The dc-Josephson effect with more than four superconducting leads

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The p-terminal dc-Josephson current is sensitive to the superconducting phase variables of p terminals. In the paper, we establish protocol for direct detection of the p-terminal dc-Josephson effect with p ≥ 3 in a device containing N superconducting leads S1, S2, ..., SN having the phase variables φ1, φ2, ..., φN. The calculated signal χ(N) is the higher-order nonlocal inverse inductance obtained from differentiating the current I1 through S1 with respect to the remaining N−2 independent phase differences φ2 − φN, φ1 − φ2, ..., φN−1 − φN. We find that the values p ≤ N = 2 do not contribute to χ(N), and that χ(N) ≠ 0 implies evidence for the p = N − 1 or the p = N-terminal dc-Josephson currents. For N = 4 superconducting leads, we demonstrate that χ(4) ≠ 0 implies evidence for the p = 3 or p = 4 dc-Josephson effect, irrespective of the p = 2-terminal dc-Josephson current. Thus, we provide a way to demonstrate the dc-Josephson effect with more than three terminals (i.e., with p ≥ 3) in a device containing at least four superconducting leads (i.e., with N ≥ 4). The paper can be viewed as generalizing the recently considered φ-junctions in Andreev molecules to arbitrary number N of the superconducting leads. Nontrivial topology and Weyl point singularities are not required.

1. INTRODUCTION

The two-terminal Josephson junction formed with the BCS superconductors S1 and S2 reveals physical relevance of the gauge-invariant difference φ1 − φ2 between their macroscopic phase variables (see figure 1). Equilibrium dissipationless supercurrent

\[ I^{(2)}_S(\phi_1 - \phi_2) = I^{(2)}_S \sin(\phi_1 - \phi_2) \]  

(1)

flows across the S1-S2 Josephson weak link, where the superscript "(2)" in Eq. (1) refers to the number N = 2 of the superconducting leads. Anderson and Rowell experimentally confirmed the prediction of the dc-Josephson effect.

Eq. (1) is valid for tunnel junctions and the current-phase relation

\[ I^{(n)}_S(\phi_1 - \phi_2) = \sum_n I^{(n)}_c \sin(n(\phi_1 - \phi_2)) \]  

(2)

with n-Cooper pair tunneling holds more generally at arbitrary interface transparency, see figure 2 for an example of the higher-order two-Cooper pair tunneling with n = 2 in Eq. (2).

Nonlocality of Andreev reflection at the scale R0 ≈ \( \tilde{\xi}_0 \) of the BCS coherence length \( \tilde{\xi}_0 \) was predicted and experimentally probed in a N1-S2-N3 three-terminal device since the early 2000’s, see figures 1 and 3. The normal leads N1 and N3 are laterally connected to the grounded S2 and the voltages V1 and V3 are applied on N1 and N3, the superconducting S2 being grounded at V2 = 0.

Elastic cotunneling (EC) over \( R_0 \approx \tilde{\xi}_0 \) transfers electrons from N1 to N3 across S2, or from N3 to N1, see figure 1. Crossed Andreev reflection (CAR) over \( R_0 \approx \tilde{\xi}_0 \) scatters spin-up electron from N1 as spin-down hole into N2, leaving a Cooper pair in S2, see figure 1. Said differently, two opposite-spin electrons from N1 and N3 are cooperatively transmitted into the central S2 and eventually join the condensate.

EC and CAR in the above mentioned N1-S2-N3 three-terminal device were generalized to dEC (see figure 3) and dCAR (see figure 4) in a laterally-connected S1-S2-S3 three-terminal Josephson junction with separation \( R_0 \approx \tilde{\xi}_0 \) between the S1-S2 and S2-S3 interfaces. Namely, double elastic cotunneling (dEC) on figure 3 transfers a Cooper pair from S1 to S3 across S2, or from S1 to S3. Double crossed Andreev reflection (dCAR) on figure 4 takes a Cooper pair from S1 and another one from S2. The two pairs from S1 and S3 exchange partners, yielding four-fermion state, i.e., the so-called quartet of electrons. The quartet eventually dissociates as two “outgoing” Cooper pairs joining the condensate of S2.

Voltage biasing the S1-S2-S3 three-terminal Josephson junction on figures 3 and 4 at the voltage differences \( V_1 - V_2 \) and \( V_3 - V_2 \) is a possibility to reveal dEC and the quartets as emergence of the \( V_1 + V_3 = 0 \) dc-Josephson resonance line, if one of the elements of the differential conductance matrix is plotted in the \( V_1, V_3 \) voltage plane while the “central” S2 is grounded at the reference voltage \( V_2 = 0 \). Other dc-Josephson resonance lines were predicted and observed such as \( V_1 = V_3 \) due to dEC.

Considering more generally \( p_1 \) Cooper pairs from S1 and \( p_3 \) from S3 yields the energy \( E_{\text{initial}} = 2p_1V_1 + 2p_3V_3 \) of the “initial state”, and \( E_{\text{final}} = 2(p_1 + p_3)V_3 \) for the final state, with \( E_{\text{final}} = 0 \) because \( V_3 = 0 \) for the grounded S2. Energy conservation \( E_{\text{initial}} = E_{\text{final}} \) implies the dc-Josephson resonance line at

\[ p_1V_1 + p_3V_3 = 0 \]  

(3)

which is sustained by the static dc-phase variable

\[ \varphi_{p_1, p_3} = p_1(\varphi_1 - \varphi_2) + p_3(\varphi_3 - \varphi_2) \]  

(4)

The following Josephson relations

\[ \varphi_1(t) = \varphi_1 + \frac{2eV_1}{h}t \]  

(6)

\[ \varphi_2(t) = \varphi_2 \]  

(7)

\[ \varphi_3(t) = \varphi_3 + \frac{2eV_3}{h}t \]  

(8)

for the superconducting phase variables \( \varphi_1(t), \varphi_2(t) \), and \( \varphi_3(t) \) as a function of the time \( t \) were combined to Eq. (3) in order to...
deduce Eq. (5) from Eq. (4). Then, the multipair supercurrent generalizing the quartets 

\[ I_{p_1,p_2}^{(3)} = I_{c,p_1,p_2} \sin \left( \frac{\phi_1 - \phi_2}{2} + \frac{\phi_3 - \phi_2}{2} \right). \]  

However, Eq. (9) also holds at equilibrium, i.e., if all superconducting leads \( S_1, S_2 \) and \( S_3 \) are grounded at \( V_1 - V_2 = V_3 - V_2 = 0 \), and biased at the phase differences \( \phi_1 - \phi_2 \) and \( \phi_2 - \phi_3 \). The present paper focuses on the equilibrium multiterminal dc-Josephson effect where all leads are grounded.

