STRUCTURE-PRESERVING MODEL REDUCTION FOR MARGINALLY STABLE LTI SYSTEMS

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Abstract. This work proposes a structure-preserving model reduction method for marginally stable linear time-invariant (LTI) systems. In contrast to Lyapunov-stability-based approaches for stable and anti-stable systems—which ensure the poles of the reduced system remain in the left-half and trial-half planes, respectively—the proposed method preserves marginal stability by reducing the subsystem with poles on the imaginary axis in a manner that ensures those poles remain purely imaginary. In particular, the proposed method decomposes a marginally stable LTI system into (1) an asymptotically stable subsystem with eigenvalues in the left-half plane and (2) a marginally stable subsystem with all eigenvalues on the imaginary axis. We propose a method based on inner-product projection and Lyapunov stability to reduce the first subsystem while preserving asymptotic stability. In addition, we propose both inner-product and symplectic balancing methods that balance the operators associated with primal and dual (quadratic) energy functionals while preserving asymptotic and pure marginal stability, respectively. We formulate a geometric perspective that enables a unified comparison of the proposed inner-product and symplectic projection methods. Numerical examples illustrate the ability of the method to reduce the dimensionality of marginally stable LTI systems while retaining accuracy and preserving marginal stability; further, the resulting reduced-order model yields a finite infinite-time energy, which arises from the pure marginally stable subsystem.

Key words. model reduction, structure preservation, marginal stability, symplectic structure, inner-product structure, inner-product balancing, symplectic balancing

AMS subject classifications. 65P10, 37M15, 34C20, 93A15, 37J25

1. Introduction. Reduced-order models (ROMs) are essential for enabling high-fidelity computational models to be used in many-query and real-time applications such as control, optimization, and uncertainty quantification. Marginally stable linear time-invariant dynamical (LTI) systems often arise in such applications; examples include inviscid fluid flow, quantum mechanics, and undamped structural dynamics. An ideal model-reduction approach for such systems would produce a dynamical-system model that is lower dimensional, is accurate with respect to the original model, and remains marginally stable, which is an intrinsic property of the dynamical system (it ensures, e.g., a finite system response at infinite time). Unfortunately, most classical model-reduction methodologies, such as balanced truncation [34], Hankel norm approximation [19], optimal $H_2$ approximation [20, 48, 32], and Galerkin projection exploiting inner-product structure [43], were originally developed for asymptotically stable LTI systems, i.e., systems with all poles in the test half-plane.

Although developed for asymptotically stable systems, balanced truncation and optimal $H_2$ approximation can be extended to unstable stable systems without poles on the imaginary axis. In particular, a reduced-order model can be obtained by balancing and truncating frequency-domain controllability and observability Gramians [41, 51]. By extending the $H_2$ norm to the $L_2$-induced Hilbert-Schmidt norm, an iteratively corrected rational Krylov algorithm was proposed for optimal $L_2$ model reduction [31]. However, the methods in Refs. [41, 51, 31] cannot be applied to marginally stable systems, as the frequency-domain controllability and observability Gramians as well as the $L_2$-induced Hilbert-Schmidt norm are not well defined when there are poles on the imaginary axis.

Although many well-known model reduction methods can be directly applied to systems with purely imaginary poles, they do not guarantee stability. These methods include proper orthogonal decomposition (POD) [24], balanced POD [42], pseudo balanced POD [30, 36], and moment matching [5, 18]. The shift-reduce-shift-back approach (SRSB) [50, 8, 44, 52, 49] reduces a $\mu$-shifted system $(A-\mu I, B, C)$ by balanced truncation. However, this approach fails to ensure stability when the balanced reduced system is shifted back by $\mu$.

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In general, stability-preserving ROMs fall into roughly two categories. The first category of methods derives a priori a stability-preserving model reduction framework, often specific to a particular equation set; the present work falls within this category. Refs. [43, 9, 25] construct ROMs in an energy-based inner product. Ref. [45] extends Ref. [9, 25] by applying the stabilizing projection to a skew-symmetric system constructed by augmenting a given linear system with its dual system. Refs. [28, 13, 12, 46, 22, 40, 21, 38, 1] construct reduced-order models to preserve the Lagrangian and (port-)Hamiltonian structures of the original systems.

The second category of methods stabilizes an unstable ROM through a posteriori stabilization step. In particular, Ref. [26] stabilizes reduced-order models via optimization-based eigenvalue reassignment. Refs. [11, 2, 6] construct reduced basis via minimal subspace rotation on the Stiefel manifold while preserving certain properties of the original system matrix. Other methods include to introduce viscosity [4, 39, 10] or penalty term [14], to enrich basis functions representing the small and energy dissipation scale [7, 35, 10], and to calibrate POD coefficients [15, 27]. In many cases, the stabilization alters the original unstable ROM and a sacrifice of accuracy is inevitable.

In this work, we propose a structure-preserving model-reduction method for marginally stable systems. The method guarantees marginal-stability preservation by executing two steps. First, the approach decomposes the original marginally stable linear system into two subsystems: one with eigenvalues in the left-half plane and one with nonzero eigenvalues on the imaginary axis. This is similar to the approach take in Ref. [33, 51] for performing model reduction of unstable systems without poles on the imaginary axis. Specifically, given a marginally stable (autonomous) LTI system \( \dot{x} = Ax \), where \( A \) is invertible and all eigenvalues have a non-positive real part, we apply a similarity transformation, which yields

\[
A = T \text{diag}(A_s, A_m) T^{-1}.
\]

Here, \( A_s \) has eigenvalues in the left-half plane (i.e., is Hurwitz) and \( A_m \) has purely imaginary eigenvalues. In this case, the subsystem \( \dot{x}_s = A_s x_s \) is asymptotically stable, while we show that the subsystem \( \dot{x}_m = A_m x_m \) is a generalized Hamiltonian system. Second, the method performs structure-preserving model reduction on the subsystems separately; namely, inner-product projection based on Lyapunov stability is employed to reduce the asymptotically stable subsystem, while symplectic projection is applied to the pure marginally stable subsystem characterized by purely imaginary eigenvalues.

Specific contributions of this work include:

1. A novel structure-preserving model reduction method for marginally stable LTI systems that preserves the asymptotic stability of the asymptotically stable subsystem via inner-product projection and the pure marginal stability of pure marginally stable subsystem via symplectic projection (Algorithm 1).
2. A general inner-product projection framework (Section 3), which we demonstrate ensures asymptotic-stability preservation if matrix used to define the inner product satisfies the Lyapunov inequality (Lemma 3.8).
3. An inner-product balancing approach that enables the operators associated with any primal or dual quadratic energy functional to be balanced (Section 3.4). If either of these satisfies a Lyapunov inequality, then asymptotic stability is additionally preserved (Corollary 3.12). We show that many existing model-reduction techniques (e.g., POD–Galerkin, balanced truncation, balanced POD) can be expressed as an inner-product projection and in fact are special cases of inner-product balancing (Table 3.2).
4. A stabilization approach that produces an asymptotically stable reduced-order model starting with a subset of the ingredients required for a stability-preserving inner-product projection, e.g., starting with an arbitrary trial basis matrix and a matrix that satisfies the Lyapunov inequality (Section 3.5).
5. Analysis that demonstrates that any pure marginally stable system is equivalent to a generalized Hamiltonian system (Theorem 4.9).
6. A novel symplectic-projection framework (Section 4) that ensures preservation of pure marginal stability (Theorem 4.16).
7. A symplectic balancing approach that enables the operators associated with any primal or negative dual quadratic energy functional to be balanced (Section 4.1) and preserve pure marginal stability (Corollary 4.19). In particular, we show that the generalized Hamiltonians associated with the primal and negative dual systems can be balanced with this approach.
8. A stabilization approach that produces a pure marginally stable reduced-order model starting with...
a subset of the ingredients required for a symplectic projection (Section 4.5).
9. A geometric framework that enables a unified analysis and comparison of inner-product and symplectic projection (Tables 2.1 and 3.1).
10. Experiments on two model problems that demonstrate that the proposed method has a small relative error in both the state and total energy (Section 5). Because symplectic model reduction is energy-conserving, the proposed method ensures that the infinite-time system energy is equal to the initial energy of the marginally stable subsystem. In contrast, the infinite-time energy of other reduced models is zero or infinity.

The remainder of the paper is organized as follows. Section 2 provides an overall view of the proposed method. Sections 3 and 4 present the methodologies to reduce the asymptotically stable subsystem and marginally stable subsystem, respectively. Section 5 illustrates the stability, accuracy, and efficiency of the proposed method through two numerical examples. Finally, Section 6 provides conclusions.

We make extensive use of the following sets in the remainder of the paper:
- SPD(n): the set of all $n \times n$ symmetric-positive-definite (SPD) matrices.
- SPSD(n): the set of all $n \times n$ symmetric-positive-semidefinite (SPSD) matrices.
- SS(n): the set of $n \times n$ nonsingular, skew-symmetric matrices.
- H(n): the set of real-valued $n \times n$ matrices whose eigenvalues have strictly negative real parts (i.e., the set of Hurwitz matrices).
- GH(n): the set of real-valued $n \times n$ diagonalizable matrices with nonzero purely imaginary eigenvalues.
- $\mathbb{R}_{n \times k}^+$: the set of full-column-rank $n \times k$ matrices with $k \leq n$ (i.e., the non-compact Stiefel manifold).
- $O(M, N)$: the set of full-column-rank $n \times k$ matrices $V$ with $k \leq n$ such that $V^T MV = N$ with $M \in \text{SPD}(n)$ and $N \in \text{SPD}(k)$. Note that $O(I_n, I_k)$ represents the Stiefel manifold.
- Sp(J), J: the set of full-column-rank $2n \times 2k$ matrices $V$ with $k \leq n$ such that $V^T J V = J$ with $J, J \in \text{SS}(2n)$ and $J \in \text{SS}(2k)$. Note that Sp(J), J represents the symplectic Stiefel manifold.

### 2. Marginally stable LTI systems

We begin by formulating the full-order model, which is a marginally stable LTI system (Section 2.1), and subsequently present the formulation for a general projection-based reduced-order model (Section 2.2). Then, we present the proposed framework based on system decomposition (Section 2.3).

#### 2.1. Full-order model

This work considers continuous-time LTI systems of the form

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx
\]

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$, $x \in \mathbb{R}^{n}$, $u \in \mathbb{R}^{p}$, and $y \in \mathbb{R}^{q}$. We denote this system by $(A, B, C)$ and focus on the particular case where the linear system is marginally stable. Because stability concerns the spectrum of the operator $A$, we focus primarily on the corresponding autonomous system

\[
\dot{x} = Ax.
\]

We now define marginal stability.

**Definition 2.1 (Marginal stability).** Linear system (2.1) is marginally stable, or Lyapunov stable, if for every initial condition $x(0) = x_0 \in \mathbb{R}^{n}$, the state response $x(t)$ of the associated autonomous system (2.2) is uniformly bounded.

The following standard lemmas (e.g., Ref. [23, pp. 66–70]) provide conditions for marginal stability.

**Lemma 2.2.** The following conditions are equivalent:
1. The system (2.1) is marginally stable.
2. All eigenvalues of $A$ have non-positive real parts and all Jordan blocks corresponding to eigenvalues with zero real parts are $1 \times 1$.

**Lemma 2.3.** The system (2.1) is marginally stable if one of the following conditions holds:
1. There exists $\Theta \in \text{SPD}(n)$ that satisfies the Lyapunov matrix inequality

\[
A^T \Theta + \Theta A \leq 0.
\]
2. For every $Q \in \text{SPSD}(n)$, there exists a unique solution $\Theta \in \text{SPD}(n)$ to the Lyapunov equation
\begin{equation}
A^T \Theta + \Theta A = -Q.
\end{equation}

3. There exists $\Theta \in \text{SPD}(n)$ such that the energy $\frac{1}{2} x^T \Theta x$ of the corresponding autonomous system is nonincreasing in time, i.e.,
\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} x^T \Theta x \right) \leq 0,
\end{equation}
for $x \in \mathbb{R}^n$ satisfying \( \Theta \).

In Lemma 2.3 it is trivial to show Conditions 1 and 2 are equal. Because $\frac{d}{dt} \left( \frac{1}{2} x^T \Theta x \right) = \frac{1}{2} x^T (A^T \Theta + \Theta A) x$, Conditions 1 and 3 are also equivalent. We note that Lemma 2.3 provides sufficient conditions for marginal stability; not all marginally stable systems have a Lyapunov matrix $\Theta$ that satisfies (2.3)-(2.4).

2.2. Reduced-order model. Let $\Psi, \Phi \in \mathbb{R}^{n \times k}$ denote test and trial basis matrices that are biorthogonal (i.e., $\Psi^T \Phi = I_k$) and whose columns span $k$-dimensional test and trial subspaces of $\mathbb{R}^n$, respectively. If the reduced-order model is constructed via Petrov–Galerkin projection performed on the full-order model, then (2.1) reduces to
\begin{align}
\dot{z} &= \tilde{A} z + \tilde{B} u \\
y &= \tilde{C} z,
\end{align}
where $\tilde{A} := \Psi^T A \Phi \in \mathbb{R}^{k \times k}$, $\tilde{B} := \Psi^T B \in \mathbb{R}^{k \times p}$, $\tilde{C} := C \Phi \in \mathbb{R}^{q \times k}$, and the state is approximated as $x \approx \Phi z$. We denote this system by $(\tilde{A}, \tilde{B}, \tilde{C})$. The corresponding autonomous system is
\begin{equation}
\dot{z} = \tilde{A} z
\end{equation}
with initial condition $z(0) = \Psi^T x_0 \in \mathbb{R}^k$.

2.3. System decomposition. If the full-order-model system (2.1) is marginally stable and the matrix $A$ has a full rank, then there exists $T \in \mathbb{R}^{n \times n}$ such that the similarity transformation satisfies
\begin{equation}
A = T \begin{bmatrix} A_s & 0 \\ 0 & A_m \end{bmatrix} T^{-1},
\end{equation}
where $A_s \in \text{H}(n_s)$ and $A_m \in \text{GH}(n_m)$ and $n_s + n_m = n$. Let $T = [T_s \ T_m]$ with $T_s \in \mathbb{R}^{n_s \times n_s}$ and $T_m \in \mathbb{R}^{n_m \times n_m}$. Then, $AT_i = T_i A_i$ (for $i \in \{s, m\}$), which implies that the columns of $T_i$ span an invariant subspace of $A$. Substituting $x = T[x_s^T \ x_m^T]^T$ into (2.1) and premultiplying the first set of equations by $T^{-1}$ yields a decoupled LTI system
\begin{align}
\frac{d}{dt} \begin{bmatrix} x_s \\ x_m \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & A_m \end{bmatrix} \begin{bmatrix} x_s \\ x_m \end{bmatrix} + \begin{bmatrix} B_s \\ B_m \end{bmatrix} u \\
y &= \begin{bmatrix} C_s & C_m \end{bmatrix} \begin{bmatrix} x_s \\ x_m \end{bmatrix},
\end{align}
where $T^{-1} B = [B_s^T \ B_m^T]^T$ and $CT = [C_s \ C_m]$. Here, the subsystem associated with $x_s$ is asymptotically stable, while the subsystem associated with $x_m$ is marginally stable.

This decomposition enables each subsystem to be reduced in a manner that preserves its particular notion of stability. In the present context, we can accomplish this by defining test and trial basis matrices for each subsystem $\Psi_i \in \mathbb{R}^{n_i \times k_i}$, $\Phi_i \in \mathbb{R}^{n_i \times k_i}$, $i \in \{s, m\}$. Applying Petrov–Galerkin projection to (2.9) with test basis matrix $\text{diag}(\Psi_s, \Psi_m)$ and trial basis matrix $\text{diag}(\Phi_s, \Phi_m)$ yields a decoupled reduced LTI system
\begin{align}
\frac{d}{dt} \begin{bmatrix} z_s \\ z_m \end{bmatrix} &= \begin{bmatrix} \tilde{A}_s & 0 \\ 0 & \tilde{A}_m \end{bmatrix} \begin{bmatrix} z_s \\ z_m \end{bmatrix} + \begin{bmatrix} \tilde{B}_s \\ \tilde{B}_m \end{bmatrix} u \\
y &= \begin{bmatrix} \tilde{C}_s & \tilde{C}_m \end{bmatrix} \begin{bmatrix} z_s \\ z_m \end{bmatrix},
\end{align}
where the full state is approximated as
\begin{equation}
x(t) \approx T \begin{bmatrix} \Phi_s z_s(t) \\ \Phi_m z_m(t) \end{bmatrix}.
\end{equation}
Within this decomposition-based approach, basis matrices $\Psi_s$ and $\Phi_s$ can be computed to preserve asymptotic
stability in the associated reduced subsystem (e.g., via balanced truncation or other Lyapunov methods). For the marginally stable subsystem, we will show that the symplectic model reduction method can be applied to obtain a low-order marginally stable system wherein all eigenvalues of $A_m$ are nonzero and purely imaginary.

