SIMULTANEOUS SMOOTHNESS AND SIMULTANEOUS STABILITY
OF A $C^\infty$ STRICTLY CONVEX INTEGRAND AND ITS DUAL

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ABSTRACT. In this paper, we investigate simultaneous properties of a convex integrand $\gamma$ and its dual $\delta$. The main results are the following three.

(1) For a $C^\infty$ convex integrand $\gamma : S^n \to \mathbb{R}^+$, its dual convex integrand $\delta : S^n \to \mathbb{R}^+$ is of class $C^\infty$ if and only if $\gamma$ is a strictly convex integrand.

(2) Let $\gamma : S^n \to \mathbb{R}^+$ be a $C^\infty$ strictly convex integrand. Then, $\gamma$ is stable if and only if its dual convex integrand $\delta : S^n \to \mathbb{R}^+$ is stable.

(3) Let $\gamma : S^n \to \mathbb{R}^+$ be a $C^\infty$ strictly convex integrand. Suppose that $\gamma$ is stable. Then, for any $i$ ($0 \leq i \leq n$), a point $\theta_0 \in S^n$ is a non-degenerate critical point of $\gamma$ with Morse index $i$ if and only if its antipodal point $-\theta_0 \in S^n$ is a non-degenerate critical point of the dual convex integrand $\delta$ with Morse index $(n-i)$.

1. INTRODUCTION

Throughout this paper, we let $n$ and $\mathbb{R}^+$ be a positive integer and the set consisting of positive real numbers respectively. Let $\text{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$ be the inversion with respect to the origin $0$ of $\mathbb{R}^{n+1}$, namely, $\text{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$ is defined as follows where $(\theta, r)$ means the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$:

$$\text{inv}(\theta, r) = \left(-\theta, \frac{1}{r}\right).$$

Let $S^n$ be the unit sphere of $\mathbb{R}^{n+1}$. For a continuous function $\gamma : S^n \to \mathbb{R}^+$, denote the boundary of the convex hull of $\text{inv}($graph$(\gamma))$ by $\Gamma_\gamma$, where graph$(\gamma)$ is the subset of $\mathbb{R}^{n+1} - \{0\}$ defined as follows:

$$\text{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\}.$$

A continuous function $\gamma : S^n \to \mathbb{R}^+$ is said to be a convex integrand if the equality $\Gamma_\gamma = \text{inv}(\text{graph}(\gamma))$ is satisfied (\text{[1]}).

**Definition 1.1.** A convex integrand $\gamma : S^n \to \mathbb{R}^+$ is called a strictly convex integrand if the convex hull of $\text{inv}(\text{graph}(\gamma))$ is strictly convex.

Define $C^\infty(S^n, \mathbb{R}^+)$ and $C^\infty_{\text{conv}}(S^n, \mathbb{R}^+)$ as follows.

$$C^\infty(S^n, \mathbb{R}^+) = \{\gamma : S^n \to \mathbb{R}^+ \mid \gamma \in C^\infty(S^n, \mathbb{R}^+)\},$$

$$C^\infty_{\text{conv}}(S^n, \mathbb{R}^+) = \{\gamma \in C^\infty(S^n, \mathbb{R}^+) \mid \gamma \text{ is a convex integrand}\}.$$

The set $C^\infty(S^n, \mathbb{R}^+)$ is a topological space endowed with Whitney $C^\infty$ topology (for details on Whitney $C^\infty$ topology, for instance see [5, 7, 13, 30]); and the set $C^\infty_{\text{conv}}(S^n, \mathbb{R}^+)$...
is a topological subspace of \( C^\infty(S^n, \mathbb{R}_+) \). Given a \( \gamma \in C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) \), the \textbf{Wulff shape associated with} \( \gamma \), denoted by \( \mathcal{W}_\gamma \), is the following intersection:

\[
\bigcap_{\theta \in S^n} \left\{ x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta) \right\},
\]

where \( x \cdot \theta \) stands for the standard scalar product of two vectors \( x \) and \( \theta \) of \( \mathbb{R}^{n+1} \). The notion of Wulff shape was firstly introduced by G. Wulff [13] in 1901 as a geometric model of a crystal at equilibrium. For details on Wulff shapes, see for instance [12, 29, 36, 41, 42]. By definition, any Wulff shape \( \mathcal{W}_\gamma \) is compact, convex and contains the origin of \( \mathbb{R}^{n+1} \) as an interior point. In order to investigate Wulff shapes, the notion of convex integrand was introduced (see [41]).

**Definition 1.2.** Let \( \gamma : S^n \rightarrow \mathbb{R}_+ \) be a convex integrand.

1. A convex integrand \( \delta : S^n \rightarrow \mathbb{R}_+ \) is called the \textit{dual convex integrand} of \( \gamma \) or just the \textit{dual} of \( \gamma \) if the equality \( \text{inv}(\text{graph}(\delta)) = \partial \mathcal{W}_\gamma \) holds, where \( \partial \mathcal{W}_\gamma \) stands for the boundary of \( \mathcal{W}_\gamma \).

2. The Wulff shape associated with \( \delta \) is called the \textit{dual Wulff shape} of \( \mathcal{W}_\gamma \) and is denoted by \( \mathcal{D}\mathcal{W}_\gamma \).

\[ \mathcal{D}\mathcal{W}_\gamma = \mathcal{W}_\delta. \]

Notice that both of the above two dual notions are involutive, namely, we have that the dual convex integrand of \( \delta \) is \( \gamma \) and the equality \( \mathcal{D}\mathcal{D}\mathcal{W}_\gamma = \mathcal{W}_\gamma \) holds (see Lemma 2.7 in Subsection 2.6). In this paper, it is focused exclusively on investigating simultaneous properties of a convex integrand \( \gamma \) and its dual \( \delta \).

**Definition 1.3.** A \( C^\infty \) function \( \gamma \in C^\infty(S^n, \mathbb{R}_+) \) is said to be \textit{stable} if the \( \mathcal{A} \)-equivalence class of \( \gamma \) is open, where two elements \( \gamma_1, \gamma_2 \in C^\infty(S^n, \mathbb{R}_+) \) are said to be \( \mathcal{A} \)-\textit{equivalent} if there exist \( C^\infty \) diffeomorphisms \( h : S^n \rightarrow S^n \) and \( H : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying \( \gamma_1 = H \circ \gamma_2 \circ h^{-1} \).

There are two reasons why we prefer stable convex integrands. It is well-known that a convex integrand represents surface energy density. And, usually, it is almost impossible to obtain the precise surface energy density, that is to say, we merely have an approximated surface energy density. Therefore, we would like to have a situation that the space consisting of \( C^\infty \) convex integrands having only non-degenerate critical points is dense in the space consisting of \( C^\infty \) convex integrands. This is one reason. Another reason is as follows. The Morse inequalities [28] are almost indispensable tools for investigating convex integrands globally from the differentiable viewpoint. In order to apply the Morse inequalities, critical points must be non-degenerate.

In [6], it has been shown that stable convex integrands form an open and dense subset of \( C^\infty_{\text{conv}}(S^n, \mathbb{R}_+) \). As the next step of [6], it is natural to investigate when and how the simultaneous stability of the convex integrand \( \gamma \) and its dual \( \delta \) occurs, which is the main purpose of this paper. Since stable functions must be of class \( C^\infty \), before carrying out the main purpose, it is necessary to show the following:

**Theorem 1.** Let \( \gamma : S^n \rightarrow \mathbb{R}_+ \) be a \( C^\infty \) convex integrand and let \( \delta \) be the dual convex integrand of \( \gamma \). Then, \( \delta \) is of class \( C^\infty \) if and only if \( \gamma \) is a strictly convex integrand.

It should be noted that weaker versions of Theorem 1 have been already shown independently in [11, 29, 30]. Notice also that by [11], the assumption of Theorem 1 implies that \( \delta \) is a strictly convex integrand. Thus, we have the following corollary.
Corollary 1. Let $\gamma : S^n \to \mathbb{R}_+$ be a convex integrand and let $\delta$ be its dual convex integrand. Then, $\gamma$ is a $C^\infty$ strictly convex integrand if and only if $\delta$ is a $C^\infty$ strictly convex integrand.

Define the functions $\hat{\gamma}, \hat{\delta} : S^n \to \mathbb{R}_+$ by
\[
\hat{\gamma}(\theta) = \frac{1}{\gamma(-\theta)} \quad \text{and} \quad \hat{\delta}(\theta) = \frac{1}{\delta(-\theta)} \quad (\forall \theta \in S^n)
\]
respectively. It is easily seen that $\partial DW_\gamma$ (resp., $\partial DW_\delta$) is the graph of the function $\hat{\gamma}$ (resp., $\hat{\delta}$) and that $\gamma$ (resp., $\delta$) is $\mathcal{A}$-equivalent to $\hat{\gamma}$ (resp., $\hat{\delta}$). Thus, as another corollary of Theorem 1, we have the following:

Corollary 2. Let $\gamma : S^n \to \mathbb{R}_+$ be a strictly convex integrand and let $\delta : S^n \to \mathbb{R}_+$ be the dual convex integrand of $\gamma$. Then, the following are equivalent.