Nonlocality of the dc-Josephson effect can be understood in different ways:

(i) Subgap propagation over the zero-energy BCS coherence length \( R_0 \approx \xi_0 \) across the “central” \( S_2 \) in a \( S_1-S_2-S_3 \) three-terminal Josephson junction made with the \( S_1-S_2 \) and \( S_2-S_3 \) lateral contacts separated by \( R_0 \), see the above discussion.

(ii) Emergence of dc-Josephson current controlled by the phase of three or more superconducting leads, see Eq. (9) with \( p_1 \neq 0 \) and \( p_3 \neq 0 \).

The considered devices involve \( N \) superconductors connected by single-channel weak links to the nonsuperconducting “central” region, see figure 2. The above item (i) for nonlocality over \( R_0 \approx \xi_0 \) is thus not directly relevant to the present work. According to the above item (ii), we evaluate the contribution of the \( p \)-terminal Josephson effect in a device containing \( N \) superconducting leads. Namely, we evaluate the sensitivity of the current \( I_1 \) through lead \( S_1 \) on the \( p \) superconducting phase variables \( \phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_p} \) where \( 1 \leq a_1 < a_2 < \ldots < a_p \leq N \). Then, we define \( \chi^{(N)} \) as the partial derivative of \( I_1 \) with respect to the \( N - 2 \) phase differences \( \phi_2 - \phi_N, \phi_3 - \phi_N, \ldots, \phi_{N-1} - \phi_N \). We show that the \( p \)-terminal dc-Josephson effect with \( p \leq N - 2 \) does not contribute to \( \chi^{(N)} \) while \( p = N \) and \( p = N - 1 \) yield nonvanishingly small contribution to \( \chi^{(N)} \). Thus, experimental evidence for \( \chi^{(N)} \neq 0 \) with \( N = 4 \) nontrivially implies the \( p = 3 \) or the \( p = 4 \) three of four-terminal dc-Josephson effects, whatever the value of the \( p = 2 \) dc-Josephson supercurrent. Thus, the following paper demonstrates the possibility of directly testing the dc-Josephson effect with \( p \geq 3 \).

The paper is organized as follows. Section II establishes connection to known results. Section III presents the model and the \( p \)-terminal Josephson current in a device containing \( N \) superconducting leads. Section IV provides examples with \( N = 3 \) and \( N = 4 \) superconducting leads. Section V generalizes the theory to arbitrary number \( N \) of the superconducting leads. Numerical results for \( N = 4 \) superconducting leads are presented in section VI. Summary and final remarks are provided in section VII.

II. CONNECTION WITH RECENT RESULTS

This section provides connection to known results on the multiterminal dc-Josephson interferometers discussed in Refs. 24, 34, 35, 36.

The recent Refs. 34, 35 considered a “Andreev molecule” in the laterally-connected \( S_1-S_2-S_3 \) three-terminal Josephson junction under equilibrium voltage biasing conditions, i.e., the three superconducting leads \( S_1, S_2 \) and \( S_3 \) are grounded at \( V_1 - V_2 = V_3 - V_2 = 0 \). Refs. 34, 35 characterize the supercurrent \( I_1(\phi_3) \) through lead \( S_1 \) as a function of the phase \( \phi_3 \neq 0 \) on \( S_3 \) which is not directly connected to \( S_1 \), while \( \phi_2 = 0 \) is the reference, and it is assumed in addition that \( \phi_1 = 0 \).

This nonlocal response 34, 35 of the supercurrent \( I_1 \) through \( S_1 \) to the phase \( \phi_3 \) on \( S_3 \) originates from the above mentioned dEC and dCAR currents. In the limit of tunnel contacts, the dEC and dCAR Josephson currents are given by

\[ I_{dEC}^{(3)} = I_{c,dEC}^{(3)} \sin(\phi_1 - \phi_3), \]

\[ I_{dCAR}^{(3)} = I_{c,dCAR}^{(3)} \sin(\phi_1 + \phi_3 - 2\phi_2), \]

where “(3)” in the superscript refers to the number \( N = 3 \) of the superconducting leads. Considering \( \phi_1 = \phi_2 = 0 \) and \( \phi_3 \neq 0 \) yields the dc-Josephson current \( I_1 \) through \( S_1 \):

\[ I_1^{(3)}(0) = I_{dEC}^{(3)} + I_{dCAR}^{(3)} = \left( -I_{c,dEC}^{(3)} + I_{c,dCAR}^{(3)} \right) \sin \phi_3. \]
In an intuitive two-terminal picture, a $\Phi$-junction is obtained. The supercurrent $I_c^{(3)}$ plotted as a function of $\phi_1 - \phi_2$ is shifted by arbitrary phase which is controlled by $\phi_3 - \phi_2 \neq 0$. Two limiting cases are considered:

(a) In the tunnel limit, the dEC and dCAR current-phase relations are 0- and π-shifted respectively. Then, $I_{c,dEC}^{(3)} < 0$ and $I_{c,dCAR}^{(3)} < 0$ are both negative, yielding the same negative sign $-I_{c,dEC}^{(3)} + I_{c,dCAR}^{(3)} < 0$ as if $-I_{c,dEC}^{(3)}$ was alone. In this item (a), we assume that the contacts have linear dimension which is large compared to the Fermi wave-length $\lambda_F$ and small compared to the superconducting coherence length $\xi_0$.

(b) The limit $R_0 < \xi_0$ of small separation between the contacts necessarily produces $I_{c,dCAR}^{(3)} \approx 0$, because the assumption $R_0 < \xi_0$ implies equivalence to the two-terminal $S_1$-$S_2$. Then, the sign of $-I_{c,dEC}^{(3)} + I_{c,dCAR}^{(3)} \approx -I_{c,dEC}^{(3)} < 0$ is necessarily negative in the absence of dCAR.

The conditions for deducing “evidence for dCAR” from positive

$$I_{c,dEC}^{(3)} + I_{c,dCAR}^{(3)} > 0 \quad (13)$$

have thus not been elucidated at present time: The above arguments show that Eq. (13) is parameter-dependent. In the following paper, we propose another possibility for directly probing the $p$-terminal de-Josephson effect with $p \geq 3$.

Considering now the $(S_1, S_2, S_3, S_4)$ four-terminal Josephson junction with $N = 4$, we recently proposed a superconducting quantum interference device (SQUID) containing two loops making four contacts on a double quantum dot defined in a semiconducting nanowire or a carbon nanotube. It was shown in Ref. [24] that dEC and dCAR can be distinguished as different peaks in the Fourier transform of the critical current with respect to the magnetic field. Thus, this double SQUID successfully proposes a way to provide evidence for dCAR and for “the three-terminal de-Josephson effect”.

