SPECTRAL ASYMPTOTICS FOR EIGENVALUES AND RESONANCES IN THE PRESENCE OF A CHANGE OF BOUNDARY CONDITIONS

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Abstract. We consider a second-order elliptic differential operator on a domain with a cylindrical end. We impose Dirichlet boundary conditions on the boundary with the exception of a small set, where we impose Neumann boundary conditions. Shrinking this set to a point we calculate the asymptotic behaviour of the resonances.

1. Introduction

Let $\Omega$ be a domain with a cylindrical end and $(\Sigma_\ell)_\ell$ a family of subsets of the boundary, which shrink smoothly to a point for $\ell \to 0$. We consider a self-adjoint elliptic operator $A(x, \nabla_x)$ with Dirichlet boundary conditions $\partial \Omega \setminus \Sigma_\ell$ and Neumann boundary conditions on $\Sigma_\ell$. We calculate the asymptotic behaviour of eigenvalues and resonances as $\ell \to 0$. The problem goes back to the study of two quantum waveguides which are laterally coupled through a small window, cf. [15]. Assuming that both waveguides have the same width one considers instead the Dirichlet Laplacian on one of the waveguides having a small Neumann window. In this case there exists a discrete eigenvalue below the essential spectrum, which becomes unique and tends to the threshold of the essential spectrum as the window size decreases. The convergence is of order $\ell^4$, where $2\ell$ denotes the window size. The first term of the asymptotic formula was given in [33] by an asymptotic matching for the eigenfunctions. Since then the subject has been extended in many directions, e.g. higher dimensional cylinders [34, 19], a finite or an infinite number of windows [34, 35, 8, 9, 10, 31], resonances [16, 17, 6, 7] or problems in elasticity theory [23] to give only a small selection. For further references for problems concerning quantum waveguides we refer e.g. to [14]. A similar question in the case of bounded domains is treated in [18], cf. also the monograph [1] and the references therein for related problems. Since we want to consider resonances we note that there is also a vast amount of literature considering scattering theory in waveguides. In this context we also refer to [13] and the references therein as well as
where an asymptotic formula for resonances was shown. However, they are mainly concerned with the Laplacian, for which the structure of the essential spectrum is well known. Closely related to the structure of the essential spectrum is the notion of ingoing and outgoing waves, which depends on the dispersion curves of the operator. In the case of the Laplacian the horizontal and transversal direction decouple and the dispersion curves may easily be calculated. For more general problems less seems to be known, we refer e.g. to the monograph [32] for the notion of ingoing and outgoing problems for general boundary value problems. We note that the structure of these curves play also an important role in numerics, cf. [5] for a study of elastic waveguides.

The aim of the article is to generalise previous results for the Laplacian in waveguides to general second-order elliptic operators and to resonances. To this end we want to study the analytic continuation of the resolvent in detail and we prove a limiting absorption principle. Then the proof of the asymptotic formula for the resonances is based on a treatment of boundary integral operators as in [11], cf. also [22, 23] for the application of the method to waveguides. More precisely we use the Dirichlet-to-Neumann operator of the problem and apply an operator-valued version of Rouché’s theorem. Finally, we want to mention that only the principal symbol of the operator and the shape of the boundary and of the window will have an influence on the first term of the asymptotic formula.

2. Statement of the problem and results

Let \( d \geq 2 \) and consider \( \Omega \subseteq \mathbb{R}^d \) such that \( \Omega \cap (\mathbb{R} \leq 0 \times \mathbb{R}^{d-1}) \) is bounded and such that
\[
\Omega \cap (\mathbb{R} \geq 0 \times \mathbb{R}^{d-1}) = \mathbb{R} \geq 0 \times G,
\]
where \( G \subseteq \mathbb{R}^{d-1} \) denotes the cross-section of the cylindrical end. We assume that \( \Omega \) has locally Lipschitz boundary and \( \partial G \) is of regularity \( C^{1,1} \). Later additional regularity conditions near the window will arise. We denote the coordinates by \( x = (y, z) \in \Omega \subseteq \mathbb{R} \times \mathbb{R}^{d-1} \) and consider a scalar second-order differential operator
\[
A(x, \nabla_x)u := -\text{div}(a(x)\nabla_x u) + a(x)u, \tag{2.1}
\]
where \( a = a^* : \overline{\Omega} \to \mathbb{R}^{d \times d} \) and \( a : \overline{\Omega} \to [0, \infty) \) are Lipschitz continuous. We note that most of the assertions will also hold true for matrix-valued differential operators. We assume that \( A(x, \nabla_x) \) is uniformly strongly elliptic, i.e., there exists a constant \( c > 0 \) such that
\[
(a(x)\eta, \eta)_{\mathbb{C}^d} \geq c\|\eta\|^2_{\mathbb{C}^d}, \quad x \in \overline{\Omega}, \eta \in \mathbb{C}^d.
\]
We suppose that on the cylindrical end the coefficients depend only in the transversal variable, i.e., for \( y \geq R \) we have

\[
a(y, z) = a^0(z), \quad a(y, z) = a^0(z)
\]

for some \( C^{1,1} \)-functions \( a^0 : \mathbb{C} \rightarrow \mathbb{R}^{d \times d} \) and \( a^0 : \mathbb{C} \rightarrow [0, \infty) \). In what follows we denote by \( \gamma_0 u = u|_{\partial \Omega} \) the boundary trace of a function \( u \) and by \( \mathbf{n} = (n_1, \ldots, n_d)^T : \partial \Omega \rightarrow \mathbb{R}^d \) the outward unit normal vector at some point of the boundary. For a smooth function \( u \in C^\infty(\Omega) \) we define its conormal derivative by

\[
\gamma_1 u = \mathbf{n} \cdot \gamma_0 (a \nabla u)
\]

Let \( \Sigma \subseteq \partial \Omega \) be a bounded, open subset of \( \partial \Omega \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}^{d-1}) \) and assume that \( \partial \Omega \) and the coefficients of \( A(x, \nabla_x) \) are smooth near \( \Sigma \). We suppose that \( \Sigma \) itself has Lipschitz boundary and consider the operator \( A(x, \nabla_x) \) in \( \Omega \) with Dirichlet boundary conditions on \( \partial \Omega \setminus \Sigma \) and Neumann boundary conditions on \( \Sigma \). It is defined by its sesquilinear form

\[
a[u, v] := \int_\Omega \langle a(x) \nabla u(x), \nabla u(x) \rangle \, \text{d}x + \int_\Omega a(x) u(x) \overline{v(x)} \, \text{d}x,
\]

which has the domain

\[
D[a] := \{ u \in H^1(\Omega) : \gamma_0 u = 0 \text{ on } \partial \Omega \setminus \Sigma \}.
\]

We denote the associated self-adjoint operator by \( A_\Sigma \) and by \( A_\varnothing \) the operator with Dirichlet boundary conditions on all of \( \partial \Omega \) corresponding to \( \Sigma = \varnothing \). In what follows we assume that \( \Sigma \) is contained in the domain of a smooth chart \( (U, \phi) \) of \( \partial \Omega \) and that \( \phi(U) \) is star-shaped with centre 0. We define \( \Sigma_\ell \) through

\[
\phi(\Sigma_\ell) := \ell \cdot \phi(\Sigma),
\]

which means that we shrink the window to the point \( s_0 := \kappa^{-1}(0) \in \partial \Omega \). We want to investigate the behaviour of resonances and eigenvalues as \( \ell \) goes to zero. We call \( \lambda \in \mathbb{C} \) a resonance of \( A_\Sigma \) if there exists an outgoing solution \( u \) of the boundary value problem

\[
(A(x, \nabla_x) - \lambda^2)u = 0 \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \partial \Omega \setminus \Sigma, \quad \gamma_1 u = 0 \text{ on } \Sigma.
\]

Note that an outgoing solution belongs to some exponentially weighted \( L_2 \)-space and satisfy a given asymptotic behaviour on the cylindrical end, cf. (3.9) for the precise definition.
1st Result (the non-threshold case): Here and subsequently we denote by $B(\lambda_0, \varepsilon)$ the ball in the complex plane with centre $\lambda_0$ and radius $\varepsilon$. Moreover, we assume that $\lambda_0$ is a resonance of $A_\varnothing$ which is contained in a sufficiently small neighbourhood of the real axis. We assume that $\lambda_0$ is simple and that $\lambda_0$ is not a threshold of the essential spectrum. Let $u_0$ be a resonance solution of $A_\varnothing$ corresponding to $\lambda_0$, which shall be chosen as in Theorem 5.7.

**Theorem 2.1.** There exist $\ell_0 > 0$ and $\varepsilon > 0$ such that for all $\ell \in (0, \ell_0)$ the operator $A_{\Sigma \ell}$ has exactly one resonance $\lambda(\ell) \in B(\lambda_0, \varepsilon)$, which satisfies the asymptotic estimate

$$
\lambda(\ell) = \lambda_0 - \nu \cdot \gamma_1 u_0(s_0)^2 \cdot \ell^d + O(\ell^{d+1}).
$$

Here $\nu > 0$ is a constant given by (5.13).

2nd Result (the threshold case): Let $\Lambda \in \mathbb{R}$ be a branching point of order 2 for the resolvent of $A_\varnothing$ and assume that $\Lambda$ is as simple resonance of $A_\varnothing$ but does not admit any square integrable solutions. Let $u_0$ be a resonance solution of $A_\varnothing$ chosen as in the remark after Theorem 5.9.

**Theorem 2.2.** There exists $\ell_0 > 0$ and $\varepsilon > 0$ such that for all $\ell \in (0, \ell_0)$ the operator $A_{\Sigma \ell}$ has exactly one resonance $\lambda(\ell) \in B(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, which satisfies the asymptotic estimate

$$
\lambda(\ell) = \Lambda - \nu^2 \cdot \gamma_1 u_0(s_0)^4 \cdot \ell^{2d} + O(\ell^{2d+1}).
$$

The constant $\nu > 0$ is chosen as in the previous theorem.

We start by considering the meromorphic continuation of the resolvent in order to define the notion of resonances.

3. The meromorphic continuation of the resolvent

3.1. The limiting absorption principle. As a first step we consider the operator

$$
A^0(z, \nabla_{(y,z)}) := -\text{div}(a^0(z)\nabla_{(y,z)}u) + a^0(z)u, \quad y, z \in \mathbb{R} \times G,
$$

acting on functions in $C^\infty(\mathbb{R} \times G)$. Here $a^0$ and $a^0$ are chosen as above. Let $A^0$ be its self-adjoint realisation in $L^2(\mathbb{R} \times G)$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial G$. The regularity assumptions on $G$ and on the coefficients $a^0$, $a^0$ imply that $D(A^0) := H^2(\mathbb{R} \times G) \cap H^1_0(\mathbb{R} \times G)$. Next we consider the family of parameter-dependent operators $(A^0(\xi))_{\xi \in \mathbb{C}}$, which act as $A^0(z, i\xi, \nabla_z)$ on $D(A^0(\xi)) = H^2(G) \cap H^1_0(G)$. As $D(A^0(\xi))$ is compactly embedded into $L^2(G)$ the spectrum of $A^0(\xi)$ consists of a discrete set of eigenvalues accumulating only at infinity. Moreover, the operators $A^0(\xi)$
form a family of type (B) in the sense of Kato, cf. [27, Chapter VII.§4]. Thus, the eigenvalues depend analytically on $\xi \in \mathbb{C}$ with the possible exception of algebraic branching points.

