1 Introduction

We have recently derived generalized spin probability amplitudes, states and operators and their eigenvectors [1-4]. The new quantities are more generalized than the quantities considered most generalized hitherto in the literature. In this paper, we show how these new results are related to the standard results, and how the new results are used to calculate the quantities of usual interest.

This paper is organized as outlined below. In Section 2, we review the standard interpretation of spin, so as to prepare the way for a comparison with the interpretation arising from the new results. Section 3 is devoted to an exposition of the generalized interpretation of spin. We explain the connection between probability amplitudes for spin projection measurements and spin states in Section 3.1. We give explicit forms for the spin states in this section. In Section 3.2, we give explicit formulas for the spin operators and their eigenvectors. We then explain the connection between spin states and the eigenvectors of spin operators in Section 3.3.

Section 4 is dedicated to the treatment of spin-projection expectation values. Thus, the contrast between the standard and the generalized interpretation of these expectation values is drawn in Section 4.1. The generalized results are presented in Section 4.2. Section 4.3 is a summary of the various possibilities that arise when we compute these expectation values by the new methods. The relation between these formulas and the standard ones is clarified in Section 4.4. The paper closes with a discussion and conclusion in Section 5.

2 Standard Interpretation of Spin

We now briefly review the standard interpretation of spin. We shall illustrate our discussion by means of the spin-1/2 case. The general results will be valid for other values of spin.

Let the spin be measured in units of $\hbar/2$. Therefore, the $z$ component of spin is given by the Pauli matrix

$$[\sigma_z] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whose eigenvectors are

$$[\xi_+] = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}$$

and

$$[\xi_-] = \begin{pmatrix} a_- \\ b_- \end{pmatrix}.$$

The spin states, which we denote by $[\psi_{\pm}]$, are of the same form and are

$$[\psi_+] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$[\psi_-] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

for eigenvalues $+1$ and $-1$ respectively. In the state $[\psi_+]$, the probability of finding the spin to be "up" upon measurement is $|a_+|^2 = 1$, while the probability of finding it to be "down" is $|b_+|^2 = 0$. Similarly, in the state $[\psi_-]$, the
The probability of finding the spin to be "up" upon measurement is \(|a_-|^2 = 0\), while the probability of finding it to be "down" is \(|b_-|^2 = 1\).

The spin vector is

\[
|\sigma| = \hat{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{j} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \hat{k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (4)

Hence the component of spin in the direction of the unit vector \(\hat{n}\) whose polar angles are \((\theta, \varphi)\) is

\[
[\sigma_\hat{n}] = \hat{n} \cdot |\sigma| = \begin{pmatrix} \cos \theta & \sin \theta e^{i\varphi} \\ \sin \theta e^{-i\varphi} & -\cos \theta \end{pmatrix}.
\] (5)

The eigenvectors of this operator are

\[
[\xi_+] = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\varphi} \end{pmatrix}
\] (6)

and

\[
[\xi_-] = \begin{pmatrix} a_- \\ b_- \end{pmatrix} = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i\varphi} \end{pmatrix}
\] (7)

for eigenvalues +1 and -1 respectively.

In the literature, Eqns. (5)-(7) are given as representing the most general description of spin [5]. We shall therefore call them the "standard generalized quantities". Since the form of the spin states is the same as the form of the eigenvectors, the "up" spin state is

\[
[\psi_+] = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\varphi} \end{pmatrix}
\] (8)

while the "down" state is

\[
[\psi_-] = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i\varphi} \end{pmatrix}.
\] (9)

If the spin state is \([\psi_+]\), then the probability of finding the spin to be "up" upon measurement is \(|a_+|^2 = \cos^2 \theta/2\), while that of finding it to be "down" is \(|b_+|^2 = \sin^2 \theta/2\). By the same token, if the spin state is \([\psi_-]\), then the probability of finding the spin to be "up" upon measurement is \(|a_-|^2 = \sin^2 \theta/2\), while that of finding it to be "down" is \(|b_-|^2 = \cos^2 \theta/2\).