1. The convex integrand $\gamma$ is of class $C^\infty$.
2. The convex integrand $\delta$ is of class $C^\infty$.
3. The function $\hat{\gamma}$, whose graph is exactly $\partial W_\delta = \partial DW_\gamma$, is of class $C^\infty$.
4. The function $\hat{\delta}$, whose graph is exactly $\partial W_\gamma = \partial DW_\delta$, is of class $C^\infty$.

Notice that for any convex integrand $\gamma : S^n \to \mathbb{R}_+$, the Wulff shape $W_\gamma$ is a convex body such that the origin is contained in its interior. In Convex Body Theory, there is the notion of dual for a convex body containing the origin as an interior point. Namely, in Convex Body Theory, the boundary of the dual of $W_\gamma$ is the following set (see for example [39]).

\[
\left\{ \left( \theta, \frac{1}{\gamma(\theta)} \right) \mid \theta \in S^n \right\}.
\]

However, the notion of dual in this sense seems to have less relations with the notion of pedal which seems to be a common background in Physics (for instance, see [36]). On the other hand, the notion of dual Wulff shapes in our sense is closely related to the notion of pedal. Moreover, via the central projection, the pedal of $C^\infty$ embedding $\Phi : S^n \to \mathbb{R}^{n+1} - \{0\}$ defined by $\Phi(\theta) = (\theta, 1/\delta(-\theta))$ relative to the origin is characterized by using the spherical dual of the corresponding embedding $\tilde{\Phi} : S^n \to S^{n+1}$; and the spherical dual is a well-known notion in Singularity Theory (for details, see Subsection 2.3). Since pedals are useful to study a Wulff shape associate with a $C^\infty$ convex integrand, we adopt $\partial W_\gamma$ as the notion of dual Wulff shape of $W_\gamma$.

The following Theorem 2 answers the question “When does the simultaneous stability of $\gamma$ and $\delta$ occur?”.

Theorem 2. Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand and let $\delta$ be the dual convex integrand of $\gamma$. Then, $\gamma$ is stable if and only if $\delta$ is stable.

Corollary 3. Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand and let $\delta : S^n \to \mathbb{R}_+$ be the dual convex integrand of $\gamma$. Then, the following are equivalent.

1. The convex integrand $\gamma$ is stable.
2. The convex integrand $\delta$ is stable.
3. The function $\hat{\gamma}$, whose graph is exactly $\partial W_\delta = \partial DW_\gamma$, is stable.
4. The function $\hat{\delta}$, whose graph is exactly $\partial W_\gamma = \partial DW_\delta$, is stable.

The following Theorem 3 answers the question “How does the simultaneous stability of $\gamma$ and $\delta$ occur?”.
Theorem 3. Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand and let $\delta : S^n \to \mathbb{R}_+$ be the dual convex integrand of $\gamma$. Suppose that $\gamma$ is stable. Then, the following hold:

1. A point $\theta_0 \in S^n$ is a non-degenerate critical point of $\gamma$ if and only if its antipodal point $-\theta_0 \in S^n$ is a non-degenerate critical point of $\delta$.

2. Suppose that a point $\theta_0 \in S^n$ is a non-degenerate critical point of $\gamma$. Then, the Morse index of $\gamma$ at $\theta_0$ is $i$ if and only if the Morse index of $\delta$ at $-\theta_0$ is $(n - i)$, where $i$ is an integer such that $0 \leq i \leq n$.

It is clear that, by Theorem 3, we have the following corollary.

Corollary 4. Let $\gamma : S^n \to \mathbb{R}_+$ be a stable convex integrand and let $\delta : S^n \to \mathbb{R}_+$ be the dual convex integrand of $\gamma$. Moreover, let $\theta_0$ be a point of $S^n$ and let $i$ be an integer such that $0 \leq i \leq n$. Then, the following are equivalent.

1. The point $\theta_0 \in S^n$ is a non-degenerate critical point of $\gamma$ with Morse index $i$.

2. The point $-\theta_0 \in S^n$ is a non-degenerate critical point of $\delta$ with Morse index $(n - i)$.

3. The point $-\theta_0 \in S^n$ is a non-degenerate critical point of $\tilde{\gamma}$ with Morse index $(n - i)$.

4. The point $\theta_0 \in S^n$ is a non-degenerate critical point of $\tilde{\delta}$ with Morse index $i$.

This paper is organized as follows. In Section 2 preliminaries are given. Theorems 1, 2, and 3 are proved in Sections 4, 5, and 6 respectively.

2. Preliminaries

2.1. Stable functions $S^n \to \mathbb{R}_+$. In this subsection, we quickly review a geometric characterization of a stable function $\gamma : S^n \to \mathbb{R}_+$ and the definition of Morse index of $\gamma$ at a non-degenerate critical point $\theta \in S^n$. Both are well-known.

Among Mather’s celebrated series [21, 22, 23, 24, 25, 26], the geometric characterization of a proper stable mapping is dealt with in [25]. It is easily seen that the following well-known geometric characterization of a stable function $S^n \to \mathbb{R}_+$ is derived from Mather’s geometric characterization.

Proposition 1 ([25]). A $C^\infty$ function $\gamma : S^n \to \mathbb{R}_+$ is stable if and only if all critical points of $\gamma$ are non-degenerate and $\gamma(\theta_1) \neq \gamma(\theta_2)$ holds for any two distinct critical points $\theta_1, \theta_2 \in S^n$.

It seems that a proper stable function is usually called a Morse function. However, a Morse function in [25] is a $C^\infty$ function having only non-degenerate critical points, and thus it is a weaker notion than the notion of stable function. Therefore, in order to avoid unnecessary confusion, stable functions and Morse functions are distinguished in this paper.

Definition 2.1 ([25]). Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ function and let $\theta \in S^n$ be a non-degenerate critical point of $\gamma$. Then, there exists a coordinate neighborhood $(U, \varphi)$ of $\theta$ such that $\varphi(\theta) = 0$ and the following equality holds:

$$
\gamma \circ \varphi^{-1}(x_1, \ldots, x_n) = \gamma(\theta) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2.
$$

The integer $i$ $(0 \leq i \leq n)$ does not depend on the particular choice of the coordinate neighborhood $(U, \varphi)$ and it is called the Morse index of $\gamma$ at $\theta$. Here, the integer $i$ is more than or equal to 0 and less than or equal to $n$. 
2.2. Pedals. Although it has been explained only for plane pedal curves in it, the reference [9] is an excellent book for pedals. The definition of higher dimensional pedal is parallel to the definition of plane pedal curve as follows.

Definition 2.2. Given a $C^\infty$ embedding $\Phi : S^n \to \mathbb{R}^{n+1} - \{0\}$, the pedal relative to the pedal point $0$ for $\Phi$, denoted by $\text{ped}_{\Phi,0} : S^n \to \mathbb{R}^{n+1}$, is the mapping which maps $\theta \in S^n$ to the unique nearest point of $\Phi(\theta) + T_{\Phi(\theta)}(S^n)$ from the origin $0$.

By the definition of Wulff shape, if the boundary of a Wulff shape $\partial \mathcal{W}_\gamma$ is the image of a $C^\infty$ embedding $\Phi : S^n \to \mathbb{R}^{n+1} - \{0\}$, the graph of the given $C^\infty$ convex integrand $\gamma$ may be considered as the pedal relative to the pedal point $0$ for $\Phi$. In this case, since $\text{graph}(\gamma)$ does not contain the origin, from the information of $\text{ped}_{\Phi,0} : S^n \to \mathbb{R}^{n+1}$, if the boundary of a Wulff shape $\partial \mathcal{W}_\gamma$ is the image of a $C^\infty$ embedding $\Phi$, the family of affine tangent hyperplanes to $\Phi(S^n)$ can be uniquely restored. In other words, $\text{ped}_{\Phi,0} : S^n \to \mathbb{R}^{n+1}$ is one method to store the family of affine tangent hyperplanes to $\Phi(S^n)$ if the boundary of a Wulff shape $\partial \mathcal{W}_\gamma$ is the image of a $C^\infty$ embedding $\Phi$. In this sense, $\text{ped}_{\Phi,0} : S^n \to \mathbb{R}^{n+1}$ itself may be considered as a sort of Legendre transform for the hypersurface $\Phi(S^n)$. Since $\text{ped}_{\Phi,0}(\theta) = (\theta, \gamma(\theta))$, it follows that if $\Phi(S^n)$ is the graph of a $C^\infty$ function $\tilde{\delta} : S^n \to \mathbb{R}_+$, then $\gamma$ may be regarded as the very Legendre transform of $\tilde{\delta}$. Moreover, it has been known that $\mathcal{W}_\gamma$ is strictly convex if and only if the convex integrand $\gamma$ is of class $C^1$ ([14]). Thus, in our situation, if both $\gamma$ and $\tilde{\delta}$ are of class $C^\infty$, then both of $\mathcal{W}_\gamma$ and $\mathcal{W}_\tilde{\delta}$ are strictly convex. Therefore, we can expect that the Legendre transform works well in our situation.