Algorithm 1 summarizes the proposed procedure for computing reduced-order-model operators $(\tilde{A}, \tilde{B}, \tilde{C})$ and $(\tilde{A}_m, \tilde{B}_m, \tilde{C}_m)$. Here, we have defined Table 2.1 lists the methods and key properties of each subsystem. The next two sections explain Algorithm 1 and Table 2.1 in detail.

Algorithm 1 Structure-preserving model reduction for marginally stable LTI systems.

**Input:** A marginally stable LTI system $(A,B,C)$.

**Output:** Reduced-order-model operators $(\tilde{A}, \tilde{B}, \tilde{C})$ and $(\tilde{A}_m, \tilde{B}_m, \tilde{C}_m)$.

1. Compute a matrix $T$ such that $A$ is transformed into block-diagonal form (2.8).
2. Select $M \in \text{SPD}(n_s)$ such that the Lyapunov inequality $A_s^T M + M A_s < 0$ is satisfied.
3. Construct trial basis matrix $\Phi_s \in O(M,N)$ for some $N \in \text{SPD}(k_s)$, $k_s < n_s$.
4. Construct test basis matrix $\Psi_s = M \Phi_s N^{-1}$.
5. Construct the reduced system $\tilde{A}_s = \Psi_s^T A_s \Phi_s$, $\tilde{B}_s = \Psi_s^T B_s$, $\tilde{C}_s = C_s \Phi_s$.
6. Select $J_1 \in \text{SS}(n_m)$ such that the generalized Hamiltonian property $A_m^T J_1 + J_1 A_m = 0$ is satisfied.
7. Construct trial basis matrix $\Phi_m \in \text{Sp}(J_1, J_1)$ for some $J_1 \in \text{SS}(k_m)$, $k_m < n_m$.
8. Construct test basis matrix $\Psi_m = J_1 \Phi_m J_1^{-1}$.
9. Construct the reduced system $\tilde{A}_m = \Psi_m^T A_m \Phi_m$, $\tilde{B}_m = \Psi_m^T B_m$, $\tilde{C}_m = C_m \Phi_m$.

Appendix B describes how this decomposition approach can be extended to general unstable LTI systems with A possibly singular.

3. Reduction of asymptotically stable subsystems. This section focuses on reducing the asymptotically stable subsystem $\dot{x}_s = A_s x_s$. Section 3.2 introduces inner-projection projection, Section 3.3 demonstrates that a model-reduction method based on inner-projection projection preserves asymptotic stability, Section 3.4 presents the inner-product-balancing framework, and Section 3.5 describes methods for constructing the basis matrices that lead to an inner-projection given a subset of the required ingredients. For notational simplicity, we omit the subscript $s$ throughout this section.

3.1. Asymptotically stable systems. We begin by defining asymptotic stability.

**Definition 3.1 (Asymptotic stability).** Linear system (2.1) is asymptotically stable if, in addition to being marginally stable, $x(t) \to 0$ as $t \to \infty$ for every initial condition $x(0) = x_0 \in \mathbb{R}^n$.

In analogue to Lemmas 2.2, 2.3 we now provide conditions for asymptotic stability.

**Lemma 3.2.** The following conditions are equivalent:

1. The system (2.1) is asymptotically stable.
2. $A \in \text{H}(n)$.
3. There exists $\Theta \in \text{SPD}(n)$ that satisfies the Lyapunov matrix inequality
   \[ A^T \Theta + \Theta A < 0. \]
   (3.1)
4. For every $Q \in \text{SPD}(n)$, there exists a unique Lyapunov matrix $\Theta \in \text{SPD}(n)$ that satisfies (2.4).
5. There exists $\Theta \in \text{SPD}(n)$ such that the energy $\frac{1}{2} x^T \Theta x$ of the corresponding autonomous system is strictly decreasing in time, i.e.,
   \[ \frac{d}{dt} \left( \frac{1}{2} x^T \Theta x \right) < 0, \]
   for $x \in \mathbb{R}^n$ satisfying (2.2).

We note that $A \in \text{H}(n)$ does not necessarily imply that the symmetric part of $A$ is negative definite. However, $A \in \text{H}(n)$ if and only if it can be transformed into a matrix with negative symmetric part by similarity transformation with a real matrix; see Lemma A.1 in Appendix A for details.

We now connect asymptotic stability of the primal system to that of its dual.
Table 2.1
Inner-product model reduction v. symplectic model reduction.

|                              | Asymptotically stable subsystem | Marginally stable subsystem |
|------------------------------|---------------------------------|----------------------------|
| Original space               | Inner-product space:            | Symplectic space:          |
|                              | $(\mathbb{R}^n, M)$ with $M \in \text{SPD}(n)$ | $(\mathbb{R}^m, J_\Omega)$ with $J_\Omega \in \text{SS}(m)$ |
| System matrix                | $A_s \in \text{H}(n_s)$        | $A_m \in \text{GH}(m)$    |
| Autonomous system            | $x_s = A_s x_s$                 | $x_m = A_m x_m$            |
| Key property of full system  | $A_s^T M + M A_s < 0$           | Generalized Hamiltonian property: |
|                              | Lyapunov inequality:            | $A_m^T J_\Omega + J_\Omega A_m = 0$ |
| Energy property of full system| $\frac{d}{dt} \left( \frac{1}{2} x_s^T M x_s \right) < 0$ | $\frac{d}{dt} \left( \frac{1}{2} x_m^T L x_m \right) = 0$ with $A_m = -J_\Omega^{-1} L$ |
| Canonical form               | $M = I_n$                      | $J_\Omega = J_{2n}$       |
|                              | $A_s^T + A_s < 0$               | $A_m^T J_{2n} + J_{2n} A_m = 0$ |
| Reduced space                | Inner-product space:            | Symplectic space:          |
|                              | $(\mathbb{R}^k, N)$ with $N \in \text{SPD}(k)$ | $(\mathbb{R}^m, J_\Omega)$ with $J_\Omega \in \text{SS}(k)$ |
| Projection                   | Inner-product projection        | Symplectic projection      |
| Trial basis matrix           | $\Phi_s \in O(\mathcal{M}, N)$ | $\Phi_m \in \text{Sp}(J_\Omega, J_{2n})$ |
| Test basis matrix            | $\Psi_s = M \Phi_s N^{-1} \in \mathbb{R}^n_{s \times k}$ | $\Psi_m = J_\Omega \Phi_m J_{2n}^{-1} \in \mathbb{R}^m_{n \times k}$ |
| Reduced-system matrix        | $A_s = \Psi_s^T A_s \Phi_s \in \text{H}(k)$ | $A_m = \Psi_m^T A_m \Phi_m \in \text{GH}(k)$ |
| Reduced autonomous system    | $\dot{z}_s = A_s z_s$           | $\dot{z}_m = A_m z_m$      |
| Key property of reduced system| $A_s^T N + N A_s < 0$           | Generalized Hamiltonian property: |
|                              | Lyapunov inequality:            | $A_m^T J_{2n} + J_{2n} A_m = 0$ |
| Energy property of reduced system| $\frac{d}{dt} \left( \frac{1}{2} z_s^T N z_s \right) < 0$ | $\frac{d}{dt} \left( \frac{1}{2} z_m^T L z_m \right) = 0$ with $A_m = -J_\Omega^{-1} L$ |
| Approximate solution         | $x_s(t) \approx \Phi_s z_s(t)$ | $x_m(t) \approx \Phi_m z_m(t)$ |

Lemma 3.3 (Dual version of Lemma 3.2). If any condition of Lemma 3.2 holds, then the following conditions hold:

1. The dual system $(A^*, C^*, B^*)$ is asymptotically stable.
2. $A^* \in \text{H}(n)$.
3. There exists $\Theta' \in \text{SPD}(n)$ that satisfies the dual Lyapunov matrix inequality
   \[(3.3)\quad A \Theta' + \Theta' A^* < 0.\]
4. For every $Q' \in \text{SPD}(n)$, there exists a unique Lyapunov matrix $\Theta' \in \text{SPD}(n)$ that satisfies
   \[(3.4)\quad A \Theta' + \Theta' A^* = -Q'.\]

Proof. Because the eigenvalues of $A$ are identical to the eigenvalues of $A^*$, $A \in \text{H}(n)$ if and only if $A^* \in \text{H}(n)$. Thus the LTI system associated with $A^*$ is asymptotically stable and satisfies the corresponding conditions of Lemma 3.2.

Remark 3.4 (Relationship with dual system: asymptotic stability). Thus, any method proposed in this work for ensuring asymptotic stability of a given (sub)system also ensures asymptotic stability of the associated dual (sub)system. However, because the trial basis $\Phi$ associated with $(\hat{A}, \hat{B}, \hat{C})$ corresponds to the test basis of $(A^*, C^*, B^*)$ (i.e., $A^* = \Phi^* A \Phi$), the proposed methods for constructing a trial basis matrix $\Phi$ should be applied to the dual system as a test basis matrix. Similarly, the proposed methods for constructing a test basis matrix $\Psi$ should be applied to the dual system as a trial basis matrix.
3.2. Inner-product projection of spaces. Let $\mathbb{V} \cong \mathbb{R}^n$ and $\mathbb{W} \cong \mathbb{R}^k$ with $k \leq n$ denote vector spaces equipped with inner products $\langle \cdot, \cdot \rangle_\mathbb{V} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_\mathbb{W} : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ respectively. These inner products can be represented by matrices $M \in \text{SPD}(n)$ and $N \in \text{SPD}(k)$, respectively, i.e.,

$$\langle \hat{x}_1, \hat{x}_2 \rangle_\mathbb{V} \equiv x_1^T M x_2, \quad \forall x_1, x_2 \in \mathbb{R}^n$$

$$\langle \hat{z}_1, \hat{z}_2 \rangle_\mathbb{W} \equiv z_1^T N z_2, \quad \forall z_1, z_2 \in \mathbb{R}^k,$n

where the operator $\hat{\cdot}$ provides the representation of an element of a vector space from its coordinates, i.e., $\hat{x} \in \mathbb{V}$, $\forall x \in \mathbb{R}^n$ and $\hat{z} \in \mathbb{W}$, $\forall z \in \mathbb{R}^k$. We represent these inner-product spaces $\mathbb{V}$ and $\mathbb{W}$ by $(\mathbb{R}^n, M)$ and $(\mathbb{R}^k, N)$ respectively.

**Definition 3.5** (Inner-product lift). An inner-product lift is a linear mapping $\phi : \mathbb{W} \to \mathbb{V}$ that preserves inner-product structure:

$$\langle \hat{z}_1, \hat{z}_2 \rangle_\mathbb{W} = \langle \phi(\hat{z}_1), \phi(\hat{z}_2) \rangle_\mathbb{V}, \quad \forall \hat{z}_1, \hat{z}_2 \in \mathbb{W}.$$ (3.5)

**Definition 3.6** (Inner-product projection). Let $\phi : \mathbb{W} \to \mathbb{V}$ be an inner-product lift. The adjoint of $\phi$ is the linear mapping $\psi : \mathbb{V} \to \mathbb{W}$ satisfying

$$\langle \psi(\hat{x}), \hat{z} \rangle_\mathbb{W} = \langle \hat{x}, \phi(\hat{z}) \rangle_\mathbb{V}, \quad \forall \hat{z} \in \mathbb{W}, \hat{x} \in \mathbb{V}.$$ (3.6)

We say $\psi$ is the inner-product projection induced by $\phi$.

In coordinate space, this inner-product lift and projection can be expressed equivalently as

$$\phi(\hat{z}) \equiv \Phi z, \quad \forall z \in \mathbb{R}^k$$

$$\psi(\hat{x}) \equiv \Psi^* x, \quad \forall x \in \mathbb{R}^n,$n

respectively, where (3.5)–(3.6) imply that $\Phi \in \mathbb{R}^{n \times k}$ and $\Psi \in \mathbb{R}^{n \times k}$ satisfy

$$\Phi^* M \Phi = N$$ (3.7)

$$\Psi N = M \Phi,$n

from which it follows that

$$\Psi = M \Phi N^{-1}.$$ (3.9)

For convenience, we write $\Phi \in O(M, N)$. Although $\Psi^*$ is not in general equal to the Moore–Penrose pseudoinverse $(\Phi^* \Phi)^{-1} \Phi^*$, it can be verified that it is indeed a left inverse of $\Phi$, which implies that $\psi \circ \phi$ is the identity map on $\mathbb{W}$.

3.3. Inner-product projection of dynamics. This section describes the connection between inner-product projection and asymptotic-stability preservation in model reduction. Namely, we show that if inner-product projection is employed to construct the reduced-order model with $M$ corresponding to a Lyapunov matrix of the original system, then the reduced-order model inherits asymptotic stability.

**Definition 3.7** (Model reduction via inner-product projection). A reduced-order model $(\hat{A}, \hat{B}, \hat{C})$ with $\hat{A} = \Psi^* A \Phi$, $\hat{B} = \Psi^* B$, and $\hat{C} = C \Phi$ is constructed by an inner-product projection if $\Phi \in O(M, N)$, $\Psi = M \Phi N^{-1}$, where $M \in \text{SPD}(n)$ and $N \in \text{SPD}(k)$.

**Lemma 3.8** (Inner-product projection preserves asymptotic stability). If the original LTI system $(A, B, C)$ has a Lyapunov matrix $\Theta$ satisfying (3.1) and the reduced-order model is constructed by inner-product projection with $M = \Theta$, then the reduced-order model $(\hat{A}, \hat{B}, \hat{C})$ is asymptotically stable with Lyapunov matrix $N$.

**Proof.** Left- and right-multiplying inequality (3.1) (with $\Theta = M$) by $\Phi^*$ and $\Phi$, respectively, yields

$$\Phi^* A^* M \Phi + \Phi^* M A \Phi < 0.$$ (3.10)

Substituting (3.8) and $\hat{A} = \Psi^* A \Phi$ in (3.10) yields

$$\hat{A}^* N + N \hat{A} < 0,$n

which implies that the reduced system is asymptotically stable by Lemma 3.2.

We note that $M$ is a Lyapunov matrix of the inner-product lifted reduced solution $\Phi z$, as

$$\frac{d}{dt} \left( \frac{1}{2} \hat{z}^T N \hat{z} \right) = \frac{d}{dt} \left( \frac{1}{2} \hat{z}^T \Phi^* M \Phi \hat{z} \right) < 0.$$ (3.12)

We note that Lemma 3.8 is a generalization of the stability-preservation property in Ref. 13, which
required the reduced space to be Euclidean (i.e., \( N = I_k \) in the present notation). Lemma 3.8 considers a more general form where the reduced space can be any inner-product space, i.e., \( N \in \text{SPD}(k) \) but otherwise arbitrary.

### 3.4. Inner-product balancing

We now describe an inner-product-balancing approach that leverages inner-product structure. Table 3.1 compares this approach with a novel symplectic-balancing approach, which will be described in Section 4.4.

**Definition 3.9** (Inner-product balancing). Given any \( \Xi \in \text{SPD}(n) \) and \( \Xi' \in \text{SPD}(n) \), the trial and test basis matrices characterizing an inner-product balancing correspond to

\[
\Phi = SV_1\Sigma_1^{-1/2} \quad \text{and} \quad \Psi = RU_1\Sigma_1^{-1/2},
\]

respectively, where \( \Xi = RR^*, \Xi' = SS^* \), and \( R^*S = USV^* \) is the singular value decomposition. Here, we have defined \( U = [U_1, U_2] \), \( \Sigma = \text{diag}(\Sigma_1, \Sigma_2) \), and \( V = [V_1, V_2] \), where \( U_1, V_1 \in O(I_n, I_k) \) and \( \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k) \).