The expectation value of the spin projection is

\[
\langle \sigma_z \rangle_\pm = [\psi_\pm]^\dagger [\sigma_z] [\psi_\pm].
\] (10)

Using the Pauli spin operator and spin states Eqn. (3), the expectation value of the spin projection is found to be

\[
\langle \sigma_z \rangle_\pm = \pm 1.
\] (11)
The same results are obtained if the standard generalized operator Eqn. (5) and standard generalized states Eqns. (8) and (9) are used in Eqn. (10) to compute this quantity.

In order to obtain more generalized expectation values, we must in Eqn. (10) use the Pauli spin states, Eqn. (3), and the standard generalized operator Eqn. (5). Alternatively, we can use the generalized spin states Eqns. (8) and (9) with the Pauli operator Eqn. (1). In either case, the expectation value is

$$\langle \sigma_z \rangle \pm = \pm \cos \theta$$  \hspace{1cm} (12)

These are the standard results for the expectation values. We shall elucidate them further when we have presented the generalized results.

3 Generalized Interpretation

3.1 Probability Amplitudes and Spin States

The generalized interpretation is inspired by the approach to quantum mechanics due to Landé [6 − 9]. In this approach, we characterize spin projection measurements by probability amplitudes describing the measurements. Let the unit vector $\hat{a}$ with polar angles $(\theta, \varphi)$ determine the initial direction of the spin projection, so that before the current measurement the projection is up or down with respect to this direction. Let the unit vector $\hat{c}$ with polar angles $(\theta', \varphi')$ be the new direction in which we seek the spin projection. If the spin projection is initially up with respect to $\hat{a}$, a measurement along $\hat{c}$ will give projections up or down with respect to $\hat{c}$ with respective probability amplitudes $\psi(m_i(\hat{a}); m_f(\hat{c}))$ where we have labelled the spin states by their magnetic quantum numbers and the corresponding direction of quantization. The subscripts $i$ and $f$ refer to initial and final respectively. The explicit expressions for these probability amplitudes depend on the choice of phase; there are many such choices [4]. For spin 1/2, one choice gives the following [1, 2] expressions:

$$\psi\left(\left(+\frac{1}{2}\right)(\hat{a}); \left(+\frac{1}{2}\right)(\hat{c})\right) = \cos \frac{\theta}{2} \cos \frac{\theta'}{2} + e^{i(\varphi-\varphi')} \sin \frac{\theta}{2} \sin \frac{\theta'}{2},$$  \hspace{1cm} (13)

$$\psi\left(\left(+\frac{1}{2}\right)(\hat{a}); \left(-\frac{1}{2}\right)(\hat{c})\right) = \cos \frac{\theta}{2} \sin \frac{\theta'}{2} - e^{i(\varphi-\varphi')} \sin \frac{\theta}{2} \cos \frac{\theta'}{2},$$  \hspace{1cm} (14)

$$\psi\left(\left(-\frac{1}{2}\right)(\hat{a}); \left(+\frac{1}{2}\right)(\hat{c})\right) = \sin \frac{\theta}{2} \cos \frac{\theta'}{2} - e^{i(\varphi-\varphi')} \cos \frac{\theta}{2} \sin \frac{\theta'}{2}$$  \hspace{1cm} (15)

and

$$\psi\left(\left(-\frac{1}{2}\right)(\hat{a}); \left(-\frac{1}{2}\right)(\hat{c})\right) = \sin \frac{\theta}{2} \sin \frac{\theta'}{2} + e^{i(\varphi-\varphi')} \cos \frac{\theta}{2} \cos \frac{\theta'}{2}.$$  \hspace{1cm} (16)

The principle that measurements should be reproducible means that if we repeat a measurement, the same value should be obtained with certainty. That
These probability amplitudes do indeed satisfy these conditions.

Eqns. (13) – (16) can be expanded according to the example below for amplitudes for measurements from the direction \( \hat{\varphi} \) in the basis \( \hat{\theta} \) and \( \hat{\phi} \).