Since Wulff shapes and pedals are defined by using perpendicular properties, $S^{n+1}$ is more suitable than $\mathbb{R}^{n+1}$ as the space where perpendicular properties are considered. In the next subsection, we investigate spherical pedals.

2.3. Spherical duals and spherical pedals. Let $\Phi : S^n \to \mathbb{R}^{n+1} - \{0\}$ be a $C^\infty$ embedding. We first construct a $C^\infty$ embedding $S^n \to S^{n+1}$ from the given $\Phi$. Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by $Id(x) = (x, 1)$. Let $N$ be the north pole of $S^{n+1}$ where $S^{n+1}$ is the unit sphere in $\mathbb{R}^{n+2}$, namely, $N = (0, \ldots, 0, 1) \in S^{n+1} \subset \mathbb{R}^{n+2}$. Let $S_{N,+}^{n+1}$ be the northern hemisphere of $S^{n+1}$, namely, $S_{N,+}^{n+1} = \{P \in S^{n+1} | N \cdot P > 0\}$ where $N \cdot P$ stands for the standard scalar product of $(n+2)$-dimensional two vectors $N, P \in \mathbb{R}^{n+2}$. Define the mapping $\alpha_N : S_{N,+}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$, called the central projection, as follows:

$$\alpha_N(P_1, \ldots, P_{n+1}, P_{n+2}) = \left( \frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1 \right),$$

where $P = (P_1, \ldots, P_{n+1}, P_{n+2}) \in S_{N,+}^{n+1}$. Then, the mapping $\tilde{\Phi} : S^n \to S_{N,+}^{n+1} \subset S^{n+1}$ is defined as follows.

$$\tilde{\Phi} = \alpha_N^{-1} \circ Id \circ \Phi.$$

Definition 2.3. For the constructed $C^\infty$ embedding $\tilde{\Phi} : S^n \to S_{N,+}^{n+1}$, the spherical pedal relative to the pedal point $N$ for $\tilde{\Phi}$, denoted by $s-ped_{\tilde{\Phi},N} : S^n \to S_{N,+}^{n+1}$, is the mapping which maps $\theta \in S^n$ to the unique nearest point of $GH_{\tilde{\Phi}(\theta)}\tilde{\Phi}(S^n)$ from the north pole $N$. Here, $GH_{\tilde{\Phi}(\theta)}\tilde{\Phi}(S^n)$ stands for the great hypersphere which is tangent to $\tilde{\Phi}(S^n)$ at $\tilde{\Phi}(\theta)$.

Next, we decompose $s-ped_{\tilde{\Phi},N}$ into two simple mappings. In order to do so, we firstly define the spherical dual $D\tilde{\Phi} : S^n \to S^{n+1}$ of $\tilde{\Phi}$. 
Definition 2.4. For any \( \theta \in S^n \), \( \tilde{D}\Phi(\theta) \) is the point in \( S_{N,+}^{n+1} \) such that \( \tilde{D}\Phi(\theta) \) is perpendicular to any \( P \in GH_{\tilde{\Phi}}(\Phi(S^n)) \). The mapping \( \tilde{D}\Phi : S^n \to S_{N,+}^{n+1} \) is called the spherical dual of \( \tilde{\Phi} \).

The notion of spherical dual for a spherical curve was firstly introduced by Arnol’d in [3]. Definition 2.4 is a natural generalization of his notion to spherical hypersurfaces.

Let \( \Psi_N : S^{n+1}_N \to S^{n+1}_N \) be the mapping defined by

\[
\Psi_N(P) = \frac{1}{\sqrt{1-(N \cdot P)^2}}(N - (N \cdot P)P).
\]

Proposition 2 ([31]). \( s\text{-ped}_{\Phi,N} = \Psi_N \circ \tilde{D}\Phi \).

In [31], Proposition 2 has been proved for spherical pedal curves. However, the proof given in [31] works well also for spherical pedal hypersurfaces. The mapping \( \Psi_N \) has the following characteristic properties.

1. For any \( P \in S^{n+1}_N \), the equality \( P \cdot \Psi_N(P) = 0 \) holds,
2. for any \( P \in S^{n+1}_N \), the property \( \Psi_N(P) \in \mathbb{R}N + \mathbb{R}P \) holds,
3. for any \( P \in S^{n+1}_N \), the property \( N \cdot \Psi_N(P) > 0 \) holds,
4. the restriction \( \Psi_N|_{S^{n+1}_N \setminus \{N\}} : S^{n+1}_N \setminus \{N\} \to S^{n+1}_N \setminus \{N\} \) is a \( C^\infty \) diffeomorphism.

By these properties, the mapping \( \Psi_N \) is called the spherical blow-up relative to \( N \). The mapping \( \Psi_N \) is quite useful for studying many topics related to perpendicularity. For instance, it was used for studying singularities of spherical pedal curves in [31, 32], for studying spherical pedal unfoldings in [33], for studying hedgehogs in [34], for studying (spherical) Wulff shapes in [35, 15, 14], and for studying the aperture of plane curves in [20]. There is also a hyperbolic version of \( \Psi_N \) ([19]). By the above properties, the following clearly holds:

Lemma 2.1 ([20]). The mapping \( \text{Id}^{-1} \circ \alpha_N \circ \Psi_N \circ \alpha_N^{-1} \circ \text{Id} \) is exactly the inversion \( \text{inv} : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\} \).

2.4. Spherical polar sets. In the Euclidean space \( \mathbb{R}^{n+1} \), the notion of polar set seems to be relatively common (for instance, see [27]). On the other hand, the notion of spherical polar set seems to be less common. Since the notion of spherical polar set plays an important role in this paper, in this subsection, properties of spherical polar sets in \( S^{n+1} \) are quickly reviewed.

For any point \( P \) of \( S^{n+1} \), we let \( H(P) \) be the following set:

\[
H(P) = \{ Q \in S^{n+1} \mid P \cdot Q \geq 0 \}.
\]

Definition 2.5. Let \( X \) be a subset of \( S^{n+1}_N \). Then, the set

\[
\bigcap_{P \in X} H(P)
\]

is called the spherical polar set of \( X \) and is denoted by \( X^\circ \).

By definition, it is clear that \( X^\circ \) is closed for any \( X \subset S^{n+1} \).

Lemma 2.2 ([35]). Let \( X, Y \) be subsets of \( S^{n+1} \). Suppose that the inclusion \( X \subset Y \) holds. Then, the inclusion \( Y^\circ \subset X^\circ \) holds.

Lemma 2.3 ([35]). For any subset \( X \) of \( S^{n+1} \), the inclusion \( X \subset X^\circ \) holds.
Definition 2.6. A subset $X \subset S^{n+1}$ is said to be hemispherical if there exists a point $P \in S^{n+1}$ such that $H(P) \cap X = \emptyset$.

Let $X$ be a hemispherical subset of $S^{n+1}$. Then, for any $P, Q \in X$, $PQ$ stands for the following arc:

$$PQ = \left\{ \frac{(1-t)P + tQ}{\| (1-t)P + tQ \|} \in S^{n+1} \mid 0 \leq t \leq 1 \right\}.$$  

Notice that $\| (1-t)P + tQ \| \neq 0$ for any $P, Q \in X$ and any $t \in [0,1]$ if $X \subset S^{n+1}$ is hemispherical.

Definition 2.7. 

(1) A hemispherical subset $X \subset S^{n+1}$ is said to be spherical convex if $PQ \subset X$ for any $P, Q \in X$.

(2) A hemispherical subset $X \subset S^{n+1}$ is said to be strictly spherical convex if $PQ - \{P, Q\}$ is a subset of the set consisting of interior points of $X$ for any $P, Q \in X$.

Notice that $X^\circ$ is spherical convex if $X$ is hemispherical and has an interior point. However, in general, $X^\circ$ is not necessarily spherical convex even if $X$ is hemispherical (for instance if $X = \{P\}$ then $X^\circ = H(P)$ is not spherical convex).

Lemma 2.4 (\cite{[35]}). Let $X_\lambda \subset S^{n+1}$ be a spherical convex subset for any $\lambda \in \Lambda$. Then, the intersection $\cap_{\lambda \in \Lambda} X_\lambda$ is spherical convex.

Definition 2.8. Let $X$ be a hemispherical subset of $S^{n+1}$. Then, the following set is called the spherical convex hull of $X$ and is denoted by $s\text{-conv}(X)$.

$$s\text{-conv}(X) = \left\{ \frac{\sum_{i=1}^{k} t_i P_i}{\| \sum_{i=1}^{k} t_i P_i \|} \mid P_i \in X, \sum_{i=1}^{k} t_i = 1, t_i \geq 0, k \in \mathbb{N} \right\}.$$  

It is clear that $s\text{-conv}(X) = X$ if $X$ is spherical convex. More generally, we have the following:

Lemma 2.5 (\cite{[35]}). For any hemispherical subset $X$, the spherical convex hull of $X$ is the smallest spherical convex set containing $X$.