The device on figure 2 contains the four superconducting leads $S_1, S_2, S_3, S_4$ which are biased at the three independent phases differences $\phi_1 - \phi_4$, $\phi_2 - \phi_4$, and $\phi_3 - \phi_4$. As it is mentioned above, the superconducting leads are grounded at $V_1 - V_4 = 0$, $V_2 - V_4 = 0$ and $V_3 - V_4 = 0$. In addition, the two loops are pierced by the magnetic field fluxes $\Phi_A$ and $\Phi_B$. Compared to Ref. [24] we complementarily propose measurement of the second-order nonlocal inverse inductance

$$\chi^{(4)}(\Phi_1 - \Phi_4, \Phi_2 - \Phi_4, \Phi_3 - \Phi_4) \quad (14)$$

$$= \frac{\partial^2 I_1^{(4)}(\Phi_1 - \Phi_4, \Phi_2 - \Phi_4, \Phi_3 - \Phi_4)}{\partial (\Phi_2 - \Phi_4) \partial (\Phi_3 - \Phi_4)}, \quad (15)$$

which is also evaluated as a function of $\phi_1 - \phi_4$ for $\phi_2 - \phi_4 = 0$ and $\phi_3 - \phi_4 = 0$:

$$\chi^{(4)(0)}(\Phi_1 - \Phi_4) \equiv \chi^{(4)(0)}(\Phi_1 - \Phi_4) \quad (16)$$

$$= \frac{\partial^2 I_1^{(4)}(\Phi_1 - \Phi_4, \Phi_2 - \Phi_4, \Phi_3 - \Phi_4)}{\partial (\Phi_2 - \Phi_4) \partial (\Phi_3 - \Phi_4)} \bigg|_{\Phi_2 - \Phi_4 = 0} \quad (17)$$

The superscript “(4)” in Eqs. (14-15) and Eqs. (16-17) refers to the number $N = 4$ of the superconducting leads, see the device on figure 2.

In the paper, we demonstrate that “Absence of the three- and four-terminal dc-Josephson effect, i.e. $I_{c,S_1-p_1}^{(4)} = 0$ in Eq. (9) for all $p_1$, $p_3$, and $I_{c,p_1-p_2-p_3}^{(4)} = 0$ for all $p_1$, $p_2$, $p_3$”.

The above mentioned four-terminal critical current $I_{c,p_1-p_2-p_3}^{(4)}$ corresponds to

$$I_{c,p_1-p_2-p_3}^{(4)}(\Phi_1 - \Phi_4, \Phi_2 - \phi_4, \Phi_3 - \Phi_4) \quad (18)$$

$$= I_{c,p_1-p_2-p_3}^{(4)}(\Phi_1 - \Phi_4, \Phi_2 - \phi_4, \Phi_3 - \Phi_4) \quad (18)$$

$$= I_{c,p_1-p_2-p_3}^{(4)}(\Phi_1 - \Phi_4) + I_{c,p_1-p_2-p_3}^{(4)}(\Phi_2 - \phi_4) + I_{c,p_1-p_2-p_3}^{(4)}(\Phi_3 - \Phi_4).$$
TABLE I. The table summarizes the values of $p$ and $N$ considered in the paper.

| $N$   | Figure number | $p = 2$ | $p = 3$ | $p = 4$ | ... | $p = N - 1$ | $p = N$ |
|-------|---------------|---------|---------|---------|-----|-------------|---------|
| 2     |               | Eqs. (1) | Eqs. (2) |         |     |             |         |
| 3     |               | Eqs. (32) - (33) | Eq. (34) |         |     |             |         |
| 4     | Fig. 2c       | Eqs. (40) - (42) | Eqs. (43) - (45) | Eq. (46) |     |             |         |
|       | Fig. 3a1      | Eq. (49) | Eq. (50) |         |     |             |         |
|       | Fig. 3a2      | Eq. (52) | Eq. (52) |         |     |             |         |
| N = 2q | Fig. 3a        | Eq. (53) | Eq. (53) |         |     |             |         |
| N = 2q + 1 | Fig. 3a      | Eq. (54) | Eq. (54) |         |     |             |         |

Conversely, we deduce a second logical link:

"Nonvanishingly small signal $\chi^{(4)}(\varphi_1 - \varphi_4, \Phi_A, \Phi_B) \neq 0$ or $\chi^{(4)(0)}(\varphi_1 - \varphi_4) \neq 0"$ generically implies "Evidence for the three- or four-terminal dc-Josephson effect, i.e. $I_{\phi_{p_1,p_2}} \neq 0$ for some $p_1, p_2$ or $I_{\phi_{p_1,p_2,p_3}} \neq 0$ for some $p_1, p_2, p_3$".

The two logical links are independent on the value of the arbitrary number $N$ of the superconducting leads.

III. $p$-TERMINAL JOSEPHSON CURRENT WITH $N$ SUPERCONDUCTING LEADS

In this section, we establish an expression for the $p$-terminal dc-Josephson current with $N \geq p$ superconducting leads. The expression of the supercurrent carried by the Andreev Bound States

In this subsection, we evaluate the Fourier transform of the ABS energies with respect to all superconducting phase variables $\varphi_1, ..., \varphi_N$ of the $N$ superconducting leads $S_1, ..., S_N$ connected to a small nonsuperconducting region. The ABS energies $E_{\text{ABS}, \lambda}^{(\psi)}(\varphi_1, ..., \varphi_N)$ depend on all values of the $\varphi_n - \varphi_0$ (where $n = 1, ..., n_{\text{ABS}}$ is used for the negative-energy ABS). Taking the Fourier transform leads to

$$E_{\text{ABS}, \lambda}^{(\psi)}(\varphi_1, \varphi_2, ..., \varphi_N) = \sum_{m_1} \sum_{m_2} ... \sum_{m_N} E_{\text{ABS}, \lambda}^{(m)}[m_1, m_2, ..., m_N] \times \exp \left( i (m_1 \varphi_1 + m_2 \varphi_2 + ... + m_N \varphi_N) \right) \delta (m_1 + m_2 + ... + m_N),$$

where the constraint

$$m_1 + m_2 + ... + m_N = 0$$

originates from gauge invariance, i.e. any of the superconducting phases $\varphi_{n_{\text{ref}}}$ can be chosen as the reference, leaving $N - 1$ independent gauge-invariant phase differences $\varphi_n - \varphi_{n_{\text{ref}}}$ with $n \neq n_{\text{ref}}$. For instance, choosing $n_{\text{ref}} = N$ leads to

$$m_1 \varphi_1 + m_2 \varphi_2 + ... + m_N \varphi_N = m_1 (\varphi_1 - \varphi_0) + m_2 (\varphi_2 - \varphi_0) + ... + m_{N-1} (\varphi_{N-1} - \varphi_0),$$

where Eq. (20) and Eq. (21) are equivalently used. The variables $m_1, m_2, ..., m_N$ are conjugate to the superconducting phases $\varphi_1, \varphi_2, ..., \varphi_N$, i.e. they are interpreted as algebraic number of Cooper pairs transmitted into $S_1, S_2, ..., S_N$.