**Lemma 3.1.** We have $\omega \in \sigma_{\text{ess}}(A_\Sigma)$ if and only if $\omega \in \sigma(A^0(\xi))$ for some $\xi \in \mathbb{R}$.

The proof is based on the first observation that

$$\sigma(A^0) = \sigma_{\text{ess}}(A^0) = \{\omega \in \mathbb{R} : \exists \xi \in \mathbb{R} \text{ with } \omega \in \sigma(A^0(\xi))\}.$$ 

This follows from the fact that $A^0$ is unitary equivalent to the direct integral operator

$$\int_{\mathbb{R}} A^0(\xi) \, d\xi.$$ 

Moreover, we have $\sigma_{\text{ess}}(A_\Sigma) = \sigma_{\text{ess}}(A^0)$. This well-known assertion goes back to Birman [4], where he considered a perturbed exterior domain and proved a weak Schatten estimate for the difference of the corresponding resolvents. In our case it is sufficient to apply the Weyl criterion and the assertion follows. Note that the assertion Lemma 3.1 remains true if we use lower regularity assumptions on $\partial G$. In this case one may use a weak notion of Weyl sequences, cf. e.g. [28].

**Remark.** Note that we have $\sigma(A^0(\xi)) = \sigma(A^0(-\xi))$ for all $\xi \in \mathbb{R}$. Indeed, if $\psi$ is an eigenfunction of $A^0(\xi)$ then $\overline{\psi}$ is an eigenfunction of $A^0(-\xi)$ for the same eigenvalue.

Now we consider the thresholds of the essential spectrum of $A^0$. To this end we look at the parameter-dependent family $A^0(\xi)$ and use the following lemma, whose proof is given [27, Theorem VIII.3.9].

**Lemma 3.2.** There exist real-analytic functions $\mu_k : \mathbb{R} \to \mathbb{R}$, $P_k : \mathbb{R} \to \mathcal{L}(L^2(G))$, $k \in \mathbb{N}$, where the $P_k(\xi)$’s are mutually orthogonal projections, such that

$$A^0(\xi) = \sum_{k=1}^{\infty} \mu_k(\xi) P_k(\xi), \quad \xi \in \mathbb{R}. $$

Note that each $\mu_k$ may be continued to an analytic function defined in some neighbourhood of the real axis. However, a common domain of analyticity for all $k \in \mathbb{N}$ does not necessarily exist, cf. e.g. the remarks in [27] VIII.3. In what follows we assume that

$$\inf_{k \in \mathbb{N}} \inf_{\xi \in \mathbb{R}} \mu_k(\xi) > 0. \quad (3.1)$$

**Definition.** A value $\omega > 0$ is called spectral threshold of $A^0$ if there exists $\xi \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $\mu_k(\xi) = \omega^2$ and $\mu_k'(\xi) = 0$. 


Theorem 3.4. Let $s$ and $\chi$ as a holomorphic mapping from $H$ spaces. Let $\chi$ to extend it to the lower half-plane one introduces exponentially weighted $\Sigma$ exist neighbourhood $r > 0$ for sufficiently large $\xi$ eigenvalues we observe that for each $A$ prove the assertion. The ellipticity of $H$ we write $\Sigma$ exist such that the function $H$ we write $\Sigma$. Using the local perturbation theory of the $H$ we put $\Sigma$. There exists an open neighbourhood $\Sigma$ such that there exists infinitely many thresholds, which we order increasingly $0 < \Lambda_1 < \Lambda_2 < \ldots$

Lemma 3.3. The thresholds form a discrete set accumulating only at infinity.

Proof. We show that for each $\omega_0 \in \mathbb{R}$ the set 
\[ \{ k \in \mathbb{N} : \mu_k(\mathbb{R}) \cap [-\omega_0, \omega_0] \neq \emptyset \} \]
has only finitely many elements. Since the $\mu_k$ are real-analytic this will prove the assertion. The ellipticity of $A(x, \nabla x)$ implies that there are constants $c_0, c_1 > 0$ such that
\[ \langle A^0(\xi)u, u \rangle \geq c_0(\xi^2\|u\|^2_{L^2(G)}) + \|\nabla u\|^2_{L^2(G)} - c_1\|u\|^2_{L^2(G)}, \quad u \in D(A^0(\xi)). \]
From the min-max principle for self-adjoint operators we get $\inf_k \mu_k(\xi) \to \infty$ as $|\xi| \to \infty$. In particular we only need to consider the set
\[ \{ k \in \mathbb{N} : \mu_k([-r, r]) \cap [-\omega_0, \omega_0] \neq \emptyset \} \]
for sufficiently large $r > 0$. Using the local perturbation theory of the eigenvalues we observe that for each $\xi \in [-r, r]$ and $\omega \in [-\omega_0, \omega_0]$ there exist neighbourhoods $U_\xi$ and $V_\omega$ such that $\{ k \in \mathbb{N} : \mu_k(U_\xi) \in V_\omega \}$ is finite. Since $[-r, r] \times [-\omega_0, \omega_0]$ is compact the assertion follows. \[ \square \]

We write $H_+ := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ and consider
\[ \omega \mapsto R^0(\omega) = (A^0 - \omega^2)^{-1} \]
as a holomorphic mapping from $H_+$ with values in $L(L_2(\mathbb{R} \times G); H^2(\mathbb{R} \times G))$. To extend it to the lower half-plane one introduces exponentially weighted spaces. Let $\chi \in C^\infty_c(\mathbb{R})$ be chosen such that $0 \leq \chi \leq 1$, $\chi = 0$ on $(-1, 1)$ and $\chi = 1$ on $(-\infty, -2) \cup (2, \infty)$. For $\beta \in \mathbb{R}$ and $s \in \mathbb{R}$ we put
\[ H^s_\beta(\mathbb{R} \times G) = \left\{ u \in H^s_{\text{loc}}(\mathbb{R} \times G) : \chi(y)e^{-|y|}u(y, z) \in H^s(\mathbb{R} \times G) \right\}. \]
For $s = 0$ we write $L_{2, \beta}(\mathbb{R} \times G) := H^0_\beta(\mathbb{R} \times G)$.

Theorem 3.4. Let $\beta > 0$. There exists an open neighbourhood $U$ of $H_+$ such that the function
\[ \omega \mapsto R^0(\omega) \in L(L_{2, -\beta}(\mathbb{R} \times G); H^2_{\beta}(\mathbb{R} \times G)) \]
may be continued to a multiple-valued holomorphic function on $U \setminus \{ \pm \Lambda_j \}$. The points $\pm \Lambda_j \in \mathbb{R}$ are branching points of finite order.

Proof. Let $\beta > 0$ and $f_1, f_2 \in L_{2,-\beta}(\mathbb{R} \times G)$. We denote by $\tilde{f}_1, \tilde{f}_2 \in L_2(\mathbb{R}; L_2(G))$ their Fourier transforms with respect to the horizontal variable. Due to the exponential weight we may continue the $\tilde{f}_i$ analytically to the strip $\{ \xi \in \mathbb{C} : |\text{Im}(\xi)| < \beta \}$ for $\omega \in \mathbb{H}_+$ we have

$$
\langle R^0(\omega)f_1, f_2 \rangle = \int_{\mathbb{R}} \langle (A^0(\xi) - \omega^2)^{-1}\tilde{f}_1(\xi), \tilde{f}_2(\xi) \rangle \, d\xi.
$$

For $\omega \in \mathbb{C}$ we define

$$
\Xi(\omega) := \{ \xi \in \mathbb{C} : \omega^2 \in \sigma(A^0(\xi)) \}
$$

and $\Xi_\beta(\omega) := \Xi(\omega) \cap \{ \xi \in \mathbb{C} : |\text{Im}(\xi)| \leq \beta \}$. Note that $\Xi(\omega)$ is discrete and $\Xi_\beta(\omega)$ is finite, cf. e.g. [21]. Let $\omega_0 \in \mathbb{R}$ be fixed and $\varepsilon > 0$ sufficiently small. Then there exist $r > 0$ such that for all $\omega \in B(\omega_0, \varepsilon)$ and $|\text{Re}(\xi)| \geq r$ and $|\text{Im}(\xi)| \leq \beta$ the operator $(A^0(\xi) - \omega^2)^{-1}$ exists and satisfies

$$
\xi^2\| (A^0(\xi) - \omega^2)^{-1}f \|_{L_2(G)}^2 + \| (A^0(\xi) - \omega^2)^{-1}f \|_{H^2(G)}^2 \leq c\|f\|_{L_2(G)}^2. \tag{3.2}
$$

The constant $c > 0$ does not depend on $f$, $\omega$ and $\xi$. For proofs and further references we refer to [29] [32]. Choosing $\varepsilon$ and $\beta$ sufficiently small we may assume that

$$
\Xi_\beta(\omega_0) \subseteq \mathbb{R} \quad \text{and} \quad \Xi(\omega) \cap \{ \xi \in \mathbb{C} : |\text{Im}(\xi)| = \beta \} = \emptyset \quad \tag{3.3}
$$

for all $\omega \in B(\omega_0, \varepsilon)$. Using the estimate (3.2) we may shift the path of integration and obtain for $\omega \in \mathbb{H}_+ \cap B(\omega_0, \varepsilon)$ that

$$
\langle R^0(\omega)f_1, f_2 \rangle = \int_{\mathbb{R}+i\beta} \langle (A^0(\xi) - \omega^2)^{-1}\tilde{f}_1(\xi), \tilde{f}_2(\xi) \rangle \, d\xi
$$

$$
+ 2\pi i \sum_{\xi \in \Xi_\beta(\omega) \setminus \{ \xi \in \mathbb{C} : |\text{Im}(\xi)| = 0 \}} \text{Res}_{\xi}(\langle (A^0(\xi) - \omega^2)^{-1}\tilde{f}_1(\xi), \tilde{f}_2(\xi) \rangle). \tag{3.4}
$$

The first term is well-defined for all $\omega \in B(\omega_0, \varepsilon)$ and gives rise to a bounded linear operator from $L_{2,-\beta}(\mathbb{R} \times G)$ to $H_2^2(\mathbb{R} \times G)$. It remains to consider the residual term and the behaviour of $\Xi_\beta(\omega)$ as $\omega$ crosses the real axis. As

$$(A^0(\xi) - \omega^2)^{-1} = \sum_{k=1}^{\infty} (\mu_k(\xi) - \omega^2)^{-1}P_k(\xi), \quad \xi \in \mathbb{R},$$

we easily obtain for $\omega \in B(\omega_0, \varepsilon)$ that

$$
\Xi_\beta(\omega) = \{ \xi \in \mathbb{C} : |\text{Im}(\xi)| \leq \beta, \mu_k(\xi) = \omega^2 \},
$$

if $\beta$ and $\varepsilon$ are chosen sufficiently small. Now we consider all pairs $(\xi_\alpha, k_\alpha) \in \mathbb{R} \times \mathbb{N}$, $\alpha = 1, \ldots, m$ such that $\mu_{k_\alpha}(\xi_\alpha) = \omega_0^2$. For each $\alpha = 1, \ldots, m$ there
exists a neighbourhood $U_{\xi}$ of $\xi_{\alpha}$ and an invertible holomorphic function $G_\alpha : U_{\xi} \to G_\alpha(U_{\xi})$ with $G_\alpha(\xi) = 0$ and

$$\mu_{k_\alpha}(\xi) - \omega_0^2 = G_\alpha(\xi)^{n_\alpha}, \quad \xi \in U_{\xi}.$$  

