Approximations of the Restless Bandit Problem

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Abstract

The multi-armed restless bandit problem is studied in the case where the pay-offs are not necessarily independent over time nor across the arms. Even though this version of the problem provides a more realistic model for most real-world applications, it cannot be optimally solved in practice since it is known to be PSPACE-hard. The objective of this paper is to characterize special sub-classes of the problem where good approximate solutions can be found using tractable approaches. Specifically, it is shown that in the case where the joint distribution over the arms is ϕ-mixing, and under some conditions on the ϕ-mixing coefficients, a modified version of UCB can prove optimal. On the other hand, it is shown that when the pay-off distributions are strongly dependent, simple switching strategies may be devised which leverage the strong inter-dependencies. To this end an example is provided using Gaussian Processes. The techniques developed in this paper apply, more generally, to the problem of online sampling under dependence.

1 Introduction

As one of the simplest examples of sequential optimization under uncertainty, multi-armed bandit problems arise in various modern real-world applications, such as online advertisement, and internet routing. These problems are typically studied under the assumption that the pay-offs are independently and identically distributed (i.i.d.), and the arms are independent. However, this assumption does not necessarily hold in many practical situations. Consider, for example, the problem of online advertisement in which the aim is to garner as many clicks as possible from a user. Grouping adverts into categories and associating with each category an arm, this problem turns into that of multi-armed bandits’. There is dependence over time and across the arms since, for example, we expect a user to be more likely to select adverts that are related to her selections in the recent past.

In this paper, we are concerned with developing algorithms that are robust with respect to such dependencies. More specifically, we consider the multi-armed bandit problem in the case where the pay-offs are dependent and each arm evolves over time regardless of whether...
or not it is played. This is an instance of the so-called restless bandit problem (Whittle, 1988; Guha, Munagala, and Shi, 2010; Ortner, Ryabko, Auer, and Munos, 2014). Note that, since in this setting an optimal policy can leverage the inter-dependencies between the samples and switch between the arms at appropriate times, it can obtain an overall pay-off much higher than that given by playing the best arm, i.e. the distribution with the highest expected pay-off, see Example 1 in (Ortner et al., 2014). However, finding the best such switching strategy is PSPACE-hard, even in the case where the process distributions are Markovian with known dynamics (Papadimitriou and Tsitsiklis, 1999).

This paper is an initial attempt to characterize special sub-classes of the problem where good approximate solutions can be found using simple, computationally tractable approaches. More specifically, we consider two extreme cases, namely, when the process distributions are weakly dependent and when they are strongly dependent; see Figure 1. Our main focus is on the former sub-problem where we demonstrate how an optimistic UCB-type approach can effectively approximate the optimal strategy for stationary \( \varphi \)-mixing restless bandits. Moreover, the proof techniques developed to address this problem can be more generally used in the context of online sampling under dependence. To address the latter sub-problem we provide an example, using stationary Gaussian Processes, where a simple switching strategy can be obtained to leverage the strong inter-dependencies between the samples.

Weakly dependent reward distributions. We consider the restless bandit problem in the case where the distributions of the arms are stationary \( \varphi \)-mixing. First, we show that in this case, the regret of settling for the arm with the highest stationary mean scales at most linearly with \( \varphi_1 \). Moreover, we propose a UCB-type algorithm to sample the arm with the highest stationary mean, and demonstrate that it achieves logarithmic regret with respect to the highest stationary mean. This gives rise to a natural relaxation for the case where \( \varphi_1 \leq \epsilon \) for some small \( \epsilon \). Observe that this condition translates directly to the pay-off distributions being weakly dependent.

Note that even this relaxed version of the problem is not straightforward. The main challenge lies in obtaining confidence intervals around empirical estimates of the stationary means. Since Hoeffding-type concentration bounds exist for \( \varphi \)-mixing processes, it may be tempting to use such inequalities directly with standard UCB algorithms designed for the i.i.d. setting to find the best arm; in fact, this seems to be the approach taken by Audiffren and Ralaivola (2015). However, as we show through an example in this paper, a sequence of random variables obtained by sampling a stationary \( \varphi \)-mixing process at random times, may not necessarily be \( \varphi \)-mixing. As a result, in order for Hoeffding-type concentration results (for \( \varphi \)-mixing processes) to be applicable in this setting, the sampling policy must be designed appropriately. To address this issue, the proposed approach plays its selected arms in batches of exponentially growing length, and relies on the following two key observations. First, consecutive samples \( X_{\tau;i}, \ldots, X_{\tau+\ell,i} \) where \( \tau \in \mathbb{N}_+ \) is a random starting time at which arm \( i \in 1..k \) is sampled in a batch of length \( \ell \), form a \( \varphi \)-mixing sequence. Second, for a long enough batch the average expectations over batches converge to the stationary mean and the empirical mean of the batch is concentrated around its expectation.

Strongly dependent reward distributions. At the other end of the extreme lie bandit problems with strongly dependent pay-off distributions. The intuitive reason why in this setting it may also
Figure 1: A chart of the restless bandit problem. The problem is PSPACE hard in general. However, in two extreme cases, namely, where the process distributions are either strongly dependent or weakly dependent, optimal solutions can be obtained efficiently.

be possible to efficiently obtain approximately optimal solutions, is that the strong dependencies can allow for the prediction of future rewards even from scarce observations of a sample path. As a natural starting point to study strongly dependent processes, we consider pay-off sequences that are distributed according to stationary Gaussian processes with slowly decaying covariance functions. We present a simple algorithm that iterates between an exploration and exploitation phase to maximize its pay-off. In the exploration phase it sweeps through all k-arms to determine the highest pay-off and in the exploitation phase it selects this arm and plays it for a long stretch of time. While this approach gives a much higher overall pay-off than what would be given by settling for the highest stationary mean, it is computationally efficient. We provide a regret bound for this algorithm that directly reflects the dependence between the pay-offs: the higher the dependence the lower the regret.

Organization. The remainder of the paper is organized as follows. In Section 2 we introduce preliminary notation and definitions. Section 3 presents our results for weakly dependent processes. Our results on strongly dependent processes are given in Section 4. We conclude in Section 5 with a summary and a discussion of open problems. Due to space constraints, all proofs are provided in the supplementary material.

2 Preliminaries

Let \( \mathbb{N}_+ := \{1, 2, \ldots \} \) and \( \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} \) denote the set and extended set of natural numbers respectively. We introduce the abbreviation \( a_{m..n}, m, n \in \mathbb{N}_+, m \leq n \), for sequences \( a_m, a_{m+1}, \ldots, a_n \). Given a finite subset \( C \subset \mathbb{N}_+ \) and a sequence \( a \), we let \( a_C := \{a_i : i \in C\} \) denote the set of elements of \( a \) indexed by \( C \). If \( X_C \) is a sequence of random variables indexed by \( C \subset \mathbb{N}_+ \), we denote by \( \sigma(X_C) \) the smallest \( \sigma \)-algebra generated by \( X_C \).
Stochastic Processes. Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a measurable space; we let $\mathcal{X} \subset [0, 1]$ and denote by $\mathcal{B}_\mathcal{X}^{(m)}$ the Borel $\sigma$-algebra on $\mathcal{X}^m$, $m \in \mathbb{N}_+$. A stochastic process is a probability measure over the space $(\mathcal{X}^\infty, \mathcal{B})$ where $\mathcal{B}$ denotes the $\sigma$-algebra on $\mathcal{X}^\infty$ generated by the cylinder sets. A process distribution $\rho$ is stationary if $\rho(X_{1..m} \in B) = \rho(X_{i+1..i+m} \in B)$ for all Borel sets $B \in \mathcal{B}^{(m)}$, $i \in \mathbb{N}_+, m \in \mathbb{N}_+$.

$\varphi$-mixing Processes. Part of our results concern the class of stationary $\varphi$-mixing processes which may be defined as follows (Doukhan, 1994). Let $(\mathcal{X}^\infty, \mathcal{B}, \rho)$ be a stochastic process as defined above. The $\varphi$-dependence between $X_A$ and $X_B$ is defined as

$$\varphi(X_A, X_B) := \sup\{|\rho(V) - \rho(U \cap V)/\rho(U)| : U \in \sigma(X_A), \rho(U) > 0, V \in \sigma(X_B)\}.$$ 

A process $(X_i)_{i \in \mathbb{N}_+}$ is $\varphi$-mixing if $\lim_{n \to \infty} \varphi_n(u, v) = 0$, for all $u, v \in \mathbb{N}_+$, where $\varphi_n(u, v) := \sup\{|\rho(V) - \rho(U \cap V)/\rho(U)| : U \in \sigma(X_{1..n}), \rho(U) > 0, V \in \sigma(X_{1..n})\}$. Under this assumption there could be dependence between the arms; we only require for the joint process to be $\varphi$-mixing. Note that, the assumption that each process is $\varphi$-mixing and the processes are independent seems to be insufficient for the joint process to be $\varphi$-mixing. However, a stronger mixing condition on the individual processes does allow for a jointly $\varphi$-mixing process. More specifically, the following proposition states that if we have $k$ independent $\psi$-mixing processes then the joint process is $\varphi$-mixing. The $\psi$-mixing property is defined in the same way as $\varphi$-mixing whereby the coefficient $\varphi(X_A, X_B)$ is replaced with $\psi(X_A, X_B) := \sup\{|\rho(V) - \rho(U \cap V)/\rho(U)| : U \in \sigma(X_A), \rho(U) > 0, V \in \sigma(X_B), \rho(V) > 0\}$. A $\psi$-mixing process is also $\varphi$-mixing and the $\psi$-mixing coefficients upper bound the $\varphi$-mixing coefficients.

Proposition 1. Let $(\Omega, \mathcal{A}, P)$ be some probability space with $k$ mutually independent processes defined on it. If each of these processes is $\psi$-mixing then the joint process is also $\psi$-mixing and for all $i \in \mathbb{N}$, $1 + \psi_i \leq (1 + \psi_i)^k$, where the $\psi_i$ are the mixing coefficients of the joint process and the $\psi_i$ are upper bounds on the mixing coefficients of the individual processes.

The proof is provided in Appendix A.1.

Random Times. An inherent part of the bandit problem involves working with pay-offs $X_{\tau,i}$, $i \leq k$, obtained by the player at random times $\tau : \Omega \to \mathbb{N}_+$ when arm $i$ is played. We define these as $X_{\tau,i} := \sum_{t \in \mathbb{N}_+} \chi\{\tau = t\} X_{t,i}$, where $\chi$ denotes the indicator function.

\[1\] More generally $\mathcal{X}$ can be a finite set or a closed interval $[a, b]$, for $a < b$, $a, b \in \mathbb{R}$.
3 Weakly dependent reward distributions

We address the question of when and how an optimistic approach can be effective in the case where the process distributions are weakly dependent. More specifically, we consider the restless bandit problem in a setting where the reward distributions are jointly \( \varphi \)-mixing. As discussed earlier, the optimal strategy in this case is to switch between the arms, but obtaining the best switching strategy is PSPACE-hard. In this section we consider a relaxation of the problem when the \( \varphi \)-mixing coefficients are small, and develop an algorithm to approximate the optimal policy of the relaxed problem. We characterize the approximation error in terms of \( \varphi_1 \), and show that for small \( \varphi \)-mixing coefficients, the optimum of the relaxed problem is close to that given by the optimal switching strategy.

3.1 Setting

A total of \( k < \infty \) bandit arms are given, where for each \( i \in 1..k \), arm \( i \) corresponds to a stationary process that generates a time series of pay-offs \( X_{1,i}, X_{2,i}, \ldots \). The joint process over the \( k \) arms is \( \varphi \)-mixing in the sense defined in Section 2, and \( \| \varphi \| := \sum_{i=1}^{\infty} \varphi_i < \infty \). Each process has stationary mean \( \mu_i \), \( i = 1..k \) and we denote by \( \mu^* := \max\{\mu_1, \ldots, \mu_k\} \) the highest stationary mean. At every time-step \( t \in \mathbb{N}_+ \), a player chooses one of \( k \) arms according to a policy \( \pi_t \) and receives a reward \( X_{t,\pi_t} \). The player’s objective is to maximize the sum of the pay-offs received. The policy has access only to the pay-offs gained at earlier stages and to the arms it has chosen. More formally, a policy is a sequence of mappings \( \pi_t : \Omega \rightarrow \{1, \ldots, k\}, t \geq 1 \), each of which is measurable with respect to \( F_{t-1} \) where \( \langle F_t \rangle_{t \geq 0} \) is a filtration that tracks the pay-offs obtained in the past \( t \) rounds, i.e. \( F_0 = \{\emptyset, \Omega\} \), and for \( t \geq 1 \) we let \( F_t = \sigma(X_{1,\pi_1}, \ldots, X_{t,\pi_t}) \).

This assumption is equivalent to the assumption that the policy can be written as a function of the past pay-offs and chosen arms Shiryaev (1991)[Thm. 3, p.174]. Let \( \Pi = \{\langle \pi_t \rangle_{t \geq 1} : \pi_t \) is \( F_{t-1} \)-measurable for all \( t \geq 1 \} \) denote the space of all possible policies. We define the maximal value that can be achieved in \( n \) rounds as

\[
v^*_n = \sup_{\pi \in \Pi} \sum_{t=1}^{n} E X_{t,\pi_t}.
\]

Our objective is to devise policies that achieve an expected cumulative pay-off close to \( v^*_n \) after \( n \) rounds of play.

3.2 Approximation Error

We start by translating \( \varphi \)-mixing properties to those of expectations in order to control the difference between what a switching strategy can achieve as compared to the best stationary mean. This is established by Proposition 2, which shows that if we have a random variable \( X \) that takes values in \([0, 1]\) and depends only weakly on some collected information, then the conditional expectation of \( X \) given that information is close to the expected value of \( X \).

**Proposition 2.** Let \( (\Omega, A, P) \) be a probability space, let \( X \) be a real-valued random variable taking values in \( \mathcal{X} \) and denote by \( \mathcal{G} \) some \( \sigma \)-subalgebra of \( A \). If there exists \( \varphi \geq 0 \) such that

\[
|P(A)P(B) - P(A \cap B)| \leq \varphi P(B)
\]

then...
for all $A \in \sigma(X), B \in \mathcal{G}$, then for any $B \in \mathcal{G}$ we have

$$\int_B |E(X|\mathcal{G}) - E(X)| \, dP \leq 4\varphi P(B), \text{ and } \|E(X|\mathcal{G}) - E(X)\|_{L^1(P)} \leq 4\varphi.$$ 

The result is tight in the sense that for any $0 < \varphi < 1/2$ there exists a probability space $(\Omega, \mathcal{A}, P)$, a $\sigma$-subalgebra $\mathcal{G} \subset \mathcal{A}$, and a random variable $X$, such that $|P(A)P(B') - P(A \cap B')| \leq \varphi P(B')$ for all $B' \in \mathcal{G}$ and there exists a set $B \in \mathcal{G}$ and $P(B) > 0$, with

$$\varphi P(B) \leq \int_B |E(X|\mathcal{G}) - E(X)| \, dP.$$ 

The proof is given in Appendix A.2.