Definition 2.9. Let $\{P_1, \ldots, P_k\}$ be a hemispherical finite subset of $S^{n+1}$. Suppose that $s\text{-conv}(\{P_1, \ldots, P_k\})$ has an interior point. Then, $s\text{-conv}(\{P_1, \ldots, P_k\})$ is called the spherical polytope generated by $P_1, \ldots, P_k$.

Proposition 3 (\cite{[11], [35]}). For any closed hemispherical subset $X \subset S^{n+1}$, the equality $s\text{-conv}(X) = (s\text{-conv}(X))^\circ$ holds.

Notice that for any closed hemispherical subset $X \subset S^{n+1}$, $s\text{-conv}(X)$, too, is closed and hemispherical. Notice also that by Lemma 2.8, for any subset $X \subset S^{n+1}$, the inclusion $X \subseteq X^\circ$ always holds. However, the inverse inclusion $X \supseteq X^\circ$ does not hold in general even if $X$ is closed and hemispherical.

Lemma 2.6 (\cite{[35]}). For any hemispherical finite subset $X = \{P_1, \ldots, P_k\} \subset S^{n+1}$, the following holds:

$$\left\{ \frac{\sum_{i=1}^{k} t_i P_i}{\| \sum_{i=1}^{k} t_i P_i \|} \mid P_i \in X, \sum_{i=1}^{k} t_i = 1, t_i \geq 0 \right\}^\circ = H(P_1) \cap \cdots \cap H(P_k).$$  

Lemma 2.6 is called Maehara’s lemma.
2.5. Caustics and symmetry sets. Let \( \Phi : S^n \to \mathbb{R}^{n+1} \) be a \( C^\infty \) embedding. Consider the family of functions \( F : \mathbb{R}^{n+1} \times S^n \to \mathbb{R} \) defined by
\[
F(v, \theta) = \frac{1}{2} \| \Phi(\theta) - v \|^2.
\]
Notice that \( F \) may be regarded as the map from \( \mathbb{R}^{n+1} \) to \( C^\infty(S^n, \mathbb{R}) \) which maps each \( v \in \mathbb{R}^{n+1} \) to the function \( f_v(\theta) = F(v, \theta) \in C^\infty(S^n, \mathbb{R}) \). The set consisting of vectors \( v \) for which \( f_v(\theta) \) has a degenerate critical point form the Caustic of \( \Phi \), denoted by \( \text{Caust}(\Phi) \) (or \( \text{Caust}(\Phi(S^n)) \)). For details on caustics, see for instance [2, 4, 5, 17, 18]. The set consisting of vectors \( v \) for which \( f_v(\theta) \) has a multiple critical value form the Symmetry Set of \( \Phi \), denoted by \( \text{Sym}(\Phi) \) (or \( \text{Sym}(\Phi(S^n)) \)). For details on symmetry sets, see for instance [8, 9, 10]. These two sets provide the set consisting of vectors \( v \) at which the function \( f_v(\theta) \) is not stable.

2.6. Spherical Wulff shapes, spherical caustics and spherical symmetry sets. Let \( W_\gamma \) be a Wulff shape. Then, the image of \( W_\gamma \) by \( \alpha^{-1} \circ \text{Id} : \mathbb{R}^{n+1} \to S^{n+1} \mathbb{R} \) is called the spherical Wulff shape associated with \( W_\gamma \) and is denoted by \( \tilde{W}_\gamma \). By using the spherical blow-up and the spherical polar set operation, any spherical Wulff shape \( \tilde{W}_\gamma \) can be characterized as follows:
\[
\tilde{W}_\gamma = (\Psi_N \circ \alpha^{-1} \circ \text{Id} (\text{graph}(\gamma)))^\circ.
\]
For a spherical Wulff shape \( \tilde{W}_\gamma \), the spherical polar set \( (\tilde{W}_\gamma)^\circ \) is called the spherical dual of \( \tilde{W}_\gamma \), and is denoted by \( D\tilde{W}_\gamma \).
\[
D\tilde{W}_\gamma = (\tilde{W}_\gamma)^\circ.
\]

**Lemma 2.7.** Let \( \gamma : S^n \to \mathbb{R}_+ \) be a convex integrand and let \( \delta : S^n \to \mathbb{R}_+ \) be the dual of \( \gamma \). Then, the following hold:
\[
(1) \quad D\tilde{W}_\gamma = \text{Id}^{-1} \circ \alpha_N(D\tilde{W}_\gamma).
\]
\[
(2) \quad D\tilde{W}_\gamma = \tilde{W}_\gamma.
\]
\[
(3) \quad \text{The dual of } \delta \text{ is } \gamma.
\]

**Proof.** We first show the assertion (1). By definition, it is clear that the following sub-lemma holds.

**Sublemma 2.1.** For any \( P \in S_{N,+}^{n+1} \), the following equality holds where \( \theta \in S^n \) and \( r \in \mathbb{R}_+ \) is defined by \( (\theta, r) = \text{Id}^{-1} \circ \alpha_N \circ \Psi_N(P) \).
\[
\text{Id}^{-1} \circ \alpha_N \left( H(P) \cap S_{N,+}^{n+1} \right) = \{ x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq r \}.
\]
We have the following:

\[
\begin{aligned}
\text{Id}^{-1} \circ \alpha_N(D\tilde{W}_\gamma) \\
= \text{Id}^{-1} \circ \alpha_N \left( \bigcap_{P \in \tilde{W}_\gamma} H(P) \right) \\
= \text{Id}^{-1} \circ \alpha_N \left( \bigcap_{P \in \partial \tilde{W}_\gamma} H(P) \right) \quad \text{(by Lemma 2.6)}
\end{aligned}
\]

(by Lemma 2.6)

\[
\begin{aligned}
= \bigcap_{\theta \in \mathbb{S}^n} \text{Id}^{-1} \circ \alpha_N \left( H \left( \alpha_N^{-1} \circ \text{Id} \left( \theta, \hat{\delta}(\theta) \right) \right) \right) \cap S^{n+1}_{N,+} \\
= \bigcap_{\theta \in \mathbb{S}^n} \{ x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \delta(\theta) \} \quad \text{(by Lemma 2.1 and Sublemma 2.1)}
\end{aligned}
\]

(by Lemma 2.1)

\[
= W_\delta = DW_\gamma.
\]

Hence, the assertion (1) follows.

Nextly, we show the assertion (2). By Proposition 3, the following holds:

\[
\Delta D\tilde{W}_\gamma = (\tilde{W}_\gamma)^\circ = \tilde{W}_\gamma.
\]

Hence, by the assertion (1), we have the following, which proves the assertion (2):

\[
\Delta DW_\gamma = \text{Id}^{-1} \circ \alpha_N \left( \Delta D\tilde{W}_\gamma \right) = \text{Id}^{-1} \circ \alpha_N \left( \tilde{W}_\gamma \right) = W_\gamma.
\]

Finally we show the assertion (3). Let \( \xi : \mathbb{S}^n \to \mathbb{R}_+ \) be the dual of \( \delta \). Then, by the assertion (2), we have the following:

\[
W_\xi = DW_\delta = \Delta DW_\gamma = W_\gamma.
\]

By Proposition 4, we have the following:

\[
\Gamma_\xi = \Gamma_\gamma.
\]

Since both \( \xi, \gamma \) are convex integrands, it follows that \( \xi = \gamma \). □

Suppose that the boundary of \( \tilde{W}_\gamma \) is the image of a \( C^\infty \) embedding \( \tilde{\Phi} : \mathbb{S}^n \to S^{n+1}_{N,+} \). Then, for the embedding \( \tilde{\Phi} \), the Spherical Caustic and the Spherical Symmetry Set can be defined as follows. Let \( d : S^{n+1}_{N,+} \times S^{n+1}_{N,+} \to \mathbb{R} \) be the distance squared function, i.e., \( d(P_1, P_2) \) is the square of the length of the arc \( P_1P_2 \). Consider the family of functions \( \tilde{F} : S^{n+1}_{N,+} \times S^n \to \mathbb{R} \) defined by

\[
\tilde{F}(v, \theta) = \frac{1}{2} \left( \tilde{\Phi}(\theta), v \right).
\]

Then, \( \tilde{F} \) may be regarded as the map from \( S^{n+1}_{N,+} \) to \( C^\infty(S^n, \mathbb{R}) \) which maps each \( v \in S^{n+1}_{N,+} \) to the function \( \tilde{f}_v(\theta) = \tilde{F}(v, \theta) \in C^\infty(S^n, \mathbb{R}) \). The set consisting of vectors \( v \) for which \( \tilde{f}_v(\theta) \) has a degenerate critical point form the Spherical Caustic of \( \tilde{\Phi} \), denoted by \( \text{Sph-Caust}(\tilde{\Phi}) \) or \( \text{Sph-Caust}(\tilde{\Phi}(\mathbb{S}^n)) \). The set consisting of vectors \( v \) for which \( \tilde{f}_v(\theta) \) has a
multiple critical value form the Spherical Symmetry Set of \( \tilde{\Phi} \), denoted by \( \text{Sph-Sym}(\tilde{\Phi}) \) (or \( \text{Sph-Sym}(\tilde{\Phi}(S^n)) \)).