Here $n_\alpha$ is the multiplicity of the root of $\mu_{k_\alpha}(\cdot) - \omega_0^2$ at $\xi_{\alpha}$. For $|\omega - \omega_0| < \varepsilon$ the equation $\mu_{k_\alpha}(\xi) = \omega^2$ has exactly $n_\alpha$ solutions near $\xi_{\alpha}$ which are given by

$$\xi_{\alpha,\delta}(\omega) = G_\alpha^{-1}\left(\frac{2\pi i n_\alpha}{n_\alpha} \left(\omega^2 - \omega_0^2\right)^{1/n_\alpha}\right), \quad \delta = 0, \ldots, n_\alpha - 1,$$

and we obtain $\Xi_\beta(\omega) = \{\xi_{\alpha,\delta}(\omega) : \alpha = 1, \ldots N, \delta = 1, \ldots n_\alpha\}$. Moreover,

$$\text{Res}_{\xi_{\alpha,\delta}(\omega)} \frac{(P_{k_\alpha}(\xi)\hat{f}_1(\xi), \hat{f}_2(\xi))}{\mu_{k_\alpha}(\xi) - \omega^2} = \frac{(P_{k_\alpha}(\xi_{\alpha,\delta}(\omega))\hat{f}_1(\xi_{\alpha,\delta}(\omega)), \hat{f}_2(\xi_{\alpha,\delta}(\omega)))}{\mu_{k_\alpha}'(\xi_{\alpha,\delta}(\omega))}.$$ (3.6)

If $\omega_0$ is not a threshold then we have $n_\alpha = 1$ for all $\alpha = 1, \ldots, N$. In particular, the functions $\xi_{\alpha,0}(\omega)$ depend analytically on $\omega$ and so does the expression (3.6). If $n_\alpha > 1$ for some $\alpha$ then we have a Puiseux expansion for $\xi_{\alpha,\delta}(\omega)$ and we obtain also a meromorphic Puiseux expansion for the residual terms. This proves the assertion. \hfill \Box

In what follows we denote by $\Psi_\beta$ the maximal Riemannian manifold such that the operator $R^0(\cdot) : L^2(\mathbb{R} \times G) \to H^2_\beta(\mathbb{R} \times G)$ is well-defined and analytic.

**Example.** We want to outline the procedure. We assume that the eigenvalue curves have the following form:

\[ \sigma(A(\xi)) \]

\[ \omega_0^2 \]

\[ \Lambda_1 \]

\[ \Lambda_2 \]

\[ \Lambda_3 \]

\[ \xi \]

Note that such dispersion curves appear e.g. in linear elasticity. Let $\Lambda_1, \Lambda_2$ and $\Lambda_3$ be the first spectral thresholds and let $\mu_1(\cdot), \mu_2(\cdot)$ denote the first eigenvalue curves with projections $P_1(\cdot), P_2(\cdot)$. Assume that the branching points are of order 2. We choose $\omega_0$ such that $\Lambda_2 < \omega_0 < \Lambda_3$. If $\beta > 0$ and $\varepsilon > 0$ are sufficiently small we have

$$\Xi_\beta(\omega) = \{\xi_1(\omega), \ldots, \xi_6(\omega)\},$$
for all $B(\omega_0, \varepsilon)$. We assume that for $\omega = \omega_0$ we have $\xi_i(\omega_0) < \xi_{i+1}(\omega_0)$. For $\omega \in \mathbb{H}_+ \cap B(\omega_0, \varepsilon)$ we obtain

$$\text{Im}(\xi_3(\omega)) > 0, \quad \text{Im}(\xi_5(\omega)) > 0, \quad \text{Im}(\xi_6(\omega)) > 0$$

as may be easily seen by evaluating the sign of the derivatives of the $\mu_i$. If $\omega$ crosses the real axis we will obtain the following behaviour for the functions $\xi_i(\cdot), i \in \{4, 5, 6\}$:

\begin{align*}
\Lambda_1 & \rightarrow \omega_0 \rightarrow \Lambda_2, \\
\Lambda_3 & \rightarrow \omega_0 \rightarrow \Lambda_2.
\end{align*}

For the analytic continuation we obtain that

$$R^0(\omega) - R^0(-\omega) = 2\pi i \left( \sum_{j \in \{3, 5\}} M_2(\xi_j(\omega)) + M_1(\xi_6(\omega)) \right) - 2\pi i \left( \sum_{j \in \{2, 4\}} M_2(\xi_j(\omega_1)) - M_1(\xi_1(\omega_1)) \right)$$

where $\langle M_i(\xi)f_1, f_2 \rangle = \mu_i(\xi)^{-1}(P_i(\xi)f_1(\xi), f_2(\xi))$.

Now we consider the operator $A_\Sigma$ acting in $L_2(\Omega)$. For $\beta > 0$ we define the exponentially weighted spaces

$$H^{s}_\beta(\Omega) := \{ u \in H^{s}_{\text{loc}}(\Omega) : \chi(y)e^{\beta y}u(y, z) \in H^{s}(\Omega) \},$$

where $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi(y) = 0, y \leq R + 1$ and $\chi(y) = 1$ for $y > R + 2$. Let $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots$ denote as before the thresholds of $A_\Sigma$.

**Theorem 3.5.** Let $R_\Sigma(\omega) := (A_\Sigma - \omega^2)^{-1}$. Then the mapping

$$\omega \mapsto R_\Sigma(\omega) \in \mathcal{L}(L_{2, -\beta}(\Omega), H^1_\beta(\Omega))$$

may be continued to a multiple-valued meromorphic function on the Riemann surface $\Psi_\beta$. The points $\pm \Lambda_j \in \mathbb{R}$ are branching points of finite order.

**Proof.** The proof is well known for obstacle scattering in $\mathbb{R}^d$ and applies also in the present case. We want to sketch only its basic steps. Let
\( \chi_0, \chi_1, \chi_2 \in C^\infty(\mathbb{R}) \) be chosen as above. Additionally we assume that we have \( \chi_1 \chi_2 = \chi_1, \chi_0 \chi_1 = \chi_1 \). We consider the operators

\[
Q_1(\omega) := \chi_0 R^0(\omega) \chi_1, \quad Q_2 := (1 - \chi_2) R_\Sigma(\eta_0)(1 - \chi_1),
\]

where \( \eta_0 \in \mathbb{H}_+ \) is fixed. Theorem 3.2 implies that \( Q_1(\cdot) \) may be continued to a multiple-valued holomorphic functions on \( \Psi_\beta \). A short calculation shows that \( Q_1(\omega) \) and \( Q_2 \) maps into \( D(A_\Sigma) \) and that \( (A(x, \nabla_x) - \omega^2)(Q_1(\omega) + Q_2) = I + M_1(\omega) + M_2(\omega) \), where

\[
M_1(\omega) = [A(x, \nabla_x), (1 - \chi_0)] R^0(\omega) \chi_1
\]

\[
M_2(\omega) = ([A(x, \nabla_x), (1 - \chi_2)] + (\eta_0^2 - \omega^2)(1 - \chi_2)) R_\Sigma(\eta_0)(1 - \chi_1).
\]

Note that \( M_1(\omega) : L_{2,-\beta}(\Omega) \to L_{2,\beta}(\Omega), i = 1, 2 \) are compact, which follows from elliptic regularity. Finally, we have to show that \( I + M_1(\omega) + M_2(\omega) \) is invertible for at least some \( \omega \in \mathbb{H}_+ \). Using the spectral theorem for the self-adjoint operator \( A^0 \) we have

\[
\|M_1(\omega)\|_{L_{2,-\beta} \to L_{2,\beta}} \leq C \|R^0(\omega)\|_{L_1 \to H^1} < 1/2
\]

for sufficiently large imaginary part of \( \omega \). For a suitable choice of \( \omega, \eta_0 \) we obtain likewise \( \|M_2(\omega)\|_{L_{2,-\beta} \to L_{2,\beta}} < 1/2 \), and thus, \( I + M_1(\omega) + M_2(\omega) \) is invertible. Now the meromorphic Fredholm theorem implies that the mapping is invertible for all \( \omega \in \Psi_\beta \) except for a discrete set. \( \square \)

3.2. Incoming and outgoing representations. We want to describe the meromorphic continuation of the resolvent also in terms of outgoing solutions. The distinction between ingoing and outgoing solution will be based on the limiting absorption principle. We refer also to \cite{[32], Chapter 5} for a different approach. For \( \omega \in \mathbb{C} \) and \( \xi \in \mathbb{C} \) we define

\[
\mathfrak{A}_\omega(\xi) : H^2(G) \to \begin{array}{c} L_2(G) \\ \oplus \\ H^{3/2}(\partial G) \end{array}, \quad \mathfrak{A}_\omega(\xi) = \begin{pmatrix} A^0(z, i\xi, \nabla_z) - \omega^2 \\ \gamma_0 \end{pmatrix},
\]

where \( \gamma_0 \) denotes as before the trace operator. Then \( \mathfrak{A}_\omega(\cdot) \) is a finitely meromorphic function of Fredholm type, cf. Section 5.1. The points \( \xi_0 \) with \( \ker \mathfrak{A}_\omega(\xi_0) \neq \{0\} \) are called characteristic values. Note that the set \( \Xi(\omega) \), which was introduced in the proof of Theorem 5.1, is the set of characteristic values of \( \mathfrak{A}_\omega \). Let \( \xi_0 \in \Xi(\omega) \). A family of elements \( u_0, \ldots, u_k \in H^2(G) \) is called a Jordan chain of length \( k + 1 \) for \( \mathfrak{A}_\omega \), if and only if

\[
0 = \sum_{q=0}^{j} \frac{1}{q!} \mathfrak{A}^{(q)}(\xi_0) u_{j-q}, \quad j = 0, \ldots, k.
\]
For $u_0 \in \ker \mathfrak{A}_\omega(\xi)_0$ we denote by $\text{rank } u_0$ the maximal length of a Jordan chain associated with $u_0$. A basis $v_1, \ldots, v_m$ of $\ker \mathfrak{A}_\omega(\xi)_0$ is called canonical if $\text{rank}(v_i) \geq \text{rank}(v)$ for all $v \in \text{lin}(v_{i+1}, \ldots, v_m)$. The number

$$N(\xi)_0 := \sum_{i=1}^m \text{rank } v_i$$

is called the total multiplicity of the characteristic value and does not depend on the choice of the canonical basis. Note that these notions coincide with the definitions given in Section 5.1. For $\xi_0 \in \Xi(\omega)$ and a Jordan-chain $u_0, \ldots, u_k \in H^2(G)$ we define the functions

$$U_j: \mathbb{R} \times G \to \mathbb{C},$$

$$U_j(y, z) = e^{i\xi_0 y} \sum_{q=0}^{j} \frac{(it)^q}{q!} u_{j-q}(z), \quad j = 0, \ldots, k. \quad (3.7)$$

They satisfy $(A^0(z, \nabla_{(y,z)}) - \omega^2) U_j = 0$ in $\mathbb{R} \times G$ and $\gamma_0 U_j = 0$ on $\mathbb{R} \times \partial G$. For $\beta > 0$ and $f \in L^2(\Omega)$ we want to consider solutions $u \in H^1_{\gamma}(\Omega)$ of

$$(A(x, \nabla x) - \omega^2) u = f \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \partial \Omega \setminus \Sigma, \quad \gamma_1 u = 0 \text{ on } \Sigma. \quad (3.8)$$

Here the normal derivative is defined in the weak sense, we refer also to the next section. Assume that $\beta$ is chosen such that $\Xi(\omega) \cap \{\xi \in \mathbb{C} : |\text{Im}(\xi)| = \beta\} = \emptyset$ and let $n$ be the sum of the total multiplicities of all characteristic values in the strip $\{\xi \in \mathbb{C} : |\text{Im}(\xi)| \leq \beta\}$. Choosing for each characteristic value a canonical basis we obtain functions $U_j(\omega), j = 1, \ldots, n$ as in (3.9).