The zero-temperature dc-Josephson current through the lead $S_{N_0}$ is given by
\[ S_{N_0}^{(N)}(\varphi_1, \ldots, \varphi_N) = -\frac{e}{\hbar} \sum_{\Lambda} E^{(\varphi)}_{\Lambda} (\varphi_1, \varphi_2, \ldots, \varphi_N) \]
\[ = \sum_{m_1} \sum_{m_2} \ldots \sum_{m_N} \sum_{m_{n}} I_{c, n_0}^{(N)} (m_1, \ldots, m_N) \sin (m_1 \varphi_1 + m_2 \varphi_2 + \ldots + m_N \varphi_N) \delta (m_1 + m_2 + \ldots + m_N), \]

where

\[ I_{c, n_0}^{(N)} (m_1, \ldots, m_N) = -\frac{e}{\hbar} m_{n_0} E^{(m)}_{\text{ABS}, \lambda} (m_1, m_2, \ldots, m_N). \]  

B. \textit{p-terminal dc-Josephson current with N superconducting leads}

In this subsection, we provide an expression for the \textit{p}-terminal dc-Josephson current in a device containing \( N \) superconducting leads, with \( p \leq N \).

We consider now the \( p \) integers \( a_1, \ldots, a_p \) such that \( 1 \leq a_1 < a_2 < \ldots < a_p \leq N \). They encode supercurrent flowing through the superconducting leads \( S_{a_1}, S_{a_2}, \ldots, S_{a_p} \), and absence of supercurrent through \( S_{b_1}, S_{b_2}, \ldots, S_{b_{N-p}} \), where \( a_\alpha \neq b_\beta \) for all \( \alpha = 1, 2, \ldots, p \) and \( \beta = 1, 2, \ldots, p - 1 \). Then,

\[ I_{c, n_0}^{(N)} (p) (\varphi_1, \ldots, \varphi_N) = \sum_{a_1=1}^{N} \sum_{a_2=0}^{N} \ldots \sum_{a_p=0}^{N} \delta \left( \prod_{\alpha=1}^{N} (1 - \delta (a_\alpha, n_0)) \right) \sum_{m_{a_1} \neq 0} \sum_{m_{a_2} \neq 0} \sum_{m_{a_p} \neq 0} \times \]

The \( \delta \)-function \( \delta \left( \prod_{\alpha=1}^{N} (1 - \delta (a_\alpha, n_0)) \right) \) in Eq. 26 is used to indicate that the lead \( S_{n_0} \) (through which the supercurrent is evaluated) is among \( S_{a_1}, \ldots, S_{a_p} \) (which sustain the considered \( p \)-terminal dc-Josephson process). Namely, \( a_0 = N_0 \) for one of the \( \alpha = 0 \) implies \( \prod_{\alpha=1}^{N} (1 - \delta (a_\alpha, n_0)) = 0 \) and \( a_\alpha \neq N_0 \) for all \( \alpha = 1, 2, \ldots, p \) implies \( \prod_{\alpha=1}^{N} (1 - \delta (a_\alpha, n_0)) = 1 \).

Eqs. (22)-(23) were calculated from the contribution of the ABS to the dc-Josephson supercurrent. However, the present paper is based on the general current-phase relation given by Eq. (26), which also captures the contribution of the continuum.

C. \textit{Single-level quantum dot}

In this subsection, we illustrate Eqs. (22)-(23) for a quantum dot connected to three or four superconducting leads. We denote by \( \Gamma_1, \Gamma_2, \ldots, \Gamma_N \) the line-width broadening associated to hopping between the dot and each of the superconducting leads \( S_1, S_2, \ldots, S_N \) in the normal-state. We take the infinite-gap limit, see for instance Refs. [38]-[40].

The \( 2 \times 2 \) Nambu Hamiltonian takes the form

\[ H_\varphi = \begin{pmatrix} 0 & \Sigma_{\alpha=1}^{N} \Gamma_\alpha \exp (-i \varphi_\alpha) \\ \Sigma_{\alpha=1}^{N} \Gamma_\alpha \exp (i \varphi_\alpha) & 0 \end{pmatrix}. \]

The ABS energies

\[ E_{\text{ABS}, \pm} (\varphi_1, \varphi_2, \ldots, \varphi_N) \]

\[ = \pm \left| \sum_{\alpha=1}^{N} \Gamma_\alpha \exp (i \varphi_\alpha) \right|. \]

The nonvanishingly small Fourier coefficients \( E_{\text{ABS}, \pm}^{(m)} (m_1, m_2, \ldots, m_N) \) in Eq. (19). This yields the \( p \)-terminal Josephson effect, i.e. components of the supercurrent which are sensitive to the superconducting phase variables of \( p \) superconducting leads:

\[ \Psi_p (a_1, a_2, \ldots, a_p) = m_{a_1} \varphi_{a_1} + m_{a_2} \varphi_{a_2} + \ldots + m_{a_p} \varphi_{a_p}, \]

where \( \{a_\alpha\} \) is a set of \( p \) integers such that \( 1 \leq a_1 < a_2 < \ldots < a_p \leq N \) and \( m_{a_\alpha} \neq 0 \) for all \( \alpha = 1, \ldots, p \).
FIG. 3. Panels a1, b1 and c1 show the “incoming” and “outgoing” Cooper pairs for the $p = 2, 3, 4$-terminal dc-Josephson processes respectively, with a total of $N = 4$ superconducting leads. The corresponding values of $a_\alpha$ and $m_\alpha$ are shown on panels a2, b2 and c2. The integer $a_\alpha \in \{1, \ldots, n\}$ is shown on the $x$-axis of panels a2, b2 and c2 corresponding to the parameters of panels a1, b1 and c1 respectively. The variable $m \equiv m_\alpha$ on the $y$-axis of panels a2, b2 and c2 is the algebraic number of Cooper pairs transmitted in each superconducting lead.

FIG. 4. Panels a1 and a2 are obtained by “dressing” the $p = 2$ and $p = 3$-terminal dc-Josephson current processes on figures 3-a1 and 3-a2 by an “excursion” in and out from one of the superconducting leads. Thus, the $p = 2$ and $p = 3$-terminal dc-Josephson processes on panel a1 and b1 involve three and four terminal respectively. They receive the same $a_\alpha$ and $m_\alpha$ values as the “bare” diagrams on figures 3-a1 and 3-a2.

Figures 5 and 6 show the distributions $P(H_{n_1, n_2, n_3})$ and $P(H_{n_1, n_2, n_3, n_4})$ of the Fourier harmonics of the three and four-terminal dc-Josephson currents

\[ H_{n_1, n_2, \ldots, n_p} = \int \frac{d\varphi_{a_1}}{2\pi} \int \frac{d\varphi_{a_2}}{2\pi} \ldots \int \frac{d\varphi_{a_p}}{2\pi} \times \]

\[ I^{(N)}(p) (\varphi_{a_1}, \varphi_{a_2}, \ldots, \varphi_{a_p}) \times \]

\[ \sin (m_{a_1} \varphi_{a_1} + m_{a_2} \varphi_{a_2} + \ldots + m_{a_p} \varphi_{a_p}) \]

where $a_\alpha \equiv \alpha$ on the examples of figures 5 and 6 where $p = N$.

