A logarithmic estimate for the inverse source scattering problem with attenuation in a two-layered medium

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Abstract. The paper aims a logarithmic stability estimate for the inverse source problem of the one-dimensional Helmholtz equation with attenuation factor in a two layer medium. We establish a stability by using multiple frequencies at the two end points of the domain which contains the compact support of the source functions.

Keywords: Inverse source problems, scattering theory, exact observability.

Mathematics Subject Classification(2000): 35R30; 35J05; 35B60; 33C10; 31A15; 76Q05; 78A46

1 Introduction and problem formulation

Inverse source problem arises in many areas of science. It has numerous applications to surface vibrations, acoustical and biomedical/medical imaging, antenna synthesis, geophysics, and material science ([2, 3]). It has been known that the data of the inverse source problems for Helmholtz equations with single frequency can not guarantee the uniqueness ([14], Ch.4). On the other hand, various studies, for instance in [13], showed that the uniqueness can be regained by taking multi-frequency boundary measurement in a non-empty frequency interval $(0, K)$ noticing the analyticity of wave-field on the frequency. Because of the wide applications, these problems have attracted considerable attention. For example, In the papers [9, 10] sharp results were obtained in sub-domains of $\mathbb{R}^3$ and $\mathbb{R}^2$ with a possibility of handling spatially variable coefficients. An iterative/recursive algorithm was developed for recovering unknown sources in [4, 5, 6]. In papers [11, 16], authors considered Helmholtz equation with damping factor. Authors in [21], improved the stability for the source when the domain is a disk/ball. In [19] the uniform logarithmic stability with respect to the wave numbers for continuation of the Helmholtz equation from the unit disk onto any larger disk was studied and recently [1, 15, 17] showed the increasing stability for continuation problems.
with large wave number under (pseudo) convexity conditions on the domain. We also have to mention that in [12] inverse source problem was considered for classical elasticity system.

In particular attenuation can have various reasons and in application one of the fundamental reasons of poor resolution in inverse problems is a spatial decay of the signal due in part to the damping factor. The main purpose of this paper is to study the dependence of increasing stability on the constant attenuation (damping) coefficient in the inverse scattering source problems. Our result in agreement with the result of paper [20], if the damping factor becomes zero. To achieve our goal we used analytic continuation, Carleman estimates for damped wave equation and exact observability bounds for hyperbolic equations which was recently developed in [9]. In this paper, we assume that the medium is homogeneous in the whole space. Here we try to establish a stability estimate to recover of the source functions for the inverse source problem for the one-dimensional Helmholtz equation in a two-layered medium with attenuation factor \( \alpha \). In this paper, the damping factor is considered the same for both layers of medium.

In this paper both functions \( f_0 \in H^2((-1, 1)), f_1 \in H^1((-1, 1)) \) are assumed to be zero outside our domain and \( \text{supp} f_0 \cup \text{supp} f_1 \subset (-1, 1) \). In this work for simplicity we used \( \partial \Omega \) instead of our boundary which is \( \{1, -1\} \). We consider the following attenuated Helmholtz equation in a two-layered medium

\[
\frac{d^2 u}{dx^2} + (k^2(x) + i\alpha k(x))u = -f_1 - \alpha f_0 + ik f_0, \quad x \in (-1, 1),
\]

with the exponential decay at infinity condition

\[
|u(x)| + |u'(x)| \leq C(u)e^{\delta(u)|x|},
\]

where \( \delta(u) > 0 \) and wave number \( k \) defines as follows

\[
k(x) = \begin{cases} 
    k_p & \text{if } x > 0 \\
    k_n & \text{if } x < 0,
\end{cases}
\]

Our goals are uniqueness and stability of the functions \( f_0, f_1 \) from the Dirichlet data. Now let

\[
u^*(x, \kappa) := u(x, k), \quad \kappa^2 := k^2 + \alpha ki,
\]

then the equation (1) becomes

\[
u''^* + \kappa^2 u^* = -f_1 - \alpha f_0 + ik f_0,
\]

and also we can reformulate (3) as follows

\[
\kappa(x) = \begin{cases} 
    \kappa_p = (k_p^2 + \alpha k_p \omega)^{1/2} & \text{if } x > 0 \\
    \kappa_n = (k_n^2 + \alpha k_n \omega)^{1/2} & \text{if } x < 0,
\end{cases}
\]

Using standard formulation of Helmholtz equation, \( \kappa_p = c_p \omega, \kappa_n = c_n \omega, \omega > 0 \) is the angular frequency and \( c_p, c_n \) are constants.