**Theorem 3.6** (see e.g. [29, 32]). For every solution $u \in H^1_{\gamma}(\Omega)$ of (3.8) there exist $c_j \in \mathbb{C}, j = 1, \ldots, n$, such that

$$u(y, z) - \chi(y) \sum_{j=1}^{n} c_j U_j(\omega, y, z) \in H^1_{\gamma}(\Omega). \quad (3.9)$$

Here $\chi \in C^\infty(\mathbb{R})$ is chosen such that $\chi = 0$ on $(-\infty, R+1)$ and $\chi = 1$ on $(R+2, \infty)$.

The next step is to define the notion of an outgoing solution such that $R_{\Sigma}(\omega)f$ will give the unique outgoing solution of the boundary value problem (3.8). For $\omega \in \mathbb{H}_+$ we denote

$$\Xi_{\beta, \pm}(\omega) := \{\xi \in \Xi_\beta(\omega) : \pm \text{Im}(\xi) > 0\}$$

and denote by $n_{\pm}$ the corresponding total multiplicities. We denote by $U_{j, \pm}(\omega), j = 1, \ldots, n_{\pm}$ the functions in (3.9) corresponding to characteristic values in $\Xi_{\beta, \pm}(\omega)$. Then $U_{j, \pm}(\omega)(y, z)$ is exponentially decreasing as $y \to \infty$. 

Definition. Let $\omega \in \mathbb{H}_+$. A solution $u \in H^1_\beta(\Omega)$ of (3.8) is outgoing if and only if there exist $c_j \in \mathbb{C}$, $j = 1, \ldots, n_+$, such that

$$u(y, z) - \chi(y) \sum_{j=1}^{n_+} c_j U_{j,+}(\omega, y, z) \in H^1_{-\beta}(\Omega). \quad (3.10)$$

Now let $\omega_0 \in \mathbb{R}\setminus\{\Lambda_i\}$ and assume that $\beta > 0$ and $\varepsilon > 0$ are sufficiently small. Using the results of the previous section we obtain that there exist functions $\xi_{j,+}(\cdot)$, which are analytic along any path in $B(\omega_0, \varepsilon)\setminus\{\omega_0\}$, such that

$$\Xi_{\beta,+}(\omega) = \{\xi_{j,+}(\omega) : j = 1, \ldots, n_+\} \quad \text{for} \quad \omega \in B(\omega_0, \varepsilon) \cap \mathbb{H}_+.$$ 

Let $\omega_1 \in B(\omega_0, \varepsilon)\setminus\{\omega_0\}$. We define

$$\Xi_{\beta,+}(\omega_1) = \{\xi_{j,+}(\omega_1) : j = 1, \ldots, n_+\},$$

and $U_{j,+}(\omega_1)$ by analytic continuation. We also assume that the $\xi_{j,+}(\cdot)$ and the functions $U_{j,+}(\cdot)$ may be continued analytically to $\Psi_\beta$, i.e., the maximal Riemannian manifold such that $\omega \mapsto R_0(\omega)$ is analytic.

Definition. Let $\omega_1 \in \Psi_\beta$. A solution $u \in H^1_\beta(\Omega)$ of (3.8) is called outgoing if and only if there exist $c_j \in \mathbb{C}$, $j = 1, \ldots, n_+$, such that

$$u(y, z) - \chi(y) \sum_{j=1}^{n_+} c_j U_{j,+}(\omega_1, y, z) \in H^1_{-\beta}(\Omega), \quad (3.11)$$

where $\chi$ is chosen as above.

Theorem 3.7. Let $\beta > 0$, $\varepsilon > 0$ be sufficiently small. Assume that $\omega_1 \in B(\omega_0, \varepsilon)\setminus\{\omega_0\}$ is not a pole of $R_\Sigma(\cdot)$ and let $f \in L_{2,-\beta}(\Omega)$. Then $u := R_\Sigma(\omega_1)f \in H^1_\beta(\Omega)$ is the unique outgoing solution of (3.8).

The proof is based on the ideas in one-dimensional scattering theory, see e.g. [12, Theorem 2.3].

Proof. By analytic continuation we have $(A - \omega_1^2)R_\Sigma(\omega_1)f = f$, and in the same way we obtain $\gamma_0 R_\Sigma(\omega_1)f = 0$ on $\partial \Omega \setminus \Sigma$ as well as $\gamma_1 R_\Sigma(\omega_1)f = 0$ on $\Sigma$. Thus, $u := R_\Sigma f$ is a solution of the boundary problem. It remains to show that the solution is outgoing and unique. Choosing $\chi_0, \chi_1 \in C^\infty(\mathbb{R})$ and $M_i(\omega_1)$, $i = 1, 2$, as in the proof of Theorem 3.5 we obtain that

$$R_\Sigma(\omega_1)f - \chi_1 R^0(\omega_1)\chi_0 \left( I + M_1(\omega_1) + M_2(\omega_1) \right)^{-1}f \in H^1_{-\beta_1}(\Omega),$$

for all $\beta_1 > 0$. Using (3.4) and (3.6) it easily follows that the range of $\chi_1 R^0(\omega)\chi_0$ consists of outgoing functions. Thus, $R_\Sigma(\omega_1)f$ has to be outgoing. To show uniqueness it is sufficient to prove $R_\Sigma(\omega)(A(x, \nabla_x) - \omega^2_1)u = u$
for all outgoing functions \( u \in H^1_\beta(\Omega) \). Let
\[
  u(y, z) = \chi(y) \sum_{j=1}^{n_u} c_j U_{j,+}(\omega_1, y, z) + \tilde{u}(y, z), \quad \text{with } \tilde{u} \in H^1_{-\beta}(\Omega).
\]
We define \( \tilde{f}(\omega) = (A(x, \nabla x) - \omega^2) \tilde{u} \) and \( f_j(\omega) = (A(x, \nabla x) - \omega^2) \chi U_{j,+}(\omega) \). Then \( R_\Sigma(\omega) \tilde{f}(\omega) = \tilde{u} \) and \( R_\Sigma(\omega) f_j(\omega) = \chi U_{j,+}(\omega) \) for all \( \omega \) by analytic continuation. This implies the assertion. \( \square \)

Note that an analogous assertion as in Theorem 3.7 will hold true for all \( \omega \in \Psi_\beta \). Let \( \beta > 0 \). A value \( \lambda_0 \in \Psi_\beta \) is called resonance if \( R_\Sigma(\cdot) \) has a pole in \( \lambda_0 \). The order of the resonance is given by
\[
  \dim \text{ran } \frac{1}{2\pi i} \int_{|\omega - \lambda_0| = \epsilon} R_\Sigma(\omega) \, d\omega.
\]

**Theorem 3.8.** Let \( \lambda_0 \in \Psi_\beta \). Then the following assertions are equivalent:

1. \( \lambda_0 \) is a resonance.
2. There exists a non-trivial outgoing solution of (3.8).

The proof follows as in [12, Theorem 2.4]. A threshold \( \Lambda_1 \in \mathbb{R} \) will be called a resonance of \( A_\Sigma \) if the operator \( \zeta \mapsto R_\Sigma(\Lambda_1 - \zeta^k) \) has a pole at \( \zeta = 0 \), where \( k \) is the order of the branching point.

### 4. The Dirichlet-to-Neumann Operator

In this section we introduce the notion of the Dirichlet-to-Neumann operator. We start by considering for \( \omega \in \mathbb{H}_+ \) the boundary value problem
\[
  (A(x, \nabla x) - \omega^2) u = f \text{ in } \Omega, \quad \gamma_0 u = g \text{ on } \partial \Omega, \quad (4.1)
\]
where \( u \in H^1(\Omega) \), \( f \in L^2(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \).

**Lemma 4.1.** Let \( \omega \in \mathbb{H}_+ \). Then the boundary value problem (4.1) is uniquely solvable and there exists \( c = c(\omega) \) independent of \( f \) and \( g \) such that \( \|u\|_{H^1(\Omega)} \leq c(\omega)(\|g\|_{H^{1/2}(\partial \Omega)} + \|f\|_{L^2(\Omega)}) \).

For the proof we refer e.g. to [29, Theorem 5.5.2], where the assertion was proved for cylindrical domains with smooth boundary. Putting \( f = 0 \) we denote by \( u = K_\omega g \) the unique solution of the Poisson problem
\[
  (A(x, \nabla x) - \omega^2) u = 0 \text{ in } \Omega, \quad \gamma_0 u = g \text{ on } \partial \Omega. \quad (4.2)
\]

**Lemma 4.1** implies that we have \( K_\omega : H^{1/2}(\partial \Omega) \to H^1(\Omega) \). Now let \( u \in H^1(\Omega) \) with \( A(x, \nabla x) u = f \in L^2(\Omega) \). The weak conormal derivative \( \gamma_1 u \in H^{-1/2}(\partial \Omega) \) will be defined by Green’s formula, i.e., we have
\[
  \langle \gamma_1 u, \gamma_0 v \rangle = a[u, v] - \langle f, v \rangle, \quad \text{for all } v \in H^1(\Omega).
\]
Here $a[u,v]$ is defined as in (2.3). Then the Dirichlet-to-Neumann operator is given by

$$D_\omega : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega), \quad D_\omega g := \gamma_1 K_\omega g. \quad (4.3)$$

As a next step we want to show that the operators $K_\omega$ and $D_\omega$ admit meromorphic extension to the lower half-plane. Let $\chi \in C^\infty(\mathbb{R})$, $\chi = 0$ on $(-\infty, R + 1)$ and $\chi = 1$ on $(R + 2, \infty)$. For $\beta \in \mathbb{R}$ and $s \in \mathbb{R}$ we define

$$H^s_\beta(\partial \Omega) = \{ g \in H^s_{loc}(\partial \Omega) : g(y,z)\chi(y)e^{-\beta z} \in H^s(\partial \Omega) \}.$$

Then as above we may define for $u \in H^{1/2}_\beta(\Omega) \cap \cap_\Sigma H^{s}_\beta(\partial \Omega)$ a weak conormal derivative.

**Theorem 4.2.** The mappings

$$\omega \mapsto K_\omega \in \mathcal{L}(H^{1/2}_\beta(\partial \Omega); H^1(\Omega)), \quad \omega \mapsto D_\omega \in \mathcal{L}(H^{1/2}_\beta(\partial \Omega); H^{-1/2}_\beta(\partial \Omega))$$

extend meromorphically to the Riemann surface $\Psi_\beta$. If $\omega$ is not a resonance of $A_{\emptyset}$ then $K_\omega g$ is the unique outgoing solution of the Poisson problem

$$(A(x, \nabla_x) - \omega^2)u = 0 \text{ in } \Omega, \quad \gamma_0 u = g \text{ on } \partial \Omega.$$

The proof follows the same ideas as for the resolvent and will be omitted.

