The Unruh effect for higher derivative field theory

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Abstract
We analyse the emergence of the Unruh effect within the context of a field Lagrangian theory associated with the Pais–Uhlenbeck fourth order oscillator model. To this end, we introduce a transformation that brings the Hamiltonian bounded from below and is consistent with $\mathcal{PT}$-symmetric quantum mechanics. We find that, as far as we consider different frequencies within the Pais–Uhlenbeck model, a particle together with an antiparticle of different masses are created and may be traced back to the Bogoliubov transformation associated with the interaction between the Unruh–DeWitt detector and the higher derivative scalar field. In contrast, whenever we consider the equal frequencies limit, no particle creation is detected as the pair particle/antiparticle annihilate each other. Further, following Moschella and Schaeffer, we construct a Poincaré invariant two-point function for the Pais–Uhlenbeck model, which in turn allows us to perform the thermal analysis for any of the emanant particles.

Keywords: conformal gravity, higher order, Unruh effect, Pais–Uhlenbeck

1. Introduction

Lagrangians depending on higher order derivatives appear in many areas of physics, including string theory, brane theories and remarkably gravitation-like models for which the higher order dependence is introduced by means of curvature terms, such as, for example, in the Regge–Teitelboim model for gravity à la string [1, 2], and Dirac’s membrane model [3, 4].
One may even encounter this kind of higher order derivative Lagrangian in economics and biology [5, 6]. In particular, in the gravitational context, one may consider Weyl’s conformal gravity which is a theory of gravity constructed from the Weyl tensor. Among the peculiarities of conformal gravity we emphasize that it contains higher order derivative terms and also it is a renormalizable theory. Despite the importance of higher order theories in physics, the study of this kind of system was neglected as there was some evidence of the appearance of associated negative norm states at the quantum level, commonly known as ghosts. However, in the past few years, higher order theories were analysed from different perspectives, among them the \( PT \)-symmetric Hamiltonian version, from which the ghosts may disappear under suitable conditions on the physical states [7–9]. Indeed, in \( PT \) quantum mechanics if the Hamiltonian is not Dirac Hermitian, but instead is symmetric under parity reflexion and time reversal transformations, a new dynamical inner product must be introduced. This inner product is constructed by means of isospectral similarity transformations on the \( PT \)-symmetric Hamiltonian, resulting in a positive definite Hamiltonian, whose eigenstates have a positive inner product. In this sense, \( PT \)-symmetric Hamiltonian quantization was introduced for the Pais–Uhlenbeck fourth order oscillator, avoiding the emergence of ghosts [7]. Besides, by considering a different perspective, in [10–12] the viability of higher derivative theories is studied by focusing on the analysis of their instabilities even if the Hamiltonian results are not bounded from below.

In this paper we present a field theoretical version of the Pais–Uhlenbeck oscillator in order to analyse the appearance of the Unruh effect, that is, the creation of particles as seen from an accelerated observer and corresponding to a thermal bath with temperature proportional to the acceleration. The Pais–Uhlenbeck oscillator has been proposed as a toy model to study certain issues in conformal gravity. Indeed, by considering a linear approximation around a flat spacetime background on the conformal action leads to a classical fourth order derivative theory, as discussed in [13], and also, symmetries of the Pais–Uhlenbeck model have been studied from the perspective of one of the subgroups of the conformal group [14]. In consequence, our main aim is to address the emergence of the Unruh effect associated with a higher order derivative theory. Our construction considers a transformation which brings the Pais–Uhlenbeck field model to a well-defined bounded from below Hamiltonian when decomposed into Fourier modes. This decomposition is in full agreement with the principles developed within \( PT \)-symmetric quantum mechanics [15]. Further, we note that the Unruh effect only appears in the case where the frequencies that characterize the Pais–Uhlenbeck oscillator are unequal. Within our setup, this can be easily seen as the coefficients appearing in the Bogoliubov transformation vanishing in the equal frequency limit. This confirms the intuitive picture that for the equal frequency limit the Pais–Uhlenbeck model is known to have continuous eigenvalues, in opposition to the discrete eigenvalues obtained for unequal frequencies [16, 17, 18]. As for the thermal behaviour, we adapt Moschella and Schaeffer construction [19, 20], to the model of our interest. Due to the symmetry of the Pais–Uhlenbeck decomposition into modes, we argue that the two-point function describing the system behaves as a couple of subsystems, one for a particle and one for an antiparticle, for which the temperature is associated with a factor proportional to their different masses.

