a_N(x, y, A)$ ($N > D/2 + 2$) is just the second term in the right hand side. It can be explicitly written down (see [20])

\[(F_N * F)(t, x, y) = \int_0^t d\tau \tau^{-D/2}(t - \tau)^{N-D/2} \int_\mathcal{M} d\mu_g(z) \mathcal{F}_N(\tau, x, z) F(t - \tau, z, y) \times e^{-\sigma(x, z)/2}\tau e^{-\sigma(z, y)/2(t-\tau)}. \quad (107)\]

$\mathcal{F}_N(t, x, z)$ defines a function which belongs to $C^\infty([0, +\infty) \times \mathcal{M} \times \mathcal{M}$) and vanishes smoothly whenever the geodesical distance between $x$ and $z$ is sufficiently large, i.e. $d(x, z) \geq r/2$, due to the presence of the function $\chi$ in the expression of the parametrics [103]. $\mathcal{F}$ defines an everywhere continuous function which belongs also to $C^L$ provided $N > D/2 + 2L$ and the geodesical distance between $y$ and $z$ is sufficiently short, i.e. $d(y, z) < r$, and $t \in [0, +\infty)$.

Then let us pick out a point $u \in \mathcal{M}$. We can find a geodesically spherical open neighborhood of $u$, $J_u$, with a geodesic radius $r_0 < r/8$. By the definition of the function $\chi$, it holds $\chi(\sigma(x, y)) = 1$ whenever $x, y \in J_u$ and thus the coefficient $\chi$ can be omitted in the heat-kernel expansion working with any coordinate system defined in a neighborhood of $J_u$ (e.g. normal Riemannian coordinates). From now on concerning the points $x$ and $y$ we shall work within such a coordinate system in the neighborhood $J_u$. Notice also that, by the triangular inequality $d(x, y)(= \sqrt{2\sigma(x, y)}) < r/4$ whenever $x, y \in J_u$.

Now, let us suppose $N > D/2 + 2|\alpha| + 2|\beta|$, this entails $F \in C^{1|\alpha|+|\beta|}$ and thus also $\mathcal{F} \in C^{1|\alpha|+|\beta|}$ provided the distance of its arguments defined in the manifold is less than $r$ and $t \in (0, +\infty)$. We can apply operators $D_x^\alpha$ and $D_y^\beta$ to both sides of [104]. The action of the derivatives [104] produces the first term in the right hand side of [20] at least (notice that $\chi \equiv 1$ in our hypotheses). Let us focus attention on the action of the derivatives on the remaining in [104].
question concerns the possibility to pass these under the integration symbol in (107). The action of the derivatives can be carried under integration symbols (obtaining also $x, y$-continuous final function if the derivatives of the integrand are continuous) provided, for any fixed choice of a couple of multindices $\alpha, \beta$, the derivatives of the integrand are locally $x, y$-uniformly bounded by an integrable function (dependent on the multindices in general). We shall see that this is the case after we have manipulated the integral opportunity. Notice that the derivatives (with respect to the manifold variables) of the function $F$ do exist because the second integral in the right hand side of (107) takes contribution only from the points $z$ such as both $d(y, z) < r$ and $d(x, z) < r$ are fulfilled as required above. Indeed, it must be $d(x, z) < r/2$ otherwise $F_N(\tau, x, z)$ smoothly vanishes as pointed out above, and, taking account of $d(x, y) < r/4$, the triangular inequality entails also $d(y, z) \leq d(x, z) + d(x, y) < r/2 + r/4 = 3r/4$.