It is well-known that equation (5) has the following unique solution provided \( f_1 \in L^2(-1, 1), f_0 \in H^1(-1, 1) \):
\[ u^*(x, \omega) = \int_0^1 G(x, y)(-f_1 - \alpha f_0 + ik f_0)(y)dy, \quad (7) \]

where \( G(x, y) \) is the Green function given as follows

\[
G(x, y) = \begin{cases} 
\frac{i \kappa_p - \kappa_n}{2 \kappa_p (\kappa_p + \kappa_n)} e^{i \kappa_p (x + y)} + \frac{i}{2 \kappa_p} e^{i \kappa_p |x - y|}, & \text{if } x > 0, \\
\frac{1}{\kappa_p + \kappa_n} e^{i (\kappa_p y - \kappa_n x)}, & \text{if } x < 0,
\end{cases}
\]

and

\[
G(x, y) = \begin{cases} 
\frac{i}{2 \kappa_p (\kappa_p + \kappa_n)} e^{-i \kappa_p (x + y)} + \frac{i}{2 \kappa_p} e^{i \kappa_p |x - y|}, & \text{if } x > 0, \\
\frac{1}{\kappa_p + \kappa_n} e^{i (-\kappa_n y + \kappa_p x)}, & \text{if } x < 0,
\end{cases}
\]

To see more detail for the Green function see [20]. The following logarithmic estimate states our main result.

**Theorem 1.1.** There exists a generic constant \( C \) depending on the domain \((-1,1)\) such that

\[
\| f_0 \|_2^2 (-1,1) + \| f_1 \|_2 (-1,1) \leq Ce^{C\alpha^2 \left( \epsilon^2 + \frac{(\alpha^2 + 1)M^2}{K^2 E^2 + 1} \right)}, \quad (8)
\]

for all \( u \in H^2((-1,1)) \) solving (1) with \( K > 1 \). Here

\[
\epsilon^2 = \int_0^K \omega^2 (|u(1, \omega)|^2 + |u(-1, \omega)|^2) d\omega,
\]

\( E = -\ln \epsilon \) and \( M = \max \{ \| f_0 \|_2 (-1,1) + \| f_1 \|_1 (-1,1), 1 \} \) where \( \| . \|_l ((-1,1)) \) is the standard Sobolev norm in \( H^l((-1,1)) \).

**Remark 1.1.** Estimate (8) implies that for any fixed \( \alpha \), the stability of \( f_1, f_0 \) from the boundary data is improving with growing \( K \). In another words, the problem becomes more stable when higher frequency data is used, but it also implies that larger attenuation will deteriorate this improvement. The right hand-side of the estimate (8) consists of two parts: data discrepancy and the high frequency tail. There is some numerical evidence that when \( K \) grows, the functions \( f_0, f_1 \) will have better resolution. In addition, our estimate is a proof for the uniqueness of the inverse source functions as \( \epsilon \to 0 \).