The distribution of the $\Gamma$'s in Eqs. (27)-(28) is a Gaussian centered at $\bar{\Gamma} = 1$, with root-mean-square $\delta \Gamma = 0.5$. The negative values of the $\Gamma$'s are rejected. These parameter values imply that $H_{n_1, n_2, n_3}$ and $H_{n_1, n_2, n_3, n_4}$ are roughly of order unity on figures 5 and 6. Figure 5a shows the $(1, 0, -1)$ harmonics with $N = 3$, with the expected positive sign. Figure 5b shows the expected negative sign of the $(2, 0, -2)$ harmonics. Figure 5c shows the $(1, 1, -2)$ quartets with negative sign corresponding to the negative $I_c, dC/dI < 0$ mentioned in section II. Figure 5d shows the $(2, 1, -3)$ harmonics.

Figures 6-8 further illustrate the multiterminal dc-Josephson effect on the example of $N = 4$ superconducting leads, with $(n_1, n_2, n_3, n_4) = (1, -1, 1, -1), (3, 1, -2, -2), (2, 3, -3, -1)$ and $(1, 3, -3, -1)$ respectively.

It is concluded that the multiterminal dc-Josephson effect naturally emerges if a quantum dot is connected to $N$ super-
conducting leads, in the form of the higher-order harmonics of the dc-Josephson current-phase relation, see the selected values of \((n_1, n_2, n_3)\) and \((n_1, n_2, n_3, n_4)\) for \(N = 3\) and \(N = 4\) superconducting leads on figures 2b and 2c respectively.

IV. EXAMPLES WITH \(N = 3, 4\) SUPERCONDUCTING LEADS

In this section, we present the specific examples of \(N = 3\) and \(N = 4\) superconducting leads, see the devices on figures 2b and 2c. The cases \(N = 3\) and \(N = 4\) are treated in sections IV A and IV B respectively. We show that \(\chi^{(3)}\) for \(N = 3\) superconducting leads contains the contributions of the \(p = 2\) and the \(p = 3\)-terminal dc-Josephson effect, and that \(\chi^{(4)}\) for \(N = 4\) superconducting leads contains the contributions of the \(p = 3\) and the \(p = 4\)-terminal dc-Josephson effect, whatever the value of the \(p = 2\)-terminal dc-Josephson current.

A. Device with \(N = 3\) superconducting leads on figure 2b

In this subsection, we consider the device with \(N = 3\) superconducting leads on figure 2b. We specifically evaluate the flux-\(\Phi\) sensitivity of the supercurrent through the lead \(S_1\):

\[
I^{(3)}_1(\varphi_1, \varphi_2, \varphi_3) = \sum_{n \neq 0} I^{(3)(2)(1,2)}_{e}[n] \sin[n(\varphi_1 - \varphi_2)] + \sum_{n \neq 0} I^{(3)(2)(1,3)}_{e}[n] \sin[n(\varphi_1 - \varphi_3)] + \sum_{n_1 \neq 0, n_2 \neq 0} I^{(3)(3)(1,2,3)}_{e}[n_1, n_2] \sin[n_1(\varphi_1 - \varphi_3) + n_2(\varphi_2 - \varphi_3)].
\]

Taking into account that \(\varphi_3 = \varphi_2 - \Phi\), we obtain \(\varphi_1 - \varphi_3 = \varphi_1 - \varphi_2 + \Phi\). The considered signal for \(N = 3\) is defined as
The zero-flux limit $\Phi = 0$ of Eqs. (36)-(37) is the following:

$$
\chi^{(3)(0)}(\varphi_1 - \varphi_2) \equiv \chi^{(3)}(\varphi_1 - \varphi_2, 0) \\
= \sum_{n \neq 0} n \left[ L_c^{(3)(2)}(1,2)[n] + \sum_{n_2 \neq 0} L_c^{(3)(3)}(1,2,3)[n, n_2] \right] \times \\
\cos [n_1 (\varphi_1 - \varphi_2 + \Phi) + n_2 \Phi].
$$

Both $p = 2$ and $p = 3$ contribute to $\chi^{(3)}$ in Eqs. (36)-(37) and $\chi^{(3)(0)}$ in Eqs. (38)-(39). Thus, $\chi^{(3)}$ or $\chi^{(3)(0)}$ with $N = 3$ superconducting leads cannot be used to probe the $p = 3$-terminal dc-Josephson effect independently on the contribution of the $p = 2$-terminal Josephson effect, see the discussion in section II.

**B. Device with $N = 4$ superconducting leads on figure 2**

In this subsection, we now consider the device with $N = 4$ superconducting leads, see figure 2. We evaluate the sensitiv-
ity of the supercurrent $I^{(4)}_1$ through the lead $S_1$ on the three superconducting phase differences $\phi_1 - \phi_4$, $\phi_2 - \phi_4$ and $\phi_3 - \phi_4$.

The supercurrent through the superconducting lead $S_1$ takes the form

$$I^{(4)}_1 = \sum_{n \neq 0} I^{(4)(2)(1,2)}_c[n] \sin [n(\phi_1 - \phi_2)]$$

$$+ \sum_{n \neq 0} I^{(4)(2)(1,3)}_c[n] \sin [n(\phi_1 - \phi_3)]$$

$$+ \sum_{n \neq 0} I^{(4)(2)(1,4)}_c[n] \sin [n(\phi_1 - \phi_4)]$$

$$+ \sum_{n_1 \neq 0, n_2 \neq 0} I^{(4)(3)(1,2,3)}_c[n_1, n_2] \times$$

$$\sin [n_1(\phi_1 - \phi_3) + n_2(\phi_2 - \phi_3)]$$

$$+ \sum_{n_1 \neq 0, n_2 \neq 0} I^{(4)(3)(1,2,4)}_c[n_1, n_2] \times$$

$$\sin [n_1(\phi_1 - \phi_4) + n_2(\phi_2 - \phi_4)]$$

$$+ \sum_{n_1 \neq 0, n_2 \neq 0, n_3 \neq 0} I^{(4)(4)(1,2,3,4)}_c[n_1, n_2, n_3] \times$$

$$\sin [n_1(\phi_1 - \phi_3) + n_2(\phi_2 - \phi_3) + n_3(\phi_3 - \phi_4)].$$

The fluxes $\Phi_A$ and $\Phi_B$ are taken into account with $\phi_3 = \phi_2 - \Phi_A$ and $\phi_4 = \phi_3 - \Phi_B$, which leads to

$$\phi_3 - \phi_4 = \Phi_B$$

$$\phi_2 - \phi_3 = \Phi_A + \Phi_B.$$
V. N SUPERCONDUCTING LEADS

We demonstrated in the previous section IV that $\chi^{(3)} \neq 0$ or $\chi^{(3)}(\phi_1 - \phi_3) \neq 0$ for $N = 3$, and that $\chi^{(4)} \neq 0$ or $\chi^{(4)}(\phi_1 - \phi_4) \neq 0$ for $N = 4$ imply evidence for the $p = 2, 3$ and the $p = 3, 4$ dc-Josephson effects respectively.