In order to treat the mixed problem we introduce the following function spaces

$$H^s_0(\Sigma) := \{ g \in H^s(\partial \Omega) : \text{supp}(g) \subseteq \Sigma \},$$

$$H^{-s}(\Sigma) := \{ G|_\Sigma \in C^\infty_c(\Sigma)' : G \in H^{-s}(\partial \Omega) \} = \{ G|_\Sigma \in C^\infty_c(\Sigma)' : G \in H^{-s}_\beta(\partial \Omega) \}.$$  

We denote by $r_\Sigma : \cup_\Sigma H^{-s}(\partial \Omega) \to H^{-s}(\Sigma)$ the restriction operator and let $e_\Sigma : H^s_0(\Sigma) \to \cap_\Sigma H^{s}_\beta(\partial \Omega)$ be the corresponding embedding. Since $\Sigma$ has Lipschitz boundary we have a dual pairing between $H^s_0(\Sigma)$ and $H^{-s}(\Sigma)$, which reads as

$$\langle g, h \rangle_\Sigma := \langle G, h \rangle_{\partial \Omega}, \quad g \in H^{-s}(\Sigma), \quad h \in H^s_0(\Sigma), \quad (4.4)$$

where $G \in H^{-s}_\beta(\partial \Omega)$ is an arbitrary extension of $g \in H^{-s}(\Sigma)$, cf. [30, Theorem 3.30] for the case $\beta = 0$. The case $\beta \neq 0$ follows likewise. Finally we define truncated Dirichlet-to-Neumann operator

$$D_{\Sigma,\omega} : H^{1/2}_0(\Sigma) \to H^{-1/2}(\Sigma), \quad D_{\Sigma,\omega} := r_\Sigma D_\omega e_\Sigma,$$

which extends to an meromorphic function on the Riemann surface $\cup_{\beta > 0} \Psi_\beta$.

From Theorem 4.2 and Theorem 4.2 we obtain the following result.

**Theorem 4.3.** Let $\lambda_0 \in \Psi_\beta$ be not a resonance of $A_{\emptyset}$. Then the following assertions are equivalent:
(1) \( \lambda_0 \) is a resonance of \( A_\Sigma \);  
(2) \( \ker(D_{\Sigma, \lambda_0}) \neq \{0\} \).

Finally we will need the following lemma which provides a formula for the Dirichlet-to-Neumann operator if the spectral parameter is perturbed.

**Theorem 4.4.** Let \( \omega, \eta \in \mathbb{H}_+ \). Then the following identities hold true:

1. \( K_\omega^* = \gamma_1 R_\varnothing(-\varnothing) \);
2. \( K_\omega = K_\eta + (\omega^2 - \eta^2)R_\varnothing(\omega)K_\eta \);
3. \( D_\omega = D_{\Sigma, \pi} - (\omega^2 - \eta^2)K_\eta^* (I + (\omega^2 - \eta^2)R_\varnothing(\omega))K_\eta \).

For the proof we refer e.g. to Theorem 2.6 in [3] in the context of boundary triplets. Then we obtain for the truncated Dirichlet-to-Neumann operator that

\[
D_{\Sigma, \omega} = D_{\Sigma, \pi} - (\omega^2 - \eta^2)K_\eta^* (I + (\omega^2 - \eta^2)R_\varnothing(\omega))K_\eta
\]

We want to extend the formula to \( \omega \in \Psi_\beta \). Let \( \eta \in \mathbb{H}_+ \) be fixed and choose \( \beta \) such that \( \Xi_\beta(-\varnothing) = \emptyset \). From Theorem 3.6 we obtain that \( R_\varnothing(-\varnothing) : L^2_{2, \beta}(\Omega) \to H^{1/2}_{\beta}(\Omega) \). Since the boundary and the coefficients of \( A(x, \nabla x) \) are smooth in some neighbourhood of the window local regularity estimates imply that

\[
r_\Sigma K_\eta^* = r_\Sigma \gamma_1 R_\varnothing(-\varnothing) : L^2_{2, \beta}(\Omega) \to H^{1/2}(\Sigma),
\]

continuously. This implies \( K_\eta e_\Sigma = (r_\Sigma K_\eta^*)^* : H^{-1/2}_0(\Sigma) \to L^2_{2, \beta}(\Omega) \), and thus, we observe that assertion (3) from the previous theorem holds also true for \( \omega \in \Psi_\beta \).

5. **An asymptotic formula**

In what follows let \( \Sigma \subseteq \partial \Omega \) be the Neumann window. We assumed that \( \Sigma \) is contained in the domain \( U \) of some smooth chart \( \kappa : U \to \kappa(U) \) and that \( \kappa(U) \) is star-shaped with centre 0. Let \( s_0 := \kappa^{-1}(0) \). We denote by \( \Sigma^* := \kappa(\Sigma) \) the window in local coordinates and the scaled window \( \Sigma_\ell \) shall satisfy \( \kappa(\Sigma_\ell) = \ell \cdot \Sigma^* =: \Sigma^*_\ell \) for \( \ell \in (0, 1) \). We want to use the Dirichlet-to-Neumann operator to investigate the behaviour of the resonances of \( A_{\Sigma_\ell} \) as \( \ell \to 0 \). Since for different \( \ell \) the operators \( D_{\Sigma_\ell, \omega} \) are each acting in a different Hilbert space, we define the unitary operator

\[
T_\ell : L^2(\Sigma^*) \to L^2(\Sigma_\ell), \quad T_\ell g(x) := \ell^{-(d-1)/2}(\sqrt{\alpha} \cdot g)(\kappa(x)/\ell),
\]

where \( \alpha(t) \) denotes the Jacobian of the chart. For \( T_\ell \) and its adjoint we have

\[
T_\ell : H^{1/2}_0(\Sigma^*) \to H^{1/2}_0(\Sigma_\ell) \quad \text{and} \quad T_\ell^* : H^{-1/2}(\Sigma_\ell) \to H^{-1/2}(\Sigma^*).
\]
We define the scaled Dirichlet-to-Neumann operator by
\[
Q(\ell, \omega) : H^{1/2}(\Sigma^*) \to H^{-1/2}(\Sigma^*), \quad Q(\ell, \omega) := T^*_\ell \mathcal{D}_{\ell, \omega} T_\ell.
\] (5.1)

Then \( \lambda_0 \in \Psi_\beta \) is a resonance of \( A_\Sigma \) if and only if \( \ker Q(\ell, \lambda_0) \) is non-trivial. Next we want to investigate the behaviour of \( Q(\ell, \omega) \) as \( \ell \to 0 \) for fixed spectral parameter. To this end we use the following result. The proof is given in the appendix.

**Theorem 5.1.** Let \( \omega \in \mathbb{H}_+ \) and let \( V \subseteq \mathbb{R}^d \) be a suitable neighbourhood of \( s_0, U = V \cap \Omega \). Let \( \phi, \psi \in C_c^\infty(U) \), \( \chi \in C_c^\infty(V) \). Then the operators
\[
\psi D_\omega \phi : C_c^\infty(U) \to C_c^\infty(U) \quad \text{and} \quad \chi K_\omega \phi : C_c^\infty(U) \to C_c^\infty(V \cap \overline{\Omega}),
\]
belong to the Boutet-de-Monvel calculus, i.e., \( \chi K_\omega \phi \) is a classical potential operator of order \(-1/2\) and \( \psi D_\omega \phi \) is a classical pseudo-differential operator of order 1.

For an introduction to the Boutet de Monvel’s calculus we refer e.g. to [24]. Let \( \kappa^* : C_c^\infty(\kappa(U)) \to C_c^\infty(U) \), \( \kappa^* g := g \circ \kappa \) be the corresponding pullback operator. We denote by \( p(t, \theta) \in S^1(\kappa(U) \times \kappa(U) \times \mathbb{R}^{d-1}) \) the complete symbol of the operator
\[
\frac{1}{\sqrt{\alpha}} (\kappa^* \psi D_\omega \phi) \kappa \sqrt{\alpha} : C_c^\infty(\kappa(U)) \to C_c^\infty(\kappa(U)).
\]

Here \( S^j(\kappa(U) \times \mathbb{R}^{d-1}) \) are the standard symbol space, see e.g. [24] for their definition. Choosing \( \phi = 1 \) and \( \psi = 1 \) on \( \Sigma_\ell \) for all \( \ell \in (0, 1) \) we obtain
\[
\langle Q(\ell, \omega)g, h \rangle = \frac{\ell^{1-d}}{(2\pi)^{d-1}} \int_{\Sigma^*_\ell} \int_{\mathbb{R}^{d-1}} \int_{\Sigma^*_\ell} e^{i(t-s)\theta} p(t, \theta)g(s/\ell)h(t/\ell) \, ds \, d\theta \, dt
\]
for all \( g, h \in H_0^{1/2}(\Sigma^*) \). As \( p(t, \theta) \) is a classical symbol we have the following expansion into homogeneous symbols
\[
p(t, \theta) \sim \sum_{j=-1}^{\infty} p_j(t, \theta),
\]
where \( p_j \) is homogeneous of order \(-j\).

**Theorem 5.2.** There exist operators \( Q_j, j = 0, \ldots d-2 \) and \( Q^{(1)}_j, Q^{(2)}_j, j \geq d-1 \), such that we have the following asymptotic expansion
\[
\ell Q(\ell, \omega) \sim \sum_{j=0}^{d-2} \ell^j Q_j + \sum_{j=d-1}^{\infty} \ell^j \left( Q^{(1)}_j + (\ln \ell)Q^{(2)}_j \right),
\]
where for the first term we obtain
\[
\langle Q_0g, h \rangle := \int_{\Sigma^*} \int_{\mathbb{R}^{d-1}} \int_{\Sigma^*} e^{i(t-s)\theta} p_{-1}(0, \theta)g(s)h(t) \, ds \, d\theta \, dt.
\]
Remark. The expansion should be interpreted as follows: for \( n \leq d - 3 \) we have
\[
\| \ell Q(\ell, \omega) - \sum_{j=0}^{n} \ell^j Q_j \| = O(\ell^{n+1}),
\]
whereas for \( n \geq d - 2 \) we have
\[
\| \ell Q(\ell, \omega) - \sum_{j=0}^{d-2} \ell^j Q_j - \sum_{j=d-1}^{n} \ell^j (Q_j^{(1)} + (\ln \ell) Q_j^{(2)}) \| = O(\ell^{n+1} \ln \ell)
\]
as operators mapping \( H^{1/2}_{1/2}(\Sigma^*) \) into \( H^{-1/2}(\Sigma) \).

