Stability for Semivectorial Bilevel Programs

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Abstract. This paper studies the stability for bilevel program where the lower-level program is a multiobjective programming problem. As we know, the weakly efficient solution mapping for parametric multiobjective program is not generally lower semicontinuous. We first obtain this semicontinuity under a suitable assumption. Then, a new condition for the lower semicontinuity of the efficient solution mapping of this problem is also obtained. Finally, we get the continuities of the value functions and the solution set mapping for the upper-level problem based on the semicontinuities of solution mappings for the lower-level parametric multiobjective program.

1. Introduction. Bilevel program (BP) is an active research area in mathematical programs at present. This model has a framework to deal with decision processes involving two decision makers with a hierarchical nested structure. The upper-level decision maker (leader) has the first choice, and the lower-level decision maker (follower) reacts optimal solution to the leader’s selection. In recent decades, the bilevel program has been widely studied [3, 7, 8, 10, 12, 13, 16, 20, 22, 23, 26, 28].

In the present article, we consider a bilevel problem where the upper-level problem is a single objective program and the lower-level problem is a multiobjective program. In the last two decades, many scholars considered this model. A penalty approach was suggested to solve this model in the case when the follower react weakly efficient solutions to the leader[4]. Another penalty method was developed...
by Zheng and Wan [27] in the case when the lower-level problem is linear multiobjective program. Bonnel [5] derived necessary conditions for the semivectorial bilevel program in very general Banach spaces, while considering the properly efficient solutions and weakly efficient solutions of the lower-level problem. Considering the scalarization approach for the lower-level multiobjective program, Dempe et al. [9] first transform the semivector bilevel program into a single objective program with inequality constraints by the means of the well-known optimal value reformulation, and then a new optimality conditions for this problem is proposed by this method, finally completely detailed first-order necessary optimality conditions are derived in the smooth and nonsmooth settings while using the generalized differentiation calculus of Mordukhovich. In [6] the convex scalarization theorem is adapted to Riemannian setting. Based on that, optimality conditions are presented for this problem under the weakly efficient case as well as the properly efficient case. In [8] a theoretical existence result for semivectorial bilevel programming problems is obtained via scalar bilevel programming.

Some other theorems and algorithms for semivectorial bilevel programs can be found in [1, 2, 14]. As far as we know, there is hardly any results for the stability on this problem. The aim of our paper is to discuss the stability on semivectorial bilevel program while considering the (weakly) efficient solutions of the lower-level multiobjective program. The difficulty faced by this paper is that the weakly efficient solution mapping for the lower-level multiobjective problem is not generally lower semicontinuous, so some suitable conditions should be considered for it.

The rest of this paper is organized as follows. In Section 2, we first give some well-known definitions and some basic notations of multiobjective program needed in the sequel. In Section 3, we mainly consider the lower semicontinuities of the (weakly) efficient solution mapping for the lower-level parametric multiobjective problem. In Section 4, we discuss the continuities for the value function of the upper-level optimization problem. In Section 5, we consider the existence and stability of solutions.

2. Preliminaries. Throughout the paper, we let

\[ R^p_\geq := \{ a = (a_1, \ldots, a_p)^\top \in R^p | a_i \geq 0, \forall i \in \{1, \ldots, p\} \}, \]

\[ R^p_\succ := \{ a = (a_1, \ldots, a_p)^\top \in R^p | a_i > 0, \forall i \in \{1, \ldots, p\} \}. \]

Next, we recall the basic concepts of multiobjective programming problem and the basic properties of set-valued mapping.

2.1. Parametric multiobjective program. In this article we consider the following parametric multiobjective programming problem:

\[
\begin{align*}
\min_y & \quad f(x, y) = (f_1(x, y), \ldots, f_p(x, y))^\top \\
\text{s.t.} & \quad g_j(x, y) \leq 0, j = 1, \ldots, q,
\end{align*}
\]

(1)

here \( y \in \mathbb{R}^n \), \( x \) is a perturbation parameter vector in \( \mathbb{R}^m \), \( f \) is a \( p \)-dimensional objective function, and \( g = (g_1, \ldots, g_q)^\top \) is a \( q \)-dimensional constraint function. The partial orders \( \preceq \) and \( \prec \) considered in this paper are introduced by \( R^p_\geq \), and defined as follows: for every \( z_1, z_2 \in \mathbb{R}^p \)

\[ z_1 \preceq z_2 \iff z_2 - z_1 \in R^p_\geq, \]

\[ z_1 < z_2 \iff z_2 - z_1 \in \text{int} R^p_\geq. \]
Let $K$ be a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by

$$K(x) := \left\{ y \in \mathbb{R}^n \mid g_j(x, y) \leq 0, j = 1, 2, \ldots, q \right\}.\]

Similarly, let $N$ be a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^p$ defined by

$$N(x) := \left\{ z \in \mathbb{R}^p \mid z = f(x, y), y \in K(x) \right\} = f(x, K(x)).$$

The $\mathbb{R}^p_{\geq}$-minimal points of $N(x)$ is defined by

$$M(x) := \left\{ \hat{z} \in N(x) \mid \text{there exists no other } z \in N(x) \text{ such that } z \preceq \hat{z} \right\}$$

$$= \left\{ \hat{z} \in N(x) \mid (N(x) - \hat{z}) \cap (-\mathbb{R}^p_{\geq}) = \{0\} \right\}. \quad (2)$$