Here \( \hat{\theta} \), whose polar angles are \( (\theta'', \varphi'') \), is another vector with respect to which we can measure the spin projection. The \( \chi \)'s are probability amplitudes for measurements from the direction \( \hat{\alpha} \) to the direction \( \hat{\beta} \) and the \( \phi \)'s are probability amplitudes for measurements from the direction \( \hat{\beta} \) to the direction \( \hat{\gamma} \). Therefore in the basis \( \phi((+\frac{1}{2})(\hat{\beta}); (+\frac{1}{2})(\hat{\gamma})) \) and \( \phi((-\frac{1}{2})(\hat{\beta}); (+\frac{1}{2})(\hat{\gamma})) \), the probability amplitude \( \psi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma})) \) is represented by the vector

\[
[\psi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma}))] = \begin{pmatrix}
\chi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\beta})) \\
\chi((+\frac{1}{2})(\hat{\alpha}); (-\frac{1}{2})(\hat{\beta}))
\end{pmatrix}.
\]  

Using the expressions Eqns. (13) – (16) for the probability amplitudes, we deduce that the generalized spin state \( [\psi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma}))] \) has the form [1]

\[
[\psi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma}))] = \begin{pmatrix}
\cos \frac{\theta}{2} \cos \frac{\theta''}{2} + e^{i(\varphi - \varphi'')} \sin \frac{\theta}{2} \sin \frac{\theta''}{2} \\
\cos \frac{\theta}{2} \sin \frac{\theta''}{2} - e^{i(\varphi - \varphi'')} \sin \frac{\theta}{2} \cos \frac{\theta''}{2}
\end{pmatrix}
\]  

\[
= [\psi(+\frac{1}{2})(\hat{\alpha})].
\]  

The expansions for the other probability amplitudes show that

\[
[\psi((+\frac{1}{2})(\hat{\alpha}); (-\frac{1}{2})(\hat{\gamma}))] = [\psi((+\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma}))]
\]  

and

\[
[\psi((-\frac{1}{2})(\hat{\alpha}); (+\frac{1}{2})(\hat{\gamma}))] = [\psi((-\frac{1}{2})(\hat{\alpha}); (-\frac{1}{2})(\hat{\gamma}))]
\]
with respect to the direction \( \hat{e} \) respectively. The elements of \( \sigma_c \) and general forms are a special case of the generalized forms. We observe that owing to the fact that the quantities \( \psi, \chi \) and \( \phi \) have exactly the same structure, we can as well use one symbol \( \chi \) to represent them in the expansion Eqn. (19).

### 3.2 Spin Operators

The most general form of the "z component" of the spin operator is \( [\sigma_c] \), defined with respect to the direction \( \hat{e} \), and is

\[
[\sigma_c] = \begin{pmatrix}
(\sigma_c)_{11} & (\sigma_c)_{12} \\
(\sigma_c)_{21} & (\sigma_c)_{22}
\end{pmatrix}
\]  

(24)

Here, the unit vectors \( \hat{e} \) and \( \hat{b} \) have the polar angles \( (\theta', \varphi') \) and \( (\theta'', \varphi'') \) respectively. The elements of \( [\sigma_c] \) are \([1] \)

\[
(\sigma_c)_{11} = \cos \theta'' \cos \theta' + \sin \theta'' \sin \theta' \cos(\varphi'' - \varphi'),
\]

(25)

\[
(\sigma_c)_{12} = \sin \theta'' \cos \theta' - \sin \theta'' \cos \theta' - \sin \theta' [\cos \theta'' \cos(\varphi'' - \varphi') + i \sin(\varphi'' - \varphi')],
\]

(26)

\[
(\sigma_c)_{21} = \sin \theta'' \cos \theta' - \sin \theta'' \cos \theta' - \sin \theta' [\cos \theta'' \cos(\varphi'' - \varphi') - i \sin(\varphi'' - \varphi')]
\]

(27)

and

\[
(\sigma_c)_{11} = -\cos \theta'' \cos \theta' - \sin \theta'' \sin \theta' \cos(\varphi'' - \varphi').
\]

(28)