Using Proposition 2 we have Proposition 3 below which quantifies the loss of settling for the arm with the best stationary mean instead of devising the best switching strategy.

**Proposition 3.** Consider a $k$-armed stationary $\varphi$-mixing bandit problem. Let $\mu_1, \ldots, \mu_k$ be the means of the stationary distributions and let $\mu^* = \max\{\mu_1, \ldots, \mu_k\}$. Then for every $n \geq 1$ we have

$$v^* - n\mu^* \leq n\varphi.$$ 

The proof given in Appendix A.3.

**Remark.** Note that this relaxation introduces an inevitable linear component to the regret as shown by Proposition 3. However, we argue that if the reward distributions are weakly dependent in the sense that $\varphi_1 \leq \epsilon$ for some small $\epsilon$, then we are guaranteed to only lose a factor of $n\epsilon$ after $n$ rounds of play, if we settle for the arm with the highest stationary mean instead of using the best possible switching policy.

### 3.3 Policies and the $\varphi$-mixing Property

The policies we consider have information about the whole (observed) past which may lead to stronger couplings between past and future pay-offs. As a result, depending on the policy used, the pay-off sequences obtained by playing a set of jointly $\varphi$-mixing bandit arms are not guaranteed to be $\varphi$-mixing. This is demonstrated by the following example.

**Example 1.** Consider a two-armed bandit problem, where the first arm is deterministically set to 0, i.e. $X_{t,1} = 0, \ t \in \mathbb{N}_+$ and the second arm has a process distribution described by a two state Markov chain with the following transition matrix,

$$T = \begin{pmatrix} 1 - \epsilon & \epsilon \\ \epsilon & 1 - \epsilon \end{pmatrix}, \text{ with some } \epsilon \in (0, 1).$$

Observe that for this process, if $\epsilon$ is small, with high probability the Markov chain stays in its current state. It is easy to check that the arms are jointly $\varphi$-mixing. Now consider a policy $\pi$, and denote by $\tau_1, \tau_2, \ldots$ the sequence of random times at which $\pi$ samples the second arm according
to the following simple rule. Set $\tau_1 = 1$. For subsequent random times, if $X_{\tau_n,2} = X_{1,2}$ for $n \in \mathbb{N}_+$ then $\tau_{n+1} = \tau_n + 1$. Otherwise, $\tau_{n+1}$ is set to be significantly larger than $\tau_n$ to guarantee that the distribution of $X_{\tau_{n+1},2}$ given $X_{\tau_n,2}$ is close to the stationary distribution of the Markov chain, during which time the first arm is sampled. The sequence $X_{\tau_1,2}, X_{\tau_2,2}, \ldots$ so generated is highly dependent on $X_{1,2}$, does not have a stationary distribution, and is not $\varphi$-mixing. In fact the expectations $EX_{\tau_n,2}, n \in \mathbb{N}_+$ are very different from the stationary mean $EX_{1,2}$.

A more detailed treatment of this example is given in Appendix A.4.

### 3.4 An optimistic approach

In this section we describe a UCB-type algorithm to identify the arm with the highest stationary mean in a jointly $\varphi$-mixing bandit problem.

The main challenge in achieving this objective lies in building confidence intervals around empirical estimates of the stationary means. Indeed, as shown in Example 1, the sampling process may introduce long range dependencies in such a way that the resulting pay-off sequence may neither be stationary nor $\varphi$-mixing. This is the reason why a standard UCB algorithm designed for the i.i.d. setting may not be suitable here, even when equipped with a Hoeffding-type concentration bound for $\varphi$-mixing processes. Thus, care must be taken when devising a sampling policy in order to allow for Hoeffding-type concentration results (for $\varphi$-mixing processes) to be applicable. We propose Algorithm 1, which given $\|\varphi\|$ works as follows. First, each arm is sampled once for initialization. Next, from $t = k + 1$ on, arms are played in batches of exponentially growing length. Specifically, at each round arm $j$ with the highest upper-confidence on its empirical mean is selected, and played for $2^{s_j}$ consecutive time-steps, where $s_j$ denotes the

\[ X_j + \sqrt{\frac{8\xi(\frac{1}{\xi} + \ln t)}{2s_j} + \frac{\|\varphi\|}{2^{s_j-1}}} \]

where $\xi := 1 + 4 \|\varphi\|$

- **Update:** $t_j \leftarrow t$, $t \leftarrow t + 2^{s_j}$, $X_j \leftarrow \frac{1}{2^{s_j}} \sum_{t'_j = t_j}^{t_j + 2^{s_j} - 1} X_{t'_j,j}$, $s_j \leftarrow s_j + 1$

\[ \textbf{Algorithm 1: A UCB-type Algorithm for } \varphi\text{-mixing bandits.} \]
number of times that arm \(j\) has been selected so far. The upper confidence bound is calculated based on a Hoeffding-type bound for \(\varphi\)-mixing processes given by Corollary 2.1 of Rio (1999). The \(2^s\) samples obtained by playing the selected arm are used in turn to calculate (from scratch) the arm’s empirical mean.

To address the challenge induced by the inter-dependent reward sequences obtained at random times, this approach relies on two key observations. First, consecutive samples \(X_{\tau,i}, X_{\tau+1,i}, \ldots, X_{\tau+2^s,i}\), where \(\tau \in \mathbb{N}_+\) is a random starting time at which arm \(i \in 1..k\) is sampled in a batch of length \(2^s\), form a \(\varphi\)-mixing though not necessarily stationary sequence. Second, for a long enough batch the average expectations \(\frac{1}{n} \sum_{j=0}^{n-1} EX_{\tau+j,i}\) converge to the stationary mean \(\mu_i\) and the empirical mean of the batch is concentrated around its expectation. More formally we have Lemma 1 below.

**Lemma 1.** Consider a \(k\)-armed bandit problem described in 3.1, where we have \(k\) arms each with a bounded stationary pay-off sequence such that the joint process is \(\varphi\)-mixing. For a fixed \(i \in 1..k\) and \(s \in \mathbb{N}\), consider the consecutive samples \(X_{\tau,i}, X_{\tau+1,i}, \ldots, X_{\tau+2^s,i}\), where \(\tau : \Omega \rightarrow \mathbb{N}_+\) is a random time at which the \(i\)th arm is sampled. Let \(\mu_i\) denote the stationary mean of arm \(i\). We have

\[
\left| \mu_i - \frac{1}{2^s} \sum_{j=0}^{2^s-1} EX_{\tau+j,i} \right| \leq \frac{1}{2^{s-1}} \| \varphi \|
\]

The proof is given in Appendix A.5.

Consider the pseudo-regret of Algorithm 1 with respect to the arm with the highest stationary mean given by \(R(n) := n\mu^* - \sum_{t=1}^{n} EX_{t,\pi_t}\). Note that in an i.i.d. setting we trivially have \(R(n) = n\mu^* - \mu_j \sum_{t=1}^{k} ET_j(n)\) where \(T_j(n)\) is the total number of times that arm \(j\) is played by the algorithm. In our framework, this equality does not necessarily hold due to the interdependencies between the pay-offs. However, as shown in Proposition 4 below, an analogous result in the form of an upper-bound holds for our algorithm.

**Proposition 4.** Let \(R(n) := n\mu^* - \sum_{t=1}^{n} EX_{t,\pi_t}\) be the expected regret of Algorithm 1 with respect to the best stationary mean \(\mu^*\) after \(n\) rounds of play. We have

\[
R(n) \leq \sum_{j=1}^{k} \Delta_j ET_j(n) + 2k \left( \sum_{t=1}^{n} \varphi_t \right) \log n
\]

where \(T_j(n)\) denotes the number of times that arm \(j\) has been played in \(n\) rounds.

The proof is given in Appendix A.5.

An upper-bound on the algorithm’s regret is given by Theorem 1, whose proof relies in part on the technical results outlined above and is given in Appendix A.5.

**Theorem 1 (Regret Bound.).** Let \(R(n) := n\mu^* - \sum_{t=1}^{n} EX_{t,\pi_t}\) be the expected regret of Algorithm 1 compared to the best stationary mean \(\mu^*\) after \(n\) rounds of play. We have,

\[
R(n) \leq \sum_{i=1}^{k} \frac{32(1 + 4 \| \varphi \|) \ln n}{\Delta_i} + (1 + 2\pi^2/3)(\sum_{i=1}^{k} \Delta_i) + \| \varphi \| \log n
\]

where \(\Delta_i := \mu^* - \mu_i\).
4 Strongly dependent reward distributions

At the other end of the extreme lie bandit problems with strongly dependent pay-off distributions. Our objective in this section is to give an example, where a simple switching strategy can be obtained in this case to leverage the strong inter-dependencies between the samples. While this approach gives a much higher overall pay-off than what would be given by settling for the highest stationary mean, it is computationally efficient. The intuition is that in many cases strong dependencies may allow for the prediction of future rewards even from scarce observations of a sample path.

We consider a class of stochastic processes for which we can easily control the level of dependency. A natural choice is to use stationary Gaussian processes on \( \mathbb{N}_+ \). Recall that a Gaussian process on \( \mathbb{N}_+ \) is a stochastic process \( X_t, t \in \mathbb{N}_+ \), such that for all \( n \in \mathbb{N}_+, a_1, \ldots, a_n \in \mathbb{R}, t_1, \ldots, t_n \in \mathbb{N}_+ \), the random variables \( \sum_{i=1}^n a_i X_{t_i} \) are normally distributed. Given a non-negative definite covariance function \( \text{cov} : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R} \) it follows from Kolmogorov’s consistency theorem that there exists a Gaussian process which has \( \text{cov} \) as its covariance function (see, for instance, Giné and Nickl (2016) for more details). A Gaussian process is stationary if it has constant mean on \( \mathbb{N}_+ \) and its covariance can be written as \( \text{cov}(s) = \text{cov}(t, t + s) \) for all \( t \in \mathbb{N}_+, s \in \mathbb{N} \). We measure the degree of smoothness by means of Hölder-continuity of the covariance function, i.e. we assume that there exists some \( \alpha > 0 \) such that for all \( s, t \in \mathbb{N} \), \( |\text{cov}(s) - \text{cov}(t)| \leq c |s - t|^\alpha \). A low \( c \) and \( \alpha \) correspond to highly dependent processes since the covariance decreases slowly over time. A slowly decreasing covariance also implies large \( \phi \)-mixing coefficients since, due to Rio (1999)[Thm. 1.4],

\[
\frac{|\text{cov}(t)|}{2 \text{cov}^2(0)} \leq (\varphi(X_0, X_t)\varphi(X_t, X_0))^{1/2}
\]

and the results of Section 3 are of limited use here.

If we have two arms and we play arm one then the regret incurred at time \( t \) is \( E(X_{t,2} - X_{t,1})^+ \) where \( z^+ = \max \{0, z\} \). This implies that in general, we need to control \( E(X - Y)^+ \) for normal random variables \( X \) and \( Y \). To this end, we use the bounds given by (1) below. Consider two independent and normally distributed random variables \( X, Y \) with mean \( \mu_X > \mu_Y \) and variance \( \sigma_X^2, \sigma_Y^2 \). Let \( \Delta = \mu_X - \mu_Y > 0 \) and \( \sigma^2 = \sigma_X^2 + \sigma_Y^2 \) and \( \phi \) be the density function of the standard normal distribution then the following bounds hold.

\[
0 \leq E(Y - X)^+ \leq \sigma \phi(\Delta/\sigma)
\]
\[
\Delta \leq E(X - Y)^+ \leq \sigma \phi(\Delta/\sigma) + \Delta.
\]

The derivation is given in the supplementary material on page 30.

If the processes change slowly over time then a lot of information can be gained about the pay-offs by looking at past pay-offs of the process. To make use of this information we need to consider conditional distributions of \( X_{t,i} \) given \( X_{s,i} \) for some \( 1 \leq s < t \) and any \( i \leq k \). We recall that \( X_{t,i} \) conditioned on \( X_{s,i} = x \) has a normal distribution with mean \( \tilde{\mu}_i = \mu_i + (x - \mu_i)\text{cov}_i(t-s)/\text{cov}_i(0) \) and variance \( \tilde{\sigma}_i^2 = \text{cov}_i(0) - \text{cov}_i^2(t-s)/\text{cov}_i(0) \).
4.1 Setting

We consider the bandit problem where we have \( k < \infty \) arms, each of which is distributed according to a stationary Gaussian process with stationary mean \( \mu_i, \ i = 1..k \). We assume that the processes are mutually independent with the same covariance function \( \text{cov} \) which is in turn Hölder continuous with known constants \( c \) and \( \alpha \). We also assume that bounds on the stationary means of the processes are known and that \( \text{cov}(0) = 1 \). We use \( \langle X_{t,i} \rangle \) to denote the sample paths of the processes and \( X_{t,i} \) represents the pay-off gained by the \( i \)'th process at time \( t \). Let \( \Delta_i = \max_{j \leq k} \mu_j - \mu_i \) and \( \Delta = \max_{i \leq k} |\Delta_i| \). We measure the regret with respect to the best switching strategy. As in the \( \varphi \)-mixing case let \( \Pi = \{ \langle \pi_t \rangle_{t \geq 1} : \pi_t \text{ is } F_{t-1} - \text{measurable for all } t \geq 1 \}, F_0 = \{ \emptyset, \Omega \} \) and for \( t \geq 1, F_t = \sigma(X_{1,\pi_1}, \ldots, X_{t,\pi_t}) \). The regret that builds up over \( T \) rounds for any strategy \( \pi \) is

\[
R_1(T) = \sup_{\pi' \in \Pi} \sum_{t=1}^{T} (E(X_{t,\pi'_t}) - E(X_{t,\pi_t})).
\]

We start by analyzing a related quantity which is

\[
R_2(T) = \sum_{t=1}^{T} E(\max_{i \leq k} (X_{t,i} - X_{t,\pi_t})) = \sum_{t=1}^{T} E(\max_{i \leq k} (X_{t,i} - X_{t,\pi_t})^+).
\]

The regret \( R_2(T) \) is always larger than \( R_1(T) \) since \( R_2(T) \) compares to the best choice in hindsight while \( R_1(T) \) compares to the best policy that bases its decision only on observations received in the past. We provide upper-bounds on \( R_2(T) \) using a simple arm selection strategy. This approach is similar to the usual approach in the i.i.d. setting where one measures the regret with respect to a strategy that has more information about the problem. Observe that only in cases where the process is degenerate or completely deterministic is it possible to achieve sub-linear regret since, otherwise, there is always some remaining uncertainty about the future pay-offs. Therefore, we aim for minimizing the scaling factor of \( T \).