The weakly $\mathbb{R}^p_{\geq}$-minimal points of $N(x)$ can be defined by

$$WM(x) := \left\{ \hat{z} \in N(x) \mid \text{there exists no } z \in N(x) \text{ such that } z \prec \hat{z} \right\}$$

$$= \left\{ \hat{z} \in N(x) \mid (N(x) - \hat{z}) \cap (-\text{int}\mathbb{R}^p_{\geq}) = \emptyset \right\}. \quad (3)$$

The efficient solution mapping $E$ and the weakly efficient solution mapping $WE$ of problem (1) can be defined as

$$E(x) := \left\{ y \in K(x) \mid f(x, y) \in M(x) \right\},$$

$$WE(x) := \left\{ y \in K(x) \mid f(x, y) \in WM(x) \right\},$$

respectively.

2.2. Basic known definitions. The material presented here is essentially taken from [11, 17, 21, 25]. Let $\Xi$ be a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, and we recall the definitions of the lower\"upper semicontinuity of this mapping firstly.

**Definition 2.1** (Hogan). $\Xi$ is said to be upper semicontinuous at $x \in \mathbb{R}^m$ in the sense of Hogan (Hogan u.s.c. in short) if, $x_k \rightarrow x$, $y_k \in \Xi(x_k)$, and $y_k \rightarrow y$ all imply that $y \in \Xi(x)$.

**Definition 2.2** (Hogan). $\Xi$ is said to be lower semicontinuous at $x \in \mathbb{R}^m$ in the sense of Hogan (Hogan l.s.c. in short) if, $x_k \rightarrow x$, $y \in \Xi(x)$ imply the existence of an integer $K > 0$ and a sequence $\{y_k\} \subset \mathbb{R}^n$ such that $y_k \in \Xi(x_k)$ for $k > K$ and $y_k \rightarrow y$.

**Definition 2.3.** For any real number $\epsilon > 0$, the open $\epsilon$-neighborhood of $U \subset \mathbb{R}^p$ is defined as

$$U_{+\epsilon} := \left\{ z \in \mathbb{R}^p \mid \text{there exists a } x \in U \text{ such that } \|z - x\| < \epsilon \right\}.$$ 

And we define $M(U) := \bigcup_{x \in U} M(x)$.

**Definition 2.4** (Hausdorff). $\Xi$ is said to be lower semicontinuous at $\hat{x} \in \mathbb{R}^m$ in the sense of Hausdorff (Hausdorff l.s.c. in short) if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \{\hat{x}\}_{+\delta}$, $\Xi(\hat{x}) \subset \Xi(x)_{+\epsilon}$.

The following remark considering the relationship or equivalence between the two definitions of lower semicontinuity comes from [25].

**Remark 1.** If $\Xi(x)$ is a compact set, then Definition 2.2 is consistent with Definition 2.4.
Definition 2.5. $\Xi$ is said to be continuous at $x \in \mathbb{R}^m$ if it is both Hogan u.s.c. and Hogan l.s.c. at $x$.

Definition 2.6. $\Xi$ is said to be locally bounded at $x \in \mathbb{R}^m$ if there is a neighborhood $U$ of $x$ such that the set $\Xi(U)$ is bounded.

Definition 2.7. $\Xi$ is said to be uniform compact at $x \in \mathbb{R}^m$ if there is a neighborhood $U$ of $x$ such that the closure of the set $\Xi(U)$ is compact.

3. Semicontinuities of solution mappings for multiobjective program.

In this section we investigate sufficient conditions for the semicontinuities of the (weakly) efficient solution mapping for the parametric multiobjective program (1). The difficulty faced by this section is that, the weakly efficient solution mapping of the lower-level multiobjective problem is not generally lower semicontinuous, so some suitable conditions need to be found. First of all, we recall some existing classic theorems about the semicontinuity of the (weakly) efficient solution mapping.

Theorem 3.1 ([18]). $E$ is Hogan u.s.c. at $\hat{x}$ if the following conditions are satisfied:

(i) $K$ is continuous and locally bounded at $\hat{x}$;
(ii) $f$ is continuous on $\{\hat{x}\} \times K(\hat{x})$;
(iii) $M(\hat{x}) = WM(\hat{x})$.

Theorem 3.2 ([18]). $E$ is Hogan l.s.c. at $\hat{x}$ if the following conditions are satisfied:

(i) $K$ is continuous and locally bounded at $\hat{x}$;
(ii) $f$ is continuous on $O_1 \times clK(O_1)$ with a compact neighborhood $O_1$ of $\hat{x}$, and one-to-one on $\{\hat{x}\} \times K(\hat{x})$;
(iii) $N(x) \subset M(x) + \mathbb{R}_+^p$ for every $x \in O_2$, where $O_2$ is a neighborhood of $\hat{x}$.