The eigenvectors of this operator are \([1]\)

\[
\chi^\dagger = \begin{pmatrix}
\chi((+\frac{1}{2})\hat{c}); (+\frac{1}{2})\hat{b}) \\
\chi((+\frac{1}{2})\hat{c}); (-\frac{1}{2})\hat{b})
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta'}{2} \cos \frac{\theta''}{2} + e^{i(\varphi'' - \varphi')} \sin \frac{\theta'}{2} \sin \frac{\theta''}{2} \\
\cos \frac{\theta'}{2} \sin \frac{\theta''}{2} - e^{i(\varphi'' - \varphi')} \sin \frac{\theta'}{2} \cos \frac{\theta''}{2}
\end{pmatrix}
\]

(29)

for eigenvalue +1 and

\[
\chi^{\dagger} = \begin{pmatrix}
\chi((-\frac{1}{2})\hat{c}); (+\frac{1}{2})\hat{b}) \\
\chi((-\frac{1}{2})\hat{c}); (-\frac{1}{2})\hat{b})
\end{pmatrix} = \begin{pmatrix}
\sin \frac{\theta'}{2} \cos \frac{\theta''}{2} - e^{i(\varphi'' - \varphi')} \cos \frac{\theta'}{2} \sin \frac{\theta''}{2} \\
\sin \frac{\theta'}{2} \sin \frac{\theta''}{2} + e^{i(\varphi'' - \varphi')} \cos \frac{\theta'}{2} \cos \frac{\theta''}{2}
\end{pmatrix}
\]

(30)

for eigenvalue -1.

In order to obtain the standard generalized forms Eqns. (1) and (2) from these formulas, we only need to set \( \theta'' = 0 \), and \( \varphi'' = \pi \). Thus, the standard generalized forms are a special case of the generalized forms.
3.3 Spin States and Eigenvectors

The matrix state of a wave function is obtained by first expanding the wave function in terms of a complete set. When the expansion coefficients are arranged in a column or row vector, they form the matrix state of the wave function. This is what we have done to obtain the generalized states and operators from a probability amplitude basis.

In standard quantum mechanics, the expansion coefficients are viewed as probability amplitudes which are constants. But if we use the Landé formula [6–9] to perform the expansion, then we straight away recognize that the expansion coefficients are probability amplitudes with a structure. This insight allows us to deduce the eigenvectors of the generalized spin operator without any calculation [1–4].

Spin states and spin eigenvectors have the same structure. Both are each characterized by two reference directions. In addition, the spin operators are also characterized by being functions of two reference directions. But the difference between spin states and spin eigenvectors is this. The eigenvectors have the same reference directions as the operator they are eigenfunctions of. But the spin states in the general case have at least one reference vector different from those characterizing the operator.

4 Expectation Values

4.1 Standard Generalized and Generalized Results

The standard way of talking about spin concentrates on the current measurement to the exclusion of the measurement that brought about the spin state that obtains prior to the current measurement. Thus, this way of talking implicitly ignores the original direction of the spin projection. This failure to explicitly mention the precise state that obtains before the current measurement derives from the belief that in many cases, a quantum system is in a superposition state before a measurement is made. In such a state, certain properties of the system are undefined.

In the present formalism, however, the spin state is always well defined. Probability amplitudes always involve an index or label corresponding to the state that pertains before the current measurement, and then another characterizing the state brought about by this measurement. Therefore spin states are characterized by four angles: one pair of angles defines the direction of the initial reference vector, while the other pair describes the direction of the final reference vector. When spin states involve only two of the angles, as in the standard prescription, then, as we shall explain later, we are dealing with a special instance of the general case.
4.2 Generalized Results