## 2 Increasing Stability of Continuation to higher frequencies

To proof our main theorem, let’s define the following functions:

\[
f_{1p} = \begin{cases} 
f_1 & \text{if } x > 0, \\
0 & \text{if } x < 0,
\end{cases} \quad f_{1n} = \begin{cases} 
0 & \text{if } x > 0, \\
f_1 & \text{if } x < 0,
\end{cases}
\]

\[
f_{0p} = \begin{cases} 
f_0 & \text{if } x > 0, \\
0 & \text{if } x < 0,
\end{cases} \quad f_{0n} = \begin{cases} 
0 & \text{if } x > 0, \\
f_0 & \text{if } x < 0.
\end{cases}
\]
And also defining

\[ I(k) = I_1(k) + I_2(k) \]

where

\[ I_1(k) = \int_0^1 \omega^2 |u(-1, \omega)|^2 d\omega, \quad I_2(k) = \int_0^1 \omega^2 |u(1, \omega)|^2 d\omega, \]

using (13), we can show that

\[ \omega u(-1, \omega) = \int_0^1 \frac{i}{c_1 + c_2} e^{i(c_1 \omega y + c_2 \omega)} (-f_{1p} - \alpha f_{0p} + ikf_{0p})(y) dy \]

\[ + \int_{-1}^0 \frac{i(c_2 - c_1)}{2c_2(c_1 + c_2)} e^{-ic_2 \omega(-1+y)} (-f_{1n} - \alpha f_{0n} + ikf_{0n})(y) dy \]

\[ + \int_{-1}^0 \frac{i}{2c_2} e^{-ic_2 \omega(-1-y)} (-f_{1n} - \alpha f_{0n} + ikf_{0n})(y) dy \]

and

\[ \omega u(1, \omega) = \int_0^1 \frac{i(c_1 - c_2)}{2c_1(c_1 + c_2)} e^{ic_1 \omega(1+y)} (-f_{1p} - \alpha f_{0p} + ikf_{0p})(y) dy \]

\[ + \int_0^1 \frac{i}{2c_1} e^{ic_2 \omega(1-y)} (-f_{1p} - \alpha f_{0p} + ikf_{0p})(y) dy \]

\[ + \int_{-1}^0 \frac{i}{c_1 + c_2} e^{i(-c_2 \omega y - c_1 \omega)} (-f_{1n} - \alpha f_{0n} + ikf_{0n})(y) dy. \]

Functions \( I_1 \) and \( I_2 \) are both analytic with respect to the wave number \( k \in \mathbb{C} \setminus [0, -i\alpha] \) and play important roles in relating the inverse source problems of the Helmholtz equation and the Cauchy problems for the wave equations.

**Lemma 2.1.** Let \( \text{supp} f_0, \text{supp} f_1 \in (-1, 1) \) and \( f_0 \in \mathcal{H}^0(-1, 1), f_1 \in \mathcal{H}^0(-1, 1) \). Then

\[ |I_1(k)| \leq C \left( |k| \| f_1 \|_0^2 (-1, 1) + (|k| |\alpha|^2 + \frac{|k|^3}{3}) \| f_0 \|_0^2 (-1, 1) \right) e^{4c_{\max} (4k_1 + \alpha)}, \]

\[ |I_2(k)| \leq C \left( |k| \| f_1 \|_0^2 (-1, 1) + (|k| |\alpha|^2 + \frac{|k|^3}{3}) \| f_0 \|_0^2 (-1, 1) \right) e^{4c_{\max} (4k_1 + \alpha)}, \]

**Proof.** Since we have \( \kappa = \kappa_1 + \kappa_2 i = \sqrt{k^2 + \alpha k} \) is complex analytic on \( \mathbb{C} \setminus [0, -\alpha i] \) and in particular on the set \( S \setminus [0, k] \), where \( S \) is the sector \{ \arg k < \pi/4 \} with \( k = k_1 + ik_2 \). It is easy to see that \( |\kappa| = |k| \frac{1}{2} |k + \alpha i| \frac{1}{2} \leq 2k_1^{1/2}(k_1^{1/2} + \alpha)^{1/2} \) and \( |\kappa| \leq \sqrt{2}k_1 \leq \sqrt{2} |\kappa| \) for any \( k \) in \( S \). By a simple calculation, we can show that

\[ I_1(k) = \int_0^1 \left( \int_0^1 \frac{1}{c_1 + c_2} e^{i(c_1 \omega y + c_2 \omega)} (-f_{1p} - \alpha f_{0p} + ikf_{0p})(y) dy \right) \]