Moreover, for any \( t \in \mathbb{R} \) such that \(|t| < \pi\), we define a \( C^\infty \) mapping, denoted by \( \tilde{\Phi}_t : S^n \to S^{n+1} \), as follows. For any \( \theta \in S^n \), let \( GC_{\tilde{\Phi}(\theta)} \) is the great circle passing through \( \tilde{\Phi}(\theta) \) which is perpendicular to \( \tilde{\Phi}(S^n) \) at \( \tilde{\Phi}(\theta) \). For any \( t \in \mathbb{R} \) \((0 < |t| < \pi)\), inside \( GC_{\tilde{\Phi}(\theta)} \), there exist exactly two distinct points \( P_1(\theta), P_2(\theta) \) such that \( d(P_1(\theta), \tilde{\Phi}(\theta)) = d(P_2(\theta), \tilde{\Phi}(\theta)) = t^2 \). Notice that one of \( P_1(\theta), P_2(\theta) \) is inside the connected component of \( S^{n+1} - \tilde{\Phi}(S^n) \) containing \( N \). Without loss of generality, we may assume that \( P_1(\theta) \) is inside the region. Then, for any \( t \) \((0 < |t| < \pi)\), the mapping \( \tilde{\Phi}_t : S^n \to S^{n+1} \) is defined by \( \tilde{\Phi}_t(\theta) = P_1(\theta) \) (resp., \( \tilde{\Phi}_t(\theta) = P_2(\theta) \)) if \( t \) is positive (resp., \( t \) is negative). For \( t = 0 \), the point \( \tilde{\Phi}_0(\theta) \) is defined as the point \( \tilde{\Phi}(\theta) \). The mapping \( \tilde{\Phi}_t : S^n \to S^{n+1} \) is called the spherical wave front of \( \tilde{\Phi} \). It is clear that the spherical wave front of \( \tilde{\Phi} \) is a \( C^\infty \) mapping for any \( t \) such that \(|t| < \pi\). Notice that \( \tilde{\Phi}_{\pi/2} \) is exactly \( D\tilde{\Phi} \). It follows that \( D^2\tilde{\Phi} = \tilde{\Phi} \). It is easily seen that the following proposition holds (see also FIGURE 1 where the dot dash curve is the spherical caustic, dotted curves are the spherical wave fronts. Two images by orthogonal projections also are depicted for the sake of clearness.).

**Proposition 5.**

\[
\text{Sph-Caust}(\tilde{\Phi}) = \bigcup_{|t|<\pi} \left\{ \tilde{\Phi}_t(\theta) \mid \theta \text{ is a singular point of } \tilde{\Phi}_t \right\}.
\]

\[
\text{Sph-Sym}(\tilde{\Phi}) = \bigcup_{|t|<\pi} \left\{ \tilde{\Phi}_t(\theta_1) = \tilde{\Phi}_t(\theta_2) \mid \theta_1, \theta_2 \in S^n \ (\theta_1 \neq \theta_2) \right\}.
\]

By Proposition 5, if we consider inside the sphere \( S^{n+1} \), it is expected that everything is clearly understood. Moreover, by [16], the angle \( \pi/2 \) is closely related to the self-dual Wulff shapes. Thus, we may consider that \( \pi/2 \) is a significant number for studying Wulff shapes, although we have no such significant real numbers if we restrict ourselves to consider Wulff shapes only in \( \mathbb{R}^{n+1} \).

**Definition 2.10.** A \( C^\infty \) map-germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+1}, 0) \) is said to be Legendrian if there exists a germ of \( C^\infty \) vector field \( \nu_f \) along \( f \) satisfying the following two:

\[
\begin{align*}
(1) & \quad \frac{\partial f}{\partial x_1}(x) \cdot \nu_f(x) = \cdots = \frac{\partial f}{\partial x_n}(x) \cdot \nu_f(x) = 0. \\
(2) & \quad \text{The map-germ } L_f : (\mathbb{R}^n, 0) \to T_1 \mathbb{R}^{n+1} \text{ defined as follows is non-singular, where } T_1 \mathbb{R}^{n+1} \text{ is the unit tangent bundle of } \mathbb{R}^{n+1}. \\
& \quad L_f(x) = (f(x), \nu_f(x)).
\end{align*}
\]

It is well-known that for any \( t \) such that \(|t| < \pi \) and any \( \theta \in S^n \), the germ of spherical wave front \( \tilde{\Phi}_t : (S^n, \theta) \to S^{n+1} \setminus \{N\} \) is Legendrian. For details on Legendrian map-germs, see for instance [2, 4, 5, 17, 18].
2.7. Andrews formulas. Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand and let $\delta : S^n \to \mathbb{R}_+$ be the dual convex integrand of $\gamma$. Notice that $\partial \mathcal{W}_\gamma$ (resp., $\partial \mathcal{D}_\gamma$) is the image of the embedding $\Phi_\delta$ (resp., $\Phi_\gamma$) defined by $\Phi_\delta(\theta) = \left(\theta, \tilde{\delta}(\theta)\right)$ (resp., $\Phi_\gamma(\theta) = (\theta, \tilde{\gamma}(\theta))$). Define the mapping $h_\delta : S^n \to S^n$ (resp., $h_\gamma : S^n \to S^n$) so that $\gamma(\theta)$ (resp., $\delta(\theta)$) is the perpendicular distance from the origin $0$ to the affine tangent hyperplane to $\Phi_\delta(S^n)$ (resp., $\Phi_\gamma(S^n)$) at $\Phi_\delta(h_\delta(\theta))$ (resp., $\Phi_\gamma(h_\gamma(\theta))$). Then, it turns out that both of $h_\delta, h_\gamma$ are $C^\infty$ diffeomorphisms (see Remark 3.1 in Section 3). Moreover, since both of $\Phi_\delta(S^n), \Phi_\gamma(S^n)$ are strictly convex by [14], as shown in [1], the simultaneous equations for

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1Subsection 2.7 is used only for the proof of Theorem 3. For the proofs of Theorems 1 and 2 Subsection 2.7 is unnecessary.
envelopes leads to the following equalities for any \( \theta \in S^n \) (see FIGURE 2 where \( T_1 \) (resp., \( T_2 \)) denotes the affine tangent hyperplane to \( \Phi_{\delta}(S^n) \) (resp., \( \Phi_{\gamma}(S^n) \)) at \( \Phi_{\delta}(h_{\delta}(\theta)) \) (resp., \( \Phi_{\gamma}(h_{\gamma}(\theta)) \)).

\[
\begin{align*}
\Phi_{\delta}(h_{\delta}(\theta)) &= \gamma(\theta)\theta + \nabla\gamma(\theta), \\
\Phi_{\gamma}(h_{\gamma}(\theta)) &= \delta(\theta)\theta + \nabla\delta(\theta).
\end{align*}
\]

Here, \( \nabla\gamma(\theta) \) (resp., \( \nabla\delta(\theta) \)) stands for the gradient vector of \( \gamma \) (resp., \( \delta \)) at \( \theta \in S^n \) with respect to the standard metric on \( S^n \).

**Figure 2.** Andrews formulas

### 3. Proof of Theorem 1

The notations, terminologies and notions introduced in Sections 1 and 2 are used without explaining them explicitly again. Firstly, we show that \( \delta \) is a \( C^\infty \) convex integrand under the assumption that its dual \( \gamma \) is a \( C^\infty \) strictly convex integrand.

Define the \( C^\infty \) embedding \( \Psi : S^n \to \mathbb{R}^{n+1} - \{0\} \) by

\[
\Psi(\theta) = (\theta, \gamma(\theta)).
\]

Since \( \delta \) and \( \hat{\delta} \) are \( \mathcal{A} \)-equivalent, it is sufficient to show that the function \( \hat{\delta} : S^n \to \mathbb{R}_+ \) is of class \( C^\infty \) so that \( \text{ped}_{\Phi_{\delta}, \theta} = \Psi \) where \( \Phi_{\delta} \) is the \( C^\infty \) embedding defined in Subsection 2.7.