Now, let us fix a new open neighborhood of $u, I_u$, such that its closure is contained in $I_u$, and fix $T > 0$. Barring $\tau \mapsto \tau - D/2$, all functions of $\tau, x, y, z$ and all their $(x, y, z)$-derivatives we shall consider are bounded in the compact $[0, T] \times I_u \times I_u \times \mathcal{M}$ where we are working because these are continuous therein. We can rearrange the expression (107) into

$$
(F_N * F)(t, x, y) = \int_0^t d\tau \tau^{-D/2}(t - \tau)^N - D/2 \int_{\mathcal{S}^{D-1}} d\nu \int_0^{r/4} d\rho \rho^{-D-1} J(x, \nu, \rho)
\times e^{-\rho^2/2} F_M(\tau, x, z(x, \rho, \nu)) F(t - \tau, z(x, \rho, \nu), y)
\times e^{-\sigma(z(x, \rho, \nu), y)/2(t - \tau)}.
$$

(108)

where, to determine the position of $z$, we have employed a spherical system of normal coordinates $\rho, \nu$ centered in any $x$, $\rho$ is the distance of $z$ from $x$, its range is maximized in the integrals above because the integrand vanishes smoothly for $\rho > r/2$, and thus all the functions contained in the integrand are well-defined within $\{\rho \in [0, +\infty]\}$. $\nu$ is a unitary $D - 1$ dimensional vector and $d\mu_\rho(z) = dpd\nu \rho^{-D-1} J(x, \nu, \rho)$, $d\nu$ is the natural measure in $\mathcal{S}^{D-1}$. The function $J$ is continuous and bounded in $I_u \times \mathcal{S}^{D-1} \times \{\rho \in [0, r/2]\}$ together with all derivatives.

Then, we can change variables $\rho \mapsto \rho/\sqrt{\tau} =: \rho'$ obtaining

$$
(F_N * F)(t, x, y) = \int_0^t d\tau (t - \tau)^N - D/2 \int_{\mathcal{S}^{D-1}} d\nu \int_0^{r/4} d\rho \rho^{1/2} \rho'^{-D-1} J(x, \nu, \rho'^{1/2}/2)
\times e^{-\rho'^2/2} F_M(\tau, x, z(x, \rho'^{1/2}/2, \nu)) F(t - \tau, z(x, \rho'^{1/2}/2, \nu), y)
\times e^{-\sigma(z(x, \rho'^{1/2}/2, \nu), y)/2(t - \tau)}.
$$

(109)

The formal action of the operators $D^\alpha_x$ and $D^\beta_y$ under the integration produces a sum of continuous and bounded functions (now the function $\tau \mapsto \tau^{-D/2}$ has disappeared and the remaining functions and their $x, y, z$-derivatives are bounded since they are product of bounded functions). Also, it changes $(t - \tau)^N - D/2$ into several terms of the form $(t - \tau)^N - D/2 - L_i$ (where each $L_i \leq |\alpha| + |\beta|$), because of the derivatives of the second exponential. These function of $\tau$ are continuous and bounded being $N > D/2 + |\alpha| + |\beta| \geq L_i$ in our hypotheses. We can bound the absolute value of these functions by $Ce^{-\rho'^2/2}$, where $C$ is a sufficiently large constant. This function is trivially integrable in the measure we are considering. This $x, y, t$-uniform bound assures
that, concerning the \(x, y\)-derivatives of \(F_N \ast F\), one can interchange the symbols of derivatives with those of integrals and also that the derivative of \((t, x, y) \mapsto (F_N \ast F)(t, x, y)\) are continuous functions in \((0, +\infty) \times I_u \times I_u\).

In order to finish this proof let us consider a finer estimate of \(O^{(\alpha, \beta)}_{\eta, N}(x, y)\). We have the inequality \([20]\), for \(\tau \in [0, t]\)

\[
\frac{d^2(x, y)}{t} \leq \frac{d^2(x, z)}{\tau} + \frac{d^2(z, y)}{t - \tau},
\]

and thus, picking out any \(\eta \in (0, 1)\) and posing \(\delta := 1 - \eta \in (0, 1)\) we get (notice that \(t - \tau \geq 0\))

\[
e^{-\sigma(x, z)/2\tau} e^{-\sigma(z, y)/2t} \leq e^{-\eta \sigma(x, y)/2t} \left( e^{-\delta \sigma(x, z)/2t} e^{-\delta \sigma(z, y)/2(t-\tau)} \right) \leq e^{-\eta \sigma(x, y)/2t} e^{-\delta \sigma(x, z)/2t}.
\]