**Lemma 3.10.** If the test and trial basis matrices \((\Psi, \Phi)\) characterize an inner-product balancing with \( \Xi \) and \( \Xi' \), then they correspond to an inner-product projection with \( M = \Xi \) and \( N = \Sigma_1 \). Further, these basis matrices \((\Phi, \Psi)\) correspond to an inner-product projection performed on the dual system with \( \Xi' = \Xi \) and \( \Sigma_1 = \Sigma_1 \). Thus, these basis matrices balance \( \Xi \) and \( \Xi' \), as \( \Phi \in O(\Xi, \Sigma_1) \) and \( \Psi \in O(\Xi', \Sigma_1) \).

**Proof.** From the definition of the inner-product-balancing test and trial basis (3.13), we first verify that \( \Phi^T\Xi\Phi = \Sigma_1^{-1/2}V_1S^*RR^*SV_1\Sigma_1^{-1/2} = \Sigma_1 \), thus \( \Phi \in O(\Xi, \Sigma_1) \). Furthermore, it can be verified that \( \Psi = \Xi\Psi\Sigma_1^{-1} = \Xi SV_1\Sigma_1^{-1/2}\Sigma_1^{-1} = RU_1\Sigma_1^{-1/2} \). Thus, \( \Phi \) and \( \Psi \) satisfy the conditions for an inner-product projection; note that \( \Psi^T\Phi = I_k \). For the dual system, recall from Remark 3.4 that the test basis of the dual system corresponds to \( \Phi \), while the trial basis corresponds to \( \Psi \). Thus, we aim to verify that \( \Psi \in O(\Xi', \Sigma_1) \) and \( \Phi \in O(\Xi', \Sigma_1) \). We can verify the first of these, as \( \Psi^T\Xi'\Psi = \Sigma_1^{-1/2}V_1S^*RR^*SV_1\Sigma_1^{-1/2} = \Sigma_1 \), thus \( \Psi \in O(\Xi', \Sigma_1) \). Furthermore, it can be verified that \( \Phi = \Xi\Psi\Sigma_1^{-1} = \Xi SV_1\Sigma_1^{-1/2}\Sigma_1^{-1} = RU_1\Sigma_1^{-1/2} \). Thus, \( \Psi \) and \( \Phi \) satisfy the conditions for an inner-product projection on the dual system.

**Lemma 3.11.** If test and trial basis matrices \((\Psi, \Phi)\) characterize an inner-product projection with \( M \) and \( N \), then they balance \( M \) and \( M' = \Psi_{ext}^{-1}N_{ext}\Psi_{ext}^{-1} \), i.e., \( \Phi \in O(M, N) \) and \( \Psi \in O(M', N) \). Here,
Comparison of different model-reduction methods with inner-product-balancing structure defined by $\Xi$ and $\Xi'$. In all cases, $N = \Sigma_1$ is defined by the inner-product balancing. The quantities defined are in Appendix A.

| Method                         | POD–Galerkin | Balanced truncation | Balanced POD | Proposed inner-product projection |
|-------------------------------|--------------|---------------------|--------------|-----------------------------------|
| $M = \Xi$                     | $XX'$        | $W_o$               | $W_o$        | $\Theta$ satisfying (3.1)          |
| $M' = \Xi'$                   | $XX'$        | $W_e$               | $W_e$        | $\Theta'$ satisfying (3.3)         |
| Stability preservation?       | No           | Yes¹                | No           | Yes                               |

Further, there exists a realization of that reduced-order model that corresponds to an inner-product balancing with $\Xi = M$, $\Xi' = M'$, and $\Sigma_1 = \Lambda$ characterized by basis matrices $(\hat{\Psi}, \tilde{\Psi})$ that satisfy $\hat{\Psi} \in O(M, \Lambda)$ and $\tilde{\Psi} \in O(M', \Lambda)$ with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k)$. Here, $N = U\Lambda U^T$ is the symmetric eigenvalue decomposition, $M' = \hat{\Psi}_{\text{ext}}^T N_{\text{ext}} \hat{\Psi}_{\text{ext}}^{-1}$, $N_{\text{ext}} = \text{diag}(\Lambda, N)$, and $\tilde{\Psi}_{\text{ext}} = [\hat{\Psi} \tilde{\Psi}] \in \mathbb{R}^{n \times n}$.

Proof. From the definition of an inner-product projection with $M$ and $N$, we have $\Phi \in O(M, N)$ and $\Psi = M\Phi N^{-1}$. Using $M' = \hat{\Psi}_{\text{ext}}^T N_{\text{ext}} \hat{\Psi}_{\text{ext}}^{-1}$, we have $[\hat{\Psi} \tilde{\Psi}] M' [\hat{\Psi} \tilde{\Psi}] = \text{diag}(N, \tilde{\Psi})$, whose $(1,1)$ block gives $\tilde{\Psi} \in O(M', \Lambda)$. Similarly, $\Phi^T M \Phi = N$ implies $U^T \hat{\Psi}^T M \Phi U = \Lambda$, thus $\hat{\Psi} \in O(M, N)$ implies $\tilde{\Psi} \in O(M, \Lambda)$ with $\tilde{\Psi} = \tilde{\Phi} U$. Using $M' = \hat{\Psi}_{\text{ext}}^T N_{\text{ext}} \hat{\Psi}_{\text{ext}}^{-1}$, we have $[\hat{\Psi} \tilde{\Psi}] M' [\hat{\Psi} \tilde{\Psi}] = \text{diag}(\Lambda, \tilde{\Psi})$, whose $(1,1)$ block gives $\tilde{\Psi} \in O(M', \Lambda)$. Note, to ensure $(\Psi, \Phi)$ corresponds to an inner-product projection with $M$ and $N = \Lambda$, we set $\Psi = M \Phi \Lambda^{-1} = M \Phi U \Lambda^{-1} = M \Phi N^{-1} N U \Lambda^{-1} = \Psi N U \Lambda^{-1}$. Noting that $\text{Ran}(\hat{\Phi}) = \text{Ran}(\Phi)$ and $\text{Ran}(\tilde{\Phi}) = \text{Ran}(\Psi)$ as well as $\Phi^T \Phi = \tilde{\Phi}^T \tilde{\Phi} = I_k$, we conclude that basis matrices $(\Psi, \Phi)$ and $(\tilde{\Psi}, \tilde{\Phi})$ yield different realizations of the same reduced-order model.

Corollary 3.12. A inner-product-balancing reduced-order model preserves asymptotic stability if $\Xi = \Theta$ with $\Theta$ a Lyapunov matrix satisfying (3.1), or if $\Xi' = \Theta'$ with $\Theta'$ a dual Lyapunov matrix satisfying (3.3).

Proof. The result follows directly from Lemmas 3.8 and 3.10 as Lemma 3.3 for the dual system.

We note that many existing model-reduction methods correspond to an inner-product balancing; these methods are reported in Table 3.2, Appendix A these model-reduction methods in more detail.

3.5. Construction of basis matrices given subset of ingredients. Lemma 3.8 demonstrated that a ROM will preserve asymptotic stability if it is constructed via inner-product projection with $M = \Theta$ a Lyapunov matrix satisfying (3.1). Unfortunately, as reported in Table 3.2, while many typical model-reduction techniques associate with an inner-product projection (and an inner-product balancing), the associated operator $M$ does not often satisfy the Lyapunov inequality, which precludes assurances of stability preservation (e.g., POD–Galerkin and Balanced POD).

We propose three methods (including inner-product balancing) for constructing a stability-preserving inner-product projection satisfying the conditions of Lemma 3.8. Table 3.3 summarizes these methods; this corresponds to Steps 6–8 in Algorithm 1. Methods 2 and 3 assume that we are given a subset of the required ingredients, which can be computed by any technique. For example, the trial basis can be computed by POD, balanced POD, or rational approximation; the metric $\Theta$ can be obtained by solving Lyapunov equation (2.4) with $Q \in \text{SPD}(n)$ but otherwise arbitrary. Thus, these methods can be viewed as stabilization techniques applied to the provided inputs.

Method 2 constructs a stability-preserving inner-product projection starting with any arbitrarily chosen trial basis matrix $\Phi \in \mathbb{R}^{n \times k}$ and a Lyapunov matrix $\Theta$ satisfying (3.1). Method 3 constructs a stability-preserving inner-product projection starting with a basis $\Phi_0 \in O(M_0, N_0)$, where $M_0$ might not satisfy the Lyapunov inequality (3.1), and a Lyapunov matrix $\Theta$ satisfying (3.3). For simplicity, we can choose $G = M^{-1/2} M_0^{1/2}$ and $\tilde{G} = N^{-1/2} N_0^{1/2}$. Lemma 3.13 demonstrates that we can compute the trial basis matrix in this context as $\Phi = G \Phi_0 \tilde{G}^{-1} \in O(M, N)$, which constitutes step 4 of the algorithm.

Lemma 3.13. Let $\phi: (\mathbb{R}^k, N) \to (\mathbb{R}^n, M), x \mapsto \Phi x$ denote an inner-product lift with $\Phi \in O(M, N)$. Let

¹With the current framework, we can only show that balanced truncation preserves marginal stability, as the right-hand-side matrices in the Lyapunov equations are symmetric positive semidefinite. However, additional analyses based on controllability and observability demonstrate that balanced truncation does preserve asymptotic stability (e.g., Ref. [3] pp. 213–215).
Table 3.3
Algorithms for constructing an inner-product projection that ensure the conditions of Definition 3.14.

| Method 1 (inner-product balancing) | Method 2 | Method 3 |
|------------------------------------|----------|----------|
| **Input**                          | **Output** | **Algorithm** |
| \(\Xi, \Xi' \in \text{SPD}(n)\) with \(\Xi = \Theta\) satisfying (3.1) or \(\Xi' = \Theta'\) satisfying (3.3) | \(M \in \text{SPD}(n), N \in \text{SPD}(k), \Phi \in O(M, N), \Psi \in O(M', N)\) | 1. Compute symmetric factorization \(\Xi = RR^\top, \Xi' = SS^\top\)  
2. Compute SVD \(R^\top S = U\Sigma V^\top\)  
3. \(\Phi = SV_1\Sigma_1^{-1/2}\)  
4. \(\Psi = RU_1\Sigma_1^{-1/2}\)  
5. \(M = \Xi, M' = \Xi', N = \Sigma_1\) |
| \(\Phi \in \mathbb{R}^n_k\), \(\Theta\) satisfying (3.1) | \(M \in \text{SPD}(n), N \in \text{SPD}(k), \Phi \in O(M, N), \Psi \in \mathbb{R}^n_{k \times k}\) | 1. Set \(M = \Theta\)  
2. Construct \(G \in O(M, M_0)\)  
3. Construct \(\tilde{G} \in O(N, N_0)\)  
4. \(\Phi = G\Phi_0\tilde{G}^{-1}\)  
5. \(\Psi = M\Phi N^{-1}\) |

Let \(g : (\mathbb{R}^n, M_0) \to (\mathbb{R}^n, M), x \mapsto Gx\) and \(\tilde{g} : (\mathbb{R}^k, N_0) \to (\mathbb{R}^k, N), x \mapsto \tilde{G}x\) represent (invertible) inner-product transformations, i.e., \(G \in O(M, M_0) \subseteq \mathbb{R}^{n \times n}\) and \(\tilde{G} \in O(N, N_0) \subseteq \mathbb{R}^{k \times k}\), respectively. Then, there exists a unique inner-product lift \(\phi_0 : (\mathbb{R}^k, N_0) \to (\mathbb{R}^n, M_0), x \mapsto \Phi_0x\) with \(\Phi_0 \in O(M_0, N_0)\), such that the following diagram commutes:

\[
\begin{array}{ccc}
(\mathbb{R}^n, M_0) & \xrightarrow{g} & (\mathbb{R}^n, M) \\
\phi_0 & \uparrow \Phi & \downarrow \phi \\
(\mathbb{R}^k, N_0) & \xleftarrow{\tilde{g}^{-1}} & (\mathbb{R}^k, N)
\end{array}
\]

Equivalently, for all \(z \in (\mathbb{R}^k, N)\),

\[
(3.14) \quad \phi(z) = g(\phi_0(\tilde{g}^{-1}(z)))
\]

and \(\Phi = G\Phi_0\tilde{G}^{-1}\) in matrix representation.

Proof. Because \(G \in O(M, M_0)\), we have \(G^\top MG = M_0\). It follows that \(G^\top G = M_0G^{-1} = M\). By the same argument, \(\tilde{G} \in O(N, N_0)\) implies that \(\tilde{G}^\top \tilde{G} = N_0\). Because \(\Phi \in O(M, N)\), we have \(\Phi^\top M\Phi = N\). Because \(g\) is invertible, we can define \(\phi_0 : (\mathbb{R}^k, N_0) \to (\mathbb{R}^n, M_0)\) by \(g^{-1} \circ \phi \circ \tilde{g}\) with matrix representation \(\Phi_0 = G^{-1}\Phi\tilde{G}\). It follows that \(\Phi_0^\top M_0\Phi_0 = G^\top \Phi^\top (G^{-1}\Phi\tilde{G})\Phi \tilde{G} = G^\top \Phi^\top M_0\Phi \tilde{G} = G^\top NG = N_0\). The last equation implies that \(\Phi_0 \in O(M_0, N_0)\). Finally, if \(\phi_0\) satisfies \(\phi = g \circ \phi_0 \circ \tilde{g}^{-1}\), \(\phi_0\) is uniquely determined by \(\phi_0 = g^{-1} \circ \phi \circ \tilde{g}\).

We now show that if the original trial basis matrix \(\Phi_0\) exhibits a POD-like optimality property, then \(\Phi\) computed by Method 3 in Table 3.3 will inherit a related optimality property. Given a set of snapshots \(\{x_i\}_{i=1}^N\) with \(x_i \in (\mathbb{R}^n, M)\), we define the projection error of the ensemble in the \(M\)-induced norm by

\[
\sum_{i=1}^N \|x_i - \Phi^\top x_i\|^2_M = \sum_{i=1}^N \left\|x_i - \Phi N^{-1}\Phi^\top Mx_i\right\|^2_M,
\]

where we have used \(\Psi = M\Phi N^{-1}\).

Theorem 3.14. Let \(M_0, M, N_0, N, G, \tilde{G}, \Phi_0\) and \(\Phi\) be as defined in Lemma 3.13. If \(\Phi_0\) minimizes the projection of the snapshot ensemble \(\{y_i\}_{i=1}^N\) with \(y_i \in (\mathbb{R}^n, M_0)\), i.e.,

\[
(3.15) \quad \Phi_0 = \arg \min_{V_0 \in O(M_0, N_0)} \sum_{i=1}^N \|y_i - V_0N_0^{-1}V_0^\top M_0y_i\|^2_{M_0},
\]

then \(\Phi = G\Phi_0\tilde{G}^{-1}\) minimizes the projection of the snapshot ensemble \(\{x_i\}_{i=1}^N\) with \(x_i \in (\mathbb{R}^n, M)\), i.e.,

\[
(3.16) \quad \Phi = \arg \min_{V \in O(M, N)} \sum_{i=1}^N \|x_i - VN^{-1}V^\top Mx_i\|^2_M,
\]
where \( x_i = G y_i \). Moreover, the cost function in (3.15) and (3.16) achieves the same minimal value.

Proof. By Lemma 3.13 for any \( V \in O(M,N) \), there exists a unique \( V_0 \in O(M_0,N_0) \) such that \( V = GV_0 \tilde{G}^{-1} \).