4.2 The Algorithm and an Upper Bound on the Regret

We provide a simple algorithm, namely, Algorithm 2, that exploits the dependence between the pay-offs. Starting from an exploration phase, the algorithm alternates between exploration and exploitation, denoted Phase I and Phase II respectively. In Phase I it sweeps through all \( k \)-arms to observe the corresponding pay-offs. In Phase II it plays the arm with the highest observed pay-off for \( m - k \) rounds, where \( m \) is a (large) constant given by (2) which reflects the degree of dependence between the samples in the processes. It is worth mentioning that we do not estimate the stationary distributions in this algorithm but use bounds on the differences between the stationary means. These are only of minor relevance unless the differences in the stationary means are high as compared to the dependence in the individual processes. We have the following regret bound for Algorithm 2; the derivation is given in Appendix B.2.

**Proposition 5.** Given a \( k \)-armed bandit problem with independent arms where each arm evolves according to a stationary Gaussian process on \( \mathbb{N}_+ \) with stationary mean \( \mu_i, i \leq k, \) and common
**Input:** a bound $\Delta$ on the difference between the stationary means, $\max_{i,j \leq k} |\mu_i - \mu_j| \leq \Delta$; Hölder coefficients $\alpha, c$ such that $|1 - \text{cov}(t)| \leq c t^\alpha$ for all $t \geq 0$.

Set

$$m^* = \left[ \left( \frac{2\Delta + \sqrt{8/\pi}}{\sqrt{8c\alpha}} \right)^{\frac{2}{\alpha}} \right].$$

If $m^* < 2k$ set $m^* = 2k$.

If $\Delta < \sqrt{8(m^* - k)^\alpha/(\sqrt{c}(m^* - k)^\alpha + k^\alpha))}$ set

$$m^* = \arg\max_{m \in \mathbb{N}} \{ m : \Delta \geq \sqrt{8(m - k)^\alpha}/(\sqrt{c}(m - k)^\alpha + k^\alpha)) \}.$$  

Loop over $l = 0, 1, \ldots$

- **Phase I:** observe pay-offs $X_{lm^*+1,1} = x_1, \ldots, X_{lm^*+k,k} = x_k$ and choose $i^* \in \arg\max_{i \leq k} \{ x_1, \ldots, x_k \}$.
- **Phase II:** play arm $i^*$ for $m^* - k$ steps.

**Algorithm 2:** An Algorithm for highly dependent arms.

Covariance function $\text{cov}(\cdot)$. Given $\Delta$ such that $\max_{i,j} |\mu_i - \mu_j| \leq \Delta$ and $\alpha, c$ such that $|1 - \text{cov}(t)| \leq c t^\alpha$ for all $t \geq 0$, Algorithm 2 has after $T$ rounds a regret $R_2(T)$ of at most

$$(T + m^*)k(k - 1) \left( \frac{\Delta + \sqrt{2/\pi}}{m^*} + \frac{\sqrt{a_{m^*}}}{8\pi(1 - b_{m^*})} \left( \sqrt{8\pi} - (1 - \sqrt{b_{m^*}}, \Delta) \exp \left( -\frac{b_{m^*} \Delta^2}{8} \right) \right) \right),$$

with $a_{m^*} = 8c(m^* - k)^\alpha, b_{m^*} = c(m^* - k)^\alpha + k^\alpha$.

For comparison, if we chose to play the arm with the highest stationary mean at all steps then the standard bounds on the normal distribution in Equation 1 give us a bound on the regret of

$$\sqrt{2T} \sum_{i=1}^{k} \phi(\Delta_i/2) = \pi^{-1/2} T \sum_{i=1}^{k} \exp \left( -\frac{\Delta_i^2}{4} \right) \geq k \pi^{-1/2} T \exp \left( -\frac{\Delta^2}{4} \right).$$

The bound on Algorithm 2 is about

$$k^2 T \left( 4c\alpha + c^{1/2} \alpha^{-\frac{\alpha}{2+\alpha}} \pi^{-1/2} (2^{-1/2} \Delta + \pi^{-1/2})^{\frac{2}{2+\alpha}} \right).$$

The term $\alpha^{-\frac{\alpha}{2+\alpha}}$ term is about 1 for small $\alpha$ and the regret is of order $T c^{1/2}$. Hence, if the dependence is high in the process as compared to the difference in stationary means then our simple switching algorithm outperforms the algorithm that stays with the best stationary mean.
5 Outlook

We studied two regimes of the restless bandit problem for which efficient approximations can be provided. For the weakly dependent case we developed a UCB-style algorithm that accounts for the dependence between the pay-offs to estimate the arm with the highest stationary mean and to build confidence intervals. We provided a regret bound for this algorithm that shows that it attains a logarithmic regret in comparison to a strategy that plays the best stationary mean throughout. We also provided an upper bound on the difference in performance of the best switching strategy compared to playing the best stationary mean. This bound shows that one can effectively approximate the restless bandit problem with a simple strategy if the dependence is not too high. We complemented this analysis with results on highly dependent restless bandits. Specifically, we demonstrated that for the case of Gaussian process bandits, there exists a simple algorithm that can achieve a significantly higher pay-off than an algorithm that plays the arm with the highest stationary mean.

An open problem for the $\phi$-mixing bandit setting is the estimation of $\phi$-mixing coefficients to make the algorithm adaptive to the dependence of the process. We conjecture that in the i.i.d. setting such an algorithm can perform as well as the standard UCB algorithm (up to constants) while being robust to potential dependencies. Another open problem is the derivation of a lower bound for the $\phi$-mixing bandit problem. In the strongly dependent setting an open problem concerns how to weaken the assumptions on the process, and obtain an analogous formulation of the dependence as to the weakly dependent setting.
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A Results on $\varphi$-mixing bandits

A.1 Various useful lemmas

The first lemma is simple but fundamental to our derivations. It allows us to control the $\varphi$-mixing coefficient of disjoint events.

Lemma 2. Let $(\Omega, A, P)$ be a probability space and let $B, C$ be two $\sigma$-subalgebras of $A$. If there exists a $\varphi \geq 0$ such that for all $B \in B$ and $C \in C$ it holds that $|P(B)P(C) - P(B \cap C)| \leq \varphi P(C)$ then for any disjoint sequence $(B_n)_{n \in \mathbb{N}}, B_n \in B$ for all $n \in \mathbb{N}$, and any $C \in C$, we have

$$\sum_{n=0}^{\infty} |P(B_n)P(C) - P(B_n \cap C)| \leq 2\varphi P(C).$$

Proof. Let $c_n = P(B_n)P(C) - P(B_n \cap C)$, $I_+ = \{n \in \mathbb{N} : c_n \geq 0\}$, $I_- = \{n \in \mathbb{N} : c_n < 0\}$. Now, since $\bigcup_{n \in I_+} B_n \in B$ and $\bigcup_{n \in I_-} B_n \in B$ we have

$$P(C)\varphi \geq P\left(\bigcup_{n \in I_+} B_n\right)P(C) - P\left(\bigcup_{n \in I_+} B_n \cap C\right)$$

$$= \sum_{n \in I_+} (P(B_n)P(C) - P(B_n \cap C))$$

$$= \sum_{n \in I_+} |P(B_n)P(C) - P(B_n \cap C)|$$

and $P(C)\varphi \geq \sum_{n \in I_-} |P(B_n)P(C) - P(B_n \cap C)|$. Hence

$$2P(C)\varphi \geq \sum_{n \in I_+} |P(B_n)P(C) - P(B_n \cap C)| + \sum_{n \in I_-} |P(B_n)P(C) - P(B_n \cap C)|$$

$$= \sum_{n \in \mathbb{N}} |P(B_n)P(C) - P(B_n \cap C)|.$$

$\blacksquare$

Proposition 6 (this is Proposition 1 in the main text.). Let $(\Omega, A, P)$ be some probability space with $k$ mutually independent processes defined on it. If each of these processes is $\psi$-mixing then the joint process is also $\psi$-mixing and for all $i \in \mathbb{N}, 1 + \tilde{\psi}_i \leq (1 + \psi_i)^k$, where the $\tilde{\psi}_i$ are the mixing coefficients of the joint process and the $\psi_i$ are upper bounds on the mixing coefficients of the individual processes.

Proof. Fix some $n, a, u, v > 0$, $a \geq u + n$ and let us denote the individual processes with $(X_{n,i})_{n \in \mathbb{N}_+}, i \leq k$. Furthermore, let $A = \{1, \ldots, u\}, B = \{a, \ldots, a + v - 1\}$ and $\mathcal{G} = \sigma(X_{A,1}, \ldots, X_{A,k}), \mathcal{H} = \sigma(X_{B,1}, \ldots, X_{B,k})$. 

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(i) Consider a set $U \in \mathcal{G}$ of the form $U = \bigcap_{i \leq k} U_i$ where $U_i \in \sigma(X_{A,i})$ for all $i \leq k$ and a set $V \in \mathcal{H}$ of the form $V = \bigcap_{i \leq k} V_i, V_i \in \sigma(X_{B,i})$, for all $i \leq k$. The mutual independence of the processes implies that

$$|P(U)P(V) - P(U \cap V)| = \left| \prod_{i=1}^{k} P(U_i)P(V_i) - \prod_{i=1}^{k} P(U_i \cap V_i) \right|$$

$$\leq P(U_1)P(V_1) \left| \prod_{i=2}^{k} P(U_i)P(V_i) - \prod_{i=2}^{k} P(U_i \cap V_i) \right|$$

$$+ |P(U_1)P(V_1) - P(U_1 \cap V_1)| \prod_{i=2}^{k} P(U_i \cap V_i)$$

$$\leq \left( \prod_{i=1}^{2} P(U_i)P(V_i) \right) \left| \prod_{i=3}^{k} P(U_i)P(V_i) - \prod_{i=3}^{k} P(U_i \cap V_i) \right|$$

$$+ P(U_1)P(V_1) |P(U_2)P(V_2) - P(U_2 \cap V_2)| \prod_{i=3}^{k} P(U_i \cap V_i)$$

$$+ \psi_n P(U_1)P(V_1) \prod_{i=2}^{k} P(U_i \cap V_i)$$

$$\leq \psi_n \sum_{i=1}^{k} \left( \prod_{j=1}^{i} P(U_j)P(V_j) \right) \left( \prod_{j=i+1}^{k} P(U_j \cap V_j) \right)$$

$$\leq \psi_n P(U)P(V) \sum_{i=1}^{k-1} \prod_{j=i+1}^{k} \frac{P(U_j \cap V_j)}{P(U_j)P(V_j)}$$

$$\leq \psi_n P(U)P(V) \sum_{i=0}^{k-1} (1 + \psi_n)^{k-i-1} = ((1 + \psi_n)^k - 1) P(U)P(V).$$

(ii) Next, we consider general $U \in \mathcal{G}$ and $V \in \mathcal{H}$. We use here a product measure approach. To make use of this let the index set $C = A \cup B$ and consider the independent $\sigma$-algebras $\sigma(X_{C,1}), \ldots, \sigma(X_{C,k})$. Let $P_i$ be the restriction of $P$ to $\sigma(X_{C,i})$ for all $i \leq k$ and define the product space $(\Omega^k, \otimes_{i \leq k} \sigma(X_{C,i}), \mu)$, where $\mu$ is the product measure of $P_1, \ldots, P_k$. The map $\phi : \Omega \rightarrow \Omega^k$, $\phi(\omega)(i) = \omega_i$ for all $i \leq k$, is inverse-measure preserving due to Fremlin (2010)[272J,254Xc]. In particular, $P(\bigcap_{i \leq k} U_i) = P^{\phi^{-1}[U_1 \times \ldots \times U_k]} = \mu(U_1 \times \ldots \times U_k)$ for $U_i \in \sigma(X_{A,i})$. The important property is the following: if $U \in \mathcal{G}$ then there exists an $F \in \otimes_{i \leq k} \sigma(X_{A,i}) \subseteq \otimes_{i \leq k} \sigma(X_{C,i})$ such that $U = \phi^{-1}[F]$. To see this consider the set

$$\mathcal{S} := \{ \phi^{-1}[F] : F \in \otimes_{i \leq k} \sigma(X_{A,i}) \}. $$

A standard argument shows that $\mathcal{S}$ is a $\sigma$-algebra, i.e. $\emptyset \in \mathcal{S}$, if $U \in \mathcal{S}$ then $\Omega \backslash U = \phi^{-1}[\Omega^k \backslash U] \in \mathcal{S}$ and if $(U_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{S}$ then $\bigcup_{n \in \mathbb{N}} U_n = \phi^{-1}[(\bigcup_{n \in \mathbb{N}} F_n)] \in \mathcal{S}$ for suitable $F_n$. Furthermore, $\mathcal{S} \supseteq \{ \bigcap_{i \leq k} U_i : U_i \in \sigma(X_{A,i}), i \leq k \}$ and the latter set is closed under intersections. Hence, the monotone class theorem tells us that $\mathcal{G} = \sigma\{ \bigcap_{i \leq k} U_i : U_i \in \sigma(X_{A,i}), i \leq k \} \subset \mathcal{S}$ and the result follows. Similarly, we can link $V \in \mathcal{G}$ to $\otimes_{i \leq k} \sigma(X_{B,i}).$
(iii) We like to demonstrate that the $\psi$ mixing property as stated in (i) carries over to arbitrary elements $U \in \bigotimes_{i,k} \sigma(X_{A,i})$ and $V \in \bigotimes_{i,k} \sigma(X_{B,i})$. First, observe that if $U = U_1 \times \ldots \times U_k$ and $V = V_1 \times \ldots \times V_k$, $U_i \in \sigma(X_{A,i})$, $V_i \in \sigma(X_{B,i})$, $i \leq k$, then using $U_1 \times \ldots \times U_k \cap V_1 \times \ldots \times V_k = (U_1 \cap V_1) \times \ldots \times (U_k \cap V_k)$ with the inverse-measure preserving property of $\phi$ and (i) it follows

$$
|\mu(U_1 \times \ldots \times U_k)\mu(V_1 \times \ldots \times V_k) - \mu(U_1 \times \ldots \times U_k \cap V_1 \times \ldots \times V_k)|
\leq ((1 + \psi_n)^k - 1) P(\cap_{i \leq k} U_i) P(\cap_{i \leq k} V_i)
= ((1 + \psi_n)^k - 1) \mu(U_1 \times \ldots \times U_k)\mu(V_1 \times \ldots \times V_k).
$$

The advantage of the product approach is that we can approximate the sets $U$ and $V$ with cylinders of the form $U_1 \times \ldots \times U_k$ and this allows us to carry the $\psi$-mixing property to $G$ and $H$. Due to Fremlin (2010) [Thm. 251Ie, 251W] for $\epsilon > 0$ there exist sequences $U_{1,1}, \ldots, U_{m_1,1}, \ldots, U_{1,k}, \ldots, U_{m_1,k}$ and $V_{1,1}, \ldots, V_{m_2,1}, \ldots, V_{1,k}, \ldots, V_{m_2,k}$, with $U_{i,j}, V'_{i,j} \in \sigma(X_{G,j})$ for all $i \leq m_1, i' \leq m_2, j \leq k$, such that

$$
\mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right) \leq \epsilon,
\mu \left( \bigcup_{i \leq m_2} V_{i,1} \times \ldots \times V_{i,k} \right) \leq \epsilon/m_1
$$

and

$$
\mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right) \geq \sum_{i \leq m_1} \mu(U_{i,1} \times \ldots \times U_{i,k}) - \epsilon.
$$