We can use this relation in the \(x, y\)-derivatives of \([105]\) obtaining

\[
|D_x^\alpha D_y^\beta (F_N \ast F)(t, x, y)| \leq \sum_i e^{-\eta \sigma(x, y)/2t} \int_0^t d\tau (t - \tau)^{N - D/2 - L_i} \int_{SD-1} d\vec{v} \times \int_0^{+\infty} d\rho \; \rho^{D-1} e^{-\delta \rho^2/2} C_i,
\]

where the coefficients \(C_i\) are upper bounds of the absolute values of the continuous functions missed in the integrand and \(L_i \leq |\alpha| + |\beta|\). We can execute the integral in \(\tau\) obtaining, for \(0 < t \leq T\) and \(x, y \in I_u\) (remind that \(N > D/2 + |\alpha| + |\beta|\))

\[
|D_x^\alpha D_y^\beta (F_N \ast F)(t, x, y)| \leq e^{-\eta \sigma(x, y)/2t} \sum_i C_i \delta^{N+1-L_i-D/2} \leq \frac{C_\delta}{(4\pi)^{D/2}} e^{-\eta \sigma(x, y)/2t} t^{N+1-D/2-|\alpha|-|\beta|},
\]

\(C_\delta\) is a positive constant sufficiently large which depends on \(T, \alpha, \beta\) in general. This proves the remaining part of the thesis posing \(K^{(\alpha, \beta)}_{\eta, N} := T\) and \(M^{(\alpha, \beta)}_{\eta, N} := C_\delta\). Indeed, the remaining \(O^{(\alpha, \beta)}_{\eta, N}\) we wanted to compute coincides with \(D_x^\alpha D_y^\beta (F_N \ast F)\) just up to the factor \((4\pi t)^{-D/2} t^{N-|\alpha|-|\beta|} \exp(-\eta \sigma/2t)\). \(O^{(\alpha, \beta)}_{\eta, N}\) can be defined in \(t = 0\) as \(O^{(\alpha, \beta)}_{\eta, N}(0; x, y) = 0\), obtaining a continuous function in \((0, +\infty) \times I_u \times I_u\). \(\Box\)

**Proof of Lemma 2.2.** Let us consider an eigenvector \(\phi_j\) and fix \(T \in (0, +\infty)\) and consider a neighborhood of \(u \in \mathcal{M}\), \(J_u\) where a coordinate system is defined. In the following, \(x\) and \(y\) are points in a new neighborhood of \(u\), \(I_u\), such that its closure is contained in \(J_u\). These points are represented by the coordinate system given above and the derivative operators are referred to these coordinates. From **Theorem 1.3** of [1], it holds

\[
e^{-T\lambda_j \phi_j(x)} = \int d\mu_g(z) K(T, x, z|A) \phi_j(z).
\]

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We can derive both sides of the equation above employing operators $D^\alpha_x$. Since, for a fixed $T$ the derivatives of $K$ are bounded ($(x, z) \mapsto K(T, x, z|A)$ is $C^\infty$ and $I_u \times \mathcal{M}$ is compact in our hypotheses [1]), we can pass the derivative operator under the integral symbol obtaining

$$\left| D^\alpha_x \phi_j(x) \right| = \left| e^{\lambda_j t} \int_{\mathcal{M}} \mu_y(z) D^\alpha_x K(t, x, z|A) \phi_j(z) \right| \leq e^{\lambda_j t} \left| D^\alpha_x K(T, x, \cdot|A) \right|_{L^2(M, d\mu_g)} (115)$$

where we have made use of the Cauchy-Schwarz inequality and we have taken account of $||\phi_j|| = 1$ (from now on we omit the index $L^2(M, d\mu_g)$ in the norms because there is no ambiguity). The function $x \mapsto ||D^\alpha_x K(T, x, \cdot|A)||$, for $x \in I_u$ is continuous from Lebesgue’s dominate convergence theorem since $D^\alpha_x K(T, x, y|A)$ defines a continuous function in $x$ and $y$ and there is a constant (dependent on $T$ in general) $M_T$ such that $||D^\alpha_x K(T, x, z|A)||^2 \leq M_T$ for $(x, z)$ which belong in the compact $I_u \times \mathcal{M}$ and the measure of the manifold is finite. The same results holds whenever one keeps fixed $y$ in $I_u$ and integrates over $x$. Therefore, let us define