Theorem 3.3 ([24]). $WE$ is Hogan u.s.c. at $\hat{x}$ if the following conditions are satisfied:

(i) $K$ is continuous at $\hat{x}$;
(ii) $K$ is uniform compact at $\hat{x}$ and closed for $x$ near $\hat{x}$;
(iii) $f$ is continuous on $\{\hat{x}\} \times K(\hat{x})$.

It is obvious that the weakly efficient solution mapping is generally upper semicontinuous in the sense of Hogan. But it is a hard work to obtain the lower semicontinuity. For the case where $WE(\hat{x})$ is singleton, the following theorem shows that the lower semicontinuity of $WE$ is automatically satisfied.

Theorem 3.4. If the following conditions hold, then $WE$ is Hogan l.s.c. at $\hat{x}$.

(i) $K$ is continuous at $\hat{x}$;
(ii) $K$ is uniform compact at $\hat{x}$ and closed for $x$ near $\hat{x}$;
(iii) $f$ is continuous on $\{\hat{x}\} \times K(\hat{x})$;
(iv) $WE(\hat{x})$ is singleton.

Proof. According to the Theorems 1 and 2 of [24], we can obtain this result easily. $\square$

For the case where $WE(\hat{x})$ is not singleton. In order to obtain the lower semicontinuity of the weakly efficient solution mapping, we give the following assumption.

Assumption 3.1. For any $\epsilon > 0$, there exist $\alpha \in \mathbb{R}_+^p$ and $\delta > 0$ such that for any $x \in \{\hat{x}\} + \delta$ and any $y \notin WE(x) + \epsilon$, one has that $z + \alpha \leq f(x, y)$, for at least one $z \in WM(x)$. 

Theorem 3.5. \( WE \) is Hogan l.s.c. at \( \hat{x} \) if the following conditions hold:

(i) \( K \) is Hogan l.s.c. at \( \hat{x} \);

(ii) Assumption 3.1 holds at \( \hat{x} \);

(iii) \( WM \) is Hogan u.s.c. and locally bounded at \( \hat{x} \);

(iv) \( WE \) is Hogan u.s.c. at \( \hat{x} \), and \( WE(\hat{x}) \) is compact.

Proof. We will prove the lower semicontinuity of \( WE \) at \( \hat{x} \) in the sense of Hogan by contradiction. Supposing not, since \( WE(\hat{x}) \) is compact, from Remark 1, we know that \( WE \) is also not lower semicontinuous at \( \hat{x} \) in the sense of Hausdorff. According to Definition 2.4, it follows that we can find \( \epsilon > 0 \), sequence \( \{ x_k \} \subset \mathbb{R}^m \) and \( \{ \hat{y}_k \} \subset \mathbb{R}^n \) satisfying \( x_k \to \hat{x} \), such that \( \hat{y}_k \in WE(\hat{x}) \) and \( \hat{y}_k \not\in WE(x_k) + \epsilon \). For convenience, we write \( \{ WE(x_k) + \epsilon \}^c \) to denote the complementary set of \( WE(x_k) + \epsilon \). We also denote by \( \text{cl}\{ WE(x_k) + \epsilon \}^c \) the closure of \( \{ WE(x_k) + \epsilon \}^c \). It is obvious that

\[
\{ WE(x_k) + \epsilon \}^c \subset \text{cl}\{ WE(x_k) + \epsilon \}^c \subset \{ WE(x_k) + \epsilon \}^c. \tag{4}
\]

According to the above relation (4), it is evident that

\[ \hat{y}_k \in \text{cl}\{ WE(x_k) + \epsilon \}^c \subset \{ WE(x_k) + \epsilon \}^c. \]

Since \( K \) is Hogan l.s.c. at \( \hat{x} \), from Definition 2.2, it is easy to see that we can find a subsequence \( \{ x_{k_l} \} \) of \( \{ x_k \} \), \( \{ \hat{y}_{k_l} \} \) of \( \{ \hat{y}_k \} \), \( \{ y_l \} \subset \mathbb{R}^n \) and an open neighborhood \( \{ \hat{y}_{k_l} \} + \delta_l \) of \( \hat{y}_{k_l} \), such that

\[ y_l \in \{ \hat{y}_{k_l} \} + \delta_l \subset \text{cl}\{ WE(x_{k_l}) + \epsilon \}^c \]

and \( y_l \in K(x_{k_l}) \), where \( \delta_l > 0 \), and \( \delta_l \to 0 \) as \( l \to \infty \). According to the Assumption 3.1 it is easy to know that we can find \( \alpha \in \mathbb{R}^n_+ \) and \( L > 0 \) such that for all \( l > L \) one has

\[ z_l + \alpha \preceq f(x_{k_l}, y_l), \]

for at least one \( z_l \in WM(x_{k_l}). \tag{5} \]

Without loss of generality we assume that as \( l \to \infty \), \( \hat{y}_{k_l} \to \hat{y} \), \( y_l \to \hat{y} \). Due to \( WE \) is Hogan u.s.c. at \( \hat{x} \), one has \( \hat{y} \in WE(\hat{x}) \).