For this purpose, the ambient space is changed to \( S^{n+1} \). Let \( \tilde{\Psi}_\gamma : S^n \to S^{n+1}_{\gamma, +} \) be the \( C^\infty \) embedding defined by

\[
\tilde{\Psi}_\gamma(\theta) = \alpha_{\mathcal{N}}^{-1} \circ \text{Id}(\theta, \tilde{\gamma}(\theta)) \quad (\theta \in S^n).
\]

Then, since \( \gamma \) is a convex integrand, the image \( \tilde{\Psi}_\gamma(S^n) \) is the boundary of a spherical convex set \( \tilde{\mathcal{W}}_{\hat{\delta}} \). Set

\[
\tilde{\Psi}_\gamma(\theta) = \left( \tilde{\Psi}_1(\theta), \ldots, \tilde{\Psi}_{n+2}(\theta) \right).
\]

Define the mapping \( \tilde{\Phi} : S^n \to S^{n+1} \) by

\[
\tilde{\Phi} = D\tilde{\Psi}_\gamma.
\]
then, by Proposition 2, we have the following:

\[ s-ped_{\tilde{\Phi},N} = \Psi_N \circ \tilde{\Psi}_\gamma. \]

Set

\[ \tilde{\Phi}(\theta) = \left( \tilde{\Phi}_1(\theta), \ldots, \tilde{\Phi}_{n+2}(\theta) \right), \]

where \( \left( \tilde{\Phi}_1(\theta), \ldots, \tilde{\Phi}_{n+2}(\theta) \right) \) is the standard Euclidean expression of the point \( \tilde{\Phi}(\theta) \). Define the mapping \( \tilde{h} : S^n \to S^n \) by

\[ \tilde{h}(\theta) = \frac{\left( \tilde{\Phi}_1(\theta), \ldots, \tilde{\Phi}_{n+1}(\theta) \right)}{\left\| \left( \tilde{\Phi}_1(\theta), \ldots, \tilde{\Phi}_{n+1}(\theta) \right) \right\|}. \]

Then, by the definition of dual \( D\tilde{\Psi}_\gamma \), it follows that \( \tilde{h} \) is a well-defined \( C^\infty \) mapping.

Since \( \gamma \) is a strictly convex integrand, by the two equalities \( DD\tilde{\Psi}_\gamma = \tilde{\Psi}_\gamma \) and

\[ \frac{\left( \tilde{\Psi}_1(\theta), \ldots, \tilde{\Psi}_{n+1}(\theta) \right)}{\left\| \left( \tilde{\Psi}_1(\theta), \ldots, \tilde{\Psi}_{n+1}(\theta) \right) \right\|} = -\theta, \]

it follows that \( \tilde{h} \) is bijective.

Next, we show that \( \tilde{h} \) is a \( C^\infty \) diffeomorphism. Since \( \tilde{\Phi} \) is the spherical dual of the \( C^\infty \) embedding \( \tilde{\Psi}_\gamma : S^n \to S^{n+1}_N, \) it is the spherical wave front \( \left( \tilde{\Psi}_\gamma \right)_\pi/2 \). Thus, \( \tilde{\Phi} \) is Legendrian. Hence, for any singular point \( \theta_0 \in S^n \) of \( \tilde{\Phi} \), there exists a germ of \( C^\infty \) normal vector field \( \nu_{\tilde{\Phi}} \) along \( \tilde{\Phi} \) such that the map-germ \( L_{\tilde{\Phi}} : (S^n, \theta_0) \to T_1S^{n+1} \) defined as follows is non-singular.

\[ L_{\tilde{\Phi}}(\theta) = \left( \tilde{\Phi}(\theta), \nu_{\tilde{\Phi}}(\theta) \right). \]

In particular, even at the critical value \( \tilde{\Phi}(\theta_0) \), the normal great circle to \( \tilde{\Phi}(S^n) \) at \( \tilde{\Phi}(\theta_0) \) must be unique. By this fact, it is easily seen that the spherical Wulff shape \( \tilde{W}_{\delta} \) is strictly spherical convex. Hence, by [14], it follows that the image \( \tilde{\Phi}(S^n) \) is the graph of a \( C^1 \) function. This implies that the derivative \( d\tilde{h}_\theta \) is bijective for any \( \theta \in S^n \). Therefore, by the inverse function theorem, \( \tilde{h} \) is a \( C^\infty \) diffeomorphism.

Notice that \( \Phi(\theta) \) can be expressed as follows:

\[ \Phi(\theta) = Id^{-1} \circ \alpha_N \circ \tilde{\Phi} \circ h^{-1}(\theta) = \left( \theta, \tan \left( \cos^{-1} \left( \tilde{\Phi}_{n+2} \circ h^{-1}(\theta) \right) \right) \right). \]

Hence, we have the following:

\[ \tilde{\delta}(\theta) = \tan \left( \cos^{-1} \left( \tilde{\Phi}_{n+2} \circ h^{-1}(\theta) \right) \right). \]

Since \( \tilde{\Phi}_{n+2} \) is of class \( C^\infty \) and all of \( \tilde{h} : S^n \to S^n, \cos : (0, \pi/2) \to (0, 1) \) and \( \tan : (0, \pi/2) \to \mathbb{R}_+ \) are \( C^\infty \) diffeomorphisms, it follows that \( \tilde{\delta} \) is of class \( C^\infty \).

Next, we show that \( \gamma \) is a strictly convex integrand under the assumption that its dual \( \delta \) is a \( C^\infty \) convex integrand. In [14], it is shown that a Wulff shape is strictly convex if and only if its convex integrand \( S^n \to \mathbb{R}_+ \) is of class \( C^1 \). Since \( \delta \) is a \( C^\infty \) convex integrand, we have that the Wulff shape

\[ \mathcal{W}_\delta = \text{the convex hull of } \text{inv}(\text{graph}(\gamma)) \]
is strictly convex. This implies that the convex integrand $\gamma$ is a $C^\infty$ strictly convex integrand. \hfill \Box

**Remark 3.1.**

1. Notice that $\tilde{h} : S^n \to S^n$ is exactly the same mapping as $h_\delta : S^n \to S^n$ given in Subsection 2.7. Thus, $h_\delta$ is a $C^\infty$ diffeomorphism. Similarly, $h_\gamma$ in Subsection 2.7 also is a $C^\infty$ diffeomorphism.

2. FIGURE 3 explains that $\tilde{h} : S^n \to S^n$ is not bijective if $\gamma$ is not a $C^\infty$ strictly convex integrand but a $C^\infty$ convex integrand.

**Figure 3.** Left: The image $\text{inv}(\text{graph}(\gamma))$ for a $C^\infty$ not strictly convex integrand $\gamma$. Right: $\mathcal{W}_\gamma$.

4. **Proof of Theorem 2**

The notations, terminologies and notions introduced in Sections 1, 2 and 3 are used without explaining them explicitly again.

Define the function $\tilde{\gamma} : S^n \to \R_+$ by $\tilde{\gamma}(\theta) = d(\tilde{\Psi}_\gamma(\theta), N)$.

**Lemma 4.1.** There exists a point $\theta$ of $S^n$ which is a degenerate critical point of $\tilde{\gamma}$ if and only if the north pole $N$ is contained in the spherical caustic of $\tilde{\Psi}_\gamma$.

**Proof.** Let $S(\Phi)$ be the set consisting of singular points of the map $\Phi$ defined by

$$\Phi : S^n \times S_{N,+}^{n+1} \to \R \times S_{N,+}^{n+1}$$

$$(\theta, P) \mapsto (d(\tilde{\Psi}_\gamma(\theta), P), P).$$

Then, it is well-known that $S(\Phi)$ is an $(n + 1)$-dimensional $C^\infty$ submanifold of $S^n \times S_{N,+}^{n+1}$ (for instance, see [37, 38]). Denote the restriction to $S(\Phi)$ of the canonical projection $\pi : S^n \times S_{N,+}^{n+1} \to S_{N,+}^{n+1}$ by $\Pi$. Then, the set $\Pi(S(\Pi))$ is the spherical caustic of $\tilde{\Psi}_\gamma$.

Notice that when $P = N$, we have that $\Phi(\theta, N) = (\tilde{\gamma}(\theta), N)$. Thus, we have the following:

$$N \in Spherical-Caust(\tilde{\Psi}_\gamma) \iff \exists \theta \in S^n \text{ such that } \nabla\tilde{\gamma}(\theta) = 0 \text{ and } \det(\text{Hess}\tilde{\gamma})(\theta) = 0.$$

For the symmetry set of $\tilde{\Psi}_\gamma$, we have the following.

Let $\mathcal{W}$ be a $C^\infty$ Wulff shape and $\tilde{\mathcal{W}}$ be the spherical Wulff shape of $\mathcal{W}$. 

Lemma 4.2. The origin 0 is a point of symmetry set of ∂W if and only if the north polar N is a point of spherical symmetry set of ∂W̃.

Proof. First we give the proof of the “only if” part. Suppose that the origin 0 of \( \mathbb{R}^{n+1} \) is a point of symmetry set of ∂W. Then, there exist two point \( x_1, x_2 \) of ∂W such that \( |x_10| = |x_20| \) and \( x_10 \) (resp., \( x_20 \)) is a subset of the affine normal line \( \ell_1 \) (resp. \( \ell_2 \)) to ∂W at \( x_1 \) (resp., \( x_2 \)). Set \( \tilde{X} = \alpha_N^{-1} \circ Id(X) \), where \( X \) is a subset of \( \mathbb{R}^{n+1} \). Since \( \alpha_N : S_{n,+}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \) is the central projection at \( N \), we have that the half of great circle \( \alpha_N^{-1} \circ Id(\ell_1) \) (resp., \( \alpha_N^{-1} \circ Id(\ell_2) \)) is the half of spherical normal to ∂W̃ at \( \alpha_N^{-1} \circ Id(x_1) \) (resp., \( \alpha_N^{-1} \circ Id(x_2) \)) and \( d(\alpha_N^{-1} \circ Id(x_1), N) = d(\alpha_N^{-1} \circ Id(x_2), N) \). Therefore, if the origin is a point of \( Sym(\partial W) \), then the north pole \( N \) is a point of \( Spherical-Sym(\partial W) \).