$$P_T^{(\alpha, \beta)} := \left[ \sup_{x \in I_u} ||D^\alpha_x K(T, x, \cdot|A)|| \right] \left[ \sup_{y \in I_u} ||D^\beta_y K(T, \cdot, y|A)|| \right] (116)$$

and we have, for any $x, y \in I_u$, the $\lambda_j$-uniform upper bound

$$|e^{-\lambda_j t} D^\alpha_x \phi_j(x) D^\beta_y \phi_j^*(y)| \leq P_T^{(\alpha, \beta)} e^{-\lambda_j (t-2T)} \cdot (117)$$

The found inequality proves that the absolute values of the terms of the series

$$\sum_{j \in \mathbb{N}} e^{-\lambda_j t} D^\alpha_x \phi_j(x) D^\beta_y \phi_j^*(y) (118)$$

are $x, y$-uniformly bounded, for $(x, y, t) \in I_u \times I_u \times (2T, +\infty)$, by terms of the convergent series (see (30) in [1])

$$\sum_{j \in \mathbb{N}} e^{-\lambda_j (t-2T)} P_T^{(\alpha, \beta)} = P_T^{(\alpha, \beta)} \int_{\mathcal{M}} \mu_g(z) \{ K(t - 2T, z, z|A) - P_0(z, z|A) \}$$

$$= P_T^{(\alpha, \beta)} Tr \left\{ K_{(t-2T)} - P_0 \right\} .$$

This holds for any choice of the multindices $\alpha, \beta$ and this entails [2], (25) and (28). The final upper bound (24) is a trivial consequence of (99) in [1] and the fact that the manifold has a finite measure. □

**Sketch of Proof of Theorem 3.1.** Let us fix a coordinate system in a neighborhood $I_u$ of a point $u \in \mathcal{M}$, all the following considerations will be referred to these coordinates, and in particular to a couple of points $x, y$ within that neighborhood. Then, let us consider the expression (58) for the $\zeta$ function of the stress tensor. Employing the eigenvalue equation $A \phi_j = \lambda_j \phi_j$ one can rearrange (58) into

$$Z_{ab}(s, y|A/\mu^2) = \sum_{j \in \mathbb{N}} \sqrt{\frac{2}{s^2}} \left( \frac{\lambda_j}{\mu^2} \right)^{-(s+1)} T_{ab}[\phi_j, \phi_j^*, g](y) \cdot (119)$$

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where (C.C. means the complex conjugation of the terms already written)

\[
T_{ab}[\phi_j, \phi_j^*; g](y) = \left(1 - 2\xi\right) \frac{1}{2} \left(\nabla_a \phi_j(y) \nabla_b \phi_j^*(y) + \phi_j(y) \nabla_a \nabla_b \phi_j^*(y) + C.C.\right)
\]

\[
+ \left(2\xi - \frac{1}{2}\right) \frac{g_{ab}(x)}{2} \left(\nabla_c \phi_j(y) \nabla^c \phi_j^*(y) + \phi_j(y) \Delta \phi_j^*(y) + C.C.\right)
\]

\[
+ \frac{1}{2} \left(\frac{g_{ab}(y)}{D} \phi_j(y) \Delta \phi_j^*(y) - \phi_j(y) \nabla_a \nabla_b \phi_j^*(y) + C.C.\right)
\]

\[
+ \xi \left(R_{ab}(y) - \frac{g_{ab}(y)}{D} R(y)\right) |\phi_j(y)|^2 - \frac{V'(y) + m^2}{D} g_{ab}(y) |\phi_j(y)|^2
\]

\[
+ \frac{\lambda_j}{D} g_{ab}(y) |\phi_j(y)|^2.
\]

(120)