On the other hand, since \( WM \) is locally bounded at \( \hat{x} \), \( \{ z_l \} \) has an accumulation point. Assume without loss of generality that \( z_l \to z \). It follows from the upper semicontinuity of \( WM \) in the sense of Hogan that \( z \in WM(\hat{x}) \). This together with (5) and by letting \( l \to \infty \) follows that, for the point \( z \in WM(\hat{x}) \) the following formula holds:

\[ z + \alpha \preceq f(\hat{x}, \hat{y}), \]

which contradicts the fact that \( \hat{y} \in WE(\hat{x}). \) This completes the proof. \( \square \)

Remark 3. The conditions of Theorem 3.5 imply the continuity of \( WE \).
To understand the Assumption 3.1 and show the reasonableness of Theorem 3.5 we consider the following example.

**Example 1.** For \((x, y) \in \mathbb{R} \times \mathbb{R}\), the objective function \(f(x, y) = (f_1(x, y), f_2(x, y))\) is defined as

\[
\begin{align*}
  f_1(x, y) &:= \begin{cases} -2y + 2x + 4, & 0 \leq y < 1 + x, \\ 2, & 1 + x \leq y < 3 + x, \\ 2y - 2x - 4, & y \geq 3 + x. \end{cases} \\
  f_2(x, y) &:= \begin{cases} -2y + 2x + 5, & y \leq 2 + x, \\ 1, & 2 + x \leq y < 4 + x, \\ 2y - 2x - 7, & y \geq 4 + x. \end{cases}
\end{align*}
\]

We define the feasible set mapping as \(K(x) := [0, 8]\). Through some simple calculations, it follows that \((2, 1)^\top \in WM(x)\), and \(WE(x) = [1 + x, 4 + x]\). Next we will show that Assumption 3.1 holds at \(\hat{x} = 0\). In fact, for any \(\epsilon > 0\), there exist \(\alpha = (\epsilon, \epsilon)^\top\) and \(\delta = \epsilon\) such that for any \(x \in \{0\}_{+\delta}\) and any \(y \notin WE(x)_{+\epsilon}\) one has

\[
\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix} < \begin{bmatrix} 2 + 2\epsilon \\ 1 + 2\epsilon \end{bmatrix} \preceq f(x, y).
\]

Furthermore, it is easy to verify that all the other conditions are valid. And it is obvious that for any sequence \(\{x_n\}\) converging to 0, and any \(\tilde{y} \in WE(0)\), there exists \(y_k = x_k + \tilde{y}\), such that \(y_k \in WE_k(x_k)\) for \(k\) large. Thus, \(WE\) is Hogan l.s.c. at \(\hat{x}\).

In what follows, we give the the following assumption, similarly to Assumption 3.1, which can be used to obtain new conditions for the lower semicontinuity of the efficient solution set mapping \(E\).

**Assumption 3.2.** For any \(\epsilon > 0\), there exist \(\alpha \in \mathbb{R}^2 \setminus \{0\}\) and \(\delta > 0\) such that for any \(x \in \{\hat{x}\}_{+\delta}\) and any \(y \notin E(x)_{+\epsilon}\), one has

\[
z + \alpha \preceq f(x, y), \text{ for at least one } z \in M(x).
\]

With the help of the preceding Assumption 3.2 we can now obtain a new lower semicontinuity of \(E\) without the second half of condition (ii) and condition (iii) of Theorem 3.2.

**Theorem 3.6.** \(E\) is Hogan l.s.c. at \(\hat{x}\) if the following conditions are satisfied:

(i) \(K\) is continuous at \(\hat{x}\);
(ii) \(K\) is locally bounded at \(\hat{x}\);
(iii) \(f\) is continuous on \(\{\hat{x}\} \times K(\hat{x})\);
(iv) \(M(\hat{x}) = WM(\hat{x})\).
(v) Assumption 3.2 holds at \(\hat{x}\), and \(E(\hat{x})\) is compact.

**Proof.** We suppose to the contrary that \(E\) is not l.s.c. at \(\hat{x}\) in the sense of Hogan. Since \(E(\hat{x})\) is compact, it follows from Remark 1 that \(E\) is also not l.s.c. at \(\hat{x}\) in the sense of Hausdorff. By Definition 2.4, there exist \(\epsilon > 0\), sequence \(\{x_k\} \subset \mathbb{R}^n\) with \(x_k \to \hat{x}\), and \(\{\tilde{y}_k\} \subset \mathbb{R}^n\) such that, \(\tilde{y}_k \in E(\hat{x})\) and \(\tilde{y}_k \notin E(x_k)_{+3\epsilon}\). Similar to the proof process of Theorem 3.5, since \(K\) is l.s.c. at \(\hat{x}\), from Definition 2.2, we know that there exist subsequence \(\{x_{k_i}\}\) of \(\{x_k\}\), \(\{\tilde{y}_{k_i}\}\) of \(\{\tilde{y}_k\}\), \(\{y_i\} \subset \mathbb{R}^n\), and an open neighborhood \(\{\tilde{y}_{k_i}\}_{+\delta_i}\) of \(\tilde{y}_{k_i}\), such that

\[
y_i \in \{\tilde{y}_{k_i}\}_{+\delta_i} \subset cl\{E(x_{k_i})_{+2\epsilon}\}^c
\]
and \( y_l \in K(x_k) \), where \( \delta_l > 0 \), and \( \delta_l \to 0 \) as \( l \to \infty \). By Assumption 3.2, there exist \( \alpha \in \mathbb{R}^2 \setminus \{0\} \) and \( L > 0 \) such that for all \( l > L \), one has

\[
\bar{x}_l + \alpha \preceq f(x_{k+1}, y_l), \text{ for at least one } z_l \in M(x_{k+1}).
\]

Without loss of generality we assume that \( y_{k+1} \to \hat{y}, \bar{y}_l \to \hat{y} \) as \( l \to \infty \). The conditions (i)-(iv) imply that \( E \) is Hogan u.s.c. at \( \hat{x} \), so \( \hat{y} \in E(\hat{x}) \).