The rest of the article is organized as follows. In section 2 we start by describing the Lagrangian and Hamiltonian formalisms for the Pais–Uhlenbeck field model. In section 3 we analyse extensively the Unruh effect adapted to our original setup. In opposition to the standard analysis of the Unruh effect, we discuss the creation of a pair of particles of different masses. Finally, in section 4 we include some concluding remarks.
2. Pais–Uhlenbeck field model

We consider an action described in a four dimensional Minkowski spacetime with signature \((+−−−)\) corresponding to the field version of the Pais–Uhlenbeck oscillator [21] with a current term, representing the Unruh–DeWitt detector,

\[
S = \int d^4x \left[ \frac{1}{2} \phi(\Box + m_1^2 + m_2^2)\phi + J\phi \right] = \int d^4x \left[ \frac{1}{2} (\Box \phi)^2 - \frac{1}{2} (m_1^2 + m_2^2)\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m_1^2 m_2^2 \phi^2 + \epsilon \phi \right],
\]

(1)

where the first term in the first line represents the Pais–Uhlenbeck oscillator while the second term stands for the Unruh–DeWitt detector, from the last line we identify \(J = \epsilon Q\), \(\epsilon\) being a free parameter associated with the detector efficiency and \(Q = Q(t)\) an arbitrary function described by a point-like object endowed with an internal structure that couples with a scalar field by means of a monopole moment, as described in [22–24]. The function \(\phi = \phi(t, \vec{x})\) stands for the Pais–Uhlenbeck scalar field, and the masses \(m_1\) and \(m_2\) are positive definite parameters characterizing the system. We emphasize that, in order to analyse the Unruh effect, we will consider that the current term \(J\) is only active for a certain period of time, that is,

\[
J(t) := \begin{cases} 
0 & t < 0 \\
\epsilon Q(t) & 0 < t < \tau \\
0 & t > \tau
\end{cases}
\]

(2)

Despite the arbitrariness of the function \(Q(t)\), we will only focus our attention on the regions before and after it interacts with the field \(\phi(x)\). In the regions without interaction \(J(t) = 0\), and by expanding the field \(\phi(x)\) in spatial Fourier modes

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3} \phi_k e^{i\vec{k} \cdot \vec{x}},
\]

(3)

the field equation of motion

\[
(\Box + m_1^2 + m_2^2)\phi = 0
\]

(4)

reduces to

\[
0 = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{d^4\phi_k}{dt^4} + (2k^2 + m_1^2 + m_2^2) \frac{d^2\phi_k}{dt^2} + (k^4 + k^2(m_1^2 + m_2^2) + m_1^2 m_2^2) \phi_k \right].
\]

(5)

and from this expression we obtain, by comparison with the equation of motion of the Pais–Uhlenbeck oscillator, the dispersion relations

\[
\omega_1^2 = k^2 + m_1^2, \quad \omega_2^2 = k^2 + m_2^2,
\]

(6)

where \(\omega_1\) and \(\omega_2\) stand for the frequencies of the Pais–Uhlenbeck oscillator. We note that each Fourier mode represents an independent Pais–Uhlenbeck fourth order oscillator. From now on, and without loss of generality, we will consider \(\omega_1 > \omega_2\). As we will see, the equal frequency limit becomes inadequately defined when obtained from the general perspective \(\omega_1 = \omega_2\) as the quantum behaviour becomes completely different [16, 17]. As shown in these references, in the equal frequency limit case the quantum Hamiltonian may be decomposed.
into a part proportional to an angular momentum term with discrete spectrum and a part proportional to a vector norm term with continuous spectrum, in opposition to our area interest for which only a discrete spectrum is obtained. However, by taking the equal frequency limit in our formulation we will show that the Unruh effect will not emerge in this case. Further, as discussed in [17], even though for the classical equal frequency limit the canonical transformation diverges, at the quantum level the canonical transformation becomes well-defined in this limit allowing the wave functions to pass from the discrete to the continuous case, thus supporting our claims about the Unruh effect in the following sections. We will include appropriate comments whenever necessary in order to clarify this issue.