Now, we generalize this statement to arbitrary values of the number $N$ of the superconducting leads. Subsection V A provides expression of the higher-order nonlocal inverse inductance $\chi^{(N)}$ at arbitrary $N$. Subsections V B and V C present the calculations for even and odd values of $N$ respectively.

A. Higher-order nonlocal inverse inductance

In this subsection, we discuss how $N = 3, 4$ in the previous sections IV A and IV B respectively can be generalized to arbitrary number $N$ of superconducting leads.

Eqs. (25)–(26) for the dc-Josephson current are written as

$$
\delta \left( \prod_{\alpha=1}^{N-1} (1 - \delta (d'_\alpha, 1)) \right) \sum_{m'_{a_1} \neq 0} \sum_{m'_{a_2} \neq 0} \ldots \sum_{m'_{a_p} \neq 0} \delta (N-1) \left( N \prod_{\alpha=1}^{N-1} (\sum_{a'_\alpha = a'_\alpha + 1} (\sum_{a'_\alpha = a'_\alpha + 1} \ldots (\sum_{a'_p = a'_p + 1} \ldots \sum_{a'_p = a'_p + 1} I_{c}(\phi_{a'_1}, \ldots, \phi_{a'_p}) \sin \left( \sum_{\alpha=1}^{p} m'_{a'_\alpha} (\phi_{a'_\alpha} - \phi_N) \right) \right),
$$

where we evaluated the supercurrent through the lead labeled by $N_0 = 1$, and used the phase $\phi_N$ of the superconducting lead $\delta_N$ as a reference. The variable $a'_\alpha$ in Eq. (53) is defined in the interval $1 \leq a'_\alpha \leq N - 1$ while $a_\alpha$ in Eq. (26) is such that $1 \leq a_\alpha \leq N$.

We define the order-$N-2$ nonlocal inductance as the sig-
superconducting phase variable $\varphi_N$ is the reference for evaluating the partial derivatives of $I_1^{(S)}$ with respect to all of the $\varphi_n - \varphi_N$, with $n = 2, ..., n - 1$. Now, we provide a calculation of Eq. (54).

B. Even number $N = 2q$ of superconducting leads

In this subsection, we assume that $N = 2q$ is even. In addition, we assume in subsection $\text{V B 1}$ that $\sum_{\alpha=1}^{2q-1} m'_{\alpha} \neq 0$ (with $m'_{\beta} \neq 0$ for all $\beta = 1, ..., 2q - 1$). Conversely, subsection $\text{V B 2}$ deals with $\sum_{\alpha=1}^{2q-1} m'_{\alpha} = 0$.

1. $p = 2q$-terminal Josephson processes with $N = 2q$ superconducting leads

In this subsection, we assume that $N = 2q$ is even and $\sum_{\alpha=1}^{2q-1} m'_{\alpha} \neq 0$ (with $m'_{\beta} \neq 0$ for all $\beta = 1, ..., 2q - 1$).

This results in the $p = 2q$-terminal dc-Josephson contribution $\left[L_{2q-2}^{-1}\right]'(2q)$ to $\chi^{(2q)} = L_{2q-2}^{-1}'$ in Eq. (54). Taking into account that $b'_\alpha = \alpha$ for a $p = 2q = N$-terminal dc-Josephson process, we find

$$\left[L_{2q-2}^{-1}\right]'(2q) = (-)^{q+1} \sum_{m_1 \neq 0, m_2 \neq 0} \prod_{\alpha=2}^{2q-1} m'_{\alpha} \sin \left( \sum_{\alpha=1}^{2q-1} m'_{\alpha} (\varphi_{\alpha} - \varphi_{2q}) \right).$$

(55)

Specializing to $\varphi_2 - \varphi_{2q} = ... = \varphi_{2q-1} = 0$ and $\varphi_1 - \varphi_{2q} \neq 0$ leads to

$$\left[L_{2q-2}^{-1}\right]'(2q)^{(0)} = (-)^{q+1} \sum_{m_1 \neq 0, m_2 \neq 0} \prod_{\alpha=2}^{2q-1} m'_{\alpha} \sin \left( m'_1 (\varphi_1 - \varphi_{2q}) \right).$$

(56)

2. $p = (2q - 1)$-terminal Josephson processes with $N = 2q$ superconducting leads

Now, we assume in this subsection that $N = 2q$ is even and

$$\sum_{\alpha=1}^{2q-1} m'_{\alpha} = 0,$$

(57)

with $c'_\alpha = \alpha$ for all $\alpha = 1, ..., 2q - 1$. Then, we make use of the identities

$$\sin \left( \sum_{\alpha=1}^{2q-1} m'_{\alpha} (\varphi_{\alpha} - \varphi_{2q}) \right) = \sin \left( \sum_{\alpha=1}^{2q-1} m'_{\alpha} \varphi_{\alpha} \right) = \sin \left( \sum_{\alpha=2}^{2q-1} m'_{\alpha} (\varphi_{\alpha} - \varphi_1) \right)$$

$$= \sin \left( \sum_{\alpha=1}^{2q-1} m'_{\alpha} (\varphi_{\alpha} - \varphi_{2q}) - \sum_{\alpha=2}^{2q-1} m'_{\alpha} (\varphi_{2q} - \varphi_1) \right) = \sin \left( \sum_{\alpha=2}^{2q-1} m'_{\alpha} (\varphi_{\alpha} - \varphi_{2q}) + m'_1 (\varphi_{2q} - \varphi_1) \right).$$

(58)
Specializing to \( \varphi_1 - \varphi_{2q} \neq 0 \) and \( \varphi_2 - \varphi_{2q} = \ldots = \varphi_{2q-1} - \varphi_{2q} = 0 \), we deduce the contribution \( L_{2q-1}^{-1}(2q-1) \) of the \( p = 2q - 1 \)-terminal dc-Josephson processes to \( L_{2q-2}^{-(2q-2)} \) in the presence of \( N = 2q \) superconducting leads:

\[
L_{2q-2}^{-(2q-1)(0)} = (-)^q \sum_{m_1' \neq 0} \sum_{m_2' \neq 0} \ldots \sum_{m_{2q-1}' \neq 0} I_{c,1}^{(2q-1)(1,2,\ldots,2q-1)} \left[ - \left( \sum_{\alpha=2}^{2q-1} m_{\alpha}' \right), m_2', \ldots, m_{2q-1}' \right] \times (59)
\]

where we used \( m_{2q-1}' = -\sum_{\alpha=1}^{2q-2} m_{\alpha}' \) deduced from Eq. (57).