\[ + \int_{-1}^0 \frac{(c_2 - c_1)}{2c_2(c_1 + c_2)} e^{-ic_2 \omega(1+y)} (-f_{1n} - \alpha f_{0n} + ikf_{0n})(y) dy \]
\[ + \int_{-1}^{0} \frac{1}{2c_2} e^{-ic_2\omega(-1-y)} (-f_{0n} - \alpha f_0 + k f_0)(y) dy \]

and

\[
I_2(k) = \int_{0}^{k} \left| \int_{0}^{1} \frac{(c_1 - c_2)}{2c_1(c_1 - c_2)} e^{i(c_1\omega y + c_2\omega)} (-f_{1p} - \alpha f_0 + k f_0)(y) dy \right| + \int_{0}^{1} \frac{1}{2c_1} e^{-ic_2\omega(1-y)} (-f_{1p} - \alpha f_0 + k f_0)(y) dy
\]
\[
+ \int_{-1}^{0} \frac{1}{2c_2} e^{-ic_2\omega(-1-y)} (-f_{0n} - \alpha f_0 + k f_0)(y) dy \right| d\omega.
\]

Since the integrands in (13) and (14) are analytic functions of \(k\) in \(S\), their integrals with respect to \(\omega\) can be taken over any path in \(S\) joining points 0 and \(k\) in the complex plane. Using the change of variable \(\omega = ks\), \(s \in (0, 1)\) in the line integral (14) we obtain

\[
|I_1(k)| \leq \int_{0}^{1} |k| \int_{0}^{1} \frac{1}{c_1 + c_2} e^{i(c_1\omega y + c_2\omega)} (-f_{1p} - \alpha f_0 + ik f_0)(y) dy
\]
\[
+ \int_{-1}^{0} \frac{(c_2 - c_1)}{2c_2(c_1 + c_2)} e^{-ic_2\omega(1+y)} (-f_{0n} - \alpha f_0 + k f_0)(y) dy
\]
\[
+ \int_{-1}^{0} \frac{1}{2c_2} e^{-ic_2\omega(-1-y)} (-f_{0n} - \alpha f_0 + k f_0)(y) dy \right| d\omega,
\]

using the following inequalities for \(y \in (-1, 1)\)

\[
|e^{\pm i\omega(c_1 y + c_2)}| \leq e^{2c_{\text{max}}|\kappa_2|}, \quad |e^{\pm i\omega(\pm y-1)}| \leq e^{2c_{\text{max}}|\kappa_2|},
\]

it is easy to drive that

\[
|I_1(k)| \leq \int_{0}^{1} |k| \int_{-1}^{1} \left| |f_1(y)| + (\alpha + s|k|)|f_0(y)|e^{2c_{\text{max}}|\kappa_2|} dy \right|^2 ds,
\]

integrating with respect to \(s\), using the bound for |\(\kappa\)| in \(S\) and trivial inequality \(2ab \leq a^2 + b^2\), we complete the proof of (13).

Similarly for \(y \in (-1, 1)\) we have

\[
|e^{\pm i\omega c_2(\pm y-1)}| \leq e^{2c_{\text{max}}|\kappa_2|}, \quad |e^{\pm i\omega(c_1 - c_2 y)}| \leq e^{2c_{\text{max}}|\kappa_2|},
\]

using the same technique for \(I_2(k)\), proof for (12) is complete.

The following steps are essential to link the unknown \(I_1(k)\) and \(I_2(k)\) for \(k \in [K, \infty)\) to the known value \(\epsilon\) in (11). Obviously

\[
|I_1(k)e^{-16c_{\text{max}}k}| \leq C \left( |k| \| f_1 \|_{(0)}^2 (-1, 1) + \left( |k| |\alpha|^2 + \frac{|k|^3}{3} \right) \| f_0 \|_{(0)}^2 (-1, 1) \right)e^{4c_{\text{max}}\alpha},
\]
\[
\leq C \alpha^2 e^{4c_{\text{max}}\alpha} M^2,
\]