Since the Pais–Uhlenbeck field theory possesses higher derivative terms, in order to get the Hamiltonian, we proceed using the standard Ostrogradski formalism. To implement this method, the phase space involves, in addition to the canonical coordinates \( (\phi, \pi_\phi) \), an extra pair of canonical variables associated with its higher order nature, namely \( \psi = \partial_0 \phi \), with corresponding canonical momentum \( \pi_\psi \). The momenta are defined explicitly as

\[
\pi_\phi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} - \partial_0 \left( \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \right) = - \left[ (m_1^2 + m_2^2) \partial_0 \phi + \partial_0 (\partial_0 \phi) \right]
\]

\[
\pi_\psi = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = \Box \phi,
\]

where \( \mathcal{L} \) denotes the Lagrangian density appearing in the action (1), and \( \partial_0 \) denotes the partial time derivative. The canonical Hamiltonian, \( \mathcal{H} = \int d^3 x \mathcal{H} \), is thus obtained in a standard manner through the Legendre transformation, \( \mathcal{H} = \partial_0 \phi \pi_\phi + \partial_0 \psi \pi_\psi - \mathcal{L} \). Hence, the Hamiltonian density reads

\[
\mathcal{H} = \psi \pi_\phi + \frac{1}{2} \pi_\phi^2 + \frac{1}{2} (m_1^2 + m_2^2) \psi^2 - \frac{1}{2} m_1^2 m_2^2 \phi^2 - (\partial_\mu \nabla^\mu \phi) \partial_0 \phi
\]

\[-\frac{1}{2} (m_1^2 + m_2^2) \nabla \phi \cdot \nabla \phi + \partial_\mu (\nabla^\mu \phi \partial_0 \phi) - \alpha Q \phi,
\]

where the next to last term corresponds to a boundary term. If we expand the field into spatial Fourier modes we get independent Pais–Uhlenbeck oscillators interacting with the Unruh–DeWitt detector for each mode

\[
H = \int \frac{d^3 k}{(2\pi)^3} \left[ \psi_k \pi_{\psi_k} + \frac{1}{2} \pi_{\psi_k}^2 + \frac{1}{2} (\omega_k^2 + \omega_{\psi_k}^2) \psi_k^2 - \frac{1}{2} \omega_{\psi_k}^2 \phi_k^2 - \alpha Q \phi \right],
\]

as expected.

In the Fock space representation, we introduce two different pairs of creation and annihilation operators for each mode

\[
\phi_k^+ := \frac{1}{\Delta} \left[ \frac{1}{\sqrt{2\omega_k}} (b_k + b_k^\dagger) - i \frac{1}{\sqrt{2\omega_k}} (a_k^\dagger - a_k) \right]
\]

\[
\pi_{\phi_k} := \frac{\omega_k}{\Delta} \left[ i \frac{\omega_k}{\sqrt{2\omega_k}} (b_k - b_k^\dagger) - \frac{\omega_k}{\sqrt{2\omega_k}} (a_k^\dagger + a_k) \right]
\]

\[
\psi_k^+ := \frac{1}{\Delta} \left[ \frac{\omega_k}{\sqrt{2}} (a_k^\dagger + a_k) - i \sqrt{2} (b_k - b_k^\dagger) \right]
\]

\[
\pi_{\psi_k} := \frac{1}{\Delta} \left[ i \omega_k \sqrt{2} (a_k^\dagger - a_k) - \omega_k \sqrt{2} (b_k + b_k^\dagger) \right]
\]

(10)
where \( \Delta := \sqrt{\omega_1^2 - \omega_2^2} \). These operators satisfy the equal time commutation relations
\[
[a_{\vec{k}}^+, a_{\vec{k}}] = \delta(\vec{k} - \vec{k'}) = -[b_{\vec{k}}^+, b_{\vec{k}}],
\]
and, otherwise they vanish. The choice of the sign in front of the commutation relation between the \( b_{\vec{k}} \) and \( b_{\vec{k}}^+ \) operators is necessary in order to guarantee that the energy spectrum of the Hamiltonian is real and bounded from below [25].