The stress tensor is then given by (100) in Definition 2.4. after the analytic continuation in the variable \(s\) of \(Z_{ab}(s, y/A/\mu^2)\) given in (119). Employing Theorem 2.2 and (120) and (119), we can write down the expression of \(Z_{ab}(s, y/A/\mu^2)\) employing also functions \(\zeta^{[\alpha, \beta]}(s, y/A/\mu^2)\) defined as in Definition 2.2 with the difference that covariant derivatives are employed instead of ordinary derivatives. We get, omitting the arguments \(y\) and \(A/\mu^2\) in the various \(\zeta\) functions for sake of brevity,

\[
Z_{ab}(s, y/A/\mu^2) =
\]

\[
(1 - 2\xi) \frac{s}{\mu^2} \left(\zeta^{(1a, 1b)}(s + 1) + \zeta^{(1b, 1a)}(s + 1) + \zeta^{[1a + 1b, 0]}(s + 1) + \zeta^{[0, 1a + 1b]}(s + 1)\right)
\]

\[
+ \left(2\xi - \frac{1}{2}\right) \frac{sg_{ab}(y)g^{cd}(y)}{\mu^2} \left(2\zeta^{(1c, 1d)}(s + 1) + \zeta^{[0, 1c + 1d]}(s + 1) + \zeta^{[1c + 1d, 0]}(s + 1)\right)
\]

\[
+ \frac{s}{\mu^2} \left[\frac{g_{ab}(y)g^{cd}(y)}{D} \left(\zeta^{[0, 1c + 1d]}(s + 1) + \zeta^{[1c + 1d, 0]}(s + 1)\right) - \zeta^{[0, 1a + 1b]}(s + 1)\right]
\]

\[
- \zeta^{[1a + 1b, 0]}(s + 1)
\]

\[
+ \frac{2s\xi}{\mu^2} \left(R_{ab}(y) - \frac{g_{ab}(y)}{D} R(y)\right) \zeta(s + 1) - \frac{V'(y) + m^2}{D} \frac{2sg_{ab}(y)}{\mu^2} \zeta(s + 1)
\]

\[
+ \frac{2sg_{ab}(y)}{D} \zeta(s).
\]

(121)

First of all, we notice that the term proportional to \(g_{ab}(y)\) in (100) arises from the last term above via item (c) of Theorem 2.2 in [1].

Let us consider the contribution to the stress tensor due to the terms \(\zeta^{(1a, 1b)}(s + 1, y/A/\mu^2)\). Similarly to (101) in [1], we can define, for any \(\mu_0^2 > 0\) fixed and \(N\) integer \(> D/2 + 4\), taking account of Lemma 2.1 above

\[
\zeta^{(1a, 1b)}(N, s + 1, x, y/A/\mu^2, \mu_0^2) := \frac{\mu_0^{2s}}{\Gamma(s + 1)} \int_0^{\mu_0^{-2}} dt t^{s - \nu(s, y)/2} \frac{e^{-\nu(t, s, y)/2t}}{(4\pi t)^{D/2}} t^{N-2} O^{(1a, 1b)}_{\eta, N}(t; x, y)
\]

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Similarly to Lemma 2.1 in [1], one can prove that the function of \( s, x, y \) defined above is continuous in a neighborhood \( I \times I_u \times I_u \), where \( I \) is a complex neighborhood of \( s = 0 \) with all of its \( s \) derivatives, in particular, it is \( s \)-analytic therein. Employing Lemma 2.1 and the item (a) of Theorem 2.1 one can write also, for \( \Re s + 1 > D/2 + 4 \),

\[
\zeta^{(1_{a,1_b})}(s + 1, x, y|A/\mu^2) = \zeta^{(1_{a,1_b})}(N, s + 1, x, y|A/\mu^2, \mu_0^{-2}) - \frac{(\mu_0^2)}{(s + 1)\Gamma(s + 1)} \sum_{j=0}^{N} \nabla_{(x)_a} \nabla_{(y)_b} (\int_0^{\mu_0^{-2}} dt t^{s-D/2+j} e^{-\sigma/2} a_j(x, y|A)).
\]

(123)