\[
\text{5}
\]
with $M = \max \{ \| f_1 \|_{(0)}^2 (-1, 1) + \| f_0 \|_{(0)}^2 (-1, 1), 1 \}$. With the similar argument bound (15) is true for $I_2(k)$. Observing that
\[
|I_1(k) e^{-16c_{\max} k}| \leq \epsilon^2, \quad |I_2(k) e^{-16c_{\max} k}| \leq \epsilon^2 \text{ on } [0, K].
\]
Let $\mu(k)$ be the harmonic measure of the interval $[0, K]$ in $\mathbb{S}\backslash [0, K]$, then as known (for example see [13], p.67), from two previous inequalities and analyticity of the function $I_1(k) e^{-16c_{\max} k}$ and $I_2(k) e^{-16c_{\max} k}$ we conclude that
\[
|I_1(k) e^{-16c_{\max} k}| \leq C \alpha e^{-4c_{\max} \alpha} \epsilon^2 \mu(k) M^2, \quad (16)
\]
when $K < k < +\infty$. Similar arguments also yield for
\[
|I_2(k) e^{-16c_{\max} k}| \leq C \alpha e^{-4c_{\max} \alpha} \epsilon^2 \mu(k) M^2, \quad (17)
\]
hence
\[
|I(k) e^{-16c_{\max} k}| \leq C \alpha e^{-4c_{\max} \alpha} \epsilon^2 \mu(k) M^2. \quad (18)
\]

To achieve a lower bound of the harmonic measure $\mu(k)$, we use the following technical lemma. The proof can be found in [9].

**Lemma 2.2.** If $0 < k < 2^{\frac{1}{4}} K$, then
\[
\frac{1}{2} \leq \mu(k). \quad (19)
\]
If $2^{\frac{1}{4}} K < k$, then
\[
\frac{1}{\pi} \left( \left( \frac{k}{K} \right)^4 - 1 \right) ^{\frac{1}{4}} \leq \mu(k). \quad (20)
\]

### 2.0.1 Exact observability bound for wave equation with damping factor

In order to use the bound for higher frequency, we consider the hyperbolic initial value problem
\[
\partial_t^2 U(x, t) - U''(x, t) + \alpha \partial_t U(x, t) = 0 \text{ on } (-1, 1) \times (0, \infty), \quad U(x, 0) = f_0, \quad \partial_t U(x, 0) = f_1 \quad \text{on } (-1, 1). \quad (21)
\]
By assuming $\alpha = 0$, the exact observability bounds for the hyperbolic equation can be found in [9] [10]. The following theorem presents a generalized result which is of its own interest.

**Theorem 2.3.** Let the observation time $4(D + 1) < T < 5(D + 1)$. Then there exists a generic constant $C$ depending on the domain $\Omega$ such that
\[
\| f_0 \|_{(1)}^2 (\Omega) + \| f_1 \|_{(0)}^2 (\Omega) \leq C e^{C \alpha^2} \left( \| \partial_t U \|_{(0)}^2 (\partial \Omega \times (0, T)) + \| U \|_{(0)}^2 (\partial \Omega \times (0, T)) \right), \quad (22)
\]
for all $U \in H^2((-1, 1) \times (0, \infty))$ solving (3.1).

**Proof.** Proof is in ([16], Lemma 3.3 and Theorem 3.1).
3 Logarithmic stability for inverse source problem

To proceed, we consider the hyperbolic initial value problem

\[ \partial_t^2 U - U'' + \alpha \partial_t U = 0 \text{ on } \mathbb{R} \times (0, \infty), \quad U(x, 0) = f_0, \quad \partial_t U(x, 0) = f_1 \text{ on } \mathbb{R} \]  

(23)

Defining \( U(x, t) = 0 \) for \( t < 0 \). We claim that the solution of (1) coincides with the Fourier transform of \( U \);

\[ u(x, k) = \int_{-\infty}^{\infty} U(x, t) e^{ikt} dt. \]

(24)

Known results in [7] (see Theorem 1.1. and Theorem 1.2.), [22] and the assumption on the functions \( f_0, f_1 \) imply that

\[ \| U(., t) \|_{(0)} \leq C(f_0, f_1)(1 + t)^{-\frac{1}{4}}, \]

so the Fourier transform (24) is well defined. To prove the claim, the following steps are essential.