Similarly, the “if” part of Lemma 4.2 follows.

Remark 4.1. Notice that \( \gamma(\theta) = \tan(\tilde{\gamma}(\theta)) \) \( \gamma(\theta) = \tan \left( \frac{\pi}{2} - \tilde{\gamma}(\theta) \right) \) for any \( \theta \in S^n \).

Since the function \( \tan : (0, \pi/2) \to \mathbb{R}_+ \) is a \( C^\infty \) diffeomorphism, it follows that \( \theta \) is a non-degenerate critical point of \( \gamma \) if and only if \( \tilde{\gamma} \) is a non-degenerate critical point of \( \gamma \).

Proposition 6. Let \( W_\gamma \) be the Wulff shape associated with \( \gamma \) and let \( W_{\tilde{\gamma}} \) be the spherical Wulff shape associated with \( W_\gamma \). Then the following holds:

1. \( Spherical-Caust \left( \partial W_{\tilde{\gamma}} \right) = Spherical-Caust \left( \partial D\tilde{W}_{\tilde{\gamma}} \right) \).
2. \( Spherical-Sym \left( \partial W_{\tilde{\gamma}} \right) = Spherical-Sym \left( \partial D\tilde{W}_{\tilde{\gamma}} \right) \).

Proof. By Theorem 1 it follows that \( \delta \) is of class \( C^\infty \) if \( \gamma \) is of class \( C^\infty \). Thus, under the assumption of Theorem 2 the mapping \( \tilde{\Psi}_\delta : S^n \to S_{n,+}^{n+1} \) defined as follows is a \( C^\infty \) embedding.

\[
\tilde{\Psi}_\delta(\theta) = \alpha_N^{-1} \circ Id \left( \theta, \tilde{\delta}(\theta) \right) \quad (\theta \in S^n).
\]

By Proposition 2 the following holds.

\[
\tilde{\Psi}_\delta = D\tilde{\Psi}_\tilde{\delta}.
\]

By the definition of spherical wave fronts, the following holds:

\[
\left( \tilde{\Psi}_\tilde{\delta} \right)_{t} = \left( \tilde{\Psi}_\delta \right)_{\pi/2 - t}.
\]

Thus, by Proposition 5 we have the following.

\[
Spherical-Caust \left( \tilde{\Psi}_\tilde{\delta} \right) = Spherical-Caust \left( D\tilde{\Psi}_\tilde{\delta} \right)
\]

\[
Spherical-Sym \left( \tilde{\Psi}_\tilde{\delta} \right) = Spherical-Sym \left( D\tilde{\Psi}_\tilde{\delta} \right).
\]

Therefore, Proposition 6 follows.

Now we start to prove Theorem 2. Firstly, recall the definition of the caustic of \( \partial W_\gamma \).

\[
Caust(\partial W_\gamma) = \{ v \mid \exists \theta \in S^n ; \nabla \delta_v(\theta) = 0 \text{ and } \det(\text{Hess}(\delta_v))(\theta) = 0 \},
\]

where \( \delta_v = \frac{1}{2} ||(\theta, 1/\delta(-\theta)) - v||^2 \). Suppose that the origin is a point of \( Caust(\partial W_\gamma) \).

Then, there exists a point \( \theta \in S^n \) such that \( \nabla \delta(\theta) = 0 \) and \( \det(\text{Hess}(\delta))(\theta) = 0 \). By Remark 11, it follows that \( \nabla \delta(\theta) = 0 \) and \( \det(\text{Hess}(\tilde{\delta}))(\theta) = 0 \). Thus, by Lemma 11 it follows that the north pole \( N \) is contained in the spherical caustic of \( \partial W_{\tilde{\gamma}} \). Then, by
Proposition 6, it follows that $N$ is contained in the spherical caustic of $\partial \widetilde{DW}_\gamma$. Thus, there exist $\tilde{\theta} \in S^n$ such that both $\nabla \tilde{\gamma}(\tilde{\theta}) = 0$ and $\det(\text{Hess}(\tilde{\gamma}))(\tilde{\theta}) = 0$ hold. Hence, again by Remark 4.1, the origin is contained in the caustic of $\partial \widetilde{DW}_\gamma$.

Next, suppose that the origin is a point of $\text{Sym}(\partial W_\gamma) - \text{Caust}(\partial W_\gamma)$. Then, by Lemma 4.2, the north pole $N$ is contained in the spherical symmetry set of $\partial \widetilde{W}_\gamma$. Then, by Proposition 6, it follows that $N$ is contained in the spherical symmetry set of $\partial \widetilde{D}_\gamma W_\gamma$. Hence, again by Lemma 4.2, the origin is contained in the symmetry set of $\partial \widetilde{D}_\gamma W_\gamma$. □

Remark 4.2. As a by-product of the proof of Theorem 2, we have the following:

Theorem 4. (1) Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand having only non-degenerate critical points. Then, $\delta$ is a $C^\infty$ function having only non-degenerate critical points.

(2) Let $\gamma : S^n \to \mathbb{R}_+$ be a $C^\infty$ strictly convex integrand having only non-degenerate critical points. Then, the restriction of $\gamma$ to the set consisting critical points of $\gamma$ is injective if and only if the restriction of $\delta : S^n \to \mathbb{R}_+$ to the set consisting of critical points of it is injective.

5. Proof of Theorem 3

We first show the assertion (1) of Theorem 3. The following two equalities have been given in Subsection 2.7.

$$(5.1) \quad \hat{\delta}(h_\delta(\theta))h_\delta(\theta) = \Phi_\delta(h_\delta(\theta)) = \gamma(\theta)\theta + \nabla \gamma(\theta).$$

$$(5.2) \quad \hat{\gamma}(h_\gamma(\theta))h_\gamma(\theta) = \Phi_\gamma(h_\gamma(\theta)) = \delta(\theta)\theta + \nabla \delta(\theta).$$

By elementary geometry, for any $\theta \in S^n$ the following inequalities hold.

$$(5.3) \quad \frac{1}{\delta(-h_\delta(\theta))} \geq \gamma(\theta).$$

$$(5.4) \quad \frac{1}{\gamma(-h_\gamma(\theta))} \geq \delta(\theta).$$

On the other hand, since $\Phi_\delta(S^n)$ and $\Phi_\gamma(S^n)$ are strictly locally convex, for any $\theta \in S^n$ the following equivalent inequalities hold.

$$(5.5) \quad \gamma(\theta) \geq \frac{1}{\delta(-\theta)}.$$ 

$$(5.6) \quad \delta(\theta) \geq \frac{1}{\gamma(-\theta)}.$$ 

The above equalities (5.1) and (5.2) imply that $h_\delta(\theta_0) = \theta_0$ if and only if $\nabla \gamma(\theta_0) = 0$, and $h_\gamma(\theta_0) = \theta_0$ if and only if $\nabla \delta(\theta_0) = 0$. It follows that $\theta_0 \in S^n$ is a critical point of $\gamma$ if and only if the following equality holds for $\theta_0 \in S^n$

$$(5.7) \quad \frac{1}{\delta(-\theta_0)} = \gamma(\theta_0),$$

and equivalently that $\theta_0 \in S^n$ is a critical point of $\delta$ if and only if the following equality holds for $\theta_0 \in S^n$

$$(5.8) \quad \frac{1}{\gamma(-\theta_0)} = \delta(\theta_0).$$
Hence, the assertion (1) of Theorem 3 follows.