We now proceed to give the general treatment of spin expectation values \([1]\). Let the spin projection be initially \(m_i\) with respect to \(\hat{\mathbf{a}}\). It is then measured with respect to the vector \(\hat{\mathbf{c}}\). Its value is found to be \(m^i_f\). Here \(m^1_f = m^1_c = +1\), and \(m^2_f = m^2_c = -1\). Since the probability amplitude for obtaining +1 is \(\psi(m^1_f; +\frac{1}{2})\), and that for obtaining −1 is \(\psi(m^2_f; -\frac{1}{2})\), the expectation value of the spin projection, which we shall denote by \(\langle \sigma_c^z \rangle\), is

\[
\langle \sigma_c^z \rangle = \psi^* (m^i_f; +\frac{1}{2}) \psi (m^i_f; +\frac{1}{2})(+1) \\
+ \psi^* (m^i_f; -\frac{1}{2}) \psi (m^i_f; -\frac{1}{2})(-1) . 
\]

(31)

We now expand both \(\psi(m^i_f; +\frac{1}{2})\) and \(\psi(m^i_f; -\frac{1}{2})\) as in Eqn. \([19]\). We find that \([1]\)

\[
\langle \sigma_c^z \rangle = [\psi(m^i_f)]^\dagger [\sigma_z] [\psi(m^i_f)], 
\]

(32)

where

\[
[\psi(m^i_f)] = \begin{pmatrix} 
\chi(m^i_f; +\frac{1}{2}) & \\
\chi(m^i_f; -\frac{1}{2}) 
\end{pmatrix} 
\]

(33)

and the elements of \([\sigma_c^z]\) are

\[
(\sigma_c^z)_{11} = \left| \phi(\frac{1}{2}; \frac{1}{2}) \right|^2 - \left| \phi(\frac{1}{2}; \frac{1}{2}) \right|^2 ,
\]

(34)

\[
(\sigma_c^z)_{12} = \phi^*(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2}) \\
- \phi^*(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2}),
\]

(35)

\[
(\sigma_c^z)_{21} = \phi^*(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2}) \\
- \phi^*(\frac{1}{2}; \frac{1}{2}) \phi(\frac{1}{2}; \frac{1}{2})
\]

(36)

and

\[
(\sigma_c^z)_{22} = \left| \phi(\frac{1}{2}; \frac{1}{2}) \right|^2 - \left| \phi(\frac{1}{2}; \frac{1}{2}) \right|^2 .
\]

(37)

As has already been pointed out, the vector \(\hat{\mathbf{b}}\) is arbitrary. Therefore, depending on the choice of \(\hat{\mathbf{b}}\), the state, Eqns. \([33]\), and the operator, Eqn. \([37]\)
take different forms. Further variation results from whether \( \hat{c} \) equals \( \hat{a} \) or not. We expect to find among these various forms the specialized ones that correspond to the standard results. The full complement of various cases is as given below.

4.3 Summary of Possibilities for the Reference Vectors

4.3.1 Case (a): \( \hat{b} \neq \hat{a} \) and \( \hat{c} \neq \hat{a} \).

This is the most general case. In this case, the matrix representations are

\[
[\psi((+1/2)\hat{a}; (+1/2)\hat{c})] = [\psi((+1/2)\hat{a}; (-1/2)\hat{c})] = \begin{pmatrix} \chi((+\frac{1}{2})\hat{a}; (+\frac{1}{2})\hat{b}) \\ \chi((+\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{b}) \end{pmatrix} \tag{38}
\]

and

\[
[\psi((-1/2)\hat{a}; (-1/2)\hat{c})] = [\psi((-1/2)\hat{a}; (+1/2)\hat{c})] = \begin{pmatrix} \chi((-\frac{1}{2})\hat{a}; (+\frac{1}{2})\hat{b}) \\ \chi((-\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{b}) \end{pmatrix} \tag{39}
\]

while the operator \([\sigma_z]\) has the elements given by Eqns. \((34) - (37)\).