For any \( i \in \{1, \ldots, N\} \), we have
\[
\| x_i - VN^{-1}V^* M x_i \|_M^2 = \| x_i - (GV_0 \tilde{G}^{-1}) N^{-1} (GV_0 \tilde{G}^{-1})^* M x_i \|_M^2 \quad (V = GV_0 \tilde{G}^{-1})
\]
\[
= \| Gy_i - GV_0 \tilde{G}^{-1} N^{-1} G^\gamma V_0^* (G^T MG) y_i \|_M^2 \quad (x_i = G y_i)
\]
\[
= \| Gy_i - GV_0 N_0^{-1} V_0^* M_0 y_i \|_M^2 \quad (\tilde{G}^T N \tilde{G} = N_0, G^T MG = M_0)
\]
\[
= \| y_i - V_0 N_0^{-1} V_0^* M_0 y_i \|_M^2 \quad (G^T MG = M_0)
\]

Then, the cost function in (3.15) and (3.16) have the same minimal value when \( V = GV_0 \tilde{G}^{-1} \). Thus, if \( \Phi_0 \) is given by (3.15), then \( \Phi = G \Phi_0 \tilde{G}^{-1} \) is the optimal value in (3.16). Moreover, two cost functions achieve the same minimal value.

We note that (typical) POD satisfies optimality property (3.15) with \( M_0 = I_n \), \( N_0 = I_k \), and \( \{y_i\}_{i=1}^N \) corresponding to snapshots of the system state, while balanced POD (BPOD) [47] [42] satisfies this property with \( M_0 = \tilde{W}_0 \) and \( N_0 = \Sigma_1 \), and \( y_i, i = 1, \ldots, N \) corresponding to snapshots arising from an impulse response. Because \( \Phi \) constructed by Method 3 in Table 3.3 satisfies \( \Phi = G \Phi_0 \tilde{G}^{-1} \), Theorem 3.15 implies that \( \Phi \) inherits the optimality to minimize the projection error.

4. Reduction of pure marginally stable subsystems. This section focuses on reducing the pure marginally stable subsystem \( \dot{x}_m = A_m x_m \). While the inner-product-projection approach could be applied to the marginally stable subsystem if it has Lyapunov structure (i.e., if (2.3)–(2.4) hold), not all marginally stable subsystem exhibits this structure; further, such a reduction would not guarantee the poles remain nonzero and purely imaginary. Instead, we pursue an approach that is valid for all pure marginally stable subsystems. It is based on the key observation that all pure marginally stable systems are equivalent to a generalized Hamiltonian system.

Section 4.1 introduces LTI Hamiltonian systems and demonstrates that the marginally stable subsystem has symplectic structure (Theorem 4.9). Subsequently, Section 4.2 introduces symplectic projection. Section 4.3 demonstrates that a model-reduction method based on symplectic projection preserves symplectic structure of generalized LTI Hamiltonian systems and thus preserves pure marginal stability. Section 4.4 presents the symplectic-balancing framework, and Section 4.5 describes methods for constructing the basis matrices that lead to a symplectic projection given a subset of the required ingredients. For notational simplicity, we omit the subscript \( m \) throughout this section.

4.1. Pure marginally stable systems. We first introduce the concept of symplectic spaces, and subsequently introduce the LTI Hamiltonian and generalized LTI Hamiltonian equations. Then, Theorem 4.9 proves the key result: any pure marginally stable system (with purely nonzero imaginary eigenvalues) is a generalized Hamiltonian system.

Let \( V \cong \mathbb{R}^{2n} \) denote a vector space. A symplectic form \( \Omega : V \times V \to \mathbb{R} \) is a skew-symmetric, nondegenerate, bilinear function on the vector space \( V \). The pair \((V, \Omega)\) is called a symplectic vector space. Assigning a symplectic form \( \Omega \) to \( V \) is referred to as equipping \( V \) with symplectic structure.

By choosing canonical coordinates on \( V \), the symplectic vector space can be represented by \((\mathbb{R}^{2n}, J_{2n})\), where \( J_{2n} \in \{0,1\}^{2n \times 2n} \) is a Poisson matrix defined as
\[
J_{2n} := \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}
\]
that satisfies \( J_{2n} J_{2n}^T = J_{2n}^T J_{2n} = I_{2n} \), and \( J_{2n} J_{2n}^T = J_{2n}^T J_{2n} = -I_{2n} \). The symplectic form \( \Omega \) can be represented by the Poisson matrix as
\[
\Omega(\hat{x}_1, \hat{x}_2) = x_1^T J_{2n} x_2, \quad \forall x_1, x_2 \in \mathbb{R}^{2n},
\]
where (as before) the operator \( \hat{\cdot} \) provides the representation of an element of a vector space from its coordinates, i.e., \( \hat{x} \in V, \forall x \in \mathbb{R}^{2n} \).

Definition 4.1 (LTI Hamiltonian system). An LTI system \((A, B, C)\) is an LTI Hamiltonian system if its corresponding autonomous system is given by
(I)
\[
\dot{x} = J_{2n} \nabla_x H_0(x) = J_{2n} L_0 x,
\]
where $L_0 \in \mathbb{R}^{2n \times 2n}$ is symmetric and defines the (quadratic) Hamiltonian
\begin{equation}
H_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}, \ x \mapsto \frac{1}{2} x^T L_0 x.
\end{equation}

**Definition 4.2** (Hamiltonian matrix). A Hamiltonian matrix is given by
\begin{equation}
A_0 = J_{2n}L_0 \in \mathbb{R}^{2n \times 2n},
\end{equation}
where $L_0 \in \mathbb{R}^{2n \times 2n}$ is symmetric.

Thus, the $A$ matrix characterizing an LTI Hamiltonian system $(A, B, C)$ is a Hamiltonian matrix.

**Lemma 4.3.** $A_0 \in \mathbb{R}^{2n \times 2n}$ is a Hamiltonian matrix if and only if it satisfies
\begin{equation}
A_0^T J_{2n} + J_{2n} A_0 = 0.
\end{equation}

**Proof.** Suppose the matrix $A_0 \in \mathbb{R}^{2n \times 2n}$ is Hamiltonian. Substituting $A_0$ with $J_{2n}L_0$, (4.4) holds for any symmetric $L_0$. Conversely, suppose (4.4) holds. This implies that $J_{2n}^T A_0$ is symmetric. So $A_0 = J_{2n}L_0$ with $L_0 = J_{2n}^T A_0$ is a Hamiltonian matrix.

We note that the Hamiltonian is constant in time, as $$\frac{d}{dt} H_0(x(t)) = \frac{d}{dt} \left( \frac{1}{2} x^T L_0 x \right) = x^T L_0 \dot{x} = x^T L_0 A_0 x = x^T L_0 J_{2k} L_0 x = 0.$$ More generally, if non-canonical coordinates are chosen, the symplectic vector space $(\mathbb{V}, \Omega)$ can be represented by $(\mathbb{R}^{2n}, J_\Omega)$, where $J_\Omega \in \text{SS}(2n)$. Then, the symplectic form can be represented as
$$\Omega(\dot{x}_1, \dot{x}_2) = x_1^T J_\Omega x_2, \ \forall x_1, x_2 \in \mathbb{R}^{2n}.$$

**Definition 4.4** (Generalized LTI Hamiltonian system). An LTI system $(A, B, C)$ is a generalized LTI Hamiltonian system if its corresponding autonomous system is given by
\begin{equation}
\dot{x} = -J_\Omega^{-1} \nabla_x H(x) = -J_\Omega^{-1} L x,
\end{equation}
where $L \in \mathbb{R}^{2n \times 2n}$ is symmetric and defines the (quadratic) generalized Hamiltonian
\begin{equation}
H : \mathbb{R}^{2n} \rightarrow \mathbb{R}, \ x \mapsto \frac{1}{2} x^T L x.
\end{equation}

**Definition 4.5** (Generalized Hamiltonian matrix). A generalized Hamiltonian matrix is given by
\begin{equation}
A = -J_\Omega^{-1} L \in \mathbb{R}^{2n \times 2n},
\end{equation}
where $J_\Omega \in \text{SS}(2n)$ and $L \in \mathbb{R}^{2n \times 2n}$ is symmetric.

Thus, the $A$ matrix characterizing a generalized LTI Hamiltonian system $(A, B, C)$ is a generalized Hamiltonian matrix.

**Lemma 4.6.** $A \in \mathbb{R}^{2n \times 2n}$ is a generalized Hamiltonian matrix if and only if it satisfies
\begin{equation}
A^T J_\Omega + J_\Omega A = 0
\end{equation}
for some $J_\Omega \in \text{SS}(2n)$.

The proof is similar to that of Lemma 4.6.

As with the Hamiltonian, the generalized Hamiltonian is constant in time, as
$$\frac{d}{dt} H(x(t)) = \frac{d}{dt} \left( \frac{1}{2} x^T L x \right) = x^T L \dot{x} = x^T L A x = x^T L (-J_\Omega^{-1}) L x = 0$$
due to skew-symmetry of $J_\Omega$.

We now derive the transformation between the coordinates defining the Hamiltonian and generalized Hamiltonian systems.

**Lemma 4.7.** $J_\Omega \in \text{SS}(2n)$ if and only if there exists a $G \in \mathbb{R}^{2n \times 2n}$ such that
\begin{equation}
G^T J_\Omega G = J_{2n}.
\end{equation}

The proof is provided in [17, Corollary 5.4.4]. Note that $\text{Sp}(J_\Omega, J_{2n})$ is not empty for $J_\Omega \in \text{SS}(2n)$ by Lemma 4.7. This set represents the set of (invertible, linear) symplectic transformations $g : (\mathbb{R}^{2n}, J_{2n}) \rightarrow \mathbb{R}^{2n}$.
\((\mathbb{R}^n, J_\Omega), \ y \mapsto Gy\) with \(G \in \text{Sp}(J_\Omega, J_{2n})\) such that 
\[
x_1^T J_{2n} x_2 = (Gx_1)^T J_\Omega (Gx_2), \quad \forall x_1, x_2 \in (\mathbb{R}^n, J_{2n}).
\]

**Lemma 4.8.** \(A\) is a generalized Hamiltonian matrix if and only if it can be transformed into a Hamiltonian matrix \(A_0\) by a similarity transformation with a matrix \(G \in \mathbb{R}^{2n \times 2n}\), i.e.,
\[
A_0 = G^{-1}AG.
\]

**Proof.** Assume \(A = -J_\Omega^{-1}L\) is a generalized Hamiltonian matrix. Choose \(G \in \text{Sp}(J_\Omega, J_{2n})\) such that 
\[
G^* J_\Omega G = J_{2n}.
\]
It follows that 
\[
G^{-1}AG = G^{-1}(-J_\Omega^{-1}L)G = -G^{-1}(G^{-1} J_{2n} G^{-1})^{-1} L G = J_{2n} G^{-1} L G.
\]
By setting \(L_0 = G^* L G\) and \(A_0 = J_{2n} L_0\), the last expression implies that \(G^{-1}AG = A_0\), which is a Hamiltonian matrix.

Conversely, suppose that \(A_0 = G^{-1}AG\), where \(A_0 = J_{2n} L_0\) is a Hamiltonian matrix. Substituting \(A_0 = G^{-1}AG\) in (4.4) yields 
\[
(G^{-1}A)^T J_{2n} + J_{2n}(G^{-1}AG) = 0.
\]
Left-multiplying by \(G^{-T}\) and right-multiplying by \(G^{-1}\) yields 
\[
A^G G^{-T} J_{2n} G^{-1} + G^{-T} J_{2n} G^{-1} A = 0.
\]
Letting \(G^* J_\Omega G = J_{2n}\), the above equation is equivalent to (4.8), which implies that \(A\) is a generalized Hamiltonian matrix.

By Lemma \[4.8\] an autonomous LTI system \(\dot{x} = Ax\) can be transformed into an LTI Hamiltonian system \[4.1\] if and only if the autonomous LTI system is a generalized LTI Hamiltonian system \[4.5\], as substituting \(x = Gy\) in (4.8) yields 
\[
\dot{y} = G^{-1} \dot{x} = G^{-1}Ax = G^{-1}AGy = A_0 y.
\]

The next theorem shows that the any pure marginally stable system \(\dot{x} = Ax\) with \(A \in \text{GH}(2n)\) is a generalized LTI Hamiltonian system with \(L \in \text{SPD}(2n)\), where \(\text{GH}(n)\) denotes the set of real-valued \(n \times n\) diagonalizable matrices with nonzero purely imaginary eigenvalues.

**Theorem 4.9.** The following conditions are equivalent:
1. \(A \in \text{GH}(2n)\).
2. \(A\) is a generalized Hamiltonian matrix whose corresponding generalized LTI Hamiltonian system is marginally stable.
3. There exists \(G \in \mathbb{R}^{2n \times 2n}\) such that \(G^{-1}AG = J_{2n} L_0\), where \(L_0 = \text{diag}(\beta, \beta)\) with \(\beta = \text{diag}(\beta_1, \ldots, \beta_n)\) and \(\beta_i > 0\). Further, we have 
\[
A = -J_\Omega^{-1}L, \quad J_\Omega = G^{-T} J_{2n} G^{-1}, \quad L = G^{-T} L_0 G^{-1}.
\]
Thus, \(L_0, L \in \text{SPD}(2n)\).

**Proof.** 3 \(\Rightarrow\) 1. Recall from Lemma \[2.2\] that the marginal-stability assumption is equivalent to assuming that eigenvalues of \(A\) have non-positive real parts and all Jordan blocks corresponding to eigenvalues with zero real parts are \(1 \times 1\). Now, assume that \(A\) is a generalized Hamiltonian matrix whose eigenvalues have non-positive real parts. By definition, a generalized Hamiltonian matrix \(A\) satisfies \(A^T J_\Omega + J_\Omega A = 0\) for some \(J_\Omega \in \text{SS}(2n)\). Thus, \(A = J_\Omega^{-1}(-A^T)J\), i.e., \(A\) is similar to \(-A^T\). So, if \(\lambda\) is an eigenvalue of \(A\), it is an eigenvalue of \(-A^T\) and thus an eigenvalue of \(-A\). This implies that \(-\lambda\) is also an eigenvalue of \(A\). Thus, the eigenvalues of \(A\) would have positive real parts unless the real part is zero, i.e., the eigenvalues of \(A\) are purely imaginary. Due to the marginal-stability assumption on \(A\), every Jordan block for purely imaginary eigenvalues must have dimension \(1 \times 1\). Therefore, \(A\) is diagonalizable and has only nonzero pure imaginary eigenvalues, i.e., \(A \in \text{GH}(2n)\).