Proof. Choose \( \delta(\theta) \in C^\infty(\mathbb{R}^{d-1}) \) such that \( \delta = 0 \) in some neighbourhood of 0 and \( \delta = 1 \) outside some compact set. Then we have
\[
p(t, \theta) = \delta(\theta)(p_{-1}(t, \theta) + p_0(t, \theta)) + r(t, \theta),
\]
for some \( r(t, \theta) \in S^{-1}(\kappa(U) \times \mathbb{R}^{d-1}) \). For \( g, h \in C^\infty_c(\Sigma^*) \) we have
\[
\langle Q(\ell, 0)g, h \rangle = \frac{\ell^{1-d}}{(2\pi)^{d-1}} \sum_{j \in \{-1,0\}} \int_{\Sigma^*_t} \int_{\Sigma^*_s} e^{i(t-s)\theta} p_j(t, \theta) \delta(\theta) g(s/\ell) h(t/\ell) \, ds \, d\theta \, dt
\]
\[
+ \ell^{d-1} \int_{\Sigma^*_t} \int_{\Sigma^*_s} k_r(\ell t, \ell(t-s)) g(s) h(t) \, ds \, dt,
\]
where \( k_r(t, u) \) the Schwartz kernel of the operator \( r(t, D) \) with \( D = -i\nabla \). We note that \( k_r \) is integrable since it admits an expansion
\[
k_r(t, u) \sim \sum_{j \geq -d+2} k_{r,j}(t, u),
\]
into pseudo-homogeneous functions \( k_{r,j} \). For the corresponding definitions and proofs we refer to [26, Chapter 7]. We note that for \( j = -(d-2), \ldots, -1 \) the functions \( k_{r,j}(t, u) \) are homogeneous of degree \( j \) in the second component. For these \( j \) we have
\[
\int_{\Sigma^*_t} \int_{\Sigma^*_s} k_{r,j}(\ell t, \ell(t-s)) g(s) h(t) \, ds \, dt
\]
\[
= \ell^j \int_{\Sigma^*_t} \int_{\Sigma^*_s} k_{r,j}(\ell t - s) g(s) h(t) \, ds \, dt
\]
and we may apply a Taylor expansion with respect to the first variable. The case \( j \geq 0 \) follows likewise, however additional logarithmic terms will
appear. Next we want to treat the terms involving $p_{-1}(t, \theta)$ and $p_0(t, \theta)$. For $j \in \{-1, 0\}$ we have
\[
\frac{\ell^{-(d-1)}}{(2\pi)^{d-1}} \int_{\Sigma^*} \int_{\mathbb{R}^{d-1}} e^{i(t-s)\theta} p_j(t, \theta)\delta(\theta)g(s/\ell)h(t/\ell) \, ds \, d\theta \, dt \\
= \frac{\ell^j}{(2\pi)^{d-1}} \int_{\Sigma^*} \int_{\mathbb{R}^{d-1}} e^{i(t-s)\theta} p_j(\ell t, \theta)g(s/h) \, ds \, d\theta \, dt \\
+ \frac{\ell^j}{(2\pi)^{d-1}} \int_{\Sigma^*} \int_{\mathbb{R}^{d-1}} e^{i(t-s)\theta} k^{(j)}(\ell t, \ell(t-s))g(s/h) \, ds \, d\theta \, dt,
\]
which is well-defined since $p_j(t, \cdot)$ is bounded in some neighbourhood of $\theta = 0$. Note that $k^{(j)} \in C^\infty(\kappa(U) \times \mathbb{R}^{d-1})$, and we easily obtain an asymptotic expansion for this term. For $n \geq 0$ we have
\[
p_j(\ell t, \theta) = \sum_{|\alpha| \leq n} \ell^{(|\alpha|)} \frac{\partial^\alpha p_j(0, \theta)}{\alpha!} \ell^n + \ell^{n+1} r_{j,n,\ell}(t, \theta),
\]
where $\{r_{j,n,\ell}(t, \theta) : \ell \in (0, 1)\}$ is bounded in the space of homogeneous symbols. The assertion follows now from the next lemma. \qed

**Lemma 5.3.** Let $\phi, \psi \in C^\infty(\kappa(U))$ and let $q \in \mathcal{S}_k^j(\kappa(U) \times \mathbb{R}^{d-1})$, where we assume that $j \in \{0, 1\}$. We define for $g \in C^\infty(\kappa(U))$
\[
q(t, D)g(t) := \frac{1}{(2\pi)^{d-1}} \int_{\kappa(U)} \int_{\mathbb{R}^{d-1}} e^{i(t-s)\theta} q(t, \theta)g(s) \, ds \, d\theta.
\]
Then the following assertions hold true:

1. If $j = 0$ then $\psi q(t, D)\phi : L_2(\kappa(U)) \to L_2(\kappa(U))$ defines a bounded operator.
2. If $j = 1$, then $\psi q(t, D)\phi : H^{1/2}(\kappa(U)) \to H^{-1/2}(\kappa(U))$ defines a bounded operator.

Moreover, the mapping $q(t, \theta) \mapsto \psi q(t, D)\phi$ is continuous with respect to the underlying topologies.

The proof is given in the appendix.

### 5.1. Meromorphic operator valued functions

In order to prove the main results we need elements from the theory of operator-valued meromorphic functions as in [21]. We want to recall the basic notions. Let $X, Y$ Banach spaces and $\mathcal{O} \subseteq \mathbb{C}$ be an open set. A mapping $A : \mathcal{O} \setminus K \to \mathcal{L}(X, Y)$, where $K \subseteq \mathcal{O}$ is a discrete subset, is called finitely meromorphic of Fredholm type if for all $\omega_0 \in \mathcal{O}$ the function $A$ admits a Laurent expansion near
$\omega_0$,

$$A(\omega) = \sum_{k=-N}^{\infty} A_k(\omega - \omega_0)^k,$$

(5.2)

where the operators $A_{-1}, \ldots, A_{-N} \in \mathcal{L}(X,Y)$ are finite-rank operators and $A_0$ is Fredholm. A point $\omega_0 \in \mathcal{O}$ is called characteristic value of $A$ if there exists a holomorphic function $\phi : U \subseteq \mathcal{O} \rightarrow X$ with $\phi(\omega_0) = x \neq 0$ such that $A(\omega)\phi(\omega) \rightarrow 0$ as $\omega \rightarrow \omega_0$. The order of the zero of $A(\omega)\phi(\omega)$ at $\omega_0$ is called the multiplicity of the root functional $\phi$ corresponding to the eigenvector $x = \phi(\omega_0)$ and is denoted by $\text{rank } x$. Let

$$\ker A(\omega_0) := \{x : x \text{ eigenvector of } A(\omega_0)\}.$$  

(5.3)

The rank of an eigenvector $x_0$ is defined as the supremum taken over the multiplicities of all root functional $\phi$ corresponding to $x_0$.

Let $\dim \ker A(\omega_0) =: n < \infty$ and assume that every eigenvector has finite multiplicity. Then we may choose a basis $x_1, \ldots, x_n$ of $\ker A(\omega_0)$ such that $\text{rank } x_j$ is the maximum of the ranks of all eigenvectors in some direct complement of the linear span of the vectors $x_1, \ldots, x_{j-1}$. The value

$$N(A; \omega_0) := \sum_{j=1}^{n} \text{rank } x_j$$

(5.4)

is called null multiplicity of the characteristic value $\omega_0$. If $\omega_0$ is not a characteristic value then we put $N(A; \omega_0) = 0$. Assuming that $A(\omega)$ is invertible with the possible exception of a discrete subset we observe that $A^{-1}$ is also finitely meromorphic and of Fredholm type. The number

$$P(A; \omega_0) := N(A^{-1}; \omega_0)$$

(5.5)

is called the polar multiplicity of $\omega_0$. We denote by

$$M(A; \omega_0) = N(A; \omega_0) - P(A; \omega_0)$$

(5.6)

the total multiplicity of $\omega_0$. In what follows let $\Gamma$ denote a Jordan curve in $\mathcal{O}$, whose interior is contained in $\mathcal{O}$. Then $M(A; \Gamma)$ denotes the sum of the total multiplicities of all characteristic values of either $A$ or $A^{-1}$ in the interior of $\Gamma$.

**Theorem 5.4** (see also Theorem 2.1 in [21]). Let $A : \mathcal{O} \rightarrow \mathcal{L}(X,Y)$ be a finitely meromorphic function of Fredholm type and assume that $A$ is invertible on $\text{ran}(\Gamma)$. For an analytic function $f : \mathcal{O} \rightarrow \mathbb{C}$ we have

$$\sum_{\omega_0 \in \text{Int}(\Gamma)} f(\omega_0)M(A; \omega_0) = \frac{1}{2\pi i} \text{tr} \int_{\Gamma} f(\omega)A(\omega)^{-1} \frac{d}{d\omega} A(\omega) \, d\omega.$$
Theorem 5.5 (Theorem 2.2 in [21]). Let $A : \Omega \to \mathcal{L}(X,Y)$ be chosen as above and let $B : \Omega \to \mathcal{L}(X,Y)$ be a finitely meromorphic function such that

$$\|A(\omega)^{-1}B(\omega)\|_{\mathcal{L}(X)} < 1 \quad \text{for} \ \omega \in \text{ran}(\Gamma).$$

Then

$$M(A; \Gamma) = M(A + B; \Gamma).$$

5.2. Proof of the main results. The following lemma serves as a first preparation.

Lemma 5.6. Let $\beta > 0$. Then for $\omega \in \Psi_\beta$ the function

$$\Psi_\beta \ni \omega \mapsto Q(\ell, \omega) \in \mathcal{L}(H^{1/2}_0(\Sigma); H^{-1/2}(\Sigma))$$

is a finitely meromorphic function of Fredholm type.

Proof. It easily follows that $\omega \mapsto R_\emptyset, \omega \mapsto K_\omega$ and $\omega \mapsto D_\omega = \gamma_1 K_\omega$ are all finitely meromorphic functions. Then $\omega \mapsto Q(\ell, \omega)$ will be also finitely meromorphic and we only have to show that it is of Fredholm type. We use that the operator

$$D_{\Sigma,\omega} - D_{\Sigma,\eta} : L_2(\Sigma) \to L_2(\Sigma)$$

is compact for all $\omega, \eta \in \mathbb{H}_+$, which follows from Theorem 5.1. As the compact operators form a closed subset it follows by analytic continuation that $D_{\Sigma,\omega} - D_{\Sigma,\eta}$ is compact for each $\omega \in \Psi_\beta$. As $D_{\Sigma,\eta}$ is bijective the operator family $\omega \mapsto Q(\ell, \omega)$ has to be of Fredholm type. \[\square\]

Now we want to prove Theorem 2.1. The main ingredient will be the perturbation formula given in Theorem 4.4.

$$D_\omega = D_{\eta} - (\omega^2 - \eta^2)K_\eta^*(I + (\omega^2 - \eta^2)R_\emptyset(\omega))K_\eta$$

which holds a priori only for $\eta, \omega \in \mathbb{H}_+$. Using the remarks after Theorem 4.4 this formula still holds true for all $\omega \in \Psi_\beta$ if $\beta$ is small enough. Here and subsequently let $\eta \in i\mathbb{R}$. Then we have

$$Q(\ell, \omega) = Q(\ell, \eta) - (\omega^2 - \eta^2)T_\ell^* r_{\Sigma,\eta} K_\eta^*(I + (\omega^2 - \eta^2)R_\emptyset(\omega))K_\eta e^{\Sigma,\ell} T_\ell,$$

(5.7)

Thus, the function $Q(\ell, \cdot)$ will be singular at most at resonance points of $A_\emptyset$. Now let $\lambda_0 \in \Psi_\beta \setminus \{\Lambda_j\}$ be a resonance of order 1 for $A_\emptyset$. We assume that $\lambda_0 \neq 0$. Then $R_\emptyset(\cdot)$ has a pole in $\lambda_0$ and we have

$$1 = \dim \text{ran} \left\{ \frac{1}{2\pi i} \int_{|\omega - \lambda_0| = \varepsilon} R_\emptyset(\omega) \ d\omega. \right\}$$
Theorem 5.7. For $\lambda_0 \neq 0$ we have

$$R_{\gamma}(\omega) = \frac{1}{2\lambda_0} \cdot \frac{\Pi_0}{\lambda_0 - \omega} + O(1) \quad \text{as } \omega \to \lambda_0,$$

where $\Pi_0$ is a rank-one projection. The integral kernel of $\Pi_0$ satisfies

$$\Pi_0(x, \bar{x}) = (u_0 \otimes u_0)(x, \bar{x}),$$

where $u_0$ is suitably normalised outgoing solution of

$$(A(x, \nabla_x) - \lambda_0^2)u = f \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \partial\Omega.$$ 