3. Summary

The zero-flux limit of the predicted experimental signal with \( N = 2q \) generalizes the above Eq. (52) with \( q = 2 \):

\[
\chi^{(2q)(0)} (\varphi_1 - \varphi_{2q}) = [56] + [59].
\]

C. Odd number \( N = 2q' + 1 \) of superconducting leads

In this subsection, we assume now that \( N = 2q' + 1 \) is odd. We assume in subsection IV C 1 that \( \sum_{\alpha=1}^{2q-1} m_{\alpha}' \neq 0 \) (with \( m_{\beta}' \neq 0 \) for all \( \beta = 1, \ldots, 2q - 1 \)) and we assume in subsection IV C 2 that \( \sum_{\alpha=1}^{2q-1} m_{\alpha}' = 0 \).

1. \( p = (2q' + 1) \)-terminal Josephson processes with \( N = 2q' + 1 \) superconducting leads

In this subsection, we assume that \( N = 2q' + 1 \) is odd and \( \sum_{\alpha=1}^{2q-1} m_{\alpha}' \neq 0 \) (with \( m_{\beta}' \neq 0 \) for all \( \beta = 1, \ldots, 2q' \)). We find

\[
L_{2q-1}^{-(2q'+1)} = (-)^{q'+1} \sum_{m_1' \neq 0} \sum_{m_2' \neq 0} \ldots \sum_{m_{2q}' \neq 0} I_{c,1}^{(2q'+1)(1,2,\ldots,2q') \left[ m_1', \ldots, m_{2q}' \right] \times (61)
\]

Specializing Eq. (61) to \( \varphi_2 - \varphi_{2q'+1} = \ldots = \varphi_{2q'} - \varphi_{2q'+1} = 0 \) and \( \varphi_1 - \varphi_{2q'+1} \neq 0 \) leads to

\[
L_{2q-1}^{-(2q'+1)(0)} = (-)^{q'+1} \sum_{m_1' \neq 0} \sum_{m_2' \neq 0} \ldots \sum_{m_{2q}' \neq 0} I_{c,1}^{(2q'+1)(1,2,\ldots,2q') \left[ m_1', \ldots, m_{2q}' \right] \times (63)
\]

2. \( p = 2q' \)-terminal Josephson processes with \( N = 2q' + 1 \) superconducting leads

In this subsection, we assume that \( N = 2q' + 1 \) is odd and \( \sum_{\alpha=1}^{2q-1} m_{\alpha}' = 0 \) (with \( m_{\beta}' \neq 0 \) for all \( \beta = 1, \ldots, 2q' \)).
We specialize to $\varphi_2 - \varphi_{2q' + 1} = \ldots = \varphi_{2q'} - \varphi_{2q' + 1} = 0$ and $\varphi_1 - \varphi_{2q' + 1} \neq 0$, we deduce

$$\left[ L_{2q'-1}^{-1} \right]^{(2q')'(0)} = (-)^{q' + 1} \sum_{m_1 \neq 0} \sum_{m_2 \neq 0} \ldots \sum_{m_{2q'} \neq 0} \left( \sum_{\alpha=2}^{2q'} m_\alpha' \right) \left( \varphi_1 - \varphi_{2q' + 1} \right) \times \left( - \left( \sum_{\alpha=2}^{2q'} m_\alpha' \right), m_2', \ldots, m_{2q'}' \right) \times \left( L_{N-2}^{-1} \right)^{(p)} = 0 \text{ if } p \leq N - 2. \tag{69}$$

In addition, the $(N - 2)$-th order derivative of $\sin \left( m_1', \varphi_1' + \ldots + m_{2q'}, \varphi_{2q'}' \right)$ in Eq. (54) produces $\sin$ $\cos$ according to the parity of $N$.

The Appendix details examples with $N = 6$ superconducting leads, see figure 7.

VI. NUMERICAL RESULTS

Now, we present a numerical illustration that $\chi^{(4)}(\varphi_1 - \varphi_4)$ defined in Eq. (16) is vanishingly small if $I_c^{(4)}(1,3,4) = 0$ and $I_c^{(4)}(1,2,3,4) = 0$ in Eqs. (40)–(46).

This implies that “measurement of $\chi^{(4)}(\varphi_1 - \varphi_4) \neq 0$” is evidence for the $p = 3$ or $p = 4$-terminal dc-Josephson critical currents $I_c^{(4)}(1,3,4) \neq 0$ or $I_c^{(4)}(1,2,3,4) \neq 0$.

We use the following phenomenological modeling of the ABS energies:

$$E_{ABS}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \alpha_0 + \alpha_{2T} \left( \cos(\varphi_1 - \varphi_2) + \cos(\varphi_1 - \varphi_3) + \cos(\varphi_1 - \varphi_4) + \cos(\varphi_2 - \varphi_3) + \cos(\varphi_2 - \varphi_4) + \cos(\varphi_3 - \varphi_4) \right)$$

where $\alpha_0$ is a constant energy shift, and $\alpha_{2T}$, $\alpha_{3T}$ and $\alpha_{4T}$ parameterize the $p = 2, 3, 4$ dc-Josephson effect.

The current $I_1$ through the lead $S_1$ is deduced from Eq. (22), and $\chi^{(4)}(\varphi_1 - \varphi_4)$ is obtained from Eqs. (16)–(17).

Figure 8 shows $\chi^{(4)}(\varphi_1 - \varphi_4)$ for $\alpha_{2T} = 1$, $\alpha_{4T} = 0$ and $\alpha_{3T}$ between 0 and 1. It is concluded that $\alpha_{2T} = \alpha_{4T} = 0$ implies $\chi^{(4)}(\varphi_1 - \varphi_4) = 0$. Conversely, $\alpha_{3T} \neq 0$ with $\alpha_{2T} = 0$ implies nonvanishingly small $\chi^{(4)}(\varphi_1 - \varphi_4) \neq 0$, whatever $\alpha_{2T}$. Thus, Figure 8 confirms that $\chi^{(4)}(\varphi_1 - \varphi_4) \neq 0$ is evidence for the $p = 3$ or $p = 4$ dc-Josephson effect in a device with $N = 4$ superconducting leads.

VII. CONCLUSIONS

Now, we present summary of the main results and final remarks.

We considered multiterminal dc-Josephson interferometers and investigated devices with $N = 4$ (or with $N \geq 4$) superconducting leads, in a way which is different from the previous Refs. 24, 34, 56.

Specifically, we proposed to detect the $p \geq 3$-terminal Josephson effect in a device containing $N = 4$ superconducting leads, see figure 5. The experimental signal $\chi^{(4)}(\varphi_1 - \varphi_4)$ is given by Eqs. (14)–(15) or Eqs. (16)–(17).
respectively. We obtained the following result:

“Nonvanishingly small second-order nonlocal inverse inductance $\chi^{(4)}(\rho_1 - \rho_4)$ (i.e., $\chi^{(4)}(\rho_1 - \rho_4)$) generically implies

“Evidence for the three or four-terminal dc-Josephson effect”.