Eq. (3) also holds for the $V_{i,j}$ series if we replace $m_1$ with $m_2$. Hence,

$$
|\mu(U)\mu(V) - \mu(U \cap V)|
\leq 8\epsilon + |\mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right)\mu \left( \bigcup_{i \leq m_2} V_{i,1} \times \ldots \times V_{i,k} \right) - \mu \left( \bigcup_{i \leq m_1} \bigcup_{j \leq m_2} U_{i,1} \times \ldots \times U_{i,k} \cap V_{j,1} \times \ldots \times V_{j,k} \right)|.
$$

Furthermore,

$$
|\sum_{i \leq m_1} \sum_{j \leq m_2} \mu(U_{i,1} \times \ldots \times U_{i,k})\mu(V_{j,1} \times \ldots \times V_{j,k})
- \mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right)\mu \left( \bigcup_{i \leq m_2} V_{i,1} \times \ldots \times V_{i,k} \right)|
\leq \left( \mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right) + \epsilon \right) \sum_{j \leq m_2} \mu(V_{j,1} \times \ldots \times V_{j,k})
- \mu \left( \bigcup_{i \leq m_1} U_{i,1} \times \ldots \times U_{i,k} \right)\mu \left( \bigcup_{i \leq m_2} V_{i,1} \times \ldots \times V_{i,k} \right)
\leq 3\epsilon.
$$
Observe that if for some sequence $U_1, \ldots, U_m$, $\mu(\bigcup_{i \leq m} U_i) \geq \sum_{i \leq m} \mu(U_i) - \epsilon$ then

\[
\mu\left(\bigcup_{i \leq m} U_i\right) = \mu\left(\bigcup_{i \leq m} U_i \setminus \bigcup_{i < j} U_j \cap U_i\right) = \sum_{i \leq m} \mu(U_i) \setminus \bigcup_{i < j} U_j \cap U_i) \\
= \sum_{i \leq m} \mu(U_i) - \sum_{i \leq m} \mu(\bigcup_{j=i} U_j \cap U_i) \\
\]

and $\sum_{i \leq m} \mu(\bigcup_{j<i} U_j \cap U_i) \leq \epsilon$. This implies that for any other measurable set $V$ we have the following relation

\[
\mu\left(\bigcup_{i \leq m} U_i \cap V\right) = \mu\left(\bigcup_{i \leq m} U_i \cap V \setminus \bigcup_{j<i} U_j \cap U_i\right) = \sum_{i \leq m} \mu(U_i \cap V \setminus \bigcup_{j<i} U_j \cap U_i) \\
= \sum_{i \leq m} \mu(U_i \cap V) - \sum_{i \leq m} \mu(\bigcup_{j<i} U_j \cap U_i) \geq \sum_{i \leq m} \mu(U_i \cap V) - \epsilon.
\]

Hence, using eq. (3), we can infer that

\[
\left| \sum_{i \leq m_1} \sum_{j \leq m_2} \mu(U_{i_1} \times \ldots \times U_{i_k} \cap V_{j_1} \times \ldots \times V_{j_k}) \right| \\
- \mu\left(\bigcup_{i \leq m_1} \bigcup_{j \leq m_2} U_{i_1} \times \ldots \times U_{i_k} \cap V_{j_1} \times \ldots \times V_{j_k}\right) \leq \sum_{i \leq m_1} \sum_{j \leq m_2} \mu(U_{i_1} \times \ldots \times U_{i_k} \cap V_{j_1} \times \ldots \times V_{j_k}) \\
- \sum_{i \leq m_1} \mu(U_{i_1} \times \ldots \times U_{i_k} \cap (\bigcup_{j \leq m_2} V_{j_1} \times \ldots \times V_{j_k})) + \epsilon \\
\leq 2\epsilon.
\]

We gain the following upper bound on $|\mu(U)\mu(V) - \mu(U \cap V)|$ by combining the different inequalities

\[
13\epsilon + \sum_{i \leq m_1} \sum_{j \leq m_2} \left| \mu(U_{i_1} \times \ldots \times U_{i_k} \cap V_{j_1} \times \ldots \times V_{j_k}) \right| \\
- \mu(U_{i_1} \times \ldots \times U_{i_k} \cap V_{j_1} \times \ldots \times V_{j_k}) \leq 13\epsilon + ((1 + \psi_n)^k - 1) \sum_{i \leq m_1} \mu(U_{i_1} \times \ldots \times U_{i_k}) \sum_{j \leq m_2} \mu(V_{j_1} \times \ldots \times V_{j_k}) \\
\leq ((1 + \psi_n)^k - 1) \mu(U)\mu(V) + 16\epsilon.
\]

This last step is the only part for which $\phi$-mixing seems insufficient and we need $\psi$-mixing. Since $\epsilon$ is arbitrary we gain the upper bound for arbitrary elements $U \in \hat{\otimes}_{i \leq k} \sigma(X_{A,i})$ and $V \in \hat{\otimes}_{i \leq k} \sigma(X_{B,i})$. Now, if $U \in \mathcal{G}, V \in \mathcal{H}$ then we have an element $E \in \hat{\otimes}_{i \leq k} \sigma(X_{A,i}), F \in \hat{\otimes}_{i \leq k} \sigma(X_{B,i})$ such that

\[
|P(U \cap V) - P(U)P(V)| = |P\phi^{-1}[E \cap F] - P\phi^{-1}[E]P\phi^{-1}[F]| \\
= |\mu(E \cap F) - \mu(E)\mu(F)| \leq ((1 + \psi_n)^k - 1) P(U)P(V).
\]

(iv) The joint process is $\psi$-mixing since $\lim_{n \to \infty} (1 - \psi_n)^k - 1 = 0$. \hfill \qed
A.2 $\varphi$-mixing Property & Expectations

We translate the $\varphi$-mixing property into a statement about expected values. Lemma 2 allows us to state some results about conditional expectations. The conditional expectation $E(X|\mathcal{G})$ is here a random variable $Z$ which is $\mathcal{G}$-measurable and for which $\int_B Z = \int_B X$ for all $B \in \mathcal{G}$. The following Proposition and the example that follows after it imply Proposition 2 in the main text.

**Proposition 7** (this is Proposition 2 in the main text.). Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $X$ be a real-valued random variable with $\|X\|_{\infty} < \infty$ and let $\mathcal{G}$ be some $\sigma$-subalgebra of $\mathcal{A}$. If there exists $\varphi \geq 0$ such that for all $A \in \sigma(X), B \in \mathcal{G}$ we have $|P(A)P(B) - P(A \cap B)| \leq \varphi P(B)$ then for any $B \in \mathcal{G}$

$$\left| \int_B (E(X|\mathcal{G}) - E(X)) \right| \leq 2\varphi \|X\|_{\infty} P(B),$$

$$\int_B |E(X|\mathcal{G}) - E(X)| \leq 4\varphi \|X\|_{\infty} P(B)$$

and

$$\|E(X|\mathcal{G}) - E(X)\|_{L^1(P)} \leq 4\varphi \|X\|_{\infty}.$$

**Proof.** Consider a simple function $X = \sum_{i=1}^n a_i \chi_{A_i}$ with $\langle A_i \rangle_{i=1}^n$ a disjoint sequence, $A_i \in \sigma(X)$ and $a_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$ then

$$\left| \int_B (E(X|\mathcal{G}) - E(X)) \right| = \left| \int_B X - P(B)E(X) \right|$$

$$\leq \sum_{i=1}^n |a_i| \left| \int_B \chi_{A_i} - P(B)P(A_i) \right|$$

$$= \sum_{i=1}^n |a_i| |P(A_i \cap B) - P(B)P(A_i)|$$

$$\leq 2\varphi \|X\|_{\infty} P(B).$$

For a general real-valued random variable $X$ and any $\epsilon > 0$ there exists a random variable $Z = \sum_{i=1}^n a_i \chi_{A_i}, n \in \mathbb{N}, a_i \in \mathbb{R}, A_i \in \mathcal{A}$, for all $i$, such that $\|Z - X\|_{L^1(P)} \leq \epsilon$ (Fremlin, 2010)[242M]. Furthermore, $Z$ can be chosen such that $Z = \sum_{i=1}^n a_i \chi_{A_i}$ and the $A_i$ are disjoint and elements of $\sigma(X)$: consider the probability space $(\Omega, \sigma(X), Q)$ where $Q$ is the restriction of $P$ to $\sigma(X)$. $X$ is $\sigma(X)$ measurable and, hence, a random variable for this new probability space. Also, since $X$ is bounded and measurable the expectation is well defined and Fremlin (2010)[242M] tells us now that there exists for each $\epsilon > 0$ a simple function $Z = \sum_{i=1}^n a_i \chi_{A_i}$ with disjoint $A_i \in \sigma(X), i \in \{1, \ldots n\}$, such that $\|X - Z\|_{L^1(P)} \leq \epsilon$. Furthermore, $Z$ can be chosen such that $\|Z\|_{\infty} \leq \|X\|_{\infty}$. We can also observe that for any $B \in \mathcal{G}$

$$\left| \int_B (E(X|\mathcal{G}) - E(Z|\mathcal{G})) \right| = \left| \int_B (X - Z) \right| \leq \|X - Z\|_{L^1(P)} \leq \epsilon.$$
Hence,

$$
\left| \int_B (E(X|\mathcal{G}) - E(X)) \right| \\
\leq \left| \int_B (E(X|\mathcal{G}) - E(Z|\mathcal{G})) \right| + \left| \int_B (E(Z|\mathcal{G}) - E(Z)) \right| + \left| \int_B E(Z - X) \right| \\
\leq 2\epsilon + 2\varphi \|Z\|_\infty P(B) \leq 2\epsilon + 2\varphi \|X\|_\infty P(B).
$$

Because this holds for any $\epsilon > 0$ the first result follows. The other two claims can be verified by modifying a simple argument from Fremlin (2010)[246F]: $E(X|\mathcal{G}) - E(X)$ is $\mathcal{G}$-measurable and, hence, $F = \{\omega : (E(X|\mathcal{G}) - E(X))(\omega) \geq 0\} \in \mathcal{G}$. Now,

$$
\int_B |E(X|\mathcal{G}) - E(X)| = \left| \int_{B\cap F} (E(X|\mathcal{G}) - E(X)) \right| + \left| \int_{B\setminus F} (E(X|\mathcal{G}) - E(X)) \right| \\
\leq 2\sup_{C\in\mathcal{G}} \int_{B\cap C} (E(X|\mathcal{G}) - E(X)) \leq 4\varphi \|X\|_\infty P(B \cap C) \leq 4\varphi \|X\|_\infty P(B).
$$

The third claim follows by using $B = \Omega \in \mathcal{G}$. Observe that if $X$ attains only finite values then $X$ is a simple function and the argument above simplifies since we can avoid the $\epsilon$-approximation. All other steps carry through in the same way. \qed

The restriction that $\|X\|_\infty < \infty$ is necessary as the example below shows. In fact, the factor $\|X\|_\infty$ is tight and we cannot have guarantees of the above form for unbounded random variables without further assumptions.

**Example:** We construct a simple example where two discrete random variables $X$ and $Y$ are weakly coupled and fulfill the $\varphi$-mixing assumption. Despite fulfilling this assumption the conditional expectation of $X$ given $Y$ attains in certain cases values that are very different from $E(X)$.

Given some $0 < \varphi < 1/2$ let $\epsilon$ be some real number in $(0, \varphi)$, $u$ some positive real value, $\Omega = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, $\mathcal{A} = \mathcal{P}\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and $P$ be such that $P(1, 1) = \epsilon/2$, $P(1, 0) = \epsilon/2$, $P(0, 1) = (\varphi + \epsilon)\epsilon/2$, $P(0, 0) = 1 - (\epsilon + \epsilon/(\varphi + \epsilon))/2$. $P$ is a probability measure since $P(\Omega) = 1$. Let $X(\omega) = u$ if $\omega \in \{(1, 0), (1, 1)\}$ and 0 otherwise. Hence, $P(X = u) = \epsilon$ and $P(X = 0) = 1 - \epsilon$ and $E(X) = \epsilon u$. Similarly, let $Y(\omega) = 1$ if $\omega \in \{(0, 1), (1, 1)\}$. We can observe that $\sigma(X) = \{\emptyset, \Omega, \{(1, 0), (1, 1)\}, \{(0, 0), (0, 1)\}\}$ and $\sigma(Y) = \{\emptyset, \Omega, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}\}$. Now, consider

$$
\{P(V) - P(U \cap V)/P(U) : P(U) > 0, U \in \sigma(Y), V \in \sigma(X)\}.
$$

For $V = \{\Omega\}$ and any $U \in \sigma(Y) \setminus \{\emptyset\}$, we have that $P(V) - P(U \cap V)/P(U) = 0$. Similarly, for $V = \{\emptyset\}$ and any $U \in \sigma(Y) \setminus \{\emptyset\}$ the difference is 0. The difference is also zero if $U = \{\Omega\}$ and $V$ is any element in $\sigma(X)$. The differences for the remaining cases is bounded in absolute value by

- $\varphi$ if $U = \{(0, 1), (1, 1)\}, V = \{(1, 0), (1, 1)\}$;
- $\varphi$ if $U = \{(0, 1), (1, 1)\}, V = \{(0, 0), (0, 1)\}$;
• $\varphi/(2\varphi + \epsilon) \leq \varphi/2$ if $U = \{(1,0), (0,0)\}, V = \{(1,0), (1,1)\};$

• $\varphi/(2\varphi + \epsilon) \leq \varphi/2$ if $U = \{(1,0), (0,0)\}, V = \{(0,0), (0,1)\}.$