Next, we show the assertion (2) of Theorem 3. We first show the “only if” part. Let \( \theta_0 \in S^n \) be a non-degenerate critical point of \( \gamma \) with Morse index \( i \). Then, there exists a coordinate neighborhood \((U, \varphi)\) of \( \theta_0 \) such that \( \varphi(\theta_0) = 0 \) and the following equality holds:

\[
\gamma \circ \varphi^{-1}(x_1, \ldots, x_n) = \gamma(\theta_0) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2.
\]

By Theorem 2 and the assertion (1) of Theorem 3, \(-\theta_0 \in S^n\) is a non-degenerate critical point of \( \delta \). Thus, there exist an integer \( j \) \((0 \leq j \leq n)\) and a coordinate neighborhood \((V, \psi)\) of \(-\theta_0\) such that \( \psi(-\theta_0) = 0 \) and the following equality holds:

\[
\delta \circ \psi^{-1}(x_1, \ldots, x_n) = \delta(-\theta_0) - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2.
\]

We show that \( j = n - i \). By (5.7), it follows that

\[
\gamma(\theta_0)\delta(-\theta_0) = 1.
\]

Set \( x = (x_1, \ldots, x_n) \) and

\[
\begin{align*}
U_1 &= \{ x \in \varphi(U) \mid x_1^2 + \cdots + x_i^2 \geq x_{i+1}^2 + \cdots + x_n^2 \}, \\
U_2 &= \{ x \in \varphi(U) \mid x_1^2 + \cdots + x_i^2 \leq x_{i+1}^2 + \cdots + x_n^2 \}, \\
V_1 &= \{ x \in \psi(V) \mid x_1^2 + \cdots + x_j^2 \geq x_{j+1}^2 + \cdots + x_n^2 \}, \\
V_2 &= \{ x \in \psi(V) \mid x_1^2 + \cdots + x_j^2 \leq x_{j+1}^2 + \cdots + x_n^2 \}.
\end{align*}
\]

For any \( x \in U_2 \), by (5.9), we have the following:

\[
\gamma \circ \varphi^{-1}(x) \geq \gamma(\theta_0).
\]

Hence, by (5.3), (5.10) and (5.11), we have the following for any \( x \in U_2 \):

\[
\frac{1}{\delta(-h_\delta \circ \varphi^{-1}(x))} \geq \gamma \circ \varphi^{-1}(x) \geq \gamma(\theta_0) = \frac{1}{\delta(-\theta_0)}.
\]

This implies

\[
\psi(-h_\delta \circ \varphi^{-1}(x)) \in V_1.
\]

Since \( x \mapsto \psi(-h_\delta \circ \varphi^{-1}(x)) \) is a \( C^\infty \) diffeomorphism, it follows that \( n - i \leq j \). Notice that (5.3) can be replaced with (5.4) to obtain the same inequality \( n - i \leq j \). On the other hand, for any \( x \in U_1 \), by (5.9), we have the following:

\[
\gamma \circ \varphi^{-1}(x) \leq \gamma(\theta_0).
\]

Hence, by (5.5), (5.10) and (5.11), we have the following for any \( x \in U_1 \):

\[
\frac{1}{\delta(-\theta_0)} = \gamma(\theta_0) \geq \gamma \circ \varphi^{-1}(x) \geq \frac{1}{\delta(-\varphi(x))}.
\]

This implies

\[
\psi(-\varphi^{-1}(x)) \in V_2.
\]

Since \( x \mapsto \psi(-\varphi^{-1}(x)) \) is a \( C^\infty \) diffeomorphism, it follows that \( i \leq n - j \). Therefore, we have \( j = n - i \).

The “if” part can be proved by the same method. Thus, the assertion (2) of Theorem 3 follows.
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REFERENCES

[1] B. Andrews, *Harnack inequalities for evolving hypersurfaces*, Math. Z., 217 (1994), no. 2, 179–197.
[2] V.I. Arnol’d, *Singularities of caustics and wave fronts*, Kluwer Academic Publishers Group, Dordrecht, 1990.
[3] V. I. Arnol’d, *The geometry of spherical curves and the algebra of quaternions*, Russian Math. Surveys, 50 (1995), 1–68.
[4] V. I. Arnol’d, V. V. Goryunov, O. V. Lyashko and V. A. Vasil’ev, *Dynamical Systems VIII*. Encyclopaedia of Mathematical Sciences, 39, Springer-Verlag, Berlin Heidelberg New York, 1989.
[5] V.I. Arnol’d, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps I*, Monographs in Mathematics, 82, Birkhäuser, Boston Basel Stuttgart, 1985.
[6] E. B. Batista, H. Han and T. Nishimura, *Stability of $C^\infty$ convex integrands*, Kyushu J. Math., 71 (2017), 187-196.
[7] Th. Bröcker, *Differential Germs and Catastrophes*, London Mathematical Society Lecture Note Series, 17, Cambridge University Press, Cambridge, 1975.
[8] J.W. Bruce and P.J. Giblin, *Growth, motion and 1-parameter families of symmetry sets*, Proc. Roy. Soc. Edinburgh Sect. A, 104 (1986), 179-204.
[9] J.W. Bruce and P.J. Giblin, *Curves and Singularities* (second edition), Cambridge University Press, Cambridge, 1992.
[10] J.W. Bruce, P.J. Giblin and C.G. Gibson, *Symmetry sets*, Proc. Roy. Soc. Edinburgh Sect. A, 101 (12) (1985), 163-186.
[11] S. Gao, D. Hug and R. Schneider, *Intrinsic volumes and polar sets in spherical space*, Math. Notae, 41 (2001/02), 159–176(2003).
[12] Y. Giga, *Surface Evolution Equations*, Monographs of Mathematics, 99, Springer, 2006.
[13] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, New York, 1973.
[14] H. Han and T. Nishimura, *Strictly convex Wulff shapes and $C^1$ convex integrands*, Proc. Amer. Math. Soc., 145 (2017), 3997–4008.
[15] H. Han and T. Nishimura, *The spherical dual transform is an isometry for spherical Wulff shapes*, to appear in Studia Math. (available from arXiv:1504.02845 [math.MG]).
[16] H. Han and T. Nishimura, *Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$*, to appear in J. Math. Soc. Japan (available from arXiv:1511.04165 [math.MG]).
[17] S. Izumiya, *Differential geometry from the viewpoint of Lagrangian or Legendrian singularity theory*, in Singularity Theory, Proceedings of the 2005 Marseille Singularity School and Conference, ed., D. Chéniot et al., World Scientific (2007), 241-275.
[18] S. Izumiya and M. Takahashi, *Caustics and wave front propagations: Applications to differential geometry*, Banach Center Publ., 82 (2008), 125-142.
[19] S. Izumiya and F. Tari, *Projections of hypersurfaces in the hyperbolic space to hyperhorospheres and hyperplanes*, Rev. Mat. Iberoam., 24 (2008), 895–920.
[20] D. Kagatsume and T. Nishimura, *Aperture of plane curves*, J. Singul., 12 (2015), 80–91.
[21] J. Mather, *Stability of $C^\infty$ mappings, I. The division theorem*, Ann. of Math., 87 (1968), 89–104.
[22] J. Mather, *Stability of $C^\infty$ mappings, II. Infinitesimal stability implies stability*, Ann. of Math., 89 (1969), 254–291.
[23] J. Mather, *Stability of $C^\infty$ mappings, III. Finitely determined map-germs*, Publ. Math. Inst. Hautes Études Sci., 35 (1969), 127–156.
[24] J. Mather, *Stability of $C^\infty$ mappings, IV. Classification of stable map-germs by $\mathbb{R}$-algebras*, Publ. Math. Inst. Hautes Études Sci., 37 (1970), 223–248.
[25] J. Mather, *Stability of $C^\infty$-mappings V. Transversality*, Adv. in Math., 4 (1970), 301–336.
[26] J. Mather, *Stability of $C^\infty$-mappings VI. The nice dimensions*, Lecture Notes in Math., 192 (1971), 207–253.
[27] J. Matousek, *Lectures on Discrete Geometry*, Springer, 2002.
[28] J. Milnor, Morse Theory, Annals of Mathematics Studies 51, Princeton University Press, Princeton, New Jersey, 1973.

[29] F. Morgan, The cone over the Clifford torus in $\mathbb{R}^4$ is $F$-minimizing, Math. Ann., 289 (1991), 341–354.

[30] R. Narasimhan, Analysis on real and complex manifolds, Masson and Cie, Paris, and North-Holland, Amsterdam, 1968.

[31] T. Nishimura, Normal forms for singularities of pedal curves produced by non-singular dual curve germs in $S^n$, Geom. Dedicata, 133 (2008), 59–66.

[32] T. Nishimura, Singularities of pedal curves produced by singular dual curve germs in $S^n$, Demonstratio Math., 43 (2010), 447–459.

[33] T. Nishimura, Singularities of one-parameter pedal unfoldings of spherical pedal curves, J. Singul., 2 (2010), 160–169.

[34] T. Nishimura and Y. Sakemi, View from inside, Hokkaido Math. J., 40 (2011), 361–373.

[35] T. Nishimura and Y. Sakemi, Topological aspect of Wulff shapes, J. Math. Soc. Japan, 66 (2014), 89–109.

[36] A. Pimpinelli and J. Villain, Physics of Crystal Growth, Monographs and Texts in Statistical Physics, Cambridge University Press, Cambridge New York, 1998.

[37] I. R. Porteous, The normal singularities of a submanifold, J. Diff. Geom., 5 (1971), 543–564.

[38] M. C. Romero-Fuster and M. A. S. Ruas, Some stability questions concerning caustics for different propagation laws, Portugal. Math., 51 (1994), 595–605.

[39] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory 2nd Edition, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 2013.

[40] H. M. Soner, Motion of a set by the curvature of its boundary, J. Differential Equations, 101 (1993), 313–372.

[41] J. E. Taylor, Crystalline variational problems, Bull. Amer. Math. Soc., 84 (1978), 568–588.

[42] J. E. Taylor, J. W. Cahn and C. A. Handwerker, Geometric models of crystal growth, Acta Metallographia et Materialia, 40 (1992), 1443–1474.

[43] G. Wulff, Zur frage der geschwindindigkeit des wachstrums und der auflösung der krystallflachen, Z. Kristallographine und Mineralogie, 34 (1901), 449–530.

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