1 \(\Rightarrow\) 2. Assume \(A \in \text{GH}(2n)\). Let \(\lambda\) be an eigenvalue of \(A\). Then \(\lambda\) is a root of the characteristic polynomial \(\det(\lambda I_{2n} - A) = 0\). Because the matrix \(A\) is a real matrix, the characteristic polynomial only contains real coefficients of \(\lambda\). Thus, if \(i\beta\) with \(\beta \in \mathbb{R}\) is a root of \(\det(\lambda I_{2n} - A) = 0\), so is \(-i\beta\). Moreover, \(i\beta\) and \(-i\beta\) must have the same algebraic multiplicity. It follows that \(A\) contains eigenvalues of the form.
\{\pm i\beta_1, \ldots, \pm i\beta_n\}, \text{ where } \beta_1 \geq \ldots \geq \beta_n > 0. \text{ Because the system matrix } A \text{ is assumed to be diagonalizable, there exists a matrix } P_1 \in \mathbb{C}^{2n \times 2n} \text{ such that}

\begin{equation}
A = P_1 \text{ diag}(i\beta_1, -i\beta_1, \ldots, i\beta_n, -i\beta_n) P_1^{-1}.
\end{equation}

Let \( \beta = \text{ diag}(\beta_1, \ldots, \beta_n) \), it is straightforward to verify that the matrix \( J_{2n} \text{ diag}(\beta, \beta) \) also contains eigenvalues \( \{\pm i\beta_1, \ldots, \pm i\beta_n\} \) and is diagonalizable. Thus, there exists a matrix \( P_2 \in \mathbb{C}^{2n \times 2n} \) such that

\begin{equation}
J_{2n} \text{ diag}(\beta, \beta) = P_2 \text{ diag}(i\beta_1, -i\beta_1, \ldots, i\beta_n, -i\beta_n) P_2^{-1}.
\end{equation}

With \( P_3 = P_1 P_2^{-1} \in \mathbb{C}^{2n \times 2n} \), we have

\begin{equation}
P_3^{-1} A P_3 = P_2 (P_1^{-1} A P_1) P_2^{-1}
= P_2 \text{ diag}(i\beta_1, -i\beta_1, \ldots, i\beta_n - i\beta_n) P_2^{-1}
= J_{2n} L_0.
\end{equation}

where \( L_0 = \text{ diag}(\beta, \beta) \in \mathbb{R}^{2n} \) is symmetric. Equation (4.14) implies that \( A \) is similar to the Hamiltonian matrix \( J_{2n} L_0 \) via a complex matrix \( P_3 \). Let \( A_0 = J_{2n} L_0 \), we can also rewrite (4.14) as

\[ A P_3 = P_3 A_0. \]

Decomposing this matrix as \( P_3 = P_4 + iP_5 \) with \( P_4, P_5 \in \mathbb{R}^{2n \times 2n} \) and noting that both \( A \) and \( A_0 \) are real matrices, the above equation implies that

\[ A P_4 = P_4 A_0, \quad A P_5 = P_5 A_0. \]

This implies that \( A \) is similar to \( A_0 \) via a real matrix \( P_4 + \alpha P_5 \) for any \( \alpha \in \mathbb{R} \). Because \( \text{det}(P_4) = \text{det}(P_4 + iP_5) \neq 0 \), \( P(\alpha) := \text{det}(P_4 + \alpha P_5) \) is a nonzero polynomial of \( \alpha \) with degree no greater than \( 2n \). Thus, the equation \( P(\alpha) = 0 \) contains \( 2n \) roots at most. Thus, we can choose \( \alpha_0 \in \mathbb{R} \) such that \( G = P_4 + \alpha_0 P_5 \) is invertible. Thus, we obtain \( G^{-1} A G = J_{2n} L_0 \) with \( G \in \mathbb{R}^{2n \times 2n} \). It follows that

\[ A = G J_{2n} L_0 G^{-1} = (G J_{2n} G^T)(G^{-T} L_0 G^{-1}) = -J_{14} L, \]

where \( J_{14} = G^{-T} J_{2n} G^{-1} \) and \( L = G^{-T} L_0 G^{-1} \).

2 \( \Rightarrow \) 3. Suppose \( G^{-1} A G = J_{2n} L_0 \in \mathbb{R}^{2n \times 2n} \) with a matrix \( G \in \mathbb{R}^{2n \times 2n} \). Lemma 4.8 implies that \( A \) is a generalized Hamiltonian matrix. Because \( J_{2n} \text{ diag}(\beta, \beta) \) also contains eigenvalues \( \{\pm i\beta_1, \ldots, \pm i\beta_n\} \), so is \( A \). Thus, the corresponding system of \( A \) is marginally stable.

The part 1 \( \Rightarrow \) 2 is a constructive proof. Algorithm 2 lists the detailed procedure.

**Algorithm 2** Transform \( A \in \text{GH}(2n) \) into a canonical Hamiltonian matrix.

**Input:** \( A \in \text{GH}(2n) \).

**Output:** \( G \in \mathbb{R}^{2n \times 2n} \) and \( \beta = \text{ diag}(\beta_1, \ldots, \beta_n) \) satisfying \( G^{-1} A G = J_{2n} L_0, L_0 = \text{ diag}(\beta, \beta) \).

1. Compute the eigenvalue decomposition (4.12) of \( A \) to obtain the eigenvalues \( \{\pm i\beta_1, \ldots, \pm i\beta_n\} \) and the transformation matrix \( P_1 \in \mathbb{C}^{2n \times 2n} \).
2. Construct the matrix \( A_0 = J_{2n} L_0 \), where \( L_0 = \text{ diag}(\beta, \beta) \) with \( \beta = \text{ diag}(\beta_1, \ldots, \beta_n) \).
3. Compute the eigenvalue decomposition (4.13) of \( A_0 \) to obtain the transformation matrix \( P_2 \in \mathbb{C}^{2n \times 2n} \).
4. Compute \( P_3 = P_1 P_2^{-1} \in \mathbb{C}^{2n \times 2n} \).
5. Decompose \( P_3 = P_4 + iP_5 \) with \( P_4, P_5 \in \mathbb{R}^{2n \times 2n} \) and define \( P(\alpha) := P_4 + \alpha P_5 \).
6. \( \alpha \leftarrow 0 \)
7. **while** \( \text{det}(P(\alpha)) = 0 \) and \( \alpha < 2n \) **do**
8. \( \alpha \leftarrow \alpha + 1 \).
9. **end while**
10. \( G = P(\alpha) \).

Theorem 4.9 implies that performing model reduction in a manner that preserves generalized Hamiltonian structure and marginal stability will ensure that the reduced-order model retains pure marginal stability. We will accomplish this via symplectic projection.

**Corollary 4.10** (Dual version of Theorem 4.9). If any condition of Theorem 4.9 holds, then the following conditions hold:

1. \( -A^\tau \in \text{GH}(2n) \).
2. $-A^\tau$ is a generalized Hamiltonian matrix whose corresponding generalized LTI Hamiltonian system is marginally stable.

3. With $G$ and $L_0$ defined in Theorem 4.4, we have $G^\tau(-A^\tau)G^{-\tau} = J_{2n}L_0$. Moreover, we can have
   \begin{equation}
   -A^\tau = -J_{\Omega}^{-1}L', \quad J_{\Omega} = GJ_{2n}G^\tau, \quad L' = GL_0G^\tau.
   \end{equation}

Proof. Suppose $A \in \text{GH}(2n)$. Then, we have (4.12). It follows that
   \begin{equation}
   -A^\tau = P_1^{-1} \text{diag}(-i\beta_1, i\beta_1, \ldots, -i\beta_n, i\beta_n)P_1^*$
   \end{equation}
   with $P_1 \in \mathbb{R}_{2n \times 2n}^*$. This implies $-A^\tau \in \text{GH}(2n)$, i.e., the first condition holds. By Theorem 4.9, conditions 1 and 2 in this corollary are equivalent.

Using condition 3 in Theorem 4.9, i.e., $G^{-1}AG = J_{2n}L_0$ with $L_0 = \text{diag}(\beta, \beta)$, we have
   \begin{equation}
   G^{-1}AG = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}.
   \end{equation}
and thus
   \begin{equation}
   G^\tau(-A^\tau)G^{-\tau} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} = J_{2n}L_0.
   \end{equation}

Defining $J_{\Omega} = GJ_{2n}G^\tau$ and $L' = GL_0G^\tau$, we can verify $-A^\tau = -J_{\Omega}^{-1}L'$.

**Remark 4.11 (Relationship with dual system: pure marginal stability).** Thus, any method proposed in this work for ensuring pure marginal stability of a given (sub)system also ensures pure marginal stability of the associated negative dual (sub)system. However, as before, the proposed methods for constructing a trial basis matrix $\Phi$ should be applied to the negative dual system as a test basis matrix. Similarly, the proposed methods for constructing a test basis matrix $\Psi$ should be applied to the negative dual system as a trial basis matrix.

### 4.2. Symplectic projection of spaces.

Let $(\mathbb{V}, \Omega)$ and $(\mathbb{W}, \Pi)$ be two symplectic vector spaces with coordinate representations $(\mathbb{R}^{2n}, J_\Omega)$ and $(\mathbb{R}^{2k}, J_\Pi)$, respectively, $\dim(\mathbb{V}) = 2n$, $\dim(\mathbb{W}) = 2k$, and $k \leq n$.

**Definition 4.12 (Symplectic lift).** A symplectic lift is a linear mapping $\phi : (\mathbb{W}, \Pi) \to (\mathbb{V}, \Omega)$ that preserves symplectic structure:
   \begin{equation}
   \Pi(\hat{z}_1, \hat{z}_2) = \Omega(\phi(\hat{z}_1), \phi(\hat{z}_2)), \quad \forall \hat{z}_1, \hat{z}_2 \in \mathbb{W}.
   \end{equation}

**Definition 4.13 (Symplectic projection).** Let $\phi : (\mathbb{W}, \Pi) \to (\mathbb{V}, \Omega)$ be a symplectic lift. The adjoint of $\phi$ is the linear mapping $\psi : (\mathbb{V}, \Omega) \to (\mathbb{W}, \Pi)$ satisfying
   \begin{equation}
   \Pi(\psi(\hat{x}), \hat{z}) = \Omega(\hat{x}, \phi(\hat{z})), \quad \forall \hat{x} \in \mathbb{V}, \hat{z} \in \mathbb{W}.
   \end{equation}
We say $\psi$ is the symplectic projection induced by $\phi$.

As in the case of the inner-product lift and projection, the symplectic lift and projection can be expressed in coordinate space as
   \begin{equation}
   \phi(\hat{z}) \equiv \Phi z, \quad \forall z \in \mathbb{R}^{2k}, \quad \psi(\hat{x}) \equiv \Psi^\tau x, \quad \forall x \in \mathbb{R}^{2n},
   \end{equation}
respectively, where (4.16)–(4.17) imply that $\Phi \in \mathbb{R}^{2n \times 2k}$ and $\Psi \in \mathbb{R}^{2n \times 2k}$ satisfy
   \begin{align}
   &\Phi^\tau J_\Omega \Phi = J_\Pi \\
   &\Psi J_\Pi = J_\Omega \Phi,
   \end{align}
from which it follows that
   \begin{equation}
   \Psi = J_\Omega \Phi J_\Pi^{-1}.
   \end{equation}

When (4.18) holds, we say $\Phi$ is a symplectic matrix with respect to $J_\Omega$ and $J_\Pi$, which we denote by $\Phi \in \text{Sp}(J_\Omega, J_\Pi)$. As in the inner-product projection case, it can be verified that $\Psi^\tau$ is a left inverse of $\Phi$, as
   \begin{equation}
   \Psi^\tau \Phi = (J_\Omega \Phi J_\Pi^{-1})^\tau \Phi = J_\Pi^{-1} (\Phi^\tau J_\Pi \Phi) = J_\Pi^{-1} J_\Pi = I_{2k},
   \end{equation}
which implies that $\psi \circ \phi$ is the identity map on $\mathbb{W}$.

### 4.3. Symplectic projection of dynamics.

This section first defines symplectic projection of dynamics. We show that if the original system is a generalized Hamiltonian LTI system, then the reduced system
constructed by symplectic projection is also a generalized Hamiltonian LTI system.

**Definition 4.14** (Model reduction via symplectic projection). A reduced-order model \((\hat{A}, \hat{B}, \hat{C})\) with \(\hat{A} = \Psi^T A \Phi, \hat{B} = \Psi^T B, \text{ and } \hat{C} = C \Phi\) is constructed by a symplectic projection if \(\Phi \in \text{Sp}(J_\Omega, J_\Pi)\) and \(\Psi = J_\Omega \Phi J_\Pi^{-1}\), where \(J_\Omega \in \text{SS}(2n)\) and \(J_\Pi \in \text{SS}(2k)\).

**Lemma 4.15.** If the original LTI system \((A, B, C)\) is a generalized LTI Hamiltonian system—i.e., \(A = -J_\Omega^{-1}L\) with \(J_\Omega \in \text{SS}(2n)\) and \(L \in \mathbb{R}_{2n \times 2n}^\ast\) is symmetric—and the reduced-order model is constructed by symplectic projection, then the reduced-order model \((\hat{A}, \hat{B}, \hat{C})\) remains a generalized LTI Hamiltonian system.

**Proof.** Because \(A = -J_\Omega^{-1}L\) and \(\Phi \in \text{Sp}(J_\Omega, J_\Pi)\), we have from (4.20) that \(\hat{A} = \Psi^T A \Phi = (J_\Pi^{-1} \Phi^T J_\Pi)(-J_\Omega^{-1}L)\Phi = -J_\Pi^{-1}(\Phi^T L \Phi)\). Noting that \(J_\Pi \in \text{SS}(2k)\) and \(\Phi^T L \Phi \in \mathbb{R}_{2k \times 2k}^\ast\) is symmetric, we conclude that \(\hat{A}\) is also a generalized Hamiltonian matrix.

**Theorem 4.16** (Preservation of pure marginal stability). Suppose the original system \((A, B, C)\) is pure marginally stable, i.e., \(A \in \text{GH}(2n)\). Then the reduced system \((\hat{A}, \hat{B}, \hat{C})\) constructed by symplectic projection remains pure marginally stable, i.e., \(\hat{A} = -J_\Pi^{-1}L \in \text{GH}(2k)\), with \(L = \Phi^T L \Phi\).

**Proof.** The proof of Theorem 4.9 demonstrated that if \(A \in \text{GH}(2n)\), then there exists a similarity transformation \(A = G K G^{-1}\) with \(G \in \mathbb{R}_{2n \times 2n}^\ast\), such that the system matrix \(A\) can be transformed into a Hamiltonian matrix \(A_0 = J_{2n} L_0\) with \(L_0 = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n)\) and \(\beta_1 \geq \ldots \geq \beta_n > 0\). Theorem 4.9 also states that \(A \in \text{GH}(2n)\) implies that \(A\) is a generalized Hamiltonian matrix with marginal stability.

Thus, we can also write \(A = -J_\Omega^{-1}L\), where \(J_\Omega = G^{-T} J_{2n} G^{-1}\) and \(L = G^{-T} L_0 G^{-1}\). By Lemma 4.15, the reduced system matrix \(\hat{A}\) constructed by symplectic projection preserves the symplectic structure. In particular, \(\hat{A} = -J_\Pi^{-1}L\), where \(J_\Pi = \Phi^T J_{2n} L_0\), \(L = \Phi^T L \Phi\), and \(\Phi \in \text{Sp}(J_\Omega, J_\Pi)\). It follows that

\[
\hat{L} = \Phi^T L \Phi = \Phi^T (G^{-T} L_0 G^{-1}) \Phi = (G^{-1} \Phi)^T L_0 (G^{-1} \Phi).
\]

Because \(L_0 \in \text{SPD}(2n)\), we have \(\hat{L} \in \text{SPD}(2k)\).

Let \(\hat{H} : x \mapsto \frac{1}{2} x^T \hat{L} x\) denote the Hamiltonian function of the reduced system \(\dot{z} = \hat{A} z\). Because \(\hat{L} \in \text{SPD}(2k)\), there exists a \(\delta > 0\) such that \(\hat{H}(z) > \hat{H}(z_0)\) for all \(\|z\| = \delta\), where \(z_0\) is the initial condition. Because the reduced system is a generalized Hamiltonian system, the Hamiltonian function satisfies \(\hat{H}(z(t)) = \hat{H}(z_0)\) for all \(t \geq 0\). Thus, \(\|z(t)\| < \delta\) for all \(t \geq 0\), i.e., the reduced-order-model solution is uniformly bounded. Because the reduced system is also linear, it is marginally stable.

Finally, since \(\hat{A}\) is a generalized Hamiltonian matrix with marginal stability, we obtain \(\hat{A} \in \text{GH}(2k)\), by Theorem 4.9.

**4.4. Symplectic balancing.** In analogue to Section 3.4, we now describe a symplectic-balancing approach that leverages symplectic structure. Recall that Table 3.1 compares the proposed symplectic-balancing approach with inner-product balancing.

**Definition 4.17** (Symplectic balancing). Given any \(\Xi, \Xi' \in \text{SPD}(n)\) and \(G \in \text{Sp}(J_\Omega, J_{2n})\), the trial and test bases characterize a symplectic balancing correspond to

\[
\Phi = G \text{ diag}(\bar{\Phi}, \bar{\Psi}) \quad \text{and} \quad \Psi = G^{-T} \text{ diag}(\bar{\Psi}, \bar{\Phi}),
\]

where matrices \((\bar{\Phi}, \bar{\Psi})\) characterize an inner-product balancing on matrices \(\Xi\) and \(\Xi',\ i.e.,

\[
\bar{\Phi} = S V_1 \Sigma_1^{-1/2} \quad \text{and} \quad \bar{\Psi} = R U_1 \Sigma_1^{-1/2},
\]

where quantities \((R, S, U_1, \Sigma_1, V_1)\) are defined in Definition 3.9.