Hence, the conditional distribution changes the probability that a particular $V \in \sigma(X)$ occurs by at maximum a value of $\varphi$. The expected value of $X$ given $Y = 1$ is now $E(X|Y = 1) = uP(1,1)/(P(1,1) + P(0,1)) = (\varphi + \epsilon)u$ and

$$E(X|Y = 1) - E(X) = \varphi u = \varphi \|X\|_\infty.$$  

Furthermore, let $G = \sigma(Y)$ and $B = \{Y = 1\} \in G$ then

$$\int_B (E(X|G) - E(X)) = \int_B X - P(B)E(X)$$

$$= uP(\{Y = 1\} \cap \{X = 1\}) - P(B)E(X)$$

$$= P(B)(E(X|Y = 1) - E(X)) = \varphi P(B) \|X\|_\infty.$$  

### A.3 Best Arm versus Best Policy: Upper Bound

We start with the case of a single stationary bounded real-valued time series $(X_t)_{t \in \mathbb{N}^+}$, $X_t \in \mathcal{X}$ for all $t \in \mathbb{N}$, that is $\varphi$-mixing and which is governed by the probability space $(\Omega, \mathcal{A}, P)$ to build some intuition before embarking on the general case. Let $c$ be an upper bound on the absolute value of the stochastic process. In this section, let us use $(\mathcal{F}_t)_{t \geq 1}$ for the filtration $\mathcal{F}_t = \sigma(\sigma(X_1, \ldots, X_t) \cup \mathcal{N})$ where $\mathcal{N}$ is the family of sets of measure zero. We consider a sequence of random times (which will later describe the points in time at which a particular arm is played) $\tau_1 < \tau_2 < \tau_3 \ldots$ where each $\tau_i : \Omega \to \mathbb{N}^+$ is $\mathcal{A} - \mathcal{P}(\mathbb{N}^+)$ measurable and $\tau_i < \infty$ almost surely. We define these inductively together with a filtration that tracks our observed information. In this section let $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and let $\tau_1$ be $\mathcal{G}_0$ measurable. Then, for given $\tau_1, \ldots, \tau_i$ and $\mathcal{G}_1, \ldots, \mathcal{G}_i$ let $\tau_{i+1}$ be some $\mathcal{G}_i$-measurable random variable such that $\tau_{i+1} > \tau_i$ almost surely and define $\mathcal{G}_{i+1} = \sigma(X_{\tau_1}, \ldots, X_{\tau_{i+1}})$. We can observe that $\sigma(X_{\tau_1}, \ldots, X_{\tau_i}) = \sigma(X_{\tau_1}, \ldots, X_{\tau_i}, \tau_1, \ldots, \tau_i)$: $\tau_i$ is $\sigma(X_{\tau_1}, \ldots, X_{\tau_i})$ measurable hence $\sigma(X_{\tau_1}, \ldots, X_{\tau_i}, \tau_1, \ldots, \tau_i) = \sigma(X_{\tau_1}, \ldots, X_{\tau_i}, \tau_2, \ldots, \tau_i).$ $\tau_2$ is $\sigma(X_{\tau_1}, \tau_1) = \sigma(X_{\tau_1})$-measurable and so $\sigma(X_{\tau_1}, \ldots, X_{\tau_i}, \tau_2, \ldots, \tau_i) = \sigma(X_{\tau_1}, \ldots, X_{\tau_i}, \tau_3, \ldots, \tau_i)$. By induction the claim follows.

The following technical result is important:

**Lemma 3.** For all $i, t \geq 1$ and $A \in \mathcal{G}_{i-1}$ we have that $A \cap \{\tau_i = t\} \in \mathcal{F}_t$. In particular $\{\tau_i = t\} \in \mathcal{F}_t$.

**Proof.** For $i = 1$ result holds trivially since $A \cap \{\tau_1 = t\}$ is either $\Omega$ or $\emptyset$ and since $\mathcal{F}_t$ is a $\sigma$-algebra it contains $\Omega$ and $\emptyset$. For any other $i \geq 2$ observe that if the claim holds for all $1 \leq j \leq i - 1$ then since $\tau_i$ is $\mathcal{G}_{i-1}$-measurable and because $\tau_i > \tau_{i-1}$ almost surely implies that $U := \{\tau_i = t\} \cap \{\tau_{i-1} \geq t\} \in \mathcal{N} \subset \mathcal{F}_t$ and we have

$$\{\tau_i = t\} = U \cup \bigcup_{s=1}^{t-1} \{\tau_i = t\} \cap \{\tau_{i-1} = s\} \in \mathcal{F}_t.$$  

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We can also observe that \( \{ B \cap \{ \tau_i = t \} : B \in \sigma(X_{\tau_i}) \} \subset \mathcal{F}_t \) because

\[
\{ B \cap \{ \tau_i = t \} : B \in \sigma(X_{\tau_i}) \} = \{ X_t^{-1}[B'] \cap \{ \tau_i = t \} : B' \in \mathcal{B} \}
\]

and both \( X_t^{-1}[B'] \) and \( \{ \tau_i = t \} \) lie in \( \mathcal{F}_t \).

Also, for any \( j \in \{1, \ldots, i-1\} \) we have \( \{ B \cap \{ \tau_i = t \} : B \in \sigma(X_{\tau_j}) \} \subset \mathcal{F}_t \): let \( U = \{ \tau_i = t \} \cap \{ \tau_j \geq t + 1 - (i - j) \} \) then \( U \in \mathcal{N} \) and

\[
\{ B \cap \{ \tau_i = t \} : B \in \sigma(X_{\tau_j}) \} = U \cup \bigcup_{s=1}^{t-(i-j)} \{ X_s^{-1}[B'] \cap \{ \tau_i = t \} \cap \{ \tau_j = s \} : B' \in \mathcal{B} \}
\]

and \( X_s^{-1}[B'] \cap \{ \tau_j = s \} \in \mathcal{F}_s \subset \mathcal{F}_t \).

We have shown that

\[
\left\{ \{ \tau_i = t \} \cap \bigcap_{j=1}^{i-1} B_j : B_j \in \sigma(X_{\tau_j}) \right\} \subset \mathcal{F}_t.
\]

This implies directly that

\[
\sigma\left\{ \{ \tau_i = t \} \cap \bigcap_{j=1}^{i-1} B_j : B_j \in \sigma(X_{\tau_j}) \right\} \subset \mathcal{F}_t.
\]

It remains to show that \( \{ A \cap \{ \tau_i = t \} : A \in \sigma(X_{\tau_1}, \ldots, X_{\tau_{i-1}}) \} \) is included in the left side. One can verify this in the same way that we verified that the different forms of \( \sigma(X_1, \ldots, X_n) \) are equal. \[ \square \]

We also need the following observation: Let \( s, t \in \mathbb{N}_+, s < t \) then for any \( B \in \mathcal{F}_s \) we have \( |\int_{B}(E(X_t|\mathcal{F}_s) - E(X_t))| \leq 2\varphi_{t-s}P(B) \). This follows from the following lemma by remembering that \( \mathcal{F}_s = \sigma(\sigma(X_1, \ldots, X_s) \cup \mathcal{N}) \).

**Lemma 4.** Given some probability space \((\Omega, \mathcal{A}, P)\), three \( \sigma \)-subalgebras \( \mathcal{B}, \mathcal{C}, \mathcal{D} \subset \mathcal{A} \) and \( \varphi > 0 \) such that \( |P(U)P(V) - P(U \cap V)| \leq \varphi P(V) \) for all \( U \in \mathcal{B}, V \in \mathcal{C} \) and \( P(V) = 0 \) for all \( V \in \mathcal{D} \) then it holds that \( |P(U)P(V) - P(U \cap V)| \leq \varphi P(V) \) for all \( U \in \mathcal{B}, V \in \sigma(\mathcal{C} \cup \mathcal{D}) \).

**Proof.** Consider the set

\[
\mathcal{E} = \{ V : \sup_{U \in \mathcal{U}} |P(U)P(V) - P(U \cap V)| \leq \varphi (P(V) \wedge P(\Omega \setminus V)), V \in \sigma(\mathcal{C} \cup \mathcal{D}) \}.
\]

We have that \( \mathcal{C}, \mathcal{D} \subset \mathcal{E} \) because for any \( V \in \mathcal{C} \cup \mathcal{D} \)

\[
|P(U \cap (\Omega \setminus V)) - P(U)P(\Omega \setminus V)| = |P(U) - P(U \cap V) - P(U)P(\Omega \setminus V)| \\
\leq |P(U) - P(U)P(V)| - |P(U \cap V) - P(U)P(V)| \\
= |P(U \cap V) - P(U)P(V)| \leq \varphi P(V)
\]

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and applying this to $\Omega \setminus V$ shows $V \in \mathcal{E}$. In fact, $\{V \cap V' : V, V' \in \mathcal{C} \cup \mathcal{D}\} \supseteq \mathcal{C} \cup \mathcal{D}$ is a subset of $\mathcal{E}$ since for $V \in \mathcal{C}, V' \in \mathcal{D}$ we have $P(U \cap V \cap V') - P(U)P(V \cap V') = 0$. Also, the set

$$\mathcal{E}' = \{V : \sup_{U \in \mathcal{U}} |P(U)P(V) - P(U \cap V)| \leq \varphi(P(V) \wedge P(\Omega \setminus V)), \quad V \in \mathcal{C} \cup \mathcal{D} \cup \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}\}$$

is closed under intersection, $\sigma(\mathcal{E}') = \sigma(\mathcal{C} \cup \mathcal{D})$ and $\mathcal{E}' \subseteq \mathcal{E}$.

Furthermore, $\mathcal{E}$ is a Dynkin system: (1) $\emptyset \in \mathcal{E}$, (2) if $V \in \mathcal{E}$ then with the argument above

$$|P(U \cap (\Omega \setminus V)) - P(U)P(\Omega \setminus V)| \leq |P(U \cap V) - P(U)P(V)| \leq \varphi(P(V) \wedge P(\Omega \setminus V)),$$

and $\Omega \setminus V \in \mathcal{E}$, (3) if $\{V_n\}_{n \in \mathbb{N}}$ a disjoint family of sets in $\mathcal{E}$ with union $V$ then

$$|P(U \cap V) - P(U)P(V)| \leq \sum_{n \in \mathbb{N}} |P(U \cap V_n) - P(U)P(V_n)| \leq \varphi \sum_{n \in \mathbb{N}} P(V_n) = \varphi P(V).$$

The Monotone Class Theorem gives the claimed result.

Lemma 5. Let $\tau_1, \tau_2, \ldots$ be a sequence of random times as defined above and assume that $\langle X_t \rangle_{t \in \mathbb{Z}}$ is a bounded stationary $\varphi$-mixing process with mean $\mu$, upper bound $c$ on the absolute value of the process and mixing coefficients $\varphi_n$ then

$$\sum_{i=1}^{n} |EX_{\tau_i} - \mu| \leq 2cn\varphi_1.$$

Proof. Consider first $\tau_1$. $\tau_1$ is independent of any $X_t$ since for $U \in \sigma(X_t), V \in \sigma(\tau_1) = \{\emptyset, \Omega\}$ we have either $V = \Omega$ and $P(U \cap V) = P(U) = P(U)P(V)$ or $V = \emptyset$ and $P(U \cap V) = 0 = P(U)P(V)$. Independence and stationarity of the process $\langle X_t \rangle_{t \in \mathbb{Z}}$ give us

$$EX_{\tau_1} = EX_t \times \chi\{\tau_1 = t\} = P(\tau_1 = t)E(X_t) = P(\tau_1 = t)E(X_1).$$

Now, since the $X_t$ are bounded and $\tau_1$ attains a value in $\mathbb{N}_+$ we know that

$$\sum_{i=1}^{\infty} E|X_i| \times \chi\{\tau_1 = t\} = \sum_{i=1}^{\infty} P(\tau_1 = t)E|X_1|$$

is finite and B. Levi’s Theorem tells us that $\sum_{i=1}^{\infty} |X_i| \times \chi\{\tau_1 = t\}$ is integrable. Also $|\sum_{t=1}^{t'} X_t \times \chi\{\tau_1 = t\}|$ is upper bounded by the integrable function $\sum_{t=1}^{\infty} |X_t| \times \chi\{\tau_1 = t\}$ for all $t' \in \mathbb{N}_+$ and Lebesgue’s Dominated Convergence Theorem gives us

$$EX_{\tau_1} = E\sum_{t=1}^{\infty} X_t \times \chi\{\tau_1 = t\} = \sum_{t=1}^{\infty} EX_t \times \chi\{\tau_1 = t\} = \sum_{t=1}^{\infty} P(\tau_1 = t)EX_t = EX_1.$$

We perform induction over $\tau_i, i \geq 2$. The set $\{\tau_i = t\}$ is an element of $\mathcal{G}_{t-1}$ and for any $s < t, s, t \in \mathbb{N}_+$ we have that $B = \{\tau_i = t\} \cap \{\tau_{i-1} = s\} \in \mathcal{G}_{t-1}$. Observe that Lemma 3 tells us that $B \in \mathcal{F}_s$ and, hence, that

$$\int_B E(X_{\tau_i}|\mathcal{G}_{t-1}) = \int_B X_t = \int_B E(X_t|\mathcal{F}_s).$$
This implies because of stationarity that
\[
\left| \int_B (E(X_{\tau_i}|G_{t-1}) - E(X_t)) \right| = \left| \int_B (E(X_t|F_s) - E(X_1)) \right| \leq 2c \varphi_{t-s} P(B).
\]

Because \(\tau_i\) is \(G_{t-1}\) measurable we can now write
\[
EX_{\tau_i} = EE(X_{\tau_i}|G_{t-1}) = \sum_{t=1}^{\infty} EE(X_t \times \chi\{\tau_i = t\}|G_{t-1})
\]
\[
= \sum_{t=1}^{\infty} E(E(X_t|G_{t-1}) \times \chi\{\tau_i = t\})
\]
\[
= \sum_{t=1}^{\infty} E(E(X_t|G_{t-1}) \times \chi\{\tau_i = t\} \cap \bigcup_{s=1}^{t-1} \{\tau_{i-1} = s\})
\]
\[
= \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} E(E(X_t|G_{t-1}) \times \chi\{\tau_i = t\} \cap \{\tau_{i-1} = s\})
\].

The infinite sum can be moved outside with the same argument as above using B. Levi’s Theorem and Lebesgue’s Dominated Convergence Theorem for the expectation operator and for the conditional expectation operator. Finally, \(|EX_{\tau_i} - EX_1|\) equals
\[
\left| \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} (E(E(X_t|G_{t-1}) \times \chi\{\tau_i = t\} \cap \{\tau_{i-1} = s\}) - P(\tau_i = t, \tau_{i-1} = s) E(X_1)) \right|
\]
\[
\leq 2c \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \varphi_{t-s} P(\tau_i = t, \tau_{i-1} = s) \leq 2c \varphi_1.
\]

The main result of this section is given by Proposition 8 below.