Defining function \( u_*(x, k) \) as the right hand side of (24);

\[ u_*(x, k) = \frac{1}{\sqrt{\pi}} \int_{|x|}^{\infty} U(x, t) e^{ikt} dt. \]

(25)

We observe that due to speed of the propagation, \( U(x, t) = 0 \) when \( 1 + tc_{\text{max}} < |x|, \quad x \in (-1, 1) \) (see [18]). Using integration by parts and (23), it is easy to see that

\[
0 = \int_0^{\infty} (\partial_t U(., t) - U(., t)'' + \alpha \partial_t U(., t)) e^{ikt} dt
\]

\[
= -\partial_t U(., 0) - \int_0^{\infty} ik \partial_t U(., t) e^{ikt} dt - \int_0^{\infty} U(., t)'' e^{ikt} dt - \alpha U(., 0)
\]

\[
-\alpha ik \int_0^{\infty} U(., t) e^{ikt} dt
\]

\[
= -\partial_t U(., 0) - \alpha U(., 0) + ik U(., 0) - \int_0^{\infty} (k^2 U(., t) + U(., t)'' + iak U(., t)) e^{ikt} dt.
\]

All above holds when \( k = k_1 + ik_2, \quad k_2 > 0 \). Considering the well known integral representation of the solution \( U \) when \( k_2 > 0 \) for large \( |x| \) as in [8], p. 695 and [22], \( U \) decays for large \( t \) (Bessel functions of first kind are bounded functions).

Hence above bound for \( U \) combined with the exponential decay of \( e^{ikt} \) with respect to \( t \) for \( k_2 > 0 \) implies an exponential decay of \( u_*(x, k) \) when \( |x| \) is getting large. By standard argument from stationary scattering theory, function \( u(x, k) \) decays exponentially. Hence we conclude that \( u(x, k) = u_*(x, k) \) or (24) for all \( k_2 > 0 \). Due to the \( L^2 \)-continuity of both side with respect to \( k_2 \) we conclude (24) for \( k_2 = 0 \).

To proceed the estimate for reminders of the whole integrands in (13) and (14) for \((k, \infty)\), we need the following lemma.

**Lemma 3.1.** Let \( u \) be a solution to the forward problem (1) with \( f_1 \in H^1((-1, 1)) \) and \( f_0 \in H^2((-1, 1)) \) with \( \text{supp} f_0, \text{supp} f_1 \subset (-1, 1) \), then

\[
\int_k^{\infty} \omega^2 |u(-1, \omega)|^2 d\omega + \int_k^{\infty} \omega^2 |u(1, \omega)|^2 d\omega 
\]

\[
\leq Ck^{-1} \left( (1 + \alpha^2) \| f_0 \|_{L^2} (-1, 1) + \| f_1 \|_{L^1} (-1, 1) \right),
\]

(26)
Proof. Using (9) and (10), we obtain

\[
\int_{k}^{\infty} \omega^2 |u(-1, \omega)|^2 d\omega + \int_{k}^{\infty} \omega^2 |u(1, \omega)|^2 d\omega
\]

(27)

\[
\leq \int_{k}^{\infty} \int_{0}^{1} e^{ic_1 \omega y} (-f_{1p} - \alpha f_{0p} + k f_{0p})(y) dy \bigg| d\omega + \int_{k}^{\infty} \int_{0}^{1} e^{-ic_1 \omega y} (-f_{1p} - \alpha f_{0p} + k f_{0p})(y) dy \bigg| d\omega
\]

\[
+ \int_{k}^{\infty} \int_{-1}^{0} e^{ic_2 \omega y} (-f_{1n} - \alpha f_{0n} + k f_{0n})(y) dy \bigg| d\omega + \int_{k}^{\infty} \int_{-1}^{0} e^{-ic_2 \omega y} (-f_{1n} - \alpha f_{0n} + k f_{0n})(y) dy \bigg| d\omega
\]