In particular, for \( \Re s + 1 > D/2 + 4 \), the left hand side above is continuous in \( x, y \) and thus we can take the coincidence limit for \( x \to y \). Noticing that one can also pass the derivatives under the sign of integration in the right hand side and that \( \nabla_{(x)_a} \nabla_{(y)_b}(x, y)|_{x=y} = -g_{ab}(y) \) and \( \nabla_{\sigma}(x, y)|_{x=y} = 0 \), we get for the right hand side of the expression above multiplied by \( 2s/\mu^2 \) and evaluated for \( x = y \)

\[
\frac{2s}{\mu^2} \zeta^{(1_{a,1_b})}(N, s + 1, y, y|A/\mu^2, \mu_0^{-2}) - \frac{2s}{\mu^2} \frac{(\mu_0^2)}{(s + 1)\Gamma(s + 1)} \sum_{j=0}^{N} \frac{a_{j}(y, y|A)\mu_0^{-2}(s-D/2+j+1)}{s-D/2+j+1} + \frac{g_{ab}(y) a_{j}(y, y|A)\mu_0^{-2}(s-D/2+j)}{2}
\]

where \( a_{j}(y, y|A) := \nabla_{(x)_a} \nabla_{(y)_b} a_{j}(x, y|A)|_{x=y} \), and \( P_{j}(y, y|A) := \nabla_{(x)_a} \nabla_{(y)_b} P_{0}(x, y|A)|_{x=y} \).

The contribution to the stress tensor, namely, to \( \frac{d}{ds}|_{s=0} Z_{ab}(s, y|A/\mu^2)/2 \) of the considered term is then obtained by continuing the result above as far as \( s = 0 \), executing the \( s \) derivative and multiplying for \( (1 - 2\xi)/2 \) the final result. Taking account that \( \zeta(N, s, x, y|A/\mu^2, \mu_0^{-2}) \) is smooth in a neighborhood of \( s = 0 \), this lead to, apart from the unessential factor \( (1 - 2\xi) \),

\[
\frac{d}{ds}|_{s=0} \frac{2s}{\mu^2} \zeta^{(1_{a,1_b})}(s + 1, 1, y|A/\mu^2)
\]

\[
= \frac{1}{\mu^2} \zeta^{(1_{a,1_b})}(N, 1, y, y|A/\mu^2, \mu_0^{-2}) - \frac{P_{ab}(y, y|A)}{\mu^2}
\]

\[
+ \frac{1}{(4\pi)^{D/2}} \sum_{j=0, j \neq D/2-1}^{N} \frac{a_{j}(y, y|A)}{2j-D+2} \left( j - D/2 + 1 \right) \delta_D \left( \gamma + 2 \ln \frac{\mu_0}{\mu} \right) \frac{a(D/2-1)_{ab}(y, y|A)}{(4\pi)^{D/2}}
\]

\[
+ \frac{g_{ab}(y)}{(4\pi)^{D/2}} \sum_{j=0, j \neq D/2}^{N} \frac{a_{j}(y, y|A)}{2j-D} \delta_D g_{ab}(y) \left( \gamma + 2 \ln \frac{\mu_0}{\mu} \right) \frac{a(D/2)_{ab}(y, y|A)}{(4\pi)^{D/2}}.
\]

(124)
Let us consider the first line in the right hand of (123) side for a moment. The other terms different from $\zeta^{(1,1,1)}(s+1)$ can be undertaken to a procedure similar to that developed above. The important point is that, once one has performed such a procedure, all terms with a factor $g_{ab}(y)$ similar to the terms in the last line of (124) cancels out each other, and thus, in the final expression of the first line of the right hand side of (120), no term with a factor $g_{ab}(y)$ survives. The same fact happens for the second and third lines of (121). Since $\zeta^{(1,1,1)}(N, 1, y|A/\mu^2, \mu_0^{-2})$ and the derivatives of heat-kernel coefficients are continuous in $x, y$ we can compute the right hand side of (124) as a limit of coincidence

$$\frac{d}{ds}|_{s=0} \frac{2s}{\mu^2} \zeta^{(1,1,1)}(s+1, y|A/\mu^2)$$

$$= \lim_{x \to y} \left\{ \frac{1}{\mu^2} \zeta^{(1,1,1)}(N, 1, x, y|A/\mu^2, \mu_0^{-2}) - \frac{P_{ab}(x, y|A)}{\mu^2} \right\}$$