In terms of the creation and annihilation operators (10), the Hamiltonian for the Pais–Uhlenbeck model is given by
\[
H = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \omega_1 \left( a_{\vec{k}}^a a_{\vec{k}} + \frac{1}{2} \right) - \omega_2 \left( b_{\vec{k}}^a b_{\vec{k}} - \frac{1}{2} \right) + i \epsilon \frac{Q (a_{\vec{k}}^a - a_{\vec{k}})}{\sqrt{2\omega_1}} - \epsilon \frac{Q (b_{\vec{k}}^a + b_{\vec{k}})}{\sqrt{2\omega_2}} \right],
\]
The first two terms in this Hamiltonian correspond to the diagonal operator representation of an infinite sum of Pais–Uhlenbeck oscillators, while the terms in the second line comprise the different way in which the Unruh–DeWitt detector interacts with the introduced creation and annihilation operators. In the case where \( \epsilon = 0 \), by considering the basic commutators (11), and the property that \( a_{\vec{k}}^a \) and \( b_{\vec{k}}^a \) annihilate the 0-particle state \( \Omega \),
\[
a_{\vec{k}}^a |\Omega\rangle = 0, \quad b_{\vec{k}}^a |\Omega\rangle = 0,
\]
we can observe that the state \(|\Omega\rangle\) corresponds to the ground state with energy \( \frac{1}{2}(\omega_1 + \omega_2) \) for each mode. As noted in [25], although the Hamiltonian (12) is not manifestly Dirac Hermitian, it becomes invariant under \( PT \) transformations. This means, that the ghost states can be reinterpretated as positive quantum states with respect to a suitable \( PT \) inner product. This product is constructed by introducing a dynamical reflection symmetry operator \( C \), which commutes with the Hamiltonian and the \( PT \) operator. Then, under this new inner product the norm of the states proves to be strictly positive. Similarly, in order to analyse the thermal behaviour of the Pais–Uhlenbeck particles, we will show that a positive inner product defined on the space of solutions must be introduced with the aim of avoiding the emergence of ghosts.

Finally, we note that the term \( \Delta = \sqrt{\omega_1^2 - \omega_2^2} \) appearing in (10) becomes singular in the equal frequencies limit, this limit corresponds to the case of equal masses for a given mode. As stated in [16, 17], in the equal frequencies case, the Pais–Uhlenbeck system behaves completely differently as the spectrum of the Hamiltonian is composed of a discrete spectrum part coming from an angular momentum term, and of a continuous part originated from a different term given by the squared norm of the position variables.

### 3. The Unruh effect for the Pais–Uhlenbeck field

In this section we will explore the Unruh effect construction for the field theory developed so far, demonstrating the way in which particles are created for such a model, and we will also explore the thermal behaviour of the system by introducing an appropriate two-point function.
3.1. Creation of particles and Unruh–DeWitt detector

Following Heisenberg prescription, \( i\langle df/dt \rangle = [f, H] \) \((\hbar = 1)\), we may determine how the system evolves in time. In this way, using the commutation relations (11), the evolution of the operators \( a_k^\tau \) and \( b_k^\tau \) is given by

\[
a_k^\tau(t) = a_k^{(\text{in})} e^{-i\omega_1 t} + \frac{\epsilon}{\Delta\sqrt{2\omega_1}} \int_0^t e^{i\omega_1(t-t')} Q(t') dt' \\
b_k^\tau(t) = b_k^{(\text{in})} e^{i\omega_1 t} - \frac{\epsilon}{\Delta\sqrt{2\omega_2}} \int_0^t e^{i\omega_2(t-t')} Q(t') dt',
\]

where \( a_k^{(\text{in})} \) and \( b_k^{(\text{in})} \) are constants determined by the initial conditions at \( t = 0 \). Assuming that the current \( J(t) = \epsilon Q(t) \) only acts in the interval \([0, \tau]\) (see definition (2)), we define the interaction terms

\[
J_a := \frac{\epsilon}{\Delta\sqrt{2\omega_1}} \int_0^\tau e^{i\omega_1 t} Q(t') dt', \quad J_b := -\frac{\epsilon}{\Delta\sqrt{2\omega_2}} \int_0^\tau e^{i\omega_2 t} Q(t') dt',
\]

where the integrals are considered in the interval \([0, \tau]\) as only in this interval the current interacts with the Pais–Uhlenbeck field. Note that the non-symmetric manner in which \( J_a \) and \( J_b \) appear in the expressions above results from the higher order structure of the model. Next, we separate our operators \( a_k^\tau(t) \) and \( b_k^\tau(t) \) into two parts each, one before and one after the current interaction. Thus, we have

\[
a_k^\tau(t) = \begin{cases} a_k^{(\text{in})} e^{-i\omega_1 t}, & t < 0 \\ a_k^{(\text{out})} e^{-i\omega_1 t}, & t > \tau \end{cases}, \quad b_k^\tau(t) = \begin{cases} b_k^{(\text{in})} e^{i\omega_1 t}, & t < 0 \\ b_k^{(\text{out})} e^{i\omega_1 t}, & t > \tau \end{cases},
\]

where the operators \( a_k^{(\text{out})} \) and \( b_k^{(\text{out})} \) are straightforwardly obtained from (14) and (15), and explicitly may be written as \( a_k^{(\text{out})} := a_k^{(\text{in})} + J_a \), and \( b_k^{(\text{out})} := b_k^{(\text{in})} + J_b \), respectively.