4.3.2 Case (b): \( \hat{b} = \hat{a} \).

The matrix representations are

\[
[\psi((+1/2)\hat{a}; (\pm 1/2)\hat{c})] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{40}
\]

and

\[
[\psi((-1/2)\hat{a}; (\pm 1/2)\hat{c})] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{41}
\]

while the elements of \([\sigma_c]\) are

\[
(\sigma_c)_{11} = \left| \phi((+1/2)\hat{a}; (+1/2)\hat{c}) \right|^2 - \left| \phi((+1/2)\hat{a}; (-1/2)\hat{c}) \right|^2, \tag{42}
\]

\[
(\sigma_c)_{12} = \phi^*((+1/2)\hat{a}; (+1/2)\hat{c})\phi((-1/2)\hat{a}; (+1/2)\hat{c}) - \phi^*((+1/2)\hat{a}; (-1/2)\hat{c})\phi((-1/2)\hat{a}; (-1/2)\hat{c}), \tag{43}
\]

\[
(\sigma_c)_{21} = \phi^*((-1/2)\hat{a}; (+1/2)\hat{c})\phi((+1/2)\hat{a}; (+1/2)\hat{c}) - \phi^*((-1/2)\hat{a}; (-1/2)\hat{c})\phi((+1/2)\hat{a}; (-1/2)\hat{c}) \tag{44}
\]
and

\[(\sigma^c_{\text{e}})_{22} = \left| \phi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{c}) \right|^2 - \left| \phi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{c}) \right|^2. \tag{45}\]

4.3.3 Case (c): $\hat{b} = \hat{c}$.

The matrix representations are

\[\begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{c}) \end{bmatrix} = \begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \end{bmatrix} \begin{bmatrix} \chi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \\ \chi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix}. \tag{46}\]

and

\[\begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \end{bmatrix} = \begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix} \begin{bmatrix} \chi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \\ \chi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix}. \tag{47}\]

while the operator is

\[\begin{bmatrix} \sigma^c_{\text{e}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{48}\]

4.3.4 Case (d): $\hat{c} = \hat{a}$.

The matrix representations are

\[\begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{a}) \end{bmatrix} = \begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \end{bmatrix} \begin{bmatrix} \chi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \\ \chi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix}. \tag{49}\]

and

\[\begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \end{bmatrix} = \begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix} \begin{bmatrix} \chi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{b}) \\ \chi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{b}) \end{bmatrix}. \tag{50}\]

while the elements of $[\sigma^c_{\text{e}}]$ are given by Eqns. (34) - (37).

4.3.5 Case (e): $\hat{b} = \hat{a}$ and $\hat{c} = \hat{a}$.

The matrix representations are

\[\begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (+\frac{1}{2})^\text{a}) \end{bmatrix} = \begin{bmatrix} \psi((-\frac{1}{2})^\text{a}; (-\frac{1}{2})^\text{a}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{51}\]

and
\[
[\psi((-\frac{1}{2})\hat{a}); (+\frac{1}{2})\hat{a})] = [\psi((-\frac{1}{2})\hat{a}); (-\frac{1}{2})\hat{a})] = \begin{pmatrix}
0 \\
1
\end{pmatrix},
\]
while the operator is
\[
[\sigma_c] = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

4.4 Comparison With Standard Results

As Eqns. (5) - (7) show, the generalized standard forms of the various quantities explicitly involve only one direction, represented here by the angles \((\theta', \varphi')\). There would appear to be no case among the possibilities (a) - (e) corresponding to the operator being a function of one direction only. But in fact, if we set \(\theta'' = 0\), and \(\varphi'' = \pi\), so that \(\hat{b} = \hat{k}\) in Case (a), we achieve this situation. Thus, using the actual expressions for the probability amplitudes, Eqns. (21) and (23), and for the elements of \([\sigma_c]\), Eqns. (25) - (28) we obtain for Case (a):

\[
\psi((+\frac{1}{2})\hat{a})] = \begin{pmatrix}
\cos \theta / 2 e^{i\varphi} \\
\sin \theta / 2 e^{i\varphi}
\end{pmatrix},
\]

\[
\psi((-\frac{1}{2})\hat{a})] = i \begin{pmatrix}
\sin \theta / 2 e^{-i\varphi} \\
\cos \theta / 2 e^{-i\varphi}
\end{pmatrix}
\]
and

\[
[\sigma_c] = \begin{pmatrix}
\cos \theta' & \sin \theta' e^{-i\varphi'} \\
\sin \theta' e^{i\varphi'} & -\cos \theta'
\end{pmatrix},
\]

As in this case \(\hat{a} \neq \hat{c}\), using these quantities ensures that we get the most general expectation value. In view of this result, it is clear that the standard generalized results are just a special case of the generalized results. This special form is here obtained from the generalized form Eqns. (34) - (37) by fixing the angles of \(\hat{b}\) to \(\theta'' = 0\) and \(\varphi'' = \pi\).