We generalized this statement to a device containing arbitrary number $N \geq 4$ of the superconducting leads, thus with $N - 1$ independent phase differences. One of those phase differences (e.g., $\rho_1 - \rho_N$) is reserved for controlling the current-phase relation of the overall two-terminal device. The remaining $N - 2$ independent superconducting phase differences are used in the definition Eq. (54) of the higher-order nonlocal inverse inductance $\chi^{(N)}$ and they are related to the $N - 2$ independent fluxes.

Practically, the partial derivative of the supercurrent $I_1$ through the superconducting lead $S_1$ with respect to $\rho_2 - \rho_N$, $\rho_3 - \rho_N$, ..., $\rho_{N-1} - \rho_N$ [see Eqs. (14) - (15), (16) - (17) and Eq. (54)] can be evaluated by changing variables from the $N - 1$ independent phase differences $\rho_1 - \rho_N$, $\rho_2 - \rho_N$, ..., $\rho_{N-1} - \rho_N$ to $\phi_1 - \phi_N$ and the $N - 2$ fluxes $\Phi_{A_1}$, $\Phi_{A_2}$, ..., $\Phi_{A_{N-2}}$.

We demonstrated that finite values for $\chi^{(N)}(\rho_1 - \rho_N) \neq 0$ or $\chi^{(N+1)}(\rho_1 - \rho_N) \neq 0$ necessarily originate from the $p = N - 1$ and $p = N$-terminal dc-Josephson effects: the $p$-terminal dc-Josephson effect does not contribute to $\chi^{(N)}(\rho_1 - \rho_N)$ if $p \leq N - 2$.

Finally, a device with number $N = 4$ of the superconducting leads and two loops seems to be already quite interesting from the point of view of possible experiments, see figure 7a. Detection of $\chi^{(N)}(\rho_1 - \rho_N) \neq 0$ or $\chi^{(N+1)}(\rho_1 - \rho_N) \neq 0$ can be realized in microwave experiments, see also the recent Refs. [40, 42].

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APPENDIX: EXAMPLES WITH $N = 6$ SUPERCONDUCTING LEADS

In this Appendix, we detail the examples shown on figure 7 with $N = 6$ superconducting leads.

Figure 8 shows example of a $p = 3$-terminal dc-Josephson process with $N = 6$. Figure 7b shows a $p = 6$-terminal process with $N = 6$. Figures 7d and 7e show $p = 5$ with $N = 6$. Not all values of $p$ contribute to $\chi^{(N)} = [L^{-1}_N]_T$ in Eq. (54):

(i) The $p = 3$-terminal dc-Josephson current $i_1(\phi_1 - \rho_1, \phi_3 - \rho_1)$ shown on figure 7b depend on the two independent phase differences $\phi_1 - \rho_1$ and $\phi_3 - \rho_1$, thus it cannot contribute to $\chi^{(6)} = [L^{-1}_4]_T$ in Eq. (54) where nonvanishingly small value for $\chi^{(6)} = [L^{-1}_4]_T \neq 0$ requires additional sensitivity of the supercurrent through the lead $S_1$ on both the superconducting phase variables $\phi_2$ and $\phi_4$.

(ii) The $p = 6$-terminal dc-Josephson current $i_2$ shown on figure 7e is sensitive to all phase differences between $S_n$ and $S_m$ with $n, m = 1, \ldots, 6$, thus it contributes for a finite value to
\( \chi^{(N)} = [L_{N-2}^{-1}]^t \) in Eq. (54), see below.

(iii) The \( p = 5\)-terminal dc-Josephson current \( i_3(\varphi_2 - \varphi_1, \varphi_4 - \varphi_1, \varphi_5 - \varphi_1) \) shown on figure 7d depends on \( N - 2 = 4 \) phase differences. However, this dc-Josephson process produces \( \chi^{(6)} = [L_4^{-1}]^t = 0 \) because \( i_3 \) on figure 7d is independent on \( \varphi_3 \).

(iv) The \( p = 5\)-terminal dc-Josephson current \( i_4(\varphi_2 - \varphi_1, \varphi_3 - \varphi_1, \varphi_4 - \varphi_1, \varphi_5 - \varphi_1) \) shown on figure 7d is sensitive on all the phase variables \( \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\} \). Then, \( i_4 \equiv i_4(\varphi_2 - \varphi_1, \varphi_3 - \varphi_1, \varphi_4 - \varphi_1, \varphi_5 - \varphi_1) \) contributes for a finite value to \( \chi^{(6)} = [L_4^{-1}]^t \), see below.

Now, we detail the above items (ii) and (iv) corresponding to figures 7c and 7d.

First, concerning item (ii) and figure 7c, the lowest-order supercurrent \( i_2 \) through the lead \( S_1 \) is given by

\[
i_2(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) = j_2^{(0)} \sin[(\varphi_2 - \varphi_1) + (\varphi_3 - \varphi_1) + (\varphi_4 - \varphi_1) + (\varphi_5 - \varphi_1)] = j_2^{(0)} \sin[-5\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6]
\]

(71)

Thus,

\[
\frac{\partial^4 i_2(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)}{\partial (\varphi_2 - \varphi_6) \partial (\varphi_3 - \varphi_6) \partial (\varphi_4 - \varphi_6) \partial (\varphi_5 - \varphi_6)} = i_2(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)
\]

(74)

and we find:

\[
\frac{\partial^4 i_2(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)}{\partial (\varphi_2 - \varphi_6) \partial (\varphi_3 - \varphi_6) \partial (\varphi_4 - \varphi_6) \partial (\varphi_5 - \varphi_6)} \bigg|_{\varphi_2 - \varphi_6 = 0, \varphi_3 - \varphi_6 = 0, \varphi_4 - \varphi_6 = 0, \varphi_5 - \varphi_6 = 0} = -j_2^{(0)} \sin[5(\varphi_1 - \varphi_6)].
\]

(75)

Second, we consider item (iv) and figure 7e:

\[
i_4(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) = j_4^{(0)} \sin[(\varphi_2 - \varphi_1) + (\varphi_4 - \varphi_1) + (\varphi_5 - \varphi_1)]
\]

(76)

\[
= j_4^{(0)} \sin[-4\varphi_1 + \varphi_2 + \varphi_4 + \varphi_5 + \varphi_6]
\]

(77)

\[
= j_4^{(0)} \sin[-4(\varphi_1 - \varphi_6) + (\varphi_2 - \varphi_6) + (\varphi_4 - \varphi_6) + (\varphi_5 - \varphi_6)],
\]

(78)

which implies that the following is vanishingly small:

\[
\frac{\partial^4 i_4(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6)}{\partial (\varphi_2 - \varphi_6) \partial (\varphi_3 - \varphi_6) \partial (\varphi_4 - \varphi_6) \partial (\varphi_5 - \varphi_6)} = 0.
\]

(79)

The above Eqs. (74) and (79) confirm the general theory presented in the paper.
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