Before the interaction occurs we assume, for any value of \( \tilde{k} \) (thus, from now on we will omit the \( \tilde{k} \) index), that there exist no particle states \([\Omega_{0,b}]_\text{in} \) and \([\Omega_{a,0}]_\text{in} \), such that

\[
a^{(\text{in})} [\Omega_{0,b}]_\text{in} = 0, \quad b^{(\text{in})} [\Omega_{a,0}]_\text{in} = 0,
\]

where states \([\Omega_{0,b}]_\text{in} \) and \([\Omega_{a,0}]_\text{in} \) represent the absence of particles of type \( a \) and \( b \), respectively, while the state \([\Omega_{0,0}]_\text{in} \) stands for the Minkowski vacuum state in the \( t < 0 \) region. Once we have defined our vacuum, we can construct from the no particle state all the other states by the repeated use of creation operators \( a^{(\text{in})} \) and \( b^{(\text{in})} \).

\[
[\Omega_{n,b}]_\text{in} = \frac{1}{\sqrt{n!}} (a^{(\text{in})})^n [\Omega_{0,b}]_\text{in}, \quad [\Omega_{a,n}]_\text{in} = \frac{1}{\sqrt{n!}} (b^{(\text{in})})^n [\Omega_{a,0}]_\text{in}.
\]

Analogously, after the interaction took place we also assume the existence of no particle states \([\Omega_{0,b}]_\text{out} \) and \([\Omega_{a,0}]_\text{out} \), such that

\[
a^{(\text{out})} [\Omega_{0,b}]_\text{out} = 0, \quad b^{(\text{out})} [\Omega_{a,0}]_\text{out} = 0,
\]

where, similarly, states \([\Omega_{0,b}]_\text{out} \) and \([\Omega_{a,0}]_\text{out} \) represent the absence of particles of type \( a \) and \( b \), respectively, while the state \([\Omega_{0,0}]_\text{out} \) stands for the Minkowski vacuum state in the \( \tau < t \) region. Of course, analogous construction to (18) occurs for any state in the \( \tau < t \) region.
As mentioned before, we have two sets of states, $\Omega_{a,b}^{\text{in}}$ corresponding to the region $t < 0$, and $\Omega_{a,b}^{\text{out}}$ corresponding to the region $\tau < t$, respectively. As both sets are described in Minkowski spacetime, the idea is to compare both sets and see how they are related. This relation is described by the Bogoliubov transformation

$$\rho_{a,b}^{\text{in}} = \sum_{n,m} \Lambda_{n,m} \Omega_{n,m}^{\text{out}}$$

(20)

where $\Lambda_{n,m} \in \mathbb{C}$. For the case we are interested in, the explicit form of the Bogoliubov transformation reads

$$\Lambda_{n,m} = \frac{1}{\sqrt{n! \; m!}} e^{-\frac{1}{2} (|J_a|^2 + |J_b|^2) J_a^m J_b^m}.$$  

(21)

where $J_a$ and $J_b$ were defined in (15). Indeed, by considering this Bogoliubov transformation one may easily verify relations (17) by taking into account the expressions $d^{\text{out}} = d^{\text{in}} + J_a$ and $b^{\text{out}} = b^{\text{in}} + J_b$ described above.

As time passes, one may wonder if the state $\Omega_{0,0}^{\text{in}}$ may evolve until it reaches the state $\Omega_{a,b}^{\text{out}}$. As in ordinary quantum mechanics, the probability of this happening is

$$\rho^2 = \left| \Omega_{a,b}^{\text{out}} \right|^2 = \left| \sum_{n,m} \Lambda_{n,m} \Omega_{n,m}^{\text{out}} \right|^2 = \sum_{n,m} |\Lambda_{n,m}|^2.$$  

(22)