**Lemma 4.18.** A symplectic balancing characterized by the test and trial basis matrices \((\Psi, \Phi)\) with \(\Xi, \Xi' \in \text{SPD}(n)\) has the following properties:

1. The test and trial subsystem basis matrices \((\bar{\Psi}, \bar{\Phi})\) balance \(\Xi\) and \(\Xi'\), i.e., \(\bar{\Phi} \in O(\Xi, \Sigma_1)\) and \(\bar{\Psi} \in O(\Xi', \Sigma_1)\).

2. The test and trial (full-system) basis matrices \((\Psi, \Phi)\) balance \(M = G^{-T} \text{ diag}(\Xi, \Xi') G^{-1}\) and \(M' = G \text{ diag}(\Xi', \Xi) G^T\), i.e., \(\Phi \in O(M, \text{ diag}(\Sigma_1, \Sigma_1))\) and \(\Psi \in O(M', \text{ diag}(\Sigma_1, \Sigma_1))\).

3. The basis matrices \((\Psi, \Phi)\) correspond to a symplectic projection with \(J_\Omega\) given by the generalized Hamiltonian matrix \(A\) and \(J_{2n} = J_{2k}\).
The basis matrices \((\Phi, \Psi)\) correspond to a symplectic projection performed on the negative dual system with \(J_{1n}\) given by the generalized Hamiltonian matrix \(-A^T\) and \(J_{W} = J_{2k}\).

**Proof.** For Property 1, we have from the definition of inner-product balancing that basis matrices \((\Psi, \Phi)\) balance \(\Xi\) and \(\Xi'\), as \(\Phi \in O(\Xi, \Sigma_1)\) and \(\Psi \in (\Xi', \Sigma_1)\). Thus, we have \((\Phi)^T \Xi \Psi = (\Psi)^T \Xi' \Psi = \Sigma_1\) and \((\Phi)^T \Phi = (\Psi)^T \Psi = I_k\).

For Property 2, we verify that basis matrices \((\Psi, \Phi)\) balance \(M\) and \(M'\), as
\[
\Phi^T M \Phi = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} \Phi^T \Xi \Phi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_1 \end{pmatrix},
\]
and
\[
\Psi^T M' \Psi = \begin{pmatrix} G^- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G^- & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} (\Phi)^T \Xi' \Phi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_1 \end{pmatrix},
\]
thus \(\Phi \in O(M, \mathrm{diag}(\Sigma_1, \Sigma_1))\) and \(\Psi \in O(M', \mathrm{diag}(\Sigma_1, \Sigma_1))\).

For Property 3, we verify that \(\Phi \in \text{Sp}(J_{1n}, J_{2k})\) and \(\Psi = J_{1n} \Phi J_{2k}^{-1}\). Using \(G^TJ_{1n}G = J_{2n}\) and \((\Phi)^T \Phi = (\Psi)^T \Psi = I_k\), we obtain
\[
\Phi^T J_{1n} \Phi = \begin{pmatrix} \Phi^T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G^T J_{1n} G & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (\Phi)^T \Xi \Phi & 0 \\ 0 & 0 \end{pmatrix} = J_{2k},
\]
and
\[
J_{1n} \Phi J_{2k}^{-1} = (G^T J_{2n} G^{-1}) \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix} = (-J_{2k}) = G^T \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix} = \Psi.
\]
Thus, the conditions for a symplectic projection are satisfied.

We next consider the negative dual system characterized by matrix \(-A^T\). From Theorem 4.9, this system is a generalized Hamiltonian satisfying (4.15). Thus, we must verify that these bases lead to a symplectic projection on this negative dual system, i.e., \(\Psi \in \text{Sp}(J_{1n}', J_{2k})\) and \(\Phi = J_{1n} \Psi J_{2k}^{-1}\). From the definition of the symplectic-balancing test and trial basis (4.22)–(4.23), we verify that
\[
\Psi^T J_{1n} \Psi = \begin{pmatrix} (\Psi)^T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (\Psi)^T \Xi \Psi & 0 \\ 0 & 0 \end{pmatrix} = J_{2k},
\]
and
\[
J_{1n} \Psi J_{2k}^{-1} = -G J_{2n} G^T \begin{pmatrix} \Psi & 0 \\ 0 & 0 \end{pmatrix} = -G J_{2n} \begin{pmatrix} \Psi & 0 \\ 0 & 0 \end{pmatrix} = J_{2k}.
\]
Thus, the conditions for a symplectic projection on the negative dual system are satisfied.

**Corollary 4.19.** A symplectic-balancing reduced-order model preserves pure marginal stability.

**Proof.** The result follows directly from Theorem 4.9 and Lemma 4.18.

**Theorem 4.20.** Performing symplectic balancing with \(\Xi = \Xi' = \beta = \text{diag}(\beta_1, \ldots, \beta_n)\) preserves pure marginal stability and balances the Hamiltonians of the primal and the negative dual systems, i.e., \(\Phi \in O(L, \Sigma_1), \Psi \in O(L', \Sigma_1)\).

**Proof.** The result follows trivially from Lemma 4.18 and Corollary 4.19 verifying that \(M = G^{-T} \text{diag}(\Xi, \Xi') G^{-1} = L\) and \(M' = G \text{diag}(\Xi', \Xi) G'^{-1} = L'\) when \(\Xi = \Xi' = \beta\).

Thus, in analogue to inner-product balancing for asymptotically stable systems, performing symplectic balancing with \(M = L\) and \(M' = L'\) (i.e., \(\Xi = \Xi' = \beta = \text{diag}(\beta_1, \ldots, \beta_n)\)) not only preserves stability in the appropriate (i.e., pure marginal) sense, it also balances the quadratic energy functionals that characterize the primal and the dual systems.

In analogue to Section 3.5, the next section presents three algorithms (including symplectic balancing) for constructing basis matrices that ensure symplectic projection given a subset of the required ingredients.

**4.5. Construction of basis matrices given a subset of ingredients.** Theorem 4.16 demonstrated that a ROM will preserve marginally stable if it is constructed via symplectic projection when the original system has a symplectic structure. Unfortunately, most other model reduction methods, such as POD–Galerkin and balanced POD, does not preserves the symplectic structure, consequently the reduced model can be unstable.

We propose three methods (including symplectic balancing) for constructing a stability-preserving symplectic projection satisfying the conditions of Definition 4.14. Table 3.2 summarizes these methods; this corresponds to Steps 6–8 in Algorithm 1.
Method 2 constructs a stability-preserving symplectic projection starting with any trial-basis matrix satisfying $\Phi \in \text{Sp}(J_1, J_1)$. Method 3 constructs a stability-preserving symplectic projection starting with a basis $\Phi_0 \in O(J_{2n}, J_{2n})$, $J_1 \in \text{SS}(2k)$, and $J_3 \in \text{SS}(2n)$ satisfying (4.8). Method 3 follows the same steps as Method 2 with $g$ defined in terms of $\Phi_0 = G\Phi_0 G^{-1}$ as described in Lemma 4.21.

**Lemma 4.21.** Let $\phi : (\mathbb{R}^{2k}, J_{11}) \to (\mathbb{R}^{2n}, J_{21})$ denote a symplectic lift with matrix presentation $\Phi \in \text{Sp}(J_{11}, J_{11})$. Let $g : (\mathbb{R}^{2n}, J_{21}) \to (\mathbb{R}^{2n}, J_{11})$ and $g : (\mathbb{R}^{2k}, J_{2k}) \to (\mathbb{R}^{2k}, J_{11})$ represent symplectic transformations, represented by $G \in \text{Sp}(J_{11}, J_{21})$ and $\tilde{G} \in \text{Sp}(J_{11}, J_{2k})$ respectively. Then, there exists a unique symplectic lift $\phi_0 : (\mathbb{R}^{2k}, J_{2k}) \to (\mathbb{R}^{2n}, J_{21})$, represented by $\Phi_0 \in \text{Sp}(J_{2n}, J_{2k})$, such that the following diagram commutes:

\[
\begin{array}{ccc}
(\mathbb{R}^n, J_{2n}) & \xrightarrow{g} & (\mathbb{R}^{2n}, J_{21}) \\
\phi_0 \downarrow & & \downarrow \phi \\
(\mathbb{R}^k, J_{2k}) & \xleftarrow{g^{-1}} & (\mathbb{R}^{2k}, J_{11})
\end{array}
\]

Equivalently, for all $z \in (\mathbb{R}^{2k}, J_{11})$,

\[(4.24)\] 

$\phi(z) = g(\phi_0(\tilde{g}^{-1}(z)))$,

and $\Phi = G\Phi_0 \tilde{G}^{-1}$ in matrix representation.

**Proof.** Because $G \in \text{Sp}(J_{11}, J_{21})$, we have $G^* J_{11} G = J_{21}$. It follows that $G^{-1} J_{21} G^{-1} = J_{11}$. By the same argument, $\tilde{G} \in \text{Sp}(J_{11}, J_{2k})$ implies that $G^* J_{11} \tilde{G} = J_{2k}$ and $G^{-1} J_{2k} G^{-1} = J_{11}$. Because $\Phi \in \text{Sp}(J_{11}, J_{11})$, we have $\Phi^* J_{11} \Phi = J_{11}$. Because $g$ is invertible, we can define $\phi_0 : (\mathbb{R}^{2k}, J_{2k}) \to (\mathbb{R}^{2n}, J_{21})$ by $\tilde{g}^{-1} \circ \phi \circ \tilde{g}$ with matrix representation $\Phi_0 = G^{-1} \Phi \tilde{G}$. It follows that $\Phi_0 J_{2k} \Phi_0 = G^* \Phi^* G^{-1} \Phi_0 \tilde{G} = G^* (\Phi^* J_{11} \Phi) \tilde{G} = G^* J_{11} \tilde{G} = J_{2k}$. The last equation implies that $\Phi_0 \in \text{Sp}(J_{2n}, J_{2k})$. Finally, if $\phi_0$ satisfies $\phi = g \circ \phi_0 \circ \tilde{g}^{-1}$, $\phi_0$ is uniquely determined by $\phi_0 = g^{-1} \circ \phi \circ \tilde{g}$. 

Apart from the symplectic-balancing approach we propose, there is no standard method to construct a trial basis matrix satisfying $\Phi \in \text{Sp}(J_{11}, J_{11})$. However, Ref. [39] proposed several empirical methods to construct $\Phi_0 \in \text{Sp}(J_{2n}, J_{2k})$, including cotangent lift (reviewed in Appendix C), the complex SVD, and nonlinear optimization. Alternatively, we can also use a greedy algorithm [40] to construct $\Phi_0 \in \text{Sp}(J_{2n}, J_{2k})$ from empirical data.

**5. Numerical examples.** This section illustrates the performance of the proposed structure-preserving method (SP) using two numerical examples. We compare the full-order model with reduced-order models
constructed by POD–Galerkin (POD) (Appendix E.1), shift-reduce-shift-back method (SRSB) (Appendix E.4), balanced POD (BPOD) (Appendix E.3), as well as the proposed structure-preserving (SP) method. For reference, Table E.1 reports the algorithms for the existing model-reduction methods. For simplicity, we focus on autonomous systems \( \dot{x} = Ax \) and employ the analytical solution \( x(t) = \exp(At)x_0 \) as the ‘truth’ solution. When applying BPOD and SRSB—which require a full \((A, B, C)\) description—we set \( B = C^\tau = x_0 \). For POD–Galerkin (see Appendix E.4), we employ \( N_s \) snapshots \( \{x(i\Delta t)\}_{i=0}^{N_s-1} \) with \( \Delta t \) the specified snapshot interval. For balanced POD, we compute the primal and dual snapshots according to (E.4) and (E.5), respectively, with the same snapshot interval \( \Delta t \). For SRSB, we must define only the shift margin \( \mu \). For each example, we compare two different SP methods: a POD-like method and a balancing-based method.

For time discretization, we define a uniform grid \( t_i, i = 0, \ldots, N \) with \( t_0 = 0 \) and \( t_N = t_f \), which employs a uniform time step \( \delta t \) such that \( t_i = t_{i-1} + \delta t \), \( i = 1, \ldots, N \). We apply the midpoint rule \( x(t_{i+1}) = x(t_i) + \frac{\delta t}{2}A(x(t_i) + x(t_{i+1})) \) for performing time integration of both the full-order and reduced-order models. When \( A \) is a Hamiltonian matrix, this scheme corresponds to a symplectic integrator; this ensures that the time-discrete system will inherit any Hamiltonian structure that exists in the time-continuous system.

To assess the accuracy of each method, we define the relative state-space error as
\[
\eta = \left( \frac{\sum_{i=1}^{N} ||x(t_i) - \hat{x}(t_i)||^2_2}{\sum_{i=1}^{N} ||x(t_i)||^2_2} \right)^{1/2},
\]
(5.1)
where \( x(t_i) \) and \( \hat{x}(t_i) \) denote the full-order-model and reduced-order-model solutions computed at time instance \( t_i \). We also consider the relative system-energy error as
\[
\eta_E = \left( \frac{\sum_{i=1}^{N} (E(x(t_i)) - E(\hat{x}(t_i)))^2}{\sum_{i=1}^{N} E(x(t_i))^2} \right)^{1/2},
\]
(5.2)
where the energy \( E \) is defined in (D.1) of Appendix D. Here, the Lyapunov matrix \( M = \Theta \) satisfies the Lyapunov equation (2.4) with \( Q = I_n \), and \( L \) is derived directly from the Hamiltonian matrix of the pure marginally stable subsystem.

### 5.1. A 1D example
To provide a simple illustration of the merits of the proposed technique, we first consider a simple linear system with \( n = 8 \), where
\[
A = \begin{bmatrix} -8 & -29 & -72 & -139 & -192 & -171 & -128 & -60 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]
(5.3)
The eigenvalue decomposition of \( A \) gives \( A = P_1 \Lambda P_1^{-1} \), where \( \Lambda = \text{diag}(-3, -2 + i, -2 - i, -1, 2i, -2i, i, -i) \) and \( P_1 \in \mathbb{C}^{8 \times 8} \). Thus, the original system is marginally stable and thus (2.8) holds with
\[
A_s = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_m = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.
\]
(5.4)
For this problem, we obtain \( M = \Theta = \text{diag} \left( \frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right) \) (which satisfies the Lyapunov equation (2.4) with \( Q = I_4 \) and \( J_3 = J_4 \) (which satisfies the Hamiltonian property (4.8)). Because \( \text{diag}(A_s, A_m) = P_2 \Lambda P_2^{-1} \) with \( P_2 \in \mathbb{C}^{8 \times 8} \), we can define \( P_3 = P_1 P_2^{-1} \in \mathbb{C}^{8 \times 8} \). Then, \( A = P_3 \text{diag}(A_s, A_m) P_3^{-1} \) holds. Then, we can choose \( \alpha \) such that \( T = \text{Re}(P_3) + \alpha \text{Im}(P_3) \in \mathbb{R}^{8 \times 8} \). In our test, \( \alpha = 0 \) gives an nonsingular \( T \). Note that this procedure is similar to Algorithm 2 where \( A \in \text{GH}(2n) \) is transformed into a canonical form.

We test two SP methods; both of them reduce \( A_s \) and \( A_m \) from dimension \( n/2 \) to dimension \( k/2 \). The
first SP method, SP1, is a POD-like method. SP1 applies Method 3 in Table 3.3 for the asymptotically stable subsystem, where $\Phi_0$ is computed via POD with snapshots $\{x_m(i\Delta t)\}_{i=0}^{N-1}$; thus, $N_0$ and $M_0$ are identity matrices of appropriate dimension. $M$ is computed by solving the Lyapunov equation (2.4) with $Q = I_n$. For the pure marginally stable subsystem, SP1 applies Method 2 in Table 4.1 to compute $\Psi$, where $\Phi$ is constructed via cotangent lift (see Algorithm 3 of Appendix C) with snapshots $\{x_m(i\Delta t)\}_{i=0}^{N-1}$, $J_D$ and $J_O$ the Poisson matrix of appropriate dimension. The second SP method, SP2, is a balancing-based method. SP2 applies the balanced truncation for the asymptotically stable subsystem and symplectic balancing with $\Sigma = \Sigma' = \text{diag}(\beta, \beta)$ for the pure marginally stable subsystem; this approach balances the primal and negative dual Hamiltonians.