The proof follows as in [12, Theorem 2.4]. Note that one has to use that the coefficients of $A(x, \nabla_x)$ are real-valued. If $\omega^2$ is a discrete eigenvalue of $A_\beta$ then we may choose $u_0$ as any real-valued normalised eigenfunction. From (5.7) it follows that $\lambda_0$ is a pole of $Q(\ell, \omega)$ if and only if $r_{\Sigma_\ell}K^*_\eta \Pi K_\eta e_{\Sigma_\ell} \neq 0$. We have

$$\langle T^*_\ell r_{\Sigma_\ell} K^*_\eta \Pi K_\eta e_{\Sigma_\ell} T_{\ell} g, h \rangle = \langle T^*_\ell K^*_\eta u_0, h \rangle \cdot \langle K_\eta T_{\ell} g, \overline{u_0} \rangle,$$

where $u_0$ be chosen as in Theorem 5.7. We have

$$r_{\Sigma_\ell} K^*_\eta u_0 = r_{\Sigma_\ell} \gamma_1 R(\eta) u_0 = (\lambda_0^2 - \eta^2)^{-1} r_{\Sigma_\ell} \gamma_1 u_0.$$

Note that $(A(x, \nabla_x) - \lambda_0^2)u_0 = 0$ and $\gamma_0 u = 0$ on $\Sigma_\ell$. Then the unique continuation principle (see e.g. [36]) implies that

$$C^\infty(\Sigma_\ell^0) \ni \phi_\ell := r_{\Sigma_\ell} \gamma_1 u_0 \neq 0.$$  

(5.9)

Finally,

$$\langle T^*_\ell r_{\Sigma_\ell} K^*_\eta \Pi K_\eta e_{\Sigma_\ell} T_{\ell} g, h \rangle = (\lambda_0^2 - \eta^2)^{-2} \langle T^*_\ell \phi_\ell, h \rangle \cdot \langle g, \overline{T^*_\ell \phi_\ell} \rangle,$$

and thus, $\lambda_0$ is a pole of $Q(\ell, \cdot)$. Choosing $\varepsilon > 0$ small enough we observe that $\omega \mapsto Q(\ell, \omega)$ is holomorphic for $\omega \in B(\lambda_0, 2\varepsilon) \setminus \{\lambda_0\}$. Let $\Gamma(t) := \lambda_0 + \varepsilon e^{2\pi it}$. We use Rouché’s theorem to show that

$$M(Q(\ell, \cdot); \Gamma) = 0.$$  

Then there exists a unique $\lambda(\ell) \in B(\lambda_0, \varepsilon)$ such that $\ker Q(\ell, \lambda(\ell)) \neq \{0\}$. Let $Q_0$ be chosen such that

$$\|\ell Q(\ell, \eta) - Q_0\| = O(\ell) \quad \text{as } \ell \to 0,$$

cf. Theorem 5.2. Using Formula 5.7 together with an estimate on the perturbation term we obtain that

$$\|\ell Q(\ell, \omega) - Q_0\| = O(\ell)$$

uniformly in $\omega \in \partial B(\lambda_0, \varepsilon)$ for sufficiently small $\varepsilon > 0$. Recall that we have $K_\eta e_{\Sigma_\ell} T_{\ell} : L^2(\Sigma_\ell) \to L^2_{2,\beta}(\Omega)$ and $r_{\Sigma_\ell} K^*_\eta : L^2_{2,\beta}(\Omega) \to L^2(\Sigma_\ell)$ continuously. Now Rouché’s Theorem implies that

$$M(\ell Q(\ell, \cdot); \Gamma) = M(Q(0); \Gamma) = 0.$$
for sufficiently small $\ell > 0$. This proves the existence of a unique resonance $\lambda(\ell)$ of $A_{\Sigma_\ell}$ near $\lambda_0$. Next we want to prove the asymptotic formula. Using Theorem 5.4 we have

$$\lambda(\ell) - \lambda_0 = \frac{1}{2\pi i} \text{tr} \int_{|\omega - \lambda_0| = \varepsilon} (\omega - \lambda_0) Q(\ell, \omega)^{-1} \frac{d}{d\omega} Q(\ell, \omega) \, d\omega.$$  

Let $R(\ell, \omega) := Q(\ell, \omega) - \ell^{-1} Q_0$. Then we have $\|R(\ell, \omega)\| = O(1)$ and we obtain

$$Q(\ell, \omega)^{-1} = (\ell^{-1} Q_0 + R(\ell, \omega))^{-1} = \sum_{k=0}^{\infty} (-1)^k \ell^{k+1} (Q_0^{-1} R(\ell, \omega))^k Q_0^{-1},$$  

where the sum converges for sufficiently small $\ell > 0$. Since $R(\ell, \cdot)$ is meromorphic we have

$$R(\ell, \omega) = \sum_{k=-1}^{\infty} (\omega - \lambda_0)^k R_k(\ell)$$  

for operators $R_k(\ell) : H_0^{1/2}(\Sigma^*) \to H^{-1/2}(\Sigma^*)$, $k \geq -1$. Note that $R_{-1}$ is a rank-one operator. Then we have

$$\lambda(\ell) - \lambda_0 = \frac{1}{2\pi i} \text{tr} \int_{|\omega - \lambda_0| = \varepsilon} (\omega - \lambda_0) Q(\ell, \omega)^{-1} \frac{d}{d\omega} R(\ell, \omega) \, d\omega$$

$$= \text{tr} \sum_{k=0}^{\infty} (-1)^k \ell^{k+1} B_k(\ell), \quad (5.10)$$

where

$$B_k(\ell) := \sum_{\alpha_1 + \ldots + \alpha_{k+1} = -1, \alpha_i \geq -1} \alpha_{k+1} \cdot Q_0^{-1} R_{\alpha_1}(\ell) \ldots R_{\alpha_k} Q_0^{-1} R_{\alpha_{k+1}}(\ell).$$

Next we want to interchange the trace with the summation. Note that the sum in (5.10) converges in the operator norm. Moreover, the $B_k(\ell)$ are all rank-one operators since $R_{-1}(\ell)$ is a rank-one operator. Thus, the operator norm of $B_k(\ell)$ coincide with its trace norm and we obtain

$$\lambda(\ell) - \lambda_0 = \sum_{k=0}^{\infty} (-1)^k \ell^{k+1} \text{tr} B_k(\ell).$$

**Lemma 5.8.** There exists constants $c, d > 0$ such that for all $k \geq 0$ we have $\|B_k(\ell)\| \leq cd^k \ell^{d-1}$ as $\ell \to 0$.

**Proof.** For $k \geq 0$ we have

$$R_k(\ell) = \frac{1}{2\pi i} \int_{|\omega - \lambda_0| = \varepsilon} \frac{R(\ell, \omega)}{(\omega - \lambda_0)^{k+1}} \, d\omega,$$
and thus, \( \| \mathcal{R}_k(\ell) \| \leq C \varepsilon^{-(k+1)} \) for \( C \) independent of \( \ell \). In the case \( k = -1 \) we obtain for \( g, h \in H^{1/2}(\Sigma^*) \) that

\[
\langle \mathcal{R}_{-1}(\ell) g, h \rangle = \frac{1}{2\lambda_0} \langle T^*_\ell \phi_\ell, h \rangle \cdot \langle g, T^*_\ell \phi_\ell \rangle,
\]

where we have set \( \phi_\ell = \tau_{\Sigma^*} K_{\rho_0} u_0 \in C^\infty(\Sigma_\ell) \). Then we obtain \( \| \mathcal{R}_{-1}(\ell) \| \leq \| \phi_\ell \|_{L^2(\Sigma_\ell)}^2 \leq C \varepsilon^{d-1} \), and finally

\[
\| \mathcal{B}_k(\ell) \| \leq (k - 1) C^{k+1} \varepsilon^{-k} \ell^{d-1} \| Q^{-1}_0 \|^k \cdot \# \{ \alpha \in \mathbb{N}^{k+1} : |\alpha| = k \}.
\]

Note that

\[
\# \{ \alpha \in \mathbb{N}^{k+1} : |\alpha| = k \} = \binom{2k}{k} = \frac{(2k)!}{k!k!} \leq 4 \left( \frac{2(k - 1)}{k - 1} \right) \leq \ldots \leq 4^k.
\]

This proves the lemma. \( \square \)

Finally, we have \( \lambda(\ell) - \lambda_0 = \ell \text{ tr } \mathcal{B}_0(\ell) + \mathcal{O}(\ell^{d+1}) \), where \( \mathcal{B}_0 = -Q_0 \mathcal{R}_{-1}(\ell) \).

Since

\[
\langle \mathcal{B}_0(\ell) g, h \rangle = -\frac{1}{2\lambda_0} \langle Q^{-1}_0 T^*_\ell \phi_\ell, h \rangle \cdot \langle g, T^*_\ell \phi_\ell \rangle
\]

we obtain

\[
\text{tr } \mathcal{B}_0(\ell) = -(2\lambda_0)^{-1} \langle Q^{-1}_0 T^*_\ell \phi_\ell, T^*_\ell \phi_\ell \rangle.
\]

Note that

\[
T^*_\ell \phi_\ell(t) := \ell^{d-1/2} \frac{\phi_\ell(\kappa^{-1}(\ell t))}{\sqrt{\alpha(\ell t)}} = \ell^{(d-1)/2} \frac{\gamma u(\kappa(0))}{\sqrt{\alpha(0)}} + \mathcal{O}(\ell^{(d+1)/2}).
\]

Let \( 1 \in L^2(\Sigma^*) \) denote the constant function. Setting \( s_0 := \kappa(0) \) we obtain

\[
\text{tr } \mathcal{B}_0(\ell) = -\frac{\gamma u(s_0)^2}{2\lambda_0 \alpha(0)} \cdot \langle Q^{-1}_0 1, 1 \rangle \ell^d + \mathcal{O}(\ell^{d+1}),
\]

which proves Theorem 2.1 with

\[
\nu := \frac{\langle Q^{-1}_0 1, 1 \rangle}{2\lambda_0 \alpha(0)} = \frac{\langle Q^{-1/2}_0 1, Q^{-1/2}_0 1 \rangle}{2\lambda_0 \alpha(0)} > 0.
\]

Finally, we consider the case where \( \lambda_0 = \Lambda_i \) is a threshold of the essential spectrum. We assume that the branching point is of second order. Thus, the functions \( \zeta \mapsto R_{\overline{\phi}}(\Lambda_i - \zeta^2) \) and \( \zeta \mapsto R_{\Sigma}(\Lambda_i - \zeta^2) \) are meromorphic near \( \zeta = 0 \). In this case we obtain the following result. Its proof follows as in [12] Theorem 2.5 and Theorem 3.13.

**Theorem 5.9.** We have

\[
R_{\overline{\phi}}(\Lambda_i - \zeta^2) = \frac{H_1}{\zeta^2} + \frac{H_0}{\zeta} + \mathcal{O}(1) \quad \text{as } \zeta \to 0.
\]
Here $\Pi_1$ is a bounded operator in $L^2(\Omega)$ mapping onto the space of square integrable solutions of

$$(A(x, \nabla_x) - \lambda_0^2)u = f \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \partial \Omega \setminus \Sigma, \quad \gamma_1 u = 0 \text{ on } \Sigma,$$

and the range of $\Pi_1$ consists of possibly non-square integrable solutions of the above boundary value problem.