This result tells us that the probability that the vacuum state in the region $t < 0$ evolves into a state with particles in the region $\tau < t$ may be different from zero. Thus, we can conclude that in the process of interaction at the region $0 < t < \tau$ particles were created. This particle creation is due to the interaction term with the Pais–Uhlenbeck field by means of the Unruh–DeWitt detector. In order to associate this particle creation with the Unruh effect we have to calculate the transition probability of the Unruh–DeWitt detector from a ground state to an excited one [24]. By choosing a hyperbolic trajectory in Rindler coordinates, one may see that the transition probability is different from zero as far as we consider a massless Klein–Gordon field. However, in the massive case this procedure becomes non-analytic [23]. An alternative approach, developed in [19, 20], takes advantage of the fact that we are dealing with systems possessing an infinite number of degrees of freedom. Indeed, in this case one may construct uncountably unitarily inequivalent Hilbert space representations of the canonical commutation relations. Thus the choice of a specific representation depends on appropriate physical considerations such as renormalizability, thermal equilibrium, positivity conditions, etc. The thermal behaviour developed in the next section will follow this latter approach by implementing a Poincaré invariant and positive definite representation necessary to obtain analytic expressions for the massive case which is an essential issue in our higher order derivative model.

Also note that the particle creation may be related to the discrete energy spectrum of the Hamiltonian as changes in energy levels are interpreted as emission or absorption of particles. In contrast, for the equal frequencies limit, $\Delta \to 0$, the probability density $\rho$ goes to zero, as in this case the coefficients appearing in the Bogoliubov transformation vanish, that is, $\Lambda_{n,m} \to 0$. As a consequence, one may interpret this as the absence of the Unruh effect in the equal frequency limit. As proved in [17, 18], although it is possible to construct a well defined equal frequency limit for the quantum canonical transformations, hence strengthening our claim on the Unruh effect, a more carefully analysis must be carried out from the very beginning considering the equal frequency Pais–Uhlenbeck Hamiltonian for this case.
3.2. Thermal radiation

For the Pais–Uhlenbeck field model (1) we start by solving the field equation
\[ (\Box + m_1^2)(\Box + m_2^2)\phi = 0. \]  
(23)
which, by symmetry, may be obtained by considering the linear complex combination
\[ \phi(x) = \alpha u(x) + \beta v(x), \]
where \( \alpha, \beta \) are constants and \( u(x) \) and \( v(x) \) are independent solutions to Klein–Gordon equations with masses \( m_1 \) and \( m_2 \), respectively, \[26\]
\[ (\Box + m_1^2)u(x) = 0, \quad (\Box + m_2^2)v(x) = 0. \]  
(24)
From Pais–Uhlenbeck action (1), and by means of the Noether theorem, we define an inner product on the complex solutions to our field equations as
\[ \langle \phi, \psi \rangle = i \int d\Sigma^\mu \left[ - (m_1^2 + m_2^2)(\phi^* \partial_\mu \psi - \psi \partial_\mu \phi^*) + (\partial_\mu \phi^* \Box \psi - \partial_\mu \psi \Box \phi^*) \right], \]  
(25)
where \( d\Sigma^\mu = n^\mu d\Sigma \), with \( n^\mu \) a unitary vector orthogonal to the spatial Cauchy hypersurface \( \Sigma \), and \( d\Sigma \) represents the volume element of \( \Sigma \). Under this inner product the independent solutions \( u(x) \) and \( v(x) \) are orthogonal, that is, \( \langle u, v \rangle = 0 \). In this manner, we can express the Pais–Uhlenbeck field operator as a formal expansion in terms of a family of complex independent classical solutions \( \{ u_i(x) \} \) and \( \{ v_i(x) \} \), in the following way
\[ \phi(x) = \sum_{i=0}^{\infty} [u_i(x) a_i + u_i^*(x) a_i^+] + v_i(x) b_i + v_i^*(x) b_i^+], \]  
(26)
where \( a \) and \( b \) are the annihilation operators and \( a^\dagger \) and \( b^\dagger \) are the creation operators, respectively, and they follow the commutation relations (11).