**Proposition 8** *(this is Proposition 3 in the main text.).* Assume that we have \(k\) arms such that each is associated to a bounded stationary pay-off sequence and the joint process is \(\varphi\)-mixing. Let \(\mu_1, \ldots, \mu_k\) be the means of the stationary distributions of the \(k\) stochastic processes and \(c_1, \ldots, c_k\) upper bounds on their absolute value. Furthermore, let \(\mu^* = \max\{\mu_1, \ldots, \mu_k\}\) and \(c^* = \max\{c_1, \ldots, c_k\}\) then for every \(n \geq 1\) we have
\[
v_n^* - n\mu^* \leq n\varphi_1 c^*.
\]

**Proof.** Denote by \(G_t\) the filtration that keeps track of all the information available up to time \(t\). More specifically, let \(\langle G_t \rangle_{t \geq 0}, G_0 = \mathcal{N}\) and \(G_t = \sigma(\bigcup_{i=1}^{t} \bigcup_{s=1}^{t} \sigma(X_{s,i}) \cup \mathcal{N})\) for \(t \geq 1\), where \(\mathcal{N} \subset \mathcal{A}\) is the family of sets of \(P\)-measure zero. Consider an arbitrary policy \(\langle \pi_t \rangle_{t \in \mathbb{N}_+}\) and an arbitrary \(t \in \mathbb{N}_+\) then
\[
EX_{t,\pi_t} = \sum_{j=1}^{k} \sum_{i=1}^{t} EX_{t,j} \times \chi\{\tau_{i,j} = t\}
\]
and since \( \tau_{i,j} \) is \( \mathcal{G}_{t-1} \)-measurable we have with \( B = \{ \tau_{i,j} = t \} \) that

\[
EX_{t,j} \times \chi\{\tau_{i,j} = t\} = EE(X_{t,j}|\mathcal{G}_{t-1}) \times \chi\{\tau_{i,j} = t\} = \int_B E(X_{t,j}|\mathcal{G}_{t-1}).
\]

We can extend the \( \varphi \)-mixing property of the joint process to \( \mathcal{G}_t \) by applying Lemma 4 and we get

\[
\left| \int_B (E(X_{t,j}|\mathcal{G}_{t-1}) - EX_{1,j}) \right| \leq 2c^*\varphi_1 P(B).
\]

Since the different sets \( \{\tau_{i,j} = t\}, j \in \{1, \ldots, k\}, i \in \{1, \ldots, t\} \) are disjoint

\[
EX_{t,\pi_t} - \mu^* \leq \sum_{j=1}^k \sum_{i=1}^t (EX_{t,j} \times \chi\{\tau_{i,j} = t\} - P(\tau_{i,j} = t) EX_{1,j}) \leq 2c^*\varphi_1 \sum_{j=1}^k \sum_{i=1}^t P(\tau_{i,j} = t)
\leq 2c^*\varphi_1.
\]

\( \square \)

### A.4 Details of Example 1

We give in this section the details of the construction in Example 1. This example demonstrates that the sequence \( \{X_{\tau_n}\}_{n \in \mathbb{N}} \) of samples of one of the arms of a two armed bandit problem does not need to be \( \varphi \)-mixing even though the original process is \( \varphi \)-mixing. The argument uses standard results from Markov chains as they can be found in Levin et al. (2008) and a well known perturbation result for Markov chains. We provide the details of the argument for completeness.

Assume we have a two arm bandit problem where the pay-off for arm two is zero at all time and the pay-off distribution of arm one is described by a Markov chain with two states, transition probabilities \( p_{11} = p_{22} = 1 - \epsilon, p_{12} = p_{21} = \epsilon \), for some \( \epsilon \in (0,1) \), and probability \( 1/2 \) to be in state 1 at time \( t = 0 \). The player gains a pay-off of 1 if the Markov chain is in state 1 and a pay-off of 0 if the Markov chain is in state 2. The Markov chain is irreducible and aperiodic. Furthermore, the Markov chain induces a stationary pay-off distribution and the bandit problem is \( \varphi \)-mixing with mixing coefficients \( \varphi_k \) being upper bounded by \( \varphi_k \leq (1 - 2\epsilon)^k \) : one can derive the particular bound by considering an eigendecomposition of the transition matrix \( T \) which yields eigenvalues \( \lambda_1 = 1, \lambda_2 = 1 - 2\epsilon \) and eigenvectors \( u_1 = \sqrt{2}(1/2 \quad 1/2)^T, u_2 = \sqrt{2}(1/2 \quad -1/2)^T \), i.e. with \( U = (u_1 \quad u_2) \) and \( \Lambda \) being the diagonal matrix with entries \( \lambda_1 \) and \( \lambda_2 \) we have \( T = U\Lambda U^T \). The stationary distribution over the states is \( s = (1/2 \quad 1/2)^T \) and for any vector \( v = (v_1 \quad v_2)^T, v_1, v_2 \geq 0, v_1 + v_2 = 1 \), we have \( \Lambda^kU^Tv = (1/\sqrt{2})(1 \quad (1 - 2\epsilon)^k(v_1 - v_2))^T \). Hence, \( T^kv - s = (1/2)(1 - 2\epsilon)^k(v_1 - v_2)(1 \quad -1)^T \) and \( \|T^kv - s\|_\infty \leq (1/2)(1 - 2\epsilon)^k \). The mixing coefficients can now be bounded in the following way. Let \( X_{t,i}, t \geq 1, i \in \{1,2\} \), be random variables that represent the pay-off of arm \( i \) gained at time \( t \). Consider a particular realisation where \( X_{1,1} = x_1, \ldots, X_{n,1} = x_n \) and \( X_{n+k,1} = x_{n+k}, \ldots, X_{n+k+m,1} = x_{n+k+m} \) for some \( x_1, \ldots, x_n, x_{n+k+1}, \ldots, x_{n+k+m} \in \{0,1\} \) then \( P(X_{1,1} = x_1, \ldots, X_{n,1} = x_n, X_{n+k,1} = x_{n+k}, \ldots, X_{n+k+m,1} = x_{n+k+m}) = P(X_{1,1} = x_1, \ldots, X_{n,1} = x_n, X_{n+k,1} = x_{n+k}, \ldots, X_{n+k+m,1} = x_{n+k+m}) \).
\begin{align*}
\pi(x_1, \ldots, x_{n+1}) = \pi(x_1, \ldots, x_{n+1}) \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n) \text{ and } \\
|\mathbb{P}(X_{n+1} = x_{n+1}, \ldots, X_{n+k+m,1} = x_{n+k+m} \mid X_n = x_n) - \mathbb{P}(X_{n+1} = x_{n+k}, \ldots, X_{n+k+m,1} = x_{n+k+m} \mid X_n = x_n)| \\
= \mathbb{P}(X_{n+k+1,1} = x_{n+k+1}, \ldots, X_{n+k+m,1} = x_{n+k+m} \mid X_{n+k,1} = x_{n+k}) \\
\times |\mathbb{P}(X_{n+k,1} = x_{n+k}) - \mathbb{P}(X_{n+k,1} = x_{n+k} \mid X_n = x_n)| \\
\leq (1/2)(1 - 2\epsilon)^k \mathbb{P}(X_{n+k+1,1} = x_{n+k+1}, \ldots, X_{n+k+m,1} = x_{n+k+m} \mid X_{n+k,1} = x_{n+k}) \\
= (1 - 2\epsilon)^k \mathbb{P}(X_{n+k,1} = x_{n+k}, \ldots, X_{n+k+m,1} = x_{n+k+m}).
\end{align*}

since \( \mathbb{P}(X_{n+k,1} = x_{n+k}) = 1/2 \). Let \( A = \{1, \ldots, n\}, B = \{n + k, \ldots, n + k + m\} \) and consider \( \sigma(X_A), \sigma(X_B) \). The events in these \( \sigma \)-algebras are finite unions of events of the form \( \{X_{1,1} = x_1, \ldots, X_{n,1} = x_n\} \) and \( \{X_{n+k,1} = x_k, \ldots, X_{n+k+m,1} = x_{n+k+m}\} \).

For any \( U \in \sigma(X_A) \) we know that \( U \) consists at most of finite many such events \( U_1, \ldots, U_l \), \( U_i \cap U_j = \emptyset \), for all \( i, j \leq l \). Similarly for \( V \in \sigma(X_B) \) we know that \( V = V_1 \cup \ldots \cup V_o, V_i \cap V_j = \emptyset \), for all \( i, j \leq o \). The above argument allows us to conclude that for any \( U \in \sigma(X_A) \) and \( V \in \sigma(X_B) \)

\[ |\mathbb{P}(U \cap V) - \mathbb{P}(U \cap V)| = \sum_{i \leq l} \sum_{j \leq o} |\mathbb{P}(U_i)\mathbb{P}(V_j) - \mathbb{P}(U_i \cap V_j)| \]

\[ \leq (1 - 2\epsilon)^k \sum_{i \leq l} \sum_{j \leq o} \mathbb{P}(U_i)\mathbb{P}(V_j) \leq (1 - 2\epsilon)^k. \]

The argument works in the same way for sets \( A \) and \( B \) that do not consist of consecutive indices. Hence the Markov chain is \( \varphi \)-mixing and since the second arm is producing a constant reward of 0 we also know that the bandit problem is \( \varphi \)-mixing with the same mixing coefficient.

Now fix some \( \delta > 0 \) and consider the following policy \( \pi^\delta \). At \( t = 1 \) the policy plays arm 1 receiving pay-off \( X_{1,1} \). Then at any other \( t \geq 2 \) the arm is selected according to the following rules: if \( t - 1 \) arm 1 has been played and \( X_{t-1,1} = X_{1,1} \) then the policy chooses at \( t \) arm 1; if at \( t - 1 \) arm 1 has been played and \( X_{t-1,1} \neq X_{1,1} \) then the policy chooses at \( t \) arm 2 and plays arm 2 for the next \( k := \lceil \log(2\delta) / \log(1 - 2\epsilon) \rceil \) rounds before switching back to arm 1. We like to show that the sequence of pay-offs generated by policy \( \pi^\delta \) at arm 1 is not \( \varphi \)-mixing. Let \( \tau_1, \tau_2, \ldots \) be the sequence of random times at which arm 1 is played (by construction \( \tau_1 = 1 \) and \( \tau_2 = 2 \)). The evolution of the process \( \langle X_{\tau_n} \rangle_{n \geq 1} \) can also be described by transition matrices. The transition probabilities to move from a state at \( t = 1 \) to a state at \( t = 2 \) are just the probabilities summarized in \( T \). For all other \( t \) \((t \geq 2)\) the transition matrix is either

\[ \tilde{T} = \begin{pmatrix} 1 - \epsilon & \epsilon \\ \frac{T^k}{21} & \frac{T^k}{22} \end{pmatrix} \text{ or } \quad \bar{T} = \begin{pmatrix} (T^k)_{11} & (T^k)_{12} \\ \epsilon & 1 - \epsilon \end{pmatrix} \]

depending on \( X_{\tau_1} \), i.e. if \( X_{\tau_1} = 1 \) then the former is describing the evolution and if \( X_{\tau_1} = 0 \) the latter is the one describing the evolution of the Markov chain. In the following we discuss the case that \( X_{\tau_1} = 1 \), but the same arguments apply to the case \( X_{\tau_1} = 0 \). We can observe that

\[ \|v^\top (\tilde{T} - \bar{T})\|_\infty \leq \delta \text{ for all } v \text{ with non-negative entries and } v_1 + v_2 = 1, \]

where

\[ \tilde{T} = \begin{pmatrix} 1 - \epsilon & \epsilon \\ 1/2 & 1/2 \end{pmatrix} \]
in case that $X_{\tau_1} = 1$. The claim can be verified through

$$
\|v^T \tilde{T} - v^T \hat{T}\|_\infty = \|v^T \left( \begin{array}{cc} 0 & 0 \\ \tilde{T}_{11} - 1/2 & \tilde{T}_{22} - 1/2 \end{array} \right) \|_\infty
= v_2 \left\| T^k \left( \begin{array}{c} 0 \\ 1/2 \end{array} \right) \right\|_\infty \leq v_2 \delta \leq \delta.
$$

The Markov chains associated to $\tilde{T}$ and $\hat{T}$ are both irreducible and aperiodic. This implies in particular the existence of stationary distributions, with the associated probabilities to be in state one and two summarized in vectors $\bar{s}, \bar{s} \in [0, 1]^2$, and the convergence of $\tilde{T}^l, \hat{T}^l$ to $\bar{s}, \bar{s}$ in $l$ (measured in $\|\cdot\|_\infty$, Levin et al. (2008)(Thm. 4.9)). In particular, there exist constants $\bar{c}, \bar{c} > 0$ and $\bar{\alpha}, \bar{\alpha} \in (0, 1)$ such that for all $l \geq 1$ and any vector $v \in [0, 1]^2$ with $v_1 + v_2 = 1$

$$
\|v^T \tilde{T}^l - \bar{s}\|_\infty \leq \bar{c} \bar{\alpha}^l \quad \text{and} \quad \|v^T \hat{T}^l - \bar{s}\|_\infty \leq \bar{c} \bar{\alpha}^l.
$$

We can also calculate the stationary distribution of $\hat{T}$ explicitly to get $\hat{s} = (1/(1+2\epsilon), 2\epsilon/(1+2\epsilon))^T$.

It is known that Markov chains with slightly perturbed transition matrices have similar stationary distributions. Due to Cho and Meyer (2001) there exists a constant $c > 0$ that is only dependent on $\hat{T}$ (and independent of $\tilde{T}$) such that $\|\tilde{s} - \hat{s}\|_\infty \leq c\|\tilde{T} - \hat{T}\|_{\infty, 1} \leq 2\delta c$ where $\|\tilde{T} - \hat{T}\|_{\infty, 1} := \max_{i \in \{1, 2\}} \sum_j |\tilde{T}_{i,j} - \hat{T}_{i,j}| \leq 2\delta$. Combining these inequalities yields $\|v^T \hat{T}^l - \bar{s}\|_\infty \leq 2\delta c + \bar{c} \bar{\alpha}^l$ for any $v$ with non-negative entries and $v_1 + v_2 = 1$.

Consider now the event $U = \{X_{\tau_1} = 1\}$ and $V = \{X_{\tau_2} = 1\}$ for $n \geq 2$. $P(U) = 1/2 = P(V)$ where the second equality follows from $2P(X_{\tau_1} = 0) = P(X_{\tau_1} = 0|X_{\tau_1} = 0) + P(X_{\tau_1} = 0|X_{\tau_1} = 1) = P(X_{\tau_1} = 1|X_{\tau_1} = 1) + P(X_{\tau_1} = 1|X_{\tau_1} = 0) = 2P(X_{\tau_1} = 1)$. Furthermore, with $\tilde{T}$ being the matrix defined on the left side of (4)

$$
\left| \frac{P(U \cap V)}{P(U)} - \frac{1}{1+2\epsilon} \right| = \left| (1-\epsilon) \tilde{T}^{n-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) - \frac{1}{1+2\epsilon} \right|
\leq \left\| (1-\epsilon) \tilde{T}^{n-1} - \frac{1}{1+2\epsilon} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\|_\infty
$$

and the last term is upper bounded by $2\delta c + \bar{c} \bar{\alpha}^{n-1}$. Recalling that $c$ does not depend on $\tilde{T}$ and, hence, not on $\delta$ we see that we can make the term $2\delta c$ arbitrary small. Furthermore, by considering a large $n$ we can make the second term arbitrary small. In particular, let $\epsilon = 1/10$ then there exists a $\pi^4$ and an $N \in \mathbb{N}$ such that for all $n \geq N$

$$
\left| \frac{P(U \cap \{X_{\tau_1} = 1\})}{P(U)} - \frac{1}{1+2\epsilon} \right| \leq 1/10.
$$

Hence, for all $n \geq N$

$$
|P(U)P(X_{\tau_1} = 1) - P(U \cap \{X_{\tau_1} = 1\})| \geq P(U) \left( \frac{1}{1+2\epsilon} - \frac{1}{1+2\epsilon} - 1/10 \right)
\geq P(U)/5
$$

and the process is not $\varphi$-mixing.
A.5 Regret Analysis of Algorithm 1

In this Section we give a proof for Theorem 1. The proof depends on some technical lemmas stated and proved below. The following definition will be used in the proofs. Let \( G_t \) denote the filtration that keeps track of all the information available up to time \( t \). That is, \( \langle G_t \rangle_{t \geq 0} \), \( G_0 = 0 \) and \( G_t = \sigma(\bigcup_{i=1}^k \bigcup_{s=1}^{t} \sigma(X_{s,i}) \cup N) \) for \( t \geq 1 \), where \( N \subset A \) is the family of sets of \( P \)-measure zero.