We assume that $\Pi_1 = 0$, which means $\Lambda_2^2$ is not an embedded eigenvalue of $A_\Sigma$, and assume that $\Pi_0$ is one-dimensional. As in Theorem 5.7 the projection $\Pi_0$ has an integral kernel $\Pi_0(x, \tilde{x}) = (u_0 \otimes u_0)(x, \tilde{x})$ with some suitably chosen function $u_0$. We consider now the operator $\zeta \mapsto Q(\ell, \Lambda_i - \zeta^2)$. As above we obtain the existence of a unique $\zeta(\ell)$ near $\Lambda_i$ such that $\ker Q(\ell, \Lambda_i - \zeta^2) \neq 0$. Moreover, we have as before

$$\zeta(\ell) = -\nu \cdot \gamma_1 u_0(s_0)^2 \cdot \ell^d + O(\ell^{d+1}),$$

where $\nu$ is given as before. Finally, we have

$$\lambda(\ell) = \Lambda_i - \zeta(\ell)^2 = \Lambda_i - \nu^2 \cdot \gamma_1 u_0(s_0)^4 \cdot \ell^{2d} + O(\ell^{2d+1}),$$

which proves Theorem 2.2.

**Remark.** Analogous results hold true in the case of several cylindrical ends. For the special case $\Omega = \mathbb{R} \times G$ we may easily calculate the behaviour of the projections $\Pi_0$ at a branching point $\Lambda > 0$. Indeed, as in Theorem 3.4 we choose $\mu_k$ and $P_k$ such that

$$A^0(\xi) = \sum_{k=1}^{\infty} \mu_k(\xi) P_k(\xi).$$

We consider $\xi_1, \ldots, \xi_n$ and $k_1(\xi_i), \ldots, k_{m_i}(\xi_i)$ such that $\mu_{k_1(\xi_i)}(\xi_i) = \Lambda^2$ and $\mu_{k_2(\xi_i)}(\xi_i) = 0$. Assume that $\mu''_{k_i(\xi_i)}(\xi_i) \neq 0$. We denote by $\psi^{(i,l)}_1, \ldots, \psi^{(i,l)}_{\ell_ij} \in H^2(G)$ an orthonormal basis of $\text{ran} \, P_{k_i(\xi_i)}(\xi_i)$. From the proof of Theorem 3.4 it easily follows that

$$\Pi_0(y, z, \tilde{y}, \tilde{z}) = \pi \Lambda^{-1/2} \sum_{i=1}^{n} \sum_{l=1}^{\ell_i} \sum_{p=1}^{\ell_{ij}} \rho_{i,l} e^{i\xi_i y \psi^{(i,l)}_p(\xi)} e^{-i\xi_i \tilde{y} \psi^{(i,l)}_p(\xi)} \frac{1}{\sqrt{|\mu''_{k_i(\xi_i)}(\xi_i)|}}$$

where $\rho_{i,l} = 1$ if $\mu''_{k_i(\xi_i)}(\xi_i) > 0$ and $\rho_{i,l} = i$ if $\mu''_{k_i(\xi_i)}(\xi_i) < 0$. Note that we have $\sigma(A_0(-\xi)) = \sigma(A(\xi))$, and thus, $\Pi_0$ is one-dimensional only if $n = 1$ and $\xi_1 = 0$. If $\Pi_0$ is not a rank-one operator it also possible to prove an asymptotic formula in some cases using the symmetries of the domain and the operator, cf. e.g. 23 for the elastic case.
Remark. We briefly indicate the necessary changes in the case of matrix-valued operators. The assertions in Chapter 2, 3, 4 and 5.1 may easily be adapted to elliptic systems. In Chapter 5.2 it will be necessary to prove a corresponding unique continuation principle for elliptic systems in order to show the existence of a resonance, cf. Formula 5.9. Then we obtain as before

\[ \lambda(\ell) - \lambda_0 = \ell \text{tr} \mathcal{B}_0(\ell) + O(\ell^{d+1}), \]

where \( \text{tr} \mathcal{B}_0(\ell) \) is again given as in Formula (5.11). Using Formula (6.12) a corresponding asymptotic formula may also be deduced in this case.

6. Appendix

6.1. Proof of Theorem 5.1. Choosing \( U, V \) as in Theorem 5.1 we consider the boundary value problem

\[
\left( A(x, \nabla_x) - \omega^2 \right) \gamma_0 : H^s_{\text{loc}}(V \cap \Omega) \to H^{s-2}_{\text{loc}}(V \cap \Omega) \oplus H^{s-3/2}_{\text{loc}}(U) \quad \text{for} \ s > 2.
\]

Note that boundary value problem is elliptic, and thus, there exists a corresponding parametrix (cf. e.g. [24]). There is potential operator

\[
L_\omega : H^{s-1/2}_{\text{comp}}(U) \to H^s_{\text{loc}}(V \cap \Omega), \quad s \in \mathbb{R},
\]

such that \( (A(x, \nabla_x) - \omega^2) L_\omega + \gamma_0 L_\omega - \text{Id} \) are smoothing. In particular,

\[
(A(x, \nabla_x) - \omega^2) L_\omega : H^{1/2}_{\text{comp}}(U) \to C^\infty(V \cap \Omega),
\]

and \( \gamma_0 L_\omega - \text{Id} : H^{1/2}_{\text{comp}}(U) \to C^\infty(U) \) are continuous. Let \( \phi \in C^\infty_c(U) \) and \( \chi \in C^\infty_c(V) \). Then for \( g \in H^{1/2}(U) \) we have

\[
(A(x, \nabla_x) - \omega^2) (L_\omega - K_\omega) \phi g \in C^\infty(V \cap \Omega), \quad \gamma_0 (L_\omega - K_\omega) \phi g \in C^\infty(U),
\]

and local regularity theorems imply \( \chi (L_\omega - K_\omega) \phi : H^{1/2}(U) \to C^\infty(V \cap \Omega) \) continuously. We consider \( (\chi (L_\omega - K_\omega) \phi)^* \) which maps \( L_2(V \cap \Omega) \) into \( H^{-1/2}(U) \). From Theorem 5.1 we obtain \( (\chi K_\omega \phi)^* = \phi \gamma_1 R_\omega(-\overline{\omega}) \chi \). Let \( \chi_1 \in C^\infty_c(V) \) be chosen such that \( \chi_1 = 1 \) on the support of \( \chi \) and on the support of \( \phi \). For \( f \in L^2(V \cap \Omega) \) and \( g \in H^{1/2}(U) \) we have

\[
\langle g, (\chi L_\omega \phi)^* f \rangle = \langle \chi_1 L_\omega \phi g, (A(x, \nabla_x) - \overline{\omega}^2) R_\omega(-\overline{\omega}) \chi f \rangle
\]

\[
= \langle \gamma_0 \chi_1 L_\omega \phi g, \gamma_1 R_\omega(-\overline{\omega}) \chi f \rangle + \langle (A(x, \nabla_x) - \omega^2) \chi_1 L_\omega \phi g, R_\omega(-\overline{\omega}) \chi f \rangle
\]

\[
= \langle \phi g, \gamma_1 R_\omega(-\overline{\omega}) \chi f \rangle + \langle S_1 g, \gamma_1 R_\omega(-\overline{\omega}) \chi f \rangle + \langle S_2 g, R_\omega(-\overline{\omega}) \chi f \rangle
\]

\[
+ \langle (A(x, \nabla_x) - \overline{\omega}^2) \chi_1 L_\omega \phi g, R_\omega(-\overline{\omega}) \chi f \rangle.
\]
where \( S_1 \) and \( S_2 \) are a smoothing operators. Due to the support assumptions we easily see that \([A(x, \nabla_x), \chi_1]L_\omega \phi \) is smoothing. Thus, we have
\[
\chi(L_\omega - K_\omega) \phi : H^{1/2}(U) \rightarrow C^\infty(V \cap \overline{\Omega}),
\]
\[
(\chi(L_\omega - K_\omega) \phi)^* : L_2(V \cap \overline{\Omega}) \rightarrow C^\infty(U).
\]
Using a similar approach as in [24, Theorem 2.4.87] we obtain that there exists a function \( c_\omega \in C^\infty_c((V \cap \overline{\Omega}) \times U) \) such that
\[
(\chi(L_\omega - K_\omega) \phi)(x) = \int_U c_\omega(x, x')g(x') \, dx'.
\]
Thus, \( \chi(L_\omega - K_\omega) \phi \) is smoothing, which implies the assertion.

6.2. Proof of Lemma 5.3. We only prove the second assertion of the lemma, the first assertion follows in the same way. The proof follows the ideas of [25, Theorem 18.1.11]. Let \( g \in C^\infty_c(\kappa(U)) \) and let us denote by \( Fg \) its Fourier transform. We put \( \langle \theta \rangle = (1 + |\theta|^2)^{1/2} \). Then the Fourier transform of \( \psi p(t, D) \phi g \) is given by
\[
\eta \mapsto \int_{\mathbb{R}^{d-1}} \hat{q}(\eta - \theta, \theta) \langle \theta \rangle^{1/2} F(\phi g)(\theta) \, d\theta,
\]
where
\[
\hat{q}(\eta - \theta, \theta) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{-i(\eta - \theta)t} \psi(t)q(t, \theta) \, dt.
\]
For all \( n \in \mathbb{N} \) we have \( |\hat{q}(\eta - \theta, \theta)| \leq c_n \langle \theta \rangle^{-n} \langle \eta - \theta \rangle^{-n} \), where the constant \( c_n \) may be expressed in terms of seminorms of the symbol \( q \in S^d_{\text{hom}}(V \times \mathbb{R}^{d-1}) \). Using the Schur-test we will show that there exist functions \( \alpha, \beta \) and constants \( C_1, C_2 > 0 \) such that
\[
\int_{\mathbb{R}^{d-1}} \frac{|\hat{q}(\eta - \theta, \theta)|}{\langle \eta \rangle^{1/2} \langle \theta \rangle^{1/2}} \alpha(\eta) \, d\eta \leq C_1 \beta(\theta),
\]
\[
\int_{\mathbb{R}^{d-1}} \frac{|\hat{q}(\eta - \theta, \theta)|}{\langle \eta \rangle^{1/2} \langle \theta \rangle^{1/2}} \beta(\theta) \, d\eta \leq C_2 \alpha(\eta).
\]
Then the assertion follows. Choosing \( \alpha(\eta) := \langle \eta \rangle^{-n} \) and \( \beta(\theta) := \langle \theta \rangle^{1/2} \) we obtain
\[
\int_{\mathbb{R}^{d-1}} \frac{|\hat{q}(\eta - \theta, \theta)|}{\langle \eta \rangle^{1/2} \langle \theta \rangle^{1/2}} \alpha(\eta) \, d\eta \leq c_n \langle \theta \rangle^{1/2} \int_{\mathbb{R}^{d-1}} \langle \eta - \theta \rangle^{-n} \, d\eta = C_1 \langle \theta \rangle^{1/2}
\]
for \( n \) sufficiently large. Moreover, we have
\[
\int_{\mathbb{R}^{d-1}} \frac{|\hat{q}(\eta - \theta, \theta)|}{\langle \eta \rangle^{1/2} \langle \theta \rangle^{1/2}} \beta(\theta) \, d\theta \leq \frac{c_n}{\langle \eta \rangle^{1/2}} \int_{\mathbb{R}^{d-1}} \langle \theta \rangle \langle \eta - \theta \rangle^{-n} \, d\theta = \frac{c_n}{\langle \eta \rangle^{1/2}} \int_{\mathbb{R}^{d-1}} \langle \eta - \theta \rangle^{-n} \, d\theta.
\]
Now Peetre’s inequality implies that $\langle \eta - \theta \rangle \leq \sqrt{2} \langle \theta \rangle \langle \eta \rangle$, and the assertion follows.

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