(28)

Using integration by parts and the fact that \( f_0 \) and \( f_1 \) are vanished at the endpoints, we have

\[
\int_{0}^{1} e^{\pm ic_1 \omega y} f_{1p}(y) dy = \frac{1}{\pm ic_1 \omega} \int_{0}^{1} e^{\pm ic_1 \omega y} f'_{1p}(y) dy
\]

\[
(\alpha + k) \int_{0}^{1} e^{\pm ic_1 \omega y} f_{0p}(y) dy = \frac{(\alpha + k)}{(\pm ic_1 \omega)^2} \int_{0}^{1} e^{\pm ic_1 \omega y} f''_{0p}(y) dy
\]

\[
\int_{-1}^{0} e^{\pm ic_2 \omega y} f_{1n}(y) dy = \frac{1}{\pm ic_2 \omega} \int_{-1}^{0} e^{\pm ic_2 \omega y} f'_{1n}(y) dy
\]

\[
(\alpha + k) \int_{-1}^{0} e^{\pm ic_2 \omega y} f_{0n}(y) dy = \frac{(\alpha + k)}{(\pm ic_2 \omega)^2} \int_{-1}^{0} e^{\pm ic_2 \omega y} f''_{0n}(y) dy,
\]

consequently for the first and second terms in (28) we obtain

\[
\left| \int_{0}^{1} e^{\pm ic_1 \omega y} f_{1p}(y) dy \right|^2 \leq \frac{C}{\omega^2} \| f_{1p} \|_{(1)}^2 (0, 1) \leq \frac{C}{\omega^2} \| f_{1p} \|_{(1)}^2 (-1, 1)
\]

and

\[
\left| \int_{0}^{1} (\alpha + k) e^{ic_1 \omega y} f_{0p}(y) dy \right|^2 \leq \frac{C(\alpha^2 + k^2)}{\omega^4} \| f_{0p} \|_{(2)}^2 (0, 1) \leq \frac{C(\alpha^2 + k^2)}{\omega^4} \| f_{0p} \|_{(2)}^2 (-1, 1),
\]

repeating the argument for the other terms in (28) and integrating with respect to \( \omega \) the proof is complete.

\[\square\]
Remark 2.1. Obviously, the following inequality holds
\[ \int_0^\infty \omega^2 \| u(\omega) \|^2_{(0)} (\partial \Omega) d\omega \leq k^{-2} \int_{k<|\omega|} \omega^4 \| u(\omega) \|^2_{(0)} (\partial \Omega) d\omega \leq \]
\[ k^{-2} \int_{\mathbb{R}} \omega^4 \| u(\omega) \|^2_{(0)} (\partial \Omega) d\omega = 2\pi k^{-2} \int_{\mathbb{R}} \| \partial_t^2 U(t) \|^2_{(0)} (\partial \Omega) dt \]
by the Parseval’s identity.

Finally, we are ready to establish the increasing stability estimate of Theorem 1.1

**Proof of Theorem 1.1.**

Proof. Without loss of generality, we can assume that \( \varepsilon < 1 \) and \( 3\pi E^{-\frac{2}{3}} < 1 \), otherwise the bound (1.1) is obvious. Let

\[ k = \begin{cases} K \frac{4}{3} E^{\frac{1}{3}} & \text{if } 2^{\frac{2}{3}} K^{\frac{4}{3}} < E^{\frac{1}{3}} \\ K & \text{if } E^{\frac{2}{3}} \leq 2^{\frac{2}{3}} K^{\frac{4}{3}}, \end{cases} \]  

(29)

if \( E^{\frac{2}{3}} \leq 2^{\frac{2}{3}} K^{\frac{4}{3}} \), then \( k = K \) and

\[ |I_1(k)| \leq 2\varepsilon^2. \]  

(30)