**Lemma 6** (this is Lemma 1 in the main text.). Consider a \( k \)-armed bandit problem described in 3.1, where we have \( k \) arms each with a bounded stationary pay-off sequence such that the joint process is \( \varphi \)-mixing. For a fixed \( i \in 1..k \) and \( s \in \mathbb{N} \), consider the consecutive samples

\[
X_{\tau,i}, X_{\tau+1,i}, \ldots, X_{\tau+2^s,i},
\]

where \( \tau : \Omega \to \mathbb{N}_+ \) is a random time at which the \( i \)-th arm is sampled. Let \( \mu_i \) denote the stationary mean of arm \( i \). We have

\[
|\mu_i - \frac{1}{2^s} \sum_{j=0}^{2^s-1} E X_{\tau+j,i}| \leq \frac{1}{2^{s-1}} \| \varphi \|
\]

**Proof.** Observe that the event \( \{ \tau = t \} \) is thus measurable with respect to \( G_{t-1} \) for all \( t \in \mathbb{N}_+ \), where \( G_{t-1} \) defined above is the filtration that keeps track of all the information available up to time \( t \). Note that \( X_{\tau,i} = \sum_{t \in \mathbb{N}_+} X_{t,i} \chi\{\tau = t\} \). For simplicity of notation we denote \( X_{t,i} \) by \( X_t \) and \( X_{\tau,i} \) by \( X_\tau \). Note that by stationarity \( \mu_i = E X_t \).

We have,

\[
2^s \mu_i - \sum_{j=0}^{2^s-1} E X_{\tau+j,i} = \sum_{t \in \mathbb{N}_+} \sum_{j=0}^{2^s-1} E \chi\{\tau = t\} X_{t+j} - E \chi\{\tau = t\} E X_t \leq \sum_{t \in \mathbb{N}_+} \sum_{j=0}^{2^s-1} E(\chi\{\tau = t\} X_{t+j} | G_{t-1}) - E \chi\{\tau = t\} E X_t \leq \sum_{t \in \mathbb{N}_+} \sum_{j=0}^{2^s-1} E(\chi\{\tau = t\} | E(X_{t+j} | G_{t-1}) - E X_t) \leq \sum_{t \in \mathbb{N}_+} \sum_{j=0}^{2^s-1} 2 \varphi_j E \chi\{\tau = t\} \| X \|_\infty \leq 2 \| \varphi \|
\]

where (5) follows from the fact that the event \( \{ \tau = t \} \) is \( G_{t-1} \)-measurable, and (6) follows from Proposition 7 and the assumption that \( \| X \|_\infty = 1 \).

\( \square \)
Proposition 9 (this is Proposition 4 in the main text.). Let \( R(n) := n\mu^* - \sum_{t=1}^{n} E X_{t,\pi_t} \) be the expected regret of Algorithm 1 with respect to the best stationary mean \( \mu^* \) after \( n \) rounds of play. We have
\[
R(n) \leq \sum_{j=1}^{k} \Delta_j E T_j(n) + 2k \left( \sum_{\ell=1}^{n} \varphi_{\ell} \right) \log n
\]
where \( T_j(n) \) denotes the number of times that arm \( j \) has been played in \( n \) rounds.

Proof. Denote by \( \tau_{i,j} : \Omega \to \mathbb{N}_+ \) the random time at which the \( j^{th} \) arm is sampled for the \( i^{th} \) time. Note that for any \( t \in \mathbb{N} \) the event \( \{ \tau_{i,j} = t \} \) is measurable with respect to the filtration \( \mathcal{G}_{t-1} \) that keeps track of all the information available up to time \( t \). First note that
\[
E(\chi_{\{\tau_{i,j} = t\}} X_{j,t+1}) = E E(\chi_{\{\tau_{i,j} = t\}} X_{j,t+1} | \mathcal{G}_{t-1}) = E (\chi_{\{\tau_{i,j} = t\}} E (X_{j,t+1} | \mathcal{G}_{t-1})) \\
\geq (\mu_j - 2 \|X_j\|_{\infty} \varphi_t) P(\{\tau_{i,j} = t\})
\]
where the second equality follows from the fact that the event \( \{\tau_{i,j} = t\} \) is \( \mathcal{G}_{t-1} \)-measurable and (7) follows from Proposition 7. We have,
\[
R(n) = n\mu^* - \sum_{t=1}^{n} E X_{t,\pi_t} \\
= n\mu^* - \sum_{t=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{\log n \min\{2^m,n-t\}} \sum_{\ell=1}^{\log n \min\{2^m,n-t\}} E(\chi_{\{\tau_{m,j} = \ell\}} X_{j,t+\ell}) \\
\leq n\mu^* - \sum_{t=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{\log n \min\{2^m,n-t\}} \sum_{\ell=1}^{\log n \min\{2^m,n-t\}} P(\{\tau_{m,j} = \ell\}) (\mu_j - 2 \|X_j\|_{\infty} \varphi_t) \\
\leq n\mu^* - \sum_{t=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{\log n \min\{2^m,n-t\}} \sum_{\ell=1}^{\log n \min\{2^m,n-t\}} P(\{\tau_{m,j} = \ell\}) \mu_j + 2k \|X\|_{\infty} \left( \sum_{\ell=1}^{n} \varphi_{\ell} \right) \log n \\
= n\mu^* - \sum_{j=1}^{k} \mu_j E T_j(n) + 2k \|X\|_{\infty} \left( \sum_{\ell=1}^{n} \varphi_{\ell} \right) \log n
\]
where the third inequality follows from (7).

of Theorem 1. We assume that \( \max_{j \in 1..k} \|X_j\|_{\infty} = 1 \); extension to other bounded-valued processes is straightforward. Thanks to Proposition 9, to bound the regret \( R(n) \) it suffices to calculate the expected number of times \( T_i(n) \) that a suboptimal arm is played in \( n \) rounds. For any \( s, t \in \mathbb{N}_+ \), let \( c_{t,s} := \sqrt{\frac{8(\frac{1}{2^s} + \ln t)}{2^s}} + \|\varphi\|_{2^s} \), where \( \zeta = 1 + 4 \|\varphi\| \). Recall that Algorithm 1 plays its selected arms in batches of exponentially growing length so that if arm \( j \) for \( j \in 1..k \) is selected at round \( t \), it is played for \( 2^s_j(t) \) consecutive time-steps, where \( s_j(t) \) denotes the number of times that arm \( j \) has been selected up-to time \( t \). Below, we denote by \( X_{j,s} := \frac{1}{s} \sum_{t'=t}^{t+2^s-1} X_{j,t'} \) with \( s := s_j(t) \), the Algorithm’s estimate of the stationary mean of arm \( j \) selected at time \( t \). As usual,
a superscript "\(^{*}\)" refers to the quantities for the arm with the highest stationary mean. Fix some \(l \in \mathbb{N}_+\). We have,

\[
T_i(n) = 1 + \sum_{t=k+1}^{n} \chi\{\pi_t = i\}
\]

\[
\leq 2^{l+1} + \sum_{t=2^{l+1}+1}^{n} \log t \sum_{m=t+1}^{2^l} 2^m \chi\{\tau_{m,i} = t\}
\]

\[
\leq 2^{l+1} + \sum_{t=2^{l+1}+1}^{n} \log t \sum_{m=t+1}^{2^l} 2^m \chi\{\min_{s=1..\log t} \overline{X}_s^* + c_{t,s} \leq \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} + c_{t,m-1}\}
\]

\[
\leq 2^{l+1} + \sum_{t=1}^{\infty} \sum_{m=t+1}^{2^l} \sum_{s=1}^{2^m} 2^m \chi\{\overline{X}_s^* + c_{t,s} \leq \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} + c_{t,m-1}\}
\]

For every \(t \in \mathbb{N}\) and every \(s \in 1..\log t\) we have that \(\overline{X}_s^* + c_{t,s} \leq \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} + c_{t,m-1}\) implies that

\[
\overline{X}_s^* \leq \mu^* - c_{t,s} \quad (8)
\]

\[
\frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} \geq \mu_i + c_{t,m-1} \quad (9)
\]

\[
\mu^* < \mu_i + 2c_{t,m-1} \quad (10)
\]

Now, observe that for a fixed \(t \in 1..n\) we have,

\[
P(\overline{X}_s^* \leq \mu^* - c_{t,s}) \leq P\left(\overline{X}_s^* - \mu^* \geq c_{t,s}\right)
\]

\[
\leq P\left(\sum_{j=0}^{2^s-1} X_{\tau_{s+j}}^* - E X_{\tau_{s+j}}^* \geq c_{t,s}\right) + \frac{2^{s-1}}{2} \left(\sum_{j=0}^{2^s-1} (E X_{\tau_{s+j}}^* - \mu^*) \geq 2^s c_{t,s}\right)
\]

\[
\leq P\left(\sum_{j=0}^{2^s-1} X_{\tau_{s+j}}^* - E X_{\tau_{s+j}}^* \geq 2^s c_{t,s} - 2 \|\varphi\|\right) \quad (11)
\]

\[
\leq \sqrt{e} \exp\left\{-\frac{(2^s c_{t,s} - 2 \|\varphi\|)^2}{2^{s+1} \zeta}\right\} \quad (12)
\]

\[
\leq t^{-4} \quad (13)
\]

where, (11) follows from Lemma 6 and (12) follows from a Hoeffding-type bound for \(\varphi\)-mixing processes given by Corollary 2.1 of Rio (1999). Moreover, noting that \(\|\varphi\| \geq 0\) we similarly
obtain,
\[
P\left( \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} \geq \mu_i + c_{t,m-1} \right) \leq P\left( \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} \geq \mu_i + c_{t,m-1} - \frac{2}{2^{m-1}} \|\varphi\| \right)
\]
\[
\leq P\left( \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} - 2^{m-1} \mu_i \geq 2^{m-1} c_{t,m-1} - 2 \|\varphi\| \right)
\]
\[
\leq t^{-4}.
\]
(14)

Let \( L := \log \frac{32(1+4\|\varphi\|)\ln n}{\Delta_i^2} \). Since, for \( t \geq 2^L + 1 \) and every \( m \geq L + 1 \) we have \( \mu^* - \mu_i - 2c_{t,m-1} \geq 0 \), it follows that (10) is false for all \( m \geq L + 1 \). Therefore we have,

\[
E[T_i(n)] \leq 2^{L+1} + \sum_{t=1}^{\infty} \sum_{m=L+1}^{\infty} \sum_{s=1}^{\log t} P\left( \overline{X}_s \leq \mu^* - c_{t,s}, \frac{1}{2^{m-1}} \sum_{u=t}^{t+2^{m-1}-1} X_{i,u} \geq \mu_i + c_{t,m-1} \right)
\]
\[
\leq 2^{L+1} + \sum_{t=1}^{\infty} \sum_{m=L+1}^{\infty} \sum_{s=1}^{\log t} 2^{m+1} t^{-4}
\]
\[
\leq \frac{32(1 + 4\|\varphi\|)\ln n}{\Delta_i^2} + 1 + 2\pi^2/3
\]

where (15) follows from (13) and (14).

\[ \square \]

### B Strongly dependent reward distributions

#### B.1 Basic Bounds

Consider two independent and normal distributed random variables \( X, Y \) with mean \( \mu_X > \mu_Y \) and variance \( \sigma_X^2, \sigma_Y^2 \). Let \( \Delta = \mu_X - \mu_Y > 0 \) and \( \sigma^2 = \sigma_X^2 + \sigma_Y^2 \). The following bounds are used in the main text.

\[
E(Y - X)^+ \leq \sigma \phi(\Delta/\sigma),
E(Y - X)^+ \geq 0,
E(X - Y)^+ \leq \sigma \phi(\Delta/\sigma) + \Delta,
E(X - Y)^+ \geq \Delta.
\]

The derivation is based on basic properties of Gaussian random variables. Recall that \( Z = X - Y \) is normal distributed with mean \( \Delta \) and variance \( \sigma^2 \). Therefore,

\[
\sqrt{2\pi\sigma} E(X - Y)^+ = \int_{0}^{\infty} z \exp \left( -\frac{(z - \Delta)^2}{2\sigma^2} \right) dz
\]
\[
= \int_{-\Delta}^{\infty} z \exp \left( -\frac{z^2}{2\sigma^2} \right) + \Delta \int_{-\Delta}^{\infty} \exp \left( -\frac{z^2}{2\sigma^2} \right) dz
\]
\[
= \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) + \Delta \sigma \int_{-\infty}^{\Delta/\sigma} \exp \left( -\frac{z^2}{2} \right) dz.
\]
Standard bounds on the cdf, as can be found in Dudley (2002)[Lem. 12.1.6], yield the following lower bound
\[
\sqrt{2\pi\sigma} E(X - Y)^+ = \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) + \Delta \sigma \sqrt{2\pi} \left( 1 - \frac{1}{\sqrt{2\pi}} \int_{\Delta/\sigma}^{\infty} \exp \left( -\frac{z^2}{2} \right) \right)
\geq \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) + \Delta \sigma \sqrt{2\pi} \left( 1 - \frac{\sigma}{\sqrt{2\pi}\Delta} \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) \right)
= \Delta \sigma \sqrt{2\pi}.
\]
Similarly, if we consider \( Z = Y - X \) which has mean \(-\Delta\)
\[
\sqrt{2\pi\sigma} E(Y - X)^+ = \int_0^\infty z \exp \left( -\frac{(z + \Delta)^2}{2\sigma^2} \right) = \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) - \Delta \sigma \int_\Delta/\sigma^\infty \exp \left( -\frac{z^2}{2} \right).
\]
Applying the result from Dudley (2002)[Lem. 12.1.6] leads here to the trivial lower bound 0.
We use the following inequalities to gain upper bounds on \( E(Y - X)^+ \) and \( E(X - Y)^+ \). Let \( Z \) be a standard normal random variable then
\[
\Pr(Z \geq c) \geq \begin{cases} 
\phi(c)/(2c) & \text{if } c \geq 1, \\
\phi(c)(1-c)/2 & \text{if } 0 \leq c < 1.
\end{cases}
\]
The first bound can be found in Dudley (2014). The second bound is a straightforward adaptation of the techniques used to derive the first bound. Applying these bounds we get the following upper bounds on \( E(Y - X)^+ \). If \( \Delta/\sigma \geq 1 \) then
\[
\sqrt{2\pi\sigma} E(Y - X)^+ \leq \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) - \Delta \sigma \frac{\sigma}{2\Delta} \exp \left( -\frac{\Delta^2}{2\sigma^2} \right)
= \frac{\sigma^2}{2} \exp \left( -\frac{\Delta^2}{2\sigma^2} \right)
\]
and if \( 0 \leq \Delta/\sigma < 1 \)
\[
\sqrt{2\pi\sigma} E(Y - X)^+ \leq \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) - \Delta \sigma \frac{1 - \Delta/\sigma}{2} \exp \left( -\frac{\Delta^2}{2\sigma^2} \right)
= (\sigma^2 - \Delta\sigma/2 + \Delta^2/2) \exp \left( -\frac{\Delta^2}{2\sigma^2} \right)
\leq \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right)
\]
The upper bounds on \( E(X - Y)^+ \) are gained in the same way. For \( \Delta/\sigma \geq 1 \) these are
\[
\sqrt{2\pi\sigma} E(X - Y)^+ \leq \sigma^2 \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) - (\sigma^2/2) \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) + \Delta \sigma \sqrt{2\pi}
= \frac{\sigma^2}{2} \exp \left( -\frac{\Delta^2}{2\sigma^2} \right) + \Delta \sigma \sqrt{2\pi}
\]
and for $0 \leq \Delta/\sigma \leq 1$

$$\sqrt{2\pi\sigma}E(X - Y)^+ \leq \sigma^2 \exp \left(-\frac{\Delta^2}{2\sigma^2}\right) - \frac{\Delta\sigma(1 - \Delta/\sigma)}{2} \exp \left(-\frac{\Delta^2}{2\sigma^2}\right) + \Delta\sigma\sqrt{2\pi}$$

$$= (\sigma^2 - \Delta\sigma/2 + \Delta^2/2) \exp \left(-\frac{\Delta^2}{2\sigma^2}\right) + \Delta\sigma\sqrt{2\pi}.$$