On the other hand, conditions (ii) and (iii) imply that \( \{z_l\} \) has an accumulation point. We assume without loss of generality that \( z_l \to z \). Combining Theorem 3.1 of [18] and conditions (i), (iii) and (iv), it follows that \( M \) is Hogan u.s.c. at \( \hat{x} \), so \( z \in M(\hat{x}) \). This together with (6) follows that the following formula (7) holds by letting \( l \to \infty \).

\[
z + \alpha \preceq f(\hat{x}, \hat{y}).
\]

This result contradicts the fact that \( \hat{y} \in E(\hat{x}) \). This completes the proof. \( \square \)

**Remark 4.** The conditions of Theorem 3.6 imply the continuity of \( E \).

In order to understand the Assumption 3.2 and show the reasonableness of Theorem 3.6 we give the following example.

**Example 2.** For \( x \in \mathbb{R} \), we define the feasible set mapping as

\[
K(x) := \left\{ y = (y_1, y_2) \in \mathbb{R}^2 \left| \begin{array}{l}
-y_1 + x - 2 \leq 0, \\
y_1 - x - 2 \leq 0, \\
y_2 + 2x - 2 \leq 0, \\
y_2 - 2x - 2 \leq 0,
\end{array} \right. \right\},
\]

The objective function is defined as \( f(x, y) = (f_1(x, y), f_2(x, y))^\top \), here

\[
f_1(x, y) := \begin{cases}
(1 - x)^2 + (y_2 - 2x)^2, & (y_1 - x)^2 + (y_2 - 2x)^2 \leq \frac{1}{4}, \\
\frac{1}{2}, & (y_1 - x)^2 + (y_2 - 2x)^2 > \frac{1}{4},
\end{cases}
\]

\[
f_2(x, y) := \begin{cases}
(1 - x)^2 + (y_2 - 2x)^2, & (y_1 - x)^2 + (y_2 - 2x)^2 \leq \frac{1}{4}, \\
\frac{1}{2}, & (y_1 - x)^2 + (y_2 - 2x)^2 > \frac{1}{4}.
\end{cases}
\]

It is easy to verify that

\[
K(x) = \left\{ y \in \mathbb{R}^2 \left| \begin{array}{l}
-2 \leq y_1 - x \leq 2, \\
-2 \leq y_2 - 2x \leq 2,
\end{array} \right. \right\}.
\]

Through some simple calculations, it follows that \( WM(\hat{x}) = M(\hat{x}) = (\frac{1}{2}, \frac{1}{3})^\top \) and

\[
WE(\hat{x}) = E(\hat{x}) = \{ y \in \mathbb{R}^2 \left| (y_1 - x)^2 + (y_2 - 2x)^2 \leq \frac{1}{3} \}. \]

Obviously, \( E(0) \) has more than one element. Next, we show that for point \( \bar{x} = 0 \), the Assumption 3.2 is valid. In fact, for any \( \frac{1}{6} > \epsilon > 0 \), there exist \( \alpha = (0, \frac{2}{3})^\top \) and \( \delta = \epsilon \) such that for any \( x \in \{0\} + \delta \) and any \( y \notin E(x) + \epsilon \) one has that

\[
\left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{3}
\end{array} \right) + \left( \begin{array}{c}
0 \\
\frac{2}{3} + \epsilon
\end{array} \right) \preceq f(x, y).
\]

It is not difficult to verify that all other conditions of Theorem 3.6 also hold at \( \bar{x} \). For any sequence \( \{x_n\} \) converging to 0, and any \( (y_1, y_2) \in E(0) \), there exists \( y_1^k = x_k + y_1, \ y_2^k = 2x_k + y_2, \) such that \( y_k = (y_1^k, y_2^k) \in E_k(x_k) \) for \( k \) large. Thus, \( E \) is Hogan l.s.c. at \( \bar{x} \).
4. Stability for semivectorial bilevel programs. In this section we apply the semicontinuities of the efficient solution mapping and weakly efficient solution mapping of parametric multiobjective program to the stability of the following parametric semivectorial bilevel programming problem:

\[
\min_{x \in X(t)} \min_{y \in \psi(t, x)} F(t, x, y),
\]

where \( t \in T \subset \mathbb{R}^r \), \( F \) is a function from \( T \times \mathbb{R}^m \times \mathbb{R}^n \) to \( \mathbb{R} \), \( X \) is a set-valued mapping from \( T \) to \( \mathbb{R}^m \), and \( \psi(t, x) \) may denote the efficient solution mapping or weakly efficient solution mapping of the lower-level parametric multiobjective program:

\[
\min_y \left\{ f(t, x, y) \mid y \in K(t, x) \right\},
\]

here \( f \) is a parameterized vector-valued function from \( T \times \mathbb{R}^m \times \mathbb{R}^n \) to \( \mathbb{R}^p \) and \( K \) is a set-valued mapping from \( T \times \mathbb{R}^m \) to \( \mathbb{R}^n \). For convenience, we define the following notations:

\[
\eta : (t, x) \in T \times \mathbb{R}^m \to \min_{y \in E(t, x)} F(t, x, y),
\]

\[
\bar{\eta} : (t, x) \in T \times \mathbb{R}^m \to \min_{y \in W_{E(t, x)}} F(t, x, y),
\]

\[
\theta : t \in T \to \min_{x \in X(t)} \eta(t, x),
\]

\[
\bar{\theta} : t \in T \to \min_{x \in X(t)} \bar{\eta}(t, x).
\]

Before proceeding further, we give a general scheme for obtaining the semicontinuities of these marginal functions. Let \( \bar{\eta} \) and \( \bar{\theta} \) be the functions defined by

\[
\bar{\eta} : (t, x) \in T \times \mathbb{R}^m \to \min_{y \in \psi(t, x)} F(t, x, y),
\]

\[
\bar{\theta} : t \in T \to \min_{x \in X(t)} \bar{\eta}(t, x).
\]

4.1. Semicontinuities of the value functions. In this subsection, we consider the semicontinuities of the value functions for the parametric semivectorial bilevel programming problem. According to Theorem 6.1 of [15], it is not difficulty to check the following lemma.

**Lemma 4.1.** (a) If \( F \) is l.s.c. on \( \hat{i} \times \mathbb{R}^m \times \mathbb{R}^n \), and \( \psi \) is u.s.c. on \( \hat{i} \times \mathbb{R}^m \), then \( \eta \) is l.s.c. on \( \hat{i} \times \mathbb{R}^m \).

(b) If \( \bar{\eta} \) is l.s.c. on \( \hat{i} \times \mathbb{R}^m \), and \( X \) is u.s.c. at \( \hat{i} \), then \( \bar{\theta} \) is l.s.c. at \( \hat{i} \).

(c) If \( F \) is u.s.c. on \( \hat{i} \times \mathbb{R}^m \times \mathbb{R}^n \), and \( \psi \) is l.s.c. on \( \hat{i} \times \mathbb{R}^m \), then \( \bar{\eta} \) is u.s.c. on \( \hat{i} \times \mathbb{R}^m \).

(d) If \( \bar{\eta} \) is u.s.c. on \( \hat{i} \times \mathbb{R}^m \), and \( X \) is l.s.c. at \( \hat{i} \), then \( \bar{\theta} \) is u.s.c. at \( \hat{i} \).

Case \( \psi(t, x) = E(t, x) \). For any fixed point \( t \), we consider the semicontinuities of \( \eta \) and \( \theta \).

**Theorem 4.2.** Assume that \( F \) is l.s.c on \( \hat{i} \times \mathbb{R}^m \times \mathbb{R}^n \), and \( X \) is Hogan u.s.c. at \( \hat{i} \), and for any \( \hat{x} \in \mathbb{R}^m \), the following conditions hold:

(i) \( K \) is continuous and locally bounded at \( \hat{i}, \hat{x} \);

(ii) \( f \) is continuous on \( \hat{i} \times \{ \hat{x} \} \times K(\hat{i}, \hat{x}) \);

(iii) \( M(\hat{i}, \hat{x}) = WM(\hat{i}, \hat{x}) \).

Then, \( \eta \) is l.s.c. on \( \hat{i} \times \mathbb{R}^m \) and \( \theta \) is l.s.c. at \( \hat{i} \).
Theorem 4.3. Assume that $F$ is u.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and $X$ is Hogan l.s.c. at $\hat{t}$, and for any $\hat{x} \in \mathbb{R}^m$, the following conditions hold:

(i) $K$ is continuous and locally bounded at $(\hat{t}, \hat{x})$;

(ii) $f$ is continuous on $\{\hat{t}\} \times O_1 \times \partial K(\{\hat{t}\} \times O_1)$ with a compact neighborhood $O_1$ of $\hat{x}$, and one-to-one on $\{\hat{t}\} \times \{\hat{x}\} \times K(\hat{t}, \hat{x})$;

(iii) $N(\hat{t}, x) \subset M(\hat{t}, x) + \mathbb{R}_+^p$ for every $x \in O_2$, where $O_2$ is a neighborhood of $\hat{x}$.

Then, $\eta$ is u.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$ and $\theta$ is u.s.c. at $\hat{t}$.

Proof. From condition (i)-(iii) and Theorem 3.2, it follows that $E$ is Hogan l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Due to $F$ is u.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, it follows from (c) of Lemma 4.1 that $\eta$ is u.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. By the lower semicontinuity of $X$ at $\hat{t}$ and (d) of Lemma 4.1, it is easy to see that $\theta$ is u.s.c. at $\hat{t}$. This completes the proof. □

Remark 5. According to Theorem 3.6 it is easy to verify that, the condition (i)-(iii) of Theorem 4.3 can be replaced by the following conditions:

(i) $K$ is continuous and locally bounded at $(\hat{t}, \hat{x})$;

(ii) $f$ is continuous on $\{\hat{t}\} \times \hat{x} \times K(\hat{t}, \hat{x})$;

(iii) $M(\hat{t}, \hat{x}) = WM(\hat{t}, \hat{x})$;

(iv) Assumption 3.2 holds at $(\hat{t}, \hat{x})$, and $E(\hat{t}, \hat{x})$ is compact.