We set the initial condition to the first canonical unit vector, i.e., $x_0 = e_1$. For the purpose of snapshot collection, we set the final time to $t_f = 5$ and the time step to $\delta t = 0.001$. We compute $N = 11$ snapshots with snapshot interval $\Delta t = 0.5$. For SRSB, we set the shift margin to $\mu = 0.01$.

All experiments consider reduced-order models of dimension $k = 4$. Table 5.1 compares the performance of different methods for this example; we compute the infinite-time energy via eigenvalue analysis. We choose a longer time interval characterized by $t_f = 50$ (and time step $\delta t = 0.001$) to compute the errors $\eta$ and $\eta_E$; thus, $N = 5 \times 10^4$ in (5.1) and (5.2). Figure 5.1 plots the $L^2$-norm of the state-space error $e(t):=x(t) - \hat{x}(t)$ as well as the system energy $E(t)$ as a function of time for all considered methods.

First, note that POD yields the largest state and system-energy errors, and its energy grows rapidly,
even within the considered time interval. This can be attributed to its eigenvalues \( \lambda = 0.0828 \pm 1.9679 i \), which correspond to unstable modes. Because the POD reduced-order model is unstable (its matrix \( \bar{A} \) has eigenvalues with positive real parts), its infinite-time energy will be unbounded. While SRSB yields lower errors \( \eta \) and \( \eta_E \) than POD, it has larger errors over the first part of the time interval. This can be attributed to the relatively large error in the projected initial condition, which occurs because SRSB does not employ snapshot data (which includes the initial condition). Also, even though SRSB does not preserve marginal stability, its instability margin is only \( 2 \times 10^{-4} \), which is relatively small and precludes instabilities from becoming apparent over the finite time interval considered; however, its infinite-time energy is unbounded. BPOD has smaller average errors than both POD and SRSB; however, the associated reduced-order model is asymptotically stable, which implies that its infinite-time energy is zero; thus, the reduced-order model does not have a pure marginal subsystem. Not only do the proposed SP methods produce the smallest average errors over all reduced-order models, they are also the only methods that preserve marginal stability, including the pure marginally stable subsystem. As a result, two proposed SP methods yield a finite infinite-time energy; in fact, this energy incurs a sub-1% error with respect to the infinite-time energy of the full-order model. Critically, note that extreme pure imaginary eigenvalues are exact in the case of SP2; this results from the fact that it balances the Hamiltonians directly.

5.2. 2D mass-spring system. We now consider a 2D mass-spring system. Each mass is located on a grid point of an \((\bar{n} + 2) \times (\bar{n} + 2)\) grid with \( \bar{n} = 50 \). The governing equations associated with mass \((i,j), \ i, j = 1, \ldots, \bar{n}\) is given by

\[
\begin{aligned}
m \ddot{u}_{i,j} &= k_x (u_{i+1,j} + u_{i-1,j} - 2u_{i,j}) - 2b \dot{u}_{i,j}, \\
m \ddot{v}_{i,j} &= k_y (v_{i,j+1} + v_{i,j-1} - 2v_{i,j}),
\end{aligned}
\]  

(5.5)

where \( u_{i,j} \) and \( v_{i,j} \) are state variables representing the \( x \)- and \( y \)-displacements of mass \((i,j), \ m = 1 \) denotes the mass, \( k_x \) and \( k_y \) denote spring constants with \( k_x = k_y = 2500 \), and \( b = 1 \) denotes the damping coefficient in the \( x \)-direction. We apply homogeneous Dirichlet boundary conditions \( u_{0,j} = u_{\bar{n}+1,j} = v_{i,0} = v_{i,\bar{n}+1} = 0 \).

Define canonical coordinates: \( q_{i,j} = u_{i,j}, \ p_{i,j} = m \dot{q}_{i,j}, r_{i,j} = v_{i,j}, \) and \( s_{i,j} = m \dot{r}_{i,j} \). The system Hamiltonian \( H \) is given by \( H = H_x + H_y \), where

\[
\begin{aligned}
H_x &= \frac{1}{2m} \sum_{i,j=1}^{\bar{n}} p_{i,j}^2 + \frac{k_x}{2} \sum_{i,j=0}^{\bar{n}} (q_{i+1,j} - q_{i,j})^2, \\
H_y &= \frac{1}{2m} \sum_{i,j=1}^{\bar{n}} s_{i,j}^2 + \frac{k_y}{2} \sum_{i,j=0}^{\bar{n}} (r_{i,j+1} - r_{i,j})^2.
\end{aligned}
\]

(5.6)

Now, the original system (5.5) can be represented by dissipative Hamiltonian ODEs,

\[
\begin{aligned}
\dot{q}_{i,j} &= \frac{\partial H}{\partial p_{i,j}}, \\
\dot{p}_{i,j} &= -\frac{\partial H}{\partial q_{i,j}} - \frac{2b}{m} p_{i,j}, \\
\dot{r}_{i,j} &= \frac{\partial H}{\partial s_{i,j}}, \\
\dot{s}_{i,j} &= -\frac{\partial H}{\partial r_{i,j}}.
\end{aligned}
\]

(5.7)

Let \( q = [q_{1,1} \ \cdots \ q_{1,\bar{n}} \ \cdots \ q_{\bar{n},1} \ \cdots \ q_{\bar{n},\bar{n}}]^{T} \) and \( r = [r_{1,1} \ \cdots \ r_{1,\bar{n}} \ \cdots \ r_{\bar{n},1} \ \cdots \ r_{\bar{n},\bar{n}}]^{T} \) denote the generalized coordinates. Let \( p = [p_{1,1} \ \cdots \ p_{1,\bar{n}} \ \cdots \ p_{\bar{n},1} \ \cdots \ p_{\bar{n},\bar{n}}]^{T} \) and \( s = [s_{1,1} \ \cdots \ s_{1,\bar{n}} \ \cdots \ s_{\bar{n},1} \ \cdots \ s_{\bar{n},\bar{n}}]^{T} \) denote the generalized momenta. With \( \psi_x = [q^T \ p^T]^{T} \in \mathbb{R}^{2\bar{n}^2} \) and \( \psi_y = [r^T \ s^T]^{T} \in \mathbb{R}^{2\bar{n}^2} \), the above equation can be written as matrix form, i.e.,

\[
\begin{bmatrix}
\dot{\psi}_x \\
\dot{\psi}_y
\end{bmatrix} =
\begin{bmatrix}
A_x & 0 \\
0 & A_m
\end{bmatrix}
\begin{bmatrix}
\psi_x \\
\psi_y
\end{bmatrix},
\]

where \( \dot{\psi}_x = A_x \psi_x \) represents an asymptotically stable system and \( \dot{\psi}_y = A_m \psi_y \) represents a (pure marginally stable) Hamiltonian system. Thus, the dimension of the full-order model is \( n = 4\bar{n}^2 = 1 \times 10^4 \).

Let \( \alpha(x) = \frac{|x-l/2|}{l/10} \) with \( l = 1 \) the length of the spatial interval in each direction and \( h(\alpha) \) be a cubic
Let \( x = 5.9 \). Equations AM cannot be directly used for this subsystem as well. Thus, we also compute Gramians by solving SP1 method employed in the previous example. The second SP method, SP2, is a balancing-based method.

\[ n/ \text{dimension} \text{snapshot interval } \Delta \text{M} \]  

It requires solvability of Lyapunov equations (E.6) and (E.7). Instead, we compute \((\text{the first method in Table 4.1})\) is employed with \( \Xi = \) stable subsystem, we collect snapshot ensemble \((A, q_0, \ldots, q_{N-1}) = h(\alpha(\xi, y_0))h(\alpha(y_j)), p_{i,j}(0) = s_{i,j}(0) = 0. \)  

We employ a time step of \( \delta t = 0.002 \) and set the final time to \( t_f = 15 \). Figure 5.2 depicts the initial condition and final state computed by the full-order model. In the experiment, we compute \( N_s = 101 \) snapshots with snapshot interval \( \Delta t = 0.05 \).

Because the original system is neither controllable nor observable, SRSB cannot be directly used, as it requires solvability of Lyapunov equations [E.6] and [E.7]. Instead, we compute \( M_c \) and \( M_o \) in this experiment by solving modified Lyapunov equations \( (\Delta - \mu I)M_c + M_c(\Delta - \mu I)^* = -(BB^* + \varepsilon I) \) and \( (\Delta - \mu I)^*M_o + M_o(\Delta - \mu I) = -(C^*C + \varepsilon I) \), and let \( \varepsilon = 10^{-4} \). We set the shift margin to \( \mu = 1 \).

For the reduced-order model, we again test two SP methods; both of them reduce \( A_s \) and \( A_m \) from dimension \( n/2 \) to dimension \( k/2 \). The first SP method, SP1, is a POD-like method, which is identical to the SP1 method employed in the previous example. The second SP method, SP2, is a balancing-based method. Because the asymptotically stable subsystem is neither controllable nor observable, balanced truncation cannot be directly used for this subsystem as well. Thus, we also compute Gramians by solving \( \varepsilon \)-Lyapunov equations \( AM_c + M_cA^* = -(BB^* + \varepsilon I) \) and \( A^*M_o + M_oA = -(C^*C + \varepsilon I) \). For the pure marginally stable subsystem, we collect snapshot ensemble \((x_m(i\Delta t))_{i=0}^{N-1} \) and construct two snapshot matrices \( Q = [q_0 \cdots q_{N-1}] \) and \( P = [p_0 \cdots p_{N-1}] \), where \( x_m(i\Delta t) = [q_i^T, p_i^T]^T \). Then, the symplectic balancing method (the first method in Table 4.1) is employed with \( \Xi = QQ^* \) and \( \Xi^* = PP^* \). We note that if the pure marginally stable subsystem was a generalized Hamiltonian, then the snapshot ensemble would require premultiplication.
Table 5.2
2D mass–spring example. Comparison of different model-reduction methods for reduced dimension $k = 40$.

|                  | POD | SRSB | BPOD | SP1 | SP2 | Full-order model |
|------------------|-----|------|------|-----|-----|------------------|
| Number of unstable modes | 8   | 16   | 18   | 0   | 0   | 0                |
| Instability margin max(Re($\lambda$)) | 50.480 | 10.586 | 3.695 | 0   | 0   | 0                |
| Marginal-stability preservation | No | No   | No   | Yes | Yes | Yes              |
| Relative state-space error $\eta$ | $+\infty$ | $+\infty$ | $+\infty$ | 0.11156 | 0.10214 | 0.04358 |
| Relative system-energy error $\eta_E$ | $+\infty$ | $+\infty$ | $+\infty$ | $8.6868 \times 10^{-5}$ | $4.8843 \times 10^{-3}$ | $3.413 \times 10^{-5}$ |
| Infinite-time energy | $+\infty$ | $+\infty$ | $+\infty$ | $1.9958 \times 10^{-3}$ | $1.9959 \times 10^{-3}$ | $1.9959 \times 10^{-3}$ |

Figure 5.3. 2D mass–spring example. The evolution of the state-space error $\|e(t)\| = \|x(t) - \hat{x}(t)\|$ and system energy $E(t)$ for all tested methods and reduced dimension $k = 40$.

by $G^{-1}$ to represent them in canonical coordinates.

Table 5.2 compares the performance of different reduced-order models (all of dimension $k = 40$, while Figure 5.3 plots the $\ell^2$-norm of the state-space error $e(t):=x(t) - \hat{x}(t)$ and the system energy $E(x(t))$ for those reduced-order models as a function of time. Here, the system energy is defined by the total Hamiltonian, i.e., $E(t) = H_z(\psi_x(t)) + H_y(\psi_y(t))$, and its infinite-time value is computed by eigenvalue analysis.

First, note that among all the tested methods, only the full-order model and the proposed SP reduced-order models preserve marginal stability and have finite errors $\eta$ and $\eta_E$. Further, the SP methods ensure that the reduced-order model has a pure marginally stable subsystem, and thus a finite infinite-time energy that is nearly identical to that of the full-order model. Because POD, SRSB, and BPOD have unstable modes, they yield unbounded infinite-time energy. Further, due to their relatively large instability margins, their errors and energy grow rapidly within the considered time interval, leading to significant errors.

Finally, we vary the reduced dimension between $k = 4$ to $k = 40$ to assess the effect of subspace dimension on method performance. Figure 5.2 plots the relative state-space error $\eta$ of state variable and the relative system-energy error $\eta_E$ as a function of $k$. Only the full-order model and the SP reduced-order models yield finite values of $\eta$ and $\eta_E$ for all the tested values of subspace dimension $k$. 

23
6. Conclusions. This work proposed a model-reduction method that preserves marginal stability for linear time-invariant (LTI) systems. The method decomposes the LTI system into asymptotically stable and pure marginally stable subsystems, and subsequently performs structure-preserving model reduction on the subsystems separately. Advantages of the method include

- its ability to preserve marginal stability,
- its ability to ensure finite infinite-time energy,
- its ability to balance primal and dual energy functionals for both subsystems.

A geometric perspective enabled a unified comparison of the proposed inner-product and symplectic projection methods.

Two numerical examples demonstrated the stability and accuracy of the proposed method. In particular, the proposed method yielded a finite infinite-time energy, while all other tested methods (i.e., POD–Galerkin, shift-reduce-shift-back, and balanced POD) produced an infinite (unstable) or zero (asymptotically stable) response.

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Appendix A. Canonical form of Lyapunov equation. We now prove a claim at the end of Section 2.1, which states that any Hurwitz matrix can be transformed by a similarity transformation to a matrix with negative-definite symmetric part; in other words, a similarity transform enables any Hurwitz matrix to satisfy the canonical Lyapunov inequality $A_0^\tau + A_0 \prec 0$. This is in anologue to Lemma 4.8, which shows that a generalized Hamiltonian matrix can be transformed into a Hamiltonian matrix that satisfies the canonical Hamiltonian property.

**Lemma A.1.** A Hurwitz matrix $A$ can be transformed into a matrix $A_0$ with negative symmetric part by similarity transformation with a real matrix $G$. Conversely, if the system matrix $A$ can be transformed into a matrix $A_0$ with negative symmetric part by a similarity transformation, then $A$ is a Hurwitz matrix.

**Proof.** Because $A$ is a Hurwitz matrix, there exists an SPD matrix $\Theta$ such that (3.1) holds. Defining $\Theta = \Theta^{1/2} \Theta^{-\tau/2}$ and left- and right-multiplying (3.1) by $\Theta^{-1/2}$ and $\Theta^{-\tau/2}$, respectively, yields

$$(\Theta^{-1/2} A \Theta^{1/2})(\Theta^{-1/2} \Theta \Theta^{-\tau/2}) + (\Theta^{-1/2} \Theta \Theta^{-\tau/2})(\Theta^{\tau/2} A \Theta^{-1/2}) \prec 0.$$ 

Let $A_0 = \Theta^{\tau/2} A \Theta^{-\tau/2}$. Noting that $\Theta^{-1/2} \Theta = I_n$, the above equation and implies $A_0^\tau + A_0 \prec 0$, i.e., $A_0$ has negative symmetric part.

Conversely, suppose that $A = \Theta^{-\tau/2} A_0 \Theta^{\tau/2}$, where $A_0$ has negative symmetric part and $\Theta^{1/2}$ is nonsin-
gular. Substituting $A_0 = \Theta^{\tau/2}A\Theta^{\tau/2}$ into $A_0^T + A_0 < 0$ yields

$$\Theta^{\tau/2}A\Theta^{\tau/2} + \Theta^{\tau/2}A\Theta^{\tau/2} < 0.$$  

Left- and right-multiplying the above equation by $\Theta^{1/2}$ and $\Theta^{\tau/2}$, respectively, yields

$$A^T(\Theta^{1/2}Q^{\tau/2}) + (\Theta^{1/2}Q^{\tau/2})A < 0.$$  

Because $\Theta = \Theta^{1/2}Q^{\tau/2} \in \text{SPD}(n)$, the above equation gives $A \in \text{H}(n)$ by Lemma 3.2.