### B.2 Proof of Proposition 5

We bound the regret of the two phases individually. Some technical steps are moved further below to streamline the discussion.

**Regret of Phase I.** We sweep through all $k$ arms at times $1$ to $k$, $m + 1$ to $m + k$, etc. During each of these phases we build up regret. This regret can be bounded by

$$\sum_{i=1}^{k} \sum_{j \leq k} E(X_{m+i,i} - X_{m+j,j})^+ \leq (k - 1) \sum_{i=1}^{k} \Delta_i + 2(k - 1) \sum_{i=1}^{k} \phi(\Delta_i/2).$$

where we used the bounds from Equation 1 and the observation that these bounds are maximized if we replace $j$ with the arm with the highest stationary mean.

**Regret of Phase II.** To control the regret building up in the second phase we condition on the observations in a sweep at time $lm$, $l \in \mathbb{N}$, i.e. on the observed pay-offs $x_1, \ldots, x_k$. Arm $i^*$ is selected such that $x_{i^*} \geq \max_{i \leq k} x_i$ and this arm is played for $m - k$ steps. We need to control $E(\max_{i \leq k}(X_{lm+i,i} - X_{lm+i',i'})^+)$ for $lm + k + 1 \leq t' \leq (l + 1)m$. Due to stationarity this is equal to $E(\max_{i \leq k}(X_{t,i} - X_{t,i})^+), k + 1 \leq t \leq m$. The following holds for the latter expression.

$$E(\max_{i \leq k}(X_{t,i} - X_{t,i})^+) \leq \sum_{i=1}^{k} \sum_{i \neq i^*} \int X\{u = i^*\} \int (X_{t,i} - X_{t,u})^+ dP_{x_i,x_u} dP$$

where we used $P_{x_i,x_u}$ for the conditional distribution of $X_{t,i}$ and $X_{t,u}$ given the values $X_{t,i} = x_i$ and $X_{t,u} = x_u$ (see below in Appendix B.2.1 for details). The inner integral can be bounded by

$$\int (X_{t,i} - X_{t,u})^+ dP_{x_i,x_u} \leq \phi \left(\frac{\Delta_{t,i}}{\sigma_{t,i}}\right),$$

where $\Delta_{t,i} = E(X_{t,u} - X_{t,i}|X_{t,i} = x_i, X_{t,u} = x_u)$. Similarly, $\sigma_{t,i}^2$ denotes the conditional variance, i.e. $\sigma_{t,i}^2 = E(X_{t,u}|X_{t,u} = x_u) - E(X_{t,u}|X_{t,u} = x_u)^2 + E(X_{t,i}|X_{t,i} = x_i) - E(X_{t,i}|X_{t,i} = x_i)^2$. With this notation and with $\tilde{\Delta}_i = x_u - x_i$, we have the following lower bound on $\Delta_{t,i}$.

$$\Delta_{t,i} = (\mu_u - \mu_i)(1 - \text{cov}(t-k)) + \tilde{\Delta}_i \text{cov}(t-k) + \varepsilon(t) \geq \tilde{\Delta}_i - c(t-k)^\alpha (\Delta + \tilde{\Delta}_i) - |\varepsilon(t)|.$$
with \( \varepsilon(t) \) being a term which can be bounded by \( |\varepsilon(t)| \leq (\Delta + \tilde{\Delta}_i)ck^\alpha \) (see Appendix B.2.2). We also have an upper bound on \( \sigma^2_{t,i} \). Using \( \text{cov}(u) \geq 1 - cu^\alpha \),

\[
\sigma^2_{t,i} = 2 - \text{cov}^2(t - u) - \text{cov}^2(t - i) \leq 2(1 - \text{cov}^2(t - k)) \leq 4c(t - k)^\alpha.
\]

Combining these we gain

\[
\int (X_{t,i} - X_{t,u})^+ dP_{x_i,x_u} \leq (2\pi)^{-1/2} \exp \left( -\frac{(\tilde{\Delta}_i - c((t - k)^\alpha + k^\alpha)(\tilde{\Delta}_i + \Delta))^2}{8c(t - k)^\alpha} \right)
\]

\[
= (2\pi)^{-1/2} \exp \left( -\frac{\tilde{\Delta}_i^2}{8c(t - k)^\alpha} \left( 1 - c((t - k)^\alpha + k^\alpha)(1 + \frac{\Delta}{\tilde{\Delta}_i}) \right)^2 \right).
\]

The bound is maximized for \( t = m \). Finally, by integrating over \( x_i \) and \( x_u \) we gain

\[
\sum_{i=k+1}^{m} E\left( \max_{i\leq k} (X_{t,i} - X_{t,\pi_t})^+ \right) \leq (m-k)k(k-1) \frac{\sqrt{a}}{8\pi(1-b)} \left( \sqrt{8\pi} - (1 - \sqrt{b}\Delta) \exp \left( -\frac{b\Delta^2}{8} \right) \right),
\]

where \( a = 8c(m-k)^\alpha \) and \( b = c((m-k)^\alpha + k^\alpha) \). See Appendix B.2.3 for the calculations.

**Combined Regret.** The combined regret for each \( m \) steps, using \( \Delta_i \leq \Delta \), is bounded by

\[
k(k-1) \left( \Delta + \sqrt{2/\pi} + (m-k) \frac{\sqrt{a_m}}{8\pi(1-b_m)} \left( \sqrt{8\pi} - (1 - \sqrt{b_m}\Delta) \exp \left( -\frac{b_m\Delta^2}{8} \right) \right) \right),
\]

where we added the index \( m \) to \( a \) and \( b \) to highlight the dependence. Given a time horizon \( T \) we have \([T/m]\) many iterations of phase I and II and the overall regret is bounded by

\[
(T/m + 1)k(k-1) \left( \Delta + \sqrt{2/\pi} + \frac{(m-k)\sqrt{a_m}}{8\pi(1-b_m)} \left( \sqrt{8\pi} - (1 - \sqrt{b_m}\Delta) \exp \left( -\frac{b_m\Delta^2}{8} \right) \right) \right)
\]

\[
= (T + m)k(k-1) \left( \Delta + \sqrt{2/\pi} \right) \left( \frac{\sqrt{a_m}}{8\pi(1-b_m)} \left( \sqrt{8\pi} - (1 - \sqrt{b_m}\Delta) \exp \left( -\frac{b_m\Delta^2}{8} \right) \right) \right).
\]

Instead of trying to find the \( m \) that minimizes this equation we optimize \( m \) over the considerably simpler expression

\[
\frac{\Delta + \sqrt{2/\pi}}{m} + \frac{\sqrt{a_m}}{\sqrt{8\pi}}.
\]

(16)

The \( (1 - b_m) \) term that we leave out is of minor relevance since \( b_m \) is small. The negative term in the bracket is not larger than 1 and by ignoring this term we lose only another constant. Minimizing eq. 16 with respect to \( m \) yields

\[
m^* = \left\lceil \left( \frac{2\Delta + \sqrt{8/\pi}}{\sqrt{8c\alpha}} \right)^{2/\pi} \right\rceil.
\]

Combining the above steps proves the following proposition.
B.2.1 Conditioning

Due to stationarity and since our policy depends only on the observations at the last sweep we have that $E(\max_{i \leq k} (X_{t',i} - X_{t', \pi_t})) = E(\max_{i \leq k} (X_{t,i} - X_{t, \pi_t}))$ for any $t'$, $lm + k + 1 \leq t' \leq lm$, $l \in \mathbb{N}$, and corresponding $t = t' - lm$. Now, consider any $t$, $k + 1 \leq t \leq m$ and, using $i^*$ for the choice of arm given the observations $X_{1,1}, \ldots, X_{k,k}$, rewrite the regret in the following way

$$E\left(\max_{i \leq k} (X_{t,i} - X_{t, \pi_t})^+ \right) = \sum_{u=1}^{k} E\left(\max_{i \neq u} \chi\{u = i^*\} \times (X_{t,i} - X_{t,i^*})^+ \right)$$

$$= \sum_{u=1}^{k} E\left(\max_{i \neq u} \chi\{u = i^*\} \times (X_{t,i} - X_{t,u})^+ \right) \leq \sum_{u=1}^{k} \sum_{i \neq u} E\left(\chi\{u = i^*\} \times (X_{t,i} - X_{t,u})^+ \right)$$

$$= \sum_{u=1}^{k} \sum_{i \neq u} \int \chi\{u = i^*\} \times (X_{t,i} - X_{t,u})^+ dP_{x_i, x_u} d\mu(x_i, x_u)$$

$$= \sum_{u=1}^{k} \sum_{i \neq u} \int \chi\{u = i^*\} \int (X_{t,i} - X_{t,u})^+ dP_{x_i, x_u} dP.$$

B.2.2 Bound on $\varepsilon(t)$.

The term $\varepsilon(t)$ that we introduced is

$$\varepsilon(t) = \mu_{i^*} (\text{cov}(t - k) - \text{cov}(t - i^*)) - \mu_i (\text{cov}(t - k) - \text{cov}(t - i)) + x_{i^*} (\text{cov}(t - i^*) - \text{cov}(t - k)) - x_i (\text{cov}(t - i) - \text{cov}(t - k)).$$

Since, due to our Hölder assumption, $|\text{cov}(t - k) - \text{cov}(t - i^*)| \leq ck^\alpha$ and $|\text{cov}(t - k) - \text{cov}(t - i)| \leq ck^\alpha$ we have the following bound.

$$|\varepsilon(t)| \leq (\Delta + \tilde{\Delta}) ck^\alpha.$$

B.2.3 Integration.

Recall that $\tilde{\Delta}_i = X_u - X_i$ is normal distributed with mean $\mu_u - \mu_i$ and variance 2. The multiplication with $\chi\{u = i^*\}$ implies, in particular, that we need to integrate $\tilde{\Delta}_i$ only over the positive real line. Writing

$$f(X_u, X_i) = (2\pi)^{-1/2} \exp \left( -\frac{\tilde{\Delta}_i^2}{8c(t - k)^\alpha} \left( 1 - c((t - k)^\alpha + k^\alpha) \left( 1 + \frac{\Delta}{\tilde{\Delta}_i} \right) \right)^2 \right),$$

we have

$$\int \chi\{u = i^*\} \int (X_{t,i} - X_{t,u})^+ dP_{x_i, x_u} dP \leq \int \chi\{X_u \geq X_i\} \times f(X_u, X_i) dP.$$
Multiplying the density of $\tilde{\Delta}_i$ with $f$ and using $a = 8c(t - k)^\alpha$, $b = c((t - k)^\alpha + k^\alpha)$, $d = (2\pi)^{-1}2^{-1/2}$, and $\Delta_{u,i} = \mu_u - \mu_i$, yields for $\Delta < \sqrt{a/(b\sqrt{2})}$ that

$$d \int_0^\infty \exp \left( -\frac{(\tilde{\Delta}_i(1 - b) - \Delta b)^2}{a} - \frac{(\tilde{\Delta}_i - \Delta_{u,i})^2}{4} \right) d\tilde{\Delta}_i$$

$$= \frac{d}{1 - b} \int_{-\Delta_b}^\infty \exp \left( -\frac{\tilde{\Delta}_i}{a} \right) d\tilde{\Delta}_i$$

$$= \frac{\sqrt{a}d}{(1 - b)\sqrt{2}} \int_{-\infty}^{\sqrt{2}\Delta_b/\sqrt{a}} \exp \left( -\frac{\tilde{\Delta}_i^2}{2} \right) d\tilde{\Delta}_i$$

$$= \frac{\sqrt{a}d\sqrt{2}\pi}{(1 - b)\sqrt{2}} \left( 1 - \frac{1}{\sqrt{2}\pi} \int_{\sqrt{2}\Delta_b/\sqrt{a}}^\infty \exp \left( -\frac{\tilde{\Delta}_i^2}{2} \right) d\tilde{\Delta}_i \right)$$

$$\leq \frac{\sqrt{a}d\sqrt{2}\pi}{(1 - b)\sqrt{2}} \left( 1 - \frac{1 - \sqrt{2}\Delta b/\sqrt{a}}{2\sqrt{2}\pi} \exp \left( -\frac{\Delta_{u,i}^2}{a} \right) \right)$$

$$\leq \frac{\sqrt{a}}{8\pi(1 - b)} \left( \sqrt{8\pi} - \left( 1 - \frac{\sqrt{2b}\Delta}{\sqrt{a}} \right) \exp \left( -\frac{b\Delta_{u,i}^2}{8} \right) \right)$$

and for $m \geq 2k$ this can be further bounded by

$$\frac{\sqrt{a}}{8\pi(1 - b)} \left( \sqrt{8\pi} - \left( 1 - \sqrt{2b}\Delta \right) \exp \left( -\frac{b\Delta_{u,i}^2}{8} \right) \right).$$