As previously discussed, the ambiguity in the representations allows us to look for a general Poincaré invariant extension of the standard two-point function \( W(x, x') = \langle \Omega | \phi(x) \phi(x') | \Omega \rangle \) which, due to the symmetry of the Pais–Uhlenbeck model, is explicitly decomposed into modes as
\[ W(x, x') = W_0(x, x') - W_1(x, x'). \]  
(27)
The separation of the two-point function \( W(x, x') \) into two-point functions for each mode resulted as a consequence of the orthogonality of the \( u \) and \( v \) modes within the inner product (25). In order to obtain the exact expression for the normalized \( u \) and \( v \) modes we need to solve the pair of Klein–Gordon equation (24). We will solve these equations in Rindler spacetime in order to describe the Unruh effect. The coordinates for this spacetime are explicitly given by the mapping \((t, x, y, z) \mapsto (\rho \sinh \eta, \rho \cosh \eta, y, z)\). Thus, focusing on the \( \rho \) and \( \eta \) components of the coordinates, the Klein–Gordon equation (24) read
\[ \left( \frac{1}{\rho^2} \frac{\partial^2}{\partial \eta^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + m_1^2 \right) u = 0, \]
\[ \left( \frac{1}{\rho^2} \frac{\partial^2}{\partial \eta^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + m_2^2 \right) v = 0, \]  
(28)
from which we obtain the wave-like solutions,
respectively. Note that the difference in signs in the exponential of the two modes is necessary to guarantee the positiveness of the inner product defined in (25). Furthermore, this positivity condition can be traced back to the non-Hermiticity property of the Pais–Uhlenbeck Hamiltonian as, for these kind of systems, in order to ensure a positive inner product and normalized wave functions $\mathcal{PT}$-symmetric eigenstates must be introduced [25]. Similarly, in the present case, these states are given by the solutions $v(\eta, \rho)$, which become positive definite under the inner product derived from the conserved current (25). Note that in (29), the functions $K_{\omega_n}(m_\rho)$, for $n = 1, 2$, stand for the Macdonald functions [27] that are solutions to the modified Bessel equation of the second kind with imaginary index $\rho$.

At this point, it is important to mention that we choose the solution $K_{\omega_n}(m_\rho)$ to (30) due to its asymptotic behaviour $K_{\omega_n}(m_\rho) \approx (\pi / 2m_\rho)^{1/2}e^{-m_\rho(1 + O((m_\rho)^{-1}))}$ which tends to zero as the argument tends to infinity. This issue is relevant in order to analyse the Unruh effect as we assume an adiabatic interaction for a static spacetime [24].

We also note that, by considering the inner product (25), the normalization of the $u$ and $v$ modes is given by

\begin{equation}
(u_\omega(\eta, \rho), u_{\omega'}(\eta, \rho)) = \frac{\Delta^2 \pi^2}{\sinh(\pi \omega_1)} \delta(\omega_1 - \omega'),
\end{equation}

\begin{equation}
(v_\omega(\eta, \rho), v_{\omega'}(\eta, \rho)) = \frac{\Delta^2 \pi^2}{\sinh(\pi \omega_2)} \delta(\omega_2 - \omega').
\end{equation}

Again, we may note that at the equal frequencies limit $\Delta \to 0$ this product automatically vanishes. We interpret, in a simple manner as in this limit, that the Pais–Uhlenbeck model is invariant under a different symmetry, and thus, the inner product for the equal frequency limit may be obtained from an appropriate Noether current. In view of the inner products (31), we finally propose the normalized modes

\begin{equation}
u_{\omega}(\eta, \rho) = \frac{\sqrt{\sinh(\pi \omega_1)}}{\Delta \pi} e^{-i\omega_1 \rho} K_{\omega_1}(m_\rho),
\end{equation}

\begin{equation}
u_{\omega}(\eta, \rho) = \frac{\sqrt{\sinh(\pi \omega_2)}}{\Delta \pi} e^{-i\omega_2 \rho} K_{\omega_2}(m_\rho).
\end{equation}

The different signs of the exponential functions may be interpreted as particles of mass $m_1$ and antiparticles of mass $m_2$, respectively. This interpretation is a consequence of the equal frequency limit discussed above, as within this case no particles are observed. From this point of view, we may simply infer that in the equal frequency limit particles and antiparticles annihilate each other.