### Appendix B. System decomposition in the general case.

We now extend the decomposition method (in Section 2.3) to a general case where the original system is unstable and $A$ is singular. Let $T \in \mathbb{R}^{n \times n}$ be a nonsingular matrix such that a similarity transformation gives $A = T \begin{bmatrix} A_s & 0 & 0 & 0 \\ 0 & A_m & 0 & 0 \\ 0 & 0 & A_u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1}$, where all eigenvalues of $A_u \in \mathbb{R}^{n_u \times n_u}$ have a positive real part. Substituting $x = T[x_s^T, x_m^T, x_u^T, x_f^T]^T$ into (2.1) and premultiplying the first set of equations by $T^{-1}$ yields a decoupled LTI system

$$\frac{d}{dt} \begin{bmatrix} x_s \\ x_m \\ x_u \\ x_f \end{bmatrix} = \begin{bmatrix} A_s & 0 & 0 & 0 \\ 0 & A_m & 0 & 0 \\ 0 & 0 & A_u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ x_m \\ x_u \\ x_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_m \\ B_u \\ B_f \end{bmatrix} u$$

(B.2)

$$y = [C_s \ C_m \ C_u \ C_f] \begin{bmatrix} x_s \\ x_m \\ x_u \\ x_f \end{bmatrix},$$

where $T^{-1}B = [B_s^T \ B_m^T \ B_u^T \ B_f^T]^T$ and $CT = [C_s \ C_m \ C_u \ C_f]$. Here, the subsystem associated with $x_u$ is antistable, and the subsystem associated with $x_f$ has $0$ as system matrix and is marginally stable.

In the general case characterized by decomposition (B.2), we can perform this reduction by defining test and trial basis matrices for each subsystem $\Psi_i \in \mathbb{R}^{n_i \times k_i}$, $\Phi_i \in \mathbb{R}^{n_i \times k_i}$, $i \in \{s, m, u, f\}$. Applying Petrov–Galerkin projection to (B.2) with test basis matrix $\text{diag}(\Psi_i)$ and trial basis matrix $\text{diag}(\Phi_i)$ yields a decoupled reduced LTI system

$$\frac{d}{dt} \begin{bmatrix} z_s \\ z_m \\ z_u \\ z_f \end{bmatrix} = \begin{bmatrix} \tilde{A}_s & 0 & 0 & 0 \\ 0 & \tilde{A}_m & 0 & 0 \\ 0 & 0 & \tilde{A}_u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_s \\ z_m \\ z_u \\ z_f \end{bmatrix} + \begin{bmatrix} \tilde{B}_s \\ \tilde{B}_m \\ \tilde{B}_u \\ \tilde{B}_f \end{bmatrix} u$$

(B.3)

$$y = [\tilde{C}_s \ \tilde{C}_m \ \tilde{C}_u \ \tilde{C}_f] \begin{bmatrix} z_s \\ z_m \\ z_u \\ z_f \end{bmatrix},$$

where $\tilde{A}_i = \Psi_i^T A \Phi_i \in \mathbb{R}^{k_i \times k_i}$, $\tilde{B}_i = \Psi_i^T B \in \mathbb{R}^{k_i \times p}$, $\tilde{C}_i = C \Phi_i \in \mathbb{R}^{q \times k_i}$, $i \in \{s, m, u, f\}$.

The techniques proposed in this work can be employed to construct $(\Psi_i, \Phi_i)$, $i \in \{s, m\}$, while bases $(\Psi_u, \Phi_u)$ can be computed to preserve antistability in the associated reduced subsystem (e.g., via techniques proposed in Refs. [33, 51, 41, 31]). When $A_f = 0$, $\Psi_f^T A \Phi_f = 0$ holds for any $\Psi_f$ and $\Phi_f$. Thus, we can choose any existing method, (e.g., POD, balanced truncation, and balanced POD) and the reduced subsystem associated with $x_f$ always preserves marginally stability.

### Appendix C. Cotangent lift.

The end of Section 4.5 mentions that there is no general way to construct a trial basis matrix satisfying $\Phi \in \text{Sp}(J_{Q_1}, J_{H_1})$. This section briefly reviews the cotangent lift method, which is an SVD-based method to construct $\Phi_0 \in \text{Sp}(J_{2n}, J_{2k})$; from this matrix, a trial basis matrix satisfying $\Phi \in \text{Sp}(J_{Q_1}, J_{H_1})$ can then be computed from Method 3 in Table 4.1 using $\Phi_0$ as an input.

The cotangent lift method [37] assumes that $\Phi_0$ has a block diagonal form, i.e., $\Phi_0 = \text{diag}(\bar{\Phi}, \bar{\Phi})$ for some
with a Lyapunov matrix in all cases, which precludes some methods from ensuring asymptotic-stability. Assume we have snapshots of a pure marginally stable system \( \{x_i\}_{i=1}^N \); then, we apply the inverse symplectic transformation to obtain the associated snapshots in the canonical coordinates \( \{y_i\}_{i=1}^N \) with \( y_i = G^{-1}x_i, \ i = 1, \ldots, N \). Writing the decomposition \( y_i = [q_i^T \ p_i^T]^T \in \mathbb{R}^{2n} \) with \( q_i, p_i \in \mathbb{R}^n \), \( \Phi \) can be computed by the SVD of an extended snapshot matrix \( M_{\text{cot}} = [q_1, \ldots, q_N, p_1, \ldots, p_N] \in \mathbb{R}^{n \times 2N} \).

Algorithm 3 Cotangent lift

**Input:** Snapshots \( \{x_i\}_{i=1}^N \subset \mathbb{R}^{2n} \) with and a symplectic transformation matrix \( G \) associated with a pure marginally stable system.

**Output:** A symplectic matrix \( \Phi_0 \in \text{Sp}(J_{2n}, J_{2k}) \) in block-diagonal form.

1. Apply inverse symplectic transformation to snapshots \( y_i = G^{-1}x_i, \ i = 1, \ldots, N \).
2. Form the extended snapshot matrix \( M_{\text{cot}} = [q_1, \ldots, q_N, p_1, \ldots, p_N] \), where \( y_i = [q_i^T \ p_i^T]^T \).
3. Compute the SVD of \( M_{\text{cot}} \); the basis matrix \( \Phi \) comprises the first \( k \) test singular vectors.
4. Construct the symplectic matrix \( \Phi_0 = \text{diag}(\Phi, \Phi) \).

Algorithm 3 lists the detailed procedure of the cotangent lift. Although the cotangent lift method can only find a near optimal solution to fit empirical data, we can prove that the projection error of cotangent lift is no greater than the projection error of POD with a constant factor.

**Appendix D. Generalized system energy.** If the original system is asymptotically stable, we can define a quadratic function as the system energy \([43]\). When it is marginally stable, we can extend the definition; the system energy is used in Section 5 to measure the performance of several model reduction methods. In particular With \((x_s^T, x_m^T)^T = T^{-1}x\), the system energy can be defined as

\[
E(x) = \frac{1}{2} \begin{bmatrix} x_s^T & x_m^T \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} x_s \\ x_m \end{bmatrix} = \frac{1}{2} \|x_s\|^2_M + H(x_m),
\]

where \( H \) is the Hamiltonian function of the marginally stable subsystem defined in (4.12). The time evolution of the system energy is given by

\[
\frac{d}{dt} E(x(t)) = \frac{1}{2} (x_s^T M x_s + x_s^T M x_s) + \frac{1}{2} (x_m^T L x_m + x_m^T L x_m) = \frac{1}{2} x_s^T (A_s^T M + M A_s) x_s + \frac{1}{2} x_m^T (A_m^T L + L A_m) x_m
\]

\[
= -\frac{1}{2} x_s^T Q x_s + \frac{1}{2} x_m^T (L J_{11}^{-1} L + (L J_{11})^{-1} L) x_m = -\frac{1}{2} x_s^T Q x_s.
\]

In the third equality, we use the Lyapunov equation \( A_s^T M + M A_s = -Q \), the definition \( A_m = (L J_{11})^{-1} L \), and the fact that \((L J_{11})^{-1}\) is skew-symmetric. Because \( Q \) is SPD, the system energy is decreasing for all \( x_s \neq 0 \).

**Appendix E. Review of existing model reduction methods.** In this section, we briefly review a few existing model reduction methods, including POD, balanced truncation, balanced POD, and shift-reduce-shift-back, as listed in Table 3.2. Section 3 numerically compares the performance of these methods with the proposed structure-preserving technique. We show that each of these methods exhibits an inner-product structure (see Table 3.2 in Section 3.5); however, the associated inner-product matrix \( M \) does not associate with a Lyapunov matrix in all cases, which precludes some methods from ensuring asymptotic-stability preservation.
Table E.1

| Algorithms for computing test and trial basis matrices using existing model-reduction methods |
|-------------------------------------------------------------|
| **POD–Galerkin** | Balanced truncation | Balanced POD | SRSB |
| **Input** | Snapshots X in (E.1) | (A, B, C) | Primal snapshots S in (E.4) | Dual snapshots R in (E.5) | (A, B, C) | Shift margin μ |
| **Output** | Ψ, Φ ∈ O(I_n, I_k) | Φ ∈ O(W_o, Σ_o); Ψ ∈ O(W_e, Σ_e) | Φ ∈ O(W_c, Σ_c) | Ψ ∈ O(W_e, Σ_e) | Φ ∈ O(W_c, Σ_c) |
| **Algorithm** | 1. Compute SVD X = USV^T. 2. Ψ = U. | 1. Compute W_o by (E.2) 2. Compute W_e by (E.3). 3. Compute Cholesky factors W_o = SS^T, W_e = RR^T. 4. Compute SVD R^T S = USV^T. 5. Φ = SV_1Σ_1^{-1/2}. 6. Ψ = RU_1Σ_1^{-1/2}. | 1. Compute SVD W^T_o = USV^T. 2. Φ = SY_1Σ_1^{-1/2}. 3. Ψ = RU_1Σ_1^{-1/2}. | 1. Compute W_o by (E.6) 2. Compute W_e by (E.7). 3. Compute Cholesky factors W_o = SS^T, W_e = RR^T. 4. Compute SVD R^T S = USV^T. 5. Φ = SV_1Σ_1^{-1/2}. 6. Ψ = RU_1Σ_1^{-1/2}. |

E.1. **POD–Galerkin.** Proper orthogonal decomposition (POD) [24] computes a basis Φ that minimizes the mean-squared projection error of a set of snapshots \( \{x_i\}_{i=1}^N \), i.e., satisfies optimality property (3.16) with \( M = I_n \) and \( N = I_k \). Algebraically, POD computes the singular value decomposition (SVD)
\[
X = [x_1 \cdots x_N] = USV^T,
\]
where \( U \in O(I_n, I_k) \), \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r) \) with singular values \( \sigma_1 \geq \ldots \geq \sigma_r \geq 0 \), \( V \in O(I_N, I_r) \), and \( r = \min(n, N) \). Typically, the reduced-order-model dimension \( k \leq r \) is chosen to satisfy the energy criterion, i.e., \( k \) is set to the smallest integer in the set \( \{ i : \sum_{j=1}^i \sigma_j^2 / \sum_{k=1}^r \sigma_k^2 \geq \nu \} \) for some chosen threshold \( \nu \in [0, 1] \). Finally, the trial basis \( \Phi \) is set to the first \( k \) columns of \( U \) and the test basis is set to \( \Psi = \Phi \), which is equivalent to enforcing the Galerkin orthogonality condition (i.e., performing Galerkin projection).

As reported in Table 3.2, it can be verified that POD–Galerkin corresponds to an inner-product balancing with \( \Xi = \Xi' = XX^T \). Thus, \( \Psi, \Phi \in O(XX^T, \Sigma^2) \); however, note that we also have \( \Psi, \Phi \in O(I_n, I_k) \).

E.2. **Balanced truncation.** Balanced truncation [34] can be applied to LTI systems that are asymptotically stable, controllable, and observable. In the present framework, balanced truncation corresponds to a specific type of inner-product balancing with \( \Xi = W_o \) and \( \Xi' = W_c \), where \( W_o \) and \( W_c \) represent observability and controllability Gramians that satisfy primal and dual Lyapunov equations
\[
A^TW_o + W_oA = -C^TC,
\]
\[
AW_c + W_cA^T = -BB^T,
\]
respectively, which are defined for observable and controllable asymptotically stable LTI systems. While the present framework cannot prove that balanced truncation preserves asymptotic stability (the right-hand-side matrices in the Lyapunov equations (E.2)–(E.3) are positive semidefinite), it can be shown using observable and controllable conditions that balanced truncation does in fact preserve asymptotic stability [3], pp. 213–215.

E.3. **Balanced POD.** Several techniques exist to solve the Lyapunov equations (E.2) and (E.3) for the controllability and observability Gramians [3]; however, they are prohibitively expensive for large-scale systems. For this reason, several methods have been developed that instead employ empirical Gramians [29] that approximate the analytical Gramians. One particular method is balanced POD (BPOD) [47, 42], which relies on collecting primal snapshots for \( N \) timesteps (one impulse response of the forward system per column in \( B \)):
\[
S = \begin{bmatrix} B & e^{A\Delta t}B & \ldots & e^{A(N-1)\Delta t}B \end{bmatrix}
\]
and dual snapshots for \( N \) timesteps (one impulse response of the dual system per row in \( C \)):
\[
R = \begin{bmatrix} C^T & e^{A^T\Delta t}C^T & \ldots & e^{A^T(N-1)\Delta t}C^T \end{bmatrix};
\]
the empirical observability and controllability Gramians are then set to \( \hat{W}_o = RR^T \) and \( \hat{W}_c = SS^T \), respectively. Then, BPOD corresponds to inner-product balancing with \( \Xi = \hat{W}_o \) and \( \Xi' = \hat{W}_c \).

Critically, empirical Gramians \( \hat{W}_o \) and \( \hat{W}_c \) may not be Lyapunov matrices, i.e., they may not satisfy
associated algorithm, which amounts to computing an inner-product balancing with \( \Xi = \) but applying the test and trial basis matrices to the original (unshifted) system. Table E.1 provides the test and trial basis matrices can be computed by performing balanced truncation on the shifted system, thus, even if \( A \) is marginally stable or unstable, the shifted matrix \( A - \mu I \) will be asymptotically stable and the test and trial basis matrices can be computed by performing balanced truncation on the shifted system, but applying the test and trial basis matrices to the original (unshifted) system. Table E.1 provides the associated algorithm, which amounts to computing an inner-product balancing with \( \Xi = W_o^\mu \) and \( \Xi' = W_e^\mu \), where the shifted observability and controllability Gramians satisfy

\[
(A - \mu I)^T W_o^\mu + W_o^\mu (A - \mu I) = -C^T C, \\
(A - \mu I) W_e^\mu + W_e^\mu (A - \mu I)^T = -BB^T.
\]

While SRSB ensures that the shifted reduced system \((\Psi^*(A - \mu I)\Phi, \Psi^* B, C\Phi) = (\tilde{A} - \mu I_k, \tilde{B}, \tilde{C})\) remains asymptotically stable, this guarantee does not extend to the reduced system \((\tilde{A}, \tilde{B}, \tilde{C})\) that is used in practice. In particular, there is no assurance that the reduced system will retain the marginal stability or instability that characterized the original system \((A, B, C)\).

### E.4. SRSB

The shift-reduce-shift-back (SRSB) method aims to extend the applicability of balanced truncation to marginally stable and unstable systems. By Lemma 3.2 the real parts of the eigenvalues of \( A \) are less than the shift margin \( \mu \in \mathbb{R} \) if and only if given any \( Q \in \text{SPD}(n) \), there exists \( M \in \text{SPD}(n) \) that is the unique solution to

\[
(A - \mu I)^T M + M(A - \mu I) = -Q.
\]

Thus, even if \( A \) is marginally stable or unstable, the shifted matrix \( A - \mu I \) will be asymptotically stable and the test and trial basis matrices can be computed by performing balanced truncation on the shifted system, but applying the test and trial basis matrices to the original (unshifted) system. Table E.1 provides the associated algorithm, which amounts to computing an inner-product balancing with \( \Xi = W_o^\mu \) and \( \Xi' = W_e^\mu \), where the shifted observability and controllability Gramians satisfy

\[
(A - \mu I)^T W_o^\mu + W_o^\mu (A - \mu I) = -C^T C, \\
(A - \mu I) W_e^\mu + W_e^\mu (A - \mu I)^T = -BB^T.
\]

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