It is also possible using Case (b) to obtain the standard generalized expression for the expectation value. In this case, the state is either

\[
[\psi((+\frac{1}{2})\hat{a})] = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
or
\[
[\psi((-\frac{1}{2})\hat{a})] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
while the operator is given by the elements Eqns. (25) - (28); however, these elements have to be brought to those for the standard generalized form by setting the angles of \(\hat{b} = \hat{a}\) to \(\theta = 0\) and \(\varphi = \pi\). Then, the operator becomes identical with Eqn. (5), save for the fact that the angles are double primed..

Similarly, Case (c) can also yield the standard results. We can get the generalized expectation value by setting the angles of \(\hat{b} = \hat{c}\) to \(\theta'' = 0\) and
ϕ'' = π in the expressions for the states. We then obtain Eqns. (8) and (9) for the states, save for phase changes. The operator is the Pauli operator.

What all the three methods of obtaining the standard generalized expectation value have in common is that the vector \( \hat{b} \) is set equal to the unit vector \( \hat{k} \) which defines the positive \( z \) direction. All these methods give the standard generalized expectation value. However, the form that results from Case (a) is a new form; at any rate we have not seen it in the literature. The two forms resulting from Cases (b) and (c) both lead to the results Eqn. (12). Both these cases take one of the reference directions for spin projection measurements as the \( z \) axis. Case (b) takes the initial direction \( \hat{a} \) to equal \( \hat{k} \), while Case (c) takes the final direction \( \hat{c} \) to equal \( \hat{k} \). Thus, the standard treatment of the expectation value completely satisfies the Landé-approach requirement that spin projection measurements must refer to well-defined initial and final states.

We must mention that we have not seen any evidence that the literature recognizes that the angular arguments of the spin state and the operator can differ in the standard generalized expression for the expectation value. But, considering that the standard description of spin measurements is vague about the initial state, this is not surprising. Making these arguments different gives a different final reference direction from the initial, thus making the expectation value truly general.

For the case where the spin projection is measured with respect to the same vector twice in succession, the formulas for expectation value are given by Cases (d) and (e). These cases correspond closely to what is in the literature. Case (e), of course, involves the standard Pauli quantities. Both cases lead to the results Eqn. (11). To correspond this case to what the literature contains, we again have to make \( \hat{b} \) correspond to the positive \( z \) direction. This is accomplished by setting \( \theta' = 0 \) and \( \varphi' = \pi \).

If we use the generalized approach to compute the expectation value, we confirm that

\[
\langle \sigma_z \rangle = \pm \cos \Theta,
\]

(59)

where \( \Theta \) is the angle between \( \hat{a} \) and \( \hat{c} \), so that

\[
\cos \Theta = \hat{a} \cdot \hat{c} = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi').
\]

(60)

5 Discussion and Conclusion

In this paper, we have explained how the new generalized spin quantities we have introduced relate to the usual quantities which we have called the "standard generalized quantities". We have shown how despite the seeming difference in the two kinds of quantities, the "standard generalized quantities" are special forms of the new quantities.

Our success in obtaining the spin quantities from first principles [1–4], and in using them to elucidate standard ideas is strong indication that the Landé approach to quantum mechanics could be fruitful if pursued logically.
and applied even to seemingly well-understood results. The idea of attaching two labels to a probability amplitude is one which seems to be of general validity. This idea is applicable to eigenfunctions. These are just probability amplitudes corresponding to the measurement of an observable with a continuous eigenvalue spectrum. For an eigenfunction, the characteristic eigenvalue defines the state that obtains before the current measurement.

It is our belief that many more interesting results and elucidations of standard quantum theory will be obtained by means of the Landé approach.

6 References

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