If \( 2^{\frac{2}{3}} K^{\frac{4}{3}} < E^{\frac{1}{3}} \), we can assume that \( E^{-\frac{1}{3}} < \frac{1}{4\pi} \), otherwise \( C < E \) and hence \( K < C \) and the bound (3) is straightforward. From (29), (20), Lemma 2.2, (1.6) and the equality \( \varepsilon = e^{-E} \) we obtain

\[ |I_1(k)| \leq CM^2 \alpha^2 e^{4\alpha} e^{\frac{2\pi}{\varepsilon} \left( (\frac{1}{4})^4 - 1 \right)^{-\frac{1}{2}}} \]

\[ \leq CM^2 e^{4\alpha} \alpha^2 e^{-\frac{2\pi}{\varepsilon} K^{\frac{4}{3}} E^{\frac{1}{3}} (1 - \frac{5\pi}{2} E^{-\frac{1}{3}})}, \]

using the trivial inequality \( e^{-t} \leq \frac{6}{t} \) for \( t > 0 \) and the assumption at the beginning of the proof, we conclude that

\[ |I_1(k)| \leq CM^2 \alpha^2 e^{2\alpha} \frac{1}{K^2 E^{\frac{4}{3}} (1 - \frac{5\pi}{2} E^{-\frac{1}{3}})^3}. \]  

(31)

Using (29), (30), (31), and Lemma 4.1 we obtain

\[ \int_0^\infty \omega^2 |u(-1, \omega)|^2 d\omega + \int_0^\infty \omega^2 |u(1, \omega)|^2 d\omega \]

\[ \leq I(k) + \int_k^\infty \omega^2 |u(-1, \omega)|^2 d\omega + \int_k^\infty \omega^2 |u(1, \omega)|^2 d\omega \]

\[ \leq 2\varepsilon^2 + C \left( \frac{(\alpha^2 + 1) M^2 e^{4\alpha}}{K^2 E^{\frac{4}{3}}} + \frac{\alpha^2 + 1}{K^{\frac{4}{3}} E^{\frac{1}{3}} + 1} \right) \cdot \]

Using (32) and Theorem 3.1, we finally obtain

\[ \| f_1 \|^2_{(0)} (\Omega) + \| f_0 \|^2_{(1)} (\Omega) \leq Ce^{\alpha^2} \left( \| \partial_t U \|^2_{(0)} (\partial \Omega \times (0,T)) + \| U \|^2_{(0)} (\partial \Omega \times (0,T)) \right) \]
\[
\leq C e^{\alpha^2} \left( \| \partial_t U \|_{(0)}^2 (\partial \Omega \times (0, \infty)) + \| U \|_{(0)}^2 (\partial \Omega \times (0, \infty)) \right)
\leq C e^{\alpha^2} \left( \varepsilon^2 + \frac{(\alpha^2 + 1) M^2 e^{8\alpha}}{K^2 E^{\frac{2}{3}}} + \frac{(\alpha^2 + 1) || f_0 ||_2^2 + || f_1 ||_1^2}{K^2 E^{\frac{2}{3}} + 1} \right),
\]
due to the Parseval’s identity. Since \( K^2 E^{\frac{2}{3}} < K^2 E^{\frac{2}{3}} \) for \( 1 < K, 1 < E \), the proof is complete.

4 Concluding Remarks

In this paper, we studied the inverse source scattering problem with attenuation and many frequencies of the one-dimensional Helmholtz equation in a two-layered medium using multi-frequency Dirichlet data at the two end points of an interval which contains the compact support of the source. Our results showed a deterioration of stability with growing attenuation/damping constant \( \alpha \).

Due to the \( \alpha \) and term \( e^{C\alpha} \), the result of theorem 1.1 is not sufficiently sharp. The quadratic dependence on \( \alpha \) in (8) is a consequence of Carleman estimates for the hyperbolic equation to prove Theorem 2.3. In particular, we used Carleman estimates to trace the dependence of exact observability bounds on the factor \( \alpha \). In [16], they provided numerical evidence which agreed with our result.

5 Acknowledgment:

This research is supported in part by NSF Award HRD-1824267.

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