Case $\psi(t, x) = WE(t, x)$. For any fixed point $\hat{t}$, we consider the semicontinuities of $\eta$ and $\theta$.

Theorem 4.4. Assume that $F$ is l.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and $X$ is Hogan u.s.c. at $\hat{t}$, and for any $\hat{x} \in \mathbb{R}^m$ the following conditions hold:

(i) $K$ is continuous at $(\hat{t}, \hat{x})$;

(ii) $K$ is uniform compact at $(\hat{t}, \hat{x})$ and closed for $(t, x)$ near $(\hat{t}, \hat{x})$;

(iii) $f$ is continuous on $\{\hat{t}\} \times \{\hat{x}\} \times K(\hat{t}, \hat{x})$.

Then, $\eta$ is l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$ and $\theta$ is l.s.c. at $\hat{t}$.

Proof. According to condition (i)-(iii) and Theorem 3.3, it follows that the weakly efficient solutions mapping $WE$ is Hogan u.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Since $F$ is l.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, from (a) of Lemma 4.1 we know that $\bar{\eta}$ is l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Due to $X$ is u.s.c. at $\hat{t}$, from (b) of Lemma 4.1, it is evident that $\theta$ is l.s.c. at $\hat{t}$. This completes the proof. □

Theorem 4.5. Assume that $F$ is u.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and $X$ is Hogan l.s.c. at $\hat{t}$, and for any $\hat{x} \in \mathbb{R}^m$ the following conditions hold:

(i) $K$ is continuous at $(\hat{t}, \hat{x})$;

(ii) $K$ is uniform compact at $(\hat{t}, \hat{x})$ and closed for $(t, x)$ near $(\hat{t}, \hat{x})$;

(iii) $f$ is continuous on $\{\hat{t}\} \times \{\hat{x}\} \times K(\hat{t}, \hat{x})$;

(iv) $WE(\hat{t}, \hat{x})$ is singleton.

Then, $\eta$ is u.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$ and $\theta$ is u.s.c. at $\hat{t}$.

Proof. Combining conditions (i)-(iv) and Theorem 3.4, it can easily be seen that the weakly efficient solution set mapping $WE$ is Hogan l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Since $F$ is u.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, from (c) of Lemma 4.1 we know that $\bar{\eta}$ is u.s.c. on
the following conditions hold: Assume that Theorem 4.8.

(2) $t \times \mathbb{R}^m$. Due to $X$ is l.s.c. at $\hat{t}$, it follows from (d) of Lemma 4.1 that $\hat{\theta}$ is u.s.c. at $\hat{t}$. This completes the proof.

**Remark 6.** According to Theorem 3.5 it is easy to verify that, the condition (i)-(iv) of Theorem 4.5 can be replaced by the following conditions:

(i) $K$ is Hogan l.s.c. at $(\hat{t}, \hat{x})$;
(ii) Assumption 3.1 holds at $(\hat{t}, \hat{x})$;
(iii) $WM$ is Hogan u.s.c. and locally bounded at $(\hat{t}, \hat{x})$;
(iv) $WE$ is Hogan u.s.c. at $(\hat{t}, \hat{x})$, and $WE(\hat{t}, \hat{x})$ is compact.

### 4.2. Existence and stability of solutions

In this subsection we consider the existence and stability of solutions for the upper-level programming problem. First of all, we introduce the solution mapping for the upper-level problem when $\psi(t, x)$ is the (weakly) efficient solution mapping for the lower-level parametric multiobjective program. For simplicity, we let $\theta(t) > -\infty$, $\hat{\theta}(t) > -\infty$, for any $t \in T$.

$$
\Omega : t \in T \rightarrow \left\{ x \in X(t) \mid \eta(t, x) \leq \theta(t) \right\}.
$$

$$
\bar{\Omega} : t \in T \rightarrow \left\{ x \in X(t) \mid \bar{\eta}(t, x) \leq \bar{\theta}(t) \right\}.
$$

With the help of Theorem 7.3 of [15], we can obtain the following theorems easily.

**Theorem 4.6.** Assume that $F$ is l.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and $K$ is compact on $\{\hat{t}\} \times \mathbb{R}^m$, and $X$ is continuous at $\hat{t}$, and for any $\hat{x} \in \mathbb{R}^m$ conditions(i)-(iii) of Theorem 4.2 hold at $(\hat{t}, \hat{x})$. Then, $\Omega(\hat{t}) \neq \emptyset$.

**Proof.** From Theorem 4.2, it follows that $\eta$ is l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$ and $\hat{\theta}$ is l.s.c. at $\hat{t}$. Combining this with the continuity of $X$ at $\hat{t}$, it is easy to verify that $\Omega(\hat{t}) \neq \emptyset$. This completes the proof.