The generalized Poincaré invariant two-point functions in terms of the normalized modes are given by (see [19, 20] for details)

\begin{equation}
W_\omega(x, x') = \frac{1}{\pi^2 \Delta} \int_0^\infty \sinh(\pi \omega_1) \left[ \frac{e^{-i\omega_1(x - x')}}{1 - e^{-2\pi \omega_1}} - \frac{e^{i\omega_1(x - x')}}{1 - e^{2\pi \omega_1}} \right] K_{\omega_1}(m_\rho) K_{\omega_1}(m_\rho') d\omega_1,
\end{equation}

\begin{equation}
W_\omega(x, x') = \frac{-1}{\pi^2 \Delta} \int_0^\infty \sinh(\pi \omega_2) \left[ \frac{e^{-i\omega_2(x - x')}}{1 - e^{-2\pi \omega_2}} - \frac{e^{i\omega_2(x - x')}}{1 - e^{2\pi \omega_2}} \right] K_{\omega_2}(m_\rho) K_{\omega_2}(m_\rho') d\omega_2.
\end{equation}
The terms in the square brackets may be written as hyperbolic cosines, respectively, and thus, by considering the integral identity [28]

\[
\frac{2}{\pi} \int_0^\infty K_0(a)K_0(b) \cosh \left[ y(\pi - \phi) \right] dy = K_0 \left[ \sqrt{a^2 + b^2 - 2ab \cos(\phi)} \right],
\]

(34)

we finally obtain the expressions

\[
W_0(x, x') = \frac{1}{2\pi \Delta} K_0(m_1\|x - x'\|)
\]

\[
W_i(x, x') = -\frac{1}{2\pi \Delta} K_0(m_2\|x - x'\|),
\]

(35)

where the terms \(\|\cdot\|\) in the argument of the Macdonald \(K_0\) functions stand for the Minkowski norm. In this way, we have obtained the complete generalized Poincaré invariant two-point function (27) associated to the Pais–Uhlenbeck model

\[
W(x, x') = \frac{K_0(m_1\|x - x'\|) + K_0(m_2\|x - x'\|)}{2\pi \Delta^2}.
\]

(36)

Further, from (33) we note that any of the functions \(W_0(x, x')\) and \(W_i(x, x')\) may be regarded as the inverse Fourier transform of the functions \(F(\omega_n)\), \(n = 1, 2\), identified as

\[
F(\omega_n) = \frac{1}{2\pi \Delta^2} \frac{K_{in}(m_1\omega)K_{in}(m_2\omega') \sinh(\pi \omega_n)}{e^{2\pi \omega_n} - 1}.
\]

(37)

The function \(F(\omega_n)\) thus represents the Fourier transform of the generalized two-point function \(W(x, x')\). From the expression (37), we may see that the Planck factor \((e^{2\pi \omega_n} - 1)^{-1}\) emerges naturally, thus concluding that each of our particles follows a Bose–Einstein distribution with temperature related to their respective frequencies \(\omega_n\).

4. Conclusions

In this article, we analysed the Unruh effect that emerges from a field theoretical version of the Pais–Uhlenbeck fourth order oscillator. Even though, higher derivative theories are commonly associated with the presence of ghost states and energy spectra that are not bounded by below resulting in the loss of unitarity, the Pais–Uhlenbeck oscillator does not suffer from these kinds of disadvantages. This last result may be seen as a consequence of the discrete \(\mathcal{PT}\)-symmetry invariance of the Hamiltonian, obtained by introducing an appropriate dynamical inner product in order to preserve positive norm states, instead of using the standard Dirac inner product. Within this setup, we derived the Bogoliubov transformations associated with the interaction between the Unruh–DeWitt detector and the higher derivative scalar field. Afterwards, in order to describe the thermal behaviour present in the Unruh effect, our strategy was to obtain the most general Poincaré invariant two point function. Due to the symmetries associated with the Pais–Uhlenbeck Hamiltonian, the quantum theory possesses a positive definite inner product and, therefore, unitarity is guaranteed. We also showed that, within the unequal frequencies case and as a consequence of the Bogoliubov transformation for the normalized modes, particles of mass \(m_1\) together with antiparticles of mass \(m_2\) were created in the process. This interpretation results physically as a consequence of the equal frequency limit, as within this case the energy spectrum is now continuous, and no particles are observed at all. Finally, by using the generalized Poincaré invariant two point function, we conclude that each kind of particle created follows a Bose–Einstein distribution with temperature related
to their respective frequencies, which in turn are associated with the value of their respective masses. Further studies are necessary to guarantee that our affirmations on the absence of the Unruh effect in the equal frequency limit may be valid as we may consider a different Hamiltonian from scratch within this limit. This construction will effectively change most of the structures involved as, for example, the inner product definitely corresponds to a different Noether current, thus affecting the two-point functions required for the description of the thermal behaviour of the system. Also, a more general analysis must be carried out in order to discern the generality of the results obtained here for a generic higher derivative model. This will be done elsewhere.

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