$$+ \frac{1}{(4\pi)^{D/2}} \sum_{j=0, j\neq D/2-1}^{N} \frac{a_{j}(x, y|A)}{\mu_0^{2j-D+2}(j-D/2+1)} + \delta_{D} \left( \gamma + 2 \ln \frac{\mu}{\mu_0} \right) \frac{a_{(D/2-1)ab}(x, y|A)}{(4\pi)^{D/2}}$$

$$+ \frac{g_{ab}(y)}{(4\pi)^{D/2}} \sum_{j=0, j\neq D/2}^{N} \frac{a_{j}(x, y|A)}{\mu_0^{2j-D}(j-D/2)} + \delta_{D} g_{ab}(y) \left( \gamma + 2 \ln \frac{\mu}{\mu_0} \right) \frac{a_{D/2}(x, y|A)}{(4\pi)^{D/2}} . \quad (125)$$

Moreover, since the function in the limit is continuous, the same limit can be computed by identifying the tangent space at $x$ with the tangent space at $y$ and thus introducing the the bitensor $I_{a'}^{(x)a} = I_{(y)a}^{(x)a'}(y, x)$ of parallel displacement from $y$ to $x$ as usual. Employing (123) we finally get

$$(1 - 2\xi) \frac{d}{ds}|_{s=0} \frac{2s}{\mu^2} \zeta^{(1,1,1)}(s+1, y|A/\mu^2)$$

$$= \lim_{x \to y} \left\{ (1 - 2\xi) I_{a'}^{(x)a} \nabla_{(x)a'} \nabla_{(y)ab} \left\{ \frac{1}{\mu^2} \zeta^{(1,1,1)}(1, x, y|A/\mu^2) - H_{N}(x, y) \right\} \right\}$$

$$+ (1 - 2\xi) g_{ab}(y) H'(y) \quad (126)$$

where, as we said above, the final term proportional to $g_{ab}(y)$ gives no contribution to the final stress tensor because it cancels against similar terms in the first line of (120). The explicit form of $H_{N}$ reads

$$H_{N}(x, y) = \sum_{j=0}^{N} \frac{a_{j}(x, y|A)}{(4\pi)^{D/2}} \int_{0}^{\mu_0^{-2}} e^{-\sigma(x,y)/2t} \right\}$$

$$- \frac{1}{(4\pi)^{D/2}} \sum_{j=0, j\neq D/2-1}^{N} \frac{a_{j}(x, y|A)}{\mu_0^{2j-D+2}(j-D/2+1)}$$

$$- \delta_{D} \frac{a_{D/2-1}(x, y|A)}{(4\pi)^{D/2}} \left[ \gamma + \ln \left( \frac{\mu}{\mu_0} \right)^2 \right] . \quad (127)$$
The same procedure has to be used for each term in the right hand side of (120) except for the last term which, as it stands, produces the last term in the right hand side of (90). Summing all contributions, one gets (90) with $H_N$ in place of $H_{\mu^2}$. Anyhow, executing the integrations above using the results (52) - (58) in [1] ($D > 1$), expanding $H_N$ in powers and logarithm of $\sigma$ and taking account of Comments (1) and (2) after Theorem 3.1 above, we have that, in the expansion of $H_N$ one can take account only of the terms pointed out in the item (a) of Theorem 3.1; these are the only terms which do not contain the arbitrary parameter $\mu^2_0$ (which cannot remain in the final result). Therefore, the part of $H_N$ which gives contributions to the final stress tensor coincides with $H_{\mu^2}$ given in (91) and (92).

This proves the point (a) of Theorem 3.1. The point (b) is trivially proven by a direct comparison between (23), (24), (25) in [1] and the equation for the coefficients of the Hadamard local solution given in Chapter 5 of [17] which determine completely the coefficients $u_j$ and $v_j$ of the local solution once the values of the coefficients of the leading divergences are fixed for $x \to y$, and the coefficients $w_j$ once $w_0$ has been fixed. In performing this comparison, concerning the normalization conditions (94) and (95) in particular, notice that the measure used in the integrals employed in [17] is the Euclidean one $d^n x$ instead of our measure $\sqrt{g(x)} d^n x$. □

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