**Theorem 4.7.** Assume that $F$ is l.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and $X$ is continuous at $(\hat{t}, \hat{x})$, and for any $\hat{x} \in \mathbb{R}^m$, conditions(i)-(iii) of Theorem 4.4 hold. Then, $\bar{\Omega}(\hat{t}) \neq \emptyset$.

**Proof.** From Theorem 4.4, it follows that $\bar{\eta}$ is l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$ and $\bar{\theta}$ is l.s.c. at $\hat{t}$. Since $X$ is continuous at $\hat{t}$, $\Omega(\hat{t}) \neq \emptyset$. This completes the proof.

In what follows, we assume $X(t) = U$ for any $t \in T$ and $U$ is compact, and consider the stability of the solution set mapping when $\psi(t, x)$ is the (weakly) efficient solution mapping for the lower-level parametric multiobjective program.

**Theorem 4.8.** Assume that $F$ is continuous on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, and for any $\hat{x} \in \mathbb{R}^m$, the following conditions hold:

(i) $K$ is continuous and locally bounded at $(\hat{t}, \hat{x})$;
(ii) $f$ is continuous on $\{\hat{t}\} \times O_1 \times cK((\hat{t}) \times O_1)$ with a compact neighborhood $O_1$ of $\hat{x}$, and one-to-one on $\{\hat{t}\} \times \{x\} \times K(\hat{t}, \hat{x})$;
(iii) $\forall \hat{t}, \hat{x} \in M(\hat{t}, \hat{x}) + \mathbb{R}^2$ for every $x \in O_2$, where $O_2$ is a neighborhood of $\hat{x}$;
(iv) $M(\hat{t}, \hat{x}) = WM(\hat{t}, \hat{x})$.

Then, $\Omega$ is closed graph at $\hat{t}$.

**Proof.** By condition (i)-(iii) and Theorem 3.2, it follows that the efficient solution set mapping $E$ is Hogan l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Since $F$ is l.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, from Lemma 4.1 we know that $\eta$ is l.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Similarly, from condition (i), (ii) and (iv) and Theorem 3.1, it follows that the efficient solution set mapping $E$ is Hogan u.s.c. on $\{\hat{t}\} \times \mathbb{R}^m$. Since $F$ is u.s.c on $\{\hat{t}\} \times \mathbb{R}^m \times \mathbb{R}^n$, from (e) of
Lemma 4.1 we have \( \eta \) is u.s.c. on \( \{ \hat{t} \} \times \mathbb{R}^m \), that is, \( \eta \) is continuous on \( \{ \hat{t} \} \times \mathbb{R}^m \). Combining this with Theorem 3.1 of [15], it follows that \( \Omega \) is closed graph at \( \hat{t} \). This completes the proof.

**Theorem 4.9.** Assume that \( F \) is continuous on \( \{ \hat{t} \} \times \mathbb{R}^m \times \mathbb{R}^n \), and for any \( \hat{x} \in \mathbb{R}^m \), the following conditions hold:

(i) \( K \) is continuous at \( (\hat{t}, \hat{x}) \);
(ii) \( K \) is uniform compact at \( (\hat{t}, \hat{x}) \) and closed for \( (t, x) \) near \( (\hat{t}, \hat{x}) \);
(iii) \( f \) is continuous on \( \{ \hat{t} \} \times \{ \hat{x} \} \times K(\hat{t}, \hat{x}) \);
(iv) \( WM \) is Hogan u.s.c. and locally bounded at \( (\hat{t}, \hat{x}) \);
(v) \( WE(\hat{t}, \hat{x}) \) is compact, and Assumption 3.1 holds at \( (\hat{t}, \hat{x}) \).

Then, \( \bar{\Omega} \) is closed graph at \( \hat{t} \).

**Proof.** According to conditions (i)-(iii) and Theorem 3.3, it follows that \( WE \) is Hogan u.s.c. on \( \{ \hat{t} \} \times \mathbb{R}^m \). Combining this with conditions (i), (iv) and (v) imply that the set mapping \( WE \) is Hogan l.s.c. on \( \{ \hat{t} \} \times \mathbb{R}^m \). Since \( F \) is continuous on \( \{ \hat{t} \} \times \mathbb{R}^m \times \mathbb{R}^n \), it follows from (a) and (c) of Lemma 4.1 that \( \bar{\eta} \) is continuous on \( \{ \hat{t} \} \times \mathbb{R}^m \). From Theorem 3.1 of [15], it is evident that \( \bar{\Omega} \) is closed graph at \( \hat{t} \). This completes the proof.

5. **Conclusions.** As we know, the weakly efficient solution mapping of parametric multiobjective program is not generally lower semicontinuous. Since this semicontinuity is the key to consider the stability for semivectorial bilevel programming problem, we obtain this semicontinuity under a suitable assumption. Under a similar assumption, new conditions for the lower semicontinuity of the efficient solution mapping of this problem is also obtained. In this paper, we considered the stability of the solution mapping of semivectorial bilevel program. Similarly, it is not easy to obtain the lower semicontinuous of the set-valued mapping \( \Omega \) and \( \bar{\Omega} \), we will discuss these properties in our future work.

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